The spinning Kerr-black-hole-mirror bomb: A lower bound on the radius of the reflecting mirror

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
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The intriguing superradiant amplification phenomenon allows an orbiting scalar field to extract rotational energy from a spinning Kerr black hole. Interestingly, the energy extraction rate can grow exponentially in time if the black-hole-field system is placed inside a reflecting mirror which prevents the field from radiating its energy to infinity. This composed Kerr-black-hole-scalar-field-mirror system, first designed by Press and Teukolsky, has attracted the attention of physicists over the last four decades. Previous numerical studies of this spinning black-hole bomb have revealed the interesting fact that the superradiant instability shuts down if the reflecting mirror is placed too close to the black-hole horizon. In the present study we use analytical techniques to explore the superradiant instability regime of this composed Kerr-black-hole-linearized-scalar-field-mirror system. In particular, it is proved that the lower bound $r_m > \frac{1}{2} \left( \frac{1}{\sqrt{1 + \frac{8M}{r_+}} - 1} \right)$ provides a necessary condition for the development of the exponentially growing superradiant instabilities in this composed physical system, where $r_m$ is the radius of the confining mirror and $r_\pm$ are the horizon radii of the spinning Kerr black hole. We further show that, in the linearized regime, this analytically derived lower bound on the radius of the confining mirror agrees with direct numerical computations of the superradiant instability spectrum which characterizes the spinning black-hole-mirror bomb.

I. INTRODUCTION

Within the framework of classical general relativity, the horizon of a black hole acts as a one-way membrane which irreversibly absorbs matter and radiation fields. And yet, the intriguing superradiance phenomenon, first discussed by Zel’dovich [1] more than four decades ago, allows an orbiting cloud made of bosonic particles to extract rotational energy from a spinning black hole. In particular, an incident bosonic (integer-spin) field of azimuthal harmonic index $m$ can be amplified (that is, can gain rotational energy and angular momentum) as it scatters off a spinning Kerr black hole. This superradiant scattering phenomenon occurs in the black-hole spacetime if the proper frequency of the orbiting bosonic field lies in the bounded regime

$$0 < \omega < m\Omega_H ,$$

where $\Omega_H$ is the black-hole angular velocity [see Eq. (2) below].

Interestingly, the energy and angular momentum extraction rates from the spinning Kerr black hole can grow exponentially in time if the co-rotating bosonic cloud is prevented from radiating its (growing) energy to infinity. In particular, Press and Teukolsky [2] have explicitly shown that, by placing the black-hole-bosonic-field system inside a closed cavity (a reflecting mirror), one can build a powerful explosive device which continuously extracts energy and angular momentum from the spinning black hole [5–7]. This composed Kerr-black-hole-bosonic-field-mirror system is known as the spinning black-hole bomb [2, 8–10].

In a very interesting work, Cardoso et. al. [11] have used a combination of numerical and analytical techniques to explore the physical properties of the composed Kerr-black-hole-scalar-field-mirror system. In particular, the numerical results presented in [11] have revealed the interesting fact that, in the linearized regime [12], the superradiant instability shuts down if the reflecting mirror, which is used to confine the Kerr-black-hole-scalar-field system, is placed too close to the horizon of the spinning black hole.

The critical (minimum) radius of the reflecting mirror, $r_m^{c}$, marks the onset of the superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system [11, 13, 14]. In particular, this critical mirror radius corresponds to stationary (marginally stable) confined field modes, which are characterized by the critical (threshold) frequency

$$\omega_c = m\Omega_H$$

for the superradiant scattering phenomenon of bosonic fields in the spinning Kerr black-hole spacetime.

The physical significance of the critical mirror radius $r_m^{c}$ stems from the fact that it is the smallest (innermost) radius of the reflecting mirror which allows one to extract rotational energy from the spinning Kerr black hole. In particular, composed Kerr-black-hole-mirror systems with $r_m < r_m^{c}$ can only support stable (decaying in time) bosonic field
configuration, whereas composed Kerr-black-hole-mirror systems with \( r_m > r^c_m \) can support explosive (exponentially growing in time) bosonic field configuration.

The main goal of the present paper is to study analytically the superradiant instability regime of the composed Kerr-black-hole-scalar-field-mirror system (the spinning black-hole bomb of Press and Teukolsky \[2\]). In particular, we would like to provide a rigorous analytical proof for the existence of a critical (minimum) radius \( r^c_m \) for the reflecting mirror, below which there is no extraction of rotational energy from the spinning black hole. Interestingly, we shall show below that one can use analytical techniques in order to derive an explicit lower bound [see Eq. (37) below] on the critical radius \( r^c_m \) of the confining mirror which marks the onset of the exponentially growing superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system.

II. DESCRIPTION OF THE SYSTEM

We shall analyze the dynamics of a massless scalar field \( \Psi \) which is linearly coupled to a spinning Kerr black hole. The curved black-hole spacetime is described by the line element \[15–17\]

\[
ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\sin^2 \theta}{\rho^2} [a dt - (r^2 + a^2) d\phi]^2 ,
\]

where

\[
\Delta \equiv r^2 - 2Mr + a^2 ; \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta .
\]

Here \( M \) and \( a \) are the mass and angular momentum per unit mass of the Kerr black hole, respectively. The zeros of \( \Delta \),

\[
r_\pm = M \pm \sqrt{M^2 - a^2} ,
\]
determine the horizon radii of the spinning Kerr black hole. The black-hole angular velocity is given by \[15, 16\]

\[
\Omega_H = \frac{a}{r_+ + a} .
\]

The dynamics of a linearized massless scalar field \( \Psi \) in the black-hole spacetime is described by the familiar Klein-Gordon wave equation \[18\]

\[
\nabla^\nu \nabla_\nu \Psi = 0 .
\]

It is convenient to write the scalar eigenfunction \( \Psi \) in the form

\[
\Psi(t, r, \omega, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; m, a\omega) R_{lm}(r; M, a, \omega)e^{-i\omega t} ,
\]

where \( \omega, l, \) and \( m \) are respectively the conserved frequency of the field mode and its angular (spheroidal and azimuthal) harmonic indices.

Substituting the field decomposition \[8\] into the Klein-Gordon wave equation \[17\], one finds that the angular eigenfunctions \( S_{lm} \) satisfy the characteristic angular equation \[18, 23\]

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_{lm}}{d\theta} \right) + \left( \frac{\omega}{r^2} - a \omega \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right) S_{lm} = 0 .
\]

Assuming a regular behavior of the eigenfunctions \( S_{lm}(\theta) \) at the two poles \( \theta = 0 \) and \( \theta = \pi \), one finds that these angular functions are characterized by a discrete family \( \{ K_{lm} \} \) of angular eigenvalues (see \[24, 25\] and references therein). Interestingly, it was proved in \[24\] that the angular eigenvalues \( \{ K_{lm} \} \) associated with the angular eigenfunctions \( \{ S_{lm} \} \) are bounded from below by the characteristic inequality \[26\]

\[
K_{lm} \geq m^2 + a^2 \omega^2 .
\]

Substituting the scalar field decomposition \[8\] into the Klein-Gordon wave equation \[14\], one finds that the radial eigenfunctions \( R_{lm} \) are determined by the differential equation \[18, 19, 27\]

\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR_{lm}}{dr} \right) + \left\{ \omega (r^2 + a^2) - ma \right\}^2 + \Delta (2ma \omega - K_{lm}) \right\} R_{lm} = 0 .
\]
The ordinary differential equation (11), which determines the radial behavior of the scalar eigenfunctions $R_{lm}(r)$, should be supplemented by the physical boundary condition of purely ingoing waves (as measured by a comoving observer) at the horizon of the spinning black hole [6, 7, 28]:

$$R \sim e^{-i(\omega-m\Omega_H)g} \quad \text{for} \quad r \to r_+ \quad (y \to -\infty), \quad (12)$$

where the radial coordinate $y$ is determined by the differential relation $dy = (r^2/\Delta)dr$ [see Eq. (15) below]. In addition, the presence of the reflecting mirror which surrounds the composed Kerr-black-hole-scalar-field system imposes the boundary condition [2, 11, 13, 14]

$$R(r = r_m) = 0 \quad (13)$$
on the spatial behavior of the confined scalar field modes, where $r_m$ is the characteristic radius of the confining mirror.

It is worth emphasizing that the sign of $\Im \omega$ in (8) determines the (in)stability properties of the confined scalar field modes. In particular, stable modes (modes decaying exponentially in time) are characterized by $\Im \omega < 0$, whereas superradiantly unstable modes (modes growing exponentially in time) are characterized by $\Im \omega > 0$. The boundary between stable and unstable solutions of the composed Kerr-black-hole-scalar-field-mirror system is marked by the presence of stationary (marginally stable, with $\Im \omega = 0$) field modes, which are characterized by the critical (threshold) frequency [see Eq. (2)]

$$\omega_c = m\Omega_H \quad (14)$$

for the superradiant scattering phenomenon in the spinning Kerr black-hole spacetime.

### III. THE EFFECTIVE RADIAL POTENTIAL OF THE COMPOSED KERR-BLACK-HOLE-SCALAR-FIELD SYSTEM

The differential equation (11) for the radial scalar eigenfunctions, together with the boundary conditions (12) and (13), determine a discrete set of complex resonant frequencies $\{\omega_n(r_m; M, a)\}$ which characterize the composed Kerr-black-hole-scalar-field-mirror system [2, 11, 13, 14, 29, 30]. As discussed above, Cardoso et. al. [11] have performed a very interesting numerical study of the complex resonant spectrum which characterizes the spinning black-hole bomb. In particular, Cardoso et. al. [11] have revealed the interesting fact that the superradiant instability shuts down if the reflecting mirror is placed too close to the horizon of the spinning black hole.

The main goal of the present paper is to explore the physical properties of the marginally stable (stationary) confined field configurations which characterize the composed Kerr-black-hole-scalar-field-mirror system. In particular, we would like to provide a rigorous analytical proof for the existence of a critical (minimum) radius $r = r_c^m$, which characterizes the reflecting mirror, below which there is no extraction of rotational energy from the spinning Kerr black hole. Furthermore, below we shall derive analytically an explicit lower bound on the critical radius $r_c^m(M, a)$ of the confining mirror which marks the onset of the exponentially growing superradiant instabilities in this composed physical system.

In order to study the physical properties which characterize the composed Kerr-black-hole-scalar-field-mirror system, we shall first transform the radial equation (11) into the more familiar form of a Schrödinger-like differential equation.

Using in Eq. (11) the differential relation [31]

$$dy = \frac{r^2}{\Delta}dr, \quad (15)$$

one obtains the characteristic Schrödinger-like differential equation

$$\frac{d^2\psi}{dy^2} - V(y)\psi = 0 \quad (16)$$

for the radial scalar eigenfunction

$$\psi = rR \quad, (17)$$

where the effective radial potential in (16) is given by

$$V = V(r; \omega, M, a, l, m) = \frac{2\Delta}{r^6}(Mr - a^2) + \frac{\Delta}{r^4}(K_{lm} - 2ma\omega) - \frac{1}{r^2}[\omega(r^2 + a^2) - ma]^2. \quad (18)$$

In the next section we shall study the near-horizon behavior of the radial eigenfunction $\psi$ which characterizes the spatial properties of the confined scalar fields in the spinning Kerr black-hole spacetime. To that end, we shall first explore the near-horizon properties of the effective radial potential $V(r)$ which governs the dynamics of the linearized scalar fields in the black-hole spacetime.
IV. THE NEAR-HORIZON SPATIAL BEHAVIOR OF THE RADIAL SCALAR EIGENFUNCTIONS

The main goal of the present paper is to study the onset of the superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system. Thus, we shall henceforth explore the physical properties of the marginally stable scalar modes, which are characterized by the critical (threshold) frequency \( \bar{\omega} \) for the superradiant scattering phenomenon in the spinning Kerr black-hole spacetime. As emphasized above, the boundary between stable \((\Im \bar{\omega} < 0)\) and unstable \((\Im \bar{\omega} > 0)\) composed Kerr-black-hole-scalar-field-mirror configurations is marked by the presence of these stationary (marginally stable, with \( \Im \bar{\omega} = 0 \)) confined scalar field modes.

For later purposes, we shall prove in this section that the radial scalar eigenfunction \( \psi \), which characterizes the stationary (marginally stable) confined scalar field modes of the composed Kerr-black-hole-scalar-field-mirror system, is a positive \( \psi \geq 0 \), increasing, and convex function in the near-horizon region of the spinning Kerr black-hole spacetime, where

\[
x \ll \tau
\]

are a dimensionless radial coordinate and the dimensionless black-hole temperature, respectively.

Substituting the resonant oscillation frequency \( \bar{\omega} \), which characterizes the marginally stable (stationary) confined field modes, into the expression (18) for effective radial potential of the composed black-hole-scalar-field system, one finds the near-horizon behavior

\[
r^2 V(x \to 0) = F \tau \cdot x + O(x^2),
\]

where the expansion coefficient in (21) is given by

\[
F \equiv K_{lm} - \frac{2(ma)^2}{\bar{r}_+^2 + a^2} + \tau.
\]

Furthermore, substituting the lower bound \( \bar{\alpha} \) on the angular eigenvalues into (22), one finds the characteristic inequality

\[
F > m^2 \left( \frac{\bar{r}_+}{\bar{r}_+ + \bar{r}_-} \right)^2 + \tau > 0
\]

for the near-horizon expansion coefficient of the effective radial potential. We therefore conclude that, in the near-horizon region \( x \ll \tau \), the effective radial potential \( V \) which characterizes the stationary (marginally stable) Kerr-black-hole-scalar-field-mirror configurations has the form of an effective potential barrier with \( V \geq 0 \) for \( x \ll \tau \).

Taking cognizance of the differential relation (15), one finds the near-horizon \( x \ll \tau \) relation

\[
y = \frac{\bar{r}_+}{\tau} \ln(x) + O(x),
\]

which implies \( x = e^{\tau y/\bar{r}_+} [1 + O(e^{\tau y/\bar{r}_+})] \).

Substituting (21) and (20) into (16), one obtains the Schrödinger-like differential equation

\[
\frac{d^2 \psi}{d\tilde{y}^2} - \frac{4F}{\tau} e^{2\tilde{y}} \psi = 0 \quad \text{with} \quad \tilde{y} \equiv \frac{\tau}{2\bar{r}_+} y
\]

in the near-horizon region \( x \ll \tau \). The physically acceptable solution [that is, the solution which respects the physical boundary condition \( \bar{\psi}(\bar{r}_+) = 0 \) at the black-hole horizon] of the Schrödinger-like radial equation (27) is given by

\[
\psi(y) = J_0 \left( 2 \sqrt{\frac{F}{\tau}} e^{\tau y/2\bar{r}_+} \right).
\]

[32]
Using the well-known mathematical properties of the modified Bessel function of the first kind $I_0$ \cite{22}, one finds from the radial solution \cite{22} that the scalar eigenfunction $\psi$, which characterizes the spatial behavior of the stationary (marginally stable) Kerr-black-hole-scalar-field-mirror configurations, is a positive, increasing, and convex function in the near-horizon region \cite{19}. That is,

$$\{ \psi > 0 ; \frac{d\psi}{dy} > 0 ; \frac{d^2\psi}{dy^2} > 0 \} \quad \text{for} \quad 0 < x \ll \tau . \quad (29)$$

Taking cognizance of Eqs. \cite{13} and \cite{29}, one concludes that the radial scalar eigenfunction $\psi$, which characterizes the stationary confined scalar field modes, must have (at least) one maximum point, $x = x_{\text{max}}$, between the horizon of the spinning Kerr black hole [where $\psi$ is a positive and increasing function, see \cite{29}] and the radial location $r = r_m$ of the reflecting mirror [where $\psi$ vanishes, see \cite{13}]. For later purposes, it is important to stress the fact that, at the maximum point $x = x_{\text{max}}$, the scalar eigenfunction $\psi$ is characterized by the inequalities

$$\{ \psi > 0 \quad \text{and} \quad \frac{d^2\psi}{dy^2} < 0 \} \quad \text{for} \quad x = x_{\text{max}} . \quad (30)$$

\section{V. The Superradiant Instability Regime of the Composed Kerr-Black-Hole-Scalar-Field-Mirror System}

The analysis presented in the previous section has revealed the important fact that the eigenfunction $\psi$, which characterizes the radial behavior of the stationary (marginally stable) confined scalar field modes in the spinning Kerr black-hole spacetime, must have (at least) one maximum point, $r = r_{\text{max}}$, within the interval

$$r_+ < r_{\text{max}} < r_m . \quad (31)$$

In particular, taking cognizance of Eqs. \cite{13} and \cite{29}, one realizes that the radial potential \cite{13}, which governs the dynamics of the scalar field in the spinning Kerr black-hole spacetime, must have a negative value at this maximum point. That is,

$$V(r = r_{\text{max}}) < 0 . \quad (32)$$

As we shall now show, one can use this characteristic property of the effective radial potential in order to derive a generic lower bound on the critical radius $r_{\text{c}}^m(M, a)$ of the confining mirror which marks the onset of the superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system.

Substituting the critical oscillation frequency \cite{22} of the stationary (marginally stable) confined field modes into \cite{13}, and using the characteristic lower bound \cite{10} on the angular eigenvalues, one finds that the effective radial potential of the composed black-hole-field-mirror system is characterized by the inequality

$$V(r; \omega = \omega_c) \geq m^2 \cdot \frac{r - r_+}{r^3(r_+^2 + a^2)^2}(-a^2 r^2 - a^2 r_+ r + 2Mr_+^3) + \frac{2\Delta}{r^6}(Mr - a^2) . \quad (33)$$

Furthermore, using the relation $Mr - a^2 \geq 0$ \cite{22}, one finds from \cite{33} the characteristic inequality

$$V(r; \omega = \omega_c) \geq m^2 \cdot \frac{r - r_+}{r^3(r_+^2 + a^2)^2}(-a^2 r^2 - a^2 r_+ r + 2Mr_+^3) \quad (34)$$

for the effective radial potential.

The inequalities \cite{32} and \cite{34} imply that the maximum point $r = r_{\text{max}}$ of the radial scalar eigenfunction $\psi$ is characterized by the inequality

$$-a^2 r_{\text{max}}^2 - a^2 r_+ r_{\text{max}} + 2Mr_+^3 < 0 , \quad (35)$$

which yields the simple lower bound

$$r_{\text{max}} > \frac{r_+}{2} \left( \sqrt{1 + \frac{8Mr_+}{a^2}} - 1 \right) . \quad (36)$$

Finally, taking cognizance of the inequalities \cite{31} and \cite{36}, one obtains the lower bound \cite{36}

$$r_{\text{c}}^m > \frac{r_+}{2} \left( \sqrt{1 + \frac{8Mr_+}{a^2}} - 1 \right) \quad (37)$$
on the critical radius \( r^c_m (M, a) \) of the reflecting mirror which supports the stationary (marginally stable) confined field configurations in the Kerr black-hole spacetime.

Interestingly, the expression on the r.h.s of (37) is a decreasing function of the black-hole rotation parameter \( a \). In particular, in the near-extremal \( a/M \to 1 \) limit one finds the simple lower bound [see Eqs. (20) and (37)]

\[
x^c_m > \frac{\tau}{3}
\]  

(38)
on the dimensionless critical radius \( x^c_m = (r^c_m - r_+)/r_+ \) of the confining mirror.

We would like to stress the fact that the physical significance of the analytically derived lower bound (37) on the critical radius \( r^c_m (M, a) \) [36] of the confining mirror stems from the fact that this inequality provides a necessary condition for the development of the exponentially growing superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system (the spinning black-hole-mirror bomb of Press and Teukolsky [2]).

VI. NUMERICAL CONFIRMATION

It is of physical interest to verify the validity of the analytically derived lower bounds (37) and (38) on the superradiant instability regime of the composed Kerr-black-hole-scalar-field-mirror system (the spinning black-hole-mirror bomb). The resonant oscillation spectrum of this composed physical system was investigated numerically in [11, 13, 14]. In Table I we present the dimensionless ratio \( x^\text{stat}_m / x^\text{bound}_m \) for near-extremal black holes and for various values of the azimuthal harmonic index \( m \) which characterizes the confined scalar mode, where \( x^\text{stat}_m \) is the numerically computed value of the dimensionless mirror radius which corresponds to the stationary (marginally stable) confined scalar fields [37], and \( x^\text{bound}_m \) is the analytically derived lower bound (38) on the dimensionless critical radius of the confining mirror. One finds from Table I that the superradiant instability regime of the composed Kerr-black-hole-scalar-field-mirror system is characterized by the inequality \( x^\text{stat}_m / x^\text{bound}_m > 1 \), in agreement with the analytically derived lower bound (38).

| \( l = m \) | 5   | 25  | 50  | 100 | 150 | 200 |
|-----------|-----|-----|-----|-----|-----|-----|
| \( x^\text{stat}_m / x^\text{bound}_m \) | 10.35 | 2.25 | 1.65 | 1.38 | 1.29 | 1.23 |

TABLE I: The superradiant instability regime of the composed Kerr-black-hole-scalar-field-mirror system (the Press-Teukolsky spinning black-hole-mirror bomb [2]). We display the dimensionless ratio \( x^\text{stat}_m / x^\text{bound}_m \) for near-extremal black holes and for various values of the azimuthal harmonic index \( m \) which characterizes the confined scalar mode, where \( x^\text{stat}_m \) is the numerically computed value of the dimensionless mirror radius which corresponds to the stationary (marginally stable) confined scalar fields [37], and \( x^\text{bound}_m \) is the analytically derived lower bound (38) on the dimensionless critical radius of the confining mirror. One finds that the composed Kerr-black-hole-scalar-field-mirror system is characterized by the inequality \( x^\text{stat}_m / x^\text{bound}_m > 1 \), in agreement with the analytically derived lower bound (38).

VII. THE CASE OF A CONFINED MASSIVE SCALAR FIELD

Our analysis can be extended to the physically interesting case of confined massive scalar fields linearly coupled to a spinning Kerr black hole. In this case one finds that Eq. (35) should be replaced by [6]

\[
\alpha \cdot a^2 r^2 _{\text{max}} - a^2 (2M + \alpha \cdot r_-) r^\text{max} + 2Mr^3 _+ < 0,
\]

(39)
where [38]

\[
\alpha \equiv \frac{\mu^2 - \omega^2_c}{\omega^2_c}.
\]

(40)
The inequality (39) yields the lower bound [see (41)]

\[
r^c_m > \frac{2M + \alpha r_- - \sqrt{(2M + \alpha r_-)^2 - 8\alpha Mr^3 _+/a^2}}{2\alpha}
\]

(41)
on the critical radius \( r^c_m (M, a, \mu) \) of the reflecting mirror which supports the stationary (marginally stable) massive scalar field configuration in the spinning Kerr black-hole spacetime.
It is worth emphasizing the fact that the inequality \( r_+ > \sqrt{\frac{8M}{r_-} - 1} \) [36], which is a necessary condition for the development of the exponentially growing superradiant instabilities in the Press-Teukolsky spinning black-hole-mirror bomb [2], can only be satisfied in the restricted regime \( 0 < \alpha < \frac{2M[2r_+^2 - r_-^2 - 2r_+(r_+^2 - r_-^2)^{1/2}]}{r_-^3} \). \( (42) \)

We therefore conclude that confined massive scalar fields whose masses lie outside the regime \( 42 \) cannot extract the rotational energy of the corresponding spinning Kerr black hole.

VIII. SUMMARY AND DISCUSSION

The superradiant instability regime of the spinning black-hole bomb [2, 11, 13, 14] was studied analytically. This physical system, first designed by Press and Teukolsky [2], is composed of a spinning Kerr black hole of horizon radii \( r_+ \) which is surrounded by a reflecting mirror of radius \( r_m \). Thanks to the intriguing superradiant amplification mechanism [11, 13], an orbiting scalar field, which is placed between the black-hole horizon and the reflecting mirror, is able to extract rotational energy and angular momentum from the spinning Kerr black hole. The role of the confining mirror is to prevent the amplified scalar field from radiating this extracted energy to infinity. As a consequence, it is well known that this composed Kerr-black-hole-scalar-field-mirror system may develop exponentially growing superradiant instabilities [2, 11, 13, 14].

In the present paper we have addressed the following physically interesting question: Is it possible to extract the rotational energy of a spinning Kerr black hole by placing a reflecting mirror (whose role is to confine the superradiantly amplified scalar fields) arbitrarily close to the black-hole horizon? An analytical answer to this important question would certainly enrich our understanding of the intriguing phenomena of superradiant amplification of bosonic fields in rotating black-hole spacetimes.

Moreover, it is worth noting that the answer to this intriguing question may one day be of practical importance [14]. Astrophysical observations made in recent years [39] have provided compelling evidence that spinning black holes are ubiquitous in our Universe. Thus, Kerr black holes may serve our civilization in the future as enormous sources of clean (and probably cheap) energy. In this physical process of energy extraction from a spinning black hole, one would like to build the reflecting mirror as close as possible to the horizon of the central black hole in order to save construction materials. It is therefore of practical importance to determine the minimum radius of the reflecting mirror which allows the extraction of rotational energy from the spinning Kerr black hole.

The main results derived in this paper and their physical implications are as follows:

1. Using analytical techniques, we have proved that, in order for the exponentially growing superradiant instabilities to develop in this composed black-hole-field-mirror system, the dimensionless radius of the reflecting mirror must be bounded from below by the relation [see Eq. (38)] [36, 40]

\[
\frac{r^c_m}{r_+} > \frac{1}{2}\left(\sqrt{1 + \frac{8M}{r_-}} - 1\right). \tag{43}
\]

It is worth stressing the fact that the physical significance of this analytically derived lower bound on the dimensionless critical radius of the confining mirror stems from the fact that the compact relation \( 39 \) provides a necessary condition for the development of the exponentially growing superradiant instabilities in the Press-Teukolsky spinning black-hole-mirror bomb [2].

2. In a very interesting work, Cardoso et. al. [11] have studied the instability properties of the spinning black-hole-scalar-field-mirror bomb. In particular, the numerical results presented in [11] have revealed the interesting fact that the critical (innermost) radius \( r^c_\tau \) of the confining mirror [38] is a decreasing function of both the black-hole angular momentum \( a \) and the azimuthal harmonic index \( m \) of the confined field mode. It is important to note, however, that the analytically derived lower bound [45] reveals the fact that, even in the double asymptotic limit \( a/M \to 1 \) with \( m \to \infty \), the reflecting mirror cannot be placed arbitrarily close to the black-hole horizon. In particular, in the extremal \( a/M \to 1 \) (\( \tau \to 0 \)) limit one finds the lower bound [see Eq. (38)]

\[
\frac{r^c_m}{\tau} > \frac{1}{3} \tag{44}
\]

on the dimensionless critical radius of the reflecting mirror.

3. It is worth noting that the numerical results presented in [14] predict the asymptotic value \( \frac{r^c_\tau}{\tau} \simeq 0.36 \) for the dimensionless critical radius of the reflecting mirror in the double asymptotic limit \( a/M \to 1 \) with \( m \to \infty \). This
numerically computed asymptotic value is consistent with (and, in fact, quite close to) the analytically derived lower bound $\Delta_{\text{c}} > 1/3$ [see Eq. (42)] on the dimensionless critical radius of the reflecting mirror. Thus, our results [and, in particular, the analytically derived lower bounds (43) and (44)] provide a quantitative analytical explanation for the numerical results presented in [14] for the asymptotic large-$m$ behavior of the composed Kerr-black-hole-scalar-field-mirror system (the Press-Teukolsky spinning black-hole-mirror bomb).

(4) It is important to stress the fact that the existence of a lower bound [see the analytically derived relation (43)] on the radius of the confining mirror is a highly non-trivial feature of the Kerr black-hole spacetime. In particular, it is physically interesting to contrast the physical properties of the spinning Kerr-black-hole-scalar-field-mirror system with the corresponding properties of the charged Reissner-Nordström-black-hole-scalar-field-mirror system studied in [10]. Our analytically derived results have revealed the fact that in the former case the radius of the confining mirror is bounded from below by (43) (that is, the confining mirror cannot be placed arbitrarily close to the horizon of the spinning Kerr black hole), whereas in the later case the confining mirror can be placed arbitrarily close to the horizon of the charged Reissner-Nordström black hole [10].

(5) We have also generalized the analysis to the physically interesting case of confined massive scalar fields linearly coupled to a spinning Kerr black hole. In particular, we have explicitly proved that confined massive scalar fields whose masses lie outside the regime [12] cannot extract the rotational energy of the corresponding spinning Kerr black hole.

Finally, we would like to stress the fact that, in the present analysis, the confined scalar fields were treated at the perturbative (linearized) level. The analytical results derived in this paper are therefore expected to be valid in the ignition stage of the superradiant instabilities, when the amplitude of the confined field is still small and its dynamics in the black-hole spacetime can still be treated at the linearized perturbative level. As we explicitly shown in this paper, the physical properties of the Press-Teukolsky spinning black-hole bomb (the composed Kerr-black-hole-scalar-field-mirror system [2]) can be studied analytically within the framework of this perturbative (linearized) approach. We believe that it would be highly interesting (and physically important) to use more sophisticated numerical techniques in order to explore the late-time non-linear development of the explosive superradiant instabilities in this composed Kerr-black-hole-scalar-field-mirror system.

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\[ K_{mm} = m^2 [1 + O(m^{-1})] + a \omega^2 \]

[27] If is worth emphasizing the fact that, due to the rotation of the black-hole spacetime, the differential equation (11) for the radial eigenfunctions \( R_{lm} \) is coupled to the differential equation (9) for the angular eigenfunctions \( S_{lm} \) by the characteristic angular eigenvalues \( K_{lm} \).

[28] We shall henceforth omit the angular harmonic indices \( \{l, m\} \) for brevity.

[29] It is worth emphasizing again that the caged (bounded) scalar field modes that we study in the present paper are characterized by the boundary condition (13). This boundary condition, which is imposed at the finite radius \( r = r_m \), is dictated by the presence of the reflecting mirror around the black hole. On the other hand, the more familiar quasinormal resonant frequencies, which characterize the linearized dynamics of scalar fields in an asymptotically flat black-hole spacetime, are characterized by an outgoing boundary condition at asymptotically large distances \( r \rightarrow \infty \) from the black hole, see [30] and references therein.

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[31] Note that, in the black-hole spacetime, the differential relation (15) maps the radial coordinate \( r \in [r_+, \infty] \) into \( g \in [-\infty, +\infty] \).

[32] One can assume \( R(r = r_+) \geq 0 \) for the radial scalar eigenfunction without loss of generality.

[33] It is worth noting that the near-horizon \( x \ll \tau \) region corresponds to \( \tau y/r_+ \rightarrow -\infty \) [see Eq. (25)]. This implies \( e^{\tau y/r_+} \rightarrow 0 \) in the near-horizon region.

[34] Here we have used equations (9.1.54) and (9.6.3) of [22].

[35] Note that this inequality follows from the series of inequalities \( M r - a^2 \geq M r_+ - a^2 = r_+(M - r) \geq 0 \) in the external region \( r \geq r_+ \) of the spinning Kerr black-hole spacetime.

[36] Here \( r_{lm}(M, a) \) is the radius of the reflecting mirror which supports the stationary (marginally stable) scalar field configurations in the spinning Kerr black-hole spacetime. It is worth emphasizing again that in the present paper we have explored the physical properties of the stationary (\( 3\omega = 0 \)) confined scalar field modes. These marginally stable field
configurations are characterized by the critical (threshold) frequency $\omega_{c}$ for the superradiant amplification phenomenon in the spinning Kerr black-hole spacetime. As emphasized earlier, the stationary confined field modes mark the onset of the superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system. Our analysis has revealed that the critical (innermost) radius of the confining mirror, which supports these stationary (marginally stable) confined field modes, is bounded from below by the characteristic inequality (37). It should be emphasized that former studies [11, 13, 14] have shown that unstable (explosive) composed Kerr-black-hole-scalar-field-mirror systems are characterized by confining mirrors whose radii are larger than the critical mirror radius $r_{m}^{c}$ which characterizes the stationary (marginally stable) Kerr-black-hole-scalar-field-mirror configuration [11, 13, 14]. Thus, explosive spinning black-hole-mirror bombs must also conform to the lower bound (37) [that is, their mirror radii are also characterized by the inequality (37)].

[37] It is worth emphasizing again that these marginally stable (stationary) confined field configurations [with $\omega = \omega_{c}$, see Eq. (2)] mark the onset of the exponentially growing superradiant instabilities in the composed Kerr-black-hole-scalar-field-mirror system. In particular, the explosive (unstable) regime of the Press-Teukolsky spinning black-hole bomb [2] is characterized by reflecting mirrors whose radii lie in the regime $r_{m} > r_{m}^{c}$ [11, 13, 14].

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