THE SORTING INDEX ON COLORED PERMUTATIONS AND EVEN-SIGNED PERMUTATIONS

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ABSTRACT. We define a new statistic sor on the set of colored permutations $G_{r,n}$ and prove that it has the same distribution as the length function. For the set of restricted colored permutations corresponding to the arrangements of $n$ non-attacking rooks on a fixed Ferrers shape we show that the following two sequences of set-valued statistics are joint equidistributed: $(\ell, Rmil_0, Rmil_1, \ldots, Rmil_{r-1}, Lmil_0, Lmil_1, \ldots, Lmil_{r-1}, Lmap_0, Lmap_1, \ldots, Lmap_{r-1})$ and $(sor, Cyc_0, Cyc_{r-1}, \ldots, Cyc_1, Lmic_0, Lmic_1, \ldots, Lmic_{r-1}, Lmap_0, Lmap_1, \ldots, Lmap_{r-1})$. Analogous results are also obtained for Coxeter group of type $D$. Our work generalizes recent results of Petersen, Chen-Gong-Guo and Poznanović.

1. Introduction

1.1.Mahonian and Stirling statistics. Let $S_n$ be the group of permutations on $n$ letters $[n] := \{1, 2, \ldots, n\}$. A pair $(\sigma_i, \sigma_j)$ is called an inversion in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ if $i > j$ and $\sigma_i < \sigma_j$. Denote by $\text{inv}(\sigma)$ the number of inversions in $\sigma$. The distribution of $\text{inv}$ over $S_n$ was first found by Rodriguez [8] to be

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = \prod_{i=1}^{n} [i]_q,$$

(1.1)

where $[i]_q := 1 + q + \cdots + q^{i-1}$.

In a Coxeter group, the length $\ell(\sigma)$ of a group element $\sigma$ is the minimal number of generators needed to express $\sigma$. It is well known [2, Chapter 8] that $S_n$ is the Coxeter group of type A, where the generators are the adjacent transpositions and $\ell(\sigma) = \text{inv}(\sigma)$. A permutation statistic is called Mahonian if it is equidistributed with $\text{inv}$ over $S_n$. Similarly in a Coxeter group a statistic is called Mahonian if it is equidistributed with the length function $\ell$.

The number of cycles $\text{cyc}$ is another important statistic, whose distribution over $S_n$ is

$$\sum_{\sigma \in S_n} t^{\text{cyc}(\sigma)} = \prod_{i=1}^{n} (t + i - 1).$$

(1.2)

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As the coefficients of this polynomial are the (unsigned) Stirling numbers of the first kind, a permutation statistic over $S_n$ is called \textit{Stirling} if it is equidistributed with $\text{cyc}$.

The \textit{reflection length} $\ell'(\sigma)$ of $\sigma$ in a Coxeter group is the minimal number of reflections (i.e., elements conjugate to generators) needed to express $\sigma$. In type A, the reflections are the transpositions and one has

$$\text{cyc}(\sigma) = n - \ell'(\sigma).$$ \hfill (1.3)

1.2. Sorting index. Petersen \cite{Pet} defined the \textit{sorting index} $\text{sor}$ over $S_n$ and proved it is Mahonian. One can uniquely decompose $\sigma \in S_n$ into a product of transpositions $\sigma = (i_1 j_1) (i_2 j_2) \cdots (i_k j_k)$ with $j_1 < j_2 < \cdots < j_k$ and $i_1 < i_2 < j_2, \ldots, i_k < j_k$. Then the sorting index of $\sigma$ is

$$\text{sor}(\sigma) := \sum_{r=1}^k (j_r - i_r).$$

Simply put, $\text{sor}(\sigma)$ counts the number of steps needed to bubble sort a permutation (i.e., the total number of steps needed to successively move $n, n-1, \ldots, 1$ back in places). For example, for $\sigma = 31524 = (1 2)(1 3)(3 4)(3 5)$, the sorting process is $31524 \rightarrow 31452 \rightarrow 31254 \rightarrow 21345 \rightarrow 12345$, and $\text{sor}(31524) = (5 - 3) + (4 - 3) + (3 - 1) + (2 - 1) = 6$ as it needs 2, 1, 2, 1 step(s) respectively to move 5, 4, 3, 2 back to its place.

By defining $r_{\text{min}}(\sigma)$ to be the number of right-to-left minima of $\sigma = \sigma_1 \cdots \sigma_n$, Petersen also showed that $(\text{inv}, r_{\text{min}})$ and $(\text{sor}, \text{cyc})$ have the same joint distribution \cite{Pet} and

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} r_{\text{min}}(\sigma) = \sum_{\sigma \in S_n} q^{\text{sor}(\sigma)} \text{cyc}(\sigma) = \prod_{i=1}^n (t + [i]_q - 1).$$ \hfill (1.4)

The sorting indices $\text{sor}_B$ (for type $B$) and $\text{sor}_D$ (for type $D$) were also defined by Petersen, and type $B, D$ analogous identities of (1.4) were found by Petersen \cite{Pet} and Chen-Gong-Guo \cite{CGG} respectively. In the case of type $B$, set-valued equidistribution results are also obtained in \cite{CGG}.

1.3. Sorting index on a Ferrers shape. Recently the above results were extended by Poznanović over the permutations corresponding to arrangements of $n$ non-attacking rooks on a fixed Ferrers shape with $n$ rows and $n$ columns \cite{Poz}. The setting can be described as follows. For a given sequence of integers $f = (f_1, f_2, \ldots, f_n)$ with $1 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq n$, we define the set of restricted permutations by

$$\mathcal{S}_{n,f} := \{ \pi \in \mathcal{S}_n : \pi(i) \leq f_i, 1 \leq i \leq n \}.$$ 

It is clear that $f$ defines a Ferrers shape and $\mathcal{S}_{n,f}$ consists of those permutations corresponding to non-attacking rook placements. For example, let $f = (2, 3, 3, 4)$, then $\mathcal{S}_{4,f} = \{1234, 1324, 2134, 2314\}$, as illustrated in Figure 1. By defining the set-valued statistics
Figure 1. The 4 permutations in $S_{4,(2,3,3,4)}$

\[\text{Cyc}(\sigma) := \{\text{the smallest number in each cycle of the cycle decomposition}\}\]
\[Rmil(\sigma) := \{\sigma_i : \sigma_i < \sigma_j \text{ for any } j > i\} \text{ (Right-to-left minimum letters)}\]
\[Lmal(\sigma) := \{\sigma_i : \sigma_i > \sigma_j \text{ for any } j < i\} \text{ (Left-to-right maximum letters)}\]
\[Lmap(\sigma) := \{i : \sigma_i > \sigma_j \text{ for any } j < i\} \text{ (Left-to-right maximum places)}\]

Poznanović [7] proved that $(\text{inv}, Rmil, Lmal, Lmap)$ and $(\text{sor}, \text{Cyc}, Lmal, Lmap)$ have the same joint distribution over $S_n$, by means of Foata-Han’s bijection in [3]. Analogous results on Coxeter groups of type $B$ and $D$ were also obtained in [7], generalizing the works of Petersen [6] and Chen-Guo-Gong [3].

In this paper we extend Poznanović’s results further in two ways. In the first part we obtain analogous new results on the colored permutations $G_{r,n}$, and in the second part we refine known results for type $D$.

1.4. Colored permutations on a Ferrers shape. The first part of our work is to generalize results on Coxeter group of type $A(= C_1 \wr S_n)$ and $B(= C_2 \wr S_n)$ to the colored permutations $G_{r,n} := C_r \wr S_n$ within a fixed Ferrers shape.

First of all, in Section 2 we will define the sorting index $\text{sor}$ on $G_{r,n}$ and prove that it is equidistributed with the length function $\ell$ (defined in the next section).

**Theorem 1.1** (Theorem 2.3). For any $r$ and $n$ the statistics $\ell$ and $\text{sor}$ have the same distribution over $G_{r,n}$. That is,

$$\sum_{\pi \in G_{r,n}} q^{\text{sor}(\pi)} = \sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q! \cdot \prod_{i=1}^{n} (1 + q^i [r-1]_q). \quad (1.5)$$

We then consider those colored permutations on a fixed Ferrers shape and seek for analogous results of (1.4) and set-valued equidistributions. Denote by $G_{r,n,f}$ the restricted version of $G_{r,n}$ determined by $f$ and define the set-valued statistics $\text{Cyc}^t, \text{Lmic}^t, \text{Rmil}^t, \text{Lmil}^t, \text{Lmal}^t$ and $\text{Lmap}^t$ for $t = 0, 1, \ldots, r-1$ (see Section 2 for detailed definition). Our first main theorem gives two rather interesting long tuples of joint equidistributed set-valued statistics over $G_{r,n,f}$.
Main Theorem A (Theorem 4.5). The two tuples of set-valued statistics

\[(\ell, \text{Rmil}_0, \text{Rmil}_1, \ldots, \text{Rmil}_{r-1}, \text{Lmil}_0, \text{Lmil}_1, \ldots, \text{Lmil}_{r-1}, \text{Lmal}_0, \text{Lmal}_1, \ldots, \text{Lmal}_{r-1}, \text{Lmap}_0, \text{Lmap}_1, \ldots, \text{Lmap}_{r-1})\]

and

\[(\text{so}, \text{Cyc}_0, \text{Cyc}_{r-1}, \ldots, \text{Cyc}_1, \text{Lmic}_0, \text{Lmic}_{r-1}, \ldots, \text{Lmic}_1, \text{Lmal}_0, \text{Lmal}_{r-1}, \ldots, \text{Lmal}_1, \text{Lmap}_0, \text{Lmap}_{r-1}, \ldots, \text{Lmap}_1)\]

have the same joint distribution over \(G_{r,n,f}\).

We emphasize that the statistics are set-valued, and the equidistribution is 'twisted', that is, the superindices for each kind of statistic are 0, 1, \ldots, \(r\), \(r-1\) in the first tuple and 0, \(r\), \(r-1\), \ldots, 1 in the second.

Our second main theorem gives the generating function for counting colored permutations by the set-valued statistics \((\ell, \text{Rmil}_0, \text{Rmil}_1, \ldots, \text{Rmil}_{r-1}, \text{Lmil}_0, \text{Lmil}_1, \ldots, \text{Lmil}_{r-1})\) or \((\text{so}, \text{Cyc}_0, \text{Cyc}_{r-1}, \ldots, \text{Cyc}_1, \text{Lmic}_0, \text{Lmic}_{r-1}, \ldots, \text{Lmic}_1)\). To understand the statement we need some notation. Given \(f\) we define

\[H(f) := (h_1, h_2, \ldots, h_n),\]

where \(h_i\) is the smallest possible index at which the letter \(i\) can appear in a colored permutation \(\sigma \in G_{r,n,f}\). In the above example \(f = (2, 3, 3, 4)\) and hence \(H(f) = (1, 1, 2, 4)\). The superindex of a statistic is taken as \(\mod r\), for example, \(\text{Rmil}^{-2}(\sigma) = \text{Rmil}^3(\sigma)\) if \(r = 5\). The checking function \(\xi_A(\omega)\) is defined by

\[\xi_A(\omega) := \begin{cases} 
\omega & \text{if the statement } A \text{ is true} \\
1 & \text{if the statement } A \text{ is false}.
\end{cases}\]

Main Theorem B (Theorem 5.2). Given \(r, n\) and \(f\). Let \(H(f) = (h_1, \ldots, h_n)\). We have

\[
\sum_{\pi \in G_{r,n,f}} q^{l(\pi)} \prod_{t=0}^{r-1} \left(\prod_{i \in \text{Rmil}^{-1}(\pi)} x_{t,i} \prod_{i \in \text{Lmil}^{-1}(\pi)} y_{t,i}\right) = \sum_{\pi \in G_{r,n,f}} q^{s\alpha(\pi)} \prod_{t=0}^{r-1} \left(\prod_{i \in \text{Cyc}^{-1}(\pi)} x_{t,i} \prod_{i \in \text{Lmic}^{-1}(\pi)} y_{t,i}\right)
\]

\[
= \prod_{j=1}^{n} \left( x_{0,j} + q + \cdots + q^{j-h_j-1} + \xi_{h_j=1}(y_{0,j})q^{j-h_j} 
+ \sum_{t=1}^{r-1} \left( x_{r-t,j}q^{2j+t-2} + q^{2j+t-3} + \cdots + q^{j+h_j+t-1} + \xi_{h_j=1}(y_{r-t,j})q^{j+h_j+t-2} \right) \right).
\]

1.5. An example. We illustrate our main theorems by an example. The set

\[G_{3,2} = \{1^{[0]}0^{[2]}, 1^{[1]}0^{[1]}, 1^{[1]}1^{[0]}, 1^{[1]}2^{[2]}, 1^{[1]}1^{[2]}, 1^{[2]}2^{[0]}, 1^{[2]}1^{[0]}, 1^{[2]}1^{[2]}, 2^{[0]}1^{[0]}, 2^{[0]}0^{[1]}, 2^{[0]}1^{[2]}, 2^{[1]}1^{[0]}, 2^{[1]}1^{[2]}, 2^{[1]}1^{[2]}, 2^{[1]}1^{[2]}, 2^{[1]}2^{[0]}, 2^{[1]}2^{[0]}, 2^{[1]}2^{[0]}, 2^{[2]}1^{[0]}, 2^{[2]}1^{[0]}, 2^{[2]}2^{[0]}\},\]

consists of 18 colored permutations of the form \(\sigma_1^{[z_1]} \sigma_2^{[z_2]}\), where \(\sigma_1, \sigma_2\) is a permutation of \(\{2\}\) and \(0 \leq z_1, z_2 \leq 2\) are the colors. The Main Theorem A says that

\[(\ell, \text{Rmil}_0, \text{Rmil}_1, \text{Rmil}_2, \text{Lmil}_0, \text{Lmil}_1, \text{Lmil}_2, \text{Lmal}_0, \text{Lmal}_1, \text{Lmal}_2, \text{Lmap}_0, \text{Lmap}_1, \text{Lmap}_2)\]
have the same joint distribution, as we can see from Table 1 and Table 2.

Moreover, if we let \( f = (1, 2) \), then \( G_{3,2f} \) consists of the first 9 colored permutations and Main Theorem A still holds.

| \( \pi \) | \( \ell \) | \( \text{Rmil}^0 \) | \( \text{Rmil}^1 \) | \( \text{Rmil}^2 \) | \( \text{Lmil}^0 \) | \( \text{Lmil}^1 \) | \( \text{Lmil}^2 \) | \( \text{Lmal}^0 \) | \( \text{Lmal}^1 \) | \( \text{Lmal}^2 \) | \( \text{Lmap}^0 \) | \( \text{Lmap}^1 \) | \( \text{Lmap}^2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1^{[0]}2^{[0]} | 0 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| 1^{[0]}2^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 1^{[1]}2^{[0]} | 4 | 1 | 1 | 1 | 2 | 1 | 2 | 1 |
| 1^{[2]}2^{[2]} | 5 | 2 | 1 | 1 | 2 | 1 | 2 | 1 |
| 2^{[0]}1^{[0]} | 3 | 1 | 1 | 1 | 2 | 1 |
| 2^{[0]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[0]} | 4 | 1 | 1 | 2 | 1 |
| 2^{[1]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[2]} | 5 | 1 | 1 | 2 | 1 |

| \( \pi \) | \( \text{sor} \) | \( \text{Cyc}^0 \) | \( \text{Cyc}^1 \) | \( \text{Lmic}^0 \) | \( \text{Lmic}^1 \) | \( \text{Lmic}^2 \) | \( \text{Lmal}^0 \) | \( \text{Lmal}^1 \) | \( \text{Lmal}^2 \) | \( \text{Lmap}^0 \) | \( \text{Lmap}^1 \) | \( \text{Lmap}^2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1^{[0]}2^{[0]} | 0 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| 1^{[0]}2^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 1^{[1]}2^{[0]} | 4 | 1 | 1 | 1 | 2 | 1 | 2 | 1 |
| 1^{[2]}2^{[2]} | 5 | 2 | 1 | 1 | 2 | 1 | 2 | 1 |
| 2^{[0]}1^{[0]} | 3 | 1 | 1 | 1 | 2 | 1 |
| 2^{[0]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[0]} | 4 | 1 | 1 | 2 | 1 |
| 2^{[1]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[2]} | 5 | 1 | 1 | 2 | 1 |

**Table 1.** \((\ell, \text{Rmil}^0, \text{Rmil}^1, \text{Rmil}^2, \text{Lmil}^0, \text{Lmil}^1, \text{Lmil}^2, \text{Lmal}^0, \text{Lmal}^1, \text{Lmal}^2, \text{Lmap}^0, \text{Lmap}^1, \text{Lmap}^2)\) for \( G_{3,2} \)

| \( \pi \) | \( \text{sor} \) | \( \text{Cyc}^0 \) | \( \text{Cyc}^1 \) | \( \text{Lmic}^0 \) | \( \text{Lmic}^1 \) | \( \text{Lmic}^2 \) | \( \text{Lmal}^0 \) | \( \text{Lmal}^1 \) | \( \text{Lmal}^2 \) | \( \text{Lmap}^0 \) | \( \text{Lmap}^1 \) | \( \text{Lmap}^2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1^{[0]}2^{[0]} | 0 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| 1^{[0]}2^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 1^{[1]}2^{[0]} | 4 | 1 | 1 | 1 | 2 | 1 | 2 | 1 |
| 1^{[2]}2^{[2]} | 5 | 2 | 1 | 1 | 2 | 1 | 2 | 1 |
| 2^{[0]}1^{[0]} | 3 | 1 | 1 | 1 | 2 | 1 |
| 2^{[0]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[0]} | 4 | 1 | 1 | 2 | 1 |
| 2^{[1]}1^{[1]} | 1 | 2 | 1 | 2 | 1 |
| 2^{[1]}1^{[2]} | 5 | 1 | 1 | 2 | 1 |

**Table 2.** \((\text{sor}, \text{Cyc}^0, \text{Cyc}^1, \text{Lmic}^0, \text{Lmic}^1, \text{Lmic}^2, \text{Lmal}^0, \text{Lmal}^1, \text{Lmal}^2, \text{Lmap}^0, \text{Lmap}^1, \text{Lmap}^2)\) for \( G_{3,2} \)
As for the generating function, if \( f = (1, 2) \), then \( H(f) = (1, 2) \). Main Theorem B says that

\[
\sum_{\pi \in G_{3,2}(1,2)} q^{\ell(\pi)} \prod_{t=0}^{2} \left( \prod_{i \in \text{Rmil}^{-t}(\pi)} x_{t,i} \prod_{i \in \text{Lmil}^{-t}(\pi)} y_{t,i} \right) = \sum_{\pi \in G_{3,2}(1,2)} q^{\text{sor}(\pi)} \prod_{t=0}^{2} \left( \prod_{i \in \text{Cyc}^{t}(\pi)} x_{t,i} \prod_{i \in \text{Lmic}^{t}(\pi)} y_{t,i} \right) = (x_{0,1}y_{0,1} + x_{2,1}y_{2,1}q + x_{1,1}y_{1,1}q^{2})(x_{0,2} + x_{2,2}q^{3} + x_{1,2}q^{4}).
\]

In the above the \( q^{4} \) term is

\[1 \cdot x_{0,1}x_{1,2}y_{0,1}q^{4} + 1 \cdot x_{2,1}x_{2,2}y_{2,1}q^{4}.\]

If we look at the second equality, this term says that there are two colored permutations \( \pi \in G_{3,2} \). The first permutation has \( 1 \in \text{Cyc}^{0}(\pi), 2 \in \text{Cyc}^{1}(\pi), 1 \in \text{Lmic}^{0}(\pi) \) and \( \text{sor}(\pi) = 4 \), namely the permutation \( 1^{[0]}2^{[1]} \), and the second one has \( 1, 2 \in \text{Cyc}^{2}(\pi), 1 \in \text{Lmic}^{2}(\pi) \) and \( \text{sor}(\pi) = 4 \), namely the permutation \( 1^{[2]}2^{[2]} \). Similary from the first equality this term says that there are two colored permutations \( \pi \): one has \( 1 \in \text{Rmil}^{0}(\pi), 2 \in \text{Rmil}^{-1}(\pi) = \text{Rmil}^{2}(\pi), 1 \in \text{Lmil}^{0}(\pi) \) and \( \ell(\pi) = 4 \), namely the permutation \( 1^{[0]}2^{[2]} \), and the other has \( 1, 2 \in \text{Rmil}^{-2}(\pi) = \text{Rmil}^{1}(\pi), 1 \in \text{Lmil}^{-2}(\pi) = \text{Lmil}^{1}(\pi) \) and \( \ell(\pi) = 4 \), namely the permutation \( 1^{[1]}2^{[1]} \).

1.6. Even-signed permutations on a Ferrers shape. The second part of the paper is to investigate Coxeter group of type D, or even-signed permutations. The sorting index \( \text{sor}_{D} \) was defined by [6] and the set-valued equidistibution result restricted to a Ferrers shape was investigated in [7]. Our two main results are Theorem 6.5 and Theorem 6.8 which are the type D version of Main Theorem A and Main Theorem B respectively, refining the results in [7].

The rest of the paper is organized as follows. In Section 2 we introduce the colored permutation groups \( G_{r,n} \) and define the sorting index and various set-valued Stirling statistics. A bijection on \( G_{r,n} \), again inspired by Foata and Han [41], is given in Section 3. Based on this bijection, in Section 4 and 5 we prove Main Theorem A and B respectively. In Section 6 we investigate even-signed permutations.

2. Sorting index of colored permutations

2.1. Colored permutations. Let \( r, n \) be positive integers. The group of colored permutations \( G_{r,n} \) of \( n \) letters with \( r \) colors is

\[ G_{r,n} := C_{r} \wr \mathfrak{S}_{n}, \]

the wreath product of the cyclic group \( C_{r} := \mathbb{Z}/r\mathbb{Z} \) with \( \mathfrak{S}_{n} \).

An elements of \( G_{r,n} \) is an ordered pair \( (\sigma, z) \), where \( \sigma = \sigma_{1}\sigma_{2} \cdots \sigma_{n} \in \mathfrak{S}_{n} \) and \( z = (z_{1}, z_{2}, \ldots, z_{n}) \) is an \( n \)-tuple of integers with \( z_{i} \in C_{r} \). The product of \( (\sigma, z) \) and \( (\rho, w) \) is \( (\sigma \rho, w + \rho(z)) \), where \( \rho(z) := (z_{\rho(1)}, z_{\rho(2)}, \ldots, z_{\rho(n)}) \) and the addition is taken as mod \( r \). It is easy to see that \( e = (12 \cdots n, (0, 0, \ldots, 0)) \) is the identity.
We can represent elements of $G_{r,n}$ in different ways. Let 

$$\Sigma := \{1, \ldots, n, \bar{1}, \ldots, \bar{n}, 1^{r-1}, \ldots, n^{r-1}\},$$

then $(\sigma, z)$ can be viewed as the bijection $\pi : \Sigma \rightarrow \Sigma$ such that $\pi(i) = \sigma_z^i$ for $i \in [n]$ and $\pi(\bar{i}) = \pi(i)$ for $i \in \Sigma$. For instance, $(3214, (2, 1, 1, 0)) \in G_{3,4}$ can be represented as the bijection

$$\begin{pmatrix}
1 & 2 & \bar{3} & 4 & \bar{1} & 2 & 3 & 4 \\
3 & 2 & 1 & \bar{4} & \bar{2} & \bar{1} & 3 & 2 & 1 & 4
\end{pmatrix},$$

which is called its two-line notation. By omitting the first row we have the one-line notation $\bar{3}21\bar{4}3\bar{2}\bar{1}4$, or more tersely the window notation $\bar{3}21$ by only recording the image of $[n]$. In this manner we write an element $(\sigma, z) \in G_{r,n}$ as a word $\sigma_1^{z_1} \sigma_2^{z_2} \cdots \sigma_n^{z_n}$, in which $\sigma_i$ and $z_i$ are respectively called the base value and color of $\pi(i)$.

The group $G_{r,n}$ can be generated by the set of generators $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$.

In the window notation, $s_i$ $(1 \leq i \leq n)$ is the transposition of swapping the $i$-th and $(i+1)$-th letters, while $s_0$ is the action of adding one more bar on the first letter (the number of bars is taken module $r$). Note that the multiplication is on the right. For example, if $r = 3, n = 4$, then

$$s_0 s_1 s_0 s_2 s_1 s_0 s_3 = \bar{3}21\bar{4}3\bar{2}\bar{1}4.$$

The generators are subject to the conditions

$$
\begin{align*}
&\begin{cases}
  s_0^r = 1, \\
  s_i^2 = 1, &1 \leq i \leq n - 1, \\
  (s_i s_j)^2 = 1, &|i - j| > 1, \\
  (s_i s_{i+1})^3 = 1, &1 \leq i \leq n - 2, \\
  (s_0 s_1)^{2r} = 1.
\end{cases}
\end{align*}
$$

2.2. Length. We define the length $\ell(\pi)$ of $\pi = \sigma_1^{z_1} \sigma_2^{z_2} \cdots \sigma_n^{z_n} \in G_{r,n}$ to be the minimal number of generators in $S_n$ needed to represent it. Bagno [1] gave the following combinatorial interpretation of $\ell(\pi)$:

$$\ell(\pi) = \text{inv}(\pi) + \sum_{z_i > 0} (\sigma_i + z_i - 1),$$

where $\text{inv}(\pi)$ is the number of inversions in the window notation of $\pi$ with respect to the linear order

$$n^{r-1} < \cdots < \bar{n} < \cdots < 1^{r-1} < \cdots < \bar{1} < 1 < \cdots < n.$$

(2.2)

For example, let $\pi = \bar{3}21\bar{4}3\bar{2}\bar{1}4 \in G_{3,4}$. Then $\ell(\pi) = 1 + (4 + 2 + 1) = 8$. The distribution of $\ell$ was also derived in [1] as

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = [n]_q ! \prod_{i=1}^n (1 + q^{|r-1|} q^i).$$

(2.3)
There is another ‘length function’ on $G_{r,n}$. Consider the one-line notation. For $1 \leq i < j \leq n$ and $0 \leq t < r$ let $(i^t j)$ be the transposition of swapping the $i^t$-th with $j$-th letters, the $i^{t+1}$-th with $j$-th letters, ..., the $i^{t+r-1}$-th with $j^{t-1}$-th letters. Also, for $1 \leq i \leq n$ and $0 < t < r$ let $(i^t i)$ be the action of adding $t$ bars on the $i$-th, $i$-th, ..., $i^{r-1}$-th letters. In the window notation, multiplying $\pi$ the $j$-th letter by $(\bar{2} 5)$ has the effect of replacing $i < j$, has the effect of replacing $\pi_j$ by $\sigma_i^{[zi+t]}$ and $\pi_i$ by $\sigma_j^{[z_i-t]}$, while multiplying $\pi$ on the right by $(i^t i)$ has the effect of replacing $\pi_i$ by $\sigma_i^{[zi+t]}$. For example, if $\pi = 25143 \in G_{3,5}$, then $\pi \cdot (\bar{2} 5) = 2\bar{3}51\bar{4}3$.

It can be seen that $G_{r,n}$ can also be generated by

$$\mathcal{T}_n := \{(i^t j) : 1 \leq i < j \leq n \text{ and } 0 \leq t < r\} \cup \{(i^t i) : 1 \leq i \leq n \text{ and } 0 < t < r\}.$$ 

Denote by $\ell'(\pi)$ the minimal number of elements in $\mathcal{T}_n$ needed to express $\pi$. The distribution of $\ell'$ will be derived in Corollary 5.4 as

$$\sum_{\pi \in G_{r,n}} \ell'(\pi) = \frac{n}{1 + (ri - 1)t}.$$ 

Note that $\ell'$ is also called the reflection length when $r = 1, 2 \in [4]$, where each element of $\mathcal{T}_n$ is a reflection. However when $r \geq 3$ elements of $\mathcal{T}_n$ are not reflections.

2.3. Sorting index. Now we come to the key definition of the whole paper. We will define a reasonable sorting index on $G_{r,n}$. Note that any $\pi \in G_{r,n}$ can be uniquely written as a product

$$\pi = (i_1^{[t_1]} j_1) (i_2^{[t_2]} j_2) \cdots (i_k^{[t_k]} j_k)$$

for some $k$ such that $0 < j_1 < \cdots < j_k$.

**Definition 2.1.** The sorting index of $\pi \in G_{r,n}$ is

$$\text{sort}(\pi) = \sum_{s=1}^k \left(j_s - i_s + \chi(t_s > 0) \cdot (2(i_s - 1) + t_s)\right),$$

where $\chi(A) = 1$ if the statement $A$ is true, or $\chi(A) = 0$ otherwise.

It can be checked that when $r = 1, 2$ our definitions meet the definitions in type A and B in [6].

$\text{sort}(\pi)$ can be computed conveniently on the labeled graph $G^{(r,n)}$ in the way we explain below. The graph $G^{(r,n)}$ has $r \times n$ vertices arranged in an rectangle shape. Two vertices are connected by an edge if they are adjacent and are of the same row or on the leftmost column. For convenience, the columns are indexed by $1, 2, \ldots, n$ and the rows by $0, 1, \ldots, r - 1$.

For $1 \leq i \leq n$ and $0 \leq j \leq r - 1$, the vertex of the $i$-th column and $j$-row is labelled by $\pi(i^t j)$. Simply put, we label vertices of the first row by $\pi$, and the labels of the $j$-th row are obtained by adding a bar on each letter of the $(j - 1)$-th row. See Fig. 2 for the graph $G^{(3,5)}$ and its labelling from $\pi = 24135$.

The sorting process of $\pi$ can be done on $G^{(r,n)}$ in the following way. The goal of the sorting is the identity permutation $(12 \cdots n, (0, 0, \ldots, 0))$, or $12 \cdots n$ for short.
413 5
2 4 1 3 5
2 4 1 3 5

Figure 2. The $G^{(3,5)}$ with the labeling with respect to $24135$.

(i) Find the largest unsorted letter $j$ in $\{1, 2, \ldots, n\}$. Suppose it is at the $i$-th column and the $t$-th row.

(ii) Exchange $j$ with the $j$-th letter of the 0-th row. The distance for $j$ needed to travel along the graph $G^{(r,n)}$ to its new position (namely, row 0, column $j$) is recorded.

(iii) If $i \neq j$, relabel the vertices of the $i$-th and $j$-th columns by fixing the two exchanged letters first and then following the labelling rule above. If $i = j$, relabel the $j$-th column by fixing $j$ first and then following the labelling rule.

(iv) Back to (i).

Then the sorting index $\text{sort}(\pi)$ is the total distances in the above process. For example, the process for sorting $\pi = \bar{2}\bar{4}1\bar{3}\bar{5} \in G^{3,5}$ can be illustrated as

\begin{align*}
\bar{2} & \overset{24135}{\rightarrow} \bar{3} \bar{4} 1 3 5 \\
2 & \overset{24135}{\rightarrow} \bar{3} \bar{4} 1 3 5 \\
2 & \overset{24135}{\rightarrow} \bar{3} \bar{4} 1 3 5 \\
\end{align*}

Therefore, $\pi = (\bar{1}1)(\bar{1}2)(\bar{2}3)(\bar{2}4)(\bar{5}5)$ and $\text{sort}(\pi) = 2 + 3 + 1 + 5 + 10 = 21$.

It is easy to see that to apply (i),(ii),(iii) once is equivalent to multiply the permutation by the transposition $(i[t] \; j)$.

2.4. Sorting index v.s. length. There is a simple way to generate elements of $G_{r,n}$ recursively. The idea is to append the letter $n$ to the end of $\pi = \pi_1 \cdots \pi_{n-1} \in G_{r,n-1}$, then pick a letter $i$ and a color $t$ arbitrary and apply the transposition $(i[t] \; n)$. More formally, we define elements $\Phi_i$ of the group algebra of $G_{r,n}$ by $\Phi_1 := 1 + (\bar{1}1) + \cdots + (1^{[r-1]} 1)$ and for $2 \leq j \leq n$

$$\Phi_j := 1 + \sum_{i=1}^{j-1} (i \; j) + \sum_{l=1}^{r-1} \sum_{i=1}^{j} (i[t] \; j).$$

Lemma 2.2. We have

$$\Phi_1 \Phi_2 \cdots \Phi_n = \sum_{\pi \in G_{r,n}} \pi.$$

Proof. Obviously the formula holds for $n = 1$. Suppose by induction that $\Phi_1 \Phi_2 \cdots \Phi_{n-1} = \sum_{\pi \in G_{r,n-1}} \pi$. Observe that $\Phi_n$ is 1 plus the sum of all transpositions in $T$ involving the
letter $n$. Thus, for $\pi = \pi_1 \cdots \pi_{n-1} n \in G_{r,n}$ in the window notation we have

$$\pi \cdot \Phi_n = \pi_1 \cdots \pi_{n-1} n + \pi_1 \cdots n \pi_n \cdots + n \pi_2 \cdots \pi_{n-1} \pi_1 + \pi_1 \cdots \pi_{n-1} n^{[r-1]} + \pi_1 \cdots n^{[r-1]} \pi_n \cdots + n^{[r-1]} \pi_2 \cdots \pi_{n-1} \pi_1 + \cdots + \pi_1 \cdots \pi_{n-1} n^{[1]} + \pi_1 \cdots n^{[1]} \pi_n \cdots + n^{[1]} \pi_2 \cdots \pi_{n-1} \pi_1^{[r-1]}.$$

It is clear now that $\pi \cdot \Phi_n = \pi' \cdot \Phi_n$ iff $\pi = \pi'$ and hence we have

$$\Phi_1 \Phi_2 \cdots \Phi_{n-1} \Phi_n = \sum_{\pi \in G_{r,n}, \pi(n) = n} \pi \cdot \Phi_n = \sum_{\pi \in G_{r,n}} \pi.$$ 

□

Now we prove the main result of this section.

**Theorem 2.3.** The statistics $\ell$ and $\text{sor}$ have the same distribution over $G_{r,n}$. That is,

$$\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} = \sum_{\pi \in G_{r,n}} q^{\text{sor}(\pi)} = [n]_q! \prod_{i=1}^{n} (1 + q^{i}[r-1]_q).$$

**Proof.** We proceed by induction. Define the linear mapping $\Phi : \mathbb{Z}(G_{r,n}) \to \mathbb{Z}(q)$ by

$$\Phi(\pi) := q^{\text{sor}(\pi)}.$$

It is obvious that

$$\Phi(\Phi_i) = (1 + q + \cdots + q^{-1}) + (q^i + q^{i+1} + \cdots + q^{2i-1}) + (q^{i+1} + q^{i+2} + \cdots + q^{2i}) + \cdots + (q^{i+r-2} + q^{i+r-1} + \cdots + q^{2i+r-3}) + (1 + q^{[r-1]_q} [i]_q).$$

Thus, by Lemma 2.2, it suffices to show that

$$\Phi(\Phi_1 \cdots \Phi_{n-1}) \Phi(\Phi_n) = \Phi(\Phi_1 \cdots \Phi_{n-1} \Phi_n).$$

Following the proof of Lemma 2.2, let $\pi = \pi_1 \cdots \pi_{n-1} n \in G_{r,n}$ be in the window notation. Since $\pi(n) = n$, by the definition of $\text{sor}$ we have $\text{sor}(\pi \cdot (i n)) = \text{sor}(\pi) + (n-i)$ for $1 \leq i \leq n,$
and \(\text{sor}(\pi \cdot (i^t) n) = \text{sor}(\pi) + (n + t - i - 2)\) for \(1 \leq i \leq n\) and \(1 \leq t < r\). This implies that

\[
\Phi(\pi \cdot \Phi_n) = \Phi(\pi) \left( 1 + \sum_{i=1}^{n-1} q^{n-i} + \sum_{t=1}^{r-1} \sum_{i=1}^{n} q^{n+t+i-2} \right)
= \Phi(\pi) \left( \sum_{i=0}^{n-1} q^i + q^n \sum_{t=1}^{r-1} \sum_{i=1}^{n} q^{i-1} \right)
= \Phi(\pi) \left( 1 + q^n[r-1]_q \right)[n]_q
= \Phi(\pi) \Phi(\Phi_n).
\]

Thus,

\[
\Phi(\Phi_1 \cdots \Phi_{n-1} \Phi_n) = \Phi \left( \sum_{\pi \in G_{r,n}, \pi(n)=n} \pi \cdot \Phi_n \right)
= \sum_{\pi \in G_{r,n}, \pi(n)=n} \Phi(\pi \cdot \Phi_n)
= \Phi(\Phi_n) \sum_{\pi \in G_{r,n}, \pi(n)=n} \Phi(\pi)
= \Phi(\Phi_n) \Phi(\Phi_1 \cdots \Phi_{n-1}),
\]

as desired.

\[\square\]

2.5. **Set-valued Stirling statistics.** We can also represent a colored permutation \(\pi = \sigma_1^{z_1} \sigma_2^{z_2} \cdots \sigma_n^{z_n} \in G_{r,n}\) as a product of disjoint colored cycles. This is done by write \(\sigma_1 \sigma_2 \cdots \sigma_n\) (taken as a permutation in \(S_n\)) in its cycle decomposition and add back bars of each letters. For example, in \(G_{3,9}\), we have

\[
563142798 = (154)(26)(3)(7)(89).
\]

We note that this is not the same with the cycle decomposition from the two-line notation.

Assume that \(C_1, C_2, \ldots, C_k\) are the disjoint colored cycles of \(\pi\). For \(C_i = (\sigma_{i_1}^{[z_{i_1}]}) \sigma_{i_2}^{[z_{i_2}]} \cdots\), let \(a_i\) be the smallest integer in \(\{\sigma_{i_1}, \sigma_{i_2}, \ldots\\}\) and \(c_i\) be the remainder of \(\sum z_{i_j}\) divided by \(r\). We define the colored cycle set of \(\pi\) by

\[
\text{Cyc}(\pi) := \{\alpha_1^{[c_1]}, \ldots, \alpha_k^{[c_k]}\},
\]

and for each \(t = 0, 1, \ldots, r-1\) the refined colored cycle set by

\[
\text{Cyc}^t(\pi) := \{\alpha_i^{[c_1]} : \alpha_i^{[c_1]} \in \text{Cyc}(\pi) \text{ and } c_i = t\}.
\]

Also we denote by \(\text{cyc}\) and \(\text{cyc}^t\), \(0 \leq t \leq r-1\), the cardinalities of \(\text{Cyc}\) and \(\text{Cyc}^t\) respectively.

In the above example, \(\pi = 563142798 \in G_{3,9}\) has \(\text{Cyc}(\pi) = \{1, 2, 3, 7, 8\}\), \(\text{Cyc}^0(\pi) = \{7\}\), \(\text{Cyc}^1(\pi) = \{2, 3\}\) and \(\text{Cyc}^2(\pi) = \{1, 8\}\).

In Corollary [1.3] we will prove that

\[
\text{cyc}^0(\pi) = n - t'(\pi),
\]

(2.5)
and in Corollary 5.4 the distribution of $\text{cyc}^0$ will be derived as

$$\sum_{\pi \in G_{r,n}} t^{\text{cyc}^0(\pi)} = \prod_{i=1}^{n} (t + ri - 1),$$

which are counterparts of (1.3) and (1.2). We may call a statistic colored Stirling if it is equidistributed with $\text{cyc}^0$ over $G_{r,n}$.

Now we define the following set-valued statistics for $\pi = \sigma_{\pi_1}^{[z_1]} \sigma_{\pi_2}^{[z_2]} \cdots \sigma_{\pi_n}^{[z_n]} \in G_{r,n}$:

1. $R_{\text{mil}}$, the set of right-to-left minimum letters:
   $$R_{\text{mil}}(\pi) := \{ \sigma_i^{[z_i]} : \sigma_i < \sigma_j \text{ for any } j > i \}.$$

2. $R_{\text{mip}}$, the set of right-to-left minimum places:
   $$R_{\text{mip}}(\pi) := \{ i^{[z_i]} : \sigma_i < \sigma_j \text{ for any } j > i \}.$$

3. $L_{\text{mal}}$, the set of left-to-right maximum letters:
   $$L_{\text{mal}}(\pi) := \{ \sigma_i^{[z_i]} : \sigma_i > \sigma_j \text{ for any } j < i \}.$$

4. $L_{\text{map}}$, the set of left-to-right maximum places:
   $$L_{\text{map}}(\pi) := \{ i^{[z_i]} : \sigma_i > \sigma_j \text{ for any } j < i \}.$$

5. $L_{\text{mil}}$, the set of left-to-right minimum letters:
   $$L_{\text{mil}}(\pi) := \{ \sigma_i^{[z_i]} : \sigma_i < \sigma_j \text{ for any } j < i \}.$$

6. $L_{\text{mic}}$, the set of left-to-right minimum letters in the first cycle:
   $$L_{\text{mic}}(\pi) := L_{\text{mil}}(\pi_1(\pi) \pi_2(\pi) \cdots).$$

Let $r_{\text{min}}, l_{\text{min}}, l_{\text{max}}$ and $l_{\text{mic}}$ denote the cardinalities of $R_{\text{mil}}, L_{\text{mil}}, L_{\text{map}}$ and $L_{\text{mic}}$ respectively. For each $t = 0, 1, \ldots, r - 1$ the statistics $R_{\text{mil}}^t, L_{\text{mil}}^t, L_{\text{map}}^t, L_{\text{mic}}^t$ and $r_{\text{min}}^t, l_{\text{min}}^t, l_{\text{max}}^t, l_{\text{mic}}^t$ are defined similarly. In Corollary 4.3 we will prove that $\text{cyc}^0, r_{\text{min}}^0, l_{\text{min}}^0, l_{\text{max}}^0$ and $l_{\text{mic}}^0$ are colored Stirling and $\text{cyc}, r_{\text{min}}, l_{\text{min}}, l_{\text{max}}$ and $l_{\text{mic}}$ are equidistributed over $G_{r,n}$.

For example, Table 3 lists these set-valued statistics for $\pi = 563142798 \in G_{3,9}$. Note that $\pi(1) \pi^2(1) \pi^3(1) \cdots = 541541541$.

| (Stat) | Cyc | Rmil | Rmip | Lmal | Lmap | Lmil | Lmic |
|--------|-----|------|------|------|------|------|------|
| Stat$(\pi)$ | 1, 2, 3, 7, 8 | 1, 2, 7, 8 | 4, 6, 7, 9 | 5, 6, 7, 8 | 1, 2, 7, 9 | 1, 3, 5 | 1, 4, 5 |
| Stat$(\pi)$ | 7 | 7 | 7 | 7 | 7 | 0 | 0 |
| Stat$(\pi)$ | 2, 3 | 1 | 4 | 5 | 1 | 1, 3, 5 | 4, 5 |
| Stat$(\pi)$ | 1, 8 | 2, 8 | 6, 9 | 6, 8 | 2, 9 | 0 | 1 |

Table 3. Set-valued statistics for $\pi = 563142798$. 
3. A bijection on $G_{r,n}$

In this section we establish a bijection $\phi$ on $G_{r,n}$, which is the key ingredient for proving the Main Theorem A. It turns out that $\phi$ is the composition

$$\phi := (\text{B-code})^{-1} \circ (\text{A-code})$$

of the A-code and B-code defined below. Note that $\phi$ is a generalization of the bijection first defined by Foata and Han [4] and extended by Chen-Guo-Gong [3] and Poznanović [7].

3.1. The A-code. For $\pi = \sigma_1^{[z_1]} \sigma_2^{[z_2]} \cdots \sigma_n^{[z_n]} \in G_{r,n}$ define its Lehmer code to be the sequence

$$\text{Leh}(\pi) = (h_1^{[-z_1]}, h_2^{[-z_2]}, \ldots, h_n^{[-z_n]}),$$

where $-z_i$ is taken modulo $r$ and for each $i$

$$h_i := |\{j : 1 \leq j \leq i \text{ and } \sigma_j \leq \sigma_i\}|.$$

And the A-code of $\pi$ is then defined as

$$\text{A-code}(\pi) := \text{Leh}(\pi^{-1}).$$

For example, for $\pi = \bar{5}\bar{6}\bar{3}1\bar{4}2\bar{7}9\bar{8} \in G_{3,9}$, we have $\pi^{-1} = \bar{4}\bar{6}\bar{3}\bar{5}1\bar{2}79\bar{8}$ and A-code($\pi$) = $(\bar{1}, \bar{2}, \bar{1}, 3, \bar{1}, 2, 7, 8, 9)$. It is obvious that the A-code is a bijection from $G_{r,n}$ to the set

$$\text{CS}_{r,n} := \{(c_1^{[e_1]}, c_2^{[e_2]}, \ldots, c_n^{[e_n]}) : 1 \leq c_i \leq i, 0 \leq e_i < r \text{ for each } 1 \leq i \leq n\}.$$

We can compute A-code by the following algorithm.

Algorithm for A-code. For $\pi$ we construct a sequence of $n$ colored permutations $\pi = \pi^{(n)}, \pi^{(n-1)}, \ldots, \pi^{(1)}$ such that $\pi^{(i)} \in G_{r,i}$ and meanwhile build up the A-code $(a_1, a_2, \ldots, a_n)$, where $a_i$ is of the form $c_i^{[e_i]}$. For $i$ from $n$ down to 2 we look at $i^{[l]}$. If $i^{[l]}$ appears at the $p$-th position in $\pi^{(i)}$, then we set $a_i = p^{[l]}$ and let $\pi^{(i-1)}$ be obtained from $\pi^{(i)}$ by deleting the element $i^{[l]}$. Finally we let $a_1 := \pi^{(1)}(1)$. It is easy to see the procedure is reversible.

For instance, for $\pi = \bar{5}\bar{6}\bar{3}1\bar{4}2\bar{7}9\bar{8} \in G_{3,9}$ we successively get

- $\pi^{(9)} = \bar{5}\bar{6}\bar{3}1\bar{4}2\bar{7}9\bar{8}$, $p = 8$, $t = 0$, $a_9 = 9$
- $\pi^{(8)} = \bar{5}\bar{6}\bar{3}1\bar{4}2\bar{7}\bar{8}$, $p = 8$, $t = 2$, $a_8 = \bar{8}$
- $\pi^{(7)} = \bar{5}\bar{6}\bar{3}1\bar{4}2\bar{7}$, $p = 7$, $t = 0$, $a_7 = \bar{7}$
- $\pi^{(6)} = \bar{5}\bar{6}\bar{3}1\bar{4}2$, $p = 2$, $t = 2$, $a_6 = \bar{2}$
- $\pi^{(5)} = \bar{5}\bar{3}1\bar{4}2$, $p = 1$, $t = 1$, $a_5 = \bar{1}$
- $\pi^{(4)} = \bar{3}\bar{1}\bar{4}\bar{2}$, $p = 3$, $t = 0$, $a_4 = \bar{3}$
- $\pi^{(3)} = \bar{3}\bar{1}\bar{2}$, $p = 1$, $t = 1$, $a_3 = \bar{1}$
- $\pi^{(2)} = \bar{1}\bar{2}$, $p = 2$, $t = 2$, $a_2 = \bar{2}$
- $\pi^{(1)} = \bar{1}$, $p = 1$, $t = 1$, $a_1 = \bar{1}$

Thus A-code($\pi$) = $(\bar{1}, \bar{2}, \bar{1}, 3, \bar{1}, \bar{2}, 7, 8, 9)$. We can read the length from A-code.
Lemma 3.1. Suppose A-code($\pi$) = ($c_1^{[e_1]}$, $c_2^{[e_2]}$, $\ldots$, $c_n^{[e_n]}$). Then we have

$$\ell(\pi) = \sum_{i=1}^{n} \left( i - c_i + \chi(e_i > 0) \cdot (2c_i - 1 + e_i) \right).$$

Proof. Consider the procedure of recovering the colored permutation from its A-code. At the $i$-th step we insert the entry $i^{[e_i]}$ into the $c_i$-th position of $\pi^{(i-1)}$. From the definition of $\ell$, it can be seen that after the $i$-th step the length function increases by $i - c_i$ when $e_i = 0$ and by $c_i - 1 + (i + e_i - 1)$ when $e_i > 0$. Hence we have

$$\ell(\pi^{(i)}) - \ell(\pi^{(i-1)}) = i - c_i + \chi(e_i > 0) \cdot (2c_i - 2 + e_i).$$

Since $\ell(\pi^{(0)}) = 0$, the result follows. \qed

For $a = (c_1^{[e_1]}, c_2^{[e_2]}, \ldots, c_n^{[e_n]}) \in CS_{r,n}$, we define the set-valued statistics

$$\text{Max}(a) := \{i^{[e_i]} : c_i = i\}, \quad \text{Min}(a) := \{i^{[e_i]} : c_i = 1\},$$

and their refined versions

$$\text{Max}'(a) := \{i : c_i = i, e_i = t\}, \quad \text{Min}'(a) := \{i : c_i = 1, e_i = t\}$$

for each $0 \leq t \leq r - 1$. $\text{Rmil}(a)$, $\text{Rmil}'(a)$, $\text{Rmip}(a)$ and $\text{Rmip}'(a)$ are defined similarly by regarding $a$ as a word.

Lemma 3.2. Let $\pi \in G_{r,n}$ and $a = A$-code($\pi$). Then for each $0 \leq t \leq r - 1$ we have

1. $\text{Rmil}(\pi) = \text{Max}(a)$ and $\text{Rmil}'(\pi) = \text{Max}'(a)$,
2. $\text{Lmil}(\pi) = \text{Min}(a)$ and $\text{Lmil}'(\pi) = \text{Min}'(a)$,
3. $\text{Lmap}(\pi) = \text{Rmil}(a)$ and $\text{Lmap}'(\pi) = \text{Rmil}'(a)$,
4. $\text{Lmal}(\pi) = \text{Rmip}(a)$ and $\text{Lmal}'(\pi) = \text{Rmip}'(a)$.

Proof. Let $\pi = (\sigma, z) = \sigma_1^{[z_1]} \cdots \sigma_n^{[z_n]}$ and $a = (c_1^{[e_1]}, c_2^{[e_2]}, \ldots, c_n^{[e_n]})$.

1. Following the algorithmic construction of A-code, $\sigma_i^{[z_i]}$ is a right-to-left minimum letter iff it is at the last position in $\pi^{(\sigma_i)}$, which implies $c_{\sigma_i} = \sigma_i$ and $e_{\sigma_i} = z_i$. Hence $\text{Rmil}(\pi) = \text{Max}(a)$.
2. Similar to (1), now $\sigma_i^{[z_i]}$ is a left-to-right minimum letter iff it is at the first position in $\pi^{(\sigma_i)}$, which implies $c_{\sigma_i} = 1$ and $e_{\sigma_i} = z_i$. Hence $\text{Lmil}(\pi) = \text{Min}(a)$.
3. Suppose $\pi^{-1} = (\rho, w)$. Since $(\sigma \rho, w + \rho(z)) = \pi \circ \sigma^{-1} = (12 \cdots n, (0, 0, \ldots, 0))$, we obtain $\rho = \sigma^{-1}$ and $w_i = -z_{\rho(i)}$ for all $i$. In other words, $\pi^{-1} = (\sigma^{-1}, -\sigma^{-1}(z))$, where

$$-\sigma^{-1}(z) = (-z_{\sigma^{-1}(1)}, -z_{\sigma^{-1}(2)}, \ldots, -z_{\sigma^{-1}(n)}).$$

Since $\text{Lmap}(\sigma) = \text{Rmil}(\sigma^{-1})$, this implies that $i^{[z_i]}$ is an element of $\text{Lmap}(\pi)$ iff $i^{[t]}$ is an element of $\text{Rmil}(\pi^{-1})$ for some $t$, hence by (3.1) we have $t = -z_{\sigma^{-1}(i)} = -z_i$. Therefore,

$$i^{[z_i]} \in \text{Lmap}(\pi) \iff i^{[-z_i]} \in \text{Rmil}(\pi^{-1}). \quad (3.2)$$

On the other hand by the definition of Lehmer code it is obvious that

$$\sigma_i^{[z_i]} \in \text{Rmil}(\pi) \iff \sigma_i^{[-z_i]} \in \text{Rmil}(\text{Leh}(\pi)). \quad (3.3)$$
Combining (3.2) and (3.3) we obtain

\[ i[z] \in \text{Lmap}(\pi) \text{ iff } j[z] \in \text{Rmil}(\text{Leh}(\pi^{-1})) = \text{Rmil}(a), \]

hence \( \text{Lmap}(\pi) = \text{Rmil}(a) \).

(4) Similar to the proof of (3). \( \square \)

3.2. The B-code. The B-code of \( \pi \in \mathcal{G}_{r,n} \) is defined in the following way. For \( i = 1, 2, \ldots, n \) let \( k_i \) be the smallest integer \( k \geq 1 \) such that the base value of \( \pi^{-k}(i) \) is less than or equal to \( i \). Then we define

\[ \text{B-code}(\pi) := (b_1, b_2, \ldots, b_n) \text{ with } b_i = \pi^{-k_i}(i). \]

It is not difficult to see that B-code is a bijection from \( \mathcal{G}_{r,n} \) to \( \mathcal{CS}_{r,n} \). A simple way to compute \( \text{B-code}(\pi) \) is from its cycle representation. For example, the cycle representation of \( \pi = 563142798 \in \mathcal{G}_{3,9} \) is

\[ (1 \, 5 \, 4 \, 1 \, 5 \, 4 \, 1 \, 5 \, 4)(2 \, 6 \, 2 \, 6 \, 2 \, 6)(3 \, 3 \, 3 \, 3)(7 \, 7 \, 7)(8 \, 9 \, 8 \, 9 \, 8 \, 9) \]

and \( \text{B-code}(\pi) = (1, 2, 3, 4, 5, 6, 7, 8, 9) \).

We can also compute B-code by the following algorithm.

**Algorithm for B-code.** For \( \pi \in \mathcal{G}_{r,n} \), we construct a sequence of colored permutations \( \pi = \pi^{(n)}, \pi^{(n-1)}, \ldots, \pi^{(1)} \) such that \( \pi^{(i)} \in \mathcal{G}_{r,i} \) and meanwhile build up the B-code \( (b_1, b_2, \ldots, b_n) \).

For \( i \) from \( n \) down to 2 we assume that \( \pi^{(i)}(p^{(i)}) = i \) for some \( p \) and \( t \). Set \( b_t = p^{(t)} \) and \( \pi' = \pi^{(i)} \cdot (p^{(t)} \, i) \), the product of \( \pi^{(i)} \) and the transposition \( (p^{(t)} \, i) \). Let \( \pi^{(i-1)} \) be obtained from \( \pi' \) by deleting the last term, which must be \( i \). Finally we set \( b_1 := 1^{(t)}, \) where \( \pi^{(1)}(1^{(t)}) = 1 \) for some \( t \).

In the above example \( \pi = 563142798 \in \mathcal{G}_{3,9} \) and we successively get

\[
\begin{align*}
\pi^{(9)} &= 563142798, & p = 8, & t = 0, & b_9 = 8; \\
\pi^{(8)} &= 56314278, & p = 8, & t = 1, & b_8 = 8; \\
\pi^{(7)} &= 56314278, & p = 7, & t = 0, & b_7 = 7; \\
\pi^{(6)} &= 56314278, & p = 2, & t = 1, & b_6 = 2; \\
\pi^{(5)} &= 52314, & p = 1, & t = 2, & b_5 = 1; \\
\pi^{(4)} &= 4231, & p = 1, & t = 2, & b_4 = 1; \\
\pi^{(3)} &= 123, & p = 3, & t = 2, & b_3 = 3; \\
\pi^{(2)} &= 12, & p = 2, & t = 2, & b_2 = 2; \\
\pi^{(1)} &= 1, & p = 1, & t = 1, & b_1 = 1.
\end{align*}
\]

We can see from the algorithm that the choice of \( p^{(t)} \) satisfies that \( i^{[-t]} \) is the \( p \)-th position in \( \pi^{(i)} \). Furthermore, it also can be seen that \( \pi \) can be written as the product \( \prod_{i=1}^{n} (b_i \, i) \), which is a key step for computing \( \text{sort}(\pi) \).

**Lemma 3.3.** Suppose that \( b = \text{B-code}(\pi) = (c_1^{[e_1]}, c_2^{[e_2]}, \ldots, c_n^{[e_n]}) \). Then we have

\[
\text{sort}(\pi) = \sum_{i=1}^{n} \left( i - c_i + \chi(e_i > 0) \cdot (2(c_i - 1) + e_i) \right) \quad (3.4)
\]
and
\[ \ell'(\pi) = n - |\text{Max}^0(b)|, \] (3.5)

Proof. The equality (3.4) can be directly obtained from (2.4) and we only prove (3.5).

Since \( \pi = \prod_{i=1}^n(b_i) = \prod_{b_i \neq 1} b_i \), we have \( \ell'(\pi) \leq n - |\text{Max}^0(b)| \). We use induction on \( n \) to prove the equality. If \( \pi_n = n \), the assertion is true by taking \( \pi \) as an element in \( G_{c,n-1} \).

Assume \( \pi_n \neq n \). Let \( \pi(p^{[t]}) = n \) for some \( p \) and \( t \) with \( (p, t) \neq (n, 0) \). Let \( \pi' = \pi(p^{[t]}) \) and then \( \ell'(\pi) = \ell'(\pi') + 1 \). As \( \pi' \) fixes \( n \), we can regard it as an element in \( G_{r,n-1} \) and by the algorithmic definition of B-code we have \( b' := \text{B-code}(\pi') = (c_1^{[t]}, \ldots, c_{n-1}^{[t]}) \). Therefore by the induction hypothesis we have \( \ell'(\pi') = (n - 1) - |\text{Max}^0(b')| = |\text{Max}^0(b)| \), which completes the proof. \( \square \)

Lemma 3.4. Let \( \pi \in G_{r,n} \) and \( b = \text{B-code}(\pi) \). Then for each \( 1 \leq t \leq r - 1 \) we have

1. \( \text{Cyc}^0(\pi) = \text{Max}^0(b) \) and \( \text{Cyc}^r(\pi) = \text{Max}^{-t}(b) \),
2. \( \text{Lmic}^0(\pi) = \text{Min}^0(b) \) and \( \text{Lmic}^r(\pi) = \text{Min}^{-t}(b) \),
3. \( \text{Lmap}^0(\pi) = \text{Rmil}^0(b) \) and \( \text{Lmap}^r(\pi) = \text{Rmil}^{-t}(b) \),
4. \( \text{Lma}^0(\pi) = \text{Rmip}^0(b) \) and \( \text{Lma}^r(\pi) = \text{Rmip}^{-t}(b) \).

Proof. The lemma holds for \( n = 1 \). Assume by induction that the lemma holds for \( n - 1 \), where \( n \geq 2 \).

Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) and \( b = (b_1, b_2, \ldots, b_n) \). By the algorithmic definition of the B-code, there is a colored permutation \( \pi' \in G_{r,n-1} \) such that \( b' := \text{B-code}(\pi') = (b_1, b_2, \ldots, b_{n-1}) \). Let \( \pi' = C_1 C_2 \cdots C_k \) be its colored cycle decomposition. In the following we prove the lemma in two cases according to the position of \( n^{[t]} \) in \( \pi \).

Case 1. \( \pi_n = n^{[t]} \) for some \( 0 \leq t < r \). In this case \( b = (b_1, b_2, \ldots, b_{n-1}, n^{[t]}) \), \( \pi' = \pi_1 \pi_2 \cdots \pi_{n-1} \) and the colored cycle decomposition of \( \pi \) is \( C_1 C_2 \cdots C_k(n^{[t]}) \). It is easy to see that
\[
\text{Cyc}(\pi) = \text{Cyc}(\pi') \cup \{n^{[t]}\} \quad \text{and} \quad \text{Max}(b) = \text{Max}(b') \cup \{n^{[t]}\},
\]
\[
\text{Lmic}(\pi) = \text{Lmic}(\pi') \quad \text{and} \quad \text{Min}(b) = \text{Min}(b'),
\]
\[
\text{Lmap}(\pi) = \text{Lmap}(\pi') \cup \{n^{[t]}\} \quad \text{and} \quad \text{Rmil}(b) = \text{Rmil}(b') \cup \{n^{[t]}\},
\]
\[
\text{Lma}(\pi) = \text{Lma}(\pi') \cup \{n^{[t]}\} \quad \text{and} \quad \text{Rmip}(b) = \text{Rmip}(b') \cup \{n^{[t]}\},
\]
and the result follows by induction.

Case 2. \( \pi_p = n^{[t]} \) for some \( 1 \leq p < n \) and \( 0 \leq t < r \). By the definition of wreath product, \( \pi(p^{[t]}) = n^{[t+p]} \) for any \( i \). Then we have \( \pi(p^{[t]}) = n \) and thus \( \pi' = \pi \cdot (p^{[t]} \cdot n) \). Therefore, \( b = (b_1, b_2, \ldots, b_{n-1}, p^{[t]} \cdot n) \) and hence
\[
\text{Max}(b) = \text{Max}(b'),
\]
\[
\text{Min}(b) = \text{Min}(b') \cup \{n^{[t]}\} \quad \text{if} \ p = 1, \quad \text{and} \ \text{Min}(b) = \text{Min}(b') \quad \text{otherwise},
\]
\[
\text{Rmil}(b) = \{i^{[j]} \in \text{Rmil}(b') : i < p\} \cup \{p^{[t]} \cdot n\},
\]
\[
\text{Rmip}(b) = \{i^{[j]} \in \text{Rmip}(b') : i < n\} \cup \{n^{[t]}\}.
\]
Now it suffices to show that
\[
\text{(i) Cyc}(\pi) = \text{Cyc}(\pi'),
\]
\[
\text{(ii) Lmic}(\pi) = \text{Lmic}(\pi') \cup \{n^{[t]}\} \quad \text{if} \ p = 1, \quad \text{and} \ \text{Lmic}(\pi) = \text{Lmic}(\pi') \quad \text{otherwise},
\]
\[
\text{(iii) Lmap}(\pi) = \{i^{[j]} \in \text{Lmap}(\pi') : i < p\} \cup \{p^{[t]}\}.
\]
(iv) \( \text{LMal}(\pi) = \{i[j] \in \text{LMal}(\pi') : i < n\} \cup \{n[t]\} \).

Assume that \( \pi_n = m^{[s]} \) for some \( 1 \leq m < n \) and \( 0 \leq s < r \). From \( \pi' = \pi \cdot (p^{-t} \cdot n) \) we obtain
\[
\pi = \pi_1 \cdots \pi_{p-1} m^{[t]} \pi_{p+1} \cdots \pi_{n-1} m^{[s]} = \\
\pi' = \pi_1 \cdots \pi_{p-1} m^{[s+t]} \pi_{p+1} \cdots \pi_{n-1} m^{[s]} \tag{3.6}
\]

The cases (iii) and (iv) can be obtained directly from (3.6). So we only consider (i) and (ii) in what follows.

(i) Assume that in the colored cycle decomposition of \( \pi' \), \( C_h \) is the cycle containing the letter \( p \), namely, \( C_h = (\cdots p^{[t]} m^{[s+t]} \cdots) \) for some color \( t' \). By (3.6), the colored cycle decomposition of \( \pi \) must be \( C_1 \cdots C_{h-1} \bar{C}_h C_{h+1} \cdots C_k \) with \( \bar{C}_h = (\cdots p^{[t]} m^{[s]} \cdots) \). Notice that the insertion of letter \( n \) does not affect the choice of the smallest letter of \( \bar{C}_h \).

Hence \( \text{Cyc}(\pi) = \text{Cyc}(\pi') \), as desired.

(ii) We denote by \( g_1 g_2 g_3 \cdots \cdot \) the word \( \pi'(1) \pi(2) \pi(3) \cdots \). When \( p = 1 \), it is easy to see that \( g_1 = m^{[s+t]} \) and the word \( \pi(1) \pi(2) \pi(4) \cdots \) will be \( n^{[t]} m^{[s+t]} \) \( g_2 g_3 \cdots \). Then we have \( \text{LMic}(\pi) = \text{LMic}(\pi') \cup \{n^{[t]}\} \). When \( p > 1 \), the word \( \pi(1) \pi(2) \pi(4) \cdots \) will be the same as \( g_1 g_2 g_3 \cdots \), or be obtained from it by inserting letters \( n, \bar{n}, \ldots \) into some places after \( g_1 \). Therefore, \( \text{LMic}(\pi) = \text{LMic}(\pi') \) in this case. This completes the proof. \( \square \)

Finally, we define
\[
\phi = (B\text{-code})^{-1} \circ (A\text{-code}).
\]

4. Main Theorem A

4.1. Colored permutations. We first prove the Main Theorem A in the case of \( G_{r,n} \).

**Theorem 4.1.** For \( \pi \in G_{r,n} \) we have
\[
(\ell, \text{Rmil}^0, \text{Rmil}^1, \ldots, \text{Rmil}^{r-1}, \text{Lmil}^0, \text{Lmil}^1, \ldots, \text{Lmil}^{r-1}, \\
\text{Lma}^0, \text{Lma}^1, \ldots, \text{Lma}^{r-1}, \text{Lmap}^0, \text{Lmap}^1, \ldots, \text{Lmap}^{r-1}) \pi = \\
(\text{sor}, \text{Cyc}^0, \text{Cyc}^{r-1}, \ldots, \text{Cyc}^1, \text{LMic}^0, \text{LMic}^{r-1}, \ldots, \text{LMic}^1, \\
\text{Lma}^0, \text{Lma}^{r-1}, \ldots, \text{Lma}^1, \text{Lmap}^0, \text{Lmap}^{r-1}, \ldots, \text{Lmap}^1) \phi(\pi).
\]

**Proof.** The proof is done by combining the Lemma 3.1, 3.2, 3.3 and 3.4. \( \square \)

A direct corollary is the following.

**Corollary 4.2.** The pair of joint (set-valued) statistics
\[
(\ell, \text{Rmil}^0, \text{Rmil}^1, \ldots, \text{Rmil}^{r-1}, \text{Lmil}^0, \text{Lmil}^1, \ldots, \text{Lmil}^{r-1}, \\
\text{Lma}^0, \text{Lma}^1, \ldots, \text{Lma}^{r-1}, \text{Lmap}^0, \text{Lmap}^1, \ldots, \text{Lmap}^{r-1})
\]
and
\[
(\text{sor}, \text{Cyc}^0, \text{Cyc}^{r-1}, \ldots, \text{Cyc}^1, \text{LMic}^0, \text{LMic}^{r-1}, \ldots, \text{LMic}^1, \\
\text{Lma}^0, \text{Lma}^{r-1}, \ldots, \text{Lma}^1, \text{Lmap}^0, \text{Lmap}^{r-1}, \ldots, \text{Lmap}^1)
\]
have the same distribution over \( G_{r,n} \).
It is interesting to see that the superindices of each kind of statistics in the first tuple are 0, 1, 2, ..., r − 1 while in the second 0, r − 1, r − 2, ..., 1.

**Corollary 4.3.** The followings hold.

(1) $\text{cyc}^0, \text{rmin}^0, \text{lmin}^0, \text{lmax}^0$ and $\text{lmic}^0$ have the same distribution with $n - \ell'$ over $G_{r,n}$.

(2) $\text{cyc}, \text{rmin}, \text{lmin}, \text{lmax}$ and $\text{lmic}$ are equidistributed over $G_{r,n}$.

**Proof.** (1) Denote by $\text{stat}_1 \sim \text{stat}_2$ the two statistics $\text{stat}_1$ and $\text{stat}_2$ having the same distribution over $G_{r,n}$. By (3.5) and Lemma 3.4(1), we have

$$\ell' (\pi) = n - \text{cyc}^0 (\pi),$$

and therefore $(n - \ell') \sim \text{cyc}^0$.

We know from corollary 4.2 that $\text{cyc}^0 \sim \text{rmin}^0$ and $\text{lmin}^0 \sim \text{lmic}^0$. Also from (3.2) we deduce that $\text{rmin}^0 \sim \text{lmax}^0$. Finally, by defining $\pi' := \pi (n) \cdots \pi (1)$, the reverse of $\pi$, it is obvious that $\text{Lmic} (\pi) = \text{Rmic} (\pi')$ and then we have $\text{lmin}^0 \sim \text{rmin}^0$. This completes the proof.

(2) The proof is similar to that of (1) and is omitted. □

### 4.2. Colored permutations on a Ferrers shape

Let $f = (f_1, f_2, \ldots, f_n)$ be a non-decreasing sequence of integers with $1 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq n$. The set of colored restricted permutations $G_{r,n,f}$ is defined as

$$G_{r,n,f} := \{ \pi = (\sigma, z) \in G_{r,n} : \sigma (i) \leq f_i, 1 \leq i \leq n \}.$$ 

Note that $G_{r,n,f} = G_{r,n}$ when $f = (n, n, \ldots, n)$. We say that $f'$ dominates $f$, denoted by $f \triangleright f'$, if $f_i \leq f_i'$ for all $i$. Note that $G_{r,n,f} \subseteq G_{r,n,f'}$ if $f \triangleright f'$.

For $\pi \in G_{r,n}$ define its minimum sequence $f (\pi)$ by

$$f (\pi) := f$$

such that $\pi \in G_{r,n,f}$ and $f \triangleright f'$ whenever $\pi \in G_{r,n,f'}$.

$f (\pi)$ can be easily obtained from $\text{Lmap} (\sigma)$ and $\text{Lmal} (\sigma)$. Namely, it is the unique non-decreasing integer sequence $f$ such that $\text{Lmap} (\sigma) = \text{Lmap} (f)$ and $\text{Lmal} (\sigma) = \text{Lmal} (f)$ by regarding $f$ as a word $f_1 f_2 \cdots f_n$.

More precisely, let $\text{Lmap} (\sigma) = \{ i_1, i_2, \ldots, i_s \}$ and $\text{Lmal} (\sigma) = \{ \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_s} \}$, where $i_1 < \cdots < i_s$. Then one has $f (\pi) = (f_1, f_2, \ldots, f_n)$ with

$$f_{i_j} = f_{i_j + 1} = \cdots = f_{i_{j+1} - 1} = \sigma_{i_j}$$

for $j = 1, \ldots, s - 1$ and $f_{i_s} = \cdots = f_n = \sigma_{i_s}$.

For example, let $\pi = (\sigma, z) = (361475928, (3, 0, 3, 2, 2, 1, 0, 3, 1)) \in G_{4,0}$. Then $\text{Lmap} (\sigma) = \{1, 2, 5, 7\}$, $\text{Lmal} (\sigma) = \{3, 6, 7, 9\}$ and $f (\pi) = (3, 6, 6, 6, 7, 7, 7, 9, 9, 9)$.

**Lemma 4.4.** Let $\pi \in G_{r,n}$ and $f = f (\pi)$. Then $\phi (\pi) \in G_{r,n,f}$.

**Proof.** Let $\pi = (\sigma, z)$ and $\phi (\pi) = (\sigma', z')$. By Theorem 4.1 and the definition of $\text{Lmap}, \text{Lmal}$, we have

$$\text{Lmap} (\sigma) = \bigcup_{t=0}^{r} \text{Lmap}^t (\pi) = \bigcup_{t=0}^{r} \text{Lmap}^t (\phi (\pi)) = \text{Lmap} (\sigma')$$

and

$$\text{Lmal} (\sigma) = \bigcup_{t=0}^{r} \text{Lmal}^t (\pi) = \bigcup_{t=0}^{r} \text{Lmal}^t (\phi (\pi)) = \text{Lmal} (\sigma').$$

(4.2)
Since \( f \) is determined by \( \text{Lmap} \) and \( \text{Lmal} \), by (4.1) and (4.2) we have \( f(\phi(\sigma)) = f(\sigma) = f \) and the result follows.

Finally we come to our main theorem.

**Theorem 4.5 (Main Theorem A).** Given \( r, n, f \). Then the pair of joint (set-valued) statistics

\[
(\ell, R\text{mil}^0, R\text{mil}^1, \ldots, R\text{mil}^{r-1}, L\text{mil}^0, L\text{mil}^1, \ldots, L\text{mil}^{r-1},
L\text{mal}^0, L\text{mal}^1, \ldots, L\text{mal}^{r-1}, L\text{map}^0, L\text{map}^1, \ldots, L\text{map}^{r-1})
\]

and

\[
(\text{sor}, \text{Cyc}^0, \text{Cyc}^{r-1}, \ldots, \text{Cyc}^1, L\text{mic}^0, L\text{mic}^{r-1}, \ldots, L\text{mic}^1,
L\text{mal}^0, L\text{mal}^{r-1}, \ldots, L\text{mal}^1, L\text{map}^0, L\text{map}^{r-1}, \ldots, L\text{map}^1)
\]

have the same distribution over \( G_{r,n,f} \).

**Proof.** By Theorem 4.1, it suffices to show that if \( \pi \in G_{r,n,f} \), then \( \phi(\pi) \in G_{r,n,f} \). And the proof is done from Lemma 4.4.

5. **Main Theorem B**

Recall that given \( f \) we define \( H(f) := (h_1, h_2, \ldots, h_n) \), where \( h_i \) is the smallest possible index at which the letter \( i \) can appear in a colored permutation \( \sigma \in G_{r,n,f} \). It is routine to verify that

\[
R\text{mil}(H(f)) = \text{Lmap}(f) \quad \text{and} \quad R\text{mip}(H(f)) = \text{Lmal}(f)
\]

by regarding \( f \) and \( H(f) \) as words.

Similar to that of \( G_{r,n} \), all colored restricted permutations of \( G_{r,n,f} \) can be generated recursively. Let \( \Psi_1 := 1 + (11) + \cdots + (11^{r-1}1) \) and for \( j \geq 2 \)

\[
\Psi_j := 1 + \sum_{i=h_j}^{j-1} (i,j) + \sum_{t=1}^{r-1} \sum_{i=h_j}^{j} (i^t,j).
\]

We omit the proof of the following as it is similar to that of Lemma 2.2

**Lemma 5.1.** Given \( r, n, f \). Then we have

\[
\Psi_1\Psi_2\cdots\Psi_n = \sum_{\pi \in G_{r,n,f}} \pi.
\]

Recall the checking function \( \xi_A(\omega) \) defined in the Introduction. Now we arrive at the Main Theorem B.
Theorem 5.2 (Main Theorem B). Given \( r, n, f \). Let \( H(f) = (h_1, \ldots, h_n) \). Then we have

\[
\sum_{\pi \in G_{r,n,f}} q^{\ell(\pi)} \prod_{i=0}^{r-1} \prod_{i \in \text{Rmil}^{-i}(\pi)} x_{t,i} \prod_{i \in \text{Lmic}^{-i}(\pi)} y_{t,i}
\]

\[
= \sum_{\pi \in G_{r,n,f}} q^{\text{sort}(\pi)} \prod_{i=0}^{r-1} \prod_{i \in \text{Cyc}^{i}(\pi)} x_{t,i} \prod_{i \in \text{Lmic}^{i}(\pi)} y_{t,i}
\]

\[
= \prod_{j=1}^{n} \left( x_{0,j} + q + \cdots + q^{j-h_j-1} + \sum_{\pi \in \text{Cyc}(\pi)} x_{r,n,\pi} \right) 
\]

\[
= \prod_{j=1}^{n} \left( x_{0,j} + q + \cdots + q^{j-h_j-1} + \sum_{\pi \in \text{Cyc}(\pi)} x_{r,n,\pi} \right) 
\]

\[
\cdot \left( 1 + \sum_{i=1}^{r-1} \left( x_{r-t,j} q_{j}^{2j+t-2} + q_{j}^{2j+t-3} + \cdots + q_{j}^{j+h_j+t-1} + \sum_{\pi \in \text{Cyc}(\pi)} x_{r-n,\pi} \right) \right) 
\]

**Proof.** We only consider the second equality as the first one is directly from Theorem 4.5.

Let \( F_n(q, x_{t,i}, y_{t,i}; 0 \leq t < r, 1 \leq i \leq n) \) denote the desired generating function (which is clearly a polynomial). Define the linear mapping \( \Psi : \mathbb{Z}(G_{r,n,f}) \to \mathbb{Z}(q, x_{t,i}, y_{t,i}; 0 \leq t < r, 1 \leq i \leq n) \) by

\[
\Psi(\pi) := q^{\text{sort}(\pi)} \prod_{i=0}^{r-1} \prod_{i \in \text{Cyc}(\pi)} x_{t,i} \prod_{i \in \text{Lmic}(\pi)} y_{t,i}.
\]

By Lemma 5.1 it suffices to show that

\[
\Psi(\Psi_1 \Psi_2 \cdots \Psi_n) = F_n(q, x_{t,i}, y_{t,i}; 0 \leq t < r, 1 \leq i \leq n).
\]

We proceed by induction. As \( \text{sort}(1) = 0 \), \( \text{sort}(1^{|f|}) = r - t \) for \( 0 < t < r \), and \( \text{Cyc}(1^{|f|}) = \text{Lmic}(1^{|f|}) = \{1^{|f|}\} \) for \( 0 \leq t < r \), it is easy to see that

\[
\Psi(\Psi_1) = x_{0,1,y_0,1} + x_{1,1,y_1,1}q_{1}^{r-1} + x_{2,1,y_2,1}q_{2}^{r-2} + \cdots + x_{r-1,1,y_{r-1},1}q_{r-1}.
\]

Let \( n \geq 2 \) and suppose that \( \Psi(\Psi_1 \cdots \Psi_{n-1}) = F_{n-1}(q, x_{t,i}, y_{t,i}; 0 \leq t < r, 1 \leq i \leq n - 1) \). Notice that \( G_{r,n-1,f} \) can be identified with the set \( \{\pi \in G_{r,n,f} : \pi_n = n\} \) in the window notation. Given an element \( \pi = \pi_1 \cdots \pi_{n-1}n \) in this set, we have

\[
\pi \cdot \Psi_n = \pi_1 \cdots \pi_{n-1}n + \pi_1 \cdots n \pi_{n-1} + \cdots + \pi_1 \cdots \pi_{h_n-1}n \pi_{h_n+1} \cdots \pi_{n-1} \pi_{h_n} 
\]

\[
+ \pi_1 \cdots \pi_{n-1}n^{[r-1]} + \pi_1 \cdots n^{[r-1]}\pi_{n-1}^{[1]} + \cdots + \pi_1 \cdots \pi_{h_n-1}n^{[r-1]}\pi_{h_n+1} \cdots \pi_{n-1} \pi_{h_n}^{[1]} 
\]

\[
+ \cdots \cdots 
\]

\[
+ \pi_1 \cdots \pi_{n-1}n^{[1]} + \pi_1 \cdots n^{[1]}\pi_{n-1}^{[r-1]} + \cdots + \pi_1 \cdots \pi_{h_n-1}n^{[1]}\pi_{h_n+1} \cdots \pi_{n-1} \pi_{h_n}^{[r-1]}.
\]

Denote by \( \pi' \) any one of the summands above. Without loss of generality, let the letter \( n^{[t]} \), for some \( t \), be at the \( i \)-th position in \( \pi' \). That is, \( \pi' = \pi \cdot (i^{[t]} n) \). Then \( \text{sort}(\pi') = \text{sort}(\pi) + n - i \) if \( t = 0 \) and \( \text{sort}(\pi') = \text{sort}(\pi) + n + i + r - t - 2 \) if \( t > 1 \). Moreover, from the proof of Lemma 3.4
we have
\[ \text{Cyc}(\pi') = \begin{cases} \text{Cyc}(\pi) \cup \{n[t]\} & \text{if } \pi' = \pi_1 \cdots \pi_{n-1} n[t] \text{ for some } t, \\
\text{Cyc}(\pi) & \text{otherwise}; \end{cases} \]
and
\[ \text{Lmic}(\pi') = \begin{cases} \text{Lmic}(\pi) \cup \{n[t]\} & \text{if } \pi' = n[t]_2 \cdots \pi_{n-1} \pi_1^{-t}\text{ for some } t, \\
\text{Lmic}(\pi) & \text{otherwise.} \end{cases} \]

Therefore,
\begin{align*}
\Psi(\pi \cdot \Psi_n) &= \Psi(\pi) \left(x_{0,n} + q + \cdots + q^{n-h_n-1} + \xi_{h_n=1}(y_{0,n})q^{n-h_n}\right) \\
&\quad + \sum_{t=1}^{r-1} \left(x_{r-t,n} q^{2n+t-2} + q^{2n+t-3} + \cdots + q^{n+h_n+t-1} + \xi_{h_n=1}(y_{r-t,n})q^{n+h_n+t-2}\right). \tag{2.3}
\end{align*}

Now it suffices to show that \(\Psi(\Psi_1 \cdots \Psi_{r-1}) \Psi(\Psi_n) = \Psi(\Psi_1 \cdots \Psi_{n-1} \Psi_n)\). This can be done by the same argument in the proof of Theorem 2.3. Hence we are done. \(\square\)

We obtain the following corollary by replacing \(x_{t,i}\) with \(x_t\) and \(y_{t,i}\) with \(y_t\), for each \(0 \leq t < r\) and \(1 \leq i \leq n\).

**Corollary 5.3.** Given \(r, n, f\). Let \(H(f) = (h_1, \ldots, h_n)\). Then we have
\begin{align*}
\sum_{\pi \in G_{r,n,f}} q^{\ell(\pi)} \prod_{t=0}^{r-1} x_t^{\text{min}^{-1}(\pi)} y_t^{\text{min}^{-1}(\pi)} &= \sum_{\pi \in G_{r,n,f}} q^{\text{Cyc}(\pi)} \prod_{t=0}^{r-1} x_t^{\text{Cyc}(\pi)} y_t^{\text{Lmic}(\pi)} \\
&= \prod_{j=1}^{n} \left(x_0 + q + \cdots + q^{j-h_j-1} + \xi_{h_j=1}(y_0)q^{j-h_j}\right) \\
&\quad + \sum_{t=1}^{r-1} \left(x_{r-t} q^{2j+t-2} + q^{2j+t-3} + \cdots + q^{j+h_j+t-1} + \xi_{h_j=1}(y_{r-t})q^{j+h_j+t-2}\right). \\
\end{align*}

In particular,
\begin{align*}
\sum_{\pi \in G_{r,n}} q^{\ell(\pi)} \prod_{t=0}^{r-1} x_t^{\text{min}^{-1}(\pi)} y_t^{\text{min}^{-1}(\pi)} &= \sum_{\pi \in G_{r,n}} q^{\text{Cyc}(\pi)} \prod_{t=0}^{r-1} x_t^{\text{Cyc}(\pi)} y_t^{\text{Lmic}(\pi)} \\
&= \prod_{j=1}^{n} \left(x_0 + y_0 q^{j-1} + \sum_{t=1}^{r-1} q^{j+r-1} (x_t q^{j-1} + y_t) + q[j-2]_q (1 + q[r-1]_q)\right). \\
\end{align*}

Since \(\ell' = n - \text{Cyc}^0\), we have the following.

**Corollary 5.4.** We have
\[ \sum_{\pi \in G_{r,n}} t^{\text{Cyc}(\pi)} = \prod_{i=1}^{n} (t + ri - 1) \quad \text{and} \quad \sum_{\pi \in G_{r,n}} t^{\ell'(\pi)} = \prod_{i=1}^{n} (1 + (ri - 1)t). \]
6. EVEN-SIGNED PERMUTATION GROUP

We turn to the case of even-signed permutation group, defined as the subgroup \( \mathcal{D}_n \) of \( G_{2,n} \) consisting of those signed permutations \( \pi \) with even number of negatives in the window notation \( \pi = \pi_1, \cdots, \pi_n \). Here we adopt the convention \( \bar{i} = -i \).

6.1. Sorting index. It is known that \( \mathcal{D}_n \) is a Coxeter group generated by
\[
S_n^D = \{s_0^D, s_1, \ldots, s_{n-1}\},
\]
where \( s_0^D \) is the transposition \((1 2)\) and \( s_i \) is the transposition \((i i + 1)\) for \( i \geq 1 \). Let \( \ell_D \) be the length function of \( \mathcal{D}_n \) with respect to \( S_n^D \).

\( \mathcal{D}_n \) can also be generated by
\[
T_n^D := \{t_{ij}^D : 1 \leq |i| < j \leq n\} \cup \{t_{ii}^D : 1 < i \leq n\},
\]
where \( t_{ij}^D = (i j) \) for \( 1 \leq |i| < j \leq n \) and \( t_{ii}^D = (1)(i i) \) for \( 1 < i \leq n \). For \( \pi \in \mathcal{D}_n \), let \( \tilde{\ell}_D(\pi) \) be the minimum number of elements in \( T_n^D \) needed to express \( \pi \). Again we note that \( \tilde{\ell}_D(\pi) \) is not the reflection length of \( \pi \) [6].

Any \( \pi \in \mathcal{D}_n \) has a unique factorization in the form
\[
\pi = t_{i_1 j_1}^D t_{i_2 j_2}^D \cdots t_{i_k j_k}^D
\]
with \( 1 < j_1 < j_2 < \cdots < j_k \leq n \). Petersen [6] defined the sorting index
\[
sor_D(\pi) = \sum_{r=1}^{k} (j_r - i_r - 2\chi(i_r < 0)).
\]
and proved that it is Mahonian. For example, \( \pi = 32451 = t_{13}^D t_{34}^D t_{45}^D \) has the sorting index
\[
sor_D(\pi) = (3 - (-1) - 2) + (4 - 3) + (5 - (-4) - 2) = 10.
\]

6.2. Set-valued Stirling statistics. For \( \pi \in \mathcal{D}_n \) we define the (set-valued) statistics
\[
\text{Cyc}_D(\pi), \text{Rmil}_D(\pi), \text{Rmilp}_D(\pi), \text{Lmal}_D(\pi), \text{Lmap}_D(\pi), \text{Lmil}_D(\pi), \text{Lmic}_D(\pi)
\]
and
\[
\text{cyc}_D(\pi), \text{rmin}_D(\pi), \text{lmax}_D(\pi), \text{lmin}_D(\pi), \text{lmic}_D(\pi)
\]
by viewing \( \pi \) as an element in \( G_{2,n} \). The subscript \( D \) is to emphasize that these statistics are considered in \( \mathcal{D}_n \). We also define some new ‘twisted’ statistics:

1. \( \text{Cyc}_D^+ \), the set of twisted balanced cycles:
\[
\text{Cyc}_D^+(\pi) := \text{Cyc}_D(\pi) \cup \{1\}.
\]
2. \( \text{Cyc}_D^- \), the set of twisted unbalanced cycles:
\[
\text{Cyc}_D^-(\pi) := \text{Cyc}_D^+(\pi) \setminus \{1\}.
\]
3. \( \text{Rmil}_D^+ \), the set of twisted positive right-to-left minimum letters:
\[
\text{Rmil}_D^+(\pi) := \text{Rmil}_D(\pi) \cup \{1\}.
\]
4. \( \text{Rmil}_D^- \), the set of twisted negative right-to-left minimum letters:
\[
\text{Rmil}_D^-(\pi) := \text{Rmil}_D(\pi) \setminus \{1\}.
\]
Denote by $\text{cyc}_D^+, \text{cyc}_D^-, \text{rmin}_D^+ , \text{rmin}_D^-$ their cardinalities respectively.

6.3. A bijection. The key ingredient in this section is the bijection $\psi : D_n \to D_n$, introduced by Chen-Guo-Gone [3], which is the composition

$$\psi = (\text{D-code})^{-1} \circ (\text{C-code})$$

of the C-code and the D-code on $D_n$.

Algorithm for C-code. For $\pi \in D_n$ we construct a sequence of $n$ even-signed permutations $\pi = \pi^{(n)}, \pi^{(n-1)}, \ldots, \pi^{(1)}$ such that $\pi^{(i)} \in D_i$ and meanwhile obtain the C-code $(c_1, c_2, \ldots, c_n)$. For $i$ from $n$ down to 2 we consider the letter $i$ or $\bar{i}$ in $\pi^{(i)}$. If $i$ appears at the $p$-th position in $\pi^{(i)}$, then we define $c_i = p$ and let $\pi^{(i-1)}$ be obtained from $\pi^{(i)}$ by deleting the letter $i$. If $\bar{i}$ appears at the $p$-th position in $\pi^{(i)}$, then we first define $c_i = -p$, let $\pi'$ be obtained by deleting $\bar{i}$ and then obtain $\pi^{(i-1)}$ from $\pi'$ by changing the sign of the first letter. Finally, set $c_1 := \pi^{(1)}(1)$. It is easy to see that $\pi^{(1)}$ is always the identity permutation and hence $c_1 = 1$.

For example, let $\pi = 5\overline{2}1\overline{3}4$, then

$$\begin{align*}
\pi^{(5)} &= 5 \overline{2} 1 \overline{3} 4, \quad c_5 = -1, \\
\pi^{(4)} &= 2 \overline{1} \overline{3} 4, \quad c_4 = 4, \\
\pi^{(3)} &= 2 \overline{1} \overline{3}, \quad c_3 = -3, \\
\pi^{(2)} &= 2 \overline{1}, \quad c_2 = -1, \\
\pi^{(1)} &= 1, \quad c_1 = 1,
\end{align*}$$

hence $\text{C-code}(\pi) = (1, -1, -3, 4, -1)$.

Let $\text{SE}^D_n$ be the set of integer sequences $(c_1, c_2, \ldots, c_n)$ such that $c_1 = 1$ and for $i \geq 2, c_i \in [-i, i] \setminus \{0\}$. It is obvious that C-code is a bijection from $D_n$ onto $\text{SE}^D_n$.

Lemma 6.1. For $\pi \in D_n$ we have

1. $(\text{Lmil}_D, \text{Lmap}_D, \text{Lmal}_D)\pi = (\text{Min}, \text{Rmil}_D, \text{Rmip}_D)\text{C-code}(\pi)$.
2. $(\text{Rmil}_D, \text{Rmil}_D)\pi = (\text{Max}^0, \text{Max}^1)\text{C-code}(\pi)$.

Proof. (1) Let $a = \text{A-code}(\pi)$ and $c = \text{C-code}(\pi)$. By definition one has

$$|a_i| = |c_i| \quad \text{for } i = 1, 2, \ldots, n.$$ 

Hence the result follows from Lemma 3.2 (2) - (4).

(2) Observe that the sign of each letter on the right of 1 or $\bar{1}$ will not change during the construction of the C-code. Assume that $\pi_p = 1$ or $\bar{1}$ for some $p$. Then we have $a_j = c_j$ for $j \in \{\pi_i : i > p\}$. Thus, $a_j = c_j$ for $j \in \text{Rmil}_D(\pi)$. Since $c_1$ is always 1, by Lemma 3.2 (1), we have

$$\text{Rmil}_D^l(\pi) = \text{Rmil}_D^l(\pi) \cup \{1\} = \text{Max}^0(a) \cup \{1\} = \text{Max}^0(c)$$

and

$$\text{Rmil}_D(\pi) = \text{Rmil}_D(\pi) \setminus \{1\} = \text{Max}^0(a) \setminus \{1\} = \text{Max}^1(c),$$

hence the lemma is proved. \qed

The D-code is defined as follows.
Algorithm for D-code. For $\pi \in D_n$, we generate a sequence of even-signed permutations $\pi = \pi^{(n)}$, $\pi^{(n-1)}$, ..., $\pi^{(1)}$ such that $\pi^{(i)} \in D_i$ and meanwhile construct the D-code $(d_1, d_2, ..., d_n)$. For $i$ from $n$ down to 2 we consider the letter $i$ or $i$ in $\pi^{(i)}$. If $i$ appears at the $p$-th position, we set $d_i = p$ and let $\pi^{(i-1)}$ be the first $i-1$ terms of $\pi^{(i)}t^D_{\rho_i}$. If $i$ appears at the $p$-th position, then we set $d_i = -i$ and let $\sigma^{(i-1)}$ be the first $i-1$ terms of $\pi^{(i)}t^D_{\rho_i}$. It can be seen that $\pi^{(1)}$ is always the identity 1 and hence $d_1 = 1$.

For example, if $\pi = 51342$, then we have

$$
\begin{align*}
\pi^{(5)} &= 51342, \quad d_5 = -1 \\
\pi^{(4)} &= 2134, \quad d_4 = 4 \\
\pi^{(3)} &= 213, \quad d_3 = -3 \\
\pi^{(2)} &= 21, \quad d_2 = -1 \\
\pi^{(1)} &= 1, \quad d_1 = 1
\end{align*}
$$

and thus $\text{D-code}(\pi) = (1, -1, -3, 4, -1)$.

Lemma 6.2. For $\pi \in D_n$ we have

1. $(\text{Lmic}_D, \text{Lmap}_D, \text{Lmai}_D)\pi = (\text{Min}, \text{Rmil}_D, \text{Rmip}_D)\text{D-code}(\pi)$.
2. $(\text{Cyc}_D^+, \text{Cyc}_D^-)\pi = (\text{Max}^0, \text{Max}^1)\text{D-code}(\pi)$.

Proof. (1) Let $b = \text{B-code}(\pi)$ and $d = \text{D-code}(\pi)$. By definition one has

$$
|b_i| = |d_i| \text{ for } i = 1, 2, ..., n.
$$

The result follows from Lemma 3.4(2) - (4).

(2) Fix an integer $1 < i < n$. Let $\sigma$ and $\rho$ denote the permutations $\pi^{(i)}$ during the construction of the B-code and D-code respectively. One can see that $|\sigma_1| = |\rho_1|$ and $\sigma_j = \rho_j$ for $j \geq 2$. That is, $d_j$ must be equal to $b_j$ whenever $|d_j| \neq 1$. Since $d_1$ is always 1, by Lemma 3.4(1) we have

$$
\text{Cyc}_D^+(\pi) = \text{Cyc}_D^0(\pi) \cup \{1\} = \text{Max}^0(b) \cup \{1\} = \text{Max}^0(d)
$$

and

$$
\text{Cyc}_D^-(\pi) = \text{Cyc}_D^1(\pi) \setminus \{1\} = \text{Max}^1(b) \setminus \{1\} = \text{Max}^1(d),
$$

and we are done. \hfill \Box

Note that in \footnote{3} Proposition 4.5] it is showed that $\tilde{\ell}_D^p(\pi) = n - \sum_{i=1}^n \chi(d_i = i)$. Hence from Lemma 6.2 we have

$$
\tilde{\ell}_D^p(\pi) = n - \text{cyc}_D^+(\pi).
$$

(6.1)

The next result extends the type $D$ main result in \footnote{7}.

Theorem 6.3. For $\pi \in D_n$ we have

$$
(\ell_D, \text{Rmil}_D^+, \text{Rmil}_D^-, \text{Lmil}_D, \text{Lmap}_D)\pi = (\text{sor}_D, \text{Cyc}_D^+, \text{Cyc}_D^-, \text{Lmic}_D, \text{Lmap}_D)\psi(\pi).
$$

Proof. It was showed in \footnote{3} that $\text{inv}_D(\pi) = \text{sor}_D(\psi(\pi))$ for $\pi \in D_n$. The theorem is proved by combining it with Lemma 6.1 and Lemma 6.2. \hfill \Box
6.4. Even-signed permutations on a Ferrers shape. We can also extend the result on the permutations restricted to a Ferrers shape. Similar to the case of $G_r n, f$, for a given integer sequence $f = (f_1, f_2, \ldots, f_n)$ with $1 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq n$ we define the set of restricted even-signed permutations by

$$D_{n, f} := \{ \pi \in D_n : |\pi_i| \leq f_i, 1 \leq i \leq n \}.$$

The minimum sequence $f(\pi)$ is similarly determined by both $L\text{map}(|\pi|)$ and $L\text{mal}(|\pi|)$, where $|\pi|$ is the permutation $|\pi_1| \cdots |\pi_n| \in \mathfrak{S}_n$. We have the following.

**Lemma 6.4.** Let $\pi \in D_n$ and $f(\pi) = f$. Then $\psi(\pi) \in D_{n, f}$.

**Proof.** From the proof of Lemma 4.4 we have

$$(L\text{map}, L\text{mal})|\pi| = (L\text{map}, L\text{mal})\psi(\pi).$$

Therefore, $f(\psi(\pi)) = f$ and thus the result follows. □

By combining Theorem 6.3 and Lemma 6.4, we obtain the first main result of this section.

**Theorem 6.5.** Given $n$ and $f$. Then the pair of (set-valued) statistics

$$(\ell_{D}, R\text{mil}_D, R\text{mil}_D, L\text{map}_D, L\text{mal}_D) \quad \text{and} \quad (s_{\text{or}D}, C_{\text{yc}D}, C_{\text{yc}D}, L\text{mic}_D, L\text{map}_D, L\text{mal}_D)$$

have the same joint distribution over $D_{n, f}$.

Now we look at the generating function. Define elements $\Theta_1, \Theta_2, \ldots, \Theta_n$ of the group algebra of $D_{n, f}$ by $\Theta_1 := 1$ and for $j \geq 2$

$$\Theta_j := 1 + \sum_{i=h_j}^{j-1} t_{ij}^D + \sum_{i=h_j}^{j} t_{ij}^D,$$

where $H(f) = (h_1, \ldots, h_n)$. The next lemma is derived by a similar argument in the proof of Lemma 2.2.

**Lemma 6.6.** We have

$$\Theta_1 \Theta_2 \cdots \Theta_n = \sum_{\pi \in D_{n, f}} \pi.$$

Our second main result is a generating function over $D_{n, f}$, analogous to Theorem 5.2.

**Theorem 6.7.** Given $n$ and $f$ with $H(f) = (h_1, \ldots, h_n)$, we have

$$\sum_{\pi \in D_{n, f}} q^{\ell_{D}(\pi)} u^{\text{min}_D(\pi)} \prod_{i \in R\text{mil}_D(\pi)} t_i \prod_{i \in R\text{mil}_D(\pi)} s_i = \sum_{\pi \in D_{n, f}} q^{s_{\text{or}D}(\pi)} u^{\text{mic}_D(\pi)} \prod_{i \in C_{\text{yc}D}(\pi)} t_i \prod_{i \in C_{\text{yc}D}(\pi)} s_i = t_1 u \prod_{j=2}^{n} \left( t_j + q + \cdots + q^{j-h_j-1} + \xi_{h_j} (2u) q^{j-h_j} + q^{j+h_j-1} + \cdots + q^{2j-3} + s_j q^{2j-2} \right).$$
Proof. The first equality follows from Theorem 6.5. For the second one, let $F_n(q, u, t_i, s_i : 1 \leq i \leq n)$ denote the desired generating function. Define the linear mapping $\theta : \mathbb{Z}(D_{n,t}) \rightarrow \mathbb{Z}(q, u, t_i, s_i : 1 \leq i \leq n)$ by

$$\theta(\pi) := q^{\text{sort}_D(\pi)} u^{\text{lmic}_D(\pi)} \prod_{i \in \text{Cyc}^+_D(\pi)} t_i \prod_{i \in \text{Cyc}^-_D(\pi)} s_i.$$ 

By Lemma 6.6 it suffices to show that

$$\theta(\Theta_1 \Theta_2 \cdots \Theta_n) = F_n(q, u, t_i, s_i : 1 \leq i \leq n).$$

We proceed by induction. It is easy to see that $\theta(\Theta_1) = t_1 u$ and

$$\theta(\Theta_1 \Theta_2) = \begin{cases} t_1 u(t_2 + s_2 q^2) & \text{if } f = (1, 2) \\ t_1 u(t_2 + 2u q + s_2 q^2) & \text{if } f = (2, 2). \end{cases}$$

Let $n \geq 3$ and suppose that $\theta(\Theta_1 \cdots \Theta_{n-1}) = F_{n-1}(q, u, t_i, s_i : 1 \leq i \leq n-1)$. Similar to the proof of Theorem 5.2, we identify elements of $D_{n-1,t}$ with the set $\{\pi \in D_{n,t} : \pi_n = n\}$. Given an element $\pi = \pi_1 \cdots \pi_{n-1} n$ in this set, we have

$$\pi \cdot \Theta_n = \pi_1 \pi_2 \cdots \pi_{n-1} n + \pi_1 \cdots n \pi_{n-1} + \cdots + \pi_1 \cdots \pi_{h_n} n \pi_{h_n+1} \cdots \pi_{n-1} \pi_{h_n}$$

$$+ \frac{n!}{1!} \sum_{i=1}^{n} \pi_1 \pi_2 \cdots \pi_i \pi_{i+1} \cdots \pi_{n-1} n \pi_{n-1} \pi_n$$

Denote by $\pi'$ any one of the summands above. Without loss of generality, let the letter $n$ or $\bar{n}$ be at the $i$-th position in $\pi'$. That is, $\pi' = \pi t_i^D$ or $\pi \bar{t}_i^D$. Then $\text{sort}_D(\pi') = \text{sort}_D(\pi) + n - i$ if $n$ has a positive sign and $\text{sort}_D(\pi') = \text{sort}_D(\pi) + n + i - 2$ otherwise. Since 1 is always counted in $\text{Cyc}^+_D$, by the proof of Lemma 3.4, we have

$$\text{Cyc}^+_D(\pi') = \begin{cases} \text{Cyc}^+_D(\pi) \cup \{n\} & \text{if } \pi' = \pi_1 \cdots \pi_{n-1} n, \\ \text{Cyc}^-_D(\pi) & \text{otherwise}. \end{cases}$$

and

$$\text{Cyc}^-_D(\pi') = \begin{cases} \text{Cyc}^-_D(\pi) \cup \{n\} & \text{if } \pi' = \pi_1 \cdots \pi_{n-1} \bar{n}, \\ \text{Cyc}^-_D(\pi) & \text{otherwise}. \end{cases}$$

Moreover, it is not hard to verify that

$$\text{lmic}_D(\pi') = \begin{cases} \text{lmic}_D(\pi) + 1 & \text{if } \pi' = n \pi_2 \cdots \pi_{n-1} \pi_1 \text{ or } \bar{n} \pi_2 \cdots \pi_{n-1} \bar{n}, \\ \text{lmic}_D(\pi) & \text{otherwise}. \end{cases}$$

Thus, we have

$$\theta(\pi \cdot \Theta_n) =$$

$$\theta(\pi) \left( t_n + q + \cdots + q^{n-h_n-1} + \xi_{h_n=1}(2u) q^{n-h_n} + q^{n+h_n-1} + \cdots + q^{2n-3} + s_n q^{2n-2} \right)$$

Therefore, we have

$$\theta(\Theta_1 \Theta_2 \cdots \Theta_n) = F_n(q, u, t_i, s_i : 1 \leq i \leq n).$$
and therefore

\[
\theta(\Theta_1 \cdots \Theta_{n-1} \Theta_n) = \theta \left( \sum_{\pi \in \mathcal{D}_{n,t}, \pi_n = n} \pi \cdot \Theta_n \right) = \sum_{\pi \in \mathcal{D}_{n,t}, \pi_n = n} \theta(\pi \cdot \Theta_n)
\]

\[
= \left( t_n + q + \cdots + q^{n-h_n-1} + \xi_{h_n} = 1(2u)q^{n-h_n} + q^{n+h_n-1} + \cdots + q^{2n-3} + s_n q^{2n-2} \right) \sum_{\pi \in \mathcal{D}_{n-1,t}} \theta(\pi)
\]

\[
= \left( t_n + q + \cdots + q^{n-h_n-1} + \xi_{h_n} = 1(2u)q^{n-h_n} + q^{n+h_n-1} + \cdots + q^{2n-3} + s_n q^{2n-2} \right) \cdot F_{n-1}(q, u, t_i, s_i : 1 \leq i \leq n - 1)
\]

\[
= F_n(q, u, t_i, s_i : 1 \leq i \leq n).
\]

By replacing \( t_i \) with \( t \) and \( s_i \) with \( s \) for all \( i \), we obtain the following result.

**Corollary 6.8.** Given \( n \) and \( f \) with \( H(f) = (h_1, \ldots, h_n) \), we have

\[
\sum_{\pi \in \mathcal{D}_{n,t}} q^{f_D(\pi)} u^{\min_D(\pi)} t^{\min_D^+(\pi)} s^{\min_D^-(\pi)} = \sum_{\pi \in \mathcal{D}_{n,t}} q^{\text{sort}_D(\pi)} u^{\text{lmic}_D(\pi)} t^{\text{cyc}_D^+(\pi)} s^{\text{cyc}_D^- (\pi)}
\]

\[
= tu \prod_{j=2}^n \left( t + q + \cdots + q^{j-h_j-1} + \xi_{h_j} = 1(2u)q^{j-h_j} + q^{j+h_j-1} + \cdots + q^{2j-3} + sq^{2j-2} \right).
\]

In particular,

\[
\sum_{\pi \in \mathcal{D}_n} q^{f_D(\pi)} u^{\min_D(\pi)} t^{\min_D^+(\pi)} s^{\min_D^-(\pi)} = \sum_{\pi \in \mathcal{D}_n} q^{\text{sort}_D(\pi)} u^{\text{lmic}_D(\pi)} t^{\text{cyc}_D^+(\pi)} s^{\text{cyc}_D^-(\pi)}
\]

\[
= tu \prod_{j=2}^n \left( t + q + \cdots + q^{j-2} + 2uq^{j-1} + q^{j} + \cdots + q^{2j-3} + sq^{2j-2} \right).
\]

Since \( \ell_D^r = n - \text{cyc}_D^+ \), we have the following.

**Corollary 6.9.** We have

\[
\sum_{\pi \in \mathcal{D}_n} t^{\text{cyc}_D^+(\pi)} = t \prod_{i=2}^n (t + 2i - 1) \quad \text{and} \quad \sum_{\pi \in \mathcal{D}_n} t^{\ell_D^r(\pi)} = n \prod_{i=2}^n (1 + (2i - 1)t).
\]

7. Concluding Remark

In the notation of complex reflection groups, \( G_{r,n} \) is denoted by \( G(r, 1, n) \) and \( D_n \) by \( G(2, 2, n) \). Hence it is natural to ask if one can have the sorting index and analogous (set-valued) equidistribution results on \( G(r, 2, n) \), or even better, \( G(r, r, n) \) or \( G(r, p, n) \). We leave these questions to the interested readers.
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