More results on greedy defining sets

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Abstract
The greedy defining sets of graphs were appeared first time in [M. Zaker, Greedy defining sets of graphs, Australas. J. Combin, 2001]. We show that to determine the greedy defining number of bipartite graphs is an NP-complete problem. This result answers affirmatively the problem mentioned in the previous paper. It is also shown that this number for forests can be determined in polynomial time. Then we present a method for obtaining greedy defining sets in Latin squares and using this method, show that any $n \times n$ Latin square has a GDS of size at most $n^2 - (n \log n)/4$. Finally we present an application of greedy defining sets in designing practical secret sharing schemes.

1 Introduction

Let $G$ be a simple graph whose vertices are ordered by an order $\sigma$ as $v_1, \ldots, v_n$. The first-fit (greedy) coloring of $G$ with respect to $\sigma$ starts with $v_1$ and assigns color 1 to $v_1$ and then goes to the next vertex. It colors $v_i$ by the first available color which is not appeared in the neighborhood of $v_i$. If the algorithm finishes coloring of $G$ by $\chi(G)$ colors then we say that it succeeds. But this is not the case in general. If we want the greedy algorithm to succeed, then we need to pre-color some of the vertices in $G$ before the algorithm is invoked. So we define a greedy defining set (GDS) to be a subset of vertices in $G$ together with a pre-coloring of $S$, that will cause the greedy algorithm to successfully color the whole graph $G$ with $\chi(G)$ colors. It is understood that the algorithm skips over the vertices that are part of the defining set. Greedy defining sets of graphs were first defined and studied by the author in [4]. This concept have also been studied for Latin squares in [5, 6] and recently in [8]. In the sequel follow the formal definitions.

Definition 1. For a graph $G$ and an order $\sigma$ on $V(G)$, a greedy defining set is a subset $S$ of $V(G)$ with an assignment of colors to vertices in $S$, such that the pre-coloring can be extended to a $\chi(G)$-coloring of $G$ by the greedy coloring of $(G, \sigma)$
and fixing the colors of $S$. The greedy defining number of $G$ is the size of a greedy defining set which has minimum cardinality, and is denoted by $\text{GDN}(G, \sigma)$. A greedy defining set for a $\chi(G)$-coloring $C$ of $G$ is a greedy defining set of $G$ which results in $C$. The size of a greedy defining set of $C$ with the smallest cardinality is denoted by $\text{GDN}(G, \sigma, C)$.

Let an ordered graph $(G, \sigma)$ and a proper vertex coloring $C$ of $G$ using $\chi(G)$ colors be given. Let $i$ and $j$ with $1 \leq i < j \leq \chi(G)$ be two arbitrary and fixed colors. Let a vertex say $v$ of color $j$ be such that all of its neighbors with color $i$ (this may be an empty set) are higher than $v$. Then $v$ together with these neighbors form a subset which we call a descent. It was proved in [4] that a subset $S$ of vertices is a greedy defining set for the triple $(G, \sigma, C)$ if and only if $S$ intersects any descent of $G$ or equivalently $S$ is a transversal for the set of all descents.

Consider the Cartesian product $K_n \square K_n$ and the lexicographic order of its vertices. Namely $(i, j) < (i', j')$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. Since any $n \times n$ Latin square is equivalent to a proper $n$-coloring of the Cartesian product $K_n \square K_n$ then we can define greedy defining set and number of Latin squares. In any Latin square we denote any cell in row $i$, column $j$ with entry $x$ by $(i, j; x)$. Now the concept of descent in the context of Latin squares is stated as follows. Given a Latin square $L$, a set consisting of three cells $(i, j; y), (r, j; y)$ and $(i, k; x)$ where $i < r, j < k$ and $x < y$, is called a descent. The following theorem proved in [5, 6] is in fact a consequence of the theorem concerning GDS and transversal of descents which was mentioned in the previous paragraph.

**Theorem 1.** A subset $D$ of entries in a Latin square $L$ is greedy defining set if and only if $D$ intersects any descent of $L$.

## 2 Greedy defining number of graphs

In [4] the computational complexity of determining $\text{GDN}(G, \sigma, C)$ has been studied.

**Theorem 2.** ([4]) Given a triple $(G, \sigma, C)$ and an integer $k$. It is an NP-complete problem to decide $\text{GDN}(G, \sigma, C) \leq k$.

Throughout the paper by the vertex cover problem we mean the following decision problem. Given a simple graph $F$ and an integer $k$, whether $F$ contain a vertex cover of at most $k$ vertices? Recall that a vertex cover is a subset $K$ of vertices such that any edge is incident with a vertex of $K$. This problem is a well-known NP-hard problem. In [4] the vertex cover problem was used to prove Theorem 2.
But because there exists a flaw in its proof, in the sequel we first fix the proof by slight modification of it and then discuss the open question posed in [4].

**Proof of Theorem 2.** It is enough to reduce the vertex cover problem to our problem. Let \((F, k)\) be an instance of the vertex cover problem where \(F\) has order \(n\). We first color arbitrarily the vertices of \(F\) by \(n\) distinct colors. Denote the color of a vertex \(v \in F\) by \(c(v)\). Now we consider the complete graph \(K_n\) (vertex disjoint from \(F\)) on vertex set \(\{1, 2, \ldots, n\}\). We order a vertex \(i \in K_n\) by the very \(i\) and a vertex \(v \in F\) by \(2n - j + 1\) if \(c(v) = j\). For any \(i \) and \(j\) with \(i < j\), we put an edge between a vertex \(v \) of \(F\) of color \(j = c(v)\) and a vertex \(i \) from \(K_n\) if and only if \(v\) is not adjacent to the vertex of color \(i\) in \(F\). Let the color of a vertex \(i \in K_n\) be \(i\). Denote the resulting ordered graph \((G, \sigma)\) and the proper coloring of \(G\) by \(C\). It is easily checked that no descent in \((G, \sigma, C)\) consists of only a single vertex. Since the colors of \(F\) are all distinct then a descent can only have two vertices and we note that any edge in \(F\) forms in fact a descent and these are the only descents of \(G\). We conclude that a transversal for the set of descents in \(G\) is a vertex cover for \(F\) and vise versa. This completes the proof.

It was asked in [4] that given an ordered graph \((G, \sigma)\), whether to determine \(GDN(G, \sigma)\) is an NP-complete problem? This problem is in fact the uncolored version of Theorem 2 where no coloring of graph is given in the input.

In the following we answer this problem affirmatively. We begin with the following lemma.

**Lemma 1.** Let \(G\) be a connected bipartite graph with a bipartition \((X, Y)\) whose vertices are colored properly by 1 and 2. Then there exists a minimum greedy defining set for \(G\) consisting of only vertices colored 1.

**Proof.** Let \(S\) be a minimum GDS which contains the minimum number of vertices colored 2. Consider a vertex \(v \in S\) of color 2. Since \(S\) is a minimum GDS, \(S \setminus \{v\}\) is not a GDS and so all neighbors of \(v\) with color 1 appear after \(v\) in the ordering of \(G\). Now it suffices to delete \(v\) from \(S\) and add any neighbor of it to \(S\). The new member has color 1 because there are only two colors in the graph. The resulting set is still a GDS and this contradicts with our choice of \(S\). Therefore there exists a minimum GDS containing no vertex of color 2. \(\square\)

**Theorem 3.** Given an ordered connected bipartite graph \(G\) and a positive integer \(k\). It is NP-complete to decide whether \(GDN(G) \leq k\).

**Proof.** We transform an instance \((F, k)\) of the vertex cover problem to an instance of our problem where \(F\) is a connected graph. Let \(V(F)\) and \(E(F)\) be the vertex
and edge set of $F$, respectively. Assume that $V_1$ and $V_2$ are two disjoint copies of $V(F)$. Namely any vertex of $F$ has two distinct copies in $V_1$ and $V_2$. Similarly let $E_1$ and $E_2$ be two disjoint copies of $E(F)$. Let $G$ be the bipartite graph consisting of the bipartite sets $X = V_1 \cup E_1$ and $Y = V_2 \cup E_2$, where $v \in V_1 \subseteq X$ is adjacent to $e \in E_2 \subseteq Y$ if $e$ (as an edge of $F$) is incident to $v$ in $F$. Also a vertex $v \in V_2$ is adjacent to $e \in E_1$ if $e$ is incident to $v$ in $F$. The only extra edges of $G$ are of the form $vv'$ where $v$ is an arbitrary vertex in $V_1$ and $v'$ its copy in $V_2$. We consider any ordering $\sigma$ of $V(G)$ in which $E_2 < E_1 < V_2 < V_1$, where for any two sets $A$ and $B$ by $A < B$ we mean any element of $A$ has lower order than any element of $B$.

The bipartite graph $G$ is connected since $F$ is so. Therefore $G$ has only two proper colorings with two colors. To determine $GDN(G)$ it is enough to determine the minimum greedy defining number of these two colorings of $G$. Consider an arbitrary coloring $C$ of $V(G)$ in which the part $X$ is colored 1. According to Lemma 1 it is enough to consider those greedy defining sets of $G$ which are contained in $X$. Based on the property of our ordering $\sigma$, we obtain that a descent in $G$ consists only of a vertex from $E_2$ together with its two endpoints in $V_1$. This shows that a greedy defining set of $G$ is a subset of $V_1$ which dominates the elements of $E_2$, i.e. a vertex cover of $F$. The converse is also true. It turns out that $GDN(G)$ is the same as the minimum size of a vertex cover in $F$. This proves the theorem for this case. The case where $X$ is colored by 2 is proved similarly in which a subset $S \subseteq Y$ is a GDS for $G$ if and only if $S \subseteq V_2$ and it dominates all elements of $E_1$. Namely in this case too the minimum GDS is the same as the smallest vertex cover of $F$. This completes the proof.  

\textbf{Theorem 4.} There exists an efficient algorithm to determine the greedy defining number of a forest.

\textbf{Proof.} It is enough to prove the theorem for trees, since suppose that a forest $F$ consists of the connected components $T_1, T_2, \ldots, T_k$. Then $GDN(G) = \sum_i GDN(T_i)$.

Now let $T$ be an ordered tree. It contains exactly two proper colorings using two colors since it is connected. It is enough to determine the greedy defining number of a 2-coloring of $T$. Let a 2-coloring be given by a bipartition $(X, Y)$ of $V(T)$ where $X$ consists of vertices colored 1. Recall that a descent is of the form a vertex colored 2 say $v$ together with its all neighbors of $v$. These neighbors are colored 1 and have higher order than $v$. Consider the subgraph $T'$ of $T$ induced by the vertices of the descents in $T$. By Lemma 1 it is enough to find a subset of vertices of color 1 with the minimum cardinality which dominates all the vertices of color 2 in $T'$. Such a subset will be denoted by $K$ and constructed gradually. Let $G$ be a connected component of $T'$. Assume first that a vertex $v$ colored 2 has degree one in $G$ and let $u$ be its neighbor of color 1. Then $\{v, u\}$ forms a descent and $u$ should be put in $K$. Delete now $\{v, u\} \cup N$ from $G$, where $N$ is the neighbors of $u$. We do the same
for other similar vertices of $G$. After this stage we obtain a subgraph of $G$ whose all leaves are colored 1. Note that since any leaf of color 1 dominates only one vertex therefore it is enough to consider only those vertices of color 1 which are not leaf as possible elements to be put in $K$. Hence at this stage we remove all leaves of color 1 from the graph. Since the earlier graph has no any cycle then at each stage of our algorithm there is at least one leaf. If it is colored by 2 then we put its neighbor (colored 1) in our GDS $K$. Otherwise we remove it and continue until no vertex in $G$ is left. We repeat this procedure for other connected components of $T'$. This completes the proof. 

\[\square\]

3 Latin squares

The minimum greedy defining number of any $n \times n$ Latin square is denoted by $g(n)$ in [5, 6] where it was shown that $g(n) = 0$ when $n$ is a power of two. The exact values of $g(n)$ for $n \leq 6$ were given in [6] and for $n = 7, 9, 10$ in [3]. But the complexity status of determining the greedy defining number of Latin squares is still unknown.

In the sequel we present a method to obtain a greedy defining set in a Latin square. For any $n \times n$ Latin square $L$ on $\{1, 2, \ldots, n\}$ we correspond three graphs $R(L)$, $C(L)$ and $E(L)$. Let $R$ be an arbitrary row of $L$. We first define a graph $G[R]$ on the vertex set $\{1, \ldots, n\}$ as follows. Two vertices $i$ and $j$ with $1 \leq i < j \leq n$ are adjacent in $G[R]$ if and only if (1) $j$ appears before $i$ in the row $R$ and (2) there is another entry $i'$ in the same column of $j$ such that it comes after $j$ (i.e. lower than $j$). In other words $i$ and $j$ are adjacent if and only if they form a descent (jointly with an additional entry $i$). The graph $R(L)$ is now defined the disjoint union $\cup G[R]$ on $n^2$ vertices where the union is taken over all $n$ rows of $L$. For any column $C$ of $L$ we define $G[C]$ similarly. The graph $C(L)$ consists of the disjoint union of $G[C]$'s. Finally, in the sequel we define $E(L)$. Let $e \in \{1, \ldots, n\}$ be any fixed entry. There are $n$ entries equal to $e$ in $L$. First, a graph denoted by $G[e]$ on these $n$ entries is defined in the following form. Two entries $e_1$ and $e_2$ (which both are the same as $e$ but in different rows and columns) are adjacent if and only if with an additional entry they form a descent in $L$. The disjoint union of $G[e]$’s form a graph which we denote by $E(L)$. The following proposition is immediate.

**Proposition 1.** A subset $D$ of entries of a Latin square $L$ is a GDS if $D$ is a vertex cover for at least one of the graphs $R(L)$, $C(L)$ and $E(L)$.

Proposition $\square$ provides some upper bounds for the greedy defining number of Latin squares. As an application, in the following we present an upper bound for the greedy defining number of any Latin square. We recall that according to Turan’s theorem any graph on $n$ vertices and with no clique of order $m$ has at most $(m-2)n^2/(2m-2)$ edges.
Theorem 5. Any $n \times n$ Latin square contains a GDS of size at most $n^2 - \frac{n \log 4n}{4}$.

Proof. It is enough to find a vertex cover for $E(L)$ of the desired cardinality. For this purpose consider the $n$ connected components of $E(L)$ which correspond to distinct entries of $L$ i.e. $G[1], G[2], \ldots, G[n]$. We obtain an upper bound for the vertex cover of each $G[i]$. The number of edges of $G[i]$ is maximized when the $n$ entries of $i$ lie in the northeast-southwest diagonal of $L$ and the maximum possible number of entries greater than $i$ are placed in the top of this diagonal. It turns out that in this case the graph has no more than $n(n - 1)/2 - i(i - 1)/2$ edges. Assume that $G[i]$ has at most $f(i)$ independent vertices. Then the complement of $G[i]$ which has at least $i(i - 1)/2$ edges, does not contain a clique of order $f(i) + 1$. Using Turan’s theorem we obtain

$$f(i) \geq \frac{n^2}{n^2 - i(i - 1)}.$$

If we write $i = n - j$ for some $0 \leq j \leq n - 1$ then

$$f(i) \geq \frac{n^2}{n(2j + 1) - (j^2 + j)} \geq \frac{n}{2j + 1}.$$

This shows that $E(L)$ contains at least $n \sum_{j=0}^{n-1} \frac{1}{2j + 1}$ independent vertices. But from other side

$$2 \sum_{j=0}^{n-1} \frac{1}{2j + 1} \geq \sum_{k=1}^{2n+1} \frac{1}{k} \geq 1 + \frac{\log(2n + 1) - 1}{2} \geq \frac{\log 4n}{2}.$$

It turns out that $E(L)$ contains at least $(n \log 4n)/4$ independent vertices. Therefore it contains a vertex cover of no more than $n^2 - (n \log 4n)/4$ vertices. This completes the proof. \qed

Finally we mention an application of GDS of Latin squares in secret sharing schemes. There is a known technique to design secret sharing schemes using critical sets in Latin squares [2]. In a Latin square $L$, a subset of entries $S$ is said to be a critical set if $L$ is the only Latin square which contains $S$ as its partial Latin square. In the model presented in [2], a dealer chooses a Latin square $L$ as the key of the scheme and then obtains a collection of critical sets of $L$. Then she shares the key among the participants in such a way that the participants of any authorized set receive the entries of a critical set $S$ and therefore by pooling their shares they can extend $S$ to obtain the key uniquely. Since $S$ is a critical set then the Latin square obtained, is nothing but $L$ itself. There are two serious practical problems for this
model. First, it is not easy to find an arbitrary number of non-trivial critical sets in a random Latin square. Second, once a set of participants pool their shares and obtain a partial Latin square $S$, it is difficult to determine whether it can be uniquely completed to a Latin square and then to obtain the value of the key. Because it was known that, given a partial Latin square $S$, whether $S$ has a unique completion to a Latin square is an NP-complete problem [1]. Now we propose the same scenario as above but in stead of critical sets we use GDS. The advantages of our model are as follow. First, there are known and convenient techniques to obtain greedy defining sets in Latin squares. In fact Theorem [1] as a general method and Proposition [1] as its refined version provide some tools to construct arbitrary greedy defining sets in Latin squares (note that to detect the descents of a square is an easy job). Second and more importantly, when we have a partial Latin square in hand, it is very easy using greedy coloring to find out whether it extends (uniquely) to a given Latin square or not.

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