On the ranks of the algebraic $K$-theory of hyperbolic groups

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Abstract Let $G$ be a word hyperbolic group. We prove that the algebraic $K$-theory groups of $\mathbb{Z}[G]$, $K_n(\mathbb{Z}[G])$, have finite rank for all $n \in \mathbb{Z}$. For a few classes of groups, we give explicit formulas for the ranks of the algebraic $K$-theory groups of their group rings.

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1 Introduction and preliminaries

Recall that the Farrell-Jones isomorphism conjecture proposes that, for any discrete group $G$, the algebraic $K$-theory of the group ring $\mathbb{Z}[G]$ is determined by the algebraic $K$-theory of the virtually cyclic subgroups of $G$ plus homological information.
Conjecture 1 (Farrell-Jones isomorphism conjecture, IC) Let $G$ be a discrete group. Then for all $n \in \mathbb{Z}$ the assembly map

$$A^\text{cyc}_{\mathcal{V}} : H^G_n(EG; \mathcal{K}) \to H^G_n(pt; \mathcal{K}) \cong K_n(\mathbb{Z}[G])$$

(1)

induced by the projection $EG \to pt$ is an isomorphism, where $H^G_*(-; \mathcal{K})$ is a suitable equivariant homology theory with local coefficients in $\mathcal{K}$, the non-connective spectrum of algebraic $K$ theory and $EG$ is a model for the classifying space for actions with isotropy in the family of virtually cyclic subgroups of $G$.

This conjecture has been verified, among others, when $G$ is a word hyperbolic group [1], or a CAT(0)-group [13]. Once we know the conjecture holds for a group $G$, we can try to compute $K_n(\mathbb{Z}[G]) \cong H^G_n(EG)$ using an Atiyah-Hirzebruch type spectral sequence.

In this paper, we use the validity of the Farrell-Jones conjecture and the corresponding spectral sequence to show that the rank of $K_n(\mathbb{Z}[G])$ is finite for all $n \in \mathbb{Z}$, where $G$ is a word hyperbolic group. Next, we give some explicit examples of computations of these ranks.

For hyperbolic groups Leary and Juan-Pineda [8], showed that

$$H^G_n(EG; \mathcal{K}) \cong H^G_n(EG; \mathcal{K}) \oplus \bigoplus_{(V)} \text{cok}_n(V),$$

(2)

where $EG$ is the classifying space for the family $\text{FIN}$, of finite subgroups of $G$, $(V)$ consists of one representative from each conjugacy class of maximal infinite virtually cyclic subgroup of $G$ and $\text{cok}_n(V)$ is the cokernel of the homomorphism $H^V_n(EG \to pt; \mathcal{K})$.

It is well known, see [5, Thm. 5.9 and page 4.] that the terms $\text{cok}_n(V)$ are torsion groups. This gives the following:

Lemma 1 Let $G$ be a discrete word hyperbolic group. Then for all $n \in \mathbb{Z}$

$$\text{rank}(K_n(\mathbb{Z}[G])) = \text{rank}(H^G_n(EG; \mathcal{K})).$$

This paper is, in part, complementary to [7] where we treated the case lower $K$ groups, namely $K_i(\cdot)$ for $i \leq 1$. Here we treat the whole spectrum of $K$ theory and a broader class of groups.

2 Ranks

In view of Lemma 1 the ranks of the algebraic $K$-theory groups of $\mathbb{Z}[G]$ are determined by the ranks of the algebraic $K$-theory of the finite subgroups of $G$ and the homology of $EG$. The ranks of the algebraic $K$ groups of the group ring of a finite group are given as follows:
Theorem 1 ([2], [3], [4], [6]) Let $H$ be a finite group with $r$ distinct real irreducible representations, $c$ of them of complex type, and $q$ distinct rational irreducible representations. For $n > 1$ we then have

$$\text{rank}(K_n(\mathbb{Z}[H])) = \begin{cases} r & \text{if } n \equiv 1 \text{ mod } 4, \\ c & \text{if } n \equiv 3 \text{ mod } 4, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

When $n \leq 1$, we have:

$$\text{rank}(K_1(\mathbb{Z}[H])) = r - q,$$
$$\text{rank}(K_0(\mathbb{Z}[H])) = 1,$$
$$\text{rank}(K_{-1}(\mathbb{Z}[H])) < \infty \text{ and }$$
$$K_{-n}(\mathbb{Z}[H]) = 0 \text{ for } n > 1.$$

Note that $r$ is equal to the number of real conjugacy classes of $H$, that is, classes of the form $C(h) = \{ghg^{-1}, gh^{-1}g^{-1} | g \in H\}$, $c$ is equal to the number of real conjugacy classes such that $C(h) \neq \{hgh^{-1} | h \in H\}$, and $q$ is the number of conjugacy classes of cyclic subgroups of $H$, see [11].

To compute the equivariant homology groups $H^G_*(EG; K)$ we may use an Atiyah-Hirzebruch type spectral sequence. Let $C_n$ denote the set of $n$-cells of the space $BG = EG/G$, then the first page of our spectral sequence is given by

$$\cdots \cdots \bigoplus_{\sigma^p \in C_n} K_q(\mathbb{Z}[G_{\sigma^p}]) \bigoplus_{\sigma^{p+1} \in C_{n+1}} K_q(\mathbb{Z}[G_{\sigma^{p+1}}]) \cdots \cdots$$

$$\cdots \cdots \bigoplus_{\sigma^p \in C_n} K_{q-1}(\mathbb{Z}[G_{\sigma^p}]) \bigoplus_{\sigma^{p+1} \in C_{n+1}} K_{q-1}(\mathbb{Z}[G_{\sigma^{p+1}}]) \cdots \cdots$$

where $G_{\sigma}$ denotes the stabilizer of a pre-image $\sigma' \in EG$ of $\sigma \in BG$, and the homomorphisms in the chain complex are induced by the natural inclusions (up to conjugacy). Note that $G_{\sigma}$ is always a finite group, hence we can apply Theorem 1 to every group appearing in our spectral sequence. As a consequence we may identify the second page of the Atiyah-Hirzebruch spectra sequence as

$$E^2_{p,q} = H_2(BG; \{K_q(\mathbb{Z}[G_{\sigma}])\}),$$
where the above is a homology theory with local coefficients given by the algebraic $K$ groups of the group rings $\mathbb{Z}[G_\sigma]$ for all the finite isotropy groups $G_\sigma$.

**Theorem 2** Let $G$ be an hyperbolic group. Then $\text{rank}(K_n(\mathbb{Z}[G]))$ is finite for all $n \in \mathbb{Z}$.

**Proof** It is known that for $G$ word hyperbolic, there exists a finite model for $EG$, i.e., such that $BG$ is compact, see for example [10]. Take this finite model for $EG$. Hence the only possible non-zero terms in the $n$th page of our spectral sequence $E^n_{p,q}$ are those terms with $0 \leq p \leq m$, $m = \text{dim } BG$, that is, they are contained in a vertical strip for all $n \in \{1, 2, 3, \ldots \} \cup \{\infty\}$. Now, since $E^\infty_{p,q}$ has finite rank because it is the quotient of a subgroup of the abelian group $E^1_{p,q}$ and

$$\text{rank}(K_n(\mathbb{Z}[G])) = \sum_{p+q=n} \text{rank}(E^\infty_{p,q})$$

the proof follows by Theorem 1 and the compactness of $BG$. $\square$

Note that using [5, Thm. 5.11] Lemma 1 is valid for every group satisfying the Farrell-Jones conjecture, hence following the proof of Theorem 2 we have:

**Theorem 3** Let $G$ be a group that admits a finite model for $BG$ and such that satisfies the Farrell-Jones conjecture. Then $\text{rank}(K_n(\mathbb{Z}[G]))$ is finite for all $n \in \mathbb{Z}$.

This last Theorem is more general and applies, for instance, to the groups that appear in [9].

### 3 Examples

In this section we give some explicit computations of $\text{rank}(K_n(\mathbb{Z}[G]))$.

#### 3.1 Finitely generated free groups

Let $F_n$ be the free group on $n$ generators, $n \in \mathbb{Z}$. Since $F_n$ is torsion free $EG = EG$, on the other hand we know that the Cayley graph of $G$ is a model for $EG$, and $BG$ with this model is a wedge of circles. Hence there is one 0-cell and $n$ 1-cells. Moreover, $G_\sigma = 1$ for all cells, hence

$$E^2_{p,q} = H_p(\vee_n S^1; \{K_q\}) = H_p(\vee_n S^1; K_q(\mathbb{Z})).$$

This gives

$$H_p(BG; K_q(\mathbb{Z})) = \begin{cases} K_q(\mathbb{Z}) & \text{for } p = 0, \\ \oplus_n (K_q(\mathbb{Z})) & \text{for } p = 1, \\ 0 & \text{for } p > 1 \text{ or } q \leq -1. \end{cases}$$
Ranks of the algebraic $K$-theory of hyperbolic groups

The graph associated to the free group on $n$ generators, the labels are the coefficients of the corresponding cell, these have all trivial stabilizers.

Notice that all the differentials vanish, hence this spectral sequence collapses at this stage giving

$$\text{rank}(K_n(\mathbb{Z}[F_n])) = \text{rank}(K_n(\mathbb{Z})) + n \cdot \text{rank}(K_{n-1}(\mathbb{Z})).$$

Applying Theorem 1 to the trivial group we have that

$$\text{rank}(K_n(\mathbb{Z})) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \text{ and } n > 1, \text{ or } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

it follows that

$$\text{rank}(K_n(\mathbb{Z}[F_n])) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \text{ and } n > 1, \text{ or } n = 0, \\ n & \text{if } n \equiv 2 \pmod{4} \text{ and } n > 2, \text{ or } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Free products of finite groups

Let $G_1$ and $G_2$ be finite groups, and let $G = G_1 * G_2$ be their free product. We can find a one-dimensional model for $EG$ such that $BG$ is a closed interval with trivial isotropy in the edge and isotropy at the vertices $G_1$ and $G_2$ [12]:

Once again our spectral sequence collapses at the second page giving

$$\text{rank}(K_n(\mathbb{Z}[G])) = \text{rank}(K_n(\mathbb{Z}[G_1])) + \text{rank}(K_n(\mathbb{Z}[G_2])) - \text{rank}(K_{n-1}(\mathbb{Z})).$$
and

\[
\text{rank}(K_n(\mathbb{Z}[G])) = \begin{cases} 
1 & n = 0, \\
r_1 + r_2 - q_1 - q_2 & i = 1, \\
r_1 + r_2 - 1 & \text{if } n \equiv 1 \mod 4, n > 1, \\
c_1 + c_2 & \text{if } n \equiv 3 \mod 4, n > 1, \\
0 & \text{otherwise.}
\end{cases}
\]

where \(r_i\) is the number of distinct real irreducible representations of \(G_i, i = 1, 2; c_i\) is the number of distinct real irreducible representations of complex type of \(G_i, i = 1, 2;\) and \(q_i\) is the number of distinct rational irreducible representations of \(G_i, i = 1, 2.\)

### 3.3 \(\text{PSL}_2(\mathbb{Z})\)

This is a particular case of the previous example since \(\text{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3.\) Using the notation from above, we set \(G_1 = \mathbb{Z}[\mathbb{Z}_2]\) and \(G_2 = \mathbb{Z}[\mathbb{Z}_3],\) hence \(r_1 = 2, r_2 = 2, c_1 = 0, c_2 = 1, q_1 = 2, q_2 = 2\) and

\[
\text{rank}(K_n(\mathbb{Z}[\text{PSL}_2(\mathbb{Z})])) = \begin{cases} 
0 & \text{if } n = -1, \\
1 & n = 0, \\
0 & n = 1, \\
3 & \text{if } n \equiv 1 \mod 4, n > 1, \\
1 & \text{if } n \equiv 3 \mod 4, n > 1, \\
0 & \text{otherwise.}
\end{cases}
\]

### 3.4 The fundamental group of a closed orientable aspherical surface

Let \(S_g\) be the orientable closed surface of genus \(g > 1.\) Since the universal covering of \(S_g\) is contractible we have that \(S_g\) is a model for \(B\pi_1(S_g)\). Furthermore, \(S_g\) is a model for \(B\pi_1(S_g)\) as well. Moreover, these groups are hyperbolic as \(S_g\) is a compact surface that admits a metric with constant curvature \(-1.\) Using the classical construction of \(S_g\) as the quotient of a \(4g\)-agon we can give \(S_g\) a CW-structure consisting of one 0-cell, \(2g\) 1-cells, and one 2-cell and they all have trivial isotropy. Hence, the second term of the Atiyah-Hirzebruch spectral sequence has constant coefficients the \(K\)-theory of the integers:

\[
H_p(S_g; K_q(\mathbb{Z})) = \begin{cases} 
K_q(\mathbb{Z}) & \text{for } p = 0, \\
\bigoplus_{2g} K_q(\mathbb{Z}) & \text{for } p = 1, \\
K_q(\mathbb{Z}) & \text{for } p = 2, \\
0 & \text{for } p > 1 \text{ or } q < 0.
\end{cases}
\]
Once again all differentials are trivial and our spectral sequence collapses. This gives

\[ \text{rank}(K_n(\mathbb{Z}[\pi_1(S_g)]) = \begin{cases} 
1 & n = 0, 2 \text{ or } n \equiv 1, 3 \mod 4, n > 1, \\
2g & n = 1 \text{ or } n \equiv 2 \mod 4, n > 1, \\
0 & \text{otherwise.}
\]  

3.5 Finitely generated virtually free groups

Let \( G \) be a finitely generated virtually free group, that is \( G \) has a finitely generated free subgroup of finite index. As finitely generated free groups are hyperbolic and a group with a \( \delta \)-hyperbolic subgroup of finite index is hyperbolic, it follows that \( G \) is \( \delta \)-hyperbolic as well. By Bass-Serre theory [12] and by the work in [7], one can find a tree \( T \) on which \( G \) acts with finite stabilizers, that is \( T \) is a model for \( EG \). Let \( E \) and \( V \) be the set of edges and vertices of the graph of groups for \( G \) determined by \( T \), this part is developed in [7] [section 2.2]. In order to calculate the second page of our spectral sequence, observe that \( T \) has only cells of dimension 0 and 1 and hence our page is

\[
E^2_{p,q} = \begin{cases} 
\text{coker} \left( \bigoplus_{e \in E} K_q(\mathbb{Z}[G_e]) \to \bigoplus_{v \in V} K_q(\mathbb{Z}[G_v]) \right), & \text{for } p = 0, \\
\text{ker} \left( \bigoplus_{e \in E} K_q(\mathbb{Z}[G_e]) \to \bigoplus_{v \in V} K_q(\mathbb{Z}[G_v]) \right), & \text{for } p = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that the differentials vanish and our spectral sequence collapses at this page hence

\[
H^n_G(EG; \{K_q\}) = \text{coker} \left( \bigoplus_{e \in E} K_n(\mathbb{Z}[G_e]) \to \bigoplus_{v \in V} K_n(\mathbb{Z}[G_v]) \right) \oplus \text{ker} \left( \bigoplus_{e \in E} K_{n-1}(\mathbb{Z}[G_e]) \to \bigoplus_{v \in V} K_{n-1}(\mathbb{Z}[G_v]) \right).
\]

In order to simplify the notation, let us define

\[
E_n = \bigoplus_{e \in E} K_n(\mathbb{Z}[G_e]), \\
V_n = \bigoplus_{v \in V} K_n(\mathbb{Z}[G_v]).
\]
\[ \text{Ker}_n = \ker(E_n \rightarrow V_n) \text{ and} \]
\[ \text{Cok}_n = \text{coker}(E_n \rightarrow V_n). \]

In this way we have that
\[ H_n^G(E_G; \{K_q\}) = \text{Cok}_n \oplus \text{Ker}_{n-1}. \]

These last groups depend on the graph structure of our tree with the stabilizers of the action, which are all finite.

### 3.6 $G = F_n \rtimes S_n$

This example is worked out in detail in [7, section 3] for other purposes. Let $G = F_n \rtimes S_n$ with the symmetric group $S_n$ on $n$ letters, acting on the free group on $n$ generators by permuting the generators. The graph of groups is a single loop with vertex group $S_{n-1}$ and edge group $S_n$. In this case the morphisms
\[ E_i \rightarrow V_i \]
are all zero, it follows that
\[ H_i^G(E_G; \{K_q\}) = K_i(\mathbb{Z}[S_n]) \oplus K_{i-1}(\mathbb{Z}[S_{n-1}]). \]

It is well known that the conjugation class of an element $x \in S_n$ is determined by its cyclic decomposition, since $x$ and $x^{-1}$ have the same cyclic decomposition we have that they belong to the same conjugacy class. Hence, the number of real conjugacy classes of $S_n$ is equal to $p(n)$, the number of partitions of $n$, and the number of real conjugacy classes of complex type is zero. Finally if two elements on $S_n$ determine the same cyclic subgroup then they are conjugate, this implies that the number of conjugacy classes of cyclic subgroups of $S_n$ is equal to $p(n)$.

\[ \text{rank}(K_i(\mathbb{Z}S_n)) = \begin{cases} p(n) & i \equiv 1 \mod 4 \ i > 1, \\ 1 & i = 0, \\ 0 & \text{otherwise}, \end{cases} \]

and
\[ \text{rank}(K_i(\mathbb{Z}G)) = \begin{cases} p(n) & i \equiv 1 \mod 4 \ i > 1, \\ p(n-1) & i \equiv 2 \mod 4 \ i > 1, \\ 1 & i = 0, \\ 0 & \text{otherwise}. \end{cases} \]

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