On Optimal Control Problems with Nonregular Mixed Constraints

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Abstract—Necessary optimality conditions for optimal control problems with mixed constraints are known when some regular conditions are satisfied. However, the nonregular case has received little attention although some interesting problems, including those involving sweeping systems and some problems arising from robotics, can be easily formulated as problems with nonregular mixed constraints. In this work we study and discuss necessary conditions for optimal control problems with nonregular mixed constraints.

I. INTRODUCTION

Optimal Control theory came up with the publication of the famous Pontryagin Maximum Principle (PMP) in the late 1950’s. For standard optimal control problem (OC), defined on a fixed time interval $[0,1]$, the PMP provides a set of necessary optimality conditions that a feasible solution $(x,u)$ of (OC) should satisfy. In the above, a process comprises an absolutely continuous function $x:[0,1] \rightarrow \mathbb{R}^n$, called the state, and a measurable and bounded function $u$, called the control. The process $(x,u)$ is a solution of (OC) if it satisfies all the constraints of the problem.

Right from the very beginning of the optimal control field, extensions of the PMP to cover optimal control problems that additionally involve other constraints on the state and control have been the focus of attention as one can see from Chapter VI in [19] (see also the references therein and in [10]). In the decades to come, the subject of optimality conditions for two kinds of state constrained problems have attracted considerable attention. These constraints mainly take the form

- Pure state constraints:
  \[ h(t,x(t)) \leq 0. \]  

- Mixed constraints:
  \[ b(t,x(t),u(t)) = 0 \quad \text{and} \quad g(t,x(t),u(t)) \leq 0. \]  

In geometric terms, (1) and (2), this one together with the control constraints $u(t) \in U(t)$, can be written as

\[ x(t) \in X(t) \quad \text{and} \quad (x(t),u(t)) \in S(t), \]  

where

\[ X(t) = \{ x \in \mathbb{R}^n : h(t,x) \leq 0 \} \]  

and

\[ S(t) = \{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^k : b(t,x,u) = 0, \quad g(t,x,u) \leq 0, \quad u \in U(t) \}. \]

While most of the attention regarding optimality conditions for state/mixed constrained problems has been concentrated on pure state constraints (1), mixed constraints have deserved study mainly focused on mixed constraints satisfying some regular conditions, called regular mixed constraints (see, for example, [3], [4] and references therein).

Regarding the derivation of optimality conditions for mixed constraints, and rephrasing [3], regularity constraint qualification imposed on (2) implicitly allows the removal of the mixed constraints, written in the form $(x(t),u(t)) \in S(t)$, and its insertion into the dynamics producing an equivalent reformulated standard optimal control problem to which the known optimality conditions applies. This sort of reformulation is well illustrated in [4] where a nonsmooth PMP is derived for mixed constrained optimal control by capturing the dynamics and the mixed constraints into a differential inclusion $\dot{x}(t) \in F(t,x(t))$ where

\[ F(t,x) = \{ v \in \mathbb{R}^n : v = f(t,x,u), \quad (x,u) \in S(t) \}. \]

A remarkable fact is that regularity of mixed constraints guarantees the Lipschitz continuity or the “pseudo- Lipschitz” continuity\(^1\) of the set valued function $F$ with respect to $x$, a property that fails to hold when no regularity conditions are imposed; in this respect, see [5] and [8].

Regular mixed constraints do not subsume pure state constraints because they involve the control $u$. Consequently, it comes as no surprise that pure state constraints have been studied separately. Different versions of the PMP for pure state constraints involve measures as multipliers associated with the pure state constraints and a possible discontinuous bounded variation function; see, for example, [3], [10], [12], [20] and references within. On the other hand, known necessary optimality conditions (in the form of PMP or as “weak” versions of such principle) for problems with regular mixed constraints have bounded measurable functions as multipliers associated with such conditions and absolutely continuous

\(^1\)For the definition of pseudo Lipschitz continuous set valued function see, for example, [5].
adjoint multipliers; see, [4], [10] and references within. Nonregular mixed constraints have been greatly neglected with exception perhaps of some isolated works; in this respect see [10], [14] and [15] to name but a few. This comes as no surprise since optimal control problems involving mixed constraints not satisfying some sort of regularity of conditions, have multipliers taking values in the dual space of the essentially bounded functions, \( (L^\infty)^* \), this being a very large and not so well known space.

Nowadays one of the most significant merits and trends of optimal control resides on its applications to many fields spreading from Biomathematics to Engineering problems. The increase of such applications we can witness today brings new challenges. In particular, it is our belief that it also empathizes the need to further research on necessary conditions of optimality for problems with nonregular mixed constraints. Necessary conditions of optimality are of the foremost importance to validate or to partially validate computational solutions. Since multipliers for nonregular mixed constraints as stated in [10], [14], [15] may seem to be quite untractable, we may nevertheless hope to extract useful information for some classes of problems.

The aim of this paper is to highlight the importance of nonregular mixed constraints and to clarify the nature of the multipliers one may expect to get when derive PMP or more weak necessary conditions for such problems. To do so, we start to present and discuss known regularity conditions in the literature in Section II. Then, we present two problems that can easily be reformulated as problems with nonregular mixed constraints together with pure state constraints in Section III. Section IV contains a summary of the main properties of the dual of \( L^\infty \). Then, in Section V, we show how an infinite dimensional optimization approach, in the vein of [14], can lead to weak forms of necessary conditions for optimal control problems with nonregular mixed constraints in the form of inequalities\(^2\). Moreover, we show how such necessary conditions lead to known conditions when regularity is imposed. We conclude this paper with a discussion on future research.

Before proceeding, it is important to alert the reader that throughout we try to keep the technical details to a minimum skipping some important points so as to focus the attention on central issues. In particular, we do not dwell on the nature of the local minimizer for the optimal control problems of interest; we refer to “local minimizer” without any proper definition. For a discussion on the nature of minimizers we refer the reader to [20].

\(^2\) We assume that equality constraints defined by the function \( b \) are absent in (2).

The distance function of a point \( x \in \mathbb{R}^n \) from a closed set \( A \subset \mathbb{R}^n \) is defined as \( d_A(x) := \inf\{|x - y| : y \in A\} \), and the signed function

\[
d^*_A(x) = \begin{cases} 
  d_A(x) & \text{if } x \in A^c, \\
  -d_A(x) & \text{if } x \in A,
\end{cases}
\]

where \( A^c \) denotes the complement of \( A \).

For a function \( h : [0, 1] \to \mathbb{R}^p \), we say that \( h \in W^{1,1}([0, 1]; \mathbb{R}^p) \) if \( h \) is absolutely continuous, \( h \in L^1([0, 1]; \mathbb{R}^p) \) if \( h \) is integrable and that \( h \in L^\infty([0, 1]; \mathbb{R}^p) \) if \( h \) is an essentially bounded function.

Let \( C^*([0, 1]; \mathbb{R}^p) \) be the dual space of the continuous functions defined from \([0, 1]\) to \( \mathbb{R}^p \) and with supremum norm, denoted by \( C([0, 1]; \mathbb{R}^p) \). The norm of \( C^*([0, 1]; \mathbb{R}^p) \) is denoted by \( \|\mu\|_{TV} \). Let \( NBV([0, 1]) \) be the space of real-valued functions of bounded variation that are right-continuous in \((0, 1)\) and vanish at \( 1 \). \( NBV^+([0, 1]) \) denotes the subspace of nondecreasing functions in \( NBV([0, 1]) \). The total variation of a function \( \mu \in NBV([0, 1]) \) is written \( \|\mu\|_{TV} \). Each \( \mu \in NBV([0, 1]) \) defines a Borel measure on \([0, 1]\).

Now, throughout this paper, and to simplify the notation, for a function \( h : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^p \), we write simply \( \bar{h}(t) \) to denote the function evaluated along \((t, \bar{x}(t), \bar{u}(t))\), i.e., \( \bar{h}(t) = h(t, \bar{x}(t), \bar{u}(t)) \). Also, \( h_x \) and \( h_u \) denote the derivative of \( h \) with respect to \( x \) and \( u \). Moreover, instead of \( W^{1,1}([0, 1]; \mathbb{R}^p), C([0, 1]; \mathbb{R}^p) \) and \( L^\infty([0, 1]; \mathbb{R}^p) \) we write simply \( W^{1,1}, C \) and \( L^\infty \).

II. REGULARITY CONDITIONS FOR MIXED CONSTRAINTS

The need to impose constraint qualifications (known as regularity conditions) on mixed constraints so as to produce PMPs for optimal control problems involving such conditions has been recognized from the very beginning of optimal control theory (see [19]). In the earlier literature and for smooth mixed constraints in the form (2) such the constraint qualification is the linear independence of the gradients of the functions \( b \) and \( g \) with respect to the control \( u \) in a neighbourhood of the the solution \((\bar{x}, \bar{u})\). Later on, linear independence of the gradients of \( g \) with respect to \( u \) is replaced by the weaker assumption of “positive” linear independence; see [10]. Such constraint qualification could be seen as expressions of the Mangasarian-Fromovitz conditions (MFC) for nonlinear programming problems. We have then witnessed many attempts to weaken such conditions over the years (in this respect, we refer the reader to [4] and [3] where some of these attempts are mentioned). Breaking new ground and using tools from the nonsmooth analysis based on the seminal work [5], a nonsmooth PMP is derived in [4] under what could be seen as a “minimal” regularity assumption on mixed constraints; the so-called bounded slope condition (BS). Denoting the local minimizer by \((\bar{x}, \bar{u})\), one can show that, in the smooth setting, (BS) reduces to the classical (MFC): For all \((t, x, u)\) such that \( g(t, x, u) \leq 0 \) and \(|x - \bar{x}(t)| \leq \epsilon \), for some \( \epsilon > 0 \), and all \( \gamma \in \mathbb{R}_+^n \), the following condition holds

\[
\langle \gamma, g(t, x, u) \rangle = 0, \nabla_u \langle \gamma, g(t, x, u) \rangle = 0 \Rightarrow \gamma = 0.
\]
Let us now consider the problem

\[ \begin{align*}
(P) & \quad \text{Minimize } \int_0^1 L(t, x(t), u(t)) \, dt \\
& \quad \text{over processes } (x, u) \in W^{1,1} \times L^\infty \text{ such that} \\
& \quad \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.,} \\
& \quad g(t, x(t), u(t)) \leq 0 \text{ a.e.,} \\
& \quad x(0) = x_0
\end{align*} \]

where \( L, f \) and \( g \) are continuously differentiable functions and \( g \) satisfies the (MFC) constraints qualification. Assume that \((\bar{x}, \bar{u})\) is a local minimizer. Then, it follows from Corollary 7.2 in [4] that there exists a scalar \( \lambda_0 \geq 0 \), an absolutely continuous \( \lambda : [0, 1] \to \mathbb{R}^n \) and an integrable function \( z : [0, 1] \to \mathbb{R}^m \) such that

(i) \( (\lambda_0, \lambda(t)) \neq 0 \);

(ii) \( -\dot{\lambda}(t) = (f_x^T \lambda - g_x^T z - \lambda_0 L_x)(t, \bar{x}(t), \bar{u}(t)) \) a.e. \( t \);

(iii) \( 0 = (f_u^T \lambda - g_u^T z - \lambda_0 L_u)(t, \bar{x}(t), \bar{u}(t)) \) a.e. \( t \);

(iv) \( \langle \lambda(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - \lambda_0 \langle L(t, \bar{x}(t), \bar{u}(t)) \rangle \leq \langle \lambda(t), f(t, x(t), u(t)) \rangle - \lambda_0 \langle L(t, x(t), u(t)) \rangle \)

for all \( u \) such that \( g(t, x(t), u(t)) \leq 0 \) a.e. \( t \);

(v) \( z(t) \geq 0 \), \( \langle z(t), g(t, x(t), u(t)) \rangle = 0 \) a.e. \( t \).

The set of conditions (i)–(iii) together with (v) is commonly designated as “weak necessary conditions” or weak version of the PMP. We remark that for specific class of problems, Corollary 7.2 in [4] has been proved to hold under weaker assumptions than (MFC). In this respect, see, for example, [17, Theorem 5.2]. Additional clarification and research on PMPs for mixed constraints can be found, for example, in [1], [3], [17] (in [3] problems with both mixed constraints and pure state constraints are studied). Such developments have also triggered further research on some problems that can be reformulated as mixed constraints like problems with implicit systems and those with differential algebraic equations (see [7] and [18]).

### III. EXAMPLES OF PROBLEMS WITH NONREGULAR MIXED CONSTRAINTS

Our research is motivated by applied problem where nonregular mixed constraint appear. Next we present two of such examples. Noteworthy, these examples have both pure state and nonregular mixed constraints. They are nevertheless of interest, because pure state constraints can be seen as particular cases of nonregular mixed constraints.

Our first example, concerns the path planning of Autonomous underwater vehicles (AUV) for complex missions. Suppose we would like to determine the minimum energy needed to drive an AUV with motion given by

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t) \]

go from \( x(0) = x_0 \) to a target set \( \mathcal{T} \) in 1 unit of time and such that the vehicle must “live” in a region \( \Omega \) for more than \( \theta_1 \) units of time (e.g., for communications) and in \( \mathcal{O} \) for less than \( \theta_2 \) units of time (e.g., if \( \mathcal{O} \) dangerous area). Here \( \Omega \) and the complement of \( \mathcal{O} \) are closed sets. To take into account these two requirements we introduce two new states and two new controls:

\[ \begin{align*}
\dot{z}_1(t) &= \beta_1(t) \\
\dot{z}_2(t) &= \beta_2(t)
\end{align*} \]

where \( \beta_i(t) \in [0, 1], i = 1, 2, \) \( z_1(1) \geq \theta_1, z_2(1) \leq \theta_2 \) and

(a) \( \beta_1(t) d^s(x(t), \Omega) \leq 0 \),

(b) \( (1 - \beta_2(t)) d^p(x(t), \mathcal{O}) \leq 0 \).

The states \( z_1 \) and \( z_2 \) count the time the vehicle “lives” in \( \Omega \) and \( \mathcal{O} \) with the help of the mixed constraints (a) and (b), which are clearly nonregular when \( x(t) \) crosses the boundary of \( \Omega \) and \( \mathcal{O}^c \). We can now easily formulate the problem of driving the AUV from \( x_0 \) to the target minimizing the energy (i.e., minimizing \( \int_0^1 u^2(t) \, dt \)) under the constraints imposed. Remarkably, the functions defining the mixed constraints are merely Lipschitz failing to be of class \( C^1 \).

Another class of problems with interest are those involving sweeping problems. Let us look at the following example that we take from [9]:

\[ \begin{align*}
(\text{SP}) & \quad \text{Minimize } \phi(x(1)) \\
& \quad \text{over processes } (x, u) \text{ such that} \\
& \quad x(t) \in f(x(t), u(t)) - N_C(x(t)) \text{ a.e.} \\
& \quad u(t) \in U \text{ a.e.} \\
& \quad x(0) = x_0, \ x(1) \in C_1
\end{align*} \]

where \( x_0 \in C \) and \( C_1 \subset C \) is closed, \( C := \{ x \in \mathbb{R}^n : \psi(x) \leq 0 \} \) for \( \psi : \mathbb{R}^n \to \mathbb{R} \) strictly convex and of class \( C^2 \). As pointed out in [9], it is an easy matter to see that the constraints of the problem can be rewritten as \( \dot{x}(t) = f(x(t), u(t)) - \lambda(t) \nabla \psi(x(t)) \), together with the state constraint \( \psi(x(t)) \leq 0 \), the nonregular mixed constraint \( -\lambda(t) \psi(x(t)) \leq 0 \) and the control set constraints \( (u(t), \lambda(t)) \in U \times \{ \lambda \in \mathbb{R} : \lambda \geq 0 \} \).

For more interesting problems that may profit from a formulation as mixed constrained optimal control problems, see [10]. We also remark that none of the problems above fall into the class of problems we cover here. Clearly, these two examples illustrate the need for further research. We will come back to this subject in the final section of this paper.

### IV. ON THE DUAL OF \( L^\infty \)

Denote the Lebesgue measure by \( m \) and the \( \sigma \)-algebra of Lebesgue measurable subsets of \([0, 1]\) by \( \Sigma \). Following [2], a finitely additive set function \( \zeta \) defined on \( \Sigma \), with the additional property that \( \zeta(\emptyset) = 0 \), is called a charge. The set of all bounded charges is denoted by \( \mathbf{ba}([0, 1], \Sigma) \), while the set of bounded measures (i.e. countably additive charges or simply measures) is denoted by \( \mathbf{ca}([0, 1], \Sigma) \).

The space \( \mathbf{ba}([0, 1], \Sigma, m) \) of weakly absolutely continuous charges with respect to the Lebesgue measure (see definition below) is of interest because it is identified with the dual space of \( L^\infty \) in [2], [13] or [21] for example. The set \( \mathbf{ca}([0, 1], \Sigma, m) \) is its subset of countably additive elements.

**Definition 4.1:** ([2], [21]) Let \( \zeta \) and \( \lambda \) be charges.

- We say the \( \zeta \) is bounded if \( \sup \{ \| \zeta(E) \| : E \in \Sigma \} < \infty \).
- We say that \( \lambda \) is a minorant of \( \zeta \), and write \( \lambda \leq \zeta \), if \( \lambda(E) \leq \zeta(E) \) for all \( E \in \Sigma \).
- We say that \( \lambda \) is weakly absolutely continuous with respect to \( \zeta \), and write \( \lambda \ll w \zeta \), if \( \zeta(E) = 0 \) implies \( \lambda(E) = 0 \).
• An element $0 \leq \zeta$ is called pure, if $0 \leq \mu \leq \zeta$ and $\mu$ countably additive implies that $\mu = 0$.
• A Lebesgue measurable function $h : [0, 1] \to \mathbb{R}$ is zero $\zeta$-a.e. if, for all $\varepsilon > 0$, $\zeta\{t \in [0, 1] : |h(t)| > \varepsilon\} = 0$. We might be tempted to assume that a function $h$ is zero $\zeta$-a.e. if and only if $\zeta\{t : h(t) \neq 0\} = 0$, as when $\zeta$ is countably additive. In general, the above equality is sufficient but not necessary for $h$ to be zero $\zeta$-a.e. (for more details see [2]).

Next we state some important properties of pure charges.

**Theorem 4.1:** ([2], [21]) For any $\zeta \in \mathfrak{ba}([0, 1], \Sigma)$, there exists $\zeta_\sigma, \zeta_\varphi \in \mathfrak{ba}([0, 1], \Sigma)$ such that $\zeta = \zeta_\sigma + \zeta_\varphi$, where $\zeta_\sigma, \zeta_\varphi \in \mathfrak{ba}([0, 1], \Sigma)$ and $\zeta_\varphi$ is a pure charge. If $\zeta$ is positive, then both $\zeta_\sigma$ and $\zeta_\varphi$ can be taken positive. This decomposition is unique.

**Theorem 4.2:** ([2], [21]) Let $0 \leq \zeta \in \mathfrak{ba}([0, 1], \Sigma, m)$. Then $\zeta$ is pure if and only if there exists a decreasing sequence of sets $\{E_n\}$ in $\Sigma$ such that

$$m(E_n) \to 0 \quad \text{and} \quad \zeta(E_n) = 0 \quad \text{for all} \quad n.$$

The following result, which follows from the above properties of pure charges, will play an essential role in the next section.

**Lemma 4.3:** Let $0 \leq \zeta \in \mathfrak{ba}([0, 1], \Sigma, m)$ be pure. Let $p$ and $q$ be functions in $L^\infty$. If

$$\int_0^1 p(t)q(t)dt = \int_0^1 q(t)v(t)d\zeta$$

for all $v \in L^\infty$, then

$$\int_0^1 p(t)v(t)dt = \int_0^1 q(t)v(t)d\zeta = 0.$$

**Proof:** Take a sequence of sets $\{E_n\}$ as in Theorem 4.2 and any $v \in L^\infty$. Then

$$\int_0^1 p(t)v(t)dt = \lim_n \int_{\cup_n E_n} p(t)v(t)dt$$

$$= \lim_n \int_{E_n} p(t)v(t)dt = \lim_n \int_{E_n} q(t)v(t)d\zeta = 0,$$

proving our claim.

**V. WEAK NECESSARY CONDITIONS**

We now turn to problem $(P)$ introduced in Section II. To keep technical details to a minimum, assume throughout that the function $L$, $f$ and $g$ are of class $C^1$. Also, suppose that $(\bar{x}, \bar{u}) \in W^{1,1} \times L^\infty$ is a local minimizer for $(P)$. To derive necessary conditions for this problem, we start by associating it to the following optimization problem:

$$(MC) \begin{cases}
\text{Minimize } J(x, u) = \int_0^1 L(t, x(t), u(t))dt \\
\text{over processes } (x, u) \text{ such that } x, u \in W^{1,1} \times L^\infty, \quad H(x, u) \in T, \\
G(x, u) \in K,
\end{cases}$$

where $T = \{0\} \subset C$, $K = \{h \in L^\infty : h(t) \leq 0 \text{ a.e.}\}$ and the operators

$H : W^{1,1} \times L^\infty \to C$ and $G : W^{1,1} \times L^\infty \to L^\infty$

are defined as

$$H(x, u)(t) = x(t) - x_0 - \int_0^t f(s, x(s), u(s))ds$$

and

$$G(x, u)(t) = g(t, x(t), u(t)).$$

Observe that $(\bar{x}, \bar{u})$ solves $(MC)$.

Before deriving necessary conditions for $(MC)$ we need the following two lemmas.

**Lemma 5.1:** The Fréchet derivatives of the operators $J, H$ and $G$ at $(\bar{x}, \bar{u}) \in W^{1,1} \times L^\infty$ exist and are given by

$$J'(\bar{x}, \bar{u}; y, v) = \int_0^1 \{\bar{L}_x(s)y(s) + \bar{L}_u(s)v(s)\}ds,$$

$$H'(\bar{x}, \bar{u}; y, v)(t) = y(t) - \int_0^t \{\bar{f}_x(s)y(s) + \bar{f}_u(s)v(s)\}ds,$$

$$G'(\bar{x}, \bar{u}; y, v)(t) = \bar{g}_x(t)y(t) + \bar{g}_u(t)v(t),$$

for all $(y, v) \in W^{1,1} \times L^\infty$.

**Lemma 5.2:** The linear map $H'(\bar{x}, \bar{u})$ is surjective.

**Proof:** For arbitrary $z \in W^{1,1}$, [16, Theorem 2.2] guarantees the existence of a solution $y \in W^{1,1}$ to the Volterra integral equation

$$z(t) = y(t) - \int_0^t \bar{f}_x(s)y(s)ds.$$

Therefore, $H'(\bar{x}, \bar{u}; y, 0) = z$ and $H'$ is surjective.

Let $X$ be a normed space and $\mathcal{S} \subset X$ a cone. Recall that $S^*$ is defined by

$$S^* = \{\zeta \in X^* : \zeta(f) \leq 0 \text{ for all } f \in S\}.$$

**Theorem 5.3:** ([6]) There exist $\lambda_0 \geq 0$, $\eta \in K^*$ and $\Lambda$ in $T^*$ such that

$$\lambda_0 J'(\bar{x}, \bar{u}; y, v) + \eta G'(\bar{x}, \bar{u}; y, v) + \Lambda H'(\bar{x}, \bar{u}; y, v) = 0$$

(5)

for all $(y, v) \in W^{1,1} \times L^\infty$ and

$$\eta G'(\bar{x}, \bar{u}) = 0.$$  

(6)

Moreover, $\lambda_0$ and $\eta$ are not both zero.

Using the representation theorems for the duals of $C$ and $L^\infty$, there exist $\lambda \in NBV[0, 1]$ and $\zeta \geq 0$ in $\mathfrak{ba}(I, \Sigma, m)$ such that

$$\lambda(y) = \int_0^1 y(t)d\lambda \quad \text{for all} \quad y \in W^{1,1},$$

and

$$\eta(h) = \int_0^1 h(t)d\zeta \quad \text{for all} \quad h \in L^\infty.$$

Accordingly, we can rewrite equations (5) and (6) as

$$\lambda_0 \int_0^1 \{\bar{L}_x(t)y(t) + \bar{L}_u(t)v(t)\}dt + \int_0^1 \left[ y(t) - \int_0^t \{\bar{f}_x(s)y(s) + \bar{f}_u(s)v(s)\}ds \right]d\lambda = 0,$$

(7)

$$+ \int_0^1 \{\bar{g}_x(t)y(t) + \bar{g}_u(t)v(t)\}d\zeta = 0.$$
for all \((y,v) \in W^{1,1} \times L^\infty\), and
\[
\int_0^1 \bar{g}(t)d\zeta = 0. \tag{8}
\]

Now, write \(\zeta = \zeta_\sigma + \zeta_p\) as in Theorem 4.1. The non-
positivity of \(\bar{g}\) implies that \((8)\) is equivalent to the pair of
equations
\[
\int_0^1 \bar{g}(t)d\zeta_\sigma = 0, \quad \int_0^1 \bar{g}(t)d\zeta_p = 0. \tag{9}
\]
By the Radon-Nikodym Theorem, there exists an integrable function \(z\) such that
\[
\int_0^1 z(t)y(t)dt = \int_0^1 y(t)d\zeta_\sigma \quad \text{for all } y \in L^\infty.
\]

The fact that \(z\) is nonnegative \(m\text{-a.e.}\) can be seen by setting \(y\) as the indicator function of the set \(\{t : z(t) < 0\}\) in the
equation above and keeping in mind the nonnegativity of \(\zeta_\sigma\). Thus, the first equality in \((9)\) is equivalent to
\[
\int_0^1 z(t)\bar{g}(t)dt = 0
\]
where the integrand is nonpositive \(a.e.\). This last equality, together with the second equality of \((9)\), yield the comple-
mentary slackness conditions
\[
z(t)\bar{g}(t) = 0 \quad \text{-a.e. and } \quad \bar{g}(t) = 0 \quad \zeta_p\text{-a.e.} \quad \text{(CS)}
\]

On the other hand, using properties of the Stieltjes integral, equation \((7)\) can be broken down into the two equations:
\[
\int_0^1 y(t)d\psi(t) + \int_0^1 \bar{g}_x(t)y(t)d\zeta_p = 0, \quad \forall y \in W^{1,1}, \tag{10}
\]
where \(\psi \in NBV\) is
\[
\psi(t) = -\int_t^1 \{\lambda_0 \bar{L}_x(s) + z(s)\bar{g}_x(s) + \lambda(s)\bar{f}_x(s)\}ds + \lambda(t),
\]
and
\[
\int_0^1 \{\lambda_0 \bar{L}_u(t) + z(t)\bar{g}_u(t) + \lambda(t)\bar{f}_u(t)\}v(t)dt
+ \int_0^1 \bar{g}_u(t)v(t)d\zeta_p = 0 \quad \text{for all } v \in L^\infty. \tag{11}
\]

Notice that the situation in equality \((11)\) is precisely the one in Lemma 4.3. Hence, \((11)\) leads to
\[
\int_E \{\lambda_0 \bar{L}_u(t) + z(t)\bar{g}_u(t) + \lambda(t)\bar{f}_u(t)\}dt = 0, \tag{12}
\]
\[
\int_E \bar{g}_u(t)d\zeta_p = 0, \tag{13}
\]
for all \(E \in \Sigma\). Equality \((12)\) is equivalent to the usual
stationarity condition
\[
\lambda_0 \bar{L}_u(t) + z(t)\bar{g}_u(t) + \lambda(t)\bar{f}_u(t) = 0 \quad \text{a.e.}, \tag{SC}
\]
while \((13)\) is equivalent to \(\bar{g}_u = 0 \quad \zeta_p\text{-a.e.}, \quad \text{i.e.}
\[
\zeta_p\{\{t : |\bar{g}_u(t)| > \epsilon\}\} = 0 \quad \text{for all } \epsilon > 0. \tag{14}
\]

Now, let us turn our attention to equation \((10)\). The mapping
\[
y \mapsto \int_0^1 \bar{g}_x(t)y(t)d\zeta_p
\]
is a linear bounded functional and, in accordance to Riesz’s
representation theorem, there exists a function \(\beta \in NBV\) such that
\[
\int_0^1 y(t)d\beta(t) = \int_0^1 \bar{g}_x(t)y(t)d\zeta_p \tag{15}
\]
for all \(y \in C([0,1];\mathbb{R})\). Consequently, \((10)\) becomes
\[
\int_0^1 y(t)d\Psi = 0 \quad \text{for all } y \in W^{1,1}, \tag{16}
\]
where \(\Psi \in NBV\) is the function
\[
\Psi(t) = \psi(t) + \beta(t).
\]
The following auxiliary result is of importance.

**Lemma 5.4:** Let \(\Gamma \in NBV\) be such that
\[
\int_0^1 h(t)d\Gamma(t) = 0 \quad \text{for all } h \in W^{1,1}.
\]
Then \(\Gamma = 0 \; a.e.

**Proof:** Let \(0 < a < b < 1\) and consider the indicator function \(\chi_{[a,b]}\) of the interval \([a,b]\). The function \(h\) defined by
\[
h(t) = \int_a^t \chi_{[a,b]}(s)ds
\]
is absolutely continuous and \(\dot{h}(t) = \chi_{[a,b]}(t)\) a.e.. Integration by parts of \(h\) and \(\Gamma\) yields
\[
\int_a^b \Gamma(t)dt = 0 \quad \text{for all } \{a,b\} \subset [0,1],
\]
which implies that \(\Gamma = 0\) almost everywhere. \(\blacksquare\)

State equations in integral form appear when we apply the
previous lemma to equation \((16)\):
\[
-\lambda(t) = -\int_t^1 \{\lambda_0 \bar{L}_x(s) + z(s)\bar{g}_x(s) + \lambda(s)\bar{f}_x(s)\}ds
+ \beta(t), \tag{17}
\]
or equivalently
\[
-d\lambda(t) = \lambda(t)\bar{f}_x(t) + \lambda_0 \bar{L}_x(t) + z(t)\bar{g}_x(t) + d\beta(t) \quad \text{a.e.}
\]
Summarizing our findings we have

**Theorem 5.5:** There exist a constant \(\lambda_0 \geq 0\), functions \(\lambda, \beta \in NBV\), a nonnegative integrable function \(z\) and a
nonnegative pure charge \(\zeta_p \in \mathbf{ba}([0,1],\Sigma,m)\) such that
\[
\text{(NT) } \lambda_0 + \|z\|_\infty + \zeta_p([0,1]) > 0;
\]
\[
\text{(CS) } z(t)\bar{g}(t) = 0 \; \text{-a.e., and } \; \bar{g}(t) = 0 \; \zeta\text{-a.e.}.
\]
(SE) $d\lambda(t) = \lambda(t) \bar{f}_x(t) + \lambda_0 \bar{L}_x(t) + z(t) \bar{g}_x(t) + d\beta(t) \text{ a.e.};$
(SC) $\lambda_0 \bar{L}_u(t) + z(t) \bar{g}_u(t) + \lambda(t) \bar{f}_u(t) = 0 \text{ a.e.}.$

Notice how the transversality condition $\lambda(1) = 0$ is a consequence of the contention $\lambda \in NVB.$

Finally, we show that under (MFC) we can recover the classical result for regular mixed constraints from Theorem 5.5:

**Theorem 5.6:** Suppose that (MFC) holds in a compact set $C$ containing all the point $(t, \tilde{x}(t), \tilde{u}(t))$. Then, there exist a constant $\lambda_0 > 0$, a function $\lambda \in W^{1,1}$ and a nonnegative integrable function $z$ such that conditions (i), (ii), (iii) and (v) in Section II are satisfied.

**Proof:** First, we show that $\zeta_p$ in Theorem 5.5 vanishes. The condition $\bar{g}(\tilde{t}, \tilde{x}, \tilde{u}) = 0$ a.e. in (CS) entails

$$\zeta_p([0,1]) = \zeta_p\{t : |\bar{g}(t)| \leq \alpha \} \quad \text{for all } \alpha > 0.$$

So, to see that $\zeta_p$ vanishes reduces to showing that, for some $\alpha > 0$,

$$\zeta_p\{t : |\bar{g}(t)| \leq \alpha \} = 0.$$

We claim that there exist $\tilde{\alpha}, \tilde{\beta} > 0$ such that

$$\{t, x, u \in C : |g(t, x, u)| \leq \tilde{\alpha}\} = \{t, x, u \in C : |g(t, x, u)| \leq \tilde{\alpha}, |g_u(t, x, u)| > \tilde{\beta}\}.$$

If this were not the case, we could find a sequence $(t_n, x_n, u_n)$ in $C$ with limit $(\tilde{t}, \tilde{x}, \tilde{u})$ also in $C$ satisfying

$$g(\tilde{t}, \tilde{x}, \tilde{u}) = 0 \quad \text{and} \quad g_u(\tilde{t}, \tilde{x}, \tilde{u}) = 0,$$

which would contradict the (MFC). Now, from equations (18) and (14) we deduce that $\zeta_p([0,1]) = 0$.

Finally, to see that $\lambda$ is absolutely continuous, notice that, from (15), we have $\int_0^1 g(t) d\beta = 0$ for all $g \in W^{1,1}$. Hence, applying Lemma 5.4, we conclude that $\beta = 0$ a.e.. The conclusion follows from (17).

VI. FUTURE WORK

Here we discussed how weak necessary conditions for nonregular mixed constrained optimal control problems can be derived using an infinite optimization approach. Our discussion is far from over; we concentrate our attention in a very simple optimal control problems. Although this was done to keep technical details to a minimum (they are anyhow complicated as it is), it is important to discuss more general cases. It would be also interesting to see how such approach would work when we additionally have pure state constraints. Additionally, we would like to get necessary conditions in the form of a PMP. Indeed, note that we have not produced any Weierstrass conditions (or maximization condition) for our very simple problem. Another possible approach to these problem would be to use tools from nonsmooth analysis. We believe this is a greatly unexplored subject that deserves attention and that future work on these two subjects may add some light on the nature of necessary conditions for these mysterious problems.

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