Canonical bases arising from quantum symmetric pairs of Kac–Moody type

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Compositio Math. 157 (2021), 1507–1537.

doi:10.1112/S0010437X2100734X
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Abstract

For quantum symmetric pairs \((U, U')\) of Kac–Moody type, we construct \(i\)-canonical bases for the highest weight integrable \(U\)-modules and their tensor products regarded as \(U'\)-modules, as well as an \(i\)-canonical basis for the modified form of the \(i\)-quantum group \(U'\). A key new ingredient is a family of explicit elements called \(i\)-divided powers, which are shown to generate the integral form of \(U'\). We prove a conjecture of Balagovic–Kolb, removing a major technical assumption in the theory of quantum symmetric pairs. Even for quantum symmetric pairs of finite type, our new approach simplifies and strengthens the integrality of quasi-\(K\)-matrix and the constructions of \(i\)-canonical bases, by avoiding a case-by-case rank-one analysis and removing the strong constraints on the parameters in a previous work.

Contents

1. Introduction 1507
2. Quantum groups and canonical bases 1512
3. The \(i\)-quantum groups \(U_i\) 1515
4. Fundamental lemma for QSP 1521
5. The \(i\)-divided powers 1522
6. \(i\)-Canonical bases for modules 1529
7. Canonical bases on the modified \(i\)-quantum groups 1533

Acknowledgements 1535
References 1535

1. Introduction

1.1 Background

Drinfeld–Jimbo quantum groups \(U = U(g)\) are deformations of universal enveloping algebras of simple or symmetrizable Kac–Moody Lie algebras \(g\), associated with a generalized Cartan matrix \((a_{ij})_{i \in I}\). The theory of canonical bases for quantum groups has been developed by Lusztig and also by Kashiwara \([Lus90, Kas91, Lus91, Lus92, Lus10, Kas94]\). It has applications to Kazhdan–Lusztig theory and geometric representation theory; it was also a major motivation for several exciting active research directions in the past decade including categorification and cluster algebras.
The classification of real simple Lie algebras or, equivalently, the symmetric pairs \((\mathfrak{g}, \mathfrak{g}^\theta)\), was achieved by É. Cartan, and they are in bijection with (bicolored) Satake diagrams, \(I = I_\circ \cup I_\bullet\); cf. [Ara62]. As a deformation of symmetric pairs, the quantum symmetric pair (QSP) \((U, U^\dagger)\) was formulated by Letzter [Let99, Let02] in finite type with the Satake diagrams as inputs; this construction has been generalized by Kolb [Kol14] to the Kac–Moody setting. Here \(U^\dagger\) is a coideal subalgebra of \(U\) and will be referred to as an \(\dagger\)-quantum group on its own. A subtle and deep feature of QSPs is that \(U^\dagger = U_{\varsigma, \kappa}\) depends on a multiple of parameters \(\varsigma = (\varsigma_i)_{i \in I_{\circ}}\) and \(\kappa = (\kappa_i)_{i \in I_{\circ}}\) subject to some constraints; specializing at \(q = 1\) and \(\varsigma_i = 1\) allows one to recover the symmetric pairs.

In recent years, it has become increasingly clear that a number of fundamental constructions for quantum groups admit highly nontrivial generalizations to the setting of QSP. In [BW18b] (also cf. [BW18a]), the authors developed a theory of \(\dagger\)-canonical bases for the QSP \((U, U^\dagger)\) of finite type, on \(U^\dagger\)-modules as well as a modified form \(\dot{U}^\dagger\). The constructions of quasi-\(K\)-matrix and universal \(K\)-matrix for QSP [BW18a, BK19] have also played a significant role.

The \(\dagger\)-quantum group \(U^\dagger\) of quasi-split type AIII and its \(\dagger\)-canonical basis have applications to super Kazhdan–Lusztig theory of type BCD [BW18a, Bao17], admit a geometric realization [BKLW18, LW18] and a KLR-type categorification [BSWW18]. The \(\dagger\)-quiver varieties introduced more recently by Li [Li19] have intimate connections with QSPs.

1.2 Obstacles
Let us expand on some of the main obstacles toward canonical bases arising from QSP before the current work. Set \(A = \mathbb{Z}[q, q^{-1}]\). Similar to the canonical bases for quantum groups, the \(\dagger\)-canonical bases on modules require three ingredients:

1. a bar involution \(\psi_\dagger\);
2. an integral \(A\)-form;
3. a \(\mathbb{Z}[q^{-1}]\)-lattice (with some distinguished basis).

The bar involution on \(U^\dagger\) in general was proposed in [BW18a] and then constructed in great generality by Balagovic–Kolb in [BK15] who also specified the constraints on the parameters \(\varsigma\) and \(\kappa\). On the other hand, the new bar involution \(\psi_\dagger\) at the level of \(U\)-modules (regarded as \(U^\dagger\)-modules by restriction) requires the notion of quasi-\(K\)-matrix due to the authors of [BW18a], which is a QSP generalization of Lusztig’s quasi-\(R\)-matrix; the existence of quasi-\(K\)-matrix in much generality has been subsequently established by [BK19].

These general constructions in [BK15, BK19] largely rely on two crucial assumptions. The first (and somewhat mild) assumption is that the Cartan integers are bounded, i.e. \(|a_{ij}| \leq 3\). The construction of a bar involution on \(U^\dagger\) relies on explicit Serre-type presentations of \(U^\dagger\), which are only available under the bound assumption on \((a_{ij})\) until very recently. In a recent work [CLW20], a Serre presentation was obtained for the quasi-split \(\dagger\)-quantum group \(U^\dagger\) (i.e. associated to the Satake diagram with \(I_{\bullet} = \emptyset\)) without any bound constraint on \((a_{ij})\), and the existence of the bar involution on such \(U^\dagger\) follows. Note the QSP of finite and affine types are all covered.

The second (and a more serious) assumption is the validity of a conjecture in [BK15] that certain numbers \(v_i \in \{1, -1\}\) for \(i \in I_{\circ}\) should always be \(v_i = 1\); for a precise formulation see (3.9) and (3.10) and Conjecture 3.5. It was shown in [BK15] that \(v_i = 1\) always holds for \(U\) of finite type. For \(U\) of Kac–Moody type, the conjecture is only known for \(U\) of locally finite type (cf. [BW18b, §5.4]), and in this case it follows from the counterpart in the finite type because
the definition of $\nu_i$ is local. We shall refer to this Balagovic–Kolb conjecture as the fundamental lemma of QSP, because although this sounds like a mere technicality, several major constructions in great generality depend on it. For example, the quasi-$K$-matrix as well as universal $K$-matrix in [BK19] (which was first constructed in [BW18a, § 2.5] for quasi-split QSP of type AIII) both rely on the validity of the fundamental lemma of QSP.

The approach developed in [BW18a] on the integrality issue for the quasi-$K$-matrix $\Upsilon$ and the modified form $\dot{\U}$ of finite type is rather tedious and technical as it is based on a case-by-case analysis in the eight real rank-one finite-type cases. Such an approach toward the integrality is not generalizable to $\U$ of Kac–Moody type, as it is probably impossible to classify the real rank-one $\i$-quantum groups of Kac–Moody type (there is already a zoo of real rank one $\i$-quantum groups of affine type; cf. [RV16]).

1.3 Goal

The goal of this paper is to develop a general theory of $\i$-canonical bases for the QSP of Kac–Moody type. We shall construct $\i$-canonical bases on integrable highest weight $U$-modules and their tensor products. We shall also construct the $\i$-canonical basis on the modified form $\dot{U}$. (Note that the canonical basis on $\bar{U}$ [Lus92] can be viewed as the $\i$-canonical basis for the $\i$-quantum group for the QSP of diagonal type.) Our goal is achieved by building on the foundational works [Lus10, BK19], following closely several constructions in [BW18a] when applicable (including a projective system of based $U'$-modules) and, more crucially, introducing several new ideas to overcome the major obstacles as mentioned previously.

Even for QSP of finite type, the results in this paper here strengthen the main results in [BW18b] by allowing general integral parameters $\varsigma_i$ and simplifying the approach in [BW18b] by substituting the tedious case-by-case real rank-one analysis therein with a conceptual $\i$-divided powers construction.

We further provide a proof of the fundamental lemma of QSP in full generality. This removes a major technical assumption in [BK15, BK19] toward the existence of bar involution, quasi-$K$-matrix and universal $K$-matrix, and as well for the constructions of $\i$-canonical bases in this paper.

1.4 Fundamental lemma of QSP

The fundamental lemma of QSP is actually a statement for quantum groups, motivated by the study of QSPs. By [BK15, Proposition 2.5] $\sigma\tau(Z_i) = \nu_i Z_i$, where $\nu_i \in \{1, -1\}$, $Z_i$ is defined in (3.9) and $\sigma$ is the anti-involution of $U$ which fixes $E_i, F_i$ for all $i \in \I$, and $\tau$ is a diagram automorphism. We show that $Z_i$ descends to a certain cell quotient of $\bar{U}$ as a nonzero scalar multiple of a canonical basis element, and it follows that $\nu_i = 1$ as the canonical basis is preserved by $\sigma$ and $\tau$. Our proof relies in an essential way on Lusztig’s theory of based modules and cells on quantum groups [Lus10, Chapters 27, 29]. Some old results of Joseph and Letzter [JL94, JL96] also play a role.

1.5 The $\i$-divided powers

A key new construction in this paper is a family of explicit elements called $\i$-divided powers, for each $i \in \I$, which by definition satisfy the three properties: bar invariant, integral and having a leading term the standard divided powers embedded as elements in $\bar{U}$. The $\i$-divided powers associated to $i \in \I_+$ are the standard divided powers $F_{\i}^{(n)}, E_{\i}^{(n)}$ as in [Lus10]. The $\i$-divided powers associated to $i \in \I_0$ with $\tau i \neq i$ or to $i \in \I_\circ$ with $w_\bullet i = i = \tau i$ were introduced and studied
earlier in [BW18a, BW18b, BeW18]. For the class of \( i \in \mathbb{I}_0 \) with \( w_i \neq i = \tau i \), our formula for the \( \iota \)-divided powers, denoted by \( B_{i, \zeta}^{(n)} \) for \( n \geq 1 \), is explicit and universal. (Note that five out of eight \( \iota \)-quantum groups of real rank one in finite type belong to this class; cf. [BW18b, §3.2, Table 1].)

The \( \iota \)-divided powers \( B_{i, \zeta}^{(n)} \) should be regarded as a leading term of a corresponding \( \iota \)-canonical basis element. We eventually show that the \( A \)-form \( \dot{\mathcal{A}} \mathcal{U}_\iota \), which is defined in a more conceptual way, is actually generated by the \( \iota \)-divided powers, and \( \dot{\mathcal{A}} \mathcal{U}_\iota \) is a free \( A \)-submodule of \( \dot{\mathcal{U}}_\iota \) such that \( \dot{\mathcal{U}}_\iota = Q(q) \otimes_{\dot{\mathcal{A}}} \dot{\mathcal{U}}_\iota \).

1.6 Integrality of \( \Upsilon \) in action

It is neither expected nor needed that the quasi-\( K \)-matrix for \( \mathcal{U}^\iota \) beyond finite type is integral on its own, based on the knowledge from quasi-\( R \)-matrix for a quantum group \( \mathcal{U}^\iota \) beyond finite type (cf. [BW16]). With the help of the aforementioned \( \iota \)-divided powers, we show that \( \Upsilon \) preserves the integral \( A \)-forms on integrable highest weight \( \mathcal{U} \)-modules and their tensor products. The proof is in part inspired by our argument in [BW16] that Lusztig’s quasi-\( R \)-matrix preserves the integral \( A \)-forms of these \( \mathcal{U} \)-modules; actually the approach developed in [BW16] toward canonical bases on tensor product modules can in turn be viewed as dealing with the special case of QSP of diagonal type.

For QSP of finite type, our new approach via the \( \iota \)-divided powers allow us to establish the integrality of \( \Upsilon \) for general integral parameters \( \zeta_i \) (in contrast to \( \zeta_i \in \pm q\mathbb{Z} \) in [BW18b]) and to bypass the tedious case-by-case real rank-one analysis in [BW18a, Appendix A.4–A.7]. (The complete detail takes 23 pages and can be found in Appendix A.4–A.11 in the arXiv version 1 of the paper.)

Following [BW18a, BW18b], we define a new bar involution \( \psi_i = \Upsilon \circ \psi \) on the based \( \mathcal{U} \)-modules such as integrable highest weight \( \mathcal{U} \)-modules and their tensor products. As \( \psi_i \) preserves the integral \( A \)-forms, we are able to construct the \( \iota \)-canonical bases on these modules using their canonical bases from [BW16] (cf. [Lus10, Part IV]). The \( \iota \)-canonical basis spans the same \( \mathbb{Z}[q^{-1}] \)-lattices as the usual canonical basis: this is the characterization property (3) for \( \iota \)-canonical basis in §1.1.

We further extend the construction of based \( \mathcal{U}^\iota \)-modules to tensor products of a based \( \mathcal{U}^\iota \)-module with a based \( \mathcal{U} \)-module, using a construction of \( \Theta^\iota \) in [BW18a] and [Kol20]; this was recently carried out for QSP of finite type in [BWW20].

1.7 \( \iota \)-Canonical bases on based modules and \( \dot{\mathcal{U}}^\iota \)

Note we have established the \( \iota \)-canonical bases on modules for \( \mathcal{U}^\iota \) with general parameters \( \zeta_i \in A \) (in contrast to the very strict constraint that \( \zeta_i \in \pm q\mathbb{Z} \) in [BW18b]). Recall the condition on \( \zeta_i \) in [BW18b] was imposed to ensure that an anti-involution \( \varphi \) on \( \mathcal{U} \) (see Proposition 2.1) restricts to an involution to the subalgebra \( \mathcal{U}^\iota \) (see [BW18b, Proposition 4.6]).

We make a crucial observation here that the general construction of \( \iota \)-canonical basis on the modified form \( \dot{\mathcal{U}}^\iota \) does not rely on the fact that \( \varphi \) preserves \( \mathcal{U}^\iota \) (this was rather a mental block for us for quite some time). In all our constructions toward the \( \iota \)-canonical basis on \( \dot{\mathcal{U}}^\iota \), we only need to use the twist of \( \varphi \) on \( \mathcal{U} \)-modules. Recall that any \( \mathcal{U} \)-module is automatically a \( \mathcal{U}^\iota \)-module by restriction.

The construction for the canonical basis on the modified \( \iota \)-quantum group \( \dot{\mathcal{U}}^\iota \) relies crucially on the based \( \mathcal{U} \)-submodules \( L(w\lambda, \mu) \subset L(\lambda) \otimes L(\mu) \), for a Weyl group element \( w \); cf. (2.5). In finite type, we proved that \( L(w\lambda, \mu) \) is a based module using an identification
Canonical bases arising from quantum symmetric pairs

\[\Lambda(\lambda) \otimes \Lambda(\mu) \cong \omega L(-w_0 \lambda) \otimes \Lambda(\mu),\]
where \(w_0\) is the longest element in the Weyl group. Instead, Kashiwara’s theory of extreme weight modules [Kas94] is used to prove that \(L(w \lambda, \mu)\) is a based submodule of \(\Lambda(\lambda) \otimes \Lambda(\mu)\) over \(U\) of Kac–Moody type; see Theorem 2.2.

With all these preparations and observations, we follow [BW18a] closely to build a projective system of based \(U\)-modules (which generalizes Lusztig’s construction [Lus10, Part IV]), and establish the \(\nu\)-canonical basis of \(\hat{U}\) and of \(\mathcal{A}\hat{U}\).

For QSP of affine type AIII, a geometric realization of \(\hat{U}\) and its \(\nu\)-canonical bases was first given in [FLLLW20].

1.8 Applications
For the existence of bar involution, we shall make the basic assumption that \(|a_{ij}| \leq 3\) or \(I_\bullet = \emptyset\) throughout the paper. It seems possible that the new \(\nu\)-divided powers introduced in this paper might help to obtain a Serre presentation for \(U\) (and then a bar involution) by weakening or removing this bound assumption on the generalized Cartan matrices in the long run. This is exactly how the (old) \(\nu\)-divided powers helped solving this very problem for the quasi-split \(\nu\)-quantum groups \(U\) in [CLW20].

The \(\nu\)-divided powers are expected to play a key role in computational aspects of \(\nu\)-canonical bases and related combinatorics. They will help to shed new light on the categorification and geometric realization of QSP, and in addition have applications in the study of QSPs at roots of 1.

In [RV16], several crucial constructions for QSP were carried over for a more general class of quantum algebras associated to generalized Satake diagrams. It is interesting to explore if the technique of \(\nu\)-divided powers introduced in this paper allows a possible further generalization of integral forms and \(\nu\)-canonical basis in this generalized QSP setting.

1.9 The organization
The paper is organized as follows. Sections 4–6 contain the main new ideas of this paper.

In §2, we review and set up notation for quantum groups and canonical basis. The new result in this section is the existence of a family of based \(U\)-modules \(L(w \lambda, \mu)\).

In §3, we set up notation for QSPs \((U, U')\). We recall some earlier work [BK19] and summarize various results in [BW18b] which remain valid in the Kac–Moody setting. The existence of an anti-involution \(\sigma_\nu\) on \(U\) (analogous to an anti-involution \(\sigma\) on \(U\)) is new. We assume in §3 the validity of the fundamental lemma of QSP, so the results in this section work in great generality. The fundamental lemma of QSP, which is a conjecture of Balagovic and Kolb, is then proved in §4.

In §5, we define explicitly and study in depth the \(\nu\)-divided powers in three separate classes, including a major new class of \(i \in I_\circ\) with \(w_\bullet i \neq i = \tau i\). We prove that the \(\nu\)-divided powers are integral, bar invariant with suitable leading term; we show in §7 that they generate the \(A\)-form \(\mathcal{A}\hat{U}\).

In §6, we show that the quasi-\(K\)-matrix \(\Upsilon\) preserves the \(A\)-forms of highest weight integrable \(U\)-modules and their tensor products, and then construct \(\nu\)-canonical bases on these modules. We further construct \(\nu\)-canonical bases on a tensor product of a based \(U\)-module and a based \(U\)-module.

In §7, we establish the \(\nu\)-canonical basis on \(\hat{U}\), and show that the \(A\)-subalgebra \(\mathcal{A}\hat{U}\) is indeed a free \(A\)-module such that \(\mathcal{Q}(q) \otimes \mathcal{A} \hat{U} = \hat{U}\). This is essentially a summary of [BW18b, §6], now in the Kac–Moody setting, based on the results in
previous sections. A different (and more elementary) partial order inspired by [BWW20] is used here.

2. Quantum groups and canonical bases

In this section, we review some basic constructions and set up notation in quantum groups. We then establish a family of based U-modules \( L(w\lambda, \mu) \).

2.1 Let \( q \) be an indeterminate. Consider a free \( \mathbb{Q}(q) \)-algebra \( 'f \) generated by \( \theta_i \) for \( i \in \mathbb{I} \) associated with the Cartan datum of type \( (\mathbb{I}, \cdot) \). As a \( \mathbb{Q}(q) \)-vector space, \( 'f \) has a weight space decomposition as \( 'f = \bigoplus_{\mu \in \mathbb{N}[\mathbb{I}]} f_{\mu} \), where \( \theta_i \) has weight \( i \) for all \( i \in \mathbb{I} \). For \( \mu = \sum_{i \in \mathbb{I}} a_i i \), the height of \( \mu \) is denoted by \( \text{ht}(\mu) = \sum_{i \in \mathbb{I}} a_i \). For any \( x \in 'f_{\mu} \), we set \( |x| = \mu \). For any \( i \in \mathbb{I} \), we set \( q_i = q^{i/2} \).

Let \( W \) be the corresponding Weyl group generated by simple reflections \( s_i \) for \( i \in \mathbb{I} \).

For each \( i \in \mathbb{I} \), we define \( r_i, i r : 'f \to 'f \) to be the unique \( \mathbb{Q}(q) \)-linear maps such that

\[
    r_i(1) = 0, \quad r_i(\theta_j) = \delta_{ij}, \quad r_i(f x') = x r_i(f) x', \quad (2.1)
\]

for all \( x \in 'f_{\mu} \) and \( x' \in 'f_{\mu'} \).

Let \( \langle \cdot, \cdot \rangle \) be the symmetric bilinear form on \( 'f \) defined in [Lus10, 1.2.3]. Let \( \mathcal{I} \) be the radical of the symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( 'f \). For \( i \in \mathbb{I}, n \in \mathbb{Z} \) and \( s \in \mathbb{N} \), we define

\[
    [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \quad \text{and} \quad [s]_i^* = \prod_{j=1}^{s} [j]_i.
\]

We shall also use the notation

\[
    \left[ \frac{n}{s} \right]_i = \frac{[n]_i^s}{[s]_i^n [n - s]_i}, \quad \text{for } 0 \leq s \leq n.
\]

It is known [Lus10] that \( \mathcal{I} \) is generated by the quantum Serre relations \( S(\theta_i, \theta_j) \), for \( i \neq j \in \mathbb{I} \), where

\[
    S(\theta_i, \theta_j) = \sum_{s=0}^{1 - a_{ij}} (-1)^s \left[ 1 - a_{ij} \right]_i \theta_i^s \theta_j \theta_i^{1 - a_{ij} - s}. \quad (2.2)
\]

Let \( \mathfrak{f} = 'f / \mathcal{I} \). We introduce the divided power \( \theta_i^{(a)} = \theta_i^a / [a]_i! \) for \( a \geq 0 \). Let

\[
    \mathcal{A} = \mathbb{Z}[q, q^{-1}].
\]

Let \( \mathcal{A} \mathfrak{f} \) be the \( \mathcal{A} \)-subalgebra of \( \mathfrak{f} \) generated by \( \theta_i^{(a)} \) for various \( a \geq 0 \) and \( i \in \mathbb{I} \).

2.2 Let \( (Y, X, \langle \cdot, \cdot \rangle, \ldots) \) be a root datum of type \( (\mathbb{I}, \cdot) \); cf. [Lus10]. We define a partial order \( \leq \) on the weight lattice \( X \) as follows: for \( \lambda, \lambda' \in X \),

\[
    \lambda \leq \lambda' \text{ if and only if } \lambda' - \lambda \in \mathbb{N}[\mathbb{I}]. \quad (2.3)
\]

The quantum group \( \mathcal{U} \) associated with this root datum \( (Y, X, \langle \cdot, \cdot \rangle, \ldots) \) is the associative \( \mathbb{Q}(q) \)-algebra generated by \( E_i, F_i \) for \( i \in \mathbb{I} \) and \( K_\mu \) for \( \mu \in Y \), subject to the following relations:
for all $\mu, \mu' \in Y$ and $i \neq j \in I$,

$$K_0 = 1, \ K_\mu K_{\mu'} = K_{\mu + \mu'},$$

$$K_\mu E_j = q^{(\mu, \mu')} E_j K_\mu, \ K_\mu F_j = q^{-(\mu, \mu')} F_j K_\mu,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{q_i - q_i^{-1}},$$

$$S(F_i, F_j) = S(E_i, E_j) = 0,$$

where $K_{\pm i} = K_{\pm (i/2)}$ and $S(\cdot, \cdot)$ is defined as (2.2).

Let $U^+, U^0$ and $U^-$ be the $\mathbb{Q}(q)$-subalgebra of $U$ generated by $E_i$ ($i \in I$), $K_\mu (\mu \in Y)$, and $F_i$ ($i \in I$), respectively. We identify $f \cong U^-$ by matching the generators $\theta_i$ with $F_i$. This identification induces a bilinear form $(\cdot, \cdot)$ on $U^-$ and $\mathbb{Q}(q)$-linear maps $r_i, i r(i \in I)$ on $U^-$. Under this identification, we let $U^-_\mu$ be the image of $f_\mu$. Similarly, we have $f \cong U^+$ by identifying $\theta_i$ with $E_i$. We let $A U^-$ (respectively, $A U^+$) denote the image of $A f$ under this isomorphism, which is generated by all divided powers $F_i^{(\alpha)} = F_i^a/[a]^1_i$ (respectively, $E_i^{(\alpha)} = E_i^a/[a]^1_i$). The coproduct $\Delta : U \to U \otimes U$ is defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes \tilde{K}_{-i}, \quad \Delta(K_\mu) = K_\mu \otimes K_\mu. \quad (2.4)$$

2.3 We recall several symmetries of $U$; cf. [Lus10].

**Proposition 2.1.**

1. There is an involution $\omega$ of the $\mathbb{Q}(q)$-algebra $U$ such that $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, and $\omega(K_\mu) = K_{-\mu}$ for all $i \in I$ and $\mu \in Y$.

2. There is an anti-involution $\varphi$ of the $\mathbb{Q}(q)$-algebra $U$ such that $\varphi(E_i) = q_i^{-1} F_i \tilde{K}_i$, $\varphi(F_i) = q_i^{-1} E_i \tilde{K}_i^{-1}$ and $\varphi(K_\mu) = K_{-\mu}$ for all $i \in I$ and $\mu \in Y$.

3. There is an anti-involution $\sigma$ of the $\mathbb{Q}(q)$-algebra $U$ such that $\sigma(E_i) = E_i$, $\sigma(F_i) = F_i$ and $\sigma(K_\mu) = K_{-\mu}$ for all $i \in I$ and $\mu \in Y$.

4. There is a bar involution $^\ast$ of the $\mathbb{Q}(q)$-algebra $U$ such that $q \mapsto q^{-1}$, $\tilde{E}_i = E_i$, $\tilde{F}_i = F_i$, and $\tilde{K}_\mu = K_{-\mu}$ for all $i \in I$ and $\mu \in Y$. (Sometimes we denote the bar involution on $U$ by $\psi$.)

5. There are automorphisms $T_{i, e}^n$ (for $e = \pm 1$, $i \in I$) of the $\mathbb{Q}(q)$-algebra $U$ such that

$$T_{i, e}^n(E_i) = -F_i \tilde{K}_{-e i}, \quad T_{i, e}^n(F_i) = -\tilde{K}_e i E_i, \quad T_{i, e}^n(K_\mu) = K_{s_i(\mu)};$$

$$T_{i, e}^n(E_j) = \sum_{r + s = -(i, j')} (-1)^r q_i^{sr} E_i^{(s)} E_j E_i^{(r)} \quad \text{for } j \neq i.$$

As $T_{i, 1}^n$ satisfies the braid group relation, we can define the automorphism $T_{w, 1}^n$ of $U$, associated to $w \in W$, in a standard fashion. To simplify the notation, throughout the paper we often write

$$T_i = T_{i, 1}^n, \quad \text{and} \quad T_w = T_{w, 1}^n, \quad \text{for } w \in W.$$

2.4 Let $M(\lambda)$ be the Verma module of $U$ with highest weight $\lambda \in X$ and with a highest weight vector denoted by $\eta_\lambda$. We define a lowest weight $U$-module $^\omega M(\lambda)$, which has the same underlying vector space as $M(\lambda)$ but with the action twisted by the involution $\omega$ given in Proposition 2.1. When considering $\eta_\lambda$ as a vector in $^\omega M(\lambda)$, we shall denote it by $\xi_{-\lambda}$.
Let $$X^+ = \{ \lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N}, \forall i \in \mathbb{I} \}$$ be the set of dominant integral weights. By $$\lambda \gg 0$$ we shall mean that the integers $$\langle i, \lambda \rangle$$ for all $$i$$ are sufficiently large. The Verma module $$M(\lambda)$$ associated to $$\lambda \in X$$ has a unique simple quotient $$U$$-module, denoted by $$L(\lambda)$$. We shall abuse the notation and denote by $$\eta_\lambda \in L(\lambda)$$ the image of the highest weight vector $$\lambda \in M(\lambda)$$. Similarly, we define the $$U$$-module $$\omega L(\lambda)$$ of lowest weight $$\lambda$$ with lowest weight vector $$\xi_{-\lambda}$$. For $$\lambda \in X^+$$, we let $$\mathcal{A}L(\lambda) = \mathcal{A}U^-\eta_\lambda$$ and $$\mathcal{A}U^\tau \xi_{-\lambda}$$ be the $$\mathcal{A}$$-submodules of $$L(\lambda)$$ and $$\omega L(\lambda)$$, respectively.

There is a canonical basis $$B$$ on $$\mathfrak{f}$$, a canonical basis $$\{b^+ | b \in B\}$$ on $$U^+$$ and a canonical basis $$\{b^- | b \in B\}$$ on $$U^-$$. For each $$\lambda \in X^+$$, there is a subset $$B(\lambda)$$ of $$B$$ so that $$\{b^- \eta_\lambda | b \in B(\lambda)\}$$ (respectively, $$\{b^+ \xi_{-\lambda} | b \in B(\lambda)\}$$) forms a canonical basis of $$L(\lambda)$$ (respectively, $$\omega L(\lambda)$$). For any Weyl group element $$w \in W$$, let $$\eta_{w \lambda}$$ denote the unique canonical basis element of weight $$w \lambda$$.

Let $$\hat{U} = \bigoplus_{\zeta \in X} \hat{U}_1 \zeta$$ be the idempotented quantum group and $$\mathcal{A}\hat{U}$$ be its $$\mathcal{A}$$-form. Then $$\hat{U}$$ admits a canonical basis $$B = \{ b_1 \otimes b_2 \mid (b_1, b_2) \in B \times B, \zeta \in X \}$$; cf. [Lus10, Part IV].

2.5 For any $$I_{\bullet} \subset I$$, let $$U_{I_{\bullet}}$$ be the $$\mathbb{Q}(q)$$-subalgebra of $$U$$ generated by $$F_i \ (i \in I_{\bullet})$$, $$E_i \ (i \in I_{\bullet})$$ and $$K_i (i \in I_{\bullet})$$. Let $$B_{I_{\bullet}}$$ be the canonical basis of $$f_{I_{\bullet}}$$ (here $$f_{I_{\bullet}}$$ is simply a version of $$f$$ associated to $$I_{\bullet}$$), which induces canonical bases on $$U^-_{I_{\bullet}}$$ and $$U^+_{I_{\bullet}}$$. Let $$\hat{P} = P_{I_{\bullet}}$$ be the $$\mathbb{Q}(q)$$-subalgebra of $$U$$ generated by $$U_{I_{\bullet}}$$ and $$U^+$$. We denote by $$L_{I_{\bullet}}(\lambda)$$ the simple $$U_{I_{\bullet}}$$-module of highest weight $$\lambda$$.

We introduce the following subalgebra of $$\hat{U}$$:

$$\hat{P} = \bigoplus_{\lambda \in X} P_{1_{\lambda}}.$$ 

We further set $$\mathcal{A}\hat{P} = \hat{P} \cap \mathcal{A}\hat{U}$$.

2.6 Based submodules $$L(w_{\lambda}, \mu)$$

Recall the theory of based $$U$$-modules of Lusztig [Lus10, Chapter 27] for $$U$$ of finite type, which was extended by the authors in [BW16] for $$U$$ of Kac–Moody type. For $$\lambda, \mu \in X^+$$ and $$w \in W$$, we introduce the following $$U$$-submodule:

$$L(w_{\lambda}, \mu) = U(\eta_{w \lambda} \otimes \eta_\mu) \subset L(\lambda) \otimes L(\mu).$$ (2.5)

The following theorem is the main result of this section, the proof of which was kindly communicated to us by Kashiwara.

Theorem 2.2. Let $$\lambda, \mu \in X^+$$ and $$w \in W$$. Then the $$U$$-submodule $$L(w_{\lambda}, \mu)$$ is a based $$U$$-submodule of $$L(\lambda) \otimes L(\mu)$$.

Proof. Write $$w_{\lambda} = \lambda_1 - \nu$$, for some $$\lambda_1, \nu \in X^+$$. Thanks to [BW16, Theorem 2.9, Proposition 2.11], $$L(\lambda_1) \otimes L(\mu)$$ is a based $$U$$-module, and the map $$\chi : L(\lambda_1 + \mu) \to L(\lambda_1) \otimes L(\mu)$$, which sends $$\eta_{\lambda_1 + \mu} \mapsto \eta_{\lambda_1} \otimes \eta_\mu$$, is a based $$U$$-module homomorphism. Therefore,

$$\chi' := \text{id}_{L(\nu)} \otimes \chi : \omega L(\nu) \otimes L(\lambda_1 + \mu) \longrightarrow \omega L(\nu) \otimes L(\lambda_1) \otimes L(\mu),$$

which sends $$\xi_{\nu} \otimes \eta_{\lambda_1 + \mu} \mapsto \xi_{\nu} \otimes \eta_{\lambda_1} \otimes \eta_\mu$$, is a based module homomorphism.

Recall the notion of extremal weight modules [Kas94, §8], and in particular the extremal weight module $$L(w_{\lambda})$$ coincides with $$L(\lambda)$$, thanks to $$\lambda \in X^+$. 

1514
It follows from [Lus10, Proposition 23.3.6] that there exists a $U$-module homomorphism $\phi : \omega L(\nu) \otimes L(\lambda) \rightarrow L(w \lambda) = L(\lambda)$, which sends $\xi \otimes \eta_{\lambda,1} \mapsto \eta_{w \lambda}$. By Kashiwara [Kas94, Lemma 8.2.1], the map $\phi : \omega L(\nu) \otimes L(\lambda) \rightarrow L(w \lambda) = L(\lambda)$ is a based module homomorphism. Thus,

$$\phi' := \phi \otimes \text{id}_{L(\mu)} : \omega L(\nu) \otimes L(\lambda) \otimes L(\mu) \rightarrow L(\lambda) \otimes L(\mu),$$

which sends $\xi \otimes \eta_{\lambda,1} \otimes x \mapsto \eta_{w \lambda} \otimes x$, for $x \in L(\mu)$, is a based $U$-module homomorphism.

Therefore, the composition homomorphism

$$\phi' \chi' : \omega L(\nu) \otimes L(\lambda + \mu) \rightarrow L(\lambda) \otimes L(\mu),$$

which sends $\xi \otimes \eta_{\lambda,1+\mu} \mapsto \eta_{w \lambda} \otimes \eta_{\mu}$, is a based $U$-module homomorphism. As $\xi \otimes \eta_{\lambda,1+\mu}$ is a cyclic vector (i.e. it generates the $U$-module $\omega L(\nu) \otimes L(\lambda_1 + \mu)$), the $U$-module $L(w \lambda, \mu)$ is the image of the based module homomorphism $\phi' \chi'$. Hence $L(w \lambda, \mu)$ is a based $U$-submodule of $L(\lambda) \otimes L(\mu)$. \hfill $\square$

### Remark 2.3.

Theorem 2.2 for $U$ of finite type appeared as [BW18b, Theorem 2.6] with a different proof.

### 3. The $\iota$-quantum groups $U^\iota$

In this section, we review the basic definitions and constructions of QSPs, including the bar involution and quasi-$K$-matrix. We also formulate various (old and new) symmetries on $U^\iota$ and $U^\iota$-modules. Our general setup assumes the validity of the fundamental lemma of QSP, which is to be established in §4.

#### 3.1 Let $\tau$ be an involution of the Cartan datum $(\mathbb{I}, \cdot)$; we allow $\tau = \text{id}$. We further assume that $\tau$ extends to an involution on $X$ and an involution on $Y$, respectively, such that the perfect bilinear pairing is invariant under the involution $\tau$. For any $\lambda \in X$ (or $Y$), we shall write $\lambda^\tau = \tau(\lambda)$.

From now on, we assume $\mathbb{I}_\bullet \subset \mathbb{I}$ is a Cartan subdatum of finite type. Let $W_\bullet$ be the parabolic subgroup of $W$ with $w_\bullet$ as its longest element. Let $\rho_\bullet^\vee$ be the half sum of all positive coroots in the set $R_\bullet^\vee$, and let $\rho_\bullet$ be the half sum of all positive coroots in the set $R_\bullet$. We shall write

$$\mathbb{I}_0 = \mathbb{I} \setminus \mathbb{I}_\bullet. \quad (3.1)$$

A pair $(\mathbb{I}_\bullet, \tau)$ is called admissible (cf. [Kol14, Definition 2.3]) if the following conditions are satisfied:

1. $\tau(\mathbb{I}_\bullet) = \mathbb{I}_\bullet$;
2. the action of $\tau$ on $\mathbb{I}_\bullet$ coincides with the action of $-w_\bullet$;
3. if $j \in \mathbb{I}_0$ and $\tau(j) = j$, then $\langle \rho_\bullet^\vee, j' \rangle \in \mathbb{Z}$.

In this paper, all pairs $(\mathbb{I}_\bullet, \tau)$ considered are admissible.

Note that $\theta = -w_\bullet \circ \tau$ is an involution of $X$ and $Y$. Following [BW18b], we introduce the $\iota$-weight lattice and $\iota$-root lattice

$$X_\iota = X/\bar{X}, \quad \text{where} \quad \bar{X} = \{ \lambda - \theta(\lambda) \mid \lambda \in X \},$$

$$Y_\iota = \{ \mu \in Y \mid \theta(\mu) = \mu \}. \quad (3.2)$$

For any $\lambda \in X$ denote its image in $X_\iota$ by $\bar{\lambda}$. 

1515
The involution $\tau$ of $I$ induces an isomorphism of the $\mathbb{Q}(q)$-algebra $U$, denoted also by $\tau$, which sends $E_i \mapsto E_{\tau i}$, $F_i \mapsto F_{\tau i}$ and $K_\mu \mapsto K_{\tau \mu}$.

### 3.2 Definition

We recall the definition of QSP $(U, U')$, where $U'$ is a coideal subalgebra of $U$ from [Let99, Kol14, BK15, BK19]; also cf. [BW18b, §3.3].

**Definition 3.1.** The algebra $U'$, with parameters

$$\varsigma_i \in \mathbb{Z}[q, q^{-1}], \quad \kappa_i \in \mathbb{Z}[q, q^{-1}], \quad \text{for } i \in I_0,$$

is the $\mathbb{Q}(q)$-subalgebra of $U$ generated by the following elements:

$$F_i + \varsigma_i T_{w_i}(E_{\tau i})K_i^{-1} + \kappa_i K_i^{-1} (i \in I_0),$$

$$K_\mu (\mu \in Y^\vee), \quad F_i (i \in I_0), \quad E_i (i \in I_0).$$

The parameters are required to satisfy Conditions (3.4)–(3.7):

$$\kappa_i = 0 \quad \text{unless } \tau(i) = i, \quad \langle i, j' \rangle = 0 \quad \forall j \in I_0,$$

and $\langle k, j' \rangle \in 2\mathbb{Z}$ $\forall k = \tau(k) \in I_0$ such that $\langle k, j' \rangle = 0$ for all $j \in I_0$; (3.4)

$$\kappa_i = \kappa_i;$$ (3.5)

$$\varsigma_i = \varsigma_{\tau i} \text{ if } i \cdot \theta(i) = 0;$$ (3.6)

$$\varsigma_{\tau i} = (-1)^{\langle 2\rho^\vee, i' \rangle} q_i^{-(i, 2\rho^\vee + w_0w^\vee i')} \varsigma_i.$$ (3.7)

By definition, the algebra $U'$ contains $U_{I_0}$ as a subalgebra.

**Remark 3.2.** Note that the conditions (3.4)–(3.7) for the parameters are as general as in [BK19], except the integral requirement in (3.3) which is necessary for an integral form of $U'$. We are going to develop the theory of $s$-canonical basis for $U'$ in this full generality. This is a significant improvement than the constraint that $\varsigma_i \in \pm q^{\mathbb{Z}}$ in [BW18b, Definition 3.5] even in finite type, where (3.7) was written as $\varsigma_{\tau i} \varsigma_i = (-1)^{\langle 2\rho^\vee, i' \rangle} q_i^{-(i, 2\rho^\vee + w_0w^\vee i')} \varsigma_i$.

**Remark 3.3.** Our parameter $\varsigma_i$ is related to the notation of parameters in [Kol14] and [BK15] via $\varsigma_i = -s(\tau(i))\varsigma_i$; we shall never need these additional parameters separately. The parameters $\varsigma_i$ can always be chosen to be $\varsigma_i \in q^{\mathbb{Z}}$ by [BK15, Remark 3.14], once [BK15, Conjecture 2.7] (i.e. Conjecture 3.5 below) is established in Theorem 4.1. This allows the specialization at $q = 1$ of the QSP $(U, U')$ to the corresponding symmetric pair, justifying the terminology of QSP.

Set

$$\mathcal{A}^{\text{inv}} = \{ f \in \mathcal{A} \mid \bar{f} = f \}.$$

**Remark 3.4.** The coefficient $(-1)^{\langle 2\rho^\vee, i' \rangle} q_i^{-(i, 2\rho^\vee + w_0w^\vee i')}$ has been computed explicitly in [BW18b, Lemma 3.10] in finite type. The Satake diagrams of symmetric pairs of real rank one in finite type are listed in [BW18b, §3, Table 1]. The values of [BW18b, §3, Table 3] of $\varsigma_i$ for QSPs of real rank one are now updated to become the following table, taking into account the relaxed conditions on parameters $\varsigma_i$ in (3.3).
Canonical bases arising from quantum symmetric pairs

| AI | AIII | AIIV, n ≥ 2 |
|----|------|-------------|
| \(q^{-1} \cdot A^{\text{inv}}\) | \(q \cdot A^{\text{inv}}\) | \(c_1 = c_2 \in A^{\text{inv}}\) |
| BII, n ≥ 2 | CII, n ≥ 3 | DII, n ≥ 4 |
| \(q^{2n-3} \cdot A^{\text{inv}}\) | \(q^{n-1} \cdot A^{\text{inv}}\) | \(q^{n-2} \cdot A^{\text{inv}}\) |
| FII | |
| \(q^5 \cdot A^{\text{inv}}\) |

It is sometimes convenient to set

\[
B_i = \begin{cases} 
F_i + q_i T_{\omega_i}(E_{\tau_i}) K_i^{-1} + \kappa_i K_i^{-1} & \text{if } i \in I_0; \\
F_i & \text{if } i \in I_. \end{cases} \quad (3.8)
\]

3.3 For \(i \in I_0\), we define

\[
Z_i = \frac{1}{q_i - q_i} r_i (T_{\omega_i}(E_i)) \in U^+_{\omega_0}. \quad (3.9)
\]

It is known (cf. [BK15]) that \(Z_i \neq 0\) (we thank Kolb for explaining this fact in detail). Balagovic–Kolb showed in [BK15, Proposition 2.5] that

\[
\sigma \tau (Z_i) = \nu_i Z_i \text{ with } \nu_i \in \{1, -1\}, \quad \forall i \in I_0. \quad (3.10)
\]

Conjecture 3.5 [BK15, Conjecture 2.7]. We have \(\sigma \tau (Z_i) = Z_i\), that is, \(\nu_i = 1\), for all \(i \in I_0\).

This conjecture will be established in full generality as Theorem 4.1 in § 4. It is known [BK15, Proposition 2.3] that \(\nu_i = 1\) (that is, [BK15, Conjecture 2.7] holds) for \((U, U^0)\) of finite type.

We shall assume that Theorem 4.1 holds in the remainder of this section.

3.4 Throughout the paper, we make the following basic assumptions:

\[|\langle i, j \rangle| \leq 3, \quad \forall i, j \in I, \quad \text{or} \quad I_0 = \emptyset. \quad (3.11)\]

Remark 3.6. Condition (3.11) is imposed so that explicit Serre-type defining relations for \(U^0\) are available, and then the bar involution on \(U^0\) can be verified [BK15, CLW20]. It is generally expected that the assumptions (3.11) can be removed eventually.

The existence of the bar involution on \(U^0\) below was predicted in [BW18a].

Lemma 3.7 [BK15, CLW20]. There is a unique anti-linear bar involution of the \(Q\)-algebra \(U^0\), denoted by \(\bar{\psi}\), such that

\[
\psi_i(q) = q^{-1}, \quad \psi_i(B_i) = B_i (i \in I), \quad \psi_i(E_i) = E_i (i \in I_0), \quad \psi_i(K_\mu) = K_{-\mu} (\mu \in Y^0).
\]

We recall the following theorem (cf. [BW18a, Theorem 2.3], [BK19, Theorem 6.10] and [BW18b, Theorem 4.8, Remark 4.9]).

Theorem 3.8. There exists a unique family of elements \(Y_\mu \in U^0_{\mu}\), such that \(Y_0 = 1\) and \(Y = \sum_\mu Y_\mu\) satisfies the following identity (in \(\hat{U}\)):

\[
\psi_i(u)Y = Y \psi_i(u), \quad \text{for all } u \in U^0. \quad (3.12)
\]

Moreover, \(Y_\mu = 0\) unless \(\mu^0 = -\mu \in X\).
The formulation of the quasi-$K$-matrix $\Upsilon$ (called sometimes an intertwiner) was due to the present authors [BW18a]; its existence in full generality has been established in [BK19] (also cf. [BW18b, Remark 4.9]).

Remark 3.9. Lemma 3.7 and Theorem 3.8 were established in the literature under the assumption that $\nu_i = 1$ for $i \in \mathbb{I}_0$; this assumption is now removed unconditionally thanks to Theorem 4.1. Theorem 3.8 for $U^+$ with $\mathbb{I}_* = \emptyset$ is new as its bar involution has only recently been established in [CLW20], the usual proof in [BK19]–[BW18a] carries over.

3.5 We define the modified version (i.e. idempotented version) $\hat{\mathcal{U}}^+$ of the $t$-quantum groups following [Lus10, Part IV]. Some extra care is required because the pairing between $X_i$ and $Y^*$ is not perfect in general, and we expand and correct slightly the definition given in [BW18b]. We thank Hideya Watanabe for helpful remarks and suggestions.

We define an $X_i$-grading on $\mathcal{U}^+$ by assigning $B_i$ ($i \in \mathbb{I}$), $E_j$ ($j \in \mathbb{I}_*$) and $K_{\mu}$ ($\mu \in Y^*$) degree $-\overline{t}$, $\overline{t}$ and $0$, respectively. One can easily check this is well-defined via Definition 3.1. We denote $\pi_{X_i,\lambda''} : \mathcal{U}^+ \to \mathcal{U}_i^+$ the natural quotient map, where $\pi_{X_i,\lambda''}(x) = 0$ if $x \notin \mathcal{U}^+(\lambda'' - \lambda')$. We write $1_{X_i} = \pi_{X_i,\lambda''}(1)$ for the orthogonal idempotents.

Following [Lus10, Part IV], we then define an associative $\mathbb{Q}(q)$-algebra structure (without unit) on

$$\hat{\mathcal{U}}^+ = \bigoplus_{\lambda',\lambda'' \in X_i} \mathcal{U}_{\lambda''}^+.$$

Moreover, the modified quantum group $\hat{\mathcal{U}}$ is naturally a $(\hat{\mathcal{U}}^+, \hat{\mathcal{U}}^+)$-bimodule. For any $u \in \hat{\mathcal{U}}^+$ (or $\hat{\mathcal{U}}^+$) and $1_{X_i} \in \hat{\mathcal{U}}$, we shall denote by $u1_{X_i} \in \hat{\mathcal{U}}$ the action of the $u$ on $1_{X_i}$. The coproduct of $\mathcal{U}^+$ induces a similar structure on $\hat{\mathcal{U}}^+$ similar to [Lus10, 23.1.5].

The bar involution on $\mathcal{U}^+$ in Lemma 3.7 induces a bar involution $\psi_i$ on the $\mathbb{Q}$-algebra $\hat{\mathcal{U}}^+$ such that $\psi_i(q) = q^{-1}$ and

$$\psi_i(B_i1_{\zeta}) = B_i1_{\zeta} \quad (i \in \mathbb{I}), \quad \psi_i(E_i1_{\zeta}) = E_i1_{\zeta} \quad (i \in \mathbb{I}_*), \quad \psi_i(1_{\zeta}) = 1_{\zeta} \quad (\zeta \in X_i).$$

Recall $\mathcal{P} = \mathcal{P}_1$ and $\hat{\mathcal{P}}$ from §2.5. Let $\mathcal{U}^+(w^*)_<$ be the two-sided ideal of $\mathcal{U}^+$ generated by $E_i$ for $i \in \mathbb{I}_0$. The composition map, denoted by $p_i = p_i,\lambda$, $\hat{\mathcal{U}}^+1_{\lambda} \longrightarrow \hat{\mathcal{U}}1_{\lambda} \longrightarrow \hat{\mathcal{U}}1_{\lambda}/\mathcal{U}^+(w^*)_>1_{\lambda} \longrightarrow \hat{\mathcal{P}}1_{\lambda}$, (3.13)

is a $\mathbb{Q}(q)$-linear isomorphism; cf. [BW18b, §3.8].

Definition 3.10. We define $\mathcal{A}\hat{\mathcal{U}}^+$ to be the set of elements $u \in \hat{\mathcal{U}}^+$, such that $u \cdot m \in \mathcal{A}\hat{\mathcal{U}}^+$ for all $m \in \mathcal{A}\hat{\mathcal{U}}^+$. Then $\mathcal{A}\hat{\mathcal{U}}^+$ is clearly a $\mathcal{A}$-subalgebra of $\hat{\mathcal{U}}^+$ which contains all the idempotents $1_{\zeta} (\zeta \in X_i)$, and $\mathcal{A}\hat{\mathcal{U}}^+ = \bigoplus_{\zeta \in X_i} \mathcal{A}\hat{\mathcal{U}}^+1_{\zeta}$. 

1518
3.6 Symmetries of $U^*$ and $U^3$-modules

3.6.1 Recall the braid group operators $T_{i,e}^,$ and $T_{i,e}''$, for $e = \pm 1$, from \cite{lus10}.

**Proposition 3.11** \cite{bw18b, Theorem 4.2 and Propositions 4.6 and 4.13}. Let $i \in \mathbb{I}_*$, $j \in \mathbb{I}_0$, and $e = \pm 1$.

1. The braid group operators $T_{i,e}^,$ and $T_{i,e}''$ restrict to isomorphisms of $U^*$.
2. We have $T_{i,e}''(Y) = Y$ and $T_{i,e}''(Y) = Y$.
3. Assume $\varsigma_j = q_j^{-1}$ if $\kappa_j \neq 0$ and $\varsigma_j = q_j$ for the parameters. Then the anti-involution $\varphi$ on $U$ restricts to an anti-involution $\varphi$ on $U^*$.

3.6.2 We define the anti-linear involution $\sigma'_i$ of $U$ as

$$\sigma'_i = \sigma \circ \tau \circ \psi : U \rightarrow U.$$ 

**Lemma 3.12**. We have $\sigma'_i(T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1}) = \varsigma_{\tau i}T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1})$, for $i \in \mathbb{I}_0$.

**Proof.** We have

$$\sigma'_i(T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1}) = \sigma \circ \tau(\varsigma_i T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i})) = \sigma(\varsigma_i T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i})) = \varsigma_i \tilde{K}_{\tau i}^{-1} \sigma T_{\mathbb{I}_*,+1}(E_{\tau i}) = \varsigma_i \tilde{K}_{\tau i}^{-1} \frac{1}{\varsigma_i q_i^{-1}} T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1}) = \varsigma_i \tilde{K}_{\tau i}^{-1} \frac{1}{\varsigma_i q_i^{-1}} T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1}) = \varsigma_i \tilde{K}_{\tau i}^{-1} \frac{1}{\varsigma_i q_i^{-1}} T_{\mathbb{I}_*,+1}(E_{\tau i})(\tilde{K}_{\tau i}^{-1}).$$

Here the equality $(a)$ follows from \cite{lus10, § 37.2.4}, $(b)$ follows from \cite{lus10, § 37.2.4} and a similar computation as in \cite{bw18b, Corollary 4.5} and, finally, $(c)$ follows from (3.7).

**Proposition 3.13.** We have a $\mathbb{Q}(q)$-linear anti-involution $\sigma_i = \psi_i \circ \sigma'_i$ of the algebra $U^*$, such that $\sigma_i(F_i) = F_{\tau i}$, $\sigma_i(E_i) = E_{\tau i}$ ($i \in \mathbb{I}_*$), $\sigma_i(B_i) = B_{\tau i}$ ($i \in \mathbb{I}_0$), and $\sigma_i(K_{\mu}) = K_{-\tau \mu}$ ($\mu \in Y'$).

**Proof.** Note that $\sigma'_i(F_i) = F_{\tau i}$, $\sigma'_i(E_i) = E_{\tau i}$ and $\sigma'_i(K_{\mu}) = K_{-\tau \mu}$, for all $i \in \mathbb{I}_*, \mu \in Y$. It follows that

$$\sigma'_i(q) = q^{-1}, \sigma'_i(B_i) = B_{\tau i} (i \in \mathbb{I}_0), \quad \sigma'_i(K_{\mu}) = K_{-\tau \mu} (\mu \in Y'),$$

and, hence, $\sigma'_i$ restricts to an anti-linear anti-involution on the subalgebra $U^*$.

The proposition follows immediately from the above and the definition of $\psi_i$ in Lemma 3.7.

Note $\sigma_i$ takes a particular neat form when $\tau = \text{id}$, and it is strikingly similar to the anti-involution $\sigma$ on $U$. 

1519
3.6.3 Following [BW18b, § 4.5], we consider the automorphism obtained by the composition

\[ \vartheta = \sigma \circ \varphi \circ \tau : U \longrightarrow U, \]

which sends

\[ \vartheta(E_i) = q_{\tau_i}F_{\tau_i}\bar{K}_{-\tau_i}, \quad \vartheta(F_i) = q_{\tau_i}E_{\tau_i}\bar{K}_{\tau_i}, \quad \vartheta(K_\mu) = K_{-\tau_\mu}. \] (3.14)

For any U-module \( M \), we define a new U-module \( \vartheta M \) as follows: \( \vartheta M \) has the same underlying \( \mathbb{Q}(q) \)-vector space as \( M \), but we shall denote a vector in \( \vartheta M \) by \( \vartheta m \) for \( m \in M \), and the action of \( u \in U \) on \( \vartheta M \) is now given by \( u^\vartheta m = \vartheta(\vartheta^{-1}(um)) \). Hence, we have

\[ \vartheta(u)^\vartheta m = \vartheta(um), \quad \text{for } u \in U, m \in M. \]

As \( \vartheta M \) is simple if the U-module \( M \) is simple, one checks by definition that

\[ \vartheta L(\lambda) \cong \omega(L(\lambda^\tau)). \]

**Remark 3.14.** Note that \( \vartheta \) is not an automorphism of the subalgebra \( \tilde{U} \) in general (as we allow more general parameters \( \varsigma_i \)). Nevertheless, \( M \) and \( \vartheta M \) are both \( U \)-modules and, hence, \( \tilde{U} \)-modules by restriction.

Let \( g : X \longrightarrow \mathbb{Q}(q) \) be such that [BW18b, (4.16) and (4.17)] hold. The function \( g \) induces a \( \mathbb{Q}(q) \)-linear map from any weight \( U \)-module \( M = \oplus_{\mu \in X} M_\mu \) to itself:

\[ \tilde{g} : M \longrightarrow M, \quad \tilde{g}(m) = g(\mu)m, \quad \text{for } m \in M_\mu. \] (3.15)

Recall that we denote by \( \eta_\lambda \) the highest weight vector in \( L(\lambda) \). Let \( \eta_\lambda^* \) be the unique canonical basis element in \( L(\lambda) \) of weight \( w_\lambda \). Recall \( \lambda^\tau = \tau(\lambda) \). Let \( \mathcal{C}_{hi} \) be the BGG category of \( U \)-modules with weights in \( X \) and \( \mathcal{C}' \) be the full subcategory of \( \mathcal{C}_{hi} \) consisting of integrable \( U \)-modules [Lus10, 3.4.7 and 3.5.1].

**Theorem 3.15** (cf. [BW18b, Theorem 4.18] and [BK19, Theorem 7.5]). For any integrable \( U \)-module \( M = \oplus_{\mu \in X} M_\mu \) in \( \mathcal{C}' \), we have the following isomorphism of \( \tilde{U} \)-modules

\[ \mathcal{T} := \Upsilon \circ \tilde{g} \circ T_{\omega}^{-1} : M \longrightarrow \vartheta M. \]

In particular, we have the isomorphism of \( \tilde{U} \)-modules

\[ \mathcal{T} : L(\lambda) \longrightarrow \omega(L(\lambda^\tau)), \quad \eta_\lambda^* \mapsto \xi_{-\lambda^\tau}. \]

We note that the function \( g \) can be chosen such that \( \mathcal{T} \) is an isomorphism of the \( A \)-form \( \mathcal{A}L(\lambda) \longrightarrow \omega(\mathcal{A}L(\lambda^\tau)) \) once Corollary 6.8 is established.

**Proof.** The same proof as [BW18b, Theorem 4.18] applies with minor modifications as specified in the following.

The definition of the weight function \( g \) in [BW18b, (4.15)] remains the same. However, we should replace \( \varsigma_i^{-1} \) by \( \varsigma_i^\tau \) for the first identity in [BW18b, Lemma 4.16], thanks to our relaxed conditions on parameters (3.3). Otherwise, the proof of the lemma remains identical, and the original proof for [BW18b, Theorem 4.18] applies here verbatim. \( \square \)
4. Fundamental lemma for QSP

The goal of this section is to establish the Balagovic–Kolb conjecture [BK15, Conjecture 2.7] in full generality. The main tool here is Lusztig’s theory of based modules and cells for $U_{\mathfrak{i}}$ as developed in [Lus10, Chapters 27 and 29].

Recall $Z_i$ from (3.9), and recall from (3.10) that $\sigma\tau(Z_i) = \nu_i Z_i$ for some $\nu_i \in \{1, -1\}$ and any $i \in \mathbb{I}_0$. Balagovic–Kolb conjectured [BK15, Conjecture 2.7] (which is recalled in Conjecture 3.5) that $\nu_i = 1$ for all $i$ for $U$ of Kac–Moody type. This conjecture looks rather technical and innocent, but has been critical in several advances in the theory of QSP; for such reasons, we referred to the Balagovic–Kolb conjecture as the fundamental lemma of QSP.

Several crucial results, such as the existence of bar involution with the parameters $\varsigma_i$ chosen to be in $q^\mathbb{Z}$ and the existence of quasi-$K$-matrix, are established only under the assumption of this conjecture. Without the validity of the conjecture, it is unclear if suitable parameters for the QSP $(U, U')$ can be chosen to ensure the bar involution, quasi-$K$-matrix and a meaningful specialization at $q = 1$ to the usual symmetric pair. The conjecture was only known (based on results in [BK15]) to hold for $U$ of locally finite type, in the sense that all the real rank-one Levi subalgebras of $U'$ are of finite type.

**Theorem 4.1** (Fundamental lemma of QSP). For $U$ of an arbitrary Kac–Moody type, we have $\nu_i = 1$, that is, $\sigma\tau(Z_i) = Z_i$, for all $i \in \mathbb{I}_0$.

Let us prepare several lemmas. Denote

$$X_{\mathfrak{i}}^+ = \{ \lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N}, \ \forall i \in \mathbb{I}_0 \}.$$  

Let us fix $i \in \mathbb{I}_0$. We write $\alpha_i = i' \in X$ for notational consistency with [BK15]. Then we may and shall regard $-\alpha_i \in X_{\mathfrak{i}}^+$ thanks to $i \cdot j \leq 0$ for $j \in \mathbb{I}_0$. Recall that $\xi_{\alpha_i}$ denotes the lowest weight vector in $\omega L_{\mathfrak{i}}(-\alpha_i) = L_{\mathfrak{i}}(w\alpha_i)$.

**Lemma 4.2.** The element $Z_i \in U_{\mathfrak{i}}^+$ acts on $L_{\mathfrak{i}}(w\alpha_i)$ as a nonzero map. In particular, we have $Z_i(\xi_{\alpha_i}) \neq 0$. (As $Z_i \in U_{\mathfrak{i}}^{w\alpha_i-\alpha_i}$ by (3.9), $Z_i(\xi_{\alpha_i})$ is of highest weight.)

**Proof.** We owe this proof to Gail Letzter and her suggestions and references. The statement follows from a very special case of general results of Joseph and Letzter.

Let $\lambda \in X_{\mathfrak{i}}^+$. Denote by $K \in U_{\mathfrak{i}}^0$ such that $KE_j K^{-1} = q_j^{\langle j, \lambda \rangle} E_j$, for all $j \in \mathbb{I}_0$. (This $K$ is relevant to [BK15, Proposition 2.4].) Note the notation $\tau(\lambda)$ in [JL94, JL96] translates to $K^2$ here. It follows by [JL94, Corollary 3.3] and [JL96, (8.6)] that $\text{ad}(U_{\mathfrak{i}}^+)(K^2) \longrightarrow \text{End}(L_{\mathfrak{i}}(\lambda))$ is injective. (In fact, this is an isomorphism for dimension reasons.)

Setting $\lambda = w\alpha_i$, we have $K = \tilde{K}_i$, and $Z_i\tilde{K}_i^2 \in \text{ad}(U_{\mathfrak{i}}^+)(\tilde{K}_i^2)$ by [Kol14, (4.4)] (or [BK15, (2.21)]); by the injectivity above $Z_i\tilde{K}_i^2$ and, hence, $Z_i$ have nonzero images in $\text{End}(L_{\mathfrak{i}}(w\alpha_i))$. The lemma is proved.

Recall from [Lus10, 29.1.2] the two-sided ideals $\hat{U}_{\mathfrak{i}}[w\alpha_i]$ and $\hat{U}_{\mathfrak{i}}[-w\alpha_i]$ of $\hat{U}_{\mathfrak{i}}$ are based submodule of $\hat{U}_{\mathfrak{i}}$. In addition, we have that $\hat{U}_{\mathfrak{i}}[w\alpha_i] \subset \hat{U}_{\mathfrak{i}}[-w\alpha_i]$ and, moreover, $\hat{U}_{\mathfrak{i}}[w\alpha_i]/\hat{U}_{\mathfrak{i}}[w\alpha_i]$ admits a canonical basis. The natural action of $\hat{U}_{\mathfrak{i}}[w\alpha_i]$ on $L(w\alpha_1)$ factor through the projection

$$\text{pr} : \hat{U}_{\mathfrak{i}}[w\alpha_i] \longrightarrow \hat{U}_{\mathfrak{i}}[w\alpha_i]/\hat{U}_{\mathfrak{i}}[w\alpha_i].$$
and it further induces the following \( \mathbb{Q}(q) \)-algebra isomorphism (cf. [Lus10, Proposition 29.2.2]).

\[
\begin{array}{ccc}
\hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] & \xrightarrow{\text{pr}} & \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i]/\hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \\
\end{array}
\]

(4.1)

**Lemma 4.3.** We have:

1. \( Z_i 1_{\alpha_i} \in \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \);
2. \( \text{pr}(Z_i 1_{\alpha_i}) \neq 0 \) in \( \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i]/\hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \);
3. there exists a unique canonical basis element, denoted by \( b_\bullet \), in the based module \( \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i]/\hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \) of weight \( w_\bullet \alpha_i - \alpha_i \); hence, \( \text{pr}(Z_i 1_{\alpha_i}) = c \cdot b_\bullet \), where \( c \in \mathbb{Q}(q)^\times \).

**Proof.** Assume \( Z_i 1_{\alpha_i} \) acts on \( L(\lambda_2) \), for some \( \lambda_2 \in X_\mathfrak{sl}_n^+ \), by a nonzero map. Then we must have \( \alpha_i \geq w_\bullet \lambda_2 \) or, equivalently, \( \lambda_2 \geq w_\bullet \alpha_i \). Hence, by [Lus10, Lemma 29.1.3], we have \( Z_i 1_{\alpha_i} \in \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \), whence part (1).

By Lemma 4.2, we have \( (Z_i 1_{\alpha_i})(\xi_{\alpha_i}) \neq 0 \), and it follows by (4.1) that \( \text{pr}(Z_i 1_{\alpha_i}) \neq 0 \), whence part (2).

As the weight subspace of \( \text{End}(L(w_\bullet \alpha_i)) \) of weight \( w_\bullet \alpha_i - \alpha_i \) is clearly one-dimensional, by the isomorphism in (4.1) we see that \( \hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i]/\hat{U}_\mathfrak{sl}_n[\geq w_\bullet \alpha_i] \) has a one-dimensional weight subspace of weight \( w_\bullet \alpha_i - \alpha_i \), and part (3) follows. \( \square \)

Now we are ready to prove the fundamental lemma of QSP.

**Proof of Theorem 4.1.** The anti-involution \( \sigma \) sends \( \hat{U}_\mathfrak{sl}_n[\geq \lambda] \) to \( \hat{U}_\mathfrak{sl}_n[\leq -w_\bullet \lambda] \), for any \( \lambda \in X_\mathfrak{sl}_n^+ \), according to [Lus10, Lemma 29.3.1]. By the admissible condition (2) in §3.1, the action of the diagram involution \( \tau \) on \( \mathfrak{I}_\bullet \) coincides with the action of \( -w_\bullet \) and, thus, \( \tau \) induces an isomorphism \( \hat{U}_\mathfrak{sl}_n[\geq -w_\bullet \lambda] \rightarrow \hat{U}_\mathfrak{sl}_n[\geq \lambda] \), for any \( \lambda \in X_\mathfrak{sl}_n^+ \).

Hence, the composition \( \sigma \tau \) preserves \( \hat{U}_\mathfrak{sl}_n[\geq \lambda] \), for each \( \lambda \in X_\mathfrak{sl}_n^+ \). It follows from [Kas94, Theorem 4.3.2] that \( \sigma \) preserves the canonical basis of \( \hat{U}_\mathfrak{sl}_n \). The diagram automorphism \( \tau \) preserve the canonical basis of \( \hat{U}_\mathfrak{sl}_n \) as well. Hence, the canonical basis of \( \hat{U}_\mathfrak{sl}_n[\geq \lambda] \) is stable under the action of \( \sigma \tau \).

Therefore, by Lemma 4.3(3), we must have \( \sigma \tau(b_\bullet) = b_\bullet \) since \( \sigma \tau \) is weight-preserving.

Hence, we have \( \text{pr}(\sigma \tau(Z_i 1_{\alpha_i})) = \sigma \tau(\text{pr}(Z_i 1_{\alpha_i})) = \sigma \tau(c \cdot b_\bullet) = c \cdot b_\bullet \). On the other hand, by (3.10), we have \( \text{pr}(\sigma \tau(Z_i 1_{\alpha_i})) = \text{pr}(\nu_i \cdot Z_i 1_{\alpha_i}) = \nu_i c \cdot b_\bullet \). By comparison, we conclude that \( \nu_i = 1 \). \( \square \)

**5. The \( \nu \)-divided powers**

**5.1** This section is devoted to a constructive proof of the existence of the so-called \( \nu \)-divided powers.

**Theorem 5.1.** For any \( i \in \mathfrak{I} \) and \( \zeta \in X_\mathfrak{sl}_n \), there exists an element \( B_{i,\zeta}^{(n)} \in \mathcal{A}\hat{U}^i 1_\zeta \) satisfying the following properties:

1. \( \psi_i(B_{i,\zeta}^{(n)}) = B_{i,\zeta}^{(n)} \);
2. \( B_{\hat{i},\zeta}^{(n)} 1_\lambda = F_{i}^{(n)} 1_\lambda + \sum_{a < n} F_{i}^{(a)} \mathcal{A} U^+ 1_\lambda \), for \( 1_\lambda \in \mathcal{A}\hat{U}^i \) with \( \hat{\lambda} = \zeta \).
When $i \in \mathbb{I}_*$, we can simply set $B_{i,\zeta}^{(n)} = F_{i,\zeta}^{(n)} 1_\zeta$.

We shall explicitly construct the elements $B_{i,\zeta}^{(n)} \in \mathcal{A} \tilde{U}^i 1_\zeta$ with the desired properties in Theorem 5.1 for $i \in \mathbb{I}_0$ by separating the rank-one into three classes (this is a much rougher division than [BW18a], where eight cases of quantum groups of finite type of real rank one are enumerated). Two classes are essentially known from [BW18a, BW18b] are treated in §5.5. Most of this section (§§5.2–5.4) deals with the most nontrivial (potentially nonfinite type) class when $\tau(i) = i \neq w_i i$ (Theorem 5.1 in this case is summarized below as Theorem 5.13); the formula for this class is new even in finite type and allows a major simplification of [BW18b].

**Remark 5.2.** The elements $B_{i,\zeta}^{(n)} \in \mathcal{A} \tilde{U}^i 1_\zeta$ in Theorem 5.1 will be called $\iota$-divided powers. The element $B_{i,\zeta}^{(n)}$ should be regarded as a leading term of the $\iota$-canonical basis element $(1^{\otimes \iota} F_{i}^{(n)})$ established in Theorem 7.2, because the $\iota$-canonical basis is hard to compute and the equality $(1^{\otimes \iota} F_{i}^{(n)}) = B_{i,\zeta}^{(n)}$ remains uncertain in general.

We also refer to $E_{i}^{(n)} 1_\zeta$, for $i \in \mathbb{I}_*$, as $\iota$-divided powers.

### 5.2 A q-boson algebra

Let $i \in \mathbb{I}_0$ be such that $\tau(i) = i \neq w_i i$. Then we have $T_{w_i}(E_i) \neq E_i$. This implies that $i \in I_{ns}$ in the notation of [Kol14], and it follows that $\kappa_i = 0$ always.

#### 5.2.1 For such an $i \in \mathbb{I}_0$, recalling $Z_i$ from (3.9), for convenience in the following we define $\mathfrak{Z}_i = c_i Z_i$, that is,

$$3_i = \frac{s_i}{q_i - q_i} r_i(T_{w_i}(E_i)) \in \mathcal{U}_i^+.$$  \hspace{1cm} (5.1)

**Remark 5.3.** An element $Z_i = -s(\tau(i)) r_i(T_{w_i}(E_i))$ (in case $\tau = \text{id}$) was introduced and studied in depth in [Kol14] (see [BK15, (3.10)]). Our $3_i$ is related to $Z_i$ by

$$3_i = \frac{1}{q_i - q_i} c_i Z_i.$$  \hspace{1cm} (5.2)

Indeed, for the theory of QSP throughout [BK15] and this paper, one only needs to use $Z_i$ instead of $3_i$.

**Lemma 5.4.** Let $i \in \mathbb{I}_0$ be such that $\tau(i) = i \neq w_i i$. We have:

1. $[F_i, s_i T_{w_i}(E_i)] = 3_i \tilde{K}_i$;
2. $i r_i(T_{w_i}(E_i)) = 0$;
3. $T_i T_{w_i}(E_i) \in \mathcal{U}_i^+$.

**Proof.** Recall [Lus10, 3.1.6] that, for $x \in \mathcal{U}_i^+$,

$$F_i x - x F_i = \frac{r_i(x) \tilde{K}_i - \tilde{K}_i^{-1} i r(x)}{q_i - q_i}.$$  \hspace{1cm} (5.3)

The equivalence between parts (1) and (2) follows from this. On the other hand, by [Lus10, Proposition 38.1.6], parts (2) and (3) are equivalent.

Thus, it suffices to prove part (3). The assumption $T_{w_i}(E_i) \neq E_i$ is equivalent to that $w_i \alpha_i \neq \alpha_i$, an so we have $w_i \alpha_i = \alpha_i + \sum_{j \in i} k_j \alpha_j \in \Phi_+ \setminus \{\alpha_i\}$, where $\Phi_+$ denotes the set of positive roots...
of the Kac–Moody algebra $\mathfrak{g}$. Hence, $s_i w_i \alpha_i = w_i \alpha_i - \langle i, w_i \alpha_i \rangle \alpha_i \in \Phi_+$. This implies $T_i T_{w_i}(E_i) = T_{s_i w_i}(E_i) \in U^+$. \hfill \Box

5.2.2 Set

$$Y_i = s_i T_{w_i}(E_i) \tilde{K}_i^{-1}.$$ 

Lemma 5.5. Let $i \in \mathbb{I}_o$ be such that $\tau(i) = i \neq w_i i$. Then:

1. $3_i$ commutes with $F_i, Y_i$;
2. $F_i Y_i - q_i^{-2} Y_i F_i = 3_i$.

Proof. (1) It follows from the presentation of $U^+$ that $[3_i, B_i] = 0$ (cf. [BK15, (3.15)] or [Kol14, (7.7)]). Hence, it follows by (3.8) that

$$[3_i, F_i] + [3_i, Y_i] = [3_i, B_i] = 0.$$ 

As $[3_i, F_i]$ and $[3_i, Y_i]$ have distinct weights, we must have $[3_i, F_i] = [3_i, Y_i] = 0$.

(2) This follows because $F_i Y_i - q_i^{-2} Y_i F_i = [F_i, s_i T_{w_i}(E_i)] \tilde{K}_i^{-1} = 3_i$. \hfill \Box

We focus on two algebras $H_i$ and $T_i$ in the following.

(i) Denote by $H_i$ the $\mathbb{Q}(q)$-subalgebra with 1 of $U$ generated by $F_i, Y_i, 3_i$; we shall call $H_i$ a $q$-boson algebra.

(ii) Denote by $T_i$ the $\mathbb{Q}(q)$-subalgebra with 1 of $H_i$ generated by $B_i = F_i + Y_i$ and $3_i$.

Clearly $T_i$ is also a $\mathbb{Q}(q)$-subalgebra of $U^+$, and the bar map $\bar{\psi}$ on $U^+$ preserves the algebra $T_i$ thanks to [BK15, Theorem 3.11]. The algebras $H_i$ and $T_i$ contain various integral elements of interest.

5.3 Integral elements

We continue to assume $i \in \mathbb{I}_o$ such that $\tau(i) = i \neq w_i i$.

5.3.1 Denote $3_i^{(n)} = 3_i^n/[n]!$ and $Y_i^{(n)} = Y_i^n/[n]!$, for $i \in \mathbb{I}_0$.

Proposition 5.6. We have $Y_i^{(n)} \in \mathcal{A}U$ and $Y_i^{(n)} \in \mathcal{A}U_{i^*}$.

Proof. Note

$$Y_i^{(n)} := \frac{k^k (T_{w_i}(E_i) \tilde{K}_i^{-1})^n}{[n]!} = \frac{k^k q^n T_{w_i}(E_i^{(n)}) \tilde{K}_i^{-n} \in \mathcal{A}U,$$

where $*$ denotes a suitable integer.

It remains to show that $3_i^{(n)} \in \mathcal{A}U$, because this implies that $3_i^{(n)} \in U_{i^*} \cap \mathcal{A}U = \mathcal{A}U_{i^*}$.

Assume $xy - q_i^{-2} yx = z$ and $z$ commutes with both $x$ and $y$. Denote $x^{(n)} = x^n/[n]!, y^{(n)} = y^n/[n]!$, and $z^{(n)} = z^n/[n]!$. Then one checks

$$x^{(n)} y^{(m)} = \sum_{a=0}^{n} q_i^{-2nm+(n+m)a-(1/2)a(a-1)} y^{(m-a)} x^{(n-a)} z^{(a)}. \quad (5.3)$$
This is a variant of [Kas91, (3.1.2)], which corresponds to our formula by specializing $z = 1$ and $z^{(n)} = [a]_i^{-1}$ for all $a$. We rewrite the formula (5.3) for $m = n$ as

$$z^{(n)} = d_i^{(1/2)n(n-1)} \left( \sum_{a=0}^{n-1} q_i^{-2n^2+2na-(1/2)a(a-1)} y(n-a) x(n-a) z^{(a)} - x^{(n)} y^{(n)} \right).$$  

(5.4)

By induction on $n$ and (5.4), we conclude that $z^{(n)} \in \mathcal{A} \mathbf{U}$ if $x^{(k)}$ and $y^{(k)}$ lie in $\mathcal{A} \mathbf{U}$, for all $k$.

The above general formalism is applicable to $x = F_i$, $y = Y_i$ and $z = \mathbf{3}_i$, thanks to Lemma 5.5(2). Therefore, we conclude that $\mathbf{3}_i^{(n)} \in \mathcal{A} \mathbf{U}$. \hfill $\square$

**Lemma 5.7.** We have $Z_i/(q - q^{-1}) \in \mathbf{U}_1^s \cap \mathcal{A} \mathbf{U}$, and $\mathbf{3}_i/(q - q^{-1}) \in \mathbf{U}_1^s \cap \mathcal{A} \mathbf{U}$.

**Proof.** If suffices to prove the first statement as $\mathbf{3}_i = c_i \mathbf{Z}_i$. By [BK15, Proposition 3.5], (recalling $Z_i = -s(\tau(i)) r_i(T_{w_i}(E_i))$), we have

$$\psi(r_i(T_{w_i}(E_i))) = \nu_i \ell_i r_i(T_{w_i}(E_i)) = \ell_i r_i(T_{w_i}(E_i)),$$  

(5.5)

where $\nu_i = 1$ thanks to Theorem 4.1. By definition (3.9), $Z_i = (1/(q_i^{-1} - q_i)) r_i(T_{w_i}(E_i))$ and, hence,

$$\psi(Z_i) = -\ell_i Z_i.$$  

(5.6)

We recall $\ell_i = q^{(\alpha_i, \alpha_i - w_i \alpha_i - 2 \rho_i)} = q^2 - (i, 2 \rho_i + w_i \ell_i)$ is always an even power of $q$ by [BK15, Remark 3.14]. Set $m = (\alpha_i, \alpha_i - w_i \alpha_i - 2 \rho_i)/2 \in \mathbb{Z}$, and $Z_i' = q_m Z_i$. Then

$$\psi(Z_i') = -Z_i'.$$

As $Z_i' \in \mathcal{A} \mathbf{U}_1^s$, we write $Z_i' = \sum_k d_k b_k$, a finite $\mathcal{A}$-linear combination of canonical basis elements $b_k$, where $d_k \in \mathcal{A}$. It follows by $\psi(Z_i') = -Z_i'$ that $\sum_k d_k b_k = -\sum_k d_k b_k$. Therefore, $\overline{d_k} = -d_k$ for each $k$. Writing $d_k = \sum_{n \in \mathbb{Z}} a_{k,n} q^n$ with $a_{k,n} \in \mathbb{Z}$, we conclude that $a_{k,-n} = -a_{k,n}$ for all $n$. Hence $d_k = \sum_{n > 0} a_{k,n} (q^n - q^{-n})$, and $d_k/(q - q^{-1}) \in \mathcal{A}$. The lemma is proved. \hfill $\square$

**Example 5.8.** Consider the following Satake diagram of type $BII$.

$$\begin{array}{c}
1 \\
\circ \\
2 \circ
\end{array}$$

We have $i(B_1) = F_1 + c_1 T_2(E_1) \tilde{K}_1^{-1}$. Recall $T_2(E_1) = E_2^{(2)} E_1 - q^{-1} E_2 E_1 E_2 + q^{-2} E_1 E_2^{(2)}$. A direct computation shows that $[F_1, T_2(E_1)] = (q^{-4} - q^{-2}) E_2^{(2)} \tilde{K}_1$. Hence, by Lemma 5.4 we have that

$$Z_1 = (q^{-4} - q^{-2}) E_2^{(2)}, \quad \frac{Z_1}{q - q^{-1}} = -q^{-3} E_2^{(2)} \in \mathcal{A} \mathbf{U}.$$  

5.3.2 For $n \geq 0$, we define

$$b_i^{(n)} = \sum_{a=0}^{n} q_i^{-a(n-a)} y_i^{(a)} f_i^{(n-a)} \in \mathcal{A} \mathbf{U}.$$  

(5.7)

**Lemma 5.9.** We have $b_i^{(n)} \in \mathbf{U} \cap \mathcal{A} \mathbf{U}$, for $i \in \mathbb{I}_0$.  

1525
Proposition 5.10. We have

$$[n]_q b_i^{(n)} = b_i^{(n-1)} B_i - q_i^{2-n} Z_i b_i^{(n-2)} ,$$

(5.8)

thanks to $B_i, Z_i \in U^i$.

It follows by Lemma 5.5 and an induction on $m$ that

$$F_i^{(m)} Y_i - q_i^{-2m} Y_i F_i^{(m)} = q_i^{1-m} Z_i F_i^{(m-1)} .$$

(5.9)

Recalling $B_i = F_i + Y_i$ and using (5.9) with $m = n - a - 1$, we have

$$b_i^{(n-1)} B_i = \sum_{a=0}^{n-1} q_i^{-a(n-a-1)} Y_i^{(a)} F_i^{(n-a-1)} B_i$$

$$= \sum_{a=0}^{n-1} q_i^{-a(n-a-1)} [n-a]_i Y_i^{(a)} F_i^{(n-a)}$$

$$+ \sum_{a=0}^{n-1} q_i^{-a(n-a-1)} Y_i^{(a)} (q_i^{2+2a-2n} Y_i F_i^{(n-a-1)} + q_i^{2+a-n} Z_i F_i^{(n-a-2)})$$

$$= \sum_{a=0}^{n} (q_i^{-a(n-a-1)} [n-a]_i + q_i^{-a(n-a-1)} [a]_i) Y_i^{(a)} F_i^{(n-a)}$$

$$+ q_i^{2-n} Z_i \sum_{a=0}^{n-2} q_i^{-a(n-a-2)} Y_i^{(a)} F_i^{(n-a-2)}$$

$$= \sum_{a=0}^{n} [n]_i q_i^{-a(n-a)} Y_i^{(a)} F_i^{(n-a)} + q_i^{2-n} Z_i \sum_{a=0}^{n-2} q_i^{-a(n-a-2)} Y_i^{(a)} F_i^{(n-a-2)}$$

$$= q_i^{2-n} Z_i b_i^{(n-2)} + [n]_i b_i^{(n)} .$$

To obtain the equality ($\ast$), we have shifted the index of the (first half of) the second summand from $a$ to $a - 1$. This proves the lemma. 

5.3.3 Recall the $\mathbb{Q}(q)$-subalgebra $T_i$ of $U^i \cap H_i$ from §5.2.2. It follows by induction on $n$ using (5.8) that $b_i^{(a)} \in T_i$ for all $n$. Denote by $A_i T_i$ the $A$-subalgebra (with 1) of $T_i$ generated by $\mathfrak{z}_i^{(a)}, b_i^{(a)}$, for all $n \geq 1$. It follows by (5.7) and Proposition 5.6 that $A_i T_i \subset A_i U$.

Proposition 5.10. We have

$$\psi_i(b_i^{(n)}) = \sum_{k \geq 0} q_i^{k(k+1)/2} \mathfrak{z}_i^{(k)} b_i^{(n-2k)} .$$

In particular, we have $\psi_i(b_i^{(n)}) \in U^i \cap A_i T_i \subset U^i \cap A_i U$.

Proof. By Lemma 5.9, the integrality statement for $\psi_i(b_i^{(n)})$ follows from the explicit formula in the lemma.
To prove the formula, we proceed by induction on \( n \). The case \( n = 1 \) is clear as \( b_i^{(1)} = B_i \) is \( \psi_i \)-invariant.

By [BK15, Theorem 3.11(2)], we have \( \psi_i(c_i Z_i) = q^{(i, \tau_i)} c_i Z_{\tau_i} \); this equality is transformed via (5.2) to be

\[
\psi_i(3_i) = \psi(3_i) = -q_i^2 3_i,
\]

thanks to \( 3_i \in U^+_i \).

Assume the statement holds for the cases of \( \psi_i(b_i^{(k)}) \) with \( k \leq n \), and we shall prove the formula for \( \psi_i(b_i^{(n+1)}) \). By (5.8), we have

\[
[n + 1] b_i^{(n+1)} = b_i^{(n)} B_i - q_i^{1-n} 3_i b_i^{(n-1)}.
\]

Applying the bar map \( \psi_i \) to the above identity and using inductive assumptions on \( \psi_i(b_i^{(n)}) \) and \( \psi_i(b_i^{(n-1)}) \), we have

\[
[n + 1] \psi_i(b_i^{(n+1)}) \]

\[
\overset{(5.10)}{=} \psi_i(b_i^{(n)}) B_i + q_i^{n-1} q_i^2 3_i \psi_i(b_i^{(n-1)})
\]

\[
= \sum_{k \geq 0} q_i^{k(k+1)/2} 3_i^{(k)} b_i^{(n-2k)} B_i + q_i^{n+1} \sum_{k \geq 0} q_i^{k(k+1)/2} 3_i^{(k)} b_i^{(n-1-2k)}
\]

\[
\overset{(*)}{=} \sum_{k \geq 0} q_i^{k(k+1)/2} ([n + 1 - 2k] 3_i^{(k)} b_i^{(n+1-2k)} + q_i^{1+2k-n} [k + 1] 3_i^{(k)} b_i^{(n-1-2k)})
\]

\[
+ q_i^{n+1} \sum_{k \geq 0} q_i^{k(k+1)/2} [k + 1] 3_i^{(k)} b_i^{(n-1-2k)}
\]

\[
= \sum_{k \geq 0} q_i^{k(k+1)/2} ([n + 1 - 2k] q_i^{-k} q_i^{-2k-n} + q_i^{n+1} [k] q_i) 3_i^{(k)} b_i^{(n+1-2k)}
\]

\[
= [n + 1] \sum_{k \geq 0} q_i^{k(k+1)/2} 3_i^{(k)} b_i^{(n+1-2k)}.
\]

In the identity \((*)\), we have used (5.8) for \( b_i^{(n-2k)} B_i \). The proof is complete.

\[ \square \]

**5.4 The \( \tau \)-divided powers \( B_i^{(n)} \)**

We continue to assume \( i \in \mathbb{S} \) such that \( \tau(i) = i \neq \omega_s i \). We define

\[
B_i^{(n)} = \frac{q \psi_i(b_i^{(n)}) - q^{-1} b_i^{(n)}}{q - q^{-1}}.
\]

It follows from Proposition 5.10 that

\[
B_i^{(n)} = b_i^{(n)} + \frac{q}{q - q^{-1}} \sum_{k \geq 1} q_i^{k(k+1)/2} 3_i^{(k)} b_i^{(n-2k)}.
\]

**Lemma 5.11.** Let \( f, g \in \mathcal{A} \) be relatively prime. Assume \( u \in U \) satisfies that \( u/f, u/g \in \mathcal{A} U \). Then we have \( u/(fg) \in \mathcal{A} U \).
Proof. Let \( \mathfrak{B} \) be an \( \mathcal{A} \)-basis of \( \mathcal{A}U \). Then \( u = \sum_{b \in \mathfrak{B}} h_b b \), for \( h_b \in \mathbb{Q}(q) \) and a finite subset \( \mathfrak{S} \subseteq \mathfrak{B} \). By assumption on \( u/f \) and \( u/g \), we have \( h_b/f, h_b/g \in \mathcal{A} \), for \( b \in \mathfrak{S} \). As \( f, g \in \mathcal{A} \) are relatively prime and \( \mathcal{A} \) is a unique factorization domain, we have \( h_b/(fg) \in \mathcal{A} \). Hence, \( u/(fg) = \sum_{b \in \mathfrak{B}} h_b/(fg) \cdot b \in \mathcal{A}U \).

**Lemma 5.12.** For \( n \geq 1 \), we have \( \psi_i(B_i^{(n)}) = B_i^{(n)} \) and, moreover,

\[
B_i^{(n)} \in U^i \cap A T_i \subset U^i \cap \mathcal{A}U.
\]

Proof. It follows by definition (5.11) that \( \psi_i(B_i^{(n)}) = B_i^{(n)} \).

Clearly, \([n]_q!\) and \((q - q^{-1})\) are relatively prime in \( \mathcal{A} \). By Lemma 5.7, we have \( 3^q_i/(q - q^{-1}) \in \mathcal{A}U \). Since \( 3^q_i = 3^q_i/[n]_q! \in \mathcal{A}U \) by Proposition 5.6, we conclude by Lemma 5.11 that \( 3^q_i/[n]_q!(q - q^{-1}) \in \mathcal{A}U \), i.e. \( 3^q_i/(q - q^{-1}) \in \mathcal{A}U \). Then it follows by (5.12) and Lemma 5.9 that \( B_i^{(n)} \in \mathcal{A}T_i \subset \mathcal{A}U \).

Summarizing the discussions in §§ 5.2–5.4, we have arrived at the following.

**Theorem 5.13.** Assume \( i \in \mathbb{I}_0 \) such that \( \tau(i) \neq i \neq w_0i \), and let \( \zeta \in X_i \). Then \( B_i^{(n)} = B_i^{(n)}1_\zeta \in \mathcal{A}U^i \) satisfies conditions (1) and (2) in Theorem 5.1.

Proof. Condition (1) in Theorem 5.1 is satisfied by \( B_i^{(n)} = B_i^{(n)}1_\zeta \in \mathcal{A}U^i \), thanks to Lemma 5.12. Condition (2) in Theorem 5.1 follows from the formula (5.12) for \( B_i^{(n)} \) and the definition (5.7) of \( b_i^{(n)} \) (note that \( 3_i^{(k)} \in \mathcal{A}U^i \) and \( Y_i^{(a)}1_{\mu} \in \mathcal{A}U^{i+1}1_{\mu} \)).

### 5.5 Additional \( i \)-divided powers

We consider the remaining two classes for \( i \)-divided powers associated to \( i \in \mathbb{I}_0 \).

#### 5.5.1 The class with \( \tau(i) \neq i \)

Let \( i \in \mathbb{I}_0 \) be such that \( \tau(i) \neq i \). Then \( \kappa_i = 0 \), and \( B_i = F_i + \varsigma_i T_{\varsigma_i}(E_{\varsigma_i})\tilde{K}_i^{-1} \). We write again \( Y_i := \varsigma_i T_{\varsigma_i}(E_{\varsigma_i})\tilde{K}_i^{-1} \). We have \( F_i Y_i - q_i^{-2} Y_i F_i = [F_i, Y_i]\tilde{K}_i^{-1} = 0 \).

Thanks to the \( q \)-binomial theorem, we define

\[
B_i^{(n)} = B_n^{(n)}1_\zeta = \sum_{a=0}^{n} q_i^{-a}\left(n-a\right)Y_i^{(a)} F_i^{(n-a)}1_\zeta \in \mathcal{A}U^i.
\]

#### 5.5.2 The class with \( \tau(i) = i = w_0i \)

Then \( B_i = F_i + \varsigma_i E_i\tilde{K}_i^{-1} + \kappa_i\tilde{K}_i^{-1} \), for such \( i \in \mathbb{I}_0 \).

This real rank-one QSP is of local type AI. When \( \varsigma_i = q_i^{-1} \), the existence of the \( i \)-canonical basis (= \( i \)-divided powers) in \( \tilde{U}^i \) parametrized by \( E_i^{(n)}1_\zeta \), for \( \zeta \in X_i \), was established in [BW18a].

With the more general parameter in the current setting, we can still obtain the precise inductive formula for the intertwiner \( \Upsilon \) in the real rank-one case as [BW18a, Lemma 4.6]. Afterwards, we can establish the \( i \)-canonical basis of \( \tilde{U}^i \) as in [BW18a], which will serve as the elements \( B_i^{(n)} \) desired in Theorem 5.1. Indeed, if we assume \( \Upsilon = \sum_{k \geq 0} c_k E_i^{(k)} \), then we have

\[
c_{k+1} = -q^{-k}(q - q^{-1})(q^2\varsigma_i[k]c_{k-1} + \kappa_i c_k), \quad \text{where } c_0 = 1, c_{-1} = 0.
\]

We clearly have \( c_k \in \mathcal{A} \) thanks to \( \varsigma_i, \kappa_i \in \mathcal{A} \).
The precise formulas for the $i$-divided powers ($=i$-canonical basis) in this real rank-one case can be found in [BeW18] for distinguished parameters $q_i = q_i^{-1}$ and $\kappa_i$ a $q_i$-integer. The explicit formulas for the $n$th $i$-divided powers with a general parameter $q_i$ (and $\kappa_i = 0$), for $n = 2$, can be computed, though they remain to be computed in general for $n > 2$.

6. $i$-Canonical bases for modules

In this section, we develop a theory of based $U^-$-modules. To that end, we establish a key property that the quasi-$K$-matrix $T$ preserves the integral forms of various based modules and their tensor products.

6.1 The $A$-forms

**Definition 6.1.** Let $A \hat{U}^i$ be the $A$-subalgebra of $A \hat{U}^a$ generated by the $i$-divided powers $B_{i,\xi}^{(n)}$ ($i \in \mathbb{I}$) and $E_j^{(n)} 1_\xi (j \in \mathbb{I})$, for all $n \geq 1$ and $\xi \in X_i$.

**Remark 6.2.** We shall see later that $A \hat{U}^i = A \hat{U}^a$ in Corollary 7.5.

Recall for $\lambda \in X$, we denote by $M(\lambda)$ the Verma modules of highest weight $\lambda$. We denote the highest weight vector by $\eta_\lambda$. The following is an $i$-analogue of [BW16, Lemma 2.2(1)].

**Lemma 6.3.** Let $(M, B(M))$ be a based $U$-module. Let $\lambda \in X$. Then:

1. for $b \in B(M)$, the $\mathbb{Q}(q)$-linear map $\pi_b : \hat{U}^1[1]_{\lambda} \rightarrow M \otimes M(\lambda), u \mapsto u(b \otimes \eta_\lambda)$, restricts to an $A$-linear map $\pi_b : A \hat{U}^1[1]_{\lambda} \rightarrow A M \otimes_A A M(\lambda)$;
2. we have $\sum_{b \in B(M)} \pi_b(A \hat{U}^1[1]_{\lambda}) = A M \otimes_A A M(\lambda)$.

**Proof.** Recall $A \hat{U}^i \subset A \hat{U}^a$. Part (1) follows from Definition 3.10. We prove part (2) here following [BW16, Lemma 2.2]. We have $\sum_{b \in B(M)} \pi_b(A \hat{U}^1[1]_{\lambda}) \subset A M \otimes_A A M(\lambda)$ by part (1).

Recall $A_u^-\hat{U}^i$ has an increasing filtration

$$A = A U_{\leq 0} \subset A U_{\leq 1} \subset \cdots \subset A U_{\leq N} \subset \cdots$$

where $A U_{\leq N}$ denotes the $A$-span of $\{F_{i_1}^{(a_1)} \cdots F_{i_n}^{(a_n)} | a_1 + \cdots + a_n \leq N, i_1, \ldots, i_n \in I\}$. This induces an increasing filtration $\{A M(\lambda)_{\leq N}\}$ on $A M(\lambda)$. We shall prove by induction on $N$ that $A M \otimes_A A M(\lambda)_{\leq N} \subset \sum_{b \in B(M)} \pi_b(A \hat{U}^1[1]_{\lambda})$.

Let $b \otimes (F_{i_1}^{(a_1)} \cdots F_{i_n}^{(a_n)} \eta_\lambda) \in A M \otimes_A A M(\lambda)_{\leq N}$. Recall the element $B_{i_1,\xi}^{(a_1)}$ in Theorem 5.1 with appropriate $\xi \in X_i$. Thanks to [BW16, Lemma 2.2] and Theorem 5.1, we have

$$B_{i_1,\xi}^{(a_1)} (b \otimes (F_{i_2}^{(a_2)} \cdots F_{i_n}^{(a_n)} \eta_\lambda)) \in b \otimes (F_{i_2}^{(a_2)} \cdots F_{i_n}^{(a_n)} \eta_\lambda) + A M \otimes_A A M(\lambda)_{\leq N-1}.$$

The lemma follows. \(\square\)

For $\lambda \in X^+$, we abuse the notation and denote also by $\eta_\lambda$ the image of $\eta_\lambda$ under the projection $p_\lambda : M(\lambda) \rightarrow L(\lambda)$. Note that $p_\lambda$ restricts to $p_\lambda : A M(\lambda) \rightarrow A L(\lambda)$. The next corollary follows from Lemma 6.3.
Corollary 6.4. Let $\lambda \in X^+$, and let $(M, B(M))$ be a based $U$-module. Then:

1. for $b \in B(M)$, the $\mathbb{Q}(q)$-linear map $\pi_b : \hat{U}^1_{[\lambda]} M \otimes L(\lambda), u \mapsto u(b \otimes \eta_\lambda)$, restricts to an $A$-linear map $\pi_b : 'A \hat{U}^1_{[\lambda]} M \otimes A L(\lambda)$;

2. we have $\sum_{b \in B(M)} \pi_b('A \hat{U}^1_{[\lambda]} M) = A M \otimes A L(\lambda)$.

6.2 Integrality of actions of $\Upsilon$

6.2.1 The quasi-$K$-matrix $\Upsilon \in \hat{U}^+$ induces a well-defined $\mathbb{Q}(q)$-linear map on $M \otimes L(\lambda)$:

$$\Upsilon : M \otimes L(\lambda) \rightarrow M \otimes L(\lambda),$$

for any $\lambda \in X^+$ and any weight $U$-module $M$ whose weights are bounded above.

Recall [BW18b, §5.1] that a $U^+$-module $M$ equipped with an anti-linear involution $\psi$, is called involutive (or $i$-involutive) if

$$\psi(um) = \psi(u)\psi(m), \quad \forall u \in U^+, \ m \in M.$$

Proposition 6.5. Let $(M, B)$ be a based $U$-module whose weights are bounded above. We denote the bar involution on $M$ by $\check{\psi}$. Then $M$ is an $i$-involutive $U^+$-module with involution

$$\check{\psi} := \Upsilon \circ \psi.$$  

Proof. Note that because the weights of $M$ are bounded above, the action $\Upsilon : M \rightarrow M$ is well defined. The rest of the argument follows from [BW18b, Proposition 5.1].

6.2.2

Proposition 6.6. Let $(M, B)$ be a based $U$-module whose weights are bounded above such that $\Upsilon$ preserves the $A$-submodule $\hat{A}M$. Then the $\mathbb{Q}(q)$-linear map $\check{\psi} := \Upsilon \circ \psi$ preserves the $A$-submodule $\hat{A}M \otimes A L(\lambda)$, for any $\lambda \in X^+$.

Proof. The $U$-module $M \otimes L(\lambda)$ is involutive with the involution $\psi := \Theta \circ (\cdot \otimes \cdot)$; see [Lus10, 27.3.1]. It follows by [BW16, Proposition 2.4] that $\psi$ preserves the $A$-submodule $\hat{A}M \otimes A L(\lambda)$.

Regarded as a $U^+$-module, $M \otimes L(\lambda)$ is $i$-involutive with the involution $\check{\psi} := \Upsilon \circ \psi$. We now prove that $\check{\psi}$ preserves the $A$-submodule $\hat{A}M \otimes A L(\lambda)$.

By Corollary 6.4(2), for any $x \in \hat{A}M \otimes A L(\lambda)$, we can write $x = \sum_k u_k(b_k \otimes \eta_\lambda)$, for $u_k \in \hat{A}U^+$ and $b_k \in B$. As $M \otimes L(\lambda)$ is $i$-involutive, we have

$$\check{\psi}(x) = \sum_k \check{\psi}(u_k)\check{\psi}(b_k \otimes \eta_\lambda) = \sum_k \check{\psi}(u_k)\Upsilon \psi(b_k \otimes \eta_\lambda) = \sum_k \check{\psi}(u_k)(\Upsilon b_k \otimes \eta_\lambda),$$

where we have used $\Delta(\Upsilon) \in U \otimes U^\perp$ by [Lus10, 3.1.4]. By assumption, we have $\Upsilon b_k \in \hat{A}M$ and it follows by definition of $\hat{A}U^+$ that $\check{\psi}(u_k) \in \hat{A}U^+$. Applying Corollary 6.4(1) again to (6.3), we obtain that $\check{\psi}(x) \in \hat{A}M \otimes A L(\lambda)$. The proposition follows.

Corollary 6.7. Let $(M, B)$ be a based $U$-module whose weights are bounded above such that $\Upsilon$ preserves the $A$-submodule $\hat{A}M$. Then $\Upsilon$ preserves the $A$-submodule $\hat{A}M \otimes A L(\lambda)$. In particular, $\Upsilon$ preserves the $A$-submodule $A L(\lambda)$ of $L(\lambda)$.

Proof. Recall $\Upsilon = \check{\psi} \circ \psi$. The corollary follows from Proposition 6.6 and the fact that $\psi$ preserves the $A$-submodule $\hat{A}M \otimes A L(\lambda)$.

□
Corollary 6.8. Let $\lambda_i \in X^+$ for $1 \leq i \leq \ell$. The involution $\psi_i$ on the $i$-involutive $U^i$-module $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ preserves the $A$-submodule $\mathcal{A} L(\lambda_1) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} L(\lambda_\ell)$.

Proof. Thanks [BW16, Theorem 2.9] we know that $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ is a based $U$-module whose weights are bounded above. Then the corollary follows by applying Proposition 6.6 consecutively.

Theorem 6.9. Assume $(U, U^i)$ is of finite type. Write $\Upsilon = \sum_{\mu \in \mathbb{N}} \Upsilon_\mu$. Then we have $\Upsilon_\mu \in \mathcal{A} U^+$ for each $\mu$.

Proof. Follows by Corollary 6.7 and applying $\Upsilon$ to the lowest weight vector $\xi_{-\omega_\lambda} \in \mathcal{A} L(\lambda)$, for $\lambda \gg 0$ (i.e. $\lambda \in X^+$ such that $\langle i, \lambda \rangle \gg 0$ for each $i$).

Remark 6.10. Theorem 6.9, which allows general parameters $\varsigma_i \in \mathbb{Z}[q, q^{-1}]$ as in (3.3), generalizes [BW18b, Theorem D], which required $\varsigma_i \in \pm q^\mathbb{Z}$. Our current approach (which is based on the new $i$-divided powers developed in §5) avoids the tedious case-by-case verification in the eight cases of real rank-one QSP in [BW18b, Appendix A].

6.3 $i$-Canonical bases on modules

6.3.1 Let us first recall the following definition of based $U^i$-modules in [BWW20, Definition 1]. Let $A = \mathbb{Q}[[q^{-1}]] \cap \mathbb{Q}(q)$.

We call a $U^i$-module $M$ a weight $U^i$-module, if $M$ admits a direct sum decomposition $M = \bigoplus_{\lambda \in X_i} M_\lambda$ such that, for any $\mu \in Y^i$, $\lambda \in X_i$, $m \in M_\lambda$, we have $K_\mu m = q^{(\mu, \lambda)} m$.

Definition 6.11. Let $M$ be a weight $U^i$-module over $\mathbb{Q}(q)$ with a given $\mathbb{Q}(q)$-basis $B^i$. The pair $(M, B^i)$ is called a based $U^i$-module if the following conditions are satisfied:

1. $B^i \cap M_\nu$ is a basis of $M_\nu$, for any $\nu \in X_i$;
2. the $A$-submodule $\mathcal{A} M$ generated by $B^i$ is stable under $\mathcal{A} U^i$;
3. $M$ is $i$-involutive; that is, the $A$-linear involution $\psi_i : M \to M$ defined by $\psi_i(q) = q^{-1}, \psi_i(b) = b$ for all $b \in B^i$ is compatible with the $U^i$-action, i.e. $\psi_i(um) = \psi_i(u)\psi_i(m)$, for all $u \in U^i, m \in M$;
4. let $L(M)$ be the $A$-submodule of $M$ generated by $B^i$; then the image of $B^i$ in $L(M)/q^{-1}L(M)$ forms a $Q$-basis in $L(M)/q^{-1}L(M)$.

We shall denote by $L(M)$ the $\mathbb{Z}[q^{-1}]$-span of $B^i$; then $B^i$ forms a $\mathbb{Z}[q^{-1}]$-basis for $L(M)$. We also have the obvious notions of based $U^i$-submodules and based quotient $U^i$-modules.

6.3.2 We have the following generalization of [BW18b, Theorem 5.7] to QSP of Kac–Moody type.

Theorem 6.12. Let $(M, B)$ be a based $U$-module whose weights are bounded above. Assume the involution $\psi_i$ of $M$ from Proposition 6.5 preserves the $A$-submodule $\mathcal{A} M$.

1. The $U^i$-module $M$ admits a unique basis (called $i$-canonical basis) $B^i := \{b^i \mid b \in B\}$ which is $\psi_i$-invariant and of the form

$$b^i = b + \sum_{b' \in B, b < b'} t_{b,b'} b'^i, \quad \text{for } t_{b,b'} \in q^{-1}\mathbb{Z}[q^{-1}].$$ (6.4)
(2) The basis $B^i$ forms an $A$-basis for the $A$-lattice $AM$ (generated by $B$), and forms a $\mathbb{Z}[q^{-1}]$-basis for the $\mathbb{Z}[q^{-1}]$-lattice $M$ (generated by $B$).

(3) The module $(M, B^i)$ is a based $U^i$-module, where we call $B^i$ the $i$-canonical basis of $M$.

6.3.3 Recall from Theorem 2.2 the based $U$-submodule $L(w, \mu)$, for $\lambda, \mu \in X^+$ and $w \in W$.

**Theorem 6.13.** Let $\lambda, \mu, \lambda_i \in X^+$ for $1 \leq i \leq \ell$ and $w \in W$:

1. $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ is a based $U^i$-module, with the $i$-canonical basis defined as Theorem 6.12;
2. $L(w, \mu)$ is a based $U^i$-submodule of $L(\lambda) \otimes L(\mu)$.

**Proof.** It suffices to verify the assumptions of Theorem 6.12. Indeed, it is clear that both $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ and $L(w, \mu)$ have weights bounded above. It follows from Corollary 6.8 that $\psi_i$ preserves the $A$-submodule $A L(\lambda_1) \otimes A \cdots \otimes A L(\lambda_\ell)$ and, hence, preserves $A L(w, \mu)$ too. Therefore, both $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ and $L(w, \mu)$ are based $U^i$-modules. It is obvious that $L(w, \mu)$ is a based $U^i$-submodule of $L(\lambda) \otimes L(\mu)$. The theorem is proved. □

For $\lambda, \mu \in X^+$, we shall denote by $L^i(\lambda, \mu) = L(w \cdot \lambda, \mu)$, which is a based $U$-module and also a based $U^i$-module thanks to Theorem 2.2.

### 6.4 The element $\Theta^i$

We define

$$\Theta^i = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes \text{id}).$$

(6.5)

Note that $\Theta^i$ only lies in a completion of $U \otimes U$, whose definition can be found in [Kol20, §3.1]. Even though only finite types were considered in [Kol20], the generalization to Kac–Moody is straightforward.

We can write

$$\Theta^i = \sum_{\mu \in \mathcal{N}} \Theta^i_{\mu}, \quad \text{where } \Theta^i_{\mu} \in U \otimes U^i_\mu.$$  

(6.6)

The following result first appeared in [BW18a, Proposition 3.5] for the QSPs of (quasi-split) type AIII/AIV.

**Lemma 6.14** [Kol20, Proposition 3.10]. We have $\Theta^i_{\mu} \in U^i \otimes U^i_\mu$, for all $\mu$. In particular, we have $\Theta^i_0 = 1 \otimes 1$.

The proof of [Kol20, Proposition 3.10] remains valid in the Kac–Moody setting. (We thank Kolb for the confirmation.)

**Theorem 6.15.** Let $M$ be a based $U^i$-module, and $\lambda \in X^+$. Then $\psi_i \equiv \Theta^i \circ (\psi_i \otimes \psi)$ is an anti-linear involution on $M \otimes L(\lambda)$, and $M \otimes L(\lambda)$ is a based $U^i$-module with a bar involution $\psi_i$.

**Proof.** The anti-linear operator $\psi_i = \Theta^i \circ (\psi_i \otimes \psi) : M \otimes L(\lambda) \rightarrow M \otimes L(\lambda)$ is well defined thanks to Lemma 6.14 and the fact that the weights of $L(\lambda)$ are bounded above. Then entirely similar to [BW18a, Proposition 3.13], we see that $\psi_i^2 = 1$ and $M \otimes L(\lambda)$ is $i$-involutive in the sense of Definition 6.11(3).
We now prove that $\psi_i$ preserves the $A$-submodule $AM \otimes_A AL(\lambda)$. By assumption, $(M, B'(M))$ is a based $U^i$-module. For any $b \in B'(M)$, we define

$$\pi_b : A \hat{U}^i \to AM \otimes_A AL(\lambda), \quad u \mapsto u(b \otimes 1).$$

Indeed, $\pi_b$ is well-defined, because $u \in A \hat{U}^i$ if and only if $u \cdot 1_\mu \in A \hat{U}$ for each $\mu \in X$; cf. [BW18b, Lemma 3.20] (see Definition 3.10 for $\hat{U}^i$).

Note that $\psi_i(\delta \otimes \eta_\lambda) = (\delta \otimes \eta_\lambda)$ for any $\delta \in B'(M)$. Following the proof of Lemma 6.3, we have $\sum_{b \in B'(M)} \pi_b'(A \hat{U}^i) = A M \otimes_A AL(\lambda)$. Hence, we also have $\sum_{b \in B'(M)} \pi_b'(A \hat{U}^i) = A M \otimes_A AL(\lambda)$, because $A \hat{U}^i \subset A \hat{U}^i$. Then the same strategy of Proposition 6.6 implies that $\psi_i$ preserves the $A$-submodule $A M \otimes_A AL(\lambda)$.

We write $B = \{ b - \eta_\lambda : b \in B(\lambda) \}$ for the canonical basis of $L(\lambda)$. Following the same argument as for [BW20, Theorem 4], we conclude that:

(1) for $b_1 \in B^1, b_2 \in B$, there exists a unique element $b_1 \otimes b_2$ which is $\psi_i$-invariant such that $b_1 \otimes b_2 \in b_1 \otimes b_2 + q^{-1}Z[q^{-1}]B^1 \otimes B$;
(2) we have $b_1 \otimes b_2 \in b_1 \otimes b_2 + \sum_{b' \neq b} q^{-1}Z[q^{-1}]b'_1 \otimes b'_2$;
(3) $B^1 \otimes B := \{ b_1 \otimes b_2 : b_1 \in B^1, b_2 \in B \}$ forms a $\mathbb{Q}(q)$-basis for $M \otimes L(\lambda)$, an $A$-basis for $A M \otimes_A AL(\lambda)$, and a $\mathbb{Z}[q^{-1}]$-basis for $L(M) \otimes_{\mathbb{Z}[q^{-1}]} L(\lambda)$;
(4) $(M \otimes L(\lambda), B^1 \otimes B)$ is a based $U^i$-module.

\[\square\]

7. Canonical bases on the modified $\nu$-quantum groups

In this section, we formulate the main definition and theorems on canonical bases on the modified $\nu$-quantum groups. The formulations are straightforward generalizations of the finite type counterparts in [BW18b, §6] (with mild modifications), and the reader is encouraged to be familiar with [BW18b, §6] first. Thanks to the new results established in the previous sections, they are now valid in the setting of QSP of Kac–Moody type.

7.1 The modified $\nu$-quantum groups

Recall the partial order $\leq$ on the weight lattice $X$ in (2.3). The following proposition generalizes [BW18b, Propositions 6.8, 6.12, 6.13 and 6.16] to Kac–Moody types.

**Proposition 7.1.** Let $\lambda, \mu \in X^+$ and let $\zeta = w_\lambda \lambda + \mu$ and $\zeta_i = \zeta$.

1. The $\nu$-canonical basis of $L^\nu(\lambda, \mu)$ is the basis $B^\nu(\lambda, \mu) = \{ (b_1 \otimes \zeta)_{\nu_\lambda, \mu} : (b_1, b_2) \in B^1 \times B \} \{0\}$, where $(b_1 \otimes \zeta)_{\nu_\lambda, \mu}$ is $\psi_i$-invariant and lies in

$$(b_1 \otimes \zeta)(\eta_\lambda \otimes \eta_\mu) + \sum_{|b'_1| + |b'_2| \leq |b_1| + |b_2|} q^{-1}Z[q^{-1}] (b'_1 \otimes \zeta)(\eta_\lambda \otimes \eta_\mu).$$

2. We have the projective system $\{ L^\nu(\lambda + \nu^\tau, \mu + \nu) \}_{\nu \in X^+}$ of $U^i$-modules, where

$$\pi_{\nu + \nu_1, \nu_1} : L^\nu(\lambda + \nu^\tau, \mu + \nu + \nu_1) \to L^\nu(\lambda + \nu^\tau, \mu + \nu), \quad \nu, \nu_1 \in X^+,$$

is the unique homomorphism of $U^i$-modules such that

$$\pi(\eta_{\lambda \nu^\tau + \nu_1} \otimes \eta_{\mu + \nu + \nu_1}) = \eta_{\lambda \nu^\tau + \nu_1} \otimes \eta_{\mu + \nu}.$$
We consider the QSP of type AIV of rank one. We have essential role here as we are taking $\nu$. Theorem 7.2 [BW18b, (6.5)]. The rest is exactly the same. Note that the (fixed) parameters plays no
essential role here as we are taking $\nu \gg 0$.

Proof. Claim (1) is just a reformulation of Theorem 6.13(2). Claim (2) follows by the same proof as [BW18b, Proposition 6.12].

Claim (3) is essentially the same as [BW18b, Proposition 6.16] with a mild modification which we now explain. We used the finite-dimensionality of the module $L(\nu^r, \nu)$ four lines below
[BW18b, (6.5)]. However, this can be replaced by the fact that only finitely many $a(b', b'')$ are nonzero [BW18b, (6.5)]. The rest is exactly the same. Note that the (fixed) parameters plays no
essential role here as we are taking $\nu \gg 0$.

\begin{theorem} [BW18b, Theorem 6.17] \label{thm:7.2} Let $\zeta_i \in X_i$ and $(b_1, b_2) \in B_i \times B$.

\begin{enumerate}
\item There is a unique element $u = b_1 \hat{\otimes}_i b_2 \in \hat{U}^\iota$ such that
\begin{equation}
\pi(\nu \otimes \eta) = (b_1 \hat{\otimes}_i b_2)^{\nu \otimes \eta} \in L(\nu, \eta),
\end{equation}
for all $\lambda, \mu \gg 0$ with $\nu \otimes \eta$.

\item The element $b_1 \hat{\otimes}_i b_2$ is $\psi_i$-invariant.

\item The set $\hat{B}^i = \{b_1 \hat{\otimes}_i b_2 \mid \zeta_i \in X_i, (b_1, b_2) \in B_i \times B\}$ forms a $Q(q)$-basis of $U^\iota$ and an $\mathcal{A}$-basis of $\mathcal{A}U^\iota$.
\end{enumerate}
\end{theorem}

Proof. The proof is the same as for [BW18b, Theorem 6.17] once Proposition 7.1 is available.

\begin{remark} \label{rem:7.3} Let us illustrate the dependence on parameters for small $\nu$ by a simple example. We consider the QSP of type AIV of rank one. We have $i(B) = F + \xi E K^{-1}$ (with $\kappa = 0$). Let us write $\xi = q^{-2} = q^{-1} \sum_{i \in \mathbb{Z}} a_i q^i$ with $a_i = a_{-i} \in \mathbb{Z}$. We have the contraction map
\begin{equation}
\pi : L(2) \longrightarrow L(0)
\end{equation}
\begin{equation}
\eta_2 \mapsto \eta_0, \quad F \eta_2 \mapsto 0, \quad F^{(2)} \eta_2 \mapsto -\xi \eta_0.
\end{equation}

The $\iota$-canonical basis of $L(2)$ is of the following form
\begin{equation}
\eta_2^i = \eta_2, \quad (F \eta_2)^i = F \eta_2, \quad (F^{(2)} \eta_2)^i = F^{(2)} \eta_2 + \left( \sum_{i < 1} a_i q^{i-1} - \sum_{i > 1} a_i q^{1-i} \right) \eta_2.
\end{equation}

Therefore, we see that
\begin{equation}
\pi((F^{(2)} \eta_2)^i) = -\sum_{i > 1} a_i (q^i + q^{-i}).
\end{equation}

Hence, the map $\pi : L(2) \longrightarrow L(0)$ is generally not a morphism of based $U^\iota$-modules.

Recall the linear map $p_{\iota, \lambda} : \hat{U}^\iota 1_\lambda \longrightarrow \hat{P}^i 1_\lambda$ from (3.13).

\begin{corollary} [BW18b, Corollaries 6.19 and 6.20] \label{cor:7.4} The $\mathcal{A}$-algebra $\mathcal{A}\hat{U}^\iota$ is generated by $1 \hat{\otimes}_i E_i^{\jmath} (i \in I)$ and $E_j^{\jmath} 1_\lambda$ for $\zeta_i \in X_i$ and $a \geq 0$. Moreover, $\mathcal{A}\hat{U}^\iota$ is a free $\mathcal{A}$-module such that $\hat{U}^\iota = Q(q) \otimes_{\mathcal{A}} \mathcal{A}\hat{U}^\iota$.

For $\lambda \in X$, the map $p_{\iota, \lambda} : \mathcal{A}\hat{U}^\iota 1_\lambda \longrightarrow \mathcal{A}\hat{P}^i 1_\lambda$ is an isomorphism of (free) $\mathcal{A}$-modules.

\end{corollary}
Recall from Definition 6.1 the $\mathcal{A}$-subalgebra $\mathcal{A}' \bar{U}^i \subset \mathcal{A} \bar{U}^i$.

**Corollary 7.5.** We have $\mathcal{A}' \bar{U}^i = \mathcal{A} \bar{U}^i$; that is, the $\mathcal{A}$-algebra $\mathcal{A} \bar{U}^i$ is generated by the $i$-divided powers $B^{(a)}_{i,\zeta_\omega} (i \in \mathbb{I})$ and $E_{j}^{(a)} 1_{\zeta_\omega} (j \in \mathbb{I})$ for $\zeta_\omega \in X_i$ and $a \geq 0$.

**Proof.** The equality follows from Corollary 7.4 and the surjectivity of the restriction $p_{i,\lambda} : \mathcal{A}' \bar{U}^i 1_{\lambda} \rightarrow \mathcal{A} \bar{P} 1_{\lambda}$. The rephrase follows by Definition 6.1 of $\mathcal{A}' \bar{U}^i$. \hfill $\Box$

### 7.2 The bilinear form

Following [BW18b, Definition 6.15], we define a symmetric bilinear form $(\cdot, \cdot) : \bar{U}^i \times \bar{U}^i \rightarrow \mathbb{Q}(q)$ via a limit of the corresponding bilinear forms (defined using the anti-involution $\varphi$) on the projective system of $\bar{U}^i$-modules $L'(\lambda, \mu)$.

The next theorem generalizes [BW18b, Theorem 6.27] to $\bar{U}^i$ associated with more general parameters $\zeta_\omega$ and to the Kac–Moody setting.

**Theorem 7.6.** The $i$-canonical basis $\tilde{B}^i$ of $\bar{U}^i$ is almost orthonormal in the following sense: for $\zeta_\omega, \zeta'_\omega \in X_i$ and $(b_1, b_2), (b'_1, b'_2) \in \tilde{B}^i \times \tilde{B}^i$, we have

$$ (b_1 \diamond \zeta_\omega, b_2, b'_1 \diamond \zeta'_\omega, b'_2) \equiv \delta_{\zeta_\omega, \zeta'_\omega} \delta_{b_1, b'_1} \delta_{b_2, b'_2}, \mod q^{-1}A. $$

In particular, the bilinear form $(\cdot, \cdot)$ on $\bar{U}^i$ is nondegenerate.

**Remark 7.7.** A notable feature in the definition of the bilinear forms on $\bar{U}$, or on $\bar{U}^i$ as in [BW18b, §6.6] (where it was assumed that $\zeta_\omega \in \pm q \mathbb{Z}$), is the adjunction $\varphi$.

For $\zeta_\omega$ in (3.3) satisfying the general relation (3.7) in general, the $\bar{U}$-automorphism $\varphi$ does not restrict to an automorphism of $\bar{U}'$, or $\bar{U}^i$. However, because $L'(\lambda, \mu)$ is a $\bar{U}$-module, the $\varphi$-twisted action of $\bar{U}^i$ is always well defined. Hence, the definition of the bilinear form $(\cdot, \cdot)$ on $\bar{U}^i$ based on the projective system of $\bar{U}^i$-modules $L'(\lambda, \mu)$ remains valid, and the proof of [BW18b, Theorem 6.27] can be carried here.

Although the adjunction induced by $\varphi$ no longer holds in general, if the anti-involution $\varphi$ preserves the algebra $\bar{U}^i$ as in Proposition 3.11(3), we shall still have the induced adjunction.

**Acknowledgements**

This project could not have been completed without the generous help from many people over the years. We are indebted to Masaki Kashiwara for supplying a proof of Theorem 2.2. We thank Gail Letzter for her expertise and help regarding Lemma 4.2, and thank Stefan Kolb for helpful clarifications of his work with Balagovic. We thank Hideya Watanabe for enlightening discussions regarding the modified $i$-quantum groups. We thank Collin Berman (supported by REU in W.W.’s NSF grant) for various computer explorations on $i$-divided powers, and Thomas Sale for a verification of a crucial example. We would also like to thanks an anonymous referee for helpful comments.

H.B. is supported by NUS-MOE grant R-146-000-294-133 and R-146-001-294-133. W.W. is partially supported by NSF grant DMS-1702254 and DMS-2001351.

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