Exact linear modeling using Ore algebras

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Abstract

Linear exact modeling is a problem coming from system identification: Given a set of observed trajectories, the goal is find a model (usually, a system of partial differential and/or difference equations) that explains the data as precisely as possible. The case of operators with constant coefficients is well studied and known in the systems theoretic literature, whereas the operators with varying coefficients were addressed only recently. This question can be tackled either using Gröbner bases for modules over Ore algebras or by following the ideas from differential algebra and computing in commutative rings. In this paper, we present algorithmic methods to compute “most powerful unfalsified models” (MPUM) and their counterparts with variable coefficients (VMPUM) for polynomial and polynomial-exponential signals. We also study the structural properties of the resulting models, discuss computer algebraic techniques behind algorithms and provide several examples.

Key words: Ore algebra, noncommutative Gröbner basis, annihilator, syzygies, linear exact modeling.

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1 Introduction

Linear exact modeling is a problem of system identification that leads to interesting algebraic questions. We start with some motivation from the systems theoretic point of view: The problem of linear exact modeling was formulated for one-dimensional behaviors in [1], see also [9, 10]. Starting with an observed set of polynomial-exponential signals, the aim is to find a linear differentiation-invariant model for these. Evidently, the whole signal set is a behavior that is not falsified by observation. But such a model has no significance. Making the behavior larger than necessary, the accuracy of the explanation decreases. So besides the condition that the desired model should be unfalsified, we are searching for the most powerful one. This means that the model does not admit more solutions than necessary. A model satisfying all conditions is abbreviated called continuous MPUM (most powerful unfalsified model).

In [16], the modeling was extended to multidimensional behaviors [3, 14], and in [17] to the discrete framework, that is, instead of the requirement that the model should contain all derivatives of the signals, it is required that all shifts of the signals are contained.

In other words, the problem is to find a homogeneous system of partial differential equations with constant coefficients that is as restrictive as possible with the property of possessing the observed signals as solutions.

In [16] a different approach was introduced. There the goal is to find all partial differential equations with polynomial coefficients that are solved by the signals. Thus the new aspect of this approach is the choice of a different model class. Indeed the properties of the resulting model depend strongly on the model class. For instance, by the transition from the MPUM to the VMPUM, the time-invariance vanishes. In this paper, we continue this approach. But since the continuous case is not the only interesting one, we will consider a more general problem comprising both the continuous and the discrete situation. Later some special model classes will be discussed in more detail.

Let us particularize our goal. Let $K$ be a field and $O$ be an operator algebra over $K$. Further let $A_O$ be a function space over $K$ possessing an $O$-module structure.

A model or a so-called behavior $B$ is the solution set of a homogeneous linear system, given by finitely many equations. These equations are defined in terms of the operator algebra $O$. Thus $B$ is characterized by

$$B = \text{Sol}(O^{1 \times r} R) = \{ \omega \in A_O^m \mid R \bullet \omega = 0 \}, \quad \text{where} \quad R \in O^{r \times m}$$

and $\bullet$ denotes the natural extension of the module action $o \bullet \omega$ of $o \in O$ on $\omega \in A_O$ to the matrix $R \in O^{r \times m}$ and the vector $\omega \in A_O^m$. In most cases of interest, we have $K \subseteq O$ and $ok = ko$ for all $k \in K$, $o \in O$. Then $B$ is a $K$-vector space, and thus the introduced model class is linear. Within such a model class we want to perform modeling now. Suppose we observe a set of signals $\Omega \subseteq A_O^m$. The aim is to find a model $B_\Omega$ in the model class such that

1. $B_\Omega$ is unfalsified by $\Omega$, i.e. $\Omega \subseteq B_\Omega$.
2. $B_\Omega$ is most powerful, i.e. for every behavior $B$ with $\Omega \subseteq B$, it follows that $B_\Omega \subseteq B$.

If $B_\Omega$ is invariant under the action of $O$, that is, if we have for all $o \in O$

$$\omega \in B \Rightarrow o \bullet \omega \in B,$$
it is called **most powerful unfalsified model**, short MPUM of $\Omega$. Else, if $B_\Omega$ varies under $O$ it is called **variant most powerful unfalsified model**, short VMPUM of $\Omega$. We denote the VMPUM of $\Omega$ by $B^V_\Omega$.

The following example shows how the choice of the model class affects the model.

**Example 1.1** Consider the signal set consisting of a single signal

\[ \Omega = \{ \omega \}, \quad \text{where } \omega(t) = t \text{ for all } t \in \mathbb{R}. \]

1. Let $O = \mathbb{C}[\partial]$ and $A_O = \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, where $\partial \bullet f := \frac{df}{dt}$. Using the commutative structure of the operator ring, the underlying system is invariant under differentiation:

\[ R \bullet w = 0 \implies R(\partial \bullet w) = (R\partial) \bullet w = (\partial R) \bullet w = \partial(R \bullet w) = 0. \]

Since we are searching for a differentiation-invariant model, we obtain that besides $\omega$, also its derivative, the constant function 1, belongs to $B_\Omega$. Using that the model is $\mathbb{C}$-linear, we get that

\[ B_\Omega = \{ w | \exists a, b \in \mathbb{C} : \forall t \in \mathbb{R} : w(t) = at + b \}. \]

An element $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ is contained in $B_\Omega$ if and only if

\[ \partial^2 \bullet w = 0, \]

i.e. the MPUM is specified by a single ordinary differential equation with constant coefficients.

2. Now let $O = \mathbb{C}[t]((\partial))$, where $\partial \bullet f := \frac{df}{dt}$ and $A_O$ is defined as above. We want to describe $\omega$ as a solution of homogeneous ordinary differential equations with polynomial coefficients. The equations

\[ \partial^2 \bullet w = 0 \quad \text{and} \quad t\partial \bullet w - w = 0 \]

are satisfied by $\omega$. We will see later that these two generate a kernel representation of the VMPUM of $\Omega$. The corresponding solution space equals

\[ B^V_\Omega = \{ w | \exists a \in \mathbb{C} : \forall t \in \mathbb{R} : w(t) = at \}. \]

Notice that this example demonstrates the variance under $\partial$, since we have $\partial \bullet \omega \notin B^V_\Omega$. Another property that should be pointed out is that the VMPUM yields a more precise description of $\Omega$ than MPUM.

## 2 Ore algebras

The example above deals with continuous signals. But in applications, there are also discrete phenomena or combinations of discrete and continuous signals that are of great interest too. Many of the relevant operator algebras have the structure of an Ore algebra, as studied e.g. in [3, 2, 4]. We give a definition that is motivated by [4]. Moreover, this simplifies more general setup of [8]. Hence first consider skew polynomial rings, a generalization of polynomial rings to the noncommutative framework.
Definition/Remark 2.1

(1) Let $A$ be a ring and $\sigma : A \to A$ be a ring endomorphism.

(a) The map $\delta : A \to A$ is called a $\sigma$-derivation if it is $K$-linear and satisfies the skew Leibniz rule
\[
\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \quad \text{for all } a, b \in A. \tag{1}
\]

(b) For a $\sigma$-derivation $\delta$, the ring $A[\partial; \sigma, \delta]$ which consists of all polynomials in $\partial$ with coefficients in $A$ with the usual addition and a product defined by the commutation rule
\[
\partial a = \sigma(a)\partial + \delta(a) \quad \text{for all } a \in A,
\]
is called a skew polynomial ring or an Ore extension of $A$ with $\sigma$ and $\delta$.

If $A$ is a domain and $\sigma$ is injective, the skew polynomial ring $A[\partial; \sigma, \delta]$ is a domain by degree arguments. Then the definition can be iterated to the so-called Ore algebras [4].

(2) Let $A = K[t_1, \ldots, t_n]$. An iterated skew polynomial ring
\[
O = K[t_1, \ldots, t_n][\partial_1; \sigma_1, \delta_1] \cdots [\partial_s; \sigma_s, \delta_s]
\]
is called a (polynomial) Ore algebra if the $\sigma_i$’s and $\delta_j$’s commute for $1 \leq i, j \leq s$, the $\partial_i$’s commute with $\partial_j$’s and further for all $1 \leq i \leq s$ the map $\sigma_i : O \to O$ is an injective $K$-algebra endomorphism and $\delta_i : O \to O$ is a $\sigma_i$-derivation satisfying $\sigma_i(\partial_j) = \partial_j$ and $\delta_i(\partial_j) = 0$.

Using multi-index notation, every element of an Ore algebra can be expressed into the normal form
\[
\sum_{\alpha \in \mathbb{N}_0^s} p_\alpha \partial^\alpha = \sum_{\alpha \in \mathbb{N}_0^s} p_\alpha \partial_1^{\alpha_1} \cdots \partial_s^{\alpha_s} \quad \text{where } p_\alpha \in A.
\]

For our issues the most interesting examples of Ore algebras are the following ones.

Example 2.2 Let $n = 1$, thus $A = K[t]$. The algebras can be iterated to $n \in \mathbb{N}$.

1. The first Weyl algebra is defined by $W_1 := A[\partial; \text{id}_W, \frac{\partial}{\partial t}]$ with the commutation rule $\partial t = t\partial + 1$.

2. The first difference algebra is defined by $S_1 := A[\Delta; \sigma, \delta]$, where $(\sigma p)(t) = p(t + 1)$ and $\delta(p) = \sigma(p) - p$ for all $p \in S_1$. The commutation rule is $\Delta t = t\Delta + \Delta + 1$.

3. The following Ore algebra is a combination of the first and second one. Define $SW_1 := A[\Delta; \sigma_1, \delta_1] [\partial; \sigma_2, \delta_2]$, where $\sigma_2 := \text{id}_{SW_1}$, $\delta_2 := \frac{\partial}{\partial t}$ and $(\sigma_1 p)(t) = p(t + 1)$, $\delta_1(p) = \sigma_1(p) - p$ for all $p \in SW_1$. Then $\partial t = t\partial + 1$, $\Delta t = t\Delta + \Delta + 1$ and $\partial\Delta = \Delta\partial$. 


Let \( K \) be a field, \( \phi \) be a \( \varphi \)-derivation such that \( \phi(a) = \varphi(a) + \alpha(a) \).

We obtain the commutation rule \( \partial t = qt\partial + (q - 1)t \).

**Lemma 2.3** Let \( A \) be a ring, and \( A[\partial; \sigma, \delta] \) be an Ore extension of \( A \). For any \( \alpha \in A \) there exists an Ore extension \( A[\Delta_\alpha; \sigma, \delta'] \) with \( \delta'(a) = \sigma(a)\alpha - \alpha a + \delta(a) \), such that \( A[\partial; \sigma, \delta] \cong A[\Delta_\alpha; \sigma, \delta'] \) as rings.

**Proof** For all \( a \in A \), the equality \( \partial a = \sigma(a)\partial + \delta(a) \) holds. For \( a \in A \) define \( \Delta_\alpha := \partial - \alpha \). Then it obeys the relation \( \Delta_\alpha a = \sigma(a)\Delta_\alpha + \sigma(a)\alpha - \alpha a + \delta(a) = \sigma(a)\Delta_\alpha + \delta'(a) \). The map \( \delta' \) is linear and it is a \( \varphi \)-derivation since

\[
\delta'(ab) = \sigma(a)\sigma(b)\alpha - \sigma(a)\alpha b + \sigma(a)\delta(b) - \delta(a)ab + \alpha a + \delta(a)b = \\
\sigma(a)\sigma(b)\alpha - \alpha ab + \sigma(a)\delta(b) - \delta(a)b = \sigma(ab)\alpha - \alpha ab + \delta(ab).
\]

Define the ring homomorphism \( \varphi_\alpha : A[\partial; \sigma, \delta] \to A[\Delta_\alpha; \sigma, \delta'] \), \( \varphi_\alpha(a) = a \) for all \( a \in A \), \( \varphi_\alpha(\partial) = \Delta_\alpha = \partial - \alpha \). Then \( \varphi_\alpha \) is an isomorphism. \( \square \)

Let \( O := A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m] \) be an Ore algebra. With the action

\[
\partial_i \cdot p := \delta_i(p) \quad \text{and} \quad o \cdot p := a \cdot p \quad \text{for all } p \in A \text{ and } a \in A
\]

the \( K \)-algebra \( A \) becomes an \( O \)-module. For this, we have to show that

1. \((o_1 \cdot o_2) \cdot p = o_1 \cdot (o_2 \cdot p) \) for all \( o_1, o_2 \in O \) and \( p \in A \)
2. \((o_1 + o_2) \cdot p = o_1 \cdot p + o_2 \cdot p \) for all \( o_1, o_2 \in O \) and \( p \in A \)
3. \( o \cdot (p + q) = o \cdot p + o \cdot q \) for all \( o \in O \) and \( p, q \in A \).

To show 1. it suffices to consider \( o_1 = a\partial_i \) and \( o_2 = b\partial_j \) with \( a, b \in A \). Then

\[
(o_1 \cdot o_2) \cdot p = (a\sigma_i(b)\partial_i + \delta_i(b))\partial_j \cdot p = (a\sigma_i(b)\partial_i + \delta_i(b))\partial_j \cdot p = a\sigma_i(b)\delta_j(p) + a\delta_i(b)\delta_j(p) = a\delta_i(b\delta_j(p)) = a\delta_i(b\partial_j \cdot p) = a\partial_i \cdot (o_2 \cdot p).
\]

The equality in 2. and 3. holds by similar arguments.

Using this action, we can define the kernel of a linear operator \( f \) from the Ore algebra \( O \) over a ring \( A \) to be \( \ker_A f := \{ a \in A \mid f \cdot a = 0 \} \), which is a \( K \)-vector space.

**Lemma 2.4** Let \( K \) be a field, \( A \) be a \( K \)-algebra, \( \partial \) be a \( K \)-linear operator, acting on \( A \) and \( B = A[\partial; \sigma, \delta] \) be the corresponding operator algebra (that is, for all \( a \in A \) we have \( \partial a = \sigma(a)\partial + \delta(a) \)). Then the following holds:

(i) \( \ker_A \partial = A \iff \delta = 0 \iff B = A[\partial; \sigma, 0] \).

(ii) If \( \ker_A \partial = A \), then we have for \( \Delta := \partial - 1 : A[\partial; \sigma, 0] \) is isomorphic as \( K \)-algebra to operator algebra \( A[\Delta; \sigma, \delta'] \) with \( \delta' := \sigma - 1 \). Moreover, \( \ker_A \Delta = \{ a \in A \mid \sigma(a) = a \} = \operatorname{const}_A A \subseteq A \) with the equality if and only if \( A \) is invariant under \( \sigma \), what is the case if \( \sigma = 1_A \).
Remark 2.5 Using Lemmata 2.3 and 2.4 we pass to the new setting of operators, which action \( \bullet \) is nontrivial on \( A \). We call such an operator nontrivial and from now on, we work with such operators only.

Example 2.6 Consider the two most important operator algebras, built from operators having zero kernels. The first forward shift algebra is defined by \( K[t][s;\sigma,0] \) with \( (\sigma f)(t) = f(t+1) \) for all \( f \in K[t] \). The commutation rule is \( st = ts + s \). There is a natural operator associated to \( s \), namely the difference operator \( \Delta = s - 1 \), already defined in 2.2, obeying the relation \( \Delta t = t\Delta + \Delta + 1 \). Applying Lemma 2.4 we see by degree argument, that \( \ker \Delta = K \) and the two algebras are isomorphic both as Ore extensions and \( K \)-algebras.

Let \( q \) be transcendental over \( K \). Then the first \( q \)-commutative algebra (or Manin’s quantum plane) is defined as \( K_q[x,y] := K_q[[\sigma,x,0]] \) with \( (\sigma f)(x) = f(qx) \) for \( f \in K[t] \). Again, there is a natural \( q \)-difference operator \( \Delta_q := \partial - 1 \) and the corresponding operator algebra has been already described in 2.2 as the first continuous \( q \)-difference algebra. Its commutation rule reads as \( \partial t = qt\partial + (q - 1)t \).

For \( o_1,\ldots,o_k \in O^n \), we denote by \( O\langle o_1,\ldots,o_k \rangle \) the left submodule of \( O^n \), generated by \( o_1,\ldots,o_k \).

Theorem 2.7 Let \( O \) be an Ore \( A \)-algebra, built from operators \( \partial_1,\ldots,\partial_s \) which have non-zero kernels. Then there is an isomorphism of left \( O \)-modules

\[
O/\langle \partial_1,\ldots,\partial_s \rangle \cong A.
\]

Proof There is a left \( O \)-module homomorphism

\[
\varphi : O \rightarrow A, \quad a = \sum_{a \in \mathbb{N}_0^s} a \cdot \partial^a \mapsto a \bullet 1
\]

since \( \varphi(b \cdot a) = (b \cdot a) \bullet 1 = b \bullet \varphi(a) \). Due to Def. 2.1 (1) we have \( \delta(1) = 0 \) and thus \( a \bullet 1 = a_0 \). The kernel of \( \varphi \) is given by the left ideal \( \langle \partial_1,\ldots,\partial_s \rangle \). Further, \( \varphi \) is clearly surjective. So the claim follows from the homomorphism theorem.

Following Theorem 2.4 every polynomial \( p \in A \) can be viewed as an element of the left \( O \)-module \( O/\langle \partial_1,\ldots,\partial_s \rangle \) by identifying \( p \) with \( p + \langle \partial_1,\ldots,\partial_s \rangle =: [p] \).

Then the action of \( \partial_i \) is exactly the \( \sigma_i \)-derivation \( \delta_i \), since

\[
\partial_i[p] = \partial_i[p] = [\sigma_i(p)\partial_i + \delta_i(p)] = [\delta_i(p)] = [\partial_i \bullet p].
\]

Remark 2.8 Let \( p \in A \) and \( o \in O \). Then there is the following equivalence

\[
o \bullet p = 0 \quad \text{if and only if} \quad o \cdot p \in \langle \partial_1,\ldots,\partial_s \rangle.
\]

Proof By Theorem 2.4 we have an \( O \)-module isomorphism \( A \cong O/\langle \partial_1,\ldots,\partial_s \rangle \) given by

\[
A \xrightarrow{\cong} O/\langle \partial_1,\ldots,\partial_s \rangle, \quad p \mapsto [p].
\]

Since the \( O \)-module structure is respected, \( o \bullet p \) maps to \( [o \cdot p] \) and hence the claim follows.
Remark 2.8 gives the possibility to describe and to compute the \textit{annihilator} of an element $p \in A$. Consider the map

$$\kappa_p : O \to O/O(\partial_1, \ldots, \partial_s), \quad o \mapsto o \cdot [p],$$

which is clearly a left $O$-module homomorphism with the kernel

$$\ker(\kappa_p) = \operatorname{Ann}_O(p) := \{o \in O \mid o \cdot p = 0\},$$

which is a left ideal in $O$. See Corollary 3.2 for its algorithmic computation.

This construction lifts to the case of vectors. Suppose $p = [p_1, \ldots, p_m]^T \in A_m$. An element of $o \in O^{1 \times m}$ naturally acts on $p$ by

$$o \cdot p := \sum_{i=1}^m o_i \cdot p_i.$$  

A subset $B \subseteq A_m$ is called \textbf{invariant} under $G \subseteq O^{1 \times m}$ if and only if $o \cdot p = 0$ for all $o \in G$ and $p \in B$. The set of elements under which $p$ is invariant has an $O$-module structure and equals to the kernel of

$$\kappa_p : O^{1 \times m} \to O/O(\partial_1, \ldots, \partial_s), \quad o = [o_1, \ldots, o_m] \mapsto \sum_{i=1}^m o_i \cdot [p_i].$$

Moreover the following isomorphism holds

$$O^{1 \times m} / \ker(\kappa_p) \cong O\langle p_1, \ldots, p_m \rangle / O\langle p_1, \ldots, p_m \rangle \cap O(\partial_1, \ldots, \partial_s).$$

The image of $\kappa_p$ equals $(o(p_1, \ldots, p_m) + o(\partial_1, \ldots, \partial_s)) / O(\partial_1, \ldots, \partial_s)$. This is isomorphic to $O\langle p_1, \ldots, p_m \rangle / O\langle p_1, \ldots, p_m \rangle \cap O(\partial_1, \ldots, \partial_s)$. So the claim follows, since $\kappa_p$ is a homomorphism.

\textbf{Remark 2.9} If $O$ is Noetherian (see [13]), then the left submodule $\ker(\kappa_p) \subseteq O^{1 \times m}$ is finitely generated.

For a polynomial $m$-tuple $p \in A_m$, we consider

$$\operatorname{Ann}_O(p) = \{o \in O \mid o \cdot p = 0\} = \{o \in O \mid o \cdot p_i = 0 \quad \forall i\} = \bigcap \operatorname{Ann}_O(p_i),$$

which is a left ideal in $O$. As we see immediately, $\operatorname{Ann}_O(p)^{1 \times m}$ is a (usually strict) submodule of $\ker(\kappa_p)$ and hence, the latter typically has more interesting structure, see Example 6.3. It is always possible to recover $\operatorname{Ann}_O(p)$ from $\ker(\kappa_p)$. In our opinion, using $\ker(\kappa_p)$ is more natural in the context of vectors of signals.

\section{Algorithmic computations}

For the concrete calculations used in this article, we need algorithms for the following computational tasks over (polynomial) Ore algebras:

1. syzygy module of a tuple of vectors
2. elimination of module components from a submodule of a free module
3. annihilator ideal of an element in a finitely presented module
4. kernel of a homomorphism of modules
5. intersection of a finite number of submodules of a free module.

Let $O$ be a Noetherian Ore algebra. Moreover, let $M$ be a finitely presented left $O$-module, that is, there exists a matrix $P \in O^{m \times n}$ such that there is the following exact sequence of left $O$-modules:

$$O^{1 \times m} \xrightarrow{P} O^{1 \times n} \rightarrow M \rightarrow 0.$$ 

Recall that for a tuple $F = (f_1, \ldots, f_s), f_i \subset O^{1 \times n}$, the set $\text{LeftSyz}(F) := \{ [a_1, \ldots, a_s] \in O^{1 \times s} \mid \sum_i a_if_i = 0 \}$ carries the structure of a left $O$-module and is called the \textit{left syzygy module} of $F$. Since $O$ is Noetherian, $\text{LeftSyz}(F)$ is finitely generated. Computation of syzygies over Noetherian Ore algebras can be accomplished with several algorithms and requires Gröbner basis techniques; see [8] for Ore algebras and [7] for the commutative case.

Let $\{e_i\}$ be the canonical basis of the free module $O^{1 \times \ell} = \bigoplus_{i=1}^{\ell} Oe_i$.

**Proposition 3.1** 1. “Elimination of module components”.

Let $S \subset O^{1 \times \ell}$ be a submodule. Moreover, let $<_O$ be a monomial ordering on $O$ and $<_m = (c, <_O)$ be a position over term monomial module ordering on the free module $O^{1 \times \ell}$, defined as follows. The components are ordered in a descending way $e_1 > \cdots > e_\ell$ and for any monomials $o_1, o_2 \in O$

$$o_1e_i <_m o_2e_j \iff j < i \text{ or } (j = i \text{ and } o_1 <_O o_2).$$

Let $G$ be a Gröbner basis of $S$ with respect to $<_m$. Then $\forall 1 \leq k < \ell$ $G \cap \bigoplus_{i=k}^{\ell} Oe_i$ is a Gröbner basis of $S \cap \bigoplus_{i=k}^{\ell} Oe_i$.

2. “Kernel of a homomorphism of modules”.

Consider an $O$-module homomorphism $O^{1 \times s} \xrightarrow{\psi} O^{1 \times n}/O^{1 \times m}P, e_i \mapsto [\Psi_i]$, where $\Psi_i \in O^{1 \times n}$. Let $P_i$ be the $i$-th row of the matrix $P$. Then

$$\text{ker } \psi = \text{LeftSyz}( (\Psi_1, \ldots, \Psi_s, P_1, \ldots, P_m) ) \cap \bigoplus_{i=1}^{s} Oe_i.$$ 

**Proof** 1. Define $W = \bigoplus_{i=k}^{\ell} Oe_i$. Since $G$ is a Gröbner basis of $S$, for any $s \in S$ there exists $g \in G$ such that $\text{lm}(g)$ divides $\text{lm}(s)$. If $s \in S \cap W$, then $\text{lm}(g) \in W$ and hence, by definition of $<_m$, we have $g \in W$ and $g \in G \cap W$. So, $G \cap W$ is a Gröbner basis of $S \cap W$. 

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2. We have
\[ [b_1, \ldots, b_s] \in \ker \psi \iff \exists a \in O : \sum_{i=1}^s b_i \Psi_i + \sum_{k=1}^m a_k P_k = 0 \iff \]
\[ [b_1, \ldots, b_s] \in \text{LeftSyz}(\left(\Psi_1, \ldots, \Psi_s, P_1, \ldots, P_m\right)) \cap \bigoplus_{i=1}^s O e_i. \]

Corollary 3.2 1. “Annihilator of an element in a module”.

Let \( M = \mathbb{O}^{1 \times n} / \mathbb{O}^{1 \times m} P \) and let \( P_1, \ldots, P_m \) be the rows of \( P \). Moreover, let \( v \in \mathbb{O}^{1 \times n} \). Then the left ideal \( \text{Ann}_M^O(v) := \{a \in O \mid a[v] = 0 \in M\} \subseteq O \)
can be computed as
\[ \text{Ann}_M^O(v) = \ker(O \rightarrow M) = \text{LeftSyz}(v, P_1, \ldots, P_m) \cap O e_1. \]

2. “Intersection of finitely many submodules”.

Let \( N_1, \ldots, N_m \subset \mathbb{O}^{1 \times r} \) be submodules. Then
\[ \bigcap_{i=1}^m N_i = \ker(O^{1 \times r} \rightarrow O^{1 \times r}/N_1) \oplus \cdots \oplus (O^{1 \times r}/N_m), \quad e_i \mapsto ([e_i], \ldots, [e_i]). \]

Remark 3.3 For an \( O \)-module homomorphism \( O^{1 \times s} / O^{1 \times r} Q \xrightarrow{\psi} O^{1 \times n} / O^{1 \times m} P \),
its kernel is the image of \( \ker \psi \) (as in Theorem 3.1) under the natural projection \( O^{1 \times s} \rightarrow O^{1 \times s} / O^{1 \times r} Q \). A left Gröbner basis can be obtained by reducing a
left Gröbner basis of \( \ker \psi + O^{1 \times r} Q \) with a left Gröbner basis of \( O^{1 \times r} Q \), see [11].

Note that in practical computations, elimination of module components is usually not complicated. This stands in distinct contrast with the elimination of algebra variables, which is often very hard to achieve. The algorithms used in this article involve only the elimination of module components and thus are feasible in practice.

The algorithms we have discussed are implemented in computer algebra systems like e.g. SINGULAR::Plural [6] or MAPLE [3, 4] with the package OreModules. More background on these algorithms can be found in e.g. [8], [11].

In particular, a set of generators of \( \ker(\kappa_p) \) from the previous section can be calculated explicitly.
By Proposition 3.1(2) \( \ker(\kappa_p) \) is obtained via the kernel of a module homomorphism, that is, by one Gröbner basis computation with respect to module monomial ordering eliminating components. The monomial part of this ordering can be chosen arbitrarily to be e. g. a fast one.
4 Application to linear exact modeling

We will now use the results from above to define an unfalsified and most powerful model over an Ore algebra.

Assumptions and notations: Suppose $\mathcal{O}$ to be a Noetherian Ore algebra with the additional property that $\partial_i$ acts nontrivially on $A$ for all $1 \leq i \leq s$.

Recall that $\mathcal{A}_O$ denotes a function space over $K$ possessing an $O$-module structure. Suppose further that $A \subseteq \mathcal{A}_O$.

Remark 4.1 [13, Theorem 1.2.9.] Since $A$ is Noetherian, $\mathcal{O}$ is Noetherian if $\sigma_i$ is an automorphism for all $1 \leq i \leq s$ on $A$.

Thus all Ore algebras considered in Example 2.2 are Noetherian.

Starting with a single signal $p \in A^m$, we want to find the VMPUM of $p$, that is a behavior, invariant under some finitely generated submodule of $O^1 \times m$.

Theorem 4.2 Let $p \in A^m$ be given. Consider the map $\kappa_p$ from (2). Let $\ker(\kappa_p) = O\langle k_1, \ldots, k_r \rangle$ and let $R \in O^r \times m$ be a matrix whose $i$-th row equals $k_i$. Then the VMPUM of $\{p\}$ is given by

$$B^V_{\{p\}} = \{ g \in A^m_0 \mid R \cdot g = 0 \}.$$

Proof By the definition of $R$ and Remark 2.8 it is clear that $\{p\} \subseteq B^V_{\{p\}}$.

It remains to show that $B^V_{\{p\}}$ is most powerful. Suppose there exists another behavior $B'$ unfalsified by $p$. The behavior $B'$ possesses a kernel representation $R' \in O'^r \times m$. By the definition of $R$, there exists a matrix $X \in O'^r \times r$ such that $R' = XR$. But since $(X \cdot R) \cdot p = X \cdot (R \cdot p)$, it follows that $B^V_{\{p\}} \subseteq B'$.

□

Example 4.3 Let us consider a more interesting example than Example 1.1 with respect to our favorite algebras from Example 2.2. Let $\Omega = \{\omega\}$ consists of the cuspidal cubic

$$\omega(t_1, t_2) = t_1^3 - t_2^2.$$

Let us denote by $\mathcal{A}_O = C[[t_1, t_2]]$ the ring of formal power series and consider the VMPUM $B^V_{\{\omega\}} = \{ f \in \mathcal{A}_O \mid R_{\text{VMPUM}} \cdot f = 0 \}$ of $\Omega$ with respect to several operator algebras $O$.

1. Suppose $O$ to be the second Weyl algebra (see Example 2.2). Then by using SINGULAR we obtain:

$$R_{\text{VMPUM}} = \begin{bmatrix}
\partial_1^3 \\
\partial_1 \partial_2 \\
\partial_2^3 + 3\partial_2^2 \\
t_2 \partial_2^2 - \partial_2 \\
t_2 \partial_1^2 + 3t_1 \partial_2 \\
2t_1 \partial_1 + 3t_2 \partial_2 - 6
\end{bmatrix}.$$

Now let us determine $B^V_{\{\omega\}}$ to see how precise the description given by the VMPUM is. Let $f \in \mathcal{A}_O$. 10
2. Suppose \( O \) to be the second difference algebra see Example 2.2 Then by using SINGULAR we obtain:

\[
\begin{pmatrix}
\Delta_2^3 \\
\Delta_1 \Delta_2 \\
\Delta_1^3 + 3 \Delta_1^2 \\
2 \Delta_2 \Delta_1^2 + \Delta_1^2 + 2 \Delta_1 \\
8 \Delta_1^2 + 21 \Delta_2 + 24 t_1 \Delta_1 + 36 t_2 \Delta_2 - 24 \Delta_1 - 18 \Delta_2 - 72
\end{pmatrix}
\]

Similar arguments as above lead us to

\[
B^V_{\{\omega\}} = \{c(t_1^3 - t_2^2) \mid c \in \mathbb{C}\}.
\]

3. Suppose \( O \) to be the second SW algebra see Example 2.2 Then by using SINGULAR we obtain:

\[
\begin{pmatrix}
\Delta_2^3 \\
\Delta_1 \Delta_2 \\
\Delta_1^3 + 3 \Delta_1^2 \\
2 \partial_1 + \Delta_1^2 - 2 \Delta_1 + 2 \Delta_2^2 \\
2 t_2 \Delta_2^3 + \Delta_2^2 - 2 \Delta_2 \\
8 \Delta_1^2 + 21 \Delta_2 + 24 t_1 \Delta_1 + 36 t_2 \Delta_2 - 24 \Delta_1 - 18 \Delta_2 - 72
\end{pmatrix}
\]

Note that generators in the output depend on the monomial ordering of the operators. In this example \( \Delta_{1,2} \) were chosen to be greater that \( \partial_{1,2} \). Taking a reverse ordering produces different (but equivalent) answer.

Comparing this matrix with the matrix belonging to the difference case appear also here. We conclude that

\[
B^V_{\{\omega\}} = \{c(t_1^3 - t_2^2) \mid c \in \mathbb{C}\}.
\]
Thus, taking $SW$ as operator algebra, we have got more equations than with the difference algebra. However, we have obtained very interesting mixed differential-difference equations, which show the interplay of two different operator settings.

4. The second $q$-difference algebra see Example 2.2

$$R_{\text{VMPUM}} = \left\{ \begin{array}{c}
\partial_2^2 + (-q^2 + 1)\partial_2 \\
(q^4 + q^3 - q - 1) t_1^2 - t_2^2 \partial_2 + (q^2 - 1)t_2^2
\end{array} \right\} .$$

(a) The first equation yields:

$$\sum_{i,j} c_{i,j}(q^i - 1)^2t_1^i t_2^j + (-q^2 + 1) \sum_{i,j} c_{i,j}(q^i - 1)t_1^i t_2^j = 0$$

$$\iff (q^j - 1)^2 + (-q^2 + 1)(q^j - 1) = 0$$

$$\iff j = 0 \vee j = 2.$$

(b) Now consider the second equation.

i. Suppose $j = 2$, then

$$(-q - 1) \sum_{i,j} c_{i,j}(q^i - 1)t_1^i t_2^j + (-q^2 - q - 1) \sum_{i,j} c_{i,j}(q^i - 1)t_1^i t_2^j$$

$$+ (q^4 + q^3 - q - 1) \sum_{i,j} c_{i,j} t_1^i t_2^j = 0$$

$$\iff (-q - 1)(q^i - 1) + (-q^2 - q - 1)(q^2 - 1) + (q^4 + q^3 - q - 1) = 0$$

$$\iff i = 0 .$$

ii. Suppose $j = 0$, then

$$(-q - 1) \sum_{i,j} c_{i,0}(q^i - 1)t_1^i + (q^4 + q^3 - q - 1) \sum_i c_{i,0} t_1^i = 0$$

$$\iff i = 0.$$ Thus $f = c_{00} t_1^3 + c_{02} t_2^2$.

(c) Applying the last equation, we get

$$t_1^3 c_{02}(q^2 - 1)t_2^2 - t_2^2 c_{02}(q^2 - 1)t_2^2 + (q^2 - 1)t_2^2(c_{30} t_1^3 + c_{02} t_2^2) = 0$$

$$\iff t_1^2 t_2^2(q^2 - 1)(c_{30} + c_{02}) = 0 \iff c_{30} = -c_{02} .$$

Thus we obtain once more $B^V_{\omega} = \{ c(t_1^3 - t_2^2) \mid c \in \mathbb{C} \}$.

**Remark 4.4** As we have seen in the previous example, the number of equations giving the VMPUM depends strongly on the underlying Ore algebra. In all cases, with Gröbner bases we get more equations than it might be actually necessary. However, it is possible to compute a smaller generating set, which is usually not a Gröbner basis. Namely, one computes a left syzygy module of a given system and almost directly deduces a smaller generating set from it. As an example, we show that only 3 of 6 equations from the first example of 4.3 generate the whole ideal, namely $\partial_1 \partial_2, \partial_1^3 + 3\partial_2^2, 2t_1 \partial_1 + 3t_2 \partial_2 - 6$. Analogous smaller generating sets can be obtained for other examples.
Theorem 4.2 can be generalized to a set of several signals directly. A kernel representation of the VMPUM of $\Omega = \{\omega_1, \ldots, \omega_N\}$ is determined by stacking a set of generators of
$$\bigcap_{i=1}^N \ker(\kappa_{\omega_i})$$
row-wise into a matrix $R$.

**Theorem 4.5** Using the notation from above, the VMPUM of $\Omega$ equals
$$B^V_\Omega = \{ g \in A^O_{\Omega} \mid R \bullet g = 0 \}.$$  

**Proof** By the definition of $R$, it is clear that $\Omega \subseteq B^V_\Omega$. Also the property of being most powerful follows by the same arguments as used in the proof of Theorem 4.2. \(□\)

**Example 4.6** Suppose $O$ to be the first Weyl algebra and $A_O = \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{C})$. Consider the signal set $\Omega = \{t, v_0 t - v_1 t^2\}$, where $v_0, v_1 \in \mathbb{C} \setminus \{0\}$. The second trajectory will appear in Example 5.4 again. Since
$$\ker(\kappa_t) \cap \ker(\kappa_{v_0 t - v_1 t^2}) = W_1 \langle t \partial \sim 1, \partial^2 \rangle \cap W_1 \langle -v_0^2 \partial^2 + (4v_1^2 t - 2v_0 v_1) \partial - 8v_1^2, \partial^3 \rangle$$
$$= W_1 \langle t^2 \partial^2 - 2t \partial + 2, \partial^3 \rangle,$$
the VMPUM of $\Omega$ is given by $B^V_\Omega = \{c_1 t + c_2 t^2 \mid c_1, c_2 \in \mathbb{C}\}$. The intersection of submodules of a free module over a Noetherian Ore algebra can be computed as in Corollary 3.2, for instance with the system **SINGULAR::PLURAL**.\[6\]

5 VMPUM by using the polynomial Weyl algebra

In this section, we suppose $O$ to be the $n$-th Weyl algebra
$$O = W_n := C[t_1, \ldots, t_n][\partial_1; id_{W_n}, \frac{\partial}{\partial t_1}] \cdots [\partial_n; id_{W_n}, \frac{\partial}{\partial t_n}].$$

Thus for $p \in \mathbb{C}[t_1, \ldots, t_n]$, we obtain $\partial_i \bullet p := \frac{\partial p}{\partial t_i}$. Further suppose $A_O$ to be $C^{\infty}(\mathbb{R}^n, \mathbb{C})$, the space of smooth functions. Identifying a polynomial with the corresponding polynomial function, we obtain $A \subseteq A_O$.

In this context, the VMPUM was already introduced in [15]. Here, we will recall some results and additionally point out a new interesting property.

5.1 $C$-dimension

A known result is that the VMPUM is a finite-dimensional vector space over $\mathbb{C}$, since it is contained in the corresponding MPUM [15]. In some cases, we can determine the dimension more precisely. We claim that the VMPUM of a single non-zero signal has $\mathbb{C}$-dimension one.
Suppose $p \in A^n$. Every polynomial $p_i$ can be written as $\sum_{k=1}^{h_i} c_{ik} t_{ik}$, where $c_{ik} \in \mathbb{C}$ for all $i, k$. Let $\mathcal{E}_i := \{\beta_{1i}, \ldots, \beta_{ik}\} \subseteq \mathbb{N}_0^n$ denote the set of all exponent multi-indices occurring in $p_i$ and let

$$d_{ij} := \max_{1 \leq k \leq h_i} \{(\beta_{ik})_j : \beta_{ik} \in \mathcal{E}_i\}$$

be the highest degree in $t_j$ of $p_i$. Recall that by $e_i$ we denote the $i$-th canonical vector of the free module $A^n$. The set

$$\mathcal{E}_{p_i} = \{\alpha \in \mathbb{N}_0^n : \alpha_j \leq d_{ij} + 1 \text{ for } 1 \leq j \leq n\}$$

is finite, that is, $\mathcal{E}_{p_i} = \{\alpha_{i1}, \ldots, \alpha_{it}\}$. Define for $p \in A^n$

$$\text{Der}_p = \{p_1, \frac{\partial^{\alpha_{11}}|p_1}{\partial x_1}, \ldots, \frac{\partial^{\alpha_{11}}|p_1}{\partial x_1}, \ldots, p_m, \frac{\partial^{\alpha_{m1}}|p_m}{\partial x_1}, \ldots, \frac{\partial^{\alpha_{m1}}|p_m}{\partial x_1}\}.$$

Let $\text{Syz}(\text{Der}_p)$ denote the module of polynomial syzygies. Define for the matrix $M = [e_1, \partial^{\alpha_{11}}e_1, \ldots, \partial^{\alpha_{11}}e_1, \ldots, e_m, \partial^{\alpha_{m1}}e_m, \ldots, \partial^{\alpha_{m1}}e_m]^T$ the $A$-module homomorphism

$$\Phi_p : \text{Syz}(\text{Der}_p) \to \ker(\kappa_p), \quad (q_1, \ldots, q_t) \mapsto (q_1, \ldots, q_t) \cdot M,$$

which is clearly injective.

**Lemma 5.1** $\mathcal{W}_n(\text{Im}(\Phi_p)) = \ker(\kappa_p)$.

**Proof** Evidently $\mathcal{W}_n(\text{Im}(\Phi_p)) \subseteq \ker(\kappa_p)$. Now suppose that $a \in \ker(\kappa_p)$. Since every element in $\mathcal{W}_n$ can be written in normal form, we obtain

$$a \cdot p = \sum_k a_k \cdot p_k = \sum_k \left(\sum_j c_{kj} t^{\beta_{kj}} \partial^{\gamma_{kj}}\right) \cdot p_k = \sum_k \left(\sum_j c_{kj} t^{\beta_{kj}} \partial^{\gamma_{kj}} \cdot p_k\right).$$

Let us split the element $a$ in $a_z$ and $a_{nz}$ such that $a = a_z + a_{nz}$ and $(a_z)_k$ consists of the parts of $a_k$ where $\partial^{\gamma_{kj}} \cdot p_k$ is zero. By the choice of $d_{ij}$, the set $\{\partial^{(d_{ij}+1)} | 1 \leq j \leq n, 1 \leq i \leq m\}$ generates the set of $\partial^T$ with the property that there exists $1 \leq i \leq m$ such that $\partial^T \cdot p_i = 0$. Then $a_z$ is contained in $\mathcal{W}_n(\partial^{(d_{ij}+1)}) | 1 \leq j \leq n, 1 \leq i \leq m\)$. But by the choice of $\text{Der}_p$, the element $a_z$ is in the image of $\Phi_p$. Suppose $\partial^{\gamma_{kj}} \cdot p_k \neq 0$, then $\gamma_{kj}$ is equal or smaller than $(d_{ik1}, \ldots, d_{ikn})$ in each component and again by the choice of $\text{Der}_p$, the element $a_{nz}$ is contained in the image of $\Phi_p$. Thus it follows that $a \in \text{Im}(\Phi_p)$.

**Theorem 5.2** The VMPUM of $p \neq 0$ is a one-dimensional vector space over $\mathbb{C}$.

**Proof** We use the notation of the Lemma which reduces to commutative calculations. It is easy to see that the equivalence

$$s \cdot (f_1, \partial^{\alpha_{11}} \cdot f_1, \ldots, \partial^{\alpha_{m1}} \cdot f_m)^T = 0 \iff \Phi_p(s) \cdot f = 0 \quad (4)$$
holds for every \( s \in \text{Syz}(\text{Der}_p) \). Now let us discuss the left hand side. More precisely, let us consider the solution space \( \text{Sol}(\text{Syz}(\text{Der}_p)) \) in \( \mathcal{A}_O \) belonging to \( \text{Syz}(\text{Der}_p) \). Since \( \text{Der}_p \) contains all non-zero derivatives of \( p_i \) for all \( i \), there exists a non-zero constant \( \mathbb{C} \ni k \in \text{Der}_p \). We can suppose \( k = 1 \) and without loss of generality let \( \partial^{\alpha_{mh} \cdot p_m} = 1 \). Then

\[
(-1, 0, \ldots, 0, p_1), \ldots, (0, \ldots, 0, -1, \frac{\partial^{\alpha_{mh} \cdot p_m}}{\partial^{\alpha_{mh} \cdot p_m}}) \in \text{Syz}(\text{Der}_p)
\]

and thus

\[
\text{Sol}(\text{Syz}(\text{Der}_p)) = \{ c \cdot (p_1, \frac{\partial^{\alpha_{11} \cdot p_1}}{\partial^{\alpha_{11} \cdot p_1}}, \ldots, \frac{\partial^{\alpha_{mh} \cdot p_m}}{\partial^{\alpha_{mh} \cdot p_m}}) | c \in \mathbb{C} \}. \quad (5)
\]

Now suppose \( f \in \text{VMPUM}_p \). From Lemma 5.1 together with (4) and (5) we deduce the claim.

**Remark 5.3** In the case of a single non-zero signal, the VMPUM gives the most precise description one can get for a linear system.

**Example 5.4** Consider the trajectory \( \omega(t) = v_0 t - v_1 t^2 \), where \( v_0, v_1 \in \mathbb{C} \setminus \{0\} \). Then the MPUM of \( \omega \) is given by

\[
B_{\{\omega\}} = \{ \alpha(v_0 t - v_1 t^2) + \beta(v_0 - 2v_1 t) + \gamma(-2v_1) | \alpha, \beta, \gamma \in \mathbb{R} \}
\]

\[
= \{ at^2 + bt + c | a, b, c \in \mathbb{R} \}
\]

\[
= \{ w \in \mathcal{A}_O | \partial^3 \cdot w = 0 \}.
\]

Thus there are three free parameters to choose. The VMPUM of \( \omega \) is given by

\[
B_{\{\omega\}}^V = \{ w \in \mathcal{A}_O | \left[ -v_0^2 \partial^2 + (4v_1^2 t - 2v_0 v_1) \partial - 8v_1^2 \right] \cdot w = 0 \}
\]

\[
= \{ c(v_0 t - v_1 t^2) | c \in \mathbb{R} \},
\]

that is, two degrees of freedom vanish when we consider the time-variant model.

### 5.2 Structural properties

Let us discuss some structural properties of the \( W_n \)-module \( \ker(\kappa_p) \). Since every element of \( W_n \) can be transformed into normal form, the degree of an element \( a \in W_n \) can be introduced as

\[
\deg(a) := \max\{ \sum_{i=1}^{n} \alpha_i + \beta_i \mid a = \sum_{\alpha, \beta \in \mathbb{N}_0} a_{\alpha, \beta} t^\alpha \partial^\beta, \ a_{\alpha, \beta} \in \mathbb{C} \}.
\]

Then \( F'(W_n) := \{ a \in W_n \mid \deg(a) \leq i \} \) induces a filtration on \( W_n \). The corresponding associated graded ring \( \text{Gr}(W_n) \) is isomorphic to \( \mathbb{C}[t_1, \ldots, t_n, \partial_1, \ldots, \partial_n] \) as a graded \( \mathbb{C} \)-algebra. For every finitely generated \( W_n \)-module \( M \), we define the Hilbert polynomial \( \text{HP}_{W_n}^{\text{Gr}(W_n)} := \text{HP}_{\text{Gr}(M)} \). The dimension of \( M \) is defined as \( \text{dim}_{W_n}(M) := \deg(\text{HP}_{M}) + 1 \). Furthermore \( M \) is called **holonomic** if it has dimension \( n \). A holonomic module is of minimal dimension, since the dimension
of $W_n$-modules is bounded below by $n$ and bounded above by $2n$. Holonomic $W_n$-modules are additionally cyclic and torsion modules. For details see [5].

As usual, we write $A = \mathbb{C}[t_1, \ldots, t_n] \subset W_n$.

**Theorem 5.5** There is an isomorphism of $W_n$-modules $W_n^{1 \times m} / \ker(\kappa_p) \cong A$. In particular, $W_n^{1 \times m} / \ker(\kappa_p)$ is simple holonomic $W_n$-module.

**Proof** Since $\kappa_p$ is a homomorphism of $W_n$-modules, we get

$$W_n^{1 \times m} / \ker(\kappa_p) \cong \text{Im}(\kappa_p) \subseteq W_n / W_n(\partial_1, \ldots, \partial_n) \cong A.$$ 

Thus $W_n^{1 \times m} / \ker(\kappa_p)$ is isomorphic to a submodule of $A$. Due to the fact that $A$ is a simple holonomic $W_n$-module and $W_n^{1 \times m} / \ker(\kappa_p) \neq 0$ the claim follows. □

**Corollary 5.6** Since $W_n^{1 \times m} / \ker(\kappa_p)$ is holonomic, there exists a left ideal $L_p$, depending on $p$, such that $W_n^{1 \times m} / \ker(\kappa_p)$ is isomorphic to the cyclic left $W_n$-module $W_n / L_p$.

An algorithm, using Gröbner bases, to compute a generator of a holonomic module is given in [12]. On the other hand, since $W_n^{1 \times m} / \ker(\kappa_p)$ is simple holonomic module, any non-zero element can be taken as a generator for a cyclic presentation.

**Example 5.7** Suppose $\omega = [c_1, c_2, c_3]^T$ for $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$. Then

$$\ker(\kappa_\omega) = W_1([0, c_3, -c_2], [c_3, 0, -c_1], [0, 0, \partial]).$$

Since

$$\begin{bmatrix} 0 & c_3 & -c_2 \\ c_3 & 0 & -c_1 \\ 0 & 0 & \partial \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/c_3 & c_1/c_3 \\ 1/c_3 & 0 & c_2/c_3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \partial \end{bmatrix},$$

we obtain $W_1^3 / \ker(\kappa_\omega) \cong W_1^3 / \ker(\kappa_\omega)C \cong W_1 / W_1(\partial) \cong \mathbb{C}[t]$.

### 5.3 VMPUM of polynomial-exponential signals

In this section, we extend the signal space that should be modeled. The goal is to compute the VMPUM of

$$p = [p_1 \exp{\lambda_1}, \ldots, p_m \exp{\lambda_m}]^T,$$

where for all $1 \leq i \leq m$, we have $p_i \in A, \lambda^i \in \mathbb{C}^n$ and

$$\exp{\lambda} := \exp(\lambda_1 t_1 + \cdots + \lambda_n t_n) \quad \text{for} \quad \lambda \in \mathbb{C}^n.$$

By the action $\partial_j \cdot \exp{\lambda} = \lambda_j \exp{\lambda}$ for all $1 \leq j \leq n$ the space of polynomial-exponential functions becomes a $W_n$-module.
Consider the scalar setting first, that is, \( m = 1 \). Define for \( \lambda \in \mathbb{C}^n \) the \( W_n \)-homomorphism

\[
\sigma_\lambda : W_n \to W_n, \quad \partial_i \mapsto (\partial_i - \lambda_i), \quad t_i \mapsto t_i.
\]

It is easy to see that \( \sigma_\lambda \) is a \( W_n \)-automorphism. We claim that for \( a \in W_n \) and \( f \in A \)

\[
a \cdot p = 0 \quad \text{if and only if} \quad \sigma_\lambda(a) \cdot (p \exp_\lambda) = 0. \tag{7}
\]

For the proof suppose \( a = \sum_i c_i t^\alpha \partial^\beta \).

Using the identity \((\partial - \lambda i)(p \exp_\lambda) = (\partial_i \cdot p) \exp_\lambda\), the claim follows by

\[
\sigma_\lambda(a) \cdot (p \exp_\lambda) = \sum_i c_i t^\alpha (\partial_1 - \lambda_1)^{\beta_1} \cdots (\partial_n - \lambda_n)^{\beta_n} \cdot (p \exp_\lambda)
\]

\[
= \sum_i c_i t^\alpha (\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}) \cdot p \exp_\lambda = (a \cdot p) \exp_\lambda.
\]

Extending the dimension, there are two special cases requiring attention. First suppose \( \lambda_1, \ldots, \lambda_m \) to be equal, that is, \( p = [p_1, \ldots, p_m]^T \exp_\lambda \), where \( \lambda := \lambda_1 \).

Then claim (7) can be generalized directly and it follows that

\[
\sum_{j=1}^m a_j \cdot (p_j \exp_\lambda) = 0 \quad \text{if and only if} \quad [a_1, \ldots, a_m] \in \sigma_\lambda(\ker(\kappa_p)). \tag{8}
\]

Assume now that \( \lambda_1, \ldots, \lambda_m \) are pairwise different. Then

\[
\sum_{j=1}^m a_j \cdot (p_j \exp_\lambda) = 0 \quad \text{if and only if} \quad [a_1, \ldots, a_m] \in \bigoplus_{j=1}^m \sigma_{\lambda_j}(\ker(\kappa_{p_j})). \tag{9}
\]

Since \( \exp_\lambda, \ldots, \exp_{\lambda_m} \) are algebraically independent over \( A \), the claim follows from

\[
\sum_{j=1}^m a_j \cdot (p_j \exp_\lambda) = 0
\]

\[
\iff \sum_{j=1}^m \left( \sum_{i=1}^{h} c_{ji} t_i^{\alpha_i} (\partial_1 + \lambda_1^{(\beta_i)}) \cdots (\partial_n + \lambda_n^{(\beta_i)}) \cdot p_j \right) \exp_\lambda = 0
\]

\[
\iff \sigma_{\lambda_i}^{-1}(a_j) \in \ker(\kappa_{p_j}) \quad \text{for all} \quad 1 \leq j \leq m.
\]

Recapitulating we get:

**Theorem 5.8** Let \( f \) be of the form (6). Further let

\[
K_i := \{ j \mid \lambda_j = \lambda_i \} = \{ k_{i1}, \ldots, k_{ih_i} \}
\]

and let \( l \) be chosen minimal such that we have a disjoint union \( K_1 \cup \ldots \cup K_l = \{ k_{11}, \ldots, k_{1h_1}, \ldots, k_{11}, \ldots, k_{1h_1}, \ldots, k_{li}, \ldots, k_{li} \} = \{ 1, \ldots, m \} \). Further define the vector \( h_i := \)
component-wise order on $N_m$ case, where $g$ that $g$

First, the coefficients $K_{1\cdots l}$ for $p$ and moreover, each element $p$ to write $\nu$ and $\kappa$ for $p$

Similar to the continuous case, the kernel of $\kappa_p$ is given by $\bigoplus_{i=1}^l \phi_i(H_i)$.

**Proof** After choosing a suitable projection, the claim follows by (7) and (8).

6 VMPUM via the polynomial difference algebra

Suppose that $|K| = \infty$. Recall the definition of the $n$-th difference algebra:

$$S_n := K[t_1, \ldots, t_n][\Delta; \sigma_1, \delta_1] \cdots [\Delta_n; \sigma_n, \delta_n].$$

For $p \in K[t_1, \ldots, t_n]$, we have

$$\Delta \bullet p = \delta_i(p) = \sigma_i(p) - p = p(t + e_i) - p(t).$$

Further suppose that $A_0 = K^{N_0^n}$. Identifying a polynomial with the corresponding polynomial function, we obtain $A \subseteq A_0$.

Similar to the continuous case, the kernel of $\kappa_p$ can be computed in a completely commutative framework. For this we choose a special representation of the polynomials that is adapted to the action of $\Delta$, see [17]. For $t \in N_0^n$ and $\nu = (\nu_1, \ldots, \nu_n)$, we consider the binomial functions

$$p_\nu : N_0^n \to K, \ t \mapsto \binom{t_1}{\nu_1} \cdots \binom{t_n}{\nu_n},$$

where $\binom{t_i}{\nu_i} = 1$ for all $i$. Then $\nu! p_\nu = t_1 \cdots (t_1 - \nu_1 + 1) \cdots t_n \cdots (t_n - \nu_n + 1)$ and moreover, each element $p \in A^n$ can be written as

$$p = \sum_{\nu \in N_0^n, \nu \leq \nu'} c_\nu p_\nu$$

for $\nu \in N_0^n$, some suitable coefficient vectors $c_\nu \in K^m$ and $\leq_{cw}$ denoting the component-wise order on $N_0^n$, that is, $\nu_i \leq g_i$ for all $1 \leq i \leq n$. Let us describe how to find this representation. We restrict to the scalar and one-dimensional case, where $m = n = 1$. The general case can be treated similarly. For $p \in A = K[t]$ we show how to find the introduced representation. Usually, a polynomial $p$ is given in the form

$$p(x) = d_0 x^\nu + d_{\nu-1} x^\nu - 1 + \cdots + d_1 t + d_0, \quad \text{where} \ d_i \in K.$$

To write $p$ in the form (10), the occurring coefficients $c_\nu$ have to be determined.

We will show how this can be done for a monomial $d_\nu x^\nu$. Since $\nu! p_\nu = t \cdot (t - 1) \cdots (t - \nu + 1)$, we define

$$g_\nu(x) := t \cdot (t - 1) \cdots (t - \nu + 1) = t^\nu + g_{\nu-1}(x) t^{\nu - 1} + \cdots + g_1(x) t.$$

First, the coefficients $g_\nu(x)$ will be determined for $1 \leq \nu \leq \nu$ by using the fact that $g_\nu(x) = g^{(\nu-1)}(t - \nu + 1)$.

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Finally, we observe where by using

\[ g^{(v)}_1 = \begin{cases} 
1 & \text{for } \nu = 1 \\
(-1)^{\nu-1} \prod_{k=1}^{\nu-1} k & \text{for } \nu > 1.
\end{cases} \]

2. Determine \( g^{(v)}_2 \): Using \( g^{(v)} = g^{(v-1)} \cdot (tv + 1) \), we get

\[ g^{(v)}_2 = g^{(v-1)} - (v-1) \cdot g^{(v-1)} = (-1)^{\nu-2} \prod_{k=1}^{\nu-2} k = (\nu-1)g^{(v-1)}_2 \]

Since \( g^{(2)}_2 = 1 \), we get a recursive formula.

3. Determine \( g^{(v)}_j \) for \( j \leq \nu \):

A similar consideration as in the previous point yields

\[ g^{(v)}_j = g^{(v-1)}_j - (\nu-1) \cdot g^{(v-1)}_j. \]

Finally, we observe

\[ d_v t^v = d_v \left( g(v) - g^{(v)}_{v-1} \cdot g(v-1) - (g^{(v)}_{v-2} - g^{(v)}_{v-1}) g(v-2) - \cdots \right) \]

\[ = d_v \left( g(v) + \sum_{i=1}^{v-1} k_v(i) \cdot g(v-i) \right) = d_v \left( v!p_v + \sum_{i=1}^{v-1} k_v(i) \cdot (v-i)! \cdot p_{v-i} \right), \]

where

\[ k_v(1) := -g^{(v)}_{v-1}, \quad \text{and} \quad k_v(l) = \begin{cases} 
-g^{(v)}_{v-l} + \sum_{i=1}^{l-2} k_v(i) \cdot g^{(v-i)}_{v-l}, & \text{if } l < v \\
0, & \text{if } l \geq v.
\end{cases} \]

Consider for example \( p(t) = t^3 + t^2 + 1 \). The bounding value \( v \) equals three, so by using

| \( j \) | \( p_1^{(j)} \) | \( p_2^{(j)} \) | \( p_3^{(j)} \) | \( k_3(j) \) | \( k_2(j) \) |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 3 | 1 |
| 2 | -1 | 1 | 0 | 1 | 0 |
| 3 | 2 | -3 | 1 | 0 | 0 |

we finally get

\[ t^3 = p(3) + k_3(1) \cdot p(2) + k_3(2) \cdot p(1) = 6 \cdot p_3 + 3 \cdot 2 \cdot p_2 + 1 \cdot p_1 \]

\[ t^2 = p(2) + k_2(1) \cdot p(1) = 2 \cdot p_2 + 1 \cdot p_1 \]

\[ 1 = p_0 \]

\[ \Rightarrow p(t) = 6 \cdot p_3 + 8 \cdot p_2 + 2 \cdot p_1 + p_0 \]

In the following we show the advantage of this notation. Since

\[ (\delta_t p_{\nu})(t_i) = \begin{cases} 
\binom{t_i + 1}{\nu_i} - \binom{t_i}{\nu_i} & \text{if } \nu_i \geq 1 \\
\frac{\nu_i!}{0!} & \text{if } \nu_i = 0
\end{cases} \]

we get

\[ (\delta_t p_{\nu})(t_i) = \begin{cases} 
\binom{t_i + 1}{\nu_i} - \binom{t_i}{\nu_i} & \text{if } \nu_i \geq 1 \\
\frac{\nu_i!}{0!} & \text{if } \nu_i = 0
\end{cases} \]
Finally we get that \( \ker(\kappa) \) is given by
\[
\kappa = \max \{ (v_1, \ldots, v_n) \in \mathbb{N}_0^n \mid v_j = (\mu_k)_{ij} \text{ for } 1 \leq k \leq d_i \}.
\]
Then the bounding multi-index \( \varrho \) belonging to the binomial representation \( \binom{10}{0} \) is given by
\[
\varrho = \max \{ (v_1, \ldots, v_n) \mid v_i = (\varrho)_{ij} \text{ for } 1 \leq j \leq m \}.
\]
From now on suppose that \( p = \sum_{\nu \in \mathbb{N}_0^n, \varrho \leq c \varrho} c_{\varrho} p_{\nu} \).

**Remark 6.2** Connecting remark 6.1 and (11) we get that \( \delta^\mu p = 0 \) for all \( \mu \) with \( \mu_i > q_i \) for at least one \( 1 \leq i \leq n \). Now consider the finitely generated left \( A \)-module generated by
\[
\text{Shift}_p = A(\delta^\mu p \mid \mu \leq c \varrho \varrho).
\]
The corresponding syzygy module \( \text{Syz} (\text{Shift}_p) \) is finitely generated too, since \( A \) is a Noetherian ring. Analogously to the continuous case, we can give an \( A \)-module homomorphism from \( \text{Syz} (\text{Shift}_p) \) to \( \ker (\kappa_p) \), such that the image of \( s_1, \ldots, s_d \) under this map generates \( \ker (\kappa_p) \), that is, \( \ker (\kappa_p) \) is finitely generated as an \( A \)-module. This implies that \( \ker (\kappa_p) \) is finitely generated as an \( S_n \)-module.

**Example 6.3** Let \( p = [t^3, t]^T \). Then the continuous VMPUM is the same as the discrete VMPUM, that is, equal to \( \{ c [t^3, t]^T \mid c \in K \} \). Direct computation over \( S_1 \) yields
\[
\ker_{S_1} (\kappa_p) = \mathcal{S}_1 ([0, \Delta^2], [0, t\Delta - 1], [1, -t^2])
\]
and this means that
\[
\begin{bmatrix}
0 & \Delta^2 \\
0 & t\Delta - 1 \\
1 & -t^2
\end{bmatrix}
\]
is a kernel representation of the VMPUM of \( p \). Note that over \( A_1 \), we have
\[
\ker_{A_1} (\kappa_p) = A_1 ([0, \Delta^2], [0, t\Delta - 1], [1, -t^2]).
\]
Alternatively, we can compute \( \ker_{S_1} (\kappa_p) \) in the commutative framework, using the analogue of “difference algebra” approach. At first, we observe that
\[
\text{Shift}_{[t^3, t]^T} = \mathcal{K}|_{[t^3, 3t^2 + 3t + 1, 6t + 6, 6, t, 1]}
\]
so
\[
\text{Syz} (\text{Shift}_{[t^3, t]^T}) = \mathcal{K}|_{[t^3, 3t^2 + 3t + 1, 6t + 6, 6, t, 1]}
\]
Finally we get that \( \ker (\kappa_p) = \mathcal{S}_1 ([0, -\Delta^2 + 1], [\Delta^3, -6\Delta], [\Delta^2, (-6t-6)\Delta], [\Delta, (-3t^2 - 3t - 1)\Delta], [1, -t^4\Delta], [\Delta^3, 0], [0, \Delta^2]) = \mathcal{S}_1 ([0, \Delta^2], [0, t\Delta - 1], [1, -t^2]).
\]
6.1 VMPUM of polynomial-exponential signals

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in K^n \) the discrete exponential function is given by

\[
\exp_\lambda : \mathbb{N}_0^n \to K, \quad t \mapsto \lambda^t = \lambda_1^t \cdots \lambda_n^t.
\]

First suppose that \( m = 1 \), that is, we want to construct the VMPUM of a scalar polynomial exponential trajectory of the form \( p \exp_\lambda \), where \( p \in A \). Without loss of generality, we can assume \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq n \), since otherwise if \( \lambda_j = 0 \)

\[
p \exp_\lambda(t) = \begin{cases} 
0 & \text{if } t_j \neq 0 \\
g & \text{if } t_j = 0
\end{cases},
\]

where

\[
g : \mathbb{N}^{n-1} \to K, \quad t \mapsto (p \exp_\lambda)(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n).
\]

Consider the automorphism of \( S_n \)

\[
\chi_\lambda : S_n \to S_n, \quad \{ t_i \mapsto t_i, \Delta_i \mapsto \frac{1}{\lambda_i} (\Delta_i - \lambda_i + 1) \}.
\]

Since the equality

\[
\chi_\lambda(\Delta_i) \bullet (p \exp_\lambda) = \frac{1}{\lambda_i} (\Delta_i - \lambda_i + 1) \bullet (p \exp_\lambda)
\]

\[
= \frac{1}{\lambda_i} (\lambda_i \exp_\lambda \sigma_i(p) - p \exp_\lambda - \lambda_i p \exp_\lambda + p \exp_\lambda)
\]

\[
= \frac{1}{\lambda_i} (\lambda_i \exp_\lambda(\sigma_i(p) - p)) = \exp_\lambda \Delta_i \bullet p
\]

holds, we obtain the identity

\[
\chi_\lambda(\Delta^k_i) \bullet (p \exp_\lambda) = \exp_\lambda \Delta^k_i \bullet p
\]

that finally extends to

\[
\chi_\lambda(\Delta^\mu) \bullet (p \exp_\lambda) = \chi^\mu_\lambda(\Delta) \bullet (p \exp_\lambda) = \exp_\lambda \Delta^\mu \bullet p. \quad (12)
\]

Now using (12) we can deduce for \( a = \sum_{i=1}^h a_i \Delta^\alpha_i \in S_n \) the equivalence

\[
a \bullet p = 0 \iff \chi_\lambda(a) \bullet (\exp_\lambda p) = 0, \quad (13)
\]

since

\[
\chi_\lambda(a) \bullet (\exp_\lambda p) = \sum_{i=1}^h a_i \chi_\lambda(\Delta^\alpha_i) \bullet (\exp_\lambda p) = \sum_{i=1}^h a_i (\Delta^\alpha_i \bullet p) \exp_\lambda
\]

\[
= \exp_\lambda \sum_{i=1}^h a_i (\Delta^\alpha_i \bullet p) = \exp_\lambda a \bullet p.
\]

Summarizing, we obtain

**Theorem 6.4** Let \( R \in S_n^{l \times 1} \) be a kernel representation matrix of the VMPUM of \( p \). Then the kernel representation matrix of \( p \exp_\lambda \) is given by \((\chi_\lambda(R))_i\).
Now consider
\[
p = \begin{bmatrix}
p_1 \exp_{\lambda^{(1)}} \\
\vdots \\
p_m \exp_{\lambda^{(m)}}
\end{bmatrix}, \quad \text{where } \lambda^{(i)} \in K^n \setminus \{0\}, \quad \text{and } p_i \in A
\]
and suppose \(\lambda^{(1)}, \ldots, \lambda^{(m)}\) to be pairwise different and without loss of generality \(\lambda^{(i)} \neq 0\) for all \(1 \leq i \leq m\) and \(1 \leq j \leq n\). Then
\[
\sum_{j=1}^{m} a_j \cdot (p_j \exp_{\lambda^{(j)}}) = 0 \quad \text{if and only if } [a_1, \ldots, a_m] \in \bigoplus_{j=1}^{m} \chi_{\lambda^{(j)}}(\ker(\kappa_{p_j})),
\]
which follows by
\[
\sum_{j=1}^{m} a_j \cdot (p_j \exp_{\lambda^{(j)}}) = 0 \iff \sum_{j=1}^{m} \left( \sum_{i=1}^{h_j} c_{ji} t^{\alpha_{ji}} \Delta^{\beta_{ji}} \cdot (p_j \exp_{\lambda^{(j)}}) \right) = 0
\]
\[
\iff \sum_{j=1}^{m} \left( \chi_{\lambda^{(j)}}^{-1}(a_j) \cdot p_j \right) \exp_{\lambda^{(j)}} = 0 \iff \chi_{\lambda^{(j)}}^{-1}(a_j) \in \ker(\kappa_{p_j}) \quad \text{for all } 1 \leq j \leq m.
\]
Choosing a suitable projection, we obtain by (13) and (15)

**Theorem 6.5** Let \(p\) be of the form (14). Further let \(K_i := \{ j \mid \lambda^j = \lambda^i \} = \{ k_{i1}, \ldots, k_{ih_i} \}\) and \(l\) chosen minimal such that the disjoint union \(K_1 \cup \ldots \cup K_l = \{ k_{11}, \ldots, k_{1h_1}, \ldots, k_{lh_l} \} = \{ 1, \ldots, m \}\). Further define the vector \(h_i := [f_{k_{i1}}, \ldots, f_{k_{ih_i}}]^T\) and \(H_i := \chi_{\lambda^{(i)}}(\ker(\kappa_{k_{ii}}))\). Let \(e_{k_i j}\) denote the \(k_{ij}\)-th standard generator of \(S_n^{1 \times m}\) for \(1 \leq i \leq l\) and \(1 \leq j \leq h_i\). Defining for \(1 \leq i \leq l\)
\[
\phi_i : H_i \rightarrow S_n, \quad [a_1, \ldots, a_{h_i}] \mapsto \sum_{j=1}^{h_i} a_j e_{k_{ij}}
\]
the VMPUM of \(p\) is given by \(\bigoplus_{i=1}^{l} \phi_i(H_i)\).

**Conclusion**

Generalizing ideas from systems theory, we have defined a “varying most powerful unfalsified model” (VMPUM) over polynomial Ore algebras such as the Weyl algebra or the difference algebra. Mathematically, this amounts to computing kernels of module homomorphisms over these algebras. On the one hand, this can be achieved using Gröbner bases techniques, and on the other, by translating the problem to an associated syzygy computation over a commutative polynomial ring, thus mimicking ideas of differential algebra. We have also studied some structural properties of the resulting models, and we have seen, in terms of examples, that models with polynomial coefficients provide a much better (and more precise) description of the data than models with constant coefficients. Further future work concerns, for instance, a characterization of the
vector space dimension of the VMPUM of several trajectories, thus generalizing
Theorem 5.2. Let \( p = [p_1, \ldots, p_m] \) consist of \( \mathbb{C} \)-linear independent signals. We
conjecture that \( \text{dim}_\mathbb{C}(\text{VMPUM}(p)) = m \). Moreover, it seems possible to us to
develop VMPUM with polynomial coefficients for data, represented by rational
and by rational-exponential functions.

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