FRAMED DEFORMATION OF GALOIS REPRESENTATION

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Abstract. We studied framed deformations of two dimensional Galois representation of which the residue representation restrict to decomposition groups are scalars, and established a modular lifting theorem for certain cases. We then proved a family version of the result, and used it to determine the structure of deformation rings over characteristic zero fields. As a corollary, we obtain the $L$-invariant of adjoint square representation associated to a Hilbert Hecke eigenform.

1. Introduction

Given a Hilbert modular Hecke eigenform $f$ (over some totally real field $F$), one can associated a two dimensional continuous Galois representation $\rho_f$ (into $GL_2(K)$ for some finite extension $K$ over $\mathbb{Q}_p$). When $f$ is ordinary, $\rho_f$ is nearly ordinary at $p$. Conversely, assume the residue representation of a given representation $\rho$ is modular and nearly ordinary, and assume it satisfies a technical distinguishedness condition, the representation can be proved to be modular by Taylor-Wiles [23], Fujiwara [4] and Skinner-Wiles [25]. The main idea is to study certain types of deformation problems, and identify the universal deformation ring to the localization of the Hecke algebra at the maximal ideal determined by $\rho$. This type of result is often called "$R = T$" theorem.

When the Distinguishedness condition of a nearly ordinary representation fails, the deformation functor with the prescribed local conditions is no longer representable. This phenomena also appeared in the deformation of Barsotti-Tate representations studied by Kisin [17]. To make a representable functor, instead consider the deformations with prescribed local conditions, Kisin consider the functor associated to each ring a deformation of Barsotti-Tate representation, together with a basis lifting a fixed chosen basis of the residue representation. The basis eliminate automorphisms of the functor, and thus form a representable functor. As a result, Kisin proved certain framed version of the "$R = T$" theorem, up to some finite torsion due to the Barsotti-Tate condition, which is enough to prove the modularity.

In this paper, we will consider the local condition that the restriction of $\rho$ to the decomposition groups are scalers, so the distinguishedness condition fails. Inspired by Kisin’s work, we invent a deformation ring $\mathcal{R}_p^{\Delta,\psi,s}$, which represent the functor associated each ring $A$ a deformation of the representation of a decomposition group into the Borel subgroup (upper triangular matrices) together with a basis, which is transformed to the standard basis by an element of the Borel subgroup. The reason we make such choice is the Schlessinger criterion, which ensure the functor is representable. Using these rings, we proved a framed version of the "$R = T$" theorem in section 5 and the modularity assuming the residue representation is modular in Theorem 6.1. We then generalize this result to Hida’s family of Hilbert modular forms. As an application, we prove Hida’s conjecture that $\mathcal{R}_K$ is a power series ring, where $\mathcal{R}_K$ is the universal deformation ring representing the functor of deformations into representations over Artinian $K$-algebras. This result implies Hida’s conjectural formula [16] of the $L$-invariant of the adjoint representation, in Corollary 8.4.

In this paper, we provide some well known facts on Hilbert modular forms. In section 2 we study the framed deformation (without local conditions) of Kisin [17], and calculate their tangential dimensions using Galois cohomology. The ring $\mathcal{R}_p^{\psi,s}$ and the deformations of the scaler representation of the decomposition group into Borel subgroups are studied in section 3 which is the most original part of the paper. Section 4 and 5 are the Taylor-Wiles system and the modularity. We generalize these result to Hida’s family in section 6 and obtained the $L$-invariant in section 8.

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2. Hilbert Modular Forms

Fix a totally real number field $F$ of degree $d$. $I$ is the set of embedding of $\sigma : F \hookrightarrow \mathbb{Q}$. Denote by $\mathbb{A}_F$ its ring of adeles, which decompose into finite and infinite parts as $\mathbb{A}_F = \mathbb{A}_F^f \times \mathbb{A}_F^\infty$. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $E \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_E$. Fix the embedding $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ once and for all.

As Hida in [5], we consider the following type of continuous “Neben” characters

$$\varepsilon = (\varepsilon_1, \varepsilon_2 : \hat{O}^\times \rightarrow \mathbb{C}^\times \varepsilon_+ : \mathbb{A}_F / \mathbb{F} \rightarrow \mathbb{C}^\times)$$

and the weights $k = (k_1, k_2) \in \mathbb{Z}[I]^2$ such that $k_1 + k_2 = (n + 1) \cdot I$ for some integer $n$, $\varepsilon_+ |_{\hat{O}} = \varepsilon_1 \varepsilon_2$ and $\varepsilon_+(x) = x^{-(k_1 + k_2) + I}$. Let $N$ be an integral ideal of $\mathcal{O}$ and define the $\Gamma_0$ type congruence subgroup

$$\hat{\Gamma}_0(N) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\hat{\mathcal{O}}) \mid c \in N \hat{\mathcal{O}} \}$$

and the $\Gamma_1$ type congruence subgroup

$$\hat{\Gamma}_1(N) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\hat{\mathcal{O}}) \mid a - 1, b, d - 1 \in N \hat{\mathcal{O}} \}.$$ 

Let $\varepsilon^- = \varepsilon_2^{-1} \varepsilon_1$ and assume its conductor $c(\varepsilon^-) \supset N$, then the character

$$\varepsilon : \hat{\Gamma}_0(N) \rightarrow \mathbb{C}^\times$$

defined by $\varepsilon \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = \varepsilon_2(ad - bc) \varepsilon-(a)$ is a continuous character of $\hat{\Gamma}_0(N)$.

Under the above notation, define the automorphy factor of weight $k$ as

$$J_k(g, z) = \det(g)^{k_1 - I} f(g, z)^{k_2 - k_1 + I} = \prod_{\sigma \in I} \det(g_{\sigma})^{k_1 \cdot \varepsilon(\sigma z_\sigma + d_\sigma)_{k_2 + k_2 - k_1, \sigma}}$$

for $g = (g_{\sigma}) \in GL_2(\mathbb{A}_F^\infty) = GL_2(\mathbb{R})^I$ and $z = (z_\sigma) \in \mathbb{H}^I$, where $\mathbb{H}^I$ is the $d$ upper half plane as usual. Define the Hilbert cusp form $S_k(N, \varepsilon; \mathbb{C})$ of weight $k$, level $N$ and “Neben” type $\varepsilon$ to be the functions $f$ with the following three conditions:

(A1) For all $\alpha \in GL_2(F)$, $z \in \mathbb{H}(\mathbb{A}_F)$ and $u \in \hat{\Gamma}_0(N) C_1$, where $C_1$ is the stabilizer of $i = (\sqrt{-1}, \sqrt{-1}, \cdots, \sqrt{-1}) \in \mathbb{H}^I$ in $GL_2^{+}(\mathbb{A}_F^\infty) \subset GL_2^{+}(\mathbb{A}_F^\infty)$, $f(\alpha x u z) = \varepsilon_+(z) \varepsilon(\sigma f) f(x) J_k(u_{\infty}, 1)^{-1}$.

(A2) For each $z \in \mathbb{H}^I$, choose an element $u \in GL_2(\mathbb{A}_F^\infty)$, then the functions $f_g : \mathbb{H}^I \rightarrow \mathbb{C}$ defined by $f_g(z) = f(gu_{\infty}) J_k(u_{\infty}, 1)$ are holomorphic for all $g \in GL_2(\mathbb{A}_F^\infty)$.

(A3) $\int_{\mathbb{A}_F^F} f_{\mathbb{F}}(\left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) x) du = 0$ for all $x \in GL_2(\mathbb{A}_F^\infty)$.

As usual, one can define the Hecke operators $T_v$ and $S_v$ for $v \nmid N$ and $U_l$ for $l|N$ acting on the finite dimensional complex vector space $S_k(N, \varepsilon; \mathbb{C})$. Let $\mathcal{W}$ be the integer ring of some number field containing the values of $\varepsilon_1, \varepsilon_2, \varepsilon_+ +$ and all the conjugates of $O$ in $\overline{\mathbb{Q}}$. It’s well known that there is a $\mathcal{W}$ lattice $S_k(N, \varepsilon; \mathcal{W}) \subset S_k(N, \varepsilon; \mathbb{C})$ stable under the action of the Hecke operators mentioned above such that

$$S_k(N, \varepsilon; \mathcal{W}) \otimes \mathcal{W} \mathbb{C} = S_k(N, \varepsilon; \mathbb{C}).$$

For each $\mathcal{W}$-algebra $A$, define $S_k(N, \varepsilon; A) = S_k(N, \varepsilon; \mathcal{W}) \otimes_\mathcal{W} A$. Define $h_k(N, \varepsilon; A)$ to be the $A$-subalgebra of $\text{End}_A(S_k(N, \varepsilon; A))$ generated by the operators $T_v$ for $v \nmid N$ and $U_l | N$. Then there are isomorphisms

$$\text{Hom}_A(S_k(N, \varepsilon; A), A) \cong h_k(N, \varepsilon; A)$$

and

$$\text{Hom}_A(h_k(N, \varepsilon; A), A) \cong S_k(N, \varepsilon; A)$$

given by the perfect pairing $(\cdot, \cdot) : h_k(N, \varepsilon; A) \times S_k(N, \varepsilon; A) \rightarrow A$ such that $(h, f) = a(1, f|h)$ for $a(1, f)$ the first coefficient in the $q$-expansion of $f$. 
Let $W$ be the completion of $W$ at a prime above $p$, so $W$ is a complete discrete valuation ring with residue field $k$ of characteristic $p$. The following theorem is well known and should be attributed to many people including Shimura, Deligne, Serre, Wiles, Blasius, Rogawski and Taylor. The version we present here is partially adopted from Hida’s book [3].

**Theorem 2.1.** Suppose $k_2 - k_1 + I \geq 2I$. Let $P$ be a prime ideal of $h = h_k(N, \varepsilon; W)$ and assume that the characteristic of the fraction field of $h/P$ is different from $2$. Then, there is a continuous semisimple Galois representation $\rho_P : G_F \to GL_2(h/P)$ unramified outside $pN$ such that

$(1)$. \( \operatorname{tr}(\rho_P)(Frob_\ell) = T_\ell \) for all prime ideals $\ell \nmid pN$ and $\det(\rho_P) = \varepsilon_+ N^n$ for the $p$-adic cyclotomic character $N$.

$(2)$. Let $m$ be the unique maximal ideal containing $P$ and assume that $T_P \notin m$ for all primes $p|m$, then we have $\rho_P|_{D_p} \cong \begin{pmatrix} \ell_p & \star \\ 0 & \delta_p \end{pmatrix}$ for the restriction of the representation $\rho_P$ at the decomposition group $D_p$.

Moreover, $\delta_p([\varpi_p; F_p]) = U_p(\varpi_p)$ and $\delta_p([u, F_p]) = \varepsilon_{1, p}(u)u^{-k_1 \cdot p}$ for $u \in O_F^\times$.

$(3)$. Write $N = N_0(\varepsilon^-)$ and suppose that $N_0$ is square free and prime to $c$. If $\ell$ is a prime factor of $N_0$ which prime to $p$ and $\ell^2 \nmid N$, then $\rho_P|_{D_\ell} \cong \begin{pmatrix} \ell_\ell & \star \\ 0 & \delta_\ell \end{pmatrix}$ such that $\delta_\ell([\varpi_\ell, F_\ell]) = U_\ell$ and $\delta_\ell([u, F_\ell]) = \varepsilon_{1, \ell}(u)$ for $u \in O_F^\times$.

Here in the statement (2) of the above theorem, the symbol $u^{-k_1 \cdot p}$ denote the product \( \prod_{\sigma \in I_p} \sigma(u)^{-k_1 \cdot p} \) where $I_p$ is the subset of $I$ consisting of the embedding $\sigma : F \to \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ which give rise to the completion $F_p$. Since we choose $W[\varpi]$ contains all the image of the embedding of $\mathcal{O}_F$, we may regard the above embedding actually into $W[\varpi]$. Let $m$ be a maximal ideal of $h$, then the localization $h_m$ is a direct summand of $h$. The tensor product $S_k(N, \varepsilon; A)_m = S_k(N, \varepsilon; A) \otimes_h h_m$ for a $W$-algebra $A$ is a direct summand of $S_k(N, \varepsilon; A)$ and can be identified with $\operatorname{Hom}_A(h_m \otimes W A, A)$.

If the representation $\overline{\varphi} = \rho_m$ constructed as above with the extra property that $\overline{\varphi}_\ell \neq \bar{\delta}_\ell$, we call this condition (ds) at $\ell$. From now on, we assume the square free hypothesis in (3) of the above theorem.

Let $x$ be a prime ideal of $F$ such that $x \nmid pN$ and write $\varpi_x$ the uniformizer of $x$. Consider the map

$$i_x : S_k(N, \varepsilon; A)^2 \to S_k(Nx, \varepsilon; A)$$

sends $(f_1, f_2) \in S_k(N, \varepsilon; A)^2$ to $f_1 + f_2[\eta_x]$ for $\eta_x = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_x \end{pmatrix}$. We denote $S_k(Nx, \varepsilon; A)_{m}^{old}$ the image of $S_k(N, \varepsilon; A)^2_m$ under the map $i_x$ and denote $h_{m}^{old} \subset \operatorname{End}_A(S_k(Nx, \varepsilon; A)_{m}^{old})$ generated by all the Hecke operators.

**Lemma 2.2.** Define a linear operator $U$ acting on $S_k(N, \varepsilon; A)^2$ by the matrix $T_x \begin{pmatrix} \varepsilon_+ (\varpi_x) N(x)^n & 1 \\ 0 & 0 \end{pmatrix}$ multiplication on the right of $(f_1, f_2) \in S_k(N, \varepsilon; A)^2$. Then we have $i_x \circ U = U_x \circ i_x$. Furthermore, if the residue representation $\overline{\varphi} : G_F \to GL_2(h/m)$ associate to the maximal ideal $m$ of the Hecke algebra $h = h_k(N, \varepsilon; W)$ restrict to the decomposition group $D_x$ satisfies the (ds) condition, then there is an isomorphism

$$h_{m}^{old} \cong h_m[X]/(X^2 - T_x X + \varepsilon_+ (\varpi_x) N(x)^n) \cong h_2.$$

**Proof.**

$$f[T_x = [\eta_x](f) + \sum_{a \mod x} [1 \ a \ \varpi_x] \eta_x](f) = [\eta_x](f) + U_x(f).$$

which means that on $S_k(Nx, \varepsilon; A)^{old}_m$, we have $T_x = U_x + [\eta_x]$. On the other hand, $[\eta_x][\eta_x] = [\varpi_x]$, so $U_x + [\eta_x] = \varepsilon_+ (\varpi_x) N(x)^n$ on $S_k(Nx, \varepsilon; A)^{old}_m$. 

for \((f_1, f_2) \in S_k(N, \varepsilon; A)_{m}^2\). The determinant det\((U) = \varepsilon_+(\varpi)N(x)^n \in W^\times\) is invertible, which implies that \(i_x\) is an isomorphism. \(U_x\) is invertible in \(h^\text{old}_m\) and \(T_x = U_x + \varepsilon_+(\varpi)N(x)^nU_x^{-1} \in h^\text{old}_m\), thus 
\[ h^\text{old}_m = h_m[U_x] \cong h_m[Y]/(Y^2 - T_xY + \varepsilon_+(\varpi)N(x)^n). \]
If further assume \(\mathfrak{p}\) satisfies (ds) condition at \(x\), the equation \(Y^2 - T_xY + \varepsilon_+(\varpi)N(x)^n = 0\) has two distinct roots modulo \(m\), so has two distinct roots by Hensel lemma, which implies \(h_m[Y]/(Y^2 - T_xY + \varepsilon_+(\varpi)N(x)^n) \cong h^2_m\), as algebras. □

Let \(Q\) be a finite set of finite primes of \(F\) such that \(\mathfrak{p}\) satisfies (ds) at \(v\) and \(N(v) \equiv 1(\text{mod} p)\) for all \(v \in Q\). Let \(NQ\) be the ideal \(N \prod_{v \in Q} (\varpi_v)\) and let \(\hat{\Gamma}_1(Q) = \prod_{v \in Q} \hat{\Gamma}_1(\varpi_v)\). For each \(v \in Q\), choose a solution \(\alpha_v\) of the equation \(Y^2 - T_vY + \varepsilon_+(\varpi_v)N(x)^n = 0\), we thus has a maximal ideal \(m_Q\) of \(h_k(NQ, \varepsilon; A)\) in the old part generated by \(x, T_x - \alpha V(Frob_x)\) for \(x \in Q NQ\) and \(U_v - \alpha_v\) for \(v \in Q\). The above lemma implies \(h_k(NQ, \varepsilon; A)_{m} \cong h_k(N, \varepsilon, A)_{m}\).

For each \(v \in Q\), let \(\Delta_v\) be the maximal \(p\)-power quotient of \((O_v/\varpi_vO_v)^\times\), and let \(\Delta = \prod_{v \in Q} \Delta_v\). Define \(U_v\) to be the kernel of the projection \((O_v/\varpi_vO_v)^\times \to \Delta_v\), i.e., \(U_v\) consists of elements in \((O_v/\varpi_vO_v)^\times\) with order prime to \(p\). Denote by \(S_k(NQ, \varepsilon; A)\) the forms in \(S_k(\Gamma_0(N) \cap \hat{\Gamma}_1(Q), \varepsilon; A)\) fixed by the action of \(\prod_{v \in Q} U_v\). Let \(h_k(NQ, \varepsilon; A)\) be the elements in \(End(A(S_k(NQ, \varepsilon; A)) \) generated by Hecke operators. \(h_k(NQ, \varepsilon; W)\) is then a \(W[\Delta]\)-algebra and \(h_k(NQ, \varepsilon; W)/\langle \delta - 1, \delta \in \Delta \rangle \cong h_k(NQ, \varepsilon; W)\). Let \(m_Q\) be the inverse image of \(m_Q\) under the projection \(h_k(NQ, \varepsilon; W) \to h_k(NQ, \varepsilon; W)\), so \(m_Q\) is a maximal ideal of \(h_k(NQ, \varepsilon; W)\).

**Lemma 2.3.** The group \(\Delta\) acting on \(S_k(NQ, \varepsilon; A)_{m_Q}\) induces an isomorphism between the invariants of the action, and the space \(S_k(N, \varepsilon; A)_{m}\). The localization \(h_k(NQ, \varepsilon; W)_{m_Q}\) is a \(W[\Delta]\)-algebra, free of finite rank over \(W[\Delta]\).

For a proof of this lemma, see Hida section 3.2.3.

## 3. Galois Cohomology

Fix a finite extension \(K\) of \(\mathbb{Q}_p\) with integer ring \(W\), uniformiser \(\pi_K\) and residue field \(k\). Denote \(CNL_W\) the category of complete neotherian local \(W\)-algebras with residue field \(k\). Let \(S\) be a finite set of primes of \(F\) containing all the primes dividing \(p\). Denote the Galois group of the maximal extension of \(F\) unramified outside \(S\) by \(G_{F,S}\). Choose a decomposition group \(G_{F,v}\) for each \(v \in S\) once and for all. Let \(\psi : (A_f^\times) / F^\times \to (W^\times)^\times\) be a continuous character unramified outside \(S\), and we regard it as a Galois character via class field theory

\[
Gal(F^{\text{ab}} / F) \simeq (F \otimes_{\mathbb{Z}} \mathbb{R})^\times / (A_f^\times / F^\times) \to (A_f^\times / F^\times) /
\]

\[
(3.1) \quad \mathfrak{p} : G_{F,S} \to GL_2(k)\] is a continuous representation on a two dimensional \(k\)-vector space \(V\). After possibly replacing \(k\) by a quadratic extension, we may and do assume that \(k\) contains all the eigenvalues of the image of \(\mathfrak{p}\).

We will consider the framed deformations of representations for both the decomposition groups and the global Galois group \(G_{F,S}\). For each prime \(p\) of \(F\) dividing \(p\), we choose a basis \(\beta_k\) of the Galois module \(k^2\), and consider the functor from the category \(CNL_W\) to the category of sets, sending a \(W\)-algebra \(A\) with residue field \(k\) to the set of equivalence classes of the pairs \((\rho_A, \beta_A)\), where \(\rho_A : G_{F,p} \to GL_2(A)\) is a deformation of \(\mathfrak{p}|_{G_{F,p}}\) and \(\beta_A\) becomes \(\beta_k\) via the isomorphism \(A^2 \otimes A / A \cong k^2\). We may regard \(\beta_A\) as a two by two matrix with entries in \(A\) which, after modulo the maximal ideal \(m_A\), became the matrix \(\beta_k\) in \(M_2(k)\). Two such pairs \((\rho_A, \beta_A)\) and \((\rho'_A, \beta'_A)\) are equivalent, if there exist a matrix \(T \in GL_2(A)\) such that \(T \equiv 1(\text{mod} m_A)\) and \(T(\rho_A)T^{-1} = \rho'_A, T\beta_A = \beta'_A\). This functor is representable by an \(W\)-algebra in \(CNL_W\), which we denoted by \(\mathcal{R}^p\).

In the special case when \(\mathfrak{p}|_{G_{F,p}}\) is a scaler, this functor has another description. The set \(Spec\mathcal{R}^p(A)\) can also be regarded as \(\{\rho_A, \beta_A\} / \sim \) for \(\rho_A : G_{F,p} \to GL_2(A)\) a lifting of \(\mathfrak{p}|_{G_{F,p}}\) and \(\beta_A\) an arbitrary basis of...
the rank two module $A^2$ (not necessary a lifting of $\beta_k^i$), while $(\rho_A, \beta_A) \sim (\rho_A', \beta_A')$ if there exist a matrix $T \in GL_2(A)$ (not necessary a lifting of the identity matrix) such that $T(\rho_A)T^{-1} = \rho_A'$ and $T\beta_A = \beta_A'$. One can check these two descriptions of the set $\text{Spec} R_p^\square(A)$ are the same, if the representation $\overline{\rho}_{|G_{F_p}}$ is a scaler. Both these two descriptions are need.

Let $R_p^\square,\psi$ be the quotient of $R_p^\square$ corresponding to the deformations with fixed determinant $\psi$, and write $R_p^{\square,\psi} = \otimes_{|p}\overline{R}_p^\square$ and $R_p^\square,\psi = \otimes_{|p}\overline{R}_p^\square$.

Next we consider the functor from the category $\text{CNL}_W$ to the category of sets sending $A$ to the set of equivalence classes of the pairs $(\rho_A, \beta_A|_{p|p})$, consists of a deformation $\rho_A : G_{F,S} \to GL_2(A)$ of $\overline{\rho}$, and an $A$-basis $\beta_A$ lifting the chosen basis $\beta_k^i$ for each $p|p$. Two pairs $(\rho_A, \beta_A|_{p|p})$ and $(\rho_A', \beta_A'|_{p|p})$ are equivalent, if there exist a matrix $T \in GL_2(A)$ such that $T \equiv \text{id}(\text{mod } m_A)$, $T(\rho_A)T^{-1} = \rho_A'$ and $T\beta_A = \beta_A'$ for every $p|p$. This functor is representable by some $W$-algebra $R_{F,S}^\square$. If $\overline{\rho}$ is absolutely irreducible, the universal (non-framed) deformation functor is also representable by a $W$-algebra $R_{F,S}$. Similarly, write $R_{F,S}^{\square,\psi}$ for the quotient of $R_{F,S}^\square$ corresponding to deformations with fixed determinant $\psi$.

Denote by $\Sigma$ the places $p$ above $p$ and write $r = |\Sigma|$. For any representation $(\rho, V)$ of a group $G_{F,S}$ (resp. $G_v$), over a rank two free module $V$ over some ring $A$, denote $ad(\rho)$ (resp. $ad^0(\rho)$) or $ad(V)$ (resp. $ad^0(V)$) the adjoint action of $G$ on $\text{End}(V)$ (resp. the trace zero elements in $\text{End}(V)$). We have the following relation between the rings $R_{F,S}^\square$ and $R_{F,S}$.

**Proposition 3.1.** Assume $\overline{\rho}$ is absolutely irreducible. The morphism $R_{F,S} \to R_{F,S}^\square$ is of relative dimension $j = 4|\Sigma| - 1 = 4r - 1$. The ring $R_{F,S}^\square$ can be identified to a power series ring $R_{F,S}[[w_1, \cdots, w_r]]$.

**Proof.** Since we assume that the residue representation $\overline{\rho}$ is absolutely irreducible, we have the tangent dimension

$$\dim(R_{F,S}/(\pi_k)R_{F,S}) = \dim H^1(G_{F,S}, ad(\overline{\rho})) - \dim H^0(G_{F,S}, ad(\overline{\rho}))$$

$$= \dim H^1(G_{F,S}, ad(\overline{\rho})) - 1.$$  

These are computed in the following way. Each deformation is regarded as $\rho : G_{F,S} \to GL_2(A)$, determined by a matrix valued function $A(g)$ of the Galois group $G_{F,S}$. Let $A(g) = A_0(g) + \varepsilon A_1(g)$, where $A_0(g)$ is the representation $\overline{\rho}$. Then $A_1(g)A_0(g)^{-1}$ is a cocycle in the module $ad(\overline{\rho})$. The deformation is trivial, if there is a matrix $T = T_0 + \varepsilon T_1$ such that $TA(g)T^{-1} = A_0(g)$, i.e., $T_0 \in H^0(G_{F,S}, ad(\rho_0))$ and $T_1T_0^{-1}$ is a coboundary.

For the framed deformation, two pairs $(\rho_1, \beta_1^1, \cdots, \beta_r^1)$ and $(\rho_2, \beta_2^1, \cdots, \beta_r^2)$ are equivalent, if there is some $T$ as above, such that $T\rho_1(g)T^{-1} = \rho_2(g)$ and $T\beta_1^i = \beta_2^i$. Now assume we have a framed deformation $(\rho_A, \beta_1, \cdots, \beta_r)$, we have a morphism $\text{R}_{F,S} \to A$ such that the composition $G \to GL_2(\text{R}_{F,S}) \to GL_2(A)$ is equivalent to the given $\rho_A$, say they are up to conjugation by $T$. Replace the original pair $(\rho_A, \beta_1, \cdots, \beta_r)$ by $(T\rho_AT^{-1}, T\beta_1, \cdots, T\beta_r)$. Then there is no way to change the $\rho_A$. However, a multiplication by a scaler matrix $T$ won’t change the equivalence class of the framed deformation pairs, and this is the only way to get an equivalent pair while preserving the shape of the representation. Thus, we may also fix one of the coordinates of the frames, say the right bottom corner of $\beta_r$ have value 1. Then we can see that the $R_{F,S}[[w_1, \cdots, w_r]]$ has the universal property and the uniqueness, while the $r$ pairs of basis are given by $\beta_i = \left( \begin{array}{cc} 1 + w_{i-3} & w_{i-2} \\ w_{i-1} & 1 + w_{i} \end{array} \right)$ for $i = 1, 2, \cdots, r$, where the last one $w_{4r} = 0$ as we fixed.

Restrict the universal representation $G_{F,S} \to GL_2$ to the decomposition groups, $\text{R}_{F,S}^\square,\psi$ become an algebra over $\text{R}_{F,S}^\square,\psi$ by the universality. Both of these two rings describe deformations without any local conditions. To encode the local description, later we will consider certain quotients of these rings. The following lemma of Kisin [17] described the relative tangential dimensions of these two rings in terms of Galois cohomology.

**Lemma 3.2.** (Kisin) Let $\delta_p = \dim_k H^0(G_{F_p}, ad(\overline{\rho}))$ for $p|p$ and $\delta_F = \dim_k H^0(G_{F,S}, ad(\overline{\rho}))$. Define

$$H^1_\Sigma(G_{F,S}, ad^0(\overline{\rho})) = \ker(\theta^1 : H^1(G_{F,S}, ad^0(\overline{\rho})) \to \prod_{v|p} H^1(G_{F_v}, ad^0(\overline{\rho}))).$$
Then $R_{F,S}^{\square,\psi}$ is a quotient of a power series ring over $R_p^{\square,\psi}$ in $g = \dim H^1_{\text{dR}}(G_{F,S}, \text{ad}^{\psi}(\overline{\rho})) + \sum_{v \mid p} \delta_v - \delta_F$ variables.

From now on, we will make the following assumptions.

1. $\overline{\rho}$ has odd determinant and the restriction to $G_{F(\psi)}$ is absolutely irreducible.
2. $\varpi$ is an element of $\mathbb{Q}$.
3. $p \geq 5$. If $p = 5$ and $\overline{\rho}$ has projective image isomorphic to $PGL_2(\mathbb{F}_5)$, then $[F(\varpi) : F] = 4$.
4. For every $v \in S \setminus \Sigma$, we have

\begin{equation}
(1 - N(v))(1 + N(v))^2 \det \varphi(Frob_v) - N(v)(\text{tr} \varphi(Frob_v))^2 \in k^\times.
\end{equation}

Here $Frob_v$ means the arithmetic Frobenius at $v$. The scaler condition (1) is the deformation problems we want to discuss in this paper. This type of deformation problems are not covered by Fujiwara [4] and Skinner-Wiles [25], because the deformation functor is no longer representable if the restriction of $\overline{\rho}$ to decomposition groups are scaler. This is the main problem we will solve in this paper by inventing a new type of framed deformation functor. Assumptions (2) and (3) are standard requirements for the Taylor-Wiles argument. (4) is to simplify the argument of bad primes outside $p$, which may be removed if one use Fujiwara’s more general argument.

The Taylor-Wiles argument needs to enlarge the set of ramification primes. The relevant result here we quote Kisin’s version in [17] section 3.

**Proposition 3.3.** (Kisin) Write $g = \dim_h H^1(G_{F,S}, \text{ad}^{\psi}(\overline{\rho})) - [F : Q] + |\Sigma| - 1$. For each positive integer $n$, there exist a finite set of primes $Q_n$ of $F$, disjoint from $S$, such that

1. For each $v \in Q_n$, $N(v) \equiv 1 \pmod{p^n}$ and $\varphi(Frob_v)$ has distinct eigenvalues.
2. $|Q_n| = \dim_h H^1(G_{F,S}, \text{ad}^{\psi}(\overline{\rho}))$.
3. Let $S_{Q_n} = S \cup Q_n$, then $R_{F,S_{Q_n}}^{\square,\psi}$ is topologically generated over $R_p^{\square,\psi}$ by $g$ elements.

For each $n$, fix a set of primes $Q_n$ and denote by $\Delta_{Q_n}$ the Sylow $p$-subgroup of $(O/\prod_{v \in Q_n} \varpi_v O)^\times = \prod_{v \in Q_n}(O/\varpi_v O)^\times$. For $v \in Q_n$, let $\xi_v = \Delta_{Q_n}$ be the image of some fixed generator of $(O_F/\varpi_v O)^\times$. Write $h^1 = |Q_n| = \dim_h H^1(G_{F,S}, \text{ad}^{\psi}(\overline{\rho}))$ and order the elements of $Q_n$ as $v_1, v_2, \ldots, v_{h^1}$. $W[\Delta_{Q_n}]$ becomes a quotient of $W[y_1, y_2, \ldots, y_{h^1}]$ by mapping $y_i$ to $\xi_v - 1$, say $W[\Delta_{Q_n}] \cong W[y_1, y_2, \ldots, y_{h^1}]\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq ((y_1 + 1)^{p^n} - 1, (y_2 + 1)^{p^n} - 1, \ldots, (y_{h^1} + 1)^{p^n} - 1)$. As explained in [2], the ring $R_{F,S_{Q_n}}$ is a $W[\Delta_{Q_n}]$-algebra, and $R_{F,S} \cong R_{F,S_{Q_n}}/(y_1 - 1, y_2 - 1, \ldots, y_{h^1} - 1)$ via the above isomorphism.

4. Framed deformation rings with local conditions

Recall in section 2, if we take the set $S$ to be the places divide the ideal $NP$, there is a continuous Galois representation

$$\rho_m : G_{F,S} \rightarrow GL_2(h_k(N, \varepsilon, W)_m)$$

such that for each place $v \notin S$, the characteristic polynomial of $\rho_m(Frob_v)$ is given by $X^2 - T_v X + \varepsilon_+ N(\varpi_v)^n$. We take the character $\psi = \varepsilon_+ N^n$.

Denote by $\overline{\varphi} : G_{F,S} \rightarrow GL_2(k)$ the representation obtained by reducing $\rho_m$ modulo $m$ and we assume that $\overline{\varphi}$ is absolutely irreducible. We further assume the ideal $m$ is nearly ordinary at all the places $p|p$. Thus by theorem 2.1 in section 2,

$$\rho_m|_{G_{F,p}} \sim \begin{pmatrix} \delta_p & \ast \\ 0 & \delta_p \end{pmatrix},$$

where $\delta_p(\varpi_p) = U_p(\varpi_p)$ and $\delta_p([u, F_p]) = \varepsilon_1 p^{-k_1+1} p^{-1}$. When the reduction $\overline{\varphi}$ satisfies the distinguished condition, i.e., $\overline{\varphi} \neq \overline{\varphi}_p$, the usual (non-framed) deformation functor is representable, and the lifting problem has been studied by many authors, cf Wiles [28], Taylor-Wiles [23], Fujiwara [3] and Skinner-Wiles [25]. In this paper, we focus on the worst situation that

$$\overline{\varphi}|_{G_{F,p}} \sim \begin{pmatrix} \chi_p & 0 \\ 0 & \chi_p \end{pmatrix},$$

where $\chi_p = \varepsilon_p = \overline{\delta}_p$, in which case the usual deformation functor is no longer representable.
Proposition 4.1. \( \dim \mathcal{R}_{\hat{\rho}, \psi, s}^p / m_W \mathcal{R}_{\hat{\rho}, \psi, s}^p = 1 + \dim \text{Hom}(G_{F_p}, k) = 2 + [F_p : Q_p] \).

Proof. The above dimension is equal to \( \dim F_{p, \hat{\rho}, \psi, s}(k[\varepsilon]) \), where \( \varepsilon^2 = 0 \). The restriction of \( \chi_{2, p} \) on the inertia subgroup \( I_p \) is fixed, only the image of Frobenius element has a one dimensional deformation. The determinant of \( \rho \) is fixed, which implies \( \chi_{1, p} = \psi \chi_{2, p} \) is uniquely determined by \( \chi_{2, p} \). For fixed \( \chi_{1, p} \) and \( \chi_{2, p} \), a deformation is given by an extension of \( \chi_{2, p} \) by \( \chi_{1, p} \), ie, the vector space \( \mathcal{H}(G_{F_p}, k) = \text{Hom}(G_{F_p}, k) \), which is of dimension \( [F_p : Q_p] + 1 \). The allowed ways of choosing basis are precisely the equivalence relation \( \sim \), thus, they have no more contribution. 

Write \( T = h_k(N, \varepsilon, W)_m \) and write \( \rho_T \) for \( \rho_m \). Take a basis \( \beta^0_i \) of \( T^2 \) for each for each \( p | p \), under which \( \rho_{T/G_{F_p}} \) is upper triangular. Then we can consider the framed deformation of the pair \( (\mathcal{H}, \beta^0_q)_{i=1,2,\ldots, r} \), where we fix the basis \( \beta^0_i \) to be \( \beta^0_i \) modulo \( m_T \). Choose a basis \( \beta^i \) of \( (\mathcal{R}_{F, S})^2 \) which lifting \( \beta^0_i \) via the surjective homomorphism \( \mathcal{R}_{F, S} \rightarrow T \) by the universality. We normalize the isomorphism

\[
R_{F, S} \cong R_{F, S}[w_1, \ldots, w_{4j-1}]
\]

such that for each ring \( A \) in the category \( CLN_W \) and a homomorphism of \( W \)-algebra \( i : R_{F, S} \rightarrow A \), the corresponding element in the set \( \text{Spec} R_{F, S}(A) \) is \( (i \circ \rho_{F, S}, i(\left( \begin{array}{cc} 1 + w_{4i-3} & w_{4i-2} \\ w_{4i-1} & w_{4i} \end{array} \right), \beta^i))_{i=1,2,\ldots, r} \), where again we agree on that \( w_{4r} = 0 \) as in the proof of proposition 3.1.

Define \( T^\square = T \otimes_{R_{F, S}} R_{F, S} \), which is isomorphic to a power series ring over \( T \). The representation \( \rho_T \) gives a surjective homomorphism of \( W \)-algebra \( \theta : R_{F, S} \rightarrow T \), hence a surjective homomorphism \( \theta : R_{F, S} \rightarrow T^\square \). The basis determined by the mapping \( \theta \) is thus given by

\[
\beta^i_{T^\square} = \left( \begin{array}{cc} 1 + w_{4i-3} & w_{4i-2} \\ w_{4i-1} & 1 + w_{4i} \end{array} \right) \beta^i_T
\]

for each prime \( p_i \). Write \( \rho_{T^\square} \) for the composition of \( \rho_T \) and the natural inclusion \( T \rightarrow T^\square \). The pair \( (\rho_{T^\square}|G_{F_p}, \beta^i_{T^\square}) \) is not in the set \( F_{p, \hat{\rho}, \psi, s}(T^\square) \), since there is a lower left corner in the basis \( \beta^i_{T^\square} \). Define a quotient \( T^\circ \) of \( T^\square \) by \( T^\square/(w_{4i-1}, i = 1, 2, \ldots, j) \) via the isomorphism in proposition 3.1 which is also the ring removed the variables \( w_{4i-1} \) for \( i = 1, 2, \ldots, r \). The projection \( T^\square \) to \( T^\circ \) push the pair \( (\rho_{T^\square}|G_{F_p}, \beta^i_{T^\square}) \) to the pair \( (\rho_{T^\circ}|G_{F_p}, \beta^i_{T^\circ}) \) where the representation \( \rho_{T^\circ} \) is again the composition of \( \rho_T \) and the natural inclusion \( T \rightarrow T^\circ \) while \( \beta^i_{T^\circ} \), as the image of \( \beta^i_{T^\square} \), now becomes \( \left( \begin{array}{cc} 1 + w_{4i-3} & w_{4i-2} \\ 0 & 1 + w_{4i} \end{array} \right) \beta^i_T \).
Proposition 5.1. We slightly modified Hida’s simplified proof [13] of this proposition. Similarly, replace $S$ by $S_{Q_n}$, we can define $\mathcal{R}_n^\Lambda$ to be the tensor product $\mathcal{R}_n^F \otimes_{\mathcal{R}_p^F} \mathcal{R}_p^F$. Take $Q = Q_n$ in section [2] write $T_n = S_k(NQ_n, \varepsilon, W)_{mQ_n}$, $T_n^\Lambda = T_n \otimes_{\mathcal{R}_n^F} \mathcal{R}_n^F$ and $T_n^\Lambda = T_n^\Lambda / (w_{4i-1}, i = 1, 2, \cdots, j)$, then there is a surjection $R_n^\Lambda \twoheadrightarrow T_n^\Lambda$ of $W[[y_1, \cdots, y_h]]$ algebras.

5. Patching

In the section we present a patching criterion of the Taylor-Wiles system after Fujiwara and Kisin. Write

\[ \text{Proof.} \]

Similarly, replace $B$ by $B_0$, we can define $\mathcal{R}_n^\Lambda$ to be the tensor product $\mathcal{R}_n^F \otimes_{\mathcal{R}_p^F} \mathcal{R}_p^F$. Take $Q = Q_n$ in section [2] write $T_n = S_k(NQ_n, \varepsilon, W)_{mQ_n}$, $T_n^\Lambda = T_n \otimes_{\mathcal{R}_n^F} \mathcal{R}_n^F$ and $T_n^\Lambda = T_n^\Lambda / (w_{4i-1}, i = 1, 2, \cdots, j)$, then there is a surjection $R_n^\Lambda \twoheadrightarrow T_n^\Lambda$ of $W[[y_1, \cdots, y_h]]$ algebras.

Proposition 5.1. Let $B$ be a complete local noetherian $W$-algebra generated by $d$ elements in the maximal ideal over $W$, i.e., $B$ is a quotient of the power series ring $W[[z_1, z_2, \cdots, z_d]]$. $\theta : R \to T$ is a surjective homomorphism of $B$-algebras. $h$ and $j$ are two fixed non-negative integers. Assume for each positive integer $n$, there exist two rings $R_n$ and $T_n$ fitting into a commutative diagram of $W$-algebras:

\[ B[[x_1, \cdots, x_{h+j-d}]] \xrightarrow{\phi_n} R_n \xrightarrow{\theta_n} T_n \]

which satisfies the following conditions:

1. The above diagram consists of surjective $B$-algebra homomorphisms.
2. $R_n$ is a $\Lambda = W[[y_1, \cdots, y_h, t_1, \cdots, t_j]]$-algebra, and $(y_1, \cdots, y_h)R_n = \ker(R_n \to R), (y_1, \cdots, y_h)T_n = \ker(T_n \to T)$.
3. The kernel $b_n = \ker(W[[y_1, \cdots, y_h, t_1, \cdots, t_j]] \to T_n)$ is contained in the ideal $((1 + y_1)^{p^n} - 1, \cdots, (1 + y_h)^{p^n} - 1)$, and $T_n$ is finite free over $\Lambda/b_n$. In particular, $T$ is finite free over $\Lambda_f := W[[t_1, \cdots, t_j]]$.

Then, $\theta$ is an isomorphism and $B \cong W[[z_1, z_2, \cdots, z_d]]$.

Proof. Write $s = \text{rank}_\Lambda T_n = \text{rank}_\Lambda b_n T_n$ and $r_n = sn(h + j)p^n$. Consider the ideal $c_n = (mW)^n + ((1 + y_1)^{p^n} - 1, \cdots, (1 + y_h)^{p^n} - 1, t_1, \cdots, t_j)$ of $\Lambda$. As the residue field $k$ of $W$ is finite of $q$ elements, we have $|T_n' / c_n T_n'| = |T_n / c_n T_n| = q^{c_n}$ for all $n' \geq n$ and so $\text{length}_{R_n} T_n / c_n T_n < r_n$. Hence $m_{t_1}^{\tau_n} T_n / c_n T_n = 0$ and the composition $\theta : R_n' \to T_n' \to T_n / c_n T_n$ factors through the homomorphism $\theta : R_n' / (c_n + m_{t_1}^{\tau_n}) \to T_n'/c_n T_n'$. In particular, $\theta : R \to T \to T / c_n T$ factors through the homomorphism $\theta : R / (c_n + m_{t_1}^{\tau_n}) \to T / c_n T$.

Call $(D, A)$ a patching datum of level $n$, if there is a commutative diagram consisting of surjective homomorphism of $B$-algebras

\[ B[[x_1, \cdots, x_{h+j-d}]] \longrightarrow D \longrightarrow A \]

\[ R / (c_n + m_{t_1}^{\tau_n}) \longrightarrow T / c_n T \]

and $D$ is a $\Lambda / c_n$-algebra (thus also a $\Lambda$-algebra) with the property $m_{t_1}^{\tau_n} = 0$. Two patching datum $(D, A)$ and $(D', A')$ are isomorphic, if there exist isomorphisms $D \cong D'$ and $A \cong A'$ which commute with the two diagrams. For a given level $n$ of patching datum, the order of $D$ and so as well as $A$ is bounded. Thus there are only finitely many patching datum of level $n$. On the other hand, the above argument provide a patching datum $\theta : R_n' / (c_n + m_{t_1}^{\tau_n}) \to T_n' / c_n T_n'$, for each integer $n' \geq n$. By Dirichlet’s drawer principle, there is an infinite subset $I$ of the natural numbers $\mathbb{N}$, such that for any two integers $n < n'$ in $I$, the patching datum $(\theta : R_n' / (c_n + m_{t_1}^{\tau_n}), T_n' / c_n T_n')$ is isomorphic to $(\theta : R_n' / (c_n + m_{t_1}^{\tau_n}), T_n / c_n T_n)$. Take a projective limit $R_\infty = \lim_{\xrightarrow{\xleftarrow{n \in I}}} : R_n / (c_n + m_{t_1}^{\tau_n})$ and $T_\infty = \lim_{\xrightarrow{\xleftarrow{n \in I}}} : T_n / (c_n + m_{t_1}^{\tau_n})$, then we have
a commutative diagram of surjective morphisms

\[
\begin{array}{ccc}
B[[x_1, \ldots, x_{h+j-d}]] & \xrightarrow{\phi_{\infty}} & R_\infty \\
\downarrow & & \downarrow \\
R & \xrightarrow{\theta} & T
\end{array}
\]

\(R_\infty\) becomes a \(\Lambda\)-algebra and \(T_\infty\) is free of rank \(s\) over \(\Lambda = \lim_{\rightarrow n \in \mathbb{N}} \Lambda/\beta_{n}\Lambda\). The Krull dimension of \(T_\infty\) is then equal to \(h+j+1\), the Krull dimension of \(\Lambda\). However, we have a chain of surjective homomorphisms

\[(5.1)\quad W[[z_1, \ldots, z_d, x_1, \ldots, x_{h+j-d}]] \to B[[x_1, \ldots, x_{h+j-d}]] \to R_\infty \to T_\infty.
\]

\(\text{Spec}(T_\infty)\) becomes a closed sub scheme of \(\text{Spec}(W[[z_1, \ldots, z_d, x_1, \ldots, x_{h+j-d}]])\); it can’t be a proper sub scheme since the latter has the same dimension \(h+j+1\) as the former, so \(T_\infty \cong W[[z_1, \ldots, z_d, x_1, \ldots, x_{h+j-d}]]\) which forces all the arrows in (5.1) are isomorphisms. In particular, we have \(B \cong W[[z_1, \ldots, z_d]]\) by the fact that \(W[[x_1, \ldots, x_{h+j-d}]]\) is faithfully flat over \(W\), \(R_\infty \cong T_\infty\) and \(R \cong R_\infty/(y_1, y_2, \ldots, y_h)R_\infty \cong T_\infty/(y_1, y_2, \ldots, y_h)\).

In our case, \(\mathcal{R}^{\Delta, \psi, s}_p\) is a \(W\)-algebra of relative tangent dimension \(d = \sum_{p | \mathfrak{p}} (2 + [F_p : \mathbb{Q}_p]) = 2|\Sigma| + [F : \mathbb{Q}]\). In other words, \(\mathcal{R}^{\Delta, \psi, s}_p\) is a quotient of \(W[[z_1, z_2 \cdot \cdot \cdot, z_d]]\). Apply this patching lemma to our setting that \(B = \mathcal{R}^{\Delta, \psi, s}_p, R = \mathcal{R}^{\Delta}_p, R_n = \mathcal{R}^{\Delta}_{Q_n}, T = T^\circ\) and \(T_n = T_{Q_n}\), we get the following

**Theorem 5.2.** \(\mathcal{R}^{\Delta} = T^\circ\).

One byproduct of this Taylor-Wiles argument is the following corollary, which seems not follow directly from the definition of the deformation ring.

**Corollary 5.3.** \(\mathcal{R}^{\Delta, \psi, s}_p \cong W[[z_1, z_2 \cdot \cdot \cdot, z_d]]\).

### 6. Modularity

Let \(f\) be a cuspidal Hilbert modular eigenform over \(F\) and let \(E_{f, \lambda}\) be the \(\lambda\) completion of its coefficient field for some place \(\lambda | p\). Denote by \(O_{f, \lambda}\) the integer ring of \(E_{f, \lambda}\), \(\pi_{f, \lambda}\) the uniformiser and \(k\) the residue field. A Galois representation \(\rho_{f, \lambda} : G_{F,S} \to GL_2(O_{f, \lambda})\) is attached to \(f\), where \(S\) is a finite set of places containing the infinite places, the primes above \(p\) and the primes where \(f\) is ramified.

Let \(E\) be a finite extension of \(\mathbb{Q}_p\) with integer ring \(G_E, \text{uniformiser } \pi\) and residue field \(k\). If a representation \(\rho : G_{F,S} \to GL_2(E)\) is equivalent to a some \(\rho_{f, \lambda}\), then we call it modular. After conjugation, we may assume \(\rho : G_{F,S} \to GL_2(O_E)\) and denote \(\mathcal{E}\) the composition \(G_{F,S} \to GL_2(O_E) \to GL_2(k)\). Under the assumption that \(\mathcal{E}\) is irreducible, then it’s independent of the choice of the element we used to conjugate the image of \(\rho\). Call \(\rho\) residually modular if \(\mathcal{E} \sim \mathcal{E}^{f, \lambda}\) for some Hilbert eigenform \(f\) over \(F\). We assume \(E\) is large enough that it contains all the conjugates of \(F\) and its residue field \(k\) contains all the eigenvalues of the images of \(\mathcal{E}\).

To prove the modular lifting theorem, we need more descriptions on the shape of the restriction of \(\rho\) to the decomposition groups above \(p\). Write \(\beta_0 = (e_1, e_2)\) for the standard basis \(e_1 = \text{e}(1, 0)\) and \(e_2 = \text{e}(0, 1)\) of \(k^2\). Recall that in section 2 when we define the framed deformation rings, we have chosen the basis \(\beta_k\) of \(k^2\) for each \(p_i\), according to the representation \(\rho^\circ : G_{F,S} \to GL_2(T)\) into the Hecke algebra. A matrix \(M_i \in GL_2(k)\) is uniquely determined by \(M_i \cdot \beta_0 = \beta_k\) for each \(i\).

**Theorem 6.1.** Let \(\rho : G_{F,S} \to GL_2(O_E)\) be a continuous representation with the following conditions hold

1. The determinant of \(\rho = \varepsilon_+ N^n\) is odd for some finite order character \(\varepsilon_+\) and positive integer \(n\), where \(N\) is the \(p\)-adic cyclotomic character.

2. \(\rho\) is nearly ordinary at all place \(p\) above \(p\), \(\rho|_{G_{p_i}} = M_i \begin{pmatrix} \varepsilon_{p_i} & 0 \\ 0 & \delta_{p_i} \end{pmatrix} \) for some matrix \(M_i \in GL_2(O)\) such that \(M_i \equiv \mathcal{M}_i(\text{mod}(\pi))\), where \(\delta_{p_i}(u, F_p) = \varepsilon_{p_i}(u) u^{-k_1} \) for \(u \in O_p^\times, k_1 \in \mathbb{Z}[\text{I}]\) and \(k_2 = (n+1)I - k_1 + I \geq 2I\).
For every $\mathfrak{p}$ above $p$, there is a character $\overline{\chi}_\mathfrak{p} : G_{F,S} \to k^\times$ such that $\overline{\chi}_\mathfrak{p} = \overline{\rho}_\mathfrak{p} = \overline{\chi}_\mathfrak{p}$, and the residue representation $\overline{\rho}|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \overline{\chi}_\mathfrak{p} & 0 \\ 0 & \overline{\chi}_\mathfrak{p} \end{pmatrix}$.

The restriction of $\overline{\rho}$ to $G_{F_{(v)}}$ is absolutely irreducible. If $p = 5$ and $\overline{\rho}$ has projective image isomorphic to $\text{PGL}_2(\mathbb{F}_5)$, then $[F(\zeta_5) : F] = 4$.

6.4. The representation of $\text{GL}_2(F)$ corresponding to $\rho|_{G_v}$ under local Langlands correspondence is not special at any prime $v \neq p$.

If $\overline{\rho} \sim \overline{\rho}_f$ for some automorphic form $f$ of weight $k = (k_1, k_2)$, then $\rho$ is associated to an automorphic form of weight $k$, too.

Remark 6.1. The deformation problem characterized by the local conditions (2) and (3) is not covered by Fujiwara’s work [4], which assumes the distinguished condition on nearly ordinary primes. We make the additional description of the matrices $M_i$’s in (2), because under the situation (3), unlike the situation in the condition (ds) holds, only assuming $\rho$ lifting $\overline{\rho}$ is not enough to make sure that $\rho$ is in the category parameterized by our (framed) deformation rings.

Remark 6.2. The condition (5) can be removed either by consider the minimal lifting problem at such $v$ as Fujiwara [4], or by introducing automorphic forms on a suitable quaternion algebra as in Kisin [17].

Remark 6.3. Under the circumstance $\rho|_{G_{\mathfrak{p}_i}} = \begin{pmatrix} \chi_{\mathfrak{p}_i} & 0 \\ 0 & \chi_{\mathfrak{p}_i} \end{pmatrix}$ for all $i$, the hypothesis on the existence of $M_i$ is automatic.

$\rho$ is modular by descent. We first prove the theorem assuming the following extra condition

\[(*) \quad f \in S_k(p, \varepsilon, W) \text{ for the level } N = p \text{ and the “Neben” } \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_+) .

Let $m$ be the maximal ideal of $h_k(p, \varepsilon, W)$ associated to $f$. The local ring $T = h_k(p, \varepsilon, W)_m$ is in the nearly ordinary part of $h_k(p, \varepsilon, W)$ and thus reduced and generated by $T_{\ell}$ for $\ell \mid p$. Take the set $S$ of bad primes to be those above $p$ and the character $\psi = \varepsilon_++\omega^n$.

Associated to each $p_i$ the basis $\beta_{E_i} = M_i \cdot (e_1, e_2) \in D_{E_i}$, where $e_1 = \varepsilon_0(1, 0)$ and $e_2 = \varepsilon^t(0, 1)$. The basis $\beta_{E_i}$ congruent to the universal basis $\beta$ in section 4, so the pair $(\rho, \beta_{E_i})$ gives a homomorphism

$$R^\Delta_{F,S} \to O_E$$

by the universality, which factor through the map

$$R^\Delta = R^\Delta_{F,S} \otimes_{R^\Delta_{F,\varepsilon}} R^\Delta_{\psi, s} \to O_E,$$

since $\rho|_{G_{\mathfrak{p}_i}}$ is upper triangular under the basis $\beta_{E_i}$. So we have a homomorphism

$$h_k(p, \varepsilon, W)_m = T \hookrightarrow T^\Delta \cong R^\Delta \to W$$

which, by the duality of Hecke algebra and modular form, implies that $\rho$ is modular.

In the general case, the base change technique developed by Skinner-Wiles [25], (see also (3.5) of [17]). As we assumed that $\rho$ is non special for all primes $\ell$ outside $p$, the only possibilities for $\rho|_{G_{\ell}}$ are either principal or induced from a character of a quadratic extension of $F$. Both of these two cases will become unramified after a suitable base change to a totally real field $F'$. For the primes in the level above $p$, we can put any powers on them as the Hecke algebra is nearly ordinary, thus, this reduce to the case $(*)$. The local descriptions (2) and (3) of $\overline{\rho}$ are obviously preserved under restriction to a subgroup. Then the above argument implies $\rho|_{G_{\mathfrak{p}_i}}$ is modular, so $\rho$ is modular by descent.

7. Locally cyclotomic deformation

In this section, we will prove a family version of the lifting theorem. Assume the initial weight $k_0 = (0, I)$ in this section and we will write $\varepsilon_0$ for the initial “Neben” instead of using the letter $\varepsilon$. As
before, we again assume that $\rho : G_{F,S} \rightarrow GL_2(W)$ is a representation satisfies all the assumptions of theorem 6.1 for the weight $k = k_0$ and “Neben” $\varepsilon = \varepsilon_0$.

We consider a new local framed deformation $\hat{\Phi}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$ sending a ring $A$ in the category $\text{CNL}_W$ to the set of equivalence classes of pairs $(\rho_A, \beta_A)$ consisting of a deformation $\rho_A : G_{F_p} \rightarrow GL_2(A)$ of $\bar{\rho}|_{G_{F_p}}$ with fixed determinant $\psi$, and an $A$-basis $\beta_A$ lifting the chosen $\beta_k$, under which $\rho_A$ is given by $\begin{pmatrix} \chi_1(p) & \ast \\ 0 & \chi_2(p) \end{pmatrix}$ for some characters $\chi_1(p)$ and $\chi_2(p)$ of $G_{F_p}$ such that the character $\chi_2(p)|_{I_p} \neq 1$ factors through $\text{Gal}(\mathcal{F}_p, p\mathbb{Z}_p)/\mathbb{Z}_p$ where $\mathcal{F}_p$ is the maximal unramified extension of $F_p$. Two pairs $(\rho_A, \beta_A)$ are equivalent, if there exist an upper triangular matrix $T \equiv I_2 \pmod m_A$, such that $T \rho_A T^{-1} = \rho_A$ and $T \beta_A = \beta_A$. Notice that the only difference between the above definition and the functor $F_{\text{cyc}}^{\varepsilon,\varepsilon_0}$ studied in section 4 is that in the latter functor, we require that $\chi_2(p)|_{I_p} \neq 1$ (recall that we have have the initial weight $k = k_0 = (0, I)$).

The tangent dimension of the local cyclotomic deformation functor $\Phi_{\text{cyc}}^{\varepsilon,\varepsilon_0}$ is equal to $3 + [\mathbb{F}_p : \mathbb{Q}_p]$, one more than that of $F_{\text{cyc}}^{\varepsilon,\varepsilon_0}$. Again by checking the Schlessinger criterions, $\Phi_{\text{cyc}}^{\varepsilon,\varepsilon_0}$ is also representable by some complete local noetherian $W$-algebra $\mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$.

Write $\Gamma_p$ for the $p$-Sylow subgroup of $\text{Gal}(\mathcal{F}_p, p\mathbb{Z}_p)/\mathbb{Z}_p$, which is embedded into $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p$ via the $p$-adic cyclotomic character. It is isomorphic to $\mathbb{Z}_p$ by choosing some character $\gamma_p$. Let $\Gamma_p = \prod_{p|\mathbb{P}} \Gamma_p$. We then have an isomorphism $\mathbb{W}[\Gamma_p] \cong \mathbb{W}[X_1, X_2, \ldots, X_t]$ by sending $\gamma_p$ to $1 + X_i$, where we ordered the primes $\mathbb{P}$ as in section 3.

The universal representation $\rho_{\text{cyc}} : G_{F_p} \rightarrow GL_2(\mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}) \cong \begin{pmatrix} \chi_1(p) & \ast \\ 0 & \chi_2(p) \end{pmatrix}$ with $\chi_2(p) \equiv 1$ factors through $\Gamma_p$ and we get a $W[\Gamma_p]$-algebra structure on $\mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$ via the character $\chi_2(p)|_{I_p} \neq 1 : \Gamma_p \rightarrow \mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$. Define $\mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0} := \bigotimes_{p|\mathbb{P}} \mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$. $\mathcal{R}_{\text{cyc}}$ is a $W[\Gamma_p]$-algebra and there is an isomorphism $\mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}/(X_1, \ldots, X_t) \cong \mathcal{R}_{\text{cyc}}^{\varepsilon,\varepsilon_0}$.

Identify $\widehat{\Pi}_0(p^s)/\widehat{\Pi}_1(p^s)$ with $((\mathcal{O}/p^{s\mathbb{O}})^2 \times$ by sending $\begin{pmatrix} a \\ c \\ d \end{pmatrix} \in \widehat{\Pi}_0(p^s)/\widehat{\Pi}_1(p^s)$ to $(a, d) \in ((\mathcal{O}/p^{s\mathbb{O}})^2 \times$. We have an isomorphism $((\mathcal{O}/p^{s\mathbb{O}})^2 \times \cong \prod_{p|\mathbb{P}} (\mathcal{O}_p/p^{s\mathbb{O}_p})^2 \times$ and the local norm maps $N_p : \mathcal{O}_p^2 \rightarrow \mathbb{Z}_p$ form a surjective homomorphism $N_p = \prod_{p|\mathbb{P}} N_p : (\mathcal{O}_p/p^{s\mathbb{O}_p})^2 \rightarrow \prod_{p|\mathbb{P}} N_p(\mathbb{Z}/p^{s\mathbb{Z}})^2 \times$ for each integer $s > 0$.

Let $\widehat{\Pi}_p(p^s)$ be the subgroup of $GL_2(\mathcal{H}_p)$ such that $\widehat{\Pi}_1(p^s) \subset \widehat{\Pi}_p(p^s) \subset \widehat{\Pi}_0(p^s)$ and

$$\widehat{\Pi}_p(p^s)/\widehat{\Pi}_1(p^s) = \text{Ker}(N_p^2 : ((\mathcal{O}/p^{s\mathbb{O}})^2 \times) \rightarrow (\prod_{p|\mathbb{P}} (\mathcal{O}_p/p^{s\mathbb{O}_p})^2 \times).$$

Put $\widehat{\Pi}_s = \widehat{\Pi}_{\text{cyc}}(p^s) \cap \widehat{\Pi}_0(N)$, then there is an inclusion $S_{k_0}(\widehat{\Pi}_n, \varepsilon_0, W) \rightarrow S_{k_0}(\widehat{\Pi}_m, \varepsilon_0, W)$ for each pair of positive integers $m > n$, which is compatible with the Hecke algebra structure. Thus there is a surjective $W$-algebra homomorphism $S_{k_0}(\widehat{\Pi}_m, \varepsilon_0, W) \rightarrow S_{k_0}(\widehat{\Pi}_m, \varepsilon_0, W)$ by restriction, and we define the projective limit

$$\text{h}_{\text{cyc}}^n(\varepsilon, \varepsilon_0 ; W[[\Gamma_p]]) := \lim_{\rightarrow} \text{h}_{k_0}^n(\varepsilon_0 ; W),$$

where for any level group $\widehat{\Pi}$ and “Neben” $\varepsilon$, the nearly ordinary Hecke algebra is defined by $\text{h}_{p_0}^n(\varepsilon ; W) := e_p \text{h}_{p_0}^n(\varepsilon ; W)$ for Hida’s idempotent

$$e_p = \lim_{n \rightarrow \infty} (\prod_{p|\mathbb{P}} U(p))^{n!}$$

where $U(p)$ is the Hecke operator normalized as [8] section 3.1.2, page 168.

It’s known that $\text{h}_{p_0}^n(\varepsilon ; W)$ is a torsion free $W[[\Gamma_p]]$-module of finite type ([8], theorem 3.53), so $\text{h}_{\text{cyc}}^n(\varepsilon, \varepsilon_0 ; W[[\Gamma_p]])$ is a $p$-profinite semilocar ring, i.e. a direct sum of the localizations at its maximal ideals. Let $f_0$ be an Hecke eigen form in $S_{k_0}(N\mathbb{P}, \varepsilon, W)$ such that the associated Galois representation $\rho|_{I_p}$ modulo $MW$ equal to $\overline{p}$. Let $m$ be the maximal ideal of $\text{h}_{\text{cyc}}^n(\varepsilon, \varepsilon_0 ; W[[\Gamma_p]])$ given by $f_0$, and denote
by $T_{cyc}$ the localization $h_{cyc}^{\text{ord}}(N, \varepsilon; W[\Gamma_f])_m$. Then there is Galois representation (8, Proposition 3.49)

$$\rho_{T_{cyc}} : \text{Gal}(\overline{F}/F) \to GL_2(T_{cyc})$$

unramified outside $pN$ with the following properties:

1) $\text{tr}(\rho_{T_{cyc}} (\text{Frob}_\ell)) = \ell \mod pN$.
2) $\det \rho_{T_{cyc}} = \varepsilon_+N$.
3) For each $p|p$, there is a basis $\beta_{T_{cyc}}^i$ of $T_{cyc}^2$, such that the pair $(\rho_{T_{cyc}|G_{F_p}}, \beta_{T_{cyc}})$ is in $\Phi_{cyc,p}^\phi/(T_{cyc})$, where we take the character $\phi = \varepsilon_+N$.

Define $T_{cyc}^\beta := T_{cyc}([w_1, \ldots, w_{4r-1}])(w_{4i-1}, i = 1, \ldots, r)$ and define $\beta_{T_{cyc}}^i = \left(\begin{array}{cc} 1 + w_{4i-3} & w_{4i-2} \\ 0 & 1 + w_{4i} \end{array}\right)$.

Theorem 7.1. The surjective $W[[\Gamma_f]]$-algebra homomorphism $R_{cyc}^\beta \to T_{cyc}^\beta$ is an isomorphism. $R_{cyc}^\beta$ is free of finite rank over $W[[\Gamma_f]][[w_1, \ldots, w_{4r-1}])(w_{4i-1}, i = 1, \ldots, r)$. For any locally cyclotomic $(k, \varepsilon)$ such that

1) $k_2 - k_1 \geq 1$,
2) $-k_1, p \varepsilon_j(\varepsilon_p) = -k_0, p \varepsilon_j(\varepsilon_p)$,
3) $\varepsilon_+ = \varepsilon_0, +$ and $\varepsilon_j|_{\mathcal{O}_{\varepsilon_0}} = \varepsilon_0, j|_{\mathcal{O}_{\varepsilon_0}}$ for $j = 1, 2$,
4) $c(\varepsilon_0, j|_{\mathcal{O}_{\varepsilon_0}})(\varepsilon_j)|_{\mathcal{O}_{\varepsilon_0}}$ for $j = 1, 2$.

There is a unique local factor $T_{k,\varepsilon} \subset h^{\text{ord}}(N \cap \varepsilon(\varepsilon_-), \varepsilon; W)$ such that we have the isomorphisms $R_{cyc}^\beta/P_{k,\varepsilon} R_{cyc}^\beta \cong T_{k,\varepsilon}$ induced by $\pi$.

Proof. $R_{cyc}^\phi/p_{k,\varepsilon} R_{cyc}^\phi$ is the maximal quotient of $R_{cyc}^\phi$ on which $\delta_{p, u} \to N_{p}^{-k_1, p} \varepsilon_1, p(u)$ for $u \in \mathcal{O}_{\varepsilon}$. Thus $R_{cyc}^\phi/p_{k,\varepsilon} R_{cyc}^\phi$ is the universal framed deformation ring of $\varphi_{G_{F_p}}$ for the deformations of type $(k, \varepsilon)$ instead of $(k_0, \varepsilon_0)$. The previous proved proposition 5.1 states that $R_{cyc}^\phi/p_{k_0,\varepsilon_0} R_{cyc}^\phi \cong \mathbb{R} \cong T_{\phi}$. Write $a$ for the rank of $T$ over $W$, there is a surjection $W[[\Gamma_f]^a \to T_{cyc}$, which induces $W^a \cong T$ by modulo $P_{k_0,\varepsilon_0}$ by Nakayama’s lemma. Since $T_{cyc}$ is torsion free $W[[\Gamma_f]]$-module, the kernel must be trivial, and we have $T_{cyc} \cong W[[\Gamma_f]]^a$. As $R_{cyc}^\phi/p_{k,\varepsilon} R_{cyc}^\phi \cong T_{\phi}$, $R_{cyc}^\phi$ is generated by at most $a$ elements as a $W[[\Gamma_f]][[w_1, \ldots, w_{4r-1}])(w_{4i-1}, i = 1, \ldots, r)$-module. In addition, $R_{cyc}^\phi$ surjectively cover the rank $a$ free $W[[\Gamma_f]][[w_1, \ldots, w_{4r-1}])(w_{4i-1}, i = 1, \ldots, r)$ module $T_{cyc}$. We must have $R_{cyc}^\phi \cong T_{cyc}$.

8. Characteristic zero deformation and $L$-invariant

There are many different ways to define $L$-invariant, which is expected to measure the difference between $p$-adic $L$-function at an exceptional zero and the archimedean $L$-function. The $L$-invariant of an elliptic curve $E/\mathbb{Q}$ is studied by Mazur-Tate-Teitelbaum [19] and Greenberg-Stevens [7]. Hida studied the $L$-invariants of Tate curves [14, 15] and make a vast generalization of Mazur-Tate-Teitelbaum conjecture to the symmetric powers of Galois representations associated to Hilbert modular forms [16], which gave explicit predictions of the $L$-invariants of these representations. Hida proved certain cases of this conjecture, assuming a conjectural shape of certain universal deformation rings over characteristic zero fields.

Let $K$ be the fraction field of $W$ and $\rho : G_{F,S} \to GL_2(W) \to GL_2(K)$ is the representation associated to some nearly ordinary Hilbert modular form $f$. We further assume that $\epsilon_p \neq \delta_p$ for all $p|p$, i.e., the two characters $\epsilon_p$ and $\delta_p$ of $G_{F_p}$ are distinct over $W$ for each $p$, but become equal after modulo $m_W$.
Conjecture 8.1. (Hida, Hida, [6]) The ring \( R_K \) is isomorphic to \( K[[t_p]] \). 

In the case of representations associated to nearly ordinary Hilbert modular forms with Distinguished-

\( w \) and \( w \), we have obtained a homomorphism \( \delta' : R_{\text{cyc}} \rightarrow A \) corresponding to the deformation \((\rho_A, \beta_1^A, \ldots, \beta_r^A)\) such that the kernel of the homomorphism \( \delta' : R_{\text{cyc}} \rightarrow A \) corresponding to the deformation \((\rho_A, \beta_1^A, \ldots, \beta_r^A)\) contains the ideal \((w_1, w_2, \ldots, w_{4r-1})\). This can be done by taking \( \beta_i^A = \left( 1 + \theta(w_{4i-3}) \theta(w_{4i-2}) \right)^{-1} \cdot \beta_i^A \).

As we have obtained a homomorphism \( T_{\text{cyc}} : T \rightarrow W \), the mapping \( \theta_0 : R_{\text{cyc}} \rightarrow W \) sends \( w_1, \ldots, w_{4r-1} \) to zero. Denote \( P \) the kernel of \( \theta_0 \), we can see \( w_i \)'s and \( (\gamma - 1) \) for \( \gamma \in \Gamma_F \) are in \( P \). The localization completion \( \hat{R}^\gamma_p := \lim_{\rightarrow} (R_{\text{cyc}}^\gamma) / P^n \) is a pro-Artinian local K-algebra and \( \rho_p : G_F \rightarrow GL_2(\hat{R}^\gamma_p) \) is obviously in the set \( \Phi_K(\hat{R}^\gamma_p) \). There is a \( W[[T]] \)-algebra homomorphism \( \vartheta : R_K \rightarrow \hat{R}^\gamma_p \) by the universality. To describe the image of this map, we have the following theorem.

Theorem 8.2. Let \( \pi \) be the projection \( \hat{R}^\gamma_p \rightarrow \hat{R}^\gamma_p/(w_1, \ldots, w_{4r-1}) \). The composition \( \pi \circ \vartheta : R_K \rightarrow \hat{R}^\gamma_p/(w_1, \ldots, w_{4r-1}) \) is an isomorphism.

Proof. It's well known that the ring \( R_{F,S} \) and \( R_K \) are topologically generated by the trace of the universal Galois representations, for example, page 244 of [6]. The framed ring \( R_{\text{cyc}}^\gamma \), and hence the completion localization \( \hat{R}^\gamma_p \) are then generated by the trace of image the universal representation, together with the images of \( w_i \)'s. In particular, the quotient \( \hat{R}^\gamma_p/(w_1, \ldots, w_{4r-1}) \) is generated only by the trace of \( \rho_p \) too, and \( \pi \circ \vartheta \) is surjective. We prove the ring \( \hat{R}^\gamma_p/(w_1, \ldots, w_{4r-1}) \) represents the functor \( \Phi^K_F \) directly.

For a local artinian K-algebra \( A \) and a representation \( \rho_A \in \Phi^K_F(A) \), there is a W-lattice \( L \) in the finite dimensional K-vector space \( A^2 \) stable under \( \rho_A(G_F) \). The W-algebra \( A_0 = A \cap \End_{\mathbb{Q}}(L) \) is compact and contains the trace of the image of \( \rho_A \). \( A_0 \) is a local W-algebra free of finite rank over \( W \) with maximal ideal \( m_{A_0} = m_A \cap A_0 \). One can construct a representation \( \rho_A : G_F \rightarrow GL_2(\mathbb{Q}_p) \) by means of pseudo representation which, after composing with the inclusion \( A_0 \hookrightarrow A \), is isomorphic to \( \rho_A \). For each \( p \mid \rho \), the two distinct characters \( \epsilon_p \) and \( \delta_p \) having values in \( A_0 \), so the local representation \( \rho_{A_0} \mid G_{p_0} \) is isomorphic to a representation into upper-triangular matrices over \( GL_2(A_0) \). The reduction of \( \rho_{A_0} \) modulo \( m_{A_0} \) is
isomorphic to $\rho_0$. Choose a basis $\beta^j_{A_0}$ such that $(\rho_{A_0}|_{G_{F_p}}, \beta^j_{A_0})$ is in the set $\Phi_{cyc,p}^\delta, \phi, s$. The universaliy gives a homomorphism $\theta_0 : R_{cyc}^{\delta} \to A_0$ corresponding to the pair $(\rho_{A_0}, \beta^1_{A_0}, \cdots, \beta^r_{A_0})$. We can further assume the $\theta_0$ maps $w_i$ to 0 by the remark before this theorem. The composition $R_{cyc}^{\delta} \to A_0 \to A$, which we again denoted by $\theta_0$, factors through the localization completion $\widehat{R}_p^{\delta}$, and further factor $\widehat{R}_p^{\delta}/(w_1, \cdots, w_{4r-1})$. $\widehat{R}_p^{\delta}/(w_1, \cdots, w_{4r-1})$ must be universal, and $\pi \circ \theta$ is an isomorphism.

**Theorem 8.3.** If we normalize the isomorphism $W[[\Gamma_F]] \cong W[[X_p]]_{p/p}$ sending the generators $\gamma_p$ to $1+X_p$. Then Hida’s conjecture hold, i.e, there is an isomorphism $R_{K'} \cong K[t_p]_{p/p}$ such that $t_p = X_p - p$.

Proof. By the above theorem, we have the isomorphisms $R_K \cong \widehat{R}_p^{\delta}/(w_1, \cdots, w_{4r-1}) \cong \widehat{T}_{cyc,p}$, where $P_0$ is the image of $P/(w_1, \cdots, w_{4r-1})$ under the isomorphism $R_{cyc}^{\delta}$ and $\widehat{T}_{cyc,p} := \lim_{\leftarrow} (\widehat{T}_{cyc,p_0}/P_0^r)$ is the localization completion of $T_{cyc}$ at $P_0$.

The Hecke ring $T_{cyc}/P_0 T_{cyc} = T$, the local Hecke algebra for forms of weight $k_0 = (0, I)$ and Neben $\varepsilon_0$, is reduced, since the level $N$ is square free. $T \otimes W K$ is reduced and so unramified and étale over $K$. By Proposition 3.8 of chapter I in [20], $T_{cyc}$ is étale over $W[[\Gamma]]$ in an open neighborhood of $P_0$. The pull back of $P_0$ to $W[[\Gamma]]$ is the prime ideal $(X_p - p)_{p/p}$. The étaleness implies that $\widehat{T}_{cyc,p}$ coincide to the completion localization of $W[[\Gamma]]$ at $(x_p - p)_{p/p}$ (see Theorem 4.2 in the chapter I of [20]), which is isomorphic to $R_K \cong K[t_p]_{p/p}$ via the map $t_p = X_p - p$.

Ordered the primes $p|p$ such that $\rho|_{G_{F_p}} \cong \begin{pmatrix} w & \xi_{p,i} \\ 0 & 1 \end{pmatrix}$, $\delta_p$, for $i \leq b$ and $\epsilon_p/\delta_p, \not\equiv \omega$ for $b < i \leq r$. A cocycle $\xi_q : \text{Gal}(\overline{F_p}/F_p) \to K(1)$ labelled by $q \in F_p^\times$ is given by $\xi_q = \lim_{\leftarrow} \xi_q,n$ for $\xi_q,n(\sigma) = (q^n/q^*)^{\sigma-1}$. Put $Q_j = N_{F_p/q_0}(q_j), F_i = F_p$, and $\delta_{R,K,i} = \delta_{R,K,p,i}$. Theorem 8.3 have the following corollary, by Hida’s Theorem 0.3 in [10].

**Corollary 8.4.** The Greenberg $L$-invariant of $\text{Ind}^G_{\Lambda}(ad^0(\rho))$ is given by

\[
\det \left( \frac{\partial \delta_{R,K,i}(p,F_i)}{\partial t_j} \right)_{i,j \geq b \mid t_1 = \cdots = t_r = \prod_{i \geq b} \frac{\log_p(\xi_i)}{\delta_i(p,F_i)} \prod_{i \leq b} \frac{\log_p(Q_i)}{\delta(p,F_i)}
\]

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