THE BIOT–SAVART OPERATOR OF A BOUNDED DOMAIN

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Abstract. We construct the analog of the Biot–Savart integral for bounded domains. Specifically, we show that the velocity field of an incompressible fluid with tangency boundary conditions on a bounded domain can be written in terms of its vorticity using an integral kernel $K_{\Omega}(x, y)$ that has an inverse-square singularity on the diagonal.

1. Introduction

The Biot–Savart operator,

$$\text{BS}(\omega)(x) := \int_{\mathbb{R}^3} \frac{\omega(y) \times (x - y)}{4\pi |x - y|^3} \, dy,$$

plays a key role in fluid mechanics and electromagnetism as a sort of inverse of the curl operator. More precisely, if $\omega$ is a well-behaved divergence-free vector field on $\mathbb{R}^3$, then $u := \text{BS}(\omega)$ is the only solution to the equation

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0$$

that falls off at infinity. Consequently, in fluid mechanics the Biot–Savart operator maps the vorticity $\omega$ of a fluid into its associated velocity field $u$.

In this paper we will be concerned with the problem of mapping the vorticity of a fluid contained in a bounded domain $\Omega$ of $\mathbb{R}^3$ into its velocity field. To put it differently, given a vector field $\omega$ we want to solve the problem

$$(1) \quad \nabla \times u = \omega, \quad \nabla \cdot u = 0, \quad u \cdot \nu = 0$$

in $\Omega$, where the tangency condition $u \cdot \nu = 0$ means that the fluid stays inside the domain. We will assume throughout that the boundary of this domain is smooth, although one could relax this condition.

It is easy to see that, for this equation to admit a solution, the field $\omega$ must satisfy several hypotheses. Firstly, since the divergence of a curl is zero, it is obvious that $\omega$ must be divergence-free. Secondly, it is easy to see that if $\Gamma_1, \ldots, \Gamma_m$ denote the connected components of $\partial \Omega$, then one must have

$$(2) \quad \int_{\Gamma_j} \omega \cdot \nu \, d\sigma = 0 \quad \text{for all } 1 \leq j \leq m.$$

To see why this is true, it is enough to take the harmonic functions $\psi_j$ defined by the boundary value problems

$$\nabla^2 \psi_j = 0 \quad \text{in } \Omega, \quad \psi_j|_{\Gamma_k} = \delta_{jk}.$$
and observe that
\[
\int_{\Gamma_j} \omega \cdot \nu \, d\sigma = \int_{\Omega} \nabla \cdot (\psi_j \omega) \, dx = \int_{\Omega} \nabla \psi_j : (\nabla \times u) \, dx = \int_{\partial \Omega} u \cdot (\nabla \psi_j \times \nu) \, d\sigma = 0.
\]

In addition to using that \( \omega \) is divergence-free and integrating by parts, we have exploited that the gradient \( \nabla \psi_j \) is proportional to \( \nu \) at the boundary.

The problem (1) has been considered by a number of people, who have shown the existence of solutions for domains of different regularity and derived estimates in \( L^p \) or Hölder spaces. Up to date accounts of the problem can be found e.g. in [1,2,4] and references therein. However, the question of whether the solution is given by an integral formula generalizing the classical Biot–Savart law remains wide open.

Our objective in this paper is to fill this gap by showing that one can construct a solution to the problem (1) through a generalized Biot–Savart operator with a reasonably well-behaved integral kernel. We will only state our result for the flat-space problem, but it will be apparent from the construction that the result holds true (mutatis mutandis) for bounded domains in any Riemannian 3-manifold. The existence of a Biot–Savart operator on a compact Riemannian 3-manifold without boundary can be obtained from the Green’s function of the Hodge Laplacian, computed in [5], although to the best of our knowledge the only result available in the literature [14] is a weaker analog of the Biot–Savart operator on a closed manifold that provides a solution to the equation up to a gradient field (i.e., a vector field satisfying \( \nabla \times u = \omega + \nabla \varphi \), where \( \varphi \) is some smooth function). It is should be stressed that this connection between the Green’s function of the Laplacian in a domain (with certain boundary conditions) and the Biot–Savart operator is no longer true in the presence of a boundary condition. Indeed, the combination of the boundary condition with the fact that the field must be divergence-free makes the structure of the generalized Biot–Savart kernel rather involved.

Our motivation for this is threefold. Firstly, the integral kernel of the Biot–Savart operator on a compact 3-manifold without boundary has been recently employed [9] to show that the helicity is the only regular integral invariant of volume-preserving transformations. Proving a similar result for the case of manifolds with boundary presents additional difficulties, but in any case it is important to have a thorough understanding of the associated Biot–Savart operator. A particular case has been recently established in [11]. Applications to electrodynamics of the Biot–Savart operator for domains in the 3-sphere (whose existence as an integral operator is not discussed, however) can be found in [12]. Secondly, the existence of a generalized Biot–Savart integral in domains ensures that the celebrated connection between the helicity of a field and the linking number, unveiled by Arnold in the full space [3], remains valid in bounded domains. A third motivation to construct the Biot–Savart operator in domains is that the structure of the integral kernels of the inverses of operators has been key to develop certain approximation theorems that we have exploited in different contexts [6,7,8].

To state the existence of a Biot–Savart integral operator on \( \Omega \), let us define the function
\[
\ell(y) := \log \left( 2 + \frac{1}{\text{dist}(y, \partial \Omega)} \right).
\]
Let us also recall that a vector field $h$ on $\Omega$ is said to be harmonic and tangent to the boundary if
\[
\nabla \times h = 0, \quad \nabla \cdot h = 0, \quad h \cdot \nu = 0.
\]
By Hodge theory, the dimension of the linear space of tangent harmonic fields is the genus of $\partial \Omega$ (if $\partial \Omega$ is disconnected, this is defined as the sum of the genus of the connected components of the boundary). The main result of the paper can then be presented as follows:

**Theorem 1.** Let $\omega \in W^{k,p}(\Omega)$ be a divergence-free vector field satisfying the hypothesis (2), with $k \geq 0$ and $1 < p < \infty$. Then the boundary-value problem (1) has a solution $u \in W^{k+1,p}(\Omega)$ that satisfies the estimate
\[
\|u\|_{W^{k+1,p}(\Omega)} \leq C \|\omega\|_{W^{k,p}(\Omega)}
\]
and which can be represented as an integral of the form
\[
u(x) = \text{BS}_{\Omega}(\omega)(x) := \int_{\Omega} K_{\Omega}(x,y) \omega(y) \, dy,
\]
where $K_{\Omega}(x,y)$ is a matrix-valued integral kernel that is smooth outside the diagonal and satisfies the pointwise bound
\[
|K_{\Omega}(x,y)| \leq C \frac{\ell(y)}{|x-y|^2}.
\]
Furthermore, the solution is unique modulo the addition of a harmonic field tangent to the boundary.

Let us emphasize that the core of the result is not the estimate (4), which is not new, but the existence of an integral kernel $K_{\Omega}(x,y)$ with the above properties. Of course, a serious difficulty that arises in the analysis of the integral kernel $K_{\Omega}(x,y)$ of the Biot–Savart operator on a domain is that it is strongly non-unique. Indeed, it is easy to see that if $\phi$ is a scalar function on $\Omega$ satisfying the Dirichlet boundary condition $\phi|_{\partial \Omega} = 0$, then its gradient is orthogonal to any divergence-free field $\omega$ in the sense that
\[
\int_{\Omega} \nabla \phi \cdot \omega \, dx = 0
\]
for any vector field with $\nabla \cdot \omega = 0$. Hence if $K'(x,y)$ is a matrix-valued function of the form
\[
K'_{ij}(x,y) = \partial_{y_j} A_i(x,y)
\]
with $A_i(x,\cdot)|_{\partial \Omega} = 0$, then $K_{\Omega}(x,y) + K'(x,y)$ is also an admissible kernel for the Biot–Savart operator of the domain.

The paper is organized as follows. In Section 2 we will present some estimates and identities for the usual Biot–Savart integral that are needed later. In Section 3 we construct an extension operator for vector fields we employ during the construction of the kernel $K_{\Omega}(x,y)$ to deal with boundary terms. In Section 4 we construct the solution $u$ to the problem (1) using layer potentials. This is convenient because it leads to quite explicit formulas, which we carefully analyze in Section 5 to establish the existence of the desired integral kernel. The proof works without any major modifications on any Riemannian 3-manifold, using the layer potentials and the Biot–Savart operator associated with the metric. To conclude, for completeness we discuss in Section 6 the uniqueness of the solutions (possibly with nonzero but prescribed divergence and normal component on the boundary).
2. Estimates and identities for the Biot–Savart integral

This section is a brief but reasonably self-contained presentation of several results and identities for the Biot–Savart integral that will be used later. These results are essentially standard.

Given a vector field $F \in C^1(\Omega)$, we will state the results in terms of the field

$$w(x) := \int_{\Omega} F(y) \times \frac{(x - y)}{4\pi|x - y|^3} \, dy.$$  

In what follows, $\epsilon_{ijk}$ will denote Levi-Civita’s permutation symbol and $B_r(x)$ (resp. $B_r(\text{centered at the point } x$) will denote the three-dimensional ball of radius $r$ centered at the point $x$ (resp. at the origin).

**Lemma 2.** The derivative of the field $w$ at any point $x \in \Omega$ is

$$\partial_j w_k(x) = \epsilon_{klm} \text{PV} \int_{\Omega} F(y) \left[ \frac{|x - y|^2 \delta_{jm} - 3(x_j - y_j)(x_m - y_m)}{4\pi|x - y|^3} \right] dy$$

$$- \epsilon_{klm} F_i(x) A_m^j(x),$$

where $A_m^j$ stands for the $m^{th}$ component of a certain continuous vector field $A^j \in L^\infty(\Omega)$ and $\text{PV}$ denotes the principal value.

**Proof.** Let us take a smooth function $\eta(r)$ which vanishes for $r < \frac{1}{2}$ and is equal to 1 for $r > 1$ and set $\eta_\delta(r) := \eta(r/\delta)$, with $\delta$ a small positive constant. Then let us define the vector field

$$w^\delta(x) := \int_{\Omega} \eta_\delta(|x - y|) \frac{F(y) \times (x - y)}{4\pi|x - y|^3} \, dy.$$  

It is apparent that $w^\delta$ is a smooth vector field on $\mathbb{R}^3$. Moreover, it is not hard to see that $w^\delta$ converges to $w$ uniformly, since for any $x \in \Omega$ one has

$$|w(x) - w^\delta(x)| = \left| \int_{B_\delta(x)} (1 - \eta_\delta(|x - y|)) \frac{F(y) \times (x - y)}{4\pi|x - y|^3} \, dy \right|$$

$$\leq (1 + \|\eta\|_{L^\infty}) \|F\|_{L^\infty(\Omega)} \int_{B_\delta} \frac{dz}{4\pi|z|^2}$$

$$\leq C\delta.$$  

The derivative of the $k^{th}$ component of $w^\delta$ can be readily computed as

$$\partial_j w^\delta_k(x) = \epsilon_{klm} \int_{\Omega} F_i(y) \partial_j \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy$$

$$= \epsilon_{klm} \int_{\Omega} \left[ F_i(y) - F_i(x) \right] \partial_j \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy$$

$$- \epsilon_{klm} \int_{\Omega} \partial_j \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy$$

$$= \epsilon_{klm} \int_{\Omega} \left[ F_i(y) - F_i(x) \right] \partial_j \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy$$

$$- \epsilon_{klm} \int_{\Omega} \partial_j \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy$$

$$- \epsilon_{klm} F_i(x) A_m^j(x),$$  

$$\quad (6)$$
where for each \( x \in \Omega \) we have set
\[
A_{m,h}^i(x) := \int_{\partial \Omega} \nu_j(y) \eta_h(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \, d\sigma(y).
\]

As \( \delta \to 0 \), the first term converges to the principal value integral of the statement, since the difference
\[
M := \int_{\Omega} [F_i(y) - F_i(x)] \partial_{x_j} \left( \eta_h(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) \, dy
\]
\[
- \text{PV} \int_{\Omega} F_i(y) \frac{|x - y|^2 \delta_m - 3(x_j - y_j)(x_m - y_m)}{4\pi|x - y|^5} \, dy
\]
can be estimated using the fact that \( \eta_h(r) = 1 \) for \( r > \delta \) and the mean value theorem as
\[
|M| \leq \int_{B_{\delta}(x)} |F_i(y) - F_i(x)| \left| \partial_{x_j} \left( 1 - \eta_h(|x - y|) \right) \frac{x_m - y_m}{4\pi|x - y|^3} \right| \, dy
\]
\[
\leq C \int_{B_{\delta}(x)} \frac{|F_i(y) - F_i(x)|}{|x - y|^3} \, dy
\]
\[
\leq C\|\nabla F_i\|_{L^\infty(\Omega)} \int_{B_{\delta}(x)} \frac{1}{|x - y|^2} \, dy
\]
(7)
\[
\leq C\delta.
\]

Since \( A_{i,\delta}^j(x) \) obviously converges to the field
\[
A^j(x) := \int_{\partial \Omega} \nu_j(y) \frac{x - y}{4\pi|x - y|^3} \, d\sigma(y)
\]
for all \( x \in \Omega \), Equations (6) and (7) show that the derivative \( \partial_j w_\varepsilon \) is indeed given by the formula (5) of the statement. As \( A^j \) is obviously smooth in \( \Omega \), it only remains to show that \( A^j(x) \) is bounded when \( x \to \partial \Omega \).

For this, let us take an arbitrary point of the boundary, which we can assume to be the origin. Rotating the coordinate axes if necessary, one can parametrize \( \partial \Omega \) in a neighborhood \( U_\rho \) of the origin as the graph
\[
Y \in D_\rho \mapsto (Y, h(Y)) \in \mathbb{R}^3,
\]
where \( D_\rho := \{Y \in \mathbb{R}^2 : |Y| < \rho \} \) is the two-dimensional disk of a small radius \( \rho \) and the function \( h \) satisfies
\[
h(0) = 0, \quad \nabla h(0) = 0.
\]
Let us now analyze the behavior of \( A^j(x) \) when \( x = (0,0,-t) \) and \( t \to 0^+ \). Since the unit normal and the surface measure can be written in terms of \( h \) as
\[
\nu = \frac{(-\nabla h, 1)}{\sqrt{1 + |\nabla h|^2}}, \quad d\sigma = \sqrt{1 + |\nabla h|^2} \, dY
\]
and \( x \in U_\rho \) for small enough \( t \), it follows that
\[
|A_j(x)| \leq \left| \int_{\partial Y \cap U_\rho} \nu_j(y) \frac{x - y}{4\pi|x - y|^3} \, d\sigma(y) \right| + \left| \int_{\partial Y \setminus U_\rho} \nu_j(y) \frac{x - y}{4\pi|x - y|^3} \, d\sigma(y) \right|
\]
(9)
\[
\leq \frac{1}{4\pi} \int_{D_\rho} \tilde{\nu}_j(Y) \frac{(Y, t + h(Y))}{(|Y|^2 + (t + h(Y))^2)^{3/2}} \, dY + C,
\]
where we have used that the second integral is bounded by a constant independent of \( t \) and

\[
\bar{\nu}_j := \sqrt{1 + |\nabla h|^2} \nu_j.
\]

It is clear then that

\[
\left| \int_{D_\rho} \bar{\nu}_j(Y) \frac{Y}{(|Y|^2 + (t + h(Y))^2)^{3/2}} dY \right| = \left| \int_{D_\rho} \bar{\nu}_j(0) Y + O(|Y|^2) + tO(Y) dY \right|
\]

\[
\leq \left| \bar{\nu}_j(0) \right| \int_{D_\rho} \frac{Y}{(|Y|^2 + t^2)^{3/2}} dY + C \int_{D_\rho} \frac{dY}{|Y|}
\]

\[
\leq C \rho,
\]

where we have used that the first integral in the second line vanishes by parity. Likewise,

\[
\left| \int_{D_\rho} \bar{\nu}_j(Y) \frac{t + h(Y)}{(|Y|^2 + (t + h(Y))^2)^{3/2}} dY \right| = \left| \int_{D_\rho} \bar{\nu}_j(0) t + O(|Y|^2) + tO(Y) dY \right|
\]

\[
\leq 2\pi |\bar{\nu}_j(0)| \int_0^\rho \frac{r \, dr}{(r^2 + t^2)^{3/2}} + C \int_{D_\rho} \frac{dY}{|Y|}
\]

\[
\leq 2\pi |\bar{\nu}_j(0)| \int_0^\infty \frac{r \, dr}{(r^2 + 1)^{3/2}} + C \rho
\]

\[
\leq C,
\]

Hence we infer from (19) that \( \|A^i\|_{L^\infty(\Omega)} \leq C \) and the lemma follows. \( \square \)

Lemma 2 immediately yields the following characterization of the divergence and curl of \( w \):

**Proposition 3.** The vector field \( w \) is divergence-free in \( \mathbb{R}^3 \) and its curl at a point \( x \in \Omega \) is given by

\[
\nabla \times w(x) = F(x) + \nabla \int_\Omega \frac{\nabla \cdot F(y)}{4\pi |x-y|} \, dy - \nabla \int_{\partial \Omega} \frac{F(y) \cdot \nu(y)}{4\pi |x-y|} \, d\sigma(y).
\]

**Proof.** It follows from the proof of Lemma 2 that \( \nabla \cdot w = \lim_{\delta \to 0} \nabla \cdot w^\delta \), with \( w^\delta \) defined as before. For simplicity, let us write

\[
E := \frac{1}{4\pi |x-y|}.
\]

Since Equation (20) shows that

\[
\nabla \cdot w^\delta = \epsilon_{klm} \int_\Omega F_i(y) \partial_{x_k} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^4} \right) \, dy
\]

\[
= \epsilon_{klm} \int_\Omega F_i(y) \left( \eta_\delta(|x-y|) \frac{x_k - y_k}{4\pi |x-y|^4} \right) - \eta_\delta(|x-y|) \partial_{x_k} \partial_{x_m} E \, dy
\]

\[
= 0,
\]

it follows that \( \nabla \cdot w = 0 \) everywhere.
To compute $\nabla \times w$, let us begin by computing the $i^{th}$ component of $\nabla \times w$ using again Equation (5) and the fact that $\epsilon_{ijk} \delta_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$:

$$\epsilon_{ijk} \partial_j w_k^i(x) = \int_{\Omega} F_i(y) \partial_x \left( \eta_{\delta}(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} \right) \, dy$$

$$+ \int_{\Omega} F_j(y) \partial_x \left( \eta_{\delta}(|x-y|) \partial_x E \right) \, dy$$

$$= \partial_x \int_{\Omega} F_i(y) \eta_{\delta}(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} \, dy$$

$$- \int_{\Omega} F_j(y) \partial_y \left( \eta_{\delta}(|x-y|) \partial_x E \right) \, dy$$

$$= \partial_x \int_{\Omega} F_i(y) \eta_{\delta}(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} \, dy$$

$$- \int_{\partial\Omega} F \cdot \nu(y) \eta_{\delta}(|x-y|) \partial_x E \, d\sigma(y) + \int_{\Omega} \nabla \cdot F(y) \eta_{\delta}(|x-y|) \partial_x E \, dy .$$

Taking the limit $\delta \to 0$ and using that $E$ is a fundamental solution of the Laplacian one then obtains

$$(\nabla \times w)_i(x) = \lim_{\delta \to 0} \epsilon_{ijk} \partial_j w_k^i(x)$$

$$= \partial_x \int_{\Omega} F_i(y) \frac{x_j - y_j}{4\pi|x-y|^3} \, dy - \int_{\partial\Omega} F \cdot \nu(y) \partial_x E \, d\sigma(y)$$

$$+ \int_{\Omega} \nabla \cdot F(y) \partial_x E \, dy$$

$$= F_i(x) - \partial_x \int_{\partial\Omega} F \cdot \nu E \, d\sigma(y) + \partial_x \int_{\Omega} \nabla \cdot F E \, dy .$$

The proposition then follows. \hfill \Box

We are now ready to state the basic $L^p$ estimate for $w$:

**Proposition 4.** For any nonnegative integer $n$ and $1 < p < \infty$, the field $w$ can be estimated as

$$\|w\|_{W^{n+1,p}(\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)} .$$

**Proof.** Notice that the principal value integral appearing in the first term of the formula (5) is the action of a Calderón–Zygmund operator in $\mathbb{R}^3$ on $F_1 1_{\Omega}$, where $1_{\Omega}$ denotes the indicator function of the domain. As the field $A^j$ is bounded, it is then standard that

$$\|w\|_{W^{1,p}(\Omega)} \leq C \|F\|_{L^p(\Omega)} .$$

This is the bound of the statement with $n = 0$.

We claim that, if $Z^1, \ldots, Z^n$ are smooth vector fields in $\overline{\Omega}$ which are tangent to $\partial\Omega$, then

$$\|Z^1 \cdots Z^n w\|_{W^{1,\frac{p}{n}}(\partial\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)} ,$$

where, as it is customary, we are regarding the vector field $Z^j$ as a first-order differential operator (namely, $Z^j = Z^j_i(x) \partial_{x_i}$) and the function $F$ can be taken smooth. It is easy to see that the proposition readily follows from this estimate.
Indeed, Proposition 3 ensures that \( w \) is divergence-free, so we can take the curl of (10) to find that

\[
\Delta w = -\nabla \times F
\]
in \( \Omega \). Since (12) means that

\[
\|w\|_{W^{n+1,p}(\Omega)} \leq C\|F\|_{W^{n,p}(\Omega)},
\]
standard elliptic estimates then yield

\[
\|w\|_{W^{n+1,p}(\Omega)} \leq C\left(\|\nabla \times F\|_{W^{n-1,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)} + \|w\|_{W^{n+1-\frac{1}{p}}(\partial \Omega)}\right)
\]

\[
\leq C\|F\|_{W^{n,p}(\Omega)},
\]
as claimed.

Hence it only remains to prove (12). In view of the trace inequality

\[
\|f\|_{W^{s,q}(\partial \Omega)} \leq C\|f\|_{W^{s+\frac{1}{q}}(\partial \Omega)},
\]
it suffices to show that for any \( j \) and \( n \) one can write

\[
\partial_j(Z^1 \cdots Z^n w_k) = I_n + J_n,
\]
where the terms \( I_n \) and \( J_n \) (which depend on \( j \), \( k \) and \( n \)) are respectively bounded as

\[
\|I_n\|_{W^{s,q}(\partial \Omega)} \leq C\|F\|_{W^{n-1,q}(\partial \Omega)}, \quad \|J_n\|_{L^p(\Omega)} \leq C\|F\|_{W^{n,p}(\Omega)}
\]
for all real \( s \) and all \( 1 < q < \infty \). In fact, it is slightly more convenient to prove an analogous estimate for the quantity

\[
Z^1 \cdots Z^n \partial_j w_k = I_1 + I_2;
\]
this clearly suffices for our purposes as the commutator term

\[
\partial_j(Z^1 \cdots Z^n w_k) - Z^1 \cdots Z^n \partial_j w_k
\]
only involves \( n \)th order derivatives of \( w \), of which at least \( n - 1 \) are taken along tangent directions on the boundary. The ideas of the proof are mostly standard, but we will provide a sketch of the proof as we have not found a suitable reference in the literature.

Let us start with the case \( n = 1 \). We can differentiate the formula (9) to obtain, for any tangent vector field \( Z \),

\[
Z \partial_j w_k^l(x) = \epsilon_{klm} Z_i(x) \int_{\Omega} F_i(y) \partial_{x_i} \partial_{x_j} \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy
\]

\[
= -\epsilon_{klm} Z_i(x) \int_{\Omega} F_i(y) \partial_{y_i} \partial_{x_j} \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy
\]

\[
=: I_1 + J_1,
\]
where

\[
I_1 := -\epsilon_{klm} \int_{\partial \Omega} Z(x) \cdot \nu(y) F_i(y) \partial_{x_i} \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) d\sigma(y),
\]

\[
J_1 := \epsilon_{klm} Z_i(x) \int_{\Omega} \partial_i F_i(y) \partial_{x_j} \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{4\pi|x - y|^3} \right) dy.
\]
Writing the volume integral as
\[ J_1 = \epsilon_{klm} Z_i(x) \int_{\Omega} [\partial_i F_l(y) - \partial_i F_l(x)] \partial_{x_j} \left( \eta \delta \left( \frac{x_m - y_m}{4\pi|x - y|^3} \right) \right) dy \]

- \[ Z_i(x) = Z \eta \delta \left( \frac{x_m - y_m}{4\pi|x - y|^3} \right) \]

- \[ \epsilon_{klm} A^l_m(x) \]

to obtain a principal-value-type formula, it is clear in view of the boundedness of \( A^l_m \) that the \( L^p \) norm of \( J_1 \) is controlled in terms of \( F \) with a \( \delta \)-independent constant:
\[ \|J_1\|_{L^p(\Omega)} \leq C\|F\|_{W^{1,p}(\Omega)}. \]

To analyze the boundary term, \( I_1 \), we will next restrict our attention to points \( x \) lying on \( \partial \Omega \). We can perform the analysis locally, parametrizing a portion of the boundary as a graph using the notation (8), which amounts to writing \( x = (X, h(X)) \), possibly after a rotation of the coordinate axes. A basis for the space of tangent vectors at the point \( x \) is
\[ T_1(X) := (1, 0, \partial_1 h(X)), \quad T_2(X) := (0, 1, \partial_2 h(X)). \]

Abusing the notation to denote by \( f(X) \) the value of a function \( f(x) \) at the point \( x = (X, h(X)) \in \partial \Omega \), one can therefore write
\[ Z(X) = a_1(X) T_1(X) + a_2(X) T_2(X) \]
in terms of two smooth functions \( a_j(X) \). This implies that
\[ Z(X) \cdot \nu(Y) = \frac{a_1(X) [\partial_1 h(X) - \partial_1 h(Y)] + a_2(X) [\partial_2 h(X) - \partial_2 h(Y)]}{\sqrt{1 + |\nabla h(Y)|^2}}. \]

Using the cancellation that stems from this formula it is not hard to see that, as the singular part of \( I_1 \) is
\[ \int_{D_\rho} g(Y) F_l(Y) \left[ Z(X) \cdot \nu(Y) \eta \delta(|X - Y|) \frac{|X - Y|^5}{|X - Y|^5} \right] dY \]
with \( g(Y) \) a smooth function, \( I_1 \) defines a singular integral operator on the boundary, so we have
\[ \|I_1\|_{L^q(\partial \Omega)} \leq C\|F\|_{L^q(\partial \Omega)} \]
for all \( 1 < q < \infty \). Furthermore, the coefficients are smooth and the derivatives have the right singularities, so a straightforward computation shows that \( I_1 \) behaves like a zeroth order pseudodifferential operator on the boundary, leading to the estimate
\[ \|I_1\|_{W^{s,q}(\partial \Omega)} \leq C\|F\|_{W^{s,q}(\partial \Omega)}. \]

The case \( n = 1 \) then follows.

In the case of \( n \geq 2 \), the proof goes along the same lines but there is another kind of term that one needs to consider. To see this, we next consider the case \( n = 2 \), which illustrates all the difficulties that appear in the general case. For this
we take another vector field \( Z' \) that is tangent to the boundary and differentiating the formula for \( Z \partial_j w_k \) to get

\[
Z' Z \partial_j w_k^m(x) = \epsilon_{klm} Z'_n(x) \partial_{x_n} \int_{\Omega} Z(x) \partial_i F_i(y) \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) dy
\]

\[
- \epsilon_{klm} Z'_n(x) \partial_{x_n} \int_{\partial \Omega} Z(x) \cdot \nu(y) F_i(y) \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) d\sigma(y).
\]

The volume integral can be dealt with just as in the case of \( n = 1 \). Indeed, the action of \( \partial_{x_n} \) on \( Z_i(x) \) is harmless, while when \( \partial_{x_n} \) acts on the singular term, one just replaces the derivative \( \partial_{x_n} \) by \( -\partial_{y_n} \) and integrates by parts. This yields another volume integral that can be related to a principal value as above, leading to \( W^{2,p}(\Omega) \rightarrow L^p(\Omega) \) bounds, and a boundary integral with the same structure as above (with \( DF \) playing the role of \( F \)), which leads to \( W^{s+1,p}(\partial \Omega) \rightarrow W^{s,q}(\partial \Omega) \) bounds.

The estimates for the surface integral are also as above when the derivative \( \partial_{x_n} \) acts on the field \( Z(x) \). When it acts on the singular term, however, one needs to refine the argument a little bit. Again we start by replacing the \( \partial_{x_n} \) by \( \partial_{y_n} \), so we have to control the integral

\[
I := \int_{\partial \Omega} Z(x) \cdot \nu(y) F_i(y) \partial_{y_n} \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) d\sigma(y).
\]

The point is that, as one can decompose the \( n \)th unit vector as

\[
e_n = T(y) + b(y) \nu(y),
\]

where \( T \) is a tangent vector and \( \nu \) is the unit normal, one can also write

\[
\partial_{y_n} = T(y) + b(y) \nu(y) \cdot \nabla_y,
\]

where the vector field \( T \) is here interpreted as a differential operator as before.

We can now integrate the tangent field by parts to arrive at

\[
I = \int_{\partial \Omega} |Z(x) \cdot \nu(y)| TF_i(y) \partial_{y_n} \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) d\sigma(y)
\]

\[
+ \int_{\partial \Omega} |Z(x) \cdot \nu(y)| TF_i(y) \partial_{y_n} \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) d\sigma(y)
\]

\[
+ \int_{\partial \Omega} |Z(x) \cdot \nu(y)| TF_i(y) b(y) \nu(y) \cdot \nabla_y \partial_{x_j} \left( \eta_b(|x-y|) \frac{x_m - y_m}{4\pi |x-y|^3} \right) d\sigma(y).
\]

The first term admits bounds \( W^{s+1,q}(\partial \Omega) \rightarrow W^{s,q}(\partial \Omega) \) just as before, the second term (which we get from tangential derivatives that act on the unit normal and from the divergence of \( T \) as a vector field on \( \partial \Omega \)) is clearly bounded \( W^{s,q}(\partial \Omega) \rightarrow W^{s,q}(\partial \Omega) \), and we just have to control the last integral. For this we need another cancellation, which hinges on the well-known fact that

\[
\nu(y) \cdot (x-y)
\]
is of order $|x - y|^2$ when $x, y \in \partial \Omega$. This appears here because the singular term in
the last integral is

$$
[Z(x) \cdot \nu(y)][\nu(y) \cdot \nabla] \partial_x \left( \eta_3(|x - y|) \frac{x_m - y_m}{|x - y|^3} \right) = 
\eta_3(|x - y|) [Z(x) \cdot \nu(y)] [\nu(y) \cdot (x - y)] - \frac{9|x - y|^2 \delta_{jm} + 15(x_j - y_j)(x_m - y_m)}{|x - y|^7} 
\eta_3 \delta \left( \frac{1}{|x - y|} \right)
$$

This readily yields the estimate $W^{s+1,q}(\partial \Omega) \to W^{s,q}(\partial \Omega)$. The general case is
handled by repeatedly applying these ideas. \qed

3. The Extension Operator $E_T$

In this section we will construct an extension operator that will be of use in the
construction of the kernel $K_{\Omega}(x, y)$. For this, let us denote by $\rho : \mathbb{R}^3 \to \mathbb{R}$ the signed
distance to the set $\partial \Omega$, which is smooth in the set $\tilde{U} := \rho^{-1}((0, \rho_0))$ provided that
$\rho_0$ is small enough. Notice that $\tilde{U}$ is a tubular neighborhood of the boundary $\partial \Omega$,
which one can then identify $\tilde{U}$ with $\partial \Omega \times (-\rho_0, \rho_0)$ via a diffeomorphism

$$
x \in \tilde{U} \mapsto (x', \rho) \in \partial \Omega \times (-\rho_0, \rho_0).
$$

We will often write $\rho_x \equiv \rho(x)$.

Taking local normal coordinates $X \equiv (X_1, X_2)$ on $\partial \Omega$, the Euclidean metric
reads as

$$
ds^2|_{\tilde{U}} = G_\rho + d\rho^2,
$$

where

$$
G_\rho := h_{ij}(X, \rho) \, dX_i \, dX_j
$$

defines a $\rho$-dependent metric on $\partial \Omega$ that coincides with the induced surface metric
on $\partial \Omega$ at $\rho = 0$. Hence the volume reads as

$$
dx = d\sigma_\rho(x') \, d\rho,
$$

where $d\sigma_\rho$ is a $\rho$-dependent metric on $\partial \Omega$ that can be written in local coordinates as

$$
d\sigma_\rho = \sqrt{\det(h_{ij}(X, \rho))} \, dX_1 \, dX_2.
$$

Obviously the connection with the surface measure on $\partial \Omega$ is

$$
d\sigma_\rho = (1 + O(\rho)) \, d\sigma.
$$

Consider the portion of the tubular neighborhood $\tilde{U}$ contained in $\Omega$,

$$
\tilde{U} := \tilde{U} \cap \Omega = \rho^{-1}((0, \rho_0)).
$$

Let us denote by $\omega^\perp$ the $\rho$-component of a vector field $\omega$, so that one can decom-
pose $\omega$ in $\tilde{U}$ as

$$
\omega = \omega^\parallel + \omega^\perp \partial_\rho,
$$

where $\omega^\parallel$ is orthogonal to $\partial_\rho$. To put it differently,

$$
\omega^\perp := \omega \cdot \nabla \rho, \quad \omega^\parallel := \omega - \omega^\perp \nabla \rho.
$$
Proposition 5. Let $T$ be an operator of the form \eqref{eq:operator}. For any divergence-free vector field $\omega \in C^1(\Omega)$ one has

$$T(\omega \cdot \nu) = (E_T \omega) \cdot \nu,$$

where $E_T \omega$ is defined by \eqref{eq:extension-operator}, and for each $x \in \Omega$ the divergence of $E_T \omega$ can be written as

$$\nabla \cdot E_T \omega(x) = \int_{\partial \Omega} K_{T, \text{div}}(x, y) \omega(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

with a kernel of the form

\begin{align*}
K_{T, \text{div}}(x, y) \omega(y, \rho_x) &= \chi(\rho_x) \left[ \frac{3(x' - y) \cdot \nu(x') (y - x) \cdot \nu(x)}{4\pi|x - y|^5} \omega^+(y, \rho_x) \\
&+ \frac{3(x' - y) \cdot \nu(x') (x - y) - |x - y|^2 \nu(x')}{|x - y|^5} \omega^\perp(y, \rho_x) \right] + \tilde{K}_T(x, y) \omega(y, \rho_x).
\end{align*}

where $|\tilde{K}_T(x, y)| \leq C/|x - y|$ and is supported in $\overline{U} \times \partial \Omega$.

Proof. Since $x \notin \partial \Omega$, one can easily compute the divergence of $\bar{\omega} := E_T \omega$ as

\begin{align*}
\nabla \cdot \bar{\omega}(x) &= \partial_{\rho_x} \left[ \chi(\rho_x) \int_{\partial \Omega} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} + K_T(x', y) \right) \omega^+(y, \rho_x) \, d\sigma_{\rho_x}(y) \right] \\
&= \chi(\rho_x) \int_{\partial \Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \omega^+(y, \rho_x) \right) \, d\sigma_{\rho_x}(y) + J_2,
\end{align*}
where $J_2$ denotes a term of the form

$$J_2 = \int_{\partial \Omega} \tilde{K}_T(x, y) \omega(y, \rho_x) \, d\sigma(y)$$

with a kernel $\tilde{K}_T$ as in the statement. Let us denote by $J_1$ the integral that appears in the second line of (19). To simplify the expression of $J_1$, we shall use that, by (13), one can write the divergence of $\omega$ as

$$0 = \nabla \cdot \omega(y, \rho_x) = \partial_{\rho_x} \omega^\perp(y, \rho_x) + \nabla^\parallel \cdot \omega^\parallel(y, \rho_x),$$

where $\nabla^\parallel \cdot \omega^\parallel(y, \rho_x)$ is the divergence of the field $\omega^\parallel$ (understood as a tangent vector field on $\partial \Omega$) with respect to the divergence operator on $\partial \Omega$ associated with the measure $d\sigma_{\rho_x}$. This allows us to write

$$J_1 := \int_{\partial \Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{{4\pi|x - y|^3}} \right) \omega^\perp(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

$$= \int_{\partial \Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{{4\pi|x - y|^3}} \right) \omega^\perp(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

$$+ \int_{\partial \Omega} \frac{(x' - y) \cdot \nu(x')}{{4\pi|x - y|^3}} \partial_{\rho_x} \omega^\perp(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

$$= \int_{\partial \Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{{4\pi|x - y|^3}} \right) \omega^\perp(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

$$- \int_{\partial \Omega} \frac{(x' - y) \cdot \nu(x')}{{4\pi|x - y|^3}} \nabla^\parallel \cdot \omega^\parallel(y, \rho_x) \, d\sigma_{\rho_x}(y)$$

$$= \int_{\partial \Omega} \left( \omega^\perp(y, \rho_x) \partial_{\rho_x} + \omega^\parallel(y, \rho_x) \cdot \nabla \right) \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \, d\sigma_{\rho_x}(y)$$

$$= \int_{\partial \Omega} \left[ 3(x' - y) \cdot \nu(x') (y - x) \cdot \nu(x') \frac{\omega^\perp(y, \rho_x)}{4\pi|x - y|^5} \right.$$

$$+ \left. \frac{3(x' - y) \cdot \nu(x') (y - x) - |x - y|^2 \nu(x')}{4\pi|x - y|^5} \cdot \omega^\parallel(y, \rho_x) \right] \, d\sigma_{\rho_x}(y),$$

where we have integrated by parts the tangential divergence $\nabla^\parallel \cdot \omega^\parallel$. The proposition then follows. \qed

4. Construction of the solution

Given the divergence-free field $\omega$ in $W^{k,p}(\Omega)$, our objective in this section is to construct a vector field $u$ that satisfies the equations

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0$$

in $\Omega$ and is tangent to the boundary. To this end, let us start by considering the vector field $v$ on $\Omega$ defined as

$$v(x) := v_1(x) + v_2(x),$$

$$v_1(x) := \int_{\Omega} \frac{\omega(y) \times (x - y)}{4\pi|x - y|^3} \, dy,$$

$$v_2(x) := -\int_{\Omega} \frac{\nabla S f(y) \times (x - y)}{4\pi|x - y|^3} \, dy,$$
where for \( x \in \Omega \) we define the scalar function \( Sf \) through the single layer potential
\[
Sf(x) := -\int_{\partial\Omega} \frac{f(y)}{4\pi|x-y|} \, d\sigma(y),
\]
with \( f \) a function on \( \partial\Omega \) to be determined.

Since \( \Delta(Sf) = 0 \) in \( \Omega \) for any function \( f \), it follows from Proposition 3 that \( \nabla \times v \) is divergence-free and that its curl is given by
\[
\nabla \times v(x) = \omega(x) - \nabla Sf(x) + \nabla \int_{\partial\Omega} \frac{\partial_{\nu} Sf(y) - \omega \cdot \nu(y)}{4\pi|x-y|} \, d\sigma(y)
\]
\[= \omega(x) + \nabla \int_{\partial\Omega} f(y) + \frac{\partial_{\nu} Sf(y) - \omega \cdot \nu(y)}{4\pi|x-y|} \, d\sigma(y). \tag{22}\]

As a side remark, observe that since we have only defined \( Sf \) on \( \Omega \), the fact that the derivative of the extension of \( Sf \) to the whole space \( \mathbb{R}^3 \) is discontinuous across \( \partial\Omega \) does not play a role here. Consequently, \( \partial_{\nu} Sf(x) \) will necessarily stand for the interior derivative of \( Sf \) at the boundary point \( x \) in the direction of the outer normal, that is,
\[
\partial_{\nu} Sf(x) := \lim_{y \to x} \nu(x) \cdot \nabla Sf(y),
\]
where for all \( y \in \Omega \) one has
\[
\nabla Sf(y) = \int_{\partial\Omega} f(z) - \frac{y-z}{4\pi|y-z|^3} \, d\sigma(z).
\]

Consider the operator \( T_0 \) defined in (14), which appears in the analysis of the normal derivative of \( Sf \) at the boundary through the formula
\[
\partial_{\nu} Sf = (T_0 - \frac{1}{2}I) f.
\]
In view of Equation (22), our goal now is to choose the function \( f \) so that
\[
f + \partial_{\nu} Sf - \omega \cdot \nu = 0,
\]
or equivalently
\[
(\frac{1}{2}I + T_0) f = \omega \cdot \nu. \tag{23}\]

Since \( \frac{1}{2}I + T_0 \) is precisely the operator that one needs to invert in order to solve the exterior Neumann boundary value problem for the Laplacian, it is well known (see e.g. [10, Section 3.E]) that there is a function \( f \) satisfying (23) if and only if
\[
\int_{\partial\Omega_j} \omega \cdot \nu \, d\sigma = 0, \quad 1 \leq j \leq m-1. \tag{24}\]

Here \( \Omega_1, \ldots, \Omega_{m-1} \) are the bounded connected components of \( \mathbb{R}^3 \setminus \overline{\Omega} \). Since each \( \partial\Omega_j \) is a connected component of \( \partial\Omega \), the hypothesis (2) ensures that the condition (24) holds, so one can find a function \( f \) on \( \partial\Omega \) satisfying (23). Notice, moreover, that (23) implies that \( f \) can be written as
\[
f = (2 - 4T)(\omega \cdot \nu), \tag{25}\]

with \( T \) an operator of the form (15).

Since \( T \) is a pseudodifferential operator on \( \partial\Omega \) of order \(-1\), Equation (25) yields the estimate
\[
\|f\|_{W^{k,\frac{1}{p}}(\partial\Omega)} \leq C\|\omega \cdot \nu\|_{W^{k,\frac{1}{p}}(\partial\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)},
\]
and therefore, by the properties of the single layer potential,
\[ \|Sf\|_{W^{k+1,p}(\Omega)} \leq C\|f\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}. \]

With this function \( f \), by construction one then has that
\[ \nabla \times v = \omega, \quad \nabla \cdot v = 0, \]
while Proposition 4 ensures that \( v \) is bounded as
\[ \|v\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)} + C\|\nabla Sf\|_{W^{k,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}. \]
Hence one can now find a solution to the problem (1) by setting
\[ u := v - \nabla \varphi, \]
where \( \varphi \) is a solution to the Neumann boundary value problem
\[ \Delta \varphi = 0 \quad \text{in } \Omega, \quad \partial_n \varphi = v \cdot \nu. \]

It is well-known that the solution exists and is unique up to an additive constant because
\[ \int_{\partial \Omega} v \cdot \nu \, d\sigma = \int_{\Omega} \nabla \cdot v \, dx = 0. \]
Moreover, the function \( \varphi \) can be written as a single layer potential
\[ \varphi = Sg, \]
where the function \( g \) and \( v \cdot \nu \) are related through the operator \( T_0 \) as
\[ \left( \frac{1}{2}I - T_0 \right) g = -v \cdot \nu. \]
Therefore,
\[ g = -\left( 2 + 4\bar{T} \right)(v \cdot \nu) \]
with \( \bar{T} \) a pseudodifferential operator on \( \partial \Omega \) of the form (15), so
\[ \|g\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|v \cdot \nu\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|v\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}. \]
By the properties of single layer potentials it then follows that
\[ \|\nabla \varphi\|_{W^{k+1,p}(\Omega)} \leq \|Sg\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}. \]
Hence in this section we have proved the following:

**Theorem 6.** The field \( u \) given by (26) solves the problem (1) and satisfies the estimate (4).

### 5. Existence and bounds for the integral kernel

In this section we shall show that the solution \( u \) is actually obtained by integrating the field \( \omega \) against an integral kernel. Specifically, in terms of the function \( \ell \) defined in (3), we aim to prove the following:

**Theorem 7.** The solution \( u \) constructed in Theorem 6 (see Equation (26)) is of the form
\[ u(x) = \int_{\Omega} K_\Omega(x,y) \omega(y) \, dy \]
for some matrix-valued kernel satisfying \( |K_\Omega(x,y)| \leq C\ell(y)/|x - y|^2 \).
Without loss of generality, we can assume that \( \omega \in C^1(\Omega) \). We will construct the kernel by treating separately the three summands appearing in the decomposition

\[
u = v_1 + v_2 - \nabla \varphi
\]
presented in Equations (20), (21) and (26). Since

\[v_1(x) = \int_\Omega K_{v_1}^1(x, y) \omega(y) \, dy\]

with

\[K_{v_1}^1(x, y) := \frac{\omega \times (x - y)}{4\pi |x - y|^3},\]

we will only need to show that (28) and using the fact that the divergence of \( \omega \) is zero, one readily obtains that

\[v_2(x) := \frac{1}{8\pi^2} \int_\Omega \int_{\partial \Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} \left[ \omega \cdot \nu(z) - 2T(\omega \cdot \nu)(z) \right] d\sigma(z) \, dy\]

\[=: V_1(x) - 2V_2(x),\]

with \( T \) a pseudodifferential operator on \( \partial \Omega \) of order \(-1\) and of the form (15).

In this subsection we will work out the details for \( V_1(x) \). Integrating by parts and using the fact that the divergence of \( \omega \) is zero, one readily obtains that

\[V_1(x) := \int_\Omega \int_{\partial \Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} \omega \cdot \nu(z) \, d\sigma(z) \, dy = I_1 + I_2,\]

where

\[I_1 := \int_\Omega \int_{\partial \Omega \setminus B_{\delta}(y)} \frac{x}{z - y} \times \frac{z - y}{|z - y|^3} \omega_j(z) \, dz \, dy\]

\[= - \int_\Omega \int_{\partial \Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \left[ \frac{1}{|z - y|^2} I - 3 \frac{(z - y) \otimes (z - y)}{|z - y|^3} \omega(z) \right] \, dz \, dy,\]

\[I_2 := \int_\Omega \int_{\partial \Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \frac{z - y}{|z - y|^3} \omega \cdot \nu(z) \, d\sigma(z) \, dy,\]

\( \nu \) denotes the outward normal of \( \partial B_{\delta}(y) \) and the dot in the second lines denotes the multiplication of the vector field \( \omega(z) \) by a symmetric matrix.

Let us begin by showing that \( I_2 \) tends to zero as \( \delta \to 0 \). For this, observe that for \( \omega \in C^1(\Omega) \) one has

\[
\int_{\Omega \cap \partial B_{\delta}(y)} \omega(z) \cdot \nu(z) \frac{z - y}{|z - y|^3} \, d\sigma(z) = \omega(y) \cdot \int_{\Omega \cap \partial B_{\delta}(y)} \frac{\nu(z) \otimes (z - y)}{|z - y|^3} \, d\sigma(z) + O(\delta).
\]

To analyze the remaining integral, we shall begin by noticing that, in terms of the variable \( \Theta := \frac{1}{3}(z - y) \), the set \( \Omega \cap \partial B_{\delta}(y) \) can be described as

\[\Omega \cap \partial B_{\delta}(y) = \{ \Theta \in S_{y,\delta} \},\]

where \( S_{y,\delta} \) is a subset of the unit sphere \( S^2 \subset \mathbb{R}^3 \) with

\[S_{y,\delta} = S^2 \quad \text{if dist}(y, \partial \Omega) > \delta.\]
Denoting by $d\Theta$ the canonical area form on the unit sphere, it is clear that
\[
\int_{\Omega \cap \partial B_\delta(y)} \frac{v(z) \otimes (z-y)}{|z-y|^3} d\sigma(z) = \int_{S^2_\gamma} (\Theta \otimes \Theta) d\Theta.
\]
Since the norm of the latter integral is obviously bounded by a constant that does not depend on $y$ or $\delta$ and the integral is zero when $S_{y,\delta} = S^2$, it then follows that
\[
|I_2| \leq C \int_{\Omega_\delta} \frac{\omega(y)}{|x-y|^2} dy + C\delta \leq C\delta,
\]
with $\Omega_\delta := \{ y \in \Omega : \text{dist}(y, \partial \Omega) < \delta \}$, thereby proving that the boundary term vanishes in the limit $\delta \to 0$:
\[
\lim_{\delta \to 0} I_2 = 0.
\]
Our next step will be to show that
\[
|\tilde{I}_1| \leq C f(z) \frac{|z-x|^2}{|x-z|^2}
\]
with
\[
\tilde{I}_1 := \int_{\Omega \setminus B_a(z)} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3 (z-y) \otimes (z-y)}{|z-y|^5} dy,
\]
where the cross product of a vector field with a matrix has the obvious meaning. Let us take some small but fixed number $a > 0$. Clearly,
\[
\left| \int_{\Omega \setminus B_a(z)} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3 (z-y) \otimes (z-y)}{|z-y|^5} dy \right| \leq C,
\]
so it suffices to study the behavior of the integral
\[
I_3 := \int_{\Omega \setminus (B_a(z) \setminus B_\delta(z))} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3 (z-y) \otimes (z-y)}{|z-y|^5} dy.
\]
If the distance $\rho_z$ between the point $z$ and the boundary is greater than $a$, setting
\[
e := \frac{x-z}{R} \quad \text{and} \quad R := |x-z|
\]
one has
\[
I_3 = \int_{B_a(z) \setminus B_\delta(z)} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3 (z-y) \otimes (z-y)}{|z-y|^5} dy
\]
\[
= \frac{1}{R^2} \int_{B_{|e-w|/R} \setminus B_{|w|/R}} \frac{e-w}{|e-w|^3} \times \frac{|w|^2 I - 3 w \otimes w}{|w|^5} dw,
\]
where we have defined $w := (y-z)/R$. The integral is uniformly convergent as $R \to 0$ because the integrand is bound by $C |w|^{-5}$ for large $w$. For small $R_0$ and $\delta < R R_0$ one has the asymptotic behavior
\[
I_3 = \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} (e + O(w)) \times \frac{|w|^2 I - 3 w \otimes w}{|w|^5} dw
\]
\[
e = e \times \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} \frac{|w|^2 I - 3 w \otimes w}{|w|^5} dw + \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} O(|w|^{-2}) dw
\]
\[
= \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} O(|w|^{-2}) dw < \frac{C}{R^2}
\]
with a constant independent of \(\delta\). Here we have used that the first integral in the second line vanishes and the average of the matrix \(w \otimes w\) over any sphere is \(-\frac{1}{3} I\).

Together with (31), this shows that

\[ |I_3| \leq \frac{C}{|x-z|^2} \]

whenever the distance between \(z\) and \(\partial \Omega\) is at least \(a\).

If \(\rho_z := \text{dist}(z, \partial \Omega) < a\), a similar argument yields the same bound but with a constant that can grow as \(\ell(z)\). In this case, one begins by straightening out the boundary, so we locally identify the boundary with a vertical plane. This amounts to saying that there is a diffeomorphism \(\Phi_z\) of the ball \(B_a\) such that

\[ \Omega \cap (B_a(z) \setminus B_{\delta}(z)) = \{ z + \Phi_z(w) : w \in B_{\delta, \rho_z} \}, \]

with \(B_{\delta, r}\) being the intersection of the annulus of outer radius \(a\) and inner radius \(\delta\) with the half-space of points whose third coordinate is at most \(r\):

\[ B_{\delta, r} := \{ w \in \mathbb{R}^3 : \delta < |w| < a, -r < w_3 \} \]

Furthermore, upon choosing a suitable orientation of the axes one can take

\[ \|\Phi_z - I\|_{C^k(B_a)} < Ca \]

and one can assume without loss of generality that

\[ \Phi_z(0) = 0. \]

In this case one can write

\[
I_3 = \int_{B_{\delta, \rho_z}} \frac{x - z - \Phi_z(w)}{|x - z - \Phi_z(w)|^3} \frac{|\Phi_z(w)|^2 I - 3 \Phi_z(w) \otimes \Phi_z(w)}{|\Phi_z(w)|^5} \det D\Phi_z(w) \, dw
\]

\[
= \int_{B_{\delta, \rho_z}} \frac{x - z - w}{|x - z - w|^3} \frac{|w|^2 I - 3 w \otimes w}{|w|^5} \, dw + \int_{B_{\delta, \rho_z}} \frac{x - z - w}{|x - z - w|^3} \times O(1) \, dw
\]

=: \(I_{31} + I_{32}\).

One can introduce the variable \(q := w/R\) and define \(e\) and \(R\) as in Equation (33). The second integral \(I_{32}\) can be readily bounded by \(C/R^2\), while one can argue as before to obtain

\[
I_{31} = \frac{1}{R^2} \int_{B_{\delta/R, \rho_z/R}} \frac{e - q}{|e - q|^3} \times \frac{|q|^2 I - 3 q \otimes q}{|q|^5} \, dq
\]

\[
= e \times \frac{1}{R^2} \int_{B_{\delta/R, \rho_z/R}} \frac{|q|^2 I - 3 q \otimes q}{|q|^5} \, dq.
\]

The point now is that

\[ I_4 := \int_{B_{\delta/R, \rho_z/R}} \frac{|q|^2 I - 3 q \otimes q}{|q|^5} \, dq \]

is bounded by \(C \log(2 + 1/\rho_z)\), with \(C\) a constant that does not depend on \(\delta\), which proves the \(\delta\)-independent bound

\[ |I_{31}| < C \ell(z) \frac{1}{|x - y|^2}. \]

In order to see this, let us take spherical coordinates \((r, \theta, \phi)\):

\[ q := \frac{r}{R} \Theta(\theta, \phi), \quad \Theta(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]
and use again the shorthand notation
\[ d\Theta := \sin \theta \, d\theta \, d\phi \]
for the surface measure on the unit sphere. With \( \theta \in (0, \frac{\pi}{2}) \) defined by
\[ \cos \theta = \frac{\rho}{a} , \]
the integral (35) can be written as
\[ I_4 = \int_{0}^{2\pi} \int_{0}^{\theta_z} \int_{\delta}^{\theta_z} I - 3 \Theta \otimes \Theta \, dr \, \sin \theta \, d\theta \, d\phi \]
\[ + \int_{0}^{2\pi} \int_{\theta_z}^{\pi} \int_{\delta}^{a} I - 3 \Theta \otimes \Theta \, dr \, \sin \theta \, d\theta \, d\phi \]
Since
\[ \int_{\delta}^{\theta_z} dr = \log \frac{a}{\delta} + \log \frac{\rho}{a \cos \theta} , \quad \int_{\delta}^{a} dr = \log \frac{a}{\delta} , \]
one obtains
\[ |I_4| = \left| \log \frac{a}{\delta} \int_{\delta}^{\theta_z} (I - 3 \Theta \otimes \Theta) \, d\theta + \int_{0}^{2\pi} \int_{0}^{\theta_z} (I - 3 \Theta \otimes \Theta) \, \log \frac{\rho}{a \cos \theta} \sin \theta \, d\theta \, d\phi \right| \]
\[ = \int_{0}^{2\pi} \int_{0}^{\theta_z} (I - 3 \Theta \otimes \Theta) \, \log \frac{\rho}{a \cos \theta} \sin \theta \, d\theta \, d\phi \]
\[ \leq C(1 + |\log \rho|) \leq C \ell(z) , \]
where \( C \) is independent of \( \delta \) and to pass to the second line we have used again that the first integral after the equality sign vanishes. This completes the proof of (36).

The bounds for \( I_{31} \) and \( I_{32} \) then imply that the kernel defined by setting
\[ \int_{\Omega} K_1(x, z) \omega(x) \, dz := \frac{1}{8\pi^2} \lim_{\delta \to 0} I_1 , \]
which is formally given by
\[ K_1(x, z) = \frac{1}{8\pi^2} \text{PV} \int_{\Omega} \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 I - 3 (z - y) \otimes (z - y)}{|z - y|^5} \, dy , \]
is well-defined kernel bounded as
\[ |K_1(x, z)| \leq \frac{C \ell(z)}{|x - z|^2} . \]
Moreover, as \( I_2 \to 0 \) as \( \delta \to 0 \) by (30), it then follows from (29) that \( K_1(x, z) \) satisfies
\[ \lim_{\delta \to 0} V_1(x) = \int_{\Omega} K_1(x, z) \omega(z) \, dz . \]

**Step 2:** The second integral associated with \( v_2 \). To complete our analysis of \( v_2 \), it remains to consider the term in Equation (28) having the quantity \( T(\omega \cdot \nu) \), which we have called \( V_2(x) \). Our goal is to show that
\[ \lim_{\delta \to 0} V_2(x) = \int_{\Omega} K_2(x, y) \omega(y) \, dy \]
for some kernel that we will bound as \( |K_2(x, y)| \leq C \ell(y)/|x - y| . \)
Let us begin by using Proposition 5 to write
\[ T(\omega \cdot \nu) = \tilde{\omega} \cdot \nu, \]
where
\[ \tilde{\omega} := \mathcal{E}_T \omega \]
is the extension of \( T(\omega \cdot \nu) \) associated to the operator \( T \) defined by the integral (17).

To analyze the term \( V_2(x) \) in Equation (28), let us write
\[ V_2(x) = \int_{|y - z| > \delta} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} (\tilde{\omega} \cdot \nu)(z) \, d\sigma(z) \, dy, \]
where of course the integration variables \((y, z)\) range over \( \Omega \times \partial \Omega \). Integrating by parts one then finds
\[ V_2 = I_5 + I_6 - I_7, \]
where
\begin{align*}
I_5 &:= \int_{|y - z| > \delta} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} |z - y|^2 I - 3 \frac{(z - y) \otimes (z - y)}{|z - y|^5} \cdot \tilde{\omega}(z) \, dy \, dz, \\
I_6 &:= \int_{|y - z| > \delta} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} \nabla \cdot \tilde{\omega}(z) \, dy \, dz, \\
I_7 &:= \int_{|y - z| = \delta} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} (\tilde{\omega} \cdot \nu)(z) \, d\sigma(z) \, dy.
\end{align*}

Let us discuss the structure of these integrals. Arguing as in the case of (30), it is easy to show that
\[ \lim_{\delta \to 0} I_7 = 0. \]

To study \( I_5 \) notice that, identifying \( \mathcal{U} \) with \( \partial \Omega \times (0, \rho_0) \) via the coordinates
\[ z \mapsto (z', \rho_z) \]
as in Section 3, one can write
\[ \tilde{\omega}(z) = \nabla \rho(z) \int_{\partial \Omega} K_T(z, w) \omega^\perp(w, \rho_z) \, d\sigma(w) \]
with a scalar kernel of the form (16). It stems from this formula that
\[ \lim_{\delta \to 0} I_5 = \int_{\Omega} \mathcal{K}(x, \bar{w}) \omega(\bar{w}) \, d\bar{w}, \]
if we define \( \mathcal{K}(x, \bar{w}) \) as the limit as \( \delta \to 0 \) of
\[ \mathcal{K}_\delta := \int_\delta K_T(z, w) \left( \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 \nabla \rho(z) - 3 (z - y) \cdot \nabla \rho(z) (z - y)}{|z - y|^5} \right) \otimes \nabla \rho(\bar{w}) \, dy \, d\sigma(z'). \]

Provided that the latter exists. As before, we are denoting by \( \bar{w} \) the point in \( \mathcal{U} \) of coordinates \( (w, \rho_z) \), and the subscript \( \delta \) in the integral will henceforth mean that we integrate over points \((y, z')\) such that
\[ |y - z| > \delta \quad \text{and} \quad |w - z'| > \delta. \]

Of course, \((y, z') \in \Omega \times \partial \Omega\), but in what follows we shall not explicitly write the domain of integration (which will be apparent) to keep the notation simple.
To analyze the behavior of this integral, in addition to exploiting the diffeomorphism \( \mathcal{U} \to \partial \Omega \times (0, \rho_0) \) we will take local normal coordinates on \( \partial \Omega \), thereby identifying a point \( x \) in \( \mathcal{U} \) (with \( x' \) certain open subset of \( \partial \Omega \)) with the triple \((X, \rho_x)\), with \( X = (X_1, X_2) \). Since we are using normal coordinates on \( \partial \Omega \), it is standard that, given two points \( x, y \) of coordinates \((X, \rho_x), (Y, \rho_y)\), one has

\[
|x - y|^2 = (\rho_x - \rho_y)^2 + |X - Y|^2 + \text{h.o.t.},
\]

where in what follows we write h.o.t. to denote higher order terms. Hence by Equation (16) the kernel \( K(z, w) \) can be written in these coordinates as

\[
K_T(z, w) = \frac{q(\zeta)}{4\pi (\rho_z^2 + |\zeta|)^{3/2}} + \text{h.o.t.},
\]

where \( \zeta := Z - W \) and \( q(\zeta) := q_{ij}\zeta_i\zeta_j \) is a quadratic form (here we have used that \((z' - w) \cdot \nu(z') \) vanishes to second order). Let us set \( \tilde{y} := y - z \) and write

\[
|x - y| = |x - z - \tilde{y}| = |x - \tilde{w} - \zeta - \tilde{y}| + \text{h.o.t.},
\]

where we have identified \( \zeta \) with the point of coordinates \((0, \zeta)\) and made use of (41). This permits us to write

\[
\mathcal{K}_\delta = \int_{\delta R} \frac{q(\zeta)}{(\rho_z^2 + |\zeta|)^{3/2}} \frac{e - \zeta - \tilde{y} (\tilde{y}^2 e_3 - 3\tilde{y}\tilde{y}_3)}{4\pi |\tilde{y}|^5} \otimes e_3 d\tilde{y} d\zeta + \text{h.o.t.},
\]

where the subscript \( \delta \) again means that

\[
|\tilde{y}| > \delta, \quad |x - \tilde{w} - \zeta - \tilde{y}| > \delta.
\]

Let us now write

\[
x - \tilde{w} =: R e, \quad R := |x - \tilde{w}|
\]

and observe that, for small \( R \), one can rescale the variables as

\[
\tilde{y} := \frac{y}{R}, \quad \zeta' := \frac{\zeta}{R}, \quad \rho_z' := \frac{\rho_z}{R}
\]

and write the above integral as

\[
\mathcal{K}_\delta = \frac{1}{R} \int_{\delta R} \frac{q(\zeta')}{(\rho_z'^2 + |\zeta'|)^{3/2}} \frac{e - \zeta' - \tilde{y}' (\tilde{y}'^2 e_3 - 3\tilde{y}'\tilde{y}_3')}{4\pi |\tilde{y}'|^5} \otimes e_3 d\tilde{y}' d\zeta' + \text{h.o.t.},
\]

where of course the rescaled domains of integration become unbounded in the limit \( R \to 0 \).

Our goal now is to show that

\[
I_R := \int_{\delta R} \frac{q(\zeta')}{(\rho_z'^2 + |\zeta'|)^{3/2}} \frac{e - \zeta' - \tilde{y}' (\tilde{y}'^2 e_3 - 3\tilde{y}'\tilde{y}_3')}{|\tilde{y}'|^5} d\tilde{y}' d\zeta'
\]

is bounded as

\[
|I_R| \leq C \ell(z)
\]

uniformly as \( \delta \to 0 \). Observe that this will show that

\[
|\mathcal{K}_\delta| \leq \frac{C \ell(z)}{R},
\]

which will in turn ensure the existence of the kernel \( \mathcal{K}(x, \tilde{w}) \) with a bound

\[
|\mathcal{K}(x, \tilde{w})| \leq \frac{C \ell(z)}{|x - \tilde{w}|}.
\]
To prove the bound for $I_8$ it suffices to analyze the behavior of the integrand at the points where it can be not uniformly in $L^1_{\text{loc}}$ (that is, in a neighborhood of the regions $e - \zeta' - \tilde{y}' = 0$ and $\tilde{y} = 0$). Let us start with this first case. Around $e - \zeta' - \tilde{y}' = 0$ one can write
\[
e - \tilde{y}' = (B, b),
\]
with $B = (B_1, B_2) \in \mathbb{R}^2$. It is clear from that it is enough to consider values of $\tilde{y}$ that are close to zero, say with $|\tilde{y}| < \frac{1}{4}$. It then follows that
\[
|B|^2 + b^2 \geq \frac{1}{2},
\]
so that the integral in $\zeta'$ can be written in terms of the variable
\[
E := \zeta' - B
\]
as
\[
I_0 := \int_{\delta/R} \frac{q(\zeta')}{(\rho^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}' |e - \zeta' - \tilde{y}'|^3}{d\zeta'}
\]
\[
= \int_{D_{\delta,R} \setminus D_{\delta/R}} \frac{q(\zeta')}{(\rho^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}' |e - \zeta' - \tilde{y}'|^3}{d\zeta'} + O(1)
\]
\[
= \int_{D_{\delta,R} \setminus D_{\delta/R}} \frac{q(E + B)}{(\rho^2 + |E + B|^2)^{3/2}} \frac{(E, b)}{(b^2 + |E|^2)^{3/2}} dE + O(1).
\]
The possible problem can arise as $b \to 0$. Since in this case $B$ is bounded away from zero by (45), however, one can write $I_0$ as
\[
I_0 = \frac{q(B)}{(\rho^2 + |B|^2)^{3/2}} \int_{D_{\delta,R} \setminus D_{\delta/R}} \frac{(E, b)}{(b^2 + |E|^2)^{3/2}} dE + O(1)
\]
\[
= \int_{D_{\delta,R} \setminus D_{\delta/R}} \frac{E dE}{(b^2 + |E|^2)^{3/2}} \int_{D_{\delta,R} \setminus D_{\delta/R}} \frac{b dE}{(b^2 + |E|^2)^{3/2}} + O(1).
\]
The first integral vanishes by parity, so one can estimate $I_0$ as
\[
|I_0| \leq C \int_{\mathbb{R}^2} \frac{b dE}{(b^2 + |E|^2)^{3/2}} + O(1) = C \int_{\mathbb{R}^2} \frac{dE}{(1 + |E|^2)^{3/2}} + O(1) \leq C.
\]
It remains now to consider the behavior of the integral with respect to $\tilde{y}$ around $\tilde{y} = 0$. This can be handled exactly as in the case of $I_1$, which yields to a bound of the form $C \log(2 + \rho^{-1})$. Combining both results one immediately obtains that
\[
|I_0| \leq C \ell(z),
\]
thereby establishing (41).

**Step 3: The remaining integrals.** The rest of the proof of Theorem 1 follows by repeatedly applying the ideas that we have used above. Let us sketch the remaining steps.

To complete our treatment of $v_2$, one has to show that, just as in (39),
\[
\lim_{\delta \to 0} I_6 = \int_{\Omega} K'(x, \bar{w}) \omega(\bar{w}) d\bar{w}
\]
for some suitable integral kernel, where \( I_6 \) is given by (38). In view of the formula for the divergence of \( \tilde{\omega} \) given in Proposition 5, this kernel arises as the limit as \( \delta \to 0 \) of the integral

\[
K_\delta' := \int_\delta \left( \frac{x - y}{|x - y|^3} \times \frac{z - y}{|z - y|^3} \right) \otimes K_{T, \text{div}}(z, w) \, dy \, d\sigma(z'),
\]

where as before the subscript \( \delta \) means that one only integrates over \((y, z')\) with

\[
|y - z| > \delta, \quad |z' - w| > \delta.
\]

Note that the kernel \( K_{T, \text{div}} \), which was introduced in (18), diverges as the inverse square of the distance. Setting, in local coordinates \((Z, \rho_z)\) as above,

\[
\zeta := Z - W, \quad \tilde{y} := y - z, \quad x - \bar{w} =: R e
\]

with \( R := |x - \bar{w}| \), and rescaling the integral as before, one finds that

\[
K_\delta' = \frac{1}{R} I_{10} + \text{h.o.t.},
\]

where \( I_{10} \) is a certain integral with respect to rescaled variables \((\tilde{y}', \zeta')\) that is shown to be bounded by \( C \ell(z) \) by tediously repeating the steps taken before, with only minor modifications. This completes the proof of the existence of a kernel bounded as

\[
|K^2(x, z)| \leq \frac{C \ell(z)}{|x - z|^2}
\]

and such that

\[
\lim_{\delta \to 0} V_2(x) = \int_{\Omega} K^2(x, z) \omega(z) \, dz.
\]

The analysis of \( \nabla \varphi \) is similar. Since

\[
\nabla \varphi(x) = -\int_{\partial \Omega} \frac{x - y}{4\pi|x - y|^3} g(y) \, d\sigma(y)
\]

with \( g \) given in terms of \( v \cdot \nu \) by (27). In order to prove that one can write

\[
\nabla \varphi(x) = \int_{\Omega} K^3(x, z) \omega(z) \, dz
\]

with a kernel bounded as

\[
|K^3(x, z)| \leq \frac{C \ell(z)}{|x - z|^2}
\]

one starts off by writing

\[
\nabla \varphi(x) = -\int_{\partial \Omega} \frac{x - y}{2\pi|x - y|^3} v \cdot \nu(y) \, d\sigma(y) - \int_{\partial \Omega} \frac{x - y}{\pi|x - y|^3} T(v \cdot \nu)(y) \, d\sigma(y)
\]

\[
= -\frac{J_1 + 2J_2}{2\pi},
\]

with

\[
J_1 := \int_{\partial \Omega} \frac{x - y}{|x - y|^3} \nu(y) \, d\sigma(y), \quad J_2 := \int_{\partial \Omega} \frac{x - y}{|x - y|^3} T(v \cdot \nu)(y) \, d\sigma(y).
\]
where \( \tilde{T} \) is an operator of the form \( (15) \). Integrating by parts and arguing as before, one can readily infer that

\[
J_1 = - \lim_{\delta \to 0} \int_{|y-x|>\delta} v(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dy
\]

\[
= - \frac{1}{8\pi^2} \lim_{\delta \to 0} \left( \int_{|y-x|>\delta} V_1(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^3} dy + 2 \int_{|y-x|>\delta} V_2(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dy \right)
\]

\[
= - \frac{1}{8\pi^2} \lim_{\delta \to 0} (J_{11} + 2J_{12}).
\]

Given the expression of \( V_1 \), it turns out that one can integrate by parts to write

\[
\lim_{\delta \to 0} J_{11} = \lim_{\delta \to 0} \int_{\delta} \omega(z) \frac{|z-w|^2 I - 3(z-w) \otimes (z-w)}{|z-w|^3} \times \frac{y-w}{|y-w|^3} \cdot \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dw dy dz.
\]

Hence essentially the same reasoning that we used with \( I_3 \) allows us to show that

\[
J_{11} = \int K^{31}(x, z) \omega(z) \, dz
\]

with a kernel satisfying the bound

\[
|K^{31}(x, z)| \leq \frac{C\ell(z)}{|x-z|^3}
\]

that is given by the limit as \( \delta \to 0 \) of the integral

\[
\lim_{\delta \to 0} \int_{\delta} \frac{|z-w|^2 I - 3(z-w) \otimes (z-w)}{|z-w|^3} \times \frac{y-w}{|y-w|^3} \cdot \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dw dy dz.
\]

Likewise, arguing exactly as in the analysis of \( V_2 \) one can show that in the limit \( \delta \to 0 \) (which will not be written explicitly for the ease of notation) one has

\[
J_{12} = \int_{\delta} \frac{z-w}{|z-w|^3} \times \frac{y-w}{|y-w|^3} \cdot \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} \cdot \tilde{\omega}(z) \cdot \nu(z) \, d\sigma(z) \, dw dy dz
\]

\[
= \int_{\delta} \frac{z-w}{|z-w|^3} \times \frac{y-w}{|y-w|^3} \cdot \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} \cdot \tilde{\omega}(z) \cdot \nu(z) \, dz \, dw dy dz
\]

Writing \( \tilde{\omega} \) and \( \nabla \cdot \tilde{\omega} \) in terms of \( \omega \) using Lemma 5 as above one finds that

\[
\lim_{\delta \to 0} J_{12} = \int K^{32}(x, q) \omega(q) \, dq,
\]

where the kernel \( K^{32} \) is bounded as

\[
|K^{32}(x, q)| \leq \frac{C\ell(q)}{|x-q|}.
\]
The treatment of \( J^2 \) can be accomplished using a completely analogous reasoning, yielding

\[
J^2 = \int K^{33}(x, z) \omega(z) \, dz
\]

with

\[
|K^{32}(x, z)| \leq \frac{C \ell(z)}{|x-z|}.
\]

The details, which are tedious but now straightforward, are omitted. This completes the proof of Theorem 7.

6. Uniqueness and an Application to the Full Div-Curl System

As a last step in the proof of Theorem 1, in this easy section we will consider the uniqueness of the solution. For completeness, we will do so in the context of the general div-curl system

\[
\nabla \times v = \omega, \quad \nabla \cdot v = f, \quad v \cdot \nu = g
\]

in \( \Omega \). Here \( \omega \) is a divergence-free field satisfying \( 2 \) and the functions \( f \) and \( g \) satisfy the well known compatibility conditions

\[
\int_{\Omega} f \, dx = \int_{\partial \Omega} g \, d\sigma.
\]

**Proposition 8.** The system \( 46 \), with \( \omega \), \( f \) and \( g \) as above, admits a solution \( v \), which is bounded as

\[
\|v\|_{W^{k+1,p}(\Omega)} \leq C(\|\omega\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-rac{1}{q},p}(\partial\Omega)}).
\]

Furthermore, the solution is unique modulo the addition of a harmonic field tangent to the boundary, and the dimension of this linear space equals the genus of \( \partial \Omega \).

**Proof.** Uniqueness is immediate: if \( v \) and \( v' \) are two solutions to the problem, their difference \( w := v - v' \) satisfies

\[
\nabla \times w = 0, \quad \nabla \cdot w = 0, \quad w \cdot \nu = 0,
\]

so it is a harmonic field on \( \Omega \) tangent to the boundary. The dimension of the linear space of solutions to \( 48 \) is known to be given by the genus of \( \partial \Omega \) by Hodge theory.

To prove the existence of a solution, let \( \phi \) be the only solution to the problem

\[
\Delta \phi = f, \quad \partial_\nu \phi = g, \quad \int_{\Omega} f \, dx = 0
\]

in \( \Omega \), which is granted to exist by the hypothesis \( 47 \) and satisfies

\[
\|\phi\|_{W^{k+2,p}(\Omega)} \leq C(\|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-rac{1}{q},p}(\partial\Omega)}).
\]

The field \( u := v - \nabla \phi \) then satisfies

\[
\nabla \times u = \omega, \quad \nabla \cdot u = 0, \quad u \cdot \nu = 0,
\]

so Theorem \( 5 \) ensures that there is a solution satisfying

\[
\|u\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}.
\]

The statement then follows. \( \square \)
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