QUANTIZATION OF A PARTICLE IN A BACKGROUND YANG-MILLS FIELD

YIHREN WU

Department of Mathematics, Hofstra University, Hempstead, NY 11550

Abstract

Two classes of observables defined on the phase space of a particle are quantized, and the effects of the Yang-Mills field are discussed in the context of geometric quantization.

PACS: 03.65.Bz, 02.40.Vh
Short title: Particle in background Yang-Mills field.
I. Introduction.

Let $Q$ be a Riemannian manifold considered as the configuration space of a particle, the purpose of this paper is to discuss the quantization of the observables on the phase space $T^*Q$ of this particle when it is moving under the influence of a background Yang-Mills field, the Yang-Mills potential is a connection $\alpha$ on a principal bundle $N$ over $Q$.

The free $G$–action on $N$ can be lifted to a Hamiltonian $G$–action on $T^*N$ with an equivariant moment map $J : T^*N \rightarrow g^*$. Let $\mu \in g^*$, and denote by $O_\mu$ the coadjoint orbit through $\mu$. Then $J^{-1}(O_\mu)/G$ has a canonical symplectic structure given by the Marsden-Weinstein reduction [1]. This reduced phase space is the appropriate phase space of a particle in a background Yang-Mills field $\alpha$ of charge $\mu$ [2].

We will denote by $Q(X)$ the quantization of the phase space $X$, suppressing in our notation the choices of polarizations and pre-quantization line bundles etc, via the standard procedure of geometric quantization [3, 4]. Suppose we choose the vertical polarization on $T^*N$ so that the quantization $Q(T^*N)$ of $T^*N$ gives $L^2(N)$. Moreover, suppose the co-adjoint orbit $O_\mu$ is integral, so that the quantization of this coadjoint orbit gives a irreducible representation space $H_\mu$ of $G$ [5] [6]. A theorem of Guillemin-Sternberg [7] (see also [8]) then suggests that the quantization of $J^{-1}(O_\mu)/G$ is given by $\text{Hom}_G(H_\mu, L^2(N))$, the space of $G$–equivariant linear maps from $H_\mu$ to $L^2(N)$. And this result holds independent of whether there is a Yang-Mills field present in the background. Thus when some technical assumptions are made so that the procedure of geometric quantization can be carried out smoothly, the Yang-Mills field plays no role in the quantization of the phase space $J^{-1}(O_\mu)/G$.

We will discuss the effect of the Yang-Mills field in quantizing observables that are lifted from functions on $T^*Q$. We will show that the resulting quantum operators are expressed in terms of the covariant derivatives, which is defined by the connection $\alpha$. In particular, we will show that the quantum operators for $f$ of the form $\frac{1}{2}||p||^2 + V(q)$ are expressed in terms of the covariant Laplace operator and the Ricci curvature. This is obtained by a standard Blattner-Kostant-Sternberg (BKS) pairing approach [9]. Our results are in agreement with those of Landsman [10] who arrived at the conclusion via deformation quantization.

Outline of this paper is as follows; In section 2 we give a detailed exposition of our problem in order to standardize the notations used throughout this paper. As a prelude to our result, we note that if the gauge group is abelian, we will recover the Dirac quantization of a charge particle in the presence of an electro-magnetic field. In section 3 we introduce local coordinates to facilitate our calculation, and state some results concerning the Hamiltonian vector fields for our observables. We follow closely the treatment of [8] on the polarization chosen for the phase space in section 4. Our results when $f$ is polarization preserving are given in section 5, and the quantization of $\frac{1}{2}||p||^2 + V(q)$ using BKS pairing appears in section 6.

II. Preliminary discussions.

Let $N$ be a principal $G$–bundle over $Q$ where $G$ is compact, with Lie algebra $\mathfrak{g}$, and the group action is on the right. We define two functions $R_g : N \rightarrow N$ and $\hat{n} : G \rightarrow N$, where

$$R_g(n) = \hat{n}(g) = ng,$$

and we denote by $F$ the Jacobian of $F$. A connection is a linear map $\alpha(n) : T_n N \rightarrow \mathfrak{g}$,
\( g \) for each \( n \in N \) satisfying

\begin{align*}
   i. \quad & \text{Ad}_{g^{-1}} \alpha(n) = \alpha(ng)R_{g^{-1}} : T_n N \to g, \\
   ii. \quad & \alpha(n) \hat{n}_* = \text{Id} : g \to g.
\end{align*}

The free \( G \)-action on \( N \) can be lifted to a Hamiltonian \( G \)-action on \( T^*N \) with moment map \( J : T^*N \to g^* \) given by \( J(\xi, n) = \hat{n}_* \cdot \xi \).

Let \( N^\# \) be the pullback bundle over \( T^*Q \): Explicitly,

\[ N^\# = \{ (p, n) \mid p \in T^*_q Q \text{ where } q = \pi(n) \}. \]

Define a diffeomorphism

\[ \chi : N^\# \times g^* \to T^*N, \quad (p, n, \mu) \mapsto (\xi, n) \quad \text{where} \quad \xi = \pi^*(p) + \langle \mu, \alpha \rangle \in T^*_n N. \]

This map in turn induces an \( \alpha \)-dependent projection \( \pi_\alpha : T^*N \to T^*Q \), and their corresponding symplectic forms are related by

\[ \Omega_{T^*N} = \pi^*_\alpha \Omega_{T^*Q} + d\langle \mu, \alpha \rangle. \tag{3} \]

One shows that the moment map is simply the projection \( N^\# \times g^* \to g^* \), i.e., \( \chi^{-1}(J^{-1}(\mu)) = N^\# \times \{ \mu \} \). Thus for each \( \mu \), the \( G \)-action on \( N^\# \) induces a \( G \)-action on \( J^{-1}(\mu) \). This action is non-canonical (\( R^*_g \Omega_{T^*N} \neq \Omega_{T^*N} \)) in general:

\[ (\xi, n) \mapsto (\xi_g, ng) \quad \text{where} \quad \xi_g = R^*_{g^{-1}} \xi + R^*_{g^{-1}} [(\text{Ad}_{g^{-1}} - \text{Id})\mu] \alpha(n). \]

However, this action coincides with the canonical \( G \)-action when restricted to the isotropy subgroup

\[ H = \{ g \in G \mid \text{Ad}_g \mu = \mu \} \]

of \( \mu \). (There \( \text{Ad}_{g^{-1}} \mu = \mu \) and \( \xi_g = R^*_{g^{-1}} \xi \).) The relevant phase space becomes \( J^{-1}(\mathcal{O}_\mu)/G = J^{-1}(\mu)/H = N^\#/H \times \{ \mu \} \).

Given an observable on the phase space of the particle \( f : T^*Q \to \mathbb{R} \), by the projection \( \pi_\alpha \), the pullback map, which we continue to denote by \( f \),

\[ f = \pi^*_\alpha f : T^*N \simeq N^\# \times g^* \longrightarrow \mathbb{R}, \tag{4} \]

is invariant with respect to both the canonical \( G \)-action and the non-canonical one. In particular, \( f \) is independent on the charge variables \( \mu \in g^* \).

We assume that \( \mathcal{O}_\mu \) is integral, so that \( Q(\mathcal{O}_\mu) = \mathcal{H}_\mu \) is an irreducible representation space of \( G \) induced by \( \rho_\mu : H \to U(1) \). Choose an orthonormal basis \( \{ \phi_i \} \) for \( \mathcal{H}_\mu \), where \( \phi_i \) is a holomorphic function on the Kähler manifold \( G/H \) [5]. Then \( \Psi \in Q(J^{-1}(\mu)/H) = \text{Hom}_G(\mathcal{H}_\mu, L^2(N)) \) is determined by \( \Psi(\phi_i) = \Psi_i \in L^2(N) \).

Using orthonormality of \( \phi_i \) and \( G \)-equivariance of \( \Psi \), we write

\[ \Psi = \sum \Psi_i \phi_i : N \times G/H \to N \times_G G/H \to \mathbb{C} \]

which is uniquely determined by \( \psi = \Psi(\cdot, eH) : N \to \mathbb{C} \) with the condition \( \psi(\phi_h) = e^{i(H-\bar{H})/2} \psi(\phi) \) for all \( h \in H \). So we see that the Yang-Mills potential plays...
no role in quantizing the relevant phase space, it simply picks up the multiplicity of the charge sector \( \mathcal{O}_\mu \) in \( L^2(N) \) (cf. [7], [8]).

If we are to quantize an observable that is a pullback of \( f \) on the phase space of the particle \( T^*Q \), the connection \( \alpha \) plays an important role. As an illustration, suppose the charge \( \mu \in \mathfrak{g}^* \) is \( G \)-invariant, then \( H = G \). This will be the case if for instance \( G \) is abelian. Under this condition, \( J^{-1}(\mu)/H \) is diffeomorphic to \( T^*Q \) via the projection \( \pi_\alpha \), and the canonical symplectic form on the reduction space pushes forward onto \( T^*Q \). So \( J^{-1}(\mu)/H \) is symplectomorphic to \( T^*Q \) if \( T^*Q \) is equipped with the “effective” symplectic form \( \Omega_{\text{eff}} = \Omega_{T^*Q} + \langle \mu, \Omega_\nabla \rangle \) where \( \Omega_\nabla \) is a two-form on \( Q \) which pulls back to the curvature form \( d\alpha \) on \( N \) [11]. It is with respect to this effective symplectic form that the quantization procedure must be carried out. Since the adjustment is a two form on \( Q \), quantization of \( T^*Q \) using the vertical polarization still gives \( L^2(Q) \). However, Hamiltonian vector fields \( \mathcal{H}_f \), associated with observables \( f \) and the Poisson bracket are defined in terms of the effective form:

\[
\Omega_{\text{eff}}(\mathcal{H}_f, -) = -df, \quad \{f_1, f_2\} = \Omega_{\text{eff}}(\mathcal{H}_{f_1}, \mathcal{H}_{f_2}).
\]

The quantization of observables must preserve Poisson bracket

\[
[\mathcal{Q}(f_1), \mathcal{Q}(f_2)] = i\hbar \mathcal{Q}(\{f_1, f_2\}).
\]

When carry out the geometric quantization with respect to \( \Omega_{\text{eff}} \), the result is the Dirac quantization with \( \alpha \) as the vector potential associated with a electro-magnetic field (cf. [3]).

### III. The Hamiltonian vector fields.

Let us first introduce local canonical coordinates \((\xi_i, n_i)\) on \( T^*N \) so that \( \Omega_{T^*N} = d\xi_i dn_i \), similarly \((p_a, q_a)\) on \( T^*Q \) with \( \Omega_{T^*Q} = dp_a dq_a \), and let \( \alpha = A_s dn_i \) where \( i = 1 \ldots \dim N, a = 1 \ldots \dim Q, s = 1 \ldots \dim G \), and repeated indices are summed. For each \( n \in N \), let us denote the horizontal lift \( T^*_nQ \rightarrow T_nN \) by the matrix \( M_{ia}(n) \), \( i = 1 \ldots \dim N, \sigma = 1 \ldots \dim Q \). We have

\[
\frac{\partial q_a}{\partial n_i} M_{ib} = \delta_{ab}
\]

and the covariant derivative is the horizontal lift of \( \frac{\partial}{\partial q_a} \):

\[
D_a = M_{ia} \frac{\partial}{\partial n_i}
\]

In these coordinates, the canonical one-form and the symplectic two-form of \( T^*N \) can be calculated using (3)

\[
\chi^*\xi_i dn_i = \left( p_a \frac{\partial q_a}{\partial n_i} + \mu_s A_s i \right) dn_i
\]

\[
\chi^*\Omega_{T^*N} = \frac{\partial q_a}{\partial n_i} dp_a dn_i + \mu_s \frac{\partial A_s i}{\partial n_j} dn_j dn_i + A_s i d\mu_s dn_i.
\]

Let \( f : N^\# \times \mathfrak{g}^* \rightarrow \mathbb{R} \) be a pullback function from \( T^*Q \), and let \( \mathcal{H}_f \) be its Hamiltonian vector field

\[
\mathcal{H}_f = B_a \frac{\partial}{\partial q_a} + C_i \frac{\partial}{\partial n_i} + U_s \frac{\partial}{\partial \mu_s}.
\]
Using $\chi^*\Omega_{T^*N}(\mathcal{H}_f, -) = -df$ we get

$$
\frac{\partial f}{\partial q_a} \frac{\partial q_a}{\partial n_i} = -B_a \frac{\partial q_a}{\partial n_i} + \mu_s \left( \frac{\partial A_{aj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j - A_{si} U_s,
$$

$$
\frac{\partial f}{\partial p_a} = \frac{\partial q_a}{\partial n_i} C_i,
$$

$$
\frac{\partial f}{\partial \mu_s} = A_{si} C_i.
$$

Since $f$ is invariant with respect to the canonical $G$–action, $\mathcal{H}_f$ is tangent to the subspace $J^{-1}(\mu) \cong N^\# \times \{\mu\}$, thus $U_s = 0$. As remarked after (4), $f$ is independent of $\mu$, thus $A_{si} C_i = 0$, which implies $\mathcal{H}_f$ is horizontal. Moreover, letting $B_a = -\frac{\partial f}{\partial q_a} + E_a$, we have

$$
\mathcal{H}_f = \left[ -\frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + C_i \frac{\partial}{\partial n_i} \right] + E_a \frac{\partial}{\partial p_a}.
$$

The terms in the bracket is the horizontal lift of the Hamiltonian vector field of $f$ with respect to the usual symplectic form $\Omega_{T^*Q}$ on $T^*Q$. $E_a$ satisfies

$$
E_a \frac{\partial q_a}{\partial n_i} = \mu_s \left( \frac{\partial A_{aj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j.
$$

So we summarize the properties of $\mathcal{H}_f$ needed for our purpose.

**Proposition 1.** If $f : T^*N \to \mathbb{R}$ is a pullback of a function on $T^*Q$, then for all $\mu \in \mathfrak{g}$ the Hamiltonian vector field $\mathcal{H}_f$ is a vector field on the subspace $J^{-1}(\mu)$, and as the total space of a principal bundle over $T^*Q$ with connection $\alpha$, $\mathcal{H}_f$ is horizontal. If differs from the horizontal lift of the standard Hamiltonian vector field on $T^*Q$ by a field in the vertical direction. With respect to the local coordinates chosen, we have explicitly:

$$
\mathcal{H}_f = \left[ -\frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + M_{ia} \frac{\partial f}{\partial p_a} \frac{\partial}{\partial n_i} \right] + \mu_s M_{ia} M_{jb} \left( \frac{\partial A_{aj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) \frac{\partial f}{\partial p_b} \frac{\partial}{\partial p_a}.
$$

Furthermore, we have

$$
\langle \chi^* \xi_i dn_i, \mathcal{H}_f \rangle = p_a \frac{\partial f}{\partial p_a}.
$$

**IV. Polarization.**

We first state some well known results concerning the quantization of integral coadjoint orbits $O_\mu$. Let $\mathfrak{h}$ be the Lie algebra of $H$, we say that $O_\mu$ is integral if the map

$$
v \in \mathfrak{h} \to 2\pi i \langle v, \mu \rangle
$$

is the derivative of a global character, i.e., there is a group homomorphism $\rho_\mu : H \to U(1)$ such that $\rho_{\mu*}$ is the map given in (11). A version of the Borel-Weil theorem, due to Kirillov [5] and Kostant [6] asserts that there is a one-to-one correspondence between the integral orbits of $C$ and its unitary irreducible representations, and

$$
\langle \chi^* \xi_i dn_i, \mathcal{H}_f \rangle = p_a \frac{\partial f}{\partial p_a}.
$$
these representations can be construction by the method of geometric quantization applied to the coadjoint orbit $O_\mu$, which we will briefly explain.

Let $L$ be the prequantization line bundle $G \times_{\rho_\mu} \mathbb{C}$ over $O_\mu \simeq G/H$ with connection induced by the map $\rho_\mu$. It is known that $O_\mu$ is a Kähler manifold with complex coordinates with respect to which the $G-$action is holomorphic. There is a standard $G-$equivariant polarization quantizing with respect to the line bundle and this polarization gives $\mathcal{H}_\mu$ whose elements are holomorphic functions on $O_\mu$. The polarization, known as the positive Kähler polarization, is given by left translation of a set of $v_k \in g \otimes \mathbb{C}$, the complexification of $g$, so that the polarization is generated by

$$V_k(\text{Ad}_g \mu) = g_* v_k \in T_{\text{Ad}_g \mu} O_\mu.$$  \hspace{1cm} (12)

As a polarization, $v_k$ thought of as vectors in $T_\mu O_\mu$ satisfies

$$\Omega_{O_\mu}(v_h, v_k) = 0$$ \hspace{1cm} (13)

where $\Omega_{O_\mu}$ is the canonical symplectic form on $O_\mu$. The specifics of the choices of $v_k$ will not be important in what follows. It is worth mentioning that $V_k$ is contained in the vertical polarization on $T^*N$.

Consider the complex distribution on $N^\# \times O_\mu$ generated by $\{ \frac{\partial}{\partial p_a}, V_k \}$, it is $G-$equivariant thus projects onto $N^\# \times_G O_\mu \simeq N^\# / H \times \{ \mu \}$. One checks that the image is a polarization which we denote by $\mathcal{P}$ \[8\].

It is easy to represent $\mathcal{P}$ in local coordinates on $N^\#; \frac{\partial}{\partial p_a}$ are vector fields on $N^\#$, and $V_k$ corresponds to $\hat{n}_* v_k$ as complex vector fields on $N$, $\hat{n}_*$ as in (1). This is so since the assignment $(p, n, \text{Ad}_g \mu) \mapsto (p, ng, \mu)$ defines the projection $N^\# \times O_\mu \rightarrow N^\# \times_G O_\mu \simeq N^\# / H \times \{ \mu \}$. Thus vector field generated by $G-$action on $O_\mu$ translates to vector field generated by $G-$action on $N$.

The Hilbert space structure on $L^2(N)$ is given by integration with respect to the measure $dn = d\sigma \sqrt{\text{det}(g)} \, dq$, where $d\sigma$ is a Haar measure on $G$ which we transfer to a measure on the fiber in the projection $N \rightarrow Q$ and $g$ is the metric on $Q$. Using the half-form bundle formalism the wavefunctions are of the form $\psi(n) \sqrt{dn}$. It is clear the Haar measure will play no role in our consideration as the polarization and all Hamiltonians in question are $G-$invariant. To keep the half-form bundle formalism to a minimum, we may identify the wavefunctions as $\psi(n) \det g^{1/4}$. We will determine explicitly the differential operators corresponding to $f$ so that

$$\psi(n) \det g^{1/4} \mapsto [Q(f) \psi(n)] \det g^{1/4}. \hspace{1cm} (14)$$

In quantizing $f$ that is linear in the momentum variables, the $\det g^{1/4}$ term will give rise to the covariant divergence, and for $f = \frac{1}{2} ||p||^2 + V(q)$, it results in the Ricci curvature. The appearance of the Ricci curvature is also reported in [4].

V. Polarization preserving case.

If $f : T^*Q \rightarrow \mathbb{R}$ is linear in $p$, $f = K_a(q)p_a$, then one easily checks, using (9), that $\exp t \mathcal{H}_f \mathcal{P} = \mathcal{P}$. In fact, we have

\textbf{Proposition 2.}

$$\left[ \mathcal{H}_f, \frac{\partial}{\partial p_a} \right] = \frac{\partial K_a}{\partial q_b} \frac{\partial}{\partial p_b}, \hspace{1cm} (15)$$

$$\left[ \mathcal{H}_f, V_k \right] \in \text{span} \left\{ \frac{\partial}{\partial a} \right\}. \hspace{1cm} (16)$$
where the brackets refer to the Lie algebra bracket on vector fields.

Proof. Equation (15) is by direct computation. We have

\[ \mathcal{H}_f = \left( E_b - p_c \frac{\partial K_c}{\partial q_b} \right) \frac{\partial}{\partial p_b} + K_b M_{ib} \frac{\partial}{\partial n_i} \]  

(17)

where \( E_b \) is independent of \( p_a \), and (15) results.

To show (16), we first realize from Proposition 1 that \( \mathcal{H}_f = W_1 + W_2 \) where \( W_1 \) is the horizontal lift of a vector field on \( T^*Q \), thus \([W_1, V_h] = 0\) as \( V_h \) is generated by the group action on \( N \). \( W_2 \) is of the form \( F_a \frac{\partial}{\partial p_a} \). Since \( V_h \) is independent of \( p \), \([W_2, V_h] = V_h (F_a) \frac{\partial}{\partial p_a} \), where \( V_h (F_a) \) refers to applying the vector field as a differential operator to the coefficient function \( F_a \). □

The importance of (16) is that \([\mathcal{H}_f, V_h]\) is a combination of vectors fields in \( P \) which does not involve the \( V_h \) vector fields. According to (7.12) of [3], the quantization of \( f \) is then given by:

\[ Q(f)\psi = -\frac{i\hbar}{\det g^{1/4}} \left[ \mathcal{H}_f(\psi(n) \det g^{1/4}) + \frac{1}{2} \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a} (\psi(n) \det g^{1/4}) \right] \]  

(18)

The Hamiltonian vector field \( \mathcal{H}_f \) projects to a vector field \( V^\# \) on \( N \), which is the horizontal lift to the projection of \( \mathcal{H}_f \) onto \( Q \) where \( V = K_b \frac{\partial}{\partial q_b} \). Note that \( \mathcal{H}_f(\sqrt{\det g}) = V(\sqrt{\det g}) \). The divergence of the vector field \( V \) on \( Q \) is defined [12] through the relation

\[ d \ast V = \text{div} V \sqrt{\det g} dq \]

The covariant divergence on \( N \) is defined as the divergence of the horizontal lift \( V^\# \). We have

\[ \text{div} V = \frac{1}{\sqrt{\det g}} V(\sqrt{\det g}) + \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a} \]  

Using (17), (18) and the fact that \( \psi \) is independent of \( p \), we have

**Proposition 3.** \( Q(f)\psi = -i\hbar (K_a D_a + \frac{1}{2} \text{div} V)\psi \) where \( D \) is the covariant derivative with respect to the connection \( \alpha \).

Since \( \det g \) is a function of \( n \) through \( q \), the divergence and the covariant divergence are the same.

VI. BKS pairing case.

Let \( P \) and \( P' \) be transversal polarizations on \( T^*N \), denote their associated quantum spaces by \( Q \) and \( Q' \). The BKS pairing gives rise to a map \( B : Q' \to Q \) such that

\[ \langle B(\psi), \phi \rangle = \int \psi \bar{\phi} (\det \omega)^{1/2} d\ell \]
where $d\ell$ is the Liouville form $d\xi_1 \ldots d\xi_n \, dn_1 \ldots dn_n$. Since the volume form on $N$ is $dn = \sqrt{\det g} \, dn_1 \ldots dn_n$, we have

$$B\psi(n) = \frac{1}{\sqrt{\det g}} \int_{T^*_N} \psi(\det \omega)^{1/2} d\xi_1 \ldots d\xi_n$$  \hspace{1cm} (19)

Let $f = \frac{1}{2} g^{ab} p_a p_b + V(q)$, where $g^{ab}$ is the inverse of the metric $g_{ab}$. From (9) we have

$$\mathcal{H}_f = - \left( \frac{\partial V}{\partial q_a} + \frac{1}{2} g^{bc} \frac{\partial g_{pc}}{\partial p_a} \right) \frac{\partial}{\partial p_a} + M_{ia} g^{ab} p_b \frac{\partial}{\partial n_i}$$

$$+ \mu_s M_{ia} M_{jb} \left( \frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) g^{bc} p_c \frac{\partial}{\partial p_a},$$

and note the linear dependence of the coefficients on the $p$ variables. We denote by $\mathcal{P}_t = \exp t \mathcal{H}_f, \mathcal{P}$, here the two polarizations $\mathcal{P}$ and $\mathcal{P}_t$ do not intersect transversely. We claim

**Proposition 4.** Vector fields generated by the group action are in $\mathcal{P} \cap \mathcal{P}_t$.

**Proof.** It suffices to show that

$$\Omega(\exp t \mathcal{H}_f, V_k, \frac{\partial}{\partial p_a}) = 0$$  \hspace{1cm} (20)

and

$$\Omega(\exp t \mathcal{H}_f, V_k, V_h) = 0$$  \hspace{1cm} (21)

with $\Omega$ as in (7). Let $\tilde{p}(p, n, t)$ and $\tilde{n}(p, n, t)$ denote the flow generated by $\exp t \mathcal{H}_f$ at $(p, n) \in N^\#$ with $\mu$ fixed, $(\tilde{p}, \tilde{n}, \mu) = \exp t \mathcal{H}_f(p, n, \mu)$. Then $\exp t \mathcal{H}_f, V_k = V_k(\tilde{p}_0) \frac{\partial}{\partial p_a} + V_k(\tilde{n}_0) \frac{\partial}{\partial n_i}$. So $\Omega(\exp t \mathcal{H}_f, V_k, \frac{\partial}{\partial n_i}) = \frac{\partial q_a}{\partial n_i} V_k(\tilde{n}_i)$. Recall $V_k = \tilde{n}_* v_k$ is vertical and $\frac{\partial q_a}{\partial n_i}$ is the Jacobian of the projection $\Pi : N \to Q$. Then $\tilde{n}_* \tilde{n}_* = 0$ implies (20) holds.

Equation (21) follows from general principle; Since $f$ is $G$–invariant (with respect to the non-canonical $G$–action), $V_h$ is equivariant with respect to the flow: $\exp t \mathcal{H}_f, V_h(p, n) = V_h(\exp t \mathcal{H}_f(p, n))$. Since the flow of a Hamiltonian vector field preserves the symplectic form, we have

$$\Omega(\exp t \mathcal{H}_f, V_k(p, n), V_h(\exp t \mathcal{H}_f(p, n))) = \Omega(V_k, V_h) = \Omega_{O_\mu}(v_h, v_k) = 0,$$

where $v_h$ and $v_k$ belongs to a polarization on $O_\mu$ to begin with (13).

This being the case, quantization of $f$ via BKS pairing involves integrating only over the $p$ variables, i.e., the fiber coordinates of the projection $\Pi : N^\# \to N$. According to (7.20) of [3], together with the similarity transform (14) adjustment and the adjustment in the BKS pairing described in (19),

$$Q(f)\psi(n) = \frac{1}{\det g^{1/4} \det g^{1/4} i\hbar} \frac{d}{dt} \bigg|_{t=0} \Psi_t(n)$$  \hspace{1cm} (22)

where $\Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi^{-1}(n)} [\det \omega_{ab}]^{1/2} \exp(i\hbar^{-1} L) \Psi(p, n, t) \, dp,$

$$\Psi(p, n, t) = \psi(\tilde{n}(p, n, t)) \times [\det g(\tilde{n}(p, n, t))]^{1/4},$$

$$\omega_{ab} = \Omega \left( \frac{\partial}{\partial p_a}, \exp t \mathcal{H}_f \frac{\partial}{\partial p_b} \right),$$

$$L = t \left( \frac{1}{4} \| \cdot \|^2 + V(q) \right) - 2 \int^t V(\tilde{n}(p, n, s)) \, ds.$$  \hspace{1cm} (26)
The manipulation follows closely that of Sniatycki [3]. Making the substitution $x_a = tp_a$, we have results analogues to (7.26) and (7.27) of [3]:

**Proposition 5.**

$$\lim_{t \to 0^+} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right) = (2\pi \hbar)^{\dim Q/2} e^{\pi \text{sgn}(g)/4} \sqrt{\det g} \delta(x). \quad (27)$$

$$\frac{\partial}{\partial t} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right) = \frac{i\hbar}{2} g_{ab} \frac{\partial^2}{\partial x_a \partial x_b} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right). \quad (28)$$

**Proof.** The first equation follows from the method of stationary phase (cf. [13]), which for $n-$dimensional space reads

$$\int_{\mathbb{R}^n} a(y) e^{ik\phi(y)} dy = \left( \frac{2\pi}{k} \right)^{n/2} \sum_{y|d\phi(y)=0} e^{\pi \text{sgn}H(y)/4} \frac{e^{ik\phi(y)} a(y)}{\sqrt{|\det H(y)|}} + O(k^{-n/2-1}).$$

The $\frac{1}{t}$ factor in (27) plays the role of the large parameter $k$. $H$ is the Hessian of $\phi$ which in our case is $g^{\mu
u}$, thus $\det H = \det g^{-1}$ and sgn is the signature of the metric. The only stationary point in (27) is $x = 0$, thus the right hand side of (27) has a ($\dim Q-$dimensional) delta function at $x = 0$.

The second equation is a straightforward computation. We need the fact that $g_{a\mu} g^{\mu b} = \delta_{ab}$, and $\sum_a \sum_b g_{ab} g^{ab} = \sum_a \delta_{aa} = \dim Q$ in the course of the computation. \(\square\)

One checks that $\omega_{ab}$ in (29) is

$$\frac{\partial q_a}{\partial n_i} \frac{\partial \tilde{n}_i}{\partial p_b} = tg^{ab} + \text{higher order terms in } t$$

using (30) below. Then $[\det \omega_{ab}]^{1/2} dp \sim t^{-\dim Q/2} dx$, providing us with the needed factor to apply the results of Proposition 5. Thus in determining $\frac{d}{dt} \big|_{t=0} \Psi_t(n)$, we need only to consider terms involving $t, tp_a$ and $t^2p_ap_b$ while ignoring terms of the form $t^2p_a$ and all higher order terms. Using (9), the expansion of $\tilde{n}$, expressed in the $t$ and $x_a$ variables, up to the relevant terms are

$$\tilde{n}_i(n, x) = n_i + M_{ia} g^{\mu a} x_\mu + \frac{1}{2} \left[ M_{jb} \left( \frac{\partial M_{ia}}{\partial n_j} g^{\nu b} g^{\mu a} + M_{ia} \frac{\partial g^{\mu a}}{\partial q_b} g^{\nu b} - \frac{1}{2} M_{ia} \frac{\partial g^{\mu \nu}}{\partial q_b} g^{ba} \right) \right] x_\mu x_\nu \quad (30)$$

which is independent of $t$. And $\Psi_t(n)$ in (23) is reduced to

$$\Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi^{-1}(n)} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right) \Phi(n, x, t) dx,$$

where $\Phi(n, x, t) = \exp(-i\hbar^{-1} tV(q)) \times \psi(\tilde{n}(n, x)) \det g(\tilde{n}(n, x))^{1/4}$. \(\quad (31)\)
By applying Proposition 5, integration by parts yields
\[
Q(f)\psi(n) = \frac{(-2\pi i)^{\dim Q/2}e^{\pi i\text{sign}(g)/4}}{\det g^{1/4} \frac{\partial^2 \Phi}{\partial x_a \partial x_b} \bigg|_{x,t=0} + \frac{\partial g_{\mu\nu}}{\partial q_\alpha} \bigg|_{x,t=0} \Phi} (32)
\]

The \(\sqrt{\det H}\) term that appears in the method of stationary phase formula results in a \(g^{1/2}\) factor (27) that cancels with the \(\det g^{1/2}\) on the right hand side of (22).

Since \(n\) is fixed, we can choose a normal coordinate system [12] around \(q = \pi(n)\) so that \(\frac{\partial g_{\mu\nu}}{\partial q_\alpha} = 0\) for all \(\mu, \nu\) and \(q_\alpha\) when evaluated at \(n\), (i.e., at \(x = t = 0\)). A direct computation shows
\[
\frac{d}{dt} \bigg|_{x=t=0} \Phi(n, x, t) = -i\hbar V(q)\psi(n) \det g^{1/4} (33)
\]

Here we have made repeated use of the identity [12 p.302],
\[
\frac{d}{\partial q_\alpha} \det g = \frac{\partial g_{\mu\nu}}{\partial q_\alpha} g^{\mu\nu} = g^\mu_\alpha \partial_\mu q_\alpha.
\]

In normal coordinates, the covariant Laplace operator reduces to
\[
\Delta_\alpha \psi = \frac{1}{\det g^{1/2}} \partial_\mu (\det g^{1/2} g^{\mu\nu} \partial_\nu \psi) = g^\mu_\nu M_{i\mu} M_{j\nu} \frac{\partial^2 \psi}{\partial n_i \partial n_j} + g^\mu_\nu M_{i\mu} \frac{\partial M_{i\mu}}{\partial n_j}, (35)
\]

and the Ricci curvature becomes
\[
R = g^{ik} (\partial_k \Gamma^j_{ji} - \partial_j \Gamma^i_{ki} + \Gamma^k_{km} \Gamma^m_{ji} - \Gamma^j_{jm} \Gamma^m_{ki}) = (g^{ik} g^{\mu\nu} - g^{i\nu} g^{k\mu}) \partial_i \partial_k g_{\mu\nu} = \frac{3}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu}, (36)
\]

Here \(\Gamma^k_{ij}\) are the Christoffel symbols, and the identity
\[
\Gamma^m_{ij,k} + \Gamma^m_{jk,i} + \Gamma^m_{ki,j} = 0
\]

is used to show \(g^{i\nu} g^{k\mu} \partial_i \partial_k g_{\mu\nu} = -\frac{1}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu}\). We must caution the readers that these expressions only hold in normal coordinates. However, by combining (32–36) we can express our final result in an coordinate invariant form:

**Proposition 6.** Quantization of \(f = \frac{1}{2} ||p||^2 + V(q)\) gives
\[
Q(f)\psi(n) = \frac{(-2\pi i)^{\dim Q/2}e^{\pi i\text{sign}(g)/4}}{\Delta_\alpha + \frac{1}{6} R} + V(q) \psi(n).
\]

We conclude with a final remark. The Yang-Mills field is defined [14] as the curvature \(D\alpha\) of the Yang-Mills potential \(\alpha\), whereas the contribution of this connection in the local expression of the symplectic form is \(d\alpha\). They are related by
\[
D\alpha(v, w) = d\alpha(\text{hor} v, \text{hor} w)
\]
where hor denotes the horizontal projection. Since the vector fields of concern are all horizontal, the effect of \(d\alpha\) is equivalent to the curvature.
Acknowledgement. We wish to thank N.P. Landsman for helpful comments on the preliminary version of this work.
References

1. J. Marsden and A. Weinstein, Rep. Math. Phys. 5, 121(1974).
2. A. Weinstein, Lett. Math. Phys. 2, 417(1978).
3. J. Śniatycki, Geometric quantization and quantum mechanics, (Springer-Verlag, New York, 1980).
4. N.M.J. Woodhouse, Geometric Quantization, (Clarendon Press, Oxford, second edition, 1992).
5. A.A. Kirillov, Elements of the theory of representation, (Springer-Verlag, Berlin, 1976).
6. B. Kostant, Quantization and unitary representation. In: Modern analysis and applications, Lecture Notes in Math. 170 (1970), 87-207.
7. V.W. Guillemin and S. Sternberg, Invent. Math. 67, 515(1982).
8. M.A. Robson, J. Geom. Phys. 19, 207(1996).
9. R.J. Blattner, Proc. Symp. Pure Math. 26, 87(1973).
10. N.P. Landsman, J. Geom. Phys. 12, 93(1993).
11. M. Kummer, Indiana Univ. Math. J. 30, 281(1981).
12. Y. Choquet-Bruhat, C. de Witt-Morette and M. Dillard-Bleick, Analysis, manifolds and physics, (North-Holland, Amsterdam, 1977).
13. V.W. Guillemin and S. Sternberg, Geometric asymptotics, (Amer.Math.Soc., Providence, 1977).
14. T.T. Wu and C.N. Yang, Phys. Rev. D 12, 3845(1975).