TRIPLE AND MULTIPLE COLLISIONS
OF COMPETING BROWNIAN PARTICLES

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Abstract. Consider a finite system of competing Brownian particles. They move as Brownian motions with drift and diffusion coefficients depending on their ranks. This includes the case of asymmetric collisions, when the local time of any collision is distributed unevenly between the two colliding particles, see Karatzas, Pal and Shkolnikov (2012). A triple collision occurs if three particles occupy the same site at a given moment. This is sometimes an undesirable phenomenon. Continuing the work of Ichiba, Karatzas and Shkolnikov (2013), we find necessary and sufficient condition for absence of triple collisions. We also prove sufficient conditions for absence of quadruple collisions, of quintuple collisions, and so on. Our method is reduction to reflected Brownian motion in the positive multidimensional orthant hitting non-smooth parts of the boundary and, more generally, edges of the boundary of certain low dimension.

1. Introduction

The topic of this paper is the multidimensional semimartingale reflected Brownian motion (SRBM) in the orthant. Let us loosely describe it; for the precise definition, see the next section. Fix $d \geq 1$, the dimension. Let $\mathbb{R}_+ := [0, \infty)$. Let $S = \mathbb{R}^d_+$ be the positive orthant in $\mathbb{R}^d$. Fix the parameters of an SRBM: a drift vector $\mu$, a $d \times d$-positive definite symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq d}$, and another $d \times d$-matrix $R = (r_{ij})_{1 \leq i,j \leq d}$ with $r_{ii} = 1$, $i = 1, \ldots, d$. The matrix $A$ is called a covariance matrix, and the matrix $R$ - a reflection matrix. Then an SRBM in the orthant $S$ with parameters $R, \mu, A$, denoted by $\text{SRBM}^d(R, \mu, A)$, is a Markov process with state space $S$ which:

(i) behaves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$ in the interior of the orthant $S$;

(ii) is reflected on the face $S_i = \{ x \in S \mid x_i = 0 \}$ of the boundary $\partial S$ in the direction of $r_i$ for each $i = 1, \ldots, d$. Here, $r_i$ is the $i$th column of $R$.

If $r_i = e_i$, where $e_i$ is the $i$th standard basis vector in $\mathbb{R}^d$, then the reflection is called normal. Otherwise, it is called oblique.

We are interested whether this process hits non-smooth parts of the boundary $\partial S$, that is, intersections of faces $S_i \cap S_j$, $i \neq j$. The most important case for applications is when the reflection matrix is a nonsingular $M$-matrix, see Definition 3 and Lemma 5.5 below. For this case, we find a necessary and sufficient condition.

Theorem 1.1. Let $R$ be a nonsingular $M$-matrix. Then the $\text{SRBM}^d(R, \mu, A)$ a.s. does not hit non-smooth parts of the boundary if and only if

$$r_{ij}a_{jj} + r_{ji}a_{ii} \geq 2a_{ij}, \quad i, j = 1, \ldots, d. \quad (1)$$

If this condition is violated for some $1 \leq i < j \leq d$, then with positive probability there exists $t > 0$ such that $Z_i(t) = Z_j(t) = 0$. 

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We can also write the condition (1) in the matrix form:

\[ RD + DR^T \geq 2A, \]

where \( D \) is a \( d \times d \)-diagonal matrix with the same diagonal entries as \( A \). We compare matrices elementwise.

We are also interested in whether the SRBM avoids edges of higher order, that is, intersections of three faces, intersections of four faces, etc. We find a sufficient condition for this, and another sufficient condition for it to hit the edges of given order with positive probability. The method of proof is (i) reduction of this problem to hitting the corner of the orthant (that is, the origin); and (ii) comparing this process with a Bessel process of appropriate dimension.

We apply our results to finite systems of competing Brownian particles. Let us informally describe the basic model of competing Brownian particles; the precise definition will be in the next section. Take \( N \geq 2 \) particles on the real line. Suppose the particle which currently occupies the \( k \)th largest position among all particles (we say: has rank \( k \)) moves as a one-dimensional Brownian motion with drift \( g_k \) and variance \( \sigma_k^2 > 0 \), \( k = 1, \ldots, N \). A particular case is the Atlas model, when \( b_N = 1, b_1 = \ldots = b_{N-1} = 0, \sigma_1 = \ldots = \sigma_N = 1 \). The gaps between adjacent particles form an SRBM in the \( N-1 \)-dimensional orthant. This is called the gap process.

When the gap process hits \( S_i \cap S_j \), in other words, the non-smooth parts of the boundary, this corresponds to triple collisions (when three particles occupy the same site) or simultaneous collisions (when two particles collide and at the same moment another two particles collide). Triple collisions is generally an undesirable phenomenon, see for example [25]. The condition under which SRBM avoids hitting non-smooth parts of the boundary translates into another condition when these particles avoid triple collisions.

In particular, we get the following result: lack of triple collisions implies lack of simultaneous collisions.

**Theorem 1.2.** For finite systems of CBP, there are a.s. no triple collisions and a.s. no simultaneous collisions if and only if

\[ \sigma_{k+1}^2 - \sigma_k^2 \leq \sigma_k^2 - \sigma_{k-1}^2, \quad k = 2, \ldots, N - 1. \]

If this condition is violated for some \( k \), then with positive probability there is a triple collision at some time \( t > 0 \) of particles with ranks \( k-1, k, k+1 \).

We are also interested in collisions of higher order (when four, five or more particles collide). The result for SRBM avoiding edges of higher order implies the following result for competing Brownian particles avoiding collisions of higher order.

**Theorem 1.3.** Consider a system of \( M \) competing Brownian particles with symmetric collisions. If for any \( L = 1, \ldots, M - N + 1 \) we have:

\[ \sigma_L^2 \leq k_N(\sigma_{L+1}^2 + \ldots + \sigma_{L+N}^2), \]

\[ \sigma_{L+N-1}^2 \leq k_N(\sigma_L^2 + \ldots + \sigma_{L+N-2}^2), \]

where

\[ k_N := 1 - \frac{2}{(N-1)^2}, \]

then a.s. there are no collisions of order \( N \).

The results for an SRBM can also be applied to competing Brownian particles with asymmetric collisions. This model was introduced in [29] as a diffusion limit of exclusion-type interacting particle systems. Let us loosely describe it: in the previous model, particles collide in a symmetric fashion, that is, each collision local time is split evenly between the two colliding particles. The new model exhibits a different behavior: when two particles collide, they are pushed apart with different
speed (as if they had different masses). The corresponding collision local time is split between the upper and lower particle in a different proportion: the part \( q_{k+1}^+ \) belongs to the lower particle with rank \( k + 1 \), and the part \( q_k^- \) belongs to the upper particle with rank \( k \), where \( q_{k+1}^+, q_k^- > 0 \) are constants with \( q_{k+1}^+ + q_k^- = 1 \).

The motivation comes from particle systems relevant in random matrix theory and random surfaces evolving according to the KPZ equation. For example, see the recent article [14]. Typically, such particle systems are in infinite dimension, although the theory of SRBM has been developed in finite dimension. This provides impetus to our current work.

**Theorem 1.4.** There are a.s. no triple and no simultaneous collisions if and only if

\[
(q_{k-1}^- + q_{k+1}^+) \sigma_k^2 \geq q_k^- \sigma_{k+1}^2 + q_k^+ \sigma_{k-1}^2, \quad k = 2, \ldots, N - 1.
\]

If this condition is violated for some \( k = 2, \ldots, N - 1 \), then with positive probability there is a triple collision between particles with ranks \( k - 1, k, k + 1 \).

1.1. **Historical review.** We refer the reader to an excellent survey [12]. The concept of an SRBM in the orthant was introduced in the papers [19], [17]. Existence and uniqueness results were proved in [38], [39], [40]. The article [38] contains a necessary and sufficient condition when this process does not hit the corner of the wedge. It is done for a reflected Brownian motion in a wedge with identity covariance matrix, but the case we are interested in can be easily reduced to it by linear transformation, see [29] Proposition 2 for details. In the case of general dimension, an important partial result was obtained in [41]: if the skew-symmetry condition \( r_{ij}a_{jj} + r_{ji}a_{ii} = 2a_{ij}, 1 \leq i, j \leq d \), is satisfied, then the SRBM a.s. does not hit non-smooth parts of the boundary.

Some important results about stationary distributions for SRBM are in [10], [15], [18], [41]. Let us also mention some other papers on SRBM in the orthant: [10], [7], [20], [4] (positive recurrence); [5], [36] (exponential convergence to the stationary distribution).

The concept of competing Brownian particles (with symmetric collisions) was introduced in the paper [1] in connection with financial modeling. It was further studied in [27], [23], [33], [6], [26], [25], [13], [36]. The paper [24] gives some partial results about triple collisions; the paper [12] gives some other partial results, including the proof that the condition which we give is sufficient. (We prove it is also necessary.) The model with asymmetric collisions was introduced in [29], and they found some sufficient conditions for absence of triple collisions, which are weaker than our necessary and sufficient condition. They also resolved the problem for the case of three particles. See also applications to financial modeling: [11], [13].

Infinite systems of CBP with symmetric collisions were also considered recently: [33], [25].

1.2. **Organization of the paper.** Section 2 contains definitions and notational conventions used in this paper. Section 3 is devoted to triple collisions of CBP and hitting non-smooth parts of the origin by an SRBM. Section 4 contains the result of an SRBM hitting edges of higher order. Section 5 contains a proof of the main theorem about multiple collisions of CBP. The Appendix contains a technical lemma.

2. **Definitions**

2.1. **Notation.** For \( a \in \mathbb{R} \), let \( a_+ := \max(a, 0) \) and \( a_- := \max(-a, 0) \). For \( a, b \in \mathbb{R} \), let \( a \wedge b = \min(a, b) \) and \( a \vee b = \max(a, b) \). We think of vectors \( a \in \mathbb{R}^d \) as column-vectors, i.e. \( d \times 1 \)-matrices. For a vector or matrix \( a \), we refer to its transpose as \( a^T \). Let \( I_k \) be the \( k \times k \)-identity matrix. As mentioned before, \( S = \mathbb{R}_+^d \) and \( S_i = \{ x \in S \mid x_i = 0 \}, \quad i = 1, \ldots, d \). In addition, \( S^0 := S \setminus \partial S \) is the interior of \( S \).
For a vector \( x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \), let \( \|x\| := (x_1^2 + \ldots + x_d^2)^{1/2} \) denote the Euclidean norm. For a \( d \times d \)-matrix \( C = (c_{ij})_{1 \leq i, j \leq d} \), let us define its spectral norm:

\[
\|C\| := \sup_{\|x\| = 1} \|Cx\| = \max \left\{ \sqrt{\lambda}, \ \lambda \text{ is an eigenvalue of } C^T C \right\}.
\]

For any two vectors \( x = (x_1, \ldots, x_d)^T \), \( y = (y_1, \ldots, y_d)^T \in \mathbb{R}^d \), let \( x \cdot y = x_1y_1 + \ldots + x_dy_d \) be their dot product. As noted above, we compare points \( x, y \in S \) componentwise: \( x \geq y \) means that \( x_i \geq y_i \) for each \( i = 1, \ldots, d \); similarly for \( x \leq y \), \( x > y \), \( x < y \).

Fix \( d \geq 1 \) and suppose \( I \subseteq \{1, \ldots, d\} \) is a nonempty subset, with elements ordered in the increasing order. For a vector \( b = (b_i)_{1 \leq i \leq d} \in \mathbb{R}^d \), let \( |b|_I = (b_i)_{i \in I} \). For a \( d \times d \)-matrix \( R = (r_{ij}) \), let \( |R|_I = (r_{ij})_{i,j \in I} \).

We call any set of indices of the type \( \{i, i+1, \ldots, i+k-1\} \) a discrete interval of \( k \) elements.

### 2.2. Semimartingale reflected Brownian motion in the orthant

Let \( d \geq 1 \) be the number of dimensions. Assume we have the usual setting: \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})\), where \((\Omega, \mathcal{F})\) is a measurable space, and \((\mathcal{F}_t)_{t \geq 0}\) is a right-continuous filtration. Take the parameters \( R, \mu, A \) described in the Introduction.

**Definition 1.** Fix \( T \geq 0 \). Assume \( \mathcal{X} : [0, T] \to \mathbb{R}^d \) is a continuous function. A solution to the Skorohod problem in the positive orthant \( \mathbb{R}^d_+ \) with reflection matrix \( R \) and driving function \( \mathcal{X} \) is a pair \((\mathcal{Y}, \mathcal{Z})\) of continuous functions \([0, T] \to \mathbb{R}^d\) which satisfy the following conditions:

(i) for every \( t \in [0, T] \) we have:

\[
\mathcal{Z}(t) = \mathcal{X}(t) + R\mathcal{Y}(t) \in S;
\]

(ii) for every \( i = 1, \ldots, d \), the function \( \mathcal{Y}_i \) is nondecreasing, \( \mathcal{Y}_i(0) = 0 \), and \( \int_0^\infty \mathcal{Z}_i(t)d\mathcal{Y}_i(t) = 0 \). The last equality shows that \( \mathcal{Y}_i \) can increase only when \( \mathcal{Z}_i = 0 \), that is, when \( \mathcal{Z} \) is on the face \( S_i \) of the boundary \( \partial S \).

One can state the same definition for the infinite time-horizon, that is, for \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) from \( C(\mathbb{R}_+, \mathbb{R}^d) \).

**Definition 2.** Let \( B = (B(t), t \geq 0) \) be an \( \mathbb{R}^d \)-valued stochastic process on \((\Omega, \mathcal{F})\). Then an SRBM in the positive orthant \( \mathbb{R}^d_+ \) with reflection matrix \( R \), drift vector \( \mu \) and covariance matrix \( A \), shorty SRBM\(^d\)(\(R, \mu, A\)), is a continuous \((\mathcal{F}_t)_{t \geq 0}\)-adapted \( \mathbb{R}^d_+ \)-valued process \( Z = (Z(t), t \geq 0) \) such that, for every \( x \in \mathbb{R}^d_+ \), there exists an \((\mathcal{F}_t)_{t \geq 0}\)-adapted \( \mathbb{R}^d \)-valued process \( L = (L(t), t \geq 0) \) and a probability measure \( \mathbf{P}_x \) on \((\Omega, \mathcal{F})\) such that:

(i) the pair \((L, Z)\) of functions is \( \mathbf{P}_x \)-a.s. a solution to the Skorohod problem in the orthant \( \mathbb{R}^d_+ \) with driving function \( B \) and reflection matrix \( R \);

(ii) the filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfies the usual conditions with respect to \( \mathbf{P}_x \), and the process \( B \) is an \((\mathcal{F}_t)_{t \geq 0}, \mathbf{P}_x\)-Brownian motion with drift vector \( \mu \) and covariance matrix \( A \), starting from \( x \).

In this case, the Brownian motion \( B \) and the process \( L \) are called the driving Brownian motion and the local time for the process \( Z \).

We can normalize the matrix \( R \) by scaling the process \( Y \) so that \( r_{ii} = 1 \) for \( i = 1, \ldots, d \), see [4] Appendix B]. In the sequel, we always implicitly assume this.

**Definition 3.** A \( d \times d \)-matrix is called a reflection matrix if \( r_{ii} = 1, \ i = 1, \ldots, d \). A \( d \times d \)-matrix is called nonnegative if all its elements are nonnegative. A \( d \times d \)-matrix \( R \) is called an \( S \)-matrix if there exists \( u \in \mathbb{R}^d \), \( u > 0 \), such that \( Ru > 0 \). A principal submatrix of a \( d \times d \)-matrix \( R \) is obtained by deleting all rows and columns with indices in some subset (possibly empty) of \( \{1, \ldots, d\} \). A \( d \times d \)-matrix is called completely-\( S \) if all its principal submatrices are \( S \)-matrices. A \( d \times d \)-matrix is called a \( Z \)-matrix if its off-diagonal elements are nonpositive. It is called a nonsingular \( M \)-matrix if it is both completely-\( S \) and a \( Z \)-matrix. A \( d \times d \)-matrix \( R \) is called strictly inverse-nonnegative if \( R \) is invertible and \( R^{-1} \geq 0 \) with the diagonal elements of \( R^{-1} \) strictly positive.
An equivalent characterization of nonsingular $M$-matrices is in the Appendix. The following necessary and sufficient condition for weak existence of an SRBM $^d((R, \mu, A)$ was shown in [35, 37, 9]. See also the survey [12].

**Proposition 2.1.** There exists an SRBM $^d((R, \mu, A)$ if and only if the matrix $R$ is completely-$. In this case, this process is unique in law and is Feller continuous strong Markov.

**Definition 4.** We say that a process $Z = (Z(t), t \geq 0)$ in $\mathbb{R}_+^d$ hits edges of order $k \leq d$ at time $t \geq 0$ if there exists a subset $I \subseteq \{1, \ldots, d\}$ of $k$ elements such that $Z_i(t) = 0$ for $i \in I$. In particular, if $k = 2$, then we say that this process *hits non-smooth parts of the boundary*. If $k = d$, we say that it *hits the corner*.

**Definition 5.** We say that the process $Z$ a.s. does *not hit non-smooth parts of the boundary* if the event that there exists $t > 0$ such that the process $Z$ hits non-smooth parts of the boundary at time $t$ has probability zero. The statement that the process $Z$ a.s. does *not hit edges of order $K$* has similar meaning.

In other words, if $Z$ hits non-smooth parts of the boundary at time $t \geq 0$, this means that there exist $i, j = 1, \ldots, d$, $i \neq j$, such that $Z_i(t) = Z_j(t) = 0$. If $Z$ hits the corner at time $t \geq 0$, this means that for each $i = 1, \ldots, d$ we have: $Z_i(t) = 0$.

### 2.3. Systems of Competing Brownian Particles

Fix an integer $N \geq 1$. Consider a system of $N$ particles moving on the real line, formally written as one $\mathbb{R}^N$-valued process

$$X = (X(t), t \geq 0), \quad X(t) = (X_1(t), \ldots, X_N(t))^T.$$  

For any vector $x \in \mathbb{R}^N$, denote by $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(N)}$ its *ranked components*. We resolve ties in favor of the lowest index, see [1, 27]. Let $W = (W(t), t \geq 0)$, $W(t) = (W_1(t), \ldots, W_N(t))^T$, be the standard $N$-dimensional Brownian motion. Consider a few market models from Stochastic Portfolio Theory.

**Definition 6.** A finite system of CBP with symmetric collisions is an $\mathbb{R}^N$-valued process governed by the following system of stochastic differential equations:

$$dX_i(t) = \sum_{k=1}^{N} [g_k dt + \sigma_k dW_k(t)]1_{\{X_i(t) = X_{(k)}(t)\}}, \quad i = 1, \ldots, N, \quad t \geq 0.$$  

Here, $g_1, \ldots, g_N$ are fixed real numbers, and $\sigma_1, \ldots, \sigma_N$ are fixed positive real numbers. The processes $Y_k = (Y_k(t), t \geq 0)$, $Y_k(t) = X_{(k)}(t)$, $k = 1, \ldots, N$, are called *ranked particles*. The original processes $X_i$, $i = 1, \ldots, N$, are called *named particles*. We say that the *rank* of $Y_k$ is $k$, and the *name* of $X_i$ is $i$.

Informally, the $k$th largest particle moves as a one-dimensional Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$. Let $L^{(k,k+1)} = (L^{(k,k+1)}(t), t \geq 0)$ be the local time of $Y_k - Y_{k+1}$ at zero. For notational convenience, let $L^{(0,1)} = L^{(N,N+1)} = 0$ for all $t \geq 0$. From [27, Lemma 1], we get:

$$dY_k(t) = g_k dt + \sigma_k dW_k(t) + \frac{1}{2} dL^{(k,k+1)}(t) - \frac{1}{2} dL^{(k-1,k)}(t), \quad t \geq 0.$$  

**Definition 7.** For the system above, the *gap process* is defined as the $\mathbb{R}_+^{N-1}$-valued process $Z = (Z(t), t \geq 0)$, $Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))^T$, where $Z_k(t) = Y_k(t) - Y_{k+1}(t)$, $k = 1, \ldots, N-1$, $t \geq 0$. 
It was proved in [27, Lemma 1] (see also [1]) that the gap process is an SRBM^{N-1}(R, \mu, A), where

\begin{equation}
R = \begin{bmatrix}
1 & -1/2 & 0 & \ldots & 0 & 0 \\
-1/2 & 1 & -1/2 & \ldots & 0 & 0 \\
0 & -1/2 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1/2 & 1
\end{bmatrix}
\end{equation}

\begin{equation}
A = \begin{bmatrix}
\sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \ldots & 0 \\
-\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \sigma_{N-1}^2 + \sigma_N^2
\end{bmatrix},
\end{equation}

\begin{equation}
\mu = (g_1 - g_2, \ldots, g_{N-1} - g_N)^T \in \mathbb{R}^{N-1}.
\end{equation}

Let us formally define a generalization of this model of CBP with asymmetric collisions from [29].

**Definition 8.** Take the parameters \(g_k, \sigma_k, k = 1, \ldots, N\), from the model with symmetric collisions. In addition, fix positive real numbers \(q_k^+, k = 1, \ldots, N\), which satisfy \(q_{k+1}^+ + q_k^- = 1, k = 1, \ldots, N - 1\). Consider a continuous \(\mathbb{R}^N\)-valued semimartingale \(Y = (Y(t), t \geq 0)\) with values in the set \(W := \{y \in \mathbb{R}^N \mid y_1 \geq \ldots \geq y_N\}\). Suppose it is governed by the following system of equations:

\[
Y_k(t) = g_k t + \sigma_k W_k(t) + q_k^- \Lambda^{(k,k+1)}(t) - q_{k+1}^+ \Lambda^{(k-1,k)}(t), \quad k = 1, \ldots, N, \ t \geq 0.
\]

For each \(k = 1, \ldots, N - 1\), the process \(\Lambda^{(k,k+1)} = (\Lambda^{(k,k+1)}(t), t \geq 0)\) is the local time at the origin of the nonnegative semimartingale \(Y_k - Y_{k+1}\). We let \(\Lambda^{(0,1)} = \Lambda^{(N,N+1)} = 0\) for notational convenience.

The regulating role of these local times is to make sure that the process \(Y\) stays in \(W\). This can be regarded as an SRBM in the domain \(W\), see [16], [29]. In the previous model, particles collide in a symmetric fashion, that is, each collision has local time split evenly between the two colliding particles. In contrast, here this local time is split between the upper and lower particle in a different proportion: the part \(q_{k+1}^+\) belongs to the lower particle \(Y_{k+1}\), and the part \(q_k^-\) belongs to the upper particle \(Y_k\).

The gap process is defined similarly to the symmetric case. Let \(R\) be the following \((N - 1) \times (N - 1)\)-matrix:

\[
R = \begin{bmatrix}
1 & -q_2^- & 0 & \ldots & 0 & 0 \\
-q_2^+ & 1 & -q_3^- & \ldots & 0 & 0 \\
0 & -q_3^+ & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -q_{N-1}^+ & 1
\end{bmatrix}.
\]

Take \(A\) and \(\mu\) from (6) and (7). Then it is easy to show that \(Z = \text{SRBM}^{N-1}(R, \mu, A)\).

**Remark 1.** It is shown in [29, Section 2.1] that the matrix (2.3), in particular, the matrix (5), is a nonsingular reflection \(M\)-matrix.

**Definition 9.** Consider a system of CBP with symmetric or asymmetric collisions. Let \(k = 2, \ldots, N - 1\). We say that a triple collision of particles with ranks \(k-1, k, k+1\) occurs at time \(t \geq 0\) if

\[
Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t).
\]
Let \( 2 \leq k < l \leq N \). We say that a simultaneous collisions of particles with ranks \( k-1, k, l-1, l \) occurs at time \( t \geq 0 \) if
\[
Y_k(t) = Y_{k-1}(t) \quad \text{and} \quad Y_l(t) = Y_{l-1}(t).
\]
So a triple collision is a particular case of a simultaneous collision.

**Definition 10.** Consider a system of \( N \) competing Brownian particles with symmetric or asymmetric collisions. Let \( K = 3, \ldots, N \). We say that a collision of order \( K \) occurs at time \( t \geq 0 \) if there exists a discrete interval \( I \subseteq \{1, \ldots, N\} \) of \( K \) elements such that \( Y_j(t) \) is the same for \( j \in I \). We say that a simultaneous collision of order \( K \) occurs at time \( t \geq 0 \) if there exist nonintersecting discrete intervals \( I_1, \ldots, I_t \subseteq \{1, \ldots, N\} \) such that the value \( Y_j(t) \) is the same for all \( j \in I_1 \), the value \( Y_j(t) \) is the same for all \( j \in I_2 \), etc., and together these subsets contain \( K + l - 1 \) elements. A collision of order \( K = N \) is called a total collision.

**Definition 11.** We say that there are a.s. no triple collisions if a.s. there does not exist \( t > 0 \) and \( k = 2, \ldots, N - 1 \) such that there is a triple collision of particles with ranks \( k-1, k, k+1 \) at time \( t \). The phrases a.s. no simultaneous collisions, a.s. no collisions of order \( K \), a.s. no simultaneous collisions of order \( K \), a.s. no total collisions have similar meaning.

**Lemma 2.2.** (i) A simultaneous collision of order \( K \) at time \( t \geq 0 \) is equivalent to the gap process hitting edges of order \( K - 1 \) at time \( t \geq 0 \).

(ii) A collision of order \( K \) at time \( t \geq 0 \) is a particular case of a simultaneous collision of order \( K \).

(iii) A total collision is the same as the gap process hitting the corner.

(iv) A total collision at time \( t \geq 0 \) is equivalent to \( Y_1(t) = Y_2(t) = \ldots = Y_N(t) \).

**Proof.** The proof is trivial. \( \square \)

2.4. Girsanov removal of drift.

**Lemma 2.3.** Consider two processes: \( Z = \text{SRBM}^d(R, \mu, A) \) and \( Z' = \text{SRBM}^d(R, 0, A) \), where \( R \) is a completely-\( \mathcal{S} \) reflection \( d \times d \)-matrix, and \( A \) is a positive definite symmetric \( d \times d \)-matrix. Take a subset \( I \subseteq \{1, \ldots, d\} \). There is a.s. no \( t > 0 \) such that \( Z_i(t) = 0 \) for \( i \in I \) if and only if there is a.s. no \( t > 0 \) such that \( Z'_i(t) = 0 \) for \( i \in I \).

**Proof.** This follows from the Girsanov theorem. Assume we have the usual setting: a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with the filtration satisfying the usual conditions. If \( B = (B(t), t \geq 0) \) is an \( (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \)-Brownian motion in \( \mathbb{R}^d \) with drift vector \( \mu \) and covariance matrix \( A \), then we can construct a locally equivalent probability measure \( \mathbb{Q} \) on \( (\Omega, \mathcal{F}) \) such that \( B \) is an \( (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}) \)-Brownian motion in \( \mathbb{R}^d \) with drift vector zero and covariance matrix \( A \). The statement that \( \mathbb{Q} \) is locally equivalent to \( \mathbb{P} \) means that the restrictions of \( \mathbb{Q} \) and \( \mathbb{P} \) to every \( \mathcal{F}_t, t \geq 0 \), are equivalent.

If \( Z = \text{SRBM}^d(R, \mu, A) \) under \( \mathbb{P} \), then \( Z = \text{SRBM}^d(R, 0, A) \) under \( \mathbb{Q} \). Suppose there is \( \mathbb{P} \)-a.s. no \( t \geq 0 \) such that \( Z_i(t) = 0 \) for \( i \in I \). Then for every \( T \geq 0 \) there is \( \mathbb{P} \)-a.s. no \( t \in [0, T] \) such that \( Z_i(t) = 0 \) for \( i \in I \). Events of probability one remain such when we switch to an equivalent measure; so for every \( T \geq 0 \) there is \( \mathbb{Q} \)-a.s. no \( t \in [0, T] \) such that \( Z_i(t) = 0 \) for \( i \in I \). Since \( \mathbb{Q} \) is countably additive, there is \( \mathbb{Q} \)-a.s. no \( t \geq 0 \) such that \( Z_i(t) = 0 \) for \( i \in I \). So if \( \text{SRBM}^d(R, \mu, A) \) has the stated property, then \( \text{SRBM}^d(R, 0, A) \) also has this property. The converse statement is proved similarly. \( \square \)

**Corollary 2.4.** Consider two processes: \( Z = \text{SRBM}^d(R, \mu, A) \) and \( Z' = \text{SRBM}^d(R, 0, A) \), where \( R \) is a completely-\( \mathcal{S} \) reflection \( d \times d \)-matrix, and \( A \) is a positive definite symmetric \( d \times d \)-matrix. Fix \( K = 2, \ldots, N \). Then \( Z \) a.s. does not hit edges of order \( K \) if and only if \( Z' \) a.s. does not hit edges of order \( K \).
Corollary 2.5. Consider a system of $N$ competing Brownian particles with symmetric or asymmetric collisions. Consider another system with the same parameters as the first one, except that it has zero drifts $g_1 = 0, \ldots, g_N = 0$. Take any $K = 3, \ldots, N$. Then the first system a.s. does not have collisions of order $K$ if and only if the second system a.s. does not have collisions of order $K$. Also, the first system a.s. does not have simultaneous collisions of order $K$ if and only if the second system a.s. does not have simultaneous collisions of order $K$.

2.5. Comparison results for SRBM. The following proposition is taken from [34]; see also [36, Proposition 2.3], [31], and [30, Theorem 1.1] and a remark immediately thereafter.

Definition 12. Consider two $S$-valued processes $Z_1 = (Z_1(t), t \geq 0)$ and $Z_2 = (Z_2(t), t \geq 0)$. We say that $Z_1$ is stochastically dominated by $Z_2$ and write $Z_1 \preceq Z_2$ if for all $t \geq 0$ and for all $y \in S$ we have: $\mathbf{P}(Z_1(t) \geq y) \leq \mathbf{P}(Z_2(t) \geq y)$.

Proposition 2.6. Assume $R_1, R_2$ are $d \times d$-reflection matrices which are nonsingular $M$-matrices. Suppose $R_1 \preceq R_2$. Let $\mu$ be a drift vector, and let $A$ be a covariance matrix. Fix $x_1, x_2 \in S$, $x_1 \leq x_2$. Let $Z_k$ be an SRBM$^d(R_k, \mu, A)$, starting from $x_k$, $k = 1, 2$. Then $Z_1 \preceq Z_2$.

We should not expect these results to hold when the reflection matrices are not nonsingular $M$-matrices. For example, let $R_1 = R_2 = R$. Suppose some of the off-diagonal elements of $R$ are positive, say $r_{21} > 0$. Start two copies of SRBM$^d(R, \mu, A)$: from the origin and from $e_1$. The second component of the first copy receives an additional push in the form of a term $r_{21}Y_1(t)$ in the positive direction. The second component of the second copy does not receive it (at least not from the very beginning of its movement), and it may fall behind.

According to [28, Theorem 5], see also [36, Proposition 7.4], we can proceed from stochastic ordering to pathwise ordering, possibly by changing the probability space.

Lemma 2.7. Suppose $R$ is a $d \times d$-reflection nonsingular $M$-matrix. Let $\mu \in \mathbb{R}^d$ be a drift vector, and let $A$ be a $d \times d$-covariance matrix. Let $Z = \text{SRBM}^d(R, \mu, A)$. Take a nonempty subset $I \subseteq \{1, \ldots, d\}$. Then

$$[Z]_I \preceq \text{SRBM}^d|R|([R]_I, [\mu]_I, [A]_I),$$

if these processes start from the same point.

Proof. Consider the process $Z' = \text{SRBM}^d(R', \mu, A)$, where $R'$ is a $d \times d$-matrix defined as follows: $R' = (r'_{ij})_{1 \leq i, j \leq d}$, where

$$r'_{ij} = r_{ij}, \ i, j \in I \text{ or } i, j \notin I; \quad r'_{ij} = 0 \text{ otherwise.}$$

Then $R'$ is a reflection matrix, because $r'_{ii} = 1$, $i = 1, \ldots, d$. Also, $R'$ is a nonsingular $M$-matrix. Indeed, all off-diagonal elements of $R'$ are either zero or $r_{ij} \leq 0$, so it is a $Z$-matrix. Take a vector $u \in \mathbb{R}^d, u > 0$ such that $Ru > 0$; then $R' \geq R$, so $R'u \geq Ru > 0$. Therefore, $R'$ is an $S$-matrix. The same argument applies to any principal submatrix of $R'$. Therefore, $R'$ is completely-S, so it is a nonsingular $M$-matrix. Since $R' \geq R$ and these processes start from the same point, we have: $Z \preceq Z'$. Therefore, $[Z]_I \preceq [Z']_I = \text{SRBM}^d|R|([R]_I, [\mu]_I, [A]_I)$.

3. Hitting Non-Smooth Parts of the Boundary and Triple Collisions

3.1. Proof of Theorem 1.1. First, assume a.s. $Z(0) \in S^d$. Suppose we have the equality in (1) instead of inequality. Then the skew-symmetry condition holds, and by a result from [11] the process SRBM$^d(R, \mu, A)$ does not hit non-smooth parts of the boundary. A remark is in order: This fact was proven for a bit different version of an SRBM, the one in a general polyhedral domain, but with the identity covariance matrix. Reduction of the SRBM$^d(R, \mu, A)$ in the orthant to this case is done
in [15]. Now, suppose the condition (11) holds. Let us find another reflection matrix \( \tilde{R} = (\tilde{r}_{ij})_{1 \leq i, j \leq d} \) such that \( R \geq \tilde{R} \), and \( \tilde{R}D + D\tilde{R}^T = 2A \). We need:

\[
\tilde{r}_{ij}a_{jj} + \tilde{r}_{ji}a_{ii} = 2a_{ij}, \quad i, j = 1, \ldots, d.
\]

We must have \( \tilde{r}_{ij} = 1 \) for \( i = j \). Then this equality is true for \( i = j \). It suffices to establish it for \( i < j \). To this end, let

\[
\tilde{r}_{ij} = \frac{1}{a_{jj}} [2a_{ij} - r_{ji}a_{ii}], \quad \tilde{r}_{ji} = a_{ji}.
\]

This is well defined, since \( a_{ii} > 0 \) (indeed, the matrix \( A \) is positive definite). Also, \( \tilde{r}_{ij} \leq r_{ij} \), because \( r_{ij}a_{jj} + r_{ji}a_{ii} \geq 2a_{ij} \). Since \( \tilde{r}_{ij} \leq r_{ij} \leq 0 \) for \( i \neq j \), \( \tilde{R} \) is a \( \mathcal{M} \)-matrix. We have: \( \tilde{R}D + D\tilde{R}^T = 2A \), so by [21, Theorem 2.5] (compare conditions 12 and 16), \( \tilde{R} \) is a nonsingular \( \mathcal{M} \)-matrix. Let \( Z = \text{SRBM}^d(R, \mu, A, \tilde{r}) \), \( Z = \text{SRBM}^d(R, \mu, A) \), starting from the same initial condition. Then we have: \( R \) and \( \tilde{R} \) are nonsingular \( \mathcal{M} \)-matrices, and \( R \geq \tilde{R} \). By [34], we have: \( Z \) is stochastically smaller than \( Z \). By [28, Theorem 5] (see also [36, Proposition 2.9]), we can claim that a.s. for all \( t > 0 \) we have: \( Z(t) \leq Z(t) \) (possibly after changing the probability space). But the process \( \tilde{Z} \) does not hit non-smooth parts of the boundary, so for every \( 1 \leq i < j \leq d \), a.s. for all \( t > 0 \) we have: \( \tilde{Z}_i(t) + \tilde{Z}_j(t) > 0 \). Therefore, a.s. for all \( t > 0 \) we have: \( Z_i(t) + Z_j(t) > 0 \). Thus, the process \( Z \) does not hit non-smooth parts of the boundary.

Assume now that (11) fails, and for some \( 1 \leq i < j \leq d \) we have:

\[
r_{ij}a_{jj} + r_{ji}a_{ii} < 2a_{ij}.
\]

Without loss of generality, assume \( i = 1, j = 2 \). Consider the following two-dimensional SRBM:

\[
Z' = \text{SRBM}^2([R]_I, [\mu]_I, [A]_I), \quad I = \{1, 2\}.
\]

By Lemma [26], we know that \( |Z'| \leq |Z| \). Let us show that \( Z' \) hits non-smooth parts of the boundary of \( \mathbb{R}^2_+ \) (that is, the origin) with positive probability. We can rewrite (8) as

\[
n_iq_2 + n_jq_1 > 0,
\]

which is the condition from [24, Lemma 3.2]. We use the notation from this latter paper. See the details in [29, Section 2, Proposition 2]; in this article, this is done for a particular case, it can be transferred to the general case without any substantial adjustments. From [24, Lemma 3.2], it follows that \( Z' \) indeed hits the origin with positive probability. So with positive probability there exists \( t > 0 \) such that \( Z'_i(t) = Z'_j(t) = 0 \). Therefore, with positive probability there exists \( t > 0 \) such that \( Z_i(t) = Z_j(t) = 0 \).

The statement is proved for the case when \( Z(0) \in S^0 \) a.s. If not, then we proceed as follows. For every \( s > 0 \) a.s. \( Z(s) \in S^0 \). By the Markov property, the process \( (Z(t), t \geq s) \) is itself an \( \text{SRBM}^d(R, \mu, A) \), starting from \( Z(s) \). Let \( \alpha(x) \) be the probability that an \( \text{SRBM}^d(R, \mu, A) \), starting from \( x \), hits non-smooth parts of the boundary at a certain time \( t > 0 \). Suppose that (11) holds. Then \( \alpha(x) = 0 \) for all \( x \in S^0 \). Therefore, the probability that \( Z \) hits non-smooth parts of the boundary at some moment \( t > s \) is \( \mathbb{E}\alpha(Z(s)) = 0 \). It suffices to let \( t \downarrow 0 \). When (11) fails, then for every \( x \in S^0 \) we have: \( \alpha(x) > 0 \), so \( \mathbb{E}\alpha(Z(s)) > 0 \), which is the probability that \( Z \) hits non-smooth parts of the boundary at some moment \( t > s \). It follows trivially that the probability that \( Z \) hits non-smooth parts of the boundary at any moment \( t > 0 \) is positive. The proof is complete.

### 3.2. Proof of Theorem 1.2

There is a triple collision or simultaneous collision if and only if the gap process hits non-smooth parts of the boundary. Here, the gap process is an \( \text{SRBM}^{N-1}(R, \mu, A) \), where \( R, \mu, A \) are given by [5], [6] and [7]. Recall from Remark 1 that \( R \) is a nonsingular \( \mathcal{M} \)-matrix. Let us check the condition (11). For \( i = j \) and for \( |i - j| \geq 2 \), it is automatic. By symmetry, we need just to check this condition for \( i = j - 1, j = 2, \ldots, N - 1 \). We can rewrite it as \( \sigma_{j-1}^2 - \sigma_j^2 \leq \sigma_j^2 - \sigma_{j+1}^2 \). If this is true for all \( j = 2, \ldots, N - 1 \), then the gap process does not hit non-smooth parts of the boundary. By Lemma 2.2, this is equivalent to the absence of simultaneous collisions (and, in particular, triple collisions). If for some \( j = 2, \ldots, N - 1 \) we have:
Theorem 4.1. \[ \sigma_j^2 - \sigma_{j-1}^2 < \sigma_{j+1}^2 - \sigma_j^2, \] then with positive probability for some \( t > 0 \) we have: \( Z_j(t) = Z_{j-1}(t) = 0. \) Therefore, with positive probability at some time \( t > 0 \) there is a triple collision between particles with ranks \( j - 1, j, j + 1. \) The proof is complete.

3.3. Proof of Theorem 4.1. The proof is similar to the previous one, except that the matrix \( R \) is given by \( (2.3) \).

4. SRBM HITTING EDGES OF HIGHER ORDER

4.1. SRBM hitting corners.

Theorem 4.1. Consider the process \( \text{SRBM}^d(R, \mu, A) \), where \( R \) is a \( d \times d \)-reflection nonsingular \( M \)-matrix. Let \( C \) be a \( d \times d \)-diagonal matrix with positive entries on the main diagonal such that \( \tilde{R} = RC \) is symmetric. If

\[
(9) \quad \text{tr}(\tilde{R}^{-1}A) < 2 \min_{i=1,\ldots,d} \frac{(\tilde{R}^{-1}A\tilde{R}^{-1})_{ii}}{(\tilde{R}^{-1})_{ii}},
\]

then this process hits the corner with positive probability. If

\[
(10) \quad \text{tr}(\tilde{R}^{-1}A) \geq 2 \max_{i=1,\ldots,d} \frac{(\tilde{R}^{-1}A\tilde{R}^{-1})_{ii}}{(\tilde{R}^{-1})_{ii}},
\]

then this process does not hit the corner a.s.

Proof. By Corollary 2.4 we can assume \( \mu = 0. \) Consider the function \( F : S \to \mathbb{R}, \) \( F(x) := x^T \tilde{R}^{-1}x. \) Since \( \tilde{R}^{-1} \) is strictly inverse-nonnegative, the matrix \( \tilde{R}^{-1} = C^{-1}R^{-1} \) is also strictly inverse-nonnegative, and \( F(x) > 0 \) for \( x \in S \setminus \{0\}. \) By the generalization of Itô - Tanaka formula from [19], we have:

\[
dF(Z(t)) = DF(Z(t)) \cdot dB(t) + \sum_{i=1}^d D_iF(Z(t)) \, dL_i(t) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij} \frac{\partial^2 F(Z(t))}{\partial x_i \partial x_j} \, dt.
\]

Here, \( B = (B(t), t \geq 0) \) is the driving Brownian motion for \( Z. \) This is a \( d \)-dimensional Brownian motion with drift zero and covariance matrix \( A. \) Also, we define \( D_iF(x) := r_i \cdot DF(x), \) \( i = 1, \ldots, d. \) Finally, \( L = (L_1, \ldots, L_d) \) is the local time for \( Z. \) But \( DF(x) = 2\tilde{R}^{-1}x, \) so for \( x \in S_i, \) \( i = 1, \ldots, d, \) we have:

\[
D_iF(x) = 2\tilde{R}^{-1}x \cdot r_i = 2x \cdot \tilde{R}^{-1}r_i = 2x \cdot C^{-1}R^{-1}r_i = 2xc_i^{-1}e_i = 2xc_i^{-1} = 0.
\]

Here, \( C = \text{diag}(c_1, \ldots, c_d), \) \( e_i \) is the \( i \)th standard unit vector in \( \mathbb{R}^d. \) Since \( L_i(t) \) increases only when \( Z_i(t) = 0, \) we have:

\[
D_iF(Z(t)) \, dL_i(t) = 0 \quad \text{for all} \quad i = 1, \ldots, d.
\]

Also,

\[
\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij} \frac{\partial^2 F(Z(t))}{\partial x_i \partial x_j} = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij} 2(\tilde{R}^{-1})_{ij} = \text{tr}(A\tilde{R}^{-1}).
\]

Finally, consider the local martingale \( M = (M(t), t \geq 0) \), where

\[
M(t) = \int_0^t DF(Z(s)) \cdot dB(s) = \sum_{i=1}^d \int_0^t \frac{\partial F(Z(s))}{\partial x_i} \, dB_i(s) = \sum_{i=1}^d \int_0^t 2(\tilde{R}^{-1}Z(s))_i \, dB_i(s).
\]
Let $M_i$ denote the $i$th summand in the above formula. We have:
\[
\begin{align*}
    d < M_i >_t & := \sum_{i=1}^{d} \sum_{j=1}^{d} d < M_i, M_j >_t = 4 \sum_{i=1}^{d} \sum_{j=1}^{d} (\tilde{R}^{-1} Z(s))_i (\tilde{R}^{-1} Z(s))_j d < B_i, B_j >_s \\
    &= 4 \sum_{i=1}^{d} \sum_{j=1}^{d} (\tilde{R}^{-1} Z(s))_i (\tilde{R}^{-1} Z(s))_j a_{ij} d t \\
    &= 4 (\tilde{R}^{-1} Z(s))^T A (\tilde{R}^{-1} Z(s)) d s \\
    &= 4 Z^T(s) \tilde{R}^{-1} A \tilde{R}^{-1} Z(s) d s.
\end{align*}
\]

We have the following equation:
\[
F(Z(t)) = F(Z(0)) + tr(\tilde{A} \tilde{R}^{-1} t + M(t), t \geq 0.
\]

Let $\tau(s) := \inf\{t \geq 0 | < M >_t \geq s\}, s \geq 0$. Then we have:
\[
d\tau(s) = \frac{1}{4Z^T(\tau(s)) \tilde{R}^{-1} A \tilde{R}^{-1} Z(\tau(s))} d s.
\]

Let $K(s) = F(Z(\tau(s)))$. Then
\[
K(s) = K(0) + \hat{B}(s) + tr(\tilde{A} \tilde{R}^{-1}) \tau(s),
\]
so
\[
dK(s) = d\hat{B}(s) + \frac{tr(\tilde{A} \tilde{R}^{-1})}{4Z^T(\tau(s)) \tilde{R}^{-1} A \tilde{R}^{-1} Z(\tau(s))} d s.
\]

Let
\[
c_- := \min_{i=1, \ldots, d} \frac{(\tilde{R}^{-1} A \tilde{R}^{-1})_i}{(\tilde{R}^{-1})_i} = \inf_{z \in S \setminus \{0\}} \frac{z^T \tilde{R}^{-1} A \tilde{R}^{-1} z}{z^T \tilde{R}^{-1} z},
\]
\[
c_+ := \max_{i=1, \ldots, d} \frac{(\tilde{R}^{-1} A \tilde{R}^{-1})_i}{(\tilde{R}^{-1})_i} = \sup_{z \in S \setminus \{0\}} \frac{z^T \tilde{R}^{-1} A \tilde{R}^{-1} z}{z^T \tilde{R}^{-1} z}.
\]

If $\text{tr}(\tilde{R}^{-1} A) \geq 2c_+$, then for $x \in S \setminus \{0\}$ we have:
\[
\frac{\text{tr}(\tilde{A} \tilde{R}^{-1})}{4x^T \tilde{R}^{-1} A \tilde{R}^{-1} x} \geq \frac{\text{tr}(\tilde{A} \tilde{R}^{-1})}{4c_+ x \tilde{R}^{-1} x} \geq \frac{1}{2x^T \tilde{R}^{-1} x}.
\]

Therefore, $K(s) \geq K'(s)$, where the process $K' = (K'(s), s \geq 0)$ satisfies the following equation:
\[
dK'(s) = d\hat{B}(s) + \frac{1}{2K'(s)}.
\]

This is the Bessel process of dimension two, and it does not hit zero. Therefore, the processes $K(s)$ and $F(Z(t))$ do not hit zero, and $Z(t)$ does not hit the corner (because $x^T \tilde{R}^{-1} x = 0 \iff x = 0$ for $x \in S$). The proof of the second statement of the theorem is similar. \(\square\)

4.2. **Connection between hitting corners and hitting edges.** The next proposition is a generalization of [12] Lemma 6. It is proved similarly.

**Theorem 4.2.** Consider an SRBM$^d(R, \mu, A)$ with a reflection nonsingular $M$-matrix $R$. Fix $K = 2, \ldots, d$. If for every subset $I \subseteq \{1, \ldots, d\}$ with $|I| \geq K$ we have: SRBM$^{|I|}([R]_I, [\mu]_I, [A]_I)$ a.s. does not hit the corner, then SRBM$^d(R, \mu, A)$ a.s. does not hit edges of order $K$.

**Proof.** Induction by $d \geq K$. Base: $d = K$, the statement is trivial. Induction step: suppose the statement is true for $d - 1$ instead of $d$; let us show this statement for the given value of $d$. Assume the process $Z = \text{SRBM}^d(R, \mu, A)$ has the following property: for every subset $I \subseteq \{1, \ldots, d\}$ with $|I| \geq K$ the process SRBM$^{|I|}([R]_I, [\mu]_I, [A]_I)$ does not hit the corner.
Suppose \( \tau \) is the moment when \( Z = \text{SRBM}^d(R, \mu, A) \) hits edges of order \( K \). We shall prove that \( \tau = \infty \). For every \( \varepsilon > 0 \), let \( K_\varepsilon := \{ x \in S \mid \varepsilon \leq \|x\| \leq \varepsilon^{-1} \} \). Note that \( K_\varepsilon \uparrow S \setminus \{0\} \) as \( \varepsilon \downarrow 0 \). Let \( \eta_\varepsilon \) be the exit moment of \( Z \) from \( K_\varepsilon \). Since \( Z \) does not hit the origin, we have: \( \eta_\varepsilon \to \infty \). If suffices to show that \( \tau \geq \eta_\varepsilon \), since this would immediately imply \( \tau = \infty \).

For every \( x \in K_\varepsilon \), there exists an open ball \( U(x) \) such that for some coordinate \( j(x) = 1, \ldots, d \), we have: \( y_{j(x)} > 0 \) for all \( y \in U(x) \). The open sets \( \{ U(x) \}_{x \in K_\varepsilon} \) form an open cover of \( K_\varepsilon \); we can extract a finite subcover \( U(y_j), j = 1, \ldots, N \), because \( K_\varepsilon \) is a compact set.

Let \( I(x) := \{1, \ldots, d\} \setminus \{j(x)\} \). While \( Z \) is in \( U(x) \), we have:

\[
Z' = [Z]_{I(x)} = \text{SRBM}^{d-1}(\{R\}_{I(x)}, [\mu]_{I(x)}, [A]_{I(x)}).
\]

This process does not hit the corner, and for every subset \( I' \subseteq I(x) \) of \( K \) or more elements we have: the process \( \text{SRBM}^{d'}([R]_{I(x)}), [\mu]_{I(x)}, [A]_{I(x)} \) does not hit the corner. By the induction hypothesis, the process \( \text{SRBM}^{d-1}(\{R\}_{I(x)}, [\mu]_{I(x)}, [A]_{I(x)}) \) does not hit edges of order \( K \). In other words, the process \( [Z]_{I(x)} \) does not hit edges of order \( K \) while \( Z(t) \in U(x) \). Since \( Z_t(x)(t) > 0 \) while \( Z(t) \in U(x) \), the process \( Z(t) \) also does not hit edges of order \( K \). More precisely, we can construct a sequence \( \tau_0 := 0 \leq \tau_1 \leq \tau_2 \leq \ldots \) of stopping times such that for every \( k \geq 1 \), for all \( t \in [\tau_{k-1}, \tau_k) \), the process \( Z(t) \) lies in a certain \( U(y_j) \), and it does not hit edges of order \( K \). Therefore, \( \tau \geq \tau_k \). But \( \tau_k \uparrow \eta_\varepsilon \), so \( \tau \geq \eta_\varepsilon \). The proof is complete. \( \square \)

**Lemma 4.3.** Consider an \( \text{SRBM}^d(R, \mu, A) \) with reflection nonsingular \( M \)-matrix \( R \). Take a subset \( I \subseteq \{1, \ldots, d\} \).

(i) If an \( Z' = \text{SRBM}^{|I|}(\{R\}_{I}, [\mu]_{I}, [A]_{I}) \) hits edges of order \( K \) with positive probability, then \( Z = \text{SRBM}^d(R, \mu, A) \) hits edges of order \( K \) with positive probability.

(ii) Fix a subset \( J \subseteq I \). If with positive probability there exists \( t > 0 \) such that \( Z'_j(t) = 0 \) for \( j \in J \), then with positive probability there exists \( t > 0 \) such that \( Z_j(t) = 0 \) for \( j \in J \).

**Proof.** This follows from Lemma 2.7: \( [Z]_{I} \leq Z' \), where

\[
Z = \text{SRBM}^d(R, \mu, A), \ Z' = \text{SRBM}^{|I|}(\{R\}_{I}, [\mu]_{I}, [A]_{I}).
\]

\( \square \)

5. Multiple Collisions of Competing Brownian Particles

5.1. **Reduction of multiple collisions to total collisions.** Consider a system of \( N \) competing Brownian particles with symmetric or asymmetric collisions. Suppose that the \( k \)th largest particle moves as a Brownian motion with drift \( g_k \) and variance \( \sigma_k^2 \), where \( k = 1, \ldots, N \). Suppose that \( a_k^\pm \), \( k = 1, \ldots, N \) are parameters of collision. Let \( Y_1, \ldots, Y_N \) be the ranked particles. Take a discrete interval \( I = \{l+1, \ldots, m\} \subseteq \{1, \ldots, N\} \). Then a subsystem of competing Brownian particles corresponding to the interval \( I \) is, by definition, a system of \( |I| = m-l \) competing Brownian particles such that the \( k \)th largest particle moves as a Brownian motion with drift \( g_{k+l} \) and variance \( \sigma_{k+l}^2 \), \( k = 1, \ldots, m-l \), and the parameters of collision are \( a_k^\pm \), \( k = 1, \ldots, m-l \).

**Lemma 5.1.** Consider a system of \( N \) competing Brownian particles with symmetric or asymmetric collisions. Suppose that every subsystem corresponding to a connected discrete interval \( I \subseteq \{1, \ldots, N\} \) of \( K \) or more elements a.s. does not have total collisions. Then the original system a.s. does not have collisions of order \( K \), and a.s. does not have simultaneous collisions of order \( K \).

**Proof.** Let \( Z = \text{SRBM}^{N-1}(R, \mu, A) \) be the gap process of the original system. Then the gap process of the subsystem corresponding to a discrete interval \( I \subseteq \{1, \ldots, N\} \) is given by \( \text{SRBM}^{N'}([R]_{I'}, [\mu]_{I'}, [A]_{I'}) \), where \( I' = I \setminus \{\max I\} \). For example, if \( I = \{l+1, \ldots, m\} \), then \( I' = \{l+1, \ldots, m-1\} \). The total collision corresponds to the gap process hitting the corner; it suffices to apply Theorem 4.2. \( \square \)
Lemma 5.2. Consider a system of $N$ competing Brownian particles with symmetric or asymmetric collisions. (i) Suppose that a subsystem corresponding to a discrete interval $I \subseteq \{1, \ldots, N\}$ has collisions of order $K$ with positive probability. Then the original system also has collisions of order $K$ with positive probability.

(ii) If the subsystem mentioned above has simultaneous collisions of order $K$ with positive probability, then the original system also has simultaneous collisions of order $K$ with positive probability.

Proof. Follows from Lemma 2.2 and Lemma 4.3.

5.2. Calculation for symmetric collisions. Now, let us write the condition (9) and (10) for the case of $N$ competing Brownian particles with symmetric collisions. By Corollary 2.5, we can remove drifts. The matrices $R$ and $A$ are given by (5) and (6).

Theorem 5.3. For the gap process of systems of CBP with symmetric collisions, the condition (10) takes the following form:

\[
\frac{N-1}{2} \sum_{k=1}^{N} \sigma_k^2 \geq \max_{k=1, \ldots, N} \left[ \frac{N-k}{k} (\sigma_1^2 + \ldots + \sigma_k^2) + \frac{k}{N-k} (\sigma_{k+1}^2 + \ldots + \sigma_N^2) \right].
\]

The condition (9) takes the following form:

\[
\frac{N-1}{2} \sum_{k=1}^{N} \sigma_k^2 < \min_{k=1, \ldots, N} \left[ \frac{N-k}{k} (\sigma_1^2 + \ldots + \sigma_k^2) + \frac{k}{N-k} (\sigma_{k+1}^2 + \ldots + \sigma_N^2) \right].
\]

Proof. Since the matrix $R$ from (5) is already symmetric, we have: $C = I_{N-1}$, $\tilde{R} = R$. The inverse matrix $R^{-1} = \tilde{R}^{-1}$ has the form

\[
(R^{-1})_{ij} = \begin{cases} 
\frac{2i(N-j)}{N}, & i \leq j; \\
\frac{2j(N-i)}{N}, & i \geq j
\end{cases}
\]

This can be found in [8] or [22] (the latter article deals with a slightly different matrix, from which one can easily find the inverse of the given matrix $R$). Therefore,

\[
\text{tr}(R^{-1}A) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (R^{-1})_{ij} a_{ij} = \sum_{i=1}^{N-1} (\sigma_i^2 + \sigma_{i+1}^2) \frac{2i(N-i)}{N} + \frac{2(1-N)}{N} \sum_{i=2}^{N-1} (-\sigma_i^2) 2(i-1)(N-i) + \frac{2(N-1)}{N} \sum_{i=2}^{N-1} (-\sigma_i^2) 2(i-1)(N-i) \frac{2N-2}{N}
\]

In addition, for $k = 1, \ldots, N-1$, we have:

\[
(R^{-1}AR^{-1})_{kk} = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (R^{-1})_{ik}(R^{-1})_{jk}A_{ij} = \sum_{i=1}^{N-1} (R^{-1})_{ik}^2 (\sigma_i^2 + \sigma_{i+1}^2) - 2 \sum_{i=2}^{N-1} (R^{-1})_{ik}(R^{-1})_{i-1,k} \sigma_i^2.
\]

The coefficient near $\sigma_i^2$ is equal to

\[
(R^{-1})_{ik}^2 + (R^{-1})_{i-1,k}^2 - 2(R^{-1})_{ik}(R^{-1})_{i-1,k} = [(R^{-1})_{ik} - (R^{-1})_{i-1,k}]^2.
\]

Here, we take $(R^{-1})_{0k} = (R^{-1})_{Nk} = 0$ for simplicity of notation. If $i > k$, then

\[
(R^{-1})_{ik} = \frac{2k(N-i)}{N}, \quad (R^{-1})_{i-1,k} = \frac{2k(N-i+1)}{N}.
\]
This is consistent with our convention that \((R^{-1})_{Nk} = 0\). Therefore, the coefficient is equal to 
\[
\frac{2k(N-i)}{N} - \frac{2k(N-i+1)}{N} = \frac{4k^2}{N^2}.
\]

Similarly, if \(i \leq k\), then 
\[
(R^{-1})_{ik} = \frac{2i(N-k)}{N}, \quad (R^{-1})_{i-1,k} = \frac{2(i-1)(N-k)}{N}.
\]

This is consistent with our convention that \((R^{-1})_{0k} = 0\). Therefore, the coefficient is equal to 
\[
\left( \frac{2i(N-k)}{N} - \frac{2(i-1)(N-k)}{N} \right)^2 = \frac{4(N-k)^2}{N^2}.
\]

We can write it as 
\[
(R^{-1}AR^{-1})_{kk} = \frac{4(N-k)^2}{N^2} \left( \sigma_1^2 + \ldots + \sigma_k^2 \right) + \frac{4k^2}{N^2} \left( \sigma_{k+1}^2 + \ldots + \sigma_N^2 \right).
\]

Therefore, 
\[
\max_{k=1, \ldots, N-1} \frac{(R^{-1}AR^{-1})_{kk}}{(R^{-1})_{kk}} = \max_{k=1, \ldots, N-1} \frac{2(N-k)}{Nk} \left( \sigma_1^2 + \ldots + \sigma_k^2 \right) + \frac{2k}{N(N-k)} \left( \sigma_{k+1}^2 + \ldots + \sigma_N^2 \right).
\]

The rest of the proof is trivial. \(\square\)

**Example 1.** Consider a system of three particles, \(N = 3\), the condition (\ref{III}) takes the form 
\[
\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \geq \max(2\sigma_1^2 + \frac{1}{2}(\sigma_2^2 + \sigma_3^2), 2\sigma_2^2 + \frac{1}{2}(\sigma_1^2 + \sigma_3^2)),
\]
which is equivalent to 
\[
\sigma_2^2 + \sigma_3^2 \geq 2\sigma_1^2, \quad \sigma_1^2 + \sigma_2^2 \geq 2\sigma_3^2.
\]

This condition is sufficient for absence of triple collisions for the system of three particles. It is stronger than \(\sigma_3^2 - \sigma_2^2 \leq \sigma_2^2 + \sigma_1^2\), which is a necessary and sufficient condition for absence of triple collisions, proved in Theorem (\ref{1.2}). The condition (\ref{12}) is equivalent to 
\[
\sigma_2^2 + \sigma_3^2 < 2\sigma_1^2, \quad \sigma_1^2 + \sigma_2^2 < 2\sigma_3^2.
\]

This is weaker than \(\sigma_3^2 - \sigma_2^2 > \sigma_2^2 + \sigma_1^2\), which is a necessary and sufficient condition for existence of triple collisions with positive probability, proved in Theorem (\ref{1.2}).

**Example 2.** For \(N \geq 4\), we always have: for \(i = 2, \ldots, N-1\),
\[
(13) \quad \frac{N-1}{2} \sum_{k=1}^{N} \sigma_k^2 \geq \frac{N-i}{i} (\sigma_1^2 + \ldots + \sigma_i^2) + \frac{i}{N-i} (\sigma_{i+1}^2 + \ldots + \sigma_N^2).
\]

So the condition (\ref{12}) is never true.

5.3. **Proof of Theorem (\ref{1.3}).** From (\ref{13}), we get: the condition (\ref{III}) is equivalent to
\[
(14) \quad \frac{N-1}{2} \sum_{k=1}^{N} \sigma_k^2 \geq \max \left[ (N-1)\sigma_1^2 + \frac{1}{N-1}(\sigma_2^2 + \ldots + \sigma_N^2), (N-1)\sigma_N^2 + \frac{1}{N-1}(\sigma_1^2 + \ldots + \sigma_{N-1}^2) \right].
\]

We can rewrite it as 
\[
\frac{N-1}{2} \sigma_1^2 \leq \left( \frac{N-1}{2} - \frac{1}{N-1} \right) (\sigma_2^2 + \ldots + \sigma_N^2),
\]
\[
\frac{N-1}{2} \sigma_N^2 \leq \left( \frac{N-1}{2} - \frac{1}{N-1} \right) (\sigma_1^2 + \ldots + \sigma_{N-1}^2).
\]
This is a sufficient condition for the system of $N$ competing Brownian particles to avoid corners. Now, the final step: consider the system of $M > N$ competing Brownian particles with variances $\sigma_1^2, \ldots, \sigma_M^2$. Suppose that the condition (14) holds for every subsystem of $N$ particles, which is equivalent to (2) and (3). Then the condition (14) holds for every subsystem of $N' \geq N$ particles. The proof of this is straightforward. Now we can use Lemma 5.2 to finish the proof of Theorem 1.3.

**Corollary 5.4.** Consider a system of $N$ competing Brownian particles with symmetric collisions. The following is a sufficient condition for this system to avoid collisions of order 4: for $k = 1, \ldots, N - 3$,

$$\sigma_k^2 \leq \frac{7}{9}(\sigma_{k+1}^2 + \sigma_{k+2}^2 + \sigma_{k+3}^2), \quad \sigma_{k+3}^2 \leq \frac{7}{9}(\sigma_k^2 + \sigma_{k+1}^2 + \sigma_{k+2}^2).$$

**Example 3.** Take a system of four competing Brownian particles with symmetric collisions and with $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$, $\sigma_4^2 = 10$. Then it has triple collisions between the particles with ranks 2, 3, 4 with positive probability, since $\sigma_2^2 - \sigma_3^2 > \sigma_3^2 - \sigma_4^2$. But it does not have total collisions, since

$$\sigma_1^2 \leq \frac{7}{9}(\sigma_2^2 + \sigma_3^2 + \sigma_4^2), \quad \sigma_4^2 \leq \frac{7}{9}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2).$$

**Appendix**

The most important case for applications is when the reflection matrix is a nonsingular $M$-matrix. This is a characterization of when a reflection matrix is a nonsingular $M$-matrix. On this topic, see also [3, Chapter 6], [32, Chapter 8], [2, 21, Section 2.5].

**Lemma 5.5.** For a $d \times d$-matrix $R = (r_{ij})$ with $r_{ii} = 1$, $i = 1, \ldots, d$, the following statements are equivalent: (i) The matrix $R$ is a nonsingular $M$-matrix.

(ii) The matrix $R$ is a $Z$-matrix which is strictly inverse-nonnegative.

(iii) The matrix $R$ can be written as $R = I_d - Q$, where $Q$ is a nonnegative matrix with zeros on the main diagonal and spectral radius less than one.

**Proof.** (i) $\Rightarrow$ (iii). Use [21, Theorem 2.5.3]. Since $R$ is completely-$S$, it satisfies condition 12 from this theorem. Therefore, it satisfies condition 2. We have the representation $R = \alpha I_d - Q$, where $\alpha = \max_{1 \leq i \leq d} r_{ii} = 1$, and a $d \times d$-matrix $Q$ is nonnegative with spectral radius less than one. (See the beginning of [21, Section 2.5.4].)

(iii) $\Rightarrow$ (ii). By [32, Section 7.10], $R^{-1} = I_d + Q + Q^2 + \ldots \geq 0$, and the diagonal elements of $R^{-1}$ are strictly positive (and even greater than or equal to 1).

(ii) $\Rightarrow$ (i). Again apply [21, Theorem 2.5.3]: condition 17 implies condition 12. There exists $x > 0$ such that $Rx > 0$. Take a principal submatrix $\tilde{R}$ of $R$. Without loss of generality, assume this is an upper left corner $d' \times d'$-submatrix. Let $\tilde{x}$ consist of the first $d'$ components of $x$. Then

$$(\tilde{R}\tilde{x})_i = \sum_{j=1}^{d'} r_{ij} x_j \geq \sum_{i=1}^{d} r_{ij} x_j = (Rx)_i > 0, \quad i = 1, \ldots, d'.$$

Therefore, $\tilde{x} > 0$ and $\tilde{R}\tilde{x} > 0$. So every principal submatrix of $R$ is an $S$-matrix, which proves that $R$ is completely-$S$. $\square$

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