The momentum analyticity of two point correlators from perturbation theory and AdS/CFT

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Abstract: The momentum plane analyticity of two point function of a relativistic thermal field theory at zero chemical potential is explored. A general principle regarding the location of the singularities is extracted. In the case of the $N = 4$ supersymmetric Yang-Mills theory at large $N_c$, a qualitative change in the nature of the singularity (branch points versus simple poles) from the weak coupling regime to the strong coupling regime is observed with the aid of the AdS/CFT correspondence.

Keywords: AdS/CFT, analyticity, Super Yang-Mills
1. Introduction

Analyticity of the Green’s functions with respect to energy and momentum is an important property of a quantum field theory. Extensive investigations have been made mostly on a complex energy plane either at zero temperature or at nonzero temperature. The location of the singularities is dictated by the unitarity and the causality of the underlying field theory and the nature of the singularities reflects the character of the excitation spectrum with poles associated to bound states and the branch cuts to the continuum spectrum of asymptotic states. By means of the techniques of complex analysis, general relation between different physical observables (such as Kramers-Kronig relation) can be extracted without resorting to a perturbative expansion.

The analyticity of a Green’s function with respect to momentum variables is relevant to the spatial correlation of different operators and static inter-particle potentials. It is less explored in literature, probably because of the complication of vector character of momenta and the lack of general guidelines. For a two point Green’s function of a relativistic field theory, however, simple properties may be deduced even at a nonzero temperature and this is the main issue addressed in this report. Our results are two folds: One concerns the location of the singularity on the complex momentum plane of a two point Green’s function at a nonzero temperature but zero chemical potential. Our statement applies to any relativistic field theory. The other is about the nature of the singularities in the weak coupling limit and in the strong coupling limit and we focus on $N=4$ super-symmetric Yang-Mills theory for which both limits are under control.

For a relativistic field theory at zero temperature and chemical potential, a two point Green’s function at four-momentum $Q \equiv (iE, \vec{q})$ depends linearly on a number of scalar form factors which are nontrivial functions of the four-momentum square $\vec{q}^2 - E^2$. Therefore, the analyticity with respect to momentum follows from that with respect to energy. This is no longer the case at a nonzero temperature because of the lack of the Lorentz invariance and the momentum analyticity
becomes a separate issue. On the other hand, the partition function of a relativistic field theory at a nonzero temperature $T$ can be expressed in terms of a path integral with the same Lagrangian density as zero temperature formulated in Euclidean space-time $S^1 \times R^3$ with the radius of $S^1$ equal to $1/T$ so the non-invariance is only imposed by the boundary condition. Because the $O(4)$ invariance (Euclidean continuation of the original Lorentz invariance) of the Lagrangian density, which dimension is associated to the Euclidean time makes no difference mathematically. Identifying $S^1$ with the Euclidean time leads to the Matsubara formulation of the thermal field theory while identifying one of $R^3$ dimensions with the Euclidean time would end up with a field theory with the same Lagrangian density but formulated in $S^1 \times R^2$ at zero temperature. Consequently, we can map the momentum plane of the former to the energy plane to the latter and infer the momentum analyticity of the former from the unitarity and causality of the later. This way, we deduce that the two point function with a Euclidean energy is analytic throughout the complex momentum plane except the imaginary axis for any thermal relativistic field theory at zero chemical potential. Perturbatively there is a branch cut running along the imaginary axis, reflecting the continuum spectrum of the asymptotic states in the interaction representation of the field theory in $S^1 \times R^2$.

Perturbation series of a thermal Yang-Mills theory of three space dimensions suffers from severe infrared divergence. Even with the hard-thermal-loop resummation, the perturbation series ceases to approximate beyond certain orders of the coupling constant $g$. For the high temperature QCD, the perturbative expansion of the pressure cannot go beyond $g^6$ and that of a static two point function fails for small momentum no matter how weak the coupling $g$ is. The nonperturbative effect behind the infrared problems may also impact on the analyticity of Green’s functions with respect to energy and momentum. The advent of AdS/CFT duality enable us to explore the $N = 4$ supersymmetric Yang-Mills theory at large $N_c$ and large ’t Hooft coupling. It would be interesting to examine the energy-momentum analyticity of its Green’s functions at nonzero temperature and to compare with the perturbative results. Such a project has been carried out in the literature for the energy variable of two point functions. It was found that, the singularity on the energy plane remains a cut along the real axis. The analytic continuation through the cut hits a set of poles associated to the quasi-normal modes of a black hole. We shall supplement their result with a study of the analyticity with respect to momentum. Although the differential equation satisfied by the two point Green’s function are of Heun type which cannot be solved explicitly, the polynomial dependence of the coefficients of its power series solution on momentum enable us to make rigorous statements regarding the analyticity with the aid of Weierstrass theorem. Here we witness a qualitative change in the nature of the singularities along the imaginary axis: The branch cut in the weak coupling evolves to a set of simple poles in the strong coupling. To pinpoint the mechanism of this transition may be highly nontrivial. In this paper, we merely report our discovery without offering deeper insights.

In the next section, we shall start with the momentum analyticity of a simple example of a one-loop self-energy and generalize it to arbitrary number of loops. Then, we shall extract the correspondence between the thermal field theory in $R^3$ and the zero temperature field theory in $S^1 \times R^2$. The strong coupling limit of the analyticity will be examined in section III via AdS/CFT duality. In the last section, we discuss our results together with miscellaneous implications and generalizations. We also speculate the relation of the poles we discovered to the confinement in two space dimensions. Without special declaration, the signature of our metric is always Euclidean and we write the four-momentum of a two point function as $Q = (q_0, \vec{q})$ with $\vec{q} = (0, 0, q)$.

2. The two point Green’s function of perturbation theory

In this section, we shall demonstrate the momentum plane analyticity of a two point Green’s function from perspective of perturbation theory. We take the $R$-charge correlator of the $N = 4$
SYM as an example in order to compare with the nonperturbative results at strong coupling in the next section. In the fundamental representation of the $R$-symmetry group, $SU(4)$, the $R$-charge operator corresponds to the one of the diagonal generators and we will focus on the generator
\begin{equation}
Q_R = \text{diagonal} \left( \frac{1}{2}, -\frac{1}{2}, 0, 0 \right),
\end{equation}
following Ref. [13], which assigns nonzero charges $\pm \frac{1}{2}$ to Weyl fermion fields of the theory. It follows from the antisymmetric representation of (2.1), two complex scalar fields of the theory carry a nonzero charges $\frac{1}{2}$ each. If the Abelian transformation generated by (2.1) is gauged by a $R$-photon field via a coupling constant $e$, the correlator we are interested in corresponds to the Coulomb component of the self energy tensor of the photon to the lowest order in $e$, i.e.
\begin{equation}
e^2 F(q) = \Pi_{00}(0, \vec{q}),
\end{equation}
where we consider the static case with $q_0 = 0$ and $q \equiv |\vec{q}|$. Its analog in QED is related to the dielectric function in a medium through
\begin{equation}
\epsilon(q) = 1 + e^2 \frac{F(q)}{q^2}.
\end{equation}

2.1 The one-loop diagrams

To the zeroth order of the Yang-Mills coupling, this correlator is given by the one-loop diagram of Fig.1. We have
\begin{equation}
F(q) = (N_c^2 - 1) \times \left[ 2 \times \left( \frac{1}{2} \right)^2 F_f(q) + 2 \times \left( \frac{1}{2} \right)^2 F_b(q) \right] = \frac{N_c^2 - 1}{2} [F_f(q) + F_b(q)],
\end{equation}
where $F_f(q)$ accounts for the contribution of each Weyl fermion of unit charge and $F_b(q)$ accounts for the contribution of each complex scalar of unit charge. It follows from the Feynman rule at nonzero temperature that
\begin{equation}
F_f(q) = T \sum_n \mu^{4-D} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \text{tr} \gamma_4 \frac{1}{p^2 + \omega_n^2} \frac{1}{p^2 + Q^2},
\end{equation}
with $P = (\nu_n, \vec{p})$ and
\begin{equation}
F_b(q) = -4T \sum_n \mu^{4-D} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{\omega_n^2}{(\omega_n^2 + p^2)(\omega_n^2 + (\vec{p} + q)^2)} + 2T \sum_n \mu^{4-D} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{1}{\omega_n^2 + p^2},
\end{equation}
where Matsubara frequencies $\nu_n = 2\pi T(n + \frac{1}{2})$ and $\omega_n = 2\pi Tn$. The dimensional regularization is applied with $D \to 4^-$ and $\mu$ an energy scale to compensate the dimensionality.
standard procedure in textbooks, we obtain that
\[
F_f(q) = \frac{q^2}{12\pi^2(4-D)} + \frac{q^2}{24\pi^2} \left( -\ln \frac{q^2}{\mu^2} + \ln 4\pi - \gamma_E + \frac{5}{3} \right)
\]
\[
+ \frac{1}{\pi^2} \int_0^\infty dp \frac{p}{e^{\beta p} + 1} \left[ 1 + \frac{p}{q} \left( 1 - \frac{q^2}{4p^2} \right) \ln \frac{2p+q}{2p-q} \right]
\]
\[(2.6)\]
and
\[
F_b(q) = \frac{q^2}{24\pi^2(4-D)} + \frac{q^2}{48\pi^2} \left( -\ln \frac{q^2}{\mu^2} + \ln 4\pi - \gamma_E + \frac{4}{3} \right) \]
\[
+ \frac{1}{\pi^2} \int_0^\infty dp \frac{p}{e^{\beta p} + 1} \left( \frac{1}{p} + \frac{1}{q} \ln \frac{2p+q}{2p-q} \right)
\]
\[(2.7)\]
with \(\gamma_E = 0.5772\ldots\) the Euler constant. To explore the analyticity for a complex \(q\), it is more convenient to perform the integrations \((2.6)\) and \((2.7)\) differently. Upon Feynman parametrization of the integrand according to
\[
\frac{1}{ab} = \int_0^1 dx \frac{1}{(ax + b(1-x))^2}
\]
we find that
\[
F_f(q) = 2T \sum_n \int_0^1 dx \mu^{4-D} F D^{-1} \int \frac{\nu^{D-1} p}{(2\pi)^{D-1}} \frac{1}{\nu^2 + p^2 + q^2 x(1-x)}
\]
\[(2.8)\]
and
\[
F_b(q) = -4T \sum_n \int_0^1 dx \mu^{4-D} \frac{q^{D-1} p}{(2\pi)^{D-1}} \frac{1}{\nu^2 + p^2 + q^2 x(1-x)} + 2T \sum_n \int \mu^{4-D} \frac{q^{D-1} p}{(2\pi)^{D-1}} \frac{1}{\nu^2 + p^2}
\]
\[(2.9)\]
It follows that \(F_f(q)\) and \(F_b(q)\) are analytic on the complex \(q\)-plane cut along the imaginary axis. The discontinuities acrossing the cut are given by
\[
\Delta_f(\kappa) \equiv F_f(i\kappa + 0^+) - F_f(i\kappa - 0^-) = 2\text{Im}F_f(i\kappa + 0^+)
\]
\[
= \frac{T}{\pi^2} \sum_n \int_0^1 dx \int_0^\infty dp \nu^2 \left[ 1 \right] \frac{1}{\nu^2 + p^2 - \kappa^2 x(1-x) - i0^+} - \frac{1}{\nu^2 + p^2 - \kappa^2 x(1-x) + i0^+}
\]\n\[
= 2i(N_f + 1)T \kappa \left[ 1 - \frac{\pi^2 T^2}{3\kappa^2} (4N_f^2 + 8N_f + 3) \right]
\]
\[(2.10)\]
and
\[
\Delta_b(\kappa) \equiv F_b(i\kappa + 0^+) - F_b(i\kappa - 0^-) = 2\text{Im}F_b(i\kappa + 0^+)
\]
\[
= -\frac{2T}{\pi^2} \sum_n \omega_n^2 \int_0^1 dx \int_0^\infty dp \nu^2 \left[ 1 \right] \frac{1}{\omega_n^2 + p^2 - \kappa^2 x(1-x) + i0^+} - \frac{1}{\omega_n^2 + p^2 - \kappa^2 x(1-x) - i0^+}
\]
\[
= \frac{i}{3\kappa} \pi^2 T^{3} N_b(N_b + 1)(2N_b + 1)
\]
\[(2.11)\]
where \(N_f \geq 0\) and \(N_b \geq 0\) are the maximum integers such that \(\kappa^2 \geq 4\nu_n^2\) and \(\kappa^2 \geq 4\omega_n^2\). The same analyticity including the discontinuities can be extracted from \((2.4)\) and \((2.7)\) with a careful treatment of the logarithm of the integrands. See appendix A for details. We notice that \(\text{Im}F_{f,b}(i\kappa + 0^+) > 0(0^-)\) for \(\kappa > 0(0^-)\), consistent with the identity
\[
F_{f,b}(q^*) = F_{f,b}(q)
\]
\[(2.12)\]
for an analytic function that is real for real \(q\). Substituting \((2.6)\) and \((2.7)\) into \((2.3)\) we obtain that
\[
F(q) = \frac{(N_f^2 - 1)}{16\pi^2} \left\{ q^2 \ln \frac{1}{q^2} + 2 \int_0^\infty dp \frac{p}{e^{\beta p} + 1} \left[ 1 + \frac{p}{q} \left( 1 - \frac{q^2}{4p^2} \right) \ln \frac{2p+q}{2p-q} \right]
\]
\[
+ 2 \int_0^\infty dp \frac{p}{e^{\beta p} + 1} \left( \frac{1}{p} + \frac{1}{q} \ln \frac{2p+q}{2p-q} \right) \right\}
\]
\[(2.13)\]
where we have introduced a UV cutoff length \(l\) via
\[
\frac{1}{4 - D} = \ln \frac{1}{\mu l} + \frac{1}{2} \left( -\ln 4\pi + \gamma_E - \frac{14}{3} \right).
\]
(2.15)
The discontinuity across the cut along the imaginary axis is given by
\[
\Delta(\kappa) \equiv F(i\kappa + 0^+) - F(i\kappa - 0^+) = \frac{N^2 - 1}{2} [\Delta_f(\kappa) + \Delta_b(\kappa)].
\]
(2.16)

### 2.2 A multi-loop diagram

The above analyticity on the momentum plane is generic to a Feynman diagram of a two point Green’s function consisting of an arbitrary finite number of loops. For a self-energy diagram of \(I\) internal lines and \(L\) loops, we have
\[
\mathcal{G}(Q) = \left( \prod_{i=1}^L \sum_{\nu_i} \int \frac{d^{D-1}p_i}{(2\pi)^{D-1}} \right) \frac{N(P, Q)}{\prod_{i=1}^L k_i^2}
\]
(2.17)
where \(K_i = (\omega_i, \vec{k}_i)\) is the fourth momentum of the i-th internal line and \(N(P, Q)\) is a polynomial of loop momenta \(P_i = (\nu_i, \vec{p}_i)'s\) and \(Q = (q_0, 0, 0, q)\). Introducing Feynman parameters, we find that
\[
\mathcal{G}(q) = (I - 1)! \int_0^1 \prod_{i=1}^I dx_i \delta(1 - \sum_{i=1}^I x_i) \left( \prod_{i=1}^L \sum_{\nu_i} \int \frac{d^{D-1}p_i}{(2\pi)^{D-1}} \right) \frac{N(P, Q)}{\left( \sum_{i=1}^I (\omega_i^2 + k_i^2)x_i \right)^I}
\]
(2.18)
where
\[
\sum_{i=1}^I k_i^2 x_i = \sum_{l'} M_{ll'} \vec{p}_l \cdot \vec{p}_{l'} + \sum_l u_l \vec{p}_l \cdot \vec{q} + v q^2
\]
(2.19)
with \(M_{ll'}, u_l\) and \(v\) functions of the Feynman parameters \(x_i's\). \(M_{ll'}\) forms a \(L \times L\) symmetric and positive matrix. Upon an orthogonal transformation and a translation,
\[
\sum_{i=1}^I k_i^2 x_i = \sum_{l} \lambda_l p_l^2 + R q^2
\]
(2.20)
with all \(\lambda_l's\) positive. Regarding the self-energy part as a DC circuit with the i-th internal line carrying a resistance of \(x_i\), the function \(R\) equals to the total resistance between the two terminals of the external lines. We obtain that
\[
\mathcal{G}(q) = (I - 1)! \int_0^1 \prod_{i=1}^I dx_i \delta(1 - \sum_{i=1}^I x_i) \left( \prod_{i=1}^L \sum_{\nu_i} \int \frac{d^{D-1}p_i}{(2\pi)^{D-1}} \right) \frac{N(P, Q)}{\left( \sum_{i=1}^I \omega_i^2 x_i + \sum_{l} \lambda_l p_l^2 + R q^2 \right)^I}
\]
(2.21)
which is analytic throughout the complex \(q\)-plane except a cut along the imaginary axis. Note that we did not assume the static limit in the analysis and the result is not restricted to the Coulomb component of the photon self energy. While the discontinuity acrossing the cut may be subject to the UV divergence in general, it should be finite for \(N = 4\) SYM. On the other hand, the infrared divergence cannot be removed perturbatively beyond certain orders, leaving rooms for nonperturbative modifications of the analyticity.

### 2.3 A general correspondence

The location of the singularity is actually dictated by a general principle. The path integral of a relativistic thermal field theory at zero chemical potential reads
\[
Z = \text{const.} \int [d\Phi] \exp \left( - \int_0^\beta d\tau \int_{-\infty}^\infty dx_3 \int d^3 \vec{x} \mathcal{L}_E[\Phi] \right)
\]
(2.22)
where $\Phi$ is the collection of the field variables and $\mathcal{L}_E$ is the Lagrangian density with a Euclidean time, which is of $O(4)$ invariance (For a gauge theory, manifest invariance can be implemented in the covariant gauge). The field theory at nonzero temperature $T = 1/\beta$ is formulated on $R^3$. Switching the role of the Euclidean time $\tau$ and the spatial coordinate $x_3$, the same path integral describes a Euclidean field theory at zero temperature formulated on $S^1 \times R^2$ with the same Lagrangian density, described by the partition function

$$Z = \text{const.} \int [d\Phi] \exp \left( - \int_{-\infty}^{\infty} d\tau \int_0^\beta d\tilde{x}_3 \int d^2 \vec{x}_\perp L_E[\Phi] \right) \quad (2.23)$$

The original complex momentum plane associated to the partition function $Z$ becomes the complex energy plane associated to the partition function $Z$ with the positive imaginary axis of the former corresponding to the positive real axis of the latter. To see this mapping clearly, we notice that the spatial coordinate $x_3$ in $Z$ goes to the Euclidean time $\tau$ of $Z$, which is related to the Minkowski time via $\tau = it$. Accordingly, the spatial momentum $q$ associated to $Z$ becomes the negative Euclidean energy $\omega$ associated to $Z$ so that $e^{iqx_3} \rightarrow e^{-i\omega \tau}$ and $\omega$ is related to the Minkowski energy via $\omega = -iE$. It follows then that the spatial momentum $q$ of $Z$ maps to the Minkowski energy $E$ of $Z$ according to $q \rightarrow iE$. The unitarity of the field theory specified by $Z$ dictates singularities along the real axis of the energy plane only, which corresponds to the imaginary axis of the momentum plane of the original thermal field theory. The spectral representation of the self energy function implies that its imaginary part is negative when approaching the cut along the positive real axis from below and the residues of its poles are all positive along the positive real axis. The signs of RHS of (2.11) and (2.12) as well as the residues of the asymptotic poles discussed in the next section confirm the above statement.

3. The AdS/CFT implied correlators

AdS/CFT duality relates the $N = 4$ SYM at large $N_c$ and large 't Hooft coupling to the following super-gravity action with a R-photon gauge field reads\[15, 16, 13\]

$$S_{\text{sugr}} = \frac{N_c^2}{8\pi^2 L^3} \int d^5 x \sqrt{g} \left( R + \frac{12}{L^2} \right) + \frac{N_c^2}{64\pi^2 L} \int d^5 x \sqrt{g} g^{\mu\rho} g^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda} + \text{Gibbons-Hawking term}, \quad (3.1)$$

where $R$ is the scalar curvature and

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}, \quad (3.2)$$

The metric $g_{\mu\nu}$ is slightly perturbed from the Schwarzschild-AdS$_5$ background, given by

$$ds^2 = \frac{L^2}{z^2} \left( f d\tau^2 + d\vec{x}^2 + \frac{dz^2}{f} \right) + h_{\mu\nu} dx^\mu dx^\nu \quad (3.3)$$

where $f = 1 - \frac{\pi^4}{4} T^4 z^4$ with $T$ the temperature of the corresponding field theory. The action $S_{\text{sugr}}$ is a functional of the metric fluctuation $h_{\mu\nu}(x,z)$ and the R-photon gauge potential $A_\mu(x,z)$ with $x = (\tau, \vec{x})$. Solving the Maxwell equation

$$\frac{\partial}{\partial x^\rho} \sqrt{g} g^{\mu\rho} g^{\nu\lambda} F_{\nu\lambda} = 0 \quad (3.4)$$

and the linearized Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{6}{L^2} g_{\mu\nu} = 0 \quad (3.5)$$
subject to the boundary conditions
\[ h_{\mu\nu}(x, 0) = \tilde{h}_{\mu\nu}(x) \quad A_\mu(x, 0) = \tilde{A}_\mu(x) \] (3.6)
under the gauge conditions \( A_z(x, z) = h_{\mu z}(x, z) = h_{zz}(x, z) = 0 \), and substituting the solutions back to the action, we end up with
\[
S_{\text{sugr}} = S_{\text{sugr}}^{(0)} + \frac{1}{2} \int_{u=0} \int d^4y \left[ C_{\mu\nu}(x - y) \tilde{A}^\mu(x) \tilde{A}^\nu(y) + \frac{1}{4} \tilde{C}_{\mu\nu,\rho\lambda}(x - y) \tilde{h}^{\mu\rho}(x) \tilde{h}^{\nu\lambda}(y) \right] d^4y
\]
\[
= \frac{1}{2} \int \frac{d^4Q}{(2\pi)^4} \left[ C_{\mu\nu}(Q) \tilde{A}^\mu(Q) \tilde{A}^\nu(Q) + \frac{1}{4} \tilde{C}_{\mu\nu,\rho\lambda}(Q) \tilde{h}^{\mu\rho}(Q) \tilde{h}^{\nu\lambda}(Q) \right].
\] (3.7)
where the 4D momentum representation has been introduced in the 2nd line of (3.7). The coefficients \( C_{\mu\nu} \) and \( C_{\mu\nu,\rho\lambda} \) give rise to the R-photon self-energy tensor and the stress tensor correlators. For the rest of this section, we shall examine the Coulomb and the transverse components of the R-photon self energy and the shear component of the stress tensor correlator in detail. The Matsubara energy \( \omega \) will be set to zero so that the equation of motion of \( C_{00} \) decouples. The generalization to a nonzero Euclidean energy will be discussed in the next section. A UV cutoff is introduced by pulling the 3-brane slightly off the AdS boundary, i.e. \( z = l \to 0^+ \). Upon the coordinate transformation, \( u = \pi^2 T^2 z^2 \), the metric (3.3) becomes
\[
d s^2 = \frac{\pi^2 L^2 T^2}{u} (d\tau^2 + dz^2) + \frac{L^2 du^2}{4u^2 f} + h_{\mu\nu} dx^\mu dx^\nu
\] (3.8)
with \( f = 1 - u^2 \). The horizon is at \( u = 1 \) and the UV cutoff corresponds to \( u = \varepsilon \equiv \pi^2 T^2 l^2[17] \).

We shall work with this metric below and set the AdS radius \( L = 1 \). The equations of motion (3.4) and (3.5) becomes ordinary differential equations in the 4D momentum representation and the solution of each equation that satisfies the boundary conditions (3.6) and gives rise to a finite action is unique. The analyticity of these solutions implies that the two point Green’s function we examine are all meromorphic functions on the complex \( q \)-plane with infinite number of poles along the imaginary axis.

3.1 The correlator of the R-charge density

It follows from the recipe just stated that the Coulomb component of the R-photon self energy is given by (3.4)
\[
F(q) \equiv C_{00}(0, q) = -\frac{N_c^2 T^2}{8} \frac{A_0(\varepsilon|q)}{A_0(\varepsilon|q)}
\] (3.9)
where \( A_0(u|q) \) solves the equation
\[
\frac{d^2 A_0}{du^2} - \frac{\hat{q}^2}{u(1-u^2)} A_0 = 0
\] (3.10)
with \( \hat{q} = q/(2\pi T) \). The indexes of this equation at the canonical singularity \( u = 1 \) are 0 and 1, which give rise to two linearly independent behaviors as \( u \to 0 \),
\[
A_0(u|q) \simeq 1 - u
\] (3.11)
or
\[
A_0(u|q) \simeq 1 + \frac{1}{2} \hat{q}^2 \ln(1-u) + \frac{1}{2} \Big[ \ln(1-u) + \frac{1}{2} \Big].
\] (3.12)
The second solution, eq.(3.12) will make the action \( S_{\text{sugr}} \) diverge and should be discarded and the eq.(3.11) generates a power series power series solution
\[
A_0(u|q) = (1 - u) \sum_{n=0}^{\infty} a_n(q)(1-u)^n
\] (3.13)
where the coefficients $a_n(q)$ satisfy the recursion relation

$$a_{n+1}(q) = \frac{1}{2(n+1)(n+2)} \left[ 3n(n+1) + q^2 a_n(q) - n(n-1)a_{n-1}(q) \right]$$  \hspace{1cm} (3.14)$$

with $a_0(q) = 1$. Notice that the coefficients are all polynomials in $q^2$ and is therefore analytic everywhere on the complex $q$-plane. The indexes at $u = 0$ are also 0 and 1 so the series (3.13) converges for $0 \leq 1 - u \leq 1$ uniformly with respect to $q$. It follows from the Weierstrass theorem [13], that the sum $f(q) = A_0(0|q)$ is an analytic function throughout the complex $q$-plane and is therefore an entire function.

Next, we consider the derivative $A'_0(u|q)$. The indexes at $u = 0$ implies two linearly independent behaviors there,

$$A_0^{(1)} \sim u \left( 1 + \frac{1}{2} \hat{q}^2 u \right)$$  \hspace{1cm} (3.15)$$

and

$$A_0^{(2)} \sim 1 + \hat{q}^2 u \left( \ln u + \frac{1}{2} \right)$$  \hspace{1cm} (3.16)$$

The solution (3.13) is a superposition of $A_0^{(1)}$ and $A_0^{(2)}$ and is dominated by $A_0^{(2)}$ as $u \to 0$, i.e.

$$A_0(u|q) = f(q)[1 + \hat{q}^2 u \ln u + O(u)].$$  \hspace{1cm} (3.17)$$

and its derivative becomes logarithmically divergent at $u = 0$. This is also consistent with the asymptotic behavior of the coefficients of (3.13) at large $n$,

$$a_n(q) \sim \frac{\hat{q}^2 f(q)}{n(n+1)}$$  \hspace{1cm} (3.18)$$

where the overall factor $\hat{q}^2 f(q)$ follows from (3.17). We find then

$$A'_0(\varepsilon|q) = \hat{q}^2 f(q) \ln \varepsilon + g(q) + O(\varepsilon)$$  \hspace{1cm} (3.19)$$

where

$$g(q) = -1 + \sum_{n=1}^{\infty} \left[ - (n+1)a_n(q) + \frac{1}{n} \frac{\hat{q}^2 f(q)}{n(n+1)} \right]$$  \hspace{1cm} (3.20)$$

Each coefficient term of the infinite series above is analytic function of $q$ and the series converges uniformly with respect to $q$. The same Weierstrass theorem applied to the infinite series of $A_0(0|q)$ implies that $g(q)$ is also an entire function of $q$. Substituting (3.19) and $A_0(\varepsilon|q) = f(q) + O(\varepsilon \ln \varepsilon)$ into (3.9) and dropping the terms that vanish in the limit $\varepsilon \to 0$, we obtain that

$$F(q) \equiv C_{00}(0, q) = \frac{N^2}{16\pi^2} \hat{q}^2 \ln \frac{1}{\varepsilon} + \frac{N^2 T^2}{16\pi^2} \frac{g(q)}{f(q)}$$  \hspace{1cm} (3.21)$$

which is a meromorphic function of $q$.

The poles of $F(q)$ are determined by the zeros of $f(q)$, i.e.

$$A_0(0|q) = 0$$  \hspace{1cm} (3.22)$$

It is easy to rule out the zeros off real and imaginary axes, where, $\text{Im} q^2 \neq 0$. Indeed, taking the imaginary part of the product of $A_0^*(u|q)$ and the equation (3.10) and integrating $u$ from 0 to 1 yield

$$\left( A_0 \frac{dA_0}{du} - \frac{dA_0^*}{du} A_0 \right)_{u=0} - \text{Im} \hat{q}^2 \int_0^1 du \frac{A_0^* A_0}{u(1-u^2)} = 0$$  \hspace{1cm} (3.23)$$
If there were a complex \( q^2 \) satisfying (3.22), we would have

\[
\int_0^1 \frac{du}{u(1-u^2)} = 0,
\]

(3.24)

which requires that \( A_0(u|q) \equiv 0 \). It follows that there cannot be a nontrivial solution satisfying (3.22) with \( \text{Im} q^2 \neq 0 \). Turning to a real \( q \), the equation (3.10) implies that the solution \( A_0(u|q) \) is a convex (concave) function of \( u \) if \( A_0(u|q) > 0(<0) \). Therefore starting with a nonzero \( A_0(u|q) \) at \( u \neq 0 \) there is no way the solution cannot bent towards zero at \( u = 0 \) and therefore there is no nontrivial solution satisfying (3.22) in this case either. The only location for the roots of \( A_0(0|q) \) is the imaginary \( q \)-axis.

For large and imaginary \( q \), i.e. \( q = i\kappa \) with \( \kappa >> T \), the WKB approximation of the appendix B yields

\[
F(i\kappa) \simeq -\frac{N_c^2}{16\pi^2\kappa^2} \left( \ln \frac{2}{\kappa l} - \gamma_E + \frac{1}{2\pi} \tan \hat{\kappa} \delta \right),
\]

(3.25)

where

\[
\delta = \frac{1}{2\sqrt{2\pi}} \Gamma^2 \left( \frac{1}{4} \right)
\]

(3.26)

with \( \hat{\kappa} \equiv \frac{\kappa}{2\pi T} \). Therefore the asymptotic locations of the poles are given by

\[
\kappa \simeq (2n + 1) \frac{\pi^2 T}{\delta}
\]

(3.27)

and the number of them is infinite.

### 3.2 The self-energy of the transverse \( R \)-photon

If follows from the AdS/CFT principle, the self-energy of the transverse \( R \)-photon is given by the coefficient of the term quadratic in \( \tilde{A} \) of the supergravity action (3.7). We have [16]

\[
G(q) \equiv C_{11}(0,q) = C_{22}(0,q) = -\frac{N_c^2 T^2}{8} \frac{\mathcal{A}'(\varepsilon|q)}{\mathcal{A}(\varepsilon|q)}
\]

(3.28)

with \( \varepsilon \to 0 \), where \( \mathcal{A}(u|q) \) solves the differential equation

\[
-\frac{d^2}{du^2} \left[ (1-u^2) \frac{d\mathcal{A}}{du} \right] - \frac{\hat{q}^2}{u} \mathcal{A} = 0.
\]

(3.29)

Both indexes of (3.29) at \( u = 1 \) are zero. Only the power series solution leaves the UV regularized action finite, which reads

\[
\mathcal{A}(u|q) = \sum_{n=0}^{\infty} b_n(q)(1-u)^n
\]

(3.30)

where the coefficients are given recursively by

\[
b_{n+1}(q) = \frac{[\hat{q}^2 + n(3n + 1)]b_n(q) - n(n-1)b_{n-1}(q)}{2(n+1)^2}
\]

(3.31)

with \( b_0 = 1 \). The radius of convergence equal to one since none of the indexes, 0 and 1, at \( u = 0 \) gives rise to divergent behavior there. It follows from the Weierstrass theorem of the previous subsection that \( G(q) \) is a meromorphic function of \( q \).

Upon a transformation \( \mathcal{A} = \frac{\psi}{\sqrt{1-u^2}} \), the eq.(3.29) becomes a one-dimensional Schroedinger equation

\[
-\frac{d^2\psi}{du^2} + V \psi = 0
\]

(3.32)
where the potential
\[ V = -\frac{1}{(1 - u^2)^2} + \frac{\hat{q}^2}{u(1 - u^2)}. \quad (3.33) \]

and \( \psi(u|q) = 0 \) is required to vanish faster than \( \sqrt{1 - u} \) as \( u \to 1 \). The same argument that rules out the poles of \( F(q) \) with a nonzero \( \text{Im} q^2 \) can be applied to the present case but the argument ruling out the poles on the real \( q \) axis need to be modified because of the negative term of the potential. The attractiveness of the potential is maximized at \( q = 0 \). If there were a nontrivial solution such that \( \psi(0|q) = 0 \) (i.e. \( \mathcal{A}(0|q) = 0 \)) for a nonzero real \( q \), the zero of \( \mathcal{A}(u|q) \) at \( u = 0 \) would shift towards \( u = 1 \) when \( q^2 \) is reduced towards zero. Consequently, there would be a nontrivial solution at \( q = 0 \), that vanishes at least once within the domain \( u \in (0, 1) \). This is, however, not the case. The exact solution of (3.29) for a finite action is \( \mathcal{A} = 1 \). For a large and imaginary \( q \), \( q = i\kappa \), the WKB approximation yields
\[ G(i\kappa) \simeq -\frac{N_v^2}{16\pi^2} \kappa^2 \left( \ln \frac{2}{\kappa l} - \gamma_E - \frac{1}{2} \pi \cot \hat{k}\delta \right). \quad (3.34) \]

Therefore, the self-energy of a transverse \( R \)-photon is a meromorphic function of the momentum \( q \) with infinite number of poles along the imaginary axis.

### 3.3 The shear component of the stress tensor correlator

The shear component of the stress tensor correlator is extracted from the coefficients of the quadratic term in \( h_1^2 \), \( C_1^1 \chi^2(Q) \) of (3.27). It follows from the rotational symmetry about \( x_3 \)-axis that \( h_1^2 = h_2^1 \) does not couple with other components of the metric fluctuations and the quadratic action in \( h_1^2 \) is identical to that of a scalar field [16]. We have
\[ \Pi(q) = C_1^1 \chi^2(0, q) = -\frac{N_v^2\pi^2 T^4}{4} - \frac{\varepsilon^2}{\varepsilon} \phi'(\varepsilon|q), \quad (3.35) \]

where \( \phi(u|q) \) solves the ordinary differential equation
\[ u^2 \frac{d}{du} \left[ \frac{1 - u^2}{u} \frac{d\phi}{du} \right] - \hat{q}^2 \phi = 0, \quad (3.36) \]

which is the Fourier transform of the scalar Laplace equation the background metric (3.8). Between the two linearly independent solutions, only the power series solution
\[ \phi(u|q) = \sum_{n=0}^{\infty} c_n(q)(1 - u)^n \quad (3.37) \]

leaves the action (3.1) finite upon UV regularization, where the coefficients are given by the recursion relation
\[ c_{n+1}(q) = \frac{[\hat{q}^2 + n(3n - 1)]c_n(q) - (n - 1)^2 c_{n-1}(q)}{2(n + 1)^2} \quad (3.38) \]

and are polynomials in \( q^2 \) again. The power series (3.37) is convergent at \( u = 0 \). Making the transformation \( \phi = \sqrt{\frac{u}{1 - u^2}} \psi \), the equation (3.36) can be converted into an one-dimensional Schroedinger equation of zero energy in the potential
\[ V = \frac{3}{4u^2} - \frac{u}{(1 - u^2)^2} + \frac{\hat{q}^2}{u(1 - u^2)} \quad (3.39) \]

which is the least repulsive at \( q = 0 \). The exact solutions of (3.36) at \( q = 0 \) is \( \phi(u|q) = 1 \). For \( q = i\kappa \) with \( \kappa \gg T \), we find that
\[ \Pi(i\kappa) \simeq -\frac{N_v^2}{16\pi^2 T^2} \kappa^2 - \frac{N_v^2}{32\pi^2} \kappa^4 \left( \ln \frac{2}{\kappa l} - \gamma_E + \frac{1}{2} \pi \tan \hat{k}\delta \right). \quad (3.40) \]
Therefore the arguments applied in the last subsection for the transverse $R$ photon self-energy implies that $\Pi(q)$ is a meromorphic function with an infinite number of poles along the imaginary axis of the $q$-plane, like the other two cases.

4. Discussions

Let us recapitulate what we have done in previous sections. We started with the example of the $R$-charge density correlator of the $N=4$ SYM in the static limit and worked out its analyticity on the complex momentum plane. Through the correspondence between a relativistic field theory at a nonzero temperature formulated in $R^3$ and another relativistic field theory at zero temperature formulated in $S^1 \times R^2$, we have found the universality regarding the location of the singularities on the momentum plane of a two point Green’s function. Perturbatively, the singularities form a branch cut along the imaginary axis, reflecting the continuum of asymptotic states of the field theory on $S^1 \times R^2$. Our conclusion in this regard, however, is not limited to the $N=4$ SYM being considered and is also valid with a nonzero Euclidean energy.

A by-product of our analysis is to rule out the Yukawa oscillations of certain static potential in a hot medium at zero chemical potential, as was suggested in the literature\cite{19, 20}. Taking the Coulomb potential as an example, which takes the form

\begin{equation}
U(r) = \frac{e^2}{4\pi^2 r} \Im \int_{-\infty}^{\infty} dq \frac{qe^{iqr}}{q^2 + e^2 F(q)}. \tag{4.1}
\end{equation}

with $e^2 << 1$. Here both $e$ and $F(q)$ refer to renormalized quantities which amounts to replace the cutoff length $l$ of (2.14) with a renormalization length scale comparable to $1/T$. A complex pole at $q = q_R + iq_l$ with $q_R \neq 0$ will make the function $U(r)$ oscillates in $r$. Because $F(q)$ grows like $\ln |q|$ for large $|q| >> T$, there is a tachyonic pole $q = K_T$ of the integrand on the real axis with $\ln K_T \sim \frac{1}{r}$. The oscillation it causes can be eliminated easily by smearing the source charge of the potential. To explore the poles with $|q| \sim T$, we write $d(q) \equiv q^2 + e^2 F(q)$ and look for its roots within a circle $|q| = K$ with $T << K << K_T$, given by the intersection of the lines $\text{Re} d(q) = 0$ and $\text{Im} d(q) = 0$. Because of the symmetry of $F(q)$, we may restrict our searching scope within the first quadrant. A line of $\text{Im} d(q) = 0$ cannot cross the cut along the imaginary axis since $\text{Im} d(q) \neq 0$ there. It cannot form a loop within the region where $d(q)$ is analytic. So it has to cross the circle just introduced. On the circle $d(q) \approx q^2$ and we need only to examine the neighborhood where the circle cuts the real and imaginary axes. The real axis itself is a line where $\text{Im} d(q) = 0$ and $d(q) > 0$ for $|q| \leq K$. There is another one near the imaginary axis which can be located by treating $\text{Im} F(q)$ perturbatively. The positivity of $\text{Im} F(q)$ for $q$ approaching the positive imaginary axis from right pushes the line $\text{Im} d(q) = 0$ off the first quadrant. Therefore there is no complex zero of $d(q)$ that gives rise to the Yukawa oscillation\cite{18, 21, 22}.

Next we moved to the strong coupling limit where exact expression of the two point Green’s function exists for $N=4$ SYM at large $N_c$ and large ‘t Hooft coupling through AdS/CFT duality. There we examined the $R$-charge density correlator, the self-energy of a transverse $R$-photon and the correlator of the shear component of the stress tensor, by solving the equation of motion of their gravity dual. In the static limit it follows from the series representation and the Weierstrass theorem that these functions are all analytic on the momentum plane except the imaginary axis. But instead of a branch cut running there, we have found infinite number of poles, which signify a bona fide nonperturbative effect. More impressive is to view the energy plane of the $N=4$ SYM in $S^1 \times R^2$ at zero temperature, show in Fig.2. The continuum of asymptotic states is strongly distorted into a set of bound states and their asymptotic distribution $E_n \simeq \frac{2n\pi^2 T}{s}$ for $n \gg 1$ resembles that of the mass spectrum of AdS/QCD with a hard wall\cite{23}. Here is a possible mechanism: The composite states excited by the R-charge current operators or the stress tensors are
all color singlet. The continuous energy spectrum would correspond to a composite wave function whose constituents can be separated indefinitely. While this is the case for a cluster of free particles but may be prohibited when Yang-Mills interaction is turned on. If the extension of the wave function in $R^2$ dimensions is much longer than the radius of $S^1$, the binding dynamics is essentially 2D and a Yang-Mills theory is expected to confine with two spatial dimensions [24]. Also the color singlet intermediate states cannot decay into separate color singlet objects since the corresponding Feynman diagrams are suppressed at large $N_c$ [24].

The generalization to a nonzero Euclidean energy is straightforward for the transverse $R$-photon self energy and the shear component of the stress tensor correlator, since their equation of motion remains uncoupled. For the transverse $R$-photon of a Euclidean energy $\omega$, the EOM (3.29) becomes

$$-\frac{d}{du}\left[(1-u^2)\frac{dA}{du}\right] - \left(q^2 + \frac{\hat{\omega}^2}{1-u^2}\right)\frac{A}{u} = 0$$

(4.2)

with $\hat{\omega} = \frac{\omega}{2\pi T}$ and solution (3.30) takes the form

$$A(u|Q) = (1+u)^{\alpha}(1-u)^{\beta}\sum_{n=0}^{\infty} b_n(Q)(1-u)^n$$

(4.3)

with $\alpha = -\frac{i}{2}\omega$ and $\beta = \frac{1}{2}|\omega|$. The recursion formula of the coefficients reads

$$b_{n+1}(Q) = \frac{1}{2(n+1)(n+1+2\beta)} \left\{ [\omega^2 + q^2 + \alpha + \beta + 2\alpha\beta + n(3n+1+2\alpha+6\beta)]b_n(Q) - [(n-1)(n+2\alpha+2\beta) + \alpha + \beta + 2\alpha\beta]b_{n-1}(Q) \right\}$$

(4.4)

with $b_0(Q) = 1$ and all coefficients remain polynomials of $q^2$. Furthermore, the indexes at $u = 0$ remains 0 and 1. Upon transforming eq. (4.2) into a Schroedinger equation of the form (3.32), we have the potential

$$V = -\frac{1}{(1-u^2)^2} + \frac{q^2}{u(1-u^2)} + \frac{\hat{\omega}^2}{u(1-u^2)^2}$$

(4.5)
which is more repulsive than the static case. Therefore the same analysis of the last section leads to the conclusion that the self-energy is a meromorphic function of a complex momentum at $\omega \neq 0$ and has infinite number of poles along the imaginary axis. Likewise is the shear component of the stress tensor correlator at $\omega \neq 0$. Notice that the coefficients of the power series depends on $|\omega|$ because only the vanishing solution at the horizon is selected. This also reflects the cut along the real axis of the energy plane.

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A. Analyticity of self-energy in complex $q$-plane

A direct analytical continuation to complex $q$-plane of the integral representation of the one-loop self energy functions $F_f(q)$ and $F_b(q)$ of Eqs. (2.6) and (2.7) is obscured by the logarithm of the integrand. In this appendix, we shall show how this is carried out for $F_f(q)$ in details.

Let us break $F_f(q)$ into two the contributions from the vacuum polarization and the thermal fluctuations,

$$F_f(q) = F_{\text{vac}}(q) + F_{\text{med}}$$

(A.1)

with

$$F_{\text{vac}}(q) = \frac{q^2}{12\pi^2(4-D)} + \frac{q^2}{24\pi^2} \left( -\ln \frac{q^2}{\mu^2} + \ln 4\pi - \gamma_E + \frac{5}{3} \right)$$

(A.2)

and

$$F_{\text{med}}(q) = \frac{1}{\pi^2} \int_0^\infty dp \frac{pe^{\beta p}}{e^{\beta p} + 1} \left[ 1 + \frac{p}{q} \left( 1 - \frac{q^2}{4p^2} \right) \ln \left( \frac{2p + q}{2p - q} \right) \right].$$

(A.3)

We draw the branch cut of the logarithm from $-q/2$ to $q/2$ on $p$-plane. To accomplish the continuation, one could first implement the absolute value under the logarithm with two integral paths, one above and one below the cut for a real $q$ and then deform the paths accordingly as $q$ becomes complex, i.e.

$$F_{\text{med}}(q) = \frac{1}{\pi^2} \int_0^\infty dp \frac{pe^{\beta p}}{e^{\beta p} + 1} \left( 1 + \frac{4p^2 - q^2}{4pq} \ln \left( \frac{2p + q}{2p - q} \right) \right)$$

(A.4)

$$= \frac{1}{2\pi^2} \left( \int_{L_+} + \int_{L_-} \right) dp \frac{pe^{\beta p}}{e^{\beta p} + 1} \left[ 1 + \frac{4p^2 - q^2}{4pq} \ln \left( \frac{2p + q}{2p - q} \right) \right]$$

(A.5)

where the two integral path $L_+$ and $L_-$ are defined slightly above and below the positive real axis as shown in Fig. 3(a). The sections with $p > q/2$ of $L_+$ make no difference to the integral. To find out the discontinuity of $F_{\text{med}}$ across the imaginary axis, one has to calculate $F_{\text{med}}(i\kappa + 0^+) - F_{\text{med}}(i\kappa - 0^+)$ where $\kappa$ is an arbitrary real positive number and $0^+$ the positive infinitesimal. The case of negative $\kappa$ can be obtained by symmetry. Now we are ready to make the analytical continuation for both positive and negative real $q$.

• For real $q > 0 \rightarrow q = q_R + iq_I$ where $q_R > 0, q_I > 0$. 

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Figure 3: Analytical continuation of complex $q$. The two integral paths in (a) are deformed as (b) and (c) for real $q > 0$ and $q < 0$ respectively.

We rotate the branch cut of $p$-plane counter-clockwise and deform the $L_+$ path along the cut as shown in Fig.3(b). Then $F_{med}$ contains contribution both from the path along positive real axis and that around the cut, i.e.,

$$ F_{med}(q) = F_1^+(q) + F_2^+(q) \quad (A.6) $$

$$ F_1^+(q) = \frac{1}{\pi^2} \int_0^\infty dp \frac{p}{e^{bp} + 1} \left[ 1 + \frac{4p^2 - q^2}{4pq} \ln \left( \frac{2p + q}{2p - q} \right) \right] \quad (A.7) $$

$$ F_2^+(q) = -\frac{i}{4\pi q} \int_0^\infty dp \frac{4p^2 - q^2}{e^{bp} + 1} \quad (A.8) $$

- For real $q < 0 \rightarrow q = q_R + iq_I$ where $q_R < 0$, $q_I > 0$.

Similarly, we rotate the cut clockwise to realize the continuation as shown in Fig.3(c). Notice that now real $q < 0$ and $F_{med}$ is decomposed as

$$ F_{med}(q) = F_1^-(q) + F_2^-(q) \quad (A.9) $$
Let $q = i\kappa \pm 0^+$ and consider that $F_1^+(i\kappa) - F_1^-(i\kappa) = 0$, then the discontinuity across the imaginary $q$ axis reads

$$F_{med}(i\kappa + 0^+) - F_{med}(i\kappa - 0^+) = F_2^+(i\kappa + 0^+) - F_2^-(i\kappa - 0^+)$$ (A.12)

Rewriting

$$F_2^+(i\kappa + 0^+) = -\frac{1}{4\pi\kappa} \int_{0^+}^{0^+} dp e^{i\kappa + 0^+} dp e^{i\kappa - 0^+} - \frac{1}{4\pi\kappa} \int_{0^-}^{0^-} dp e^{i\kappa + 0^+} dp e^{i\kappa - 0^+} + \Delta$$ (A.13)

where

$$\Delta = -\frac{1}{4\pi\kappa} \left( \int_{0^+}^{0^+} - \int_{0^-}^{0^-} \right) dp e^{i\kappa + 0^+} dp e^{i\kappa - 0^+}$$

$$= \frac{i}{2}\kappa \sum_{\nu_\alpha > \kappa/2} \text{Res} \left( \frac{4p^2 + \kappa^2}{e^{i\nu_\alpha} + 1} : p = i\nu_\alpha \right)$$

$$= 2iT\kappa \sum_{\nu_\alpha > 0} \left( \frac{1}{4} - \frac{\nu_\alpha^2}{\kappa^2} \right)$$

$$= 2(N_f + 1)T\kappa \left[ -\frac{4}{3}\kappa^2 (4N_f^2 + 8N_f + 3) \right],$$

where $N_f \geq 0$ is the maximum integer such that $\kappa^2 \geq 4\nu_\alpha/N_f$. Combining (A.2) with (A.3), one finds that all terms except $\Delta$ cancel out, leading the discontinuity of Eq.(2.6) to

$$F_f(i\kappa + 0^+) - F_f(i\kappa - 0^+) = -\frac{(i\kappa + 0^+)^2}{24\pi^2} \ln \frac{(i\kappa + 0^+)^2}{\mu^2} + \frac{(i\kappa - 0^+)^2}{24\pi^2} \ln \frac{(i\kappa - 0^+)^2}{\mu^2}$$

$$- \frac{1}{4\pi\kappa} \left( \int_{0^+}^{0^+} - \int_{0^-}^{0^-} \right) dp e^{i\kappa + 0^+} dp e^{i\kappa - 0^+} + \Delta$$ (A.17)

$$= \frac{i\kappa^2}{12\pi} - \frac{1}{4\pi\kappa} \int_{0^-}^{0^+} dp (4p^2 + \kappa^2) + \Delta$$ (A.18)

$$= \Delta$$ (A.19)

in agreement with Eq.(2.11).

**B. WKB approximation for a large and imaginary $q$**

For imaginary momentum, we set $\hat{q} = i\kappa$, then Eq.(3.10) becomes

$$\frac{d^2 A_0}{du^2} + \frac{\kappa^2}{u (1 - u^2)} A_0 = 0$$ (B.1)

We are going to find out $A_0(0|\kappa)$ by solving this equation for $\kappa \gg 1$ with WKB approximation.

Near the $u = 0$ and $u = 1$ regions, the equation of motion can be rewritten as

$$\frac{d^2 A_0}{du^2} + \frac{\kappa^2}{u} A_0 = 0$$ (B.2)

$$\frac{d^2 A_0}{du^2} + \frac{\kappa^2}{2(1 - u)} A_0 = 0$$ (B.3)
respectively, which can be reduced to the standard Bessel equations with the solutions

\[ A_0(u \to 0|\kappa) = \sqrt{u}[C_1 H_1^{(1)}(2\kappa \sqrt{u}) + C_2 H_1^{(2)}(2\kappa \sqrt{u})] \]  
\[ A_0(u \to 1|\kappa) = \sqrt{1-u} J_1(\kappa \sqrt{2(1-u)}) \]

where \( H_1^{(1)} \) and \( H_1^{(2)} \) are the first and second kind of Hankel functions respectively, and \( J_1 \) is the Bessel function. \( C_1 \) and \( C_2 \) are the combination coefficients. Notice that in the solution near \( u = 1 \), considering the vanishing contribution from the horizon, one must reserve only one independent solution.

When \( \kappa \sqrt{u(1-u^2)} \gg 1 \), one could implement the WKB approximation and obtain

\[ A_0^{WKB} = [u(1-u^2)]^{\frac{1}{4}} \left[ C_+ \exp \left( i\kappa \int_u^1 \frac{dx}{\sqrt{x(1-x^2)}} \right) + C_- \exp \left( -i\kappa \int_u^1 \frac{dx}{\sqrt{x(1-x^2)}} \right) \right] \]

In the matching region of validity where \( 1-u << 1 \) but \( \kappa \sqrt{2(1-u)} \gg 1 \), both \( A_0^{WKB} \) and \( B.4 \) work as the solution of \( B.1 \), one could expand \( B.5 \) as

\[ A_0(u|\kappa) \approx \frac{2^\frac{1}{4}}{\sqrt{\pi\kappa}} (1-u)^{\frac{1}{4}} \sin \left[ \kappa \sqrt{2(1-u)} - \frac{\pi}{4} \right] \]

\[ \approx \frac{2^\frac{1}{4}}{i\sqrt{\pi\kappa}} (1-u)^{\frac{1}{4}} \left[ \exp \left( i\kappa \sqrt{2(1-u)} - \frac{\pi}{4} \right) - \exp \left( -i\kappa \sqrt{2(1-u)} + \frac{\pi}{4} \right) \right] \]

and rewrite the WKB solution \( B.6 \) as

\[ A_0^{WKB}(u|\kappa) = 2^{\frac{1}{4}}(1-u)^{\frac{1}{4}} \left[ C_+ \exp(i\kappa \sqrt{2(1-u)}) + C_- \exp(-i\kappa \sqrt{2(1-u)}) \right] \]

Then one obtains

\[ C_+ = \frac{1}{2i\sqrt{\pi\kappa}} e^{-i\frac{\pi}{4}}, C_- = C_+^* \]

and the WKB solution \( B.6 \) becomes

\[ A_0^{WKB} = \frac{u^{\frac{1}{4}}(1-u^2)^{\frac{1}{4}}}{2i\sqrt{\pi\kappa}} \left[ \exp \left( i\kappa \int_u^1 \frac{dx}{\sqrt{x(1-x^2)}} - i\frac{\pi}{4} \right) - \exp \left( -i\kappa \int_u^1 \frac{dx}{\sqrt{x(1-x^2)}} + i\frac{\pi}{4} \right) \right] \]

This expression remains efficient in the region where \( u << 1 \) but \( 2\kappa \sqrt{u} \gg 1 \) and we have

\[ A_0^{WKB}(u|\kappa) \approx \frac{u^{\frac{1}{4}}}{2i\sqrt{\pi\kappa}} \left[ \exp \left( i\kappa \delta - i\frac{\pi}{4} - 2i\kappa \sqrt{u} \right) \right] - \left[ \exp \left( -i\kappa \delta + i\frac{\pi}{4} + 2i\kappa \sqrt{u} \right) \right] \]

where \( \delta \equiv \frac{1}{2\sqrt{\pi} \Gamma^2(\frac{1}{4})} \). In Eq.\( B.12 \) we have used the approximation

\[ \int_u^1 \frac{dx}{\sqrt{x(1-x^2)}} = \int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} - \int_0^u \frac{dx}{\sqrt{x(1-x^2)}} \approx \delta - 2\sqrt{u} \]

when \( u << 1 \). We match the WKB solution to the analytic form of Eq.\( B.4 \) in \( u << 1 \), and the solution all the way to \( u = 0 \) is

\[ A_0^{WKB}(u|\kappa) = \frac{\sqrt{u}}{2i} \left[ e^{-i\kappa \delta} H_1^{(1)}(2\kappa \sqrt{u}) - e^{i\kappa \delta} H_1^{(2)}(2\kappa \sqrt{u}) \right] = A_0(u \to 0|\kappa). \]

In the calculation we used the approximation formulae of Hankel function when \( 2\kappa \sqrt{u} \gg 1 \)

\[ H_1^{(1)}(2\kappa \sqrt{u}) \approx -\frac{u^{-\frac{1}{4}}}{\sqrt{\pi\kappa}} \exp \left[ i \left( 2\kappa \sqrt{u} + \frac{\pi}{4} \right) \right], \]

\[ H_1^{(2)}(2\kappa \sqrt{u}) \approx -\frac{u^{-\frac{1}{4}}}{\sqrt{\pi\kappa}} \exp \left[ -i \left( 2\kappa \sqrt{u} + \frac{\pi}{4} \right) \right]. \]
According to Eq. (3.9), the correlator of R-charge density with imaginary momentum reads

\[ F(i\kappa) = -\frac{N_c^2 T^2 \kappa e^{-i\delta H_0^{(1)}(2\kappa \sqrt{\varepsilon})} - e^{i\delta H_0^{(2)}(2\kappa \sqrt{\varepsilon})}}{8\sqrt{\varepsilon}} \]

\[ \approx \frac{N_c^2}{16\pi^2} \left[ -\kappa^2 \left( \ln \frac{2}{\kappa l} - \gamma_E \right) - \frac{1}{2} \pi \kappa^2 \tan(\delta \kappa) \right] \]

where \( l = \varepsilon/(\pi^2 T^2) \).

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\[ \sqrt{g} = \pi T \left( \frac{\kappa^2}{4\varepsilon} \right)^{D-1} \Rightarrow = \frac{1}{2} (\pi T)^D u^{-D-1} \text{ with } D \to 4^- \]

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