Scattering Problem for Klein–Gordon Equation with Cubic Convolution Nonlinearity

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Abstract The scattering problem for the Klein–Gordon equation with cubic convolution nonlinearity is considered. Based on the Strichartz estimates for the inhomogeneous Klein–Gordon equation, we prove the existence of the scattering operator, which improves the known results in some sense.

Keywords Asymptotic of solution, Klein–Gordon equation, scattering operator

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1 Introduction

This paper is concerned with the scattering problem for the nonlinear Klein–Gordon equation of the form

\[
\begin{aligned}
\partial_t^2 u - \Delta u + u &= F_\gamma(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
|u|_{t=0} &= f(x), \quad |\partial_t u|_{t=0} = g(x),
\end{aligned}
\]  

(1.1)

where \( u \) is a real-valued or a complex-valued unknown function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\). The nonlinearity is a cubic convolution term \( F_\gamma(u) = -(V_\gamma(x) \ast |u|^2)u \) with \(|V_\gamma(x)| \leq C|x|^{-\gamma}\). Here, \( 0 < \gamma < n \) and \( \ast \) denotes the convolution in the space variables. The term \( F_\gamma(u) \) is an approximative expression of the nonlocal interaction of specific elementary particles. The equation (1.1) was studied by Menzala and Strauss in [4].

In order to define the scattering operator for (1.1), we first give some Banach spaces. The usual Lebesgue space is given by \( L^p = \{ \phi \in S' : \|\phi\|_{L^p} < +\infty \} \), where the norm \( \|\phi\|_{L^p} = \left( \int_{\mathbb{R}^n} |\phi(x)|^p dx \right)^{1/p} \) if \( 1 \leq p < +\infty \) and \( \|\phi\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |\phi(x)| \) if \( p = +\infty \). The weighted Sobolev space \( H^{\beta,k}_p \) is defined by

\[
H^{\beta,k}_p = \{ \phi \in S' : \|\phi\|_{H^{\beta,k}_p} = \|\langle x \rangle^k (i\nabla)^{\beta} \phi\|_{L^p} < +\infty \},
\]

with \( \langle x \rangle = \sqrt{1 + x^2} \) and \( \langle i\nabla \rangle = \sqrt{1 - \Delta} \). We also write \( H^{\beta,k}_p = H^{\beta,k}_2 \) if it does not cause a confusion. A Hilbert space \( X^{\beta,k}_\rho \) is denoted by \( H^{\beta,k}_2 \oplus H^{\beta-1,k}_2 \). Let \( X^{\beta,k}_\rho \) be a ball of a radius \( \rho > 0 \) with a center in the origin in the space \( X^{\beta,k}_2 \). The scattering operator of (1.1) is defined as the mapping \( S : X^{\beta,k}_\rho \ni (f_-, g_-) \rightarrow (f_+, g_+) \in X^{\beta,0}_\rho \) if the following condition holds:

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For \((f_-, g_-) \in X^\beta_{\rho, k}\), there uniquely exists a time-global solution \(u \in C(\mathbb{R}; H^\beta)\) of (1.1), and data \((f_+, g_+) \in X^{\beta, 0}\) such that \(u(t)\) approaches \(u_{\pm}(t)\) in \(H^\beta\) as \(t \to \pm \infty\), where \(u_{\pm}(t)\) are solutions of linear Klein–Gordon equations whose initial data are \((f_{\pm}, g_{\pm})\), respectively.

We say that \((S, X^{\beta, k})\) is well defined if we can define the scattering operator \(S : X^\beta_{\rho, k} \to X^{\beta, 0}\) for some \(\rho > 0\). In [5], Mochizuki proved that if \(n \geq 3\), \(\beta \geq 1, \gamma < n\) and \(2 \leq \gamma \leq 2\beta + 2\), then \((S, X^{\beta, 0})\) is well defined. By using the Strichartz estimate for pre-admissible pair and the complex interpolation method for the weighted Sobolev space, Sasaki [8] showed that \((S, X^{\beta, k})\) is well defined if \(n \geq 2\), \(\beta \geq 1, 4/3 \leq \gamma < 2\) and \(k > (2 - \gamma)/2\). Our aim of this article is to show that \((S, X^{\beta, 1})\) is well defined if \(n \geq 2\).

More precisely, we prove the following theorem.

**Theorem 1.1**  Let the function \(V_\gamma (x)\) satisfy

\[
|V_\gamma (x)| \leq C|x|^{-\gamma}, \quad |\nabla V_\gamma (x)| \leq C|x|^{-(1+\gamma)}.
\]

Assume that \(n \geq 2\), \(\gamma\) and \(\beta\) satisfy (1.2). Then there exists a positive number \(\delta_0 > 0\) satisfying the following properties:

1. For \((f, g) \in X^{\beta, 1}\) with \(\| (f, g) \|_{X^{\beta, 1}} \leq \delta_0\), there uniquely exist final states \((f_{\pm}, g_{\pm}) \in X^{\beta, 0}\) and a solution \(u(t) \in C(\mathbb{R}; H^\beta)\) of (1.1) such that \(u(t)\) approaches \(u_{\pm}(t)\) in \(X^{\beta, 0}\) as \(t \to \pm \infty\), where \(u_{\pm}(t)\) are solutions of the linear Klein–Gordon equation with initial data \((f_{\pm}, g_{\pm})\), respectively. Moreover, as \(\varepsilon t\) large enough we have

\[
\|(u(t), \partial_t u(t)) - (u_{\pm}(t), \partial_t u_{\pm}(t))\|_{X^{\beta, 0}} \leq C(t)^{-\delta},
\]

with \(\delta = \frac{2n\beta}{n+2} - 2 > 0\).

2. For \((f_-, g_-) \in X^{\beta, 1}\) with \(\| (f_-, g_-) \|_{X^{\beta, 1}} \leq \delta_0\), there uniquely exist a final state \((f_+, g_+) \in X^{\beta, 0}\) and a solution \(u(t) \in C(\mathbb{R}; H^\beta)\) of (1.1) such that \(u(t)\) approaches \(u_{\pm}(t)\) in \(X^{\beta, 0}\) as \(t \to \pm \infty\), where \(u_{\pm}(t)\) are solutions of the linear Klein–Gordon equation with initial data \((f_{\pm}, g_{\pm})\), respectively. Moreover, as \(\varepsilon t\) large enough we have

\[
\|(u(t), \partial_t u(t)) - (u_{\pm}(t), \partial_t u_{\pm}(t))\|_{X^{\beta, 0}} \leq C(t)^{-\delta},
\]

with \(\delta = \frac{2n\beta}{n+2} - 2 > 0\).

In this article we denote by \(J_\varepsilon = (i\nabla) x + i\varepsilon t\nabla, L_\varepsilon = i\partial_t - \varepsilon (i\nabla)\) and \(P = t\nabla + x\partial_t\) with \(\varepsilon \in \{+, -\}\). For a given Banach space with norm \(\| \cdot \|\) and a vector \(v = (v^+, v^-)\), denote

\[
\|v\| = \|v^+\| + \|v^-\|, \quad \|Pv\| = \|Pv^+\| + \|Pv^-\|, \quad \|Jv\| = \|Jv^+\| + \|Jv^-\|, \quad \|Lv\| = \|Lv^+\| + \|Lv^-\|.
\]

We also denote the space-time norm by

\[
\|\phi\|_{L^2(I, L^2)} = \|\phi(t)\|_{L^2(P^\beta)} \|L^2(I),
\]

where \(I\) is a bounded or unbounded time interval, and denote different positive constants by the same letter \(C\).

The rest of the article is organized as follows. In Section 2, we give some preliminary calculations. Section 3 is devoted to the proof of Theorem 1.1.
2 Preliminaries

In this section, we prove some lemmas for our main results. Let \( w^\varepsilon = i\partial_t (i\nabla)^{-1} u - \varepsilon u \) with \( \varepsilon \in \{+,-\} \). Then the Klein–Gordon equation (1.1) can be rewritten as a system of equations

\[
\begin{aligned}
L_\varepsilon w^\varepsilon &= (i\nabla)^{-1} F_\gamma(u), \\
\left| w^\varepsilon \right|_{t=0} &= w^\varepsilon_0,
\end{aligned}
\]

where \( L_\varepsilon = i\partial_t - \varepsilon (i\nabla) \), \( w^\varepsilon_0 = i(\nabla)^{-1} g + \varepsilon f \). By the fact that

\[
u = \frac{1}{2}(w^+ - w^-), \quad \partial_t u = -\frac{i}{2}(i\nabla)(w^+ + w^-),
\]

we can rewrite the term \( F_\gamma(u) \) as

\[
F_\gamma(u) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma \ast \omega^{\varepsilon_1} w^{\varepsilon_2}) \omega^{\varepsilon_3}
\]

with some constants \( C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \). Denote \( U_\varepsilon(t)\varphi = e^{-\varepsilon(i\nabla)t} \varphi \) and for given \( T \in \mathbb{R} \),

\[
\Psi_{\varepsilon}[g] = \int_T^t U_\varepsilon(t - \tau)(i\nabla)^{-1} g(\tau) d\tau.
\]

**Lemma 2.1** Let \( 2 \leq q < \frac{2n}{n-2} \), \( \frac{2}{r} = \frac{n}{q} - \frac{2}{3} \). Then for any time interval \( I \) and for any given \( T \in I \), the following estimates are true:

\[
\| \Psi_{\varepsilon}[g] \|_{L^1(I,L^q)} \leq \| g \|_{L^r_\varepsilon(I,H^{2\mu})},
\]

\[
\| \Psi_{\varepsilon}[g] \|_{L^r_\varepsilon(I,L^2)} \leq \| g \|_{L^r_\varepsilon(I,H^{2\mu})},
\]

and

\[
\| U_\varepsilon(t)\varphi \|_{L^1(I,L^q)} \leq \| \varphi \|_{H^\mu},
\]

where \( r' = \frac{r}{r-1}, \ q' = \frac{n}{q-1} \) and \( \mu = \frac{1}{2}(1 + \frac{n}{2})(1 - \frac{2}{q}) \).

The proof of Lemma 2.1 is based on the duality argument along with the \( L^p - L^q \) time decay estimates. Similar results can be found in [1].

**Lemma 2.2** Assume \( 2 \leq p < \frac{2n}{n-2} \) for \( n \geq 3 \) (\( 2 \leq p < +\infty \) for \( n = 2 \)), denote \( \alpha = (1 + \frac{n}{2})(1 - \frac{2}{p}) \). The estimate

\[
\| \phi \|_{L^p} \leq C(1 - \frac{2}{p})^{(1 - \frac{2}{p})}(\| \phi \|_{H^\alpha} + \| J_{\varepsilon_{\phi}} \|_{H^\alpha - 1})
\]

is valid for all \( t \in \mathbb{R} \), provided that the right-hand side is finite.

This lemma comes from [1, Lemma 2.1] and the fact that \( \| \phi \|_{L^p} \leq C\| \phi \|_{H^\alpha} \) when \( p \geq 2 \).

**Lemma 2.3** Assume \( |V_\gamma(x)| \leq |x|^{-\gamma} \) with \( 0 < \gamma < \eta \), \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\} \).

1. For \( 1 < r < +\infty \), \( 1 < p_1, p_2 < +\infty \) and \( p_3 > r \) satisfying \( 1 + \frac{1}{r} = \frac{2}{n} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \), we have

\[
\| (V_\gamma \ast \omega^{\varepsilon_1} w^{\varepsilon_2}) w^{\varepsilon_3} \|_{L^r} \leq \| w^{\varepsilon_1} \|_{L^{p_1}} \| w^{\varepsilon_2} \|_{L^{p_2}} \| w^{\varepsilon_3} \|_{L^{p_3}}.
\]

2. For \( \rho > 0 \), \( 1 < r < +\infty \), \( 1 < p_{jk} < +\infty \) for \( j,k \in \{1,2\} \) and \( p_{13}, p_{23} > r \) satisfying \( 1 + \frac{1}{r} = \frac{2}{n} + \frac{1}{p_{13}} + \frac{1}{p_{23}} + \frac{1}{p_{13}} \), we have

\[
\| (V_\gamma \ast \omega^{\varepsilon_1} w^{\varepsilon_2}) w^{\varepsilon_3} \|_{H^\rho} \leq \| w^{\varepsilon_1} \|_{H^\rho_{p_{13}}} \| w^{\varepsilon_2} \|_{L^{p_{12}}} \| w^{\varepsilon_3} \|_{L^{p_{13}}} + \| w^{\varepsilon_1} \|_{L^{p_{12}}} \| w^{\varepsilon_2} \|_{H^\rho_{p_{11}}} \| w^{\varepsilon_3} \|_{L^{p_{13}}}.
\]
Proof To prove (1), put $\frac{1}{p_4} = \frac{1}{r} - \frac{1}{p_3}$. By the Hölder inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\| (V_\gamma * w^{f_1} w^{f_2}) w^{f_3} \|_{L^r} \leq \| V_\gamma * w^{f_1} w^{f_2} \|_{L^p_{p_4}} \| w^{f_3} \|_{L^p_{p_3}}$$

since $1 + \frac{1}{p_4} = \frac{2}{n} + \frac{1}{p_1} + \frac{1}{p_2}$.

To prove (2), we set $\frac{1}{p} = \frac{1}{p_4} + \frac{1}{p_3}$ and $\frac{1}{r} = \frac{1}{p_4} + \frac{1}{p_3}$. For $\rho > 0$, the generalized Hölder inequality in [7] implies

$$\| (V_\gamma * w^{f_1} w^{f_2}) w^{f_3} \|_{H^p_{p_4}} \leq \| V_\gamma * w^{f_1} w^{f_2} \|_{H^p_{p_4}} \| w^{f_3} \|_{L^p_{p_3}} + \| V * w^{f_1} w^{f_2} \|_{L^p_{p_3}} \| w^{f_3} \|_{H^p_{p_4}}.$$ 

By the generalized Hölder inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\| V_\gamma * w^{f_1} w^{f_2} \|_{H^p_{p_1}} \leq \| V_\gamma * \langle i \nabla \rangle^\beta (w^{f_1} w^{f_2}) \|_{L^p_{p_1}} \leq \| \langle i \nabla \rangle^\beta (w^{f_1} w^{f_2}) \|_{L^p_{p_1}}$$

since $1 + \frac{1}{p_4} = \frac{2}{n} + \frac{1}{p_1}$ and $\frac{1}{p_1} = \frac{1}{p_1} + \frac{1}{p_2}$. Similarly, we have

$$\| V_\gamma * w^{f_1} w^{f_2} \|_{L^p_{p_2}} \leq \| w^{f_1} \|_{L^p_{p_2}} \| w^{f_2} \|_{L^p_{p_2}}.$$ 

3 Proof of Theorem 1.1

For $1 < \gamma < \min \left\{ \frac{2(n+1)}{n+2}, \frac{3n-2}{n+2} \right\}$, we choose

$$\frac{(n+2)(\gamma+1)}{4n} + \frac{1}{2} < \beta < \frac{(n+2)(\gamma+1)}{2n}, \quad q = \left( \frac{2\beta}{n+2} + \frac{1}{2} - \frac{\gamma+1}{n} \right)^{-1}.$$ 

They satisfy

$$1 \leq \beta \leq 2, \quad 2 < q < \frac{2n}{n + 2(1 - \gamma)}, \quad 1 < \gamma < \frac{3n\beta}{n + 2}.$$ 

Letting $\mu = \frac{1}{2} (1 + \frac{n}{2}) (1 - \frac{2}{q})$, we also have

$$\mu + \beta - 2 \leq 0, \quad \mu \leq \beta - 1, \quad \text{and} \quad 0 < \mu \leq \frac{1}{2}.$$ 

Let $r, p$ and $s$ be chosen as

$$\frac{2}{r} = \frac{n}{2} \left( 1 - \frac{2}{q} \right), \quad \frac{2}{p} + \frac{2}{n} = 2 - \frac{2}{q}, \quad \frac{2}{s} = 1 - \frac{2}{r}.$$ 

Proof of Theorem 1.1 (1) Introduce the function space

$$X = \{ v = (v^+, v^-) \in C(\mathbb{R}; (L^2(\mathbb{R}^n))^2); \quad \| v \|_X < +\infty \}$$

with the norm

$$\| v \|_X = \| v \|_{L^p(\mathbb{R}, H^\beta)} + \| v \|_{L^p(\mathbb{R}, H^\beta - \mu)} + \| \partial_1 v \|_{L^p(\mathbb{R}, H^\beta - 1)} + \| \partial_2 v \|_{L^p(\mathbb{R}, L^q)} + \| P v \|_{L^p(\mathbb{R}, H^\beta - 1)} + \| J v \|_{L^p(\mathbb{R}, H^\beta - 1)}.$$ 

Denote by $X_\rho$, a ball of a radius $\rho > 0$ with a center in the origin in the space $X$. Let us consider the linearized version of (2.1)

$$\begin{cases}
L_\phi w^\varepsilon = (i \nabla)^{-1} F_\gamma(v), \\
\| w^\varepsilon \|_{L^p} = \| w_0^\varepsilon \|,
\end{cases}$$

(3.1)
with a given vector $v = (v^+, v^-) \in X_p$, where

$$F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3},$$

with some given constants $C_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$. The integration of the linearized Cauchy problem (3.1) with respect to time yields

$$w^\varepsilon = U_\varepsilon(t)w_0^\varepsilon + \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}). \quad (3.2)$$

Taking the $L_t^\infty(\mathbb{R}; H_\beta^\mu)$-norm of (3.2), applying the Hölder inequality, Lemmas 2.1 and 2.3, we find

$$\|w^\varepsilon\|_{L_t^\infty(\mathbb{R}; H_\beta^\mu)} \leq \|U_\varepsilon(t)w_0^\varepsilon\|_{L_t^\infty(\mathbb{R}; H_\beta^\mu)} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|\Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3})\|_{L_t^\infty(\mathbb{R}; H_\beta^\mu)}$$

$$\leq \|w_0^\varepsilon\|_{H_\beta^\mu} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}\|_{L_t^\infty(\mathbb{R}; H_\beta^\mu+\mu-1)}$$

$$\leq \|w_0\|_{H_\beta^\mu} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|v^\varepsilon\|_{H_\beta^\mu+\mu-1} \|v^{\varepsilon_1}\|_{L^p} \|v^{\varepsilon_2}\|_{L^p} \|v^{\varepsilon_3}\|_{L_t^\infty(\mathbb{R})}$$

$$\leq \|w_0\|_{H_\beta^\mu} + C \rho \|v^\varepsilon\|^2_{L_t^2(\mathbb{R}; L^p)}, \quad (3.3)$$

since $p > 2 > q'$, $q > 2 > q'$, $\mu \leq \frac{1}{2}$ and $2 - \frac{2}{q} = \frac{2}{n} + \frac{2}{p}$. Similarly, taking the $L_t^1(\mathbb{R}; H_q^{\beta-\mu})$, we obtain

$$\|w^\varepsilon\|_{L_t^1(\mathbb{R}; H_q^{\beta-\mu})} \leq \|U_\varepsilon(t)w_0^\varepsilon\|_{L_t^1(\mathbb{R}; H_q^{\beta-\mu})} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|\Psi_\varepsilon((V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3})\|_{L_t^1(\mathbb{R}; H_q^{\beta-\mu})}$$

$$\leq \|w_0^\varepsilon\|_{H_\beta^\mu} + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} \|(V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}\|_{L_t^\infty(\mathbb{R}; H_q^{\beta+\mu-1})}$$

$$\leq \|w_0\|_{H_\beta^\mu} + C \|v^\varepsilon\|_{L_t^1(\mathbb{R}; H_q^{\beta+\mu-1})} \|v^\varepsilon\|^2_{L_t^2(\mathbb{R}; L^p)}$$

$$\leq \|w_0\|_{H_\beta^\mu} + C \rho \|v^\varepsilon\|^2_{L_t^2(\mathbb{R}; L^p)}, \quad (3.4)$$

since $\mu \leq \frac{1}{2}$, $p > 2 > q'$, $q > 2 > q'$ and $2 - \frac{2}{q} = \frac{2}{n} + \frac{2}{p}$. Applying the operator $\partial_t$ to (3.1), we deduce that $\partial_t w^\varepsilon$ satisfies the following system

$$\begin{cases} L_\varepsilon \partial_t w^\varepsilon = (i\nabla)^{-1}\partial_t F_\gamma(v), \\ \partial_t w^\varepsilon|_{t=0} = -i\varepsilon(i\nabla)w_0^\varepsilon - i(\nabla)^{-1}F_\gamma(v)|_{t=0}, \end{cases}$$

with

$$F_\gamma(v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+, -\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma * \overline{v^{\varepsilon_1} v^{\varepsilon_2}}) v^{\varepsilon_3}.$$
Applying the operator \( H^{\alpha - 1} + \| \partial_t F_\gamma (v) \|_{L_t' (\mathbb{R}; H^{\rho + \beta - 2})} \)
\[ \leq \| \partial_t w^\varepsilon \|_{t=0} \| H^{\beta - 1} + \| \partial_t w^\varepsilon \|_{L_t' (\mathbb{R}; H^{\rho + \beta - 2})} \]
\[ + C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}} \|(V_\gamma \ast (\partial_t v^{\varepsilon_1} v^{\varepsilon_2} + v^{\varepsilon_1} \partial_t v^{\varepsilon_2})) v^{\varepsilon_3} + (V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) \partial_t v^{\varepsilon_3} \|_{L_t' (\mathbb{R}; L^{\rho})} \]
\[ \leq \| \partial_t w^\varepsilon \|_{t=0} \| H^{\beta - 1} + C \rho \| v \|_{L_t' (\mathbb{R}; L^p)}^2 \]

On the other hand, we have
\[ \| \partial_t w^\varepsilon \|_{t=0} \| H^{\beta - 1} \leq \| w_0^\varepsilon \|_{H^\beta} + \| F_\gamma (v) \|_{L_t^\infty (\mathbb{R}; H^{\beta - 2})} \]
and for \( p_1 \) satisfying \( \frac{3}{2} = \frac{2}{n} + \frac{3}{p_1} \),
\[ \| F_\gamma (v) \|_{L_t^\infty (\mathbb{R}; H^{\beta - 2})} \leq C \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}} \|(V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) v^{\varepsilon_3} \|_{L_t^\infty (\mathbb{R}; L^2)} \]
\[ \leq C \| v \|_{L_t^\infty (\mathbb{R}; L^p)}^3 \leq C \| v \|_{L_t^\infty (\mathbb{R}; H^\beta)} \leq C \rho^3 \]

since \( \beta \leq 2, \gamma \leq 3 \beta \) and \( \| v \|_{L_t^p} \leq C \| v \|_{H^\beta} \). Then
\[ \| \partial_t w^\varepsilon \|_{L_t^\infty (\mathbb{R}; H^{\beta - 1})} + \| \partial_t w^\varepsilon \|_{L_t (\mathbb{R}; L^p)} \leq C \| u_0 \|_{H^\beta} + C \rho^3 + C \rho \| v \|_{L_t^\infty (\mathbb{R}; H^\beta)} \| v \|_{L_t^\infty (\mathbb{R}; L^p)}^2 \]
(3.5)

Notice that \( P = t \nabla + x \partial_t, J_\varepsilon = (i \nabla) x + i \varepsilon t \nabla \) and \( L_\varepsilon = i \partial_t - \varepsilon (i \nabla) \). We get
\[ J_\varepsilon = i \varepsilon P - \varepsilon L_\varepsilon, \quad [L_\varepsilon, P] = -i \varepsilon (i \nabla)^{-1} \nabla L_\varepsilon, \]
\[ [x, (i \nabla)] = (i \nabla)^{-1} \nabla, \quad [P, (i \nabla)^{-1}] = (i \nabla)^{-3} \nabla \partial_t \]
and
\[ P ((V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) v^{\varepsilon_3}) = (V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) P (v^{\varepsilon_3}) + (t \nabla V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) v^{\varepsilon_3}. \]

Applying the operator \( P \) to (3.1) yields
\[ \begin{cases} L_\varepsilon P w^\varepsilon = i \varepsilon (i \nabla)^{-2} \nabla F_\gamma (v) - (i \nabla)^{-1} P F_\gamma (v) - (i \nabla)^{-3} \nabla \partial_t F_\gamma (v), \\
\quad P w^\varepsilon |_{t=0} = x \partial_t w^\varepsilon |_{t=0} = x (i \varepsilon (i \nabla) w_0^\varepsilon - i (i \nabla)^{-1} F_\gamma (v)) |_{t=0}, \end{cases} \]
with
\[ P F_\gamma (v) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+,-\}} C_{\varepsilon_1 \varepsilon_2 \varepsilon_3} (V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) P v^{\varepsilon_3} + (t \nabla V_\gamma \ast v^{\varepsilon_1} v^{\varepsilon_2}) v^{\varepsilon_3}. \]

Integrating with respect to time, we get
\[ P w^\varepsilon = U_\varepsilon (t) (P w^\varepsilon |_{t=0}) - \Psi_\varepsilon (i \varepsilon (i \nabla)^{-1} \nabla F_\gamma (v)) + \Psi_\varepsilon (P F_\gamma (v)) + \Psi_\varepsilon (i \varepsilon (i \nabla)^{-2} \nabla \partial_t F_\gamma (v)). \]
(3.6)
Taking the \( L_t^\infty (\mathbb{R}; H^{\beta - 1}) \)-norm and the \( L_t^1 (\mathbb{R}, L^q) \)-norm of (3.6), applying the H"older inequality and Lemma 2.1, we find
\[ \begin{align*}
\| P w^\varepsilon \|_{L_t^\infty (\mathbb{R}; H^{\beta - 1})} + \| P w^\varepsilon \|_{L_t^1 (\mathbb{R}, L^q)} & \leq \| P w^\varepsilon |_{t=0} \| H^{\beta - 1} + \| (i \nabla)^{-1} \nabla F_\gamma \|_{L_t^1 (\mathbb{R}, L^{q'})} \\
& + \| P F_\gamma (v) \|_{L_t^1 (\mathbb{R}; L^{q'})} + \| (i \nabla)^{-2} \nabla \partial_t F_\gamma (v) \|_{L_t^1 (\mathbb{R}; L^{q'})} \\
& \leq \| P w^\varepsilon |_{t=0} \| H^{\beta - 1} + \| F_\gamma (v) \|_{L_t^1 (\mathbb{R}; L^{q'})} \\
& + \| P F_\gamma (v) \|_{L_t^1 (\mathbb{R}; L^{q'})} + \| \partial_t F_\gamma (v) \|_{L_t^1 (\mathbb{R}; L^{q'})},
\end{align*} \]
(3.7)
since $\beta \geq 1$ and $\mu + \beta - 2 \leq 0$ and $\mu \leq \beta - 1$. As in the proof of (3.5), we deduce
\[
\|F_\gamma(v)\|_{L_t^\infty(\mathbb{R};L_x^2)} + \|\partial_t F_\gamma(v)\|_{L_t^{\infty}(\mathbb{R};L_x^2)} \\
\leq C\|v\|_{L_t^4(\mathbb{R};L_x^4)}\|v\|_{L_t^8(\mathbb{R};L_x^8)}^2 + C\|\partial_t v\|_{L_t^{4}(\mathbb{R};L_x^4)}\|v\|_{L_t^8(\mathbb{R};L_x^8)}^2 \leq C\rho\|v\|_{L_t^{4}(\mathbb{R};L_x^2)}^2. \tag{3.8}
\]
Let $p_3 > 2$ and $s_3 > 2$ satisfy
\[
\frac{3}{2} - \frac{1}{q} = \frac{\gamma + 1}{n} + \frac{2}{p_3}, \quad 1 - \frac{1}{r} = \frac{2}{s_3}.
\]
The Hölder inequality and Lemma 2.3 imply
\[
\|PF_\gamma(v)\|_{L_t^{4}(\mathbb{R};L_x^2)} \\
\leq C \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+, -\}} \left[ \|\left(\nabla v_\gamma * \frac{1}{\varepsilon_1^q} v^{\varepsilon_2}\right)\|_{L_t^{4}(\mathbb{R};L_x^2)} + \left(\int_{\mathbb{R}} F_\gamma v^{\varepsilon_3}\right)\|_{L_t^{4}(\mathbb{R};L_x^2)} \right] \\
\leq C\|Pv\|_{L_t^{4}(\mathbb{R};L_x^2)}\|v\|_{L_t^{8}(\mathbb{R};L_x^8)} + C\|v\|_{L_t^{4}(\mathbb{R};L_x^2)}\|t^{1/2} v\|_{L_t^{4}(\mathbb{R};L_x^2)} \\
\leq C\rho\|v\|_{L_t^{4}(\mathbb{R};L_x^2)}^2 + C\rho\|t^{1/2} v\|_{L_t^{4}(\mathbb{R};L_x^2)}^2, \tag{3.9}
\]
here we use the condition $\|\nabla v_\gamma\| \leq C|x|^{-(1+\gamma)}$. By Lemma 2.2, we have
\[
\|v\|_{L_t^{4}(\mathbb{R};L_x^2)} \leq C\left(\|t^{-\frac{\beta}{2}}(1 - \frac{1}{p})\|_{H^{\alpha}} + \|Jv\|_{H^{\alpha-1}}\right) \|L_t^{4}(\mathbb{R})\| \\
\leq C\left(\|v\|_{L_t^{\infty}(\mathbb{R};H^{\beta})} + \|Jv\|_{L_t^{\infty}(\mathbb{R};H^{\beta-1})}\right) \leq C\rho, \tag{3.10}
\]
since $\alpha = (1 + \frac{\beta}{2})(1 - \frac{2}{p}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p}) > \frac{3}{2}$. Similarly,
\[
\|t^{1/2} v\|_{L_t^{4}(\mathbb{R};L_x^2)} \leq C\left(\|t^{-\frac{\beta}{2}}(1 - \frac{1}{p})\|_{H^{\alpha+1}} + \|Jv\|_{H^{\alpha-1}}\right) \|L_t^{4}(\mathbb{R})\| \\
\leq C\left(\|v\|_{L_t^{\infty}(\mathbb{R};H^{\beta})} + \|Jv\|_{L_t^{\infty}(\mathbb{R};H^{\beta-1})}\right) \leq C\rho, \tag{3.11}
\]
since $\alpha_3 = (1 + \frac{n}{2})(1 - \frac{2}{p_3}) \leq \beta$ and $\frac{n}{2}(1 - \frac{2}{p_3}) > \frac{1}{s_3}$. Then we obtain, from (3.7)–(3.11),
\[
\|Pw^{\varepsilon}\|_{L_t^{\infty}(\mathbb{R};H^{\beta-1})} + \|Pw^{\varepsilon}\|_{L_t^{4}(\mathbb{R};L_x^2)} \leq \|Pw^{\varepsilon}\|_{t=0}\|H^{\beta-1} + C\rho^3, \tag{3.12}
\]
\[
\|w^{\varepsilon}\|_{L_t^{\infty}(\mathbb{R};H^{\beta})} + \|w^{\varepsilon}\|_{L_t^{4}(\mathbb{R};L_x^2)} \leq \|w^{\varepsilon}\|_{H^{\beta}} + C\rho^3, \tag{3.13}
\]
\[
\|\partial_t w^{\varepsilon}\|_{L_t^{\infty}(\mathbb{R};H^{\beta-1})} + \|\partial_t w^{\varepsilon}\|_{L_t^{4}(\mathbb{R};L_x^2)} \leq \|w^{\varepsilon}\|_{H^{\beta}} + C\rho^3. \tag{3.14}
\]
To estimate the term $\|Pw^{\varepsilon}\|_{t=0}\|H^{\beta-1}$, we give some estimates. It follows from the Sobolev embedding theorem that
\[
\|F_\gamma(v)\|_{L_t^{\infty}(\mathbb{R};L_x^2)} \leq C \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+, -\}} \|\left(\nabla v_\gamma * \frac{1}{\varepsilon_1^q} v^{\varepsilon_2}\right)\|_{L_t^{\infty}(\mathbb{R};L_x^2)} \\
\leq \|v\|^3_{L_t^{\infty}(\mathbb{R};L_x^2)} \leq C\|v\|^3_{L_t^{\infty}(\mathbb{R};H^{\beta})} \leq C\rho^3, \tag{3.15}
\]
where $p_5 = \frac{6n-2}{3n-2\gamma}$, which satisfies $p_5 \leq \frac{2n}{n-2\gamma}$ because of $\gamma \leq 3\beta$. Using the relation $x = (i\nabla)^{-1}J_v - i\varepsilon_1\langle i\nabla\rangle^{-1}\nabla$, we deduce
\[
\|xF_\gamma(v)\|_{L_t^{\infty}(\mathbb{R};L_x^2)} \\
\leq C \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+, -\}} \|\left(\nabla v_\gamma * \frac{1}{\varepsilon_1^q} v^{\varepsilon_2}\right)\|_{L_t^{\infty}(\mathbb{R};L_x^2)} \\
\leq C \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{+, -\}} \|v^{\varepsilon_1}\|_{L_t^{4}(\mathbb{R};L_x^4)}\|v^{\varepsilon_2}\|_{L_t^{4}(\mathbb{R};L_x^4)}\|\langle i\nabla\rangle^{-1}J_v v^{\varepsilon_3}\|_{L_t^{4}(\mathbb{R};L_x^4)} + t\|\langle i\nabla\rangle^{-1}\nabla v^{\varepsilon_3}\|_{L_t^{4}(\mathbb{R};L_x^4)} \\
\leq C\|v\|^2_{L_t^{\infty}(\mathbb{R};L_x^4)}\|\langle i\nabla\rangle^{-1}J_v v\|_{L_t^{\infty}(\mathbb{R};L_x^4)} + C\|t^{1/3} v\|^3_{L_t^{\infty}(\mathbb{R};L_x^2)}
\[ \leq C\|v\|_{L^\infty_t(H^\alpha)}^2 \|Jv\|_{L^\infty_t(H^\beta - 1)} + C \left(\|v\|_{L^\infty_t(H^\alpha)} + \|Jv\|_{L^\infty_t(H^\beta - 1)}\right)^3 \]
\[ \leq C \left(\rho + \|Jv\|_{L^\infty_t(H^\beta - 1)}\right)^3 \leq C \rho^3, \quad (3.16) \]

where \( p_4 = \frac{6n}{3n - 2\gamma} \), which satisfies
\[ 2 < p_4 \leq \frac{2n}{n - 2\beta}; \quad n \left(1 - \frac{1}{p_4}\right) \geq \frac{1}{3}, \quad \left(1 + \frac{n}{2}\right) \left(1 - \frac{2}{p_4}\right) \leq \beta, \]
because of \( 1 < \gamma \leq \frac{3n\beta}{n^2 + 2}. \) Using the relation \([i\nabla]^{-1}, x] = -(\beta - 1)[i\nabla]^3, \) we deduce
\[ \|Pw^\epsilon\|_{t=0} \leq \|x(i\nabla)w_0^\epsilon\|_{H^\beta - 1} + \|x(i\nabla)^{-1}F_\gamma(v)\|_{L^\infty_t(H^\beta - 1)} \leq \|x(i\nabla)w_0^\epsilon\|_{H^\beta - 1} + \|x(i\nabla)^{-1}F_\gamma(v)\|_{L^\infty_t(H^\beta - 1)} \leq \|x(i\nabla)^{-1}w_0^\epsilon\|_{L^2} + C\|xF_\gamma(v)\|_{L^\infty_t(H^\beta - 1)} \leq \|w_0\|_{H^\beta - 1} + C\rho^3, \]
which, combining with (3.12), yields
\[ \|Pw^\epsilon\|_{L^\infty_t(H^\beta - 1)} + \|Pw^\epsilon\|_{L^1_t(L^q)} \leq \|w_0\|_{H^\beta - 1} + C\rho^3. \quad (3.17) \]

Notice that
\[ [L_\epsilon, x] = -\epsilon(i\nabla)^{-1}x, \quad [x, (i\nabla)^{-1}x] = -(i\nabla)^{-3}x. \]

Then we deduce that \( xw^\epsilon \) satisfies
\[ L_\epsilon(xw^\epsilon) = -\epsilon(i\nabla)^{-1}xw^\epsilon - (i\nabla)^{-1}(xF_\gamma(v)) + (i\nabla)^{-1}xF_\gamma(v). \]

Using \( J_\epsilon = i\epsilon P - \epsilon L_\epsilon x \) and (3.13) yields
\[ \|J_\epsilon w^\epsilon\|_{L^\infty_t(H^\beta - 1)} \leq \|Pw^\epsilon\|_{L^\infty_t(H^\beta - 1)} + \|\epsilon L_\epsilon(xw^\epsilon)\|_{L^\infty_t(H^\beta - 1)}, \]

with
\[ \|\epsilon L_\epsilon(xw^\epsilon)\|_{L^\infty_t(H^\beta - 1)} \leq \|w^\epsilon\|_{L^\infty_t(H^\beta - 1)} + \|\epsilon(i\nabla)^{-2}F_\gamma(v)\|_{L^\infty_t(H^\beta - 1)} + \|\epsilon(i\nabla)^{-1}(xF_\gamma(v))\|_{L^\infty_t(H^\beta - 1)} \leq \|w^\epsilon\|_{L^\infty_t(H^\beta)} + \|F_\gamma(v)\|_{L^\infty_t(H^\beta - 1)} \leq C\|w_0\|_{H^\beta} + C\rho^3. \]

Then we get
\[ \|J_\epsilon w^\epsilon\|_{L^\infty_t(H^\beta - 1)} \leq C\|w_0\|_{H^\beta - 1} + C\rho^3. \quad (3.18) \]

A combination of (3.12) with (3.13), (3.14), (3.17) and (3.18) yields
\[ \|w\|_{X} \leq C\|w_0\|_{H^\beta - 1} + C\rho^3. \quad (3.19) \]

Therefore, the map \( M : w = M(v) \) defined by the problem (3.1) transforms a ball \( X_\rho \) with a small radius \( \rho = C\|w_0\|_{H^\beta - 1} \) into itself. Denote \( \tilde{w} = M(\tilde{v}) \). Then in the same way as in the proof of (3.19) we have
\[ \|M(v) - M(\tilde{v})\|_{X} \leq C\rho^2\|v - \tilde{v}\|_{X}. \]

Thus \( M \) is a contraction mapping in \( X_\rho \) and so there exists a unique solution \( w = M(w) \) of (3.1) if the norm \( \|w_0\|_{H^\beta - 1} \) is small enough.
To prove the asymptotic of the solution \( w(t,x) \), we use the equation, for \( |t| > |t'| \),
\[
U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t') = \int_{t'}^t U_\varepsilon(-\tau)(i\nabla)^{-1}F_\gamma(w(\tau))d\tau.
\]
Taking the \( H^\beta \)-norm of this equation, using similar proofs as that of (3.3) and (3.4), we deduce
\[
\|U_\varepsilon(-t)w^\varepsilon(t) - U_\varepsilon(-t')w^\varepsilon(t')\|_{H^\beta} \leq C\rho^2\langle t'\rangle^{-\delta},
\]
with \( \delta \frac{2n\beta}{n+2} - 2 > 0 \), since we have \( \|w\|_{X} \leq \rho \) and
\[
\|\langle t\rangle^{-\frac{n-1}{2}(1-\frac{1}{T})}\|_{L_1^1([t',t])} \leq C\langle t'\rangle^{-\delta}.
\]
Then there uniquely exist final states \( w^\pm_\varepsilon \in H^\beta \) satisfying, for \( \pm t \) large enough,
\[
\|w^\varepsilon(t) - U_\varepsilon(t)w^\pm_\varepsilon\|_{H^\beta} \leq C\rho^2\langle t\rangle^{-\delta}.
\]
Set \( u(t) = \frac{1}{\varepsilon} (w^+(t) - w^-(t)), f_\pm(x) = \frac{1}{\varepsilon} (w^+_\varepsilon - w^-_\varepsilon), g_\pm(x) = -\frac{1}{\varepsilon} (i\nabla)(w^+_\varepsilon + w^-_\varepsilon) \) and \( u_\pm(t) = \frac{1}{2} (U_\varepsilon(t)w^+_\varepsilon - U_\varepsilon(t)w^-_\varepsilon) \). Then \( u(t) \) and \( u_\pm(t) \) satisfy Theorem 1.1(1).

**Proof of Theorem 1.1(2)** For given \( (f_-,g_-) \in X^{\beta,1} \) and \( v = \{v^+,v^-\} \in X_{\rho} \), we consider the linearized version of the final state problem of (3.1)
\[
\begin{cases}
L_\varepsilon w^\varepsilon = -\langle i\nabla \rangle^{-1}F_\gamma(v), \\
\|U_\varepsilon(t)w^\varepsilon - w^\varepsilon_\varepsilon(x)\|_{H^\beta} \to 0, \text{ as } t \to -\infty,
\end{cases}
\]
with \( w^\varepsilon_\varepsilon(x) = i\langle i\nabla \rangle^{-1}g_-(x) - \varepsilon f_-(x) \in H^{\beta,1} \). The integration with respect to time yields
\[
w^\varepsilon(t) = U_\varepsilon(t)w^\varepsilon_\varepsilon + \int_{-\infty}^t U_\varepsilon(t - \tau)\langle i\nabla \rangle^{-1}F_\gamma(v(\tau))d\tau.
\]
In the same way as in the proof of Theorem 1.1(1), we find that, if \( \|(f_-,g_-)\|_{X^{\beta,1}} \leq \rho \) is small, there uniquely exist a global solution \( w^\varepsilon(t) \in C(\mathbb{R},H^\beta) \) and a final state \( w^\varepsilon_\varepsilon \in H^\beta \) such that, as \( t \to +\infty \),
\[
\|w^\varepsilon(t) - U_\varepsilon(t)w^\varepsilon_\varepsilon\|_{H^\beta} \leq C\langle t\rangle^{-\delta},
\]
with \( \delta \frac{2n\beta}{n+2} - 2 > 0 \). Set \( u(t) = \frac{1}{\varepsilon} (w^+(t) - w^-(t)), f_\pm(x) = \frac{1}{\varepsilon} (w^+_\varepsilon - w^-_\varepsilon), g_\pm(x) = -\frac{1}{\varepsilon} (i\nabla)(w^+_\varepsilon + w^-_\varepsilon) \) and \( u_\pm(t) = \frac{1}{2} (U_\varepsilon(t)w^+_\varepsilon - U_\varepsilon(t)w^-_\varepsilon) \). Then \( u(t) \) and \( u_\pm(t) \) satisfy Theorem 1.1(2).

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