Relic gravitational waves and the generalized second law

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Abstract

The generalized second law of gravitational thermodynamics is applied to the present era of accelerated expansion of the Universe. In spite of the fact that the entropy of matter and relic gravitational waves inside the event horizon diminish, the mentioned law is fulfilled provided the expression for the entropy density of the gravitational waves satisfies a certain condition.
I. INTRODUCTION

As is well-known, when the Universe makes a transition from a stage of expansion dominated by a given energy source to the next (e.g., inflation–radiation era, radiation–matter era, matter–dark energy era), relic gravitational waves (RGWs), i.e., the gravitational waves generated from the quantum vacuum, suffer amplification [1, 2, 3, 4]. Likewise, their wavelength get stretched as the Universe expands. For a wave mode \( k \) to fit in the Hubble horizon \( (H^{-1}) \) it must fulfill the obvious condition \( k > aH \) (where \( a \) is the scale factor of the Robertson-Walker metric). During decelerated eras of cosmic expansion the quantity \( aH \) decreases and gravitational waves continuously enter the horizon. By contrast, in accelerated eras -like the present one, see e.g. [5]- \( aH \) increases whereby gravitational waves continuously leave the horizon. Those most recently created are the first to exit, those created in the inflationary era are the last to exit.

On the other hand, gravitational waves whose wavelength exceeds the horizon size do not contribute to the energy density in the horizon [3, 4] and so they do not add to the entropy inside the horizon. Therefore, in the current accelerated era of expansion the gravitational wave entropy in the horizon is steadily diminishing and, as we shall see, the entropy of matter is decreasing as well. As for the entropy of the dark energy field responsible for the acceleration we can only say that it is likely zero (as is the case of the cosmological constant) or undetermined. There still remains the entropy of the horizon itself. Properly speaking, the entropy of the event horizon is given by \( S_H = (k_B/4)A/\ell_P^2 \) -where \( A \) is the area of the horizon, \( k_B \) the Boltzmann constant, and \( \ell_P \) the Planck length-, see Ref. [6].

By extending the “generalized second law” (GSL) of black hole spacetimes [7] to cosmological settings, several authors have considered the interplay between ordinary entropy and the entropy associated to cosmic event horizons [8, 9]. According to this law, the entropy of matter and/or radiation within the horizon plus the entropy of the event horizon cannot diminish with time. In this Brief Report we study, with the help of the GSL, the entropy budget of matter, gravitational waves and event horizon in the current accelerated stage of cosmic expansion. As it turns out, the GSL is fulfilled provided a proportionality constant entering the expression of the gravitational wave entropy satisfies a certain upper bound.

Section II gives the power spectrum of the relic gravitational waves at the beginning of the present era of accelerated expansion. Section III considers the evolution of the entropy.
of the gravitational waves during that era. Finally, in section IV application of the GSL is seen to imply an upper bound on the expression of the entropy density of gravitational waves.

II. THE RELIC GRAVITATIONAL WAVES IN THE DARK ENERGY ERA

The current era of cosmic acceleration is believed to be dominated by some sort of energy (generically called “dark energy”) characterized by violating the strong energy condition \[10\]. In a recent paper we studied the spectrum and energy density of RGWs \[11\]. We assumed the present era was successively preceded by an inflationary era, a radiation dominated era and a matter dominated era. The dependence of the scale factor on the conformal time in these eras is given by

\[
a(\eta) = \begin{cases} 
-\frac{1}{H_1 \eta} & (-\infty < \eta < \eta_1 < 0), \\
\frac{1}{H_1 \eta_1^2} (\eta - 2\eta_1) & (\eta_1 < \eta < \eta_2), \\
\frac{1}{4H_1 \eta_1^2} \frac{(n_3 - 4\eta_1)^2}{n_2 - 2n_1} & (\eta_2 < \eta < \eta_3), \\
\left(\frac{1}{2}\right)^{-l} \frac{(n_3 + 2\eta_1 - 4\eta_1)^{2-l}}{4H_1 \eta_1^2 (n_2 - 2n_1)} (\eta_2)^l & (\eta_3 < \eta), 
\end{cases}
\]

where \(l < -1, \eta_i = \eta + \frac{l}{2} \left[-\frac{2}{l+1} \eta_3 + \eta_2 - 4\eta_1\right]\). The subindexes 1, 2, 3 correspond to sudden transitions from De Sitter era to radiation era, from radiation to dust era, and from dust era to dark energy era, respectively, \(H_i\) is the Hubble factor at the instant \(\eta = \eta_i\). The present time \(\eta_0\) lies in the dark energy era. The Hubble factor during the dark energy era obeys

\[
H(\eta) = \left(\frac{a_3}{a(\eta)}\right)^{1+\frac{1}{l}} H_3.
\]

The evolution of the quantity \(aH\) in terms of the conformal time \(\eta\) is sketched in Fig. \[\text{Fig. 1}\].

The modes’ solution to the gravitational wave equation during the De Sitter era are related with those of the final dark energy era by a Bogoliubov transformation with coefficients \(\alpha_{T_r}\) and \(\beta_{T_r}\). Specifically, \(\langle N_\omega \rangle = |\beta_{T_r}|^2\) gives the number of RGWs created from the initial vacuum state, and from it we get the power spectrum \(P(\omega(\eta)) = (\hbar \omega^3(\eta)/\pi^2 c^3) \langle N_\omega \rangle\), where \(\omega\) denotes the frequency. At the beginning of the dark energy era, \(\eta = \eta_3\), the power
FIG. 1: Evolution of $a(\eta)H(\eta)$ in a universe with scale factor given by Eq. (1).

spectrum (obtained in the adiabatic vacuum approximation [12]) was [11]

$$P(\omega) \simeq \begin{cases} 0 & (\omega(\eta) > 2\pi(a_1/a(\eta))H_1), \\ \frac{h}{16\pi^2 c^3} \left(\frac{a_2}{a(\eta)}\right)^4 H_1^4 \omega^{-1} & (2\pi(a_2/a(\eta))H_2 < \omega(\eta) < 2\pi(a_1/a(\eta))H_1), \\ \frac{h}{16\pi^2 c^3} \left(\frac{a(\eta)}{a_2}\right)^2 \left(\frac{a_1}{a(\eta)}\right)^8 H_1^6 \omega^{-3} & (2\pi H(\eta) < \omega(\eta) < 2\pi(a_2/a(\eta))H_2). \end{cases}$$

(3)

During the radiation and dust eras $a(\eta)H(\eta)$ decreases with $\eta$ and increases during the De Sitter and dark energy eras. Consequently, RGWs are continuously leaving the Hubble radius during the accelerated dark energy era [11, 13]. At some instant $\eta_{f0}$, defined by $a(\eta_{f0})H(\eta_{f0}) = a_2H_2$, the third term in (3) ceases to contribute to the power spectrum since the wavelengths of the corresponding RGWs exceed the size of the horizon. Finally, at $\eta_{f1}$, defined by $a(\eta_{f1})H(\eta_{f1}) = a_1H_1$, all RGWs have their wavelength longer than the Hubble radius and the power spectrum vanishes altogether.
III. ENTROPY OF THE RGWS IN THE DARK ENERGY ERA

There are different expressions in the literature for the entropy density of gravitational waves -see e.g. [14, 15, 16]. All of them are based on the assumption that the gravitational entropy is associated with the amount of RGWs inside the horizon. We shall adopt the proposal of Nesteruk and Ottewill [16], namely: the gravitational entropy is proportional to the number of RGWs, i.e.,

$$ \displaystyle s_g = A n_g, \quad (4) $$

where $n_g$ is the number density of gravitational waves, and $A$ is an unknown positive–definite constant of proportionality.

We are interested in the evolution of $n_g$ during the dark energy era. The number density of RGWs created from the initial vacuum state is

$$ \displaystyle dn_g(\eta) = \frac{\omega^2(\eta)}{2\pi^2 c^3} d\omega(\eta) \langle N_\omega \rangle = \frac{P(\omega(\eta))}{2\hbar\omega(\eta)} d\omega(\eta), \quad (5) $$

where the term in square brackets is the density of states. We can obtain $n_g(\eta)$ by inserting Eq. (3) into Eq. (5) and integrating over the frequency.

At the beginning of the dark energy era the RGWs number density is

$$ \displaystyle n_g(\eta_3) = n_g(\eta_2) \left( \frac{a_2}{a_3} \right)^3 + \frac{1}{168\pi^5 c^3} \left( \frac{a_1}{a_2} \right)^2 \left( \frac{a_2}{a_3} \right)^3 H_1^3 \left[ \left( \frac{a_3}{a_2} \right)^2 - 1 \right], \quad (6) $$

where

$$ \displaystyle n_g(\eta_2) = \frac{1}{16\pi^3 c^3} \left( \frac{a_1}{a_2} \right)^3 H_1^3 \left[ \frac{a_2}{a_1} - 1 \right] \quad (7) $$

is the number density at the transition radiation era–dust era.
For \( \eta < \eta_{f0} \), i.e. \( \frac{a(\eta)}{a_3} < \left( \frac{a_1}{a_2} \right)^{-\frac{l}{2}} \), one has

\[
n_g(\eta_3 < \eta < \eta_{f0}) = n_g(\eta_2) \left( \frac{a_2}{a(\eta)} \right)^3 + \frac{1}{768\pi^5c^3} \left( \frac{a_1}{a_2} \right)^2 \left( \frac{a_2}{a_3} \right)^3 H_1^3 \left( \frac{a_3}{a(\eta)} \right)^3 \left[ \left( \frac{a_3}{a(\eta)} \right)^{-\frac{l}{2}} \left( \frac{a_3}{a_2} \right)^\frac{l}{2} - 1 \right].
\]

The density number given by Eq. (8) decreases with the scale factor because of two effects: (i) the evolution of the volume considered, which grows as \( a^3 \), and (ii) the exit of those waves whose wavelength becomes longer than the Hubble radius, which appears in the term in square brackets. As \( \frac{a(\eta)}{a_3} \) approaches \( \left( \frac{a_3}{a_2} \right)^{-\frac{l}{2}} \), this term tends to zero. From this instant on, the number density reads

\[
n_g(\eta_{f0} < \eta < \eta_{f1}) = \frac{1}{16\pi^3c^3} \left( \frac{a_1}{a_3} \right)^3 H_1^3 \left( \frac{a_3}{a(\eta)} \right)^3 \left[ \left( \frac{a_2}{a_1} \right)^\frac{l}{2} \left( \frac{a_3}{a_1} \right)^\frac{l}{2} \left( \frac{a_3}{a(\eta)} \right)^{-\frac{l}{2}} - 1 \right]. \tag{9}
\]

As time goes on, the term in brackets tends to zero and at the instant \( \eta_{f1} \), \( n_g \) vanishes.

Consequently, the entropy density, proportional to the number density, decreases during the dark energy era not just because of the variation of the volume considered but also because of the disappearance of the RGWs from the Hubble volume.

The RGWs entropy inside the event horizon is

\[
S_g = \frac{4\pi}{3} R_H^3 s_g, \tag{10}
\]

where \( R_H = a(t) \int_t^\infty dt'/a(t') \) is the radius of the event horizon and \( t \) the cosmic time. For the horizon to exist \( R_H \) must not diverge. For expansions of the general form \( a(\eta) \propto \eta^n \) (i.e., \( a(t) \propto t^n \) with \( n = l/(1 + l) \) and \( l < -1 \)), the horizon exists and can be expressed as \( R_H = -cH^{-1}(\eta) \). Since it is of the order of magnitude of the Hubble radius, \( cH^{-1} \), we will neglect \( -l \) and use both terms interchangeably.

Making use of Eq. (2) and Eqs. (8)-(9), we obtain

\[
S_g(\eta_3 < \eta < \eta_{f0}) = \frac{4\pi c^3}{3} H_3^{-3/2} A \left( \frac{a_3}{a(\eta)} \right)^{-3/l} \times \left\{ n_g(\eta_2) \left( \frac{a_2}{a_3} \right)^3 + \frac{1}{768\pi^5c^3} \left( \frac{a_1}{a_2} \right)^2 \left( \frac{a_2}{a_3} \right)^3 H_1^3 \left[ \left( \frac{a_3}{a(\eta)} \right)^{-\frac{l}{2}} \left( \frac{a_3}{a_2} \right)^\frac{l}{2} - 1 \right] \right\} \tag{11}
\]
and

\[
S_g(\eta_f_0 < \eta < \eta_f_1) = \frac{4\pi c^3}{3} H_{f_1}^{-3} A \left( \frac{a_3}{a(\eta)} \right)^{-3/l} \frac{1}{16\pi^3 c^3} \left( \frac{a_1}{a_3} \right)^3 H_1^3 \\
\times \left[ \left( \frac{a_2}{a_1} \right)^{\frac{2}{3}} \left( \frac{a_3}{a_1} \right)^{\frac{1}{3}} \left( \frac{a_3}{a(\eta)} \right)^{-\frac{1}{3}} - 1 \right].
\]

(12)

The RGWs entropy is a decreasing function of the scale factor and consequently of conformal time. The next section explores whether this entropy descent can be compensated by an increase of the entropy of the other contributors, namely, matter and horizon.

IV. THE GENERALIZED SECOND LAW

According the generalized second law of gravitational thermodynamics the entropy of the horizon plus its surroundings (in our case, the entropy in the volume enclosed by the horizon) cannot decrease. Consequently, we must evaluate the total entropy to see how it evolves during the present dark energy era.

The entropy of the horizon (proportional to its area, \(4\pi c^2 H^{-2}\)),

\[
S_H = \frac{k_B c^2}{\ell_{Pl}^2} H^{-2}(\eta) = \frac{k_B c^2}{\ell_{Pl}^2} H_3^{-2} \left( \frac{a_3}{a(\eta)} \right)^{2 - \frac{2}{3}},
\]

(13)

increases with expansion. (Bear in mind that \(l < -1\), i.e., we are assuming that the dark energy behind the acceleration is not of “phantom” type [17]).

We must also consider the non-relativistic matter fluid. Assuming the latter consists in particles of mass \(m\) and that each of them contributes \(k_B\) to the matter entropy, we get

\[
S_m = k_B \rho_m \frac{4\pi c^3}{3} H^{-3} = k_B \frac{c^3}{2mG} H_3^{-1} \left( \frac{a_3}{a(\eta)} \right)^{-\frac{1}{3}},
\]

(14)

for the entropy of the non-relativistic fluid. Here, we made use of the conservation equation

\[
\rho_m(\eta) = \rho_m(\eta_3) \left( \frac{a_3}{a(\eta)} \right)^3 = \frac{3}{8\pi G} H_3^2 \left( \frac{a_3}{a(\eta)} \right)^3 \frac{3}{2} \text{ with } \rho_m \text{ the energy density of matter. From (14), it is apparent that } S_m \text{ decreases with expansion.}
In virtue of the above equations, the GSL, \( S'_m + S'_g + S'_H \geq 0 \), where the prime indicates derivation with respect to \( \eta \), can be written as

\[
\frac{3}{l} \left[ \frac{c^3}{2mG} H_3^{-1} + \frac{4\pi c^3}{3} AH_3^{-3} \left( \frac{a_2}{a_3} \right)^3 n_g(\eta) \right] + \left( 2 + \frac{2}{l} \right) \frac{c^2 \pi}{l^2 P_l} H_3^{-2} \left( \frac{a_3}{a(\eta)} \right)^{-2+\frac{1}{l}} + \frac{3}{l} \frac{A}{576\pi^4} \left( \frac{a_1}{a_2} \right)^2 \left( \frac{a_2}{a_3} \right)^3 H_1^3 H_3^{-3} \left[ 2 \left( \frac{a_3}{a(\eta)} \right)^{-\frac{1}{l}} \left( \frac{a_3}{a_2} \right)^{\frac{1}{l}} - 1 \right] \geq 0, \tag{15}
\]

for \( \frac{a(\eta)}{a_3} < \left( \frac{a_3}{a_2} \right)^{-\frac{1}{l}} \), and

\[
\frac{3}{l} \frac{c^3}{2mG} H_3^{-1} + \left( 2 + \frac{2}{l} \right) \frac{\pi c^2}{l^2 P_l} H_3^{-2} \left( \frac{a_3}{a(\eta)} \right)^{-2+\frac{1}{l}} + \frac{3}{l} \frac{A}{12\pi^2} \left( \frac{a_1}{a_3} \right)^3 H_1^3 H_3^{-3} \left[ \frac{4}{3} \left( \frac{a_2}{a_1} \right)^{\frac{1}{l}} \left( \frac{a_3}{a_1} \right)^{\frac{1}{l}} \left( \frac{a_3}{a(\eta)} \right)^{-\frac{1}{l}} - 1 \right] \geq 0, \tag{16}\]

for \( \frac{a(\eta)}{a_3} > \left( \frac{a_3}{a_2} \right)^{-\frac{1}{l}} \).

For \( l < -1 \), both conditions are of the type \( f \left( \frac{a(\eta)}{a_3} \right) \geq 0 \), \( f \) being an increasing function of \( a(\eta) \). Therefore, if the condition holds true at the beginning of the dark energy era, \( \eta = \eta_3 \), it will hold for \( \eta > \eta_3 \).

By setting \( a(\eta) = a_3 \) in Eq. (15), a restriction over the unknown constant of proportionality \( A \) follows

\[
A \leq \frac{- (2l + 2) \frac{k_B c^2 \pi}{3l P_l} H_3 - \frac{k_B c^3}{2mG c^2} H_3^2}{\frac{4\pi c^3}{3} \left( \frac{a_1}{a_3} \right)^3 n_g(\eta_2) + \frac{1}{576\pi^4} \left( \frac{a_1}{a_2} \right)^2 \left( \frac{a_2}{a_3} \right)^3 H_1^3 H_3^{-3} \left[ 2 \left( \frac{a_3}{a_2} \right)^{\frac{1}{l}} - 1 \right]} \tag{17},
\]

implying that for the GSL to be satisfied the above upper bound must be met. In this case the event horizon soon comes to dominate the total entropy and steadily augments with expansion. So, even though the entropy of matter and RGWs within the horizon decrease during the present dark energy era, the GSL is preserved provided Eq. (17) holds. Note that since Nesteruk and Ottewill left the constant \( A \) unspecified, restriction (17) turns to be all the more important: it is the only knowledge we have about how much big \( A \) may be.

Obviously, our conclusions hang on the expression adopted for the entropy density of the gravitational waves. Here we have chosen (4) since, on the one hand, it is the simplest one
based on particle production in curved spacetimes [12], and on the other hand, $s_g$ cannot fail to be an increasing function of $n_g$. We believe, that any sensible expression for $s_g$ should not run into conflict with the GSL, and that the latter may impose restrictions on the parameters entering the former.

As mentioned above, we have left aside models of late acceleration driven by dark energy of “phantom type” (i.e., $-1 < l < 0$) [17]. In this case, owing to the fact that the dominant energy condition is violated, the event horizon decreases with expansion whereby it is rather doubtful that the GSL may be satisfied at all. We shall focus our attention on this issue in a future research.

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