Does generalization performance of $l^q$ regularization learning depend on $q$? A negative example

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Abstract

$l^q$-regularization has been demonstrated to be an attractive technique in machine learning and statistical modeling. It attempts to improve the generalization (prediction) capability of a machine (model) through appropriately shrinking its coefficients. The shape of a $l^q$ estimator differs in varying choices of the regularization order $q$. In particular, $l^1$ leads to the LASSO estimate, while $l^2$ corresponds to the smooth ridge regression. This makes the order $q$ a potential tuning parameter in applications. To facilitate the use of $l^q$-regularization, we intend to seek for a modeling strategy where an elaborative selection on $q$ is avoidable. In this spirit, we place our investigation within a general framework of $l^q$-regularized kernel learning under a sample dependent hypothesis space (SDHS). For a designated class of kernel functions, we show that all $l^q$ estimators for $0 < q < \infty$ attain similar generalization error bounds. These estimated bounds are almost optimal in the sense that up to a logarithmic factor, the upper and lower bounds are asymptotically identical. This finding tentatively reveals that, in some modeling contexts, the choice of $q$ might not have a strong impact in terms of the generalization capability. From this perspective, $q$ can be arbitrarily specified, or specified merely by other no generalization criteria like smoothness, computational complexity, sparsity, etc..

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1 Introduction

Contemporary scientific investigations frequently encounter a common issue of exploring the relationship between a response and a number of covariates. In machine learning research, the subject is typically addressed through learning a underlying rule from the data that accurately predicates future values of the response. For instance, in banking industry, financial analysts are interested in building a system that helps to judge the risk of a loan request. Such a system is often trained based on the risk assessments from previous loan applications together with the empirical experiences. An incoming loan request is then viewed as a new input, upon which the corresponding potential risk (response) is to be predicted. In such applications, the predictive accuracy of a trained rule is of the key importance.

In the past decade, various strategies have been developed to improve the prediction (generalization) capability of a learning process, which include $l^q$ regularization as an well-known example [33]. The $l^q$ regularization learning prevents over-fitting by shrinking the model coefficients and thereby attains a higher predictive value. To be specific, suppose that the data $\mathbf{z} = \{x_i, y_i\}$ for $i = 1, \ldots, m$ are collected independently and identically according to an unknown but definite distribution, where $y_i$ is a response of ith unit and $x_i$ is the corresponding $d$-dimensional covariates. Let

$$
\mathcal{H}_{K, \mathbf{z}} := \left\{ \sum_{i=1}^{m} a_i K_{x_i} : a_i \in \mathbb{R} \right\}
$$

be a sample dependent space (SDHS) with $K_t(\cdot) = K(\cdot, t)$ and $K(\cdot, \cdot)$ being a positive definite kernel function. The coefficient-based $l^q$ regularization strategy ($l^q$ regularizer) takes the form of

$$
f_{\mathbf{z}, \lambda, q} = \arg \min_{f \in \mathcal{H}_{K, \mathbf{z}}} \left\{ \frac{1}{m} \sum_{i=1}^{m} (f(x_i) - y_i)^2 + \lambda \Omega_{\mathbf{z}}^q(f) \right\},
$$

where $\lambda = \lambda(m, q) > 0$ is a regularization parameter and $\Omega_{\mathbf{z}}^q(f)$ ($0 < q < \infty$) is defined by

$$
\Omega_{\mathbf{z}}^q(f) = \sum_{i=1}^{m} |a_i|^q \text{ when } f = \sum_{i=1}^{m} a_i K_{x_i} \in \mathcal{H}_{K, \mathbf{z}}.
$$
With different choices of order $q$, (1) leads to various specific forms of the $l_q$ regularizer. In particular, when $q = 2$, $f_{x,\lambda,q}$ corresponds to the ridge regressor [23], which smoothly shrinks the coefficients toward zero. When $q = 1$, $f_{x,\lambda,q}$ leads to the LASSO [29], which sets small coefficients exactly at zero and thereby also serves as a variable selection operator. When $0 < q < 1$, $f_{x,\lambda,q}$ coincides with the bridge estimator [8], which tends to produce highly sparse estimates through a non-continuous shrinkage.

The varying forms and properties of $f_{x,\lambda,q}$ make the choice of order $q$ crucial in applications. Apparently, an optimal $q$ may depend on many factors such as the learning algorithms, the purposes of studies and so forth. These factors make a simple answer to this question infeasible in general.

To facilitate the use of $l^q$-regularization, alternatively, we intend to seek for a modeling strategy where an elaborate selection on $q$ is avoidable. Specifically, we attempt to reveal some insights for the role of $q$ in $l^q$-learning via answering the following question:

**Problem 1.** Are there any kernels such that the generalization capability of (1) is independent of $q$?

In this paper, we provide a positive answer to Problem 1 under the framework of statistical learning theory. Specifically, we provide a featured class of positive definite kernels, under which the $l_q$ estimators for $0 < q < \infty$ attain similar generalization error bounds. We then show that these estimated bounds are almost essential in the sense that up to a logarithmic factor the upper and lower bounds are asymptotically identical. In the proposed modeling context, the choice of $q$ does not have a strong impact in terms of the generalization capability. From this perspective, $q$ can be arbitrarily specified, or specified merely by other no generalization criteria like smoothness, computational complexity, sparsity, etc..

The reminder of the paper is organized as follows. In Section 2, we provide a literature review and explain our motivation of the research. In Section 3, we present some preliminaries including spherical harmonics, Gegenbauer polynomials and so on. In Section 4, we introduce a class of well-localized needlet type kernels of Petrushev and Xu [22] and show some crucial properties of them which will play important roles in our analysis. In Section 5, we then study the generalization capabilities of $l^q$-regularizer associated with the constructed kernels for different $q$. In Section 6, we provide the proof of the main results. We conclude the paper with some useful remarks in the
last section.

2 Motivation and related work

2.1 Motivation

In practice, the choice of $q$ in (1) is critical, since it embodies certain potential attributions of the anticipated solutions such as sparsity, smoothness, computational complexity, memory requirement and generalization capability of course. The following simple simulation illustrates that different choice of $q$ can lead to different sparsity of the solutions.

The samples are identically and independently drawn according to the uniform distribution from the two dimensional Sinc function pulsing a Gaussian noise $N(0, \delta^2)$ with $\delta^2 = 0.1$. There are totally 256 training samples and 256 test samples. In Fig. 1, we show that different choice of $q$ may deduce different sparsity of the estimator for the kernel $K_{0.1}(x) := \exp\left\{-\|x - y\|^2/0.1\right\}$. It can be found that $l^q$ ($0 < q \leq 1$) regularizers can deduce sparse estimator, while it impossible for $l^2$ regularizer.

![Figure 1: Sparsity for $l^q$ learning schemes](image)

Therefore, for a given learning task, how to choose $q$ is an important and crucial problem for $l^q$ regularization learning. In other words, which standards should be adopted to measure the quality
of $l^q$ regularizers deserves study. As the most important standard of statistical learning theory, the generalization capability of $l^q$ regularization scheme $l^q$ may depend on the choice of kernel, the size of samples $m$, the regularization parameter $\lambda$, the behavior of priors, and, of course, the choice of $q$. If we take the generalization capability of $l^q$ regularization learning as a function of $q$, we then automatically wonder how this function behaves when $q$ changes for a fixed kernel. If the generalization capabilities depends heavily on $q$, then it is natural to choose the $q$ such that the generalization capability of the corresponding $l^q$ regularizer is the smallest. If the generalization capabilities is independent of $q$, then $q$ can be arbitrarily specified, or specified merely by other no generalization criteria like smoothness, computational complexity, sparsity.

However, the relation between the generalization capability and $q$ depends heavily on the kernel selection. To show this, we compare the generalization capabilities of $l^2$, $l^1$, $l^{1/2}$ and $l^{2/3}$ regularization schemes for two kernels: $\exp\{-\|x - y\|^2/0.1\}$ and $\exp\{-\|x - y\|/10\}$ in the simulation. The one case shows that the generalization capabilities of $l^q$ regularization schemes may be independent of $q$ and the other case shows that the generalization capability of $l^q$ depends heavily on $q$. In the left of Fig. 2, we report the relation between the test error and regularization parameter for the kernel $\exp\{-\|x - y\|^2/0.1\}$. It is shown that when the regularization parameters are appropriately tuned, all of the aforementioned regularization schemes may possess the similar generalization capabilities. In the right of Fig. 2, for the kernel $\exp\{-\|x - y\|/10\}$, we see that the generalization capability of $l^q$ regularization depends heavily on the choice of $q$.

From these simulations, we see that finding kernels such that the generalization capability of $l^q$ is independent of $q$ is of special importance in theoretical and practical applications. In particular, if such kernels exist, with such kernels, $q$ can be solely chosen on the basis of algorithmic and practical
considerations for \( l^q \) regularization. Here we emphasize that all these conclusions can, of course only be made in the premise that the obtained generalization capabilities of all \( l^q \) regularizers are (almost) optimal.

2.2 related work

There have been several papers that focus on the generalization capability analysis of the \( l^q \) regularization scheme \([1]\). Wu and Zhou \([33]\) were the first, to the best of our knowledge, to show a mathematical foundation of learning algorithms in SDHS. They claimed that the data dependent nature of the algorithm leads to an extra error term called hypothesis error, which is essentially different from regularization schemes with sample independent hypothesis spaces (SIHSs). Based on this, the authors proposed a coefficient-based regularization strategy and conducted a theoretical analysis of the strategy by dividing the generalization error into approximation error, sample error and hypothesis error. Following their work, Xiao and Zhou \([34]\) derived a learning rate of \( l^1 \) regularizer via bounding the regularization error, sample error and hypothesis error, respectively. Their result was improved in \([24]\) by adopting a concentration inequality technique with \( l^2 \) empirical covering numbers to tackle the sample error. On the other hand, for \( l^q \) (\( 1 \leq q \leq 2 \)) regularizers, Tong et al. \([30]\) deduced an upper bound for generalization error by using a different method to cope with the hypothesis error. Later, the learning rate of \([30]\) was improved further in \([11]\) by giving a sharper estimation of the sample error.

In all those researches, some sharp restrictions on the probability distributions (priors) have been imposed, say, both spectrum assumption of the regression function and concentration property of the marginal distribution should be satisfied. Noting this, for \( l^2 \) regularizer, Sun and Wu \([28]\) conducted a generalization capability analysis for \( l^2 \) regularizer by using the spectrum assumption to the regression function only. For \( l^1 \) regularizer, by using a sophisticated functional analysis method, Zhang et al. \([36]\) and Song et al. \([25]\) built the regularized least square algorithm on the reproducing kernel Banach space (RKBS), and they proved that the regularized least square algorithm in RKBS is equivalent to \( l^1 \) regularizer if the kernel satisfies some restricted conditions. Following this method, Song and Zhang \([26]\) deduced a similar learning rate for the \( l^1 \) regularizer and eliminated the concentration property assumption on the marginal distribution.
Limiting $q$ within $[1, 2]$ is certainly incomplete to judge whether the generalization capability of $l^q$ regularization depends on the choice of $q$. Moreover, in the context of learning theory, to intrinsically characterize the generalization capability of a learning strategy, the essential generalization bound \cite{10} rather than the upper bound is required, that is, we must deduce a lower and an upper bound simultaneously for the learning strategy and prove that the upper and lower bounds can be asymptotically identical. We notice, however, that most of the previously known estimations on generalization capability of learning schemes \cite{11} are only concerned with the upper bound estimation. Thus, their results can not serve the answer to Problem 1. Different from the previous work, the essential bound estimation of generalization error for $l^q$ regularization schemes \cite{11} with $0 < q < \infty$ will be presented in the present paper. As a consequence, we provide an affirmative answer to Problem 1.

\section{Preliminaries}

In this section, we introduce some preliminaries on spherical harmonics, Gegenbauer polynomial and orthonormal basis construction., which will be used in the construction of the positive definite needlet kernel.

\subsection{Gegenbauer polynomial}

The Gegenbauer polynomials are defined by the generating function \cite{31}

$$(1 - 2tz + z^2)^{-\mu} = \sum_{n=0}^{\infty} G_n^\mu(t) z^n,$$

where $|z| < 1, |t| \leq 1$, and $\mu > 0$. The coefficients $G_n^\mu(t)$ are algebraic polynomials of degree $n$ which are called the Gegenbauer polynomials associated with $\mu$. It is known that the family of polynomials $\{G_n^\mu\}_{n=0}^{\infty}$ is a complete orthogonal system in the weighted space $L^2(I, w)$, $I := [-1, 1]$, $w_\mu(t) := (1 - t^2)^{\mu - \frac{1}{2}}$ and there holds

$$\int_I G_m^\mu(t) G_n^\mu(t) w_\mu(t) dt = \begin{cases} 0, & m \neq n \\ h_{n,\mu}, & m = n \end{cases}$$

with $h_{n,\mu} = \frac{\pi^{1/2}(2\mu)_n \Gamma(\mu + \frac{1}{2})}{(n + \mu)n! \Gamma(\mu)}$,

where

$$\langle a \rangle_0 := 0, \langle a \rangle_n := a(a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}.$$
Define
\[ U_n := (h_n d/2)^{-1/2} G_n d/2, \quad n = 0, 1, \ldots. \] (2)

Then it is easy to see that \( \{U_n\}_{n=0}^\infty \) is a complete orthonormal system for the weighted \( L^2 \) space \( L^2(I, w) \), where \( w(t) := (1 - t^2)^{d-1}. \) Let \( B^d \) be the unit ball in \( \mathbb{R}^d \), \( S^{d-1} \) be the unit sphere in \( \mathbb{R}^d \) and \( P_n \) be the set of algebraic polynomials of degree not larger than \( n \) defined on \( B^d \). Denote by \( d\omega_{d-1} \) the aero element of \( S^{d-1} \). Then \( \Omega_{d-1} := \int_{S^{d-1}} d\omega_{d-1} = \frac{2\pi}{(d/2)} \). The following important properties of \( U_n \) are established in [21].

**Lemma 1.** Let \( U_n \) be defined as above. Then for each \( \xi, \eta \in S^{d-1} \) we have
\[
\int_{B^d} U_n(\xi \cdot x) P(x) dx = 0 \quad \text{for} \quad P \in P_{n-1},
\] (3)

\[
\int_{B^d} U_n(\xi \cdot x) U_n(\eta \cdot x) dx = \frac{U_n(\xi \cdot \eta)}{U_n(1)},
\] (4)

\[
K_n^* + K_{n-2}^* + \cdots + K_{\epsilon_n}^* = \frac{v_n^2}{U_n(1)} U_n,
\] (5)

and
\[
\int_{S^{d-1}} U_n(\xi \cdot x) U_n(\xi \cdot \eta) d\omega_{d-1}(\xi) = \frac{U_n(1)}{v_n^2} U_n(\eta \cdot x),
\] (6)

where \( v_n := \left( \frac{(n+1)(d-1)}{2(2\pi)^{d/2}} \right)^{1/2} \), and \( K_n^* := \frac{2k+d-2}{(d-2)\Omega_{d-1}} G_{k-2}^{d-2}(\xi \cdot \eta) \).

### 3.2 Spherical harmonics

For any integer \( k \geq 0 \), the restriction to \( S^{d-1} \) of a homogeneous harmonic polynomial with degree \( k \) is called a spherical harmonic of degree \( k \). The class of all spherical harmonics with degree \( k \) is denoted by \( H_k^{d-1} \), and the class of all spherical polynomials with total degrees \( k \leq n \) is denoted by \( \Pi_k^{d-1} \). It is obvious that \( \Pi_k^{d-1} = \bigoplus_{k=0}^{n} H_k^{d-1} \). The dimension of \( H_k^{d-1} \) is given by
\[
D_k^{d-1} := \dim H_k^{d-1} := \begin{cases} 
\frac{2k+d-2}{k+d-2} \binom{k+d-2}{k}, & k \geq 1; \\
1, & k = 0,
\end{cases}
\]

and that of \( \Pi_k^{d-1} \) is \( \sum_{k=0}^{n} D_k^{d-1} = D_n^{d} \sim n^{d-1} \), where \( A \sim B \) denotes that there exist absolute constants \( C_1 \) and \( C_2 \) such that \( C_1 A \leq B \leq C_2 A \).
The well known addition formula is given by (see [20] and [31])

\[ \sum_{l=1}^{D_{k+1}} Y_{k,l}(\xi) Y_{k,l}(\eta) = 2k + d - 2 \frac{d-2}{\Omega_{d-1}} \eta^2 (\xi \cdot \eta), \]  

(7)

where \( \{Y_{k,l} : l = 1, \ldots, D_{k+1}\} \) is arbitrary orthonormal basis of \( H_{k+1} \).

For \( r > 0 \) and \( a \geq 1 \), we say that a finite subset \( \Lambda \subset S^{d-1} \) is an \((r, a)\)-covering of \( S^{d-1} \) if

\[ S^{d-1} \subset \bigcup_{\xi \in \Lambda} D(\xi, r) \quad \text{and} \quad \max_{\xi \in \Lambda} |\Lambda \cap D(\xi, r)| \leq a, \]

where \( |A| \) denotes the cardinality of the set \( A \) and \( D(\xi, r) \subset S^{d-1} \) denotes the spherical cap with the center \( \xi \) and the angle \( r \). The following positive cubature formula can be found in [2].

**Lemma 2.** There exists a constant \( \gamma > 0 \) depending only on \( d \) such that for any positive integer \( n \) and any \((\delta/n, a)\)-covering of \( S^{d-1} \) satisfying \( 0 < \delta < a^{-1}\gamma \). There exists a set of numbers \( \{\eta_\xi\}_{\xi \in \Lambda} \) such that

\[ \int_{S^{d-1}} Q(\zeta) d\omega_{d-1}(\zeta) = \sum_{\xi \in \Lambda} \eta_\xi Q(\zeta) \quad \text{for any} \quad Q \in \Pi_{4n}^{d-1}. \]

### 3.3 Basis and reproducing kernel for \( \mathcal{P}_n \)

Define

\[ P_{k,j,i}(x) = v_k \int_{S^{d-1}} Y_{j,i}(\xi) U_k(x \cdot \xi) d\omega(\xi). \]  

(8)

Then it follows from [13] (or [21]) that

\( \{P_{k,j,i} : k = 0, \ldots, n, j = k, k-2, \ldots, \varepsilon_k, i = 1, 2, \ldots, D_{j}^{d-1}\} \)

consists an orthonormal basis for \( \mathcal{P}_n \), where

\[ \varepsilon_k := \begin{cases} 0, & k \text{ even,} \\ 1, & k \text{ odd.} \end{cases} \]

Of course, \( \{P_{k,j,i} : k = 0, 1, \ldots, j = k, k-2, \ldots, \varepsilon_k, i = 1, 2, \ldots, D_{j}^{d-1}\} \)

is an orthonormal basis for \( L^2(B^d) \). The following Lemma 3 defines a reproducing kernel of \( \mathcal{P}_n \), whose proof will be presented in Appendix A.
Lemma 3. The space $(\mathcal{P}_n, \langle \cdot, \cdot \rangle_{L^2(B_d)})$ is a reproducing kernel Hilbert space. The unique reproducing kernel of this space is

$$K_n(x, y) := \sum_{k=0}^{n} v_k^2 \int_{S^{d-1}} U_k(\xi \cdot x)U_k(\xi \cdot y)d\omega(\xi).$$

(9)

4 The needlet kernel: Construction and Properties

In this section, we construct a concrete positive definite needlet kernel [22] and show its properties. A function $\eta$ is said to be admissible if $\eta \in C^{\infty}[0, \infty)$, $\eta(t) \geq 0$, and $\eta$ satisfies the following condition [22]:

$$\text{supp}\eta \subset [0, 2], \eta(t) = 1 \text{ on } [0, 1], \text{ and } 0 \leq \eta(t) \leq 1 \text{ on } [1, 2].$$

Such a function can be easily constructed out of an orthogonal wavelet mask [7]. We define a kernel $L_{2n}(\cdot, \cdot)$ as the following

$$L_{2n}(x, y) := \sum_{k=0}^{\infty} \eta \left( \frac{k}{n} \right) v_k^2 \int_{S^{d-1}} U_k(x \cdot \xi)U_k(y \cdot \xi)d\omega(\xi).$$

(10)

As $\eta(\cdot)$ is admissible, the constructed kernel $L_{2n}(x, y)$, called the needlet kernel (or localized polynomial kernel) [22] henceforth, is positive definite. We will show that so defined kernel function $L_{2n}(x, y)$, deduces the $l^q$ regularization learning whose learning rate is independent of the choice of $q$. To this end, we first show several useful properties of the needlet kernel.

The following Proposition [1] which can be deduced directly from Lemma 3 and the definition of $\eta(\cdot)$ reveals that $L_{2n}$ possesses reproducing property for $\mathcal{P}_n$.

Proposition 1. Let $L_{2n}$ be defined as in (10). For arbitrary $P \in \mathcal{P}_n$, there holds

$$P(x) = \int_{B_d} L_{2n}(x, y)P(y)dy.$$  

(11)

Since $\eta(\cdot)$ is an admissible function by definition, it follows that $L_{2n}(x, \cdot)$ is an algebraic polynomial of degree not larger than $2n$ for any fixed $x \in B_d$. At the first glance, as a polynomial kernel, it may have good frequency localization property while have bad space localization property. The following Proposition [2] which can be found in [22, Theorem 4.2], however, advocates that $L_{2n}$ is actually a polynomial kernel possessing very good spatial localized properties. This makes it widely applicable in approximation theory and signal processing [12, 22].
Proposition 2. Let $L_{2n}$ be defined as in (10). For arbitrary $l \in \mathbb{N}$, there exists a constant $c_l$ depending only on $l$, $d$ and $\eta$ such that

$$\max_{x,y \in B^d} |L_{2n}(x,y)| \leq c_l \frac{n^d}{(\sqrt{1 - |x|^2} + n^{-1})(\sqrt{1 - |y|^2} + n^{-1})(1 + d(x,y))^l}. \quad (12)$$

Let

$$E_n(f)_p := \inf_{P \in \mathcal{P}_n} \|f - P\|_{L^p(B^d)}$$

be the best approximation error of $\mathcal{P}_n$. Define

$$(L_{2n}f)(x) := \int_{B^d} L_{2n}(x,y)f(y)dy. \quad (13)$$

It has been shown in [22, Remak 4.8] that the integral operator $L_{2n}f$ possesses the following compressive property:

Proposition 3. If $L_{2n}f$ is defined as in (13), then, for arbitrary $f \in L^p(B^d)$, there exists a constant $C$ depending only on $d$ and $p$ such that

$$\|L_{2n}f\|_{L^p(B^d)} \leq C \|f\|_{L^p(B^d)}. \quad (14)$$

By Propositions 1, 2 and 3, a standard method in approximation theory [9] yields the following best approximation property of $L_{2n}f$.

Proposition 4. Let $1 \leq p \leq \infty$, and $L_{2n}$ be defined in (13), then for arbitrary $f \in L^p(B^d)$, there exists a constant $C$ depending only on $d$ and $p$ such that

$$\|f - L_{2n}f\|_{L^p(B^d)} \leq CE_n(f)_p. \quad (14)$$

5 Almost essential learning rate

In this section, we conduct a detailed generalization capability analysis of the $l^q$ regularization scheme (1) when the kernel function $K$ is specified as $L_{2n}(x,y)$. Our aim is to derive an almost essential learning rate of $l^q$ regularization strategy (1). We first present a quick review of learning theory. Then, we given the main result of this paper, where a $q$-independent learning rate of $l^q$ regularization schemes (1) is deduced. At last, we present some remarks on the main result.
5.1 Statistical learning theory

Let $X \subseteq \mathbb{B}^d$ be an input space and $Y \subseteq \mathbb{R}$ an output space. Assume that there exists a unknown but definite relationship between $x \in X$ and $y \in Y$, which is modeled by a probability distribution $\rho$ on $Z := X \times Y$. It is assumed that $\rho$ admits the decomposition

$$\rho(x, y) = \rho_X(x) \rho(y|x).$$

Let $z = (x_i, y_i)_{i=1}^m$ be a set of finite random samples of size $m$, $m \in \mathbb{N}$, drawn identically, independently according to $\rho$ from $Z$. The set of examples $z$ is called a training set. Without loss of generality, we assume that $|y_i| \leq M$ almost everywhere.

The aim of learning is to learn from a training set a function $f : X \to Y$ such that $f(x)$ is an effective estimate of $y$ when $x$ is given. One natural measurement of the error incurred by using $f$ of this purpose is the generalization error,

$$\mathcal{E}(f) := \int_Z (f(x) - y)^2 d\rho,$$

which is minimized by the regression function $[3, 4]$ defined by

$$f_\rho(x) := \int_Y y d\rho(y|x).$$

We do not know this ideal minimizer $f_\rho$, since $\rho$ is unknown, but we have access to random examples from $X \times Y$ sampled according to $\rho$.

Let $L^2_{\rho_X}$ be the Hilbert space of $\rho_X$ square integrable functions on $X$, with norm $\| \cdot \|_\rho$. In the setting of $f_\rho \in L^2_{\rho_X}$, it is well known that, for every $f \in L^2_{\rho_X}$, there holds

$$\mathcal{E}(f) - \mathcal{E}(f_\rho) = \| f - f_\rho \|_\rho^2. \quad (15)$$

The goal of learning is then to construct a function $f_z$ that approximates $f_\rho$, in the norm $\| \cdot \|_\rho$, using the finite sample $z$.

One of the main points of this paper is to formulate the learning problem in terms of probability estimates rather than expectation estimates. To this end, we present a formal way to measure the performance of learning schemes in probability. Let $\Theta \subseteq L^2_{\rho_X}$ and $\mathcal{M}(\Theta)$ be the class of all Borel measures $\rho$ on $Z$ such that $f_\rho \in \Theta$. For each $\varepsilon > 0$, we enter into a competition over
all estimators established in the hypothesis space \( \mathcal{H} \), \( \Psi_m : Z^m \rightarrow \mathcal{H}, z \mapsto f_z \), and we define the accuracy confidence function by

\[
AC_m(\Theta, \mathcal{H}, \varepsilon) := \inf_{f_z \in \Psi_m, \rho \in \mathcal{M}(\Theta)} \sup_{\rho} \mathbb{P}_m \{ \| f_\rho - f_z \|^2_\rho > \varepsilon \}.
\]

Furthermore, we define the accuracy confidence function for all possible estimators based on \( m \) samples \( \Phi_m : z \mapsto f_z \) by

\[
AC_m(\Theta, \varepsilon) := \inf_{f_z \in \Phi_m, \rho \in \mathcal{M}(\Theta)} \sup_{\rho} \mathbb{P}_m \{ \| f_\rho - f_z \|^2_\rho > \varepsilon \}.
\]

From these definitions, it is obvious that

\[
AC_m(\Theta, \varepsilon) \leq AC_m(\Theta, \mathcal{H}, \varepsilon)
\]

for all \( \mathcal{H} \).

### 5.2 \( q \)-independent learning rate

The sample dependent hypothesis space (SDHS) associated with \( L_{2n}(\cdot, \cdot) \) is then defined by

\[
\mathcal{H}_{L, z} := \left\{ \sum_{i=1}^m a_i L_{2n}(x_i, \cdot) : a_i \in \mathbb{R} \right\}
\]

and the corresponding \( l^q \) regularization scheme is defined by

\[
f_{z, \lambda, q} = \arg \min_{f \in \mathcal{H}_{L, z}} \left\{ \frac{1}{m} \sum_{i=1}^m (f(x_i) - y_i)^2 + \Omega^q_z(f_{z, \lambda, q}) \right\},
\]

where

\[
\Omega^q_z(f) := \lambda \sum_{i=1}^m |a_i|^q, \text{for } f = \sum_{i=1}^m a_i L_{2n}(x_i, \cdot).
\]

The projection operator \( \pi_M \) from the space of measurable functions \( f : X \rightarrow \mathbb{R} \) to \([-M, M]\) is defined by

\[
\pi_M(f)(x) := \begin{cases} M, & \text{if } f(x) > M, \\ f(x), & \text{if } -M \leq f(x) \leq M, \\ -M, & \text{if } f(x) \leq -M. \end{cases}
\]

As \( y \in [-M, M] \) by assumption, it is easy to check that

\[
\| \pi_M f_{z, \lambda, q} - f_\rho \|_\rho \leq \| f_{z, \lambda, q} - f_\rho \|_\rho.
\]
Also, for arbitrary $H \subset L^2(B^d)$, we denote $\pi_M H := \{ \pi_M f : f \in H \}$.

We also need to introduce the class of priors. For any $f \in L^2(B^d)$, denote by $\mathcal{F}(f)$ or $\hat{f}$ the Fourier transformation of $f$,

$$\hat{f}(u) := (2\pi)^{d/2} \int_{\mathbb{R}^d} f(x) e^{iu \cdot x} dx,$$

where $u \in B^d$. The inverse Fourier transformation will be denoted by $\mathcal{F}^{-1}$. In the space $L^2(B^d)$, the derivative of $f$ with order $\alpha$ is defined as

$$D^\alpha f := \mathcal{F}^{-1}\{|u|^\alpha \mathcal{F}(u)\},$$

where $|u| := \sqrt{u_1^2 + \cdots u_d^2}$. Here, Fourier transformation and derivatives are all taken sense in distribution. Let $r$ be any positive number. We consider the Sobolev class of functions

$$W^r_2 := \left\{ f : \max_{0 \leq \alpha \leq r} \|D^\alpha f\|_{L^2(B^d)} < \infty \right\}.$$

It follows from the well known Sobolev embedding theorem that $W^r_2 \subset C(B^d)$ provided $r > d/2$.

Now, we state the main result of this paper, whose proof will be given in the next section.

**Theorem 1.** Let $f_\rho \in W^r_2$ with $r > d/2$, $m \in \mathbb{N}$, $\varepsilon > 0$ be any numbers, and $n \sim \varepsilon^{-r/d}$. If $f_{z,\lambda,q}$ is defined as in (16) with $\lambda = m^{-1} \varepsilon$ and $0 < q < \infty$, then there exist positive constants $C_i$, $i = 1, \ldots, 4$, depending only on $M$, $\rho$, $q$ and $d$, $\varepsilon_0 > 0$ and $\varepsilon_-, \varepsilon_+$ satisfying

$$C_1 m^{-2r/(2r+d)} \leq \varepsilon_- \leq \varepsilon_+ \leq C_2 (m/\log m)^{-2r/(2r+d)},$$

such that for any $\varepsilon < \varepsilon_-$,

$$\sup_{f_\rho \in W^r_2} P^m \{ z : \|f_\rho - \pi_M f_{z,\lambda,q}\|_\rho > \varepsilon \} \geq C_3 m (W^r_2,\varepsilon) \geq \varepsilon_0,$$

and for any $\varepsilon \geq \varepsilon_+^+$,

$$e^{-C_3 m \varepsilon} \leq C_3 m (W^r_2,\varepsilon) \leq C_3 m (W^r_2,\pi_M H_{L,\varepsilon}) \leq \sup_{f_\rho \in W^r_2} P^m \{ z : \|f_\rho - \pi_M f_{z,\lambda,q}\|_\rho > \varepsilon \} \leq e^{-C_3 m \varepsilon}.$$

### 5.3 Remarks

We explain Theorem 1 below in more detail. At first, we explain why the accuracy function is used to characterize the generalization capability of the $l^q$ regularization schemes (16). In applications,
we are often faced with the following problem: There are \( m \) data available, and we are asked to produce an estimator with tolerance at most \( \varepsilon \) by using these \( m \) data only. In such circumstance, we have to know the probability of success. It is obvious that such probability depends on \( m \) and \( \varepsilon \). For example, if \( m \) is too small, we can not construct an estimator within small tolerance. This fact is quantitatively verified by Theorem 1. More specifically, (18) shows that if there are \( m \) data available and \( f_\rho \in W^2_r \) with \( r > d/2 \), then \( l_q^r \) \((0 < q < \infty)\) regularization scheme (16) is impossible to yield an estimator with tolerance error smaller than \( \varepsilon_m^- \). This is not a negative result, since we can see in (18) also that the main reason of impossibility is the lack of data rather than inappropriateness of the learning scheme (19). More importantly, Theorem 1 reveals a quantitative relation between the probability of success and the tolerance error based on \( m \) samples. It says in (19) that if the tolerance error \( \varepsilon \) is relaxed to \( \varepsilon_m^+ \) or larger, then the probability of success of \( l_q^r \) regularization is at least \( 1 - e^{-C_4m\varepsilon} \). The first inequality (lower bound) of (19) implies that such confidence can not be improved further. That is, we have presented an optimal confidence estimation for \( l_q^r \) regularization scheme (16) with \( 0 < q < \infty \). Thus, Theorem 1 basically concludes the following thing: If \( \varepsilon < \varepsilon_m^- \), then every estimator deduced from \( m \) samples by \( l_q^r \) regularization can not approximate the regression function with tolerance smaller than \( \varepsilon \), while if \( \varepsilon \geq \varepsilon_m^+ \), then the \( l_q^r \) regularization schemes with any \( 0 < q < \infty \) can definitely yield the estimators that approximate the regression function with tolerance \( \varepsilon \).

The values \( \varepsilon_m^- \) and \( \varepsilon_m^+ \) thus are critical for indicating the generalization error of a learning scheme. Indeed, the upper bound of generalization error of a learning scheme depends heavily on \( \varepsilon_m^+ \), while the lower bound of generalization error is relative to \( \varepsilon_m^- \). Thus, in order to have a tight generalization error estimate of a learning scheme, we naturally wish to make the interval \([\varepsilon_m^-, \varepsilon_m^+] \) as short as possible. Theorem 1 shows that, for \( l_q^r \) regularization scheme (16), \( \varepsilon_m^- \geq C_1m^{-2r/(2r+d)} \), and \( \varepsilon_m^+ \leq C_2(m/\log m)^{-2r/(2r+d)} \), which shows that the interval \([\varepsilon_m^-, \varepsilon_m^+] \) is almost the shortest one in the sense that up to a logarithmic factor, the upper bound and lower bound are asymptotical identical. Noting that the learning rate established in Theorem 1 is independent of \( q \), we thus can conclude that the generalization capability of \( l_q^r \) regularization does not depend on the choice of \( q \). This gives an affirmative answer to Problem 1.

The other advantage of using the accuracy confidence function to measure the generalization capability is that it allows to expose some phenomenon that can not be founded if the classical expectation standard is utilized. For example, Theorem 1 shows a sharp phase transition phe-
nomenon of $l^q$ regularization learning, that is, the behavior of the accuracy confidence function changes dramatically within the critical interval $[\varepsilon_m^-, \varepsilon_m^+]$. It drops from a constant $\varepsilon_0$ to an exponentially small quantity. We might call $[\varepsilon_m^-, \varepsilon_m^+]$ the interval of phase transition for a corresponding learning scheme. To make this more intuitive, let us conduct a simulation on the phase transition of the confidence function below. Without loss of generality, we implemented the $l^2$ regularization strategy (16) associated with the kernel (10) for $d = 1$ and $n = 8$ to yield the estimator. The regularization parameter $\lambda$ was chosen as $\varepsilon/m$. The training samples were drawn independently and identically according to the uniform distribution from the well known Sinc function, that is $f(x) := \sin x/x$. The number of the training samples $m$ was chosen from 1 to 100 and the tolerance $\varepsilon$ was chosen from $10^{-4}$ to 1 with step-length $10^{-4}$. Then, there were totally 1000 test data $(s_i, t_i)_{i=1}^{1000}$ drawn i. i. d according to the uniform distribution from $\sin C$. The test error was defined as $\delta_{\text{test}} := \sqrt{\frac{1}{100} \sum_{i=1}^{1000} (f_{s, \lambda}(s_i) - t_i)^2}$. We repeated 100 times simulations at each point, and labeled its value as 1 if $\delta_{\text{test}}$ is smaller than the tolerance error and 0 otherwise. Simulation result is shown in Fig.3. We can see from Fig.3 that in the upper right part, the colors of all points are red, which means that in those setting, the probability that $\delta_{\text{test}}$ is smaller than the tolerance is approximately 0. Thus, if the number of samples is small, then $l^2$ regularization schemes can not provide an estimation with very small tolerance. In the lower left area, the colors of all points are blue, which means that the probability of $\delta_{\text{test}}$ smaller than the tolerance is approximately 1. Between these two areas, there exists a band, that could be called the phase transition area, in which the colors of points vary from red to blue dramatically. It is seen that the length of phase transition interval monotonously decreases with $m$. All these coincide with the theoretical assertions of Theorem 1.

For comparison, we also present a generalization error bound result in terms of expectation error. Corollary 1 below can be directly deduced from Theorem 1 and [10, Chapter 3], if we notice the identity:

$$E_{\rho^m}(\mathcal{E}(f_{\rho}) - \mathcal{E}(f_{s, \lambda, q})) = \int_0^{\infty} P^m\{\mathcal{E}(f_{\rho}) - \mathcal{E}(f_{s, \lambda, q}) > \varepsilon\}d\varepsilon.$$

**Corollary 1.** Let $f_{\rho} \in W_r^d$ with $r > d/2$, $q_0 > 0$, $m \in \mathbb{N}$, and $n \sim \varepsilon^{-r/d}$. If $f_{s, \lambda, q}$ is defined as in (11) with $\lambda \sim \frac{m^{-2/(2r+d)}}{m+1}$ and $0 < q < \infty$, then there exist constants $C_5$ and $C_6$ depending only on...
Figure 3: The phase transition phenomenon of generalization with $l^2$ regularization

$M, d, q$ and $\rho$ such that

$$C_5m^{-2r/(2r+d)} \leq \inf_{\rho \in \Phi_m} \sup_{\rho \in M(W^r_2)} E_{\rho}^m \{\|f_{\rho} - f_\zeta\|_\rho^2\} \leq \inf_{\rho \in \Phi_m} \sup_{\rho \in M(W^r_2)} E_{\rho}^m \{\|f_{\rho} - f_\zeta\|_\rho^2\} \leq \sup_{\rho \in W^r_2} E_{\rho}^m \{\|f_{\rho} - f_\zeta, \lambda, q\|_\rho^2\} \leq C_6(m/ \log m)^{-2r/(2r+d)},$$  

(20)

where $\Phi_m$ is the set of all possible estimators based on $m$ samples.

It is noted that the representation theorem in learning theory [27] implies that the generalization capability of an optimal learning algorithm in SDHS is not worse than that of learning in RKHS with convex loss function. Corollary [1] then shows that if $f_{\rho} \in W^r_2$, then the generalization capability of an optimal learning scheme in SDHS associated with $L_2m$ is not worse than that of any optimal learning algorithms in the corresponding RKHS. More specifically, (20) shows that as far as the learning rate is concerned, all $l^q$ regularization schemes (16) for $0 < q < \infty$ can realize the same almost optimal theoretical rate. That is to say, the choice of $q$ has no influence on the generalization capability of the learning schemes (16). This also gives an affirmative answer to Problem 1 in the sense of expectation. Here, we emphasize that the independence of generalization of $l^q$ regularization on $q$ is based on the understanding of attaining the same almost optimal generalization error. Thus,
in application, $q$ can be arbitrarily specified, or specified merely by other no generalization criteria (like complexity, sparsity, etc.).

6 Proof of Theorem 1

6.1 Methodology

The methodology we adopted in the proof of Theorem 1 seems of novelty. Traditionally, the generalization error of learning schemes in SDHS is divided into the approximation, hypothesis and sample errors (three terms) [33]. All of the aforementioned results about coefficient regularization in SDHS falled into this style. According to [33], the hypothesis error has been regarded as the reflection of nature of data dependence of SDHS (sample dependent hypothesis space), and an indispensable part attributed to an essential characteristic of learning algorithms in SDHS, compared with the learning in SIHS (sample independent hypothesis space). With the specific kernel function $L^2_n$, we will divide the generalization error of $l^q$ regularization in this paper into the approximation and sample errors (two terms) only. Both of these two terms are dependent of the samples. The success in this paper then reveals that for at least some kernels, the hypothesis error is negligible, or can be avoided in estimation when $l^q$ regularization learning are analyzed in SDHS. We show that such new methodology can bring an important benefit of yielding an almost optimal generalization error bound for a large types of priors. Such benefit may reasonably be expected to beyond the $l^q$ regularization.

We sketch the methodology to be used as follows. Due to the sample dependent property, any estimators constructed in SDHS may be a random approximant. To bound the approximation error, we first deduce a probabilistic cubature formula for algebraic polynomial. Then we can discretize the near-best approximation operator $L_{2n}f$ based on the probabilistic cubature formula. Thus, the well known Jackson-type error estimate [9] can be applied to derive the approximation error. To bound the sample error, we will use a different method from the traditional approaches [3, 32]. Since the constructed approximant in SDHS is a random approximant, the concentration inequality such as Bernstein inequality [1] can not be available. In our approach, based on the prominent property of the constructed approximant, we will bound the sample error by using the
concentration inequality established in 3 twice. Then the relation between the so-called Pseudo-dimension and covering number 18 yields the sample error estimate for \( l^q \) regularization schemes (16) with arbitrary \( 0 < q < \infty \). Hence, we divide the proof into four subsections. The first subsection is devoted to establish the probabilistic cubature formula. The second subsection is to construct the random approximant and study the approximation error. The third subsection is to deduce the sample error and the last subsection is to derive the final learning rate. We present the details one by one below.

### 6.2 A probabilistic cubature formula

In this subsection, we establish a probabilistic cubature formula. At first, we need several lemmas. The weighted \( L^p \) norm on the \( d + 1 \)-dimensional unit sphere \( S^d \) is defined as follows. Let \( \alpha = (\alpha(1), \ldots, \alpha(d+1)) \in S^d \) and \( w_\alpha = |\alpha(d+1)|. \) Define

\[
\|f\|_{p,w_\alpha} := \begin{cases} 
(f_{S^d} |f(\alpha)|^p w_\alpha d\omega_d(\alpha))^{1/p}, & 1 \leq p < \infty, \\
\max_{\alpha \in S^d} |f(\alpha)| w_\alpha, & p = \infty.
\end{cases}
\]

The following [6, Lemma 2.3] gives a weighted Nikolskii inequality for spherical polynomial.

**Lemma 4.** Let \( 1 \leq p \leq q \leq \infty \). Then for any \( Q \in \Pi^d_n \),

\[
\|Q\|_{q,w_\alpha} \leq C n^{d(1/p-1/q)} \|Q\|_{p,w_\alpha},
\]

where \( C \) is a positive constant depending only on \( d, p \) and \( q \).

Lemma 5 establishes a relation between cubature formula on the unit sphere and cubature formula on the unit ball, which can be found in [35, Theorem 4.2].

**Lemma 5.** If there is a cubature formula of degree \( n \) on \( S^d \) given by

\[
\int_{S^d} f(\alpha) w_\alpha d\omega_d(\alpha) = \sum_{i=1}^m a_i f(\alpha_i),
\]

whose nodes are all located on \( S^d \), then there exists a cubature formula of degree \( n \) on \( B^d \), that is,

\[
2 \int_{B^d} f(x) dx = \sum_{i=1}^m a_i f(x_i),
\]

where \( x_i \in B^d \) are the first \( d \) components of \( \alpha_i \).
The following Lemma 6 is known as the Bernstein inequality for random variables, which can be found in [3].

**Lemma 6.** Let \( \xi \) be a random variable on a probability space \( Z \) with mean \( E(\xi) \), variance \( \sigma^2(\xi) = \sigma^2 \). If \( |\xi(z) - E(\xi)| \leq M_\xi \) for almost all \( z \in Z \). then, for all \( \varepsilon > 0 \),
\[
\Pr_{z \in Z} \left\{ \left| \frac{1}{m} \sum_{i=1}^{m} \xi(z_i) - E(\xi) \right| \geq \varepsilon \right\} \leq 2 \exp \left\{ -\frac{m\varepsilon^2}{2(\sigma^2 + \frac{3}{4}M_\xi \varepsilon)} \right\}.
\]

We also need a lemma showing that if \( \Xi := \{\alpha_i\}_{i=1}^m \subset S^d \) is a set of independent random variables drawn identically according to a distribution \( \mu \), then with high confidence the cubature formula holds.

**Lemma 7.** Let \( 0 < \varepsilon < 1 \), and \( 1 \leq p \leq \infty \). If \( \{\alpha_i\}_{i=1}^m \) are i.i.d. random variables drawn according to arbitrary distribution \( \mu \) on \( S^d \), then there exits a set of real numbers \( \{a_i\}_{i=1}^m \) such that
\[
\int_{S^d} Q_n(\alpha)w_\alpha d\omega_\alpha(\xi) = \sum_{i=1}^m a_i Q_n(\alpha_i)
\]
holds with confidence at least
\[
1 - 2 \exp \left\{ -\frac{Cm\varepsilon^2}{nd(1 + \varepsilon)} \right\},
\]
subject to
\[
\sum_{i=1}^m |a_i|^p \leq \frac{\Omega_d}{1 - \varepsilon} m^{1-p}.
\]

**Proof.** For the sake of brevity, we write \( w = w_\alpha \) in the following. Since the sampling set \( \Xi \) consists of a sequence of i.i.d. random variables on \( S^d \), the sampling points are a sequence of functions \( \alpha_j = \alpha_j(\omega) \) on some probability space \( (\Omega, P) \). Without loss of generality, we assume \( \|Q_n\|_{p,w} = 1 \) for arbitrary fixed \( p \). If we set \( \xi_j^p = |Q_n(\alpha_j)|^p w_\alpha \), then we have
\[
\frac{1}{m} \sum_{i=1}^m |Q_n(\alpha_i)|^p w_\alpha(\alpha_i) - E\xi_j^p = \frac{1}{m} \sum_{i=1}^m |Q_n(\alpha_i)|^p w_\alpha(\alpha_i) - \|Q_n\|_{p,w}^p,
\]
where we have used the equality
\[
E\xi_j^p = \int_{\Omega} |Q_n(\alpha_j(\omega))|^p w_\omega d\omega = \int_{S} |Q_n(\alpha)|^p w_\alpha d\omega_\alpha = \|Q_n\|_{p,w}^p = 1.
\]
Furthermore,
\[
|\xi_j^p - E\xi_j^p| \leq \sup_{\omega \in \Omega} |Q_n(\alpha(\omega))|^p w(\omega) - \|Q_n\|_{p,w}^p \leq \|Q_n\|_{\infty,w}^p - \|Q_n\|_{p,w}^p.
\]
It follows from Lemma 4 that
\[ \|Q_n\|_{\infty, w} \leq Cn^\frac{d}{p} \|Q_n\|_{p, w} = Cn^\frac{d}{p}. \]

Hence
\[ |\xi_j^p - E\xi_j^p| \leq Cn^{d-1}. \]

On the other hand, we have
\[ \sigma^2 = E((\xi_j^p)^2) - (E(\xi_j^p))^2 \leq \int_\Omega |Q_n(\alpha(\omega))|^{2p}w(\alpha)d\omega - \left( \int_\Omega |Q_n(\alpha(\omega))|^p w(x)d\omega \right)^2 \]
\[ = \|Q_n\|_{2p, w}^{2p} - \|Q_n\|_{p, w}^{2p}. \]

Then using Lemma 4 again, there holds
\[ \sigma^2 \leq Cn^{2dp\left(\frac{1}{p} - \frac{1}{2}\right)} \|Q_n\|_{p, w}^{2p} - \|Q_n\|_{p, w}^{2p} = Cn^{d-1}. \]

Thus it follows from Lemma 6 that with confidence at least
\[ 1 - 2\exp\left\{-\frac{m\varepsilon^2}{2\left(\sigma^2 + \frac{1}{2}M\varepsilon\right)}\right\} \geq 1 - 2\exp\left\{-\frac{m\varepsilon^2}{2\left((Cn^{d-1}) + \frac{1}{2}(Cn^{d-1})\varepsilon\right)}\right\}, \]
there holds
\[ \left| \frac{1}{m} \sum_{i=1}^m |Q_n(\alpha_i)|^p w(\alpha_i) - \|Q_n\|_{p, w}^p \right| \leq \varepsilon. \]

This means that if \( \Xi \) is a sequence of i.i.d. random variables, then the Marcinkiewicz-Zygmund inequality
\[ (1 - \varepsilon)\|Q_n\|_{p, w}^p \leq \frac{1}{m} \sum_{i=1}^m |Q_n(\alpha_i)|^p w(x) \leq (1 + \varepsilon)\|Q_n\|_{p, w}^p \quad \forall Q_n \in \Pi_n^d \]
holds with probability at least
\[ 1 - 2\exp\left\{-\frac{Cm\varepsilon^2}{n^{d-1}(1 + \varepsilon)}\right\}. \]

Then, almost same argument as that in [19, Theorem 4.1] or [5, Theorem 4.2] implies Lemma 7. \( \square \)

By virtue of the above lemmas, we can prove the following Proposition 5.

**Proposition 5.** Let \( 1 \leq p \leq \infty \) and \( x := (x_i)_{i=1}^m \subset B^d \) be a set of random variables independently and identically drawn according to arbitrary distribution \( \mu \). Then there exits a set of real numbers \( \{a_i\}_{i=1}^m \) and a constant \( C \) depending only on \( d \) such that the equality
\[ \int_{B^d} P_n(x)dx = \sum_{i=1}^m a_i P_n(x_i), \quad P_n \in \mathcal{P}_n \]
holds with confidence at least

\[ 1 - 2 \exp \left\{ -\frac{Cm}{n^d} \right\}, \]

subject to

\[ \sum_{i=1}^{m} |a_i|^p \leq C m^{1-p}. \]

### 6.3 Error decomposition and an approximation error estimate

To estimate the upper bound of

\[ \mathcal{E}(\pi_M f_{x,\lambda,q}) - \mathcal{E}(f_\rho), \]

we first introduce an error decomposition strategy. It follows from the definition of \( f_{x,\lambda,q} \) that, for arbitrary \( f \in \mathcal{H}_{L,z} \),

\[
\begin{align*}
\mathcal{E}(\pi_M f_{x,\lambda,q}) - \mathcal{E}(f_\rho) & \leq \mathcal{E}(\pi_M f_{x,\lambda,q}) - \mathcal{E}(f_\rho) + \lambda \Omega^q_z(f_{x,\lambda,q}) \\
& \leq \mathcal{E}(\pi_M f_{x,\lambda,q}) - \mathcal{E}_z(\pi_M f_{x,\lambda,q}) + \mathcal{E}_z(f) - \mathcal{E}(f) \\
& + \mathcal{E}_z(\pi_M f_{x,\lambda,q}) + \lambda \Omega^q_z(f_{x,\lambda,q}) - \mathcal{E}_z(f) - \lambda \Omega^q_z(f) \\
& + \mathcal{E}(f) - \mathcal{E}(f_\rho) + \lambda \Omega^q_z(f) \\
& \leq \mathcal{E}(\pi_M f_{x,\lambda,q}) - \mathcal{E}_z(\pi_M f_{x,\lambda,q}) + \mathcal{E}_z(f) - \mathcal{E}(f) \\
& + \mathcal{E}(f) - \mathcal{E}(f_\rho) + \lambda \Omega^q_z(f).
\end{align*}
\]

Since \( f_\rho \in W^r_2 \) with \( r > \frac{d}{2} \), it follows from the Sobolev embedding theorem that \( f_\rho \in C(B^d) \). Thus, it can be deduced from Proposition 3 and Proposition 4 that there exists a \( P_\rho \in \mathcal{P}_n \) such that

\[
\|P_\rho\| \leq c\|f_\rho\| \quad \text{and} \quad \|f_\rho - P_\rho\| \leq CE_{[n/2]}(f_\rho),
\]

where \([t]\) denotes the largest integer not larger than \( t \) and \( \| \cdot \| \) denotes the uniform norm on \( B^d \).

The above inequalities together with the well known Jackson inequality imply that there exists a \( P_\rho \in \mathcal{P}_n \) such that for all \( f_\rho \in W^r_2 \) with \( r > \frac{d}{2} \), there holds

\[
\|P_\rho\| \leq c\|f_\rho\| \quad \text{and} \quad \|f_\rho - P_\rho\|^2 \leq C n^{-2r}.
\]

Let \( \mathcal{H}^*_{L,z} := \{ f \in \mathcal{H}_{L,z} : \|f\| \leq cM \} \), where \( c \) is defined as in (22). Define

\[
f^*_z := \arg \min_{f \in \mathcal{H}^*_{L,z}} \| f - f_\rho \|_\rho^2 + \lambda \Omega^q_z(f).
\]
Then we have
\[
E(\pi_M f_{z,\lambda,q}) - E(f_\rho) \leq \{E(f_\rho^*) - E(f_\rho) + \lambda \Omega_2^q(f_\rho^*)\} \\
+ \{E(\Pi_M f_{z,\lambda,q}) - E_\rho(\Pi_M f_{z,\lambda,q}) + E_\rho(f_\rho^*) - E(f_\rho^*)\} \\
=: \mathcal{D}(z, \lambda, q) + S(z, \lambda, q),
\]
where \(\mathcal{D}(z, \lambda, q)\) and \(S(z, \lambda, q)\) is called the approximation error and sample error, respectively.

**Proposition 6.** Let \(m, n \in \mathbb{N}, r > d/2\) and \(f_\rho \in W_r^r\). Then, with confidence at least \(1 - 2 \exp\{-cm/n^q\}\), there holds
\[
\mathcal{D}(z, \lambda, q) \leq C (n^{-2r} + \lambda m),
\]
(25)
where \(C\) and \(c\) are constants depending only on \(d\) and \(r\).

**Proof.** From Proposition 1, it is easy to deduce that
\[
P_\rho(x) = \int_{\mathbb{B}^d} P_\rho(y)L_2n(x, y)dy.
\]
Thus, Lemma 5 with \(\varepsilon = \frac{1}{2}\) yields that with confidence at least \(1 - 2 \exp\{-cm/n^q\}\), there exists a set of real numbers \(\{a_i\}_{i=1}^m\) satisfying \(\sum_{i=1}^m |a_i|^q \leq 2\Omega_d m^{1-q}\) for \(q \geq 1\) such that
\[
P_\rho(x) = \sum_{i=1}^m a_i P_\rho(x_i)L_2n(x_i, x).
\]
The above observation together with (23) implies that with confidence at least \(1 - 2 \exp\{-cm/n^q\}\), there exists a \(g^*(x) := \sum_{i=1}^m a_i P_\rho(x_i)L_2n(x_i, x) \in \mathcal{H}_L^*\) such that for arbitrary \(f_\rho \in W_r^r\), there holds
\[
\|g^* - f_\rho\|_\rho^2 \leq \|g^* - f_\rho\|^2 \leq Cn^{-2r},
\]
and
\[
\Omega_2^q(g^*) = \sum_{i=1}^m |a_i P_\rho(x_i)|^q \leq (cM)^q \sum_{i=1}^m |a_i|^q \leq Cm,
\]
where \(C\) is a constant depending only on \(d\) and \(M\). Indeed, if \(q \geq 1\), we have \(\sum_{i=1}^m |a_i|^q \leq 2\Omega_d m^{1-q}\). Without loss of generality, we assume \(m \geq cM\). Then there holds
\[
\sum_{i=1}^m |a_i P_\rho(x_i)|^q \leq (cM)^q 2\Omega_d m^{1-q} \leq 2\Omega_{d-1} m.
\]
If \(0 < q < 1\), it follows from the Hölder inequality that
\[
\sum_{i=1}^{m} |a_i|^q \leq \left( \sum_{i=1}^{m} |a_i| \right)^q \left( \sum_{i=1}^{m} 1 \right)^{1-q} \leq m^{1-q}(2\Omega_d)^q \leq 2\Omega_dm.
\]
Thus, for all \(q_0 \leq q \leq \infty\), there holds
\[
\sum_{i=1}^{m} |a_i P_{\rho}(x_i)|^q \leq 2cM\Omega_{d-1}m.
\]
It thus follows from the definition of \(f^*_z\) that the inequalities
\[
D(z, \lambda, q) \leq \|g^* - f_\rho\|_\rho^2 + \lambda\Omega^2_2(g^*) \leq C (n^{-2r} + \lambda m)
\]
holds with confidence at least \(1 - 2 \exp\{-cm/n^d\}\). \(\square\)

### 6.4 A sample error estimate

For further use, we also need introducing some quantities to measure the complexity of a space \[14\], \[16\]. Let \(B\) be a Banach space and \(V\) a compact set in \(B\). The quantity \(H_\varepsilon(V, B) = \log_2 N_\varepsilon(V, B)\), where \(N_\varepsilon(V, B)\) is the number of elements in least \(\varepsilon\)-net of \(V\), is called \(\varepsilon\)-entropy of \(V\) in \(B\). The quantity \(N_\varepsilon(V, B)\) is called the \(\varepsilon\)-covering number of \(V\). For any \(t \in \mathbb{R}\), define
\[
\operatorname{sgn}(t) := \begin{cases} 
1, & \text{if } t \geq 0, \\
-1, & \text{if } t < 0. 
\end{cases}
\]
If a vector \(t = (t_1, \ldots, t_n)\) belongs to \(\mathbb{R}^n\), then we denote by \(\operatorname{sgn}(t)\) the vector \((\operatorname{sgn}(t_1), \ldots, \operatorname{sgn}(t_n))\).

The VC dimension of a set \(V\) over \(B^d\), denoted as \(VCdim(V, B^d)\), is the maximal natural number \(m\) such that there exists a collection \((\mu_1, \ldots, \mu_m)\) in \(B^d\) such that the cardinality of the \(\operatorname{sgn}\)-vectors set
\[
S = \{ (\operatorname{sgn}(v(\mu_1)), \ldots, \operatorname{sgn}(v(\mu_m)) ) : v \in V \}
\]
equals to \(2^m\), that is, the set \(S\) coincides with the set of all vertexes of unit cube in \(\mathbb{R}^m\). The quantity
\[
Pdim(V, B^d) := \max_g VCDim(V + g, B^d),
\]
is called pseudo-dimension of the set \(V\) over \(B^d\), where \(g\) runs all functions defined on \(B^d\) and \(V + g = \{ v + g : v \in V \} \).

Mendelson and Vershinin \[18\] (see also \[16\]) has established the following important relation between Pseudo-dimension and \(\varepsilon\)-entropy.
Lemma 8. Let $V(B^d)$ be a class of functions which consists of all functions $f \in V$ satisfying $|f(x)| \leq R$ for all $x \in B^d$. Then,

$$H_\varepsilon(V(B^d), L^2(B^d)) \leq cPdim(V, B^d) \log_2 \frac{R}{\varepsilon},$$

where $c$ is an absolute positive constant.

The following Lemma 9 further shows that the pseudo-dimension of arbitrary $m$-dimensional vector space is $m$.

Lemma 9. Let $\mathcal{H}$ be an $m$-dimensional vector space of functions from $B^d$ into $\mathbb{R}$. Then $Pdim(\mathcal{H}, B^d) = m$.

We also need to apply the following concentration inequality.

Lemma 10. Let $G$ be a set of functions on $\mathbb{Z}$ such that, for some $c \geq 0$, $|g - E(g)| \leq B$ almost everywhere and $E(g^2) \leq cE(g)$ for each $g \in G$. Then, for every $\varepsilon > 0$,

$$\text{Prob}_{x \in \mathbb{Z}^m} \left\{ \sup_{f \in G} \frac{E(g) - \frac{1}{m} \sum_{i=1}^{m} g(z_i)}{\sqrt{E(g) + \varepsilon}} \leq \sqrt{\varepsilon} \right\} \leq N_\varepsilon(G, C(B^d)) \exp \left\{ -\frac{m \varepsilon}{2c + 2B^3} \right\}.$$ 

The following Proposition gives an upper bound of sample error.

Proposition 7. Let $m, n \in \mathbb{N}$, $\varepsilon > 0$, and $f_{z, \lambda, q}$ be defined as in (16). Then with confidence at least

$$1 - \exp \left\{ cn^d \log \frac{Cn^{d+2} \max\{m^{1-1/q}, 1\} M^{2+1}}{\lambda^{1/q} \varepsilon} - \frac{3m \varepsilon}{128M^2} \right\}$$

there holds

$$S(z, \lambda, q) \leq \frac{1}{2}(E((\pi_M f_{z, \lambda, q}) - E(f_\rho)) + \frac{1}{2}D(z, \lambda, q) + 2\varepsilon.$$ 

Proof. If we set $\xi_1 := (\pi_M f_{z, \lambda, q}(x) - y)^2 - (f_\rho(x) - y)^2$, and $\xi_2 := (f^*_z(x) - y)^2 - (f_\rho(x) - y)^2$, then

$$E(\xi_1) = E((\pi_M f_{z, \lambda, q}) - E(f_\rho), \text{ and } E(\xi_2) = E(f^*_z) - E(f_\rho),$$

both of which are random variables. Hence, we can rewrite the sample error as

$$S(z, \lambda, q) = \left\{ E(\xi_1) - \frac{1}{m} \sum_{i=1}^{m} \xi_1(z_i) \right\} + \left\{ \frac{1}{m} \sum_{i=1}^{m} \xi_2(z_i) - E(\xi_2) \right\} =: S_1 + S_2.$$
Define
\[ B^q_R := \left\{ f = \sum_{i=1}^{m} a_i L_{2n}(x_i, x) : \sum_{i=1}^{m} |a_i|^q \leq R \right\}. \]

As \( f_{z, \lambda, q} := \sum_{i=1}^{m} b_i L_{2n}(x_i, x) \), it follows from (16) that
\[ \lambda \sum_{i=1}^{m} |b_i|^q \leq \frac{1}{m} \sum_{i=1}^{m} (0 - y_i)^2 + 0 \leq M^2, \]
which implies \( f_{z, \lambda, q} \in B^q_{M^2/\lambda} \). Let
\[ F_\lambda := \left\{ g = (\pi_M f(x) - y)^2 - (f_\rho(x) - y)^2 : f \in B^q_{M^2/\lambda} \right\}. \]

Then, for any fixed \( g \in F_\lambda \), there exists \( f \in B^q_{M^2/\lambda} \) such that \( g(z) = (\pi_M f(x) - y)^2 - (f_\rho(x) - y)^2 \).

It is easy to deduce that
\[ E(g) = E(\pi_M f) - E(f_\rho) \geq 0, \]
\[ \frac{1}{m} \sum_{i=1}^{m} g(z_i) = E(\pi_M f) - E(f_\rho), \]
and
\[ g(z) = (\pi_M f(x) - f_\rho(x)) \left[ (\pi_M f(x) - y) + (f_\rho(x) - y) \right]. \]

Since \(|y| \leq M \) and \(|f_\rho(x)| \leq M \) almost everywhere, we find that
\[ |g(z)| \leq (M + M)(M + 3M) \leq 8M^2. \]

Of course, we have
\[ |g(z) - E(g)| \leq B := 16M^2 \]
almost everywhere and
\[ E(g^2) = E \left[ (\pi_M f(x) - f_\rho(x))^2 \right] \leq 16M^2 \left\| \pi_M f - f_\rho \right\|_\rho^2 = 16M^2 E(g), \]

Therefore, we can apply Lemma 10 to the set of functions \( F_\lambda \) with \( B = c = 16M^2 \), yielding
\[ \sup_{f \in B^q_{M^2/\lambda}} \frac{E(\pi_M f) - E(f_\rho) - (E_{\pi_M}(\pi_M f) - E_{\pi_M}(f_\rho))}{\sqrt{E(\pi_M f) - E(f_\rho) + \varepsilon}} \leq \sqrt{\varepsilon} \quad (27) \]
with confidence at least
\[ 1 - N_{\varepsilon/4} \left( F_\lambda, C(B^d) \right) \exp \left\{ -\frac{m\varepsilon}{2B + \frac{2}{3}c} \right\} \geq 1 - N_{\varepsilon/4} \left( F_\lambda, C(B^d) \right) \exp \left\{ -\frac{3m\varepsilon}{128M^2} \right\}. \]
For every \( f_1, f_2 \in B^q_{M^2/\lambda} \), we have
\[
| (\pi_M f_1(x) - y)^2 - (\pi_M f_2(x) - y)^2 | \leq 4M \| f_1 - f_2 \|.
\]
Thus, a \( (\frac{1}{16M}) \)-covering of \( B^q_{M^2/\lambda} \) provides an \( \varepsilon \)-covering of \( \mathcal{F}_\lambda \) for any \( \varepsilon > 0 \). This implies
\[
\mathcal{N}_{\varepsilon/q}(\mathcal{F}_\lambda, C(B^d)) \leq \mathcal{N}_{\varepsilon/(16M)}(B^q_{M^2/\lambda}, C(B^d)) \leq \mathcal{N}_{\varepsilon/(16M)}(B^q_{M^2/\lambda}, L^2(B^d)).
\]
It is also needed to derive an upper bound estimation for \( \mathcal{N}_{\varepsilon/(16M)}(B^q_{M^2/\lambda}, L^2(B^d)) \). For \( q \geq 1 \), and \( f \in B^q_{M^2/\lambda} \), it follows from Proposition 2 and the Hölder inequality that
\[
\left| \sum_{i=1}^m b_i L_{2n}(x_i, x) \right| \leq \max_{x, y \in B^d} L_{2n}(x, y) \sum_{i=1}^m |b_i| \leq Cn^{2+d}(M^2/\lambda)^{\frac{1}{q}} m^{1-1/q}.
\]
For \( 0 < q < 1 \), and \( f \in B^q_{M^2/\lambda} \), using (12) again we can obtain
\[
\left| \sum_{i=1}^m b_i L_{2n}(x_i, x) \right| \leq \max_{x, y \in B^d} L_{2n}(x, y) \sum_{i=1}^m |b_i| \leq Cn^{2+d}(M^2/\lambda)^{\frac{1}{q}}.
\]
Consequently, for arbitrary \( f \in B^q_{M^2/\lambda} \) and arbitrary \( 0 < q < \infty \), there holds
\[
\left| \sum_{i=1}^m b_i L_{2n}(x_i, x) \right| \leq Cn^{2+d} \max\{m^{1-1/q}, 1\} (M^2/\lambda)^{\frac{1}{q}}.
\]
Noting that \( \mathcal{H}_{L, z} \) is a finite dimensional linear space with its dimension not larger than \( cn^d \), it follows from Lemma 9 and Lemma 8 that
\[
\log \mathcal{N}_{\varepsilon/(16M)}(B^q_{M^2/\lambda}, L^2(B^d)) \leq cn^d \log \frac{Cn^{d+2} \max\{m^{1-1/q}, 1\} M^{\frac{2}{q}+1}}{\lambda^{1/q_\varepsilon}}.
\]
Accordingly,
\[
\mathcal{N}_{\varepsilon/q}(\mathcal{F}_\lambda, C(B^d)) \leq \exp \left\{ cn^d \log \frac{Cn^{d+2} \max\{m^{1-1/q}, 1\} M^{\frac{2}{q}+1}}{\lambda^{1/q_\varepsilon}} \right\},
\]
which together with (27) further yields
\[
\mathcal{S}_1 \leq \frac{1}{2} (\mathcal{E}(\pi_M f_{x, \lambda, q}) - \mathcal{E}(f_\rho)) + \varepsilon \tag{28}
\]
with confidence at least
\[
1 - \exp \left\{ cn^d \log \frac{Cn^{d+2} \max\{m^{1-1/q}, 1\} M^{\frac{2}{q}+1}}{\lambda^{1/q_\varepsilon}} - \frac{3m\varepsilon}{128M^2} \right\}.
\]
Now, we turn to estimate \( \mathcal{S}_2 \). By definition of \( f^*_x \), we have \( \| f^*_x \| \leq cM \). Let
\[
\mathcal{G} := \{ g = (f(x) - y)^2 - (f_\rho(x) - y)^2 : f \in \mathcal{H}_{L, z} \}.
\]
Then for any fixed $g \in \mathcal{G}$, there exists an $f \in H^*_L$ such that $g(z) = (f(x) - y)^2 - (f_\rho(x) - y)^2$.

Similarly, we have

$$E(g) = \mathcal{E}(f) - \mathcal{E}(f_\rho) \geq 0 \quad \text{and} \quad \frac{1}{m} \sum_{i=1}^m g(z_i) = \mathcal{E}_z(f) - \mathcal{E}_z(f_\rho).$$

Since $|y| \leq M$, $|f_\rho(x)| \leq M$ and $\|f\| \leq cM$ almost everywhere, we get

$$|g(z)| \leq (c + 3)^2M^2 \quad \text{and} \quad |g(z) - E(g)| \leq B := 2(c + 3)^2M^2$$

almost everywhere. Furthermore,

$$E(g^2) \leq 2(c + 3)^2M^2\|f - f_\rho\|_\rho^2 = 2(c + 3)^2M^2E(g).$$

Then we apply Lemma [10] again to the set of functions $\mathcal{G}$ with $B = c = 2(c + 3)^2M^2$ and obtain

$$\sup_{f \in H^*_L} \mathcal{E}(f) - \mathcal{E}(f_\rho) \leq \sqrt{\varepsilon}$$

with confidence at least

$$1 - \mathcal{N}_{\varepsilon/4}(\mathcal{G}, C(\mathbb{B}^d)) \exp \left\{-\frac{m\varepsilon}{2B + \frac{3}{2}c}\right\} \geq 1 - \mathcal{N}_{\varepsilon/4}(\mathcal{G}, C(\mathbb{B}^d)) \exp \left\{-\frac{3m\varepsilon}{16(c + 3)^2M^2}\right\}.$$

For every $f_1, f_2 \in H^*_L$, we have

$$|(f_1(x) - y)^2 - (f_2(x) - y)^2| \leq (2c + 2)M\|f_1 - f_2\|.$$

Thus, for any $\varepsilon > 0$, a $(\frac{\varepsilon}{2M + 2M})$-covering of $H^*_L$ provides an $\varepsilon$-covering of $\mathcal{G}$. This means

$$\mathcal{N}_{\varepsilon/4}(\mathcal{G}, C(\mathbb{B}^d)) \leq \mathcal{N}_{\varepsilon/(8M + 8\varepsilon M)}(H^*_L, C(\mathbb{B}^d))$$

By definition of $H^*_L$, we then deduce from [14] Theorem 5.3 that

$$\log \mathcal{N}_{\varepsilon/(8M + 8\varepsilon M)}(H^*_L, C(\mathbb{B}^d)) \leq Cn^d \log \left(\frac{32M + 32cM}{\varepsilon}\right).$$

Hence,

$$\mathcal{N}_{\varepsilon/4}(\mathcal{G}, C(\mathbb{B}^d)) \leq \exp \left\{Cn^d \log \left(\frac{32M + 32cM}{\varepsilon}\right)\right\},$$

which together with [29] yields

$$S_2 \leq \frac{1}{2}(\mathcal{E}(f_\rho^*) - \mathcal{E}(f_\rho)) + \varepsilon$$

with confidence at least

$$1 - \exp \left\{Cn^d \log \left(\frac{32M + 32cM}{\varepsilon}\right) - \frac{3m\varepsilon}{16(c + 3)^2M^2}\right\};$$

This finishes the proof of Proposition [7].
6.5 Learning rate analysis

Now we are in a position to deduce the final learning rate of \( l^q \) regularization schemes (16). Firstly, it follows from Propositions 6 and 7 that

\[
\mathcal{E}(\pi_M f_{z, \lambda, q}) - \mathcal{E}(f_\rho) \leq \mathcal{D}(z, \lambda, q) + \mathcal{S}_1 + \mathcal{S}_2 \leq C \left( n^{-2r} + \lambda m \right) + \frac{1}{2}(\mathcal{E}(\pi_M f_{z, \lambda, q}) - \mathcal{E}(f_\rho)) + \varepsilon + \frac{1}{2}(\mathcal{E}(f^*_z) - \mathcal{E}(f_\rho)) + \varepsilon
\]

holds with confidence at least

\[
1 - 2 \exp\left\{ -cn^d/n^d \right\} - \exp\left\{ cn^d \log \frac{Cn^{d+2} \max\{m^{1-1/q}, 1\} M^{2q_0+1}}{\lambda^{1/q_0} \varepsilon} - \frac{3m \varepsilon}{128M^2} \right\}
\]

Then, by setting \( \varepsilon \geq \varepsilon_m^+ \geq C(m/\log m)^{-2r/(2r+d)} \), \( n = \left[ c_0 \varepsilon^{-1/(2r)} \right] \) and \( \lambda = m^{-1} \varepsilon \), it follows from \( r > d/2 \) that

\[
1 - 2 \exp\left\{ -Cm \varepsilon^{d/(2r)} \right\} - \exp\left\{ C \varepsilon^{-d/(2r)} \log \frac{1}{\varepsilon} - 3m \varepsilon/(16(c+3)^2 M^2) \right\}
\]

That is, for \( \varepsilon \geq \varepsilon_m^+ \)

\[
\mathcal{E}(\pi_M f_{z, \lambda, q}) - \mathcal{E}(f_\rho) \leq 6 \varepsilon
\]

holds with confidence at least \( 1 - \exp\{-Cm \varepsilon\} \).

The lower bound can be more easily deduced. Actually, it follows from [10, Equation (3.27)] (see also [17]) that for any estimator \( f_z \in \Phi_m \), there holds

\[
\sup_{f_\rho \in \Phi} P_m \{ z : \| f_z - f_\rho \|_2 \geq \varepsilon \} \geq \begin{cases} \varepsilon_0, & \varepsilon < \varepsilon^-, \\ e^{-c m \varepsilon}, & \varepsilon \geq \varepsilon^-, \end{cases}
\]

where \( \varepsilon_0 = \frac{1}{2} \) and \( \varepsilon^- = cm^{-2r/(2r+d)} \) for some universal constant \( c \). With this, the proof of Theorem 1 is completed.
7 Further discussion and conclusion

In studies and applications, regularization is a fundamental skill to improve on performance of a learning machine. The $l^q$ regularization schemes with $0 < q < \infty$ are well known to be central in use. In this paper, we have studied the dependency problem of the generalization capability of $l^q$ regularization with the choice of $q$. Through formulating a new methodology of estimation of generalization error, we have shown that there is at least a positive definite kernel, say, $L_{2n}$, such that associated with such a kernel, the learning rate of the $l^q$ regularization schemes is independent of the choice of $q$. (To be more precise, we verified that with the kernel $L_{2n}$, all $l^q$ regularization schemes can attain the same almost optimal learning rate in the following sense: up to a logarithmic factor, the upper and lower bounds of generalization error of the $l^q$ regularization schemes are asymptotically identical). This implies that for some kernels, the generalization capability of $l^q$ regularization may not depend on $q$. Therefore, as far as the generalization capability is concerned, for those kernels, the choice of $q$ is not important, which then relaxes the model selection difficulty in applications. The problem is, however, far complicated. We have also illustrated in Section 2 that there exists a kernel with which the generalization capability of $l^q$ regularization heavily depends on the choice of $q$. Thus, answering completely whether or not the choice of $q$ affects the generalization of $l^q$ regularization is by no means easy and completed.

Though we have constructed a concrete kernel example, the localized polynomial kernel $L_{2n}$, with which implementing the $l^q$ regularization in SDHS can realize the almost optimal learning rate, and this is independence of the choice of $q$, we have not provided a practically feasible algorithm to implement the learning with the almost optimal generalization capability. This is because the kernel $L_{2n}$ we have constructed is not easily computed in practice, even though we can use the cubature formula (Lemma 2) to discretize it. Thus, seeking the kernels that possesses the similar property as that of $L_{2n}$ and can be implemented easily deserve study. This is under our current investigation.

Appendix A: Proof of Lemma 3

To prove Lemma 3 we need the following Aronszajn Theorem (see [1]).
Lemma 11. Let $\mathcal{H}$ be a separable Hilbert space of functions over $X$ with orthonormal basis $\{\phi_k\}_{k=0}^\infty$. $\mathcal{H}$ is a reproducing kernel Hilbert space if and only if
\[ \sum_{k=0}^\infty |\phi_k(x)|^2 < \infty \]
for all $x \in X$. The unique reproducing kernel $K$ is defined by
\[ K(x, y) : = \sum_{k=0}^\infty \phi_k(x)\phi_k(y). \]

Proof of Lemma 31 Since
\[ \{P_{k,j,i} : k = 0, \ldots, n, j = k, k - 2, \ldots, \varepsilon_k, i = 1, 2, \ldots, D_j^{d-1}\} \]
is an orthonormal basis for $\mathcal{P}_n$, for arbitrary $P \in \mathcal{P}_n$, there exists a set of real numbers $a_{k,j,i}$ such that
\[ P(x) = \sum_{k=0}^n \sum_{j}^{D_j^{d-1}} a_{k,j,i} P_{k,j,i}(x), \]
where the summation concerning the index $j$ is $k, k - 2, \ldots, \varepsilon_k$. On the other hand, it follows from (8) that
\[ \sum_{j}^{D_j^{d-1}} a_{k,j,i} P_{k,j,i}(x) P_{k,j,i}(y) = v_k^2 \sum_{j}^{D_j^{d-1}} U_k(x \cdot \xi) \int_{S^{d-1}} U_k(y \cdot \eta) K_j^\ast (\xi \cdot \eta) d\omega_{d-1}(\eta). \]
Thus, the addition formula (7) yields
\[ \sum_{j}^{D_j^{d-1}} a_{k,j,i} P_{k,j,i}(y) = v_k^2 \sum_{j}^{D_j^{d-1}} U_k(x \cdot \xi) \int_{S^{d-1}} U_k(y \cdot \eta) K_j^\ast (\xi \cdot \eta) d\omega_{d-1}(\eta). \]
The above equality together with (5) and (6) implies
\[
\sum_j D_{d-1} \sum_{i=1}^{D_{d-1}} P_{k,j,i}(x)P_{k,j,i}(y)
= v_k^2 \int_{S^{d-1}} U_k(x, \xi) \int_{S^{d-1}} U_k(y, \eta) \sum_j K_j^*(\xi, \eta) d\omega_{d-1}(\xi) d\omega_{d-1}(\eta)
= \frac{v_k^4}{U_k(1)} \int_{S^{d-1}} U_k(x, \xi) \int_{S^{d-1}} U_k(y, \eta) U_k(\xi, \eta) d\omega_{d-1}(\xi) d\omega_{d-1}(\eta)
= v_k^2 \int_{S^{d-1}} U_k(\xi, x) U_k(\xi, y) d\omega_{d-1}(\xi).
\]
Therefore, there holds
\[
K_n(x, y) = \sum_{k=0}^{\infty} \sum_j \sum_{i=1}^{D_{d-1}} P_{k,j,i}(x)P_{k,j,i}(y).
\]
The above equality together with Lemma 11 yields Lemma 3.

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