Multi-functional Twisted-Kagome lattices: Tuning by Pruning Mechanical Metamaterials

Danilo B. Liarte,1 O. Stenull,2 and T. C. Lubensky2

1Cornell University, Ithaca, NY, USA
2University of Pennsylvania, Philadelphia, PA, USA

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This article investigates phonons and elastic response in randomly diluted lattices constructed by combining (via the addition of next-nearest-bonds) a twisted kagome lattice, with bulk modulus \( B = 0 \) and shear modulus \( G > 0 \), with either a generalized untwisted kagome lattice with \( B = 0 \) and \( G > 0 \) or with a honeycomb lattice with \( B > 0 \) and \( G = 0 \). These lattices exhibit jamming-like critical end-points at which both \( B \) and \( G \) jump discontinuously from zero while the remaining moduli (if any) begin to grow continuously from zero. Pairs of these jamming points are joined by lines of continuous rigidity percolation transitions at which both \( B \) and \( G \) begin to grow continuously from zero. The Poisson ratio and \( G/B \) can be continuously tuned throughout their physical range via random dilution in a manner analogous to “tuning by pruning” in random jammed lattices. These lattices can be produced with modern techniques, such as 3D printing, for constructing metamaterials.

I. INTRODUCTION

Ball-and-spring networks provide useful and generally accurate models for the elastic properties of solids from periodic crystals to disordered glasses. These networks undergo a transition from an elastically rigid state to a floppy one when their coordination number \( z \) undergo a transition from an elastically rigid state to a periodic crystals to disordered glasses. These networks accurate models for the elastic properties of solids from ability periodic lattices \([1–3]\) with springs removed with prob-

ability \( p \) is roughly analogous to a critical end-

point in which a second-order \( RP \) line \( (J_B Y) \) meets a first-order line \([12]\). Both \( B \) and \( G \) grow with distance from the RP line, but along paths like \( CJ_B D \) that pass through \( J_B \), \( B \) jumps discontinuously and \( G \) grows continuously from zero at \( J_B \) as in jamming. Paths starting at \( J_B \) and ending at \( Y \) cover the range of \( G/B \) from 0 to 1/2 (or Poisson ratio from 1 to 1/3) without reaching any negative values.

This paper introduces and, using both effective medium theory (EMT) and numerical simulations, explores the elastic response of two periodic lattice models \([Fig. 2]\), both of which have average \( C_3 \) symmetry and macroscopic elastic energies in the isotropic class charac-

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\*liarte@cornell.edu
\†tom@physics.upenn.edu
terized by nonvanishing $B$ and $G$ with no moduli arising from anisotropy. Both models access negative values of the Poisson ratio $\sigma$, one of which accesses the full range from $\sigma = -1$ to $\sigma = +1$. The starting point of both is the twisted kagome lattice (TwKL) [Fig. 2(b)], which by itself has a non-zero shear modulus but a vanishing bulk modulus \cite{13} and, thus, a Poisson ratio of $-1$. In the first model, the TwK/GK model [Fig. 2(c)], springs are placed on NNN bonds of the TwK with probability $p_b$. When $p_b = 1$, these bonds form three independent untwisted generalized kagome lattices (GKL), composed of two different-sized rather than single-sized triangles, for which $B/G = 2$ and $\sigma = 1/3$. Thus points in the rigid regime cover the range of $\sigma$ from $-1$ to $1/3$. In the second, the TwK/H model, bonds connecting a collection of NNN and third-neighbor points of the TwK lattice form three independent honeycomb lattices [Fig. 2(d)]. Thus, in this lattice, bonds in the TwK lattice are the analog of $G$-bonds in the TbP model and those in HL the analog of $B$-bonds; those in the GK lattice form both $B$ and $G$ bonds. Figure 3 displays the phase diagrams of these models, to be explained more fully in the next section.

In principle, these models provide a simple algorithm for creating, via 3D printing or related methods, physical 2D materials with arbitrary Poisson ratios. They do, however, suffer from a technical drawback in that the added bonds cross each other and necessarily introduce additional nodes in a purely 2D geometry. This drawback can be addressed in two ways. In the first, all but one of the extra, GK bond-lattices introduced in the TwK/GK lattice by the further neighbor bonds can be eliminated. Lattices constructed in this way have no bond crossings, and they have $C_3$ symmetry and thus isotropic elasticity \cite{14}. An alternative approach is to stack different lattices formed by further-neighbor bonds in different layers connected by rigid vertical bonds between identical realizations of the original TwK lattice as shown in Fig. 4.

\section{RESULTS}

Both of our models are built on the TwK lattice, whose bulk modulus is zero, but because of the different geometries imposed by the further-neighbor bonds, they have different-size unit cells. The unit cell of the TwK/GK model has the same number of sites ($j = 3$) and NN $a$-bonds ($\tilde{z}_a = 6$) as the TwK lattice, and it has the same number of $b$-bonds ($\tilde{z}_b = 6$) as $a$-bonds. The unit cell of the TwK/H lattice is three times as large as that of the TwK lattice with $j = 9$ and $\tilde{z}_a = 18$ but with $\tilde{z}_b = 9$. In what follows, Sec. \text{II} reviews our principal results, Sec. \text{III} defines our model energies and their elastic limits and auxetic response, Sec. \text{IV} discusses our numerical simulations, Sec. \text{V} presents our effective medium theory (EMT) and its scaling predictions at critical points, and Sec. \text{VI} presents a summary discussion. The appendices provide details of the lattice structures, dynamical matrices, dispersion relations, asymptotic behavior of the EMT integrals and additional three-dimensional plots of the moduli.
The black-dashed arrows $\sigma$ and the Poisson ratio $\nu$ lines are respectively paths toward jamming, and double-jamming points. The red, green, and black lines are perpendicular to that line. The lines of RP transitions between the EMT points $J_G$ and $J_B$, and $J_{BG}$, denote jamming, shear-jamming, and double-jamming points. The red, green, and black lines are respectively paths toward $J_G$, $J_B$ or $J_{BG}$, and RP lines along paths perpendicular to them. The ratio $\nu / \mu$ and the Poisson ratio $\nu$ along these lines are plotted in Fig. 3. The black-dashed arrows $[CJ_G D$ in (a)] depict a path along which the shear modulus varies discontinuously at the rigidity transition. Finally, $(p_a, 1)$ and $(p_b, 1)$ are the boundary $A$- and $B$-lines, respectively.

FIG. 3. Phase Diagrams of the TwK/GK (a) and TwK/H models (b) showing auxetic rigid regions (red), floppy regions (white) and regular rigid regions with $\sigma > 0$ (blue). $J_B$, $J_G$ (with $G = G1$ or $G2$) and $J_{BG}$ denote jamming, shear-jamming, and double-jamming points. The red, green, and black lines are respectively paths toward $J_G$, $J_B$ or $J_{BG}$, and RP lines along paths perpendicular to them. The ratio $\nu / \mu$ and the Poisson ratio $\nu$ along these lines are plotted in Fig. 3. The black-dashed arrows $[CJ_G D$ in (a)] depict a path along which the shear modulus varies discontinuously at the rigidity transition. Finally, $(p_a, 1)$ and $(p_b, 1)$ are the boundary $A$- and $B$-lines, respectively.

FIG. 4. One can design a 3D-printable version of the TwK/H lattice by adding pins to the sites of the twisted kagome lattice (a), and sequentially stacking the remaining three honeycomb lattices (b)-(d) on top of it.

[Note: $z_0$ is the number of $r$-bonds in a unit cell, which is half the coordination number $z_c$ associated with these bonds]. These lattice characteristics are summarized in Table 1. The Maxwell count, setting the number of degrees of freedom per unit cell ($jd = 2j$ in two dimensions) equal to the average number of constraints per cell ($p_a z_a + p_b z_b$), predicts the EMT RP phase boundary in the $p_a - p_b$ EMT phase diagram shown in Fig. 3 to occur at $\Delta p_{RP} = 0$.

\[ \Delta p_{RP} = p_a z_a + p_b z_b - jd \]  \hspace{1cm} (1)

measures the distance from the RP line along a path perpendicular to that line. The lines of RP transitions $[lines J_G - J_{BG}$ and $J_G - J_B$] terminate at critical points at their intersections with the boundary lines $A = (p_a = 1, p_b)$ and $B = (p_a, p_b = 1)$. In both cases, Eq. (1) sets the intersection with the $A$-line at

\[ p_b^A = (jd - z_a) / z_b = 0. \]  \hspace{1cm} (2)

The points $J_G = (1, p_b^A)$ (with $G = G1$ or $G2$) are “shear-jamming” points [5, 15, 16], at which $G$ jumps discontinuously from zero in paths (such as $CJ_G D$ in Fig. 3(a)) from the floppy region [17]. The second intersection at $p_b^{RP} = 1$ occurs at $p_a^{BG} = 0$, i.e., at $J_{BG} = (0, 1)$ in the TwK/GK model and at $p_b^{BG} = (jd - z_b) / z_a = (2 \times 9 - 9) / 18 = 1/2$, i.e., at $J_{BG} = (1, 2)$ in the TwK/H model. The HL of the TwK/H model at $J_B$ is fully formed and resists compression, but the system is still on the RP-line along which $G = 0$. Thus $J_B$ is a jamming point at which $B$ jumps discontinuously. At the point $J_{BG} = (0, 1)$, only the three GKLs survive, each consisting of three grids of sample-traversing lines of parallel bonds with two rather than a single spacing between lines. These lines provide states of self-stress that lead to both $B$ and $G$ being positive [18]. As a consequence, $J_{BG}$ is a double “jamming” point at which both $B$ and $G$ jump continuously from zero.

Figure 3 also shows data simulation points that indicate an RP-transition line that lies mostly below, but close to, the EMT RP-line and terminates within numerical error at the EMT points $J_G$ and $J_{BG}$. The difference between the EMT RP-lines is greatest in the TwK/H model near $J_B$.

In the vicinity of the “jamming” critical points in the EMT, all of the elastic moduli $K$ that undergo a discontinuous jump and satisfy the simple scaling equation,

\[ K / K_0 = \frac{\Delta p_{RP}}{\Delta p_{RP} + C_M \Delta p_M} = \left( 1 + C_M \frac{\Delta p_M}{\Delta p_{RP}} \right)^{-1}, \]  \hspace{1cm} (3)

where $C_M$ is a numerical constant that depends on the jamming point [19], $\Delta p_M$ equals $1 - p_a$ for the two $J_G$ points, $1 - p_b$ for the $J_B$, and $J_{BG}$ points and where $\Delta p_{RP}$ is defined in Eq. (1). This scaling form predicts $K = K_0$ when $\Delta p_M = 0$ for any $\Delta p_{RP} > 0$. Thus, for example $G$ undergoes a discontinuous jump at the point $J_G$ along a path such as $CJ_G D$ in Fig. 3(a). Away from the jamming points and near the RP line, all moduli grow linearly with $\Delta p_{RP}$ with a coefficient that changes with distance along the RP line. This behavior is clearly indicated in Eq. (3).

Figure 5 shows numerical evaluation of the full EMT equations in the vicinity of jamming points collapse onto

| TwK/GK | $\tilde{z}_a$ | $\tilde{z}_b$ | $\gamma$ | $\delta$ |
|-------|--------|--------|--------|--------|
| TwK/H | 18     | 9      | 9      | 9      |

TABLE 1. Table of basic parameters of the TwK/GK and TwK/H lattices.
the analytical form of Eq. 3, with the coefficient $c_J$ depending on the jamming point. The simulation data collapses onto a modification of the Eq. 3 that takes account of the fact that the RP transition-line lies below the RP line. As required, the numerical solution to the EMT equations also show linear growth of the bulk modulus $B$ near the $J_G$ points of both the TwK/GK and TwK/H models and of the shear modulus $G$ at the $J_B$ point of the TwK/H model. The simulation data are consistent with linear growth of $B$ near the $J_G$ points but are more consistent with quadratic behavior, which may be due to finite-size effects, of $G$ very near the $J_B$ point of the TwK/H model. Figure 6 shows the variation of $G/B$ and the Poisson ratio along the paths shown in Fig. 4. Note that dilution of our lattices induces changes in the network geometry and hence strongly affects the Poisson ratio, in agreement with the results of Ref. [20]. Three-dimensional plots of $B$ and $G$ obtained both from our EMT and our numerical simulations are shown in Appendix B.

FIG. 5. Scaling behavior of the TwK/GK ((a) and (b)) and TwK/H ((c) and (d)) models. Filled and open circles represent the shear and bulk moduli, respectively. Gray and blue symbols correspond, respectively, to full EMT solutions and to numerical simulations for a set of points in a rigid region in the neighborhood of $J_G$ ((a) and (c)), $J_B$ (d), and $J_{BC}$ (c). The dashed lines correspond to our analytical predictions (Eq. 3 normalized near the critical points).

FIG. 6. $G/B$ ((a) and (c)) and Poisson ratio ((b) and (d)) as a function of $\Delta z \equiv 2\Delta p_{RP}/(mu)$ for the TwK/GK ((a) and (b)) and the TwK/H ((c) and (d)) models, along the three paths depicted in Figure 4. The discrepancy between simulation and EMT at low $\Delta z$ for some paths is largely due to the discrepancy for the value of the phase boundary $z_c$, which is largest near $J_B$ of the TwK/H model.

III. MODEL ENERGIES AND ELASTIC ENERGIES

We consider the harmonic interaction energy arising from central force springs:

$$E = \sum_{\alpha \in \{a,b\}} \frac{k_{\alpha}}{2} \sum_{\{i,j\} \in C_{\alpha}} g_{ij}^{\alpha} [(u_i - u_j) \cdot \hat{r}_{ij}]^2,$$

where $u_i$ is a displacement vector, $\hat{r}_{ij} = (r_i - r_j) / |r_i - r_j|$, with $r_i$ giving the position of site $i$ in the reference lattice, and $C_{\alpha}$ is a set of neighbor pairs of sites for sub-lattice $\alpha$. In EMT, $g_{ij}^{\alpha} = 1, \forall i,j$, and bonds in lattices $a$ and $b$ are populated with springs with spring constants $k_a$ and $k_b$ satisfying a set of self-consistent equations depending on probabilities $p_a$ and $p_b$. In the simulations, $k_a = 1$ and $g_{ij}^{\alpha}$ is a bimodal random variable equal to one with probability $p_a$ and zero with complementary probability $1 - p_a$. In Appendix A we provide details about the lattice structures, dynamical matrices of our models, and details of phonon dispersion relations of the TwK/GK model.

In the long-wavelength limit, Eq. (4) reduces to the elastic isotropic limit

$$\frac{E}{V} = \frac{B}{2} (u_{xx} + u_{yy})^2 + 2G \left( u_{xy}^2 + \frac{1}{4} (u_{xx} - u_{yy})^2 \right),$$

(5)
where $u_{ij}$ are components of the linearized strain tensor, $B$ and $G$ are the bulk and shear moduli, respectively, and $V$ is the volume.

For the TwK/GK model, analytical expressions for $B$ and $G$ in terms of $k_a$, $k_b$ and $\alpha$ can easily be derived:

\begin{align}
B &= \frac{3}{4} k_b 2 k_a + 3 k_b - k_a \cos 2\alpha, \\
G &= \frac{3}{16} (k_a + 3 k_b).
\end{align}

Note that $B \rightarrow 0$ and $G > 0$ for $k_b \rightarrow 0$ and $k_a > 0$, except at $\alpha = 0$, where $B \rightarrow (3/8) k_a > 0$. The Poisson ratio,

\[ \sigma = \frac{B - G}{B + G}, \]

is negative (auxetic structure) for

\[ \sin^2 \alpha > \frac{k_b (k_a + 3 k_b)}{k_a (k_a - k_b)}. \]

The phase diagram of Fig. 3 shows auxetic regions in red and and non-auxetic regions in blue for $\alpha = \pi/12$.

Calculation of the moduli for the TwK/H model poses a greater challenge than it does for the TwK/GK model, and we present only numerical solutions for $\alpha = \pi/12$. To evaluate $B$ and $G$ for arbitrary numerical values of $k_a$ and $k_b$, we first project affine deformations into the states of self stress of our lattice model. We then use Eq. (3.10) of Ref. [18] to express the elastic free energy as a quadratic form in terms of strain components [21]. To extract $B$ and $G$, we compare the resulting free energy with the isotropic elastic energy given by Eq. (5). As expected, our numerical evaluations show that $B \rightarrow 0$ and $G > 0$ when $k_b \rightarrow 0$ and $k_a > 0$ (as in the region near $J_G$ in Fig. 3b), whereas $G \rightarrow 0$ and $B > 0$ for $k_a \rightarrow 0$ and $k_b > 0$ (as in the region near $J_B$ in Fig. 3b). To find the threshold for auxetic behavior, we numerically solve the equation $B = G$ (corresponding to $\sigma = 0$) for $\eta \equiv k_b / k_a$, and find that the poisson ratio is negative (auxetic structure) for $k_b \gtrsim 0.37 k_a$.

**IV. NUMERICAL SIMULATIONS**

In this section, we briefly describe our numerical simulations. As a first step, we generate supercells composed of $N_{\text{cell}} = L \times L$ unit cells of our two model lattices. For the TwK/GK we use $L = 64$, and for TwK/H with its unit cell three times larger, we use $L = 32$. The resulting number of sites per supercell is 12,288 for the TwK/GK and 9,216 and the TwK/H. Next, we randomly remove $a$ and $b$ bonds from the supercells with probability $1 - p_a$ and $1 - p_b$, respectively. Care is taken that the removal of the exterior bonds is consistent with periodic boundary conditions.

To calculate the elastic moduli of the resulting diluted supercells, we apply affine deformations via multiplying the site positions with the deformation tensor,

\[ \Lambda_{\text{bulk}} = \begin{pmatrix} 1 + \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 + \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \Lambda_{\text{shear}} = \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

for bulk and pure shear deformation, respectively. We set $\chi$, specifying the magnitude of the deformation, to 0.01. In addition to the affine deformation, the displacement $u_i$ is given a non-affine component $\delta u_i$, $u_i \rightarrow u_i + \delta u_i$, to allow for a relaxation of the supercell. Then, we minimize the resulting energy as given in Eq. (4) over the $\delta u_i$ using a conjugate gradient algorithm adapted from Numerical Recipes [22]. Denoting the minima of the elastic energy density $f = E/V$ with respect to the two applied deformations by $f_{\text{bulk}}^{\min}$ and $f_{\text{shear}}^{\min}$, the bulk and shear moduli of the TwK/GK are then obtained as

\[ B = \frac{2 f_{\text{bulk}}^{\min}}{N_{\text{cell}} X^2} \quad \text{and} \quad G = \frac{f_{\text{shear}}^{\min}}{2 N_{\text{cell}} X^2}. \]

For the TwK/H, we divide the right hand sides by an extra factor of 3 to compensate for fact that the unit cell is three times larger than that of the TwK/GK. Finally, the so-obtained moduli are averaged over a number (usually ten) lattice realizations for any fixed given pair of $p_a$ and $p_b$.

**V. EMT AND CRITICAL SCALING**

This section provides details of our EMT calculations and their results. We assign occupancy probabilities $p_a$ and $p_b$ for bonds on sub-lattices $a$ (the TwK sub-lattice) and $b$ (the GK sub-lattice in the TwK/GK model and the H sub-lattice in the TwK/H model, respectively). The effective spring constants $k_a$ and $k_b$ satisfy a set of self-consistent equations given by the EMT [11, 23, 25]:

\[ k_a = \frac{p_a h_a}{1 - h_a}, \quad k_b = \frac{p_b h_b}{1 - h_b}, \]

where

\[ h_\alpha = \frac{k_a}{\tilde{z}_\alpha N_c} \sum_q \text{Tr} K_\alpha(q) \cdot D^{-1}(q), \quad \alpha = a, b, \]

where $\tilde{z}_\alpha$ is the number of $\alpha$-bonds per unit cell [See Table 1, and $N_c$ is the number of unit cells. $K_\alpha$ is the normalized stiffness matrix, $D = k_b K_a + k_a K_b$ is the dynamical matrix, and the trace is taken over $jd$-dimensional matrices (see Appendix A for details). The integrals $h_\alpha$ satisfy the index summation rule [11, 111]:

\[ \tilde{z}_a h_a + \tilde{z}_b h_b = jd, \]

which establishes that $h_a$ and $h_b$ are not independent.
The functions $h_a$ and $h_b$ depend upon which lattice they are associated with. They can be evaluated numerically for any $k_a$ and $k_b$, and we provide graphs of them in Appendix B. Here we derive analytical expressions for these functions in the vicinity of each of the jamming points. Before proceeding, however, it is useful of introducing the concept of majority and minority lattices associated with these critical points. The majority lattice is the one whose bond occupation probability is exactly one at the jamming point in question, and the minority lattice is the one whose bond occupation probability is less that one at the same point.

### Jamming points $J_{G1}$ and $J_{G2}$ (see Fig. [3]):
In both cases, the majority lattice is the TwKL, whose stiffness matrix $K_0 \equiv K_M$ is fully gapped, and thus invertible, for all $q$ except $q = 0$. The subscript $M$ refers to the majority lattice. The evaluation of the expansion of $K_M$ in powers of $k_m/k_M = k_b/k_a$, where $m$ refers to the minority lattice proceeds as follows:

$$h_M = h_a = \frac{1}{\tilde{z}_MN_c} \sum_q \text{Tr} K_M(q) \left( K_M + \frac{k_m}{k_M}K_m \right)^{-1}$$

$$= \frac{1}{\tilde{z}_M} \sum_q \text{Tr}[K_M \cdot \Delta^{-1}(k_m/k_M)K_M^{-1}K_m \cdots]$$

$$= \frac{jd}{\tilde{z}_M} \cdot \frac{k_m}{c_M \tilde{z}_M k_M} = 1 - \frac{1}{c_M \tilde{z}_M k_M} \equiv 1 - \Delta h_M,$$

where $\Delta h_M = 1 - h_M$ and

$$c_M = \left( \frac{1}{N_c} \sum_q \text{Tr}K_M^{-1} \cdot K_m \right)^{-1}$$

with the numerical constant $c_M$ (see Table II) depending on the jamming point. Note that in both cases, $h_M \to 1$ as $k_m/k_M \to 0$. The value of $h_m$, the minority field then follows directly from Eq. (14):

$$h_m = h_b = \frac{1}{\tilde{z}_m} (jd - \tilde{z}_M h_m) = \frac{\tilde{z}_M}{\tilde{z}_m} \Delta h_M,$$

because $md - \tilde{z}_M = 6 - 6 = 0$.

### Jamming point $J_{BG}$:
In this case, the majority lattice is the $b$-lattice, which consists of three distinct GKLs that decouple from each other and from the minority TwKL or $a$-lattice. The stiffness matrix $K_M = K_b$ has two zero modes for each wavenumber $q$ along the symmetry lines $\Gamma K$ and $KM$ in the Brillouin zone. The result is that the calculation of $h_a$ and $h_b$ is considerably more complicated than it is at the $J_G$ points. Fortunately, the “heavy lifting” for this calculation has already been done in Ref. [24] with the result

$$h_{MBG} = 1 - \frac{1}{\tilde{z}_a} \left( \frac{1}{c_M} \frac{k_a}{k_b} \right)^{1/2}$$

$$h_{MBG} = \frac{jd - \tilde{z}_a h_m}{\tilde{z}_M} = 1 - \frac{1}{\tilde{z}_b} \left( \frac{1}{c_M} \frac{k_a}{k_b} \right)^{1/2}.$$

We reemphasize at this point that a nonzero $k_a$ at $J_{BG}$ produces both a nonzero $B$ and a nonzero $G$, and both undergo a discontinuous jump. Also note that the constants $c_M$ appearing in Eq. (19) and later in Eq. (20) are numerically estimated using the definition of the $h$ integrals; they cannot be evaluated using Equation (16).

### Jamming point $J_B$:
The majority lattice is again the $b$-lattice and the minority lattice the $a$-lattice. Now $K_M$ has several zero modes for each wavenumber in the Brillouin zone and is thus non-invertible, which considerably complicates the calculation of the $h$‘s. The count of zero modes in $K_M$ is obtained as follows: When $k_m = k_a = 0$, there are three sites per unit cell (or equivalently per wavenumber) that are unattached to the network and unconstrained in their motion. This gives $3 \times 2 = 6$ zero modes per wavevector $q$. In addition when $k_a = 0$, the three $H$ lattices are not attached to each other nor to the TwK lattice, and each of the three $H$ lattices has one zero mode per wavenumber $q$. In Eq. (15a), $[K_M + (k_m/k_M)K_m]$ is projected onto the range of $K_M$ whose dimension is $d_R = jd - d_{M0} = 2 \times 9 - 9 = 9$. The limit of $k_m \to 0$ gives $h_M = h_b = d_R/\tilde{z}_b = 1$. In addition though it may not be immediately obvious, $h_M$ has a well-behaved power series in $k_m/k_M$. As a result, $h_M$ has the same functional form as it has in the vicinity of the $J_G$ points. $h_m$, however is different in that its value $k_m \to 0$ is not zero, as follows from the application of Eq. (14):

$$h_{MB} = \frac{jd - \tilde{z}_b h_m}{\tilde{z}_M} = \frac{jd - \tilde{z}_b}{\tilde{z}_a} + \frac{\tilde{z}_b}{\tilde{z}_a} \Delta h_M$$

$$= \frac{1}{2} + \frac{1}{c_M} \frac{k_a}{k_b}.$$

We are now ready to calculate the effective spring constants near all of the jamming points. Following Eqs. (12), (11) and (14), we can express $k_M$ and $k_m$ as

$$k_M = \frac{\tilde{z}_M \Delta h_M - \tilde{z}_M \Delta p_M}{\tilde{z}_M \Delta h_M}$$

$$k_m = \frac{\Delta p_R + \tilde{z}_M \Delta p_M - \tilde{z}_M \Delta h_M}{s - \tilde{z}_M \Delta h_M}.$$
where
\[ s = \tilde{z}_a + \tilde{z}_b - jd. \] (23)

Taking the ratio of \( k_m \) to \( k_M \) and using Eqs. (15c) and (19), we obtain
\[ c_M (\tilde{z}_M \Delta h_M)^n \approx \frac{\Delta p_{RP} + \tilde{z}_M \Delta p_M - \tilde{z}_M \Delta h_M}{s - \tilde{z}_M \Delta h_M + \frac{1}{\tilde{z}_M \Delta h_M - \tilde{z}_M \Delta p_M}}. \] (24)

where \( n = 0 \) applies to the \( J_G \) and \( J_B \) points and \( n = 1 \) applies to the \( J_{BG} \) point. Solving this equation for \( \Delta h_M \) when \( n = 0 \), we obtain
\[ \tilde{z}_M \Delta h_M - \tilde{z}_M \Delta p_M \approx \frac{\Delta p_{RP}}{1 + s c_m}, \] (25)
and then from Eqs. (21) and (22),
\[ k_M \approx \frac{\Delta p_{RP}}{\Delta p_{RP} + (1 + s c_m) \tilde{z}_M \Delta p_M}. \] (26)

and
\[ k_m \approx \frac{c_M \Delta p_{RP}}{1 + s c_m}. \] (27)

Finally when \( n = 1 \) (\( J_{BG} \)), the equation for \( \Delta h_M \) is quadratic rather than linear with a solution to second order in \( \Delta p_{RP} \) and \( \Delta p_M \) of
\[ \tilde{z}_M \Delta h_M \approx (\Delta p_{RP} + \tilde{z}_M \Delta p_M) (1 - s c_m \Delta p_{RP}), \] (28)
\[ k_M \approx \frac{\Delta p_{RP}}{\Delta p_{RP} + \tilde{z}_M \Delta p_M}, \] (29)
\[ k_m \approx c_M \Delta p_{RP} (\Delta p_{RP} + \tilde{z}_M \Delta p_M). \] (30)

VI. REVIEW AND FUTURE QUESTIONS

This paper has presented an analysis, via Effective-Medium Theory (EMT) and numerical simulations, of the varied elastic and phonon properties of model lattices of central-force harmonic springs that tune continuously from a twisted kagome lattice with \( B = 0 \) and \( G > 0 \) to either a honeycomb lattice with \( B > 0 \) and \( G = 0 \) or to a generalized untwisted kagome lattice with both \( B \) and \( G \) greater than zero. In each case the two extreme lattices share the same lattice sites but have a different and mutually exclusive set of bonds, which can be occupied with springs with probabilities \( p_a \) and \( p_b \). The phase diagrams in the 2D \( p_a - p_b \) space [Fig. 3] exhibit jamming critical-end-points, at which one of or both \( B \) and \( G \) jump discontinuously from zero, that terminate lines of second-order rigidity-percolation transitions separating the rigid from the floppy regime. EMT provides a semi-quantitative picture, verified by simulations, of the various transitions and, in particular, an analytic representation of elastic moduli in the vicinity of the jamming points.

The values of \( G/B \) and the Poisson ratio \( \sigma \) vary continuously with \( p_a \) and \( p_b \), which can be tuned to reach arbitrarily close to physical limits such as \( \sigma = \pm 1 \). Our algorithm for reaching these limits is less complicated than “tuning by pruning” (TbP) [7] in that it involves only the variation of \( p_a \) and \( p_b \) rather than the testing of the effects of removing each individual spring in the lattice. On the other hand, our algorithm only calculates the average effect of dilution. For a given average coordination number \( z \) after dilution, there are certainly specific spring configurations that get closer to physical limits than does the average configuration. By construction TbP takes the system as close as possible to a given goal such as the maximum value of \( G/B \) or \( \sigma \). This presumably explains why references [7, 8] access more extreme values of \( G/B \) or \( \sigma \) for a given \( z \) than does our approach. It would be interesting to investigate in more detail the statistical distributions of \( G/B \) and \( \sigma \) arising from random dilution, or to apply the TbP to our system.

It would also be interesting to create laboratory versions of our lattices, which can certainly be done using modern fabrication techniques like 3D printing, and to measure their elastic and mechanical properties. These synthetic lattices will necessarily have bending forces that favor particular angles between bonds and thereby increase their rigidity relative to that of simple central-force models. The effect of these bending forces has yet to be studied in detail. Their effect on surfaces states of topological mechanical lattices and on auxetic transitions have been studied in Refs. [26] and [27], respectively.

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Appendix A: Lattice structures, dynamical matrices, and dispersion relations

In this section we provide additional information relating to the lattice structures and dynamical matrices of both the TwK/GK and TwK/H models, as well as dispersion relations for the TwK/GK model.
1. Lattice structures

Figure 7(a) shows the unit cell of the TwK/GK lattice, its three-point basis and a set of unit vectors used in our calculations. We set the origin of each cell at the position of the first atom of the unit cell, so that atoms of the three-point basis are located at \( a_1 = (1/\cos \alpha) R(\alpha) \cdot (0, 0), \) \( a_2 = (1/\cos \alpha) R(\alpha) \cdot (1/2, 0) \) and \( a_3 = (1/\cos \alpha) R(\alpha) \cdot (1/4, \sqrt{3}/4), \) where

\[
R(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}
\]  

is a rigid rotation matrix and the \( \cos \alpha \) factor in the denominator ensures that the cell size does not change with twist angle. The lattice translation vectors are given by \( a_1 = (-1/2, -\sqrt{3}/2), \) \( a_2 = (1, 0) \) and \( a_3 = (-1/2, \sqrt{3}/2). \) The vectors \( e_i(\alpha) = R(\alpha) \cdot a_i \) determine the directions of bonds for the \( a \)-sublattice. The vectors \( b_1 = (0, 1), \) \( b_2 = (-\sqrt{3}/2, -1/2) \) and \( b_3 = (\sqrt{3}/2, -1/2) \) are perpendicular to \( a_2, a_3 \) and \( a_1, \) respectively, and determine the directions of bonds of the \( b \)-sublattice. See Fig. 7(b) for an illustration of the \( a, b \) and \( e \) vectors. Figure 7(c) shows the unit cell of the TwK/H lattice with its nine-point basis. Note that here \( b_i \) are lattice translation vectors for the TwK/H model.

2. Dynamical matrices

Equation 1 can be written in Fourier space as

\[
E = \frac{1}{2N_c^2} \sum_{q,q'} \mathbf{u}(q) \cdot D(-q,q') \cdot \mathbf{u}(q'),
\]

where \( N_c \) is the number of cells, \( \mathbf{u}(q) \) is the Fourier transform of \( \mathbf{u}(r) \), and the dynamical matrix is given by,

\[
D(-q,q') = N_c \delta_{q,q'} D(q)
\]

with

\[
D(q) = \sum_{\alpha \in \{a,b\}} k_\alpha K_\alpha(q),
\]

where \( K_\alpha \) is the stiffness matrix,

\[
K_\alpha = \sum_{n=1}^{\tilde{n}_\alpha} B_n^\alpha(q) B_n^\alpha(-q).
\]

\( \tilde{n}_\alpha \) is the number of bonds per unit cell of sub-lattice \( \alpha \).

For the TwK/GK model, the \( B \)-vectors are given by:

\[
\begin{align*}
B_1^a(q) &= (e_1(\alpha), 0, -e_1(\alpha)), \\
B_2^a(q) &= (-e_2(\alpha), e_2(\alpha), 0), \\
B_3^a(q) &= (0, -e_3(\alpha), e_3(\alpha)), \\
B_4^a(q) &= (-e_1(-\alpha), 0, e^{-iq \cdot a_1} e_1(-\alpha)), \\
B_5^a(q) &= (e^{-iq \cdot a_2} e_2(-\alpha), -e_2(-\alpha), 0), \\
B_6^a(q) &= (0, e^{-iq \cdot a_3} e_3(-\alpha), -e_3(-\alpha)),
\end{align*}
\]

\[
\begin{align*}
B_1^b(q) &= (-b_1, e^{-iq \cdot a_2} b_1, 0), \\
B_2^b(q) &= (0, -b_2, e^{-iq \cdot a_1} b_2), \\
B_3^b(q) &= (e^{-iq \cdot a_2} b_3, 0, -b_3), \\
B_4^b(q) &= (-b_1, e^{-iq \cdot a_2} b_1, 0), \\
B_5^b(q) &= (0, -b_2, e^{-iq \cdot a_1} b_2), \\
B_6^b(q) &= (e^{-iq \cdot a_2} b_3, 0, -b_3).
\end{align*}
\]
For the TwK/H model, the B-vectors are given by:

\[
\begin{align*}
B_1^a(q) &= (e_1(\alpha), 0_1, -e_1(\alpha), 0_3), \\
B_2^a(q) &= (-e_2(\alpha), e_2(\alpha), 0_7), \\
B_3^a(q) &= (0_1, -e_3(\alpha), e_3(\alpha), 0_6), \\
B_4^a(q) &= (-e_1(-\alpha), 0_4, e^{-i q \cdot b_2} e_1(-\alpha), 0_3), \\
B_5^a(q) &= (0_1, -e_2(-\alpha), 0_1, e_2(-\alpha), 0_5), \\
B_6^a(q) &= (0_2, -e_3(-\alpha), 0_1, e^{-i q \cdot b_3} e_3(-\alpha), 0_4), \\
B_7^a(q) &= (0_3, e_1(\alpha), 0_1, -e_1(\alpha), 0_3), \\
B_8^a(q) &= (0_3, -e_2(\alpha), e_2(\alpha), 0_1), \\
B_9^a(q) &= (0_4, -e_3(\alpha), e_3(\alpha), 0_4), \\
B_{10}^a(q) &= (0_3, -e_1(-\alpha), 0_4, e_1(-\alpha)), \\
B_{11}^a(q) &= (0_4, -e_2(-\alpha), 0_1, e^{-i q \cdot b_2} e_2(-\alpha), 0_2), \\
B_{12}^a(q) &= (0_5, -e_3(-\alpha), 0_1, e^{-i q \cdot b_1} e_3(-\alpha), 0_1), \\
B_{13}^a(q) &= (0_6, e_1(\alpha), 0_1, -e_1(\alpha)), \\
B_{14}^a(q) &= (0_6, -e_2(\alpha), e_2(\alpha), 0_1), \\
B_{15}^a(q) &= (0_7, -e_3(\alpha), e_3(\alpha)), \\
B_{16}^a(q) &= (0_8, e^{-i q \cdot b_1} e_1(-\alpha), 0_3, -e_1(-\alpha), 0_2), \\
B_{17}^a(q) &= (e^{-i q \cdot b_2} e_2(-\alpha), 0_6, -e_2(-\alpha), 0_1), \\
B_{18}^a(q) &= (0_1, e_3(-\alpha), 0_6, -e_3(-\alpha)),
\end{align*}
\]

with \(0_n\) denoting a \((2n)\)-dimensional null vector, and

\[
\begin{align*}
B_1^b(q) &= (-a_1, 0_2, e^{-i q \cdot b_2} a_1, 0_5), \\
B_2^b(q) &= (-a_2, 0_2, a_2, 0_5), \\
B_3^b(q) &= (-a_3, 0_2, e^{-i q \cdot b_3} a_3, 0_5), \\
B_4^b(q) &= (0_4, -a_1, 0_2, a_1, 0_1), \\
B_5^b(q) &= (0_4, -a_2, 0_2, e^{-i q \cdot b_2} a_2, 0_1), \\
B_6^b(q) &= (0_4, -a_3, 0_2, e^{-i q \cdot b_3} a_3, 0_1), \\
B_7^b(q) &= (0_2, e^{i q \cdot b_1} a_1, 0_5, -a_1), \\
B_8^b(q) &= (0_2, a_3, 0_5, -a_3), \\
B_9^b(q) &= (0_2, a_3, 0_5, -a_3).
\end{align*}
\]

All vectors \((b_i, e_i)\) are defined in Section \(\ref{app1}\).

3. Dispersion curves

Figure 8 shows dispersion curves of the TwK/GK model along symmetry lines \(\Gamma K, \Gamma M\) and \(MK\); (a) \(k_a = 1, \alpha = 0\) and \(k_b = 0\) (blue dashed), \(\alpha = \pi/24\) and \(k_b = 0\) (red); (c) \(k_a = 1, \alpha = 0, k_a = 0\) (blue dashed) and \(k_a = 0.02\) (blue solid). Near \(J_G\), twist of the KL and NNN bonds have similar effects on the phonon structure. (b) and (d): Density plot of the six Eigenfrequencies over the first Brillouin zone for \(\alpha = 0, k_a = 1\) and \(k_b = 0\) ((b), corresponding to the Kagome lattice), \(k_a = 0\) and \(k_b = 1\) ((d), corresponding to the generalized Kagome lattices).

Appendix B: Asymptotic limit of the EMT Integrals and global behavior of the elastic moduli

Here we discuss plots showing the asymptotic behavior of the EMT integrals \(h_n\) near the jamming points, for both the TwK/GK and the TwK/H models. We also show 3D plots of the moduli as a function of \(p_a\) and \(p_b\) for both models.

In Section \(\ref{app2}\) we have shown that \(\Delta h_M \equiv 1 - h_M \propto k_m/k_M\) near the \(J_{G1}\), \(J_{G2}\) and \(J_B\) points, and that \(\Delta h_M \propto \sqrt{k_m/k_M}\) near the \(J_{BG}\) point. Figure 9 shows full numerical calculations of \(\Delta h_M\) near the four jamming points and confirms our analytical predictions. In (a) we show \(\Delta h_M\) as a function of \(k_m/k_M\) for the TwK/GK model near \(J_G\) (black, with \(M\) and \(m\) representing the twisted Kagome and generalized Kagome lattice, respectively) and near \(J_{BG}\) (red, with \(M\) and \(m\) representing the generalized Kagome and twisted Kagome lattice, respectively). Note that \(\Delta h_M \propto \sqrt{k_m/k_M}\) near \(J_{BG}\). In (b) we show \(\Delta h_M\) as a function of \(k_m/k_M\) for the TwK/H model near \(J_G\) (black, with \(M\) and \(m\) representing the twisted Kagome and honeycomb lattices, respectively) and near \(J_B\) (red, with \(M\) and \(m\) representing the honey-
We have used Eqs. (16) to calculate $c_m$ near the $J_{C2}$ and $J_{C2}$ points, and a numerical fit to calculate $c_M$ near the $J_{BG}$ and $J_B$ points (see Table I).

Finally, Figure 10 shows three-dimensional plots of $B$ (blue) and $G$ (red) as a function of $p_a$ and $p_b$ for the TwK/GK (a) and TwK/H (b) models. The dots and surfaces represent results from simulations and EMT, respectively. As it should be anticipated (see Figure 3), the agreement between EMT and simulations is best near the shear-jamming points $J_G$.

FIG. 9. Showing the asymptotic behavior of the EMT integrals $\Delta h_M$ as a function of $k_m/k_M$ for the TwK/GK (a) and TwK/H (b) models. In (a), the black and red curves emphasize the asymptotic behavior near the $J_C$ (with $M$ and $m$ representing the TwKL and GKL, respectively) and $J_{BG}$ (with $M$ and $m$ representing the GKL and TwKL, respectively). In (b), the black and red curves emphasize the asymptotic behavior near the $J_C$ (with $M$ and $m$ representing the TwKL and HL, respectively) and $J_B$ (with $M$ and $m$ representing the HL and TwKL, respectively). The dashed lines correspond to our asymptotic analytic predictions.

FIG. 10. Bulk (blue) and shear (red) moduli as a function of $p_a$ and $p_b$ for the TwK/GK (a) and TwK/H (b) models. The dots and surfaces correspond to numerical simulations and full solutions of the EMT equations, respectively.
[21] Note that our matrix of spring constants $k$ is not (in general) proportional to the identity matrix, so we cannot use the limiting form $(k/2V)\sum\alpha(E_{\text{aff}} \cdot \hat{t}_\alpha)^2$ on the right of Eq. (3.10) of Ref. [18].

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