Virtual bound levels in a gap of the essential spectrum of the Schrödinger operator with a weakly perturbed periodic potential

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Abstract
In the space $L^2(\mathbb{R}^d)$ we consider the Schrödinger operator $H_\gamma = -\Delta + V(x) \cdot + \gamma W(x)$, where $V(x) = V(x_1, x_2, \ldots, x_d)$ is a periodic function with respect to all the variables, $\gamma$ is a small real coupling constant and the perturbation $W(x)$ tends to zero sufficiently fast as $|x| \to \infty$. We study so called virtual bound levels of the operator $H_\gamma$, that is those eigenvalues of $H_\gamma$ which are born at the moment $\gamma = 0$ in a gap $(\lambda_-, \lambda_+)$ of the spectrum of the unperturbed operator $H_0 = -\Delta + V(x)$ from an edge of this gap while $\gamma$ increases or decreases. For a definite perturbation ($W(x) \geq 0$) we investigate the number of such levels and an asymptotic behavior of them and of the corresponding eigenfunctions as $\gamma \to 0$ in two cases: for the case where the dispersion function of $H_0$, branching from an edge of $(\lambda_-, \lambda_+)$, is non-degenerate in the Morse sense at its extremal set and for the case where it has there a non-localized degeneration of the Morse-Bott type. In the first case in the gap there is a finite number of virtual eigenvalues if $d < 3$ and we count the number of them, and in the second case in the gap there is an infinite number of ones, if the codimension of the extremal manifold is less than 3. For an indefinite perturbation we estimate the multiplicity of virtual bound levels. Furthermore, we show that if the codimension of the extremal manifold is at least 3 at both edges of the gap $(\lambda_-, \lambda_+)$, then under additional conditions there is a threshold for the birth of the impurity spectrum in the gap, that is $\sigma(H_\gamma) \cap (\lambda_-, \lambda_+)$ for a small enough $|\gamma|$.

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Contents
1 Introduction 3
   1.1 Background 3
   1.2 Description of methods and results 5
## 2 Basic notation

## 3 Preliminaries

3.1 Spectral characteristics of the unperturbed operator

3.2 The notion of a virtual eigenvalue

## 4 Formulation of main results

4.1 The birth of virtual eigenvalues from a non-degenerate edge

4.2 The birth of virtual eigenvalues from a degenerate edge

4.3 The estimate of the multiplicity of virtual eigenvalues for an indefinite perturbation in the case of a non-degenerate edge

4.4 A threshold for the birth of virtual eigenvalues for an indefinite perturbation in the case of a degenerate edge

## 5 Proof of main results

5.1 General results on the birth of virtual eigenvalues

5.2 Perturbation of a compact operator

5.3 The case of non-degenerate edges of a gap of the spectrum of the unperturbed operator

5.3.1 Compactness of Birman-Schwinger operator

5.3.2 Representation of Birman-Schwinger operator in the case of a non-degenerate edge

5.3.3 Linear independence of weighted Bloch functions

5.3.4 Proof of Theorem 4.1

5.4 The case of degenerate edges of a gap of the spectrum of the unperturbed operator

5.4.1 Representation of Birman-Schwinger operator in the case of a degenerate edge

5.4.2 Spectrum of the operator $G_W^+ (G_W^-)$

5.4.3 Proof of Theorem 4.2

5.5 Estimate for the multiplicity of virtual eigenvalues in the case of an indefinite perturbation

5.5.1 General results on the multiplicity of virtual eigenvalues

5.5.2 Proof of Theorem 4.3

5.5.3 Proof of Theorem 4.4

## 6 Representation of the resolvent of the unperturbed operator $H_0$ near the edges of a gap of its spectrum

### 6.1 Main claims

### 6.2 Auxiliary claims

6.2.1 Estimates of some integrals

6.2.2 Some geometric claims

### 6.3 Proof of claim (i) of Proposition 6.1

### 6.4 Proof of claim (ii) of Proposition 6.1

### 6.5 Proof of claim (iii) of Proposition 6.1

### 6.6 Proof of Proposition 6.2
Appendix: $C(\Omega)$-holomorphy of Bloch functions

A.1 Main claims ........................................ 54
A.2 Domains and self-adjointness of the operators $H(p)$ and $H_0$ .... 55
A.3 Fundamental solution of the Helmholtz’s equation in $\mathbb{R}^d$ .... 58
A.4 Green’s function of the operator $H(p)$ .......................... 60
A.5 $C(\Omega \times \Omega)$-holomorphy of a compositional power of the Green’s function of $H(p)$ ........................................... 63
A.6 Proof of Theorem A.1 ....................................... 70
A.7 Proof of Corollary A.2 ........................................ 71

Bibliography 72

1 Introduction

1.1 Background

In this paper we consider the Schrödinger operator

$$H_\gamma = -\Delta + V(x) \cdot + \gamma W(x),$$

acting in the space $L_2(\mathbb{R}^d)$, where $V(x) = V(x_1, x_2, \ldots, x_d)$ is a periodic function with respect to all the variables and satisfying some mild condition which will be pointed below, the function $W(x)$ is measurable and bounded in $\mathbb{R}^d$ and $\gamma$ is a small real coupling constant. In what follows we shall impose on the perturbation $W(x)$ some conditions, which mean that it tends to zero sufficiently fast as $|x| \to \infty$ in an integral sense. We study so called virtual bound levels of the operator $H_\gamma$, that is those eigenvalues of $H_\gamma$ which are born at the moment $\gamma = 0$ in a gap $(\lambda_-, \lambda_+)$ of the spectrum of the unperturbed operator $H_0 = -\Delta + V(x)$ from an edge of this gap while $\gamma$ increases or decreases. In physics they are called ”resonance levels” or ”trapped levels”.

A wide literature is devoted to the study of discrete spectrum in gaps of the essential spectrum. In [Rof] and [Rof1] for the one-dimensional case ($d=1$) the tests for finiteness and infiniteness of the number of eigenvalues of the operator $H_1 = H_\gamma|_{\gamma=1}$ in the gaps of the spectrum of the unperturbed operator $H_0 = -\Delta + V(x)$ from an edge of this gap while $\gamma$ increases or decreases. In physics they are called ”resonance levels” or ”trapped levels”.

A.7 Proof of Corollary A.2 ........................................ 71

Bibliography 72
spectrum is well studied for a "strong coupling", that is for $|\gamma| \to \infty$ \cite{Bi2, Bi-Sc, Soh}.

But there is a comparatively small number of results concerning the behavior of the discrete part of the spectrum of $H_\gamma$ for a "weak coupling", that is for $|\gamma| \to 0$. In the paper of M. Sh. Birman \cite{Bi1} (1961) a variational approach has been worked out for the study of birth of negative eigenvalues under a small perturbation in the case $V(x) \equiv 0$. In the 1970’s in a series of papers the asymptotic behavior as $\gamma \to 0$ of negative eigenvalues and the corresponding eigenfunctions of the Schrödinger operator $H_\gamma$ was studied for $V(x) \equiv 0$ with the help of analytical methods \cite{Re-Si, S, S1, Kl, B-G-S}. These investigations were based on the explicit form of the Green function for the unperturbed operator $H_0 = -\Delta$ and on the Birman-Schwinger principle, which describes the discrete spectrum of the perturbed operator in the gaps of the spectrum of the unperturbed one with the help of so called Birman-Schwinger operator, defined by \cite{Bi3, Sc, Re-Si, S}.

The interest in this subject was renewed in the last two decades. In \cite{Wei} T. Weidl has developed the Birman approach for the study of the existence of virtual eigenvalues for a wide class of elliptic differential operators of high order and even for indefinite perturbations. In the papers \cite{Ar-Zl1} and \cite{Ar-Zl2} the negative virtual eigenvalues were studied for the perturbation $(-\Delta)^l + \gamma W(x)$ of the polyharmonic operator $(-\Delta)^l$ with the help of an analytical method. The Green function of the unperturbed operator $(-\Delta)^l$ was not constructed there explicitly, but by using the Fourier transform (the "momentum representation") a representation for the resolvent $((-\Delta)^l - \lambda I)^{-1}$ of the unperturbed operator was obtained near the bottom $\lambda = 0$ of its spectrum, which permit to get asymptotic formulas for the negative virtual eigenvalues of the perturbed operator with the help of the Birman-Schwinger principle. Observe that for the unperturbed operator $(-\Delta)^l$ the dispersion function (dependence of the energy $\lambda$ on the momentum $p$) has the form $\lambda = |p|^{2l}$, hence it has one minimum point $p = 0$. Thanks this fact, for a short range perturbation $W(x)$ (that is, or it has a compact support, or tends to zero sufficiently fast as $x \to \infty$) the perturbed operator $(-\Delta)^l + \gamma W(x)$ has a finite number of negative virtual eigenvalues. In particular (for $l = 1$), this property holds for the perturbation of the Laplacian. But in \cite{Ch-M} for $d = 2$ an axially symmetric Hamiltonian describing a spin-orbit interaction was considered such that the minima set of its dispersion function is a circle (non-localized degeneration of the dispersion function). By the use of the method of separating variables it was shown in \cite{Ch-M} that in the presence of an arbitrarily shallow axially symmetric potential well $W(x)$ an infinite number of negative eigenvalues of the perturbed Hamiltonian appear. With the physical point of view in this case an infinite number of electrons have the minimal energy level of the unperturbed Hamiltonian, hence an infinite number of bound energy levels can appear beneath this level in the presence of some small impurities in the system. For a more general situation the analogous result was established in \cite{Br-G-P} and \cite{Pan} by using the variational method. In \cite{H-S} a general situation of unperturbed Hamiltonian ("kinetic energy") is
considered such that the minima set of its dispersion function is a submanifold of codimension one. The existence of an infinite number of virtual negative eigenvalues is proved there and asymptotic formulas are obtained for them and corresponding eigenfunctions for a small coupling constant. The method used in [H-S] is close to one used in [Ar-Zl1], [Ar-Zl2].

In [P-L-A-J] with the help of the variational method the existence of virtual eigenvalues in gaps of the essential spectrum of the operator $H_\gamma$, defined by \eqref{1.1}, is proved. But this method does not permit to investigate the asymptotic behavior of virtual eigenvalues and of the corresponding eigenfunctions for $\gamma \to 0$.

1.2 Description of methods and results

In the present paper we investigate the number of virtual eigenvalues in a gap of the essential spectrum of the operator $H_\gamma$ and obtain asymptotic formulas for them and for the corresponding eigenfunctions for $\gamma \to 0$. The method we use is close to one used in [Ar-Zl1], [Ar-Zl2] and [H-S], but instead of the Fourier transform we apply the so called Gelfand-Fourier-Floquet transform \eqref{5.23} (a “quasi-momentum representation” of the Hamiltonian), which realizes a unitary equivalence between the unperturbed operator $H_0 = -\Delta + V(x)$, generated on a fundamental domain $\Omega$ of the lattice of periodicity of $V(x)$ by the operation $-\Delta + V(x)$, and some cyclic boundary conditions (Gel, Wil, Kuch, Zl). In this situation the role of the dispersion function for the operator $H_0$ plays the dependence of the energy $\lambda$ (the eigenvalue of $H(p)$) on the quasi-momentum $p$: $\lambda = \lambda(p)$. Like in [Ar-Zl1], [Ar-Zl2], our considerations are based on a representation of the resolvent $(H_0 - \lambda I)^{-1}$ of the unperturbed operator near the edge $\lambda = \lambda_+ (\lambda = \lambda_-)$ of a gap $\lambda_+ = \lambda_-$ of its spectrum as a sum of a singular (w.r.t. $\lambda$) part and a regular remainder. We consider this representation in two cases of behavior of the dispersion function $\lambda(p)$ branching from the edge $\lambda_+ (\lambda_-)$ of the gap $\lambda_+ (\lambda_-)$ and having the extremal set $F^+ = \lambda^{-1}(\lambda_+)$ ($F^- = \lambda^{-1}(\lambda_-)$): the case of a non-degenerate edge $\lambda_+ (\lambda_-)$ (Proposition 6.2) of the Morse’s type in the sense that at any point of the set $F^+ (F^-)$ the Hessian operator of $\lambda(p)$ is non-degenerate, and the case of a non-localized degeneration of the Bott-Morse type (Proposition 6.1) in the sense that the set $F^+ (F^-)$ is smooth submanifold of a non-zero dimension and at each point of $F^+ (F^-)$ the Hessian operator of $\lambda(p)$ is non-degenerate along the normal subspace to $F^+ (F^-)$. In both cases we assume that $\lambda(p)$ is a simple eigenvalue of the operator $H(p)$ for any $p \in F^+ (p \in F^-)$.

As it was shown in [K-R], this situation is generic in the sense that all the edges of gaps of the spectrum of the operator $H_0$ are simple for a dense $G_δ$-set of periodic potentials $V \in L_\infty(\Omega)$.

\footnote{As it was shown in [K-R], this situation is generic in the sense that all the edges of gaps of the spectrum of the operator $H_0$ are simple for a dense $G_δ$-set of periodic potentials $V \in L_\infty(\Omega)$.}
inite perturbation, from the edge \( \lambda \) character of this birth depends on the codimension \( \text{codim}(F) \) which has a non-localized degeneration of the Morse-Bott type. The virtual eigenvalues of \( H_\lambda \) from the extremal submanifold \( F \) points of the extremal set of the spectral gap of \( H_\lambda \) is non-degenerate even in the multi-dimensional case. Taking the periodic potential in the form \( V(x) = \sum_{k=1}^d V_k(x_k) \) \( (x = (x_1, x_2, \ldots, x_d)) \) and using the method of separating variables, it is not difficult to construct an example of the multi-dimensional operator \( H_0 \) having a finite spectral gap with non-degenerate edges.

In Theorem 4.1 we count the number of virtual eigenvalues of \( H_\lambda \) in a spectral gap \( (\lambda_-, \lambda_+) \) of \( H_0 \) being born from its non-degenerate edge \( \lambda_+ \) \( (\lambda_-) \) and obtain asymptotic formulas for them as \( |\gamma| \to 0 \) for a definite perturbation (that is, \( W(x) \geq 0 \) on \( \mathbb{R}^d \)) under the assumption that \( W(x) \) tends to zero sufficiently fast in an integral sense. For \( d = 1 \) there is only one virtual eigenvalue being born from the edge \( \lambda_+ \) \( (\lambda_-) \) of the spectral gap of \( H_0 \) for \( \gamma < 0 \) \( (\gamma > 0) \) and the leading term of the asymptotic formula for the distance between this virtual eigenvalue and the edge \( \lambda_+ \) \( (\lambda_-) \) has the order \( O(\gamma^2) \) as \( \gamma \uparrow 0 \) \( (\gamma \downarrow 0) \). For \( d = 2 \) the number of virtual eigenvalues being born from the edge \( \lambda_+ \) \( (\lambda_-) \) of the spectral gap of \( H_0 \) for \( \gamma < 0 \) \( (\gamma > 0) \) coincides with the number of points of the extremal set \( F^+ \) \( (F^-) \) of the dispersion function branching from \( \lambda_+ \) \( (\lambda_-) \) and the leading terms of the asymptotic formulas for the distances between these virtual eigenvalues and the edge \( \lambda_+ \) \( (\lambda_-) \) has an exponential order as \( \gamma \uparrow 0 \) \( (\gamma \downarrow 0) \). Furthermore, Theorem 4.1 claims that for \( d \leq 2 \) the eigenfunctions corresponding to the virtual eigenvalues of \( H_\lambda \) converge in some sense as \( \gamma \uparrow 0 \) \( (\gamma \downarrow 0) \) with the rate \( O(\gamma) \) to linear combinations of Bloch functions of \( H_0 \) corresponding to the energy level \( \lambda_+ \) \( (\lambda_-) \) and the quasi-momenta from \( F^+ \) \( (F^-) \). For \( d \geq 3 \) there is a threshold for the birth of the impurity spectrum of \( H_\lambda \) in the spectral gap \( (\lambda_-, \lambda_+) \) of \( H_0 \) for \( \gamma < 0 \) \( (\gamma > 0) \), that is for a small enough \( \gamma < 0 \) \( (\gamma > 0) \) there is no eigenvalue of \( H_\lambda \) in \( (\lambda_-, \lambda_+) \). Furthermore, for \( d = 2 \) Theorem 4.1 yields an asymptotic formula of Lieb-Thirring type for the sum of inverse logarithms of distances between the virtual eigenvalues and the edge \( \lambda_+ \) \( (\lambda_-) \), and its leading term is expressed explicitly via the perturbation \( W(x) \) and spectral characteristics of the unperturbed operator \( H_0 \) at this edge: Bloch functions and effective masses of electrons at the energy level \( \lambda_+ \) \( (\lambda_-) \).

Theorem 4.2 describes the birth of virtual eigenvalues of \( H_\lambda \), under a definite perturbation, from the edge \( \lambda_+ \) \( (\lambda_-) \) of a spectral gap \( (\lambda_-, \lambda_+) \) of \( H_0 \), which has a non-localized degeneration of the Morse-Bott type. The character of this birth depends on the codimension \( \text{codim}(F^+) \) \( (\text{codim}(F^-)) \) of the extremal submanifold \( F^+ \) \( (F^-) \) of the dispersion function branching from \( \lambda_+ \) \( (\lambda_-) \). For \( \text{codim}(F^+) \leq 2 \) \( (\text{codim}(F^-) \leq 2) \) there is an infinite number of virtual eigenvalues of \( H_\lambda \) being born from this edge, and the asymptotic be-
behavior of them as $\gamma \uparrow 0$ ($\gamma \downarrow 0$) for $\text{codim}(F^+) = 1$ ($\text{codim}(F^-) = 1$) and $\text{codim}(F^+) = 2$ ($\text{codim}(F^-) = 2$) is analogous to one in the non-degenerate case for $d = 1$ and $d = 2$ respectively, and furthermore, the asymptotic behavior of the corresponding eigenfunctions is analogous to one in the non-degenerate case for $d \leq 2$. If $\text{codim}(F^+) \geq 3$ ($\text{codim}(F^-) \geq 3$), then for $\gamma < 0$ ($\gamma > 0$) there is a threshold for the birth of the impurity spectrum of $H_\gamma$ in the spectral gap $(\lambda_-, \lambda_+)$ of $H_0$ like in the non-degenerate case for $d \geq 3$. Furthermore, for $\text{codim}(F^+) \leq 2$ ($\text{codim}(F^-) \leq 2$) Theorem 4.2 yields a weak version of the asymptotic formula of Lieb-Thirring type, mentioned above, for the sum of square roots of distances between virtual eigenvalues and the edge $\lambda_+$ ($\lambda_-$), and for the sum of inverse logarithms of ones if $\text{codim}(F^+) = 2$ ($\text{codim}(F^-) = 2$). The leading terms of these formulas have the form of the integrals over $F^+$ ($F^-$), whose integrands are expressed explicitly via the perturbation $W(x)$ and spectral characteristics of the unperturbed operator $H_0$ at the edge $\lambda_+$ ($\lambda_-$), mentioned above (merely in this case the effective masses are computed in the directions normal to the extremal submanifold).

Theorem 4.3 treats the case of an indefinite perturbation ($W(x)$ may change the sign) and the non-degenerate edge $\lambda_+$ ($\lambda_-$). It yields an estimate of the multiplicity of virtual eigenvalues if $d \leq 2$, and for $d \geq 3$ it claims the existence of a threshold for the birth of the impurity spectrum from this edge.

Theorem 4.4 claims the existence of a threshold for the birth of the impurity spectrum from the degenerate edge $\lambda_+$ ($\lambda_-$) for an indefinite perturbation, if $\text{codim}(F^+) \geq 3$ ($\text{codim}(F^-) \geq 3$).

In the Appendix we prove Theorem A.1, which yields some kind of an elliptic regularity result: under a mild condition for the periodic potential $V(x)$ it claims that if a branch of eigenvalues $\lambda(p)$ of the family of operators $H(p)$, mentioned above, is holomorphic and a branch of corresponding eigenfunctions $b(\cdot, p)$ is holomorphic in the $L^2(\Omega)$-norm, then the latter branch is holomorphic in the $C(\Omega)$-norm. For $d \leq 3$ this claim follows immediately from results of the paper [Wil], but for $d \geq 4$ the arguments used there fail and we use a modification of them. Corollary A.2 of Theorem A.1 is used in the proof of the main results.

The paper is organized as follows. After this Introduction, in Section 2 we introduce some basic notation, in Section 3 (Preliminaries) we recall some known facts concerning the operator $H_0$ with the periodic potential and define the notion of a virtual eigenvalue. In Section 4 we formulate the main results. In Section 5 we prove the main results. In Section 6 we obtain the representation of the resolvent of the operator $H_0$, mentioned above. Section 7 is the Appendix. We add the label “A” to numbers of claims and formulas from the Appendix.

2 Basic notation

$x \cdot y$ ($x, y \in \mathbb{R}^d$) is the canonical inner product in the real vector space $\mathbb{R}^d$; $|x| = \sqrt{x \cdot x}$ is the Euclidean norm in $\mathbb{R}^d$;
$S^d$ is the $d$-dimensional unit sphere: $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$; $s_d$ is the $d$-dimensional volume of $S^d$;

$T^d = \times_{k=1}^d S^1$ is the $d$-dimensional torus;

$\mathbb{Z}$ is the ring of integers;

$\mathbb{Z}^d = \times_{k=1}^d \mathbb{Z}$;

$\text{Dom}(f)$ is the domain of a mapping $f$;

$(f,g)$ $(f,g \in \mathcal{H})$, $\|f\|$ are the inner product and the norm in a complex Hilbert space $\mathcal{H}$ (in particular, in $L_2(\mathbb{R}^d)$); the norm of linear bounded operators, acting in $\mathcal{H}$, is denoted in the same manner;

$L_2(\mathbb{R}^d)$ is the set of all functions from $L_2(\mathbb{R}^d)$ having compact supports;

If $A$ is a closed linear operator acting in a Hilbert space $\mathcal{H}$, then:

$\ker(A)$ is the kernel of $A$, i.e. $\ker(A) = \{x \in \mathcal{H} : Ax = 0\}$;

$\text{Im}(A)$ is the image of $A$;

$\sigma(A)$ is the spectrum of $A$;

$\mathcal{R}(A)$ is the resolvent set of $A$, i.e. $\mathcal{R}(A) = \mathbb{C} \setminus \sigma(A)$;

$R_\lambda(A)$ ($\lambda \in \mathcal{R}(A)$) is the resolvent of $A$, i.e. $R_\lambda(A) = (A - \lambda I)^{-1}$.

$\mathcal{B}(E)$ is the Banach space of linear bounded operators, acting in a Banach space $E$.

Some specific notation will be introduced in what follows.

### 3 Preliminaries

#### 3.1 Spectral characteristics of the unperturbed operator

Consider the unperturbed operator $H_0 = -\Delta + V(x)$. We assume that the potential $V(x)$ is periodic on the lattice $\Gamma = \{l \in \mathbb{R}^d | 1 = (l_1T_1, l_2T_2, \ldots , l_dT_d), k = (l_1,l_2,\ldots,l_d) \in \mathbb{Z}^d \}$ ($T_k > 0 (k = 1, 2, \ldots , d)$), that is $V(x + l) = V(x)$ for any $x \in \mathbb{R}^d$ and $l \in \Gamma$. For the simplicity we shall assume that $T_1 = T_2 = \cdots = T_d = 1$, that is $\Gamma = \mathbb{Z}^d$. Denote by $\Omega$ the fundamental domain of the lattice $\Gamma$: $\Omega := \times_{k=1}^d [-1/2, 1/2]$. Furthermore, assume that

$$V \in \begin{cases} L_2(\Omega), & \text{if } d \leq 3 \\ \bigcup_{q \geq 2} L_q(\Omega), & \text{if } d \geq 4. \end{cases} \quad (3.1)$$

By claim (ii) of Proposition A.5, the operator $H_0$ with the domain $W^2_2(\mathbb{R}^d)$ is self-adjoint and bounded below.

For any $p \in T^d$ consider the operator $H(p)$, generated by the operation $h = -\Delta + V(x)$ in the Hilbert space $\mathcal{H}_p$ of functions $u \in L_{2,\text{loc}}(\mathbb{R}^d)$ satisfying the condition

$$u(x + l) = \exp(ip \cdot l)u(x) \quad \forall x \in \mathbb{R}^d, l \in \Gamma \quad (3.2)$$

8
with the inner product and the norm, defined by
\[
(f, g) = \int_\Omega f(x)g(x) \, dx \quad (f, g \in H_p), \quad \|f\| = \sqrt{(f, f)}.
\] (3.3)

The domain of \(H(p)\) is the linear set \(D_0(\Gamma) = W^2_{2,loc}(\mathbb{R}^d) \cap H_p\). By claim (i) of Proposition \(\breve{A}, 5\) and Proposition \(\breve{A}, 6\) the operator \(H(p)\) is self-adjoint, bonded below uniformly w.r. to \(p \in \mathbb{T}^d\) and its spectrum is discrete. As it is easy to check, the operator \((E_p u)(x) := \exp(-ip \cdot x)u(x)\), acting from \(H_p\) onto \(H_0 = L_2(\mathbb{R}^d/\Gamma)\), realizes a unitary equivalence between the operator \(H(p)\) and the operator
\[
\hat{H}(p) = -\Delta_p + V(x),
\] (3.4)
with the domain \(W^2_2(\mathbb{R}^d/\Gamma)\), where
\[
\Delta_p = \sum_{j=1}^d (D_j + ip_j)^2.
\] (3.5)

Let \(\lambda_1(p) \leq \lambda_2(p) \leq \ldots \lambda_n(p) \leq \ldots\) be the eigenvalues of the operator \(\hat{H}(p)\) (counting their multiplicities), and \(e_1(x, p), e_2(x, p), \ldots, e_n(x, p), \ldots\) be the corresponding eigenfunctions of this operator which form an orthonormal basis in the space \(L_2(\mathbb{R}^d/\Gamma)\). Using the physical terminology, we shall call the vector \(p\) the \textit{quasi-momentum} and each branch of the eigenvalues \(\lambda_n(p)\) will be called the \textit{dispersion function}. It is known that \(\lambda_n(p)\) are continuous functions on \(\mathbb{T}^d\) and \(\sigma(H_0) = \bigcup_{p \in \mathbb{T}^d} \sigma(\hat{H}(p)) = \bigcup_{n=1}^\infty \{\lambda_n(p)\}\) (\cite{Gel, Eas, Eas1, Kuc, Zl}). We shall consider also the \textit{Bloch function} corresponding to a dispersion function \(\lambda_0(p)\) and a quasi-momentum \(p \in \mathbb{T}^d\):
\[
b_l(x, p) = \exp(ip \cdot x)e_l(x, p).
\] (3.6)

It is clear that for each natural \(l\) \(b_l(x, p)\) is an eigenfunction of the operator \(H(p)\), corresponding to its eigenvalue \(\lambda_l(p)\) and for any fixed \(p \in \mathbb{T}^d\) the sequence \(\{b_l(x, p)\}_{l=1}^\infty\) forms an orthonormal basis in the space \(H_p\).

It is easy to check that for any \(p \in \mathbb{T}^d\) \(JH(p)J = \hat{H}(-p)\), where \(J\) is the conjugation operator \((Jf)(x) := \overline{f(x)}\) (the property of a “time reversibility”). Hence in this case \(\sigma(H(p)) = \sigma(H(-p))\) and for any \(\mu \in \sigma(H(p))\) the corresponding eigenprojections \(Q_\mu(p)\) of \(H(p)\) and \(Q_\mu(-p)\) of \(H(-p)\) are connected in the following manner: \(Q_\mu(-p) = JQ_\mu J(p)\). The same property is valid for the operator \(\hat{H}(p)\).

Assume that \((\lambda_-, \lambda_+)\) is a gap of \(\sigma(H_0)\), that is for some \(j \geq 1\) \(\lambda_- = \max_{p \in \mathbb{T}^d} \lambda_j(p) < \lambda_+ = \min_{p \in \mathbb{T}^d} \lambda_{j+1}(p)\), and \(\lambda_- = \min_{p \in \mathbb{T}^d} \lambda_1(p), \lambda_+ = -\infty\) for \(j = 0\). Since \(j\) will be fixed in our considerations, we shall denote \(\lambda^-(p) := \lambda_j(p)\) and \(\lambda^+(p) := \lambda_{j+1}(p)\). In other words, \(\lambda^-(p)\) and \(\lambda^+(p)\) are the dispersion functions branching from the edges \(\lambda_-\) and \(\lambda_+\) respectively. We shall denote by \(b^-(x, p)\) and \(b^+(x, p)\) the eigenfunctions of \(H(p)\) (Bloch functions), corresponding to \(\lambda^-(p)\) and \(\lambda^+(p)\) respectively, i.e. \(b^-(x, p) = b_j(x, p)\) and
where

\[ b^+(x,p) = b_{j+1}(x,p). \]

In the analogous manner we denote \( e^-(x,p) = e_j(x,p) \) and \( e^+(x,p) = e_{j+1}(x,p) \).

Consider the following subsets of \( \mathbb{T}^d \):

\[ F^- := \{ p \in \mathbb{T}^d : \lambda^-(p) = \lambda_- \}, \]
\[ F^+ := \{ p \in \mathbb{T}^d : \lambda^+(p) = \lambda_+ \}, \]

which are the extremal sets of the functions \( \lambda^-(p) \) and \( \lambda^+(p) \) respectively. Assume that for the edge \( \lambda_- (\lambda_- > -\infty) \) or for the edge \( \lambda_+ \) of the gap \( (\lambda_-, \lambda_+) \) the condition is fulfilled:

(A) The edge \( \lambda_+ (\lambda_-) \) is non-degenerate in the Morse's sense, that is

(a) it is simple in the sense that for any \( p_0 \in F^+ (p_0 \in F^-) \) there exists a neighborhood \( \mathcal{O}^+(p_0) (\mathcal{O}^-(p_0)) \) of \( p_0 \) such that the function \( \lambda^+(p) (\lambda^-(p)) \) is real-analytic in \( \mathcal{O}^+(p_0) (\mathcal{O}^-(p_0)) \) (hence in particular there exists the second differential of this function, taking part in the condition (A)-(b)), and furthermore, the corresponding branch of eigenfunctions \( b^+(x,p) (b^-(x,p)) \) of \( H(p) \) can be chosen such that for each fixed \( p \in \mathcal{O}^+(p_0) (p \in \mathcal{O}^-(p_0)) \), \( \| b^+(\cdot,p) \|_2 = 1 (\| b^-(\cdot,p) \|_2 = 1) \), the function \( b^+(x,p) (b^-(x,p)) \) is continuous and the mapping \( p \rightarrow b^+(x,p) \in C(\Omega) (p \rightarrow b^-(x,p) \in C(\Omega)) \) is real-analytic in \( \mathcal{O}^+(p_0) (\mathcal{O}^-(p_0)) \). It is clear that the corresponding eigenfunction \( e^+(x,p) (e^-(x,p)) \) of the operator \( \tilde{H}(p) \) has the same properties.

It is clear that if the condition (A) is satisfied for \( \lambda_+ (\lambda_-) \), then the set \( F^+ (F^-) \) is finite, that is \( F^+ = \{ p^+_1, p^+_2, \ldots, p^+_n \} (F^- = \{ p^-_1, p^-_2, \ldots, p^-_n \}). \) In particular, it is known \([\text{Lect}]\) that in the case where \( d = 1 \) condition (A) is always satisfied, all the functions \( \lambda_l(p) (l = 1, 2, \ldots) \) are even, \( n_+ = n_- = 1 \) and or \( p^+_1 = p^-_1 = 0 \), or \( p^+_k = p^-_1 = \pi^2 \). Hence in this case we shall write \( p_1 \) instead of \( p^+_1 \) and \( p^-_1 \).

Denote

\[ m^+_k = (J \text{Hes}_p(\lambda^+))^{-1} \quad (m^-_k = -(J \text{Hes}_p(\lambda^-))^{-1}) \]  

where

\[ J \text{Hes}_p(\lambda^+) = \det \left( \frac{\partial^2 \lambda^+(p)}{\partial p_\mu \partial p_\nu} \right)_{\mu,\nu=1}^d \]  

\[ J \text{Hes}_p(\lambda^-) = \det \left( \frac{\partial^2 \lambda^-(p)}{\partial p_\mu \partial p_\nu} \right)_{\mu,\nu=1}^d. \]  

\(^2\)For \( d = 1 \) the gaps of the spectrum of the operator \( H_0 \) are or \( (-\infty, \lambda_0) \), or \( (\lambda_{k}, \lambda_{k+1}) \), or \((\mu_k, \mu_{k+1}) (k = 0, 1, 2, \ldots) \), where \( \lambda_k \) and \( \mu_k \) are the eigenvalues of the operators \( H(0) \) and \( H(\pi) \) respectively, and \( \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \ldots \)
In particular, in the case where \( d = 1 \)

\[
(m_+^+)^{-1} = \frac{d^2 \lambda^+(p)}{dp^2} |_{p=p_1}, \quad (m_-^-)^{-1} = -\frac{d^2 \lambda^-(p)}{dp^2} |_{p=p_1} \tag{3.9}
\]

\((p_1 \in \{0, \pi\})\). With the physical point of view the quantity \( m_+^+ (m_-^-) \) is (up to a physical constant multiplier) the modulus of the determinant of the effective-mass tensor, that is the product of effective masses (in the principal directions) of an electron having the quasi-momentum \( \mathbf{p}_k^+ (\mathbf{p}_k^-) \) at the energy level \( \lambda_+ (\lambda_-) \).

Along with the Bloch functions \( b^+(\mathbf{x}, \mathbf{p}), b^- (\mathbf{x}, \mathbf{p}) \) we shall consider the \textit{weighted Bloch functions}

\[
v^+(\mathbf{x}, \mathbf{p}) = \sqrt{W(\mathbf{x})} b^+(\mathbf{x}, \mathbf{p}), \quad v^-(\mathbf{x}, \mathbf{p}) = \sqrt{W(\mathbf{x})} b^- (\mathbf{x}, \mathbf{p}), \tag{3.10}
\]

corresponding to them, with \( W(\mathbf{x}) \geq 0 \) a.e. on \( \mathbb{R}^d \), and denote

\[
v^+_k (\mathbf{x}) = v^+(\mathbf{x}, \mathbf{p}_k^+), \quad v^-_k (\mathbf{x}) = v^- (\mathbf{x}, \mathbf{p}_k^-). \tag{3.11}\]

In particular, in the case where \( d = 1 \)

\[
v^+_{1} (x) = v^+(x, p_1), \quad v^-_{1} (x) = v^- (x, p_1) \quad (p_1 \in \{0, \pi\}). \tag{3.12}\]

A part of our results concerns the case of degenerate edges of the gap of the spectrum of the unperturbed operator \( H_0 \), where condition (A)-(b) does not fulfilled. Let \( \text{Hess}_p(f) \) be the Hessian operator of a function \( f : T^d \rightarrow \mathbb{R} \) at a point \( \mathbf{p} \in T^d \), that is this is a linear operator acting in the tangent space \( T_p(T^d) \) to the torus \( T^d \) at the point \( \mathbf{p} \) and defined by

\[
\forall \ s, t \in T_p(T^d) : \quad d^2 f(p)[s, t] = \text{Hess}_p(f)s \cdot t. \tag{3.13}\]

The more general condition than (A), which we shall consider, is following:

(B) The function \( \lambda^+ (\lambda^-) \) and the set \( F^+ (F^-) \) satisfy the Morse-Bott type conditions (\textit{Ban-Hur}): 
(a) condition (A)-(a) is satisfied;
(b) the set \( F^+ (F^-) \) consists of a finite number of disjoint connected components: \( F^+ = \bigcup_{k=1}^{k_+} F^+_k \) \( (F^- = \bigcup_{k=1}^{k_-} F^-_k) \), such that each of \( F^+_k \) \( (F^-_k) \) is a \( C^\infty \)-smooth submanifold of \( T^d \) of the dimension \( d^+_k \) \( (d^-_k) \);
(c) for any point \( \mathbf{p} \in F^+ (\mathbf{p} \in F^-) \) the normal Hessian \n
\[
JN \text{Hess}_p(\lambda^+) := \det(\text{Hess}_p(\lambda^+)|_{N_p}) \tag{3.14}
\]

\[
JN \text{Hess}_p(\lambda^-) := \det(\text{Hess}_p(\lambda^-)|_{N_p})
\]

is not equal to zero. Here \( N^+_p \subseteq T_p(T^d) \) is a normal subspace to \( F^+ (F^-) \) at \( \mathbf{p} \), that is \( N^+_p = T_p(T^d) \cap T_p(F^+) \) \( (N^-_p = T_p(T^d) \cap T_p(F^-)) \).

If \( d^+_k \geq 1 \) \( (d^-_k \geq 1) \) for at least one \( k \), we shall say that the edge \( \lambda_+ (\lambda_-) \) is \textit{degenerate in the Morse-Bott sense}.
Denote
\[
\begin{align*}
m^+(p) &= (J N \text{Hes}_p(\lambda^+))^{-1} \quad (p \in F^+) \\
(m^-(p) &= -(J N \text{Hes}_p(\lambda^-))^{-1} \quad (p \in F^-).
\end{align*}
\]

In the similar manner as above the quantity \( m^+(p) \) \((m^+(p)) \) is (up to a physical constant multiplier) the product of effective masses (in the principal directions normal to \( F^+ (F^-) \)) of an electron having the quasi-momentum \( p \in F^+ (p \in F^-) \) at the energy level \( \lambda_+ (\lambda_-) \).

Assuming that condition (\textit{A})-(\textit{a}) is satisfied and taking \( p_0 \in F^+ \), consider the integral kernel \( \mathcal{Q}^+(x, s, p) \) of the eigenprojection \( \mathcal{Q}^+(p) \) of \( H(p) \), corresponding to the dispersion function \( \lambda^+(p) \) branching from the edge \( \lambda_+ \) and defined in a neighborhood \( \mathcal{O}^+(p_0) \) of \( p_0 \). We shall call it the \textit{eigenkernel} of \( H(p) \), corresponding to \( \lambda^+(p) \). By claims (ii) and (iii) of Corollary [A.2] after a suitable choice of \( \mathcal{O}^+(p_0) \) this kernel acquires the form \( \mathcal{Q}^+(x, s, p) = b^+(x, p) \xi(p, s, p) \), it does not depend on the choice of a branch of Bloch functions \( b^+(x, p) \) having the properties mentioned above and the mapping \( p \to \mathcal{Q}^+(x, s, p) \) is \( C(\Omega \times \Omega) \)-analytic in \( \mathcal{O}^+(p_0) \). The analogous properties has the eigenkernel \( \mathcal{Q}^-(x, s, p) \) of \( H(p) \), corresponding to \( \lambda^-(p) \). It is clear that the eigenkernel of \( \tilde{H}(p) \), corresponding to \( \lambda^+(p) \) \((\lambda^-(p)) \) has the form
\[
\begin{align*}
\tilde{Q}^+(x, s, p) &= \exp(-ip \cdot (x - s)) \mathcal{Q}^+(x, s, p) \\
\tilde{Q}^-(x, s, p) &= \exp(-ip \cdot (x - s)) \mathcal{Q}^-(x, s, p)
\end{align*}
\]
and it has the same properties as \( \mathcal{Q}^+(x, s, p) \) \((\mathcal{Q}^-(x, s, p)) \), but in addition it is \( \Gamma \)-periodic w.r.t. \( x \) and \( s \). We shall consider also the \textit{weighted eigenkernels}
\[
\begin{align*}
\mathcal{Q}_W^+(x, s, p) &= \sqrt{W(x)} \mathcal{Q}^+(x, s, p) \sqrt{W(s)}, \\
\mathcal{Q}_W^-(x, s, p) &= \sqrt{W(x)} \mathcal{Q}^-(x, s, p) \sqrt{W(s)}.
\end{align*}
\]

### 3.2 The notion of a virtual eigenvalue

Before formulating the main results, let us recall some notions and facts from [Ar-ZI]. Consider an operator \( H_\gamma = H_0 + \gamma W \) acting in a Hilbert space \( \mathcal{H} \), where \( H_0 \) and \( W \) are self-adjoint operators and \( \gamma \) is a real coupling constant. We assume that the following conditions are satisfied:

1. \((\lambda_-, \lambda_+) \) \((-\infty \leq \lambda_- < \lambda_+ \leq +\infty) \) is a gap of the spectrum \( \sigma(H_0) \) of the unperturbed operator \( H_0 \).

2. The operator \( W \) is bounded\(^3\) and for some \( \lambda_0 \in \mathcal{R}(H_0) \) the operator \( R_{\lambda_0}(H_0)|W|^{1/2} \) is compact.

\(^3\)In [S1] and [B32] the perturbation \( W \) is not bounded operator in general, it is supposed to be only relatively compact w.r.t. \( H_0 \) in the sense of quadratic forms. But in the present paper we consider for the simplicity only the case of a bounded perturbation and we think that all our results can be obtained without difficulties under a more general assumption.
By Proposition 5.1 of the present paper, the set $\sigma(H_\gamma) \cap (\lambda_-, \lambda_+)$ consists of at most countable number of eigenvalues having finite multiplicities which can cluster only to the edges $\lambda_+$ and $\lambda_-$. Let us return to the Schrödinger operator $H_\gamma = H_0 + \gamma W$ ($H_0 = -\Delta + V(x)$), $W = W(x)$, considered in Section 3. Recall that $V(x)$ is measurable, bounded and periodic on some lattice $\Gamma$. In this section we consider the case of a definite perturbation and of a non-degenerate edge of a gap $\bar{\omega}$, that is at least one of the edges $\lambda_+$ or $\lambda_-$ satisfies condition (A) of Section 3.1. Consider the finite rank operator

$$G_W^+ = \sum_{k=1}^{n_+} \sqrt{m_k^+} \langle \cdot, v_k^+ \rangle v_k^+ \quad (G_W^- = \sum_{k=1}^{n_-} \sqrt{m_k^-} \langle \cdot, v_k^- \rangle v_k^-),$$  

(4.1)

$m_k^+$ ($m_k^-$) is defined by (3.7) and (3.8) and $v_k^+(x)$ ($v_k^-(x)$) is the weighted Bloch function defined by (3.11), (3.10) and (3.6). We shall show in what follows (Lemma 5.13) that $G_W^+$ ($G_W^-$) has $n_+$ $(n_-)$ positive eigenvalues

$$\nu_1^+ \geq \nu_2^+ \geq \cdots \geq \nu_{n_+}^+ > 0 \ (\nu_1^- \geq \nu_2^- \geq \cdots \geq \nu_{n_-}^- > 0)$$  

(4.2)

(counting their multiplicities), which are eigenvalues of the matrix

$$\left( (m_k^+ m_l^+) \langle v_i^+, v_k^+ \rangle \right)_{k,l=1}^{n_+} \quad (m_k^- m_l^-) \langle v_i^-, v_k^- \rangle_{k,l=1}^{n_-}$$  

(4.3)

Let $g_1^+(x), g_2^+(x), \ldots, g_{n_+}^+(x)$ ($g_1^-(x), g_2^-(x), \ldots, g_{n_-}^-(x)$) be an orthonormal sequence of eigenfunctions of the operator $G_W^+$ ($G_W^-$) corresponding to its eigenvalues 123.
If \( d \leq 2 \), we shall impose on the non-negative perturbation \( W(x) \) the following conditions of its fast decay as \( |x| \to \infty \): for \( d = 1 \)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x)(x-s)^2 W(s) \, dx \, ds < \infty
\]  

(4.4)

and for \( d = 2 \)

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W(x)(\ln(1 + |x-s|)^2 W(s) \, dx \, ds < \infty.
\]  

(4.5)

Our result about virtual eigenvalues in the non-degenerate case is following:

**Theorem 4.1.** Assume that the unperturbed potential \( V(x) \) satisfies the strengthened version of condition (3.1):

\[
V \in \begin{cases} 
L^2(\Omega), & \text{if } d = 1 \\
\bigcup_{q>d} L^q(\Omega), & \text{if } d \geq 2,
\end{cases}
\]  

(4.6)

the perturbation \( W(x) \) is measurable and bounded in \( \mathbb{R}^d \), \( \lim_{|x| \to \infty} W(x) = 0 \), \( W(x) \geq 0 \) a.e. on \( \mathbb{R}^d \) and \( W(x) > 0 \) on a set of positive measure. Furthermore, assume that if the edge \( \lambda_+ (\lambda_-) \) is non-degenerate, for \( d = 1 \) the condition (4.4) is satisfied and for \( d = 2 \) the condition (4.5) is satisfied. Then

(i) for \( \gamma < 0 \) the operator \( H_\gamma \) can have in \( (\lambda-, \lambda_+) \) a virtual eigenvalue only at the edge \( \lambda_+ ; \)

(ii) if the edge \( \lambda_+ \) is non-degenerate, \( d \leq 2 \) and \( \gamma < 0 \), the operator \( H_\gamma \) has in \( (\lambda-, \lambda_+) \) virtual eigenvalues at \( \lambda_+ \) having the properties:

(a) if \( d = 1 \), there is a unique virtual eigenvalue \( \rho^+_1(\gamma) \) in \( (\lambda-, \lambda_+) \), having the following asymptotic representation for \( \gamma \uparrow 0 \):

\[
\sqrt{\lambda_+ - \rho^+_1(\gamma)} = |\gamma|(\sqrt{m^+_1} \frac{\|v^+_1\|^2}{\sqrt{2}} + O(\gamma)),
\]  

(4.7)

where \( m^+_1 \) is defined by (3.7a) and \( v^+_1(x) \) is defined by (3.12a);

(b) if \( d = 2 \), there are \( n_+ \) virtual eigenvalues

\[
\rho^+_1(\gamma) \leq \rho^+_2(\gamma) \leq \cdots \leq \rho^+_n(\gamma)
\]  

(4.8)

in \( (\lambda-, \lambda_+) \) (counting their multiplicities), and the following asymptotic representation is valid for them for \( \gamma \uparrow 0 \):

\[
\left( \ln \left( \frac{1}{\lambda_+ - \rho^+_k(\gamma)} \right) \right)^{-1} = |\gamma| \left( \frac{\nu^+_k}{2\pi} + O(\gamma) \right) \quad (k = 1, 2, \ldots, n_+),
\]  

(4.9)

and furthermore, the asymptotic formula of Lieb-Thirring type is valid for \( \gamma \uparrow 0 \):

\[
\sum_{k=1}^{n_+} \left( \ln \left( \frac{1}{\lambda_+ - \rho^+_k(\gamma)} \right) \right)^{-1} = \frac{|\gamma|}{2\pi} \sum_{k=1}^{n_+} \|v^+_k\|^2 \sqrt{m^+_k} + O(\gamma^2);
\]  

(4.10)
(iii) if the edge \( \lambda_{+} \) is non-degenerate and \( d \leq 2 \), the eigenfunctions corresponding to the virtual eigenvalues, considered above, have the properties:

(a) if \( d = 1 \), there exists \( \bar{\gamma} > 0 \) such that for any \( \gamma \in [\bar{\gamma}, 0) \) it is possible to choose an eigenfunction \( \psi_{\gamma,1}^{+}(x) \) of the operator \( H_{\gamma} \) corresponding to its eigenvalue \( \rho_{1}^{+}(\gamma) \) such that \( \| \sqrt{W} \psi_{\gamma,1}^{+} - g_{1}^{+} \| = O(\gamma) \) for \( \gamma \uparrow 0 \), where

\[
 g_{1}^{+} = \frac{\psi_{1}^{+}}{\| \psi_{1}^{+} \|}.
\]

(b) if \( d = 2 \) and \( m(j) \ (j \in \{1, 2, \ldots, n_{+}\}) \) is the multiplicity of an eigenvalue \( \nu_{j}^{+} \) of the operator \( G_{W}^{+} \) and \( \rho_{1(j)}^{+}(\gamma) \leq \rho_{2(j+1)}^{+}(\gamma) \leq \cdots \leq \rho_{(j+m(j)−1)}^{+}(\gamma) \) \( (l(j) \in \{1, 2, \ldots, n_{+}\}) \) is the group of virtual eigenvalues of \( H_{\gamma} \), for which

\[
 \lim_{\lambda_{0} \uparrow 0} \ln \left( \frac{1}{\lambda_{0} - \rho_{k}^{+}(\gamma)} \right) |\gamma| = \frac{2\pi}{\gamma_{j}^{+}},
\]

then there exists \( \bar{\gamma} > 0 \) such that for any \( \gamma \in [\bar{\gamma}, 0) \) there are numbers \( \gamma_{0}(\gamma) = \gamma \), \( \{\gamma_{k}(\gamma)\}_{k=1}^{m(j)} \) having the properties:

\[
 \gamma - \gamma_{k}(\gamma) = O(\gamma^{2}) \quad \text{as} \quad \gamma \uparrow 0, \quad \text{for each} \quad k \in \{0, 1, \ldots, m(j) - 1\}
\]

the number \( \rho_{1(j)}^{+}(\gamma) \) is an eigenvalue of the operator \( H_{\gamma_{k}(\gamma)} \), and it is possible to choose a basis

\[
 \psi_{\gamma_{k},l(j)}^{+}(x), \psi_{\gamma_{k},l(j)+1}^{+}(x), \ldots, \psi_{\gamma_{k},l(j)+m(j)−1}^{+}(x)
\]

in the linear span of eigenspaces of all the operators \( H_{\gamma_{k}(\gamma)} \) \( (k \in \{0, 1, \ldots, m(j)−1\}) \), corresponding to their eigenvalue \( \rho_{1(j)}^{+}(\gamma) \), for which the property

\[
 \| \sqrt{W} \psi_{\gamma,k}^{+} - g_{k}^{+} \| = O(\gamma) \quad \text{as} \quad \gamma \uparrow 0 \quad (4.11)
\]

is valid for any \( k \in \{l(j), l(j)+1, \ldots, l(j)+m(j)−1\} \);

(iv) if the edge \( \lambda_{+} \) is non-degenerate, \( d \geq 3 \), \( W \in L_{1}(\mathbb{R}^{d}) \) and \( \gamma < 0 \) the operator \( H_{\gamma} \) has in \( (\lambda_{−}, \lambda_{+}) \) no virtual eigenvalue at \( \lambda_{+} \);

(v) for \( \gamma > 0 \) the claims (i)-(iv) are valid with \( \lambda_{−}, n_{−}, \rho_{k}^{−}(\gamma), \nu_{k}^{−}, m_{k}^{−}, \psi_{\gamma,k}^{−}(x), g_{k}^{−}(x), \gamma > 0 \) and \( \gamma \downarrow 0 \) instead of, respectively, \( \lambda_{+}, n_{+}, \rho_{k}^{+}(\gamma), \nu_{k}^{+}, m_{k}^{−}, \psi_{\gamma,k}^{+}(x), g_{k}^{−}(x), \gamma < 0 \) and \( \gamma \uparrow 0 \);

(vi) if both edges \( \lambda_{+} \) and \( \lambda_{−} \) are non-degenerate, \( d \geq 3 \) and \( W \in L_{1}(\mathbb{R}^{d}) \), there is a threshold for the birth of the impurity spectrum in the gap \( (\lambda_{−}, \lambda_{+}) \), that is \( \sigma(H_{\gamma}) \cap (\lambda_{−}, \lambda_{+}) = \emptyset \) for a small enough \( |\gamma| \).

4.2 The birth of virtual eigenvalues from a degenerate edge

Assume that at least one of the edges \( \lambda_{+} \) or \( \lambda_{−} \) of a gap \( (\lambda_{−}, \lambda_{+}) \) of the spectrum of the unperturbed operator \( H_{0} \) satisfies the condition (B) of Section 3.1 such that the dimension of at least one of the connected components of the extremal set \( F^{+} = (\lambda_{+})^{-1}(\lambda_{+}) \) \( F_{−} = (\lambda_{−})^{-1}(\lambda_{−}) \) of the dispersion function \( \lambda^{\pm}(p) \) \( (\lambda^{\pm}(p)) \), branching from the edge \( \lambda_{+} \) \( (\lambda_{−}) \), is non-zero. In this case we have a non-localized degeneration of the dispersion function \( \lambda_{j+1}(p) \) \( (\lambda_{j}(p)) \) at the edge \( \lambda_{+} \) \( (\lambda_{−}) \) of the forbidden zone \( (\lambda_{−}, \lambda_{+}) \). For the simplicity we shall
assume in this section that the following condition is satisfied for at least one of the edges of $(\lambda_-, \lambda_+)$:

(C) The edge $\lambda_+ (\lambda_-)$ of a gap $(\lambda_-, \lambda_+)$ of the spectrum of the unperturbed operator $H_0$ satisfies condition (B) of Section 3.1 with $n_+ = 1 \ (n_- = 1)$ (that is $F^+ (F^-)$ is a connected smooth submanifold of $T^d$) and $d_+ = \text{dim}(F^+) \geq 1 \ (d_- = \text{dim}(F^-) \geq 1)$.

Consider the integral operator acting in $L_2(\mathbb{R}^d)$

$$G^+_W f = \int_{\mathbb{R}^d} G^+_W(x,s)f(s)\,ds \quad \left( G^-_W f = \int_{\mathbb{R}^d} G^-_W(x,s)f(s)\,ds \right) \quad (4.12)$$

with

$$G^+_W(x,s) = \int_{F^+} Q^+_W(x,s,p)\sqrt{m^+(p)}\,dF(p) \quad (4.13)$$

$$\left( G^-_W(x,s) = \int_{F^-} Q^-_W(x,s,p)\sqrt{m^-(p)}\,dF(p) \right),$$

where $Q^+_W(x,s,p) \ (Q^-_W(x,s,p))$ is the weighted eigenkernel, corresponding to $\lambda^+(p) \ (\lambda^-(p))$ and defined by (3.14), $m^+(p) \ (m^-(p))$ is defined by (3.15) and $dF(p)$ is the volume form on the submanifold $F^+ (F^-)$. In what follows we shall prove (Lemma 5.15) that if $W \in L_1(\mathbb{R}^d)$, $W(x) \geq 0$ a.e. on $\mathbb{R}^d$ and $W(x) > 0$ on a set of the positive measure, then the integral operator $G^+_W \ (G^-_W)$ is self-adjoint, nonnegative, belongs to the trace class and has an infinite number of positive eigenvalues

$$\nu^+_1 \geq \nu^+_2 \geq \cdots \geq \nu^+_n \geq \cdots \quad (4.14)$$

(each eigenvalue is repeated according to its multiplicity). Let

$$g^+_1(x), g^+_2(x), \ldots, g^+_n(x), \ldots \quad (g^-_1(x), g^-_2(x), \ldots, g^-_n(x), \ldots)$$

be an orthonormal sequence of eigenfunctions of the operator $G^+_W \ (G^-_W)$ corresponding to its eigenvalues (4.14).

If $d-d_+ \leq 2 \ (d-d_- \leq 2)$, we shall impose on the non-negative perturbation $W(x)$ the following conditions: for $d-d_+ = 1 \ (d-d_- = 1)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x)(x-s)^2W(s)\,dx\,ds < \infty \quad (4.15)$$

and for $d-d_+ = 2 \ (d-d_- = 2)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x)(\ln(1+|x-s|)W(s)\,dx\,ds < \infty \quad (4.16)$$

Our result about virtual eigenvalues in the degenerate case is following:

**Theorem 4.2.** Assume that the unperturbed potential $V(x)$ satisfies the condition (4.6), the perturbation $W(x)$ is measurable and bounded in $\mathbb{R}^d$, $\lim_{|x| \to \infty}$
$W(x) = 0$, $W(x) \geq 0$ a.e. on $\mathbb{R}^d$ and $W(x) > 0$ on a set of positive measure. Furthermore, assume that if the edge $\lambda_+$ ($\lambda_-$) satisfies the condition \((C)\), for $d - d_+ = 1$ ($d - d_- = 1$) the condition \((4.13)\) is satisfied and for $d - d_+ = 2$ ($d - d_- = 2$) the condition \((4.16)\) is satisfied. Then

(i) for $\gamma < 0$ the operator $H_\gamma = H_0 + \gamma W$ can have in $(\lambda_-, \lambda_+)$ a virtual eigenvalue only at the edge $\lambda_+$;

(ii) if the edge $\lambda_+$ satisfies the condition \((C)\), $d - d_+ \leq 2$ and $\gamma < 0$, the operator $H_\gamma$ has in $(\lambda_-, \lambda_+)$ an infinite number of virtual eigenvalues $\rho^+_n(\gamma) \leq \rho^+_1(\gamma) \leq \ldots \rho^+_n(\gamma) \leq \ldots$ at $\lambda_+$, and moreover the following asymptotic representation is valid for $\gamma \uparrow 0$:

$$\Psi(\lambda_+ - \rho^+_n(\gamma)) = |\gamma| (\nu^+_n + O(\gamma)), \quad (4.17)$$

where

$$\Psi(s) = \left\{ \begin{array}{ll}
\frac{(2\pi)^d}{\sqrt{2\pi}}\sqrt{s}, & \text{if } d - d_+ = 1, \\
(2\pi)^{d-1}(\ln \left(\frac{s}{\bar{s}}\right))^{-1}, & \text{if } d - d_+ = 2
\end{array} \right. \quad (s \in (0, \lambda_+ - \lambda_-)), \quad \text{and furthermore, the asymptotic formula of Lieb-Thirring type is valid:}$

$$\lim_{n \to \infty} \lim_{\gamma \to 0} \frac{1}{|\gamma|} \sum_{k=1}^{n} \Psi(\lambda_+ - \rho^+_k(\gamma)) =$$

$$\int_{F^+} \int_{\mathbb{R}^d} \mathcal{Q}^+_W(s, s, p) \, ds \sqrt{m^+(p)} \, dF(p); \quad (4.18)$$

(iii) if the edge $\lambda_+$ satisfies the condition \((C)\) and $d - d_+ \leq 2$, the eigenvalues corresponding to the virtual eigenvalues, considered above, have the property: if $m(j)$ is the multiplicity of an eigenvalue $\nu^+_j$ of the operator $G^+_W$ and $\rho^+_j(\gamma) \leq \rho^+_j(\gamma) + 1(\gamma) \leq \cdots \leq \rho^+_j(\gamma) + m(j) - 1(\gamma)$ is the group of virtual eigenvalues of $H_\gamma$, for which $\lim_{\gamma \to 0} \frac{1}{|\gamma|} \Psi(\lambda_+ - \rho^+_k(\gamma)) = \nu^+_j$ (if $k \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\}$), then there exists $\gamma_0 > 0$ such that for any $\gamma \in [-\bar{\gamma}, 0)$ there are numbers $\gamma_0(\gamma) = \gamma$ such that they and the eigenvectors of the operators $H_{\gamma_k(\gamma)} (k \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\})$, corresponding to their eigenvalue $\rho^+_j(\gamma)$, have the same asymptotic properties as in claim \(iii-b)$$ of Theorem \(4.1\).

(iv) if the edge $\lambda_+$ satisfies the condition \((C)\), $d - d_+ \geq 3$, $W \in L_1(\mathbb{R}^d)$ and $\gamma < 0$, the operator $H_\gamma$ has in $(\lambda_-, \lambda_+)$ no virtual eigenvalue at $\lambda_+$;

(v) for $\gamma > 0$ the claims (i)-(iv) are valid with $d_-, \lambda_-, \rho^-_k(\gamma), \nu^-_k, \mathcal{Q}^-_W(x, s, p), m^-(p), \psi^-_k(x), g_k(x), \gamma > 0$ and $\gamma \downarrow 0$ instead of, respectively, $d_+, \lambda_+, \rho^+_k(\gamma), \nu^+_k, \mathcal{Q}^+_W(x, s, p), m^+(p), \psi^+_k(x), g_k(x), \gamma < 0$ and $\gamma \uparrow 0$;

(vi) if both edges $\lambda_+$ and $\lambda_-$ satisfy the condition \((C)\), $d - d_+ \geq 3$, $d - d_- \geq 3$ and $W \in L_1(\mathbb{R}^d)$, there is a threshold for the birth of the impurity spectrum in the gap $(\lambda_-, \lambda_+)$, that is $\sigma(H_\gamma) \cap (\lambda_-, \lambda_+) = \emptyset$ for a small enough $|\gamma|$. 

17
4.3 The estimate of the multiplicity of virtual eigenvalues for an indefinite perturbation in the case of a non-degenerate edge

In the case of an indefinite perturbation $W(x)$ we only can, under some conditions, estimate from above the multiplicity of virtual eigenvalues of $H_\gamma$ in the gap $(\lambda_-, \lambda_+)$ and, in particular, establish the existence of a threshold for the birth of the impurity spectrum for $d \geq 3$. Denote by $W_+(x)$ and $W_-(x)$ the positive and negative parts of $W(x)$, that is $W_+(x) = \frac{1}{2}(W(x) + |W(x)|)$ and $W_-(x) = \frac{1}{2}(W(x) - |W(x)|)$. We shall consider the following conditions: in the case where $d = 1$

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_+(x)(x-s)^2W_+(s) \, dx \, ds < \infty, \quad (4.19)
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_-(x)(x-s)^2W_-(s) \, dx \, ds < \infty, \quad (4.20)
\]

and in the case where $d = 2$

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_+(x)(\ln(1 + |x-s|)^2W_+(s) \, dx \, ds < \infty, \quad (4.21)
\]

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_-(x)(\ln(1 + |x-s|)^2W_+(s) \, dx \, ds < \infty. \quad (4.22)
\]

Recall that if the edge $\lambda_+ (\lambda_-)$ is non-degenerate, then the extremal set $F^+ (F^-)$ of the dispersion function $\lambda^+ (p) (\lambda^- (p))$, branching from this edge, is finite and we denote by $n_+ (n_-)$ the number of its points.

The following result is valid:

**Theorem 4.3.** Let $(\lambda_-, \lambda_+)$ be a gap of the spectrum of the unperturbed operator $H_0$. Assume that the unperturbed potential $V(x)$ satisfies the condition $(4.20)$, the perturbation $W(x)$ is measurable and bounded in $\mathbb{R}^d$ and $\lim_{|x| \to \infty} W(x) = 0$. Let us take $\delta \in (0, \lambda_+ - \lambda_-)$. Then

(i) for $d = 1$: if the edge $\lambda_+$ is non-degenerate and condition $(4.19)$ is satisfied, then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (-\bar{\gamma}, 0)$ all the virtual eigenvalues of the operator $H_\gamma$, being born from the edge $\lambda_+$ and lying in $(\lambda_+ - \delta, \lambda_+)$, are simple; if the edge $\lambda_-$ is non-degenerate and condition $(4.20)$ is satisfied, then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (-\bar{\gamma}, 0)$ all the virtual eigenvalues of the operator $H_\gamma$, being born from the edge $\lambda_-$ and lying in $(\lambda_-, \lambda_- + \delta)$ are simple;

(ii) for $d = 2$: if the edge $\lambda_+$ is non-degenerate the condition $(4.21)$ is satisfied, then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (-\bar{\gamma}, 0)$ the multiplicity of each virtual eigenvalue of the operator $H_\gamma$, being born from the edge $\lambda_+$ and lying in $(\lambda_+ - \delta, \lambda_+)$ is not bigger than $n_+; \text{ if the edge } \lambda_- \text{ is non-degenerate and condition } (4.22) \text{ is satisfied, then there exists } \bar{\gamma} > 0 \text{ such that for any } \gamma \in (-\bar{\gamma}, 0)$
the multiplicity of each virtual eigenvalue of the operator $H_\gamma$, being born from the edge $\lambda_-$ and lying in $(\lambda_-, \lambda_- + \delta)$, is not bigger than $n_-$;

(iii) for $d \geq 3$: if the edge $\lambda_+$ is non-degenerate and $W_+ \in L_1(\mathbb{R}^d)$, then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (\bar{\gamma}, 0)$ no eigenvalue of the operator $H_\gamma$ lies in $(\lambda_+ - \delta, \lambda_+)$; if the edge $\lambda_-$ is non-degenerate and $W_- \in L_1(\mathbb{R}^d)$, then there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (\bar{\gamma}, 0)$ no eigenvalue of the operator $H_\gamma$ lies in $(\lambda_-, \lambda_- + \delta)$;

(iv) for $\gamma > 0$ all the above claims are valid with $\lambda_-, \lambda_+, W_-(x), W_+(x)$, $\gamma \in (0, \bar{\gamma})$ $(\lambda_-, \lambda_- + \delta)$, $(\lambda_+ - \delta, \lambda_+)$, $n_-$ and $n_+$ instead of, respectively, $\lambda_-, \lambda_+, W_-(x), W_+(x)$, $\gamma \in (-\bar{\gamma}, 0)$, $(\lambda_+ - \delta, \lambda_+)$, $(\lambda_-, \lambda_- + \delta)$, $n_-$ and $n_+$;

(v) for $d \geq 3$: if both edges $\lambda_-$ and $\lambda_+$ are non-degenerate and $W \in L_1(\mathbb{R}^d)$, then there is a threshold for the birth of the impurity spectrum in the gap $(\lambda_-, \lambda_+)$, that is $\sigma(H_\gamma) \cap (\lambda_-, \lambda_+) = \emptyset$ for a small enough $|\gamma|$.

4.4 A threshold for the birth of virtual eigenvalues for an indefinite perturbation in the case of a degenerate edge

Theorem 4.4. Let $(\lambda_-, \lambda_+)$ be a gap of the spectrum of the unperturbed operator $H_0$. Assume that the unperturbed potential $V(x)$ satisfies the condition (4.3), the perturbation $W(x)$ is measurable and bounded in $\mathbb{R}^d$ and $\lim_{|x| \to \infty} W(x) = 0$. Let us take $\delta \in (0, \lambda_+ - \lambda_-)$. Then

(i) if the edge $\lambda_+$ satisfies condition (C) of Section 4.2, $d - d_+ \geq 3$ and $W_+ \in L_1(\mathbb{R}^d)$, there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (-\bar{\gamma}, 0)$ no eigenvalue of the operator $H_\gamma$ lies in $(\lambda_+ - \delta, \lambda_+)$; if the edge $\lambda_-$ satisfies condition (C), $d - d_- \geq 3$ and $W_- \in L_1(\mathbb{R}^d)$, there exists $\bar{\gamma} > 0$ such that for any $\gamma \in (-\bar{\gamma}, 0)$ no eigenvalue of the operator $H_\gamma$ lies in $(\lambda_-, \lambda_- + \delta)$;

(ii) for $\gamma > 0$ claim (i) is valid with $d_-, d_+, \lambda_-, \lambda_+, W_-(x), W_+(x)$, $\gamma \in (0, \bar{\gamma})$ $(\lambda_-, \lambda_- + \delta)$ and $(\lambda_+ - \delta, \lambda_+)$ instead of, respectively, $d_+, d_-, \lambda_+, \lambda_-, W_+(x), W_-(x)$, $\gamma \in (-\bar{\gamma}, 0)$, $(\lambda_+ - \delta, \lambda_+)$ and $(\lambda_-, \lambda_- + \delta)$;

(iii) if condition (C) is satisfied for both edges $\lambda_-$ and $\lambda_+$, $d - d_- \geq 3$, $d - d_+ \geq 3$ and $W \in L_1(\mathbb{R}^d)$, there is a threshold for the birth of the impurity spectrum in the gap $(\lambda_-, \lambda_+)$, that is $\sigma(H_\gamma) \cap (\lambda_-, \lambda_+) = \emptyset$ for a small enough $|\gamma|$.

5 Proof of main results

5.1 General results on the birth of virtual eigenvalues

Let $H_0$, $H_\gamma = H_0 + \gamma W$ and $(\lambda_-, \lambda_+)$ are the same as in Section 4.1. For $\lambda \in (\lambda_-, \lambda_+)$ consider the operator

$$X_W(\lambda) = W^{\frac{1}{2}} R_\lambda(H_0)|W|^{\frac{1}{2}},$$

(5.1)
which is called in the literature the Birman–Schwinger operator (S1, Sc). Here

\[ W^{1/2} x = \begin{cases} W(|W| (\text{ker}(W))^{-1})^{-1/2} x, & \text{if } x \in (\text{ker}(W))^{-1}, \\ 0, & \text{if } x \in \text{ker}(W), \end{cases} \quad (5.2) \]

hence \( W = |W|^{1/2} W^{1/2} \). In S1 (Lemma 7.1) a connection between the spectrum of the operator pencil \( I + \gamma X_W(\lambda) (\lambda \in (-\infty, 0)) \) and the spectrum of \( H_\gamma \) in the gap \((-\infty, 0)\) of \( \sigma(H_0) \) was established for the case where \( H = L_2(\mathbb{R}^d) \) and \( H_0 = -\Delta \). For the case of a definite perturbation \( (W \geq 0) \) of a periodic potential the analogous result was obtained in Bi2 (Proposition 1.5). The following claim is a generalization of this claim to the case of an indefinite perturbation. We also have indicated an operator which realizes a linear isomorphism between \( \text{ker}(H_\gamma - \lambda I) \) and \( \text{ker}(I + \gamma X_W(\lambda)) \).

**Proposition 5.1.** Assume that for some \( \lambda_0 \in \mathcal{R}(H_0) \) the operator \( R_{\lambda_0}(H_0)|W|^{1/2} \) is compact and \( (\lambda_-, \lambda_+) \) is a gap of the spectrum of the operator \( H_0 \). Then the set \( \sigma_c(\lambda_-, \lambda_+) = \sigma(H_0) \cap (\lambda_-, \lambda_+) \) consists of at most countable number of eigenvalues of finite multiplicity of the operator \( H_\gamma \) and they can cluster only to the edges of the gap \((\lambda_-, \lambda_+)\). Furthermore, \( \sigma_c(\lambda_-, \lambda_+) \) coincides with the spectrum of the pencil of operators \( \Phi(\gamma) = I + \gamma X_W(\lambda) (\lambda \in (\lambda_-, \lambda_+)) \). Moreover, the operator function \( X_W(\lambda) \) is holomorphic in \( \mathcal{R}(H_0) \) in the operator norm, each of the operators \( X_W(\lambda) \) is compact and any point \( \lambda \in \sigma_c(\lambda_-, \lambda_+) \) is an eigenvalue of the pencil \( \Phi(\gamma) \) such that the operator \( W^{1/2}(E_\lambda) \) realizes a linear isomorphism between the subspaces \( E_\lambda = \text{ker}(H_\gamma - \lambda I) \) and \( L_\lambda = \text{ker}(I + \gamma X_W(\lambda)) \), hence \( \text{dim}(E_\lambda) = \text{dim}(L_\lambda) \).

**Proof.** Assume that \( \lambda \in \mathcal{R}(H_0) \). The subspace \( E_\lambda \) is the set of solutions of the equation \( H_0 x - \lambda x + \gamma W x = 0 \), which is equivalent to the equation \( x + \gamma R_\lambda(H_0) W x = 0 \). Denote \( y = P_W x, z = (I - P_W) x \), where \( P_W \) is the orthogonal projection on the subspace \( \mathcal{H}_W = \mathcal{H} \ominus \text{ker}(W) \). Since \( W(I - P_W) = 0 \), the last equation is equivalent to the system

\[ y + \gamma P_W R_\lambda(H_0) W y = 0, \quad z + \gamma \Xi y = 0, \quad (5.3) \]

where

\[ \Xi = (I - P_W) R_\lambda(H_0) W. \quad (5.4) \]

Hence the operator \( \Psi = (I - \gamma \Xi)|_{G_\lambda} \) realizes a linear isomorphism between the subspace \( G_\lambda \) of solutions of equation (5.3-a) and the subspace \( E_\lambda \). As it is clear, \( \Psi^{-1} = P_W|_{E_\lambda} \). Consider the operator \( S = W^{1/2}|_{\mathcal{H}_W} \). Since \( \text{ker}(W^{1/2}) = \text{ker}(W), \mathcal{H}_W \) is an invariant subspace for the operator \( W^{1/2} \). Hence \( S \) maps injectively \( \mathcal{H}_W \) into itself. Let us show that \( S(G_\lambda) = L_\lambda \). Assume that \( y \in G_\lambda \), that is \( y \) is a solution of the equation (5.3-a). Multiplying the both sides of (5.3-a) by \( W^{1/2} \), we get that \( v = S y = W^{1/2} y \) is a solution of the equation

\[ v + \gamma W^{1/2} P_W R_\lambda(H_0) W^{1/2} v = 0. \]

Since \( W^{1/2}(I - P_W) = 0 \), the last equation is equivalent to

\[ v + \gamma W^{1/2} R_\lambda(H_0) W^{1/2} v = 0. \]

(5.5)
This means that \( v \in \ker(I + \gamma X_W(\lambda)) = L_\lambda \), hence \( S(G_\lambda) \subseteq L_\lambda \). Let us prove the inverse inclusion. Assume that \( v \in L_\lambda \), that is \( v \) is a solution of equation (5.3a). Then we see that \( v \in \text{Im}(W^{1/2}) \subseteq H_W \). Hence there exists \( y \in H_W \) such that \( v = S y = W^{1/2} y \) and \( W^{1/2} y + \gamma W^{1/2} R_\lambda(H_0) W y = 0 \). The last equality and \( W^{1/2}(I - P_W) = 0 \) imply that \( S y + \gamma S P_W R_\lambda(H_0) W y = 0 \). Since the operator \( S \) is injective, we get that \( y \) satisfies equation (5.3a), that is \( y \in G_\lambda \). We have shown that \( L_\lambda \subseteq S(G_\lambda) \). Since we have proved above the inverse inclusion, we get: \( L_\lambda = S(G_\lambda) \). Observe that \( W^{1/2}(I - P_W) = 0 \). Thus, the operator \( S\Psi^{-1} = W^{1/2} P_W|_{E_\lambda} = W^{1/2}|_{E_\lambda} \) realizes a linear isomorphism between \( E_\lambda \) and \( L_\lambda \). Hence, in particular, \( \dim(E_\lambda) = \dim(L_\lambda) \).

The compactness of the operator \( R_\lambda(H_0)|W^{1/2} \), the boundedness of the operator \( W^{1/2} \) and the Hilbert identity \( R_\lambda(H_0) - R_{\lambda_0}(H_0) = (\lambda - \lambda_0) R_\lambda(H_0) R_{\lambda_0}(H_0) \) imply that for any \( \lambda \in \mathcal{R}(H_0) \), the operator \( X_W(\lambda) \) is compact and it is a holomorphic on \( \mathcal{C} \setminus \sigma(H_0) \) operator function w.r.t. the operator norm. Then, as it is known, \( \alpha(\lambda) = \dim(\ker(I + \gamma X_W(\lambda))) \) \( < \infty \) for any \( \lambda \in \mathcal{R}(H_0) \) and \( \alpha(\lambda) \) takes a constant value \( m \) at any point \( \lambda \in \mathcal{R}(H_0) \), except a set of isolated points in \( \mathcal{C} \setminus \sigma(H_0) \) at which \( \alpha(\lambda) > m \) (Goh-Kr., Chapt. 1, Sect. 5, Theorem 5.4). On the other hand, since \( H_0 \) is self-adjoint, for \( \Im(\lambda) \neq 0 \) the estimate \( ||X_W(\lambda)|| \leq ||W|| ||R_{\lambda_0}(H_0)|| \leq \frac{||W||}{\Im(\lambda)} ||R_{\lambda}(H_0)|| \) is valid, hence for a large enough \( |\Im(\lambda)| \) the operator \( I + \gamma X_W(\lambda) \) is continuously invertible. This means that \( m = 0 \). Therefore, in particular, \( \alpha(\lambda) = 0 \) for any \( \lambda \in (\lambda_-, \lambda_+) \), except a set \( \Lambda \subset (\lambda_-, \lambda_+) \) of isolated points in \( (\lambda_-, \lambda_+) \). Taking into account that \( \dim(E_\lambda) = \dim(L_\lambda) \), \( \lambda \in (\lambda_-, \lambda_+) \), we obtain that each point of \( \Lambda \) is an isolated eigenvalue of the operator \( H_\gamma \), of a finite multiplicity. On the other hand, if \( \lambda \in (\lambda_-, \lambda_+) \setminus \Lambda \), the operator \( I + \gamma X_W(\lambda) \) is continuously invertible, because the operator \( X_W(\lambda) \) is compact. Hence \( \Im(I + \gamma X_W(\lambda)) = \mathcal{H} \). Therefore, in particular, for any \( f \in \mathcal{H} \) the equation \( v + \gamma X_W(\lambda)v = W^{1/2} P_W f \) has a solution \( v \), as it is clear, belongs to \( \text{Im}(W^{1/2}) \). Hence the vector \( y = (W^{1/2}|_{\mathcal{H}_W})^{-1} v \in \mathcal{H}_W \) satisfies the equation \( W^{1/2} y + \gamma X_W(\lambda) W^{1/2} y = W^{1/2} P_W f \). Since the operator \( (W^{1/2}|_{\mathcal{H}_W}) \) is injective, the last equality implies that \( y + \gamma P_W R_\lambda(H_0) W y = P_W f \). Hence the vector \( x = (I - \gamma \Xi)y + (I - P_W) f \) satisfies the equation \( x + R_\lambda(H_0) W x = f \), which is equivalent to \( H_\gamma x - \lambda x = f \). Recall that the operator \( \Xi \) is defined by (5.4). Thus, we have proved that for any \( \lambda \in (\lambda_-, \lambda_+) \setminus \Lambda \), \( \Im(H_\gamma - \lambda I) = \mathcal{H} \). Since \( H_\gamma \) is self-adjoint, this means that \( (\lambda_-, \lambda_+) \setminus \Lambda \subseteq \mathcal{R}(H_\gamma) \). Thus, the set \( \Lambda \) which is the spectrum of the pencil \( X_W(\lambda) \) (\( \lambda \in (\lambda_-, \lambda_+) \)), coincides with the part of the spectrum of the operator \( H_\gamma \) lying in \( (\lambda_-, \lambda_+) \) and it consists of isolated eigenvalues of \( H_\gamma \) having finite multiplicities.

In what follows we shall assume that the condition of Proposition 5.1 is fulfilled.

Consider the case where \( W \geq 0 \), hence \( X_W(\lambda) = W^{1/2} R_\lambda(H_0) W^{1/2} \). Let us recall some notions and results from [Ar-Z1, II] used in this section. By Proposition 5.1, for each \( \lambda \in (\lambda_-, \lambda_+) \) the operator \( X_W(\lambda) \) is compact. Then since it is self-adjoint, its spectrum consists of at most a countable number of real eigenvalues which can cluster only to the point 0. Furthermore, each its non-zero eigenvalue
Definition 5.2. \cite{Ar-Z1} For any fixed \( \lambda \in (\lambda_-, \lambda_+ \) let us number all the positive eigenvalues \( \mu_k^+ (\lambda) \) of the operator \( X_W (\lambda) \) in the non-increasing ordering \( \mu_1^+ (\lambda) \geq \mu_2^+ (\lambda) \geq \cdots \geq \mu_k^+ (\lambda) \geq \cdots \) and all the negative ones \( \mu_k^- (\lambda) \) in the non-decreasing ordering \( \mu_1^- (\lambda) \leq \mu_2^- (\lambda) \leq \cdots \leq \mu_k^- (\lambda) \leq \cdots \) (each eigenvalue is repeated according to its multiplicity). So, by such ordering we have chosen one-valued branches of eigenvalues of the operator function \( X_W (\lambda) \). We call these branches the characteristic branches (positive and negative) of the operator \( H_0 \) with respect to the operator \( W \) on a gap \( (\lambda_-, \lambda_+) \) of \( \sigma (H_0) \).

Remark 5.3. As it was shown in \cite{Ar-Z1} (Proposition 3.7), if \( W \geq 0 \) the functions \( \mu_k^+ (\lambda) \) and \( \mu_k^- (\lambda) \) are continuous and increasing. Since they can “go to zero” at some points of the gap \( (\lambda_-, \lambda_+) \), each of them has its domain \( \text{Dom}(\mu_k^+) \) and \( \text{Dom}(\mu_k^-) \) which have the form \( \text{Dom}(\mu_k^+) = (\eta_k^+, \lambda_+) \), \( \eta_k^+ \in [\lambda_-, \lambda_+] \), \( \text{Dom}(\mu_k^-) = (\lambda_-, \eta_k^-) \), \( \eta_k^- \in [\lambda_-, \lambda_+] \) and the following property is valid:

\[ \eta_k^- \in (\lambda_-, \lambda_+) \Rightarrow \lim_{\lambda \to \eta^-_k} \mu_k^- (\lambda) = 0, \eta_k^- \in (\lambda_-, \lambda_+) \Rightarrow \lim_{\lambda \to \eta^-_k} \mu_k^- (\lambda) = 0. \]

Furthermore, the sequence \( \{\eta_k^+\} \) is non-decreasing and the sequence \( \{\eta_k^-\} \) is non-increasing.

Denote \( \bar{\mu}_k^+ = \lim_{\lambda \to \lambda_-} \mu_k^+ (\lambda) \), \( \bar{\mu}_k^- = \lim_{\lambda \to \lambda_-} \mu_k^- (\lambda) \) (the values \(+\infty\) and \(-\infty\) are allowed for these limits).

Definition 5.4. Consider \( l(\lambda_-) \), \( l(\lambda_+) \in \mathbb{Z}_+ \cup \{+\infty\} \) defined by the conditions:

\[ -\infty = \bar{\mu}_1^- = \bar{\mu}_2^- = \cdots = \bar{\mu}_{l(\lambda_-)}^- < \bar{\mu}_{l(\lambda_-)+1}^- \leq \bar{\mu}_{l(\lambda_-)+2}^- \leq \cdots, \]

if \( \lambda_- > -\infty \),

\[ +\infty = \bar{\mu}_1^+ = \bar{\mu}_2^+ = \cdots = \bar{\mu}_{l(\lambda_+)}^+ > \bar{\mu}_{l(\lambda_+)+1}^+ \geq \bar{\mu}_{l(\lambda_+)+2}^+ \geq \cdots, \]

if \( \lambda_+ < +\infty \).

We call \( l(\lambda_-) \) and \( l(\lambda_+) \) the asymptotic multiplicities of the edges \( \lambda_- \) and \( \lambda_+ \) of the gap \( (\lambda_-, \lambda_+) \) of \( \sigma (H_0) \) with respect to the operator \( W \) and denote them \( l(\lambda_-) = M(\lambda_-; H_0, W) \), \( l(\lambda_+) = M(\lambda_+, H_0, W) \). The branches \( \{\mu_k^+ (\lambda)\}_{k=1}^{l(\lambda_-)}, \{\mu_k^+ (\lambda)\}_{k=1}^{l(\lambda_+)} \) are called the main characteristic branches of the operator \( H_0 \) with respect to the operator \( W \) near the edges \( \lambda_- \) and \( \lambda_+ \), respectively.

We shall use the following results from \cite{Ar-Z1}:

Proposition 5.5. The set \( \sigma (H_0) \cap (\lambda_-, \lambda_+) \) has the representation:

\[ \sigma (H_0) \cap (\lambda_-, \lambda_+) = \left\{ \bigcup_{k=1}^{l(\lambda_-)} \{\rho_k^+ (\gamma)\} \right\} \quad \text{for} \quad \gamma < 0, \]

\[ \left\{ \bigcup_{k=1}^{l(\lambda_+)} \{\rho_k^- (\gamma)\} \right\} \quad \text{for} \quad \gamma > 0, \]

where \( \rho_k^+ (\gamma) = (\mu_k^+)^{-1} \left( -\frac{1}{\gamma} \right) \), \( \rho_k^- (\gamma) = (\mu_k^-)^{-1} \left( -\frac{1}{\gamma} \right) \) and \( (\mu_k^+)^{-1}, (\mu_k^-)^{-1} \) are the inverses of the functions \( \mu_k^+ (\lambda) \), \( \mu_k^- (\lambda) \), the positive and negative characteristic branches of \( H_0 \) with respect to \( V \) on the gap \( (\lambda_-, \lambda_+) \).
Proof. Claim (a) follows from the monotony and continuity of \( \rho_k^+(\lambda) \) of virtual eigenvalues near the edge \( \lambda_+ \) of \((\lambda_-, \lambda_+)\). Furthermore, for any \( 1 \leq k \leq l(\lambda_+) \) and \( \gamma \in \text{Dom}(\rho_k^+), \)

\[
\rho_k^+(\gamma) = (\mu_k^+)^{-1} \left( -\frac{1}{\gamma} \right),
\]

where \((\mu_k^+)^{-1}\) is the inverse of the function \(\mu_k^+ (\lambda)\) (the main characteristic branch of \(H_0\) with respect to \(W\) near the edge \(\lambda_+\)). If \( l(\lambda_+) < \infty \) and \( \lambda_- = -\infty \), then \( \text{Dom}(\rho_k^+) = (-\infty, 0) \) \( \forall k \in \{1, 2, \ldots, l(b)\} \). Hence the operator \(H_\gamma\) has at least \(l(\lambda_+)\) eigenvalues in the gap \((-\infty, \lambda_+)\) for any \( \gamma < 0 \).

(ii) If \( M(\lambda_+, H_0, W) = \infty \), then for \( \gamma < 0 \) the number of the branches of eigenvalues \(\{\lambda_k^+(\gamma)\}\) of the operator \(H_\gamma\), which enter the gap \((\lambda_-, \lambda_+)\) across the edge \(\lambda_+\), is infinite, each of them is a virtual eigenvalue and the property is valid for them \((-\theta_k, 0) \subseteq \text{Dom}(\rho_k^+)\), where \(\theta_k \uparrow +\infty\) for \(k \to \infty\). The latter fact means that the operator \(H_\gamma\) has an infinite number of eigenvalues in the gap \((\lambda_-, \lambda_+)\) for any \( \gamma < 0 \). These eigenvalues cluster to the edge \(\lambda_+\) only and formula \((5.6)\) with \(l(\lambda_+) = \infty\) is valid for them.

(iii) If \( \gamma < 0 \), for the branches of virtual eigenvalues of the operator \(H_\gamma\) at the edge \(\lambda_-\) of \((\lambda_-, \lambda_+)\) the analogous claims are valid like in the case of the edge \(\lambda_+\).

In order to get in what follows an asymptotic representation for virtual eigenvalues, we need the following claim:

Lemma 5.7. If \( \mu : (a, b) \to \mathbb{R} \) be an increasing continuous function having the asymptotic representation for \( \lambda \uparrow b\): \(\mu(\lambda) = \frac{A}{\lambda^\gamma} + O(1)\), where \(A > 0\) and \(f : (a, b) \to \mathbb{R}\) is a decreasing continuous function such that \(\lim_{\lambda \uparrow b} f(\lambda) = 0\), then

(a) there is \(\bar{\gamma} > 0\) such that for any \(\gamma \in (0, \bar{\gamma}]\) the equation \(\mu(\lambda) = \frac{A}{\lambda^\gamma}\) has in \((a, b)\) a unique solution \(\lambda = \rho(\gamma)\);

(b) the asymptotic representation \(f(\rho(\gamma)) = \gamma(A + O(\gamma))\) is valid for \(\gamma \downarrow 0\).

Proof. Claim (a) follows from the monotony and continuity of \(\mu\) and the fact that \(\lim_{\lambda \uparrow b} \mu(\lambda) = \infty\). Then in view of the equality \(\frac{A}{f(\rho(\gamma))} + O(1) = \frac{1}{\gamma}\), we have: \(f(\rho(\gamma)) = \gamma A + \gamma f(\lambda(\gamma)) O(1)\), hence \(f(\rho(\gamma)) = O(\gamma)\) for \(\gamma \downarrow 0\). The last two equalities imply the desired asymptotic representation.

5.2 Perturbation of a compact operator

In what follows we shall use the following result about a perturbation of a compact operator in a Hilbert space.
Proposition 5.8. Let $\Phi(t)$ $(t \in [0, \delta], \delta > 0)$ be a family of linear compact operators acting in a Hilbert space $\mathcal{H}$ such that

$$
\Phi(t) = \Phi(0) + t\theta(t),
$$

where

$$
\bar{\theta} = \sup_{t \in [0, \delta]} ||\theta(t)|| < \infty
$$

and the operator $\Phi(0)$ has positive eigenvalues. Let $\mu_1^0 \geq \mu_2^0 \geq \cdots \geq \mu_N^0$ $(N \leq \infty)$ be such eigenvalues of $\Phi(0)$ arranged counting their multiplicities and $e_1^0, e_2^0, \ldots, e_N^0$ be an orthonormal system of eigenvectors corresponding to them. Then

(i) if $N < \infty$, there exists $\tilde{\delta} \in [0, \delta]$ such that

(a) for any $t \in [0, \tilde{\delta}]$ the operator $\Phi(t)$ has $N$ positive eigenvalues $\mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_N(t)$ (counting their multiplicities) having the asymptotic representation

$$
\mu_k(t) = \mu_k^0 + O(t) \quad \text{as } t \to 0 \quad (k = 1, 2, \ldots, N))
$$

and all the rest of positive eigenvalues of $\Phi(t)$ (if they exist) have the asymptotic representation

$$
\mu_k(t) = O(t) \quad \text{as } t \to 0 \quad (k > N);
$$

(b) if $m(j)$ is the multiplicity of the positive eigenvalue $\mu_j^0$ of the operator $\Phi(0)$ $(j \in \{1, 2, \ldots, N\})$ and

$$
\mu_{l(j)}(t) \geq \mu_{l(j)+1}(t) \geq \cdots \geq \mu_{l(j)+m(j)-1}(t) \quad (l(j) \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\})
$$

is the group of the eigenvalues of the operator $\Phi(t)$ which tend to $\mu_j^0$ as $t \to 0$, then for any $t \in [0, \tilde{\delta}]$ it is possible to choose an orthonormal basis $e_k(t)$ $(k \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\})$ in the subspace spanned by eigenvectors of $\Phi(t)$ corresponding to these eigenvalues, such that for any $k \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\}$ $\|e_k(t) - e_k^0\| = O(t)$ for $t \to 0$.

(ii) If $N = \infty$, for any natural $n$ there exists $\tilde{\delta} \in [0, \delta]$ such that for any $t \in [0, \tilde{\delta}]$ the operator $\Phi(t)$ has $n$ positive eigenvalues $\mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_n(t)$ (counting their multiplicities). Furthermore, these eigenvalues and eigenvectors, corresponding to them, have the same properties as in claim (i) except (5.10), with the number $n$ instead of $N$.

Proof. (i) In view of (5.7) and (5.8), $\Phi(0) - \bar{\theta}tI \leq \Phi(t) \leq \Phi(0) + \bar{\theta}tI$ for any $t \in [0, \delta]$. Hence by Lemma 3.4 of [Ar-Zl1] (a modification of the comparison theorem on the base of the minimax principle), if $\tilde{\delta} < \min\{\delta, \frac{\bar{\theta}t_0}{\mu_1^0}\}$, then for any $t \in [0, \tilde{\delta}]$ the operator $\Phi(t)$ has $N$ positive eigenvalues $\mu_1(t) \leq \mu_2(t) \leq \cdots \leq \mu_N(t)$ (counting their multiplicities), for which the estimates

$$
\forall t \in [0, \tilde{\delta}]: \quad \mu_k^0 - \bar{\theta}t \leq \mu_k(t) \leq \mu_k^0 + \bar{\theta}t \quad (k = 1, 2, \ldots, N)
$$

are valid and all the rest of positive eigenvalues of $\Phi(t)$ (if they exist) satisfy the estimates $\mu_k(t) \leq \bar{\theta}t$ $(k > N)$ for any $t \in [0, \tilde{\delta}]$. These estimates imply the desired asymptotic formulas (5.9) and (5.10). Claim (i-a) is proven.
Let \( \tilde{\mu}_1^0 > \tilde{\mu}_2^0 > \cdots > \tilde{\mu}_N^0 \) (\( N \leq N \)) be all the mutually different positive eigenvalues of the operator \( \Phi(0) \). The sum of their multiplicities is equal to \( N \). Let \( E_1^0, E_2^0, \ldots, E_N^0 \) be the eigenspaces of \( \Phi(0) \) corresponding to these eigenvalues. In order to prove the second part of claim (i), we shall show that there exists an operator function \( U(\cdot) : [0, \delta] \to \mathcal{B}(\mathcal{H}) \) taking unitary values such that for any \( j \in \{1, 2, \ldots, N\} \) the subspace \( E_j(t) = U(E_j^0) \) is spanned by the eigenspaces of those eigenvalues of the operator \( \Phi(t) \) which tend to the eigenvalue \( \tilde{\mu}_j^0 \) of the operator \( \Phi(0) \) as \( t \to 0 \) and moreover,

\[
\|I - U(t)\| = O(t) \quad \text{for} \quad t \to 0; \tag{5.13}
\]

Using (5.12), we can choose \( r > 0 \) such that for any \( j \in \{1, 2, \ldots, N\} \) and \( t \in [0, \delta] \) the circle \( C^r_j \) in the complex plane with the radius \( r \) and the center at the eigenvalue \( \tilde{\mu}_j^0 \) lies in the resolvent set of the operator \( \Phi(t) \) and surrounds the group of eigenvalues \( \{5.11\} \) of the operator \( \Phi(t) \). Consider the total orthogonal projection \( P_j(t) \) on the subspace \( E_j(t) \) spanned by the eigenspaces of the above group of eigenvalues of \( \Phi(t) \): \( P_j(t) = -\frac{1}{2\pi i} \oint_{C^r_j} R_\lambda(\Phi(t)) \, d\lambda \). In particular, \( P_j^0 = P_j(0) \) is the orthogonal projection on the eigenspace \( E_j^0 \) of the operator \( \Phi(0) \) corresponding to its eigenvalue \( \tilde{\mu}_j^0 \). Then

\[
P_j(t) - P_j^0 = -\frac{1}{2\pi i} \oint_{C^r_j} (R_\lambda(\Phi(t)) - R_\lambda(\Phi(0))) \, d\lambda. \tag{5.14}
\]

From (5.7) we get easily the representation \( R_\lambda(\Phi(t)) - R_\lambda(\Phi(0)) = -tR_\lambda(\Phi(0))\theta(t)R_\lambda(\Phi(t)) \). Then choosing \( \delta > 0 \) such that

\[
\forall \ t \in [0, \delta], \ \forall \ \lambda \in \bigcup_{j=1}^N C^r_j : \ |tR_\lambda(\Phi(0))\theta(t)| \leq \delta \theta \|R_\lambda(\Phi(0))\| < 1
\]

and using standard arguments, we obtain from representation (5.14) that

\[
\exists C > 0, \ \forall \ t \in [0, \delta], \ \forall \ j \in \{1, 2, \ldots, N\} : \ \|P_j(t) - P_j^0\| \leq Ct.
\]

Then after a suitable restriction of the interval \([0, \delta]\) the inequality \( \|P_j(t) - P_j^0\| \leq 1 \) is valid for any \( t \in [0, \delta] \), which implies easily that the operator \( Q(t) = P_j(t)|_{E_j^0} \) realizes a linear isomorphism between the subspaces \( E_j^0 \) and \( E_j(t) \). Since for any \( x \in E_j^0 \) \( \|Q(t)x - x\| \leq \|P_j(t) - P_j^0\|\|x\| \leq Ct\|x\| \), then for the sequence of vectors \( e_k(t) = Q(t)e_k^0 \) \( (k \in \{k(j), k(j) + 1, \ldots, k(j) + m(j) - 1\}) \) claim (i-b) is valid.

(ii) Claim (ii) follows straightforwardly from all the arguments used in the proof of claim (i). \( \square \)

5.3 The case of non-degenerate edges of a gap of the spectrum of the unperturbed operator

In what follows we need some results about the Birman-Schwinger operator corresponding to the Schrödinger operator \( H_0 = -\Delta + V(x) \) with a periodic potential \( V(x) \) and the perturbation \( W(x) \) of this potential (see Section 4).
5.3.1 Compactness of Birman-Schwinger operator

Proposition 5.9. If the periodic potential \( V(x) \) satisfies the condition (5.17) and the perturbation \( W(x) \) is measurable and bounded in \( \mathbb{R}^d \) and \( \lim_{|x| \to \infty} W(x) = 0 \), then for some \( \lambda_0 \in \mathbb{R} \setminus \sigma(H_0) \) the operator \( R_{\lambda_0}|W|^\frac{1}{2} \) is compact, hence for the operator \( H_0 \) and the Birman-Schwinger operator \( X_W(\lambda) = W^\frac{1}{2} R_{\lambda}(H_0)|W|^\frac{1}{2} \) all the claims of Proposition 5.1 are valid.

Proof. By claim (ii) of Proposition A.5 the operator \( H_0 \) is self-adjoint, bounded below and its domain is \( W_2^2(\mathbb{R}^d) \). It is enough to prove that the operator \( W^\frac{1}{2} R_{\lambda_0}(H_0)^\frac{1}{2} = (R_{\lambda_0}(H_0)^\frac{1}{2}|W|^\frac{1}{2})^* \) is compact for some \( \lambda_0 < \inf(\sigma(H_0)) \), that is the set

\[
K = \{ u \in W_2^2(\mathbb{R}^d) \mid ((H_0 - \lambda_0 I)u, u) \leq 1 \} \tag{5.15}
\]

is compact, hence for the semi-norm \( \sqrt{(W|u,u)} \). By claims (ii) and (iii) of Lemma A.3 and claims (ii) and (iii) of Lemma A.4, it is possible to choose \( \lambda_0 < \inf(\sigma(H_0)) \) such that for some \( m, m_1 > 0 \) and any \( u \in W_2^2(\mathbb{R}^d) \), \( 1 \in \Gamma \)

\[
\int_{\Omega} (H_0 u - \lambda_0 u)^2 \, dx \geq m \int_{\Omega} (\nabla u)^2 \, dx \quad \text{and} \quad ((H_0 - \lambda_0 I)u, u) \geq m_1 (\|u\|_2^2 + \|u\|_2^2),
\]

where \( \Omega = \Omega + \{1\} \) (recall that \( \Gamma \) is the lattice of periodicity of \( V(x) \)). Then

\[
\int_{\Omega} |W(x)||u(x)|^2 \, dx \leq m \sup_{x \in G_N} |W(x)||((H_0 - \lambda_0 I)u, u)|, \quad \text{where} \quad G_N = \bigcup_{l \geq 0} (\Omega_l \setminus \Omega) \quad \text{and} \quad \|l\| = \max_{l} \|l\| \leq \|l\|_G. \]

These circumstances and the compactness of the embedding of \( W_2^1(\mathbb{R}^d \setminus G_N) \) into \( L_2(\mathbb{R}^d \setminus G_N) \) imply that the set \( K \), defined by (5.15), is compact in the semi-norm \( \sqrt{(W|u,u)} \). \( \square \)

5.3.2 Representation of Birman-Schwinger operator in the case of a non-degenerate edge

Proposition 5.10. Let \( (\lambda_-, \lambda_+) \) be a gap of the spectrum of the unperturbed operator \( H_0 \). Assume that \( W(x) \) is measurable and bounded in \( \mathbb{R}^d \), \( W \in L_1(\mathbb{R}^d) \) and \( W(x) \geq 0 \) a.e. on \( \mathbb{R}^d \). Let us take \( \delta \in (0, (\lambda_+ - \lambda_-)) \).

(i) If \( d = 1 \), the edge \( \lambda_+ \) is non-degenerate and condition (4.4) is satisfied, then for \( \lambda \in ((\lambda_+ - \delta, \lambda_+) \) the representation is valid:

\[
X_W(\lambda) = \Phi_+(\lambda) + \Xi_+(\lambda),
\]

where \( \Phi_+(\lambda) \) is a rank one operator of the form

\[
\Phi_+(\lambda) = \frac{\sqrt{m_1^+}}{\sqrt{2(\lambda_+ - \lambda)}} \langle \cdot, v_1^+ \rangle v_1^+,
\]

\( m_1^+ \) and \( v_1^+ \) are defined by (3.24) and (3.12) and \( \Xi_+(\lambda) \) is a bounded operator in \( L_2(\mathbb{R}) \) such that

\[
\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \|\Xi_+(\lambda)\| < \infty \tag{5.18}
\]

(ii) If \( d = 2 \), the edge \( \lambda_+ \) is non-degenerate and condition (4.5) is satisfied, then for \( \lambda \in ((\lambda_+ - \delta, \lambda_+) \) the representation (5.16) is valid, where \( \Theta_+(\lambda) \) is a
bounded operator in $L_2(\mathbb{R}^d)$ such that condition (5.18) is satisfied, $\Phi_+(\lambda)$ is a finite rank operator of the form

$$\Phi_+(\lambda) = \frac{1}{2\pi} \ln \left( \frac{1}{\lambda_+ - \lambda} \right) G_W^+, \quad (5.19)$$

and $G_W^+$ is defined by (4.11), in which $v_k^-$ is defined by (3.11), (3.10) and (3.8).

(iii) If $d \geq 3$ and the edge $\lambda_+$ is non-degenerate, then

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \| X_W(\lambda) \| < \infty. \quad (5.20)$$

(iv) The claims analogous to (i)-(iii) are valid with $\lambda_-$, $(\lambda_+, \lambda_+ + \delta)$, $m_1^-$, $G_W^-$ and $v_k^+$ instead of, respectively, $\lambda_+$, $(\lambda_+ - \delta, \lambda_+)$, $m_1^+$, $G_W^+$ and $v_k^-$.

Proof. Using claim (i) of Proposition 6.2 we get: $X_W(\lambda) = \sqrt{W}K(\lambda)\sqrt{W} + \sqrt{W}\Theta(\lambda)\sqrt{W}$, where, in view of (6.2) and the boundedness of $W(x)$,

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \| \sqrt{W}\Theta(\lambda)\sqrt{W} \| \leq \sup_{x \in \mathbb{R}^d} |W(x)| \times \sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \| \Theta(\lambda) \| < \infty. \quad (5.21)$$

Assume that $d = 1$ and $\lambda \in (\lambda_+ - \delta, \lambda_+)$. Observe that the multiplication operator $\sqrt{W}$ maps the set $L_{2,0}(\mathbb{R})$ into itself. In view of claim (ii,a) of Proposition 6.2, $\sqrt{W}K(\lambda)\sqrt{W}|_{L_{2,0}(\mathbb{R}^d)}$ is an operator with the integral kernel

$$\sqrt{W(x)}F^+(x, s, \lambda)\sqrt{W(s)} + \sqrt{W(x)}K^+(x, s, \lambda)\sqrt{W(s)},$$

and

$$(\Phi_+(\lambda)f)(x) = \frac{1}{2\mu^+\sqrt{\lambda_+ - \lambda}} \int_{-\infty}^{\infty} \sqrt{W(x)}b^+(x, p_1)\sigma^+(s, p_1)\sqrt{W(s)}f(s) \, ds.$$
This means that for any \( \lambda \in (\lambda_+ - \delta, \lambda_+) \) the operator \( \tilde{K}^+(\lambda) \) with the integral kernel \( \sqrt{W(x)}\tilde{K}^+(x,s,\lambda)\sqrt{W(s)} \) belongs to the Hilbert-Schmidt class (hence it is bounded in \( L_2(\mathbb{R}) \)), and furthermore \( \sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \| \tilde{K}^+(\lambda) \| < \infty \). Then, in view of \( \text{(5.21)} \) and the density of \( L_{2,0}(\mathbb{R}) \) in \( L_2(\mathbb{R}) \), we obtain representation \( \text{(5.16)} \), where the operator \( \Theta \) is linearly independent. Recall that the functions \( S_b \) is dual to the Schwartz space \( S \).

As it is known, the Gelfand-Fourier-Floquet transform maps the space \( S' \) onto the space of distributions \( \mathcal{L}' \), which is dual to the set \( \mathcal{L} \) of all \( C^\infty \)-sections of the direct integral \( \int_{\mathbb{T}^d} \mathcal{H}_p \, dp \) \( \text{(Kučh)} \). Recall that \( \mathcal{H}_p \) is the Hilbert space of functions \( u \in L_{2,loc}(\mathbb{R}^d) \) satisfying the condition \( \text{(3.2)} \) with the inner product defined by \( \text{(3.3a)} \). Using the \( \Gamma \)-periodicity of \( e_k(x) \), we have:

\[
\hat{b}_k(x,p) = (U b_k)(x,p) = \frac{1}{(2\pi)^d} \sum_{l \in \mathbb{Z}^d} \exp(i(x - l) \cdot p) \| e_k(x) = \frac{\exp(i(x - l) \cdot p)}{(2\pi)^d} \sum_{l \in \mathbb{Z}^d} \exp(i(x - l) \cdot p) e_k(x) \] \quad (5.23)

\[
\hat{b}_{k_0}(x,p) = \sum_{k \in \{1,2,...,N\} \setminus \{k_0\}} c_k \hat{b}_k(x,p).
\]

5.3.3 Linear independence of weighted Bloch functions

For the proof of Theorem \( \text{[4.1]} \) we need three lemmas.

**Lemma 5.11.** (\textit{Kučh}) A finite sequence of Bloch functions

\[
\{b_k(x) = \exp(i(p_k \cdot x) e_k(x))\}_{k=1}^N,
\] \quad (5.22)

corresponding to mutually different quasi-momenta \( p_k \in \mathbb{T}_d \) \((k = 1, 2, ..., N)\), is linearly independent. Recall that the functions \( e_k(x) \) are \( \Gamma \)-periodic.

**Proof.** Consider the functions \( b_k(x) \) as distributions from the space \( S' \), which is dual to the Schwartz space \( S \). As it is known, the Gelfand-Fourier-Floquet transform

\[
\hat{f}(x,p) = (U f)(x,p) = \frac{1}{(2\pi)^d} \sum_{l \in \Gamma} \exp(i(x - l) \cdot p) f(x - l)
\] \quad (5.23)

maps the space \( S' \) onto the space of distributions \( \mathcal{L}' \), which is dual to the set \( \mathcal{L} \) of all \( C^\infty \)-sections of the direct integral \( \int_{\mathbb{T}^d} \mathcal{H}_p \, dp \) \( \text{(Kučh)} \). Recall that \( \mathcal{H}_p \) is the Hilbert space of functions \( u \in L_{2,loc}(\mathbb{R}^d) \) satisfying the condition \( \text{(3.2)} \) with the inner product defined by \( \text{(3.3a)} \). Using the \( \Gamma \)-periodicity of \( e_k(x) \), we have:

\[
\hat{b}_k(x,p) = (U b_k)(x,p) = \frac{1}{(2\pi)^d} \sum_{l \in \mathbb{Z}^d} \exp(i(x - l) \cdot p) e_k(x) = \frac{\exp(i(x - l) \cdot p)}{(2\pi)^d} \sum_{l \in \mathbb{Z}^d} \exp(i(x - l) \cdot p) e_k(x) \] \quad (5.24)

\((k = 1, 2, ..., N)\). Here \( \delta(p) \) is the Dirac delta function. Assume, on the contrary, that \( \{b_k(x)\}_{k=1}^N \) are linearly dependent. Then

\[
b_{k_0}(x) = \sum_{k \in \{1,2,...,N\} \setminus \{k_0\}} c_k b_k(x),
\]

hence

\[
\hat{b}_{k_0}(x,p) = \sum_{k \in \{1,2,...,N\} \setminus \{k_0\}} c_k \hat{b}_k(x,p).
\]

In view of \( \text{(5.24)} \), the support of the distribution in l.h.s. of the last equality is \( \{p_{k_0}\} \), but the distribution in its r.h.s. vanishes in a neighborhood of \( p_{k_0} \). We have a contradiction. Thus, the sequence \( \{b_k(x)\}_{k=1}^N \) is linearly independent. \( \square \)
Lemma 5.12. Assume that the unperturbed potential $V(x)$ satisfies condition (4.6). Let $\{b_k(x)\}_{k=1}^N$ be a finite sequence of Bloch functions of the form (5.23) corresponding to the same energy level $\lambda$ of the unperturbed Hamiltonian $H_0 = -\Delta + V(x)$ and having mutually different quasi-momenta $p_k \in T^d \ (k = 1, 2, \ldots, N)$. If $W \in L_1(R^d)$, $W(x) \geq 0 \ a.e. \ on \ R^d$ and $W(x) > 0$ on a set of positive measure, then the sequence of functions $v_k(x) = \sqrt{W(x)} b_k(x) \ (k = 1, 2, \ldots, N)$ belongs to $L_2(R^d)$ and it is linearly independent.

Proof. Since all the Bloch functions are bounded on $R^d$, the inclusion $v_k \in L_2(R^d)$ is obvious. Since $b_k(x)$ correspond to the same energy level $\lambda$, they are solutions of the Schrödinger equation $H_0 b = \lambda b$ belonging to $W_{2, loc}^2(R^d)$. Assume, on the contrary, that $\{v_k(x)\}_{k=1}^N$ are linearly dependent, that is there exist constants $c_1, c_2, \ldots, c_N$ such that $\sum_{k=1}^N |c_k| > 0$ and $\sum_{k=1}^N c_k v_k(x) = \sqrt{W(x)} \sum_{k=1}^N c_k b_k(x) = 0$ for almost all $x \in R^d$. The latter equality implies that the function $b(x) = \sum_{k=1}^N c_k b_k(x)$ vanishes on a set of positive measure. On the other hand, since by Lemma 5.11 the functions $b_k(x)$ are linearly independent, then the function $b(x)$ is a non-trivial solution of the equation $H_0 b = \lambda b$ belonging to $W_{2, loc}^2(R^d)$. Observe that its first generalized derivatives $w_j = D_j b (j = 1, 2, \ldots, d)$ belong to $W_{2, loc}^1(R^d)$ and satisfy the equations $\Delta w_j + \lambda w_j = D_j (V(x) b)$. Observe that by claim (i) of Theorem A.1, the function $b(x)$ is continuous on $R^d$. Then, in view of condition (4.6) and by the elliptic regularity theorem (Gilbarg-Trudinger, Theorem 8.22), the function $b(x)$ is locally $C^{1_\alpha}$, ($\alpha \in (0, 1)$). Then it follows from Theorem 1.7. of [Har-Stri] that the set of zeros of $b(x)$ (the nodal set) has a locally finite $d - 1$-dimensional Hausdorff measure, hence it has a zero Lebesgue measure. We have a contradiction. Thus, the sequence $\{v_k(x)\}_{k=1}^N$ is linearly independent.

It is not difficult to prove the following consequence of Lemma 5.12.

Lemma 5.13. Under all the conditions of Lemma 5.12 the finite rank operator $G = \sum_{k=1}^N \theta_k (\cdot, v_k b_k) (\theta_k > 0 \ for \ any \ k \in \{1, 2, \ldots, N\})$ is non-negative, it has exactly $N$ positive eigenvalues (counting their multiplicities) and they coincide with the eigenvalues of the matrix $\left(\left(\theta_k \theta_l \right)_{k,l=1}^N(v_l, v_k)\right)_{k,l=1}^N$.

5.3.4 Proof of Theorem 4.1

Proof. Since each positive characteristic branch $\mu_k^+ (\lambda)$ of the operator $H_0$ w.r. to $W$ on the gap $(-\lambda, \lambda)$ is an increasing function (see Remark 5.3), then by Proposition 5.5 each eigenvalue $\rho_k^+ (\gamma)$ of the operator $H_\gamma$ appearing in the gap $(-\lambda, \lambda)$ for $\gamma < 0$ increases, hence it cannot approach the edge $\lambda_-$ as $\gamma \uparrow 0$. Hence it cannot be a virtual eigenvalue of $H_\gamma$ at the edge $\lambda_-$, that is claim (i) is valid (see Definitions 4.1, 4.2 and 4.4).

Observe that, as it is easy to show, for $d \leq 2$ each of the conditions (4.3) and (4.5) imply that $W \in L_1(R^d)$. On the other hand, by claim (i) of Theorem A.1 the Bloch functions $b_k^+(x)$ are bounded on $R^d$. Hence the corresponding weighted Bloch functions $v_k^+(x) = \sqrt{W(x)} b_k^+(x)$ belong to $L_2(R^d)$.
We now turn to the proof of the claim (iia) (the case where \( d = 1 \)). Recall that for any fixed \( \lambda \in (\lambda_-, \lambda_+) \) \( \{\mu_k^\ell(\lambda)\}_{k=1}^\infty(\lambda) \) is the set of all positive eigenvalues of the Birman-Schwinger operator \( X_W(\lambda) \) arranged in the non-decreasing ordering \( \mu_1^\ell(\lambda) \geq \mu_2^\ell(\lambda) \geq \cdots \geq \mu_{N_+}^\ell(\lambda) > 0 \), were each \( \mu_k^\ell(\lambda) \) is repeated according to its multiplicity (the value \( N_+ = \infty \) is allowed). Denote \( t = \sqrt{\lambda_+ - \lambda} \) and \( \Phi(t) = tX_W(\lambda_+ - t^2) \). In view of the representation \( (5.16) \), definition \( (5.17) \) and the relation \( (5.18) \) (Proposition \( 5.10 \)), for some \( \delta > 0 \) and any \( t \in [0, \delta] \)
\[
\Phi(t) = \Phi(0) + t\theta(t), \quad \text{where} \quad \Phi(0) = \sqrt{\frac{m^2 - \lambda_+^2}{2}} \quad \text{and} \quad \sup_{t \in [0, \delta]} ||\theta(t)|| < \infty.
\]
As it is clear, the one-rank operator \( \Phi(0) \) has the unique non-zero eigenvalue \( \mu_1^0 = \sqrt{\frac{m^2 - \lambda_+^2}{2}} \) and it is positive. Then by claim (i-a) of Proposition \( 5.8 \) there exists \( \delta \in [0, \delta] \) such that for any \( t \in [0, \delta] \) the operator \( \Phi(t) \) has a unique simple positive eigenvalue \( \mu_1(t) \) having the asymptotic representation \( \mu_1(t) = \mu_1^0 + O(t) \) as \( t \to 0 \) and all the rest of positive eigenvalues of \( \Phi(t) \) (if they exist) have the asymptotic representation \( \mu_k(t) = O(t) \) as \( t \to 0 \) \( (k > 1) \).

Returning from the variable \( t \) to the variable \( \lambda \) and taking into account that all the eigenvalues of \( X_W(\lambda) \) are obtained from the eigenvalues of \( \Phi(t) \) via multiplication by \( \frac{1}{t} = \frac{1}{\sqrt{\lambda_+ - \lambda}} \), we obtain that for some \( \delta \in (0, \lambda_+ - \lambda_-) \) and for any \( \lambda \in [\lambda_+ - \delta, \lambda_+] \) the operator \( X_W(\lambda) \) has a unique positive eigenvalue \( \mu_1^\ell(\lambda) \) having the asymptotic representation

\[
\mu_1^\ell(\lambda) = \frac{\sqrt{m^2 - \lambda^2}}{2(\lambda_+ - \lambda)} + O(1) \quad \text{for} \quad \lambda \uparrow \lambda_+
\]

(hence \( \lim_{\lambda \uparrow \lambda_+} \mu_1^\ell(\lambda) = \infty \) ) and all the rest of its positive eigenvalues \( \mu_k^\ell(\lambda) \) \( (k > 1) \) (if they exist) are bounded in \( (\lambda_+ - \delta, \lambda_+) \). This means that \( \mu_1^\ell(\lambda) \) is a unique main characteristic branch of \( H_0 \) w.r.t. to \( W \) near the edge \( \lambda_+ \) of the gap \( (\lambda_-, \lambda_+) \), hence the corresponding asymptotic multiplicity \( M(\lambda_+, H_0, W) \) of \( \lambda_+ \) is one (see Definition \( 5.4 \)). By Proposition \( 5.6 \) the latter circumstances mean that for \( \gamma < 0 \) there exists a unique branch \( \rho_1^\ell(\gamma) \) of virtual eigenvalues of the operator \( H_\gamma \) at the edge \( \lambda_+ \) (see Definition \( 5.1 \)) and it is the solution of the equation \( \mu_1^\ell(\lambda) = \frac{1}{\rho} \) for a sufficiently small \( |\gamma| \). From \( (5.22) \) and Lemma \( 5.7 \) we obtain the desired asymptotic formula \( (4.7) \). Claim (iia) is proven.

Let us prove claim (iib) (the case where \( d = 2 \)). By Proposition \( 5.10 \) in this case the finite rank operator \( \Phi_\lambda(\lambda) \), taking part in the representation \( (5.16) \) for \( X_W(\lambda) \) has the form \( (5.19) \), where \( G_W^\lambda \) is defined by \( (4.11) \). Like above, let \( \mu_k^\ell(\lambda) \) \( (k = 1, 2, \ldots, N_+) \) be positive characteristic branches of \( H_0 \) w.r.t. \( W \) in the gap \( (\lambda_-, \lambda_+) \), that is they are all the positive eigenvalues of the operator \( X_W(\lambda) \). Denote

\[
t = t(\lambda) = \left( \ln \left( \frac{1}{\lambda_+ - \lambda} \right) \right)^{-1} \quad \text{and} \quad \Phi(t) = tX_W(\lambda + \exp(-t^{-1})).
\]

In view of representation \( (5.16) \), definition \( (5.19) \) and condition \( (5.18) \), for some \( \delta > 0 \) and any \( t \in [0, \delta] \) representation \( (5.7) \) is valid with \( \Phi(0) = \frac{1}{2\pi} G_W^\lambda \) and
Let \( L \) such that \( \mu \) exist) have the asymptotic representation \( \mu_k(t) = O(t) \) for \( t \to 0 \) and all the rest of positive eigenvalues of \( \Phi(t) \) (if they exist) have the asymptotic representation \( \mu_k(t) = O(t) \) for \( t \to 0 \) \((k > n_+)\). Recall that \( \theta(t) \) satisfying the condition (5.3). Then by claim (i-a) of Proposition 5.8 and Lemma 5.13, there exists \( \tilde{\delta} \in [0, \delta] \) such that for any \( t \in [0, \tilde{\delta}] \) the operator \( \Phi(t) \) has \( n_+ \) positive eigenvalues \( \mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_{n_+}(t) \) (counting their multiplicities), having the asymptotic representation \( \mu_k(t) = \frac{1}{2\pi} n_k^+ + O(t) \) for \( t \to 0 \). Observe that each of these eigenvalues \( \lambda \), we obtain that there exists \( \tilde{\delta} \in (0, \lambda_+ - \lambda_-) \) such that \( \mu^+_{i+1}(\lambda), \mu^+_{i+2}(\lambda), \ldots, \mu^+_n(\lambda) \) exist for any \( \lambda \in (\lambda_+ - \tilde{\delta}, \lambda_+) \), they have the asymptotic representation \( \mu^+_k(\lambda) = \frac{1}{2\pi} \ln \left( \frac{1}{\lambda_k - \lambda} \right) n_k^+ + O(1) \) for \( \lambda \uparrow \lambda_+ \) \((k = 1, 2, \ldots, n_+)\), and for \( k > n_+ \) \( \mu_k^+(\lambda) \) are bounded in \((\lambda_+ - \tilde{\delta}, \lambda_+)\) if they exist. This means that \( \mu_k^+(\lambda) \) is the set of all main characteristic branches of \( H_0 \) w.r. to \( W \) near the edge \( \lambda_+ \) of \((\lambda_-, \lambda_+)\), hence \( M(\lambda_+, H_0, W) = n_+ \). By Proposition 5.6 the latter circumstances mean that for \( \gamma < 0 \) there exist exactly \( n_+ \) branches \( \rho^+_1(\gamma) \leq \rho^+_2(\gamma) \leq \cdots \leq \rho^+_n(\gamma) \) of virtual eigenvalues of the operator \( H_\gamma \) near the edge \( \lambda_+ \) and each of them is the solution of the equation \( \mu_k^+(\lambda) = \frac{1}{2\pi} \) for any \( \gamma \in (-\tilde{\gamma}, 0) \) and for some \( \tilde{\gamma} > 0 \). Using Lemma 5.7 we obtain from (5.27) the desired asymptotic formula (4.9). Formula (4.10) follows from (4.9) and the fact that \( \text{tr}(G^+_W) = \sum k=1^n n_k^+ \). Claim (ii-b) is proven.

We shall prove only claim (iii-b) (the case where \( d = 2 \)), because claim (iii-a) (the case where \( d = 1 \)) is proved analogously. For any fixed \( j \in \{1, 2, \ldots, n_+\} \) consider the group of virtual eigenvalues \( \rho_{l(j)}(\gamma) \leq \rho_{l(j)+1}(\gamma) \leq \cdots \leq \rho_{l(j)+m(j)-1}(\gamma) \) \((l(j) \in \{1, 2, \ldots, n_+\}, \gamma \in [-\tilde{\gamma}, 0]\) of the operator \( H_\gamma \) in \((\lambda_-, \lambda_+)\), for which the quantities \( \frac{\mu_{l(j)}^+(\gamma)}{\gamma} \) \((k \in \{l(j), l(j) + 1, \ldots, l(j) + m(j) - 1\}) \) tend as \( \gamma \uparrow 0 \) to the eigenvalue \( \frac{\nu^+}{2\pi} \) of the operator \( \Phi(0) \), whose multiplicity is \( m(j) \). Recall that the function \( t(\lambda) \) and the operator function \( \Phi(t) \) are defined by (5.20) and \( g^+_1(x), g^+_2(x), \ldots, g^+_n(x) \) is an orthonormal sequence of eigenfunctions of the operator \( G^+_W = 2\pi \Phi(0) \) corresponding to its positive eigenvalues \( \nu^+_1, \nu^+_2, \ldots, \nu^+_n \). Denote \( \tilde{\rho}(\gamma) = \rho_{l(j)}(\gamma) \). Let \( \tilde{\rho}_0(\gamma) = \mu_{l(j)}^+(\tilde{\rho}(\gamma)) > \tilde{\mu}_1^+(\gamma) > \cdots > \tilde{\mu}_n(\gamma) \) \((n(\gamma) \leq m(j) - 1) \) be all the mutually different numbers from the sequence \( \{\rho_{l(j)+k}(\tilde{\rho}(\gamma))\}^{m(j)-1}_{k=0} \). These numbers are positive eigenvalues of the self-adjoint operator \( X_\gamma \). Let \( L_0(\gamma), L_1(\gamma), \ldots, L_n(\gamma) \) be the eigenspaces of \( X_\gamma(\tilde{\rho}(\gamma)) \), corresponding to these eigenvalues. Observe that each of \( L_k(\gamma) \) is the eigenspace of the operator \( \Phi(\gamma) = \Phi(t(\tilde{\rho}(\gamma))) \), corresponding to its eigenvalue \( t(\tilde{\rho}(\gamma))) \). Denote

31
\[ L(\gamma) = \bigoplus_{k=0}^{n(\gamma)} L_k(\gamma) \]. By claim (i-b) of Proposition 5.8 for a suitable \( \hat{\gamma} \in (0, \gamma] \) and any \( \gamma \in [0, \hat{\gamma}] \) it is possible to choose a basis

\[ e_{l(j)}(\gamma), e_{l(j)+1}(\gamma), \ldots, e_{l(j)+m(j)-1}(\gamma) \]

in the subspace \( L(\gamma) \) such that \( \|e_k(\gamma) - g^\pm_k\| = O(t(\tilde{\rho}(\gamma))) \) for \( \gamma \uparrow 0 \). Then taking into account the asymptotic formula (4.19), we get that

\[ \|e_k(\gamma) - g^\pm_k\| = O(\gamma) \quad \text{for} \quad \gamma \uparrow 0 \quad (k = l(j), l(j)+1, \ldots, l(j)+m(j)-1), \quad (5.28) \]

For any \( \gamma \in [0, \hat{\gamma}] \) consider the sequence \( \gamma_0(\gamma) = \gamma, \gamma_k(\gamma) = -\frac{1}{\mu^i_{l(j)+k}(\tilde{\rho}(\gamma))} (k = 1, 2, \ldots, m(j)-1) \). Since the r.h.s. of the asymptotic formula (5.27) is the same for all the functions from the sequence \( \{\mu^i_{l(j)+k}(\lambda)\}_{k=0}^{m(j)-1} \), then

\[ \mu^i_{l(j)}(\tilde{\rho}(\gamma)) - \mu^i_{l(j)+k}(\tilde{\rho}(\gamma)) = -\gamma^{-1} + (\gamma_k(\gamma))^{-1} = O(1) \quad \text{for} \quad \gamma \uparrow 0 \]

\((k = 1, 2, \ldots, m(j)-1)\). This fact implies easily that \( \gamma - \gamma_k(\gamma) = O(\gamma^2) \) as \( \gamma \uparrow 0 \) \((k = 0, 1, \ldots, m(j)-1)\). Observe that \( \tilde{\gamma}_k(\gamma) = -\frac{1}{\tilde{\mu}^i_k(\gamma)} (k = 0, 1, \ldots, n(\gamma)) \) are all the different numbers from the sequence \( \{\gamma_k(\gamma)\}_{k=0}^{m(j)-1} \). It is clear that for any \( k \in \{0, 1, \ldots, n(\gamma)\} \) the number \( \tilde{\rho}(\gamma) \) is an eigenvalue of the operator \( H_{\tilde{\gamma}_k}(\gamma) \). Let \( E_k(\gamma) \) be the eigenspace of \( H_{\tilde{\gamma}_k}(\gamma) \), corresponding to the eigenvalue \( \tilde{\rho}(\gamma) \).

By Proposition 5.1 the multiplication operator \( W^{1/2} \cdot |E_k(\gamma) \) realizes a linear isomorphism between \( E_k(\gamma) \) and \( L_k(\gamma) \). This circumstance and the fact that the subspaces \( L_k(\gamma) \) are mutually orthogonal imply that the subspaces \( E_k(\gamma) \) are linearly independent and the operator \( Q(\gamma) = W^{1/2} \cdot |E(\gamma) \) realizes a linear isomorphism between the subspace \( E(\gamma) = E_0(\gamma) + E_1(\gamma) + \cdots + E_{n(\gamma)}(\gamma) \) and the subspace \( L(\gamma) \). Let us define the following basis in \( E(\gamma) \): \( \psi^+_k = (Q(\gamma))^{-1} e_k(\gamma) \) \((k \in \{l(j), l(j)+1, \ldots, l(j)+m(j)-1\})\). Then we obtain from (5.28) the desired property (4.11). Claim (iii-b) is proven.

Let us prove claim (iv). In view of (5.20) (claim (iii) of Proposition 5.10), all the positive characteristic branches of \( H_0 \) w.r.t. \( W \) on the gap \((\lambda_-, \lambda_+)\) are bounded, hence \( M(\lambda_-, H_0, W) = 0 \). This fact, Proposition 5.6 and claim (i) imply the desired claim (iv).

Claim (v) is proved in the same manner as the previous ones. Claim (vi) follows from claims (iv) and (v).

Theorem 4.1 is proven. \( \Box \)

5.4 The case of degenerate edges of a gap of the spectrum of the unperturbed operator

As above, \((\lambda_-, \lambda_+)\) is a gap of the spectrum of the unperturbed operator \( H_0 \) and we assume that the perturbation \( W(x) \) of the periodic potential \( V(x) \) is measurable and bounded in \( \mathbb{R}^d \), \( W \in L_1(\mathbb{R}^d) \) and \( W(x) \geq 0 \) a.e. on \( \mathbb{R}^d \).
5.4.1 Representation of Birman-Schwinger operator in the case of a degenerate edge

In the case of a degenerate edge of a gap of the spectrum of the unperturbed operator we have the following modification of Proposition 5.10:

**Proposition 5.14.** Let us take \( \delta \in (0, (\lambda_+ - \lambda_-)) \).

(i) If \( d - d_+ = 1 \), the edge \( \lambda_+ \) satisfies condition (C) of Section 4.2 and condition (4.13) is satisfied, then for \( \lambda \in (\lambda_+ - \delta, \lambda_+) \) the representation is valid:

\[
X_W(\lambda) = \Phi_+(\lambda) + \Theta_+(\lambda),
\]

where

\[
\Phi_+(\lambda) = \frac{1}{\sqrt{\lambda_+ - \lambda}} \sqrt{2\pi} d_G^+, \quad (5.30)
\]

the operator \( G_W^+ \) is defined by (4.12)-(4.13) and \( \Theta_+(\lambda) \) is a bounded operator in \( L^2(\mathbb{R}^2) \) satisfying the condition

\[
\Theta_+ = \sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \|\Theta_+(\lambda)\| < \infty; \quad (5.31)
\]

(ii) if \( d - d_+ = 2 \), the edge \( \lambda_+ \) satisfies condition (C) and condition (4.16) is satisfied, then for \( \lambda \in ((\lambda_+ - \delta, \lambda_+)) \) the representation (5.29) is valid with \( \Theta_+(\lambda) \) satisfying condition (5.31) and with \( \Phi_+(\lambda) \) having the form:

\[
\Phi_+(\lambda) = \ln \left( \frac{1}{\lambda_+ - \lambda} \right) \sqrt{2\pi} d_G^+; \quad (5.32)
\]

(iii) if \( d - d_+ \geq 3 \) and the edge \( \lambda_+ \) satisfies condition (C), then

\[
\sup_{\lambda \in ((\lambda_+ - \delta, \lambda_+))} \|X_W(\lambda)\| < \infty;
\]

(iv) The claims analogous to (i)-(iii) are valid with \( d_-, \lambda_- \), \( (\lambda_- + \delta, \lambda_-) \) and \( G_W^- \) instead of, respectively, \( d_+, \lambda_+, \lambda_+ - \delta, \lambda_+ \) and \( G_W^+ \).

*Proof.* The proof is the same as the proof of Proposition 5.10 merely instead of Proposition 6.3 it is supported by Proposition 6.1 with \( n_+ = n_- = 1 \). \( \square \)

5.4.2 Spectrum of the operator \( G_W^+ (G_W^-) \)

In what follows we need the following

**Lemma 5.15.** If the edge \( \lambda_+ (\lambda_-) \) satisfies condition (C) of Section 4.2, the unperturbed potential \( V(x) \) satisfies condition (4.3) and \( W(x) > 0 \) on a set of the positive measure, then the integral operator \( G_W^+ \), defined by (4.12)-(4.13),

(i) is self-adjoint, non-negative, belongs to the trace class and

\[
\text{tr}(G_W^+) = \int_{\mathbb{R}^3} G_W^+(s,s) ds = \int_{F^+} \int_{\mathbb{R}^4} Q_W^+(s,s,p) ds \sqrt{m^+(p)} dF(p) \quad (5.33)
\]
(ii) has an infinite number of positive eigenvalues having finite multiplicities.

The analogous properties has the operator $G_W$.

Proof. We shall restrict ourselves only on the operator $G_W^+$, since the operator $G_W^-$ has the same structure. Let us prove claim (i). Using claim (i) of Corollary A.2 and compactness of the submanifold $F^+$, it is possible to construct a finite open covering $\{U_k\}_{k=1}^L$ of $F^+$ such that for any $k \in \{1, 2, \ldots, L\}$ $Q_{U_k}^+(x, s, p) = v_k^+(x, p)v_k^+(s, p)$ for any $p \in U_k$, where $v_k^+(x, p) = \sqrt{W(x)b_k^+(x, p)}$ and $b_k^+(x, p)$ $(p \in U_k)$ is a branch of Bloch functions, corresponding to the energy level $\lambda_+$ of the family of the operators $H(p)$ $(p \in U_k)$ such that the mapping $p \to b_k^+(\cdot, p)$ belongs to the class $C(U_k, C(\Omega))$. Hence since $W \in L_1(\mathbb{R}^d)$, the inclusion

$$
(p \to v_k^+(\cdot, p)) \in C(U_k, L_2(\mathbb{R}^d)).
$$

is valid. Let $\{\phi_k(p)\}_{k=1}^L$ be a decomposition of unit, corresponding to the covering $\{U_k\}_{k=1}^L$ such that each $\phi_k$ is continuous on $F^+$ and positive on $U_k$. Then the operator $G_W^+$ can be represented in the form $G_W^+ = \sum_{k=1}^L G_k$, where $G_k \tilde{f} = \int_{\mathbb{R}^d} G_k(x, s) f(s) \, ds$ ($f \in L_2(\mathbb{R}^d)$) and

$$
G_k(x, s) = \int_{U_k} v_k^+(x, p)v_k^+(s, p)\phi_k(p)\sqrt{m^+(p)} \, dF(p).
$$

In order to prove that $G_W^+$ belongs to the trace class, it is enough to show that each of $G_k$ belongs to the trace class. We see that

$$
G_k(x, s) = \int_{U_k} A_k(x, p)A_k(s, p) \, dF(p),
$$

where $A_k(x, p) = v_k^+(x, p)(\phi_k(p))^{1/2}(m^+(p))^{1/4}$. Consider the integral operator

$$
(A_k \phi)(x) = \int_{U_k} A_k(x, p)\phi(p) \, dF(p),
$$

which acts from $L_2(U_k)$ to $L_2(\mathbb{R}^d)$, hence $G_k = A_k(A_k)^*$. Formally in order to prove that the operator $G_k$ belongs to the trace class, it is enough to show that $A_k$ belongs to the Hilbert-Schmidt class. But since the domain and the range of $A_k$ are different, we are forced to use a direct sum argument. Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^d) \oplus L_2(F^+)$ and the operator $A_W$ acting in it and defined by the operator matrix:

$$
A_k = \begin{pmatrix}
0 & A_k \\
A_k^* & 0
\end{pmatrix}
$$

(5.35)

Since the dispersion function $\lambda^+(p)$ is real-analytic and satisfies the Morse-Bott condition in a neighborhood of $F^+$, then in view of (3.15)

$$
m^+ \in C(F^+, \mathbb{R}).
$$

(5.36)
This fact and inclusion (5.34) imply easily that \( \int_{\mathbb{R}^d} \int_{U_k} |A^+_k(x, p)|^2 \, dx \, dF(p) < \infty \), hence in view of (5.35) the matrix integral kernel of the operator \( \hat{A}_k \) is square integrable on \( \mathbb{R}^d \times U_k \). Therefore \( \hat{A}_k \) belongs to the Hilbert-Schmidt class and it is self-adjoint. On the other hand, since \( \hat{A}_k^2 = \text{diag}(G^+_N, A^*_k A_k) \), the operator \( G_k \) is self-adjoint, non-negative and belongs to the trace class. In view of Theorem 3.1 of [Bris], \( \text{tr}(G_k) = \int_{\mathbb{R}^d} \hat{G}_k(s, s) \, ds \), where \( \hat{G}_k(x, s) \) is the local average of \( \hat{G}_k(x, s) \) defined in [Bris]. On the other hand, using the decomposition \( G^+_k = A_k(A_k)^* \) and the arguments of the proof and Theorem 3.5 from [Bris], it is not difficult to show that \( \hat{G}_k(x, x) = G_k(x, x) \) a.e. on \( \mathbb{R}^d \).

Hence, taking into account that \( \text{tr}(G^+_N) = \sum_{k=1}^L \text{tr}(G_k) \), we obtain using (5.39) the local average of \( \hat{G}_k(x, s) \) defined in [Bris]. On the other hand, using the arguments of the proof and Theorem 3.1 of [Bris], \( \text{tr}(G^+_N) = \sum_{j=1}^N \nu_j^+ = v_k^+ \). We have for \( f \in L^2(\mathbb{R}^d) \):

\[
(G^+_N f, f) \geq F(\epsilon)(\hat{G}_N f, f) + (\Theta_{N, \epsilon} f, f),
\]

where

\[
\hat{G}_N = \sum_{j=1}^N (\cdot, \hat{v}_j^+ (\cdot, p_j)) \hat{v}_j^+ (\cdot, p_j) \sqrt{m^+(p_j)}
\]

and

\[
\Theta_{N, \epsilon} = \sum_{k=1}^N \int_{\mathcal{O}_k} (\cdot, \hat{v}_j^+ (\cdot, p)) \hat{v}_j^+ (\cdot, p) \sqrt{m^+(p)} - (\cdot, \hat{v}_j^+ (\cdot, p_j)) \hat{v}_j^+ (\cdot, p_j) \sqrt{m^+(p_j)} \, dF(p).
\]

The last equality and the inclusions (5.34) and (5.36) imply that

\[
\lim_{\epsilon \to 0} \frac{\|\Theta_{N, \epsilon}\|}{F(\epsilon)} = 0.
\]

In view of (5.38) and Lemma 5.13 the operator \( \hat{G}_N \) has exactly \( N \) positive eigenvalues \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_N > 0 \) (counting their multiplicities). Then in the same manner as in the proof of Proposition 5.8 we obtain using (5.39) and a modification of the comparison theorem on the base of the minimax principle (Lemma 3.4 of [At-Zi]) that for a small enough \( \epsilon > 0 \) the operator \( F(\epsilon) \hat{G}_N + \Theta_{N, \epsilon} \) has at least \( N \) positive eigenvalues \( \nu_1(\epsilon) \geq \nu_2(\epsilon) \geq \cdots \geq \nu_N(\epsilon) > 0 \) (counting their multiplicities). Using again the minimax principle, we obtain from (5.37) that the operator \( G^+_N \) has at least \( N \) positive eigenvalues (counting their multiplicities). Since \( N \) is arbitrary, claim (ii) is proven. \( \square \)
5.4.3 Proof of Theorem 4.2

Proof. Claim (i) is proved in the same manner as claim (i) of Theorem 4.1.

We now turn to the proof of the claim (ii) in the case where \( d - d_+ = 2 \).

By claim (ii) of Proposition 5.14, for the Birman-Schwinger operator \( X_W(\lambda) \) representation (5.29) is valid for \( \lambda \in ((\lambda_+ + \lambda_-)/2, \lambda_+) \), in which \( \Phi_+(\lambda) \) has the form (5.32) and \( \Theta_+(\lambda) \) satisfies the condition (5.31). Let \( \mu^+_1(\lambda) \geq \mu^+_2(\lambda) \geq \cdots \geq \mu^+_n(\lambda) \geq \cdots (\lambda \in (\lambda_-, \lambda_+)) \) be the positive characteristic branches of the operator \( H_0 \) w.r. to \( W \) on the gap \( (\lambda_-, \lambda_+) \), that is they are positive eigenvalues of the operator \( X_W(\lambda) \) (Definition 5.2). By Lemma 5.15, the operator \( G^+_W \) has an infinite number of positive eigenvalues \( \nu^+_1 \geq \nu^+_2 \geq \cdots \geq \nu^+_n \geq \cdots \). Let \( \Phi(t) \) be the operator function defined by (5.26). Then in view of the representation (5.29), definition (5.32) and the condition (5.31), for some \( \delta > 0 \) and any \( t \in [0, \delta] \) representation (5.7) is valid with \( \Phi(0) = \frac{1}{2}G_W \) and \( \theta(t) \) satisfying the condition (5.8). Then claim (ii) of Proposition 5.8 implies claim (ii) of Theorem 4.2 for \( d - d_+ = 2 \) in the same manner as the corresponding claim was obtained in the non-degenerate case (see proof of claim (ii-b) of Theorem 4.1). Formula (4.18) follows from (4.17), the formula \( \sum_{n=1}^{\infty} \nu^+_n = \text{tr}(G^+_W) \) and claim (i) of Lemma 5.15. In the analogous manner claim (ii) for \( d - d_+ = 1 \) is proved by the use of representation (5.29) and definition (5.30). Claim (iii) follows from claim (ii) of Proposition 5.8 in the same manner as in the proof of Theorem 4.1 claim (iii-b) follows from claim (i-b) of Proposition 5.8.

Claim (iv) is proved in the same manner as the corresponding claim of Theorem 4.1 using claim (iii) of Proposition 5.14.

Claim (v) is proved in the same manner as the previous ones. Claim (vi) follows from claims (iv) and (v).

Theorem 4.2 is proven. □

5.5 Estimate for the multiplicity of virtual eigenvalues in the case of an indefinite perturbation

5.5.1 General results on the multiplicity of virtual eigenvalues

Before proving Theorem 4.3 we shall prove some auxiliary claims. If \( A \) is a linear operator acting in a Hilbert space, we denote \( \Re(A) := \frac{1}{2}(A + A^*) \); if \( A \) is self-adjoint, we denote \( A_+ := \frac{1}{2}(|A| + A) \), \( A_- := \frac{1}{2}(|A| - A) \).

**Lemma 5.16.** Let \( (\lambda_-, \lambda_+) \) be a gap of the spectrum of a self-adjoint operator \( H_0 \) acting in a Hilbert space \( \mathcal{H} \) be and \( W \) be a bounded self-adjoint operator in \( \mathcal{H} \). Then for any \( \lambda \in (\lambda_-, \lambda_+) \) the real part of the Birman-Schwinger operator \( X_W(\lambda) = W^\dagger R_\lambda(H_0)|W|^\dagger \) admits the representation:

\[
\Re(X_W(\lambda)) = X_{W_+}(\lambda) - X_{W_-}(\lambda). \tag{5.40}
\]

Recall that \( W^\dagger \) is defined by (5.2).
Proof. We have:

\[
(W^{\frac{1}{2}})^+ R_\lambda(H_0)(W^{\frac{1}{2}})^- - (W^{\frac{1}{2}})^- R_\lambda(H_0)(W^{\frac{1}{2}})^+ = \\
\frac{1}{4} \left( (|W|^\frac{1}{2} + W^{\frac{1}{2}})R_\lambda(H_0)(|W|^\frac{1}{2} + W^{\frac{1}{2}}) - \\
(|W|^\frac{1}{2} - W^{\frac{1}{2}})R_\lambda(H_0)(|W|^\frac{1}{2} - W^{\frac{1}{2}}) \right) = \frac{1}{2} \left( W^{\frac{1}{2}} R_\lambda(H_0)|W|^\frac{1}{2} - \\
|W|^\frac{1}{2} R_\lambda(H_0)W^{\frac{1}{2}} \right) = \frac{1}{2} \left( W^{\frac{1}{2}} R_\lambda(H_0)W^{\frac{1}{2}} + (W^{\frac{1}{2}} R_\lambda(H_0)|W|^\frac{1}{2})^* \right). \tag{5.41}
\]

Let \(E_\lambda(\lambda \in \mathbb{R})\) be the spectral resolution of identity, corresponding to the operator \(W\). Since \(|W|^\frac{1}{2} = \int_\mathbb{R} |\lambda|^\frac{1}{2} dE_\lambda\) and, in view of \([5.2]\),

\[
W^{\frac{1}{2}} = \int_\mathbb{R} \text{sign}(\lambda)|\lambda|^\frac{1}{2} dE_\lambda,
\]

the equalities are valid: \((W^{\frac{1}{2}})^+ = (W^+)^\frac{1}{2}\) and \((W^{\frac{1}{2}})^- = (W^-)^\frac{1}{2}\). Then \([5.41]\) can be written in the form \([5.40]\).

**Proposition 5.17.** Assume that all the conditions of Lemma \([5.16]\) are satisfied and furthermore assume that for some \(\lambda_0 \in \mathcal{R}(H_0)\) the operator \(R_{\lambda_0}|W|^\frac{1}{2}\) is compact. Let us take \(\delta \in (0, \lambda_+ - \lambda_-)\). Then

(i) if \(M(\lambda_+, H_0, W_+) < \infty\), there exists \(\bar{\gamma} > 0\) such that for any \(\gamma \in (-\bar{\gamma}, 0)\) the multiplicity of each eigenvalue of the operator \(H_\gamma\) lying in \((\lambda_+ - \delta, \lambda_+\) is at most \(M(\lambda_+, H_0, W_+))\);

(ii) the claim analogous to (i) is valid for \(\gamma > 0\) with \(W_-\) and \(\gamma \in (0, \bar{\gamma})\) instead of, respectively, \(W_+\) and \(\gamma \in (-\bar{\gamma}, 0)\);

(iii) the claims analogous to (i), (ii) are valid near the edge \(\lambda_-\) with \(\lambda_-\), \((\lambda_-, \lambda_+ + \delta)\), \(W_-\), \(\gamma \in (0, \bar{\gamma})\) and \(\gamma \in (-\bar{\gamma}, 0)\) instead of, respectively, \(\lambda_+\), \((\lambda_+ - \delta, \lambda_+)\), \(W_+\), \(\gamma \in (-\bar{\gamma}, 0)\) and \(\gamma \in (0, \bar{\gamma})\).

**Proof.** We shall prove only claim (i), because (ii) and (iii) are proved analogously. Denote \(l_+ = M(\lambda_+, H_0, W_+)\). By Proposition \([5.1]\) we should prove that

\[
\forall \lambda \in (\lambda_+ - \delta, \lambda_+) : \quad \dim(\ker(I + \gamma X_W(\lambda))) \leq l_+. \tag{5.42}
\]

In view of Lemma \([5.11]\) we have for \(u \in \mathcal{H}\):

\[
\Re((I + \gamma X_W(\lambda))u, u) = (u, u) + \gamma((X_{W_+}(\lambda)u, u) - (X_{W_-}(\lambda)u, u)). \tag{5.43}
\]

Let \(\mu_+^k(\lambda)\) and \(\mu_-^k(\lambda)\) be positive and negative characteristic branches of the operator \(H_0\) w.r.t. \(W_+\), that is they are continuous branches of eigenvalues of the operator \(X_{W_+}(\lambda)\) such that \(\mu_+^k(\lambda)\) are numbered in the non-increasing ordering \(\mu_+^k(\lambda) \geq \mu_+^{k+1}(\lambda) \geq \ldots \mu_+^1(\lambda) \geq 0\) and \(\mu_-^k(\lambda)\) are numbered in the non-increasing ordering \(\mu_-^k(\lambda) \leq \mu_-^{k+1}(\lambda) \leq \ldots \mu_-^1(\lambda) \leq -\infty < 0\), \(\text{Dom}(\mu_+^k) = (\eta_k^+, \lambda_+), \eta_k^+ \in [\lambda_-, \lambda_+), \text{Dom}(\mu_-^k) = (\lambda_-, \eta_k^-), \eta_k^- \in [\lambda_-, \lambda_+]\) and they are
increasing functions (Definition 5.2, Remark 5.3). Let \( \{\mu_k^+(\lambda)\}_{k=1}^{l_+} \) be the main characteristic branches of \( H_0 \) w.r.t. \( W_+ \) near the edge \( \lambda_+ \), that is

\[
\lim_{\lambda \uparrow \lambda_+} \mu_k^+(\lambda) = \begin{cases} 
\infty & \text{for } 1 \leq k \leq l_+ \\
< \infty & \text{for } k > l_+.
\end{cases}
\]

(5.44)

As it is clear, \( \eta^+_1 \leq \eta^+_2 \leq \cdots \leq \eta^+_k \). For each \( \lambda \in (\eta^+_k, \lambda_+) \) and \( j \in \{1, 2, \ldots, l_+\} \) consider the eigenspace \( E_j^+(\lambda) \) of the operator \( X_{W_+}(\lambda) \), corresponding to its eigenvalue \( \mu_j^+(\lambda) \). Consider the following family of subspaces:

\[
\mathcal{F}(\lambda) = \left\{ \mathcal{H} \ominus \left( \bigoplus_{j=1}^{l_+} E_j^+(\lambda) \right) \right\} \quad \text{for } \lambda \in (\lambda_+ - \delta, \lambda_+) \cap (\eta^+_k, \lambda_+),
\]

(5.45)

Then we obtain:

\[
\forall u \in \mathcal{F}(\lambda):
\]

\[
(X_{W_+}(\lambda)u, u) \leq \begin{cases} 
\max_{j \geq l_+} \mu_j^+(\lambda)||u||^2, & \text{if } \lambda \in (\lambda_+ - \delta, \lambda_+) \cap (\eta^+_k, \lambda_+), \\
\mu_1^+(\eta^+_k)||u||^2, & \text{if } \lambda \in (\lambda_+ - \delta, \lambda_+) \setminus (\eta^+_k, \lambda_+).
\end{cases}
\]

Hence, in view of (5.44), we have:

\[
\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), u \notin \mathcal{F}(\lambda) \setminus \{0\}} \frac{(X_{W_+}(\lambda)u, u)}{||u||^2} < \infty.
\]

(5.46)

Since the operator function \( X_{W_+}(\lambda) \) increases in \( (\lambda_-, \lambda_+) \) in the sense of comparison of quadratic forms (\([Ar-Zl1, \text{Lemma 3.5}]\), we have:

\[
\inf_{\lambda \in (\lambda_+ - \delta, \lambda_+), u \in \mathcal{H} \setminus \{0\}} \frac{(X_{W_+}(\lambda)u, u)}{||u||^2} \geq
\]

\[
\inf_{u \in \mathcal{H} \setminus \{0\}} \frac{(X_{W_+}(\lambda_+ - \delta)u, u)}{||u||^2} > -\infty.
\]

(5.47)

Then we conclude from (5.43), (5.46) and (5.47) that there exists \( \bar{\gamma} > 0 \) such that

\[
\forall \gamma \in (-\bar{\gamma}, 0) : \inf_{\lambda \in (\lambda_+ - \delta, \lambda_+), u \notin \mathcal{F}(\lambda) \setminus \{0\}} \mathbb{R}((I + \gamma X_{W}(\lambda))u, u) \geq \frac{1}{2}
\]

Taking into account that \( \mathbb{R}((I + \gamma X_{W}(\lambda))u, u)) \leq (I + \gamma X_{W}(\lambda))u||u|| \), we obtain from the last estimate that

\[
\forall \gamma \in (-\bar{\gamma}, 0) : \inf_{\lambda \in (\lambda_+ - \delta, \lambda_+), u \notin \mathcal{F}(\lambda) \setminus \{0\}} \frac{||(I + \gamma X_{W}(\lambda))u||}{||u||} \geq \frac{1}{2}
\]

(5.48)

Let us prove estimate (5.42). Assume, on the contrary, that

\[
\exists \gamma_0 \in (-\bar{\gamma}, 0), \exists \lambda_0 \in (\lambda_+ - \delta, \lambda_+) : \dim(\ker(I + \gamma_0 X_{W}(\lambda_0))) > l_+.
\]

(5.49)
On the other hand, in view of (5.45),
\[ \forall \lambda \in (\lambda_+ - \delta, \lambda_+) : \quad \text{codim}(\mathcal{F}(\lambda)) \leq 1. \]

This inequality together with (5.49) imply that
\[ \mathcal{F}(\lambda_0) \cap \ker(I + \gamma_0 X_W(\lambda_0)) \neq \{0\}. \]

The last fact contradicts the inequality (5.48). So, estimate (5.42) is proven. \( \square \)

5.5.2 Proof of Theorem 4.3

**Proof.** Assume that the edge \( \lambda_+ \) is non-degenerate. If in the in the proof of Theorem 4.1 we take \( W_+ \) instead of \( W \), we obtain that for \( d = 1 \) \( M(\lambda_+, H_0, W_+) = 1 \), for \( d = 2 \) \( M(\lambda_+, H_0, W_+) = n_+ \) and for \( d \geq 3 \) \( M(\lambda_+, H_0, W_+) = 0 \). Then by Proposition 5.17 all the claims (i)-(iii) are valid in the case where \( \lambda_+ \) is non-degenerate. The case where \( \lambda_- \) is non-degenerate is treated analogously.

Claim (iv) is proved in the same manner as the previous ones. Claim (v) follows from the claims (iii) and (iv).

Theorem 4.3 is proven. \( \square \)

5.5.3 Proof of Theorem 4.4

**Proof.** The proof is the same as the proof of Theorem 4.3 in the case where \( d \geq 3 \) and it is supported by Proposition 5.17 and claim (iv) of Theorem 4.2. \( \square \)

6 Representation of the resolvent of the unperturbed operator \( H_0 \) near the edges of a gap of its spectrum

6.1 Main claims

**Proposition 6.1.** Let \((\lambda_-, \lambda_+)\) be a gap of the spectrum of the unperturbed operator \( H_0 \) such that its edge \( \lambda_+ \) satisfies condition (B) of Section 3.1, and \( \lambda^+(p) \) be the dispersion function, branching from the edge \( \lambda_+ \). Let us take \( \delta \in (0, (\lambda_+ - \lambda_-)) \).

(i) For \( \lambda \in (\lambda_-, \lambda_+) \) the resolvent \( R_\lambda(H_0) \) has the form:
\[ R_\lambda(H_0) = K(\lambda) + \Theta(\lambda), \quad (6.1) \]

where \( \Theta(\lambda) \) is a bounded operator in \( L_2(\mathbb{R}^d) \) for any \( \lambda \in (\lambda_-, \lambda_+) \) such that the condition is satisfied
\[ \sup_{\lambda \in (\lambda_+ - \delta, \lambda_+)} \|\Theta(\lambda)\| < \infty, \quad (6.2) \]
$K(\lambda) = \sum_{k=1}^{n_+} K^+_k(\lambda)$ and for any $k \in \{1, 2, \ldots, n_+\}$ the restriction $K^+_k(\lambda)|_{L^2_0(\mathbb{R}^d)}$

of $K^+_k(\lambda)$ on $L^2_0(\mathbb{R}^d)$ is an operator with a continuous in $\mathbb{R}^d \times \mathbb{R}^d \times (\lambda_-, \lambda_+)$ integral kernel $K^+_k(x, s, \lambda);

(ii) if $d - d^+_k \geq 3$, then for any $k \in \{1, 2, \ldots, n_+\}$ the function $K^+_k(x, s, \lambda)$ satisfies the condition

$$\sup_{\lambda \in ((\lambda_+ - \delta, \lambda_+), x, s \in \mathbb{R}^d} |K^+_k(x, s, \lambda)| < \infty; \quad (6.3)$$

(iii) If $d - d^+_k \leq 2$, then for any $k \in \{1, 2, \ldots, n_+\}$, $\lambda \in (\lambda_-, \lambda_+)$ the function $K^+_k(x, s, \lambda)$ admits the representation $K^+_k(x, s, \lambda) = G^+_k(x, s, \lambda) + \hat{K}^+_k(x, s, \lambda)$, where $\hat{K}^+_k(x, s, \lambda)$ is continuous in $\mathbb{R}^d \times \mathbb{R}^d \times (\lambda_-, \lambda_+)$ and

(a) for $d - d^+_k = 1$

$$G^+_k(x, s, \lambda) = \frac{\sqrt{2\pi}}{(2\pi)^d \sqrt{\lambda_+ - \lambda}} \int_{F^+_k} Q^+(x, s, p) \sqrt{m^+(p)} dF(p) \quad (6.4)$$

where $Q^+(x, s, p)$ is the eigenkernel of $H(p)$ corresponding to the eigenvalue $\lambda^+(p)$, defined in Section 3.1, $m^+(p)$ is defined by $(3.13)$, $(3.14)$, $(3.15)$, and

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), x, s \in \mathbb{R}^d} |\hat{K}^+_k(x, s, \lambda)| < \infty; \quad (6.3)$$

(b) for $d - d^+_k = 2$

$$G^+_k(x, s, \lambda) = \frac{1}{(2\pi)^{d-1}} \ln \left(\frac{1}{\lambda_+ - \lambda}\right) \int_{F^+_k} Q^+(x, s, p) \sqrt{m^+(p)} dF(p), \quad (6.5)$$

and

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), x, s \in \mathbb{R}^d} \frac{|\hat{K}^+_k(x, s, \lambda)|}{1 + \ln(1 + |x - s|)} < \infty. \quad \text{(6.3)}$$

The claims analogous to above ones are valid with $\lambda_-, \lambda^-(p), n_-, d^-_k, F^-_k, Q^-(x, s, p), m^-(p)$ and $(\lambda_-, \lambda_+ + \delta)$ instead of, respectively, $\lambda_+, \lambda^+(p), n_+, d^+_k, F^+_k, Q^+(x, s, p), m^+(p)$ and $(\lambda_+ - \delta, \lambda_+)$. In particular, in the case of non-degenerate edges of the gap of $\sigma(H_0)$ the previous proposition acquires the form:

**Proposition 6.2.** Let $(\lambda_-, \lambda_+)$ is a gap of the spectrum of the unperturbed operator $H_0$ such that its edge $\lambda_+$ is non-degenerate and $\lambda^+(p)$ be the dispersion function, branching from the edge $\lambda_+$. Let us take $\delta \in (0, (\lambda_+ - \lambda_-))$.

(i) The resolvent $R_\lambda(H_0)$ has the form for $\lambda \in (\lambda_-, \lambda_+)$: $R_\lambda(H_0) = K(\lambda) + \Theta(\lambda)$, where $\Theta(\lambda)$ is a bounded operator in $L^2(\mathbb{R}^d)$ for any $\lambda \in (\lambda_-, \lambda_+)$, such
that condition (6.2) is satisfied and the restriction $K(\lambda)|_{L_{2,0}(\mathbb{R}^d)}$ of $K(\lambda)$ on $L_{2,0}(\mathbb{R}^d)$ is an operator with a continuous in $\mathbb{R}^d \times \mathbb{R}^d \times (\lambda_-, \lambda_+)$ integral kernel $K(x, s, \lambda)$;

(ii) if $d \geq 3$, then the function $K(x, s, \lambda)$ satisfies the condition

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), \mathbf{x}, \mathbf{s} \in \mathbb{R}^d} |K(x, s, \lambda)| < \infty;$$

(iii) if $d \leq 2$ and $\lambda \in (\lambda_-, \lambda_+)$, then the function $K(x, s, \lambda)$ admits the representation $K(x, s, \lambda) = F^+(x, s, \lambda) + \tilde{K}^+(x, s, \lambda)$, where $\tilde{K}^+(x, s, \lambda)$ is continuous in $\mathbb{R}^d \times \mathbb{R}^d \times (\lambda_-, \lambda_+)$ and

(a) for $d = 1$

$$F^+(x, s, \lambda) = \sqrt{m_1^+} \frac{b^+(x, p_1)b^+(s, p_1)}{\sqrt{2(\lambda_+ - \lambda)}},$$

where $m_1^+$ is defined by (3.9) and

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), \mathbf{x}, \mathbf{s} \in \mathbb{R}^2} \frac{|\tilde{K}^+(x, s, \lambda)|}{1 + |x - s|} < \infty; \tag{6.6}$$

(b) for $d = 2$

$$F^+(x, s, \lambda) = \frac{1}{2\pi} \ln \left( \frac{1}{\lambda_+ - \lambda} \right) \sum_{k=1}^{n_+} \sqrt{m_k^+ b^+_k(x) b^+_k(s)},$$

where $m_k^+$ is defined by (3.7), (3.8), $b^+_k(x)$ is the Bloch function, corresponding to the dispersion function $\lambda^+(\mathbf{p})$ and the quasi-momentum $\mathbf{p} = \mathbf{p}_k^+$, i.e. $b^+_k(x) = b^+(\mathbf{p}_k^+)$; furthermore

$$\sup_{\lambda \in (\lambda_+ - \delta, \lambda_+), \mathbf{x}, \mathbf{s} \in \mathbb{R}^2} \frac{|\tilde{K}^+(x, s, \lambda)|}{1 + \ln(1 + |x - s|)} < \infty.$$

The claims analogous to above ones are valid with $\lambda_-, \lambda^- (\mathbf{p}), n_-, \mathbf{p}_k^-, b^-_k(x), m^-_k$ and $(\lambda_-, \lambda_- + \delta)$ instead of, respectively, $\lambda_+, \lambda^+ (\mathbf{p}), n_+, \mathbf{p}_k^+, b^+_k(x), m^+_k$ and $(\lambda_+ - \delta, \lambda_+)$.  

6.2 Auxiliary claims

For the proof of the propositions, formulated above, we need some auxiliary claims.
6.2.1 Estimates of some integrals

Lemma 6.3. If \( \rho > 0, \alpha > 1, \mu > 0, \beta \in \mathbb{R}, n \in \{1, 2\} \) and \( \phi : [0, \infty) \to \mathbb{R} \) is a continuous function such that \( \tilde{\phi} = \sup_{u \in [0, \rho]} \frac{\phi(u)}{u^2} < \infty \), then the following estimates are valid:

\[
I_1 = \int_0^\rho \frac{u^{n-1} |\exp(i\phi(u)) - 1|}{u^2 + \mu} \, du \leq \quad (6.7)
\]

\[
\begin{cases}
\tilde{\phi} + 2 \rho^{-1} (1 - \alpha^{-1}) & \text{for } n = 1, \\
\tilde{\phi} \rho^2 / 2 + 2 \ln \alpha & \text{for } n = 2;
\end{cases}
\]

\[
I_2 = \int_0^\rho \frac{u^{n-1} (1 - \cos(\beta u))}{u^2 + \mu} \, du \leq \quad (6.8)
\]

\[
\begin{cases}
\beta \rho^2 / 2 + 2 \rho^{-1} (1 - \alpha^{-1}) & \text{for } n = 1, \\
\beta \rho^2 / 4 + 2 \ln \alpha & \text{for } n = 2;
\end{cases}
\]

\[
I_3 = \left| \int_0^\rho \frac{u^n \sin(\beta u)}{u^2 + \mu} \, du \right| \leq \left\{ \begin{array}{ll}
|\beta| \rho + \ln \alpha & \text{for } n = 1, \\
|\beta| \rho^2 / 2 + \rho (\alpha - 1) & \text{for } n = 2. \end{array} \right. \quad (6.9)
\]

Proof. We have for \( u \in [0, \rho] \):

\[
|\exp(i\phi(u)) - 1| = \left| \int_0^1 \partial_\tau \exp(i \tau u) \right| = \left| \int_0^1 \exp(i \tau \phi(u)) \, d\tau \right| \leq \tilde{\phi} u^2,
\]

hence

\[
I_1 = \int_0^\rho \frac{u^{n-1} |\exp(i\phi(u)) - 1|}{u^2 + \mu} \, du + \int_\rho^\rho \frac{u^{n-1} |\exp(i\phi(u)) - 1|}{u^2 + \mu} \, du \leq \\
\tilde{\phi} \int_0^\rho u^n \, du + 2 \int_\rho^\rho \alpha u^{n-3} \, du.
\]

Therefore, (6.7) is valid. Estimates (6.8) and (6.9) are proved analogously. \( \square \)

6.2.2 Some geometric claims

Assume that the edge \( \lambda^+ (\lambda^-) \) of the gap \( \lambda^- (\lambda^+) \) of the spectrum of the unperturbed operator \( H_0 \) satisfies condition (B) of Section 5.1. Taking into account that \( F^+ (F^-) \) is the set of all minimum (maximum) points of the dispersion function \( \lambda^+ (\lambda^-) \) and using the Morse-Bott lemma ([Ban-Hur]), we obtain that for any \( k \in \{1, 2, \ldots, n_+\} \) \( (k \in \{1, 2, \ldots, n_-\}) \) and \( p_\ast \in F_k^+ (p_\ast \in F_k^-) \) on the torus \( \mathbb{T}_d \) there exists a smooth chart \( (U, \phi) (p_\ast \in U) \) such that \( \phi : U \to \mathbb{R}^{d_k^+} \times \mathbb{R}^{d_k^-} \) \( (\phi : U \to \mathbb{R}^{d_k^+} \times \mathbb{R}^{d_k^-} \) and

(a) \( \phi(p_\ast) = 0; \)

(b) \( \phi(U \cap F_k^+) = \{(x, y) \in \mathbb{R}^{d_k^+} \times \mathbb{R}^{d_k^-} \mid y = 0\} \) \( \phi(U \cap F_k^-) = \{(x, y) \in \mathbb{R}^{d_k^+} \times \mathbb{R}^{d_k^-} \mid y = 0\}; \)

(c) \( (\lambda^+ \circ \phi^{-1})(x, y) = \sum_{l=1}^{d_k^+} y_l^2 + \lambda_+ \) \( ((\lambda^- \circ \phi^{-1})(x, y) = -\sum_{l=1}^{d_k^-} y_l^2 + \lambda_-). \)

The chart \( (U, \phi) \) is called the reducing chart at the point \( p_\ast \in F_k^+ (p_\ast \in F_k^-). \)
Lemma 6.4. Let \((U, \phi)\) be a reducing chart at a point \(p_\ast \in F^+_k (p_\ast \in F^-_k)\) defined above and \(\Phi = P \circ \phi\), where \(P : \mathbb{R}^d \to \mathbb{R}^{d-d_u^+} (P : \mathbb{R}^d \to \mathbb{R}^{d-d_u^-})\) is the orthogonal projection on the subspace \(\mathbb{R}^{d-d_u^+} (\mathbb{R}^{d-d_u^-})\) of \(\mathbb{R}^d\). Then for any \(p \in F^+_k \cap U (p \in F^-_k \cap U)\) the normal Jacobian of \(\Phi\) \(N \mathbb{J} \Phi(p) = \det (\Phi'(p)|_{Np})\) is equal to \(\sqrt{\frac{NJ \text{Hes}_p(\lambda^+)}{2^d-d_u^+}} (\sqrt{\frac{NJ \text{Hes}_p(\lambda^-)}{2^d-d_u^-}})\). Hence, in particular, for any \(p \in F^+_k \cap U (p \in F^-_k \cap U)\) \(\text{Im}(\Phi'(p)) = \mathbb{R}^{d-d_u^+} (\text{Im}(\Phi'(p)) = \mathbb{R}^{d-d_u^-})\) and the mapping \(\Phi'(p)|_{Np} : Np \to \mathbb{R}^{d-d_u^+} (\Phi'(p)|_{Np} : Np \to \mathbb{R}^{d-d_u^-})\) is a linear isomorphism. Recall that \(NJ \text{Hes}_p(\lambda^+) (NJ \text{Hes}_p(\lambda^-))\) is the normal Hessian of \(\lambda^+ (\lambda^-)\) at the point \(p\) defined by (3.13), and \(\text{Hes}_p(\lambda^+) (\text{Hes}_p(\lambda^-))\) is the Hessian operator of \(\lambda^+ (\lambda^-)\) at the point \(p\), defined by (3.13).

**Proof.** We shall prove the claim only for the function \(\lambda^+(p)\), because for \(\lambda^-(p)\) it is proved analogously. In view of condition (c) for the reducing chart, for \(p \in F^+_k \cap U\) and \(y = \phi(p) \lambda^+ \circ \phi^{-1}(y) = Py \cdot y\). Then for \(v, w \in \mathbb{R}^d\)

\[
\begin{align*}
  d(\lambda^+ \circ \phi^{-1}(y))[v] &= d\lambda^+(\phi^{-1}(y))[d(\phi^{-1}(y))[v]] = 2v \cdot Py, \\
  d^2(\lambda^+ \circ \phi^{-1}(y))[v, w] &= d^2\lambda^+(\phi^{-1}(y))[d(\phi^{-1}(y))[v], d(\phi^{-1}(y))[w]] + \\
  &d\lambda^+(\phi^{-1}(y))[d^2(\phi^{-1}(y))[v, w]] = 2Pv \cdot w
\end{align*}
\]

(we identify the tangent and the second tangent bundles over \(\mathbb{R}^d\) and the flat torus \(T^d\) with \(\mathbb{R}^2 \times \mathbb{R}^d\) and \(\mathbb{R}^d \times \mathbb{R}^2 \times \mathbb{R}^d\) respectively). Since \(p = \phi^{-1}(y)\) is a minimum point of the function \(\lambda^+, d\lambda^+(\phi^{-1}(y)) = 0\). Hence we get, in view of definition (3.13) of the Hessian operator: \(\text{Hes}_p(\lambda^+)|\) \(d(\phi^{-1}(y))[v], d(\phi^{-1}(y))[w] = 2Pv \cdot w\), that is

\[
\forall s, t \in \mathbb{R}^d : \quad \text{Hes}_p(\lambda^+)s \cdot t = 2d\Phi(p)[s] \cdot d\Phi(p)[t]. \quad (6.10)
\]

In view of condition (b) for the reducing chart, \(\Phi(F^+_k \cap U) = \{0\}\), hence for any \(p \in F^+_k \cap U T_p(F^+_k) \subseteq \ker(\Phi'(p))\). The last fact and (6.10) imply that \(T_p(F^+_k) \subseteq \ker(\text{Hes}_p(\lambda^+))\). Hence since the operator \(\text{Hes}_p(\lambda^+)\) is self-adjoint, it maps the subspace \(N_p^+\) into itself. Then by (6.10) \(\text{Hes}_p(\lambda^+)|_{N_p^+} = 2(\Phi'(p)|_{N_p})^* (\Phi'(p)|_{N_p})\). The last equality implies the desired claim.

**Lemma 6.5.** Let \((U, \phi)\) be a reducing chart at a point \(p_\ast \in F^+_k (p_\ast \in F^-_k)\) defined above and \(\Phi\) is the same mapping as in Lemma 6.4. Consider the differential equation in \(U\):

\[
\frac{dP}{dt} = (\Phi'(p)|_{Np})^{-1} y, \quad y \in \mathbb{R}^{d-d_u^+} (y \in \mathbb{R}^{d-d_u^-})\). \quad (6.11)
\]

Let \(p(t, p_0, y)\) be the flow of this equation, that is the solution of it satisfying the initial condition \(p(0, p_0, y) = p_0\). Then it is possible to restrict the neighborhood \(U\) of \(p_\ast\) such that
(i) for some $r > 0$, any $t \in [0, r]$ and any $y$ belonging to the unit sphere $S^{d - d_k - 1} \subset \mathbb{R}^{d - d_k}$ ($S^{d - d_k - 1} \subset \mathbb{R}^{d - d_k}$) the set $F_y = \Phi^{-1}(ty)$ is a $d_k^+$ ($d_k^+$)-dimensional smooth submanifold of $T^d$ (in particular, $F_0 = F_k^+ \cap U$ ($F_0 = F_k^- \cap U$)), $p(t, F_k^+ \cap U, y) = F_{ty}$ $(p(t, F_k^+ \cap U, y) = F_{ty})$ and the mapping $p(t, \cdot, y) : F_k^+ \cap U \to F_y$ $(p(t, \cdot, y) : F_k^- \cap U \to F_y)$ is a diffeomorphism;

(ii) for $q \in F_k^+ \cap U$ ($q \in F_k^- \cap U$) the functions $\partial_t p(t, q, y)|_{t=0}$ and $\partial_t T_j p(t, q, y)|_{t=0}$ are odd w.r.t. $y$. Here $T_j p(t, q, y)$ is the tangential Jacobian of the mapping $p(t, \cdot, y)$, that is

$$
T_j p(t, q, y) = \left( \partial q p(t, q, y)|_{T_0(F_k^+)} \right) \left( T_j p(t, q, y) = \partial q p(t, q, y)|_{T_0(F_k^-)} \right).
$$

Proof. Consider only the case where $p_* \in F_k^+$, because the case $p_* \in F_k^-$ is treated analogously. Using the fact that $\Phi(F_k^+ \cap U) = \{0\}$, we see that if $q \in F_k^+ \cap U$, then for any $y \in S^{d - d_k - 1}$, $t > 0$ the solution $p(\cdot, q, y)$ of equation (6.11) exists in the interval $[0, t]$, then $\Phi(p(\tau, q, y)) = y$ $(\tau \in [0, t])$. Integrating both sides of the last equality by $\tau$ over $[0, t]$, we get: $\Phi(p(t, q, y)) = ty$, that is $p(t, q, y) = \Phi^{-1}(ty)$. Using the theorem on the local existence and uniqueness of solution of the Cauchy problem for a dynamical system in a Banach space, we can restrict the neighborhood $U$ of $p_*$ such that claim (i) is valid.

Let us prove claim (ii). The fact that the function $\partial_q p(t, q, y)|_{t=0}$ is odd w.r.t. $y$ for $q \in F_k^+ \cap U$ follows immediately from (6.11) and the equality $p(0, q, y) = q$. As it is known, the derivative $Y(t) = \partial_q p(t, q, y)$ satisfies the following linear equation, which is the linearization of equation (6.11) at its solution $p(t) = p(t, q, y)$:

$$
\frac{dY}{dt} = (A(t)Y)y,
$$

where $A(t) = d_p(\Phi'(p(t)))|_{\mathbb{R}^n(0)}$. Furthermore, $Y(0) = I$. In view of claim (i), for any $q \in F_k^+$ and $t \in [0, r]$ the operator $Y(t)|_{T_0(F_k^+)}$ realizes a linear isomorphism between $T_0(F_k^+)$ and $T_{p(t)}(F_y)$. Let $dv = dp_1 \wedge dp_2 \wedge \cdots \wedge dp_d$ be the volume form on the flat torus $T^d$. As it is known, the volume form $dF_{ty}$ on the submanifold $F_{ty}$ has the form $dF_{ty}(p) = dv|_{T_0(F_y)}$ $(p \in F_y)$. Let $d^* F_{ty}$ be the pullback of the form $dF_{ty}$ w.r.t. the mapping $p(t, \cdot, y)|_{T_0(F_k^+) \cap U}$. Taking into account equation (6.12) and the equalities $F_0 = F_k^+ \cap U$, $Y(0) = I$ and $p(0) = p(0, q, y) = q$ $(q \in F_k^+ \cap U)$, we have for $s = (s_1, s_2, \ldots, s_d) \in T_q(F_k^+)$:

$$
\frac{d}{dt} d^* F_{ty}[s]|_{t=0} = \sum_{i=1}^d (Y(t)s) s_1 \wedge (Y(t)s) s_2 \wedge \ldots \wedge (Y(t)s) s_d = \sum_{i=1}^d s_1 \wedge s_2 \wedge \cdots \wedge s_{d-1} \wedge ((A(0)s) \wedge (A(0)s) s_1 \wedge s_{d-1} \wedge \cdots \wedge s_d.
$$
The last representation implies that \( \frac{d}{dt} d^* F_{ty} [s] \big|_{t=0} = -\frac{d}{dt} d^* F_{-ty} [s] \big|_{t=0} \). On the other hand, it is known that \( d^* F_{ty} (q) = TJp(t,q,y) dF_0(q) \), where \( dF_0 = dF_{ty} \big|_{t=0} \) is the volume form on \( F_t^+ \cap U \). These circumstances imply that the function \( \frac{d}{dt} TJp(t,q,y) \big|_{t=0} \) is odd w.r.t. \( y \). Claim (ii) is proven. \( \square \)

6.3 Proof of claim (i) of Proposition 6.1

Proof. Let us choose a suitable neighborhood of each of the submanifolds \( F_k^+ (k \in \{1, 2, \ldots, n_+\}) \). Let \( (U_{p_1}, \phi) \) be a reducing chart at a point \( p_1 \in F_k^+ \). Since \( F_k^+ \) are closed disjoint subsets of the torus \( T^d \), we can choose it such that \( U_{p_1} \cap F_k^+ = \emptyset \) for any \( i \neq k \). Let \( \Phi : U_{p_1} \to \mathbb{R}^{d-d_k^+} \) be the same as in Lemmas 6.4 and 6.5. Then by Lemma 6.5 it is possible to restrict the neighborhood \( U_{p_1} \) of \( p_1 \) such that for the flow \( \Phi(t,q,y) \) of the differential equation (6.11) claim (i) of this lemma is valid. Recall that the dispersion function \( \lambda^+(p) \) branching from the edge \( \lambda_+ \) of the gap \( \lambda_- \) in \( \sigma(H_0) \) has the form \( \lambda^+(p) = \lambda_{j+1}(p) \) for some \( j \geq 0 \). Like in Section 3.1, we can restrict the neighborhood \( U_{p_1} \) of \( p_1 \) taking into account condition (A)-(a) and Corollary A.2 such that for any \( p \in U_{p_1} \), \( \lambda^+(p) \) is a simple eigenvalue of \( \hat{H}(p) \), i.e.

\[
\forall p \in U_{p_1} : \quad \lambda^+(p) < \lambda_{j+2}(p),
\]

(6.13)

the function \( \lambda^+(p) \) is real-analytic in \( U_{p_1} \) and furthermore, the mapping \( p \to \tilde{Q}^+(\cdot, p) \in C(\Omega \times \Omega) \) is real-analytic in \( U_{p_1} \). Recall that \( \tilde{Q}^+(x,s,p) \) is the eigenkernel of \( \tilde{H}(p) \), corresponding to \( \lambda^+(p) \). Since each \( F_k^+ \) is compact, then it is possible to select a finite subcovering \( \{ U_{k,i} \}_{i=1}^{L_k} \) from its open covering \( \{ U_{p_1} \}_{p_1 \in F_k^+} \). Consider the neighborhood \( U_{k,i} = \bigcup_{i=1}^{L_k} U_{k,i} \) of \( F_k^+ \) for each \( k \in \{1, 2, \ldots, n_+\} \). As it is clear, these neighborhoods are disjoint. Denote \( S_+ = \bigcup_{k=1}^{n_+} U_k \).

As it is known (\[\text{Gel, Kuch, Zl}\]), the operator \( \tilde{U} = \exp(-ip \cdot x) \cdot U \), where \( U \) is defined by (5.23), maps the space \( L_2(\mathbb{R}^d) \) on the direct integral \( \int_{\mathbb{R}^d} L_2(\mathbb{R}^d/\Gamma) \, dp \). \( L_2(\mathbb{R}^d/\Gamma) \) and \( L_2(\mathbb{T}^d) \) realizes a unitary equivalence between the operator \( H_0 \) and the direct integral \( \tilde{H} = \int_{\mathbb{T}^d} \hat{H}(p) \, dp \) of the operators \( \hat{H}(p) \). Recall that the operator \( \hat{H}(p) \) is defined by (3.4), (3.5) and its domain is \( W_2^2(\mathbb{R}^d/\Gamma) \). Formula (5.23) implies that the operator inverse to \( \tilde{U} \), has the form: for any \( \phi \in L_2(\mathbb{R}^d/\Gamma) \otimes L_2(\mathbb{T}^d) \)

\[
(\tilde{U}^{-1} \phi)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \exp(ip \cdot x) \phi(x, p) \, dp.
\]

(6.14)

Let us take \( f \in L_2(\mathbb{R}^d) \) and denote \( \tilde{f} = U f \). Since \( R_{\lambda}(H_0) = \tilde{U}^{-1} R_{\lambda}(\tilde{H}) \tilde{U} \) for any \( \lambda \in \sigma(H_0) \), we get using (6.14):

\[
(R_{\lambda}(H_0) f)(x) = \left( \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \exp(ip \cdot x) (R_{\lambda}(\tilde{H}(p)) \tilde{f}(\cdot, p))(x) \, dp \right)
\]

\[
(R^*_p f)(x) + (\Theta^*_p f)(x),
\]

(6.15)
where

\[
(R_\lambda^+ f)(x) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{S_+} \exp(ip \cdot x)(R_\lambda(\tilde{H}(p))\tilde{f}(\cdot, p))(x) \, dp = \\
(\tilde{U}^{-1} \, R_\lambda(\tilde{H}(p))\chi_{_{S_+}}(p)\tilde{f}(\cdot, p))(x), \quad (6.16)
\]

\[
(\Theta_\lambda^+ f)(x) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{T^d \setminus S_+} \exp(ip \cdot x)(R_\lambda(\tilde{H}(p))\tilde{f}(\cdot, p))(x) \, dp = \\
(\tilde{U}^{-1} \, R_\lambda(\tilde{H}(p))\chi_{_{T^d \setminus S_+}}(p)\tilde{f}(\cdot, p))(x). \quad (6.17)
\]

Here we denote by \(\chi_A\) the characteristic function for a set \(A \subset T^d\). Since \(F^+ = \bigcup_{n=1}^\infty F^+_n = (\lambda^+)^{-1}(\lambda)\) and \(\lambda_+ = \min_{p \in T^d} \lambda^+_+(p)\), then \(\lambda^+_+(p) > \lambda_+\) for any \(p \notin S_+\). Hence there exists \(\delta > 0\) such that for any \(\lambda \in ((\lambda_+ + \lambda_-)/2, \lambda_+)\) and \(p \in T^d \setminus S_+\) \(dist(\lambda, \sigma(\tilde{H}(p))) \geq \delta\), and hence \(||R_\lambda(\tilde{H}(p))||_0 \leq \frac{1}{\delta}\). Denote by \(||\cdot||\) the norm of elements in \(L_2(\mathbb{R}^d/\Gamma) \otimes L_2(T^d)\). Then using the isometry of the operator \(\tilde{U}\), we get from (6.17):

\[
||\Theta_\lambda^+ f||^2 = ||R_\lambda(\tilde{H}(p))\chi_{_{T^d \setminus S_+}}(p)\tilde{f}(\cdot, p)||^2 = \\
\frac{1}{\delta^2} \int_{T^d} ||\tilde{f}(\cdot, p)||^2 \, dp \leq \\
\frac{1}{\delta^2} \int_{T^d} ||\tilde{f}(\cdot, p)||^2 \, dp = \frac{1}{\delta^2} ||\tilde{U} f||^2 = \frac{1}{\delta^2} ||f||^2.
\]

Recall that we denote by \(||\cdot||_2\) and \((\cdot, \cdot)_2\) the norm and the inner product in the space \(\mathcal{H}_0 = L_2(\mathbb{R}^d/\Gamma)\) (see (3.3)). Thus, we get that

\[
\forall \lambda \in ((\lambda_+ + \lambda_-)/2, \lambda_+) : \quad ||\Theta_\lambda^+|| \leq \frac{1}{\delta}. \quad (6.18)
\]

Now consider the case where \(p \in S_+\). As it is known, the resolvent \(R_\lambda(\tilde{H}(p))\) can be represented in the form:

\[
R_\lambda(\tilde{H}(p))g = \int_\Omega \frac{\tilde{Q}(x, s, p)\tilde{f}(s, p) \, ds}{\lambda^+(p) - \lambda} + \tilde{R}(\lambda, p)g \quad (g \in L_2(\mathbb{R}^d/\Gamma)), \quad (6.19)
\]

where

\[
\tilde{R}(\lambda, p) = \sum_{l \in \mathbb{N} \setminus \{j+1\}} \frac{(\cdot, e_l(\cdot, p))_2 e_l(\cdot, p)}{\lambda_l(p) - \lambda}. \quad (6.20)
\]

Recall that \(\{e_l(x, p)\}\) is the orthonormal sequence of the eigenfunctions of the operator \(H(p)\). Then by (6.16), (6.19), \(R_\lambda^+ = K(\lambda) + \Theta_\lambda^+\), where

\[
(K(\lambda)f)(x) = \\
\frac{1}{(2\pi)^\frac{d}{2}} \sum_{k=1}^{n_+} \int_{U_k} \exp(ip \cdot x) \int_\Omega \frac{\tilde{Q}(x, s, p)\tilde{f}(s, p) \, ds}{\lambda^+(p) - \lambda} \, dp,
\]

46
In view of (6.13), there exists \( \delta_1 > 0 \) such that for any \( p \in S_+ \) and \( \lambda \in ((\lambda_+ + \lambda_-)/2, \lambda_+) \): \( \text{dist}(\lambda, \sigma(\tilde{R}(p))) \setminus \{\lambda_{j+1}(p)\} > \delta_1 \), hence in view of (6.20), \( \|\tilde{R}(\lambda, p)\|_0 \leq \frac{1}{\delta_1} \). Then in the same manner as estimate (6.18), we obtain the following estimate:

\[
\forall \lambda \in ((\lambda_+ + \lambda_-)/2, \lambda_+) : \quad \|\tilde{\Theta}_\lambda\| \leq \frac{1}{\delta_1}. \tag{6.22}
\]

Taking \( f \in L_{2,0}(\mathbb{R}^d) \) and using the \( \Gamma \)-periodicity of \( \tilde{Q}^+(x, s, p) \) by \( s \) and the fact that \( \hat{f}(x, p) = \exp(-ip \cdot x)\hat{f}(x, p) \) (with \( \hat{f}(x, p) \) defined by (6.23)), we obtain from (6.21):

\[
(K(\lambda)f)(x) = \frac{1}{(2\pi)^d} \sum_{k=1}^{\infty} \int_{U_k} \exp(ip \cdot x) \times \left( \frac{1}{\lambda^+(p)} - \lambda \sum_{l \in \Gamma} \int_{\Omega(l)} \exp(-ip \cdot s)\tilde{Q}^+(x, s, p)f(s)ds \right) dp.
\]

Since the inner sum is finite, we obtain after a permutation of sums and integrals:

\[
(K(\lambda)f)(x) = \int_{\mathbb{R}^d} K(x, s, \lambda)f(s)ds, \quad \text{where} \quad K(x, s, \lambda) = \sum_{k=1}^{\infty} K_k^+(x, s, \lambda),
\]

\[
K_k^+(x, s, \lambda) = \frac{1}{(2\pi)^d} \int_{U_k} \exp(ip \cdot (x - s))\frac{\tilde{Q}^+(x, s, p)}{\lambda^+(p) - \lambda} dp. \tag{6.23}
\]

We see that each \( K_k^+(x, s, \lambda) \) is continuous in \( \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_-, \lambda_+) \). From (6.15) and (6.16) we obtain the representation (6.1) with \( \Theta(\lambda) = \Theta_\lambda^+ + \tilde{\Theta}_\lambda^+ \). In view of (6.18) and (6.22), \( \Theta(\lambda) \) satisfies condition (6.2). Claim (i) of Proposition 6.1 is proven.

\[\square\]

### 6.4 Proof of claim (ii) of Proposition 6.1

**Proof.** For each \( k \in \{1, 2, \ldots, n_+\} \) consider a decomposition of the unit

\[
\{\phi_{k,l}(p)\}_{l=1}^{L_k}, \quad (\phi_{k,l} \in C^\infty(U_k)),
\]

corresponding to the covering \( \{U_{k,l}\}_{l=1}^{L_k} \) of \( U_k \), constructed above. Then we have the representation for the function \( K_k^+(x, s, \lambda) \), defined by (6.23):

\[
K_k^+(x, s, \lambda) = \sum_{l=1}^{L_k} K_{k,l}(x, s, \lambda), \quad \tag{6.24}
\]

47
where

\[
K_{k,l}(x,s,\lambda) = \frac{1}{(2\pi)^d} \int_{U_{k,l}} \phi_{k,l}(p) \exp(i p \cdot (x - s)) \frac{\tilde{Q}^+(x,s,p)}{\lambda_{j+1}(p) - \lambda} \, dp.
\]

Let \(\Phi_{k,l}\) be the mapping \(\Phi\) defined in Lemma 6.4 and corresponding to the neighborhood \(U_{k,l}\). Like in Lemma 6.5 consider the differential equation in \(U_{k,l}\):

\[
\frac{d}{dt} = \left(\Phi_{k,l}^{-1}(p) \right) N_F y.
\]

Let \(p_{k,l}(t, p_0, y)\) be the flow of this equation. Taking into account definition of the reducing chart and using claim (i) of Lemma 6.6 and the coarea formula, we have:

\[
K_{k,l}(x,s,\lambda) = \frac{1}{(2\pi)^d} \int_{B_{k,l}} \frac{dy}{|y|^2 + \lambda_+ - \lambda} \int_{F_y} \phi_{k,l}(p) \exp(i p \cdot (x - s)) \times \frac{\tilde{Q}^+(x,s,p)}{N_F \Phi_{k,l}^{-1}(p)} \, dF_y(p),
\]

where \(B_{k,l} = \{ y \in \mathbb{R}^{d-d^+_k} | |y| \leq r_{k,l} \} \) \((r_{k,l} > 0)\), \(dF_y(p)(\cdot)\) is the volume form of the submanifold \(F_y = \Phi_{k,l}^{-1}(y)\) \((y \in B_{k,l})\), and \(N_F \Phi_{k,l}^{-1}(p)\) is the normal Jacobian of \(\Phi_{k,l}\). Since by claim (i) of Lemma 6.5 for any \(t \in [0, r_{k,l}]\) and \(y \in S^{d-d^+_k-1}\), the mapping \(p_{k,l}(t, \cdot, y)\) is a diffeomorphism between \(U_{k,l}^0 = U_{k,l} \cap F_{k,l}^+\) and \(F_{t,y}\), then (6.28) can be written in the form after the change of the variable \(p = p_{k,l}(t, q, y)\) \((q \in U_{k,l}^0)\):

\[
K_{k,l}(x,s,\lambda) = \frac{1}{(2\pi)^d} \int_{U_{k,l}^0} E_{k,l}(q,x,s,\lambda) \, dF_0(q),
\]

where

\[
E_{k,l}(q,x,s,\lambda) = \int_{S^{d-d^+_k-1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d^+_k-1}}{t^2 + \lambda_+ - \lambda} \phi_{k,l}(p_{k,l}(t, q, y)) \exp(i p_{k,l}(t, q, y) \cdot (x - s)) \times \frac{\tilde{Q}^+(x,q,p_{k,l}(t, q, y))}{N_F \Phi_{k,l}^{-1}(p_{k,l}(t, q, y))} \, dt,
\]

\(dS(\cdot)\) is the volume form of the unit sphere \(S^{d-d^+_k-1}\), \(dF_0(\cdot)\) is the volume form of the submanifold \(U_{k,l}^0\) and \(T_j \Phi_{k,l}(t, q, y)\) is the tangential Jacobian of the mapping \(\Phi_{k,l}(t, \cdot, y): U_{k,l}^0 \to F_{t,y}\). Since for \(d-d^+_k \geq 3\)

\[
\int_0^{r_{k,l}} \frac{t^{d-d^+_k-1}}{t^2 + \lambda_+ - \lambda} \, dt \leq \frac{r_{k,l}^{d-d^+_k-2}}{d-d^+_k-2} < \infty \quad \text{for any } \lambda \in (\lambda_-, \lambda_+),
\]

then by (6.28), (6.27) and (6.24) the property (6.3) is valid. Hence claim (ii) of Proposition 6.1 is proven. \(\square\)
6.5 Proof of claim (iii) of Proposition 6.1

Proof. Assume that \(1 \leq d - d_k^+ \leq 2\). Taking into account the connection (3.16) between the eigenkernels \(Q^+(x,s,p)\) and \(\tilde{Q}^+(x,s,p)\), let us represent the function \(E_{k,l}(q,x,s,\lambda)\), defined by (6.28), in the form:

\[
E_{k,l}(q,x,s,\lambda) = E_{k,l}^{(1)}(q,x,s,\lambda) + E_{k,l}^{(2)}(q,x,s,\lambda) + E_{k,l}^{(3)}(q,x,s,\lambda),
\]

(6.29)

where

\[
E_{k,l}^{(1)}(q,x,s,\lambda) = \phi_{k,l}(q) 2^{(d-d_k^+)/2} Q^+(x,s,q) \times
\]

\[
\sqrt{m^+(q)} \int_{S^{d-d_k^+}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} dt
\]

(6.30)

\[
E_{k,l}^{(2)}(q,x,s,\lambda) = \phi_{k,l}(q) 2^{(d-d_k^+)/2} Q^+(x,s,p) \sqrt{m^+(q)} \tilde{E}_{k,l}^{(2)}(q,x,s,\lambda),
\]

(6.31)

\[
E_{k,l}^{(3)}(q,x,s,\lambda) = \int_{S^{d-d_k^+}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} \times
\]

\[
(\exp(i(p_{k,l}(t,q,y) - q) \cdot (\tilde{x} - \tilde{s})) - 1) dt,
\]

(6.32)

\[
E_{k,l}^{(3)}(q,x,s,\lambda) = \int_{S^{d-d_k^+}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} \times
\]

\[
\exp(i(p_{k,l}(t,q,y) \cdot (x - s)) D_{k,l}(t,q,x,s) dt,
\]

(6.33)

\[
D_{k,l}(t,y,q,x,s) = \phi_{k,l}(p_{k,l}(t,q,y)) \frac{\tilde{Q}^+(x,s,p_{k,l}(t,q,y))TJp_{k,l}(t,q,y)}{NJH_{e8q}(\lambda_j+1)} - \phi_{k,l}(q) 2^{(d-d_k^+)/2} \frac{\tilde{Q}^+(x,s,q)}{\sqrt{NJH_{e8q}(\lambda_j+1)}}.
\]

(6.34)

After simple calculations we have:

\[
\int_0^{r_{k,l}} \frac{t^{d-d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} dt = \begin{cases} \frac{\pi}{2\sqrt{\lambda_+ - \lambda}} + \theta_1(\lambda) \quad \text{for} \quad d - d_k^+ = 1, \\ \frac{1}{2} \ln \left( \frac{1}{\lambda_+ - \lambda} \right) + \theta_2(\lambda) \quad \text{for} \quad d - d_k^+ = 2, \end{cases}
\]

(6.35)

where

\[
\theta_\nu(\lambda) = O(1) \quad \text{for} \quad \lambda \uparrow \lambda_+ \quad (\nu = 1,2).
\]

(6.36)

Then taking into account that \(\{\phi_{k,l}[F_k^+]'\}_{l=1}^{L_k}\) is a decomposition of the unit for the submanifold \(F_k^+\), we get from (6.30), (6.35) and (6.36):

\[
\sum_{l=1}^{L_k} \int_{F_k^+} E_{k,l}^{(1)}(q,x,s,\lambda) dF_0(q) = G_k^+(x,s,\lambda) + \tilde{F}_k^{(1)}(x,s,\lambda),
\]

(6.37)

49
where \( G_k^+(x, s, \lambda) \) is defined by (6.4) for \( d - d_k^+ = 1 \) and by (6.5) for \( d - d_k^+ = 2 \), and the function \( K_k^{(1)}(x, s, \lambda) \) is continuous in \( \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+) \) and

\[
\sup_{(x, s, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} |K_k^{(1)}(x, s, \lambda)| < \infty. \tag{6.38}
\]

Recall that \( \delta \in (0, \lambda_+ - \lambda_-) \).

Let us estimate the function \( \tilde{E}_{k,l}^{(2)}(q, x, s, \lambda) \), defined by (6.31). To this end consider the Taylor representation of the flow of the equation (6.25) in a neighborhood of the point \( t = 0 \):

\[
\begin{align*}
p_{k,l}(t, q, y) &= q = p_{k,l}(t, q, y) - p_{k,l}(0, q, y)t + \partial_t p_{k,l}(0, q, y)t + \\
r_{k,l}(t, q, y),
\end{align*}
\tag{6.39}
\]

where \( r_{k,l}(t, q, y) = \int_0^1 (1 - s) \partial^2_t p_{k,l}(s, q, y) ds \). We have from (6.25):

\[
\partial^2_t p_{k,l}(s, q, y) = \partial_p \left( \left( \Phi_{k,l}^{(1)}(p) \right)^{-1} y \right) \cdot \left( \Phi_{k,l}^{(1)}(p) \right)^{-1} y \big|_{p = p_{k,l}(t, q, y)}.
\]

These circumstances imply that

\[
\tilde{r}_{k,l} = \sup_{(t, q, y) \in [0, r_{k,l}] \times U_{k,l} \times S^{d - d_k^+ - 1}(0)} \frac{|r_{k,l}(t, q, y)|}{t^2} < \infty. \tag{6.40}
\]

Using (6.33), let us represent the function \( \tilde{E}_{k,l}^{(2)}(q, x, s, \lambda) \), defined by (6.32), in the form:

\[
\tilde{E}_{k,l}^{(2)}(q, x, s, \lambda) = \tilde{E}_{k,l}^{(2,1)}(q, x, s, \lambda) + \tilde{E}_{k,l}^{(2,2)}(q, x, s, \lambda), \tag{6.41}
\]

where

\[
\tilde{E}_{k,l}^{(2,1)}(q, x, s, \lambda) =
\int_{S^{d - d_k^+ - 1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d - d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} \left( \exp(i \partial_t p_{k,l}(0, q, y)t \cdot (x - s)) - 1 \right) dt,
\tag{6.42}
\]

and

\[
\tilde{E}_{k,l}^{(2,2)}(q, x, s, \lambda) =
\int_{S^{d - d_k^+ - 1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d - d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} \left( \exp(i \partial_t p_{k,l}(0, q, y)t \cdot (x - s)) - 1 \right) \times
\exp(\partial_t p_{k,l}(0, q, y)t \cdot (x - s)) dt,
\tag{6.43}
\]

Let us estimate \( \tilde{E}_{k,l}^{(2,1)}(q, x, s, \lambda) \). Observe that, in view of (6.25), \( \partial_t p_{k,l}(0, q, y) = -\partial_s p_{k,l}(0, q, -y) \). Then, since the integral of the odd part (w.r. to \( y \)) of the integrand in (6.42) vanishes, we have:

\[
\tilde{E}_{k,l}^{(2,1)}(q, x, s, \lambda) =
\int_{S^{d - d_k^+ - 1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d - d_k^+ - 1}}{t^2 + \lambda_+ - \lambda} \left( \cos(\partial_t p_{k,l}(0, q, y)t \cdot (x - s)) - 1 \right) dt,
\tag{6.44}
\]

50
Deriving the change of the variable \( u = (1 + |x - s|)t \) in the inner integral, we get:

\[
\hat{E}_{k,l}^{(2,1)}(q, x, s, \lambda) = (1 + |x - s|)^{2-(d-d_k^+)} \times \\
\int_{S^{d-d_k^+-1}} dS(y) \int_0^{r_{k,l}(1+|x-s|)} u^{d-d_k^+-1} \\
\left( \cos (\partial_t p_{k,l}(0, q, y) u \cdot \frac{x - s}{1 + |x-s|}) - 1 \right) du.
\]

Using estimate (6.38) of Lemma 6.3 with \( \rho = r_{k,l}, \alpha = 1 + |x - s| \) and \( \beta = \partial_t p_{k,l}(0, q, y) \cdot \frac{x - s}{1 + |x-s|} \), we get:

for \( d - d_k^+ = 1 \):

\[
\sup_{(q, x, s, \lambda) \in U_{k,l}^0 \times \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+, \lambda_+)} \frac{|\hat{E}_{k,l}^{(2,1)}(q, x, s, \lambda)|}{1 + |x - s|} < \infty
\]

and

for \( d - d_k^+ = 2 \):

\[
\sup_{(q, x, s, \lambda) \in U_{k,l}^0 \times \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+, \lambda_+)} \frac{|\hat{E}_{k,l}^{(2,1)}(q, x, s, \lambda)|}{1 + \ln(1 + |x - s|)} < \infty.
\]

Now let us estimate the function \( \hat{E}_{k,l}^{(2,2)}(q, x, s, \lambda) \), defined by (6.43). To this end let us derive the change of the variable \( u = \sqrt{1 + |x - s|}t \) in the inner integral of (6.43):

\[
\left| \int_{0}^{r_{k,l}} \frac{t^{d-d_k^+-1}}{t^{2} + \lambda_+ - \lambda} \left( \exp(i r_{k,l}(t, q, y) \cdot (x - s)) - 1 \right) \times \\
\exp(i \partial_t p_{k,l}(0, q, y) t \cdot (x - s)) \right| dt \leq \\
(1 + |x - s|)^{2-(d-d_k^+)} \int_0^{r_{k,l}} \frac{u^{d-d_k^+-1}}{u^{2} + (\lambda_+ - \lambda)(1 + |x - s|)} \times \\
\left| \exp \left( i r_{k,l} \left( \frac{u}{\sqrt{1 + |x - s|}}, q, y \right) \cdot (x - s) \right) - 1 \right| du
\]

Then using (6.40) and estimate (6.7) of Lemma 6.3 with \( \rho = r_{k,l}, \alpha = \sqrt{1 + |x - s|} \) and \( \phi(u) = r_{k,l} \left( \frac{u}{\sqrt{1 + |x - s|}}, q, y \right) \cdot (x - s) \), we obtain:

for \( d - d_k^+ = 1 \):

\[
\sup_{(q, x, s, \lambda) \in U_{k,l}^0 \times \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+, \lambda_+)} \frac{|\hat{E}_{k,l}^{(2,2)}(q, x, s, \lambda)|}{\sqrt{1 + |x - s|}} < \infty
\]

51
and

\[ \text{for } d - d_k^+ = 2 : \]

\[ \sup_{(q,x,s,\lambda) \in U_{0,k}^0 \times \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} \frac{|E_{k,l}^{(2,2)}(q,x,s,\lambda)|}{1 + \ln(1 + |x - s|)} < \infty. \tag{6.47} \]

Let us estimate the function \( E_{k,l}^{(3)}(q,x,s,\lambda) \), defined by \( \text{(6.33)} \). To this end consider the Taylor representation of the function \( D_{k,l}(t,y,q,x,s) \), defined by \( \text{(6.34)} \) in a neighborhood of \( t = 0 \), taking into account that in view of Lemma \( a \) and the equalities \( p_{k,l}(0,q,y) = q \) and \( T_0(p_{k,l}(0,q,y)) = 1 \), the equality \( D_{k,l}(0,y,q,x,s) = 0 \) is valid. We have: \( D_{k,l}(t,y,q,x,s) = g_{k,l}(y,q,x,s) t + \zeta_{k,l}(t,y,q,x,s) \), where \( g(y,q,x,s) = \partial_t D_{k,l}(0,y,q,x,s) \). We see from \( \text{(6.34)} \) that the functions \( g_{k,l}(y,q,x,s) \) and \( \zeta_{k,l}(t,y,q,x,s) \) are smooth and bounded in \([0,r_{k,l}] \times U_{0,k} \times S^{d-d_k^*-1} \times \mathbb{R}_+^d \times \mathbb{R}_+^d \). In the same manner as \( \text{(6.40)} \), we obtain the following property of \( \zeta_{k,l}(t,y,q,x,s) \):

\[ \tilde{\zeta}_{k,l} = \sup_{(t,q,y,x) \in [0,r_{k,l}] \times U_{0,k} \times S^{d-d_k^*-1} \times \mathbb{R}_+^d \times \mathbb{R}_+^d} \frac{|\zeta_{k,l}(t,q,y,x,s)|}{t^2} < \infty. \tag{6.48} \]

Using the above representation and \( \text{(6.33)} \), let us represent the function \( E_{k,l}^{(3)}(q,x,s,\lambda) \) in the form:

\[ E_{k,l}^{(3)}(q,x,s,\lambda) = E_{k,l}^{(3,1)}(q,x,s,\lambda) + \exp(iq \cdot (x - s)) \times \]

\[ \left( E_{k,l}^{(3,2)}(q,x,s,\lambda) + E_{k,l}^{(3,3)}(q,x,s,\lambda) \right), \tag{6.49} \]

where

\[ E_{k,l}^{(3,1)}(q,x,s,\lambda) = \int_{S^{d-d_k^*-1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^*-1}}{t^2 + \lambda_+ - \lambda} \times \exp(iq \cdot (x - s)) \zeta_{k,l}(t,y,q,x,s) dt, \tag{6.50} \]

\[ E_{k,l}^{(3,2)}(q,x,s,\lambda) = \int_{S^{d-d_k^*-1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^+}}{t^2 + \lambda_+ - \lambda} \times \exp(it\partial_q p_{k,l}(0,q,y) \cdot (x - s)) g_{k,l}(y,q,x,s) dt, \tag{6.51} \]

\[ E_{k,l}^{(3,3)}(q,x,s,\lambda) = \int_{S^{d-d_k^*-1}} dS(y) \int_0^{r_{k,l}} \frac{t^{d-d_k^+}}{t^2 + \lambda_+ - \lambda} \times \exp(it\partial_q p_{k,l}(0,q,y) \cdot (x - s)) g_{k,l}(y,q,x,s) \times \]

\[ \left( \exp(it\partial_q p_{k,l}(0,q,y) \cdot (x - s)) - 1 \right) dt. \tag{6.52} \]

Using \( \text{(6.48)} \), we obtain from \( \text{(6.50)} \):

\[ \text{for } d - d_k^+ \in \{1,2\} : \]

\[ \sup_{(q,x,s,\lambda) \in U_{0,k}^0 \times \mathbb{R}^d \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} |E_{k,l}^{(3,1)}(q,x,s,\lambda)| < \infty. \tag{6.53} \]
Now we turn to the estimation of the function $E_{k,l}^{(3,2)}(q, x, s, \lambda)$. Deriving the change of the variable $u = (1 + |x - s|)t$ in the inner integral of (6.51) and taking into account that by claim (ii) of Lemma 6.3, the functions $\partial_l p_{k,l}(0, q, y)$ and $g_{k,l}(y, q, x, s) = \partial_l D_{k,l}(0, y, q, x, s)$ are odd w.r.t. $y$, we get:

$$E_{k,l}^{(3,2)}(q, x, s, \lambda) = (1 + |x - s|)^{1 - (d - d_k^+)} \int_{S^{d - d_k^+ + 1}} g_{k,l}(y, q, x, s) dS(y) \times$$

$$\int_0^{r_k,(1 + |x - s|)^{d - d_k^+}} u^{d - d_k^+} \sin \left( u \partial_l p_{k,l}(0, q, y) \cdot \frac{x - s}{1 + |x - s|} \right) \frac{u^2 + (\lambda_+ - \lambda)(1 + |x - s|)^2}{du}$$

Using estimate (6.3) of Lemma 6.3 with $\rho = r_k, \alpha = 1 + |x - s|$ and $\beta = \partial_l p_{k,l}(0, q, y) \cdot \frac{x - s}{1 + |x - s|}$, we get:

for $d - d_k^+ = 1$:

$$\sup_{(q, x, s, \lambda) \in U_{\rho, \lambda}^0 \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} \frac{|E_{k,l}^{(3,2)}(q, x, s, \lambda)|}{1 + \ln(1 + |x - s|)} < \infty$$

and

for $d - d_k^+ = 2$:

$$\sup_{(q, x, s, \lambda) \in U_{\rho, \lambda}^0 \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} |E_{k,l}^{(3,2)}(q, x, s, \lambda)| < \infty$$

Now we turn to the estimation of the function $E_{k,l}^{(3,3)}(q, x, s, \lambda)$. Deriving the change of the variable $u = \sqrt{1 + |x - s|}t$ in the inner integral of (6.52), we get:

$$|E_{k,l}^{(3,3)}(q, x, s, \lambda)| \leq (1 + |x - s|)^{1 - (d - d_k^+)} \int_{S^{d - d_k^+ + 1}} |g_{k,l}(y, q, x, s)| dS(y) \times$$

$$\int_0^{r_k,\sqrt{1 + |x - s|}} u^{d - d_k^+} \exp \left( \frac{u}{\sqrt{1 + |x - s|}} \cdot \frac{y, q, \cdot (x - s)}{1 - 1} \right) \frac{u^2 + (\lambda_+ - \lambda)(1 + |x - s|)}{du}$$

Using, as above, estimate (6.7) of Lemma 6.3 we obtain:

for $d - d_k^+ = 1$:

$$\sup_{(q, x, s, \lambda) \in U_{\rho, \lambda}^0 \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} \frac{|E_{k,l}^{(3,3)}(q, x, s, \lambda)|}{1 + \ln(1 + |x - s|)} < \infty$$

and

for $d - d_k^+ = 2$:

$$\sup_{(q, x, s, \lambda) \in U_{\rho, \lambda}^0 \times \mathbb{R}^d \times (\lambda_+ - \delta, \lambda_+)} |E_{k,l}^{(3,3)}(q, x, s, \lambda)| < \infty.$$

The representations (6.24), (6.27), (6.29), (6.31), (6.33), (6.34), (6.37) and the properties (6.38), (6.41), (6.43), (6.47), (6.50), (6.52), (6.55) and (6.56) and (6.57) imply claim (iii) of Proposition 6.1
6.6 Proof of Proposition 6.2

Proof. The proof is the same as the proof of Proposition 6.1 only we should take into account that since in our case each of \( F^+_k \) is a singleton \( \{ p^+_k \} \), then the set \( U^0_{k,l} = U_{k,l} \cap F^+_k \) is or this singleton, or it is empty. Hence in the first case in the r.h.s. of (6.27) it will be the value of the integrand at \( p \), the integral along \( U^0_{k,l} \) value of the integrand at \( L \). Hence in the first case in the r.h.s. of (6.4) and (6.5) it will be the value of the integrand at \( p = p^+_k \) instead of the integral along \( F^+_k \).

\[ \square \]

7 Appendix : \( C(\Omega) \)-holomorphy of Bloch functions

A.1 Main claims

In this Appendix we prove that under some assumptions the Bloch functions can be chosen to be holomorphic w.r.t. the quasi-momentum in the \( C(\Omega) \)-norm.

In this section we shall denote by \( \| \cdot \|_q \) and \( \| \cdot \|_{\alpha,t} \) the norms in the spaces \( L_q(\Omega) \) and \( W^d_\alpha(\Omega) \) respectively. Observe that for \( q = 2 \) the notation \( \| \cdot \|_q \) is compatible with the notation given by (3.3).

The main results of this section are following:

Theorem A.1. Assume that the periodic potential \( V(x) \) satisfies the condition (A.1). Then

(i) if \( \lambda(p) \) is an eigenvalue of the operator \( H(p) \), then any eigenfunction \( b(x,p) \) of \( H(p) \) corresponding to \( \lambda(p) \) belongs to \( C(\Omega) \);

(ii) if the family of operators \( H(p) \) has a holomorphic branch of eigenvalues \( \lambda(p) \) in a connected neighborhood \( \mathcal{O}(p_0) \subset \mathbb{C}^d \) of a point \( p_0 \in \mathbb{R}^d \), and for each \( p \in \mathcal{O}(p_0) \) it is possible to choose an eigenfunction \( b(x,p) \neq 0 \) of \( H(p) \) corresponding to \( \lambda(p) \) such that the mapping \( p \to b(x,p) \in L_2(\Omega) \) is holomorphic in \( \mathcal{O}(p_0) \), then the mapping \( p \to b(x,p) \in C(\Omega) \) is holomorphic in \( \mathcal{O}(p_0) \).

Corollary A.2. If \( V(x) \) is as in Theorem A.1 and for some \( p_0 \in \mathbb{R}^d \) an eigenvalue \( \lambda_0 \) of \( H(p_0) \) is simple, then

(i) for some neighborhood \( \mathcal{O}(p_0) \subset \mathbb{C}^d \) of \( p_0 \) there exists a branch \( \lambda(p) \) of eigenvalues of the family \( H(p) \) (\( p \in \mathcal{O}(p_0) \)) such that \( \lambda(p_0) = \lambda_0 \), \( \lambda(p) \) is simple for any \( p \in \mathcal{O}(p_0) \), the function \( \lambda(p) \) is holomorphic in \( \mathcal{O}(p_0) \) and for any \( p \in \mathcal{O}(p_0) \) it is possible to choose an eigenfunction \( b(x,p) \) of \( H(p) \) corresponding to \( \lambda(p) \) such that \( \| b(x,p) \|_2 = 1 \), the function \( b(x,p) \) is continuous in \( \Omega \), and the mapping \( p \to b(\cdot,p) \in C(\Omega) \) is real-analytic in \( \mathcal{O}(p_0) \cap \mathbb{R}^d \); vskip2mm

(ii) for any \( p \in \mathcal{O}(p_0) \cap \mathbb{R}^d \) the eigenkernel, corresponding to \( \lambda(p) \), has the form \( Q(x,s,p) = b(x,p)b(s,p) \) and the mapping \( p \to Q(\cdot,\cdot,p) \in C(\Omega \times \Omega) \) is real analytic in \( \mathcal{O}(p_0) \cap \mathbb{R}^d \); vskip2mm

(iii) for any \( p \in \mathcal{O}(p_0) \cap \mathbb{R}^d \) the eigenkernel \( Q(x,s,p) \) does not depend on the choice of the branch \( b(x,p) \) of eigenfunctions of the family \( H(p) \) (\( p \in \mathcal{O}(p_0) \)) \( \cap \mathbb{R}^d \) satisfying the conditions imposed in claim (i).
Notice that for $d = 3$ the claim of Theorem A.1 follows from the results of the paper [Wil]. It has been shown there ([Wil], Lemmas 3.7, 3.8) that the square $R^2(p)$ of the resolvent $R(p)$ of the operator $H(p)$ is an integral operator with an integral kernel $K(x, s, p)$, such that for some $\gamma > 0$ for any fixed $p$ with $|\Im(p)| \leq \gamma K(x, s, p) \in C(\Omega \times \Omega)$ and the mapping $p \to K(x, s, p) \in C(\Omega \times \Omega)$ is holomorphic for $|\Im(p)| \leq \gamma$. This fact implies easily the claim of Theorem A.1. Observe that the arguments of [Wil] are true also in the case where $d < 3$.

But in the case $d \geq 4$ the integral kernel of the resolvent $R(p)$ has a stronger singularity at its diagonal, hence we need to deal with a higher power $R^l(p)$ of it in order to get an integral kernel having the property mentioned above.

A.2 Domains and self-adjointness of the operators $H(p)$ and $H_0$

In this section we generalize to the case of an arbitrary dimension $d$ the results on domains and self-adjointness of the operators $H(p)$ and $H_0$, obtained in [Wil] for $d = 3$ (Lemmas 1.2 and 1.4). The arguments used in [Wil] are true also for $d < 3$. These arguments are based on the fact that for $d \leq 3$ the continuous embedding $W_2^2(\Omega) \hookrightarrow C(\Omega)$ holds. For $d \geq 4$ this embedding is not true, but we use in this case the Sobolev’s theorem on embedding of $W_2^p(\Omega)$ into $L_q(\Omega)$.

First of all, let us prove the lemma, whose first claim is an analog of Lemma 1.3 from [Wil]:

Lemma A.3. (i) If $d \geq 4$ and $V \in L_s(\Omega)$ with $s > \frac{d}{2}$, then there exists $C > 0$ such that for any $u \in W_2^2(\Omega)$ and any $\epsilon > 0$

$$\|Vu\|_2^2 \leq C\|Vu\|_2^2(\epsilon\|u\|_{2,2}^2 + \epsilon^{-\mu}\|u\|_2^2),$$

(A.1)

where

$$\mu = \mu(s) = \frac{(p(s))^{-1} - (q(s))^{-1}}{(q(s))^{-1} - (q(\tilde{s}))^{-1}},$$

(A.2)

$$q(s) = \frac{2s}{s - \frac{d}{2}},$$

(A.3)

$$p(s) = \frac{2ds}{(d + 4)s - 2d}$$

(A.4)

and

$$\tilde{s} = \frac{1}{2}\left(\frac{d}{2} + s\right);$$

(A.5)

(ii) If $d \geq 2$ and $V \in L_s(\Omega)$ with $s > \frac{d}{2}$, then there exists $\tilde{C} > 0$ such that for any $u \in W_2^1(\Omega)$ and any $\epsilon > 0$

$$\left|\int_\Omega V(x)|u(x)|^2\,dx\right| \leq \tilde{C}\|u\|_s(\epsilon\|u\|_{2,1} + \epsilon^{-\tilde{\mu}}\|u\|_2^2),$$

where

$$\tilde{\mu} = \tilde{\mu}(s) = \frac{(\tilde{p}(s))^{-1} - (\tilde{q}(s))^{-1}}{(\tilde{q}(s))^{-1} - (\tilde{q}(\tilde{s}))^{-1}}, \quad \tilde{p}(s) = \frac{2ds}{(d + 2)s - d},$$

(A.6)
\[ q(s) = \frac{2s}{s - 1} \]  
(A.7)

and \( \tilde{s} \) is defined by (A.5);

(iii) If \( d = 1 \) and \( V \in L_1(\Omega) \), then for any \( u \in W_2^1(\Omega) \) and any \( \epsilon > 0 \)
\[ \left| \int_\Omega V(x) |u(x)|^2 \, dx \right| \leq ||V||_1(\epsilon ||u||_{2,1}^2 + (\epsilon^{-1} + T^{-1}) ||u||_2^2) \), where \( T = \text{tenth}(\Omega) \).

Proof. (i) Using Hölder’s inequality, we have for any \( u \in W_2^2(\Omega) \):
\[ ||Vu||_2^2 = \int_\Omega (V(x))^2 |u(x)|^2 \, dx \leq \left( \int_\Omega (V(x))^p \, dx \right)^{\frac{2}{p}} \left( \int_\Omega |u(x)|^{q(s)} \, dx \right) \frac{2}{q(s)} = ||V||^2_s ||u||_{q(s)}^2, \]  
(A.8)

where \( q(s) \) is defined by (A.3). We see from the representation \( q(s) = 2 + \frac{1}{s-2} \) that \( q(s) \) is decreasing and if \( s \) runs over \( \left( \frac{d}{2}, \infty \right) \), \( q(s) \) runs over \( (2, \frac{2d}{d+4}) \) for \( d > 4 \) and over \( 2, (2, \infty) \) for \( d = 4 \). By Sobolev’s embedding theorem, if
\[ \frac{1}{p} = \frac{1}{q(s)} + \frac{2}{d} \]  
(A.9)

then \( W_2^p(\Omega) \hookrightarrow L_{q(s)}(\Omega) \). Taking into account (A.3), we get that the number \( p = p(s) \), for which (A.9) holds, has the form (A.4). Observe that the representation \( p(s) = \frac{2d}{d+4} \left( 1 + \frac{2d}{d+4} \right) \) implies that \( p(s) \) is decreasing and when \( s \) runs over \( \left( \frac{d}{2}, \infty \right), p(s) \) runs over \( \left( \frac{2d}{d+4}, 2 \right) \). Observe that, in view of (A.3), \( p(s) < p(\tilde{s}) < 2 < q(s) < q(\tilde{s}) \), hence the interpolation inequality \( ||u||_{q(s)} \leq ||u||_{p(s)}^\lambda ||u||_{q(\tilde{s})}^{1-\lambda} \) is valid with \( \lambda = \frac{(q(s))-1(q(\tilde{s}))-1}{q(s)-q(\tilde{s})} \). Then the Young’s inequality implies that for any \( \epsilon > 0 \)
\[ ||u||_{q(s)}^2 \leq \epsilon ||u||_{q(\tilde{s})}^2 + \epsilon^{-\mu} ||u||_{p(s)}^2, \]  
(A.10)

where \( \mu = \mu(s) \) is defined by (A.2) (see [Gil-Tr], Chapt. 7). By the Sobolev’s embedding theorem, \( W_2^p(\Omega) \hookrightarrow L_{q(s)}(\Omega) \), hence since \( W_2^p(\Omega) \hookrightarrow W_2^p_{p(s)}(\Omega) \), we get: \( W_2^p(\Omega) \hookrightarrow L_{q(\tilde{s})}(\Omega) \). This fact, the embedding \( L_2(\Omega) \hookrightarrow L_{p(s)}(\Omega) \) and the inequalities (A.8) and (A.10) imply that for some \( C > 0 \) and for any \( u \in W_2^2(\Omega) \), \( \epsilon > 0 \) the desired inequality (A.11) is valid. Claim (i) is proven.

(ii) We have for \( u \in W_2^1(\Omega) \):
\[ \left| \int_\Omega V(x) |u(x)|^2 \, dx \right| \leq ||V||_{s} ||u||_{\tilde{q}(s)}^2, \]  
where \( \tilde{q}(s) \) is defined by (A.7). Further we continue the proof like the proof of claim (i), using the fact that \( W_1^p(\Omega) \hookrightarrow L_{\tilde{q}(s)}(\Omega) \), if \( \frac{1}{p} = \frac{1}{q(s)} + \frac{4}{d} \).

(iii) Using the Newton-Leibnitz formula, we get easily that for any \( u \in W_2^1(\Omega) \), \( \epsilon > 0 \) and \( x \in \Omega \) the inequality holds \( |u(x)|^2 \leq \epsilon ||u||_{2,1}^2 + (\epsilon^{-1} + T^{-1}) ||u||_{\tilde{q}(s)}^2 \), from which the claim follows immediately.

Using Lemma (A.3) and the arguments of the proof of Lemma 1.1 from [Wil], it is not difficult to prove the following claim:
Lemma A.4. (i) If \( d \geq 4 \) and \( V \in L_s(\Omega) \) with \( s > \frac{d}{2} \), then there exists \( C > 0 \) such that for any \( u \in W_2^2(\mathbb{R}^d) \) and any \( \epsilon > 0 \)
\[
\|Vu\|_{L_2(\mathbb{R}^d)} \leq C\|V\|_{s}(\epsilon\|u\|_{W_2^2(\mathbb{R}^d)} + \epsilon^{-\mu}\|u\|_{L_2(\mathbb{R}^d)}),
\]
where \( \mu = \mu(s) \) is defined by (A.2)-(A.4);

(ii) If \( d \geq 2 \) and \( V \in L_s(\Omega) \) with \( s > \frac{d}{2} \), then there exists \( \tilde{C} > 0 \) such that for any \( u \in W_2^1(\mathbb{R}^d) \) and any \( \epsilon > 0 \)
\[
\left| \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, dx \right| \leq \tilde{C}\|V\|_{s}(\epsilon\|u\|^2_{W_2^2(\mathbb{R}^d)} + \epsilon^{-\tilde{\mu}}\|u\|^2_{L_2(\mathbb{R}^d)}),
\]
where \( \tilde{\mu} = \tilde{\mu}(s) \) is defined by (A.6);

(iii) If \( d = 1 \) and \( V \in L_1(\Omega) \), then for any \( u \in W_2^1(\Omega) \) and any \( \epsilon > 0 \)
\[
\left| \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, dx \right| \leq \|V\|_{1}(\epsilon\|u\|^2_{W_2^1(\mathbb{R}^d)} + (\epsilon^{-1} + T^{-1})\|u\|^2_{L_2(\mathbb{R}^d)}),
\]
where \( T = \text{lenth}(\Omega) \).

We now turn to the main claim of this section.

Proposition A.5. If \( V(x) \) satisfies the condition (3.1), then

(i) for any \( p \in T^d \) the operator \( H(p) \), generated in the space \( \mathcal{H}_p \) by the operation \( h = -\Delta + V(x) \cdot \) and having the domain \( \mathcal{D}_p = \mathcal{H}_p \cap W_2^2,\text{loc}(\mathbb{R}^d) \), is self-adjoint and bounded below uniformly w.r.t. \( p \in T^d \);

(ii) the operator \( H_0 \), generated in the space \( L_2(\mathbb{R}^d) \) by the operation \( h = -\Delta + V(x) \cdot \) and having the domain \( W_2^2(\mathbb{R}^d) \), is self-adjoint and bounded below.

Recall that \( \mathcal{H}_p \) is the Hilbert space of functions \( u \in L_2,\text{loc}(\mathbb{R}^d) \) satisfying the condition (3.3) with the inner product, defined by (5.3).

Proof. For \( d = 3 \) the claims are proved in [Will] (Lemmas 1.2 and 1.4) and the arguments used there are true also in the case \( d < 3 \). Let us prove claim (i) for the case \( d \geq 4 \). Assume that \( V \in L_s(\Omega) \) with \( s > \frac{d}{2} \). Then using claim (iii) of Lemma A.3 and the arguments from the proof of Lemma 1.3 from [Will], we get that for some \( \tilde{C} > 0 \) and for any \( u \in \mathcal{D}_p \), \( \epsilon > 0 \) \( \|Vu\|_2 \leq \tilde{C}\|V\|_{s}(\epsilon\|\Delta u\|_2 + \epsilon^{-\mu}\|u\|_2) \) and the operator \( -\Delta \) with the domain \( \mathcal{D}_p \) is self-adjoint and non-negative. Hence claim (i) follows from the Kato’s theorem ([Kat], p. 287, Theorem 4.3). In the analogous manner claim (ii) for \( d \geq 4 \) follows from claim (i) of Lemma A.4 arguments from the proof of Lemma 1.1 of [Will] and the Kato’s theorem mentioned above.

In the same manner as in [Will] (Lemma 1.5) the following claim is proved:

Proposition A.6. If \( V(x) \) satisfies the condition (3.1), then the resolvent operator \( R_\zeta(H(p)) = (H(p) - \zeta I)^{-1} \) is compact for every \( \zeta \) in the resolvent set of \( H(p) \). Hence, in particular, \( H(p) \) has a discrete spectrum \( \sigma(H(p)) \) for every \( p \in T^d \).
A.3 Fundamental solution of the Helmholtz’s equation in \( \mathbb{R}^d \)

First of all, let us study the structure of the fundamental solution \( E(x, \gamma_0) \) \((\gamma_0 > 0)\) of the Helmholtz equation in the space \( \mathbb{R}^d \), that is the generalized solution of the equation

\[
- \Delta E + \gamma_0^2 E = \delta(x),
\]

(A.11)

belonging to the space \( S'(\mathbb{R}) \) of slowly growing distributions.

**Proposition A.7.** (i) Equation (A.11) has a unique solution \( E_d \in S'(\mathbb{R}^d) \);

(ii) it is spherically symmetric, that is \( E_d(x, \gamma_0) = E_d(|x|, \gamma_0), E(x, \gamma_0) > 0 \) for \( x \neq 0 \), and it has the form:

(iii) for \( d = 1 \)

\[
E_d(x, \gamma_0) = \frac{1}{2\gamma_0} e^{-\gamma_0 |x|};
\]

(A.12)

(iv) for \( d = 3 \)

\[
E_d(x, \gamma_0) = \frac{1}{4\pi |x|} e^{-\gamma_0 |x|};
\]

(v) for \( d = 2m + 1 \) \((m = 2, 3, \ldots)\)

\[
E_d(x, \gamma_0) = \frac{s_d-2}{2(2\pi)^{d-1}} \sum_{k=0}^{m-1} \left( \begin{array}{c} m-1 \\ k \end{array} \right) (-1)^{m-1-k} \sum_{j=0}^{2k} \left( \begin{array}{c} 2k \\ j \end{array} \right) \gamma_0^{2(m-1-k)+j} \times \\
\frac{(2k-j)!}{|x|^{2k-j+1}} e^{-\gamma_0 |x|};
\]

(vi) for \( d = 2 \)

\[
E_d(x, \gamma_0) = \frac{1}{2\pi} K_0(\gamma_0 |x|),
\]

where

\[
K_\nu(x) := \int_1^\infty \frac{e^{-xt}}{t^{\nu+1}} \, dt
\]

\((x > 0, \nu \geq 0)\) is the MacDonald’s function;

(vii) for \( d = 2m \) \((m = 2, 3, \ldots)\)

\[
E_d(x, \gamma_0) = \frac{s_d-1}{(2\pi)^d} \sum_{k=0}^{m-1} \left( \begin{array}{c} m-1 \\ k \end{array} \right) (-1)^{m-1-k} \sum_{j=0}^{2k} \left( \begin{array}{c} 2k \\ j \end{array} \right) \gamma_0^{2(m-1-k)+j} \times \\
\frac{(2k-j)!}{|x|^{2k-j}} K_{2k-j}(\gamma_0 |x|).
\]

**Proof.** We see from (A.11) that the Fourier transform \( \hat{E}_d \) of \( E_d \) satisfies the equation

\[
(|p|^2 + \gamma_0^2) \hat{E}_d = \frac{1}{(2\pi)^{d/2}},
\]

(A.14)
hence $\hat{E}_d(p, \gamma_0) = \frac{1}{(2\pi)^d} \int_{|p|^2 + \gamma_0} e^{ip \cdot x} dp$, and, as it is clear, this is a unique solution of (A.12) belonging to $S'(\mathbb{R}^d)$. This proves claim (i) of the proposition. Let us reconstruct $E_d$ from $\hat{E}_d$:

$$E_d(x, \gamma_0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ip \cdot x} dp. \quad (A.15)$$

for $d = 1$ we have using Jordan’s lemma:

$$E_d(x, \gamma_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ipx}}{p^2 + \gamma_0} dp = \begin{cases} i \text{Res}_{p=i\gamma_0} e^{ipx}, & \text{for } x > 0, \\ -i \text{Res}_{p=-i\gamma_0} e^{ipx}, & \text{for } x < 0 \end{cases} = \frac{1}{2\gamma_0} e^{-\gamma_0|x|}.$$

So, we have proved claim (iii). Rotating the space $\mathbb{R}^d$ such that the direction of the vector $x$ transforms to the direction of the axis $p_d$, we have from (A.15) for $d > 1$:

$$E_d(x, \gamma_0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-1}} dp \int_{-\infty}^{\infty} \frac{e^{ip \cdot |x|}}{|p|^2 + p_d^2 + \gamma_0} dp_d,$$

where $\hat{p} = (p_1, p_2, \ldots, p_{d-1})$. Applying Jordan’s lemma to the inner integral, we have:

$$E_d(x, \gamma_0) = \frac{1}{2\pi (2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \exp(-\gamma_0|x|) d\hat{p} = \frac{s_{d-2}}{2(2\pi)^{d-1}} \int_0^{\infty} \frac{r^{d-2} \exp(-\sqrt{r^2 + \gamma_0^2}|x|)}{\sqrt{r^2 + \gamma_0^2}} dr \quad (A.16)$$

We see from the last equality and (A.12) that $E_d(x, \gamma_0)$ is spherically symmetric, that is $E_d(x, \gamma_0) = E(|x|, \gamma_0)$, it is finite for $x \neq 0$ and $E_d(x, \gamma_0) > 0$. We have proved claim (ii). Taking in (A.16) $d = 3$, we get easily claim (iv).

For $d \geq 4$ let us write (A.16) in the form:

$$E_d(x, \gamma_0) = \frac{s_{d-2}}{2(2\pi)^{d-1}} \left( \int_0^{\infty} \frac{r^{d-4} (p^2 + \gamma_0^2) \exp(-\sqrt{r^2 + \gamma_0^2}|x|)}{\sqrt{r^2 + \gamma_0^2}} dr - \gamma_0^2 \int_0^{\infty} \frac{r^{d-4} \exp(-\sqrt{r^2 + \gamma_0^2}|x|)}{\sqrt{r^2 + \gamma_0^2}} dr \right) = \frac{s_{d-2}}{2(2\pi)^{d-1}} \left( \frac{d^2}{d\rho^2} - \gamma_0^2 \right) E_{d-2}(\rho, \gamma_0)|_{\rho = |x|},$$

From this recursive formula and claim (iv) we get easily claim (v).

Using the Hadamard’s descent principle:

$$E_{2m}(x, \gamma_0) = \int_{-\infty}^{\infty} E_{2m+1}((x, \xi), \gamma_0) d\xi$$

(see [W]), it is not difficult to prove claims (vi) and (vii) with the help of claims (iv) and (v).
Corollary A.8. For any natural $d$ there exists $M = M(d, \gamma_0) > 0$ such that

(i) if $d = 1$, then for any $x \in \mathbb{R}$, $\mathcal{E}_d(x, \gamma_0) \leq M e^{-\gamma_0|x|}$;

(ii) if $d \geq 3$ is odd, then

\[
\mathcal{E}_d(x, \gamma_0) \leq \begin{cases} \frac{M}{|x|^{d-1}} & \text{for } |x| \leq 1 \\ Me^{-\gamma_0|x|} & \text{for } |x| > 1; \end{cases}
\]

(iii) if $d$ is even, then

\[
\mathcal{E}_d(x, \gamma_0) \leq \begin{cases} \frac{M}{|x|^{d-1}} \ln \left( \frac{1}{|x|} \right) & \text{for } |x| \leq 1 \\ Me^{-\gamma_0|x|} & \text{for } |x| > 1; \end{cases}
\]

(iv) in particular, $\mathcal{E}_d(x, \gamma_0) \in L_q(\mathbb{R}^d)$ for $d = 1, 2$, $q \in [1, \infty)$ and for $d > 2$, $q \in [1, \frac{d}{d-2})$.

The following claims are proved easily:

Lemma A.9. The following equality is valid: $\mathcal{E}_d(x, \gamma_0) = \gamma_0^{d-2} \mathcal{E}_d(\gamma_0 x, 1)$.

Lemma A.10. The restriction of the operator $R_{-\gamma_0^2}(-\Delta) = (-\Delta + \gamma_0^2)^{-1}$ on the set $L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ is represented as the integral operator:

\[
R_{-\gamma_0^2}(-\Delta)f = \int_{\mathbb{R}^d} \mathcal{E}_d(x - s, \gamma_0) f(s) ds \quad (f \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)).
\]

A.4 Green’s function of the operator $H(p)$

Let $H_0(p)$ be the operator generated by the operation $-\Delta$ in the space $\mathcal{H}_p$ (defined in Section 3) with the domain $\mathcal{P}_p(\Gamma) = W_{2,\text{loc}}^2(\mathbb{R}^d) \cap \mathcal{H}_p$. Consider the function

\[
G_0(x, p, \gamma_0) := \sum_{m \in \mathbb{Z}^d} \mathcal{E}_d(x - m, \gamma_0) \exp(i p \cdot m). \quad (A.17)
\]

Furthermore, we shall assume in what follows that $d \geq 4$, because the case $d \leq 3$ have been studied in [11]. Using the function $G_0(x, p, \gamma_0)$, we shall construct below the integral kernel of the operator $R_{-\gamma_0^2}(H(p)) = (H(p) + \gamma_0^2)^{-1}$, where the operator $H(p)$ have been defined in Section 3. But first of all we shall study some properties of the function $G_0(x, p, \gamma_0)$.

Denote by $L_{c, q}(\Omega \times \Omega)$ ($q \geq 1$) the set of all measurable functions $f : \Omega \times \Omega \to \mathbb{C}$ such that for any fixed $x \in \Omega$ the function $f(x, \cdot)$ belongs to $L_q(\Omega)$ and the function $x \to f(x, \cdot) \in L_q(\Omega)$ is continuous. This is a Banach space w. r. to the norm

\[
\|f\|_{c, q} := \|f(x, \cdot)\|_{L_q(\Omega)} \|C(\Omega) = \max_{x \in \Omega} \left( \int_{\Omega} |f(x, s)|^q ds \right)^{\frac{1}{q}}.
\]

In the analogous manner the space $L_{q, c}(\Omega \times \Omega)$, having the norm

\[
\|f\|_{q, c} := \|f(\cdot, s)\|_{L_q(\Omega)} \|C(\Omega) = \max_{x \in \Omega} \left( \int_{\Omega} |f(x, s)| dx \right)^{\frac{1}{q}},
\]

60
is defined.

If \( q \geq 1 \), we shall denote by \( q' \) the number conjugate to \( q \), that is \( 1/q + 1/q' = 1 \). As it is easy to check, \( q \in (1, \frac{2}{d}) \) if and only if \( q' > \frac{2}{d} \).

Denote \( \mathcal{H}_{p,q'} := \mathcal{H}_p \cap L_{q',loc}(\mathbb{R}^d) \), \( \mathcal{H}_{p,\infty} := \mathcal{H}_p \cap L_{\infty,loc}(\mathbb{R}^d) \), \( \mathbb{Z}_+^d = \{ m \in \mathbb{Z}^d \mid m_k \geq 0 \ (k = 1, 2, \ldots, d) \} \). For \( m \in \mathbb{Z}^d \) we denote \( |m| := \max_{1 \leq k \leq d} |m_k| \), \( |m|_1 := \sum_{k=1}^d |m_k| \) (\( m = (m_1, \ldots, m_d) \)).

Applying Corollary A.8 of Proposition A.7 and using the same arguments that have been used for \( d = 3 \) in [Wil] (Lemmas 3.1, 3.2, 3.5), we can prove the following claims, using the \( L_{c,q}(\Omega \times \Omega) \)-norm for integral kernels of operators instead of the \( L_2(\Omega \times \Omega) \)-norm used in [Wil]:

**Lemma A.11.**

(i) The series in (A.17) converges for \( (x, p) \in (\mathbb{R}^d/\mathbb{Z}^d) \times \{ p \in \mathbb{C}^d \mid |\Im p| < \gamma_0 \} \) and the convergence is uniform on compact subsets. Moreover, if \( G_0(x, p, \gamma_0) \) is defined by

\[
G_0(x, p, \gamma_0) = \sum_{m \in \mathcal{N}} \mathcal{E}_d(x - m, \gamma_0) \exp(i p \cdot m) + G'_0(x, p, \gamma_0), \tag{A.18}
\]

where \( \mathcal{N} := \{ m \in \mathbb{Z}^d \mid |m| \leq \sqrt{d} \} \), then \( G'_0(\cdot, p, \gamma_0) \in C(2\Omega) \) and the mapping \( p \to G'_0(\cdot, p, \gamma_0) \) is holomorphic for \( |\Im p| < \gamma_0 \).

Furthermore, if \( g \geq 1 \) for \( d \leq 2 \) and \( q \in [1, \frac{2}{d-2}) \) for \( d > 2 \), then

(ii) each term of the series \( \sum_{m \in \mathbb{Z}^d} \mathcal{E}_d(x - s - m, \gamma_0) \exp(i p \cdot m) \) belongs to the space \( L_{c,q}(\Omega \times \Omega) \) for any fixed \( p \in \mathbb{C}^d \) and it is holomorphic in \( \mathbb{C}^d \) w.r. to \( p \) in the \( L_{c,q}(\Omega \times \Omega) \)-norm;

(iii) this series converges in the \( L_{c,q}(\Omega \times \Omega) \)-norm to a kernel \( G_0(x, s, p, \gamma_0) = G_0(x - s, p, \gamma_0) \in L_{c,q}(\Omega \times \Omega) \) uniformly w.r. to \( p \in \Pi_q = \{ p \in \mathbb{C}^d \mid |\Im p| < \gamma \} \ (\gamma \in (0, \gamma_0)) \), hence the mapping \( p \to G_0(x, s, p, \gamma_0) \) is holomorphic in the strip \( \Pi_{\gamma_0} \);

(iv) for any \( p \in \mathbb{T}^d \) the restriction of the operator \( R_0(p) = (-\Delta + \gamma_0^2 I)^{-1} \) on the set \( \mathcal{H}_{p,q'} \) has the form:

\[
R_0(p)f = \int_\Omega G_0(x, s, p, \gamma_0)f(s) \, ds;
\]

(v) the operator \( R_0(p) \) maps the set \( \mathcal{H}_{p,q'} \) into \( \mathcal{H}_{p,\infty} \).

**Lemma A.12.** Let \( \Pi \) be a domain in \( \mathbb{C}^d \). If \( V \in L_{q'}(\Omega) \) \((q' \in (1, \infty]) \) and \( G_k(x, s, p) \) \((k = 0, 1) \) be functions defined in \( \Omega \times \Omega \times \Pi \) such that for any fixed \( p \in \Pi \) \( G_k(\cdot, \cdot, p) \in L_{c,q}(\Omega \times \Omega) \) \((k = 0, 1) \) are holomorphic in \( \Pi \). Then the function \( G_2(x, s, p) = \int_0^1 G_0(x, \xi, p)V(\xi)G_1(\xi, s, p) \) has the same properties, i.e. for any fixed \( p \in \Pi \) \( G_2(\cdot, \cdot, p) \in L_{c,q}(\Omega \times \Omega) \) and the mapping \( p \to G_2(\cdot, \cdot, p) \in L_{c,q}(\Omega \times \Omega) \) is holomorphic in \( \Pi \).

**Proof.** The claim follows from the estimates:

\[
\|G_2(x, \cdot, p)\|_q \leq \left( \int_\Omega \left( \int_\Omega |G_0(x, \xi, p)||V(\xi)||G_1(\xi, s, p)| \, d\xi \right)^q \, ds \right)^{\frac{1}{q}} \leq \|V\|_q \|G_0(x, \cdot, p)\|_q \|G_1(\cdot, \cdot, p)\|_{c,q},
\]
\[ \|G_2(x, \cdot, p) - G_2(x_0, \cdot, p)\|_q \leq \]
\[ \left( \int_\Omega \left( \int_\Omega |G_0(x, \xi, p) - G_0(x_0, \xi, p)| \|V(\xi)\| G_1(\xi, s, p) \, ds \right)^q \, ds \right)^{\frac{1}{q}} \leq \]
\[ \|V\|_q \|G_0(x, \cdot, p) - G_0(x_0, \cdot, p)\| \|G_1(\cdot, \cdot, p)\|_{c,q}, \]

\[ \|G_2(\cdot, \cdot, p_0 + \bf{h}) - G_2(\cdot, \cdot, p_0)\| - \int_\Omega (\nabla_p G_0(\cdot, \xi, p))_{\|p=p_0\cdot h}\, d\xi \leq \]
\[ \|V\|_q (\|G_0(\cdot, \cdot, p_0 + \bf{h}) - G_0(\cdot, \cdot, p_0)\| - \nabla_p G_0(\cdot, \cdot, p))_{\|p=p_0\cdot h\|} \times \]
\[ \|G_1(\cdot, \cdot, p_0) - G_1(\cdot, \cdot, p_0)\|_{c,q} + \|G_0(\cdot, \cdot, p_0 + \bf{h}, \gamma_0)\|_{c,q} + \]
\[ \|G_1(\cdot, \cdot, p_0 + \bf{h}) - G_1(\cdot, \cdot, p_0)\|_{c,q} - \nabla_p G_1(\cdot, \cdot, p)|_{p=p_0\cdot h}|_{c,q}, \]

where \(x, x_0 \in \Omega\) and \(p, p_0, p_0 + \bf{h} \in \Pi. \)

The proof of the following proposition is based on the previous claims and it is analogous to the proof of Lemma 3.5 from \([\text{Wil}]\), only again instead of the \(L_2(\Omega \times \Omega)\)-norm one should use the \(L_{c,q}(\Omega \times \Omega)\)-norm:

**Proposition A.13.** If \(d \geq 4\) and \(V \in L_{q'}(\Omega)\) with \(q' \in (\frac{4}{d}, \infty)\), then for a large enough \(\gamma_0:\)

(i) \(-\gamma_0^2 \in \mathcal{R}(H(p))\) for any \(p \in \mathbb{R}^d\) and the resolvent \(R(p) = (H(p) + \gamma_0^2 I)^{-1}\) is represented by the Neumann's series

\[ R(p) = (I + L_0(p))^{-1} R_0(p) = \sum_{n=0}^{\infty} (-1)^n (L_0(p))^n R_0(p), \]

where \(R_0(p) = (-\Delta + \gamma_0^2 I)^{-1}\), \(L_0(p) := R_0(p)V\) and this series converges to \(R(p)\) in \(B(L_2(\Omega))\)-norm;

(ii) for any \(p \in \mathbb{R}^d\) the restriction of \(R(p)\) on \(\mathcal{H}_{p, \infty}\) is the integral operator of the form: \(R(p)f = \int_\Omega G(x, s, p, \gamma_0) f(s) \, ds \) \(\in \mathcal{H}_{p, \infty}\), where \(G(\cdot, \cdot, p, \gamma_0) \in L_{c,q}(\Omega \times \Omega)\) \((q^{-1} + q')^{-1} = 1)\),

\[ G(x, s, p, \gamma_0) = G_0(x, s, p, \gamma_0) + \sum_{n=1}^{\infty} G_n(x, s, p, \gamma_0), \tag{A.19} \]

\[ G_{n+1}(x, s, p, \gamma_0) = \]
\[ -\int_\Omega G_0(x, \xi, p, \gamma_0) V(\xi) G_n(\xi, s, p, \gamma_0) \, d\xi \quad (n = 0, 1, \ldots), \]

\(G_n(x, s, p, \gamma_0) \in L_{c,q}(\Omega \times \Omega)\) and the series in \(\tag{A.19}\) converges to \(G(x, s, p, \gamma_0)\) in the \(L_{c,q}(\Omega \times \Omega)\)-norm;
(iii) each mapping \( p \to G_n(\cdot, \cdot, p, \gamma_0) \in L_{c,q}(\Omega \times \Omega) \) admits a holomorphic continuation from \( \mathbb{R}^d \) to the strip \( \Pi_{\frac{d}{2}} := \{ p \in \mathbb{C}^d \mid |\Re p| < \frac{d}{2} \} \) and the series in \( A.19 \) converges in the \( L_{c,q}(\Omega \times \Omega) \)-norm uniformly in \( \Pi_{\frac{d}{2}} \). Hence the mapping \( p \to G(\cdot, \cdot, p, \gamma_0) \in L_{c,q}(\Omega \times \Omega) \) admits a holomorphic continuation from \( \mathbb{R}^d \) to \( \Pi_{\frac{d}{2}} \):

(iv) for any \( p \in \Pi_{\frac{d}{2}} \) the operator \( R(p) \) with the integral kernel \( G(x,s,p,\gamma_0) \) maps the set \( \mathcal{H}_{p,\infty} \) into itself;

(v) for any \( p \in \Pi_{\frac{d}{2}} \) and \( x, s \in \Omega \) the equalities are valid: \( G_0(x,s,p,\gamma_0) = G_0(s,x,-p,\gamma_0) \), \( G(x,s,p,\gamma_0) = G(s,x,-p,\gamma_0) \).

**Corollary A.14.** If \( d \geq 4 \) and \( V \in L_{q'}(\Omega) \) with \( q' > \frac{d}{2} \), then for a large enough \( \gamma_0 \) and any \( p \in \Pi_{\frac{d}{2}} \), \( G_0(\cdot,\cdot,p,\gamma_0) \in L_{c,q}(\Omega \times \Omega) \cap L_{q,c}(\Omega \times \Omega) \) and the mappings \( p \to G_0(\cdot,\cdot,p,\gamma_0) \in L_{c,q}(\Omega \times \Omega) \), \( p \to G_0(\cdot,\cdot,p,\gamma_0) \in L_{q,c}(\Omega \times \Omega) \), \( p \to G(\cdot,\cdot,p,\gamma_0) \in L_{c,q}(\Omega \times \Omega) \) and \( p \to G(\cdot,\cdot,p,\gamma_0) \in L_{q,c}(\Omega \times \Omega) \) are holomorphic in \( \Pi_{\frac{d}{2}} \).

**A.5 \( C(\Omega \times \Omega) \)-holomorphic of a compositional power of the Green’s function of \( H(p) \)**

We shall use the following notation. By \( K_1(\cdot,\cdot) \circ K_2(\cdot,\cdot) \) we denote the composition of integral kernels \( K_1(x,\cdot) \) and \( K_2(x,\cdot) \): \( (K_1(\cdot,\cdot) \circ K_2(\cdot,\cdot))(x,s) := \int_{\Omega} K_1(x,\xi)K_2(\xi,s)d\xi \). The composition of several integral kernels \( K_1(\cdot,\cdot) \circ K_2(\cdot,\cdot) \circ \cdots \circ K_N(\cdot,\cdot) \) will be denoted briefly by \( \circ_{j=1}^{N} K_j(\cdot,\cdot) \); if \( K_1(\cdot,\cdot) = K_2(\cdot,\cdot) = \cdots = K_N(\cdot,\cdot) = K(\cdot,\cdot) \), we shall denote it as a compositional power \( K^\circ_{\cdot,\cdot} \). We shall denote by \( Hol(D,B) \) the set of abstract holomorphic functions \( \phi : D \to B \), where \( D \) is an open domain in \( \mathbb{C}^d \) and \( B \) is a complex Banach space. Like in \( [\text{Wil}] \), let us define the following equivalence relation between the functions \( p \to K(x,s,p) \) (\( x, s \in \Omega \)): we shall write that \( K(x,s,p) \sim 0 \), if \( (p \to K(\cdot,\cdot,p)) \in Hol(\Pi_{\frac{d}{2}}, C(\Omega \times \Omega)) \) and \( K_1(x,s,p) \sim K_2(x,s,p) \), if \( K_1(x,s,p) - K_2(x,s,p) \sim 0 \).

**Lemma A.15.** Assume that \( d \geq 4 \) and \( V \in L_{q'}(\Omega) \) with \( q' > \frac{d}{2} \). Consider the composition of integral kernels

\[
K(x,s,p) = \circ_{j=1}^{N} K_j(\cdot,\cdot,p)(x,s) \quad (p \in \Pi_{\frac{d}{2}}), \tag{A.20}
\]

where or \( K_j(x,s,p) \sim 0 \) and the composition contains at least one term of this kind, or \( K_j(x,s,p) = L_j(x,s,p) \), or for \( j < N \) \( K_j(x,s,p) = L_j(x,s,p)V(s) \) with

\[
(p \to L_j(\cdot,\cdot,p)) \in Hol(\Pi_{\frac{d}{2}}, L_{c,q}(\Omega \times \Omega)) \cap Hol(\Pi_{\frac{d}{2}}, L_{q,c}(\Omega \times \Omega)), \tag{A.21}
\]

where \( q^{-1} + (q')^{-1} = 1 \). Then \( K(x,s,p) \sim 0 \).

**Proof.** We shall prove the lemma in two steps.
Step 1. Consider the kernel of the form $K_{1}(\cdot, \cdot, p) = L(\cdot, \cdot, p) V(\cdot) \circ K(\cdot, \cdot, p)$, where the mapping $p \to L(\cdot, \cdot, p)$ satisfies the condition (A.21) and $K(x, s, p) \sim 0$. Let us prove that $K(x, s, p) \sim 0$. We have for $x, x + h, s \in \Omega$:

$$
|K_{1}(x + h, s, p) - K_{1}(x, s, p)| = \\
\left|\int_{\Omega} (L(x + h, \xi, p) - L(x, \xi, p)) V(\xi) K(\xi, s, p) d\xi \right| \leq \\
\|V\|_{q'} \|K(\cdot, \cdot, p)\|_{C(\Omega \times \Omega)} \|L(x + h, \cdot, p) - L(x, \cdot, p)\|_{q}.
$$

This estimate and the inclusion $L(\cdot, \cdot, p) \in L_{c,q}(\Omega \times \Omega)$ imply that for any $p \in \prod \Omega$ the function $K_{1}(x, s, p)$ is continuous w.r. to $x$ at each point $x \in \Omega$ uniformly w.r. to $s \in \Omega$. Let us estimate for $x, s, s + h \in \Omega$:

$$
|K_{1}(x, s + h, p) - K_{1}(x, s, p)| \leq \\
\|V\|_{q'} \|L(\cdot, \cdot, p)\|_{c,q} \max_{\xi \in \Omega} |K(\xi, s + h, p) - K(\xi, s, p)|.
$$

This estimate and the continuity of $K(\xi, s, p)$ in $\Omega \times \Omega$ imply that for any fixed $x \in \Omega$ $K_{1}(\xi, s, p)$ is continuous at each point $s \in \Omega$. So, we have proved that for any $p \in \prod \Omega$ $K_{1}(\cdot, \cdot, p) \in C(\Omega \times \Omega)$. The estimate

$$
\|K_{1}(\cdot, \cdot, p + t) - K_{1}(\cdot, \cdot, p) - \nabla_{p} L(\cdot, \cdot, p) \cdot t V(\cdot) \circ K(\cdot, \cdot, p)\|_{C(\Omega \times \Omega)} \leq \|V\|_{q'} \times \\
\left(\|L(\cdot, \cdot, p + t) - L(\cdot, \cdot, p)\|_{c,q} \|K(\cdot, \cdot, p)\|_{C(\Omega \times \Omega)} + \\
\|L(\cdot, \cdot, p + t) - L(\cdot, \cdot, p)\|_{c,q} \|K(\cdot, \cdot, p)\|_{C(\Omega \times \Omega)} + \|L(\cdot, \cdot, p)\|_{c,q} \right)
$$

and the holomorphy of the mappings $p \to L(\cdot, \cdot, p) \in L_{c,q}(\Omega \times \Omega)$ and $p \to K(\cdot, \cdot, p) \in C(\Omega \times \Omega)$ in the strip $\prod \Omega$ imply that the mapping $p \to K_{1}(\cdot, \cdot, p) \in C(\Omega \times \Omega)$ is holomorphic there, that is $K_{1}(x, s, p) \sim 0$. In the analogous manner we can prove that if $L$ satisfies the condition (A.21) and $K(x, s, p) \sim 0$, then $L(\cdot, \cdot, p) \circ K(\cdot, \cdot, p) \sim 0, K(\cdot, \cdot, p) \circ V(\cdot) \circ L(\cdot, \cdot, p) \sim 0$ and $K(\cdot, \cdot, p) \circ L(\cdot, \cdot, p) \sim 0$.

Step 2. By the assumption of the lemma, in the composition (A.20) for some $j_{0} \in \{1, 2, \ldots, N\}$ $K_{j_{0}}(x, s, p) \sim 0$. Observe that if a term of (A.20) is equivalent to zero, it belongs to the set from the right hand side of (A.21). Then using inductively the results of Step 1, we obtain that the composition $T(x, s, p)$ of kernels from (A.20) which are after $K_{j_{0}}(x, s, p)$ and of $K_{j_{0}}(x, s, p)$ itself has the property $T(x, s, p) \sim 0$. On the other hand, the composition $T(x, s, p)$ of terms from (A.20) which are after $K_{j_{0}}(x, s, p)$ can be represented in the form $\hat{T}(x, s, p) = (\phi_{j}^{N})_{j=j_{0}+1}^{N}(\cdot) (x, s)$, where the kernels $\hat{K}_{j}(\cdot, \cdot, p)$ have one of the forms: or $\hat{K}_{j}(\cdot, \cdot, p) = V(\cdot) \hat{L}_{j}(\cdot, \cdot, p)$, or $\hat{K}_{j}(\cdot, \cdot, p) = \hat{L}_{j}(\cdot, \cdot, p)$ and each $\hat{L}_{j}(\cdot, \cdot, p)$ belongs to the set from the right hand side of (A.21). Again using inductively the results of Step 1, we obtain that $K(x, s, p) = (T(\cdot, \cdot, p) \circ \hat{T}(\cdot, \cdot, p))(x, s) \sim 0$.

In what follows we need the following result about Fourier’s transform of a function with a polar singularity:
Lemma A.16. For the Fourier’s transform \( \hat{F}(\omega) \) on \( \mathbb{R}^d \) of the function \( F(x) = \exp(-|x|^\beta) \) (\( \beta \in [0, d), \gamma > 0 \)) the property is valid:

\[
\sup_{\omega \in \mathbb{R}^d} \psi(\omega)|\hat{F}(\omega)| < \infty, \quad (A.22)
\]

where

\[
\psi(\omega) = \begin{cases} 
|\omega|, & \text{if } \beta \in [0, d-1), \\
\frac{\omega}{1 + |\omega|}, & \text{if } \beta = d-1, \\
|\omega|^{d-\beta}, & \text{if } \beta \in (d-1, d).
\end{cases} \quad (A.23)
\]

Proof. Since \( \beta < d \), \( F \in L^1(\mathbb{R}^d) \), hence \( \hat{F}(\omega) \) is bounded on \( \mathbb{R}^d \). Hence it is sufficient to prove \( (A.22) \) with \( \{|x| \geq 1\} \) instead of \( \mathbb{R}^d \). Along with the Euclidean norm \( |\omega| \) on \( \mathbb{R}^d \) consider the equivalent norm \( |\omega|_{\infty} = \max_{1 \leq j \leq d} |\omega_j| \) (\( \omega = (\omega_1, \omega_2, \ldots, \omega_d) \)). Let us take \( \omega \neq 0 \) and \( j_0 \in \{1, 2, \ldots, d\} \) such that \( |\omega_{j_0}| = |\omega|_{\infty} \). If the vector \( h(\omega) \) is defined by

\[
(h(\omega))_j = \begin{cases} 
\pi / |\omega|_{\infty} & \text{for } j = j_0, \\
0 & \text{for } j \neq j_0
\end{cases}
\]

\( (j \in \{1, 2, \ldots, d\}) \), then the equality \( \exp(-i\omega \cdot (x + h(\omega))) = -\exp(-i\omega \cdot x) \) is valid, hence we have the representation

\[
\hat{F}(\omega) = \frac{1}{2(2\pi)^{d/2}} \left( \int_{\mathbb{R}^d} F(x) \exp(-i\omega \cdot x) \, dx + \int_{\mathbb{R}^d} F(x + h(\omega)) \exp(-i\omega \cdot (x + h(\omega))) \, dx \right) = \frac{1}{2(2\pi)^{d/2}} \int_{\mathbb{R}^d} (F(x) - F(x + h(\omega))) \exp(-i\omega \cdot x) \, dx,
\]

which implies the estimate:

\[
|F(\omega)| \leq \frac{1}{2(2\pi)^{d/2}} \|F(x) - F(x + h(\omega))\|_1. \quad (A.24)
\]

Here we denote by \( \| \cdot \|_1 \) the norm in the space \( L^1(\mathbb{R}^d) \). We have:

\[
\|F(x) - F(x + h(\omega))\|_1 \leq I_1(\omega) + I_2(\omega), \quad (A.25)
\]

where

\[
I_1(\omega) = \int_{|x| \leq 1} |F(x) - F(x + h(\omega))| \, dx,
\]

\[
I_2(\omega) = \int_{|x| \geq 1} |F(x) - F(x + h(\omega))| \, dx. \quad (A.26)
\]
Let us derive the change of the variable in the integral $I_1(\omega)$: $y = \frac{x}{|h(\omega)|}$, denote $\tilde{h} = \frac{h(\omega)}{|h(\omega)|}$ and rotate the space $\mathbb{R}^d$ such that $\tilde{h} = (1, 0, \ldots, 0)$. Then we obtain:

$$I_1(\omega) = |h(\omega)|^{d-\beta} \int_{|y| \leq \frac{1}{|h(\omega)|}} \left| \frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^\beta} - \frac{\exp(-\gamma |h(\omega)||y|)}{|y|^\beta} \right| \, dy \leq \pi|\omega|_\infty (I_{1,1} + I_{1,2}(\omega)), \quad (A.27)$$

where

$$I_{1,1} = \int_{|y| \leq 2} \left( \frac{1}{|y + \tilde{h}|^\beta} + \frac{1}{|y|^\beta} \right) \, dy, \quad (A.28)$$

$$I_{1,2}(\omega) = \int_{2 \leq |y| \leq \frac{1}{|h(\omega)|}} \left| \frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^\beta} - \frac{\exp(-\gamma |h(\omega)||y|)}{|y|^\beta} \right| \, dy. \quad (A.29)$$

Let us represent:

$$\frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^\beta} - \frac{\exp(-\gamma |h(\omega)||y|)}{|y|^\beta} =$$

$$\int_0^1 \frac{\partial}{\partial t} \left( \frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^\beta} \right) \, dt =$$

$$- \int_0^1 \left( \beta \frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^{\beta+1}} + \gamma |h(\omega)| \frac{\exp(-\gamma |h(\omega)||y + \tilde{h}|)}{|y + \tilde{h}|^\beta} \right) \times$$

$$\frac{(y + \tilde{h}) \cdot \tilde{h}}{|y + \tilde{h}|} \, dt.$$

Then we have from (A.29):

$$I_{1,2}(\omega) \leq \int_{2 \leq |y| \leq \frac{1}{|h(\omega)|}} \left( \frac{\beta}{(|y| - 1)^{\beta+1}} + \frac{\gamma |h(\omega)|}{(|y| - 1)^\beta} \right) \, dy =$$

$$s_{d-1} \left( \beta \int_2^{1/|h(\omega)|} r^{d-1} \, dr + \gamma |h(\omega)| \right) \int_2^{1/|h(\omega)|} r^{d-1} \, dr,$$

hence for $\beta \in [0, d) \setminus \{d-1\}$

$$I_{1,2}(\omega) \leq s_{d-1} 2^{d-1} \left( |h(\omega)|^{\beta-d+1} + 1 \right) \left( \frac{\beta}{\beta - d + 1} + \frac{\gamma}{d - \beta} \right) \quad (A.31)$$

and for $\beta = d - 1$

$$I_{1,2}(\omega) \leq s_{d-1} 2^{d-1} \left( \beta \ln \left( \frac{1}{|h(\omega)|} \right) + \gamma \right). \quad (A.32)$$

Let us estimate the integral $I_2(\omega)$, defined by (A.20), using for the difference under it the representation, analogous to (A.30):

$$I_2(\omega) \leq |h(\omega)| \int_{|x| \geq 1} \left( \frac{\beta}{(|x| - |h(\omega)|)^{\beta+1}} + \frac{\gamma}{(|x| - |h(\omega)|)^\beta} \right) \times$$

$$\exp \left( -\gamma (|x| - |h(\omega)|) \right) \, dx,$$
if $|h(\omega)| < 1$. From this estimate and from (A.24), (A.25), (A.27), (A.28), (A.31) and (A.32) we obtain the desired claim.

In what follows we need also the following claim:

**Lemma A.17.** Assume that $V \in L_{q'}(\Omega)$ with $q' > \frac{d}{2}$ and denote

$$l := \left\lfloor \frac{d}{\theta} \right\rfloor + 2,$$

(A.33)

where

$$\theta = \min\{1, d - q(d - 2)\}.$$  

(A.34)

Then the composition of kernels

$$\Theta(x, s) = (\circ_{j=1}^{l} E_j(\cdot, \cdot))(x, s),$$

(A.35)

where $\tilde{E}_l(x, s) = E_d(x - s - m_l, \gamma_0)$ and for $j < l$

$$\tilde{E}_j(x, s) = \begin{cases} \text{or} & E_d(x - s - m_j, \gamma_0), \\ \text{or} & E_d(x - s - m_j, \gamma_0)V(s) \end{cases}$$

(A.36)

is continuous in $\Omega \times \Omega$.

**Proof.** Denote $\tilde{\Theta}(x, s) = \left(\circ_{j=1}^{l-2} \tilde{E}_j(\cdot, \cdot)\right) \circ E_d(x - s - m_{l-1}, \gamma_0)(x, s)$ and

$$\tilde{\Theta}(x, s) = \left(\circ_{j=2}^{l-1} \tilde{E}_j(\cdot, \cdot)\right)(x, s).$$

(A.37)

Then we have from (A.35):

$$\Theta(x, s) = (\tilde{\Theta}(\cdot, \cdot) \circ W_{l-1}(\cdot) \circ E_d(\cdot, \cdot, \cdot, \gamma_0))(x, s),$$

(A.38)

where

$$W_j(\xi_j) = \begin{cases} \text{or} & V(\xi_j), \\ \text{or} & 1, \end{cases}$$

(A.39)

Assume that $d$ is odd. From the explicit formula

$$\tilde{\Theta}(x, s) = \int_{\Omega} E_d(x - \xi_1 - m_1, \gamma_0)W_1(\xi_1) d\xi_1 \int_{\Omega} E_d(\xi_1 - \xi_2 - m_2, \gamma_0)W_2(\xi_2) d\xi_2 \ldots$$

$$W_{l-2}(\xi_{l-2}) \int_{\Omega} E_d(\xi_{l-2} - s - m_{l-1}, \gamma_0) d\xi_{l-2}$$

(A.40)
and claims (ii)-(v) of Proposition \ref{prop:A.7} we obtain that there exists $A > 0$ such that the estimate is valid:

$$|\hat{\Theta}(x,s)| \leq \max\{1||q'||,|V||q'\}\left(\int_{\Omega} |E_d(x - \xi_1 - m_1, \gamma_0)|^q d\xi_1 \times \right. $$

$$\left. \left| \int_{\Omega} E_d(\xi_1 - \xi_2 - m_2, \gamma_0)W_2(\xi_2) d\xi_2 \ldots W_{l-2}(\xi_{l-2}) \times \right. \right.$$ 

$$\left. \left. \int_{\Omega} E_d(\xi_{l-2} - s - m_{l-1}, \gamma_0) d\xi_{l-2} \right|^{\frac{q}{q'}} \right) \leq \left( \max\{1||q'||,|V||q'\}\right)^2 \times \right.$$ 

$$\left(\int_{\Omega} |E_d(x - \xi_1 - m_1, \gamma_0)|^q d\xi_1 \times \right.$$ 

$$\left. \int_{\Omega} |E_d(\xi_1 - \xi_2 - m_2, \gamma_0)|^q d\xi_2 \left| \int_{\Omega} d\xi_3 \ldots W_{l-2}(\xi_{l-2}) \times \right. \right.$$ 

$$\left. \int_{\Omega} E_d(\xi_{l-2} - s - m_{l-1}, \gamma_0) d\xi_{l-2} \right|^{\frac{q}{q'}} \right) \leq \cdots \leq \right.$$ 

$$A \cdot \left( \max\{1||q'||,|V||q'\}\right)^{l-2} (\phi(x - s)^\frac{1}{q'}, \text{ (A.39)}$$

where

$$\phi(x) = (\mathcal{F}(\cdot - m_1) \ast \mathcal{F}(\cdot - m_2) \ast \cdots \ast \mathcal{F}(\cdot - m_{l-1}))(x) \text{ (A.40)}$$

and $\mathcal{F}(x) = \exp(-\frac{r^2}{|x|^2})$ (here $\ast$ is the sign of the convolution of functions defined on $\mathbb{R}^d$). Let us show that the function $\phi(\cdot, \gamma)$ is bounded on $\mathbb{R}^d$. To this end let us calculate the Fourier transform on $\mathbb{R}^d$ of the function $\phi(x)$. We have from (A.40):

$$\hat{\phi}(\omega) = \exp \left( -i\omega \cdot \sum_{j=1}^{l-1} m_j \right) (\hat{\mathcal{F}}(\omega))^{l-1} (\omega \in \mathbb{R}^d),$$

hence by Lemma \ref{lem:A.10} $\sup_{\omega \in \mathbb{R}^d} |\psi(\omega)|^{l-1} |\hat{\phi}(\omega)| < \infty$, where $\psi(\omega)$ is defined by (A.23) with $\beta = q(d - 2)$. Since in view of (A.33) and (A.34), $(l - 1)\theta > d$, these circumstances imply that $\hat{\phi} \in L_1(\mathbb{R}^d)$. Hence the function $\phi(x)$ is bounded in $\mathbb{R}^d$ and we obtain from (A.39) that for an odd $d$ the function $\hat{\Theta}(x,s)$ is bounded in $\Omega \times \Omega$. Consider the case of an even $d$. From the formula (A.13) for the MacDonald’s function, we get that for any $\epsilon > 0$

$$K_{\nu}(\gamma_0|x|) \leq C(\epsilon) \frac{\exp(-\frac{\gamma_0|x|}{2})}{|x|^\nu + \epsilon} \quad \text{((\nu \geq 0))},$$

where $C(\epsilon) = \sup_{\tau \in [0,\infty)} \left( \tau^\epsilon \exp(-\gamma_0\tau/2) \right) \int_1^\infty \frac{dt}{t^{\nu+\epsilon} \sqrt{\tau - 1}}$. Using claims (vi)-(vii) of Proposition \ref{prop:A.7} and choosing a small enough $\epsilon > 0$, we can show in the same manner as above that also for an even $d$ the function $\hat{\Theta}(x,s)$ is bounded in $\Omega \times \Omega$. In the analogous manner we obtain that the function $\Theta(x,s)$, defined by (A.36), is bounded in $\Omega \times \Omega$. 

68
Let us prove that the function $\Theta(x,s)$ is continuous in $\Omega \times \Omega$. We have from (A.37) for $x,s,s+h \in \Omega$:

$$
|\Theta(x,s+h) - \Theta(x,s)| = \\
\left|\int_{\Omega} \tilde{\Theta}(x,\xi)(\mathcal{E}_d(\xi - s - h - m, \gamma_0) - \mathcal{E}_d(\xi - s - m, \gamma_0)) \, d\xi\right| \\
\leq \|\tilde{\Theta}(\cdot,\cdot)\|_{L^\infty(\Omega \times \Omega)} \\
x \max\{\|1\|_{q'}, \|V\|_{q'}\}\|\mathcal{E}_d(\cdot - s - h - m, \gamma_0) - \mathcal{E}_d(\cdot - s - m, \gamma_0)\|_{q'}.
$$

Since by claim (ii) of Lemma A.11 the function $\mathcal{E}_m(\cdot,\cdot) = \mathcal{E}_d(\cdot - \cdot - m, \gamma_0)$ belongs to the class $L_{q,c}(\Omega \times \Omega)$, then the latter estimate imply that the function $\Theta(x,s)$ is continuous w.r. to $s$ at each point $s \in \Omega$ uniformly w.r. to $x \in \Omega$. In the analogous manner we prove, using (A.38), that for any fixed $s \in \Omega$ the function $\Theta(x,s)$ is continuous in $\Omega \times \Omega$. The lemma is proven. 

We now turn to the main result of this section.

**Proposition A.18.** If $d \geq 4$, $V \in L_{q'}(\Omega)$ with $q' > \frac{d}{2}$, $R(p) = (H(p) + \gamma_0^2)^{-1}$ and the natural $l$ is defined by (A.33), (A.34), then for a large enough $\gamma_0$ the operator $K(p) = (R(p))^l$ has the integral kernel $K(x,s,p)$ such that $K(x,s,p) \sim 0$.

**Proof.** By Proposition A.12 for a large enough $\gamma_0$ and $p \in \Pi_{\omega}$ the operator maps the set $H_{p,\infty}$ into itself and for any $f \in H_{p,\infty}$ $R(p)f = \int_{\Omega} G(x,s,p)f(s) \, ds$, where

$$
G(x,s,p) = G_0(x,s,p) + \sum_{k=1}^{\infty} (G_0(\cdot,\cdot,p)V(\cdot))^{\circ k} \circ G_0(\cdot,\cdot,p) \right) (x,s) \quad (A.41)
$$

and the latter series converges in the $L_{c,q}(\Omega \times \Omega)$-norm uniformly w.r. to $p \in \Pi_{\omega}$. Then for any $f \in H_{p,\infty}$

$$
K(p)f = \int_{\Omega} K(x,s,p)f(s) \, ds, \quad (A.42)
$$

where

$$
K(x,s,p) = (G(\cdot,\cdot,p)^{\circ l})(x,s). \quad (A.43)
$$

Let us represent (A.41) in the form:

$$
G(x,s,p) = G_0(x,s,p) + \left(\sum_{k=1}^{l-1} (G_0(\cdot,\cdot,p)V(\cdot))^{\circ k} \circ G_0(\cdot,\cdot,p) \right) (x,s) + \\
((G_0(\cdot,\cdot,p)V(\cdot))^{\circ (l-1)} \circ G_0(\cdot,\cdot,p) \circ V(\cdot)G(\cdot,\cdot,p))(x,s). \quad (A.44)
$$

On the other hand, by claim (i) of Lemma A.11 the representation (A.18) is valid, in which $G_0(x,s,p) \sim 0$ and $\mathcal{N} := \{m \in \mathbb{Z}^d \mid |m| \leq \sqrt{d}\}$. Substituting
The latter expression is a trigonometric polynomial w.r.t. $p$, whose coefficients are kernels satisfying the condition of Lemma A.17, hence they are continuous in $\Omega \times \Omega$. Hence $K(x,s,p) \sim 0$. Since for any fixed $p \in \Pi_{m,p}$ the operator $K(p)$ is bounded in $L_2(\Omega)$, the operator with the continuous integral kernel $K(x,s,p)$ in the right hand side of (A.42) is bounded in $L_2(\Omega)$ too and the set $\mathcal{H}_{p,\infty}$ is dense in $L_2(\Omega)$, then the representation (A.42) is valid for any $f \in L_2(\Omega)$. The proposition is proven.

A.6 Proof of Theorem A.1

Proof. (i) Let us choose $\gamma_0 > 0$ and consider the domain $\Pi_{m,p} = \{ p \in \mathbb{C}^d : |\Im(p)| \leq \gamma_0/2 \}$. Furthermore, let us take $l = 2$ for $d \leq 3$ and assume that $l$ is
defined by \[\text{(A.58)}\] , \[\text{(A.54)}\] for \(d \geq 4\). Like in [Wil] (Lemmas 3.6 and 3.7), it is easy to show that if \(\gamma_0\) is large enough, for \(p \in \mathbb{R}^d\) the number \(\nu(p) = (\lambda(p) + \gamma_0^2)^d\) is a singular value of the operator \(K(p) = (R(p))^d (R(p) = (H(p) + \gamma_0^2)^{-1})\) with the eigenfunction \(b(x, p)\). This means that for \(p \in \mathbb{R}^d\) the equality is valid:

\[
b(\cdot, p) = \nu(p) K(p) b(\cdot, p),
\]

(A.45)

By Lemma 3.7 from [Wil] (for \(d = 3\) and Proposition \[\text{A.18}\] (for \(d \geq 4\)), if \(\gamma_0\) is large enough, for each \(p \in \Pi_{\gamma_0/2}\) \(K(p)\) is an operator with a continuous integral kernel \(K(x, s, p)\) such that the mapping \(p \to K(\cdot, p) \in C(\Omega \times \Omega)\) is holomorphic in \(\Pi_{\gamma_0/2}\). Observe that this property is valid also for \(d \leq 2\), because all the arguments used in [Wil] are true also in this case. Then for \(p \in \mathbb{R}^d\) the equality (A.45) acquires the form:

\[
b(x, p) = \int_{\Omega} \nu(p) K(x, s, p) b(s, p) \, ds.
\]

(A.46)

This equality and the inclusion \(b(\cdot, p) \in L_2(\Omega)\) imply that \(b(\cdot, p) \in C(\Omega)\). Claim (i) is proven.

(ii) Let us choose \(\gamma_0\) such that \(\mathcal{O}(p_0) \subset \Pi_{\gamma_0}\). In view of the properties of \(K(x, s, p)\) mentioned above, the mapping \(p \to K(p) \in \mathcal{B}(L_2(\Omega))\) is holomorphic in \(\mathcal{O}(p_0)\). Taking into account that the function \(\nu(p)\) and the mapping \(p \to b(\cdot, p) \in L_2(\Omega)\) are holomorphic in \(\mathcal{O}(p_0)\), we obtain by the principle of analytic continuation that the equality (A.45) is valid for any \(p \in \mathcal{O}(p_0)\). Furthermore, the holomorphy in \(\mathcal{O}(p_0)\) of \(\nu\), of \(p \to K(x, s, p) \in C(\Omega \times \Omega)\) and of \(p \to b(x, p) \in L_2(\Omega)\) imply that for any \(p \in \mathcal{O}(p_0)\) the function \(\nu(p) K(x, \cdot, p) b(\cdot, p)\) belongs to the class \(C(\Omega, L_2(\Omega))\) and the mapping \(p \to \nu(p) K(x, \cdot, p) b(\cdot, p) \in C(\Omega, L_2(\Omega))\) is holomorphic in \(\mathcal{O}(p_0)\). These circumstances and equality (A.46) imply claim (ii) of the theorem. \(\square\)

A.7 Proof of Corollary A.2

Proof. (i) Consider the family of operators \(K(p)\) defined in the proof of Theorem A.1. Since \(\lambda_0\) is a simple eigenvalue of the operator \(H(p_0)\), then \(\nu_0 = (\lambda(p_0) + \gamma_0^2)^d\) is a simple singular number of the operator \(K(p_0)\). Let \(b_0(x) \neq 0\) be an eigenfunction of \(K(p_0)\), corresponding to \(\nu_0\) (hence it is an eigenfunction of \(H(p_0)\), corresponding to \(\lambda_0\)). Then, as it has been proved in Bau (Corollary 1, p. 376), there exists a neighborhood \(\mathcal{O}(p_0) \subset \mathbb{C}^d\) of \(p_0\) such that in it there is a branch \(\nu(p)\) of singular numbers of the family \(K(p) (p \in \mathcal{O}(p_0))\) having the properties: \(\nu(p_0) = \nu_0\), \(\nu(p)\) is simple for any \(p \in \mathcal{O}(p_0)\), the function \(\nu(p)\) is holomorphic in \(\mathcal{O}(p_0)\) and the corresponding eigenprojections \(Q(p)\) of \(K(p)\) have the property: the mapping \(p \to Q(p) \in \mathcal{B}(L_2(\Omega))\) is holomorphic in \(\mathcal{O}(p_0)\). Hence the eigenfunctions \(b(\cdot, p) = Q(p) b_0\) of \(K(p)\), corresponding to \(\nu(p)\), have the property: the mapping \(p \to b(x, p) \in L_2(\Omega)\) is holomorphic in \(\mathcal{O}(p_0)\). Furthermore, we can restrict the neighborhood \(\mathcal{O}(p_0)\) such that \(b(\cdot, p) \neq 0\) for any \(p \in \mathcal{O}(p_0)\). Hence by Theorem A.1 \(b(x, p) \in C(\Omega)\) for any \(p \in \mathcal{O}(p_0)\) and the mapping \(p \to b(x, p) \in C(\Omega)\) is holomorphic in \(\mathcal{O}(p_0)\). Observe that \(\nu(p)\) is
real for any $p \in O(p_0) \cap \mathbb{R}^d$, because in this case $K(p)$ is self-adjoint. Recall that for any $p \in \mathbb{R}^d$, $JH(p)J = H(-p)$, where $J$ is the conjugation operator $(Jf)(x) := \overline{f(x)}$. Hence $JK(p)J = K(-p)$ for any $p \in \mathbb{R}^d$, therefore in this case $\sigma(K(p)) = \sigma(K(-p))$ and for any $\mu \in \sigma(K(p))$ the corresponding eigenprojections $Q_\mu$ of $K(p)$ and $\tilde{Q}_\mu$ of $K(-p)$ are connected in the following manner: $	ilde{Q}_\mu = JQ_\mu J$. In particular, for any $p \in O(p_0)$ the number $\nu(p)$ is a simple characteristic number of $K(-p)$ and the operator $JQ(p)J$ is the eigen-projection of $K(-p)$ corresponding to $\nu(p)$, hence $b(x, p)$ is an eigenfunction of $K(-p)$ corresponding to $\nu(p)$. These circumstances and the arguments used above imply that the mapping $p \rightarrow b(x, p) \in C(\Omega)$ is real analytic in $O(p_0) \cap \mathbb{R}^d$.

Therefore the function $\|b(\cdot, p)\|_2 = \left( \int_\Omega \overline{\overline{b(s, p)}b(s, p) \overline{dx}} \right)^{1/2}$ is real analytic in $O(p_0) \cap \mathbb{R}^d$. Hence the mapping $p \rightarrow b(x, p) = \frac{\overline{b(x, p)}}{\|b(\cdot, p)\|_2} \in C(\Omega)$ is real-analytic in $O(p_0) \cap \mathbb{R}^d$. Returning from the family $K(p)$ to the family $H(p)$, we obtain the desired claim (i) of the corollary.

(ii) The claim follows from the representation

$$(Q(p)f)(x) = (f, b(\cdot, p))_2 b(x, p) = \int_\Omega \overline{\overline{b(s, p)}b(s, p)f(s) \overline{dx}} \quad (f \in L^2(\Omega))$$

and the results obtained in the process of the proof of claim (i).

(iii) Since for any $p \in O(p_0) \cap \mathbb{R}^d$, the eigenvalue $\lambda(p)$ is simple, then another choice $b_1(x, p)$ of the branch of the eigenfunctions corresponding to $\lambda(p)$ (with $\|b_1(\cdot, p)\|_2 = 1$) is connected with the previous one in the following manner: $b_1(x, p) = e^{i \theta(p)}b(x, p)$ with a real-valued function $\theta(p)$. Then $b_1(x, p)b_1(s, p) = b(x, p)b(s, p)$. This proves claim (iii). \qed

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