Cosmic Strings on the Lattice

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We develop a formalism for the quantization of topologically stable excitations in the 4-dimensional abelian lattice gauge theory. The excitations are global and local (Abrikosov-Nielsen-Olesen) strings and monopoles. The operators of creation and annihilation of string states are constructed; the string Green functions are represented as a path integral over random surfaces. Topological excitations play an important role in the early universe. In the broken symmetry phase of the $U(1)$ spin model, closed global cosmic strings arise, while in the Higgs phase of the noncompact gauge-Higgs model, local cosmic strings are present. The compact gauge-Higgs model also involves monopoles. Then the strings can break if their ends are capped by monopoles. The topology of the Euclidean string world sheets are studied by numerical simulations.

1. INTRODUCTION

During cooling, the early universe has undergone different phase transitions. Depending on the symmetry that is spontaneously broken, various excitations, for example, domain walls, cosmic strings or monopoles, arise as topologically stable objects \cite{1,2}. The $U(1)$ symmetric scalar fields in the 4-dimensional field theories give rise to string-like topological excitations. In a broken $U(1)$ symmetry phase, the strings are stable solutions of the classical equations of motion. One distinguishes two types of cosmic strings: global and local ones. Global strings arise when a global $U(1)$ symmetry breaks spontaneously, whereas local strings are due to the breakdown of the $U(1)$ gauge symmetry. A well known example is the local Abrikosov-Nielsen-Olesen string present in the Higgs phase of the noncompact abelian gauge-Higgs model with a local $U(1)$ symmetry \cite{3,4}. An even simpler example is the 4-dimensional global $U(1)$ scalar field theory with a field $\Phi = |\Phi| \exp(i\varphi)$. In two dimensions, zeros of the complex scalar field, $\Phi(x) = 0$, are located at isolated points $x$. Going around $x$, the phase $\varphi$ may change by $2\pi n$, where $n \in \mathbb{Z}$ is the topological characteristic of the vorticity of the scalar field. The relevant homotopy group is $\Pi_1[U(1)] = \mathbb{Z}$. In three dimensions, the zeros of the scalar field are located on lines (the global cosmic strings), and in four dimensions, on surfaces (the world sheets swept out by the strings during their time evolution). The strings are topologically stable excitations, i.e., they are insensitive to small deformations of the field $\Phi$.

The dynamics of cosmic strings and monopoles is usually investigated in semiclassical physics. In the early universe, however, quantum effects played a crucial role. The classical description is inadequate when complicated dynamical situations arise in which strings or monopoles are condensed. The description of phase transitions may also require the inclusion of quantum effects. Due to the topological nature of the problem, it is essential to formulate it in a nonperturbative framework, as provided by the lattice regularization. For global cosmic strings, we start with a complex scalar field $\Phi = |\Phi|^2 \exp(i\varphi)$ with a quartic potential $V(\Phi) = \lambda(|\Phi|^2 - v^2)^2$, and consider the limit as $\lambda \to \infty$. Then only the compact variable $\varphi \in [-\pi, \pi]$ remains and the theory is reduced to the global $U(1)$ lattice spin model (the
four-dimensional XY model). A cosmic string then manifests itself by the vorticity of a plaquette and the corresponding dual plaquette belongs to the Euclidean string world sheet. The local cosmic strings results from the gauging of the $U(1)$ symmetry. On the lattice, this leads to the noncompact and the compact abelian gauge-higgs models. The theory with compact gauge fields has monopoles as additional topological excitations. Since the string theory is formulated on the lattice, one has complete nonperturbative control of the dynamics. In particular, one can study the string tension or the question of string condensation in the high high temperature phase by standard lattice techniques like numerical simulations or strong coupling expansions.

In the present publication, we only give the explicit form of the creation operators of the strings, the details of the derivation will be published in the subsequent paper. The quantization of the global cosmic strings is discussed in ref. [5].

2. GLOBAL STRINGS

Let us consider the 4-dimensional $U(1)$ lattice spin model in the Villain formulation [3]. Its partition function is given by

$$Z = \sum_{l \in \mathbb{Z}(C_1)} \int_{-\pi}^{+\pi} D\varphi \exp\left( -\frac{\kappa}{2} \|d\varphi + 2\pi l\|^2 \right). \quad (1)$$

We use the notations of the calculus of differential forms on the lattice [3]. $D\varphi$ denotes the integral over all site variables, $\varphi$; $d$ is the exterior differential operator, $d\varphi$ is the link variable constructed as usual in terms of the site angles $\varphi$. The scalar product is defined in the standard way, e.g., if $\varphi$ and $\psi$ are the site variables, then $(\varphi, \psi) = \sum_s \varphi(s)\psi(s)$, where $\sum_s$ is the sum over all sites $s$. The norm is defined as: $\|a\|^2 = (a, a)$; therefore $\|d\varphi + 2\pi l\|^2$ implies summation over all links. $\sum_{l \in \mathbb{Z}(C_1)}$ denotes the sum over all configurations of the integers $l$ attached to the links $C_1$. It occurs that the partition function (1) can be represented as follows [3]:

$$Z^K = \sum_{k \in \mathbb{Z}(C_2), \Delta^*k = 0} \exp\left(-2\pi^2\kappa(*k, \Delta^{-1}\ast k)\right). \quad (2)$$

Where $*x$ denotes the object dual to $x$, the codifferential $\delta = *d$ satisfies the rule for partial integration $(\varphi, \delta\psi) = (d\varphi, \psi)$. The summation in (2) is carried over the integer variables $k$, which belong to plaquettes $C_2$. The condition $\delta k = 0$ means that the summation is performed over the closed two-dimensional objects defined by $k$. The surface elements interact with each other via long range forces described by the inverse Laplacian. Physically, these forces are due to the massless Goldstone bosons in the broken phase of the original spin model. The action of the random surface model is nonlocal and differs from the Nambu action which is proportional to the surface area. It can be expected that the random surface model is equivalent to a lattice theory of closed strings, and that the closed surfaces are actually the string world sheets. To justify this statement one must construct creation and annihilation operators of string states.

3. STRING CREATION OPERATORS

The creation of a cosmic string (as a nonlocal object) requires the use of nonlocal operators. Global (local) cosmic strings are surrounded by a cloud of Goldstone (gauge) bosons, just as charged particles are surrounded by their photon cloud. Creation operators for charged particles were first constructed by Dirac [3], whose idea was to compensate the gauge variation of a charged field, $\Phi(x') = \Phi(x)\exp(ia(x))$, by a contribution of the gauge field representing the photon cloud:

$$\Phi_e(x) = \Phi(x)\exp(i \int d^3 y B_i(x - y) A_i(y)), \quad (3)$$

where $\partial_i B_i(x) = \delta(x)$, and $A_i(x') = A_i(x) + \partial_i a(x)$ is the photon field. The gauge invariant operator $\Phi_e(x)$ creates a scalar charged particle at the point $x$, together with the photon cloud surrounding it. Our construction of string creation operators [3] is based on the same idea, and it is very similar to the construction of soliton creation operators suggested by Fröhlich and Marchetti [3]. In fact, it is a generalization of their construction of monopole sectors in 4-dimensional $U(1)$ lattice gauge theory. We perform the duality transformation of the original theory defined by
the partition function \([\hat{Z}]\), and obtain the hypergauge theory of integer valued fields on the dual lattice. The Wilson loop \(W(C)\), constructed from the auxiliary gauge fields, is gauge invariant, but not hypergauge invariant. We make \(W(C)\) hypergauge invariant by surrounding it by the cloud of the hypergauge field. This hypergauge invariant operator can be shown to create the closed string on the curve \(C\). In terms of the variables \(k \in \mathbb{Z}\), the expectation value of the string creation operator has the form:

\[
< U_C > = \frac{1}{2K} \sum_{k \in \mathbb{Z}(C_2) \atop \delta^* k = -\delta_C} \exp\{-2\pi^2 \kappa^* (k + D_C), \Delta^{-1}* (k + D_C)\};
\]

\(D_C\), being the counterpart of the function \(B_i(x)\) in eq.\(\[6\]\), depends on the curve \(C\): \(\delta^{(3)}* D_C = \delta_C\), since the operator of the creation of the string should act at a definite time slice, we use the three-dimensional operator of the codifferentiation \(\delta^{(3)}\); \(\delta_C\) is the lattice delta function which is equal to unity on the links belonging to the curve \(C\), and equal to zero on the other links. It is important that because of the condition \(\delta^* k = -\delta_C\), the summation in \(\[4\]\) is performed over closed surfaces, and those bounded by the curve \(C\). This is exactly what one would expect intuitively: a string world sheet opens up on curve along which the string is created. In the general case, the curve \(C\) may consist of several closed loops: \(\delta_C = \sum_i \delta C_i\); then \(D_C = \sum_i D C_i\). Moreover, if we set \(\delta C_i = -1\) on several closed loops, then the string is annihilated on these loops. Placing the loops \(C_i\) and \(C_j\) on different time slices, we may construct operators corresponding to string scattering and decay processes.

The expectation value of the string creation operator in terms of the original fields \(\varphi\) is:

\[
< U_C > = \frac{1}{2Z} \sum_{l \in \mathbb{Z}(C_1)} \int_{-\pi}^{\pi} D\varphi \exp(-\beta\|d\varphi + 2\pi\delta\Delta^{-1}(D_C - \rho C) + 2\pi l\|^2).
\]

Here the integer valued field \(\rho C\) satisfies the equation \(\delta^{(3)}(\rho C^* - \rho^* C) = 0\). It can be shown that \(\rho^* C\) is the analog of the (invisible) Dirac string. The Dirac string connected to the monopole is a one-dimensional object, while \(\rho C\), being defined on the plaquettes, is a two-dimensional object.

## 4. LOCAL COSMIC STRINGS

Local cosmic strings are topological excitations in the abelian gauge-Higgs model whose partition function in the Villain form is given by:

\[
\mathcal{Z} = \sum_{n \in \mathbb{Z}(C_2)} \int D\theta \int_{C(C_1)}^{\pi} D\varphi \exp(-\beta\|d\theta + 2\pi n\|^2 - 2\frac{\kappa}{2}\|d\varphi - \theta + 2\pi l\|^2).
\]

If the gauge field is noncompact (the integration over \(\theta\) is from \(-\infty\) to \(+\infty\), then the excitations are closed local strings. Below we discuss a more interesting case, when the gauge fields are compact \((-\pi < \theta \leq \pi)\) and the monopoles are present in addition to strings. It occurs that the strings may be open, with monopoles at their ends. The general Green function for this case consists of the creation and annihilation operators of strings and monopoles:

\[
\left\langle \prod_i U_{C_i} \prod_j \Phi_{x_j} \prod_k \bar{\Phi}_{x_k} \right\rangle = \frac{1}{2Z} \sum_{n \in \mathbb{Z}(C_2)} \int^{\pi} D\theta \int_{C(C_1)}^{\pi} D\varphi \exp(-\beta\|d\theta + 2\pi n + 2\pi\delta\Delta^{-1}(B - \omega)\|^2) \exp(-\frac{\kappa}{2}\|d\varphi + 2\pi l - \theta + 2\pi\delta\Delta^{-1}(D - \rho)\|)^2).
\]

It follows from the derivation of this formula (which is skipped) that \(\delta \sum_i \delta C_i + \sum_j \delta x_j - \sum_k \delta x_k = 0\); therefore open strings carry monopoles and antimonopoles at their ends. In the case of creation of monopoles at points \(x_j\) and antimonopoles at points \(x_k\), the field \(B\) satisfies the equation: \(\delta^{(3)}* B = \sum_j \delta x_j - \sum_k \delta x_k\); Dirac string \(\delta^*- \omega \in \mathbb{Z}\) is such that \(\delta^{(3)}(\delta^*- B - \omega) = 0\); \(D\) and \(\rho\) are defined by the equations: \(\delta^{(3)}* D = \delta C + \delta B; \delta^{(3)}(D - \rho) = B - \omega\). It is easy
to show that the Green function is invariant under the deformation of the Dirac string:
\[ \omega' = \omega + d\xi, \quad \rho' = \rho + \xi. \]

5. COSMIC STRING DYNAMICS AND NUMERICAL SIMULATIONS

In the high temperature phase of the early universe, the \( U(1) \) symmetry was unbroken and strings were condensed. This was possible because the string tension vanished and string creation required no energy. As a consequence, one expects that the strings formed clusters percolating through the universe. After the phase transition, the \( U(1) \) symmetry gets spontaneously broken and the string network freezes. Because the string tension no longer vanishes, the strings become massive. Small strings shrink and decay, whereas large cosmic strings survive as stable massive structures catalyzing the formation of galaxies. Such scenario can be studied within the formalism discussed above.

The string condensate can be calculated numerically by direct measurement of the expectation value \( < U_C > \). A similar measurement of the monopole condensate in the compact electrodynamics was performed in refs.\[1,0]. Another quantity that can be studied numerically is the string tension of the cosmic strings. We are performing these calculations at present.

The simplest objects related to the cosmic strings are the defects in the four-dimensional \( XY \) model. In the given configuration of the spins \( \varphi \), each plaquette carries vorticity
\[ *j = \frac{1}{2\pi}([\varphi_1 - \varphi_2]_{2\pi} + [\varphi_2 - \varphi_3]_{2\pi} + [\varphi_3 - \varphi_4]_{2\pi} + [\varphi_4 - \varphi_1]_{2\pi}), \]
where \( \varphi_k \) are the angles on the corners of the plaquette, and \( [\alpha]_{2\pi} \) is \( \alpha \mod 2\pi \). If \( *j \neq 0 \), then we may ascribe vorticity \( j \) to the plaquette dual to the original one. The surfaces formed by these dislocations are closed. The proof is very simple:
\[ *j = \frac{1}{2\pi}d([d\varphi]_{2\pi}) = \frac{1}{2\pi}d(\varphi + 2\pi p) = dp, \]
where the integer \( p \) is such that \( (d\varphi + 2\pi p) \in (-\pi, \pi] \). Now, using equalities \( d = *d^* \) and \( d^2 = 0 \), we get \( \delta = 0 \). These world sheets are characterized by the Euler number:
\[ N_E = n_p - n_l + n_s, \]
where \( n_p \), \( n_l \) and \( n_s \) are the number of the plaquettes, the links and the sites belonging to the world sheet respectively. The number of handles is defined by
\[ g = 1 - N_E/2. \]

We have studied numerically the topology of the world sheets in the four-dimensional \( XY \) model on the lattices \( 6^4 - 8^4 \) for the different values of \( \kappa \). Usually, there are several disconnected closed world sheets in each configuration of the spins. It occurs that below the phase transition \( (\kappa < \kappa_C) \), in the region of the expected condensation of the strings, there exist one world sheet of the size comparable to that of the lattice; this world sheet contains a lot of handles. There are also satellites: small disconnected objects with a simple topology (no handles). After the phase transition \( (\kappa > \kappa_C) \), almost all world sheets have no handles. It turns out that the number of handles \( g_i \) taken into account with the weight proportional to the area \( S_i \) of the corresponding world sheet \( i \) (i.e. \( < g > = \sum \frac{g_i S_i}{\sum_i S_i} \)) is the order parameter of the system. Numerical simulations show that \( < g > \neq 0 \) for \( \kappa < \kappa_C \) and \( < g > = 0 \) for \( \kappa > \kappa_C \).

Two of the authors (MIP and LP) would like to thank the HLRZ in Jülich for hospitality. The work of MIP and AVP has been partially supported by the grant of the American Physical Society.

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