Research Article

Exceptional Sets for Sums of Prime Cubes in Short Interval

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Abstract

This paper is to obtain $E(N, X) = o(X)$ for $X > N^{1/30}$. The main result in the paper is as follows.

Theorem 1. Let $\theta > 1/36$ and $E(N, X)$ be defined as above. For $X > N^\theta$, we have

$$E(N, X) = o(X).$$

We shall prove Theorem 1 by means of the Hardy–Littlewood method. The treatment of the integrals on the major arcs is standard, and we will focus on the treatment of the integrals on the minor arcs.

1. Introduction

The Waring–Goldbach problem is to study the representation of positive integers as sums of powers of prime numbers. In this paper, we shall focus on the cubic Waring–Goldbach problem. This topic can be traced back to the work of Hua [1]. He proved that almost all integers satisfying certain congruence conditions can be written as $s$ cubes of primes, where $s = 5, 6, 7, 8$, and the abovementioned congruence conditions are

$$N_5 = \{n \in \mathbb{N} : n \not\equiv 1 \pmod{2}, n \not\equiv 0 \pmod{9}, n \not\equiv 0 \pmod{7}\},$$
$$N_6 = \{n \in \mathbb{N} : n \not\equiv 0 \pmod{2}, n \not\equiv 1 \pmod{9}\},$$
$$N_7 = \{n \in \mathbb{N} : n \not\equiv 1 \pmod{2}, n \not\equiv 0 \pmod{9}\},$$
$$N_8 = \{n \in \mathbb{N} : n \not\equiv 0 \pmod{2}\}.$$  

Let $E_s(N)$ denote the number of positive numbers $n \in N_s$ not exceeding $N$ and cannot be represented as $s$ cubes of primes; Hua showed that, for any $A > 0$,

$$E_s(N) \ll N (\log N)^{-A}.$$  

Throughout, we assume that $N$ is a large natural number, and $X < N$. In this paper, we consider the exceptional set of even integers $n$ in the short interval $N - X \leq n \leq N$, which cannot be represented as sum of eight cubes of primes. Precisely, let $\xi(N, X)$ denote the set of natural numbers $n \in N_8$ with $N - X \leq n \leq N$, such that $n$ cannot be written as the following expression:

$$n = p_1^3 + p_2^3 + \cdots + p_8^3,$$

where $p_i$ are primes. Moreover, we set

$$E(N, X) = \text{card} \xi(N, X).$$

The purpose of this paper is to obtain $E(N, X) = o(X)$ with $X$ as small as possible. Zhao [2] proved that $E_8(N) \ll N^{1/60\varepsilon}$, which implies $E(N, X) = o(X)$ if $X > N^{1/60\varepsilon}$. The main result in the paper is as follows.

Theorem 1. Let $\theta > 1/36$ and $E(N, X)$ be defined as above. For $X > N^\theta$, we have

$$E(N, X) = o(X).$$

We shall prove Theorem 1 by means of the Hardy–Littlewood method. The treatment of the integrals on the major arcs is standard, and we will focus on the treatment of the integrals on the minor arcs.

In this paper, we make use of the method of Vaughan [3] to deal with the equation $x_1^3 + y_1^3 + y_2^3 + z_1 = x_2^3 + y_3^3 + y_4^3 + z_2$ (see Lemma 3). We also make use of the result of Zhao [2] on the $10^{th}$ moment of the Weyl sums over cubes of primes restricted on minor arcs.

Notation. Throughout the paper, $\varepsilon$ denotes a sufficiently small positive number, and let $A, c$ denote a positive constant. We need to point out that $A, c$ and $\varepsilon$ are allowed to change at different occurrences. With or without subscript, $p$ denotes a prime number. Denote by $d(n)$ the number of divisors of $n$, and as usual, we write $e(x)$ for $e^{2\pi i x}$. 

2. Preliminaries

Before we prove Theorem 1, we introduce the following theorem.

**Theorem 2.** Let \( E = E(N, N^{5/6}) \) be defined in Section 1. Then, we have

\[
E \ll N^{1/36 + \varepsilon}.
\]  

(6)

We can get Theorems 1 from 2 immediately. Therefore, our task is to prove Theorem 2.

Let

\[
U = \left( \frac{N}{16} \right)^{1/3}.
\]  

(7)

Write

\[
g(a) = \sum_{U \leq p \leq 2U} (\log p)e(p^3a),
\]

\[
f(a) = \sum_{U^{5/8} \leq p \leq 2U^{5/6}} (\log p)e(p^3a).
\]  

(8)

Let \( R(n) \) be the weighted number of solutions of \( n = p_1^3 + \cdots + p_b^3 \) with \( U \leq p_1, \ldots, p_b \leq 2U \) and \( U^{5/6} \leq p_7, \ldots, p_b \leq 2U^{5/6} \). By orthogonality, one has

\[
R(n) = \int_0^1 g(a)^6 f(a)^2 e(-na) da.
\]  

(9)

We write

\[
P = U^{2/5},
\]

\[
Q = U^3 p^{-1}.
\]  

(10)

We define the set of major arcs \( \mathcal{M} \) as the union of the intervals

\[
\mathcal{M}(q \cdot a) = \{ a \in (Q^{-1}, 1 + Q^{-1}]: |qa - a| \leq Q^{-1} \},
\]  

(11)

with \( 1 \leq a \leq q \leq P \) and \( (a, q) = 1 \). We then denote the corresponding set of minor arcs by \( \mathcal{M} = (Q^{-1}, 1 + Q^{-1}]) \setminus \mathcal{M} \). So, we can get the lemma.

**Lemma 1.** For \( a \in \mathcal{M} \), we have

\[
\int_{\mathcal{M}} g(a)^6 f(a)^2 e(-na) da \gg U^{14/3}.
\]  

(12)

We will prove Lemma 1 in Section 4.

**Lemma 2.** For \( a \in \mathcal{m} \), we have

\[
\int_{\mathcal{m}} |g(a)|^{10} da \ll U^{7-1/4 + \varepsilon}.
\]  

(13)

We can find the proof in ([2], Section 9).

**Lemma 3.** Let \( S \) be the number of solutions of

\[
x_1^3 + y_1^3 + z_1 = x_2^3 + y_2^3 + z_2,
\]

with \( U \leq x_i \leq 2U, U^{5/6} \leq y_i \leq 2U^{5/6}, \) and \( z_i \in \xi(N, N^{5/6}) \). We have

\[
S \ll U^{1+\varepsilon}(U^{5/3} E + U^{55/36} E^2).
\]  

(15)

We will prove Lemma 3 in Section 3.

**Proof of Theorem 2.** On recalling the set \( \xi(N, N^{5/6}) \) defined in Section 1 and by means of an argument of Wooley (see [4]), we have

\[
\sum_{m \in \xi(N, N^{5/6})} \int_0^1 g(a)^6 f(a)^2 e(-na) da = 0.
\]  

(16)

Let

\[
R(\mathcal{M}) = \sum_{m \in \xi(N, N^{5/6})} \int_{\mathcal{M}} g(a)^6 f(a)^2 e(-na) da,
\]

\[
R(\mathcal{m}) = \int_{\mathcal{m}} g(a)^6 f(a)^2 h(-a) da,
\]

where

\[
h(a) = \sum_{m \in \xi(N, N^{5/6})} e(na).
\]  

(18)

So, one has

\[
|R(\mathcal{m})| = |R(\mathcal{M})|.
\]  

(19)

With the help of Lemma 1, we can get

\[
|R(\mathcal{m})| \gg U^{14/3} E.
\]  

(20)

With an application of the Cauchy–Schwarz inequality, one has

\[
\int_{\mathcal{m}} |g(a)^6 f(a)^2 h(-a)| da \leq \left( \int_{\mathcal{m}} |g(a)|^{10} da \right)^{1/2} \left( \int_0^1 |g(a)f(a)^2 h(-a)|^2 da \right)^{1/2}.
\]  

(21)

By Lemma 2, we have

\[
\int_{\mathcal{m}} |g(a)|^{10} da \ll U^{7-1/4 + \varepsilon}.
\]  

(22)

By considering the underlying Diophantine equations and Lemma 3, we get

\[
\int_0^1 |g(a)f(a)^2 h(-a)|^2 da \ll U^{1+\varepsilon}(U^{5/3} E + U^{55/36} E^2).
\]  

(23)

From (20) to (23), we have

\[
U^{14/3} E \ll U^{4-1/8+5/6+\varepsilon} E^{1/2} + U^{4-1/8+55/72+\varepsilon} E.
\]  

(24)

Therefore, we conclude that
3. Proof of Lemma 3

We write $S_1$ as the number of solutions of (14) with $x_i = x_i^1$, and let $S_2$ denote the number of solutions of (14) with $x_i^3 \neq x_i^2$. It is easy to find that

$$S = S_1 + S_2.$$  \hspace{1cm} (26)

**Lemma 4.** We denote by $T(k)$ the number of solutions of the Diophantine equation

$$y_1^2 + y_2^2 - y_3^2 = k,$$  \hspace{1cm} (27)

with $1 \leq y_i \leq U^{5/6}$ and $k \neq 0$. Then, we have

$$T(k) \ll k^{1}U^{36/55+\varepsilon}.$$  \hspace{1cm} (28)

The proof of Lemma 1 can follow from an argument of Hooley [5] (see the proof of Lemma 1 of Parsell [6] for a sketch of the necessary adjustments to Hooley’s argument).

To calculate $S_0$, by symmetry, we can assume that $x_1 > x_1$. Write $x_1 = x_1^1 + h$. Then, (14) becomes

$$h \left( 3x_1^2 + 3x_1h + h^2 \right) = y_1^2 + y_2^2 - y_3^2 - y_3^1 - z_1 - z_2.$$  \hspace{1cm} (29)

Since $y_1^2 + y_2^2 \leq 16U^{5/2}$, $z_1 - z_2 \leq 16U^{5/2}$, and $x_1^2 \geq U^2$, it follows that $h \leq 11U^{1/2}$. Let

$$g_h(n) = \sum_{U \leq h < 2U} \left( ah(3x^2 + 3xh + h^2) \right),$$

$$G(a) = \sum_{h < 11U^{1/2}} g_h(n).$$

Then,

$$S_2 \leq \int_0^1 G(a)|F(a)|^4|\delta_h(a)|^2 \, da,$$  \hspace{1cm} (31)

where $h(\alpha)$ is defined in (18), and

$$F(a) = \sum_{U^{5/2} \leq y \leq 2U^{1/6}} e(y^3a).$$  \hspace{1cm} (32)

Let $\mathfrak{R}(a)$ denote the interval $[a/q - q^{-1}U^{-3/2}, a/q + q^{-1}U^{-3/2}]$ and $\mathfrak{U} = (U^{12}, 1 + U^{-2})$. We may suppose that $U \geq 4$. Then, $\mathfrak{R}(a)$ with $1 \leq a \leq U$, $(a, q) = 1$ contained in $\mathfrak{U}$. Let $\mathfrak{R}$ denote the union of $\mathfrak{R}(a, q)$ with $1 \leq a \leq U$, $(a, q) = 1$, and let $n = \mathfrak{U}\setminus\mathfrak{R}$. Then, we have

$$S_2 \leq \int_{\mathfrak{R}} G(a)|F(a)|^4|\delta_h(a)|^2 \, da + \int_n G(a)|F(a)|^4|\delta_h(a)|^2 \, da.$$  \hspace{1cm} (33)

By the same method of the lemma of Vaughan [3], we can get the following lemma.

**Lemma 5.** Suppose that $|a - a/q| \leq q^{-2}$ and $(a, q) = 1$, then

$$G(a) \ll U^{t/2}(U^{3/2} - q^{-1/2} + U + U^{1/4}q^{-1/2}).$$  \hspace{1cm} (34)

Moreover, we can get the following lemma.

**Lemma 6.** Let $\mathfrak{R}$ be denoted as above. Then, we have

$$\int_{\mathfrak{R}} G(a)|F(a)|^4|\delta_h(a)|^2 \, da \ll U^{5/2+\varepsilon}.$$  \hspace{1cm} (35)

The proof can be also found in Vaughan [3].

**Proof of Lemma 3.** When $y_1^3 + y_2^3 + z_1 = y_3^3 + y_4^3 + z_2$, the number $R$ of solutions of it satisfies

$$R = \int_0^1 |F(a)|^4|\delta_h(a)|^2 \, da.$$  \hspace{1cm} (36)

Given any one of the $O(E^2)$ possible choices for $z_1$ and $z_2$ with $z_1 \neq z_2$, it follows from Lemma 4 that the number of permissible choices for $y_1$ is $O(U^{55/36+\varepsilon})$, and thus the contribution arising from this class of solutions is $O(U^{55/36+\varepsilon}E^2)$. When $z_1 = z_2$, on the other hand, the variables $y_1$ satisfy the equation $y_1^3 + y_2^3 = y_3^3 + y_4^3$; the number of the solutions is $O(U^{55/36+\varepsilon})$. Thus, we conclude that the number of solutions of this type is $O(U^{55/36+\varepsilon}E)$. So, we can get

$$R \ll U^{t/2}(U^{5/3} + U^{55/36}E^2).$$  \hspace{1cm} (37)

Then, we have

$$S_1 \ll U^{t/2}(U^{5/3} + U^{55/36}E^2).$$  \hspace{1cm} (38)

We consider $n$. By Dirichlet’s theorem on Diophantine approximation, once $\alpha \in \mathfrak{R}$, we may choose $a$, $q$ with $(a, q) = 1$, $|a - a/q| \leq q^{-1}U^{-3/2}$ and $q \leq U^{3/2}$. It is easily verified that $1 \leq a \leq q$. Moreover, since $a \notin \mathfrak{R}$, we have $q > U$. Therefore, by Lemma 5, we have

$$G(a) \ll U^{1/\varepsilon}.$$  \hspace{1cm} (39)

Hence, we have

$$\int \mathfrak{R} G(a)|F(a)|^4|\delta_h(a)|^2 \, da \ll U^{1/\varepsilon} \int_0^1 |F(a)|^4|\delta_h(a)|^2 \, da$$

$$\ll U^{1/\varepsilon}(U^{5/3} + U^{55/36}E^2).$$  \hspace{1cm} (40)

When $\alpha \in \mathfrak{R}$, by Lemma 6, we can get

$$\int \mathfrak{R} G(a)|F(a)|^4|\delta_h(a)|^2 \, da \ll E \int \mathfrak{R} G(a)|F(a)|^4 \, da$$

$$\ll (U^{5/2+\varepsilon}E^2).$$  \hspace{1cm} (41)

By (40) and (41),

$$S_2 \ll U^{1/\varepsilon}(U^{5/3} + U^{55/36}E^2).$$  \hspace{1cm} (42)

In view of (38) and (42), we complete the proof of Lemma 3. \hspace{1cm} \Box
4. Proof of Lemma 1

Let \( L = (\log U)^A \) for \( A > 0 \). We write
\[
P' = L, \\
Q' = U^{2/3} L^{-1}.
\]

We define the set of major arcs \( \mathcal{M} \) as the union of the intervals
\[
\mathcal{M} (q \cdot a) = \left\{ \alpha \in \left( Q^{-1}, 1 + Q^{-1} \right] : |q\alpha - a| \leq Q^{-1} \right\},
\]
with \( 1 \leq a \leq q \leq P \) and \( (a \cdot q) = 1 \). It is easy to find that \( \mathcal{M} \subset \mathcal{M} \). We then denote the corresponding set of minor arcs by \( \mathcal{M}' = (Q^{-1}, 1 + Q^{-1}) \setminus \mathcal{M} \).

Lemma 7. Suppose that the integer \( n \) satisfies \( N/2 \leq n \leq N \) and \( n \in \mathcal{N}_g \). Then, one has
\[
\int_{\mathcal{M}} g(\alpha)^6 f(\alpha) e(-\alpha) d\alpha \gg U^{3}.
\]
(45)

The proof can be found in ([3], Lemma 3.1). We define the multiplicative function \( w(q) \) by taking
\[
w(q) = \begin{cases} 3^p^{3 + q}, & \text{when } u \geq 0 \text{ and } v = 1, \\
p^{-u - 1}, & \text{when } u \geq 0 \text{ and } v = 2, 3.
\end{cases}
\]
(46)

We have
\[
q^{-1/2} \leq w(q) \ll q^{-1/3}.
\]
(47)

Therefore, we have the following result.

Lemma 8. Let \( c \) be a constant. When \( m \geq 2 \), one has
\[
\sum_{q \leq m} w(q)^2 d(q)^c \ll (\log m)^A,
\]
(48)

for some constant \( A \).

This is due to Zhao ([2], Lemma 1). Before we deal with the integrals on the minor arcs, we should give an upper bound of \( g(\alpha) \). We also quote the following estimate proved by Ren [7].

Lemma 9. Suppose that \( \alpha \) is a real number and that \( 1 \leq a \leq q \) with \( (a, q) = 1 \). Let \( \beta = \alpha - a/q \). Then, one can have
\[
g(\alpha) \ll d(q)^c (\log U)^c (U^{1/2} V + U^{1/2} + U V^{-1}),
\]
(49)

where \( c_0 = 1/2 + \log 3/\log 2 \) and \( V = \sqrt{q(1 + U^3 |\alpha - a/q|)} \) and \( c \) is a constant.

Proof of Lemma 1. When \( \alpha \in \mathcal{M} \), we have
\[
\int_{\mathcal{M}} g(\alpha)^6 f(\alpha) e(-\alpha) d\alpha = \sum_{U^{5/6} \leq p \leq 2U^{5/6}} \log p
\cdot \int_{\mathcal{M}} g(\alpha)^6 e(-(n - p^3)\alpha) d\alpha.
\]

Moreover, one has \( n - p^3 \in \mathcal{N}_g \) when \( n \in \mathcal{N}_g \), and when \( N - 2N^{5/6} \leq n \leq N \), we have \( N/2 \leq n - p^3 \leq N \), and by Lemma 7, we have
\[
\int_{\mathcal{M}} g(\alpha)^6 f(\alpha) e(-\alpha) d\alpha \gg U^{3} \sum_{U^{5/6} \leq p \leq 2U^{5/6}} \log p \gg U^{23/6}.
\]

By the same measure, when \( N - N^{5/6} \leq n \leq N \) and \( n \in \mathcal{N}_g \), we can get
\[
\int_{\mathcal{M}} g(\alpha)^6 f(\alpha)^2 e(-\alpha) d\alpha \gg U^{14/3}.
\]

When \( a \in m' \cap \mathcal{M} \), by Lemma 9, we have
\[
g(\alpha) \ll \left( \frac{U^{4/5} + U}{\sqrt{q(1 + U^3 |\alpha - a/q|)}} \right)^c (\log U)^c
\cdot \int_{\mathcal{M}} d(q)^c (\log U)^c U^2 |f(\alpha)|^2 d\alpha.
\]

Therefore, we have
\[
\int_{m' \cap \mathcal{M}} |g(\alpha)^6 f(\alpha)^2|^2 d\alpha \ll U^{6} L^{-2}.
\]

Let \( \alpha = a/q + \beta \). We have
\[
\sum_{1 \leq a \leq q} \left| \frac{f(a/q + \beta)}{q} \right|^2 = q \sum_{p_1, p_2 \mod q \atop U^{5/6} \leq p_1, p_2 \leq 2U^{5/6}} e((p_1^3 - p_2^3)\beta).
\]
(50)

Since \( q \ll U^{5/6} \), we have
\[
\sum_{1 \leq a \leq q} \left| \frac{f(a/q + \beta)}{q} \right|^2 \ll U^{5/3} \sum_{b_1^3 = b_2^3 \mod q \atop 1 \leq b_1, b_2 < q \atop b_1 b_2 a = 1} 1 \ll U^{5/3} \sum_{b \mid q \atop 1 \leq b \leq q} d(q)^c.
\]
(51)

Therefore, by Lemma 8, we obtain
\[
\int_{\mathcal{M}} g(\alpha)^6 f(\alpha) e(-\alpha) d\alpha = \sum_{U^{5/6} \leq p \leq 2U^{5/6}} \log p
\cdot \int_{\mathcal{M}} g(\alpha)^6 e(-(n - p^3)\alpha) d\alpha.
\]
\[
\int_{m'\mod{m}} |g(\alpha)|^2 f(\alpha)^2 \, d\alpha \\
\ll U^{5+5/3} L^{-2} \sum_{q \leq U^{2/3}} w(q) d(q)^{3+\varepsilon} \int_{|\beta| \leq U^{2/3}} \frac{1}{1 + |\beta| U^3} \, d\beta \\
\ll U^{14/3} L^{-2+\varepsilon} (\log U)^A,
\]
(57)

for some absolute constant \(A\).

In view of (52) and (57), we complete the proof of Lemma 1. \(\square\)

**Data Availability**

Except references, no data were used to support this study.

**Conflicts of Interest**

The author declares no conflicts of interest.

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