Large eddy simulation of two-dimensional isotropic turbulence

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ABSTRACT:

Large eddy simulation (LES) of forced, homogeneous, isotropic, two-dimensional (2D) turbulence in the energy transfer subrange is the subject of this paper. A difficulty specific to this LES and its subgrid scale (SGS) representation is in that the energy source resides in high wave number modes excluded in simulations. Therefore, the SGS scheme in this case should assume the function of the energy source. In addition, the controversial requirements to ensure direct enstrophy transfer and inverse energy transfer make the conventional scheme of positive and dissipative eddy viscosity inapplicable to 2D turbulence. It is shown that these requirements can be reconciled by utilizing a two-parametric viscosity introduced by Kraichnan (1976) that accounts for the energy and enstrophy exchange between the resolved and subgrid scale modes in a way consistent with the dynamics of 2D turbulence; it is negative on large scales, positive on small scales and complies with the basic conservation laws for energy and enstrophy. Different implementations of the two-parametric viscosity for LES of 2D turbulence were considered. It was found that if kept constant, this viscosity results in unstable numerical scheme. Therefore, another scheme was advanced in which the two-parametric viscosity depends on the flow field. In addition, to extend simulations beyond the limits imposed by the

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finiteness of computational domain, a large scale drag was introduced. The resulting LES exhibited remarkable and fast convergence to the solution obtained in the preceding direct numerical simulations (DNS) by Chekhlov et al. (1994) while the flow parameters were in good agreement with their DNS counterparts. Also, good agreement with the Kolmogorov theory was found. This LES could be continued virtually indefinitely. Then, a simplified SGS representation was designed, referred to as the stabilized negative viscosity (SNV) representation, which was based on two algebraic terms only, negative Laplacian and positive biharmonic ones. It was found that the SNV scheme performed in a fashion very similar to the full equation and it was argued that this scheme and its derivatives should be applied for SGS representation in LES of quasi-2D flows.

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1. INTRODUCTION

Homogeneous and isotropic turbulence has been a traditional idealization of real turbulent flows which are usually neither homogeneous nor isotropic. However, this idealization provided wealth of information on the physics of turbulence and it still remains one of the main tools of theoretical and numerical turbulence research (Monin and Yaglom, 1975). The same can be said about the dimensionality of the problem; indeed, many natural flows that span large number of scales possess features of both three- and two-dimensional turbulence and can be classified somewhere between the purely 3D and 2D extremes. Thus, even though the focus of this paper is isotropic 2D turbulence one should keep in mind that applications of the results to quasi-2D flows are thought. Quasi-2D turbulent flows are widely found in geophysics and engineering. Although under normal circumstances all flows are unstable to three-dimensional instabilities (Batchelor, 1969), there exist natural situations when a flow may attain a quasi-2D configuration or even become quasi-two-dimensionalized on certain scales. There are two major factors that may cause a flow to become quasi-2D: geometry of the flow boundaries and/or certain body forces (or ‘extra strains’) whose action leads to smoothing of the velocity fluctuations in a preferred direction. While in the geophysical context both factors are equally important (i.e., the small aspect ratio, density stratification, rotation), in the engineering context the second factor usually pre-dominate (for instance, the so called mechanism of ‘magnetic friction’ in magnetohydrodynamic flows with low magnetic Reynolds number; Sommeria and Moreau, 1982).

Although mathematical modeling of quasi-2D flows has important practical applications, particularly in the atmospheric and oceanic sciences, it has not received as much attention in the literature as the modeling of the 3D flows. Partly, it can be explained by the fact that the quasi-2D problems are less computationally intense than their 3D counterparts. Thus there exists a hope that in the near future, practically important quasi-2D problems can be
solved using DNS in which all scales are resolved (Lesieur, 1990).

In addition, despite the specific peculiarities of quasi-2D flows related to the energy and vorticity dynamics, their subgrid scale representation has not received sufficient attention so far. There have been attempts to parameterize the SGS processes in quasi-2D flows similarly to those in 3D flows using Laplacian or biharmonic dissipation, the most advanced method being the anticipated potential vorticity method (Sadourny and Basdevant, 1985). However, such methods can only perform well in the vorticity dissipation subrange when energy is injected on relatively large scales. Being applied in the energy transfer subrange, they will lead to energy dissipation, contradicting basic energy and vorticity transfer dynamics of quasi-2D turbulence. Moreover, LES of quasi-2D flows in the energy transfer subrange has never been attempted, despite the fact that such flows would bear strong analogy to large scale oceanic and atmospheric circulation and that DNS of such flows cannot be expected in the foreseeable future. Thus, there exists a need to improve our understanding of the SGS processes in the energy transfer subrange of quasi-2D flows and to successfully simulate such flows when their energy sources reside in the SGS region. These both issues are addressed in the present paper. In the next section, the basic difficulties of the SGS representation of the quasi-2D flows in the energy transfer subrange are discussed. Then, the following section elaborates on the notion of the two-parametric viscosity and explains how this viscosity resolves the conflict between inverse transfer of energy and direct transfer of enstrophy. In section 4, advantages and deficiencies of various implementations of the two-parametric viscosity for LES of 2D turbulence in the energy subrange are described. In section 5, simplified SGS representations for LES of 2D turbulence are considered and the notion of the stabilized negative viscosity (SNV) is introduced. Finally, section 6 discusses the results and provides some conclusions.
2. BASIC PROBLEMS OF THE SGS REPRESENTATION OF QUASI-2D FLOWS IN THE ENERGY TRANSFER SUBRANGE

Confined to two dimensions, turbulent flows become non-vortex-stretching and undergo dramatic structural changes (Kraichnan and Montgomery, 1980). The most profound modifications take place in the dynamics of energy and vorticity transfer. It is well known that in isotropic homogeneous 3D turbulence, the direct energy cascade from large to small scales facilitates efficient energy dissipation by molecular viscosity. This process is accompanied by and closely related to the production of enstrophy (mean square vorticity) through vortex stretching mechanism. Since in 2D flows vortex stretching can not occur, the enstrophy then is conserved. Thus, 2D inviscid fluids possess two nontrivial integrals of motion: the energy and the enstrophy. The enstrophy conservation prevents cascade of energy from large to small scales because such cascade would increase enstrophy (Kraichnan, 1967, 1971; Kraichnan and Montgomery, 1980).

Mathematically, this important feature is illustrated by the Fjørtoft theorem (Lesieur, 1990). However, the direct cascade of enstrophy from large to small scales is possible. It results in molecular dissipation of large scale vorticity at small scales. Drawing analogy to eddy viscosity in 3D flows, one can infer that small scale processes in 2D flows generate an effective, or eddy viscosity for the vorticity of large scales. However, introducing an eddy viscosity concept in 2D flows seems to be intrinsically inconsistent and self-defeating because the dissipation of enstrophy will be accompanied by the dissipation of energy, which is physically incorrect. This controversy calls for modification of the eddy viscosity concept for quasi-2D flows; the issue in the focus of the present paper.

More detailed consideration of the transport processes in 2D turbulence reveals that they depend on the wave numbers of the energy injection, \( k_f \). For \( k < k_f \), the energy cascades up scales (inverse cascade), while the enstrophy flux is zero. For \( k > k_f \), the energy flux is zero,
but there exists the direct enstrophy flux (Kraichnan, 1971). If LES of a quasi-2D flow is thought, the proper SGS parameterization should depend on the wave number of the energy source, $k_f$, i.e., whether $k_f$ belongs in the resolved (or explicit) or unresolved (or SGS) region. In the former case, a simple hyperviscous SGS representation may suffice, because it should only account for the enstrophy dissipation due to the direct cascade. However, if the forcing is located in the subgrid scales, then the hyperviscous SGS representation would lead to erroneous results since it will constitute energy dissipation in non-energy-dissipating flows. To sustain such flows, one would need to introduce a large scale energy source in the energy cascade subrange. A possible solution to this problem would be to replace an SGS forcing by a forcing located in the explicit region near $k_c$, where $k_c$ is the cutoff wave number corresponding to the grid resolution. However, this solution is not only quite cumbersome but it also significantly distorts the explicit scales near $k_c$. In addition, this approach is difficult for implementation in the physical space, particularly for bounded systems and/or systems with spatially non-uniform energy sources.

Another solution would be to introduce a negative eddy viscosity, first extensively discussed by Starr (1968), as an SGS parameterization of the unresolved energy source. Recent studies of flows with negative viscosity were conducted by Gama et al. (1991, 1994) in 2D and Yakhot and Sivashinsky (1985) in 3D. Although such an SGS representation could address the issue of the inverse energy cascade, it would not satisfy the constraint of the zero enstrophy flux. In addition, equations of motion with negative viscosity produce ill-posed problems. It appears therefore that addressing the issue of SGS representation for quasi-2D flows in self-consistent and comprehensive way would require full consideration of energy and enstrophy dynamics and should be based upon the corresponding transport equations. Such an approach was first outlined by Kraichnan (1976) who introduced the notion of two-parametric viscosity. This approach and its implications will be elaborated in the next section.
Two-dimensional incompressible turbulent flows are described by the vorticity equation

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (\nabla^{-2} \zeta, \zeta)}{\partial (x, y)} = \nu_0 \nabla^2 \zeta + f,
\]

(1)

where \( \zeta \) is fluid vorticity, \( \nu_0 \) is molecular viscosity and \( f \) is external force.

The introduction of the classical eddy viscosity concept for LES with Eq. (1) implies that there is a distinct scale separation between the resolvable and SGS modes. Indeed, only if such a separation exists, the eddy viscosity would be \( k \)-independent and a function of the cutoff wave number \( k_c \) only. However, the assumption of scale separation fails in all turbulent flows, particularly in 2D flows, such that, strictly speaking, an SGS representation should depend on two parameters, \( k \) and \( k_c \). Such two-parametric viscosity, denoted by \( \nu(k|k_c) \), was first introduced by Kraichnan (1976). It describes the energy exchange between given resolved vorticity mode with the wave number \( k \) and all SGS modes with \( k > k_c \). The two-parametric viscosity is derived from the evolution equation for the spectral enstrophy density \( \Omega(k, t) \equiv (4\pi)^{-1}k \langle \zeta(k, t)\zeta(-k, t) \rangle \), where \( \langle \ldots \rangle \) denotes ensemble averaging:

\[
\left( \partial_t + 2\nu k^2 \right) \Omega(k, t) = \mathcal{T}_\Omega(k, t).
\]

(2)

The enstrophy transfer function in (2), \( \mathcal{T}_\Omega(k, t) \), is given by

\[
\mathcal{T}_\Omega(k, t) = \frac{k}{2\pi} \Re \left\{ \int_{p+q=k} \frac{P \times Q}{p^2} \langle \zeta(p, t)\zeta(q, t)\zeta(-k, t) \rangle \frac{dp dq}{(2\pi)^2} \right\}.
\]

(3)

For a system in statistical steady state the two-parametric transfer \( \mathcal{T}_\Omega(k|k_c) \) and viscosity \( \nu(k|k_c) \) are calculated from (3) by extending integration only over all such triangles \( (k, \ p, \ q) \) that \( |k - p| < q < k + p \) and \( p \) and/or \( q \) are greater than \( k_c \):
\[ \nu(k|k_c) = -\frac{T_\Omega(k|k_c)}{2k^2\Omega(k)}. \] (4)

In a wide class of quasi-normal approximations \( T_\Omega(k|k_c) \) in two dimensions is given by

\[
T_\Omega(k|k_c) = \frac{2k}{\pi} \int \int_{\Delta} \Theta_{-k,p,q}(p^2 - q^2) \sin \alpha \left[ \frac{p^2 - q^2}{p^3 q^3} \Omega(p)\Omega(q) \right. \\
- \left. \frac{k^2 - q^2}{k^3 q^3} \Omega(q)\Omega(k) + \frac{k^2 - p^2}{k^3 p^3} \Omega(p)\Omega(k) \right] dpdq, \] (5)

where \( \Theta_{-k,p,q} \) is the triad relaxation time. Also, the angle \( \alpha \) is formed by the vectors \( p \) and \( q \), and \( \int \int_{\Delta} \) denotes integration over the area defined above.

Different spectral closure models provide different specification of \( \Theta_{-k,p,q} \). Chekhlov et al. (1994) compared \( \nu(k|k_c) \) calculated from their DNS data with 512\(^2 \) resolution with those evaluated by Kraichnan (1976) using his Test Field Model (TFM) and obtained from the Renormalization Group (RG) theory (see Appendix). The results of this comparison are shown in Figs. 1 and 2. Figure 1 presents the DNS-inferred normalized two-parametric viscosity,

\[ N(k/k_c) \equiv \frac{\nu(k|k_c)}{|\nu(0|k_c)|}, \] (6)

along with the TFM- and RG-based analytical predictions. The results are in a very good agreement with each other over the entire energy transfer range, up to the wave numbers close to \( k_c \), where the DNS data saturates, while TFM and RG curves exhibit sharp cusp. This theoretical cusp is due to the fact that as \( k \to k_c \), more and more elongated triads with either \( p \) or \( q \ll k_c \) become involved in the energy exchange between the mode \( k \) and the subgrid scale modes. The contribution from these triads to the energy exchange near \( k_c \) is very significant and results in the cusp behavior. However, in finite box DNS with large-scale energy removal, the energy of small wave number modes is reduced and \( \nu(k|k_c) \) is expected to saturate near \( k_c \). Indeed, when the RG-based \( \nu(k|k_c) \) was re-calculated based upon the
DNS energy spectrum, the unnormalized DNS- and RG-based two-parametric viscosities were found to be in very good agreement for all wave numbers, as shown in Fig. 2.

Figures 1 and 2 show that for the large scale modes, for which \( k \ll k_c \) and scale separation exists, the effect of the SGS modes is represented by a negative and constant viscosity, such that these modes gain energy from their SGS counterparts by means of the inverse transfer. On the other hand, \( \nu(k|k_c) > 0 \) for \( k \to k_c \) such that the modes close to \( k_c \) lose their energy to the SGS modes. The difference between the large scale gain and the small scale loss is equal to \( \epsilon \), the rate of the energy input due to the forcing \( f \) in Eq. (1); see also Eq. (13) below. If enstrophy balance is considered, recall that the enstrophy transfer is most efficient at small scales, such that the resulting balance for the resolvable scales turns out to be zero [Kraichnan, 1971; see also Eq. (18) below], i.e., the enstrophy is conserved. This explains how the two-parametric viscosity resolves the controversy of the inverse cascade of energy and conservation of enstrophy in the energy subrange of 2D flows. It appears therefore that the only physically correct way to represent SGS processes in 2D turbulence would be through the two-parametric viscosity. Such an approach has become quite popular in simulations of 3D flows (Domaradzki et al., 1987; Galperin and Orszag, 1993) but has not yet been fully explored for LES of quasi-2D flows. The implementation of the two-parametric viscosity for LES of 2D turbulent flows, the arising problems, their solutions and results are described in the following sections.
4. IMPLEMENTATION OF THE TWO-PARAMETRIC VISCOSITY FOR LES OF 2D TURBULENCE

To test the two-parametric viscosity-based SGS parameterization in the energy transfer subrange, a series of LES of 2D turbulence in Fourier space was designed. These LES were based upon Eq. (1) in which all subgrid scale processes including the forcing were represented by the two-parametric viscosity $\nu(k|k_c)$,

$$\frac{\partial \zeta(k)}{\partial t} + \int_{|p|,|k-p|<k_c} \frac{p \times k}{p^2} \zeta(p) \zeta(k-p) \frac{dp}{(2\pi)^2} = \nu(k|k_c)k^2 \zeta(k), \quad 0 < k < k_c. \quad (7)$$

It is important to reiterate that in LES of 2D turbulence in the energy subrange, the source of energy resides on the unresolved scales, such that Eq. (7) appears unforced. However, as was explained earlier, the negative part of $\nu(k|k_c)$ serves as the only energy source for the resolved modes. In the course of the present LES it was found that numerical results critically depend on the way $\nu(k|k_c)$ is introduced in the solver. Thus, a series of simulations was designed with the purposes of understanding the nature of the problems associated with the implementation of the two-parametric viscosity and of identifying the most viable and robust ways to use this viscosity in LES of 2D flows.

4.1. Description of numerical method

The numerical solver used in the present calculations was based upon Fourier-Galerkin pseudo-spectral formulation (Orszag, 1969) in the periodic box $[0,2\pi] \times [0,2\pi]$ and was essentially the same as that utilized in the preceding DNS (Chekhlov et al., 1994). The present LES employed $162^2$ resolution with complete dealiasing based on the 2/3-rule; the cutoff wave number was set at $k_c = 50$, which is about half of the resolution used in DNS of Chekhlov et al. (1994). The preprocessing was done using the second order Runge–Kutta scheme, while marching in time was accomplished using an implicit, second order, stiffly stable Adams scheme (Karniadakis et al., 1991). The initial flow field was set to zero.
everywhere except for a narrow band of wave numbers in the middle part of the spectrum where it was assigned random Gaussian distribution. The DNS inferred value of $\tau$ was about $5.19 \times 10^{-10}$; the same $\tau$ was used in LES. The time step in LES was set to $\delta t = 0.5$ satisfying both convective and viscous necessary conditions for linear stability. Based upon the size of the largest energy containing eddy with the wave number $k_{\text{min}}$, $2\pi/k_{\text{min}}$, and the total energy of the steady-state $E(t)$, the maximum large scale eddy turnover time defined as $\tau_{tu} = 2\pi/(k_{\text{min}}\sqrt{2E})$ was about $\tau_{tu} \simeq 3600$ for LES of cases 1 and 2 below, where $k_{\text{min}} = 1$, and $\tau_{tu} \simeq 900$ for cases 3, 4, and 5, for which $k_{\text{min}} = 4$.

4.2. Case 1. Flow independent $\nu(k|k_c)$

With $\nu(k|k_c)$ known and flow independent, Eq. (7) can be solved directly. According to Fig. 1, $\nu(k|k_c)$ can be obtained from DNS or from some statistical theory of turbulence. Thus, in the first LES numerical solver for Eq. (7) utilized $\nu(k|k_c)$ derived from the renormalization group (RG) theory of turbulence (see Appendix for the details). As shown in the Appendix,

$$\nu(k|k_c) = 0.327\tau^{1/3}k_c^{-4/3}N(k/k_c),$$

(8)

where $N(k/k_c)$ is given by (6), and since $\tau$ is a constant, $\nu(k|k_c)$ is a function of $k$ and $k_c$ only. It was assumed that thus defined $\nu(k|k_c)$ would be capable of supporting inverse energy cascade with constant SGS energy input $\tau$. To verify this assumption, one needs to examine the evolution of total energy and enstrophy of the resolved modes, $\overline{E}(t)$ and $\overline{\Omega}(t)$, respectively. By definition, $\overline{E}(t) = \int_0^{k_c} E(k,t)dk$, where $E(k,t) \equiv (4\pi k)^{-1}\langle \zeta(k,t)\zeta(-k,t) \rangle$ is the spectral energy density, and $\overline{\Omega}(t) = \int_0^{k_c} \Omega(k,t)dk$. The basic requirement to LES would be that $\overline{E}(t)$ and $\overline{\Omega}(t)$ of LES have the same behavior as those derived from Eq. (1) for which the evolution laws are $\overline{E}(t) \propto \tau t$ and $\overline{\Omega}(t) = \text{const}$, due to the conservation of the inviscid integrals for $\overline{E}(t)$ and $\overline{\Omega}(t)$ (recall that in the energy subrange of 2D turbulence the rate of the enstrophy flux $\eta = 0$). Figures 3a,b show that in the first LES, both $\overline{E}(t)$ and
\( \Omega(t) \) exhibit nonlinear growth indicating that not only the rate of the energy transfer to the resolvable scales \( \tau_{\text{LES}} \neq \text{const} \), but also the rate of the enstrophy transfer \( \eta_{\text{LES}} \neq 0 \). Figure 4 shows that instantaneous spectrum \( E(k, t) \) also reveals tendency to growing up with time without stabilizing around any universal distribution. At some point of its evolution, \( E(k, t) \) crossed the Kolmogorov law

\[
E(k) = C_K \bar{\epsilon}^{2/3} k^{-5/3},
\]

where \( C_K \approx 6 \) is the Kolmogorov constant, and \( \bar{\epsilon} \) was close to its prescribed DNS value. However, at larger times the Kolmogorov scaling (9) was lost, while \( \tau_{\text{LES}} \) kept growing. The roots of the problem are revealed when one calculates the rate of the energy input into all resolved modes, \( \tau_{\text{LES}} \). The energy equation derived from the definition of spectral energy density and Eq. (7) yields

\[
\tau_{\text{LES}}(t) \equiv \frac{\partial E(t)}{\partial t} = -2 \int_0^{k_c} \nu(k|k_c) E(k, t) k^2 dk.
\]

Since in the first series of LES, \( \nu(k|k_c) \) depends on \( k \) and \( k_c \) only, and \( E(k, t) \) is a dynamic variable that depends on the evolution of the flow field, \( \tau_{\text{LES}}(t) \) also turns out to be time dependent. This is not only in direct conflict with the requirement that \( \tau_{\text{LES}} = \tau = \text{const} \), but also leads to a positive feedback between the energy input and total energy of the system, which results in numerical instability. To correct this problem, one must ensure that \( \tau_{\text{LES}} = \text{const} \). This can be achieved by allowing \( \nu(k|k_c) \) to become time dependent and related to resolved variables. The philosophy of using actual flow field characteristics to determine energy input and dissipation would be analogous to a standard practice of 3D LES.

Inspection of Eq. (10) reveals that \( \tau_{\text{LES}}(t) \) is proportional to the total enstrophy of the resolvable field, such that stipulating \( \tau_{\text{LES}} = \text{const} \) would require \( \nu(k|k_c) \) to become time dependent and inversely proportional to \( \Omega(t) \) (the time dependency of \( \nu(k|k_c) \) will be implied in the following discussion but suppressed in notations). To find the explicit dependency of
\( \nu(k|k_c) \) on \( \Omega(t) \), let us assume that \( \nu(k|k_c) \) can be represented by

\[
\nu(k|k_c) = F(\Omega)N(k/k_c), \tag{11}
\]

where \( F(\cdot) \) is some function of \( \Omega(t) \) and \( N(k/k_c) \) is still given by (6). According to Fig. 1, \( N(k/k_c) \) can be split into negative and positive terms,

\[
N(k/k_c) = -1 + \phi(k/k_c), \quad \phi(k/k_c) > 0, \quad 0 \leq k/k_c \leq 1. \tag{12}
\]

One can now calculate integral in (10) using Eqs. (11) and (12):

\[
\bar{\epsilon}_{\text{LES}} = 2F(\Omega) \int_0^{k_c} E(k, t)k^2 dk - 2F(\Omega) \int_0^{k_c} \phi(k/k_c)E(k, t)k^2 dk \\
= 2F(\Omega)\Omega(t) - 2F(\Omega) \int_0^{k_c} \phi(k/k_c)E(k, t)k^2 dk. \tag{13}
\]

Integration of (13) for the Kolmogorov spectrum with \( \phi(k/k_c) \) evaluated from the RG theory (see Appendix) yields

\[
\bar{\epsilon}_{\text{LES}} \simeq 0.8F(\Omega)\Omega(t). \tag{14}
\]

Equation (14) shows that to satisfy the requirement \( \bar{\epsilon}_{\text{LES}} = \bar{\tau} = \text{const} \), one has to impose

\[
F(\Omega) = \frac{\bar{\tau}}{[0.8\Omega(t)]}, \tag{15}
\]

such that the two-parametric viscosity (11) becomes

\[
\nu(k|k_c) = \frac{\bar{\tau}}{0.8\Omega(t)}N(k/k_c) = -\frac{\bar{\tau}}{0.8\Omega(t)} + \frac{\bar{\tau}}{0.8\Omega(t)}\phi(k/k_c). \tag{16}
\]

The first term in the right hand side of (16) accounts for the SGS energy input while the second term represents the high wave number dissipation as \( k \to k_c \). As was argued earlier, to ensure \( \bar{\epsilon}_{\text{LES}} = \text{const} \), the energy source term must be time dependent and inversely proportional to \( \Omega(t) \). Thus, the negative feedback between energy input and enstrophy of the resolved modes is the mechanism that stabilizes numerical process. Note that the SGS formulation based upon Eq. (16) complicates Eq. (7) because its right hand side now
depends on the functional of the solution, $\Omega(t)$. However, on the one hand, it is clear from the presented analysis that LES of 2D turbulence based upon Eq. (7) is impossible if $\nu(k|k_c)$ depends on $k$ and $k_c$ only. On the other, the SGS representation (16), though complicated, is in line with the eddy viscosity approach, in which the eddy viscosity coefficient is usually solution dependent (Smagorinsky, 1963, 1993; Yakhot and Orszag, 1986; Yakhot et al., 1989).

4.3. Case 2. Flow dependent $\nu(k|k_c)$ with no large scale drag

The SGS formulation (16) was used in the second LES and considerable improvement over Case 1 was observed. Figures 5a,b show that up to the simulated time $t = 3\tau_{tu}$, $\overline{E}(t)$ grows linearly, while $\overline{\Omega}(t)$ attains a constant value. Figure 6 shows that during the same $t$, $E(k, t)$ quickly approaches steady state Kolmogorov distribution (9). However, for $t > 3\tau_{tu}$ the flow field undergoes irreversible modifications; the behavior of $\overline{E}(t)$ and $\overline{\Omega}(t)$ changes, while $E(k, t)$ begins to deviate from the Kolmogorov law. All these changes reflect the basic problem of the present LES that simulates the behavior of an infinite system in a finite computational box (Smith and Yakhot, 1993a,b). In this box, the smallest wave number modes become energy saturated at $t \simeq 3\tau_{tu}$, and, if the energy of these modes is not removed, they begin to alter the behavior of the entire flow field. Therefore, to extend LES beyond $t \simeq 3\tau_{tu}$, one needs to prevent the accumulation of energy at the lowest modes, which was accomplished in LES of Case 3. Note however that by the time $t \simeq 3\tau_{tu}$ the inverse cascade swept through all the resolved modes such that they became energy saturated and attained the steady state. Therefore, one should expect that in LES with $t > 3\tau_{tu}$ both $\overline{E}(t)$ and $\overline{\Omega}(t)$ remain nearly constant.

4.4. Case 3. Flow dependent $\nu(k|k_c)$ with large scale drag

The simplest way to withdraw energy from the lowest modes would be by mere setting to zero the amplitudes of those modes. However, such a “chopping” alone is known to produce
unsatisfactory results (Browning and Kreiss, 1989). Therefore, in addition to the chopping, one needs to introduce a mechanism that would account for the energy exchange between the resolved modes and the low wave number modes excluded in LES. Such a mechanism, the large scale drag, was introduced in this study in analogy to the two-parametric viscosity. However, the derivation of this large scale drag is beyond the scope of this paper and will be presented elsewhere.

The large scale drag was implemented in the third LES, whereas the amplitudes of all modes with $k < k_{\text{min}} = 4$ were set to zero. As was explained in section 3 and shown in Fig. 2, such a chopping results in flattening of the cusp in $N(k/k_c)$ as $k \to k_c$, such that this function had to be recalculated which in turn led to modification of the coefficient in (14) from 0.8 to 0.87. The results of the third LES are shown in Figs. 7a,b and 8a,b. One can see that this simulation could virtually be extended indefinitely, with $\bar{E}(t)$ and $\bar{\Omega}(t)$ slightly oscillating around their steady state values (the source of these oscillations is probably the self-adjustment of the numerical scheme to the mismatch between the small scale energy forcing and large scale withdrawal). The instantaneous energy spectrum, Fig. 8a, exhibits steady and nearly perfect Kolmogorov scaling. Since the large scale drag enables one to dramatically increase the integration time in LES, it will be retained in all further simulations. Note however that these simulations will pertain to steady state rather than time developing flows.

An important characteristic of 2D turbulence in the energy transfer subrange is the energy flux $\Pi_E(k) = \int_0^k \mathcal{T}_\Omega(n)n^{-2}dn$ which theoretically should be equal to $\bar{\epsilon}$ for any $k$. Note that the enstrophy transfer function $\mathcal{T}_\Omega(n)$ is defined by (3) where integration is extended over all triangles $k + p + q = 0$ including the SGS ones for which $k$ and/or $p$ and/or $q > k_c$. However, if $\Pi_E(k)$ is calculated for LES results, then the SGS triangles should be excluded such that the integration area in (3) is reduced. On the other hand, the energy flux into the interval $0 < n < k$ in LES consists of two contributions, $\int_0^k \mathcal{T}'_\Omega(n)n^{-2}dn$
(where $T_\Omega'$ implies the reduced integration area in (3)), and the direct flux from the SGS modes,

$$-2 \int_0^k \nu(n|k_c)n^2 E(n)dn.$$  

Figure 8b shows that thus calculated $\Pi_E(k)$ remains nearly constant for $k > 16$ and is equal to about $5 \times 10^{-10}$, which is very close to the specified value of $\tau$ obtained from DNS by Chekhlov et al. (1994). The decrease of $\Pi_E(k)$ at small $k$ is due to the large scale energy removal.

### 4.5. Case 4. Flow dependent energy input with flow independent dissipation

It would be tempting to simplify (16) by relaxing the time dependency in the dissipation term. It is not clear a priori whether or not this time dependency is critical, and to find out about it a fourth LES was conceived in which SGS representation (16) was modified by replacing $\tau/0.8 \overline{\Omega}(t)$ in the dissipation term by the RG derived expression $0.32\tau^{1/3}k_c^{-4/3}$. Figures 9a,b and 10a,b,c show that this simplified SGS scheme performs in a very robust way with no oscillations at all for a relatively long time, $t \simeq 30\tau_{tu}$. Similarly to Case 3, there exists a mismatch between the small scale forcing and large scale energy removal, but since the dissipation in Case 4 cannot self-adjust, the solution begins to deteriorate when this mismatch accumulates significantly. Still, Case 4 LES could be extended to many more turnover times than the corresponding DNS, the result quite remarkable for its own sake.

During this time, the instantaneous energy spectrum, Fig. 10, exhibits steady and nearly perfect Kolmogorov scaling almost indistinguishable from that in Fig. 8. Figure 10b shows that the Kolmogorov constant $C_K$ in (9) is about 5, in good agreement with the RG derived value of 5.12 (see Appendix). As in Case 3, the energy flux $\Pi_E(k)$ shown in Fig. 10c is almost constant for $k > 16$ and is about $-5 \times 10^{-10}$.

Although the SGS formulation (16) is relatively easy to implement in spectral LES, in practical applications, particularly in the physical space, it would be far more useful to approximate $N(k/k_c)$ analytically. An additional benefit of such an approach would be a possibility to carry out further analytical studies of this SGS representation; the direction
pursued in the next section.
5. STABILIZED NEGATIVE VISCOSITY (SNV) FORMULATION

For practical implementation of SGS formulation (16) it is convenient to approximate $N(k/k_c)$ in (6) by a series in powers of $k^2$. It was found that even the first two terms of this series,

$$N(k/k_c) \simeq -1 + \alpha (k/k_c)^2,$$

(17)

where $\alpha$ is a constant, are sufficient to perform successful LES of 2D turbulence. To find $\alpha$ recall that representation (17) must ensure zero enstrophy transfer in the energy subrange,

$$\overline{\eta}_{\text{LES}} = 2 \int_0^{k_c} \nu(k|k_c) E(k,t)k^4 dk = 2 F(\Omega) \int_0^{k_c} N(k/k_c) E(k,t)k^4 dk = 0.$$

(18)

Substituting (17) into (18) and assuming that $E(k,t)$ is Kolmogorovian, one finds that $\alpha = \frac{8}{5}$ such that (13) becomes

$$\overline{\epsilon}_{\text{LES}} = -2 F(\Omega) \int_0^{k_c} \left[ -1 + \frac{8}{5} (k/k_c)^2 \right] E(k,t)k^2 dk$$

$$= 2 F(\Omega) \overline{\Omega}(t) - \frac{16}{5} F(\Omega) k_c^2 P(t),$$

(19)

where $P(t) \equiv \int_0^{k_c} E(k,t)k^4 dk$ is the total palinstrophy of the resolved modes. For Kolmogorovian $E(k,t)$, (19) can be integrated analytically to yield $P(t) = \frac{2}{5} \overline{\Omega}(t) k_c^2$ and

$$F(\Omega) = \frac{25}{18} \frac{\overline{\epsilon}}{\overline{\Omega}(t)}.$$

(20)

Equation (20) completes the two-term SGS parameterization for LES of 2D turbulence in which the right hand side of Eq. (7) takes the form

$$\nu(k|k_c)k^2 \zeta(k) = \frac{25}{18} \overline{\epsilon}_{\text{LES}} \left[ -1 + \frac{8}{5} \left( \frac{k}{k_c} \right)^2 \right] k^2 \zeta(k).$$

(21)

As in (16), the first term in the right hand side of (21) accounts for the energy flux from the unresolved modes and the inverse cascade of this energy, while the second term represents the energy dissipation near the cutoff. Using (21) and following the philosophy of Cases
3 and 4 LES, two more simulations were designed, with flow dependent and independent dissipation term in (21).

5.1. Case 5. Two-term LES with flow dependent dissipation

Simulations performed with the formulation (21) slightly adjusted to account for the finiteness of the computational domain are shown in Figs. 11a,b and 12. They exhibit very little difference compared to the third LES that employed the full curve $N(k/k_c)$ given by (6) and shown in Figs. 7a,b and 8. One infers therefore that (21) is a viable two term SGS representation for LES of 2D turbulence; obviously, (21) is significantly simpler than (16).

5.2. Case 6. Two-term LES with flow independent dissipation

For practical purposes, it would be most appealing to use formulation (21) with the dissipation term constant. A numerical experiment analogous to that of Case 4 LES was conducted with the energy source in (21) not changed but in the dissipation term, $\Omega(t)$ was replaced by its value calculated for the Kolmogorov spectrum (9). Such an approach yields the dissipation term in (21) in the form

$$Ak^4 \zeta(k),$$

where $A$ is a constant given by

$$A = 0.511\tau^{1/3}k_c^{-10/3},$$

which corresponds to $C_K=5.8$. The results of this case 5 LES are presented in Figs. 13a,b and 14a,b; there is a very good agreement with the corresponding results obtained with the full curve $N(k/k_c)$ up to $t \simeq 40\tau_{tu}$ after which, similarly to Case 4, the solution begins to deteriorate. The Kolmogorov constant $C_K$ shown in Figs. 14b is close to 5.8, which is consistent with derivation of (23). One can therefore infer that (21), (23) is probably the simplest SGS representation for LES of 2D turbulence possible.
Further advantages of the SGS representation (23) are revealed when LES of quasi-2D turbulence is sought in the physical space, where Eq. (21) combined with (23) leads to the following SGS representation:

$$\frac{\partial}{\partial x_i} \left( A_2 \frac{\partial}{\partial x_i} \zeta(x) \right) - A_4 \frac{\partial^4}{\partial x_i^2 \partial x_j^2} \zeta(x),$$

(24)

where

$$A_2 = \frac{25}{18} \frac{\tau}{\Omega(x)},$$

(25)

$$A_4 = \frac{A}{0.511 \varepsilon^{1/3} (\Delta/2\pi)^{10/3}} = \text{const},$$

(26)

and where $\Omega(x)$ denotes the enstrophy averaged over an area adjacent to the grid cell, and $\Delta$ is the grid resolution (note that the Laplacian term in (24) is written in the conservative form). Equation (24) thus includes two terms, the negative Laplacian and positive (in the sense of dissipation) biharmonic, and structurally resembles the Kuramoto–Sivashinsky equation widely known from combustion theory (Sivashinsky, 1977) and flows with chemical reactions (Kuramoto and Tsuzuki, 1976; Kuramoto, 1984). However, the SGS representation (24)-(26) combined with the explicit equation for the resolved scales produces far more complicated equation than the Kuramoto–Sivashinsky equation because generally its coefficients are not constant but, as in the eddy viscosity approach, are functions of the flow. There have been previous attempts to use formulation similar to (24)-(26) but with constant coefficients (dubbed the Kuramoto–Sivashinsky–Navier–Stokes equation, see Gama et al., 1991) to perform LES of 2D turbulence. However, they were not overly successful even in reproducing the Kolmogorov spectrum, mostly because they used constant “eddy viscosity” coefficients. Since, on the one hand, SGS representation (24)-(26) includes a negative Laplacian viscosity term and a positive, stabilizing, dissipation term, but, on the other hand, it is quite different from the Kuramoto–Sivashinsky equation in which “viscosity” coefficients are constant, it will be referred to as the Stabilized Negative Viscosity (SNV) formulation. The SNV formulation is expected to be particularly useful in simulations of atmospheric
and oceanic flows where large scale motions are quasi-two-dimensional by their nature while small scale forcing is significant (Monin, 1986).
6. DISCUSSION AND CONCLUSIONS

LES of homogeneous and isotropic 2D turbulence in the energy transfer subrange and appropriate SGS representation have been the central subject of this paper. Conservation of energy and enstrophy in 2D turbulent flows leads to coexistence of two spectral transfer processes, upscale for energy and downscale for enstrophy. Conventional eddy viscosity formulations are purely dissipative and fail to accommodate both transfers simultaneously. It is argued that proper SGS parameterization for LES of 2D flows is given by the two-parametric viscosity $\nu(k|k_c)$ introduced by Kraichnan (1976) that accounts for the energy (or enstrophy) exchange between given resolved and all SGS modes and which includes negative and positive branches; the negative branch of $\nu(k|k_c)$ represents the unresolved, small scale forcing and inverse cascade of energy, while the positive one represents the dissipation. It was shown that the negative and positive parts of the two-parametric viscosity play vital role in ensuring that all conservation and spectral transfer laws of 2D turbulence are satisfied. Then, $\nu(k|k_c)$ was used in LES of 2D turbulence in the energy transfer subrange where the negative part of $\nu(k|k_c)$ was the only energy source. The sensitivity of numerical results to the way of implementation of $\nu(k|k_c)$ in numerical schemes was studied. It was found that if $\nu(k|k_c)$ is specified as a flow independent parameter then a positive feedback is established between the forcing and the total energy of the system leading to numerical instability. Thus, another scheme was designed in which $\nu(k|k_c)$ was flow dependent but the rate of the energy input was kept constant. This LES exhibited a very stable behavior consistent with analytical theories and DNS; Kolmogorov scaling was evident and robust, and all conservation and spectral transfer laws were fulfilled. Then, a simplified SGS representation was advanced, in which only two terms were retained: one negative and proportional to $k^2$, and one positive and proportional to $k^4$. The viscosity coefficient in the negative term served as the energy source and had to be flow dependent, to ensure that the energy input remains constant.
However, since the $k^4$ term is mostly active at high wave number region near the cutoff wave number $k_c$, and its main role is energy and enstrophy dissipation at $k \to k_c$, this term did not need to be flow dependent and could essentially be kept constant and evaluated from analytical theories of 2D turbulence. Indeed, LES with constant and flow dependent $k^4$ terms gave very similar results.

The application of the two term parameterization to simulations in physical space results in the so called Stabilized Negative Viscosity (SNV) representation which includes negative Laplacian and positive biharmonic viscosities. Numerical implementation of the scheme with the constant biharmonic viscosity is obviously much simpler than that with the flow dependent viscosity.

On the one hand, the negative viscosity term is essential in SNV scheme; on the other, this scheme substantially differs from the Kuramoto-Sivashinsky equation with its constant viscosity because in SNV, the negative viscosity not only is not constant but is a nonlinear functional of the solution. In fact, the SNV representation is a peculiar case of the eddy viscosity approach; one normally expects that if all scales of a turbulent flow are resolved then eddy viscosity would become equal to the molecular viscosity which is true in 3D turbulence. In 2D turbulence, however, resolution of all scales must be accompanied by restoration of the explicit small scale forcing which would result in disappearance of both negative Laplacian and positive biharmonic viscosities and appearance of one positive, Laplacian, molecular viscosity.

It is believed that the SNV representation should be especially useful in quasi-2D flows in which considerable amount of energy resides on the unresolved scales. Such flows are typical in geophysical fluid dynamics where, due to the topographic and other constraints, flows are quasi-2D and where the small scale forcing is the predominant source of energy.
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Fourier transformed Eq. (1) reads
\[ \zeta(\hat{k}) = G^o(\hat{k}) \int \frac{k \times q \zeta(\hat{q})\zeta(\hat{k} - \hat{q})}{(2\pi)^3} d\hat{q} + G^o(\hat{k})f, \] (A1)
where \( \hat{k} \equiv (k, \omega), \hat{q} \equiv (q, \Omega) \), and \( G^o(\hat{k}) \equiv (i\omega + \nu_0 k^2)^{-1} \) is the bare propagator. The random, zero-mean, white in time Gaussian stirring force \( f \) in the right hand side of (A1) accounts for the steady energy input localized in the vicinity of a given wave number \( k_f \). In the RG formalism (Yakhot and Orszag, 1986; Staroselsky and Sukoriansky, 1993) the effect of a localized random stirring on the bare equation of motion is equivalent to the effect of a spatially-distributed forcing with the correlation function
\[ < f(k, \omega)f(k', \omega') > = D(k)(2\pi)^3 \delta(k + k')\delta(\omega + \omega') \] (A2)
on the renormalized equation. Here, \( D(k) = 2D_0 k^{-y+2} \), where \( y \) and \( D_0 \) are different for \( k \ll k_f \) and \( k \gg k_f \). The region \( k \ll k_f \) corresponds to the inverse cascade of energy for which \( y = 2 \) and \( D_0 \propto \bar{\tau} \), \( \bar{\tau} \) being the constant energy injection rate; the region \( k \gg k_f \) corresponds to the direct cascade of enstrophy with the constant rate \( \eta \); in that region \( D_0 \propto \eta \) and \( y = 4 \). Note that in the framework of the 2D RG theory, the regions \( k \ll k_f \) and \( k \gg k_f \) are treated as separate asymptotic regimes. Equation (A1) is defined in the interval \( 0 < k < \Lambda_0 \).

Fourier coefficients of the vorticity field can be separated into small scale, or fast modes, \( \zeta^>(k) \), for which \( \Lambda_0 - \delta\Lambda_0 < k < \Lambda_0 \), and large scale, or slow modes, \( \zeta^<(k) \), for which \( 0 < k < \Lambda_0 - \delta\Lambda_0 \), respectively. The RG procedure detailed in Forster et al. (1977) and Yakhot and Orszag (1986) consists of successive elimination of infinitesimal shells of the small scale modes from the equation of motion for the large scale modes. Essentially, \( \zeta \) field decomposed into \( \zeta^>(k) \) and \( \zeta^<(k) \) parts is substituted into Eq. (A1); then, the one-loop
approximation is invoked and the average is taken over the small scale modes. Equation (A1) is then rewritten in terms of $\zeta(<k)$ and is defined in the abridged interval $0 < k < \Lambda_0 - \delta\Lambda_0$; it now contains a new term represented by the correction to the Green function,

$$\delta G(\hat{k})^{-1} = \int^> k^2 q^2 - (k \cdot q)^2 k^4 \left( \frac{1}{q^2} - \frac{1}{|q-k|^2} \right) G(\hat{k} - \hat{q}) |G(\hat{q})|^2 D_0 q^{-y+2} \frac{d\hat{q}}{(2\pi)^3}, \quad (A3)$$

that accounts for the effect of the unresolved scales. Here, $\int^>$ denotes an integration over the band of wave numbers being removed.

The integral in (A3) is calculated in the limit $k \rightarrow 0$, $\omega \rightarrow 0$. The $O(k^2)$ terms provide correction to $\nu_0$. Iterating the scale elimination procedure, one can remove a finite band of modes and obtain a differential equation for the parameter of renormalization $\nu(k)$,

$$\frac{d\nu}{dk} = -\frac{D_0}{16\pi \nu^2 k^{\epsilon+1}} , \quad (A4)$$

where $\epsilon = 4 + y - d$. For the energy inertial subrange considered below, $y = 2$ and $\epsilon = 4$.

The solution to Eq. (A4) is

$$\nu(k) = \nu_0 \left\{ 1 + \frac{3D_0}{64\pi \nu_0^2 \Lambda_0^4} \left[ \left( \frac{k}{\Lambda_0} \right)^{-4} - 1 \right] \right\}^{1/3} . \quad (A5)$$

Assuming that $k/\Lambda_0 \ll 1$ and the molecular viscosity can be neglected [$\nu_0 \ll \nu(k)$], one finds:

$$\nu(k) = \left( \frac{3D_0}{64\pi} \right)^{1/3} k^{-4/3} . \quad (A6)$$

It is argued (Forster et al., 1977; Yakhot and Orszag, 1986) that iterated indefinitely, the RG procedure of small scale elimination converges to a fixed point solution whereas $\zeta(k)$ is described by the Langevin equation

$$\zeta(k, \omega) = G(k, \omega) f(k, \omega), \quad (A7)$$

where $G \equiv [i\omega + \nu(k)k^2]^{-1}$ is the renormalized propagator. The existence of the fixed point for RG procedure in 2D has been proven by Staroselsky et al. (1995) for $\epsilon \ll 1$. Although the
feasibility of continuation of these results into the region of large \( \epsilon \) is still an open question, it is assumed here that the fixed point solution (A7) exists at \( \epsilon = 4 \).

Equation (A7) allows one to calculate the vorticity correlator, \( U(k, \omega) \equiv \langle \zeta(k, t)\zeta(-k, t) \rangle \), as well as the kinetic energy spectrum,

\[
E(k) = \frac{1}{4\pi} k^{-1} \int \frac{d\omega}{(2\pi)} U(k, \omega) = (3\pi^2)^{-1/3} D_0^{2/3} k^{-5/3}.
\] (A8)

Although \( \nu(k) \) describes renormalization of the bare viscosity \( \nu_0 \), it is not what is often comprehended as an eddy viscosity. This is merely a response function of the nonlinear dynamical system described by the renormalized Eq. (A7). As such, it allows for calculating vorticity correlator and energy spectrum but does not directly relate to enstrophy and energy transfer and dissipation. Furthermore, \( [\nu(k) k^2]^{-1} \) can be viewed as a characteristic time scale of information loss at given \( k \) caused by nonlinear scrambling of all other modes (Dannevik et al., 1987). Therefore, \( \nu(k) \) has a meaning of an eddy damping parameter and is substantially one-point turbulence characteristic.

To analyze energy and enstrophy transfer, one needs to consider two-point characteristics that account for interaction between a given explicit mode \( k < k_c \) and all modes \( k > k_c \); \( k_c \) is identified with the moving dissipation cutoff. For this purpose, following Dannevik et al. (1987) the energy evolution equation should be derived at lowest nontrivial order of nonlinear coupling using fully renormalized propagator \( G(k) \). The resulting dynamical closure is similar to the Eddy-Damped, Quasi-Normal, Markovian (EDQNM) approximation where the lowest order RG analysis is invoked to obtain the eddy damping function. The energy equation then reads

\[
\frac{\partial}{\partial t} E(k, t) = \int \int_{\Delta} T(k, p, q, t) dp dq,
\] (A9)

where

\[
T(k, p, q, t) \equiv \frac{2}{\pi k} \theta_{kpq}(t) (p^2 - q^2) \left[ \frac{p^2 - q^2}{pq} E(p, t) E(q, t) \right]
\]
\[-\frac{k^2 - q^2}{kq} E(q, t) E(k, t) + \frac{k^2 - p^2}{kp} E(p, t) E(k, t) \right] \sin \alpha, \tag{A10}\]

and where \( \nu_k \equiv \nu(k)k^2 \), \( \alpha \) is an angle opposite to the vector \( k \) in the triangle \( k + p + q = 0 \), and the integration domain \( \Delta \) is defined by the triangular inequalities \( |k - p| < q < k + p \).

The function \( \nu(k) \) given by (A6) is used to compute the relaxation time \( \theta_{kpq}(t) \equiv (1 - e^{(\nu_k + \nu_p + \nu_q)k})/(\nu_k + \nu_p + \nu_q) \) in (A10).

Following Kraichnan (1976) one can now define the two-parametric, or effective eddy viscosity at wave number \( k \) in terms of the energy transfer from all subgrid scale modes with \( k > k_c \) to the given explicit mode \( k \),

\[ \nu(k|k_c) = -T(k|k_c)/[2k^2E(k)], \tag{A11} \]

where

\[ T(k|k_c) \equiv \int \int_\Delta T(k, p, q) dp dq, \quad k < k_c, \tag{A12} \]

and where the integration is extended over all \( p \) and \( k \) such that \( p \) and/or \( q > k_c \). Assuming that the limit \( t \to \infty \) in (A10) is considered the time argument in \( T(k, p, q) \) has been omitted.

In that limit, \( \theta_{kpq} = (\nu_k + \nu_p + \nu_q)^{-1} \).

For \( k \ll k_c \) the two-parametric viscosity \( \nu(k|k_c) \) can be calculated analytically (Kraichnan, 1976). In this case, the triangular inequality becomes \( |p - q| \leq k \ll q \). Therefore, all the quantities that enter \( T(k, p, q) \) can be expanded in powers of \( p - q \). Then, the \( p \) integration can be performed resulting in

\[ \nu(k|k_c) = \frac{1}{4} \int_{k_c}^{\infty} \theta_{kqq} \frac{d}{dq} [qE(q)] dq, \quad k \ll k_c. \tag{A13} \]

Substitution of (A6) and (A8) into (A13) gives the asymptotic eddy viscosity for the largest scales,

\[ \nu(0|k_c) = -\frac{1}{3} \left( \frac{3D_0}{64\pi} \right)^{1/3} k_c^{-4/3}. \tag{A14} \]
For arbitrary \( k \), \( \nu(k|k_c) \) was calculated via numerical integration of (A10); the resulting normalized two-parametric viscosity \( N(k/k_c) \), Eq. (6), is shown in Fig. 1. This function is negative for \( k \ll k_c \) and positive for \( k \to k_c \).

Noting that in the energy transfer subrange the energy injection rate \( \tau \) is equal to the rate of energy transfer from all the subgrid modes \( k > k_c \) to all explicit modes \( k < k_c \), one can find the relation between the forcing amplitude \( D_0 \) in (A2) and \( \tau \). Indeed, as follows from (A11) and (A12),

\[
\tau = -2 \int_0^{k_c} \nu(k|k_c)E(k)k^2 dk. \tag{A15}
\]

Substituting (A8) and (A14) into (A15) and performing numerical integration one finds

\[
D_0 \simeq 63\tau. \tag{A16}
\]

Using (A16) one can now calculate the Kolmogorov constant \( C_K \) in (A8) and the numerical factor in (A14), which are respectively,

\[
E(k) = C_K \bar{\tau}^{2/3} k^{-5/3}, \quad C_K \simeq 5.12, \tag{A17}
\]

and

\[
\nu(0|k_c) = -0.327\bar{\tau}^{1/3} k_c^{-4/3}. \tag{A18}
\]

These results have been used in Eqs. (8) and (14).
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9. LIST OF THE FIGURE CAPTIONS

Figure 1. Normalized two-parametric eddy viscosity from DNS (dots), from TFM (dashed line), and from RG (solid line) (from Chekhlov et al., 1994).

Figure 2. Actual two-parametric eddy viscosity from DNS (dots) and from RG (solid line). In RG calculations, the energy spectrum for $k < 5$ was corrected in accordance with the DNS data (from Chekhlov et al., 1994).

Figure 3. The evolution of the total energy $E(t)$ (a) and total enstrophy $\Omega(t)$ in Case 1 LES. Figure 3a also shows the evolution of $E(t)$ with the energy of the 1st, 2nd, 3rd, 4th, 5th, 6th and 7th modes removed.

Figure 4. The evolution of the instantaneous energy spectrum for $t/\tau_{tu} = 0.56, 1.11, 1.67, and 2.78$. The solid line shows the Kolmogorov $-5/3$ slope.

Figure 5. Same as Fig. 3 but for Case 2 LES.

Figure 6. Same as Fig. 4 but for Case 2 LES. Note that after $t/\tau_{tu} \approx 2$ all instantaneous profiles $E(k, t)$ become close to Kolmogorov law (9).

Figure 7. Same as Fig. 3 but for Case 3 LES. Because the amplitudes of the first four modes are set to zero, only the evolution of $E(t)$ with the energy of the 4th, 5th, 6th and 7th modes removed is shown.

Figure 8a. Same as Fig. 6 but for Case 3 LES. Note that since the Kolmogorov scaling is attained after about $t/\tau_{tu} = 2$, only time average energy spectrum is shown.

Figure 8b. The energy flux $\Pi_E(k)$ for Case 3 LES.

Figure 9. Same as Fig. 7 but for Case 4 LES.

Figure 10a. Same as Fig. 8 but for Case 4 LES.

Figure 10b. The time averaged Kolmogorov constant $C_K$ for Case 4 LES.
Figure 10c. The energy flux $\Pi_E(k)$ for Case 4 LES.

Figure 11. Same as Fig. 7 but for Case 5 LES.

Figure 12. Same as Fig. 8 but for Case 5 LES.

Figure 13. Same as Fig. 7 but for Case 6 LES.

Figure 14a. Same as Fig. 8 but for Case 6 LES.

Figure 14b. The time averaged Kolmogorov constant $C_K$ for Case 6 LES.
Figure (a) shows the evolution of \( \Pi(t) \times 10^6 \) over time \( t/\tau_{tu} \). Figure (b) displays the variation of \( \Omega(t) \times 10^4 \) with time.
