Riemann-Lagrange geometry for starfish/coral dynamical system

Mircea Neagu

Abstract. In this paper we develop the Riemann-Lagrange geometry, in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy, associated with the dynamical system concerning social interaction in colonial organisms. Some possible trophodynamic interpretations are derived.

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Key words: tangent spaces; least squares Lagrangian functions; Riemann-Lagrange geometry; starfish/coral dynamics.

1 Social interactions in colonial organisms

Let \( m \geq 2 \) be an integer. We introduce social interactions for starfish/coral dynamics as follows (see Antonelli et al. [1]):

\[
\begin{align*}
\frac{dN^1}{dt} &= \lambda_1 N^1 - \alpha_1 (N^1)^2 - \alpha_2 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \\
&\quad + \frac{\alpha_1}{m-1} \left( \frac{N^2}{N^1} \right)^{m-2} \cdot (N^2)^2 - \delta_1 F N^1 \\
\frac{dN^2}{dt} &= \lambda_2 N^2 - \alpha_2 (N^2)^2 - \alpha_1 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 + \\
&\quad + \frac{\alpha_2}{m-1} \left( \frac{N^1}{N^2} \right)^{m-2} \cdot (N^1)^2 - \delta_2 F N^2 \\
\frac{dF}{dt} &= \beta F (N^1 + N^2) + \gamma F^2 - \rho F,
\end{align*}
\]

where

- \( \alpha_1, \alpha_2, \lambda_1, \lambda_2, \delta_1, \delta_2, \beta, \gamma, \rho \) are positive coefficients;
- \( N^1, N^2 \) are coral densities;
• $F$ is the starfish density;
• $\lambda_1$ and $\lambda_2$ are growth rates;
• $\lambda_1/\alpha_1$ and $\lambda_2/\alpha_2$ are single species carrying capacities;
• $\beta$, $\delta_1$ and $\delta_2$ are the interaction coefficients for starfish preying on corals;
• $\gamma$ is the coefficient of starfish aggregation.

Note that $m$ is the effect of increasing the social parameter. If we set $m = 2$, we obtain the (2 corals/1 starfish)-model of Antonelli and Kazarinoff [2], in which every term of degree greater than one is quadratic. It is $m \geq 3$ which forces the social interaction terms to be nonquadratic.

By differentiation, the dynamical system (1.1) can be extended to a dynamical system of order two coming from a first order Lagrangian of least squares type. This extension is called in the literature in the field as geometric dynamical system (see Udriște [7]).

2 The Riemann-Lagrange geometry

The system (1.1) can be regarded on the tangent space $T\mathbb{R}^3$, whose coordinates are

$$
\left( x^1 = N^1, x^2 = N^2, x^3 = F, y^1 = \frac{dN^1}{dt}, y^2 = \frac{dN^2}{dt}, y^3 = \frac{dF}{dt} \right).
$$

**Remark 2.1.** We recall that the transformations of coordinates on the tangent space $T\mathbb{R}^3$ are given by

$$
(2.1) \quad \bar{x}^i = \bar{x}^i(x^j), \quad \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} y^j,
$$

where $i, j = 1, 3$.

In this context, the solutions of class $C^2$ of the system (1.1) are the global minimum points of the least squares Lagrangian function (see [7], [6])

$$
(2.2) \quad L = (y^1 - X^1(N^1, N^2, F))^2 + (y^2 - X^2(N^1, N^2, F))^2 +
$$

$$
+ (y^3 - X^3(N^1, N^2, F))^2 \geq 0,
$$

where

$$
X^1(N^1, N^2, F) = \lambda_1 N^1 - \alpha_1 (N^1)^2 - \alpha_2 \left( \frac{m}{m-1} \right) \cdot N^1 N^2 +
$$

$$
+ \frac{\alpha_1}{m-1} \left( \frac{N^2}{N^1} \right)^{m-2} \cdot (N^2)^2 - \delta_1 F N^1,
$$
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\[
X^2 (N^1, N^2, F) = \lambda_2 N^2 - \alpha_2 (N^2)^2 - \alpha_1 \left( \frac{m}{m - 1} \right) \cdot N^1 N^2 + \frac{\alpha_2}{m - 1} \left( \frac{N^1}{N^2} \right)^{m-2} \cdot (N^1)^2 - \delta_2 FN^2,
\]

\[
X^3 (N^1, N^2, F) = \beta F (N^1 + N^2) + \gamma F^2 - \rho F.
\]

**Remark 2.2.** The solutions of class \(C^2\) of the system (1.1) are solutions of the Euler-Lagrange equations attached to the least squares Lagrangian (2.2), namely (geometric dynamics, in Udrişte’s terminology)

\[
\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}, \quad \forall \ i = 1, 3, \Leftrightarrow
\]

\[
\frac{d^2 x^i}{dt^2} + 2G^i(x^k, y^k) = 0 \Leftrightarrow \frac{d^2 x^i}{dt^2} + \frac{1}{2} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right) = 0 \Leftrightarrow
\]

\[
\frac{d^2 x^i}{dt^2} = \left( \frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i} \right) y^k + \frac{\partial X^k}{\partial x^i} X^k,
\]

where

\[
G^i(x^k, y^k) = \frac{1}{4} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right) = -\frac{1}{2} \left[ \left( \frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i} \right) y^k + \frac{\partial X^k}{\partial x^i} X^k \right]
\]

is endowed with the geometrical meaning of **semispray** of \(L\) (for more geometrical details, see Miron and Anastasiei book [5] and Udrişte’s book [7]).

But, the least squares Lagrangian (2.2), together with its Euler-Lagrange equations (2.3), provide us with an entire Riemann-Lagrange geometry on the tangent space \(T \mathbb{R}^3\), in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy. These geometrical objects are naturally associated with the trophodynamical system (1.1).

Let us recall the main geometrical ideas developed in the Miron and Anastasiei book [5]. The canonical nonlinear connection \(\tilde{N} = (\tilde{N}^j^i)_{i,j=1}^{3} \) produced by the semispray (2.4) is given by the components

\[
\tilde{N}^j^i = \frac{\partial G^i}{\partial y^j} = -\frac{1}{2} \left( \frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i} \right).
\]

**Remark 2.3.** We recall that, under a transformation of coordinates (2.1), the local components of the nonlinear connection obey the rules [4], [5]

\[
\tilde{N}^k_l = \tilde{N}^j^i \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{y}^k}{\partial \tilde{x}^j}.
\]

From a geometrical point of view, we point out that the coefficients \(N^j_i\) of the above nonlinear connection have not a global character on \(T \mathbb{R}^3\).
Remark 2.4. Using the well-known Cartan-Kosambi-Chern (KCC) theory, used also in the paper of Böhmer, Harko and Sabău [3], we can remark that the deviation curvature tensor associated with the dynamical system \((1.1)\) is given by the formula

\[ P_{ij} = -2\frac{\partial G^i}{\partial x^j} - 2G^i\frac{\partial N^j_l}{\partial y^l} + \frac{\partial N^j_i}{\partial x^j} y^l + N^i_l N^l_j. \]

It is important to note that the solutions of the Euler-Lagrange equations \((2.3)\) are Jacobi stable iff the real parts of the eigenvalues of the deviation tensor \(P_{ij}\) are strictly negative everywhere, and Jacobi unstable, otherwise. For more details, see [3] and references therein.

The canonical nonlinear connection defines the adapted bases of vector fields and covector fields on the tangent space \(T\mathbb{R}^3\), namely

\[ \left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\} \subset \mathcal{X}(T\mathbb{R}^3), \]

\[ \left\{ dx^i, \delta y^i = dy^i + N^i_j dx^j \right\} \subset \mathcal{X}^*(T\mathbb{R}^3). \]

The adapted local components of the Cartan \(N\)-linear connection \(\mathcal{C}(N) = \left( L^i_{jk}, C^i_{jk} \right)\) are given by the formulas

\[ L^i_{jk} = \frac{g^{ir}}{2} \left( \frac{\partial g_{rj}}{\partial x^k} + \frac{\partial g_{rk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r} \right), \quad C^i_{jk} = \frac{g^{ir}}{2} \left( \frac{\partial g_{rj}}{\partial y^k} + \frac{\partial g_{rk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r} \right), \]

where

\[ g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = \delta_{ij}. \]

The only non-vanishing d-torsion adapted component associated with the Cartan \(N\)-linear connection \(\mathcal{C}(N)\) is given by the coefficient

\[ R^r_{ij} = \frac{\delta N^r_i}{\delta x^j} - \frac{\delta N^r_j}{\delta x^i} = \frac{\partial N^r_i}{\partial x^j} - \frac{\partial N^r_j}{\partial x^i}. \]

At the same time, all the adapted components of the curvature attached to the Cartan \(N\)-linear connection \(\mathcal{C}(N)\) are zero (for all curvature formulas, see [5]).

The electromagnetic-like distinguished 2-form attached to the Lagrangian \(L\), defined via its deflection d-tensors (for more details, see Miron and Anastasiei book [5]), is given by \(\mathcal{F} = F_{ij} dy^i \wedge dx^j\), where

\[ F_{ij} = \frac{1}{2} \left( g_{ir} N^r_j - g_{jr} N^r_i \right) = \frac{1}{2} \left( N^r_j - N^r_i \right) = N^r_j. \]

In this context, let us use the notation

\[ J(X) = \left( \frac{\partial X^i}{\partial x^j} \right)_{i,j=1,3} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}, \]

\[ \left( \begin{array}{ccc} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{array} \right). \]
where
\[
J_{11} = \lambda_1 - 2\alpha_1 N^1 - \alpha_2 \left(\frac{m}{m-1}\right) \cdot N^2 - \alpha_1 \left(\frac{m-2}{m-1}\right) \frac{(N^2)^m}{(N^1)^{m-1}} - \delta_1 F,
\]
\[
J_{12} = -\alpha_2 \left(\frac{m}{m-1}\right) \cdot N^1 + \alpha_1 \left(\frac{m}{m-1}\right) \frac{(N^2)^{m-1}}{(N^1)^{m-2}},
\]
\[
J_{13} = -\delta_1 N^1, \quad J_{21} = -\alpha_1 \left(\frac{m}{m-1}\right) \cdot N^2 + \alpha_2 \left(\frac{m}{m-1}\right) \frac{(N^1)^{m-1}}{(N^2)^{m-2}},
\]
\[
J_{22} = \lambda_2 - 2\alpha_2 N^2 - \alpha_1 \left(\frac{m}{m-1}\right) \cdot N^1 - \alpha_2 \left(\frac{m-2}{m-1}\right) \frac{(N^1)^m}{(N^2)^{m-1}} - \delta_2 F,
\]
\[
J_{23} = -\delta_2 N^2, \quad J_{31} = \beta F, \quad J_{32} = \beta F, \quad J_{33} = \beta (N^1 + N^2) + 2\gamma F - \rho.
\]

Following the above Miron and Anastasiei’s geometrical ideas, we obtain the following geometrical results:

**Theorem 2.1.** (i) The canonical nonlinear connection on $T\mathbb{R}^3$, produced by the system (1.1), has the local components $N = (N^i)_{i,j=1,3}$, where $N^i$ are the entries of the skew-symmetric matrix
\[
N = -\frac{1}{2} [J(X) - \text{T} J(X)] = \begin{pmatrix}
N^1_1 & N^1_2 & N^1_3 \\
N^2_1 & N^2_2 & N^2_3 \\
N^3_1 & N^3_2 & N^3_3
\end{pmatrix},
\]

where
\[
N^1_1 = N^2_2 = N^3_3 = 0,
\]
\[
N^1_2 = -N^2_1 = -\frac{1}{2} \left\{ \left(\frac{m}{m-1}\right) \left(\alpha_1 N^2 - \alpha_2 N^1\right) + \left(\frac{m}{m-1}\right) \left[ \alpha_2 \left(\frac{(N^1)^{m-1}}{(N^2)^{m-2}} - \alpha_1 \left(\frac{(N^2)^{m-1}}{(N^1)^{m-2}}\right) \right] \right\},
\]
\[
N^1_3 = -N^3_1 = \frac{1}{2} (\beta F + \delta_1 N^1), \quad N^2_3 = -N^3_2 = \frac{1}{2} (\beta F + \delta_2 N^2).
\]

(ii) All adapted components of the canonical Cartan connection $C\Gamma(N)$, produced by the system (1.1), are zero.

(iii) The effective adapted components $R^i_{jkl}$ of the torsion $d$-tensor $T$ of the canonical Cartan connection $C\Gamma(N)$, produced by the system (1.1), are the entries of the following skew-symmetric matrices:
\[
R_i = (R^i_{jkl})_{i,j,k,l=1,3} = \frac{\partial N}{\partial N^i} = \begin{pmatrix}
0 & \frac{\partial N^2}{\partial N^i} & \frac{\delta_1}{2} \\
-\frac{\partial N^1}{\partial N^i} & 0 & 0 \\
-\frac{\delta_1}{2} & 0 & 0
\end{pmatrix},
\]
where

\[ \frac{\partial N_1^2}{\partial N^2} = \frac{1}{2} \left( \frac{m}{m-1} \right) \left[ \alpha_2 - \alpha_2 (m-1) \left( \frac{N_1}{N^2} \right)^{m-2} - \alpha_1 (m-2) \left( \frac{N^2}{N_1} \right)^{m-1} \right]; \]

\[ R_2 = (R^i_j)_{i,j=1,3} = \frac{\partial N}{\partial N^2} = \begin{pmatrix} 0 & \frac{\partial N_2^1}{\partial N^2} & 0 \\ -\frac{\partial N_1^1}{\partial N^2} & 0 & \delta_2 \frac{m}{2} \\ 0 & -\delta_2 \frac{m}{2} & 0 \end{pmatrix}, \]

where

\[ \frac{\partial N_1^2}{\partial N^2} = \frac{1}{2} \left( \frac{m}{m-1} \right) \left[ -\alpha_1 + \alpha_2 (m-2) \left( \frac{N_1}{N^2} \right)^{m-1} + \alpha_1 (m-1) \left( \frac{N^2}{N_1} \right)^{m-2} \right]; \]

\[ R_3 = (R^i_j)_{i,j=1,3} = \frac{\partial N}{\partial F} = \begin{pmatrix} 0 & 0 & \beta \frac{m}{2} \\ 0 & 0 & \beta \frac{m}{2} \\ -\beta \frac{m}{2} & -\beta \frac{m}{2} & 0 \end{pmatrix}. \]

(iv) All adapted components of the curvature d-tensor $R$ of the canonical Cartan connection $\Gamma(N)$, produced by the system (1.1), vanish.

(v) The geometric electromagnetic-like distinguished 2-form, produced by the system (1.1), is given by $F = F_{ij} dy^i \wedge dx^j$, where the adapted components $F_{ij}$ are the entries of the skew-symmetric matrix $F = (F_{ij})_{i,j=1,3} = N$.

(vi) The geometric Yang-Mills-like energy, produced by the system (1.1), is given by the formula

\[ E_{YM}(t) = F_{12}^2 + F_{13}^2 + F_{23}^2 = \]

\[ = \frac{1}{4} \left( \frac{m}{m-1} \right)^2 \left[ \alpha_1 N^2 - \alpha_2 N^1 + \alpha_2 \left( \frac{N^2}{N_1} \right)^{m-1} - \alpha_1 \left( \frac{N_2}{N^1} \right)^{m-1} \right]^2 + \]

\[ + \frac{1}{4} \left( \beta F + \delta_1 N^1 \right)^2 + \frac{1}{4} \left( \beta F + \delta_2 N^2 \right)^2. \]

Remark 2.5. In the author’s opinion, from a trophodynamic point of view the zero level of the jet geometric Yang-Mills energy produced by the system (1.1) is important. The jet geometric Yang-Mills trophodynamical energy produced by the system (1.1) is zero iff

\[ \beta F + \delta_1 N^1 = 0, \quad \beta F + \delta_2 N^2 = 0, \]

\[ (\alpha_1 N^2 - \alpha_2 N^1) + \left[ \alpha_2 \left( \frac{N^2}{N_1} \right)^{m-1} - \alpha_1 \left( \frac{N_2}{N^1} \right)^{m-1} \right] = 0. \]
If $\delta_1 \neq \delta_2$, these conditions imply the impossible fact that $F = N^1 = N^2 = 0$, and if $\delta_1 = \delta_2 = \delta$, then we obtain $N^1 = N^2 = -\beta F/\delta$. In this last case, we find a Bernoulli differential equation as the last equation of the system (1.1), namely

$$\frac{dF}{dt} = -\rho F + \left(\gamma - 2\frac{\beta^2}{\delta}\right) F^2.$$

This equation can be integrated by using the changing of variable $z = F^{-1}$. The solution of the above Bernoulli differential equation is

$$F(t) = \frac{1}{a \exp (\rho t) + b},$$

where $a \in \mathbb{R}$ is an arbitrary constant, and we have

$$b = \frac{1}{\rho} \left(\gamma - 2\frac{\beta^2}{\delta}\right).$$

At the same time, we consider that the constant level surfaces of the jet geometric Yang-Mills trophodynamical energy $\mathcal{E}_{YM}(t) = C$, $C > 0$, could contain important trophodynamic connotations. Consequently, the graphical representation of these surfaces in the system of axes $O\mathcal{F}N^1N^2$ could be a fruitful and open problem in trophodynamics.

**Remark 2.6.** The deviation curvature tensor components $P_{ij}^k$ can be obtained by contracting with $y^k$ the nonzero components of the torsion tensor $R_{ijk}^l$, that is $P_{ij}^k = R_{ijk}^l y^l = (\partial N^l_j/\partial x^k) y^k$. Consequently, the matrix of the deviation curvature tensor is given by

$$P = R_k y^k = \begin{pmatrix}
0 & \frac{\partial N^1_j}{\partial N^1} & \delta_1 \\
-\frac{\partial N^1_j}{\partial N^1} & 0 & 0 \\
-\delta_1 & 0 & 0
\end{pmatrix} y^1 + \begin{pmatrix}
0 & \frac{\partial N^2_j}{\partial N^2} & 0 \\
-\frac{\partial N^2_j}{\partial N^2} & 0 & \delta_2 \\
0 & -\delta_2 & 0
\end{pmatrix} y^2 +
\begin{pmatrix}
0 & 0 & \beta \\
0 & 0 & \beta \\
-\beta & -\beta & 0
\end{pmatrix} y^3 = \begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix},$$

where

$$a = \frac{\partial N^1_j}{\partial N^1} y^1 + \frac{\partial N^1_j}{\partial N^2} y^2, \quad b = \frac{\delta_1}{2} y^1 + \beta y^2, \quad c = \frac{\delta_2}{2} y^2 + \beta y^3.$$ 

The eigenvalues of the matrix $P$ are the real values

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm \sqrt{a^2 + b^2 + c^2}.$$ 

In conclusion, the behavior of neighboring solutions of the Euler-Lagrange equations (2.3) is Jacobi unstable.
Open problem. The trophodynamic interpretations associated with the geometrical objects constructed in this paper still represent an open problem.

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Author’s address:

Mircea Neagu
Department of Mathematics and Informatics,
Transilvania University of Brașov,
Blvd. Iuliu Maniu, No. 50, Brașov 500091, Romania.
E-mail: mircea.neagu@unitbv.ro