On characterizations of nondicritical generalized curve foliations

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Abstract

We characterize nondicritical generalized curve foliations with fixed reduced separatrix. Moreover, we give sufficient conditions when a plane analytic curve is its reduced separatrix. For that, we introduce a distinguished expression for a given 1-form, called Weierstrass form. Then, using Weierstrass forms, we characterize the nondicritical generalized curve foliations: first, for foliations with monomial separatrix using toric resolution; second, for foliations with reduced separatrix, using the GSV-index. In this last case the characterization, which is our main result, could be interpreted in function of a polar of the foliation and a polar of its reduced separatrix.

1 Introduction

In the study of holomorphic foliations to determine topological information is an important and a non trivial question. A special class of foliations gives some information about this problem: the generalized curve foliations. A foliation is said generalized curve if no saddle-nodes appear in its desingularization process. Moreover, a nondicritical generalized curve foliation and the union of its separatrices have the same process of reduction of singularities (see [Cam-LN-S], Theorem 2).

In this paper, we consider this class of foliations with the aim to present some new characterizations of nondicritical generalized curve foliation by means the 1-form that defines it. First, we characterize, in Theorem 3.4, the nondicritical generalized curve foliations with monomial separatrix using toric resolution, the prenormal form given by Loray in [Lo, page 157] and the notion of weighted order associated with a foliation and a curve (see Definition 3.1). This allows us to give, in Corollary 3.6, the condition in order that a foliation of second type becomes a generalized curve foliation following [FS-GB-SM, Theorem 1.2 (b)]. Second, using the Weierstrass division of power series, we introduce, in Definition 4.2, the notion of Weierstrass form of any 1-form with respect to any polynomial in one variable and coefficients in the complex power ring in one variable with a unit as principal coefficient. We can consider the Weierstrass form as a generalization, to any 1-form, of the prenormal form given by Loray for foliations with monomial separatrix.

In [B], Brunella established relations among the Baum-Bott, Camacho-Sad and Gómez-Mont-Seade-Verjovsky indices associated to one foliation. He proved that if the foliation is generalized curve then the Baum-Bott and Camacho-Sad indices are equal and the Gómez-Mont-Seade-Verjovsky index is zero. This paper goes in this direction.

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Our main result is Theorem 5.3, where we characterize the nondicritical generalized curve foliations using the GSV-index and the notion of Weierstrass form associated with a 1-form defining a foliation. This characterization is given in terms of intersection numbers and it could be interpreted in the language of polar curves of the foliation and its union of separatrices.

We precise, in Proposition 5.10, this characterization for foliations with a single separatrix and for foliations with a single separatrix of genus 1 in Corollary 5.11, this last one in terms of the weighted order.

The Weierstrass form has been very useful throughout the development of this work and we consider that it may be interesting in the future for foliation studies. Example of this is Proposition 4.3, where given $f \in \mathbb{C}\{x\}[y]$ with $\text{ord} f = \text{deg}_y(f)$ and principal coefficient a unit in $\mathbb{C}\{x\}$, and using the notion of Weierstrass form of a 1-form $W$ (defining a non-dicritical foliation) with respect to $f$, we give sufficient conditions when $f$ is the equation of the union of separatrices of the foliation determined by $W$.

2 Preliminaries

Let $f(x,y) \in \mathbb{C}\{x,y\}$ be a non unit power series, where $\mathbb{C}\{x,y\}$ is the set of convergent power series. A plane curve $C : \{f(x,y) = 0\}$ is by definition the zeroes set determined by $f(x,y) \in \mathbb{C}\{x,y\}$. The curve $C$ is irreducible (respectively reduced) if $f$ is irreducible in $\mathbb{C}\{x,y\}$ (respectively $f$ has no multiple factors).

An irreducible plane curve is called branch. The multiplicity of $C$, denoted by $\text{mult} C$, is by definition the order of the power series $f(x,y)$, that is $\text{mult} C = \text{ord} f$. Remember that the order of $f$ is the minimum of degrees of terms of $f$.

Consider a branch $C : \{f(x,y) = 0\}$ of multiplicity $n$. After a change of coordinates we can suppose that $x = 0$ is transversal (not tangent) to $C$ at 0. From Newton’s Theorem, $C$ admits an expansion with rational exponents $y(x^{1/n})$, such that $f(x,y(x^{1/n})) = 0$. According to Puiseux, the branch $C$ admits $n$ different expansions $\{y_i(x^{1/n})\}_{1 \leq i \leq n}$, where $y_i(x^{1/n}) = y(\varepsilon_i x^{1/n})$ and $\{\varepsilon_i\}_{1 \leq i \leq n}$ are the $n$th roots of unity in $\mathbb{C}$. We can write $f$, up to product by a unit in $\mathbb{C}\{x,y\}$, as the product

$$f(x,y) = \prod_{i=1}^{n} \left(y - y_i(x^{1/n})\right).$$

The expansions $\{y_i(x^{1/n})\}_{1 \leq i \leq n}$ are called Newton-Puiseux roots of the branch $C$ (or equivalently of $f$). Any Newton-Puiseux root $y_i(x^{1/n})$ is of the form

$$y_i(x^{1/n}) = \sum_{j \geq n} a_j^{(i)} x^{j/n},$$

where $j \in \mathbb{N}$, $a_j^{(i)} \in \mathbb{C}$ and it determines a parametrization of $C$ as follows

$$(x_i(t), y_i(t)) = \left(t^n, \sum_{j \geq n} a_j^{(i)} t^j\right).$$

It is well-known that if the multiplicity of a branch is bigger than one, then there exist $g \in \mathbb{N}\setminus\{0\}$ and positive integers $\beta_0 = n$ and $\beta_k = \min \{k : a_k^{(i)} \neq 0 \text{ and } \gcd(\beta_0, \beta_1, \beta_2, \ldots, \beta_{k-1}) \text{ does not divide } k\}$ for $1 \leq k \leq g$. In the sequel we put $\varepsilon_k = \gcd(\beta_0, \beta_1, \beta_2, \ldots, \beta_k)$ for $0 \leq k \leq g$. We get $\varepsilon_0 = n > \varepsilon_1 > \cdots > \varepsilon_g = 1$. Set $n_k := \varepsilon_{k+1}/\varepsilon_k$. In particular, $n = \beta_0 = n_1 \cdots n_g$.

The sequence $(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ is called the Puiseux characteristic exponents of the branch $C$ and the number $g$ is called the genus of the branch $C$. We denote by $K(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$ the set of plane branches with Puiseux characteristic exponents $(\beta_0, \beta_1, \beta_2, \ldots, \beta_g)$. It is well-known, after Brauner-Zariski, that the characteristic exponents determine the topological class of the branch $C$.
Let \( h_1(x, y), h_2(x, y) \in \mathbb{C}\{x, y\} \) be two power series and \( I = (h_1, h_2) \) the ideal generated by \( h_1, h_2 \in \mathbb{C}\{x, y\} \).

The intersection number of the curves \( C_i : \{h_i(x, y) = 0\}, \; 1 \leq i \leq 2, \) is \( i_0(h_1, h_2) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/I. \)

Given \( f \in K(\beta_0, \beta_1, \beta_2, \ldots, \beta_g) \) the semigroup \( \Gamma(f) \) associated to \( f \) is

\[
\Gamma(f) = \{i_0(f, h) : \; h \in \mathbb{C}\{x, y\} \setminus \{f\}\}.
\]

The semigroup \( \Gamma(f) \) admits a unique minimal system of generators, given by \( \{v_0, v_1, v_2, \ldots, v_g\}, \) that is, \( \Gamma(f) = \langle v_0, v_1, v_2, \ldots, v_g \rangle = \{\sum_{i=0}^{g} a_i v_i : \; a_i \in \mathbb{N}\}. \) It is a well know fact that the semigroup \( \Gamma(f) \) and the characteristic exponents are mutually determined. Moreover, we can obtain the minimal system of generators by the Puiseux characteristic exponents using the relations (see [Z], Theorem 3.9):

\[
v_0 = n = \beta_0, \quad v_1 = m = \beta_1, \quad v_{i+1} = n_i v_i + \beta_{i+1} - \beta_i \quad \text{for} \quad i = 1, \ldots, g - 1.
\]

Observe that \( e_i = \gcd(v_0, \ldots, v_i) \) for \( 0 \leq i \leq g. \)

For any reduced plane curve (not necessary irreducible) \( C : \{f(x, y) = 0\}, \) an important topological invariant, useful in this paper, is the Milnor number, that is the intersection number \( \mu(f) := i_0(f, f_g), \) where \( f_g \) (respectively \( f_x \)) denotes the partial derivative of \( f \) with respect to \( y \) (respectively \( x \)).

By Teissier’s Lemma (see [Te] Chapter II, Proposition 1.2]) we have

\[
i_0(f, f_g) = \mu(f) + i_0(f, x) - 1. \tag{2}
\]

We also have

\[
i_0(f, f_y) = \mu(f) + i_0(f, y) - 1. \tag{3}
\]

For more details on plane curves see for example [Hef] or [W].

Let \( \Omega^1_{\mathbb{C}^2, 0} := \mathbb{C}\{x, y\}dx + \mathbb{C}\{x, y\}dy \) be the \( \mathbb{C}\{x, y\} \)-module of holomorphic 1-form.

A holomorphic foliation singular at the origin is defined, in a neighbourhood of the origin, by an equation \( \mathcal{F}_W : \{W = 0\}, \) where \( W \) is a 1-form \( W = A(x, y)dx + B(x, y)dy, \) with \( A, B \in \mathbb{C}\{x, y\} \) without common factors and such that \( A(0, 0) = B(0, 0) = 0. \) The polar of \( \mathcal{F}_W : \{W = 0\} \) with respect the direction \( (b : -a) \in \mathbb{P}^1 \) is the curve \( aA(x, y) + bB(x, y) = 0. \)

The multiplicity of \( W \) is \( \text{mult}(W) := \min\{\text{ord}(A(x, y)), \text{ord}(B(x, y))\}. \) More precisely if \( \text{mult}(W) = n_0 \) then we can write \( W = \sum_{i+j \geq n_0} a_{i,j} x^i y^j dx + \sum_{i+j \geq n_0} b_{i,j} x^i y^j dy, \) and for some \( i, j \) such that \( i + j = n_0 \) we get \( a_{i,j} \neq 0 \) or \( b_{i,j} \neq 0. \) The multiplicity of the foliation \( \mathcal{F}_W : \{W = 0\} \) is by definition the multiplicity of the 1-form \( W. \)

Let \( \mathcal{F}_W \) be a singular foliation at the origin. The Jacobian matrix of the linear part of \( W \) is the matrix

\[
\begin{pmatrix}
-B_x(0, 0) & -B_y(0, 0) \\
A_x(0, 0) & A_y(0, 0)
\end{pmatrix}.
\]

Denote by \( \lambda_1, \lambda_2 \) its complex eigenvalues. We say that the origin is an irreducible singularity of \( \mathcal{F}_W \) if one of the following conditions is satisfied:

1. \( \lambda_1 \lambda_2 \neq 0 \) and \( \frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+, \)

2. \( \lambda_1 = 0 \) and \( \lambda_2 \neq 0; \) or \( \lambda_2 = 0 \) and \( \lambda_1 \neq 0. \)

If the first condition holds we say that the origin is a simple or non-degenerate singularity of \( \mathcal{F}_W. \) Nevertheless if the second condition holds then we say that the origin is a saddle-node singularity of \( \mathcal{F}_W. \)

Example 2.1. If \( W = y^2 dx + xdy \) then the Jacobian matrix of the linear part of \( W \) is \( \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \) Its eigenvalues are \(-1 \) and \( 0, \) so the origin is a saddle-node singularity of \( \mathcal{F}_W. \)
Let $\Pi : (M, D) \to (\mathbb{C}^2, 0)$ the process of reduction of singularities of the foliation $\mathcal{F}_W$ (see [Sei]), given by a finite composition of quadratic transformations (blow-ups) having $D = \Pi^{-1}(0)$ as the exceptional divisor, which is a finite union of projective lines with normal crossings; and where $M$ is a non-singular analytic manifold. There is a 1-form $W$ given by a finite composition of quadratic transformations (blow-ups) having $\mathcal{F}_W$ branches $N \cdot P$ separatrices $\mathcal{S} = (\Pi : (M, D) \to (\mathbb{C}^2, 0))$. We will denote it $F_{\mathcal{F}_W}$ and put $f_{\mathcal{F}_W} = 0$ and $N \cdot P$ separatrices of $\mathcal{F}_W$. According to the number of separatrices that pass through the singular point of the foliation we can classify the foliations in dicritical in the case that we have infinite separatrices or nondicritical when we have a finite number of separatrices. As a consequence of [Cam-LN-S, Theorem 1] we get

**Theorem 2.2.** Let $\mathcal{F}_W$ be a nondicritical foliation with union of separatrices $\{f(x, y) = 0\}$. Then $\text{mult}(W) \geq \text{ord}(f) - 1$.

**Example 2.3.** If $W_1 = nx \cdot dy - my \cdot dx$ with $n$ and $m$ coprime positive integers, then the branches $\{x = 0\}, \{y = 0\}$ and $\{y^n \cdot cx^m = 0\}$, $c \neq 0$ are separatrices of $W_1 = 0$, so the foliation $\mathcal{F}_{W_1}$ is dicritical. Nevertheless if $W_2 = x \cdot dy + y \cdot dx$ then the foliation $\mathcal{F}_{W_2}$ is nondicritical since its only separatrices are $\{x = 0\}$ and $\{y = 0\}$.

Let $C_i : \{f_i(x, y) = 0\}, 1 \leq i \leq r$, be the set of different separatrices of a nondicritical foliation $\mathcal{F}_W : \{W = 0\}$ and put $f(x, y) := f_1(x, y) \cdots f_r(x, y)$. The reduced curve $C = \{f(x, y) = 0\}$ will be called the union of the separatrices of the nondicritical foliation $\mathcal{F}_W$.

Other important notion we will use is the Newton polygon that we introduce in the sequel. Let $S \subset \mathbb{N}^2$. We consider the convex hull $\text{conv}(S)$ of the Minkowski sum $S + \mathbb{R}^2_{\geq 0}$, where $\mathbb{R}^2_{\geq 0}$ denote the non-negative real numbers. By definition, the Newton polygon of $S$, denoted by $\mathcal{N}P(S)$, is the union of the compact edges of the boundary of $\text{conv}(S)$.

The support of a power series $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ is supp $f := \{(i, j) \in \mathbb{N}^2 : a_{i,j} \neq 0\}$. The support of a foliation $\mathcal{F}_W : \{W = 0\}$, where $W = A(x, y) \cdot dx + B(x, y) \cdot dy$, is the union of the supports of $x \cdot A(x, y)$ and $y \cdot B(x, y)$, and we will denote it supp $W$, that is supp $W = \text{supp}(x \cdot A(x, y)) \cup \text{supp}(y \cdot B(x, y))$. The Newton polygon of a power series $f(x, y) \in \mathbb{C}\{x, y\}$ is by definition the Newton polygon of supp $f$ and we will denote it $\mathcal{N}P(f)$. The Newton polygon of a foliation $\mathcal{F}_W$ is by definition the Newton polygon of supp $W$, and we will denote it $\mathcal{N}P(W)$. Observe that the Newton polygon depends on coordinates. Remark that $\mathcal{N}P(u \cdot f) = \mathcal{N}P(f)$ for any $u, f \in \mathbb{C}\{x, y\}$, where $u$ is a unit. Hence, we can define the Newton polygon of the curve $C : \{f(x, y) = 0\}$ as the Newton polygon of any of its equations.

**Proposition 2.4.** ([Re, Proposition 3.8]) Let $W = A(x, y) \cdot dx + B(x, y) \cdot dy$ be a 1-form. If $\mathcal{F}_W$ is a nondicritical generalized curve foliation and $C : \{f(x, y) = 0\}$ is its union of separatrices then the Newton polygons of $\mathcal{F}_W$ and $C$ are the same.

Saravia, in her PhD thesis [SM] (see also [FS-GB-SM, Example 3.3]), shows that the foliation $\mathcal{F}_W$, where $W = ((b-1)xy - y^3)dx + (xy - bx^2 + xy^2)dy$, with $b \not\in \mathbb{Q}$, is not a generalized curve foliation but its Newton polygon equals to the Newton polygon of its union of separatrices $f(x, y) = xy(x - y)$.

Milnor number can be also defined in the foliation context. Let $A(x, y), B(x, y) \in \mathbb{C}\{x, y\}$ with $A(0, 0) = B(0, 0) = 0$. Let $W = A(x, y) \cdot dx + B(x, y) \cdot dy$ be a 1-form and consider the singular foliation at the origin
The Milnor number at the origin of the foliation \( \mathcal{F}_W \) is \( \mu(\mathcal{F}_W) := i_0(A(x, y), B(x, y)) \).

Following [Cam-LN-S, Theorem 4] we have the next characterization of nondicritical generalized curve foliations using Milnor numbers:

**Theorem 2.5.** Let \( W = A(x, y)dx + B(x, y)dy \) be a 1-form. Suppose that \( \mathcal{F}_W \) is a nondicritical generalized curve foliation and \( C : \{ f(x, y) = 0 \} \) is its union of separatrices. Then \( \mu(\mathcal{F}_W) \geq \mu(f) \). The equality holds if and only if \( \mathcal{F}_W \) is a generalized curve foliation.

Let \( C : \{ f(x, y) = 0 \} \) be a reduced plane curve. In what follows we consider:

- \( \text{Fol}(f) \) the set of all 1-forms \( W \), defining a nondicritical foliation \( \mathcal{F}_W \), such that \( f \) divides \( W \wedge df \).
- \( \text{Fol}(f) \) the set of nondicritical foliations defined by elements of \( \text{Fol}(f) \) which union of separatrices is \( C : \{ f(x, y) = 0 \} \).

In this paper we will present new characterizations for generalized curve foliations.

### 3 Characterization of generalized curve foliations with monomial separatrix

In this section we characterize the nondicritical generalized curve foliations with monomial separatrix, by means of the weighted order defined as following

**Definition 3.1.** Let \( p, q \in \mathbb{Z}^+ \). The weighted order \( v_{p,q} \) for power series and differential forms is:

\[
v_{p,q} \left( \sum_{i,j} a_{i,j} x^i y^j \right) = \min \{ ip + jq : a_{i,j} \neq 0 \},
\]

and

\[
v_{p,q} \left( \sum_{i,j} A_{i,j} x^{i-1} y^j dx + \sum_{i,j} B_{i,j} x^i y^{j-1} dy \right) = \min \{ ip + jq : A_{i,j} \neq 0 \text{ or } B_{i,j} \neq 0 \}.
\]

Let \( f \in \mathbb{C}\{x, y\} \) and \( n, m \) positive integers not necessarily coprime. For \( f = y^n - x^m \), Frank Loray obtained a prenormal form of an arbitrary element \( \mathcal{F}_W \in \text{Fol}(f) \).

**Theorem 3.2.** (Lo, page 157) If \( f = y^n - x^m \) and \( \mathcal{F}_W \in \text{Fol}(f) \) then

\[
W = df + \sum_{0 \leq i \leq m-2, 0 \leq j \leq n-2} P_{i,j}(f)x^iy^j(nx\,dy - my\,dx),
\]

for some \( P_{i,j}(z) \in \mathbb{C}\{z\} \).

For \( f(x, y) = y^n - x^m \) with \( 0 < n \leq m \) not necessary coprime, we will give a characterization when the foliation \( \mathcal{F}_W \in \text{Fol}(f) \) is generalized curve.

The 1-form \( W \) in (4) can be rewritten as

\[
W = df + (\Delta(x, y) + f(x, y)g(x, y))(nx\,dy - my\,dx),
\]

where \( g \in \mathbb{C}\{x, y\} \) and \( \Delta(x, y) = \sum_{0 \leq i \leq m-2, 0 \leq j \leq n-2} a_{i,j} x^i y^j \).

Let \( \gcd(n, m) = d \). Since \( m \geq n \) there are integers \( p, q \in \mathbb{Z}^+ \) with \( mp - nq = d \).
On the other hand, the hypothesis \( \sum \) using the representation given in (5) we have \( N \) and by Proposition 2.4, we get the equality \( W \). Notice that \( \Pi \) where \( r, s \) and the Jacobian matrix at the point \((0, 0)\) is

\[
\begin{pmatrix}
\frac{mn}{d} & 0 \\
\ast & -d(1 + \sum_{r,s} a_{r,s} \xi^{p(r+1)+q(s+1)-q_n})
\end{pmatrix},
\]

(7)

where \( r, s \) verify \( m(r + 1) + n(s + 1) = mn \).

**Proof.** Notice that \( \Pi^* W = u^{m-1}v^{m_n}d^{-1} \cdot W' \), where

\[
W' = \left[ (nq-mpu^d) + (mq-mp)S + h(u,v)(nqu^{p+q}v^{\frac{n+m}{d}} - pmu^{d+p+q}v^{\frac{n+m}{d}}) \right] du + \left[ \frac{mn}{d}(1-u^d)u + \frac{m}{d}(u^{p+q}v^{\frac{n+m}{d}} - u^{p+q+d}v^{\frac{n+m}{d}}) \right] dv,
\]

(8)

with \( S = \sum a_{i,j}u^{p(i+1)+j}v^{q(j+1)} - q_n v^{m(j+1)+n(s+1)-mn} \) and \( h(u, v) \in \mathbb{C}\{u, v\} \). By hypothesis, if \((i, j) \in \text{supp} \Delta \) then \( ni + jm \geq mn - m - n \), and we have \( \text{ord}_v(S) \geq 0 \). On the other hand we claim that \( \text{ord}_u(S) > 0 \), that is \( p(i+1) + q(j+1) > q_n \). In effect, from \( ni + jm \geq mn - m - n \) and \( mp - nq = d \) we get \( p(i+1)+q(j+1) = \frac{pm(i+1)+pm(j+1)}{n} - \frac{d(j+1)}{n} \geq \frac{pmn-d(j+1)}{n} \). Since \( 0 \leq j \leq n-2 \) we have \( \frac{pmn-d(j+1)}{n} > \frac{pmn-dn}{n} \), so \( p(i+1)+q(j+1) > pm - d - nq \).

Therefore the singular points of \( \mathcal{F}_{W'} \) are \((0, 0)\) and \((\xi, 0)\), with \( \xi^d = 1 \). The point \((0, 0)\) corresponds to the intersection of the exceptional divisors, the other points correspond to the irreducible components of \( y^n - x^m \).

Rewrite (8) as \( W' = A(u,v)du + B(u,v)dv \). We get \( B_u(0, 0) = \frac{mn}{d} \), \( B_v(0, 0) = A_u(0, 0) = 0 \) and \( A_v(0, 0) = nq \), so the Jacobian matrix at \((0, 0)\) is (6) and the origin is not a saddle-node. On the other hand \( B_u(\xi, 0) = \frac{mn}{d} \), \( B_v(\xi, 0) = A_u(\xi, 0) = 0 \) and \( A_v(\xi, 0) = 1 + \sum a_{r,s} \xi^{p(r+1)+q(s+1)-q_n} \). Hence the Jacobian matrix at \((\xi, 0)\) is (7).

Using Lemma 3.3 we get:

**Theorem 3.4.** Let \( \mathcal{F}_W \in \text{Fol}(f) \) be a foliation defined by \( W \) as in (5). Then \( \mathcal{F}_W \) is a generalized curve foliation if and only if \( v_{n,m}(\Delta(x, y)) \geq mn - n - m \), and \( (1 + \sum_{r,s} a_{r,s} \xi^{p(r+1)+q(s+1)-q_n}) \notin \mathbb{Q}^- \cup \{0\} \), where \( m(r+1) + n(s+1) = mn \) and \( \xi^d = 1 \).

**Proof.** Suppose that the foliation \( \mathcal{F}_W \) is generalized curve. Since its union of separatrices is \( \{f(x, y) = 0\} \) and by Proposition 2.1 we get the equality \( \mathcal{N}(f) = \mathcal{N}(df) \) and we have \( v_{n,m}(W) \geq mn \). In addition, using the representation given in (5) we have \( v_{n,m}(\sum a_{i,j}x^iy^jdx) \geq mn \) and \( v_{n,m}(\sum a_{i,j}x^iy^jdy) \geq mn \); so \( v_{n,m}(\Delta(x, y)) + n + m = n(i+1) + m(j+1) \geq mn \). Moreover the points \((\xi, 0)\) are not saddle-nodes, hence by (7) we get \( (1 + \sum_{r,s} a_{r,s} \xi^{p(r+1)+q(s+1)-q_n}) \notin \mathbb{Q}^- \cup \{0\} \).

On the other hand, the hypothesis \( v_{n,m}(\Delta(x, y)) \geq mn - n - m \) allows us to apply Lemma 3.3. Since \( (1 + \sum_{r,s} a_{r,s} \xi^{p(r+1)+q(s+1)-q_n}) \notin \mathbb{Q}^- \cup \{0\} \) we conclude that \( \mathcal{F}_W \in \text{Fol}(f) \) is a generalized curve foliation. □
Example 3.5. Suppose that \( W = d(y^3 - x^6) + axy(3x dy - 6y)dx \), where \( a \in \mathbb{C} \). In this case \( p = q = 1 \), and the toric morphism is \( x = uv, y = uv^2 \). The total transform of \( W \) is

\[
\Pi^*W : u^2v^5[(3 - 6u^3) - 3au)vdu + 6(1 - u^3)udv] = u^2v^5(Adx + Bdy),
\]

where \( A = (3 - 6u^3) - 3au \) and \( B = 6(1 - u^3)u \). Then the singularities of the strict transform of \( W \) are \((0, 0), (\xi, 0)\) with \( \xi^3 = 1 \). Now \( 6r + 2 + 3(s + 1) = 18 \) if and only if \((r, s) = (1, 1)\). Hence, if \((1 + a\xi) \in \mathbb{Q}^-\), for some \( \xi \) with \( \xi^3 = 1 \), then the foliation \( \mathcal{F}_W \) is not generalized curve.

Mattei and Salem \([\text{Ma-Sal}]\) consider a family of foliations, more general than the generalized curved foliations, called foliations of the \( \xi \) for some \( \xi > \mathcal{F}_W \). It is a consequence of \([\text{FS-GB-SM}, \text{Theorem 1.2}\) and Theorem 3.4.\]

Proof. It is a consequence of \([\text{FS-GB-SM}, \text{Theorem 1.2}\) and Theorem 3.4.\]

Remark 3.7. From Theorem 3.4 we can deduce Proposition 5.7 of \([\text{FS-GB-SM}\) : let \( n, m \) be two positive integers which are not coprime. Consider \( f(x, y) = y^n - x^m \) and \( \mathcal{F}_W \in \text{Fol}(f) \), where \( W = df + \Delta'(x, y)(nxdy - mydx) \) with \( \Delta'(x, y) \in \mathbb{C}\{x, y\} \). If we suppose that \( i_0(\Delta', f) > mn - m - n \) then from the proof of Lemma 3.3 we get that \( \text{ord}_v(S) > 0 \). Hence \(-d(1 + \sum a_{r,s}^{p(r+1) + q(s+1)} - qn) = -d \). In particular, the foliation \( \mathcal{F}_W \) is generalized curve.

In Example 3.4 we have a family of nondicritical generalized curve foliations \( \mathcal{F}_W \) with \( i_0(\Delta', f) = mn - m - n \) when \((1 + a\xi) \not\in \mathbb{Q}^-\).

\section{Weierstrass form of a 1-form}

In this section we introduce, using the Weierstrass division of power series, a distinguished equation for a given 1-form, with respect to any Weierstrass polynomial, called \textit{Weierstrass form}. The Weierstrass form is very useful for computations and it can be considered as a generalization, to any 1-form of the prenormal form given by Loray for foliations with monomial separatrix. The Weierstrass form is well-defined for any 1-form. Nevertheless, in this paper we are using the Weierstrass forms associated with 1-form \( W \) defining a nondicritical foliation.

Let \( f = \sum_{i=0}^n a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y] \) be a reducible polynomial (not necessarily irreducible) of degree \( \deg_y(f) = n, a_0(0) \neq 0 \) and \( W \in \Omega^{1}_{\mathbb{C}^2,0} \) be a 1-form. Here we do not suppose that \( C : \{f(x, y) = 0\} \) is the union of separatrices of the foliation \( \mathcal{F}_W \). In that follows we will obtain an equation of \( W \) in function of \( f \) and \( df \).

Lemma 4.1. If \( W \in \Omega^{1}_{\mathbb{C}^2,0} \) and \( f(x, y) = \sum_{i=0}^n a_i(x)y^{n-i} \in \mathbb{C}\{x\}[y] \) with \( \deg_y(f) = n > 1 \) and \( a_0(0) \neq 0 \), then there exist unique \( h, p \in \mathbb{C}\{x, y\} \) and \( A, B \in \mathbb{C}\{x\}[y] \) with \( B = 0 \) or \( \deg_y(B) < n - 1 \) and \( A = 0 \) or \( \deg_y(A) < n \) such that

\[
W = hd + pf \frac{dx}{y} + Adx + Bdy. \tag{9}
\]
Proof. Put \( W = R(x, y)dx + S(x, y)dy \in \Omega^1_{C^2, 0} \). After the Weierstrass division of \( S \) by \( f_y \), there exist \( h \in \mathbb{C}[x, y] \) and \( B \in \mathbb{C}[x][y] \) such that \( S = hf_y + B \), where \( B = 0 \) or \( \deg_y B < \deg_y f_y = \deg_y f - 1 = n - 1 \) and \( W = Rdz + hf_ydy + Bdy \). Since \( df = f_xdx + f_ydy \) then \( W = (R - hf_x)dx + hdf + Bdy \). Now, by the Weierstrass division of \( R - hf_x \) by \( f \) there exist \( p \in \mathbb{C}[x, y] \) and \( A \in \mathbb{C}[x][y] \) such that \( R - hf_x = pf + A \) with \( A = 0 \) or \( \deg_y A < \deg_y f = n \) and

\[
W = hdf + pf dx + Adx + Bdy.
\]

Suppose that \( h_1 df + p_1 f dx + A_1 dx + B_1 dy = hdf + p_2 f dx + A_2 dx + B_2 dy \) with \( h_i, p_i \in \mathbb{C}[x, y] \) and \( A_i, B_i \in \mathbb{C}[x][y] \) with \( \deg_y(B_i) < n - 1 \) and \( \deg_y(A_i) < n \) for \( i = 1, 2 \). Then \( (h_1 - h_2)f_y = B_2 - B_1 \) and

\[
(h_1 - h_2)f_x + (p_1 - p_2)f = A_2 - A_1.
\]

As \( \deg_y(B_1) < n - 1 = \deg_y(f_y) \), we get \( h_1 = h_2 \) and \( B_1 = B_2 \). Similarly, \( \deg_y(A_i) < n = \deg_y(f) \) implies that \( p_1 = p_2 \) and \( A_1 = A_2 \).

**Definition 4.2.** Let \( W \in \Omega^1_{C^2, 0} \) and \( f = \sum_{i=0}^n a_i(x) y^{n-i} \in \mathbb{C}[x][y] \) with \( \deg_y(f) = n > 1 \) and \( a_0(0) \neq 0 \). The **Weierstrass form** of \( W \) with respect to \( f \) is \( hdf + pf dx + Adx + Bdy \), where \( h, p \in \mathbb{C}[x, y] \) and \( A, B \in \mathbb{C}[x][y] \) are unique and satisfying \( B = 0 \) or \( \deg_y(B) < n - 1 \) and \( A = 0 \) or \( \deg_y(A) < n \).

We can rewrite (10) as

\[
W = hdf + pf dx + \omega, \quad (10)
\]

where \( \omega = \sum A_{i,j} x^i y^j dx + \sum B_{i,j} x^i y^j dy = Adx + Bdy \) for some \( A_{i,j}, B_{i,j} \in \mathbb{C} \). Since \( B = 0 \) or \( \deg_y B < n - 1 \) then \( B_{0,0} = 0 \).

Observe that, in (10), \( h, p \) and \( w \) depend on \( f \). Moreover if \( W = R(x, y)dx + S(x, y)dy \in \Omega^1_{C^2, 0} \), where \( R(x, y), S(x, y) \in \mathbb{C}[x][y] \) and \( f \in \mathbb{C}[x][y] \) with \( \deg_y(f) > \max\{(\deg_y(R), \deg_y(S) + 1) \} \) then, in (10), \( h = p = 0 \) and the Weierstrass form of \( W \) is \( w \).

**Proposition 4.3.** Let \( W \in [\text{Fol}(f)] \) as (10), where \( h \) is a unit of \( \mathbb{C}[x, y] \) and \( f(x, y) = \sum_{i=0}^n a_i(x) y^{n-i} \in \mathbb{C}[x][y] \) with \( n = \deg_y(f) = \text{ord}(f) \) and \( a_0(0) \neq 0 \). Then \( \mathcal{F}_W \in \text{Fol}(f) \), that is \( C \) is the union of separatrices of \( \mathcal{F}_W \).

**Proof.** Given that \( \deg_y(f) = \text{ord}(f) \), the multiplicity of \( W \) equals the multiplicity of \( hdf + \omega \). Since \( h \) is a unit we get \( \text{mult}(W) = \min\{\text{ord}(f_x + A), \text{ord}(f_y + B)\} \leq \text{ord}(f_y + B) \). But \( B_{0,0} = 0 \) and \( \text{ord}(f_y + B) \leq n - 1 \). Hence

\[
\text{mult}(W) \leq n - 1. 
\]

(11)

Suppose that \( \mathcal{F}_W \) has other separatrix \( g(x, y) = 0 \). By Theorem 2.2 we have \( \text{mult}(W) \geq \text{ord}(fg) - 1 > n - 1 \), which is a contradiction after the inequality (11).

In Section 5 we will see how the Weierstrass forms allow us to give new characterizations of generalized curve foliations. We can also characterize the second type foliations using the Weierstrass forms. To do this, we first remember the characterization given by Mattei and Salem:

**Theorem 4.4.** ([Ma-Sa, Théorème 3.1.9]) Let \( W = A(x, y)dx + B(x, y)dy \) be a 1-form. Suppose that \( \mathcal{F}_W \) is a non dicritical foliation and \( C : \{f(x, y) = 0\} \) is its union of separatrices. Then \( \mathcal{F}_W \) is a second type foliation if and only if \( \text{mult}(W) = \text{mult}(df) \) is a 1-form.
• If $h(0,0) = 0$ then $\text{mult} (W) = \text{mult} (df)$ if and only if $\text{mult} (\omega) = \text{mult} (df)$.
• If $h(0,0) \neq 0$ then $\text{mult} (W) = \text{mult} (df)$ if and only if $\text{mult} (\omega) \geq \text{mult} (df)$.

As, by Theorem 4.4, the equality $\text{mult} (W) = \text{mult} (df)$ characterizes the 1-forms that define second type foliations, we can read this property using $\omega$.

In \cite{[10]} we remarked that for any Weierstrass form $W = hdf + pf dx + \omega$, where $\omega = \sum A_{i,j} x^{i-1} y^j dx + \sum B_{i,j} x^i y^{j-1} dy = Adx + Bdy$ for some $A_{i,j}, B_{i,j} \in \mathbb{C}$, we get $B_{0,0} = 0$. If $f \in K(n, m)$ and $W \in [\text{Fol}(f)]$ we can obtain additional information about the coefficient $A$.

We can write, without lost of generality

$$f(x, y) = y^n - x^m + \sum_{i+j \geq n} a_{i,j} x^i y^j.$$  \hspace{1cm} (12)

**Lemma 4.5.** Let $f \in K(n, m)$. If $W \in [\text{Fol}(f)]$, with $W$ as in \cite{[10]} then $A_{m,0} = 0$.

**Proof.** Consider $f(x, y)$ as in \cite{[12]}. Since $C : \{ f(x, y) = 0 \}$ is the separatrix of $F_W$ and $\omega = W - hdf - pfdx$ then $f(x, y) = 0$ is also a separatrix of $w$. Consider a parametrization of $f(x, y) = 0$ given by $(x(t), y(t)) = (t^m, t^m + \cdots)$, where $\cdots$ means terms of greater degree. Therefore $\sum A_{i,j} x(t)^{i-1} y(t)^{j} dx(t) + \sum B_{i,j} x(t)^i y(t)^{j-1} dy(t) = 0$. Suppose that $A_{m,0} \neq 0$. Since $A_{m,0} x(t)^{m-1} dx(t) = n A_{m,0} t^{m-1}$ then there is $(i_0, j_0) \in \text{supp} A \cup \text{supp} B$ such that $n i_0 + m j_0 = nm$, where $A = \sum A_{i,j} x(t)^{i-1} y(t)^{j} dx(t) - n A_{m,0} t^{m-1}$ and $B = \sum B_{i,j} x(t)^i y(t)^{j-1} dy(t)$. But $n$ and $m$ are coprime, so $(i_0, j_0) \in \{ (m, 0), (0, n) \}$, which is a contradiction since these points are not in $\text{supp} A \cup \text{supp} B$. \hfill $\square$

## 5 Characterization of generalized curve foliations: general case

In this section we present our main result. For that we need the notion of GSV-index.

Let $f(x, y) \in \mathbb{C}[x, y]$ and $W \in [\text{Fol}(f)]$. By \cite{[L]} page 198] in the irreducible case and \cite{[S]} (1.1) [Lemma] in the reduced case, there are $g, k \in \mathbb{C}[x, y]$ and a 1-form $\eta$ such that $gW = kdf + f \eta$, with $f$ and $k$ coprime.

**Definition 5.1.** With the above notations, the GSV-index of $W$ with respect to $C : \{ f(x, y) = 0 \}$ is

$$\text{GSV}(W, C) := \frac{1}{2 \pi i} \int_{\partial C} \frac{g}{k} \log \left( \frac{k}{g} \right).$$

Suppose that $f(x, y)$ is irreducible and consider a parametrization $(x(t), y(t))$ of $C$. Write $W = A(x, y) dx + B(x, y) dy$ and $\eta = p(x, y) dx + q(x, y) dy$. We have

$$W = \left( \frac{k}{g} f_x + \frac{f}{g} p \right) dx + \left( \frac{k}{g} f_y + \frac{f}{g} q \right) dy.$$  \hspace{1cm} (13)

On the other hand,

$$\frac{k}{g} (x(t), y(t)) = \frac{(k/f_y) f_y + (f/y) f_y}{f_y} (x(t), y(t)),$$

so the number of zeroes of $k$ restricted to $C$ minus the number of zeroes of $g$ restricted to $C$ equals $\text{ord}_t \frac{k}{f_y} (x(t), y(t))$. Hence by Rouché-Hurwitz theorem we have

$$\text{GSV}(W, C) = \text{ord}_t \frac{B}{f_y} (x(t), y(t)).$$  \hspace{1cm} (13)

A similar calculation as before shows that
\[
GSV(W, C) = \text{ord}_t \frac{A}{f_x}(x(t), y(t)).
\]

Now, consider \(C : \{f(x, y) = 0\}\) for \(f = f_1 \cdot f_2\) with \(f_1\) and \(f_2\) irreducible. After [13] page 532 we have

\[
GSV(W, C) = GSV(W, C_1) + GSV(W, C_2) - 2i_0(f_1, f_2), \tag{14}
\]

where \(C_i : \{f_i(x, y) = 0\}\). The equality (14) is also true when \(f_1, f_2\) are reduced, not necessary irreducible.

Cavalier and Lehmann gave a characterization of generalized curve foliations using the GSV-index:

**Theorem 5.2.** ([Cav-Le, Théorème 3.3]) Let \(C : \{f(x, y) = 0\}\) be a reduced curve and \(\mathcal{F}_W \in \text{Fol}(f)\) a nondicritical foliation. Then \(\mathcal{F}_W\) is generalized curve if and only if \(GSV(W, C) = 0\).

In the remainder of the section we will present our main result and consequences of it.

Let \(f \in \mathbb{C}\{x\}[y]\), where \(f = f_1 \cdots f_r\) is the factorization of \(f\) into irreducible factors. Consider \(C : \{f(x, y) = 0\}\).

**Theorem 5.3.** Let \(\mathcal{F}_W \in \text{Fol}(f)\) and \(hdf + pf dx + Adx + Bdy = Adx + Bdy\) the Weierstrass form of \(W\) with respect to \(f\). Then \(\mathcal{F}_W\) is a generalized curve foliation if and only if \(h \in \mathbb{C}\{x, y\}\) is a unit and \(i_0(\mathcal{B}, f) = \mu(f) + i_0(f, x) - 1\).

**Proof.** First, observe that \(\mathcal{B} = hf_y + B\). Hence,

\[
i_0(\mathcal{B}, f) = i_0(hf_y + B, f) = \sum_{j=1}^{r} i_0\left(h \prod_{i \neq j} f_i(f_j)_y + B, f_j\right).
\]

Suppose that \(\sum_{j=1}^{r} i_0(h \prod_{i \neq j} f_i(f_j)_y + B, f_j) = \mu(f) + i_0(f, x) - 1\) and \(h(x, y)\) is a unit.

If \(C_i : \{f_i(x, y) = 0\}\) and \(C : \{f_1 \cdots f_r = 0\}\) then, by (13), the GSV-index of \(W\) with respect to \(C\) is

\[
GSV(W, C) = \sum_{i=1}^{r} GSV(W, C_i) - 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).
\]

By hypothesis,

\[
W = \left(h \sum_{i=1}^{r} \prod_{j \neq i} f_j(f_i)_x + fp + A\right)dx + \left(h \sum_{i=1}^{r} \prod_{j \neq i} f_j(f_i)_y + B\right)dy = Adx + Bdy,
\]

hence \(A = h \sum_{i=1}^{r} \prod_{j \neq i} f_j(f_i)_x + fp + A\) and \(B = h \sum_{i=1}^{r} \prod_{j \neq i} f_j(f_i)_y + B\). Then by (13) the GSV-index of \(W\) with respect to the separatrix \(C_j : \{f_j(x, y) = 0\}\) is

\[
GSV(W, C_j) = \text{ord}_t \left(\frac{\mathcal{B}(x_j(t), y_j(t))}{(f_j)_y(x_j(t), y_j(t))}\right) = i_0(\mathcal{B}, f_j) - i_0((f_j)_y, f_j),
\]

where \((x_j(t), y_j(t))\) is a parametrization of \(C_j\). By definition of \(\mathcal{B}\) and using (2) we have

\[
GSV(W, C_j) = i_0\left(h \prod_{i \neq j} f_i(f_j)_y + B, f_j\right) - \left(\mu(f_j) + i_0(f_j, x) - 1\right).
\]

Hence

\[
GSV(W, C) = \sum_{j=1}^{r} i_0\left(h \prod_{i \neq j} f_i(f_j)_y + B, f_j\right) - \sum_{j=1}^{r} \left(\mu(f_j) + i_0(f_j, x) - 1\right) - 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).
\]
Now, by the formula of Milnor for a reduced curve (see for example [W, Theorem 6.5.1]) we obtain
\[
\text{GSV}(W,C) = \sum_{j=1}^{r} i_0 \left( h \prod_{i \neq j} f_i(f_j)_y + B, f_j \right) - \left( \mu(f) + i_0(f,x) - 1 \right).
\] (15)

So, by Theorem 5.2 we conclude that \( F_W \) is a generalized curve foliation.

Suppose now that \( F_W \) is a generalized curve foliation which union of separatrices is \( C : \{ f(x,y) = 0 \} \). Hence \( N \mathcal{P}(W) = N \mathcal{P}(f) \). Moreover, since \( h d f + p f dx + Adx + Bdy \) is the Weierstrass form of \( W \) with respect to \( f \) we get \( B_{0,n} = 0 \) and we conclude that \( h \) is a unit. In particular, we have again the equality (15).

Finally, since \( F_W \) is a generalized curve foliation, again by Theorem 5.2 we get \( 0 = \text{GSV}(W,C) \). This finishes the proof. \( \square \)

Theorem 5.3 gives a characterization of a generalized curve foliation \( F_W \), where \( W = Adx + Bdy \), using the polar \( B \) of the foliation \( F_W \) and the polar \( f_g \) of the separatrix \( C : \{ f(x,y) = 0 \} \).

**Corollary 5.4.** Let \( F_W \in \text{Fol}(f) \) and \( h d f + p f dx + Adx + Bdy = Adx + Bdy \) the Weierstrass form of \( W \) with respect to \( f \). Then \( F_W \) is a generalized curve foliation if and only if \( h \in \mathbb{C}[x,y] \) is a unit and \( i_0(B,f) = i_0(f_g,y,f) \). In particular \( i_0(B,f) \geq i_0(f_g,f) \).

Suppose now that \( f \in \mathbb{C}[x,y] \) is an irreducible monic polynomial of degree \( n = \text{ord} f \) with semigroup \( \Gamma(f) = \{ v_0, v_1, v_2, \ldots, v_g \} \) \( (v_0 = n) \). Remember that we denote \( e_i = \text{gcd}(v_0, \ldots, v_i) \) for \( i \in \{0, \ldots, g\} \) and \( n_i = \frac{v_i}{e_i} \) for \( i \in \{1, \ldots, g\} \). By convention we put \( n_0 = 1 \). We say that \( f_k \in \mathbb{C}[x,y] \) is a \( k \)-semiroot of the polynomial \( f \) if \( f_k \) is monic, \( \text{deg}_y(f_k) = \frac{m}{e_{k-1}} = n_0 n_1 \cdots n_{k-1} \) and \( i_0(f_k,f) = v_k \). The notion of semi-root is a generalization of the characteristic approximate roots introduced and studied by Abhyankar and Moh in [A-M].

Applying [Pa] Corollary 5.4 to the polynomial \( B \in \mathbb{C}[x,y] \) in (9), it can be uniquely written as a finite sum of the form
\[
B = \sum_{\text{finite}} a_\alpha(x) f_1^{\alpha_1} \cdots f_g^{\alpha_g},
\] (16)

where \( a_\alpha(x) \in \mathbb{C}[x] \), \( f_k \) are \( k \)-semiroots of \( f \) and \( 0 \leq \alpha_k < n_k \). Since \( \text{deg}_y B < \text{deg}_y f - 1 \) the polynomial \( f \) does not appear as a factor in the terms of the right-hand side of (16).

As a \( k \)-semiroot \( f_k \) is irreducible and admits semigroup \( \{ \frac{n_0}{e_{k-1}}, \frac{n_1}{e_{k-1}}, \ldots, \frac{n_{k-1}}{e_{k-1}} \} \) it follows that its Newton polygon has a single compact face with vertices \( (0, \frac{m}{e_{k-1}}) \) and \( (\frac{n}{e_{k-1}}, 0) \) and consequently \( v_{n,m}(f_k) = \frac{nm}{e_{k-1}} \).

In this way, using the relations (1) we get
\[
i_0(f,f_k) = v_k = \frac{nm}{e_{k-1}} + \sum_{i=2}^{k} \frac{n_i \cdots n_k}{n_k} (\beta_i - \beta_{i-1}) = v_{n,m}(f_k) + \sum_{i=2}^{k} \frac{e_{i-1}}{e_{k-1}} (\beta_i - \beta_{i-1}).
\] (17)

In particular, \( i_0(f,f_k) \geq v_{n,m}(f_k) \) with equality if and only if \( k = 1 \).

**Remark 5.5.** For any two distinct terms \( T_\alpha := a_\alpha(x) f_1^{\alpha_1} \cdots f_g^{\alpha_g} \) and \( T_\alpha' := a_\alpha'(x) f_1^{\alpha'_1} \cdots f_g^{\alpha'_g} \) of (16) we get \( i_0(T_\alpha,f) \neq i_0(T_\alpha',f) \).

In addition, remark that we can not have \( \alpha_i = n_i - 1 \) for all \( i = 1, \ldots, g \). Indeed, if this is the case we get
\[
\text{deg}_y(B) = \sum_{i=1}^{g} (n_i - 1) \text{deg}_y(f_i) = \sum_{i=1}^{g} (n_i - 1) n_0 n_1 \cdots n_{i-1} = \sum_{i=1}^{g} n_0 n_1 \cdots n_{i-1} n_i \sum_{i=1}^{g} n_0 n_1 \cdots n_{i-1} - n_0 = n_0 n_1 \cdots n_g - n_0 = v_0 - 1 = n - 1.
\]
which is a contradiction.

As a consequence of (17) and Remark 5.5 we have that \( i_0(f, B) \geq v_{n,m}(f) \) with equality if and only if \( f \in K(n, m) \).

**Lemma 5.6.** With the above notations, if \( f \) is irreducible then \( i_0(B, f) \neq i_0(f, f) \).

**Proof.** Suppose that \( i_0(f, f) = i_0(B, f) \). After Remark 5.5 there exists a unique \( g \)-tuple \((\alpha_1, \ldots, \alpha_g)\) with \( 0 \leq \alpha_i < n_i \) such that

\[
i_0(B, f) = i_0(a_\alpha(x)f_1^{\alpha_1} \cdots f_g^{\alpha_g}, f) = \sum_{i=1}^g \alpha_i v_i + \lambda_0 v_0,
\]

where \( \lambda_0 = \text{ord}_x a_\alpha(x) \). Now, by (2) we get \( i_0(f, f) = \mu(f) + v_0 - 1 = \sum_{i=1}^g (n_i - 1) v_i \) (see for example [Hel Proposition 7.5 (ii), page 102] for the last equality). In this way, we have

\[
\sum_{i=1}^g (n_i - 1) v_i = i_0(f, f) = i_0(B, f) = \sum_{i=1}^g \alpha_i v_i + \lambda_0 v_0,
\]

that is \( \sum_{i=1}^g (n_i - 1 - \alpha_i) v_i = -\lambda_0 v_0 = 0 \). But, this implies that \( \lambda_0 = 0 \) and \( \alpha_i = n_i - 1 \) for all \( i = 1, \ldots, g \), which is a contradiction. Hence, \( i_0(f, f) \neq i_0(B, f) \).

**Remark 5.7.** Observe that Lemma 5.6 is not true for \( f \) reduced (non-irreducible): consider \( f(x, y) = y^2 - x^2 \) and \( B = x^e \). We have \( \deg_B(B) = 1 < 1 = \deg_B(f) - 1 \), \( i_0(f, f) = 2 \) and \( i_0(B, f) = 2e \). So, for \( e = 1 \) we get \( i_0(f, f) = i_0(B, f) \) and \( i_0(f, f) \neq i_0(B, f) \) for \( e \neq 1 \).

**Corollary 5.8.** With the above notations, for \( f \) irreducible we have, \( i_0(B, f) > i_0(f, f) \) if and only if \( i_0(f + B, f) = \mu(f) + i_0(f, x) - 1 \).

**Proof.** Suppose that \( i_0(B, f) > i_0(f, f) \). Then

\[
i_0(f + B, f) = \min\{i_0(f, f), i_0(B, f)\} = i_0(f, f),
\]

and by Teissier’s Lemma (see (2)) we have \( i_0(f + B, f) = \mu(f) + i_0(f, x) - 1 \).

Now we suppose that \( i_0(f + B, f) = \mu(f) + i_0(f, x) - 1 \), that is, by Teissier’s Lemma, \( i_0(f + B, f) = i_0(f, f) \) and by Lemma 5.6 we conclude \( i_0(B, f) > i_0(f, f) \).

**Remark 5.9.** If \( W \) is written as \([9]\) and \( W \in [\text{Fol}(f)] \) then \( A(x, y)dx + B(x, y)dy \in \text{Fol}(f) \). We must have \( i_0(A, f) + v_0 = i_0(B, f) + v_1 \). Hence by (2) and (3) we get \( i_0(A, f) - i_0(f, f) = i_0(B, f) - i_0(f, f) \).

So, \( i_0(B, f) > i_0(f, f) \) if and only if, \( i_0(A, f) > i_0(f, f) \). Moreover, by Lemma 5.6 we have \( i_0(A, f) \neq i_0(f, f) \). So, \( i_0(f + A, f) = \min\{i_0(f, f), i_0(A, f)\} \). Consequently \( i_0(f + B, f) = \mu(f) + i_0(f, x) - 1 \) if and only if \( i_0(f + A, f) = \mu(f) + i_0(f, y) - 1 \).

Next proposition gives us a characterization of generalized curve foliations with a single separatrix.

**Proposition 5.10.** Let \( f(x, y) \in \mathbb{C}\{x\}[y] \) be irreducible, \( \mathcal{F}_W \in \text{Fol}(f) \), where \( hdf + pf dx + Adx + Bdy \) is the Weierstrass form of \( W \) with respect to \( f \). Then \( \mathcal{F}_W \) is a generalized curve foliation if and only if \( h \in \mathbb{C}\{x, y\} \) is a unit and \( i_0(B, f) > i_0(f, y) \).

**Proof.** It is a consequence of Corollaries 5.4 and 5.8.

Next corollary gives us a characterization of generalized curve foliations with a single separatrix of genus 1 in terms of the weighted order (see Definition 5.1).

**Corollary 5.11.** Let \( f \in K(n, m) \), \( \mathcal{F}_W \in \text{Fol}(f) \), where \( hdf + pf dx + Adx + Bdy \) is the Weierstrass form of \( W \) with respect to \( f \). Then \( \mathcal{F}_W \) is a generalized curve foliation if and only if \( v_{n,m}(\omega) > nm \), where \( \omega = Adx + Bdy \).
Proof. By Remark 5.9 the equality $i_0(f_y + B, f) = \mu(f) + i_0(f, x) - 1$ is equivalent to $i_0(f_y + A, f) = \mu(f) + i_0(f, y) - 1$. It follows, by Corollary 5.8, that this is equivalent to claim $i_0(f_y, f) < i_0(B, f)$ and $i_0(f_x, f) < i_0(A, f)$. As $f \in K(n, m)$, we have $\mu(f) = nm - n - m + 1$ if $i_0(f_y, f) = nm - m$ and $i_0(f_x, f) = nm - n$ (see [2] and [3]). So, $nm < i_0(B, f) + m$ and $nm < i_0(A, f) + n$. Since $i_0(H, f) = v_{n,m}(H)$, for any $H \in \mathbb{C}\{x, y\}$ then $i_0(f_y + B, f) = \mu(f) + i_0(f, x) - 1$ is equivalent to $nm < \min\{v_{n,m}(B) + m, v_{n,m}(A) + n\} = v_{n,m}(\omega)$. We finish the proof using Proposition 5.10 and Corollary 5.8.

Example 5.12. Let $W_c = d(y^3 - x^4) + cx^2y(3xdy - 4ydx)$, $c \in \mathbb{C}^*$. By Proposition 4.3 we get $F_{W_c} \in \text{Fol}(y^3 - x^4)$. In this case $B(x, y) = -4cx^2y^2$ and $i_0(B, f) = 14 > \mu(f) + i_0(f, x) - 1 = 9$, then $F_{W_c}$ is a generalized curve foliation.

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