A NONASPERICAL CELL-LIKE 2-DIMENSIONAL SIMPLY CONNECTED CONTINUUM AND RELATED CONSTRUCTIONS

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ABSTRACT. We prove the existence of a 2-dimensional nonaspherical simply connected cell-like Peano continuum (the space itself was constructed in one of our earlier papers). We also indicate some relations between this space and the well-known Griffiths’ space from the 1950’s.

1. INTRODUCTION

It is well-known (see [10, 12]) that every \( n \)-dimensional compactum is weakly homotopy equivalent to an \((n + 1)\)-dimensional cell-like compactum (i.e. a compactum with the trivial shape). Therefore there exist nonaspherical cell-like simply connected compacta in all dimensions \( \geq 3 \).

It was heretofore unknown whether such compacta also exist in dimension 2. In this paper we give the affirmative answer to this question. We show that the space \( SC(S^1) \) which we constructed in our earlier paper [9], is in fact, a nonaspherical cell-like 2-dimensional simply connected Peano continuum (i.e. locally connected continuum).

We also modify our original construction of the space \( SC(S^1) \) and show that the modified construction gives a space which has the homotopy type of the classical well-known space [11] from the 1950’s, which is a non-simply connected one-point union of two contractible spaces.

Our main result concerns \( SC(X) \) for a non-simply connected path-connected space \( X \). To analyze the singular homology \( H_2(SC(X)) \), we use infinitary words and a result from [5]. Although infinitary words have already been introduced in [1], they may not be a familiar notion. In the special case \( X = S^1 \), we can prove the result only by using finitary words - we present it at the end of Section 3.

2. PRELIMINARIES

We recall the construction of the space \( SC(S^1) \) from [9]. Consider the so-called Topologist sine curve \( T \) and embed \( T \) into the square \( \mathbb{I}^2 = \mathbb{I} \times \mathbb{I} \) as in Figure 1, i.e. \( T \) is embedded as the union of \( A_1B_1A_2B_2 \cdots \) and \( AB \). Let \( S^1 \) be the circle and \( s_0 \) any of its points which we consider as the base point. Consider the topological sum of \( \mathbb{I}^2 \) and \( T \times S^1 \). The space \( SC(S^1) \) is now defined as the quotient space of this sum, obtained by identification of the points \((t, s_0)\) with \( t \in T \subset \mathbb{I}^2 \), and by identification of each set \( \{t\} \times S^1 \) with \( t \), when \( t \in \{0\} \times \mathbb{I} \). For an arbitrary
compactum \( X \), one defines the space \( SC(X) \) by replacing \( S^1 \) everywhere above by \( X \). For the details of the definition of \( SC(X) \) we refer the reader to [9].

The subspace \( \mathbb{H} = \bigcup_{m=1}^{\infty}\{(x,y) : (x - 1/m)^2 + y^2 = 1/m^2\} \) of the Euclidean plane \( \mathbb{R}^2 \) is called the Hawaiian earring. Denote \( \theta = (0,0) \in \mathbb{H} \) and let \( C(\mathbb{H}) \) be the cone over \( \mathbb{H} \). We consider \( \mathbb{H} \) as the subspace of \( C(\mathbb{H}) \). A space \( \mathcal{G} \) is then defined as the one-point union of two copies of \( C(\mathbb{H}) \), obtained by identifying two copies of \( \theta \) at the point \( \theta \). This space is a well-known example of a non-contractible space which is a one-point union of contractible spaces – Griffiths was the first to investigate this kind of spaces [11, p.190], where he also acknowledges ideas by James. The fact that \( \mathcal{G} \) is aspherical was proved in [8]. For further information of this space and its generalizations we refer the reader to [4, 6, 7].

Throughout the paper, we shall denote the singular homology with integer coefficients by \( H_*(\cdot) \).

3. On nonasphericity of \( SC(S^1) \) and \( SC(X) \)

Obviously, \( SC(S^1) \) is a cell-like Peano continuum. It was shown in [9] that this space is simply connected. Therefore it suffices to show that \( SC(S^1) \) is nonaspherical. In order to prove this it certainly suffices to verify that there exists a nontrivial 2-dimensional singular cycle in \( SC(S^1) \). We shall prove this as a corollary of the following general result – Theorem 3.1 below – in the sense of [9]. Our notation for \( SC(X) \) is the same as in [9].

Consider Figure 1: the piecewise linear line \( A_1B_1A_2B_2 \cdots \) with the segment \( AB \) in this figure is the PL Topologist sine curve which was used to build \( SC(X) \), i.e. along which we attached the “infinite tube”.

**Theorem 3.1.** Let \( X \) be any path-connected space. Then the following assertions hold:

1. If \( X \) is not simply connected, then \( H_2(SC(X)) \) is not trivial; and
(2) If \( \pi_1(X) \) and \( \pi_2(X) \) are trivial, then \( H_2(SC(X)) \) is also trivial.

**Corollary 3.2.** The space \( SC(S^1) \) is a nonaspherical cell-like 2-dimensional simply connected Peano continuum.

For the proof of Theorem 3.1 we recall a notion of the free \( \sigma \)-product of groups and a lemma from [5]. Let \((X_i, x_i)\) be any family of pointed spaces such that \( X_i \cap X_j = \emptyset \), for \( i \neq j \). The underlying set of a pointed space \((\bigvee_{i \in I} (X_i, x_i), x^*)\) is the union of all \( X_i \) s obtained by identifying all \( x_i \) to a point \( x^* \) and the topology is defined by specifying the neighborhood bases as follows:

1. If \( x \in X_i \setminus \{x_i\} \), then the neighborhood base of \( x \) in \( \bigvee_{i \in I} (X_i, x_i) \) is the one of \( X_i \);
2. The point \( x^* \) has a neighborhood base, each element of which is of the form: \( \bigvee_{i \in I \setminus F} (X_i, x_i) \lor \bigvee_{j \in F} U_j \), where \( F \) is a finite subset of \( I \) and each \( U_j \) is an open neighborhood of \( x_j \) in \( X_j \) for \( j \in F \).

**Lemma 3.3.** [5 Theorem A.1] Let \( X_i \) be locally simply-connected and first countable at \( x_n \) for each \( i \). Then

\[
\pi_1(\bigvee_{i \in I} (X_i, x_i), x^*) \simeq \bigvee_{i \in I} \pi_1(X_i, x_i).
\]

In particular \( I = \mathbb{N} \),

\[
\pi_1(\bigvee_{n \in \mathbb{N}} (X_n, x_n), x^*) \simeq \bigvee_{n \in \mathbb{N}} \pi_1(X_n, x_n).
\]

We also need basic descriptions of paths and loops. A loop \( f : \mathbb{I} \rightarrow X \) is a continuous map with \( f(0) = f(1) \). For a loop \( f \), \( f^- \) denotes the loop defined by: \( f^-(t) = f(1 - t) \). For loops \( f, g \) with the same base point, the concatenation \( fg \) is a loop defined by: \( fg(t) = f(2t) \) for \( 0 \leq t \leq 1/2 \) and \( fg(t) = g(2t - 1) \) for \( 1/2 \leq t \leq 1 \). We denote the homotopy class relative to end points of a loop \( f \) by \([f]\) and the homology class of \( f \) by \([f]_s\).

**Proof of Theorem 3.1** Let \( p \) be the natural projection of \( SC(X) \) onto \( \mathbb{I}^2 \) which we consider as a subspace of the plane \( \mathbb{R}^2 \).

Let \( Y_0 = p^{-1}(\mathbb{I} \times [0, 2/3]) \) and \( Y_1 = p^{-1}(\mathbb{I} \times (1/3, 1]) \). Then \( SC(X) = Y_0 \cup Y_1 \) and \( Y_0 \cap Y_1 \) is open in \( SC(X) \).

Consider the following Mayer-Vietoris homology exact sequence:

\[
H_2(SC(X)) \xrightarrow{\partial} H_1(Y_0 \cap Y_1) \xrightarrow{h} H_1(Y_0) \oplus H_1(Y_1).
\]

We let \( i_0 : Y_0 \cap Y_1 \rightarrow Y_0 \) and \( i_1 : Y_0 \cap Y_1 \rightarrow Y_1 \) be the inclusion maps. Then \( h = i_{0*} - i_{1*} \).

We now present the proof of property (1) above. We first observe that non-injectivity of \( h \) implies that \( H_2(SC(X)) \) is non-trivial.

Since \( p^{-1}(\mathbb{I} \times \{0\}), p^{-1}(\mathbb{I} \times \{1/2\}), p^{-1}(\mathbb{I} \times \{1\}) \) are strong deformation retracts of \( Y_0, Y_0 \cap Y_1 \) and \( Y_1 \) respectively, the homotopy types of \( Y_0, Y_1 \) and \( Y_0 \cap Y_1 \) have the same homotopy type as \( p^{-1}(\mathbb{I} \times \{0\}) \). We denote the deformation retractions by \( r_0 : Y_0 \rightarrow p^{-1}(\mathbb{I} \times \{0\}) \) and \( r_1 : Y_1 \rightarrow p^{-1}(\mathbb{I} \times \{1\}) \).

Choose a point \( x^\# \in X \) and form a one point union \((X, x^\#) \lor (\mathbb{I}, 0)\) under the identification of \( x^\# \) and 0. Let \( X_n \) s be copies of \((X, x^\#) \lor (\mathbb{I}, 0)\) and \( x_n \) s copies of 1 in \( \mathbb{I} \). Then the space \( p^{-1}(\mathbb{I} \times \{0\}) \) has the same homotopy type \( Y = \bigvee_{n \in \mathbb{N}}(X_n, x_n) \).
Hence \((X_n, x_n)\) is locally simply connected and first countable at \(x_n\). Lemma 3.3 implies that \(\pi_1(Y) \simeq \times_{n \in \mathbb{N}} \pi_1(X_n, x_n)\).

Since \(X\) is not simply connected, we can find an essential loop \(f\) in \(X\) whose base point is \(x^\#\). Observe that \(p^{-1}(\{P\})\) is a copy of \(X\) for each point \(P\) on \(A_1B_1A_2B_2 \cdots\). A point \(P\) on \(A_1B_1A_2B_2 \cdots\) is written as \(P = (x, y)\) for \(x, y \in I\).

Define

\[
f_P(t) = \begin{cases} 
(3xt, y), & \text{for } 0 \leq t \leq 1/3 \\
(P, f(3(t - 1/3))), & \text{for } 1/3 \leq t \leq 2/3 \\
(3(1 - t)x, y), & \text{for } 2/3 \leq t \leq 1.
\end{cases}
\]

Then for \(n \geq 1\), \(f_{A_n}\) is a loop in \(P^{-1}(I \times \{0\}) \subseteq Y_0\) with the base point \(A\) and \(f_{B_n}\) one in \(P^{-1}(I \times \{1\}) \subseteq Y_1\) with the base point \(B\) and \(f_{C_n}\) one in \(P^{-1}(I \times \{1/2\}) \subseteq Y_0 \cap Y_1\) with the base point \(C\) respectively. Since the images of \(f_{C_n}\) are the commutator subgroup of \(\pi\) and is well-ordered from the right to the left. But this is impossible, because \(f_{C_n}\) is of infinite length. Therefore, \(W_{2m}\) is of infinite length and is well-ordered from the right to the left. But this is impossible, because \(f_{C_n}\) is well-ordered from the left to the right.

Next we show the second statement (2). Suppose that \(\pi_1(X)\) and \(\pi_2(X)\) are trivial. Consider another part of the Mayer-Vietoris sequence:

\[
H_2(Y_0) \oplus H_2(Y_1) \to H_2(SC(X)) \to H_1(Y_0 \cap Y_1).
\]

By \(\pi_1(Y_0 \cap Y_1) \simeq \times_{n \in \mathbb{N}} \pi_1(X_n, x_n)\), we conclude that \(\pi_1(Y_0 \cap Y_1)\) is trivial. Hence \(H_1(Y_0 \cap Y_1)\) is trivial. Since \(\pi_1(Y_0)\) is trivial, it follows that \(H_2(Y_0)\) is isomorphic to \(\pi_2(Y_0)\). Now we have \(\pi_2(Y_0) = \pi_2(Y) \simeq \Pi_{n \in \mathbb{N}} \pi_2(X_n, x_n) = \{0\}\) by [7] Theorem 1.1. Similarly, \(H_2(Y_1) = 0\) and \(H_2(Y_0) \oplus H_2(Y_1) = \{0\}\). Now the above exact sequence implies that \(H_2(SC(X))\) is trivial.

We denote the commutator generator of the fundamental group of \(X_{C_n}\), which is isomorphic to \(Z\). As in

Alternative proof of Corollary 3.2 For the case \(X = S^1\) we take \(c_n\) as the generator of the fundamental group of \(X_{C_n}\), which is isomorphic to \(Z\). As in
the preceding proof of Theorem 3.1 it suffices to show that the element \( c = c_1c_2^{-1}c_3c_4^{-1} \cdots c_i^{-1}c_{i+1} \cdots \) does not belong to the commutator subgroup of the group \( \pi_1(Y_0 \cap Y_1) \). To prove this by contradiction, suppose that \( c \) belongs to the commutator subgroup, i.e. \( c \) is a product of \( m \) commutators for some \( m \).

Consider natural homomorphism \( f : \pi_1(Y_0 \cap Y_1) \to \pi_1(\bigvee_{1 \leq i \leq 2m+2}(X_{C_i}, C_i)) \), where \( X_{C_i} = S^1 \). The group \( \pi_1(\bigvee_{1 \leq i \leq 2m+2}(X_{C_i}, C_i)) \) is a free group with \( 2m + 2 \) generators \( \langle c_1, c_2, \cdots, c_{2m+1}, c_{2m+2} \rangle \). We have

\[
f(c) = c_1c_2^{-1} \cdots c_{2m+1}c_{2m+2}c_1^{-1}c_2^{-1} \cdots c_{2m+1}c_{2m+2}.
\]

Let \( d_1 = c_1, d_2 = c_2^{-1}, d_{2k-1} = c_{2k-2}c_{2k-3} \cdots c_2^{-1}c_1c_{2k-1} \) and \( d_{2k} = c_{2k}^{-1}c_{2k-1} \).

It is easy to prove by induction the equality \( c_1c_2^{-1} \cdots c_{2k-1}c_{2k-1}^{-1}c_2^{-1} \cdots c_{2k-1}c_{2k} = [d_1, d_2] \cdots [d_{2k-1}, d_{2k}] \).

Since \( (d_1, d_2, \cdots, d_{2m+1}, d_{2m+2}) \) is obtained by a Nielsen transformation [13, p.5] from \( (c_1, c_2, \cdots, c_{2m+1}, c_{2m+2}) \), the set \( \{d_0, d_1, \cdots, d_{2m}, d_{2m+2}\} \) generates the free group \( \langle c_1, c_2, \cdots, c_{2m+1}, c_{2m+2} \rangle \). It follows from this and by [13, Proposition 6.8, p.55] (see also [2, p.137]) that \( f(c) \) cannot be presented as a product of less than \( m + 1 \) commutators. This contradicts our assumption.

\[\square\]

4. A PL Model for \( SC(S^3) \) and Some Related Constructions

In this section we demonstrate piecewise linear constructions which are similar to \( SC(S^3) \), using parameters for oscillations of a tube. Actually we prove in Theorem 4.3 that they are homotopy equivalent to the point, \( SC(S^3) \), or \( G \) depending on their parameters.

For \( 0 \leq y \leq 1 \) and \( \varepsilon \geq 0 \) with \( 0 < y + \varepsilon \leq 1 \), we construct a space \( S(y, \varepsilon) \subseteq \mathbb{R}^3 \) as follows. Consider the following points on \( \mathbb{I}^2 \) for \( n \in \mathbb{N} \) (see Figure 2), where we regard \( \mathbb{I}^2 \subseteq \mathbb{R}^2 \) as \( \mathbb{I}^2 \times \{0\} \):

\[
A_n = (\frac{1}{2n-1}, 0), \quad B_n = (\frac{1}{2n}, 1), \quad C_n = (\frac{1}{2n-1}, y + \frac{\varepsilon}{2n-1}),
\]

\[
D_n = (\frac{1}{2n}, 1 - y - \frac{\varepsilon}{2n}), \quad E_n = (\frac{1}{2n-1}, \frac{1}{2}y + \frac{\varepsilon}{2n-1}),
\]

\[
F_n = (\frac{1}{2n+1}, \frac{1}{2}(2y - \varepsilon - \frac{\varepsilon}{2n})).
\]

Let \( \overline{E_n} \) and \( \overline{F_n} \) be points on the plane \( \{(x, y, z) \in \mathbb{R}^3| z = \frac{1}{2}x\} \) the projections of which to the plane \( \mathbb{I}^2 \) are points \( E_n \) and \( F_n \) respectively, i.e.,

\[
\overline{E_n} = (\frac{1}{2n-1}, \frac{1}{2}(y + \frac{\varepsilon}{2n-1}), \frac{1}{2n-1}), \quad \overline{F_n} = (\frac{1}{2n+1}, \frac{1}{2}(2y - \varepsilon), \frac{1}{4n}).
\]

Let \( H_{2n} \) be the convex hull of the points \( A_n, B_n, C_n, D_n, E_n \) and \( F_n \) and \( H_{2n} \) the convex hull of the points \( A_{n+1}, B_n, C_{n+1}, D_n, F_n \) and \( E_n \).

Let \( H_{\infty} \) be the set \( \bigcup_{n=1}^{\infty} H_{2n} \) and \( \partial H_{\infty} \) its boundary. Let \( \triangle A_1C_1E_1 \) be an open triangle in \( \partial H_{\infty} \). Finally, define \( S(y, \varepsilon) \) to be the subspace \( (\mathbb{I}^2 \times \{0\}) \cup \partial H_{\infty} \setminus \triangle A_1C_1E_1 \) of \( \mathbb{R}^3 \).
The first lemma is easy to prove and we therefore omit its proof.

**Lemma 4.1.** Let \( \varepsilon, \varepsilon' \in (0, 1) \). Then the spaces \( S(0, \varepsilon) \) and \( S(0, \varepsilon') \) are homeomorphic and \( S(0, 1) \) is homotopy equivalent to \( S(0, \varepsilon) \).

**Lemma 4.2.** If \( 0 < y \leq 1/2 \) and \( 0 < y + \varepsilon \leq 1 \), the space \( S(y, \varepsilon) \) is homotopy equivalent to \( S(1/2, 0) \).

**Proof.** It is easy to see that \( S(y, \varepsilon) \) and \( S(y, 0) \) are homeomorphic and so we only need to prove that \( S(y, 0) \) for \( 0 < y < 1/2 \) and \( S(1/2, 0) \) are homotopy equivalent. (Without any loss of generality we may assume that \( y = 1/3 \).)

Since there might be some confusion regarding the homotopy equivalence, we explain this first. Let \( A_n, B_n, C_n, D_n, \ldots \) be the notation for \( S(1/2, 0) \) and \( C'_n, D'_n, \ldots \) be the corresponding notation for \( S(1/3, 0) \).

If we remove \( \{0\} \times I \) from \( S(1/2, 0) \) and \( S(1/3, 0) \), then the resulting spaces are homeomorphic, that is, \( S(1/2, 0) \setminus \{0\} \times I \) and \( S(1/3, 0) \setminus \{0\} \times I \) are homeomorphic. However, this homeomorphism cannot be extended over to \( S(1/2, 0) \), since the homeomorphism maps \( C_n \) to \( C'_n \) and \( D_n \) to \( D'_n \), that is, upwards for \( C_n \) and downwards for \( D_n \), with respect to the \( y \)-coordinate. Conversely, if we construct a homotopy on \( S(1/2, 0) \setminus \{0\} \times I \) or \( S(1/3, 0) \setminus \{0\} \times I \), whose projection to the \( y \)-coordinate only depends on the \( y \)-coordinate on the domain, it extends on \( SC(1/2, 0) \) or \( SC(1/3, 0) \).

We define \( \varphi : S(1/2, 0) \rightarrow S(1/3, 0) \) and \( \psi : S(1/2, 0) \rightarrow S(1/3, 0) \) piecewise linearly as follows:

Let \( \varphi(x, y, 0) = (x, y, 0) \) and \( \varphi(x, y, z) = (x, y, \varphi_2(x, y, z)) \), for \( z > 0 \), where \( \varphi_2(x, y, z) > 0 \) if and only if \( z > 0 \) and there exists \( z' > 0 \) such that \( (x, y, z') \in \)
$S(1/3, 0)$. Let

$$\psi_1(y) = \begin{cases} 
3y/2, & \text{for } 0 \leq y \leq 1/3 \\
1/2, & \text{for } 1/3 \leq y \leq 2/3 \\
3y/2 - 1/2, & \text{for } 2/3 \leq y \leq 1.
\end{cases}$$

and $\psi(x, y, z) = (\psi_0(x, y, z), \psi_1(y), \psi_2(x, y, z))$, where $\psi_2(x, y, 0) = 0$ and $\psi_2(x, y, z) > 0$, for $z > 0$ and $\psi_0(x, y, z)$ is defined as we explain using Figure 3 in the sequel.

Figure 3 demonstrates how $[\frac{1}{2n+1}, \frac{1}{2n}] \times I$ of $S(1/2, 0)$ and $S(1/3, 0)$ are mapped by $\varphi$ and $\psi$.

![Figure 3: Parts of $S(1/2, 0)$ and $S(1/3, 0)$](image)

First we explain the map $\psi$. The two shadowed triangles are mapped to $C_{n+1}$ or $D_n$, respectively. Accordingly, the segments $B'_n C'_{n+1}$ and $D'_n A'_{n+1}$ are mapped onto $B_n C_{n+1}$ and $D_n A_{n+1}$ respectively. The segments $N'_n D_n$ and $C'_{n+1} M'_n$ are mapped bijectively to $C_{n+1} D_n$.

Next we explain the map $\varphi \psi$. The two shadowed triangles are mapped to $\varphi(C_{n+1})$ or $\varphi(D_n)$, which are the dotted point. The two bending segments are mapped onto $C'_{n+1} B'_n$ or $A'_{n+1} D'_n$.

Last we explain the map $\psi \varphi$. The two shadowed triangles are mapped to $C_{n+1}$ or $D_n$. The two segments having slope greater than 1 are mapped to $C_{n+1} B_n$ or $A_{n+1} D_n$.

We have a homotopy $H(x, y, z, t)$ on $S(1/2, 0) \setminus \{0\} \times I$ such that:

1. $H(x, y, z, 0) = (x, y, z)$ and $H(x, y, z, 1) = \psi \varphi(x, y, z)$;
2. for the $y$-coordinate $H_1(x, y, z, t)$ of $H(x, y, z, t)$,

$$H_1(x, y, z, t) = \begin{cases} 
y + yt/2, & \text{for } 0 \leq y \leq 1/3 \\
y + t/2 - yt, & \text{for } 1/3 \leq y \leq 2/3 \\
y - t/2 + yt/2, & \text{for } 2/3 \leq y \leq 1;
\end{cases}$$

3. $H(*, *, *, t)$ maps $p^{-1}(I \times I \times I)$ onto itself for each $n$. 

$\varphi$ maps $B_n$ onto $A_{n+1}$ and $D_n$ onto $B_n$. The two segments having slope greater than 1 are mapped to $C_{n+1}$ or $D_n$.
Then we can extend $H(\ast, \ast, \ast, t)$ to $S(1/2, 0)$ uniquely and continuously.

Concerning $S(1/3, 0)$ with $\varphi \psi$, we have a homotopy with the same properties as above and we now see that $S(1, 0)$ and $S(1/3, 0)$ are homotopy equivalent. □

**Theorem 4.3.** Suppose that $0 \leq y \leq 1$, $\varepsilon \geq 0$ and $0 < y + \varepsilon \leq 1$. Then the following assertions hold:

1. For every $1/2 < y \leq 1$, the spaces $S(1, 0)$ and $S(y, \varepsilon)$ are contractible;
2. For $y = 0$, the space $S(y, \varepsilon)$ is homotopy equivalent to $SC(S^1)$; and
3. For every $0 < y \leq 1/2$, the space $S(y, \varepsilon)$ is homotopy equivalent to the space $G$.

**Proof.** The statements (1) and (2) are easy to verify. Therefore we shall only prove (3).

By Lemma 4.2 it suffices to show that $S(1/2, 1/4)$ is homotopy equivalent to the space $G$. Let $\Delta$ be the half-open triangle, defined as $\Delta = \{(x, y) \mid x \in (0, 1), y \in (-x/4 + 1/2, x/4 + 1/2)\}$. Then $p^{-1}(\mathbb{I}^2 \setminus \Delta)$ is a strong deformation retract of $S(1/2, 1/4)$.

Identifying $\{(x, y) \mid y = a + (1 - a)x/4, x \in \mathbb{I}\}$ as one point for $a \in [1/2, 1]$ and $\{(x, y) \mid y = a - ax/2, x \in I\}$ as one point for $a \in [0, 1/2]$, we get the quotient space of $p^{-1}(\mathbb{I}^2 \setminus \Delta)$, which is homeomorphic to $G$. Now the homotopy equivalence between $p^{-1}(\mathbb{I}^2 \setminus \Delta)$ and $G$ is evident and so $S(1/2, 1/4)$ is indeed homotopy equivalent to $G$. □

**Remark 4.4.** The space $SC(S^1)$ is simply connected (see [9]), whereas the space $G$ is not simply connected (see [11]). We remark that $H_2(G) = \{0\}$, which contrasts with Theorem 4.2.

To show this, we introduce some notation. Since the cone $C(X)$ over the space $X$ is the quotient space of $X \times I$, obtained by identifying $X \times \{1\}$ to a point, we let $p : X \times I \to C(X)$ be the canonical projection.

For a subset $A$ of $I$, let $C_A(X) = p(X \times A) \subset C(X)$. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be copies of the Hawaiian earring $\mathbb{H}$ and $G = C(\mathbb{H}_1) \vee C(\mathbb{H}_2)$ be the one point union of $C(\mathbb{H}_1)$ and $C(\mathbb{H}_2)$ defined in Section 2. Let $X_1$ be the disjoint union of $C_{1/3, 1}(\mathbb{H}_1)$ and $C_{1/3, 1}(\mathbb{H}_2)$ and $X_2$ be $C_{0, 2/3}(\mathbb{H}_1) \vee C_{0, 2/3}(\mathbb{H}_2)$.

Then $G = X_1 \cup X_2$ and we have the following part of the Mayer-Vietoris sequence:

$$H_2(X_1) \oplus H_2(X_2) \longrightarrow H_2(G) \overset{\partial}{\longrightarrow} H_1(X_1 \cap X_2) \overset{h}{\longrightarrow} H_1(X_1) \oplus H_1(X_2).$$

Obviously, $H_2(X_1) = \{0\}$. Since $X_2$ is homotopy equivalent to $\mathbb{H}_1 \vee \mathbb{H}_2$ which is a 1-dimensional compact metric space, $H_2(X_2)$ is trivial [3]. Hence $\partial$ is injective. We observe that $X_1 \cap X_2$ is the disjoint union of $C_{1/3, 2/3}(\mathbb{H}_1)$ and $C_{1/3, 2/3}(\mathbb{H}_2)$.

Since $C_{1/3, 2/3}(\mathbb{H}_1)$ and $C_{1/3, 2/3}(\mathbb{H}_2)$ are retracts of $C_{0, 2/3}(\mathbb{H}_1) \vee C_{0, 2/3}(\mathbb{H}_2)$ and are homotopy equivalent to $C_{1/3, 2/3}(\mathbb{H}_1)$ and $C_{1/3, 2/3}(\mathbb{H}_2)$ respectively, it follows that $h$ is injective. Therefore we obtain that $H_2(G) = \{0\}$.

**Problem 4.5.** Does there exist a finite-dimensional non-contractible Peano continuum all homotopy groups of which are trivial?

**Remark 4.6.** Recently we have strengthened Theorem 3.1(2) by proving the following: If $X$ is simply connected, then $\pi_2(SC(X))$ is trivial. We have proved earlier that $SC(X)$ is also simply connected [9]. Therefore by Theorem 3.1(1) the following statements are equivalent for any path-connected space $X$:

1. $X$ is simply connected;
(2) $\pi_2(\text{SC}(X))$ is trivial; and
(3) $H_2(\text{SC}(X))$ is trivial.

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