Crossover between various initial conditions in KPZ growth: flat to stationary

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We conjecture the universal probability distribution at large time for the one-point height in the 1D Kardar-Parisi-Zhang (KPZ) stochastic growth universality class, with initial conditions interpolating from any one of the three main classes (droplet, flat, stationary) on the left, to another on the right, allowing for drifts and also for a step near the origin. The result is obtained from a replica Bethe ansatz calculation starting from the KPZ continuum equation, together with a “decoupling assumption” in the large time limit. Some cases are checked to be equivalent to previously known results from other models in the same class, which provides a test of the method, others appear to be new. In particular we obtain the crossover distribution between flat and stationary initial conditions (crossover from Airy$_1$ to Airy$_{stat}$) in a simple compact form.

I. INTRODUCTION

The one-dimensional Kardar-Parisi-Zhang (KPZ) equation\textsuperscript{1} describes, in the continuum, the stochastic growth of an interface, of height $h(x,t)$ at point $x \in \mathbb{R}$, as a function of time $t$

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda_0}{2} (\partial_x h(x,t))^2 + \sqrt{D} \xi(x,t)$$ \hspace{1cm} (1)

driven by a unit white noise $\xi(x,t)\xi(x',t') = \delta(x-x')\delta(t-t')$. It has a number of experimental realizations\textsuperscript{2-5} and is at the center of large (and growing) universality class, which contains exactly solvable models in discrete settings, studied both in physics and mathematics. Recently there was progress in finding exact solutions for the continuum KPZ equation itself. While the scaling exponents $h \sim t^{1/3}$, $x \sim t^{2/3}$ have been known for a while\textsuperscript{6}, the present interest is to characterize the full statistics of the height field $h(x,t)$. The KPZ equation can be mapped to the continuous directed polymer (DP) in a quenched random potential, such that $h(x,t) = \ln Z(x,t)$ is proportional to the free energy of the DP of length $t$ with one fixed endpoint at $x$.

Interestingly, the KPZ interface retains some memory of the initial condition, and a few main universal statistics are found to emerge at large time, depending on the type of initial conditions. Remarkably, these are also related to the universality of large random matrices. This was first obtained from discrete models in the KPZ universality class, i.e. expected to share all its (rescaled) large time properties, such as the PNG growth model\textsuperscript{7-9}, the TASEP particle transport model\textsuperscript{10,12} or discrete DP models\textsuperscript{11}. Recently it was obtained more directly, from exact solutions of the KPZ equation: on the infinite line there are three main classes

- **The droplet (or hard wedge) initial condition** (DP with two fixed endpoints) leads to height fluctuations governed at large time by the Tracy Widom (TW) distribution $F_2$, the CDF (cumulative distribution function) of the largest eigenvalue of the GUE random matrix ensemble\textsuperscript{13}. It was solved by two methods (i) as a limit from an ASEP model with weak asymmetry\textsuperscript{14} leading to a rigorous derivation\textsuperscript{16,17} (ii) using the replica Bethe ansatz (RBA) method\textsuperscript{18,19} by calculating the integer moments of $Z = e^{\theta}$ from the known exact solution of the Lieb-Liniger delta Bose gas\textsuperscript{20} (also derived recently from the sine-Gordon field theory\textsuperscript{21}). Both methods obtained the CDF for all times $t$, as a Fredholm determinant, displaying convergence to $F_2$ as $t \to +\infty$.

- **The flat initial condition** (point to line DP), solved with the RBA at all times\textsuperscript{22-25} and at large time\textsuperscript{26}. Rigorous calculations within ASEP have not yet led to a proof of the finite time results for the KPZ equation (see\textsuperscript{27} for the present status of rigorous calculations). The convergence of the one-point CDF is now to $F_1$, associated with the GOE ensemble of random matrices.

- **The stationary i.e. Brownian initial condition** solved first at all times using the RBA\textsuperscript{28} exhibits convergence at large time to the Baik-Rains $F_0$ distribution. Recently it was solved rigorously as a limit of discrete directed polymer models using tools from the theory of Macdonald processes\textsuperscript{29}.

The RBA also allowed to solve the KPZ equation on the half-line\textsuperscript{30} which relates to the GSE random matrix ensemble\textsuperscript{31}. Although non-rigorous (since the integer moments $Z_m$ of the continuum KPZ equation grow too fast with $n$, as $\sim e^{m^3}$, to determine uniquely the distribution) the RBA has shown impressive heuristic value, often preceding rigorous results, still not available in all cases. A number of the latter have been obtained recently as limits (e.g. $q \to 1$)
from a hierarchy of (often novel) integrable discrete models (including \(q\)-TASEP, \(q\)-bosons, semi-discrete DP, vertex models) and new mathematical tools (e.g. Macdonald processes\(^{33-36,49}\)).

Besides the three main classes, one expects universal crossover classes (also called transition classes) with initial conditions interpolating from one of the three classes at \(x = -\infty\) to another one at \(x = +\infty\), see e.g. Fig. 4 in Ref.\(^{12}\). If the two limits are distinct classes, there are three possibilities as follows:

- **Droplet to stationary:** The KPZ equation with half-Brownian initial conditions was solved for all times using the RBA\(^{44}\). Although the precise form of the obtained kernel is different, it is found equivalent to the result obtained by taking the weak asymmetry limit\(^{26}\) of the general result for the ASEP with half-Bernouilli initial conditions obtained in\(^{39}\). At large time this leads to the universal GUE to stationary crossover distribution. It admits an interesting generalization where the initial condition on the half-space is the partition sum of an O’Connor-Yor directed polymer with \(N\) layers, equivalently the highest eigenvalue of the GUE(\(N\)) Dyson Brownian motion (\(N = 1\) corresponds to the half-Brownian)\(^{44}\).

- **Droplet to flat:** We studied recently using RBA\(^{45}\) the transition from GUE to GOE in the KPZ equation, realized for the so-called ”half-flat” initial condition, which is flat for \(x < 0\) and droplet-like for \(x > 0\). From the ”half-flat” formula obtained in\(^{24}\), we could produce a conjecture for the PDF in the large time limit. We obtained a new formula for the transition Kernel and showed that it is equivalent to the one obtained in Ref.\(^{45}\) from a solution of the TASEP with initial condition of particles on even sites for \(x \leq 0\) and empty for \(x > 0\). This is a mark of the expected universality at large time of this transition class. The corresponding Airy process was defined and characterized in Ref.\(^{45}\) and called \(A_{2\to 1}\).

- **Flat to stationary:** At present there is no derivation of the flat to stationary distribution directly for the KPZ equation. In the large time limit, the corresponding distribution was obtained in\(^{27}\) for TASEP with initial conditions \(2\mathbb{Z}\), i.e. particles on even sites for \(x \leq 0\), no particles for \(x > 0\), with the \(M\) first particles endowed with a slower speed \(\alpha < 1\). With this setting there is a point in the phase diagram of the model where the crossover flat to stationary can be attained (the corresponding kernel is given by Eq. (5.4)-(5.8) there with \(M = 1\) and \(\kappa = 0\), for \(\alpha = 1/2\)). In terms of process this is called \(A_{1\to stat}\).

The aim of this paper is to revisit these crossover classes starting from the KPZ equation \(\Box\) and using the replica Bethe ansatz method. We will in particular obtain the lacking result for the third universal crossover class in the KPZ equation, the flat to stationary. Note that for each class of initial condition there are two degrees of freedom which can be varied, corresponding to two known invariances of the KPZ equation, namely (in units such that \(\lambda_0 = D = 2\), \(\nu = 1\) - see below) a shift in height by a constant \(h(x, t) \to h(x, t) + \Delta\), and a tilt by a finite slope \(h(x, t) \to h(x + \Delta, t) = h(x + 2\Delta t, t) + wx + w^2 t\), also known as Galilean invariance on the associated Burgers equation (the derivative of the KPZ equation). Hence it is quite natural to study the crossover between initial conditions with different slopes \(w_{L,R}\) and a height mismatch (step size) \(2\Delta\) on each side. By scaling the slopes and step size appropriately with time one obtains universal crossover distributions at large time (it then makes sense as a crossover distribution even when the class is the same on each side). We also obtain for example the universal distribution for the wedge initial condition (flat to flat crossover) and for the Brownian to Brownian. In summary we expect our results to apply to any initial condition \(h(x, t = 0)\) which interpolates between a left initial condition \(h_{0,\text{left}}(x)\) for \(x < 0\), and a right initial condition \(h_{0,\text{right}}(x)\), for \(x > 0\), each belonging to one the three main classes, with possibly a mismatch in height, \(2\Delta\), and in slopes \(w_L - w_R\), merging within an interpolation region of size \(x_0 = L_0^{2/3}\), e.g. an initial condition of the form

\[
h(x, t = 0) = (h_{0,\text{left}}(x) + \Delta)\theta(-x) + L_0^{1/3} \int f(x/L_0^{1/3}) + (h_{0,\text{right}}(x) - \Delta)\theta(x)
\]

where \(f(x)\) is a bounded function which decays to 0 at infinity, and \(\theta(x)\) the Heaviside unit step function. We will consider the large time limit \(t \gg t_0\), in which the precise form \(f(x)\) of the interpolating region becomes irrelevant, and study the result as a scaling function of the scaled parameters \(\Delta/L_0^{1/3}\) and \(w_{L,R} L_0^{1/3}\).

We obtain these results within the RBA using an assumption in the large time limit, sometimes called a ”decoupling assumption”\(^{26,48-53}\). The method is similar from the one we applied to study the droplet to flat crossover in Ref.\(^{45}\) (some results of that work are recovered here in some limit) but is significantly more involved as it requires working with the combinatorics of groups of replica, as pioneered by Dotsenko\(^{28}\) (an approach that we will test and slightly extend here). For the Brownian to Brownian crossover, our results take a different form, but agree with known results, e.g. the one of\(^{28}\), which produces yet another a non-trivial check of the method.

The outline of the paper is as follows. In section \(\S\) we recall the model, the units and the mapping to the directed polymer. In section \(\S\) we describe the initial conditions studied in this paper. In section \(\S\) we first recall standard results, then we summarize the main results of the present work. In section \(\S\) we give the definitions of...
the generating functions. Section III contains the main replica calculations. The quantum mechanical method and the Bethe ansatz are recalled, then in section IV we display the combinatorics identity which is used, leading to a general formula for the moments and generating function in III. The large time limit is studied in section IV and using the decoupling assumption leads to an expression for the generating function as a Fredholm determinant, involving a kernel which is given in two equivalent forms. The section V details these two equivalent forms of the kernel in each of the various cases, with their limit forms and comparison with known results. Finally, the section V is the conclusion. The first Appendix details the combinatorial identity. The next two appendices detail the calculations of the kernels associated to the standard generating function, and the last one for the generalized generating function.

II. MODEL AND MAIN RESULTS

A. KPZ equation, directed polymer and units

Let us consider the KPZ equation (1), and define the scales

\[ x^* = \frac{(2\nu)^3}{D\lambda_0}, \quad t^* = \frac{2(2\nu)^5}{D^2\lambda_0}, \quad h^* = \frac{2\nu}{\lambda_0} \]

which we will use as units, i.e. we set units and the KPZ equation becomes:

\[ \partial_t h(x,t) = D^2 \eta(x,t) + (\partial_x h(x,t))^2 + \sqrt{2} \eta(x,t) \]

where \( \eta \) is also a unit white noise \( \eta(x,t) \eta(x',t') = \delta(x-x')\delta(t-t') \). As is well known the Cole-Hopf mapping solves the KPZ equation from an arbitrary initial condition as follows. The solution at time \( t \) can be written as:

\[ e^{h(x,t)} = Z(x,t) : = \int dy Z_\eta(x,t|y,0) e^{h(y,t=0)}. \]

where here and below we denote \( \int dy = \int_{-\infty}^{\infty} dy \). Here \( Z_\eta(x,t|y,0) \) is the partition function of the continuum directed polymer in the random potential \(-\sqrt{2} \eta(x,t)\) with fixed endpoints at \( (x,t) \) and \( (y,0) \):

\[ Z_\eta(x,t|y,0) = \int_{x(0)=y}^{x(t)=x} Dx e^{-\int_0^t d\tau \left(\frac{\nu}{4\eta} \eta(x(\tau))\right)} \]

which is the solution of the (multiplicative) stochastic heat equation (SHE):

\[ \partial_t Z = \nabla^2 Z + \sqrt{2} \eta Z \]

with Ito convention and initial condition \( Z_\eta(x=0,t|y,0) = \delta(x-y) \). Equivalently, \( Z(x,t) \) is the solution of (7) with initial conditions \( Z(x,t=0) = e^{h(x,t=0)} \). We will adopt the notation (for the solution of the droplet initial condition started in \( y \)):

\[ h_\eta(x,t|y,0) = \ln Z_\eta(x,t|y,0) \]

although it is somewhat improper (it requires a regularization, see below). We will sometimes omit the "environment" index \( \eta \). Here and below overbars denote averages over the white noise \( \eta \).

B. Initial conditions

We will study the KPZ equation (1) with the following initial condition:

\[ h(y,t=0) = h_0(y) = (w_L y + a_L B_L(-y))\theta(-y) + (-w_R y + a_R B_R(y))\theta(y) \]

where \( \theta(y) \) is the unit step Heaviside function, \( B_L(y) \) and \( B_R(y) \) are independent one-sided unit centered Brownians, with \( B_L(0) = B_R(0) = 0 \), \( B_{L,R}(y) \) being defined for \( y \geq 0 \). The correlator is \( < B_R(y)B_R(y') >= \min(y,y') \) and
similarly for $B_L(y)$. The parameters $w_L, w_R$ (usually chosen positive) measure the bias of the Brownian, i.e. the slopes of the KPZ initial profile on each side.

The parameters $a_L, a_R$ are chosen in $\{0, 1\}^2$ to allow to study the four "solvable" cases (in fact three distinct ones, by symmetry). The wedge initial condition corresponds to $a_L = a_R = 0$ and contains left (resp. right) half-flat initial condition as limits $w_R \to +\infty$ (resp. $w_L \to +\infty$). The Brownian to Brownian (with drifts) corresponds to $a_L = a_R = 1$, and contains the stationary case as a limit (when $w_L = w_R = 0$). The flat to Brownian (with drifts) corresponds to $a_L = 0, a_R = 1$ and contains as limits the half-Brownian $w_L \to +\infty$ and half-flat $w_R \to +\infty$. By symmetry it is also realized for $a_L = 1, a_R = 0$. In addition, at little further expense in the calculation, we will be able to add a step $\Delta$ in the initial height, i.e. study the initial condition

$$h_0^\Delta(y) = h_0(y) - \Delta \sgn(y)$$

where $h_0(y)$ is any of the above cases. With no loss of generality we will consider $\Delta \geq 0$, i.e. a downward step.

C. Results

In order to obtain the most interesting large time limit, we need to scale the original slopes and position with time so that the following rescaled parameters (denoted by hat)

$$\hat{w}_{L,R} = t^{1/3} w_{L,R} \quad , \quad \hat{x} = \frac{x}{2t^{2/3}}$$

are kept fixed as time becomes large. This is consistent with the standard KPZ scaling. Clearly this also contains the (less interesting) case where the limit is done instead with fixed $x, w_L, w_R$, which is equivalent to set $\hat{w}_{L,R} \to \infty$ and $\hat{x} \to 0$ in all formula below.

At large time the KPZ field grows linearly in time plus $\sim t^{1/3}$ fluctuations

$$h(x,t) \simeq v_\infty t + O(t^{1/3})$$

and for the continuum KPZ solution $v_\infty = -1/12$. To get rid of this part linear in time we will, from now on redefine the KPZ field, and the DP partition sum, at all times, as

$$h(x,t) = -\frac{t}{12} + \hat{h}(x,t) \quad , \quad Z(x,t) = e^{-t/12} \hat{Z}(x,t)$$

and for notational simplicity, we will omit the tilde in what follow.

1. Recall of standard results

Let us first recall the standard results. The first is for $h_{\text{drop}}(x,t)$ corresponding to the "droplet" or wedge initial condition $h_0(y) = -w|y|$ (i.e. here to $w_{L,R} = w$, $a_{L,R} = 0$). Strictly speaking, its exact solution at all times is valid only for the "hard" wedge limit, i.e. $w \to +\infty$. However here we will be interested only in the large time limit, hence $w$ can be chosen arbitrary but fixed. At large time the one-point fluctuations of the height are governed by the GUE Tracy Widom (cumulative) distribution $F_2(s)$ as

$$h_{\text{drop}}(0,t) = \ln Z_{\text{drop}}(0,t) \simeq t^{1/3} \chi_2 + o(t^{1/3}) \quad , \quad \text{Prob}(\chi_2 < \sigma) = F_2(\sigma)$$

where $F_2(\sigma)$ is given by a Fredholm determinant involving the Airy Kernel:

$$F_2(\sigma) = \text{Det}[I - P_\sigma K_{Ai} P_\sigma] \quad , \quad K_{Ai}(v,v') = \int_{y>0} dy Ai(y+v)Ai(y+v') = \frac{Ai(v)Ai'(v') - Ai'(v)Ai(v')}{v-v'}$$

and $P_\sigma(v) = \theta(v-\sigma)$ is the projector on $[\sigma, +\infty]$. Note that the solution is $h(x,t|y,0) \equiv h_{\text{drop}}(x-y,t) + \ln(\frac{w}{x})$ (for large $w$), corresponding to a hard wedge centered in $y$. Everywhere in this paper $\equiv$ means equivalent in law. The additive constant $\ln(\frac{w}{x})$ is necessary for regularization, but we will ignore below all time-independent constants.

More generally, for droplet initial conditions, the multi-point correlation of the field $h(x,t)$ is believed to converge to the ones of the Airy$_2$ process $A_2(\hat{x})$ with, in our units:

$$h(x,t) \simeq t^{1/3}(A_2(\hat{x}) - \hat{x}^2) + o(t^{1/3})$$
where \( A_2(0) \equiv \chi_2 \). Here \( \simeq \) means in law, as a process as \( x \) is varied. The process \( A_2(\hat{x}) \) is stationary, i.e. statistically translationally invariant in \( \hat{x} \), and well characterized: its correlations can be expressed in terms of, larger, Fredholm determinants in terms of the so-called extended Airy kernel. More generally, at large time

\[
h(x, t|y, 0) \simeq t^{1/3}(A_2(\hat{x} - \hat{y}) - (\hat{x} - \hat{y})^2) + o(t^{1/3})
\]

where \( h(x, t|y, 0) \) is the droplet solution with arbitrary endpoints \([\hat{y}][3]\). In terms of processes, this equivalence is only valid at either fixed \( x \) or fixed \( x \). The process as \( (x, y) \) are both varied is called the Airy sheet and is not yet characterized.

The second standard result is for the flat initial condition \( h(x, t = 0) = 0 \). There it was found\(^{22,23}\) that:

\[
h_{\text{flat}}(0, t) = \ln Z_{\text{flat}}(0, t) = 2^{-2/3}t^{1/3}\chi_1 + o(t^{1/3}), \quad \text{Prob}(\chi_1 < s) = F_1(s)
\]

where \( F_1(s) \) is the GOE Tracy Widom (cumulative) distribution. It is expressed as a Fredholm determinant

\[
F_1(s) = \text{Det}[1 - P_{s/2}K_{\text{GOE}}P_{s/2}], \quad K_{\text{GOE}}(v, v') = Ai(v + v')
\]

In that case, it is believed that the joint distribution of the heights \( \{h_{\text{flat}}(x, t)\}_x \) is governed by the so-called Airy\(_1\) stationary process \( A_1(u) \):

\[
h(x, t) \simeq 2^{1/3}t^{1/3}A_1(2^{-2/3}\hat{x}) + o(t^{1/3})
\]

where \( A_1(0) = \frac{1}{2}\chi_1 \). For definition and normalizations of the Airy\(_1\) process see e.g. Ref.\(^{11,12,59}\).

Note that there is a connection between these results. Indeed from the definition one expects, in the large time limit:

\[
h_{\text{flat}}(x, t) = \ln \int dy e^{h(x, t|y, 0)} \equiv \ln \int dy e^{h(y, t|x, 0)} \simeq t^{1/3} \max_{\hat{y}}(A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2)
\]

where we have used that the sets \( \{h(x, t|y, 0)\} \equiv \{h(y, t|x, 0)\} \) are statistically equivalent and that, since height fluctuations grow as \( t^{1/3} \), the integral is dominated by its maximum. Since one can shift \( \hat{y} \) by \( \hat{x} \), the maximum of the Airy\(_2\) process minus a parabola is given by the Airy\(_1\) process at one point

\[
\max_{\hat{y}}(A_2(\hat{y} - \hat{x})^2) = 2^{-2/3}\chi_1 = 2^{1/3}A_1(0)
\]

as proved in\(^{59}\).

2. Main results of the present work

Let us summarize some of the results of the present work, more results, i.e. equivalent kernels, various limits, comparison with known results, and more cases, are presented in Section \([11\] and Appendix [12\].

Here we obtain that, for the various initial conditions detailed in Section [11\], the following CDF is given in the limit of large time by a Fredholm determinant

\[
\text{Prob}(t^{-1/3}(h(x, t) + \frac{\sigma^2}{4t}) < \sigma) = \text{Det}[I - P_{\sigma}K_{P_{\sigma}}]
\]

where the kernel \( K \) takes the following forms in the various cases. We need to define the function

\[
\mathcal{B}_w(v) = e^{w^3/3}e^{-vw} - \int_{-\infty}^{\infty} dy Ai(v + y)e^{wy} = \int_{-\infty}^{0} dy Ai(v + y)e^{wy}
\]

where the second expression is only valid for \( w > 0 \), while the first is valid for any real \( w \). We find:

1. wedge initial condition \((a_L = a_R = 0)\)

\[
K(v_i, v_j) = \int_{0}^{+\infty} dy Ai(v_i + y)Ai(v_j + y)(1 - e^{-2(\hat{w}_L + \hat{w}_R)y}) + \int_{-\infty}^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2\hat{w}_L y}e^{-2\hat{w}_R y} + \theta(-y)e^{2\hat{w}_R y}
\]

which interpolates between flat \((\hat{w}_{L,R} \rightarrow 0^+)\) and droplet \((\hat{w}_{L,R} \rightarrow +\infty)\) initial conditions, and contains the half-flat \((\text{crossover } \mathcal{A}_{2 \rightarrow 1})\) as a special case for \( \hat{w}_L = 0^+ \), \( \hat{w}_R = +\infty \).
2. wedge-Brownian initial condition \((a_L = 0, a_R = 1)\)

\[
K(v_i, v_j) = \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j + y) - A_i(v_j) \int_0^{+\infty} dy A_i(v_i + y) e^{-2y} e^{-(2\hat{w}_L + \hat{w}_R)y} \\
+ B_{\hat{w}_R - \hat{x}_i} A_i(v_j) + \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j - y) e^{-2y(\hat{x} + \hat{w}_L)}
\]  

(27)

which for \(\hat{w}_L = +\infty\) reproduces the half-Brownian case. In the limit \(\hat{w}_L, \hat{w}_R \to 0^+\) we obtain the flat to stationary transition kernel given in (131), which is one main result of this paper.

3. Brownian-Brownian initial condition \((a_L = 1, a_R = 1)\)

\[
K(v_i, v_j) = \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j) - A_i(v_i) A_i(v_j) / \hat{w}_L + \hat{w}_R + A_i(v_j) B_{\hat{w}_L + \hat{x}_i} + B_{\hat{w}_R - \hat{x}_i} A_i(v_j)
\]

(29)

which, as is shown in Section [IV] reproduces the known result for the stationary case, although in a compact form (as a single Fredholm determinant) to our knowledge not presented before.

4. Finally we display the result for the step initial condition, here for simplicity for \(\hat{w}_L, \hat{w}_R = 0\) and \(a_L, a_R = 0\), i.e. a flat initial condition plus a (descending) step of amplitude \(\Delta > 0\). For the large time limit to be non-trivial we must scale the step size as \(t^{1/3}\), hence we define

\[
\hat{\Delta} = \Delta / t^{1/3}
\]

(30)

as the quantity kept fixed in the large time limit. In practice, since the KPZ equation has been derived, and is valid, only for small height gradients \(\partial_x h \ll 1\), we can think of smoothing the step over a scale \(\delta x \sim t^n\). The condition of small gradient only requires \(a \geq 1/3\), and we need \(a < 2/3\) for our result to hold (equivalent to \(t \gg \tau \gg 1\) in the formulation (23)). With this scaling, the kernel reads

\[
K(v_i, v_j) = \int_0^{+\infty} dy A_i(v_i + y) (A_i(v_j + y) - A_i(v_j + y + 4\hat{\Delta}) e^{4\hat{\Delta}}) \\
+ \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j) e^{-2y\hat{x}} + \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j + 4\hat{\Delta} + y) e^{2y\hat{x}} e^{4\hat{\Delta}}
\]

(31)

with however replacing the projector \(P_n \to P_{n - \hat{\Delta}}\) in (23), see Eq. (152) and (IV D) for more details. The generalization to arbitrary slopes \(\hat{w}_L, \hat{w}_R > 0\) is given in the Appendix, equation (D11).

Finally the result for a step on top of the Brownian-Brownian initial condition is given in (D13). The result for a step on top of the wedge-Brownian initial condition is given in (D15) and (D10) (and includes the flat to stationary plus a step for \(\hat{w}_L, \hat{w}_R = 0^+\)).

D. Generating functions

To obtain these results we will define and calculate some generating functions. For notational convenience we introduce a second set of rescaled parameters

\[
\lambda := 2^{-2/3} t^{1/3} , \quad s = 2^{2/3} (\sigma - \hat{x}^2) , \quad \hat{w}_{L,R} = \lambda w_{L,R} = 2^{-2/3} \hat{w}_{L,R} , \quad \hat{x} = x / \lambda^2 = \hat{\Delta}^{7/3} \hat{x}
\]

(32)

where the parameter \(\lambda\) was introduced in Refs.18,19,22. As in these works we define the standard generating function

\[
g_\lambda(s) := \exp(-e^{-\lambda s} Z(x,t)) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{e^{-n \lambda s}}{n!} Z(x,t)^n
\]

(33)

where \(Z(x,t) = e^{h(x,t)}\) and the second equality is only formal, as always in the RBA method for the continuum KPZ equation, since the sum is a divergent series. Examination of this series, however, will allow to obtain (or conjecture) the first average. In the large time limit it identifies with the CDF of the rescaled height

\[
\lim_{\lambda \to +\infty} g_\lambda(s) = \text{Prob} \left( \frac{1}{\lambda} h(x,t) < s \right)
\]

(34)
In view of the initial condition \[ \text{III} \] it is natural to split, in each realization of \((\eta, B_L, B_R)\), the DP partition sum into the set of paths starting at \(x\) and ending either left or right of \(y = 0\), as
\[
Z(x, t) = Z^L + Z^R
\]
\[
Z^L = Z^L(x, t) = \int_{-\infty}^{0} dy Z_\eta(x, t|y, 0) Z_0(y), \quad Z^R = Z^R(x, t) = \int_{0}^{+\infty} dy Z_\eta(x, t|y, 0) Z_0(y)
\]
with \(Z_0(y) = Z(x, t = 0)\), and to introduce the corresponding generalized generating function:
\[
g_\lambda(s_L, s_R) = e^{-s_L \lambda L - s_R \lambda R} Z^\lambda = \sum_{n_L, n_R \geq 0} (-1)^{n_L + n_R} \frac{e^{-n_L \lambda s_L - n_R \lambda s_R}}{n_L!n_R!} (Z^L)^{n_L} (Z^R)^{n_R}
\]
the standard generating function being recovered for equal arguments
\[
g_\lambda(s) = g_\lambda(s_L = s, s_R = s)
\]
This generalized generating function allows to study the initial condition \[ \text{III} \] in presence of a step, which can be written as
\[
Z_0^\Delta(y) = Z_0(y)e^{\Delta \theta(-y)} + Z_0(y)e^{-\Delta \theta(y)}
\]
Hence the standard generating function for this problem, denoted \(g_\lambda^\Delta(s)\), can be expressed as
\[
g_\lambda^\Delta(s) = g_\lambda^{\Delta = 0}(s_L = s - \frac{\Delta}{\lambda}, s_R = s + \frac{\Delta}{\lambda})
\]

### III. REPLICA CALCULATIONS

#### A. Averaging and quantum mechanics

The initial condition for the DP partition sum \(Z(x, t)\) is:
\[
Z(y, t = 0) = Z_0(y) = e^{h_0(y)} = e^{w_L y + a_L B_L(-y) \theta(-y) + e^{-w_R y + a_R B_R(y) \theta(y)}}
\]
The solution of the SHE with this initial condition can be written:
\[
Z(x, t) = \int dy Z_\eta(x, t|y, 0) Z_0(y)
\]
and we will calculate its positive integer moments with respect to the joint measure on \(\eta\) and \((B_L, B_R)\), denoted here more explicitly by overline \[ \cdot \cdot \cdot \] \(\eta\) and bracket \(<>_{B_L, B_R}\) respectively
\[
Z_n := < Z(x, t)^n >_{B_L, B_R}
\]
Since we have chosen \(\eta\) and \(B_L, B_R\) to be independent it can be written as
\[
Z_n = \int dy_1...dy_n \frac{1}{Z_\eta(x, t|y_1, 0) ... Z_\eta(x, t|y_n, 0)} < \prod_{\alpha=1}^{n} Z_0(y_\alpha) >_{B_L, B_R}
\]
Let us recall the STS symmetry. Using appendix A of Ref.\[23\] one easily sees that for all \(n\)
\[
< Z_{w^L, w^R}(x, t) >_{B_L, B_R} = e^{-\alpha^2} < Z_{w^L, w^R - \alpha}(0, t) >_{B_L, B_R}
\]
equivalently
\[
\ln Z_{w^L, w^R}(x, t) + \frac{\alpha^2}{4t} \equiv \text{inlaw} \ln Z_{w^L, w^R - \alpha}(0, t)
\]
The STS symmetry thus also fixes how the generating function depends on some combination of variables:

\[ g_{\lambda}(s; w_L, w_R, x) = \tilde{g}_{\lambda}(s + \frac{x^2}{16}; \tilde{w}_L + \frac{x}{8}, \tilde{w}_R - \frac{x}{8}) \]  

(47)

As is now well known\textsuperscript{24,35}, the \( \eta \) average in the middle of (41) can be rewritten as the expectation value between initial and final states of the quantum-mechanical evolution operator associated to the attractive Lieb-Liniger (LL) Hamiltonian for \( n \) identical particles\textsuperscript{20}.

\[ H_n = - \sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x^2_{\alpha}} - 2\bar{c} \sum_{1 \leq \alpha < \beta \leq n} \delta(x_\alpha - x_\beta), \quad \bar{c} = 1 \]  

(48)

We can thus rewrite (44) in quantum mechanical notations:

\[ Z_n = \langle x \ldots x | e^{-tH_n} | \Phi_0 \rangle \]  

(49)

where \( |x \ldots x \rangle \) is the state will all particles at the same point \( x \). Since this state is fully symmetric in exchanges of particles, only symmetric eigenfunctions will contribute and we can consider particles as bosons. The wavefunction of the initial replica state is:

\[ \Phi_0(Y) = \langle y_1 \ldots y_n | \Phi_0 \rangle = \langle \prod_{\alpha=1}^{n} e^{w_L y_\alpha} e^{a_L B_L(-y_\alpha)} \theta(-y_\alpha) + e^{-w_R y_\alpha} e^{a_R B_R(y_\alpha)} \theta(y_\alpha) \rangle_{B_L,B_R} \]  

(50)

where here and below coordinate multiplets are denoted by capital letters, e.g. \( Y \equiv y_1 \ldots y_n \). This state is also clearly symmetric in the replica, hence the argument about bosons can also be made with this state alone.

We now introduce the decomposition into eigenstates \( \Psi_{\mu} \) and eigenenergies \( E_{\mu} \) of \( H_n \) and rewrite the moment as a sum over eigenstates

\[ Z_n = \sum_{\mu} \Psi_{\mu}(x, \ldots, x) e^{-tE_{\mu}} \langle |\mu| |\Phi_0 \rangle = \sum_{\mu} \Psi_{\mu}^{*}(x, \ldots, x) e^{-tE_{\mu}} \langle \Phi_0 | \Psi_{\mu} \rangle \]  

(51)

where we have used that \( Z_n \) is real, and for convenience we will work with the second (i.e. complex conjugate) expression.

We can now use the explicit form of the eigenfunctions known from the Bethe ansatz\textsuperscript{20}. They are parameterized by a set of rapidities \( \mu \equiv \{\lambda_1, \ldots, \lambda_n\} \) which are solution of a set of coupled equations, the Bethe equations (see below). The eigenfunctions are totally symmetric in the \( x_\alpha \), and in the sector \( x_1 \leq x_2 \leq \cdots \leq x_n \), take the (un-normalized) form

\[ \Psi_{\mu}(x_1, \ldots, x_n) = \sum_{P \in S_n} A_P \prod_{j=1}^{n} e^{i \sum_{\alpha=1}^{n} \lambda_\alpha x_\alpha}, \quad A_P = \prod_{1 \leq \alpha < \beta \leq n} \left( 1 + \frac{i}{\lambda_\beta - \lambda_\alpha} \right). \]  

(52)

They can be deduced in the other sectors from their full symmetry with respect to particle exchanges. The sum runs over all \( n! \) permutations \( P \) of the rapidities \( \lambda_\alpha \). The corresponding eigenenergies are \( E_{\mu} = \sum_{\alpha=1}^{n} \lambda_\alpha^2 \). In the formula (51) we first need:

\[ \Psi_{\mu}^{*}(x, \ldots, x) = n! e^{-ix \sum_{\alpha} \lambda_\alpha}. \]  

(53)

and second, we need the overlap. Since both states are symmetric, their overlap can be rewritten as:

\[ \langle \Phi_0 | \Psi_{\mu} \rangle = n! \int_{y_1 < y_2 < \cdots < y_n} \Psi_{\mu}(Y) \Phi_0(Y) = n! \sum_{P \in S_n} A_P \int_{y_1 < y_2 < \cdots < y_n} e^{i \sum_{\alpha=1}^{n} \lambda_\alpha y_\alpha} \Phi_0(Y) \]  

(54)

using the explicit form of the Bethe eigenstates.

Introducing the numbers of replica \( n_{L,R} \) on each side of \( y = 0 \), it can be expressed as:

\[ \langle \Phi_0 | \Psi_{\mu} \rangle = n! \sum_{P \in S_n} A_P \sum_{n_{L,R} \geq 0} G_{n_{L,R} \alpha L}^{L} [\lambda_{P_1}, \ldots, \lambda_{P_{n_L}}] G_{n_{L,R} \alpha R}^{R} [\lambda_{P_n-n_{R}+1}, \ldots, \lambda_{P_{n_R}}] \]  

(55)
with $G^L_{a,0,\alpha} = G^R_{a,0,\alpha} = 1$. We have defined the integrals over the left and right half-axis

\begin{align}
G^L_{p,w,a}[\lambda_1, \ldots, \lambda_p] &:= \int_{y_1 < y_2 < \ldots < y_p < 0} e^{\sum_{j=1}^{p} (w + i\lambda_j) y_j} < e^{\sum_{j=1}^{p} a B_L(-y_j)} > B_L \\
G^R_{p,w,a}[\lambda_1, \ldots, \lambda_p] &:= \int_{0 < y_1 < y_2 < \ldots < y_p} e^{\sum_{j=1}^{p} (-w + i\lambda_j) y_j} < e^{\sum_{j=1}^{p} a B_R(y_j)} > B_R
\end{align}

We have taken advantage that $B_L$ and $B_R$ are independent and of the factorized form of each term in the wavefunction in each sector. To evaluate these blocks we now use the following averages over a one-sided Brownian, valid for ordered coordinates, as indicated:

\begin{align}
< e^{\sum_{j=1}^{p} B(-y_j)} > B &= e^{\frac{i}{2} \sum_{j=1}^{p} (2j-1)y_j}, \quad y_1 < \ldots < y_p < 0 \label{eq:bl}
< e^{\sum_{j=1}^{p} B(y_j)} > B &= e^{\frac{i}{4} \sum_{j=1}^{p} (2p-2j+1)y_j}, \quad 0 < y_1 < \ldots < y_p \label{eq:br}
\end{align}

and the integration identities, valid in the domains where the integrals converge:

\begin{align}
\int_{y_1 < y_2 < \ldots < y_p < 0} e^{\sum_{j=1}^{p} z_j y_j} = \prod_{j=1}^{p} \frac{1}{z_1 + \ldots + z_j}, \quad \int_{0 < y_1 < y_2 < \ldots < y_p} e^{\sum_{j=1}^{p} z_j y_j} = \prod_{j=1}^{p} \frac{-1}{z_{p+\ldots+z_{j-1}+1}}
\end{align}

It leads to:

\begin{align}
G^L_{p,w,a}[\lambda_1, \ldots, \lambda_p] &= \int_{y_1 < y_2 < \ldots < y_p < 0} e^{\sum_{j=1}^{p} (w + i\lambda_j) y_j} - a \frac{1}{2} (2j-1)y_j = \prod_{j=1}^{p} \frac{1}{jw + i\lambda_1 + \ldots + i\lambda_j - aj^2/2}, \label{eq:gl}
G^R_{p,w,a}[\lambda_1, \ldots, \lambda_p] &= \int_{0 < y_1 < y_2 < \ldots < y_p} e^{\sum_{j=1}^{p} (-w + i\lambda_j) y_j} + a \frac{1}{2} (2p-2j+1)y_j = \prod_{j=1}^{p} \frac{-1}{jw + i\lambda_1 + \ldots + i\lambda_{p-1} + aj^2/2}.
\end{align}

Note that:

\begin{align}
G^R_{p,w,a}[\lambda_1, \ldots, \lambda_p] = G^L_{p,w,a}[-\lambda_p, \ldots, -\lambda_1] \label{eq:mirror}
\end{align}

For our two "solvable" cases, $a = 0, 1$ a "miracle" occurs upon performing the summation over the permutations, leading to a factorized form $\ref{eq:gl}$.\ref{eq:mirror}.

\begin{align}
H^R_{p,w,a=1}[\{\lambda_1, \ldots, \lambda_p\}] := \sum_{P \in S_p} A_p G^R_{p,w,a=1}[\lambda_{P_1}, \ldots, \lambda_{P_p}] = \frac{2^p}{\prod_{j=1}^{p} (2w - 1 - 2i\lambda_j)} \label{eq:hr}
H^L_{p,w,a=0}[\{\lambda_1, \ldots, \lambda_p\}] := \sum_{P \in S_p} A_p G^L_{p,w,a=0}[\lambda_{P_1}, \ldots, \lambda_{P_p}] = \frac{1}{\prod_{\alpha=1}^{p} (w + i\lambda_{\alpha})} \prod_{1 \leq \alpha < \beta \leq p} \frac{2w + i\lambda_{\alpha} + i\lambda_{\beta} - 1}{2w + i\lambda_{\alpha} + i\lambda_{\beta}} \label{eq:hl}
\end{align}

where we have introduced two new functions which depend only on the set of rapidities, not on their order. They obey now

\begin{align}
H^R_{p,w,a}[\{\lambda_1, \ldots, \lambda_p\}] = H^L_{p,w,a}[\{-\lambda_1, \ldots, -\lambda_p\}] \label{eq:her}
\end{align}

Note that these miracle identities allow to obtain simple expressions for the terms where either $n_L$ or $n_R$ is zero in $\ref{eq:hr}$ but (a priori) not for the general term since there are then permutations which exchange rapidities in the left and right groups of rapidities. These simpler cases were used to obtain solutions for the half-flat and half-Brownian initial conditions $\ref{eq:gl}$.\ref{eq:mirror} (formally obtained by taking one of the slopes to infinity).

In the present case one does not seem able to proceed further without specifying the eigenfunctions $\ref{eq:her}$. We now recall that in the large $L$ limit one can work with string eigenstates. These possess specific properties which allow to obtain explicit expressions. This is based on combinatorial properties which were first claimed by Dotsenko $\ref{ref:22}$, and used by him in the case of the wedge (mostly for infinitesimal $w_{L,R}$). We will re-formulate, check, and slightly generalize these combinatorial identities and apply them to other cases.
B. Strings and combinatorial identities

Let us recall the spectrum of $H_n$ in the limit of infinite system size, i.e. the rapidities solution to the Bethe equations\(^{\text{56}}\). A general eigenstate is built by partitioning the $n$ particles into a set of $1 \leq n_s \leq n$ bound states called *strings* each formed by $m_j \geq 1$ particles with $n = \sum_{j=1}^{n_s} m_j$. The rapidities associated to these states are written as

$$\lambda_{j,a} = k_j - \frac{i}{\hbar}(m_j + 1 - 2a)$$

(64)

where $k_j$ is a real momentum, the total momentum of the string being $K_j = m_j k_j$. Here, $a = 1, ... , m_j$ labels the rapidities within the string $j = 1, ... , n_s$. We will denote $|\mu\rangle \equiv |k, m\rangle$ these states, labelled by the set of $k_j, m_j$, $j = 1, ... , n_s$. Here and below the boldface represents vectors with $n_s$ components.

Inserting these rapidities in (52) leads to the Bethe eigenfunctions of the infinite system, and their corresponding eigenenergies:

$$E_\mu = \frac{1}{12}nt + \tilde{E}(k, m)$$

(65)

$$\tilde{E}(k, m) := \sum_{j=1}^{n_s} m_j k_j^2 - \frac{1}{12}m_j^3$$

We have separated a trivial part of the energy, which can be eliminated by defining

$$Z_n = e^{-\frac{\alpha}{12}nt} \tilde{Z}_n$$

(66)

$$Z(x, t) = e^{-\frac{\alpha}{12}t} \tilde{Z}(x, t)$$

and

$$h(x, t) = -\frac{1}{12}t + \tilde{h}(x, t)$$

i.e. leading to a simple shift in the KPZ field. We will implicitly study in the remainder of the paper $\tilde{Z}_n$, $\tilde{Z}(x, t)$ and $\tilde{h}(x, t)$ but will remove the tilde in these quantities for notational simplicity (as already mentioned in the introduction).

The formula for the norm of the string states has been obtained as\(^{\text{57}}\):

$$\frac{1}{||\mu||^2} = \frac{1}{n!n_s} \Phi(k, m) \prod_{j=1}^{n_s} \frac{1}{m_j^2}$$

(67)

$$\Phi(k, m) = \prod_{1 \leq i < j \leq n_s} \Phi_{k_i, m_i, k_j, m_j}, \quad \Phi_{k_i, m_i, k_j, m_j} = \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2}$$

(68)

so that the formula\(^{\text{58}}\) for the moments becomes for $L \to +\infty$ (provided all limits exist)

$$Z_n = \sum_{n_s=1}^{n} \frac{1}{n!} \sum_{(m_1, ... , m_{n_s}), n_s}^{n_s!} \prod_{j=1}^{n_s} \int \frac{dk_j}{2\pi m_j} \Phi(k, m) e^{-t\tilde{E}(k, m)} e^{-i \sum_{j=1}^{n_s} m_j k_j x} \langle \Phi_0 | k, m \rangle$$

(69)

where the second sum is over the set of partitions, denoted $(m_1, ... , m_{n_s}), n$, of the integer $n = \sum_{j=1}^{n_s} m_j$ into $n_s$ parts, with each $m_j \geq 1$.

It remains to calculate the overlap, formula\(^{\text{55}}\). If the states are strings, the sum over permutations can be performed, using a general combinatoric identity which is detailed in the Appendix. The result is:

$$\langle \Phi_0 | k, m \rangle = n! \sum_{n_s=0}^{n} \sum_{m_1, ... , m_{n_s}} \tilde{H}_{L}^{n_s, w_L, a_L}[k, m, \mathbf{m}^L] \tilde{H}_{R}^{n_s, w_R, a_R}[k, m, \mathbf{m}^R] G[k, \mathbf{m}^L, \mathbf{m}^R]$$

(70)

for any string state. The sum corresponds to all possible ways to split rapidities in two groups associated to particles on $x < 0$ (left) and $x > 0$ (right). The combinatoric factor $G$ is a complicated product of Gamma functions and given in the Appendix\(^{\text{3}}\). It coincides with the one obtained by Dotsenko\(^{\text{20}}\). It will not play a role in the following as it will be set to unity in the large time limit. The $\tilde{H}_{L,R}$ terms are obtained by evaluating $H_{L}$ and $H_R$ on the two following complementory sets of rapidities

$$\tilde{H}_{L}^{n_s, w_L, a_L}[k, m, \mathbf{m}^L] = H_{L}^{n_s, w_L, a_L}[\lambda_1, \lambda_{1, m_1}, ..., \lambda_{n_s, 1}, ...]$$

(71)

$$\tilde{H}_{R}^{n_s, w_R, a_R}[k, m, \mathbf{m}^R] = H_{R}^{n_s, w_R, a_R}[\lambda_{1, m_1}, ... , \lambda_{n_s, 1}, ...]$$
where the first set contains the \( n_L \) rapidities on the left, and the second the \( n_R \) on the right, with the following notation for the string rapidities in the two groups

\[
\lambda_j, r_j = k_j - \frac{i}{2}(m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m^L_j
\]

\[
\tilde{\lambda}_j, r_j = k_j + \frac{i}{2}(m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m^R_j
\]

The functions \( \tilde{H}^{L,R} \) are thus only functions of the set of \( \{ k_j, m^L_j, m^R_j \}_{j=1,..,n} \), equivalently \( \{ k_j, m^L_j, m^R_j \}_{j=1,..,n} \) or \( \{ k_j, m^L_j, m^R_j \}_{j=1,..,n} \) since \( m^L_j + m^R_j = m_j \). They satisfy the symmetry relations

\[
\tilde{H}^{R}_{n_R,n_W,a_R} [k, m, m^R] = \tilde{H}^{L}_{n_L,n_W,a_L} [-k, m, m^R] = \left( \tilde{H}^{L}_{n_R,n_W,a_R} [k, m, m^R] \right)^* \tag{73}
\]

C. calculation of the factors \( \tilde{H} \)

We can now evaluate the functions \( \tilde{H}^{L,R} \) by injecting the string rapidities \( \{ k_j, m^L_j, m^R_j \}_{j=1,..,n} \), into the formula \( (71) \) and \( (72) \), according to the rule \( (71) \). This leads to expressions involving Pochhammer symbols, equivalently Gamma functions. We have:

\[
\tilde{H}^{L}_{n_L,n_W,a_L} [k, m, m^L] = \prod_{j=1}^{n_L} S_{m^L_j,m^R_j,k_j}^{w_{L,j},a_{L,j}} \prod_{1 \leq i < j \leq n_L} D_{m^L_i,m^R_i,k_i,m^L_j,m^R_j,k_j}^{w_{L,i},a_{L,i}} \tag{74}
\]

\[
\tilde{H}^{R}_{n_R,n_W,a_R} [k, m, m^R] = \prod_{j=1}^{n_R} S_{m^R_j,m^L_j,-k_j}^{w_{R,j},a_{R,j}} \prod_{1 \leq i < j \leq n_R} D_{m^R_i,m^L_i,-k_i,m^R_j,m^L_j,-k_j}^{w_{R,i},a_{R,i}} \tag{75}
\]

The single string factors are:

\[
S_{m^L,m^R,k}^{w,0} = \frac{(-2)^m \Gamma \left( 1 - 2w - 2ik - m^L - m^R \right)}{\Gamma \left( 1 - 2w - 2ik - m^R \right)} = 2^m \frac{\Gamma \left( 2w + 2ik + m^R \right)}{\Gamma \left( 2w + 2ik + m \right)} \tag{76}
\]

\[
S_{m^L,m^R,k}^{w,1} = \frac{(-1)^m \Gamma \left( 1 - w - ik - \frac{m^L}{2} \right)}{\Gamma \left( 1 - w - ik - \frac{m^L}{2} + m^R \right)} = \frac{\Gamma \left( w + ik + \frac{m^L}{2} \right)}{\Gamma \left( w + ik + \frac{m^L}{2} \right)} \tag{77}
\]

On the first line it reproduces, for \( m_R = 0 \), the one obtained in Ref.\(^{22}\) (Section 5.1) for the half-flat initial condition. The factors involving two strings are:

\[
D_{m^L_i,m^R_i,k_i,m^L_j,m^R_j,k_j}^{w_{L,i},a_{L,i}} = \frac{\Gamma \left( 1 - z_{j,j'} + \frac{i}{2} \left( -m^L_j - m^L_j - m^R_j - m^R_j \right) \right) \Gamma \left( 1 - z_{j,j'} + \frac{i}{2} \left( m^L_j + m^L_j - m^R_j - m^R_j \right) \right)}{\Gamma \left( 1 - z_{j,j'} + \frac{i}{2} \left( m^L_j - m^L_j - m^R_j + m^R_j \right) \right) \Gamma \left( 1 - z_{j,j'} + \frac{i}{2} \left( -m^L_j + m^L_j - m^R_j - m^R_j \right) \right)} \tag{77}
\]

and \( D_{m^L_i,m^R_i,k_i,m^L_j,m^R_j,k_j}^{w_{L,i},a_{L,i}} = 1 \), with \( z_{j,j'} = 2w_L + ik_j + ik_{j'} \). In the Brownian case there is thus no inter-string factors. For the half-flat case, \( 77 \) reproduces the one of Ref.\(^{22}\) (Section 5.1) when \( m^R_j = m^R_j = 0 \).

D. General formula for the moments and generating function

Putting all together we thus obtain the general formula for the moments

\[
Z_n = n! \sum_{n_L,n_R \geq 0} \prod_{j=1}^{n_L} \int \frac{dk_j}{2\pi m_j} \Phi(k,m)e^{-i\tilde{E}(k,m)}e^{-i\sum_j m_j k_j} x \times \sum_{n_L,n_R \geq 0} \tilde{H}^{L}_{n_L,n_W,a_L} [k, m, m^L] \tilde{H}^{R}_{n_R,n_W,a_R} [k, m, m^R] \mathcal{G}[k, m^L, m^R] \tag{78}
\]
Note that in this sum the term with fixed $n_L, n_R$ has a simple interpretation. Consider \[36\), \[37\], where in each realization of $(\eta, B_L, B_R)$, the DP partition sum is split into the set of paths starting at $x$ and ending either left or right of $y = 0$, The moments then split as:

$$Z_n = (Z^L + Z^R)^n = \sum_{n_L=0}^n \sum_{n_R=0}^n \delta_{n_L+n_R,n} \frac{n!}{(Z^L)^{n_L} (Z^R)^{n_R}}$$ (79)

Hence, by simple identification of the terms with fixed $n_L, n_R$ in \[38\] and multiplication by the factor $\frac{n_L! n_R!}{n!}$ we obtain an expression for each joint moment $(Z^L)^{n_L} (Z^R)^{n_R}$.

Let us come back to the generating function \[37\]. The sums over the variables $m, m_L, m_R$ now become free summations leading to

$$g_\lambda(s) = 1 + \sum_{n_s \geq 1} \frac{1}{n_s!} Z(n_s, s)$$ (80)
$$g_\lambda(s_L, s_R) = 1 + \sum_{n_s \geq 1} \frac{1}{n_s!} Z(n_s, s_L, s_R)$$ (81)

with

$$Z(n_s, s_L, s_R) = \prod_{j=1}^{n_s} \sum_{m_{jL}} \int \frac{dk_j}{2\pi m_j} (-1)^{m_j} e^{-\sum k_{jL} x_j} \Phi(k, m) e^{-tE(k, m)} \sum_{m_L, m_R} e^{-\sum (\lambda m_{jL}s_L + \lambda m_{jR}s_R)}$$ (82)

$$\times \prod_{j=1}^{n_s} S^{w_{L}, a_{L}}_{m_{jL}, m_{jR}, k_j} S^{w_{R}, a_{R}}_{m_{jR}, k_j} \times \prod_{1 \leq i < j \leq n_s} D^{w_{L}, a_{L}}_{m_{L}^i, m_{R}^i, k_j} D^{w_{R}, a_{R}}_{m_{R}^i, m_{L}^i, -k_i, m_{L}^j, m_{R}^j, -k_j} G[k, m^L, m^R]$$

Although this is an exact and explicit expression, apart from the case $a_L = a_R = 1$, it is unclear how to handle it for arbitrary time. We thus now turn to the large time limit.

**E. large time limit**

In the large time limit we will assume that one can set the product of factors $D$ and $G$ to unity. This is of course a highly non-trivial and radical assumption, however it is justified a posteriori by the results. It will be checked in all cases where the solution is known by other means. This procedure follows what has been done in other works, where it was also checked against other methods.\[36,45,48,50,51\]

1. **Determinantal form**

Let us first obtain a closed expression once these factors are set to unity, and take the large $\lambda$ limit in a second stage:

$$Z(n_s, s_L, s_R) = \prod_{j=1}^{n_s} \sum_{m_{jL}} \int \frac{dk_j}{2\pi m_j} (-1)^{m_j} e^{-im_j k_j x_j} \Phi(k, m) e^{-tE(k, m)}$$ (83)

$$\times \prod_{m_L, m_R} \sum_{m_{jL}} S^{w_{L}, a_{L}}_{m_{jL}, m_{jR}, k_j} S^{w_{R}, a_{R}}_{m_{jR}, k_j} e^{-\lambda m_{jL}s_L - \lambda m_{jR}s_R}$$ (84)

We now use the standard determinantal double-Cauchy identity:

$$\Phi(k, m) = \prod_{j=1}^{n_s} (2m_j) \det_{1 \leq i, j \leq n_s} \left[ \frac{1}{2i(k_i - k_j) + m_i + m_j} \right]$$ (85)
and perform the rescaling $k_j \to k_j / \lambda$. Denoting $\tilde{x} = x / \lambda^2$ we obtain:

\[
Z(n_s, s_L, s_R) = 2^{n_s} \prod_{j=1}^{n_s} \left[ \sum_{m_j=1}^{\infty} \frac{dk_j}{2\pi} (-1)^{m_j} e^{-i\lambda m_j k_j \tilde{x} + \frac{i}{2} \lambda m_j^2 - 4\lambda m_j k_j^2} \right] \det_{1 \leq i,j \leq n_s} \left[ \frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j} \right] \quad (86)
\]

\[
\times \sum_{m^L+m^R=m} \prod_{j=1}^{n_s} \frac{S^{w_L,a_L}}{m^L_{i},m^L_{j},k_{i}} \frac{S^{w_R,a_R}}{m^R_{i},m^R_{j},k_{j}} e^{-\lambda m_j^2 s_L - \lambda m_j^2 s_R} \quad (87)
\]

We now perform the Airy trick, i.e. use the representation $e^{\frac{i}{2} \lambda m^3} = \int dy Ai(y) e^{\lambda m y}$ to obtain

\[
Z(n_s, s_L, s_R) = 2^{n_s} \prod_{j=1}^{n_s} \left[ \sum_{m_j=1}^{\infty} \frac{dk_j}{2\pi} dy_j Ai(y_j) (-1)^{m_j} e^{-i\lambda m_j k_j \tilde{x} - 4\lambda m_j k_j^2 + \lambda m_j y_j} \right] \quad (88)
\]

\[
\det_{1 \leq i,j \leq n_s} \left[ \frac{1}{2i(k_i - k_j) + \lambda m_i + \lambda m_j} \right] \times \sum_{m^L+m^R=m} \prod_{j=1}^{n_s} \frac{S^{w_L,a_L}}{m^L_{i},m^L_{j},k_{i}} \frac{S^{w_R,a_R}}{m^R_{i},m^R_{j},k_{j}} e^{-\lambda m_j^2 s_L - \lambda m_j^2 s_R} \quad (89)
\]

Using standard manipulations the partition sum at fixed number of string $n_s$ can thus be expressed itself as a determinant:

\[
Z(n_s, s_L, s_R) = \prod_{j=1}^{n_s} \int_{v_j > 0} \det_{1 \leq i,j \leq n_s} M_{s_L,s_R}(v_i,v_j) \quad (90)
\]

with the Kernel:

\[
M_{s_L,s_R}(v_i,v_j) = \int \frac{dk}{2\pi} dy Ai(y + 4k^2 + ik\tilde{x} + v_i + v_j) e^{-2ik(v_i - v_j)} \phi_\lambda(k, y - s_L, y - s_R) \quad (92)
\]

\[
\phi_\lambda(k, y_L, y_R) = 2 \sum_{m^L \geq 0, m^R \geq 0, m^L + m^R \geq 1} (-1)^{m^L+m^R} \frac{S^{w_L,a_L}}{m^L_{i},m^L_{j},k_{i}} \frac{S^{w_R,a_R}}{m^R_{i},m^R_{j},k_{j}} e^{\lambda m^L y_L + \lambda m^R y_R} \quad (93)
\]

where the $S$ factors are given explicitly in (76). The generating function is thus a Fredholm determinant:

\[
g_\lambda(s_L, s_R) = \text{Det}[I + P_0 M_{s_L,s_R} P_0] \quad (94)
\]

where, again, this expression is valid as soon as the factors $D$ and $G$ are set (arbitrarily) to unity.

We must now study the function $\phi_\lambda(k, y_L, y_R)$ in the large time limit $\lambda \to +\infty$.

### 2. large time limit

We first rewrite:

\[
\phi_\lambda(k, y_L, y_R) = -2 + 2 \sum_{m^L \geq 0, m^R \geq 0} (-1)^{m^L+m^R} \frac{S^{w_L,a_L}}{m^L_{i},m^L_{j},k_{i}} \frac{S^{w_R,a_R}}{m^R_{i},m^R_{j},k_{j}} e^{\lambda m^L y_L + \lambda m^R y_R} \quad (95)
\]

and we use the Mellin-Barnes identity:

\[
\sum_{m=0}^{\infty} (-1)^m f(m) = -\frac{1}{2i} \int_C \frac{dz}{\sin \pi z} f(z) \quad (96)
\]

where $C = \kappa + i\mathbb{R}$, $-1 < \kappa < 0$, valid provided $f(z)$ is meromorphic, with no pole for $z > \Re(\kappa)$, and sufficient decay at infinity. It allows to rewrite (for $2w_L, w_R > \kappa > 0$)

\[
\phi_\lambda(k, y_L, y_R) = -2 + 2 \frac{(-1)^2}{2i} \int_C \frac{dz_L}{\sin \pi z_L} \int_C \frac{dz_R}{\sin \pi z_R} e^{\lambda z_L y_L + \lambda z_R y_R} \quad (97)
\]
Here the analytic continuation \( f(m) \rightarrow f(z) \) has been performed using the second expression in (76) as

\[
S_{w,z}^{w,0} = 2 \frac{\Gamma(2w + 2ik + zR)}{2 \Gamma(2w + 2ik + zR + 2)}
\]

We now rescale \( z_{L,R} \rightarrow z_{L,R}/\lambda \), and we study the large time limit \( \lambda \rightarrow +\infty \). We first recall the definition of the rescaled drifts:

\[
\tilde{w}_L = w_L \lambda \quad , \quad \tilde{w}_R = w_R \lambda
\]

and we use that for \( a = 0, 1 \):

\[
\lim_{\lambda \rightarrow +\infty} S_{\tilde{w}^a/\lambda, a}^{w, a} = 1 + \frac{(1 + a)z_L}{2w + 2ik + zL - az_L}
\]

as can be seen from (98). Thus we obtain the infinite \( \lambda \) limit in the form of a double contour integral:

\[
\phi_{+\infty}(k, y_L, y_R) = -2 \int_{C'} \frac{dz_L}{2\pi iz_L} \int_{C'} \frac{dz_R}{2\pi iz_R} (1 + \frac{(1 + a)z_L}{2\tilde{w}_L + 2ik + zL - az_L})(1 + \frac{(1 + a)z_R}{2\tilde{w}_R + 2ik + zR - az_R}) e^{z_Ly_L + z_Ry_R}
\]

where \( C' = 0^- + i\mathbb{R} \). The calculation of this integral is performed in Appendix B and the result is displayed in (B6).

For now, we focus on the simpler generating function, i.e. we set \( s_L = s_R = s \). In the infinite time limit it takes the form

\[
g_{+\infty}(s) = Det[I + P_0 M_s P_0]
\]

where \( P_0(v) = \theta(v) \) is the projector on \([0, +\infty]\) and with the Kernel

\[
M_s(v_i, v_j) = \int \frac{dk}{2\pi} dyAi(y + s + 4k^2 + ik \hat{x} + v_i + v_j) e^{-2ik(v_i - v_j)} \phi_{+\infty}(k, y)
\]

\[
-\frac{1}{2} \phi_{+\infty}(k, y) = \theta(y)[1 - (1 + a_L + a_R - 3a_La_R)e^{-(2\tilde{w}_L + 2ik)(1 + a_L)(2\tilde{w}_R - 2ik)(1 + a_L)y}]
\]

\[
+20(-y)(a_L e^{(2\tilde{w}_L + 2ik)y} + a_R e^{(2\tilde{w}_R - 2ik)y} + \delta(y)) \frac{1 - a_L}{2\tilde{w}_L + 2ik} + \frac{1 - a_R}{2\tilde{w}_R - 2ik} - \frac{a_R a_L}{\tilde{w}_L + \tilde{w}_R}
\]

The function \( \phi_{+\infty}(k, y) := \phi_{+\infty}(k, y_L = y, y_R = y) \) being obtained from the more general result (106) in the Appendix.

We now use Airy function identities in order to rewrite the result in terms of an alternative kernel. The calculation is performed in Appendix C. The final result is:

\[
g_{+\infty}(s) = Det[I + P_0 K_\sigma P_0] \quad , \quad K_\sigma(v_1, v_2) = K(v_1 + \sigma, v_2 + \sigma)
\]

\[
\sigma = 2^{-2/3}(s + \frac{z^2}{16}) \quad , \quad \tilde{w} = 2^{2/3} \tilde{w} \quad , \quad \tilde{x} = 2^{2/3} \frac{\tilde{x}}{8}
\]

\[
K(v_1, v_2) = \int_{-\infty}^{+\infty} dyAi(v_i + y)Ai(v_j + y) - \frac{a_R a_L}{\tilde{w}_L + \tilde{w}_R} Ai(v_i)Ai(v_j)
\]

\[
-(1 + a_L + a_R - 3a_La_R) \int_{-\infty}^{+\infty} dyAi(v_i + (1 - a_L + a_R)y)Ai(v_j)\big[1 - (1 + a_L - a_R)\big]
\]

\[
\times e^{-2y(\tilde{w}_L + \tilde{w}_R + a_R(\tilde{w}_L + \tilde{x}) + a_L(\tilde{w}_R - \tilde{x}))}
\]

\[
+a_LAi(v_i) \int_{-\infty}^{0} dyAi(v_i + y)e^{(\tilde{x} + \tilde{w}_L)y} + a_RAi(v_j) \int_{-\infty}^{0} dyAi(v_i + y)e^{(\tilde{w}_R - \tilde{x})y}
\]

\[
+(1 - a_L) \int_{0}^{+\infty} dyAi(v_i + y)Ai(v_j - y)e^{-2y(\tilde{x} + \tilde{w}_L)} + (1 - a_R) \int_{0}^{+\infty} dyAi(v_i - y)Ai(v_j + y)e^{-2y(\tilde{w}_R - \tilde{x})}
\]

which, we note depends only on \( \sigma \) and the combinations \( \tilde{w}_L + \tilde{x} \) and \( \tilde{w}_R - \tilde{x} \) (and their sum), as required by the STS symmetry (47).
IV. RESULTS FOR THE VARIOUS CROSSOVER KERNELS

We now discuss in details the results for the various initial conditions. We give both the result in the first form \((102)\), with kernel \(M_s\) from \((103)\).

\[
\lim_{t \to +\infty} \text{Prob}\left(t^{-1/3} h(x = 2^{-4/3} t^{2/3} \hat{x}, t) < 2 - 2^{2/3} s\right) = g_{+\infty}(s; \hat{x}, \hat{w}_L, \hat{w}_R) = \text{Det}[I + P_0 M_s P_0] \tag{108}
\]

naturally expressed in the variables \(\hat{x}, \hat{w}_{L,R} = 2^{-2/3} L^{1/3} w_{L,R}\), and the second, equivalent form of the result \((105)\), with kernel \(K_\sigma\) and \(K\) from \((107)\).

\[
\text{Prob}\left(t^{-1/3} h(x = 2t^{2/3} \hat{x}, t) + \frac{z^2}{4t} < \sigma\right) = g_{+\infty}(s) = \text{Det}[I - P_0 K_\sigma P_0] = \text{Det}[I - P_0 K P_0] \tag{109}
\]

naturally expressed in the variables \(\sigma, \hat{x}, \hat{w}_{L,R} = t^{1/3} w_{L,R}\).

A. the wedge initial condition

Let us start with the wedge initial condition \((11)\) with \(a_L = a_R = 0\).

1. first form of the wedge kernel

From \((103)\) we find

\[
M_s(v_i, v_j) = \int \frac{dk}{2\pi} dy A_i(y + s + 4k^2 + ik\hat{x} + v_i + v_j) e^{-2ik(v_i-v_j)}\phi_{+\infty}(k,y) \tag{110}
\]

\[
\phi_{+\infty}(k,y) = -2\theta(y)[1 - e^{-2(\hat{w}_L + \hat{w}_R)y}] - \delta(y)\left[\frac{1}{\hat{w}_L + ik} + \frac{1}{\hat{w}_R - ik}\right] \tag{111}
\]

Let us discuss several limits.

**Half-flat initial condition and GUE-GOE crossover:** For \(\hat{w}_R \to +\infty\) one recovers the Kernel for the half-flat case obtained in Ref.\(^{25}\) (formula (80-81) there). As shown there it interpolates between the GOE (flat) for \(\hat{x} \to -\infty\) and the GUE (droplet) kernels for \(\hat{x} \to +\infty\).

**Symmetric wedge:** In that case one chooses \(\hat{w}_L = \hat{w}_R = w\). We obtain

\[
M_s(v_i, v_j) = \int \frac{dk}{2\pi} dy A_i(y + s + 4k^2 + ik\hat{x} + v_i + v_j) e^{-2ik(v_i-v_j)}\phi_{+\infty}(k,y) \tag{112}
\]

\[
\phi_{+\infty}(k,y) = -2\theta(y)(1 - e^{-4\hat{w}y}) + \delta(y)\left[\frac{\hat{w}}{\hat{w}^2 + k^2}\right] \tag{113}
\]

This kernel also provides an interpolation from GUE (droplet) to GOE (flat) as \(\hat{w}\) is decreased from \(+\infty\) to 0. The two limits are particularly immediate on that form of the kernel. For \(\hat{w} \to +\infty\) one has:

\[
M_s(v_i, v_j) \to M_s^{\text{GUE}}(v_i, v_j) = -2 \int \frac{dk}{2\pi} \int_0^{+\infty} dy A_i(y + s + 4k^2 + ik\hat{x} + v_i + v_j) e^{-2ik(v_i-v_j)} \tag{114}
\]

This is identical to the GUE kernel in the form given in\(^{18}\). In the other limit \(\hat{w} \to 0^+\) we can replace

\[
\frac{\hat{w}}{\hat{w}^2 + k^2} \to \hat{w} \to 0^+ \pi\delta(k) \tag{115}
\]

leading to:

\[
M_s(v_i, v_j) \to M_s^{\text{GOE}}(v_i, v_j) = -Ai(s + v_i + v_j) \tag{116}
\]

which is the simplest form of the GOE kernel.
2. second form of the wedge kernel

The second form of the wedge kernel reads:

\[ K(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y)(1 - e^{-2(\tilde{w}_L + \tilde{w}_R)y}) + \int_{-\infty}^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y\tilde{x}}(\theta(y)e^{-2\tilde{w}_L y} + \theta(-y)e^{2\tilde{w}_R y}) \]

(117)

In the limit \( \tilde{w}_R \to +\infty \) one recovers the Kernel for the half-flat case in the second form obtained in\(^{45} \) (formula (89-91) there), namely

\[ K^{\text{half-flat}}(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y) + \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y\tilde{x}} \]

(119)

which, as discussed there, is equivalent to the result of Ref.\(^{46} \) for TASEP. As shown there it interpolates between the GOE (flat) for \( \tilde{x} \to -\infty \) and the GUE (droplet) kernels for \( \tilde{x} \to +\infty \), i.e. it is the (one-point) kernel associated to the \( A_{2\to1} \) interpolation process.

More generally in the double limit \( (\tilde{w}_L, \tilde{w}_R) \to (0^+, 0^+) \) we obtain, using another Airy identity:

\[ K(v_i, v_j) \simeq \int_{-\infty}^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y\tilde{x}} = 2^{-1/3}Ai(2^{-1/3}(v_i + v_j - 2\tilde{x}^2))e^{2\tilde{x}(v_i - v_j)} \]

(120)

(121)

Hence using that \( \sigma - \tilde{x}^2 = 2^{-2/3}s \):

\[ K_\sigma(v_i, v_j) = K(v_i + \sigma, v_j + \sigma) \simeq 2^{-1/3}Ai(2^{-1/3}(v_i + v_j + s))e^{2\tilde{x}(v_i - v_j)} \]

(122)

Under a similarity transformation this is equivalent to the GOE kernel:

\[ K_\sigma(v_i, v_j) \equiv Ai(v_i + v_j + s) \]

(123)

B. the wedge-Brownian initial condition

Let us now consider now the wedge-Brownian initial condition \(^{41} \), with \( a_L = 0 \) and \( a_R = 1 \). This case contains the flat to stationary crossover as a limit, see below.

1. first form of the kernel

From \(^{103} \) we find:

\[ M_\sigma(v_i, v_j) = \int \frac{dk}{2\pi} dy Ai(y + s + 4k^2 + ik\tilde{x} + v_i + v_j)e^{-2ik(v_i - v_j)}\phi_{+\infty}(k, y) \]

(124)

\[ \phi_{+\infty}(k, y) = -2\theta(y)[1 - 2e^{-2(\tilde{w}_L + \tilde{w}_R + ik)y}] - 4\theta(-y)e^{(2\tilde{w}_R - 2ik)y} - \frac{1}{\tilde{w}_L + ik} \]

(125)

2. second form of the kernel

The second form of the kernel reads:

\[ K(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y) - Ai(v_j) \int_0^{+\infty} dy Ai(v_i + y)e^{-2y\tilde{x}}e^{-(2\tilde{w}_L + \tilde{w}_R)y} \]

\[ +Ai(v_j) \int_{-\infty}^{0} dy Ai(v_i + y)e^{(\tilde{w}_R - \tilde{x})y} + \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y(\tilde{x} + \tilde{w}_L)} \]
where we have performed the change of variable \( y \to y/2 \) in the second term. Note that the second integral is convergent only for \( \hat{w}_R - \hat{x} > 0 \). It is however easily extended to arbitrary values (see below).

**Half-Brownian limit:** In the limit \( \hat{w}_L \to +\infty \) one should recover the half-Brownian initial condition. In that limit

\[
K(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y) + Ai(v_j)\int_{-\infty}^{0} dy Ai(v_i + y)e^{(\hat{w}_R-\hat{x})y} \tag{126}
\]

Note that the second integral is convergent only for \( \hat{w}_R - \hat{x} > 0 \). To obtain a more general expression, we can use the identity, valid for \( u > 0 \):

\[
\int_{-\infty}^{+\infty} dy Ai(v_i + y)e^{uy} = e^{\frac{u^3}{3} - u v_i} \tag{127}
\]

and replace

\[
K(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y) + Ai(v_j)(e^{\frac{1}{3}(\hat{w}_R-\hat{x})^3}e^{-(\hat{w}_R-\hat{x})v_i} - \int_{0}^{\infty} dy Ai(v_i + y)e^{(\hat{w}_R-\hat{x})y} \tag{128}
\]

and expression where now the integrals are convergent for any \( \hat{w}_R - \hat{x} \) and which coincides with the asymptotic large time formula (2.23) in Ref.41 (the correspondence is that \( X, \gamma \) there are \( X = \hat{x} - \hat{w}_R, \gamma_t = t^{1/3} \)). Thus the above replacement (127) is legitimate (it can in fact be shown also from the first form of the kernel, repeating the calculation of Appendix C) and we will use it repeatedly in the following.

We can now go back to the general case of the wedge-Brownian initial condition (129) and note that it can be written as the sum of the half-flat kernel (which interpolates between GUE and GOE) and a projector

\[
K(v_i, v_j) = K_{\text{half-flat}}(v_i, v_j) + \Phi(v_i)Ai(v_j) \tag{129}
\]

where

\[
\Phi(v_i) = e^{\frac{1}{3}(\hat{w}_R-\hat{x})^3}e^{-(\hat{w}_R-\hat{x})v_i} - \int_{0}^{\infty} dy Ai(v_i + y)e^{\hat{x}y}[e^{\hat{w}_Ry} + e^{-(2\hat{w}_L+\hat{w}_R)y}] \tag{130}
\]

and \( K_{\text{half-flat}}(v_i, v_j) \) is given in (131).

**Flat to stationary crossover:** It is now possible to consider the limit \( \hat{w}_L, \hat{w}_R \to 0^+ \). One obtains

\[
K(v_i, v_j) = \int_{0}^{+\infty} dy Ai(v_i + y)Ai(v_j + y) + \int_{0}^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y\hat{x}} \tag{131}
\]

\[
+ Ai(v_j)(e^{-\frac{1}{3}\hat{x}^3}e^{\hat{x}v_i} - 2\int_{0}^{\infty} dy Ai(v_i + y)e^{-\hat{x}y})
\]

which is the main result of this paper. It has the form of the \( A_{2\to1} \) transition kernel plus a projector.

### C. the Brownian-Brownian initial condition

Consider now the Brownian-Brownian initial condition (41), with \( a_L = a_R = 1 \), i.e. a double sided Brownian initial condition.

1. **first form of the kernel**

From (103) we find:

\[
M_s(v_i, v_j) = \int \frac{dk}{2\pi} dy Ai(y + s + 4k^2 + ik\hat{x} + v_i + v_j)e^{-2ik(v_i-v_j)}\phi_{+\infty}(k, y) \tag{132}
\]

\[
\phi_{+\infty}(k, y) = -2\theta(y) - 4\theta(-y)(e^{(2\hat{w}_L+2ik)y} + e^{(2\hat{w}_R-2ik)y}) + \frac{2}{\hat{w}_L + \hat{w}_R}\delta(y) \tag{133}
\]
2. second form of the kernel

The second form of the kernel reads:

\[
K(v_1, v_2) = \int_0^{\infty} dy Ai(v_1 + y) Ai(v_2 + y) - \frac{1}{\hat{w}_L + \hat{w}_R} Ai(v_1) Ai(v_2) + Ai(v_1) B_{\hat{w}_L + \hat{w}_R}(v_2) + Ai(v_2) B_{\hat{w}_L + \hat{w}_R}(v_1)
\]  

(134)

where we have defined

\[
B_w(v) = e^{w^3/3} e^{-vw} - \int_0^{\infty} dy Ai(v + y) e^{wy} = \int_{-\infty}^{0} dy Ai(v + y) e^{wy}
\]  

(135)

where the second form is valid for \( w > 0 \), while the first one is valid for arbitrary \( w \) (see discussion above).

We now show that this result is equivalent to the result of Ref.\(^{28}\) in the large time limit. The notations of that paper are \( X = \hat{x}, \gamma_t = t^{1/3}, v_\pm = w_{R,L} \) hence \( \omega_\pm = v_\pm \gamma_t = \hat{w}_{R,L} \) (with \( \alpha = 1 \) in our units \( \nu = 1, D = 2, \lambda_0 = 2 \)). The CDF of the height was obtained in formula (6.21-22) at at large time (correcting the misprint \( X \to -X \) there):

\[
F_{\hat{w}_R, \hat{w}_L}(\sigma) = \text{Det}(1 - P_0 B_\sigma P_0) \quad (136)
\]

\[
B(v_1, v_2) = K_{\hat{w}_R}(v_1, v_2) + (\hat{w}_R + \hat{w}_L) B_{\hat{w}_R - \hat{w}}(v_1) B_{\hat{w}_L + \hat{w}}(v_2)
\]  

(137)

To show that they are the same, let us first express explicitly the derivative

\[
F_{\hat{w}_R, \hat{w}_L}(\sigma) = \text{Det}(1 - P_0 B_\sigma P_0) (1 - \frac{1}{\hat{w}_R + \hat{w}_L} \text{Tr} ((1 - P_0 B_\sigma P_0)^{-1} P_0 \partial_\sigma B_\sigma P_0))
\]

(138)

we have used that \( \text{Det}(1 - P_0 B_\sigma P_0) = \text{Det}(1 - P_0 B_\sigma P_0) \) where \( B_\sigma(v_1, v_2) = B(v_1 + \sigma, v_2 + \sigma) \). To obtain \( \partial_\sigma B_\sigma \) we calculate the following derivatives

\[
\partial_\sigma K_{\hat{w}_R}(v_1 + \sigma, v_2 + \sigma) = \int_{\lambda > 0} \lambda [Ai(v_1 + \sigma + \lambda) Ai(v_2 + \sigma + \lambda) - Ai(v_1 + \sigma) Ai(v_2 + \sigma)]
\]

\[
\partial_\sigma B_w(v + \sigma) = \int_{\lambda > 0} d\lambda \lambda [Ai(v + \sigma + \lambda)] e^{\lambda w} = Ai(v + \sigma) - w B_w(v + \sigma)
\]

(139)

(140)

where we used integration by parts. Hence we obtain

\[
\frac{1}{\hat{w}_R + \hat{w}_L} \partial_\sigma B_\sigma(v_1, v_2) = -\frac{Ai(v_1 + \sigma) Ai(v_2 + \sigma)}{\hat{w}_R + \hat{w}_L} + Ai(v_1 + \sigma) B_{\hat{w}_L + \hat{w}}(v_2 + \sigma) + Ai(v_2 + \sigma) B_{\hat{w}_L + \hat{w}}(v_1 + \sigma)
\]

\[
-(\hat{w}_R + \hat{w}_L) B_{\hat{w}_R - \hat{w}}(v_1 + \sigma) B_{\hat{w}_L + \hat{w}}(v_2 + \sigma)
\]

(141)

Now we note that it can be written as a product

\[
\frac{1}{\hat{w}_R + \hat{w}_L} \partial_\sigma B_\sigma(v_1, v_2) = -\phi_1(v_1 + \sigma) \phi_2(v_2 + \sigma)
\]

(142)

where we have defined

\[
\phi_1(v_1) = \frac{Ai(v_1)}{\sqrt{\hat{w}_L + \hat{w}_R} B_{\hat{w}_R - \hat{w}}(v_1)}
\]

\[
\phi_2(v_2) = \frac{Ai(v_2)}{\sqrt{\hat{w}_L + \hat{w}_R} B_{\hat{w}_L + \hat{w}}(v_2)}
\]

(143)

(144)

Hence \( \frac{1}{\hat{w}_R + \hat{w}_L} \partial_\sigma B_\sigma(v_1, v_2) \) is a projector, which implies that (138) can be rewritten as

\[
F_{\hat{w}_R, \hat{w}_L}(\sigma) = \text{Det}(1 - P_0 (B_\sigma - \phi_1^2 \phi_2^2) P_0) = \text{Det}(1 - P_0 K_\sigma P_0)
\]

(145)

since one can check that our Kernel (134) can be written as:

\[
K(v_1, v_2) = B(v_1, v_2) - \phi_1(v_1) \phi_2(v_2)
\]

\[
K_\sigma(v_1, v_2) = B_\sigma(v_1, v_2) - \phi_1(v_1 + \sigma) \phi_2(v_2 + \sigma)
\]

(146)

(147)

As is discussed in Ref.\(^{28}\) the expression (139) is equivalent to the one in Theorem 5.1. of Ref.\(^{29}\) derived for the PNG model with external sources. The relation between (139) and the result of Baik and Rains in Ref.\(^{28}\) (in terms of the solution of a Painleve II equation) is discussed in Proposition 5.2 of Ref.\(^{28}\). The result also agrees with the one for stationary TASEP\(^{10}\) and a rigorous derivation was given in Ref.\(^{40}\). Note that in Ref.\(^{28}\) the solution is given for arbitrary time, which is possible in that case. This provides a test of our more direct (but more empirical) method to obtain directly the large time limit.
D. Adding a step to the initial condition

1. first form of the kernel

Consider now the step initial condition \( \{10\} \). As discussed in previous sections, to obtain the solution for that case, in the large time limit, we need to calculate the generalized generating function in the large time limit

\[
\lim_{t \to +\infty} \text{Prob}\left(t^{-1/3} h(x = 2t^{2/3} \hat{x}, t) < s\right) = g_{+\infty}^\Delta(s) = g_{+\infty}(s_L = s - \hat{\Delta}, s_R = s + \hat{\Delta})
\]

(148)

where \( \hat{\Delta} = \Delta/\lambda \). We will specify to \( a_L = a_R = 0 \), i.e. step initial condition on top of the wedge. The solutions for the two other cases, the step plus half-Brownian (or step on top of flat to stationary), and step on top of two-sided Brownian are given in Appendices \[\ref{D.5}\] and \[\ref{D.4}\] respectively.

From \[\ref{D.2}\], \[\ref{D.3}\] and the result \[\ref{D.6}\] in Appendix \[\ref{D.4}\] we can write that

\[
g_{+\infty}(s_L, s_R) = \text{Det}[I + P_0 M_{s_L, s_R} P_0]
\]

(149)

Let us give the result for \( \hat{\Delta} > 0 \), i.e. \( s_R > s_L \)

\[
M_{s_L, s_R}(v_i, v_j) = \int \frac{dk}{2\pi} dy A_i(y + 4k^2 + ik \hat{x} + v_i + v_j) e^{-2ik(v_i - v_j)} \phi_{+\infty}(k, y - s_L, y - s_R)
\]

(150)

with

\[
\phi_{+\infty}(k, y - s_L, y - s_R) = -2 + 2\theta(s_L - y) + 2\theta(y - s_R) e^{-2\hat{\Delta} \hat{\Delta} \hat{\Delta}} e^{2ik(s_R - s_L)}
\]

(151)

at this stage we have also kept arbitraty slopes \( \hat{w}_{L,R} \).

2. second form of the kernel

We now obtain the second form for the result. The details are given in Appendix \[\ref{D.9}\] and \[\ref{D.3}\]. We find

\[
\text{Prob}\left(t^{-1/3} h(x = 2t^{2/3} \hat{x}, t) < \sigma\right) = g_{+\infty}^\Delta(s) = \text{Det}[I - P_\sigma - \hat{\Delta} K_\Delta P_\sigma - \hat{\Delta}], \quad \hat{\Delta} = \Delta/t^{1/3}
\]

(152)

with the kernel

\[
K_\Delta(v_i, v_j) = \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j + y) - \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j + y + 4\hat{\Delta}) e^{4\hat{\Delta} \hat{\Delta} \hat{\Delta}}
\]

(153)

\[
+ \int_0^{+\infty} dy A_i(v_i + y) A_i(v_j - y) e^{-2y \hat{\Delta} \hat{\Delta} \hat{\Delta}} + \int_0^{+\infty} dy A_i(v_i - y) A_i(v_j + 4\hat{\Delta} + y) e^{2y \hat{\Delta} \hat{\Delta} \hat{\Delta}}
\]

The generalization to arbitrary slopes \( \hat{w}_{L,R} > 0 \) is given in the Appendix, equation \[\ref{D.11}\].

Note that \[\ref{D.2}\] can also be written, denoting \( \sigma' = \sigma - \Delta \), as

\[
\text{Prob}\left(t^{-1/3} h(x = 2t^{2/3} \hat{x}, t) - \Delta + \frac{x^2}{4t} < \sigma'\right) = \text{Det}[I - P_\sigma, K_\Delta P_\sigma]
\]

(154)

Hence \( \sigma' \) measure the fluctuatin w.r.t the height level of the step on the left \( x < 0 \). Thus, if \( \hat{\Delta} \to +\infty \) the step size becomes infinite and the height level on the right goes to \( -\infty \) (relatively to the left). Thus one must find the half-flat kernel, and indeed one can check that the second and fourth term in \[\ref{D.3}\] vanish in that limit, i.e. \( K_{\Delta=+\infty}(v_i, v_j) \to K_{\text{half-flat}}(v_i, v_j) \), as given in \[\ref{D.10}\].

In the limit \( \hat{\Delta} \to 0 \) the first two terms in \[\ref{D.3}\] cancel and one finds

\[
K_{\Delta=0^+}(v_i, v_j) = \int_{-\infty}^{+\infty} dy A_i(v_i + y) A_i(v_j - y) e^{-2y \hat{\Delta} \hat{\Delta} \hat{\Delta}} = 2^{-1/3} A_i(2^{-1/3}(v_i + v_j - 2\hat{x}^2)) e^{2\hat{x}(v_i - v_j)}
\]

(155)
Hence
\[ K_{\lambda=0^+}(v_i + \sigma + \hat{x}^2, v_j + \sigma + \hat{x}^2) = 2^{-1/3} \hat{K}_{\text{GOE}}(2^{-1/3} v_i + s/2, 2^{-1/3} v_j + s/2) e^{\tilde{z}(v_i - v_j)} \]  
\[ \equiv K^{\text{GOE}}(v_i + s/2, v_j + s/2) \]  
with \( s = 2^{2/3} \sigma \) and we recall \( K^{\text{GOE}}(v_i, v_j) = Ai(v_i + v_j) \). Hence we recover the result for the flat initial condition \[15\].

V. CONCLUSION

In conclusion we have used the replica Bethe ansatz method to study the distribution of the the scaled interface height at one space time point in the 1D KPZ equation, with a set of initial conditions which are different on the negative and the positive half line. This set contains all standard crossover classes between respectively flat, droplet and stationary on each side, as well as in presence of slopes (i.e. drifts). The method also allows to add a step at the origin for each of these initial conditions. The slopes and step parameters, as well as the coordinate of the observation point, are properly scaled with time so that the result is non-trivial in the large time limit and interpolates between various classes of initial conditions, as they are varied. This generalizes our previous work on the crossover between flat and droplet. In all cases the one point CDF of the height can be expressed as a Fredholm determinant with various kernels depending on the parameters. All these expressions, although obtained starting from the KPZ equation, are conjectured to be universal for all models in the 1D KPZ class.

The method contains some heuristics, following previous works, as it assumes that in the large time limit, a decoupling occurs, so that some terms can be set to unity in the complicated sum over string eigenstates, allowing for an exact calculation. The calculation is performed by using, and further testing and extending, a combinatorics method introduced by Dotsenko. We test the validity of the method in cases where the answer is known, such as flat, droplet and their crossover, as well as Brownian and half Brownian. In these cases, it reproduces the known result, although sometimes naturally leading to new, equivalent, forms for the kernels. In all other cases, it produces some conjectures for the kernels. Among them, the flat to stationary crossover kernel is directly obtained. It would be interesting to confirm all the present results by different methods.

Note added: while this work was in the last stages of completion, we learned of the recent work of Quastel and Remenik\[61\], who obtained a general formula for a very large class of initial conditions. Although these do not yet allow to average over random initial conditions (such as Brownian) it would be interesting, in the deterministic case, to compare their formula (obtained for Airy processes) and the present results (obtained starting from the KPZ equation). In an even more recent work\[59\], they prove the convergence to such formula starting from TASEP.

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Appendix A: General identity

1. Preliminaries

Consider the Bethe wave function \[52\], \( \Psi_{\mu}(X) \), which is a symmetric function of its arguments, and which, for \( x_1 \leq \ldots \leq x_n \) reads
\[
\Psi_{\mu}(X) = \sum_{P \in S_n} A_P e^{i \sum_{\alpha} \lambda_{P_{\alpha}} x_\alpha} \quad , \quad A_P = \prod_{1 \leq \alpha < \beta \leq n} a_{\lambda_{P_{\beta}}, \lambda_{P_{\alpha}}} \quad , \quad a_{\lambda, \lambda'} = 1 + \frac{i}{\lambda - \lambda'} \tag{A1}
\]
where \( \lambda_1, \ldots, \lambda_n \) are the rapidities. Split the coordinates \( x_\alpha \) into two groups, \( N_L \) with \( \alpha = 1, \ldots, n_L \), i.e. \( x_1 \leq x_2 \leq \ldots \leq x_{n_L} \) and \( N_R \) with \( \alpha = n_L + 1, \ldots, n_L + n_R = n \), i.e. \( x_{n_L + 1} \leq \ldots \leq x_n \), such that \( x_\alpha \leq x_b \) for all \( a \in N_L \) and \( b \in N_R \). In (A1), for each permutation \( P \) a rapidity \( \lambda_{P_{\alpha}} \) is associated to each coordinate \( x_\alpha \), hence for each permutation \( P \) a first \( n_L \)-plet of rapidities
$(\lambda_{P_1}, \ldots, \lambda_{P_{n_L}})$ is associated to the group $N_L$ and a second $n_R$-uplet, $(\lambda_{P_{n_L+1}}, \ldots, \lambda_{P_n})$ is associated to the group $N_R$.

In a number of applications one needs to calculate

$$\sum_{P \in S_n} \prod_{1 \leq \alpha < \beta \leq n} a_{\lambda_{P_\alpha}, \lambda_{P_{\beta}}} F_{n_L}^{L}[\lambda_{P_1}, \ldots, \lambda_{P_{n_L}}] F_{n_R}^{R}[\lambda_{P_{n_L+1}}, \ldots, \lambda_{P_n}]$$  \hspace{1cm} (A2)

where here we will consider $F_{n_L}^{L}$ and $F_{n_R}^{R}$ to be arbitrary functions of $n_L$, respectively $n_R$, variables, with a priori no symmetry (i.e. functions of the $n_L$-uplet and $n_R$-uplet, respectively). This is the case for instance for the calculation of the overlap of $\Psi_\mu(X)$ with any other wave function which splits into a product over $N_L$ and $N_R$, see [5] as an example. Note that there we eventually sum over $n_L, n_R = n - n_L$ but here we will consider the more general question of evaluation of (A2) for any fixed $n_L, n_R = n - n_L$.

Let us consider this question when the rapidities are strings. So consider now a Bethe state with $n_s$ strings specified by $k_j, m_j, j = 1, \ldots, n_s$, i.e. rapidities labeled as:

$$\lambda_\alpha \rightarrow \lambda_j, r_j = k_j - \frac{i}{2} (m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m_j$$  \hspace{1cm} (A3)

In notations of the text such a state is denoted as $[k, m]$. As is well known, and clear from the definition (A1), the only permutations $P$ which have a non vanishing amplitude $A_P$, are those such that for each string the intra-string order of increasing imaginary part is maintained. Hence if one is given the set of $2n_s$ integers $(m_j^L, m_j^R), j = 1, \ldots, n_s$:

$$0 \leq m_j^L \leq m_j \quad 0 \leq m_j^R \leq m_j \quad m_j^L + m_j^R = m_j$$  \hspace{1cm} (A4)

which specifies how many particles in each string belongs to each of the two groups, then one knows (bijectively) the two sets of rapidities which belong of each group. For instance one knows that the first set of rapidities is:

$$\Lambda_L = \{\lambda_{j, r_j}, j = 1, \ldots, n_s, r_j = 1, \ldots, m_j^L\}$$  \hspace{1cm} (A5)

and the second set is the complementary $\Lambda_R = \{\lambda_{j, r_j}, j = 1, \ldots, n_s, r_j = m_j^L + 1, \ldots, m_j\}$. To treat these two sets on equal footing, it is convenient to introduce the notation:

$$\lambda_{j, r_j} = k_j - \frac{i}{2} (m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m_j^L$$  \hspace{1cm} (A6)

$$\bar{\lambda}_{j, r_j} = k_j + \frac{i}{2} (m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m_j^R$$  \hspace{1cm} (A7)

Note that the sets are now specified, but that within each set, one still needs to sum over all possible orders, i.e. permutations. That is, to each of these two sets, one can associate $n_L!$ possible $n_L$-uplets (respectively $n_R!$ possible $n_R$-uplets) of rapidities.

Consider now the quantity defined by (A2) for a given string state $[k, m]$. One can guess that the sum over $P \in S_n$ in (A2) can now be made in two stages. In a first stage one fixes the $m^R = \{m_{j}^{R}\}_{j=1, \ldots, n_s}$, equivalently the $m^L = \{m_{j}^{L}\}_{j=1, \ldots, n_s}$, and perform the sum over permutations inside each set, and then, in a second stage sum over the variables $\{m_{j}^{R}\}$. One takes advantage that one can factor $A_P$ as

$$\prod_{1 \leq \alpha < \beta \leq n} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}} \prod_{1 \leq \alpha < \beta \leq n_L} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}} \prod_{n_L + 1 \leq \alpha < \beta \leq n} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}} \prod_{\alpha = 1}^{n} \prod_{\beta = n_L + 1}^{n} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}}$$  \hspace{1cm} (A8)

and one defines

$$H_{n_L}^{L}\{\lambda_{1}, \ldots, \lambda_{n_L}\} = \sum_{P \in S_{n_L}} (\prod_{1 \leq \alpha < \beta \leq n_L} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}}) F_{n_L}^{L}[\lambda_{P_1}, \ldots, \lambda_{P_{n_L}}]$$  \hspace{1cm} (A9)

$$H_{n_R}^{R}\{\lambda_{n_L+1}, \ldots, \lambda_{n}\} = \sum_{P \in S_{n_R}} (\prod_{n_L + 1 \leq \alpha < \beta \leq n} a_{\lambda_{P_{\alpha}}, \lambda_{P_{\beta}}}) F_{n_R}^{R}[\lambda_{P_{n_L+1}}, \ldots, \lambda_{P_n}]$$  \hspace{1cm} (A10)

Clearly $H_{n_L}^{L}$ and $H_{n_R}^{R}$ are now fully symmetric functions of their arguments. One can then evaluate $H_{n_L}^{L}$ on the set $\Lambda_L$ and $H_{n_R}^{R}$ on the set $\Lambda_R$. One thus defines:

$$\tilde{H}^{L}[k, m, m^L] = H_{n_L}^{L}[\lambda_{1,1}, \ldots, \lambda_{1,m_1^L}, \ldots, \lambda_{n_s,1}, \ldots, \lambda_{n_s,m_s^L}]$$  \hspace{1cm} (A11)

$$\tilde{H}^{R}[k, m, m^R] = H_{n_R}^{R}[\bar{\lambda}_{1,1}, \ldots, \bar{\lambda}_{1,m_1^R}, \ldots, \bar{\lambda}_{n_s,1}, \ldots, \bar{\lambda}_{n_s,m_s^R}]$$  \hspace{1cm} (A12)
where we have defined the function $G$ and the factors $G$ can be expressed using Pochhammer symbols $(x)_n = \Gamma(x + n)/\Gamma(x)$ as follows

$$G_{j,j'} = \prod_{r=1}^{m^L_j} \prod_{r'=1}^{m^R_{j'}} a_{\lambda_j,\lambda_{j'},\lambda_{j'},r} = \frac{(m^L_j + m^R_{j'})!}{m^L_j!m^R_{j'}!}$$

$$G_{j,j''} = \prod_{r=1}^{m^L_j} \prod_{r'=1}^{m^R_{j''}} a_{\lambda_j,\lambda_{j''},\lambda_{j''},r} = \frac{(i(k_j - k_{j''}) + \frac{1}{2}(-m^L_j + m^R_{j} + m^L_{j''} + m^R_{j''}) + 1)m^L_j}{\Gamma(i(k_j - k_{j''}) + \frac{1}{2}(-m^L_j + m^R_{j} + m^L_{j''} - m^R_{j''}) + 1)}$$

Note that using the identity $\Gamma(1 - x) = \pi/(\Gamma(x) \sin(\pi x))$ we can rewrite this function differently. One can check that for integers $m^L_{j,R}$ the factors containing the sinus functions all together simplify to unity. Hence the function $G$ can equivalently be written as

$$G = \Gamma(-ik_j + ik_{j'} + \frac{m^L_j - m^R_{j'} + m^R_{j} - m^L_{j'}}{2}) \Gamma(ik_j - ik_{j'} + \frac{-m^L_j - m^R_{j'} + m^R_{j} - m^L_{j'}}{2}) \prod_{1 \leq j \neq j' \leq n_s} \frac{(m^L_j + m^R_{j})!}{m^L_j!m^R_{j}!}$$

which shows that the question of its analytic continuation to $m^L_{j,R}$ complex is non-trivial (non-unique). Indeed if one were to attempt calculations including this factor using Mellin Barnes formula, one could argue from the form $A18$ that the standard scaling at large time $k_j \to k_j/\lambda$, $m^L_{j,R} \to z^L_{j,R}/\lambda$ leads to $G \to 1$, however on the second form such a property does not seem to hold.

2. **Main identity**

Our main result is the following general identity, for the evaluation of $(A2)$ for a given string state $|k,m\rangle$, valid for any fixed $(n_L, n_R, n_s, m_j, k_j)$ and arbitrary functions $F^L_{n_L}$, $F^R_{n_R}$

$$\sum_{P \in S_n} \prod_{1 \leq j \leq n_s} a_{\lambda_{P_j}, \lambda_{P_n}} F^L_{n_L}|\lambda_{P_1}, \ldots, \lambda_{P_{n_L}}\rangle F^R_{n_R}|\lambda_{P_{n_L+1}}, \ldots, \lambda_{P_n}\rangle$$

$$= \prod_{j=1}^{n_s} \sum_{m^L_{j}+m^R_{j}=m_j} \delta_{\sum_{j=1}^{n_s} m^L_{j}=n_L} \delta_{\sum_{j=1}^{n_s} m^R_{j}=n_R} \tilde{H}^L[k,m,m^L]\tilde{H}^R[k,m,m^R]G[k,m,m^L,m^R]$$

where the functions $\tilde{H}^L$, $\tilde{H}^R$ and $G$ are given above. We have not attempted to prove this identity, but we have checked it using mathematica for a large set of values of the parameters $n_R, n_L, n_s$. Setting the functions $F^L_{n_L}$, $F^R_{n_R}$
to unity we have also checked the (quite non-trivial) "normalisation identity":

$$
\prod_{j=1}^{n_L} \sum_{m_j=0}^{n_j} \delta_{\sum_{j=1}^{n_L} m_j} \delta_{\sum_{j=1}^{n_R} m_j} G[k, m_L, m_R] = n!/(n_L! n_R!)
$$

(A21)

which can be seen as an identity involving Gamma functions. We have also checked that the above expression for $G$ (once multiplied by its symmetric) is consistent with the formula given by Dotsenko.

**Appendix B: Calculation of the auxiliary function $\phi_{+\infty}(k, y_L, y_R)$**

To perform the integrals in (101) we expand the product, leading to four terms. We use the elementary integrals

$$
\left(-\frac{1}{2\pi i}\right) \int_{C'} \frac{dz}{z} e^{zy} = \theta(-y)
$$

(B1)

$$
\left(-\frac{1}{2\pi i}\right) \int_{C'} \frac{dz}{z} A + z \frac{e^{zy}}{A} = \int_0^{+\infty} dve^{-Av} \theta(-y + v) = \frac{1}{A} (\theta(-y) + \theta(y)e^{-Ay}), \quad \text{Re}(A) > 0
$$

(B2)

This allows to show, assuming everywhere Re$(A) > 0$:

$$
\left(-\frac{1}{2\pi i}\right)^2 \int_{C'} \frac{dz_L}{z_L} \int_{C'} \frac{dz_R}{z_R} e^{z_L y_L + z_R y_R} = \int_{v_2 > 0} e^{-Av_2} \theta(-a_R v_2 - y_R) \theta(v_2 - y_L)
$$

(B3)

Taking a derivative w.r.t. $y_R$ one obtains:

$$
\left(-\frac{1}{2\pi i}\right)^2 \int_{C'} \frac{dz_L}{z_L} \int_{C'} \frac{dz_R}{z_R} \frac{z_R e^{z_L y_L + z_R y_R}}{A + z_L - a_R z_R} = -\theta(-y_R) \theta(-y_R - y_L) e^{A y_R} \delta_{a_R 1} - \delta(y_R) \frac{1}{A} (\theta(-y_R) + \theta(y_R) e^{-A y_R}) \delta_{a, 0}
$$

which allows to evaluate the two cross-terms in (101). We also need:

$$
\left(-\frac{1}{2\pi i}\right)^2 \int_{C'} \frac{dz_L}{z_L} \int_{C'} \frac{dz_R}{z_R} \frac{e^{z_L y_L + z_R y_R}}{(A_L + z_R - a_L z_L)(A_R + z_L - a_R z_R)}
$$

(B4)

$$
= \int_{v_1, v_2 > 0} e^{-A_L v_1 - A_R v_2} \theta(v_1 - y_R - a_R v_2) \theta(v_2 - y_L - a_L v_1)
$$

and taking two derivatives we obtain:

$$
\left(-\frac{1}{2\pi i}\right)^2 \int_{C'} \frac{dz_L}{z_L} \int_{C'} \frac{dz_R}{z_R} \frac{z_L z_R e^{z_L y_L + z_R y_R}}{(A_L + z_R - a_L z_L)(A_R + z_L - a_R z_R)}
$$

(B5)

$$
= \theta(y_L + a_L y_R) \theta(y_R + a_R y_L) e^{-A_L(y_R + a_L y_R) - A_R(y_L + a_R y_L)} (1 - a_R a_L)
$$

$$
+ a_R a_L \delta(y_L + y_R) \frac{1}{A_L + A_R} (\theta(y_L) e^{-A_R y_L} + \theta(-y_L) e^{A_L y_L})
$$

Putting all together, denoting $A_L = 2\tilde{w}_L + 2ik$ and $A_R = 2\tilde{w}_R - 2ik$ and slightly simplifying using that $(a_L, a_R) \in \{0, 1\}^2$ we obtain from (101)

$$
\frac{1}{2} \phi_{+\infty}(k, y_L, y_R) = 1 + \theta(-y_R) \theta(-y_L)
$$

(B6)

$$
-2a_L \theta(-y_R) \theta(-y_R - y_L) e^{-A_L y_L} - (1 - a_L) \delta(y_L) \frac{1}{A_L} (\theta(-y_R) + \theta(y_R) e^{-A_L y_R})
$$

$$
-2a_R \theta(-y_R) \theta(-y_R - y_L) e^{-A_R y_R} - (1 - a_R) \delta(y_R) \frac{1}{A_R} (\theta(-y_R) + \theta(y_R) e^{-A_R y_R})
$$

$$
+ (1 + a_L + a_R - 3a_R a_L) \theta(y_L + a_L y_R) \theta(y_R + a_R y_L) e^{-A_L(y_R + a_R y_L) - A_R(y_L + a_L y_R)}
$$

$$
+ 4a_R a_L \delta(y_L + y_R) \frac{1}{A_L + A_R} (\theta(y_L) e^{-A_R y_L} + \theta(-y_L) e^{A_L y_L})
$$

If we set $y_L = y_R = y$ we obtain the formula (104) in the text.
Appendix C: Airy function identities and second form of the Kernel

We use the Airy function identities (see e.g. Section 9 in Ref. 45 and references therein)

\begin{align}
2 \int \frac{dk}{2 \pi} A_i(4k^2 + a + b + i k \bar{\omega}) e^{2ik(b-a)} &= 2^{-1/3} A_i(2^{1/3}(a + \frac{\bar{\omega}^2}{32})) A_i(2^{1/3}(b + \frac{\bar{\omega}^2}{32})) e^{\frac{\bar{\omega}}{2}} (b-a) \quad \text{(C1)} \\
2 \int \frac{dk}{2 \pi} A_i(4k^2 + a + b + i k \bar{\omega}) \frac{e^{2ik(b-a)}}{\bar{\omega} + ik} &= \int_0^{+\infty} dr 2^{-1/3} A_i(2^{1/3}(a + \frac{r}{4} + \frac{\bar{\omega}^2}{32})) A_i(2^{1/3}(b - \frac{r}{4} + \frac{\bar{\omega}^2}{32})) e^{\frac{\bar{\omega}}{2}} (b-a) - r (\frac{\bar{\omega}}{2} + \bar{\omega}) \quad \text{(C2)} \\
2 \int \frac{dk}{2 \pi} A_i(4k^2 + a + b + i k \bar{\omega}) e^{2ik(b-a)} \frac{1}{\bar{\omega} - ik} &= \int_0^{+\infty} dr 2^{-1/3} A_i(2^{1/3}(a + \frac{r}{4} + \frac{\bar{\omega}^2}{32})) A_i(2^{1/3}(b - \frac{r}{4} + \frac{\bar{\omega}^2}{32})) e^{\frac{\bar{\omega}}{2}} (b-a) - r (\frac{\bar{\omega}}{2} + \bar{\omega}) \quad \text{(C3)}
\end{align}

where we assumed $\bar{\omega} > 0$.

Consider now the expression [103] for the kernel $M_s(v_i, v_j)$ and enumerate the terms upon expanding the products in [104]. In the same order as they appear there, we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows. In the first four terms we use the above identities as follows.

In the final Fredholm determinant the common factor $e^{\frac{\bar{\omega}}{2}(v_j - v_i)}$ can be discarded, since $\text{Det}[I + P_0 M_s P_0] = \text{Det}[I + P_0 \hat{M}_s P_0]$. We now rescale $y \rightarrow 2^{2/3} y$ in the first and second term, $y \rightarrow 2^{-1/3} y$ in the third and fourth term, and $r \rightarrow 2^{5/3} y$ in the last two terms, and we use the similarity transformation $M_s(v_1, v_2) = -2^{1/3} K_\sigma (2^{1/3} v_1, 2^{1/3} v_2)$ and we obtain the result [103] [107] displayed in the text.

Appendix D: Generalized kernels and step initial conditions

1. General case: first form of kernel

Here we obtain the kernels associated to the generalized generating function [37]. We start with the first form

\begin{align}
g_{+\infty}(s_L, s_R) &= \text{Det}[I + P_0 M_{s_L} s_R P_0] \quad \text{(D1)}
\end{align}
To express $\phi_+(k, y - s_L, y - s_R)$ let us insert $y_L = y - s_L$ and $y_R = y - s_R$ in (D2). We obtain

$$M_{s_L, s_R}(v_i, v_j) = \int \frac{dk}{2\pi} dy \frac{A_i(y + 4k^2 + ikx + v_i + v_j)e^{-2ik(v_i-v_j)}}{2\pi} \phi_+(k, y - s_L, y - s_R)$$  \hspace{2cm} (D2)

$$\frac{1}{2} \phi_+(k, y - s_L, y - s_R) = -1 + \theta(s_L - y)\theta(s_R - y)$$  \hspace{2cm} (D3)

$$-2a_L\theta(s_L - y)\theta(s_L + s_R - 2y)e^{A_L(y - s_L)} - (1 - a_L)\delta(y - s_L) \frac{1}{A_L}(\theta(s_R - s_L) + \theta(s_L - s_R)e^{-A_L(s_L - s_R)})$$

$$-2a_R\theta(s_R - y)\theta(s_L + s_R - 2y)e^{A_R(y - s_R)} - (1 - a_R)\delta(y - s_R) \frac{1}{A_R}(\theta(s_R - s_L) + \theta(s_L - s_R)e^{-A_R(s_R - s_L)})$$

$$(1 + a_L + a_R - 3a_R\delta_L)\delta((1 + a_l)y - s_L - a_Ls_L)\theta((1 + a_R)y - s_R - a_Rs_R)$$

$$\times e^{-A_L((1 + a_R)y - s_R - a_Rs_R)} - A_L((1 + a_L)y - s_L - a_Ls_L)$$

$$\times (1 + a_L + a_R - 3a_R\delta_L)\delta((1 + a_l)y - s_L - a_Ls_L)\theta((1 + a_R)y - s_R - a_Rs_R)$$

$$+ 2a_R\frac{a_L\delta(y - \frac{s_L + s_R}{2})}{A_L + A_R}(\theta(s_R - s_L)e^{\frac{A_L}{2}(s_R - s_L)} + \theta(s_L - s_R)e^{\frac{A_R}{2}(s_R - s_L)})$$

with $A_L = 2\bar{w}_L + 2ik$ and $A_R = 2\bar{w}_R - 2ik$.

2. General case: second form of kernel

We now rewrite the kernel using the Airy function identities given in the previous section. This gives $M_{s_L, s_R}(v_i, v_j) = e^{\frac{1}{2}(v_j - v_i)}M_s(v_i, v_j)$ with:

$$\tilde{M}_{s_L, s_R}(v_i, v_j) = -\int dy \left(1 - \theta(s_L - y)\theta(s_R - y)\right)2^{1/3}Ai(2^{1/3}(v_i + \frac{y}{2} + \frac{\tilde{w}_R}{32})Ai(2^{1/3}(v_j + \frac{y}{2} + \frac{\tilde{w}_R}{32}))$$

$$- (1 + a_L + a_R - 3a_R\delta_L)\theta((1 + a_L)y - s_L - a_Ls_L)\theta((1 + a_R)y - s_R - a_Rs_R)$$

$$\times e^{(2\bar{w}_L + \frac{\tilde{w}_R}{32})(y - s_L)}(1 + a_L)y - s_L - a_Ls_L)$$

$$\times 2^{1/3}Ai(2^{1/3}(v_i + \frac{1}{2}(y - 1 + a_L - a_R) + s_L)(1 - a_R) - s_R(1 - a_R) + \frac{\tilde{w}_R}{32}))$$

$$\times Ai(2^{1/3}(v_j + \frac{1}{2}(y - 1 + a_L - a_R) - s_L)(1 - a_R) + s_R(1 - a_R) + \frac{\tilde{w}_R}{32}))$$

$$+ 2a_L\theta(s_L - y)\theta(s_L + s_R - 2y)e^{(2\bar{w}_L + \frac{\tilde{w}_R}{32})(y - s_L)}2^{1/3}Ai(2^{1/3}(v_i + \frac{s_L}{2} + \frac{\tilde{w}_R}{32}))Ai(2^{1/3}(v_j + \frac{s_R}{2} + \frac{\tilde{w}_R}{32}))$$

$$+ 2a_R\theta(s_R - y)\theta(s_L + s_R - 2y)e^{(2\bar{w}_R - \frac{\tilde{w}_L}{32})(y - s_R)}2^{1/3}Ai(2^{1/3}(v_i + \frac{s_L}{2} + \frac{\tilde{w}_L}{32}))Ai(2^{1/3}(v_j + \frac{s_R}{2} + \frac{\tilde{w}_L}{32}))$$

$$+ \frac{1 - a_L}{2}\delta(y - s_L)\theta(s_R - s_L)\int_0^{\infty} dr 2^{1/3}Ai(2^{1/3}(v_i + \frac{y}{2} + \frac{\tilde{w}_L}{32}))Ai(2^{1/3}(v_j + \frac{y}{2} - \frac{\tilde{w}_L}{32}))e^{-r(\frac{\tilde{w}_L}{2} + \tilde{w}_R)}$$

$$+ \frac{1 - a_L}{2}\delta(y - s_L)\theta(s_R - s_L)e^{-(2\tilde{w}_L + \frac{\tilde{w}_R}{32})(s_R - s_L)}$$

$$\times \int_0^{\infty} dr 2^{1/3}Ai(2^{1/3}(v_i + \frac{y + s_L - s_R}{2} + \frac{\tilde{w}_L}{32}))Ai(2^{1/3}(v_j + \frac{y + s_R - s_L}{2} - \frac{\tilde{w}_L}{32}))e^{-r(\frac{\tilde{w}_L}{2} + \tilde{w}_R)}$$

$$+ \frac{1 - a_R}{2}\delta(y - s_R)\theta(s_L - s_R)\int_0^{\infty} dr 2^{1/3}Ai(2^{1/3}(v_i + \frac{y}{2} - \frac{\tilde{w}_R}{32}))Ai(2^{1/3}(v_j + \frac{y}{2} + \frac{\tilde{w}_R}{32}))e^{-r(\frac{\tilde{w}_R}{2} - \frac{\tilde{w}_L}{32})}$$

$$+ \frac{1 - a_R}{2}\delta(y - s_R)\theta(s_L - s_R)e^{-(2\tilde{w}_R - \frac{\tilde{w}_L}{32})(s_R - s_L)}$$

$$\times \int_0^{\infty} dr 2^{1/3}Ai(2^{1/3}(v_i + \frac{y + s_L - s_R}{2} - \frac{\tilde{w}_R}{32}))Ai(2^{1/3}(v_j + \frac{y + s_R - s_L}{2} + \frac{\tilde{w}_R}{32}))e^{-r(-\frac{\tilde{w}_R}{2} + \tilde{w}_L)}$$

$$- a_R\frac{AL}{\tilde{w}_L + \tilde{w}_R}\delta(y - \frac{s_L + s_R}{2})\theta(s_L - s_R)e^{\frac{1}{2}(2\tilde{w}_L + \frac{\tilde{w}_R}{32})(s_R - s_L)} + \theta(s_L - s_R)e^{\frac{1}{2}(2\tilde{w}_R + \frac{\tilde{w}_L}{32})(s_R - s_L)}$$

$$\times 2^{1/3}Ai(2^{1/3}(v_i + \frac{y}{2} + \frac{s_L - s_R}{4} + \frac{\tilde{w}_L}{32}))Ai(2^{1/3}(v_j + \frac{y}{2} + \frac{s_R - s_L}{4} + \frac{\tilde{w}_R}{32}))$$

a sum of ten terms. The identities are used with $a = v_i + \frac{y + R}{2}$ and $b = v_j + \frac{y + R}{2}$ where $R$ is as follows. In term 1 of (D4) (which comes from term 1, 2 of (D3) we use $R = 0$ and identity 1, then we list similarly: term 2 (term
We now want rewrite the generating function using the second kernel
\[ g_{+\infty}(s_L, s_R) = \text{Det}[I - P_0K_{\sigma L, \sigma R}P_0] \] (D6)
and we define
\[ \sigma_L - \tilde{\Delta}^2 = 2^{-2/3}s_L \quad , \quad \sigma_R - \tilde{\Delta}^2 = 2^{-2/3}s_R \] (D7)
where we use that \(2^{1/3}\tilde{\Delta}^2 = \tilde{x}^2\). We obtain
\[
K_{\sigma L, \sigma R}(v_i, v_j) = \int dy \left(1 - \theta(\sigma_L - y)\theta(\sigma_R - y)\right)\text{Ai}(v_i + y)\text{Ai}(v_j + y) \tag{D8}
\]
\[
-(1 + a_L + a_R - 3a_RA_L) \int dy \theta((1 + a_L)y - \sigma_L - a_L\sigma_R)\theta((1 + a_R)y - \sigma_R - a_R\sigma_L)
\]
\[
x e^{2(\bar{w}_L + \bar{w}_R)(\sigma_R + a_R\sigma_L)} + 2(\bar{w}_R^2)(\sigma_L + a_L\sigma_R) e^{-2(\bar{w}_L(1 + a_R) + \bar{w}_R(1 + a_L))y + 2\bar{y}(a_L - a_R)}
\]
\[
\times \text{Ai}(v_i + y(1 - a_L + a_R) + \sigma_L(1 - a_R) - \sigma_R(1 - a_L)) \text{Ai}(v_j + y(1 + a_L - a_R) - \sigma_L(1 - a_R) + \sigma_R(1 - a_L))
\]
\[
+ 2a_L \int dy \theta(\sigma_L - y)\theta(\sigma_L + 2y)e^{2(\bar{w}_L^2)(\sigma_L^2)} \text{Ai}(v_i + \sigma_L)\text{Ai}(v_j + 2y - \sigma_L)
\]
\[
+ 2a_R \int dy \theta(\sigma_R - y)\theta(\sigma_R + 2y)e^{2(\bar{w}_R^2)(\sigma_R^2)} \text{Ai}(v_i + 2y - \sigma_R)\text{Ai}(v_j + \sigma_R)
\]
\[
+(1 - a_L)\theta(\sigma_R - \sigma_L) \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + \sigma_L + y)\text{Ai}(v_j + \sigma_L - y)e^{-2y(\bar{w}_L + \bar{w}_R)}
\]
\[
+(1 - a_L)\theta(\sigma_L - \sigma_R) e^{-2(\bar{w}_L^2)(\sigma_L^2)} \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + 2\sigma_L - \sigma_R + y)\text{Ai}(v_j + \sigma_R - y)e^{-2y(\bar{w}_L + \bar{w}_R)}
\]
\[
+(1 - a_R)\theta(\sigma_L - \sigma_R) \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + \sigma_L + y)\text{Ai}(v_j + 2\sigma_R - \sigma_L + y)e^{-2y(\bar{w}_R - \bar{w}_L)}
\]
\[
+(1 - a_R)\theta(\sigma_L - \sigma_R) e^{-2(\bar{w}_R^2)(\sigma_R^2)} \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + 2\sigma_R - \sigma_L + y)\text{Ai}(v_j + \sigma_L - y)e^{-2y(\bar{w}_R - \bar{w}_L)}
\]
\[
- \frac{a_RA_L}{\bar{w}_L + \bar{w}_R} \theta(\sigma_R - \sigma_L) e^{-2(\bar{w}_R^2)(\sigma_R^2)} + \theta(\sigma_R - \sigma_L) e^{2(\bar{w}_L^2)(\sigma_R^2)} \theta(\sigma_R - \sigma_L) e^{(\bar{w}_L + \bar{w}_R)(\sigma_R - \sigma_L)} \text{Ai}(v_i + \sigma_L)\text{Ai}(v_j + \sigma_R)
\]
To obtain this it is more convenient to first define \(\sigma_{L,R} = 2^{-2/3}s_L\). Then, in all terms we have performed a similarity transformation \(2^{1/3}v_i, 2^{1/3}v_j \rightarrow v_i, v_j\) which multiplies the kernel by \(2^{-1/3}\). In terms 1–4 we have rescaled \(y \rightarrow 2^{2/3}y\), in terms 5–8 we have integrated over the delta functions, then changed variable \(r = 2^{5/3}y\). The last step was to make the substitution in the resulting formula, \(\sigma_{L,R} \rightarrow \sigma_{L,R} - \tilde{x}^2\) and, simultaneously change \(y \rightarrow y - \tilde{x}^2\) but only in terms 1–4.

### 3. Step on top of the wedge initial condition

We now want to apply this formula to the step initial conditions. From the text we have
\[
\lim_{t \to +\infty} \text{Prob} \left( t^{-1/3}h(x = 2^{-4/3}\tilde{x}^2, t) < s \right) = g_{+\infty}^\Delta(s) = g_{+\infty}^\Delta(s_L = s - \tilde{\Delta}, s_R = s + \tilde{\Delta}) \tag{D9}
\]
where \(\tilde{\Delta} = \Delta/\lambda\). This can be rewritten using the above results as
\[
\lim_{t \to +\infty} \text{Prob} \left( t^{-1/3}(h(x = 2t^{2/3}\tilde{x}, t) + \tilde{x}^2/4t) < s \right) = g_{+\infty}^\Delta(s) = \text{Det}[I - P_0K_{\sigma L, \sigma R = \sigma - \tilde{\Delta}, \sigma R = \sigma + \tilde{\Delta}}P_0] \tag{D10}
\]
where we recall \(\tilde{\Delta} = \Delta/t^{1/3}\).
Let us specify to the case $a_L = a_R = 0$, which represents the wedge plus a step. With no loss of generality, let us restrict to the case $\sigma_R > \sigma_L$, i.e. $\Delta > 0$. The kernel then can be written

$$K_{\sigma_L,\sigma_R}(v_i, v_j) = K_{\Delta=(\sigma_R-\sigma_L)/2}(v_i + \sigma_L, v_j + \sigma_L)$$

$$K_{\Delta}(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)[Ai(v_j + y) - e^{4\Delta(y-x)}e^{-2y(\hat{w}_L+\hat{w}_R)}Ai(v_j + y + 4\Delta)]$$

$$+ \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j - y)e^{-2y(\hat{w}_L+\hat{w}_R)} + e^{4\Delta(y-x)}e^{-2y(\hat{w}_L+\hat{w}_R)}\int_{-\infty}^{+\infty} dy Ai(v_i - y)Ai(v_j + 4\Delta + y)e^{-2y(\hat{w}_R-x)}$$

We can now consider the limit $\hat{w}_{L,R} \to 0^+$, which leads to the well defined (trace class) kernel given in [133].

4. Step on top of the Brownian-Brownian initial condition

Let us specify to the case $a_L = a_R = 1$, which represents the two-sided Brownian (plus drifts) initial condition plus a step. With no loss of generality, let us restrict to the case $\sigma_R > \sigma_L$, i.e. $\Delta > 0$. The kernel then can be written

$$K_{\sigma_L,\sigma_R}(v_i, v_j) = K_{\Delta=(\sigma_R-\sigma_L)/2}(v_i + \sigma_L, v_j + \sigma_L)$$

$$K_{\Delta}(v_i, v_j) = \int_0^{+\infty} dy Ai(v_i + y)Ai(v_j + y) + \int_{-\infty}^{0} dy e^{(\hat{w}_R-x)(y-2\Delta)}Ai(v_i + y)Ai(v_j + 2\Delta) - \frac{1}{\hat{w}_L + \hat{w}_R}e^{-2\Delta(y-x)}Ai(v_i)Ai(v_j + 2\Delta)$$

It can be rewritten in a more generally valid form

$$K_{\Delta}(v_i, v_j) = K_{Ai}(v_i, v_j) - \frac{e^{-2\Delta(y-x)}}{\hat{w}_L + \hat{w}_R}Ai(v_i)Ai(v_j + 2\Delta) + Ai(v_i)B_{\hat{w}_L+\hat{w}_R}(v_j) + \frac{e^{-2\Delta(y-x)}}{\hat{w}_L + \hat{w}_R}Ai(v_j + 2\Delta)B_{\hat{w}_L+\hat{w}_R}(v_i)$$

(D13)

using the functions $B_w(v)$ defined in [133]. On this form it is apparent that as $\Delta \to 0$ the kernel converges to the one for the Brownian-Brownian case [144]. In the opposition limit $\Delta \to +\infty$ we see that it converges as it should to the half-Brownian limit [128] (upon exchange of left and right, and kernel transposition).

5. Step on top of the wedge-Brownian initial condition

Let us specify to the case $a_L = 0, a_R = 1$, which represents an initial condition which is flat on the left (with a drift), Brownian on the right (with a drift) and, on top of it, a step. We obtain, from [DS]

$$K_{\sigma_L,\sigma_R}(v_i, v_j) = \int dy (1 - \theta(\sigma_L - y)\theta(\sigma_R - y))Ai(v_i + y)Ai(v_j + y)$$

$$-2 \int dy (1 - \theta(\sigma_L - y)\theta(\sigma_R - y))e^{2(\hat{w}_L+\hat{w}_R)(\sigma_R + \sigma_L)}e^{-2(y-x)(\hat{w}_L+\hat{w}_R)}Ai(v_i + y + 2\sigma_R - \sigma_L)$$

$$\times Ai(v_i + 2\sigma_R - \sigma_L)$$

$$+ 2 \int dy \theta(\sigma_R - y)\theta(\sigma_L + \sigma_R - 2y)e^{2(\hat{w}_R-x)(y-\sigma_R)}Ai(v_i + 2\sigma_R - \sigma_L)Ai(v_j + \sigma_R + \sigma_L)$$

$$+ \theta(\sigma_R - \sigma_L)\int_{-\infty}^{+\infty} dy Ai(v_i + \sigma_L + y)Ai(v_j + \sigma_R + \sigma_L - y)e^{-2y(\hat{w}_L+\hat{w}_R)}$$

$$+ \theta(\sigma_L - \sigma_R)e^{-2(\hat{w}_L+\hat{w}_R)(\sigma_R - \sigma_L)}\int_{-\infty}^{+\infty} dy Ai(v_j + 2\sigma_R - \sigma_R - y)Ai(v_j + \sigma_R - y)e^{-2y(\hat{w}_L+\hat{w}_R)}$$

Now we must distinguish the two cases $\Delta > 0$ and $\Delta < 0$. 

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Let us start with $\Delta > 0$ (downward step), i.e. the case $\sigma_R > \sigma_L$. The kernel then can be written

$$K_{\sigma_L,\sigma_R}(v_i, v_j) = K_{\Delta=(\sigma_R-\sigma_L)/2}(v_i + \sigma_R, v_j + \sigma_L)$$  \hspace{1cm} (D15)$$

$$K_{\Delta}(v_i, v_j) = \int_0^{+\infty} dy \, \text{Ai}(v_i + y)\text{Ai}(v_j + y) - \int_0^{-\infty} dy \, e^{2(\tilde{w}_L + \tilde{w}_R)\Delta} e^{-2(y - \tilde{w}_L + \tilde{w}_R)} \text{Ai}(v_i + y)\text{Ai}(v_j + 2\Delta)$$

$$+ \int_{-\infty}^{0} dy e^{(\tilde{w}_L - \tilde{w}_R)(y - 2\Delta)} \text{Ai}(v_i + y)\text{Ai}(v_i + 2\Delta) + \int_{0}^{+\infty} dy \text{Ai}(v_i + y)\text{Ai}(v_j - 2\Delta) e^{-2y(\tilde{w}_L + \tilde{w}_R)}$$  \hspace{1cm} (D16)$$

For $\Delta = +\infty$ it goes as expected to the half flat kernel given in \(119\). For $\Delta = 0^+$ is goes to the wedge-Brownian kernel given in \(129\).

Let us consider now $\Delta < 0$ (upward step), i.e. the case $\sigma_R < \sigma_L$. The kernel then can be written

$$K_{\sigma_L,\sigma_R}(v_i, v_j) = K_{\Delta=(\sigma_R-\sigma_L)/2}(v_i + \sigma_R, v_j + \sigma_L)$$

$$K_{\Delta}(v_i, v_j) = \int_0^{+\infty} dy \, \text{Ai}(v_i + y)\text{Ai}(v_j + y) - \int_0^{-\infty} dy \, e^{4(\tilde{w}_L + \tilde{w}_R)\Delta} e^{-4(y - \tilde{w}_L + \tilde{w}_R)} \text{Ai}(v_i + y - 4\Delta)\text{Ai}(v_j)$$

$$+ \int_{-\infty}^{0} dy e^{(\tilde{w}_L - \tilde{w}_R)y} \text{Ai}(v_i + y)\text{Ai}(v_j) + e^{4\Delta(\tilde{w}_L + \tilde{w}_R)} \int_{0}^{+\infty} dy \text{Ai}(v_i + \sigma_R - 4\Delta + y)\text{Ai}(v_j + \sigma_R - y) e^{-2y(\tilde{w}_L + \tilde{w}_R)}$$

In the limit $\Delta \to -\infty$ we see that it converges as it should to the half-Brownian limit \(128\), and for $\Delta = 0^-$ is goes to the wedge-Brownian kernel given in \(129\), being of course continuous at $\Delta = 0$.
A. Borodin, I. Corwin, P. L. Ferrari, B. Veto, arXiv:1407.6977.
30 T. Gueudre, P. Le Doussal, EPL 100 26006 (2012), arXiv:1208.5669.
31 C. A. Tracy and H. Widom, Commun. Math. Phys. 177 (1996), 727-754.
A. Borodin and I. Corwin Prob. Theor. Rel. Fields 158 (2014), no. 1-2, 225–400, arXiv:1111.4408
32 T. Gueudre, P. Le Doussal, EPL 100 26006 (2012), arXiv:1208.5669.
33 C. A. Tracy and H. Widom, Commun. Math. Phys. 177 (1996), 727-754.
A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1308.3415
34 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1407.8534
35 A. Borodin, I. Corwin, V. Gorin, arXiv:1407.6729
36 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1308.3475.
37 A. Borodin, I. Corwin, V. Gorin, arXiv:1407.6729
38 A. Borodin, I. Corwin, P. L. Ferrari, arXiv:1204.1024, Comm. Pure Appl. Math. 67 (2014), 1129.
39 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1407.8534.
40 A. Borodin, I. Corwin, V. Gorin, arXiv:1407.6729
41 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1407.8534.
42 T. Imamura, T. Sasamoto, arXiv:1105.4659, J. Phys. A : Math. Theor. 44, 385001 (2011).
43 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1308.3475.
44 Ivan Corwin, Jeremy Quastel, arXiv:1006.1338 Annals of Probability 2013, Vol. 41, No. 3A, 1243-1314.
45 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1407.8534.
46 A. Borodin, I. Corwin, V. Gorin, arXiv:1407.6729
47 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1308.3475.
48 A. Borodin, I. Corwin, L. Petrov, T. Sasamoto, arXiv:1207.5035.
49 A. Borodin, I. Corwin, P. L. Ferrari, arXiv:1204.1024, Comm. Pure Appl. Math. 67 (2014), 1129.
50 P. Le Doussal, arXiv:1401.1081, J. Stat. Mech. (2014) P04018.
51 A. Borodin, P. L. Ferrari, T. Sasamoto, arXiv:1207.5035.
52 A. Borodin, P. L. Ferrari, and T. Sasamoto, Comm. Pure Appl. Math. 61.11 (2008), 1603.
53 A. Borodin, P. L. Ferrari, T. Sasamoto arXiv:0904.4655, J. Stat. Phys. 137 (2009), 936-977.
54 S. Prolhac and H. Spohn, arXiv:1011.401, J. Stat. Mech. (2011) P10031.
55 S. Prolhac and H. Spohn, arXiv:1101.3622 J. Stat. Mech. (2011) P03020.
56 V. Dotsenko, arXiv:1304.6571 J. Phys. A: 46 (2013), 355001.
57 T. Imamura, T. Sasamoto, H. Spohn, arXiv:1305.1217
58 V. Dotsenko, arXiv:1204.6166, J. Stat. Mech. (2013) P02012.
59 I. Corwin, J. Quastel, D. Remenik, arXiv:1103.3422 J. Stat. Phys. 160 815 (2015).
60 M. Kardar, Nucl. Phys. B 290, 582 (1987).
61 E. Brunet and B. Derrida, Phys. Rev. E 61, 6789 (2000); Physica A 279, 395 (2000).
62 J. B. McGuire, J. Math. Phys. 5, 622 (1964).
63 P. Calabrese and J.-S. Caux, Phys. Rev. Lett. 98, 150403 (2007); J. Stat. Mech. (2007) P08032.
64 I. Corwin, A. Hammond, arXiv:1312.2600
65 J. Quastel, D. Remenik, arXiv:1111.2565 and arXiv:1301.0750
66 Jeremy Quastel, Daniel Remenik, arXiv:1606.09228
67 K. Matetski, J. Quastel, D. Remenik, arXiv:1701.00018
68 G. Schehr, arXiv:1203.1658 J. Stat. Phys. 149(3), 385-410 (2012)
69 Note that \( v_\infty = -1/12 \) at large time is a result of the Ito convention in [4], which implies that \( Z(x,t) \) obeys the free diffusion equation: this defines the Cole-Hopf solution to the continuum KPZ equation for white noise. In presence of a regularized noise (i.e. with spatial correlations) \( v_\infty \) becomes non-universal. However, as detailed in [44] if one considers \( \ln(Z(t)/Z(\tau)) \) then \( v_\infty = -1/12 \) in our units
70 In several works, e.g. [17,49,59], the dimensionless equation is chosen by setting \( \nu = 1/2, \alpha_0 = 1 \) and \( D = 1 \). This is equivalent to only a change of the time, i.e. it corresponds to the choice \( t^* = 2 \) (i.e. \( t' = 2t \) where \( t \) denotes the time here and \( t' \) the time there)
71 except for the case \( a_L = a_R = 1 \), i.e. Brownian on both sides where there is a further miracle identity [28].