Proof of the Conjecture that the Planar
Self-Avoiding Walk has Root Mean Square
Displacement Exponent $3/4$

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Abstract
This paper proves the long-standing open conjecture rooted in chemical physics (Flory (1949) $^3$) that the self-avoiding walk (SAW) in the square lattice has root mean square displacement exponent $3/4$. We consider (a) the point process of self-intersections defined via certain paths of the symmetric simple random walk in $\mathbb{Z}^2$ and (b) a “weakly self-avoiding cone process” relative to this point process when in a certain “shape”. We derive results on the asymptotic expected distance of the weakly SAW with parameter $\beta > 0$ from its starting point, from which a number of distance exponents are immediately collectable for the SAW as well. Our method employs the Palm distribution of the point process of self-intersection points in a cone.

1 Introduction
The self-avoiding walk serves as a model for linear polymer molecules. Polymers are of interest to chemists and physicists and are the fundamental building blocks in biological systems. A polymer is a long chain of monomers (groups of atoms) joined to one another by chemical bonds. These polymer molecules form together randomly with the restriction of no overlap. This repelling force makes the polymers more diffusive than a simple random walk. What are the properties of an average configuration of a polymer, most of all, what is the average distance between the two ends of a long polymer? This paper will give an answer to the latter question in two dimensions and a separate paper will deal with all dimensions. Note that the subject “self-avoiding walk” has been well attended, not exclusively by probabilists. Here, no attempt is made to survey the vast literature (consult e.g. Madras and Slade $^9$).

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(Weakly) Self-Avoiding Walk. We will consider the weakly self-avoiding walk in $\mathbb{Z}^2$ starting at the origin. More precisely, if $J_n = J_n(\cdot)$ denotes the number of self-intersections of a symmetric simple random walk $S_0 = 0, S_1, \ldots, S_n$ on the planar lattice starting at the origin, that is,

$$J_n = J_n(S_0, S_1, \ldots, S_n) = \sum_{0 \leq i < j \leq n} 1_{\{S_i = S_j\}},$$

(1.1)

and if $\beta \geq 0$ denotes the self-intersection parameter, then the weakly self-avoiding walk is the stochastic process, induced by the probability measure

$$Q^\beta_n(\cdot) = \frac{\exp\{-\beta J_n(\cdot)\}}{E\exp\{-\beta J_n(\cdot)\}},$$

(1.2)

where $E$ denotes expectation of the random walk. In other words, $J_n = r$ self-intersections are penalized by the factor $\exp\{-\beta r\}$. The measure $Q^\beta_n$ may be looked at as a measure on the set of all simple random walks of length $n$ which weighs relative to the number of self-intersections. This restraint walk is also being called the Domb-Joyce model in the literature (see Lawler [8], p. 170), but importantly, differs from the discrete Edwards model, which is a related repelling walk (see Madras and Slade [9], p. 367 and Lawler [8], p. 172 for some background). Thus, when setting $\beta = 0$, we recover the simple random walk (SRW), while letting $\beta \to \infty$ well mimics the self-avoiding walk (SAW). The SAW in $\mathbb{Z}^2$ is a SRW-path of length $n$ with no self-intersections. Thus, this walk visits each site of its path exactly once.

We will be interested in the expected distance of the weakly SAW from its starting point after $n$ steps, as measured by the root mean square displacement at the $n$-th step. Let $E_\beta = E_{Q^\beta_n}$ denote expectation under the measure $Q^\beta_n$ (that is, expectation wrt. to the weakly SAW). Thus, $E_0$ denotes expectation wrt. to the SRW. Also, write $S_n = (X_n, Y_n)$ for every integer $n \geq 0$. Objects of interest to us are the expectation $E_\beta$ of the distance

$$\chi_n = (X_n^2 + Y_n^2)^{1/2}$$

of the walk from the starting point $0$, the mean square displacement $E_\beta \chi_n^2$, and the root mean square displacement $(E_\beta \chi_n^2)^{1/2}$ of the weakly SAW. Shorter, we shall write MSD and RMSD (for the latter two), respectively. The main results of this paper are the following statements, valid in two dimensions.

**Theorem 1** For each $\beta > 0$, the exponent of the distance of the planar weakly self-avoiding walk equals $3/4$. Furthermore, there are some constants $0 < \rho_1 = \rho_1(\beta) \leq \rho_2 < \infty$ ($\rho_2$ : uniform in $\beta$) such that

$$\rho_1 \leq \liminf_{n \to \infty} n^{-3/4} E_\beta(\chi_n) \leq \limsup_{n \to \infty} n^{-3/4} E_\beta(\chi_n) \leq \rho_2.$$  

In particular, the planar self-avoiding walk has distance exponent $3/4$. 
The proof of Theorem 1 is collected in Propositions 4 and 5 and in Corollary 1. We remark that the constants $\rho_2$ and $\rho_4$ in Theorems 1 and 2, respectively, are independent of all $\beta > 0$, whereas the constants $\rho_1$ and $\rho_3$, respectively, in the lower bounds may depend on $\beta$, even as $\beta \to 0$ or $\beta \to \infty$.

**Theorem 2** For each $\beta > 0$, the root mean square displacement exponent of the planar weakly self-avoiding walk equals $3/4$. Moreover, there are some constants $0 < \rho_3 = \rho_3(\beta) \leq \rho_4 < \infty$ ($\rho_4$ : uniform in $\beta$) such that

$$\rho_3 \leq \liminf_{n \to \infty} n^{-3/2} E_\beta(\chi_n^2) \leq \limsup_{n \to \infty} n^{-3/2} E_\beta(\chi_n^2) \leq \rho_4.$$ 

Particularly, the planar self-avoiding walk has root mean square displacement exponent $3/4$.

See Corollary 2 for a proof. Theorem 2 solves a major several decades-old open conjecture that can be traced back to at least Flory's work [3] in the 1940ies and is one among numerous stones yet to be uncovered in the field of 2, 3, and 4-dimensional random polymers.

In contrast to popular believes, the same approach, the approach that we pursue in this article extends to dimensions 3, 4, and higher as well as 1. These cases will be discussed in detail in a separate paper [6] to come. Here, we content ourselves with stating the formula for the RMSD exponent $\nu$ of the weakly SAW in $\mathbb{Z}^d$ (with all definitions being much the same as in two dimensions). This expression coincides with the one for the RMSD exponent of the SAW, defined as the $\lim_{n \to \infty} \lim_{\beta \to \infty} \ln(E_\beta(\chi_n^2))/(2 \ln n)$ if the limits exist, in view of existing thresholds of $n$ that are uniform in $\beta$ as $\beta \to \infty$ and considerations towards exchanging the limits $\lim_{\beta \to \infty}$ and $\lim_{n \to \infty}$:

$$\nu = \begin{cases} 1 & \text{for } d = 1, \\ \max\left(\frac{1}{2}, \frac{1}{4} + \frac{1}{d}\right) & \text{for } d \geq 2. \end{cases}$$

Equivalently,

$$\nu = \begin{cases} 1 & \text{for } d = 1, \\ 3/4 & \text{for } d = 2, \\ 7/12 & \text{for } d = 3, \\ 1/2 & \text{for } d \geq 4. \end{cases}$$

We observe that, in dimension 4, the exponent $1/2$ arises for different reasons than it occurs in dimensions 5 and higher. In dimension larger than 1, the MSD is asymptotic to the sum of at least two terms, one of which is dominating in dimensions 2 and 3, the other of which is dominating for $d \geq 5$. The latter is the term that would present itself for the SRW. In this sense, the SAW in dimensions $d \geq 5$ behaves as the SRW. Note that $7/12 = 0.58333...$ differs from the value 0.59... (see e.g. Lawler [8], p. 167) that was believed more recently,
as stipulated by heuristic and “numerical evidence” (Earlier estimates included the Flory estimate 0.6).

Whereas our result is novel for $d = 2, 3, \text{and } 4$ and $\beta \in (0, \infty]$, the result on the RMSD exponent for the SAW for $d \geq 5$ is in Hara and Slade [5] and the one on the RMSD exponent for the weakly SAW for $d = 1$ is in Greven and den Hollander [4]. The former was accomplished via the perturbation technique “lace expansion” and the latter via large deviation theory. Brydges and Spencer [2] establish that the scaling limit of the weakly SAW is Gaussian for sufficiently small $\beta > 0$ and $d \geq 5$.

Here are a few words about the ideas of proof and how we came across them before we embark on the detailed arguments. The square root of the mean square displacement of the SRW up to time $n$ is of order $\sqrt{n}$. Similarly, the scaling $\sqrt{t}$ is an elementary and distinctive feature of standard Brownian motion in $\mathbb{R}^d$ run for time $t$. The latter may be calculated by integrating the appropriate expression in polar coordinates, in other words, by regarding the projection of the Brownian motion onto any fixed line. This process along a line is one-dimensional. Motivated by this observation, it is natural to similarly attempt to view a one-dimensional process that mimics the SRW and which is penalized according to the number of self-intersections of the random walk which happen on (or near) a line and to ask the questions (i) how far such a process is expected to move from the origin and (ii) how its expected distance compares to the one of the weakly SAW. Clearly, if we fix a line, the random walk that we run may not intersect it – a configuration that is far from ideal –. The situation improves if we pick a “typical” line. Of course, such a typical line would have to be chosen differently for each realization of a SRW-path. A useful concept in stochastic geometry allows us to deal with typical random geometric objects, for instance, sample points of point processes. This is the Palm distribution of a random measure.

We shall employ the Palm distribution of the point process of self-intersections, defined via certain paths of length $n$ of the symmetric SRW in $\mathbb{Z}^2$, in a cone to introduce a “weakly self-avoiding cone process” relative to this point process when in a certain “shape”. To finish the story all the way, the asymptotic expected distance of this process can be calculated rather explicitly as $n$ tends large, at least if the point process is $1/2$-shaped, in which case it can be shown to equal the expected distance of the weakly SAW from its starting point. From these results along with some considerations towards uniform bounds and estimates in $\beta$ as $\beta \to \infty$, the mean square displacement exponent of the SAW immediately derives.

Section 2 presents a characterization of the SRW-paths that are “atypical” but significant from the perspective of a weakly SAW. Section 3 makes a connection between Palm distributions and the random walk, introduces this weakly self-avoiding cone process, calculates some asymptotic mean distances of this process and links those to the ones of the weakly SAW. Some remarks on the transitions $\beta \to \infty$ and $\beta \to 0$ end Section 3.
2 Atypical SRW-Paths are the Important Ones

An elementary calculation, based on the Local Central Limit theorem (consult any graduate-level probability textbook), shows that the expected \textit{self-intersection local time} (SILT) $E_0 J_n$ of the SRW is asymptotic to $\pi^{-1} n \ln n$, with the error being no larger than order $n$. However, these typical paths are “negligible” in any analysis of the weakly SAW as the following estimates indicate. We will make use of the convenient $o(\cdot)$ notation, that is, write $f(n) = o(g(n))$ as $n \to \infty$ for two real-valued functions $f$ and $g$ if $\lim_{n \to \infty} f(n)/g(n) = 0$.

\textbf{Proposition 1 (Upper Bound for $J_n$)} Let $\beta > 0$ and let $\nu_0$ denote the exponent of the number of self-avoiding walks. Then for every $B > B_* = (\ln 4 - \nu_0)/\beta > 0$ and every integer $n \geq 0$,

$$E_0(e^{-\beta J_n} 1\{J_n > Bn\}) < E_0(e^{-\beta J_n} 1\{J_n = 0\}),$$

in particular, as $n \to \infty$,

$$E_0(e^{-\beta J_n} 1\{J_n > Bn\}) = o(E_0(e^{-\beta J_n} 1\{J_n = 0\})).$$

\textbf{Proof.} We proceed to show that the set of self-avoiding paths of the SRW contributes a term to $E_0 \exp\{-\beta J_n\}$ which has larger exponential rate than the one contributed by the SRW paths with $J_n > Bn$ for every $B > B_*$. 

For this purpose, let $\Gamma_n$ denote the set of SAW-paths $S_0 = 0, S_1, \ldots, S_n$ (with $J_n = 0$) up through time $n$. Since, for every pair $(m, n)$, concatenating two paths $\gamma_1 \in \Gamma_n$ and $\gamma_2 \in \Gamma_m$ does not always provide a path in $\Gamma_{n+m}$, we immediately gain

$$|\Gamma_{n+m}| \leq |\Gamma_n| |\Gamma_m|,$$

where $|\Gamma_n|$ denotes the cardinality of $\Gamma_n$. An easy subadditivity argument yields that the limit

$$\lim_{n \to \infty} |\Gamma_n|^{1/n} = e^{\nu_0}$$

exists for some $2 \leq e^{\nu_0} \leq 3$ and that $|\Gamma_n| \geq e^{n\nu_0}$ for every integer $n \geq 0$. Observe that in $\mathbb{Z}^d$ for $d > 1$, the \textit{upper} bound $2d - 1$ for $e^{\nu_0}$ may be seen by counting all paths of length $n$ that do not return to the most recently visited point (clearly, an overestimate), whereas the \textit{lower} bound $d$ for $e^{\nu_0}$ may be seen by counting all paths of length $n$ that take only positive steps in both coordinates, e.g. for $d = 2$, move only north or east, say.

Fix $\beta > 0$ and choose $B > (\ln 4 - \nu_0)/\beta > 0$. First, if $P_0$ denotes probability wrt. to the SRW,

$$E_0(e^{-\beta J_n} 1\{J_n > Bn\}) < e^{-\beta Bn} P_0(J_n > Bn) \leq e^{-\beta Bn}.$$  

Second, in view of the illustrated submultiplicativity property of the SAW,

$$E_0(e^{-\beta J_n} 1\{J_n = 0\}) = P_0(J_n = 0)$$

$$\geq e^{\nu_0} A^{-n} = e^{-n(\ln A - \nu_0)}.$$
Combining (2.2) and (2.3) and recalling that \(-\beta B < - (\ln 4 - \nu_0)\) yields

\[
E_0(e^{-\beta J_n} 1_{\{J_n > Bn\}}) < e^{-\beta Bn}
\]
\[
< e^{-n(\ln 4 - \nu_0)}
\]
\[
\leq E_0(e^{-\beta J_n} 1_{\{J_n = 0\}}),
\]
and thus, both advertized claims. This ends the proof.

Therefore, we learn that it suffices to focus on the SRW-paths that exhibit \(J_n \leq nB_* = n(\ln 4 - \nu_0)/\beta\). On the other hand, the paths with \(J_n\) of order less than \(n\) are not significant either.

**Proposition 2 (Lower Bound for \(J_n\))** Let \(\beta > 0\). There is some \(b_* = b_*(\beta) > 0\) (made precise below) so that for every \(\delta > 0\) and every \(b < b_*\), as \(n \to \infty\),

\[
E_0(e^{-\beta J_n} 1_{\{J_n \leq n^{1-\delta}\}}) = o(E_0(e^{-\beta J_n} 1_{\{J_n < bn\}})).
\]

**Proof.** As in the previous proof, it suffices to find an upper bound for the lefthand side and a lower bound for the righthand side of the display in such a fashion that the former is (exponentially) smaller than the latter.

Fix \(\delta > 0\). Let \(\Omega_n^\delta\) denote the set of all SRW-paths of length \(n\) that have \(J_n \leq n^{1-\delta}\). Clearly,

\[
E_0(e^{-\beta J_n} 1_{\{J_n \leq n^{1-\delta}\}}) \leq P_0(J_n \leq n^{1-\delta})
\]
\[
= \exp\{n(\ln |\Omega_n^\delta|/n - \ln 4)\}. \tag{2.4}
\]

It remains to come up with a lower bound for \(E_0(e^{-\beta J_n} 1_{\{J_n < bn\}})\) for all \(b > 0\) and to determine when this lower bound is larger than the righthand side of (2.4). It will turn out that this is the case for all sufficiently small \(b\). Note that if \(b\) is not small enough, \(e^{-\beta J_n}\) may get too small. We begin with deriving a lower bound for \(P_0(J_n < bn)\).

Pick a suitable \(s < b\) and consider the set \(\Lambda_n\) of SRW-paths of length \(n\) with \(J_n = sn\). Observe that, for all sufficiently large \(n\), the set \(\Lambda_n\) contains \(\Omega_n^\delta\). In fact, each path \(\gamma \in \Omega_n^\delta\) gives rise to a large set \(G_\gamma\) of paths in \(\Lambda_n\).

To see this, choose any path \(\gamma\) in \(\Omega_n^\delta\) and introduce \(\alpha_n n\) repetitions on that path, that is, choose \(\alpha_n n\) distinct sites \(x_j\) among the \(n\) visited sites of \(\gamma\) where \(S_{j+1} = S_{j-1}\) and \(S_{j+2} = S_j = x_j\) (Immediate backtracking and moving on). If we choose \(\alpha_n\) suitably, then this new SRW-path \(\hat{\gamma}\) in \(G_\gamma\), starting at \(0\), will have \(J_n = sn\). It will turn out that \(\alpha_n = \hat{\alpha} - \rho_n\) for some positive finite constant \(\hat{\alpha}\) and \(\rho_n \leq n^{-\delta}\). Thus, \(\alpha_n \to \hat{\alpha}\) as \(n \to \infty\).

To facilitate notation, we will drop the subscript \(n\) from \(\alpha_n\) and just write \(\alpha\) for the rest of the proof. Observe that the path \(\hat{\gamma}\) of length \(n\) has a trace which is shorter by \(2\alpha n\) units and ends at the \(n(1 - 2\alpha)\)-th site of the path \(\gamma\), as \(2\alpha n\) times were wasted revisiting
sites. In order that two distinct paths $\gamma$ and $\gamma'$ in $\Omega^\delta_n$ generate two sets $G_\gamma$ and $G_{\gamma'}$ that are disjoint, first, we do not allow to place repetitions among the $2\alpha n$ last sites of the paths, thus, let $\gamma$ in $\Omega^\delta_{n(1-2\alpha)}$, and second, add a fixed number $f$ of self-intersection points at each site where there is at least one self-intersection point of $\gamma$ and accordingly deduct the corresponding number of SILT from the $\alpha n$ repetitions to be performed. Thus, in this repetition scheme, only one repetition is allowed per site except for the prescribed repetitions at existing self-intersection points of $\gamma$, where a larger number of repetitions will be placed. The latter agreement (concerning prescribed repetitions) guarantees that the newly generated paths in $G_\gamma$ distinguish themselves from all paths in $G_{\gamma'}$ for $\gamma$ different from $\gamma'$ and that $G_\gamma \neq G_{\gamma'}$ (since each class can be uniquely identified). In other words, the classes $G_\gamma$ are disjoint. Importantly, observe that the number of prescribed repetitions is no larger than $fn^{1-\delta} = o(sn)$ as $n \to \infty$. Also, note that $\tilde{\gamma} \in G_\gamma$ leave the same trace as $\gamma \in \Omega^\delta_{n(1-2\alpha)}$.

Counting all paths in $G_\gamma$ will lead to a lower bound for $|\Lambda_n|$, thus, to a lower bound for $P_0(J_n = sn)$, and eventually, to a lower bound for $P_0(J_n \leq bn)$. Two moments’ thoughts reveal that the set of all selections of $\alpha n$ distinct sites among the $(n(1-2\alpha)(1+o(1))$ visited sites of $\gamma$ is in one-to-one correspondence with the set $G_\gamma$. The cardinality of the former equals the number of ways to distribute $\alpha n$ identical balls in $n(1-2\alpha)(1+o(1))$ urns under the restriction that there is no more than one ball per urn. (The correction $o(1)$ enters in view of the prescribed repetitions at self-intersection points, which are negligible in number in comparison to the total $\alpha n$, but will be suppressed in the calculation below.) Hence, as an appeal to the Stirling approximation $k! = \sqrt{2\pi k}e^{-k}k^{k}(1 + o(1))$ as $k \to \infty$,

$$|G_\gamma| = \binom{n(1-2\alpha)}{\alpha n} = \left[ \frac{(1-2\alpha)^{1-2\alpha}}{\alpha^\alpha(1-3\alpha)^{1-3\alpha}} \right]^n \frac{1}{2\pi n} \left[ \frac{1-2\alpha}{\alpha(1-3\alpha)} \right]^{1/2} (1 + o(1)). \quad (2.5)$$

Since $(1-2\alpha)^{1-2\alpha}/(1-3\alpha)^{1-3\alpha} \geq 1$, there is some $\xi = \xi_\alpha > 0$ with $\exp\{\xi\} \geq 1/\alpha^\alpha$ so that the righthand side of (2.5) $\geq \exp\{\xi n\}$, that is,

$$|G_\gamma| \geq \exp\{\xi n\}. \quad (2.6)$$

We conclude that for each $\gamma \in \Omega^\delta_{n(1-2\alpha)}$, we can identify at least $\exp\{\xi n\}$ SRW-paths with $J_n = sn$. Keeping in mind our earlier observations that, first, all paths in $G_\gamma$ are different SRW-paths, and second, the sets $G_\gamma$ are different from one another and combining (2.6) with the fact that each path in $\Omega^\delta_{n(1-2\alpha)}$ can be completed to render a path of length $n$ in $\Omega^\delta_n$, we obtain

$$|\Lambda_n| \geq |\Omega^\delta_{n(1-2\alpha)}| \cdot |G_\gamma| \geq \exp\{n[\ln|\Omega^\delta_{n(1-2\alpha)}|/n + \xi]\}. \quad (2.7)$$
As a consequence, for each $b > 0$ and suitably small $0 < s < b$,
\[
E_0(e^{-\beta J_n} 1_{J_n < bn}) > e^{-\beta bn} P_0(J_n < bn) \\
\geq e^{-\beta bn} P_0(J_n = sn) \\
= e^{-\beta bn} |\Lambda_n| 4^{-n} \\
\geq \exp\{n(|\Omega_{n(1-2\alpha)}^\delta|/n + \xi - \ln 4 - \beta b)\}, \quad (2.8)
\]

Thus, we arrived at a lower bound for $E_0(e^{-\beta J_n} 1_{J_n < bn})$.

Next, we will verify the claim that for $C = 2\ln 4 + 1$ and all sufficiently large $n$,
\[
\frac{1}{n} (|\Omega_n^\delta| - |\Omega_{n(1-2\alpha)}^\delta|) \leq C\alpha
\]
for every $\alpha$. For this purpose, fix $\alpha$. Each path in $\Omega_n^\delta$ arises either from a path in $\Omega_{n(1-2\alpha)}^\delta$ or from a path in the difference $\Omega_{n(1-2\alpha)}^\delta \setminus \Omega_{n(1-2\alpha)}^\delta$ for $\delta > \delta' \geq \delta_*$, where $\delta_* = \delta_s(n, \alpha)$ satisfies the equation $n^{1-\delta} = [n(1-2\alpha)]^{1-\delta_*}$. Solving for $\delta_*$ gives
\[
\delta_* = \frac{\delta \ln n + \ln(1-2\alpha)/\delta}{\ln n + \ln(1-2\alpha)}. \quad (2.9)
\]

Concatenating two paths $\gamma \in \Omega_{n(1-2\alpha)}^\delta$ or $\gamma' \in \Omega_{2\alpha n}^\delta$ will not always produce a path in $\Omega_n^\delta$ since $x^{1-\delta}$ is a concave function in $x > 0$, in particular, $(x_1 + x_2)^{1-\delta} \leq x_1^{1-\delta} + x_2^{1-\delta}$ for $x_1, x_2 \geq 0$, and, in the concatenated path, possible overlap of the subpaths $\gamma$ and $\gamma'$ may introduce additional SILT. Consequently, we collect
\[
|\Omega_n^\delta| \leq |\Omega_{n(1-2\alpha)}^\delta| \cdot |\Omega_{2\alpha n}^\delta| + |\Omega_{n(1-2\alpha)}^\delta \setminus \Omega_{n(1-2\alpha)}^\delta| \cdot |\Omega_{2\alpha n}^\delta|
\]

equivalently,
\[
\ln |\Omega_n^\delta| - \ln |\Omega_{n(1-2\alpha)}^\delta| \leq \ln |\Omega_{2\alpha n}^\delta| - \ln |\Omega_{n(1-2\alpha)}^\delta| \\
\leq 2\alpha n \ln 4.
\]

Inspecting (2.9) makes clear that letting $n$ be suitably large will bring $\delta_*$ as close to $\delta$ as desired. Moreover, since the function $\ln |\Omega_n^\delta|/n$ is bounded and nonincreasing in $\delta$, it has at most finitely many jump discontinuities of size larger than, say, $\tau > 0$. Therefore, in view of (2.4) and the fact that $|\Omega_n^\delta| \leq |\Omega_n^\delta|$ for $\delta < \delta$, it suffices to restrict our attention to those $\delta$ for which the function $\ln |\Omega_{n(1-2\alpha)}^\delta|/n$ has no jumps of size $\geq \alpha$ on the interval $[\delta_*, \delta]$ for all sufficiently large $n$. A combination of these observations leads us to conclude that, for all sufficiently large $n$, we obtain $\ln(|\Omega_{n(1-2\alpha)}^\delta|/|\Omega_{n(1-2\alpha)}^\delta|)/n \leq \alpha$. Therefore, for all sufficiently large $n$,
\[
\frac{1}{n} (\ln |\Omega_n^\delta| - \ln |\Omega_{n(1-2\alpha)}^\delta|) = \frac{1}{n} (\ln |\Omega_n^\delta| - \ln |\Omega_{n(1-2\alpha)}^\delta|) \\
+ \frac{1}{n} (\ln |\Omega_{n(1-2\alpha)}^\delta| - \ln |\Omega_{n(1-2\alpha)}^\delta|) \\
\leq 2\alpha \ln 4 + \alpha = \alpha C. \quad (2.10)
\]
Now, the bound (2.8) is larger than the upper bound for $E_0(e^{-\beta J_n} 1_{J_n \leq n^{1-\delta}})$ in (2.4) if we choose
\[
\ln |\Omega_{n(1-2\alpha)}^\delta|/n + \xi - \ln 4 - \beta b > \ln |\Omega_{n}^\delta|/n - \ln 4,
\]
equivalently,
\[
((\ln |\Omega_{n(1-2\alpha)}^\delta|) - \ln |\Omega_{n}^\delta|)/n + \xi) / \beta > b.
\]
Note that, since $\xi \geq -\alpha \ln \alpha$, choosing $\alpha < e^{-C}$ yields $\xi > \alpha C$, equivalently, $-\alpha C + \xi > 0$. Hence, by virtue of (2.10), if we choose $\alpha < e^{-C}$, it follows that there is some $\zeta_* = \zeta_*(\beta) > 0$ such that for all sufficiently large $n$,
\[
(\ln |\Omega_{n(1-2\alpha)}^\delta|) - \ln |\Omega_{n}^\delta|)/n + \xi > \zeta_*.
\]
Consequently, we can let
\[
b_* = b_*(\beta) = \zeta_*/\beta > 0,
\]
and, the announced result follows for all $b < b_*$. Observe that it follows from display (2.12) that $\zeta_*(\beta)$ is decreasing to zero as $\beta \to \infty$, equivalently, as both $b$ and $\alpha$ tend to zero. This completes the proof. \qed

In fact, minor adaptations of the arguments provide a proof for the case when the bound $n^{1-\delta}$ in the statement of Proposition 2 is replaced by $n q_n$, where $q_n \to 0$ arbitrarily slowly as $n \to \infty$. In other words, the set of paths with $J_n \in [0, n q_n)$ contributes to $E_0(e^{-\beta J_n})$ or to the $k$-th moments $E_\beta(\chi_n^k)$ merely negligibly in the sense that the contribution is $o(E_0(e^{-\beta J_n}))$ or $o(E_\beta(\chi_n^k))$, respectively, as $n$ tends large (in fact, this error term is exponentially smaller, as the proof of Proposition 2 indicates). As a consequence of Propositions 1 and 2, for all that follows, we may neglect to keep track of those error terms and assume that
\[
J_n \in [b_1 n, b_2 n]
\]
for some constants $0 < b_1 < b_2 < \infty$ such that $\beta b_2$ is a positive number independent of $\beta$.

## 3 Palm Distribution of the Point Process of Self-Intersections

Palm distributions help answer questions dealing with properties of a point process, viewed from a typical point. For example, we can make mathematically precise the perhaps heuristically clear answers to the questions (a) what is the mean number of points of a point process in the plane whose nearest neighbors are all at distance at least $r$? and (b) what is the probability that the point process has a certain property, given that the point process has a sample point at $x$?

We pause for two observations to illustrate the subtlety and the importance of the concept of Palm distributions. First, the Palm distribution represents a conditional distribution when applied to, say, simple point processes on the real line, which historically motivated
the study of Palm distributions (Palm [14] and Kallenberg [7], Chapter 10, p. 83). However, the event to be conditioned upon has probability zero. Second, notice that, given a realization of a point process, the sample point closest to the origin is not a typical point of the realization but rather special since identified as the point with the property “being closest to the origin”. Consequently, in analyzing typical points regarding some properties of the point process, conditioning upon the point closest to the origin or upon any other specified point will lead to an incorrect answer.

Interestingly, Palm distributions may be utilized for other typical random geometric objects than points. In the study at hand, the relevant typical tools will be the lines which are typical relative to the SILT $J_n$ when $J_n \in [b_1 n, b_2 n]$, more precisely, relative to the point process of self-intersections of the SRW with $J_n \in [b_1 n, b_2 n]$ in a cone that is defined via a line through the origin. This idea will be taken up in Section 3.1.

After this passage of motivation, let us introduce more notation as needed. If we let $X_n$ and $Y_n$ denote the first and second coordinate processes of the SRW, that is, $S_n = (X_n, Y_n)$ for every integer $n \geq 0$, define the distance $\chi_n = (X_n^2 + Y_n^2)^{1/2}$

or the root of the square displacement of the walk from the starting point 0. Moreover, let $P_{\chi_n}$ denote the probability distribution of the distance $\chi_n$ of the SRW. On the range of $x$ where we can invoke the Local Central Limit theorem, we will approximate $dP_{\chi_n}(x)/dx$ by the density of the corresponding Brownian motion, that is, $2(2\pi n)^{-1/2} \exp\{-x^2/2n\}$, and on the remaining range of $x$ in $(0, n]$, use some bounds on a large deviation estimate for $dP_{\chi_n}(x)$, which denotes $P_0(x \leq \chi_n < x + dx)$ when $dx$ is arbitrarily small.

Note that throughout the paper, we shall omit discussion of the obvious case $\beta = 0$. We next collect a technical lemma that relies on a condition and a couple more definitions.

The main players in this condition are bounded numbers $a_x$ that depend on $x$ and are such that there is some number $\zeta > 0$ so that $\beta a_x \geq \zeta$ for every $0 \leq x \leq n$. Then define

$$
\mu_x = (\beta a_x)^{1/2} n^{3/4} \tag{3.2}
$$

$$
q(x) = \exp\{-\beta a_x^2 n^{1/2}\} \tag{3.3}
$$

for every $n \geq 0$, $\beta > 0$, and $x$ in $[0, n]$. Since $a_x$ is bounded in $x$, for suitably small $\varepsilon \geq 0$ and for $\gamma > 0$, we may define

$$
\begin{align*}
  r_1 &= r_1(\varepsilon, \gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu_x n^{-\varepsilon}\} \\
  r_2 &= r_2(\gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu_x \}.
\end{align*} \tag{3.4}
$$

Thus, $r_2(\gamma) = r_1(0, \gamma)$.

**Condition D.** For any suitably small $\varepsilon \geq 0$, there exist some $\gamma > 0$ and $\rho_\varepsilon > 0$ such that

$$
\int_{r_1}^n x q(x) dP_{\chi_n}(x) = \rho_\varepsilon \int_0^{r_2} x q(x) dP_{\chi_n}(x) \tag{3.5}
$$
with \( \rho_n \geq \rho_* \) for all sufficiently large \( n \).

If \( \int_0^x q(x) dP_{\chi_n}(x) = o(\int_0^x q(x) dP_{\chi_n}(x)) \) as \( n \to \infty \), then \( \varepsilon = 0 \) and \( \rho_n \to \infty \).

Observe that, by virtue of the expression in (3.3) for \( q(x) \), Condition D guarantees that \( a_x \) not be constant in \( x \) and \( \beta > 0 \).

**Lemma 1 (Exponent of Expected Radial Distance equals 3/4)** Let \( \beta > 0 \). Assume that the \( a_x \) are bounded numbers that depend on \( x \), are such that there is some number \( \zeta > 0 \) so that \( \beta a_x \geq \zeta \) for every \( 0 \leq x \leq n \), and that satisfy Condition D in (3.3) for some \( \varepsilon \geq 0 \) and \( \gamma > 0 \). Define

\[
I_n = \int_0^n x q(x) dP_{\chi_n}(x) \tag{3.6}
\]

\[
g(n) = \int_0^n (a_x)^{1/2} q(x) dP_{\chi_n}(x),
\]

where \( q(x) \) is defined in (3.3). Then there are some constants \( M < \infty \) and \( c(\rho_*) > 0 \) (both independent of \( \beta \)) such that as \( n \to \infty \),

\[
\gamma c(\rho_*) \beta^{1/2} n^{3/4 - \varepsilon} (1 + o(1)) \leq \frac{I_n}{g(n)} \leq M \beta^{1/2} n^{3/4} (1 + o(1)). \tag{3.7}
\]

**Proof.** Luckily, the crudest of all estimates will serve us. Condition D will only be relevant to the lower bound for \( I_n \). We start with a number of general observations. Importantly, note that we shall not need a concrete form of the expression for \( P_{\chi_n}(\cdot) \) to prove the statement of the lemma (and ultimately, of the main result on the MSDE). However, in order to bound \( I_n \), it is crucial to recognize the exact nature of the exponential function that appears in the integrand. For this purpose, we are interested in an estimate for \( dP_{\chi_n}(x) \), more precisely, in an estimate for \( dP_{\chi_n}(x) \) beyond the range \((0, n^{2/3})\) of \( x \) where the Local Central Limit theorem is in force. A large deviation type result of Billingsley [1], Theorem 9.4, p. 149 says that if \( \tilde{S}_n \) denotes the partial sum of \( n \) independent and identically distributed random variables with mean 0 and variance 1 and \( \alpha_n \uparrow \infty \) denotes any sequence so that \( \alpha_n/n^{1/2} \to 0 \) as \( n \to \infty \), then \( P(\tilde{S}_n \geq \alpha_n n^{1/2}) = 2 \exp\{-\alpha_n^2 (1+\xi_n)/2\} \) for some sequence \( \xi_n \to 0 \). Now, consider the **diagonal** symmetric simple random walk, each coordinate of which independently takes values +1 and −1 with probability 1/2, convolute the two coordinates, and, apply Billingsley’s estimate to each coordinate separately. The distance of the diagonal random walk (up to scaling by \( \sqrt{2} \)) indicates the distance of the (non-diagonal) random walk in the square lattice since we can turn the lattice by the angle \( \pi/4 \). Applying these steps, we obtain, for \( n^{1/2} \ll x \leq n \),

\[
P_0(\chi_n \geq x) = 2 \exp\{-x^2(1 + \xi_x)/(2n)\}, \tag{3.8}
\]

with \( \xi_x = o(1) \) as \( x \uparrow \) (increasing) and \( n \to \infty \). If the distance of the SRW is measured along any **fixed** line through the origin and the endpoint of its path, then the values of \( x \) form a
discrete set (for fixed \( n \)). If this distance is measured along all lines through the origin and the possible endpoints of the SRW after \( n \) steps (for fixed \( n \)), then the values of \( x \) form a discrete set as well. Call it \( \mathcal{Z}_n \). Between the points in \( \mathcal{Z}_n \), we may interpolate the upper tail probability of \( \chi_n \) in any desired way as long as the estimate in (3.8) is not violated, thereby introducing an error to \( I_n \) which can be shown to be of order less than \( o(g(n)) \) as \( n \to \infty \). We think of embedding the set \( \mathcal{Z}_n \) in the nonnegative reals and of extending (3.8) to a differentiable function that obeys the expression for the upper tail probability for some partial sum \( \tilde{S}_n \) as prescribed by BILLINGSLEY’s estimate. In that case, for arbitrarily small \( dx \) for every \( n^{1/2} \ll x \leq n \), expanding the difference \( P_0(x \leq \chi_n < x + dx) \) into a Taylor series yields

\[
P_0(x \leq \chi_n < x + dx) = P_0(\chi_n \geq x) - P_0(\chi_n \geq x + dx)
\]

\[
= 2 \exp\left\{-x^2(1 + \xi_x)/(2n)\right\} \cdot \left[1 - \exp\left\{ -x^2(\xi_x + dx - \xi_x) + (2x(dx) + (dx)^2(1 + \xi_x + dx))/(2n)\right\}\right]
\]

\[
= n^{-1/2} \exp\{-x^2(1 + \xi'_x)/(2n)\} dx
\]

(3.9)

with \( \xi_x = o(1) \) and \( \xi'_x = o(1) \) as \( x \uparrow \) and \( n \to \infty \). We let \( dP_{\chi_n}(x) \) denote the difference \( P_0(x \leq \chi_n < x + dx) \), as specified by (3.9), when \( dx \) is arbitrarily small. Observe that (3.9) is a valid expression as well when the Central Limit theorem applies.

Next, upon completing the square

\[
\frac{x^2}{2n} + \beta \frac{\alpha_x}{2} n^{1/2} = \frac{1}{2n}(x^2 + \mu_x^2) = \frac{1}{2n}(x - \mu_x)^2 + \frac{1}{n} x \mu_x
\]

(3.10)

and by relying on (3.9), we obtain as \( n \to \infty \),

\[
I_n = \int_0^n n^{-1/2} x \exp\left\{-\frac{x^2}{2n}\right\} \exp\left\{-\beta \frac{\alpha_x}{2} n^{1/2}\right\} \exp\left\{-\frac{x^2}{2n} \xi'_x\right\} dx
\]

(3.11)

(i) Upper Bound for \( I_n \). In view of the form of the expression for the integrand in (3.11), the key contribution to the integral \( I_n \) stems from values of \( x \) for which the exponential functions are largest. It suffices to regard the first two exponential factors of the integrand. We shall argue that these exponential factors are not maximal for \( x \) of order strictly larger than \( n^{3/4} \).

Fix some \( \epsilon > 0 \). The first observation is that \( \exp\{-x^2/2n\} \) decays rapidly with \( x \) for \( x \geq n^{3/4 + \epsilon} \), and on that interval, has strictly smaller exponential rate than for \( x \leq n^{3/4} \), the latter rate essentially being \( -\mu_x^2/2n \). Another number of observations will indicate rapid decay of \( e_\ast(x) = \exp\{-x\mu_x/n\} \) with \( x \geq n^{3/4 + \epsilon} \). Indeed, since \( \alpha_x \) is bounded in \( x \), if we let \( x \leq n^{3/4 - \epsilon} < n^{3/4} < n^{3/4 + \epsilon} \leq u \), then we collect as \( n \to \infty \),

\[
e_\ast(u) < e_\ast(n^{3/4}) < e_\ast(x),
\]

(3.12)

\[
e_\ast(u) = o(e_\ast(n^{3/4})),
\]

\[
e_\ast(n^{3/4}) = o(e_\ast(x)).
\]
For some suitably large $M < \infty$ ($M = 1 + 1/\zeta^{1/2}$ should suffice) write $T_\epsilon = \inf\{x \in [0,n] : x > (M-1)\mu x n^\epsilon\}$. Thus, $x \in (0,T_\epsilon)$ implies that for all sufficiently large $n$, we have $x - \mu_x \leq (M-1)\mu x n^\epsilon$. Clearly, $T_\epsilon \geq \zeta^{1/2}(M-1)n^{3/4+\epsilon}$, in particular, $T_\epsilon \geq n^{3/4+\epsilon}$ if $\zeta^{1/2}(M-1) \geq 1$. Putting each of these pieces together and recalling that the integrand has exponential form yields as $n \to \infty$,

$$I_n = \int_0^n (x-\mu_x) q(x) d\mathbf{P}_x, \quad + \int_0^n \mu x q(x) d\mathbf{P}_x$$

$$= (1 + o(1)) \int_0^{T_\epsilon} (x-\mu_x) q(x) d\mathbf{P}_x + \int_0^n \mu x q(x) d\mathbf{P}_x$$

$$\leq (1 + o(1)) n^{3/4+\epsilon}(M-1)\beta^{1/2} \int_0^{T_\epsilon} (a_x)^{1/2} q(x) d\mathbf{P}_x$$

$$+ \beta^{1/2} n^{3/4} \int_0^n (a_x)^{1/2} q(x) d\mathbf{P}_x$$

$$\leq (1 + o(1)) n^{3/4+\epsilon}(M-1)\beta^{1/2} \int_0^n (a_x)^{1/2} q(x) d\mathbf{P}_x + \beta^{1/2} n^{3/4} g(n)$$

$$= (1 + o(1)) n^{3/4+\epsilon}(M-1)\beta^{1/2} g(n) + \beta^{1/2} n^{3/4} g(n).$$

Since $\epsilon > 0$ was arbitrary, this is the claimed upper bound, as $n \to \infty$,

$$I_n \leq M \beta^{1/2} n^{3/4} g(n) (1 + o(1)).$$

(ii) Lower Bound for $I_n$. To handle the lower bound for $I_n$, we suppose the instance of Condition D and we split the integrals $I_n$ and $g(n)$, respectively, over the three intervals $[0,r_1], (r_1,r_2)$ and $[r_2,n]$ as follows:

$$I_n = \int_0^n x q(x) d\mathbf{P}_x = J_1(n) + J_2(n) + J_3(n) \quad (3.13)$$

$$g(n) = \int_0^n (a_x)^{1/2} q(x) d\mathbf{P}_x = \hat{J}_1(n) + \hat{J}_2(n) + \hat{J}_3(n).$$

In light of the symmetric roles of $J_2(n)$ and $J_3(n)$ in Condition D, we may assume that $J_3(n) = o(J_1(n) + J_2(n))$ as $n \to \infty$ because otherwise $J_1(n)$ can be expressed in terms of $J_3(n)$ instead of in terms of $J_2(n)$ and parallel reasoning to the one employed below applies to establish the lower bound for $I_n/g(n)$. The exponential form of the integrands implies that, as $n \to \infty$, $\hat{J}_3(n) = o(\hat{J}_1(n) + \hat{J}_2(n))$. Write $J_2(n) = \rho n J_1(n)$ (here, we neglect a possible factor $(1 + o(1))$, compare to (3.13)) and $\hat{J}_2(n) = \hat{\rho} n \hat{J}_1(n)$ for some $\rho, \hat{\rho} > 0$. In addition, observe that, if $J_1(n) = o(\hat{J}_2(n))$ as $n \to \infty$, then we obtain $\rho n, \hat{\rho} n \to \infty$ as $n \to \infty$ (In particular, we can choose $c(\rho_n) = 1$ below). Thus, this case shall be covered as a special case in our treatment below. A similar remark is in force in the already excluded scenario that $J_1(n) + J_2(n) = o(J_3(n))$ as $n \to \infty$. Another consequence of the exponential form of the integrands is that there exists some $\rho > 0$ so that $\rho_n, \hat{\rho}_n \geq \rho$ for all sufficiently
large \( n \) if and only if there exists some \( \rho_* > 0 \) so that \( \rho_n \geq \rho_* \) for all sufficiently large \( n \). Thus, \( \rho_n \) is bounded away from 0 if and only if \( \hat{\rho}_n \) is bounded away from 0. Now, since we assume that Condition D is valid, it follows that there is a \( c(\rho_*) > 0 \) such that \((1 + 1/\rho_n)/(1 + 1/\hat{\rho}_n) \geq c(\rho_*)\) for all sufficiently large \( n \). Keeping these in mind, as \( n \to \infty \), we arrive at

\[
I_n = (1 + o(1)) \int_0^{r^2} x q(x) d\mathbb{P}_{\chi_n}(x)
= (1 + o(1))(1 + 1/\rho_n) J_2(n)
\geq (1 + o(1))(1 + 1/\rho_n) \int_{r_1}^{r^2} (\gamma \mu_x n^{-\varepsilon}) q(x) d\mathbb{P}_{\chi_n}(x)
= (1 + o(1)) \gamma \beta^{1/2} n^{3/4-\varepsilon} (1 + 1/\rho_n) \hat{J}_2(n)
= (1 + o(1)) \gamma \beta^{1/2} n^{3/4-\varepsilon} (1 + 1/\rho_n) (\hat{J}_1(n) + \hat{J}_2(n)) (1 + 1/\hat{\rho}_n)^{-1}
\geq (1 + o(1)) \gamma c(\rho_*) \beta^{1/2} n^{3/4-\varepsilon} \hat{g}(n),
\tag{3.14}
\]

as desired. This accomplishes the lower bound and proof.

We remark that the function \( a_x \) will emerge shortly, in \( \text{(3.19)} \) below.

### 3.1. Point Process of Self-Intersections and Cones

Next, we shall transfer the setting of Stoyan, Kendall, and Mecke [11], Chapter 4, p. 99, to SRW language and describe the particulars of the point process of self-intersections. Let \( \Phi = \Phi_n = \{ x_1, x_2, \ldots \} \) denote the point process of self-intersection points of the SRW in \( \mathbb{Z}^2 \) when \( J_n \in [b_1 n, b_2 n] \). Thus, \( |\Phi| \in [b_1 n, b_2 n] \). We allow the points \( x_i \) of \( \Phi \) to have multiplicity and count such a point exactly as many times as there are self-intersections of the SRW at \( x_i \). Observe that \( \Phi \) depends on \( n, b_1, \) and \( b_2 \), thus, on \( \beta \) and that the condition \( J_n \in [b_1 n, b_2 n] \) imposed upon \( \Phi \) moves the analysis to the large deviation range of the SRW and to the right setting for the weakly SAW. This random sequence of points \( \Phi \) in \( \mathbb{Z}^2 \) may also be interpreted as a random measure. Note that \( E_0 \Phi \) is \( \sigma \)-finite. Let \( N_\Phi \) denote the set of all point sequences, generated by \( \Phi \), \( N_\Phi \) the point process \( \sigma \)-algebra generated by \( N_\Phi \), and \( \varphi \in N_\Phi \) denote a realization of \( \Phi \). Formally, \( \Phi \) is a measurable mapping from the underlying probability space into \( (N_\Phi, \mathcal{N}_\Phi) \) that induces a distribution on \( (N_\Phi, \mathcal{N}_\Phi) \), the distribution \( \mathbb{P}_\Phi \) of the point process \( \Phi \). In light of the \( \sigma \)-finiteness of \( E_0 \Phi, \mathbb{P}_\Phi \) is a probability measure. Also, let \( E_\Phi \) denote expectation relative to \( \mathbb{P}_\Phi \).

An important intermediate tool will consist in a (weakly self-avoiding) process related to the SRW \( S_n \) which satisfies condition \( (2.14) \) and whose one-dimensional distribution of the radial component we understand well enough to calculate a rather precise expression for its expected distance from the origin. In turn, this estimate will lead to upper and lower bounds for the expected distance of the two-dimensional process, and ultimately, for the expected distance of the weakly self-avoiding walk. For this purpose, our interest will revolve around the self-intersections of the two-dimensional SRW near (half-)lines, more precisely, within certain cones, positioned at the origin.
A cone will be described by the cone that contains a certain line. Thus, let us now introduce the test set \( V \) of lines \( L \) that will be useful. Let \( V \) denote a set of half-lines (that we call ‘lines’, for ease) that emanate from the origin, spread around a circle in a way that we will not exactly specify at this point but will depend on our (optimal) choice later on (see Definition 3) and on \( n \). It will turn out to be efficient to choose the lines in \( V \) equally spaced around the unit circle. The choice of \( V \) will have strong ties with the shape of the set \( \Phi \). Eventually, nothing else will be retained about \( V \) than its cardinality \(|V|\).

While the description of \( V \), in particular, of its size \(|V|\) will be precised further in the proof of Proposition 5, no more is needed to handle Proposition 3 below, which presents a result that is valid, regardless of the number \(|V|\) of lines and of the arrangement of lines. The proofs of Propositions 4 and 5, however, will address the issue on how to choose the lines for \( V \), in particular, how many are needed to allocate the relevant self-intersection points of the walk to the corresponding cones. Next, let us turn to the restriction of the process \( \Phi \) to a line \( L \) in \( V \), more precisely, to all points that lie closer to \( L \) than to any other line in \( V \).

For any \( L \in V \), let the “cone” \( C_L \) be defined by

\[
C_L = \{ x_i \in \Phi : \text{dist}(x_i, L) \leq \text{dist}(x_i, L') \text{ for all } L \neq L' \in V \}
\]

with the convention that if equality \( \text{dist}(x_i, L) = \text{dist}(x_i, L') \) holds for two lines \( L \) and \( L' \) and a certain number of points \( x_i \), then half of them will be assigned to \( C_L \) and the other half to \( C_{L'} \). Note that no point of \( \Phi \) belongs to more than one \( C_L \) and each point to exactly one \( C_L \). Thus, \(|C_L|\) equals the number of self-intersections of the planar SRW \( S_n \) in a cone at the origin that contains the line \( L \). Moreover, for any constants \( 0 < a_1 < a_2 < \infty \), define the random set

\[
\mathcal{L}_{1/2} = \mathcal{L}_{1/2}(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^{1/2}, a_2 n^{1/2}] \},
\]

which depends on \( a_1, a_2, \) and \( V \). We will choose \( a_1 \) and \( a_2 \) such that \( a_1 \beta \) and \( a_2 \beta \) are positive numbers which are independent of \( \beta \) and \( n \).

3.2. Distance along Cones with Order \( n^{1/2} \) SILT. If \( h : \mathbb{R} \times N_\Phi \to \mathbb{R}_+ \) denotes a nonnegative measurable real-valued function and \( \mathcal{L}_*(\Phi) \) denotes any subset of lines in \( V \), then since \( E_0\Phi \) is \( \sigma \)-finite, we may disintegrate relative to the probability measure \( P_\Phi \),

\[
E_{\Phi} \left( \sum_{L \in \mathcal{L}_*(\Phi)} h(L, \Phi) \right) = \int \sum_{L \in \mathcal{L}_*(\phi)} h(L, \phi) dP_\Phi(\phi)
\]

(consult also Kallenberg [1], p. 83, and Stoyan, Kendall, and Mecke [11], p. 99). For a discussion of some examples of Palm distributions of \( P_\Phi \), the reader is referred to the Appendix.

Next, observe that the conditional distribution \( P_{\Phi|\chi_n} \) of the point process \( \Phi \), given \( \chi_n \), is a function of \( \chi_n \) and depends on condition (2.14), as explained earlier, so as to produce
realizations that satisfy the requirement $J_n \in [b_1 n, b_2 n]$. Apply formula (3.17) with
\[ h(L, \Phi) = \frac{\exp\{-\beta |C_L|\}}{|L_{1/2}(\Phi)|}, \tag{3.18} \]
with $P_{\Phi|\chi_n}(\varphi|x)$ in place of $P_{\Phi}(\varphi)$, and $L_* = L_{1/2}$ to define the numbers $a_x = a_x(L_{1/2})$ by
\[ \exp\{-\beta a_x n^{1/2}/2\} = E_{\Phi|\chi_n}(|L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |C_L|} |\chi_n = x|) \tag{3.19} \]
\[ = \int_{\mathbb{Z}^2} |L_{1/2}(\varphi)|^{-1} \sum_{L \in L_{1/2}(\varphi)} e^{-\beta |C_L|} dP_{\Phi|\chi_n}(\varphi|x) \]
for $0 \leq x \leq n$, where we set $\sum_{L \in L_{1/2}} = 0$ if $L_{1/2} = \emptyset$. Thus, conditioned on the event $\chi_n = x$, the number $a_x n^{1/2}/2$ may be interpreted as “typical” SILT relative to the lines in $L_{1/2}$, equivalently, $\exp\{-\beta a_x n^{1/2}/2\}$ represents a “typical” penalizing factor with respect to $L_{1/2}$, provided that $\chi_n = x$. Taking expectation, we arrive at the expected “typical” penalizing factor
\[ E_0(e^{-\beta J_n^{1/2}}) = E_0(\exp\{-\beta a_x n^{1/2}/2\}). \tag{3.20} \]
In the same fashion, we calculate
\[ E_0(\chi_n e^{-\beta J_n^{1/2}}) = E_0(\chi_n E_{\Phi|\chi_n}(|L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |C_L|} |\chi_n = x|)). \tag{3.21} \]
The proofs of Propositions 4 and 5 below (see also Definition 2) will throw light on the issue of this particular choice of penalizing weight. Observe in (3.19), though, that asymptotically the sum is preserved if lines were included that have larger SILT than $n^{1/2}$, in particular, lines that are typical to the SRW (as opposed to the weakly SAW) and tend to carry much larger SILT. Hence, $E_0$ might as well be employed as the expectation relative to the SRW-paths that are typical to the weakly SAW, which justifies (3.20) and (3.21).

Our first result collects an expression for the expected distance $E_0(\chi_n e^{-\beta J_n^{1/2}})$ in terms of $g(n)$ as defined in Lemma 1. A parallel derivation will provide an expression for $E_0(e^{-\beta J_n^{1/2}})$. Ultimately, we will be interested in the quotient of the two expectations. To justify the eventual transfer of the principal results to the SAW, we shall continue to be careful about whether constants in $n$ and/or $x$ depend on $\beta$ or not and often indicate this.

**Proposition 3 (Expected Distance Along Cones with Order $n^{1/2}$ SILT)** Let $\beta > 0$. If the $a_x$, specified in (3.19), satisfy Condition D in (3.5) for $\varepsilon = 0$ and $\gamma > 0$, then there are some constants $0 < \gamma_* \leq M < \infty$ (independent of $\beta$ as $\beta \to \infty$ and $M$ independent of $\beta > 0$ as well) such that as $n \to \infty$,
\[ E_0(\chi_n e^{-\beta J_n^{1/2}}) = K(n) n^{3/4} \beta^{1/2} g(n)(1 + o(1)) \]
for $\gamma_* \leq K(n) \leq M$, where $g(n)$ was defined in (3.6).
Proof. A combination of the observations preceding (3.21) together with (3.19) and (3.20) and Lemma [1] provides, as \( n \to \infty \),

\[
E_0(\chi_n e^{-\beta J_n^{1/2}}) = E_0(\chi_n E_{\Phi|\chi_n}(|L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |C_L|} |\chi_n = x|))
\]

\[
= \int_0^n x E_{\Phi|\chi_n}(|L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |C_L|} |\chi_n = x|) dP_{\chi_n}(x)
\]

\[
= \int_0^n x \left( \int_{\mathbb{Z}^2} |L_{1/2}(\varphi)|^{-1} \sum_{L \in L_{1/2}(\varphi)} e^{-\beta |C_L|} dP_{\Phi|x}(\varphi|x) \right) dP_{\chi_n}(x)
\]

\[
= \int_0^n x \exp\{-\beta a_x n^{1/2}/2\} dP_{\chi_n}(x)
\]

(3.22)

\[
= \int_0^n x g(x) dP_{\chi_n}(x)
\]

(3.23)

\[
= K(n) n^{3/4} \beta^{1/2} g(n) (1 + o(1))
\]

(3.24)

for \( \gamma_s \leq K(n) \leq M \), where to obtain the last two lines of the display, we apply Lemma [1] with \( \gamma_s = \gamma c(\rho_s) \), \( \varepsilon = 0 \), and with the \( a_x \) being bounded and such that there is some number \( \zeta > 0 \) so that \( \beta a_x \geq \zeta \) for every \( 0 \leq x \leq n \). These two properties of \( a_x \) may be seen as follows. First, since, by (3.16), \(|C_L|/n^{1/2} \) is in \([a_1, a_2] \), the average of the exponential terms \( \exp\{-\beta |C_L|\} \) over all lines in \( L_{1/2}(\varphi) \) may be rewritten as \( \exp\{-\beta a_x n^{1/2}/2\} \), say, for some number \( a_x \in [a_1, a_2] \), depending on \( x \). In particular, the \( a_x \) are bounded. Additionally, we assumed (remark following (3.16)) that \( a_1 \beta \) is a positive number independent of \( \beta \), thus, there is some number \( \zeta > 0 \) so that \( \beta a_x \geq \zeta \) for all \( x \). This completes our proof. \( \square \)

3.3. Weakly Self-Avoiding Cone Process relative to \( r \)-Shaped \( \Phi \). Once the lines are selected for \( V \), we may classify them according to the SILT that their cones carry. For any suitably small \( \delta > 0 \), define

\[
L_{1/2+} = L_{1/2+}(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^{1/2-\delta}, a_2 n^{1/2+\delta}] \}
\]

(3.25)

\[
L_- = L_-(\Phi) = \{ L \in V : 2|C_L| \in (0, a_1 n^{1/2-\delta}] \}
\]

\[
L_+ = L_+(\Phi) = \{ L \in V : 2|C_L| \in (a_2 n^{1/2+\delta}, 2b_2 n] \}
\]

\[
L_r = L_r(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^r, a_2 n^r] \}
\]

\[
L_\emptyset = L_\emptyset(\Phi) = \{ L \in V : |C_L| = 0 \}
\]

for each \( 0 \leq r \leq 1 \) and the same constants \( 0 < a_1 < a_2 < \infty \) as employed in (3.16). Thus, we here modify and extend the earlier definition \( L_{1/2} \).

In dealing with the problem to derive the expected distance with respect to the measure \( Q_n^\beta \), in other words, the expected distance of the weakly SAW, we will introduce a weakly self-avoiding process that is related to the weakly SAW. This related object that we shall
construct is suitable to calculate concrete expressions for the expected distances and attempts to “mimic” the following idea to asymptotically calculate the expected distance of the SRW after \( n \) steps from the starting point (for which process, though, the calculation is much more straightforward). In case of the latter, the basic ingredients may be sketched as follows. Rely on the Local Central Limit theorem and rewrite the density of the approximating Brownian motion to the SRW in polar coordinates. Calculating the expected distance of the SRW involves an integration over the radial part and an integration over the angle. This approximation by means of Brownian motion involves controlling an error.

In case of the former, roughly speaking, the process may be depicted as a weakly self-avoiding process whose penalizing weight takes into consideration the number of self-intersections near the line that passes through the starting point and the endpoint of the SRW-path (rather than penalizing the two-dimensional process according to \( J_n \)). Importantly, the definition of this process will depend on the choice of the set \( \mathcal{V} \), as made precise shortly, which will determine the SILT near the relevant lines. Moreover, bounds on the expected distance of this newly-defined process will be gotten by

(a) keeping track of the radial part of the SRW, penalized by the SILT in a certain cone,
(b) by integrating out over all lines in \( \mathcal{V} \).

Part of our strategy involves relating the expected distance of this process with the one of the weakly SAW. We begin to describe the “shape” of the set \( \mathcal{V} \). Note that \( \mathcal{V} \) depends on \( \Phi \) and its so-called shape reflects upon the shape of \( \Phi \).

**Definition 1 (\( \mathcal{V} \) and \( \Phi \) are \( r \)-shaped)** Let \( \rho > 0 \) be suitably small. We say that \( \mathcal{L}_r \) contributes (to \( J_n \)) essentially if

\[
\sum_{L \in \mathcal{L}_r} |C_L| \geq \frac{1}{2} J_n^{1-\rho}.
\]

In this case, we say that \( \mathcal{V} \) and \( \Phi \) are \( r \)-shaped or have shape \( r \). In particular, when \( r = 1/2 \), then we say that \( \mathcal{V} \) and \( \Phi \) have circular shape or are circular. The convention is that multiple shapes are allowed, that is, \( \Phi \) may simultaneously have shape \( 1/2 \) and shape \( 3/4 \).

**Remarks.**

(1) For our purposes and later calculations, it is not necessary that the lines contributing essentially, as explained in Definition 1, have exact SILT of order \( n^r \) in the sense that the real value \( r \) is hit precisely. Instead, it suffices to replace \( \mathcal{L}_r \) by \( \mathcal{L}_{r^*} = \{ L \in \mathcal{V} : 2|C_L| \in [a_1 n^r, a_2 n^{r+\delta}] \} \) for \( \delta > 0 \), and to ultimately let \( \delta \to 0 \) in the obtained results (because \( \delta > 0 \) was arbitrary). Hence, when applying Definition 1, we will think of \( \mathcal{L}_{r^*} \) rather than \( \mathcal{L}_r \) and refer to

\[
\sum_{L \in \mathcal{L}_{r^*}} |C_L| \geq \frac{1}{2} J_n^{1-\rho}.
\]  

(3.26)

With this meaning, it is obvious that, for sufficiently large \( n \), there must be \( 0 \leq r \leq 1 \) such that the set \( \mathcal{L}_{r^*} \) contributes essentially, and thus, the shape of \( \Phi \) and \( \mathcal{V} \) is well-defined.
Nevertheless, for the sake of not complicating our presentation, we shall not write $L_{r^*}$ and not use the extension in (3.26) but simply write $L_r$.

(2) We might as well choose $J_n \tau_n / 2$ with $\tau_n \rightarrow 0$ arbitrarily slowly as $n \rightarrow \infty$ in place of $J_n^{-\rho}/2$ in the defining inequality for the shape of $\Phi$. There is nothing special about the choice above.

Observe that if $L_r$ contributes essentially then, by (2.14) and (3.25),

$$b_1 n^{1-r-\rho} \leq |L_r| \leq \frac{2b_2}{a_1} n^{1-r}. \tag{3.27}$$

It is apparent that the upper bound in (3.27) holds even when $\Phi$ is not $r$-shaped. Since we choose $a_1$ and $a_2$ such that $\beta a_1$ and $\beta a_2$ are independent of $\beta$, it follows that $b_2/a_1$ is independent of $\beta$. Next, similarly as in (3.19), for any subset $L$ of $L_r \subset V$, define the numbers $a_x = a_x(L)$ by

$$\exp\{-\beta a_x(L)n^r/2\} = E_{\Phi|X_n}(|L(\Phi)|^{-1} \sum_{L \in L(\Phi)} e^{-\beta |C_L|} |x_n = x}) \tag{3.28}$$

for $0 \leq x \leq n$, where we set $\sum_{L \in L} = 0$ if $L = \emptyset$, and in parallel to (3.24) and (3.21), define the expected “typical” penalizing factor with respect to $L \subset L_r$ by

$$E_0(e^{-\beta J_n^L}) = E_0(\exp\{-\beta a_{x_n} n^r/2\}) \tag{3.29}$$

and $E_0(x_n \exp\{-\beta J_n^L\})$.

**Definition 2 (Weakly self-avoiding cone process relative to $r$-shaped $V$)** Define a weakly self-avoiding cone process relative to $V$ in shape $r$ by some two-dimensional process whose radial part is induced by the probability measure

$$Q_n^{\beta,V,r} = \frac{\exp\{-\beta |C_L|\}}{E_0 \exp\{-\beta J_n^L\}} \tag{3.30}$$

on the set of SRW-paths of length $n$ if $V$ has shape $r$, where $L$ denotes the line through the origin and the endpoint of the SRW after $n$ steps. Moreover, the expectation $E_{\beta,V,L_r} = E_{Q_n^{\beta,V,r}}$ relative to the radial part is calculated as in (3.28) followed by (3.29) with $L = L_r$.

Let $E_{\beta,V,*}(r)$ denote expectation of the two-dimensional weakly self-avoiding cone process relative to $V$ in shape $r$. In particular, we write $E_{\beta,V,*} = E_{\beta,V,*}(1/2)$. Thus, the definition of this process depends on the choice of $V$ and on $\Phi$. Note that there is no unique such process since only the distribution of the radial component of the process is prescribed and not even the distribution on the lines in $V$ is specified. Consequently, there will be several ways to choose the set $V$. Importantly though, the shape carries much information. It is worthwhile noting that, while $E_{\beta}(X_n)$ does not easily appear to be accessible to direct calculations, rather precise asymptotic expressions may be calculated for $E_{\beta,V,L_r}(X_n)$ for $0 \leq r \leq 1$. 


Lemma 2 (The $a_x(L_{1/2})$ satisfy Condition D) If $\Phi$ has circular shape for sufficiently large $n$, then the $a_x(L_{1/2})$, defined in (3.11), satisfy Condition D in (3.7) for $\varepsilon = 0$ and $\gamma > 0$, independent of $\beta$ as $\beta \to \infty$.

Proof. Fix some suitably small $\varepsilon > 0$ and suppose that $\Phi$ be circular for all sufficiently large $n$. Choose $\rho > 0$ sufficiently small. Let us invoke the notation that we introduced in the proof of Lemma 3, that is, write $E_0(\chi_n e^{-\beta J_n^{L_{1/2}}}) = I_n = J_1(n) + J_2(n) + J_3(n)$, and in the same spirit, $E_0(e^{-\beta J_n^{L_{1/2}}}) = J_1(n) + J_2(n) + J_3(n)$. We need to show that there is some $\rho_\ast > 0$ so that $J_2(n) + J_3(n) = \rho_\ast J_1(n)$ with $\rho_\ast \geq \rho_\ast$ for all sufficiently large $n > 0$. We begin with proving that $J_2(n) + J_3(n) \neq o(J_1(n))$ as $n \to \infty$.

For a moment, let us suppose in contrast that $J_2(n) + J_3(n) = o(J_1(n))$ as $n \to \infty$ so as to take this claim to a contradiction. Thus, $J_2(n) = o(J_1(n))$ and $J_3(n) = o(J_1(n))$ as $n \to \infty$. It would follow that $I_n = J_1(n)(1 + o(1))$ as $n \to \infty$ as well as $\sum_{i=1}^3 J_i(n) = J_1(n)(1 + o(1))$. The probability measure $Q^{eta,\varepsilon,1/2}$ induces a one-dimensional process $W_n$ which has expectation $E^W_{\beta,\varepsilon,L_{1/2}}(\chi_n)$, call it $E^W_{\beta,\varepsilon,L_{1/2}}$. Associate $W_n$ with the numbers $a_x(L_{1/2})$.

In view of the exponential form of the integrand of $I_n$, our assumption would imply that there is a number $z_n = z$ in $[0, r_1]$ that enjoys the property

$$
\frac{E_0(\chi_n e^{-\beta J_n^{L_{1/2}}})}{E_0(e^{-\beta J_n^{L_{1/2}}})} = \frac{I_n}{E_0(e^{-\beta J_n^{L_{1/2}}})} = (1 + o(1)) z
$$

(3.31)

as $n \to \infty$. In that event, the function $a_x$ is minimal at $z = z_n$, that is $a_x = \inf_{0 \leq y \leq r_1} a_y$ for all sufficiently large $n$. This can be seen as follows. Define $k(x) = \exp\{-(x^2 + \mu_x^2)/(2n)\}$, let $a_0 > 0$ and let $0 < \omega \leq a_0$ be some arbitrarily small number. If $a_{x_1} = a_0$ and $a_{x_2} = a_0 - \omega \geq 0$ for $0 \leq x_1, x_2 \leq r_1$, then it follows that $k(x_1) < k(x_2)$ for all sufficiently large $n$.

Now, for some suitably small $\omega = \omega(\beta) > 0$, define the set

$$
S_\omega = \{x \in [0, r_1] : a_x > a_x + \omega\}.
$$

Consider a modified process $\tilde{W}_n$ that is associated with numbers $\tilde{a}_x$ with $\tilde{a}_x = a_x$ for $x \in [0, r_1] \setminus S_\omega$, $\tilde{a}_x = a_x + \omega$ for $x \in S_\omega$, and $\tilde{a}_x = a_x + a(n)$ for $r_1 < x \leq n$, where $a(n) > 0$ is some suitable number, chosen so as to preserve the distribution of $J_n$. Thus, $\tilde{a}_x \leq a_x + \omega$ for $x \in [0, r_1]$. Observe that the modified process $\tilde{W}_n$ has the same expectation $E^W_{\beta,\varepsilon,L_{1/2}}(\chi_n) = E^W_{\beta,\varepsilon,L_{1/2}}(\chi_n)$ as the process $W_n$ since, firstly, $q(x)$ in (3.33) was decreased on $(r_1, n]$, and thus, $J_2(n) + J_3(n)$ and the corresponding part $J_2(n) + J_3(n)$ of the integral in the denominator of $E^W_{\beta,\varepsilon,L_{1/2}}(\chi_n)$ were both decreased, and secondly, $J_1(n)$ is as before thanks to (3.31). Note that adding a constant number of self-intersections to all realizations of this underlying weakly self-avoiding process $\tilde{W}_n$ does not change its probability measure. Subtract the number $a_z$ from $\tilde{a}_x$ for every $0 \leq x \leq n$, that is, let $\tilde{a}_x = \tilde{a}_x - a_z \geq 0$ for every
0 \leq x \leq n. Thus, \( \hat{a}_x \leq \omega \) for \( x \in [0, r_1] \) and \( \hat{a}_x \) is suitable on \( (r_1, n] \). In particular, we may choose \( \omega < a_1 b_1/b_2 \), where \( a_1 \) was introduced in (3.16). The gotten process \( \hat{W}_n \) associated with the numbers \( \hat{a}_x \) has expectation \( \mathbf{E}_{\hat{W}}^{\hat{V}, \mathcal{L}_{1/2}}(\chi_n) = \mathbf{E}_{\hat{V}, \mathcal{L}_{1/2}}(\chi_n) \), too, the same as do \( W_n \) and \( \hat{W}_n \).

The number of lines in \( \mathcal{L}_{1/2} \) that would be needed to assign the self-intersection points of the two-dimensional process with marginal \( \hat{W}_n \) is at least \( n^{1/2-\rho_1}b_1/\omega \), where \( \omega \) is suitably small. But since \( \rho > 0 \) was arbitrary and also by Remark (2) following Definition [1], this contradicts (3.27) and the assumption that \( \Phi \) is circular. We conclude that \( J_2(n) + J_3(n) \neq o(J_1(n)) \) as \( n \to \infty \). Since \( \varepsilon > 0 \) was arbitrary, it follows that, for every \( \varepsilon > 0 \), \( J_2(n) + J_3(n) \neq o(J_1(n)) \) as \( n \to \infty \).

It remains to be shown that there is no subsequence \( n_k \) such that \( J_2(n_k) + J_3(n_k) = o(J_1(n_k)) \) as \( k \to \infty \). From this it will follow that there is some number \( \rho_n > 0 \) that bounds \( \rho_n \) from below with \( n \). But the same point can be made as explained above when \( n \) is replaced by \( n_k \) everywhere. Whence, we conclude that Condition D must hold for \( \varepsilon = 0 \). Observe that this implies that \( r_1 = r_2 \) and \( J_2(n) = 0 \).

In addition, we remark that \( \gamma > 0 \) may be chosen uniformly over \( \beta > 0 \) as \( \beta \to \infty \). This can be seen as follows. Any of the asymptotic statements in Lemma [1] and in the above lines of proof depend on expressions, for example, of the form \( \beta^{1/2} n^{3/4} \). Hence, if \( N(\beta) \) is a threshold so that, for all \( n \geq N(\beta) \), a given expression in \( n \) differs from its corresponding limiting expression by at most \( \varepsilon \) (some fixed \( \varepsilon \)), it follows that \( N(\beta') \leq N(\beta) \) for \( \beta < \beta' \). As a consequence of the fact that \( \gamma > 0 \) may be chosen uniformly in \( n \), the choice of \( \gamma \) is uniformly over \( \beta > 0 \) as \( \beta \to \infty \) (yet not as \( \beta \to 0 \)). This proves the advertised claim. \( \square \)

The following two propositions collect the principal results.

**Proposition 4 (Upper Bound for \( \mathbf{E}_{\beta} \chi_n \))** Let \( \beta > 0 \). There is some constant \( M < \infty \) (independent of \( \beta \)) so that as \( n \to \infty \),

\[
\mathbf{E}_{\beta}(\chi_n) \leq (1 + o(1)) \frac{\max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \mathbf{E}_0(\chi_n e^{-\beta J_{\mathcal{L}}})}{\mathbf{E}_0(e^{-\beta J_{n^{1/2}}})} \\
\leq M n^{3/4} \beta^{1/2} (1 + o(1)) \frac{\max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} g(n)}{h(n)} \\
= M n^{3/4} \beta^{1/2} (1 + o(1)) \frac{\max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \int_0^n (a_x(\mathcal{L}))^{1/2} q(x) \, d\mathbf{P}_{\chi_n}(x)}{\int_0^n q(x) \, d\mathbf{P}_{\chi_n}(x)},
\]

where \( q(x) \) is as defined in (3.23) and \( a_x(\mathcal{L}) \) in (3.28) when \( r = 1/2 \).

**Proof.** It will be sufficient to prove that, for \( \mathcal{V} \) in circular shape, as \( n \to \infty \),

(I) \( \mathbf{E}_{\beta, \mathcal{V}, s}(\chi_n) \leq (1 + o(1)) \max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \mathbf{E}_0(\chi_n e^{-\beta J_{\mathcal{L}}})/\mathbf{E}_0(e^{-\beta J_{n^{1/2}}}) \),

(II) \( \mathbf{E}_{\beta}(\chi_n) \leq \mathbf{E}_{\beta, \mathcal{V}, s}(\chi_n)(1 + o(1)) \), and,

(III) to evaluate the expression on the righthand side in (I).
The Planar Self-Avoiding Walk

Parts (I) and (III). For this purpose, assume that \( V \) is 1/2-shaped for all sufficiently large \( n \). Fix some suitably small \( \delta > 0 \) and fix \( \rho < \delta/2 \). Since upon the assignment of all self-intersection points to cones, every line in \( V \) falls in exactly one of the four sets \( \mathcal{L}_{1/2}^+, \mathcal{L}_-, \mathcal{L}_+ \), and \( \mathcal{L}_\emptyset \), defined in (3.2) and in view of the definition of \( E_{\beta,\nu,*} \), we collect

\[
E_{\beta,\nu,*}(x_n) \leq \sum_{\tilde{L} \in \{\mathcal{L}_{1/2}^+, \mathcal{L}_-, \mathcal{L}_+ \}} \mathbf{P}(L \in \tilde{L}) \frac{\max_{\tilde{E} \subset \tilde{L}} E_0(x_n e^{-\beta J_n^{\tilde{E}}})}{E_0(e^{-\beta J_n^{\tilde{E}}})}, \tag{3.32}
\]

where \( E_0(x_n e^{-\beta J_n^{\tilde{E}}}) \) and \( E_0(e^{-\beta J_n^{\tilde{E}}}) \) for \( \tilde{L} \in \{\mathcal{L}_{1/2}^+, \mathcal{L}_-, \mathcal{L}_+, \mathcal{L}_\emptyset \} \) are to be understood in the sense of definition (3.29). Observe that the last term in (3.32), the SRW term (because it resembles the contribution that we would obtain from the SRW), is of asymptotic order no larger than \( n^{1/2} \). It will turn out that (3.32) is bounded above by the first term in (3.32) times \((1 + o(1))\) as \( n \to \infty \). Clearly, \( \mathbf{P}(L \in \mathcal{L}_{1/2}^+) \leq 1 \). Thus, the first term is bounded above by its quotient. First, we will see that the first term dominates the second and third terms and is of order no larger than \( n^{3/4} \).

To analyze \( E_{\beta,\nu,\mathcal{L}_{1/2}^+}(x_n) = E_0(x_n e^{-\beta J_n^{\mathcal{L}_{1/2}^+}})/E_0(e^{-\beta J_n^{\mathcal{L}_{1/2}^+}}) \), we shall proceed much as we did to verify Proposition 3. Let us point out the modifications required in the proofs of Lemma 4 and Proposition 3. For ease of exposition, we will shorter write \( a_x = a_x(\mathcal{L}_{1/2}^+) \).

In parallel to the handling of \( I_n \) in Lemma 4, we let \( q_r(x) = \exp\{-\beta a_x n^{1/2 + r}/2\} \) for any \( r \in [-1/2,1/2] \) and evaluate the integral

\[
I_n(r) = \int_0^n x q_r(x) \, d\mathbf{P}_{\chi_n}(x)
\]

by proceeding along the same reasoning (as in Lemma 4) with

\[
\mu_x(r) = (\beta a_x)^{1/2} n^{3/4 + r/2}
\]

in place of \( \mu_x = (\beta a_x)^{1/2} n^{3/4} \) in (3.2). Then we arrive at

\[
I_n(r) = \beta^{1/2} K_r(n) n^{3/4 + r/2} g_r(n)(1 + o(1)) \tag{3.33}
\]

for \( K_r(n) \leq M < \infty \), where \( g_r(n) = \int_0^n (a_x(\mathcal{L}_{1/2}^+))^{1/2} q_r(x) \, d\mathbf{P}_{\chi_n}(x) \), and in view of Proposition 3 and since, by Lemma 2, the \( a_x(\mathcal{L}_{1/2}^+) \) satisfy Condition D in (3.3) for \( \varepsilon = 0 \) and some \( \gamma > 0 \), we have \( 0 < \gamma_x \leq K_0(n) = K(n) \leq M < \infty \) (\( M \) independent of both \( \beta \) and \( \gamma_x \) independent of \( \beta \) as \( \beta \to \infty \)).

Following the steps in the proof of Proposition 3 line by line, with \( \mathcal{L}_{1/2}^+ \) replaced by \( \mathcal{L}_{1/2}^+ \), and keeping in mind expression (3.33), we obtain for each \( r \in [-1/2,1/2] \), as \( n \to \infty \),

\[
E_0(x_n e^{-\beta J_n^{\mathcal{L}_{1/2}^+}}) = \beta^{1/2} K_r(n) n^{3/4 + r/2} g_r(n)(1 + o(1)) \tag{3.34}
\]

and the quotient

\[
E_{\beta,\nu,\mathcal{L}_{1/2}^+}(x_n) = \frac{E_0(x_n e^{-\beta J_n^{\mathcal{L}_{1/2}^+}})}{E_0(e^{-\beta J_n^{\mathcal{L}_{1/2}^+}})} = K_r(n) \beta^{1/2} n^{3/4 + r/2} g_r(n) h_r(n)(1 + o(1))
\]

\[
= K_r(n) \beta^{1/2} n^{3/4 + r/2} \frac{\int_0^n (a_x)^{1/2} q_r(x) \, d\mathbf{P}_{\chi_n}(x)}{\int_0^n q_r(x) \, d\mathbf{P}_{\chi_n}(x)} (1 + o(1)). \tag{3.35}
\]
Hence, in view of the boundedness of the $a_x(L_{1/2+r})$ in $n$, expression (3.35), for $r \in [-1/2, -\delta)$, is maximal for $r = -\delta$, as $n \to \infty$. As a consequence,

$$\frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}}} \mathbb{1}_{\mathcal{L}_{1/2+r}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}}})} \leq \frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2+r}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2+r}}})} \quad (3.36)$$

with $r = -\delta$. However, the righthand side of (3.36) is strictly less than $\mathbf{E}_{\beta, \mathcal{V}, \mathcal{L}_{1/2}} = \mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2}}}) / \mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2}}})$. Holding on to (3.35) with $r = 0$, we conclude that

$$\frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}}})} < \frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2}}})} \leq M^{3/2} n^{3/4} \frac{g(n)}{h(n)} (1 + o(1)). \quad (3.37)$$

Since the exponents of the terms in the expression on the leftmost side of (3.37) are strictly less than the exponent of the leading term in the expression in the middle, even more is true, namely, as $n \to \infty$,

$$\frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}}})} = o \left( \frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2}}})} \right). \quad (3.38)$$

(3.38) continues to hold if the numerator of the expression on the lefthand side is maximized over subsets of $\mathcal{L}_{\mathcal{L}_{1/2}}$. We conclude that the second term in (3.32) is dominated by the rightmost side of (3.37). To accomplish the upper bound for $\mathbf{E}_{\beta, \mathcal{V}, \mathcal{L}}(\chi_n)$, it remains to be shown that the second factor of the first term in (3.32) as well dominates the third term.

We will argue that $\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2+r})$ for $r \in (\delta, 1/2]$ is small relative to $\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2})$. The probability $\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2+r})$ may be interpreted as a Palm probability (see (4.4) in the Appendix), that is,

$$\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2+r}) = \frac{\mathbf{E}_\Phi \sum_{L \in \mathcal{V}} \mathbb{1}_{\mathcal{L}_{1/2+r}}(L)}{|\mathcal{V}|} = \frac{\mathbf{E}_\Phi |\mathcal{L}_{1/2+r}|}{|\mathcal{V}|}. \quad (3.39)$$

There are, however, at most $b_2 n$ self-intersections to distribute to cones, each of which carries at least $a_1 n^{1/2+r}/2$ self-intersections. Thus, $|\mathcal{L}_{1/2+r}| \leq (2b_2/a_1) n^{1/2-r}$. Therefore on the one hand, $\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2+r}) \leq (2b_2/a_1 |\mathcal{V}|) n^{1/2-r}$. On the other hand, because we assumed $\Phi$ to be circular, we have $\mathbf{P}_\Phi(L \in \mathcal{L}_{1/2}) \geq (b_1/a_2 |\mathcal{V}|) n^{1/2-r}$. Hence, a combination of these two observations together with (3.35) yields, for every $r \in (\delta, 1/2]$,

$$\frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2+r}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2+r}}})} \mathbf{P}_\Phi(L \in \mathcal{L}_{1/2+r}) < \frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2}}})} \mathbf{P}_\Phi(L \in \mathcal{L}_{1/2})$$

because the order of the term on the left is at most $n^{5/4+r}/|\mathcal{V}|$ and the one on the righthand side is at least $n^{5/4-r}/|\mathcal{V}|$, the latter being strictly larger than the former since we picked $\rho < \delta/2$, and thus,

$$\frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_+}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_+}})} \mathbf{P}_\Phi(L \in \mathcal{L}_+) < \frac{\mathbf{E}_0(\chi_n e^{-\beta J_n^{\mathcal{L}_{1/2}}})}{\mathbf{E}_0(e^{-\beta J_n^{\mathcal{L}_{1/2}}})}. \quad (3.40)$$
Inequality (3.40) continues to hold if the numerator of the quotient of the lefthand expression is maximized over subsets of \( \mathbb{L}_+ \). Since the exponents of the terms in the expression on the lefthand side of (3.40) are strictly less than the exponent of the leading term on the righthand side, again as \( n \to \infty \),

\[
\frac{E_0(\chi_n e^{-\beta J_n^L})}{E_0(e^{-\beta J_n^L})} P_{\Phi}(L \in \mathbb{L}_+) = o \left( \frac{E_0(\chi_n e^{-\beta J_n^L_{1/2}})}{E_0(e^{-\beta J_n^L_{1/2}})} \right).
\] (3.41)

We summarize our progress as follows. Since \( \delta > 0 \) was arbitrary, combining (3.32), (3.38), and (3.41) provides as \( n \to \infty \),

\[
E_{\beta, V, s}(\chi_n) \leq (1 + o(1)) \frac{\max_{L \subset \mathbb{L}_1/2} E_0(\chi_n e^{-\beta J_n^L})}{E_0(e^{-\beta J_n^L_{1/2}})} \leq M \beta^{1/2} n^{3/4} (1 + o(1)) \frac{\max_{L \subset \mathbb{L}_1/2} g(n)}{h(n)}
\] (3.42)

for \( M < \infty \) (independent of \( \beta \)). This completes the verification of (I) along with the asymptotic evaluation of its righthand side.

**Part (II).** We turn to showing (II), which will finish our proof. Recall that \( E_{\beta, V, s}(r) \) denotes expectation of the two-dimensional weakly self-avoiding cone process relative to \( V \) in shape \( r \). In order to compare \( E_{\beta, V, s}(r)(\chi_n) \) and \( E_{\beta}(\chi_n) \), the strategy will be to show that, for fixed \( J_n \in [b_1n, b_2n] \), the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r \)-shaped is larger than the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( s \)-shaped (but not \( r \)-shaped) for \( 1/2 \leq r < s \). We will continue to show that \( \mathbb{L}_+ \) for \( 0 \leq s < 1/2 \) plays a negligible role as well. Hence, the measure \( \mathcal{Q}_n^0 \) prefers circular shape. In other words, most SRW-paths that satisfy (2.14) arise from a \( \Phi \) that is \( 1/2 \)-shaped. Finally, we shall compare the centers of mass of the weakly self-avoiding cone process and the weakly SAW.

**(a) \( \Phi \) prefers circular shape.** Fix \( J_n \) (and assume that \( J_n \in [b_1n, b_2n] \)). Partition the interval \([1/2, 1]\) into \( R \) subintervals, each of which has equal length, that is, let \( 1/2 = r_0 < r_1 < r_2 < \ldots < r_R = 1 \). We are interested in comparing the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r_{k-1} \)-shaped to the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r_k \)-shaped (but not \( r_{k-1} \)-shaped). For this purpose, we shall give an inductive argument over \( k \). Pick a SRW-path \( \gamma \) of length \( n \) with \( J_n \) whose point process \( \Phi \) has shape \( r_k \). We will show that (i) associated with \( \gamma \), there is a large set \( F_\gamma \) of SRW-paths whose realizations of \( \Phi \) have shape \( r_{k-1} \), and (ii) two sets \( F_\gamma \) and \( F_{\gamma'} \) are disjoint for \( \gamma \neq \gamma' \). To see this, we *cut and paste* the path \( \gamma \) as follows. Let \( P_\gamma \) denote the smallest parallelepiped that contains the path \( \gamma \) and let \( l_\gamma \) denote the largest integer less than or equal to the length of the long side of \( P_\gamma \). Divide \( P_\gamma \) into sub-parallelepiped whose sides are parallel to the sides of \( P_\gamma \) by partitioning the two long sides of \( P_\gamma \) into \( n_f \) subintervals in the same fashion whose endpoints are vertices of the integer lattice and by connecting the two endpoints of the subintervals that are opposite to each other on the two sides. Shift
each of the sub-parallelepipeds including the SRW-subpaths contained by a definite amount between 1 and \( K \) \((K:\text{ some constant})\) along one of the two directions of the shorter sides of \( P_\gamma \) and reconnect the SRW-subpaths where they were disconnected. In doing this, the shifts are chosen such that the new path \( \gamma' \) will have shape \( r_{k-1} \) and the total number of connections needed to reconnect those subpaths equals a number \( C_n \) that is constant in \( k \). Observe that such a choice of shifts exists. When walking through the new path \( \tilde{\gamma} \), because of the necessary extra steps to reconnect the subpaths, the last several steps of \( \gamma \) will be ignored. Note that this latter number of steps is independent of \( k \). Hence, if the pieces to reconnect are self-avoiding, then \( J_n \) is no larger after this cut-and-paste procedure than before. This is always possible for otherwise we shift apart the sub-parallelepipeds such that they are sufficiently separated from each other. Now, either we choose the reconnecting pieces such that \( J_n \) is preserved or we “shift back” (along the direction of the long sides of the parallelepiped) some or all of the sub-parallelepipeds so that any two parallelepipeds overlap sufficiently to preserve \( J_n \) and then reconnect the SRW-subpaths where they were disconnected. Again, we shift in such a fashion that the total number of connections needed to reconnect the subpaths equals \( C_n \). The number of these newly constructed paths in \( F_\gamma \) grows at least at the order that the number of ways does to choose \( n_f \) locations (to shift) among \( l_s \) sites, which is a number larger than 1 for all large enough \( n \). Hence, the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r_{k-1} \)-shaped is larger than the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r_k \)-shaped. Since this argument can be made for every \( 1 \leq k \leq R \) and the number of SRW-paths with \( J_n \) whose point process \( \Phi \) has shape \( r_R = 1 \) is at least 1, it follows that the number of SRW-paths with \( J_n \) whose point process \( \Phi \) is \( r \)-shaped is maximal for \( r = 1/2 \).

(b) It suffices to consider shapes \( r \) with \( r \geq 1/2 \). Our next point will be to reason that it suffices to consider only \( L_r \) with \( r \geq 1/2 \). Suppose that \( 0 \leq r < 1/2 \). In view of (3.33), for every \( \epsilon > 0 \), we obtain \( E_{\beta, \mathcal{V}, L_r}(\chi_n) = o(n^{1/2+r/2+\epsilon}) \) as \( n \to \infty \). Moreover, we would need of order \( n^{1-r} > n^{1/2} \) lines in \( L_r \) to allocate all points of \( \Phi \). However, again by (3.33), the points of \( \Phi \) in the cones of at least a positive fraction of these \( n^{1-r} \) lines are expected (under \( P_\Phi \)) to lie at distance of order strictly larger than \( E_{\beta, \mathcal{V}, L_r}(\chi_n) \). Therefore, we may instead use lines in \( \mathcal{V} \) along directions that are about “orthogonal” to the directions of the lines in \( L_r \), that is, lines that cross the smallest rectangle that contains the points of \( \Phi \) along the long side of the rectangle. In other words, we may use lines in \( L_s \) with \( s \geq 1/2 \). Consequently, it follows that it is sufficient to restrict attention to \( r \)-shaped \( \Phi \) for \( 1/2 \leq r \leq 1 \) and to use \( L_r \) with \( r \geq 1/2 \).

The considerations in (a) above also imply that both probability distributions decay exponentially fast around their centers of mass. Combining this observation with the fact that the shape of \( \Phi \) relates the SILT of the weakly SAW to the one of the weakly self-avoiding cone process provides that the two probability distributions asymptotically have the same centers of mass (up to error terms). Together with these, the upshot of above passages (a) and (b) is that, in comparing \( E_\beta(\chi_n) \) to \( E_{\beta, \mathcal{V}_s(r)}(\chi_n) \) for \( 0 \leq r \leq 1 \), it is
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enough to choose \( r = 1/2 \) and to study the expected distance of the weakly self-avoiding cone process relative to \( \Phi \) when in circular shape. Hence, in particular, we are led to

\[
E_\beta(\chi_n) \leq E_{\beta,V,\ast}(\chi_n)(1 + o(1))
\]
as \( n \to \infty \). This accomplishes the proof of (II), and thus, ends the proof. \( \square \)

**Proposition 5 (Lower Bound for \( E_\beta \chi_n \))** Let \( \beta > 0 \). There is a constant \( m > 0 \) (that may depend on \( \beta \)) such that as \( n \to \infty \),

\[
E_\beta(\chi_n) \geq \mathbb{P}_\Phi(L \in \mathcal{L}_{1/2}) \frac{\min_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} E_0(\chi_n e^{-\beta J_n^L})}{E_0(e^{-\beta J_n^\ast L})} \geq (1 + o(1)) mn^{3/4} \beta^{1/2} \min_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \frac{g(n)}{h(n)},
\]

where \( g(x) \) is as defined in (3.3), \( a_x(\mathcal{L}) \) in (3.28) when \( r = 1/2 \), and \( h(n) \) as in Proposition 4, and, the minimum is over subsets \( \mathcal{L} \subset \mathcal{L}_{1/2} \) that form a subset of \( \mathcal{V} \) that is circular for sufficiently large \( n \).

**Proof.** In parallel as we argued earlier in part (II)(b) of the proof of Proposition 4 (to prove that shapes \( s \) of \( \Phi \) for \( 0 \leq s < 1/2 \) are negligible), we find \( E_{\beta,V,\ast}(\chi_n) \leq E_\beta(\chi_n) \) for every \( 0 \leq s < 1/2 \). Moreover, integrating out over the lines in \( \mathcal{V} \) yields

\[
E_{\beta,V,\ast}(\chi_n) \leq E_\beta(\chi_n).
\]

This together with (3.27) for \( r = 1/2 \) yields

\[
|\mathcal{L}_{1/2}| \geq (b_1/a_2)n^{1/2-\rho}.
\]

From the fact that \( \Phi \) prefers circular shape (see part (II) of the proof of Proposition 4) and the estimates in (3.27) for \( r = 1/2 \), and, from the fact that we can choose \( a_1 \) such that \( b_2/a_1 \) is independent of \( \beta \), we also conclude that \( \mathcal{V} \) can be (optimally) constructed to have size \( |\mathcal{V}| = v_n n^{1/2} \) for
0 < v_n ≤ v_2 < ∞ for all sufficiently large n, where v_2 is independent of β. Consequently, we end up with
\[ P_\Phi(L \in \mathcal{L}_{1/2}) = \frac{\mathbb{E}_\Phi \sum_{L \in \mathcal{V}} \chi_{1/2}(L)}{|\mathcal{V}|} > m_* n^{-\rho} \] (3.45)
for m_* = b_1/(v_2a_2) > 0 and every sufficiently large n. Note that since b_1/a_2 depends on β, so does m_. In light of (3.35) with r = 0, (3.43), (3.44), and (3.45), we collect
\[ \mathbb{E}_\beta(\chi_n) \geq m_* n^{-\rho} \frac{\min_{L \subseteq \mathcal{L}_{1/2}} \mathbb{E}_0(\chi_n e^{-\beta J_n^L})}{\mathbb{E}_0(e^{-\beta J_n^L})} \]
\[ \geq m_* \gamma_n (1 + o(1)) \beta^{1/2} n^{3/4-\rho} \frac{\min_{L \subseteq \mathcal{L}_{1/2}} g(n)}{h(n)}, \] (3.46)
where \( \gamma_n \) is independent of \( \beta \) as \( \beta \to \infty \). Since \( \rho > 0 \) was arbitrary, this lower bound is as announced when \( m_* \gamma_n = m \), which finishes our proof.

In this paper, we won’t address the issue of existence of the limit \( \lim_{n \to \infty} n^{-3/4} \mathbb{E}_\beta(\chi_n) \) but only consider its lim sup and lim inf. Observe that, since \( a_1 < a_x < a_2 \) for all \( x \), we collect, for any subset \( \mathcal{L}_* \) in \( \mathcal{V} \),
\[ (a_1)^{1/2} \leq \frac{\min_{L \subseteq \mathcal{L}_*} g(n)}{h(n)} \leq \frac{\max_{L \subseteq \mathcal{L}_*} g(n)}{h(n)} \leq (a_2)^{1/2}, \] (3.47)

3.4. Distance Exponents of the Self-Avoiding Walk. Propositions 4 and 5 in combination with (3.47), as \( n \to \infty \),
\[ m (a_1 \beta)^{1/2} (1 + o(1)) \leq n^{-3/4} \mathbb{E}_\beta(\chi_n) \leq M (a_2 \beta)^{1/2} (1 + o(1)), \]
where \( 0 < m \) may depend on \( \beta \), even as \( \beta \to \infty \), and \( M < \infty \) is independent of all \( \beta > 0 \). Since we can choose \( a_1 \) and \( a_2 \) such that \( \beta a_1 \) and \( \beta a_2 \) do not depend on \( \beta \) (see the remark following (3.27)), we let \( \rho_1 = m (a_1 \beta)^{1/2} \) and \( \rho_2 = M (a_2 \beta)^{1/2} \), which establishes what we stated for the weakly SAW.

It is left to notice the following about the results in Propositions 4, 5 and 6. Suppose that, for some fixed \( \epsilon > 0 \) and \( \beta_0 \), there be an integer \( N(\beta_0, \epsilon) = N(\beta_0) \), such that, for every
$n > N(\beta_0)$, a given expression for finite $n$ is within distance $\varepsilon$ from its corresponding limiting expression in Propositions 3, 4 and 5. Since, everywhere in the calculated expectations (as exponent or multiplicative factor), $n$ shows up as $n^s \beta^t$ for some powers $s, t > 0$, the threshold $N(\beta_0)$ is valid for all $\beta \geq \beta_0$. In other words, $\beta' > \beta_0$ implies that $N(\beta') \leq N(\beta_0)$. Therefore, each of the various thresholds $N(\cdot)$ can be chosen uniformly for all $\beta \geq \beta_0$, in particular, uniformly as $\beta \to \infty$.

Therefore, each of the various thresholds $N(\cdot)$ can be chosen uniformly for all $\beta \geq \beta_0$, in particular, uniformly as $\beta \to \infty$. Together with the fact that $E_\beta(\chi_n)/n^{3/4}$ is bounded above by a constant that is independent of $\beta$ as $\beta \to \infty$ for every sufficiently large $n$, this implies that $\limsup_{n \to \infty} \lim_{\beta \to \infty} E_\beta(\chi_n)/n^{3/4} \leq \rho_2$, where $\rho_2$ is a constant that is independent of $\beta$. In other words, the limsup of the normalized expected distance of the SAW is bounded above. In particular, this establishes that $3/4$ is an upper bound for the distance exponent of the SAW.

To bound the distance exponent of the SAW from below, it suffices to let $\beta \to \infty$ in the double limit in such a way that $\limsup_{n \to \infty} |\ln \rho_1(\beta)/\ln n| = 0$ and to then exchange the limit as $\beta \to \infty$ and the limitinf as $n \to \infty$ of $\ln E_\beta(\chi_n)/\ln n$. As a consequence, the distance exponent of the SAW is no less than $3/4$. In other words, the self-avoiding walk has the same distance exponent as does the weakly SAW, as claimed.

This completes the proof of Theorem 1. Observe that we did not bound the normalized expected distance of the SAW from below, whereas we gave an upper bound that is constant.

The next result accomplishes Theorem 2.

**Corollary 2** Let $\beta > 0$. There are some constants $0 < \rho_3(\beta) \leq \rho_4 < \infty$ ($\rho_4$ independent of $\beta$) such that

$$\rho_3 \leq \liminf_{n \to \infty} n^{-3/2} E_\beta(\chi_n^2) \leq \limsup_{n \to \infty} n^{-3/2} E_\beta(\chi_n^2) \leq \rho_4.$$ 

In particular, the MSD exponent of the planar SAW equals $3/2$ and its normalized MSD is bounded above by $\rho_4$.

**Proof.** We will only address the arguments that show the statements for the weakly SAW and refer the reader to the proof of Corollary 1 for the aspects of transferring some portion of the results to the SAW. Since the pattern of proof is as before, we only list a short guide. Carry out Propositions 3, 4 and 5 with $\chi_n^2$ in place of $\chi_n$. In particular, write

$$E_\beta(\chi_n^2 e^{-\beta J_n^{1/2}}) = \int_0^n x^2 q(x) d\mathbf{P}_{\chi_n}(x)$$

and proceed along our earlier lines that derived $E_\beta(\chi_n e^{-\beta J_n^{1/2}})$ and $E_\beta(\chi_n)$. Nothing more than minor modifications are required to wind up with the advertised results.
Theorem 3 Let $\beta > 0$ and let $R_n$ denote the radius of the convex hull of the SRW-path $S_0, S_1, \ldots, S_n$. Then $R_n$ satisfies all statements in Corollaries 1 and 2 with $\chi_n$ replaced by $R_n$.

Proof. Observe that we are interested in the maximal distance of $S_0, S_1, \ldots, S_n$ along any line rather than the distance of the position of $S_n$ from the starting point. The reflection principle gives the upper bound $dP_{R_n}(x)/dx \leq 2dP_{\chi_n}(x)/dx$ whereas the lower bound $dP_{R_n}(x)/dx \geq dP_{\chi_n}(x)/dx$ is readily apparent. From this, the results are immediately collectable.

Remark (Transition $\beta \rightarrow 0$). The transition $\beta \rightarrow 0$ is quite different from the transition $\beta \rightarrow \infty$. Let us quickly look at what happens to our derived expressions when $\beta = 0$. In that fictive case (since the results were proven under the assumption $\beta > 0$), all terms in (3.32) are of asymptotic order $n^{1/2}$, and so is the term in (3.44). Because this is drastically different from the case $\beta > 0$, in which case the asymptotic order of the largest term is $n^{3/4}$, we observe a discontinuity of the expected distance measures and distance exponents of the weakly SAW as $\beta \rightarrow 0$. In contrast, the case $\beta \rightarrow \infty$ behaves as any case for fixed $\beta$.

A Appendix: Examples of Palm Distributions

The first example is the one alluded to in (a) of the introductory paragraph of Section 3, with the “typical” random objects being points. We will write down the Palm distribution of the random measure $P_\Phi$ in either case, when the underlying point process is stationary and when non-stationary. The second example will study a sum of some exponential random variables, with the random objects being points, whereas the third example will look at some sum of exponential random functionals when the “typical” random objects are lines. All examples are in $\mathbb{R}^d$ for $d \geq 1$. We borrow the notation introduced in Section 3.1.

Example 1: Number of points without nearest neighbors within distance $r$. Let $\Phi = \{x_1, x_2, \ldots\}$ denote some point process in $\mathbb{R}^d$ so that its expectation $E\Phi$ is $\sigma$-finite. Let $B_r(z)$ denote the ball of radius $r > 0$ centered at the point $z$ in $\mathbb{R}^d$ and $o$ denote the origin in $\mathbb{R}^d$. Define the set

$$Y = \{\varphi \in N_\Phi : |\varphi \cap B_r(o)| = 1\} = \{\varphi \in N_\Phi : \varphi \cap B_r(o)\text{ is a singleton}\}$$

in $N_\Phi$, let $B$ denote some Borel set in $\mathbb{R}^d$, and

$$h(z, \varphi) = 1_B(z)1_Y(\varphi - z),$$

where $1_B(\cdot)$ denotes the indicator function of $B$. We may think of the condition $1_Y(\varphi - z)$ as removing all points from a realization $\varphi$ that have any nearest neighbors at distance less
than \( r \). Keeping these in mind, we might be interested in evaluating the mean number of points of \( \Phi \) in \( B \) whose nearest neighbors are all at distance at least \( r \). Thus, in light of some version of formula (3.17),

\[
E_{\Phi} \left( \sum_{z \in \Phi} h(z, \Phi) \right) = \int \sum_{z \in \varphi} h(z, \varphi) \, dP_{\Phi}(\varphi),
\]

we obtain

\[
E_{\Phi} \left( \sum_{z \in \Phi \cap B} 1_{Y}(\Phi - z) \right) = \int \sum_{z \in \varphi \cap B} 1_{Y}(\varphi - z) \, dP_{\Phi}(\varphi).
\]

If we assume that \( \Phi \) is a stationary point process with finite nonzero intensity \( \lambda \) and \( \mu_d \) denotes Lebesgue measure in \( \mathbb{R}^d \), then the Palm distribution \( P_o \) (at \( o \)) of \( P_{\Phi} \) is a distribution on \( (N_{\Phi}, N_{\Phi}) \) defined by

\[
P_o(Y) = \int_{\mathbb{R}^d} \sum_{z \in \varphi \cap B} 1_{Y}(\varphi - z) \, dP_{\Phi}(\varphi) / \lambda \mu_d(B).
\]

This formula holds for any \( Y \in N_{\Phi} \) and any Borel set \( B \) of positive volume. Note that, by the stationarity of the point process, the definition does not depend on the choice of \( B \).

On the other hand, if the point process is not stationary, then the Palm distribution \( P_o \) of \( P_{\Phi} \) is gotten by normalizing as follows:

\[
P_o(Y, B) = \frac{\int_{\mathbb{R}^d} \sum_{z \in \varphi \cap B} 1_{Y}(\varphi - z) \, dP_{\Phi}(\varphi)}{\int_{\mathbb{R}^d} |\varphi \cap B| \, dP_{\Phi}(\varphi)}
\]

whenever this quotient is well defined. This definition depends on the choice of \( B \).

**Example 2: Average of exponential random functional from a point’s perspective.** As in the previous example, let \( \Phi = \{x_1, x_2, \ldots \} \) denote some point process in \( \mathbb{R}^d \). For some real numbers \( s_1 < s_2 \), define the set

\[
Y = \{ \varphi \in N_{\Phi} : |\varphi \cap B_r(o)| = [s_1, s_2] \}
\]

in \( N_{\Phi} \), let \( B \) denote some Borel set in \( \mathbb{R}^d \), let \( \beta > 0 \) denote some fixed parameter, and define

\[
h(z, \varphi) = 1_B(z) 1_Y(\varphi - z) \exp\{-\beta|\varphi \cap B_r(z)|\}.
\]

This functional marks or weighs each point according to the number of nearest neighbors within distance \( r \), where the penalizing weight has exponential form. The more points cluster, the less they weigh. Nicely isolated points have large weights. In fact, marking the points of the point process generates a so-called marked point process (see STOYAN, KENDALL, AND MECKE [11], p. 105). The average of \( h \) over points in \( B \) may be interpreted as the weight of points in \( B \) when the number of nearest neighbors within distance \( r \) lies in \([s_1, s_2]\). Therefore, in view of (4.1),

\[
E_{\Phi} \left( \sum_{z \in \Phi \cap B} 1_{Y}(\Phi - z) \exp\{-\beta|\Phi \cap B_r(z)|\} \right) = \int \sum_{z \in \varphi \cap B} 1_Y(\varphi - z) \exp\{-\beta|\varphi \cap B_r(z)|\} \, dP_{\Phi}(\varphi).
\]
Example 3: Average of exponential random functional from a line’s perspective. As in the previous two examples, let $\Phi = \{x_1, x_2, \ldots\}$ denote some point process in $\mathbb{R}^d$. Let $\mathcal{V}$ denote some test set of lines that depends on $\Phi$. From (3.15), recall the restriction $\mathcal{C}_L$ of $\Phi$ to a neighborhood of $L$, more precisely, those points of a realization of $\Phi$, closest to the line $L$ in $\mathcal{V}$, as opposed to other lines in $\mathcal{V}$. If $\mathcal{L}_* \subset \mathcal{V}$, $B$ denotes some Borel set in the set of all lines, the constant $\beta > 0$ denotes some fixed parameter, and

$$h(L, \varphi) = 1_B(L)1_{\mathcal{L}_*}(L) \exp\{-\beta|\mathcal{C}_L|\},$$

then we obtain, by (3.17),

$$E_{\Phi}\left(\sum_{L \in \Phi \cap \mathcal{L}_* \cap B} \exp\{-\beta|\mathcal{C}_L|\}\right) = \int \sum_{L \in \varphi \cap \mathcal{L}_* \cap B} \exp\{-\beta|\mathcal{C}_L|\} dP_{\Phi}(\varphi). \quad (4.5)$$

In this sum of $h$ over lines in $\mathcal{L}_*$, lines are penalized according to the number of points of $\Phi$ near them.

In the setting of this paper, $\Phi$ denotes the point process of self-intersections of the SRW (conditioned upon $|\Phi| \in [b_1n, b_2n]$) and the terms in the summation are associated to the lines in some $\mathcal{V}$. The calculations on $\Phi$ are carried out under the condition that the SRW-path ends at distance $x$. Hence, in that case, the point process and its Palm distribution both depend on $x$. In other words, the role of $\Phi$ is being played by $\Phi|_x$.

References

[1] BILLINGSLEY, P. (1986). *Probability and Measure*. John Wiley & Sons, New York.

[2] BRYDGES, D. AND SPENCER, T. (1985). Self-avoiding walk in 5 or more dimensions. *Communications in Math. Physics* 97, 125–148.

[3] FLORY, P. (1949). The configuration of real polymer chains. *Journal of Chemical Physics* 17, 303–310.

[4] GREVEN, A. AND DEN HOLLANDER, F. (1993). A variational characterization of the speed of a one-dimensional self-repellent random walk. *Annals of Applied Probability* 3, 1067–1099.

[5] HARA, T. AND SLADE, G. (1992). The lace expansion for self-avoiding walk in five or more dimensions. *Review in Math. Physics* 4, 235–327.

[6] HUETER, I. (2001). Formula for the mean square displacement exponent of the self-avoiding walk in 3, 4, and all dimensions. Preprint.

[7] KALLENBERG, O. (1983). *Random Measures*. Akademie Verlag, Berlin.
[8] Lawler, G.F. (1991). *Intersections of Random Walks. Probability and its Applications*. Birkhäuser Boston, Boston, MA.

[9] Madras, N. and Slade, G. (1993). *The Self-Avoiding Walk. Probability and its Applications*. Birkhäuser Boston, Boston, MA.

[10] Palm, C. (1943). Intensitätsschwankungen in Fernsprechverkehr. *Ericsson Technics* 44, 1–189.

[11] Stoyan, D., Kendall, W.S. and Mecke, J. (1995). *Stochastic Geometry and its Applications*. Wiley and Sons, Chichester, England.