Lagrangian fibrations and theta functions

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Abstract

In this thesis we study asymptotic behavior of projective embeddings of abelian varieties and their amoebas. The projective embeddings are given by theta functions. It is known that a Lagrangian fibration of the abelian variety determines a basis of theta functions. After reviewing the relation from the viewpoint of geometric quantization and mirror symmetry, we prove that the Lagrangian fibration of the abelian variety can be approximated by the moment maps of natural torus action in a suitable sense. In the final section we discuss a part of the result in more general setting.

Contents

1 Introduction 3

2 Basic facts on symplectic geometry 6
  2.1 Symplectic manifolds and automorphisms 6
  2.2 Action-angle coordinate 8
  2.3 Global structure of Lagrangian torus fibrations 10

3 Geometric Quantization 11
  3.1 Brief review of quantum mechanics 11
  3.2 Prequantization 11
  3.3 Polarizations 12
  3.4 Comparison of the spaces of wave functions for Kähler and real polarizations 15
  3.5 Geometric quantization for abelian varieties 17

4 Special Lagrangian fibrations and mirror symmetry 21
  4.1 Special Lagrangian submanifolds 21
  4.2 A local model of mirror symmetry 23
  4.3 Mirror symmetry for abelian varieties 26

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| Section | Title                                                      | Page |
|---------|------------------------------------------------------------|------|
| 5       | Projective embeddings and Kähler metrics                  | 32   |
| 5.1     | Bergman kernels and projective embeddings                 | 32   |
| 5.2     | Stability and Kähler metrics                              | 33   |
| 6       | Abelian varieties and theta functions                     | 35   |
| 6.1     | Statement of the main theorem                             | 35   |
| 6.2     | Proof of Theorem 6.1                                      | 37   |
| 7       | Lagrangian fibrations and holomorphic sections            | 47   |
| 7.1     | Asymptotic behavior of the Bergman kernel                 | 47   |
| 7.2     | Asymptotic behavior of holomorphic sections               | 48   |
| 7.3     | Proofs                                                    | 50   |
1 Introduction

The purpose of this thesis is to discuss some relations between theta functions and Lagrangian fibrations of abelian varieties.

Lagrangian fibration is a map \((M, \omega) \to B\) from a symplectic manifold such that each (smooth) fiber is a Lagrangian submanifold. Lagrangian fibrations appear in several areas of mathematics and mathematical physics such as completely integrable systems (classical mechanics), geometric quantization and mirror symmetry. In particular, the picture of mirror symmetry via special Lagrangian fibrations proposed by Strominger-Yau-Zaslow [39] has attracted much attention and a lot of works relating to this program have been carried on (e.g. [12], [13], [18], [35], [36], [37]).

However, little is known about (special) Lagrangian submanifolds and fibrations at the moment. In general, it is more difficult to deal with submanifolds compared to functions or connections. In the case of (special) Lagrangian fibrations, the main difficulties seems to be the following two points. The first one is the fact that general Lagrangian fibrations have singular fibers. Singularities appearing in Lagrangian fibrations are not completely understood yet. The other one concerns the case of special Lagrangian fibrations. To define the special Lagrangian submanifolds, we need a Ricci-flat metric. Ricci-flat metrics are difficult objects to analyze.

The cases of Abelian varieties and toric varieties are typical examples of Lagrangian fibrations which are well understood. Mirror symmetry via special Lagrangian fibrations works well for abelian varieties and theta functions play an important role in the theory ([34], [11]).

Theta functions, or more generally holomorphic sections of ample line bundles \(L\) on a smooth projective variety \(X\), are the other main characters in this paper. Of course, the space \(H^0(X, L^k)\) of holomorphic sections does not have natural basis in general. However, some projective varieties have natural basis. For example, in the case of toric varieties, monomials are natural basis of holomorphic sections. Also in the case of abelian varieties, there are natural basis of theta functions (one can find such basis in the Mumford’s book [29]). We can think that such basis are determined by Lagrangian fibrations by using the notion of geometric quantization (the case of abelian varieties is discussed in [9], [12], [44] and other cases are, for example, in [1], [4], [15], [10], [20]). The case of abelian varieties can be also interpreted as a special case of mirror symmetry.

Note that, for large \(k\), each basis of \(H^0(X, L^k)\) defines an embedding into the projective space \(\mathbb{CP}^{N_k} = \mathbb{P}H^0(X, L^k)^*\):

\[
t_k : X \mapsto \mathbb{CP}^{N_k}, \quad z \mapsto (s_0(z) : \cdots : s_{N_k}(z))
\]

We see the relation between Lagrangian fibration of \(X\) and projective embeddings defined by the basis corresponding to the Lagrangian fibration. Namely, we compare the Lagrangian fibration with the standard one of projective spaces (i.e. the moment maps of natural torus actions)

\[
\mu_k : \mathbb{CP}^{N_k} \longrightarrow \mathbb{R}^{N_k} = \text{Lie} T^{N_k}.
\]
We denote the restriction of the moment map to \( X \) by
\[
\pi_k = \mu_k \circ \iota_k : X \rightarrow B_k,
\]
where \( B_k \) is the image of \( X \) under the moment map.

The simplest example is the case of toric varieties. If we take monomials as a basis of \( H^0(X, L^k) \), the corresponding embeddings are torus equivariant. This means that \( \pi_k : X \rightarrow B_k \) coincides with the moment map of \( X \).

The main theorem of this paper deal with the case of abelian varieties. In this case, the situation is not so trivial. The rough statement is the following.

**Theorem 1.1.** Let \( X \) be an abelian variety. We consider projective embeddings \( \omega \) of \( X \) by using theta functions determined by a Lagrangian fibration of \( X \). Then the sequence \( \{ \pi_k : X \rightarrow B_k \} \) converges to the original Lagrangian fibration as \( k \rightarrow \infty \) in the “Gromov-Hausdorff topology”.

The precise statement is given in Section 6.

This theorem is closely related to the theory of approximation of Kähler metrics by Fubini-Study metrics. Tian \cite{tian1990} and Zelditch \cite{zelditch1998} proved that every Kähler metric in a fixed Kähler class \( c_1(L) \) can be approximated by restrictions of (normalized) Fubini-Study metrics
\[
\omega_k = \frac{1}{k} \iota_k^* \omega_{\text{FS}}
\]
on \( \mathbb{CP}^{N_k} = \mathbb{P}H^0(X, L^k)^* \) for large \( k \) by choosing appropriate embeddings.

There is another theory on this subject and it is related to the notion of stability in the sense of geometric invariant theory. This is a possible approach to prove “Hitchin-Kobayashi correspondence for manifolds” which claims the equivalence of stability of a projective variety and the existence of special Kähler metrics such as Kähler metric of constant scalar curvature. In this approach, we relate the choice of projective embeddings with the stability condition and study the asymptotic behavior of the restrictions of Fubini-Study metrics. In particular, this theory could be effective for the study of Ricci-flat metrics.

The projective embeddings defined by theta functions give examples of both of the above theories. The main theorem can be regarded as a prototype of approximations of Kähler metrics and Lagrangian fibrations at the same time. Our method might be useful for the study of special Lagrangian fibrations.

This paper is organized as follows. In Section 2, we recall basic facts on symplectic geometry used in this paper. In Section 3, we review the theory of geometric quantization, especially, the part related to Lagrangian fibration in detail. Section 4 is devoted to a brief summary of mirror symmetry for abelian varieties via special Lagrangian fibrations. We recall the theorem of Tian and Zelditch stated above in Section 5. In this section, we also recall the relation between projective embeddings, stability of polarized manifolds and canonical metrics. The precise statement and proof of the main theorem is given in Section 6. Finally we discuss a generalization of the above result in Section 7.
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2 Basic facts on symplectic geometry

In this section, we summarize basic facts on symplectic geometry which we need later.

2.1 Symplectic manifolds and automorphisms

Definition 2.1. Let $M$ be a smooth manifold of dimension $2n$. A symplectic structure $\omega$ on $M$ is a non-degenerate closed 2-form on $M$. A symplectic manifold is a pair $(M, \omega)$ of smooth manifold and symplectic structure on it.

We denote the group of diffeomorphisms of $M$ by $\text{Diff}(M)$. Its Lie algebra $\text{Lie Diff}(M)$ is identified formally with the Lie algebra $\mathfrak{X}(M)$ of vector fields on $M$. By using the symplectic structure $\omega$, we can identify $\mathfrak{X}(M)$ with the space of 1-forms on $M$:

$$
\mathfrak{X}(M) \longrightarrow \Omega^1(M), \quad \xi \longmapsto i_\xi \omega.
$$

Let $\text{Symp}(M, \omega) \subset \text{Diff}(M)$ be the subgroup of diffeomorphisms which preserve the symplectic structure $\omega$. Diffeomorphisms in $\text{Symp}(M, \omega)$ are called symplectomorphisms.

Proposition 2.2. Under the above identification, the Lie algebra of $\text{Symp}(M, \omega)$ corresponds to the space $Z^1(M)$ of closed 1-forms.

This proposition follows from $d \omega = 0$ and the following formula

$$
L_\xi \omega = i_\xi d \omega + d(i_\xi \omega),
$$

where $L_\xi$ is the Lie derivative.

Definition 2.3. For a smooth function $f \in C^\infty(M)$, its Hamilton vector field $\xi_f$ is defined by

$$
i_{\xi_f} \omega = df.
$$

A Hamilton diffeomorphism is a diffeomorphism generated by Hamilton vector fields. We denote the group of Hamilton diffeomorphisms by $\text{Ham}(M, \omega)$.

From the definition, the Lie algebra of $\text{Ham}(M, \omega)$ is identified with the space $B^1(M)$ of exact 1-forms. In particular, every Hamilton diffeomorphism preserves $\omega$.

Proposition 2.4. Assume that $\omega$ represents an integral cohomology class:

$$
[\omega] \in H^2(M, \mathbb{Z}) .
$$

Then there exists a Hermitian line bundle $L \rightarrow M$ with a unitary connection $\nabla$ such that $c_1(L, \nabla) = \omega$.

Such $(L, \nabla)$ is called a prequantum bundle of $(M, \omega)$. This is a fundamental ingredient in geometric quantization.
Proof. Let $\mathcal{A}$ (resp. $\mathcal{A}^*$) be the sheaf of smooth functions (resp. non-zero smooth functions) and consider the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{A} \longrightarrow \exp(2\pi\sqrt{-1} \cdot) \mathcal{A}^* \longrightarrow 0.$$ 

Recall that $H^1(M, \mathcal{A}^*)$ parameterizes the isomorphism classes of smooth complex line bundles on $M$. Since $\mathcal{A}$ is a fine sheaf, the long exact sequence becomes

$$\ldots \longrightarrow H^1(M, \mathcal{A}) \longrightarrow H^1(M, \mathcal{A}^*) \longrightarrow c_1 \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0.$$ 

This implies that there exists a line bundle $L \to M$ such that $c_1(L) = [\omega]$.

Now we take an arbitrary Hermitian metric and a unitary connection $\nabla'$ on $L$. Note that its curvature $\omega'$ represents $c_1(L)$. There exists a smooth 1-form $\alpha$ such that

$$\omega = \omega' + d\alpha.$$ 

Then $\nabla = \nabla' - 2\pi\sqrt{-1}\alpha$ gives a unitary connection such that its curvature coincides $\omega$. \hfill $\square$

We consider the automorphism group $\mathcal{G}$ of a prequantum bundle $(L, \nabla) \to (M, \omega)$. $\mathcal{G}$ consists of isomorphisms

$$L \quad \xrightarrow{\hat{F}} \quad L$$

$$\downarrow \quad \quad \downarrow$$

$$M \quad \xrightarrow{F} \quad M$$

such that $\hat{F}$ preserves the Hermitian metric and the connection $\nabla$ of $L$ (hence $F$ preserves $\omega$).

Proposition 2.5. The Lie algebra of $\mathcal{G}$ is formally identified with the space of smooth functions $C^\infty(M)$ on $M$, here the Lie algebra structure of $C^\infty(M)$ is given by the Poisson bracket.

We can see this as follows. Consider the natural projection

$$\mathcal{G} \longrightarrow \text{Symp} (M, \omega), \quad (\hat{F}, F) \mapsto F$$

and denote its image by $\mathcal{G}_0$. The kernel consists of automorphisms which preserve the base space. It is easy to see that this is isomorphic to $S^1$. Then we have the following exact sequence:

$$1 \longrightarrow S^1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_0 \longrightarrow 1.$$ 

Proposition 2.5 is a consequence of the fact that $\mathcal{G}_0 \cong \text{Ham} (M, \omega)$. In fact, the corresponding exact sequence of Lie algebras is given by

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Lie} \mathcal{G} \longrightarrow dC^\infty(M) \longrightarrow 0.$$ 

Let $\Gamma(L) = \Gamma(M, L)$ be the space of smooth sections of $L$. Then, for each $f \in C^\infty(M) \cong \text{Lie} \mathcal{G}$, its action $\hat{f} : \Gamma(L) \to \Gamma(L)$ is given by

$$\hat{f}(s) = \nabla_{\xi_f} s - \sqrt{-1}fs.$$ 

(1)
2.2 Action-angle coordinate

In this and the next subsection, we recall basic properties of Lagrangian fibrations.

**Definition 2.6.** A submanifold $S \subset M$ is called a Lagrangian submanifold if it satisfies $\omega|_S = 0$ and $\dim S = \frac{1}{2}\dim M$.

We call a fibration $\pi : M \to B$ a Lagrangian fibration if general fibers are Lagrangian.

**Remark 2.7.** It is natural to allow Lagrangian fibrations to have degenerate fibers. However we do not care about singular fibers in this section.

**Example 2.8.** Let $B$ be an $n$-dimensional manifold and $T^*B$ its cotangent bundle. Then the natural projection $T^*B \to B$ is a Lagrangian fibration with respect to the standard symplectic structure $\omega_0$ on $T^*B$.

Now we recall the notion of moment maps.

**Definition 2.9.** Let $G$ be a Lie group acting on a symplectic manifold $(M, \omega)$ as symplectomorphisms. A moment map $\mu : M \to g^* = (\text{Lie } G)^*$ is a $G$-equivariant map satisfying

$$d\langle \mu, \xi \rangle = i\xi \omega$$

for $\xi \in g$, where $\langle \cdot, \cdot \rangle$ is the natural pairing and we identified $\xi \in g$ with the vector field on $M$ defined by its action.

Remark that moment maps $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^n$ of $T^n$-action ($n = \frac{1}{2}\dim M$) are Lagrangian fibrations (with degenerate fibers in general).

**Proposition 2.10.** Let $\pi : (M, \omega) \to B$ be a Lagrangian fibration.

1. Every smooth fiber has a natural affine structure. In particular, compact fiber is a torus $T^n$.

2. Assume that $\pi^{-1}(b_0)$ is smooth and compact. Then there exist a neighborhood $U \subset B$ of $b_0$ and local coordinates $(x^1, \ldots, x^n)$ of $U$ and $(y^1, \ldots, y^n)$ of $T^n$ such that

(a) $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a coordinate of $\pi^{-1}(U) \cong U \times T^n$ with $\omega = \sum dx^i \wedge dy^i$, and

(b) there exists a $T^n$-action on $\pi^{-1}(U)$ such that

$$\left(x^1, \ldots, x^n\right) : \pi^{-1}(U) \to \mathbb{R}^n$$

gives its moment map.

Such coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ are called action-angle coordinates.
Outline of the proof. 1. Let $U$ be a neighborhood of $b_0 \in B$. We may assume that $U \subset \mathbb{R}^n$ is a small ball. Denote the coordinate of $\mathbb{R}^n$ by $(\tilde{x}^1, \ldots, \tilde{x}^n)$ and set $f_i = \tilde{x}^i \circ \pi : \pi^{-1}(U) \to \mathbb{R}$.

Claim. $\{f_i, f_j\} = 0$ for each $i, j = 1, \ldots, n$.

Proof. Since $f_i$ is constant along fibers, 

$$0 = \xi f_i = df_i(\xi) = \omega(\xi f_i, \xi)$$

for each $\xi \in T(\pi^{-1}(b))$. In particular, $\xi f_i, \xi f_j \in T(\pi^{-1}(b))$ and

$$0 = \omega(\xi f_i, \xi f_j) = \{f_i, f_j\}.$$

Note that $\xi f_i, \ldots, \xi f_n$ are linearly independent. Furthermore, we have

$$[\xi f_i, \xi f_j] = \xi(f_i, f_j) = 0$$

from the above claim. Then $\xi f_i, \ldots, \xi f_n$ generates an $\mathbb{R}^n$-action on $\pi^{-1}(U)$ and each fiber $\pi^{-1}(b)$ is preserved by this action. In particular, if fibers are compact, we have $\pi^{-1}(b) \cong T^n$.

2. Here we only recall the construction of action-angle coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. Since $U$ is contractible, the inclusion $\iota : \pi^{-1}(b_0) \hookrightarrow \pi^{-1}(U)$ induces an isomorphism

$$\iota^* : H^2(\pi^{-1}(U), \mathbb{R}) \cong H^2(\pi^{-1}(b_0), \mathbb{R}).$$

Because $\pi^{-1}(b_0)$ is Lagrangian, we have

$$\iota^*[\omega] = [\omega|_{\pi^{-1}(b_0)}] = 0.$$ 

This implies that there exists a 1-form $\theta$ on $\pi^{-1}(U)$ such that $d\theta = \omega$. We choose generators $\gamma^b_i$ of $H_1(\pi^{-1}(b), \mathbb{Z})$, $i = 1, \ldots, n$, so that they depend smoothly on $b \in U$ and set

$$x^i(b) = \frac{1}{2\pi} \int_{\gamma^b_i} \theta.$$ 

Note that $x^i(b)$ depends only on the homology classes $[\gamma^b_i] \in H_1(\pi^{-1}(b_0), \mathbb{Z})$.

Then we can show that $(x^1, \ldots, x^n)$ defines a coordinate around $b_0$. Moreover, $\exp(\xi x)(x) = x$ for $x \in \pi^{-1}(U)$. In other words, $\exp(t\xi x)$ has period 1.

Fix a Lagrangian section $\lambda : U \to \pi^{-1}(U)$ and define a map

$$\psi : T^*U \to \pi^{-1}(U)$$

by $\psi(df(b)) = \exp(\xi f) \cdot \lambda(b)$. We define the coordinate of $\exp(\xi f \cdot \psi^{-1}(\lambda) \cdot \lambda$ by $(y^1, \ldots, y^n)$. Note that the zero section of $T^*U$ corresponds to $\lambda$. Then the coordinate $(y^1, \ldots, y^n)$ has period 1. 

**Remark 2.11.** If there exists a quantum line bundle $(L, \nabla)$, we can take $\theta$ to be a connection 1-form of $\nabla$. In this case, $\exp(2\pi \sqrt{-1}x^i)$ is the holonomy of $(L, \nabla)$ on each fiber.
2.3 Global structure of Lagrangian torus fibrations

Next we discuss global structures of Lagrangian fibrations. Here we assume that
the Lagrangian fibration \(\pi : M \to B\) has no degenerate fiber. In the proof of
Theorem 2.10, we constructed a map \(\psi : T^* U \to \pi^{-1}(U)\). Define \(\Lambda_U \subset T^* U\) by
\[
\Lambda_U = \{ \alpha \in T^* U \mid \psi(\alpha) = \lambda \}
\]
Then this is a \(\mathbb{Z}^n\)-bundle spanned by \(dx^1, \ldots, dx^n\). For each \(b \in U\), \(\Lambda_b\) can be
identified with \(H_1(\pi^{-1}(b), \mathbb{Z}) \cong H^{n-1}(\pi^{-1}(b), \mathbb{Z})^*\). In other words, we have
\[
\Lambda_U \cong (\mathbb{R}^{n-1} \pi_\ast \mathbb{Z})^*.
\]
This means that \(\Lambda_U\) defines a global \(\mathbb{Z}^n\)-bundle \(\Lambda \subset T^* B\). Since \(\Lambda\) is spanned
locally by \(dx^1, \ldots, dx^n\), \(\Lambda\) is a Lagrangian submanifold in \((T^* B, \omega_0)\). Hence
\(\omega_0\) induces a symplectic structure on \(T^* B/\Lambda\) and the natural projection \(T^* B/\Lambda \to B\)
is a Lagrangian torus fibration. We remark that \(T^* B/\Lambda\) has a Lagrangian
section (i.e. the zero section). This is called the Jacobian fibration of the
Lagrangian fibration \(\pi : M \to B\) \((12)\).

Now assume that the Lagrangian fibration \(\pi : (M, \omega) \to B\) has a Lagrangian
section \(\lambda : B \to M\). Then the map \(\psi\) extends to a global map
\[
0 \to \Lambda \to T^* B \xrightarrow{\psi} M \to 0.
\]
Hence we have an isomorphism \(T^* B/\Lambda \to M\) of Lagrangian fibrations.

Next we consider the case that \(\pi : M \to B\) does not necessarily have a global
Lagrangian section. Take a open cover \(M = \bigcup U_i\) of \(M\) such that a Lagrangian
section \(\lambda_i : U_i \to \pi^{-1}(U_i)\) and an action-angle coordinate are defined on each
\(U_i\). Then we have
\[
0 \to \Lambda|_{U_i} \to T^* U_i \to \pi^{-1}(U_i) \to 0
\]
on each \(U_i\) (i.e. \(M \to B\) is locally isomorphic to \(T^* B/\Lambda \to B\)). \(M \to B\)
can be reconstructed by gluing \(T^* U_i/\Lambda|_{U_i} \to U_i\). The gluing is determined by
Lagrangian sections \(\mu_{ij} : U_i \cap U_j \to T^* B/\Lambda\) by
\[
\lambda_i(b) = \mu_{ij}(b) \cdot \lambda_j(b)
\]
for each \(b \in U_i \cap U_j\). Let \(L(T^* B/\Lambda)\) be the sheaf of Lagrangian sections of
\(T^* B/\Lambda\). Then \(\{\mu_{ij}\}\) defines a cohomology class \([\mu] \in H^1(B, L(T^* B/\Lambda))\).
We summarize as follows:

**Theorem 2.12 (Duistermaat [8]).**

1. If a Lagrangian fibration \(M \to B\) has a Lagrangian section, then \(M \to B\) is isomorphic to \(T^* B/\Lambda \to B\) as
Lagrangian fibrations.

2. The isomorphism classes of Lagrangian fibrations \(M \to B\) with Jacobian
fibration \(T^* B/\Lambda \to B\) are parameterized by \(H^1(B, L(T^* B/\Lambda))\).
3 Geometric Quantization

Geometric quantization is a method to construct representations geometrically. In the case of abelian varieties, we obtain the representations of finite Heisenberg groups. This viewpoint gives an interpretation of the relation between Lagrangian fibrations of abelian varieties and theta functions.

3.1 Brief review of quantum mechanics

First we recall a formulation of quantum mechanics on Euclidean spaces. Let
\[ M = T^*\mathbb{R}^n = \mathbb{R}^{2n} \]
be a symplectic vector space with the standard symplectic structure
\[ \omega = \frac{1}{\hbar} \sum_{i=1}^{n} dq^i \wedge dp_i, \]
where \((q^1, \ldots, q^n)\) is a coordinate on \(\mathbb{R}^n\), \((p_1, \ldots, p_n)\) the canonical coordinate on the cotangent spaces and \(\hbar\) is the Planck’s constant.

The goal is to correspond a representation of a suitable Lie subalgebra of \(C^\infty(M)\). For each function \(f\), we denote the corresponding operator by \(\hat{f} : H \to H\), where \(H\) is a Hilbert space. From the requirement that \(f \mapsto \hat{f}\) is a Lie algebra homomorphism, we obtain the canonical commutation relation
\[ [\hat{q}_i, \hat{p}_j] = \sqrt{-\frac{1}{\hbar}} \delta_{ij}. \]

This is realized as follows: Let \(H = L^2(\mathbb{R}^n)\) be the space of \(L^2\)-functions on \(\mathbb{R}^n\) with respect to the Lebesgue measure and define
\[ \hat{q}_i \varphi(q) = -\sqrt{-1}q^i \varphi(q), \]
\[ \hat{p}_j \varphi(q) = \hbar \frac{\partial \varphi(q)}{\partial q^j}, \]
for \(\varphi(q) \in L^2(\mathbb{R}^n)\). Then the relation (2) are satisfied.

Remark 3.1. From (2), \(\hat{q}_i\) and \(\hat{p}_i\) get commutative when \(\hbar \to 0\). Such a limit is called the semi-classical limit.

Definition 3.2. The Heisenberg algebra \(\text{heis}(\mathbb{R}^{2n})\) is the Lie algebra generated by \(\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n\) (and a center) satisfying the Heisenberg relations. In other words, \(\text{heis}(\mathbb{R}^{2n})\) is given by the central extension
\[ 0 \to \mathbb{R} \to \text{heis}(\mathbb{R}^{2n}) \to \mathbb{R}^{2n} \to 0, \]
where the extension class is given by \(\omega\).

Fact 3.3 (Stone, von Neumann (see, for example [30])). \(L^2(\mathbb{R}^n)\) is the unique (up to isomorphisms) irreducible representation of \(\text{heis}(\mathbb{R}^{2n})\).

3.2 Prequantization

Let \((M, \omega)\) be a symplectic manifold of dimension \(2n\) and assume that there exists a prequantum line bundle \((L, \nabla) \to M\).
As we saw in the previous section, the Lie algebra of the automorphism group \( G \) of the prequantum bundle \((L, \nabla)\) is identified with the space \( C^\infty(M) \) of smooth functions on \( M \). Hence we have a representation \( \Gamma(L) \) of \( G \) and \( \text{Lie} \mathcal{G} \cong C^\infty(M) \). However this representation is not the desired one. This is “too large” as we see in the following example (This is why we call this “prequantization”).

**Example 3.4.** We consider the case of symplectic vector space \( M = T^*\mathbb{R}^n = \mathbb{R}^{2n} \) with the symplectic form \( \omega = \frac{1}{\hbar} \sum_{i=1}^{n} dq^i \wedge dp_i \), as in the previous subsection. In this case, the prequantum bundle \( L \) is the trivial line bundle with connection \( \nabla = d + \sqrt{-1} \hbar \sum p_i dq^i \). Then \( \Gamma(L) \) is identified with \( L^2(M) = L^2(\mathbb{R}^{2n}) \). Note that \( \xi_q = -\hbar \frac{\partial}{\partial p_i}, \quad \xi_p = \hbar \frac{\partial}{\partial q^i} \).

From this and (1), the actions of \( q^i \) and \( p_i \) are given by
\[
\hat{q}^i(s) = -\hbar \frac{\partial}{\partial p_i} s - \sqrt{-1} q^i s, \\
\hat{p}_i(s) = \hbar \frac{\partial}{\partial q^i} s
\]
for \( s \in L^2(\mathbb{R}^{2n}) \). This representation is different from the one discussed in the previous subsection.

To get the desired representation, we have to eliminate half of the parameters \((q^1, \ldots, q^n, p_1, \ldots, p_n)\). This is done by choosing a “polarization”.

**Remark 3.5.** We can take \( k\omega \) as a symplectic form instead of \( \omega \). In this case, \( L^k \) is a prequantum bundle. \( \frac{1}{k} \) play the role of the Planck’s constant \( \hbar \). We also call the limit \( k \to \infty \) the semi-classical limit.

### 3.3 Polarizations

**Definition 3.6.** Extend \( \omega \) on \( TM \otimes \mathbb{C} \) complex bilinearly. A polarization \( P \) is an integrable Lagrangian subbundle in \( TM \otimes \mathbb{C} \), i.e. a complex subbundle \( P \subset TM \otimes \mathbb{C} \) of rank \( n \) satisfying \([P, P] \subset P \) and \( \omega|_P \equiv 0 \).

For a polarization \( P \), we “define” the space of polarized sections by
\[
\Gamma_P(L) = \{ s \in \Gamma(L) | \nabla_\xi s = 0, \text{ for all } \xi \in P \}.
\]
(This definition is a temporary one. For some class of polarization, we need to modify the definition. We will discuss this point later.)

**Remark 3.7.** The conditions of polarization are necessary for the integrability condition of \( \nabla_\xi s = 0 \). Since the curvature of \((L, \nabla)\) coincides with \( \omega \), the integrability condition can be written as
\[
0 = [\nabla_\xi, \nabla_\eta]s = (\nabla_{[\xi, \eta]} - \sqrt{-1} \omega(\xi, \eta)) s
\]
for every \( \xi, \eta \in P \). The integrability of \( P \) implies \([\xi, \eta] \in P \) and the Lagrangian condition implies \( \omega(\xi, \eta) = 0 \).
There are two important classes of polarizations.

**Example 3.8.** Assume that we have a Lagrangian fibration \( \pi : (M, \omega) \to B \). Then the complexified relative tangent bundle
\[
P := T_{M/B} \otimes \mathbb{C} = \ker(d\pi : TM \to TB) \otimes \mathbb{C}
\]
gives a polarization. This polarization satisfies \( \bar{P} = P \). We call polarizations satisfying this condition *real polarizations*.\(^1\)

In this case, the space \( \Gamma_P(L) \) of polarized sections consists of sections of \( L \) which are covariantly constant along fibers. Note that the restriction of \( L \) on each fiber is flat since each fiber is Lagrangian. We discuss this case later.

**Example 3.9.** Let \((M, \omega)\) be a Kähler manifold. Then \( P := T^{0,1}M + TM \otimes \mathbb{C} \) is a polarization. Such polarization is called a *Kähler polarization*. Note that \( L \) is holomorphic since the curvature \( \omega \) is of type \((1, 1)\). Kähler polarizations are characterize by
\[
\begin{align*}
P \cap \bar{P} &= \{0\}, \\
\omega|_{P \times \bar{P}} &> 0.
\end{align*}
\]
Giving a Kähler polarization is equivalent to fixing a compatible complex structure on \((M, \omega)\).

In this case, \( \Gamma_P(L) \) is nothing but the space \( H^0(M, L) \) of holomorphic sections.

**Remark 3.10.** The action of \( G \) (or \( \mathcal{C}^\infty(M) \)) does not preserve \( \Gamma_P(L) \subset \Gamma(L) \). In general, the subgroup of \( G \) which preserves the polarization is very small.

**Example 3.11.** We consider the case of symplectic vector space again. The natural projection \( \pi : T\mathbb{R}^n \to \mathbb{R}^n \) is a Lagrangian fibration. Let \( P = \ker d\pi \). Then \( \Gamma_P(L) \) is identified with \( L^2(\mathbb{R}^n) \). Furthermore, the actions of \( q^i \) and \( p_i \in \mathcal{C}^\infty(T\mathbb{R}^n) \) preserve this subspace. In fact, its action is given by
\[
\begin{align*}
\hat{q}^i \varphi(q) &= -\sqrt{-1}q^i \varphi(q), \\
\hat{p}_i \varphi(q) &= \hbar \frac{\partial}{\partial q^i} \varphi(q)
\end{align*}
\]
for \( \varphi(q) \in L^2(\mathbb{R}^n) \), as desired.

We can take a Kähler polarization on \( T\mathbb{R}^n \cong \mathbb{C}^n \). In this case, we also have an irreducible representation of Heisenberg group on a space of holomorphic functions on \( \mathbb{C}^n \). This is called the Bergman-Fock representation. (See [30].)

**Example 3.12 (Borel-Weil theory).** Let \( G \) be a compact Lie group, \( T \) a maximal torus in \( G \), \( G^\mathbb{C} \) the complexification of \( G \), and \( B \subset G^\mathbb{C} \) a Borel subgroup. We denote the flag variety by \( X = G/T = G^\mathbb{C}/B \). Then every \( G^\mathbb{C} \)-equivariant line bundle on \( X \) can be determined by a holomorphic character \( B \to \mathbb{C}^* \) by
\[
L = G^\mathbb{C} \times_B \mathbb{C}.
\]

\(^1\)In general, Lagrangian fibrations have degenerate fibers. Thus it is natural to allow real polarizations to be degenerate. However we ignore singular fibers in this section.
On the other hand, every character of $B$ descend to a character of $B/[B,B] \cong T^\mathbb{C}$:

\[
\begin{array}{ccc}
B & \longrightarrow & \mathbb{C}^* \\
\downarrow & & \\
T^\mathbb{C} & \swarrow & \\
\end{array}
\]

Hence there is a one to one correspondence

\[\{G^\mathbb{C}\text{-equivariant line bundles on } X\} \longleftrightarrow \{\text{holomorphic characters } T^\mathbb{C} \to \mathbb{C}^*\}\]

For each character $\lambda : T^\mathbb{C} \to \mathbb{C}^*$, we denote the corresponding line bundle by $L_\lambda = G^\mathbb{C} \times_B \mathbb{C} \to X$.

**Theorem 3.13 (Borel-Weil).** If $\lambda$ is a dominant weight, then $H^i(X, L_\lambda) = 0$ for $i > 0$ and $H^0(X, L_\lambda)$ is the irreducible representation of $G$ of highest weight $\lambda$. Any finite dimensional irreducible representation of $G$ is given in this way.

Next we discuss the case of real polarizations with compact Lagrangian fibers. Let $\pi : M \to B$ be a Lagrangian torus fibration. Since $\omega$ is the curvature of $L$, the restriction $L|_{\pi^{-1}(b)}$ is a flat line bundle on $\pi^{-1}(b)$ for each $b \in B$. However $L|_{\pi^{-1}(b)}$ has non-trivial holonomy in general. Therefore, there is no nontrivial smooth section of $L$ which is constant along fibers. Hence we must change the definition of $\Gamma_P(L)$.

**Definition 3.14.** A fiber $\pi^{-1}(b)$ is called a Bohr-Sommerfeld fiber if the restriction $(L, \nabla)|_{\pi^{-1}(b)}$ is trivial.

For each $k \in \mathbb{N}$, we can take $k\omega$ to be a symplectic form instead of $\omega$. In this case, $L^k$ is a prequantum bundle.

**Definition 3.15.** $\pi^{-1}(b)$ is called a Bohr-Sommerfeld fiber of level $k$ (or $k$-Bohr-Sommerfeld fiber) if the restriction $(L^k, \nabla)|_{\pi^{-1}(b)}$ is trivial.

From Proposition 2.10 and Remark 2.11, $\pi^{-1}(b)$ is a Bohr-Sommerfeld fiber of level $k$ if and only if the action coordinate $(q^1, \ldots, q^n)$ of $b$ takes its value in $\frac{1}{k}\mathbb{Z}^n$. In particular, Bohr-Sommerfeld fibers appear discretely.

Instead of smooth sections, we consider distributional sections of $L$ supported only on Bohr-Sommerfeld fibers and are covariantly constant on the support:

\[\Gamma_P(L) = \{s \mid \supp s \subset \text{BS fibers and } \nabla_\xi s = 0, \xi \in P\}\]

Since each fiber is connected, covariantly constant sections on a Bohr-Sommerfeld fiber is unique up to constants. In particular, the dimension of $\Gamma_P(L)$ is equal to the number of Bohr-Sommerfeld fibers.

**Remark 3.16.** There exists a cohomological definition of $\Gamma_P(L)$. Consider the following complex

\[
0 \longrightarrow \Gamma(L) \overset{d_F}{\longrightarrow} \Gamma(L \otimes P^*) \overset{d_F}{\longrightarrow} \Gamma(L \otimes \wedge^2 P^*) \overset{d_F}{\longrightarrow} \ldots
\]
where \( d_P^\nabla \) is the restriction of the connection to \( P \). This is in fact a complex since

\[
(d_P^\nabla)^2 = \omega|_P = 0.
\]

Then

**Theorem 3.17 (Śniatycki [38]).**

\[
H^i \left( \Gamma(L \otimes \wedge^* P^*), d_P^\nabla \right) = 0
\]

for \( i \neq n \) and

\[
H^n \left( \Gamma(L \otimes \wedge^* P^*), d_P^\nabla \right) \cong \Gamma_P(L) \quad \text{("Poincaré duality")}.
\]

### 3.4 Comparison of the spaces of wave functions for Kähler and real polarizations

Assume that we have a Lagrangian fibration \( \pi : (X, \omega) \to B \) of a compact Kähler manifold and a prequantum bundle \( L \to X \). Then we have two spaces of wave functions, i.e. the space \( H^0(X, L) \) of holomorphic sections and \( \Gamma_{TX/B}(L) \). It is natural to ask whether these spaces are isomorphic or not. In general, it is difficult to define and compute \( \Gamma_{TX/B}(L) \) since there exist degenerate fibers for general Lagrangian fibrations. However it is shown that these spaces are isomorphic for several cases.

**Example 3.18.** Let \( X = \mathbb{C}P^1 \) with the Fubini-Study metric and \( L = \mathcal{O}(1) \). The moment map of a natural \( S^1 \)-action is a Lagrangian fibration. This can be written explicitly as follows. We identify \( \mathbb{C}P^1 \) with the unit sphere

\[
S^2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \} \subset \mathbb{R}^3.
\]

Then the moment map is given by

\[
\pi : S^2 \to [-1, 1], \quad (x_1, x_2, x_3) \mapsto x_3.
\]

It is easy to see that \( \pi^{-1}(b) \) satisfies the Bohr-Sommerfeld condition of level \( k \) if and only if \( b = \frac{2i \cdot k}{k}, \ i = 0, 1, \ldots, k \) is a lattice point. Note that

\[
\text{dim } H^0(\mathbb{P}^1, \mathcal{O}(k)) = k + 1 = \text{the number of } k\text{-BS fibers}.
\]

In this case, the isomorphism between two spaces of wave functions is given by just the restrictions:

\[
H^0(\mathbb{P}^1, \mathcal{O}(k)) \to \Gamma_{TX/B}(L)^k, \quad z_0^i z_1^{k-i} \mapsto z_0^i z_1^{k-i} |_{\pi^{-1}(\frac{2i \cdot k}{k})}.
\]

The same is true for general projective toric manifolds (with a suitable Kähler form).

Guillemin-Sternberg [15] proved the case of flag varieties. This is also true for the case of abelian varieties. We will see this in detail in the next subsection.
Theorem 3.19 (Andersen [1]). Let \((X, \omega)\) be a compact Kähler manifold, \(L \to X\) a prequantum bundle. Assume that we have a Lagrangian fibration \(\pi : X \to B\) with no degenerate fiber. Then

\[
\dim H^0(X, L^k) = \dim \Gamma_{T_X/B}(L^k)
\]

for large \(k\).

Remark 3.20. In general, the restriction of holomorphic sections to BS fibers does not give covariantly constant sections.

Remark 3.21. Very few compact Kähler manifolds admit Lagrangian fibration without degenerate fibers. However it is expected that this theorem holds for more general cases such as K3 surfaces. We discuss this point in section 4 again.

The dimension of \(H^0(X, L^k)\) is given by Riemann-Roch formula and vanishing theorem of cohomologies. Thus what we need to do is to count the number of Bohr-Sommerfeld fibers. The idea is to use its “mirror”, i.e. a dual torus fibration.

Proof. From Riemann-Roch theorem,

\[
\dim H^0(X, L^k) = \int_X ch(L^k) \hat{A}(TX) = \int_X \exp(k \omega) \hat{A}(TX).
\]

Now we have

\[
0 \to T_{X/B} \to TX \to \pi^*TB \to 0.
\]

By using the symplectic form, \(T_{X/B}\) can be identified with \(\pi^*T^*B\). Since \(T^*B\) and \(TB\) contains lattices of maximal rank, these are flat. Then

\[
\hat{A}(TX) = \hat{A}(T_{X/B}) \hat{A}(\pi^*TB) = \pi^*(\hat{A}(T^*B) \hat{A}(TB)) = 1.
\]

Consequently we have

\[
\dim H^0(X, L^k) = \int_X k^n \omega^n / n! = k^n \cdot \text{vol}(X, \omega).
\]

Next we calculate the number of Bohr-Sommerfeld fibers. First we define a dual torus fibration of \(\pi : X \to B\). Let \(T^*B/\Lambda \to B\) be the Lagrangian fibration associated to \(\pi\), as in the previous section. We denote the dual lattice of \(\Lambda\) by \(\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z}) \subset TB\). Then \(\tilde{\pi} : TB/\Lambda^* \to B\) is a dual torus fibration.

Recall that the dual torus \(\tilde{T}^n\) of a torus \(T^n\) parameterizes flat line bundles \(E \to T^n\). In our situation, for each \(b \in B\), we can associate a flat line bundle \(L^k|_{\pi^{-1}(b)}\) on \(\pi^{-1}(b)\). This defines a section \(\lambda_k : B \to TB/\Lambda^*\) of the dual torus fibration. On the other hand, the zero section \(\lambda_0\) corresponds to the trivial bundle on \(X\). By definition, \(\pi^{-1}(b)\) is a Bohr-Sommerfeld fibers of level \(k\) if and only if \(\lambda_k(b) = \lambda_0(b)\). This implies

\[
\dim \Gamma_{T_X/B}(L^k) = \#(\lambda_0(B) \cap \lambda_k(B)).
\]
\( \lambda_k \) can be written explicitly as follows. Let \((x^1,\ldots,x^n,y_1,\ldots,y_n)\) be the action-angle coordinate constructed in section 2 and take the dual coordinate \((x^1,\ldots,x^n,v^1,\ldots,v^n)\). Then \( \lambda_k \) is given by

\[
\lambda_k(x^1,\ldots,x^n) = (x^1,\ldots,x^n,kx^1,\ldots,kx^n).
\]

From this expression, we see that \( \lambda_k \) and \( \lambda_0 \) intersect transversely and positively (under a suitable orientation). Therefore the number of Bohr-Sommerfeld fibers of level \( k \) coincides with the intersection number of \( \lambda_k(B) \) and \( \lambda_0(B) \). Put \( \alpha = dv^1 \wedge \cdots \wedge dv^n \in \Omega^n(TB/P). \) This is the Poincaré dual of \( \lambda_0(B) \).

\[
\dim \Gamma_{T_{X/B}}(L^k) = \int_{\lambda_k(B) \cap \lambda_0(B)} 1 = \int_{\lambda_k(B)} \alpha = \int_{B} \lambda_k^* \alpha.
\]

From (3), we have

\[
\lambda_k^* \alpha = k^n dx^1 \wedge \cdots \wedge dx^n = k^n \pi_* \omega^n/n!,
\]

hence

\[
\dim \Gamma_{T_{X/B}}(L^k) = k^n \int_B \pi_* \omega^n/n! = k^n \int_X \omega^n/n! = k^n \cdot \text{vol}(X,\omega).
\]

\[\blacksquare\]

### 3.5 Geometric quantization for abelian varieties

Let \( X = \mathbb{C}^n/(\Omega \mathbb{Z}^n + \mathbb{Z}^n) \) be an abelian variety of complex dimension \( n \), where \( \Omega \) is an \( n \times n \) complex symmetric matrix whose imaginary part \( \text{Im} \Omega \) is positive definite, and \( L \) an ample line bundle on \( X \). We assume that \( L \) is symmetric of degree 1, for simplicity. Recall that \( L \) is said to be symmetric if \((-1_X)^* L \cong L \), here \(-1_X : X \to X \) is the inverse morphism of \( X \). Explicitly, \( L \) is given by

\[
L = (\mathbb{C}^n \times \mathbb{C})/(\Omega \mathbb{Z}^n + \mathbb{Z}^n),
\]

where the action of \( \Omega \mathbb{Z}^n + \mathbb{Z}^n \) on \( \mathbb{C}^n \times \mathbb{C} \)

\[
(z,\zeta) \mapsto (z + \lambda e^{\pi i (\text{Im} \Omega)^{-1} z + \frac{\pi}{i} (\text{Im} \Omega)^{-1} \lambda} \zeta)
\]

for \( \lambda \in \Omega \mathbb{Z}^n + \mathbb{Z}^n \). Any ample bundle of degree 1 is a pull back of \( L \) by some translation of \( X \). From the definition of \( L \),

\[
h_0 = \exp(-\pi i z (\text{Im} \Omega)^{-1} \zeta)
\]
defines a Hermitian metric on \( L \) whose first Chern form
\[
c_1(L, h_0) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_0 = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j =: \omega_0
\]
gives a flat Kähler metric on \( X \), here we put \( (h_{ij}) = (\text{Im } \Omega)^{-1} \).

Take the following two Lagrangian tori
\[
X^+ = \{ \Omega x \mid x \in \mathbb{R}^n / \mathbb{Z}^n \}, \quad X^- = \{ y \mid y \in \mathbb{R}^n / \mathbb{Z}^n \} \subset X
\]
and consider the natural projection
\[
\pi : X = X^+ \times X^- \rightarrow X^-, \quad z = \Omega x + y \mapsto y.
\]

Then we have two polarizations and hence two spaces of wave functions.

Next we define finite Heisenberg groups in a similar way as in the case of vector space \( \mathbb{R}^{2n} \). For each \( k \in \mathbb{N} \), let \( X_k \) (resp. \( X_k^\pm \)) be the subgroup of points of \( X \) (resp. \( X^\pm \)) of order \( k \). Then \( X_k^\pm \cong \mathbb{Z}^n / \mathbb{Z}^n \) and \( X_k = X_k^+ \times X_k^- \). It is easy to see that \( X_k \) can be characterized by
\[
X_k = \{ w \in X \mid \tau_w^* L \cong L_k \}
\]
where \( \tau_w : X \rightarrow X \) will denote the translation \( \tau_w(z) = z + w \). Then there exists a central extension \( \mathcal{G}(L^k) \subset \text{Aut}(X, L^k) \) of \( X_k \):
\[
1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L^k) \longrightarrow X_k \longrightarrow 0,
\]
where \( \mathbb{C}^* \) acts on \( L^k \) by multiplications on each fiber. The \textit{finite Heisenberg group} \( G_k \) is defined by
\[
1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L^k) \longrightarrow X_k \longrightarrow 0
\]
\[
\cup \quad \cup \quad \| \quad \|
\]
\[
1 \longrightarrow \mu_k \longrightarrow G_k \longrightarrow X_k \longrightarrow 0
\]
where \( \mu_k \) is the group of \( k \)-th roots of 1. The group law is
\[
(c_1, a_1, b_1) \cdot (c_2, a_2, b_2) = (e^{2\pi \sqrt{-1}b_1}a_1c_2, a_1 + a_2, b_1 + b_2)
\]
for \( (c_1, a_1, b_1), (c_2, a_2, b_2) \in G_k \cong \mu_k \times \frac{1}{k} \mathbb{Z}^n \times \frac{1}{k} \mathbb{Z}^n \).

\textbf{Kähler polarization}

\( G_k \) acts naturally on \( H^0(X, L^k) \). Note that the Hermitian metric \( h_0 \) is invariant under the \( G_k \)-action. This means that the induced \( G_k \)-action on \( H^0(X, L^k) \) is unitary.

\textbf{Theorem 3.22.} \( H^0(X, L^k) \) is a unique (up to isomorphism) irreducible representation of \( G_k \), called the Heisenberg representation.
Holomorphic sections of $L^k$ can be explicitly written by using theta functions. We take a basis of $H^0(X, L^k)$ defined by

$$s_{b_i} = s_i = C_{\Omega} k^{-\frac{7}{4}} \cdot \theta_0(\Omega, z)^k \cdot \vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right](k^{-1}\Omega, z),$$  \hspace{1cm} (6)

where we denote $X_k^- = \{ b_i \}_{i=1,\ldots,k^n}$ and

$$C_{\Omega} = 2^{\frac{7}{4}} \left( \det(\text{Im } \Omega) \right)^{\frac{7}{4}},$$

$$\theta_0(\Omega, z) = \exp \left( \frac{\pi}{2} t z (\text{Im } \Omega)^{-1} z \right),$$

$$\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right](\Omega, z) = \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2} (l + a) \Omega(l + a) + t(l + a)(z + b) \right),$$

$$e(t) = \exp(2\pi\sqrt{-1}t).$$

By definition, the action of $G_k$ on $H^0(X, L^k)$ is given by

$$s(z) \mapsto c \cdot \tau^* \Omega^n + s \cdot s(z + \Omega a + b)$$

for $(c, a, b) \in G_k$. Put $\{a_j\} = X_k^+$. By direct computation, we have

$$\rho_k(1, a_j, 0) s_{b_i} = e^{2\pi \sqrt{-1} t(a_j b_i)} s_{b_i},$$

$$\rho_k(1, 0, b_j) s_{b_i} = s_{b_i - b_j}.$$  \hspace{1cm} (7)

From (7), we can show that $H^0(X, L^k)$ is an irreducible representation of $G_k$.

**Real polarizations**

Next we consider the real polarization defined by (5). In this case, Bohr-Sommerfeld fibers are characterized as follows:

**Proposition 3.23.** For $b \in X^-$, $\pi^{-1}(b) = X^+ \times \{b\}$ satisfies the Bohr-Sommerfeld condition of level $k$ if and only if $b \in X_k^-$. In fact, for each $b_i \in X_k^-$,

$$\sigma_i(x) = \exp \left( \frac{k\pi}{2} t(\Omega x + b_i)(\text{Im } \Omega)^{-1}(\Omega x + b_i) - \sqrt{-1} k\pi t x \Omega x \right)$$

defines a covariantly constant section of $L^k|_{\pi^{-1}(b_i)}$. In particular the number of Bohr-Sommerfeld fibers of level $k$ coincides with $\dim H^0(X, L^k) = k^n$.

The basis $s_i \in H^0(X, L^k)$ can be reconstructed from $\sigma_i$’s by using the Bergman kernel. The Bergman kernel $\Pi_k(z, w)$ is the integral kernel of the orthogonal projection

$$L^2(X, L^k) \longrightarrow H^0(X, L^k).$$
Proposition 3.24. Let $\Pi_k(z, w)$ be the Bergman kernel of the Hermitian line bundle $(L^k, h_0)$. Then

$$\int_{X^{+}\times \{b_i\}} \Pi_k(z, x)\sigma_i(x)\,dx = C'_\Omega k^\frac{n}{4} s_i(z),$$

where

$$C'_\Omega = \frac{2^{\frac{n}{2}} \sqrt{-1}^{\frac{n}{2}} \det(\text{Im } \Omega)^{\frac{n}{4}}}{(\det \overline{\Omega})^{\frac{n}{2}}}. $$

Proof. We will show that $s_1, \ldots, s_{k^n}$ define an orthonormal basis of $H^0(X, L^k)$ in the next subsection. Then the Bergman kernel is given by

$$\Pi_k(z, w) = \sum_{i=1}^{k^n} s_i(z)s_i(w)^*. $$

From this expression, it suffices to show that

$$\int_{X^{+}\times \{b_i\}} (\sigma_i, s_j)_{h_0}\,dx = C'_\Omega k^\frac{n}{4} \delta_{ij}. \quad (8)$$

By direct computation, we have

$$(\sigma_i, s_j)_{h_0} = C_{\Omega \overline{\Omega}} k^\frac{n}{4} \sum_{l \in \mathbb{Z}^n} e \left( -\frac{k}{2} \left( \frac{i}{k} \left( x + \frac{l}{k} \right) \right) - i l (b_i - b_j) \right),$$

and we obtain (8) by integrating this. \qed
4 Special Lagrangian fibrations and mirror symmetry

Mirror symmetry is a duality in string theory. Mathematically, this is regarded as a duality between symplectic geometry on a Calabi-Yau manifold $M$ and complex geometry on another Calabi-Yau manifold $W$. Homological mirror symmetry conjectured by Kontsevich [23] claims the equivalence of the Fukaya category on $M$ and the derived category of coherent sheaves on $W$. Strominger-Yau-Zaslow [39] proposed mirror symmetry via special Lagrangian fibrations. This picture gives a geometric construction of mirror manifolds and correspondence of two categories. This works successfully for abelian varieties and theta functions play an important role again.

4.1 Special Lagrangian submanifolds

Definition 4.1. A Calabi-Yau manifold $X$ is a Kähler manifold with trivial canonical line bundle: $K_X \cong \mathcal{O}_X$.

An important fact is:

Theorem 4.2 (Yau [45]). If $X$ is a compact Kähler manifold with $c_1(X) = c_1(K_X^{-1}) = 0$, then each Kähler class contains a unique Ricci-flat Kähler metric.

Let $X$ be a compact Calabi-Yau manifold of complex dimension $n$. By definition, $X$ carries a non-vanishing holomorphic $n$-form $\Omega \in H^0(X,K_X)$. Recall that the Ricci form of a Kähler form $\omega$ is given by $\text{Ric}(\omega) = -\partial \bar{\partial} \log \omega^n$. Hence $\omega$ is Ricci-flat if and only if $\omega^n = c \Omega \wedge \bar{\Omega}$ for some constant $c$.

Now we fix a Ricci-flat metric $\omega$ and normalize $\Omega$ so that

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left( \frac{-\sqrt{-1}}{2} \right)^n \Omega \wedge \bar{\Omega}.$$ 

This condition is satisfied for $\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$ and $\Omega = dz^1 \wedge \cdots \wedge dz^n$. Such $\Omega$ is unique up to $S^1$.

Definition 4.3. Let $S \subset X$ be an $n$-dimensional submanifold. $S$ is called a special Lagrangian submanifold if

$$\omega|_S = \text{Im}(e^{-\sqrt{-1} \theta} \Omega)|_S = 0$$

for some $\theta \in \mathbb{R}$.

An important property of special Lagrangian submanifolds is the following:

Theorem 4.4 (Harvey-Lawson [17]). Every special Lagrangian submanifold is volume minimizing in its homology class.
Note that the special Lagrangian condition is a first order differential equation. On the other hand, the condition of minimal submanifolds is second order. In addition, minimal submanifolds are not necessarily volume minimizing. Such a situation is similar to that of ASD connections and Yang-Mills connections in gauge theory.

Recall that the normal bundle of a Lagrangian submanifold $S$ is identified with the cotangent bundle $T^* S$ by using the symplectic structure. Under this identification, every small deformations of $S$ is given as a graph of a 1-form on $S$. Then the graph is also Lagrangian if and only if the 1-forms is closed.

**Theorem 4.5 (McLean [28])**. Under the above identification, infinitesimal deformations of a special Lagrangian submanifold are given by harmonic 1-forms:
\[
d\alpha = d^* \alpha = 0.
\]
Furthermore, every infinitesimal deformation is unobstructed. In particular, the tangent space of special Lagrangian submanifolds is identified with $H^1(S, \mathbb{R})$.

**Definition 4.6**. A fibration $\pi : X \to B$ is called a special Lagrangian fibration if each smooth fiber is special Lagrangian.

Since each fiber $\pi^{-1}(b)$ is a Lagrangian torus,
\[
T_b B \cong T^*_{\pi^{-1}(b)} \cong H^1(\pi^{-1}(b), \mathbb{R})
\]
for $x \in \pi^{-1}(b)$. This implies that $B$ can be considered as (a component of) the moduli space of special Lagrangian tori in $X$.

**Example 4.7**. Let $X = \mathbb{C}^n/(\tau \mathbb{Z}^n + \mathbb{Z}^n)$ be an abelian variety with a flat metric $\omega = \frac{\sqrt{-1}}{2} \sum h_{ij} dz^i \wedge d\bar{z}^j$, where $\tau$ is a $n \times n$ matrix with positive definite imaginary part and $(h_{ij}) = (\text{Im} \tau)^{-1}$. Then
\[
\pi : X \to T^n,
\]
is a special Lagrangian fibration.

**Example 4.8**. Let $X$ be a K3 surface and assume that $\pi : X \to \mathbb{C}P^1$ is an elliptic fibration with respect to a complex structure $I$. We fix a Ricci-flat Kähler metric $g$. Since $X$ is hyperKähler, there exist compatible complex structures $J$ and $K$ satisfying $I^2 = J^2 = K^2 = -1$ and $IJ = K$. We denote the corresponding Kähler form by $\omega_I$, $\omega_J$ and $\omega_K$ respectively. Then
\[
\Omega_I = \omega_K + \sqrt{-1} \omega_I \text{ is holomorphic with respect to } I,
\]
\[
\Omega_J = \omega_I + \sqrt{-1} \omega_K \text{ is holomorphic with respect to } J,
\]
\[
\Omega_K = \omega_J + \sqrt{-1} \omega_I \text{ is holomorphic with respect to } K.
\]
Since $\pi$ is holomorphic with respect to $I$, we have $\Omega_I|_{\text{fiber}} = 0$. This implies that $\pi : X \to S^2$ is a special Lagrangian fibration with respect to $J$.

In this case, we know the structure of singular fibers from the result for elliptic surfaces by Kodaira [22].
4.2 A local model of mirror symmetry

Homological mirror symmetry conjecture states the equivalence of the Fukaya category of $M$ and the derived category of coherent sheaves on $W$. Roughly, this means that each pair $(S, L)$ of a special Lagrangian submanifold in $M$ and a flat line bundle on $S$ corresponds to a coherent sheaf on $W$. In this subsection, we see this correspondence from the point of view of special Lagrangian fibrations, following Leung-Yau-Zaslow [25] and Leung [24] (see also Hitchin [18]).

Conjecture 4.9 (Strominger-Yau-Zaslow [39]). If $(M, W)$ is a mirror pair, then there exist special Lagrangian torus fibrations

$$
\pi : M \rightarrow B,
\tilde{\pi} : W \rightarrow B
$$

satisfying the following condition: there exists an open dense subset $B_0 \subset B$ such that $\pi^{-1}(b)$ and $\tilde{\pi}^{-1}(b)$ are smooth and dual to each other for each $b \in B_0$.

This picture gives a concrete description of the correspondence of (special) Lagrangian submanifolds in $M$ and coherent sheaves on $W$. In this subsection, we see the correspondence of holomorphic condition and special Lagrangian condition in a local model.

Let $B$ be a small ball in $\mathbb{R}^n$ with coordinate $(x^1, \ldots, x^n)$. Then every Lagrangian torus fibration has the form

$$
\pi : M = T^*B/\Lambda \rightarrow B
$$

for some $\mathbb{Z}^n$-bundle $\Lambda \subset T^*B$. We denote the standard symplectic form by $\omega = \sum dx^i \wedge dy_i$. As in the previous section, we construct the dual torus fibration by

$$
\tilde{\pi} : W = TB/\Lambda^* \rightarrow B,
$$

where $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice bundle of $\Lambda$. Then $W$ has a complex structure induced from

$$
TB \rightarrow \mathbb{C}^n, \quad \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i} \mapsto (z^1 = x^1 + \sqrt{-1}y^1, \ldots),
$$

where $y^i$ is the dual coordinate of $x^i$. Note that $\Omega = dz^1 \wedge \cdots \wedge dz^n$ is a holomorphic $n$-form on $X$.

Remark 4.10. Here we assume that the B-field is zero. If we consider non-zero B-fields, we obtain more general complex structures.

Next we introduce Kähler metrics on $M$ and $W$. From the definition of the complex structure on $W$, every Kähler metric and its Kähler form have the form

$$
\hat{g} = \sum g_{ij}(dx^i dx^j + dy^i dy^j),
\hat{\omega} = \frac{\sqrt{-1}}{2} \sum g_{ij} dz^i \wedge d\bar{z}^j.
$$
The induced metric on $M$ is written as

$$g = \sum (g_{ij} dx^i dx^j + g^{ij} dy_i dy_j),$$

where $(g^{ij}) = (g_{ij})^{-1}$. Then we have a compatible almost complex structure $J$ on $M$ defined by

$$\omega(\xi, \eta) = g(J\xi, \eta).$$

Recall that each fiber of $\pi$ and $\tilde{\pi}$ have a natural affine structure. Hereafter we assume that $g$ is constant along fibers with respect to the natural affine structure (semi-flat condition). Then the Kähler condition ($d\omega = 0$) implies that

$$g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$$

for some $\phi = \phi(x) \in C^\infty(B)$. Since $(g_{ij}) = \left(\frac{\partial^2 \phi}{\partial x^i \partial x^j}\right)$ is non-degenerate, we can take a new coordinate $(x_1, \ldots, x_n)$ of $B$ satisfying

$$x_i = \frac{\partial \phi}{\partial x^i}, \quad i = 1, \ldots, n.$$

Remark that the Jacobian is given by $\frac{\partial x^i}{\partial x^j} = g_{ij}$. We denote the Legendre transform of $\phi$ by

$$\psi = \sum_{i=1}^n x^i x_i - \phi.$$

Then

$$x^i = \frac{\partial \psi}{\partial x_i} \quad \text{and} \quad g^{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x^j}.$$ 

Hence we have

$$g = \sum g^{ij} (dx_i dx_j + dy_i dy_j),$$

$$\omega = \sqrt{\det(g)} \sum g^{ij} dz_i \wedge d\bar{z}_j,$$

where $z_i = x_i + \sqrt{-1} y_i$. In particular $J$ is integrable.

The Ricci-flat condition for $g$ is equivalent to the Monge-Ampère equation

$$\det \left( \frac{\partial^2 \phi}{\partial x^i \partial x^j} \right) = C$$

for some constant $C$. Since $\det(g^{ij}) = \det(g_{ij})^{-1}$, $g$ is Ricci-flat if and only if $\tilde{g}$ is Ricci-flat.

Namely, in this setting, mirror of $M$ is nothing but its Legendre transform.

We summarize the above discussion. $M = T^* B / \Lambda$ has a standard symplectic structure while $W = TB / \Lambda^*$ has a natural complex structure. Each compatible almost complex structure $J$ (or, equivalently, a metric $g$) on $M$ determines a metric $\tilde{g}$ (or $\tilde{\omega}$) on $W$. Under the semi-flat condition, closedness of $\tilde{\omega}$ corresponds to the integrability condition of $J$. Moreover, Ricci-flat condition for $g$ is equivalent to that for $\tilde{g}$. 

24
Remark 4.11. The semi-flat condition is quite strong. In fact, few compact Calabi-Yau manifolds admit semi-flat Ricci-flat metrics. Nevertheless this local model is still important because it is expected that Calabi-Yau manifolds possess structures close to this semi-flat model near the large complex structure limit (i.e., when it is close to the most degenerate Calabi-Yau’s). Gross-Wilson \[14\] analyzed the case of K3 surfaces.

Next we consider the correspondence of special Lagrangian submanifolds in $M$ and coherent sheaves on $W$.

Definition 4.12 (Leung-Yau-Zaslow \[25\]).

1. A supersymmetric A-cycle is a pair $(S, L)$ of a special Lagrangian submanifold $S$ and a flat line bundle on it.

2. A supersymmetric B-cycle is a pair $(C, E)$ of a complex submanifold $C$ and a holomorphic vector bundle $E \to C$ with a unitary connection $A$ satisfying $\text{Im} e^{\sqrt{-1} \theta} (\omega + F_A)^m = 0$ for some $\theta \in \mathbb{R}$, where $F_A$ is the curvature of $E$ and $m = \dim_C C$. We call holomorphic vector bundles satisfying this condition Generalized Hermitian Yang-Mills.

Remark 4.13. A holomorphic vector bundle $(E, A) \to C$ is said to be Hermitian-Yang-Mills if $\Lambda F_A = \lambda Id$ for some constant $\lambda$, where $\Lambda$ is the adjoint operator of $\omega \wedge$. In this case, $F_A$ satisfies the generalized Hermitian-Yang-Mills condition.

First we consider the case that $S \subset M$ is a special Lagrangian section of $\pi : M \to B$. $S$ can be given as a graph of 1-form $\alpha = \sum A_i dx^i$ on $B$. The Lagrangian condition is equivalent to $d\alpha = 0$. We set

$$\frac{\partial A_i}{\partial x^j} = \frac{\partial A_j}{\partial x^i} = F_{ij}.$$  \hspace{1cm} (9)

Since $z_i = x_i + \sqrt{-1} y_i = \frac{\partial \phi}{\partial x^i} + \sqrt{-1} y_i$, we have

$$\alpha^* dz_i = \frac{\partial}{\partial x_j} \left( \frac{\partial \phi}{\partial x^i} + \sqrt{-1} A_i \right) dx^j = (g_{ij} + \sqrt{-1} F_{ij}) dx^j .$$

Hence

$$\alpha^* \Omega = \det (g_{ij} + \sqrt{-1} F_{ij}) dx^1 \wedge \cdots \wedge dx^n .$$

The special Lagrangian condition $\text{Im} e^{\sqrt{-1} \theta} \Omega = 0$ becomes

$$\text{Im} e^{\sqrt{-1} \theta} \det (g_{ij} + \sqrt{-1} F_{ij}) = 0 .$$  \hspace{1cm} (10)

Then the corresponding supersymmetric B-cycle is given by $C = W$ and $E$ is a line trivial bundle with the connection $\nabla_A = d + \sqrt{-1} \sum A_i (x) dy^i$. Note that $(E, A)$ is holomorphic if and only if

$$F_A^{0,2} = \frac{1}{4} \sum F_{ij} d\bar{z}^i \wedge d\bar{z}^j = 0 .$$
This follows from (9) (i.e. the closedness of \(\alpha\)). Furthermore the generalized Hermitian-Yang-Mills condition is equivalent to the special Lagrangian condition (10).

Next we consider the case that \(S\) is a multisection, i.e. the restriction \(\pi|_{S} : S \to B\) is a \(r\)-fold covering. In this case, the corresponding supersymmetric B-cycle is a holomorphic vector bundle \(E \to W\) of rank \(r\). To see this, we consider the following \(r\)-fold covering:

\[
\begin{array}{c}
M \quad M \\
r:1 \\
\circ \\
S \quad S \\
r:1 \\
\pi \\
B \quad B \\
1:1
\end{array}
\]

Then the problem is reduced to the case of single sections. \(\bar{S}\) corresponds to a line bundle \(L \to \bar{W}\). \(E\) is defined by

\[
f_*L = \begin{cases} E & \text{in } W \\ L & \text{in } \bar{W} \end{cases}
\]

Since \(f : \bar{W} \to W\) is a \(r\)-fold covering, \(E\) is a vector bundle of rank \(r\).

### 4.3 Mirror symmetry for abelian varieties

The correspondence discussed in the previous subsection works in the case of abelian varieties. For elliptic curves, Polishchuk-Zaslow \[34\] proved an equivalence of the derived category \(D^b(W)\) of coherent sheaves on \(W\) and a modified version \(\mathcal{F}_0(M)\) of Fukaya category from this viewpoint. For higher dimensional case, Fukaya \[11\] constructed a functor from the \(A_{\infty}\) category of affine Lagrangian submanifolds in \(M\) to \(D^b(W)\) for abelian varieties. We recall these theory very briefly.

Let \(M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}\) be a \(2n\) dimensional torus with a constant complexified
symplectic form $\omega^C = \omega + \sqrt{-1} \beta$, where
$$
\omega = \sum dx^i \wedge dy^i,
\beta = \sum b_{ij} dx^i \wedge dy^j,
$$
and consider the following Lagrangian fibration
$$
\pi : M \rightarrow B = T^n, \quad (x, y) \mapsto x.
$$

Set $\tau = (b_{ij} + \sqrt{-1} \delta_{ij})$. Then the mirror is given by
$$
W = \mathbb{C}^n/\tau \mathbb{Z}^n + \mathbb{Z}^n
$$
with dual torus fibration $\tilde{\pi} : W \rightarrow B$.

**Theorem 4.14 (Fukaya [11], Polischuk [33]).** For each pair $(S, L)$ of an affine Lagrangian submanifold $S$ in $M$ and a flat line bundle $L \rightarrow S$, we can associate an object $E = E(S, L)$ of $D^b(W)$.

The construction in the previous subsection can be rephrased by using “the Poincaré bundle”. Let $p : M \times_B W \rightarrow W$ be the natural projection. Then the Poincaré bundle is a line bundle
$$
\mathcal{P} \rightarrow M \times_B W
$$
such that, for $\xi \in W$,
$$
\mathcal{P}|_{\pi^{-1}(\xi)} \rightarrow \pi^{-1}(\tilde{\pi}(\xi))
$$
is the flat line bundle corresponding to $\xi \in \tilde{\pi}^{-1}(\tilde{\pi}(\xi)) = (\pi^{-1}(\tilde{\pi}(\xi)))^\vee$.

Let $S \subset M$ be an affine Lagrangian submanifold and $L \rightarrow S$ a line bundle. Assume that $\pi|_S : S \rightarrow B$ is an unramified covering. Let
$$
p_S : S \times_B W \rightarrow S,
p_W : S \times_B W \rightarrow W,
(i \times id) : S \times_B W \rightarrow M \times_B W
$$
be natural maps. Then $E(S, L)$ is defined by
$$
E(S, L) = (p_W)_*((i \times id)^*\mathcal{P} \otimes p_S^*L)
$$
(“Fourier-Mukai transform” of $(S, L)$).

**Proposition 4.15 (Polischuk [33]).** $E(S, L)$ is holomorphic if and only if the curvature of $L$ coincides with $\omega^C|_S = 0$.

Newt we see the correspondence of morphisms. Morphisms in Fukaya category are given by Floer homologies $HF((S_1, L_1), (S_2, L_2))$. Roughly speaking, Floer homologies are generated by intersection points of Lagrangian submanifolds. Here we are interested in the case that Lagrangian submanifolds intersect transversally.
Fact 4.16 (Fukaya). If $S_1$ and $S_2$ are transverse,

$$HF((S_1, L_1), (S_2, L_2)) \cong \mathbb{C}^\#S_1 \cap S_2 \otimes \text{Hom}(L_1, L_2),$$

where $\text{Hom}(L_1, L_2)$ is homomorphisms of vector spaces underlying the local systems at $S_1 \cap S_2$.

Theorem 4.17 (Fukaya [11]). Let $S_1$ and $S_2$ be affine Lagrangian submanifolds in $M$ (not necessarily transverse) and $L_1$, $L_2$ flat line bundles on $S_1$ and $S_2$ respectively. Then

$$HF((S_1, L_1), (S_2, L_2)) \cong \text{Ext}(E_1, E_2),$$

where $E_i = E(S_i, L_i)$.

Example 4.18. Let $S = S_0$ be the zero section of $\pi : M = T^*B/\Lambda \to B$ with trivial bundle. This Lagrangian submanifold corresponds to the structure sheaf $E(S_0) = \mathcal{O}_W$ on $W$. A principal polarization (an ample line bundle of degree 1) $E \to W$ corresponds to an affine Lagrangian torus $S_1$ of “slope 1”. Similarly, $E^k \to W$ corresponds to an affine Lagrangian torus $S_k$ of “slope k”. Hence we have

$$\dim H^0(W, E^k) = k^n = \#S_0 \cap S_k = \dim HF(S_0, S_k).$$

These Lagrangian sections $S_k$ are just the section $\lambda_k$ appeared in the proof of Theorem 3.19. In other words, $HF(S_0, S_k)$ can be identified with the space of wave functions defined by the real polarization $\pi : M \to B$. In this case, the isomorphism of the spaces of wave functions can be regarded as a part of mirror symmetry.
The next theorem states the correspondence of the product structure (compositions) of morphisms.

**Theorem 4.19 (Fukaya [11]).** The following diagram commutes.

\[ \begin{array}{ccc}
\text{HF}\left( (S_1, L_1), (S_2, L_2) \right) \otimes \text{HF}\left( (S_2, L_2), (S_3, L_3) \right) & \longrightarrow & \text{HF}\left( (S_1, L_1), (S_3, L_3) \right) \\
\downarrow m_2 & & \downarrow m_2 \\
\text{Ext}(E_1, E_2) \otimes \text{Ext}(E_2, E_3) & \longrightarrow & \text{Ext}(E_1, E_3)
\end{array} \]

The matrix elements are given by *theta functions*. We see this in the following example.

**Example 4.20 (Polishchuk-Zaslow [34]).** Let \( M = T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) be a 2-torus with a complexified Kähler form \( \omega^C = -\sqrt{-1} \tau dx \wedge dy \). In this case, its mirror is given by \( W = C/\tau \mathbb{Z} + \mathbb{Z} \). We consider the affine Lagrangian submanifolds \( S_k \) of \( M \) corresponding to the lines \( y = kx \) in \( \mathbb{R}^2 \) for \( k = 0, 1, 2 \). Then

\[
\begin{align*}
S_0 \cap S_1 &= S_1 \cap S_2 = \{b_0\}, \\
S_0 \cap S_2 &= \{b_0, b_1\},
\end{align*}
\]

where \( b_0 \) and \( b_1 \) are the points corresponding to \( (0,0), (\frac{1}{2},0) \in \mathbb{R}^2 \) respectively.

![Diagram of Lagrangian submanifolds](image)

Then

\[
\begin{align*}
\text{HF}(S_0, S_1) &= \mathbb{C}b_0, \\
\text{HF}(S_1, S_2) &= \mathbb{C}b_0, \\
\text{HF}(S_0, S_2) &= \mathbb{C}b_0 \oplus \mathbb{C}b_1
\end{align*}
\]

and the product

\[
m_2 : \text{HF}(S_0, S_1) \otimes \text{HF}(S_1, S_2) \rightarrow \text{HF}(S_0, S_2)
\]
is defined by counting triangles bounded by \( S_0, S_1, S_2 \). More precisely,

\[
m_2(b_0 \otimes b_0) = C(b_0, b_0, b_0) \cdot b_0 + C(b_0, b_0, b_1) \cdot b_1
\]

with

\[
C(b_0, b_0, b_0) = \sum \exp(2\pi \sqrt{-1}(\text{area of triangle}))
\]

where the summation is taken over all triangles with vertices \( b_0, b_0, b_0 \), etc. Consequently we have

\[
m_2(b_0 \otimes b_0) = \vartheta \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] (2\tau, 0) \cdot b_0 + \vartheta \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] (2\tau, 0) \cdot b_1.
\]

On the other hand, \( S_k \) corresponds to a holomorphic line bundle on \( W \) of degree \( k \):

\[
E_0 = \mathcal{O}_W, \\
E_1 = E : \text{a principal polarization}, \\
E_2 = E^2.
\]

Then

\[
\begin{align*}
\text{Ext}(E_0, E_1) &= H^0(W, E) = \mathbb{C} \cdot \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, z), \\
\text{Ext}(E_1, E_2) &= H^0(W, E), \\
\text{Ext}(E_0, E_2) &= H^0(W, E^2) = \mathbb{C} \cdot \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, 2z) \oplus \mathbb{C} \cdot \vartheta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (2\tau, 2z)
\end{align*}
\]

In this case, the product

\[
H^0(E) \otimes H^0(E) \to H^0(E^2)
\]

is the natural product

\[
\vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, z) \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, z)
\]

\[
= \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, 0) \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (2\tau, 2z) + \vartheta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (2\tau, 0) \vartheta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (2\tau, 2z)
\]

(“the addition formula”).

**Remark 4.21.** For more general Calabi-Yau manifolds with (special) Lagrangian fibrations satisfying suitable conditions, and its “topological mirror” \( W \to B \), we can generalize some parts of the above discussion.

**Theorem 4.22 (Gross [12], Tyurin [41]).** Let \( M, W \) be a mirror pair of K3 surfaces and take an ample line bundle \( E \) on \( W \). Let \( S_0 \) and \( S \) be Lagrangian section in \( M \) corresponding to \( \mathcal{O}_W \) and \( E \) respectively. Then

\[
(-1)^n(n-1)/2 S_0 \cdot S = \chi(E).
\]

(11)
It is not known whether the Floer homology $HF(S_0, S)$ can be defined, and $\dim HF(S_0, S)$ coincides with the number of the intersection $S_0 \cap S$.

For higher dimensional case, \((\ref{11})\) holds asymptotically:

**Theorem 4.23 (Gross \[12\]).** Under some assumptions,

the leading term of \((-1)^{n(n-1)/2} S_0 \cdot S_k = \text{the leading term of } \chi(E^k)\).
5 Projective embeddings and Kähler metrics

By definition, every projective manifold $X$ can be embedded into projective spaces. A way to equip a Kähler metric on $X$ is to restrict the Fubini-Study metric on the projective space. On the other hand, some projective manifolds have standard Kähler metrics such as Ricci-flat metrics and Kähler-Einstein metrics. The restriction of the Fubini-Study metric is not a desired one in general. For example, the restriction of the Fubini-Study metric to cubic curves in $\mathbb{CP}^2$ is not flat. Kähler metrics obtained from the Fubini-Study metric is restricted.

A way to deal with more general Kähler metrics from the viewpoint of projective embeddings is to consider all tensor power of the polarization $L \to X$. We can think of this as an analogy of the approximation of smooth functions by polynomials. In this section, we recall two theories of approximations of Kähler metrics.

5.1 Bergman kernels and projective embeddings

Let $X$ be a smooth projective variety and $L \to X$ an ample line bundle. In this subsection, we fix a Kähler metric $\omega$ in the class $c_1(L)$ and a Hermitian metric on $L$ whose first Chern form coincides with $\omega$.

For each $k$, we take a basis $s_0, \ldots, s_{N_k}$ of the space $H^0(X, L^k)$ of holomorphic sections, and consider the embeddings defined by

$$t_k : X \hookrightarrow \mathbb{CP}^{N_k}, \quad z \mapsto (s_0(z) : \cdots : s_{N_k}(z)).$$

We denote the pull-back of the Fubini-Study metric by $\omega_k$:

$$\omega_k := \frac{1}{K} t_k^\ast \omega_{FS},$$

here we normalize the metric so that $\omega_k$ represents $c_1(L)$. Tian and Zelditch proved the following theorem:

**Theorem 5.1 (Tian [40], Zelditch [46]).** If $s_0, \ldots, s_{N_k}$ are orthonormal for each $k \gg 1$, then $\omega_k$ converges to the original metric $\omega$ as $k \to \infty$ in the $C^\infty$-topology. More precisely,

$$\|\omega - \omega_k\|_{C^r} = O(k^{-1})$$ (12)

for each $r$.

Namely, any Kähler metric representing $c_1(L)$ can be approximated by pull-back of Fubini-Study metrics on $\mathbb{CP}^{N_k} = \mathbb{P}H^0(X, L^k)$ for large $k$.

**Remark 5.2.** The case of abelian varieties is discussed by Ji [19] and Kempf [21].
This theorem follows from the asymptotic behavior of the Bergman kernels $\Pi_k(z, w)$. Recall that $\Pi_k(z, w)$ is given by

$$\Pi_k(z, w) = \sum_{i=0}^{N_k} s_i(z)s_i(w)^*$$

for orthonormal basis $s_0, \ldots, s_{N_k}$ of $H^0(X, L^k)$. Its diagonal part

$$f_k(z) = \Pi_k(z, z) = \sum_{i=0}^{N_k} |s_i(z)|^2$$

gives a distortion function, i.e.

$$\omega - \omega_k = \frac{\sqrt{-1}}{k} \partial \bar{\partial} \log f_k.$$

Theorem 5.1 follows from the following theorem.

**Theorem 5.3 (Zelditch [46]).** $f_k$ has the following asymptotic expansion:

$$f_k(z) = k^n + a_1k^{n-1} + a_2k^{n-2} + \ldots$$

with $a_i(z) \in C^\infty(X)$ and

$$\left\| f_k - \sum_{i=0}^{q} a_i k^{n-i} \right\|_{C^r} \leq C_{r,q}k^{n-q-1}$$

for some constant $C_{r,q} > 0$, here we put $a_0 = 1$.

### 5.2 Stability and Kähler metrics

The notion of stability is introduced by Mumford to construct moduli spaces. Moduli spaces are constructed as quotient spaces. In general, quotient spaces are not Hausdorff. To obtain a Hausdorff space, we consider moduli space of (semi-)stable objects.

On the other hand, stability is closely related to existences of differential geometric structures such as metrics and connections. The following theorem is a typical example, known as “Hitchin-Kobayashi correspondence”.

**Theorem 5.4 (Narasimhan-Seshadri [31], Donaldson [5, 6], Uhlenbeck-Yau [43]).** Let $(X, L)$ be a polarized manifold and $E \to X$ a holomorphic vector bundle. Then $E$ is stable in the sense of Mumford-Takemoto if and only if $E$ admits an irreducible Hermitian-Yang-Mills connection.

It is expected that such a theorem also holds for the case of manifolds.

**Conjecture 5.5 (Hitchin-Kobayashi correspondence for manifolds).** A smooth polarized projective variety $(X, L)$ admits a Kähler metric of constant scalar curvature in the class $c_1(L)$ if and only if it is stable in a suitable sense.
There are several notions of stability (these are closely related to each other but not equivalent) and it is not clear which stability is the right one at the moment.

Recently, Donaldson proved a part of the above conjecture. Here we review the result.

**Definition 5.6.** For a basis $s_0, \ldots, s_{N_k}$ of $H^0(X, L^k)$, put

$$M_{ij} = \sqrt{-1} \int_{\varphi(X)} Z^i \overline{Z}^j \sum_l Z_l^2 \omega_{FS}^n = \sqrt{-1} \int_X \sum_l |s_l|^2 (k \omega_k)_n$$

and $M = (M_{ij}) \in su(N_k + 1)$. The basis $s_0, \ldots, s_{N_k}$ is said to be balanced if $M$ is a constant multiple of the identity matrix.

Since $\omega_k$ changes as $s_0, \ldots, s_{N_k}$, it is not clear whether balanced basis exist or not. The existence of balanced basis is closely related to the stability of the polarized manifold $(X, L)$.

**Theorem 5.7 (Luo [27], Zhang [47]).** Assume that $\text{Aut}(X, L)/\mathbb{C}^*$ is discrete. If $H^0(L^k)$ has a balanced basis, then the Hilbert/Chow point of $X$ is stable.

**Theorem 5.8 (Donaldson [7]).** Assume that $X$ admits a Kähler metric $\omega_\infty$ in the class $c_1(L)$ of constant scalar curvature. Furthermore, we assume that the automorphism group $\text{Aut}(X, L)$ of the polarized variety $(X, L)$ is discrete. Then, for each $k \gg 1$, we can take a balanced embedding. In particular, $(X, L)$ is stable. Furthermore, the pull-backs of the Fubini-Study metrics $\omega_k = \frac{1}{k} \omega_{FS}$ converge to the constant scalar curvature metric $\omega_\infty$. 

34
6 Abelian varieties and theta functions

In section 3 and 4, we saw some relations between theta functions and Lagrangian fibrations on abelian varieties from the point of view of geometric quantization and mirror symmetry. It is natural to ask what happens if we embed the abelian varieties into projective spaces by these theta functions. In this section, we prove that the Lagrangian fibration of the abelian variety can be approximated by the restrictions of the moment maps of projective spaces in a certain sense.

6.1 Statement of the main theorem

We consider the same situation as in section 3.5. Let $X = \mathbb{C}^n/(\Omega \mathbb{Z}^n + \mathbb{Z}^n)$ be an abelian variety of complex dimension $n$, where $\Omega$ is an $n \times n$ complex symmetric matrix whose imaginary part $\text{Im} \, \Omega$ is positive definite, and take an ample symmetric line bundle $L$ on $X$ of degree 1 defined by

\[ L = (\mathbb{C}^n \times \mathbb{C})/(\Omega \mathbb{Z}^n + \mathbb{Z}^n), \]

\[ (z, \zeta) \sim (z + \lambda, e^{\pi^2 \lambda (\text{Im} \, \Omega)^{-1} z + \frac{1}{2} \lambda (\text{Im} \, \Omega)^{-1} \lambda \zeta}, \lambda \in \Omega \mathbb{Z}^n + \mathbb{Z}^n) \]

We fix a Hermitian metric

\[ h_0 = \exp(-\pi^2 z (\text{Im} \, \Omega)^{-1} \bar{z}) \]

on $L$ and a flat metric

\[ \omega_0 = c_1(L, h_0) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_0 = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j. \]

on $X$, where we denote $(h_{ij}) = (\text{Im} \, \Omega)^{-1}$.

We consider the following Lagrangian torus fibration as before:

\[ \pi : X = X^+ \times X^- \rightarrow X^- = T^n, \quad z = \Omega x + y \mapsto y. \]

For each $k$, we denote $\{s_1, \ldots, s_{kn}\}$ the basis of $H^0(X, L^k)$ defined in (6) and consider the embedding

\[ t_k : X \rightarrow \mathbb{CP}^{N_k}, \quad x \mapsto (s_0(x) : \cdots : s_{N_k}(x)). \]

Write $\omega_k = \frac{1}{2} t_k^* \omega_{FS}$.

We compare the Lagrangian fibration $\pi : X \rightarrow X^-$ and the restriction of a natural Lagrangian fibration (the moment map of the natural torus action) on projective spaces.

Let $T^{k-1} \subset SU(k^n)$ be the maximal torus which consists of diagonal matrices and consider its action on $\mathbb{CP}^{k^n-1}$ given by

\[ (Z_1 : \cdots : Z^{k^n}) \mapsto (\lambda_1 Z^1 : \cdots : \lambda_{kn} Z^{k^n}), \quad \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_{kn} \end{pmatrix} \in T^{k-1}. \]

35
Then this action is Hamiltonian and its moment map is given by

\[ \mu_{T^k_{n-1}}(Z^1 : \ldots : Z^k) = \frac{1}{\sum |Z_i|^2} \left( |Z_1|^2, \ldots, |Z^k|^2 \right), \]

here we identify the dual of the Lie algebra of \( T^k_{n-1} \) with

\[ \{ (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \mid \sum \xi_i = 1 \}. \]

Denote the image of \( \iota_k(X) \subset \mathbb{C}P^X \) under \( \mu_{T^k_{n-1}} \) by \( B_k \) and consider the following map

\[ \pi_k := \mu_{T^k_{n-1}} \circ \iota_k : X \to B_k. \]

Note that \( \pi_k \) has many degenerate fibers. In fact the most degenerate one is 0-dimensional. We also remark that each fiber is isotropic with respect to \( \omega_k \) (at smooth points).

Next we introduce a distance on \( B_k \). For that purpose, we define a distance on the polytope \( \Delta_k = \mu_{T^k_{n-1}}(\mathbb{C}P^{k-1}) \). Note that \( \mu_{T^k_{n-1}} \) has no degenerate fiber on the interior \( \Delta_k \) of \( \Delta_k \). We take a metric on \( \Delta_k \) so that the moment map \( \mu_{T^k_{n-1}} : (\mathbb{C}P^{k-1}, \omega_{FS}) \to \Delta_k \) is a Riemannian submersion on the interior. This is equivalent to the following definition. Consider the restriction \( \mu_{T^k_{n-1}} : \mathbb{R}P^{k-1} \to \Delta_k \) of the moment map to \( \mathbb{R}P^{k-1} \subset \mathbb{C}P^{k-1} \). This is a \( 2^{k-1} \)-sheeted covering which branches on the boundary of \( \Delta_k \). By identifying \( \Delta_k \) with a sheet of \( \mathbb{R}P^{k-1} \), we have the restriction of the Fubini-Study metric on \( \Delta_k \). We define a distance on \( B_k \) induced by the normalized metric on \( \Delta_k \) with normalized factor \( \frac{1}{k} \).

On the other hand, We equip the metric on \( X^- \) so that \( \pi : (X, \omega) \to X^- \) is a Riemannian submersion.

Under the above preparation, we can state the main theorem

**Theorem 6.1.**

1. \( \omega_k \) converges to \( \omega_0 \) in \( C^\infty \). In particular, the sequence \( (X, \omega_k) \) of compact Riemannian manifolds converges to \( (X, \omega_0) \) in Gromov-Hausdorff topology.

2. \( B_k \) converge to \( X^- \) in Gromov-Hausdorff topology.

3. The sequence of maps \( \pi_k : X \to B_k \) between metric spaces converge to \( \pi : X \to X^- \).

Before the proof of this theorem, we recall the definition of Gromov-Hausdorff distances. If \( Z \) is a metric space and \( X, Y \subset Z \), then the Hausdorff distance of \( X \) and \( Y \) is defined by

\[ d^H_H(X, Y) = \inf \{ \varepsilon > 0 \mid X \subset B(Y, \varepsilon) \text{ and } Y \subset B(X, \varepsilon) \}, \]

where we denote \( B(X, \varepsilon) \) the \( \varepsilon \)-neighborhood of \( X \) in \( Z \).

For metric spaces \( X \) and \( Y \), the Gromov-Hausdorff distance is defined by

\[ d_{GH}(X, Y) = \inf \{ d^H_H(X, Y) \mid X, Y \leftrightarrow Z \text{ are isometric embeddings} \}. \]
This is equivalent to the following definition:

\[ d_{GH}(X, Y) = \inf \{ d_{h}^{X \coprod Y}(X, Y) \} , \]

where the infimum is taken over all metrics on \( X \coprod Y \) compatible with these on \( X \) and \( Y \).

Next we define convergence of maps. Let \( X_i, Y_i, X, Y \) be metric spaces and consider the maps \( f_i : X_i \to Y_i, f : X \to Y \). Suppose that \( X_i \) and \( Y_i \) converge to \( X \) and \( Y \) respectively in Gromov-Hausdorff topology. Then, from the definition of Gromov-Hausdorff distance, there exist metrics on \( X \coprod (\coprod_i X_i) \) such that \( X_i \) converge to \( X \) in Hausdorff topology in \( X \coprod (\coprod_i X_i) \) (and the same is true for \( Y_i \) and \( Y \)). In this case, we say that \( \{ f_i \} \) converges to \( f \) if for every sequence \( x_i \in X_i \) converging to \( x \in X \), \( f_i(x_i) \) converges to \( f(x) \) in \( Y \coprod (\coprod_i Y_i) \).

6.2 Proof of Theorem 6.1

Let \( \{ s_1, \ldots, s_{k^n} \} \) be the basis of \( H^0(X, L^k) \) as above.

**Proposition 6.2.** The basis \( \{ s_1, \ldots, s_{k^n} \} \) of \( H^0(X, L^k) \) are both balanced and orthonormal with respect to \( h_0 \) and \( \omega_0 \).

**Proof.** Recall that the \( G_k \)-action on \( H^0(X, L^k) \) preserves the \( L^2 \)-inner product defined by \( \omega_0 \) and \( h_0 \). From the formula (7),

\[ \| s_{b_i} \|_{L^2}^2 = \| \rho_k(-b_i)s_{b_i} \|_{L^2}^2 = \| s_{b_i} \|_{L^2}^2, \]

i.e. all \( s_i \)'s have the same \( L^2 \)-norm, here we identify \( b_i \) with \((1, 0, b_i) \in G_k \). Furthermore, we have

\[ e^{2\pi \sqrt{-1} a b_i} (s_{b_i}, s_{b_j})_{L^2} = (\rho_k(a)s_{b_i}, s_{b_j})_{L^2} = (s_{b_i}, \rho_k(-a)s_{b_j})_{L^2} = e^{2\pi \sqrt{-1} a b_j} (s_{b_i}, s_{b_j})_{L^2} \]

for all \( a \in X^+_k \). This implies that \( (s_{b_i}, s_{b_j})_{L^2} = 0 \) if \( i \neq j \). In other words, \( s_1, \ldots, s_{k^n} \) are orthonormal basis up to a constant.

To show that each \( s_i \) has unit norm, we consider the following function

\[ f_k(z) := \sum_{i=1}^{k^n} |s_i(z)|_{h_0}^2. \]
From the definition of $h_0$ and $s_i$,

\[ |s_i(z)|^2_{h_0} = \exp(-k\pi^2z(\text{Im}(\Omega))^{-1}z) \cdot C_i^2k^{-\frac{7}{2}} \exp \left( \frac{\pi k_i z}{2}(\text{Im}(\Omega))^{-1}z \right) \cdot |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2 \]

\[ = C_i^2k^{-\frac{7}{2}} \exp \left( \frac{\pi k_i (z - \bar{z})}{2}(\text{Im}(\Omega))^{-1}(z - \bar{z}) \right) |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2 \]

\[ = C_i^2k^{-\frac{7}{2}} \exp \left( -2\pi k^i(\text{Im}(\Omega))^{-1}(\text{Im}z) \right) |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2. \]

By using $z = \Omega x + y$, we have

\[ |s_i(z)|^2_{h_0} = C_i^2k^{-\frac{7}{2}} \exp \left( -2\pi k^i x(\text{Im}(\Omega))x \right) |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2 \]

\[ = C_i^2k^{-\frac{7}{2}} e \left( \frac{k^i}{2} \Omega x - \frac{k^i}{2}(\bar{\Omega}x) \right) |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2. \]  

(14)

From the definition of theta functions,

\[ |\vartheta \left[ \begin{array}{c} 0 \\ -b_i \end{array} \right] (k^{-1}\Omega, z)|^2 \]

\[ = \sum_{l,m \in \mathbb{Z}^n} e \left( \frac{1}{2k} \Omega l + \frac{1}{2k}(\bar{\Omega}x - b_i) \right) \cdot e \left( -\frac{1}{2k}m\Omega m - \frac{1}{2k}(\bar{\Omega}x - b_i) \right) \]

\[ = \sum_{l,m \in \mathbb{Z}^n} e \left( \frac{1}{2k} \Omega l + \frac{1}{2k}m\Omega m - \frac{1}{2k}(\bar{\Omega}x - b_i) \right). \]

Therefore we have

\[ |s_i(z)|^2_{h_0} \]

\[ = C_i^2k^{-\frac{7}{2}} \sum_{l,m \in \mathbb{Z}^n} e \left( \frac{k^i}{2} \Omega x - \frac{k^i}{2}(\bar{\Omega}x) + \frac{1}{2k} \Omega l + \frac{1}{2k}m\Omega m \right) \]

\[ + \frac{1}{2k}(\bar{\Omega}x - b_i) \]

\[ = C_i^2k^{-\frac{7}{2}} \sum_{l,m \in \mathbb{Z}^n} e \left( \frac{k^i}{2} \Omega x + \frac{l}{k} \right) \Omega \left( x + \frac{l}{k} \right) - \frac{k^i}{2} \left( x + \frac{m}{k} \right) \Omega \left( x + \frac{m}{k} \right) \]

\[ + \frac{1}{2k}(\bar{\Omega}x - b_i) \]

\[ = C_i^2k^{-\frac{7}{2}} \sum_{l,m \in \mathbb{Z}^n} e \left( \frac{k^i}{2} \right) \Omega \left( x + \frac{l}{k} \right) - \frac{k^i}{2} \left( x + \frac{m}{k} \right) \Omega \left( x + \frac{m}{k} \right) \]

\[ + \frac{1}{2k}(\bar{\Omega}x - b_i) \cdot e \left( \frac{1}{2}(l-m)y \right). \]
From the fact that \( \{b_i\} = \frac{1}{k} \mathbb{Z}^n / \mathbb{Z}^n \), we have the following identity:

\[
\sum_{i=1}^{k^n} e(i l b_i) = \begin{cases} 
  k^n, & l \in k \mathbb{Z}^n, \\
  0, & \text{otherwise}.
\end{cases}
\]

By using this, we obtain

\[
f_k(z) = C_{\Omega}^2 k^\frac{n}{2} \sum_{l, m \in \mathbb{Z}^n, l - m \in k \mathbb{Z}^n} e \left( \frac{k}{2} \left( \frac{t}{k} \left( x + \frac{l}{k} \right) \Omega \left( x + \frac{l}{k} \right) \right. \right.
\]

\[
- \left. \left( x + \frac{m}{k} \right) \tilde{\Omega} \left( x + \frac{m}{k} \right) \right) + t(l - m)y \bigg) \right) dx
dx dy
\]

\[
= C_{\Omega}^2 k^\frac{n}{2} \sum_{l \in \mathbb{Z}^n} \int_{X^+} e \left( \frac{k}{2} \left( \frac{t}{k} \left( x + \frac{l}{k} \right) \Omega \left( x + \frac{l}{k} \right) \right. \right.
\]

\[
- \left. \left( x + \frac{l}{k} \right) \tilde{\Omega} \left( x + \frac{l}{k} \right) \right) \right) dx
\]

\[
= C_{\Omega}^2 k^\frac{n}{2} \sum_{l \in \mathbb{Z}^n} \int_{X^+} \exp \left( -2\pi k \left( x + \frac{l}{k} \right) \left( \text{Im} \Omega \left( x + \frac{l}{k} \right) \right) \right) dx
\]

\[
= C_{\Omega}^2 k^\frac{n}{2} \sum_{l \in \mathbb{Z}^n} \int_{X^+} \exp \left( -2\pi k \left( x + l + m \right) \left( \text{Im} \Omega \left( x + l + m \right) \right) \right) dx
\]
\[
\begin{align*}
&= C_\Omega^2 k^{\frac{n}{2}} \sum_{m \in \mathbb{Z}^n / \mathbb{Z}^n} \int_{\mathbb{R}^n} \exp \left( -2\pi k^i (x + m)(\text{Im}\Omega)(x + m) \right) dx \\
&= C_\Omega^2 k^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp \left( -2\pi k^i x(\text{Im}\Omega)x \right) dx \\
&= C_\Omega^2 k^n \sqrt{\det(\text{Im}\Omega)^{-1}} \cdot 2^{-\frac{n}{2}} = k^n.
\end{align*}
\]

Therefore, \( C = \|s_i\|_{L^2(h_0)}^2 = 1 \).

Next we prove that these basis satisfies the balanced condition. Let \( h_k \) be the pull back of the standard Hermitian metric (the Fubini-Study metric) on the hyperplane bundle \( \mathcal{O}_{\mathbb{C}P^{kn-1}} \) i.e. \( h_k = (\sum_i |s_i|^2)^{-1} \). Then \( \omega_k = -\frac{i}{2\pi} \partial \bar{\partial} \log h_k \). From (7) we have

\[
\begin{align*}

h_k &= \left( \frac{1}{|\mu_k| |X_k^+|} \sum_{g \in G_k} |\rho_k(g)s_1|^2 \right)^{-1},
\end{align*}
\]

where \( \rho_k(g) : H^0(X, L^k) \to H^0(X, L^k) \) is the Heisenberg representation of \( G_k \). This means that \( h_k \) and \( \omega_k \) are invariant under the \( G_k \)-action. Hence this action on \( H^0(X, L^k) \) is also unitary with respect to \( h_k \) and \( \omega_k \). We can prove that \( s_1, \ldots, s_{kn} \) are balanced by the same argument as above. \( \square \)

The next lemma is a key of the proof of Theorem 6.1.

**Lemma 6.3.** There exists a constant \( C > 0 \) independent of \( k \) such that for each \( z = \Omega x + y \in X \),

\[
C^{-1} k^{\frac{n}{2}} e^{-kC\text{dist}(y, b_i)^2} \leq |s_i(z)|_{h_0}^2 \leq C k^{\frac{n}{2}} e^{-kC\text{dist}(y, b_i)^2},
\]

for some distance \( \text{dist} \) on \( X^+ \).

**Proof.**

For \( z = \Omega x + y \in X \), we write \( w = \Omega^{-1}(z - b_i) = x + \Omega^{-1}(y - b_i) \). Then

\[
\begin{align*}
q \begin{bmatrix} 0 \\ -b_i \end{bmatrix} (k^{-1}\Omega, z) &= \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2k} l^i \Omega l + l^i (z - b_i) \right) \\
&= \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2k} l^i \Omega l + l^i (\Omega x + y - b_i) \right) \\
&= \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2k} \left( \frac{l}{\sqrt{k}} + \sqrt{k}w \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k}w \right) \right) \\
&\quad \cdot e \left( -\frac{1}{2k} (x + \Omega^{-1}(y - b_i)) \Omega (x + \Omega^{-1}(y - b_i)) \right).
\end{align*}
\]
From (14), we have

\[ |s_i(z)|^2_{\Omega_0} = C_1^2 k^{-\frac{2}{n}} e^{\left(\frac{k_i}{2} x(\Omega - \Omega x) \right)} \]

\[ \cdot e^{\left(-\frac{k_i}{2} (x + \Omega^{-1}(y - b_1))\Omega(x + \Omega^{-1}(y - b_1)) \right)} \]

\[ \cdot \left| \sum_{l \in \mathbb{Z}^n} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \right)} \right|^2 \]

\[ = C_1^2 k^{-\frac{2}{n}} \exp(2\pi k^t(y - b_1)(\Im\Omega^{-1})(y - b_1)) \]

\[ \cdot \left| \sum_{l \in \mathbb{Z}^n} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \right)} \right|^2. \]

Since

\[ \frac{1}{\sqrt{k}} \sum_{l \in \mathbb{Z}^n} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \right)} = \sum_{l \in \mathbb{Z}^n} \frac{1}{\sqrt{k}} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} w \right) \right)} \]

converges to \( \int_{\mathbb{R}^n} e^{\left(\frac{1}{2} \Omega u^2 \right)} du \) uniformly on a fundamental domain of \( \Omega \mathbb{Z}^n + \mathbb{Z}^n \) as \( k \to \infty \), this part can be bounded uniformly from above and below. Furthermore \( \exp(2\pi k^t(y - b_1)(\Im\Omega^{-1})(y - b_1)) \) is of the form \( e^{-kC \text{dist}(y, b_1)^2} \), since \( \Im\Omega^{-1} \) is negative definite. Hence we have the desired estimate. \[ \square \]

**Proof of Theorem 6.1 (1).**

The convergence of Kähler forms \( \omega_k \) follows from Proposition 6.2 and Theorem 5.1 or 5.8.

**Lemma 6.4.** Let \( d_0 \) and \( d_k \) denote the distance on \( X \) defined by \( \omega_0 \) and \( \omega_k \) respectively. Then there exists a constant \( C \) independent of \( k \) such that

\[ \left(1 - \frac{C}{k}\right) d_0(p, q) \leq d_k(p, q) \leq \left(1 + \frac{C}{k}\right) d_0(p, q) \]

holds for each \( p, q \in X \).

**Proof.** Set \( x^{n+i} = y^i \) for \( i = 1, \ldots, n \). From the estimate (12) by Zelditch, the equation of geodesics with respect to \( \omega_k \) is

\[ \tilde{z}^\alpha + \sum_{\beta, \gamma} \Gamma_{\beta\gamma}^\alpha \tilde{z}^\beta \tilde{z}^\gamma = 0 \text{ with } |\Gamma_{\beta\gamma}^\alpha| = O(k^{-1}). \]
Let $\gamma_k(t) = \xi_0 t + \eta(t)$, $0 \leq t \leq 1$ be a geodesic with respect to $\omega_k$, where $\xi_0$ is a constant vector and $\eta(0) = \dot{\eta}(0) = 0$. Note that $\Gamma_0(t) = \xi_0 t$ is a geodesic with respect to $\omega_0$. Then $\eta$ satisfies

$$\ddot{\eta} + \Gamma(\dot{\xi_0} + \dot{\eta}, \xi_0 + \dot{\eta}) = 0,$$

where $\Gamma(\dot{x}, \dot{x}) = \left( \sum_{\beta, \gamma} \Gamma^\alpha_{\beta \gamma} \dot{x}^\beta \dot{x}^\gamma \right)$. Hence we have

$$\frac{d}{dt}|\xi_0 + \dot{\eta}|^2 = 2(\ddot{\eta}, \xi_0 + \dot{\eta}) = -2(\Gamma(\dot{\xi_0} + \dot{\eta}, \xi_0 + \dot{\eta}), \xi_0 + \dot{\eta}).$$

From the estimate of $\Gamma$,

$$\left| \frac{d}{dt}|\xi_0 + \dot{\eta}|^2 \right| \leq \frac{C}{k} |\xi_0 + \dot{\eta}|^3$$

for some constant $C > 0$. Hereafter we denote constants independent of $k$ be the same $C$. Therefore

$$\left| \frac{1}{|\xi_0 + \dot{\eta}|} \right| \leq \frac{Ct}{k}.$$

From this, we have a uniform bound $|\xi_0 + \dot{\eta}| \leq C'$ for large $k$. Hence we have

$$|\dot{\eta}| = |\Gamma(\dot{\xi_0} + \dot{\eta}, \xi_0 + \dot{\eta})| \leq \frac{C}{k}.$$

Since $\dot{\eta}(0) = 0$, we have $|\dot{\eta}| = O \left( \frac{1}{k} \right)$. In particular

$$|\dot{\gamma}_k|_{\omega_k} = |\xi_0| + O \left( \frac{1}{k} \right).$$

Therefore the length of $\gamma_k$ is given by

$$\int_0^1 |\gamma_k|_{\omega_k} dt = |\xi_0| + O \left( \frac{1}{k} \right) \int_0^1 |\gamma_0|_{\omega_0} dt + O \left( \frac{1}{k} \right).$$

Lemma follows from this estimate. $\square$

Let $\{x_{ij} = \Omega a_i + b_j\}_{i,j=1,2,\ldots,k^n} = X_k$ with the distance induced by $d_0$ and denote the same set with distance $d_k$ by $\{y_{ij}\}$. Then

$$d_{GH}(X, \omega_0), \{x_{ij}\}) = O(k^{-1}),$$

$$d_{GH}(X, \omega_k), \{y_{ij}\}) = O(k^{-1}).$$

(16)
Define a distance on \( \{x_{ij}\} \cup \{y_{ij}\} \) by
\[
d^{(k)}(x_{ij}, y_{ij}) = \frac{C}{k},
\]
\[
d^{(k)}(x_{ij}, y_{hl}) = \min_{p,q} \left\{ d_0(x_{ij}, x_{pq}) + \frac{C}{k} + d_k(y_{pq}, y_{hl}) \right\},
\]
where \( C \) is the constant in Lemma 6.4. Then
\[
d_{GH}(\{x_{ij}\}, \{y_{ij}\}) \leq \frac{C}{k}.
\]
By combining this with (16), we have
\[
d_{GH}(X, \omega_k) = O(k^{-1}). \tag{17}
\]
This prove (1) of the theorem. \(\square\)

**Proof of Theorem 6.1 (2).**

To prove the Gromov-Hausdorff convergence, it suffices to construct \( \varepsilon \)-Hausdorff approximations \( \varphi_k : X^- \to B_k \) for large \( k \) (see [10]).

**Definition 6.5.** Let \((X,d_X), (Y,d_Y)\) be two compact metric spaces. A map \( \varphi : X \to Y \) is said to be an \( \varepsilon \)-Hausdorff approximation if the following two conditions are satisfied.

1. The \( \varepsilon \)-neighborhood of \( \varphi(X) \) coincides with \( Y \).
2. For each \( x, y \in X \),
   \[
   |d_X(x, y) - d_Y(\varphi(x), \varphi(y))| < \varepsilon.
   \]

We identify \( X^- \) with a section \( \{0\} \times X^- \subset X \) and put
\[
\varphi_k := \pi_k|_{X^-} : X^- \to B_k.
\]
We prove that \( \varphi_k \) is a \( C/\sqrt{k} \)-Hausdorff approximation.

**Lemma 6.6.** There exist a constant \( C > 0 \) such that the \( C/k \)-neighborhood of \( \varphi_k(X^-) \) is equal to \( B_k \).

**Proof.** Take any point \( \pi_k(z) \in B_k \) and put \( y = \pi(z) \in X^- \). Since \( \pi_k : (X, \omega_k) \to B_k \) is invariant under the action of \( X_k^+ \), there exists \( z' \in \pi^{-1}(y) \) such that \( \pi_k(y) = \pi_k(z') \) and \( d_k(z, z') \leq C/K \). Then
\[
d_{B_k}(\pi_k(z), \varphi_k(y)) = d_{B_k}(\pi_k(z), \pi_k(z')) \leq d_k(z, z') \leq C/K.
\]
\(\square\)

\(^2\)The proof in [32] is not correct.
For each $\xi \in T_p\mathbb{CP}^N_k$, we denote its vertical and horizontal parts by

$$T_p\mathbb{CP}^N_k = T_{\mathbb{CP}^N_k/\Delta_k, p} \uplus (T_{\mathbb{CP}^N_k/\Delta_k, p})^\perp$$

where $T_{\mathbb{CP}^N_k/\Delta_k, p}$ is the tangent space of the fiber of $\mu_k : \mathbb{CP}^N_k \to \Delta_k$ and $(T_{\mathbb{CP}^N_k/\Delta_k, p})^\perp$ is its orthogonal complement with respect to the Fubini-Study metric. Let $(Z^0 : \cdots : Z^N_k)$ be the homogeneous coordinate and write

$$\log \frac{Z^i}{Z^0} = u^i + \sqrt{-1}v^i.$$

Then $T_{\mathbb{CP}^N_k/\Delta_k}$ and $(T_{\mathbb{CP}^N_k/\Delta_k})^\perp$ are spanned by $\frac{\partial}{\partial u^i}$'s and $\frac{\partial}{\partial v^i}$'s respectively.

Let $\gamma : [0, l] \to B_k$ be a curve and take a lift $\tilde{\gamma} : [0, l] \to X$ of $\gamma$. Then the length of $\gamma$ is given by

$$\int_0^l \left| \frac{d}{dt} \tilde{\gamma} \right|^{\omega_k} dt.$$

**Lemma 6.7.**

$$s_j(z) = Ck^{\frac{k}{2}} \exp \left( \frac{\pi k}{2} z(\text{Im } \Omega)^{-1} z \right) \exp \left( \frac{\pi k}{\sqrt{-1}} (z - b_j) \Omega^{-1} (z - b_j) \right)$$

$$\times (1 + \phi)$$

with

$$|\phi| = O \left( \frac{1}{\sqrt{k}} \right), \quad |d\phi| = O(1).$$

**Proof.** From the definition of the theta function, we have

$$\frac{\partial}{\partial \left[ \begin{array}{c} 0 \\ -b \end{array} \right]} (k^{-1} \Omega, z)$$

$$= \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2} \Omega l + l(z + b) \right)$$

$$= \sum_{l \in \mathbb{Z}^n} e \left( \frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \right)$$

$$\times e \left( -\frac{k_l}{2} (z - b) \Omega^{-1} (z - b) \right)$$
and
\[
\sum_{l \in \mathbb{Z}^n} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \right)} = k^{\frac{\mathbb{Z}^n}{2}} \sum_{l \in \mathbb{Z}^n} \exp (\pi \sqrt{-1} t \Omega l) \frac{1}{\sqrt{k^n}}
\]
\[
= k^{\frac{\mathbb{Z}^n}{2}} \left( \int_{\mathbb{R}^n} \exp (\pi \sqrt{-1} t \Omega l) \, dl + O \left( \frac{1}{\sqrt{k}} \right) \right)
\]
\[
= Ck^{\frac{\mathbb{Z}^n}{2}} \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) \right).
\]
Similarly we have
\[
\left| \sum_{l \in \mathbb{Z}^n} e^{\left(\frac{1}{2} \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \Omega \left( \frac{l}{\sqrt{k}} + \sqrt{k} \Omega^{-1} (z - b) \right) \right)} \right| = O \left( k^{\frac{\mathbb{Z}^n}{2}} \right).
\]
Lemma 6.7 follows from this. \[\square\]

From this Lemma we have
\[
\frac{Z^l}{Z^0} = \frac{s_j(z)}{s_0(z)}
\]
\[
= C_j \exp \left( 2\pi k \sqrt{-1} (b_j - b_0) \Omega^{-1} z \right) + O \left( \frac{1}{\sqrt{k}} \right)
\]
\[
= C_j \exp \left( 2\pi k \sqrt{-1} (b_j - b_0)(x + \text{Re} (\Omega^{-1})y) - 2\pi k^l (b_j - b_0) \text{Im} (\Omega^{-1})y \right)
\]
\[
+ O \left( \frac{1}{\sqrt{k}} \right)
\]
(18)

for some constant \(C_j\).

Let \(\gamma : [0, l] \to X^-\) be a curve and take a horizontal lift \(\tilde{\gamma} : [0, l] \to X\), i.e.
\[
\frac{d}{dt} \tilde{\gamma} \in (T_{X/X^-})^\perp.\]

Then the length of \(\gamma\) is given by
\[
\int_0^l \left| \frac{d}{dt} \tilde{\gamma} \right|_\omega \, dt.
\]

Note that \((T_{X/X^-})^\perp\) is spanned by \(J \frac{\partial}{\partial z}^i\)'s, where \(J\) is the complex structure on \(X\). (Recall that \(\frac{\partial}{\partial z}^i\) tangents to fibers.) By direct computation, we have
\[
\left( J \frac{\partial}{\partial x^1}, \ldots, J \frac{\partial}{\partial x^n} \right) = \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) \left( -\text{Re} (\Omega^{-1})(\text{Im} \Omega^{-1})^{-1} \right)
\]
\[
+ \left( \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right) (\text{Im} \Omega^{-1})^{-1}
\]

45
and
\[
\frac{\partial}{\partial x^i} \left( t^i (b_j - b_0) \text{Im} \left( \Omega^{-1} y \right) \right) = 0, \\
J \frac{\partial}{\partial x^i} \left( t^i (b_j - b_0) (x + \text{Re} \left( \Omega^{-1} y \right)) \right) = 0.
\]

From this and (15), we have
\[
\left| \frac{d}{dt} \tilde{\gamma} - \left( \frac{d}{dt} \tilde{\gamma} \right)^H \right| = C \frac{d}{dt} \tilde{\gamma}.
\]

This implies that
\[
\left| (\text{length of } \gamma) - (\text{length of } \varphi_k(\gamma)) \right| \leq O \left( \frac{1}{\sqrt{k}} \right).
\]

From this, we obtain

**Lemma 6.8.** For \( p, q \in X^- \),
\[
\left| d^{X^-}(p, q) - d^B_k(\varphi_k(p), \varphi_k(q)) \right| \leq O \left( \frac{1}{\sqrt{k}} \right).
\]

Lemma 6.6 and 6.8 prove the second statement of Theorem 6.1. \( \Box \)

**Proof of Theorem 6.1 (3).**
From the second statement of Theorem 6.1 there is a distance on \( X^- \bigsqcup B_k \) compatible with those on \( X^- \) and \( B_k \) such that
\[
d^{X^-}(y, \varphi_k(y)) \leq \frac{C}{\sqrt{k}}
\]
for all \( y \in X^- \).

It suffices to show that \( \pi_k(p) \) converges to \( \pi(p) \) for each \( p \in X \). For \( p \in X \), we denote the corresponding point in \( X/X_k^+ \) by \( \bar{p} \). We regard that \( \pi(p) \in B \subset X/X_k^+ \). Then the distance between \( \pi(p) \) and \( \bar{p} \) with respect to \( \omega_k \) is
\[
d^{X/X_k^+}(\pi(p), \bar{p}) \leq O \left( \frac{1}{k} \right).
\]

Hence
\[
d^B_k(\varphi_k(\pi(p)), \pi_k(p)) \leq d^{X/X_k^+}(\pi(p), \bar{p}) \leq O \left( \frac{1}{k} \right)
\]
and we have
\[
d^{X^-} \bigsqcup B_k(\pi(p), \pi_k(p)) \\
\leq d^{X^-} \bigsqcup B_k(\pi(p), \varphi_k(\pi(p))) + d^B_k(\varphi_k(\pi(p)), \pi_k(p)) \\
\leq O \left( \frac{1}{\sqrt{k}} \right).
\]
\( \Box \)
7 Lagrangian fibrations and holomorphic sections

In the previous section, we proved that theta functions can be reconstructed from a Lagrangian fibration by using the Bergman kernels. This construction can be applied to more general settings. In this section, we study the asymptotic behavior of the holomorphic sections constructed in this way by using the asymptotic behavior of the Bergman kernels ([3], [26], [2]).

7.1 Asymptotic behavior of the Bergman kernels

The asymptotic behavior of the Bergman kernel \( \Pi_k(z, w) \) is studied in detail.

**Theorem 7.1** (Lindholm [26], Berndtsson [2]).

\[
|\Pi_k(z, w)| \leq C k^n e^{-c \sqrt{k} d(z, w)}
\]

for some constants \( C, c > 0 \), where \( d(z, w) \) is the distance of \( z, w \in X \).

The asymptotic behavior of \( \Pi_k(z, w) \) near the diagonal can be written as follows. Fix a point \( z_0 \in X \) and take a neighborhood \( U \) of \( z_0 \) with local holomorphic coordinate \( (z^1, \ldots, z^n) \) around \( z_0 \). We write \( \omega = \sqrt{-1} \sum g_{ij} dz^i \wedge \bar{dz}^j \) and set \( G = (g_{ij}) \). Then we can take a holomorphic local frame \( e_L \) of \( L \) satisfying

\[
\frac{\partial h}{\partial z^i} = \frac{\partial^2 h}{\partial z^i \partial \bar{z}^j} = 0 \text{ at } z_0
\]

for each \( i, j = 1, \ldots, n \). Here we put \( h(z) = h(e_L(z), e_L(z)) \). In other words,

\[
h(z) = \exp \left( - \sum_{i,j} g_{ij} z^i \bar{z}^j + O(|z|^3) \right)
\]

\[
= \exp \left( -t zG \bar{z} + O(|z|^3) \right).
\]

From now on, we fix a trivialization of \( L \) over \( U \) defined by

\[
L|_U \to U \times \mathbb{C}, \quad \zeta e_L(z) \mapsto (z, h(z) \zeta)
\]

and express sections of \( L^k \) with respect to this trivialization. Note that this trivialization is not holomorphic but unitary. Under this setting, \( \Pi_k(z, w) \) can be written down as follows:

**Theorem 7.2** (Bleher-Shiffman-Zelditch [3]).

\[
\Pi_k \left( z_0 + \frac{u}{\sqrt{k}} \right) \left( z_0 + \frac{v}{\sqrt{k}} \right) = \left( \frac{k}{2\pi} \right)^n e^{-\frac{1}{4} u \bar{G} \bar{u} - \frac{1}{4} v \bar{G} \bar{v} + \frac{1}{4} u G \bar{v}} \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) \right)
\]

\[
= \left( \frac{k}{2\pi} \right)^n e^{-\frac{1}{4} (u-v)G(u-v) + \frac{1}{4} Im(vG \bar{v})} \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) \right).
\]

**Remark 7.3.** This expression is slightly different from that in [3]. This is caused by the difference of the normalizations of Kähler metrics (in [3], \( \pi_{c_1}(L, h) = \frac{1}{2} \omega \) is used instead of \( \omega = 2\pi c_1(L, h) \)).
7.2 Asymptotic behavior of holomorphic sections

Let \((L,h) \to (X,\omega)\) be a holomorphic Hermitian line bundle on a compact Kähler manifold such that \(\frac{1}{2\pi} c_1(L,h) = \omega\).

Suppose that we have a Lagrangian fibration \(\pi : (X,\omega) \to B\). We take \(k\)-Bohr-Sommerfeld fibers \(\pi^{-1}(b_0), \ldots, \pi^{-1}(b_N) \subset X\). Here we consider only the part away from singular fiber. Thus we assume that \(\pi^{-1}(b_0), \ldots, \pi^{-1}(b_N)\) are smooth (hence \(\pi^{-1}(b_i) \cong T^n\)). For each \(i\), we take a section \(\sigma_i \in \Gamma(\pi^{-1}(b_i), L^k)\) satisfying the following conditions:

- \(\nabla_\xi \sigma_i = 0\) for each \(\xi \in T(\pi^{-1}(b_i))\) and
- \(\int_{\pi^{-1}(b_i)} |\sigma_i|^2_h = 1\).

Then

\[|\sigma_j(y)|^2_h \equiv \frac{1}{V_j},\]

where \(V_i\) is the volume of the fiber \(\pi^{-1}(b_j)\). Put

\[s_i(z) = \left(\frac{k}{2\pi}\right)^{-\frac{n}{4}} \int_{\pi^{-1}(b_i)} \Pi_k(z,y) \sigma_i(y) dvol,\]

where \(dvol \in \Omega^n(X)\) is the volume form of fibers defined by the Kähler metric on \(X\).

We study the asymptotic behavior of \(s_i\)'s by using the results quoted in the previous subsection. All the proofs of the results in this subsection are given in the next subsection.

Hereafter we fix an arbitrary constant \(\varepsilon > 0\). Let \(\rho > 0\) be a constant such that local holomorphic coordinates considered in the previous subsection can be defined in the ball of radius \(\rho\). We take a constant \(R > 0\) such that

\[\min \left\{ e^{-\frac{R}{\sqrt{k}}}, e^{-\frac{\rho}{R}}, \frac{\rho}{R} \right\} \leq \varepsilon.\]

Furthermore we take \(r > 0\) such that for each \(z \in X\) and \(b \in B\) with \(d(\pi(z), b) \leq r\), there exists \(y_0 \in \pi^{-1}(b)\) with \(d(z, y_0) = d(z, \pi^{-1}(b))\) and

\[d(z, y) \geq \frac{1}{2} d(y_0, y)\]

for all \(y \in \pi^{-1}(b)\). Let \(k\) be large enough so that

\[\frac{R}{\sqrt{k}} \leq \rho \quad \text{and} \quad \frac{1}{\sqrt{k}} \left(3R + \frac{3n}{4}\log k\right) \leq r.\]

**Proposition 7.4.** 1. If \(d(z, \pi^{-1}(b_j)) \leq \frac{1}{\sqrt{k}} \left(3R + \frac{3n}{4}\log k\right)\), then

\[|s_j(z)|_h \leq Ck^{\frac{n}{4}} e^{-\frac{n}{4} \sqrt{k} d(z, \pi^{-1}(b_j))}.\]
2. If \( d(z, \pi^{-1}(b_j)) \geq \frac{1}{\sqrt{k}} \left( 3R + \frac{2n}{k} \log k \right) \), then

\[ |s_j(z)|_h = O(\varepsilon). \]

Next we give the behavior of \( s_j \) near \( \pi^{-1}(b_j) \).

Fix \( z_0 \in \pi^{-1}(b_0) \) and take an action-angle coordinate \((x^1, \ldots, x^n, y^1, \ldots, y^n)\) around \( \pi^{-1}(\pi(z_0)) \) constructed in section 2. Then we can take a holomorphic coordinate around \( z_0 \) of the form \( z = y + \Omega x + O(|z|^2) \) with some symmetric matrix \( \Omega \) such that \( \text{Im} \, \Omega = \frac{1}{2} G^{-1} \) at \( z_0 \). Note that \( b_j \in \frac{1}{k} \mathbb{Z}^n \) with respect to this action angle coordinate.

**Theorem 7.5.** For \( z \in X \) with \( d(z, z_0) \leq R/\sqrt{k} \) and \( b = b_j \) with \( d(b_0, b_j) \leq R/\sqrt{k} \),

\[
s_j(z) = \left( \frac{k}{2\pi} \right)^{\frac{n}{2}} \frac{1}{\sqrt{V_j}} \exp \left( \frac{k}{2} z G(z - \bar{z}) - \sqrt{-1} k^2 z b - \frac{k}{4} b (\text{Im} \, \Omega) b \right) \times \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) + O(\varepsilon) \right).
\]

By using the above results, we compute \( L^2 \)-inner products of \( s_i \)'s.

**Proposition 7.6.** If \( d(b_0, b_1) \geq R/\sqrt{k} \), then

\[
|\langle s_0, s_1 \rangle_{L^2}| \leq O(\varepsilon).
\]

**Theorem 7.7.** Let \( b = b_0 \) or \( b_1 \) and assume that \( d(b_0, b) \leq R/\sqrt{k} \). Then

\[
\langle s_j, s_0 \rangle_{L^2} = \frac{1}{\sqrt{V_j}} \int_{T^n} \exp \left( -\sqrt{-1} k^2 b y - \frac{k}{2} (\Omega b) G(\Omega b) - \frac{1}{2} k^2 (\text{Re} \, \Omega) b \right) dvol + O(\varepsilon),
\]

here we take an action-angle coordinate around \( \pi^{-1}(b_0) \) as above. In particular,

\[
\|s_i\|_{L^2}^2 = 1 + O(\varepsilon).
\]

**Corollary 7.8.** Assume that the complex structure is invariant under the \( T^n \)-action along fibers (i.e. \( G \) is constant along each fiber), then

\[
\langle s_i, s_j \rangle_{L^2} = \delta_{ij} + O(\varepsilon),
\]

i.e. \( s_i \)'s are “asymptotically orthonormal”.

**Remark 7.9.** We cannot conclude that \( s_i \)'s are linearly independent, because \( \dim H^0(X, L^k) = O(k^n) \) is much bigger compared to the error term.
The next theorem is a weaker version of Theorem 5.3.

**Theorem 7.10.** There exists $k_0$ such that

$$f_k(z) := \sum_i |s_i(s)|^2_h = k^n(1 + O(\varepsilon))$$

for $k \geq k_0$. In particular, $s_i$'s are base point free.

### 7.3 Proofs

**Proof of Proposition 7.4.** 1. From the choice of $r$ and $k$, there exists $y_0 \in \pi^{-1}(b_j)$ such that

$$\begin{cases} d(z, y_0) = d(z, \pi^{-1}(b_j)) \quad \text{and} \\ d(z, y) \geq \frac{1}{4} d(y_0, y) \quad \text{for all } y \in \pi^{-1}(b_j). \end{cases}$$

Hence

$$d(z, y) \geq \frac{1}{4} \left(d(z, y_0) + d(y_0, y)\right) = \frac{1}{4} \left(d(z, \pi^{-1}(b_j)) + d(y_0, y)\right).$$

From Theorem 7.1,

$$|s_j(z)|_h \leq \left(\frac{k}{2\pi}\right)^{-\frac{4n}{2}} \int_{\pi^{-1}(b_j)} |\Pi_k(z, y)||\sigma_j(y)| d\text{vol}_y$$

$$\leq \left(\frac{k}{2\pi}\right)^{-\frac{4n}{2}} \int_{\pi^{-1}(b_j)} C k^n e^{-c\sqrt{k}d(z, y)} \frac{1}{\sqrt{V_j}} d\text{vol}_y$$

$$\leq C k^{\frac{3n}{2}} \frac{1}{\sqrt{V_j}} e^{-c\sqrt{k}d(z, \pi^{-1}(b_j))} \int_{\pi^{-1}(b_j)} e^{-c\sqrt{k}d(y, y_0)} d\text{vol}_y.$$  

Here

$$\int_{\pi^{-1}(b_j)} e^{-c\sqrt{k}d(y, y_0)} d\text{vol}_y \leq \int_{\mathbb{R}^n} e^{-c|y|} k^{-\frac{n}{2}} dy \leq C k^{-\frac{n}{2}.}$$

Then we obtain the desired result.

2. Since $d(z, \pi^{-1}(b_j)) \geq \frac{1}{\sqrt{k}} (R + \frac{3n}{4} \log k)$,

$$e^{-c\sqrt{k}d(z, \pi^{-1}(b_j))} \leq e^{-c(R + \frac{3n}{4} \log k)} = k^{-\frac{3n}{4}} e^{-cR} \leq k^{-\frac{3n}{4}} \varepsilon,$$

we have

$$|s_j(z)|_h \leq \left(\frac{k}{2\pi}\right)^{-\frac{4n}{2}} \int_{\pi^{-1}(b_j)} C k^n e^{-c\sqrt{k}d(z, \pi^{-1}(b_j))} \frac{1}{\sqrt{V_j}} d\text{vol}_y$$

$$\leq C k^{\frac{3n}{2}} \frac{1}{\sqrt{V_j}} \int_{\pi^{-1}(b_j)} k^{-\frac{3n}{4}} \varepsilon d\text{vol}_y$$

$$\leq C \sqrt{V_j} \varepsilon.$$
Proof of Theorem 7.5. To prove Theorem 7.5, we need to write down $\sigma_j$ explicitly.

**Lemma 7.11.** For $b = b_j \in \frac{1}{2}\mathbb{Z}^n$ with $|b| \leq \varepsilon$,

$$
\sigma_j(y) = \frac{1}{\sqrt{V_j}} \exp \left( -\frac{\sqrt{-1}}{2} k^t by - \frac{\sqrt{-1}}{2} k^t b(\Re \Omega) b + kO(|y|^3) \right).
$$

**Proof.** From the choice of local trivialization of $L$, the Hermitian metric $h(z)$ has the form

$$
h(z) = \exp(-t z G \bar{z} + O(|z|^3)).
$$

So the connection on $L^k$ is

$$
\nabla = d - k^t z G dz + O(|z|^2) dz,
$$

in particular, on the fiber $\pi^{-1}(b)$,

$$
\nabla = d - k^t (y + \bar{\Omega} b) G dy + kO(|y|^2) dy.
$$

From this, the solutions of $\nabla \sigma = 0$ is of the form

$$
\sigma = C \exp \left( \frac{k}{2} t(y + \bar{\Omega} b) G(y + \bar{\Omega} b) + kO(|y|^3) \right) e_L(y)
$$

for some constant $C$. This constant is determined by the normalization condition $|\sigma_j|^2_h = \frac{1}{V_j}$. □

**Proof of Theorem 7.5.** We put

$$
s_j(z) = \left( \frac{k}{2\pi} \right)^{-\frac{n}{4}} \int_{\pi^{-1}(b_j)} \Pi_k(z, y) \sigma_j(y) d\text{vol}(y)
$$

$$
= \left( \frac{k}{2\pi} \right)^{-\frac{n}{4}} \int_{\pi^{-1}(b_j) \cap \{|z_0 - y| \leq R/2 \sqrt{\tau}\}} + \left( \frac{k}{2\pi} \right)^{-\frac{n}{4}} \int_{\pi^{-1}(b_j) \cap \{|z_0 - y| \geq R/2 \sqrt{\tau}\}}
$$

$$
= \left( \frac{k}{2\pi} \right)^{-\frac{n}{4}} ((I) + (II)).
$$
First we consider the second term.

\[
|(II)| \leq \int_{|z_0-y| \geq R/2\sqrt{k}} |\Pi_k(z, y)| \cdot |\sigma_j(y)| \, d\text{vol}_y \\
\leq \int_{|z_0-y| \geq R/2\sqrt{k}} Ck^n e^{-k \cdot d(z, y) - 1 \over \sqrt{V_j}} \, d\text{vol}_y \\
\leq Ck^n {1 \over V_j} \int \{ y \in \mathbb{R}^n : |y| \geq R/2 \} e^{-|y|^2 k^{-\frac{n}{2}}} \, dy \\
\leq Ck^n {1 \over V_j} \int e^{-R^2/4} \\
\leq Ck^n {1 \over V_j} e^{-R^2/4}.
\]

Next we compute

\[
(I) = \int_{d(z, y) \leq R/\sqrt{k}} \Pi_k(z, y) \sigma_j(y) \, d\text{vol}_y.
\]

Note that \(d\text{vol}_y = \sqrt{\det G} dy^1 \wedge \cdots \wedge dy^n \) (up to \(O(1/\sqrt{k})\)). We put \(z = w/\sqrt{k}, \ b = a/\sqrt{k}, \) and \(y = u/\sqrt{k} + \Omega a/\sqrt{k} \), where \(w \in \mathbb{C}^n, \ u \in \mathbb{R}^n\). Then

\[
(I) = \int_{|u| \leq R} \Pi_k \left( \frac{w}{\sqrt{k}}, \frac{u + \Omega}{\sqrt{k}} \right) \sigma_j \left( \frac{u + \Omega}{\sqrt{k}} \right) \, d\text{vol} \\
= \int_{|u| \leq R} \left( \frac{k}{2\pi} \right)^n \\
\cdot \exp \left( -\frac{1}{2} \, t \, wG\bar{\omega} - \frac{1}{2} \, t \, (v + \Omega a)G(v + \Omega a) + t \, wG(v + \Omega a) \right) \\
\cdot \frac{1}{\sqrt{V_j}} \exp \left( -\frac{\sqrt{-1}}{2} \, t \, ya - \frac{\sqrt{-1}}{2} \, t \, a(\text{Re} \, \Omega) a \right) \\
\cdot \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) \right) k^{-\frac{n}{2}} \sqrt{\det G} \, du^1 \wedge \cdots \wedge du^n \\
= \left( \frac{1}{2\pi} \right)^n k^{\frac{n}{2}} \frac{1}{\sqrt{V_j}} \exp \left( -\frac{1}{2} \, t \, wG\bar{\omega} - \frac{\sqrt{-1}}{2} \, t \, a(\text{Re} \, \Omega) a \right) \cdot \sqrt{\det G(z)} \\
\cdot \int_{|u| \leq R} \exp \left( -\frac{1}{2} \, t \, (u + \Omega a)G(u + \Omega a) + t \, wG(u + \Omega a) - \frac{\sqrt{-1}}{2} \, t \, au \right) \, du \\
\cdot \left( 1 + O \left( \frac{1}{\sqrt{k}} \right) \right).
\]
Hereafter we omit error terms.

\[(I) = \left(\frac{1}{2\pi}\right)^n k^{n/2} \frac{1}{\sqrt{V_j}} \sqrt{\det G(z)}
\]

\[
\cdot \exp\left(-\frac{1}{2}wG\bar{w} - \frac{\sqrt{-1}}{2}a(\text{Re } \Omega)a - \frac{1}{2}(\Omega a)G(\bar{\Omega} a) + ^t\Omega (\Omega a)\right)
\]

\[
\cdot \int_{|u| \leq R} \exp\left(-\frac{1}{2}uGu - ^tG(\text{Re } \Omega)a + ^tuG\bar{w} - \frac{\sqrt{-1}}{2}ua\right) \, du
\]

\[
= \left(\frac{1}{2\pi}\right)^n k^{n/2} \frac{1}{\sqrt{V_j}} \sqrt{\det G(z)}
\]

\[
\cdot \exp\left(-\frac{1}{2}w\bar{w} - \frac{\sqrt{-1}}{2}a(\text{Re } \Omega)a - \frac{1}{2}(\Omega a)G(\bar{\Omega} a) + ^wG(\bar{\Omega} a)\right)
\]

\[
\cdot \int_{|u| \leq R} \exp\left(-\frac{1}{2}uGu + ^tG(w - \Omega a)\right) \, du,
\]

here we used \(\text{Im } \Omega = \frac{1}{2}G^{-1}\).

**Lemma 7.12.**

\[
\left| \int_{\{u \in \mathbb{R}^n : |u| \geq R\}} \exp\left(-\frac{1}{2}uGu + ^tG(w - \Omega a)\right) \, du \right| \leq O(\varepsilon).
\]

**Proof.** This Lemma follows from

\[
\int_{|u| \geq R} e^{-|u|^2} \, du \leq \pi^2 e^{-\frac{1}{4}R^2}
\]

and the choice of \(R\). \(\square\)

**Lemma 7.13.** For \(\lambda \in \mathbb{C}\),

\[
\int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + \lambda x} \, dx = \sqrt{2\pi} e^{\frac{1}{2}u^2}.
\]

This is an easy exercise.

By combining Lemma 7.12 and 7.13 we have

\[
\int_{|u| \leq R} \exp\left(-\frac{1}{2}uGu - ^tG(\Omega a - w)\right) \, du
\]

\[
= \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}uGu - ^tG(\Omega a - w)\right) \, du + O(\varepsilon)
\]

\[
= \frac{1}{\sqrt{\det G}}(2\pi)^{n/2} \exp\left(\frac{1}{2}(\Omega a - w)G(\Omega a - w)\right) + O(\varepsilon).
\]
Hence, up to error terms,

\[
s_j(z) = \left( \frac{1}{2\pi} \right)^{\frac{3n}{2}} \sqrt{V_j} \sqrt{\det G(z)} \cdot \exp \left( -\frac{1}{2} w G \bar{w} - \frac{\sqrt{-1}}{2} a (\text{Re} \Omega) a - \frac{1}{2} (\Omega a) G(\bar{\Omega} a) + t w G(\bar{\Omega} a) \right) \cdot \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} v G v + t v G (w - \Omega a) \right) dy
\]

\[
= \left( \frac{k}{2\pi} \right)^{\frac{3n}{2}} \sqrt{V_j} \sqrt{\det G(z)} \cdot \exp \left( -\frac{1}{2} w G \bar{w} - \frac{\sqrt{-1}}{2} a (\text{Re} \Omega) a - \frac{1}{2} (\Omega a) G(\bar{\Omega} a) + t w G(\bar{\Omega} a) \right) \cdot \frac{1}{\sqrt{\det G}} (\sqrt{2\pi})^n \exp \left( \frac{1}{2} (w - \Omega a) G(w - \Omega a) \right)
\]

\[
= \left( \frac{k}{2\pi} \right)^{\frac{3n}{2}} \sqrt{V_j} \quad \text{exp} \left( \sqrt{-1} w G(\text{Im} w) - \sqrt{-1} w a - \frac{1}{4} a G^{-1} a \right).
\]

\[\square\]

**proof of Proposition 7.6.** Let

\[
U_1 = \left\{ z \in X \mid d(z, \pi^{-1}(b_0)) \leq \frac{R}{2\sqrt{k}} \right\},
\]

\[
U_2 = \left\{ z \in X \mid \frac{R}{2\sqrt{k}} \leq d(z, \pi^{-1}(b_0)) \leq \frac{1}{\sqrt{k}} \left( R + \frac{3n}{4c} \log k \right) \right\},
\]

\[
U_3 = \left\{ z \in X \mid d(z, \pi^{-1}(b_1)) \leq \frac{R}{2\sqrt{k}} \right\},
\]

\[
U_4 = \left\{ z \in X \mid \frac{R}{2\sqrt{k}} \leq d(z, \pi^{-1}(b_1)) \leq \frac{1}{\sqrt{k}} \left( R + \frac{3n}{4c} \log k \right) \right\},
\]

\[
U_5 = \left\{ z \in X \mid d(z, \pi^{-1}(b_0)), d(z, \pi^{-1}(b_1)) \geq \frac{1}{\sqrt{k}} \left( R + \frac{3n}{4c} \log k \right) \right\}.
\]
Note that $U_1 \cap U_3 = \emptyset$. Then
\[
\left|(s_0, s_1)_{L^2}\right| \leq \int_X \left|(s_0, s_1)_h\right| \frac{\omega^n}{n!} \\
\leq \int_{U_1} + \int_{U_1} + \int_{U_3} + \int_{U_4} + \int_{U_5}
\]
From Proposition 7.4, we have
\[
\int_{U_5} \left|(s_0, s_1)_h\right| \frac{\omega^n}{n!} \leq \int_X O(\varepsilon) = O(\varepsilon).
\]
From the condition $d(b_0, b_1) \geq R/\sqrt{k}$, if $d(z, \pi^{-1}(b_0)) \leq R/\sqrt{k}$, then $d(z, \pi^{-1}(b_1)) \leq R/\sqrt{k}$. Therefore, from Proposition 7.4,
\[
|s_1(z)|_h \leq Ck^n e^{-\frac{d(z, \pi^{-1}(b_0))}{k}} \leq k^n O(\varepsilon).
\]
Hence
\[
\int_{U_1} \left|(s_0, s_1)_h\right| \frac{\omega^n}{n!} \leq \int_{U_1} Ck^n e^{-\frac{d(z, \pi^{-1}(b_0))}{k}} \\
\leq Ck^n \int_{x \in \pi(U_1)} e^{-\frac{\varepsilon^2}{k}} dx \\
\leq Ck^n \int_{\{x \in \mathbb{R}^n \mid |x| \leq R/2\}} e^{-\frac{\varepsilon^2}{k}} dx \\
\leq O(\varepsilon).
\]
Similarly we have
\[
\int_{U_3} \left|(s_0, s_1)_h\right| \frac{\omega^n}{n!} \leq O(\varepsilon).
\]
Finally we consider the part of $U_2$ (and $U_4$). If $z \in U_2 \setminus U_4$, then $|s_1(z)|_h \leq O(\varepsilon)$ by Proposition 7.4. If $z \in U_2 \cap U_4$, then $|s_1(z)|_h \leq k^n O(\varepsilon)$ since $d(z, \pi^{-1}(b_1)) \geq R/2\sqrt{k}$. Consequently,
\[
\int_{U_2} \left|(s_0, s_1)_h\right| \frac{\omega^n}{n!} \leq \int_{U_2} Ck^n e^{-\frac{d(x, b_0)}{k}} \\
\leq Ck^n \int_{\mathbb{R}^n} e^{-c|x|} k^{-\frac{n}{2}} dx \\
= O(\varepsilon).
\]
\[\square\]
Proof of Theorem 7.7. We put
\[(s_j, s_0)_{L^2} = \int_X (s_j(z), s_i(z)) \frac{\omega^n}{n!} \]
\[= \int_{d(z, \pi^{-1}(b_0)) \leq \frac{R}{\sqrt{k}}} + \int_{\frac{R}{\sqrt{k}} \leq d(z, \pi^{-1}(b_0)) \leq \frac{1}{\sqrt{k}}(2R + \frac{3n}{4c} \log k)} + \int_{d(z, \pi^{-1}(b_0)) \geq \frac{1}{\sqrt{k}}(2R + \frac{3n}{4c} \log k)} \]
\[= I + II + III.

First we consider the part III. Since \(d(b_0, b_1) \leq R/\sqrt{k}, \) if \(d(z, \pi^{-1}(b_0)) \geq \frac{1}{\sqrt{k}}(2R + \frac{3n}{4c} \log k), \) then \(d(z, \pi^{-1}(b_1)) \geq \frac{1}{\sqrt{k}}(R + \frac{3n}{4c} \log k). \) From Proposition 7.4, \(|s_0(z)|_h, |s_1(z)|_h = O(\varepsilon). \) Therefore,

\[III \leq O(\varepsilon).

Next we estimate II. If \(d(z, \pi^{-1}(b_0)) \leq \frac{1}{\sqrt{k}}(2R + \frac{3n}{4c} \log k), \) then \(d(z, \pi^{-1}(b_1)) \leq \frac{1}{\sqrt{k}}(3R + \frac{3n}{4c} \log k) \) In this case, we have

\[|z_j(z)|_h \leq Ck^{\frac{n}{2}}e^{-c\sqrt{k}d(z, \pi^{-1}(b_j))}.

Since \(d(z, \pi^{-1}(b_0)) \geq R/\sqrt{k}, \)

\[|s_0(z)|_h \leq Ck^{\frac{n}{2}}e^{-cR} \leq Ck^{\frac{n}{2}}\varepsilon.

Hence

\[II \leq \int_{d(z, \pi^{-1}(b_0)) \leq \frac{1}{\sqrt{k}}(R + \frac{3n}{4c} \log k)} Ck^{\frac{n}{2}}\varepsilon e^{-c\sqrt{k}d(z, \pi^{-1}(b_j))} \]
\[\leq Ck^{\frac{n}{2}}\varepsilon \int_{\mathbb{R}^n} e^{-c|w|}k^{-\frac{n}{2}}du \]
\[\leq O(\varepsilon).

56
Finally, we consider $I$. From Theorem 7.35

\[
(s_j(z), s_0(z))_h
= \left( \frac{k}{2\pi} \right)^\frac{1}{2} \frac{1}{\sqrt{V_iV_j}}
\cdot \left\{ \exp \left( \sqrt{-1k^t zG(Im z)} - 2k\sqrt{-1}b z - \frac{k}{4} t bG^{-1}b - \sqrt{-1k^t \bar{z} G(Im z)} \right) + O(\varepsilon) \right\}
\]

\[
= \left( \frac{k}{2\pi} \right)^\frac{1}{2} \frac{1}{\sqrt{V_iV_j}}
\cdot \left\{ \exp \left( -2k^t (Im z)G(Im z) - 2k\sqrt{-1}b z - \frac{k}{4} t bG^{-1}b \right) + O(\varepsilon) \right\}.
\]

We take an action-angle coordinate and write $z = \Omega x + y$ around a point $z_0$ as above. Then

\[
(s_j(z), s_0(z))_h
= \left( \frac{k}{2\pi} \right)^\frac{1}{2} \frac{1}{\sqrt{V_0V_j}}
\cdot \left\{ \exp \left( -k^t x(Im \Omega)x - \sqrt{-1k^t b\Omega x} - \sqrt{-1k^t b y} - \frac{k}{2} t b(Im \Omega)b \right)
+ O(\varepsilon) \right\}.
\]

Hereafter we omit the error terms.

\[
I = \left( \frac{k}{2\pi} \right)^\frac{1}{2} \frac{1}{\sqrt{V_0V_j}}
\cdot \int_{|x| \leq \frac{R}{\sqrt{k}}, y \in T^n} \exp \left( -k^t x(Im \Omega)x - \sqrt{-1k^t b\Omega x} - \sqrt{-1k^t b y} - \frac{k}{2} t b(Im \Omega)b \right) dx dy
\]

\[
= \left( \frac{k}{2\pi} \right)^\frac{1}{2} \frac{1}{\sqrt{V_0V_j}}
\cdot \int_{y \in T^n} \left( \int_{|x| \leq \frac{R}{\sqrt{k}}} \exp \left( -k^t x(Im \Omega)x - \sqrt{-1k^t b\Omega x} \right) dx \right)
\cdot \exp \left( -\sqrt{-1k^t b y} - \frac{k}{2} t b(Im \Omega)b \right) dy.
\]
\[
\int_{|x| \leq \frac{R}{\sqrt{k}}} \exp \left( -k^t x(\text{Im} \; \Omega) x - \sqrt{-1} k^t b \Omega x \right) \, dx \\
= \left( \frac{k}{2\pi} \right)^{\frac{-n}{2}} \left( \exp \left( -\frac{k}{2} b \Omega \Omega \right) + O(\varepsilon) \right).
\]

Consequently,
\[
I = \frac{1}{\sqrt{V_0 V_j}} \int_{y \in T^n} \exp \left( -\sqrt{-1} k^t b y - \frac{k}{2} b (\text{Im} \; \Omega) b - \frac{k}{2} b \Omega \Omega \right) \, dy + O(\varepsilon).
\]

By combining the identity
\[
\text{Im} \; \Omega + \Omega \Omega = \sqrt{-1} \text{Re} \; \Omega + \Omega \Omega,
\]
we have
\[
(s_j, s_0)_{L^2} = I + O(\varepsilon)
\]
\[
= \frac{1}{\sqrt{V_0 V_j}} \int_{T^n} \exp \left( -\sqrt{-1} k^t b y - \frac{\sqrt{-1}}{2} k^t b (\text{Re} \; \Omega) b - \frac{k}{2} b \Omega \Omega \right) \, dy + O(\varepsilon).
\]

In particular, if \( s_j = s_0 \) (i.e. \( b = 0 \)), then
\[
\|s_0\|_{L^2}^2 = \frac{1}{V_0} \int_{\pi^{-1}(b_0)} dy + O(\varepsilon)
\]
\[
= 1 + O(\varepsilon).
\]

**Proof of Theorem 7.10**

We may assume that \( z \in B_{R/\sqrt{k}}(z_0) \). Let
\[
U_1 = \left\{ x \in B \left| d(x, \pi(z_0)) \leq \frac{R}{\sqrt{k}} \right. \right\},
\]
\[
U_2 = \left\{ x \in B \left| \frac{R}{\sqrt{k}} \leq d(x, \pi(z_0)) \leq \frac{1}{\sqrt{k}} \left( R + \frac{3n}{4c} \log k \right) \right. \right\},
\]
\[
U_3 = \left\{ x \in B \left| d(x, \pi(z_0)) \geq \frac{1}{\sqrt{k}} \left( R + \frac{3n}{4c} \log k \right) \right. \right\}
\]
and write \( f_k \) as
\[
f_k(z) = \sum_{b_j \in U_1} |s_j(z)|^2_k + \sum_{b_j \in U_2} |s_j(z)|^2_h + \sum_{b_j \in U_3} |s_j(z)|^2_h
\]
\[
= (I) + (II) + (III).
\]

58
From Proposition 7.4 we have
\[(III) \leq \dim H^0(X, L^k) \cdot O(\varepsilon) \leq k^n O(\varepsilon)\]
and
\[(II) \leq \sum_{b_j \in U_1} Ck^{\frac{n}{2}} e^{-\frac{\pi}{4} \sqrt{d(\varepsilon, \pi^{-1}(b_j))}}
\leq C' k^n \sum_{b \in \frac{1}{k} \mathbb{Z}^n, |b| \geq R/\sqrt{k}} k^{-\frac{n}{2}} e^{-c \sqrt{k} |b|}.
\]
Since
\[\sum_{b \in \frac{1}{k} \mathbb{Z}^n, |b| \geq R/\sqrt{k}} k^{-\frac{n}{2}} e^{-c \sqrt{k} |b|} \rightarrow \int_{x \in \mathbb{R}^n, |x| \geq R} e^{-c|x|} = Ce^{-cR} = O(\varepsilon)\]
as \(k \to \infty\), (II) can be bounded from above by \(k^n O(\varepsilon)\).

Next we compute (I). From Theorem 7.5, we have
\[|s_j(z)|_h^2 = \left(\frac{k}{2\pi}\right)^{\frac{n}{2}} \frac{1}{V_j} \left\{\exp \left(-k^t(x - b_j)(\text{Im } \Omega)(x - b_j)\right) + O(\varepsilon)\right\}\]
and
\[(I) = \left(\frac{k}{2\pi}\right)^{\frac{n}{2}} \sum_{b_j \in U_1} \frac{1}{V_j} \exp \left(-k^t(x - b_j)(\text{Im } \Omega)(x - b_j)\right) + k^n O(\varepsilon)\]
\[= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} k^n \sum_{b \in \frac{1}{k} \mathbb{Z}^n + \frac{1}{k} \mathbb{Z}^n} \frac{1}{V_j} \exp \left(-t b(\text{Im } \Omega)b\right) + k^n O(\varepsilon).\]

Similarly,
\[\left|\sum_{b \in \frac{1}{k} \mathbb{Z}^n + \frac{1}{k} \mathbb{Z}^n, |b| \leq R} k^{-\frac{n}{2}} \frac{1}{V_j} \exp \left(-t b(\text{Im } \Omega)b\right) - \int_{u \in \mathbb{R}^n, |u| \leq R} \exp \left(-t u(\text{Im } \Omega)u\right) \frac{du}{\sqrt{2^n \det G}}\right| \leq \varepsilon\]
for large \(k\) (Note that \(|\sqrt{k}x| \leq R\)). On the other hand,
\[\left|\int_{|u| \geq R} \exp \left(-t u(\text{Im } \Omega)u\right) \frac{du}{\sqrt{2^n \det G}\right| \leq \varepsilon.\]

Hence
\[(I) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} k^n \int_{\mathbb{R}^n} \exp \left(-t u(\text{Im } \Omega)u\right) \frac{du}{\sqrt{2^n \det G}} + k^n O(\varepsilon)\]
\[= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} k^n \int_{\mathbb{R}^n} e^{-|u|^2} du + k^n O(\varepsilon)\]
\[= k^n (1 + O(\varepsilon)),\]

59
here we use $(\text{Im} \Omega)^{-1} = 2G$. □
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63
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