The Rank Four Heterotic Modular Invariant Partition Functions

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Abstract

In this paper, we develop several general techniques to investigate modular invariants of conformal field theories whose algebras of the holomorphic and anti-holomorphic sectors are different. As an application, we find all such “heterotic” WZNW physical invariants of (horizontal) rank four: there are exactly seven of these, two of which seem to be new. Previously, only those of rank \( \leq 3 \) have been completely classified. We also find all physical modular invariants for \( su(2)_{k_1} \times su(2)_{k_2} \), for \( 22 > k_1 > k_2 \), and \( k_1 = 28 \), \( k_2 < 22 \), completing the classification of ref. [9].
1. Introduction

In the study of two-dimensional conformal field theories, the partition functions play an important role. In particular, they could give us some insight into the general solution of the broader and as yet largely unsolved problem of classifying all such theories. Unfortunately, only in a few special cases are these functions completely known; much work remains to be done.

For a two-dimensional conformal field theory that has an operator algebra decomposable into a pair of commuting holomorphic and anti-holomorphic chiral algebras, \( \hat{g}_L \) and \( \hat{g}_R \), and a space of states which can be written as a finite sum of irreducible representations \((\lambda_L, \lambda_R)\) of \( \hat{g}_L \times \hat{g}_R \) with multiplicity \( N_{\lambda_L \lambda_R} \), the partition functions are combinations of bi-products of characters \( \chi_{\lambda_L} \) and \( \chi_{\lambda_R} \) of the form

\[
Z = \sum N_{\lambda_L \lambda_R} \chi_{\lambda_L} \chi_{\lambda_R}^*.
\] (1.1)

To classify all partition functions of a given algebra \((\hat{g}_L, \hat{g}_R)\) is to find all combinations (1.1) such that: (P1) \( Z \) is invariant under transformations of the modular group; (P2) all the coefficients \( N_{\lambda_L \lambda_R} \) are non-negative integers; and (P3) the vacuum exists and is non-degenerate, that is, \( N_{11} = 1 \), with \( \lambda = 1 \) denoting the vacuum.

Any function \( Z \) satisfying the modular invariance condition (P1) will be called an invariant; if in addition it satisfies the condition that every coefficient \( N_{\lambda_L \lambda_R} \geq 0 \), then it is said to be a positive invariant; and finally, if the conditions (P1), (P2) and (P3) are all met, then it is considered to be a physical invariant. Clearly any conformal theory must have at least these three properties to be physically meaningful, but often additional conditions are imposed.

In the past, most authors have assumed further that the algebras and levels of the holomorphic and anti-holomorphic sectors of the theory are identical \((\hat{g}_L = \hat{g}_R \text{ and } k_L = k_R)\); when it is necessary to emphasize this restriction we will use the qualitative ‘non-heterotic’ or ‘symmetric’. In the past decade, extensive work has been done in this direction, and has been reported in refs. [1,2,3], among others. But classification proofs, which determine all the physical invariants that belong to a certain class, exist only for a few cases. Namely: at level one, the simple Lie algebras \( A_n^{(1)} \), \( B_n^{(1)} \), \( C_n^{(1)} \), \( D_n^{(1)} \), and the five exceptional algebras [4,5]; and, at an arbitrary level, the untwisted Kac-Moody algebra \( A_1^{(1)} \) (see [1,6]) and the coset models based on it, such as the minimal unitary Virasoro models [1] and the \( N = 1 \) minimal superconformal models [7], and, also, the algebra \( A_2^{(1)} \) (ref. [8]). Finally, thanks in part to work done in this paper, the \( A_1 + A_1 \) classification has now been completed for all levels [9].

In the present paper we will address rather the heterotic case, where the (affine) algebras of the holomorphic and anti-holomorphic sectors of the theory are different, \( \hat{g}_L \neq \hat{g}_R \). We will focus on the rank four case, by which we mean \( \text{rank}(\hat{g}_L) + \text{rank}(\hat{g}_R) = 4 \). However our methods will be of value for higher ranks as well.

Relatively little work has been done in this direction (one of the few recent exceptions is ref. [10]) although, if the heterotic string model is any indication, heterotic invariants could lead to phenomenologically interesting models. The heterotic invariants of rank \( \leq 3 \)
were found in ref. [11]:

for \((A_2,1; A_1,4)\):
\[
Z = \chi_{11} (\chi_1 + \chi_5)^* + \{\chi_{21} + \chi_{12}\} \chi_3^*;
\]
(1.2a)

for \((C_2,1; A_1,10)\):
\[
Z = \chi_{11} (\chi_1 + \chi_7)^* + \chi_{21} (\chi_4 + \chi_8)^*
+ \chi_{12} (\chi_5 + \chi_{11})^*;
\]
(1.2b)

for \((G_2,1; A_1,28)\):
\[
Z = \chi_{11} (\chi_1 + \chi_{11} + \chi_{19} + \chi_{29})^*
+ \chi_{12} (\chi_7 + \chi_{13} + \chi_{17} + \chi_{23})^*;
\]
(1.2c)

for \((A_1,3A_1,1; A_1,28)\):
\[
Z = (\chi_1 \chi_1 + \chi 4 \chi_2) \cdot (\chi_1 + \chi_{11} + \chi_{19} + \chi_{29})^*
+ (\chi_2 \chi_2 + \chi_3 \chi_1) \cdot (\chi_7 + \chi_{13} + \chi_{17} + \chi_{23})^*;
\]
(1.2d)

where the subscripts of the characters are the Dynkin labels of the highest weight + \(\rho\) (\(\rho = \text{sum of the fundamental weights}\)).

The heterotic cases differ from the more familiar symmetric cases in that they are considerably rarer: for any algebra \(\hat{g}_L = \hat{g}_R\) and level \(k_L = k_R\), there is at least one physical invariant (namely the diagonal one); however, due to severe constraints, the existence of a physical invariant when \((\hat{g}_L, k_L) \neq (\hat{g}_R, k_R)\) is quite exceptional.

Heterotic invariants can be found using conformal embeddings, though not all heterotic invariants can be obtained in this manner. Given any heterotic invariant, others can be generated using simple currents. Simple currents can also take a symmetric invariant to an invariant whose maximally extended left and right chiral algebras are no longer isomorphic; however, since simple currents do not affect the underlying affine algebra, these invariants are not “heterotic” in our definition of the word.

In sect. 2 we begin by finding all solutions to the constraint that the conformal charges \(c_L\) and \(c_R\) be equal (mod 24). There are infinitely many solutions corresponding to \(g_L = g_R = A_1 + A_1\) (we will call these the “AA-types”), but only 61 corresponding to all other total rank 4 algebras (which we will call the “non-AA-types”). They are listed in Table 1. We then introduce the concept of “null augments” and use it to describe a parity test. This test will reduce the number of non-AA-types we have to consider from 61 to 14. These 14 are listed in Table 2. In sect. 3 we discuss the lattice method [12,11], which we use to find all physical heterotic invariants for some of the 14 non-AA-types. In sect. 4 another test, based on [2], is considered; it is used to eliminate all but one of the AA-types. In sect. 5 we handle all remaining types. The complete list of all heterotic rank four partition functions is given in sect. 6; there are precisely seven of them. The Appendix presents the results of a computer search (using methods developed in [13]) for the symmetric \(A_1 + A_1\) physical invariants at levels \(k_1 \neq k_2\). These results are new (formerly only those for \(k_1 = k_2\) were known), and are exploited in sect. 4. They also complete the \(A_1 + A_1\) classification.

2. The candidate types

In this section we give two strong tests [see eqs. (2.2) and (2.10b)] which heterotic types must pass in order for heterotic invariants to exist. Of course, both conditions are automatically satisfied by symmetric types.
The results of this section are summarized in Table 1. First a few remarks about our notation.

By the type \( \mathcal{T} \) of a physical invariant we mean a list of (left and right) algebras and levels with the left-moving (\textit{i.e.} holomorphic) and the right-moving (\textit{i.e.} anti-holomorphic) sectors separated by a semi-colon:

\[
\mathcal{T} = (T_L; T_R) = (g_{L1,k_{L1}} \cdots g_{Li,k_{Li}}; g_{R1,k_{R1}} \cdots g_{Rr,k_{Rr}})
\]

\[(2.1)\]

By a positive type we mean a type with all levels \( k_{Li} > 0 \), \( k_{Rj} > 0 \). By a null type we mean a type whose levels satisfy \( k_{L1} = \cdots = k_{Rr} = 0 \). Because the character of a level zero affine algebra is identically equal to 1, null types do not contribute to the partition function, so for most purposes can be avoided. They are, however, necessary for parity calculations (see subsect. 2.3 below) and the lattice method (see sect. 3 below). Positive types will be generally denoted by \( \mathcal{T}^+ \), and null ones by \( \mathcal{T}^0 \).

2.1 The central charge condition

Most types \( \mathcal{T}^+ \) cannot be realized by a heterotic physical invariant because of a number of stringent conditions. The most obvious condition constraining \( \mathcal{T}^+ \) is that the central charges of left-(right-)moving sectors satisfy \( c_L \equiv c_R \pmod{24} \), or more explicitly

\[
\sum_{i=1}^{l} \rho_{Li}^2 k_{Li}' - \sum_{j=1}^{r} \rho_{Rj}^2 k_{Rj}' \equiv \sum_{i=1}^{l} \rho_{Li}^2 h_{Li}' - \sum_{j=1}^{r} \rho_{Rj}^2 h_{Rj}' \pmod{2},
\]

\[(2.2)\]

where \( k_{Li}' = k_{Li} + h_{Li}' \) and \( k_{Rj}' = k_{Rj} + h_{Rj}' \) are the heights. The notation used here is quite standard and is explicitly described in \textit{e.g.} [11]; in particular, \( \rho \) is the sum of the fundamental weights and \( h^\vee \) the dual Coxeter number.

Eq. (2.2) must be satisfied by any type that has a modular invariant \( N \) satisfying \( N_{\rho_L, \rho_R} \neq 0 \). In particular, it must be satisfied by a physical invariant of any type. In this subsection we find all solutions \( \mathcal{T}^+ \) to eq. (2.2) that are of rank 4 (\textit{i.e.} \( 4 = \sum n_{Li} + \sum n_{Rj} \)), where \( n_{Li} = \text{rank}(g_{Li}), n_{Rj} = \text{rank}(g_{Rj}) \)). Any such type will be called a candidate.

A useful fact is the following:

(*) Any solution \( x, y \in \mathbb{Z} \) to

\[
y = \frac{ax}{a + bx},
\]

\[(2.3a)\]

where \( a = \prod_{i=1}^{m} p_i^{a_i} \) is the prime decomposition of \( a \), and where \( gcd(a, b) = 1 \), also satisfies

\[
x = \pm \prod_{i=1}^{m} p_i^{b_i} - a, \quad \text{where} \quad 0 \leq b_i \leq 2a_i.
\]

\[(2.3b)\]

This statement is proved simply by finding bounds for the powers of the prime divisors of the numerator and denominator of eq. (2.3a).

The above rule permits (2.2) to be solved for the heights, given the algebras \( g_{Li}, g_{Rj} \), whenever only two heights are unknown. When there are more than two independent
heights, the strategy is to use (2.2) to bound from above all but two of the heights, and then to apply (*). When this strategy is successful, only finitely many solutions to (2.2) will exist (for the given choice of $g_{Li}, g_{Rj}$), and (*) will permit these to be enumerated. When this strategy is not successful (i.e. there remain at least three unbounded heights), there will be an infinite family of solutions to (2.2). For the rank 4 case we are considering here, this strategy is successful for all possible choices of $g_{Li}, g_{Rj}$, except for one: $g_L = g_R = A_1 + A_1$. In that case (2.2) becomes

$$\frac{1}{k'_1} + \frac{1}{k'_2} = \frac{1}{k'_3} + \frac{1}{k'_4},$$

making a slight change of notation (the ‘≡’ in (2.2) becomes an ‘=’ in (2.4) because both sides lie between ±1/3). We will defer the analysis of this infinite class of candidates, which we call the AA-types, until sect. 4.

We now explicitly give an example of how to solve (2.2) for a non-AA-type. Consider $g_L = C_2$ and $g_R = A_1 + A_1$, then eq. (2.2) becomes

$$\frac{15 - 2k'_L}{3k'_L} = \frac{1}{k'_R1} + \frac{1}{k'_R2},$$

from which we derive $3 < k'_L < 7.5$. Choosing now $k'_L = 7$, say, gives

$$k'_R2 = \frac{21k'_R1}{k'_R1 - 21},$$

and now eq.(2.3b) tells us $k'_R1 = 21 + 3^i \cdot 7^j$ for $0 \leq i, j \leq 2$, leading to 5 different candidates, namely the types 40-44 in Table 1.

The list of all non-AA-types is given in the second column of Table 1. The table also includes all AA-types (there is only one) which survive the cardinality test described below in sect. 4.

2.2 Null augments

As mentioned earlier in this section, null types do not contribute to the partition function (1.1) and so for most purposes can be ignored. However they do have two related applications to the classification of heterotic invariants. One is the lattice method for constructing invariants, discussed in the following section. The other is an extremely useful relation between different coefficients of any modular invariant. We shall call it the parity rule, and discuss it in the following subsection. For both these reasons, it will be convenient to define null augments.

First, let $T = (T_L; T_R)$ be any type. Let $M_{Li}$ be the coroot lattice of $g_{Li}$. By $M(T_L)$ we mean the scaled coroot lattice

$$M(T_L) = (\sqrt{k'_L1}M_{L1}) \oplus \cdots \oplus (\sqrt{k'_Ll}M_{Ll}).$$

So $M(T_L)$ will be a lattice of total dimension $n_L = \sum n_{Li}$. Define $M(T_R)$ by a similar expression.
Now, let $\mathcal{T}'$ and $\mathcal{T}''$ be any two types. By their augment $\mathcal{T} = \mathcal{T}' + \mathcal{T}''$ we simply mean the concatenation of the two types (so e.g. $M(\mathcal{T}_L) = M(\mathcal{T}'_L) \oplus M(\mathcal{T}''_L)$).

Suppose we are given a positive type $\mathcal{T}^+ = (\mathcal{T}'_L; \mathcal{T}''_R)$ satisfying (2.2). By its null augment [11] we mean any null type $\mathcal{T}^0 = (\mathcal{T}'_L; \mathcal{T}''_R)$ for which the augment $\mathcal{T} = \mathcal{T}^+ + \mathcal{T}^0$ satisfies the following two equations:

$$M(\mathcal{T}_L) \sim M(\mathcal{T}_R);$$  \hspace{1cm} (2.6a) 

$$\sum_{i=1}^{l} \rho_{L_i}^2 - \sum_{j=1}^{r} \rho_{R_j}^2 \equiv \sum_{i=1}^{l} \frac{n_{L_i}}{4} - \sum_{j=1}^{r} \frac{n_{R_j}}{4} \pmod{2}. \hspace{1cm} (2.6b)$$

‘$\sim$’ in (2.6a) denotes the lattice similarity relation [15], which is closely related to rational equivalence. Its geometrical significance here will be briefly discussed in sect. 3. Among other things it requires that the product $|M(\mathcal{T}_L)| |M(\mathcal{T}_R)|$ of determinants be a perfect square. The sums in (2.6b) are defined over the whole augment $\mathcal{T}$.

There will always be infinitely many different null augments for any candidate [11], but which one is chosen will not affect any subsequent calculations. In Table 1 we list one choice for each candidate.

The task of finding a null augment for a given candidate is not difficult. In [15] a calculus is developed for determining whether two lattices are similar. We will do an example here, verifying (2.6a) is satisfied for candidate 11 of Table 1.

Here, $\mathcal{T}_L = (A_{2,3}C_{2,0}A_{1,0})$ and $\mathcal{T}_R = (A_{1,2}A_{1,10}A_{3,0})$, so $M(\mathcal{T}_L) = (\sqrt{6}A_2) \oplus (\sqrt{6}Z^2) \oplus (\sqrt{4}Z)$ and $M(\mathcal{T}_R) = (\sqrt{8}Z) \oplus (\sqrt{24}Z) \oplus (\sqrt{4}A_3)$. We will use the convenient abbreviation $\{m_1, \ldots, m_\ell\}$ for the orthogonal lattice $(\sqrt{m_1}Z) \oplus \cdots \oplus (\sqrt{m_\ell}Z)$. Since $A_2 \sim \{3, 3, 3, 1, 1, 1\}$ and $A_3 \sim \{1, 1, 1\}$, we get $\sqrt{6}A_2 \sim \{18, 18, 18, 6, 6, 6\} \sim \{3\}$ and $\sqrt{4}A_3 \sim \{4, 4, 4\} \sim \{1\}$, using the $\sim$-calculus. Also, $\{6, 6, 4\} \sim \{3, 3\}$ and $\{8, 24\} \sim \{3, 3, 3\}$, so we get $M(\mathcal{T}_L) \sim \{3, 3, 3\} \sim M(\mathcal{T}_R)$, and thus (2.6a) holds.

One thing should be mentioned. Any number of copies of the null type $(A_{1,0})$ can be added to either side of a null augment, producing another null augment. The only relevant change is to the dimensions $n_L, n_R$. In particular, for some purposes it will be most convenient (see sect. 3) to choose the null augment so that $\mathcal{T}$ satisfies

$$n_L \equiv n_R \pmod{8}, \hspace{1cm} (2.7a)$$

while for other purposes it will be most convenient (see subsect. 2.3) to choose the null augment so that $\mathcal{T}$ satisfies

$$n_L \equiv n_R \pmod{2}. \hspace{1cm} (2.7b)$$

The choices in Table 1 all satisfy (2.7b). In some cases this was accomplished by including a copy of $A_{1,0}$.

### 2.3 The parity rule

In the classification of non-heterotic physical invariants, one of the most powerful tools is the parity rule [5,16,8]. In this subsection we will obtain its heterotic counterpart,
and use it to formulate a strong condition a type must satisfy if it is to be realized by a heterotic physical invariant.

First we need a few remarks about the weights and Weyl group of affine algebras. Let $g$ be any finite dimensional Lie algebra, let $M$ denote its coroot lattice, and choose any non-negative level $k$. The set of all weights of $g$ is just the dual lattice $M^*$. Define the sets $P_+(g, k + h^\vee)$ and $P_{++}(g, k + h^\vee)$ by

$$P_+(g, k + h^\vee) = \{ \lambda \in M^* | \lambda_i \geq 0, \sum_{i=1}^{n} a_i^\vee \lambda_i \leq k + h^\vee \},$$

$$P_{++}(g, k + h^\vee) = \{ \lambda \in M^* | \lambda_i > 0, \sum_{i=1}^{n} a_i^\vee \lambda_i < k + h^\vee \}. \quad (2.8a)$$

where $n$ is the rank of $g$ and the $a_i^\vee$ are positive integers called the colabels [17] of the affinization $g^{(1)}$. Here and elsewhere, a weight $\lambda$ will be identified by its Dynkin labels $\lambda_1, \ldots, \lambda_n$. More generally, define $P_{++}(T_{L,R})$ in the obvious way for any type $T$ (it will be the cartesian product of sets in (2.8b)).

The affine Weyl group of $g^{(1)}$ acts on $M^*$ as the semi-direct product of the group of translations by vectors in $(k + h^\vee)M$, with the (finite) Weyl group of $g$. The affine Weyl orbit of any weight $\lambda$ intersects $P_+(g, k + h^\vee)$ in exactly one weight, call it $[\lambda]$. If $[\lambda]$ lies on the boundary of $P_+$, define the parity $\epsilon(\lambda) = 0$. Otherwise it lies in the interior $P_{++}(g, k + h^\vee)$, and there exists a unique affine Weyl transformation $w$ such that $[\lambda] = w(\lambda)$; in this case define the parity $\epsilon(\lambda) = det(w) = \pm 1$. Incidentally, the task of finding the values of $[\lambda]$ and $\epsilon(\lambda)$ for arbitrary $\lambda, g, k$ is not difficult, even for large ranks, and efficient algorithms exist for all algebras (see ref. [16] for $A_n$).

Let $\mathcal{T}$ be any type satisfying eqs. (2.2), (2.6) and (2.7b). So $\mathcal{T}$ will in general include a null augment. Define $M(\mathcal{T}) = M(\mathcal{T}_L) \oplus \sqrt{-1}M(\mathcal{T}_R)$, an indefinite lattice of dimension $n_L + n_R$; $x \in M(\mathcal{T})$ can be written $(x_L; x_R)$ in the usual way. Consider any $\lambda_L \in P_{++}(\mathcal{T}_L)$, $\lambda_R \in P_{++}(\mathcal{T}_R)$. By $\lambda_L/\sqrt{k_L}$ we mean the scaled vector $(\lambda_{L1}/\sqrt{k_{L1}}, \ldots, \lambda_{Li}/\sqrt{k_{Li}})$, using obvious notation; similarly for $\lambda_R/\sqrt{k_R}$. Then $(\lambda_{L1}/\sqrt{k_{L1}}; \lambda_{R1}/\sqrt{k_{R1}}) \in M(\mathcal{T})^*$. We call a positive integer $L$ the order of $(\lambda_L; \lambda_R)$ in $\mathcal{T}$ when, for any integer $m$, the vector $(m\lambda_L/\sqrt{k_L}; m\lambda_R/\sqrt{k_R})$ lies in $M(\mathcal{T})$ iff $L$ divides $m$. Let $N$ be any modular invariant of type $\mathcal{T}$. Then for each $\ell$ relatively prime to $L$, $\epsilon_L(\ell \lambda_L) \epsilon_R(\ell \lambda_R) \neq 0$ and

$$N_{\lambda_L \lambda_R} = \epsilon_L(\ell \lambda_L) \epsilon_R(\ell \lambda_R) N_{[\ell \lambda_L]_L [\ell \lambda_R]_R}. \quad (2.9)$$

Eq. (2.9) is called the parity rule for heterotic invariants. Its proof is identical to the proof for non-heterotic invariants, given in ref. [5], which is based on the lattice method. (The one difference between the lattice methods for heterotic and non-heterotic types which seems relevant is the presence in the heterotic case of the translate $v$ – see the following section. However, by augmenting $\mathcal{T}$ by an even number of $A_{1,0}$, we can get (2.7a) satisfied, in which case $v = 0$ can always be chosen; this augmenting will not affect $\epsilon_{L,R}$, and will affect $[-]_{L,R}$ only in a trivial way.) For a different discussion of the heterotic parity rule, one not involving augments, see ref. [18]. There it is also generalized to any RCFT.
The parity rule has two valuable consequences. All such \((\ell \lambda_L; \ell \lambda_R)\) form a family of essentially equivalent representations, so that the coefficient \(N_{\lambda_L \lambda_R}\) for just one representative of each family need be stored. Another important implication of (2.9) is that for any \(\ell\) coprime to \(L\),

\[\epsilon_L(\ell \lambda_L) \epsilon_R(\ell \lambda_R) = -1 \Rightarrow N_{\lambda_L \lambda_R} = 0 \quad (2.10a)\]

for any physical invariant \(N\).

The parity rule simplifies the search for physical invariants by limiting the modular invariants we need to consider. In particular, call \((\lambda_L; \lambda_R)\) a positive parity pair if \(\epsilon_L(\ell \lambda_L) = \epsilon_R(\ell \lambda_R)\) for all \(\ell\) coprime to the order \(L\). By the positive parity commutant we mean the subspace of the Weyl-folded commutant (the space of modular invariants) consisting of all invariants \(Z\) with the property that \(N_{\lambda_L \lambda_R} \neq 0\) only for positive parity pairs \((\lambda_L; \lambda_R)\). The positive parity commutant contains all positive invariants and so is the only part of the Weyl-folded commutant we need to consider. It is generally significantly smaller than the full Weyl-folded commutant. We will need this observation to simplify the analysis for some of the more complicated cases.

Now consider \(\lambda_L = \rho_L, \lambda_R = \rho_R\). Then if \(\mathcal{T}\) is to be realized by a physical invariant \(N\), (2.10a) and (P3) imply that

\[\forall \ell \text{ coprime to } L, \epsilon_L(\ell \rho_L) = \epsilon_R(\ell \rho_R). \quad (2.10b)\]

Eq. (2.10b) is a strong constraint on the candidates. In Table 1 we run through all the candidates, and find the smallest positive \(\ell\) violating (2.10b) (if one exists). This \(\ell\) is listed in the Table.

The result is that there are precisely 14 non-AA-types which pass both conditions (2.2) and (2.10b). Some of these have physical invariants, some do not. In the following sections we will consider each of these in turn.

### 3. The lattice method

In this section we will first briefly review the extension of the Roberts-Terao-Warner lattice method [12] to heterotic invariants. Their method is a means of using self-dual lattices to generate invariants of the form (1.1). It was originally designed for symmetric types, but has been generalized [11] to heterotic ones using the idea of “null augments” discussed in subsect. 2.2. The method for symmetric types is summarized below in eq. (3.1), and for heterotic types in (3.2b). Next, we apply it to several of the candidates, finding not only all physical invariants for those types, but also the entire commutant. The definitions of the few lattice concepts we need can be found in ref. [14].

The Roberts-Terao-Warner [12] lattice method is a means of using self-dual lattices to find modular invariants of the form (1.1). It was originally designed for symmetric invariants, but has been generalized [11] to heterotic invariants, using the idea of “null augments” discussed in subsect. 2.2.

We will begin by reviewing the symmetric case. For notational convenience consider \(g = g_L = g_R\) simple. Let \(M\) be the coroot lattice of \(g\). Define the indefinite lattice
\[ \Lambda_0 = (\sqrt{k'} M) \oplus (\sqrt{-k'} M), \] where \( k' = k + k' \) is the height. Consider any even self-dual lattice \( \Lambda \supset \Lambda_0 \), of equal dimension to \( \Lambda_0 \). There will only be a finite number of these \( \Lambda \). For each of them, there will only be a finite number of cosets \([x] = [x_L; x_R] \in \Lambda/\Lambda_0\). Choose any \( x = (x_L; x_R) \in \Lambda \), and put \( \lambda_L = \sqrt{k'} x_L, \lambda_R = \sqrt{k'} x_R \). Then \( \lambda_L, \lambda_R \) are weights of \( g \), i.e. \( \lambda_L, \lambda_R \in M^* \).

The Roberts-Terao-Warner method associates to every coset \([x] \in \Lambda/\Lambda_0\) the character product
\[
\epsilon(\lambda_L)\epsilon(\lambda_R)\chi_{[\lambda_L]}\chi_{[\lambda_R]}^*, \tag{3.1}
\]
The partition function \( W Z_\Lambda \) corresponding to \( \Lambda \) consists of the sum of terms (3.1) over all cosets. Because \( \Lambda \) is even and self-dual, it is easy to show \( W Z_\Lambda \) will be modular invariant. Indeed, it has been shown [5] that these \( W Z_\Lambda \) span the commutant of \( g \), level \( k \). Examples of applications of this symmetric lattice method can be found in [12,13].

Unfortunately, extending this useful method to the heterotic types presents certain complications. Again define \( \Lambda_0 = (\sqrt{k'_L} M_L) \oplus (\sqrt{-k'_R} M_R) \). In general, there will not exist any self-dual lattices \( \Lambda \supset \Lambda_0 \). The necessary and sufficient condition for that to happen is the similarity [15] of the left and right lattices: \( \sqrt{k'_L} M_L \sim \sqrt{k'_R} M_R \). Further, for a self-dual indefinite lattice \( \Lambda \) to be even also, the condition (2.7a) must be satisfied as well.

The way out of these complications is described in detail in [11]. We will state here the conclusions. First, we must augment the heterotic type we are interested in, by some level 0 algebras in such a way that the resulting type satisfies (2.6). This is discussed in subsect. 2.2. Let the base lattice \( \Lambda_0 \) be defined with respect to this augmented type. The level 0 characters are all identically equal to 1, so at the end of the calculation they do not appear explicitly, but the null augments do affect the final lattice partition function through the parities of their weights, and are necessary to ensure modular invariance.

Eq. (2.6a) guarantees we will have self-dual \( \Lambda \supset \Lambda_0 \), though none of these may be even. Find a vector \( v \in \Lambda \) such that
\[
x^2 + 2x \cdot v \equiv 0 \pmod{2} \quad \forall x \in \Lambda \tag{3.2a}
\]
there are infinitely many such \( v \), which one we choose will only affect our final modular invariant by an irrelevant global sign. Then for each coset \([x] \in \Lambda/\Lambda_0\) (again there will only be finitely many of these), associate the character product
\[
(-1)^{x^2} \epsilon(\lambda_L)\epsilon(\lambda_R)\chi_{[\lambda_L]}\chi_{[\lambda_R]}^*, \tag{3.2b}
\]
where now \( \lambda_L = \sqrt{k'_L} (x_L + v_L) \), and similarly for \( \lambda_R \) (eq. (3.2b) is meaningful because \( \sqrt{k'_L} v_L \) is also a weight – this follows immediately from (3.2a) and the fact that each coroot lattice \( M \) is even). The sum of all these terms (3.2b), one for each coset, defines the lattice partition function \( W Z_\Lambda^v \). These are in fact modular invariant [11]. It is proven in [11] that these \( W Z_\Lambda^v \) again span the commutant of the desired type. So in principle this reduces the problem of finding all heterotic invariants of a given type to finding all self-dual \( \Lambda \supset \Lambda_0 \). Examples of the heterotic lattice method can be found in [11].

One remark should be made before we can proceed. Suppose (2.7a) holds. Then there exist self-dual \( \Lambda \supset \Lambda_0 \) which are even. It can be shown [11] that the \( W Z_\Lambda^v \) associated with
these even \( \Lambda \) also span the commutant. Moreover, by (3.2b) we can choose \( v = 0 \) for these \( \Lambda \). We can always fine-tune the ranks, by augmenting by \( A_{1,0} \), so that (2.7a) is satisfied. In some cases this can make the whole procedure a little more efficient.

The remainder of this section will be devoted to applying this lattice method to candidates 1, 2, 3, 44 and 51. The task of finding all self-dual gluings \( \Lambda \) of a given base lattice \( \Lambda_0 \) is not very difficult, at least if \( \Lambda_0 \) has a reasonably small dimension and determinant. The key is to exploit all the (e.g. Weyl) symmetries present. We will discuss type 1 in detail, but we first summarize the results for all these types. See also Table 2.

For type 1, we find that the commutant is spanned by the partition functions of nine lattices \((9 = 3 \cdot 3)\), the second ‘3’ reflects the fact that \( \text{Aut}(M(B_4))/W(B_4) \) has order 3. These partition functions are not linearly independent, and the commutant turns out to have dimension 3. For type 2, five lattices span the commutant, which is only 2-dimensional. For type 3, there are \( 9 \cdot 6 \cdot 2 \) lattices (the factor of 2 corresponds to an automorphism of the lattice \( \sqrt{6}Z^4 \) of the augment \( C_{2,0}C_{2,0} \)); for type 44, 6 \cdot 2 lattices; and for type 51, 5 \cdot 2 lattices.

We will work out the type 1 case explicitly; although it is a little simpler than most of the others, it is complicated enough to include the features of the general case.

The base lattice \( \Lambda_0 \) for type 1 is \( \Lambda_0 = \sqrt{2}D_4 \), where \( D_4 \) is the \( D_4 \) root lattice \([14]\), i.e. the set of all even-normed vectors in \( Z^4 \) (no null augment is needed in this case). The determinant \( |\Lambda_0| \) equals \( 21^4 \cdot 4 \), more precisely \( \Lambda^*_0/\Lambda_0 \cong Z_4 \times Z_{21}^4 \), which means that we need an order 2 vector \( g_1 \in \Lambda_0^* \) and two independent order 21 vectors \( g_2, g_3 \in \Lambda_0^* \); \( \Lambda \) will then be defined by

\[
\Lambda = \Lambda_0[g_1, g_2, g_3] \overset{\text{def}}{=} \bigcup_{a=0}^{20} \bigcup_{b=0}^{20} \bigcup_{c=0}^{20} \{\Lambda_0 + ag_1 + bg_2 + cg_3\}. \tag{3.3}
\]

As long as \( g_1, g_2, g_3 \) are independent, i.e. they generate (mod \( \Lambda_0 \)) a group isomorphic to \( Z_2 \times Z_{21}^2 \), and have integer dot products with each other, then \( \Lambda_0[g_1, g_2, g_3] \) will be self-dual.

There is an inaccessible number of triples \( g_1, g_2, g_3 \) with these properties, but most of these yield identical partition functions \( WZ^x_\Lambda \). Firstly, we are only interested in triples that lead to different \( \Lambda \). Secondly, if two triples \( g_1, g_2, g_3 \) and \( g_1', g_2', g_3' \) give rise to self-dual lattices \( \Lambda, \Lambda' \) differing by a global Weyl-reflection, i.e. \( \exists w \in W(B_4) \) such that \( w(\Lambda) = \Lambda' \), then by the Weyl-Kac character formula \([17]\) and eq.(3.2b), \( WZ^x_\Lambda \) and \( Wz^x_\Lambda' \) will differ by at most a global sign. We will find it convenient to “modulo out” \( \text{Aut}(D_4) \), rather than its subgroup \( W(B_4) \), but at the end we must apply the three non-Weyl automorphisms to each of the triples we have obtained.

First, let us list the possibilities (mod \( \Lambda_0 \)) for \( g_1 \): \( \sqrt{2}T(\tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \sqrt{2}T(1, 0, 0, 0) \), and \( \sqrt{2}T(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}) \) in the standard orthonormal basis for \( Z^4 \supset D_4 \). These are connected by the triality of \( D_4 \), so (modulo \( \text{Aut}(D_4) \)) there is a unique choice for \( g_1 \): we may choose \( g_1 = \sqrt{2}T(1, 0, 0, 0) \). This will mean that both \( g_2 \) and \( g_3 \) will lie in \( 1/\sqrt{2}TZ^4 \).

Our task to find \( g_2 \) and \( g_3 \) is simplified a bit by noting \([14]\) that there is only one self-dual (positive definite) lattice of dimension 4, namely \( Z^4 \). So what we must find are 4 orthonormal vectors \( u_i \) with coordinates \( 1/\sqrt{2}(a, b, c, d) \) (we will drop the \( \sqrt{2} \) in the following). Up to signs and reorderings, there are only 2 different unit vectors: \((4,2,1,0)\) and \((3,2,2,2)\).
Suppose first that there is at least one unit vector in Λ of the first kind. Then up to $W(B_4)$ we may write $u_1 = (4, 2, 1, 0)$. Up to irrelevant sign differences, there are precisely 6 unit vectors orthogonal to $u_1$. Running through these possibilities we find there are only 2 different solutions: $u_2 = (-2, 4, 0, 1)$, $u_3 = (0, -1, 2, 4)$ and $u_4 = (-1, 0, 4, -2)$; and $u_2 = (0, -2, 4, 1)$, $u_3 = (-2, 3, 2, -2)$ and $u_4 = (-1, 2, 0, 4)$. For both of these solutions we may choose $g_2 = u_1$, $g_3 = u_2$ and $v = (\frac{1}{2}, \frac{5}{2}, \frac{7}{2})$.

The remaining possibility is that all four $u_i$ are of the second kind of unit vector. In this case there is (mod $W(B_4)$) only one Λ, given by $u_1 = (3, 2, 2, 2)$, $u_2 = (-2, 3, 2, -2)$, $u_3 = (-2, -2, 3, 2)$ and $u_4 = (-2, 2, -2, 3)$. Here we may choose $g_2 = u_1$, $g_3 = u_2$, and $v = (\frac{-3}{2}, \frac{3}{2}, \frac{5}{2})$.

4. Maximal chiral extensions and heterotic invariants

The lattice method is quite practical as long as the levels and ranks do not get too large. But presumably it is completely inappropriate for candidates like $n = 59$ in Table 1. And it cannot be applied to an infinite family of types, like for instance the AA-types. In these cases, we need a more theoretical approach. One such approach is suggested by the work of [2].

The techniques discussed in this section are very powerful. However, they come with two caveats. The main one is that they require a complete knowledge of the physical invariants of the symmetric types $(\mathcal{T}_L; \mathcal{T}_R)$ and $(\mathcal{T}_R; \mathcal{T}_L)$. The other is that additional conditions, beyond (P1)-(P3) of sect. 1, must be imposed. These conditions are discussed in detail in subsect. 4.1 below, and they all are physically valid, but there are reasons for preferring classifications with a minimum of imposed conditions.

4.1 The cardinality test

We will collectively call the techniques contained in this subsection the “cardinality test”, even though only one actually involves comparing cardinalities.

Let $\mathcal{C}^L, \mathcal{C}^R$ denote the maximally extended chiral algebra [2] of respectively the holomorphic, anti-holomorphic sector of the theory. Let $\chi_i$, $i = 1, \ldots, a$ and $\tilde{\chi}_j$, $j = 1, \ldots, b$, be the characters of $\mathcal{C}^L$ and $\mathcal{C}^R$, respectively. Label these so that $\chi_1$ and $\tilde{\chi}_1$ correspond to the identities. $\chi_i, \tilde{\chi}_j$ can be expressed as linear combinations, over the non-negative integers, of the affine characters $\chi^{(L)}_{\lambda}, \chi^{(R)}_{\mu}$ of $\mathcal{T}_L$ and $\mathcal{T}_R$, respectively:

$$
\chi_i = \sum_{\lambda} m_{i\lambda} \chi^{(L)}_{\lambda}, \quad \tilde{\chi}_j = \sum_{\mu} \tilde{m}_{j\mu} \chi^{(R)}_{\mu}.
$$

(4.1a)

We have $m_{i\rho_L} = \delta_{i1}$ and $\tilde{m}_{j\rho_R} = \delta_{j1}$. Let the $S$ and $T$ modular matrices for these extended algebras be denoted $S^{(e)}, T^{(e)}, \tilde{S}^{(e)}, \tilde{T}^{(e)}$. Then

$$
S^{(e)\dagger} S^{(e)} = I, \quad S^{(e)T} = S^{(e)}, \quad T^{(e)\dagger} T^{(e)} = I, \quad T^{(e)T} = T^{(e)}, \quad S^{(e)}_{11} \geq S^{(e)}_{11} > 0,
$$

(4.1b)

with similar expressions for $\tilde{S}^{(e)}, \tilde{T}^{(e)}$. 

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Finally, the partition function (1.1) of any physical theory will look like

$$Z = \sum_{i=1}^{a} \tilde{c}_i \tilde{h}_{\tilde{\sigma}_i},$$

for some bijection $\sigma$. This means that the numbers of characters in the algebras $\mathcal{C}^L$ and $\mathcal{C}^R$ must be equal, $a = b$, and that

$$S_{ij}^{(e)} = \tilde{S}_{\sigma_i, \sigma_j}^{(e)}, \quad T_{ij}^{(e)} = \tilde{T}_{\sigma_i, \sigma_j}^{(e)},$$

for all $1 \leq h, i, j \leq a$.

So far in this paper all of our arguments have assumed only the familiar properties (P1-P3). Ref. [2] assumes a little more (namely duality, which is required for any theory to be physical). To make clear precisely which set of assumptions are being made, we will call an invariant physical if it obeys (P1-P3), and strongly physical if in addition it obeys (4.1), (4.2) (and hence (4.3)). There are physical invariants which are not strongly physical, but to be physically acceptable an invariant must be strongly physical.

These are far-reaching facts. We will use them in the following way. Suppose a given type $T = (\mathcal{T}_L; \mathcal{T}_R)$ has a strongly physical invariant $Z$, i.e. maximal chiral extensions $\mathcal{C}^L, \mathcal{C}^R$ obeying (4.1), (4.2) for some bijection $\sigma$. Construct the function

$$Z_L = \sum_{i=1}^{a} |\tilde{c}_i|^2.$$  \hspace{1cm} (4.4a)

Then from the previous comments we know that $Z_L$ is a modular invariant of type $(\mathcal{T}_L; \mathcal{T}_L)$. Similarly, we can construct a modular invariant $Z_R$ of type $(\mathcal{T}_R; \mathcal{T}_R)$. Expanded in terms of affine characters, $Z_L, Z_R$ are in block form:

$$Z_L = \sum_{i=1}^{a} \sum_{\lambda} m_{i\lambda} \chi^{(L)}_{\lambda} |^2 = \sum_{\lambda, \lambda'} M^{(L)}_{\lambda\lambda'} \chi^{(L)}_{\lambda} \chi^{(L)}_{\lambda'},$$  \hspace{1cm} (4.4b)

with a similar expression for $Z_R$. Note that $M^{(L)}_{\lambda\lambda'} \geq 0$, and $M^{(L)}_{\rho \rho'} = 1$.

Suppose we know all strongly physical invariants of (non-heterotic) types $(\mathcal{T}_L; \mathcal{T}_L)$ and $(\mathcal{T}_R; \mathcal{T}_R)$. Then in order for there to be a strongly physical heterotic invariant of type $(\mathcal{T}_L; \mathcal{T}_R)$, there must be strongly physical invariants of types $(\mathcal{T}_L; \mathcal{T}_L)$ and $(\mathcal{T}_R; \mathcal{T}_R)$ with the same number of maximally extended characters. We will call this the cardinality test.

For example, consider candidate $n = 59$: $T = (A_{2,105}; G_{2,5})$. All strongly physical invariants are known [8] for $(A_{2,105}; A_{2,105})$, and all physical invariants are known [13] for $(G_{2,5}; G_{2,5})$. In particular, there are precisely 4 strongly physical invariants for $A_{2,105}$, two with $a = 106 \cdot 107/2 = 5671$ and two with $a = 35 \cdot 36/2 + 3 = 633$. There is precisely 1 physical invariant for $G_{2,5}$, and it has $a = (7 \cdot 8/2 - 4)/2 = 12$ extended characters. Thus there can be no strongly physical invariants of type 59.

This analysis will allow us to handle the infinite series of AA-types:

$$T = \{ \{A_1, k_1\}, \{A_1, k_2\}; \{A_1, k_3\}, \{A_1, k_4\} \}.$$
Here, the central charge condition (2.2) becomes (2.4). Define \( s = k'_1 + k'_2, \ p = k'_1 k'_2, \ s' = k'_3 + k'_4 \) and \( p' = k'_3 k'_4 \), then \( s, p \) uniquely specify \( k'_1, k'_2 \), up to order; in particular they are the 2 roots of \( k'^2 - sk' + p = 0 \). (2.4) can be rewritten as

\[
    s' = p's/p. \tag{4.5}
\]

The \( A_1 + A_1 \) (non-heterotic) physical invariants have been classified in ref. [9], together with some anomolous levels worked out in ref. [13] and here in the appendix. The result is that there are a number of exceptionals, which we will discuss later, along with the simple current invariants and their conjugations. Let us consider first the (maximally extended) chiral algebras of the simple current invariants (conjugations, being automorphisms, do not affect the chiral algebra). There are 5 of these: \( C^{(1)}_{k\ell} = A_{1,k} + A_{1,\ell} \), the unextended chiral algebra, defined for all levels \( k, \ell \); \( C^{(2)}_{k\ell} \), defined for \( k \equiv -\ell \equiv \pm 1 \) (mod 4), is a chiral algebra corresponding to the simple current \( J = (1,1) \); \( C^{(3)}_{k\ell} \), defined for \( k \equiv \ell \equiv 0 \) or 2 (mod 4), corresponding to \( J = (1,1); C^{(4)}_{k\ell} \), defined for \( k \equiv 0 \) (mod 4), corresponding to \( J = (1,0) \); and finally \( C^{(5)}_{k\ell} \), defined for \( k \equiv \ell \equiv 0 \) (mod 4), corresponding to \( J = (1,0) \) and \( J' = (0,1) \). These have cardinalities given by:

\[
\begin{align*}
    \text{card } C^{(1)}_{k\ell} & : (k + 1)(\ell + 1); \\
    \text{card } C^{(2)}_{k\ell} & : \frac{(k + 1)(\ell + 1)}{4}; \\
    \text{card } C^{(3)}_{k\ell} & : \frac{k\ell + k + \ell + 8}{4}; \\
    \text{card } C^{(4)}_{k\ell} & : \frac{(k + 8)(\ell + 1)}{4}; \\
    \text{card } C^{(5)}_{k\ell} & : \frac{(k + 8)(\ell + 8)}{16}.
\end{align*}
\]

We will show that there cannot be a heterotic invariant with chiral algebras \( C^L = C^{(\alpha)}_{k_1 k_2} \), \( C^R = C^{(\beta)}_{k_3 k_4} \), for any \( \alpha, \beta = 1, \ldots, 5 \). These cardinalities alone suffice to handle some cases. For example, consider \( \alpha = \beta = 1 \). Then the cardinality condition becomes \( (k_1 + 1)(k_2 + 1) = (k_3 + 1)(k_4 + 1) \), i.e. \( p - s + 1 = p' - s' + 1 \), which combined with (4.5) forces \( p = p' \), \( s = s' \), in other words the sets \( \{k_1, k_2\} \) and \( \{k_3, k_4\} \) are equal. The arguments handling \( \alpha = \beta = 2, \alpha = \beta = 3 \) and \( \alpha = \beta = 5 \) are identical.

But using cardinalities alone is too difficult in some cases. Fortunately, for all simple current extensions it is trivial to compute their S-matrix elements \( S^{(e)}_{1i} \):

\[
    S^{(e)}_{1i} = \|\mathcal{J}\|_i S_{p,\lambda_i}, \tag{4.7}
\]

where \( S \) is the S-matrix for the underlying affine algebra, \( \lambda_i \) is any affine weight satisfying \( m_{i\lambda} \neq 0 \) (see (4.1a)), \( \|\mathcal{J}\| \) is the number of simple currents in the theory, and \( F_i \) is the number of these simple currents fixing \( \lambda_i \).
From this calculation we can read off the number of solutions \(i\) to the equation \(S_{1,i}^{(\alpha)} = S_{1,1}^{(\alpha)}\) (\(S^{(\alpha)}\) here is the \(S\)-matrix for the simple current extension \(C_{k\ell}^{(\alpha)}\)): there are 4 solutions for \(\alpha = 1\) (namely, those \(i\) with \(\lambda_i = (1,1), (k+1,1), (1,\ell+1),\) and \((k+1,\ell+1)\)); there is only 1 solution for \(\alpha = 2\) (namely, \(\lambda_i = (1,1)\)); there are 2 solutions for \(\alpha = 3\) (\(\lambda_i = (1,1)\) and \(\lambda_i = (k+1,1)\)); there are 2 solutions for \(\alpha = 4\) (\(\lambda_i = (1,1)\) and \(\lambda_i = (1,\ell+1)\)); and there is only one solution for \(\alpha = 5\) (\(\lambda_i = (1,1)\)). There are some low level exceptions, due to fixed points, to these numbers: \(C_{22}^{(3)}\) has 4 solutions; \(C_{4,\ell}^{(4)}\) has 4; and \(C_{4,\ell}^{(5)}\) has 3, unless \(\ell = 4\) in which case it has 9. These exceptions can be handled separately, e.g., by explicitly solving (2.4) (there are precisely 53 different heterotic solutions to (2.4) with \(k = 4\)).

By (4.3), this leaves only the following possibilities:

(i) \(C^L = C^{(2)}; C^R = C^{(5)}\);
(ii) \(C^L = C^{(4)}; C^R = C^{(4)}\);
(iii) \(C^L = C^{(3)}; C^R = C^{(4)}\).

Consider possibility (ii) (the arguments for (i) and (iii) are similar). We may assume \(k_1, k_3 > 4\). The second smallest value of \(S_{1,i}^{(4)}\) will be realized by \(i = i'\), where \(\lambda_{i'} = (1,2)\) (unless \(\ell = 1\), or sometimes when \(k = 8\)). Then for \(k_2, k_4 > 1\) and \(k_1, k_3 > 8\), dividing \(S_{1,i}^{(4)} = \tilde{S}_{1,i}^{(4)}\) by \(S_{1,1}^{(4)} = \tilde{S}_{1,1}^{(4)}\), and using (4.7), gives \(\sin(2\pi/k_2')/\sin(\pi/k_2') = \sin(2\pi/k_4')/\sin(\pi/k_4')\), i.e. \(k_2' = k_4'\); (2.4) then forces \((k_1, k_2) = (k_3, k_4)\). If however \(k_2 = 1\), then (2.4) can be solved explicitly: there are precisely 9 heterotic solutions; of these, only one satisfies the congruences \(k_1 \equiv k_3 \equiv 0 \pmod{4}\), and that one fails the cardinality test (4.6). \(k_1 = 8\) succumbs to a similar argument (there are precisely 110 solutions to (2.4) with \(k = 8\)).

Thus there can be no (strongly) physical invariants whose extended chiral algebras are both simple current extensions. This leaves the exceptional extensions. The greatest source of these involves the \(E_{10}\) or \(E_{28}\) exceptionals [1] of \(A_1\) (we can avoid \(E_{16}\) because it corresponds to a simple current extension). It is not difficult, using eqs. (2.3), to run through on a computer all solutions to (2.4) with \(k_1 = 10\) or \(k_2 = 28\) (there are 183 and 676 different heterotic solutions for these, respectively), and then explicitly check the cardinality test for each solution (the cardinality of the \(E_{10}\) extension is 3, and that of \(E_{28}\) is 2; when \(k_1 = 10\) and \(k_2 \equiv 2 \pmod{4}\) there is another cardinality, namely \((3k_2 + 10)/4\) corresponding to \((E_{10} \otimes A_{k_2}) N_{10,k_2}(J_1 J_2)\), which also must be considered). In this way, we find that there is only one AA-type made up of any combination of \(E_{10}, E_{28}\) and the simple currents that passes the cardinality test. It is candidate 62 in Table 1.

The remaining (non-heterotic) exceptionals for \(A_1 + A_1\) (see [13,19] and the appendix below) occur at levels \((k_1, k_2) = (4,4), (6,6), (8,8), (10,10), (2,10), (3,8), (3,28)\) and \((8,28)\). We do not have to consider the \((4,4), (8,8), (2,10)\) or \((3,28)\) exceptionals since they correspond to automorphisms of chiral algebras already considered. The cardinalities of the remaining exceptional chiral algebras can be read off, and are: 3, 4, 4, 2 respectively. It is an easy task to find the solutions to (2.4) for each of these levels, and to explicitly compare cardinalities. None of these pass the cardinality test.

4.2 Heterotic vrs symmetric automorphisms

Eqs. (4.1), (4.2), and (4.3) can have more to say, even when the candidate passes
the cardinality test. In particular, suppose we have found a heterotic invariant (4.2), corresponding to chiral extensions $C^L$ of $\mathcal{T}_L$ and $C^R$ of $\mathcal{T}_R$. Then by (4.3) the modular properties of $C^L$ and $C^R$ are completely equivalent – the bijection $\sigma$ makes this equivalence explicit.

Let us now ask the question: how many other heterotic invariants are there with the given chiral extensions? In other words, how many other bijections $\sigma'$ can we find? The answer is completely known, if we know all (non-heterotic) strongly physical invariants of $\mathcal{T}_L$, say. Let $Z_1, \ldots, Z_n$ be those physical invariants of $\mathcal{T}_L$ with (maximal) LHS and RHS chiral algebra $C^L$. Each of them is given by a bijection $\sigma_i$, mapping the characters of $C^L$ to themselves (e.g. one of these bijections, the one corresponding to the physical invariant of (4.4a), will be the identity). Then there will be precisely $n$ heterotic invariants with chiral algebras $C^L$ and $C^R$, and they will be given by the bijections $\sigma_i \circ \sigma$.

The only potential problem with this idea is field identification – i.e. in some cases different chiral characters $ch_i \neq ch_j$ correspond via (4.1a) to identical expressions of affine characters. But in practice this rarely presents any complications (see the example below). In this way we can write down all heterotic strongly physical invariants for candidates 39, 57 and 60. These are given in eqs. (6.1d), (6.1f), (6.1g).

We will give one example here. Consider $n = 39$. There are conformal embeddings [20] $C_{2,3} \subset D_{5,1}$ and $A_{1,10} A_{1,10} \subset D_{5,1}$, so each physical invariant of $(D_{5,1}; D_{5,1})$ will yield a physical invariant of $(C_{2,3}; A_{1,10} A_{1,10})$. There are only two [5] physical invariants of $(D_{5,1}; D_{5,1})$, and they both correspond to the heterotic invariant given in (6.1d). We know [13] all the physical invariants of $C_{2,3}$ and $A_{1,10} A_{1,10}$, in particular each has only one corresponding to this chiral extension.

There is field identification present here, but all the bijections map the identified fields to each other, so there is only one heterotic invariant.

Incidentally, both $C_{2,3}$ and $A_{1,10} A_{1,10}$ also have physical invariants with chiral cardinalities of 10 (see eqs. (4.3j) and (4.4a) in [13]). However, there can be no bijection between them, and hence no corresponding heterotic invariant. One way to see this is (4.3), the condition $T_{ij}^{(e)} = \tilde{T}_{\sigma_i,\sigma_j}^{(e)}$ (see (5.1) below).

5. The remaining candidates

In this section we discuss two more techniques, which suffice to complete the classification of the rank 4 heterotic invariants.

5.1 Explicit calculations

For heterotic type (2.1), $T$-invariance becomes the selection rule

$$N_{\lambda \mu} \neq 0 \Rightarrow \sum_{i=1}^{\lambda^2} \frac{\lambda_i^2}{k_i L_i} - \sum_{j=1}^{r} \frac{l_j}{k_j R_j} \equiv \sum_{i=1}^{l'} \frac{\rho_{L_i}^2}{h_{L_i}} - \sum_{j=1}^{r} \frac{\rho_{R_j}^2}{h_{R_j}} \pmod{2}. \quad (5.1)$$

Some candidates have small enough levels so that they can be explicitly worked out by hand. For this purpose the parity rule can also come in handy, simplifying the work.
by reducing the numbers of independent variables. The idea is to first use \(T\)-invariance to find all possible combinations \(\chi_{\mu}^{(L)}\chi_{\mu}^{(R)^*}\); the parity rule (2.9) can then find which of these terms can be ignored, and which of the remaining coefficients \(N_{\lambda\mu}\) are independent. If this number is sufficiently small, \(S\)-invariance can be done explicitly. Outer automorphisms (i.e. simple currents) can also be used to good effect here.

For example, consider candidate \(n = 45\). \(T\)-invariance tells us the modular invariant will look like

\[
Z = a\chi_{11}\chi_{11}^* + b\chi_{11}\chi_{22}^* + c\chi_{12}\chi_{11}^* + d\chi_{12}\chi_{22}^*,
\]

where the three characters in each term correspond to \(G_{2,1}\), \(A_{1,1}\) and \(A_{1,3}\) respectively. The parity rule tells us that

\[
a = c, \quad b = d.
\]

Thus we have only two independent variables, \(a\) and \(b\). Note from (5.2a) that in each term the weight labels for \(A_{1,1}\) and \(A_{1,3}\) are always equivalent to each other mod 2. From the relation \(N = S_LNS_R^\dagger\) we then get that \(N_{\lambda;bc} = N_{\lambda;3-b,5-c}\), i.e. \(a = d\) and \(b = c\). \(S\)-invariance can be explicitly checked to verify that the resulting partition function is indeed modular invariant.

Hence for this candidate the commutant is one-dimensional, given by the \(Z\) in (5.2a) with \(a = b = c = d\), and there is exactly one physical invariant. It is listed in (6.1e).

Similar arguments (i.e. using (5.1), the parity rule, outer automorphisms, and as a last resort explicit \(S\)-matrix calculations) work for candidates 11, 12, 13 and 62.

### 5.2 Projection

Let \(N\) be any physical invariant of type \((G_{2,24}G_{2,24};0)\). Then by the usual projection argument [21],

\[
M_{\lambda\mu} = \sum_{\nu} N_{\lambda\nu}N_{\nu\mu}, \quad M'_{\lambda\mu} = \sum_{\nu} N_{\lambda\nu}N_{\mu\nu}
\]

will both be modular invariants of type \((G_{2,24};G_{2,24})\). In fact, they will be positive invariants, i.e. each \(M_{\lambda\mu}, M'_{\lambda\mu} \geq 0\), and will be nonzero because \(M_{\rho\rho}, M'_{\rho\rho} \geq N_{\rho\rho} = 1\).

In Table 1 of [13] we find that the positive parity commutant, which of necessity contains all positive invariants, is one-dimensional for \(g_L = g_R = G_{2,24}\) (indeed, for all \(G_{2,k}, 5 \leq k \leq 31\)). Thus we get

\[
M_{\lambda\mu} = a\delta_{\lambda\mu}, \quad M'_{\lambda\mu} = a'\delta_{\lambda\mu} \quad \forall \lambda, \mu
\]

for some constants \(a, a' > 0\). Hence

\[
a = \sum_{\nu} N_{\lambda\nu}N_{\nu\lambda}, \quad a' = \sum_{\nu} N_{\lambda\nu}^2, \quad \forall \lambda
\]

\[
0 = \sum_{\nu} N_{\lambda\nu}N_{\nu\mu} = \sum_{\nu} N_{\lambda\nu}N_{\mu\nu}, \quad \forall \lambda \neq \mu.
\]
Therefore, \( N_{\lambda \mu} \neq 0 \) implies \( N_{\nu \mu} = N_{\mu \nu} = 0 \ \forall \nu \neq \lambda \). From (5.4a) this forces \( a = a' = 1 \), so each \( N_{\lambda \mu} = 0 \) or 1, and there is only one 1 on each row and column of \( N \). In other words, there exists a permutation \( \sigma \) of \( P_{++}(G_{2,24}) \) such that

\[
N_{\lambda \mu} = \delta_{\mu, \sigma \lambda}, \forall \lambda, \mu. \tag{5.5}
\]

Let us investigate now the commutation of \( N \) with \( T \). Let \((m, n)\) be the Dynkin labels of \( \lambda \), and \((m', n')\) those of \( \sigma \lambda \). Then \( T \)-invariance (5.1) means \( (\lambda^2 + (\sigma \lambda)^2)/28 \equiv 1/3 \) (mod 2), i.e.

\[
m^2 + mn + n^2/3 + m'^2 + m'n' + n'^2/3 \equiv \frac{14}{3} \text{ (mod 28).} \tag{5.6}
\]

Choose \( \lambda = (1, 3) \), then (5.6) says among other things that \( n'^2 \equiv 2 \) (mod 3), which has no solution.

Thus \( T \)-invariance for (5.5) cannot be satisfied, so there are no physical invariants for candidate 9.

6. Conclusion

In this paper, we have developed several general techniques for classifying heterotic physical invariants, and have applied them to find all such invariants of total rank 4. In addition, we have also determined all non-heterotic physical invariants for \((A_1,k_1 + A_1,k_2)\) with \( k_1 \neq k_2 < 22 \) and \( k_1 = 28, k_2 < 22 \); these results are given in the Appendix.

The following is the list of all total rank 4 heterotic (strongly) physical invariants:

\[
(C_{4,10};-) : \quad Z_2 = \chi_{1111} + \chi_{11,11,1} + \chi_{1135} + \chi_{1151} + \chi_{1155} + \chi_{5113} + \chi_{5117} + \chi_{11,1,1,1} + \chi_{1371} + \chi_{1432} + \chi_{1434} + \chi_{1611} + \chi_{5251} + \chi_{5252} + \chi_{1616} + \chi_{1911} + \chi_{1913} + \chi_{2162} + \chi_{2541} + \chi_{5331} + \chi_{6124} + \chi_{2542} + \chi_{3115} + 2\chi_{3333} + \chi_{3533} + \chi_{3413} + \chi_{7313} + \chi_{3415} + \chi_{1442} + \chi_{4144} + \chi_{4522} + \chi_{4523} + \chi_{11,1,1,1}; \quad (6.1a)
\]

\[
(A_{2,3}; A_{1,2} A_{1,10}) : \quad Z_{11} = (\chi_{11} + \chi_{14} + \chi_{41})(\chi_1 \chi_1 + \chi_1 \chi_7 + \chi_3 \chi_5 + \chi_3 \chi_1)^* + \chi_{22}(\chi_1 \chi_1 + \chi_3 \chi_5 + \chi_3 \chi_3 + 2\chi_2 \chi_4 + 2\chi_2 \chi_8)^*; \quad (6.1b)
\]

\[
(A_{2,4}; A_{1,4}, A_{1,12}) : \quad Z_{13} = (\chi_{11}(\chi_1 + \chi_5)^* (\chi_1 + \chi_13)^* + \chi_{22}(\chi_1 + \chi_5)^* (\chi_5 + \chi_9)^* + (\chi_{12} + \chi_{21}) \chi_3 \chi_7^* + (\chi_{24} + \chi_{42}) \chi_3 \chi_7^* + (\chi_{14} + \chi_{41})(\chi_1 + \chi_5)^* \chi_7^* + (\chi_{13} + \chi_{31}) \chi_3^* (\chi_3 + \chi_1)^* + \chi_{33}(\chi_1 + \chi_5)^* \chi_3^* (\chi_3 + \chi_1)^* + (\chi_{15} + \chi_{51}) \chi_3^* (\chi_1 + \chi_{13})^* + (\chi_{23} + \chi_{32}) \chi_3^* (\chi_5 + \chi_9)^*; \quad (6.1c)
\]

\[
(C_{2,3}; A_{1,10} A_{1,10}) : \quad Z_{39} = (\chi_{11} + \chi_{32})(\chi_1 \chi_1 + \chi_1 \chi_7 + \chi_5 \chi_5 + \chi_5 \chi_11 + + \chi_7 \chi_1 + \chi_7 \chi_7 + \chi_{11} \chi_5 + \chi_{11} \chi_{11} + \chi_5 \chi_5 + \chi_7 \chi_7 + \chi_{11} \chi_11 + \chi_7 \chi_5) + (\chi_{14} + \chi_{31})(\chi_1 \chi_{11} + \chi_1 \chi_5 + \chi_7 \chi_{11} + \chi_7 \chi_5
\]
\begin{equation}
+ \chi_5 \chi_7 + \chi_5 \chi_1 + \chi_{11} \chi_1 + \chi_{11} \chi_7 \}^*
+ 2\chi_{22}\{\chi_4 \chi_4 + \chi_4 \chi_8 + \chi_8 \chi_4 + \chi_8 \chi_8\}^*;
(6.1d)
\end{equation}

\begin{equation}
(G_{2,1}; A_{1,1}A_{1,3}) : Z_{45} = \chi_{11}\{\chi_1 \chi_1 + \chi_2 \chi_4\}^* + \chi_{12}\{\chi_1 \chi_3 + \chi_2 \chi_2\}^*;
(6.1e)
\end{equation}

\begin{equation}
(A_{2,9}; G_{2,2}) : Z_{57} = (\chi_{11} + \chi_{1,10} + \chi_{10,1} + \chi_{55} + \chi_{52} + \chi_{25})\{\chi_{11} + \chi_{22}\}^*
+ 2(\chi_{33} + \chi_{36} + \chi_{63}) \cdot \chi_{13}^*;
(6.1f)
\end{equation}

\begin{equation}
(C_{2,7}; G_{2,4}) : Z_{60} = (\chi_{11} + \chi_{16} + \chi_{33} + \chi_{72})\{\chi_{11} + \chi_{14}\}^* + 2(\chi_{42} + \chi_{44})\chi_{22}^*
+ (\chi_{13} + \chi_{18} + \chi_{34} + \chi_{71})\{\chi_{21} + \chi_{15}\}^*.
(6.1g)
\end{equation}

The subscript ‘\(n\)’ of \(Z_n\) in these equations denotes the candidate number (see Table 1); the subscripts on the \(\chi\)’s denote the Dynkin labels. The invariant \(Z_2\) was first found in ref. [22], while \(Z_{39}, Z_{45}, Z_{57}\) and \(Z_{60}\) are due to conformal embeddings [20] applied to the diagonal invariants of \((D_{5,1}; D_{5,1})\), \((G_{2,1}; G_{2,1})\), \((E_{6,1}; E_{6,1})\) and \((D_{7,1}; D_{7,1})\), respectively \((Z_{45}\) can also be obtained from the rank 3 invariants \((1.2c), (1.2d))\). Both \(Z_{11}\) and \(Z_{13}\) seem to be new. \(Z_{11}\) can be understood within the context of [2] as a bijection between the simple current chiral extension of \(A_{2,3}\), and the chiral extension of \(A_{1,2}A_{1,10}\) associated with the physical invariant \((A_2 \otimes E_{10}) N_{(2,10)}(J_1 J_2)\); as such it is intimately connected with the exceptional \(E''_{2,10}\) given in \((A.6)\) below. However, \(Z_{13}\) is much harder to understand.

Our results demonstrate the scarcity of heterotic invariants. For example, there are between 1 and 27 non-heterotic physical invariants corresponding to each choice \((k_1, k_2)\) of level, for the algebra \(A_1 + A_1\); however, there are zero heterotic physical invariants for \(g_L = g_R = A_1 + A_1\), for any level. For total rank \(\geq 5\), there will be infinitely many heterotic invariants (just tensor non-heterotic ones with rank 3 or 4 heterotics), but their numbers will always be very small compared to the non-heterotics of similar rank. This is reflected in the severity of the constraints which must be satisfied by the algebras and levels of heterotic physical invariants (see e.g. \((2.2)\) and \((2.10b))\); these constraints will be trivially satisfied for non-heterotic invariants.

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Appendix

In this Appendix we present the results of a computer search for the non-heterotic physical invariants of the algebra $A_{1,k_1} + A_{1,k_2}$ for $k_1 \neq k_2 < 22$ and $k_1 = 28$, $k_2 < 22$. We used a method based on the notion of even self-dual lattices developed in ref. [13] (where, however, the $A_1 + A_1$ physical invariants had been calculated only at levels $k_1 = k_2 < 22$).

Most of the physical invariants of this algebra belong to one of the infinite series of invariants, some of which are obtained as tensor products of the $A_k$ and $D_k$ physical invariants of $A_{1,k}$ [1]. But a number of others not belonging to these series also occur at various levels. Together with the four exceptionals given in eqs. (4.3f)-(4.3i) of [13], and the conjugations of all these invariants if $k_1 = k_2$, the following list exhausts all $A_1 + A_1$ (non-heterotic) physical invariants, for all levels $k_1, k_2$.

We list first the infinite series of $A_1 + A_1$ invariants. If the levels $k_1$ and $k_2$ are both even, there are 6 series; otherwise, only 2. Let $p_1 = k_1 + 2$, $p_2 = k_2 + 2$, $k = (k_1, k_2)$. The invariants can be defined by their corresponding coefficient matrices $N_{ij,i'j'}$, where $(ij)$ and $(i'j')$ are the Dynkin labels of the 2 weights $\lambda_L$ and $\lambda_R$, with $0 < i, i' < p_1$ and $0 < j, j' < p_2$. We will name the invariants using simple current notation. By $J_1$ we mean the simple current $(1,0)$, etc.

For $k_1 \equiv k_2 \pmod{4}$, both odd, the 2 series are:

(i) the diagonal (identity) invariant $N_k(0) = A_{k_1} \otimes A_{k_2}$;
(ii) $N_k(J_1 J_2)$ given by

$$N_k(J_1 J_2)_{ij, i'j'} = \begin{cases} \delta_{ii'} \delta_{jj'} & \text{if } i \equiv j \pmod{2} \\ \delta_{i,p_1 - i'} \delta_{j,p_2 - j'} & \text{otherwise} \end{cases} \quad (A.1)$$

For $k_1 \neq k_2 \pmod{4}$, both odd, we have

(i) the diagonal invariant $N_k(0)$;
(ii) the invariant $N_k(J_1 J_2)$ defined by

$$N_k(J_1 J_2)_{ij, i'j'} = \begin{cases} 0 & \text{if } i \neq j \pmod{2} \\ \delta_{ii'} \delta_{jj'} + \delta_{i,p_1 - i'} \delta_{j,p_2 - j'} & \text{otherwise} \end{cases} \quad (A.2)$$

For $k_1$ even, $k_2$ odd, the 2 series (these are equal if $k_1 = 2$) are

(i) the diagonal invariant $N_k(0)$;
(ii) $N_k(J_1) = D_{k_1} \otimes A_{k_2}$.

For $k_1 \equiv k_2 \equiv 2 \pmod{4}$, there are 6 series (3 if $k_1$ or $k_2$, but not both, equal 2; 2 if both $k_1 = k_2 = 2$):

(i) the diagonal invariant $N_k(0)$;
(ii) $N_k(J_1) = D_{k_1} \otimes A_{k_2}$;
(iii) $N_k(J_2) = A_{k_1} \otimes D_{k_2}$;
(iv) $N_k(J_1 J_2)$ defined by eq. (A.2);
(v) $N_k(J_1 J_2) = D_{k_1} \otimes D_{k_2}$;

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denote by $N$ the diagonal invariant exceptional invariants.

For $k_1 \equiv k_2 \equiv 0 \pmod{4}$, the 6 series are:

(i) the diagonal invariant $N_k(0)$;
(ii) $N_k(J_1) = D_{k_1} \otimes A_{k_2}$;
(iii) $N_k(J_2) = A_{k_1} \otimes D_{k_2}$;
(iv) $N_k(J_1, J_2)$ defined by eq. (A.2);
(v) $N_k(J_1; J_2) = D_{k_1} \otimes D_{k_2}$;
(vi) $N_k(\text{aut})$ defined by

\[
N_k(\text{aut})_{ij,i'j'} = \begin{cases} 
\delta_{ii'} \delta_{jj'} & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{i,p_1-i'} \delta_{j,j'} & \text{if } i \equiv j \equiv 0 \pmod{2} \\
\delta_{i,p_1-i'} \delta_{j,p_2-j'} & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{i,p_1-i'} \delta_{j,p_2-j'} & \text{if } i \equiv j \equiv 0 \pmod{2} 
\end{cases} . \tag{A.4}
\]

For $k_1 \equiv 0, k_2 \equiv 2 \pmod{4}$, the 6 series (3 if $k_2 = 2$) are:

(i) the diagonal invariant $N_k(0)$;
(ii) $N_k(J_1) = D_{k_1} \otimes A_{k_2}$;
(iii) $N_k(J_2) = A_{k_1} \otimes D_{k_2}$;
(iv) $N_k(J_1, J_2)$ defined by eq. (A.1);
(v) $N_k(J_1; J_2) = D_{k_1} \otimes D_{k_2}$;
(vi) the invariant $N_k(J_2; J_1, J_2)$ defined by

\[
N_k(J_2; J_1, J_2)_{ij,i'j'} = \begin{cases} 
\delta_{ii'} \delta_{jj'} & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{ii'} \delta_{j,p_2-j'} & \text{if } i \equiv j \equiv 0 \pmod{2} \\
\delta_{i,p_1-i'} \delta_{j,p_2-j'} & \text{if } i \equiv j \equiv 1 \pmod{2} \\
\delta_{i,p_1-i'} \delta_{j,j'} & \text{if } i \equiv j \equiv 0 \pmod{2} 
\end{cases} . \tag{A.5}
\]

Besides these simple current invariants, we also obtain solutions built on the three $A_1$ exceptional invariants $E_{10}$, $E_{16}$, and $E_{28}$. Namely, the three isolate solutions $E_{10} \otimes E_{16}$, $E_{10} \otimes E_{28}$, and $E_{16} \otimes E_{28}$, and the following series for all $k_2$: $E_{10} \otimes A_{k_2}$, $E_{16} \otimes A_{k_2}$, and $E_{28} \otimes A_{k_2}$; and for even $k_2 \geq 4$: $E_{10} \otimes D_{k_2}$, $E_{16} \otimes D_{k_2}$, $E_{28} \otimes D_{k_2}$, and $(E_{10} \otimes A_{k_2}) N_{(10,k_2)}(J_1 J_2)$.

Finally, there exist sporadic exceptional invariants not of the above types, which we denote by $E''_{k_1,k_2}$. They are:

\[
E''_{2,10} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,7} + \chi_{3,5} + \chi_{3,11}|^2 + (\chi_{1,5} + \chi_{1,11} + \chi_{3,1} + \chi_{3,7})(\chi_{2,4} + \chi_{2,8})^* \\
+ (\chi_{2,4} + \chi_{2,8})(\chi_{1,5} + \chi_{1,11} + \chi_{3,1} + \chi_{3,7})^* + |\chi_{2,4} + \chi_{2,8}|^2 ; \tag{A.6}
\]

\[
E''_{3,8} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{3,5} + \chi_{1,9}|^2 + |\chi_{2,5} + \chi_{4,1} + \chi_{4,9}|^2 + |\chi_{1,5} + \chi_{3,3} + \chi_{3,7}|^2 \\
+ |\chi_{2,3} + \chi_{4,5} + \chi_{2,7}|^2 ; \tag{A.7}
\]

\[
E''_{3,28} \overset{\text{def}}{=} |\chi_{1,1} + \chi_{1,11} + \chi_{1,19} + \chi_{1,29}|^2 + |\chi_{2,7} + \chi_{2,13} + \chi_{2,17} + \chi_{2,23}|^2
\]
\[\begin{align*}
+ \chi_{3,7} + \chi_{3,13} + \chi_{3,17} + \chi_{3,23} &+ |\chi_{4,1} + \chi_{4,11} + \chi_{4,19} + \chi_{4,29}|^2 \\
+ [(\chi_{1,7} + \chi_{1,13} + \chi_{1,17} + \chi_{1,23})(\chi_{3,1} + \chi_{3,11} + \chi_{3,19} + \chi_{3,29})]^* &+ cc \\
+ [(\chi_{4,7} + \chi_{4,13} + \chi_{4,17} + \chi_{4,23})(\chi_{2,1} + \chi_{2,11} + \chi_{2,19} + \chi_{2,29})]^* &+ cc; \quad (A.8)
\end{align*}\]

\[\begin{align*}
\mathcal{E}_{8,28}'' \xrightarrow{\text{def}} |\chi_{1,1} + \chi_{1,11} + \chi_{1,19} + \chi_{1,29} + \chi_{9,1} + \chi_{9,11} + \chi_{9,19} + \chi_{9,29} \\
+ \chi_{5,7} + \chi_{5,13} + \chi_{5,17} + \chi_{5,23}|^2 &+ |\chi_{5,1} + \chi_{5,11} + \chi_{5,19} + \chi_{5,29} \\
+ \chi_{3,7} + \chi_{3,13} + \chi_{3,17} + \chi_{3,23} + \chi_{7,7} + \chi_{7,13} + \chi_{7,17} + \chi_{7,23}|^2; \quad (A.9)
\end{align*}\]

where the characters \(\chi_{a,b}\) are just products of the characters of \(A_1^{(1)}\): \(\chi_{a,b} = \chi_a \chi_b\).

The invariants (A.6) and (A.8) were already found in ref. [19]. The other two invariants have not appeared in the literature before: the invariant (A.7) can be obtained from the \(A_{1,3}A_{1,8}A_{1,1} \subset F_{4,1}\) conformal embedding, with \(A_{1,1}\) projected out, while the other, (A.9), arises from a \(A_{1,8}A_{1,28} \subset F_{4,1}\) conformal embedding.

Together with the results given in ref. [13] for \(k_1 = k_2\), the above equations exhaust all the \(A_1 + A_1\) physical invariants for \(k_1, k_2 \leq 21\) and \(k_1 = 28, k_2 \leq 21\). These include all of the levels where the arguments of [9] break down, so this completes the \(A_{1,k_1} + A_{1,k_2}\) classification for all levels: the complete list of all physical invariants is given by the above invariants, along with the exceptionals \((4.3f), (4.3g), (4.3h), (4.3i)\) in [13], the exceptionals \(\mathcal{E}_{10} \otimes \mathcal{E}_{10}, \mathcal{E}_{16} \otimes \mathcal{E}_{16},\) and \(\mathcal{E}_{28} \otimes \mathcal{E}_{28},\) and the conjugations [13] of all these when \(k_1 = k_2\).
Table 1. The candidate types

| n  | $\mathcal{T}^+ = (T^+_L; T^+_R)$ | $\mathcal{T}^0 = (T^0_L; T^0_R)$ | $\ell$ |
|----|---------------------------------|---------------------------------|-------|
| 1  | $(B_{4,14}; -)$                  | -                               | -     |
| 2  | $(C_{4,10}; -)$                  | -                               | -     |
| 3  | $(D_{4,36}; -)$                  | $(C_2^2; -)$                    | -     |
| 4  | $(F_{4,108}; -)$                 | $(C_2^2; -)$                    | 29    |
| 5  | $(G_{2,11}G_{2,206}; -)$        | $(C_2^2; -)$                    | 13    |
| 6  | $(G_{2,12}G_{2,108}; -)$        | $(-; G_2^2)$                    | 13    |
| 7  | $(G_{2,14}G_{2,59}; -)$         | $(C_2^2; -)$                    | 11    |
| 8  | $(G_{2,17}G_{2,38}; -)$         | $(C_2^2; -)$                    | 13    |
| 9  | $(G_{2,24}G_{2,24}; -)$         | $(-; G_2^2)$                    | -     |
| 10 | $(A_{2,1}; A_{1,1}A_{1,1})$     | $(D_7A_1; -)$                   | 5     |
| 11 | $(A_{2,3}; A_{1,2}A_{1,10})$    | $(C_2A_1; A_3)$                 | -     |
| 12 | $(A_{2,3}; A_{1,4}A_{1,4})$     | $(-; A_2)$                      | -     |
| 13 | $(A_{2,4}; A_{1,4}A_{1,12})$    | $(C_2; A_6)$                    | -     |
| 14 | $(A_{2,5}; A_{1,5}A_{1,40})$    | $(C_3; C_2^2A_3)$               | 11    |
| 15 | $(A_{2,5}; A_{1,6}A_{1,22})$    | $(-; A_2G_2^3)$                 | 5     |
| 16 | $(A_{2,5}; A_{1,7}A_{1,16})$    | $(D_5D_7; A_3A_1)$              | 5     |
| 17 | $(A_{2,5}; A_{1,8}A_{1,13})$    | $(D_5; C_2^2A_1)$               | 11    |
| 18 | $(A_{2,5}; A_{1,10}A_{1,10})$   | $(-; A_2)$                      | 7     |
| 19 | $(A_{2,6}; A_{1,8}A_{1,88})$    | $(-; A_2)$                      | 7     |
| 20 | $(A_{2,6}; A_{1,10}A_{1,34})$   | $(A_1; G_2^2C_2A_3)$            | 5     |
| 21 | $(A_{2,6}; A_{1,16}A_{1,16})$   | $(-; A_2)$                      | 7     |
| 22 | $(A_{2,7}; A_{1,14}A_{1,238})$  | $(-; A_{14}C_2G_2)$             | 7     |
| 23 | $(A_{2,7}; A_{1,16}A_{1,88})$   | $(C_2^2A_{14}; -)$              | 11    |
| 24 | $(A_{2,7}; A_{1,18}A_{1,58})$   | $(C_2A_1; A_3)$                 | 11    |
| 25 | $(A_{2,7}; A_{1,22}A_{1,38})$   | $(-; A_{14}C_2G_2)$             | 7     |
| 26 | $(A_{2,7}; A_{1,28}A_{1,28})$   | $(A_1; B_3A_{14})$              | 7     |
| 27 | $(A_{2,8}; A_{1,32}A_{1,1120})$ | $(-; A_{10}C_2^2)$              | 13    |
| 28 | $(A_{2,8}; A_{1,34}A_{1,394})$  | $(-; A_{10}C_2^2G_2^3)$         | 5     |
| 29 | $(A_{2,8}; A_{1,40}A_{1,152})$  | $(-; A_{10}C_2^2)$              | 13    |
| 30 | $(A_{2,8}; A_{1,42}A_{1,130})$  | $(A_3A_1^2; C_2^3)$             | 17    |
| 31 | $(A_{2,8}; A_{1,64}A_{1,64})$   | $(-; A_2)$                      | 5     |

$n =$ label; $\mathcal{T}^+ =$ positive type; $\mathcal{T}^0 =$ null augment (level 0 subscripts omitted); and $\ell =$ first number (if any) which violates $\rho$ parity test.
Table 1 (cont.). The candidate types

| n    | $T^+ = (T_L^+; T_R^+)$       | $T^0 = (T_L^0; T_R^0)$       | ℓ   |
|------|-----------------------------|-----------------------------|-----|
| 32   | $(C_{2,1}; A_{1,1}A_{1,2})$ | $(C_2; G_2)$                | 5   |
| 33   | $(C_{2,2}; A_{1,2}A_{1,10})$| $(C_2; G_2)$                | 7   |
| 34   | $(C_{2,2}; A_{1,4}A_{1,4})$ | $(\neg; C_2)$              | 7   |
| 35   | $(C_{2,3}; A_{1,5}A_{1,40})$| $(G_2A_1; C_3)$             | 17  |
| 36   | $(C_{2,3}; A_{1,6}A_{1,22})$| $(C_2; G_2)$                | 5   |
| 37   | $(C_{2,3}; A_{1,7}A_{1,16})$| $(\neg; C_3G_2^2A_1)$      | 5   |
| 38   | $(C_{2,3}; A_{1,8}A_{1,13})$| $(G_2A_1; C_3)$             | 7   |
| 39   | $(C_{2,3}; A_{1,10}A_{1,10})$| $(G_2^2C_2; \neg)$         | –   |
| 40   | $(C_{2,4}; A_{1,20}A_{1,460})$| $(A_2; A_6)$               | 13  |
| 41   | $(C_{2,4}; A_{1,22}A_{1,166})$| $(A_3A_1; C_2A_6)$         | 5   |
| 42   | $(C_{2,4}; A_{1,26}A_{1,82})$| $(G_2; C_6)$                | 5   |
| 43   | $(C_{2,4}; A_{1,28}A_{1,68})$| $(A_2; A_6)$                | 11  |
| 44   | $(C_{2,4}; A_{1,40}A_{1,40})$| $(\neg; C_2)$              | –   |
| 45   | $(G_{1,1}A_{1,1}A_{1,3})$ | $(C_2A_1; B_3)$            | –   |
| 46   | $(G_{2,2}; A_{1,3}A_{1,43})$| $(C_2^3G_2; \neg)$        | 13  |
| 47   | $(G_{2,2}; A_{1,4}A_{1,16})$| $(\neg; C_2^3)$            | 5   |
| 48   | $(G_{2,2}; A_{1,7}A_{1,7})$ | $(D_7; A_3)$               | 7   |
| 49   | $(A_{2,3}; C_{2,2})$        | $(A_2; \neg)$              | 7   |
| 50   | $(A_{2,5}; C_{2,3})$        | $(A_2; \neg)$              | 7   |
| 51   | $(A_{2,15}; C_{2,6})$       | $(A_2; \neg)$              | –   |
| 52   | $(A_{2,21}; C_{2,7})$       | $(C_2; A_2)$               | 11  |
| 53   | $(A_{2,30}; C_{2,8})$       | $(A_2; \neg)$              | 5   |
| 54   | $(A_{2,45}; C_{2,9})$       | $(A_2; \neg)$              | 5   |
| 55   | $(A_{2,75}; C_{2,10})$      | $(A_2; \neg)$              | 5   |
| 56   | $(A_{2,165}; C_{2,11})$     | $(A_2A_3^2; C_6)$          | 5   |
| 57   | $(A_{2,9}; G_{2,3})$        | $(A_3A_1; \neg)$           | –   |
| 58   | $(A_{2,21}; G_{2,1})$       | $(A_3A_1; \neg)$           | 5   |
| 59   | $(A_{2,105}; G_{2,5})$      | $(A_3A_1; \neg)$           | –   |
| 60   | $(C_{2,7}; G_{2,4})$        | $(A_3; A_2A_1)$            | –   |
| 61   | $(C_{2,42}; G_{2,8})$       | $(G_2^3C_2; \neg)$        | 7   |
| 62   | $(A_{1,4}A_{1,4}; A_{1,2}A_{1,10})$ | $(G_2^3; \neg)$       | –   |

$n = \text{label}; \ T^+ = \text{positive type}; \ T^0 = \text{null augment (level 0 subscripts omitted)}; \text{ and } \ell = \text{first number (if any) which violates } \rho \text{ parity test.}$
Table 2. The Parity Test Survivors

| n   | P   | Dim | Method         |
|-----|-----|-----|----------------|
| 1   | 0   | 3   | Lattice (3)    |
| 2   | 1   | 2   | Lattice (3)    |
| 3   | 0   | 9   | Lattice (3)    |
| 9   | 0   | -   | Projection (5.2) |
| 11  | 1   | 1   | Explicit (5.1) |
| 12  | 0   | 0   | Explicit (5.1) |
| 13  | 1   | 1   | Explicit (5.1) |
| 39  | 1   | -   | Cardinality (4.2) |
| 44  | 0   | 4   | Lattice (3)    |
| 45  | 1   | 1   | Explicit (5.1) |
| 51  | 0   | 0   | Lattice (3)    |
| 57  | 1   | -   | Cardinality (4.2) |
| 59  | 0   | -   | Cardinality (4.1) |
| 60  | 1   | -   | Cardinality (4.2) |
| 62  | 0   | 0   | Explicit (5.1) |

\( n = \) label; \( P = \) number of physical invariants; \( \text{Dim} = \) dimension of commutant (if known); and Method is technique used (relevant Section in brackets).
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