LATTICE CHIRAL GAUGE THEORY WITH FINELY-GRAINED FERMIONS

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Abstract

We discuss the problem of formulating the continuum limit of chiral gauge theories (χGT) in the absence of an explicitly gauge-invariant regulator for the fermions. A solution is proposed which is independent of the details of the regulator, wherein one considers two cutoff scales, \( \Lambda_f \gg \Lambda_b \), for the fermions and the gauge bosons respectively. Our recent non-perturbative lattice construction in which the fermions live on a finer lattice than do the gauge bosons, is seen to be an example of such a scheme, providing a finite algorithm for simulating χGT. The essential difference with previous (one-cutoff) lattice schemes is clarified: in our formulation the breakage of gauge invariance is small, \( O(\Lambda_b^2/\Lambda_f^2) \), and vanishes in the continuum limit. Finally, we argue against 2-D models being significant testing grounds for 4-D regulators of χGT.

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Chiral gauge theories (χGT) contain very interesting features which make them worth exploring non-perturbatively. The lattice provides an elegant method for gauge-invariant non-perturbative studies of non-abelian gauge field interactions. Unfortunately, a lattice formulation of χGT, suitable for computer simulations, has proven to be elusive, plagued by the infamous fermion doubling problem [1]. The removal of the unphysical doublet modes from the spectrum requires explicit breakage of chiral gauge symmetry, or the introduction of new fields carrying the gauge charge. One must then take care to eliminate these undesired effects in the continuum limit. For theories with a net gauge anomaly in the fermion representations this is indeed impossible.

In this paper, we discuss our recent lattice formulation [2]. We first explain the essential idea of using a two-cutoff regulator, and then briefly review our implementation of this idea in a finite computational scheme. Finally, we compare with other proposals in the literature.

1 Two-Cutoff Regulators.

We assume that we have a gauge-invariant regulator that cuts off non-abelian gauge boson momenta above Λ_b, while the regulator for the fermion loops (with cutoff scale Λ_f) breaks chiral gauge symmetry explicitly. It is not important for now that the particular cutoff method in question is the lattice. (Indeed we do not know of any continuum or lattice fermion regulator which is exactly gauge-invariant. The last reference in [1] provides good reasons why this should be so.). The usual choice is Λ_b = Λ_f, but we will soon see the virtue of taking Λ_f ≫ Λ_b.

After formally integrating over fermion fields in the path integral, we get

\[ Z = \int \mathcal{D}A_{\Lambda_b} e^{-\frac{1}{4g_0^2} tr F_{\mu\nu}^2 + J_\mu A_\mu} e^{\Gamma_{\Lambda_f}[A_\mu]} . \]  

(1)

We have written the regulated chiral fermion determinant as the exponential of the fermion one-loop effective action, \( \Gamma_{\Lambda_f}[A_\mu] \). For simplicity we omit sources for external fermion lines in this discussion. They are treated fully in ref. [2]. The gauge boson measure is taken to include gauge-fixing and ghost terms. On the lattice it is desireable to omit gauge-fixing, but it will be easy to account for this once we understand the gauge-fixed case.

The presence of the cutoff makes \( \Gamma_{\Lambda_f}[A] \) a gauge non-invariant functional of \( A_\mu \), whereas gauge-invariance (BRST-invariance) is crucial for maintaining unitarity in the continuum limit \( \Lambda_{f,b} \rightarrow \infty \). In particular, divergent
fermion loops can induce non-invariant effects which survive in the limit $\Lambda_f \to \infty$. These divergences are local in $A_\mu$ and can be legitimately subtracted (we will see that this is only true in a theory in which gauge anomalies cancel). We therefore assume that any necessary gauge non-invariant subtractions are already done in defining $\Gamma_{\Lambda_f}$, so that

$$\delta^a_x \Gamma[A] \sim O(1/\Lambda_f^n), \quad n > 0,$$

where $\delta^a_x$ is the gauge transformation generator at $x$ in the gauge-direction $a$. Naively one might think that gauge invariance is then restored to the theory in the continuum limit. However, for $\Lambda_f = \Lambda_b$ this conclusion is only valid at one-loop. When the gauge fields in $\Gamma_{\Lambda_f}[A]$ get integrated in $Z$, gauge boson loops induce BRST non-invariant terms of order $\Lambda_b^m/\Lambda_f^n$, $m \geq n$. Thus just having eq. (2) in a one-cutoff scheme is not enough: the BRST-violating divergences would have to be obtained and subtracted to all orders, thereby precluding non-perturbative investigations! On the other hand, in our two-cutoff scheme the naive expectation is really true; gauge invariance is clearly restored to all orders in the continuum limit $\Lambda_f, \Lambda_b \to \infty$ (that is $\Lambda_f \to \infty$ before $\Lambda_b \to \infty$). The importance of taking the continuum limit in this fashion was stressed earlier in ref. [3].

How can we view this less familiar two-cutoff scheme? In principle, one can integrate out the fermions between $\Lambda_f$ and $\Lambda_b$ only, thereby matching to a theory with a single cutoff $\Lambda_b$, with a local effective lagrangian which is not gauge-invariant. However we know from our two-cutoff analysis that for $\Lambda_f \gg \Lambda_b$ ($\Lambda_f$ appearing in the couplings of the one-cutoff lagrangian now) the non-invariant terms in this effective lagrangian compensate for the gauge non-invariance of the cutoff procedure to all orders when calculating amplitudes. In this matching the one-cutoff gauge coupling at $\Lambda_b$, $g_b$, is one-loop renormalized by the fermions integrated out between the two cutoffs, so that

$$1/g_b^2 = 1/g_0^2 + \frac{t_2(\psi)}{12 \pi^2} \log(\Lambda_f/\Lambda_b),$$

where $g_0$ is the two-cutoff bare coupling in eq. (1). We can rewrite the partition functional in terms of $g_b$:

$$Z = \int DA|_{\Lambda_b} e^{-\frac{1}{4 g_b^2} \text{tr} F_{\mu \nu}^2 + J_\mu A_\mu} e^{\Gamma_{\Lambda_f}^R [A_\mu]},$$

where we define a ‘fully renormalized’ fermion effective action

$$\Gamma^R = \Gamma_{\Lambda_f} + \frac{t_2(\psi)}{48 \pi^2} \log(\Lambda_f/\Lambda_b) \int d^4 x \text{tr} F_{\mu \nu} F^{\mu \nu}.$$
Γ^R is finite as \( \Lambda_f \to \infty \) with \( \Lambda_b \) fixed, because its only divergence with \( \Lambda_f \) has been subtracted (the gauge non-invariant ones were already subtracted). In this form it is simple to see how the BRST invariance of the theory emerges in our continuum limit: one first takes \( \Lambda_f \to \infty \) so the resulting (finite) \( \Gamma^R \) is exactly gauge-invariant (since \( \Gamma^R \) and \( \Gamma \) differ by a gauge invariant term).

Now let us translate the above discussion to the case where there is no gauge-fixing, by putting back the functional integration over the gauge orbits \( \{A^\Omega\} \),

\[
Z = \int D\Omega_{|\Lambda_b} \int DA_{|\Lambda_b} e^{-\int \frac{1}{4g^2} tr F_{\mu\nu}^2 e^{\Gamma^R_{\Lambda_f}[A^\Omega_{\mu}]}}. \tag{6}
\]

The crucial new feature here is that the gauge-orbit integration has introduced a new field (without a kinetic term at tree-level) into the partition functional. If \( \Gamma^R_{\Lambda_f} \) is not gauge-invariant, then \( \Gamma^R_{\Lambda_f}[A^\Omega_{\mu}] \) is dependent on \( \Omega \), and so the \( \Omega \) field in general becomes a strongly interacting field in the theory, as opposed to completely decoupling. Note that one can take the \( \Omega \) field to transform under the gauge group, in which case the theory is exactly BRST-invariant, but it is not the BRST-invariant theory we wanted! It contains extra unwanted degrees of freedom. In our two-cutoff continuum limit however, \( \Omega \) does decouple: taking the limit \( \Lambda_f \to \infty \) first, we obviously get a finite \( \Gamma^R \) which is completely independent of \( \Omega \).

### 2 A Lattice Implementation.

We implement the two-cutoff idea by having the fermions live on a lattice with spacing \( f \equiv 1/\Lambda_f \) and the gauge bosons live on a lattice with spacing \( b \equiv 1/\Lambda_b \), with \( f \ll b \). The full details were worked out in ref. [2]. Earlier related ideas were discussed in refs. [4][5]. Our target continuum theory consists of left-handed fermions, \( \psi_L \), transforming under the gauge group, and an equal number of decoupled sterile right-handed fermions, \( \psi_R \). Therefore the continuum fermion covariant derivative is given by

\[
\hat{D} \equiv (\partial + i A) \frac{1 - \gamma_5}{2} + \partial \frac{1 + \gamma_5}{2}. \tag{7}
\]

Our lattice regularization of the partition functional is given by

\[
\int DA_{\mu} e^{-\int \frac{1}{4g^2} tr F_{\mu\nu}^2 + tr \sum_s J_\mu A_\mu \ldots} \to \prod_s dU_\mu(s) e^{-S_0[U] + tr \sum_s J_\mu A_\mu \ldots},
\]

\[
e^{\Gamma[u[U]]} = \det[i \hat{D} + f D^2]^{\frac{1}{2}} \cdot \exp(i \arg \det[i \hat{D} + f D^2]). \tag{8}
\]
The lattice action for the gauge fields is entirely standard and gauge invar- 
iant: \( U_\mu(s) = e^{ibA_\mu(s)} \) is the gauge field variable on the link \((s,s+\hat{\mu})\) of a regular lattice with spacing \( b \), and \( S_g[U] \) is the standard Wilson action\(^8\). We have not used any gauge-fixing here. The fermions are however treated quite unconventionally. Their \( f \)-lattice finely subdivides the \( b \)-lattice, and the fermion effective action depends on the \( b \)-lattice gauge fields through a gauge field, \( u_\mu = e^{ifu_\mu} \) living on the links of the \( f \)-lattice. This \( f \)-lattice gauge field, \( u \), is not an independent degree of freedom in the path integral; it is a careful interpolation to the \( f \)-lattice of the \( b \)-lattice gauge field \( U \).

Before looking at the properties of the interpolation \( u[U] \), we clarify our definition of \( \Gamma[u] \) for arbitrary \( f \)-lattice gauge fields. The operators appearing in the determinants are just the naive \( f \)-lattice versions of the corresponding continuum ones. The determinants are then just ordinary finite determinants (in finite volume). Formally, if we send \( f \) to zero we see that we get the statement that the norm of the chiral fermion determinant is just the determinant for vector-like Dirac fermions. This is an identity because the norm is the product of determinants for the desired chiral fermions with that for the conjugate chiral fermions; the chiral fermions and their conjugates add up to a vector-like Dirac representation. Thus formally, as \( f \to 0 \) we have \( e^{\Gamma[u]} = \det[i\hat{D}] \), as we should. For finite \( f \) we have a legitimate regulator\(^2\) which separately regulates \( \text{Re} \Gamma \) and \( \text{Im} \Gamma \) (magnitude and phase of \( e^\Gamma \)), corresponding to the parity-even and parity-odd parts of the effective action. Earlier versions of this trick are in refs.\(^6\)\(^5\)\(^7\). This allows us to restrict the loss of gauge invariance due to the cutoff, because it is well known from lattice-QCD how to gauge invariantly regulate vector-like determinants while also eliminating the unwanted doubler poles in fermion propagators, by the addition of the gauge-invariant ‘standard Wilson’ term \( f D^2 \) (see \(8\)). Thus it is only \( \text{Im} \Gamma \) that breaks gauge invariance: the doubler poles have been eliminated by addition of a ‘chiral Wilson’ term, \( f \partial^2 \)\(^9\). Note there is no way to make this term gauge invariant (as expected from general arguments) because, in the \( \hat{D} \) determinant, \( \psi_L \) transforms under the gauge group while the \( \psi_R \) are singlets, and the Wilson term involves a chirality-flip.

An elementary power-counting\(^2\) shows that

\[
\delta_a^\nu \Gamma[u] = \text{local functional of } a_\nu(x) + O(f^2),
\]

This is in agreement with the general considerations of the previous discussion: the terms which do not vanish as \( f \to 0 \) should be local. Furthermore
they are parity-odd, because the parity-even part of $\Gamma$ has been gauge-invariantly regulated. The unique such local functional (up to a constant factor, which we simply calculated) is

$$\delta^a \Gamma[u] = -\frac{i}{12\pi^2} \epsilon_{\alpha\beta\gamma\delta} tr\left[\lambda^\alpha \partial_\alpha (a_\mu \partial_\beta a_\nu - \frac{i}{2} a_\mu a_\beta a_\nu)\right] + \mathcal{O}(f^2), \quad (10)$$

the relative coefficient between the two terms being fixed by the Wess-Zumino consistency condition \cite{10} (i.e. the requirement that the RHS is $\delta^a$ of something). The consistent anomaly term cannot be eliminated by adding suitably chosen local counterterms to $\Gamma[u]$ because it is not the result of $\delta^a$ on any local functional. This is the significance of the gauge anomaly. The potentially large ($\mathcal{O}(1)$) breakage of gauge invariance will however vanish when anomalies cancel among the fermion representations in $\psi_L$. We will assume this to be the case from now on, so that

$$\delta^a \Gamma[u] \sim \mathcal{O}(f^2). \quad (11)$$

Now let us take into account the fact that our $f$-lattice gauge field is to be obtained by interpolating the $b$-lattice gauge field. In ref. \cite{2}, inspired by ref. \cite{11}, we described how to do this, the interpolated field minimizing the ($f$-lattice) Yang-Mills action in each $b$-lattice hypercube subject to boundary conditions set by the $b$-lattice gauge field values. The details are not important here. Instead we just note that the interpolation procedure enjoys the following properties \cite{2}. (i) Gauge invariance: for any gauge transformation $\Omega$ on a $b$-lattice gauge field $U$, there exists an interpolated gauge transformation $\omega$ on $u$ such that

$$u_\mu^\omega[U] = u_\mu[U^\Omega]. \quad (12)$$

Therefore if $\Gamma[u]$ is exactly $f$-lattice gauge-invariant, $\Gamma[u,U]$ is exactly $b$-lattice gauge-invariant. (ii) Sufficient smoothness: Of course $\Gamma[u]$ is only gauge-invariant up to $\mathcal{O}(f^2)$, so in order for this to be true of $\Gamma[u,U]$, the interpolation procedure must be smooth enough to not introduce powers of $1/f$. (iii) Locality: the interpolation does not introduce any spurious singularities in $b$-lattice gauge field momenta into $\Gamma[u,U]$. (iv) Lattice spacetime symmetries are respected. The existence of this interpolation completes our construction of the regulated partition functional in the two-cutoff form.

Though we must choose chiral gauge representations with cancelled gauge anomalies, we will typically have global classical symmetries which are anomalous (such as baryon plus lepton number in the standard model). We have
checked by a fermion one-loop lattice calculation that the associated currents obey the well-known anomalous Ward identity up to $O(f^2)$ \[2\]. The one-loop result becomes exact to all orders in our continuum limit because, again, integrating over gauge boson fields cannot eliminate the $f^2$ suppression. This is a proof of the non-renormalization theorem for the one-loop anomaly. Such anomalies are the basis for the non-perturbative phenomenon of fermion-number violation (such as standard model (B+L)-violation) \[12\]. In our scheme cluster decomposition of the full theory must be carefully used to obtain fermion-number violating amplitudes from the fermion-number conserving sector, where external fermions are easily treated \[2\].

In the discussion of section 1, we formally considered the limit $\Lambda_f \to \infty$. Mathematically this was perfectly sensible, but in a finite computation we must keep $f$ small but finite. We know that there is a sufficiently small value of $f/b$ for any particular amplitude one wants, but the question is how small? We just need to take $f$ small enough so that we are insensitive to the $O(f^2)$ violations of gauge invariance. For example an induced non-invariant gauge boson mass term should have a coefficient of order $f^2/b^4$. To be insensitive to this in a physical volume $L^4$, we need $f/b^2$ to be smaller than the infrared cutoff, $1/L$. We believe this is a reasonably conservative estimate, though it is possible that even for larger values of $f/b$, the theory is already in the “symmetric phase” in the sense of ref. \[16\]. This is well worth exploring.

There are two features of our scheme that require greater computational effort compared with simulating vector-like theories such as QCD. They are the interpolation performed on each $b$-lattice gauge field, and the calculation of the determinant of a larger matrix since the fermions live on a finer lattice than the gauge bosons. Since 4-D simulations are difficult, one might imagine that it is possible to test the method in 2-D models. The chiral Schwinger model has often been considered as a test ground for $\chi$GT regulators \[5\][7][13][14]. However, we do not believe this provides a significant test for higher dimensions. 2-D is a very special case: instead of having gauge non-invariant effects of $O(\Lambda_b^m/\Lambda_f^m)$ as in 4-D, one has $g_0^m/\Lambda_f^m$ to all orders (since the coupling is dimensionful). This implies that all the non-anomalous chiral symmetries get restored in the continuum limit $\Lambda \to \infty$ in any one-cutoff construction satisfying eq. (2).
3 Discussion.

The essential improvement in our approach is that we have a regulated effective action where gauge invariance is broken by a small amount. The idea that a small breakage of gauge symmetry should not be important in the continuum limit goes back to refs. [15] [4]. Furthermore it has been tested in real simulations for a pure gauge theory [16]. The problem was however to clarify what was meant by a “small” breakage of the gauge symmetry. We have seen that in a one-cutoff construction, the breaking at one loop is typically $O(1/\Lambda^n)$, which is not in any sense small for gauge boson momenta of order the cutoff. In our two-lattice formulation, however, $(f/b)^2$ is the small parameter that controls the breaking of gauge invariance and can be made as small as necessary (obviously at some computational expense).

All previous proposals for regulating $\chi GT$ break the gauge symmetry or introduce extra degrees of freedom (eg. Higgs field). The well-known Wilson-Yukawa models [17] essentially have a construction like eq.(6) with $\Lambda_f = \Lambda_b$. These models have been extensively studied [18], with the result that there is no region in the phase diagram that succeeds in decoupling doublers and has charged chiral fermions in the spectrum. In light of the previous discussion, this is easy to understand: whenever doublers are decoupled, $\Omega$ remains strongly coupled and only sterile $\Omega - \psi$ composites are light. A similar problem afflicts the proposal of Eichten and Preskill [19].

The ‘overlap’ formulation of the chiral determinant [13] satisfies eq. (2). But because the authors take $f = b$ in their lattice proposal, beyond one-loop we expect there to be large deviations from the target theory as one attempts to take the continuum limit.

The so-called ‘gauge fixing’ approach was introduced by the Rome group [9] and also considered for different fermion lattice actions in [20] [7]. It corresponds to the form of eq.(1) with $f = b$. The $\Omega$ field is absent because of gauge-fixing. The breakage of gauge invariance is not small, but a complete set of BRST-violating counterterms is tuned so as to restore the BRST-identities in the continuum limit. The existence of this construction is on a very solid footing to all orders in perturbation theory, but it is not yet clear how the scheme will work in practice, and whether it is truly non-perturbative in the face of Gribov ambiguities in the gauge-fixing procedure [21].

Recently there has been renewed interest [11] [25] [24] [14] in the old idea of coupling interpolated lattice gauge fields to fermions regulated in the continuum, in an attempt to preserve chiral symmetries. This
was triggered by ‘t Hooft’s proposal [1] to preserve global chiral symmetries in vector-like gauge theories, using a Pauli-Villars regulator for the continuum fermions. In this case, the advantage of the Pauli-Villars regulator is clear: it does break the anomalous chiral symmetry, but it preserves the non-anomalous ones (in the context of QCD, for instance, pions would be exactly massless without the need for asymmetric counterterms). We want to stress however that, in dealing with $\chi$GT, the important point in ‘t Hooft’s proposal is not that the fermions live in the continuum, as is often thought. After all we do not know of any continuum gauge-invariant regulator for chiral fermions either. The point is that it permits the separation of the fermion and boson cutoff scales ($\Lambda_f >> \Lambda_b$), in such a way that the ratio of the two scales controls the breaking of the chiral symmetry. ‡

Refs. [3] differ from all other proposals in that the gauge fields are regulated by a higher covariant derivative procedure. However the correct two-cutoff limit is taken.

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