Representations of $ax+b$ group and Dirichlet Series

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Abstract

Let $G$ be the $ax+b$ group. There are essentially two irreducible infinite dimensional unitary representations of $G$, $(\mu, L^2(\mathbb{R}^+))$ and $(\mu^*, L^2(\mathbb{R}^+))$. In this paper, we give various characterizations about smooth vectors of $\mu$ and their Mellin transforms. Let $\mathfrak{d}$ be a linear sum of delta distributions supported on the the positive integers $\mathbb{Z}^+$. We study the Mellin transform of the matrix coefficients $\mu_{\delta,f}(a)$ with $f$ smooth. We express these Mellin transforms in terms of the Dirichlet series $L(s, \mathfrak{d})$. We determine a sufficient condition such that the generalized matrix coefficient $\mu_{\delta,f}$ is a locally integrable function and estimate the $L^2$-norms of $\mu_{\delta,f}$ over the Siegel set. We further derive an inequality which may potentially be used to study the Dirichlet series $L(s, \mathfrak{d})$.

1 Introduction

Let $G$ be the $ax+b$ group, the semidirect product $\mathbb{R} \rtimes \mathbb{R}^+$. Here $\mathbb{R}^+$ is the multiplicative group and $\mathbb{R}$ is the additive group. The group $G$ can be realized as the matrix group consisting of

$$\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad (a \in \mathbb{R}^+, t \in \mathbb{R}).$$

For simplicity, we denote the first factor by $a$ and second factor by $b_t$. There are two equivalence classes of irreducible infinite dimensional unitary representations, $\mu$ and its contragredient $\mu^*$. The representation $\mu$ can be modeled on $L^2(\mathbb{R}^+)$ by defining

$$\mu(ab_t)f(x) = a^{-\frac{1}{2}} \exp(-2\pi i a^{-1}x) f(a^{-1}x).$$

The unitary representation $(\mu, L^2(\mathbb{R}^+))$ is well-known (3). Let $\mathcal{H}^\infty$ be the Frechet space of smooth vectors in $\mu$. The first result we proved in this paper is a characterization of smooth vectors in terms of their Mellin transform $\mathcal{M}$.

**Theorem 1.1** Let $g(z)$ be an analytic function on the right half plane $\mathbb{H} = \{ \Re(z) > \frac{1}{2} \}$. Then $g(z) \in \mathcal{M}(\mathcal{H}^\infty)$ if and only if the following hold

1. For any $n > 0$ and any finite closed interval $I$ in $(\frac{1}{2}, \infty)$, $g(\sigma + is_1)$ is bounded by $C_{I,n}|\sigma + is_1|^{-n}$ for all $s_1 \in \mathbb{R}$ and $\sigma \in I$;

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2. $g(\sigma + is_1)$ has a $L^2$ limit $g(\frac{1}{2} + is_1)$ as $\sigma \to \frac{1}{2}$.

3. For any $n \geq 1$, $s_1^n g(\sigma + is_1)$ has a $L^2$ limit as $\sigma \to \frac{1}{2}$.

Let $(\mathcal{H}^*)^{-\infty}$ be the dual topological vector space of $\mathcal{H}^\infty$, equipped with the weak star topology. We can extend the representation $\mu$ to its dual topological vector space $(\mathcal{H}^*)^{-\infty}$. We obtain a representation $\mu$ on the distribution vectors. Since the Schwartz testing function $D(\mathbb{R}^+)$ are dense in $\mathcal{H}^\infty$, (Theorem 2.1), the space $(\mathcal{H}^*)^{-\infty}$ is a linear subspace of $D'(\mathbb{R})$ [3]. Hence, all our discussion can be carried out in the classical sense.

Let $d = \sum_{n=1}^{\infty} d_n \delta_n$ be a distribution in $D(\mathbb{R}^+)'$. Here $\delta_n$ is the Dirac $\delta$ function supported on $n$ with $n$ a positive integer. Suppose that $\{d_n\}$ grow slower than a polynomial. Then $d \in \mathcal{H}^{-\infty}$.

In [4], we proved that the classical definition of matrix coefficients can be extended to a pair of distributions and the resulting generalized matrix coefficients are distributions on the group $G$. Hence generalized matrix coefficient $\mu_{\phi}(ab_1)$ is defined for any $f \in (\mathcal{H}^*)^{-\infty}$.

**Theorem 1.2** Let $f$ be a locally integrable function in $\mathcal{H}^{-\infty}$ such that for some $K > 0$ and $\epsilon > 0$

$$\int_K^\infty a^\epsilon |f(a)|^2 da < \infty.$$ 

Then $\mu_{\phi}(ab_1)$ is a well-defined locally integrable function. In addition, for some positive constant $C_{\epsilon, \phi}$

$$|d_1|^2 \int_K^\infty a^\epsilon |f(a)|^2 da \leq \int_K^\infty \int_0^1 \mu_{\phi}(ab_1)|^2 a^\epsilon \frac{da}{a} \leq C_{\epsilon, \phi} \int_K^\infty a^\epsilon |f(a)|^2 da.$$ 

Our estimate is the $L^2$-norm of $\mu_{\phi}(ab_1)$ over the Siegel set.

Since $\{d_n\}$ is of polynomial growth, the Dirichlet series $L(s, d) = \sum_{n=1}^{\infty} d_n n^{-s}$ is well-defined in a right half plane. For $\phi \in (\mathcal{H}^*)^\infty$, the matrix coefficient $\mu_{\phi, \cdot}(b_T a)$ is a smooth function. It can be expressed in terms of Mellin transform.

**Theorem 1.3** Suppose that the sequence $\{d_n\}$ is bounded by a multiple of $n^\epsilon$ for any $\epsilon > 0$. Suppose that $L(s, d)$ is holomorphic and bounded by a polynomial on the strip $\{r \leq \Re(s) \leq 1 + \epsilon\}$ for some $\epsilon > 0$. Then for any $\phi \in \mathcal{H}^\infty$, we have

$$\mu_{\phi, \cdot}(b_T a) = \langle \mu(b_T a) \delta, \phi \rangle = \frac{1}{2\pi} \int \int L(r + s_1 i, \delta) \mathcal{M}\{ \phi(x) \exp -2\pi i T x\} \exp (-r - s_1 i) ds_1 dt.$$ 

The main result of this paper is the following inequality.

**Theorem 1.4** Let $d = \sum d_n \delta_n$ with $\{d_n\}$ bounded by $C_{\epsilon} n^{-\tau}$ for any $\tau > 0$. Let $T_1 \geq 1$, $\epsilon > 0$ and $C_{\epsilon} = \sum_{n=1}^{\infty} n^{-1-\epsilon} |d_n|^2$. Let $f \in L^2(\mathbb{R}, (1+x^r)dx)$. We have

$$\frac{1}{2} T_1 C_{\epsilon} \int_1^\infty a^\epsilon |f(a)|^2 da \leq \int_1^\infty \int_0^{T_1} |\mu_{\phi, \cdot}(b_T a)|^2 dT a^\epsilon \frac{da}{a} \leq 2 T_1 C_{\epsilon} \int_1^\infty a^\epsilon |f(a)|^2 da.$$ 

Let $\chi_{[1, \infty)}$ be the indicator function of $[1, \infty)$. Then there exists a constant $C_{\epsilon, T_1}$ such that

$$\int_0^{T_1} \|\mathcal{M}^\ast \{ \mu_{\phi, \cdot}(b_T a) \chi_{[1, \infty)}(a) \} (s_1) \|_{L^2(\mathbb{R})}^2 \leq C_{\epsilon, T_1} \|\mathcal{M}^\ast (f \chi_{[1, \infty)})(s_1) \|_{L^2(\mathbb{R})}^2.$$
Here \( \mathcal{M}^\pm \{ * \}(s_1) \) is Fourier-Mellin transform, a variation of \( \mathcal{M} \{ * \}(\frac{x}{2} + s_1 i) \).

Notice that Mellin transform \( \mathcal{M}^\pm \{ \mu_{\Phi, f}(b_T a) \chi_{[1, \infty]}(a) \}(s_1) \) may be obtained from \( \mathcal{M}^\pm \{ \mu_{\Phi, f}(b_T a) \}(s_1) \) by solving a Riemann-Hilbert problem. By Theorem 1.3 our inequality may then be expressed in terms of \( L(s, \Phi) \). Furthermore, if we replace the \( ax + b \) group by \( SL(2, \mathbb{R}) \) and assume that \( \{ d_n \} \) are the Fourier coefficients of an automorphic form, then the factor \( \chi_{[1, \infty]} \) can be removed. We will be able to bound certain \( L^2 \)-norm of the Mellin transform

\[
\mathcal{M}^\pm \{ \mu_{\Phi, f}(g a) \}(s_1)
\]

by certain \( L^2 \)-norm of the form \( M^{\frac{1+2s}{2}} \{ f \}(s_1) \). This will shed some lights on the behavior of the Dirichlet series \( L(s, \Phi) \).

2 Representations of \( ax + b \) group

Let us define \( ax + b \) group to be the semidirect product of \( A \cong \mathbb{R}^+ \) and \( B \cong \mathbb{R} \). More precisely, for \( a \in \mathbb{R}^+ \) and \( b_t \in \mathbb{R} \). define \( (a, b_t) = (a, 0)(1, b_t) = (1, b_{a^{-1}t})(a, 0) \). We may simply write \( ab_t \) for \( (a, b_t) \).

Then

\[
ab_t = b_{a^{-1}t}a.
\]

The product rule is given by

\[
ab_au'b_{u'} = aa'b_{u+t+u'}.
\]

Let \( G \) be the \( ax + b \) group. The group \( G \) can be parametrized by \( ab_t \) or \( b_T a \) with \( T = a^{-1}t \). The standard left invariant Haar measure is given by \( \frac{da}{a}dt = dadT \). The standard right invariant measure is given by \( \frac{da}{a}dT = a^{-1}\frac{da}{a}dt \).

The unitary dual of \( G \) is easy to describe. There are one dimensional representations parametrized by \( i\lambda \in i\mathbb{R} \):

\[
\chi_{i\lambda} : ab_t \to a^{i\lambda}.
\]

There are two infinitely dimensional representations \( (\mu, L^2(\mathbb{R}^+)) \) and \( (\mu^*, L^2(\mathbb{R}^+)) \) defined as follows

\[
\mu(a)f(x) = a^{-\frac{1}{2}}f(a^{-1}x), \quad \mu^*(a)f(x) = a^{\frac{1}{2}}f(a^{-1}x);
\]

\[
\mu(b_t)f(x) = \exp(-2\pi i tx)f(x), \quad \mu^*(b_t)f(x) = \exp(2\pi i tx)f(x);
\]

for \( f \in L^2(\mathbb{R}^+) \), where \( \mathbb{R}^+ \) is equipped with the Euclidean measure. See for example [3] for details of the unitary dual of \( G \). Notice that \( \mu^* \) is simply the contragredient representation of \( \mu \). The representations \( \mu \) and \( \mu^* \) differ by a complex conjugation on the action of \( b_t \). We will mainly focus on \( \mu \). The support of \( \mu|_{\mathbb{R}^+} \) is the negative half line \( \mathbb{R}^- \) and the support of \( \mu^*|_{\mathbb{R}^+} \) is the positive half line \( \mathbb{R}^+ \).

We start with the following lemma about the action of the Lie algebra \( \mathfrak{g} \).

Lemma 2.1 The image of the universal enveloping algebra \( \mu(U(\mathfrak{g})) \) is spanned by

\[
\{x^m \frac{d^m}{dx^n}, m \geq n \geq 0 \}.
\]

Proof: Let \( H \) be the infinitesimal generator of \( A \) and \( E \) be the infinitesimal generator of \( B \). Then \( \mu(H) = -\frac{1}{2} - x \frac{d}{dx} \) and \( \mu(E) = -2\pi ix \). Since the identity 1 is contained in the universal enveloping
algebra $U(g)$, $\mu(U(g))$ is generated by $\{x, x^\frac{d}{dx}\}$ as a ring. One can then proceed by induction on $n$ to show that $x^n\frac{d^n}{dx^n} \in \mu(U(g))$ for any $n \geq 0$. It follows that

$$\{x^m\frac{d^n}{dx^n}, m \geq n \geq 0\} \subseteq \mu(U(g)).$$

Conversely, $\mu(U(g))$ is spanned by $x^k(x\frac{d}{dx})^n$ since the Lie bracket $[x\frac{d}{dx}, x] = x$. But $(x\frac{d}{dx})^n$ is a linear combination of $x^k\frac{d^k}{dx^k}$ with $0 \leq k \leq n$. Hence

$$\mu(U(g)) \subseteq \{x^m\frac{d^n}{dx^n}, m \geq n \geq 0\}.$$ 

\[\square\]

2.1 smooth vectors

Let $(\mu, H = L^2(\mathbb{R}^+))$ be the irreducible unitary representation of $G$ defined above. Let $H^\infty$ be the space of smooth vectors equipped with the canonical Frechet topology defined by seminorms $\{|Df|; D \in \mu(U(g))\}$. Sometimes the seminorm $|Df|$ is denoted by $\|f\|_D$.

**Lemma 2.2**

$$H^\infty = \{f \in C^\infty(\mathbb{R}^+) | x^m\frac{d^n}{dx^n}f \in L^2(\mathbb{R}^+) \ \forall \ m \geq n \geq 0\}.$$ 

Proof: Let $f$ be a smooth vector. Then $Df \in L^2(\mathbb{R})$ for any $D \in \mu(U(g))$. The higher order derivatives of $f$ are all locally $L^2$, hence locally $L^1$. Therefore any order derivatives of $f$ are continuous. This show that $f \in C^\infty(\mathbb{R}^+)$. \[\square\]

We shall now give a more detailed description of $H^\infty$.

**Theorem 2.1** $f(x) \in H^\infty$ if and only if $f \in C^\infty(\mathbb{R}^+)$ and the following are satisfied

1. For any $n \geq 0$, we have $x^n\frac{d^n}{dx^n}f \in L^2(0, \delta)$ for some $\delta > 0$;
2. For any $n, m \in \mathbb{N}$ and some $k > 0$, there exists a constant $c_{m,n,k}$ such that

$$|x^m\frac{d^n}{dx^n}f(x)| < c_{m,n,k}(1 + |x|)^{-m} \quad (x > k).$$

Roughly, the derivatives of $f$ must not behave too badly near zero and must be rapidly decaying near infinity. This follows from Sobolev inequality.

Proof: To prove "if" part, assume (1) and (2) and $m \geq n \geq 0$. From (1), we have $x^m\frac{d^n}{dx^n}f \in L^2(0, \delta)$. By the rapidly decaying condition (2), $x^m\frac{d^n}{dx^n}f \in L^2(k, \infty)$. Finally, since $f \in C^\infty(\mathbb{R})$, $x^m\frac{d^n}{dx^n}f$ will be bounded between $\delta$ and $k$, hence in $L^2(\delta, k)$. Therefore, $x^m\frac{d^n}{dx^n}f \in L^2(\mathbb{R}^+)$.
Conversely, let \( m \geq n \geq 0 \), (1) is obvious. Observe that

\[
|a^m \frac{d^n f(a)}{dx^n} - a^n f(1)| = |\int_1^a (x^m \frac{d^n f}{dx^n}) \, dx|
\]

\[
= |\int_1^a mx^{m-1} \frac{d^n f}{dx^n} + x^m \frac{d^{n+1} f}{dx^{n+1}} \, dx| 
\]

\[
\leq m \int_1^a |x^{m-1} \frac{d^n f}{dx^n}| \, dx + \int_1^a |x^m \frac{d^{n+1} f}{dx^{n+1}}| \, dx 
\]

\[
\leq m \left( \int_1^a |x^m \frac{d^n f}{dx^n}|^2 \, dx \right)^{\frac{1}{2}} \left( \int_1^a |x^{-1}|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_1^a |x^m \frac{d^{n+1} f}{dx^{n+1}}|^2 \, dx \right)^{\frac{1}{2}} \left( \int_1^a |x^{-1}|^2 \, dx \right)^{\frac{1}{2}} 
\]

\[
\leq C_{n,m} 
\]

Hence \( x^m \frac{d^n f}{dx^n}(x) \) is bounded by a constant. (2) follows. \( \square \)

We shall observe that \( C^\infty_0(\mathbb{R}^+) \subseteq \mathcal{H}^\infty \).

### 2.2 2nd Characterization of \( \mathcal{H}^\infty \)

For any \( x \in \mathbb{R}^+ \), let \( h = \ln x \). Define a unitary operator \( \mathcal{I} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}) \) by

\[
\mathcal{I}(f)(h) = \left( \exp \frac{1}{2} h \right) f(\exp h) 
\]

It is easy to check that \( \mathcal{I}(\mu(H)f)(h) = -\frac{d}{dh}(\mathcal{I}(f)) \) and \( \mathcal{I}(\mu(E)f)(h) = (-2\pi i \exp h)\mathcal{I}(f)(h) \). Immediately, we have

**Lemma 2.3** \( \phi \in \mathcal{I}(\mathcal{H}^\infty) \) if and only if

1. \( \phi \in C^\infty(\mathbb{R}) \); 
2. for any \( n \geq 0 \), \( \frac{d^n \phi(h)}{dh^n} \in L^2(-\infty, k) \) for some \( k \); 
3. \( \frac{d^n \phi(h)}{dh^n} \) is bounded by \( C_{n,m}(1 + \exp h)^{-m} \) for any \( m > 0 \) and \( h > 0 \).

We need to analyze (2) a little deeper. Since \( \phi' \in L^2(\mathbb{R}) \), for \( h < -1 \),

\[
|\phi(-1) - \phi(h)| = \left| \int_{-1}^{-h} \phi'(y) \, dy \right| \leq \sqrt{-1 - h} \|\phi'\|_{L^2}. 
\]

Hence \( \phi(h) \) is bounded by a multiple of \( \sqrt{|h|} \) at \( -\infty \). If we work a little harder, we have

**Theorem 2.2** If \( \phi(h) \in \mathcal{I}(\mathcal{H}^\infty) \), then \( \phi(h) \) and all its higher order derivatives are bounded.

**Proof:** Consider the Fourier transform

\[
\mathcal{F}\phi(\xi) = \int \phi(h) \exp(-2\pi i h \xi) \, dh. 
\]

Since \( \phi^{(n)}(h) \in L^2(\mathbb{R}) \), we have \( \xi^n \mathcal{F}\phi(\xi) \in L^2(\mathbb{R}) \) for any \( n \geq 0 \). Then

\[
|\phi(h)| = \left| \int \mathcal{F}\phi(\xi) \exp(2\pi i h \xi) \, d\xi \right| \leq \int_{-\infty}^{-1} |\mathcal{F}\phi(\xi)| \, d\xi + \int_1^\infty |\mathcal{F}\phi(\xi)| \, d\xi + C. 
\]

The first two terms can all be bounded by the \( L^2 \)-norm of \( \xi \mathcal{F}\phi(\xi) \). Hence \( \phi(h) \) is bounded. Similarly, \( \frac{d^n \phi}{dh^n}(h) \) is bounded for all \( n \geq 0 \). \( \square \)

This estimate applies to all \( \phi(h) = \exp \frac{1}{2} h f(\exp h) \). Returning to \( f \), we have
Corollary 2.1 Let \( f \in \mathcal{H}^\infty \). Then all the derivatives of \( f \) will be bounded by a multiple of \( x^{-\frac{1}{2}} \) near 0.

3 Fourier-Mellin transform of \( \mathcal{H}^\infty \)

Let \( f \) be a locally integrable function on \( \mathbb{R}^+ \). Narrowly speaking, Mellin transform

\[
\mathcal{M}f(s) = \int f(x)x^s \frac{dx}{x}
\]

is required to have a band of convergence. In this paper, we will loosely define \( \mathcal{M}f(s) = \int f(x)x^s \frac{dx}{x} \) as long as the integral converges absolutely. Indeed, if \( f(x)x^{\sigma-1} \in L^1(\mathbb{R}^+) \) we can define

\[
\mathcal{M}^\sigma f(s_1) = \int f(x)x^{\sigma+is_1} \frac{dx}{x}.
\]

Throughout this paper \( s_1 \) will always be real and \( s \) will always be complex with its imaginary part \( is_1 \).

If \( x = \exp h \), then

\[
\mathcal{M}^\sigma f(s_1) = \int f(\exp h)\exp(\sigma + is_1)hdh.
\]

Notice that \( \mathcal{M}^\sigma \) can be extended to \( L^2 \)-functions and beyond, by Fourier transform. We call \( \mathcal{M}^\sigma \) the Fourier-Mellin transform. In this context, the Fourier-Mellin inversion for \( L^2 \)-functions will be given by

\[
f(x) = \frac{1}{2\pi} \int \mathcal{M}^\sigma f(s)x^{-\sigma-s_1} ds_1.
\]

Unless such interpretation is needed, we will generally retain the usage of Mellin-transform \( \mathcal{M}f(s) \) or Mellin inversion.

Theorem 3.1 Suppose that \( f \in \mathcal{H}^\infty \). Then

1. \( \mathcal{M}f(s) \) is defined and holomorphic on the open half plane \( \mathbf{H} = \{ \Re s > \frac{1}{2} \} \);
2. \( \mathcal{M}f(s) \) decays faster than any \( |s|^{-n} \) along any compact vertical stripe in \( \mathbf{H} \);
3. \( \mathcal{M}^\sigma f(s_1) \) has an \( L^2 \)-limit as \( \sigma \to \frac{1}{2} \). The \( L^2 \)-limit is the Fourier-Mellin transform \( \mathcal{M}^{\frac{1}{2}} f(s_1) \).

Proof: Since \( f \) is bounded by a multiple of \( x^{-\frac{1}{4}} \) near 0 and \( x^{-n} \) near \( \infty \), \( \mathcal{M}f(s) \) is well-defined and analytic on the open half plane \( \mathbf{H} \). (1) is proved.

Observe that

\[
\mathcal{M}f(\sigma + is_1) = \int \mathcal{I}(f)(h)\exp(\sigma - \frac{1}{2})h\exp is_1 h dh
\]

in terms of Fourier transform. By Lemma 2.3 and Theorem 2.2 if \( \sigma > \frac{1}{2} \), \( \mathcal{I}(f)(h)\exp(\sigma - \frac{1}{2})h \) and all its derivatives will be in \( L^1 \). It follows that \( \mathcal{M}f(\sigma + is_1) \) decays faster than any \( |s_1|^{-n} \) as \( |s_1| \to \infty \) for \( \sigma \) in a finite closed interval in \(( \frac{1}{2}, \infty )\). (2) is proved.

Let \( \sigma > \frac{1}{2} \). Observe that \( \mathcal{I}(f)(h)\exp(\sigma - \frac{1}{2})h \to \mathcal{I}(f)(h) \) in \( L^2 \) as \( \sigma \to \frac{1}{2} \). On the Fourier-Mellin transform side, we have \( \mathcal{M}^\sigma f(s_1) \to \mathcal{M}^{\frac{1}{2}} f(s_1) \) in \( L^2(\mathbb{R}) \). □.

We summarize how the Lie algebra act on the Mellin transform \( \mathcal{M}f(s) \).
Lemma 3.1. Let $f \in \mathcal{H}^\infty$. Then $\mathcal{M}(xf)(s) = \mathcal{M}(f)(s + 1)$ and $\mathcal{M}(\mu(H)f)(s) = (s - \frac{1}{2})\mathcal{M}(f)(s)$ for any $s \in \mathbb{H}$.

Notice that Theorem 3.1 (1)(2) are preserved under the action of Lie algebra. We can now characterize the Mellin transform of $\mathcal{H}^\infty$.

Theorem 3.2. Let $g(s)$ be an analytic function on $\mathbb{H}$. Then $g(s) \in \mathcal{M}(\mathcal{H}^\infty)$ if and only if the following hold:

1. For any $n > 0$ and any finite closed interval $I$ in $(\frac{1}{2}, \infty)$, $g(\sigma + is_1)$ is bounded by $C_{I,n}|\sigma + is_1|^{-n}$ for all $s_1 \in \mathbb{R}$ and $\sigma \in I$;
2. $g(\sigma + is_1)$ has a $L^2$ limit $g(\frac{1}{2} + is_1)$ as $\sigma \to \frac{1}{2}$.
3. For any $n \geq 1$, $s_1^n g(\sigma + is_1)$ has a $L^2$ limit as $\sigma \to \frac{1}{2}$.

Proof. The only if part can be easily seen from Theorem 3.1 and Lemma 3.1. To show "if" part, assume (1)(2)(3). We must show that $g(z) = \mathcal{M}f(z)$ for some $f(z) \in \mathcal{H}^\infty$. First the $L^2$-limit of $s_1^n g(\sigma + is_1)$ will be $s_1^n g(\frac{1}{2} + is_1)$ as $\sigma \to \frac{1}{2}$ for $s_1$ in any compact interval. Hence $s_1^n g(\sigma + is_1) \to s_1^n g(\frac{1}{2} + is_1)$ in $L^2(\mathbb{R})$ as $\sigma \to \frac{1}{2}$.

Let

$$f(x) = \frac{1}{2\pi} \int g(\sigma + s_1 i) x^{-\sigma - is_1} ds_1 \quad (\sigma > \frac{1}{2}).$$

By (1), the integral above is independent of $\sigma$. In addition, for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi} \int g(\sigma + s_1 i) \prod_{j=0}^{n-1} (-\sigma - is_1 - j) x^{-\sigma - is_1 - n} ds_1$$

converges absolutely and defines the higher order derivative $\frac{d^n f}{dx^n}(x)$. Hence $f(x) \in C^\infty(\mathbb{R}^+)$. By letting $\sigma$ arbitrarily large, $\frac{d^n f}{dx^n}(x)$ is uniformly bounded by a multiple of $|x|^{-m}$ for $x > 1$. This proves (2) of Theorem 2.1. Near 0, by letting $\sigma \to \frac{1}{2}$, $f(x)$ is bounded by a multiple of $x^{-\frac{1}{2} - \delta}$ for arbitrarily small $\delta > 0$. It follows that $\mathcal{M}f(s)$ is well-defined on $\mathbb{H}$. By Mellin inversion, $\mathcal{M}f(s) = g(s)$ for any $s \in \mathbb{H}$.

To show that $f \in \mathcal{H}^\infty$, it suffices to show (1) of Theorem 2.1 namely $x^n \frac{d^n f}{dx^n} \in L^2(0,1)$. By Fourier-Mellin inversion, $x^n f(x) \in L^2(\mathbb{R}^+, \frac{1}{2})$ and $|x^n f(x)| \in L^2(\mathbb{R}^+, \frac{1}{2})$ for $x > 1$. By (2), $|x^n f(x)|$ has a $L^2$ limit in $L^2(\mathbb{R}^+, \frac{1}{2})$ as $\sigma \to \frac{1}{2}$. This $L^2$-limit must be the pointwise limit $f(x)$. Hence $f(x) \in L^2(\mathbb{R}^+)$. Similarly, by (3) and Lemma 3.1, $x^n \frac{d}{dx} + \frac{1}{2} f(x) \in L^2(\mathbb{R}^+)$. By Theorem 2.1, $f \in \mathcal{H}^\infty$. □

4 Dirichlet series and distributions supported on $\mathbb{Z}^+$

Recall that $C^\infty(\mathbb{R}^+) \subseteq \mathcal{H}^\infty$. In addition, for any compact interval $K \subset \mathbb{R}^+$, the seminorms of $\mathcal{H}^\infty$ yield the seminorms of $C^\infty(K)$ when restricted to $C^\infty(K)$. Thus every continuous functional on the Frechet space $\mathcal{H}^\infty$ is a distribution on $\mathbb{R}^+$ in the sense of Schwartz. However a distribution on $\mathbb{R}^+$ may not be a distribution on $\mathcal{H}^\infty$. Let $\mathcal{H}^{-\infty}$ be the space of continuous linear functionals on $\mathcal{H}^*$, equipped with the weak-star topology.
Let \( d = (d_1, d_2, \ldots, d_n \ldots) \) be an infinite sequence of complex numbers. We say that \( d \) is of polynomial growth if there exists a \( k \in \mathbb{N} \) and \( C_k > 0 \) such that \( d_n < C_k n^k \) for all \( n \in \mathbb{N} \). Let \( \delta_n \) be the \( \delta \) distribution on \( n \in \mathbb{N} \). Define the distribution \( \delta = \sum_{n=1}^{\infty} d_n \delta_n \). If \( d \) is of polynomial growth, define the Dirichlet series \( L(s, \delta) = \sum_{n=1}^{\infty} d_n n^{-s} \). \( L(s, \delta) \) is well-defined for \( \Re(s) > k + 1 \).

### 4.1 Distributions supported on \( \mathbb{N} \)

**Theorem 4.1** If \( d \) is of at most polynomial growth, then \( \delta \in \mathcal{H}^{-\infty} \).

**Proof:** It suffices to show that there exists a \( F \in \mathcal{H} \) and \( D \in U(g) \) such that \( \delta = \mu(D)F \). Consider the antiderivative of \( \delta \):

\[
  f(x) = \sum_{n \leq x} d_n \quad (x > 0).
\]

Obviously, \( f(x) \leq C_k|x|^{k+1} \) for some \( C_k > 0 \) and \( k \in \mathbb{N} \). Let \( F(x) = x^{-k-2}f(x) \). Then \( F(x) \in \mathcal{H} \) and

\[
  \delta = \frac{d}{dx} x^{k+2} F(x) = x^{k+2} \frac{d}{dx} F(x) + (k+2)x^{k+1} F(x) = (x^{k+2} \frac{d}{dx} + (k+2)x^{k+1}) F(x).
\]

Since \( x^{k+2} \frac{d}{dx} + (k+2)x^{k+1} \in \mu(U(g)), \delta \in \mathcal{H}^{-\infty} \). \( \square \)

The Dirichlet series \( L(s, \delta) \) is well-defined for \( s > k + 1 \) if \( \{d_n\} \) is bounded by \( \{C_k n^k\} \) uniformly. We say that \( L(s, \delta) \in \mathcal{R} \) if \( \{d_n\} \) is bounded by \( C \epsilon n^\epsilon \) for any \( \epsilon > 0 \). This is the Ramanujan condition. Clearly, \( L(s, \delta) \) is an analytic function on the open half plan \( \mathbb{H}_1 = \{z \in \mathbb{C} \mid \Re(z) > 1\} \). In addition, if \( L(s, \delta) \in \mathcal{R} \) and \( L(s, \delta') \in \mathcal{R} \), then \( L(s, \delta) L(s, \delta') \) is also analytic on \( \mathbb{H}_1 \). In fact \( L(s, \delta) L(s, \delta') \in \mathcal{R} \). The following is well-known.

**Lemma 4.1** The product \( L(s, \delta) L(s, \delta') = L(s, j) \) with

\[
  j_t = \sum_{mn=1} d_m d'_n.
\]

If \( L(s, \delta), L(s, \delta') \in \mathcal{R} \), then \( L(s, \delta) L(s, \delta') \in \mathcal{R} \).

Hence \( \mathcal{R} \) is a ring. We write \( j \) as \( \delta \delta' \) and \( j_t \) as \( \delta \delta' \).

### 4.2 \( \langle \delta, f \rangle \) in terms of Mellin transform

Let \( \mathcal{MR} \) be the subring of \( \mathcal{R} \) consisting of \( g(z) \in \mathcal{R} \) with meromorphic continuation to \( \mathbb{C} \). We shall see how the pairing \( \langle \delta, f \rangle \) can be expressed in terms of \( L(s, \delta) \).

**Lemma 4.2** Let \( f \in \mathcal{H}^\infty \) and \( L(s, \delta) \in \mathcal{MR} \). Then for any \( r > 1 \), we have

\[
  \langle \delta, f \rangle = \frac{1}{2\pi} \int L(r + s_1 i, \delta) Mf(r + s_1 i) ds_1.
\]

**Proof:** Let \( r > 1 \). By the characterization of Mellin transform of \( \mathcal{H}^\infty \) and Mellin inversion, we have

\[
  f(x) = \frac{1}{2\pi} \int Mf(r + is_1) x^{-r-is_1} ds_1.
\]
with $\mathcal{M}f(r+s_1i)$ fast decaying. Due to absolute convergence of $L(r+s_1i, \sigma) = \sum_{k=1}^{\infty} \frac{d_k}{k^{-r-s_1i}}$, we have

$$
\langle \sigma, f \rangle = \sum_{k=1}^{\infty} d_k f(k) = \frac{1}{2\pi} \sum_{k=1}^{\infty} d_k \int \mathcal{M}f(r+is_1)k^{-r-s_1i}ds_1
$$

$$
= \frac{1}{2\pi} \int \mathcal{M}f(r+is_1) \sum_{k=1}^{\infty} d_k k^{-r-s_1i}ds_1 = \frac{1}{2\pi} \int L(r+s_1i, \sigma)\mathcal{M}f(r+s_1i)ds_1.
$$

Now we can move the integral from $r+i\mathbb{R}$ to the left of $1+i\mathbb{R}$.

**Theorem 4.2** Let $f \in \mathcal{H}^{\infty}$, $1 \geq r > \frac{1}{2}$ and $L(s, \sigma) \in \mathcal{M}R$. Suppose that $L(s, \sigma)$ is holomorphic and bounded by a polynomial on the strip $\{r \leq \Re(s) \leq 1 + \epsilon\}$ for some $\epsilon > 0$. Then

$$
\langle \sigma, f \rangle = \frac{1}{2\pi} \int L(r+s_1i, \sigma)\mathcal{M}f(r+s_1i)ds_1.
$$

We shall make some remarks here. First this result can be easily extended to the situation that $L(s, \sigma)$ has finite number of poles in the stripe $\{r \leq \Re(s) \leq 1 + \epsilon\}$ by including the residue at the poles in the equation. Care should be taken to manage the distribution of infinitely many poles. But for the applications to $L$-functions, this theorem is sufficient. Secondly, this result can also be extended to the left half plane for $L$-functions. However, in our context, the Mellin transform $\mathcal{M}f$ is only defined on the right half plane. Hence there is a nature barrier at the critical line.

For any $\sigma > \frac{1}{2}$, $x^{\sigma-1+is_1}$ is a distribution in $\mathcal{H}^{-\infty}$. We can see this by writing $x^{\sigma-1+is_1}$ as $(x+1)^{\sigma-1+is_1}$ and notice that $x+1$ is an operator in $\mu(U(g))$.

**Corollary 4.1** Let $r > \frac{1}{2}$ and $L(s, \sigma) \in \mathcal{M}R$. Suppose that $L(s, \sigma)$ is holomorphic and bounded by a polynomial on the strip $\{r \leq \Re(s) \leq 1 + \epsilon\}$ for some $\epsilon > 0$. Then as a distribution in $\mathcal{H}^{-\infty}$,

$$
\sigma = \frac{1}{2\pi} \int L(r+s_1i, \sigma)x^{r+s_1i-1}ds_1.
$$

The right hand side is a weak-star integral with test functions in $\mathcal{H}^{\infty}$.

4.3 $\langle \mu(a)\sigma, f \rangle$ in terms of Mellin transform

Observe that

$$
\mathcal{M}(\mu^*(a^{-1})f)(s) = \int_{0}^{\infty} a^{\frac{s}{2}}f(ax)x^{s}dx = a^{\frac{s}{2}-s}\mathcal{M}f(s).
$$

**Corollary 4.2** Under the same assumption as Theorem 4.3

$$
\langle \mu(a)\sigma, f \rangle = \frac{1}{2\pi} \int L(r+s_1i, \sigma)\mathcal{M}f(r+s_1i)a^{\frac{s}{2}-r-s_1i}ds_1.
$$

or equivalently

$$
\mathcal{M}(\langle \mu(a)\sigma, f \rangle)(r - \frac{1}{2} + s_1i) = L(r+s_1i, \sigma)\mathcal{M}f(r+s_1i).
$$

9
5 Fundamental Inequality

We shall now consider the (generalized) matrix coefficient \( \langle \mu(ab_t) \delta, f \rangle \) with \( f \in (\mathcal{H}^\ast)\infty \). We may still regard \( f \) as a function in \( \mathcal{H}^\infty \) and keep in mind that the action of \( G \) will be given by \( \mu^\ast \). We have

\[
\langle \mu(ab_t) \delta, f \rangle = \langle \mu(b_t) \delta, \mu^\ast(a^{-1})f \rangle = \sum_{n \in \mathbb{Z}^+} a^2 d_n f(an) \exp(-2\pi itn).
\]

Since \( f(x) \) decays very fast, \( \sum_{n \in \mathbb{N}} \) converges as a Fourier series. We write this matrix coefficient as \( \mu_d, f(ab_t) \).

5.1 Fourier Series Estimates

By Parseval’s theorem, we have

**Lemma 5.1** Let \( \delta \) be of polynomial growth and \( f \in \mathcal{H}^\infty \), we have

\[
\int_0^1 |\mu_{\delta, f}(ab_t)|^2 dt = \sum_{n=1}^\infty |f(an)d_n|^2.
\]

**Theorem 5.1** Let \( L(s, \delta) \in \mathcal{R}, f \in \mathcal{H}^\infty \), and \( \epsilon > 0 \). Set \( C_\epsilon = \sum_{n=1}^\infty n^{-1-\epsilon}|d_n|^2 \). Then for any \( \delta \geq 0 \), we have

\[
|d_1|^2 \int_\delta^\infty a^\epsilon |f(a)|^2 da \leq \int_\delta^\infty \int_0^1 |\mu_{\delta, f}(ab_t)|^2 a^\epsilon da \leq C_\epsilon \int_\delta^\infty a^\epsilon |f(a)|^2 da.
\]

Proof: Since \( \delta \) satisfies the Ramanujan condition, \( C_\epsilon \) is well-defined. The lower bound follows from

\[
\int_0^1 |\mu_{\delta, f}(ab_t)|^2 dt \geq a^\epsilon |f(a)d_1|^2.
\]

The upper bound can be established as follows.

\[
\int_\delta^\infty \int_0^1 |\mu_{\delta, f}(ab_t)|^2 t a^\epsilon da \leq \int_\delta^\infty \sum_{n=1}^\infty |f(an)d_n|^2 a^\epsilon da
\]

\[
= \sum_{n=1}^\infty (|d_n|^2 \int_\delta^\infty a^\epsilon |f(an)|^2 da)
\]

\[
= \sum_{n=1}^\infty (|d_n|^2 \int_\delta^{\infty} a^\epsilon n^{-1-\epsilon}|f(a)|^2 da)
\]

\[
\leq \left( \sum_{n=1}^\infty n^{-1-\epsilon}|d_n|^2 \right) \int_\delta^\infty a^\epsilon |f(a)|^2 da.
\]

\( \Box \).

Notice that the domain we worked with \( \{ \delta < a, 0 \leq t \leq 1 \} \) is classically known as the Siegel domain. However, the measure we use is different. In particular, if \( \delta = 0 \), we have

\[
|d_1|^2 \int_{\mathbb{R}^+} a^\epsilon |f(a)|^2 da \leq \int_{\mathbb{R}^+} \int_0^1 |\mu_{\delta, f}(ab_t)|^2 a^\epsilon da \leq C_\epsilon \int_{\mathbb{R}^+} a^\epsilon |f(a)|^2 da.
\]
5.2 Generalized Matrix Coefficients as functions

Given any $f \in \mathcal{H}^{-\infty}$, we have shown in [4] that generalized matrix coefficient $\langle \mu(g) \delta, f \rangle$ is a distribution on $G$. The question then arises, for which $f$, $\langle \mu(g) \delta, f \rangle$ will be a locally integrable function. One simple guess is $f \in \mathcal{H}$. This turns out not to be the case for general $\delta$ satisfying the Ramanujan condition (\([12]\)). However, if we make assumption that for some $\epsilon > 0$ and $K > 0$,

$$
\int_K^{\infty} a^\epsilon |f(a)|^2 da < \infty
$$

then $\mu_\delta, f(ab_1)$ will be a locally integrable function. We first prove a weaker version of this.

**Theorem 5.2** Let $f$ be a locally square integrable function in $\mathcal{H}^{-\infty}$ such that for some $\epsilon > 0$, 

$$
\int_0^{\infty} a^\epsilon |f(a)|^2 da < \infty.
$$

Then $\mu_\delta, f(ab_1)$ is a well-defined locally square integrable function. In addition,

$$
|d_1|^2 \int_0^{\infty} a^\epsilon |f(a)|^2 da \leq \int_0^{\infty} \int_0^1 |\mu_\delta, f(ab_1)|^2 a^\epsilon \frac{da}{a} \leq C \epsilon \int_0^{\infty} a^\epsilon |f(a)|^2 da.
$$

Proof: Let $\{f_n\}$ be a sequence of function in $C_c^\infty(\mathbb{R}^+) \subseteq \mathcal{H}^\infty$ such that

$$
\int |f_n(a) - f(a)|^2 a^\epsilon da \rightarrow 0 \quad (\text{as } n \rightarrow \infty).
$$

In particular, $\{f_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^+, x^\epsilon dx)$. By Theorem [5,1], $\{\mu_\delta, f_n(ab_1)\}$ is a Cauchy sequence in $L^2(G, a^\epsilon dt \frac{da}{a})$. It has a $L^2$-limit $\psi$ under the measure $\mu^\prime dt \frac{da}{a}$. Hence, as distributions in $D(G)'$, $\mu_\delta, f_n(x) \rightarrow \psi(x)$ under the weak star topology. By [4], $\mu_\delta, f(g) = \psi(g)$ a.e.. For every $n$,

$$
|d_1|^2 \int_0^{\infty} a^\epsilon |f_n(a)|^2 da \leq \int_0^{\infty} \int_0^1 |\mu_\delta, f_n(ab_1)|^2 a^\epsilon \frac{da}{a} \leq C \epsilon \int_0^{\infty} a^\epsilon |f_n(a)|^2 da.
$$

By taking $n \rightarrow \infty$, our inequalities follow. $\square$

Knowing that $\mu_\delta, f(ab_1)$ is a square integrable function, by [4], we can approximate $\mu_\delta, f(ab_1)$ by $\mu_\delta, f_n(ab_1)$ as long as $f_n \rightarrow f$ under the weak star topology.

**Theorem 5.3** Let $f$ be a locally integrable function in $\mathcal{H}^{-\infty}$ such that for some $K > 0$ and $\epsilon > 0$

$$
\int_K^{\infty} a^\epsilon |f(a)|^2 da < \infty.
$$

Then $\mu_\delta, f(ab_1)$ is a well-defined locally integrable function. In addition,

$$
|d_1|^2 \int_K^{\infty} a^\epsilon |f(a)|^2 da \leq \int_K^{\infty} \int_0^1 |\mu_\delta, f(ab_1)|^2 a^\epsilon \frac{da}{a} \leq C \epsilon \int_K^{\infty} a^\epsilon |f(a)|^2 da.
$$

Proof: We may write $f$ as a sum of two locally integrable functions $f_1 + f_2$ with $f_1(x) = 0 \ (\forall x \geq K)$ and $f_2(x) = 0 \ (\forall x < K)$. Then $\mu_\delta, f(ab_1) = \mu_\delta, f_1(ab_1) + \mu_\delta, f_2(ab_1)$. By Theorem [5,2], $\mu_\delta, f_2(ab_1)$ is well-defined, locally square integrable. Since $f_1(x) = 0$ for any $x \geq K$, $\mu_\delta, f_1(ab_1) = \sum_{n \in \mathbb{Z}^+} a^\epsilon \delta_n f(an) \exp(-2\pi i\mu)$ is locally a finite sum of locally integrable function, hence locally integrable. In addition, $\mu_\delta, f_1(ab_1) = \cdots$
0 for any \( a \geq K \). It follows that \( \mu_{\delta,f}(ab_t) = \mu_{\delta,f_2}(ab_t) \) for any \( a \geq K \). By essentially the same proof as Theorem 5.2,

\[
|d_1|^2 \int_K^\infty a^e f(a)^2 da \leq \int_K^\infty \int_0^1 |\mu_{\delta,f}(ab_t)|^2 a^e \frac{da}{a} \leq C_\epsilon \int_K^\infty a^e |f(a)|^2 da.
\]

\( \Box \)

### 5.3 Fundamental Inequality

Define the subset of \( G, \Omega_{T_1} = \{ b_T a \mid T \in [0, T_1], a \in [1, \infty) \} \). In terms of \( ab_t \) coordinates, we have

\[
\Omega_{T_1} = \{ ab_t \mid t \in [0, aT_1], a \in [1, \infty) \}.
\]

**Theorem 5.4** Let \( L(s, \delta) \in R, T_1 \geq 1 \) and \( \epsilon > 0 \). Let \( f \in L^2(\mathbb{R}^+, (1 + x^e)dx) \). Set \( C_\epsilon = \sum_{n=1} C_n^{\epsilon-\epsilon} \). We have

\[
\frac{1}{2} T_1 C_\epsilon \int_1^\infty a^e |f(a)|^2 da \leq \int_1^\infty \int_0^{T_1} |\mu_{\delta,f}(b_T a)|^2 dT a^e \frac{da}{a} \leq 2T_1 C_\epsilon \int_1^\infty a^e |f(a)|^2 da.
\]

**Proof:** If \( f(x) \in L^2(\mathbb{R}^+, (1 + x^e)dx) \), then Theorem 5.2 holds. To prove the upper bound, we observe

\[
\int_1^\infty \int_0^{T_1} |\mu_{\delta,f}(b_T a)|^2 dT a^e \frac{da}{a} = \int_1^\infty \int_0^{aT_1} |\mu_{\delta,f}(ab_t)|^2 a^{-1} dT a^e \frac{da}{a} \\
\leq \int_1^\infty [aT_1] a^{\epsilon-1} \int_0^1 |\mu_{\delta,f}(ab_t)|^2 dt \frac{da}{a} \\
\leq \int_1^\infty 2aT_1 a^{\epsilon-1} \int_0^1 |\mu_{\delta,f}(ab_t)|^2 dt \frac{da}{a} \\
= 2T_1 \int_1^\infty a^{\epsilon} \int_0^1 |\mu_{\delta,f}(ab_t)|^2 dt \frac{da}{a} \\
\leq 2T_1 C_\epsilon \int_1^\infty a^e |f(a)|^2 da
\]

Here \([aT_1]\) is the minimal integer greater or equal to \( aT_1 \). We use the fact \([aT_1] \leq 2aT_1 \). The lower bound follows similarly. \( \Box \)

If \( T_1 \) is any positive real number, the inequalities still hold if we replace \( T_1 C_\epsilon \) be a constant \( C_{T_1} C_\epsilon \). In addition, we can allow \( a \in [\delta, \infty) \), then the inequalities will depend on \( \delta \). However, we will not have any similar bounds if \( a \in (0, \infty) \) since the behavior of \( f(ab_t) \) near \( a = 0 \) is hard to control without any additional assumption. Nevertheless, If the group is \( SL(2) \) and \( \delta \) comes from the Fourier coefficients of an automorphic form, we can bound the \( L^2 \)-norm of \( \mu_{\delta,f}(B_T a) \) over the whole space \( a \in (0, \infty) \) \((\mathfrak{B})\).
5.4 Inequality in terms of Mellin Transform

For any locally integrable function \( u(x) \) on \( \mathbb{R}^+ \), write

\[
u(x) = u_+(x) + u_-(x)\]

such that \( u_+(x) \) is supported on \([1, \infty)\) and \( u_-(x) \) is supported on \((0, 1]\). Let \( f(x) \) be a function in \( L^2(\mathbb{R}^+, (1 + x^\epsilon)dx) \). Then Theorem 5.4 asserts that

\[
\int_0^{T_1} \left| \mu_\phi,\exp(-2\pi ixT) f(a) \right|^2_{L^2(\mathbb{R}^+, a^\epsilon da)} dT \leq 2T \| f \|_{L^2(\mathbb{R}^+, a^\epsilon da)}^2.
\]

By Fubini’s theorem, for almost all \( T \in [0, T_1]\), \( \mu_\phi,\exp(-2\pi ixT) f(a)_+ \in L^2(\mathbb{R}^+, a^\epsilon da) \).

Recall that for \( 0 < \sigma \leq \frac{\epsilon}{2} \), The Fourier-Mellin transform \( \mathcal{M}_\sigma f \) is well-defined and

\[
\| \mathcal{M}_\sigma f \|^2 = \frac{1}{2\pi} \int_{\mathbb{R}^+} |f(x)|^2 x^{2\sigma - 1} dx
\]

We can now restate Theorem 5.5.

**Theorem 5.5** Let \( L(s, \mathfrak{d}) \in \mathcal{R}, T_1 \geq 1 \) and \( \epsilon > 0 \). Let \( f \in L^2(\mathbb{R}^+, (1 + x^\epsilon)dx) \). There exists a constant \( C_{\epsilon, T_1, \mathfrak{d}} \) such that

\[
\int_0^{T_1} \| \mathcal{M}_\frac{\epsilon}{2}(\mu_\phi,\exp(-2\pi ixT) f(a)_+) \|_{L^2(\mathbb{R}^+)}^2 dT \leq C_{\epsilon, T_1, \mathfrak{d}} \| \mathcal{M}_\frac{\epsilon}{2+\epsilon} f(a)_+ \|_{L^2(\mathbb{R}^+)}^2
\]

Recall that

\[
\mathcal{M}_\frac{\epsilon}{2}(\mu_\phi,\exp(-2\pi ixT) f(a))(s_1) = L(\frac{1+\epsilon}{2} + s_1 i, \mathfrak{d}) \mathcal{M}_\frac{\epsilon+\frac{\epsilon}{2}}{2} \{ f(x) \exp-2\pi iTx \}(s_1)
\]

The Mellin transform \( \mathcal{M} \{ \mu_\phi,\exp(-2\pi ixT) f(a)_+ \}(\frac{\epsilon}{2} + s_1 i) \), which lives in the Hardy space of a right half plane, may be obtained explicitly from The Mellin transform

\[
\mathcal{M} \{ \mu_\phi,\exp(-2\pi ixT) f(a) \}(\frac{\epsilon}{2} + s_1 i)
\]

by solving a Riemann-Hilbert problem. We indeed get an inequality in terms of the Dirichlet series \( L(\frac{1+\epsilon}{2} + s_1 i, \mathfrak{d}) \). Nevertheless, in order to obtain a bound on \( L(s, \mathfrak{d}) \) we will have to control

\[
\int_0^{T_1} \| \mathcal{M}_\frac{\epsilon}{2}(\mu_\phi,\exp(-2\pi ixT) f(a)_-) \|_{L^2(\mathbb{R}^+)}^2 dT
\]

with \( a \in (0, 1] \). This can be achieved when \( \mathfrak{d} \) comes from an automorphic distribution for an arithmetic subgroup of \( SL(2, \mathbb{Z}) \) ([7] [1] [5]).

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