A REASONABLE NOTION OF DIMENSION FOR SINGULAR INTERSECTION HOMOLOGY

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Abstract. M. Goresky and R. MacPherson intersection homology is also defined from the singular chain complex of a filtered space by H. King, with a key formula to make selections among singular simplexes. This formula needs a notion of dimension for subspaces $S$ of an Euclidean simplex, which is usually taken as the smallest dimension of the skeleta containing $S$. Later, P. Gajer employed another dimension based on the dimension of polyhedra containing $S$. This last one allows traces of pullbacks of singular strata in the interior of the domain of a singular simplex.

In this work, we prove that the two corresponding intersection homologies are isomorphic for Siebenmann’s CS sets. In terms of King’s paper, this means that polyhedral dimension is a “reasonable” dimension. The proof uses a Mayer-Vietoris argument which needs an adapted subdivision. With the polyhedral dimension, that is a subtle issue. General position arguments are not sufficient and we introduce strong general position. With it, a stability is added to the generic character and we can do an inductive cutting of each singular simplex. This decomposition is realised with pseudo-barycentric subdivisions where the new vertices are not barycentres but close points of them.

Contents

Introduction 1
1. Some reminders 3
2. Intersection homology 5
3. Strong general position 8
4. Pseudo-barycentric subdivisions 13
5. Mayer-Vietoris sequence 16
6. Intersection homology is intersection homology 20
Acknowledgment 22
References 22

Introduction

Intersection homology was introduced by Goresky and MacPherson to restore Poincaré duality for some singular spaces called pseudomanifolds. They first defined it for PL-pseudomanifolds in [9] and extended it to topological pseudomanifolds in [10] by using a derived category of complexes of sheaves. A presentation of intersection homology directly from the singular chain complex of a filtered space and its concordance with the initial definition is made by H. King in [11] for CS...
sets (Definition 1.4). The starting point is a selection of some specific singular simplexes, called \(\mathcal{P}\)-allowable, from a perversity introduced as a sequence of integers \(\mathcal{P}(n)\) (cf. Definition 1.6) or as a map \(\mathcal{P}: SX \to \mathbb{Z}\), defined on the set of strata of \(X\) and taking the value 0 on the regular strata (cf. Definition 1.5). Before stating it formally in Definition 2.2, let’s contemplate the key formula for a stratum \(S\) and a simplex \(\sigma: \Delta^\ell \to X\),

\[
\dim \sigma^{-1} S \leq \dim \Delta - \text{codim } S + \mathcal{P}(S).
\]

(0.1)

For regular strata, this inequality reduces to \(\dim \sigma^{-1} S \leq \dim \Delta\) which is always true for any notion of dimension compatible with the inclusions of sets. So we can restrict ourselves to the singular strata. Notice that (0.1) contains three different kinds of dimension:

- \(\dim \Delta\) refers to the classical dimension of an Euclidean simplex;
- \(\text{codim } S\) comes from the filtered dimension of \(X\) as in Definition 1.1;
- \(\dim \sigma^{-1} S\) has to be specified.

Originally in [11], King replaces (0.1) by “\(\sigma^{-1} S\) is included in the \((\dim \Delta - \text{codim } S + \mathcal{P}(S))\)-skeleton of \(\Delta\)”; i.e., the dimension of \(\sigma^{-1} S\) comes from the skeleton of \(\Delta\). This definition works perfectly and this is a common choice for many works in intersection homology, see the book of G. Friedman (6) for an extensive repertoire of its properties. It is easy to notice that, for a simplex \(\Delta\) of dimension \(\ell \geq 2\) and a perversity \(\mathcal{P} \leq 7\), the set \(\sigma^{-1} S\) is included in the \((\ell - 2)\)-skeleton of \(\Delta\). In \([2, 4]\), another chain complex is built from singular simplexes \(\sigma: \Delta \to X\), with a filtration \(\Delta = \Delta_0 \ast \cdots \ast \Delta_n\) verifying \(\sigma^{-1} X_i = \Delta_0 \ast \cdots \ast \Delta_i\). They form a simplicial set denoted by Sing\(^p\)\(X\) that is not a Kan complex in general and allow a blown-up cohomology ([11]) which gives a Poincaré duality ([3]) with cap products. For any of these two choices, all the information on the singular strata are contained in the boundary of \(\Delta\). This is not suitable if we are looking for a notion of intersection homotopy groups. To achieve a presentation of intersection homology which fits with homotopy groups, Gajer uses in [7] a notion of dimension which allows traces of the pull-back of the singular strata in the interior of the singular simplexes. For that, one says that a subspace of an Euclidean simplex, \(A \subset \Delta\), is of dimension less than or equal to \(\ell\) if \(A\) is included in a polyhedron \(Q\) with \(\dim Q \leq \ell\). Call it the polyhedral dimension.

Here, a \(\mathcal{P}\)-allowable simplex is a singular simplex verifying (0.1) with the polyhedral dimension for \(\sigma^{-1} S\). A \(\mathcal{P}\)-allowable chain is a linear combination of \(\mathcal{P}\)-allowable simplexes and the \(\mathcal{P}\)-intersection chain complex is formed of chains \(\xi\) such that \(\xi\) and its boundary \(\partial \xi\) are \(\mathcal{P}\)-allowable chains. In this work, we prove that the homology of this complex coincides with the initial intersection homology, for Siebenmann’s CS sets (Definition 1.4). To do this, we use a method initiated by King (see Theorem 6.3) who gives a series of properties which guarantee such identification. On this problem, King wrote if one changes the above definition to any reasonable notion of dimension, one ends up with the same intersection homology groups. Taking this sentence as a definition for “reasonable” and the construction of King as a reference, we can state the main result of this work (Theorem 6.2) as follows.

**Main Theorem.** The notion of polyhedral dimension is reasonable.

In the proof of the Main Theorem, a “sensitive issue” is the existence of subdivisions for having a Mayer-Vietoris sequence. In classical singular homology, they are the barycentric subdivisions which induce a chain map homotopic to the identity. In intersection homology, we have to prove the existence of subdivisions keeping the \(\mathcal{P}\)-allowability property. For our present situation, this is a subtle issue. To illustrate it, consider a filtered space, \(X\), of singular set \(\Sigma\), and a singular simplex \(\sigma: \Delta^2 \to X\) such that \(\sigma^{-1} \Sigma = b\sigma\), where \(b\sigma\) is the barycentre of \(\Delta^2\). The simplex \(\sigma\) is \(\mathcal{P}\)-allowable for the top perversity \(\mathcal{P}\). But in the barycentric subdivision, the edges containing \(b\sigma\) are no more of \(\mathcal{P}\)-intersection. To avoid this phenomenon, we act in an inductive way, as in the barycentric subdivision, but now the new vertices are not necessarily the barycentres but close points to
them. How to choose these points such that any element of this pseudo-barycentric subdivision is a $\mathcal{P}$-allowable simplex? At each step, we have two simplexes and we slightly move one of them to ensure that the new simplexes are in general position. We can realize this operation if our process is generic. But, at the same time, we need that the simplexes of the previous steps remain in general position. This is a stability condition. The classical notion of general position does not meet these requirements (Example 3.2) and we introduce the notion of strong general position in Definition 3.6. Such subdivisions are called pseudo-barycentric subdivisions. In Propositions 4.4, 4.5 and 4.8, we show that pseudo-barycentric subdivisions preserving the $\mathcal{P}$-allowability exist, that they induce a chain map homotopic to the identity and give a small simplexes $\mathcal{P}$-intersection theorem.

The preservation of $\mathcal{P}$-allowability is a significative step but we also have to take care of $\mathcal{P}$-intersection chains. For that, we prove that the allowability default of the boundary of an allowable simplex is characterized by the existence of one particular face, that we call the bad face. The final stage for the existence of a Mayer-Vietoris sequence is a characterisation of these bad faces in the case of a pseudo-barycentric subdivision, cf Proposition 5.5. The architecture of the rest of the proof comes from King’s original theorem ([11, Theorem 10]). The properties are now in place and the verification of the hypotheses of this theorem follows a classic pattern.

This is time to warn the reader on a possible confusion: the main part of [7] is devoted to the definition and investigation of a simplicial set $\mathcal{I}_X$ associated to a filtered space and a perversity. It is known ([8]) that the homology of $\mathcal{I}_X$ is not the $\mathcal{P}$-intersection homology of $X$. The simplicial set $\mathcal{I}_X$ does not appear in this work. We study it in [5] with a particular attention to its homotopy groups, the possibility of their topological invariance in terms of the space $X$ and a Hurewicz theorem between them and $\mathcal{P}$-intersection homology.

Outline of the paper. Section 1 is a recall of the notions of filtered spaces, CS sets and perversities. Intersection homology defined from the polyhedral dimension is presented in Section 2. This section also contains the computation for the cone and the compatibility with the relation of homotopy. Strong general position is studied in Section 3 and applied to the existence of pseudo-barycentric subdivisions in Section 4. The existence of Mayer-Vietoris exact sequences is established in Section 5. Finally, the Main Theorem is proven in Section 6.

Notation. In the text, the letter $R$ denotes a Dedekind ring and $G$ an $R$-module. The singular chain complex of a topological space, $X$, is denoted by $C_*(X;G)$, or $C_*(X)$ if there is no ambiguity. The domain of a simplex $\sigma \in \text{Sing } X$ is denoted by $\Delta^\ell$, where $\ell$ is its dimension, or $\Delta_\sigma$, or simply $\Delta$, depending on the parameter concerned at the place where it is used. We denote by $\Delta = \Delta \setminus \partial \Delta$ the interior of $\Delta$. The notation $V \preceq W$ means that $V$ is a face of a polyhedron $W$. The word “space” means topological space.

1. Some reminders

Definition 1.1. A filtered space is a Hausdorff space, $X$, endowed with a filtration by closed subspaces,

$$X_0 \subseteq X_1 \subseteq \ldots X_{n-1} \subseteq X_n = X.$$  

The dimension of $X$ is denoted by $\dim X = n$. The connected components, $S$, of $X_i \setminus X_{i-1}$ are the strata of $X$ and we write $\dim S = i$ and $\text{codim } S = \dim X - \dim S$. (These dimensions can be formal and not necessarily related to a notion of geometrical dimension.)

The strata of $X_n \setminus X_{n-1}$ are regular strata; the other ones are singular strata. The family of non-empty strata is denoted by $\Delta_X$ (or $\mathcal{S}$ if there is no ambiguity). The subspace $\Sigma X = X_{n-1}$ is the singular set, sometimes also denoted by $\Sigma$. Its complementary subset $X \setminus \Sigma X$ is called the regular subspace. The filtered space is said of locally finite stratification if every point has a neighborhood that intersects only a finite number of strata.
An open subset $U \subset X$ is a filtered space for the **induced filtration** given by $U_i = U \cap X_i$. The product $M \times X$ with a topological space is a filtered space for the **product filtration** defined by $(M \times X)_i = M \times X_i$. If $X$ is compact, the open cone $\hat{c}X = X \times [0,1]/X \times \{0\}$ is endowed with the **conical filtration** defined by $(\hat{c}X)_i = \hat{c}X_{i-1}$, $0 \leq i \leq n+1$. By convention, $\hat{c}0 = \{v\}$, where $v = [-,0]$ is the apex of the cone.

**Definition 1.2.** A **stratified map**, $f : X \to Y$, is a continuous map between filtered spaces such that, for each stratum $S \in S_X$, there exists a unique stratum $S' \in S_Y$ with $f(S) \subset S'$ and $\text{codim} S' \leq \text{codim} S$.

A continuous map $f : X \to Y$ is stratified if, and only if, the pull-back of a stratum $S' \in S_Y$ is empty or a union of strata of $X$, $f^{-1}(S') = \cup_{i \in I} S_{i}$, with $\text{codim} S' \leq \text{codim} S_i$ for each $i \in I$.

Thus, a stratified map sends a regular stratum in a regular one but the image of a singular stratum can be included in a regular one.

Let $X$ be a filtered space. The canonical injection of an open subset $U \hookrightarrow X$, the canonical projection, $\pi : M \times X \to X$, the maps $i_m : X \to M \times X$ with $x \mapsto (m, x)$, $m \in M$, $i_t : X \to \hat{c}X$ with $x \mapsto [x, t]$, $t \neq 0$, are stratified for the filtered structures described above. In the following definition, the product $X \times [0,1]$ is endowed with the product filtration.

**Definition 1.3.** Two stratified maps $f, g : X \to Y$ are **homotopic** if there exists a stratified map, $\varphi : X \times [0,1] \to Y$, such that $\varphi(-,0) = f$ and $\varphi(-,1) = g$. Homotopy is an equivalence relation and produces the notion of homotopy equivalence between filtered spaces.

We present now a definition of the CS sets of Siebenman, [17], in which the links of singular points are not necessarily CS sets but only supposed to be non-empty filtered spaces. With this definition, the regular subspace $X \setminus \Sigma$ is a dense open subset of $X$ and the stratification is locally finite.

**Definition 1.4.** A **CS set** of dimension $n$ is a filtered space,

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} \subset X_n = X,$$

such that, for each $i$, $X_i \setminus X_{i-1}$ is a topological manifold of dimension $i$ or the empty set. Moreover, for each point $x \in X_i \setminus X_{i-1}$, $i \neq n$, there exist

(i) an open neighborhood $V$ of $x$ in $X$, endowed with the induced filtration,

(ii) an open neighborhood $U \subset V$ of $x$ in $X_i \setminus X_{i-1}$,

(iii) a compact filtered space, $L$, of dimension $n-i-1$, where the open cone, $\hat{c}L$, is provided with the conical filtration, $(\hat{c}L)_i = \hat{c}L_{i-1}$,

(iv) a homeomorphism, $\varphi : U \times \hat{c}L \to V$, such that

(a) $\varphi(u, v) = u$, for each $u \in U$,

(b) $\varphi(U \times L_j) = V \cap X_{i+j+1}$, for each $j \in \{0, \ldots, n-i-1\}$.

The pair $(V, \varphi)$ is a **conical chart** of $x$ and the filtered space $L$ is a **link** of $x$. The CS set $X$ is called **normal** if its links are connected.

Perversity is the main ingredient in intersection homology, which allows a “control of the deviation of transversality” between simplexes and strata. The original perversities introduced by Goresky and MacPherson in [9] depend only on the codimension of the strata. More general perversities ([12, 6, 14, 15]) are defined on the set of strata. They are used in [4] where we develop a topological invariance by refinement of the filtration, in [1] with a blown-up cohomology, and in [3, 16] with a development of Poincaré duality for pseudomanifolds with a cap product. We first review their definitions.

**Definition 1.5.** A **perversity on a filtered space**, $X$, is a map $\mathcal{P} : S_X \to \mathbb{Z} = \mathbb{Z} \cup \{\pm \infty\}$ taking the value $0$ on the regular strata. The pair $(X, \mathcal{P})$ is called a **perversive space**, or a **perversive CS set** if $X$ is a CS set.
A constant perversity $\overline{k}$, with $k \in \mathbb{Z}$, is defined by $\overline{k}(S) = k$ for any singular stratum $S$. The top perversity $\overline{t}$ is defined by $\overline{t}(S) = \text{codim } S - 2$, if $S$ is a singular stratum. Given a perversity $\overline{p}$ on $X$, the complementary perversity on $X$, $D\overline{p}$, is characterized by $D\overline{p} + \overline{p} = \overline{t}$.

Any map $f : N \to N$ such that $f(0) = 0$ defines a perversity $\overline{p}$ by $\overline{p}(S) = f(\text{codim } S)$. Such perversity is called codimensional. In general, we denote by the same letter the perversity $\overline{p}$ and the map $f$. The dual perversity of a codimensional perversity remains codimensional. Among the codimensional perversities we find the original perversities of [9].

**Definition 1.6.** A Goresky-MacPherson perversity (or GM-perversity) is a map $\overline{p} : \{2, 3, \ldots \} \to \mathbb{N}$ such that $\overline{p}(2) = 0$ and $\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1$ for all $i \geq 2$.

**Definition 1.7.** Let $f : X \to Y$ be a stratified map and $\overline{p}$ be a perversity on $Y$. The pull-back perversity of $\overline{p}$ by $f$ is the perversity $f^*\overline{p}$ on $X$ defined by $(f^*\overline{p})(S) = \overline{p}(S')$, for each $S \in \mathcal{S}_X$. For a canonical injection, $i$, we still denote by $\overline{p}$ the perversity $i^*\overline{p}$ and call it the induced perversity.

The product with a topological space, $X \times M$, is endowed with the pull-back perversity of $\overline{p}$ by the canonical projection $X \times M \to X$, also denoted by $\overline{p}$.

Let $X$ be compact. The strata of the cone $\mathcal{C}X$ are the singleton $\{v\}$ and the products $S \times [0, 1]$ with $S$ a stratum of $X$. A perversity $\overline{p}$ on the open cone, $\mathcal{C}X$, induces a perversity on $X$, also denoted by $\overline{p}$ and defined by $\overline{p}(S) = \overline{p}(S \times [0, 1])$.

### 2. Intersection homology

Let $(X, \overline{p})$ be an $n$-dimensional perverse space. Let us begin with some recalls from [13].

#### 2.1. Dimension of polyhedra.

By definition, a subset $P \subset \mathbb{R}^n$ is a polyhedron if each point $a \in P$ has a cone neighbourhood $a \cdot L \subset P$ with $L$ compact. A subpolyhedron $Q$ of $P$ is a subset $Q \subset P$ which is itself a polyhedron. Any polyhedron is a locally finite union of simplexes ([13, Theorem 2.2]), $P = \cup_j \Delta_j$, called a triangulation of $P$. If $Q$ is a subpolyhedron of $P$, then any triangulation of $Q$ can be extended to a triangulation of $P$. The existence of triangulations allows a definition of dimension.

**Definition 2.1.** The dimension of a polyhedron $P$ is given by $\dim P = \max \dim \Delta_j$ where the $\Delta_j$’s form a triangulation of $P$. (This notion does not depend on the choice of the triangulation.) A subspace $A \subset P$ of a polyhedron is of polyhedral dimension less than or equal to $\ell$ if $A$ is included in a polyhedron $Q$ with $\dim Q \leq \ell$. It is said of polyhedral dimension $k$ if $\dim A \leq k$ and $\dim A \not\leq k - 1$.

This definition verifies

$$\dim(A_1 \cup A_2) = \max(\dim A_1, \dim A_2). \quad (2.1)$$

With this definition, we do a selection among singular simplexes in the spirit of (0.1).

**Definition 2.2.** Let $(X, \overline{p})$ be a perverse space. A simplex $\sigma : \Delta \to X$ is $\overline{p}$-allowable if, for each singular stratum $S$, the set $\sigma^{-1}S$ verifies

$$\dim \sigma^{-1}S \leq \dim \Delta - \text{codim } S + \overline{p}(S) = \dim \Delta - 2 - D\overline{p}(S), \quad (2.2)$$

with the convention $\dim \emptyset = -\infty$. A singular chain $\xi$ is $\overline{p}$-allowable if it can be written as a linear combination of $\overline{p}$-allowable simplexes, and of $\overline{p}$-intersection if $\xi$ and its boundary $\partial \xi$ are $\overline{p}$-allowable. We denote by $C^*_\overline{p}(X; G)$ the complex of singular chains of $\overline{p}$-intersection and by $H^*_\overline{p}(X; G)$ its homology, called $\overline{p}$-intersection homology of $X$ with coefficients in a module $G$ over a Dedekind ring $R$. 
**Remark 2.3.** We specify the notion of $\overline{\sigma}$-allowable simplex, $\sigma: \Delta^k \to X$, for $k = 0, 1, 2$, in function of the value of the perversity $D\overline{\sigma}$ on each singular stratum. This simplex $\sigma$ is $\overline{\sigma}$-allowable if, for each singular stratum $S$, the following conditions hold:

| $k$ | $D\overline{\sigma}(S)$ | $-1$ | $0$ | $\geq 1$ |
|-----|-------------------------|-----|-----|---------|
| 0   | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ |
| 1   | $\sigma^{-1}(S)$ finite set | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ |
| 2   | $\dim \sigma^{-1}(S) \leq 1$ | $\sigma^{-1}(S)$ finite set | $\sigma^{-1}(S) = \emptyset$ | $\sigma^{-1}(S) = \emptyset$ |

For instance, if $D\overline{\sigma} \geq 0$, which is the case of the GM-perversities, a 0-simplex or a 1-simplex are $\overline{\sigma}$-allowable if, and only if, $\sigma^{-1}S = \emptyset$ for any singular stratum $S$: this means that they must lie in the regular part.

**Example 2.4.** Consider two 3-simplexes of the ambient stratified space $X = \mathbb{R}^3$, with two singular strata, $S_0 = \{x_0 = 0\}$, $S_1 = ]0, \infty[$ and a regular stratum $S_2 = \mathbb{R}^3 \setminus [0, \infty[$.

![Diagram of two 3-simplexes](image)

For the left-one, we have $\dim \sigma^{-1}S_0 = 0$, $\dim \Delta - 2 - D\overline{\sigma}(S_0) = 1 - D\overline{\sigma}(S_0)$, and $\dim \sigma^{-1}S_1 = 1$, $\dim \Delta - 2 - D\overline{\sigma}(S_1) = 1 - D\overline{\sigma}(S_1)$. This is a $\overline{\sigma}$-allowable simplex for any perversity $\overline{\sigma}$ such that $D\overline{\sigma}(S_0) \leq 1$ and $D\overline{\sigma}(S_1) \leq 0$, thus in particular for $D\overline{\sigma} \equiv 0$. The previous remark implies that the 1-simplex $[x_0, x_3]$ is not $\overline{\sigma}$-allowable if $D\overline{\sigma} \equiv 0$.

Let us continue with the right-one simplex. Here, we have $\sigma^{-1}S_0 = \emptyset$ and $\sigma^{-1}S_1 = 1$. We get a $\overline{\sigma}$-allowable simplex if $D\overline{\sigma} \leq 0$. We can also check that any $\ell$-face of $\Delta$, for $\ell \in \{0, 1, 2\}$, is a $\overline{\sigma}$-allowable simplex if $D\overline{\sigma} \equiv 0$.

2.2. **Chain maps induced by stratified maps.** We begin with the chain map induced by a stratified map and the compatibility of this association with homotopy. We apply it to the canonical projection $X \times \mathbb{R} \to X$.

**Proposition 2.5.** Let $f: (X, \overline{\sigma}) \to (Y, \overline{\tau})$ be a stratified map between two perverse spaces with $f^*D\overline{\tau} \leq D\overline{\sigma}$. The association $\sigma \mapsto f \circ \sigma$ sends a $\overline{\sigma}$-allowable simplex on a $\overline{\tau}$-allowable simplex and defines a chain map $f_*: C_\overline{\sigma}(X; G) \to C_\overline{\tau}(Y; G)$.

**Proof.** Let $\sigma: \Delta \to X$ be a $\overline{\sigma}$-allowable simplex. Since $f_*(\partial \sigma) = \partial f_*(\sigma)$ it suffices to prove that $f_*(\sigma)$ is a $\overline{\tau}$-allowable simplex. By definition, the simplex $\sigma$ verifies

$$\dim \sigma^{-1}S \leq \dim \Delta - D\overline{\sigma}(S) - 2,$$

for each singular stratum $S \in S_X$, and we have to prove

$$\dim \sigma^{-1}f^{-1}T \leq \dim \Delta - D\overline{\sigma}(T) - 2,$$

for each singular stratum $T \in S_Y$. Since $f$ is a stratified map, there exists a family of strata $\{S_i \mid i \in I\} \subset S_X$ with $f^{-1}(T) = \bigcup_{i \in I} S_i$, with codim $T \leq \operatorname{codim} S_i$ for each $i \in I$. In particular,
the strata \( S_i \) are singular strata. Since \( \sigma(\Delta) \) is compact, the family \( J = \{ i \in I \mid S_i \cap \sigma(\Delta) \neq \emptyset \} \) is finite. We get

\[
\dim \sigma^{-1} f^{-1}(T) = \dim \bigcup_{i \in J} \sigma^{-1}(S_i) = \max \{ \dim \sigma^{-1}(S_i) \}
\leq (1) \dim \Delta - \min \{ D\varphi(S_i) \mid i \in J \} - 2
\leq (2) \dim \Delta - D\varphi(T) - 2,
\]

where (1) comes from (2.3) and (2) is a consequence of \( f(S_i) \subset T \) and \( f^* D\varphi \leq D\varphi \). \( \square \)

**Proposition 2.6.** Let \( \varphi : (X \times [0,1], \overline{\varphi}) \rightarrow (Y, \overline{\varphi}) \) be a homotopy between two stratified maps \( f, g : (X, \overline{\varphi}) \rightarrow (Y, \overline{\varphi}) \) with \( \varphi^* D\overline{\varphi} \leq D\overline{\varphi} \). Then \( f \) and \( g \) induce the same map in homology, \( f_* = g_* : H_T(X; G) \rightarrow H_T(Y; G) \).

**Proof.** From Proposition 2.5, we get the homomorphism \( \varphi_* : H_T^T(X \times [0,1]) \rightarrow H_T^T(Y) \). The canonical injections, \( \iota_0, \iota_1 : X \rightarrow X \times [0,1] \), defined by \( \iota_k(x) = (x, k) \) for \( k = 0, 1 \), are stratified maps and induce homomorphisms \( \iota_{0*}, \iota_{1*} : H_T^T(X) \rightarrow H_T^T(X \times [0,1]) \). Since \( f = \varphi \circ \iota_0 \) and \( g = \varphi \circ \iota_1 \), it suffices to prove \( \iota_{0*} = \iota_{1*} \).

Let \( \sigma : \Delta = [e_0, \ldots, e_m] \rightarrow X \) be a simplex. The vertices of the product \( \Delta \times [0,1] \) are \( \sigma_j = (e_j, 0) \) and \( b_j = (e_j, 1) \). We define an \((m+1)\)-chain on \( \Delta \times [0,1] \) by \( P = \sum_{j=0}^m (-1)^j [a_0, \ldots, \hat{a}_j, \ldots, a_m] \). This gives a chain homotopy, \( h : C_*(X) \rightarrow C_{*+1}(X \times [0,1]) \), between \( \iota_{0*} \) and \( \iota_{1*} \), defined by \( \sigma \mapsto (\sigma \times \text{id})_* \).

Now, it remains to prove that the image \( h(\sigma) \) of a \( T \)-allowable simplex is a \( T \)-allowable chain. To any \( j \in \{0, \ldots, m\} \), we associate the simplex \( \tau_j : \nabla = [v_0, \ldots, v_{m+1}] \rightarrow \Delta \times [0,1] \), defined by \( v_0, \ldots, v_{m+1} \mapsto (a_0, \ldots, a_j, b_{j+1}, \ldots, b_m) \). By construction we have \( \Delta \times [0,1] = \bigcup_{j=0}^m \tau_j(\nabla) \). For each stratum \( S \in S_X \), we have \( \dim \sigma_{j+1}^{-1}(\sigma \times \text{id})^{-1}(S \times [0,1]) \leq \dim \sigma^{-1} S + 1 \leq m - D\varphi(S) - 1 \).

We get

\[
\dim h(\sigma)^{-1} S \leq \max \{ \dim \tau_j^{-1}(\sigma \times \text{id})^{-1}(S \times [0,1]) \mid j \in \{0, \ldots, m\} \}
\leq m - D\varphi(S) - 1 = \dim \nabla - D\varphi(S \times [0,1] - 2),
\]

which implies the desired inclusion \( h(C_T^T(X)) \subset C_T^T(X \times [0,1]) \). \( \square \)

**Corollary 2.7.** Let \( (X, \overline{\varphi}) \) be a perverse space. The inclusions \( \iota_z : X \hookrightarrow \mathbb{R} \times X, x \mapsto (z, x) \), with \( z \in \mathbb{R} \), and the projection \( \text{pr} : \mathbb{R} \times X \rightarrow X \) induce isomorphisms \( H_T^T(X; G) \cong H_T^T(\mathbb{R} \times X; G) \).

**2.3. The case of a cone.** The space \( X \) is identified with \( X \times \{1/2\} \subset \check{c}X \).

**Proposition 2.8.** Let \( X \) be a compact filtered space and \( \overline{\varphi} \) be a perversity on the cone \( \check{c}X \) endowed with the cone filtration. If we also denote by \( \overline{\varphi} \) the perversity induced on \( X \), we have:

\[
H_T^T(\check{c}X; G) = \begin{cases} 
H_T^T(X; G) & \text{if } k \leq D\overline{\varphi}(v), \\
0 & \text{if } 0 \neq k > D\overline{\varphi}(v), \\
R & \text{if } 0 = k > D\overline{\varphi}(v),
\end{cases}
\]

where \( v \) is the apex of the cone \( \check{c}X \).

**Proof.** The allowability condition for the stratum \( \{v\} \) of a simplex \( \sigma : \Delta^\ell \rightarrow \check{c}X \) is:

\[
\dim \sigma_{-1} v \leq \ell - D\overline{\varphi}(v) - 2. \tag{2.5}
\]

1) Thus, for any \( \ell \leq D\overline{\varphi}(v)+1 \), we have \( \sigma_{-1} v = 0 \) and there is an isomorphism \( C_{\leq D\overline{\varphi}(v)+1}(\check{c}X) \cong C_{\leq D\overline{\varphi}(v)+1}(X \times [0,1]) \). This gives the first part of the statement (cf. Corollary 2.7).

2) For proving the rest of the statement, we show that the inclusion \( C_{D\overline{\varphi}(v)}(\check{c}X) \hookrightarrow C_{\check{c}X} \) induces an isomorphism in homology in degrees \( \ell \geq D\overline{\varphi}(v)+1 \). Thus, let \( \sigma : \Delta^\ell \rightarrow \check{c}X \) be a
If \([x, t] \in \mathcal{c}X\), we set \(s \cdot [x, t] = [x, st]\) for \(s \in [0, 1]\). This gives a map \(c\sigma: \Delta^{t+1} = \{u\} \to \mathcal{c}X\), \(c\sigma(sx + (1-s)u) = s \cdot \sigma(x)\). We first show that \(c\sigma\) is \(\mathfrak{F}\)-allowable. For this, we distinguish between the two types of strata.

\begin{itemize}
- Let \(S \times [0, 1]\) with \(S\) a stratum of \(X\). Suppose \((c\sigma)^{-1}(S \times [0, 1]) \neq \emptyset\). From \((c\sigma)^{-1}(S \times [0, 1]) = \sigma^{-1}(S \times [0, 1][\times [0, 1]), we deduce \(\sigma^{-1}(S \times [0, 1]) \neq \emptyset\) and
  \[
  \dim((c\sigma)^{-1}(S \times [0, 1]) = 1 + \dim \sigma^{-1}(S \times [0, 1]) \leq 1 + \ell - D\mathfrak{P}(S \times [0, 1]) - 2.
  \]
- If \(\sigma^{-1}(v) \neq \emptyset\), then \(\dim((c\sigma)^{-1}(v) = 1 + \dim \sigma^{-1}(v) \leq 1 + \ell - D\mathfrak{P}(v) - 2).
- If \(\sigma^{-1}(v) = \emptyset\), the \(\mathfrak{F}\)-allowability condition of \(c\sigma\) is
  \[
  \dim((c\sigma)^{-1}(v) = \dim v = 0 \leq \ell - D\mathfrak{P}(v) - 1 = \ell + 1 - D\mathfrak{P}(v) - 2.
  \]
\end{itemize}

The isomorphism in homology now derives from
\[
\partial c\sigma = \begin{cases}
  \sigma - c\partial \sigma & \text{if } \ell \neq 0, \\
  \sigma - v & \text{if } \ell = 0.
\end{cases}
\]

\[\Box\]

3. Strong general position

If \(\sigma: \Delta \to X\) is a singular simplex, the subset \(\sigma^{-1}S\) can be very wild. We introduce a simplicial notion that can replace it.

**Definition 3.1.** Let \(\sigma: \Delta \to X\) be a singular simplex on a filtered space \(X\) and \(S \in S_X\) be a stratum. A **simplicial envelope** of \(\sigma^{-1}S\) is a finite family \(\mathcal{T}_{\sigma, S}\) of Euclidean simplices such that
\[
\sigma^{-1}S \subset \bigcup_{T \in \mathcal{T}_{\sigma, S}} T
\]
and \(\dim \sigma^{-1}S = \max \{\dim T \mid T \in \mathcal{T}_{\sigma, S}\}\). If \(\sigma^{-1}S = \emptyset\) we set \(\mathcal{T}_{\sigma, S} = \emptyset\). A **simplicial system for \(\sigma\)** is a family \(\mathcal{T}_{\sigma} = \{ \mathcal{T}_{\sigma, S} \mid S \in S_X\}\) where each \(\mathcal{T}_{\sigma, S}\) is a simplicial envelope of \(\sigma^{-1}S\). A simplicial system for \(X\) is a family \(\mathcal{T} = \{ \mathcal{T}_{\sigma} \mid \sigma \in \operatorname{Sing}X\}\) of simplicial systems for the simplices of \(X\).

Simplicial envelopes and systems always exist. Given a simplex \(\sigma: \Delta \to X\), a stratum \(S\) and a simplicial system \(\mathcal{T}_{\sigma, S}\), the allowability condition (2.2) is equivalent to the following inequalities,
\[
\dim T \leq \dim \Delta - 2 - D\mathfrak{P}(S).
\]

(3.1)

3.1. **General position.** Detecting if a simplex included in an allowable simplex is still allowable is a crucial point and moving to general position will be a useful tool. Let us recall some basic facts, sending to [13] for more information.

It is well known that two affine spaces \(E, F \subset \mathbb{R}^m\), with \(E \cap F \neq \emptyset\), are in general position if \(\dim(E \cap F) = \dim E + \dim F - m\). General position is generic in the sense that it is always possible to slightly move two affine spaces in a general position. This is also a stable situation in the sense that if two affine spaces are in general position and we slightly move one of them, then they remain in a general position.

The situation is different for polyhedra. Recall that two polyhedra \(\{P, Q\}\) included in a simplex \(\Delta\), are in general position if
\[
\dim(P \cap Q) \leq \dim P + \dim Q - \dim \Delta.
\]

(3.2)

Given two polyhedra it is always possible to slightly move one of them in order to get the general position (cf. [13]) and thus general position of polyhedra is generic. But this notion is not stable as shows the following example.
Example 3.2. Let $\dim \Delta = 2$ and $a, b, c \in \Delta$ be three distinct points aligned. Then, the 1-simplexes $P = [a, b]$ and $Q = [b, c]$ are in general position. But the simplexes $\{P, Q_\varepsilon = [\varepsilon a + (1-\varepsilon)b, c]\}$ are not in general position, for any $\varepsilon \in [0, 1]$. There is no stability.

With this lack of stability, general position is not the right situation for our purpose. Thus, we introduce in Subsection 3.2 a more robust notion called \textit{strong general position}. Before that, we show that the two situations of general position for affine spaces and polyhedra coincide with an extra hypothesis.

Lemma 3.3. Let $\{P, Q\}$ be two simplexes living in a simplex $\Delta$ and such that $\hat{P} \cap \hat{Q} \neq \emptyset$. If $\mathbb{R}^m$, $E$, $F$, $H$ denote the affine spaces generated by $\Delta$, $P$, $Q$ and $P \cap Q \cap \Delta$ respectively, we have $H = E \cap F$. Moreover, the following statements are equivalent.

i) The affine spaces $E$, $F$ are in general position in $\mathbb{R}^m$.

ii) The simplexes $P, Q$ are in general position in $\Delta$.

iii) $\dim(P \cap Q) = \dim P + \dim Q - \dim \Delta$.

Proof. We fix a point $x_0 \in \hat{P} \cap \hat{Q}$. We clearly have $H \subseteq E \cap F$. If $E \cap F = \{x_0\}$ then we get $H = E \cap F$. So, we can suppose that $E \cap F$ contains some $z \neq x_0$. Since $E$ is the affine space generated by the simplex $P$ and $x_0 \in \hat{P}$, there exists $x \in \hat{P} \cap \{x_0, z\}$. Similarly, there exists $y \in \hat{Q} \cap \{x_0, z\}$ and we may suppose $x = y$. Then, $x \in \hat{P} \cap \hat{Q} \subseteq H$ which implies $z \in H$. We have proven $E \cap F \subseteq H$ and the first assertion. The second assertion is a consequence of the equality between the dimensions of a polyhedron and of its associated affine space. \qed

Finally, we link the notions of general position and admissibility.

Proposition 3.4. Let $(X, \overline{\pi})$ be a perverse space and $\sigma: \Delta \to X$ be a $\overline{\pi}$-allowable simplex. We fix a simplicial system $T_\sigma$ and a simplex $\nabla \subseteq \Delta$. If, for any singular stratum $S$ of $X$ and any $T \in T_{\sigma,S}$, the simplexes $\{T, \nabla\}$ are in general position in $\Delta$, then the restriction $\sigma_{\nabla}: \nabla \to X$ of $\sigma$ is a $\overline{\pi}$-allowable simplex.

Proof. We have to prove $\dim \sigma_{\nabla}^{-1}S = \dim(\sigma^{-1}S \cap \nabla) \leq \dim \nabla - D\overline{\pi}(S) - 2$. By (2.1) and the definition of $T_{\sigma,S}$, it suffices to prove that $\dim(T \cap \nabla) \leq \dim \nabla - D\overline{\pi}(S) - 2$, for each $T \in T_{\sigma,S}$. Applying (3.2) and (3.1), we get

$$\dim(T \cap \nabla) \leq \dim T + \dim \nabla - \dim \Delta \leq \dim \Delta - D\overline{\pi}(S) - 2 + \dim \nabla - \dim \Delta \leq \dim \nabla - D\overline{\pi}(S) - 2.$$ \qed

3.2. Strong general position. We start with a simplex $\Delta$ together with two simplexes, $T \subseteq \Delta$ and $V \subseteq \partial \Delta$. In order to construct a pseudo-barycentric subdivision in Section 4, we have to find out a point $u \in \Delta$ such that the simplex $T$ and the cone $c_uV$ are in general position. In the main, the choice of the point $u$ requires the notions of genericity and stability.

Definition 3.5. Let $\Pi$ be a property defined on each point of the interior $\hat{\Delta}$ of an Euclidean simplex $\Delta$. We set

$$\Delta_{\Pi} = \{u \in \hat{\Delta} \mid \Pi(u) \text{ is true}\}.$$ The property $\Pi$ is \textit{generic} if $\Delta_{\Pi}$ is a dense subset of $\hat{\Delta}$ and \textit{stable} if $\Delta_{\Pi}$ is an open subset of $\hat{\Delta}$.

As a dense subset of $\hat{\Delta}$ cannot be empty, the genericity condition implies the existence of points $u$ with $\Pi(u)$ true. The stability condition means that we can move the choice of such point $u$ in in a neighborhood, and we will use that possibility in the inductive step.

The following notion of strong general position is both generic and stable, as we show in Proposition 3.9.
Definition 3.6. Let $\Delta$ be an Euclidean simplex. We consider two Euclidean simplexes $T \subset \Delta$ and $V \subset \partial \Delta$ and a point $u \in \Delta$. Denote by $c_u V = u \ast V$ the cone on $V$ of apex $u$. The simplexes $\{T, c_u V\}$ are in a strong general position if

$$T \cap c_u V \subset \partial \Delta \quad \text{or} \quad \begin{cases} \hat{T} \cap (c_u V)^o \neq \emptyset, \\
\dim (T \cap c_u V) = \dim T + \dim V + 1 - \dim \Delta. \end{cases} \quad (3.3)$$

This property is denoted $\mathcal{P}(u, T, V)$. We start with a study of the two properties appearing in Definition 3.6. Notice that $T \cap c_u V \cap \partial \Delta = T \cap V$.

Lemma 3.7. Let $\Delta$ be an Euclidean simplex. We consider two Euclidean simplexes $T \subset \Delta$ and $V \subset \partial \Delta$ and a point $u \in \Delta$.

a) If $T \cap c_u V \cap \partial \Delta \neq \emptyset$, then $u$ belongs to the closure of $\Delta_{\Pi_1}$, where $\Pi_1$ is defined by

$$\Pi_1(b) = " \hat{T} \cap (c_u V)^o \neq \emptyset."$$

b) The following property $\Pi_2$ is stable, where

$$\Pi_2(b) = " \hat{T} \cap (c_u V)^o \neq \emptyset \quad \text{and} \quad \dim (T \cap c_u V) = \dim T + \dim V + 1 - \dim \Delta."$$

Proof. a) Let $z \in T \cap c_u V \cap \partial \Delta$ and $\varepsilon > 0$. We have to find $\omega \in B(u, \varepsilon)$ such that $\hat{T} \cap (c_\omega V)^o \neq \emptyset$. We make out three cases.

i) If $z \in \hat{T} \cap (c_u V)^o$, we take $w = u$.

ii) Suppose $z \in \partial T \cap (c_u V)^o$. Since $z \in (c_u V)^o$, then there exist $\lambda \in ]0, 1[$ and $v \in V \backslash \partial V$ such that $z = \lambda u + (1 - \lambda)v$. Since $T \neq \partial T$, there exists $t \in \hat{T}$. Let $\vec{\alpha} = \frac{t - u}{\varepsilon}$ be the vector determined by $z = t + \vec{\alpha}$ in the associated affine space. We choose $\varepsilon' \in ]0, 1[$ small enough to assure that $w = u + \varepsilon'\vec{\alpha} \in B(u, \varepsilon)$. The point $\lambda \varepsilon' t + (1 - \lambda \varepsilon') z = z + \lambda \varepsilon' \vec{\alpha} = \lambda \omega + (1 - \lambda) v$ belongs to $\hat{T} \cap (c_\omega V)^o$ since $\lambda, \varepsilon' \neq 0, 1$ and $v \in V$. We have acquired $\Pi_1(\omega)$.

iii) Suppose $z \in T \cap \partial c_u V$. Since $z \in c_u V \backslash \partial \Delta$, then $z \notin c_u V \cap \partial \Delta = V$. On the other hand, from $z \in \partial c_u V$, we deduce $z \in c_u V \backslash \partial V$. This gives $z = \lambda u + (1 - \lambda)v$ with $v \in \partial V$ and $\lambda \neq 0$. Let $b \in V$. Let us denote $\vec{\alpha} = \frac{b - z}{\varepsilon}$ the vector determined by $z = b - \vec{\alpha}$. We choose $\varepsilon' \in ]0, 1[$ small enough to assure that $w = u + \varepsilon' \vec{\alpha} \in B(u, \varepsilon)$. Set $b' = \frac{1 - \lambda}{1 + \lambda \varepsilon'} v + \frac{\lambda \varepsilon'}{1 + \lambda \varepsilon'} b$. We also have,

$$z = \lambda \frac{1}{1 + \lambda \varepsilon'} v + \left(1 - \frac{\lambda}{1 + \lambda \varepsilon'}\right) b'.'$$

The point $b'$ does not belong to $\partial V$ because $\lambda \varepsilon' \neq 0$. Notice that $z \in (c_\omega V)^o$ since $\frac{\lambda}{1 + \lambda \varepsilon'} \neq 0, 1$. We deduce $T \cap (c_\omega V)^o \neq \emptyset$ and it suffices to apply i) or ii).

b) Let $u \in \Delta$ with $\hat{T} \cap (c_u V)^o \neq \emptyset$ and $\dim (T \cap c_u V) = \dim T + \dim V + 1 - \dim \Delta$. We have to find $\varepsilon > 0$ such that for any point $\omega \in B(u, \varepsilon)$ we have $\Pi_2(\omega)$. Let $\{t_1, \ldots, t_p\}$ be a basis of $T$ and $\{v_1, \ldots, v_q\}$ be a basis of $V$. We choose $t_0 \in \hat{T} \cap (c_u V)^o \neq \emptyset$ that we write

$$t_0 = \sum_{j=1}^{p} \mu_j t_j = \lambda_0 u + \sum_{i=1}^{q} \lambda_i v_i. \quad (3.4)$$

Since $t_0$ does not belong to the boundary, the coefficients are in $]0, 1[$. The generated affine spaces being in general position (Lemma 3.3), for any point $\omega \in B(u, \varepsilon)$ and any $\varepsilon > 0$, we have

$$\omega = u + \sum_{j=1}^{p} B_j t_j \mu_j + \sum_{i=1}^{q} A_i \lambda_i v_i. \quad (3.5)$$

Denote $A = A_1 + \cdots + A_q$ and $B = B_1 + \cdots + B_p$. We consider the following two points:

(i) $y = \alpha_0 \omega + \sum_{i=1}^{q} \alpha_i v_i$, with $\alpha_0 = \lambda_0/(1 - A \lambda_0)$, $\alpha_i = (\lambda_i - A_i \lambda_0)/(1 - A \lambda_0)$, $i \in \{1, \ldots, q\}$,
\begin{itemize}
\item (ii) \( s = \sum_{j=1}^{p} \beta_j t_j \), with \( \beta_j = \mu_j (1 - B \alpha_0) + B_j \alpha_0 \), \( j \in \{1, \ldots, p\} \).
\end{itemize}

If we choose \( \varepsilon > 0 \) small enough then both combinations are convex combinations verifying \( s \in T \) and \( y \in (c_\omega V)^c \). We are going to prove \( y = s \) which implies

\[
T \cap (c_\omega V)^c \neq \emptyset.
\]  

(3.6)

The equality \( y = s \) is equivalent to the equality \( \overrightarrow{ty} = \overrightarrow{ts} \). A straightforward calculation gives the claim using the following properties:

\begin{itemize}
\item \( \overrightarrow{0} = \sum_{j=1}^{p} \mu_j \overrightarrow{t_j} = \lambda_0 \overrightarrow{o} + \sum_{i=1}^{q} \lambda_i \overrightarrow{v_i} \) (cf. (3.4)),
\item \( \overrightarrow{t_\omega} = \overrightarrow{o} + \sum_{j=1}^{q} B_j \overrightarrow{t_j} + \sum_{i=1}^{q} A_i \overrightarrow{v_i} \) (cf. (3.5)).
\end{itemize}

It remains to prove \( \dim(T \cap c_\omega V) = \dim T + \dim V + 1 - \dim \Delta \). With the notation of Lemma 3.3, the hypothesis \( \dim(T \cap c_\omega V) = \dim T + \dim V + 1 - \dim \Delta \) implies \( \mathbb{R}^m = E + H \). This is equivalent to say that the family of vectors, \( \{t_0 \lambda_1, \ldots, t_0 \lambda_q, t_0 v_1, \ldots, t_0 v_q, t_0 u\} \), is of maximal rank \( m \). This means that some determinant does not vanish and it remains true if we replace the involved points by close points. So, if \( \varepsilon > 0 \) is small enough we can suppose

\[
\text{rank} \left\{ \overrightarrow{y_1}, \ldots, \overrightarrow{y_q}, \overrightarrow{y_1}, \ldots, \overrightarrow{y_q}, \overrightarrow{y} \right\} = m.
\]

(3.7)

Let us come back from vector spaces to affine spaces: let \( \mathbb{R}^m, E', F', H' \) be the affine spaces generated by \( \Delta, T, c_\omega V \) and \( T \cap c_\omega V \) respectively. Notice that \( \dim T = \dim E', \dim c_\omega V = \dim F' \) and \( \dim(T \cap c_\omega V) = \dim H' \). Moreover Lemma 3.3 implies \( H' = E' \cap F' \). Equality (3.7) gives \( \mathbb{R}^m = E' + F' \). We deduce \( m = \dim E' + \dim F' - \dim(E' \cap F') = \dim E' + \dim F' - \dim H' \), and, therefore, \( \dim(T \cap c_\omega V) = \dim T + \dim V + 1 - \dim \Delta \). \( \square \)

**Remark 3.8.** Properties \( \Pi_1 \) and \( \Pi_2 \) of Lemma 3.7 are not generic. Let us see an example with \( \Delta = \langle a_0, a_1, a_2 \rangle \), the 2-simplex generated by the points \( \{a_0, a_1, a_2\} \). We denote by \( u \) the barycentre of \( \Delta \). Let us choose \( T = \langle a_0, u \rangle \) and \( V = \langle a_0 \rangle \), which gives \( T \cap c_\omega V \). If we slightly move the vertex \( u \), we get three possibilities for \( T \cap c_\omega V \), namely \( \langle a_0 \rangle, \langle a_0, a_\omega \rangle \) or \( \langle a_0, u \rangle \). The first case gives \( T \cap (c_\omega V)^c \neq \emptyset \) while the other two correspond to \( \dim(T \cap c_\omega V) = 1 \neq 0 = \dim T + \dim V + 1 - \dim \Delta \). This motivates the strong general position concept.

**Proposition 3.9.** Let \( \Delta \) be an Euclidean simplex and \( T \subset \Delta, V \subset \partial \Delta \) be two other Euclidean simplexes. Then the property \( \mathcal{P}(\cdot, T, V) \) of Definition 3.6 is generic and stable.

**Proof.** 

1. First step: The stability. We prove that any point \( u \in \Delta, \partial (\cdot, T, V) \) is an interior point. We distinguish three cases.

   a) \( T \cap c_\omega V = \emptyset \). Since \( T \) and \( c_\omega V \) are compact subsets, the distance of \( T \) to \( c_\omega V \) is strictly positive: \( \text{dist}(T, c_\omega V) = \eta > 0 \). Therefore, it suffices to prove that \( T \cap c_\omega V = \emptyset \) if \( \text{dist}(u, \omega) < \eta \). If \( T \cap c_\omega V \neq \emptyset \) then there exists \( \lambda \in [0, 1] \) and \( v \in V \) with \( \lambda = \lambda_\omega + (1 - \lambda)v \in T \cap c_\omega V \). Let \( z = \lambda v + (1 - \lambda)v \). We obtain the contradiction

   \[
   \eta = \text{dist}(T, c_\omega V) \leq \text{dist}(t, z) = \lambda \text{dist}(u, \omega) < \eta.
   \]

   b) \( T \cap c_\omega V \neq \emptyset \) and \( T \cap (c_\omega V) \partial \Delta = \emptyset \). Let \( t_0 \in T \cap c_\omega V = T \cap V \) be a base point. We consider the infinite open cone generated by \( T, \widehat{T} = \{(1 - \lambda)t_0 + \lambda t \mid t \in T, t \neq t_0, \lambda \geq 0\} \). First, we prove

   \[
   \widehat{T} \cap c_\omega V \partial \Delta = \emptyset.
   \]

(3.8)

Let us suppose there is a point \( z = (1 - \lambda)t_0 + \lambda t \in \widehat{T} \cap c_\omega V \partial \Delta \) with \( t \in T, t \neq t_0 \) and \( \lambda \geq 0 \). Notice that \( \lambda > 0 \) and \( t \in T \setminus \partial \Delta \). There are two cases:
+ \lambda = 1 \) which corresponds to \( z = t \in T \cap (c_u V) \backslash \partial \Delta \) and which is impossible.
+ \lambda \neq 1. As \( \hat{T} \) is the space of the points of half-lines generated by \( t_0 \) and a point \( t \in T \), the three points \( t_0, t_0, z \) are collinear, with \( t_0 \) “on the left.” We thus have two situations

\[
\begin{array}{c|c|c}
 t_0 & z & t \\
\hline
\end{array} \\
\begin{array}{c|c|c}
 t_0 & t & z \\
\end{array}
\]

The left one implies \( z \in T \cap (c_u V) \backslash \partial \Delta = \emptyset \) and the right one \( t \in T \cap (c_u V) \backslash \partial \Delta = \emptyset \), which is impossible.

This gives the claim (3.8). Let us notice that the property,

\[
\text{“there exists } \varepsilon > 0 \text{ such that for any } \omega \in B(u, \varepsilon) \text{ we have } \hat{T} \cap c_u V \backslash \partial \Delta = \emptyset,”
\]

implies \( T \cap (c_u V) \backslash \partial \Delta = \emptyset \) and \( u \) is an interior point of \( \Delta_{\partial (-T, V)} \). We are reduced to prove (3.9) and, for that, we use an induction on \( \dim V \).

- \( \dim V = 0 \). We choose \( \varepsilon = d(u, \hat{T}) \), which is strictly positive since \( u \not\in \hat{T} \) (cf. (3.8)). Let \( \omega \in B(u, \varepsilon) \) and suppose there exists \( y \in \hat{T} \cap c_u V \backslash \partial \Delta \). If we find a contradiction, we get the claim \( \hat{T} \cap c_u V \backslash \partial \Delta = \emptyset \). We have \( V = \{ t_0 \} \). Since \( y \in c_u V \backslash \partial \Delta \) then \( y = \lambda t_0 + (1 - \lambda) \omega \) with \( \lambda \in [0, 1] \). Now, since \( y \not\in \hat{T} \), we have \( y = (1 - \mu) t_0 + \mu t \) with \( \mu \geq 0, t \in T \) and \( t \neq t_0 \). This gives

\[
\omega = \left( 1 - \frac{\mu}{1 - \lambda} \right) t_0 + \frac{\mu}{1 - \lambda} t, \quad t \in \hat{T}.
\]

The contradiction sought is: \( \varepsilon = \text{dist}(u, \hat{T}) \leq \text{dist}(u, \omega) < \varepsilon \).

- Inductive step. Let \( V \subset \partial \Delta \) with \( \dim V = 1 \). Let us suppose that (3.9) is proven for each face of \( \partial V \). So, there exists \( \varepsilon > 0 \) with \( \hat{T} \cap c_u V \backslash \partial \Delta = \emptyset \) for each \( \omega \in B(u, \varepsilon) \). We prove again by the absurd, assuming the existence of \( y \in \hat{T} \cap c_u V \backslash \partial \Delta \). A contradiction will give the claim \( \hat{T} \cap c_u V \backslash \partial \Delta = \emptyset \). From \( t_0 \in \partial \Delta \) and \( y \not\in \partial \Delta \), we deduce \( t_0 \neq y \). We distinguish two cases:

i) \( y \in \partial c_u V \) which gives \( y \in \hat{T} \cap (\partial c_u V) \backslash \partial \Delta = \hat{T} \cap c_u \partial(V) \backslash \partial \Delta = \emptyset \).

ii) \( y \not\in \partial c_u V \). The intersection of the simplex \( c_u V \) with the ray \([ t_0, y, \infty ] = \{(1 - \lambda) t_0 + \lambda y : \lambda \geq 0 \} \) contains the simplex \([ t_0, y, y' \] and the three points \( t_0, y, y' \) are collinear in this order. From \( t_0 \), \( y' \in \hat{T} \), we get \( y' \in \hat{T} \). From \( t_0 \in \partial \Delta \) and \( y \not\in \partial \Delta \), we deduce \( y' \not\in \partial \Delta \). As in the first case, the contradiction comes from \( y' \in \hat{T} \cap (\partial c_u V) \backslash \partial \Delta = \hat{T} \cap c_u \partial(V) \backslash \partial \Delta = \emptyset \).

- \( T \cap c_u V \backslash \partial \Delta \neq \emptyset \). By Definition 3.6, we have \( \hat{T} \cap (c_u V)^{\circ} \neq \emptyset \) and \( \text{dim}(T \cap c_u V) = \text{dim } T + \text{dim } V + 1 - \text{dim } \Delta \). Now, it suffices to apply Lemma 3.7 b).

- Second step: The genericity. For any point \( u \in \Delta \backslash \partial \Delta' \), and for any \( \varepsilon > 0 \), we show the existence of \( \omega \in B(u, \varepsilon) \) such that \( \omega \in \Delta_{\partial (-T, V)} \). If \( T \cap c_u V \backslash \partial \Delta = \emptyset \), it suffices to choose \( \omega = u \). Let us suppose \( T \cap c_u V \backslash \partial \Delta \neq \emptyset \). Following Lemma 3.7 a), there exists \( \omega' \in B(u, \varepsilon/2) \) such that \( \hat{T} \cap (c_u V)^{\circ} \neq \emptyset \). With the notation of Lemma 3.3, we have \( E + F = \mathbb{R}^m \) or \( E + F \not\subset \mathbb{R}^m \). In the first case we get \( \text{dim}(T \cap c_u V) = \text{dim } T + \text{dim } V + 1 - \text{dim } \Delta \). This gives \( \omega' \in \Delta_{\partial (-T, V)} \). We take \( \omega = \omega' \).

For the second case, let us consider a vector \( \alpha \neq 0 \in \mathbb{R}^m \backslash (E + F) \) and take \( \varepsilon' > 0 \) small enough to get \( \omega = \omega' + \varepsilon' \alpha \in B(u, \varepsilon) \). We get the claim \( \omega \in \Delta_{\partial (-T, V)} \) if we prove \( T \cap c_u V \subset \partial \Delta \).

A point \( t \in T \cap c_u V \) is of the form \( \lambda \omega + (1 - \lambda) v \) where \( \lambda \in [0, 1] \) and \( v \in V \). Let \( t_0 \in \hat{T} \cap (c_u V)^{\circ} \). We have, \( \lambda \varepsilon' \alpha = t_0 t - (1 - \lambda) t_0 \varepsilon' \) which corresponds to \( E + F \) and therefore \( \varepsilon' = 0 \). This gives \( t = v \in V \subset \partial \Delta \).
4. Pseudo-barycentric subdivisions

Given a singular simplex $\sigma: \Delta \to X$ of a filtered space $X$, a pseudo-barycentric subdivision $B$ of $\Delta$ is a subdivision similar to the barycentric subdivision, except the fact that the new vertices are not barycentres but close points of them. These points are chosen to control the relative position of the simplexes of the triangulation $B$ and the simplicial envelope (see Definition 3.1) of the singular part $\sigma^{-1}\Sigma X$.

**Definition 4.1.** A triangulation system of a space $X$ is a family $B = \{B_\sigma \mid \sigma: \Delta_\sigma \to X\}$ of triangulations $B_\sigma$ of $\Delta_\sigma$. The diameter of a triangulation is the maximum of the diameters of its simplexes.

**Definition 4.2.** Let $X$ be a filtered space, $T$ be a simplicial system and $P = \{u_\sigma \in \Delta_\sigma \mid \sigma \in \text{Sing } X\}$ be a family of points, called pseudo-barycentres. A $T$-pseudo-barycentric system, $B$, of $X$, associated to $P$, is a triangulation system verifying the properties (PB1)-(PB5), for each $\sigma: \Delta^\ell \to X$:

1. (PB1) $u_\sigma \in \Delta_\sigma$.
2. (PB2) For any $i$-th face $\partial_i \sigma: \partial_i \Delta_\sigma \to X$ of $\sigma$, we have $B_{\partial_i \sigma} = (B_\sigma) \cap \partial_i \Delta_\sigma$.
3. (PB3) $B_\sigma = u_\sigma \cap B_{\partial \sigma}$ if $\dim \Delta_\sigma > 0$.
4. (PB4) $\text{diam } B_\sigma \leq 2\ell/(2\ell + 1)$.
5. (PB5) The simplexes $\{T, c_{u_\sigma}B\}$ are in strong general position in $\Delta_\sigma$ for any $B \in B_{\partial \sigma}$ and $T \in T_\sigma$.

Observe that Property (PB2) allows the use of the triangulation $B_{\partial \sigma}$ of $\partial \Delta$ without ambiguity. Property (PB1) gives $B_\sigma = \{u_\sigma\}$ if $\dim \sigma = 0$. Before proving the existence of a pseudo-barycentric system we need the following lemma which gives a control in the case of $\mathcal{P}$-allowable simplexes.

**Lemma 4.3.** Let $(X, \mathcal{P})$ be a perverse space, with a locally finite stratification, and $\sigma: \Delta \to X$ be a $\mathcal{P}$-allowable simplex. Then, the following property, defined for each point $u \in \Delta$ by

$$\Theta(u) = \text{the simplex } \sigma_u: \Delta^0 = [a_0] \to X, \ a_0 \mapsto u, \text{ is } \mathcal{P}\text{-allowable},$$

is generic in the sense of Definition 3.5.

**Proof.** The simplex $\sigma_u$ is $\mathcal{P}$-allowable if $\dim \sigma_u^{-1}S \leq 0 - D_{\mathcal{P}}(S) - 2$ for each singular stratum $S \in \Sigma_X$. Since $\dim \sigma_u^{-1}S \leq 0$, this condition is fulfilled if $D_{\mathcal{P}}(S) \leq -2$. It remains to prove that the subset

$$\Delta_\Theta = \left\{ u \in \Delta \mid \theta(u) \text{ is true} \right\} = \Delta \setminus \bigcup_{D_{\mathcal{P}}(S) \geq -1} \sigma_u^{-1}S \quad (4.1)$$

is dense in $\Delta$. The stratification being locally finite, the previous union is finite. By dimensional reasons, it suffices to prove that $\dim \sigma_u^{-1}S \leq \dim \Delta - 1$ for each singular stratum $S$ with $D_{\mathcal{P}}(S) \geq -1$. The allowability condition for the simplex $\sigma_u$ gives

$$\dim \sigma_u^{-1}S \leq \dim \Delta - D_{\mathcal{P}}(S) - 2 \leq \dim \Delta - 1$$

and we get the claim. \hfill \Box

4.1. Existence of pseudo-barycentric subdivisions.

**Proposition 4.4.** Let $X$ be a filtered space with a locally finite stratification and let $T$ be a simplicial system. Then there exists a $T$-pseudo-barycentric system $B$ of $X$, of associated set of pseudo-barycentres $P = \{u_\sigma \in \Delta_\sigma \mid \sigma \in \text{Sing } X\}$. Moreover, given a perversity $\mathcal{P}$ on $X$, the $T$-pseudo-barycentric system $B$ can be chosen such that, for any $\mathcal{P}$-allowable simplex $\sigma$, the restriction of $\sigma$ to any $B \in B_\sigma$ with $u_\sigma \in B$ is also $\mathcal{P}$-allowable.
Proof. Let $\sigma : \Delta^\ell \to X$, we construct $B_\sigma$ by induction on $\ell$. If $\ell = 0$ then $B_\sigma$ is given by (PB1).
For the inductive step, we suppose constructed $B_{\sigma_0}$ verifying properties (PB1)-(PB5) and we have to find a point $u_\sigma \in \Delta^\ell$ such that the triangulation $u_\sigma \ast B_{\sigma_1}$ verifies the properties (PB4) and (PB5). Let us begin with (PB4). We first show that it suffices to take $u_\sigma$ in the open ball $B(b_\sigma, \frac{\ell}{2(2\ell + 1)})$, where $b_\sigma$ is the barycentre of $\Delta^\ell$. By induction, we have

$$\text{diam } B_{\sigma_0} \leq \frac{2\ell - 2}{2\ell - 1} \leq \frac{2\ell}{2\ell + 1}.$$  

Moreover, since $B_\sigma = u_\sigma \ast B_{\sigma_0}$, it suffices to verify $\text{dist}(u_\sigma, a_j) \leq \frac{\ell}{2(2\ell + 1)}$ for each vertex $a_j$ of $\Delta^\ell$. This is a consequence of

$$\text{dist}(u_\sigma, a_j) \leq \text{dist}(u_\sigma, b_\sigma) + \text{dist}(b_\sigma, a_j) \leq \frac{\ell}{(\ell + 1)(2\ell + 1)} + \frac{\ell}{2\ell + 1} = \frac{2\ell}{2\ell + 1}.$$  

Thus, the properties (PB4) and (PB5) are satisfied if

$$B \left( b_\sigma, \frac{\ell}{(\ell + 1)(2\ell + 1)} \right) \cap \bigcap_{B \in B_{\sigma_0}, T \in T_\sigma} \Delta_{\mathcal{P}(\ell, T, B)} \neq \emptyset,$$  

since this subset is included in $\Delta^\ell$. (Recall that $\mathcal{P}(u, T, B)$ is introduced in Definition 3.6 and $\Delta_{\mathcal{P}(u, T, B)}$ in Definition 3.5.) The subdivision $B_{\sigma_0}$ is finite. Since the stratification is locally finite then the family $T_\sigma$ is also finite. Following Proposition 3.9, this intersection is a non-empty open subset and it suffices to take $u_\sigma$ in it. This gives the first part of the proof.

As for the second part, we now suppose that $\sigma$ is $\mathfrak{p}$-allowable. We choose $u_\sigma$ in the subset

$$\Delta_\Theta \cap B \left( b_\sigma, \frac{\ell}{(\ell + 1)(2\ell + 1)} \right) \cap \bigcap_{B \in B_{\sigma_0}, T \in T_\sigma} \Delta_{\mathcal{P}(\ell, T, B)}$$  

(4.2)

where $\Delta_\Theta$ is defined in (4.1). This subset is not empty according to Lemma 4.3 and is included in $\Delta^\ell$. Now, it suffices to prove that the restriction $\sigma_B$ is a $\mathfrak{p}$-allowable simplex, that is,

$$\dim(B \cap \sigma^{-1} S) = \dim \sigma_B^{-1} S \leq \dim B - \dim \mathfrak{p}(S) - 2,$$  

(4.3)

for each singular stratum $S \in S_X$. If $B = \{u_\sigma\}$, the result comes from $u_\sigma \in \Delta_\Theta$. Let us suppose $B = c_{u_\sigma} V$ with $V \in B_{\sigma_0}$, the result comes from (PB5) and Proposition 3.4. $\square$

Let $\sigma : \Delta \to X$ be a singular simplex, $F \subset \partial \Delta$ a face and $u \in \Delta$ a point. We define

$$u \ast \sigma_F : c_u F = u \ast F \to X$$  

(4.4)

as the restriction of $\sigma$ to the cone. From Proposition 4.4, we deduce a subdivision adapted to $\mathfrak{p}$-allowable simplexes.

**Proposition 4.5.** Let $(X, \mathfrak{p})$ be a perverse filtered space, with a locally finite stratification, and $U$ be an open covering of $X$. Then there is a chain map, $\text{sd} : C_\ast(X; G) \to C_\ast(X; G)$, satisfying the following properties.

1) The image of a $\mathfrak{p}$-allowable chain is a $\mathfrak{p}$-allowable chain.

2) For each simplex $\sigma : \Delta \to X$, there exists $r \in \mathbb{N}$ such that the geometric support of $\text{sd}^r \sigma$ is included in an element of $U$.

**Proof.** Let $T$ be a simplicial system of $X$ and $B$ be a $T$-pseudo-barycentric subdivision of $X$ as in Proposition 4.4. In degree 0, we define $\text{sd}$ as the identity map. Suppose that we have constructed, for some $\ell \in \mathbb{N}$, a morphism $\text{sd} : C_{\ast<\ell}(X) \to C_{\ast<\ell}(X)$ verifying the two following properties.

(i) The morphism $\text{sd}$ is a chain map, i.e. $\partial \circ \text{sd} = \text{sd} \circ \partial$.
(ii) If \( \tau: \Delta^k \to X \), with \( k < \ell \), there exists a finite family \( \{ F_i \}_{i \in I} \subset B_\sigma \) with \( sd \tau = \sum_{i \in I} n_i \tau F_i \), \( n_i \in G \). Moreover, if \( \tau \) is \( \mathcal{P} \)-allowable, then so is \( sd \tau \).

We now prove (i) and (ii) for a simplex \( \sigma: \Delta^\ell \to X \). By induction, there is a family \( \{ F_i \}_{i \in I} \subset B_{\partial \sigma} \) with \( sd(\partial \sigma) = \sum_{i \in I} n_i \sigma F_i \), \( n_i \in G \). If \( u_\sigma \) is a pseudo-barycentre, from Property (PB3) of Definition 4.2, we set

\[
sd \sigma = u_\sigma * sd(\partial \sigma) := \sum_{i \in I} n_i u_\sigma * \sigma F_i,
\]

where the last equality comes from the induction and \( u_\sigma * \sigma F_i \) is defined in (4.4). Proposition 4.4 implies that \( sd \sigma \) is \( \mathcal{P} \)-allowable if \( \sigma \) is so. We have established Property (ii). Finally, Property (i) comes from the induction and

\[
\partial(sd \sigma) = \sum_{i \in I} n_i \partial(u_\sigma * \sigma F_i) = \sum_{i \in I} n_i \sigma F_i - u_\sigma * \partial \left( \sum_{i \in I} n_i \sigma F_i \right)
\]

\[
= sd(\partial \sigma) - u_\sigma * \partial(sd(\partial \sigma)) = sd(\partial \sigma).
\]

Property 2) of the statement comes from (PB4) with a classical Lebesgue number argument. \( \square \)

### 4.2. Homotopy of pseudo-barycentric subdivision

We begin with an adaptation of Proposition 4.2 in order to construct a homotopy operator in Proposition 4.8.

**Proposition 4.6.** Let \( X \) be a filtered space, with a locally finite stratification, and \( T \) be a simplicial system. Let \( B \) be a \( T \)-pseudo-barycentric subdivision. Then, for each simplex \( \sigma: \Delta \to X \) there exists a triangulation \( B_\sigma \) of \( \Delta \times [0,1] \) verifying the following properties:

\begin{align*}
& \text{(PB6)} \quad B_\sigma = \{ \Delta \times [0,1] \} \text{ if } \dim \sigma = 0, \\
& \text{(PB7)} \quad (B_\sigma)_{0,\sigma} = B_{(0,\sigma)}, \text{ for any } i\text{-face of } \sigma, \\
& \text{(PB8)} \quad B_\sigma = \left( (u_\sigma, 1) * B_{\partial \sigma} \right) \cup \left( (u_\sigma, 1) * (\Delta \times \{0\}) \right), \\
& \quad \text{with } (u_\sigma, 1) * B_{\partial \sigma} = \{ F \in B_{\partial \sigma} \} \cup \{ (u_\sigma, 1) * F \mid F \in B_{\partial \sigma} \}, \\
& \text{(PB9)} \quad \text{pr}(B_\sigma) = B_\sigma \cup \{ \Delta \}, \text{ where } \text{pr}: \Delta \times [0,1] \to \Delta \text{ is the canonical projection.}
\end{align*}

Observe that (PB7) allows the use of the triangulation \( B_{\partial \sigma} \) of \( \partial \Delta \) and justifies (PB8).

**Proof.** The construction of \( B_\sigma \) verifying (PB6)-(PB8) is straightforward by induction. It remains to prove (PB9). We proceed by induction on the dimension of \( \sigma \). If \( \dim \sigma = 0 \), the result is clear.

For the inductive case, we consider \( H \in B_\sigma \) and compute \( \text{pr}(H) \). There are three cases.

- If \( H \in B_{\partial \sigma} \), the induction hypothesis implies \( \text{pr}(H) \in B_{\partial \sigma} \subset B_\sigma \).
- If \( H = (u_\sigma, 1) * F \) with \( F \in B_{\partial \sigma} \), the induction hypothesis implies \( \text{pr}(F) \in B_{\partial \sigma} \). Thus, we have \( \text{pr}(H) = \text{pr}((u_\sigma, 1) * F) = u_\sigma * \text{pr}(F) \in u_\sigma * B_{\partial \sigma} \subset B_\sigma \).
- Finally, \( H = (u_\sigma, 1) * (\Delta \times \{0\}) \) gives \( \text{pr}(H) = \Delta \).

\( \square \)

Let us observe the following straightforward point.

**Lemma 4.7.** Let \( \Delta \) be an Euclidean simplex and \( \text{pr}: \Delta \times [0,1] \to \Delta \) be the canonical projection. Given a simplex \( H \subset \Delta \times [0,1] \) and a subset \( A \subset \text{pr} H \), we have

\[
\dim(\text{pr}^{-1}(A) \cap H) \leq \dim A + \dim H - \dim \text{pr} H.
\]

We now construct a homotopy between the chain map \( sd \) of Proposition 4.5 and the identity.

**Proposition 4.8.** Let \( (X, \mathcal{P}) \) be a perverse filtered space, with a locally finite stratification. Then there exists a morphism \( T: C_*(X; G) \to C_{*+1}(X; G) \) verifying \( \text{id} - sd = T \partial + \partial T \) and such that the image by \( T \) of a \( \mathcal{P} \)-allowable chain is a \( \mathcal{P} \)-allowable chain.
Proof. Let \( T \) be a simplicial system of \( X \) and \( B \) be a \( T \)-pseudo-barycentric subdivision as in Proposition 4.4. Finally, let \( \widetilde{B}_\sigma \) be a family given by Proposition 4.6, where \( \sigma : \Delta \to X \). Suppose that we have constructed, for some \( \ell \in \mathbb{N} \), a morphism \( T : C_{*<\ell}(X) \to C_{*+1<\ell+1}(X) \) verifying the two following properties.

(iii) \( \text{id} - \text{sd} = T\partial + \partial T \).

(iv) If \( \tau \in C_{*<\ell}(X) \), there exists a family \( \{F_i\}_{i \in I} \subset \widetilde{B}_\sigma \) with \( T(\tau) = \sum_{i \in I} n_i(\tau \circ \text{pr})_{F_i} \).

Moreover, if \( \tau \) is \( \mathcal{P} \)-allowable, then so is \( T(\tau) \).

Let \( \sigma : \Delta^\ell \to X \). By induction, there is a family \( \{F_i\}_{i \in I} \subset \widetilde{B}_\sigma \) with \( T(\partial \sigma) = \sum_{i \in I} n_i(\sigma \circ \text{pr})_{F_i} \).

For each \( F_i, i \in I \), we use the notation previously introduced in (4.4). More precisely, we set

- \( u_\sigma : (\sigma \circ \text{pr})_{F_i} : \in \sigma + (\Delta \times \{0\}) \) for the restriction of \( \sigma \circ \text{pr} : \Delta \times [0,1] \to X \).

- In the particular case where \( F_i = \Delta \times \{0\} \), we simplify the notation and write \( u_\sigma \ast \sigma \) instead of \( (u_\sigma, 1) \ast (\Delta \times \{0\}) \).

We set

\[
T(\sigma) = u_\sigma \ast (\sigma - T(\partial \sigma)) = u_\sigma - \sum_{i \in I} n_i (u_\sigma \ast (\sigma \circ \text{pr})_{F_i}).
\]

We claim that \( T(\sigma) \) is \( \mathcal{P} \)-allowable if \( \sigma \) is so, which implies Property (iv). We have two cases.

(I) Set \( H = (u_\sigma, 1) \ast (\Delta \times \{0\}) \). Since the map \( \text{pr} \) is the canonical projection and \( \text{pr}(H) = \Delta \), we can apply Lemma 4.7 and get, for each stratum \( S \subset S_X \):

\[
\dim ((\sigma \circ \text{pr})^{-1}(S) \cap H) = \dim (\text{pr}^{-1}\sigma^{-1}(S) \cap H) \leq \dim (\sigma^{-1}S) + \dim H - \dim \Delta \]

\[
\leq (1) \dim \Delta - D\overline{\sigma}(S) - 2 + \dim H - \dim \Delta 
\]

\[
= \dim H - D\overline{\sigma}(S) - 2,
\]

where the inequality (1) comes from (2.2).

(II) Set \( H = (u_\sigma, 1) \ast F_i \). From Property (PB9) of Proposition 4.6, we have \( \text{pr}(H) \in B_\sigma \) and we get:

\[
\dim (\text{pr}^{-1}(T) \cap H) \leq (2) \dim (T \cap \text{pr}(H)) + \dim H - \dim \text{pr}(H)
\]

\[
\leq (3) \dim T + \dim \text{pr}(H) - \dim \Delta + \dim H - \dim \text{pr}(H)
\]

\[
\leq (4) \dim \sigma^{-1}(S) - \dim \Delta + \dim H
\]

\[
\leq (5) \dim \Delta - D\overline{\sigma}(S) - 2 - \dim \Delta + \dim H
\]

\[
= \dim H - D\overline{\sigma}(S) - 2,
\]

where (2) comes from Lemma 4.7, (3) from (PB5), (4) from Definition 3.1 and (5) from the inequality (2.2).

We have established Property (iv). Finally, Property (iii) is a consequence of the induction and the construction of \( \text{sd} \) made in Proposition 4.5.

\[
\partial T(\sigma) = \partial (u_\sigma \ast (\sigma - T(\partial \sigma))) = \sigma - T(\partial \sigma) - u_\sigma \ast \partial (\sigma - T(\partial \sigma))
\]

\[
= \sigma - T(\partial \sigma) - u_\sigma \ast \text{sd}(\partial \sigma) = \sigma - T(\partial \sigma) - \text{sd}(\sigma).
\]

5. Mayer-Vietoris sequence

In this section, we prove the following result.

Theorem 5.1. Let \((X, \mathcal{P})\) be a perverse space and \(\{U, V\}\) be an open covering of \(X\). Then there is a long exact sequence, called Mayer-Vietoris sequence,

\[
\ldots \to H^n_c(U \cap V) \to H^n_c(U) \oplus H^n_c(V) \to H^n_c(X) \to H^n_{c-1}(U \cap V) \to \ldots
\]
Let us consider an intersection chain $\xi$ made up of a simplex. Then there exists an integer $r \in \mathbb{N}$ such that the simplexes of $s^r \xi$ belong to $C^b_T(U)$ or $C^b_T(V)$ (cf. Proposition 4.5). The same property happens if all the simplexes of $\xi$ are intersection simplexes. So we still have the case of allowable simplexes that are not intersection chains. To study them, we follow a similar method used in [2, Proposition A.14.(i)]. The key point is that the allowability defect of the boundary of an allowable simplex is concentrated in only one face, cf Proposition 5.5.

Let us notice that an allowable simplex $\sigma: \Delta \rightarrow X$ is not an intersection chain if, and only if, there exists a codimension one face $\nabla$ of $\Delta$ and a singular stratum $S \in S_X$ with

$$0 \leq \dim(\nabla \cap \sigma^{-1}S) = \dim \Delta - D\mathcal{P}(S) - 2.$$

To study them, we introduce the following definition.

**Definition 5.2.** Let $(X, \mathcal{P})$ be a perverse space and $\sigma: \Delta \rightarrow X$ be an allowable simplex. A face $F \triangleleft \Delta$ is a critical face if $F \neq \Delta$ and there exists a singular stratum $S \in S_X$ verifying

$$0 \leq \dim(F \cap \sigma^{-1}S) = \dim \Delta - D\mathcal{P}(S) - 2.$$

The set of critical faces of $\sigma$ is denoted by $\mathcal{C}_\sigma$.

The family of critical faces of a pseudo-barycentric subdivision has a nice structure that we present now. To do it, we first introduce the following definition.

**Definition 5.3.** Let $\sigma: \Delta \rightarrow X$ be a singular simplex and $B_\sigma$ be a pseudo-barycentric subdivision of $\Delta$. If $u_0, \ldots, u_\ell$ are the vertices of $\Delta$, the family of pseudo-barycentres of $B_\sigma$ is of the shape

$$\mathcal{P}_{\sigma} = \{u_{i_0 \ldots i_a} | 0 \leq i_0 < \cdots < i_a \leq \ell\},$$

where $u_{i_0 \ldots i_a}$ is the pseudo-barycentre contained in the interior of the face $[u_{i_0}, \ldots, u_{i_a}]$.

A face $B \in B_\sigma$ such that $\dim B = \dim \sigma$ is of the type $B = [u_{j_0}, u_{j_0 j_1}, \ldots, u_{j_0 \ldots j_r}]$ with $\{j_0, \ldots, j_r\} = \{i_0, \ldots, i_\ell\}$. The faces $F = [u_{j_0}, u_{j_0 j_1}, \ldots, u_{j_0 \ldots j_a}]$, with $a \in \{0, \ldots, \ell\}$, are called complete faces of $B$.

**Lemma 5.4.** Let $(X, \mathcal{P})$ be a perverse space with a locally finite stratification, $T$ be a simplicial system and $B$ be a $T$-pseudo-barycentric subdivision given by Proposition 4.4. For any allowable simplex $\sigma: \Delta \rightarrow X$ and for any $B \in B_\sigma$ with $\dim B = \dim \sigma$, the set of critical faces, $\mathcal{C}_{\sigma_B}$, is empty or has a minimum element $M$; i.e., for every $F \in \mathcal{C}_{\sigma_B}$, one has $M \subseteq F$. If it exists, this minimal element $M \triangleleft \Delta$ is called the $\mathcal{P}$-bad face of $B$. By extension, the singular simplex $\sigma_M: M \rightarrow X$ is also called the $\mathcal{P}$-bad face of $\sigma_B$.

**Proof.** We prove that a minimal element $F$ of $\mathcal{C}_{\sigma_B}$ is complete. This gives the claim since the complete faces of $B$ are totally ordered by their dimension.

- Let us begin with some calculations. A critical face $F$ of $B$ relatively to the singular stratum $S \in S_X$, verifies

$$\dim(F \cap \sigma_B^{-1}S) = \dim B - D\mathcal{P}(S) - 2 \geq 0. \quad (5.1)$$

Since the simplex $\sigma$ is $\mathcal{P}$-allowable, we also have

$$\dim B - D\mathcal{P}(S) - 2 = \dim(F \cap \sigma_B^{-1}S) \leq \dim \sigma_B^{-1}S \leq \dim \sigma^{-1}S \leq \dim \Delta - D\mathcal{P}(S) - 2,$$

and therefore

$$\dim \sigma_B^{-1}S = \dim B - D\mathcal{P}(S) - 2. \quad (5.2)$$

Using the simplicial envelope of Definition 3.1, we get $\sigma_B^{-1}S \subset \bigcup_{T \in \mathcal{T}_{\sigma_B,S}} T$ and

$$\dim \sigma_B^{-1}S = \max \{ \dim T | T \in \mathcal{T}_{\sigma_B,S} \}. \quad (5.3)$$
With the equalities (5.2), (5.3), (5.1), the inclusion $F \cap \sigma_F^{-1}S \subset \cup_{T \in T_{\sigma,S}} F \cap T$ implies
\[
\dim B - D\overline{\sigma}(S) - 2 = \max \{\dim T \mid T \in T_{\sigma,S} \}
\geq \max \{\dim (F \cap T) \mid T \in T_{\sigma,S} \}
= \dim (\cup_{T \in T_{\sigma,S}} F \cap T)
\geq \dim (F \cap \sigma_F^{-1}S)
= \dim B - D\overline{\sigma}(S) - 2.
\]
We conclude: $\max\{\dim (F \cap T) \mid T \in T_{\sigma,S} \} = \dim B - D\overline{\sigma}(S) - 2$. In the sequel, we use the decomposition
\[
T_{\sigma,S} = T_1 \cup T_2,
\]
where
\[
T_1 = \{T \in T_{\sigma,S} \mid \dim (F \cap T) = \dim \Delta - D\overline{\sigma}(S) - 2 \}
\]
and
\[
T_2 = \{T \in T_{\sigma,S} \mid \dim (F \cap T) < \dim \Delta - D\overline{\sigma}(S) - 2 \}.
\]

- We now prove that the face $F$ is complete. Without loss of generality, we can assume $B = [u_0, u_{01}, \ldots, u_{01\ldots\ell}]$. Let $a \in \{0, \ldots, \ell \}$ be the smallest number with $F \subset [u_0, u_{01}, \ldots, u_{01\ldots a}] = E$. In particular $F = c_{u_{01\ldots a}} V$ where $V \subset [u_0, u_{01}, \ldots, u_{01\ldots (a-1)}]$ or $V = \emptyset$. We distinguish these two cases.

1) $V = \emptyset$. Then $F = [u_{01\ldots a}]$. Recall from Lemma 4.3 and (4.2) that $\sigma_F$ is $\overline{\sigma}$-allowable. Using (5.1) and this allowability, we get
\[
\dim B - D\overline{\sigma}(S) - 2 = \dim (F \cap \sigma_F^{-1}S) = \dim \sigma_F^{-1}S \leq 0 - D\overline{\sigma}(S) - 2,
\]
which gives $\dim B = 0$ and therefore $a = \ell = 0$. The face $F = [u_0]$ is complete.

2) $V \neq \emptyset$. Then the simplexes $\{F = c_{u_{01\ldots a}} V, T\}$ are in strong general position in $E$ for all $T \in T_\sigma$ (cf. (PB5)). Inspired by Definition 3.6, we distinguish two possibilities.

i) There exists $T \in T_1$ with $F \cap T \not\subset \partial E$. We have
\[
\dim \Delta - D\overline{\sigma}(S) - 2 = \dim (F \cap T) =_{(1)} \dim F + \dim T - \dim E
\leq_{(2)} \dim F + \dim \sigma_F^{-1}S - \dim E
\leq_{(3)} \dim F + \dim \sigma^{-1}S - \dim E
\leq \dim F + \dim \Delta - D\overline{\sigma}(S) - 2 - \dim E,
\]
where (1) is (3.3), (2) is (5.3) and (3) comes from the $\overline{\sigma}$-allowability of $\sigma$. This implies $\dim E \leq \dim F$ and therefore $F = E$ which is a complete face.

ii) Or, we have the inclusion $F \cap T \subset \partial E$, for each $T \in T_1$. This implies,
\[
F \cap T = V \cap T \subset V.
\]
The next step is the determination of $\dim (V \cap \sigma_F^{-1}S)$. We set $Z_1 = \cup_{T \in T_1} (V \cap T)$ and $Z_2 = \cup_{T \in T_2} (V \cap T)$.
From the decomposition $V \cap \sigma_F^{-1}S \subset Z_1 \cup Z_2$, we have
\[
\min \{\dim (V \cap \sigma_F^{-1}S \cap Z_1), \dim (V \cap \sigma_F^{-1}S \cap Z_2) \},
\]

\[
(5.4)
\]
\[
(5.5)
\]
Proposition 5.5. Let \( (X, \overline{p}) \) be a perverse space with a locally finite stratification, \( \mathcal{T} \) be a simplicial system and \( B \) be a \( \mathcal{T} \)-pseudo-barycentric subdivision whose existence is guaranteed by Proposition 4.4. Consider a \( \overline{p} \)-allowable simplex \( \sigma : \Delta \to X \). Then, for any \( B \in B_\sigma \) with \( \dim B = \dim \Delta \), we have the following properties.

(a) Let \( \tau : \nabla \to X \) be a face of codimension one of \( \sigma_B \). The simplex \( \tau \) is not \( \overline{p} \)-allowable if, and only if, \( B \) has a \( \overline{p} \)-bad face included in \( \nabla \).

(b) Let \( B' \) be another simplex of \( B_\sigma \) with \( \dim B' = \dim \Delta \). If \( \sigma_B \) and \( \sigma_{B'} \) share a codimension one face \( \tau \) which is not \( \overline{p} \)-allowable, then \( \sigma_B \) and \( \sigma_{B'} \) have the same \( \overline{p} \)-bad face. Moreover, this face is a face of \( \tau \).

(c) The simplex \( \sigma_B \) is a \( \overline{p} \)-intersection chain if, and only if, it has no \( \overline{p} \)-bad face.

Proof. Recall from Proposition 4.4 that the simplex \( \sigma_B \) is \( \overline{p} \)-allowable and verifies (2.2).

(a) Let \( \tau : \nabla \to X \) be a face of codimension one of \( \sigma_B \).

- Let us suppose that the face \( \tau \) is not \( \overline{p} \)-allowable. So, there exists a stratum \( S \in S_X \) such that \( \dim \sigma^{-1}S \geq 0 \) and

\[
\dim \sigma_B^{-1}S \geq \dim \tau^{-1}S \geq 1 + \dim \nabla - D\overline{p}(S) - 2 = \dim \Delta - D\overline{p}(S) - 2 \geq \dim \sigma_B^{-1}S.
\]

We deduce \( \dim(\nabla \cap \sigma_B^{-1}S) = \dim \sigma^{-1}S = \dim \Delta - D\overline{p}(S) - 2 \geq 0 \) and \( \nabla \in \mathcal{C}_B \). So, it exists a \( \overline{p} \)-bad face \( M \) of \( B \) with \( M \in \nabla \) (cf. Lemma 5.4).

- Conversely, let us suppose that the \( \overline{p} \)-bad face \( M \) of \( B \) exists and verifies \( M \subseteq \nabla \). There is therefore a singular stratum \( S \in S_X \) with \( \dim(M \cap \sigma_B^{-1}S) = \dim B - D\overline{p}(S) - 2 \geq 0 \). So, using also that \( \sigma \) is \( \overline{p} \)-allowable, we get

\[
\dim \Delta - D\overline{p}(S) - 2 \geq \dim \sigma^{-1}S \geq \dim(\nabla \cap \sigma_B^{-1}S) \geq \dim(M \cap \sigma_B^{-1}S) = \dim B - D\overline{p}(S) - 2.
\]
This implies
\[ \dim(\nabla \cap \sigma_B^{-1} S) = \dim B - D\mathcal{P}(S) - 2 > \dim \nabla - D\mathcal{P}(S) - 2 \]
and we conclude that \( \sigma_B : \nabla \to X \) is not \( \mathcal{P} \)-allowable.

(b) Let \( M, M' \) be the minimal faces of \( C_B \) and \( C_{B'} \) respectively. Following (a) we have that \( \sigma_M, \sigma_{M'} \) are faces of \( \tau \) and thus of \( \sigma_B \). By minimality \( M = M' \) and the bad faces of \( \sigma_B \) and \( \sigma_{B'} \) are the same. Moreover this face is a face of \( \tau \).

(c) The simplex \( \sigma_B \) is an intersection chain if, and only if, its codimension one faces are \( \mathcal{P} \)-allowable. Following (a), this is equivalent to the non existence of \( \mathcal{P} \)-bad faces.

\[ \square \]

**Proof of Theorem 5.1.** Let us consider the short exact sequence
\[ 0 \longrightarrow C^p(U \cap V) \longrightarrow C^p(U) \oplus C^p(V) \xrightarrow{\varphi} C^p(U) + C^p(V) \longrightarrow 0 \quad (5.10) \]
where the chain map \( \varphi \) is defined by \( (\alpha, \beta) \mapsto \alpha + \beta \). The existence of the Mayer-Vietoris exact sequence comes from the fact that the inclusion \( \text{Im } \varphi \hookrightarrow C^p(X) \) induces an isomorphism in homology. We decompose this proof in two steps.

- *First step: The subdivision operator is homotopic to the identity.* In Propositions 4.5 and 4.8, we have constructed two operators preserving the \( \mathcal{P} \)-allowability \( \text{sd} : C_p(X) \to C_p(X) \) and \( T : C_p(X) \to C_{p+1}(X) \), verifying \( \text{sd} \circ \partial = \partial \circ \text{sd} \) and \( \partial T + T \partial = \text{id} - \text{sd} \). Therefore, \( \text{sd} : C^p(X) \to C^p(X) \) and \( T : C^p(X) \to C^{p+1}_*(X) \) are well defined.

- *Second step: We have the following implication:*
\[ \xi \in C^p(X) \Rightarrow \text{sd}^k \xi \in C^p(U) + C^p(V) \quad \text{for some } k \geq 1. \quad (5.11) \]

Let \( \xi \) be a \( \mathcal{P} \)-allowable chain of \( X \). The canonical decomposition of \( \text{sd} \xi \) is \( \text{sd} \xi = \xi_0 + \sum_{\mu \in I_\xi} \xi_\mu \) where:

- \( \xi_0 \) is the chain containing the simplexes of \( \xi \) without \( \mathcal{P} \)-bad faces;
- \( I_\xi \) is the family of the \( \mathcal{P} \)-bad faces of the simplexes of \( \text{sd} \xi \);
- \( \xi_\mu \) is the chain containing the simplexes of \( \text{sd} \xi \) having \( \mu \) as \( \mathcal{P} \)-bad face.

The boundary \( \partial \xi_0 \) is a \( \mathcal{P} \)-allowable chain. A non-\( \mathcal{P} \)-allowable face \( \tau \) of a simplex \( \sigma \) of \( \xi_\mu \) contains necessarily \( \mu \). When \( \partial \xi_0 \), and therefore \( \text{sd}(\partial \xi_0) = \text{sd} \partial \xi_0 \) is a \( \mathcal{P} \)-allowable chain (cf. Proposition 4.5) then \( \tau \) does not appear in \( \text{sd} \partial \xi_0 \). So, there exists another simplex \( \sigma' \) of \( \text{sd} \xi \) having \( \tau \) as a face. Since \( \tau \) contains \( \mu \) then \( \mu \) is the bad face of \( \sigma' \). We conclude that \( \partial \xi_\mu \) is also a \( \mathcal{P} \)-allowable chain. (These facts come from Proposition 5.5.) For each \( \mu \in I_\xi \), we have proven
\[ \xi \in C^p(X) \Rightarrow \xi_0, \xi_\mu \in C^p(X). \]

The usual subdivision argument gives the existence of an integer \( k \geq 1 \) such that the canonical decomposition of \( \text{sd}^k \xi \) verifies the following properties.

- Each simplex of \( (\text{sd}^k \xi)_0 \) lives in \( U \) or in \( V \).
- For each \( \tau \in I_{\text{sd}^k \xi} \) the chain \( (\text{sd}^k \xi)_\tau \) lives in \( U \) or in \( V \).

This gives (5.11). \[ \square \]

6. **Intersection homology is intersection homology**

In this section, we prove that the polyhedral dimension of Definition 3.1 brings an intersection homology isomorphic to that of King. Thus, this is a “reasonable dimension”. 
6.1. **Original intersection homology.** Let us specify what we mean by “original intersection homology.” We use the expression $\mathfrak{p}$-GM-allowable to make a distinction with Definition 2.2.

**Definition 6.1.** Let $(X, \mathfrak{p})$ be a perverse space. A simplex $\sigma: \Delta \to X$ is $\mathfrak{p}$-GM-allowable if, for each stratum $S$, the set $\sigma^{-1}S$ is included in the $(\dim \Delta - \text{codim} S + \mathfrak{p}(S))$-skeleton of $\Delta$. A singular chain $\xi$ is $\mathfrak{p}$-GM-allowable if it can be written as a linear combination of $\mathfrak{p}$-GM-allowable simplexes, and of $\mathfrak{p}$-GM-intersection if $\xi$ and its boundary $\partial \xi$ are $\mathfrak{p}$-GM-allowable. We denote by $\mathcal{IF}C_\ast (X; G)$ the complex of singular chains of $\mathfrak{p}$-intersection and $\mathcal{IF}H_\ast (X; G)$ its homology, called $\mathfrak{p}$-GM-intersection homology of $X$ with coefficients in an $R$-module $G$.

In the case of GM-perversities, this definition coincides with that of King. It uses the more general framework of MacPherson perversities ([12] and Definition 1.5). We will refer to [4] or [6] for the main related properties: intersection homology of a cone, Mayer-Vietoris sequence, ...

**Theorem 6.2.** Let $(X, \mathfrak{p})$ be a perverse CS set. The intersection homology $H^\mathfrak{p}_\ast (X; G)$ of Definition 2.2 is isomorphic to the GM-intersection homology, $\mathcal{IF}H_\ast (X; G)$, introduced by H.C. King in [11] and recalled in Definition 6.1.

The method of proof is a variant of [11, Theorem 10], [4, Theorem B].

**Theorem 6.3.** ([6, Theorem 5.1.4]) Let $\mathcal{F}_X$ be the category whose objects are (stratified homeomorphic to) open subsets of a given CS set $X$ and whose morphisms are stratified homeomorphisms and inclusions. Let $\mathcal{A}_\ast$ be the category of graded abelian groups. Let $F_\ast, G_\ast: \mathcal{F}_X \to \mathcal{A}_\ast$ be two functors and $\Phi: F_\ast \to G_\ast$ be a natural transformation satisfying the conditions listed below.

1. $F_\ast$ and $G_\ast$ admit exact Mayer-Vietoris sequences and the natural transformation $\Phi$ induces a commutative diagram between these sequences.
2. If $\{U_\alpha\}$ is an increasing collection of open subsets of $X$ and $\Phi: F_\ast(U_\alpha) \to G_\ast(U_\alpha)$ is an isomorphism for each $\alpha$, then $\Phi: F_\ast(\bigcup_\alpha U_\alpha) \to G_\ast(\bigcup_\alpha U_\alpha)$ is an isomorphism.
3. If $L$ is a compact filtered space such that $X$ has an open subset which is stratified homeomorphic to $\mathbb{R}^i \times \partial L$ and, if $\Phi: F_\ast(\mathbb{R}^i \times (\partial L \setminus \{v\})) \to G_\ast(\mathbb{R}^i \times (\partial L \setminus \{v\}))$ is an isomorphism, then so is $\Phi: F_\ast(\mathbb{R}^i \times L) \to G_\ast(\mathbb{R}^i \times L)$. Here, $v$ is the apex of the cone $\partial L$.
4. If $U$ is an open subset of $X$ contained within a single stratum and homeomorphic to an Euclidean space, then $\Phi: F_\ast(U) \to G_\ast(U)$ is an isomorphism.

Then $\Phi: F_\ast(X) \to G_\ast(X)$ is an isomorphism.

6.2. **Proof of Theorem 6.2.** We verify the conditions of Theorem 6.3 for the natural transformation $\Phi: \mathcal{IF}C_\ast (U) \to H^\mathfrak{p}_\ast (U)$ induced by the canonical inclusion $\mathcal{IF}C_\ast (U) \hookrightarrow C^\ast (U)$.

(i) The Mayer-Vietoris exact sequences have been constructed in Theorem 5.1 for the complex $C^\ast (X)$ and in [4, Proposition 4.1] (or [6, Theorem 4.4.19]) for the complex $\mathcal{IF}C_\ast (X)$.

(ii) This a classical argument for homology theories.

(iii) Let $L$ be a compact filtered space such that the natural inclusion induces the isomorphism

$$\Phi_{\mathbb{R}^i \times (\partial L \setminus \{v\})}: \mathcal{IF}H_\ast (\mathbb{R}^i \times (\partial L \setminus \{v\})) \isom H^\mathfrak{p}_\ast (\mathbb{R}^i \times (\partial L \setminus \{v\})).$$

Since $\mathbb{R}^i \times [0,1] \times L = \mathbb{R}^i \times (\partial L \setminus \{v\})$, we get the isomorphism

$$\Phi_{\mathbb{R}^i \times [0,1] \times L}: \mathcal{IF}H_\ast (\mathbb{R}^i \times [0,1] \times L) \isom H^\mathfrak{p}_\ast (\mathbb{R}^i \times [0,1] \times L).$$
Let us consider the following commutative diagram

\[
\begin{array}{ccc}
\varphi_{\mathcal{L}}: I\mathcal{P}H_\mathcal{L}(\mathbb{R}^d \times [0,1] \times \mathcal{L}) & \xrightarrow{\Phi_{\mathcal{L}}} & H\mathcal{P}(\mathbb{R}^d \times [0,1] \times \mathcal{L}) \\
pr_* & & \downarrow pr_* \\
\varphi_{\mathcal{L}}: I\mathcal{P}H_\mathcal{L}(\mathcal{L}) & \xrightarrow{\Phi_{\mathcal{L}}} & H\mathcal{P}(\mathcal{L}) \\
(\iota_{\mathcal{L}})_* & & \downarrow (\iota_{\mathcal{L}})_* \\
\varphi_{\mathcal{L}}: I\mathcal{P}H_\mathcal{L}(\mathcal{L}) & \xrightarrow{\Phi_{\mathcal{L}}} & H\mathcal{P}(\mathcal{L}). \\
\end{array}
\]

From Corollary 2.7 and [4, Corollary 3.14] (or [6, Example 4.1.13.]), we know that the two maps \(pr_*\), induced by the canonical projections, are isomorphisms. We conclude that \(\Phi_{\mathcal{L}}\) is an isomorphism.

If \(\ast < n - |\mathcal{P}(v)|\) then Proposition 2.8 and [4, Proposition 5.2] (or [6, Theorem 4.2.1]) imply that the two maps \((\iota_{\mathcal{L}})_*\), are isomorphisms. So, \(\Phi_{\mathcal{L}}\) is an isomorphism in these degrees.

When \(\ast \geq n - |\mathcal{P}(v)|\), the map \(\Phi_{\mathcal{L}}\) is directly an isomorphism (cf. Proposition 2.8 and [6, Section 5.4]).

(iv) The map \(\Phi: H\mathcal{P}(U) \to I\mathcal{P}H_\mathcal{L}(U)\) is the identity \(G \to G\). \(\square\)

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References

[1] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, *Blown-up intersection cohomology*, An alpine bouquet of algebraic topology, Contemp. Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 45–102. MR 3807751 2, 4

[2] ______, *Intersection cohomology, simplicial blow-up and rational homotopy*, Mem. Amer. Math. Soc. 254 (2018), no. 1214, viii+108. MR 3796432 2, 17

[3] ______, *Pontrjagin duality with cap products in intersection homology*, Adv. Math. 326 (2018), 314–351. MR 3758431 2, 4

[4] ______, *Intersection homology: general perversities and topological invariance*, Illinois J. Math. 63 (2019), no. 1, 127–163. MR 3959870 2, 4, 21, 22

[5] ______, *Perverse homotopy groups*, arXiv e-prints (2022), arXiv:2211.06996. 3

[6] Greg Friedman, *Singular intersection homology*, New Mathematical Monographs, Cambridge University Press, 2020. 2, 4, 21, 22

[7] Paweł Gajer, *The intersection Dold-Thom theorem*, Topology 35 (1996), no. 4, 939–967. MR 1404919 (97i:55013) 2, 3

[8] ______, *Corrigendum: “The intersection Dold-Thom theorem” [Topology 35 (1996), no. 4, 939–967; MR1404919 (97i:55013)]*, Topology 37 (1998), no. 2, 459–460. MR 1489215 (99k:55004) 3

[9] Mark Goresky and Robert MacPherson, *Intersection homotopy theory*, Topology 19 (1980), no. 2, 135–162. MR 572580 (82b:57010) 1, 4, 5

[10] ______, *Intersection homology. II*, Invent. Math. 72 (1983), no. 1, 77–129. MR 696691 (84i:57012) 1

[11] Henry C. King, *Topological invariance of intersection homology without sheaves*, Topology Appl. 20 (1985), no. 2, 149–160. MR 800845 (86m:55010) 1, 2, 3, 21

[12] Robert MacPherson, *Intersection homology and perverse sheaves*, Unpublished AMS Colloquium Lectures, San Francisco, 1991. 4, 21

[13] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. MR 0350744 5, 8

[14] Martin Saralegi, *Homological properties of stratified spaces*, Illinois J. Math. 38 (1994), no. 1, 47–70. MR 1245833 (95a:55011) 4

[15] Martintxo Saralegi-Aranguren, *de Rham intersection cohomology for general perversities*, Illinois J. Math. 49 (2005), no. 3, 737–758 (electronic). MR 2210257 (2006k:55013) 4
[16] Martintxo Saralegi-Aranguren and Daniel Tanré, *Poincaré duality, cap product and Borel-Moore intersection homology*, Q. J. Math. **71** (2020), no. 3, 943–958. MR 4142716

[17] L. C. Siebenmann, *Deformation of homeomorphisms on stratified sets. I, II*, Comment. Math. Helv. **47** (1972), 123–136; ibid. 47 (1972), 137–163. MR 0319207