Guarantees in Fair Division: beyond Divide & Choose and Moving Knives

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Abstract

Divide and Choose among two agents, and the Diminishing Share (DS) and Moving Knife (MK) algorithms among many, elicit parsimonious information to guarantee to each a Fair Share, worth at least 1/n-th of the whole manna.

Our n-person Divide and Choose (D&C) rule, unlike DS and MK, works if the manna has subjectively good and bad parts. If utilities are additive over indivisible items, it implements the canonical “Fair Share up to one item” approximation.

The D&C rule also offers one interpretation of the Fair Share when utilities are neither additive nor monotonic. Under a mild continuity assumption, it guarantees to each agent her minMax utility: that of her best share in the worst possible partition. This is lower than her Maxmin utility: that of her worst share in the best possible partition.

When the manna is unanimously good, or unanimously bad, better guarantees than minMax are feasible. Our Bid & Choose rules fix an additive benchmark measure of shares, and ask agents to bid the smallest size of a share they find acceptable. The resulting Guarantee is between the minMax and Maxmin utilities.

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1 Introduction

Privacy preservation is a growing concern in a world of ever expanding information flows. It matters for trading, bargaining, and any resource allocation process where the information I reveal about my true position today can be detrimental to me in the next interaction, as future adversaries could benefit from this leakage. Thus a natural design constraint for a resource allocation mechanism is that it should elicit as little information about relevant characteristics of the participants as is necessary to achieve its normative goals.

The privacy concern is implicit in much of the mechanism design literature, not least in the search for strategyproof decision rules where my optimal message does not depend on other agents’ messages: if the mechanism itself ensures the confidentiality of individual reports, as in voting by secret ballot, only an aggregate signal of the profile of reports is revealed and privacy is preserved.

We take here a more demanding view of privacy preservation, one that applies when there is no neutral intermediary (mechanism manager) to process these reports confidentially, or we fear that confidentiality cannot be preserved in the future.¹ The archetypal example of such informationally parsimonious mechanisms is the iconic Divide and Choose (D&C₂) rule to divide a cake fairly: Divider only reports, whether this is true or not, that she is indifferent between the two shares of the cake she just cut, then Chooser reveals only the binary comparison of these shares. A related advantage is that each report requires a modest cognitive effort, neither Divider nor Chooser needs to form complete preference relations over all shares of the cake. The normative “raison d’être” of D&C₂ is fairness, specifically the Proportional Guarantee property: each agent can secure a share that is worth to him at least one half of the entire manna (the resources to be divided, of which a cake is just one example).

The early literature on fair division proposes two rules preserving the informational parsimony of D&C₂ and implementing the Proportional Guarantee for an arbitrary number of participants. First the Diminishing Share, DSₙ, rule due to Steinhaus ([32])

“Agent 1 cuts a share of the cake and offers it to agent 2 who can pass, or accept it and trim some of it; the remaining share is then offered to agent 3 who can pass or trim it etc.; the share goes to the last agent who touched it. Repeat between the remaining agents and cake.”

Next the Moving Knife, MKₙ, rule due to Dubins and Spanier ([15])

“A knife cuts an increasing share of the cake and agents can stop it at any time; the first agent who does gets the share cut so far. Repeat between the remaining agents and cake.”

To secure a share worth at least \( \frac{1}{n} \)-th of the value of the manna in DSₙ, you must cut a share worth exactly \( \frac{1}{n} \)-th if called to cut, pass on a share worth less than \( \frac{1}{n} \)-th, and trim a share that is worth more so that its value becomes \( \frac{1}{n} \)-th exactly. In MKₙ it suffices to stop the knife exactly when it has cut a share worth \( \frac{1}{n} \)-th of the total.

¹The confidentiality of a physical message can be challenged for a variety of legal reasons.
But these mechanisms only make sense if we assume, as the literature on cake division does almost systematically, that individual utilities are additive over the different parts of the manna. If utilities are not additive, \( \frac{1}{n} \)-th of the value of the manna depends on the utility representation of the underlying preferences and may not be feasible.\(^2\) If the manna is a bundle \( \omega \) of divisible commodities, splitting \( \omega \) equally is feasible but may be very unfair, in particular if utilities are not monotonic. Think of two agents sharing one unit of (non disposable) commodity, with utilities \( u_1(x) = x(1-x) \) (single-peaked) and \( u_2(x) = x(x-1) \) (single-dipped): equal split is the best possible outcome for agent 1, the worst for agent 2.

We identify a concept of **Fair Guarantee**: a lower bound on welfare that is feasible in all division problems and for all agents with ordinal preferences in a domain considerably larger than the additive and monotonic one of the typical cake cutting literature. And we propose an \( n \)-person version of Divide and Choose (D&C\(_n\)) to implement this Guarantee: D&C\(_n\) has the same moves as D&C\(_2\) (cut shares of equal value, pick among proposed shares), hence the same informational parsimony.

We describe D&C\(_n\) first before explaining the Guarantee it achieves. Agent 1 cuts the manna in \( n \) shares presumed of equal value to her; every other agent reports which of these shares they find acceptable; a simple matching algorithm gives one of “her” shares to agent 1, and some other shares to other agents (possibly none), making sure that the shares thus assigned are all unacceptable to the non served agents. Repeat with the remaining agents and manna.

In D&C\(_2\) (the ordinary Divide and Choose) Divider always guarantees her **Maxmin** utility: that of her worst share in the best partition she can offer to Chooser. And Chooser always guarantees his **minMax** utility: that of his best share in the worst partition he could face. Our D&C\(_n\) rule offers precisely these guarantees for any \( n \), the only restriction is that preferences are represented by “non atomic” utilities (continuous with respect to small changes of the shares): the first Divider is guaranteed her **Maxmin** utility, and everyone else gets at least his **minMax** utility. Moreover the **minMax** utility is weakly below the **Maxmin** one.\(^3\) Loosely speaking, the D&C\(_n\) rule is as informationally parsimonious as D&C\(_2\).

An important feature of D&C\(_n\) is that, unlike DS\(_n\) and MK\(_n\), it works for a “mixed manna”, some parts of which are desirable (money, tasty cake, valuable and resalable objects), some are not (unpleasant tasks, financial liabilities, burnt parts of the cake that must still be eaten not to offend the host \([30]\)), and preferences over other parts can be single-peaked (teaching loads, volunteering time, shares of a risky project) or single-dipped, etc.. Agents may disagree over which items are good, bad, or neither, and D&C\(_n\) respects the privacy of this important information: e. g., I may not want others to know which tasks I am actually happy to perform.

\(^2\)Two agents have identical superadditive utilities: \( u(S) = |S|^2 \) (where \( | \cdot | \) is the Lebesgue measure) over the interval \([0, 1]\). They cannot both reach \( \frac{1}{2} u([0, 1]) \) simultaneously.

\(^3\)Both are equal to \( \frac{1}{n} u_i(\Omega) \) if utilities are additive.
Both DS\textsubscript{n} and MK\textsubscript{n} work for unanimously good manna (all parts are desirable), and are easily adapted to a unanimously bad one: simply replace trimming by padding in DS\textsubscript{n}, and in MK\textsubscript{n} give to the first stopping agent the part of the manna not yet touched by the knife. But they do not work when preferences are not unanimously monotonic.\footnote{With additive utilities, DS\textsubscript{n} (but not MK\textsubscript{n}) extends to the case where everyone agrees on which items are good and which are bad. We simply allocate goods and bads separately. Neither rule works for satiable commodities.}

To repeat our main result, Theorem 1 in Section 3.3: the D&C\textsubscript{n} rule implements the minMax Guarantee for non atomic but otherwise arbitrary preferences. The key ingredient of the proof is the “equi-partition” Lemma 3 in Section 3.4: each agent can partition any subset of the manna in equally valuable shares.

The rule can be used as well in the atomic model where the manna contains only indivisible objects (good or bad). Theorem 2 in Section 4.2 states that if utilities are additive, D&C\textsubscript{n} implements the natural atomic version of the Proportional Guarantee: Proportional up to one object ([3]).

In the last Section 5 of the paper, we restrict attention to the case of monotone but not necessarily additive preferences: a bigger share is weakly better for everyone, or weakly worse for everyone. In this smaller domain MK\textsubscript{n} also implements the minMax Guarantee. But each rule MK\textsubscript{n} chooses an arbitrary path for the knife, which tightly restricts the range of shares potentially available. To eliminate this restriction we propose a large family of rules similar to MK\textsubscript{n} and dubbed the Bid \& Choose (B&C\textsubscript{n}) rules. They work by choosing a benchmark additive measure of the shares, diversely interpreted as their size, their price, etc.. A bid represents the smallest size of a share that this agent finds acceptable in a good manna (or the largest size of such a share in a bad manna). The corresponding Guarantees are between the minMax and Maxmin level and in many instances they improve substantially the minMax Guarantee: Theorem 3 in Section 5.3.

Throughout the paper we speak of implementation in the very simple sense adopted by most of the cake cutting literature, and formalized in the general collective decision context by Barbera and Dutta as implementation in “protective equilibrium” ([5]). A rule implements a certain guaranteed welfare level means: each agent, no matter what her preferences, has a strategy that also depends upon the manna and the number of agents, but nothing else, such that no matter what other agents do, her share will give her at least that welfare level. In two of our three results (Theorems 1 and 3) the guaranteeing strategy is also unique.\footnote{See statement ii) in Theorem 1 and the comments after the proof of Theorem 3.}

\section{Relevant literature}

The dawn of Mechanism Design saw several papers explore the informational content of decentralized mechanisms to allocate private resources. A key result
is that the price messages coordinating competitive exchanges have the smallest possible dimension – are the most parsimonious – for implementing an efficient allocation ([16], [24], [28]). The mechanisms discussed here only require to enforce the simple rules of the game, but nothing like a Walrasian auctioneer adjusting prices dynamically. Naturally the coarse messages featuring in our D&Cn and B&Cn rules do not allow to get anywhere near an efficient division of the manna.

The large cake cutting literature following Steinhaus’s seminal paper is explicitly concerned about informational parsimony: it evaluates the complexity of mechanisms by the number of “cuts” and “queries” they involve (e. g. [12] or [29]). Going beyond the Proportional Guarantee, a key goal is to reach by cuts and queries an Envy Free division of the cake (where everyone gets her best share among all shares on offer). The algorithms proposed by Brams and Taylor ([11]), and more recently Aziz and McKenzie ([4]), do exactly this when utilities are additive and non atomic; but because they involve an astronomical number of cuts and queries they are of no practical interest and squarely contradict informational parsimony. See ([13], [17]) for some fine tuning of these general facts.

The two welfare levels Maxmin and minMax are key to our results: they are discussed respectively by Budish ([14]) and Bouveret and Lemaitre ([10]) for the fair division of indivisible items with additive utilities. It took a couple of years and many brain cells to check that the Maxmin lower bound is not always feasible for three or more agents in that context ([27]), though this is a rare occurrence ([18]). Our paper is the first application of these two bounds to the non atomic model of cake division.

A recent new idea in fair division is to allow for a mixed manna, containing subjectively good and bad parts, as is typically the case when we divide assets and liabilities of a dissolving partnership. Introduced in [9] and [8] for the classic Arrow Debreu model, it is discussed for cake division in [30], and for indivisible items in [3]. Our Divide and Choose rules are particularly well suited to divide mixed manna, in these three models and beyond.

The “equi-partition” Lemma (Subsection 3.4) is key to Theorem 1: its proof uses sophisticated arguments from algebraic geometry. In a similar vein, subtle variants of the Sperner’s Lemma yield recent results on the existence of an Envy Free division under very general preferences, where which share I like best in a given partition can depend upon the partition itself, not just upon my own share: Stromquist’s ([33]) and Woodall’s ([36]) seminal insight is considerably strengthened by the combined results in [30], [21] and [2]. However all these results assume that an agent never likes the empty share best in any partition, a configuration that is both realistic and allowed by our own model.

The microeconomic literature on fair division discusses routinely the minimal welfare level that can be guaranteed to the participants in a given resource allocation problem: equal split of a bundle of divisible private goods is key to the popular competitive division (e. g., [35], [9]); and for the fair exploitation of a commons virtual free access to a rescaled technology (as in [23], [22]) is a natural Guarantee. Such Guarantees rely on the physical characteristics of the
problem: our model is more general because the manna is a set of items with no particular structure beyond some topological assumptions.

3 Divide and Choose guarantees the minMax

3.1 The non atomic model

In the general non atomic domain, the utility of an empty share is normalized to zero and the continuity assumption ensures that no vanishingly small piece of the manna has a non zero welfare impact. No other monotonicity or even sign assumption is present. Later we discuss the subdomains where utilities are additive (Section 4), or monotone (Section 5). It should be clear that our analysis in this Section and Section 5 relies only on the underlying ordinal preferences.

The manna \( \Omega \) is a bounded measurable set in an euclidian space, a share \( S \) is an element of the set \( B \) of its measurable subsets, and \( \cdot \mid \cdot \) is the Lebesgue measure. We write \( S \Delta T \) for the symmetric difference of \( S, T \) and \( \delta(S, T) = |S \Delta T| \) for the corresponding metric in \( B \).

Definition 1 The domain of non atomic utilities \( D(\Omega) \) contains all real valued functions \( u \) on \( B \) such that

\[
 u(\emptyset) = 0 \text{ and } u \text{ is continuous and bounded on } (B, \delta)
\]

Continuity of \( u \) implies that it does not distinguish between two shares who differ only on a set of measure zero. The normalisation at zero means that the utility of a sequence of vanishingly small shares goes to zero.

The definition above is very general, and encompasses several familiar models of smaller complexity imposing restrictions on the feasible shares. The first one in particular plays a key role in the proof of Theorem 1.

Example 1: the interval model. The manna is \( \Omega = [0, 1] \) and the feasible shares are its intervals (we do not pay attention to whether the end points are included or not). A utility function \( u(x, y) \) is continuous over the triangle \( \{(x, y) | 0 \leq x \leq y \leq 1\} \).

Example 2: the micro-economic model. The manna is a bundle \( \omega \in \mathbb{R}^A \) from a set \( A \) of commodities, and a share is a vector \( z \) in \( [0, \omega] \). To fit with Definition 1, take \( \Omega \) to be the disjoint union of the intervals \([0, \omega_a]\) for \( a \in A \) and a share is the union of \(|A|\) subintervals, one for each commodity. The initial utility \( u \) defined on \([0, \omega]\) is adapted in the obvious way to such shares. Continuity of \( u \) on \([0, \omega]\) implies continuity in the sense of Definition 1.

Example 3: Special shapes. The manna \( \Omega \) is a compact in an euclidian space, and shares are polytopes of a certain type: e.g. triangles or tetrahedrons ([31]). Or shares must have topological properties like connectedness ([6], [1]).

A useful property in some of the models above is that the restricted domain of shares \( B^e \) is compact for the symmetric difference metric \( \delta \); this is clear in Examples 1 and 2, and true as well in Example 3 if a share can be any
polytope with a uniformly bounded number of vertices. A consequence is that maximizing and minimizing over shares, as we often do in the paper, has an exact solution. Although the property is not necessary in Theorems 1 and 3, it makes their statements easier to read.

3.2 Two welfare benchmarks

Given a share \( S \in B \), an \( n \)-partition of \( S \) is a list \( \{S_i\}_{i=1}^n \) such that \( \cup_{i=1}^n S_i = S \) and for any distinct \( i, j \) the set \( S_i \cap S_j \) is of measure zero. We write \( \mathcal{P}_n(S) \) for the set of these partitions.

Fix \( u \in D(\Omega) \) and an integer \( n \). Here are the two benchmark utility levels discussed in the introduction

\[
\min \max_{\Pi \in \mathcal{P}_n(\Omega)} u(S_i) \quad \text{and} \quad \max \min_{\Pi \in \mathcal{P}_n(\Omega)} u(S_i)
\]

Thus \( \min \max \) is what utility I can achieve by having first pick in the worst possible partition, and \( \max \min \) by having last pick in the best possible partition. If shares are restricted to a compact subset \( B^* \) of \( B \), the \( \max \min \) and \( \min \max \) utilities above are true maxima and minima. For the sake of simplicity we stick to the \( \min \) and \( \max \) terminology even without this restriction.

**Lemma 1** For all \( u \in D(\Omega) \) and \( n \), we have \( \min \max (u; n) \leq \max \min (u; n) \).

Lemma 1 is a simple Corollary of the (difficult) equi-partition Lemma 3, two Sections below, stating that we can choose a partition \( \{S_i\}_{i=1}^n \) in \( \mathcal{P}_n(\Omega) \) such that \( u(S_i) = u(S_j) \) for all \( i, j \). Writing \( u(\Pi) \) for this common value the inequalities

\[
\min \max (u; n) \leq u(\Pi) \leq \max \min (u; n)
\]

are clear. Note that the partitions achieving the \( \max \min \) and \( \min \max \) utilities are not necessarily equi-partitions.\(^7\) A simple microeconomic example has 6 units of a single item to split between two agents and one of them has the following “triple-peaked” utility, depicted in Figure 1:

\[
u(x) = x^2 - 4x \text{ if } 0 \leq x \leq 3 ; \quad u(x) = 2(x - 4)^2 - 5 \text{ if } 3 \leq x \leq 6
\]

By offering the partition \{0, 6\} to the other party, our agent secures the utility \( u(0) = 0 \) and this is \( \max \min (u; 2) \). If offered the partition \{2, 4\}, he picks 2 and gets \( u(2) = -4 = \min \max (u; 2) \). But the only two equipartition are \{1, 5\} and \{3, 3\}, both giving utility \( u(\Pi) = -3 \).

In our next microeconomic example preferences are monotone, and the large “duality gap” comes from strong positive or negative complementarities. We have two agents, two commodities \( a, b \) and the manna is \( \omega = (1, 1) \).

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\(^6\)Any sequence of such polytopes contains a subsequence of polytopes with the same number of vertices, then a subsequence where the first vertex (in some arbitrary labeling) converges, then one where the second vertex converges and so on.

\(^7\)However in the domain of monotone preferences, both welfare levels are achieved by equipartitions as explained at the end of Subsection 5.1.
The first agent has Leontief preferences \( u_1(z) = \min\{z_a, z_b\} \) so his worst case partition is \( \Pi = \{(1, 0), (0, 1)\} \) and his best offer is the equal split partition \( \Pi' = \{\frac{1}{2}z, \frac{1}{2}z\} \): therefore \( \minMax(u_1; 2) = 0 < \frac{1}{2} = \Maxmin(u_1; 2) \).

The second agent has anti-Leontief preferences: \( u_2(z) = \max\{z_a, z_b\} \). For her the equal split partition \( \Pi' \) is the worst and the best one is \( \Pi: \minMax(u_2; 2) = \frac{1}{2} < 1 = \Maxmin(u_2; 2) \).

In this simple economy the profile of utilities \( (\Maxmin(u_1; 2), \Maxmin(u_2; 2)) = (\frac{1}{2}, 1) \) is not feasible: therefore the natural benchmark \( \Maxmin(u; n) \) cannot be taken as a Fair Guarantee.\(^8\)

### 3.3 The Divide & Choose rule

We start by a combinatorial observation. Let \( G \) be a bilateral graph between the sets \( M \) of agents and \( R \) of objects (they will be individual shares below). We say that agent \( m \) likes object \( r \) iff \( (m, r) \in G \); and that the subset \( \tilde{M} \) of agents can be properly matched to the subset \( \tilde{R} \) of objects if \( |\tilde{M}| = |\tilde{R}| \), agents in \( \tilde{M} \) are each matched (one-to-one) to an object they like in \( \tilde{R} \), and no one outside \( \tilde{M} \) likes any object in \( \tilde{R} \).

**Lemma 2.** Assume \( |M| = |R| \), and each agent in \( M \) likes at least one object in \( R \). Then there is a (non empty) largest set \( M^* \) of properly matchable agents: if \( \tilde{M} \) is properly matched to \( R \), then \( \tilde{M} \subseteq M^* \).

**Proof.** This is a simple consequence of the Gallai-Edmonds decomposition of a bipartite graph: see e.g. [19] Chap 3 (or Lemma 1 in [7]). If \( M \) can be matched with \( R \) this is a proper match and the statement holds true. If \( M \) and \( R \) cannot be matched, then we can uniquely partition \( M \) as \( (M^+, M^0, M^-) \) and \( R \) as \( (R^+, R^0, R^-) \) such that:

1. \( |M^+| > |R^+| \), agents in \( M^+ \) do not like any object in \( R^0 \cup R^- \), and they compete for the over-demanded objects in \( R^+ \) (every subset of \( R^+ \) is liked by a strictly larger subset of agents in \( M^+ \));
2. \( |M^0| = |R^0| \), and agents in \( M^0 \) can be matched with objects in \( R^0 \) (note that \( M^0 = R^0 = \emptyset \) is possible);
3. \( |M^-| < |R^-| \), and agents in \( M^- \) are over-demanded (every subset of \( M^- \) likes a strictly larger subset of objects in \( R^- \)).

For every subset \( \tilde{R} \) of \( R^- \) that can be matched to \( M^- \), we see that \( M^* = M^0 \cup M^- \) is perfectly matched to \( R^0 \cup \tilde{R} \). Next suppose \( \tilde{M} \) is perfectly matched to \( \tilde{R} \) and suppose \( \tilde{M} = \tilde{M} \cap M^+ \) is non empty: then \( \tilde{M} \) is matched to \( \tilde{R} \) in \( R^+ \) but \( \tilde{R} \) is liked by more than \( \tilde{M} \), therefore the match is not perfect and \( \tilde{M} = \emptyset \).

In the application of the Lemma below, there is one agent, the Divider, who likes all the objects: this implies that if \( M \) cannot be matched to \( R \), there is more than one subset of objects properly matchable to \( M^* \).

\(^8\)In Section 2 we recalled the role this utility level played for the model with indivisible items and additive utilities.
Definition 2: the D&C$_n$ rule. Fix the manna $(\Omega, B)$ and the set of agents $N$, each with a utility in $D(\Omega)$, and ordered as $N = \{1, \ldots, n\}$.

Step 1. Agent 1 divides the manna as a partition $\Pi^1 \in P_n(\Omega)$; all other agents report which shares in $\Pi^1$ they like (at least one). In the resulting bipartite graph between $N$ and the shares in $\Pi^1$, where agent 1 likes all the shares, we pick the maximal properly matchable set of agents $N^1$ (it contains agent 1), and a corresponding set of shares $R$; if $N^1 \neq N$ go to Step 2. Repeat with the remaining manna $\Omega^2$ and agents in $N\setminus N^1$. Ask the first agent in the exogenous ordering to divide as $\Pi^2 \in P_{n-|T|}(\Omega^2)$, while others report which new shares they like.

And so on.

At least one agent, the Divider, is served in each step, thus the algorithm just described takes at most $n - 1$ steps. There is some flexibility in the Definition because we have typically several choices for the set $R$ of shares to assign in each step, and multiple ways to assign these to the agents. We match as many agents as possible so as to minimize the number of queries, hence of information disclosure.

Our first main result is that the D&C$_n$ rule implements the minMax Guarantee in the very large domain of Definition 1. Like Lemma 1, the result relies on the equi-partition Lemma 3 below.

Theorem 1

i) If the manna $(\Omega, B)$ is divided by the D&C$_n$ rule, an agent with utility $u$ in $D(\Omega)$ guarantees his $\min\max(u; n)$ utility level by 1) offering an equi-partition of the remaining share of manna and number of agents when asked, and 2) accepting at any step from the shares on offer all those, and only those, not worse than $\min\max(u; n)$ in the initial problem $(\Omega, N)$.

ii) Moreover the first Divider (and no one else) can guarantee her $\maxmin(u; n)$ utility (to any degree of approximation if the maximum is not reached). Other agents cannot guarantee more than $\min\max(u; n)$.

Proof. Statement i). Consider an agent $u$ who abides by the strategy in the statement. At a step where he must report which shares he likes among those cut at that step, he can for sure find one worth at least $\min\max(u; n)$: all shares previously assigned are worth strictly less than $\min\max(u; n)$, and together with the freshly cut shares they form a partition in $P_n(\Omega)$; and in any partition at least one share is worth $\min\max(u; n)$ or more.

At a step where our agent is called to cut, and offers an equi-partition of the remaining manna, each of these new shares is worth at least $\min\max(u; n)$ because in the partition made of the the previously assigned shares and the fresh ones, each old one is worth strictly less than $\min\max(u; n)$ and the new ones are all of equal value.

Statement ii). Fix an agent $i$ with utility $u$: we must show that if he is not the first Divider, for certain strategies of the other agents, agent $u$ gets exactly his $\min\max$ utility. Pick a partition $\Pi \in P_n(\Omega)$ achieving $\min\max(u; n)$ (the

9After Step 1 an agent can secure his $\maxmin$ utility for the smaller manna among fewer agents, but this may be below the $\maxmin$ level for the initial problem.
existence assumption is without loss) and suppose that the first Divider, who is not agent \( i \), offers \( S \). Let \( S \) be a share that \( i \) reports as acceptable in Step 1. Then suppose the \( n-2 \) remaining agents accept all shares in \( \Pi \) except \( S \); by Lemma 2 in Step 1 the full match is feasible and agent \( i \)'s share must be \( S \). ■

The D&C\(_n\) rule takes at most \( n-1 \) steps; if an agent is asked to partition part of the manna in \( m \) shares, \( 2 \leq m \leq n-1 \), he has to report at most \( n-m-1 \) times which shares he finds acceptable.

Most of the cake-cutting literature (e.g., [12], [29]) assumes utilities are real valued \( \sigma \)-additive measures on \( \Omega \), non atomic with respect to the Lebesgue measure. Then Lyapounov’s theorem implies the existence of an equi-partition \( \Pi \in \mathcal{P}_n(\Omega) \) of which each share is worth \( \frac{1}{n} u(\Omega) \), which implies \( \min\text{Max}(u; n) = \max\text{min}(u; n) = \frac{1}{n} u(\Omega) \). Therefore D&C\(_n\) implements in this case the Proportional Guarantee.

### 3.4 The equi-partition Lemma

**Lemma 3**

Fix the manna \((\Omega, \mathcal{B})\), an integer \( n \), a share \( S \in \mathcal{B} \), and a utility \( u \in \mathcal{D}(\Omega) \). There is a \( n \)-partition \( \Pi \in \mathcal{P}_n(S) \) such that \( u(S_i) = u(S_j) \) for all \( i, j \). We call \( \Pi \) an equi-partition of \( S \) by \( u \).

Proof in Appendix 7.1. We stress that this result does not require the set of shares \( \mathcal{B} \) to be compact.

It is easy to prove the existence of an equi-partition if we assume that the sign of \( u \) is constant (all shares are weakly preferred to the empty share, or all are weakly worse). Using the transformations of the statement detailed in Steps 1 and 2 of Appendix 7.1, simply apply the KKM Lemma to the sets of partitions \( z \) where the \( i \)-th interval \( z_i \) gives the lowest utility of all. One can also invoke the stronger results by [33] and [34] showing the existence of an Envy Free partition under this assumption.

Of course the fact that the sign of \( u \) changes arbitrarily is a key feature in the division of a mixed manna.

### 4 Indivisible items and additive utilities

#### 4.1 The Pro1 Guarantee

The manna is a finite set \( \Omega \) of indivisible items and a share \( S \) is any subset of \( \Omega \). An additive utility is a vector \( U \in \mathbb{R}^\Omega \) and \( U_S = \sum_{x \in S} U_x \) stands for the utility of share \( S \), with the convention \( U_{\emptyset} = 0 \). As before the sign of \( U_x \) is arbitrary: each item may be a good to some agents but a bad to others.

The definition (1) of our two benchmark utilities is unchanged, but the inequality in Lemma 1 is reversed:

\[
\max\text{min}(U; n) \leq \frac{1}{n} U_{\Omega} \leq \min\text{Max}(U; n) \quad \text{for all } U \in \mathbb{R}^\Omega
\]
then the three levels \( \text{Maxmin} \), \( \text{minMax} \), and \( \frac{1}{n} U_\Omega \) are empty the property reads \( \text{Maxmin} = 4 < \frac{1}{2} U_\Omega = 3.5 < \min\text{Max} = 4 \).\(^{10}\)

The Proportional Guarantee \( \frac{1}{n} U_\Omega \) is typically infeasible and must be weakened. As explained in Section 2, weakening it to \( \text{Maxmin}(U; n) \) does not work either for \( n \geq 3 \).

A successful weakening of this lower bound is the Proportional Guarantee up to one item (Pro1) introduced in \([3]\). We say that the share \( S \) guarantees Pro1 if it meets Pro, or does so by adding a good item, or by taking away a bad one:

\[
\exists a : U_{S \cup \{a\}} \geq \frac{1}{n} U_\Omega \quad \text{and/or} \quad \exists b : U_{S \setminus \{b\}} \geq \frac{1}{n} U_\Omega
\]

If \( S \) meets Pro both inequalities hold when we take \( a \) in \( S \) or \( b \) outside it. If \( S \) is empty the property reads \( \exists a : U_a \geq \frac{1}{n} U_\Omega \) and/or \( U_\Omega \leq 0 \).

We write \( \text{Pro1}(U; n) \) for the smallest utility of a share meeting this Guarantee. For instance if we have ten goods and \( U = (1, 2, \cdots, 10) \) one checks that \( \text{Pro1}(u; 2) = 18 < 27 = \text{Maxmin}(u; 2) < 27.5 = \frac{1}{2} u(\Omega) \).

### 4.2 Divide and Choose indivisible items

We show first that the lexicographic refinement of the maximization problem \( \text{Maxmin}(U; n) \) delivers a partition where all shares meet the Pro1 Guarantee. Fixing \( \Omega, U, \) and \( n \), a leximin partition of \( \Omega \) is \( \Pi^*_n = \{S_i^n\}_{i=1}^n \in \mathcal{P}_n(\Omega) \) such that \( U_{S_i} \) increases weakly with \( i \) and moreover \( (U_{S_i^n})_{i=1}^n \) maximizes the lexicographic ordering over all weakly increasing profiles \( (U_{S_i})_{i=1}^n \) when \( \Pi \) varies in \( \mathcal{P}_n(\Omega) \).

**Lemma 4:** If an agent has additive utility \( U \) over the items in \( \Omega \), every share in a \( n \)-leximin partition meets the Pro1 Guarantee. In particular \( \text{Pro1}(U; n) \leq \text{Maxmin}(U; n) \).

**Proof.** Pick a leximin \( n \)-partition \( \Pi = (S_i)_{i=1}^n \) of \( \Omega \) and keep in mind \( U_{S_i} \geq \frac{1}{n} U_\Omega \). If \( U_{S_i} = U_{S_n} \) all shares are worth \( \frac{1}{n} U_\Omega \) so they meet the Pro1 Guarantee. Assume now \( U_{S_n} = U_{S_i} + \delta \) where \( \delta > 0 \). Suppose there is some \( a \) in \( S_n \) such that \( 0 < U_a < \delta \): then shifting \( a \) from \( S_n \) to \( S_i \) gives a partition superior to \( \Pi \) for the leximin ordering, which is ruled out. Suppose there is some \( a \) in \( S_n \) such that \( U_a \geq \delta \): then \( U_{S_i \cup \{a\}} \geq U_{S_n} \geq \frac{1}{n} U_\Omega \) for all \( i \in [n-1] \) and we are done.

It remains to consider the case where \( U_a \leq 0 \) for all \( a \in S_n \). We fix \( i \in [n-1] \) and check that \( S_i \) guarantees Pro1. This is true if \( U_{S_i} \geq 0 \) because this implies \( U_{S_i} = U_{S_n} = 0 \geq \frac{1}{n} U_\Omega \). Assume now \( U_{S_i} < 0 \) and pick \( b \in S_i \) such that \( U_b < 0 \). If \( U_{S_n \cup \{b\}} > U_{S_i} \) then shifting \( b \) from \( S_i \) to \( S_n \) gives a partition superior to \( \Pi \) for the leximin ordering, which we ruled out. Thus \( U_{S_i} \geq U_{S_n \cup \{b\}} \iff U_{S_i \setminus \{b\}} \geq U_{S_n} \geq \frac{1}{n} U_\Omega \), as was to be proven. ■

\(^{10}\)Note that if we allow general utilities in this model, i.e., any function \( u \) from \( 2^\Omega \) into \( \mathbb{R} \), then the three levels \( \text{Maxmin}, \min\text{Max}, \) and \( \frac{1}{n} U_\Omega \) are no longer comparable (all six orderings are possible) and our results for the nonatomic model do not adapt.
It follows that the D&C\textsubscript{n} rule, defined exactly as above, implements the Pro1 Guarantee.

**Theorem 2**

If the finite set of items \( \Omega \) is divided by the D&C\textsubscript{n} rule, an agent with additive utility guarantees a share meeting Pro1 by 1) offering a leximin partition of the relevant set of items and number of agents when asked; 2) accepting at any step all shares meeting Pro in the remaining problem, and only those.

**Proof.** The firstDivider picks a leximin \( n \)-partition of \( \Omega \), hence guarantees Pro1 by Lemma 4. Other agents are either served a share meeting Pro, or go to step 2 with \( n_2 \) remaining agents and manna \( \Omega_2 \). For each of them \( \frac{1}{n_2} U_{\Omega_2} > \frac{1}{n} U_\Omega \) because the shares distributed in step 1 are all worth strictly less than \( \frac{1}{n} U_\Omega \). The Divider in step 2 guarantees Pro1 in \( (\Omega_2, n_2) \) by a leximin \( n_2 \)-partition: this is a stronger property than Pro1 in \( (\Omega, n) \). And Pro in \( (\Omega_2, n_2) \) is stronger than in \( (\Omega, n) \) for the non dividing agents. We repeat this argument until all agents are served: the average worth \( \frac{1}{n_k} U_{\Omega_k} \) of the agents still not served increases strictly in each step and so does the strength of properties Pro1 and Pro in \( (\Omega_k, n_k) \).

Note that accepting at any step the shares meeting Pro in the initial problem would also implement the Pro1 Guarantee for all participants, but the strategy in the statement can only improve the share our agent is finally served.

Leximin partitions are not the only ones guaranteeing Pro1 in D&C\textsubscript{n}: a simpler algorithm for the Dividing agent collects items in an arbitrary order as long as the pile is worth no more than Pro, then start a new pile with the remaining items.

5 Bid and Choose and Moving Knives for goods or bads

5.1 Monotone preferences

When the manna is unanimously good, or unanimously bad, we find that the minMax guarantee can be improved by a large family of parsimonious rules, containing the Moving Knives (MK\textsubscript{n}) and the related Bid and Choose (B&C\textsubscript{n}) rules.

**Definition 3** Given a non atomic manna \( (\Omega, \mathcal{B}, |.|) \), the sub-domain \( \mathcal{M}^+(\Omega) \) of \( \mathcal{D}(\mathcal{B}) \) contains the utility functions \( u \) (weakly) increasing w.r.t. inclusion:

\[
S \subset T \implies 0 \leq u(S) \leq u(T) \quad \text{for all} \quad S, T \in \mathcal{B}
\]

(recall \( u(\emptyset) = 0 \)). The sub-domain \( \mathcal{M}^-(\Omega) \) of functions decreasing w.r.t. inclusion is defined by the opposite inequalities.

In the rest of this Section we focus for simplicity on increasing utilities. All results translate without difficulty to decreasing utilities.

Lemma 3 about equi-partitions is much easier to prove for monotone preferences (as explained in Appendix 7.1). Moreover the interpretation of equi-
partition is richer in the domain $\mathcal{M}^+(\Omega)$. Write $\mathcal{E}_n(\Omega; u)$ for the set of equi-partitions of the manna, then

$$\min \max (u; n) = \min_{\Pi \in \mathcal{E}_n(\Omega; u)} u(\Pi) \quad ; \quad \max \min (u; n) = \max_{\Pi \in \mathcal{E}_n(\Omega; u)} u(\Pi)$$

Indeed the continuity and monotonicity of $u$ imply: if $S, T$ are two disjoints shares such that $u(S) > u(T)$, we can trim part of $S$ and add it to $T$ to get two disjoint shares $S', T'$ with equal utility in between $u(S)$ and $u(T)$. Expanding this argument we see that if $\Pi = (S_i)_{i=1}^n$ is a $n$-partition of $S$ such that $\max_{1 \leq i \leq n} u(S_i) > \min_{1 \leq i \leq n} u(S_j)$, there exists an equi-partition $Q = (R_i)_{i=1}^n$ of $S$ by $u$ such that

$$\max_{1 \leq i \leq n} u(S_i) > u(Q) > \min_{1 \leq j \leq n} u(S_j)$$

and (2) follows (we omit the details).

### 5.2 The MK$^\kappa_n$ and B&C$^\theta_n$ rules

A moving knife through $(\Omega, B, \cdot | \cdot )$ is a path $\kappa : [0, 1] \ni t \rightarrow K(t) \in B$ from $K(0) = \emptyset$ to $K(1) = \Omega$, continuous for $\delta$ and strictly inclusion increasing:

$$0 \leq t < t' \leq 1 \implies K(t) \subset K(t') \quad \text{and} \quad |K(t') \setminus K(t)| > 0$$

The moving knife $\kappa$ arranges shares of increasing value to all participants but only along the specific path of the knife. An example is $K(t) = B(t) \cap \Omega$, where $t \rightarrow B(t)$ is a path of growing balls with a fixed center and radius growing from 0 to large enough that $B(1)$ contains $\Omega$. Moving knives can take many other shapes, for instance hyperplanes.

Our Bid and Choose rules offer more choices to the agents, with the help of a benchmark measure $\theta$ of the shares at the discretion of the designer: $\theta$ is a positive $\sigma$-additive measure on $(\Omega, B)$, normalised to $\theta(\Omega) = 1$. It is absolutely continuous w.r.t. the Lebesgue measure $|\cdot|$ and vice versa: the density of $\theta$ w.r.t. $|\cdot|$ is strictly positive. In particular $\theta$ is strictly inclusion increasing:

$$\forall S, T \in B : S \subset T \quad \text{and} \quad |T \setminus S| > 0 \implies \theta(S) < \theta(T)$$

In applications $\theta$ can evaluate for instance the market value, physical size, or weight of a share.

Fixing a moving knife $\kappa$ and a measure $\theta$, we define in parallel the Moving Knife MK$^\kappa_n$ and the Bid and Choose B&C$^\theta_n$ rules. In both cases a clock $t$ runs from $t = 0$ to $t = 1$.

**Definition 4** the MK$^\kappa_n$ and B&C$^\theta_n$ rules

*Step 1.* The first agent $i_1$ to stop the clock, at $t^1$, gets the share $K(t^1)$ in MK$^\kappa_n$, or in B&C$^\theta_n$ chooses any share in $\Omega$ s.t. $\theta(S) = t^1$, say $S_{i_1}$, and leaves.

*Step k.* Whoever stops the clock first at $t^k$ gets the share $K(t^k) \setminus K(t^{k-1})$ in MK$^\kappa_n$, or in B&C$^\theta_n$ chooses any share in $\Omega \setminus \bigcup_{i=1}^{k-1} S_{i_k}$ s.t. $\theta(S) = t^k - t^{k-1}$, say $S_{i_k}$ and leaves.
In Step \( n - 1 \) the agent who does not stop the clock takes the remaining share \( \Omega \setminus K(t^{n-1}) \) or \( \Omega \setminus \bigcup_{i=1}^{n-1} S_i \).

Breaking ties between agents stopping the clock at the same time is the only indeterminacy in these rules, much less severe than in D&C\(_n\). Up to tie-breaking, B&C\(_n\) and MK\(_n^\kappa\) are anonymous (do not discriminate between agents) but not neutral (do discriminate between shares), while D&C\(_n\) is neutral but not anonymous.

The B&C\(_n^\theta\) rule solves the main defect of MK\(_n^\kappa\): its range includes any \( n \)-partition \( \Pi = (S_i)_{i=1}^n \) of the manna. To check this assume first \( |S_i| > 0 \) for all \( i \) and consider \( n \) agents deciding (cooperatively) to achieve \( \Pi \). By the strict monotonicity of \( \theta \) the sequence \( t^i = \theta(\bigcup_{j=1}^{i} S_j) \) increases strictly therefore they can stop the clock at these successive times and choose the corresponding shares in \( \Pi \). Shares of measure zero can all be distributed at time 0.

On the other hand in B&C\(_n^\theta\) all but one agent must pick a share under constraints, thus revealing more information than in MK\(_n^\kappa\). Loosely speaking, B&C\(_n^\theta\) is about as parsimonious as D&C\(_n^\theta\).

Remark: We can define a static version of MK\(_n^\kappa\) and B&C\(_n^\theta\) where agents bid all at once for potential stopping times. This would not affect the Guarantees to participants, so we do not discuss these alternative rules.

5.3 The B&C\(_n^\theta\) and MK\(_n^\kappa\) Guarantees

We fix a utility \( u \in \mathcal{M}^+(\Omega) \) for the rest of the Section and we define two sets: the triangle \( T = \{(t^1, t^2)|0 \leq t^1 \leq t^2 \leq 1\} \) in \( \mathbb{R}^2_+ \) and the set \( Y(n) \) of increasing sequences \( \tau = (t^k)_{0 \leq k \leq n} \) in \( [0,1] \) s.t.
\[
t^0 = 0 \leq t^1 \leq \cdots \leq t^{n-1} \leq t^n
\]

For a moving knife \( \kappa \), utilities of the shares in MK\(_n^\kappa\) are described by the function \( u^\kappa \) on \( T \):
\[
u^\kappa(t^1, t^2) = u(K(t^2) \setminus K(t^1)) \text{ for all } (t^1, t^2) \in T
\]

For a measure \( \theta \), the corresponding definition is the indirect utility \( u^\theta \):
\[
u^\theta(t^1, t^2) = \min_{T: \theta(T) = t^1} \max_{S: S \cap T = \emptyset, \theta(S) = t^2 - t^1} u(S) \text{ for all } (t^1, t^2) \in T
\]
(3)
(by monotonicity and continuity of \( u \) and \( \theta \) nothing changes if the minimization bears on \( \theta(T) \leq t^1 \) and the maximization on \( \theta(S) \leq t^2 - t^1 \))

Note that both \( u^\kappa \) and \( u^\theta \) decrease (weakly) in \( t^1 \) and increase (weakly) in \( t^2 \). We now define what we show below are the Guarantees \( \Gamma^\kappa_n \) and \( \Gamma^\theta_n \) implemented by MK\(_n^\kappa\) and B&C\(_n^\theta\) respectively:
\[
u^\kappa_n(u) = \max_{\tau \in Y(n)} \min_{0 \leq k \leq n-1} u^\kappa(t^k; t^{k+1}) \text{ where } \alpha \text{ is } \kappa \text{ or } \theta
\]
(4)

Lemma 5
i) The utility \( u^\kappa \) is continuous in the general model. The indirect utility \( u^\theta \) is continuous in any submodel where the set of shares \( B^\kappa \) is compact for \( \delta \); moreover both the minimum and maximum in (3) are achieved.

ii) The maximum in \( \tau \in \mathcal{Y}(n) \) of problem (4) (for both rules) is achieved at some \( \tau \in \mathcal{Y}(n) \) where the sequence \( t^k \) increases in \( k \), all the \( u^\alpha(t^k; t^{k+1}) \) are equal, and this common utility value is the optimal value of (4).

Proof. In Appendix 7.2.

To develop the intuition for the definition (4) consider the case \( n = 2 \), where \( \tau_\kappa, \tau_\theta \) are the corresponding optimal choices of \( \tau \):

\[
\Gamma_2^\kappa(u) = \max_{0 \leq t^1 \leq 1} \min \{ u(K(t^1)), u(\Omega \setminus K(t^1)) \} = u(K(t^1_\kappa)) = u(\Omega \setminus K(t^1_\kappa)) \\
\Gamma_2^\theta(u) = \max_{0 \leq t^1 \leq 1} \min \{ \max_{\theta(S) = t^1} u(S), \min_{\theta(S) = t^1} u(\Omega \setminus S) \} = \max_{\theta(S) = t^1_\theta} u(S) = \min_{\theta(S) = t^1_\theta} u(\Omega \setminus S)
\]

In MK\(^\kappa_2 \) we look for the position \( t^1_\kappa \) of the knife making our agent indifferent between the two shares. In B&K\(^\theta_2 \) for the size \( t^1_\theta \) such that the best share of this size is as good as the worst share of size \( 1 - t^1_\theta \). In both cases she is indifferent between the share she gets if she is the first stopping at \( t^1_\alpha \) and the worse she could get if the other agent stops at \( t^1_\alpha \) or before.

Observe that if \( u \) is also \( \sigma \)-additive on \( \Omega \), the last equality in (5) reads \( \max_{\theta(S) = t^1_\theta} u(S) = \frac{1}{n} u(\Omega) \) and one checks similarly \( \Gamma_2^\theta(u) = \frac{1}{n} u(\Omega) \) as expected; these equalities follow also from inequality (6) below.

Because of Lemma 5, it is more convenient to state Theorem 3 first in compact submodels where we can choose exact solutions \( \tau_\kappa, \tau_\theta \) of the two programs (4).

**Theorem 3**

i) If the manna \((\Omega, B)\) is divided by the MK\(^\kappa\) rule, an agent guarantees the utility \( \Gamma^\kappa(u; n) \) by committing to stop the knife at \( t^k_\kappa \) if it was stopped exactly \( k - 1 \) times before;

ii) If the compact manna \((\Omega, B^\kappa)\) is divided by the B&K\(^\theta\) rule, she guarantees \( \Gamma^\theta(u; n) \) by stopping the clock at \( t^k_\theta \) if it was stopped exactly \( k - 1 \) times before, and choosing then the best available share of size \( t^k - t^{k-1} \);

iii) If the set of shares \( B \) is not compact, and \( \varepsilon > 0 \) is arbitrary, she guarantees similarly \( \Gamma^\theta(u; n) - \varepsilon \) in the B&K\(^\theta\) rule, by choosing an \( \varepsilon \)-solution of program (4).

iv) Whether the manna is compact or not

\[
\min\max(u; n) \leq \Gamma^\alpha_\kappa(u) \leq \max\min(u; n) \text{ where } \alpha = \kappa \text{ or } \theta \quad (6)
\]

**Proof.**

Statement i) and iv) for MK\(^\kappa\). Recall the equi-partition \( \Pi = (K(t^k_\kappa) \setminus K(t^{k-1}_\kappa))_1^n \) has \( u(\Pi) = \Gamma^\kappa(u; n) \). Thus (2) implies inequalities (6). Next if the knife has been stopped \( k - 1 \) times before our agent is served, the last stop occurred at or before \( t^{k-1}_\kappa \) therefore if she does stop the knife at \( t^k_\kappa \) (and wins the possible tie
break) her share is at least $K(t^k_\theta) \setminus K(t^{k-1}_\theta)$. If she never gets to stop the knife, the last stop is at or before $t^{n-1}_\theta$ and she gets at least $\Omega \setminus K(t^{n-1}_\theta)$.

**Statement ii)** If she is the first to stop the clock (perhaps by winning the tie break) at step $k$, in step $k-1$ the clock stopped at $t^{k-1}_\theta \leq t^k_\theta$ and the share $T$ already distributed at that time has $\theta(T) = t^{k-1}_\theta$; therefore she can choose a share with utility $u^\theta(t^{k-1}_\theta; t^k_\theta) \geq u^\theta(t^{k-1}_\theta; t^k_\theta) = \Gamma^\theta_n(u)$. If she is the last to be served, having never stopped the clock (or lost some tie breaks) the share assigned to all other agents has $\theta(T) = t^{n-1}_\theta \leq t^{n-1}_\theta$ therefore her share is worth $u^\theta(t^{n-1}_\theta; 1) \geq u^\theta(t^{n-1}_\theta; 1) = \Gamma^\theta_n(u)$.

**Statement iii)** We omit the simple approximation argument.

**Statement iv)** We prove it when $B^*$ is compact and omit the simple limit argument in the general case.

**Right hand inequality.** It is enough construct a partition $\Pi = (S_k)_1^n$ in which the utility of every share $S_k, 0 \leq k \leq n - 1$ is at least $u^\theta(t^k_\theta, t^k_\theta)$, implying $\min_k u(S_k) \geq \Gamma^\theta_n(u)$. We proceed by induction on the steps of $B^*$ and $C^\theta_n$. First $S_1$ maximizes $u(S)$ s.t. $\theta(S) = t^1_\theta$ so $u(S_1) = u^\theta(0; t^1_\theta) = \Gamma^\theta_1(u)$ and $\theta(S_1) = t^1_\theta$. Assume the sets $S_k$ are constructed for $1 \leq k \leq k$, mutually disjoint, s.t. $\theta(S_k) = t^k_\theta - t^{k-1}_\theta$ and $u(S_k) \geq u^\theta(t^k_\theta, t^{k-1}_\theta)$: then the set $T = \bigcup_k S_k$ is of size $t^k_\theta$ and we pick $S_{k+1}$ maximizing $u(S)$ s.t. $S \cap T = \emptyset$ and $\theta(S) = t^k_\theta + 1 - t^k_\theta$. By definition (3) we have $u(S_k) \geq u^\theta(t^k_\theta; t^{k+1}_\theta)$ and the induction proceeds. Note that in fact $\min_k u(S_k) = \Gamma^\theta_n(u)$.

**Left hand inequality.** We need now construct a partition $\Pi = (R_k)_1^n$ s.t. $u(R_k) \leq u^\theta(t^{k-1}_\theta, t^k_\theta)$ for $1 \leq k \leq n$. We do this by a decreasing induction in $n$. In (the first) step $n$ of the induction we define the 2-partition $\Pi^n = (T_{n-1}, R_n)$ of $\Omega$ where $T_{n-1}$ is any solution of the program $\min_{T: \theta(T) = t^{n-1}_\theta} u(\Omega \setminus T)$, and $R_n = \Omega \setminus T_{n-1}$. Thus $u(R_n) = u^\theta(t^{n-1}_\theta; 1)$ and $\theta(T_{n-1}) = t^{n-1}_\theta$.

Assume that in step $k$ we constructed the $(n - k + 2)$-partition $\Pi^k = (T_{k-1}, R_k, R_{k+1}, \cdots, R_n)$ s.t. $\theta(T_{k-1}) = t^{k-1}_\theta$ and $u(R_k) \leq u^\theta(t^{k-1}_\theta; t^k_\theta)$ for $k \leq \ell \leq n$. Pick $\overline{T}$ a solution of
\[
\max_{T: \theta(T) = t^{k-2}_\theta} \min_{S: S \cap T = \emptyset, \theta(S) = t^{k-1}_\theta - t^{k-2}_\theta} u(S) = u^\theta(t^{k-2}_\theta; t^{k-1}_\theta)
\]
As $\theta(\overline{T} \cap T_{k-1}) \leq t^{k-2}_\theta$ and $\theta(T_{k-1}) = t^{k-1}_\theta$ we can choose $T_{k-2}$ s.t. $\overline{T} \cap T_{k-1} \subseteq T_{k-2} \subseteq T_{k-1}$ and $\theta(T_{k-2}) = t^{k-2}_\theta$. Then we set $R_{k-1} = T_{k-1} \setminus T_{k-2}$ so that $u(R_{k-1}) \leq u^\theta(t^{k-2}_\theta; t^{k-1}_\theta)$ follows from $R_{k-1} \cap T = \emptyset$ and the definition of $\overline{T}$. This completes the induction step. We note finally that each set $R^k$ thus constructed is of $\theta$-size $t^{k}_\theta - t^{k-1}_\theta$, and that $\max_k u(S_k) = \Gamma^\theta_n(u)$. 

It is easy to check that no agent can secure more utility than $\Gamma^\theta_n$ in $MK^\theta_n$ or $\Gamma^\theta_n$ in $B^* \cap C^\theta_n$. The argument is clear if the manna is compact, and straightforward if it is not.

We illustrate the Theorem in the microeconomic model (Example 2), where the manna is $\omega \in \mathbb{R}^A$, the set of shares $z$ is $B^* = [0, \omega]$, and utilities $u$ are continuous and increasing in $[0, \omega]$. A natural Moving Knife (treating all com-
modities equally) is \( S(t) = t \omega, 0 \leq t \leq 1 \), and the corresponding Guarantee \( \Gamma_n(u) = u(\frac{t}{2} \omega) \) is the familiar equal split lower bound.

Turning to B&C\(_n\) where \( \theta \) is interpreted as a price vector (perhaps an estimate of the market value of these commodities), we note that preferences coinciding with this exogenous price \( (u(z) = \theta(z) \text{ for all } z) \) yields the indirect utility \( u^\theta(t^1, t^2) = t^2 - t^1 \) and in turn the corresponding Guarantee is equal split. If \( u \) is linear but different from \( \theta \), then the Guarantee is worse than equal split.

Assume now \( n = 2 \) so equation (5) finds \( \Gamma_n^\theta(u) \) by selecting \( t^1 \) s.t.

\[
\max \{ u(z) | z_A \leq t^1 \omega_A \} = \min \{ u(z) | z_A \geq (1 - t^1) \omega_A \} = \Gamma_n^\theta(u)
\]

In the example concluding Section 3.2, we have \( A = \{ a, b \} \), \( \omega = (1, 1) \), and agent 1 has \( u_1(z) = \min \{ z_a, z_b \} \). A straightforward computation gives \( t^1 = \frac{1}{3} \) so \( \Gamma_2^\theta(u_1) = \frac{1}{3} \) and \( 0 = \min \text{Max} < \Gamma_2^\theta(u_1) < \text{Maxmin} = \frac{1}{2} \), and \( \Gamma_2^\theta(u_1) \) is worse than equal split. On the other hand agent 2 has \( u_2(z) = \max \{ z_a, z_b \} \) so \( \Gamma_2^\theta(u_2) = \frac{2}{3} \) and \( \frac{1}{2} = \text{minMax} < \Gamma_2^\theta(u_2) < \text{Maxmin} = 1 \), hence \( \Gamma_2^\theta(u_2) \) is better than equal split.

6 Conclusion

Comparing B&C\(_n\) versus D&C\(_n\) rules

The exogenous ordering of the agents greatly affects the outcome of D&C\(_n\), whereas B&C\(_n\) treats the agents symmetrically. On the other hand the choice of the benchmark measure in B&C\(_n\) is exogenous, which allows much, perhaps too much flexibility to the designer.

As discussed at the end of Section 4.2, in D&C\(_n\) the dividing agent has many different strategies guaranteeing her minMax utility. By contrast in B&C\(_n\), the solution to programs (5) and (4) is often unique. Multiple choices and the resulting indeterminacy of the outcome may be appealing for the sake of privacy preservation, less so from the implementation viewpoint.

Two challenging open questions

1). Fix the manna \((\Omega, \mathcal{B})\) as in Theorem 1, and the set \([n]\) of agents, each with a utility in \( \mathcal{D}(\Omega) \). As discussed in Section 2 and Subsection 3.4, an envy-free partition of \( \Omega \) exists if all utilities are non negative for all shares: there is some \( \Pi = \{ S_i \}_{i=1}^n \) in \( \mathcal{P}_n(\Omega) \) s.t.

\[
u_i(S_i) \geq u_i(S_j) \text{ for all } i, j \in [n]\]

Is this still true if we remove the sign assumption on utilities?

2). Consider the asymmetric version of the Proportional Guarantee, where a vector \( \lambda \) of convex weights represents individual rights: the partition \( \Pi = \{ S_i \}_{i=1}^n \) should meet the inequalities

\[
u_i(S_i) \geq \lambda_i \cdot u_i(\Omega) \text{ for all } i \in [n]\]
It is well known that we can adapt Divide and Choose among two agents to implement these Guarantees, as long as the \( \lambda_i \)-s are rational (see e.g. Robertson Webb). The difficulty is that this requires partitions with many small shares: for instance we need \( \frac{7}{2} \)-partitions if the rights are \( \left( \frac{2}{7}, \frac{5}{7} \right) \). If utilities are additive the Divider, say player 2, cuts \( \frac{7}{2} \) shares of equal value, which ensures that any \( 5 \) of them are worth \( \frac{5}{7} \) of the whole manna.

We can define an asymmetric version of D&C for non additive utilities only if our agents can divide any subset of the manna in \( \frac{7}{2} \) shares such that the utility for the union of \( 5 \) shares is independent of the chosen subset of shares. It is not at all clear that this much more demanding version of the equi-partition Lemma is true in \( \mathcal{D}(\Omega) \) or even in \( \mathcal{M}^{\pm}(\Omega) \).

7 Appendices

7.1 Proof of Lemma 3: equi-partition

Step 1. We use a familiar trick (as in [33] and [34]) projecting the initial problem defined by \((\Omega, \mathcal{B}), u \in \mathcal{D}(\Omega)\) and \( n \) to an interval model (Example 1 Section 3.1) by means of an arbitrary moving knife \( \kappa : [0,1] \to K(t) \) through \( \otimes \) (see Subsection 5.2).

The manna is now \([0,1]\) and shares are restricted to closed intervals, so a partition is described by a sequence \( t^n \) such that the utility for the union of \( 5 \) shares is independent of the chosen subset of shares. It is not at all clear that this much more demanding version of the equi-partition Lemma is true in \( \mathcal{D}(\Omega) \) or even in \( \mathcal{M}^{\pm}(\Omega) \).

Now Lemma 3 follows if we prove the following claim: if \( u \) is a continuous real valued function on \( [0,1] \times [0,1] \) such that \( u(t, t') = 0 \) for all \( t, t' \), there exists a sequence \( t^n \) s.t. \( b_{i}^{n} - b_{i}^{n-1} \) is the \( i \)-th share. The restriction of \( u \) to the triangle \( T = \{(t, t')| 0 \leq t \leq t' \leq 1\} \) (also denoted \( u \)) is continuous because \( u(t, t') = u(K(t) \setminus K(t')) \), and \( K(t') \setminus K(t) \) and \( u \) are both continuous for \( \delta \). Moreover \( u(t, t) = 0 \) for all \( t \).

In Step 2 we reformulate the claim for a function mapping the simplex of dimension \( n - 1 \) into itself.

Step 2. The system \( z_i = t^i - t^{i-1} \) for \( 1 \leq i \leq n \), is a bijection \( b \) from \( \mathcal{T}(n) \) into the simplex \( \Delta(n - 1) \) of dimension \( n - 1 \):

\[
\Delta(n - 1) = \{(z_1, \ldots, z_n) : z_i \geq 0, \sum_{i=1}^{n} z_i = 1\}
\]

Defining \( u \) from \( \Delta(n - 1) \) into \( \mathbb{R}^n \)

\[
\bar{u}(z) = (u(\sum_{i=1}^{i-1} z_j, \sum_{i=1}^{i} z_j))_{i=1}^{n} = (u(t^{i-1}, t^i))_{i=1}^{n}
\]

(7) (with the convention \( \sum_{i=1}^{0} = 0 \)) we want to show that \( \bar{u}(\Delta(n - 1)) \) intersects the diagonal of \( \mathbb{R}^n \).
Write Λ(n−1) for the hyperplane containing Δ(n−1); d be the radial projection of Λ(n−1) onto Δ(n−1) from the center e = (1/2)n of Δ(n−1); and c the orthogonal projection of Rn onto Λ(n−1). Then g = d ◦ c ◦ ̄u maps continuously Λ(n−1) into itself, and e is in the range of g if and only if ̄u(Δ(n−1)) intersects the diagonal of Rn.

Besides continuity, g has critical symmetry properties. Let Fi be the face of Δ(n−1) where zi = 0, which implies ̄u,(z) = 0 (as u(t, t) = 0). Let σ^i,j be the permutation exchanging the coordinates i and j: σ^i+1,j sends z in Fi to σ^i+1,j(z) in Fi+1 and from (7) we see that ̄u(σ^i+1,j(z)) = σ^i+1,j(̄u(z)). Iterating this observation gives ̄u(σ^j,i(z)) = σ^j,i(̄u(z)) for any z in Fi ∪ Fj. This commutativity is preserved when we compose ̄u by two symmetric maps c and d.

To sum up, g maps continuously Δ(n−1) into itself, maps a face of any dimension of Δ(n−1) into itself, and restricted to the boundary of Δ(n−1) is equivariant under the action of the symmetric group Sm+1 by permuting coordinates.

Step 3. It is convenient to set now m = n − 1. We prove that the range of g contains the center e of Δ(m).

Rather than using its full equivariance we will fix an (m + 1)-cycle in the symmetric group and look at only the equivariance under the action of this cycle.

The argument can now be viewed as a generalization of the Borsuk-Ulam theorem. It makes use of a number of techniques from algebraic topology specifically the universal coefficient theorem, the cohomology ring (and especially the cohomology ring of the cyclic group Z/(m + 1)Z. Explanations of the terminology can be found in [25], or J. R. Munkres, Elements of Algebraic Topology, Avalon (1996) or [[20]].

Let X denote the quotient of the boundary under this Z/(m + 1)Z action. If m + 1 is a prime, then this action is free and the quotient X is a manifold, but in general it will be only an orbifold. Since the map g restricted to the boundary is equivariant under this map, it quotients to give a continuous map (which to avoid complicating the notation we will continue to denote by just g) g : X → X.

Since X has fundamental group Z/(m + 1)Z, there is a classifying map f : X → BZ/(m + 1)Z and since the universal cover of X is (topologically) an m-sphere, we can view the classifying space BZ/(m + 1)Z as being built from X by adding cells of dimension (m + 1) and higher and this classifying map as being just the inclusion map. In particular, this means that

\[ f_* : H_k(X; Z) → H_k(BZ/(m + 1)Z; Z) \]

is an isomorphism for k < m and a surjection for k = m. For m odd, the action of Z/(m + 1)Z is by orientation-preserving maps and therefore X has a fundamental class and

\[ f_* : H_m(X; Z) \cong Z → H_k(BZ/(m + 1)Z; Z) \cong Z/(m + 1)Z \]

is the usual projection. If m is even, then the action is by orientation-reversing maps and both H_m(X; Z) and H_m(BZ/(m + 1)Z; Z) vanish. In either case by
the universal coefficients theorem we find that
\[ f^*: H^k(B\mathbb{Z}/(m+1)\mathbb{Z}, \mathbb{Z}/(m+1)\mathbb{Z}) \to H_k(X; \mathbb{Z}/(m+1)\mathbb{Z}) \]
is an isomorphism for \(0 \leq k \leq m\).

Recall that the cohomology ring
\[ H^*(B\mathbb{Z}/(m+1)\mathbb{Z}; \mathbb{Z}/(m+1)\mathbb{Z}) = \mathbb{Z}/(m+1)\mathbb{Z}[\alpha, \beta]/(\beta^2 = k\alpha) \]
is generated by two elements \(\beta\) of dimension 1 and \(\alpha\) of dimension 2 with the relation that if \(m + 1\) is odd then \(\beta^2 = 0\) (so \(k = 0\) in the formula above) and if \(m + 1 = 2k\) is even, then \(\beta^2 = k\alpha\). Since the map \(X \to B\mathbb{Z}/(m+1)\mathbb{Z}\) is an isomorphism on the fundamental group and hence on first homology, it follows from the universal coefficients theorem that \(f^*\) is an isomorphism on both \(H^1\) (which comes from \(\text{Hom}(H_1)\)) and on \(H^2\) (which comes from \(\text{Ext}(H_1)\)). Hence \(H^*(X; \mathbb{Z}/(m+1)\mathbb{Z})\) is generated by the images of \(\alpha\) and \(\beta\) (which again I will denote by the same symbols), with just the added restriction that all product of dimension over \(m\) vanish.

Now since \(g: X \to X\) is also the identity map on the fundamental group and hence on \(H_1\), it follows that
\[ g^*: H^*(X; \mathbb{Z}/(m+1)\mathbb{Z}) \to H^*(X; \mathbb{Z}/(m+1)\mathbb{Z}) \]
is the identity on \(H^1\) and \(H^2\) and hence it is the identity as a ring isomorphism. In particular
\[ g^*: H^m(X; \mathbb{Z}/(m+1)\mathbb{Z}) \to H^m(X; \mathbb{Z}/(m+1)\mathbb{Z}) \]
is the identity map and hence so is
\[ g_*: H_m(X; \mathbb{Z}/(m+1)\mathbb{Z}) \to H_m(X; \mathbb{Z}/(m+1)\mathbb{Z}). \]
If \(m + 1\) is odd, it follows immediately that
\[ g_*: H_m(X; \mathbb{Z}) \to H_m(X; \mathbb{Z}) \]
is multiplication by some constant congruent to 1 modulo \(m + 1\). Hence the degree of \(g\) restricted to the boundary is 1 modulo \(m + 1\), and in particular is nonzero and thus the extension to \(\Delta(m)\) must contain the central point. If \(m + 1\) is even, the same result follows but using the fact that projection induces a surjection
\[ H_m(\partial\Delta(m); \mathbb{Z}) \to H_m(X; \mathbb{Z}/(m+1)\mathbb{Z}). \]

### 7.2 Proof of Lemma 5

1. **First statement.** Recall that we can replace in definition (\(\beta\)) the equalities like \(\theta(T) = t^1\) with inequalities \(\theta(T) \leq t^1\). We check first that the correspondence \(t \to \{S \in B^1|\theta(S) \leq t\}\) is continuous. Upper hemi continuity follows by the continuity of \(\theta\). For lower hemi continuity pick a sequence \(t_n\) converging
to \( t \) and \( S \in \mathcal{B}^c \) s.t. \( \theta(S) \leq t \). If \( t_n \) has a decreasing subsequence, we set \( S_n = S \) so that \( \theta(S_n) \leq t_n \) and \( S_n \) converges to \( S \). If \( t_n \) has an increasing subsequence we construct an inclusion increasing sequence \( S_m \) converging to \( S \) and s.t. \( |S_m| < |S| \) for all \( m \): because \( \theta \) increases strictly, so does the sequence \( \theta(S_m) \) converging to \( \theta(S) \), therefore we can pick subsequences \( S_p \) of \( S_m \) and \( t_p \) of \( t_n \) s.t. \( \theta(S_p) \leq t_p \), as desired.

Next we apply the Maximum Theorem twice. The first one to show that the function \( (T, t^1, t^2) \rightarrow C(T, t^1, t^2) = \max\{u(S) | S \subseteq \Omega \cap T; \theta(T \cup S) \leq t^1 + t^2 \} \) is continuous because the correspondence \( (T, t^1, t^2) \rightarrow \{S | S \subseteq \Omega \cap T; \theta(T \cup S) \leq t^1 + t^2 \} \) is continuous. The second one to deduce that the function \( \min_{T} \theta(T) \leq t_1^2 \) is continuous.

2). Second statement. For simplicity we assume \( n = 3 \), the general proof is entirely similar. Fixing \( u \) and \( t^1 \) there is some \( t^2 \) such that \( u^\theta(t^1; t^2) = u^\theta(t^2; 1) \). This is because of the monotonicity properties of \( u^\theta \) and of the inequalities \( u^\theta(t^1; t^1) = 0 \leq u^\theta(t^1; 1) \) and \( u^\theta(t^1; 1) \geq 0 = u^\theta(1; 1) \). This common value is unique (though \( t^2 \) may not be) and defines a function \( g(t^1) = u^\theta(t^1; t^2) = u^\theta(t^2; 1) \). It is easy to check from the continuity and monotonicity properties of \( u^\theta \) that \( g \) is weakly decreasing and continuous. Then we find in the same way \( t^3 \) s.t. \( g(t^3) = u^\theta(0; t^1) \). Check finally that if \( \tau_\ast \in \Upsilon(n) \) is such that all terms \( u^\theta(t^1; t^k+1) \), \( 0 \leq k \leq n - 1 \) equal a common value \( \lambda \), then \( \tau_\ast \) solves program (4). If it does not there is a \( \tau \) such that \( u^\theta(t^1; t^k+1) > \lambda \) for \( 0 \leq k \leq n - 1 \). Applying this inequality at \( k = 0 \) gives \( t^1 > t^1 \); next at \( k = 1 \) we get \( u^\theta(t^1, t^2) > u^\theta(t^1, t^2) \) implying \( t^2 > t^2 \); and so on until we reach a contradiction with the fact that both \( \tau \) and \( \tau_\ast \) are in \( \Upsilon(n) \).

Finally, the optimal sequence \( t^k \) increases in \( k \), strictly if \( u \) is not everywhere zero because \( u(t, t) = 0 \) for all \( t \).

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