LONG-TIME BEHAVIOUR OF A RADially SYMMETRIC
FLUID-SHELL INTERACTION SYSTEM

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1. Introduction.
1.1. Description of the model. We consider an infinite circular cylinder of radius
R filled with a viscous incompressible fluid. The axis of the cylinder is directed along
the x-axis. The wall of the cylinder is rigid with a cylindrical elastic part whose
projection on the x-axis is the interval [0, L]. We assume that the motion of the
fluid and the elastic cylindrical shell is radially symmetric. In this case we can
consider the problem in a two-dimensional infinite strip of the width R representing
the half of the longitudinal section of the cylindrical domain. Namely, we consider
the domain \( \Omega = (-\infty, \infty) \times (0, R) \in \mathbb{R}^2 \) with the boundary \( \partial \Omega = \overline{\Omega} \cup \Gamma_1 \cup \Gamma_2 \), where
\( \Omega = (0, L) \times \{ R \} \), \( \Gamma_1 = (\mathbb{R} \setminus [0, L]) \times \{ R \} \) and \( \Gamma_2 = \mathbb{R} \times \{ 0 \} \) and the system of
partial differential equations
\[
V_t - \mu \Delta V + \nabla p = f(x, y, t) \quad (1)
\]
\[
div V = 0, \quad t > 0, (x, y) \in \Omega \quad (2)
\]
and
\[
\rho u_{tt} - \frac{hE}{1 - \zeta^2} (u_x + 1/2 \eta^2_x + \frac{\eta}{R}) = -\mu(u_x + w_y)|_{\Omega} + f_1(x, t) \quad t > 0, \quad 0 < x < L, (3)
\]
\[
\rho \eta_{tt} + \frac{hE}{12(1 - \zeta^2)} \eta_{xxx} + \frac{hE}{R(1 - \zeta^2)} (\zeta(u_x + 1/2 \eta^2_x) + \frac{\eta}{R})
-
\frac{Eh}{1 - \zeta^2} \left( [\eta_x(u_x + 1/2 \eta^2_x)]_x + \frac{\zeta}{R} [\eta_x \eta]_x \right) = -2\mu v_x|_{\Omega} + p|_{\Omega} + f_2(x, t) \quad (4)
\]
Here \( V(x, y, t) = (w(x, y, t), v(x, y, t)) \), where \( v(x, y, t) \) and \( w(x, y, t) \) denote the
radial and longitudinal velocity of the fluid respectively. The functions \( \eta(x, t) \) and

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u(x, t) represent the radial and longitudinal displacements of the shell, \( p \) is the pressure of the fluid, \( f(x, y, t), f_1(x, t), \) and \( f_2(x, t) \) are the external forcing terms, \( \rho, h, E, \mu \) are positive constants denoting the density, the thickness, the Young’s modulus of the shell and the kinematic viscosity coefficient respectively, \( 0 < \varsigma < 1 \) is the Poisson ratio of the shell. Equations (1)-(4) are subjected to the coupling boundary conditions

\[
v(x, R, t) = \eta_t(x, t), \ w(x, R, t) = u_t(x, t), \ x \in [0, L], \ t > 0,
\]

no-slip boundary conditions

\[
v(x, R, t) = 0, \ w(x, R, t) = 0, \ x \notin [0, L], \ t > 0,
\]

radially-symmetric boundary conditions

\[
v(x, 0, t) = \partial_y w(x, 0, t) = 0, \ x \in \mathbb{R}, \ t > 0,
\]

and Dirichlet boundary conditions for the shell components

\[
\eta(0, t) = \eta_x(0, t) = \eta(L, t) = \eta_x(L, t) = 0, \\
u(0, t) = u(L, t) = 0.
\]

We also consider the problem under initial conditions

\[
w(x, y, 0) = w_0(x, y), \ v(x, y, 0) = v_0(x, y), \ (x, y) \in \Omega \\
u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ \eta(x, 0) = \eta_0(x), \ \eta_t(x, 0) = \eta_1(x), \ x \in \Omega.
\]

satisfying the assumption \( \text{div}V_0 = 0 \).

If we assume that the velocity field \( V \) decays sufficiently fast as \( |x| \to +\infty \) and \( x \in \overline{\Omega} \), then (2) and (5) imply the following compatibility condition

\[
\int_{\Omega} \eta_t(x, t) dx = 0 \quad \text{for all} \ t \geq 0,
\]

which can be interpreted as preservation of the volume of the fluid.

1.2. Previous work. The mathematical studies of the problem of fluid–structure interaction in the case of viscous fluids and elastic plates/bodies have a long history.

The case of elastic bodies with the fixed interface \([2, 13]\) were studied from the point of view of the well-posedness and stability of the problems.

We refer to \([4, 7, 10, 16, 17]\) and the references therein for the case of shallow plates or membranes. Works \([4, 16, 17]\) are devoted to the well-posedness of the problems of fluid-structure interaction. The existence of global attractor for fluid-structure interaction problems was investigated in \([7, 10]\). While in the first paper a nonlinear system describing the interaction of a viscous incompressible fluid in a bounded vessel with a flat elastic part of the boundary moving in the in-plane directions only is considered, the second one deals with the transversal displacement on a flexible flat part of the boundary. All these sources deal with the case of bounded reservoirs \( \Omega \) and a flat elastic shallow shell or plate.

Regarding infinite reservoirs \( \Omega \) we can mention works \([8, 12]\) which establish the existence of a compact global attractors to linearized around a Poiseuille type flow Navier-Stokes systems in unbounded domains coupled with a nonlinear equation on the boundary accounting for the transverse displacements.

In the second paper the flexible part of the boundary is assumed to be flat while in the first one shells have circular cylindrical form and their motion is described by Donnell’s shell equations. In paper \([11]\) a model in a bounded domain taking into account both transversal and longitudinal deformations and involving the full
von Karman system for the shell component is considered. Results on the well-posedness and existing of a compact global attractor are proved provided the terms accounting for rotational inertia and the mechanical damping are present in the equation for the transversal displacement of the shell.

In our paper we consider for the first time, to the best of our knowledge, the radially symmetric interaction model for a Newtonian fluid and a shell whose dynamics is described by the full von Karman model. The peculiarity of the problem considered consists in the absence of any mechanical damping and rotational inertia in the shell component, unlike a 3D problem studied in [11]. Since we do not assume any kind of mechanical damping in the plate component, this means that in case of one dimensional equations for the shell the dissipation of the energy in the fluid flow due to viscosity is sufficient to stabilize the system. Moreover, we establish additional regularity of the shell displacement components due to the fluid viscosity.

We prove the well-posedness of the system considered and investigate the long-time dynamics of solutions to the coupled problem in (1)-(11). Whether the same results are valid in the 3D case is an open question and is an issue for further considerations.

1.3. Abstract results on attractors. For the readers’ convenience we recall some basic definitions and results from the theory of attractors.

Definition 1 ([1, 5, 6, 9, 22]). A global attractor of a dynamical system \( (S_t, H) \) with the evolution operator \( S_t \) on a complete metric space \( H \) is defined as a bounded closed set \( \mathfrak{A} \subset H \) which is invariant \( (S_t\mathfrak{A} = \mathfrak{A} \quad \text{for all} \ t > 0) \) and uniformly attracts all other bounded sets:

\[
\lim_{t \to \infty} \sup \{ \text{dist}_H (S_t y, \mathfrak{A}) : y \in B \} = 0 \quad \text{for any bounded set} \ B \in H.
\]

Definition 2 ([5, 6, 9]). The fractal dimension \( \dim_H M \) of a compact set \( M \) in a complete metric space \( H \) is defined as

\[
\dim_H M = \lim_{\varepsilon \to 0} \sup \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},
\]

where \( N(M, \varepsilon) \) is the minimal number of closed sets in \( H \) of diameter \( 2\varepsilon \) which cover \( M \).

To establish the existence of an attractor we use the concept of gradient systems. The main feature of these systems is that in the proof of the existence of a global attractor we can avoid a dissipativity property (existence of an absorbing ball) in the explicit form ([5]).

Definition 3 ([5, 6, 9]). Let \( Y \subseteq H \) be a forward invariant set of a dynamical system \( (S_t, H) \). A continuous functional \( L(y) \) defined on \( Y \) is said to be a Lyapunov function on \( Y \) for the dynamical system \( (S_t, H) \) if \( t \mapsto L(S_t y) \) is a nonincreasing function for any \( y \in Y \).

The Lyapunov function is said to be strict on \( Y \) if the equation \( L(S_t y) = L(y) \) for all \( t > 0 \) and for some \( y \in Y \) implies that \( S_t y = y \) for all \( t > 0 \); that is, \( y \) is a stationary point of \( (S_t, H) \).

The dynamical system is said to be gradient if there exists a strict Lyapunov function on the whole phase space \( H \).

Definition 4 ([5, 6, 9]). A dynamical system \( (X, S_t) \) is said to be asymptotically smooth if for any closed bounded set \( B \subset X \) that is positively invariant
(\(S_t B \subseteq B\)) one can find a compact set \(\mathcal{K} = \mathcal{K}(B)\) which uniformly attracts \(B\), i.e. \(\sup_{x} \{\text{dist}_{\mathcal{X}}(S_t y, \mathcal{K}) : y \in B\} \to 0\) as \(t \to \infty\).

We single out a class of the so-called quasi-stable systems that enjoy some kind of stabilizability inequalities written in some general form. These inequalities, although often difficult to establish (most often they are obtained by means of multipliers technic), once they are proved provide a number of consequences that describe various properties of attractors \([5, 9]\).

**Definition 5** \([5, 9]\). A seminorm \(n(x)\) on a Banach space \(H\) is said to be compact if any bounded sequence \(\{x_m\} \subseteq H\) contains a subsequence \(\{x_{m_k}\}\) which is Cauchy with respect to \(n\), i.e., \(n(x_{m_k} - x_{m_l}) \to 0\) as \(k, l \to \infty\).

The dynamical system \((S_t, H)\) is said to be quasi-stable on a set \(\mathcal{B} \subseteq H\) (at time \(t_*\)) if there exist (a) time \(t_* \geq 0\), (b) a Banach space \(Z\), (c) a globally Lipschitz mapping \(K : \mathcal{B} \to Z\), and (d) a compact seminorm \(n_Z(\cdot)\) on the space \(Z\), such that

\[
\|S_{t_*}y_1 - S_{t_*}y_2\|_H \leq q \cdot \|y_1 - y_2\|_H + n_Z(Ky_1 - Ky_2)
\]

for every \(y_1, y_2 \in \mathcal{B}\) with \(0 \leq q < 1\). The space \(Z\), the operator \(K\), the seminorm \(n_Z\) and the time moment \(t_*\) may depend on \(\mathcal{B}\).

The following statement collects criteria on existence, finite dimensionality and properties of attractors to gradient systems.

**Theorem 1** \([5, 9]\). Assume that \((S_t, H)\) is a gradient quasi-stable dynamical system. Assume its Lyapunov function \(L(y)\) is bounded from above on any bounded subset of \(H\) and the set \(W_R = \{y : L(y) \leq R\}\) is bounded for every \(R\). If the set \(\mathcal{N}\) of stationary points of \((S_t, H)\) is bounded, then \((S_t, H)\) possesses a compact global attractor which possesses finite fractal dimension. Moreover, the global attractor \(\mathcal{A}\) consists of full trajectories \(\gamma = \{u(t) : t \in \mathbb{R}\}\) such that

\[
\lim_{t \to -\infty} \text{dist}_H(u(t), \mathcal{N}) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \text{dist}_H(u(t), \mathcal{N}) = 0.
\]

and

\[
\lim_{t \to +\infty} \text{dist}_H(S_{t_*}x, \mathcal{N}) = 0 \quad \text{for any} \quad x \in H;
\]

that is, any trajectory stabilizes to the set \(\mathcal{N}\) of stationary points.

The paper is organized as follows. In Section 2 we introduce Sobolev type spaces and operators we need and prove the result on the well-posedness of the system. In Section 3 we establish the well-posedness of the problem. Section 4 is devoted to the existence of a compact finite-dimensional global attractor and its properties.

2. **Spaces and notations.** Let \(D\) be a sufficiently smooth domain and \(s \in \mathbb{R}\). We denote by \(H^s(D)\) the Sobolev space of order \(s\) on a set \(D\) which we define as restriction (in the sense of distributions) of the space \(H^s(\mathbb{R}^d)\) (introduced via Fourier transform). We denote by \(\|\cdot\|_{s,D}\) the norm in \(H^s(D)\) which we define by the relation

\[
\|f\|^2_{s,D} = \inf \left\{ \|g\|^2_{s,\mathbb{R}^d} : g \in H^s(\mathbb{R}^d), g = f \text{ on } D \right\}.
\]

We also use the notation \(\|\cdot\|_{D} = \|\cdot\|_{0,D}\) for the corresponding \(L_2\)-norm and, similarly, \((\cdot, \cdot)_D\) for the \(L_2\) inner product. We denote by \(H^s_0(D)\) the closure of \(C^\infty_0(D)\) in \(H^s(D)\) (with respect to \(\|\cdot\|_{s,D}\)) and introduce the spaces \(H^s_*(D) := \{f|_D : f \in H^s(\mathbb{R}^d), \text{ supp } f \subset D\}, \ s \in \mathbb{R}\).
Since the extension by zero of elements from $H^s(D)$ gives an element of $H^s(\mathbb{R}^d)$, these spaces $H^s(D)$ can be treated not only as functional spaces defined on $D$ (and contained in $H^s(D)$) but also as (closed) subspaces of $H^s(\mathbb{R}^d)$. Below we need them to describe boundary traces on $\Omega \subset \partial D$. We endow the classes $H^s(D)$ with the induced norms $\|f\|_{s,D} = \|f\|_{s,\mathbb{R}^d}$ for $f \in H^s(D)$. It is clear that $\|f\|_{s,D} \leq \|f\|_{s,D}^{\ast}$, $f \in H^s(D)$. It is known (see [23, Theorem 4.3.2/1]) that $C_0^\infty(D)$ is dense in $H^s(D)$ and

$$
H^s(D) \subset H^0_s(D) \subset H^s(D), \ s \in \mathbb{R}; \ H^0_s(D) = H^s(D), \ -\infty < s \leq 1/2;
$$

$$
H^s(D) = H^0_s(D), \ -1/2 < s < \infty, \ s - 1/2 \notin \{0, 1, 2, \ldots\}.
$$

In particular, $H^s(D) = H^0_s(D) = H^s(D)$ for $|s| < 1/2$. By [24, Remark 4.3.2/2] we also have that $H^s(D) \neq H^s(D)$ for $|s| > 1/2$. Note that in the notations of [20] the space $H^{s+1/2}_m(D)$ is the same as $H^{m+1/2}_s(D)$ for every $m = 0, 1, 2, \ldots$, and for $s = m + \sigma$ with $0 < \sigma < 1$ we have

$$
\|u\|^{\ast}_{s,D} = \left\{ \|u\|_{s,D}^2 + \sum_{|\alpha|=m} \int_D \frac{|\partial^\alpha u(x)|^2}{d(x, \partial D)^{2\sigma}} \, dx \right\}^{1/2},
$$

where $d(x, \partial D)$ is the distance between $x$ and $\partial D$. The norm $\|\cdot\|^{\ast}_{s,D}$ is equivalent to $\|\cdot\|_{s,D}$ in the case when $s > -1/2$ and $s - 1/2 \notin \{0, 1, 2, \ldots\}$, but not equivalent in general.

Understanding adjoint spaces with respect to duality between $C^\infty(\Omega)$ and $[C_0^\infty(\Omega)]'$ by Theorems 4.8.1 and 4.8.2 from [24] we also have that

$$
[H^s(D)]' = H^{-s}(D), \ s \in \mathbb{R}, \text{ and } [H^s(D)]' = H^{-s}_0(D), \ s \in (-\infty, 1/2).
$$

To describe fluid velocity fields we introduce the following spaces.

Let $C(\overline{\Omega})$ be the class of $C^\infty$ vector-valued solenoidal (i.e., divergence-free) functions on $\overline{\Omega}$ which vanish in a neighborhood of $\Gamma_1 \cup \Omega$ and whose second component and the derivative of the first component with respect to $y$ vanish in a neighborhood of $\Gamma_2$. We denote by $X$ the closure of $C(\overline{\Omega})$ with respect to the $L_2$-norm and by $Y$ the closure with respect to the $H^1(\Omega)$-norm. One can see that

$$
\bar{X} = \{ V = (w, v) \in [L_2(\overline{\Omega})]^2 : div V = 0, v|_{\partial \Omega} = 0 \} \tag{16}
$$

and

$$
\bar{Y} = \{ V = (w, v) \in [H^1(\overline{\Omega})]^2 : div V = 0, V|_{\Gamma_1 \cup \Omega} = 0, v|_{\Gamma_2} = 0 \} \tag{17}
$$

We equip $X$ with $L_2$-type norm $\|\cdot\|_0$ and denote by $(\cdot, \cdot)_0$ the corresponding inner product. The space $Y$ is endowed with the norm $\|\cdot\|_Y = \|\nabla \cdot \|_0 = \|\cdot\|_{1,0}$. For some details concerning this type spaces we refer to [23], for instance.

We also need the Sobolev spaces consisting of functions with zero average on the domain $\Omega$, namely we consider the space

$$
\tilde{L}_2(\Omega) = \{ \eta \in L_2(\Omega) : \int_\Omega \eta(x) \, dx = 0 \}
$$

and also $\tilde{H}^s(\Omega) = H^s(\Omega) \cap \tilde{L}_2(\Omega)$ for $s > 0$ with the standard $H^s(\Omega)$-norm. The notations $\tilde{H}^s_0(\Omega)$ and $\tilde{H}^s_0(\Omega)$ have a similar meaning. For the shell component $\xi = (u, \eta)$ we use the spaces

$$
\mathcal{U} = H^1_0(\Omega) \times H^2_0(\Omega). \tag{18}
$$
with the inner product
\[(\xi_1, \xi_2)_\Omega = \frac{hE}{1 - \zeta^2}(u_{1x}, u_{2x})_\Omega + \frac{h^3E}{12(1 - \zeta^2)}(\eta_{1xx}, \eta_{2xx})_\Omega\]
and its subspace \(\tilde{\Omega} = H^{1/2}_0(\Omega) \times \tilde{H}^{1/2}_0(\Omega)\). Let the operator \(N\) is defined as follows (see [12]). Let \(\Omega' \subset \Omega\) is a bounded domain with the smooth boundary such that \(\Omega \subset \partial \Omega'\). Then \(N : [\tilde{L}^2(\Omega)]^2 \rightarrow [H^{1/2}(\Omega)]^2\) is defined by the formula
\[N\xi = V \iff \begin{cases} -\mu \Delta V + \nabla p = 0, & \text{in } \Omega'; \\ V = 0 & \text{on } \partial \Omega' \setminus \Omega; \\ V = \xi & \text{on } \Omega. \end{cases} \tag{19}\]

Let \(M(\Omega) = \{V = \tilde{V} + N\xi : \tilde{V} \in C(\Omega), \, \xi \in \tilde{\Omega}\}\). Then we denote by \(X\) the closure of \(M(\Omega)\) with respect to the \(L_2\)-norm and by \(Y\) the closure of \(M(\Omega)\) with respect to the \(H^1\)-norm. One can see that
\[X = \{V = (w, v) \in [L_2(\Omega)]^2 : divV = 0, \, v = |r_1 |_{\Gamma_1} = 0, \, v = |r_2 |_{\Gamma_2} = 0\} \tag{20}\]
and
\[Y = \{V = (w, v) \in [H^1(\Omega)]^2 : divV = 0, \, V|_{\Gamma_1} = 0, \, v|_{\Gamma_2} = 0\} \tag{21}\]

Operator \(N\) possesses properties
\[N : [\tilde{H}^s_0(\Omega)]^2 \rightarrow [H^{1/2+s}(\Omega)]^3 \cap X \tag{22}\]
continuously for every \(s \geq -1/2\) and
\[||N\xi||_{1/2+s, \Omega} \leq C||\xi||_{s, \Omega}, \quad \xi \in [H^s_0(\Omega)]^2. \tag{23}\]

We define the Hilbert spaces
\[H = \{W = (V, \xi, \tilde{\xi}) \in X \times \tilde{\Omega} \times \tilde{L}_2(\Omega) \times \tilde{L}_2(\Omega)\} \tag{24}\]
endowed with the inner product \((W_1, W_2)_H = (V_1, V_2)_\Omega + (\xi_1, \xi_2)_\Omega + \rho h(\xi_1, \xi_2)_\Omega\).

We define the operators \(A_1h = -\hat{h}_{xx}\) on \(L_2(\Omega)\) and \(A_2g = \hat{g}_{xxx}\) on \(\tilde{L}_2(\Omega)\) with the domains \(\mathcal{D}(A_1) = (H^2 \cap H^1_0)(\Omega)\) and \(\mathcal{D}(A_2) = (H^4 \cap \tilde{H}^2_0)(\Omega)\). This operators are positive and self-adjoint with \(\mathcal{D}(A_1^{1/2}) = H^1_0(\Omega), \mathcal{D}(A_2^{1/4}) = \tilde{H}^2_0(\Omega)\) and \(\mathcal{D}(A_2^{1/2}) = \tilde{H}^2_0(\Omega)\). By \(C\) we denote a generic positive constant.

### 3. Well-posedness.

We define the spaces of test functions
\[\mathcal{L}_T = \left\{ \psi = (\psi_1, \psi_2) : \begin{array}{c} \psi \in L_2(0, T; [H^1(\Omega)]^2), \, \psi_t \in L_2(0, T; [L_2(\Omega)]^2), \\ \text{div} \psi = 0, \, \psi|_{\Gamma_1} = 0, \, \psi|_{\Gamma_2} = 0, \, \psi|_{\Omega} = \zeta = (\zeta_1, \zeta_2), \\ \zeta \in \tilde{L}_2(0, T; \tilde{\Omega}), \, \zeta_t \in L_2(0, T; \tilde{L}_2(\Omega) \times \tilde{L}_2(\Omega)) \end{array} \right\} \tag{\psi(T) = 0}\]
an\[\mathcal{L}_T^0 = \{ \psi \in \mathcal{L}_T : \psi(T) = 0 \} \]

In order to make our statements precise we need to introduce the definition of weak solutions to problem (1)-(11).

**Definition 6.** A pair of functions \((V(t), \xi(t))\) is said to be a weak solution to problem (1)-(11) on a time interval \([0, T]\) if
\[
\begin{align*}
&V \in L_\infty(0, T; X) \cap L_2(0, T; Y), \, \xi \in L_\infty(0, T; \tilde{\Omega}), \, \xi_t \in L_\infty(0, T; L_2(\Omega) \times \tilde{L}_2(\Omega)) \\
&\xi_t \in L_2(0, T; [H^{1/2}(\Omega)]^2) \\
&\xi(0) = \xi_0 = (u_0, \eta_0) \\
&V(t)|_{\Omega} = \xi_t(t), \text{ for almost all } t \in [0, T] \tag{25}
\end{align*}
\]
Lemma 1. There exists a positive constant such that for any \(\xi = (u; \eta) \in H_0^1(\Omega) \times H_0^3(\Omega)\) we have that
\[
\|\xi\|_{H_0^1(\Omega)} \leq C \left( Q(\xi) + \|\eta\|_{H^{5/4}(\Omega)}^4 \right).
\]

Theorem 2. Assume that
\[
f(t) \in L_2(0, T; Y'), \quad F(t) \in L_2(0, T; \left[H^{-1/2}(\Omega)\right]^2),
\]
then for any interval \([0, T]\) there exists a unique weak solution \((V(t); \xi(t))\) to (1)-(11) with the initial data \(W_0\). This solution possesses the following properties:

(i) \(\xi \in L_2(0, T; H^{3/2}(\Omega) \times H^{5/2}(\Omega))\).

(ii) \(W(t; W_0) = W(t) = (V(t); \xi(t); \xi_t(t)) \in C(0, T; H)\).

(iii) The solution depends continuously on initial data, i.e., if \(W_n \to W_0\) in the norm of \(H\), then \(W(t; W_n) \to W(t; W_0)\) in \(H\) for each \(t > 0\).

(iv) The energy equality
\[
E(V(t), \xi(t), \xi_t(t)) + \mu \int_0^t \|\nabla V\|^2 \, dt
= E(V_0, \xi_0, \xi_1) + \int_0^t (f, V)_\nabla \, dt + \int_0^t (F, \xi_t)_\nabla \, dt
\]
holds for every \(t > 0\), where the energy functional \(E\) is defined by the relation
\[
E(V, \xi, \xi_t) = \frac{1}{2} \left[ \|V\|^2_{H_0^1} + \rho h \|\xi_t\|_{H_0^3}^2 + \frac{h^3 E}{12(1 - \zeta^2)} \|\partial_{xx} \eta\|_{H_0^3}^2 + \frac{hE}{1 - \zeta^2} Q(\xi) \right]
\]
with
\[
Q(\xi) = \int_0^L \left[ \left( u_x + \frac{\eta_x^2}{2} \right)^2 + \frac{2 \xi}{R} \left( u_x + \frac{\eta_x^2}{2} \right) \eta + \frac{1}{R^2} \eta^2 \right] \, dx.
\]

We will need the following auxiliary results.

Lemma 1. There exists a positive constant \(C\) such that for any \(\xi = (u; \eta) \in H_0^1(\Omega) \times H_0^3(\Omega)\) we have that
\[
\|\xi\|_{H_0^1(\Omega)} \leq C \left( Q(\xi) + \|\eta\|_{H^{5/4}(\Omega)}^4 \right).
\]
Proof. Using the embedding $H^{1/4}(\Omega) \subset L_4(\Omega)$ we obtain the assertion of the lemma. \hfill \Box

**Lemma 2.** There exists a positive constant $C$ such that for any $g \in H_0^1(\Omega)$ we have

$$\max_{\Omega} |g| \leq C\|g\|_{1/2+\delta,\Omega}, \text{ for any } \delta > 0.$$  \hspace{1cm} (32)

The proof is standard (see e.g. [3]) for similar arguments so we omit it here.

**Lemma 3.** There exists a positive constant $C$ such that for any functions $g \in H_0^1(\Omega)$ and $h \in H^{1/2-\delta}(\Omega)$

$$\|gh\|_{1/2,\Omega}^* \leq C\|g\|_{1-\sigma,\Omega} \left( \|h\|_{1/2-\delta,\Omega} \|h\|_{2-\sigma,\Omega} \right)^{1/2} + C\|h\|_{1/2-\delta,\Omega} \|h\|_{\frac{3-12\delta}{3-4\delta}}$$  \hspace{1cm} (33)

for any $\sigma < 1/8$ and $\delta < 1/6$.

**Proof.** By the definition of equivalent norms in spaces $H^s(\Omega)$ (see, e.g. [24]) we have

$$\|gh\|_{1/2,\Omega}^* \leq C \left( \int_0^L \left| g(x)h(x) \right|^2 dx + \int_0^L \left| g(x)h(x) - g(z)h(z) \right|^2 \frac{dx}{|x-z|^2} dz ight).$$  \hspace{1cm} (34)

Relying on the Hölder's inequality we can estimate the second term in (34) as follows:

$$\int_0^L \int_0^L \frac{|g(x)h(x) - g(z)h(z)|^2}{|x-z|^2} dx dz \leq C \left( \int_0^L \int_0^L \frac{|g(x)|^2 h(x) - h(z)^2}{|x-z|^2} dx dz + \int_0^L \int_0^L \frac{|h(z)|^2 g(x) - g(z)^2}{|x-z|^2} dx dz \right).$$

$$\leq C \left( \int_0^L \int_0^L \frac{|g(x)|^2}{|x-z|^2} dx dz \int_0^L \int_0^L \frac{|h(x) - h(z)|^4}{|x-z|^4} dx dz \right)^{1/2}$$

$$+ \left( \int_0^L \int_0^L \frac{|h(z)|^6}{|x-z|^6} dx dz \right)^{1/3} \left( \int_0^L \int_0^L \frac{|g(x) - g(z)|^3}{|x-z|^3} dx dz \right)^{2/3}.$$  \hspace{1cm} (35)

By Lemma 2 and standard embedding theorems [24] we have

$$\int_0^L \int_0^L \frac{|g(x)h(x) - g(z)h(z)|^2}{|x-z|^2} dx dz \leq C \max_{\Omega} |g(x)| \|h\|_{W_4^{1/2}(\Omega)}^2$$

$$\times \left( \int_0^L \frac{|g(x)|^2}{|x|^{5/3} + |L-x|^{5/3}} dx \right)^{1/2} + C\|h\|_{L_6(\Omega)} \|g\|_{W_3^{2/3}(\Omega)}^2.$$
Arguing in the same way for the last term in (34) we get

\[
\int_0^L |g| \left( \frac{1}{x} + \frac{1}{L-x} \right) dx \leq \left( \int_0^L |h|^2 \left( \frac{1}{x^{1/4}} + \frac{1}{(L-x)^{1/4}} \right) dx \right)^{1/2} \max_{\Omega} |g| 
\]

\[
\leq C||h||_L^2_{1/4,\Omega}||h||_{1/4,\Omega}^2 ||g||_{7/8,\Omega} \leq C||h||_L^{3/4}_{1/3,\Omega}||h||_{1/3,\Omega}^{1/2} ||g||_{7/8,\Omega}^2 
\]

\[
\leq C||h||_L^{3/4}_{1/2,-5,\Omega}||h||_{1/2,-5,\Omega}^{3/4-\frac{12\delta}{5}} ||g||_{7/8,\Omega}^2 \quad (37)
\]

This proves the lemma.

Now we return to the proof of Theorem 2.

**Proof of Theorem 2.** In our argument we use the ideas and methods developed in papers [4, 10, 11, 12, 18].

**Step 1. Existence.**

For the construction of Galerkin’s approximations we use ideas presented in [4, 10, 11, 12].

Let \( \{e_i = (e_{1i}, e_{2i})\}_{i \in \mathbb{N}} \) be the orthonormal basis in \( \tilde{X} \) consisting of the eigenvectors of the Stokes problem:

\[-\mu \Delta e_i + \nabla p_i = \lambda_i e_i \quad \text{in} \; \Omega, \; \text{div} e_i = 0, \; e_i|_{\Gamma_1,\partial \Omega} = 0, \; e_{i2}|_{\Gamma_2} = 0, \; \partial_y e_{i1}|_{\Gamma_2} = 0, (38)\]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) are the corresponding eigenvalues. The existence of solutions to (38) can be shown in the same way as in [23].

Denote by \( \{g_i\}_{i \in \mathbb{N}} \) the orthonormal basis in \( \tilde{L}_2(\Omega) \) which consists of eigenfunctions of the operator \( A_2 \)

\[A_2 g_i = \tilde{\kappa}_i g_i\] (39)

with the eigenvalues \( 0 < \tilde{\kappa}_1 \leq \tilde{\kappa}_2 \leq \cdots \).

Further, let \( \{h_i\}_{i \in \mathbb{N}} \) be the orthonormal basis in \( L_2(\Omega) \) which consists of eigenfunctions of the operator \( A_1 \)

\[A_1 h_i = \kappa_i h_i\] (40)

with the eigenvalues \( 0 < \kappa_1 \leq \kappa_2 \leq \cdots \).

Let \( \tilde{\varphi}_i = N(h_i; 0) \) and \( \varphi_i = N(0; g_i) \), where the operator \( N \) is defined by (19).

In what follows we suppose \( \varphi_i = \tilde{\varphi}_i, \; \zeta_i = (0; g_i), \; \kappa_i = \frac{h E}{\gamma^2 (1-c^2)^2} \tilde{\kappa}_i, \; \varphi_{n+i} = \tilde{\varphi}_i, \; \zeta_{n+i} = (h_i; 0), \; \kappa_{n+i} = \frac{h E}{1-c^2} \kappa_i \) for \( i = 1, \ldots, n \).

We define an approximate solution as a pair of functions \( (V_{n,m}; \xi_n) \):

\[V_{n,m}(t) = \sum_{i=1}^m \alpha_i(t)e_i + \sum_{j=1}^{2n} \beta_j(t)\varphi_j, \; \xi_n(t) = \sum_{j=1}^{2n} \beta_j(t)\zeta_j\] (41)
which satisfy the relations

\[ \dot{\alpha}_k(t) + \sum_{j=1}^{2n} \beta_j(t)(\varphi_j, e_k) = \mu \lambda_k \alpha_k(t) + \mu \sum_{j=1}^{2n} \beta_j(t)(\nabla \varphi_j, \nabla e_k) = (f, e_k) = 0 \quad (42) \]

for \( k = 1, \ldots, m \), and

\[
\sum_{i=1}^{m} \dot{\alpha}_i(t)(e_i, \varphi_k) + \sum_{j=1}^{2n} \beta_j(t)(\varphi_j, \varphi_k) + \rho \beta_k(t) + \\
+ \mu \sum_{i=1}^{m} \alpha_i(t)(\nabla e_i, \nabla \varphi_k) + \mu \sum_{j=1}^{2n} \beta_j(t)(\nabla e_j, \nabla \varphi_k) + \kappa \beta_k(t) + \\
\xi(t, \varphi_k) = (f(t), \varphi_k) + (F(t), \varphi_k) \quad (43)
\]

for \( k = 1, \ldots, 2n \). Here

\[
d(\xi_n(t), \varphi_k) = \frac{he}{1 - \zeta^2} \left( (u_{nx}\eta_{nx}, \zeta_{2kx}) + \frac{1}{2}(\eta_{nx}, \zeta_{2kx}) + \frac{1}{2}(\eta_{nx}, \zeta_{1kx}) \right) \\
+ \frac{s}{R} (u_{nx} + \frac{\eta_{nx}}{2}, \zeta_{2k}) + \frac{s}{R} (\eta_n, \zeta_{1kx}) + \frac{1}{R} (\eta_{nx}, \zeta_{2kx}) + \frac{1}{R^2} (\eta_n, \zeta_{2k})
\]

(44)

This system of ordinary differential equations (42)–(43) is endowed with the initial data

\[
V_{n,m}(0) = \Pi_m(V_0 - N(0; \eta_0) - N(u_1; 0)) + N(0; P_n \eta_1) + N(R_n u_1; 0),
\]

\[
u_n(0) = R_n u_0, \quad \eta_n(0) = P_n \eta_0, \quad \dot{u}_{n}(0) = R_n u_1, \quad \dot{\eta}_{n}(0) = P_n \eta_1,
\]

where \( \Pi_m \) is an orthoprojector on \( \text{Lin}\{e_j : j = 1, \ldots, m\} \) in \( \tilde{X} \), \( P_n \) is an orthoprojector on \( \text{Lin}\{g_i : i = 1, \ldots, n\} \) in \( L_2(\Omega) \), \( R_n \) is an orthoprojector on \( \text{Lin}\{\eta_i : i = 1, \ldots, n\} \) in \( L_2(\Omega) \). Since \( \Pi_m \), \( R_n \) and \( P_n \) are spectral projectors we have that

\[
(V_{n,m}(0); u_n(0); \eta_n(0); \dot{u}_n(0); \dot{\eta}_n(0)) \rightarrow (V_0; u_0; \eta_0; u_1; \eta_1), \quad m, n \rightarrow \infty \quad (45)
\]

strongly in \( H \). Arguing as in [10] we infer that system (42) and (43) has a unique solution on any time interval \([0, T]\).

It follows from (41) that \( V_{n,m}(t) = \sum_{i=1}^{m} \alpha_i(t)e_i + N[\partial_t \xi_n(t)] \), where \( N \) is given by (19). This implies the following boundary compatibility condition

\[
V_{n,m}(t) = \partial_t \xi_n(t) \quad \text{on} \quad \Omega \quad (46)
\]

Multiplying (42) by \( \alpha_k(t) \) and (43) by \( \beta_k(t) \), after summation we obtain an energy relation of the form (28) for the approximate solutions \( (V_{n,m}; \xi_n) \) (for a similar argument we refer to [10]). Lemma 1 together with the energy relation and the trace theorem implies the following a priori estimate:

\[
\sup_{t \in [0, T]} \left[ \left\| V_{n,m}(t) \right\|_{\Omega}^2 + \left\| \partial_t \xi_n(t) \right\|_{\Omega}^2 + \left\| \partial_{xx} \eta_n(t) \right\|_{\Omega}^2 + \left\| \partial_x u_n(t) \right\|_{\Omega}^2 \right] \\
+ \int_{0}^{T} \left\| \nabla V_{n,m}(t) \right\|_{\Omega}^2 dt + \int_{0}^{T} \left\| \partial_t \xi_n(t) \right\|_{H_{1/2}(\Omega)}^2 dt \leq C(T, \|W_0\|_{H}^2) \quad (47)
\]
for any existence interval \([0, T]\) of approximate solutions, where the constant \(C(T, \|W_0\|_{L^2})\) does not depend on \(n\) and \(m\). In particular, this implies that any approximate solution can be extended on any time interval by the standard procedure, i.e., the solution is global.

It also follows from (47) that the sequence \(\{(V_{n,m}; \xi_n; \partial_t \xi_n)\}\) contains a subsequence such that
\[
(V_{n,m}; \xi_n; \partial_t \xi_n) \rightharpoonup (V; \xi; \partial_t \xi) \quad \text{*-weakly in } L_\infty(0, T; H),
\]
\[
V_{n,m} \rightharpoonup V \quad \text{weakly in } L_2(0, T; Y).
\]
Moreover, by the Aubin-Dubinsky theorem (see, e.g., [21, Corollary 4]) we can assert that
\[
\xi_n \rightarrow \xi \quad \text{strongly in } C(0, T; H^{1-\epsilon}(\Omega) \times H^2(\Omega))
\]
for every \(\epsilon > 0\). Besides, the trace theorem yields
\[
\partial_t \xi_n \rightharpoonup \partial_t \xi \quad \text{weakly in } L_2(0, T; [H^{1/2}(\Omega)]^2).
\]
One can see that \((V_{n,m}; \xi_n; \partial_t \xi_n)(t)\) satisfies (42)–(43) with the test function \(\phi\) of the form
\[
\phi = \phi_{l,q} = \sum_{i=1}^l \gamma_i(t)e_i + \sum_{j=1}^q \delta_j(t)\varphi_j,
\]
where \(l \leq m, q \leq n\) and \(\gamma_i, \delta_j\) are scalar absolutely continuous functions on \([0, T]\) such that \(\gamma_i(T) = \delta_j(T) = 0\). Thus using (48)–(49) we can pass to the limit and show that \((V; \xi; \partial_t \xi)(t)\) satisfies (42)–(43) with \(\phi = \phi_{l,q}\), where \(l\) and \(q\) are arbitrary. By (45) and (50) we have \(W(0) = W_0\). Compatibility condition (26) follows from (47) and (51).

To conclude the proof of the existence of weak solutions we only need to show that any function \(\psi\) in \(\mathcal{C}_0^1\) can be approximated by a sequence of functions of the form (52). This can be done in the following way. We first approximate the corresponding boundary value of \(b\) by a finite linear combination \(h\) of \(x_j\), then we approximate the difference \(\psi - Nh\) (with \(N\) define by (19)) by finite linear combination of \(e_i\). Limit transition in nonlinear terms is quite standard, so we omit it here. Thus the existence of weak solutions is proved.

**Step 2.** Higher regularity of the shell component.

Taking in (26) \(\psi(t) = \int_t^T \chi(\tau)d\tau \cdot \phi\), where \(\chi\) is a smooth scalar function and \(\phi\) belongs to the space
\[
\hat{Y} = \left\{ \phi \in Y \mid \phi|_\Omega = \zeta \equiv (\zeta_1; \zeta_2) \in H^1_0(\Omega) \times H^2(\Omega) \right\},
\]
one can see that the weak solution \((V(t); \xi(t))\) satisfies the relation
\[
(V(t), \phi)_0 + \rho h(\xi_t(t), \zeta_\Omega) = (V_0, \phi)_0 + \rho h(\xi_1, \zeta_\Omega) - \int_0^t \left\{ \mu(\nabla V, \nabla \phi)_0 + \frac{h^3E}{12(1-\zeta^2)} (\eta_{xx}, \zeta_{xx})_\Omega + \frac{hE}{1-\zeta^2} r(\xi, \zeta) - (F, \phi)_0 - (F, \zeta)_\Omega \right\} d\tau
\]
for all \(t \in [0, T]\) and \(\phi \in \hat{Y}\) with \(\phi|_\Omega = \zeta = (\zeta_1, \zeta_2)\).
The vector \((V(t), \xi(t), \zeta_t(t))\) is weakly continuous in \(H\) for any weak solution \((V(t), \xi(t))\) to problem (1)-(11). Indeed, it follows from (54) that \((V(t), \xi(t))\) satisfies the relation

\[
(V(t), \phi)_0 = (V_0, \phi)_0 + \int_0^t [-\mu(\nabla V, \nabla \phi) + (f(\tau), \phi)]_0 \, d\tau
\]

for almost all \(t \in [0,T]\) and for all \(\phi \in Y_0 = \{V \in Y : V|_\Omega = 0\} \subset \hat{Y} \subset Y\), where \(\hat{Y}\) is given by (53). This implies that \(V(t)\) is weakly continuous in \(Y_0'\). Since \(X \subset Y_0'\), we can apply Lions lemma (see [20, Lemma 8.1]) and conclude that \(V(t)\) is weakly continuous in \(X\). The same lemma give us weak continuity of \(\xi(t)\) in \(U\). Now using (54) again with \(\phi \in \hat{Y}\) we conclude that \(t \mapsto (\xi_t(t), \zeta_t)\) is continuous for every \(\zeta \in \hat{U}\). This implies that \(t \mapsto \xi_t(t)\) is weakly continuous in \([L_2(\Omega)]^3\). Using weak continuity of weak solutions, we can extend the variational relation in (26) on the class of test functions from \(\mathcal{L}_T\) (instead of \(\mathcal{L}_T^0\)) by an appropriate limit transition. More precisely, one can show that any weak solution \((V; \xi)\) with \(V = (\omega; \nu)\) and \(\xi = (\nu; \eta)\) satisfies the relation

\[
- \int_0^T (V, \psi_t)_0 \, dt + \mu \int_0^T (\nabla V, \nabla \psi)_0 \, dt - \rho h \int_0^T (\xi_t, \zeta_t)_0 \, dt + \frac{hE}{1 - \xi^2} \int_0^T r(\xi, \zeta) \, dt
\]

\[
+ \frac{h^3 E}{12(1 - \xi^2)} \int_0^T (\eta_{xx}, \zeta_{xx})_0 \, dt = (V_0, \psi(0))_0 + \rho h (\xi_1, \zeta(0))_\Omega - (V(T), \psi(T))_0
\]

\[
- \rho h (\xi_T(T), \zeta(T))_\Omega + \int_0^T (f, \psi)_0 \, dt + \int_0^T (F, \zeta)_0 \, dt,
\]

(55)

for every \(\psi \in \mathcal{L}_T\) with \(\psi|_\Omega = \zeta = (\xi_1, \zeta_2)\).

Let \((V(t), \xi(t))\) be a weak solution to problem (1)-(11). Then \(\xi\) can be represented as a Fourier expansion \(\xi = \sum_{j=1}^\infty \beta_j(t) \xi_j\) and then \(V(t) = \sum_{i=1}^\infty \alpha_i(t) e_i + N \left( \sum_{j=1}^\infty \hat{\beta}_j(t) \xi_j \right)\). In what follows we use the notations \(\nu_{j/2} = \hat{\kappa}_j/2\) and \(\nu_{j+1/2} = \hat{\kappa}_j^{1/4}\) for \(j = \{1, n\}\), where \(\hat{\kappa}_j, \hat{\kappa}_j\) are defined in (39)-(40). Now we choose in (55)

\[
\psi = \psi_n = \sum_{j=1}^{2n} \beta_j(t) \nu_{j/2}^1 N \xi_j
\]

and, consequently, \(\zeta = \zeta_n = \sum_{j=1}^{2n} \beta_j(t) \nu_{j/2}^1 \xi_j\)

(56)

for any fixed \(n \in \mathbb{N}\). Then, we obtain

\[
- \int_0^T (V, \psi_n)_0 \, dt + \mu \int_0^T (\nabla V, \nabla \psi_n)_0 \, dt - \rho h \int_0^T \sum_{j=1}^{2n} \hat{\beta}_j(t) \nu_{j/2}^1 \, dt
\]

\[
+ \frac{h^3 E}{12(1 - \xi^2)} \int_0^n \sum_{j=1}^{j+n} \beta_{j+n}(t) \nu_{j/4}^5 \, dt + \frac{hE}{1 - \xi^2} \int_0^n \sum_{j=1}^n \beta_j(t) \nu_{j/2}^3 \, dt = (V_0, \psi_n(0))_0
\]

\[
+ \rho h (\xi_1, \zeta_n(0))_\Omega - (V(T), \psi_n(T))_0 - \rho h (\xi_T(T), \zeta_n(T))_\Omega
\]
\[
+ \int_0^T (f, \psi^n)_{\Omega} dt + \int_0^T (F, \zeta^n)_{\Omega} dt + \int_0^T d(\xi, \zeta^n) dt. 
\]

(57)

If we denote \(\xi_n = (\xi_{1n}, \xi_{2n}) = \sum_{j=1}^{2n} \beta_j(t) \xi_j\), then

\[
\sum_{j=1}^{2n} \beta_j(t)^2 \nu_j^{1/2} = \|\xi_n\|_{1/2}^2, \quad \sum_{j=1}^{2n} \beta_{j+n}(t)^2 \kappa_j^{5/4} = \|\xi_n\|^2_{5/2}, \\
\sum_{j=1}^{n} \beta_j(t)^2 \kappa_j^{3/2} = \|\xi_{1n}\|_{3/2}^2, 
\]

(58)

and

\[
\xi_n \to \xi \text{ strongly in } \hat{U}, \\
\xi_{tn} \to \xi_t \text{ strongly in } L_2(\Omega) \times \hat{L}_2(\Omega),
\]

(59)

(60)

this implies that there exists \(C > 0\) independent of \(n\) such that \(\|\xi_n\|_U + \|\xi_n\|_\Omega \leq C\), moreover, \(\psi_n = N\xi_n\).

The operators \(A_1\) and \(A_2\) are positive self-adjoint operators with discrete spectrums, therefore (see, e.g. [6])

\[
A_1^0 \xi_{1n} = \sum_{j=1}^{n} \beta_j(t) \kappa_j^{\alpha} h_j, \quad \|A_1^0 \xi_{1n}\|_\Omega = \|\xi_{1n}\|_{\alpha/2, \Omega}, \\
A_2^0 \xi_{2n} = \sum_{j=1}^{n} \beta_{j+n}(t) \kappa_j^{\alpha} g_j, \quad \|A_2^0 \xi_{2n}\|_\Omega = \|\xi_{2n}\|_{\alpha/4, \Omega}. 
\]

(61)

(62)

Using the properties of the operator \(N\) and estimate (47) we come to

\[
\int_0^T (f(t), N\xi_n)_{\Omega} dt + \int_0^T (F(t), \zeta_n)_{\Omega} dt \\
\leq C \left( \int_0^T \|f(t)\|_{1/2}^2 dt + \int_0^T \|F(t)\|^{2}_{-1/2, \Omega} dt + \int_0^T \|\xi_n\|_{1, \Omega} dt \right) \leq C. 
\]

(63)

Using (22) it is easy to see that

\[
\int_0^T \left\| \nabla V, \nabla \left( \sum_{j=1}^{2n} \beta_j(t) \nu_j^{1/2} \xi_j \right) \right\|_{\mathcal{O}} dt \\
\leq C \int_0^T \|\nabla V\|_{\mathcal{O}}^2 dt \int_0^T \|\sum_{j=1}^{2n} \beta_j(t) \nu_j^{1/2} \xi_j\|_{1/2, \Omega}^2 dt \\
\leq C \int_0^T \|\nabla V\|_{\mathcal{O}}^2 dt \int_0^T \|\sum_{j=1}^{2n} \beta_j(t) \nu_j^{3/4} \xi_j\|_{\Omega}^2 dt \leq C \epsilon \int_0^T \|\nabla V\|_{\mathcal{O}}^2 dt
\]

(64)
Taking into account (22) we get

\[ |(V_0, \psi^n(0))_\Omega + \rho h(\xi_1, \zeta^n(0))_\Omega + (V(T), \psi^n(T))_\Omega + \rho h(\xi(T), \zeta^n(T))_\Omega \leq C \sup_{[0,T]} (\|\xi_n\|^2 + \|\xi_t\|^2 + \|V(t)\|^2) \leq C. \quad (65) \]

and

\[
\left| \int_0^T (V, N\xi_t) dt \right| \leq C \left( \int_0^T \|\nabla V\|^2 dt + \int_0^T \|\nabla_\xi\|^2 dt \right) \leq C \left( \int_0^T \|\nabla V\|^2 dt + \int_0^T \|\xi_t\|^2 dt \right) \leq C. \]

Now we estimate the last term in the right-hand side of (57). Taking into account the structure of this term given by (44) and Lemma 2 we obtain

\[
\int_0^T \left( u_x \eta_x, \left( \sum_{j=1}^n \beta_j + n(t) \tilde{\kappa}_j^{1/4} g_j \right) x \right)_\Omega + \left( \eta_{xx} \eta_x, \sum_{j=1}^n \beta_j(t) \tilde{\kappa}_j^{1/2} h_j \right)_\Omega \ dt \\
\leq C \int_0^T \|\eta\|_2 \Omega(\|u\|_1 \Omega, \|\xi_2\|_2 \Omega, \|\eta\|_2 \Omega, \|\xi_1\|_1 \Omega) dt \leq C(T) \quad (66) \]

and

\[
\int_0^T \left( (\eta_x^2 + \eta) (\eta_{xx} + 1), \sum_{j=1}^n \beta_j + n(t) \tilde{\kappa}_j^{1/4} g_j \right)_\Omega \ dt \leq C \int_0^T \|\eta_x^2 + \eta\|_{L^\infty, \Omega} \ dt \\
\|\eta_{xx} + 1\|_2 \Omega \|\xi_2\|_1 \Omega \ dt \leq C \int_0^T (\|\eta\|_2^3 + 1) \|\xi_2\|_2 \Omega dt \leq C(T). \quad (67) \]

It is also obvious that

\[
\int_0^T \left( u_x, \sum_{j=1}^n \beta_j(t) \tilde{\kappa}_j^{1/2} h_j \right)_\Omega + \left( \eta_x, \sum_{j=1}^n \beta_j(t) \tilde{\kappa}_j^{1/2} h_j \right)_\Omega \ dt \\
\leq \int_0^T (\|u\|_1 \Omega, \|\xi_2\|_1 \Omega, \|\eta\|_1 \Omega, \|\xi_1\|_1 \Omega) dt \leq C(T). \quad (68) \]

Collecting (57), (63)–(68) and using (48), (49) we infer \( \int_0^T (\|\xi_2\|_{3/2, \Omega}^2 + \|\xi_1\|_{3/2, \Omega}^2) \) \( \leq C(T) \), which implies that there exists a subsequence \( \xi_{nk} \) which converges to a function \( \xi \) weakly in \( L_2([0,T]; H^{3/2}(\Omega) \times H^{5/2}(\Omega)) \). Together with the strong convergence \( \xi_n \to \xi \) in the space \( L_\infty([0,T]; H_0^1(\Omega) \times H_0^2(\Omega)) \) as \( n \to \infty \) this gives
Indeed, by Lemma 3 we obtain proof of Lemma 4.1 [18]. The only difference lies in the proof of the fact that the shell component only. The arguments are analogous to those presented in the fluid component in our model are the same as in [11], and we need to treat \(Dh\) as a test function in (55). For the shell component we have test function \(\psi\). The \(\tilde{\xi}\) collected in Proposition 4.3 [18].

This makes it possible to prove the energy equality in (28).

Continuity of weak solutions with respect to \(t\), initial data and the energy equality.

To prove the energy equality for a weak solution we follow the scheme presented in [11] and [18]. We introduce a finite difference operator \(Dh\), depending on a small parameter \(h\). Let \(g\) be a bounded function on \([0, T]\) with values in some Hilbert space. We extend \(g(t)\) for all \(t \in \mathbb{R}\) by defining \(g(t) = g(0)\) for \(t < 0\) and \(g(t) = g(T)\) for \(t > T\). With this extension we denote \(g_h^+ (t) = g(t + h) - g(t), g_h^- (t) = g(t) - g(t - h), D_h g(t) = \frac{1}{2h} (g_h^+ (t) + g_h^- (t))\). Properties of the operator \(D_h\) are collected in Proposition 4.3 [18].

Now we use

\[
\psi = \frac{1}{2h} \int_{t-h}^{t+h} V(\tau) d\tau
\]

as a test function in (55). For the shell component we have test function \(\zeta = \psi|_\Omega = D_h \xi\) – the same one that used in [18] for the full Karman model. All the arguments for the fluid component in our model are the same as in [11], and we need to treat the shell component only. The arguments are analogous to those presented in the proof of Lemma 4.1 [18]. The only difference lies in the proof of the fact that

\[
\lim_{h \to 0} \int_0^T \left( u_x + \frac{\eta^2}{2} + \frac{\eta}{R}, D_h \eta (\partial_x \eta_h^+ - \partial_x \eta_h^-) \right)_\Omega dt = 0.
\] (70)

Indeed, by Lemma 3 we obtain

\[
\left| \int_0^T \left( u_x + \frac{\eta^2}{2} + \frac{\eta}{R}, D_h \eta (\partial_x \eta_h^+ - \partial_x \eta_h^-) \right)_\Omega dt \right| \\
\leq C \int_0^T \left\| u_x + \frac{\eta^2}{2} + \frac{\eta}{R} \right\|_{L^{1/2-\delta, \Omega}} \left\| h D_h \eta \right\|_{L^2, \Omega} \left\| \eta_h^+ - \eta_h^- \right\|_{H^{1/2, \Omega}} dt \\
\leq C \sup_{[0, T]} \left\| \eta \right\|_{L^2, \Omega} \int_0^T \left\| u \right\|_{L^{3/2, \Omega}} \left\| \eta \right\|_{H^{1/2, \Omega}} dt.
\] (71)

This makes it possible to prove the energy equality in (28).

Continuity of weak solutions with respect to \(t\) can be obtained in the standard way from the energy equality and weak continuity (see [20, Ch. 3] and also [18]).

Continuity with respect to initial data can be shown by the same method as in [11, 18].

Let \((\tilde{V}(t); \tilde{\xi}(t))\) and \((\tilde{V}(t); \tilde{\xi}(t))\) be two solutions to the problem considered with the same initial data. Their difference \((V(t); \xi(t)) = (\tilde{V}(t) - V(t); \tilde{\xi}(t) - \xi(t))\) possesses properties

\[
V = (u, v) \in L_\infty(0, T; X) \cap L_2(0, T; Y); \\
\xi = (u; \eta) \in L_\infty(0, T; \tilde{U}) \cap L_2(0, T; H^{3/2}(\Omega) \times H^{5/2}(\Omega));
\] (72) (73)
and the relation
\[
- \int_0^T (V, \psi) dt + \mu \int_0^T (\nabla V, \nabla \psi) dt - \rho h \int_0^T (\xi, \partial_t \psi) dt + \frac{h^3 E}{12(1 - \varsigma^2)} \int_0^T (\eta, \partial_x \xi) dt
\]
\[
= \frac{h E}{1 - \varsigma^2} \int_0^T (d(\tilde{\xi}, \zeta) - d(\tilde{\zeta}, \xi)) dt - (V(T), \psi(T)) - \rho h (\xi(T), \zeta(T)) dt,
\]
for every \( \psi \in \mathcal{L}_T \) with \( \psi|_{\Omega} = \zeta = (\xi, \zeta) \). Using again (69) as a test function and relying on Proposition 4.3 [18] we get
\[
\|V(T)\|_{\Omega}^2 + \|\xi(T)\|_{\Omega}^2 + \|\eta(T)\|_{\Omega}^2 + \|\eta(T)\|_{\Omega}^2 + \int_0^T \|\nabla V(T)\|_{\Omega}^2 dt
\]
\[
\leq C \lim_{h \to 0} \sum_{i=1}^6 \int_0^T q_i(t) dt,
\]
where
\[
q_1(t) = (u_x, D_h \eta) + (\eta_x, D_h u)_\Omega
\]
\[
q_2(t) = (\eta_x(\tilde{\eta}_x + \tilde{\eta}_x), D_h \eta) + (\eta_s(\tilde{\eta}_x^2 + \tilde{\eta}_x \tilde{\eta}_s + \tilde{\eta}_s^2), D_h \eta)_\Omega
\]
\[
q_3(t) = (\tilde{\eta}_x u_x, D_h \eta)_\Omega
\]
\[
q_4(t) = (\eta_x \tilde{\eta}_x, D_h \eta)_\Omega
\]
\[
q_5(t) = (\eta_x \tilde{\eta}_x, D_h \eta) + (\tilde{\eta}_x \eta_x, D_h \eta)_\Omega
\]
\[
q_6(t) = (\eta x \tilde{\eta}_x, D_h u)_\Omega + (\tilde{\eta}_x \eta_x, D_h u)_\Omega.
\]
Now we estimate the terms in the right-hand side of (75). Obviously,
\[
\lim_{h \to 0} \int_0^T |q_1(t)| dt \leq C \lim_{h \to 0} \left( \int_0^T |u_x, D_h \eta| dt + \int_0^T |\eta_x, D_h u| dt \right)
\]
\[
\leq C \left( \int_0^T \|u_x\|^2 dt + \int_0^T \|\eta_x\|^2 dt + \int_0^T \|u_t\|^2 dt + \int_0^T \|\eta_t\|^2 dt \right).
\]
Applying Lemma 3 with \( g = \tilde{\eta}_x \) and \( h = u_x \) we obtain that for any \( \varepsilon > 0 \) and \( 0 < \delta < 1/6 \)
\[
\lim_{h \to 0} \int_0^T |q_3(t)| dt \leq C \lim_{h \to 0} \int_0^T |(\tilde{\eta}_x u_x, D_h \eta)| dt
\]
\[
\leq C \lim_{h \to 0} \int_0^T \|\tilde{\eta}_x u_x\|_{1/2, \Omega}^2 \|D_h \eta\|_{1/2, \Omega} dt.
\]
Since it is easy to see that we come to let us also observe that by Lemma 3 we also have applying Lemma 3 for $g = \eta_x$ and $h = \hat{u}_x$ we get for any $\varepsilon > 0$

$$
\lim_{h \to 0} \frac{1}{T} \int_0^T |q_4(t)| dt \leq \lim_{h \to 0} \frac{1}{T} \int_0^T (|\eta_x \hat{u}_x, D_h \eta_x|) dt \leq C \lim_{h \to 0} \frac{1}{T} \int_0^T (\|\eta_x \hat{u}_x\|_{1/2, \Omega}^* \|D_h \eta\|_{1/2, \Omega}^*) dt
$$

$$
\leq C(\varepsilon) \int_0^T \|\hat{u}\|_{3/2, \Omega}^* \|\eta_x\| dt + \varepsilon \int_0^T \|\eta\|_{1/2, \Omega}^* dt. \quad (83)
$$

Let us also observe that by Lemma 3 we also have

$$
\lim_{h \to 0} \frac{1}{T} \int_0^T |q_2(t)| dt
$$

$$
\leq C \lim_{h \to 0} \frac{1}{T} \int_0^T (\|\eta_x (\hat{\eta}_x + \eta_x)\|_{1/2, \Omega}^* \|\eta_x (\hat{\eta}_x^2 + \hat{\eta}_x \hat{\eta}_x + \eta_x^2)\|_{1/2, \Omega}^* ) \|D_h \eta\|_{1/2, \Omega}^* dt
$$

$$
\leq C \int_0^T (\|\eta_x \hat{\eta}_x + \eta_x \hat{\eta}_x + \eta_x \eta_x\|_{1/2, \Omega}^* \|\eta_x \hat{\eta}_x + \eta_x \hat{\eta}_x + \eta_x \eta_x\|_{1/2, \Omega}^* ) \|\eta_x \hat{\eta}_x + \eta_x \hat{\eta}_x + \eta_x \eta_x\|_{1/2, \Omega}^* dt.
$$

Since

$$
\|\hat{\eta}_x^2 + \hat{\eta}_x \hat{\eta}_x + \eta_x^2\|_{1/2, \Omega}^* \leq C \int_0^L (\hat{\eta}_x^2 + \hat{\eta}_x^2)^2 (\eta_x^2 + \eta_x^2)^2 dx
$$

$$
\leq C \|\eta_x + \hat{\eta}_x\|_{\infty, \Omega}^* \|\eta_x + \hat{\eta}_x\|_{\Omega}^* \leq C \|\eta_x + \hat{\eta}_x\|_{\Omega}^*,
$$

we come to

$$
\int_0^T |q_2(t)| dt \leq C(\varepsilon, T, \|\hat{\omega}_0\|_{H^2}, \|\omega_0\|_{H^2}) \int_0^T \|\eta_x\|^* dt + \int_0^T \|\eta\|_{1/2, \Omega}^* dt. \quad (85)
$$

It is easy to see that

$$
\lim_{h \to 0} \frac{1}{T} \int_0^T |q_5(t)| dt = \lim_{h \to 0} \frac{1}{T} \int_0^T (|\eta_x \hat{\eta}, D_h \eta|) dt + \frac{1}{T} \int_0^T (|\eta_x \hat{\eta}, D_h \eta|) dt
$$

$$
\leq C \lim_{h \to 0} \frac{1}{T} \int_0^T (\|\eta_x\|_{1/2, \Omega}^* \|\eta_x\|_{1/2, \Omega}^* \|D_h \eta\| d
$$

\[
\leq C(T, \|\cdot\|_{H_0}^2, \|\cdot\|_{H_0}^2) \left( \int_0^T \|\eta_{xx}\|^2 dt + \int_0^T \|\xi_t\|^2 dt \right)
\]  

(86)

and

\[
\lim_{h \to 0} \int_0^T |q_0(t)| dt \leq C \lim_{h \to 0} \left( \int_0^T |\eta_{xx}\tilde{\eta}_x, D_h u| \Omega| dt + \int_0^T (\eta_{xx}, D_h u| \Omega) dt \right)
\]
\[
\leq C \lim_{h \to 0} \left( \|\eta_{xx}\|\|\tilde{\eta}_x\| + \|\eta_{xx}\|\|\eta_{xx}\|\|D_h u\| dt \right)
\]
\[
\leq C(T, \|\cdot\|_{H_0}^2, \|\cdot\|_{H_0}^2) \left( \int_0^T \|\eta_{xx}\|^2 dt + \int_0^T \|u_t\|^2 dt \right). \quad (87)
\]

Collecting (82)–(87) we arrive at

\[
\|V(t)\|_{H_0}^2 + \|\xi_t(T)\|_{H_0}^2 + \|\eta(T)\|_{H_0}^2 + \|u(T)\|_{H_0}^2 + \int_0^T \|\nabla V(t)\|_{H_0}^2 dt
\]
\[
\leq C(T, \|\cdot\|_{H_0}^2, \|\cdot\|_{H_0}^2, \varepsilon) \left( \int_0^T \|u_x\|^2 dt + \int_0^T \left(1 + \|\tilde{u}\|_{H_0}^2\right) \|\eta_{xx}\|^2 \right)
\]
\[
+ \int_0^T \|u_t\|^2 dt + \int_0^T \|\eta_t\|^2 dt + \varepsilon \int_0^T \|u\|_{H_0}^2 dt. \quad (88)
\]

Arguing as in Step 2 and taking into account that estimates

\[
\int_0^T \left| \sum_{j=1}^n \beta_{j+n}(t)\tilde{k}_j^{1/2}h_j \right| dt \leq C \int \left( \|\eta\|_{L_2, \Omega} \|\tilde{\eta}\|_{L_1, \Omega} \|\xi_{2n}\|_{L_2, \Omega} + \|\tilde{\eta}\|_{L_2, \Omega} \|u\|_{L_1, \Omega} \|\xi_{2n}\|_{L_2, \Omega} \right)
\]
\[
+ \|\tilde{\eta}\|_{L_2, \Omega} \|\eta\|_{L_1, \Omega} \|\xi_{2n}\|_{L_2, \Omega} + \|\eta\|_{L_2, \Omega} \|\tilde{\eta}\|_{L_1, \Omega} \|\xi_{2n}\|_{L_2, \Omega} \|\xi_{2n}\|_{L_2, \Omega} dt
\]

\[
\leq C(T, \|\cdot\|_{H_0}^2, \|\cdot\|_{H_0}^2) \int \left( \|\eta\|_{H_0}^2 + \|u\|_{H_0}^2 + \|\xi_{2n}\|_{H_0}^2 + \|\xi_{2n}\|_{H_0}^2 dt, \quad (89)
\]

and

\[
\int_0^T \left( \eta_x(\tilde{\eta}_x^2 + \tilde{\eta}_x\tilde{\eta}_x + \tilde{\eta}_x^2) \right) dt \leq C \int \left( \|\tilde{\eta}\|_{H_0}^2 + \|\tilde{\eta}\|_{H_0}^2 \right)
\]
\[
\leq C(T, \|\cdot\|_{H_0}^2, \|\cdot\|_{H_0}^2) \int \left( \|\eta\|_{H_0}^2 + \|\xi_{2n}\|_{H_0}^2 \right) dt \quad (90)
\]
as soon as
\[
\int_0^T \left( \eta_{xx} \tilde{\eta} + \eta_{xx} \eta, \sum_{j=1}^n \beta_j(t) \tilde{h}_j^{1/2} h_j \right) \, dt \leq \int_0^T \left( \|\tilde{\eta}\|_{2,\Omega} + \|\tilde{\eta}\|_{2,\Omega} \right) \eta_{1n} \, dt
\]
\[
\leq C(T, \|\tilde{W}_0\|^2_{H^r}, \|\tilde{W}_0\|^2_{H^r}) \int_0^T \left( \|\eta\|^2_{2,\Omega} + \|\xi_{1n}\|^2_{1,\Omega} \right) \, dt. \quad (91)
\]

hold true and passing to the limit we arrive at
\[
\int_0^T \|u\|^2_{3/2,\Omega} \, dt + \int_0^T \|\eta\|^2_{3/2,\Omega} \leq \frac{C}{\epsilon} \int_0^T \|\xi_t\|^2_{1/2,\Omega} \, dt + \epsilon \int_0^T \|\xi\|^2_{3/2,\Omega} \, dt
\]
\[
+ C \left( \int_0^T \|V\|^2_0 \, dt + \|\xi_T(t)\|^2_{\Omega} + \|\xi_{T(\Omega)}\|^2_{1,\Omega} + \|\|V(T)\|^2_{\Omega} \right)
\]
\[
+ C(T, \|\tilde{W}_0\|^2_{H^r}, \|\tilde{W}_0\|^2_{H^r}) \left( \int_0^T \|u\|^2_{1,\Omega} \, dt + \int_0^T \|\eta\|^2_{2,\Omega} \, dt \right). \quad (92)
\]

Choosing \( \epsilon = \frac{C}{\epsilon} \) in (88) and \( \epsilon \) small enough and adding to estimate (88) estimate (92) multiplied by \( 2\epsilon \) we get
\[
\|V(T)\|^2_{\Omega} + \|\xi_T(T)\|^2_{\Omega} + \|\eta(T)\|^2_{2,\Omega} + \|u(T)\|^2_{1,\Omega} + \int_0^T \|\nabla V(t)\|^2_0 \, dt +
\]
\[
\leq C(T, \|\tilde{W}_0\|^2_{H^r}, \|\tilde{W}_0\|^2_{H^r}, \epsilon) \left( \int_0^T \|u_x\|^2 \, dt + \int_0^T \left( 1 + \|\tilde{u}\|^2_{3/2,\Omega} \right) \|\eta_{xx}\|^2 \right.
\]
\[
+ \int_0^T \|u_t\|^2 \, dt + \int_0^T \|\eta_t\|^2 \, dt + \int_0^T \|V\|^2_0 \, dt \right). \quad (93)
\]

Taking into account that \( \int_0^T \|\tilde{u}\|^2_{3/2,\Omega} \, dt \leq C(T, \|\tilde{W}_0\|^2_{H^r}) \) and using the Gronwall’s lemma we can infer from (93) that
\[
\|V(T)\|^2_{\Omega} + \|\xi_T(T)\|^2_{\Omega} + \|\eta(T)\|^2_{2,\Omega} + \|u(T)\|^2_{1,\Omega} \leq 0. \quad (94)
\]

Estimate (94) together with (48) immediately gives the uniqueness of the weak solution. \( \square \)

**Remark 1.** It can be shown (see [11]) that for Galerkin approximations \((V_n, u_n, \eta_n)\) of a weak solution \((V, u, \eta)\) to (1)-(11) the following limits for any \( \epsilon > 0 \) remain true by the Aubin-Dubinsky theorem (see [21])
\[
V^n \rightarrow V \quad \text{strongly in } C(0, T; H^{-\epsilon}(\Omega)) \quad (95)
\]
\[
u^n \rightarrow u \quad \text{strongly in } C(0, T; H_0^{-\epsilon}(\Omega)), \quad (96)
\]
\[
\eta^n_t \rightarrow \eta_t \quad \text{strongly in } C(0, T; H_0^{2-\epsilon}(\Omega)). \quad (97)
\]
Remark 2. It follows trivially from Theorem 2 that problem (1)-(11) generates a dynamical system \((S_t, \hat{H})\), where \(S_t\) is the evolution operator acting by the formula \(S_t W_0 = W(t)\). Here \(W(t)\) is the weak solution to (1)-(11) with initial data \(W_0\).

4. Long-time behaviour. Our first step in order to investigate the long-time behavior of the dynamical system \((S_t, \hat{H})\) is the following result:

**Proposition 1.** We assume that
\[
f = 0, \quad f_1 = 0 \quad \text{and} \quad f_2 \equiv g \in H^{-1/2}(\Omega).
\]
Then any stationary solution has the form \((0; \xi)\), where \(\xi = (u; \eta) \in \hat{U}\) satisfies the equation
\[
(\eta_{xx}, \zeta_{2xx})\Omega + r(u, \zeta) - (g, \zeta_2)\Omega = 0, \quad \forall \zeta = (\zeta_1; \zeta_2) \in \hat{U}.
\]
Moreover, there exists at least one solution \(\xi = (u; \eta)\) to (99) in the space \(\hat{U}\). In addition, the set \(\mathcal{N}_0\) of all solutions to (99) from \(\hat{U}\) is bounded in the space \(\hat{U}\), which means the set of all stationary points of \((S_t, \hat{H})\) is nonempty bounded set and has the form
\[
\mathcal{N} = \left\{ (0; \xi; 0) : \xi \in \hat{U} \text{ solve (99)} \right\}.
\]

The proof of Proposition 1 is similar to the proof of Proposition 4.2 [11], so we omit it.

Now we are in position to establish quasy-stability of the system \((S_t, \hat{H})\).

Let \(\hat{W}(t) = (\hat{V}(t), \hat{\xi}(t))\) and \(\tilde{W}(t) = (\tilde{V}(t), \tilde{\xi}(t))\), where we use the notations \(\hat{V}(t) = (\hat{w}(t), \hat{\nu}(t))\), \(\tilde{V}(t) = (\tilde{w}(t), \tilde{\nu}(t))\) and \(\hat{\xi}(t) = (\hat{\eta}(t), \hat{\nu}(t))\), \(\tilde{\xi}(t) = (\tilde{\eta}(t), \tilde{\nu}(t))\) be weak solutions to problem (1)-(11) with initial data \(\hat{W}_0 = (\hat{V}_0, \hat{\xi}_0)\) and \(\tilde{W}_0 = (\tilde{V}_0, \tilde{\xi}_0)\). Here \(\hat{V}_0 = (\hat{w}_0, \hat{\nu}_0)\), \(\tilde{V}_0 = (\tilde{w}_0, \tilde{\nu}_0)\) and \(\hat{\xi}_0 = (\hat{\eta}_0, \hat{\nu}_0)\), \(\tilde{\xi}_0 = (\tilde{\eta}_0, \tilde{\nu}_0)\). We assume that initial data \(\hat{W}_0, \tilde{W}_0\) belong to a bounded forward invariant with respect to the evolution operator \(S_t\) set \(\mathcal{B}\). Due to this assumption and Lemma 1 we have that there exists \(R > 0\) such that
\[
\|\hat{W}(t)\|_H \leq R, \|\tilde{W}(t)\|_H \leq R \quad \text{for any } t > 0.
\]

The difference of two solutions \(W(t) = \hat{W}(t) - \tilde{W}(t)\) satisfies the initial data \(\xi(0) = \xi_0 = \hat{\xi}_0 - \tilde{\xi}_0\) and the equality
\[
- \int_0^T (V, \psi)\Omega dt + \mu \int_0^T (\nabla V, \nabla \psi)\Omega dt - \rho h \int_0^T (\xi, \xi_t)\Omega dt + \frac{h^2 E}{1 - \sigma^2} \int_0^T (u_x, \xi_{1x})\Omega dt
\]
\[
+ \frac{h^3 E}{12(1 - \sigma^2)} \int_0^T (\eta_{xx}, \zeta_{2xx})\Omega dt + \frac{h E}{1 - \sigma^2} \int_0^T (d(\hat{\xi}, \xi) - d(\hat{\xi}, \zeta)) dt
\]
\[
= (V_0, \psi(0))\Omega + \rho h (\xi_1(0), \zeta_1(0))\Omega - (V(T), \psi(T))\Omega - \rho h (\xi_T(T), \zeta(T))\Omega,
\]
for any \(T > 0\) and \(\psi \in \mathcal{L}_T\) with \(\psi|_{\Omega_T} = \zeta\), where \(d(\hat{\xi}, \xi)\) is given by (44).

The calculations below are made on the Galerkin approximations of solutions. First, by energy argument and integration over intervals \([t, T]\) and then \([0, T]\) we establish the following energy type inequality:
Lemma 4. For any $T > 0$

$$T \Phi(V(T), \xi(T), \xi_T(T)) + \mu \int_0^T \int \| \nabla V \|_b^2 d\tau dt$$

$$= \int_0^T \Phi(V(t), \xi(t), \xi_t(t)) ds + \int_0^T \Psi(\xi, \xi) d\tau dt,$$  \hspace{1cm} (103)

where

$$\Phi(V(t), \xi(t), \xi_t(t))$$

$$= \| V \|_o^2 + \rho \| \xi_t \|_{1, \Omega}^2 + \frac{hE}{1 - \varsigma^2} \left( \frac{h^2}{12} \| \eta_{xx} \|^2 + \frac{1}{R^2} \| \eta \|^2_{1, \Omega} + \| u_x \|^2_{1, \Omega} \right)$$  \hspace{1cm} (104)

and

$$\Psi(\xi, \xi) = \frac{hE}{1 - \varsigma^2} \int_0^L \left( \frac{\varsigma}{R} u_x \eta_t - \frac{\varsigma}{R} \eta_x u_t + \frac{1}{2} \eta_x u_x (\tilde{\eta}_x + \tilde{\eta}_x) + \frac{\varsigma}{2R} \eta_x \eta_t (\tilde{\eta}_x + \tilde{\eta}_x) \right) dx.$$  \hspace{1cm} (105)

To estimate the right-hand side of (103) we need the following auxiliary result:

Lemma 5. For any $0 < \sigma < 1/2$ the following estimate holds true for $g, h \in H^{1-\sigma}(\Omega)$

$$\| gh \|_{1-\sigma, \Omega} \leq C \| g \|_{1-\sigma, \Omega} \| h \|_{1-\sigma, \Omega}$$  \hspace{1cm} (106)

Proof. Relying on Lemma 2 we obtain

$$\| gh \|_{1-\sigma, \Omega}^2 \leq C \int_0^L \int_0^L \frac{|g(x)h(x) - g(z)h(z)|^2}{|x-z|^{3-2\sigma}} dz dx \leq C \int_0^L \int_0^L \frac{|g(x)|^2|h(x) - h(z)|^2}{|x-z|^{3-2\sigma}} dz dx$$

$$+ C \int_0^L \int_0^L \frac{|g(x) - h(z)|^2|h(x)|^2}{|x-z|^{3-2\sigma}} dz dx \leq C \| g \|_{1, \Omega}^2 \int_0^L \int_0^L \frac{|h(x) - h(z)|^2}{|x-z|^{3-2\sigma}} dz dx$$

$$+ \| h \|_{1, \Omega}^2 \int_0^L \int_0^L \frac{|g(x) - g(z)|^2}{|x-z|^{3-2\sigma}} dz dx \leq C \| g \|_{1-\sigma, \Omega}^2 \| h \|_{1-\sigma, \Omega}^2.$$  \hspace{1cm} (107)

Now we estimate the first term in the right-hand side of (103).

Proposition 2. Given $\varepsilon > 0$ for any $0 < \sigma < 1/2$ there exists positive constants $C(R, T, \varepsilon) > 0$ and $C > 0$ such that the estimate

$$\int_0^T \Phi(V(t), \xi(t), \xi_t(t)) dt \leq C \Phi(V_0, \xi_0, \xi_1) + \varepsilon \int_0^T \| \nabla V \|_b^2 dt$$

$$+ C(R, T, \varepsilon) \max_{[0, T]} (\| V \|_{2-\sigma, \Omega}^2 + \| \eta \|_{2-\sigma, \Omega}^2 + \| u_t \|_{2-\sigma, \Omega}^2 + \| \eta_t \|_{2-\sigma, \Omega}^2 + \| \eta_t \|_{2-\sigma, \Omega}^2)$$  \hspace{1cm} (108)
holds true for every \( T > 0 \).

**Proof.** Choosing \( \psi = N \xi \) in (102) and relying on the properties of the operator \( N \) we get

\[
\int_0^T \left[ \left( \| \eta_{xx} \|_{\Omega}^2 + \| u_x \|_{\Omega}^2 \right) \right] dt \leq C(\Phi(V_0, \xi_0, \xi_1) + \| V(T) \|_{2-\sigma, \Omega}^2 + \| \xi(T) \|_{1-\sigma, \Omega}^2
\]

\[
+ \| \xi_T(T) \|_{2-\sigma, \Omega}^2 + \int_0^T \| \xi \|_{2-\sigma, \Omega}^2 dt + \epsilon \int_0^T \| \nabla V \|_{\Omega}^2 dt + \int_0^T \left| d(\hat{\xi}, \xi) - d(\tilde{\xi}, \xi) \right| dt. \quad (109)
\]

From the structure of \( d(\hat{\xi}, \xi) \) given by (44) it is readily seen that

\[
\int_0^T \left| d(\hat{\xi}, \xi) - d(\tilde{\xi}, \xi) \right| dt \leq C \sum_{j=1}^4 |J_j|, \quad (110)
\]

where

\[
J_1 = \int_0^T \left[ \int_0^L \eta_x u_x (\hat{\eta}_x + \tilde{\eta}_x) dx dt, \right.
\]

\[
J_2 = \int_0^T \left[ \int_0^L \eta_x^2 \hat{\eta}_x dx dt + \int_0^T \eta_x u dx dt, \right.
\]

\[
J_3 = \int_0^T \left[ \int_0^L \eta_x \eta_x u_x dx dt, \right.
\]

\[
J_4 = \int_0^T \left[ \int_0^L \eta_x^2 (\hat{\eta}_x^2 + \hat{\eta}_x \tilde{\eta}_x + \tilde{\eta}_x^2) dx dt + \int_0^T \eta_x \eta (\hat{\eta}_x + \tilde{\eta}_x) dx dt
\]

\[
+ \int_0^T \eta_x^2 \hat{\eta}_x dx dt + \int_0^T \eta_x \eta_x u_x dx dt. \right]
\]

Let us observe that Lemma 5 yields

\[
\| \eta_x (\hat{\eta}_x + \tilde{\eta}_x) \|_{2-\sigma, \Omega}^2 \leq C \| \eta \|_{2-\sigma, \Omega}^2 \| \hat{\eta}_x + \tilde{\eta}_x \|_{1-\sigma, \Omega}^2 \leq C(R) \| \eta \|_{2-\sigma, \Omega}^2
\]

for \( 0 < \sigma < 1/2 \). Consequently,

\[
|J_1| \leq C \int_0^T \| u \|_{\sigma} \| \eta_x (\eta_1x + \eta_{2x}) \|_{1-\sigma} dt \leq C(R) T \max_{[0,T]} (\| u \|_{\sigma, \Omega}^2 + \| \eta \|_{2-\sigma, \Omega}^2). \quad (111)
\]

Analogously we find that

\[
|J_3| \leq C \int_0^T \| \eta_x \|_{1-\sigma, \Omega} \| u_x \|_{\sigma-1, \Omega} dt \leq C \int_0^T \| \eta_x \|_{\sigma, \Omega} \| \eta_x \|_{1-\sigma, \Omega} \| u \|_{\sigma, \Omega} ds
\]
For any \( \text{Lemma 6.} \)

\[
\|\eta\|_{0,T}^2 + \|\eta\|_{2-\sigma,\Omega}^2.
\] (112)

It follows trivially from Lemma 2

\[
|J_2| \leq C \int_0^T \|\eta\|_2 \|\eta\|_{\infty,\sigma} \|\tilde{u}\|_2 + \|u\|_2 \, dt
\]

\[
\leq C(R) T \max_{[0,T]} \left( \|u\|_{\sigma,2}^2 + \|\eta\|_{2-\sigma,2}^2 \right) \quad (113)
\]

and

\[
|J_4| \leq C \int_0^T \|\eta\|_2^2 \left( \|\tilde{\eta}\|_{\infty,\sigma}^2 + \|\tilde{\eta}\|_{\infty,\Omega}^2 + 1 \right) dt \leq C(R) T \max_{[0,T]} \|\eta\|_{2-\sigma,2}^2. \quad (114)
\]

Collecting (109)–(114) we come to the assertion of the lemma. \(\square\)

Now we estimate the second term in the right-hand side of (103).

**Proposition 3.** Given arbitrarily small \(\varepsilon > 0\) there exists a positive constant \(C(R,T,\varepsilon)\) such that for any \(0 < \sigma < 1/8\) and \(T > 0\)

\[
\left| \int_0^T \int_0^T \Phi(\tilde{\eta}, \tilde{\xi}) \, d\tau \right| \leq \varepsilon \left( \int_0^T \int_0^T \|\nabla V\|_{0,T}^2 \, d\tau + \Phi(V_0, \xi_0, \xi_1) + T \Phi(V, \xi, \xi_l) \right)
\]

\[
+ \int_0^T \|\nabla V\|_{0,T}^2 \, dt \right) + C(R, T, \varepsilon) \max_{[0,T]} \left( \|\eta\|_{2-\sigma,\Omega}^2 + \|\eta\|_{1-\sigma,\Omega}^2 \right.
\]

\[
+ \|\eta_l(t)\|_{2-\sigma,\Omega}^2 + \|u_l(t)\|_{2-\sigma,\Omega}^2 + \|V(T)\|_{2-\sigma,\Omega}^2. \right) \quad (115)
\]

In order to prove Proposition 3 we shall make use of the following auxiliary results.

**Lemma 6.** For any \(T > 0\) there exists \(C > 0\) such that the estimate

\[
\left| \int_0^T \int_0^T (u_x \eta_r - \eta_x u_r) \, dx \, d\tau \right| \leq C T \max_{[0,T]} (\|u\|_2 + \|u_t\|_2) \quad (116)
\]

holds true.

**Proof.** Integrating by parts with respect to \(x\) in the first term and with respect to \(\tau\) in the second term we come to

\[
\left| \int_0^T \int_0^T (u_x \eta_r - \eta_x u_r) \, dx \, d\tau \right| \leq C \int_0^T \int_0^L |\eta_x(t)u(t) - \eta_x(T)u(T)| \, dx \, dt
\]

\[
\leq C T \max_{[0,T]} (\|\eta\|_{0,T}^2 + \|u\|_{2,T}^2). \quad (117)
\]

\(\square\)

**Lemma 7.** For any \(\varepsilon > 0\) there exists a positive constant \(C(R, \varepsilon)\) such that the estimate

\[
\int_0^T \int_0^T \eta_x \eta_r (\tilde{\eta}_x + \tilde{\eta}_x) \, dx \, d\tau \leq \varepsilon \int_0^T \|\nabla V\|_{0,T}^2 \, d\tau + C(R, \varepsilon) T^2 \max_{[0,T]} \|\eta_x\|_{0,T}^2 \quad (118)
\]
holds true for any \( T > 0 \).

**Proof.** Relying on (101), Lemma 2 and the trace theorem it is easy to obtain

\[
\int_0^T \int_0^t \eta_x \eta_t (\tilde{\eta}_x + \tilde{\eta}_x) dx dt \leq \int_0^T \int_0^t ||\eta_x||_{\Omega} ||\eta_t||_{\Omega} (||\tilde{\eta}_x||_{\infty, \Omega} + ||\tilde{\eta}_x||_{\infty, \Omega}) dt
\]

\[
\leq \varepsilon \int_0^T ||\nabla V||_{L^2}^2 dt + C(R) T^2 \max_{[0,T]} ||\eta_x||_{\Omega}^2.
\]

(119)

To proceed we need the following

**Lemma 8.** For any \( \delta \geq 1/4 \) and \( 0 < \sigma < 1 \) such that \( 0 < \sigma < \delta \), the estimate

\[
||gh||_{\sigma, \Omega} \leq C||h||_{\delta, \Omega} ||g||_{1/2 + \sigma, \Omega}
\]

holds true for \( h \in H^\delta(\Omega) \), \( g \in H^{1/2 + \sigma}(\Omega) \).

**Proof.** Regarding the Hölder’s inequality we obtain

\[
||gh||_{\sigma, \Omega}^2 \leq C \int_0^L \int_0^L |g(x)h(x) - g(z)h(z)|^2 dx dz
\]

\[
\leq C \left( \int_0^L \int_0^L \frac{|g(z)|^2 |h(x) - h(z)|^2}{|x - z|^{1 + 2\sigma}} dx dz + \int_0^L \int_0^L \frac{|g(x) - g(z)|^2 |h(x)|^2}{|x - z|^{1 + 2\sigma}} dx dz \right) \leq C \left( \int_0^L \int_0^L \frac{|h(x) - h(z)|^2}{|x - z|^{1 + 2\sigma}} dx dz \right.
\]

\[
\left. + \left( \int_0^L |h(z)|^4 dz \right)^{1/2} \left( \int_0^L \frac{|g(x) - g(z)|^4}{|x - z|^{2 + 4\sigma}} dx dz \right)^{1/2} \right) \leq C \left( ||g||_{L^\infty, \Omega}^2 ||h||_{\sigma, \Omega}^2 + ||h||_{W^{1/4 + \sigma}(\Omega)}^2 \right) (121)
\]

Relying on Lemma 2 and the embeddings

\[
H^\delta(0, L) \subset L_4(0, L), \quad \text{for any } \delta \geq 1/4
\]

\[
H^{1/2 + \sigma}(0, L) \subset W^{1/4 + \sigma}_4(0, L).
\]

we obtain the statement of the lemma.

\[
\square
\]

**Lemma 9.** For any \( \varepsilon > 0 \) there exists a positive constant \( C(R, \varepsilon) \) such that the estimate

\[
\left| \int_0^T \int_0^t \eta_x u_{xx} (\tilde{\eta}_x + \tilde{\eta}_x) dx dt \right| \leq \varepsilon \int_0^T \int_0^t ||\nabla V||_{L^2}^2 + C(R, \varepsilon) T^2 \max_{[0,T]} ||\eta_x||_{L^2_{-\sigma, \Omega}}^2
\]

holds true for any \( T > 0 \) and \( 0 < \sigma < 1/2 \).
Proof. Integrating by parts with respect to $x$ and taking into account boundary conditions, it is easy to see that

$$
\left| \int_0^T \int_0^L \eta_x u_{xx}(\tilde{\eta}_x + \bar{\eta}_x) dx d\tau dt \right| \leq \left| \int_0^T \int_0^L \eta_x u_{x\tau}(\tilde{\eta}_x + \bar{\eta}_x) dx d\tau dt \right|
$$

$$
+ \left| \int_0^T \int_0^L \eta_x u_{\tau}(\tilde{\eta}_{xx} + \bar{\eta}_{xx}) dx d\tau dt \right| \ (125)
$$

Obviously, by the trace theorem and (101) we have that

$$
\left| \int_0^T \int_0^L \eta_x u_{\tau}(\tilde{\eta}_x + \bar{\eta}_x) dx d\tau dt \right| \leq C \int_0^T \int_0^T \left| \eta_x \right| \| u_{\tau} \|_1 \| \tilde{\eta}_{xx} \|_\Omega + \| \bar{\eta}_{xx} \|_\Omega) d\tau dt
$$

$$
\leq \varepsilon \int_0^T \int_0^T \left| \nabla V \right|^2 + C(R) T^2 \max_{[0, T]} \| \eta \|_{2, -\sigma, \Omega}^2 \ (126)
$$

Applying Lemma 8 with $h = u_t$ and $g = \tilde{\eta}_x + \bar{\eta}_x$, it is easy to infer from from (101) that for any $0 < \sigma < 1/2$ and $1/4 \leq \delta < 1/2$

$$
\left| \int_0^T \int_0^L \eta_{xx} u_{\tau}(\tilde{\eta}_x + \bar{\eta}_x) dx d\tau dt \right| \leq C \int_0^T \int_0^T \left| \eta \right| \| \eta \|_{-\sigma, \Omega} \| u_{\tau} \|_\Omega (1 + \| \tilde{\eta}_{xx} + \bar{\eta}_{xx} \|_\Omega) d\tau dt
$$

$$
\leq \varepsilon \int_0^T \int_0^T \left| \nabla V \right|^2 + C(R) T^2 \max_{[0, T]} \| \eta \|_{2, -\sigma, \Omega}^2 \ (127)
$$

The lemma is proved.

**Lemma 10.** Let $d \in H^{1/2 + \sigma}(\Omega)$, $g \in H^{1-\sigma}(\Omega)$, and $h \in H^\delta(\Omega)$. Then for any $0 < \sigma < 1/2$ and $\delta \geq 1/4$

$$
\|dh\|_{\sigma, \Omega} \leq C \|g\|_{1-\sigma, \Omega}^2 \|h\|_{\delta, \Omega} \|d\|_{1/2 + \sigma, \Omega}^2 \ (127)
$$

**Proof.** Obviously,

$$
\|dh\|_{\sigma, \Omega}^2 \leq C \left( \int_0^L \int_0^L \frac{|g(x)|^2 |d(x)|^2 |h(x) - h(z)|^2}{|x - z|^{1+2\sigma}} dxdz + \int_0^L \int_0^L \frac{|g(x)|^2 |h(z)|^2 |d(x) - d(z)|^2}{|x - z|^{1+2\sigma}} dxdz + \int_0^L \int_0^L \frac{|d(z)|^2 |h(z)|^2 |g(x) - g(z)|^2}{|x - z|^{1+2\sigma}} dxdz \right) \ (128)
$$
To estimate the first term in the right-hand side of (128) we use again Lemma 2:
\[
\int_0^L \int_0^L \frac{|g(x)|^2|h(x)|^2}{|x-z|^{1+2\sigma}} \, dx \, dz \leq C \|g\|_{2,\Omega}^2 \|h\|_{2,\Omega}^2 \leq C \|g\|_{2,-\sigma,\Omega}^2 \|h\|_{2,-\sigma,\Omega}^2 \tag{129}
\]
Using embeddings (122) and (123) we come to
\[
\int_0^L \int_0^L \frac{|g(x)|^2|h(z)|^2|d(x) - d(z)|^2}{|x-z|^{1+2\sigma}} \, dx \, dz \leq C \|g\|_{2,\Omega}^2 \left( \int_0^L |h|^4 \, dx \right)^{1/2}
\]
\[
\left( \int_0^L \int_0^L \frac{|d(x) - d(z)|^4}{|x-z|^{1+4\sigma}} \, dx \, dz \right)^{1/2} \leq C \|g\|_{2,-\sigma,\Omega}^2 \|h\|_{2,\Omega}^2 \|d\|_{W^{1/2+\sigma}_4(\Omega)}^2 \leq C \|g\|_{1,-\sigma,\Omega}^2 \|h\|_{1,\Omega}^2 \|d\|_{1/2+\sigma,\Omega}^2 \tag{130}
\]
where \( \delta \geq 1/4 \) and
\[
\int_0^L \int_0^L \frac{|d(z)|^2|h(x)|^2|g(x) - g(z)|^2}{|x-z|^{1+2\sigma}} \, dx \, dz \leq C \|d(x)\|_{2,-\sigma,\Omega}^2 \|h\|_{2,\Omega}^2 \|g\|_{2,\Omega}^2 \|d\|_{W^{1/2+\sigma}_4(\Omega)}^2 \leq C \|d(x)\|_{1,-\sigma,\Omega}^2 \|h\|_{1,\Omega}^2 \|g\|_{1/2+\sigma,\Omega}^2 \tag{131}
\]
This proves the lemma. \( \square \)

**Lemma 11.** There exists a constant \( C(R) > 0 \) such that for any \( 0 < \sigma < 1/2 \) the estimate
\[
\left| \int_0^T \int_0^L \eta_x \eta_{xx} (\tilde{\eta}_x^2 + \tilde{\eta}_z \tilde{\eta}_x + \tilde{\eta}_z^2) \, dx \, dt \right| \leq C(R) (T^{3/2} + T) \|\eta\|_{2,-\sigma,\Omega}^2 \tag{132}
\]
holds true for every \( T > 0 \).

**Proof.** After integration by parts with respect to \( t \) we obtain
\[
\left| \int_0^T \int_0^T \eta_x \eta_{xx} (\tilde{\eta}_x^2 + \tilde{\eta}_z \tilde{\eta}_x + \tilde{\eta}_z^2) \, dx \, dt \right| \leq C \left( \left| \int_0^T \int_0^L \tilde{\eta}_x^2 ((2\tilde{\eta}_x + \tilde{\eta}_z) \tilde{\eta}_{xx} + (\tilde{\eta}_z + 2\tilde{\eta}_x) \tilde{\eta}_{xx}) \, dx \, dt \right| 
\right. \\
+ T \max_{[0,T]} \left| \int_0^L \tilde{\eta}_x^2 (\tilde{\eta}_x^2 + \tilde{\eta}_z \tilde{\eta}_x + \tilde{\eta}_z^2) \, dx \right| \right) . \tag{133}
\]
Integrating with respect to \( x \) in the first term in the right hand side of (133) we have for any \( 0 < \sigma < 1/2 \)
\[
\left| \int_0^T \int_0^T \eta_x \eta_{xx} (\tilde{\eta}_x^2 + \tilde{\eta}_z \tilde{\eta}_x + \tilde{\eta}_z^2) \, dx \, dt \right| \leq C \left( T \max_{[0,T]} \|\tilde{\eta}_x^2 + \tilde{\eta}_z \tilde{\eta}_x + \tilde{\eta}_z^2\|_{\infty,\Omega} \|\eta\|_{2,-\sigma,\Omega}^2 \right)
\]
Lemma 12. There exists a constant $C(R) > 0$ such that for any $0 < \sigma < 1/2$ and $T > 0$

$$\left| \int_0^T \int_0^T \int_0^L \eta_x \eta_{xx} \tilde{u}_x dxd\tau dt \right| \leq C(R)(T^{3/2} + T)\|\eta\|^2_{2-\sigma,\Omega}. \quad (137)$$

On account of Lemma 10 applied for $d = \tilde{\eta}_x + \tilde{\eta}_x$, $g = \eta_x$, and $h = \tilde{\eta}_x$ or $h = \tilde{\eta}_x$, we obtain the following estimate for $0 < \sigma < 1/2$ and $1/4 \leq \delta < 1/2$

$$\left| \int_0^T \int_0^T \int_0^L \eta_x \eta_{xx} ((2\tilde{\eta}_x + \tilde{\eta}_x) \tilde{\eta}_r + (\tilde{\eta}_x + 2\tilde{\eta}_x) \tilde{\eta}_r) dxd\tau dt \right| +$$

$$+ \left| \int_0^T \int_0^T \int_0^L \eta_x \eta_{xx} ((2\tilde{\eta}_x + \tilde{\eta}_x) \tilde{\eta}_r + (\tilde{\eta}_x + 2\tilde{\eta}_x) \tilde{\eta}_r) dxd\tau dt \right|. \quad (134)$$

$$\int_0^T \int_0^T \int_0^T \int_0^T \eta_x \eta_{xx} ((2\tilde{\eta}_x + \tilde{\eta}_x) \tilde{\eta}_r + (\tilde{\eta}_x + 2\tilde{\eta}_x) \tilde{\eta}_r) dxd\tau dt \leq C \int_0^T \int_0^T \|\eta\|_{2-\sigma,\Omega} d\tau dt \leq C \int_0^T \int_0^T \|\eta\|_{2-\sigma,\Omega} d\tau dt. \quad (135)$$

Consequently,

$$\left| \int_0^T \int_0^T \int_0^L \eta_x \eta_{xx} ((2\tilde{\eta}_x + \tilde{\eta}_x) \tilde{\eta}_r + (\tilde{\eta}_x + 2\tilde{\eta}_x) \tilde{\eta}_r) dxd\tau dt \right| \leq C(R) \max_{[0,T]} \|\eta\|^2_{2-\sigma,\Omega} T^{3/2} \left( \int_0^T (\|	ilde{\eta}_t\|_{\sigma,\Omega}^2 + \|\tilde{\eta}_r\|_{\delta,\Omega}^2) d\tau dt \right)^{1/2} \leq C(R) T^{3/2} \max_{[0,T]} \|\eta\|^2_{2-\sigma,\Omega}. \quad (135)$$

$$\int_0^T \int_0^T \int_0^L \eta_x^2 ((2\tilde{\eta}_x + \tilde{\eta}_x) \tilde{\eta}_r + (\tilde{\eta}_x + 2\tilde{\eta}_x) \tilde{\eta}_r) dxd\tau dt \leq CT^2 \max_{[0,T]} \left( \|\eta\|^2_{2-\sigma,\Omega}(\|	ilde{\eta}_x\|_{\Omega} + \|	ilde{\eta}_{xx}\|_{\Omega})(\|	ilde{\eta}_r\|_{\Omega} + \|	ilde{\eta}_r\|_{\Omega}) \right) \text{ or } \leq C(R) T^2 \max_{[0,T]} \|\eta\|^2_{2-\sigma,\Omega}. \quad (136)$$

Collecting (134)–(136) and taking into account (101) we infer the statement of the lemma. \qed

Lemma 12. There exists a constant $C(R) > 0$ such that for any $0 < \sigma < 1/2$ and $T > 0$

$$\left| \int_0^T \int_0^T \int_0^L \eta_x \eta_{xx} \tilde{u}_x dxd\tau dt \right| \leq C(R)(T^{3/2} + T)\|\eta\|^2_{2-\sigma,\Omega}. \quad (137)$$
Proof. Integrating by parts with respect to $t$ and relying on Lemma 2 and (101) we have

$$
\left| \int_0^T \int_0^T \eta_x \eta_{xz} \tilde{u}_x dx \, dt \right| \leq C \left( \int_0^T \int_0^T \eta_x^2 \tilde{u}_{xz} dx \, dt + T \max_{[0,T]} \int_0^L \eta_{xz}^2 dx \right) 
$$

$$
\leq C \left( \int_0^T \int_0^T \eta_x \eta_{xx} \tilde{u}_x dx \, dt \right) + T \max_{[0,T]} \| \eta \|_{\infty, \Omega}^2 \| \tilde{u}_x \|_{\Omega}
$$

$$
\leq C(R) \left( \int_0^T \int_0^T \eta_x \eta_{xx} \tilde{u}_x dx \, dt \right) + T \max_{[0,T]} \| \eta \|_{2-\sigma, \Omega}^2 . \quad (138)
$$

Applying Lemma 8 to $g = \eta_x$ and $h = \tilde{u}_x$ we get for any $0 < \sigma < 1/2$ and $1/4 \leq \delta < 1/2$ that

$$
\int_0^T \int_0^T \eta_x \eta_{xx} \tilde{u}_x dx \, dt \leq \int_0^T \| \eta \|_{-\sigma, \Omega} \| \eta_x \tilde{u}_x \|_{\sigma, \Omega} dx \, dt
$$

$$
\leq CT \max_{[0,T]} \| \eta \|_{2-\sigma, \Omega}^2 \int_0^T \| \tilde{u}_x \|_{\delta, \Omega} dx dt \leq C(R) T^{3/2} \max_{[0,T]} \| \eta \|_{2-\sigma, \Omega}^2 . \quad (139)
$$

Together with (138) this gives the statement of the lemma. \qed

**Lemma 13.** Given $\varepsilon > 0$ there exists a positive constant $C(R, T, \varepsilon)$ such that for any $0 < \sigma < 1/8$

$$
\left| \int_0^T \int_0^T \tilde{\eta}_x \eta_{xz} u_x dx \, dt \right| \leq \varepsilon \left( \int_0^T \int_0^T \| \nabla V \|_{0}^2 dt \right) + T \Phi(V, \xi, \xi)
$$

$$
+ \Phi(V_0, \xi_0, \xi_1) + \int_0^T \| \nabla V \|_{0}^2 dt \right) + C(R, T, \varepsilon) \max_{[0,T]} (\| \eta(t) \|_{2-\sigma, \Omega} + \| u(t) \|_{1-\sigma, \Omega})
$$

$$
+ \| V(t) \|_{2-\sigma, \Omega} + \| u(t) \|_{2-\sigma, \Omega} . \quad (140)
$$

Proof. Integrating by parts with respect to $\tau$ we obtain

$$
\left| \int_0^T \int_0^T \tilde{\eta}_x \eta_{xz} u_x dx \, dt \right| \leq \left| \int_0^T \int_0^T \tilde{\eta}_{xz} \eta_x u_x dx \, dt \right|
$$

$$
+ \left| \int_0^T \int_0^T \tilde{\eta}_x \eta_{xx} u_x dx \, dt \right| + \left| \int_0^T \int_0^L \tilde{\eta}_{xx} \eta_x u_x dx \, dt \right|
$$

$$
+ T \left| \int_0^L \tilde{\eta}_x \eta_x u_x dx \right| + \left| \int_0^T \int_0^L \tilde{\eta}_{xz} \eta_x u_x dx \, dt \right| . \quad (141)
$$
By Lemma 8 applied to $g = \tilde{\eta}_x$ and $h = u_\tau$ we have for any $0 < \sigma < 1/2$ and $1/4 \leq \delta < 1/2$

$$
\left| \int_0^T \int_0^L \tilde{\eta}_x \eta_{xx} u_\tau \, dx \, dt \right| \leq \int_0^T \int_0^T \| \eta \|_{2-\sigma, \Omega} \| \tilde{\eta}_x u_\tau \|_{\sigma, \Omega} \, dx \, dt \\
\leq C \int_0^T \int_0^T \| \eta \|_{2-\sigma, \Omega} \| \tilde{\eta}_x u_\tau \|_{\sigma, \Omega} \, dx \, dt \\
\leq C(R) T^2 \max_{[0, T]} \| \eta \|^2_{2-\sigma, \Omega} + \varepsilon \int_0^T \int_0^T \| \nabla \tilde{V} \|^2_0 \, dt \, dt. \quad (142)
$$

Relying on Lemma 2 and (101) it is easy to have for any $0 < \sigma < 1/2$

$$
\left| \int_0^T \int_0^L \tilde{\eta}_{xx} \eta_{xx} u_\tau \, dx \, dt \right| \leq \int_0^T \int_0^T \| \tilde{\eta}_{xx} \|_{\infty, \Omega} \| \tilde{\eta}_{xx} \|_{\Omega} \| u_\tau \|_{\Omega} \, dx \, dt \\
\leq C(R) T^2 \max_{[0, T]} \| \eta \|^2_{2-\sigma, \Omega} + \varepsilon \int_0^T \int_0^T \| \nabla \tilde{V} \|^2_0 \, dt \, dt. \quad (143)
$$

Using Lemma 3 with $g = \eta_x$ and $h = u_x$ and the trace theorem we get for any $0 < \sigma < 1/8$

$$
\left| \int_0^T \int_0^L \tilde{\eta}_{xx} \eta_{xx} u_x \, dx \, dt \right| \leq C \int_0^T \int_0^T \| \tilde{\eta}_{xx} \|_{1/2, \Omega} \| \eta \|_{2-\sigma, \Omega} \| u \|_{3/2-\sigma, \Omega} \, dx \, dt \\
\leq C \int_0^T \int_0^T \| \nabla \tilde{V} \|^2_0 \| \eta \|_{2-\sigma, \Omega} \| u \|^3_{3/2-\sigma, \Omega} \, dt \, dt. \quad (144)
$$

Then, the Hölder’s inequality and (101) yield

$$
\left| \int_0^T \int_0^L \tilde{\eta}_{xx} \eta_{xx} u_x \, dx \, dt \right| \leq C \int_0^T \left( \int_0^T \| \nabla \tilde{V} \|^2_0 \, dt \right)^{1/2} \left( \int_0^T \| \eta \|^2_{2-\sigma, \Omega} \| u \|^3_{3/2-\sigma, \Omega} \, dt \right)^{1/2} \, dt \\
\leq C(R) \max_{[0, T]} \| \eta \|^2_{2-\sigma, \Omega} \int_0^T \left( \int_0^T \| u \|^3_{3/2-\sigma, \Omega} \, dt \right)^{1/2} \, dt \\
\leq C(R) T^{1/2} \max_{[0, T]} \| \eta \|^2_{2-\sigma, \Omega} \left( \int_0^T \int_0^T \| u \|^3_{3/2-\sigma, \Omega} \, dx \, dt \right)^{1/2}. \quad (145)
$$
It is easy to see that
\[
\int_0^T \int_I \|u\|^2_{3/2-\sigma, \Omega} d\tau dt + \int_0^T \int_I \|\eta\|^2_{3/2-\sigma, \Omega} d\tau dt \leq C \left( \int_0^T \int_I \|\nabla V\|^2_0 d\tau dt + T\Phi(V, \xi, \xi_t) + \int \Phi(V, \xi, \xi_t) dt \right) + \int_0^T \int_I |d(\xi, A^{1/2-\sigma} \xi) - d(\tilde{\xi}, A^{1/2-\sigma} \xi)| d\sigma dt,
\]
(145)

where by \(A^\alpha \xi\) we understand \(A^\alpha_1 \xi_1, A^\alpha_2 \xi_2\) defined as in (61), (62). Here we remind that \(\xi = \xi_n\) are Galerkin approximations.

Due to the structure of the last term in (145) given by (44) we have
\[
\int_0^T \left| d(\xi, A^{1/2-\sigma} \xi) - d(\tilde{\xi}, A^{1/2-\sigma} \xi) \right| d\tau \leq C \int_0^T \int_I \left| (u_x + \eta_x (\tilde{\eta}_x + \eta_x))A^{1/4-\sigma/2}_1 \right| \left| u_x + \eta_x (\tilde{\eta}_x + \eta_x) \right| A^{1/4-\sigma/2}_1 \eta_x \left( \eta_x^2 + \tilde{\eta}_x \eta_x + \eta_x^2 \right) (A^{1/4-\sigma/2}_1 \eta_x) x + \eta_x u_x (A^{1/4-\sigma/2}_1 \eta_x) x + \tilde{\eta}_x \eta_x (A^{1/4-\sigma/2}_1 \eta_x) x + \eta_x (\tilde{\eta}_x + \eta_x) (A^{1/4-\sigma/2}_1 \eta_x) x \right| d\tau dt.
\]
Consequently, for any \(\varepsilon > 0\) there exists \(C(\varepsilon) > 0\) such that
\[
\int_0^T \left| d(\xi, A^{1/2-\sigma} \xi) - d(\tilde{\xi}, A^{1/2-\sigma} \xi) \right| d\tau \leq \varepsilon \int_0^T \left( \|\eta\|^2_{3/2-\sigma, \Omega} + \|u\|^2_{3/2-\sigma, \Omega} + \|\tilde{\eta}_x u_x\|^2_{\sigma, \Omega} \right) d\tau + C(\varepsilon) \int_0^T \left( \|\eta_x (\tilde{\eta}^2_x + \tilde{\eta} x \tilde{\eta}_x + \eta_x^2)\|^2_{\sigma, \Omega} + \|u\|^2_{1-\sigma, \Omega} + \|\eta\|^2_{1/2-\sigma, \Omega} \right) d\tau + \|\eta_x \tilde{u}_x\|^2_{1/2, \Omega} + \|\tilde{\eta}_x \eta + \eta_x \tilde{\eta}^2 + \|\eta_x (\tilde{\eta}_x + \eta_x)\|^2_{1/2, \Omega}) d\tau.
\]
Taking into account Lemmas 3, 8 and bound (101) we obtain
\[
\int_0^T \int_I \left| d(\xi, A^{1/2-\sigma} \xi) - d(\tilde{\xi}, A^{1/2-\sigma} \xi) \right| d\tau dt \leq C(R, T, \varepsilon) \max_{[0, T]} \left( \|u\|^2_{1-\sigma, \Omega} + \|\eta\|^2_{2-\sigma, \Omega} \right)
\]
+ \varepsilon \int_0^T \int_I \left( \|\eta\|^2_{3/2-\sigma, \Omega} + \|u\|^2_{3/2-\sigma, \Omega} \right) d\tau dt.
(146)

Then it follows from (145), (146) that
\[
\int_0^T \int_I \|u\|^2_{3/2-\sigma, \Omega} d\tau dt \leq C \left( \int_0^T \int_I \|\nabla V\|^2_0 d\tau dt + T\Phi(V, \xi, \xi_t) + \int \Phi(V_0, \xi_0, \xi_1) + \int \|\nabla V\|^2_0 dt \right) + C(R, T, \varepsilon) \max_{[0, T]} \left( \|V\|^2_{1-\sigma, \Omega} + \|\eta\|^2_{2-\sigma, \Omega} \right)
\]

Proof. Energy inequality (28) implies that

\[ + \| u \|_{1-\sigma, \Omega}^2 + \| u_t \|_{-\sigma, \Omega}^2 + \| \eta \|_{-\sigma, \Omega}^2, \] (147)

which inserted in (144) gives

\[ \left| \int_0^T \int_0^L \eta_{tx} u_x \, dx \, dr \right| \leq \varepsilon \left( \int_0^T \int_0^T \| \nabla V \|_0^2 \, dr \, dt + T \Phi(V, \xi, \xi_T) \right. \]

\[ + \Phi(V_0, \xi_0, \xi_1) + \int_0^T \| \nabla V \|_0^2 \, dt \left. \right) + C(R, T, \varepsilon) \max_{[0, T]} \| \eta \|_{2-\sigma, \Omega}^2 \]

\[ + \| u \|_{1-\sigma, \Omega}^2 + \| u_t \|_{-\sigma, \Omega}^2 + \| \eta \|_{-\sigma, \Omega}^2. \] (148)

The assertion of the lemma is a straightforward consequence of (142), (143), and (148).

\[ \square \]

Lemma 14. For any \( \varepsilon > 0 \) there exists \( C(R) > 0 \) such that for \( 0 < \sigma < 1/8 \)

\[ \left| \int_0^T \int_0^T (\eta_x \eta_t \hat{\eta} + \eta_x \eta_{tx}) \, dx \, dr \right| \leq \varepsilon \left( \int_0^T \int_0^T \| \nabla V \|_0^2 \, dr \, dt + C(R)T^2 \max_{[0, T]} \| \eta \|_{2-\sigma, \Omega}^2 \right) \] (149)

for any \( T > 0 \).

Proof. Using Lemma 3 and the trace theorem one can readily check that

\[ \left| \int_0^T \int_0^T (\eta_x \eta_t \hat{\eta} + \eta_x \eta_{tx}) \, dx \, dr \right| \leq \int_0^T \int_0^T \| \eta \|_1^2 \| \nabla V \|_0 \| \eta_{tx} \| \, dr \, dt \]

\[ \leq \varepsilon \left( \int_0^T \int_0^T \| \nabla V \|_0^2 \, dr \, dt + C(R)T^2 \max_{[0, T]} \| \eta \|_{2-\sigma, \Omega}^2 \right. \] (150)

This proves the lemma.

\[ \square \]

The statement of Proposition 3 follows readily from Lemmas 6, 7, 9, 11–13.

The key ingredient for the proof of the quasi-stability is the following statement.

Lemma 15. There exist \( T > 1 \) and \( C(R, T) > 0 \) such that

\[ \Phi(V(T), \xi(T), \xi_T(T)) \leq \gamma \Phi(V_0, \xi_0, \xi_1) \]

\[ + C(R, T) \max_{[0, T]} \| V \|_{2-\sigma, \Omega}^2 + \| \eta \|_{2-\sigma, \Omega}^2 + \| u \|_{-\sigma, \Omega}^2 + \| u_t \|_{-\sigma, \Omega}^2 + \| \eta \|_{-\sigma, \Omega}^2, \] (151)

where \( \gamma = \gamma(R, T) \) satisfies \( 0 < \gamma < 1 \) and \( 0 < \sigma < 1/8 \).

Proof. Energy inequality (28) implies that

\[ \int_0^T \| \nabla V \|_0^2 \, dt \]

\[ \leq C \left( \Phi(V_0, \xi_0, \xi_1) - \Phi(V(T), \xi(T), \xi_T(T)) + \int_0^T \Psi(\bar{\xi}(t), \bar{\xi}(t)) \, dt \right), \] (152)

where \( \Psi(\bar{\xi}(t), \bar{\xi}(t)) \) is given by (105).
We estimate separately the term

\[
\left| \int_0^T \int_0^L \hat{\eta}_x \eta_x u_x \, dx \, dt \right| \leq \left| \int_0^T \int_0^L \hat{\eta}_x \eta_x u_x \, dx \, dt \right| + \left| \int_0^T \int_0^L \hat{\eta}_{xx} \eta_x u_t \, dx \, dt \right| + \left| \int_0^T \int_0^L \hat{\eta}_x (T) \eta_x (T) u_x \, dx \right| + \left| \int_0^T \hat{\eta}_0 \eta_0 u_x \, dx \right|. \tag{153}
\]

Arguing as in Lemma 13 we obtain

\[
\left| \int_0^T \int_0^L \hat{\eta}_x \eta_x u_t \, dx \, dt \right| + \left| \int_0^T \int_0^L \hat{\eta}_{xx} \eta_x u_t \, dx \, dt \right| \leq C(R, T, \varepsilon) \max_{[0, T]} \| \eta \|^2_{2, \sigma, \Omega} + \varepsilon \int_0^T \| \nabla V \|^2_0 \, dt. \tag{154}
\]

Using Lemma 3 we have for \(0 < \sigma < 1/8\)

\[
\left| \int_0^T \int_0^L \hat{\eta}_x \eta_x u_x \, dx \, dt \right| \leq \left( \int_0^T \| \nabla \hat{V} \|^2_0 \| \eta_x \|^2_{1/2, \Omega} \right)^{1/2} \left( \int_0^T \| u \|^2_{3/2, \sigma, \Omega} \right)^{1/2}. \tag{155}
\]

Substituting

\[
\int_0^T \| u \|^2_{3/2, \sigma, \Omega} \leq C \left( \int_0^T \| \nabla V \|^2_0 \, dt + \Phi(V, \xi, \xi_T) + \Phi(V_0, \xi_0, \xi_1) \right) + C(R, T, \varepsilon) \max_{[0, T]} (\| \eta \|^2_{2, \sigma, \Omega} + \| u \|^2_{1, \sigma, \Omega}),
\]

into (155), taking into account (153)-(154), and arguing as in Lemmas 6-12 we get the estimate

\[
\left| \int_0^T \Psi(\xi(t), \xi_T(t)) \, dt \right| \leq C(R, T, \varepsilon) \max_{[0, T]} (\| \eta \|^2_{2, \sigma, \Omega} + \| u \|^2_{1, \sigma, \Omega}) + \varepsilon \int_0^T \| \nabla V \|^2_0 \, dt \]

\[
+ C(\varepsilon) \int_0^T \| \nabla \hat{V} \|^2_0 \| \eta_{xx} \|^2_{\Omega} \, dt + C(\Phi(V, \xi, \xi_T) + \Phi(V_0, \xi_0, \xi_1)), \tag{156}
\]

which together with (152) implies

\[
\int_0^T \| \nabla V \|^2_0 \, dt \leq C(R, T) \max_{[0, T]} (\| \eta \|^2_{2, \sigma, \Omega} + \| u \|^2_{1, \sigma, \Omega})
\]
Theorem 3. Let \( W^i(t) = (V^i(t),\xi^i(t)) \), \( i = 1, 2 \), be two weak solutions to (1)-(11) with initial data \( W^0 = (V^0,\xi^0) \) from \( \bar{H} \) such that \( \|W^i_0\|_{\bar{H}} \leq R \), \( i = 1, 2 \), then their difference \( \|Z(t)\|_{\bar{H}}^2 \leq M_Re^{-\gamma t}\|Z_0\|_{\bar{H}}^2 + \text{lot}_t(V(t);\xi(t);\xi_t(t)) \) for some positive constant \( M_R \) and \( \gamma \), and
\[
\text{lot}_t(V(t);\xi(t);\xi_t(t)) = C(R,T)\max_{[0,T]}(\|\hat{V}\|_{-\sigma,\Omega}^2 + \|\hat{\eta}\|_{-\sigma,\Omega}^2 + \|u\|_{-\sigma,\Omega}^2 + \|\zeta\|_{-\sigma,\Omega}^2),
\]
for \( 0 < \sigma < 1/8 \) i.e. system (1)-(11) is quasi-stable.

Remark 3. Remark 1 immediately gives that \( \text{lot}_t(V(t);\xi(t);\xi_t(t)) \) is a compact seminorm on the space \( \bar{H} \). For this reason one can pass to the limit in (159) for Galerkin approximations and obtain the same estimate for the weak solutions.

Our main result is the following theorem:

Theorem 4. Assume that external forces satisfy (98). Then the system \((S_t,\bar{H})\) possesses a compact global attractor \( 2 \) of finite fractal dimension. The global attractor \( 2 \) consists of full trajectories \( \{(V(t);\xi(t);\xi_t(t)) : t \in \mathbb{R}\} \) which are homoclinic to the set \( N \) given by (100), i.e.
\[
\lim_{t \to \pm \infty} \inf_{\xi^* \in \mathcal{N}_0} \left( \|V(t)\|_{\bar{H}}^2 + \|\xi - \xi^*\|_{\bar{H}}^2 + \|\xi_t\|_{\bar{H}}^2 \right) = 0,
\]
where \( \mathcal{N}_0 \) consists of all \( \xi^* = (u^*,\eta^*) \in \bar{U} \) solving (99). In addition we have
\[
\lim_{t \to +\infty} \text{dist}_{\bar{H}}(S_t y,\mathcal{N}) = 0 \text{ for any initial data } y \in \bar{H}.
\]

Proof. It follows from energy inequality in (28) that for each \( R > 0 \) and \( Q \) as in (98) the set \( W_R = \{W : L(W) = E(W) - (g,\eta)_{\Omega} \leq R\} \) is forward invariant with respect to \( S_t \). Here \( W = (V;\xi;\xi_t) \) with \( V = (u;v) \) and \( \xi = (u;\eta) \). Using (98) one can see that \( L(W^n) \to +\infty \) if and only if \( \|W^n\|_{\bar{H}} \to +\infty \) (see, e.g. [11] for details).

Therefore the set \( W_R \) is bounded and any bounded set belongs to \( W_R \) for some \( R \). Moreover, it follows from energy inequality (28) that the continuous functional \( L(W) \) on \( \bar{H} \) possesses the properties (i) \( L(S_t W) \leq L(W) \) for all \( t \geq 0 \) and \( W \in \bar{H} \); (ii) the equality \( L(W) = L(S_t W) \) holds for all \( t > 0 \) only if \( W \) is a stationary point of \( S_t \). This means that \( L(W) \) is a strict Lyapunov function and \((S_t,\bar{H})\) is a gradient dynamical system. Together with the quasi-stability and Proposition 1 by Theorem 1 this gives the statement of the theorem.
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