Dynamical symmetries of the Klein-Gordon equation

Fu-Lin Zhang∗ and Jing-Ling Chen†

Theoretical Physics Division, Chern Institute of Mathematics,
Nankai University, Tianjin 300071, People’s Republic of China

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Abstract

The dynamical symmetries of the two-dimensional Klein-Gordon equations with equal scalar and vector potentials (ESVP) are studied. The dynamical symmetries are considered in the plane and the sphere respectively. The generators of the $SO(3)$ group corresponding to the Coulomb potential, and the $SU(2)$ group corresponding to the harmonic oscillator potential are derived. Moreover, the generators in the sphere construct the Higgs algebra. With the help of the Casimir operators, the energy levels of the Klein-Gordon systems are yielded naturally.

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∗Email: flzhang@mail.nankai.edu.cn
†Email: chenjl@nankai.edu.cn
I. INTRODUCTION

Recently, many works about the Dirac or the Klein-Gordon (KG) equation with scalar and vector potentials of equal magnitude (SVPEM) are reported [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. When the potentials are spherical, the Dirac equation is said to have the spin or pseudospin symmetry corresponding to the same or opposite sign. These symmetries, which have been observed in the hadron and nuclear spectroscopies for a long time [13, 14], are derived from the investigation of the dynamics between a quark and an antiquark [15, 16]. The very lately studies [9, 10] have revealed that, the motion of a spin-1/2 particle with SVPEM satisfies the same differential equation and has the same energy spectrum as a scalar particle. When both scalar and vector potentials are spherical, Alberto et. al. [9] have indicated that the spin-orbit and Darwin terms of either the upper component or the lower component of the Dirac spinor vanish, which made it equivalent, as far as energy is concerned, to a spin-0 state. In this case, besides energy, a scalar particle will also have the same orbital angular momentum as the (conserved) orbital angular momentum of either the upper or lower component of the corresponding spin-1/2 particle. These results suggest that, one can image the spin-1/2 particle with SVPEM as a relativistic scalar particle with an additional spin but without the spin-orbit coupling. From this point of view, we speculate that, the kinetic characteristics of the Dirac equation with SVPEM should exist in the KG equation with the same potentials.

Dynamical symmetries are essential and prevalent both in non-relativistic classical and quantum mechanics [17]. Until the work by Ginocchio [18, 19], there were no models in the relativistic quantum mechanics with dynamical symmetry reported. Ginocchio has found the $U(3)$ and pseudo-$U(3)$ symmetry in the Dirac equation with SVPEM when the potential takes the harmonic oscillator form. And in Ref. [20] we have established the dynamical symmetries in the two-dimensional Dirac equation (for hydrogen atom as well as harmonic oscillator) with SVPEM when the signs are the same (or say ESVP). The goal of this work is to show the dynamical symmetries of the KG equation with ESVP. Of course, the following discussion also holds when the signs are opposite, by making some corrections.

Since there is no an explicit defined Hamiltonian for a spin-0 particle, instead we introduce a quasi-Hamiltonian of the KG equation with ESVP in section III and consequently provide an operable criterion for what kind of the dynamical symmetry exists. The KG equation in
a plane with the Coulomb and the harmonic oscillator form potentials will be discussed as the examples. In section III as a generalization of the plane, we will investigate the motion of a scalar particle in a sphere with equal scalar and vector Coulomb or harmonic oscillator potentials. Conclusion and discussion will be made in the last section.

II. DYNAMICAL SYMMETRIES IN A PLANE

The KG equation with scalar potential $V_s$ and vector potential $V_v$ is given by

$$\left\{ p^2 + [m + V_s]^2 - \left[ i \frac{\partial}{\partial t} - V_v \right]^2 \right\} \psi = 0. \quad (1)$$

For the time-independent potentials, and $V_s = V_v = V(r)/2$, the KG equation (1) becomes

$$[p^2 + (m + \epsilon)V(r) - (\epsilon^2 - m^2)]\psi = 0, \quad (2)$$

where $\epsilon$ is the relativistic energy. If we set

$$\epsilon + m = 2\tilde{m}, \quad \epsilon - m = E, \quad (3)$$

when $\epsilon \neq -m$, we obtain from Eq. (2) that

$$\left[ \frac{p^2}{2\tilde{m}} + V(r) - E \right] \psi = 0. \quad (4)$$

When $m \to \infty$, $\epsilon \to m$, Eq. (4) returns to the usual Schrödinger equation.

Suppose $U$ is a operator of a symmetry group of the spin-0 system with ESVP, with the generators denoted as $L_i$. Generally, if $\psi$ fulfills Eq. (4), the state $\psi' = U\psi$ should also fulfills it, namely

$$\left[ \frac{p^2}{2\tilde{m}} + V(r) - E \right] U\psi = 0. \quad (5)$$

Multiplying Eq. (4) by $U$ from the left-hand side and comparing it with Eq. (5), we obtain

$$[U, \tilde{H}] = U\tilde{H} - \tilde{H}U = 0, \quad (6)$$

where $\tilde{H} = \frac{p^2}{2\tilde{m}} + V(r)$ is called the *quasi-Hamiltonian* in this paper. Immediately, the generators of the symmetry group $L_i$ also commute with $\tilde{H}$

$$[L_i, \tilde{H}] = 0. \quad (7)$$
This result provide an operable criterion for what kind of the dynamical symmetry exists in the KG equation with ESVP.

As examples, let us consider the motion of a scalar particle constrained in a two-dimensional (2D) plane.

*Coulomb potential.* When the potential takes the Coulomb form $V(r) = -\frac{k}{r}$, then the quasi-Hamiltonian reads

$$\tilde{H} = \frac{\vec{p}^2}{2\tilde{m}} - \frac{k}{r}. \quad (8)$$

It is easy to obtain the generators from the non-relativistic results as

$$L = x_1p_2 - x_2p_1,$$
$$\tilde{R}_1 = \frac{1}{2\tilde{m}k}(Lp_2 + p_2L) - \frac{x_1}{r}, \quad (9)$$
$$\tilde{R}_2 = \frac{1}{2\tilde{m}k}(-Lp_1 - p_1L) - \frac{x_2}{r},$$

which commute with $\tilde{H}$, and satisfy the commutation relations

$$[L, \tilde{R}_1] = i\tilde{R}_2, \quad [L, \tilde{R}_2] = -i\tilde{R}_1, \quad [\tilde{R}_1, \tilde{R}_2] = \frac{-i2\tilde{H}}{mk^2}L, \quad (10)$$

and $\tilde{R}_1^2 + \tilde{R}_2^2 = 2\frac{\tilde{p}^2}{\tilde{m}k^2}(L^2 + \frac{1}{4}) + 1$. These results show that the 2D KG equation with equal scalar and vector potentials has the $SO(3)$ symmetry based on the above criterion. The relations of the generators can be also used to determine the energy levels of this system. After defining the normalized generators

$$A_3 = L, \quad A_i = \left[-\frac{2E}{mk^2}\right]^{-\frac{1}{2}}\tilde{R}_i, \quad (i = 1, 2), \quad (11)$$

one then obtains

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad (i, j, k = 1, 2, 3). \quad (12)$$

The $SO(3)$ Casimir operator is given by

$$C_{so3} = A_1^2 + A_2^2 + A_3^2 = j(j + 1), \quad j = 0, 1, 2, ... \quad (13)$$

Inserting Eq. (11) into Eq. (13), one can have

$$E = -\frac{2k^2}{n^2\tilde{m}}, \quad n = 2j + 1 = 1, 3, 5, ... \quad (14)$$
Considering the relation in Eq. (3), the relativistic energy levels of this system are given by

\[ \epsilon = E + m = \frac{n^2 - k^2}{n^2 + k^2} m, \]

which coincides with the results in 2D Dirac system [20].

In the non-relativistic limit \((m \to \infty, \epsilon \to m)\), \(\tilde{m}\) and \(\tilde{H}\) reduce to the mass \(m\) and the Hamiltonian of the non-relativistic hydrogen atom, respectively. In the meantime, the generators \(\tilde{R}_i\) reduce to the components of the Rung-Lenz vector, and Eq. (14) is nothing but the spectrum of a non-relativistic hydrogen atom.

Harmonic oscillator potential. When the potential takes the harmonic oscillator form, \(V(r) = \frac{1}{2} m \omega^2 r^2\), in comparison with the non-relativistic harmonic oscillator, we set

\[ \frac{1}{2} \tilde{m} \tilde{\omega}^2 = \frac{1}{2} m \omega^2, \quad \tilde{\omega} = \sqrt{\frac{m \omega^2}{\tilde{m}}}, \]

then the quasi-Hamiltonian becomes

\[ \tilde{H} = \frac{p^2}{2 \tilde{m}} + \frac{1}{2} \tilde{m} \tilde{\omega}^2 r^2. \]

For a fixed energy eigenvalue, \(\tilde{m}\) and \(\tilde{\omega}\) are constants. Thus the generators commuting with \(\tilde{H}\) are given by

\[
\begin{align*}
J_1 &= \frac{1}{2} \left( \frac{1}{\tilde{m} \tilde{\omega}} p_1 p_2 + \tilde{m} \tilde{\omega} x_1 x_2 \right), \\
J_2 &= \frac{1}{2} \left( x_1 p_2 - x_2 p_1 \right), \\
J_3 &= \frac{1}{2} \left( \frac{1}{\tilde{m} \tilde{\omega}} \frac{p_1^2 - p_2^2}{2} + \tilde{m} \tilde{\omega} \frac{x_1^2 - x_2^2}{2} \right).
\end{align*}
\]

They satisfy the \(SU(2)\) commutation relations

\[ [J_i, J_j] = i \epsilon_{ijk} J_k, (i, j, k = 1, 2, 3). \]

The Casimir operator of the \(SU(2)\) group is

\[
C_{su2} = J_1^2 + J_2^2 + J_3^2 = \frac{1}{4} \left( \left( \frac{\tilde{H}}{\tilde{\omega}} \right)^2 - 1 \right)
= s(s + 1), \quad s = 0, \frac{1}{2}, 1, \ldots
\]

which yields the energy levels as

\[ E = (n + 1)\tilde{\omega}, \quad n = 2s = 0, 1, 2, \ldots \]
Combining with the relations (16) and (3), it is straightforward to prove the relativistic energy is the real root of the cubic equation

$$(\epsilon - m)^2(\epsilon + m) = 2m\omega^2(n + 1)^2,$$  \hspace{2cm} (22)

which is the same as the spectrum of the Dirac equation given in [20]. In non-relativistic limit, $\bar{m} \to m$ and $\bar{\omega} \to \omega$, the above formulae return to the results of the non-relativistic harmonic oscillator.

### III. DYNAMICAL SYMMETRIES IN A SPHERE

In recent years, polynomial angular momentum algebra and its increasing applications have been the focus of very active research. The first special case, called the Higgs algebra now, is found by Higgs [21] in the motion of a non-relativistic particle in a 2D curved space with the Coulomb or harmonic oscillator potential. It have been shown in the above section that, the KG system with ESVVP has the same dynamical symmetry as the non-relativistic system with the similar potential. This gives rise to an interesting question: Does the Higgs algebra exists in the KG equation in a sphere with ESVVP, when the potential takes the Coulomb or harmonic oscillator form?

To solve this problem, we first construct the classical Hamiltonian for a relativistic particle in a sphere. In the gnomic projection, as introduced in [21], the metric is given by

$$(ds)^2 = \frac{d\vec{x} \cdot d\vec{x}}{1 + \lambda r^2} - \frac{\lambda(\vec{x} \cdot d\vec{x})^2}{(1 + \lambda r^2)^2},$$  \hspace{2cm} (23)

where $\lambda$ is the curvature of the sphere. The Lagrangian for the free motion of a relativistic scalar particle is $\mathcal{L} = -m\sqrt{1 - \dot{s}^2}$, where $m$ is the mass and $s^2$ is defined in Eq. (23). The momentum conjugate to $x_i$ ($i = 1, 2$) is

$$p_i = -\frac{m^2}{\mathcal{L}} \left[ \frac{\dot{x}_i}{1 + \lambda r^2} - \frac{\lambda(\vec{x} \cdot \dot{\vec{x}})x_i}{(1 + \lambda r^2)^2} \right],$$  \hspace{2cm} (24)

and the Hamiltonian is given by

$$H^2 = m^2 + \pi^2 + \lambda L^2,$$  \hspace{2cm} (25)

where $L = x_1 p_2 - x_2 p_1$, $\pi^2 = \pi_1^2 + \pi_2^2$, and

$$\pi_i = p_i + \frac{1}{2}\lambda[x_i(\vec{x} \cdot \vec{p}) + (\vec{p} \cdot \vec{x})x_i], \quad i = 1, 2.$$  \hspace{2cm} (26)
Thus, in the relativistic quantum mechanics, the KG equation of a free particle in a sphere is

\[- \frac{\partial^2}{\partial t^2} \psi = (m^2 + \pi^2 + \lambda L^2) \psi. \quad (27)\]

Suppose the particle coupling to a scalar potential \( V_s \) and a vector potential \( V_v \) (only the time component is nonzero), and \( V_s = V_v = \frac{V(r)}{2} \), the KG equation (27) becomes

\[\left[ \frac{1}{2\tilde{m}} (\pi^2 + \lambda L^2) + V(r) - E \right] \psi = 0, \quad (28)\]
on the premise that the energy \( \epsilon \neq -m \), and \( \tilde{m} \) and \( E \) take the definitions in Eq. (3). Then, the quasi-Hamiltonian in a sphere is defined as \( \tilde{H} = \frac{1}{2\tilde{m}} (\pi^2 + \lambda L^2) + V(r) \). Unlike the equations (2) and (4), the KG equation in a sphere with ESVP hasn’t an equivalent Dirac equation, because the spin-orbit coupling is kept by the curvature.

Coulomb potential. When \( V(r) = -\frac{k}{r} \), the quasi-Hamiltonian \( \tilde{H} = \frac{1}{2\tilde{m}} (\pi^2 + \lambda L^2) - \frac{k}{r} \) commutes with the angular momentum \( L \) and

\[\tilde{R}_1 = \frac{1}{2\tilde{m}k} (L\pi_2 + \pi_2 L) - \frac{x_1}{r}, \quad (29)\]
\[\tilde{R}_2 = \frac{1}{2\tilde{m}k} (-L\pi_1 - \pi_1 L) - \frac{x_2}{r}. \]

Set \( \tilde{R}_\pm = \tilde{R}_1 \pm i\tilde{R}_2 \), one can find the commutators

\[[L, \tilde{R}_\pm] = \pm \tilde{R}_\pm, \quad [\tilde{R}_+, \tilde{R}_-] = c_3 L^3 + c_1 L, \quad (30)\]

where \( c_3 = \frac{4\lambda}{\tilde{m}^2 k^2} \) and \( c_1 = -\frac{4\tilde{H}}{\tilde{m}k^2} + \frac{\lambda}{2\tilde{m}^2 k^2} \). These relations indicate that the generators construct the Higgs algebra, and the system has the \( SO(3) \) symmetry. To get the energy levels, we should calculate the anticommutator

\[\{ \tilde{R}_+, \tilde{R}_- \} = 2 + \frac{\tilde{H}}{\tilde{m}k^2} + (8\tilde{m}\tilde{H} - 5\lambda)L^2 - \frac{2\lambda L^4}{\tilde{m}^2 k^2}. \quad (31)\]

The Casimir of the Higgs algebra is given by \[22, 23\]

\[C = \{ \tilde{R}_+, \tilde{R}_- \} + \left( c_1 + \frac{c_3}{2} \right) L^2 + \frac{c_3}{2} L^4 \]
\[= 2 + \frac{\tilde{H}}{\tilde{m}k^2} \]
\[= c_1 C_{so3} + \frac{c_3}{2} C_{so3}^2, \quad (32)\]
where $C_{so3} = j(j + 1)$, $j = 0, 1, 2, ...$, is the $SO(3)$ Casimir operator. Based on which, the energy eigenvalues satisfy

$$
\epsilon - m = -\frac{\epsilon + m}{(2j + 1)^2} + \frac{\lambda}{\epsilon + m}j(j + 1).
$$

(33)

When $\lambda \to 0$, it reduces to the result in the plane as Eq. (15). In the non-relativistic limit, when $m \to \infty$ and $\epsilon \to m$, Eq. (33) leads to the energy levels given in [21].

**Harmonic oscillator potential.** For the harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2r^2$, we also deal with it by using Eq. (16). Then, the quasi-Hamiltonian becomes

$$
\tilde{H} = \frac{1}{2m}(\pi^2 + \lambda L^2) + \frac{1}{2}\tilde{m}\omega^2 r^2,
$$

(34)

which commutes with the angular momentum $L$ and the second order tensors

$$
\tilde{s}_1 = \frac{1}{\tilde{m}\omega^2} \left( \frac{\pi_1 \pi_2 + \pi_2 \pi_1}{2} + \tilde{m}\omega x_1 x_2 \right),
$$

$$
\tilde{s}_2 = \frac{1}{\tilde{m}\omega^2} \left( \frac{\pi_1^2 - \pi_2^2}{2} + \tilde{m}\omega x_1^2 - x_2^2 \right).
$$

(35)

Set $J_\pm = \frac{1}{2}(\tilde{s}_2 \pm i\tilde{s}_1)$ and $J_3 = \frac{1}{2}L$. They satisfy the Higgs algebra relations $[J_3, J_\pm] = \pm J_\pm$, $[J_+, J_-] = a_3 J_3^2 + a_1 J_3$, with $a_1 = 2(1 - \frac{\lambda^2}{4m^2\omega^2} + \frac{\lambda}{m\omega^2}\tilde{H})$ and $a_3 = -4\frac{\lambda^2}{m^2\omega^2}$.

The anticommutation relation is

$$
\{J_+, J_-\} = 2\frac{\lambda^2}{m^2\omega^2} J_3^4 + \left(-\frac{2\lambda}{m}\tilde{H} - 2 + \frac{5\lambda^2}{2m^2\tilde{\omega}^2}\right) J_3^2 + \left(\frac{\tilde{H}^2}{2\omega^2} - \frac{1}{2} - \frac{\lambda}{2m\omega^2}\tilde{H}\right),
$$

(36)

and the Casimir of the Higgs algebra reads

$$
C = \{J_+, J_-\} + \left(a_1 + \frac{a_3}{2}\right) J_3^2 + \frac{a_3}{2} J_3^4
$$

$$
= \frac{\tilde{H}^2}{2\omega^2} - \frac{1}{2} - \frac{\lambda}{2m\omega^2}\tilde{H}
$$

$$
= a_1 C_{su2} + \frac{a_3}{2} C_{su2}^2,
$$

(37)

where $C_{su2} = s(s + 1)$, $s = 0, \frac{1}{2}, 1, ...$, is the $SU(2)$ Casimir operator. Thus, we obtain the equation which the energy levels satisfy as

$$
\epsilon - m = \frac{\lambda(n + 1)^2}{\epsilon + m} + \sqrt{\frac{2m}{\epsilon + m}\omega^2 + \frac{\lambda^2}{(\epsilon + m)^2}}(n + 1),
$$

(38)

where $n = 2s = 0, 1, 2, ...$. This result shows that the KG equation with equal scalar and vector harmonic oscillator potential has the $SU(2)$ symmetry. When $\lambda \to 0$, Eq. (38) reduces to the energy in Eq. (21). And the non-relativistic limit give the result in [21], when $m \to \infty$ and and the coefficient of elasticity $m\omega^2$ keeps unchangeably.
IV. CONCLUSION AND DISCUSSION

To discuss the symmetry of the K-G equation with ESVP, we have introduced a quasi-Hamiltonian \( \tilde{H} \). The generators of the symmetry group commute with \( \tilde{H} \). We have investigated the motion of a relativistic scalar particle both in a plane and a sphere by providing some examples. The symmetry is shown to be the \( SO(3) \) for the Coulomb potential and the \( SU(2) \) for the harmonic oscillator potential. Specially, in the sphere, the generators according to the two types of potentials construct the Higgs algebra respectively. The Casimir operators of these systems can be used to calculate the energy spectra straightway.

The procedure, to study the dynamical symmetry of the KG with ESVP in this work, is applicable in not only 2D systems but also three-dimensional or N-dimensional (ND) systems. We can foretell the ND KG equation with ESVP has the \( SO(N+1) \) symmetry for the Coulomb potential and the \( SU(N) \) for harmonic potential. These results exhibit a new visual angle to understand the dynamical symmetries in the Dirac systems with spin or pseudospin symmetry. From the results of the above examples, we can put forward a uniform approach to solve the spectra of the KG equation or the Dirac equation with ESVP, i.e., if the Schrödinger equation with a certain potential is integrable, the energy level is expressed as a function of the mass \( m \), the parameters \( \{k_\alpha\} \) involving in the potential \( V(r) \) and a set of good quantum numbers \( \{n_j\} \) as \( \varepsilon = \varepsilon(m,k_\alpha,n_j) \), then the corresponding relativistic energy spectra of the KG equation with ESVP satisfies \( \epsilon - m = \varepsilon(\frac{1}{2}(\epsilon + m),k_\alpha,n_j) \).

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