Some properties of second order theta functions on Prym varieties

E. Izadi and C. Pauly
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Abstract

Let $P \cup P'$ be the two component Prym variety associated to an étale double cover $\tilde{C} \rightarrow C$ of a non-hyperelliptic curve of genus $g \geq 6$ and let $|2\Xi_0|$ and $|2\Xi'_0|$ be the linear systems of second order theta divisors on $P$ and $P'$ respectively. The component $P'$ contains canonically the Prym curve $\tilde{C}$. We show that the base locus of the subseries of divisors containing $\tilde{C} \subset P'$ is exactly the curve $\tilde{C}$. We also prove canonical isomorphisms between some subseries of $|2\Xi_0|$ and $|2\Xi'_0|$ and some subseries of second order theta divisors on the Jacobian of $C$.

INTRODUCTION

Let $C$ be a curve of genus $g \geq 5$ with an étale double cover $\pi : \tilde{C} \rightarrow C$. Let $Nm : Pic(\tilde{C}) \rightarrow Pic(C)$ be the norm map. Consider the Prym varieties

$$Nm^{-1}(O) = P \cup P'$$

which are characterized by the facts that $O \in P$, $O \notin P'$. Let $\sigma : \tilde{C} \rightarrow \tilde{C}$ be the involution of the cover $\pi : \tilde{C} \rightarrow C$. The curve $\tilde{C}$ admits a natural embedding in $P'$ given by the morphism

$$i : \tilde{C} \rightarrow P'$$

$$\tilde{p} \mapsto \mathcal{O}_C(\tilde{p} - \sigma \tilde{p}).$$

A symmetric Riemann theta divisor $\tilde{\Theta}_0$ on the Jacobian $J\tilde{C}$ of $\tilde{C}$ induces twice a symmetric principal polarization $\Xi_0$ on $P$ (resp. $\Xi'_0$ on $P'$). Let $\Gamma_\tilde{C}$ be the space of sections of $\mathcal{O}_{P'}(2\Xi'_0)$ vanishing on the image of $i$. In his work on the Schottky problem, Donagi proved in [Dol] (Lemma 4.8 page 597) that the base locus $Bs(\mathbb{P}\Gamma_\tilde{C})$ of $\mathbb{P}\Gamma_\tilde{C}$ is $i(\tilde{C})$ for a Wirtinger cover $\pi : C \rightarrow C$. Since he proves that for a Wirtinger cover the equality between $Bs(\mathbb{P}\Gamma_\tilde{C})$ and $i(\tilde{C})$ is scheme-theoretical outside the double points of $i(\tilde{C})$, it follows from his proof that, for a general double cover, the base locus is the union of $i(\tilde{C})$ and possibly a finite set of points. We prove (Sections 2 and 5)

1. Theorem. If $g \geq 6$ and $C$ is non-hyperelliptic, the scheme-theoretical base locus in $P'$ of the linear system $\mathbb{P}\Gamma_\tilde{C}$ is $i(\tilde{C})$.

The proof of theorem 1 has two steps. First we show that $Bs(\mathbb{P}\Gamma_\tilde{C})$ equals $i(\tilde{C})$ set-theoretically (Section 2). In order to prove the scheme-theoretic equality, we introduce and study divisors $D := \Delta(E)$ in the linear systems $|2\Xi_0|$ and $|2\Xi'_0|$ associated to certain semi-stable rank 2 vector bundles $E$ over the curve $C$ (Prop. 3.2). We calculate the tangent spaces to the divisors $\Delta(E)$ along the curve $i(\tilde{C})$ and show that at any given point of $i(\tilde{C})$ their intersection is equal to the tangent space to $i(\tilde{C})$. 

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Let $\Theta_0$ be a symmetric theta divisor on the Jacobian $JC$ and let $\alpha$ be the square-trivial invertible sheaf associated to the double cover $\tilde{C} \to C$. Translation by $\alpha$ induces an involution $T_\alpha$ on $JC$, which lifts canonically to a linear involution acting on $H^0(JC, \Theta_0 + T_\alpha^* \Theta_0)$. Mumford constructs in [M2] (see also [vGP] Proposition 1) canonical isomorphisms

$$\mu_+: H^0(P, 2\Xi_0) \sim \rightarrow H^0(JC, \Theta_0 + T_\alpha^* \Theta_0) \quad \mu_-: H^0(P, 2\Xi_0) \sim \rightarrow H^0(JC, \Theta_0 + T_\alpha^* \Theta_0)$$

where the subscript $\pm$ denotes the $\pm$eigenspaces of the involution. We are interested in some naturally defined subspaces of these vector spaces. In connection with the Schottky problem, van Geemen and van der Geer [vGvdG] introduced the subspace

$$\Gamma_{00} = \{ s \in H^0(A, 2\Theta) \mid \text{mult}_0(s) \geq 4 \}$$

for any abelian variety $A$ with symmetric principal polarization $\Theta$. It was conjectured by van Geemen, van der Geer and Donagi ([vGvdG] and [Do2] page 110) that if $(A, \Theta)$ is a Jacobian, then the base locus $B_s(p_{\Gamma_0})$ of $p_{\Gamma_0}$ is the surface $C - C = \{ \mathcal{O}_C(p - q) \mid p, q \in C \} \subset JC$ as a set and, if $(A, \Theta)$ is not in the closure of the locus of Jacobians, then $B_s(p_{\Gamma_0}) = \{ \mathcal{O} \}$. For Jacobians, the conjecture was proved by Welters [W1]. For non-Jacobians, the conjecture was proved in dimension 4 by the first author [I1]. Some evidence was also given for non-Jacobian Pryms by the first author in [I2]. Consider the subspaces $\Gamma^{(2)}_C$ of $H^0(P', 2\Xi_0')$ of elements vanishing with multiplicity $\geq 2$ along $i(\tilde{C})$ and the subspace

$$\Gamma^\alpha_{-C} := \{ s \in H^0(JC, \Theta_0 + T_\alpha^* \Theta_0) \mid C - C \subset Z(s) \}$$

where $Z(s)$ denotes the zero divisor of the section $s$. This space splits into $\pm$eigenspaces $\Gamma^\alpha_{-C}$ under the involution induced by $T_\alpha$.

The infinitesimal study of the above mentioned divisors $\Delta(E)$ at the origin $\mathcal{O} \in P$ and along the curve $i(\tilde{C})$ allows us to prove the following result (Section 4).

2. **Theorem.** Assume $C$ non-hyperelliptic of genus $g \geq 5$. Via the canonical isomorphisms ([J1]), we have equalities among the following subspaces

1. $\Gamma^{\alpha^+}_{-C} = \Gamma_{00}$, i.e., $\forall s \in H^0(P, 2\Xi_0)$, $\text{mult}_0(s) \geq 4 \iff C - C \subset Z(\mu_+(s))$

2. $\Gamma^{\alpha^-}_{-C} = \Gamma^{(2)}_C$, i.e., $\forall s \in H^0(P', 2\Xi_0)$, $\left( \forall \tilde{p} \in \tilde{C} \text{ mult}_{i(\tilde{p})}(s) \geq 2 \right) \iff C - C \subset Z(\mu_-(s))$

One can view statement 1 as an analogue for Prym varieties of the equivalence (see e.g. [F] page 489, [W1] prop. 4.8 or [vGvdG])

$$\forall s \in H^0(JC, 2\Theta_0), \quad \text{mult}_0(s) \geq 4 \iff C - C \subset Z(s).$$

Alternatively, one can derive equality 1 from an analytic identity between Prym and Jacobian theta functions (formula (41) [F]). Equality 2, however, seems to be new.

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1 Preliminaries and notation

In this section we introduce the notation and recall some well-known facts on Prym varieties. Throughout the paper we will suppose the genus of $C$ to be at least 5. Let $\omega$ be the dualizing sheaf of $C$ and consider the two additional Prym varieties

$$Nm^{-1}(\omega) = P_{\text{even}} \cup P_{\text{odd}}$$

which are characterized by the fact that $\dim H^0(S^1C, \lambda)$ is even (resp. odd) for $\lambda \in P_{\text{even}}$ (resp. $P_{\text{odd}}$). The variety $P_{\text{even}}$ carries the naturally defined reduced Riemann theta divisor

$$\Xi := \{ \lambda \in P_{\text{even}} \mid h^0(\lambda) > 0 \}$$

a translate of which is $\Xi_0$. Let $SU_C(2, \alpha)$ and $SU_C(2, \omega \alpha)$ be the moduli spaces of semi-stable vector bundles of rank 2 with determinant $\alpha$ and $\omega \alpha$ respectively. Taking direct image gives morphisms

$$\varphi : P \cup P' \longrightarrow SU_C(2, \alpha) \quad \varphi : P_{\text{even}} \cup P_{\text{odd}} \longrightarrow SU_C(2, \omega \alpha).$$

Let $L_\alpha$ (resp. $L_{\omega \alpha}$) be the generator of the Picard group of $SU_C(2, \alpha)$ (resp. $SU_C(2, \omega \alpha)$). It is known that

$$(\varphi|_P)^*L_\alpha = \mathcal{O}(2\Xi_0) \quad (\varphi|_{P_{\text{even}}})^*L_{\omega \alpha} = \mathcal{O}(2\Xi).$$

We denote by $\mathcal{O}(2\Xi_0)$ (resp. $\mathcal{O}(2\Xi')$) the pull-back of the line bundle $L_\alpha$ (resp. $L_{\omega \alpha}$) to the Prym $P'$ (resp. $P_{\text{odd}}$), i.e., $(\varphi|_P)^*L_\alpha = \mathcal{O}(2\Xi_0)$ and $(\varphi|_{P_{\text{odd}}})^*L_{\omega \alpha} = \mathcal{O}(2\Xi')$. We consider the following morphisms

$$\psi : JC \longrightarrow SU_C(2, \alpha) \quad \xi \longmapsto \xi \oplus \alpha \xi^{-1}$$

$$\psi : \text{Pic}^{g-1}(C) \longrightarrow SU_C(2, \omega \alpha) \quad \xi \longmapsto \xi \oplus \omega \alpha \xi^{-1}.$$ 

One computes the pull-backs

$$\psi^*L_\alpha = \Theta_0 + T_{\alpha}^*\Theta_0 \quad \psi^*L_{\omega \alpha} = \Theta + T_{\alpha}^*\Theta$$

where

$$\Theta := \{ L \in \text{Pic}^{g-1}(C) \mid h^0(L) > 0 \}$$

and $\Theta_0$ is a symmetric theta divisor in the Jacobian $JC$, i.e., a translate of $\Theta$ by a theta-characteristic. By abuse of notation, we will also write $L_\alpha$ and $L_{\omega \alpha}$ for $\psi^*L_\alpha$ and $\psi^*L_{\omega \alpha}$ respectively. Note that $\psi$ induces linear isomorphisms at the level of global sections:

$$\psi^* : H^0(SU_C(2, \alpha), L_\alpha) \cong H^0(JC, L_\alpha), \quad \psi^* : H^0(SU_C(2, \omega \alpha), L_{\omega \alpha}) \cong H^0(\text{Pic}^{g-1}(C), L_{\omega \alpha}).$$

(1.1)

There is a well-defined morphism

$$D : SU_C(2, \alpha) \longrightarrow |L_{\omega \alpha}| \quad (\text{resp. } D : SU_C(2, \omega \alpha) \longrightarrow |L_\alpha|)$$

where the support of $D(E)$ (reduced for $E$ general) is

$$D(E) = \{ \xi \in JC \ (\text{resp. Pic}^{g-1}(C)) \mid h^0(C, E \otimes \xi) > 0 \}.$$ 

The two involutions of the Jacobian $JC$ given by

$$T_\alpha : \xi \longmapsto \xi \otimes \alpha \quad (-1) : \xi \longmapsto \xi^{-1}$$

induce (up to $\pm 1$) linear involutions $T_{\alpha}^*$ and $(-1)^*$ on the spaces of global sections $H^0(JC, L_\alpha)$ and $H^0(\text{Pic}^{g-1}(C), L_{\omega \alpha})$. 

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1.1. Lemma. The projective linear involutions $T^*_\alpha$ and $(-1)^*$ acting on $\mathbb{P}H^0(JC, L_\alpha)$ are equal.

Proof. We observe that the composite map $T_\alpha \circ (-1) : \xi \mapsto \alpha \xi^{-1}$ verifies $\psi \circ (T_\alpha \circ (-1)) = \psi$. Since $\psi^*$ is a linear isomorphism \[1.1\], we have $(T_\alpha \circ (-1))^* = \pm id_{H^0}$. Therefore $T^*_\alpha = \pm (-1)^*$.

Thus the two spaces decompose into $\pm$eigenspaces. Note that in order to distinguish the two eigenspaces, we need a lift of the 2-torsion point $\alpha$ into the Mumford group. We will take the following convention: the $+$eigenspace (resp. $-$eigenspace) contains the Prym varieties $P$ and $P_{even}$ (resp. $P'$ and $P_{odd}$), i.e., we have canonical (up to multiplication by a nonzero scalar) isomorphisms.

\[
H^0(JC, L_\alpha)_+ = H^0(P, 2\Xi_0), \quad H^0(JC, L_\alpha)_- = H^0(P', 2\Xi'_0), \quad (1.2)
\]

\[
H^0(\text{Pic}^g(C), L_{\omega\alpha})_+ = H^0(P_{even}, 2\Xi), \quad H^0(\text{Pic}^g(C), L_{\omega\alpha})_- = H^0(P_{odd}, 2\Xi'). \quad (1.3)
\]

Since the surface $C - C$ is invariant under the involution $(-1) : \xi \mapsto \xi^{-1}$, the subspace $\Gamma^\alpha_{C-C}$ is invariant under $(-1)^*$ and decomposes into a direct sum of $\pm$eigenspaces for $(-1)^* = T^*_\alpha$:

$$
\Gamma^\alpha_{C-C} = \Gamma^\alpha_{C-C}^+ \oplus \Gamma^\alpha_{C-C}^-.
$$

Prym-Wirtinger duality

For the details see \[32\] lemma 2.3. There exists an integral Cartier divisor on the product $SU_C(2, \alpha) \times SU_C(2, \omega\alpha)$ whose support is given by

$$
\{(E, F) \in SU_C(2, \alpha) \times SU_C(2, \omega\alpha) \mid h^0(C, E \otimes F) > 0\}.
$$

Its associated section can be viewed as an element of the tensor product

$$
H^0(SU_C(2, \alpha), L_\alpha) \otimes H^0(SU_C(2, \omega\alpha), L_{\omega\alpha})
$$

and it can be shown that the corresponding linear map

$$
H^0(SU_C(2, \alpha), L_\alpha)^* \longrightarrow H^0(SU_C(2, \omega\alpha), L_{\omega\alpha}) \quad (1.4)
$$

is an isomorphism and is equivariant for the linear involutions induced by the map $E \mapsto E \otimes \alpha$. Hence using the identifications \[1.2\] and \[1.3\] we obtain canonical isomorphisms,

$$
H^0(P, 2\Xi_0)^* \overset{\sim}{\longrightarrow} H^0(P_{even}, 2\Xi) \quad H^0(P', 2\Xi'_0)^* \overset{\sim}{\longrightarrow} H^0(P_{odd}, 2\Xi'). \quad (1.5)
$$

2 The base locus of $\mathbb{P}\Gamma_{\tilde{C}}$

In this section we compute the set-theoretical base locus of the subseries $\mathbb{P}\Gamma_{\tilde{C}}$ on the Prym variety $P'$. Suppose $C$ non-hyperelliptic. We need some additional notation: denote by $\tilde{C}_m$ the $m$-th symmetric power of $\tilde{C}$ and let $S$ be the subvariety of $\tilde{C}_{2g-2}$ defined as

$$
S = \{D \in \tilde{C}_{2g-2} \mid Nm(D) \in \mid \omega \mid \text{ and } h^0(D) \equiv 1 \mod 2\}.
$$

Then, by \[31\] Corollaire page 365, the variety $S$ is normal and irreducible of dimension $g - 1$. The variety $S$ comes equipped with two natural surjective morphisms

$$
Nm : S \longrightarrow \mid \omega \mid \quad u : S \longrightarrow P_{odd}
$$
where $u$ associates to an effective divisor $D$ its line bundle $O_C(D)$. Note that $u$ is birational and $Nm$ is finite of degree $2^{2g-3}$. Also denote by $u$ the extended morphism $u: \bar{C}_{2g-2} \to \text{Pic}^{2g-2}(\bar{C})$ and consider the commutative diagram

$$
\begin{array}{ccc}
S & \hookrightarrow & \bar{C}_{2g-2} \\
\downarrow^u & & \downarrow^u \\
\text{P}_{\text{odd}} & \hookrightarrow & \text{Pic}^{2g-2}(\bar{C}).
\end{array}
$$

(2.1)

Consider the Brill-Noether locus in $\text{P}_{\text{odd}}$ which is defined set-theoretically by

$$
\Xi_3 := \{ \lambda \in \text{P}_{\text{odd}} \mid h^0(\lambda) \geq 3 \}.
$$

The scheme structure on $\Xi_3$ is defined by taking the scheme-theoretical intersection $[W2]$

$$
\Xi_3 := W_{2g-2}(\bar{C}) \cap \text{P}_{\text{odd}}
$$

where $W_{2g-2}(\bar{C}) \subset \text{Pic}^{2g-2}\bar{C}$ is the Brill-Noether locus of line bundles having at least 3 sections (see [ACGH]).

**2.1. Lemma.** The subscheme $\Xi_3 \subset \text{P}_{\text{odd}}$ is not empty and is of pure codimension 3.

**Proof.** Theorem 9 [DCP] asserts that $\Xi_3$ is not empty and every irreducible component has dimension at least $g - 4$. Suppose that there is an irreducible component $I$ of dimension $\geq g - 3$. Then its inverse image $u^{-1}(I)$ has dimension $\geq g - 1$, hence, since $S$ is irreducible, $u^{-1}(I) = S$ and $\Xi_3 = \text{P}_{\text{odd}}$. The last equality can not happen, since otherwise, using translation by an element of the form $O_C(\bar{p} - \sigma\bar{p})$, we would have $\Xi = \text{P}_{\text{even}}$. $\square$

Observe that $u$ is equivariant for the action of $\sigma$ on $S$ and $\text{P}_{\text{odd}}$. Denote by $Z = u^{-1}(\Xi_3)$ the inverse image of the subscheme $\Xi_3$. By the previous lemma $Z$ is of pure codimension 1 in $S$. We will see in a moment that there is a Cartier divisor $D$ on $S$ whose support is the support of $Z$. Let $\omega_C$ be the dualizing sheaf of $\bar{C}$. Consider the following divisors in $\bar{C}_{2g-2}$

$$
U_\bar{p} := \{ D \in \bar{C}_{2g-2} \mid \exists D' \in \bar{C}_{2g-3} \text{ with } D = D' + \bar{p} = \bar{p} + \bar{C}_{2g-3} \}
$$

$$
V_\bar{p} := \{ D \in \bar{C}_{2g-2} \mid h^0(\omega_C(-D - \bar{p})) \geq 1 \}
$$

and let $\bar{U}_\bar{p}$ and $\bar{V}_\bar{p}$ be their intersections with $S$. A straightforward calculation involving Zariski tangent spaces then shows that $\bar{U}_\bar{p}$ is a reduced divisor. We denote by $O_S(1)$ the pull-back by the norm map of the hyperplane line bundle on $|\omega|$. Then it is easily seen that, for any $\bar{p} \in \bar{C}$,

$$
Nm^*(|\omega(-\bar{p})|) = \bar{U}_\bar{p} + \bar{V}_\bar{p} \in |O_S(1)|.
$$

(2.2)

Let $\bar{\Theta}_\lambda$ denote the translate of $\bar{\Theta}$ by $\lambda$. Then, for any points $\bar{p}, \bar{q} \in \bar{C}$, we have an equality among divisors on $\bar{C}_{2g-2}$ (see [W1] page 6)

$$
u^*(\bar{\Theta}_{\bar{p} - \bar{q}}) = U_{\bar{p}} + V_{\bar{q}}.
$$

(2.3)

The analogue on the even Prym variety of the following lemma was previously proved by R. Smith and R. Varley. In the case of genus 3 it is in their paper [SV1] (Prop. 1 page 358) and for higher genus it will be published in their upcoming paper [SV2].
2.2. Lemma. There exists an effective Cartier divisor \( D \) on \( S \) whose support is equal to

\[
\text{supp } Z = \{ D \in S \mid h^0(\mathcal{O}_\tilde{C}(D)) \geq 3 \}.
\]

Moreover, we have the following equality among effective Cartier divisors

\[
u^*(\Xi_{\tilde{p} - \sigma \tilde{p}} + \Xi_{\tilde{p} - \tilde{p}}) = \tilde{U}_{\tilde{p}} + \tilde{U}_{\sigma \tilde{p}} + D \quad \forall \tilde{p} \in \tilde{C}.
\]

(2.4)

In particular, \( u^*\mathcal{O}_{P_{odd}}(2\Xi') = \mathcal{O}_S(1) \otimes \mathcal{O}_S(D) \).

Proof. We are going to define \( D \) as the residual divisor of the restricted divisor \( \tilde{V}_\tilde{q} \), for a given point \( \tilde{q} \in \tilde{C} \) and then show that it does not depend on the choice of \( \tilde{q} \). We first observe that we have an equality of sets

\[
\tilde{V}_\tilde{q} = \tilde{U}_{\sigma \tilde{q}} \cup Z
\]

which can be seen as follows: for \( D \in \tilde{C}_{2\tilde{q} - 2} \) such that \( h^0(D) = h^0(\omega_\tilde{C}(-D)) = 1 \) the assumption \( D \in \tilde{V}_\tilde{q} \) and the formula \( D + \sigma D = \pi^*(Nm(D)) \) imply that \( \tilde{q} \in \text{supp } \sigma D \iff D \in \tilde{U}_{\sigma \tilde{q}} \). If \( h^0(D) = h^0(\omega_\tilde{C}(-D)) \geq 2 \), then \( D \in \text{supp } Z \). Again a calculation involving Zariski tangent spaces shows that \( \tilde{V}_\tilde{q} \) is reduced generically on \( U_{\sigma \tilde{p}} \). Hence we can define \( D \) by \( \tilde{V}_\tilde{q} = \tilde{U}_{\sigma \tilde{q}} + D \). Now we substitute this expression into (2.3), which we restrict to \( S \), which we take the limit when \( \tilde{p} \to \tilde{q} \). Since \( \mathcal{O}_{P_{odd}}(\tilde{\Theta}) = \mathcal{O}_{P_{odd}}(2\Xi') \), we see that \( \tilde{U}_{\tilde{p}} + \tilde{U}_{\sigma \tilde{p}} + D \in |u^*\mathcal{O}_{P_{odd}}(2\Xi')| \). So by (2.2) we get the line bundle equality claimed in the lemma and we see that the scheme-structure on \( D \) does not depend on the point \( \tilde{q} \). To prove (2.4), we compute using (2.3)

\[
u^*(\tilde{\Theta}_{\tilde{p} - \sigma \tilde{p}} + \tilde{\Theta}_{\tilde{p} - \tilde{p}}) = \tilde{U}_{\tilde{p}} + \tilde{V}_{\sigma \tilde{p}} + \tilde{U}_{\sigma \tilde{p}} + \tilde{V}_{\tilde{p}}.
\]

Now we restrict to \( S \) and use the commutativity of diagram (2.1) and the divisorial equality \( \tilde{\Theta}_{\tilde{p} - \sigma \tilde{p}} \cap P_{odd} = 2\Xi_{\tilde{p} - \sigma \tilde{p}} \) to obtain

\[
u^*(2\Xi_{\tilde{p} - \sigma \tilde{p}} + 2\Xi_{\tilde{p} - \tilde{p}}) = 2\tilde{U}_{\tilde{p}} + 2\tilde{U}_{\sigma \tilde{p}} + 2D.
\]

Since \( \tilde{U}_{\tilde{p}} + \tilde{U}_{\sigma \tilde{p}} + D \in |u^*\mathcal{O}_{P_{odd}}(2\Xi')| \) we can divide this equality by 2 and we are done.

Let \( \mu \) be a point of \( Bs(\mathbb{P} \Gamma_\tilde{C}) \). By lemma 1.2 the linear map \( i^*: |\omega|^* \longrightarrow |2\Xi'|^* \) is injective and, since \( |\omega|^* \) is the span of the image of \( \tilde{C} \) in \( |2\Xi'|^* \), the space \( \mathbb{P} \Gamma_\tilde{C} \) is the annihilator of \( |\omega|^* \subset |2\Xi'|^* \).

So \( Bs(\mathbb{P} \Gamma_\tilde{C}) = |\omega|^* \cap Kum(P') \) and \( \mu \) corresponds to a hyperplane \( H_\mu \in |\omega|^* \). Since \( \mu \in Kum(P') \), the image of \( \mu \) by Wirtinger duality is the divisor \( \Xi_\mu + \Xi_{\mu^{-1}} \in |2\Xi'| \).

2.3. Lemma. With the previous notation, we have an equality

\[
\forall \mu \in Bs(\mathbb{P} \Gamma_\tilde{C}) \quad Nm^*(H_\mu) + D = u^*(\Xi_\mu + \Xi_{\mu^{-1}}).
\]

(2.5)

Proof. The equality follows from the commutativity of the right-hand square of the diagram

\[
\begin{array}{cccccccc}
\tilde{C} & \xrightarrow{\pi} & C & \xrightarrow{\varphi_{can}} & |\omega|^* & \xrightarrow{Nm^*} & |\mathcal{O}_S(1)| & \xrightarrow{+D} & |\mathcal{O}_S(1) \otimes \mathcal{O}_S(D)| \\
\downarrow i & & \downarrow i^* & & \downarrow u^* & & \\
\end{array}
\]

\[
P' & \longrightarrow & |2\Xi'|^* & \cong & |2\Xi'|.
\]

The commutativity of the right-hand square follows from that of the outside square because \( \varphi_{can}(C) \) generates \( |\omega|^* \). In other words we need to check the assertion of the lemma only for hyperplanes of the form \( |\omega(-p)| \) for \( p \in C \). This follows immediately from (2.2) and (2.4).
2.4. Corollary. For every \( \mu \in Bs(\mathbb{P} \Gamma_C) \), the hyperplane \( Nm^*(H_\mu) \) is reducible.

Proof. By the above Lemma we have

\[
u^*(\Xi_\mu + \Xi_{\mu-1}) - \mathcal{D} = Nm^*(H_\mu).
\]

If \( Nm^*(H_\mu) \) is irreducible, then the support of one of the divisors \( u^*(\Xi_\mu) \) or \( u^*(\Xi_{\mu-1}) \), say \( u^*(\Xi_\mu) \), is contained in the support of \( \mathcal{D} \). This is impossible because \( u^*(\Xi_\mu) \) is the inverse image of a divisor in \( P_{add} \) and \( supp \mathcal{D} \) is the inverse image of the codimension 3 support of \( \Xi_3 \).

The set-theoretical assertion of theorem \( \ref{thm:1} \) now follows from the following lemma.

2.5. Lemma. If \( C \) is not bi-elliptic, we have a set-theoretical equality

\[
\{ H \in |\omega|^* : Nm^*(H) \text{ is reducible} \} = \varphi_{can}(C).
\]

If \( C \) is bi-elliptic, the LHS is contained in the union of \( \varphi_{can}(C) \) and the finite set of points \( t \in |\omega|^* \) such that the projection from \( t \) induces a morphism of degree 2 from \( C \) onto an elliptic curve.

Proof. Suppose that \( Nm^*(H) \) is reducible. Then a local computation shows that the hyperplane \( H \) is everywhere tangent to the branch locus of \( Nm \). It is immediately seen that the branch locus \( B \) of \( Nm \) is the dual hypersurface of the canonical curve. The components of the singular locus \( Sing(B) \) of \( B \) are of two different types which can be described as follows

- **type 1** whose points are hyperplanes tangent to \( \varphi_{can}(C) \) in more than one point.
- **type 2** whose points are hyperplanes osculating to \( \varphi_{can}(C) \).

To prove that \( \mu \in \varphi_{can}(C) \), we need to prove that there is a point on \( H \cap B \) which is smooth on \( B \) because the dual variety of \( B \) is the closure of the set of hyperplanes tangent to \( B \) at a smooth point and this is equal to \( \varphi_{can}(C) \). In other words we need to show that \( H \cap B \) is not contained in \( Sing(B) \). Since \( H \cap B \) has pure codimension 2, it suffices to show that no codimension 2 component of \( Sing(B) \) is contained in a hyperplane.

Suppose a codimension 2 component \( B_i \) of type \( i \) \( (i = 1 \text{ or } 2) \) is contained in a hyperplane \( H \) in \( |\omega| \) and let \( t \in |\omega|^* \) be the corresponding point. Then the set of hyperplanes in \( |\omega|^* \) through \( t \) and doubly tangent (resp. osculating) to \( \varphi_{can}(C) \) has dimension \( g - 3 \). We have

2.6. Lemma. For any \( t \in \varphi_{can}(C) \) the restriction \( \rho \) of the projection from \( t \) to \( \varphi_{can}(C) \) is birational onto its image. If \( t \in |\omega|^* \setminus \varphi_{can}(C) \), then \( \rho \) is either birational onto its image or of degree two onto an elliptic curve.

Proof. First note that the degree of the image \( C_t \) of \( C \) by the projection is at least \( g - 2 \) because \( C_t \) is a non-degenerate curve in a projective space of dimension \( g - 2 \). If \( t \in \varphi_{can}(C) \), then the degree of \( \rho \) is equal to \( 2g - 3 \). The degree \( r \) of the restriction of \( \rho \) to \( C_t \) verifies \( r \cdot \deg(C_t) = 2g - 3 \). Therefore \( \frac{2g - 3}{r} \geq g - 2 \). Or \( r \leq 2 + \frac{1}{g - 2} \) which implies \( r \leq 2 \). However, \( r \) cannot be equal to 2 because \( 2g - 3 \) is odd. If \( t \notin \varphi_{can}(C) \), then the same argument gives again \( r \leq 2 \) because \( g \geq 5 \). Hence, if \( \rho \) is not generically injective, then \( r = 2 \) and \( \deg(C_t) = g - 1 \). Therefore \( C_t \) is either smooth rational or an elliptic curve. Since \( C \) is not hyperelliptic, we have that \( C_t \) is an elliptic curve.

First suppose that \( C \rightarrow C_t \) is birational. If \( i = 1 \), projecting from \( t \), we see that the set of hyperplanes in \( |\omega|^*/t \) doubly tangent to \( C_t \) has dimension \( (g - 3) \) = dimension of the dual variety of \( C_t \) which is impossible. If \( i = 2 \), then the set of hyperplanes in \( |\omega|^*/t \) osculating to \( C_t \) has dimension \( g - 3 \) which is also impossible.
If $C \to C_t$ is of degree 2, then indeed every hyperplane tangent to $C_t$ pulls back to a hyperplane twice tangent (or osculating if the point of tangency is a branch point of $C \to C_t$) to $\varphi_{can}(C)$ and we have a codimension 2 family of type $B_1$ contained in the hyperplane corresponding $H$ to $t$. Then $N\!m^*(H)$ could be reducible.

The previous lemma proves theorem 1 set-theoretically for a non bi-elliptic curve. In the bi-elliptic case, we have to work a little more. By Lemma 2.5 a hyperplane $H \not\in \varphi_{can}(C)$, such that $N\!m^*(H)$ might be reducible, corresponds to a point $e \in |\omega|^*$ such that the projection from $e$ induces a morphism $\gamma$ of degree 2 from $C$ to an elliptic curve $E$. In other words, $e$ is the common point of all chords $\langle \gamma^*q \rangle$; ($q \in E$). In that case there exists a 1-dimensional family (parametrized by $E$) of trisecants, namely the chords $\langle \gamma^*q \rangle$, to the Kummer variety $Kum(P')$. By [De] the Prym variety is a Jacobian and by [S] (see also [B3] page 610) the double cover $\pi : \tilde{C} \to C$ is of the following two types

1. $C$ is trigonal
2. $C$ is a smooth plane quintic and $h^0(\mathcal{O}_C(1) \otimes \alpha) = 0$

2.7. Lemma. No double cover of a bi-elliptic curve $C$ of genus $g \geq 6$ is of the above two types.

Proof. For a bi-elliptic curve $C$, the Brill-Noether locus $W_{g-1}^1(C)$ has two irreducible components, which are fixed by the reflection in $\omega$ ([W1] Corollary 3.10). For a smooth plane quintic this Brill-Noether locus is irreducible, ruling out 1. For a trigonal curve this Brill-Noether locus has two irreducible components, which are interchanged by reflection in $\omega$, ruling out 2.

2.8. Remark. If $g = 5$ and $C$ is bi-elliptic, we do not know whether the common point of all the chords for a given bi-elliptic structure lies on $Kum(P')$ (see also [B3] Remark (1) page 611). We expect it not to be on $Kum(P')$.

3 Rank 2 bundles and $2\Xi$-divisors

Consider the induced action of the involution $\sigma$ on the moduli space $SU_C(2, 0)$ given by $\tilde{E} \mapsto \sigma^*\tilde{E}$. Since the covering $\pi$ is unramified, the fixed point set for the $\sigma$-action

$$Fix_{\sigma}SU_C(2, 0) = \{[\tilde{E}] \in SU_C(2, 0) \mid \exists \theta : \sigma^*\tilde{E} \sim \tilde{E}\}$$

has two connected components which are the isomorphic images of $SU_C(2, \omega)$ and $SU_C(2, \omega \alpha)$ by $\pi^*$. Similarly, since $\sigma^*\omega \sim \omega_\tilde{C}$, the involution $\sigma$ acts on $SU_\tilde{C}(2, \omega_\tilde{C})$ and

$$Fix_{\sigma}SU_\tilde{C}(2, \omega_\tilde{C}) = \pi^*SU_C(2, \omega) \cup \pi^*SU_C(2, \omega \alpha)$$

3.1. Proposition. Consider a bundle $E \in SU_C(2, \omega \alpha)$ such that $E \not\in \varphi(P_{odd})$ and put $\tilde{E} = \pi^*E$. Then there is a divisor $\Delta(E) \in |2\Xi_0|$ with the following properties.

1. If $D(\tilde{E})$ does not contain $P$, then

$$D(\tilde{E}) = 2\Delta(E).$$

For $E$ general, $P$ is not contained in $D(\tilde{E})$ and $\Delta(E)$ is reduced.
2. Let $pr_+$ be the projection $|\mathcal{L}_\alpha| \to |2\Xi_\alpha|$ with center $|2\Xi_0|$ (see (1.2)). Then we have a commutative diagram
\[
\begin{array}{ccc}
SU_C(2, \omega\alpha) & \xrightarrow{D} & |\mathcal{L}_\alpha| \\
\downarrow \Delta & & \downarrow pr_+ \\
|2\Xi_\alpha| = |\mathcal{L}_\alpha|_+ 
\end{array}
\]

3.2. Remark. Similarly, when $E \in SU_C(2, \omega\alpha)$ such that $E \notin \varphi(P_{\text{even}})$, we get divisors $\Delta(E) \in |2\Xi_0|$ as described in prop. 3.2 by projecting on the $-\text{eigenspace}$ $pr_- : |\mathcal{L}_\alpha| \to |\mathcal{L}_\alpha|_- = |2\Xi_0|$.

Proof. 1. Given a bundle $F \in Fix_{\sigma}SU_C(2, \omega\tilde{C})$ and a line bundle $\xi \in J\tilde{C}$ which is anti-invariant under $\sigma$, i.e., $\sigma^*\xi \sim \xi^{-1}$, we have a natural non-degenerate quadratic form with values in the canonical bundle $\omega_{\tilde{C}}$

\[
q : F \otimes \xi \longrightarrow \omega_{\tilde{C}} \\
s \longmapsto s \wedge \sigma^*s
\]

where $s$ is a local section of $F \otimes \xi$. Note that we have canonical isomorphisms

\[
\sigma^*(F \otimes \xi) = F \otimes \xi^{-1} = \text{Hom}(F \otimes \xi, \omega_{\tilde{C}})
\]

Therefore we are in a position to apply the Atiyah-Mumford lemma \cite{M1} to the family of bundles (here $F$ is fixed, with $\sigma^*F \sim F$)

\[
\{F \otimes \xi\}_{\xi \in P}
\]

which states that the parity of $h^0(\tilde{C}, F \otimes \xi)$ is constant when $\xi$ varies in $P$.

\[\text{From now on, we suppose } F = \tilde{E} = \pi^*E, \text{ with } E \in SU_C(2, \omega\alpha), \text{ then}\]

\[
h^0(\tilde{C}, \tilde{E}) = 2h^0(C, E) \equiv 0 \mod 2.
\]

For the first equality we use the fact that $H^0(\tilde{C}, \tilde{E}) = H^0(C, E) \oplus H^0(C, E\alpha)$ and, by Riemann-Roch and Serre duality, $h^0(C, E) = h^1(C, E) = h^0(C, \omega \otimes E^*) = h^0(C, E\alpha)$.

First suppose that $E \in SU_C(2, \omega\alpha)$ is general. Then the divisor $D(\tilde{E})$ does not contain the Prym variety $P$ (e.g. because, for general $E$, $h^0(E) = 0 \iff h^0(\tilde{E}) = 0 \iff O \notin D(\tilde{E})$), so the restriction of the divisor $D(\tilde{E}) \in |2\Theta_{\tilde{C}}|$ to $P$ is a divisor in the linear system $|4\Xi_0|$. Moreover, for $\xi \in D(\tilde{E}) \cap P$

\[
\text{mult}_\xi D(\tilde{E}) \geq h^0(\tilde{C}, \tilde{E} \otimes \xi) \geq 2
\]

because $h^0(\tilde{C}, \tilde{E} \otimes \xi) \equiv h^0(\tilde{C}, \tilde{E}) \equiv 0 \mod 2$. Hence any point $\xi \in D(\tilde{E}) \cap P$ is a singular point of $D(\tilde{E})$, which implies that $D(E) \cap P$ is an everywhere non-reduced divisor. We have

3.3. Lemma. Suppose that $D(\tilde{E}) \cap P$ is a divisor in $P$. Then there is a divisor $\Delta(E) \in |2\Xi_0|$ such that $D(E) \cap P = 2\Delta(E)$.

Proof. A local equation of $\Delta(E)$ is given by the pfaffian of a skew-symmetric perfect complex of length one $\Lambda \longrightarrow L^*$ representing the perfect complex $\text{Rpr}_{1*}(\mathcal{P} \otimes pr_2^*\tilde{E})$ where $\mathcal{P}$ is the Poincaré line bundle over the product $P \times \tilde{C}$ and $pr_1, pr_2$ are the projections on the two factors. The construction of the complex $\Lambda \longrightarrow L^*$ is given in the proof of Proposition 7.9 \cite{LS}.

\[\square\]
If $E$ is of the form $E = \pi_*L$ for some $L \in P_{\text{even}}$, we have $\Delta(E) = T^*_{\alpha\omega} \Xi + T_{\omega L^{-1}} \Xi$. It follows from this equality that $\Delta(E)$ is reduced for general $E$.

So far we have defined a rational map $\Delta : SU_C(2, \omega) \longrightarrow |2\Xi_0|$. It will follow from part 2 of the proposition that $\Delta$ can be defined away from $\varphi(P_{\text{odd}})$.

2. First we consider the composite (rational) map

$$\text{Pic}^{g-1}(C) \xrightarrow{\psi} SU_C(2, \omega) \xrightarrow{\Delta} |2\Xi_0|.$$  

A straightforward computation shows that for all $\xi \in \text{Pic}^{g-1}(C)$ such that $\pi^* \xi \notin P_{\text{odd}}$ the divisor $\Delta(\psi(\xi)) = \Delta(\xi \oplus \omega \alpha \xi^{-1})$ equals the translated divisor $T_{\pi^* \xi} \tilde{\Theta}$ restricted to $P$. Hence, by [M2], the map $\Delta \circ \psi$ is given by the full linear system $|\mathcal{L}_{\omega \alpha}|_+$ of invariant elements of $|\mathcal{L}_{\omega \alpha}|$. By Prym-Wirtinger duality ($\mathcal{L}_\alpha$) and (1.3) $|\mathcal{L}_{\omega \alpha}|_+ \cong |\mathcal{L}_\alpha|_+ \cong |2\Xi_0|$ and we obtain the commutative diagram in the proposition. Geometrically, $\Delta$ is obtained by restricting the projection with center the $-\varepsilon$-space $|\mathcal{L}_\alpha|_-$ to the embedded moduli space $SU_C(2, \omega) \subset |\mathcal{L}_\alpha|$. Since by [NR] $|\mathcal{L}_\alpha|_+ \cap SU_C(2, \omega) = \varphi(P_{\text{odd}})$ we see that $\Delta$ is well-defined for $E \notin \varphi(P_{\text{odd}})$ even if $D(E) \supset P$.

3.4. Remark. We observe that we obtain by the same construction a rational map

$$\Delta : SU_C(2, \omega) \longrightarrow |2\Xi_0|.$$  

The images under $\Delta$ of the two moduli spaces $SU_C(2, \omega)$ and $SU_C(2, \omega \alpha)$ coincide, which is easily deduced from the following formula. Let $\beta$ be a 4-torsion point such that $\beta^\otimes 2 = \alpha$ and $\pi^* \beta \in P[2]$. Then, for any $E \in SU_C(2, \omega)$, we have $E \otimes \beta \in SU_C(2, \omega \alpha)$ and

$$T^*_{\pi^* \beta} \Delta(E) = \Delta(E \otimes \beta).$$

Similar statements hold for $SU_C(2, \alpha)$.

4 Proof of theorem 2

4.1 Proof of $\Gamma_{C-C}^{\alpha+} = \Gamma_{00}$

The strategy is to show that the two linear maps

$$\phi_1 : H^0(P, 2\Xi_0) \longrightarrow \text{Sym}^2 T^*_p P = \text{Sym}^2 H^0(\omega \alpha)$$

and

$$\phi_2 : H^0(JC, \mathcal{L}_\alpha)_{+0} \longrightarrow H^0(C \times C, \delta^* \mathcal{L}_\alpha - 2\Delta)_+ = \text{Sym}^2 H^0(\omega \alpha)$$

differ by multiplication by a scalar under the isomorphism (1.2) $H^0(JC, \mathcal{L}_\alpha)_{+0} \cong H^0(P, 2\Xi_0)$. Here the subscript 0 denotes the subspace (on $P$ or $JC$) consisting of global sections vanishing at the origin. The map $\phi_1$ sends $s \in H^0(P, 2\Xi_0)$ to the quadratic term of its Taylor expansion at the origin $O \in P$ and $\phi_2$ is the pull-back of invariant sections of $\mathcal{L}_\alpha$ under the difference map

$$\delta : C \times C \longrightarrow JC$$

$$(p, q) \longmapsto \mathcal{O}_C(p - q).$$

By restricting to the fibers of the two projections $p_i : C \times C \to C$ and using the See-saw Theorem, we compute that $\delta^* \mathcal{L}_\alpha = p^*_1(\omega \alpha) \otimes p^*_2(\omega \alpha)(2\Delta_C)$ where $\Delta_C \subset C \times C$ is the diagonal. Since $\phi_2^{-1}(0) = \Delta_C$ and the sections of $\mathcal{L}_\alpha$ are symmetric, we see that $\text{im} \phi_2 \subset \text{Sym}^2 H^0(\omega \alpha) \subset H^0(\omega \alpha)^{\otimes 2}$.
\[ H^0(p_1^*(\omega_\alpha) \otimes p_2^*(\omega_\alpha)) \subset H^0(p_1^*(\omega_\alpha) \otimes p_2^*(\omega_\alpha)(2\Delta_C)). \] So if \( \phi_1 \) and \( \phi_2 \) are proportional, we will have
\[ \Gamma_{00} = \ker \phi_1 = \ker \phi_2 = \Gamma^+_{C-C}. \]

To show that \( \phi_1 = \lambda \phi_2 \) for some \( \lambda \in \mathbb{C}^* \), we compute \( \phi_1(s_E) \) and \( \phi_2(s_E) \) for special sections, namely those with divisor of zeros \( Z(s_E) = \Delta(E) \) for some vector bundle \( E \in SU_C(2, \omega_\alpha) \) with \( h^0(E) = h^0(E \otimes \alpha) = 2 \). Recall that by Riemann-Roch and Serre duality we have for \( h^0(E) = h^0(E \otimes \alpha) \) for \( E \in SU_C(2, \omega_\alpha) \). Now to compute \( \phi_1(s_E) \), we need to determine the tangent cone to \( \Delta(E) \) at \( \mathcal{O} \in P \). As before we put \( \bar{E} = \pi^*E \). By [I] prop. V.2, this tangent cone is the intersection of the anti-invariant part \( H^0(\omega_C^-) \) of \( H^0(\omega_C^-) = T^*_0\mathcal{J}C \) with the affine cone over the projective cone over the Grassmannian \( Gr(2, H^0(\bar{E})^*) \subset \mathbb{P}^2H^0(\bar{E})^* \) under the linear map
\[ \mu^* : H^0(\omega_C^+) \longrightarrow \Lambda^2H^0(\bar{E})^* \] which is the dual of the map \( \mu : \Lambda^2H^0(\bar{E}) \rightarrow H^0(\omega_C^-) \) obtained from exterior product by the isomorphism \( \Lambda^2\bar{E} \cong \omega_C^- \). Note that the \( \sigma \)-invariant part \( [\Lambda^2H^0(\bar{E})^+]_\sigma \) is canonically isomorphic to the 2-dimensional subspace \( \Lambda^2H^0(\bar{E})^* \oplus \Lambda^2H^0(\omega_C^-) \subset \Lambda^2H^0(\bar{E})^* \) because \( H^0(\bar{E})_\sigma = H^0(E) \) and \( H^0(\bar{E})_\sigma = H^0(\omega_C^-) \). Since \( \Lambda^2\bar{E} \cong \Lambda^2(E \otimes \alpha) \cong \omega_\alpha \), the restriction of \( \mu \) to \( \Lambda^2H^0(E) \) (resp. \( \Lambda^2H^0(E \otimes \alpha) \)) which is obtained from exterior product by the isomorphism \( \Lambda^2E \cong \omega_\alpha \) (resp. \( \Lambda^2(E \otimes \alpha) \cong \omega_\alpha \) maps into \( H^0(\omega_\alpha) \). Therefore the linear map \( \mu^* \) maps \( \sigma \)-anti-invariant sections into \( \sigma \)-invariant sections, i.e.,
\[ \mu^*_+ : H^0(\omega_\alpha) \longrightarrow \Lambda^2H^0(\bar{E})^* \oplus \Lambda^2H^0(\omega_\alpha)^* \] Since the intersection \( \mathbb{P}(\Lambda^2H^0(E)^* \oplus \Lambda^2H^0(\omega_\alpha)^*) \cap Gr(2, H^0(\bar{E})^*) \) consists of the two points \( \mathbb{P}(\Lambda^2H^0(E)^*) \) and \( \mathbb{P}(\Lambda^2H^0(\omega_\alpha)^*) \), it follows that the intersection of \( H^0(\omega_C^-) \subset H^0(\omega_C^-) \) with the cone over \( Gr(2, H^0(\bar{E})^*) \) is the union of the two lines \( \Lambda^2H^0(E) \) and \( \Lambda^2H^0(E \otimes \alpha) \). Therefore the tangent cone of \( \Delta(E) \) at the origin is the union of the two hyperplanes in \( |\omega_\alpha|^* \) which are the zeros of \( a, b \in H^0(\omega_\alpha) \) such that
\[ a\mathcal{C} = \text{im} (\Lambda^2H^0(E) \longrightarrow H^0(\omega_\alpha)) \quad b\mathcal{C} = \text{im} (\Lambda^2H^0(\omega_\alpha) \longrightarrow H^0(\omega_\alpha)). \]

In other words, up to multiplication by a nonzero scalar,
\[ \phi_1(s_E) = a \otimes b + b \otimes a \in \text{Sym}^2H^0(\omega_\alpha). \]

We now compute \( \phi_2(s_E) \). First we note that the pull-back map induced by \( \delta \) is equivariant for the involution \( (-1) : \xi \mapsto \xi^{-1} \) acting on \( JC \) and the involution \( (p, q) \mapsto (q, p) \) acting on \( C \times C \). Since \( \Delta(E) = pr_+(D(E)) \) by Proposition 3.1, this implies that
\[ \phi_2(s_E) = \phi_2(pr_+(s_E)) = pr_+(\delta^*(s_E)) \] On the RHS \( pr_+ \) denotes the projection \( H^0(\omega_\alpha) \otimes H^0(\omega_\alpha) \rightarrow \text{Sym}^2H^0(\omega_\alpha) \). Therefore we compute
\[ \delta^*(D(E)) = \{(p, q) \in C \times C \mid h^0(E(p-q)) > 0\} \]
and take its symmetric part. It follows from [VGI] lemma 3.2 that
\[ \delta^*(D(E)) = C \times Z_a + Z_b \times C + 2\Delta_C \] where \( Z_a \) (resp. \( Z_b \)) is the divisor of zeros of \( a \) (resp. \( b \)). Hence it follows from (4.3) and (4.4) that \( \phi_2(pr_+(s_E)) = a \otimes b + b \otimes a \) up to multiplication by a nonzero scalar. We can now conclude that \( \phi_1 = \lambda \phi_2 \) for some \( \lambda \in \mathbb{C}^* \) because, by the following lemma (prop. 3.7 [VGI]), we have enough bundles \( E \in SU_C(2, \omega_\alpha) \) with \( h^0(E) = 2 \) to generate linearly the image \( \text{Sym}^2H^0(\omega_\alpha) \) of \( \phi_1 \) and \( \phi_2 \).
4.1. Lemma. (prop. 3.7 [G]) For general sections \(a, b \in H^0(\omega \alpha)\), we can find a semi-stable bundle \(E \in SU_C(2, \omega \alpha)\) with \(h^0(E) = 2\) such that (4.3) holds.

4.2 Proof of \(\Gamma_C^0 \subset C = \Gamma_C^{(2)}\)

First note that any anti-invariant section of \(L_{\alpha}\) vanishes at \(O \in JC\). Denote by \(\tau : H^0(JC, L_{\alpha})_\to \to T^*_0JC = H^0(\omega)\)

the map which sends an element \(s\) of \(H^0(JC, L_{\alpha})_\to\) to the linear term of its Taylor expansion at the origin (Gauss map). Recall the natural embedding of the curve \(\bar{C}\) into the Prym variety \(P'_{\bar{C}}\): \(\bar{C} \to \to P'_{\bar{C}}\)

Then \(i^*\mathcal{O}(2\Xi'_0) \cong \omega_{\bar{C}}\) and since all \(2\Xi'_0\)-divisors are symmetric and \(i\) is equivariant for the involution, \(i\) induces a linear map

\[i^* : H^0(P', 2\Xi'_0) \to H^0(C, \omega) = H^0(\bar{C}, \omega_{\bar{C}})_+\] (4.7)

4.2. Lemma. The linear maps \(\tau\) and \(i^*\) are proportional via the isomorphism (1.2) and are surjective.

Proof. It will be enough to show that the canonical divisors \(i^*(D(\pi_*\lambda))\) and \(\tau(D(\pi_*\lambda))\) are equal for a general element \(\lambda \in P_{\text{odd}}\). In both cases the divisor coincide with the divisor \(Nm(\delta)\), where \(\delta\) is the unique effective divisor in the linear system \(|\lambda|\). The computations are straight-forward and left to the reader. \(\square\)

Therefore we can conclude that

\[H^0(JC, L_{\alpha})_{(3)_-}^{(3)} = \ker \tau = \ker i^* = \Gamma_{\bar{C}}.\]

where \(H^0(JC, L_{\alpha})_{(3)_-}^{(3)}\) denotes the subspace of \(H^0(JC, L_{\alpha})_\to\) of elements with multiplicity \(\geq 2\) (hence \(\geq 3\) by anti-symmetry) at the origin. We now proceed as in the proof of part 1 of Theorem 2. We consider the two linear maps

\[\phi_1 : \Gamma_{\bar{C}} \to \Lambda^2 H^0(\omega \alpha)\]

\[\phi_2 : H^0(JC, L_{\alpha})_{(2)_-}^{(2)} \to H^0(C \times C, \delta^* L_{\alpha}(-2\Delta))_\to = \Lambda^2 H^0(\omega \alpha)\]

which are defined as follows. As in part 1, \(\phi_2\) is the map given by pull-back under the difference map \(\delta\). To define \(\phi_1\), let \(N_{\bar{C}/P'}\) denote the normal bundle of \(i(\bar{C})\) in \(P'\). Then \(\phi_1\) is obtained by restricting a section \(s \in \Gamma_{\bar{C}}\) to the first infinitesimal neighborhood of \(\bar{C}\). In other words

\[\Gamma_{\bar{C}}^{(2)} = \ker \{\phi_1 : \Gamma_{\bar{C}} \to H^0(\bar{C}, N_{\bar{C}/P'}^* \otimes i^* \mathcal{O}(2\Xi'_0))_\to = H^0(\bar{C}, N_{\bar{C}/P'}^* \otimes \omega_{\bar{C}})_-\}\]

The vector bundle \(N_{\bar{C}/P'}^*\) fits into the exact sequence

\[0 \to N_{\bar{C}/P'}^* \to H^0(\omega \alpha) \otimes \mathcal{O}_{\bar{C}} \to \omega_{\bar{C}} \to 0\] (4.8)
where the right-hand map is the embedding $H^0(\omega) \otimes O_C \hookrightarrow H^0(\omega_C) \otimes O_C$ followed by evaluation $H^0(\omega_C) \otimes O_C \rightarrow \omega_C$. Therefore this map is the pull-back of evaluation $H^0(\omega) \otimes O \xrightarrow{ev} \omega$. Let $M$ be the kernel of the latter, i.e., we have the exact sequence

$$0 \rightarrow M \rightarrow H^0(\omega) \otimes O \xrightarrow{ev} \omega \rightarrow 0,$$

whose pull-back by $\pi$ is (4.8).

We twist (4.9) by $\omega$ and take cohomology

$$0 \rightarrow H^0(C, M \otimes \omega) \rightarrow H^0(\omega) \otimes H^0(\omega) \xrightarrow{m} H^0(\omega^2) \rightarrow \ldots$$

where $m$ is the multiplication map. We deduce that

$$H^0(\tilde{C}, N^*_C \otimes \omega_{\tilde{C}}) = H^0(C, M \otimes \omega) = \ker m = \Lambda^2 H^0(\omega) \oplus I^P_C(2)$$

where $I^P_C(2)$ is the space of quadrics through the Prym-canonical curve. It remains to show that $\im \phi_1 = \Lambda^2 H^0(\omega)$. This will follow from the next two lemmas. First we will compute, as in part 1, the image under $\phi_1$ of some special sections $s_E \in \Gamma_C$, namely $s_E$ such that $Z(s_E) = \Delta(E)$ with $E$ a general bundle in $SU_C(2, \omega)$ with $h^0(E) = 2$, i.e., we determine the tangent spaces to $\Delta(E)$ along the curve $i(C)$. This is done in the following lemma.

**4.3. Lemma.** Let $a, b$ be the sections defined by (4.3). Then we have

$$\phi_1(s_E) = a \wedge b \in \Lambda^2 H^0(\omega)$$

up to multiplication by a nonzero scalar.

**Proof.** First we need to show that for a general semi-stable bundle $E$ with $h^0(E) = 2$ the divisor $\Delta(E)$ is smooth at a general point $i(\tilde{p}) \in \Delta(E)$. For this decompose a general Prym-canonical divisor into two effective divisors of degree $g - 1$, i.e., $D + D' \in |\omega|$. Put $L = O(D)$. Then $h^0(D) = 1 = h^0(\omega(-D)) = h^0(\omega\alpha(-D)) = h^0(\alpha(D))$. If $E = L \oplus \omega\alpha L^{-1}$, then $\tilde{E} = \pi^* E = \pi^* L \oplus \omega_C \pi^* L^{-1}$, $D(\tilde{E}) = \Theta_{\pi^* L} + \Theta_{\omega_C \pi^* L^{-1}}$ and $\Delta(\tilde{E}) = \Theta_{\pi^* L|P' \cap \Theta_{\omega_C \pi^* L^{-1}|P'}$. At a general point $i(\tilde{p}) \in \Theta_{\pi^* L}$, we see immediately that the tangent space to $\Theta_{\pi^* L}$ does not contain the tangent space to $P'$, i.e., $\Delta(E)$ is smooth at $i(\tilde{p})$. Next we compute the tangent space to the divisor $\Delta(E)$ at a smooth point $i(\tilde{p}) \in \Delta(E)$. The smoothness of $\Delta(E)$ at $i(\tilde{p})$ implies that $h^0(\tilde{C}, \tilde{E}(\tilde{p} - \sigma\tilde{p})) = 2$. We choose a basis $\{u, v\}$ of the 2-dimensional vector space $H^0(\tilde{C}, \tilde{E}(\tilde{p} - \sigma\tilde{p}))$. Then by [1] prop. V.2 and the same reasoning as in the proof of part 1 of Theorem 2, we see that the projectivized tangent space $T_{i(\tilde{p})} \Delta(E)$ to $\Delta(E)$ at $i(\tilde{p})$, which is a hyperplane in $\mathbb{P} T_{i(\tilde{p})} P' \cong |\omega'|$ is the zero locus of the section in $\gamma(\tilde{p}) \in H^0(\omega)$, which is the image of $u \wedge \sigma^* v := u \otimes \sigma^* v - v \otimes \sigma^* u$ under the exterior product map

$$H^0(\tilde{E}(\tilde{p} - \sigma\tilde{p})) \otimes \sigma^* H^0(\tilde{E}(\tilde{p} - \sigma\tilde{p})) = H^0(\tilde{E}(\tilde{p} - \sigma\tilde{p})) \otimes H^0(\tilde{E}(\sigma\tilde{p} - \tilde{p})) \xrightarrow{\mu} H^0(\omega_C)$$

Since $\det E = \omega$, we see that $\gamma(\tilde{p}) = \mu(u \wedge \sigma^* v) \in H^0(\omega) \subset H^0(\omega_C)$. We will now describe the map $\gamma : \tilde{C} \rightarrow |\omega| : \tilde{p} \mapsto \gamma(\tilde{p})$. Note that, since $h^0(\tilde{E}) = 4$, we have $h^0(\tilde{E}(-\sigma\tilde{p})) = 2$ for $\tilde{p}$ general. Hence $\{u, v\}$ is also a basis for $H^0(\tilde{E}(-\sigma\tilde{p}))$. Consider the inclusion

$$H^0(\tilde{E}(-\sigma\tilde{p})) \subset H^0(\tilde{E}) = H^0(E) \oplus H^0(E\alpha)$$

and decompose $u = u_+ + u_-, v = v_+ + v_-$ with $u_+, v_+ \in H^0(E) = H^0(\tilde{E})_+$ and $u_-, v_- \in H^0(E\alpha) = H^0(\tilde{E})_-$. Then the element $\gamma(\tilde{p})$ is the image of $(u_+ \wedge v_+, -u_- \wedge v_-) \in \Lambda^2 H^0(\tilde{E}) \oplus \Lambda^2 H^0(E\alpha)$ under the exterior product map $\Lambda^2 H^0(E) \oplus \Lambda^2 H^0(E\alpha) \rightarrow H^0(\omega)$, i.e., $\gamma(\tilde{p}) \in \mathbb{P}(Ca \oplus Cb) \subset |\omega|$. Since $\tilde{C} \subset \Delta(E)$, we have $\varphi_{\text{can}}(p) \in T_{i(\tilde{p})}(\Delta(E))$. So for general $\tilde{p}$, $\gamma(\tilde{p})$ is the unique divisor of the pencil $\mathbb{P}(Ca \oplus Cb)$ containing $\tilde{p}$. Hence we can conclude that the section $\phi_1(s_E) \in H^0(M \otimes \omega)$ considered as a tensor in $H^0(\omega) \otimes H^0(\omega)$ is $a \wedge b$. \hfill \Box
Since, a priori, we do not know that $\mathbb{P}\Gamma_C$ is spanned by divisors of the form $\Delta(E)$, we need to establish a symmetry property for any divisor $D \in \mathbb{P}\Gamma_C$. This is done as follows.

Let $s, t \in \tilde{C}$ be two points of $\tilde{C}$ with respective images $s, t \in C$ and let $D$ be an element of $\mathbb{P}\Gamma_C$. Assume that $i(\tilde{s}), i(\tilde{t}) \in D$ are smooth points of $D$ and let $\mathbb{T}_s D$ and $\mathbb{T}_t D$ denote the projectivized tangent spaces to the divisor $D$ at the points $i(\tilde{s})$ and $i(\tilde{t})$. Since we can identify the projectivized tangent space to the Prym variety $P'$ at any point with the Prym-canonical space $|\omega_\alpha|^*$, we may view $\mathbb{T}_s D$ and $\mathbb{T}_t D$ as hyperplanes in $|\omega_\alpha|^*$.

Note that $\mathbb{T}_s D$ only depends on $s \in C$ and not on the lift $\tilde{s} \in \tilde{C}$. Then we have

**4.4. Lemma.** With the preceding notation, we have an equivalence

\[ \varphi_{\text{can}}(s) \in \mathbb{T}_t D \iff \varphi_{\text{can}}(t) \in \mathbb{T}_s D \]

**Proof.** Consider the invertible sheaf $x = \mathcal{O}_C(\tilde{s} - \sigma \tilde{s} + \tilde{t} - \sigma \tilde{t}) \in P$ and the corresponding embedding

\[ i_x : \tilde{C} \to P' \quad \tilde{p} \mapsto \mathcal{O}_C(\tilde{p} - \sigma \tilde{p}) \otimes x. \]

The curve $i_x(\tilde{C})$ is the curve $i(\tilde{C})$ translated by $x$. A straight-forward computation shows that $i_x^{-1}(\mathcal{O}_{P'}(2\Xi_0)) = \omega_C x^{-2}$ and by a result of Beauville (see [KnS] page 569) the induced linear map on global sections $H^0(P', 2\Xi_0) \to H^0(\omega_C x^{-2})$ is surjective. We observe that

\[ i_x(\sigma \tilde{t}) = i(\tilde{s}), \quad i_x(\sigma \tilde{s}) = i(\tilde{t}), \]

and that the projectivized tangent line to the curve $i_x(\tilde{C})$ at the point $i_x(\sigma \tilde{t})$ (resp. $i_x(\sigma \tilde{s})$) is the point $\varphi_{\text{can}}(t)$ (resp. $\varphi_{\text{can}}(s)$) in $|\omega_\alpha|^* \cong \mathbb{P}T_i(\tilde{s}) P'$ (resp. $\cong \mathbb{P}T_i(\tilde{t}) P'$). Let $\mathbb{T}_s$ (resp. $\mathbb{T}_t$) denote the embedded tangent line in $|2\Xi_0|^*$ to the curve $i_x(\tilde{C})$ at the point $i_x(\sigma \tilde{t})$ (resp. $i_x(\sigma \tilde{s})$), so that $\mathbb{T}_s$ (resp. $\mathbb{T}_t$) passes through the point $i(\tilde{s})$ (resp. $i(\tilde{t})$) with tangent direction $\varphi_{\text{can}}(t)$ (resp. $\varphi_{\text{can}}(s)$). Then the lemma will follow if we show that these two tangent lines intersect in a point $I(\tilde{s}, \tilde{t})$, i.e.

\[ \mathbb{T}_s \cap \mathbb{T}_t = I(\tilde{s}, \tilde{t}) \in |2\Xi_0|^*. \]  (4.10)

This property follows from a dimension count: since $C$ is non-hyperelliptic, we have $x^{-2} \neq \mathcal{O}_C$, so $h^0(\omega_C x^{-2}) = 2g - 2$. Since $h^0(\omega_C x^{-2}(-2\tilde{s} - 2\tilde{t})) = h^0(\omega_C(-2\tilde{s} - 2\tilde{t})) \geq 2g - 5$, the tangent lines $\mathbb{T}_s$ and $\mathbb{T}_t$ are contained in a projective 2-plane, hence intersect. To get the equivalence stated in the lemma, let $H_D$ denote the hyperplane in $|2\Xi_0|^*$ corresponding to the divisor $D \in \mathbb{P}\Gamma_C$. Assume e.g. that $\varphi_{\text{can}}(s) \in \mathbb{T}_t D$. This means that $H_D$ contains $\mathbb{T}_t$. Since $i(\tilde{s}) \in H_D$, it follows from (4.10) that $H_D$ also contains $\mathbb{T}_s$, so $\varphi_{\text{can}}(t) \in \mathbb{T}_s D$. \hfill $\Box$

At this stage we can conclude: by lemma [4] we know that for all $s \in \Gamma_C$, $\phi_1(s) \in H^0(\omega_\alpha) \otimes H^0(\omega_\alpha)$ lies either in the symmetric or skew-symmetric eigenspace, i.e. im $\phi_1 \subset I_C^{\text{kr}}(2) \subset \text{Sym}^2 H^0(\omega_\alpha)$ or im $\phi_1 \subset \Lambda^2 H^0(\omega_\alpha)$. Lemma [3] asserts that im $\phi_1 \subset \Lambda^2 H^0(\omega_\alpha)$.

As in [4], we have that $\phi_2(\text{pr}_-(s_E)) = \text{pr}_-(\delta(s_E))$, where $\text{pr}_-$ denotes the projection $H^0(\omega_\alpha) \otimes H^0(\omega_\alpha) \to \Lambda^2 H^0(\omega_\alpha)$ and $s_E$ is as above. Hence we see that $\phi_2(\text{pr}_-(s_E)) = a \wedge b$.

By lemma [3] the projectivizations of $\phi_1$ and $\phi_2$ coincide on all divisors of the form $\Delta(E)$ whose images generate $\mathbb{P}\Lambda^2 H^0(\omega_\alpha)$. Hence $\phi_1 = \phi_2$ up to a nonzero scalar and $\phi_1$ and $\phi_2$ are surjective.

**4.5. Remark.** An alternative way of proving that im $\phi_1 \subset \Lambda^2 H^0(\omega_\alpha)$ would be to twice take the derivative of the quadriseccant identity for Prym varieties [F] prop. 6 (fix two points and consider the other two as canonical coordinates on the universal cover of $\tilde{C}$.)
5 The scheme-theoretical base locus of $\mathbb{P}^\Gamma_C$

From section 2 we know that the sets $Bs(\mathbb{P}^\Gamma_C)$ and $i(\bar{C})$ are equal. To prove the scheme-theoretical equality, it will be enough to show that, $\forall \bar{p} \in \bar{C}$, the projectivized tangent spaces at $i(\bar{p})$ to divisors $D \in \mathbb{P}^\Gamma_C$ cut out the projectivized tangent space at $i(\bar{p})$ to $i(\bar{C})$, which is $\mathcal{F}_{\text{can}}(p) \in |\omega\alpha|^* = \mathbb{P}T_{i(\bar{p})}P'$, i.e.,

$$\bigcap_{D \in \mathbb{P}^\Gamma_C} T_{i(\bar{p})}D = \mathcal{F}_{\text{can}}(p). \quad (5.1)$$

If we take $D = \Delta(E)$ for some semi-stable vector bundle $E$ with $h^0(E) = 2$ (see section 4.2) then the hyperplane $T_{i(\bar{p})}(\Delta(E)) \subset |\omega\alpha|^*$ corresponds to the unique section of the pencil $\mathbb{P}((Ca \oplus Cb)$ vanishing at $p$ (proof of lemma 4.3). Since for general $a, b \in |\omega\alpha|$ we can find a vector bundle $E$ (lemma 4.1) such that equality in lemma 4.3 holds, we can conclude (5.1).

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Elham Izadi  
Department of Mathematics  
Boyd Graduate Studies Research Center  
University of Georgia  
Athens, GA 30602-7403  
USA  
email: izadi@math.uga.edu

Christian Pauly  
Laboratoire J.-A. Dieudonné  
Université de Nice Sophia Antipolis  
Parc Valrose  
06108 Nice Cedex 02  
France  
email: pauly@math.unice.fr