Quantum Tunnelings with Global Charge

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Abstract

We investigate quantum tunneling in the theory of a complex scalar field with a global $U(1)$ symmetry when the charge density of the initial configuration does not vanish. We discuss the possible final configurations and set up the Euclidean path integral formalism to find the bubble nucleation and to study the bubble evolution. For the stationary path, or the bounce solution, in the Euclidean time, the phase variable becomes pure imaginary so that the charge density remains real. We apply this formalism to examples when the initial charge density is small. While the phase transition considered here occurs in zero temperature, the bubble dynamics is richly complicated, involving conserved charge, the sound wave and the supersonic bubble wall.

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1. Introduction

Recently, there have been some interests in the first order phase transition involving nonzero global charge. The finite temperature effective potential and the phase structure in this type of the phase transition have been extensively studied.\(^1\) However, how the phase transition proceeds has not been discussed in detail. Here we investigate the phase transition in a model which involves nonzero charge and a nontrivial bubble wall dynamics. This model is the theory of a complex scalar field with a global abelian symmetry. Even in the zero temperature phase transition, this model has a rather rich dynamics in the phase transition, depending on the initial configuration and the potential energy. Since there are only two field degrees of freedom in this model, this model can be rather easily approached analytically and numerically. We hope that our toy model illuminates some aspects of the phase transition involving global charge and that some of insights gained would be applicable to the QCD phase transition and the electroweak phase transition.

The general formalism of the Euclidean phase intergal involving nonzero charge in our model has been developed sometime ago.\(^2\) While this formalism has been applied to wormhole physics, there has been no direct attempt to apply for the first order phase transition. We extend this formalism to the first order phase phase transition at zero temperature, following the standard formalism.\(^3\) One interesting aspect of our formalism is that the stationary path of the angle variable of the complex scalar field becomes pure imaginary. From this Euclidean path integral, one can find the bounce solution and calculate the bubble nucleation rate. In addition, one can gain some insights on the bubble evolution.

The initial configuration we are interested in here is a homogeneous configuration which is classically stable but not quantum mechanically. The charge density of the initial configuration is nonzero and uniform. The final configuration after the phase transition, it turns out, could be more complicated than the configuration of the lowest potential energy. In our model, there could be an attractive force between charges and charges clump together forming Q-balls.\(^4\) Thus, the fi-
nal configuration could be inhomogeneous with Q-balls floating in the symmetric phase.

The Euclidean path integral allows us to calculate the imaginary part of the energy for the metastable initial configuration by the semiclassical method. The contribution to the path integral is dominated by bounce solutions. Contrasted to the usual case, the initial charge density now breaks the $O(4)$ symmetry to the $O(3)$ symmetry. In this paper we focus the case of small initial charge density, where the bounce solution is close the $O(4)$ bounce solution of zero charge. When the charge density is small, we can look at the perturbative correction to the $O(4)$ symmetric solution. The current conservation equation in this background turns out to be a boundary value problem in the classical electrodynamics and can be solved in the thin wall limit. This leads to some insights on the current flow in the bounce solution and the deformation of the $O(4)$ symmetric bounce solution. This in turn leads to an understanding of the bubble evolution via the analytic continuation.

When there is nonzero charge density and the initial configuration is metastable, there is always the sound wave of the speed less than the speed of light. When the bubble of a “true vacuum” is nucleated, it will expand. The bubble wall speed could reach the sound speed of the initial configuration in finite time, becoming supersonic. In addition, some bubbles speed up to the speed of light in finite time, which implies some sort of a new instability.

The plan of this paper is as follows. In Sec.2, we introduce the theory of a complex scalar field. We study the stability condition of the possible initial configurations and discuss the final configurations we expect after the phase transition. In Sec.3, we study the Euclidean path integral. We offer a formal discussion of the bubble nucleation and evolution. In Sec.4, we examine in detail various examples when the initial charge density is small, and present the qualitative pictures of the bubble nucleation and evolution. In Sec.5, we conclude with some remarks and questions.
2. Model

We study the theory of a complex scalar field \( \phi = f e^{i \theta} / \sqrt{2} \) with a global \( U(1) \) symmetry. The lagrangian is given by

\[
\mathcal{L} = |\partial_{\mu} \phi|^2 - U(\sqrt{2}\phi) = \frac{1}{2} (\partial_{\mu} f)^2 + \frac{1}{2} f^2 (\partial_{\mu} \theta)^2 - U(f) \tag{2.1}
\]

The global symmetry arising from a constant shift of \( \theta \) leads to the conserved current

\[
J_{\mu} = -i (\phi^* \partial_{\mu} \phi - \partial_{\mu} \phi^* \phi) = f^2 \partial_{\mu} \theta \tag{2.2}
\]

The total charge is \( Q = \int d^3 x f^2 \dot{\theta} \). While there could be global strings in this theory, they seem not to play any essential role in our discussion and will be neglected here.

In this paper we are interested in quantum evolution of a metastable initial configuration with nonzero charge density. To start, we should have a proper description of possible initial configurations. While some of them could be inhomogeneous and localized, here we will focus on the static homogeneous initial configurations \( (f, \theta = wt) \) with a given charge density \( \rho = f^2 \dot{\theta} = f^2 w \). The energy density for these configurations is given by

\[
e(\rho, f) = \frac{\rho^2}{2f^2} + U(f) \tag{2.3}
\]

The first term in the right hand side of Eq.(2.3) is the centrifugal term due to the conserved charge. An initial configuration would lie at the local minimum of \( e(\rho, f) \) to be stable under homogeneous fluctuations, and satisfies

\[
\frac{\partial e}{\partial f} = -\frac{\rho^2}{f^3} + U'(f) = 0 \tag{2.4}
\]
and \( \frac{\partial^2 e}{\partial f^2} > 0 \) or

\[
3\frac{\rho^2}{f^4} + U'' > 0
\]  

(2.5)

We denote a local minimum by \( f_0 \) and \( w_0 = \rho/f_0^2 \). Fig.1 shows an example of \( e(\rho, f) \) and \( U(f) \) for nonzero \( \rho \).

A metastable initial configuration should be stable, even under small inhomogeneous fluctuations, \( f_0 + \delta f \) and \( \theta = w_0 t + \delta \theta \). From the classical field equation, we get the linear equation for the fluctuations,

\[
-\delta \ddot{f} + \partial_\xi^2 \delta f + w_0^2 \delta f + 2w_0f_0\dot{\theta} - U''(f_0)\delta f = 0 \\
-2w_0f_0\dot{\delta \phi} - f_0^2\delta \dot{\theta} + f_0^2\partial_\xi^2 \delta \theta = 0
\]  

(2.6)

With \( \delta f, \delta \theta \sim e^{i(\alpha t - \vec{k} \cdot \vec{r})} \), Eq.(2.6) yields a dispersion relation

\[
[\alpha^2 - \vec{k}^2][\alpha^2 - \vec{k}^2 + w_0^2 - U''(f_0)] - 4\alpha^2w_0^2 = 0
\]  

(2.7)

or

\[
\alpha^2 = \vec{k}^2 + \frac{3w_0^2 + U''}{2} \pm \left\{ \left( \frac{3w_0^2 + U''}{2} \right)^2 + 4w_0^2\vec{k}^2 \right\}^{1/2}
\]  

(2.8)

One can see that there are one massive and one massless modes. The massless mode can be interpreted as the sound wave and satisfies the dispersion relation \( \alpha^2 = v_s^2\vec{k}^2 \) for the long wave length, or for small \( \vec{k} \), where the sound speed is given by

\[
v_s^2 = \frac{f_0^4U''(f_0) - \rho^2}{f_0^4U''(f_0) + 3\rho^2}
\]  

(2.9)

Thus, the configuration is stable under small fluctuations only if

\[
f_0^4U''(f_0) - \rho^2 > 0
\]  

(2.10)

which is a stronger condition than Eq.(2.5).
When the charge density is high, the centrifugal term would be balanced by the highest order term in the potential, say \( U \approx \epsilon f^n/n \). From Eqs.\((2.4)\) and \((2.9)\) we see that the sound speed in this limit is given as

\[
v_s = \sqrt{\frac{n - 2}{n + 2}} \quad (2.11)
\]

It is interesting to note that the potentials of \( n = 4 \) in four dimensions, \( n = 6 \) in three dimensions and \( n = \infty \) in two dimensions are renormalizable interactions and that the corresponding sound speeds \( 1/\sqrt{3} \), \( 1/\sqrt{2} \), 1 are those of hot relativistic gases in the corresponding dimensions.

The stability condition \((2.10)\) can be examined more concretely in the potential

\[
U(f) = \frac{m^2}{2} f^2 + \frac{g}{4} f^4 + \frac{\lambda}{6} f^6 + \frac{\kappa}{8} f^8 \quad (2.12)
\]

(Here we are not concerned about the renormalizability of the theory. We are interested in the general characteristics of the tunneling when global charge is involved.) Assume that the charge density is very small compare with other scale. When \( m^2 > 0 \), the condition \((2.4)\) can be satisfied for \( f_0 \) near the symmetric phase,

\[
f_0 = \sqrt{\rho/m} + \mathcal{O}(\rho^2) \quad (2.13)
\]

The stability condition \((2.10)\) becomes

\[
f_0^4 U''(f_0) - \rho^2 = 3g(\frac{\rho}{m})^3(1 + \mathcal{O}(\rho)) > 0 \quad (2.14)
\]

which is satisfied only if \( g > 0 \). The configuration is thus stable only if there is a short range repulsion due to the self interaction. The velocity of the sound in the case \( g > 0 \) is given as

\[
v_s^2 = \frac{3g\rho}{4m^3}(1 + \mathcal{O}(\rho)) \quad (2.15)
\]

which is much smaller than the unity when the charge density is small.
In the case where the potential (2.12) takes a local minima at \( f = v \), \( U \approx \frac{1}{2} m^2 (f - v)^2 \) with \( m^2 = U''(v) \). The condition (2.4) implies \( \rho^2 \approx m^2 v^3 (f_0 - v) \).

The stability condition (2.10) becomes

\[
m^2 v^4 + 3\rho^2 + v\rho^2 U'''(v)/m^2 + \mathcal{O}(\rho^3) > 0
\]

which is automatically satisfied when the charge density is small. The sound velocity becomes

\[
v_s^2 = 1 - \frac{4\rho^2}{m^2 v^4} + \mathcal{O}(\rho^4)
\]

This shows that the massless Goldstone boson becomes the sound wave as the nonzero charge density is introduced in the broken phase.

We have examined the stability condition on the possible initial configurations. If an initial configuration is metastable, it will evolve quantum mechanically so that the potential energy is converted to the kinetic energy of bubbles and the radiation energy. The important question is how we know whether an initial configuration is metastable quantum mechanically? We would say a configuration is not stable quantum mechanically if we can find a field configuration of a lower energy and the same quantum number so that there is no superselection rule preventing the transition between these two configurations.

In the case of the theory of a real scalar field, the answer comes immediately from the potential energy density.\[^3\] The potential energy density for the final configuration would be the lowest. A metastable initial configuration can decay via nucleations of bubbles whose interior is in the true vacuum. We know well how this phase transition proceeds.

For the theory of a complex scalar field, the story is more complicated as there are two degrees of freedom which work together or against each other. If we regularize the system in a large finite box, we expect that the possible final configuration has the minimum energy for a given total charge. The excessive energy of the initial configuration would be channeled into elementary excited modes in the
final configuration through the radiations and the bubble collisions. In general, it is not easy to find such a final configuration. A possible final configuration could be inhomogeneous. In this paper, we concentrate on the cases where the potential $U(f)$ has a local minimum at $f = 0$ and the phase transition occurs between the $f \approx 0$ and $f \neq 0$ phases. Fig.2 shows such potentials. In addition, we assume that the initial charge density is small compared with other scales in the problem.

In these limits, we can analyze the problem in a somewhat satisfactory way. As far as the $f$ field is concerned, this field wants to settle at the ground state of $e(\rho, f)$ in Eq.(2.3). However, we will see that the charge will move so that the energy per unit charge $e(\rho, f)/\rho$ takes the lowest value. Since we concern the phase transition between the symmetric and asymmetric phases, we choose $U(f = 0) = 0$. It turns out that there are three different cases of the phase transitions to consider: Case A) The ground state of $U(f)$ is an asymmetric phase where $U(f) < 0$ and the initial configuration is in the symmetric phase; Case B) The potential $U(f)$ has a local minima at an asymmetric phase where $U(f) > 0$ and the initial configuration is in the asymmetric phase; Case C) The potential $U(f)$ is identical to Case B but the initial configuration is in the symmetric phase. In Fig.2 $U_1(f)$ corresponds to Case A and $U_2(f)$ does to Cases B,C.

Let us first study the first Cases B,C where the symmetric phase is the ground state of $U(f)$. To find out the final configuration, let us recall that in these cases there could be Q-balls in the theory for an appropriate potential. Let us here recapitulate the Q-ball physics briefly. In the symmetric phase charged particles have mass $m$ and so the ratio $e/\rho$ would be $m$. For a given large charge $Q$, the lowest energy state however does not need to be made of a collection of these charged particles at rest, whose energy is $m|Q|$. Rather, that could be a Q-ball if there is an enough attraction between the charged particles.

To find the condition on the potential for Q-balls to exist, let us examine a homogeneous configuration whose energy per unit charge is lowest. We first
minimize $e/\rho$ with respect to $\rho$,

$$\frac{\partial (e/\rho)}{\partial \rho} = \frac{1}{2f^2} - \frac{1}{\rho^2} U(f) = 0. \quad (2.18)$$

The charge density is then fixed as a function of $f$,

$$\rho = 2f^2 U \quad (2.19)$$

The energy per unit charge is given by

$$\frac{e}{\rho} = \sqrt{\frac{2U}{f^2}} \quad (2.20)$$

We have to minimize $e/\rho$ in Eq.(2.20) with respect to the $f$ field. Fig.3 shows $\sqrt{2U/f^2}$ for various potentials. When $\sqrt{2U/f^2}$ takes the local minimum value $w_*$ at the nonzero $f = f_*$ field, such configuration is called Q-matters whose charge density is fixed to be $\rho_0 = f_*^2 w_*$. A Q-ball is a sphere whose inside is made of Q-matters and outside is just the symmetric phase. When the total charge is large, the size of a Q-ball would be large and the surface energy will be negligible compared with the volume energy and the energy per charge would be very close to $w_*$. Thus, if $w_*$ is less than $m$, the Q-balls are stable against decaying into charged particles. Such Q-balls are possible with $U_1, U_2$ of Fig.3.

For Case B, the initial configuration should be classically stable, which means $g > 0$ as shown in Eq.(2.15). $U_2$ in Fig.3 represents such a potential.

When the stable Q-balls are possible, the final configuration for Cases B, C is inhomogeneous. There will be a region with Q-matters and the rest as the symmetric phase of zero charge density. The domain walls separating two regions would slowly evolve to reduce the surface energy. The ratio of the volumes between two regions would vary depending on the initial charge density. If the initial charge density is small, Q-balls will float in the symmetric vacua. If the initial charge density is large,
density is large, the balls of the symmetric vacua will float in the Q-matter. In Case B the phase transition would proceed with the nucleations of Q-balls and in Case C it would proceed with the nucleations of bubbles with less charge density.

If Q-balls are impossible because \( w_\ast > m \) as in \( U_3 \) of Fig.3, then the final configuration would be in the symmetric phase with nonzero charge density. If the initial charge density is larger than \( \rho_\ast \), the final configuration could be more complicated would not be discussed here.

To understand the condition (2.18) better, consider the pressure of a homogeneous configuration given by

\[
p = \frac{\rho^2}{2f^2} - U(f)
\]  

We see Eq.(2.18) is identical to the zero pressure condition. When Q-balls can exist, we have argued that the final configuration could be inhomogeneous where Q-balls float in the symmetric vacuum. The zero pressure condition means that there is no pressure difference between Q-balls and the symmetric vacuum, leading to an equilibrium situation.

Having analyzed the last two cases where the symmetric phase is the ground state of \( U(f) \), let us now consider Case A where the asymmetric phase is the ground state of \( U(f) \). For this case, both the \( f \) field and charge prefer the asymmetric phase because the energy per charge in the asymmetric phase would be negative compare with that in the symmetric phase. The final configuration would be in the asymmetric phase with uniform charge density. An interesting observation on this case has been made when the initial state is not classically stable because \( g < 0 \) in the potential (2.12)\(^{[5]}\), Note that the initial configuration of zero charge density is classically stable. When we introduce a small charge density, charges get concentrated in several regions and these regions would evolve classically into the true vacuum. For the initial configuration at the symmetric phase to be stable, there should be a repulsive force between charges \( (g > 0) \) in long distance.
We note that the energy per unit charge and the pressure are sensitive to
the shift of the potential energy. In the above arguments, \( U(0) = 0 \) was used
crucially because we are dealing with tunneling between the symmetric phase and
the asymmetric phase. If we attempted to understand the tunneling between two
asymmetric phases, we would have needed another device to figure out the final
state. We will try to investigate this example elsewhere.

3. Quantum Tunneling

We have studied the general characteristics of the phase transition from a
homogeneous initial configuration with nonzero charge density. We found out
qualitatively what will be the final configuration after the quantum tunneling.
We now want to approach this problem more analytically by using the Euclidean
path integral formalism.\cite{2} When a nonzero global charge is involved, the standard
formalism should be extended to accommodate the nontrivial boundary term.\cite{2}
Here we summerize and expand the known results.

We start with the Euclidean generating functional,

\[
< F | e^{-HT} | I > = \int [df d\theta] \Psi^*_F e^{-SE} \Psi_I
\]  

(3.1)

where the Euclidean action is given by

\[
SE = \int d^4x \left\{ \frac{1}{2} (\partial_\mu f)^2 + \frac{1}{2} f^2 (\partial_\mu \theta)^2 + U(f) \right\}
\]  

(3.2)

The initial and final states \( \Psi_{I,F} \) describe the configurations of charge density, \( \rho_{I,F} \),
or

\[
\Psi_{I,F}(\theta, f) \sim \exp \left\{ i \int d^3x \rho_{I,F} \theta \right\}
\]  

(3.3)

In our case, the initial and final states are identical and describe the initial metastable
configuration.
By calculating the imaginary correction to the energy of the metastable initial configuration, we can find out the tunneling rate or the bubble nucleation rate. When we sum over the multibounce contributions to the energy,

\[ e^{-\mathcal{E}VT} = \langle i | e^{-HT} | i \rangle \]

\[ = K_0 e^{-\mathcal{E}_0 VT} \left\{ 1 + VT Ke^{-B} + \frac{1}{2} (VT Ke^{-B})^2 + \ldots \right\} \]

\[ = \exp\{-(\mathcal{E}_0 - Ke^{-B})VT\} \]

(3.4)

where \( \mathcal{E}_0 \) is the energy density of the initial configuration, \( K_0 \) is the prefactor from the small fluctuations around the initial configuration, the \( B \) factor is the difference between the bounce action and the background action, \( \mathcal{E}_0 VT \), and \( KK_0 \) is the prefactor arising from the small fluctuations around the bounce solution. There is a negative mode around the bounce solution which implies the factor \( K \) pure imaginary. The bubble nucleation rate per unit volume is then given by

\[ 2|K|e^{-B}. \]

The path integral will be dominated by the stationary configurations of the path integral. In our case the boundary condition on the \( f \) field of such stationary configuration is fixed to be the initial classical configuration. However, Eq.(3.3) implies that the boundary condition on the \( \theta \) field is free. Thus, the stationary configuration in the path integral (3.1) satisfies the Euler equation for the action \( S_E + \Sigma \), where \( \Sigma \) is the boundary term

\[ \Sigma = -i \int d^3r \left\{ \rho_F \theta(\vec{r},\tau_F) - \rho_I \theta(\vec{r},\tau_I) \right\} \]

(3.5)

From the Euclidean equation from the combined action \( S_E + \Sigma \), one can see that the stationary path of the angle variable \( \theta \) should be pure imaginary,

\[ \theta \equiv -i\eta \]

(3.6)

The Euclidean field equations for the \( f, \eta \) field are then

\[ \partial_\mu^2 f + f(\partial_\mu \eta)^2 - U'(f) = 0 \]

\[ \partial_\mu (f^2 \partial_\mu \eta) = 0 \]

(3.7)
The boundary condition on $\eta$ becomes

$$f^2 \frac{\partial \eta}{\partial \tau}(\vec{r}, \tau_{I,F}) = \rho_{I,F}(\vec{r}) \quad (3.8)$$

The $f$ field should approach the time independent $f$ given by the initial configuration. We assume that there is no vortex in the initial configuration. Then the classically stable configuration for given charge should satisfy

$$\partial_t^2 f + \frac{\rho^2}{f^3} - U'(f) = 0 \quad (3.9)$$

since $\partial_\tau f = 0$ and $\partial_t \eta$ at the boundary. In our case, the initial configuration is homogeneous in space and so Eq.(3.9) becomes identical to Eq.(2.4).

The solution of Eq(3.7) is the so-called bounce for quantum tunneling. When the initial charge density vanishes, we know that the $O(4)$ symmetric solution of Eq.(3.7) has been shown to exist by the undershoot-overshoot method. In our case, the boundary condition reduces this $O(4)$ symmetry to the $O(3)$ symmetry because the charge density selects a preferred time direction. Thus, we are interested here the bounce which is $O(3)$ symmetric invariant under the spatial rotation. This makes Eq.(3.7) a partial differential equation, whose solution is much harder to find. We can either use some analytic tools or numerical analysis to find the bounce solution. In the next section, we use the perturbation method to get an approximate bounce solution when the initial charge density is very small.

Once we find the $O(3)$ bounce solution $f_b, \eta_b$, we can calculate its action $S_E + \sigma$ from Eqs. (3.2) and (3.5). By using Eqs.(3.7) and (3.8), we can see the combined action becomes

$$S_E + \sigma = \int d^4x \left\{ \frac{1}{2} (\partial_t f_b)^2 + \frac{1}{2} f_b^2 (\partial_\gamma \eta_b)^2 + U(f_b) \right\} \quad (3.10)$$

The bubble nucleation per unit volume is then given by $Ke^{-B}$ where

$$B = (S_E + \sigma)(\text{bounce}) - (S_E + \sigma)(\text{background}) \quad (3.11)$$

While the action for the bounce $S_E + \sigma$ could be infinite, the difference $B$ between
that of the bounce and that of the initial configuration should be finite when one expects a finite tunneling rate. While we will not attempt to calculate the effect of the fluctuations around the bounce solution, we note that the field fluctuations $\delta f, \delta \theta$ should be kept real in the functional integral. This is exactly what happens in a gaussian integral

$$\int dx e^{-x^2+ipx}$$

To find out the escape point or the bubble configuration at the nucleation moment, we use the time translation and reflection symmetries of the action under $\tau \rightarrow -\tau$ and $\eta \rightarrow -\eta$ of $S_E + \sigma$ to choose the origin to be the center of the bounce so that

$$\frac{\partial f_b}{\partial \tau}(\vec{r}, \tau = 0) = 0$$
$$\eta_b(\vec{r}, \tau = 0) = 0$$  \hspace{1cm} (3.12)

As in our problem charge density remains real in Minkowski and Euclidean times, it is natural to identify the initial charge density of the bubble to be given by that of the bounce. The initial bubble configuration is then

$$f(\vec{r}, t = 0) = f_b(\vec{r}, \tau = 0)$$
$$\theta(\vec{r}, t = 0) = 0$$
$$\frac{\partial f}{\partial t}(\vec{r}, t = 0) = 0$$  \hspace{1cm} (3.13)
$$f^2 \frac{\partial \theta}{\partial t}(\vec{r}, t = 0) = f_b^2 \frac{\partial \eta_b}{\partial \tau}(\vec{r}, \tau = 0)$$

In usual quantum tunneling, momenta are imaginary and coordinates are real under the potential barrier, and we find the escape point in the coordinate space. In our case, the charge or momentum density and the angle variable have changed their role. Under the barrier the angle variable is imaginary and the charge density is real, and we find the escape point in the charge density.

Once we know the initial bubble configuration (3.13), we can solve the field equation in the Minkowski time to find out how a given bubble evolve. The bounce
solution \((f_b, \eta_b)\) can also be analytically continued to a solution in Minkowski time,

\[
\begin{align*}
  f(\vec{r}, t) &= f_b(\vec{r}, it) \\
  \theta(\vec{r}, t) &= -i\eta_b(\vec{r}, it)
\end{align*}
\] (3.14)

A further insight about bubble nucleation with global charge can be gained by using the dual formulation.\(^2\) It is well known that Goldstone bosons can be described by an antisymmetric tensor field, \(B_{\mu\nu}\). In the Minkowski time, the dual lagrangian is given by

\[
\mathcal{L}_{DM} = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{12f^2}H_{\mu\nu\rho}^2 - U(f)
\] (3.15)

where \(H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\). The field strength of the antisymmetric tensor is related to the original current by

\[
f^2 \partial^\mu \theta = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\mu\nu\rho}
\] (3.16)

The uniform initial charge density becomes the condition of the uniform ‘magnetic’ field \(H_{123}\). In Euclidean time, there will be no boundary term arising from the wave function \(\Psi(f, B_{ij})\). The Euclidean lagrangian becomes

\[
\mathcal{L}_{DE} = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{12f^2}H_{\mu\nu\rho}^2 + U(f)
\] (3.17)

The bounce equation becomes

\[
\begin{align*}
  \partial_\mu^2 f + \frac{1}{6f^3}H_{\mu\nu\rho}^2 - U'(f) &= 0 \\
  \partial^\mu(\frac{1}{f^2}H_{\mu\nu\rho}) &= 0
\end{align*}
\] (3.18)

We can relate the Euclidean fields either through the Euclidean time dual transformation or by comparing the current. The relation between the antisymmetric
tensor field and the angle variable in the Euclidean time is given by

\[ f^2 \partial^\mu \eta = \frac{1}{6} \epsilon^{\mu \nu \rho \sigma} H_{\nu \rho \sigma} \]  

(3.19)

Thus, the bounce solution in terms of the \( B_{\mu \nu} \) field would be real and there would be no contribution to the bounce action from the boundary.

4. Examples

We are now in position to examine in more detail three cases of the phase transition which we have discussed in Sec.2. To be more specific we choose the potential to be given by Eq.(2.12). Case A has the initial configuration in the symmetric phase. The potential energy has the absolute minimum at the asymmetric phase. For Cases B, C the potential energy \( U(f) \) has the absolute minimum at the symmetric phase and the local minimum at the asymmetric phase. The initial configuration of Case B is at the asymmetric phase and the initial configuration of Case C is at the symmetric phase. In Cases A, B the tunneling would proceed even when there is no initial charge density because the initial configurations are at the metastable points of \( U(f) \). Introducing a small amount of charge density would not change much of the original bubble nucleation. Thus we would expand perturbatively the bounce solution by the initial charge density and see how the zeroth order \( O(4) \) symmetric solution deforms. These are the cases we will examine closely in this section. For Case C, we do not have the zeroth order bounce solution because the quantum tunneling occurs solely due to the charge density. However, we can still get some insight for this case as we will see later.

Let us consider first the cases where the initial configuration is unstable even without any charge density. The symmetric phase of the potential \( U_1 \) and the asymmetric phase of the potential \( U_2 \) in Fig.2 are such initial configurations. When there is no initial charge density, the bounce solution can be obtained by the \( O(4) \) symmetric ansatz, \( \tilde{f}(s \equiv \sqrt{\tau^2 + \tau^2})^{18} \). Let us assume that the thin wall approximation works. We call that inside the wall, \( f = f_i \) and outside the wall \( f = f_e \).
The wall radius $a$ can be determined as follows. Suppose that the potential energy difference $\Delta U = U(f_e) - U(f_i) > 0$ between $f_e$ and $f_i$ phase is small. Then the bubble radius will be large and we can approximate the bubble wall as a domain wall separating two phases. This wall satisfies the equation, $\partial_x^2 f + U'(f) = 0$, neglecting the potential energy difference. Define the tension of the wall to be the action density per unit three volume, $T = \int dx [(df/dx)^2/2 + U(f)]$. The gain of the action due to this true vacuum bubble of the radius $a$ is then

$$S(a) = 2\pi^2 T a^3 - \frac{\pi^2}{2} \Delta U a^4$$

At $a = 3T/\Delta U$, $S(a)$ takes the maximum value $27\pi^2 T^4/2(\Delta U)^3$, which is the $B$ factor in the bubble nucleation rate.

We ask what happens to this $O(4)$ symmetric thin-wall bounce solution if we introduce a small initial charge density. From Eq.(3.7), we see that the equation of the angle variable is the first order in the charge density and the $f$ field equation has a second order correction to the bounce equation of the zero charge density. Thus, we can solve Eq.(3.7) by a perturbative expansion around this $O(4)$ symmetric background. The phase variable will be first order in charge density and satisfies the current conservation

$$\partial_t (\bar{f}^2(s) \partial_t \eta) = 0$$

with the boundary condition $f_\pm^2 \eta(\vec{r}, \tau = \pm \infty) = \rho_0$ with the initial charge density $\rho_0$. The above equation can be interpreted as a boundary problem of a dielectric media in four space dimensions. The electric field is $\partial_{\mu} \eta$ with the potential $\eta$ and the electric displacement is $J_\mu = f^2 \partial_{\mu} \eta$ with the dielectric constant $f^2$. Eq.(4.2) implies the boundary condition at the wall that the normal component of $J_\mu$ and the tangential component of $\partial_{\mu} \eta$ should be continuous. The boundary condition at infinity is that there is a constant external electric displacement field $J_\tau = \rho_0$. For a given $O(4)$ symmetric configuration described before, it is trivial to find the
potential $\eta_{e,i}$ outside and inside the thin wall,

$$\eta_e = \frac{\rho_0}{f_e^2} \left(1 + \frac{f_e^2 - f_i^2}{3f_e^2 + f_i^2} \frac{a^4}{s^4}\right) \tau$$

$$\eta_i = \frac{\rho_0}{f_e^2} \left(\frac{4f_e^2}{3f_e^2 + f_i^2}\right) \tau$$

The charge density $J_i$ at the $\tau = 0$ would be the charge profile of the bubble at the moment of nucleation as shown in Eq.(3.13). Let us now examine the implications of this solution (4.3) in various cases. The correction to the $f$ field would be second order and will be considered in each case.

**Case A: from the symmetric phase to the asymmetric phase**

Let us first consider the case when the asymmetric phase is the ground state and the initial configuration is near the symmetric phase. Thus, $f_e \sim 0 \ll f_i \sim v$. Since there is no initial charge density, the previous argument would imply that $f_e = 0$. When we introduce uniform charge density in the initial configuration, the initial value of $f$ would be given by Eq.(2.13) with $\theta = mt$, invalidating our assumption $\tilde{f} \sim O(1)$. Here let us assume simply that $f_e$ is nonzero even when there is no charge, say due to a small bump in the potential at $f = 0$. This would not change the physics of tunneling under consideration much and allows us to use Eq.(4.3).

The global current $J_\mu$ around the $O(4)$ symmetric bounce solution can be obtained from Eq.(4.3). Outside the thin wall ($\zeta > a$), $J_\mu = f_e^2 \partial_\mu \eta_e$ and inside the thin wall ($\zeta < a$), $J_\mu = f_i^2 \partial_\mu \eta_i$. The energy per charge inside the wall is small and so the charge is attracted to the interior region, making the charge density inside the bubble be higher than that outside. The charge density profile of the bubble at the moment of nucleation would be given by $J_\tau$ at $\tau = 0$,

$$J_{0 \text{ exterior}}^\tau = (1 - \frac{a^4}{r^4})\rho_0$$

$$J_{0 \text{ interior}}^\tau = 4\rho_0$$

when $f_i \gg f_e$. The charge density inside the bubble is four times larger than
the initial charge density. From Eq.(4.4), we can see $\int d^3 x (J^0 - \rho_0) = 0$, implying that the charge inside the bubble came from the region near the bubble wall. Fig.4 shows this global current on the $O(4)$ symmetric solution background.

Let us now consider the effect of the charge flow on the $f$ field. The $f$ field equation (3.7) can be expanded around the $O(4)$ symmetric solution as

$$\partial^2 \delta f - U''(f) \delta f = -\frac{J^2_\mu}{f^3}$$

(4.5)

where $J^2_\mu$ can be obtained from Eq.(4.3) and $\tilde{f}$ would be given by the $O(4)$ symmetric solution. Rather than try to solve this partial differential equation, let us approach the problem more qualitatively. When $f_e << f_i$, one can show that at the north and south poles $J^2_\mu \sim 16\rho_0^2$ both inside and outside and that at the equator $J^2_\mu \sim 0$ outside and $J^2_\mu \sim 16\rho_0^2$ inside. The centrifugal term $J^2_\mu / 2f^2$ would be then important at the poles but not at the equator. Directly from the $f$ equation (3.7) and the previous argument about the thin-wall approximation, we see the centrifugal term reduces the tension on the domain wall by reducing the effective potential energy barrier. Since the tensions at the two poles are lower while the tensor at the equator remain constant, the curvature at poles would be larger than $1/a$ and that at the equator will remain $1/a$ where $a = 3T / \Delta U$ is the radius of the $O(4)$ symmetric shell. ( In addition $\Delta U$ in increased at the pole and remain unchanged at the equator, amplifying the curvature change between the poles and the equator.) Consequently, the $O(4)$ symmetric wall would be shrunk at the equator.

We take a liberty to choose this wall configuration to be an ellipsoid

$$\frac{r^2}{b^2} + \frac{\tau^2}{ab} = 1$$

(4.6)

where $b < a$ takes a complicated function of $\rho_0$. Fig.4 shows the deformed bounce wall.

Let us now think about the bubble nucleation and evolution. The bubble of true vacuum will nucleate with a radius $b$ and a higher charge density $4\rho_0$ and then
expand. As argued in Eq.(3.14), we can analytically continue the bounce solution to find out how the bubble will evolve. From Eq.(4.6), we see that the bubble wall trajectory will be given by $R(t) = \sqrt{\frac{b}{a}} \sqrt{t^2 + ab}$. Since $b < a$, the terminal velocity $v_t = \sqrt{b/a}$ is less than 1.

How do we understand the finite terminal velocity? Energy conservation implies that the change of the wall energy comes from the potential energy difference,

$$d \left( \frac{4\pi T(R)R^2}{\sqrt{1 - R^2}} \right) = d \left( \frac{4\pi}{3} \Delta U R^3 \right)$$ \hspace{1cm} (4.7)

The tension of the bubble wall surface could depend on $R$. Integrating Eq.(4.7), we get

$$\dot{R} = \sqrt{1 - \frac{3T(R)}{\Delta UR}}$$ \hspace{1cm} (4.8)

When there is no charge density, $T, \Delta U$ are fixed and we see that the terminal velocity is the light speed. If the tension grows linearly with the radius for large $R$, $T(R) \sim \alpha R$, the terminal velocity $v_t = \sqrt{b/a}$ will be

$$v_t = \sqrt{1 - \frac{3\alpha}{\Delta U}} < 1$$ \hspace{1cm} (4.9)

The growth of tension, or the energy density of wall per unit area, can be understood by considering the phase variable $\theta$. While the charge density $f_i^2 \dot{\theta}$ inside the bubble is larger than the charge density $f_e^2 \dot{\theta}$ outside, it is but not large enough to keep the phase variable space independent since $f_i >> f_e$. The phase increases by $mt$ outside the bubble and more slowly inside the bubble, leading to the increasing its space gradient at the bubble wall. This is what we think is the source of the increasing tension or energy density at the bubble wall.

Let us now remind ourselves that the sound speed (2.15) in the symmetric phase is much smaller than the terminal speed (4.9) when the initial charge density is small. Thus, the bubble wall forms a sort of supersonic front in the symmetric
phase. However, our analysis is not accurate enough to compare the terminal speed and the sound speed (2.17) at the inside asymmetric phase. (When the initial charge density is large enough, there is a possibility that the terminal speed is less than the sound speed in the symmetric phase.) The thin wall approximation would fail eventually, because there is not enough charge lying outside the expanding bubble to keep the charge density inside the bubble to be four times larger than the initial value. The charge density profile around the expanding bubble should become more smoothly changing.

Case B: from the asymmetric phase to the symmetric phase

Let us now consider the case where the initial state is the broken phase which has the higher potential energy than that of the symmetric phase. When there is no initial charge density, there will be an O(4) symmetric bounce solution, inside which the scalar field takes value near the symmetric phase, \( f_i \ll f_e \). Again we ask what is the consequence of the small initial charge density. Eq.(4.3) implies how the current flows around this bounce solution. The charge density at the moment of bubble nucleation would be given by

\[
J_{\text{exterior}}^0 = (1 + \frac{a^4}{3r^4})\rho_0 \\
J_{\text{interior}}^0 \approx 0
\]

when \( f_i \ll f_e \). The charge is excluded from the symmetric vacuum region. Fig.5 shows the charge flow around the O(4) symmetric bounce solution.

The effect of the charge flow on the \( f \) field is given by the centrifugal term \( J^2_\mu/2f^2 \) as in Eq.(4.5). We can calculate \( J^2_\mu \) for our bounce solution. Since \( f_e >> f_i \), Eq.(4.3) implies that at north and south poles \( J^2_\mu \approx 0 \) both inside and outside and that at the equator \( J^2_\mu \approx 0 \) inside and \( J^2_\mu \approx \frac{16\rho_0^2}{9} \) outside. This raises the energy density of the false vacuum at the equator, lowering the barrier energy and increasing \( \Delta U \). This in turn lowers the tension of wall at the equator. The tension at the poles would remain unchanged and so the bounce solution would be
elongated at the equator. Now for the sake of the argument, we again approximate
the wall as an ellipsoid,
\[
\frac{r^2}{ab} + \frac{\tau^2}{b^2} = 1
\]  
(4.11)
where \( b < a \) is a complicated function of \( \rho_e \). The deformed bounce solution is
shown in Fig.5.

In the Minkowski time, the bubble wall trajectory is given by \( R(t) = \sqrt{a/b}\sqrt{t^2 + b^2} \),
with the terminal velocity \( v_t = \sqrt{a/b} > 1 \), which clearly violates the causality.
Something should happen before the wall speed becomes the light speed. When
there is a tachyonic mode, we say there is an unstable or exponentially growing
mode. There are many possibilities. Since charge is pushed out from the bubble
and is accumulated at the wall, the wall could stop expanding. Or the thin wall
approximation could break down before the wall reaches the light speed.

In Case B, the charge is pushed away from the bubbles and accumulated at the
initial asymmetric phase. At the end of phase transition, we would be left with the
islands of the original phase with high charge density, which are exactly Q-balls
floating in the symmetric phase.

Case C: forming Q-balls by quantum tunneling

If the minimum of \( \sqrt{2U/f^2} \) is lower than the mass of the charged particles in
the symmetric phase, charge likes to clump to Q-balls. Suppose the initial config-
uration lies at the symmetric phase with very small charge density and is stable
under local fluctuation. Since the minimum of the potential is chosen to be the
symmetric phase, the initial configuration would be stable if there is no charge
density. After small uniform charge density is introduced, the initial configuration
however becomes unstable under the quantum mechanical tunneling transition to
form Q-balls. Since we start from the minimum of the potential, we do not have
the bounce solution at zero charge density. However, we can still gain some un-
derstanding of the general features of the bounce solution from what we learned in
Case A.
First, the bubble at the moment of nucleation would be a Q-Ball of the minimum size, where the surface energy is as important as the volume energy. The charge density $\rho_*$ inside the Q-ball would be much larger than the initial charge density, $\rho_0$. The minimum size of a Q-ball could be rather large if $w_*$ is very close to $m$ so that the energy gain by the charge is small and so a lot of charge is needed to compensate the surface energy. We know that there will be a large current will flow into inside the bounce wall from outside in this case because the interior charge density $\rho_i = f_i^2 w_*^2$ is much larger than $\rho_0$. With a similar argument given to Case A, the current would be large inside and outside wall at the poles, and would be zero outside and large inside the equator. The energy density outside the bounce wall at the pole is larger than that outside the bounce wall at the equator. The barrier energy at the poles would in turn be lower than that at the equator, and so the wall tension at pole will be lower than that at the equator. Thus, the bounce solution in Case C would also be elongated along the $\tau$ direction in Case A. Fig.6 shows such a bounce solution for Q-ball nucleation.

Once a Q-ball is nucleated, it will grow but very slowly. The reason is that a Q-ball can grow only when it swallows the charge from outside and that there is not much charge around it because the formation itself have already diluted the initial charge density around its neighborhood. This is consistent with the picture that the bounce solution is elongated along the $\tau$ direction, which also implies a slow terminal velocity as argued after Eq.(4.9). The explicit nature of the Q-ball nucleation and expansion would however require the better analysis and would not be attempted here.
5. Discussion

We have studied the phase transitions in the theory of a complex scalar field with a global U(1) symmetry when there is non zero initial charge density. We have discussed the metastability condition on the possible initial configurations and the possible inhomogeneous final configurations. We argued that there are many cases of the phase transitions to be studied in the theory. We have set the Euclidean formalism of the bubble nucleation when there is nonzero charge density. We applied our formalism to the case when the initial charge density is small and when the phase transitions involve the symmetric phase as the initial configuration or a part of the final configuration. Here we studied the characteristics of the bounce solutions and the bubble evolution. Our system is shown to have a rich variety of the possible phase transitions, and could be a good simple toy model of the phase transition involving charges, the supersonic bubble wall, and the sound wave.

However, there are still many loose ends and questions we have not attempt here. One of the interesting questions seems what is the later development of the bubbles. Depending on the cases of the phase transitions, there is a possibility of rich dynamics. Additional interesting questions to be explored are about the phase transition between the asymmetric phases and about the phase transition when the initial charge density is not small. Finally, we note that it is rather straightforward to extend our formalism in Sec.3 to the case involving the nonzero local gauge charges.

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FIGURE CAPTIONS

1) Plot of $e(\rho, f)$ and $U(f)$ for a typical case.

2) Plot of two potentials $U_1(f)$ and $U_2(f)$.

3) Plot of $\sqrt{2U(f)/f^2}$ for three potentials $U_1, U_2, U_3$.

4) The bounce solution for Case A. The dashed circle is the wall of the $O(4)$ symmetric bounce solution. The dotted lines indicate the charge flow. The solid ellipse is the wall of the deformed bounce.

5) The bounce solution for Case B. The dashed circle is the wall of the $O(4)$ symmetric bounce solution. The dotted lines indicate the charge flow. The solid ellipse is the wall of the deformed bounce.

6) The bounce solution for Case C. The dotted lines indicate the charge flow. The solid ellipse is the wall of the bounce for Q-ball nucleation.
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