I. INTRODUCTION

Much of our insight about ultrarelativistic heavy-ion reactions comes from hydrodynamic modeling [2–4]. An inevitable ingredient in these calculations, if comparison to experiments is sought, is the conversion of the fluid to particles, called “particlization” [5]. As earlier works, e.g., [1, 6] have emphasized, such a conversion is ambiguous for fluids with nonzero shear and/or bulk viscosity because infinitely many choices for the particle phase space densities can match the given hydrodynamic fields. Analogous ambiguity is present even for ideal fluids [7], if one relaxes the implicit assumption of canonical statistics.

Phase space densities in particlization are often postulated using a convenient ansatz, for example, Grad’s quadratic corrections [8, 9]. This, however, ignores the microscopic dynamics that governs how the fluid departs from local equilibrium. In contrast, covariant kinetic theory provides a self-consistent theoretical framework that relates dissipative corrections ($\delta f$) to scattering rates, as long as the fluid can be modeled as a gas mixture near freezeout. The approach has been demonstrated in [1] for shear viscous corrections, where it was found that self-consistent corrections have a weaker momentum dependence than the typically assumed quadratic terms. This difference affects identified particle harmonic flow ($v_n$) and the extraction of shear viscosity as well [10] from data. The self-consistent results for shear were also validated [11] in a comparison of several shear $\delta f$ models against actual near-local-equilibrium evolution from covariant transport.

In this work, self-consistent bulk viscous phase space corrections are calculated from covariant transport for a one-component system of particles interacting with isotropic $2 \to 2$ interactions. Accurate results for the bulk viscosity of such a system are also obtained.

The self-consistent corrections are compared to the Grad ansatz (used, e.g., in the Hirano-Monnai bulk corrections [4]), and the bulk $\delta f$ obtained from the relaxation time approach [12, 13] (Dusling-Schäfer corrections). These are both formulated in terms of additive contributions to the local equilibrium distribution, just like the self-consistent results. The comparison is not exhaustive - particle distributions matching a set of hydrodynamic fields can also be postulated via rescaling temperature, chemical potential, and momenta in a thermal distribution (Pratt-Torrieri corrections [14], or Tinti’s approach [15]). Nevertheless, valuable insights are gained into the systematic errors made when bulk viscous fluids are particlized based on simplified models.

II. BULK VISCOS PHASE SPACE CORRECTIONS

The classification of dissipative corrections to the energy-momentum tensor and conserved currents is well known in the literature. Here we follow Sec. II of Ref. [1], and focus on single-component systems.

A. Bulk pressure and $\delta f$

In the presence of bulk viscous corrections, the local energy-momentum tensor of the fluid gets modified compared to its local equilibrium form by the presence of bulk pressure:

$$T^{\mu\nu}_{\text{eq}} = (e + p)u^\mu u^\nu - pg^{\mu\nu} \quad \rightarrow \quad T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \Pi(u^\mu u^\nu - g^{\mu\nu}) \quad \ldots \quad (1)$$
Here, \( u^\mu(x) \) is the local flow velocity, \( e(x) \) is the local energy density, \( p(x) \) is the equilibrium pressure that is determined by the equation of state \( p(e) \) of the fluid, and \( \Pi(x) \) is the bulk pressure that reflects spatially isotropic local deviations from the equilibrium pressure in the local rest frame of the fluid (LR frame\(^1\)).

By definition, the bulk correction does not contribute to the local energy density, i.e.,

\[
\delta e = u_\mu \delta T^{\mu\nu} u_\nu = 0 .
\] (2)

Furthermore, if there are conserved charges, then the bulk correction also leaves the corresponding charge densities unchanged. I.e., in the LR frame,

\[
\delta n_c = u_\mu \delta N_c^\mu = 0 \quad (\forall \text{ conserved charge } c) ,
\] (3)

where \( N_c^\mu \) is the charge current corresponding to conserved charge \( c \). The equation of state in this case is typically a function of the LR-frame charge densities as well, i.e., \( p(e, \{n_c\}) \).

For small departures from equilibrium, in the absence of transients (cf. Sec. II D), \( \Pi \) is given by the divergence\(^2\) of the flow velocity via the constitutive relation

\[
\Pi = -\zeta(\nabla u) ,
\] (4)

where \( \zeta \) is the bulk viscosity, while \( \nabla^\mu \equiv \partial^\mu - u^\mu(u\partial) \) denotes the component of the gradient orthogonal\(^3\) to flow.

However, this relation is insufficient for our purposes because it does not give the phase space density of the particles that make up the fluid.

In the presence of bulk pressure, particle phase space densities get modified\(^4\) in accordance with \( \Pi \):

\[
f^{eq}(x, p) \equiv \frac{g}{(2\pi)^3} e^{[u(x) - p_\nu u^\nu(x)]/T(x)} \quad \rightarrow \quad f(x, p) = f^{eq}(x, p) + \delta f(x, p) .
\] (5)

The general challenge for particlization is that infinitely many choices for \( \delta f \) can reproduce a given bulk correction to the energy-momentum tensor. That is because knowledge of the bulk pressure only constrains an integral of \( \delta f \). In general,

\[
T^{\mu\nu}(x) = \int \frac{d^3p}{E} p^\mu p^\nu f(x, p) ,
\] (6)

where \( E \equiv \sqrt{p^2 + m^2} \), so

\[
\delta T^{\mu\nu} = \Pi(u^\mu u^\nu - g^{\mu\nu}) = \int \frac{d^3p}{E} p^\mu p^\nu \delta f ,
\] (7)

and projecting out the spatial diagonal elements in the LR frame yields

\[
\Pi = -\frac{1}{3}(T_{LR})^i_i = \frac{1}{3} \int \frac{d^3p}{E} p_{LR}^2 \delta f .
\] (8)

Using the constraint \( [2] \), one can also write the above as

\[
\Pi = -\frac{1}{3} \delta T^\mu_\mu = -\frac{m^2}{3} \int \frac{d^3p}{E} \delta f .
\] (9)

It is customary in practice to ignore the ambiguity and take an ansatz for \( \delta f \). In contrast, the self-consistent approach in Sec. III D gives \( \delta f \) as a solution to an integral equation.

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1 In the LR frame, \( u^\mu_{LR} = (1, 0) \).

2 Minkowski scalar product of four-vectors \( a \) and \( b \) will be denoted by the shorthand \( (ab) \equiv a_\mu b^\mu \).

3 Due to the constraint \( u^2 = 1 \), one could equivalently write \( \Pi = -\zeta(u\partial) \).

4 Here, Boltzmann statistics is considered but extension to the Bose or Fermi statistics is straightforward.
B. Grad ansatz for bulk corrections

A convenient ansatz [9] comes from Grad’s 14-moment approximation (Ch. VII.2 of [16]). For bulk corrections, the approximation is

\[
\delta f_{\text{Grad}} = \left( A + B \frac{E_{LR}}{T} + C \frac{E_{LR}^2}{T^2} \right) f_{\text{eq}},
\]

where \( E_{LR} \equiv (pu) \) is the energy of the particle in the LR frame. The prefactor in (10) has a quadratic momentum dependence at high momenta. Here \( A, B, \) and \( C \) are dimensionless constants that only depend on the particle mass to temperature ratio \( z \equiv m/T, \) and are constrained by the conditions that the bulk \( \delta f \) contributes neither to the local energy density, nor to the local particle density:

\[
\delta e = \int d^3p \frac{E}{(pu)^2} \delta f_{\text{Grad}} = 0 \text{ ,}
\]

(11)

\[
\delta n = \int d^3p \frac{E}{(pu)} \delta f_{\text{Grad}} = 0 \text{ (12)}
\]

(the latter appears because particle density is conserved in \( 2 \rightarrow 2 \) scattering). This means that only one of the coefficients is independent, and thus the full \( \delta f_{\text{Grad}} \) is fixed by the bulk pressure \( \Pi. \) The thermal integrals that appear in matching \( A, B, C, \) to \( \Pi \) are collected in Appendix A.

The Grad ansatz is also commonly employed to provide analytic estimates of the bulk viscosity (cf. Sec. III B).

C. Relaxation time approximation

Bulk viscous corrections may also be obtained in the relaxation time approximation [13, 17, 18]. As shown in [12, 13], replacing the Boltzmann transport equation with the simplified linear equation

\[
p^\mu \partial_\mu f(x, p) = (pu) \frac{f_{\text{eq}}(x, p) - f(x, p)}{\tau},
\]

(13)

which is no longer an integral equation, one obtains the bulk correction

\[
\delta f_{\text{DS}} = \text{const} \times \left( \frac{p_{LR}^2}{3E_{LR}T} - c_s^2 \frac{E_{LR}}{T} \right) f_{\text{eq}}.
\]

(14)

Here, \( \tau \) is a constant of time dimension that controls the rate of scattering, and thus the dissipative corrections, while

\[
c_s^2 = \frac{\partial p}{\partial e} = \frac{K_3(z)}{zK_2(z) + 3K_3(z)}
\]

(15)

is the speed of sound that is given explicitly in the last step for a gas in thermal and chemical equilibrium (in terms of modified Bessel functions, \( K_n \)).

The correction (14) will be referred to here as the Dusling-Schäfer (DS) form for brevity. By construction, it satisfies the constraint (2), just like Grad’s ansatz, but its momentum dependence is quite different. At low momenta \( \delta_{DS}/f_{\text{eq}} \) is quadratic, while at asymptotically high momenta it depends linearly on momentum.

D. Self-consistent corrections from covariant transport theory

Self-consistent viscous corrections can be obtained from covariant transport theory. The approach has been discussed in depth in [1] for shear viscous corrections from \( 2 \rightarrow 2 \) scattering. Here we apply it to calculate bulk viscous corrections for a single-component system.

1. Covariant transport equation

The starting point is the fully nonlinear Boltzmann transport equation equation

\[
p^\mu \partial_\mu f(x, p) = S(x, p) + C[f](x, p),
\]

(16)
where the source term $S$ encodes the initial conditions, and the two-body collision term is

$$C[f](x, p_1) = \underbrace{(f_s f_4 - f_1 f_2)}_{234} W_{12\rightarrow34} \delta^4(12 - 34)$$ (17)

with shorthands $f \equiv \int \frac{d^3 p_a}{(2\pi)^3}$, $f_a \equiv f(x, p_a)$, and $\delta^4(ab - cd) \equiv \delta^4(p_a + p_b - p_c - p_d)$. The transition probability $W_{12\rightarrow34}$ for the $2 \rightarrow 2$ process with momenta $p_1 + p_2 \rightarrow p_3 + p_4$ is invariant under interchange of incoming or outgoing particles,

$$W_{12\rightarrow34} \equiv W_{21\rightarrow34} \equiv W_{12\rightarrow43} \equiv W_{21\rightarrow43} \ ,$$ (18)

satisfies detailed balance

$$W_{34\rightarrow12} \equiv W_{12\rightarrow34} \ ,$$ (19)

and is given by the corresponding unpolarized scattering matrix element or differential cross section as

$$W_{12\rightarrow34} = \frac{1}{16\pi^2 \left| W_{12\rightarrow34} \right|^2} \equiv \frac{4}{\pi} \frac{W_{12\rightarrow34}}{M_{12\rightarrow34}} \equiv \frac{4}{d\sigma_{12\rightarrow34}} \ ,$$ (20)

Here $s \equiv (p_1 + p_2)^2$ and $t \equiv (p_1 - p_3)^2$ are standard Mandelstam variables, while

$$p_{cm} = \sqrt{\frac{(p_1 p_2)^2 - m^4}{s}} = \frac{\sqrt{(p_3 p_4)^2 - m^4}}{\sqrt{s}}$$ (21)

is the magnitude of incoming (and, in our case, also outgoing) particle momenta in the center of mass frame of the microscopic two-body collision.

2. Self-consistent bulk viscous corrections

For small departure from local equilibrium one can split the phase space density into a local equilibrium part and a dissipative correction, and linearize (16) in $\delta f$:

$$p^\mu \partial_\mu f^{eq} + p^\mu \partial_\mu \delta f = \delta C[f^{eq}, \delta f]$$ (22)

with

$$\delta C[f^{eq}, \delta f](x, p_1) = \underbrace{\int \int (f_s^{eq} \delta f_4 + f_4^{eq} \delta f_3 - f_1^{eq} \delta f_2 - f_2^{eq} \delta f_1)}_{234} W_{12\rightarrow34} \delta^4(12 - 34)$$ (23)

(the source term was dropped and space-time and momentum arguments are suppressed). Typical systems quickly relax on microscopic scattering timescales to an asymptotic solution dictated by gradients of the equilibrium distribution on the left hand side of (22), which is uniquely determined by the interactions in the system. In this so-called Navier-Stokes regime the time derivative of $\delta f$ and also the spatial derivatives of $\delta f$ can both be neglected, resulting in a linear integral equation to solve at each space-time point $x$. The same considerations appear in the standard calculation of transport coefficients in kinetic theory (see, e.g., Ch. VI of [16], or [19]).

Bulk viscous correction are the response of the system to a nonuniform flow velocity field with nonzero divergence but vanishing shear, while temperature and chemical potentials are kept constant:

$$(\partial u) \neq 0 \ , \ \ \ \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} (\partial u) = 0 \ , \ \ T = const \ , \ \ \mu_c = const \ ,$$ (24)

where the tensor $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ projects out vector components orthogonal to the flow velocity. Under these conditions, the derivative on the LHS of (22) is

$$(p\partial) f^{eq} = \underbrace{-\frac{f^{eq}}{3T} p_\alpha p_\beta \Delta^{\alpha\beta} (\partial u)}_{(25)} .$$

Upon decomposition into irreducible tensors in momentum space according to $SO(3)$ representations in the LR frame (see App. A of [1] and Refs. [16, 19]), the bulk viscous driving term (25) that is on the LHS of (22) corresponds to the
scalar \((\ell = 0)\) representation, so the RHS of (22) must be in the same \(\ell = 0\) representation. This means that \(\delta f\) itself must correspond to \(\ell = 0\) because these representations are invariant subspaces of the linearized collision operator. Thus, bulk viscous corrections are constrained to the form

\[
\delta f(x, p) = \chi(|\tilde{p}|) \frac{\partial u}{T} f_{\text{eq}}(x, p) \quad \text{with} \quad \frac{1}{T} \Delta^{\mu\nu} p_{\nu} \bigg|_{LR} \equiv (0, \tilde{p}) ,
\]

where \(\tilde{p}\) is the LR frame three-momentum normalized by temperature, while \(\chi\) is a real, dimensionless, scalar function of the rescaled momentum. Substituting (26) into (22) yields, with the help of

\[
f_{\text{eq}} \propto \chi(|\tilde{p}|) \frac{\partial u}{T} f_{\text{eq}}(x, p)\]

the integral equation

\[
\frac{1}{3} \tilde{p}^2 f_1^\text{eq} = \frac{1}{T^2} \int_3 \int_4 f_1^\text{eq} f_2^\text{eq} \bar{W}_{12\rightarrow 34} \delta^4(12 - 34) (\chi_3 + \chi_4 - \chi_1 - \chi_2) ,
\]

with the shorthand

\[
\chi_n \equiv \chi(|\tilde{p}_n|) .
\]

It is straightforward to show with the help of (18), (19) and (27) that (28) is equivalent to the extremization of the functional

\[
Q[\chi] = -\frac{1}{3T^2} \int \int_1^2 f_1^\text{eq} \chi_1 \tilde{p}_1^2 + \frac{1}{2T^4} \int \int_3 \int_4 f_1^\text{eq} f_2^\text{eq} \bar{W}_{12\rightarrow 34} \delta^4(12 - 34) (\chi_3 + \chi_4 - \chi_1 - \chi_2) \chi_1
\]

\[
\equiv B + (Q_{31} + Q_{41} - Q_{11} - Q_{21}) ,
\]

i.e., (28) is reproduced by the usual variational procedure imposing \(\delta Q[\chi] = 0 + O(\delta^2)\). This allows one to estimate \(\chi\) variationally using a finite basis \(\{\Psi_n\}\) as

\[
\chi(|\tilde{p}|) = \sum_n c_n \Psi_n(|\tilde{p}|) \quad (31)
\]

and finding optimal coefficients \(\{c_n\}\) that maximize \(Q\). If the basis is complete, the limit \(n \to \infty\) reproduces the exact solution. Numerical evaluation of \(Q\) is discussed in Appendix B.

Unlike shear corrections, the general expansion (31) for bulk corrections does not automatically satisfy the matching conditions (2) and (3). This is because the densities involved are also scalars \((\ell = 0)\) under rotations of momenta in the LR frame, just like the bulk correction\(^5\). Therefore, \(Q[\chi]\) must be extremized under the constraints (2) and (3), i.e.,

\[
\int \frac{d^3 p}{E} (pu)^2 \chi f_{\text{eq}} = 0 ,
\]

\[
\int \frac{d^3 p}{E} (pu) \chi f_{\text{eq}} = 0 .
\]

These can be straightforwardly implemented using Lagrange multipliers, i.e., via extremizing the extended functional

\[
Q'[\chi] = Q[\chi] + \alpha \int \frac{d^3 p}{E} (pu)^2 \chi f_{\text{eq}} + \beta \int \frac{d^3 p}{E} (pu) \chi f_{\text{eq}}
\]

with respect to both \(\chi\), and the parameters \(\alpha\) and \(\beta\). Although the constraints affect the solution for \(\chi\), both \(Q'\) and \(Q\) evaluate to the same value at the solution.

\(^5\) Whereas shear viscous corrections correspond to angular momentum \(\ell = 2\), and thus contribute to neither the energy density nor any charge densities.
3. Relation to bulk viscosity

The form of the self-consistent bulk correction implies the constitutive relation, therefore, one can calculate the bulk viscosity from the solution $\chi$. With the help of (8) and (30) one has

$$\zeta = -\frac{\Pi}{(\partial u)} = -2T \int \frac{p_2^2}{3} \chi f_{1}^{eq} = 2T^{3} B[\chi],$$

(34)

where the last equality applies when $Q[\chi]$ is maximized. The maximal value of the quadratic functional $Q$, on the other hand, is one-half of its linear term, so $Q_{\text{max}} = B/2$. Thus, the bulk viscosity is given by the maximum of the functional $Q[\chi]$ as

$$\zeta = 4Q_{\text{max}}T^{3}.$$  

(35)

Any ansatz for $\chi$ can be used to estimate the bulk viscosity from below as $\zeta > 4T^{3}Q[\chi]$, as long as $\chi$ satisfies the constraints (32). Bulk viscosity results from the Grad ansatz are discussed in Sec. III.B.

III. RESULTS

Self-consistent bulk corrections and bulk viscosities have been computed, numerically, for a wide range of particle masses and temperatures $10^{-2} \leq m/T \leq 70$. A variety of different basis sets have been explored, such as

$$\phi_{n}^{(1)}(x) = x^n, \quad n = 0, 1, 2, ...,$$

(36)

$$\phi_{n}^{(2)}(x) = x^{n/2}, \quad n = 0, 1, 2, ...,$$

(37)

$$\phi_{n}^{(3)}(x) = x^{n/4}, \quad n = 0, 1, 2, ...,$$

(38)

$$\phi_{n}^{(4)}(x) = (x^2 + z^2)^{n/2}, \quad n = 0, 1, 2, ... .$$

(39)

Sets 1-3 correspond to integer and fractional powers of the rescaled LR-frame momentum, while basis set 4 to integer powers of the rescaled LR-frame energy. Fastest convergence was observed for basis set 2 but all basis choices converged eventually to the same answer.

The Grad ansatz corresponds to set 4 with three basis functions, $n = 0, 1, 2$.

A. Bulk viscous corrections for one-component system

Figure II shows results for the bulk viscous correction $\chi$ as a function of rescaled LR-frame momentum, for two different mass to temperature ratios $m/T = 1$ (left panel) and 7 (right panel). Given typical freeze out temperatures of $T \sim 120 - 140$ MeV in heavy-ion collisions, the former is relevant for pions at freezeout, while the latter for protons. Self-consistent corrections are proportional to the dimensionless mean free path

$$T\lambda_{MF} = \frac{T}{n\sigma},$$

(40)

so those are shown with dimensionless mean free path scaled out. The same factor has been scaled out from the Grad results as well because those correspond to a self-consistent calculation performed in a limited variational basis.

Unlike self-consistent shear viscous corrections, which are monotonic in momentum and exhibit approximate power-law dependence with an exponent close to $3/2$, self-consistent bulk corrections have a more complicated structure. This is not surprising because certain positively-weighted integrals of the bulk $\chi$ vanish, therefore, $\chi$ must switch sign at least once. In fact, the self-consistent result (solid green) switches sign twice - it is positive at low momenta and later at high momenta as well but goes negative in between. The Grad ansatz (dashed red), which is quite similar, though not identical, exhibits the same behavior. For comparison, the Dusling-Schäfer form (dotted blue), which comes from the relaxation time approximation, switches sign only once.

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6 The bulk correction $\delta f$, and thus $\chi$ as well, encodes per-particle viscous effects. The bulk pressure is the sum of such contributions from all particles, therefore, density dependence drops out from the bulk viscosity - only $T/\sigma$ remains (cf. 15)).
FIG. 1: Self-consistent bulk viscous corrections (solid green lines) vs \( p/T \) contrasted with the Grad approximation (dashed red lines), for \( m/T = 1 \) (left panel) and 7 (right panel) calculated with constant, isotropic \( 2 \rightarrow 2 \) cross section. Corrections are normalized by the dimensionless mean free path \( T\lambda \equiv T/n\sigma \). The value \( Q \) in the legends indicates the corresponding variational result for the bulk viscosity, \( \zeta = 10^{-3} \times QT/\sigma \). For comparison, the shape of the Dusling-Schäfer bulk viscous correction is also shown (dotted blue lines).

Due to its restricted variational basis, the Grad result noticeably overpredicts the correction at high momenta, especially for the lower \( m/T = 1 \) (pions). This is also reflected in the lower maximum value achieved for \( Q\chi \) (shown in the legends multiplied by a factor \( 10^3 \times 2\sigma T^2 \)), which translates into an underestimated bulk viscosity from the Grad ansatz.

B. Mass dependence of bulk viscosity

For a system of point-like particles with \( 2 \rightarrow 2 \) interactions, bulk viscosity vanishes on general grounds in both the massless \( m \rightarrow 0 \) (UR) and nonrelativistic \( m \rightarrow \infty \) (NR) limits. In the massless limit, this is because the energy-momentum tensor is traceless. So, in local equilibrium

\[
(T_{eq})^\mu_\mu = e - 3p = 0 ,
\]

i.e., the equilibrium pressure is \( p = e/3 \), while in the presence of any bulk correction \(^{11}\),

\[
T^\mu_\mu = e - 3(p + \Pi) = 0 ,
\]

which is inconsistent with \(^{11}\) unless \( \Pi = 0 \).

In the nonrelativistic limit, the reason why bulk viscosity vanishes lies in the matching conditions \(^2\) and \(^3\), namely, that the bulk correction leaves the local comoving energy density and particle density unchanged. This implies that the LR-frame kinetic energy density is also unchanged

\[
\delta(e - mn) = \delta e - m\delta n = \int \frac{d^3p}{E} (E_{LR}^2 - mE_{LR})\delta f = 0 .
\]

For large \( z \), the factor in the integrand can be written as

\[
E_{LR}^2 - mE_{LR} = p_{LR}^2 + m^2 - m^2 \sqrt{1 + \frac{p_{LR}^2}{m^2}} = \frac{p_{LR}^2}{m^2} \left[ 1 + O \left( \frac{p_{LR}^2}{m^2} \right) \right] ,
\]

so \(^3\) becomes

\[
0 = \int \frac{d^3p}{E} \left[ \frac{p_{LR}^2}{2} \left[ 1 + O \left( \frac{p_{LR}^2}{m^2} \right) \right] \delta f .
\]
Comparison to \cite{8} tells that the first term is $3\Pi/2$, and in the NR limit the correction term (coming from the relativistic $O(p^4/m^3)$ correction to the energy) is dropped straight away, so $\Pi = 0$. If in doubt, the correction term can be estimated based on the observation that in the NR limit

$$f_{\text{NR}}^{\text{eq}}(p) = \frac{n}{(2\pi m T)^{3/2}} e^{-p^2/2mT} \tag{46}$$

is Gaussian in momentum, and the thermal average of each power of $p^2$ gives a factor proportional to $mT$. With the reasonable assumption that at high momenta $\chi$ is dominated by a power-law, the additional factor of $p^2_{LR}/m^2$ in the integrand contributes a factor on the order of $(mT)/m^2$ to the integral. Thus, \cite{15} gives

$$0 = \frac{3\Pi}{2} \left[ 1 + O\left(\frac{T}{m}\right) \right], \tag{47}$$

which in the $m \to \infty$ limit indeed leads to $\Pi \to 0$ and, therefore, $\zeta \to 0$.

The analytic result in the Grad approximation \cite{16}, App. XI matches these expectations:

$$\zeta_{\text{Grad}} = \frac{z^2 K_2(z)((5 - 3\gamma)\hat{h} - 3\gamma)^2 T}{16[2K_2(2z) + zK_3(2z)]} \frac{T}{\sigma}, \tag{48}$$

where

$$\hat{h} \equiv \frac{h}{n} = \frac{zK_3(z)}{K_2(z)} \tag{49}$$

is the enthalpy per particle, while

$$\gamma \equiv \frac{c_p}{c_v} = 1 + \frac{1}{z^2 + 5h - h^2 - 1} \tag{50}$$

is the ratio of constant pressure and constant volume heat capacities. Near the UR and NR limits,

$$\zeta_{\text{Grad}}(z \ll 1) \approx \frac{z^4 T}{72 \sigma}, \quad \zeta_{\text{Grad}}(z \gg 1) \approx \frac{25\sqrt{\pi} T}{64z^{3/2} \sigma}, \tag{51}$$

which both vanish in their respective limits.

Figure 2 shows the numerically calculated bulk viscosity obtained with the self-consistent bulk viscous corrections. The left panel shows $\zeta$ as a function of the mass to temperature ratio $m/T$, normalized to the analytic Grad result \cite{15}. As expected, the Grad approximation generally underpredicts the bulk viscosity. Near the NR limit, i.e., for large masses, the error is modest, less than about 10% for $z > 10$. On the other hand, in the opposite (small mass) limit, the Grad ansatz underpredicts the bulk viscosity by a progressively larger factor that exceeds 2 for $z < 0.1$. To illustrate the accuracy of the calculations, numerical results are also shown from the limited variational basis used in the Grad approximation (red dashed line), with excellent agreement with the analytic result in the full mass range $0.01 \leq m/T \leq 70$ studied here.

The right panel of Fig. 2 compares the shear and bulk viscosities for a one-component system with energy-independent, isotropic $2 \to 2$ interactions, to the observation $\zeta \propto (1 - 3c_s^2)^2 \eta$ made by Weinberg\cite{20} near the conformal limit. The same relationship was also studied in \cite{13}. As can be seen (solid green line), the proportionality indeed holds within good accuracy for $m/T \lesssim 0.03$, in fact,

$$\zeta \approx 3.75(1 - 3c_s^2)^2 \eta. \tag{52}$$

The numerical coefficient 3.75 changes to about 1.67 if one compares the approximate shear and bulk viscosities from the Grad approach (dashed red line), in good agreement with the analytic expression

$$\zeta_{\text{Grad}} \approx \frac{5}{3}(1 - 3c_s^2)^2 \eta_{\text{Grad}}. \tag{53}$$

\footnote{The analytic result is a combination of \cite{51}, $\eta_{\text{Grad}} = 6T/5\sigma$ for $z = 0$, and $c_s^2 \approx 1/3 - z^2/36$ from \cite{15}.}
C. Effect on differential elliptic flow

Finally, we give an estimate for how much the choice of bulk viscous correction model affects differential elliptic flow in ultrarelativistic nucleus-nucleus collisions. To that end, we employ a bulk viscous generalization of the simple four-source model in Ref. [21], which models a snapshot of the system via four fireballs moving back-to-back along the \(x\) and \(y\) directions with velocities \(\pm v_x\) and \(\pm v_y\), respectively, in the transverse plane of the collision. The values and temperatures of all four sources are set to be the same but with \(v_x > v_y > 0\), which leads to a positive elliptic flow as a function of transverse momentum (except for heavy particles at low momenta). Bulk viscous corrections are introduced here analogously to the shear viscous generalization of the model in Ref. [7] via assuming that in the rest frame of each fireball viscous parameters are the same. Bulk pressure is a Lorentz scalar, so we set the bulk pressure over energy density ratio to the same value \(\Pi/e = -0.001\) in all four fireballs (in heavy-ion collisions, \((\partial u)/\epsilon\) is typically positive, so \(\Pi\) is typically negative).

The combined momentum distribution of particles from all four fireballs, is then

\[
f^{(4s)} = f_{(v_x,0)} + f_{(-v_x,0)} + f_{(0,v_y)} + f_{(0,-v_y)},
\]

where subscripts denote the velocity of the sources in the transverse plane. For a single source, the momentum distribution of particles at midrapidity, i.e., at momenta

\[
p = (p_x, p_y, 0) \equiv p_T(\cos \varphi, \sin \varphi, 0),
\]

is

\[
f_T(p_T, \varphi) = \text{const} \times \left[1 + a \frac{\Pi}{e} \chi \left(\frac{p_{LR}}{T}\right)\right] e^{-(pu)/T},
\]

where

\[
(pu) = \frac{1}{\sqrt{1-v^2}} \left[\sqrt{p_T^2 + m^2} - p_T(v_x \cos \varphi + v_y \sin \varphi)\right],
\]

\[
p_{LR} = \sqrt{(pu)^2 - m^2},
\]

and the scaling factor \(a\) is set such that the bulk pressure of the fireball reproduces the \(\Pi/e\) ratio with the given bulk viscous correction model (\(a = 0\) gives back the thermal sources in the original four-source model). Differential elliptic...
flow can now be calculated, numerically, directly from its definition as
\[
v_2(p_T) \equiv \frac{\int_0^{2\pi} d\varphi \cos 2\varphi f^{(4s)}(p_T, \varphi)}{\int_0^{2\pi} d\varphi f^{(4s)}(p_T, \varphi)}.
\] (59)

Figure 3 shows the results for differential elliptic flow for parameters \(T = 0.14 \text{ GeV}, v_x = 0.6, v_y = 0.5\) that approximate mid-central Au+Au collisions at top RHIC energy \(\sqrt{s_{NN}} = 200 \text{ A GeV}\) for \(m/T = 1\) (left panel) and 7 (right panel), appropriate for pions and protons at freezeout, respectively. In both cases, rather small differences are seen between the full self-consistent results (solid green lines) and the Grad ansatz (red dashed line); the largest difference is an about 10% smaller \(v_2\) from the Grad approach at the highest \(p_T = 2 \text{ GeV}\) plotted. In contrast, bulk corrections from the relaxation time approximation (dotted blue line) give significantly larger \(v_2\) than the other two \(\delta f\) models at high \(p_T > 1.5 \text{ GeV}\). In fact, this latter model gives almost identical \(v_2\) to that from pure thermal sources (not shown). This means that the bulk viscous correction to pion \(v_2\) is about 50% percent larger (more negative) with the Grad ansatz, in this calculation, than the bulk correction from the self-consistent approach.

A reliable assessment of bulk viscous effects on heavy-ion observables will, of course, have to take into account interactions in the multicomponent hadron gas (i.e., mixtures). As has been shown for the case of shear \([1]\), this leads to intricate inter-species dependences. For example, even when one restricts the calculation to the Grad ansatz for each species, viscous corrections acquire species-dependent scaling factors that cannot be obtained from calculations for single-component systems.

IV. CONCLUSIONS

In this work self-consistent bulk viscous phase space corrections (\(\delta f\)) are calculated from covariant kinetic theory, for one-component systems with isotropic, energy-independent \(2 \to 2\) interactions. Compared to self-consistent shear viscous corrections\([1]\), which show an approximate power-law momentum dependence, bulk corrections exhibit more complicated behavior and change sign twice as a function of momentum.

The corrections are contrasted with the Grad ansatz, which postulates a quadratic polynomial of the particle energy in the frame comoving with the fluid, and also with bulk viscous corrections based on the relaxation time approximation \([13]\) (DS model). While the Grad bulk \(\delta f\) is quite similar to the self-consistent results, except at high momenta, the relaxation time approximation leads to bulk viscous corrections with markedly different functional shapes.

The above findings are reflected in estimates for bulk viscous corrections to differential elliptic flow in \(A + A\) reactions at RHIC energies, obtained from a simple semi-analytic four-source model motivated by Refs. \([21]\) and \([7]\). The self-consistent and Grad bulk \(\delta f\) models give nearly identical proton \(v_2(p_T)\) in the \(0 < p_T < 2 \text{ GeV}\) window studied, while
for pions at high $1.5 < p_T < 2$ GeV the Grad approach overestimates bulk viscous effects by up to 50%. Under the same conditions, the bulk $\delta f$ from the DS approach generates negligible bulk viscous corrections to $v_2$.

The bulk viscosity of the system is also calculated and compared to known analytic formulas (16, App. XI) derived in the Grad approximation. For large masses $m > 10T$, the Grad result is accurate to within 10%, however, for small $m < 0.1T$ it underestimates the bulk viscosity by more than a factor of 2. We also find that near the massless limit the Weinberg relation[20] between bulk and shear viscosities $\zeta = \text{const} \times (1 - c_s^2)\eta$ does hold with a coefficient of about 3.75, which is roughly a factor of two larger than the coefficient of 5/3 in the Grad approximation.

While the above results are intriguing, they are limited to single-component systems with $2 \rightarrow 2$ interactions. It would be interesting to follow up this investigation, in the future, with a calculation of self-consistent bulk viscous corrections for hadronic mixtures.

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Appendix A: Grad correction matching

This section lists the thermal integrals that appear while matching the Grad correction (10) to a given bulk pressure. For a local local equilibrium distribution

$$f_{eq} = \frac{g}{(2\pi)^3} e^{[\mu-(pu)/T]} , \quad (A1)$$

the constraints (11) and (12) can be written as

$$j_0 A + j_1 B + j_2 C = 0 , \quad (A2)$$
$$j_1 A + j_2 B + j_3 C = 0 , \quad (A3)$$

where

$$j_n = \int_0^\infty dx x^2 y^n e^{-y} \int \frac{d^3 p}{E} (pu)^{n+1} f_{eq} \quad (A4)$$

with $y \equiv \sqrt{x^2 + z^2}$ and $z \equiv m/T$. After switching to $y$ as the integration variable, the integrals yield modified Bessel functions of the second kind:

$$j_0 = z^2 K_2(z) , \quad (A5)$$
$$j_1 = z^3 K_1(z) + 3z^2 K_2(z) , \quad (A6)$$
$$j_2 = z^4 K_2(z) + 3z^3 K_3(z) , \quad (A7)$$
$$j_3 = 2z^4 K_2(z) + (15 + z^2)z^3 K_3(z) . \quad (A8)$$

Matching to the bulk pressure gives the third constraint. Specifically, from (10):

$$\Pi = \left(-z^2\right) \frac{3}{2} \frac{g}{2\pi^2} T^4 e^{\mu/T} \int_0^\infty dx x^2 y (A + By + Cy^2) e^{-y}$$
$$= \frac{g}{2\pi^2} T^4 e^{\mu/T} \left(-z^2\right) \frac{3}{2} \left(j_{-1} A + j_0 B + j_1 C\right) , \quad (A9)$$

where $j_{-1} = z K_1(z)$. Substitution of the local equilibrium energy density

$$e \equiv \int \frac{d^3 p}{E} (pu)^2 f_{eq} = \frac{g}{2\pi^2} T^4 e^{\mu/T} j_1$$

(A10)
leads to
\[
\frac{\Pi}{e} = -\frac{z^2}{3} \left( \frac{j_1 - 1}{j_1} A + \frac{j_0}{j_1} B + C \right) .
\]  
(A11)

The linear system (A2), (A3), and (A11) can now be solved in a straightforward manner for \( A, B, \) and \( C \) in terms of \( \Pi/e. \)

**Appendix B: Calculation of momentum integrals in \( Q[\chi] \)**

The integrals in (B3) can be evaluated using analogous steps to the shear viscous case discussed in App. B of \( \Pi \) (see Sec. III for the definitions of the usual shorthands). The only difference is that, for bulk viscous corrections, the contraction
\[
P_a \cdot P_b = P_a^{\mu \nu} P_{b, \mu \nu} = (\hat{p}_a \cdot \hat{p}_b)^2\frac{1}{3} |\hat{p}_a|^2 |\hat{p}_b|^2
\]  
(B1)
does not arise, and the source \( B \) comes from the divergence of the flow instead of its shear.

For a one-component system the flavor indices play no role and, therefore, will be dropped (we have \( ij \rightarrow k\ell \) scattering with \( i = j = k = \ell \)). Thus,
\[
B = \frac{2\pi}{3T^4} \int m dE_1 p_1^3 f_{1}^{eq}(p_1/T) .
\]  
(B2)

Next, \( Q_{11} \) and \( Q_{21} \) are given by Eqs. (B4) and (B5) in Ref. \( \Pi, \) but without the extra factor \( P_1 \cdot P_1 = 2p_1^4/3T^4 \) in \( Q_{11} \) and \( P_2 \cdot P_1 = p_1^2 p_2^2 (t_{12}^2 - 1/3)/T^4 \) in \( Q_{21}: \)
\[
Q_{11} = \frac{2\pi^2}{T^4} \int m dE_1 p_1 f_{1}^{eq}(\chi_1^2) \int dE_2 p_2 f_{2}^{eq} \int_{-1}^{1} dt_{12} F(s) \sigma_{TOT}(s) ,
\]  
(B3)
\[
Q_{21} = \frac{2\pi^2}{T^4} \int m dE_1 p_1 f_{1}^{eq}(\chi_1^2) \int dE_2 p_2 f_{2}^{eq} \int_{-1}^{1} dt_{12} F(s) \sigma_{TOT}(s) ,
\]  
(B4)
where
\[
F(s) \equiv p_{cm} \sqrt{s} = \frac{1}{2} \sqrt{s(s - 4m^2)} ,
\]  
(B5)
and \( \sigma_{TOT} \) is the total cross section. The integration variables \( E_1 \) and \( E_2 \) correspond to the LR-frame energies of the incoming particles in \( 2 \rightarrow 2 \) scattering, while \( t_{12} \) is the cosine of the angle between incoming momenta in the LR frame.

Interchange symmetry (15) with \( 3 \leftrightarrow 4 \) implies \( Q_{41} = Q_{31}, \) so the last remaining contribution to discuss is \( Q_{31}. \) For isotropic cross section, it can be reduced to four integrals using the steps in App. B.2 of Ref. \( \Pi. \) In fact, in the bulk viscous case the averaging over the c.m. frame angle \( \phi_3 \) is trivial because the integrand does not depend on \( \phi_3 \) at all. Therefore, for general energy-dependent cross sections we have
\[
\int_{\Omega_3} d\Omega_3 (...) = \frac{4 \pi \cdot 2 \pi \cdot 2 \pi}{4} \int m dE_1 p_1 \int m dE_2 p_2 \int_{-1}^{1} dt_{12} \int_{-1}^{1} dt_{3}(...) ,
\]  
(B6)
i.e.,
\[
Q_{31} = \frac{\pi^2}{T^4} \int m dE_1 p_1 f_{1}^{eq}(\chi_1^2) \int dE_2 p_2 f_{2}^{eq} \int_{-1}^{1} dt_{12} F(s) \sigma_{TOT}(s) \int_{-1}^{1} dt_{3} \chi(\frac{p_3}{T}) ,
\]  
(B7)
where
\[
p_3 = |p_3| = \sqrt{E_3^2 - m^2} = \sqrt{(\gamma_3 E_T + \beta_3 p_T t_3)^2 - m^2}
\]  
(B8)
with
\begin{align}
\beta_3 & \equiv \frac{p_{cm}}{\sqrt{s}}, \quad \gamma_3 \equiv \frac{E_{3,cm}}{\sqrt{s}} = \sqrt{\beta_3^2 + \frac{m^2}{s}}, \\
E_T & \equiv E_1 + E_2, \quad p_T \equiv |p_1 + p_2| = \sqrt{p_1^2 + p_2^2 + 2p_1p_2t_{12}}.
\end{align}

The variable $t_3$ is the cosine of the angle of deflection in the microscopic scattering (in the center-of-mass frame).

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