Criteria Determining the Number of Estimate Points of Moment Method for Performance Functions Including Non-differentiable Points

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Abstract: Moment method is famous for its calculation efficiency. Moments of the performance function are obtained by point estimates based on Hermite Integration. The reliability index and probability of failure can be obtained using existing standardization function or existing distribution systems. The precision of moment method increases monotonously with the increasing number of estimating points. The performance functions with non-differentiable points are frequently encountered when elasto-plastic behavior dominates for a structure under loading. Thus the accuracy and efficiency of moment method should be addressed for performance functions with non-differentiable points. In this paper, an example of structural performance functions is presented to show the existence of non-differentiable points. Two example performance functions are used to verify the accuracy and efficiency of moment method. The MCS method is employed to obtain the accurate results of probability of failure. The relative errors of probability of failure from moment method are obtained through a comparison with the results from MCS method. The fluctuation of the relative errors with the number of estimate points is also presented. The required number of estimate points for performance functions with non-differentiable points is discussed through the above two examples. Finally, criteria to determine the estimate points for performance functions with non-differentiable points are proposed and verified the applicability

1. Introduction

Multi-fold probability integral computation is a fundamental problem in structural reliability theory

\[ P_f = \text{Prob} [G(X) \leq 0] = \int_{G(X) \leq 0} f(X) dX \]  

(1)

Where, \( X = [X_1, X_2, \ldots, X_n]^T \) is a vector of random variables representing uncertain structural quantities, \( f(X) \) denotes the joint probability density function of \( X \), \( G(X) \) is the performance function, and \( P_f \) is the probability of failure.

It is quite difficult to obtain the probability of failure by directly calculating the equation (1). Monte-Carlo Simulation Method (MCSM) can achieve the high accuracy with the increase of samples. However, large amount of computation is inevitable, for the probability of failure \( P_f \), 100/\( P_f \) samples is needed at least [6,7]. Up to now the time consumption of MCSM is unacceptable for most design practice. Therefore various approximate methods have been developed in the past several decades. Taylor series expansion...
for the performance function and searching for design point by iteration are necessary for traditional methods such as the first-order reliability method (FORM\textsuperscript{[1,2]}), the second-order reliability method (SORM\textsuperscript{[3,4,5]}). Furthermore, FORM goes into practice although its shortcoming of low accuracy is recognized\textsuperscript{[6,7]} Zhao\textsuperscript{[9,13]} and his cooperators have developed various moment methods (MM). Moments of the performance function are obtained by point estimates based on Hermite Integration for MM. Then the reliability index and failure probability of the performance function can be obtained using some existing standardization function or some existing distribution systems. According to Zhao\textsuperscript{[9]}, MM is applicable with better precision using only 5-point or 7-point estimates, and the precision of moment method increases monotonously with the increasing number of estimating points. However, this conclusion might not be verified by more research works\textsuperscript{[14]}.

As known, structures will be seriously damaged and go into strong nonlinearity under strong earthquakes, violent collisions, or tremendous explosions, which will lead to non-differentiable performance functions. In this paper, for structural performance functions, the changing relationship between errors and estimate points of MM at first, and then the criteria to determine the required estimate points are proposed.

2. Review of moment method

2.1. Calculation of moments

Follow Zhao\textsuperscript{[11]}, for a performance function \(Z=G(X)\), with mean value \(\mu_G\), standard deviation \(\sigma_G\), third moment (skewness) \(\alpha_{3G}\), fourth moment (kurtosis) \(\alpha_{4G}\). The moments of the performance function which has a single random variable can be calculated as,

\[
\mu_G = \sum_{i=1}^{m} P_i \times G[T^{-1}(u_i)], \quad \sigma_G = \sqrt{\sum_{i=1}^{m} P_i \times (G[T^{-1}(u_i)] - \mu_G)^2}, \quad \alpha_3 = \sum_{i=1}^{m} P_i \times (G[T^{-1}(u_i)] - \mu_G)^3, \quad \alpha_4 = \sum_{i=1}^{m} P_i \times (G[T^{-1}(u_i)] - \mu_G)^4
\]  

(2)

Where \(T^{-1}(u_i)\) is the inverse Rosenblatt transformation. The estimating points \(u_i, P_i\) and their corresponding weights \(P_i\) can readily be obtained as,

\[
u_i = \sqrt{2x_i}, \quad P_i = \omega_{3i}/\pi
\]  

(3)

Where \(x_i\) is the abscissas of weight function \(exp(-x_i^2)\), and \(\omega_i\) are weights for Hermite integration.

2.2. Second-moment method

If the first two moments of performance function, \(Z=G(X)\), are known, and assuming that \(Z=G(X)\) obeys normal distribution, the reliability index and failure probability based on second-moment method are expressed as\textsuperscript{[11]},

\[
\beta_{SM} = \mu_G / \sigma_G, \quad P_{fSM} = \Phi(-\beta_{SM})
\]  

(4)

2.3. Third-moment method

If the first three moments of performance function, \(Z=G(X)\), are known, and assuming that, \(z_0=(Z-\mu_G)/\sigma_G\) obeys the three-parameter lognormal distribution, the reliability index and failure probability based on second-moment method are expressed as,

\[
\beta_{TM} = -\frac{\text{Sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}(1 + \frac{\beta_{SM}}{u_p})\right], \quad P_{fTM} = \Phi(-\beta_{TM})
\]  

(5)

Where, \(A=1+1/u_p^a\), and,

\[
u_p = (a+b)^{a/(a+b)} + (a-b)^{b/(a-b)}, \quad a = \frac{1}{\alpha_{3G}} \left(\frac{1}{2} + \frac{1}{\alpha_{3G}}\right), \quad b = \frac{1}{2\alpha_{3G}}\left(\sqrt{\alpha_{3G}^2 + 4}\right)
\]  

(6)

2.4. Fourth moment method

Approximating the first four moments of the performance function using Pearson system, and then calculate the probability of failure.
\[
\frac{1}{f} \frac{df}{dZ_u} = -\frac{aZ_u + b}{c + bZ_u + dZ_u^2}
\]

(7)

Where, \(a = 10\alpha_{4g} - 12\alpha_{4g} - 18\), \(b = \alpha_{4g}(\alpha_{4g} + 3)\), \(c = 4\alpha_{4g} - 3\alpha_{4g}^2\), and \(d = 2\alpha_{4g} - 3\alpha_{4g}^2 - 6\).

The reliability index and failure probability based on second-moment method are:

\[
\beta_{FM} = -\Phi^{-1}(\int_{-\infty}^{\beta_{FM}} f(z)dz), P_{f,FM} = \Phi(-\beta_{FM})
\]

(8)

3. Case of the performance function curves with non-differentiable points

As shown in Figure 1[15], it is a simple bridge pier system under impact, say, under ship impact. Section properties of pier and pile are \(E1 = 6.225 \times 10^0\) Nm², \(E2 = 1.645 \times 10^9\) Nm² respectively. Plastic hinges are set on the foot of pier and the top of piles. The skeleton curve \((M-\theta)\) of plastic hinges is as the same as shown in Figure 2. The yield moment, \(M_y\), of pier is 22687kNm, and the yield and ultimate rotation angle of pier are \(\theta_y = 0.0033381\) rad, \(\theta_u = 0.126385\) rad, respectively. The yield moment of pile is \(M_p = 1752\) kNm, and the yield and ultimate rotation angle of pile are \(\theta_y = 0.003767\) rad, \(\theta_u = 0.0321\) rad, respectively.

Fig.1. Schematic diagram of ship bridge collision analysis

Fig.2. A Simple of Time history of impact force

Fig.3. The performance function of a simple bridge pier under ship impacts

It is assumed that a ship of 400DWT impacts the pier at the middle point of the pile cap. The probability characteristics of basic random variable impact velocity \(V\) is assumed as normal distribution \(V \sim N(1.2, 0.3)\), and the ship impact load can be expressed as Figure 2. The failure bound of pier is that the rotation angle of pile reaches 0.0015 rad. The limit state equation \((Z=G(V))\) is shown as Figure 3. There are one non-differentiable point in Figure 3, the coordinates of non-differentiable point is \((0.7993586, 0.0090485)\).

As known, the MC simulation method can achieve a high accuracy with enough samples. Therefore the relative error, \(err\), of moment method may be defined as,

\[
err = \frac{P_f - P_{f0}}{P_{f0}} \times 100
\]

(9)

Where \(P_f\) and \(P_{f0}\) means the failure probability by moment method and MC simulation respectively.

4. Required Numbers of Estimate Points for Performance Function Curves with Non-differentiable Points

4.1. Example 1: one non-differentiable point

A case of performance function with one non-differentiable point is expressed as,

\[
Z = G(x) = \begin{cases} 
-\frac{1}{2}x + \frac{1}{2} & x < 1 \\
-\frac{1}{2}x & x \geq 1 
\end{cases}
\]

(10)

Basic random variable \(x\) obeys standard normal distribution, i.e., \(x \sim N(0, 1)\). The first four moments of \(Z\) obtained by moment method with different number of estimate points is compared with the MC simulation method, showed as Table 1. \(n\) is the number of estimate points in the Table 1 and the following Tables.
The relative error, $err$, as a function of number of estimate points, is shown in Figure 4. It is observed that the precision of moment method don’t increase monotonously with the number of estimate points, but fluctuates nearby the exact value. Finally, the relative error, $err$, gradually approaches to zero when the number of estimate points is enough. For case 1, 15 more than estimate points are needed if it is required that $err$ is less than 5%.

![Relative error of Example 1](image)

**Fig.4 Relative error of Example 1**

Table 2 lists the probability of failure calculated by moment methods with 15 estimate points and by MC simulation. For example 1, the MC simulation method needs 200 samples at least. However moment method needs 63 number of calculation, which is just 1/3 of the MC simulation method.

Table.1 First four moment of Example 1

| $n$ | mean value | standard deviation | skewness | kurtosis |
|-----|-------------|---------------------|----------|----------|
| 3   | -0.061      | 0.8985              | -0.4042  | 3.0547   |
| 5   | -0.0499     | 0.9206              | -0.3142  | 2.8147   |
| 7   | -0.0403     | 0.9313              | -0.3058  | 2.7116   |
| 9   | -0.0328     | 0.9377              | -0.3093  | 2.6534   |
| 11  | -0.0355     | 0.9332              | -0.3271  | 2.6636   |
| 13  | -0.0391     | 0.9286              | -0.338   | 2.6851   |

| $n$ | mean value | standard deviation | skewness | kurtosis |
|-----|-------------|---------------------|----------|----------|
| 15  | -           | 0.9259              | -0.3425  | 2.7008   |
| 17  | -           | 0.9245              | -0.3438  | 2.7112   |
| 19  | -           | 0.9239              | -0.3433  | 2.7175   |
| 21  | -           | 0.9238              | -0.342   | 2.7205   |
| 23  | -           | 0.924               | -0.3403  | 2.7212   |
| 25  | -           | 0.9268              | -0.3359  | 2.7066   |

Table.2 the probability of failure of Example 1

|      | SM       | TM       | FM       | MCS      |
|------|----------|----------|----------|----------|
| $P_f$| 0.518934 | 0.49659  | 0.491692 | 0.5003   |
| $err$| 3.72     | 0.74     | 1.72     | -        |

4.2. Example 2: ship-bridge collision

Example 2 is just as the same as the above case in section 3. The basic random variable is the impact velocity $V \sim N(1.2, 0.3)$. The first four moments of $Z$ obtained by moment method with different number of estimate points and by MC simulation method, are shown in Table 3

Table.3 First Four Moment of Example 2

| $n$ | mean value | standard deviation | skewness | kurtosis |
|-----|-------------|---------------------|----------|----------|
| 3   | 0.0061     | 0.00209             | -0.1426  | 3.00678  |
| 5   | 0.00613    | 0.00213             | -0.1127  | 2.75254  |
| 7   | 0.00612    | 0.00212             | -0.1523  | 2.73372  |
| 9   | 0.00612    | 0.0021              | -0.166   | 2.75225  |
| 11  | 0.00611    | 0.0021              | -0.1675  | 2.76768  |
| 13  | 0.00611    | 0.0021              | -0.1647  | 2.7756   |

| $n$ | mean value | standard deviation | skewness | kurtosis |
|-----|-------------|---------------------|----------|----------|
| 15  | 0.00611    | 0.00211             | -0.1607  | 2.77722  |
| 17  | 0.00612    | 0.00211             | -0.1566  | 2.77441  |
| 19  | 0.00612    | 0.00211             | -0.1566  | 2.77441  |
| 21  | 0.00612    | 0.00211             | -0.1501  | 2.76121  |
| 23  | 0.00612    | 0.00211             | -0.1521  | 2.75889  |
| 25  | 0.00612    | 0.00211             | -0.1573  | 2.76634  |
As shown in Figure 5, $err$ fluctuates with number of estimate points, which is the same as observed in Example 1. $err$ gradually reaches zero with the increase of the number of estimate points. For Example 2, it needs 15 estimate points at least to make $err$ less than 5%.

$P_f$ calculated by moment methods with 13 estimate points and by the MCS method, is listed in Table 4. The error of fourth-moment method is within 10%. For case 2, the MC simulation method needs $4.8 \times 10^4$ number of calculation at least. However moment methods need 48 number of calculation, which is about 1/1000 of the MC simulation method.

Table 4 the probability of failure of Example 3

|        | SM    | TM    | FM    | MCS   |
|--------|-------|-------|-------|-------|
| $P_f$  | 0.00183259 | 0.00313487 | 0.00189153 | 0.0021 |
| $err$  | -12.73 | 49.28 | 9.93  | —     |

The key problem for moment method is to obtain the moments of performance function. Generally, the moments of performance function must be obtained by FEM method. Therefore, moment method might be considered as a MC method that is of extremely high computational efficiency.

For the two cases in this section, the computational efforts of moment method under various precision requirements and the least computational efforts of MC simulation method are listed in Table 5.

Table 5 computational efforts of moment method and Computational efforts of MCSM

| Case name | Computational efforts of moment method | Computational efforts of MCSM |
|-----------|--------------------------------------|-----------------------------|
|           | Error limit 2% | Error limit 5% | Error limit 10% | $P_f$ | Computational consumption |
| Case1     | 120 | 80 | 48 | 0.5003 | 200 |
| Case2     | 80 | 48 | 15 | 0.0021 | $4.8 \times 10^4$ |

The performance functions of structures in elastic range are always smooth curves, and a lot of examples have proved that moment method possesses a high precision, which is applicable using only 5-point or 7-point estimates. It can be take that moment method need more calculation amount for the performance function curves with non-differentiable points, but moment method is still superior to the MC simulation method in calculation amount.

5. Criteria Determining Required Numbers of Estimate Points

In section 4, the relative errors of $P_f$ for moment methods are estimated based on the results from MCS method. The estimate points required may be determined by a given relative error limit (e.g. 5%). However, to make the moment method a valuable one, moment method itself must have the ability to judge if the required accuracy of $P_f$ is reached, and then to determine the required numbers of estimate points. As shown in section 4, relative error, $err$, fluctuates with the number of estimate points, therefore it is necessary to define suitable criteria to end the computation. In this paper, the proposed termination criteria for computation of moment method are,

$$
\epsilon_i = \frac{P_{f,k-1} - P_{f,k}}{P_{f,k}} \leq \epsilon_0
$$

$$
\epsilon_m = \frac{P_{f,m} - P_{f,k}}{P_{f,k}} \leq \epsilon_0
$$

Fig.5 Relative Error of Example 3
Where, $P_{f,k}$ is the failure probability at $k$-th computational step; $P_{f,(k-1)}$ is the failure probability at $(k-1)$-th computational step; $P_{f,m}$ is the mean value of $P_f$ from previous $m$ computational steps as,

$$P_{f,m} = \frac{1}{m} \sum_{j=0}^{m-1} P_{f,j}$$  \hspace{1cm} (12)

For case 1, 17 points are required to meet the criterion 1 (equation (11a)), and 15 points are required to meet the criterion 2 (equation (11b)); For case 2, 11 points are required to meet the criterion 1, and 17 points are required to meet the criterion 2. Follow the proposed criteria, $err$ and the results of first four moments are listed in Table 6; and the results of $P_f$ and relative errors are listed in Table 8.

Table 6 err and result of the First Four Moments calculated based the proposed criteria

| Case | Data Type | $\mu$  | $\sigma$ | Skewness | Kurtosis |
|------|-----------|--------|----------|----------|----------|
| 1    | 17 err    | 0.611  | 0.245    | -2.339   | -0.171   |
| 17   | Result    | -0.04283 | 0.92453 | -0.34376 | 2.71123  |
| 17   | err       | 0.030  | 0.069    | 0.403    | -0.292   |
| 2    | Result    | 0.00612 | 0.00211 | -0.1566  | 2.77441  |

Table 7 Results of $P_f$ and Relative Errors based the proposed criteria

| Case | Data Type | SM     | TM     | FM     | MCS     |
|------|-----------|--------|--------|--------|---------|
| 1    | $P_f$     | 0.518475 | 0.495906 | 0.490832 | 0.5003   |
|      | $err$     | 3.633  | -0.878 | -1.892 | 0       |
| 2    | $P_f$     | 0.001863 | 0.003178 | 0.001927 | 0.0021  |
|      | $err$     | -11.29 | 51.333 | -8.238 | 0       |

It is observed from Table 6 that each value of $err$ for moments is less than 5%, and the maximum value of $err$ is $-2.339\%$. From Table 7 and Figure 6, It is observed that for case 1, $P_f$ is large, and $err$ is less than 5% for SM, TM, and FM; for case 2, $P_f$ is smaller, $err$ for moment methods become larger although FM gives a better result of $P_f$. The reason is that the tail of PDF(probabilistic density function) of a random variable contribute more to the probability of failure, and it needs the fourth moment and more high moments if the tail shape of the PDF of a random variable is to determined exactly. This is the next problem to be addressed.

6. Conclusions

The structural performance function curves with non-differentiable points are frequently encountered in the elasto-plastic analysis of structures. A study on the accuracy and efficiency of moment method for performance functions including non-differentiable points has been conducted. The MCS method is employed to obtain the accurate results of failure probability, which is used to calculate the relative errors of moment method. The results and conclusions reached in this paper are as follows.

(1). Differ from the performance functions, the accuracy of moment method does not increase monotonously with the number of estimating points for the performance functions including non-differentiable points. However, moment method may also achieve stable and reliable results if the
number of estimate points is large enough.

(2). The computational efforts of moment method for the performance functions including non-differentiable points are far larger than the performance functions without non-differentiable points, which is related to the required error limit.

(3). Two criteria are proposed to determine the estimate points for performance functions with non-differentiable points, and the applicability of the proposed criteria is demonstrated by comparison with the results of failure probability from MCS method for the three example performance functions.

(4). For a required error limit of 5%, the computational efforts the standard MCS method is 2~1000 times of the efforts by the moment method, which is related to the value of probability of failure. Generally, high efficiency of moment method may be expected for a smaller probability of failure.

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