FINITE SUPPORT OF TENSOR PRODUCTS

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ABSTRACT. We compute the submodule of finite support of the tensor product of two modules $M$ and $N$ and estimate its length in terms of $M$ and $N$. Also, we compute some higher local cohomology modules of tensor products.

1. INTRODUCTION

In this note $(R, m)$ is a commutative, noetherian and local ring of dimension $d$. Also, all modules are finitely generated. By $H_0^m(M)$ we mean the elements of $M$ that are annihilated by some power of $m$. We consider to $H_0^m(M \otimes_R N)$ and denote its length by $h^0(M \otimes_R N)$.

Question 1.1. (See [11, Page 704]) Can one estimate $h^0(M \otimes_R N)$ in terms of $M$ and $N$?

Under various assumptions on the ring and on the modules, V asconcelos proved several bounds on $h^0(M \otimes_R N)$. For example, when $R$ is regular $N$ is locally free and $\text{pd}(M) < \dim(R)$. He asked for a similar extension when the ring is Gorenstein with isolated singularity, see [12, Question 8.2]. In §2 we slightly extend V asconcelos’ bounds. Also, we present results in the singular case, see Proposition 2.7 and 2.8.

In the case the ring is Gorenstein of dimension $d \geq 1$ and $M$ has a presentation $0 \to R^n \xrightarrow{\varphi} R^{n+d-1} \to M \to 0$ where $I_n(\varphi)$ is $m$-primary, V asconcelos proved $h^0(M \otimes_R M) \leq d \left( (d-1) \deg(M) + \ell\left( \frac{R}{I^n(\varphi)} \right) \right)^2$, here $\ell(-)$ is the length function. In [12, Question 8.1], he asked how good is the estimate compared to $h^0(M \otimes_R M)$? In §3 we present some explicit computations. For example, there is a situation for which $d((d-1) \deg(M) + \ell\left( \frac{R}{I^n(\varphi)} \right))^2 > h^0(M \otimes_R M)^2$, see Proposition 3.2.

In §4 we partially answer V asconcelos’ question on the torsion part of tensor products. For example over a 3-dimensional Cohen-Macaulay local domain and a reflexive module $M$ such that $\text{pd}(M) < \infty$, we show $M$ is free provided $M \otimes 3$ is torsion-free.

Suppose $M$ and $N$ are vector bundles over a Cohen-Macaulay ring. This in turn implies that the length of $H^d_{\infty}^i(M \otimes_R N)$ is finite. In §5 we try to understand $\ell(H^d_{\infty}^i(M \otimes_R N))$. We do this in three subsections. §5.A deals with the low-dimensional cases. Also, Remark 5.2 slightly extends a result of Auslander. By a result of Auslander (see [2, Theorem 3.7]) and Huneke-Wiegand (see [6, §4]) vanishing of $H^i_{m}(M \otimes_R M^*)$ for some $i$ implies freeness of $M$ if we impose some conditions on $i, d$ and $M$. Here, the notation $M^*$ stands for $\text{Hom}_R(M, R)$. What can say in a more general setting? This is subject of §5.B. In subsection §5.C we compute $H^d_{\infty}^i(M \otimes_R N)$ in some singular cases.

2010 Mathematics Subject Classification. Primary: 13D45; Secondary: 13H10; 13H15; 13D07.
Key words and phrases. Cohomological degree; local cohomology; tensor products; torsion module; vector bundle.
2. Bounds on $h^0(-\otimes \sim)$: After Vasconcelos

By $\mu(-)$ we mean the minimal number of elements that need to generate $(-)$.

**Lemma 2.1.** Let $M$ be of finite length. Then $h^0(M \otimes_R N) \leq \ell(M)\mu(N)$.

**Proof.** The proof is by induction on $\ell(M)$. Suppose $\ell(M) = 1$. Then $M = R/m$. By definition, $H^0_m(M \otimes_R N) = M \otimes_R N = \frac{N}{mN}$ and so $h^0(M \otimes_R N) = \mu(N) = \ell(M)\mu(N)$. We look at the exact sequence $0 \to R/m \to M \to \overline{M} \to 0$ where $\ell(\overline{M}) = \ell(M) - 1$. By induction, $\ell(\overline{M} \otimes_R N) \leq \ell(\overline{M})\mu(N)$. The sequence induces $R/m \otimes_R N \xrightarrow{f} M \otimes_R N \xrightarrow{\beta} \overline{M} \otimes_R N \to 0$. Since $R/m \otimes_R N \to \ker(\beta) \to 0$ is surjective, $\ell(\ker(\beta)) = \ell(\im(\beta)) \leq \mu(N)$. We have

$$\ell(M \otimes_R N) = \ell(\overline{M} \otimes_R N) + \ell(\ker(\beta)) \leq \ell(\overline{M} \otimes_R N) + \ell(N/mN) \leq \ell(\overline{M})\mu(N) + \mu(N).$$

So, $\ell(H^0_m(M \otimes_R N)) = \ell(M \otimes_R N) \leq (\ell(M) - 1)\mu(N) + \mu(N) = \mu(N)\ell(M)$. $\square$

The particular case in the next result stated in [12, Proposition 2.1] without proof:

**Lemma 2.2.** One has $h^0(M \otimes_R N) \leq h^0(M)\mu(N) + h^0(M/\im(M))\mu(N) + h^0(M/H^0_m(M) \otimes_R N/H^0_m(N))$. In particular,

$$h^0(M \otimes_R N) \leq h^0(M)\mu(N) + h^0(\im(M))\mu(N) + h^0(M/H^0_m(M) \otimes_R N/H^0_m(N)).$$

**Proof.** We may assume neither $M$ nor $N$ are of finite length (see Lemma 2.1). We look at $0 \to H^0_m(M) \to M \to \tilde{M} := \frac{M}{H^0_m(M)} \to 0$. Apply $- \otimes_R N$ to it and look at the induced long exact sequence

$$\Tor^R_1(\tilde{M}, N) \to H^0_m(M) \otimes_R N \xrightarrow{f} M \otimes_R N \to \tilde{M} \otimes_R N \to 0.$$
Proposition 2.3. Let $R$ be an equi-dimensional and generalized Cohen-Macaulay local ring, and $N$ be locally free and of constant rank over the punctured spectrum. If $\text{pd}(M) < \text{depth}(R)$, then $h^0(M \otimes_R N) \leq \sum_{i=0}^{\text{pd}(M)} h^i(N)$.

Proof. Let $p := \text{pd}(M)$. We may assume $N$ is not of finite length (see Lemma 2.1). The assumptions implies that $N$ is generalized Cohen-Macaulay and of dimension equal to $\dim(R)$. We look at $0 \to \text{Syz}_1(M) \to R^{\text{pd}(M)} \to M \to 0$. Apply $- \otimes_R N$ to it and look at the induced long exact sequence

$$0 \to \text{Tor}_1^R(M, N) \to \text{Syz}_1(M) \otimes_R N \xrightarrow{f} R^{\text{pd}(M)} \otimes_R N \to M \otimes_R N \to 0.$$ 

We have $0 \to \text{ker}(f) \to R^{\text{pd}(M)} \otimes_R N \to M \otimes_R N \to 0$ and $0 \to \text{Tor}_1^R(M, N) \to \text{Syz}_1(M) \otimes_R N \to \text{ker}(f) \to 0$. Since $N$ is locally free, $\text{Tor}_1^R(M, N)$ is of finite length. Thus, $H_m^0(\text{Tor}_1^R(M, N)) = \text{Tor}_1^R(M, N)$ and $H_m^1(\text{Tor}_1^R(M, N)) = 0$. We apply $\Gamma_m$ to these sequences to deduce the long exact sequences:

$$0 \to H_m^0(\text{Tor}_1^R(M, N)) \to H_m^0(\text{Syz}_1(M) \otimes_R N) \to H_m^0(\text{ker}(f)) \to H_m^1(\text{Tor}_1^R(M, N)) = 0,$$

$$0 \to H_m^1(\text{ker}(f)) \to H_m^1(\text{Tor}_1^R(M, N)) \to H_m^1(M \otimes_R N) \to H_m^1(\text{ker}(f)).$$

Also, $H_m^+(\text{Syz}_1(M) \otimes_R N) \simeq H_m^+(\text{ker}(f))$. We use these to conclude that:

$$h^0(M \otimes_R N) \leq \ell(H_m^0(\text{ker}(f))) + \beta_0(M) h^0(N) = \ell(H_m^0(\text{Syz}_1(M) \otimes_R N)) + \beta_0(M) h^0(N).$$

In the same vein, $\ell(H_m^0(\text{Syz}_2(M) \otimes_R N)) \leq \ell(H_m^0(\text{Syz}_2(M) \otimes_R N)) + \beta_1(M) h^1(N)$. Thus

$$h^0(M \otimes_R N) \leq \ell(H_m^0(\text{Syz}_1(M) \otimes_R N)) + \beta_0(M) h^0(N) \leq \ell(H_m^0(\text{Syz}_2(M) \otimes_R N)) + \beta_1(M) h^1(N) + \beta_0(M) h^0(N).$$

Repeating this, $h^0(M \otimes_R N) \leq \ell(H_m^0(\text{Syz}_p(M) \otimes_R N)) + \sum_{i=0}^{p-1} \beta_i(M) h^i(N) = \sum_{i=0}^{p} \beta_i(M) h^i(N)$. □

By $h_{\text{deg}}(M)$ we mean the cohomological degree, see [11] for its definition. The following contains more data than [12] Theorem 4.2] via dealing with $\text{pd}(A) = \dim(R)$.

Proposition 2.4. Let $R$ be a $d$-dimensional regular local ring, $M$ a module and $N$ be locally free over the punctured spectrum. Then

$$h^0(M \otimes_R N) \leq \begin{cases} d \cdot h_{\text{deg}}(M) h_{\text{deg}}(N) & \text{if } \text{pd}(M) < d \\ (d + 1) \cdot h_{\text{deg}}(M) h_{\text{deg}}(N) - 1 & \text{if } \text{pd}(M) = d \end{cases}$$

Proof. Due to Lemma 2.1 we can assume that neither $M$ nor $N$ are artinian. The claim in the case $\text{pd}(M) < d$ is in [12] Theorem 4.2]. Suppose $\text{pd}(M) = d$. Since $M$ is not artinian, $M \neq \Gamma_m(M)$. We denote $M/\Gamma_m(M)$ by $\tilde{M}$. Note that $\text{depth}(\tilde{M}) > 0$. Due to Auslander-Buchsbaum formula, $\text{pd}(\tilde{M}) < d$.

We combine Lemma 2.2 with the first part to see

$$h^0(M \otimes_R N) \leq h^0(M) \mu(N) + h^0(\tilde{M} \otimes_R N) \leq h^0(M) \mu(N) + d \cdot h_{\text{deg}}(\tilde{M}) h_{\text{deg}}(N).$$

Recall from definition that $h^0(M) \leq h_{\text{deg}}(M)$. By [11] Theorem 1.10], $\beta_i(N) \leq \beta_i(k) h_{\text{deg}}(N)$. We use this for $i = 0$ to see $\mu(N) \leq h_{\text{deg}}(N)$. In view of [11] Proposition 2.8(a)] we have $h_{\text{deg}}(\tilde{M}) = h_{\text{deg}}(M) - \ell(\Gamma_m(M)) < h_{\text{deg}}(M)$. We put all of these together to see

$$h^0(M \otimes_R N) \leq h^0(M) \mu(N) + d \cdot h_{\text{deg}}(\tilde{M}) h_{\text{deg}}(N) \leq h^0(M) \mu(N) + d \cdot h_{\text{deg}}(M) h_{\text{deg}}(N).$$
The claim is now clear.

Corollary 2.5. Let $R$ be a $d$-dimensional regular local ring. Assume one of the following items hold: i) $d = 1$, ii) $d = 2$ and $M$ is torsion-free, iii) $d = 3$ and $M$ is reflexive. Then $h^0(M \otimes_R N) < (d + 1) \deg(M) \deg(N)$ for any finitely generated module $N$.

Proof. It follows that $M$ is locally free. In view of Proposition 2.4 we get the desired claim.

The next result slightly extends [12, Proposition 3.4]:

Corollary 2.6. Let $(R, m)$ be a 1-dimensional complete local integral domain containing a field, $M$ and $N$ be finitely generated. Let $J$ be the Jacobian ideal. Then

$$h^0(M \otimes_R N) \leq \deg(M) \deg(N) (2 + \deg(R) \ell(R)) - \rank(M) \deg(N) \deg(R) \ell(R).$$

In particular, $h^0(M \otimes_R N) \leq (2 + \deg(R) \ell(R)) \deg(M) \deg(N)$.

Proof. Due to Lemma 2.4 neither $M$ nor $N$ are artinian. Let $\tilde{M} := \frac{M}{H^0_{\mathfrak{m}}(M)}$. This is nonzero and of positive depth. Thus, $\tilde{M}$ is maximal Cohen-Macaulay. Over any 1-dimensional reduced local ring, the category of maximal Cohen-Macaulay modules coincides with the category of torsion free modules. Hence $\tilde{M}$ and $\tilde{N}$ are torsion free. In view of [12], we see $\Ext^1_R(\tilde{M}, \tilde{N}) = 0$. We combine this with the proof of [12] Proposition 3.4] to see $h^0(M \otimes_R \tilde{N}) \leq (\mu(\tilde{M}) \mu(\tilde{N}) - \rank(\tilde{M}) \rank(\tilde{N})) \deg(R) \ell(R)$. Recall that $\mu(\tilde{M}) \leq \mu(M)$. Denote the fraction field of $R$ by $K(R)$. Recall that $\Ext^1_R(M \otimes_R \tilde{N}) = 0$. We apply the exact functor $- \otimes_R Q(R)$ to $0 \to \Ext^1_R(M \otimes_R \tilde{N}) \to M \otimes_R \tilde{N} \to 0$ to see the sequence $0 = \Ext^0_R(M \otimes_R \tilde{N}) \to M \otimes_R \tilde{N} \to M \otimes_R \tilde{N} \to 0$ is exact. From this rank$(M) = \rank(\tilde{M})$. Therefore, $h^0(M \otimes_R \tilde{N}) \leq (\mu(M) \mu(N) - \rank(M) \rank(N)) \deg(R) \ell(R)$. In view of Lemma 2.2 we have

$$h^0(M \otimes_R N) \leq h^0(M) \mu(N) + h^0(N) \mu(M) + h^0(\tilde{M} \otimes_R \tilde{N})$$

$$\leq h^0(M) \mu(N) + h^0(N) \mu(M) + (\mu(M) \mu(N) - \rank(M) \rank(N)) \deg(R) \ell(R)$$

$$\leq \deg(M) \deg(N) (2 + \deg(R) \ell(R)) - \rank(M) \rank(N) \deg(R) \ell(R).$$

□

Proposition 2.7. Let $R$ be a Gorenstein ring with isolated singularity and $M$ be maximal Cohen-Macaulay. Then $h^0(M \otimes_R N)$ can estimate in terms of $M$ and $N$.

Proof. Maximal Cohen-Macaulay modules over Gorenstein rings are reflexive, e.g., $M$ is reflexive. We may assume $N$ is of finite length (see Lemma 2.1). In view of Lemma 2.2 we can replace $N$ with $N/\Gamma_m(N)$ and assume in addition that depth$(N) > 0$. This implies that Hom$_R(\tilde{N}, -)$ has positive depth provided Hom$_R(\tilde{N}, -) \neq 0$. Let $D(\tilde{N})$ be the Auslander’s transpose. We look at the exact sequence

$$\Tor^R_2(D(M^*), N) \to M^{**} \otimes_R N \to \Hom_R(M^*, N) \to \Tor^R_1(D(M^*), N) \to 0.$$

Without loss of the generality we can assume that Hom$_R(\tilde{N}, -) \neq 0$. Note that $M^*$ is maximal Cohen-Macaulay and so locally free over punctured spectrum. Since $D(\tilde{N})$ behaves nicely with respect to localization, we see that $D(M^*)$ is of finite length. Hence $\Tor^R_2(D(M^*), N)$ is of finite length. Due to
Let \( A \) be a Cohen-Macaulay local ring of dimension \( d \geq 1 \), and \( N \) be Buchsbaum of dimension \( d \). Then \( h^0(M \otimes_R N) < 3h\deg(M)h\deg(N) \). Suppose in addition that \( \text{depth}(N) > 0 \). Then \( h^0(M \otimes_R N) \leq 2h\deg(M)h\deg(N) \).

**Proposition 2.8.** Let \( R \) be a Cohen-Macaulay local ring of dimension \( d \geq 1 \), \( M \) be perfect of projective dimension one and \( N \) be Buchsbaum of dimension \( d \). Then \( h^0(M \otimes_R N) < 3h\deg(M)h\deg(N) \). Suppose in addition that \( \text{depth}(N) > 0 \). Then \( h^0(M \otimes_R N) \leq 2h\deg(M)h\deg(N) \).

**Proof.** Let \( \tilde{N} := N^{\alpha_d}_{H^0_m(N)} \). In view of [10] Proposition 1.2.22, \( \tilde{N} \) is Buchsbaum. Since \( \text{dim}(N) = d > 0 \), we deuce that \( \tilde{N} \neq 0 \). It follows by definition that \( \text{depth}(\tilde{N}) > 0 \), \( H^m_\alpha(N) \simeq H^m_\alpha(N) \) and that \( \text{dim}(N) = \text{dim}(\tilde{N}) \). Recall from [13] Proposition 2.7:

Fact A) Let \( A \) be a Cohen-Macaulay local ring of dimension \( d > 1 \) and \( P \) be perfect of depth one. If \( Q \) is Buchsbaum of positive depth and maximal dimension, then \( h^0(P \otimes_A Q) = \mu(P)(h^0(Q) + h^1(Q)) \).

Recall that \( h\deg(\tilde{N}) = h\deg(N) - \ell(\Gamma_m(N)) \), \( \mu(-) \leq h\deg(-) \) and that \( h^{<d}(-) \leq h\deg(-) \). In view of Lemma 2.2 we have

\[
h^0(M \otimes_R N) \leq h^0(N)\mu(M) + h^0(M \otimes_R \tilde{N}) = h^0(N)\mu(M) + \mu(M)(h^0(\tilde{N}) + h^1(\tilde{N})) \leq h\deg(M)h\deg(N) + 2h\deg(M)h\deg(\tilde{N}) = h\deg(M)h\deg(N) + 2h\deg(M)(h\deg(N) - \Gamma_m(N)) \leq 3h\deg(M)h\deg(N),
\]

and we remark that if \( \Gamma_m(N) \neq 0 \), then the last inequality is strict. This completes the proof. \( \square \)

3. Toward sharpening the bound on \( h^0(M \otimes_R M) \)

We look at \( M \) with a presentation of the form \( 0 \rightarrow R^n \xrightarrow{\varphi} R^{n+d-1} \rightarrow M \rightarrow 0 \) where \( d \geq \text{dim} R \). Question 1.2 deals with the sharpness of \( h^0(M \otimes_R M) \leq d \left( (d-1)\deg(M) + \ell\left( \frac{R}{I_1(\varphi)} \right) \right)^2 \). Suppose \( d = 2 \) and \( n = 1 \). Let us repeat the assumption: \( M \) has a presentation of the form \( 0 \rightarrow R \xrightarrow{\varphi} R^2 \rightarrow M \rightarrow 0 \) where the ideal \( I_1(\varphi) \) is \( m \)-primary. The bound translates to \( h^0(M \otimes_R M) \leq 2(\deg(M) + \ell\left( \frac{R}{I_1(\varphi)} \right))^2 \).

**Example 3.1.** Let \((R, m, k)\) be a 2-dimensional regular local ring. Then \( h^0(m \otimes_R m) = 1 \).

**Proof.** Let \( x \) and \( y \) be a generating set of \( m \) and look at \( \zeta := x \otimes y - y \otimes x \). We have

\[
x\zeta = x(x \otimes y - y \otimes x) = x^2 \otimes y - xy \otimes x = xy \otimes x - xy \otimes x = 0.
\]

Similarly, \( y\zeta = 0 \), so that \( m\zeta = 0 \). By definition, \( \zeta \in H^0_n(m \otimes_R m) \). Again due to definition, \( H^0_n(m \otimes_R m) \) is submodule of the torsion part of \( m \otimes_R m \). On the other hand, the torsion part of \( m \otimes_R m \) is \( \text{Tor}_2^R(k, k) \) (see [5] Lemma 1.4) which is a vector space of dimension equal to \( \beta_2(k) = 1 \). From these, \( H^0_n(m \otimes_R m) = \zeta R \cong k \). In particular, \( h^0(m \otimes_R m) = \ell(H^0_n(m \otimes_R m)) = 1 \).

\( \square \)
The difference $2(\deg(M) + \ell(R_{\fra_i(M)}))^2 - h^0(M \otimes_R M)$ may be large:

**Proposition 3.2.** Let $(R, m, k)$ be a 2-dimensional Cohen-Macaulay local domain and $I$ be an ideal generated by a full parameter sequence. Then $h^0(I \otimes_R I) = \hdeg(R/I)$. In particular,

$$h^0(I \otimes_R I) = \ell(R/I) \leq 2(\deg(I) + \ell(R/I))^2.$$  

**Proof.** Let $x$ and $y$ be a generating set of $I$. The notation $\mathcal{K}(I; R)$ stands for the Koszul complex of $R$ with respect to $I$. That is

$$\mathcal{K}(I; R) := 0 \longrightarrow R \xrightarrow{(x,y)} R^2 \xrightarrow{(x,y)} R \longrightarrow R/I \longrightarrow 0.$$  

This is a minimal free resolution of $R/I$. In view of definition,

$$\mathcal{K}(I; R) \otimes_R R/I \cong 0 \longrightarrow R/I \longrightarrow R/I \otimes_R R/I \longrightarrow R/I \otimes_R R/I \longrightarrow 0.$$  

By definition, $\Tor(I \otimes_I I) \cong \Tor^\mathcal{K}(R/I, R/I) \cong H_2(\mathcal{K}(I; R) \otimes_R R/I) \cong \mathcal{K}(I; R) \otimes_R R/I = 0$. We look at the exact sequence

$$0 \longrightarrow \Tor(I \otimes_I I) \longrightarrow I \otimes_R I \longrightarrow \frac{\mathcal{K}(I; R) \otimes_R R/I}{\Tor(\mathcal{K}(I; R) \otimes_R R/I)} \longrightarrow 0.$$  

Since $\Tor(I \otimes_I I)$ is torsion-free, $\Tor^0_\mathcal{K}(R/I, R/I) = 0$. We put this in $0 \rightarrow H^n_m(\Tor(I \otimes I)) \rightarrow H_0^0(I \otimes I) \rightarrow H_0^0(\Tor(I \otimes I))$ to see that $H_0^0(I \otimes I) \cong H_0^0(I \otimes I)$. Since $\ell(I) < \infty$, $H_0^0(I \otimes I) \cong H_0^0(\Tor(I \otimes I)) \cong H_0^0(\Tor(I \otimes I)) \cong R/I$. Thus, $h^0(I \otimes I) = \ell(R/I)$. \hfill \Box

**Proposition 3.3.** Let $(R, m, k)$ be a 2-dimensional regular local ring and $0 \neq M$ be torsion-free. Then $h^0(M \otimes_R M) = 0$ if and only if $M$ is free.

**Proof.** The if part is trivial. Suppose $M$ is not free. Since $M$ is $(S_1)$ it follows that $\pd(M) = 1$. We claim that $\Tor_1^R(M, M) \neq 0$. Suppose on the contradiction that $\Tor_1^R(M, M) = 0$. Let $\mathfrak{p}$ be any height one prime ideal. Since $R_{\mathfrak{p}}$ is a discreet valuation ring and $M_{\mathfrak{p}}$ is torsion-free, it follows that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. From this, $\Tor_1^R(M, M)$ is of finite length. Thus, $\depth(\Tor_1^R(M, M)) = 0$. We recall the following result of Auslander (see [2] Theorem 1.2.1):

Fact A) Let $S$ be a local ring, $\pd(A) < \infty$. Let $q$ be the largest number such that $\Tor_1^R(M, M) \neq 0$. If $\depth(\Tor_1^R(M, M)) \leq 1$, then $\depth(B) = \depth(\Tor_1^R(M, M)) + \pd(A) - q$.

We use this for $A = B = M$ and $q = 1$, to see $1 = \depth(M) = \depth(\Tor_1^R(M, M)) + \pd(M) - q = 0 + 1 - 1 = 0$, a contradiction. Thus, $\Tor_1^R(M, M) = 0$. This vanishing result allow us to use:

Fact B) (see [2] Corollary 1.3.) Let $S$ be a local ring, $A$ and $B$ be of finite projective dimension. If $\Tor_1^R(A, B) = 0$, then $\pd(A) + \pd(B) = \pd(A \otimes_S B)$.

From this, $\pd(M \otimes_R M) = 2$. By Auslander-Buchsbaum, $\depth(M \otimes_R M) = 0$. Consequently, $h^0(M \otimes_R M) \neq 0$.

\hfill \Box

It may be natural to extend the above result to the 3-dimensional case by replacing torsion-free with reflexive. This is not the case:

**Proposition 3.4.** Let $(R, m, k)$ be a 3-dimensional regular local ring. Then $h^0(M \otimes_R N) = 0$ for any reflexive modules $M$ and $N$. 

**Proof.** We put \( L := M \oplus N \). It is enough to show \( h^0(L \otimes_R L) = 0 \). Without loss of the generality, \( L \) is not free. This implies that \( \text{pd}(L) = 1 \). By Auslander-Buchsbaum, \( \text{depth}(L) = 2 \). This implies that \( H^0_m(L) = H^1_m(L) = 0 \). Let \( 0 \to R^n \to R^m \to L \to 0 \) be a free resolution of \( L \). It follows that \( 0 \to \text{Tor}^R_1(L,L) \to L^n \to L^m \to L \otimes_R L \to 0 \). Denote the fraction field of \( R \) by \( Q(R) \). Recall that \( \text{Tor}^R_1(L,L) \otimes_R Q(R) = 0 \), i.e., \( \text{Tor}^R_1(L,L) \) is torsion. Since \( \text{Tor}^R_1(L,L) \) is torsion, \( \text{Tor}^R_1(L,L) \subset L^n \) and \( L^m \) is torsion-free, we get that \( \text{Tor}^R_1(L,L) = 0 \). We put this in the above sequence to see \( 0 \to L^n \to L^m \to L \otimes_R L \to 0 \) is exact. This sequence induces \( 0 = H^0_m(L_m) \to H^0_m(L \otimes_R L) \to H^1_m(L_m) = 0 \). So, \( h^0(L \otimes_R L) = 0 \). \( \square \)

Let us consider to another situation:

**Observation 3.5.** Let \( (R, m, k) \) be a \( d \)-dimensional regular local ring with \( d > 2 \) and \( I \) be a Gorenstein ideal of height two. Then \( h^0(I \otimes_R I) = 0 \).

**Proof.** Due to a result of Serre, \( I \) generated by a regular sequence \( x \) and \( y \). Since \( H^0_m(I \otimes_R I) \subset \text{tor}(I \otimes_R I) \), we deduce that \( H^0_m(I \otimes_R I) \subset H^0_m(\text{tor}(I \otimes_R I)) \). The Koszul complex of \( R \) with respect to \( x \) and \( y \) is a free resolution of \( R/I \). Then, \( \text{tor}(I \otimes_R I) = \text{Tor}^R_2(R/I, R/I) \simeq H^0(R(I; R) \otimes_R R/I) = R/I \). Recall that depth of \( R/I \) is positive. By the cohomological characterization of depth, \( H^0_m(R/I) = 0 \). We put all things together to deduce that \( H^0_m(I \otimes_R I) \simeq H^0_m(\text{tor}(I \otimes_R I)) = H^0_m(R/I) = 0 \). So, \( h^0(I \otimes_R I) = 0 \). \( \square \)

**4. TORSION IN TENSOR PRODUCTS**

In [12, Question 8.4] Vasconcelos posed some questions. For example, let \( R \) be a one-dimensional domain and \( M \) a torsion-free module such that \( M \otimes_R M \) is torsion-free. Is \( M \) free?

**Example 4.1.** (See [5, 4.7]) Let \( R \) be a one-dimensional local domain with a canonical module which is not Gorenstein. Then there is a non-free and torsion-free module \( M \) such that \( M \otimes_R M \) is torsion-free.

Also, he asked the following:

**Question 4.2.** Let \( R \) be a local domain and \( M \) be torsion-free. Is there an integer \( e \) guaranteeing that if \( M \) is not free, then the tensor power \( M^e \otimes_R \) has nontrivial torsion?

**Proposition 4.3.** Let \( R \) be a 3-dimensional Cohen-Macaulay local domain and \( M \) be a reflexive module such that \( \text{pd}(M) < \infty \). If \( M^3 \otimes R \) is torsion-free, then \( M \) is free.

**Proof.** Since \( M \) is torsion-free it is a submodule of a free module \( F \). Let \( C := \frac{F}{M} \). There is nothing to prove if \( C = 0 \). Without loss of the generality we assume that \( C \neq 0 \). Note that \( \text{pd}(M) \leq 1 \). Suppose on the contradiction that \( \text{pd}(M) \neq 0 \), i.e., \( \text{pd}(M) = 1 \). We look at the exact sequence \( 0 \to M \to F \to C \to 0 \) \( (*) \). The induced long exact sequence, presents the natural isomorphisms \( \text{Tor}^R_i(M, M) \simeq \text{Tor}^R_i(C, M) \) for all \( i > 0 \). Since \( \text{pd}(M) = 1 \), \( \text{Tor}^R_1(C, M) = 0 \) and so \( \text{Tor}^R_i(M, M) = 0 \). This vanishing result allow us to compute \( \text{pd}(M \otimes_R M) \), see Fact [3.3]. By Auslander-Buchsbaum formula, \( \text{depth}(M) + \text{depth}(M) = \text{depth}(R) + \text{depth}(M \otimes_R M) \). From \( \text{depth}(M) = 2 \) we see \( \text{depth}(M \otimes_R M) = 1 \). Again, \( (*) \) yields the following exact sequence

\[
0 \to \text{Tor}^R_1(C, M) \to M^2 \otimes_R F \to M^2 \to M \otimes_R C \to 0
\]
and the natural isomorphisms $\text{Tor}_{i+1}^R(C, M^{\otimes 2}) \simeq \text{Tor}_i^R(M, M^{\otimes 2})$ for all $i > 0$. Recall that $\text{Tor}_i^R(C, M^{\otimes 2})$ is torsion. Since $\text{Tor}_1^R(C, M^{\otimes 2})$ is torsion, $\text{Tor}_1^R(C, M^{\otimes 2}) \subset M^{\otimes 3}$ and $M^{\otimes 3}$ is torsion-free, we get that $\text{Tor}_1^R(C, M^{\otimes 2}) = 0$. In order to show $\text{Tor}_2^R(C, M^{\otimes 2}) = 0$ we use a trick of Peskine-Szpiro. Since the assumptions are not the same, we present the details. First, we show $\text{Tor}_2^R(C, M^{\otimes 2})$ is of finite length. Indeed, let $p \neq m$ be in support of $M$. Since $M_p$ is reflexive and of finite projective dimension, it is $(S_2)$. Since $\text{depth}(R_p) = \dim R_p < 3$ it follows that $\text{pd}(M_p) = \text{depth}(R_p) - \text{depth}(M_p) = 0$. That is $M$ is locally free over the punctured spectrum. From this, $\text{Tor}_1^R(M, M^{\otimes 2})$ is of finite length. By the natural isomorphism, $\text{Tor}_2^R(C, M^{\otimes 2}) \simeq \text{Tor}_1^R(M, M^{\otimes 2})$. Hence $\ell(\text{Tor}_2^R(C, M^{\otimes 2})) < \infty$. By $(\ast)$, we have $\text{pd}(C) = 2$. Let $0 \to F_2 \to F_1 \to F_0 \to C \to 0$ be a free resolution of $C$. Apply $\otimes_R M^{\otimes 2}$ to it we have

$$\text{Tor}_2^R(C, M^{\otimes 2}) = \ker \left( F_2 \otimes_R M^{\otimes 2} \to F_1 \otimes_R M^{\otimes 2} \right) \subset \bigoplus_{\text{rank}(F_2)} M^{\otimes 2}. $$

Note that $M^{\otimes 2}$ is of positive depth. Any non-zero submodule of a module of positive depth is a module of positive depth. We apply this for the pair $\text{Tor}_2^R(C, M^{\otimes 2}) \subset \bigoplus_{\text{rank}(F_2)} M^{\otimes 2}$ to deduce that $\text{Tor}_2^R(C, M^{\otimes 2}) = 0$. Since $\text{pd}(C) = 2$, $\text{Tor}_2^R(C, M^{\otimes 2}) = 0$. This allows us to apply Fact 3.3(B) to see $\text{depth}(C) + \text{depth}(M^{\otimes 2}) \geq \text{depth}(R) + \text{depth}(M^{\otimes 2} \otimes_R C)$. By Auslander-Buchsbaum formula, $\text{depth}(C) = 1$. Recall that $\text{depth}(M^{\otimes 2}) = 1$. We see left side of $(\ast)$ is 2 and the right hand side is at least 3. This is a contradiction. In sum, $M$ is free. □

**Remark 4.4.** Let $R$ be a local ring of depth 2 and $M$ be torsion-free such that $\text{pd}(M) < \infty$. If $M^{\otimes 2}$ is torsion-free, then $M$ is free.

**Proof.** Suppose on the contradiction that $M$ is not free. Since $M$ is torsion-free it is a submodule of a free module $F$. Let $C := F_R$. Without loss of the generality we assume that $C \neq 0$. We look at the exact sequence $0 \to M \to F \to C \to 0$. The induced long exact sequence, presents the natural isomorphisms $\text{Tor}_{i+1}^R(C, M) \simeq \text{Tor}_i^R(M, M)$ for all $i > 0$. It follows by Auslander-Buchsbaum that $\text{pd}(M) = 1$. We conclude that $\text{Tor}_2^R(C, M) = 0$. Thus $\text{Tor}_1^R(M, M) = 0$. We recall from Fact 3.3(B) that $\text{depth}(M) + \text{depth}(M) \geq \text{depth}(R) + \text{depth}(M \otimes_R M)$. Also, $\text{depth}(M \otimes_R M) > 0$ because it is torsion-free. The left hand side of $(\ast)$ is 2 and the right hand side is at least 3. This contradiction says that $M$ is free. □

5. HIGHER COHOMOLOGY OF TENSOR PRODUCTS

Subsection 5.A: The low-dimensional approach. We need the following result:

**Fact 5.1.** (See [6, Theorem 2.4]) Let $R$ be such that its completion is a quotient of equicharacteristic regular local ring by a nonzero element. Let $r$ be such that $0 \leq r < \dim R$. Assume $M \otimes N$ is $(S_{r+1})$ over the punctured spectrum and at least one them is of constant rank and $\text{pd}(M) < \infty$. Then $H^n_{\mathfrak{m}}(N \otimes_R M) = 0$ and both of $M$ and $N$ has depth at least $r$ if and only if $\text{depth}(N) + \text{depth}(M) \geq \dim R + r + 1$.

It may be nice to determine the case for which $\text{depth}(M) + \text{depth}(N)$ is minimum. Recall that $M$ is called $p$-spherical if $\text{pd}(M) = p$ and $\text{Ext}_R^i(M, R) = 0$ for $i \neq 0$ and $i \neq p$. 

Remark 5.2. Let $R$ be such that its completion is a quotient of equicharacteristic regular local ring by a nonzero element and $M$ be torsion-free of constant rank, of projective dimension $p \in \mathbb{N}$ and locally free. Then $\text{depth}(M) + \text{depth}(M^\ast) = \dim R + 1$ if and only if $M$ is $p$-spherical.

Proof. Suppose $\text{depth}(M) + \text{depth}(M^\ast) = \dim R + 1$. In view of Fact 5.1 $M \otimes_R M^\ast$ is torsion-free. Let $j$ be the smallest positive integer such that $\text{Ext}_R^j(M, R) \neq 0$. Such a thing exists. Set $f : R_{p_j} \to R_{p_{j-1}}$. We look at $L := \text{coker}(f^\ast)$ and the inclusion $\text{Ext}_R^n(M, R) \subset L$. Since $\text{pd}(M) < \infty$, $M$ is generically free. Thus, $\text{Tor}_R^0(M, -)$ is torsion. We put this along with the proof of [2, 3.8(e)] to see that $\text{Tor}_R^0(L, M) = 0$. By the rigidity theorem of Lichtenbaum [8, Theorem 3], $\text{Tor}_R^0(L, M) = 0$ for all $i > j$. Since $\text{depth}(L) = 0$ this says that $\text{pd}(M) \leq j$. By definition, $M$ is $p$-spherical. Conversely, assume that $M$ is $p$-spherical. There is an exact sequence $0 \to M^\ast \to (R_{p_j})^\ast \to \ldots \to (R_{p_1})^\ast \to L \to 0$. Since $\text{Ext}_R^0(M, R) \subset L$ and $\ell(\text{Ext}_R^0(M, R)) < \infty$ we deduce that $\text{depth}(L) = 0$. It turns out that $\text{depth}(M^\ast) = p + 1$. Due to Auslander-Buchsbaum formula, $\text{depth}(M) + \text{depth}(M^\ast) = \dim R + 1$. □

Corollary 5.3. Let $R$ be a regular local ring of dimension 3 and $M$ a reflexive module.

i) Always $H^n_{\mathfrak{m}}(M \otimes_R M) = 0$.

ii) If $H^n_{\mathfrak{m}}(M \otimes_R M) = 0$ for some $0 < i < 3$, then $M$ is free.

Proof. The first item is in Observation 5.4. We may assume that $i > 0$ and that $M \neq 0$. Reflexive modules over 2-dimensional regular local rings are free. From this, $M$ is locally free over the punctured spectrum. We apply Fact 5.1 for $r = i$, to see that $2 \text{depth}(M) \geq \dim R + i + 1 \geq 5$. That is $2 < 5/2 \leq \text{depth}(M) \leq \dim(M) \leq 3$. Thus, $\text{depth}(M) = 3$. Due to Auslander-Buchsbaum, $M$ is free. □

In view of [8, Example 1.8] there is a non-free ideal $I$ of $R := \left[ \left[ \left[ \left[ x, y, z, w \right] \right] \right] \right]$ such that $I \otimes I^\ast$ is torsion-free.

Example 5.4. Let $(R, m, k)$ be a local ring of depth at least 3. Then i) $m \otimes_R m^\ast$ is torsion-free, ii) $m$ is locally free, and iii) $H^n_{\mathfrak{m}}(m \otimes_R m^\ast) = 0$.

Proof. Clearly $m$ is non-free and locally free, and that $\text{Ext}_R^{<3}(k, R) = H^{<3}_m(R) = 0$. We look at $0 \to m \to R \to k \to 0 \quad (\ast)$. It yields that $0 = k^\ast \to m^\ast \to R^\ast \to \text{Ext}_k^3(k, R) = 0$, i.e., $m^\ast \simeq R$. Also, $(\ast)$ implies that $0 = H^3_m(k) \to H^2_{\mathfrak{m}}(m) \to H^2_m(R) = 0$. So, $H^n_{\mathfrak{m}}(m \otimes_R m^\ast) \simeq H^n_{\mathfrak{m}}(m) = 0$. □

Subsection 5.B: The regular case.

Proposition 5.5. Let $(R, m, k)$ be a regular local ring and $M$ be an indecomposable Buchsbaum module of dimension $d := \dim(R)$ which is not Cohen-Macaulay.

i) If $\text{depth}(M) = 1$, then

$$h^i(M \otimes_R M) = \begin{cases} \frac{d}{2} & \text{if } i = 0 \\ d + 1 & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i < d \end{cases}$$

In particular, $M \otimes_R M$ is not Buchsbaum.
ii) If \( d > 3 \) and \( M \) is almost Cohen-Macaulay, then 

\[
h^i(M \otimes_R M^*) = \begin{cases} 
0 & \text{if } i \in \{0\} \cup [3, d - 2] \\
1 & \text{if } i = 1 \\
d & \text{if } i = 2 \text{ or } i = d - 1 
\end{cases}
\]

In particular, \( M \otimes_R M^* \) is quasi-Buchsbaum. Against to \( M \) and \( M^* \), \( M \otimes_R M^* \) is not Buchsbaum.

**Proof.** i) First, we state a more general claim:

**Claim A)** Let \((A, n, k)\) be a Cohen-Macaulay local ring of dimension at least two and \( I \triangleleft A \) be \( n \)-primary. Then

\[
h^i(I \otimes_A n) = \begin{cases} 
\beta_2(A/I) & \text{if } i = 0 \\
\mu(I) + \ell(A/I) & \text{if } i = 1 \\
0 & \text{if } 2 \leq i < \dim A 
\end{cases}
\]

Indeed, let \( d := \dim A \). We look at \( 0 \to n \to A \to k \to 0 \) and we drive the following exact sequence

\[
0 \to \text{Tor}_1^A(k, I) \to I \otimes_A n \to I \to I \otimes_A k \to 0 \quad (*)
\]

Recall that \( I \otimes_A k \simeq \frac{I}{m} \simeq k^\mu(I) \) and \( \text{Tor}_1^A(k, I) \simeq \text{Tor}_2^A(k, A/I) \simeq k^{\beta_2(A/I)} \). We break down \((*)\) into

a) \( 0 \to k^{\beta_2(A/I)} \to I \otimes_A n \to L \to 0 \) and

b) \( 0 \to L \to I \to k^\mu(I) \to 0 \).

We conclude from a) the exact sequence \( 0 \to H_n^0(k^{\beta_2(A/I)}) \to H_n^0(I \otimes_A n) \to H_n^0(L) \). It follows from b) that the sequence \( 0 \to H_n^1(L) \to H_n^1(I) = 0 \) is exact. We combine these to see \( \ell(H_n^0(I \otimes_A n)) = \ell(H_n^0(k^{\beta_2(A/I)})) = \beta_2(A/I) \). From a) we have \( H_n^0(I \otimes_A n) \simeq H_n^1(L) \). From b),

\[
0 = H_n^0(I) \to H_n^0(k^\mu(I)) \to H_n^0(L) \simeq H_n^1(I \otimes_A n) \to H_n^1(I) \to H_n^1(k^\mu(I)) = 0.
\]

In order to compute \( H_n^1(I) \), we look at \( 0 \to I \to A \to A/I \to 0 \). This induces \( 0 = H_n^0(A) \to H_n^0(A/I) \to H_n^1(I) \to H_n^1(A) = 0 \). Thus, \( H_n^1(A) \simeq H_n^0(A/I) = A/I \). We put all of these together to see \( 0 \to k^\mu(I) \to H_n^0(I \otimes_A n) \to A/I \to 0 \). We conclude that \( h^1(I \otimes_A n) = \mu(I) + \ell(A/I) \). Let \( 2 \leq i < d \). Recall that \( H_n^0(I \otimes_A n) \simeq H_n^0(L) \simeq H_n^1(I) \). We look at \( 0 = H_n^{i-1}(A/I) \to H_n^i(I) \to H_n^i(A) = 0 \) to deduce that \( H_n^i(I \otimes_A n) \simeq H_n^i(I) = 0 \). This completes the proof of Claim A). Recall from [4 Corollary (3.7)] that:

**Fact A)** Let \((A, n)\) be a regular local ring and \( P \) be an indecomposable Buchsbaum module of maximal dimension. Then \( P \simeq \text{Syz}_i(A/n) \) where \( i = \text{depth}(P) \).

In the light of Fact A) we see \( M = \text{Syz}_k(1/m) \). Note that \( \beta_2(k) \) is equal to \( \binom{d}{2} \) and \( \mu(m) = d \). It follows by the assumptions that \( \dim(R) \geq 2 \). Claim A) yields that:

\[
h^i(M \otimes_R M) = \binom{d}{2} \quad \text{if } i = 0 \\
\quad d + 1 \quad \text{if } i = 1 \\
\quad 0 \quad \text{if } 2 \leq i < d
\]

To see the particular case, we recall from [4 Theorem (1.1)] that:

**Fact B)** Let \((A, n)\) be a regular local ring and \( P \) be Buchsbaum. Then \( P \simeq \bigoplus_{0 \leq i \leq \dim(A)} \text{Syz}_i(A/n)^{h_i} \) where \( h_i := h^i(P) \) for all \( 0 \leq i < \dim A \).
Suppose on the contradiction that $M \otimes_R M$ is Buchsbaum. Due to Fact B), $M \otimes_R M \simeq \bigoplus_{0 \leq i \leq d} \text{Sy}_z(k)^{h'}$ where $h' := h'(M \otimes_R M)$ for $i \neq d$. It turns out that $M \otimes_R M \simeq \bigoplus_{0 \leq i \leq d} \text{Sy}_z(k)^{\delta + (d+1)} \oplus R^d$ for some $n \geq 0$. Since $M \simeq m$, we see the rank of left hand side of $(\ast)$ is one. The rank of right hand side is $0 + (d + 1) + n$. Since $n \geq 0$, we get to a contradiction. So, $M \otimes_R M$ is not Buchsbaum.

ii) We recall that $M$ is called almost Cohen-Macaulay if $\text{depth}(M) \geq \dim(M) - 1$. Since $M$ is not Cohen-Macaulay, $\text{depth}(M) = \dim(M) - 1 = d - 1$. In the light of Fact A), $M := \text{Sy}_{d-1}(k)$. Since $M$ is locally free, $\text{Tor}_i^R(M, M^*)$ is of finite length. We look at $0 \to R \to R^d \to M \to 0$ and we drive the following exact sequence

$$0 \to \text{Tor}_i^R(M, M^*) \to M^* \to (M^*)^d \to M \otimes_R M^* \to 0.$$ 

We break down it into $0 \to \text{Tor}_i^R(M, M^*) \to M^* \to L \to 0$ and $0 \to L \to (M^*)^d \to M \otimes_R M^* \to 0$. It follows from the first exact sequence that $0 = H^i_m(\text{Tor}_i^R(M, M^*)) \to H^i_m(M^*) \to H^i_m(L) \to H^i_m(\text{Tor}_i^R(M, M^*)) = 0$. Similarly, $H^i_m(M^*) \simeq H^i_m(L)$. Recall that $M^*$ is reflexive. In particular it is $(S_2)$. So, $H^i_m(L) \simeq H^i_m(M^*) = 0$. It follows from the second short exact sequence that $0 = H^i_m((M^*)^d) \to H^i_m(M \otimes_R M^*) \to H^i_m(L) = 0$. From this, $h^i(M \otimes_R M^*) = 0$.

Fact C) (See [1] Proposition A.1) Let $A$ be a ring, a necessarily and sufficient condition for which $P$ be projective is that $\varphi_P : P \otimes_A P^* \to \text{Hom}_A(P, P)$ is (surjective) an isomorphism.

Since $M$ is locally free, it follows from Fact A) that $K := \ker(\varphi_M)$ and $C := \coker(\varphi_M)$ are of finite length and that $C \not\simeq 0$. From this, $H^0_m(C) = C \neq 0$, $H^1_m(C) = H^1_m(K) = 0$. We look at $0 \to K \to M \otimes_R M^* \to \text{im}(\varphi_M) \to 0$ and $0 \to \text{im}(\varphi_M) \to \text{Hom}_R(M, M) \to C \to 0$. Since $\text{depth}(M) > 1$ another result of Auslander-Goldman ([11] Proposition 4.7]) says that depth$(\text{Hom}_R(M, M)) > 1$, i.e., $H^0_m(\text{Hom}_R(M, M)) = H^1_m(\text{Hom}_R(M, M)) = 0$. We apply this along with the long exact sequences of local cohomology modules to see

$$0 = H^1_m(K) \to H^1_m(M \otimes_R M^*) \to H^1_m(\text{im}(\varphi_M)) \to H^2_m(K) = 0$$

$$0 = H^0_m(\text{Hom}_m(M, M)) \to H^0_m(C) \to H^0_m(\text{im}(\varphi_M)) \to H^1_m(\text{Hom}_R(M, M)) = 0,$$

e.g., $H^1_m(M \otimes_R M^*) \simeq H^1_m(\text{im}(\varphi_M)) \not\simeq H^0_m(C) \simeq C \simeq \text{Tor}_1^R(D(M), M)$, because $\text{coker}(\varphi_M) = \text{Tor}_1^R(D(M), M)$. Let $m = (x_1, \ldots, x_d)$. In view of $0 \to R \xrightarrow{(x_1, \ldots, x_d)} R^d \to M \to 0$ we see $D(M) = \text{coker}(R^d \xrightarrow{(x_1, \ldots, x_d)} R) \simeq R^d / \langle m \rangle$. Also, $\text{Tor}_1^R(D(M), M) \simeq \text{Tor}_1^R(k, \text{Sy}_{d-1}(k)) = \text{Tor}_2^R(k, k) = k$. Combining these, $h^1(M \otimes_R M^*) = \ell(\text{Tor}_1^R(D(M), M)) = 1$. Also, $m H^2_m(M \otimes_R M^*) = 0$.

Fact D) (See [3] Proposition 4.1] Let $(A, n)$ be a local ring, $L$ be locally free and $N$ be of depth at least 3.

Then $\text{Ext}_A^i(L, N) \simeq H^i_{\text{m}}(N \otimes_A L^*)$ for all $1 \leq i \leq \text{depth}(N) - 2$.

By this $H^2_m(M \otimes_R M^*) \simeq \text{Ext}_R^1(M, M)$, because $\text{depth}(M) = d - 1 \geq 3$. Apply $\text{Hom}_R(\_ , M)$ to $0 \to R \to R^d \to M \to 0$ to see $0 \to \text{Hom}_R(M, M) \to \text{Hom}_R(R^d, M) \to \text{Hom}_R(R, M) \to \text{Ext}_R^1(M, M) \to 0$. Thus, $H^2_m(M \otimes_R M^*) \simeq \text{Ext}_R^1(M, M) = \text{coker}(M^{(x_1, \ldots, x_d)}) = \frac{M}{\text{m}M}$. Hence, $h^2(M \otimes_R M^*) = \ell(\frac{M}{\text{m}M}) = \mu(M) = \beta_{d-1}(k) = d$. Also, $m H^3_m(M \otimes_R M^*) = 0$.

Let $3 \leq i \leq d - 2$. Due to Fact D) we know that $H^i_m(M \otimes_R M^*) \simeq \text{Ext}_R^{i-1}(M, M) = 0$, because $\text{pd}(M) = 1$. Thus, $h^i(M \otimes_R M^*) = 0$. 


Fact E) Let \( A \) and \( B \) be locally free over a regular local ring \((S,n)\) of dimension \(d \geq 3\) and let \(2 \leq j \leq d-1\). Then \(H^n_m(A \otimes_S B)^p \simeq H^{d+1-j}(A^* \otimes_S B^*)\), where \((-)^p\) is the Matlis duality.
Since \(d-1 \geq 2\), \(\text{Syz}_{d-1}(k)\) is a second syzygy, it is reflexive. Also, \(\ell((-)^p) = \ell(-)\). We use these to see
\[
H^{d-1}_m(M \otimes_R M^*) = \ell(H^{d-1}_m(M \otimes_R M^*)^p) = \ell(H^{2}_m(M^* \otimes_R M^*)) = \ell(H^{2}_m(M^* \otimes_R M)) = d.
\]
Since Matlis duality preserves the annihilator we deduce that \(mH^{d-1}_m(M^* \otimes_R M) = 0\).

We proved that \(mH^{d}_m(M \otimes_R M^*) = 0\). By definition, \(M \otimes_R M^*\) is quasi-Buchsbaum. In view of \(0 \rightarrow R \rightarrow R^d \rightarrow M \rightarrow 0\) we see \(0 \rightarrow M^* \rightarrow R^d \rightarrow R\) is exact. Thus, \(M^* = \text{Syz}_2(R/m)\) is Buchsbaum. Note that \(\text{rank}(M) = \text{rank}(M^*) = d - 1\), because \(0 \rightarrow M^* \rightarrow R^d \rightarrow m \rightarrow 0\). Thus, \(\text{rank}(M \otimes_R M^*) = (d - 1)^2\). Also, \(\text{rank}(\text{Syz}_1(k)) = 1\), because \(\text{Syz}_1(k) = m\). Suppose on the contradiction that \(M \otimes_R M^*\) is Buchsbaum. Due to Fact B) there is an \(n \geq 0\) such that
\[
M \otimes_R M^* = \text{Syz}_1(k) \bigoplus \text{Syz}_2(k) \oplus \text{Syz}_{d-1}(k) \oplus R^n.
\]
The left hand side is a vector bundle of rank \((d - 1)^2\). The right hand side is a vector bundle of rank \(1 + d(d - 1) + d(d - 1) + n\). Since \(n \geq 0\), we get to a contradiction.

Over a regular local ring \((R,m,k)\) of dimension \(d > 1\), Auslander was looking for a vector bundle \(M\) without free summand of dimension \(d\) such that \(\text{pd}(M) = \text{pd}(M^*)\) and \(H^n_m(M \otimes_R M^*) = 0\). He proved the existence of \(M\) is equivalent to the oddness of \(d\).

**Corollary 5.6.** Let \((R,m,k)\) be a regular local ring of odd dimension \(d\) and \(M\) be as above. If \(M\) is Buchsbaum, then \(M \simeq \text{Syz}_{d+1}(k) \oplus m\) for some \(m\).

**Proof.** Suppose first that \(M\) is indecomposable. By Fact 5.5(A) \(M \simeq \text{Syz}_i(k)\) where \(i := \text{depth}(M)\). Since \(M\) has no free direct summand, \(i < d\). This allow us to use [4] Lemma 3.2] to see \(M^* = \text{Syz}_{d-i+1}(k)\).

We deduce from \(d - i = \text{pd}(M) = \text{pd}(M^*) = \text{pd}(\text{Syz}_{d-i+1}(k)) = d - (d - i + 1)\) that \(i = \frac{d-1}{2}\). In particular, \(M = \text{Syz}_{d+1}(k)\). As a second case, suppose \(M\) is decomposable and has a direct summand other than \(\text{Syz}_{d+1}(k)\). In view of Fact 5.5(B) there is an \(I \subset [1,d-1]\) such that \(M \simeq \bigoplus_{i \in I} \text{Syz}_i(k)^{b_i}\).

Note that \(\text{pd}(M) = \sup_{i \in I} \{\text{pd}(\text{Syz}_i(k))\} = \sup_{i \in I} \{d - i\} = d - \inf \{i : i \in I\}\). Let \(j\) be such that \(j = d - \inf \{i : i \in I\}\). Recall that \(\text{Syz}_j(k)^{s} = \text{Syz}_{d-j+1}(k)\). Since \(\text{pd}(M) = \text{pd}(M^*)\) it follows that \(\text{Syz}_{d-j+1}(k)\) is a direct summand of \(M\). One of \(j\) and \(d - j\) is smaller than \(\frac{d-1}{2}\). Without loss of the generality, we assume that \(j < \frac{d-1}{2}\) (one may use [3] Theorem 2.4) to get a contradiction. Here, we follow our simple reasoning:) We look at \(0 \rightarrow \text{Syz}_j(k) \rightarrow R^{j-1}(k) \rightarrow \text{Syz}_{j-1}(k) \rightarrow 0\). This induces
\[
0 \rightarrow \text{Tor}^R_1(\text{Syz}_j(k), \text{Syz}_{j-1}(k)) \rightarrow \text{Syz}_j(k) \otimes_R \text{Syz}_{j-1}(k) \rightarrow R^{j-1}(k) \otimes_R \text{Syz}_j(k) \rightarrow \text{Syz}_j(k) \otimes_R \text{Syz}_{j-1}(k) \rightarrow 0.
\]
Note that \(\text{Tor}^R_1(\text{Syz}_j(k), \text{Syz}_{j-1}(k)) \simeq \text{Tor}^R_1(\text{Syz}_j(k), k) \simeq \text{Tor}^R_1(k, k) \simeq k \otimes_R k\). Since \(j < \frac{d-1}{2}\) we have \(2j \leq d\). We conclude from this that \(\text{Tor}^R_1(\text{Syz}_j(k), \text{Syz}_{j-1}(k))\) is nonzero and of finite length. Since
\[
k \subset \text{Tor}^R_1(\text{Syz}_j(k), \text{Syz}_{j-1}(k)) \subset \text{Syz}_j(k) \otimes_R \text{Syz}_{j-1}(k) \subset M \otimes_R M^*.
\]
we see that \(H^m_m(M \otimes_R M^*) \neq 0\), a contradiction. □
Subsection 5.C: The singular case. Recall that vanishing of $H^2_\mathfrak{m}(M \otimes_R M^*)$ over regular local rings implies freeness of $M^*$. This can’t be extended into hypersurfaces: Let $R := \mathbb{C}[xy, z, w]/(xy - wz)$ and $I := (x, y)$. Then $H^2_\mathfrak{m}(I \otimes_R I^*) = 0$ but $I^*$ is not free. In fact, the following stated implicitly in [6].

**Remark 5.7.** Let $R$ be a hyper-surface of dimension $d \geq 2$ and $M$ be torsion-free, locally free and of constant rank. Assume one of the following holds: i) $H^1_\mathfrak{m}(M \otimes_R M^*) = 0$ or ii) $H^0_\mathfrak{m}(M \otimes_R M^*) = H^1_\mathfrak{m}(M \otimes_R M^*) = 0$. Then $M^*$ is free.

**Observation 5.8.** Let $R$ be a Cohen-Macaulay local ring with isolated Gorenstein singularity and possessing a canonical module. Let $i > 0$. Then $H^i_\mathfrak{m}(\omega_R \otimes_R \omega_R^i) \neq 0$ if and only if $i = 1$ or $i = \dim(R)$.

**Proof.** By isolated Gorenstein singularity we mean a non Gorenstein ring which is Gorenstein over the punctured spectrum. From this, $d := \dim(R) \neq 0$. Since $(\omega_R)_R \simeq \omega_R$, $\text{Supp}(\omega_R) = \text{Spec}(R)$. Also, $\text{Ass}(\text{Hom}_R(\omega_R, R)) = \text{Supp}(\omega_R) \cap \text{Ass}(R) = \text{Spec}(R) \cap \text{Ass}(R) = \text{Ass}(R)$. From this, $\text{Supp}(\omega_R^i) = \text{Spec}(R)$. It follows that $\text{Supp}(\omega_R \otimes \omega_R^i) = \text{Spec}(R)$. Thus, $\dim(\omega_R \otimes \omega_R^i) = d$. By Gotthendieck’s non-vanishing theorem, $H^d_\mathfrak{m}(\omega_R \otimes \omega_R^i) \neq 0$. This completes the proof in the case $d = 1$. We assume that $d \geq 2$. Let $\varphi_{\omega_R} : \omega_R \otimes \omega_R^i \to \text{Hom}_R(\omega_R, \omega_R)$. Recall that $\text{Hom}_R(\omega_R, \omega_R) \simeq R$ and that $H^1_\mathfrak{m}(R) = H^1_\mathfrak{m}(M = 0)$. Since $\omega_R$ is locally free, it follows from Fact [5.C] that $K := \ker(\varphi_{\omega_R})$ and $C := \text{coker}(\varphi_{\omega_R})$ are of finite length and that $C \neq 0$. From this, $H^0_\mathfrak{m}(C) = C \neq 0$, $H^1_\mathfrak{m}(C) = H^1_\mathfrak{m}(K) = 0$. We look at $0 \to K \to \omega_R \otimes \omega_R^i \to \text{im}(\varphi_{\omega_R}) \to 0$ and $0 \to \text{im}(\varphi_{\omega_R}) \to R \to C \to 0$. It follows that $H^1_\mathfrak{m}(\omega_R \otimes \omega_R^i) \simeq H^1_\mathfrak{m}(\text{im}(\varphi_{\omega_R})) \simeq H^1_\mathfrak{m}(C) \neq 0$. This completes the proof in the case $d = 2$. Assume that $d > 2$. We proved that $H^d_\mathfrak{m}(\omega_R \otimes \omega_R^i) \neq 0$ and $H^d_\mathfrak{m}(\omega_R \otimes \omega_R^i) \neq 0$. Let $2 \leq i \leq d - 1$. Then $H^i_\mathfrak{m}(\omega_R \otimes \omega_R^i) \simeq H^i_\mathfrak{m}(\text{im}(\varphi_{\omega_R})) \simeq H^i_\mathfrak{m}(C) = 0$.

**Conjecture 5.9.** (Part of [13 Conjecture 3.4]) Let $R$ be a Cohen-Macaulay local ring, $M$ be perfect and $N$ be Buchsbaum and of maximal dimension. If $\text{pd}(M) \leq \text{depth}(N)$, then $h^i(M \otimes_R N) = \sum_{j=0}^{\text{pd}(M)} \beta_j(M)h^{i+j}(N)$ for all $i < \dim(M)$.

**Proposition 5.10.** Let $R$ be a Cohen-Macaulay local ring, $M$ be perfect and $N$ be locally free and of constant rank. Then $h^i(M \otimes_R N) \leq \sum_{j=0}^{\text{pd}(M)} \beta_j(M)h^{i+j}(N)$ for all $i < \dim(M)$.

**Proof.** For every module $L$ of finite projective dimension, we have $\text{grade}(L) + \dim(L) = \dim(R)$. In particular, if $L$ is perfect then $\dim(L) = \dim(R) - \text{pd}(L)$. Therefore, things reduced to show $h^i(M \otimes_R N) \leq \sum_{j=0}^{\text{pd}(M)} \beta_j(M)h^{i+j}(N)$ for all $i < \dim(R) - \text{pd}(M)$. We may assume that $\text{pd}(M) > 0$. There is nothing to prove if $\text{pd}(R) - \text{pd}(M) = 0$. Without loss of the generality, $\text{pd}(M) < \dim(R) = \text{depth}(R)$. Now, the case $i = 0$ is in Proposition 2.3. We may assume that $i > 0$. Let $f : \text{Syz}_1(M) \otimes_R N \to R^{ \beta_0(M)} \otimes_R N$ and recall from Proposition 2.3 that $H^i_\mathfrak{m}(\text{Syz}_1(M) \otimes_R N) \simeq H^i_\mathfrak{m}(\ker(f))$ and there is an exact sequence $H^i_\mathfrak{m}(R^{ \beta_0(M)} \otimes_R N) \to H^i_\mathfrak{m}(M \otimes_R N) \to H^{i+1}_\mathfrak{m}(\ker(f))$. Hence

$$h^i(M \otimes_R N) \leq \ell(H^{i+1}_\mathfrak{m}(\ker(f))) + \beta_0(M)h^j(N) = \ell(H^{i+1}_\mathfrak{m}(\text{Syz}_1(M) \otimes_R N)) + \beta_0(M)h^j(N).$$

In the same vein, $\ell(H^{i+1}_\mathfrak{m}(\text{Syz}_2(M) \otimes_R N)) \leq \ell(H^{i+2}_\mathfrak{m}(\text{Syz}_2(M) \otimes_R N)) + \beta_1(M)h^{i+1}(N)$. Therefore,

$$h^i(M \otimes_R N) \leq \ell(H^{i+1}_\mathfrak{m}(\text{Syz}_1(M) \otimes_R N)) + \beta_0(M)h^j(N) \leq \ell(H^{i+2}_\mathfrak{m}(\text{Syz}_2(M) \otimes_R N)) + \beta_1(M)h^{i+1}(N) + \beta_0(M)h^j(N).$$
Repeating this, $h^i(M \otimes_R N) \leq \ell(H^i_{\mathfrak{m}}(\text{Syz}_\ell(M) \otimes_R N)) + \sum_{j=0}^{\ell-1} \beta_j(M) h^{i+j}(N)$. Putting $\ell := \text{pd}(M) - i$, $h^i(M \otimes_R N) \leq \ell(H^{\text{pd}(M)}_{\mathfrak{m}}(\text{Syz}_{\text{pd}(M)}(M) \otimes_R N)) + \sum_{j=0}^{\ell-1} \beta_j(M) h^{i+j}(N)$. □

The same proof shows that: Let $R$ be equi-dimensional and generalized Cohen-Macaulay local ring and $N$ be locally free and of constant rank. If $\text{pd}(M) < \text{depth}(R)$, then $h^i(M \otimes_R N) \leq \sum_{j=0}^{\text{pd}(M)} \beta_j(M) h^{i+j}(N)$ for all $i < \text{depth}(R) - \text{pd}(M)$.

REFERENCES

[1] M. Auslander and O. Goldman, Maximal orders, Trans. AMS 97 (1960), 1–24.
[2] M. Auslander, Modules over unramified regular local rings, Illinois J. Math. 5 (1961) 631-647.
[3] W. Bruns and U. Vetter, Length formulas for the local cohomology of exterior powers, Math. Z. 191 (1986), 145–158.
[4] S. Goto, Maximal Buchsbaum modules over regular local rings and a structure theorem for generalized Cohen-Macaulay modules, Commutative algebra and combinatorics, Adv. Stud. Pure Math. 11, Kinokuniya, Tokyo, North-Holland, Amsterdam (1987), 39–64.
[5] C. Huneke and R. Wiegand, Tensor products of modules and the rigidity of Tor, Math. Ann. 299 (1994), 449–476.
[6] C. Huneke and R. Wiegand, Tensor products of modules, rigidity and local cohomology, Math. Scand. 81 (1997), 161-183.
[7] S.B. Iyengar and R. Takahashi, The Jacobian ideal of a commutative ring and annihilators of cohomology, J. algebra, to appear.
[8] S. Lichtenbaum, On the vanishing of Tor in regular local rings, Ill. J. Math. 10 (1966), 220–226.
[9] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Publ. Math. IHES. 42 (1973), 47-119.
[10] J. Stuckrad, and W. Vogel, Buchsbaum rings and applications. An interaction between algebra, geometry, and topology, Mathematische Monographien 21, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.
[11] Wolmer Vasconcelos, Cohomological degrees and applications, Commutative algebra, 667-707, Springer, New York, 2013.
[12] Wolmer V. Vasconcelos, Length complexity of tensor products, Comm. Algebra 38 (2010), no. 5, 1743-1760.
[13] K.I. Yoshida, A note on multiplicity of perfect modules of codimension one, Comm. Algebra 25 (1997), no. 9, 2807-2816.

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