SOME SYMMETRY PROPERTIES OF FOUR-DIMENSIONAL WALKER MANIFOLDS

ABDOUL SALAM DIALLO*, FORTUNÉ MASSAMBA**

ABSTRACT. In this paper, we investigate geometric properties of some curvature tensors of a four-dimensional Walker manifold. Some characterization theorems are also obtained.

1. INTRODUCTION

A Walker $n$-manifold is a pseudo-Riemannian manifold, which admits a field of parallel null $r$-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [5]. Of special interest are the even-dimensional Walker manifolds ($n = 2m$) with fields of parallel null planes of half dimension ($r = m$).

It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Among these, the significant Walker manifolds are the examples of the non-symmetric and non-homogeneous Osserman manifolds [2,3]. Recently, Banyaga and Massamba derived in [1] a Walker metric when studying the non-existence of certain Einstein metric on some symplectic manifolds.

Our purpose is to study a restricted 4-Walker metrics by focusing on their curvature properties. The main results of this paper are the characterization of Walker metrics which are Einstein, locally symmetric Einstein and locally conformally flat. The paper is organized as follows. In section 2 we recall some basic facts about Walker metrics by explicitly writing its Levi-Civita connection and the curvature tensor. Walker metrics which are Einstein are investigated in section 3 (Theorem 3.1). In section 4 we study the Walker metrics which are locally symmetric Einstein (Theorem 4.1). Finally, we discuss in section 5 the conformally locally flat property of Walker metric (Theorem 5.1).

2. THE CANONICAL FORM OF A WALKER METRIC

Let $M$ be a pseudo-Riemannian manifold of signature $(n, n)$. We suppose given a splitting of the tangent bundle in the form $TM = D_1 \oplus D_2$ where $D_1$ and $D_2$ are smooth subbundles which are called distribution. This define two complementary projection $\pi_1$ and $\pi_2$ of $TM$ onto $D_1$ and $D_2$. We say that $D_1$ is parallel distribution

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if $\nabla \pi_1 = 0$. Equivalently this means that if $X_1$ is any smooth vector field taking values in $D_1$, then $\nabla X_1$ again takes values in $D_1$. If $M$ is Riemannian, we can take $D_2 = D_1^\perp$ to be the orthogonal complement of $D_1$ and it that case $D_2$ is again parallel. In the pseudo-Riemannian setting, $D_1 \cap D_2$ need not be trivial. We say that $D_1$ is a null parallel distribution if $D_1$ is parallel and the metric restricted to $D_1$ vanish identically. Manifolds which admit null parallel distribution are called Walker manifolds.

A neutral $g$ on an 4-manifold $M$ is said to be a Walker metric if there exists a 2-dimensional null distribution $D$ on $M$ which is parallel with respect to $g$. From Walker theorem [5], there is a system of coordinates $(u_1, u_2, u_3, u_4)$ with respect to which $g$ takes the local canonical form

$$
(g_{ij}) = \begin{pmatrix} 0 & I_2 \\ I_2 & B \end{pmatrix},
$$

where $I_2$ is the $2 \times 2$ identity matrix and $B$ is a symmetric $2 \times 2$ matrix whose the coefficients are the functions of the $(u_1, \cdots, u_4)$. Note that $g$ is of neutral signature $(++--)$ and that the parallel null 2-plane $D$ is spanned locally by $\{\partial_1, \partial_2\}$, where $\partial_i = \frac{\partial}{\partial u_i}, i = 1, 2, 3, 4$.

Let $M_{a,b,c} := (O, g_{a,b,c})$, where $O$ be an open subset of $\mathbb{R}^4$ and $a, b, c \in C^\infty(O)$ be smooth functions on $O$, then

$$
(g_{a,b,c})_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix},
$$

where $a, b$ and $c$ are functions of the $(u_1, \cdots, u_4)$. We denote, $h_i = \frac{\partial h(u_1, \cdots, u_4)}{\partial u_i}$, for any function $h(u_1, \cdots, u_4)$. In [3], Einsteinian, Osserman or locally conformally flat Walker manifolds were investigated in the restricted form of metric when $c(u_1, u_2, u_3, u_4) = 0$. In this paper, following [3], we consider the specific Walker metrics on a 4-dimensional manifold with

$$
a = a(u_1, u_2), \quad b = b(u_1, u_2) \quad \text{and} \quad c = c(u_1, u_2),
$$

and investigate conditions for a Walker metric (2.1) to be Einsteinian, locally symmetric Einstein and locally conformally flat.

A straightforward calculation shows that the Levi-Civita connection of a Walker metric (2.1) is given by

$$
\nabla_{\partial_i} \partial_3 = \frac{1}{2}a_1 \partial_1 + \frac{1}{2}c_1 \partial_2, \quad \nabla_{\partial_i} \partial_4 = \frac{1}{2}c_1 \partial_1 + \frac{1}{2}b_1 \partial_2, \\
\nabla_{\partial_j} \partial_3 = \frac{1}{2}a_2 \partial_1 + \frac{1}{2}c_2 \partial_2, \quad \nabla_{\partial_j} \partial_4 = \frac{1}{2}c_2 \partial_1 + \frac{1}{2}b_2 \partial_2, \\
\nabla_{\partial_i} \partial_3 = \frac{1}{2}(a a_1 + c a_2) \partial_1 + \frac{1}{2}(b a_2 + c a_1) \partial_2 - \frac{1}{2}a_1 \partial_3 - \frac{1}{2}a_2 \partial_4,
$$
\[ \nabla_{\partial_3} \partial_4 = \frac{1}{2} (ac_1 + cc_2) \partial_1 + \frac{1}{2} (bc_2 + cc_1) \partial_2 - \frac{1}{2} c_1 \partial_3 - \frac{1}{2} c_2 \partial_4, \]
\[ \nabla_{\partial_4} \partial_4 = \frac{1}{2} (ab_1 + cb_2) \partial_1 + \frac{1}{2} (bb_2 + cb_1) \partial_2 - \frac{1}{2} b_1 \partial_3 - \frac{1}{2} b_2 \partial_4. \]

From relations above, after a long but straightforward calculation we get that the nonzero components of the \((0, 4)\)-curvature tensor of any Walker metric (2.1) are determined by

\[
R_{1313} = \frac{1}{2} a_{11}, \quad R_{1314} = \frac{1}{2} c_{11}, \quad R_{1323} = \frac{1}{2} a_{12}, \quad R_{1324} = \frac{1}{2} c_{12},
\]
\[ R_{1334} = \frac{1}{4} (a_{2} b_1 - c_1 c_2), \]
\[ R_{1414} = \frac{1}{2} b_{11}, \quad R_{1423} = \frac{1}{2} c_{12}, \quad R_{1424} = \frac{1}{2} b_{12}, \]
\[ R_{1434} = \frac{1}{4} (c_1^2 - a_1 b_1 + b_1 c_2 - b_2 c_1) \]
\[ R_{2323} = \frac{1}{2} a_{22}, \quad R_{2324} = \frac{1}{2} c_{22}, \]
\[ R_{2334} = \frac{1}{4} (-c_2^2 + a_2 b_2 + a_1 c_2 - a_2 c_1), \]
\[ R_{2424} = \frac{1}{2} b_{22}, \quad R_{2434} = \frac{1}{4} (-a_2 b_1 + c_1 c_2), \]
\[ R_{3434} = \frac{1}{4} (a c_1^2 + b c_2^2 - a a_1 b_1 - c a_1 b_2 - c a_2 b_1 - b a_2 b_2 + 2 c c_1 c_2). \] (2.2)

Next, let \( \rho(X, Y) = \text{trace}\{Z \rightarrow R(X, Z)Y\} \) and \( S_c = \text{tr}(\rho) \), be the Ricci tensor and the scalar curvature respectively. Then from (2.2) we have

\[
\rho_{13} = \frac{1}{2} (a_{11} + c_{12}), \quad \rho_{14} = \frac{1}{2} (b_{12} + c_{11}),
\]
\[ \rho_{23} = \frac{1}{2} (a_{12} + c_{22}), \quad \rho_{24} = \frac{1}{2} (b_{22} + c_{12}), \]
\[ \rho_{33} = \frac{1}{2} (-c_2^2 + a_1 c_2 + a_2 b_2 - a_2 c_1 + a a_{11} + 2 c a_{12} + b a_{22}), \]
\[ \rho_{34} = \frac{1}{2} (-a_2 b_1 + c_1 c_2 + a c_{11} + 2 c c_{12} + b c_{22}), \]
\[ \rho_{44} = \frac{1}{2} (-c_1^2 + a_1 b_1 - b_1 c_2 + b_2 c_1 + a b_{11} + 2 c b_{12} + b b_{22}) \] (2.3)

and

\[ S_c = \sum_{i, j=1}^{4} g^{ij} \rho_{ij} = a_{11} + b_{22} + 2 c_{12}. \] (2.4)
The nonzero components of the Einstein tensor \( G_{ij} = \rho_{ij} - \frac{Sc}{4} g_{ij} \) are given by

\[
\begin{align*}
G_{13} &= \frac{1}{4} a_{11} - \frac{1}{4} b_{22}, & G_{14} &= \frac{1}{2} c_{11} + \frac{1}{2} b_{12}, \\
G_{23} &= \frac{1}{2} a_{12} + \frac{1}{2} c_{22}, & G_{24} &= \frac{1}{4} b_{22} - \frac{1}{4} a_{11}, \\
G_{33} &= \frac{1}{4} a a_{11} + c a_{12} + \frac{1}{2} b a_{22} - \frac{1}{2} a_{2} c_{1} + \frac{1}{2} a_{1} c_{2} \\
&\quad+ \frac{1}{2} a_{2} b_{2} - \frac{1}{2} c_{2} - \frac{1}{2} a_{1} c_{12} - \frac{1}{4} a b_{22}, \\
G_{34} &= \frac{1}{2} a c_{11} + \frac{1}{2} c_{2} c_{12} - \frac{1}{2} a_{2} b_{1} + \frac{1}{2} c_{1} c_{2} + \frac{1}{2} b_{2} c_{22} - \frac{1}{4} c a_{11} - \frac{1}{4} c b_{22}, \\
G_{44} &= \frac{1}{2} a b_{11} + c b_{12} - \frac{1}{2} c_{2} c_{1} + \frac{1}{2} a_{1} b_{1} - \frac{1}{2} b_{1} c_{2} + \frac{1}{2} b_{2} c_{1} \\
&\quad+ \frac{1}{4} b b_{22} - \frac{1}{4} b a_{11} - \frac{1}{2} b c_{12}.
\end{align*}
\]

(2.5)

3. EINSTEIN WALKER METRICS

We now turn our attention to be Einstein conditions for the Walker metric (2.1). A Walker metric is said to be Einstein Walker metric if its Ricci tensor is a scalar multiple of the metric at each point i.e., there is a constant \( \lambda \) so that

\[ \rho = \lambda g. \]

The main result in this section is the following

**Theorem 3.1.** A Walker metric (2.1) is Einstein if and only if the defining functions \( a, b \) and \( c \) are solution of the following PDES:

\[
\begin{align*}
a_{11} - b_{22} &= 0, & b_{12} + c_{11} &= 0, & a_{12} + c_{22} &= 0, \\
a_{1} c_{2} + a_{2} b_{2} - a_{2} c_{1} - c_{2}^{2} + 2 c a_{12} + b a_{22} - a c_{12} &= 0, \\
a_{2} b_{1} - c_{1} c_{2} + a_{11} - a c_{11} - c c_{12} + b c_{22} &= 0, \\
a_{1} b_{1} - b_{1} c_{2} + b_{2} c_{1} - c_{2} + a b_{11} + 2 c b_{12} - b c_{12} &= 0.
\end{align*}
\]

**Proof.** From (2.5), the Einstein condition is equivalent to \( G_{ij} = \rho_{ij} - \frac{S c}{4} g_{ij} \). We get the following PDEs:

\[
\begin{align*}
a_{11} - b_{22} &= 0, & b_{12} + c_{11} &= 0, & a_{12} + c_{22} &= 0, \\
a_{1} c_{2} + a_{2} b_{2} - a_{2} c_{1} - c_{2}^{2} + 2 c a_{12} + b a_{22} - a c_{12} &= 0, \\
a_{2} b_{1} - c_{1} c_{2} + a_{11} - a c_{11} - c c_{12} + b c_{22} &= 0, \\
a_{1} b_{1} - b_{1} c_{2} + b_{2} c_{1} - c_{2} + a b_{11} + 2 c b_{12} - b c_{12} &= 0.
\end{align*}
\]

(3.1)

This system of partial differential equations (3.1) is hard to solve. \( \square \)

In [4], the authors apply the Lie symmetry group method to determine the Lie point symmetry group and provide example of solution of the system of partial differential equations (3.1).
Example 3.2. Let \((M, g_{a,b,c})\) be a Walker metric with
\[
a = -\frac{r_1}{r_2} e^{r_1 u_1} e^{u_2}, \quad b = -r_1 r_2 e^{r_1 u_1} e^{u_2}, \quad \text{and} \quad c = r_2 e^{r_1 u_1} e^{u_2}
\]
where \(r_i\)'s are arbitrary constants. Then the Walker metric \((M, g_{a,b,c})\) is Ricci flat and Einstein.

For the following restricted Walker metric
\[
a = a(u_1, u_2), \quad b = b(u_1, u_2) \quad \text{and} \quad c = 0 \quad (3.2)
\]
we have the following result.

**Theorem 3.3.** A Walker metric \((3.2)\) is Einstein if and only if the functions \(a(u_1, u_2)\) and \(b(u_1, u_2)\) are as follows:
\[
a = a(u_1, u_2) = Ku_1^2 + Au_1 + B(u_2),
b = b(u_1, u_2) = Ku_2^2 + Cu_2 + D(u_1),
\]
where \(K, A\) and \(C\) are constants and \(B, D\) are smooth functions satisfying the following PDE’s:
\[
B D_1 = 0, \quad (D_1(u_2^2 K + u_1 A + B))_1 = 0, \quad (B_2(u_2^2 K + u_2 C + D))_2 = 0.
\]

**Proof.** The Einstein condition is equivalent to the following:
- (i) \(a_{12} = 0\) and \(b_{12} = 0\);
- (ii) \(a_{11} - b_{22} = 0\);
- (iii) \(a_2 b_1 = 0\);
- (iv) \(a_1 b_1 + a b_{11} = 0\) and \(a_2 b_2 + b a_{22} = 0\).

We divide the proof of the proposition into two steps.

(1) Step 1. The PDE system (i) imply that \(a\) and \(b\) take the following forms:
\[
a = \bar{a}(u_1) + \hat{a}(u_2) \quad \text{and} \quad b = \bar{b}(u_1) + \hat{b}(u_2). \quad (3.3)
\]
Substituting these functions \(a\) and \(b\) from (3.3) in the equation (ii), we get
\[
a_{11} = \hat{b}_{22}.
\]
Therefore we have the following:
\[
\bar{a}_{11}(x) = K \quad \text{and} \quad \hat{b}_{22}(y) = K
\]
where \(K\) is a constant. Then \(\bar{a}\) (respectively \(\hat{b}\)) is a quadratic function of \(u_1\) (respectively \(u_2\)). Therefore we have the following
\[
a = Ku_1^2 + Au_1 + B(u_2) \quad b = Ku_2^2 + Cu_2 + D(u_1) \quad (3.4)
\]
where \(A\) and \(C\) are constants, \(B = \hat{a}\) (respectively \(D = \hat{b}\)) are smooth functions of \(u_2\) (respectively \(u_1\)).

(2) Step 2. The functions \(a\) and \(b\) in (3.4) satisfy the (i) and (ii) PDEs in the Einstein conditions. We must consider further conditions for \(a\) and \(b\) to satisfy the (iii) and (iv) PDE in the Einstein condition.
From the (iii) PDE’s in the Einstein condition, we get the following condition:

\[ B_2 D_1 = 0. \]

From (iv), the two equations \( a_1 b_1 + a b_{11} = 0 \) and \( a_2 b_2 + b a_{22} = 0 \) gives

\[
\begin{align*}
(D_1(u_1^2 K + u_1 A + B))_1 &= 0, \\
(B_2(u_2^2 K + u_2 C + D))_2 &= 0,
\end{align*}
\]

which complete the proof.

\[ \square \]

Example 3.4. Let us consider, a Walker metric \((M, g_{a,b,c})\) given by (3.2).

- If \( b = 0 \) and \( c = 0 \), then \((M, g_{a,b,c})\) is Einstein if and only if \( a = K u_1 + A \), where \( K, A \) are constant. Such a metric is Ricci flat.
- If \( a = 0 \) and \( c = 0 \), then \((M, g_{a,b,c})\) is Einstein if and only if \( b = K u_2 + B \), where \( K, B \) are constant. Such a metric is Ricci flat.
- The Walker metric with \( a = K u_1^2 \), \( b = K u_2^2 \) and \( c = 0 \) is Einstein with non-zero scalar curvature equal to \( 4K \).

Remark 3.5. The description of the Einstein for the Walker metrics with \( c = 0 \) can be also found in [3], obtaining two different families for the scalar curvature being zero or not. However the problem in the general case remains open.

Four-dimensional Einstein Walker manifolds form the underlying structure of many geometric and physical models such as: \(hh\)-space in general relativity, \(pp\)-wave models and others areas.

4. Locally Symmetric Einstein-Walker Metrics

This section deals with the locally symmetry of Einstein Walker manifolds. Let us consider the Einstein-Walker metric given in Theorem 3.3

\[
a = K u_1^2 + A u_1 + B(u_2) \quad \text{and} \quad b = K u_2^2 + C u_2 + D(u_1),
\]

where \( K, A \) and \( C \) are constants and \( B, D \) are smooth functions satisfying the following PDE’s:

\[
\begin{align*}
B_2 D_1 &= 0, \\
(D_1(u_1^2 K + u_1 A + B))_1 &= 0, \\
(B_2(u_2^2 K + u_2 C + D))_2 &= 0.
\end{align*}
\]

An Einstein-Walker manifold \((M, g_{a,b})\) is said to be locally symmetric if the curvature tensor \( R \) of \((M, g_{a,b})\) satisfies \((\nabla_X R)(Y, Z, T, W) = 0\), for any \( X, Y, Z, T, W \in \Gamma(TM) \). By a straightforward calculation, we can see that the condition for the Einstein-Walker metric (Theorem 3.3) to be locally symmetric is equivalent to the
following PDEs

\[
\begin{align*}
 a_1 a_2 b_2 &= 0, \quad a_1 b_1 b_2 = 0, \quad a_1 a_{22} = 0, \quad a_1 b_{11} = 0, \\
 a_2 b_{11} &= 0, \quad b_1 a_{22} = 0, \quad b_2 a_{22} = 0, \quad b_2 b_{11} = 0, \\
 a a_{11} b_1 &= 0, \quad b a_2 b_{22} = 0.
\end{align*}
\]

From the PDEs, We have the following results

**Theorem 4.1.** A Walker metric given in Theorem 3.3 is locally symmetric Einstein if and only if the functions \(a(u_1, u_2)\) and \(b(u_1, u_2)\) are constant.

## 5. Locally Conformally Flat Walker Metrics

Let \(W\) denote the Weyl conformal curvature tensor given by

\[
W(X, Y, Z, T) : = R(X, Y, Z, T) + \frac{Sc}{(n - 1)(n - 2)} \left\{ g(Y, Z)g(X, T) - g(X, Z)g(Y, T) \right\}
\]

\[
+ \frac{1}{n - 2} \left\{ \rho(Y, Z)g(X, T) - \rho(X, Z)g(Y, T) - \rho(Y, T)g(X, Z) + \rho(X, T)g(Y, Z) \right\}.
\]

A pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes. The nonzero components of Weyl tensor of the Walker metric defined by (2.1) are given by

\[
\begin{align*}
 W_{1313} &= \frac{a_{11}}{6} + \frac{b_{22}}{6} - \frac{c_{12}}{6}, \quad W_{1314} = -\frac{b_{12}}{4} + \frac{c_{11}}{4}, \\
 W_{1323} &= \frac{a_{12}}{4} - \frac{c_{22}}{4}, \quad W_{1324} = \frac{c_{12}}{2}, \\
 W_{1334} &= \frac{a a_{11}}{12} - \frac{a b_{12}}{4} - \frac{c b_{22}}{6} + \frac{5 c c_{12}}{12} + \frac{b c_{22}}{4}, \\
 W_{1414} &= \frac{b_{11}}{2}, \quad W_{1423} = -\frac{a_{11}}{12} - \frac{b_{22}}{12} + \frac{c_{12}}{3}, \quad W_{1424} = \frac{b_{12}}{4} - \frac{c_{11}}{4}, \\
 W_{1434} &= \frac{b a_{11}}{12} + \frac{a b_{11}}{4} + \frac{c b_{12}}{4} + \frac{b b_{22}}{12} - \frac{c c_{11}}{4} - \frac{b c_{12}}{12}, \\
 W_{2323} &= \frac{a_{22}}{2}, \quad W_{2324} = -\frac{a_{12}}{4} + \frac{c_{22}}{4}, \\
 W_{2334} &= -\frac{a a_{11}}{12} - \frac{a c_{12}}{4} - \frac{b a_{22}}{4} - \frac{a b_{22}}{12} + \frac{a c_{12}}{12} - \frac{c c_{22}}{4}, \\
 W_{2424} &= \frac{a_{11}}{6} + \frac{b_{22}}{6} - \frac{c_{12}}{6}, \quad W_{2434} = \frac{c a_{11}}{6} + \frac{b a_{12}}{4} - \frac{c b_{22}}{12} - \frac{a c_{11}}{4} - \frac{5 c c_{12}}{12}.
\end{align*}
\]
Now it is possible to obtain the form of a locally conformally flat Walker metric as follows:

**Theorem 5.1.** A Walker metric (2.1) is locally conformally flat if and only if the functions
\[ a = a(u_1, u_2), \quad b = b(u_1, u_2) \text{ and } c = c(u_1, u_2) \]
take the form
\[
\begin{align*}
  a &= \frac{I}{2} u_1^2 + Ju_1 + Eu_1 u_2 + Fu_2 + K, \\
  b &= -\frac{I}{2} u_2^2 + Lu_2 + Mu_1 u_2 + Nu_1 + R, \\
  c &= \frac{M}{2} u_2^2 + Pu_1 + \frac{E}{2} u_2^2 + Gu_2 + (Q + H),
\end{align*}
\]

where the constants \( E, F, G, H, I, J, K, L, M, N, P, Q \) and \( R \) satisfy the following relations
\[
\begin{align*}
  0 &= EN - JM + IP, \\
  0 &= EL - FM + IG, \\
  0 &= ER - KM + I(H + Q), \\
  0 &= K(LP - NG) + R(JG - FP) + (Q + H)(FN - JL).
\end{align*}
\]

**Proof.** Since the locally conformally flat is equivalently to the vanishing of the Weyl tensor, let consider (5.1) as a system of PDEs. We will prove the theorem in three steps.

(1) Step 1. Considering the following components of the Weyl tensor of (5.1):
\[
\begin{align*}
  W_{1324} = \frac{c_{12}}{2} &= 0, \quad W_{1414} = \frac{b_{11}}{2} = 0 \quad \text{and} \quad W_{2323} = a_{22} = 0. \quad (5.2)
\end{align*}
\]

The PDEs (5.2) imply that the functions \( a, b \) and \( c \) take the form
\[
\begin{align*}
  a(u_1, u_2) &= u_2 \hat{a}(u_1) + \hat{a}(u_1), \\
  b(u_1, u_2) &= u_1 \hat{b}(u_2) + \hat{b}(u_2), \\
  c(u_1, u_2) &= \hat{c}(u_1) + \hat{c}(u_2).
\end{align*}
\]
Considering the result of the step 1, the Weyl equations of (5.1) reduce to

\[
\begin{align*}
W_{1313} &= \frac{1}{6}(a_{11} + b_{22}), & W_{1314} &= \frac{1}{4}(-b_{12} + c_{11}), & W_{1323} &= \frac{1}{4}(a_{12} - c_{22}), \\
W_{1334} &= \frac{ca_{11}}{12} - \frac{ab_{12}}{4} - \frac{cb_{22}}{6} + \frac{bc_{22}}{4}, & W_{1423} &= -\frac{1}{12}(a_{11} + b_{22}), \\
W_{1424} &= \frac{b_{12} - c_{11}}, & W_{1434} &= \frac{b}{12}(a_{11} + b_{22}) + \frac{c}{4}(b_{12} - c_{11}), \\
W_{2324} &= \frac{1}{4}(-a_{12} + c_{22}), & W_{2334} &= -\frac{a}{12}(a_{11} + b_{22}) - \frac{c}{4}(a_{12} - c_{22}), \\
W_{2424} &= \frac{1}{6}(a_{11} + b_{22}), & W_{2434} &= \frac{ca_{11}}{6} + \frac{ba_{12}}{4} - \frac{cb_{22}}{12} - \frac{ac_{11}}{4}, \\
W_{3434} &= \frac{c^2}{6}(a_{11} + b_{22}) + \frac{ab}{12}(a_{11} + b_{22}) + \frac{bc}{2}(a_{12} - c_{22}) + \frac{ac}{2}(b_{12} - c_{11}) \\
&\quad + \frac{ba_{12} c_2}{4} - \frac{ca_{11} b_2}{4} + \frac{ca_{11} b_1}{4} - \frac{ba_{12} c_1}{4} - \frac{ab_{12} c_2}{4} + \frac{ab_{12} c_1}{4}. 
\end{align*}
\]  

(5.3)

(2) Step 2. Considering the following components of the PDEs (5.3):

\[
\begin{align*}
W_{1313} &= 0, & W_{1314} &= 0, & \text{and} & & W_{1323} &= 0.  
\end{align*}
\]  

(5.4)

The PDEs (5.4) imply that the functions \(a\), \(b\) and \(c\) take the form

\[
\begin{align*}
a &= u_2(Eu_1 + F) + \frac{I}{2}u_1^2 + Ju_1 + K, \\
b &= u_1(Mu_2 + N) - \frac{I}{2}u_2^2 + Lu_2 + R, \\
c &= \frac{M}{2}u_1^2 + Pu_1 + Q + \frac{E}{2}u_2^2 + Gu_2 + H. 
\end{align*}
\]

Considering the result of the step 2, the Weyl equations (5.3) reduce to

\[
\begin{align*}
W_{1334} &= \frac{ca_{11}}{12} - \frac{ab_{12}}{4} - \frac{cb_{22}}{6} + \frac{bc_{22}}{4}, \\
W_{2434} &= \frac{ca_{11}}{6} + \frac{ba_{12}}{4} - \frac{cb_{22}}{12} - \frac{ac_{11}}{4}, \\
W_{3434} &= \frac{ba_{12} c_2}{4} - \frac{ca_{11} b_2}{4} + \frac{ca_{11} b_1}{4} - \frac{ba_{12} c_1}{4} - \frac{ab_{12} c_2}{4} + \frac{ab_{12} c_1}{4}. 
\end{align*}
\]  

(5.5)

(3) Step 3. From (5.5), after some straightforward calculations, the following PDES

\[
\begin{align*}
W_{1334} &= 0, & W_{2434} &= 0 & \text{and} & & W_{3434} &= 0
\end{align*}
\]

gives

\[
\begin{align*}
0 &= EN - JM + IP, \\
0 &= EL - FM + IG, \\
0 &= ER - KM + I(H + Q), \\
0 &= K(LP - NG) + R(JG - FP) + (Q + H)(FN - JL).
\end{align*}
\]

This finish the proof. \(\square\)
Remark 5.2. From (2.4) and Theorem 5.1, we see that the locally conformally flat metric (2.1) has vanishing scalar curvature.

REFERENCES

[1] A. Banyaga and F. Massamba, Non-existence of certain Einstein metrics on some symplectic manifolds, Forum Math. to appear.
[2] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikević and R. Vázquez-Lorenzo, The Geometry of Walker Manifolds, Synthesis Lectures on Mathematics and Statistics, 5. (Morgan and Claypool Publishers, Williston, VT, 2009).
[3] M. Chaichi, E. García-Río and Y. Matsushita, Curvature properties of four-dimensional Walker metrics, Classical Quantum Gravity, 22 (2005), 559-577.
[4] Medhi Nadjafikhah and Medhi Jafari, Some general new Einstein Walker manifolds, Adv. Math. Phys. 2013, Art. ID 591852, 8 pp.
[5] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, Quart. J. Math. Oxford 1 (2) (1950), 69-79.