Abstract

Stochastic PDE eigenvalue problems are useful models for quantifying the uncertainty in several applications from the physical sciences and engineering, e.g., structural vibration analysis, the criticality of a nuclear reactor or photonic crystal structures. In this paper we present a multilevel quasi-Monte Carlo (MLQMC) method for approximating the expectation of the minimal eigenvalue of an elliptic eigenvalue problem with coefficients that are given as a series expansion of countably-many stochastic parameters. The MLQMC algorithm is based on a hierarchy of discretisations of the spatial domain and truncations of the dimension of the stochastic parameter domain. To approximate the expectations, randomly shifted lattice rules are employed. This paper is primarily dedicated to giving a rigorous analysis of the error of this algorithm. A key step in the error analysis requires bounds on the mixed derivatives of the eigenfunction with respect to both the stochastic and spatial variables simultaneously. Under stronger smoothness assumptions on the parametric dependence, our analysis also extends to multilevel higher-order quasi-Monte Carlo rules. An accompanying paper [Gilbert and Scheichl, 2022], focusses on practical extensions of the MLQMC algorithm to improve efficiency, and presents numerical results.

1 Introduction

Consider the following elliptic eigenvalue problem (EVP)

\[-\nabla \cdot (a(x, y) \nabla u(x, y)) + b(x, y) u(x, y) = \lambda(y) c(x, y) u(x, y), \quad \text{for } x \in D,\]
\[u(x, y) = 0 \quad \text{for } x \in \partial D,\]

(1.1)

where the differential operator \(\nabla\) is with respect to the physical variable \(x\), which belongs to a bounded, convex domain \(D \subset \mathbb{R}^d\) \((d = 1, 2, 3)\), and where the stochastic parameter \(y = (y_j)_{j \in \mathbb{N}} \in \Omega := [-\frac{1}{2}, \frac{1}{2}]^\mathbb{N}\),

is an infinite-dimensional vector of independently and identically distributed (i.i.d.) uniform random variables on \([-\frac{1}{2}, \frac{1}{2}]\).

The dependence of the coefficients on the stochastic parameters carries through to the eigenvalues \(\lambda(y)\), and corresponding eigenfunctions \(u(y) := u(\cdot, y)\), and as such, in this

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paper we are interested in computing statistics of the eigenvalues and of linear functionals of the corresponding eigenfunction. In particular, we would like to compute the expectation, with respect to the countable product of uniform densities, of the smallest eigenvalue $\lambda$, which is an infinite-dimensional integral defined as
\[
E_y[\lambda] = \int_{\Omega} \lambda(y) \, dy := \lim_{s \to \infty} \int_{[-\frac{1}{2}, \frac{1}{2}]^s} \lambda(y_1, y_2, \ldots, y_s, 0, 0, \ldots) \, dy_1 \, dy_2 \cdots dy_s.
\]

The multilevel Monte Carlo (MLMC) method \cite{22, 30} is a variance reduction scheme that has been successfully applied to many stochastic simulation problems. When applied to stochastic PDE problems (see, e.g., \cite{6, 8}), the MLMC method is based on a hierarchy of $L + 1$ increasingly fine finite element meshes $\{T_\ell\}_{\ell=0}^{L}$ (corresponding to a decreasing sequence of meshwidths $h_0 > h_1 > \cdots > h_L > 0$), and an increasing sequence of truncation dimensions $s_0 < s_1 < \cdots < s_L \leq \infty$. Letting the dimension-truncated FE approximation on level $\ell$ be denoted by $\lambda_{\ell} := \lambda_{h_\ell, s_\ell}$, by linearity, we can write the expectation on the finest level as
\[
E_y[\lambda_L] = E_y[\lambda_0] + \sum_{\ell=1}^{L} E_y[\lambda_{\ell} - \lambda_{\ell-1}].
\]  
(1.2)

Each expectation $E_y[\lambda_{\ell} - \lambda_{\ell-1}]$ is then approximated by an independent Monte Carlo method. Defining $u_{\ell} := u_{h_\ell, s_\ell}$ we can write a similar telescoping sum for $E_y[\mathcal{G}(u_L)]$ for any linear functional $\mathcal{G}(u)$.

Quasi-Monte Carlo (QMC) methods are equal-weight quadrature rules where the samples are deterministically chosen to be well-distributed, see \cite{13}. Multilevel quasi-Monte Carlo (MLQMC) methods, whereby a QMC quadrature rule to approximate the expectation on each level, were first developed in \cite{23} for path simulation with applications in option pricing and then later applied to stochastic PDE problems (e.g., \cite{34, 33}). For certain problems, MLQMC methods can be shown to converge faster than their Monte Carlo counterpart, and for most problems the gains from using multilevel and QMC are complementary.

In this paper we present a rigorous analysis of the error of a MLQMC algorithm for approximating the expectation of the smallest eigenvalue of (1.1) in the case where the coefficients are given by a Karhunen–Loève type series expansion. The main result proved in this paper is that under some common assumptions on the summability of the terms in the coefficient expansion, the root-mean-square error (RMSE) of a MLQMC approximation of $E_y[\lambda]$, which on each level $\ell = 0, 1, \ldots, L$ uses a randomly shifted lattice rule with $N_\ell$ points, a FE discretisation with meshwidth $h_\ell > 0$ and a fixed truncation dimension, is bounded by
\[
\text{RMSE} \lesssim h_L^2 + \sum_{\ell=0}^{L} N_{\ell}^{-1+\delta} h_\ell^2, \quad \text{for } \delta > 0,
\]  
(1.3)

with a similar result for the eigenfunction (see Theorems 3.1, 3.2 and Remark 5.1). This error bound is clearly better than the corresponding result for a MLMC method, which has $N_{\ell}^{-1+\delta}$ replaced by $N_{\ell}^{-1/2}$, and in terms of the overall complexity to achieve a RMSE less than some tolerance $\varepsilon > 0$ the total cost compared to a single level QMC approximation is reduced by a factor of $\varepsilon^2$ in spatial dimensions $d \geq 2$ (see Corollary 3.1). Under equivalent assumptions, the convergence rates in (1.3) coincide with the rates in the corresponding error bound for source problems from \cite{34, 33}. Although it is not unexpected that we are able to obtain the same convergence rates as for source problems, the analysis here is completely new and because of the nonlinear nature of eigenvalue problems presents several added difficulties not encountered previously in the analysis of source problems.
Indeed, the key intermediate step is an in-depth analysis of the mixed regularity of the eigenfunction, simultaneously in both the spatial and stochastic variables. The result, presented in Theorem 4.1, is a collection of explicit bounds on the mixed derivatives of the eigenfunction, where the derivatives are second order with respect to the spatial variable $x$ and arbitrarily high order with respect to the stochastic variable $y$. The proof of these bounds forms a substantial proportion of this paper, and requires a delicate multistage induction argument along with a considerable amount of technical analysis (see Section 4 and the Appendix). These bounds significantly extend the previous regularity results for stochastic EVPs from [1], which didn’t give any bounds on the derivatives, and [19], which gave bounds that were first-order with respect to $x$ and higher order with respect to $y$. Furthermore, many other multilevel methods require similar mixed regularity bounds for their analysis, e.g., multilevel stochastic collocation [42]. Hence, the bounds are of independent interest and open the door for further research into methods for uncertainty quantification for stochastic EVPs. In particular, we show how the mixed regularity bounds can be immediately applied to extend the analysis also to multilevel quasi-Monte Carlo methods for EVPs based on higher-order interlaced polynomial lattice rules [9, 24], following the papers [11, 12] for source problems (see Section 5.4).

The focus of this paper is the theoretical analysis of our MLQMC algorithm for EVPs. As such numerical results and practical details on how to efficiently implement the algorithm will be given in a separate paper [21].

EVPs provide a useful way to model problems from a diverse range of applications, such as structural vibration analysis [43], the nuclear criticality problem [15, 31, 44] and photonic crystal structures [14, 18, 32, 36]. More recently, interest in stochastic EVPs has been driven by a desire to quantify the uncertainty in applications such as nuclear physics [2, 5, 13, 30], structural analysis [25] and aerospace engineering [39]. The most widely used numerical methods for stochastic EVPs are Monte Carlo methods [40]. More recently stochastic collocation methods [1] and stochastic Galerkin/polynomial chaos methods [17, 45, 46] have been developed. In particular, to deal with the high-dimensionality of the parameter space, sparse and low-rank methods have been considered, see [1, 16, 26, 28, 29]. Additionally, the present authors (along with colleagues) have applied quasi-Monte Carlo methods to (1.1) and proved some key properties of the minimal eigenvalue and its corresponding eigenfunction, see [19, 20].

Although we consider the smallest eigenvalue, the MLQMC method and analysis in this paper can easily be extended to any simple eigenvalue that is well-separated from the rest of the spectrum for all parameters $y$. If the quantity of interest depends on a cluster of eigenvalues, or on the corresponding subspace of eigenfunctions, then, in principle, the method in this paper could be used in conjunction with a subspace-based eigensolver. Again, one important point for the theory would be that the eigenvalue cluster is well-separated from the rest of the spectrum, uniformly in $y$.

The structure of the paper is as follows. In Section 2 we give a brief summary of the required mathematical material. Then in Section 3 we present the MLQMC algorithm along with a cost analysis. Section 4 proves the key regularity bounds, which are then required for the error analysis in Section 5. Finally, in the appendix we give the proof of the two key lemmas from Section 5.

### 2 Mathematical background

In this section we briefly summarise the relevant material on variational EVPs, finite element methods and quasi-Monte Carlo methods. For further details we refer the reader to the references indicated throughout, or [19].
As a start, we make the following assumptions on the coefficients, which will ensure that the problem (1.1) is well-posed and admits fast convergence rates of our MLQMC algorithm. In particular, we assume that all coefficients are bounded from above and below, independently of \( x \) and \( y \).

**Assumption A1.**

1. \( a \) and \( b \) are of the form
   \[
   a(x, y) = a_0(x) + \sum_{j=1}^{\infty} y_j a_j(x) \quad \text{and} \quad b(x, y) = b_0(x) + \sum_{j=1}^{\infty} y_j b_j(x),
   \]
   where \( a_j, b_j \in L^\infty(D) \), for all \( j \geq 0 \), and \( c \in L^\infty(D) \) depend on \( x \) but not \( y \).
2. There exists \( a_{min} > 0 \) such that \( a(x, y) \geq a_{min} \), \( b(x, y) \geq 0 \) and \( c(x) \geq a_{min} \), for all \( x \in D, y \in \Omega \).
3. There exist \( p, q \in (0, 1) \) such that
   \[
   \sum_{j=1}^{\infty} \max \left( \|a_j\|_{L^\infty}, \|b_j\|_{L^\infty} \right)^p < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \|\nabla a_j\|_{L^\infty(D)}^q < \infty.
   \]

For convenience, we define \( a_{max} < \infty \) so that
\[
\max \left\{ \|a(y)\|_{L^\infty}, \|\nabla a(y)\|_{L^\infty}, \|b(y)\|_{L^\infty}, \|c\|_{L^\infty} \right\} \leq a_{max} \quad \text{for all} \quad y \in \Omega.
\] (2.2)

### 2.1 Variational eigenvalue problems

To introduce the variational form of the PDE (1.1), we let \( V := H^1_0(D) \), the first order Sobolev space of functions with vanishing trace, and equip \( V \) with the norm \( \|v\|_V := \|\nabla v\|_{L^2(D)} \). The space \( V \) together with its dual, which we denote by \( V^* \), satisfy the well-known chain of compact embeddings \( V \subset L^2(D) \subset V^* \), where the pivot space \( L^2(D) \) is identified with its own dual.

For \( v, w \in V \), define the inner products \( A(y; \cdot, \cdot), M(\cdot, \cdot) : V \times V \rightarrow \mathbb{R} \) by
\[
A(y; w, v) := \int_D a(x, y) \nabla w(x) \cdot \nabla v(x) \, dx + \int_D b(x, y) w(x) v(x) \, dx,
\]
\[
M(w, v) := \int_D c(x) w(x) v(x) \, dx,
\]
and let their respective induced norms be given by \( \|v\|_A(y) := \sqrt{A(y; v, v)} \) and \( \|v\|_M := \sqrt{M(v, v)} \). Further, let \( M(\cdot, \cdot) \) also denote the duality paring on \( V \times V^* \).

In the usual way, multiplying (1.1) by \( v \in V \) and performing integration by parts with respect to \( x \), we arrive at the following variational EVP, which is equivalent to (1.1). Find \( \lambda(y) \in \mathbb{R}, u(y) \in V \) such that
\[
A(y; u(y), v) = \lambda(y) M(u(y), v) \quad \text{for all} \quad v \in V,
\]
\[
\|u(y)\|_M = 1.
\] (2.3)

The classical theory for symmetric EVPs (see, e.g., [3]) ensures that the variational EVP (2.3) has countably many strictly positive eigenvalues, which, counting multiplicities, we label in ascending order as
\[
0 < \lambda_1(y) \leq \lambda_2(y) \leq \cdots.
\]
The corresponding eigenfunctions,

\[ u_1(y), u_2(y), \ldots, \]

can be chosen to form a basis of \( V \) that is orthonormal with respect to the inner product \( M(\cdot, \cdot) \), and, by (2.3), also orthogonal with respect to \( A(y; \cdot, \cdot) \).

**Proposition 2.1.** The smallest eigenvalue is simple for all \( y \in \Omega \). Furthermore, there exists \( \rho > 0 \), independent of \( y \), such that

\[ \lambda_2(y) - \lambda_1(y) \geq \rho \quad \text{for all } y \in \Omega. \]  

(2.4)

**Proof.** The Krein–Rutmann Theorem and [19, Proposition 2.4].

Henceforth, we will let the smallest eigenvalue and its corresponding eigenfunction be simply denoted by \( \lambda = \lambda_1 \) and \( u = u_1 \).

It is often useful to compare the eigenvalues \( \lambda_k \) to the eigenvalues of the negative Laplacian on \( D \), also with homogeneous Dirichlet boundary conditions and with respect to the standard \( L^2 \) inner product. These are denoted by

\[ 0 < \chi_1 < \chi_2 \leq \chi_3 \leq \cdots, \]  

(2.5)

and will often simply be referred to as Laplacian eigenvalues or eigenvalues of the Laplacian, without explicitly stating the domain or boundary conditions.

The following form of the Poincaré inequality will also be useful throughout this paper

\[ \|v\|_{L^2} \leq \chi_1^{-1/2} \|v\|_V, \quad \text{for } v \in V. \]  

(2.6)

It follows by the min-max representation for the Laplacian eigenvalue \( \chi_1 \).

The upper and lower bounds on the coefficients (2.2), along with the Poincaré inequality (2.6), ensure that the \( A(y) \)- and \( M \)-norms are equivalent to the \( V \)- and \( L^2 \)-norms, respectively, with

\[ \sqrt{a_{\min}} \|v\|_V \leq \|v\|_A(y) \leq \sqrt{a_{\max}} \left( 1 + \frac{1}{\chi_1} \right) \|v\|_V, \]  

(2.7)

\[ \sqrt{a_{\min}} \|v\|_{L^2} \leq \|v\|_M \leq \sqrt{a_{\max}} \|v\|_{L^2}. \]  

(2.8)

Finally, as is to be expected, for our finite element error analysis we require second-order smoothness with respect to the spatial variables, which we characterise by the space \( Z = H^2(D) \cap V \), equipped with the norm

\[ \|v\|_Z := (\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2)^{1/2}. \]

In particular, the eigenfunctions belong to \( Z \), see [19, Proposition 2.1].

### 2.2 Stochastic dimension truncation

The first type of approximation we make is to truncate the infinite dimensional stochastic domain to finitely many dimensions, which, for a truncation dimension \( s \in \mathbb{N} \), we do by simply setting \( y_j = 0 \) for all \( j > s \). The result is that the coefficients \( a \) and \( b \) now only depend on \( s \) terms. We define the following notation: \( y_s = (y_1, y_2, \ldots, y_s) \),

\[ a^s(x, y) := a_0(x) + \sum_{j=1}^{s} y_j a_j(x), \quad b^s(x, y) := b_0(x) + \sum_{j=1}^{s} y_j b_j(x), \]
and
\[ A_s(y; w, v) := \int_D a^s(x, y) \nabla w(x) \cdot \nabla v(x) \, dx + \int_D b^s(x, y) w(x) v(x) \, dx. \]

So that the truncated approximations, denoted by \((\lambda_s(y), u_s(y))\), satisfy
\[ A_s(y; u_s(y), v) = \lambda_s(y) M(u_s(y), v) \quad \text{for all } v \in V. \tag{2.9} \]

### 2.3 Finite element methods for EVPs

To begin with, we first describe the finite element (FE) spaces used to discretise the EVP (2.3). Let \(\{V_h\}_{h>0}\) be a family of conforming FE spaces of dimension \(M_h\), where each \(V_h\) corresponds to a shape regular triangulation \(\mathcal{T}_h\) of \(D\) and the index parameter \(h = \max \{\text{diam}(\tau) : \tau \in \mathcal{T}_h\}\) is called the meshwidth. Since we have only assumed that the domain \(D\) is convex and \(a \in W^{1,\infty}(D)\), throughout this paper we only consider continuous, piecewise linear FE spaces. However, under stricter conditions on the smoothness of the domain and the coefficients, one could easily extend our algorithm to higher-order FE methods. Furthermore, we assume that the number of FE degrees of freedom is of the order of \(h^{-d}\), so that \(M_h \simeq h^{-d}\). This condition is satisfied by quasi-uniform meshes and also allows for local refinement.

For \(h > 0\), each \(y \in \Omega\) yields a FE (or discrete) EVP, which is formulated as: Find \(\lambda_h(y) \in \mathbb{R}, u_h(y) \in V_h\) such that
\[ A(y; u_h(y), v_h) = \lambda_h(y) M(u_h(y), v_h) \quad \text{for all } v_h \in V_h, \tag{2.10} \]
\[ \|u_h(y)\|_M = 1. \]

The discrete EVP (2.10) has \(M_h\) eigenvalues
\[ 0 < \lambda_{1,h}(y) \leq \lambda_{2,h}(y) \leq \cdots \leq \lambda_{M_h,h}(y), \]
and corresponding eigenfunctions
\[ u_{1,h}(y), u_{2,h}(y), \ldots, u_{M_h,h}(y), \]
which are known to converge to the first \(M_h\) eigenvalues and eigenfunctions of (2.3) as \(h \to 0\), see, e.g., [5] or [19] for the stochastic case.

From [19, Theorem 2.6] we have the following bounds on the FE error for the minimal eigenpair, which we restate here because they will be used extensively in our error analysis in Section 5.

**Theorem 2.2.** Let \(h > 0\) be sufficiently small and suppose that Assumption A1 holds. Then, for all \(y \in \Omega\), \(\lambda_h\) satisfies
\[ |\lambda(y) - \lambda_h(y)| \leq C_\lambda h^2, \tag{2.11} \]
the corresponding eigenfunction \(u_h\) can be chosen such that
\[ \|u(y) - u_h(y)\|_V \leq C_u h, \tag{2.12} \]
and for \(G \in H^{-1+t}(D)\) with \(t \in [0, 1]\)
\[ |G(u(y)) - G(u_h(y))| \leq C_G h^{1+t}, \tag{2.13} \]
where \(0 < C_\lambda, C_u, C_G\) are positive constants independent of \(y\) and \(h\).
We have already seen that the minimal eigenvalue of the continuous problem (2.3) is simple for all \( y \), and that the spectral gap is bounded independently of \( y \). It turns out that the spectral gap of the FE eigenproblem (2.10) is also bounded independently of \( y \) and \( h \), provided that the FE eigenvalues are sufficiently accurate. Specifically, if

\[
h \leq \bar{h} := \sqrt{\frac{\rho}{2C\lambda}},
\]

then

\[
\lambda_{2,h}(y) - \lambda_{1,h}(y) \geq \lambda_2(y) - \lambda_1(y) - (\lambda_{1,h}(y) - \lambda_1(y)) \geq \rho - C\lambda h^2 \geq \frac{\rho}{2},
\]

where we have used the FE error estimate (2.11) and that \( \lambda_{1,h}(y) \) converges from above.

In fact, it is well known that for conforming methods all of the FE eigenvalues converge from above, so that \( \lambda_{k,h}(y) \geq \lambda_k(y) \). Then, as in [19], we can use the eigenvalues of the Laplacian (or rather their FE approximations) to bound the FE eigenvalues and eigenfunctions independently of \( y \). Hence, for \( k = 1, 2, \ldots, M \) and for all \( y \in \Omega \), there exist \( \lambda_k \) and \( u_k \), which are independent of both \( y \) and \( h \), such that

\[
\lambda_k := \frac{a_{\min}}{a_{\max}} \chi_k \leq \lambda_k(y) \leq \frac{a_{\max}}{a_{\min}} (\chi_k + 1) \leq \overline{\lambda_k},
\]

\[
\max \left\{ \|u_k(y)\|_V, \|u_{k,h}(y)\|_V \right\} \leq \frac{a_{\max}(\chi_k + 1)}{a_{\min}} \leq \overline{u_k},
\]

where \( \chi_{k,h} \) is the FE approximation of the \( k \)th Laplacian eigenvalue \( \chi_k \). In addition to converging from above, for the Laplacian eigenvalues it is known that \( \chi_k \leq \chi_{k,h} \leq \chi_k + C_k h^2 \), for some constant that is independent of \( h \) (see [7, Theorem 10.4]). As such, for \( h \) sufficiently small there exists an upper bound on \( \overline{\lambda_k} \) and \( \overline{u_k} \) so that they are independent of both \( y \) and \( h \).

To conclude this section we introduce some notation and some properties of \( V_h \) that will be useful later on. First, the spaces \( V_h \) satisfy the best approximation property:

\[
\inf_{v_h \in V_h} \|w - v_h\|_V \lesssim h\|w\|_Z, \quad \text{for all } w \in Z.
\]

Then, for \( h > 0 \), let \( P_h(y) : V \to V_h \) denote the \( A(y) \)-orthogonal projection of \( V \) onto \( V_h \), which satisfies

\[
A(y; w - P_h(y)w, v_h) = 0, \quad \text{for all } w \in V, \ v_h \in V_h,
\]

and hence also

\[
\|w - P_h(y)w\|_{A(y)} = \inf_{v_h \in V_h} \|w - v_h\|_{A(y)}.
\]

### 2.4 Quasi-Monte Carlo integration

Quasi-Monte Carlo (QMC) methods are a class of equal-weight quadrature rules that can be used to efficiently approximate an integral over the \( s \)-dimensional (translated) unit cube

\[
\mathcal{I}_s f := \int_{[-\frac{1}{2}, \frac{1}{2}]^s} f(y) \, dy.
\]

There are several different flavours of QMC rules, however in this paper we focus on randomly shifted rank-1 lattice rules. In Section 5.4 we will also briefly discuss how to
extend our method to higher-order interlaced polynomial lattice rules, see [9, 24]. For further details on different QMC methods see, e.g., [13].

A randomly shifted rank-1 lattice rule approximation to $I$ using $N$ points is

$$Q_{s,N}(\Delta)f := \frac{1}{N} \sum_{k=0}^{N-1} f(t_k - \frac{1}{2}),$$

(2.20)

where for a generating vector $z \in \mathbb{N}^s$ and a uniformly distributed random shift $\Delta \in [0,1)^s$, the points $t_k$ are given by

$$t_k = t_k(\Delta) = \left\{ \frac{kz}{N} + \Delta \right\} \quad \text{for } k = 0, 1, \ldots, N - 1.$$

Here $\{ \cdot \}$ denotes taking the fractional part of each component of a vector and $\frac{1}{N} := (\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$.

The standard spaces for analysing randomly shifted lattices rules are the so-called weighted Sobolev spaces that were introduced in [11]. Here the term “weighted” is used to indicate that the space depends on a collection of positive numbers called “weights” that model the importance of different subsets of variables and enter the space through its norm. To be more explicit, given a collection of weights $\gamma := \{\gamma_u > 0 : u \subseteq \{1, 2, \ldots, s\}\}$, let $W_{s}\gamma$ be the $s$-dimensional weighted Sobolev space of functions with square-integrable mixed first derivatives, equipped with the (unanchored) norm

$$\|f\|^2_{W_s,\gamma} = \sum_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \int_{[-\frac{1}{2}, \frac{1}{2}]^{\{1:s\}\setminus u}} \left( \int_{[-\frac{1}{2}, \frac{1}{2}]^{\{1:s\}\setminus u}} \frac{\partial |u|}{\partial y_u} f(y) \, dy \right)^2 \, dy_u. \quad (2.21)$$

Here $y_u := (y_j)_{j \in u}$ and $y_{-u} := (y_j)_{j \in \{1:s\}\setminus u}$. Note also that we have used here set notation to denote the mixed first derivatives, as this is the convention in the QMC literature. However, when we later give results for higher-order mixed derivatives we will switch to multi-index notation.

A generating vector that leads to a good randomly shifted lattice rule in practice can be constructed using the component-by-component (CBC) algorithm, or the more efficient fast CBC construction [37, 38]. In particular, it can be shown (see, e.g., [13, Theorem 5.10]) that the root-mean-square (RMS) error of a randomly shifted lattice rule using a generating vector constructed by the CBC algorithm satisfies

$$\sqrt{\mathbb{E}_\Delta [ |I f - Q_{s,N} f|^2 ]} \leq \left( \frac{1}{\varphi(N)} \sum_{\emptyset \neq u \subseteq \{1:s\}} \gamma_u \left( \frac{2\zeta(2\xi)}{(2\pi^2)^{\xi}} \right)^{|u|} \right)^{1/2} \|f\|^2_{W_s,\gamma} \quad \text{for all } \xi \in \left(\frac{1}{2}, 1\right]. \quad (2.22)$$

Here $\varphi$ is the Euler totient function, $\zeta$ is the Riemann zeta function and $\mathbb{E}_\Delta$ denotes the expectation with respect to the random shift $\Delta$. For $N$ prime one has $\varphi(N) = N - 1$ or for $N$ a power of 2 one has $\varphi(N) = N/2$, and so in both cases taking $\xi$ close to 1/2 in (2.22) results in the RMS error converging close to $O(N^{-1})$.

In practice, it is beneficial to perform several independent QMC approximations corresponding to a small number of independent random shifts, and then take the final approximation to be the average over the different shifts. In particular, let $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(R)}$ be $R$ independent uniform random shifts, and let the average over the QMC approximations with random shift $\Delta^{(r)}$ be denoted by

$$\hat{Q}_{s,N,R} f := \frac{1}{R} \sum_{r=1}^{R} Q_{s,N}(\Delta^{(r)}) f.$$
Then, the sample variance,

\[
\hat{V}[\hat{Q}_{s,N,R}] := \frac{1}{R(R-1)} \sum_{r=1}^{R} \left( \hat{Q}_{s,N,R} - Q_{s,N}(\Delta^{(r)} f) \right)^2,
\]
(2.23)
can be used as an estimate of the mean-square error of \( \hat{Q}_{s,N,R} f \).

3 MLQMC for random EVPs

Applying a QMC rule to each term in the telescoping sum (1.2), using a different number \( N_\ell \) of samples on each level, a simple MLQMC approximation of \( E_y[\lambda] \) is given by

\[
Q_{ML, L}^{(\Delta)}(\lambda) := \sum_{\ell=0}^{L} Q_{\ell}(\Delta_\ell)(\lambda_\ell - \lambda_{\ell-1}).
\]
(3.1)

Here, we define \( Q_{\ell}(\Delta_\ell) := Q_{s_\ell,N_\ell}(\Delta_\ell) \) (see (2.20)) and we treat the \( L + 1 \) independent random shifts, \( \Delta_\ell \in [0,1)^s_\ell \), as a single vector of dimension \( \sum_{\ell=0}^{L} s_\ell \), denoted by \( \Delta = (\Delta_0, \Delta_1, \ldots, \Delta_L) \). Recall also that \( \lambda_\ell = \lambda_{h_\ell,s_\ell} \) for \( \ell = 0, 1, \ldots, L \), and for simplicity denote \( \lambda_{-1} = 0 \). By using a different random shift for each level, the approximations across different levels will be statistically independent. For a linear functional \( G \in V^* \), the MLQMC approximation to \( E_y[G(u)] \) is defined in a similar fashion.

As for single level QMC rules, it is beneficial to use multiple random shifts, so that we can estimate the variance on each level. Letting \( \Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(R)} \) be \( R \) independent random shifts of dimension \( \sum_{\ell=0}^{L} s_\ell \), the shift-averaged MLQMC approximation is

\[
\hat{Q}_{ML,R}^{(\Delta)}(\lambda) := \sum_{\ell=0}^{L} \frac{1}{R} \sum_{r=1}^{R} Q_{\ell}(\Delta^{(r)}_\ell)(\lambda_\ell - \lambda_{\ell-1}).
\]
(3.2)

If in practice the parameters are not specified beforehand, then we set \( h_\ell \approx 2^{-\ell}, s_\ell \approx 2^\ell \) and use the adaptive algorithm from [23] to choose the number of QMC points \( N_\ell \).

The mean-square error (with respect to the random shift(s) \( \Delta \)) of the MLQMC estimator can be written as the sum of the bias and the total variance as follows

\[
\mathbb{E}_\Delta[|E_y[\lambda] - \hat{Q}_L(\Delta)\lambda|^2] = |E_y[\lambda - \lambda_L]|^2 + \sum_{\ell=0}^{L} \mathbb{V}_\Delta[Q_{\ell}(\lambda_\ell - \lambda_{\ell-1})].
\]
(3.3)

In the equation above, we have simplified the first term (corresponding to the bias) by the telescoping property, and the variance on each level is defined by

\[
\mathbb{V}_\Delta[Q_{\ell}(\lambda_\ell - \lambda_{\ell-1})] := \mathbb{E}_\Delta[|E_y[\lambda_{\ell} - \lambda_{\ell-1}] - Q_{\ell}(\Delta_\ell)(\lambda_\ell - \lambda_{\ell-1})|^2],
\]
where the cross-terms have vanished because randomly shifted QMC rules are unbiased. By the linearity of \( G \in V^* \), the error for the eigenfunction approximation can be decomposed in the same way.

Assuming that the total bias and the variance on each level decay at some given rates, then the decomposition of the mean-square error (3.3) gives the following abstract complexity theorems (one each, for the eigenvalue and for functionals of the eigenfunction). As is usual with the analysis of multilevel algorithms, the difficult part is to verify the assumptions on the decay of the variance and to determine the corresponding parameters. This analysis will be performed in Section 5.
Theorem 3.1 (Eigenvalues). Suppose that \( \mathbb{E}[Q_\ell(\lambda_t - \lambda_{t-1})] = \mathbb{E}_y[\lambda_t - \lambda_{t-1}] \), and that there exist positive constants \( \alpha, \alpha', \beta, \beta', \eta \) such that

1. \( |\mathbb{E}_y[\lambda - \lambda_L]| \lesssim h_\ell^{\alpha} + s_\ell^{-\alpha'} \), and

2. \( \forall \mathbb{E}[Q_\ell(\lambda_t - \lambda_{t-1})] \lesssim R^{-1} N_\ell^{-\eta} \left( h_{\ell-1}^{\beta} + s_{\ell-1}^{-\beta'} \right), \) for all \( \ell = 0, 1, 2, \ldots, L \).

Then

\[
\mathbb{E}_\Delta \left[ \left| \mathbb{E}_y[\lambda] - \hat{Q}^{\text{ML}}_{L,R}(\lambda) \right|^2 \right] \lesssim h_\ell^{\alpha} + s_\ell^{-\alpha'} + \frac{1}{R} \sum_{\ell=0}^{L} \frac{1}{N_\ell^{\eta}} \left( h_{\ell-1}^{\beta} + s_{\ell-1}^{-\beta'} \right).
\]

Theorem 3.2 (Functionals). For \( \mathcal{G} \in \mathbb{V}^* \), suppose \( \mathbb{E}_\Delta[\mathcal{G}(u_t - u_{t-1})] = \mathbb{E}_y[\mathcal{G}(u_t - u_{t-1})] \), and that there exist positive constants \( \alpha, \alpha', \beta, \beta', \eta \) such that

1. \( |\mathbb{E}_y[\mathcal{G}(u - u_L)]| \lesssim h_\ell^{\alpha} + s_\ell^{-\alpha'} \), and

2. \( \forall \mathbb{E}[Q_\ell(\mathcal{G}(u_t - u_{t-1})]) \lesssim R^{-1} N_\ell^{-\eta} \left( h_{\ell-1}^{\beta} + s_{\ell-1}^{-\beta'} \right), \) for all \( \ell = 0, 1, 2, \ldots, L \).

Then

\[
\mathbb{E}_\Delta \left[ \left| \mathbb{E}_y[\mathcal{G}(u)] - \hat{Q}^{\text{ML}}_{L,R}(\mathcal{G}(u)) \right|^2 \right] \lesssim h_\ell^{\alpha} + s_\ell^{-\alpha'} + \frac{1}{R} \sum_{\ell=0}^{L} \frac{1}{N_\ell^{\eta}} \left( h_{\ell-1}^{\beta} + s_{\ell-1}^{-\beta'} \right).
\]

Remark 3.1. In the case of a single truncation dimension, \( s_\ell = s_L \) for all \( \ell = 1, 2, \ldots, L \), the terms \( h_\ell^{\alpha} + s_\ell^{-\alpha'} \) can be dropped from the theorems above.

In Section 3 we verify that if Assumption A1 on the coefficients holds, then Assumptions M1 and M2 above are satisfied, and we give explicit values of the rates.

To better illustrate the power of our MLQMC algorithm, we give here the following complexity bound for the special case of geometrically decaying meshwidths and a fixed truncation dimension. We only give the eigenvalue result, but an analogous result holds also for linear functionals \( \mathcal{G} \in L^2(D) \). For less smooth functionals, \( \mathcal{G} \in H^{-1+t}(D) \) for \( t \in [0, 1] \), similar results hold but with slightly adjusted rates.

Corollary 3.1. Let \( 0 < \varepsilon \leq c^{-1} \) and suppose that Assumption A1 holds with \( p, q \leq 2/3 \). Also, let \( h_\ell \approx 2^{-\ell} \) with \( h_0 \) sufficiently small and let \( s_\ell = s_L \approx h_L^{2p/(2-p)} \). Finally, suppose that each \( Q_\ell \) is an \( N_\ell \)-point lattice rule corresponding to a CBC-constructed generating vector. If there exists \( 0 < \gamma < d + 1 \) such that the cost on each level \( \ell \in \mathbb{N} \) satisfies

\[
\text{cost}(Q_\ell(\lambda_t - \lambda_{t-1})) \lesssim R N_\ell \left( s_\ell h_\ell^{-d} + h_\ell^{-\gamma} \right),
\]

then, \( L \) and \( N_\ell = 2^\nu \ell \), for \( n_\ell \in \mathbb{N} \), can be chosen such that

\[
\mathbb{E}_\Delta \left[ \left| \mathbb{E}_y[\lambda] - \hat{Q}^{\text{ML}}_{L,R}(\lambda) \right|^2 \right] \lesssim \varepsilon^2
\]

and for \( \delta > 0 \)

\[
\text{cost}(\hat{Q}^{\text{ML}}_{L,R}(\lambda)) \lesssim \begin{cases} 
\varepsilon^{-1-p/(2-p)-\delta} & \text{if } d = 1, \\
\varepsilon^{-1-p/(2-p)-\delta} \log_2(\varepsilon^{-1})^{3/2+\delta} & \text{if } d = 2, \\
\varepsilon^{-d/2-p/(2-p)} & \text{if } d > 2.
\end{cases}
\]

Proof. In Section 3 (cf., Theorem 3.1 and Theorem 3.3) we verify that Assumptions M1, M2 from Theorem 3.1 hold with \( \alpha_1 = 2, \alpha' = 2/p - 1, \beta_1 = 2\alpha_1 = 4 \) and \( \eta = 2 - \delta \). The remainder of the proof follows by a standard minimisation argument as in, e.g., [23, Cor. 2].

Remark 3.2. In [24] we verify that the cost does indeed satisfy Assumption M3 with \( \gamma \approx d \), which is the same order cost as the source problem.
4 Stochastic regularity

In order for a randomly shifted lattice rule approximation to achieve the error bound (2.22), we require that the integrand belongs to \( W_{s,\gamma} \), which in turn requires bounds on the mixed first derivatives. For the eigenproblem (1.11), this means that we need to study the regularity of eigenvalues (and eigenfunctions) with respect to the stochastic parameter \( y \). In order to bound the variance on each level of our MLQMC estimator, it is necessary to also study the FE error in \( W_{s,\gamma} \) (cf. (5.4)), whereas the single level analysis in [19] only required the expected FE error. This analysis of the FE error in a stronger norm requires mixed regularity of the solution with respect to both \( x \) and \( y \) simultaneously, which has not been shown previously. The theorem below presents the required bounds for \( u \), along with the bounds from [19] with respect to \( y \) only, which are included here for completeness. Analyticity of simple eigenvalues and eigenfunctions with respect to \( y \) was shown in [1], however, explicit bounds on the derivatives were not given there and they also did not consider the mixed \( x \) and \( y \) regularity required for the ML analysis.

Although the analysis of randomly shifted lattice rules requires only the mixed first derivatives (cf., (2.21)), we also give results for arbitrary higher-order mixed derivatives. We do this because the proof technique is the same, and also since these bounds may be useful for the analysis of higher-order methods, e.g., higher-order QMC (see Section 5.4) or sparse grid rules (see, e.g., [25, 47]). As such, to simplify notation we will write mixed higher-order derivatives using multi-index notation instead of the set notation used in Section 2.4. For a multi-index \( \nu = (\nu_j)_{j\in\mathbb{N}} \) with \( \nu_j \in \mathbb{N} \cup \{0\} \) and only finitely-many nonzero components, let \( \partial_y^\nu \) denote the mixed partial differential operator where the order of derivative with respect to the variable \( y_j \) is \( \nu_j \). Define \( |\nu| := \sum_{j\geq 1} \nu_j \) and denote the set of all admissible multi-indices by \( \mathcal{F} := \{ \nu \in \mathbb{N}^\mathbb{N} : |\nu| < \infty \} \). All operations and relations between multi-indices will be performed componentwise, e.g., for \( \nu, m \in \mathcal{F} \) addition is given by \( \nu + m = (\nu_j + m_j)_{j\in\mathbb{N}} \), and \( \nu \leq m \) if and only if \( \nu_j \leq m_j \) for all \( j \in \mathbb{N} \). Similarly, for \( \nu, m \in \mathcal{F} \) and a sequence \( \beta \in \ell^\infty \) define the following shorthand for products

\[ \left( \nu \right) := \prod_{j=1}^\infty \left( \nu_j \right), \quad \beta^\nu := \prod_{j=1}^\infty \beta_j^{\nu_j}. \]

Note that since \( \nu, m \in \mathcal{F} \) have finite support these products have finitely-many terms.

**Theorem 4.1.** Let \( \nu \in \mathcal{F} \) be a multi-index, let \( \varepsilon \in (0, 1) \), and suppose that Assumption A7 holds. Also, define the sequences \( \beta = (\beta_j)_{j\in\mathbb{N}} \) and \( \overline{\beta} = (\overline{\beta}_j)_{j\in\mathbb{N}} \) by

\[ \beta_j := C_{\beta} \max \left( \|a_j\|_{L^\infty}, \|b_j\|_{L^\infty} \right), \quad (4.1) \]

\[ \overline{\beta}_j := C_{\overline{\beta}} \max \left( \|a_j\|_{L^\infty}, \|b_j\|_{L^\infty}, \|\nabla a_j\|_{L^\infty} \right), \quad (4.2) \]

where \( C_{\beta} \geq 1 \), given explicitly below in (4.6), is independent of \( y \) but depends on \( \varepsilon \).

Then, for all \( y \in \Omega \), the derivative of the minimal eigenvalue with respect to \( y \) is bounded by

\[ |\partial_y^\nu \lambda(y)| \leq \overline{\lambda} |\nu|^{1+\varepsilon} \beta^\nu, \quad (4.3) \]

and the derivative of the corresponding eigenfunction satisfies both

\[ ||\partial_y^\nu u(y)||_V \leq \overline{\pi} |\nu|^{1+\varepsilon} \beta^\nu, \quad (4.4) \]

\[ ||\partial_y^\nu u(y)||_Z \leq C |\nu|^{1+\varepsilon} \overline{\beta}^\nu, \quad (4.5) \]

where \( \overline{\lambda}, \overline{\pi} \) are as in (2.16), (2.17), respectively, and \( C \) in (4.5) is independent of \( y \) but depends on \( \varepsilon \).

Moreover, for \( h > 0 \) sufficiently small, the bounds (4.3) and (4.4) are also satisfied by \( \lambda_h(y) \) and \( u_h(y) \), respectively.
Proof. To facilitate the proof with a single constant for both sequences \( \beta \) and \( \overline{\beta} \) we define

\[
C_\beta := \frac{2 \lambda_2}{\rho} \frac{a_{\min}}{a_{\max}^2} \left( \frac{3 \rho}{\lambda} C_\epsilon + 1 \right),
\]

where \( C_\epsilon \) from [19, Lemma 3.3] is given by

\[
C_\epsilon := \frac{2^{1-\epsilon}}{1-2^{-\epsilon}} \left( \frac{c^2}{\sqrt{2\pi}} \right) ^\epsilon,
\]

which is independent of \( y \) and \( h \). Then clearly it follows that \( C_\beta \) is independent of \( y \) and \( h \). Later we will use that \( 1/a_{\min} \leq C_\beta/2 \) and \( 1/(a_{\min} \chi^{1/2}) \leq C_\beta/2 \), which both follow from the lower bounds \( C_\epsilon \geq 1 \) for all \( \epsilon \in (0,1) \) and \( \chi/\lambda \geq a_{\max}^2/a_{\min}^2 (1+1/\chi) \).

The proof for the bounds (4.3) and (4.4) is given in [19, Theorem 3.4]. If \( h \) is sufficiently small such that the FE eigenvalues resolve the spectral gap (i.e., (2.13) holds) then the bounds also hold for \( \lambda_h (y) \) and \( u_h (y) \) because \( V_h \subset V \), cf. [19, Rem. 3.2 and 3.5].

For the bound (4.5), we first prove a recursive bound on \( \| \partial^\nu u(y) \|_2 \) and then use an induction result from [12] to prove the final bound. Consider the strong form of the eigenproblem (1.1) for the pair \( (\lambda(y), u(y)) \), which, omitting the \( x \) and \( y \) dependence, is given by

\[-\nabla \cdot (a \nabla u) + bu = c \lambda u.
\]

The \( \nu \)th derivative with respect to \( y \) commutes with the spatial derivatives \( \nabla \). Thus, using the Leibniz general product rule we have

\[-\nabla \cdot (a \nabla \partial^\nu y u) + b \partial^\nu y u + \sum_{j=1}^\infty \nu_j ( -\nabla \cdot (a_j \nabla (\partial^{\nu - e_j} y u) - b_j \partial^{\nu - e_j} y u) )
\]

\[= c \sum_{m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) \partial^m y \lambda \partial^{\nu - m} y u,
\]

where \( e_j \) is the multi-index that is 1 in the \( j \)th entry and zero elsewhere. Then we can use the identity \( \nabla \cdot (\phi \psi) = \phi \nabla \cdot \psi + \nabla \phi \cdot \psi \) to simplify this to

\[a \Delta \partial^\nu y u = -\nabla a \cdot \nabla \partial^\nu y u + b \partial^\nu y u - c \sum_{m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) \partial^m y \lambda \partial^{\nu - m} y u
\]

\[+ \sum_{j=1}^\infty \nu_j ( -a_j \Delta \partial^{\nu - e_j} y u - \nabla a_j \cdot \nabla \partial^{\nu - e_j} y u + b_j \partial^{\nu - e_j} y u ).
\]

Since \( a \geq a_{\min} > 0 \); \( a, a_j \in W^{1,\infty} \) and \( b, b_j \in L^\infty \) for all \( j \in \mathbb{N} \); and \( \partial^m y u \in V \) for all \( m \in \mathcal{F} \), it follows by induction on \( |\nu| \) that \( \Delta \partial^\nu y u \in L^2 \). This allows us to take the \( L^2 \)-norm of both sides, which, after using the triangle inequality and the bounds in (2.2), gives the following recursive bound for \( \Delta \partial^\nu y u \)

\[
\| \Delta \partial^\nu y u \|_{L^2} \leq \frac{\| \nabla a \|_{L^\infty}}{a_{\min}} \| \partial^\nu y u \|_V + \frac{\| b \|_{L^\infty}}{a_{\min}} \| \partial^\nu y u \|_{L^2}
\]

\[+ \frac{\| c \|_{L^\infty}}{a_{\min}} \sum_{m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) \| \partial^m y \lambda \| \| \partial^{\nu - m} y u \|_{L^2} + \sum_{j=1}^\infty \nu_j \frac{\| a_j \|_{L^\infty}}{a_{\min}} \| \Delta \partial^{\nu - e_j} y u \|_{L^2}
\]

\[+ \frac{1}{a_{\min}} \sum_{j=1}^\infty \nu_j ( \| \nabla a_j \|_{L^\infty} \| \partial^{\nu - e_j} y u \|_V + \| b_j \|_{L^\infty} \| \partial^{\nu - e_j} y u \|_{L^2} ).
\]
Adding \(\|\partial_y^{\nu} u\|_{L^2}\) to both sides and then using the definition of \(\mathcal{B}_j\), we can write this bound in terms of the \(Z\)-norm as
\[
\|\partial_y^{\nu} u\|_Z \leq \|\partial_y^{\nu} u\|_{L^2} + \|\Delta \partial_y^{\nu} u\|_{L^2} \leq \sum_{j=1}^{\infty} \nu_j \mathcal{B}_j \|\partial_y^{\nu - \epsilon_j} u\|_Z + B_\nu, 
\] (4.7)
where we used that \(1/\alpha_{\min} \leq C_\beta\), and then defined
\[
B_\nu := \frac{\|\nabla a\|_{L^\infty}}{\alpha_{\min}} \|\partial_y^{\nu} u\|_V + \frac{\|c\|_{L^\infty}}{\alpha_{\min}} \sum_{m \leq \nu} \left( \nu \frac{\nu m}{\lambda} \right) \|\partial_y^{\nu - m} u\|_{L^2}
\]
\[
\left( \frac{\|b\|_{L^\infty}}{\alpha_{\min}} + 1 \right) \|\partial_y^{\nu} u\|_{L^2} + \frac{1}{\alpha_{\min}} \sum_{j=1}^{\infty} \nu_j \left( \|\nabla a_j\|_{L^\infty} \|\partial_y^{\nu - \epsilon_j} u\|_V + \|b_j\|_{L^\infty} \|\partial_y^{\nu - \epsilon_j} u\|_{L^2} \right)
\]
Now, the sum on the right of (4.7) only involves lower-order versions of the object we are interested in bounding (namely, \(\|\partial_y^{\nu} u\|_Z\)), whereas the terms in \(B_\nu\) only involve derivatives that can be bounded using one of (4.3) or (4.4).

We bound the remaining \(L^2\)-norms in \(B_\nu\) by the Poincaré inequality (2.6) to give
\[
B_\nu \leq \left( \frac{\|\nabla a\|_{L^\infty}}{\alpha_{\min}} + \frac{\|b\|_{L^\infty}}{\alpha_{\min}} + \frac{\alpha_{\min}}{\sqrt{\chi_1}} \right) \|\partial_y^{\nu} u\|_V + \frac{\|c\|_{L^\infty}}{\alpha_{\min} \sqrt{\chi_1}} \sum_{m \leq \nu} \left( \nu \frac{\nu m}{\lambda} \right) \|\partial_y^{\nu - m} u\|_V
\]
\[
\frac{1}{\alpha_{\min}} \sum_{j=1}^{\infty} \nu_j \left( \|\nabla a_j\|_{L^\infty} + \frac{\|b_j\|_{L^\infty}}{\sqrt{\chi_1}} \right) \|\partial_y^{\nu - \epsilon_j} u\|_V
\]
where in the last inequality we have bounded the \(L^\infty\)-norms on the second line using (2.2), and then simplified. Then, substituting in the bounds (4.3) and (4.4) gives
\[
B_\nu \leq \frac{\alpha_{\max}}{\alpha_{\min}} \left( 1 + \frac{2}{\sqrt{\chi_1}} \right) \left( \pi |\nu|^{1+\epsilon} \beta^{\nu} + \sum_{m \leq \nu} \left( \nu \frac{\nu m}{\lambda} \right) \pi |m|^{1+\epsilon} \beta^{m} \cdot \pi |\nu - m|^{1+\epsilon} \beta^{m} \right)
\]
\[
\frac{1}{\alpha_{\min}} \sum_{j=1}^{\infty} \nu_j \left( \|\nabla a_j\|_{L^\infty} + \frac{\|b_j\|_{L^\infty}}{\sqrt{\chi_1}} \right) \pi (|\nu| - 1)^{1+\epsilon} \beta^{\nu - \epsilon_j}
\]
\[
= \frac{\alpha_{\max}}{\alpha_{\min}} \left( 1 + \frac{2}{\sqrt{\chi_1}} \right) \beta^{\nu} \left( |\nu|^{1+\epsilon} + \sum_{m \leq \nu} \left( \nu \frac{\nu m}{\lambda} \right) |m|^{1+\epsilon} |\nu - m|^{1+\epsilon} \right)
\]
\[
+ \frac{\pi}{\alpha_{\min}} \left( |\nu| - 1 \right)^{1+\epsilon} \sum_{j=1}^{\infty} \nu_j \left( \|\nabla a_j\|_{L^\infty} + \frac{\|b_j\|_{L^\infty}}{\sqrt{\chi_1}} \right) \beta^{\nu - \epsilon_j}.
\]
Using the fact that \((1 + \chi^{-1/2})/a_{\min} \leq C_{\beta}\) and also that clearly \(\beta_{j} \leq \bar{\beta}_{j}\), we have

\[
B_{\nu} \leq \frac{a_{\max}}{a_{\min}} \left( 1 + \frac{2}{\sqrt{\lambda}1} \right) \bar{\beta}' \left( |\nu|!^{1+\epsilon} + \chi \sum_{m \leq \nu} \left( \frac{\nu}{m} \right) |m|!^{1+\epsilon} |\nu - m|!^{1+\epsilon} \right)
\]

\[+ \frac{\nu}{a_{\min}} (|\nu| - 1)!^{1+\epsilon} \sum_{j=1}^{\infty} \nu_{j} (1 + \chi^{-1/2}) \max (\|\nabla a_{j}\|_{L^{\infty}}, \|b_{j}\|_{L^{\infty}}) \bar{\beta}'^{-\epsilon},\]

\[
\leq \frac{a_{\max}}{a_{\min}} \left( 1 + \frac{2}{\sqrt{\lambda}1} \right) \bar{\beta}' \left( |\nu|!^{1+\epsilon} + \chi \sum_{m \leq \nu} \left( \frac{\nu}{m} \right) |m|!^{1+\epsilon} |\nu - m|!^{1+\epsilon} \right)
\]

\[+ \nu |\nu|^{1+\epsilon} \bar{\beta}'.\]

The sum that remains can be bounded using the same strategy as in the proof of [19, Lemma 3.4], as follows

\[\sum_{m \leq \nu} \left( \frac{\nu}{m} \right) |m|!^{1+\epsilon} |\nu - m|!^{1+\epsilon} = 2|\nu|!^{1+\epsilon} + \sum_{k=1}^{|\nu|-1} k!^{1+\epsilon} (|\nu| - k)!^{1+\epsilon} \sum_{m \leq \nu, |m| = k} \left( \frac{\nu}{m} \right)
\]

\[= 2|\nu|!^{1+\epsilon} + \sum_{k=1}^{|\nu|-1} k!^{1+\epsilon} (|\nu| - k)!^{1+\epsilon} \left( \frac{|\nu|}{k} \right)
\]

\[= |\nu|!^{1+\epsilon} \left( 2 + \sum_{k=1}^{\infty} \left( \frac{|\nu|}{k} \right)^{-}\epsilon \right)
\]

\[\leq |\nu|!^{1+\epsilon} \left( 2 + \frac{2^{1-\epsilon}}{1 - 2^{-\epsilon}} \left( \frac{e^{2}}{\sqrt{2\pi}} \right)^{\epsilon} \right). \tag{4.8}\]

where for the inequality on the last line we have used [19, Lemma 3.3].

Hence, \(B_{\nu}\) is bounded above by

\[B_{\nu} \leq C_{B} |\nu|!^{1+\epsilon} \bar{\beta}',\]

where

\[C_{B} := \pi \left[ \frac{a_{\max}}{a_{\min}} \left( 1 + \frac{2}{\sqrt{\lambda}1} \right) (1 + \chi(2 + C_{\epsilon})) + 1 \right] < \infty\]

is clearly independent of \(y\) and \(\nu\).

Now we can bound the recursive formula (4.7) using the bound above on \(B_{\nu}\), which gives

\[\|\partial_{y}^{\nu} u\|_{Z} \leq \sum_{j=1}^{\infty} \nu_{j}\bar{\beta}_{j} \|\partial_{y}^{\nu-\epsilon_{j}} u\|_{Z} + C_{B} |\nu|!^{1+\epsilon} \bar{\beta}'.\]

Finally, by [12, Lemma 4] we can bound this above by

\[\|\partial_{y}^{\nu} u\|_{Z} \leq \sum_{m \leq \nu} \left( \frac{\nu}{m} \right) |m|!^{\bar{\beta} m} C_{B} |\nu - m|!^{1+\epsilon} \bar{\beta}'^{-m}
\]

\[= C_{B} \bar{\beta}' \sum_{m \leq \nu} \left( \frac{\nu}{m} \right) |m|!^{\bar{\beta} m} C_{B} |\nu - m|!^{1+\epsilon} \leq C_{B} (2 + C_{\epsilon}) |\nu|!^{1+\epsilon} \bar{\beta}',\]

where to obtain the final result we have again used (4.8).
5 Error analysis

We now provide a rigorous analysis of the error for (3.1), which we do by verifying the assumptions from Theorems 3.1 and 3.2.

Recall that we use the shorthand $\lambda_\ell := \lambda_{h_\ell,s_\ell}$ for the dimension-truncated FE approximation of the minimal eigenvalue on level $\ell$, whereas $\lambda_s$ denotes the minimal eigenvalue of the dimension-truncated version of the continuous EVP (2.9). The bias (the first term) in (3.3) can be bounded by the triangle inequality to give

$$|E_y[\lambda - \lambda_{h_\ell,s_\ell}]| \leq |E_y[\lambda - \lambda_s]| + |E_y[\lambda_s - \lambda_{h_\ell,s_\ell}]|,$$

and similarly for the eigenfunction. Now, both terms on the right can be bounded above using the results from the single level algorithm. Explicitly, for $G \in H^{-1+t}(D)$ with $t \in [0,1]$ using Theorem 4.1 from [19] and then Theorem 2.2 gives the bounds

$$|E_y[\lambda - \lambda_s]| \lesssim s_L^{-2/p+1} + h_L^2,$$

$$|E_y[G(u - u_L)]| \lesssim s_L^{-2/p+1} + h_L^{1+t},$$

with constants independent of $s_L$ and $h_L$. That is, we have verified Assumptions M1 from both Theorems 3.1 and 3.2 with $\alpha = 2$, $\alpha_G = 1 + t$ and $\alpha' = 2/p - 1$.

For the variance terms on each level in (3.3) (alternatively to verify Assumption M2), we must study the QMC error of the differences $\lambda_\ell - \lambda_{\ell-1}$. Since $\lambda_\ell - \lambda_{\ell-1} \in W_{s_\ell,\gamma}$ for all $\ell = 0, 1, 2, \ldots, L$ and each QMC rule $Q_\ell$ uses CBC-constructed generating vector $z_\ell$, by (2.22) we have the upper bound

$$\mathbb{V}_\Delta(Q_\ell(\lambda_\ell - \lambda_{\ell-1})) \leq \frac{C_{\xi,\ell}^2}{\varphi(N_\ell)^{1/\xi}}||\lambda_\ell - \lambda_{\ell-1}||^2_{W_{s_\ell,\gamma}} \text{ for all } \xi \in (1/2, 1],$$

where $C_{\xi,\ell}$ is the constant from (2.22) with $s = s_\ell$. Thus, in Assumption M2 we can take $\eta = 1/\xi \in [1, 2)$ and for the other parameters we must study the norm of the difference on each level.

By the triangle inequality, we can separate truncation and FE components of the error

$$||\lambda_\ell - \lambda_{\ell-1}||_{W_{s_\ell,\gamma}} \leq ||\lambda_{s_\ell} - \lambda_{s_{\ell-1}}||_{W_{s_{\ell},\gamma}} + ||\lambda_{s_{\ell}} - \lambda_{h_{s_\ell,s_\ell}}||_{W_{s_{\ell},\gamma}} + ||\lambda_{s_{\ell-1}} - \lambda_{h_{s_{\ell-1},s_{\ell-1}}}||_{W_{s_{\ell-1},\gamma}},$$

In contrast to the single level setting [19], here we need to study the truncation and FE errors in the weighted QMC norm (2.21) instead of simply the expected truncation and FE errors. Each term will be handled separately in the subsections that follow.

The key ingredient in the error analysis are the bounds of the derivatives of the minimal eigenvalue and its eigenfunction that were given in Section 4.

5.1 Estimating the FE error

As a first step towards bounding the FE errors in the $W_{s,\gamma}$-norm, we bound their derivatives with respect to $y$, which are given below in Theorem 5.1. The bulk of the work to bound the FE error in $W_{s,\gamma}$ is dedicated to proving these regularity bounds. As in Theorem 4.1 we also present bounds on higher-order mixed derivatives instead of simply the mixed first derivatives required in the $W_{s,\gamma}$ norm.

The strategy for proving these bounds is similar to the proof [19, Lemma 3.4], except in the current multilevel setting we need to bound the derivatives of the FE errors of the eigenvalue and eigenfunction, in addition to the derivatives of the eigenvalue and
eigenfunction themselves. First, we differentiate variational equations involving the errors to obtain a recursive formula for each of the eigenvalue and eigenfunction errors, and then prove the bounds by induction on the cardinality of $|\nu|$. Once we have proved the bound for the eigenfunction in (5.11), the result for any functional $G(u(y))$ in (5.12) follows by a duality argument. Throughout the proofs in this section we will omit the $x$ and $y$ dependence. Note also that throughout we must explicitly track the constants to ensure that they are independent of $y$ and $h$, but also to make sure that in both of the inductive steps the constants are not growing, since this could interfere with the summability of $\hat{\beta}$.

Also, the results in this section are all shown for $h$ sufficiently small, where here sufficiently small means that the FE eigenvalues resolve the spectral gap. Explicitly, we assume that $h \leq h_\epsilon$ (see (2.14) and (2.15)) for some $h_\epsilon > 0$ that is independent of $y$. This ensures that the condition that $h$ is sufficiently small (i.e., $h \leq h_\epsilon$) is also independent of $y$.

In the following key lemma, we bound the derivative of the difference between the eigenfunction and its projection $P_h u(y)$ onto $V_h$, which is not equal to the FE eigenfunction $u_h(y)$, but is easier to handle. The proof relies on the new mixed regularity estimate (4.5).

**Lemma 5.1.** Let $\nu \in \mathcal{F}$ be a multi-index, let $h > 0$ be sufficiently small and suppose that Assumption $A[1]$ holds. Then

$$\|\partial^\nu_y u - P_h \partial^\nu_y u\|_V \leq C_P h |\nu|^{1+\epsilon} \beta^\nu,$$  \hspace{1cm} (5.5)

where $\beta$ is as defined in (4.2), and $C_P$ is independent of $y$, $h$ and $\nu$.

**Proof.** Using the equivalence of the $V$-norm and the induced $A$-norm in (2.7), along with the $A$-orthogonality of the projection and the best approximation property (2.18), we get

$$\|\partial^\nu_y u - P_h \partial^\nu_y u\|_V \leq \sqrt{\frac{a_{\text{max}}}{a_{\text{min}}} \left( 1 + \frac{1}{\chi_1} \right)} \inf_{v_h \in V_h} \|\partial^\nu_y u - v_h\|_V \leq \sqrt{\frac{a_{\text{max}}}{a_{\text{min}}} \left( 1 + \frac{1}{\chi_1} \right)} C h \|\partial^\nu_y u\|_Z \leq C_P h |\nu|^{1+\epsilon} \beta^\nu,$$

where for the last inequality we have used the bound (4.5). The final constant $C_P$ is independent of $y$, $h$ and also $\nu$. \hfill $\Box$

The three recursive formulae presented in the next two lemmas are the key to the induction proof to bound the derivatives of the FE error. The general strategy is to differentiate variational equations involving the FE errors. However, the proofs are quite long and technical, and as such are deferred to the Appendix.

**Lemma 5.2.** Let $\nu \in \mathcal{F}$ be a multi-index, let $h > 0$ be sufficiently small and suppose that Assumption $A[1]$ holds. Then, for all $y \in \Omega$, the following two recursive bounds hold

$$|\partial^\nu_y (\lambda - \lambda_h)| \leq C_1 \left( h|\nu|^{1+\epsilon} \beta^\nu + \sum_{j=1}^{\infty} \nu_j \beta_j |\partial^\nu_y - \epsilon_j (u - u_h)|_V \right) + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \frac{\nu}{m} \right) m!^{1+\epsilon} \beta^m \left( \|\partial^\nu_y - m(u - u_h)\|_V + |\partial^\nu_y - m(\lambda - \lambda_h)| \right)$$  \hspace{1cm} (5.6)
where $\beta, \bar{\beta}$ are defined in (4.1), (4.2), respectively, and $C_1, C_{II}$ are independent of $y, h$ and $\nu$.

**Lemma 5.3.** Let $\nu \in F$ be a multi-index, let $h > 0$ be sufficiently small and suppose that Assumption $A_1$ holds. Then, for all $y \in \Omega$,

$$\|\partial_y^\nu(u - u_h)\|_V \leq C_{III} \left( h|\nu|! + \sum_{j=1}^{\infty} \nu_j \beta_j \|\partial_y^{\nu - e_j}(u - u_h)\|_V \right)$$

$$+ \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \right) \|m|! + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \right) \|m|! + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \right) \|m|! + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \right) \|m|!$$

where $\beta, \bar{\beta}$ are defined in (4.1), (4.2), respectively, and $C_{III}$ is independent of $y, h$ and $\nu$.

The astute reader may now ask, why do we need both the bounds in (5.7) and (5.8) on the derivative of the eigenvalue error? The reason is that the upper bound in (5.7) depends only on derivatives with order strictly less than $\nu$, whereas the bound in (5.8) depends on $\partial_y^\nu(u - u_h)$. Hence the inductive step for the eigenfunction (see (5.11) below) only works with (5.6). On the other hand, (5.6) cannot be used for the inductive step for the eigenvalue result (see (5.10) below), because it will only result in a bound of order $O(h)$. Hence, the second bound (5.7) is required to maintain the optimal rate of $O(h^2)$ for the eigenvalue error.

We now have the necessary ingredients to prove the following bounds on the derivatives of the FE error.

**Theorem 5.1.** Let $\nu \in F$ be a multi-index, let $h > 0$ be sufficiently small and suppose that Assumption $A_1$ holds. Define the sequence $\hat{\beta} = (\hat{\beta}_j)_{j \in \mathbb{N}}$ by

$$\hat{\beta}_j := C_{\hat{\beta}} \max \left( \|a_j\|_{L^\infty}, \|b_j\|_{L^\infty}, \|\nabla a_j\|_{L^\infty} \right),$$

where $C_{\hat{\beta}}$, given explicitly below in (5.14), is independent of $y, h$ and $j$. Then

$$\|\partial_y^\nu(\lambda(y) - \lambda_h(y))\| \leq C_1 |\nu|! \hat{\beta}^\nu h^2,$$

$$\|\partial_y^\nu(u(y) - u_h(y))\|_V \leq C_2 |\nu|! \hat{\beta}^\nu h,$$

and for $G \in H^{-1+\nu}(D)$

$$\|\partial_y^\nu G(u(y) - u_h(y))\| \leq C_3 |\nu|! \hat{\beta}^\nu h^{1+\nu},$$

with $C_1, C_2, C_3$ all independent of $y, h$ and $\nu$. 

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Proof. Throughout we use the convention that 0! = 1. Then, due to the error bound (2.11) for the FE eigenvalue error, the base case of the induction (ν = 0) for the eigenvalue result (5.10) holds provided C_1 ≥ C_λ. Thus, let

\[ C_1 := \max \{ C_\lambda, C_\Pi C_u^2(2 + C_c)(4 + C_c) \}. \]  

Similarly, defining C_2 = C_u the base case of the induction for (5.11) also clearly holds due to (2.12).

For the inductive step, let ν be such that |ν| ≥ 1 and assume that (5.10) and (5.11) hold for all m with |m| < |ν|. Now, since the recursive bound for the eigenvalue (5.7) still depends on a term of order ν, whereas the recursive bound for the eigenfunction (5.8) only depends on strictly lower order terms, for our inductive step to work we first prove the result (5.11) for the eigenfunction, before proving the result (5.10) for the eigenvalue.

Substituting the induction assumptions (5.10) and (5.11) for |m| < |ν| into (5.8) gives

\[ \| \partial_y^{\nu}(u - u_h) \|_V \leq C_{\PiIII} \left( |\nu|^{1 + \epsilon} \tilde{\beta}^\nu h + \sum_{j=1}^{\infty} \nu_j \beta_j C_2 |\nu| - 1)^{1 + \epsilon} \tilde{\beta}^{\nu - j} h \right. \]

\[ + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \atop m \right) |m|^{1 + \epsilon} \beta^m (C_2 + C_1 h) |\nu - m|^{1 + \epsilon} \tilde{\beta}^{\nu - m} h \]

\[ \leq \beta^\nu h C_{\PiIII} \left( \left[ \frac{\beta}{\tilde{\beta}} \right]^{\nu} + \frac{C_2 C_\beta}{C_{\beta}} \right) |\nu|^{1 + \epsilon} \]

\[ + \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \atop m \right) |\nu - m|^{1 + \epsilon} |m|^{1 + \epsilon} (C_1 h + C_2) \left( \frac{C_\beta}{\tilde{\beta}} \right)^{|m|} \]

where we have rescaled each product by using the definitions of β, \tilde{\beta} and \hat{\beta} in (4.11), (4.12) and (5.9). The constants can be simplified by defining

\[ \hat{\beta} := \frac{1}{C_2} C_\beta \max(1, C_{\PiIII}) \left[ 1 + C_2 + C_c(C_1 h + C_2) \right], \]  

(5.14)

which is independent of y, h and ν. This guarantees that C_β/\hat{C}_β ≤ 1, and thus since |m|, |\nu - m| ≥ 1, we have the bound

\[ \| \partial_y^{\nu}(u - u_h) \|_V \leq \hat{\beta}^\nu h C_{\PiIII} \frac{C_\beta}{\hat{C}_\beta} \left( (1 + C_2) |\nu|^{1 + \epsilon} + (C_1 h + C_2) \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \nu \atop m \right) |\nu - m|^{1 + \epsilon} |m|^{1 + \epsilon} \right) \]

\[ \leq C_{\PiIII} \frac{C_\beta}{\hat{C}_\beta} \left[ 1 + C_2 + C_c(C_1 h + C_2) \right] |\nu|^{1 + \epsilon} \hat{\beta}^\nu h \leq C_2 |\nu|^{1 + \epsilon} \hat{\beta}^\nu h, \]

where we have used [19] Lemma 3.3 to bound the sum from above by C_c|\nu|^{1 + \epsilon} (see also (4.8)), as well as (5.14) to give the final result.

For the inductive step for the eigenvalue, we substitute the result (5.11), which has just been shown to hold for all multi-indices of order up to and including |ν|, into (5.7).
and then simplify, to give

$$|\nu^\nu(\nu - \nu h)| \leq C_{II}(C_2)^2 \beta^\nu h^2 \left( \sum_{\nu \leq m} \left( \frac{\nu}{m} \right) |\nu - m|^{1+\epsilon} |m|^{1+\epsilon} \right. $$

$$+ \sum_{j=1}^\infty \nu_j \sum_{\nu \leq -\nu_j} \left( \frac{\nu - \nu_j}{m} \right) |\nu - m|^{1+\epsilon} |m|^{1+\epsilon} $$

$$+ \sum_{m \leq \nu} \nu |\nu|^{1+\epsilon} \sum_{k \leq m} \left( \frac{m}{k} \right) |m - k|^{1+\epsilon} |k|^{1+\epsilon} \right),$$

where we have used the fact that $\beta_j \leq \hat{\beta}_j$. The sums can again be bounded using (4.8) (using it twice for the double sum on the last line), to give

$$|\nu^\nu(\nu - \nu h)| \leq C_{II}(C_2)^2 \beta^\nu h^2 (2 + C_\epsilon)(4 + C_\epsilon) |\nu|^{1+\epsilon},$$

which, with $C_1$ as defined in (5.13) and $C_2 = C_n$, gives our desired result (5.10).

The final result for the derivative of the error of the linear functional $G(u)$ (5.12) follows by considering the same dual problem (A.16) as in [19]. But instead, here we let $w = \nu^\nu(u - u_h)$ and then use the upper bound (5.11) in the last step.

We can now simply substitute these bounds on the derivatives of the FE error into (5.4), in order to bound the FE component of the error. For the second and third term in (5.4), in the case of the eigenvalue, this gives

$$\|\lambda_s(y) - \lambda_{s_h}(y)\|_{W_{s,\gamma}} \leq h^2 \left( C_1 \sum_{u \subseteq \{1:s\}} \frac{|u|^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^{2} \right)^{1/2}.$$  

(5.15)

Similar results hold for $u(y)$ and $G(u(y))$.

To ensure that the constant on the RHS of (5.15), and the constants in the bounds that follow, are independent of the dimension, for $\xi \in (\frac{1}{2},1]$ to be specified later, we will choose the weights $\gamma$ by

$$\gamma_j = \max \left( \hat{\beta}_j, \beta_j^{p/q} \right), \quad \gamma_u = \left( (|u| + 3)^{2(1+\epsilon)} \prod_{j \in u} (2 \pi)^{\frac{\epsilon}{2(2\xi)}} \gamma_j^2 \right)^{1/(1+\xi)},$$  

(5.16)

where $p, q$ are the summability parameters from Assumption A1.3 so that $(\gamma_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{R})$.

### 5.2 Estimating the truncation error

It remains to estimate the first term in (5.4) — the truncation error.

**Theorem 5.2.** Suppose that Assumption A1 holds and let $s, s \in \mathbb{N}$ with $s > \tilde{s}$. Additionally, suppose that the weights $\gamma$ are given by (5.16), then

$$\|\lambda_s - \lambda_{\tilde{s}}\|_{W_{s,\gamma}} \lesssim \tilde{s}^{-1/p+1/q} \left( \sum_{u \subseteq \{1:\tilde{s}\}} \frac{(|u| + 3)^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^{2} \right)^{1/2},$$  

(5.17)

with the constant independent of $\tilde{s}$ and $s$. 

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Proof. Since $\lambda_s$ is analytic we can expand it as a Taylor series about 0 in the variables \( \{y_{\bar{s}+1}, \ldots, y_{s}\} \):

\[
\lambda_s(y_s) = \lambda_s(y_{\bar{s}}; 0) + \sum_{i=\bar{s}+1}^s y_i \int_0^1 \frac{\partial}{\partial y_i} \lambda_s(y_{\bar{s}}; ty_{\{\bar{s}+1:s\}}) \, dt,
\]

where we use the notation \( y_s = (y_1, y_2, \ldots, y_s) \), \( (y_{\bar{s}}; 0) = (y_1, y_2, \ldots, y_{\bar{s}}, 0, \ldots 0) \) and \( (y_{\bar{s}}; t y_{\{\bar{s}+1:s\}}) = (y_1, y_2, \ldots, y_{\bar{s}}, ty_{\bar{s}+1}, ty_{\bar{s}+2}, \ldots, ty_s) \).

Since $\lambda_{\bar{s}}(y_{\bar{s}}) = \lambda_s(y_{\bar{s}}; 0)$ (this is simply different notation for the same object), this can be rearranged to give

\[
\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}}) = \sum_{i=\bar{s}+1}^s y_i \int_0^1 \frac{\partial}{\partial y_i} (y_{\bar{s}}; ty_{\{\bar{s}+1:s\}}) \, dt.
\]

Let $u \subseteq \{1, 2, \ldots, \bar{s}\}$, then differentiating (5.18) with respect to $y_u$ gives

\[
\frac{\partial |u|}{\partial y_u} (\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})) = \sum_{i=\bar{s}+1}^s y_i \int_0^1 \frac{\partial |u|+1}{\partial y_u, (i)} \lambda_s(y_{\bar{s}}; ty_{\{\bar{s}+1:s\}}) \, dt.
\]

Taking the absolute value, using the triangle inequality and the fact that $|y_j| \leq 1/2$, we have the upper bound

\[
\left| \frac{\partial |u|}{\partial y_u} (\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})) \right| \leq \frac{1}{2} \sum_{i=\bar{s}+1}^s \int_0^1 \left| \frac{\partial |u|+1}{\partial y_u, (i)} \lambda_s(y_{\bar{s}}; ty_{\{\bar{s}+1:s\}}) \right| \, dt.
\]

Now, substituting in the upper bound on the derivative of $\lambda_s$ from [19 Lemma 3.4, equation (3.6)] gives

\[
\left| \frac{\partial |u|}{\partial y_u} (\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})) \right| \leq \frac{\lambda}{2} \sum_{i=\bar{s}+1}^s (|u|+1)! \beta_i \prod_{j \in u} \beta_j
\]

\[
= \frac{\lambda}{2} \left( \sum_{i=\bar{s}+1}^s \beta_i \right) (|u|+1)! \prod_{j \in u} \beta_j,
\]

with $\beta_j$ as in [19].

Letting $u \subseteq \{1, 2, \ldots, s\}$ with $u \cap \{\bar{s}+1, \bar{s}+2, \ldots, s\} \neq \emptyset$, the derivative $\lambda_s - \lambda_{\bar{s}}$ is simply

\[
\left| \frac{\partial |u|}{\partial y_u} (\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})) \right| = \left| \frac{\partial |u|}{\partial y_u} \lambda_s(y_s) \right| \leq \frac{\lambda}{2} |u|! \prod_{j \in u} \beta_j,
\]

where we have again used the upper bound [19 equation (3.6)].

We now bound the norm (2.21) of $\lambda_s - \lambda_{\bar{s}}$ in $W_{s, \gamma}$. Splitting the sum over $u \subseteq \{1, 2, \ldots\}$ by whether $u$ contains any of $\{\bar{s}+1, \bar{s}+2, \ldots, s\}$, we can write

\[
\|\lambda_s - \lambda_{\bar{s}}\|_{W_{s, \gamma}}^2 = \sum_{u \subseteq \{1: \bar{s}\}} \frac{1}{\gamma_u} \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right] [u]} \left( \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right] [\bar{s}-|u|]} \left| \frac{\partial |u|}{\partial y_u} [\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})] \right| \, dy_{\bar{s}-|u|} \right)^2 \, dy_u + \sum_{u \subseteq \{1: \bar{s}\} \atop u \cap \{\bar{s}+1:s\} \neq \emptyset} \frac{1}{\gamma_u} \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right] [u]} \left( \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right] [\bar{s}-|u|]} \left| \frac{\partial |u|}{\partial y_u} [\lambda_s(y_s) - \lambda_{\bar{s}}(y_{\bar{s}})] \right| \, dy_{\bar{s}-|u|} \right)^2 \, dy_u.
\]

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Substituting in the bounds (5.19) and (5.20) then yields
\[ \|\lambda_s - \lambda_s\|_{\mathcal{W}_{s,\gamma}} \leq \left( \sum_{i=s+1}^{s} \beta_i \right)^2 \sum_{u \subseteq \{1:s\}} \frac{|u|^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \]
\[ + \sum_{u \subseteq \{1:s\} \neq \emptyset} \frac{|u|^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \] \]
\[ \cdot \left( \sum_{u \subseteq \{1:s\}} \frac{|u|^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \right)^{1/2} \tag{5.21} \]

Next, we can bound the sum over \( i \) in (5.21) by the whole tail of the sum, which can then be bounded using (19) eq. (4.7), to give
\[ \sum_{i=s+1}^{s} \beta_i \leq \sum_{i=s+1}^{\infty} \beta_i \leq \min\left( \frac{p}{1-p}, 1 \right) \|\beta\|_{\ell^p} \tilde{s}^{-(1/p-1)} \tag{5.22} \]
Then for weights given by (5.16), following the proof of [34, Theorem 11] we can bound
\[ \sum_{u \subseteq \{1:s\} \neq \emptyset} \frac{|u|^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \leq \tilde{s}^{-2(1/p-1/q)} \sum_{u \subseteq \{1:s\}} \frac{(|u| + 3)^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \tag{5.23} \]
with a constant that is independent of \( \tilde{s} \) and \( s \).

Since \( 1/q > 1 \), after substituting the bounds (5.22) and (5.23) into (5.21) we obtain the final result (5.17).

### 5.3 Final error bound

In the previous two sections we have successfully bounded the FE and truncation error in the \( \mathcal{W}_{s,\gamma} \) norm, now these bounds can simply be substituted into (5.4) to bound the variance on each level.

**Theorem 5.3.** Let \( L \in \mathbb{N} \), let \( 1 = h_{-1} > h_0 > h_1 > \cdots > h_L > 0 \) with \( h_0 \) sufficiently small, let \( 1 = s_{-1} = s_0 \leq s_1 \leq \cdots \leq s_L \), and suppose that Assumption A7 holds with \( p < q \). Also, let each \( Q_\ell \) be a lattice rule using \( N_\ell = 2^{n_\ell}, n_\ell \in \mathbb{N} \), points corresponding to a CBC-constructed generating vector with weights \( \gamma \) given by (5.15). Then, for all \( \ell = 0, 1, 2, \ldots L \),
\[ \forall \Delta [Q_\ell (\lambda_\ell - \lambda_{\ell-1})] \leq C_1 N_\ell^{-\eta} (h_{\ell-1}^4 + s_{\ell-1}^{-2(1/p-1/q)}) \tag{5.24} \]
and, for \( G \in H^{-1+t}(D) \) with \( 0 \leq t \leq 1 \),
\[ \forall \Delta [Q_\ell (G(u_\ell) - G(u_{\ell-1}))] \leq C_2 N_\ell^{-\eta} (h_{\ell-1}^{2(1+t)} + s_{\ell-1}^{-2(1/p-1/q)}) \tag{5.25} \]
where, for \( 0 < \delta < 1 \),
\[ \eta = \begin{cases} 2 - \delta & \text{if } q \in (0, \frac{2}{3}], \\ \frac{2}{q} - 1 & \text{if } q \in (\frac{2}{3}, 1). \end{cases} \]

The second term in (5.24) and (5.25) can be dropped if \( s_\ell = s_L \), for \( \ell = 1, 2, \ldots L \).

**Proof.** We prove the result for the eigenvalue, since the eigenfunction result follows analogously. For \( \ell \geq 1 \), substituting the bounds (5.15) and (5.17) into (5.4) gives
\[ \|\lambda_\ell - \lambda_{\ell-1}\|_{\mathcal{W}_{s,\gamma}} \leq C \left( \sum_{u \subseteq \{1:s_{\ell}\}} \frac{(|u| + 3)^{2(1+\epsilon)}}{\gamma_u} \prod_{j \in u} \beta_j^2 \right)^{1/2} (h_{\ell-1}^2 + s_{\ell-1}^{-(1/p-1/q)}), \tag{5.26} \]
where we have simplified by using that \( h_\ell < h_{\ell-1} \), \( s_{\ell-1} < s_\ell \), \( \beta_j \leq \hat{\beta}_j \), and also merged all constants into a generic constant \( C \), which may depend on \( \epsilon \).

Substituting the bound above into (5.3), then using that \( N_\ell = 2^n \epsilon \) and thus \( \varphi(N_\ell) = N_\ell/2 \), the variance on level \( \ell \) can be bounded by

\[
\mathbb{V} \Delta [Q_\ell(\lambda_\ell - \lambda_{\ell-1})] \leq C_{\ell, \gamma, \xi} N_\ell^{-1/\xi} (s_{\ell-1}^{-2(1/p-1/q)} + h_\ell^{-1}).
\] (5.26)

The constant is given by

\[
C_{\ell, \gamma, \xi} := C 2^{1/\xi} \left( \sum_{u \subseteq \{1:s_\ell\}} (|u| + 3) \sum_{j=1}^{2j} \prod_{\theta \neq u \subseteq \{1:s_\ell\}} \gamma_u \right) \left( \sum_{\emptyset \neq u \subseteq \{1:s_\ell\}} \gamma_u \left( \frac{2\zeta(2\xi)}{(2\pi^2)^{\xi}} \right)^{|u|} \right)^{1/\xi}.
\]

For \( \ell = 0 \) we can similarly substitute (4.3) into the CBC bound (2.22), and then since \( h_{-1} = s_{-1} = 1 \) and \( \beta_j \leq \hat{\beta}_j \) it follows that (5.26) also holds for \( \ell = 0 \) with the constant \( C_{0, \gamma, \xi} \) as above.

All that remains to be shown is that this constant can be bounded independently of \( s_{\ell-1} \) and \( s_\ell \). To this end, substituting the formula (5.16) for \( \gamma_u \) and using the fact that \( \hat{\beta}_j \leq \gamma_j \) then simplifying, we can bound \( C_{\ell, \gamma, \xi} \) above by

\[
C_{\ell, \gamma, \xi} \leq C \left( \sum_{|u| < \infty} (|u| + 3)! \prod_{j \in u} \gamma_j^{2j} \frac{2^j \zeta(2\xi)}{(2\pi^2)^{\xi}} \right)^{1/\xi}.
\]

where again \( C \) is a generic constant, which may depend on \( \epsilon \).

We now choose the exponents \( \xi \) and \( \epsilon \) so that the sum \( S_\xi \) is finite. For \( 0 < \delta < 1 \), let

\[
\xi = \begin{cases} 
\frac{1}{2} - \frac{\delta}{q} & \text{if } q \in (0, \frac{2}{3}) , \\
\frac{1}{2} - \frac{\delta}{q} & \text{if } q \in (\frac{2}{3}, 1) ,
\end{cases} \quad \text{and} \quad \epsilon = \frac{1 - \xi}{4\xi} > 0.
\] (5.27)

With this choice of \( \xi \) we have \( (4\xi - q - 3q\xi)/(1 - \xi) \geq q \) for any \( q \in (0, 1) \), and so

\[
\sum_{j=1}^{\infty} \gamma_j^{4\xi - q - 3q\xi} < \infty.
\] (5.28)

Then define the sequence

\[
\alpha_j = \left( 1 + \sum_{i=1}^{\infty} \gamma_i^q \right)^{-1} \gamma_i^q, \quad \text{so that} \quad \sum_{j=1}^{\infty} \alpha_j < 1.
\] (5.29)

Substituting in our choice (5.27) for \( \epsilon \), and multiplying and dividing each term by the product of \( \alpha_j^{(1+3\xi)/(2(1+\xi))} \), we can write

\[
S_\xi = \sum_{|u| < \infty} (|u| + 3)! \prod_{j \in u} \alpha_j^{\frac{1+3\xi}{2(1+\xi)}} \left( \prod_{j \in \emptyset} \gamma_j^{\frac{2j}{\xi}} \alpha_j^{\frac{1+3\xi}{2(1+\xi)}} \frac{2\zeta(2\xi)}{(2\pi^2)^{\xi}} \right)^{1/\xi}.
\]
Applying Hölder’s inequality with exponents \(2(1+\xi)/(1+3\xi) > 1\) and \(2(1+\xi)/(1-\xi) > 1\) gives

\[
S_\xi \leq \left( \sum_{|u|<\infty} (|u|+3)! \prod_{j \in \mathbb{N}} \alpha_j \right)^{\frac{1+3\xi}{1+6\xi}} \left( \sum_{|u|<\infty} \prod_{j \in \mathbb{N}} \gamma_j^{\frac{4\xi}{1-\xi}} \alpha_j^{-\frac{1+3\xi}{1-\xi}} \left( \frac{2\zeta(2\xi)}{(2\pi^2)^\xi} \right)^{\frac{2-\xi}{1+6\xi}} \right)^{\frac{2-\xi}{2(1+\xi)}}
\]

\[
\leq \left[ 6 \left( 1 - \sum_{j=1}^{\infty} \alpha_j \right)^{-4} \right]^{\frac{1+3\xi}{2(1+\xi)}} \cdot \exp \left[ \frac{2-\xi}{2(1+\xi)} \frac{2\zeta(2\xi)}{(2\pi^2)^\xi} \pi^{\frac{2-\xi}{1+6\xi}} \left( 1 + \sum_{j=1}^{\infty} \gamma_j^{q} \right)^{\frac{1+3\xi}{1-\xi}} \sum_{j=1}^{\infty} \gamma_j^{\frac{4\xi}{1-\xi} - \frac{3\xi}{1+6\xi}} \right],
\]

where we have used \([35, \text{Lemma 6.3}].\) From \((5.28)\) and \((5.29)\) it follows that \(S_\xi < \infty,\) and so \(C_{\ell,T,\xi}\) can be bounded independently of \(s_\ell.\) Finally, letting \(\eta = 1/\xi\) for \(\xi\) as in \((5.27)\) gives the desired result with a constant independent of \(s_\ell.\)

\[\square\]

**Remark 5.1.** Hence, we have verified that Assumptions M2 from Theorems 3.1 and 3.2 hold with \(\beta_\lambda = 2\alpha_\lambda = 4, \beta_\nu = 2\alpha_\nu = 2(1+t), \beta' = 1/p + 1/q,\) and \(\eta\) as given above.

The upper bounds in Theorem 5.1, 5.1.5 and Theorem 5.2 are the same as the corresponding bounds from the MLQMC analysis for the source problem (see [34, Theorems 7, 8, 11]), the only differences are in the values of the constants and in the extra \(1 + \epsilon\) factor in the exponent of \(|u|!\). As such the final variance bounds in Theorem 5.3 also coincide with the bounds for the source problem from [34] for all \(q < 1.\) The only difference is that our result does not hold for \(q = 1,\) whereas the results for the source problem do.

### 5.4 Extension to higher-order QMC

As mentioned earlier, the bounds on the higher-order derivatives that we proved in Section 4 imply higher order methods can also be used for the quadrature component of our ML algorithm, which will provide a faster convergence rate in \(N_\ell.\) We now provide a brief discussion of how to extend our ML algorithm, and the error analysis, to higher-order QMC (HOQMC) rules. From an algorithm point of view, one can simply use HOQMC points instead of lattice rules for the quadrature rules for MLHOQMC approximation by \(Q_\ell^{MLHO}\). To extend the error analysis to HOQMC we can again use a general framework as in Theorems 5.1 and 3.2. We stress that the difficult part is to verify the assumptions, and in particular to show the required mixed higher-order derivative bounds that we have already proved in Theorem 4.1. The remainder of the analysis then follows the same steps as in the previous sections with only slight modifications to handle the higher-order norm as in [12], where ML-HOQMC methods were applied to PDE source problems. As such, we don’t present the full details here but only an outline.

A HOQMC rule is an equal-weight quadrature rule of the form \((2.20)\) that can achieve faster than \(1/N\) convergence for sufficiently smooth integrands. A popular class of deterministic HOQMC rules are *interlaced polynomial lattice rules*, see [9, 24] and [11, 12] for their application to PDE source problems. Loosely speaking, a polynomial lattice rule is a QMC rule similar to a lattice rule, except the points are generated by a vector of polynomials instead of integers, the number of points \(N\) is a prime power and the points are not randomly shifted. Higher order convergence in \(s\) dimensions is then achieved by taking a polynomial lattice rule in a higher dimension, \(\nu \cdot s\) for \(\nu \in \mathbb{N}_+\), and cleverly interlacing the digits across the dimensions of each \((\nu s)\)-dimensional point to produce an \(s\)-dimensional point. The factor \(\nu \in \mathbb{N}_+\) is called the *interlacing order* and it determines
the convergence rate. Good interlaced polynomial lattice rules can also be constructed by a CBC algorithm. See [24] for the full details.

Following [12], for $\nu \in \mathbb{N}$ and $1 \leq r \leq \infty$ we introduce the Banach space $W^{p,r}_{s,\gamma}$, which is a higher-order analogue of the first-order space $W_{s,\gamma}$, with the norm

$$\|f\|_{W^{p,r}_{s,\gamma}} = \max_{u \subseteq \{1:s\}} \frac{1}{\gamma_u} \left(\sum_{v \subseteq N} \sum_{\tau_u, \nu \in \{1:s\}^{u \setminus v}} \int_{[-\frac{1}{2}, \frac{1}{2}]^{|v|}} \left| \int_{[-\frac{1}{2}, \frac{1}{2}]^{-|v|}} \partial_y^{(\nu \cdot \tau_u \cdot \nu \cdot 0)} f(y) \, dy \right|^r \, dy \right)^{1/r}. \quad (5.30)$$

Here $(\nu, \tau_u \cdot \nu, 0) \in \mathcal{F}$ is the multi-index with $j$th entry given by $\nu$ if $j \in \nu$, $\tau_j$ if $j \in u \setminus \nu$ and 0 otherwise. For $f \in W^{p,r}_{s,\gamma}$, an order $\nu$ interlaced polynomial lattice rule can be constructed using a CBC algorithm such that the (deterministic) error converges at a rate $N^{-\eta}$ for $1 \leq \eta < \nu$ (see [11, Theorem 3.10]).

Let $\nu_p = \lfloor 1/p \rfloor + 1$ for $p \leq 1$ as in Assumption A1 and $1 \leq r \leq \infty$, then it follows from (4.3) that $\lambda_s \in W^{p,r}_{s,\gamma}$ for all $s$. Hence, the error of a single level QMC approximation of $E_y[\lambda_s]$ using an order $\nu_p$ interlaced polynomial lattice rule will converge as $N^{-1/p}$. Similarly, the ML analysis can be extended to show that a ML-HOQMC method achieves higher order convergence in $N_{\ell}$, where in this case we choose the interlacing factor to be $\nu_q = \lfloor 1/q \rfloor + 1$ for $q \leq 1$ as in Assumption A1. Indeed, (5.10) implies that the bound (5.15) can easily be extended to $W^{p,r}_{s,\gamma}$ and (4.3) implies that (5.17) can also be extended to $W^{p,r}_{s,\gamma}$ for all $s$. In both cases, the sums over $u$ on the right hand sides need to be updated to account for the form of (5.30), but the exponents of $h$ and $s$ remain the same. Hence, by following the proof of Theorem 5.3 it can be shown that the following deterministic analogue of the variance bound (5.30) holds for interlaced polynomial lattice rules.

**Theorem 5.4.** Suppose that Assumption A1 holds with $p < q < 1$. For $\ell \in \mathbb{N}$, let $Q_{\ell}^\text{HO}$ be an interlaced polynomial lattice rule, constructed using a CBC algorithm with $N_{\ell}$ a prime power number of points and interlacing factor $\nu_q = \lfloor 1/q \rfloor + 1$. Then $Q_{\ell}^\text{HO}$ satisfies

$$\left| Q_{\ell}^\text{HO} (\lambda_{\ell} - \lambda_{\ell-1}) \right| \lesssim N_{\ell}^{-1/q} (h_{\ell}^2 + s_{\ell}^{-1/p+1/q}), \quad (5.31)$$

where the implied constant is independent of $h_{\ell}$, $s_{\ell}$ and $N_{\ell}$.

The fact that the implied constant in (5.31) is independent of $s_{\ell}$ can be shown by following similar arguments as in [12] using a special form of $\gamma_u$ called smoothness-driven, product and order-dependent (SPOD) weights, as introduced in [11, eq. (3.17)]. Thus the following deterministic version of Theorem 5.3 holds for the error of the ML-HOQMC approximation.

**Theorem 5.5.** Suppose that Assumption A1 holds with $p < q < 1$, let $L \in \mathbb{N}$ and for $\ell = 0, 1, \ldots, L$ let $Q_{\ell}^\text{HO}$ be an interlaced polynomial lattice rule as in Theorem 5.4. Then the multilevel HOQMC approximation $Q_{L}^\text{MLHO}$ with quadrature rule $Q_{\ell}^\text{HO}$ on each level satisfies

$$\left| E_y[\lambda] - Q_{L}^\text{MLHO} (\lambda) \right| \lesssim h_{L}^2 + s_{L}^{-2/p+1} + \sum_{\ell=0}^{L} N_{\ell}^{-1/q} (h_{\ell}^2 + s_{\ell}^{-1/p+1/q}), \quad (5.32)$$

where the implied constant is independent of $h_{\ell}, s_{\ell}$ and $N_{\ell}$ for all $\ell = 0, 1, \ldots, L$.

Similar arguments can also be used to obtain an error bound with the same convergence rates for $Q_{L}^\text{MLHO} (G(u))$, i.e., for the approximation of the expected value of smooth functionals of the eigenfunction.
6 Conclusion

We have presented a MLQMC algorithm for approximating the expectation of the eigenvalue of a random elliptic EVP, and then performed a rigorous analysis of the error. The theoretical results clearly show that for this problem the MLQMC method exhibits better complexity than both single level MC/QMC and MLMC. In the companion paper [21], we will present numerical results that also verify this superior performance of MLQMC in practice. In that paper, we will in addition present novel ideas on how to efficiently implement the MLQMC algorithm for EVPs.

Other interesting avenues for future research would be to consider non-self adjoint EVPs, e.g., convection-diffusion problems, or to use the multi-index MC framework from, e.g., [10,27] to separate the FE and dimension truncation approximations on each level. In principle, the algorithm studied in this paper can also be applied to the lognormal setting as in, e.g., [33], i.e., where each coefficient is the exponential of a Gaussian random field, by using QMC rules for integrals on unbounded domains. However, in this case, the difficulty for both single level and multilevel QMC is that the coefficients are no longer uniformly bounded from above and below. As such, it is possible that the spectral gap, $\lambda_2(y) - \lambda_1(y)$, becomes arbitrarily small for certain parameter values. Since all aspects of the method (the stochastic derivative bounds, the FE error, the performance of the eigenvalue solver etc.) depend inversely on the spectral gap, then both the method and the theory fail if the gap becomes arbitrarily small. The technique for bounding the spectral gap in [19,20] fails in this case because the stochastic parameters belong to an unbounded domain. On the other hand, we conjecture that the spectral gap only becomes small with low probability, and so probabilistic arguments may be able to be used to bound the gap from below. This is again another example of the differences between stochastic EVPs and source problems, and such analysis would make for interesting future work.

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References

[1] R. Andreev and Ch. Schwab. Sparse tensor approximation of parametric eigenvalue problems. In I. G. Graham et al., editor, Numerical Analysis of Multiscale Problems, Lecture Notes in Computational Science and Engineering, pages 203–241. Springer, Berlin, 2012.

[2] M. N. Avramova and K. N. Ivanov. Verification, validation and uncertainty quantification in multi-physics modeling for nuclear reactor design and safety analysis. Prog. Nucl. Energy, 52:601—614, 2010.

[3] D. A. F. Ayres, M. D. Eaton, A. W. Hagues, and M. M. R. Williams. Uncertainty quantification in neutron transport with generalized polynomial chaos using the method of characteristics. Ann. Nucl. Energy, 45:14—28, 2012.

[4] I. Babuška and J. Osborn. Finite element-Galerkin approximation of eigenvalues and eigenvectors of selfadjoint problems. Math. Comp., 52:275–297, 1989.

[5] I. Babuška and J. Osborn. Eigenvalue problems. In P. G. Ciarlet and J. L. Lions, editor, Handbook of Numerical Analysis, Volume 2: Finite Element Methods (Part 1), pages 641–787. Elsevier, Amsterdam, 1991.
[6] A. Barth, Ch. Schwab, and N. Zollinger. Multilevel Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numer. Math.*, 119:123–161, 2011.

[7] D. Boffi. Finite element approximation of eigenvalue problems. *Acta Numer.*, 19:1–120, 2010.

[8] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup. Multilevel Monte Carlo methods and applications to PDEs with random coefficients. *Comput. Visual. Sci.*, 14:3–15, 2011.

[9] J. Dick. Walsh spaces containing smooth functions and quasi-Monte Carlo methods of arbitrary high order. *SIAM J. Numer. Anal.*, 46:1519–1553, 2008.

[10] J. Dick, M. Feischl, and Ch. Schwab. Improved efficiency of a multi-index FEM for computational uncertainty quantification. *SIAM J. Numer. Anal.*, 57:1744–1769, 2019.

[11] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and Ch. Schwab. Higher-order QMC Petrov–Galerkin discretization for affine parametric operator equations with random field inputs. *SIAM J. Numer. Anal.*, 52:2676–2702, 2014.

[12] J. Dick, F. Y. Kuo, Q. T. Le Gia, and Ch. Schwab. Multilevel higher order QMC Petrov–Galerkin discretization for affine parametric operator equations. *SIAM J. Numer. Anal.*, 54:2541–2568, 2016.

[13] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: The quasi-Monte Carlo way. *Acta Numer.*, 22:133–288, 2013.

[14] D. C. Dobson. An efficient method for band structure calculations in 2D photonic crystals. *J. Comput. Phys.*, 149:363–376, 1999.

[15] J. J. Duderstadt and L. J. Hamilton. *Nuclear Reactor Analysis*. John Wiley & Sons, New York, NY, 1976.

[16] H. C. Elman and T. Su. Low-rank solution methods for stochastic eigenvalue problems. *SIAM J. Sci. Comp.*, 41:A2657–A2680, 2019.

[17] R. Ghanem and D. Ghosh. Efficient characterization of the random eigenvalue problem in a polynomial chaos decomposition. *Int. J. Numer. Meth. Engng*, 72:486–504, 2007.

[18] S. Giani and I. G. Graham. Adaptive finite element methods for computing band gaps in photonic crystals. *Numer. Math.*, 121:31–64, 2012.

[19] A. D. Gilbert, I. G. Graham, F. Y. Kuo, R. Scheichl, and I. H. Sloan. Analysis of quasi-Monte Carlo methods for elliptic eigenvalue problems with stochastic coefficients. *Numer. Math.*, 142:863–915, 2019.

[20] A. D. Gilbert, I. G. Graham, R. Scheichl, and I. H. Sloan. Bounding the spectral gap for an elliptic eigenvalue problem with uniformly bounded stochastic coefficients. In D. Wood et al., editor, *2018 MATRIX Annals*, pages 29–43. Springer, Cham, 2020.

[21] A. D. Gilbert and R. Scheichl. Multilevel quasi-Monte Carlo methods for random elliptic eigenvalue problems II: Efficient algorithms and numerical results. *Preprint, arXiv:2103.03407*, 2022.
[22] M. B. Giles. Multilevel Monte Carlo path simulation. *Oper. Res.*, 56:607–617, 2008.

[23] M. B. Giles and B. Waterhouse. Multilevel quasi-Monte Carlo path simulation. In *Advanced Financial Modelling, Radon Series on Computational and Applied Mathematics*, pages 165–181. De Gruyter, New York, 2009.

[24] T. Goda and J. Dick. Construction of interlaced scrambled polynomial lattice rules of arbitrary high order. *Found. Comput. Math.*, 15:1245–1278, 2015.

[25] M. Griebel, H. Harbrecht, and D. Multerer. Multilevel quadrature for elliptic parametric artifical differential equations in case of polygonal approximations of curved domains. *SIAM J. Numer. Anal.*, 58:684–705, 2020.

[26] L. Grubišić, H. Hakula, and M. Laaksonen. Stochastic collocation method for computing eigenspaces of parameter-dependent operators. *Preprint, arXiv:1909.11608*, 2019.

[27] A. L. Haji-Ali, F. Nobile, and R. Tempone. Multi-index Monte Carlo: when sparsity meets sampling. *Numer. Math.*, 312:767–806, 2016.

[28] H. Hakula, V. Kaarnioja, and M. Laaksonen. Approximate methods for stochastic eigenvalue problems. *Appl. Math. Comput.*, 267:664–681, 2015.

[29] H. Hakula and M. Laaksonen. Asymptotic convergence of spectral inverse iterations for stochastic eigenvalue problems. *Numer. Math.*, 142:577–609, 2019.

[30] S. Heinrich. Multilevel Monte Carlo methods. In *Multigrid Methods, Vol. 2179 of Lecture Notes in Computer Science*, pages 58–67. Springer, Berlin, 2001.

[31] E. Jamelota and P. Ciarlet Jr. Fast non-overlapping Schwarz domain decomposition methods for solving the neutron diffusion equation. *J. Comput. Phys.*, 241:445–463, 2013.

[32] P. Kuchment. The mathematics of photonic crystals. *SIAM, Frontiers of Applied Mathematics*, 22:207–272, 2001.

[33] F. Y. Kuo, R. Scheichl, Ch. Schwab, I. H. Sloan, and E. Ullmann. Multilevel quasi-Monte Carlo methods for lognormal diffusion problems. *Math. Comp.*, 86:2827–2860, 2017.

[34] F. Y. Kuo, Ch. Schwab, and I. H. Sloan. Multi-level quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients. *Found. Comput. Math*, 15:411–449, 2015.

[35] F. Y. Kuo, Ch. Schwab, and I. H. Sloan. Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.*, 50:3351–3374, 2012.

[36] R. Norton and R. Scheichl. Planewave expansion methods for photonic crystal fibres. *Appl. Numer. Math.*, 63:88–104, 2012.

[37] D. Nuyens and R. Cools. Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. *Math. Comp.*, 75:903–920, 2006.

[38] D. Nuyens and R. Cools. Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points. *J. Complexity*, 22:4–28, 2006.
Here, we give the proofs of the recursive bounds on the derivatives of the FE error from Section 5.1 (Lemmas 5.2 and 5.3), which were key to the inductive steps in the proofs of the explicit bounds in Theorem 5.1. Throughout we omit the \( x \) and \( y \) dependence.

**Proof of Lemma 5.2 (eigenvalue bounds).** Let \( v = v_h \in V_h \) in the variational eigenproblem \( (2.3) \), and then subtract the FE eigenproblem \( (2.10) \), with the same \( v_h \), to give the following variational relationship between the two FE errors

\[
\mathcal{A}(u - u_h, v_h) = \lambda M(u - u_h, v_h) + (\lambda - \lambda_h) M(u_h, v_h),
\]

(A.1)

which holds for all \( v_h \in V_h \).

Differentiating (A.1) using the Leibniz general product rule, gives the following recursive formula for the \( \nu \)th derivatives of the eigenvalue and eigenfunction errors

\[
0 = \mathcal{A}(\partial_y^{\nu}(u - u_h), v_h) - \lambda M(\partial_y^{\nu}(u - u_h), v_h) - (\lambda - \lambda_h) M(\partial_y^{\nu}u_h, v_h)
\]

\[
+ \sum_{j=1}^{\infty} \nu_j \left( \int_D a_j \nabla \left[ \partial_y^{-\nu_j}(u - u_h) \right] \cdot \nabla v_h + \int_D b_j \left[ \partial_y^{-\nu_j}(u - u_h) \right] v_h \right)
\]

\[
- \sum_{m \leq \nu \atop m \neq \nu} \left[ \partial_y^{\nu - m} \lambda M(\partial_y^m(u - u_h), v_h) + \partial_y^{\nu - m}(\lambda - \lambda_h) M(\partial_y^m u_h, v_h) \right].
\]
Adding extra terms and using the $\mathcal{A}$-orthogonality of $\mathcal{P}_h$, we can write this in the following more convenient form

\begin{align*}
0 &= \mathcal{A}(\mathcal{P}_h \partial_y^\nu (u - u_h), v_h) - \lambda_h \mathcal{M}(\partial_y^\nu u, v_h) \\
- (\lambda - \lambda_h) \mathcal{M}(\partial_y^\nu u, v_h) - \lambda_h \mathcal{M}(\partial_y^\nu u - \mathcal{P}_h \partial_y^\nu u, v_h) \\
+ \sum_{j=1}^{\infty} \nu_j \left( \int_D a_j \nabla [\partial_y^{\nu-e_j} (u - u_h)] : \nabla v_h + \int_D b_j [\partial_y^{\nu-e_j} (u - u_h)] v_h \right) \\
- \sum_{m \leq \nu \atop m \neq \nu} \left( \sum_{j=1}^{\infty} \nu_j \right) [\partial_y^{\nu-m} \lambda \mathcal{M}(\partial_y^m (u - u_h), v_h) + \partial_y^{\nu-m} (\lambda - \lambda_h) \mathcal{M}(\partial_y^m u_h, v_h)].
\end{align*}

(A.2)

Letting $v_h = u_h$ in (A.2) and separating out the $m = 0$ term, we obtain the following formula for the derivative of the eigenvalue error

\begin{align*}
\partial_y^\nu (\lambda - \lambda_h) &= (\lambda - \lambda_h) \mathcal{M}(\partial_y^\nu u, v_h) - \lambda_h \mathcal{M}(\partial_y^\nu u - \mathcal{P}_h \partial_y^\nu u, v_h) - (\partial_y^\nu \lambda) \mathcal{M}(u - u_h, u_h) \\
+ \sum_{j=1}^{\infty} \nu_j \left( \int_D a_j \nabla [\partial_y^{\nu-e_j} (u - u_h)] : \nabla v_h + \int_D b_j [\partial_y^{\nu-e_j} (u - u_h)] u_h \right) \\
- \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \sum_{j=1}^{\infty} \nu_j \right) [\partial_y^{\nu-m} \lambda \mathcal{M}(\partial_y^m (u - u_h), u_h) + \partial_y^{\nu-m} (\lambda - \lambda_h) \mathcal{M}(\partial_y^m u_h, u_h)],
\end{align*}

where we have used the fact that $u_h$ is normalised. Also the first two terms in (A.2) cancel because the bilinear form is symmetric and $(\lambda_h, u_h)$ satisfy the FE eigenvalue problem (2.10) with $\mathcal{P}_h \partial_y^\nu (u - u_h) \in \mathcal{V}_h$ as a test function.

Taking the absolute value, then using the triangle and Cauchy–Schwarz inequalities gives the upper bound

\begin{align*}
|\partial_y^\nu (\lambda - \lambda_h)| &\leq |\lambda - \lambda_h| \|\partial_y^\nu u\|_\mathcal{M} + \lambda_h \|\partial_y^\nu u - \mathcal{P}_h \partial_y^\nu u\|_\mathcal{M} + |\partial_y^\nu \lambda| \|u - u_h\|_\mathcal{M} \\
+ \sum_{j=1}^{\infty} \nu_j \left[ |a_j|_L^\infty \|\partial_y^{\nu-e_j} (u - u_h)\|_V \|u_h\|_V + |b_j|_L^\infty \|\partial_y^{\nu-e_j} (u - u_h)\|_L^2 \|u_h\|_L^2 \right] \\
+ \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \sum_{j=1}^{\infty} \nu_j \right) [\|\partial_y^{\nu-m} \lambda\| \|\partial_y^m (u - u_h)\|_\mathcal{M} + |\partial_y^{\nu-m} (\lambda - \lambda_h)| \|\partial_y^m u_h\|_\mathcal{M}],
\end{align*}

where we have again simplified by using $\|u_h\|_\mathcal{M} = 1$. Then, using the equivalence of norms (2.8) and the Poincaré inequality (2.6), we can bound the $\mathcal{M}$- and $L^2$-norms by the corresponding $V$-norms, to give

\begin{align*}
|\partial_y^\nu (\lambda - \lambda_h)| &\leq \sqrt{\frac{a_{\text{max}}}{\chi_1}} \left[ |\lambda - \lambda_h| \|\partial_y^\nu u\|_V + |\partial_y^\nu \lambda| \|u - u_h\|_V \right] \\
+ \pi \left( 1 + \frac{1}{\chi_1} \right) \sum_{j=1}^{\infty} \nu_j \beta_j \|\partial_y^{\nu-e_j} (u - u_h)\|_V \\
+ \sqrt{\frac{a_{\text{max}}}{\chi_1}} \sum_{0 \neq m \leq \nu \atop m \neq \nu} \left( \sum_{j=1}^{\infty} \nu_j \right) [\|\partial_y^{\nu-m} \lambda\| \|\partial_y^m (u - u_h)\|_V + |\partial_y^{\nu-m} (\lambda - \lambda_h)| \|\partial_y^m u_h\|_V],
\end{align*}

where we have also used the upper bounds (2.16) and (2.17), and the definition of $\beta_j$ (4.1).
Substituting in the upper bounds on the derivatives (4.3) and (4.4), the bound on the projection error (5.5), and then the bounds on the FE errors (2.11) and (2.12), we have the upper bound

\[
|\partial_y^{\nu}(\lambda - \lambda_h)| \leq \sqrt{\frac{d_{\text{max}}}{\lambda_1}} \left[ C_L h \pi^{\nu} + \overline{\lambda C_D \beta^{\nu}} + \overline{\lambda C_u \beta^{\nu}} \right] h|\nu|^{1 + \epsilon}
\]

\[
+ \frac{u}{C_{\beta}} \left( 1 + \frac{1}{\lambda_1} \right) \sum_{j=1}^{\infty} \nu_j \beta_j \|\partial_y^{\nu - \epsilon_j}(u - u_h)\|_V + \sqrt{\frac{d_{\text{max}}}{\lambda_1}} \sum_{0 \neq m \leq \nu \neq 0 \neq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) \cdot \left[ \lambda |\nu - m|^{1 + \epsilon} \beta^{\nu - m} \|\partial_y^m(u - u_h)\|_V + \overline{\pi} |m|^{1 + \epsilon} \beta^m |\partial_y^{\nu - m}(\lambda - \lambda_h)| \right].
\]

Note that we can simplify the sum on the last line using the symmetry of the binomial coefficient, \( \binom{n}{k} = \binom{n}{n-k} \), as follows. First, we separate it into two sums

\[
\sum_{0 \neq m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) \left[ \lambda |\nu - m|^{1 + \epsilon} \beta^{\nu - m} \|\partial_y^m(u - u_h)\|_V + \overline{\pi} |m|^{1 + \epsilon} \beta^m |\partial_y^{\nu - m}(\lambda - \lambda_h)| \right] = (\lambda + \overline{\pi}) \sum_{0 \neq m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) |m|^{1 + \epsilon} \beta^m \left[ \|\partial_y^{\nu - m}(u - u_h)\|_V + |\partial_y^{\nu - m}(\lambda - \lambda_h)| \right],
\]

(A.3)

where to obtain the last equality we have simply relabelled the indices in the first sum.

Then, since \( \beta_j \leq \overline{\beta}_j \) and \( h \) is sufficiently small (i.e., \( h \leq \overline{\theta} \) with \( \overline{\theta} \) as in (2.14)), the result (5.6) holds. The constant is given by

\[
C_1 := \max \left\{ \sqrt{\frac{d_{\text{max}}}{\lambda_1}} \left[ \pi n C_L + \overline{\lambda(C_D + C_u)} \right], \frac{u}{C_{\beta}} \left( 1 + \frac{1}{\lambda_1} \right), \sqrt{\frac{d_{\text{max}}}{\lambda_1}} \left( \lambda_1 + \pi \right) \right\}
\]

which is independent of \( y, h \) and \( \nu \).

For the second result (5.7), using [4] Lemma 3.1] the eigenvalue error can also be written as

\[
\lambda - \lambda_h = -A(u - u_h, u - u_h) + \lambda M(u - u_h, u - u_h),
\]

which after taking the \( \nu \)th derivative becomes

\[
\partial_y^{\nu}(\lambda - \lambda_h) = -\sum_{m \leq \nu} \left( \begin{array}{c} \nu \\ m \end{array} \right) A(\partial_y^{\nu - m}(u - u_h), \partial_y^m(u - u_h)) + \sum_{j=1}^{\infty} \sum_{m \leq \nu - e_j} \nu_j \left( \begin{array}{c} \nu - e_j \\ m \end{array} \right) \left[ \int_D a_j \nabla \partial_y^{\nu - e_j - m}(u - u_h) \cdot \nabla \partial_y^m(u - u_h) \right]
\]

\[
+ \int_D b_j \partial_y^{\nu - e_j - m}(u - u_h) \partial_y^m(u - u_h) \right]
\]

\[
+ \sum_{m \leq \nu} \sum_{k \leq m} \left( \begin{array}{c} \nu \\ m \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \partial_y^m \lambda \lambda M(\partial_y^{m-k}(u - u_h), \partial_y^k(u - u_h))].
\]

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Taking the absolute value, then using the triangle, Cauchy–Schwarz and Poincaré inequalities, along with the norm equivalences (2.7), (2.8), gives
\[
|\partial_y^\nu(\lambda - \lambda_h)| \leq a_{\text{max}} \left(1 + \frac{1}{\chi_1}\right) \sum_{m \leq \nu} \left(\frac{\nu}{m}\right) \|\partial_y^{\nu-m}(u - u_h)\|_V \|\partial_y^m(u - u_h)\|_V \\
+ \left(1 + \frac{1}{\chi_1}\right) \sum_{j=1}^{\infty} \sum_{m \leq \nu - e_j} \nu_j \left(\frac{\nu - e_j}{m}\right) \frac{\beta_j}{C^2} \|\partial_y^{\nu-e_j-m}(u - u_h)\|_V \|\partial_y^m(u - u_h)\|_V \\
+ \frac{a_{\text{max}}}{\chi_1} \sum_{m \leq \nu} \sum_{k \leq m} \left(\frac{\nu}{m}\right) \left(\frac{m}{k}\right) \|\partial_y^{\nu-m}\|_V \|\partial_y^{m-k}(u - u_h)\|_V \|\partial_y^k(u - u_h)\|_V.
\]

Finally, substituting in the upper bound (4.3) on the derivative of \(\lambda\) gives the desired result (5.7). The constant is given by
\[
C_{II} := \left[1/C_\beta + a_{\text{max}}(1 + \chi_1)\right] \left(1 + \frac{1}{\chi_1}\right),
\]
which is independent of \(h, y\) and \(\nu\).

**Proof of Lemma 5.3 (eigenfunction bound).** We deal with the eigenfunction error projected onto \(V_h\), as opposed to \(\partial_y^\nu(u - u_h)\), because the latter belongs to \(\mathbb{V}\) but not to \(V_h\). As such, we first separate the error as
\[
\|\partial_y^\nu(u - u_h)\|_V \leq \|P_h \partial_y^\nu(u - u_h)\|_V + \|\partial_y^\nu u - P_h \partial_y^\nu u\|_V \\
\leq \|P_h \partial_y^\nu(u - u_h)\|_V + C_P h |\nu|^{1+\epsilon_\beta},
\]
where in the second inequality we have used the bound (5.8).

Similar to the proof of [19, Lemma 3.4], the bilinear form that acts on \(P_h \partial_y^\nu(u - u_h)\) (namely, \(A - \lambda_h M\)) is only coercive on the orthogonal complement of the eigenspace corresponding to \(\lambda_h\), which we denote by \(E(\lambda_h)\). Hence, to obtain the recursive formula for the derivative of the eigenfunction error, we first make the following orthogonal decomposition. The FE eigenfunctions form an orthogonal basis for \(V_h\), and so we have
\[
P_h \partial_y^\nu(u - u_h) = M(P_h \partial_y^\nu(u - u_h), u_h)u_h + \varphi_h,
\]
where \(\varphi_h \in E(\lambda_h)\). Then we can bound the norm by
\[
\|P_h \partial_y^\nu(u - u_h)\|_V \leq \|M(P_h \partial_y^\nu(u - u_h), u_h)\|_V \|u_h\|_V + \|\varphi_h\|_V.
\]

To bound the first term in this decomposition (A.6), first observe that we can write
\[
|M(P_h \partial_y^\nu(u - u_h), u_h)| \leq |M(\partial_y^\nu u, u) - M(\partial_y^{\nu-m} u_h, u_h)| \\
+ |M(\partial_y^{\nu-m} u, u - u_h)| + |M(\partial_y^{\nu-m} u_h, u - u_h)|.
\]

The first term on the right in (A.7) can be bounded by differentiating the normalisation equations \(\|u\|_M = 1\) and \(\|u_h\|_M = 1\) (see [19, eq. (3.15)]) to give
\[
M(\partial_y^\nu u, u) - M(\partial_y^{\nu-m} u_h, u_h) = -\frac{1}{2} \sum_{0 \neq m \leq \nu} \sum_{m \neq \nu} \left(\frac{\nu}{m}\right)[M(\partial_y^{\nu-m} u_h, \partial_y^{\nu-m} u) - M(\partial_y^{\nu-m} u_h, \partial_y^m u_h)] \\
= -\frac{1}{2} \sum_{0 \neq m \leq \nu} \sum_{m \neq \nu} \left(\frac{\nu}{m}\right)[M(\partial_y^{\nu-m} (u - u_h), \partial_y^{\nu-m} u) + M(\partial_y^{\nu-m} u_h, \partial_y^m (u - u_h))].
\]
Then, using the triangle inequality, the Cauchy–Schwarz inequality, the equivalence of norms (2.8) and the Poincaré inequality (2.6), gives the upper bound

\[
|M(\partial_y^n u, u) - M(\partial_y^n u_h, u_h)| 
\leq \frac{a_{\max}}{2\chi_1} \sum_{0 \neq m \leq \nu \atop m \neq \nu} C_u \frac{(|m|!^{1+\epsilon} \beta^m \|\partial_v^m (u - u_h)\|_V)}{(m!^{1+\epsilon} \beta^m \|\partial_v^m (u - u_h)\|_V)} 
\]

where for the second last inequality we have used the upper bound (4.4) and the analogous bound for \(u_h\). For the equality on the last line, we have simplified the sum using the symmetry of the binomial coefficient as in (A.3).

To bound the second and third terms in (A.7) we use the Cauchy–Schwarz inequality, the equivalence of norms (2.8), and the Poincaré inequality (2.6), followed by the bound on the projection error (2.12) and the bound on the FE error (2.12), which gives

\[
|M(\partial_y^n u, u - u_h)| + |M(\partial_y^n u - \mathcal{P}_h \partial_y^n u, u_h)| 
\leq \|\partial_y^n u\|_M\|u - u_h\|_M + \|\partial_y^n u - \mathcal{P}_h \partial_y^n u\|_M 
\leq \frac{a_{\max}}{\chi_1} \|\partial_y^n u\|_V\|u - u_h\|_V + \frac{a_{\max}}{\chi_1} \|\partial_y^n u - \mathcal{P}_h \partial_y^n u\|_V 
\leq \frac{a_{\max}}{\chi_1} C_u \beta |\nu|!^{1+\epsilon} \beta^\nu + \frac{a_{\max}}{\chi_1} C_P h |\nu|!^{1+\epsilon} \beta^\nu 
\leq \left(\frac{a_{\max}}{\chi_1} C_u + \frac{a_{\max}}{\chi_1} C_P \right) h |\nu|!^{1+\epsilon} \beta^\nu, \quad (A.8)
\]

where in the last inequality we have used that \(\beta_j \leq \beta_j\).

Substituting these two bounds into (A.7) then multiplying by \(\|u_h\|_V\) gives the following upper bound on the first term of the decomposition (A.6)

\[
|M(\mathcal{P}_h \partial_y^n (u - u_h), u_h)| \|u_h\|_V 
\leq \pi \left(\frac{a_{\max}}{\chi_1} C_u + \frac{a_{\max}}{\chi_1} C_P \right) h |\nu|!^{1+\epsilon} \beta^\nu 
+ \frac{a_{\max}}{\chi_1} \sum_{0 \neq m \leq \nu \atop m \neq \nu} |m|!^{1+\epsilon} \beta^m \|\partial_v^m (u - u_h)\|_V. \quad (A.9)
\]

Note that we have also used (2.17).

Next, to bound the norm of \(\varphi_h\) (the second term in the decomposition (A.6), we let
\[\nu_h = \varphi_h \text{ in } (A.2)\] and then rearrange the terms to give

\[
\mathcal{A}(P_h \nu^u(u - u_h), \varphi_h) - \lambda_h \mathcal{M}(P_h \nu^u(u - u_h), \varphi_h) = (\lambda - \lambda_h) \mathcal{M}(\nu^u u, \varphi_h) \\
+ \lambda_h \mathcal{M}(\nu^u u - P_h \nu^u u, \varphi_h) + \partial^v_\nu \lambda \mathcal{M}(u - u_h, \varphi_h) + \partial^v_\nu (\lambda - \lambda_h) \mathcal{M}(u_h, \varphi_h)
\]

\[\sum_{j=1}^{\infty} \nu_j \left( \int_D a_j \nabla \left[ \partial^v_\nu \epsilon_j (u - u_h) \right] \cdot \nabla \varphi_h + \int_D b_j \left[ \partial^v_\nu \epsilon_j (u - u_h) \right] \varphi_h \right) \tag{A.10}
\]

\[+ \sum_{\nu \neq m \leq \nu} \left( \nu \over m \right) \left[ \partial^v_\nu \lambda \mathcal{M}(\nu^m u(u - u_h), \varphi_h) + \partial^v_\nu (\nu - \lambda_h) \mathcal{M}(\nu^m u_h, \varphi_h) \right].\]

Again using the decomposition \[\text{(A.5)}\] and the fact that \[u_h\] satisfies the eigenproblem \[2.10\] with \[\varphi_h \in V_h\] as a test function, the left hand side of \[\text{(A.10)}\] simplifies to \[\mathcal{A}(\varphi_h, \varphi_h) - \lambda_h \mathcal{M}(\varphi_h, \varphi_h)\]. Since \[\varphi_h \in E(\lambda_h)^\perp\], we can use the FE version of the coercivity estimate \[\text{Lemma 3.1}\] (see also Remark 3.2 that follows), to bound this from below by

\[\mathcal{A}(\varphi_h, \varphi_h) - \lambda_h \mathcal{M}(\varphi_h, \varphi_h) \geq a_{\min} \left( \frac{\lambda_2 - \lambda_h}{\lambda_2} \right) \| \varphi_h \|_V^2 \geq \frac{a_{\min} \rho}{2 \lambda_2} \| \varphi_h \|_V^2,\tag{A.11}\]

where in the last inequality we have used the upper bound \[2.10\], along the lower bound \[2.15\] on the FE spectral gap, which is applicable for \[h\] sufficiently small.

Taking the absolute value, the right hand side of \[\text{(A.10)}\] can be bounded using the triangle inequality, the Cauchy–Schwarz inequality, the equivalence of norms \[2.8\], and the Poincaré inequality \[2.6\], which, combined with the lower bound \[\text{(A.11)}\], gives

\[
\frac{a_{\min} \rho}{2 \lambda_2} \| \varphi_h \|_V^2 \leq \frac{a_{\max}}{\lambda_1} \left( \| \mathcal{M}(\nu^u u) \|_V + \lambda_h \| \partial^v_\nu u - P_h \partial^v_\nu u \|_V \\
+ \| \partial^v_\nu \lambda \| \| u - u_h \|_V \right) \| \varphi_h \|_V \\
+ \sum_{j=1}^{\infty} \nu_j \left( \| a_j \|_\infty + \frac{1}{\lambda_1} \| b_j \|_\infty \right) \| \partial^v_\nu \epsilon_j (u - u_h) \|_V \| \varphi_h \|_V \\
+ \frac{a_{\max}}{\lambda_1} \sum_{\nu \neq m \leq \nu} \left( \frac{\nu}{m} \right) \left( \| \partial^v_\nu \lambda \| \| \nu^m u(u - u_h) \|_V + \| \partial^v_\nu (\nu - \lambda_h) \| \| \nu^m u_h \|_V \right) \| \varphi_h \|_V.
\]

Dividing through by \[\frac{a_{\min} \rho}{(2 \lambda_2)} \| \varphi_h \|_V\], then using the bounds \[2.11\], \[4.3\], \[4.4\], \[5.5\], along with the fact that that \[\beta_j \leq \beta_j^\perp\] for all \[j \in \mathbb{N}\] and \[h \leq \overline{h}\], we have that the norm of \[\varphi_h\] is bounded by

\[
\| \varphi_h \|_V \leq \frac{a_{\max}}{a_{\min} \rho} \frac{2 \lambda_2}{\lambda_1} \left( \frac{1}{\lambda} \| C_{\lambda \nu} + C_{\nu \lambda} \|_V \| \nu^{1+\epsilon} \| + \frac{a_{\max}}{a_{\min} \rho} \frac{2 \lambda_2}{\lambda_1} \| \partial^v_\nu (\lambda - \lambda_h) \|_V \right) \\
+ \frac{2 \lambda_2}{a_{\min} \rho} \left( \lambda \| \overline{\nu} \| + \overline{\nu} \right) \sum_{\nu \neq m \leq \nu} \left( \frac{\nu}{m} \right) \| m \|_V^{1+\epsilon} \beta^m \left( \| \partial^v_\nu (u - u_h) \|_V + \| \partial^v_\nu (\lambda - \lambda_h) \|_V \right) \right.
\]

Note that to get the sum on the last line we have again simplified similarly to \[\text{(A.3)}\].
Substituting the bounds (A.6), followed by (A.9) and (A.12), into the decomposition (A.4) gives the following recursive bound on the derivative of the eigenfunction error

$$
\| \partial^\nu y(u - u_h) \|_V \leq \left[ \frac{a_{\text{max}}}{a_{\text{min}}} 2\lambda_2 (\pi_1 C_\lambda + \bar{\lambda} C_P + \bar{\lambda} C_u) + \pi \left( \frac{a_{\text{max}}}{\lambda_1} C_u + \sqrt{\frac{a_{\text{max}}}{\lambda_1}} C_P \right) + C_P \right] h |\nu|^{1+\epsilon} \beta^\nu \\
+ \frac{a_{\text{max}}}{a_{\text{min}}} \frac{2\lambda_2}{\rho \chi_1} |\partial^\nu y(\lambda - \lambda_h)| + \frac{2\lambda_2}{a_{\text{min}} \rho C \beta} \left( 1 + \frac{1}{\chi_1} \right) \sum_{j=1}^{\infty} \nu_j \beta_j |\partial^\nu y^{-\nu_j}(u - u_h)|_V \\
+ \frac{a_{\text{max}}}{a_{\text{min}}} \frac{2\lambda_2}{\rho \chi_1} (\bar{\lambda} + \pi) \sum_{0 \neq m \leq \nu \atop m \neq \nu} \binom{\nu}{m} |m|^{1+\epsilon} \beta^m \left[ |\partial^\nu y^{-m}(u - u_h)|_V + |\partial^\nu y^{-m}(\lambda - \lambda_h)| \right] \\
+ \frac{a_{\text{max}}}{\rho \chi_1} \sum_{0 \neq m \leq \nu \atop m \neq \nu} \binom{\nu}{m} |m|^{1+\epsilon} \beta^m |\partial^\nu y^{-m}(u - u_h)|_V.
$$

Observe that all of the constant terms are independent of $y$, $h$, and $\nu$. To obtain the final result with a right hand side that does not depend on any derivative of order $\nu$, we now substitute the recursive formula (5.6) for $\partial^\nu y(\lambda - \lambda_h)$. After grouping the similar terms and collecting all of the constants into $C_{\text{III}}$ we have the final result. Since $C_1$ from (5.6) and all of the constants above are independent of $y$, $h$, and $\nu$, the final constant $C_{\text{III}}$ is as well.