Free-boundary perturbed MHD equilibria

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Abstract. The concept of perturbed ideal MHD equilibria [Boozer A H and Nührenberg C 2006 Phys. Plasmas 13 102501] is employed to study the influence of external error-fields and of small plasma-pressure changes on toroidal plasma equilibria. In tokamak and stellarator free-boundary calculations, benchmarks were successful of the perturbed-equilibrium version of the cas3d stability code [Nührenberg C et al. 2009 Phys. Rev. Lett. 102 235001] with the ideal MHD equilibrium code NEMEC [Hirshman S P et al. 1986 Comput. Phys. Commun. 43 143].

1. Introduction
In the overlap between stellarator and tokamak research, the 3D modeling of the plasma edge for divertor studies is in need of efficient methods treating plasma equilibria under the influence of external error fields. In tokamaks, e.g. the coil ripple due to the finite number of toroidal field coils and, possibly, error fields from ferritic in-vessel components or, in the control of edge-localized modes (ELMs), the applied resonant magnetic perturbations (RMPs) destroy the axisymmetry of the ideal configuration, break up magnetic surfaces, and corrugate the plasma surface. Another example is the island divertor of the Wendelstein 7-X stellarator, where the plasma edge structure is exploited and use is made of the natural island chains conforming with the boundary rotational transform. As a complement to vacuum-field studies, it is important to know how a finite plasma pressure influences the load imbalances on the divertor plates caused by magnetic field perturbations.

Though important, such changes of the magnetic field are small and, therefore, may be treated as perturbations of an equilibrium state. Here, the concept of perturbed ideal MHD equilibria [1] is employed, which was computationally realized in the CAS3D ideal MHD stability code [2] allowing, in its latest version, the study of plasma edge distortions under the influence of external error-fields and small plasma-pressure changes.

For perturbed tokamaks, the IPEC code [3] exists which follows a similar concept, but incorporates a control surface with surface currents to account for external error-fields.

In Secs. 2 and 4 of this paper, stellarator and tokamak applications are shown together with their benchmarks to free-boundary NEMEC [4] results. In Sec. 3 of this paper, a generalization of the vacuum part of the CAS3D code will be described which is necessary for the correct treatment of perturbations which preserve the periodicity of the unperturbed equilibrium.

2. Stellarator-type plasma
This section describes the influence of (i) an external error-field, (ii) a small plasma-pressure changes, and (iii) the combination of both, on a zero-\(\beta\) stellarator.
For the free-boundary benchmarks of the perturbed-equilibrium method with the MHD equilibrium code NEMEC [4], a geometrically simple case has been chosen: a turning-ellipse \( \ell = 2 \), zero-plasma-\( \beta \) stellarator equilibrium, with large aspect ratio, \( A = R/a \approx 23.4 \), a low-shear rotational transform, \( \iota_{\text{axis}} = 0.23 \), and \( \iota_{\text{edge}} = 0.244 \), and five field periods. The vacuum field for the free-boundary equilibrium calculation is the superposition of the purely toroidal, axisymmetric field of a line-current on the z-axis and equal-magnitude, opposite-sign currents in two continuous stellarator coils each of which makes two toroidal transits before closing on itself after one poloidal transit. This basic setup is shown in Fig. 1.

A deformation of the plasma boundary may be described in terms of the ideal MHD displacement vector, \( \xi \), as

\[
\delta r = r_{\text{perturbed}} - r_{\text{unperturbed}} = \xi_n = (\xi \cdot n) n. \tag{1}
\]

In Eq. (1), the outer unit normal \( n \) points from the plasma boundary into the unbounded vacuum region. The normalized toroidal flux, \( s \), is used as magnetic-surface label, \( s = 1 \) on the plasma boundary and \( F_T(s) = s F_T(1) \) the toroidal flux enclosed by a magnetic surface. The normal displacement is \( \xi^s = \xi \cdot \nabla s = |\nabla s| \frac{\xi \cdot n}{n} \). In the perturbed-equilibrium calculations, magnetic coordinates, \((s, \theta, \phi)\), are used [2], in which the magnetic field lines are straight. In one poloidal transit, \( \theta \) increases by unity. In a toroidal transit of one of the \( N_p \) field-periods, \( \phi \) increases by unity. For stellarator-symmetric configurations including up-down symmetric tokamaks, the Fourier decomposition of an even-parity normal displacement is

\[
\xi^s(s, \theta, \phi) = \sum_j \xi_j^{s m} m_j(s) \cos 2\pi \left( m_j \theta + n_j \phi / N_p \right). \tag{2}
\]

The superposition of a constant purely vertical external field, \( \delta B_0^\text{ext} = 0.0004 \) T, leads to an inward shift of the plasma column of e.g. 0.023 m at the outside of the vertical ellipse and of 0.04 m at the outside of the horizontal ellipse (Fig. 2, bottom right). The normal component of the error-field on the plasma boundary is the boundary condition of the CAS3D calculation. The corresponding \( \xi^s \) harmonics are shown in the top left frame of Fig. 2. The perturbation preserves the periodicity of the configuration, i.e. belongs to the \( N = 0 \) mode-family. The dominant harmonics are shown and their poloidal (m) and toroidal (n) Fourier indices are indicated. Here, the results of NEMEC and CAS3D differ by 0.0005 m at the outside of the vertical ellipse and by 0.0015 m at the outside of the horizontal ellipse.

The introduction of a small plasma pressure in the \( \beta = 0 \) configuration, leading to an average plasma-\( \beta \) of \( \langle \beta \rangle \approx 0.00016 \), results in an outward shift of the plasma column of approximately
Figure 2. Results for the configuration of Fig. 1: Normal displacement harmonics versus normalized toroidal flux \( s \), describing the influence of an external, inward-shifting \( \delta B_{z}^{\text{ext}}/B_0 \) (axis) = 0.0004 (top left, log-scale for the \( \xi^s \)-axis), of \( \langle \beta \rangle = 0.00016 \) (top right, log-scale for the \( \xi^s \)-axis), and of the combination of both, external field perturbation and plasma pressure (bottom left, linear scale for the \( \xi^s \)-axis). The perturbation belongs to the \( N = 0 \) mode family, the Fourier indices of the dominant harmonics as indicated. Black: NEMEC code; red: CAS3D code; 301 radial points; 4 of 10 perturbation harmonics are shown. The effect of the \( \delta B_{z}^{\text{ext}} \) perturbation (top left) is also shown in the bottom right frame: Unperturbed and perturbed boundaries at the beginning, quarter, and middle of a field period. Green: unperturbed; black: perturbed, NEMEC; red: perturbed, CAS3D.

the same amount, as may be seen in the top right frame of Fig. 2. Consequently, applying this error-field and this plasma-pressure at the same time leaves the plasma boundary nearly unaffected as can be seen from the nearly vanishing \( \xi^s \)-harmonics in the bottom right frame of Fig. 2. The results of the NEMEC code using a non-linear minimization method and of the linear stability CAS3D stability code compare very well when applied for small perturbations.
3. Construction of a boundary-conforming vacuum magnetic field

Free-boundary ideal MHD stability studies and perturbed-equilibrium calculations employ the perturbed magnetic field in the vacuum region surrounding the toroidal plasma domain. In contrast to the vector potential, the scalar potential representing the perturbed vacuum magnetic field may be multi-valued because of the non-simply connected vacuum region.

In Ref.[5], Lüst and Martensen describe how to construct the perturbed vacuum magnetic field from the gradient of a single-valued scalar potential fulfilling the boundary condition and a, possibly multi-valued, vector field tangential to the unperturbed plasma boundary. Since it is defined in the vacuum region, the latter vector field must be divergence- and curl-free, and, for uniqueness, normalized to unity net current in the plasma domain. For convenience, some results of Ref. [5] are repeated in Appendix B.

For perturbations which do not preserve the periodicity of the plasma, the multi-valued part does not contribute to the perturbed vacuum energy, and in the case of periodicity-preserving perturbations it acts stabilizing [5, 6], so that it may be omitted from the free-boundary stability calculation.

In the tokamak application of Sec. 4, the poloidal component of the equilibrium magnetic field, $B_{0}^{\text{pol}}$, is used as Lüst-Martensen field. The plasma-current contribution to this field may be calculated using Biot-Savart’s law for a surface current density on the plasma boundary, $\mu_0^{-1} B_0 \times n$, with $n$ the outer unit normal pointing into the unbounded vacuum region and $\mu_0$ the permeability of free space. For this procedure, also known as virtual casing principle [7], implementations are available in the MFBE [8] and EXTENDER [9] codes, the latter being used.
here. In Fig. 3 an example is shown, a slightly elongated, force-free, axisymmetric case with aspect ratio \( A = R/a \approx 2.7 \), inverse safety factor \( \iota_{\text{axis}} = 0.5 \), and \( \iota_{\text{edge}} = 0.3 \).

Alternatively, or in plasma configurations without a net toroidal current, for the construction of the \( \text{Lüst-Martensen} \) field, a line current may be used positioned on the unperturbed magnetic axis. Its magnetic field will have, in general, a finite normal component on the unperturbed plasma boundary. For compensation, a vacuum magnetic field is needed with the normal component equal in magnitude, but opposite in sign. The single-valued scalar potential of this magnetic field can be obtained from the solution of Laplace’s equation in the vacuum region with its normal derivative given on the plasma boundary. The \textsc{nestor} code implementation of this \textit{exterior Neumann} problem yields the values of the scalar potential on the plasma boundary which is sufficient for the use in 3d ideal MHD equilibrium [4] and stability [10] calculations. As explained in Appendix B, Eq. (B.4), the perturbed-equilibrium application, however, needs to know the scalar potential in a part of the vacuum region. The implementation of Green’s representation formula for the exterior Neumann problem will be part of future work.

4. Tokamak-type plasma

As in Sec. 2 for the stellarator case, the perturbed-equilibrium method is used for a force-free axisymmetric case to evaluate the influence of (i) an external error-field, (ii) a small plasma-pressure change, and (iii) the combination of both.

As before, a geometrically simple case has been chosen for the benchmarks with the \textsc{nemec} code. The axisymmetric toroidal field of a line-current on the z-axis superposed with a constant field purely in z-direction are used as vacuum field for the free-boundary equilibrium calculation.

The perturbations which are used for this case preserve the axisymmetry of the plasma, as external error-field a small change of the equilibrium \( B_z \)-field is used. Therefore, the MHD displacement is axisymmetric, too.

In all \textsc{cas3d} perturbed-equilibrium computations of this section, the normalized poloidal part of the equilibrium field, i.e. the plasma-current and the coil-current fields, is used as the \( \text{Lüst-Martensen} \) field \( Y \) of Eqs. (B.1) and (B.4). Only with the implementation of the \( L \gamma^2 \)-term of Eq. (B.5) in the perturbed-vacuum energy does the perturbed-equilibrium calculation agree with the respective non-linear equilibrium result.

In Fig. 4, the normal displacements are shown resulting from the perturbed-equilibrium calculations for the axisymmetric benchmark case of Fig. 3 for the application of an external error-field (top left sub-frame, \( \Delta B_{\text{ext}}^z = 0.00055 \text{ T} \)), of a small plasma-pressure change (top right, to \( \langle \beta \rangle = 0.0003 \)), and of the combination of both (bottom left). The benchmark results from the \textsc{nemec} (black) and \textsc{cas3d} (red) codes compare well. The change of the vertical field by 1\% together with an increase of the initially zero plasma-\( \beta \) to an average of \( \langle \beta \rangle = 0.0003 \) results in an outward shift of the plasma column of \( \approx 0.022 \text{ m} \) at the high-field side, and of \( \approx 0.033 \text{ m} \) at the low-field side of the plasma region. For these shifts, the values obtained by \textsc{nemec} and \textsc{cas3d} differ by \( \approx 0.0005 \text{ m} \) at the high-field side, and \( \approx 0.0027 \text{ m} \) at the low-field side. The very good agreement between the corresponding perturbed plasma boundaries from the \textsc{nemec} non-linear equilibrium calculation and the \textsc{cas3d} linear perturbed-equilibrium calculation is shown in the bottom-right subframe of Fig. 4.

5. Summary and outlook

In stellarator and tokamak benchmark cases, the non-linear ideal MHD equilibrium code \textsc{nemec} and the linear ideal MHD stability code \textsc{cas3d} in its perturbed-equilibrium version yield results in the study of small plasma-pressure changes and of small external error-fields that are in good agreement. Therefore, the perturbed-equilibrium method appears to be a safe tool for the study of scenarios which are expensive for equilibrium codes.
Figure 4. Results for the axisymmetric plasma configuration of Fig. 3: Dominant normal displacement harmonics versus normalized toroidal flux $s$, describing the influence of an external $\delta B_{\text{ext}}^z = 0.00055$ T (top left), of $\langle \beta \rangle = 0.0003$ (top right), and of the combination of both, external field perturbation and plasma pressure (bottom left). In each of these sub-frames, a log-scale $\xi^*$-axis is used. The perturbation is purely $n = 0$. Black: nemec code; red: cas3d code; 301 radial points, poloidal perturbation harmonics: $0 \leq m \leq 14$. Unperturbed and perturbed boundaries are shown for the combined perturbation (bottom right); green: unperturbed; black: perturbed, NEMEC; red: perturbed, CAS3D.

The vacuum contribution to the potential energy of the energy principle has been complemented with the multi-valued parts in the perturbed vacuum magnetic field. The free-boundary tokamak calculations show that the inclusion of these Lüst-Martensen terms is indeed essential for the correct treatment of symmetry-preserving perturbations with the perturbed-equilibrium method.

It is planned to generalize the implementation of the Lüst-Martensen terms to the stellarator case.
Appendix A. MHD energy principle and perturbed equilibria
In this section, the basic equations of the perturbed-equilibrium method [2] are summarized. The inclusion of an external error-field is described, too.

The determination of perturbed equilibria needs the variation of the ideal MHD energy through second order, so

$$\delta W = \delta^1 W + \delta^2 W = \int \int \int (\nabla p - j \times B) \cdot \xi \, d^3 r - \frac{1}{2} \int \int \int \xi \cdot F[\xi] \, d^3 r \quad .$$  \hspace{1cm} (A.1)

In the second order term, $F$ is the ideal MHD force operator acting on the displacement vector, $\xi$. The integrand of the first order term can be rewritten to show the dependence on a change of the plasma pressure, $p$, and on an external error-field, $B^\text{ext}$,

$$\left(\nabla p - j \times B\right) \cdot \xi = \left(p'^\text{perturbed} - p'^\text{unperturbed}\right) \xi^s + \left(j_0 \times B^\text{ext}_1\right) \cdot \xi \quad .$$  \hspace{1cm} (A.2)

Equation (A.2) is valid if a perfect MHD equilibrium is perturbed. Primes denote the derivative of a scalar function constant on magnetic surfaces. The unperturbed current density is $j_0$.

Stationarity of $\delta W$ leads to the perturbed equilibrium given by $F[\xi] = g$. The left-hand side is the center-piece of ideal MHD stability calculations which minimize $\delta^2 W$. The right-hand side, $g$, derives from the first-order term, $\delta^1 W$, which is linear in the displacement. For a free-boundary perturbed equilibrium, no further boundary conditions are used.

Appendix B. Perturbed vacuum magnetic field
In ideal MHD stability, the plasma is treated as a toroidal domain, $P$, with boundary $\partial P$, being surrounded by a vacuum region, $V$. In contrast to standard tokamak stability studies, in the perturbed-equilibrium-method implementation so far no enclosing wall is present, so $V = \mathbb{R}^3 \setminus (P \cup \partial P)$.

In Ref. [5], a method is described and, for convenience, summarized here how to find the, possibly multi-valued, scalar magnetic potential, $\Phi$, for the first-order perturbed magnetic field in the vacuum region,

$$B^\text{1V} = \nabla \Phi = \nabla \Phi^* + \gamma Y \quad .$$  \hspace{1cm} (B.1)

The split-up into a sum of the gradient of a single-valued scalar potential, $\Phi^*$, and the multiple of a vector field, $Y$, can always be accomplished and may be used as the defining equation for $\Phi^*$. Like $B^\text{1V}$, the vector field $Y$ must be divergence- and curl-free, $\nabla \cdot Y = 0$ and $\nabla \times Y = 0$ in $V$. Furthermore, it must be tangential to the plasma boundary, $Y \cdot n = 0$ on $\partial P$, and normalized to unity net toroidal current, $\int Y \cdot dl = 1$, integrated along any poloidally closed curve, $C$, created by the intersection of the plasma boundary and a toroidal cut of the plasma domain. With these prerequisites, $\gamma$ is the first-order perturbed net toroidal current flowing in the plasma. $\mu_0^{-1} \gamma$ has the unit of current. Expressed in terms of the displacement vector, this so-called period is the line integral

$$\gamma = \oint_C B^\text{1P} \cdot dl = \oint_C dl \cdot \nabla \times (\xi \times B_0) \quad ,$$  \hspace{1cm} (B.2)

integrated along any poloidally closed curve, $C$, on the plasma boundary, encircling the magnetic axis once. The net perturbed toroidal flux in the vacuum region can be expressed by the constant in the normal displacement taken on the plasma boundary, $\xi_0$,

$$L \gamma_0 = F_T^{1T} \xi_0 \quad .$$  \hspace{1cm} (B.3)
In the flux constraint, Eq. (B.3), the flux linkage is used to relate the net perturbed toroidal flux to net perturbed toroidal current, Eq. (B.2). With $D$ the part of the $z = 0$ plane enclosed by the inner hole of the toroidal domain, the inductance $L$ is defined by

$$
L = \int\int\int_{V} |Y|^2 \, d^3r = \int\int_{D} Y_{z} \, d^2r .
$$

(B.4)

It depends only on the geometry of the toroidal plasma domain and has the unit of length. The change of the vacuum energy expressed in terms of the ideal MHD displacement vector $\xi$ is then given by

$$
\delta W_{V} = L \gamma^2 = \frac{1}{2} \int\int\int_{V} |\nabla \Phi^*|^{2} \, d^3r
$$

$$
= -\frac{1}{2} \int_{\partial P} \Phi^* \cdot \nabla \Phi^* \, d^2r
$$

$$
= -\frac{1}{2} \int_{\partial P} \Phi^* \cdot \nabla \times (\xi \times B_0) \, d^2r .
$$

(B.5)

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