Analytic smoothing effect of the spatially inhomogeneous Landau equations for hard potentials

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Abstract

We study the spatially inhomogeneous Landau equations with hard potential in the perturbation setting, and establish the analytic smoothing effect in both spatial and velocity variables for a class of low-regularity weak solutions. This shows the Landau equations behave essentially as the hypoelliptic Fokker-Planck operators. The spatial analyticity relies on a new time-average operator, and the proof is based on a straightforward energy estimate with a careful estimate on the derivatives with respect to the new time-average operator.

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1. Introduction and main result

As specific degenerate elliptic operators, we may expect not only the usual $C^\infty$ smoothness but also the Gevrey regularity for subelliptic operators (cf. Derridj-Zuily [25]), recalling

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Gevrey class is an intermediate space between $C^\infty$ and analytic spaces. However it is highly non-trivial to improve the Gevrey regularity to analyticity for degenerate elliptic operators. In 1972, Baouendi-Goulaouic [11] constructed a counterexample that shows analytic-hypoellipticity may fail for some degenerate elliptic operators. Since then it is a long-standing problem to explore the sufficient and necessary conditions for the analytic regularity of degenerate operators, and so far there have been extensive related works with various applications in PDEs and complex analysis; interested readers may refer to D. Tartakoff’s monograph [58] and the references therein, for the comprehensive presentation on the Treves’s conjecture and the analyticity of $\bar{\partial}$-Neumann problem. As a positive example and a classical subelliptic operator, the Kolmogorov operator indeed enjoy the analyticity regularity in the degenerate direction. Thus it is natural to ask the similar properties for the spatial inhomogeneous kinetic equations since these equations may be regarded as non-local and non-linear models of Kolmogorov operators. However up to now the analytic regularization is far from well-understood for the spatially inhomogeneous kinetic equations and it is only verified recently by Morimoto-Xu [53] for the Landau equations in the Maxwellian molecules case. Note the proof in [53] relies on the specific structure of collision operators in the Maxwellian molecules case, where the Fourier analysis techniques work well but can not apply to the cases with hard or soft potentials. In this text we aim to extend the analyticity properties established by Morimoto-Xu [53] to the case of hard potential potentials, and instead of the estimates on the usual derivatives, our proof relies crucially on a careful treatment on the derivatives with respect to a new time-average operator.

Denoting by $F = F(t, x, v) \geq 0$ the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ and position $x \in \mathbb{T}^3$, the spatially inhomogeneous Landau equation reads

$$\begin{cases} \frac{\partial}{\partial t} F + v \cdot \nabla_x F = Q_L(F, F), \\ F|_{t=0} = F_0, \end{cases}$$

(1.1)

where

$$Q_L(G, H) = \sum_{1 \leq i, j \leq 3} \partial_{v_i} \left\{ \int_{\mathbb{R}^3} a_{i,j}(v - v_*) [G(v_*) \partial_{v_j} H(v) - H(v)(\partial_{v_j} G)(v_*)] dv_* \right\},$$

(1.2)

with $(a_{i,j})_{1 \leq i, j \leq 3}$ a nonnegative definite matrix given by

$$a_{i,j}(v) = (\delta_{ij} |v|^2 - v_i v_j) |v|^\gamma, \quad \gamma \in [-3, 1].$$

(1.3)

Here and below $\delta_{ij}$ is the Kronecker delta function. By spatially inhomogeneous it means the unknown density $F$ depends on the spatial $x$ variable. Meanwhile if $F$ is independent of $x$ it then called spatially homogeneous. In this paper we are concerned with so-called hard potentials that means $0 < \gamma \leq 1$ in (1.3). The Maxwellian molecules case of $\gamma = 0$ was investigated recently by Morimoto-Xu [53]. We remark that our argument may cover the specific case of Maxwellian molecules, but can not apply to the case of soft potentials which means $-3 < \gamma < 0$ in (1.3).

The collisional operator $Q_L$ only acts velocity variable $v$ and roughly speaking it behaves as the Laplacian $\Delta_v$ and thus admits a similar diffusion properties as heat equations (cf. Desvillettes-Villani [27]). The similar properties for Boltzmann equations was established by Alexandre-Desvillettes-Villani-Wennberg [1] in the form of entropy dissipation estimates. So it is a natural conjecture that the spatially homogeneous Landau equations should enjoy a similar smoothing
effect as heat equations, and this has been well confirmed in various settings; we refer to [23, 49, 52] for the sharp smoothing effect in analytic or ultra-analytic setting, after the earlier works of [20, 21, 27]. However there are only few works on the sharp regularity of spatially Boltzmann, and here we only mention the work [12] of Barbaroux-Hundertmark-Ried-Vugalter where they established the sharp Gevrey class regularization for the specific Maxwellian molecules case with the counterpart properties for the other soft or hard potentials remaining unsolved. We refer the interested readers to [7, 26, 28, 32, 50, 51, 54] for the relevant works on the regularity issue for spatially homogeneous Boltzmann equations.

Compared with the spatially homogeneous kinetic equations, much less is known for the higher order regularity of weak solutions in the spatially inhomogeneous case. The main difficulty arises from the spatial degeneracy coupled with the nonlocal collision term, and it may ask more subtle analysis to treat the nonlinear collision operators involving rough coefficients. For general initial data with finite mass, energy and entropy, the global existence of renormalized weak solutions in $L^1$ was established first by DiPerna-Lions [29] for the Boltzmann equations under the Grad’s angular cutoff assumption, and later by Alexandre-Villani [8] for the non-cutoff case, while both uniqueness and regularity of such general global solutions are still a challenging open problem. Here we mention the recent progress [14, 37, 42–46, 56] toward the conditional regularity for Boltzmann and Landau equations with general initial data. It would be interesting to develop a self-contained theory of both existence and regularity without any extra condition on solutions.

In the perturbation setting, the existence and unique theory of mild solutions is well-explored for the Boltzmann equations. For exponential perturbations near Maxwellians, when the initial data belong to some kind of regular Sobolev spaces, the well-posedness was established independently by two groups AMUXY [3–6] and Gressman-Strain [34]. Furthermore, the $C^\infty$-smoothing effect was proven by [3, 5], and a higher order Gevrey regularity, inspired by the behaviors of Kolmogorov operators, was proven by Lerner-Morimoto-Pravda-Starov-Xu [47] and [19] for the 1D and 3D Boltzmann equations, respectively. Recently the existence and uniqueness of some mild weak solutions was established by Duan-Liu-Sakamoto-Strain [31] and the Gevrey regularity were proven by [30]. For the perturbation setting with polynomial decay, the classical solutions were constructed by Alonso-Morimoto-Sun-Yang [10]; see also the independent works of He-Jiang [36] and Hérau-Tonon-Tristani [41]. The unique existence of weak solutions in the $L^2 \cap L^\infty$ setting was initiated by Alonso-Morimoto-Sun-Yang [9], and we mention the recent Silvestre-Snelson’s work [57]. The well-posedness for Landau equations can be found in [15, 17, 18, 31, 35] and references therein.

Finally we mention two techniques used frequently when investigate the regularity property of kinetic equations, one referring to De Giorgi-Nash-Moser theory with the help of the averaging lemma and another to Hörmander’s hypoelliptic theory. Interested readers may refer to [9, 14, 33, 37, 42–46, 56] for the De Giorgi type argument, and to [2, 13, 19, 22, 24, 30, 38–40, 47, 48] for the application of hypoelliptic techniques to kinetic equations. More details on the two techniques can be found in the survey paper of C. Mouhot [55].

In this work we are concerned with the sharp regularity of weak solutions to Landau equations, and this is inspired by a natural conjecture which expects kinetic equations admit a similar smoothing effect as that for heat or Kolmogorov operators. Compared with the spatially homo-
ogeneous case, it is far from well-understood for the inhomogeneous case since the degeneracy occurs in the spatial variable and thus it is a hypoelliptic problem due to the non-trivial interaction between the transport operator and the collision operator. In general we can expect only the Gevrey regularization effect for hypoelliptic operators. On the other hand, a straightforward Fourier analysis shows the hypoelliptic Fokker-Planck operator, a local model of spatially inhomogeneous Landau equations, enjoys the analyticity regularity for both spatial and velocity variables. However the analytic regularization for the spatially inhomogeneous kinetic equations is still unclear, except some specific models; here we mention the recent positive results in Morimoto-Xu [53] on Landau equations with Maxwellian molecules. The argument in [53] depends on the Fourier analysis that does not apply to the cases of hard and soft potentials. In this work we use a more flexible energy method to analyze the sharp smoothing effect for the spatially inhomogeneous Landau equations with hard potentials, and we hope the argument presented here may help give insight on the optimal regularity of Boltzmann equations. Recall for Boltzmann equations we only have Gevrey regularity of index \((1 + 2s)/2s\) in both spatial and velocity variables (cf. [19, 30, 47]), which seems not optimal in view of the counterpart for Kolmogorov type operators with fractional diffusion in velocity.

We will restrict our attention to the fluctuation around the Maxwellian distribution \(\mu\) with

\[
\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}. 
\]

Write solutions \(F\) to (1.1) as \(F = \mu + \sqrt{\mu} f\) and accordingly \(F_0 = \mu + \sqrt{\mu} f_0\) for initial data. Then the fluctuation \(f\) satisfies the Cauchy problem

\[
\begin{aligned}
\partial_t f + v \cdot \partial_x f + \mathcal{L} f &= \Gamma(f, f), \\
f_{|t=0} &= f_0,
\end{aligned}
\]

where here and below we use the notations

\[
\mathcal{L} f = -\Gamma(\sqrt{\mu}, f) - \Gamma(f, \sqrt{\mu}), \quad \Gamma(g, h) = \mu^{-\frac{1}{2}} Q_L(\sqrt{\mu}g, \sqrt{\mu}h). 
\]

We recall the space \(L^1_m L^2_v\) introduced in [31], that consists of all functions \(u\) such that \(\|u\|_{L^1_m L^2_v} < +\infty\) with

\[
\|u\|_{L^1_m L^2_v} := \sum_{m \in \mathbb{Z}^3} \|\mathcal{F}_x u(m, \cdot)\|_{L^2_v},
\]

where \(\mathcal{F}_x u(m, v)\) stands for the partial Fourier transform of \(u(x, v)\) with respect to \(x \in \mathbb{T}^3\). Here and throughout the paper \(m \in \mathbb{Z}^3\) stands for the Fourier dual variable of \(x \in \mathbb{T}^3\).

With the above notation the main result can be stated as follows.

**Theorem 1.1.** Let \(\gamma \geq 0\) in (1.3) and let \(L^1_m L^2_v\) be defined by (1.6). Suppose the initial datum \(f_0\) of (1.4) belongs to \(L^1_m L^2_v\) such that

\[
\|f_0\|_{L^1_m L^2_v} \leq \varepsilon_0,
\]

for some constant \(\varepsilon_0 > 0\). If \(\varepsilon_0\) is small sufficiently, then the Cauchy problem (1.4) admits an unique global-in-time solution satisfying that a constant \(C\) exists such that the following estimates hold true:

\[
\forall \alpha, \beta \in \mathbb{Z}^3_+ \text{ with } |\alpha| \geq 1, \quad \sup_{t > 0} \|\mathcal{T}_t^\alpha f(t)\|_{L^2} \leq \varepsilon_0 C^{\beta} (|\alpha| + |\beta| + 1) (|\alpha| + |\beta|)!,
\]

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Finally, in the following discussion we denote by $\bar{v}$ defined by

$$\bar{v} = \sum_i \| \cdot \|_{L^2}^{\tilde{i}+1} \| \partial_x^i f(t) \|_{L^2} \leq \varepsilon_0 C^{\tilde{i}+1} \| \partial_x^i \|,$$  

where $\tilde{i} = \min\{1, t\}$. In particular, the function $(x, v) \mapsto f(t, x, v)$ is real analytic in $\mathbb{T}^3 \times \mathbb{R}^3$ for all positive time $t > 0$.

**Remark 1.2.** The global-in-time existence and unique of solutions in regular Sobolev space was proven by Y. Guo [35]. Meanwhile the unique global solution in $L^1_t L^2_x$ was established recently by Duan-Liu-Sakamoto-Strain [31]. This text aims to improve the weak $L^1_t L^2_x$ regularity to analyticity at positive time.

**Remark 1.3.** The short time decay in (1.9) for velocity estimate is not sharp, which is caused by the norm $\| \cdot \|_{H^2_{t, \xi} L^2_x}$ used to deal with trilinear terms. If using the norm $\| \cdot \|_{L^1_t L^2_x}$ in (1.6) instead, we may expect the sharp short time decay for velocity estimates, that is, the estimate (1.8) will hold true for all $\alpha \in \mathbb{Z}^3_+$ without the restriction that $|\alpha| \geq 1$.

**Notations.** For simplicity of notations, we will use $\| \cdot \|_{L^2}$ and $(\cdot, \cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\mathbb{T}^3_x \times \mathbb{R}^3_\eta)$ and use the notation $\| \cdot \|_{L^2}$ and $(\cdot, \cdot)_{L^2}$ when the variable $v$ is specified. Similar notation will be used for $H^{+\infty} = H^{+\infty}(\mathbb{T}^3_x \times \mathbb{R}^3_\eta)$. In addition, We denote by $H^{(2, 0)} = H^2(\mathbb{T}^3_x)$ the classical Sobolev space, that is

$$H^{(2, 0)} = \{ u \in L^2(\mathbb{T}^3_x \times \mathbb{R}^3_\eta); \partial_x^0 u \in L^2(\mathbb{T}^3_x \times \mathbb{R}^3_\eta), |\alpha| \leq 2 \},$$

which is complemented with the norm $\| \cdot \|_{(2, 0)}$ and product $(\cdot, \cdot)_{(2, 0)}$, that is,

$$\| u \|_{(2, 0)} = \left( \sum_{|\alpha| \leq 2} \| \partial_x^\alpha u \|_{L^2}^2 \right)^{1/2}, \quad (u, w)_{(2, 0)} = \sum_{|\alpha| \leq 2} (\partial_x^\alpha u, \partial_x^\alpha w)_{L^2}.$$  

(1.10)

Throughout the paper $\hat{u}(m, \eta)$ stands for the Fourier transform of $u$ with respect to $(x, v)$, with $(m, \eta) \in \mathbb{Z}^3 \times \mathbb{R}^3$ the Fourier dual variables of $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$.

For a vector-valued function $A = (A_1, A_2, \ldots, A_n)$, we used the convention that $\|A\|^2 = \sum_{1 \leq j \leq n} \|A_j\|^2$ for a generic norm $\| \cdot \|$. Moreover $\langle v \rangle = (1 + v^2)^{1/2}$.

Denote by $[T_1, T_2]$ the commutator between two operators $T_1$ and $T_2$, that is,

$$[T_1, T_2] = T_1 T_2 - T_2 T_1.$$  

(1.11)

We denote by $v \wedge \eta$ the cross product of two vectors $v = (v_1, v_2, v_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ which is defined by

$$v \wedge \eta = (v_2 \eta_3 - v_3 \eta_2, v_3 \eta_1 - v_1 \eta_3, v_1 \eta_2 - v_2 \eta_1).$$  

(1.12)

Finally, in the following discussion $g * h$ stands for the convolution only with respect to the velocity variable $v$ for two functions $g$ and $h$, that is,

$$g * h(x, v) = \int_{\mathbb{R}^3} g(x, v - v_*) h(x, v_*) dv_*.$$
2. Analytic regularity of smooth solutions

In this section we improve the $H^\infty$-smoothness of solutions to analyticity, and the analytic regularization effect of $L^1_0 L^2_v$ weak solutions is postponed the the last section.

In the following discussion we let $t_0 \in [0,1/2]$ be an arbitrarily fixed time, and introduce a time-average differential operator $M$ by setting
\[
M = -\int_{t_0}^t [\partial_{v_i} + (r - t_0)\partial_{x_i}]^2 dr = -(t-t_0)^2 \partial_{v_i}^2 - (t-t_0)^2 \partial_{x_1} \partial_{v_i} - \frac{(t-t_0)^3}{3} \partial_{x_1}^2,
\] (2.1)
which plays a crucial role when investigating the analytic regularity. Note $M$ is a Fourier multiplier with symbol
\[
(t-t_0)\eta_1^2 + (t-t_0)^2 m_1 \eta_1 + \frac{(t-t_0)^3}{3} m_1^2,
\]
that is,
\[
Mg(m, \eta) = \left( (t-t_0)\eta_1^2 + (t-t_0)^2 m_1 \eta_1 + \frac{(t-t_0)^3}{3} m_1^2 \right) \hat{g}(m, \eta),
\]
recalling $(m, \eta) \in \mathbb{Z}^3 \times \mathbb{R}^3$ are the Fourier dual variables of $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$. The key observation is that the spatial derivatives are not involved in the commutator between $M$ and the transport operator. Precisely, direct verification shows
\[
[M, \partial_t + v \cdot \partial_x] = \partial_{v_i}^2,
\] (2.2)
recalling $[\cdot, \cdot]$ stands for the commutator between two operators defined in (1.11). So that we may make use of the diffusion property in velocity direction, to obtain the spatial analyticity.

**Theorem 2.1.** Suppose $0 < t_0 \leq 1/2$ is an arbitrarily given time. Let $\gamma \geq 0$ in (1.3) and let $f \in L^\infty([t_0,1]; H^\infty_v)$ be any solution to the Landau equation (1.4). With the notations given at the end of the previous section, we suppose that
\[
\sup_{t_0 \leq t \leq 1} \left\| f(t) \right\|_{(2,0)} + \left( \int_{t_0}^1 \left\| \psi(v, D_v) f(t) \right\|_{(2,0)}^2 dt \right)^{1/2} \leq \epsilon
\] (2.3)
for some constant $\epsilon > 0$ and that
\[
\forall \alpha, \beta \in \mathbb{Z}^3_+, \quad \sup_{t_0 \leq t \leq 1} \left\| \partial_x^\alpha \partial_v^\beta f(t) \right\|_{(2,0)} + \left( \int_{t_0}^1 \left\| \psi(v, D_v) \partial_x^\alpha \partial_v^\beta f(t) \right\|_{(2,0)}^2 dt \right)^{1/2} < +\infty,
\] (2.4)
where and below
\[
\left\| \psi(v, D_v) g \right\|_{(2,0)}^2 := \left\| (v)^{\frac{\gamma}{2}} \partial_v g \right\|_{(2,0)}^2 + \left\| (v)^{\frac{\gamma}{2}} (v \land \partial_v) g \right\|_{(2,0)}^2 + \left\| (v)^{1+\frac{\gamma}{2}} g \right\|_{(2,0)}^2
\] (2.5)
with $v \land \partial_v$ and $\| \cdot \|_{(2,0)}$ defined by (1.12) and (1.10) respectively, and $(v) = (1 + v^2)^{1/2}$. If $\epsilon$ is small sufficiently, then there exists a constant $C_\ast \geq 1$ independent of $\epsilon$, such that
\[
\forall k \in \mathbb{Z}_+, \quad \sup_{t_0 \leq t \leq 1} \left\| M^k f(t) \right\|_{(2,0)} + \left( \int_{t_0}^1 \left\| \psi(v, D_v) M^k f(t) \right\|_{(2,0)}^2 dt \right)^{1/2} \leq \frac{\epsilon}{(2k+1)^3} C_\ast^{2k}(2k)!,
\] (2.6)
where $M$ is defined by (2.1). Moreover, the above estimate (2.6) still holds true if we replace $M$ by
\[
-(t-t_0)\partial_{v_i}^2 - (t-t_0)^2 \partial_{x_i} \partial_{v_i} - \frac{(t-t_0)^3}{3} \partial_{x_i}^2,
\]
i = 2 or 3.
To prove Theorem 2.1, we need the following two propositions.

**Proposition 2.2** (Trilinear and coercivity estimates). Let $\Gamma(g, h)$ and $\mathcal{L}$ be defined in (1.5). There exists a constant $C_1$ such that for any $g, h, \omega \in H^{\infty}$ with $\psi(v, D_v)h, \psi(v, D_v)\omega \in L^2$, we have the trilinear estimate

$$
|\langle \Gamma(g, h), \omega \rangle_{L^2}| := \left| \int_{\mathbb{R}^3} \Gamma(g, h) \omega dt \right| \leq C_1 \|g\|_{L^2} \|\psi(v, D_v)h\|_{L^2} \|\psi(v, D_v)\omega\|_{L^2},
$$

and the coercivity estimate

$$
\|\psi(v, D_v)h\|_{L^2}^2 \leq C_1 (\langle \mathcal{L} h, h \rangle_{L^2} + C_1 \|h\|_{L^2}^2).
$$

**Proposition 2.3** (Commutator estimate). Let $k \geq 1$ be a given integer and $f \in L^\infty([t_0, 1]; H^{\infty})$ be any solution to (1.4) satisfying that a constant $C_* \geq 1$ exists such that

$$
\forall \ j \leq k - 1, \quad \sup_{t_0 \leq t \leq 1} \|M^j f(t)\|_{L^2} + \left( \int_{t_0}^1 \|\psi(v, D_v)M^j f(tug)\|_{L^2}^2 dt \right)^{1/2} \leq \frac{\epsilon}{(2j + 1)^3} C_*^2(2j)!,
$$

If $C_*$ is large enough, then there exists a constant $C_2 > 0$ independent of $\epsilon$ and $C_*$ above, such that for any $\delta > 0,$

$$
\int_{t_0}^1 \left( |M^k \Gamma(f, f) - \Gamma(f, M^k f, M^k f)|_{L^2} + \int_{t_0}^1 |\{M^k, \mathcal{L}\} f, M^k f, f\} | dt \right) dt \\
\leq (\delta + eC_2) \left( \sup_{t_0 \leq t \leq 1} \|M^k f\|_{L^2}^2 \right)^2 \left( \int_{t_0}^1 \|\psi(v, D_v)M^k f\|_{L^2}^2 dt \right) + C_3 \left( \epsilon + C_*^{-1} \right) \frac{\epsilon}{(2k + 1)^3} C_*^2(2k)!,
$$

where and below $C_\delta$ stands for generic constants depending on $\delta.$ Recall $[, ]$ stands for the commutator defined by (1.11).

**Remark 2.4.** In the above proposition, by saying $C_*$ is large enough we mean the condition on $C_*$ required in Lemma 4.6 is fulfilled.

**Remark 2.5.** As to be seen in Proposition 3.1, if $f \in L^\infty([t_0, 1]; H^{\infty})$ such that $\psi(v, D_v) f \in L^2([t_0, 1]; H^{\infty}),$ then for any $k \in \mathbb{Z}_+,$

$$
\langle v \rangle^{-(1+\frac{3}{2})} M^k \Gamma(f, f), \langle v \rangle^{-(1+\frac{3}{2})} \Gamma(f, M^k f), \langle v \rangle^{1+\frac{3}{2}} M^k f \in L^2([t_0, 1]; H^{\infty}).
$$

Thus

$$
\int_{t_0}^1 \left( M^k \Gamma(f, f), M^k f \right)_{L^2} dt = \sum_{|\alpha| \leq 2} \int_{t_0}^1 \int_{\mathbb{T}_1^3 \times \mathbb{R}^3} \left[ \partial^\alpha v \partial^\alpha v \Gamma(f, f) \right] \partial^\alpha v M^k f \ dx dv dt
$$

$$
= \sum_{|\alpha| \leq 2} \int_{t_0}^1 \int_{\mathbb{T}_1^3 \times \mathbb{R}^3} \left( \langle v \rangle^{-(1+\frac{3}{2})} M^k \Gamma(f, f) \right) \langle v \rangle^{1+\frac{3}{2}} \partial^\alpha v M^k f \ dx dv dt
$$

is well-defined, and so are the other trilinear or quadratic terms in Propositions 2.3 and 2.2.

The proofs of Propositions 2.2 and 2.3 are quite lengthy and we postpone them to the next two sections. By virtue of the two propositions above we are enable to complete the proof of Theorem 2.1.
Proof of Theorem 2.1. We use induction on $k$ to derive the estimate in Theorem 2.1. The validity of (2.6) for $k = 0$ is obvious in view of (2.3). Now for given $k \geq 1$, suppose

$$\forall \, j \leq k - 1, \quad \sup_{t_0 \leq t \leq 1} \|M^j f(t)\|_{L^2(0,1)} + \left( \int_{t_0}^1 \|\chi(v, D_v) M^j f(t)\|_{L^2(0,1)}^2 \right)^{1/2} \leq \frac{\epsilon}{(2j + 1)^3} C^2_{\epsilon}(2j)! (2.10)$$

for some constant $C_{\epsilon} \geq 1$ to be determined later. We will show the validity of (2.10) for $j = k$.

Applying $M^k$ to (1.4) yields

$$\frac{\partial}{\partial t} M^k f + v \cdot \partial_x M^k f + M^k L f = M^k \Gamma(f, f) - [M^k, \partial_t + v \cdot \partial_x] f = M^k \Gamma(f, f) - k\partial_x^2 M^{k-1} f,$$

the last equality using (2.2), that is,

$$\frac{\partial}{\partial t} M^k f + v \cdot \partial_x M^k f + L M^k f = \Gamma(f, M^k f) + M^k \Gamma(f, f) - \Gamma(f, M^k f) - [M^k, L] f - k\partial_x^2 M^{k-1} f. \quad (2.11)$$

In view of (2.4), we can verify directly that

$$v \cdot \partial_x M^k f, \quad \partial_x^2 M^{k-1} f \in L^2([0, 1]; H^{2(0)}), \quad \lim_{t \to t_0} M^k f(t)_{L^2(0,1)} \leq C \lim_{t \to t_0} \|f\|_{L^\infty([0, 1]; H^{2(0)})} = 0.$$

This, with (2.11) and Remark 2.5, implies

$$\frac{1}{2} \|M^k f(t)\|_{L^2(0,1)}^2 + \int_{t_0}^t \left\{ LM^k f, M^k f \right\}_{L^2(0,1)} \, ds$$

$$\leq k \int_{t_0}^t \|\partial_t M^{k-1} f\|_{L^2(0,1)} \|\partial_x M^k f\|_{L^2(0,1)} \, dt + \int_{t_0}^t \left\{ \Gamma(f, M^k f), M^k f \right\}_{L^2(0,1)} \, dt \leq (2k + 1)^3 \left( \int_{t_0}^t \sup_{t_0 \leq t \leq 1} \|f\|_{L^2(0,1)}^2 \right)^{1/2} \left( \int_{t_0}^t \|f\|_{L^\infty([0, 1]; H^{2(0)})}^2 \right)^{1/2} \leq C_1 \int_{t_0}^t \|M^k f\|_{L^2(0,1)}^2 \, dt. \quad (2.12)$$

In view of the assumption (2.4), we use the coercivity (2.8) to conclude

$$\int_{t_0}^t \sup_{t_0 \leq t \leq 1} \|f\|_{L^2(0,1)} \, dt \leq C \int_{t_0}^t \frac{\epsilon}{(2j + 1)^3} C^2_{\epsilon}(2j)! \, dt$$

As for the terms on the right hand side of (2.12), it follows from the trilinear estimate (2.7) and the inductive assumption (2.10) that

$$\int_{t_0}^t \left| \left\{ \Gamma(f, M^k f), M^k f \right\}_{L^2(0,1)} \right| \, dt$$

$$\leq \left( \sup_{t_0 \leq t \leq 1} \|f\|_{L^2(0,1)} \right) \int_{t_0}^t \|\chi(v, D_v) M^k f\|_{L^2(0,1)}^2 \, dt \leq \epsilon \int_{t_0}^t \|\chi(v, D_v) M^k f\|_{L^2(0,1)}^2 \, dt.$$

Moreover, using again the inductive assumption (2.10) yields, for any $\delta > 0$,

$$k \int_{t_0}^t \|\partial_t M^{k-1} f\|_{L^2(0,1)} \|\partial_x M^k f\|_{L^2(0,1)} \, dt$$

$$\leq \frac{\delta}{\epsilon} \int_{t_0}^t \|\chi(v, D_v) M^k f\|_{L^2(0,1)}^2 \, dt + \frac{k^2}{\delta} \int_{t_0}^t \|\chi(v, D_v) M^{k-1} f\|_{L^2(0,1)}^2 \, dt$$

$$\leq \frac{\delta}{\epsilon} \int_{t_0}^t \|\chi(v, D_v) M^k f\|_{L^2(0,1)}^2 \, dt + \frac{1}{\delta} \left[ \frac{\epsilon}{(2k - 1)^3} C^2_{\epsilon}(2k)! \right]^2.$$
Finally by Proposition 2.3,
\[ \int_{b_0}^{1} |(M^k\Gamma(f, f) - \Gamma(f, M^k f), M^k f)|_{L^2} dt + \int_{b_0}^{1} |(M^k, L) f, M^k f)|_{L^2} dt \]
\[ \leq (\delta + \epsilon C_2) \left( \left( \sup_{b_0 \leq t \leq 1} \|M^k f\|^2_{L^2} \right)^2 + \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right) + C_\delta \left( \epsilon + C_*^{-1} \right) \frac{\epsilon}{(2k + 1)^{3/2}} C_*^{2k}(2k)! \right]^2. \]

We combine the above estimates with (2.12) to conclude
\[ \frac{1}{2} \left\| M^k f(t) \right\|^2_{L^2} + \frac{1}{C_1} \left( \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right)^{1/2} \]
\[ \leq \int_{b_0}^{1} \|M^k f\|^2_{L^2} dt + (\delta + \epsilon C_2) \left( \left( \sup_{b_0 \leq t \leq 1} \|M^k f\|^2_{L^2} \right)^2 + \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right) \]
\[ + C_\delta \left( \epsilon + C_*^{-1} \right) \frac{\epsilon}{(2k + 1)^{3/2}} C_*^{2k}(2k)! \]
which with Gronwall’s inequality yields, for any \( \delta > 0 \),
\[ \sup_{b_0 \leq t \leq 1} \|M^k f\|^2_{L^2} + \frac{1}{C_1} \left( \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right)^{1/2} \]
\[ \leq 8 (\delta + \epsilon C_2) \left( \left( \sup_{b_0 \leq t \leq 1} \|M^k f\|^2_{L^2} \right)^2 + \left( \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right) \right)^{1/2} + 8C_\delta \left( \epsilon + C_*^{-1} \right) \frac{\epsilon C_*^{2k}(2k)!}{(2k + 1)^{3/2}} \]

This implies, choosing \( \delta \) small sufficiently and using the smallness assumption on \( \epsilon \),
\[ \sup_{b_0 \leq t \leq 1} \|M^k f\|^2_{L^2} + \left( \int_{b_0}^{1} \|\psi(v, D_v) M^k f\|^2_{L^2} dt \right)^{1/2} \leq C \left( \epsilon + C_*^{-1} \right) \frac{\epsilon}{(2k + 1)^{3/2}} C_*^{2k}(2k)! \]
for some constant \( C \) depending only on the constants \( C_1 \) and \( C_2 \) given in Propositions 2.2 and 2.3 but independent of \( \epsilon \) and \( C_* \). Thus we conclude the validity of (2.10) for \( j = k \), provided \( C_* > 2C \) and \( \epsilon \) is small enough such that \( \epsilon C \leq 1/2 \). We complete the proof of Theorem 2.1. \( \square \)

3. Trilinear and coercivity estimates

In this part we will prove the quantitative estimates in Propositions 2.2. The proof is quite lengthy, and we proceed through the following subsections.

3.1. Analysis for the linear collision operator

This part is devoted to deriving the representation of the Landau collision operator in terms of differential operators involving the Laplacian \( \Delta_v \) and the Laplace-Beltrami operator \((v \wedge \partial_v)^2\) on the unite sphere \( S^2 \). This enables to complete the proof of Proposition 2.2.

Recall \( g * h \) stands for the convolution with respect to \( v \) only. Let \( a_{i,j} \) be given in (1.3) and denote
\[ \hat{a}_{i,j}(f) = a_{i,j} * f = \int_{\mathbb{R}^3} a_{i,j}(v - v_*) f(v_*) dv_*. \] (3.1)

For a given function \( g \), define
\[ a_g = |v|^\gamma * (\sqrt{m} g), \ A_g = (a_{1,g}, a_{2,g}, a_{3,g}) = |v|^\gamma * (\sqrt{m} \partial_v g) \] and \( B_g = (b_{1,g}, b_{2,g}, b_{3,g}) = |v|^\gamma * (v \sqrt{m} g) \). (3.2)
and moreover define $M_{i,j,g}, \rho_{i,j,g}, \lambda_{i,j,g}, 1 \leq i, j \leq 3$, as below.

\[
\begin{align*}
M_{i,j,g}(v) &= |v|^2 \left( \delta_{i,j} |v|^2 - v_i v_j \right) \sqrt{\mu} g, \\
\rho_{i,j,g}(v) &= \int_{\mathbb{R}^3} |v - v_i|^2 \left( \sqrt{\mu} \partial_{v_i} g(v) \right) \left( \partial_j (v_i v_j) + (v_i)_{j}(v_j) \right) dv_s + \int_{\mathbb{R}^3} |v - v_i|^2 \left( \sqrt{\mu} \partial_{v_i} g(v) \right) \left( v_i (v_i)_{j} + v_j (v_i)_{j} - (v_i)_{j} (v_j) \right) dv_s, \\
\lambda_{i,j,g}(v) &= \int_{\mathbb{R}^3} |v - v_i|^2 \left( \sqrt{\mu} v_{i,j} g(v) \right) \left( v_i (v_i)_{j} - v_j (v_i)_{j} \right) dv_s,
\end{align*}
\]

where and below $(v_s)_i$ stands for the $i$th entry of the vector $v_s \in \mathbb{R}^3$.

**Proposition 3.1.** Let $\Gamma(g,h)$ be the quadratic operator defined by (1.5). Then, with the nations given by (3.2)-(3.3),

\[
\Gamma(g,h) = \sum_{1 \leq i,j \leq 6} L_{j}(g,h),
\]

where

\[
\begin{align*}
L_1(g,h) &= \frac{1}{2} (v \wedge \partial_v)(v \wedge \partial_v) h + \sum_{1 \leq i,j \leq 3} \partial_{v_i} \left( M_{i,j,g} v_{i,j} h \right) + \frac{1}{2} \left( \partial_v \wedge B_g \right)(v \wedge \partial_v) h - \frac{1}{2} (v \wedge \partial_v) \cdot (B_g \wedge \partial_v) h, \\
L_2(g,h) &= \frac{1}{4} \left( v \wedge \partial_v \cdot (B_g \wedge v) h + (B_g \wedge v) \cdot (v \wedge \partial_v) h \right) - \frac{1}{2} \sum_{1 \leq i,j \leq 3} \left[ \partial_{v_i} (M_{i,j,g} v_{i,j} h) + v_i M_{i,j,g} v_{i,j} h \right], \\
L_3(g,h) &= \frac{1}{2} \sum_{1 \leq i,j \leq 3} v_i M_{i,j,g} v_{i,j} h,
\end{align*}
\]

and

\[
\begin{align*}
L_4(g,h) &= \frac{1}{2} (v \wedge \partial_v \cdot (v \wedge \partial_v) h - \sum_{1 \leq i \leq 3 \atop i \neq j} \partial_{v_i} (\rho_{i,j,g} h)), \\
L_5(g,h) &= -\frac{1}{4} (v \wedge \partial_v \cdot (B_g \wedge v) h + \frac{1}{2} \sum_{1 \leq i,j \leq 3 \atop i \neq j} \partial_{v_i} (\lambda_{i,j,g} h) + \frac{1}{2} \sum_{1 \leq i,j \leq 3 \atop i \neq j} v_i \rho_{i,j,g} h), \\
L_6(g,h) &= -\frac{1}{4} \sum_{1 \leq i,j \leq 3 \atop i \neq j} v_i \lambda_{i,j,g} h.
\end{align*}
\]

Recall $\xi \wedge \zeta$ stands for the cross product defined by (1.12).

To prove Proposition 3.1 we first list the representations of $L_{j}(g,h), 1 \leq j \leq 6$.

**Lemma 3.2.** Let $L_{j}(g,h), 1 \leq j \leq 3$, be the bilinear operators defined in Proposition 3.1. Then
we have

\[ L_1(g, h) = \sum_{1 \leq i, j \leq 3} \partial_i(\tilde{a}_{i,j}(\sqrt{\mu}g) \partial_j h), \]

\[ L_2(g, h) = -\frac{1}{2} \sum_{1 \leq i, j \leq 3} \partial_i(\tilde{a}_{i,j}(\sqrt{\mu}g) v_j h) - \frac{1}{2} \sum_{1 \leq i, j \leq 3} \tilde{v}_i \tilde{a}_{i,j}(\sqrt{\mu}g) \partial_j h, \]

\[ L_3(g, h) = \frac{1}{4} \sum_{1 \leq i, j \leq 3} \tilde{v}_i \tilde{a}_{i,j}(\sqrt{\mu}g) v_j h. \]

**Proof.** Recall \( a_{i,j} \) is given in (1.3). Using the fact that

\[ a_{i,i}(z) = |z|^\gamma \sum_{j \neq i} z_j^2, \quad a_{i,j}(z) = -|z|^\gamma z_i z_j \quad \text{for} \quad i \neq j, \quad (3.5) \]

we compute, observing the notation in (3.1),

\[ \sum_{1 \leq i, j \leq 3} \partial_i(\tilde{a}_{i,j}(\sqrt{\mu}g) \partial_j h(v)) = \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} [v - v_s] \partial_j \left[ (\sqrt{\mu}g)(v_s) \partial_j h(v) dv \right] \]

\[ - \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} |v - v_s| \partial_j (v_i - (v_s)_i) \partial_j h(v) dv \]

\[ := I_1 - I_2. \]

Moreover, using the notations in (3.2),

\[ I_1 = \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} [v - v_s] \partial_j \left[ (\sqrt{\mu}g)(v_s) \partial_j h(v) dv \right] \]

\[ = \sum_{1 \leq i, j \leq 3} \partial_i \left( a_g(v) \partial_j \right) - 2 \sum_{1 \leq i, j \leq 3} \partial_i \partial_j (b_i g(v) \partial_j h) \]

\[ + \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} |v - v_s| \partial_j (\sqrt{\mu}g)(v_s) dv \]

\[ := I_{1,1} + I_{1,2} + I_{1,3} \]

and

\[ I_2 = \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} [v - v_s] \partial_j \left[ (v_i j - (v_i) v_i) + (v_s)_i (v_s)_i) \right] (\sqrt{\mu}g)(v_s) \partial_j h(v) dv \]

\[ = \sum_{1 \leq i, j \leq 3} \partial_i \left( a_g(v) v_i \partial_j h \right) - \sum_{1 \leq i, j \leq 3} \partial_i \partial_j (b_i g(v) v_i \partial_j h) \]

\[ + \sum_{1 \leq i, j \leq 3} \partial_i \int_{\mathbb{R}^3} [v - v_s] \partial_j (\sqrt{\mu}g)(v_s) \partial_j h(v) dv \]

\[ := I_{2,1} + I_{2,2} + I_{2,3}. \]
Combining the above equalities we conclude
\[ I_{1,1} - I_{2,1} = \sum_{i,j \neq j}^{1,\ldots,3} (v_i \partial_v(a_g(v)v_j \partial_v h) - v_j \partial_v(a_g(v)v_i \partial_v h)) = \frac{1}{2} (v \wedge \partial_v) \cdot a_g(v) (v \wedge \partial_v) h. \]

Similarly,
\[
I_{1,2} - I_{2,2} = \sum_{i,j \neq j}^{1,\ldots,3} \partial_v(b_{j,g}(v)v_i \partial_v h) + \sum_{i,j \neq j}^{1,\ldots,3} v_j \partial_v(b_{j,g}(v) \partial_v h) - 2 \sum_{i,j \neq j}^{1,\ldots,3} v_j \partial_v(b_{j,g}(v) \partial_v h) \\
= \sum_{i,j \neq j}^{1,\ldots,3} (\partial_v(b_{j,g}(v)v_i \partial_v h) - \partial_v(b_{j,g}(v)v_j \partial_v h)) + \sum_{i,j \neq j}^{1,\ldots,3} (v_j \partial_v(b_{j,g}(v) \partial_v h) - v_j \partial_v(b_{j,g}(v) \partial_v h)) \\
= \frac{1}{2} (\partial_v \wedge B_g) \cdot (v \wedge \partial_v) h - \frac{1}{2} (v \wedge \partial_v) \cdot (B_g \wedge \partial_v) h.
\]

Finally, the fact that
\[ \delta_{ij} |v_i|^2 - (v_i)(v_j) = \sum_{i \neq j} (v_i)^2, \quad \delta_{ij} |v_i|^2 - (v_i)(v_j) = -(v_i)(v_j) \text{ for } i \neq j, \]

yields
\[ I_{1,3} - I_{2,3} = \sum_{i,j \neq j}^{1,\ldots,3} \partial_v \int_{\mathbb{R}^3} |v - v_s|^\gamma ((v_s)^2 \partial_v h + (v_s) \partial_v h) (\sqrt{\mu g}(v_s) \nu)dv_s \\
= \sum_{1 \leq i,j \leq 3} \partial_v \int_{\mathbb{R}^3} |v - v_s|^\gamma (\delta_{ij} |v_i|^2 - (v_i)(v_j)) (\sqrt{\mu g}(v_s) \partial_v h)dv_s \\
= \sum_{1 \leq i,j \leq 3} \partial_v (M_{i,j,g} \partial_v h). \]

Combining the above equalities we conclude
\[ \sum_{1 \leq i,j \leq 3} \partial_v (\tilde{a}_{i,j} (\sqrt{\mu g}) \partial_v h(v)) = \frac{1}{2} (v \wedge \partial_v) \cdot a_g(v) (v \wedge \partial_v) h + \sum_{1 \leq i,j \leq 3} \partial_v (M_{i,j,g} \partial_v h) \\
+ \frac{1}{2} (\partial_v \wedge B_g) \cdot (v \wedge \partial_v) h - \frac{1}{2} (v \wedge \partial_v) \cdot (B_g \wedge \partial_v) h \quad (3.6) \]
\[ = L_1(g, h), \]

the last line using the definition (3.4). We have proven the first assertion in Lemma 3.2.

Similarly, we replace the differential operator \( \partial_v \) in (3.6) by \( v_i \), or replace \( \partial_v \) by \( v_j \), and observe \( v \wedge v = \partial_v \wedge \partial_v = 0 \); this yields
\[ -\frac{1}{2} \sum_{1 \leq i,j \leq 3} \partial_v (\tilde{a}_{i,j} (\sqrt{\mu g}) v_j h) = -\frac{1}{2} \sum_{1 \leq i,j \leq 3} \partial_v (M_{i,j,g} v_j h) + \frac{1}{4} (v \wedge \partial_v) \cdot (B_g \wedge v) h \]
and
\[ -\frac{1}{2} \sum_{1 \leq i,j \leq 3} v_i \tilde{a}_{i,j} (\sqrt{\mu g}) \partial_v h = -\frac{1}{2} \sum_{1 \leq i,j \leq 3} v_i M_{i,j,g} \partial_v h - \frac{1}{4} (v \wedge B_g) \cdot (v \wedge \partial_v) h, \]

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and thus, in view of (3.4) and the fact that $\xi \wedge \zeta = -\zeta \wedge \xi$, 

$$- \frac{1}{2} \sum_{1 \leq i, j \leq 3} \partial_v (\tilde{a}_{i,j} (\sqrt{\mu} g) v_j h) - \frac{1}{2} \sum_{1 \leq i, j \leq 3} v_i \tilde{a}_{i,j} (\sqrt{\mu} g) \partial_v h$$

$$= \frac{1}{4} \left((v \wedge \partial_v) \cdot (B_g \wedge v) h + (B_g \wedge v) \cdot (v \wedge \partial_v) h\right) - \frac{1}{2} \sum_{1 \leq i, j \leq 3} \left(\partial_v (M_{i,j,g} v_j h) + v_i M_{i,j,g} \partial_v h\right) = L_2 (g, h)$$

Moreover, we replace the differential operator $\partial_v$ and the function $\partial_v h$ in (3.6) by $v_i$ and $v_j h$, respectively, to get 

$$\frac{1}{4} \sum_{1 \leq i, j \leq 3} v_i \tilde{a}_{i,j} (\sqrt{\mu} g) v_j h = \frac{1}{4} \sum_{1 \leq i, j \leq 3} v_j M_{i,j,g} v_j h = L_3 (g, h),$$

the last equalities using again (3.4). Combining the above equalities with (3.6) we complete the proof of Lemma 3.2. \qed

**Lemma 3.3.** Let $L_j (g, h), 4 \leq j \leq 6$, be the bilinear operators defined in Proposition 3.1. It holds that

$$L_4 (g, h) = \frac{1}{4} \sum_{1 \leq i, j \leq 3} v_i \tilde{a}_{i,j} (\sqrt{\mu} g) v_j h,$$

$$L_5 (g, h) = \frac{1}{2} \sum_{1 \leq i, j \leq 3} \partial_v (\tilde{a}_{i,j} (\sqrt{\mu} g) h) + \frac{1}{2} \sum_{1 \leq i, j \leq 3} v_i \tilde{a}_{i,j} (\sqrt{\mu} g) h,$$

$$L_6 (g, h) = \frac{1}{4} \sum_{1 \leq i, j \leq 3} v_j M_{i,j,g} v_j h.$$

**Proof.** Using the notation in (3.1) as well as (3.5), we may write 

$$- \sum_{1 \leq i, j \leq 3} \partial_v (\tilde{a}_{i,j} (\sqrt{\mu} g) h) = - \sum_{1 \leq i, j \leq 3 \neq j} \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma (v_j - (v_*)_j) \left(\sqrt{\mu} \partial_v g(v_*) h(v)dv_*\right)$$

$$+ \sum_{1 \leq i, j \leq 3 \neq j} \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma (v_j - (v_*)_j) \left((v_j - (v_*)_j) (\sqrt{\mu} \partial_v g(v_*) h(v)dv_*\right)$$

$$:= K_1 + K_2,$$

and moreover 

$$K_1 = - \sum_{1 \leq i, j \leq 3 \neq j} \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma ((v_*)_j (v_*)_j - 2v_j (v_*)_j) \left(\sqrt{\mu} \partial_v g(v_*) h(v)dv_*\right)$$

$$- \sum_{1 \leq i, j \leq 3 \neq j} v_i \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma v_j (\sqrt{\mu} \partial_v g(v_*) h(v)dv_*,$$

and 

$$K_2 = \sum_{1 \leq i, j \leq 3 \neq j} \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma ((v_*)_j (v_*)_j - v_i (v_*)_j) \left(\sqrt{\mu} \partial_v g(v_*) h(v)dv_*\right)$$

$$+ \sum_{1 \leq i, j \leq 3 \neq j} v_i \partial_v \int_{\mathbb{R}^3} |v - v_i|^\gamma v_i (\sqrt{\mu} \partial_v g(v_*) h(v)dv_*.$$
Using the nations in (3.3) and (3.2) gives

\[
\sum_{1 \leq i \leq 3} \partial_i \int_{\mathbb{R}^3} |v - v_i|^\gamma (v_i)(v_i)_j - v_j(v_i)_j - (v_i)v_j \big( \sqrt{\mu \partial_{v_i} g}(v_i)h(v)dv, \\
- \sum_{1 \leq i \leq 3} \partial_i \int_{\mathbb{R}^3} |v - v_i|^\gamma (v_i)(v_i)_j - 2v_j(v_i)_j \big( \sqrt{\mu \partial_{v_i} g}(v_i)h(v)dv, -= \sum_{1 \leq i \leq 3} \partial_i (\rho_{i,j}gh)
\]

and

\[
\sum_{1 \leq i \leq 3} v_i \partial_i \int_{\mathbb{R}^3} |v - v_i|^\gamma v_i \big( \sqrt{\mu \partial_{v_i} g}(v_i)h(v)dv, \\
- \sum_{1 \leq i \leq 3} v_i \partial_i \int_{\mathbb{R}^3} |v - v_i|^\gamma v_i \big( \sqrt{\mu \partial_{v_i} g}(v_i)h(v)dv, = \frac{1}{2} (v \land \partial_i) \cdot (A_g \land v) h.
\]

Consequently, we combine the above equalities to conclude

\[
- \sum_{1 \leq i \leq 3} \partial_i \bar{a}_{i,j} \big( \sqrt{\mu \partial_{v_i} g}h \big) = K_1 + K_2 = \frac{1}{2} (v \land \partial_i) \cdot (A_g \land v) h - \sum_{1 \leq i \leq 3} \partial_i (\rho_{i,j}gh) = L_4(g, h),
\]

the last inequality using the definition of $L_4(g, h)$ given in Proposition 3.1. Similarly, we can verify that

\[
\frac{1}{2} \sum_{1 \leq i \leq 3} \partial_i \bar{a}_{i,j} \big( \sqrt{\mu \partial_{v_i} g}h \big) + \frac{1}{2} \sum_{1 \leq i \leq 3} v_i \partial_i \bar{a}_{i,j} \big( \sqrt{\mu \partial_{v_i} g}h ig) \\
= -\frac{1}{4} (v \land \partial_i) \cdot (B_g \land v) h + \frac{1}{2} \sum_{1 \leq i \leq 3} \partial_i (\lambda_{i,j}gh) + \frac{1}{2} \sum_{1 \leq i \leq 3} v_j \rho_{i,j}gh = L_5(g, h)
\]

and

\[
-\frac{1}{4} \sum_{1 \leq i \leq 3} v_i \partial_i \bar{a}_{i,j} \big( \sqrt{\mu \partial_{v_i} g} \big) h = -\frac{1}{4} \sum_{1 \leq i \leq 3} v_j \partial_i \bar{a}_{i,j} \big( \sqrt{\mu \partial_{v_i} g} \big) h = L_6(g, h),
\]

with $L_4(g, h), L_6(g, h)$ defined in Proposition 3.1. The proof of Lemma 3.3 is completed. 

\textbf{Proof of Proposition 3.1.} Recalling $\Gamma(g, h)$ is defined by (1.5) and in view of the representation (1.2), we write

\[
\Gamma(g, h) = \mu^{-\frac{1}{2}} \sum_{1 \leq i \leq 3} \partial_i \int_{\mathbb{R}^3} a_{i,j}(v - v_i)[(\sqrt{\mu g})(v_i)\partial_{v_i}(\sqrt{\mu h})(v) - (\sqrt{\mu h})(v)\partial_{v_i}(\sqrt{\mu g})(v_i)]dv, \\
- \mu^{-\frac{1}{2}} \sum_{1 \leq i \leq 3} \partial_i \int_{\mathbb{R}^3} a_{i,j}(v - v_i)(\sqrt{\mu g})(v_i)\partial_{v_i} \big( \sqrt{\mu h}(v) - \frac{v_j}{2} h(v) \big) dv, \\
- \mu^{-\frac{1}{2}} \sum_{1 \leq i \leq 3} \partial_i \int_{\mathbb{R}^3} a_{i,j}(v - v_i) \big( \sqrt{\mu h}(v) \big) \big[ (\sqrt{\mu \partial_{v_i} g})(v_i) - (v_j \sqrt{\mu g/2})(v_i) \big] dv.
\]

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Then using the notation in (3.1) we can split the terms on the right hand side as below.

\[
\Gamma(g, h) = \sum_{1 \leq i, j \leq 3} \partial_v(\bar{a}_{i,j} (\sqrt{\mu g}) \partial_v h) - \frac{1}{2} \sum_{1 \leq i, j \leq 3} \partial_v(\bar{a}_{i,j} (\sqrt{\mu g}) v_j h)
\]

\[
- \frac{1}{2} \sum_{1 \leq i, j \leq 3} v_i \bar{a}_{i,j} (\sqrt{\mu g}) \partial_v h + \frac{1}{4} \sum_{1 \leq i, j \leq 3} v_i \bar{a}_{i,j} (\sqrt{\mu g}) v_j h
\]

\[
- \sum_{1 \leq i, j \leq 3} \partial_v(\bar{a}_{i,j} (\sqrt{\mu g}) h) + \frac{1}{2} \sum_{1 \leq i, j \leq 3} \partial_v(\bar{a}_{i,j} (\sqrt{\mu g}) v_j h)
\]

\[
+ \frac{1}{2} \sum_{1 \leq i, j \leq 3} v_i \bar{a}_{i,j} (\sqrt{\mu g}) v_j h
\]

which with Lemmas 3.2 and 3.3 yields the assertion in Proposition 3.1. The proof is thus completed.

\[
\square
\]

3.2. Proof of Proposition 2.2: trilinear and coercivity estimates

For simplicity of notations, we denote by \(C\) a generic constant. We first derive the trilinear estimate (2.7), and it suffices to prove that, in view of Proposition 3.1,

\[
\sum_{i=1}^{6} |(L_i(g, h), \omega)_{L_v^2}| \leq C\|g\|_{L_v^2}\|\psi(v, D_v)h\|_{L_v^2}\|\psi(v, D_v)\omega\|_{L_v^2},
\]

(3.7)

where \(L_j(g, h), 1 \leq j \leq 6\), are defined in Proposition 3.1. By the representation of \(L_1\) in (3.4), it follows that

\[
|(L_1(g, h), \omega)_{L_v^2}| = \left| \int_{\mathbb{R}^3} L_1(g, h) \omega dv \right|
\]

\[
\leq \frac{1}{2} \|\langle v \rangle^{-\frac{3}{2}} a_{g}(v) (v \wedge \partial_v) h\|_{L_v^2} \|\langle v \rangle^{\frac{3}{2}} (v \wedge \partial_v) \omega\|_{L_v^2} + \sum_{1 \leq i, j \leq 3} \|\langle v \rangle^{-\frac{3}{2}} M_{i,j,g}(v) \partial_v h\|_{L_v^2} \|\langle v \rangle^{\frac{3}{2}} \partial_v \omega\|_{L_v^2}
\]

\[
+ \frac{1}{2} \|\langle v \rangle^{\frac{3}{2}} (v \wedge \partial_v) h\|_{L_v^2} \|\langle v \rangle^{-\frac{3}{2}} (B_{g}(v) \wedge \partial_v) \omega\|_{L_v^2} + \frac{1}{2} \|\langle v \rangle^{\frac{3}{2}} (v \wedge \partial_v) \omega\|_{L_v^2} \|\langle v \rangle^{-\frac{3}{2}} (B_{g}(v) \wedge \partial_v) h\|_{L_v^2}.
\]

In view of (3.2)-(3.3), we have

\[
\forall v \in \mathbb{R}^3, \quad |a_g(v)| + \sum_{1 \leq i, j \leq 3} |M_{i,j,g}(v)| + |B_{g}(v)| \leq C \langle v \rangle^7 \|g\|_{L_v^2}.
\]

This with (2.5) yields

\[
|(L_1(g, h), \omega)_{L_v^2}| \leq C\|g\|_{L_v^2}\|\psi(v, D_v)h\|_{L_v^2}\|\psi(v, D_v)\omega\|_{L_v^2}.
\]

(3.8)

In view the representations of \(L_i(g, h), 1 \leq i \leq 6\), in Proposition 3.1, the above estimate (3.8) still holds true with \(L_1(g, h)\) replaced by \(L_i(g, h)\) with \(i = 2, 3\) or 6.

To complete the proof of (3.7), it remains to estimate \(|(L_i(g, h), \omega)_{L_v^2}|\) with \(i = 4\) or 5. To do so we combine (3.8) with Lemma 3.2 to conclude

\[
\sum_{1 \leq i, j \leq 3} \left| \partial_v(\bar{a}_{i,j} (\sqrt{\mu g}) \partial_v h), \omega \right|_{L_v^2} \leq |(L_1(g, h), \omega)_{L_v^2}| \leq C\|g\|_{L_v^2}\|\psi(v, D_v)h\|_{L_v^2}\|\psi(v, D_v)\omega\|_{L_v^2}.
\]
Similarly, replacing $\sqrt{\mu}g$ and $\partial_{v_j} h$ above by $v_j \sqrt{\mu}g$ and $h$, respectively,

$$\left| \sum_{1 \leq i,j \leq 3} \left( \partial_v (\tilde{a}_{i,j} (v_j \sqrt{\mu}g)h), \omega \right)_{L^2} \right| \leq C \|g\|_{L^2} \|\psi(v, D_v) h\|_{L^2} \|\psi(v, D_v) \omega\|_{L^2}.$$  \quad (3.9)

By Lemma 3.3,

$$\left| (L_4(g, h), \omega)_{L^2} \right| = \left| \sum_{1 \leq i,j \leq 3} \left( \partial_v (\tilde{a}_{i,j} (\sqrt{\mu}g)h), \omega \right)_{L^2} \right|.$$  

This with the fact that, recalling the notation in (3.1),

$$\tilde{a}_{i,j} (\sqrt{\mu}g) = a_{i,j} * (\partial_v (\sqrt{\mu}g) - g \partial_{v_j} \sqrt{\mu}) = (\partial_v a_{i,j}) * (\sqrt{\mu}g) + \frac{1}{2} a_{i,j} * (v_j \sqrt{\mu}g),$$  

yields

$$\left| (L_4(g, h), \omega)_{L^2} \right| \leq \sum_{1 \leq i,j \leq 3} \left| \left( \partial_v (\partial_{v_j} a_{i,j} (\sqrt{\mu}g)h), \omega \right)_{L^2} \right| + \frac{1}{2} \sum_{1 \leq i,j \leq 3} \left| \left( \partial_v (\tilde{a}_{i,j} (v_j \sqrt{\mu}g)h), \omega \right)_{L^2} \right|$$

$$\leq \sum_{1 \leq i,j \leq 3} \| (v)^{-\frac{1}{2}} \partial_{v_j} a_{i,j} (\sqrt{\mu}g)h \|_{L^2} \| (v)^{\frac{1}{2}} \partial_v \omega \|_{L^2} + \frac{1}{2} \sum_{1 \leq i,j \leq 3} \left| \left( \partial_v (\tilde{a}_{i,j} (v_j \sqrt{\mu}g)h), \omega \right)_{L^2} \right|$$

$$\leq C \|g\|_{L^2} \|\psi(v, D_v) h\|_{L^2} \|\psi(v, D_v) \omega\|_{L^2},$$

the last inequality using (3.9) and the fact that

$$\forall v \in \mathbb{R}^3, \quad \left| \partial_{v_j} a_{i,j} (\sqrt{\mu}g) \right| = \left| \int_{\mathbb{R}^3} (\partial_{v_j} a_{i,j})(v - v_*) \sqrt{\mu}(v_*) d\nu_* \right| \leq C \langle v \rangle^{1+\gamma} \|g\|_{L^2}$$

due to (1.3). Similarly, in view of Lemma 3.3,

$$\left| (L_5(g, h), \omega)_{L^2} \right| = \frac{1}{2} \sum_{1 \leq i,j \leq 3} \left| \left( \partial_v (\tilde{a}_{i,j} (v_j \sqrt{\mu}g)h), \omega \right)_{L^2} \right| + \sum_{1 \leq i,j \leq 3} \left| \left( v_j (a_{i,j} (\sqrt{\mu}g)h), \omega \right)_{L^2} \right|$$

$$\leq C \|g\|_{L^2} \|\psi(v, D_v) h\|_{L^2} \|\psi(v, D_v) \omega\|_{L^2}.$$  

Combining the above estimates on $(L_i(g, h), \omega)_{L^2}, 1 \leq i \leq 6$, we obtain (3.7) and thus the trilinear estimate (2.7) in Proposition 2.2.

The rest part is devoted to proving the coercivity estimate (2.8) in Proposition 2.2. Recall $\mathcal{L} h = -\Gamma(\sqrt{\mu}, h) - \Gamma(h, \sqrt{\mu})$ in view of (1.5). Using the trilinear estimate (2.7) yields, for any $\delta > 0$,

$$\left| (\Gamma(h, \sqrt{\mu}), h)_{L^2} \right| \leq C \|h\|_{L^2} \|\psi(v, D_v) h\|_{L^2} \leq \delta \|\psi(v, D_v) h\|_{L^2}^2 + C_\delta \|h\|_{L^2}^2.$$  \quad (3.10)

By Proposition 3.1 we can write

$$-\Gamma(\sqrt{\mu}, h) = - \sum_{1 \leq j \leq 6} L_j(\sqrt{\mu}, h)$$

with $L_j(\sqrt{\mu}, h)$ given in Proposition 3.1. Next we will proceed to derive the lower or upper bounds of $- \left( L_i(\sqrt{\mu}, h) \right)_{L^2}$ with $1 \leq i \leq 6$.  

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Observe $v \wedge \partial_v$ is anti-selfadjoint on $L^2_v$. Then we use integration by parts to obtain

$$\frac{1}{2} \left< (\partial_v \cdot B_g) \cdot (v \wedge \partial_v) h - (v \wedge \partial_v) \cdot (B_g \wedge \partial_v) h, \ h \right>_{L^2}$$

$$= -\frac{1}{2} \left< (v \wedge \partial_v) h, (B_g \wedge \partial_v) h \right>_{L^2} + \frac{1}{2} \left< (B_g \wedge \partial_v) h, (v \wedge \partial_v) h \right>_{L^2} = 0,$$

which, with the representation of $L_1(\sqrt{\mu}, h)$ in (3.4), yields

$$-(L_1(\sqrt{\mu}, h), h)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^3} a\sqrt{\nu}(v \wedge \partial_v) h \cdot (v \wedge \partial_v) h \, dv + \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} M_{i,j,\sqrt{\nu}}(v)(\partial_v h)\partial_v h \, dv.$$ 

Here $a\sqrt{\nu}$ and $M_{i,j,\sqrt{\nu}}$ are defined in (3.2) and (3.3) which satisfy that

$$a\sqrt{\nu}(v) = \int_{\mathbb{R}^3} |v - v_+|^2 \mu(v_+) \, dv_+ \geq \langle v \rangle^2 / C$$

and that

$$\sum_{1 \leq i, j \leq 3} M_{i,j,\sqrt{\nu}}(v)\xi_i \xi_j = \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} |v - v_+|^2 (\delta_{i,j} |v_+|^2 - (v_+)_i(v_+)_j) \mu(v_+) \, dv_+ \geq \langle v \rangle^2 |\xi|^2 / C. \quad (3.11)$$

for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, since the matrix $(\delta_{i,j} |v_+|^2 - (v_+)_i(v_+)_j)_{3 \times 3}$ is positive-defined. As a result, combining the above estimate yields

$$-(L_1(\sqrt{\mu}, h), h)_{L^2} \geq C^{-1} \left\{ \|\langle v \rangle^2 (v \wedge \partial_v) h\|_{L^2}^2 + \|\langle v \rangle^2 \partial_v h\|_{L^2}^2 \right\}. \quad (3.12)$$

Direct verification shows

$$\frac{1}{4} \left< (v \wedge \partial_v) \cdot (B_g \wedge v) + (B_g \wedge v) \cdot (v \wedge \partial_v) h, \ h \right>_{L^2}$$

$$= -\frac{1}{4} \left< (B_g \wedge v) h, (v \wedge \partial_v) h \right>_{L^2} + \frac{1}{4} \left< (v \wedge \partial_v) h, (B_g \wedge v) h \right>_{L^2} = 0.$$

This implies, recalling $L_2(\sqrt{\mu}, h)$ is given in Proposition 3.1 and the matrix $M_{i,j,\sqrt{\nu}}$ in (3.3) is symmetric,

$$-(L_2(\sqrt{\mu}, h), h)_{L^2} = \frac{1}{2} \sum_{1 \leq i, j \leq 3} \left< [\partial_v (M_{i,j,\sqrt{\nu}} v) h] + v_i M_{i,j,\sqrt{\nu}} v_j h, \ h \right>_{L^2}$$

$$= -\frac{1}{2} \sum_{1 \leq i, j \leq 3} (M_{i,j,\sqrt{\nu}} v_j h, \partial_v h)_{L^2} + \frac{1}{2} \sum_{1 \leq i, j \leq 3} (\partial_v h, M_{i,j,\sqrt{\nu}} v_j h)_{L^2} = 0. \quad (3.13)$$

By the definitions of $A_{i,j,\sqrt{\nu}}$ and $M_{i,j,\sqrt{\nu}}$ in (3.3),

$$\sum_{1 \leq i, j \leq 3, i \neq j} u_i A_{i,j,\sqrt{\nu}}(v) = \sum_{1 \leq i, j \leq 3, i \neq j} \int_{\mathbb{R}^3} |v - v_+|^2 \mu(v_+) \left( (v_+)_i(v_+)_j - v_i v_j (v_+)_j \right) \, dv_+$$

$$= \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} |v - v_+|^2 \mu(v_+) \left( (v_+)_i |v_+|^2 - (v_+)_i (v_+)_j \right) v_j \, dv_+ = \sum_{1 \leq i, j \leq 3} M_{i,j,\sqrt{\nu}} v_i \mu v_j,$$

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the second equality using a similar fact as in (3.5). As a result,

$$\sum_{1 \leq i, j \leq 3} (v_i A_{i,j} \sqrt{\mu}, h)_{L^2}^2 = \sum_{1 \leq i, j \leq 3} (v_i M_{i,j} \sqrt{\mu}, h)_{L^2}^2,$$

(3.14)

which with Proposition 3.1 yields

$$- (L_3(\sqrt{\mu}, h), h)_{L^2}^2 - (L_6(\sqrt{\mu}, h), h)_{L^2}^2 = - \frac{1}{4} \sum_{1 \leq i, j \leq 3} (v_i M_{i,j} \sqrt{\mu}, h)_{L^2}^2 + \frac{1}{4} \sum_{1 \leq i, j \leq 3} (v_i A_{i,j} \sqrt{\mu}, h)_{L^2}^2 = 0.$$  \hspace{1cm} (3.15)

In view of (3.2) and (3.3), we can verify that

$$\forall \nu \in \mathbb{R}^3, \quad A_{\sqrt{\mu}^\nu} = - \frac{1}{2} B_{\sqrt{\mu}^\nu} \text{ and } \rho_{i,j} \sqrt{\mu}(\nu) = - \frac{1}{2} A_{i,j} \sqrt{\mu}(\nu),$$

and thus, in view of the definitions of \(L_3(\sqrt{\mu}, h)\) and \(L_6(\sqrt{\mu}, h)\) in Proposition 3.1,

$$- (L_3(\sqrt{\mu}, h), h)_{L^2}^2 - (L_6(\sqrt{\mu}, h), h)_{L^2}^2 = \frac{1}{2} \left( (v \wedge \partial_i) \cdot (B_{\sqrt{\mu}^\nu} \wedge v) h, h \right)_{L^2}^2 - \frac{1}{4} \sum_{1 \leq i, j \leq 3} (\partial_i (A_{i,j} \sqrt{\mu} h), h)_{L^2}^2 + \frac{1}{4} \sum_{1 \leq i, j \leq 3} (v_i A_{i,j} \sqrt{\mu} h, h)_{L^2}^2.$$  \hspace{1cm} (3.16)

As for the last term on the right-hand side of (3.16), we use (3.14) and (3.11) to conclude

$$\frac{1}{4} \sum_{1 \leq i, j \leq 3} (v_i A_{i,j} \sqrt{\mu} h, h)_{L^2}^2 \geq C^{-1} \int_{\mathbb{R}^3} \langle \nu \rangle \| \nu \| h(v)^2 dv.$$  \hspace{1cm} (3.17)

Integrating by parts and using the fact that

$$\partial_i A_{i,j} \sqrt{\mu} = \partial_i \left[ v_i (|\nu|^2 * (\mu \omega_j)) - \nu_j (|\nu|^2 * (\mu \omega_i)) \right] \geq - C \langle \nu \rangle^{1+\gamma}$$

in view of (3.3), we have, for any \(\delta > 0,\)

$$- \sum_{1 \leq i, j \leq 3} (\partial_i (A_{i,j} \sqrt{\mu} h), h)_{L^2}^2 = \frac{1}{2} \sum_{1 \leq i, j \leq 3} (\partial_i (A_{i,j} \sqrt{\mu} h), h)_{L^2}^2 \geq - \delta \| \nu \|^{1+\gamma} h^2 - C_0 \| h \|_{L^2}^2.$$  \hspace{1cm} (3.18)

Finally for the first term on the right-hand side of (3.16), observe \(B_{\sqrt{\mu}^\nu} = - \partial_i a \) with \(a = a_{\sqrt{\mu}}\), and thus, writing \(a\) instead of \(a_{\sqrt{\mu}}\) for simplicity of notations,

$$\frac{1}{2} (v \wedge \partial_i) \cdot (B_{\sqrt{\mu}^\nu} \wedge v) h = - \frac{1}{2} \sum_{1 \leq i, j \leq 3} \left( v_i \partial_i v_j - v_j \partial_i v_i \right) \left( (\partial_i a) v_j h - (\partial_j a) v_i h \right)$$

$$= - \frac{1}{2} \sum_{1 \leq i, j \leq 3} \left( v_i (\partial_i v_j) a v_j h + v_i (\partial_i a) v_j h + v_i (\partial_j a) v_i h - v_i (\partial_j a) v_i h \right)$$

$$+ \frac{1}{2} \sum_{1 \leq i, j \leq 3} \left( v_j (\partial_j a) v_i h + v_j (\partial_i a) v_i h - v_j (\partial_i a) v_j h - v_j (\partial_i a) v_j h \right)$$

$$= \sum_{1 \leq i, j \leq 3} \left( (\partial_i a) v_j^2 h + (\partial_j a) v_i^2 h - (\partial_i a) v_i^2 h - (\partial_j a) v_j^2 h \right).$$
This, with the fact that
\[
\left((\partial_v a) v_j^2 \partial_v h, \ h\right)_{L^2} = -\frac{1}{2} \left((\partial_v^2 a) v_j^2 h, \ h\right)_{L^2}
\]
and
\[
\left(-(\partial_v a) v_j v_j \partial_v h, \ h\right)_{L^2} = \frac{1}{2} \left((\partial_v \partial_v a) v_j^2 h, \ h\right)_{L^2} + \frac{1}{2} \left((\partial_v a) v_j^2 h, \ h\right)_{L^2}
\]
for any \(i \neq j\), yields
\[
\frac{1}{2} \left((v \wedge \partial_v)(B \sqrt{\gamma} \wedge v) h, \ h\right)_{L^2} = \frac{1}{2} \sum_{1 \leq i, j \leq 3 \atop i \neq j} \left((\partial_v^2 a) v_j^2 h - (\partial_v \partial_v a) v_j^2 h - (\partial_v a) v_j^2 h, \ h\right)_{L^2}
\]
\[
\geq \frac{1}{2} \sum_{1 \leq i, j \leq 3 \atop i \neq j} \left(-(\partial_v a) v_j^2 h, \ h\right)_{L^2} \geq -C\|\langle v \rangle(1 + \gamma)/2 \ h\|^2_{L^2} \geq -\delta\|\langle v \rangle\|^2_{L^2} = -C\|h\|^2_{L^2}, \quad (3.19)
\]
where the first inequality in the last line holds true because it follows from (3.11) that, recalling \(a = a \sqrt{\gamma}\) is defined in (3.2),
\[
\sum_{1 \leq i, j \leq 3 \atop i \neq j} \left((\partial_v^2 a) v_j^2 - (\partial_v \partial_v a) v_j^2 \right) = \sum_{1 \leq i, j \leq 3 \atop i \neq j} \int_{\mathbb{R}^3} |v - v_i|^2 \mu(v) \left[(v_i)(v_i)v_j^2 - (v_i)(v_i)v_j v_j \right] dv_i
\]
\[
= \sum_{1 \leq i, j \leq 3 \atop i \neq j} \int_{\mathbb{R}^3} |v - v_i|^2 \mu(v) \left[\delta_{i,j}v_j^2 - (v_i)(v_i)v_j \right] dv_i \geq 0.
\]
Now we substitute (3.17), (3.18) and (3.19) into (3.16) to conclude
\[
-(L_4(\sqrt{\gamma}, h) - L_5(\sqrt{\gamma}, h), h)_{L^2} \geq C^{-1} \int_{\mathbb{R}^3} \langle v \rangle^2 |v|^2 h(\langle v \rangle^2 dv - \delta\|\langle v \rangle\|^2_{L^2} = -C\|h\|^2_{L^2}
\]
for any \(\delta > 0\), that is,
\[
-(L_4(\sqrt{\gamma}, h) - L_5(\sqrt{\gamma}, h), h)_{L^2} \geq C^{-1} \|\langle v \rangle\|^2_{L^2} - C\|h\|^2_{L^2}.
\]
This, with (3.12), (3.13) and (3.15) as well as (2.5), implies
\[
-(\Gamma(\sqrt{\gamma}, h), h)_{L^2} \geq C^{-1} \|\phi(\langle v \rangle D_v h)^2_{L^2} = C\|h\|^2_{L^2}.
\]
As a result, the coercivity estimate (2.8) follows by combining the above estimate with (3.10) and observing \(Lh = -\Gamma(\sqrt{\gamma}, h) = \Gamma(h, \sqrt{\gamma})\). The proof of Proposition 2.2 is completed.

4. Estimate on commutators

This part is devoted to treating the commutators between \(M^k\) and \(\Gamma(f, f)\), and completing the proof of Proposition 2.3.
4.1. Quantitative properties of the time-average operator

Let \( M \) be the time-average operator defined by (2.1), which is a Fourier multiplier with symbol
\[
(t - t_0)\eta_1^2 + (t - t_0)^2m_1\eta_1 + \frac{(t-t_0)^3}{3}m_1^2,
\]
recalling \( m = (m_1, m_2, m_3) \in \mathbb{Z}^3 \) and \( \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \) are the Fourier dual variables of \( x \in \mathbb{T}^3 \) and \( v \in \mathbb{R}^3 \), respectively.

Direct verification yields, for any \( t > t_0 \),
\[
c_0\left((t-t_0)\eta_1^2+(t-t_0)^3m_1^2\right) \leq (t-t_0)\eta_1^2+(t-t_0)^3m_1\eta_1 + \frac{(t-t_0)^3}{3}m_1^2 \leq \left((t-t_0)\eta_1^2+(t-t_0)^3m_1^2\right)/c_0 \tag{4.1}
\]
for some constant \( 0 < c_0 < 1 \). This enables to define the fractional power \( M^\sigma \) by setting
\[
\forall \sigma \geq 0, \quad M^\sigma g(m, \eta) = \rho_\sigma \hat{g}(m, \eta) \tag{4.2}
\]
with Fourier symbol
\[
\rho_\sigma := \left((t-t_0)\eta_1^2+(t-t_0)^3m_1\eta_1 + \frac{(t-t_0)^3}{3}m_1^2\right)^\sigma, \quad \sigma \geq 0, \ t \in [t_0, 1]. \tag{4.3}
\]

**Lemma 4.1** (symbolic calculus). For any given integer \( k \geq 0 \) we have
\[
\forall t \in [t_0, 1], \forall j \leq 2k, \quad |\partial_{\eta_1}^j \rho_k| \leq 2^j \frac{(2k)!}{(2j-k)!} \rho_{k-\frac{j}{2}} \tag{4.4}
\]
where \( \rho_k \) is defined by (4.3).

**Proof.** We use induction on \( k \) to prove (4.4), which holds true for \( k = 0 \) or \( k = 1 \), since direct computation shows
\[
\partial_{\eta_1} \rho_1 = 2(t-t_0)\eta_1 + (t-t_0)^2m_1, \quad \partial_{\eta_1}^2 \rho_1 = 2(t-t_0),
\]
and moreover \( 2(t-t_0)\eta_1 + (t-t_0)^2m_1 \leq 4\rho_{1/2} \) and \( 2|t-t_0| \leq 2\rho_0 \) for \( t \in [t_0, 1] \). Now suppose for any \( i \leq k-1 \) with given integer \( k \geq 2 \), we have
\[
\forall t \in [t_0, 1], \forall j \leq 2i, \quad |\partial_{\eta_1}^j \rho_i| \leq 2^j \frac{(2i)!}{(2j-i)!} \rho_{i-\frac{j}{2}} \tag{4.5}
\]
we will show the above estimate also holds for \( i = k \), that is,
\[
\forall t \in [t_0, 1], \forall j \leq 2k, \quad |\partial_{\eta_1}^j \rho_k| \leq 2^j \frac{(2k)!}{(2j-k)!} \rho_{k-\frac{j}{2}} \tag{4.6}
\]
Note (4.6) holds true for \( j = 2k \), since
\[
\forall t \in [t_0, 1], \quad |\partial_{\eta_1}^{2k} \rho_k| = (2k)!|t-t_0|^k \leq (2k)! \rho_0. \tag{4.7}
\]
Now we consider the case when \( 2 \leq j \leq 2k-1 \) and write
\[
\partial_{\eta_1}^j \rho_k = \partial_{\eta_1}^{j-1}[k\rho_{k-1}(2(t-t_0)\eta_1 + (t-t_0)^2m_1)] \tag{4.8}
\]
\[
= k(\partial_{\eta_1}^{j-1}\rho_{k-1})(2(t-t_0)\eta_1 + (t-t_0)^2m_1) + k(j-1)(\partial_{\eta_1}^{j-2}\rho_{k-1})2(t-t_0).
\]

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Moreover observe \( j - 2 \leq j - 1 \leq 2(k - 1) \), and thus we use the induction assumption (4.5) as well as the fact that \( |2(t - t_0)\eta_1 + (t - t_0)^2 m_1| \leq 4\rho_{1/2} \) to compute

\[
\left| k(\partial_{\eta_1}^{j-1}(2(t - t_0)\eta_1 + (t - t_0)^2 m_1)) \right| \leq k8^{j-1} \frac{(2(k - 1))!}{(2k - j - 1)!} \rho_{k-1-j} \times 4\rho_{1/2} = \frac{8}{2} \frac{(2k)!}{(2k - j)!} \rho_{k-1-j}.
\]

and

\[
\left| k(j - 1)(\partial_{\eta_1}^{j-2}(2(t - t_0)) \right| \leq k(j - 1)8^{j-2} \frac{(2(k - 1))!}{(2k - j)!} \rho_{k-1-j} 2(t - t_0) \leq \frac{8}{2} \frac{(2k)!}{(2k - j)!} \rho_{k-1-j}.
\]

Combining these inequalities we obtain

\[
\forall t \in [t_0, 1], \forall 2 \leq j \leq 2k - 1, \quad \left| \partial_{\eta_1}^j \rho_k \right| \leq \frac{8}{2} \frac{(2k)!}{(2k - j)!} \rho_{k-1-j}.
\]

Finally, direct verification shows

\[
\forall t \in [t_0, 1], \forall 0 \leq j \leq 1, \quad \left| \partial_{\eta_1}^j \rho_k \right| \leq \frac{8}{2} \frac{(2k)!}{(2k - j)!} \rho_{k-1-j}.
\]

Combining the above two estimates with (4.7) yields (4.6). The proof of Lemma 4.1 is completed. \( \square \)

By direct computation we have, for any \( t \in [t_0, 1] \),

\[
(t - t_0)^2 \eta_1^2 + (t - t_0)^2 m_1 \eta_1 + \frac{(t - t_0)^3}{3} m_1^2 = \frac{1}{4} (t - t_0)^{1/2} \eta_1^2 + (t - t_0)^{3/2} m_1^2.
\]

Then \( M \) can be represented as the square sum of a vector filed \( \Lambda = (\Lambda_1, \Lambda_2) \):

\[
M = \Lambda \cdot \Lambda = \Lambda_1^2 + \Lambda_2^2,
\]

where, letting \( \sqrt{-1} \) be the square root of \(-1\),

\[
\Lambda_1 = \frac{1}{2 \sqrt{-1}} \left( (t - t_0)^{1/2} \partial_{\eta_1} + (t - t_0)^{3/2} \partial_{m_1} \right), \quad \Lambda_2 = \frac{\sqrt{3}}{6 \sqrt{-1}} \left( (t - t_0)^{1/2} \partial_{\eta_1} + (t - t_0)^{3/2} \partial_{m_1} \right).
\]

Note \( \Lambda_i, 1 \leq i \leq 2 \), are self-adjoint operators in \( H^{(2,0)} \). By virtue of the vector field \( \Lambda \) we have the following Leibniz type formula.

**Lemma 4.2.** For any \( k \in \mathbb{Z}_+ \), it holds that

\[
M^k(g, h) = (\Lambda \cdot \Lambda)^k(g, h) = \sum_{j=0}^{2k} A_{j2k-j}(g, h),
\]

with

\[
A_{j2k-j}(g, h) = \sum_{\ell+2p=j, \ell+2q=2k-j} c_{\ell, \rho, q}^k (\Lambda^\ell M^p g) \cdot (\Lambda^\ell M^q h),
\]

\[ \text{21} \]
where the summation is taken over all non-negative integers $\ell, p$ and $q$ satisfying $\ell + 2p = j$ and $\ell + 2q = 2k - j$, and

$$\Lambda^\ell g \cdot \Lambda^\ell h := \sum_{j_1=1}^{2} \cdots \sum_{j_r=1}^{2} (\Lambda_{j_1} \cdots \Lambda_{j_r} g) (\Lambda_{j_1} \cdots \Lambda_{j_r} h). \quad (4.11)$$

The sequence $c_{j,\ell,p,q}^{k,j}$ of non-negative integers in (4.10) are determined by

$$c_{j,\ell,p,q}^{k+1,j} = c_{j,\ell-1,p,q}^{k,j} + c_{j,\ell,p,q-1}^{k,j} + 2 c_{j,\ell-1,p,q}^{k,j-1} \quad (4.12)$$

where we have used the convention that

$$c_{j,\ell,p,q}^{k,j} = 0 \text{ if } j > 2k \text{ or any one entry in the index } (j,\ell,p,q) \text{ is negative.} \quad (4.13)$$

Moreover

$$\sum_{\ell+2p=j, \ell+2q=2k-j} c_{j,\ell,p,q}^{k,j} = \binom{2k}{j}. \quad (4.14)$$

Proof. We use induction on $k$ to prove (4.9). The validity of formula (4.9) for $k = 0$ is obvious. For given integer $k \geq 0$, suppose that

$$\forall N \leq k, \ M^N(g,h) = \sum_{j=0}^{2N} A_{j,2N-j}(g,h), \quad (4.15)$$

with $A_{j,2N-j}(g,h)$ defined in (4.10). We will prove the above assertion still holds true for $N = k + 1$.

It follows from the inductive assumption (4.15) that

$$M^{k+1}(g,h) = MM^k(g,h) = \sum_{j=0}^{2k} MA_{j,2k-j}(g,h). \quad (4.16)$$

Moreover in view of (4.10) and the fact

$$M(g,h) = \Lambda \cdot \Lambda(g,h) = (Mg)h + g(Mh) + 2(\Lambda g) \cdot (\Lambda h),$$

we compute

$$\begin{align*}
\sum_{j=0}^{2k} MA_{j,2k-j}(g,h) &= \sum_{j=0}^{2k} \sum_{l+2p=j} c_{l,p,q}^{k,j}(\Lambda^l g) \cdot (\Lambda^l M^p g) + \sum_{j=0}^{2k} \sum_{l+2q=j} c_{l,p,q}^{k,j}(\Lambda^l M^q g) \cdot (\Lambda^l M^p h) \\
&\quad + \sum_{j=0}^{2k} \sum_{l+2q=j} c_{l,p,q}^{k,j}(\Lambda^l M^q g) \cdot (\Lambda^l M^p h) + \sum_{j=0}^{2k} \sum_{l+2q=j} c_{l,p,q-1}^{k,j}(\Lambda^l M^p g) \cdot (\Lambda^l M^q h) \\
&= \sum_{j=0}^{2(k+1)} \sum_{l+2p=j} c_{l,p,q}^{k,j-2}(\Lambda^l M^p g) \cdot (\Lambda^l M^q h) + \sum_{j=0}^{2k} \sum_{l+2q=j} c_{l,p,q-1}^{k,j}(\Lambda^l M^p g) \cdot (\Lambda^l M^q h) \\
&\quad + \sum_{j=0}^{2(k+1)} \sum_{l+2q=j} c_{l,p,q-1}^{k,j-1}(\Lambda^l M^p g) \cdot (\Lambda^l M^q h) \\
&= \sum_{j=0}^{2(k+1)} \sum_{l+2p=j} c_{l,p}^{k+1,j}(\Lambda^l M^p g) \cdot (\Lambda^l M^q h) = \sum_{j=0}^{2(k+1)} A_{j,2(k+1)-j}(g,h),
\end{align*}$$

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where in the last line we have used (4.12) with the convention (4.13). As a result, substituting the above equations into (4.16) yields the validity of (4.15) for $N = k + 1$. We have proven (4.9).

It remains to prove the last assertion (4.14). We use induction on $k$. If $k = 0$ then $j = 0$ and thus $A_{0,0}(g, h) = gh$ in view of (4.9). Meanwhile by (4.10),

$$A_{0,0}(g, h) = c_{0,0,0}^0 gh,$$

which gives $c_{0,0,0}^0 = 1$. As a result, for $k = 0$ we have

$$\sum_{\ell + 2p = j, \ell + 2q = 2k-j} c_{\ell, p, q}^k = c_{0,0,0}^0 = 1 = \binom{2k}{j}.$$  

Then (4.14) holds true for $k = 0$. Supposing (4.14) holds for any integer $k$ with $k \geq 1$, we will prove its validity for $k + 1$. In fact we use (4.12) with the convention (4.13), to compute

$$\sum_{\ell + 2p = j, \ell + 2q = 2k+1-j} c_{\ell, p, q}^{k+1} = \sum_{\ell + 2p = j, \ell + 2q = 2k+1-j} c_{\ell, p, q}^{k} + \sum_{\ell + 2p = j, \ell + 2q = 2k+1-j} c_{\ell, p, q}^{k-1} + \sum_{\ell + 2p = j, \ell + 2q = 2k+1-j} 2c_{\ell, p, q}^{k-1}$$

$$= \binom{2k}{j} + \binom{2k}{j-2} + \binom{2k}{j-1} = \binom{2(k+1)}{j}.$$  

Thus (4.14) holds for all $k \geq 0$. The proof of Lemma 4.2 is completed.

**Remark 4.3.** With $c_{\ell, p, q}^{k}$ given in the above Lemma 4.2, it follows from (4.14) that $c_{0,0,0}^{2k} = c_{0,0,k}^{2k} = 1$, and $c_{1,0,0}^{k-1} = c_{1,0,0}^{k-1} = 2k$.

**Remark 4.4.** In the following discussion, by writing $\|\Lambda^\ell g\|$ for some generic norm $\|\cdot\|$ we mean

$$\|\Lambda^\ell g\| = \left( \sum_{1 \leq j_1 \leq 2} \cdots \sum_{1 \leq j_\ell \leq 2} \|\Lambda_{j_1} \cdots \Lambda_{j_\ell} g\|^2 \right)^{1/2}.$$  

Note $\|\Lambda^2 g\| \neq \|M_2 g\|$. Moreover by Cauchy inequality it follows that

$$\|\Lambda^\ell g \cdot \Lambda^\ell h\|_{(2,0)} \leq \sum_{j_1=1}^{2} \cdots \sum_{j_\ell=1}^{2} \|\Lambda_{j_1} \cdots \Lambda_{j_\ell} g\|_{(2,0)} \cdot \|\Lambda_{j_1} \cdots \Lambda_{j_\ell} h\|_{H^2_\ell L^\infty},$$  

recalling $\Lambda^\ell g \cdot \Lambda^\ell h$ is defined by (4.11).

**Lemma 4.5.** With the notations in Remark 4.4, it holds that

$$\|\Lambda^\ell g\|^2_{(2,0)} \leq \left\{ \begin{array}{ll} \|M_{\ell/2}^2 g\|^2_{(2,0)} & \text{for even number } \ell, \\
\|M_{(\ell+1)/2}^2 g\|^2_{(2,0)} \leq \|M_{\ell/2}^2 g\|^2_{(2,0)}\|M_{(\ell-1)/2}^2 g\|^2_{(2,0)} & \text{for odd number } \ell. \end{array} \right.$$  

**Proof.** In view of Remark 4.4, we have

$$\|\Lambda^\ell g\|^2_{(2,0)} = \sum_{j_1=1}^{2} \cdots \sum_{j_\ell=1}^{2} \|\Lambda_{j_1} \cdots \Lambda_{j_\ell} g\|^2_{(2,0)} = \sum_{j_1=1}^{2} \cdots \sum_{j_\ell=1}^{2} \left( \Lambda_{j_1} \cdots \Lambda_{j_\ell} g, \Lambda_{j_1} \cdots \Lambda_{j_\ell} g \right)_{(2,0)}$$

$$= \sum_{j_1=1}^{2} \cdots \sum_{j_\ell=1}^{2} \left( \Lambda_{j_1}^2 \cdots \Lambda_{j_\ell}^2 g, g \right)_{(2,0)} = \left( M^\ell g, g \right)_{(2,0)},$$  

which implies the assertion. The proof of lemma 4.5 is completed.  

$\square$
4.2. Proof of Proposition 2.3: estimate on commutators

This part is devoted to estimating the commutator between $M^k$ and the collision operator $\Gamma(f, f)$. By Proposition 3.1,

$$\Gamma(f, f) = \sum_{1 \leq j \leq 6} L_j(f, f)$$

with $L_j(f, f)$ the bilinear operators given in Proposition 3.1. We proceed through the following lemmas to deal with the commutator between $M^k$ and $L_j(f, f)$, $1 \leq j \leq 6$.

In the following discussion we will use $C \geq 1$ to denote a generic constant independent of $\epsilon, C_*$ and the derivative orders denoted by $k$, recalling $\epsilon, C_*$ are the constants given in the inductive assumption (2.9).

**Lemma 4.6** (technical lemma). *Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Let $\phi = \phi(v)$ be a given function of $v$ variable only satisfying that

$$\exists L > 0, \forall \beta \in \mathbb{Z}_+^1, \quad |\partial_1^{\beta} \phi(v)| \leq L^{2|\beta|} |\beta|!.$$  \hspace{1cm} (4.17)

Then

$$\sup_{m \leq i \leq 1} \|M^i(\phi f)\|_{(2,0)} \leq \begin{cases} C \frac{\epsilon}{(2i + 1)^3} C_*^{2i}(2i)! , \quad \text{if } 0 \leq i \leq k - 1, \\ C\|M^k f\|_{(2,0)} + C \frac{\epsilon}{(2k + 1)^3} C_*^{2k}(2k)! , \quad \text{if } i = k, \end{cases}$$

provided $C_*$ is large enough such that $C_*^{1/2} \geq 196L$, where the constant $C$ depends on $L$.

**Proof.** The assertion for $i = 0$ is obvious in view of the inductive assumption (2.9). Now consider the case when $1 \leq i \leq k$. Using (4.2) yields

$$M^i(\phi f)(m, \eta) - \hat{\phi} M^i f(m, \eta) = \int_{\mathbb{R}^3} \hat{\phi}(\eta - \tau) \rho_1(m_1, \eta_1) f(m, \tau) \rho_1(m_1, \tau_1) \hat{f}(m, \tau) d\tau$$

$$= \int_{\mathbb{R}^3} \hat{\phi}(\eta - \tau) \sum_{1 \leq j \leq 2i} \frac{1}{j!} \left( \frac{\partial^{j_1}_1 \rho_1(m_1)}{j_1!} (\eta_1 - \tau_1) \right) \hat{f}(m, \tau) d\tau$$

$$= \sum_{1 \leq j \leq 2i} \frac{1}{j!} \int_{\mathbb{R}^3} \frac{1}{j!} \left( \frac{\partial^{j_1}_1 \phi(\eta - \tau)}{j_1!} (\partial^{j_1}_1 \rho_1)(m_1, \eta_1) \right) \hat{f}(m, \tau) d\tau.$$

This implies

$$\|M^i(\phi f)\|_{(2,0)} \leq \|M^i f\|_{(2,0)} + \sum_{1 \leq j \leq 2i} \frac{1}{j!} \|\partial^{j_1}_1 \phi P_j f\|_{(2,0)},$$  \hspace{1cm} (4.18)

where the operator $P_j$ in the last term stands for a Fourier multiplier with the symbol $\partial^{j_1}_1 \rho_1$, that is,

$$P_j \hat{f}(m, \eta) = \partial^{j_1}_1 \rho_1(m_1, \eta_1) \hat{f}(m, \eta).$$

As a result, we use (4.4) and (4.17) as well as (4.2), to conclude

$$\sum_{1 \leq j \leq 2i} \frac{1}{j!} \|\partial^{j_1}_1 \phi P_j f\|_{(2,0)} \leq \sum_{1 \leq j \leq 2i} L^{j+1} S^{j} \frac{(2j)!}{(2 - j)!} \|M^{j+1/2} f\|_{(2,0)}.$$
Substituting the above estimate into (4.18) yields
\[ \|M'(\phi f)\|_{(2,0)} \leq C\|M'f\|_{(2,0)} + 16L^2 i \|M^{i-\frac{1}{2}}f\|_{(2,0)} + \sum_{2 \leq j \leq 2i} L^{j+1} 8^j \frac{(2i)!}{(2l-j)!} \|M^{i-\frac{1}{2}}f\|_{(2,0)} \]

Next we deal with the last term on the right hand side. When \( 1 \leq i \leq k \), we use inductive assumption (2.9) to conclude that for any integer \( j \) with \( 2 \leq j \leq 2i \), if \( j \) is even then
\[ \|M^{i-\frac{1}{2}}f\|_{(2,0)} \leq \frac{\epsilon}{(2l-j+1)^3} C^{2l-j}(2l-j)!, \]
and meanwhile if \( j \) is odd then
\[ \|M^{i-\frac{1}{2}}f\|_{(2,0)} \leq \|M^{i-\frac{1}{2}}f\|_{(2,0)}^{\frac{1}{2}} \|M^{i-\frac{1}{2}}f\|_{(2,0)}^{\frac{1}{2}} \]
\[ \leq C \frac{\epsilon}{(2l-j+1)^3} [C_s^{2l-j}(2l-j-1)! C_s^{2l-j+1}(2l-j+1)!] \leq C \frac{\epsilon}{(2l-j+1)^3} C_s^{2l-j}(2l-j)! \]
due to Lemma 4.5. Thus we compute
\[ \sum_{2 \leq j \leq 2i} L^{j+1} 8^j \frac{(2i)!}{(2l-j)!} \|M^{i-\frac{1}{2}}f\|_{(2,0)} \leq C\epsilon(2i)! \sum_{2 \leq j \leq 2i} L^{j+1} 8^j \frac{1}{(2l-j+1)^3} C_s^{2l-j} \]
\[ \leq C\epsilon(2i)! \sum_{2 \leq j \leq 2i} L^{j+1} 8^j \frac{1}{(2l-j+1)^3} C_s^{2l-j} + C\epsilon(2i)! \sum_{i+1 \leq j \leq 2i} L^{j+1} 8^j \frac{1}{(2l-j+1)^3} C_s^{2l-j} \]
\[ \leq C \frac{\epsilon}{(2l+1)^3} C_s^{2i}(2i)! \left[ \sum_{2 \leq j \leq 2i} (8L)^{j+1} C_s^{-j} \right] + \sum_{i+1 \leq j \leq 2i} (8L)^{j+1} (2l+1)^3 C_s^{-j} \]
\[ \leq C \frac{\epsilon}{(2l+1)^3} C_s^{2i}(2i)!, \]
the last line using the estimates that
\[ \sum_{2 \leq j \leq i} (8L)^{j+1} C_s^{-j} \leq C \]
and
\[ \sum_{i+1 \leq j \leq 2i} (8L)^{j+1} (2l+1)^3 C_s^{-j} \leq \sum_{i+1 \leq j \leq 2i} (8L)^{2i+1} (2l+1)^3 C_s^{-j-1} \]
\[ \leq (8L)^{2i+1} (2l+1)^4 C_s^{-i-1} \leq (192L)^{2i+1} C_s^{-i-1} \leq C, \]
provided \( \sqrt{C_s} \geq 192L \). Thus combining (4.19) and (4.18) we conclude, for any \( 1 \leq i \leq k \),
\[ \|M'(\phi f)\|_{(2,0)} \leq C\|M'f\|_{(2,0)} + 16L^2 i \|M^{i-\frac{1}{2}}f\|_{(2,0)} + C \frac{\epsilon}{(2l+1)^3} C_s^{2i}(2i)! \]
\[ \leq C\|M'f\|_{(2,0)} + C\|M^{i-1}f\|_{(2,0)} + C \frac{\epsilon}{(2l+1)^3} C_s^{2i}(2i)!, \]
the last inequality using the fact that \( \|M^{i-\frac{1}{2}}f\|_{(2,0)} \leq \|M'f\|_{(2,0)}^{\frac{1}{2}} \|M^{i-1}f\|_{(2,0)}^{\frac{1}{2}} \). This with the inductive assumption (2.9) yields
\[ \forall 1 \leq i \leq k-1, \sup_{i_0 \leq i \leq 1} \|M'(\phi f)\|_{(2,0)} \leq C \frac{\epsilon}{(2l+1)^3} C_s^{2i}(2i)! \]
(4.20)
\[ |M^\ell(\phi f)|_{(2,0)} \leq C|M^\ell f|_{(2,0)} + C\frac{\varepsilon}{(2k + 1)^3}C^2(2k)!. \]

The proof of Lemma 4.6 is completed. \(\square\)

**Corollary 4.7.** Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Let \(\phi = \phi(v)\) be a given function of \(v\) variable satisfying the estimate (4.17). Then with the notations in Remark 4.4,

\[ \forall (\ell, p) \in \mathbb{Z}_+^2 \text{ with } \ell + 2p \leq 2k - 2, \quad \sup_{t_0 \leq t_1} \|\Lambda^\ell M^p(\phi f)\|_{(2,0)} \leq C\frac{\varepsilon}{(\ell + p + 1)^3}C^{\ell+2p}(\ell + 2p)!. \]

Moreover,

\[ \sup_{t_0 \leq t_1} \|\Lambda^\ell M^{k-1}(\phi f)\|_{(2,0)} \leq C\sup_{t_0 \leq t_1} |M^\ell f|_{(2,0)}^{\frac{1}{2}} \left( \frac{\varepsilon}{(2k + 1)^3}C^{2k-2}(2k-2)! \right)^{\frac{1}{2}} + C\frac{\varepsilon C^{k-1}(2k-1)!}{(2k + 1)^3}. \]

**Proof.** We prove the first assertion. If \(\ell\) is even, then by the assumption \(0 \leq \ell + 2p \leq 2k - 2\) it follows that

\[ 0 \leq \frac{\ell}{2} + p \leq k - 1. \]

As a result, we use Lemmas 4.5 and 4.6 to conclude that the following estimate

\[ \sup_{t_0 \leq t_1} \|\Lambda^\ell M^p(\phi f)\|_{(2,0)} \leq \sup_{t_0 \leq t_1} \|M^\ell \!\!^\#(\phi f)\|_{(2,0)} \leq C\frac{\varepsilon}{(2p + \ell + 1)^3}C^{2p+\ell}(2p + \ell)! \quad (4.21) \]

holds true for any pair \((\ell, p) \in \mathbb{Z}_+^2\) with \(\ell\) even and \(0 \leq \ell + 2p \leq 2k - 2\).

Now we deal with the case when \(\ell\) is odd and \(\ell + 2p \leq 2k - 3\) and thus \(p + \frac{\ell + 1}{2} \leq k - 1\). This enables us to apply again Lemmas 4.5 and 4.6 to compute

\[ \|\Lambda^\ell M^p(\phi f)\|_{(2,0)} \leq \|M^{p+\frac{\ell}{2}}\!\!^\#(\phi f)\|_{(2,0)}^{1/2} \|M^{p+\frac{\ell}{2}}\!\!^\#(\phi f)\|_{(2,0)}^{1/2} \]

\[ \leq C\frac{\varepsilon}{(2p + \ell + 1)^3} \left[ C^{2p+\ell-1}(2p + \ell - 1)!C^{2p+\ell+1}(2p + \ell + 1)! \right]^{1/2} \]

\[ \leq C\frac{\varepsilon}{(2p + \ell + 1)^3}C^{2p+\ell-1}(2p + \ell)!, \]

which holds for all pair \((\ell, p) \in \mathbb{Z}_+^2\) with \(\ell\) odd and \(\ell + 2p \leq 2k - 2\). This with (4.21) yields the first assertion in Corollary 4.7.

As for the second assertion we make use of Lemmas 4.5 and 4.6 to get

\[ \|\Lambda^\ell M^{k-1}(\phi f)\|_{(2,0)} \leq \|M^k(\phi f)\|_{(2,0)}^{1/2} \|M^{k-1}(\phi f)\|_{(2,0)}^{1/2} \]

\[ \leq C\left( |M^k f|_{(2,0)} + C\frac{\varepsilon}{(2k + 1)^3}C^{2k}(2k)! \right)^{1/2} \left( \frac{\varepsilon}{(2k + 1)^3}C^{2k-2}(2k-2)! \right)^{1/2} \]

\[ \leq C\sup_{t_0 \leq t_1} |M^k f|_{(2,0)}^{1/2} \left( \frac{\varepsilon}{(2k + 1)^3}C^{2k-2}(2k-2)! \right)^{1/2} + C\frac{\varepsilon C^{k-1}(2k-1)!}{(2k + 1)^3}. \]

The proof of Corollary 4.7 is completed. \(\square\)
Lemma 4.8 (technical lemma). Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Then
\[ \forall \ell, q \in \mathbb{Z}^2_+ \text{ with } \ell + 2q \leq 2k - 2, \quad \left( \int_{t_0}^1 \| \langle \psi \rangle_\ell^2 \Lambda^\ell M^q \mathcal{P} f \|_{(2, 0)}^2 dt \right)^{\frac{1}{2}} \leq \frac{C}{(2q + \ell + 1)} C^2 q + (2q + \ell)!, \] (4.22)
and moreover,
\[ \left( \int_{t_0}^1 \| \langle \psi \rangle_\ell^2 \Lambda^\ell M^{k-1} \mathcal{P} f \|_{(2, 0)}^2 dt \right)^{\frac{1}{2}} \leq C \left( \int_{t_0}^1 \| \langle \psi \rangle_\ell^2 \mathcal{P} M^k f \|_{(2, 0)}^2 dt \right)^{\frac{1}{2}} + \frac{C}{(2k + 1)} C^{2k-2} (2k - 1)!, \] (4.23)
where \( \mathcal{P} \) stands for any one of the operators \( v \land \partial_v, \partial_v, \text{ and } v \).

Proof. As a preliminary step we first show that
\[ \forall j \geq 1, \quad \| \langle \psi \rangle_\ell^2 M^j \mathcal{P} f \|_{(2, 0)} \leq C \| \psi(v, D_v) M^j f \|_{(2, 0)} + C \| \psi(v, D_v) M^{j-1} f \|_{(2, 0)}, \] (4.24)
recalling \( \| \psi(v, D_v) f \|_{(2, 0)} \) is defined by (2.5). Without loss of generality we only prove (4.24) for \( \mathcal{P} = v \land \partial_v \), and the other cases can be treated in the same way with simpler argument. Recall \( M = \Lambda_1^2 + \Lambda_2^2 \) with \( \Lambda_j \) given in (4.8). Then Direct verification shows
\[ [M, v \land \partial_v] = 2 \sum_{1 \leq i \leq 2} ([\Lambda_i, v] \land \partial_v) \Lambda_i = 2 \sum_{1 \leq i \leq 2} \Lambda_i ([\Lambda_i, v] \land \partial_v), \quad \text{with } [\Lambda_i, v] = c_i((t - t_0)^{\frac{1}{2}}, 0, 0), \] (4.25)
where \( c_i, 1 \leq i \leq 2, \) are constants. Thus
\[ \langle \psi \rangle_\ell^2 M^j (v \land \partial_v) = \langle \psi \rangle_\ell^2 (v \land \partial_v) M^j + 2j \sum_{1 \leq i \leq 2} \langle \psi \rangle_\ell^2 \Lambda_i ([\Lambda_i, v] \land \partial_v) M^{j-1} \]
\[ = \langle \psi \rangle_\ell^2 (v \land \partial_v) M^j + 2j \sum_{1 \leq i \leq 2} \Lambda_i \langle \psi \rangle_\ell^2 ([\Lambda_i, v] \land \partial_v) M^{j-1} + 2j \sum_{1 \leq i \leq 2} \langle \psi \rangle_\ell^2 \Lambda_i ([\Lambda_i, v] \land \partial_v) M^{j-1}. \]

Moreover observe that
\[ \sum_{1 \leq i \leq 2} \| \Lambda_i \langle \psi \rangle_\ell^2 ([\Lambda_i, v] \land \partial_v) M^{j-1} f \|_{(2, 0)} \leq C \| M^{j-1} \langle \psi \rangle_\ell^2 \partial_v M^{j-1} f \|_{(2, 0)}, \]
due to Lemma 4.5, and that
\[ \sum_{1 \leq i \leq 2} \| \langle \psi \rangle_\ell^2, \Lambda_i ([\Lambda_i, v] \land \partial_v) M^{j-1} f \|_{(2, 0)} \leq C \| M^{j-1} M^{j-1} f \|_{(2, 0)}, \]
which holds because it follows from the fact \( \gamma \leq 1 \) that
\[ \forall 1 \leq |\beta| \leq 2, \quad |\partial_v^\beta \langle \psi \rangle_\ell^2| \leq C. \] (4.26)

This yields
\[ \| \langle \psi \rangle_\ell^2 M^j (v \land \partial_v) f \|_{(2, 0)} \]
\[ \leq \| \langle \psi \rangle_\ell^2 (v \land \partial_v) M^j f \|_{(2, 0)} + C \| M^{j-1} \partial_v \langle \psi \rangle_\ell^2 M^{j-1} f \|_{(2, 0)} + C \| M^{j-1} M^{j-1} f \|_{(2, 0)}, \] (4.27)
\[ \leq C \| \psi(v, D_v) M^j f \|_{(2, 0)} + C \| \psi(v, D_v) M^{j-1} f \|_{(2, 0)} + C \| M^{j-1} \partial_v \langle \psi \rangle_\ell^2 M^{j-1} f \|_{(2, 0)}, \]

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To estimate the last term on the right hand side of (4.27), we write
\[ \|M^{1/2}\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)} \leq \|M\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)} \|\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)} \leq C\|M\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)}\|\psi(v, D_v)M^{j-1} f\|_{l(2,0)}, \]
and moreover, by direct computation and using (4.26),
\[ \|M\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)} = \|\partial_v \langle \nu \rangle^2 M^{j-1} f + \partial_v [M, \langle \nu \rangle^2] M^{j-1} f\|_{l(2,0)} \leq \|\partial_v \langle \nu \rangle^2 M^{j} f\|_{l(2,0)} + \|\partial_v [M, \langle \nu \rangle^2] M^{j-1} f\|_{l(2,0)} \leq \|\partial_v \langle \nu \rangle^2 M^{j} f\|_{l(2,0)} + C\|M^{j-1} f\|_{l(2,0)} + C\|M^{j-1/2} f\|_{l(2,0)}\].

Thus, combining the above estimates yields
\[ \|M^{1/2}\partial_v \langle \nu \rangle^2 M^{j-1} f\|_{l(2,0)} \leq C\|\psi(v, D_v)M^{j} f\|_{l(2,0)^2} \|\psi(v, D_v)M^{j-1} f\|_{l(2,0)}^{1/2} + C\|\psi(v, D_v)M^{j-1} f\|_{l(2,0)}. \]

Substituting the above inequality into (4.27) yields (4.24) for \( P = v \land \partial_v \). The treatment for \( P = \partial_v \) or \( P = v \) is similar and simpler, so we omit it for brevity.

Next we will proceed to prove (4.22) and (4.23). The first two steps are devoted to proving (4.22) by induction on \( \ell \) and the last one to proving (4.23).

(i) Initial step. This step and the next one are devoted to proving (4.22) by induction on \( \ell \). For any integer \( q \in \mathbb{Z}_+ \) with \( 2q \leq 2k - 2 \), we have \( q \leq k - 1 \) and thus using (4.24) and the inductive assumption (2.9) yields
\[ \left( \int_0^1 \|\psi(v, D_v)M^q f\|_{l(2,0)}^2 dt \right)^{1/2} \leq C \left( \int_0^1 \|\psi(v, D_v)M^q f\|_{l(2,0)}^2 + q^2 \|\psi(v, D_v)M^{q-1} f\|_{l(2,0)}^2 dt \right)^{1/2} \leq C \frac{e}{(2q + 1)^3 N^{2q} (2q)!} \leq C \frac{e}{(2q + 1)^3 N^{2q} (2q)!}. \]

We have proven the validity of (4.22) for any pair \((\ell, q) \in \mathbb{Z}_+^2\) with \( \ell = 0 \) and \( \ell + 2q = 2q \leq 2k - 2 \).

(ii) Inductive step. Let \( \ell \geq 1 \) be a given integer, and suppose the following estimate
\[ \left( \int_0^1 \|\psi(v, D_v)M^{q+1} f\|_{l(2,0)}^2 dt \right)^{1/2} \leq C \frac{e}{(N + 2q + 1)^3 N^{2q+2}(N + 2q)!} \] (4.28)
holds true for any pair \((N, q) \in \mathbb{Z}_+^2\) with \( N \leq \ell - 1 \) and \( N + 2q \leq 2k - 2 \). We will show in this step that the above estimate (4.28) still holds for any pair \((N, q) \in \mathbb{Z}_+^2\) with \( N = \ell \) and \( N + 2q = \ell + 2q \leq 2k - 2 \). In the following discussion, let \( q \) be any integer satisfying that \( \ell + 2q \leq 2k - 2 \) with \( \ell \geq 1 \).

We first consider the case when \( \ell \) is odd. Then the assumption \( \ell + 2p \leq 2k - 2 \) implies
\[ \ell + 2p \leq 2k - 3. \] (4.29)

Observe \([\Lambda, \langle \nu \rangle^{\ell/2}]\) is just the first order derivatives of \( \langle \nu \rangle^{\ell/2} \). Then using (4.26) gives
\[ \left( \int_0^1 \|\langle \nu \rangle^{\ell/2} \Lambda^{\ell} M^{q+1} f\|_{l(2,0)}^2 dt \right)^{1/2} \leq \left( \int_0^1 \|\Lambda^{1} \langle \nu \rangle^{\ell/2} \Lambda^{\ell-1} M^{q+1} f\|_{l(2,0)}^2 dt \right)^{1/2} + C \left( \int_0^1 \|\Lambda^{\ell-1} M^{q+1} f\|_{l(2,0)}^2 dt \right)^{1/2}. \] (4.30)
Moreover for the first term on the right hand side,
\[
\left( \int_{t_0}^1 \| \Lambda^1 \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\leq \left( \int_{t_0}^1 \| M \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\leq C \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} + \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2},
\]
where in the last inequality we have used the fact that
\[
\| [M, \langle \psi \rangle^{\gamma/2}] g \|_{(2,0)} \leq C \| M^{1/2} g \|_{(2,0)} \leq C (\| M g \|_{(2,0)} + \| g \|_{(2,0)})
\]
due to (4.26). As a result, combining the above inequalities gives
\[
\left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2} \leq C \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
+ C \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} + \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-1} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\]
\[
\leq C \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell-1} (2q + \ell - 1)! + C \left[ \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell+1} (2q + \ell + 1)! \right]^{1/2} \left[ \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell-1} (2q + \ell - 1)! \right]^{1/2}
\leq C \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell} (2q + \ell)!,
\]
the second inequality using the inductive assumption (4.28) for \( N = \ell - 1 \) by observing (4.29).

We have proven that (4.28) holds true for any pair \((N, q)\) with \( N = \ell \) odd and \( N + 2q \leq 2k - 2 \).

Next we deal with the case when \( \ell \) is even. Similarly as in (4.30) and (4.31), we have
\[
\left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\leq \left( \int_{t_0}^1 \| \Lambda^2 \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-2} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2} + C \left( \int_{t_0}^1 \| \Lambda^1 \Lambda^{\ell-2} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\leq C \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-2} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-2} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/4} + \left( \int_{t_0}^1 \| \langle \psi \rangle^{\gamma/2} \Lambda^{\ell-2} M^q \mathcal{P} f \|^2_{(2,0)} dt \right)^{1/2}
\]
\[
\leq C \left[ \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell} (2q + \ell) ! \right]^{1/2} \left[ \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell} (2q + \ell) ! \right]^{1/2} + C \frac{\epsilon}{(2q + \ell + 1)^3} C_{\ast}^{2q+\ell} (2q + \ell)!,
\]
the last line using again the inductive assumption (4.28) for \( N = \ell - 2 \) since \( \ell - 2 + 2(q + 1) \leq \ell + 2q \leq 2k - 2 \). Combining the above estimate for even integer \( \ell \) with the previous (4.32) for odd integer \( \ell \), we conclude (4.28) holds true for any pair \((N, q)\) \( \in \mathbb{Z}^2_+ \) with \( N = \ell \) and \( N + 2q = \ell + 2q \leq 2k - 2 \). Thus the first assertion (4.22) follows.
(iii) The remaining case of $\ell = 1$ and $q = k - 1$. It remains to prove the second assertion (4.23). Applying (4.30) and (4.31) for $\ell = 1$ and $q = k - 1$, we have
\[
\left( \int_{t_0}^{1} \| v^2 \Lambda^1 M^{k-1} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}} \leq \left( \int_{t_0}^{1} \| v \Lambda^1 \langle \phi \rangle M^{k-1} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}} + C \left( \int_{t_0}^{1} \| M^{k-1} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{t_0}^{1} \| v \Lambda^1 \langle \phi \rangle M^{k} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}} \left( \int_{t_0}^{1} \| \langle \phi \rangle M^{k-1} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}} + C \left( \int_{t_0}^{1} \| M^{k-1} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{t_0}^{1} \| \langle \phi \rangle M^{k} P f \|^2_{(2,0), dt} \right)^{\frac{1}{2}} \left[ \frac{\epsilon}{(2k-1)^3} C^2 (2k-2)! \right]^{\frac{1}{2}} + C \left[ \frac{\epsilon}{(2k+1)^3} C^2 (2k)! \right]^{\frac{1}{2}}
\]
where in the third inequality we have used (4.22) and the last line follows by combining (4.24) with (4.22). Then we obtain the second inequality (4.23), completing the proof of Lemma 4.8.

**Lemma 4.9** (Commutator between $M^k$ and $L_1$). Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Then
\[
\left| \int_{t_0}^{1} (M^k L_1(f, f) - L_1(f, M^k f), M^k f)_{(2,0)} dt \right|
\]
\[
\leq (\delta + C) \left[ \left( \sup_{t_0 \leq t \leq 1} \| M^k f \|_{(2,0)} \right) + \int_{t_0}^{1} \| \langle \phi \rangle (v, D_v) M^k f \|_{(2,0)} dt \right] C_\delta \left[ \frac{\epsilon^2}{(2k+1)^3} C^2 (2k)! \right]^2,
\]
where $\delta > 0$ is an arbitrarily small constant, and $C_\delta$ is a constant depending on $\delta$.

**Proof.** In view of the representation of $L_1$ given in Proposition 3.1 we may write
\[
M^k L_1(f, f) - L_1(f, M^k f) = \sum_{1 \leq j \leq 3} S_j f,
\]
where
\[
\begin{aligned}
S_1 &= \frac{1}{2} [M^k, (v \wedge \partial_v) \cdot a_f (v \wedge \partial_v)], \\
S_2 &= \sum_{1 \leq i, j \leq 3} [M^k, \partial_v M_{i,j,f} \partial_v], \\
S_3 &= \frac{1}{2} [M^k, (v \wedge B_f) \cdot (v \wedge \partial_v)] - \frac{1}{2} [M^k, (v \wedge \partial_v) \cdot (B_f \wedge \partial_v)],
\end{aligned}
\]
with $a_f, B_f$ and $M_{i,j,f}$ defined in (3.2)-(3.3).

We split $S_1$ as
\[
S_1 = \frac{1}{2} [M^k, (v \wedge \partial_v)] \cdot a_f (v \wedge \partial_v) + \frac{1}{2} (v \wedge \partial_v) \cdot [M^k, a_f (v \wedge \partial_v)]
\]
\[
:= S_{1,1} + S_{1,2}.
\]

We first deal with $S_{1,2}$ and conclude that, for any $\delta > 0$,
\[
\int_{t_0}^{1} |(S_{1,2} f, M^k f)_{(2,0)}| dt
\]
\[
\leq (\delta + C) \left[ \left( \sup_{t_0 \leq t \leq 1} \| M^k f \|_{(2,0)} \right) + \int_{t_0}^{1} \| \langle \phi \rangle (v, D_v) M^k f \|_{(2,0)} dt \right] C_\delta \left[ \frac{\epsilon^2}{(2k+1)^3} C^2 (2k)! \right]^2.
\]
To prove (4.35), we use the fact that

\[ (v \wedge \partial_v) \cdot [M^k, a_f (v \wedge \partial_v)] = (v \wedge \partial_v) \cdot [M^k, a_f] (v \wedge \partial_v) = a_f [M^k, v \wedge \partial_v], \]

to get

\[
\int_{t_0}^1 |(S_{1.2}, M^k f)_{(2,0)}| dt = \frac{1}{2} \int_{t_0}^1 \left| (v \wedge \partial_v) \cdot [M^k, a_f] (v \wedge \partial_v) \right| dt \\
\leq \left( \int_{t_0}^1 \|\langle v \rangle^{-\frac{\xi}{2}} [M^k, a_f] (v \wedge \partial_v) f\|_{L^2(2)}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_0}^1 \|\langle v \rangle^{\frac{\xi}{2}} (v \wedge \partial_v) M^k f\|_{L^2(2)}^2 dt \right)^{\frac{1}{2}} \\
+ \left( \int_{t_0}^1 \|\langle v \rangle^{-\frac{\xi}{2}} a_f [M^k, v \wedge \partial_v] f\|_{L^2(2)}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_0}^1 \|\langle v \rangle^{\frac{\xi}{2}} (v \wedge \partial_v) M^k f\|_{L^2(2)}^2 dt \right)^{\frac{1}{2}} .
\]  
(4.36)

We deal with the first term on the right hand side of (4.36). In view of (3.2), we can verify directly

\[ \forall v \in \mathbb{R}^3, \quad |a_f (v)| \leq C \langle v \rangle^\gamma \|f\|_{L^2} \quad \text{and} \quad |\Lambda^\ell M^p a_f (v)| \leq C \langle v \rangle^\gamma \|\Lambda^\ell M^p (\sqrt{f})\|_{L^2}, \]

which, with Lemma 4.2 as well as Remark 4.4, implies

\[
\|\langle v \rangle^{-\gamma/2} [M^k, a_f] (v \wedge \partial_v) f\|_{(2,0)} \leq \sum_{j=1}^{2k} \sum_{\ell+2q=2k-j} \langle v \rangle^{\alpha_{\ell,p,q}} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}. \]
(4.38)

Thus

\[
\left( \int_{t_0}^1 \|\langle v \rangle^{-\frac{\xi}{2}} [M^k, a_f] (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2} \\
\leq \sum_{j=1}^{2k} \sum_{\ell+2q=2k-j} c_{\ell,p,q} \langle v \rangle^{\alpha_{\ell,p,q}} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \left( \int_{t_0}^1 \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2} \]
(4.39)

\[
:= \sum_{1 \leq \ell \leq 4} J_1
\]

with

\[
J_1 = \sum_{j=2}^{2k} \sum_{\ell+2q=2k-j} c_{\ell,p,q} \langle v \rangle^{\alpha_{\ell,p,q}} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \left( \int_{t_0}^1 \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2},
\]

\[
J_2 = \sum_{\ell+2p=2k} \sup_{h_0 \leq t \leq 1} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \left( \int_{t_0}^1 \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2},
\]

\[
J_3 = \sum_{\ell+2p=2k-1} \sup_{h_0 \leq t \leq 1} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \left( \int_{t_0}^1 \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2},
\]

\[
J_4 = \sum_{\ell+2p=2k} \sup_{h_0 \leq t \leq 1} \|\Lambda^\ell M^p (\sqrt{f})\|_{(2,0)} \left( \int_{t_0}^1 \|\langle v \rangle^{\gamma/2} \Lambda^\ell M^q (v \wedge \partial_v) f\|_{(2,0)}^2 dt \right)^{1/2}.
\]
To derive the upper bounds of $J_i$, $1 \leq i \leq 4$, we need the following fact:

$$\forall \beta \in \mathbb{Z}_+^3, \quad \|\partial^p \mu_{\beta}^{1/2}\|_{L^2_\gamma} + \|\partial^p \mu\|_{L^2_\gamma} \leq 2\|\partial^p e^{-\gamma^2/4}\|_{L^2_\gamma} \leq 16\|\beta\|.$$  

(4.40)

This enables us to use the estimates in Corollary 4.7 and Lemma 4.8, to compute

$$J_1 \leq C \sum_{j=2}^{2k-2} \sum_{p=0}^{(2k-2)+2} c_{\ell,p,q}^{k,j} \frac{e}{(2p+1)^{1/2}} C_{\ell+2p}^{(2p+1)^{1/2}} C_{\ell+2p+1}^{(2p+1)} (2q + \ell)!$$

$$\leq CC_{\ell}^{2k} \sum_{j=2}^{2k-2} \sum_{p=0}^{2k-2} c_{\ell,p,q}^{k,j} \frac{e^2}{(j+1)^{(2k-j+1)}!} \leq CC_{\ell}^{2k} (2k)!$$

the last line using (4.14). By Lemma 4.6 and the inductive assumption (2.9) and the fact $c_{0,0,0} = 1$

in Remark 4.3,

$$J_2 = \sum_{\ell=2}^{2k} C_{\ell}^{2k} \sup_{0 \leq t \leq 1} \|\Lambda^\ell M^p(\sqrt{\mu f})\|_{L^2_{(2,0)}} \left( \int_{t_0}^{1} \|\langle \nu \rangle^{1/2} \Lambda^\ell M^q (\nu \wedge \partial_\nu) f\|_{L^2_{(2,0)}} dt \right)^{1/2}$$

$$= c_{1,k-1} \sup_{0 \leq t \leq 1} \|\Lambda^\ell M^p(\sqrt{\mu f})\|_{L^2_{(2,0)}} \left( \int_{t_0}^{1} \|\langle \nu \rangle^{1/2} (\nu \wedge \partial_\nu) f\|_{L^2_{(2,0)}} dt \right)^{1/2}$$

$$\leq C \|\Lambda M f\|_{L^2_{(2,0)}} + C \frac{e}{(2k+1)^{1/2}} C_{\ell}^{2k} (2k)! \leq C e \|\Lambda^\ell M f\|_{L^2_{(2,0)}} + C \frac{e^2}{(2k+1)^3} C_{\ell}^{2k} (2k)!.$$
Similarly, using Corollary 4.7 and Lemma 4.8 and observing \( k_{1,0,k-1} = 2k \),

\[
J_4 = \sum_{l+2k = 2k-1} c_{l,p,q}^{k_1} \sup_{t \leq 2k-1} \| \Lambda^l M^{p} (\sqrt{\mu} f) \|_{(2,0)} \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} \Lambda^l M^{q} ((v \wedge \partial_v) f) \|_{(2,0)}^2 dt \right)^{1/2}
\]

\[
= c_{1,0,k-1}^{k_1} \sup_{t \leq 2k-1} \| \Lambda^1 (\sqrt{\mu} f) \|_{(2,0)} \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} \Lambda^1 M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{1/2}
\]

\[
\leq CK \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} (v \wedge \partial_v) M^{k-1} f \|_{(2,0)}^2 dt \right)^{1/2} \left[ \frac{\epsilon}{(2k-1)^3} C^{2k-2}(2k-2)! \right]^{1/2} C_\epsilon
\]

\[
+ CK \left( \frac{\epsilon}{(2k-1)^3} C^{2k-2}(2k-1)! \right] C_\epsilon
\]

\[
\leq C \left( \int_{t_0}^t \| \psi(v, D_v) M^{k} f \|_{(2,0)}^2 dt \right)^{1/2} \left[ \frac{\epsilon^2}{(2k-1)^3} C^{2k}(2k)! \right] + C \left( \frac{\epsilon^2}{(2k-1)^3} C^{2k}(2k)! \right].
\] (4.41)

Next we deal with the second term on the right hand side of (4.36). By (4.25),

\[
[M^k, v \wedge \partial_v] = kM^{k-1} [M, v \wedge \partial_v] = 2kM^{k-1} \sum_{1 \leq i \leq 2} \Lambda_i ([\Lambda_i, v] \wedge \partial_v),
\]

which, with the first estimate in (4.37), implies

\[
\left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} a_j [M^k, v \wedge \partial_v] f \|_{(2,0)}^2 dt \right)^{1/2} \leq C \left( \sup_{t \leq 2k-1} \| f(t) \|_{(2,0)} \right) \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} a_j [M^k, v \wedge \partial_v] f \|_{(2,0)}^2 dt \right)^{1/2}
\]

\[
\leq CK \left( \sup_{t \leq 2k-1} \| f(t) \|_{(2,0)} \right) \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} \Lambda^1 M^{k-1} \partial_v f \|_{(2,0)}^2 dt \right)^{1/2}
\]

\[
\leq Ck \left( \int_{t_0}^t \| \langle v \rangle^{\gamma/2} \partial_v M^{k} f \|_{(2,0)}^2 dt \right)^{1/2} \left[ \frac{\epsilon^2}{(2k-1)^3} C^{2k-2}(2k-2)! \right] + Ck \left( \frac{\epsilon^2}{(2k-1)^3} C^{2k-2}(2k-1)! \right]
\]

\[
\leq C \left( \int_{t_0}^t \| \psi(v, D_v) M^{k} f \|_{(2,0)}^2 dt \right)^{1/2} + C \left( \frac{\epsilon^2}{(2k-1)^3} C^{2k}(2k)! \right].
\] (4.41)

for any \( \delta > 0 \), where in the third inequality we have used (4.23) in Lemma 4.8. As a result, we substitute the above estimate and (4.41) into (4.36), to obtain the estimate (4.35).

Next we deal with \( S_{1,1} \) in (4.34) and write

\[
S_{1,1} = \frac{k}{2} [M^k, (v \wedge \partial_v)] \cdot a_j (v \wedge \partial_v) = \frac{k}{2} [M, (v \wedge \partial_v)] \cdot M^{k-1} a_j (v \wedge \partial_v)
\]

\[
= \frac{k}{2} [M, (v \wedge \partial_v)] \cdot a_j M^{k-1} (v \wedge \partial_v) + \frac{k}{2} [M, (v \wedge \partial_v)] \cdot [M^{k-1}, a_j] (v \wedge \partial_v).
\]
This, with the fact that
\[
[M, (v \wedge \partial_v)] = 2 \sum_{1 \leq i \leq 2} ([\Lambda_i, v] \wedge \partial_v) \Lambda_i
\]
in view of (4.25), yields
\[
\int_{t_0}^1 \| (S_{1.1} f, M^k f) \|_{(2,0)} dt \\
\leq C k \left( \int_{t_0}^1 \| (v)^{\gamma/2} \Lambda^1 a_f M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_0}^1 \| (v)^{\gamma/2} \partial_v M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} + C (\int_{t_0}^1 \| (v)^{\gamma/2} \Lambda^1 [M^{k-1}, a_f] (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_0}^1 \| (v)^{\gamma/2} \partial_v M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}}. 
\] (4.42)

Moreover, writing \( \Lambda^1 (a_f g) = a_f \Lambda^1 g + (\Lambda^1 a_f) g \) and then using (4.37),
\[
k \left( \int_{t_0}^1 \| (v)^{\gamma/2} \Lambda^1 a_f M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \\
\leq C k \left( \sup_{0 \leq t \leq 1} \| f \|_{(2,0)} \right) \left( \int_{t_0}^1 \| (v)^{\gamma/2} \Lambda^1 M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \\
+ C k \left( \sup_{0 \leq t \leq 1} \| \Lambda^1 (\sqrt{\mu} f) \|_{(2,0)} \right) \left( \int_{t_0}^1 \| (v)^{\gamma/2} M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \\
\leq C k \left( \int_{t_0}^1 \| (v)^{\gamma/2} (v \wedge \partial_v) M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \left[ \frac{e}{(2k-1)^3} C_\ast 2^{k-2} (2k-2)! \right]^{\frac{1}{2}} \\
+ C k \frac{e^2}{(2k-1)^3} C_\ast 2^{k-2} (2k-2)! + C C_\ast k \frac{e^2}{(2k+1)^3} C_\ast 2^{k-2} (2k-2)! \\
\leq C \left( \int_{t_0}^1 \| \psi (v, D_v) M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \left[ \frac{e^2}{(2k-1)^3} C_\ast 2^{k-2} (2k)! \right]^{\frac{1}{2}} + C \frac{e^2}{(2k+1)^3} C_\ast 2^{k-1} (2k)! \\
\leq \delta \left( \int_{t_0}^1 \| \psi (v, D_v) M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} + C \delta \frac{e^2}{(2k+1)^3} C_\ast 2^{k} (2k)! 
\]
the second inequality using the estimates (4.22)-(4.23) in Lemma 4.8 as well as the first estimate in Corollary 4.7. As a result,
\[
k \left( \int_{t_0}^1 \| (v)^{\gamma/2} \Lambda^1 a_f M^{k-1} (v \wedge \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_0}^1 \| (v)^{\gamma/2} \partial_v M^k f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}} \\
\leq \delta \int_{t_0}^1 \| \psi (v, D_v) M^k f \|_{(2,0)}^2 dt + C \delta \left[ \frac{e^2}{(2k+1)^3} C_\ast 2^{k} (2k)! \right]^2. 
\] (4.43)

We have the upper bound of the first term on the right side of (4.42), and it remains to deal with
the second one. Similarly to (4.38) we have

\[
\| (v)^{-\gamma/2} \Lambda^1 [M^{k-1}, a_f](v \land \partial_v) f \|_{(2,0)}^2 \\
\leq \sum_{j=1}^{2k-2} \sum_{\ell+2q=2k-2-j} \ell \epsilon_{\ell,p,q} k_{1,j} \| \Lambda^{\ell+1} M^p (\sqrt{f}) \|_{(2,0)} \| (v)^{\gamma/2} \Lambda^1 M^q (v \land \partial_v) f \|_{(2,0)}
\]

Then repeating the argument for treating \( J_1 \) in (4.39) we conclude, using 4.7 and Lemma 4.8,

\[
\| (v)^{-\gamma/2} \Lambda^1 [M^{k-1}, a_f](v \land \partial_v) f \|_{(2,0)}^2 \\
\leq C C_*^{2k-2} \sum_{j=2}^{2k-2} (j+1)!(2k-2-j)\frac{e^2}{(j+2)^3(2k-j-1)^3} \sum_{\ell+2q=2k-2-j} \ell \epsilon_{\ell,p,q} k_{1,j}
\]

\[
+ C C_*^{2k-2} \sum_{j=2}^{2k-2} j!(2k-1-j)\frac{e^2}{(j+1)^3(2k-j)^3} \sum_{\ell+2q=2k-2-j} \ell \epsilon_{\ell,p,q} k_{1,j}
\]

\[
\leq (C C_*^{2k-2})(2k-2)! \sum_{j=2}^{2k-2} \frac{e^2(j+2k-j)}{(j+1)^3(2k-j-1)^3} \leq C \frac{e^2}{(2k+1)^3} C_*^{2k}(2k)!. 
\]

Thus, for any \( \delta > 0 \),

\[
k \left( \int_{t_0}^1 \| (v)^{-\gamma/2} \Lambda^1 [M^{k-1}, a_f](v \land \partial_v) f \|_{(2,0)}^2 dt \right)^{\frac{1}{2}}
\leq \delta \int_{t_0}^1 \| \psi(v,D_v) M^k f \|_{(2,0)}^2 dt + C_\delta \frac{e^2}{(2k+1)^3} C_*^{2k}(2k)!. 
\]

Substituting the above estimate and (4.43) into (4.42) we obtain

\[
\int_{t_0}^1 \| (S_{1,1} f, M^k f) \|_{(2,0)}^2 dt \leq \delta \int_{t_0}^1 \| \psi(v,D_v) M^k f \|_{(2,0)}^2 dt + C_\delta \frac{e^2}{(2k+1)^3} C_*^{2k}(2k)!. 
\]

This with (4.35) as well as (4.34) yields

\[
\int_{t_0}^1 \| (S_{1,1} f, M^k f) \|_{(2,0)} dt \leq \delta \int_{t_0}^1 \| \psi(v,D_v) M^k f \|_{(2,0)}^2 dt + C_\delta \frac{e^2}{(2k+1)^3} C_*^{2k}(2k)!.
\]

By the definition of \( M_{i,k} \) and \( B_f \) given in (3.2) and (3.3), we can verify that similar to (4.37), the following estimates

\[
|B_f(v)| \leq C (v)^\gamma \| f \|_{L^2_0} \text{ and } |\Lambda^\ell M^p B_f(v)| \leq C (v)^\gamma \| \Lambda^\ell M^p (v \sqrt{f}) \|_{L^2_0}.
\]

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and

\[ |M_{i,j}f(v)| \leq C \langle v \rangle^\gamma \|f\|_{L^2} \quad \text{and} \quad |\Lambda^\ell M^p M_{i,j}f(v)| \leq C \langle v \rangle^\gamma \|\Lambda^\ell M^p ((\delta_{i,j} |v|^2 - v_i v_j) \sqrt{f})\|_{L^2} \]

hold true for any \( v \in \mathbb{R}^3 \). This with (4.33) enables us to repeat the above argument for estimating \( S_1 \) with slight modifications, to conclude that (4.44) still holds true with \( S_j, 2 \leq j \leq 3 \). The proof of Lemma 4.9 is thus completed.

\[ \square \]

**Lemma 4.10** (Commutator between \( M^k \) and \( L_j \), \( 2 \leq j \leq 6 \)). Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Then

\[ \sum_{2 \leq j \leq 6} \left| \int_{t_0}^1 \left( M^k L_j(f, f) - L_j(f, M^k f), M^k f \right)_{(2,0)} dt \right| \leq (\delta + \epsilon C) \left[ \left( \sup_{t_0 \leq t \leq 1} \|M^k f\|_{(2,0)} \right)^2 + \int_{t_0}^1 \|\psi(v, D_v) M^k f\|_{(2,0)}^2 dt \right] + C_\delta \left[ \frac{\epsilon^2}{(2k + 1)^3} C^2(2k)! \right]^2, \]

where \( \delta > 0 \) is an arbitrarily small constant.

**Proof.** The argument is quite similar as that in the proof of Lemma 4.9. Since there is no additional difficulty, we omit it for brevity.

\[ \square \]

**Lemma 4.11** (Commutator between \( M^k \) and \( \mathcal{L} \)). Suppose the inductive assumption (2.9) in Proposition 2.3 holds. Then

\[ \int_{t_0}^1 \left| \left( M^k, \mathcal{L} \right) f, M^k f \right|_{(2,0)} dt \leq \delta \int_{t_0}^1 \|\psi(v, D_v) M^k f\|_{(2,0)}^2 dt + C_\delta \left[ \frac{\epsilon}{(2k + 1)^3} C^2(2k)! \right]^2. \]

where \( \delta > 0 \) is an arbitrarily small constant.

**Proof.** Observe \( M^\ell \mu = -(t - t_0)^i \partial_{\mu}^2 \), and thus it follows from Lemma 4.5 and (4.40) that

\[ \forall (\ell, p) \in \mathbb{Z}^2_+, \quad \|\Lambda^\ell M^p \mu\|_{L^2} \leq \|M^\ell + p + 2 \mu\|_{L^2} \leq 16^{\ell + 2p + 1}(\ell + 2p)! \quad \text{(4.45)} \]

Then following the argument in the proof of Lemma 4.9 and using (4.45) instead of Corollary 4.7, we conclude by direct computation that

\[ \sum_{1 \leq j \leq 6} \left| \int_{t_0}^1 \left( M^k L_j(\sqrt{\mu}, f) - L_j(\sqrt{\mu}, M^k f), M^k f \right)_{(2,0)} dt \right| \leq \delta \int_{t_0}^1 \|\psi(v, D_v) M^k f\|_{(2,0)}^2 dt + C_\delta \left[ \frac{\epsilon}{(2k + 1)^3} C^2(2k)! \right]^2. \]

Similarly,

\[ \sum_{1 \leq j \leq 6} \left| \int_{t_0}^1 \left( M^k L_j(f, \sqrt{\mu}) - L_j(M^k f, \sqrt{\mu}), M^k f \right)_{(2,0)} dt \right| \leq \delta \int_{t_0}^1 \|\psi(v, D_v) M^k f\|_{(2,0)}^2 dt + C_\delta \left[ \frac{\epsilon}{(2k + 1)^3} C^2(2k)! \right]^2. \]

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As a result, observing  

\[-[\operatorname{M}^k, L]f = \left(\operatorname{M}^k (\sqrt{\mu}, f) - \Gamma (\sqrt{\mu}, \operatorname{M}^k f)\right) + \left(\operatorname{M}^k \Gamma(f, \sqrt{\mu}) - \Gamma(\operatorname{M}^k f, \sqrt{\mu})\right) \]

\[= \sum_{1 \leq j \leq 6} \left(\operatorname{M}^k L_j (\sqrt{\mu}, f) - L_j (\sqrt{\mu}, \operatorname{M}^k f)\right) + \sum_{1 \leq j \leq 6} \left(\operatorname{M}^k L_j (f, \sqrt{\mu}) - L_j (\operatorname{M}^k f, \sqrt{\mu})\right)\]

due to Proposition 3.1, we obtain the assertion in Lemma 4.11, completing the proof.  

Proof of Proposition 2.3. Combining the estimates in Lemmas 4.9 and 4.10 with the presentation of \(\Gamma(f, f)\) given in Proposition 3.1, we conclude

\[
\int_{t_0}^1 \left| (M^k \Gamma(f, f) - \Gamma(f, M^k f), M^k f)_{(2,0)} \right| \leq (\delta + \epsilon C) \left[ \left( \sup_{t_0 \leq t \leq 1} \|M^k f\|_{(2,0)} \right)^2 + \int_{t_0}^1 \left\| \psi(v, D_v) M^k f \right\|_{(2,0)}^2 dt \right] + C_0 \left[ \frac{\epsilon^2}{(2k + 1)^3} C^2_{\epsilon} (2k)! \right]^2.
\]

This with Lemma 4.11 yields the assertion in Proposition 2.3. The proof is completed.  

5. Analytic regularization effect of weak solutions

In this part we complete the proof of Theorem 1.1. We begin with the existence and Gevrey regularity of weak solutions to (1.4), and then improve the Gevrey regularity to analyticity. Recall the Gevrey class, denoted by \(G^r\), consists of all \(C^\infty\) smooth functions \(g\) such that

\[\exists C > 0, \forall \alpha, \beta \in \mathbb{Z}_+^3, \quad \|\partial_\alpha^\sigma \partial_\beta^\nu g\|_{L^2} \leq C^{(|\alpha| + |\beta| + 1)((|\alpha| + |\beta|)!)^r}.
\]

Theorem 5.1. Under the same assumption as in Theorem 1.1, the Cauchy problem (1.4) admits a unique global-in-time solution \(f\) satisfying that

\[
\sup_{t > 0} \|f(t)\|_{L^2} + \left( \int_0^{+\infty} \left\| \psi(v, D_v) f(t) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \leq C_0 \epsilon_0
\]

for some constant \(C_0 \geq 1\) depending only the initial data, where \(\epsilon_0\) is the small sufficiently number given in (1.7). Moreover the following estimate

\[
\sup_{t > 0} \|\partial_\alpha^\sigma \partial_\beta^\nu f(t)\|_{L^2} + \left( \int_0^{+\infty} \left\| \psi(v, D_v) \partial_\alpha^\sigma \partial_\beta^\nu f(t) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \leq C^{(|\alpha| + |\beta| + 1)((|\alpha| + |\beta|)!)^{\frac{3}{2}}}
\]

holds true for all \(\alpha, \beta \in \mathbb{Z}_+^3\), recalling \(i := \min \{i, 1\}\).

Sketch of the proof of Theorem 5.1. The global-in-time existence and uniqueness of mild solutions \(f(t, x, v)\) in the low-regularity space \(L^1_m L^2_v \subset L^2\) was established by [31, Theorem 2.1]. Furthermore we may follow the presentation in [30] with necessary modifications, to conclude the global Gevrey smoothing effect of such low-regularity solutions in the sense that \(f(t, \cdot, \cdot) \in G^{3/2}\) for any \(t > 0\), that is, the quantitative estimate (5.1) is satisfied globally in time.

Observe the proof in [30] relies on the same trilinear and coercivity estimates for Boltzmann collision operator as that in Proposition 2.2, the pseudo-differential calculus for the symbol

\[\langle v \rangle^s \left(1 + v^2 + \eta^2 + (v \cdot \eta)^2\right)^{s}, \quad s \in [0, 1],
\]

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which corresponds to $s = 1$ in the case of Landau equations. Thus the estimate (5.1) will follow without any additional difficulty by applying the same strategy as in [30], and we refer to interested readers to [16] for the detailed derivation of (5.1).

Completing the proof of Theorem 1.1. Let $f$ be the unique global-in-time solution to (1.4) constructed in Theorem 5.1 such that the quantitative estimate (5.1) is fulfilled. In view of (5.1), it follows that $f \in L^\infty([t_0, +\infty[; H^{+\infty})$ for any $0 < t_0 \leq 1/2$ and moreover

$$\sup_{t_0 \leq t \leq 1} \|f(t)\|_{(2,0)} + \left( \int_{t_0}^{1} \|\psi(v, D_v) f(t)\|_{(2,0)}^2 \, dt \right)^{1/2} \leq (2C_0/t_0)^3 \varepsilon_0$$

and

$$\forall \alpha, \beta \in \mathbb{Z}_+^3, \quad \sup_{t_0 \leq t \leq 1} \|\partial_x^\alpha \partial_y^\beta f(t)\|_{(2,0)} + \left( \int_{t_0}^{1} \|\psi(v, D_v) \partial_x^\alpha \partial_y^\beta f(t)\|_{L^2}^2 \, ds \right)^{1/2} < +\infty.$$  

Then the assumptions in Theorem 2.1 are fulfilled with $\varepsilon = (2C_0/t_0)^3 \varepsilon_0$, and thus we apply Theorem 2.1 to conclude that there exists a constant $C_* \geq 1$, such that

$$\forall k \in \mathbb{Z}_+, \quad \sup_{t_0 \leq t \leq 1} \|M^k f(t)\|_{(2,0)} + \left( \int_{t_0}^{1} \|\psi(v, D_v) M^k f(t)\|_{(2,0)}^2 \, dt \right)^{1/2} \leq \frac{(2C_0/t_0)^3 \varepsilon_0}{(2k + 1)^3} C_* (2k)! \quad (5.2)$$

Recall $M$ is a Fourier multiplier with symbol

$$(t - t_0)\eta_1^2 + (t - t_0)^2 \eta_1 m_1 + \frac{(t - t_0)^3}{3} m_1,$$

which with (5.2) and (4.1) implies, for any $t_0 \in [0, 1/2],

$$\sup_{t \in [t_0, 1]} \left[ (t - t_0)^k \|\partial_x^{2k} f(t)\|_{(2,0)} + (t - t_0)^{3k} \|\partial_x^{2k} \partial_y^0 f(t)\|_{(2,0)} \right] \leq (2C_0/t_0)^3 \varepsilon_0 (C_* / c_0)^{2k} (2k)!$$

with $c_0$ the constant given in (4.1). In particular, letting $t = 2t_0 \in [t_0, 1]$ in the above estimate yields

$$\forall k \in \mathbb{Z}_+, \forall t_0 \in [0, 1/2], \quad t_0^k \|\partial_x^{2k} f(2t_0)\|_{(2,0)} + t_0^{3k} \|\partial_x^{2k} \partial_y^0 f(2t_0)\|_{(2,0)} \leq t_0^{-3} (2C_0)^3 \varepsilon_0 (C_* / c_0)^{2k} (2k)!,$$

that is,

$$\forall k \in \mathbb{Z}_+, \forall t \in [0, 1], \quad t^{k+3} \|\partial_x^{2k} f(t)\|_{(2,0)} + t^{3k+3} \|\partial_x^{2k} \partial_y^0 f(t)\|_{(2,0)} \leq (4C_0)^3 \varepsilon_0 (3C_* / c_0)^{2k} (2k)!.$$

By virtue of the last assertion in Theorem 2.1, the estimate (5.2) also holds with $M$ replaced by

$$-(t - t_0) \partial_\nu^2 - (t - t_0)^2 \partial_\nu \partial_\nu - \frac{(t - t_0)^3}{3} \partial_\nu^3, \quad i = 2 \text{ or } 3,$$

which implies the validity of (5.3) with $\partial_\nu$ replaced by $\partial_{x_2}$ or $\partial_{x_3}$, and $\partial_\nu$ by $\partial_{x_2}$ or $\partial_{x_3}$. As a result, we combine (5.3) with the fact

$$\forall \alpha \in \mathbb{Z}_+^3, \quad \|\partial_x^\alpha f\|_{L^2} \leq \|\partial_\nu^\alpha f\|_{L^2} + \|\partial_\nu^2 f\|_{L^2} + \|\partial_\nu^3 f\|_{L^2}$$

and similarly for $\|\partial_\nu^\alpha f\|_{L^2},$ to conclude

$$\forall \alpha \in \mathbb{Z}_+^3, \quad \sup_{0 < t \leq 1} \left[ t^{k+3} \|\partial_\nu^{2k} f(t)\|_{(2,0)} + t^{3k+3} \|\partial_\nu^{2k} \partial_\nu^0 f(t)\|_{(2,0)} \right] \leq (4C_0)^3 \varepsilon_0 (3C_* / c_0)^{2k} (2k)!.$$

(5.4)
Then, for any $0 < t \leq 1$ and any $\alpha, \beta \in \mathbb{Z}^3_+$ with $|\alpha| \geq 2$, we can write $\alpha = \tilde{\alpha} + (\alpha - \tilde{\alpha})$ with $|\alpha - \tilde{\alpha}| = 2$ and thus

$$\| t^{\frac{3}{2}\tilde{\alpha}} f(t) \|_{L^2} \leq t^{\frac{3}{2} |\tilde{\alpha}| + \frac{3}{2}} \| \tilde{\alpha}_x \tilde{\alpha}_e^{\tilde{\alpha}} f(t) \|_{L^2} \leq t^{\frac{3}{2} |\alpha| + \frac{3}{2}} \| \tilde{\alpha}_x \tilde{\alpha}_e^{\tilde{\alpha}} f(t) \|_{L^2} \leq t^{\frac{3}{2} |\alpha| + \frac{3}{2}} \| \tilde{\alpha}_x \tilde{\alpha}_e^{\tilde{\alpha}} f(t) \|_{L^2},$$

we have proven the assertions (1.8) and (1.9) in Theorem 1.1 for $0 < t \leq 1$ by choosing $C = \max \{4C_0^3, 6^C \phi(c)/c \}$.

Once the analyticity regularization effect is achieved for $0 \leq t \leq 1$, it is essentially the propagation of analyticity from $t = 1$ to $t > 1$ when deriving the analyticity for $t > 1$. This will follow by performing standard energy estimates for $\| \tilde{\alpha}_x \tilde{\alpha}_e^{\tilde{\alpha}} f(t) \|_{L^2}$ at $t \in ]1, +\infty[ \text{ and there is no additional difficulty. So we omit it for brevity. The proof of Theorem 1.1 is thus completed.} \square

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