On the Lagrange and Markov Dynamical Spectra for Anosov Flows in Dimension 3

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Abstract
We consider the Lagrange and the Markov dynamical spectra associated with a conservative Anosov flow on a compact manifold of dimension 3 (including geodesic flows of negative curvature and suspension flows). We show that for a large set of real functions and typical conservative Anosov flows, both the Lagrange and Markov dynamical spectra have a non-empty interior.

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1 Introduction

The Lagrange and Markov spectra are born in the theory of numbers which have a dynamic interpretation that can be explored in a more general context using the hyperbolicity of some systems.

Many recent results, such as [8,19], and [9], show that the Lagrange and Markov spectra are well behaved when looking at the hyperbolic world, including geodesic flows of negative curvature (cf. [23]), Teichmüller flows (cf. [14]), Veech surfaces (cf. [1]), among others.

Good references for an introduction to these spectra can be found at [7] and [17].

1.1 Dynamical Markov and Lagrange Spectra

Let $M$ be a smooth manifold, $T = \mathbb{Z}$ or $\mathbb{R}$, and $\phi = (\phi^t)_{t \in T}$ be a discrete-time ($T = \mathbb{Z}$) or continuous-time ($T = \mathbb{R}$) smooth dynamical system on $M$, that is, $\phi^t : M \to M$ are smooth diffeomorphisms, $\phi^0 = \text{id}$, and $\phi^t \circ \phi^s = \phi^{t+s}$ for all $t, s \in T$.

Given an invariant subset $\Lambda \subset M$ and a function $f : M \to \mathbb{R}$, we denote the dynamical Markov and the Lagrange spectrum, as $M(\phi, \Lambda, f)$ and $L(\phi, \Lambda, f)$, respectively. They are defined as follows

$$M(\phi, \Lambda, f) = \{ m_{\phi, f}(x) : x \in \Lambda \} \quad \text{and} \quad L(\phi, \Lambda, f) = \{ \ell_{\phi, f}(x) : x \in \Lambda \},$$

where

$$m_{\phi, f}(x) := \sup_{t \in T} f(\phi^t(x)) \quad \text{and} \quad \ell_{\phi, f}(x) := \limsup_{t \to +\infty} f(\phi^t(x)).$$

It is easy to see that $L(\phi, \Lambda, f) \subset M(\phi, \Lambda, f)$ (cf. [19]).

When $\Lambda$ is the whole manifold, we denote $M(\phi, f) := M(\phi, \Lambda, f)$ and $L(\phi, f) := L(\phi, \Lambda, f)$. Analogous notation for the Markov spectra.

The first result in the context of discrete dynamic is due to C. Moreira and S. Romaña [19], where they proved that:

**Theorem** [19, Main Theorem] Let $\Lambda$ be a horseshoe associated to a $C^2$-diffeomorphism $\varphi$ of a surface $M$ such that $HD(\Lambda) > 1$. Then there is arbitrarily close to $\varphi$, a
diffeomorphism $\varphi_0$ and a $C^2$-neighborhood $\mathcal{V}$ of $\varphi_0$ such that, if $\Lambda_\psi$ denotes the continuation of $\Lambda$ associated to $\psi \in \mathcal{V}$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$, we have

$$\text{int } L(\psi, \Lambda_\psi, f) \neq \emptyset \text{ and } \text{int } M(\psi, \Lambda_\psi, f) \neq \emptyset,$$

where $\text{int } A$ denotes the interior of $A$.

This theorem will be useful to prove our results.

1.2 Main Results

In this paper, we consider a conservative Anosov flow in dimension 3, which includes the case of geodesic flows on surfaces of negative curvature with finite volume and suspension Anosov flow. More specifically, we consider a three-dimensional connected $C^\infty$-Riemannian manifold $M$ endowed with a finite volume-form $\omega$. Let $m$ be the measure associated with this form of volume, which we call the Lebesgue measure.

Let $\mathcal{X}_r^\omega(M)$ be the space of $C^r$-conservative vector fields on $M$. Then, we prove the following theorem:

**Theorem 1.1** Let $\Phi \in \mathcal{X}_r^\omega(M)$, $r \geq 2$ such that $\phi^t$ has a compact basic set $\Lambda$ with Hausdorff dimension bigger than 2, then $C^r$-arbitrarily close to $\Phi$ there is an open set $\mathcal{V} \subset \mathcal{X}_r^\omega(M)$ such that for any $X \in \mathcal{V}$ one can find a dense and $C^r$-open subset $U_{X, \Lambda} \subset C^r(M, \mathbb{R})$, so that

$$\text{int } M(X, f) \neq \emptyset \text{ and } \text{int } L(X, f) \neq \emptyset,$$

whenever $f \in U_{X, \Lambda}$. Moreover, the above statement holds persistently: for any $Y \in \mathcal{V}$, it holds for any $(X, f)$ in a suitable neighborhood of $\{Y\} \times U_{Y, \Lambda}$ in $\mathcal{X}_r^\omega(M) \times C^r(M, \mathbb{R})$.

It is worth noting that the above theorem is valid for a transitive Anosov flow which is not necessarily conservative. In this case, the proof is similar to the proof of Theorem 1.1 and does not need the conservative family of perturbations of Sect. 4.2.2, for this reason, we will only do the proof for conservative flows.

**Remark 1** Assume that $M$ is a compact manifold, then if $\Lambda = M$ is a basic set, in particular, the flow is transitive and by Urbański’s result (cf. [26]), the flow has a non-trivial basic set with Hausdorff dimension close to 3 (compare with Lemma 4.11 of Sect. 4.4.2 and Corollary 5). Thus, Theorem 1.1 is still valid.

In the case of geodesic flow, let $N$ be a complete surface, let $g_0$ be a smooth ($C^r$, $r \geq 2$) pinched negatively curved Riemannian metric on $N$ (the curvature is bounded above and below by two negative constants). Let $\phi^t_0$ be the geodesic flow on the unit bundle $SN$ and $\Phi_0$ the derivative of the geodesic flow $\phi^t_0$. In this case, it is well known that $\phi^t_0$ is an Anosov flow (cf. [2] and [16]). Moreover, if $N$ has finite volume, then $\Phi_0 \in \mathcal{X}_r^\omega(SN)$, since the Liouville measure is invariant by the geodesic flow (cf. [22]).

In these conditions, we will construct a hyperbolic set $\Lambda$ for the geodesic flow of $\phi^t_0$ with Hausdorff dimension close 3 (cf. Sect. 4.4). So, we have our second result:
Corollary 1  Arbitrarily close to $\Phi_0$ there exists an open set $\mathcal{V} \subset \mathcal{X}^2_0(SN)$ such that for any $X \in \mathcal{V}$ one can find a dense and $C^2$-open subset $\mathcal{U}_{X,\Lambda} \subset C^2(SN, \mathbb{R})$, so that

$$\text{int} \, M(X, f) \neq \emptyset \text{ and } \text{int} \, L(X, f) \neq \emptyset$$

whenever $f \in \mathcal{U}_{X,\Lambda}$. Moreover, the above statement holds persistently: for any $Y \in \mathcal{V}$, it holds for any $(X, f)$ in a suitable neighborhood of $\{Y\} \times \mathcal{U}_{Y,\Lambda}$ in $\mathcal{X}^1_0(SN) \times C^2(SN, \mathbb{R})$.

Remark 2  It is important to note that the neighborhood $\mathcal{V}$ of the above corollary is not necessarily a neighborhood of space of vector field coming from geodesic flows or Riemannian metric, since small perturbations on the metrics do not produce small perturbation on the geodesic flows.

Another interesting class of Anosov flows is the suspension Anosov flows, which are the suspension of Anosov diffeomorphisms and defined as follows:

Let $\varphi: N \to N$ be an Anosov diffeomorphism of a compact manifold $N$ and consider the manifold

$$N_\varphi = \{(x, r) : x \in N \text{ and } 0 \leq r \leq 1\} / (x, 1) \sim (\varphi(x), 0).$$

The Anosov suspension flow of $\varphi$ is the flow $\psi^t: N_\varphi \to N_\varphi$ induced by the translated time

$$\psi^t: N \times \mathbb{R} \to N \times \mathbb{R}, \psi^t(x, s) = (x, s + t).$$

We denoted by $\psi^{\varphi}$ the derivative vector field of $\psi^t$ (cf. [15] for more details).

For this class of Anosov flows we prove:

Corollary 2  Let $\psi^t_{\varphi_0}$ be is an Anosov flow which is a suspension of a $C^2$- Anosov diffeomorphism $\varphi_0$ of a compact surface $N$. Then, arbitrarily close to $\varphi_0$ there is an open set $\mathcal{W}$ of $C^2$ Anosov diffeomorphisms such that for any $\varphi \in \mathcal{W}$ we have

$$\text{int} \, M(\psi^t_{\varphi}, f) \neq \emptyset \text{ and } \text{int} \, L(\psi^t_{\varphi}, f) \neq \emptyset$$

for any $f$ in a dense and $C^2$-open subset $\mathcal{U}_\varphi$ of $C^2(N_\varphi, \mathbb{R})$, where $\psi^t_{\varphi}$ is the suspension flow associated to $\varphi \in \mathcal{W}$.

Remark 3  If $\varphi_0$, in Corollary 2, is a $C^2$- conservative Anosov diffeomorphism, then $\mathcal{W}$ can be considered contained in the $C^2$ conservative setting (see the end of Sect. 4.4.2).

To prove Theorem 1.1, we will use the Main Theorem at [19], but we point out that its proof is not an immediate consequence of the Main Theorem at [19]. We comment on three challenges that need to be overcome to apply the Main Theorem at [19].

The first challenge to overcome is to show a separation Lemma (see Lemma 3.5) using only the $C^0$ stable and unstable foliations of the flow, which allows us to reduce the problem by one dimension. More specifically, the proof of Lemma 3.5 involves some techniques of saturation of surface by one-dimensional foliations. We first construct a finite number of $C^0$-sections “transverse” to the Anosov flow which is saturated
by the stable foliation of the basic set $\Lambda$, such that the union of the flowbox neighborhood of these sections forms a finite cover of $\Lambda$. Then manipulating the hyperbolicity of $\Lambda$, we will make small surgeries to separate this finite number of $C^0$-sections. Finally, we approximate those $C^0$-cross-sections by a $C^\infty$-cross-section but now separated. The second challenge is to produce small conservative perturbations of the flow such that we can obtain, in some way, the conditions to apply the main theorem in [19] (see Sect. 4.2.2). The third and the hardest challenge to overcome is to construct the set $U_{X,\Lambda}$ of the statement of Theorem 1.1, for which we need some non-trivial combinatorial arguments for horseshoe (see Lemma 4.6).

**Structure of Paper** The paper is organized the following way: In Sect. 2, we give a little introduction of Anosov flows, in Sect. 3, we will get the tool to reduce the Theorem 1.1 to a problem of dimension two and we will construct the ingredients to define the set $U_{X,\Lambda}$, in Sect. 4 we will prove the Theorem 4.1, which is a bi-dimensional version of Theorem 1.1, with featured for the Sects. 4.2 and 4.3.1, finally, in “Appendix” we will prove the Corollary 1 and 2.

### 2 Preliminaries

Let $M$ be a complete Riemannian manifold and $\phi^t : M \to M$ a flow on $M$. We say that a closed invariant set $\Lambda \subset M$ is hyperbolic for $\phi^t$ if: there exists a continuous splitting $T\Lambda M = E^s \oplus \langle \Phi \rangle \oplus E^u$ such that for each $\theta \in \Lambda$

$$d\phi^t_\theta(E^s_\theta) = E^s_{\phi^t(\theta)},$$

$$d\phi^t_\theta(E^u_\theta) = E^u_{\phi^t(\theta)},$$

$$||D\phi^t_\theta||_{E^s} \leq C \lambda^t,$$

$$||D\phi^{-t}_\theta||_{E^u} \leq C \lambda^t,$$

for all $t \geq 0$ with $C > 0$ and $0 < \lambda < 1$, where $\Phi$ is the vector field derivative of the flow and $\langle \Phi \rangle$ its span.

When $\Lambda = M$ we said that the flow is an *Anosov flow*, that is, the whole manifold is a hyperbolic set. The subbundles $E^s$ and $E^u$ are known to be uniquely integrable. From the Stable and Unstable Manifold Theorem [15] it follows that there is $\epsilon > 0$ such that for every $x \in \Lambda$ the set

$$W^s_\epsilon(x) = \left\{ y : d(\phi^t(x), \phi^t(y)) \leq \epsilon \text{ and } d(\phi^t(x), \phi^t(y)) \underset{t \to +\infty}{\to} 0 \right\}$$

and

$$W^u_\epsilon(x) = \left\{ y : d(\phi^t(x), \phi^t(y)) \leq \epsilon \text{ and } d(\phi^t(x), \phi^t(y)) \underset{t \to -\infty}{\to} 0 \right\}$$

are invariant $C^r$-manifolds tangent to $E^s_x$ and $E^u_x$, respectively, at $x$, where $d$ is the distance on $M$ induced by the Riemannian metric. Then, we call $W^s_\epsilon(x)$ the local *strong-stable manifold* and $W^u_\epsilon(x)$ the local *strong-unstable manifold*, which by abuse of notation we denote these local manifolds simply writing $W^s_{loc}(x)$ and $W^u_{loc}(x)$,
respectively. Moreover, the manifolds \( W_s^\epsilon(x) \) and \( W_u^\epsilon(x) \) vary continuously with \( x \) (in general, it is the best one can expect for hyperbolic sets). Also, if \( x \in \Lambda \) one has that

\[
W^s(x) = \bigcup_{t \geq 0} \phi^{-t}(W^s_\epsilon(\phi^t(x))) \quad \text{and} \quad W^u(x) = \bigcup_{t \leq 0} \phi^{-t}W^u_\epsilon(\phi^t(x))
\]

are \( C^r \)-invariant manifolds immerse in \( M \), called the **strong-stable manifold** and **strong-unstable manifold** of \( x \), respectively. Finally, the sets

\[
W^{cs}(x) = \bigcup_{t \in \mathbb{R}} W^s(\phi^t(x)) \quad \text{and} \quad W^{cu}(x) = \bigcup_{t \in \mathbb{R}} W^u(\phi^t(x))
\]

are invariant \( C^r \)-manifolds tangent to \( E^s_x \oplus \langle \Phi(x) \rangle \) and \( E^u_x \oplus \langle \Phi(x) \rangle \), respectively.

A special hyperbolic set, where fractal properties are well known, are the **basic sets**, which means:

(a) the periodic orbit contained in \( \Lambda \) are dense in \( \Lambda \),
(b) \( \phi^t|_{\Lambda} \) is transitive,
(c) There is an open set \( U \supset \Lambda \) so that \( \Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U) \).

The definition of hyperbolic sets, basic sets, and Anosov diffeomorphisms are analogous, as well as, the properties above for the stable and unstable manifolds are valid.

Non-trivial basic sets for diffeomorphisms on a surface are called **Horseshoes**. It is not difficult to prove that, for basic sets, the sub-bundles \( E^s \) and \( E^u \) are non-trivial. Moreover, if \( M \) is a basic set for \( \phi^t \), then \( \phi^t \) is a transitive Anosov flow.

Some examples of Anosov flows are geodesic flows on unit tangent bundles of compact Riemannian manifolds of negative curvature, and suspensions of an Anosov diffeomorphism (see Sect. 1.2).

We denote by \( \mathcal{M} \) the set of finite invariant measures. When an Anosov flow (diffeomorphisms) preserves a finite measure \( \tilde{m} \in \mathcal{M} \) absolutely continuous with respect to the Lebesgue measure, we call it **conservative**. In this work, we focus on three-dimensional conservative Anosov flows, which include the suspension flow of a conservative Anosov diffeomorphisms on compact surfaces and the geodesic flows on surfaces of finite volume and pinched negative curvature, since the **Liouville measure** is a natural finite measure that is invariant by the geodesic flow and absolutely continuous with respect to the Lebesgue measure (cf. [22]).

### 3 Separation Lemma and Hyperbolic Set

In this section, we will show that is possible to enclose any hyperbolic set of \( \phi^t \) into a finite number of tubular neighborhoods generated by GCS (Good Cross Sections) pairwise disjoint (see Definition 2). Using this GCS, for our basic set \( \Lambda \), we can construct a basic set with Hausdorff dimension bigger than 1, for the Poincaré map, restricted to the union of such GCS. Also, we can conclude that \( \Lambda \) and, in particular, hyperbolic sets of \( \phi^t \) have topological dimensional 1. For this section, we can assume that \( \Lambda \) is simply a non-trivial compact hyperbolic set.
3.1 Good Cross-Sections

The goal of this section is to present the Lemma 3.5 (Separation Lemma) which is a very important tool for the proof of Theorem 1.1 and whose prove will be made in “Proof of Separation Lemma 3.5”.

Let us fix the following notation, we use $F^s$ and $F^u$ the strong stable and unstable foliation, i.e., $F^i(x) = W^i(x)$ for $i = s, u$, which are continuous foliations of dimension one (not necessarily $C^1$-foliations). We also denote $F^s,u_{loc} = W^s,u_{loc}$ the local stable (unstable) foliation.

**Definition 1** A $C^0$-surface $S$ is transverse to the flow $\phi^t$, if there are $\theta, r > 0$ such that for every $z \in S$ the cone $C_z$ of angle $\theta$ centered in $\Phi(z)$ with vertex at the point $z$ satisfies $C_z \cap B_r(z) \cap S = \{z\}$ (Fig. 1).

**Lemma 3.1** For each $x \in M$, let $L$ be a $C^1$-embedded curve of dimension one, containing $x$ and $C^1$-transverse to the foliation $F^s$, then the set

$$S_L := \bigcup_{z \in L} F^s(z)$$

contains a surface $S_x$, $C^0$-embedded, which contains $x$ in its interior. Moreover, if $L$ is $C^1$-transverse to the foliation $W^c_s$ then, $S_x$ is $C^0$-transverse to the flow.

**Proof** The first part of the theorem is by definition of $C^0$-foliation. For the second part, note that $L$ is $C^1$-transverse to $F^c_s$, then $L$ is $C^1$-transverse to $\phi^t$, moreover, the flow $\phi^t$ is $C^1$-transverse to $F^s$, thus by the continuity of $F^s_{loc}$ we can construct the surface $S_x$, $C^0$-transverse to $\phi^t$. \(\square\)

In particular, taking $L = W^u_\epsilon(x)$ with $\epsilon$ given by the stable and unstable manifold theorem, we call $S_L := S_x$. Note that an analogous lemma holds for the foliation $F^u$. 

Fig. 1 $C^0$-transverse to $\phi^t$
Remark 4 Note that the surface $S_x$ is a $C^0$-surface saturated by the foliation $\mathcal{F}^s$, therefore there is a homeomorphism $h : [0, 1] \times [0, 1] \to S_x$ such that the horizontal lines $[0, 1] \times \eta$ are mapped to the stable sets $W^s(y, S_x) = W^s(y) \cap S_x$. Therefore, we can define the stable-boundary, $\partial^s S_x$, of $S_x$, as being the image of $[0, 1] \times \{0, 1\}$ by the homeomorphism $h$ and the unstable-boundary, $\partial^u S_x$, of $S_x$ as being the image of $\{0, 1\} \times [0, 1]$ by the homeomorphism $h$.

From now on, unless otherwise stated, we consider cross-section as the Lemma 3.1.

Definition 2 Let $\Lambda$ be a compact subset of $M$.

We say that a compact cross-section $\Sigma$ is a Good Cross-Section (or simply GCS) for $\Lambda$ if

$$d(\Lambda \cap \Sigma, \partial^u \Sigma) > 0 \quad \text{and} \quad d(\Lambda \cap \Sigma, \partial^s \Sigma) > 0,$$

where $d$ is the intrinsic distance in $\Sigma$.

If $\Sigma$ is a GCS and $x \in \Lambda \cap \Sigma$, we say that $\Sigma$ is a GCS at $x$.

By compactness of $\Sigma$ and $\Lambda$, the above definition implies that there is $\delta > 0$ such that (cf. Fig. 2).

$$d(\Lambda \cap \Sigma, \partial^u \Sigma) > \delta \quad \text{and} \quad d(\Lambda \cap \Sigma, \partial^s \Sigma) > \delta. \quad (1)$$

The GCS play an important role in reducing Theorem 1.1 to a two-dimensional problem since they allow to enclose the set $\Lambda$ in a tubular neighborhood far from its boundary.

Remark 5 Let $\Sigma$ be a GCS, there are two GCS $\Sigma'$ and $\Sigma''$ such that

$$\Sigma' \subset \text{int}(\Sigma), \quad \partial \Sigma' \cap \partial \Sigma = \emptyset \quad \text{and} \quad \Sigma \subset \text{int}(\Sigma''), \quad \partial \Sigma \cap \partial \Sigma'' = \emptyset.$$
Therefore, from now on, we can assume that if two GCS has nonempty intersection, then their interiors have nonempty intersection.

The GCS may not exist in general, for example, for hyperbolic sets with singularities (cf. [4, Lemma 6.22]), however, if \( \Lambda \subsetneq M \) is a hyperbolic set of a transitive three-dimensional Anosov flow, then we will prove that GCS always exist (cf. Lemma 3.4). For this sake, we prove the following lemmas:

**Lemma 3.2** Let \( \Lambda \subsetneq M \) a hyperbolic set for a transitive three-dimensional Anosov flow on \( M \). Then, for any \( x \in \Lambda \) there exist points \( x^+ \notin \Lambda \) and \( x^- \notin \Lambda \) in distinct connected components of \( W^s(x) \setminus \{x\} \).

**Proof** Note that, for a three-dimensional Anosov flow, the stable manifold is one-dimensional. Let \( x \in \Lambda \), by contradiction, assume that there is a segment of the strong stable manifold entirely contained in \( \Lambda \) and containing \( x \) in the interior, we called by \( \zeta \) this segment. Without loss of generality, we can assume that \( W^s_{\text{loc}}(x) \subset \zeta \). Now take \( t_k \) a sequence such that \( t_k \to \infty \) as \( k \to \infty \). Then, as \( \Lambda \) is a compact invariant set, we can assume that \( \phi^{-t_k}(x) \to y \in \Lambda \) as \( k \to \infty \). The point \( y \) satisfies:

Claim: \( W^s(y) \subset \Lambda \).

**Proof of Claim** Let \( z \in W^s(y) \), as \( W^s(y) = \bigcup_{t \geq 0} \phi^{-t} \left(W^s_{\text{loc}}(\phi^t(y))\right) \), then there is \( T \geq 0 \), such that \( \phi^T(z) \in W^s_{\text{loc}}(\phi^T(y)) \). Then by Stable Manifold Theorem \( W^s_{\text{loc}}(\phi^T(y)) \) is accumulated by points of \( W^s_{\text{loc}}(\phi^{-t_k+T}(x)) \), for large enough \( k \). Let \( k \) be sufficiently large such that \( -t_k + T < 0 \) and \( W^s_{\text{loc}}(\phi^{-t_k+T}(x)) \subset \phi^{-t+k+T}(\zeta) \subset \Lambda \), since \( \Lambda \) is an invariant set and \( \zeta \subset \Lambda \). Hence as \( \Lambda \) is closed, we have that \( W^s_{\text{loc}}(\phi^T(y)) \subset \Lambda \) which implies that \( z \in \Lambda \), which completes the proof of claim. \( \square \)

The above claim implies that \( \Lambda \supset W^{cs}(y) = \bigcup_{t \in \mathbb{R}} W^s(\phi^t(y)) \). In fact:

Let \( w \in W^{cs}(y) \), then there is \( t_0 \in \mathbb{R} \) such that \( w \in W^s(\phi^{t_0}(y)) \). Hence, there is \( T \geq 0 \) such that \( \phi^T(w) \in W^s_{\text{loc}}(\phi^{T+t_0}(y)) \). Since \( \phi^{T+r}(w) \in W^s_{\text{loc}}(\phi^{T+r+t_0}(y)) \) for \( r \geq 0 \), where \( C \) is the constant of the definition of hyperbolicity of \( \Lambda \). So, we can assume that \( T + t_0 > 0 \). Thus,

\[
\phi^{-t_0}(w) = \phi^{-(T+t_0)}(\phi^T(w)) \in \phi^{-(T+t_0)} \left(W^s_{\text{loc}}(\phi^{T+t_0}(y))\right) \subset W^s(y) \subset \Lambda.
\]

Since \( \Lambda \) is invariant, then we have that \( W^{cs}(y) \subset \Lambda \).

So, the \( \phi^t \) is transitive, then \( M = W^{cs}(y) \subset \Lambda \) (cf. [15]), which provides a contradiction. Thus, we conclude the proof of the lemma. \( \square \)

Analogously we have,

**Lemma 3.3** Let \( \Lambda \subsetneq M \) a hyperbolic set for a transitive three-dimensional Anosov flow on \( M \). Then, for any \( y \in \Lambda \) there are points \( y^+ \notin \Lambda \) and \( y^- \notin \Lambda \) in distinct connected components of \( W^u(x) \setminus \{x\} \).

**Proof** Similar to the proof of Lemma 3.2. \( \square \)

**Lemma 3.4** Let \( \Lambda \subsetneq M \) be a hyperbolic set for a transitive three-dimensional Anosov flow on \( M \). Then, for every \( x \in \Lambda \) there is a GCS \( \Sigma_x \) at \( x \) with \( \Sigma_x \subset S_x \).
**Proof** Fix $\epsilon > 0$ as in the stable and unstable manifold theorem, and consider the cross-section $\Sigma_x$ given by the Lemma 3.1 containing the segments of $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ and the point $x$ in its interior. By Lemmas 3.2 and 3.3, we may find points $x^\pm \notin \Lambda$ in each of the connected components of $(W^s_\epsilon(x) \cap \Sigma_x) \setminus \{x\}$ and points $z^\pm \notin \Lambda$ in each of the connected components of $(W^u_\epsilon(x) \cap \Sigma_x) \setminus \{x\}$. Since $\Lambda$ is closed, there are neighborhoods $V^\pm$ of $x^\pm$ and $V^+_1$ of $z^\pm$ respectively disjoint from $\Lambda$, (cf. Fig. 3).

In Fig. 3, $V^\pm$, $V^+_1$ may enclose a region homeomorphic to a square, in this case, there is nothing to be done. Otherwise, we prove that we can obtain open sets in the cross-section which does not intersect $\Lambda$ and enclose a region homeomorphic to a square. Indeed, let $t_k$ be a sequence such that $t_k \to +\infty$ as $k \to +\infty$ and $\phi^{t_k}(x) \to y \in \Lambda$ as $k \to +\infty$, then by Lemma 3.2, there are $y^\pm$ in each of the connected components of $W^u_\epsilon(y)$ and $y^\pm \notin \Lambda$, so there are neighborhoods $J^\pm$ of $y^\pm$, respectively, with $J^\pm \cap \Lambda = \emptyset$.

The stable and unstable manifold theorem provides the following properties for $y$.

(i) There is a neighborhood $U_y$ of $y$ such that

$$W^u_\epsilon(z) \cap J^\pm \neq \emptyset, \text{ for all } z \in U_y.$$  

(ii) There is $\delta > 0$ such that $W^s_\delta(y^\pm) \subset J^\pm$, respectively.

Note that, there is $k_0$ such that $\phi^{t_k}(x) \in U_y$, for all $k \geq k_0$. Then by the property (i) and the stable and unstable manifold theorem (increasing $k_0$, if necessary), we have that there is $w^\pm_k \in W^u_\epsilon(\phi^{t_k}(x)) \cap J^\pm$ which converge to $y^\pm$ as $k \to +\infty$, respectively (see Fig. 4).
Then, again by the stable and unstable manifold theorem and property (ii), there are $\delta_0 \leq \frac{\delta}{2}$ and $k_1 \geq k_0$ such that
\[
W^s_{\delta_0}(w^\pm_k) \subset J^\pm \text{ and } W^s_\epsilon(\phi^{-tk}(w^\pm_k)) \subset \phi^{-tk}(W^s_{\delta_0}(w^\pm_k)),
\]
whenever $k \geq k_1$ (see Fig. 4).

Observe that
\[
d(x, \phi^{-tk}(w^\pm_k)) = d(\phi^{-tk}(\phi^{+tk}(x)), \phi^{-tk}(w^\pm_k)) \leq C\lambda^k d(\phi^{tk}(x), w^\pm_k) \leq C\lambda^k \epsilon.
\]
Thus, for $k \geq k_1$ large enough, the last inequality and the stable and unstable theorem implies that $W^s_\epsilon(\phi^{-tk}(w^\pm_k))$ is $C^0$-close to $W^s_\epsilon(x)$ and therefore satisfies $W^s_\epsilon(\phi^{-tk}(w^\pm_k)) \cap V^\pm \neq \emptyset$ and $W^s_\epsilon(\phi^{-tk}(w^\pm_k)) \cap V^\pm \neq \emptyset$.

Moreover, the relation (2) implies that $W^s_\epsilon(\phi^{-tk}(w^\pm_k)) \subset \phi^{-tk}(J^\pm)$ and therefore, since $J^\pm \cap \Lambda = \emptyset$, then $W^s_\epsilon(\phi^{-tk}(w^\pm_k)) \cap \Lambda = \emptyset$, for $k \geq k_1$.

Consider $V^\pm_k$ a neighborhood of $W^s_\epsilon(\phi^{-tk}(w^\pm_k))$, respectively, such that $V^\pm_k \cap \Lambda = \emptyset$. As we know that $\phi^{-tk}(w^\pm_k) \in W^u_\epsilon(x)$, then $V^\pm \cap \Sigma_\chi$ and $V^\pm \cap \Sigma_\chi$ enclose a region homeomorphic to a square and such region contains a GCS $\Sigma_k := \Sigma_\chi$, as we wish. \qed
Remark 6 It is worth note that, the stable boundary of $\Sigma_k$, $\partial^s \Sigma_k$, is equal to $W^s_{e}(\phi^{-tk}(w^s_k))$, which converge (in the $C^0$-topology) to $W^s_{e}(x)$. Thus, we can state that the cross-section of the above lemma can be taken such that the stable boundary is as close as we want to $W^s_{e}(x)$ (similar to Palis’ $\lambda$-Lemma). Analogously, the GCS of the above lemma can be constructed using unstable saturation.

The cross-sections constructed on the Lemma 3.3 and Remark 6 are $C^0$-sections and have some important properties which will be described next.

Definition 3 Given $\theta > 0$ and $r > 0$, we say that a continuous curve $\xi \subset \mathbb{R}^2$ is $\theta$-transverse in a neighborhood of radius $r$ to a one-dimensional foliation $F$ (with $C^1$-leaves) in $\mathbb{R}^2$, if for any $z \in \xi \cap F_z$ (here $F_z$ is the leaf containing $z$) there is a cone $C$ with vertex at the point $z$ such that $\xi \cap B(z, r) \subset C$ and the angle $\angle(v, T_z F_z) \geq \theta$ for every vector $v \in C$ (Fig. 5).

As the cross-section $\Sigma_x$ is saturated by the foliation $F^s$, then we call $h_x$ the homeomorphism given in Remark 4, then we have the following definition:

Definition 4 We say that a continuous curve $\zeta \subset \Sigma_x$ is transverse to foliation $F^s$, if there are $\theta > 0$ and $r > 0$ such that $h_x^{-1}(\zeta)$ is $\theta$-transverse in a neighborhood of radius $r$ to the foliation $\{h_x^{-1}(F_z^s \cap \Sigma_x) : z \in W^u_{loc}(x)\}$.

Proposition 1 Given $x, y \in \Lambda$, such that there is a $C^0$-curve $\zeta \subset \text{int } \Sigma_x \cap \text{int } \Sigma_y$. If $\zeta$ intersects transversely the foliation $F^s$, then $\text{int } \Sigma_x \cap \text{int } \Sigma_y$ is an open set of $\Sigma_x$ and $\Sigma_y$.

Proof Since $\zeta \subset \text{int } \Sigma_x \cap \text{int } \Sigma_y$ is a $C^0$-curve transverse to $F^s$. Then for all $z \in \zeta$, there are $x' \in W^u_{e}(x)$ and $y' \in W^u_{e}(y)$ such that $z \in W^s(x') \cap \Sigma_x$ and $z \in W^s(y') \cap \Sigma_y$, thus $W^s(x') = W^s(y')$. Therefore, there is $\delta > 0$ such that the set

$$B = \bigcup_{z \in \zeta} W^s_{e}(z) \subset \text{int } \Sigma_x \cap \text{int } \Sigma_y.$$

Thus, we conclude the proof of the proposition. □
3.1.1 Separation of GCS

By Lemma 3.4, at each point of \( x \in \Lambda \), we can find a GCS \( \Sigma_x \). Since \( \Lambda \) is a compact set, then for \( \gamma > 0 \), there are a finite number of points \( x_i \in \Lambda, i = 1, \ldots, l \) such that

\[
\Lambda \subset \bigcup_{i=1}^{l} \phi^{-\gamma \cdot \gamma} (\text{int} \, \Sigma_i) := \bigcup_{i=1}^{l} U_{\Sigma_i},
\]

where \( \Sigma_i := \Sigma_{x_i} \).

The main goal of this section is the following lemma, which has very technical proof that will be presented in “Proof of Separation Lemma 3.5”.

**Lemma 3.5** There is \( m \in \mathbb{N} \) and GCS \( \tilde{\Sigma}_i, i = 1, \ldots, m \) such that

\[
\Lambda \subset \bigcup_{i=1}^{m} \phi^{-2 \gamma \cdot 2 \gamma} (\text{int} \, \tilde{\Sigma}_i)
\]

with \( \tilde{\Sigma}_i \cap \tilde{\Sigma}_j = \emptyset \).

The above lemma state that the GCS in (3) can be taken pairwise disjoint and therefore reduce our problem to the study the Lagrange and Markov spectra of the map of first return (Poincaré map) on the union of these sections (see Sects. 3.2.1 and 4.1).

**Remark 7** Since \( C^\infty \)-topology is dense in \( C^0 \)-topology, from now on, we can assume without loss of generality, that there are \( C^\infty \)-GCS, \( \Sigma_i \), pairwise disjoint which satisfies the Lemma 3.5.

We ended this section by announcing an immediate consequence of Lemma 3.5 and the definition of GCS, which will be used in Sect. 4.4 to construct a basic set for geodesic flows in pinched negative curvature.

**Corollary 3** Any hyperbolic set of a three-dimensional transitive Anosov flow has topological dimension 1.

3.2 Poincaré Map

Let \( \Xi = \bigcup_{i=1}^{m} \Sigma_i \) be a finite union of cross-sections to the flow \( \phi_t \) given by Remark 7, which are pairwise disjoint. Sometimes, abusing of notation, we consider \( \Xi = \{ \Sigma_1, \ldots, \Sigma_m \} \). Let \( \mathcal{R} : \Xi \to \Xi \) be a Poincaré map, that is, the map of first return to \( \Xi \), \( \mathcal{R}(y) = \phi^{t_+(y)}(y) \), where \( t_+(y) \) corresponds to the first time that the positive orbits of \( y \in \Xi \) encounter \( \Xi \).

**Remark 8** It is not difficult to show that \( \mathcal{R} \) is a conservative diffeomorphism, i.e., there is a volume form on \( \Xi \) which is preserved by \( \mathcal{R} \) (see Lemma 5.10).
From the compactness of $\Lambda$, it is easy to see that $t_+(y)$ is uniformly bounded. This property together with the hyperbolicity of $\Lambda$ will allow us to show the hyperbolicity of $R$, as will be explained in the next section.

### 3.2.1 Hyperbolicity of Poincaré Map

In this section, we will explore some properties of set $\Delta := \Lambda \cap \Xi$. Our first property is:

**Lemma 3.6** The set $\Delta$ is hyperbolic for $R$.

We used some arguments find at [4, ch. 6] to do the proof of Lemma 3.6, which will be presented in “Proof of Hyperbolicity of Poincaré Map”.

### 3.2.2 Hausdorff Dimension of Hyperbolic Set of $R$

In this section, we estimate the Hausdorff dimension of $\Lambda$ using the Hausdorff dimension of $\Delta$.

**Lemma 3.7** The set $\Lambda$ satisfies

$$
\Lambda \subset \bigcup_{i \in \mathbb{R}} \phi^i \left( \bigcap_{n \in \mathbb{Z}} R^{-n}(\Delta) \right) = \bigcup_{i \in \mathbb{R}} \phi^i(\Delta)
$$

**Proof** Remember that $\Lambda \subset \bigcup_{i=1}^{m} U_{\Sigma_i}$, where $U_i = \phi^{(-2\gamma,2\gamma)}(\text{int} \Sigma_i)$. Let $z \in \Lambda$, then there is $t_z$ such that $z = \phi^{t_z}(x)$ with $x \in \text{int} \Sigma_i$ for some $i$. This implies that $x \in \Lambda \cap \Xi$ and therefore, $R(x) \in \text{int}(\Sigma_j)$ for some $j$, so $R(x) \in \text{int}(\Xi)$. Analogously, $R^n(x) \in \text{int}(\Xi)$, i.e., $R^n(x) \in \Lambda \cap \Xi$ for all $n \in \mathbb{Z}$. Hence, $x \in \bigcap_{n \in \mathbb{Z}} R^{-n}(\Lambda \cap \Xi)$, therefore $z \in \phi^{t_z} \left( \bigcap_{n \in \mathbb{Z}} R^{-n}(\Lambda \cap \Xi) \right)$. \(\square\)

**Lemma 3.8** The Hausdorff dimension of Delta and $\Lambda$ satisfies,

$$
HD(\Lambda) \leq HD(\Delta) + 1.
$$

**Proof** Take a bi-infinite sequence

$$
\cdots < t_{-k} < t_{-k+1} < \cdots < t_0 < t_1 < \cdots t_k < \cdots
$$

such that $|t_k - t_{k+1}| < \alpha$ with $\alpha$ sufficiently small, then

$$
\Lambda \subset \bigcup_{k=-\infty}^{+\infty} \phi^{[t_k,t_{k+1}]}(\Lambda \cap \Xi) := \bigcup_{k=-\infty}^{+\infty} A_k.
$$

Then, $HD(\Lambda) \leq \sup_k HD(A_k)$. Moreover, if $\alpha$ is small enough, the map

$$
\psi_k : (\Lambda \cap \Xi) \times [t_k, t_{k+1}] \longrightarrow A_k \quad \text{defined by}
$$

$$(x, t) \quad \longleftrightarrow \quad \phi^t(x)$$
is Lipschitz. Therefore, if we call $I_k = [t_k, t_{k+1}]$, it is easy to see that

$$HD(A_k) \leq HD(\Delta \times I_k) \leq HD(\Delta) + D(I_k),$$

where $D$ is the upper Box-Counting Dimension of $I_k$. It is easy to see that $D(I_k) = 1$ (cf. [13]). Thus,

$$HD(\Lambda) \leq \sup_k HD(A_k) \leq HD(\Delta) + 1.$$

\[\square\]

Corollary 4 If $HD(\Lambda) > 2$, then $HD(\Delta) > 1$.

4 Lagrange and Markov Spectrum

In this section, we prove the Theorem 1.1. In this direction, we will prove an equivalent version (Theorem 4.1), which is an appropriate reduction of the problem of finding a non-empty interior for the Lagrange and Markov spectrum for discrete dynamical systems in dimension two.

4.1 Regaining the Spectrum

The dynamical Lagrange and Markov spectra of $\Lambda$ and $\Delta$ are related in the following way. Given a function $F \in C^s(M, \mathbb{R}), s \geq 1$, let us denote by $f = \max F_\Phi : D_R \to \mathbb{R}$ the function

$$\max F_\Phi(x) := \max_{0 \leq t \leq t_+(x)} F(\phi^t(x)),$$

where $D_R$ is the domain of $R$ and $t_+(x)$ is such that $R(x) = \phi^{t_+(x)}(x)$.

It is not difficult to show that

$$\limsup_{n \to +\infty} f(R^n(x)) = \limsup_{t \to +\infty} F(\phi^t(x))$$

and

$$\sup_{n \in \mathbb{Z}} f(R^n(x)) = \sup_{t \in \mathbb{R}} F(\phi^t(x))$$

for all $x \in \Delta$. In particular,

$$L(\Phi, \Lambda, F) = L(R, \Delta, f) \quad \text{and} \quad M(\Phi, \Lambda, F) = M(R, \Delta, f). \quad (5)$$

Remark 9 $f = \max F_\Phi$ might not be $C^1$ in general.
Remark 10 It is worth noting that given a vector field $X$ close to $\Phi$, the Poincaré map $R_X$ of the flow of $X$ is defined in the same cross-sections where $R$ is defined.

Thus, the relation (5) reduce the Theorem 1.1 to the following theorem:

**Theorem 4.1** Let $\Phi$ be a vector field, such that $\phi^t$ is a conservative Anosov flow, which has a compact basic set $\Lambda$ with Hausdorff dimension bigger than 2, then $C^2$-arbitrarily close to $\Phi$ there is an open set $W \subset X^2(M)$, such that for any $X \in W$, if $D_X$ is the hyperbolic continuation of $\Delta$ by the Poincaré map $R_X$, one can find a dense and $C^2$-open subset $U_{X,\Lambda} \subset C^2(M, \mathbb{R})$, so that

$$\text{int} M(R_X, D_X, \max F_X) \neq \emptyset \text{ and } \text{int} L(R_X, D_X, \max F_X) \neq \emptyset,$$

whenever $F \in U_{X,\Lambda}$. Moreover, the above statement holds persistently, i.e., for any $Y \in W$, it holds for any $(F, X)$ in a suitable neighborhood of $U_{Y,\Lambda} \times \{Y\}$ in $C^2(M, \mathbb{R}) \times X^2(M)$.

4.2 Family of Perturbations

In Sect. 3 has been proven that there is a finite number of $C^\infty$-GCS, $\Sigma_i$ pairwise disjoint and such that the Poincaré map $R: \Xi \to \Xi$, (first return to $\Xi$) where $\Xi := \cup_{i=1}^m \Sigma_i$ satisfies:

- The set $\Delta$ is a basic set for $R$, since $\Lambda$ is a basic set for $\phi^t$.
- If $HD(\Lambda) > 2$, then $HD(\Delta) > 1$.

The main goal of this section is to construct a family of perturbations of $\Phi$, which produces perturbations on $R$ so that we can apply the techniques of [19] (cf. “Intersections of Regular Cantor Sets and Property V” and [20]).

Remark 11 From now on, we will consider vector fields $X \in X^2(M)$, $C^2$-sufficiently close to $\Phi$ such that: If we denote by $R_X: \Xi \to \Xi$ the Poincaré map associated to $X$, then

1. There exists the hyperbolic continuation $D_X$ of $\Delta$ by the map $R_X$.
2. $HD(D_X) > 1$, since the Hausdorff dimension of the basic sets, is continuous for $C^2$-diffeomorphisms on $\Xi$ (cf. [24]).

4.2.1 First Perturbation for The Birkhoff Invariant

We note that the Poincaré map $R$ is a conservative diffeomorphism since the flow is conservative (see Remark 8). Thus, in order to use some techniques at [19], we need an appropriate family of perturbations of $R$ (cf. [20]). For this sake, we perform a first conservative perturbation of $\Phi$, such that the remain Poincaré map has the Birkhoff invariant non-zero in some periodic point (cf. “The Birkhoff Invariant” and [21, Section 4.3]). In other words

Remark 12 We can assume, from now on, that the Poincaré map $R$ associated to the flow $\phi^t$ has the property that the Birkhoff invariant is non-zero for some periodic orbit (see, “The Birkhoff Invariant”).
4.2.2 Family of Perturbations with the Property V

The central goal of this section is to do small conservative perturbations of $\Phi_1$, in order to produce a family of perturbations of $R$ with good properties, which allow the use of the techniques of [19], more specifically, the property $V$ which will be explained in “Intersections of Regular Cantor Sets and Property $V$”. Therefore, the following three lemmas focus on this goal.

**Lemma 4.1** Given $\mathcal{V}$ a $C^r$-neighborhood of $\Phi$ and $p \in \Delta \cap \Sigma$ with $\Sigma \in \Xi$. Let $U$ be a neighborhood of $\phi_{t_0}^{-\frac{t}{2}}(p)$, then there exists a conservative vector field $X \in \mathcal{V}$ such that:

1. $X \equiv \Phi$ outside of $U$,
2. There is $\tau > 0$, such that $X \equiv \Phi$ outside of a subset of $U$ of the form $X^{[0,\tau]}(\Sigma_0) = \{X'(x) : x \in \Sigma_0, 0 < t < \tau\}$, where $\Sigma_0$ is a neighborhood of $\phi_{t_0}^{-\frac{t}{2}}(p)$ in $\phi_{t_0}^{-\frac{t}{2}}(\Sigma)$,
3. The map $\mathcal{R}_X$ satisfies $\mathcal{R}_X(p) \neq \mathcal{R}(p)$.

The proof of this lemma is an immediate consequence of the two lemmas below. The first is about conservative trivialization and the second is about local conservative perturbations.

**Lemma 4.2** [5, Lemma 3.4 (Conservative flow box theorem)] Let $X \in \mathcal{X}_{\omega}(M)$, $p$ be a regular point of the vector field and $\Sigma$ a cross-section of $X$ which contains $p$, then there exists a $C^\infty$-coordinate system $\alpha : U \subset M \to \mathbb{R}^3$ with $\alpha(p) = 0$ and such that

1. $\alpha_x X = (0, 0, 1)$,
2. $\alpha_x \omega = dx \wedge dy \wedge dz$,
3. $\alpha(U \cap \Sigma) \subset \{z = 0\}$.

To the next lemma, let $B_\delta(x, y) \subset \mathbb{R}^2$ be the open ball of center $(x, y)$ and radius $\delta$. Similarly, $\overline{B_\delta}(x, y)$ denotes the closed ball. If $C$ is the cylinder $\partial B_\delta(x, y) \times [0, h] \subset \mathbb{R}^3$
and $0 < \beta < \delta$, we define a neighborhood of $C$ as
\[ A_\beta(C) = (B_{\delta+\beta}(x, y) \setminus \overline{B}_{\delta-\beta}(x, y)) \times [0, h] \subset \mathbb{R}^3. \]
and call it **cylinder ring** with center at $C$ and radius $\beta$.

**Lemma 4.3** [10, Lemma 3.2] Let $X : \mathbb{R}^3 \to \mathbb{R}^3$ be the constant vector field defined by $X(x, y, z) = (0, 0, 1)$. Consider the cylinder ring $C = \partial B_{\delta}(0, 0) \times [0, h] \subset \mathbb{R}^3$, $\delta > 0$, $h > 0$, and points $p \in \partial B_{\delta}(0, 0) \times [0]$ and $q \in \partial B_{\delta}(0, 0) \times \{h\}$. Let $\theta$ be the angle between the vector $p - (0, 0, 0)$ and $q - (0, 0, h)$. Given $0 < \beta < \delta$ there exists a $C^\infty$ vector field $Z$ on $\mathbb{R}^3$ with the following properties (Fig. 6):

(a) $Z$ preserves the canonical volume form $dx \wedge dy \wedge dz$,
(b) $Z \equiv X$ outside the cylinder ring $A_\beta(C)$,
(c) The positive orbit of $p$, with respect to $Z$, contains $q$,
(d) Given $r \in \mathbb{N}$ and $\epsilon > 0$, if $|\theta|$ is small enough, then $\|Z - X\|_r < \epsilon$, where $\| \cdot \|$ denotes the $C^r$ norm on the set of $C^r$ vector fields.

**Proof of Lemma 4.1** By Lemma 4.2, we can consider a coordinates systems $\alpha : U' \to V$ in a neighborhood $U' \subset U$ of $\phi^{t_{\alpha}(p)}(p)$, with $\alpha(p) = (0, 0, 0)$, $\alpha_* \Phi = (0, 0, 1)$, $\alpha_* \omega = dx \wedge dy \wedge dz$ and $\alpha(\phi^{t_{\alpha}(p)}(\Sigma) \cap U') \subset \{z = 0\}$. Let $\beta, \delta > 0$, $0 < h < 1$ and $q \in \{z = 0\}$ such that the solid cylinder $\overline{B}_{\beta+\delta}(q) \times [0, h] \subset V$ and $(0, 0, 0) \in \partial B_{\delta}(q)$.

Consider now the cylinder ring $A_\beta(C)$ defined by $\beta$, $\delta$, $h$ and $q$. Let $q' \in \partial B_{\delta}(q) \times \{h\}$, and put $\theta$ the angle between $(0, 0, 0) - q$ and $q' - (q, h)$. Now we may apply the Lemma 4.3 at the cylinder $C = \partial B_{\delta}(q) \times [0, h]$ to join $0$ to $q'$ and obtain a vector field $Z$ on $V$ such that:

(a) $Z$ preserves the canonical volume form,
(b) $Z \equiv (0, 0, 1)$ in $V \setminus A_\beta(C)$,
(c) The positive orbit of $0$ respect to $Z$, contains $q'$.

Let us define the vector field $X$ in $M$ in the following way: $X \equiv \Phi$ outside of $U'$ and $X = \alpha_*(Z)$ in $U'$. Note that $X$ is $C^r$ satisfies (1) and taking $\theta$ sufficiently small, we may assume $X \in \mathcal{V}$.

In order to prove (2), we consider $\Pi_0 \subset V \cap \{z = 0\}$ a compact neighborhood of the origin contained in $\alpha(\phi^{t_{\alpha}(p)}(\Sigma) \cap U'))$. Then just take $\Sigma_0 = \alpha^{-1}(\Pi_0)$ and $\tau = \text{sup}\{t > 0 : \alpha(X^t(x)) \in \Pi_0 \times [0, h], x \in \Sigma_0\}$. The item (3) is an immediate consequence of the properties (b) and (c) of the field $Z$.

The proof of Lemma 4.1, implies that for $p \in \Delta$ and every small $\theta$ there is $X_p^0 \in \mathcal{X}^r_{c\omega}(M)$ such that the Poincaré map $\mathcal{R}_{X^0_p}$ associated to $X^0_p$ satisfies $\mathcal{R}_{X_p^0}(p) \neq \mathcal{R}(p)$.

In particular, if $p \in \Sigma \cap \Delta$, then
\[
\mathcal{R}_{X^0_p}(W^s_{loc, \mathcal{R}}(q, \Sigma)) \cap \mathcal{R}(W^s_{loc, \mathcal{R}}(q, \Sigma)) = \emptyset, \tag{6}
\]
for $q \in \Sigma \cap \Delta$ close to $p$, where $W^s_{loc, \mathcal{R}}(q, \Sigma)$ is the local stable manifold associated to $\mathcal{R}$. 
Note that if $\theta = 0$, then $X_{\theta} = \Phi$.

As $\Delta$ is a compact hyperbolic set, then there are a finite number of points in $\Delta$, say $p_1, \ldots, p_n$ and neighborhood $U_i$ of $\phi^{t_{\frac{1}{n}}(p_i)}$, pairwise disjoint as Lemma 4.1, and such that the projection of $\bigcup_{i} U_i$ over $\Xi$ along the flow $\phi^t$ contains a small Markov partition of $\Delta$.

So, we can define the $C^r$-vector field $X_{\theta} \in \mathcal{X}^r_\omega(M)$ by

$$X_{\theta} = \begin{cases} \frac{X_{p_i}}{\Phi} & \text{if } x \in U_i; \\ \Phi & \text{otherwise}. \end{cases}$$

As $\theta$ is small, then the flow of $X_{\theta}$ is still a conservative Anosov flow, since the Anosov flows are robust.

Consider the map $H^\theta(x) := \mathcal{R}^{-1} \circ \mathcal{R}_{X_{\theta}}(x)$ defined in a small Markov partition of $\Delta$. Then by equation (6), the map $\mathcal{R}^\theta := \mathcal{R} \circ H^\theta$ satisfies

$$\mathcal{R}^\theta(W^s_{loc, \mathcal{R}}(q, \Sigma)) \cap \mathcal{R}(W^s_{loc, \mathcal{R}}(q, \Sigma)) = \emptyset, \text{ for any } q \in \Delta. \quad (7)$$

The last equation implies the following Lemma (cf. “Intersections of Regular Cantor Sets and Property $V$” and [21]).

**Lemma 4.4** The family of perturbations $\mathcal{R}^\theta$ of $\mathcal{R}$ satisfies that the pair $(\mathcal{R}^\theta, \Delta_{X_{\theta}})$ has the property $V$. Moreover, this property is persistent, i.e., there exists a $C^2$-neighborhood $\mathcal{W}_{\theta} \subset \mathcal{X}^r_\omega(M)$ of $X_{\theta}$ such that for all $X \in \mathcal{W}_{\theta}$ the pair $(\mathcal{R}_X, \Delta_X)$ also have the property $V$.

### 4.3 Description of the Set $\mathcal{U}_{X, \Lambda}

The construction of the set $\mathcal{U}_{X, \Lambda}$ is closely related to the differentiability of $\max F_{\Phi}$, however, in general, the function $\max F_{\Phi}$ is not differentiable (see Remark 9). Thus, our next sections will be devoted to showing the differentiability of $\max F_{\Phi}$, at least reducing its domain.

#### 4.3.1 Combinatorial Arguments and Differentiability of $\max F_{\Phi}$

The following lemma is combinatorial and will be used to show the Lemma 4.6.

**Lemma 4.5** Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix such that $a_{ij} \in \{0, 1\}$ for any $i$, $j$ and

$$\#(i, j : a_{ij} = 1) \geq \frac{99}{100} n^2,$$

then $tr(A^k) \geq \left(\frac{n}{2}\right)^k$ for all $k \geq 2$. Moreover, there is a set $Z \subset \{1, 2, \ldots, n\}$ with $\#Z \geq \frac{4n}{5}$ such that, for any $k \geq 2$ and any $i, j \in Z$, we have

$$(A^k)_{ij} \geq \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} n^{k-1},$$

where $\#X$ denotes the cardinal of a finite set $X$. 
Remember that if $B = (b_{ij})_{1 \leq i, j \leq n}$ is a square matrix, then $tr(B) = \sum_{i=1}^{n} b_{ii}$ denotes the trace of $B$.

**Proof** There is $X \subset \{1, 2, \ldots, n\}$ with $|X| \geq \frac{9n}{10}$ such that, for any $i \in X$,

$\#\{j \leq n : a_{ij} = 1\} \geq \frac{9n}{10}$. Indeed, if there are more than $\frac{n}{10}$ lines in the matrix, each with at least $\frac{n}{10}$ null entries, then the number of null entries of the matrix is greater that $\frac{n^2}{100}$, and so $\#\{(i, j) : a_{ij} = 1\} < n^2 - \frac{n^2}{100} = \frac{99n}{100}$ which is a contradiction.

Analogously, there is $Y \subset \{1, 2, \ldots, n\}$ with $\#Y \geq \frac{9n}{10}$ such that, for any $j \in Y$,

$\#\{i \leq n : a_{ij} = 1\} \geq \frac{9n}{10}$. Let $Z = X \cap Y$, we have $\#Z \geq \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5}$. If $i, j \in Z$, then

$$(A^2)_{ij} = \sum_{r=1}^{n} a_{ir} a_{rj} = \sum_{r \in A_i \cap B_j} a_{ir} a_{rj} = \#(A_i \cap B_j) \geq \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5},$$

where $A_i = \{j \leq n : a_{ij} = 1\}$ and $B_j = \{i \leq n : a_{ij} = 1\}$. We will show by induction that if $i, j \in Z$, then

$$(A^k)_{ij} \geq \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} \quad \text{for all } k \geq 2.$$  

In fact, the case $k = 2$ was proved above and given $k \geq 2$ for which the statement is true, we have

$$(A^{k+1})_{ij} = \sum_{r=1}^{n} (A^k)_{ir} \cdot a_{rj} \geq \sum_{r \in Z} (A^k)_{ir} \cdot a_{rj}$$

$$\geq |Z \setminus \{r \in Z : a_{rj} = 0\}| \times \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1}$$

$$\geq \left(\frac{4n}{5} - \frac{n}{10}\right) \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} > \frac{4}{5} \left(\frac{3}{5}\right)^{k-1} \cdot n^k,$$

since $\#\{r \in Z : a_{rj} = 0\} \leq \frac{n}{10}$.

Thus, for all $k \geq 2$

$$tr(A^k) \geq \sum_{i \in Z} (A^k)_{ii} \geq \frac{4n}{5} \cdot \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} > \left(\frac{3}{5}\right)^{k} \cdot n^k > \left(\frac{n}{2}\right)^k.$$

\[\square\]

**Remark 13** Suppose that the matrix $A$ as in Lemma 4.5, is the matrix of transitions for a regular Cantor set $K$ with Markov partition $R = \{R_1, R_2, \ldots, R_n\}$ defined by an expansive map $\psi$ (see definition in Sect. 3) satisfying $C^{-1}/\varepsilon < |\psi'(x)| < C/\varepsilon$, $\forall x \in \cup_{i \leq n} R_i$, for a suitable constant $C$ (with $\log C \ll \log \varepsilon^{-1}$). From Lemma 4.5, we get
a set $Z$ of indices with $\#Z \geq \frac{4n}{\varepsilon}$. Fix indices $i, j \in Z$ such that $a_{i, j} = 1$. Consider a Markov partition for $\psi^{k+2}$ corresponding to the words in the set

$$X = \{\dot{j}r_1r_2 \cdots r_k \in X : r_i \leq n \text{ and } a_{j, r_1} = a_{r_1r_2} = \cdots = a_{r_{k-1}r_k} = 1\}.$$ 

By Lemma 4.5, $\#X = (A^{k+1})_{i, j} \geq \frac{4}{\varepsilon} \left(\frac{3}{5}\right)^{k-1} n^k > \left(\frac{n}{2}\right)^k$, since $a_{i, j} = 1$, any transition between two words in $X$ is admissible.

Consider the regular Cantor set

$$\tilde{K} := \{\alpha_1\alpha_2\alpha_3 \cdots | \alpha_i \in X, \forall i \geq 1\} \subset K.$$ 

Taking $k$ large enough, since $|\psi^{k+2}| < \left(\frac{C}{\varepsilon}\right)^{k+2}$, then

$$\text{HD}(\tilde{K}) > \frac{\log \left(\frac{n}{2}\right)^k}{\log \left(\frac{C}{\varepsilon}\right)^{k+2}} = \frac{k}{k+2} \log n - 2 \frac{\log C - \log \varepsilon}{\log(\varepsilon^{-1})} = (1 - o(1)) \log n \log(\varepsilon^{-1}).$$

It follows that $\text{HD}(\tilde{K}) \sim \text{HD}(K) \sim \frac{\log n}{\log(\varepsilon^{-1})}$.

We use the above Remark to understand the behavior of the horseshoe $\Delta$ when it is intersected by a finite number of $C^1$-curves.

**Lemma 4.6** (Intersection of curves with $\Delta$) *Let $\alpha = \{\alpha_i : [0, 1] \to \Xi, i \in \{1, \ldots, m\}\}$ be a finite family of $C^1$-curves. Then for all $\varepsilon > 0$ there are sub-horseshoes $\Delta^s_{\alpha^i}, \Delta^u_{\alpha^i}$ of $\Delta$ such that $\Delta^s_{\alpha^i} \cap \alpha_i([0, 1]) = \emptyset$ for any $i \in \{1, \ldots, m\}$ and

$$\text{HD}(K^s_{\alpha^i}) \geq \text{HD}(K^s) - \varepsilon \quad \text{and} \quad \text{HD}(K^u_{\alpha^i}) \geq \text{HD}(K^u) - \varepsilon,$$

where $K^s_{\alpha^i}, K^s$ are regular Cantor sets that describe the geometry transverse of the unstable foliation $W^u(\Delta^s_{\alpha^i}), W^u(\Delta)$ respectively, and $K^u_{\alpha^i}, K^u$ are regular Cantor sets that describe the geometry transverse of the stable foliation $W^s(\Delta^s_{\alpha^i}), W^s(\Delta)$ respectively (cf. “Expanding Maps Associated to a Horseshoe”).*

Before starting the proof of the previous lemma, we introduce some definitions and remarks.

Let us fix a Markov partition $R$ of $\Delta$. Given $R(a) \in R$, for an admissible word $a = (a_1, \ldots, a_i)$, we denote $|\{a_1, \ldots, a_i\}|$ the diameter of the projection on $W^s_{loc}$ of $R(a)$ along to the foliation $F^u$ (cf. the construction of $K^s$ in “Expanding Maps Associated to a Horseshoe”). Fix $a_r, a_s$ such that the pair $(a_r, a_s)$ is admissible. Given $\varepsilon > 0$, we have the following definition.

**Definition 5** A piece $(a_1, \ldots, a_k)$ (in the construction of $K^s$) is called an $\varepsilon$-piece if

$$|\{a_1, \ldots, a_k\}| < \varepsilon \quad \text{and} \quad |\{a_1, \ldots, a_{k-1}\}| \geq \varepsilon.$$
Put
\[ X_\varepsilon = \{ \text{\varepsilon-piece } (a_{i_1}, \ldots, a_{i_k}) : i_1 = s \text{ and } i_k = r \} = \{ \theta_1, \ldots, \theta_N \}. \]

Notice that \( \theta_i \theta_j \) is an admissible word for every \( i, j \leq N \). We define the Cantor set
\[ K(X_\varepsilon) := \{ \theta_{j_1} \theta_{j_2} \cdots \theta_{j_k} \cdots | \theta_{j_i} \in X_\varepsilon, \forall i \geq 1 \} \subset K_s. \]

Notice that \( N \sim \varepsilon^{-d_\varepsilon} \), where \( d_\varepsilon = HD(K_s) \), and so \( HD(K(X_\varepsilon)) \) is close to \( HD(K_s) \) provided \( \varepsilon \) is small enough.

Dividing the curves into smaller curves if necessary, we can assume that the finite family \( \alpha \) is formed by curves that are graphs of \( C^1 \)-functions of \( W^s(\Delta) \) on \( W^u(\Delta) \) or from \( W^u(\Delta) \) on \( W^s(\Delta) \).

Denote by \( I^s_{\theta_i} \) the interval associated with \( \varepsilon \)-piece \( \theta_i = (a_{i_1}, \ldots, a_{i_k}) \) in the construction of \( K_s \). There is a constant \( C > 1 \) (which depends on the geometry of the horseshoe \( \Delta \) but not on \( \varepsilon \) or \( k \)) such that
\[ C^{-1} \varepsilon < |I^s_{\theta_i}| < C \varepsilon. \]

For each \( I^s_{\theta_i} \), with \( \theta_i = (a_{i_1}, \ldots, a_{i_k}) \), we associate the interval \( I^u_{\theta_i} \) corresponding to the transposed sequence \( \theta_i^t = (a_{i_k}, \ldots, a_{i_1}) \) in the construction of \( K^u \) (unstable Cantor set), by an abuse of language, we will say that the interval \( I^u_{\theta_i} \) is the “transposed” interval of \( I^s_{\theta_i} \) (and vice-versa). Then, since \( \Delta \) is horseshoe there exists \( \beta \geq 1 \) (which depends on the geometry of the horseshoe \( \Delta \) but not on \( \varepsilon \) or \( k \)) such that
\[ C^{-1} |I^s_{\theta_i}|^{1/\beta} < |I^u_{\theta_i}| < C |I^s_{\theta_i}|^{1/\beta}. \]

**Remark 14** In the conservative case, i.e., when the horseshoe is defined by a diffeomorphism that preserves a smooth measure, the above inequality holds with \( \beta = 1 \).

**Proof of Lemma 4.6** We prove the stable case since the unstable case is analogous. For this sake, we consider the related position of the family of curves \( \alpha \) with respect to the stable and unstable manifolds.

- **First case** (Graph of a \( C^1 \)-function from \( W^s(\Delta) \) on \( W^u(\Delta) \)). In this case, consider the image \( P \) of \( I^s_{\theta_i} \) by this function. Then, \( C \) and \( \varepsilon \) (of the above discussion) can be taken such that \( |P| \leq C^2 \varepsilon \). Let \( P' \), the smallest interval of the construction of \( K^u \) containing \( P \). Then, if \( J \in W^s(\Delta) \) is the transposed interval of \( P' \), we have \( |J| \leq (C^2 \varepsilon)^{1/\beta} \). Then
\[ \# \{ I^s_{\theta_j} : I^s_{\theta_j} \cap J \neq \emptyset \} \leq C \left( \frac{(C^2 \varepsilon)^{1/\beta}}{\varepsilon} \right)^{d_s} = \tilde{C} \varepsilon^{d_s(1/\beta - 1)}. \]
where $d_s$ is the Hausdorff dimension of stable Cantor set.
Thus,

$$\#\{(I^u_{i_1}, I^u_{i_2}) : I^u_{i_1} \times I^u_{i_2} \text{ intersects the curve} \} \leq \epsilon^{-d_s} C \epsilon^{d_s(1/\beta-1)} = \tilde{C} \epsilon^{d_s(1/\beta-2)} \ll \epsilon^{-2d_s}. $$

- Second case (Graph of a $C^1$-function from $W^u(\Delta)$ on $W^s(\Delta)$). In this case, consider the image $J'$ of $I^u_{i_1}$. Then, $|J'| \leq c |I^u_{i_1}| \leq c(C\epsilon)^{1/\beta}$, ($J'$ is the image of $I^u_{i_1}$ by a $C^1$-function), so we have, analogously,

$$\#\{(I^s_{i_1}, I^s_{i_2}) : I^s_{i_1} \cap J' \neq \emptyset \} \leq \tilde{C} \epsilon^{d_s(1/\beta-1)}$$

and

$$\#\{(I^s_{i_1}, I^s_{i_2}) : I^s_{i_1} \times I^s_{i_2} \text{ intersects the curve} \} \leq \epsilon^{-d_s} \tilde{C} \epsilon^{d_s(1/\beta-1)} = \tilde{C} \epsilon^{d_s(1/\beta-2)} \ll \epsilon^{-2d_s}. $$

Note that $\epsilon^{-2d_s} \sim N^2 = \text{the total number of transitions} \theta_i \theta_j$.

We say that $\theta_U \theta_V$ is a prohibited transition if and only if some curve of the family $\alpha$ intersects the rectangle $I^u_{i_1} \times I^u_{i_2}$. Consider the admissible word $\theta_i \theta_j \theta_k \theta_s$ with $\theta_i, \theta_j, \theta_k, \theta_s \in X_\epsilon$. This word generates an interval of the size of the order of $\epsilon^4$ in the construction of $\mathcal{K}^s$.

We say that $\theta_i \theta_j \theta_k \theta_s$ is a prohibited word, if within there is a prohibited transition $\theta_U \theta_V$.

Denote by $PW$ the set of the prohibited words $\theta_i \theta_j \theta_k \theta_s$. Now we want to estimate $|PW|$.

In fact: $|I_{\rho}| |I_\beta| \sim \epsilon^2 \sim 2^{-2n}$, then there is $t \leq 2n$ such that $|I_{\rho}| \sim 2^{-t}$ and $|I_\beta| \sim 2^{t-2n}$.

Thus, $\#\{I_{\rho}\} \sim (2^{-t})^{-d_s} = 2^{td_s}$ and $\#\{I_\beta\} \sim (2^{-(2n-t)})^{-d_s} = 2^{(2n-t)d_s}$. Therefore, for some constant $\tilde{C} > 1$ (as in the first part of the proof), we have that

$$|PW| \leq \tilde{C} \cdot (2n) \cdot 2^{td_s} 2^{(2n-t)d_s} \epsilon^{-d_s(1/\beta-2)} \leq 2\tilde{C} \log \epsilon^{-1} \epsilon^{d_s(1/\beta-4)} \ll \epsilon^{-4d_s}$$

the last inequality follows from $2\tilde{C} (\log \epsilon^{-1}) \epsilon^{d_s/\beta} \ll 1$.

Then, the total of prohibited words $\theta_i \theta_j \theta_k \theta_s$ is much less than $\epsilon^{-4d_s} \sim N^4$, the total number of words $\theta_i \theta_j \theta_k \theta_s$.

Consider $A = (a(i,j)(k,s))$ for $(i, j), (k, s) \in \{1, \ldots, N\}^2$ the matrix defined by

$$a(i,j)(k,s) = \begin{cases} 1 & \text{if } \theta_i \theta_j \theta_k \theta_s \text{ is not prohibited}; \\ 0 & \text{if } \theta_i \theta_j \theta_k \theta_s \text{ is prohibited for some } \theta_U \theta_V. \end{cases}$$
Put $\tilde{\theta}_{ij} = \theta_i \theta_j$ for $i, j \leq N$. Define $\tilde{K}$ the regular Cantor set
\[
\tilde{K} := \{\tilde{\theta}_{i_1,j_1} \tilde{\theta}_{i_2,j_2} \cdots \tilde{\theta}_{i_n,j_n} : |a_{(i_k,j_k)(i_{k+1},j_{k+1})}| = 1, \forall k \geq 1\} \subset K^s.
\]
By the previous discussion, we have $\#(a_{(i,j)(k,s)} : a_{(i,j)(k,s)} = 1) \geq \frac{99}{100} (N^2)^2$, so by the Remark 13 we have $HD(\tilde{K}) \sim HD(K(X_\epsilon)) \sim HD(K^s)$. Consider the sub-
horseshoe of $\Delta$ defined by
\[
\Delta^s_\alpha := \bigcap_{n \in \mathbb{Z}} \mathcal{R}^n \left( \bigcup_{(i,j),(k,s) \in [1,2,\ldots,N]^2, a_{(i,j)(k,s)}=1} (R(\tilde{\theta}_{ij}) \cap R^{-1}(R(\tilde{\theta}_{ks}))) \right),
\]
where $R(\tilde{\theta}_{ij})$ is the rectangle associated to the word $\tilde{\theta}_{ij}$.

Then, the stable regular Cantor set $K^s_\alpha$ describing the transverse geometry of the
unstable foliation $W^u(\Delta^s_\alpha)$ is equal to $\tilde{K}$. Then by the above discussion, we have that
\[
HD(K^s_\alpha) \sim HD(K^s)
\]
and by definition of $\Delta^s_\alpha$ we have that $\Delta^s_\alpha \cap \alpha_i = \emptyset, \forall i \leq m$. This concludes the proof.

Now we can prove some lemmas, which give some differentiability to $max F_\Phi$. To get there, we introduce some subsets of $C^2(M, \mathbb{R})$. First, we consider the family of one
parameter $\beta > 0$, $\mathcal{B}^{s(u)}_{\Phi,\beta}$ defined as follows.

**Definition 6** We say that $F \in \mathcal{B}^{s(u)}_{\Phi,\beta} \subset C^2(M, \mathbb{R})$ whenever
\begin{enumerate}
    
(i) There exists a sub-
horseshoe $\Delta^s_F$ of $\Delta$ with $HD(K^s_F) > HD(K^s(u)) - 2\beta$,
(ii) There exists a Markov partition $R^s_F$ of $\Delta^s(u)$, respectively, such that the function
\[
\max_{\Xi \cap R^s_F} F_\Phi \in C^1(\Xi \cap R^s_F, \mathbb{R}),
\]
where $K^s_F$, $K^s(u)$ are the stable (unstable) Cantor sets associated to $\Delta^s_F$ and $\Delta$, respectively.

**Lemma 4.7** For any $\beta > 0$ small enough, the sets $\mathcal{B}^{s(u)}_{\Phi,\beta}$ are dense and $C^2$-open sets.

Before we present the proof of Lemma 4.7, let us to consider an auxiliary set $\mathcal{N}_{\Phi}$ of
functions defined as follows.

Once again we cover $\Delta$ with a finite number of tubular neighborhoods $U_k, 1 \leq k \leq m$ whose boundaries are $C^\infty$-GCS pairwise disjoints (as Remark 7) with $\Xi = \bigcup_{i=1}^k \Sigma_i$.

For each $k$, let us fix coordinates $(x_1(k), x_2(k), x_3(k))$ on $U_k$ such that $x_3(k)$ is the
flow direction and $U_s \cap \Xi = \{x_3(s) = 0\} \cup \{x_3(s) = 1\}$.

**Definition 7** We say that $F \in \mathcal{N}_{\Phi} \subset C^r(M, \mathbb{R}), r \geq 4$, whenever:
\begin{enumerate}
    
(i) 0 is a regular value of the restriction of $\frac{\partial F}{\partial x_3(k)}$ to $U_k \cap \Xi$;
\end{enumerate}
(ii) 0 is a regular value of \( \frac{\partial^3 F}{\partial x_3(k)^3} \);

(iii) 0 is a regular value of the functions \( \frac{\partial^2 F}{\partial x_3(k)^2} \) and \( \frac{\partial^2 F}{\partial x_3(k)^2} |_{\{\frac{\partial^3 F}{\partial x_3(k)^3} = 0\}} \);

(iv) 0 is a regular value of the functions \( \frac{\partial F}{\partial x_3(k)} |_{\{\frac{\partial^2 F}{\partial x_3(k)^2} = 0\}} \) and \( \frac{\partial F}{\partial x_3(k)} |_{\{\frac{\partial^3 F}{\partial x_3(k)^3} = 0\}} \) for each 1 \( \leq k \leq m \).

**Remark 15** The conditions (ii), (iii) and (iv) force us to consider functions in \( C^r(M, \mathbb{R}) \), \( r \geq 4 \). Such conditions will be necessary to locate the maximum value point along orbits (see comments after the proof of Lemma 4.8).

**Lemma 4.8** The set \( \mathcal{N}_\Phi \) is dense in \( C^r(M, \mathbb{R}) \), \( r \geq 4 \).

**Proof** Given a function \( F \in C^r(M, \mathbb{R}) \), \( r \geq 4 \), let us consider the three-parameter family

\[
F_{a,b,c}(x_1, x_2, x_3) = F(x_1, x_2, x_3) - cx_3^3/6 - bx_3^2/2 - ax_3
\]

where \( a, b, c \in \mathbb{R} \).

By Sard’s theorem, we can fix first a very small regular value \( c \approx 0 \) (close enough to 0) of \( \frac{\partial F}{\partial x_3} \), then a very small regular value \( b \approx 0 \) of both \( \frac{\partial^2 F}{\partial x_3^2} \) - \( cx_3 \) and its restriction to \( \{ \frac{\partial^2 F}{\partial x_3^2} = c \} \), and finally a very small regular value \( a \approx 0 \) of \( \{ \frac{\partial F}{\partial x_3} - cx_3^2/2 - bx_3 \} |_{\{\frac{\partial F}{\partial x_3} = c\} \cap \{\frac{\partial^2 F}{\partial x_3^2} - cx_3 = b\}} \).

\[
(\frac{\partial F}{\partial x_3} - cx_3^2/2 - bx_3)_{\{x_3=0\} \cup \{x_3=1\}}.
\]

For a choice of parameters \( (a, b, c) \) as above, we have that \( F_{a,b,c} \in \mathcal{N}_\Phi \); indeed, this happens because \( \frac{\partial^3 F_{a,b,c}}{\partial x_3^3} = \frac{\partial^3 F}{\partial x_3^3} - c \), \( \frac{\partial^2 F_{a,b,c}}{\partial x_3^2} = \frac{\partial^2 F}{\partial x_3^2} - cx_3 - b \) and \( \frac{\partial F_{a,b,c}}{\partial x_3} = \frac{\partial F}{\partial x_3} - cx_3^2/2 - bx_3 - a \). Clearly, \( F_{a,b,c} \) is arbitrarily close to \( F \), which proves the lemma. \( \Box \)

By definition, if \( F \in \mathcal{N}_\Phi \), then \( \mu_k := \{ \frac{\partial F}{\partial x_3(k)} = 0 \} \cap (U_k \cap \Sigma) \) is a curve (by (i)), and \( \mu_k := \{ \frac{\partial F}{\partial x_3(k)} = 0 \} \cap \{ \frac{\partial F}{\partial x_3(k)^2} = 0 \} \) is a curve intersecting the surface \( \{ \frac{\partial^3 F}{\partial x_3^3} = 0 \} \) at a finite set \( \Pi_k \) of points (by (ii), (iii) and (iv)).

Note that if \( (x_1, x_2, 0), (x_1, x_2, 1) \notin \mu_k \) (i.e., the orbit is transverse to the cross-sections) and the piece of orbit \( (x_1, x_2, z), 0 \leq z \leq 1 \), not intersect \( J_k \), then there is a neighborhood \( V \) of \( (x_1, x_2, 0) \in U_k \cap \Sigma \) and a finite collection of disjoint graphs \( (\psi_j(x, y)) : (x, y, 0) \in V \), \( 1 \leq j \leq n \) such that if \( f(x'_1, x'_2) = \max F_{\Phi}(x'_1, x'_2) = F(x', x_2, 0) \in V \), then \( f(x'_1, x'_2) \) with \( (x'_1, x'_2, 0) \in V \), then \( f' = \psi_j(x'_1, x'_2) \) for some \( j \).

**Proof of Lemma 4.7** Proof of the openness of \( B^s(\Phi, \beta) \) is a consequence of its definition.

Indeed, given \( F \in B^s(\Phi, \beta) \), then for any \( G \in C^2(M, \mathbb{R}) \) sufficiently close to \( F \), we have
that \( \max G_\Phi |_{\Xi \cap R^s(\alpha)} \in C^1(\Xi \cap R^s(\alpha), \mathbb{R}) \), therefore, we can taken \( R_G^s(\alpha) = R_F^s(\alpha) \) and \( \Delta_G^s(\alpha) = \Delta_F^s(\alpha) \), and this concludes the proof of openness.

Let us make the proof of the density for the stable case. The unstable case is analogous.

By Lemma 4.8, it is sufficient to prove that \( B^s_{\Phi, \beta} \cap C^r(M, \mathbb{R}), r \geq 4 \), is dense in \( \mathcal{N}_\Phi \). Observe that, in the statements of the proof of Lemma 4.8, we consider the curves \( \mu_k \) and the projections of the curves \( J_k \) in the flow direction (\( x_3 \)-coordinate) is a finite union \( J \) of \( C^1 \) curves contained in \( \Xi \) such that, for each \( y \in D_R \setminus J \). The value \( \max F_\Phi(z) \) for \( z \) near \( y \) is described by the values of \( \max F_\Phi \) at a finite collection of graphs transverse to the flow direction.

From the Lemma 4.6, given \( \beta > 0 \) small there is a sub-horseshoe \( \Delta_J \) such that

\[
H D(K_J^s) \geq H D(K^s) - \beta \text{ and } \Delta_J \cap \alpha = \emptyset,
\]

for each curve \( \alpha \in J \). In other terms, using the notation in the paragraph after proof of Lemma 4.8, our task is reduced to perturb \( F \) in such a way that \( f(x'_1, x'_2) \) are given by the values of \( F \) on a unique graph \( (x'_1, x'_2, \psi(x'_1, x'_2)) \).

In this direction, let \( V \) be a small neighborhood of \( \Delta_J \) such that \( V \cap \alpha = \emptyset \) for every \( \alpha \in J \). Note that the value of \( F \) at any point \( (x, y) \in V \) is described by finitely many disjoint graphs \( \psi_j, 1 \leq j \leq k \).

Let \( g_{1j}(x_1, x_2) = F(x_1, x_2, \psi_1(x_1, x_2)) - F(x_1, x_2, \psi_j(x_1, x_2)) \) for \( j \neq 1 \) and consider \( \gamma_1 > 0 \) small regular value of \( g_{1j} \) for all \( j \neq 1 \). Take \( \xi_1 \) a \( C^\infty \)-function close to the constant function 0 and equal to \( -\gamma_1 \) in neighborhood of \( \{z = \psi_1(x_1, x_2)\} \) and 0 outside. So, the function \( F + \xi_1 \) is close to \( F \). Now we define the function

\[
g_{1j}^\gamma(x_1, x_2) = (F + \xi_1)(x_1, x_2, \psi_1(x_1, x_2))
- (F + \xi_1)(x_1, x_2, \psi_j(x_1, x_2)) = g_{1j}(x_1, x_2) - \gamma_1.
\]

Put \( F_1 := F + \xi_1 \) and define \( g_{2j}(x_1, x_2) = F_1(x_1, x_1, \psi_2(x_1, x_2)) - F_1(x_1, x_2, \psi_j(x_1, x_2)) \) for \( j \neq 2 \) and let \( \gamma_2 > 0 \) small regular value of \( g_{2j} \) for all \( j \neq 2 \). Take \( \xi_2 \) a \( C^\infty \)-function close to the constant function 0 and equal to \( -\gamma_2 \) in neighborhood of \( \{z = \psi_2(x_1, x_2)\} \) and 0 outside. So, the function \( F_1 + \xi_2 \) is close to \( F \) and again, define the function

\[
g_{2j}^\gamma(x_1, x_2) = (F_1 + \xi_2)(x_1, x_2, \psi_2(x_1, x_2))
- (F_1 + \xi_2)(x_1, x_2, \psi_j(x_1, x_2)) = g_{2j}(x_1, x_2) - \gamma_2.
\]

Inductively, define \( F_{s-1} = F_{s-2} + \xi_{s-1} \) and

\[
g_{sj}(x_1, x_2) = F_{s-1}(x_1, x_2, \psi_s(x_1, x_2)) - F_{s-1}(x_1, x_2, \psi_j(x_1, x_2))
\]

for \( j \neq s \). Let \( \gamma_s > 0 \) small regular value of \( g_{sj} \) for all \( j \neq s \). Take \( \xi_s \) a \( C^\infty \)-function close to the constant function 0 and equal to \( -\gamma_s \) in neighborhood of \( \{z = \psi_s(x_1, x_2)\} \).
and 0 outside. So, the function $F_s := F_{s-1} + \xi_s$ is close to $F$ and
\[
g_{sj}^\gamma(x_1, x_2) := F_s(x_1, x_2, \psi_s(x_1, x_2)) - F_s(x_1, x_2, \psi_j(x_1, x_2)) = g_{sj}(x_1, x_2) - \gamma_s.
\]

Therefore, for each $s = 1, \ldots, k - 1$, $\Gamma_s := \bigcup_{j \neq s} (g_{sj}^\gamma)^{-1}(0)$ is a finite collection of $C^1$ curves in $\Xi$, $1 \leq s \leq k - 1$. Put $\Gamma := \bigcup_{s=1}^{k-1} \{\Gamma_s\}$, then by Lemma 4.6 there is a sub-horseshoe $\Delta_\gamma$ of $\Delta_\sigma$ such that
\[
HD(K_\Delta^\gamma) \geq HD(K_\Delta) - \beta \geq HD(K_\Delta^\gamma) - 2\beta \quad \text{and} \quad \Delta_\gamma \cap \gamma = \emptyset, \tag{8}
\]
for each $\gamma \in \Gamma$.

To finish the proof, consider the perturbation $F + \xi^k$ of $F$, where $\xi^k := \xi_1 + \cdots + \xi_{k-1}$, then if $l < j$, we have
\[
(F + \xi^k)(x_1, x_2, \psi_j(x_1, x_2)) - (F + \xi^k)(x_1, x_2, \psi_l(x_1, x_2))
\]
\[
= (F + \xi_1 + \cdots + \xi_j)(x_1, x_2, \psi_j(x_1, x_2)) - (F + \xi_1 + \cdots + \xi_l)(x_1, x_2, \psi_l(x_1, x_2))
\]
\[
= (F + \xi_1 + \cdots + \xi_j)(x_1, x_2, \psi_j(x_1, x_2)) - (F + \xi_1 + \cdots + \xi_l + \cdots + \xi_j)
\]
\[
= g_{jl}^\gamma(x_1, x_2).
\]

Thus, if $(x_1, x_2) \in \Delta_\gamma$, then
\[
(F + \xi^k)(x_1, x_2, \psi_j(x_1, x_2)) \neq (F + \xi^k)(x_1, x_2, \psi_s(x_1, x_2)) \quad \text{for all} \ j \neq s. \tag{9}
\]

So, taking a Markov partition $R_\Gamma$ of $\Delta_\Gamma$ with a diameter small enough, for each $y \in R_\Gamma$ the values of $\max(F + \xi^k)_\phi$ near $y$ are described by the values of $F + \xi^k$ at a unique graph. Hence, for each $y \in R_\Gamma$, one has that $\max(F + \xi^k)_\phi(y) = F(\phi^t(y))$ for a unique $0 \leq t(y) \leq t_\phi(y)$ depending in a $C^1$ way on $y$. Therefore, we conclude that $\max(F + \xi^k)_\phi|_{\Xi \cap R_\Gamma}$ is a $C^1$-function. Therefore, the function $F + \xi^k \in (B'_{\phi, \beta} \cap C^T(M, \mathbb{R}))$, $r \geq 4$, which concludes the proof of the lemma. \hfill \Box

Keeping the notation of the previous Lemma we have:

**Remark 16** The definition of $B^x(u)$ depends on the vector field $\Phi$. If $X$ is a vector field $C^2$-sufficiently close to $\Phi$, then $B_{\Phi, \beta}^x(u) = B_{X, \beta}^x(u)$.

### 4.3.2 The Set $\mathcal{U}_x, \Lambda$

Given a compact hyperbolic set $K$ for $\mathcal{R}$ and a Markov partition $R$ of $K$, we define the set
\[
H_1(\mathcal{R}, K) := \left\{ f \in C^1(\Xi \cap R, \mathbb{R}) : \# M_f(K) = 1, \quad z \in M_f(K), \quad D\mathcal{R}_z(e^f_z, u) \neq 0 \right\}. \tag{10}
\]
where $M_f(K) := \{z \in K : f(z) \geq f(x) \text{ for all } x \in K\}$, the set of maximum points of $f$ on $K$ and $e_z^{\xi, \eta}$ are unit vectors in $E_z^{\xi, \eta}(z)$, respectively (cf. [19, section 3]).

Note also, by Remark 11, that for any $X \in \mathcal{X}_\omega(M)$ sufficiently close of $\Phi$, we have that $HD(\Delta_X) > 1$. Thus, we have

**Definition 8** We say that $F \in \mathcal{U}_{X, \Lambda} \subset C^2(M, \mathbb{R})$, whenever

(i) There exists a sub-horseshoe $\Delta_F$ of $\Delta_X$ with $HD(\Delta_F) > 1$ and neighborhood $R_F$ of $\Delta_F$ such that

$$\max F_X|_{\Xi \cap R_F} \in C^1(\Xi \cap R_F, \mathbb{R}).$$

(ii) $\max F_X \in H_1(\mathcal{R}_X, \Delta_F) \subset C^1(\Xi \cap R_F, \mathbb{R})$.

**Lemma 4.9** The set $\mathcal{U}_{X, \Lambda}$ is dense and $C^2$-open set.

**Proof** By definition the set $\mathcal{U}_{X, \Lambda}$ is open. By Lemma 4.7 our task is simply to prove that $\mathcal{U}_{X, \Lambda}$ is dense in $B_{X, \beta}^\xi \cup B_{X, \beta}^\eta$ for some $\beta$ small enough. Indeed, fix $\beta > 0$ small enough such that $(HD(\Delta_X) - 4\beta) > 1$ and let $F \in B_{X, \beta}^\xi \cup B_{X, \beta}^\eta$, then by Lemma 4.7, consider the sub-horseshoe $\Delta_F = \Delta_F^x \cup \Delta_F^\eta$ and $R_F = R_F^x \cup R_F^\eta$, therefore by definition of $B_{X, \beta}^\xi \cup B_{X, \beta}^\eta$ we can conclude that

$$HD(K_F^x) + HD(K_F^\eta) \geq HD(K_X^x) + HD(K_X^\eta) - 4\beta.$$ 

Thus

$$HD(\Delta_F) \geq HD(\Delta_X) - 4\beta > 1,$$

since $HD(\Delta_X) = HD(K_X^x) + HD(K_X^\eta)$ (cf. [24]) and Remark 11. To conclude the proof, we need some appropriate perturbation of $F$ to become $max F_X$ an element of $H_1(\mathcal{R}_X, \Delta_F)$. Consider a point $x \in \Delta_F$. Recall that, in a small neighborhood of $x$, the values of $max F_X$ are given by the values of $F$ on a graph $(x_1, x_2, \psi(x_1, x_2))$. Now we can employ the argument of Section 3 in [19] to find arbitrarily small function $g(x_1, x_2)$ such that the functions $F_g(x_1, x_2, t) := F(x_1, x_2, t) + g(x_1, x_2)$ near the graph $(x_1, x_2, \psi(x_1, x_2))$ (and coinciding with $F$ elsewhere) with the property that $max(F_g)_X$ is an element of $H_1(\mathcal{R}_X, \Delta_F)$, as we wished. \hfill $\square$

**Proof of Theorem 4.1** We consider a family of perturbations $X_{\theta}$ of $\Phi$ as Sect. 4.2. Note also, by Remark 11 $HD(\Delta_{X_{\theta}}) > 1$. Let $F \in \mathcal{U}_{X_{\theta}, \Lambda}$, then there exists a sub-horseshoe $\Delta_F$ of $\Delta_{X_{\theta}}$ with $HD(\Delta_F) > 1$ and neighborhood $R_F$ of $\Delta_F$ such that $max F_{X_{\theta}}|_{\Xi \cap R_F} \in H_1(\mathcal{R}_{X_{\theta}}, \Delta_F) \subset C^1(\Xi \cap R_F, \mathbb{R})$. The Lemma 4.4 provides that the pair $(X_{\theta}, \mathcal{R}_{\theta})$ has the property $V$, then by Main Theorem of [19] we can conclude

$$\text{int } M(\mathcal{R}_{\theta}, \Delta_{X_{\theta}}, max F_{X_{\theta}}|_{\Xi \cap R_F}) \neq \emptyset \text{ and } \text{int } L(\mathcal{R}_{\theta}, \Delta_{X_{\theta}}, max F_{X_{\theta}}|_{\Xi \cap R_F}) \neq \emptyset.$$ 

The property of persistence is also a consequence of Lemma 4.4 and the Main Theorem of [19]. This completes the proof of Theorem 4.1 (and, a fortiori, Theorem 1.1). \hfill $\square$
4.4 Anosov Geodesic Flow and Anosov Suspension Flow

In this section, we prove Corollary 1 and 2 using Theorem 1.1.

4.4.1 Proof of Corollary 1

We can note that when the manifold $M$ is the unitary tangent bundle of a complete Riemannian manifold $N$ endowed with a metric $g_0$ of negative pinched curvature, then the geodesic flow, $\phi^t_0$, on $SN$ is Anosov. In order to use Theorem 1.1, we need to construct a basic set for $\phi^t_0$ with a Hausdorff dimension greater than 2. For this sake we used the following theorem:

**Theorem** ([11] and [12]) Let $N$ be a complete Riemannian manifold of finite volume of dimension $n$, such that all the sectional curvatures are bounded between two negative constants. Let $C$ be the set of points in $S_N$ whose orbit through of the geodesic flow is bounded. Then the Hausdorff dimension of $C$, $HD(C)$, is equal to $2n - 1$.

As a corollary of the above theorem we have:

**Lemma 4.10** In the same conditions of the last theorem, there exists a hyperbolic set $\Lambda$ for the geodesic flow $\phi^t_0 : SM \to SM$ with $HD(\Lambda)$ arbitrarily close to $2n - 1$.

**Proof** Fixed a point $p \in SM$ and consider the family of closed balls, $\Omega_k := B_k(p)$, of center $p$ and radius $k \in \mathbb{N}$.

Put $\tilde{\Omega}_k = \bigcap_{t \in \mathbb{R}} \phi^t_0(\Omega_k)$, then we have the following statement:

$$C \subset \bigcup_{k \in \mathbb{N}} \tilde{\Omega}_k,$$

where $C$ is given in the previous theorem. Indeed, let $x \in C$, then there exists a compact set $K_x$ such that the orbit of $x$, $O(x) \subset K_x \subset \Omega_{k_0}$ for some $k_0 \in \mathbb{R}^+$. This implies that $\phi^t_0(x) \in \Omega_{k_0}$ for all $t \in \mathbb{R}$, therefore $x \in \tilde{\Omega}_{k_0}$ and the statement is proved.

Now, since $HD(C) = 2n - 1$, then $\sup_{k \in \mathbb{N}} HD(\tilde{\Omega}_k) = 2n - 1$, therefore there exists $k_1 \in \mathbb{N}$ such that $HD(\tilde{\Omega}_{k_1})$ is arbitrarily close to $2n - 1$.

Notice that $\tilde{\Omega}_{k_1}$ is a compact and $\phi^t_0$-invariant set. Moreover, since $\phi^t_0$ is an Anosov flow on $SM$, then $\tilde{\Omega}_{k_1}$ is a hyperbolic set for geodesic flow $\phi^t_0$. Thus, we can take

$$\Lambda := \tilde{\Omega}_{k_0} \text{ and } HD(\Lambda) \sim 2n - 1 \text{ (arbitrarily close to } 2n - 1). \quad (11)$$

**Remark 17** If $N$ is a surface, then the hyperbolic set $\Lambda$, given by the Lemma 4.10, has the Hausdorff dimension arbitrarily close to 3. Note also that, if $N$ is a $C^r$-Riemannian manifold (the Riemannian metric is $C^r$) with finite volume, then the Liouville measure is preserved by the geodesic flow $\phi^t_0$. Therefore, if $\Phi_0$ denotes the vector field derivative from the geodesic flow, then $\Phi_0 \in \mathfrak{X}_{w}^{r-1}(SN)$ (cf. [22]).

The proof of the following corollary is based on classical arguments used to construct basis sets. \hfill $\square$
Corollary 5 In the case of surface, let $\Lambda$ be the hyperbolic set given by Lemma 4.10. Then, there is a basic set $\tilde{\Lambda}$ with $\Lambda \subset \tilde{\Lambda}$.

**Proof** Note that, by Corollary 3, the hyperbolic set $\Lambda$ is one-dimensional, then by similar arguments of Proposition 8 at [6], which is based on the argument of Anosov [3], we can conclude the proof of corollary. □

**Proof of Corollary 1** Simply note that by Remark 17 and Corollary 5 the basic set $\tilde{\Lambda}$ fits the hypotheses of the Theorem 1.1, since $HD(\tilde{\Lambda}) \geq HD(\Lambda) > 2$. In other words, the result of the corollary is an immediate consequence of Theorem 1.1. □

4.4.2 Proof of Corollary 2

Similar to Sect. 4.4.1, to prove Corollary 2, we have to find a hyperbolic set with the Hausdorff dimension arbitrarily close to 3 and then use the Theorem 1.1. For this purpose, we used the following theorem:

**Theorem** (Urbański [26]) If $M$ is a compact Riemannian manifold and $\varphi : M \to M$ is a transitive Anosov diffeomorphism, then the Hausdorff dimension of the set of points with non-dense (full) orbit under $\varphi$ equals $\dim M$. The same statement is true for Anosov flows.

As an immediate consequence,

**Lemma 4.11** If $\varphi$ is an Anosov diffeomorphism on a compact surface $N$, then there is a basic set $\Lambda$ with a Hausdorff dimension arbitrarily close to 2.

**Proof** Consider $\{x_k\}$ an enumerable and dense set, then for each $m \in \mathbb{N}$, we define the set $A_m^k := N \setminus B_{\frac{1}{m}}(x_k)$, where $B_{\frac{1}{m}}(x_k)$ is the open ball of center $x_k$ and radius $\frac{1}{m}$. We consider the compact invariant set $\tilde{A}_m^k := \bigcap_{n \in \mathbb{Z}} \varphi^n(A_m^k)$, which is hyperbolic set for $\varphi$, since $\varphi$ is Anosov.

Claim: If $\mathcal{ND}$ is the set of points with non-dense orbit under $\varphi$, then

$$\mathcal{ND} = \bigcup_{m \geq 1} \bigcup_k \tilde{A}_m^k.$$ 

**Proof of Claim** We need to prove simply that $\mathcal{ND} \subset \bigcup_{m \geq 1} \bigcup_k \tilde{A}_m^k$, indeed: let $x \in \mathcal{ND}$, then there is an open set $U \subset N$ such that the orbit of $x$, $O(x)$ does not intersect $U$, i.e. $O(x) \cap U = \emptyset$. Thus, there are $m \geq 1$ and $x_k$ such that $B_{\frac{1}{m}}(x_k) \subset U$, therefore $\varphi^n(x) \notin B_{\frac{1}{m}}(x_k)$ or $\varphi^n(x) \in N \setminus B_{\frac{1}{m}}(x_k)$, for all $n \in \mathbb{N}$, this implies that $x \in \tilde{A}_m^k$. □

Since the Anosov diffeomorphisms on a surface are transitive ([15]), then the Urbański’s Theorem implies that $HD(\mathcal{ND}) = 2$, then by the previous Claim,

$$2 = HD\left( \bigcup_{m \geq 1} \bigcup_k \tilde{A}_m^k \right) = \sup_{k,m} HD(\tilde{A}_m^k).$$
So, there are \( m_0 \) and \( k_0 \) such that \( \Lambda D(\tilde{A}^{k_0}_{m_0}) \) is arbitrarily close to 2. Note also that \( \tilde{A}^{k_0}_{m_0} \) is a hyperbolic set that has topological dimensional zero. Therefore by [3], there is a basic set \( \Lambda \) such that \( \tilde{A}^{k_0}_{m_0} \subset \Lambda \) and therefore \( \Lambda D(\Lambda) \) is arbitrarily close to 2, as we wanted. \( \square \)

The next step is to prove Corollary 2. For this goal, our task is to use the basic set \( \tilde{\Lambda} \) of Lemma 4.11 to construct the set of functions \( U_\varphi \) of the statement of Corollary 2.

The construction of set \( U_\varphi \) will be similar to the construction of \( U_X, \Lambda \) at Sect. 4.3.2, being that we use \( N \) instead of \( /\lambda_1 \).

Let \( \varphi_0 \) be a \( C^2 \) Anosov diffeomorphism of a compact surface \( N \) and \( /\Lambda_0 \) the basic set given by the Lemma 4.11. Consider \( W_0 \) a \( C^2 \) neighborhood of \( \varphi_0 \) such that, for each \( \varphi \in W_0 \), the basic set \( /\Lambda_0 \) has a hyperbolic continuation \( /\Lambda_{\varphi} \). Note that \( /\Lambda_{\varphi} \) is basic set and, by the \( C^2 \)-topology, the \( \Lambda D(/\Lambda_{\varphi}) \) is also arbitrarily close to 2 (cf. [24]).

Let \( \varphi \in W_0 \), and we consider the suspension flow of \( \varphi \), \( \psi_t : N \to N \), induced by the translated time \( \psi^t : N \times \mathbb{R} \to N \times \mathbb{R} \), \( \psi^t(x, s) = (x, s + t) \), where \( N_\varphi = \{(x, r) : x \in N \text{ and } 0 \leq r \leq 1\}/(x, 1) \sim (\varphi(x), 0) \).

We denoted by \( \psi_\varphi \) the derivative vector field of \( \psi^t \) (cf. [15] for more details).

**Definition 9** We say that \( F \in U_\varphi \subset C^2(N_\varphi, \mathbb{R}) \), whenever

(i) There exists a sub-horseshoe \( /\Lambda_F \) of \( /\Lambda_\varphi \) with \( \Lambda D(/\Lambda_F) > 1 \) and neighborhood \( R_F \) of \( /\Lambda_F \) such that

\[
\max F_{\psi_\varphi} |_{R_F} \in C^1(R_F, \mathbb{R}),
\]

where \( \max F_{\psi_\varphi}(x) := \max_{0 \leq t \leq 1} F(\psi^t_\varphi(x)) \).

(ii) \( \max F_{\psi_\varphi} \in H_1(\varphi, /\Lambda_F) \), where \( H_1(\varphi, /\Lambda_F) \) is defined analogously to (10).

Following the same lines of Sect. 4.3.2, more precisely, the proof of Lemma 4.9 we have,

**Lemma 4.12** For each \( \varphi \in W_0 \), the set \( U_\varphi \) is dense and \( C^2 \)-open set.

**Proof of Corollary 2** By Theorem 5.1 in “Intersections of Regular Cantor Sets and Property V” (see [20] and [21] for more details), we can assume by Lemma 4.11 that for a small perturbation \( \varphi \in W_0 \) of \( \varphi_0 \) in the \( C^2 \) topology, the pair \( (\varphi, /\Lambda_\varphi) \) has the property \( V \). Let \( F \in U_\varphi \), then by the main theorem at [19], we have that

\[
\text{int } M(\psi, /\Lambda_F, \max F_{\psi_\varphi} |_{R_F}) \neq \emptyset \quad \text{and} \quad \text{int } L(\psi, /\Lambda_\varphi, \max F_{\psi_\varphi} |_{R_F}) \neq \emptyset.
\]

The proof ends simply by observing that

\[
\limsup_{n \to +\infty} \max F_{\psi_\varphi}(\varphi^n(x)) = \lim_{t \to +\infty} F(\psi^t_\varphi(x))
\]
and

$$\sup_{n \to +\infty} \max F_{\psi}^{n}(\psi^{n}(x)) = \sup_{t \to +\infty} F(\psi^{t}_{\phi}(x))$$

for all $x \in \Lambda_{F}$. \hfill $\Box$

To finish this section, we note that, if $\varphi_{0}$ is a $C^{2}$ conservative Anosov diffeomorphism, then the proof of Corollary 2 and Sect. 4.2.1, allows us to conclude that the Remark 3 is valid.

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Appendix

Proof of Separation Lemma 3.5

The main goal of this section is to prove the Lemma 3.5, so we remember the relation (3)

$$\Lambda \subset \bigcup_{i=1}^{l} \phi(-\gamma, \gamma)(\text{int} \Sigma_{i}) := \bigcup_{i=1}^{l} U_{\Sigma_{i}}.$$ 

To prove Lemma 3.5 we need to understand what happens when two GCS as in the above relation intersect.

Note that if two sections $\Sigma_{i}, \Sigma_{j}$ has nonempty intersection, then we can consider two disjoint cases:

1. The intersection $\Sigma_{i} \cap \Sigma_{j}$ is transverse to the foliation $F^{s}$, i.e. for any $x \in \text{int} \Sigma_{i} \cap \text{int} \Sigma_{j}$ there is a $C^{0}$-curve $\xi_{x} \subset \text{int} \Sigma_{i} \cap \text{int} \Sigma_{j}$ which is transverse to $F^{s}$, then the Proposition 1 implies that $\text{int} \Sigma_{i} \cap \text{int} \Sigma_{j}$ is an open set of $\Sigma_{i}$ and $\Sigma_{j}$.

2. The intersection $\Sigma_{i} \cap \Sigma_{j}$ is not transverse to the foliation $F^{s}$, i.e., there may be points in $\Sigma_{i} \cap \Sigma_{j}$ in the following two situations:

   (i) For every point $x \in \Sigma_{i} \cap \Sigma_{j}$ there is not a curve $\xi_{x} \subset \text{int} \Sigma_{i} \cap \text{int} \Sigma_{j}$ transverse to $F^{s}$, this implies $\Sigma_{i} \cap \Sigma_{j}$ does not contains open sets of $\Sigma_{i}$ and $\Sigma_{j}$.

   (ii) The intersection $\Sigma_{i} \cap \Sigma_{j}$ contains an open set of $\Sigma_{i}$ and $\Sigma_{j}$ and also contains points as in (i).

The next task is to understand the cases (i) and (ii) of item 2. First, we shall prove the separation in Lemma 3.5 in the case that all intersections of the sections $\Sigma_{i}$ satisfy item 1. After that, we will make the separation in Lemma 3.5 when appears intersections in conditions 1 or 2.

Lemma 5.1 Let $B_{i} = \{ j : \Sigma_{i} \cap \Sigma_{j} \neq \emptyset \text{ and } \Sigma_{i}, \Sigma_{j} \text{ satisfies the condition 1} \}$. Then there is $\delta' > 0$ such that for every $j \in B_{i}$, $\phi^{\delta}(\Sigma_{i}) \cap \Sigma_{j} = \emptyset$ for all $0 < \delta \leq \delta'$. 

Proof Suppose otherwise, then for all $n$ sufficiently large, there is $z^n_i \in \Sigma_i$ such that $\phi^\frac{1}{n}(z^n_i) \in \bigcup_{j \in B_i} \Sigma_j$. Passing to a subsequence if necessary, we assume that $\phi^\frac{1}{n}(z^n_i) \in \Sigma_j$ for some $j_0 \in B_i$. Since $\Sigma_j$ is a compact set, we can assume that $z^n_i$ converges to $z_i$ as $n \to \infty$, thus $\phi^\frac{1}{n}(z^n_i)$ converge to $z_i$ as $n \to \infty$. This implies that $z_i \in \Sigma_i \cap \Sigma_{j_0}$.

By Remark 5, we can assume that $\{z_i\}$ converge to $z_i$. We can continue with this process and obtain by induction a finite sequences $\phi$.

Lemma 5.2 Assuming (3) and suppose that all possible intersections of sections \{\Sigma_i : i = 1, \ldots, l\} satisfy the condition 1. Then, there are GCS $\tilde{\Sigma}_i$ such that $\Lambda \subset \bigcup_{i=1}^l \phi^{(-\gamma, \gamma)}(\tilde{\Sigma}_i)$ with the property $\tilde{\Sigma}_i \cap \tilde{\Sigma}_j = \emptyset$ for all $i, j \in \{1, \ldots, l\}$.

Proof Consider the set $B_1 = \{j : \Sigma_1 \cap \Sigma_j \neq \emptyset\}$ and $\delta_1 = \inf_{j \not\in B_1} d(\Sigma_1, \Sigma_j)$, then by Lemma 5.1, there exist $t_1 < \min\{\delta_1, \gamma\}$ such that $\phi^{t_1}(\Sigma_1) \cap \Sigma_j = \emptyset$ for all $j \geq 1$.

Put $\tilde{\Sigma}_1 := \phi^{t_1}(\Sigma_1)$ and $\beta_1 = d(\tilde{\Sigma}_1, \Sigma_2)$. Analogously, we consider the set $B_2 = \{j \geq 2 : \Sigma_2 \cap \Sigma_j \neq \emptyset\}$ and $\delta_2 = \min\{\inf_{j \not\in B_2} d(\Sigma_2, \Sigma_j), \beta_2\}$, then by Lemma 5.1, there exist $t_2 < \min\{\delta_2, \gamma\}$ such that $\phi^{t_2}(\Sigma_2) \cap \Sigma_j = \emptyset$ for all $j \geq 2$ and $\phi^{t_2}(\Sigma_2) \cap \tilde{\Sigma}_1 = \emptyset$.

We can continue with this process and obtain by induction a finite sequences of positive number $\delta_1, \beta_i$ and $t_i$ define by $\beta_i = \min_{1 \leq j < i} d(\phi^{t_j}(\Sigma_j), \Sigma_i)$, $\delta_i = \min\{\inf_{j \not\in B_i} d(\Sigma_i, \Sigma_j), \beta_i\}$, where $B_i = \{j \geq i : \Sigma_i \cap \Sigma_j \neq \emptyset\}$ and $t_i < \min\{\delta_i, \gamma\}$ with the properties $\phi^{t_i}(\Sigma_i) \cap \Sigma_j = \emptyset$ for all $j \geq i$ and $\phi^{t_i}(\Sigma_i) \cap \phi^{t_j}(\Sigma_j) = \emptyset$ for all $j \leq i$.

Put $\tilde{\Sigma}_i := \phi^{t_i}(\Sigma_i)$, then it is easy to see that the set of sections $\{\tilde{\Sigma}_1, \tilde{\Sigma}_2, \ldots, \tilde{\Sigma}_l\}$ satisfies the conditions of lemma. \qed
The following lemma shows that if two GCS satisfy the condition 2(ii), then a small translate in the time on one of two sections makes the resulting sections satisfy the condition 2(i).

**Lemma 5.3** Let \( \Sigma_i, \Sigma_j \) be as in (3) satisfying the condition 2(ii), then there is \( t' \) small such \( \phi^{t'}(\Sigma_i) \) and \( \Sigma_j \) satisfy the condition 2(i), i.e., \( \phi^{t'}(\Sigma_i) \cap \Sigma_j \) does not contain an open set of \( \phi^t(\Sigma_i) \) nor \( \Sigma_j \).

**Proof** By contradiction, assume that for all \( t \) small enough, we have that \( \text{int} \phi^t(\Sigma_i) \cap \text{int} \Sigma_j \) contains an open set of \( \phi^t(\Sigma_i) \) and \( \Sigma_j \), then there is a non-degenerate interval \( I_j^t \subset W^u_{\epsilon}(x_j) \subset \Sigma_j \) and a non-degenerate interval \( I_i^t \subset W^u_{\epsilon}(x_i) \subset \Sigma_i \) such that the set

\[
\Delta_t := \bigcup_{z \in \phi^t(I_i^j)} W^s_{\epsilon}(z) \cap \bigcup_{w \in I_i^j} W^s_{\epsilon}(w) \tag{12}
\]

contains an open set of \( \phi^t(\Sigma_i) \) and \( \Sigma_j \).

**Claim:** The family of intervals \( I_i^j \) is pairwise disjoint.

**Proof of claim** Otherwise, assume that there is \( x \in I_i^j \cap I_i^{j'} \subset W^u_{\epsilon}(x_j) \) with \( t \neq t' \), then by (12) there are \( y \in I_i^j \) and \( z \in I_i^{j'} \) such that \( \phi^t(x) \in W^s_{\epsilon}(y) \) and \( \phi^{t'}(x) \in W^s_{\epsilon}(z) \). Since \( t, t' \) are small, then we have that \( \phi^{-t'}(z) \in W^s_{\text{loc}}(\phi^{-t}(y)) \). Also, since \( y, z \in W^u_{\epsilon}(x_i) \), then \( z \in W^s_{\text{loc}}(y) \), which implies that \( \phi^{-t'}(z) \in W^s_{\text{loc}}(\phi^{-t}(y)) \). Since \( t' - t \) is also small, then we have

\[
\phi^{-t'}(z) \in \phi^{-t'}(W^u_{\text{loc}}(\phi^{-t}(y))) \cap W^s_{\text{loc}}(\phi^{-t}(y)),
\]

which implies that \( t' = t \) and \( z = y \), since the stable and unstable manifold theorem, which is a contradiction. \( \square \)

Note that the above claim provides a contradiction because does not exist uncountable many disjoint non-empty open intervals \( I_i^j \). Since each of them would contain a rational number, proving an uncountable family of distinct rational numbers. So the proof is complete. \( \square \)

**Remark 19** The last lemma implies that we can always assume that the sections \( \Sigma_i \) and \( \Sigma_j \) satisfy conditions 1) or 2(i).

The next step is to understand what happens to the cross-sections that intersect as in case 2(i). Assume that \( \Sigma_i \) and \( \Sigma_j \) satisfy the condition 2(i), then \( \Sigma_i \cap \Sigma_j \) is a family of curves, \( \Gamma_{ij} \). By construction of GCS of Lemma 3.4, each curve \( c \in \Gamma_{ij} \) is a leaf of the foliation \( \{ F^s_x \cap \Sigma_i : x \in W^u_{\epsilon}(x_i) \} \), by abuse of notation we write \( F^s \cap \Sigma_i \). Remember that \( \Sigma_i = \Sigma_{x_i} \), thus we consider the projection \( \pi^s_i : \Sigma_i \to W^u_{\epsilon}(x_i) \) along \( F^s \) (see Fig. 7).
**Proposition 2** In the above conditions \( \pi_i^s(\Sigma_i \cap \Sigma_j) = \{W^s_\epsilon(x) \cap W^u_\epsilon(x_i) : x \in \Sigma_i \cap \Sigma_j\} \) is a compact set.

**Proof** Indeed, we need only to show that \( \pi_i^s(\Sigma_i \cap \Sigma_j) \) is a closed set. Let \( x_n \in \pi_i^s(\Sigma_i \cap \Sigma_j) \), such that \( x_n \to x \), then there is \( y_n \in \Sigma_i \cap \Sigma_j \) such that \( W^s_\epsilon(y_n) \cap W^u_\epsilon(x_i) = \{x_n\} \), since \( \Sigma_i \cap \Sigma_j \) is compact, then we can assume that \( y_{nk} \to y \in \Sigma_i \cap \Sigma_j \). Moreover, by continuity of foliation \( \mathcal{F}^s \), we have that \( W^s_\epsilon(y_{nk}) \cap W^u_\epsilon(x_i) \to \pi_i^s(y) \) and \( W^s_\epsilon(y_{nk}) \cap W^u_\epsilon(x_i) = \{x_{nk}\} \to x \), so \( x = \pi_i^s(y) \in \pi_i^s(\Sigma_i \cap \Sigma_j) \).

**Remark 20** It is worth noting that the proof of the previous proposition, actually shows that \( \pi_i^s \) is a continuous map.

Let \( x \in \pi_i^s(\Sigma_i \cap \Sigma_j) \), then, by transversality of the flow with both sections, there is \( \delta > 0 \) such that

\[
\phi^\delta(W^s_\epsilon(x) \cap \Sigma_i) \cap \Sigma_j = \emptyset,
\]

and by continuity we have that there is \( U_x \) neighborhood of \( W^s_\epsilon(x) \cap \Sigma_i \) on \( \Sigma_i \) such that

\[
\phi^\delta(U_x \cap \Sigma_i) \cap \Sigma_j = \emptyset.
\]  \hspace{1cm} (13)

The neighborhood \( U_x \) can be taken in the form \( U_x = \bigcup_{z \in I_x} W^s_\epsilon(z) \cap \Sigma_i \), where \( I_x \subset W^u_\epsilon(x_i) \) is an interval centered in \( x \).

Suppose that \( \mathcal{F}^s_\epsilon \cap \Sigma_i \cap \Lambda = \emptyset \) for some \( x \in \pi_i^s(\Sigma_i \cap \Sigma_j) \), then since \( \Lambda \) is a compact set there is an open set \( V_x \) containing \( \mathcal{F}^s_\epsilon \cap \Sigma_i \) with \( V_x \cap \Lambda = \emptyset \). Therefore, \( \Sigma_i \) can be subdivided into two GCS, \( \Sigma_i^1 \) and \( \Sigma_i^2 \), such that \( \Sigma_i^r \) and \( \Sigma_j \) intersecting as the case 2(i), for \( r = 1, 2 \). In other words, if \( \mathcal{F}^s_\epsilon \cap \Sigma_i \cap \Lambda = \emptyset \) for some \( x \in \pi_i^s(\Sigma_i \cap \Sigma_j) \), then we return to the case 1) or 2(i) with one more section.

**Remark 21** The above observation implies that, without loss of generality, we can assume that for any \( x \in \pi_i^s(\Sigma_i \cap \Sigma_j) \) there is \( p_x \in \mathcal{F}^s_\epsilon \cap \Sigma_i \cap \Lambda \).
Lemma 5.4 If $\Sigma_i$ and $\Sigma_j$ are two GCS as in condition 2(i). Given $\delta > 0$, $0 < \delta < \frac{\gamma}{2}$ (with $\gamma$ as in (3)), then for $x \in \pi^s_i (\Sigma_i \cap \Sigma_j)$, there is a GCS, $\Sigma_x \subset U_x \cap \Sigma_i$ containing $\mathcal{F}_x^s \cap \Sigma_i$, such that $\Sigma_i$ is subdivided into three disjoint GCS, including $\Sigma_x$. Denoted by $\Sigma^#_i$ the set of complementary sections of $\Sigma_x$ in the above subdivision of $\Sigma_i$, then

(1) $\phi^s(\tilde{\Sigma}_x) \cap \Sigma_j = \emptyset$.

(2) $\Lambda \cap \phi((-\frac{\gamma}{2}, \frac{\gamma}{2}) (\text{int}(\Sigma_i))) \subset \Lambda \cap \left(\phi(-\gamma, \gamma) \left(\phi^s(\text{int}(\Sigma_x))\right) \cup \bigcup_{\Sigma \in \Sigma^#_i} \phi(-\frac{\gamma}{2}, \frac{\gamma}{2}) (\text{int}(\Sigma))\right)$.

**Proof** By Remark 21, we can assume that for any $x \in \pi^s_i (\Sigma_i \cap \Sigma_j)$ there is $p_x \in \mathcal{F}_x^u \cap \Sigma_i \cap \Lambda$. Consider $\mathcal{F}^u_{loc}(p_x)$, then by Remark 6 we can find open sets $V^+_{p_x}$ and $V^-_{p_x}$ in each side of $W^u_{loc}(p_x) \setminus \{p_x\}$ sufficiently close to $W^u_{loc}(p_x)$ with diameter sufficiently large and $V^+_{p_x} \cap \Sigma_i \cap \Lambda = \emptyset$. Denote by $\tilde{V}_{p_x}$ the projection along to the flow of $V^\pm_{p_x}$ over $\Sigma_i$, respectively. Therefore, by Remark 6 we can take $\tilde{V}_{p_x}^\pm$ such that $\tilde{V}_{p_x}^\pm \cap \Sigma_i \subset U_x$ and $\tilde{V}_{p_x}^\pm$ crosses $\Sigma_i$. Using $\tilde{V}_{p_x}^\pm$ we can construct the GCS $\tilde{\Sigma}_x$ such that $\tilde{\Sigma}_x \subset U_x$ and by (13), $\tilde{\Sigma}_x$ satisfies the item 1) of lemma.

To prove item 2) note simply that $\delta < \frac{\gamma}{2}$ and $\phi((-\frac{\gamma}{2}, \frac{\gamma}{2}) (\text{int}(\Sigma_i))) \cap \Lambda = \phi(-\frac{\gamma}{2}, \frac{\gamma}{2}) (\text{int}(\Sigma_i)) \cap \Lambda$, which is a consequence of $\Lambda$ be invariant by the flow. \(\square\)

As the GCS $\tilde{\Sigma}_x$ obtained in the last lemma is contained in $U_x \cap \Sigma_i$, then there is an interval centered in $x$, $\tilde{I}_x \subset I_x \subset W^u_\epsilon(x_i)$ such that

$$\tilde{\Sigma}_x = \bigcup_{z \in \tilde{I}_x} W^s_\epsilon(z) \cap \Sigma_i.$$  

Moreover, since $\pi^s_i (\Sigma_i \cap \Sigma_j) \subset \bigcup \tilde{I}_x$, then the compactness $\pi^s_i (\Sigma_i \cap \Sigma_j)$ from Proposition 2, there is a finite set of points $\{x^1, \ldots, x^m\} \subset \pi^s_i (\Sigma_i \cap \Sigma_j)$ such that

$$\pi^s_i (\Sigma_i \cap \Sigma_j) \subset \bigcup_{r=1}^m \tilde{I}_{x^r}.$$  

Thus by first part of Lemma 5.4, given $\delta > 0$ small enough, holds that

$$\phi^s(\tilde{\Sigma}_{x^r}) \cap \Sigma_j = \emptyset, \quad r = 1, \ldots, m$$  
and

$$\phi^s(\tilde{\Sigma}_{x^r}) \cap \phi^s(\tilde{\Sigma}_{x^{r'}}) = \emptyset, \quad r \neq r'.$$  

(14)

In the above conditions, we prove the following

**Lemma 5.5** If $\Sigma_i$ and $\Sigma_j$ are two GCS as in the condition 2(ii). Given $0 < \delta < \frac{\gamma}{2}$ (with $\gamma$ as in (3)) there are GCS, $\tilde{\Sigma}_{x^r} \subset U_{x^r}$ containing $\mathcal{F}_{x^r}^s \cap \Sigma_i$, $r = 1, \ldots, m$ and such that $\Sigma_i$ is subdivided into $2m + 1$ disjoint GCS, including $\tilde{\Sigma}_{x^r}$, $r \in \{1, \ldots, m\}$. Denote by $\Sigma^#_{x^r}$ the complement of the set $\{\tilde{\Sigma}_{x^r}\}_{r=1}^m$ in the above subdivision of $\Sigma_i$, then

(1) $\phi^s(\tilde{\Sigma}_{x^r}) \cap \Sigma_j = \emptyset, \quad r \in \{1, \ldots, m\}$ and $\Sigma_j \cap \Sigma = \emptyset$ for any $\Sigma \in \Sigma^#_{x^r}$.

(2) $\phi^s(\tilde{\Sigma}_{x^r}) \cap \phi^s(\tilde{\Sigma}_{x^{r'}}) = \emptyset, \quad r \neq r'$ and $\phi^s(\tilde{\Sigma}_{x^r}) \cap \Sigma_i = \emptyset, \quad r \in \{1, \ldots, m\}$.  

(3) $\Lambda \cap \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma_i)) \subset \Lambda \cap \left( \bigcup_{r=1}^{m} \phi^{(-\gamma, \gamma)}(\phi^{\delta}(\text{int}(\Sigma_{x'}))) \cup \bigcup_{\Sigma \in \Sigma_k^\#} \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma)) \right)$.

**Proof** Given $0 < \delta < \frac{\gamma}{2}$ small enough. The conditions (1) and (2) are an immediate consequence of (14). To prove item (3) note simply that $\delta < \frac{\gamma}{2}$ and $\Lambda \cap \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma_i)) = \phi^{(-\gamma, \gamma)}(\Lambda \cap \text{int}(\Sigma_i))$, which is a consequence of $\Lambda$ be invariant by the flow. $\square$

**Remark 22** Let $\Sigma'$ be such that GCS such that $\Sigma' \cap \Sigma_i = \emptyset$. Then we can take $\delta < d(\Sigma', \Sigma_i)$, such that $\phi^{\delta}(\Sigma_{x'}) \cap \Sigma' = \emptyset$, $r \in \{1, \ldots, m\}$, where $\Sigma_{x'}$ as in the Lemma 5.5.

To give a complete proof of Lemma 3.5, we must now treat the more general case of Lemma 5.5, where three or more sections intersect as in case 2(i).

To reinforce the idea, we recall the Eq. (3)

$$\Lambda \subset \bigcup_{i=1}^{l} \phi^{(-\gamma, \gamma)}(\text{int} \Sigma_i) = \bigcup_{i=1}^{l} U_{\ Sigma_i}.$$ 

Now we will prove that the GCS in (3) can be taken disjoint, even if some of the cross-sections are in condition 2(i).

**Lemma 5.6** Let $\Sigma_i$ be a GCS as in (3). Let $B_i = \{j : \Sigma_i \text{ intersects } \Sigma_j \text{ as the case 2(i)}\}$. Then, $\Sigma_i$ can be subdivided in a finite number of GCS $\Sigma_i^s : s = 1, \ldots, n$ such that for each $s$, there is $0 < \delta_s < \frac{\gamma}{2}$ such that

1. $\phi^{\delta_s}(\Sigma_i^s) \cap \Sigma_j = \emptyset$, $j \in B_i$ and $\phi^{\delta_s}(\Sigma_i^s) \cap \phi^{\delta_s}(\Sigma_i^{s'}) = \emptyset$, $s \neq s'$.
2. $\Lambda \cap \bigcup_{j \in B_i \cup \{i\}} \phi^{(-\gamma, \gamma)}(\text{int}\Sigma_j) \subset \Lambda \cap \left( \bigcup_{j \in B_i} \phi^{(-\gamma, \gamma)}(\phi^{\delta}(\text{int}\Sigma_j)) \cup \bigcup_{i=1}^{l} \phi^{(-\gamma, \gamma)}(\text{int}(\phi^{\delta}(\Sigma_i^{s})) \right)$.

**Proof** The proof is by induction on $\# B_i$. The case $\# B_i = 1$ is true by the Lemma 5.4. Suppose the statement is true for $\# B_i < q$ and we prove for $\# B_i = q$. In fact:

Let $k \in B_i$, then by Lemma 5.4, given $0 < \delta < \frac{\gamma}{2}$, there are a finite number of GCS $\{\tilde{\Sigma}_k^r \subset \Sigma_k : r \in \{1, \ldots, r_k\}\}$ such that

$$\phi^{\delta}(\tilde{\Sigma}_k^r) \cap \Sigma_k = \emptyset,$$

also $\phi^{\delta}(\tilde{\Sigma}_k^r) \cap \Sigma_i = \emptyset$ for any $r$, and $\Sigma_i \cap \Sigma = \emptyset$, $\Sigma_i \in \Sigma_k^\#$. (15)

$$\Lambda \cap \phi^{(-\gamma, \gamma)}(\text{int} \Sigma_k) \subset \Lambda \cap \left( \bigcup_{r=1}^{r_k} \phi^{(-\gamma, \gamma)}(\phi^{\delta}(\phi^{\delta}(\text{int} \Sigma_k^r))) \cup \bigcup_{\Sigma \in \Sigma_k^\#} \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma)) \right).$$ (16)

where $\Sigma_k^\#$ is as in the Lemma 5.4.

Consider now the collection of GCS

$$\left\{\Sigma_i, \Sigma_j, \phi^{\delta}(\tilde{\Sigma}_k^r), \Sigma_k^\# : j \in B_i \setminus \{k\} \text{ and } r \in \{1, \ldots, r_k\}\right\}.$$ For this new collection of GCS, we have $\# B_i < q$. Therefore, by the induction hypothesis, the lemma is true for $\left\{\Sigma_j : j \in B_i \cup \{i\} \setminus \{k\}\right\}$ and by (15) and (16) we have the lemma proved. $\square$
Remark 23 Let $\Sigma'$ be a GCS as in (3) such that $\Sigma' \cap \Sigma_i = \emptyset$. Then by Remark 22, we can take $\delta_s$ less than $d(\Sigma_i, \Sigma')$. So $\phi^{\delta_s}(\Sigma_i') \cap \Sigma' = \emptyset$ for any $s \in \{1, \ldots, m\}$, $\Sigma_i'$ as in the Lemma 5.6.

We finish this section making the proof of the Lemma 3.5.

Proof of Lemma 3.5 If all the possible intersections satisfy condition 1, the result follows from Lemma 5.2. Then, we can suppose that there is $i$, such that the set $B_i = \{j : \Sigma_i \text{ intersects } \Sigma_j \text{ as the case 2(i)}\}$ is non-empty. Without loss of generality, assume that $B_1 \neq \emptyset$. Let us will conclude the proof by induction on $l$ at (3).

Note that Lemma 5.4 implies the result in the case $l = 2$. Therefore, suppose it is true for $k < l$ and we will prove for $k = l$. Indeed, fix $\Sigma_1$ and consider the set

$$T_1 = \{j : \Sigma_j \text{ intersects } \Sigma_1 \text{ as the case 1}\}.$$

Then by Lemma 5.1, there is $0 < \delta < \frac{\gamma}{2}$ small enough, such that $\phi^{\delta}(\Sigma_1) \cap \Sigma_j = \emptyset$ for any $j \in T_1$.

Abusing the notation, let’s still call $B_1 = \{j : \phi^{\delta}(\Sigma_1) \text{ intersects } \Sigma_j \text{ as the case 2(i)}\}$. Then by Lemma 5.5, $\phi^{\delta}(\Sigma_1)$ can be subdivided in a finite number of GCS $\{\Sigma_i' : s = 1, \ldots, m_0\}$ and for each $s$ there is $0 < \delta_s < \frac{\gamma}{2}$ such that holds 1) and 2) of Lemma 5.5. Also by Remark 23, we can assume that $\phi^{\delta_s}(\Sigma_i') \cap \Sigma_j = \emptyset$ for any $s \in \{1, \ldots, m_0\}$ and for any $j \in T_1 \setminus \{1\}$.

Since the cardinal $\#(T_1 \setminus \{1\} \cup B_1) < l$, then the set $\{\Sigma_j : j \in T_1 \setminus \{1\}\} \cup \{\Sigma_k : k \in B_1\}$ satisfies the induction hypothesis, therefore there are $n(l) - 1$ GCS, $\widehat{\Sigma}_i$ with $\widehat{\Sigma}_i \cap \widehat{\Sigma}_j = \emptyset$ for $i \neq j$, such that

$$\Lambda \cap \bigcup_{i \in T_1 \cup B_1 \setminus \{1\}} \left(\phi^{-(\gamma, \gamma)}(\text{int } \Sigma_i)\right) \subset \Lambda \cap \bigcup_{i = 2}^{n(l)} \phi^{(-2\gamma, 2\gamma)}(\text{int } \widehat{\Sigma}_i).$$  \hspace{1cm} (17)

Since $\phi^{\delta_s}(\Sigma_i') \cap \Sigma_j = \emptyset$ for any $j \in T_1 \cup B_1 \setminus \{1\}$ and any $s \in \{1, \ldots, m\}$, then the $\widehat{\Sigma}_j$ may be taken such that $\phi^{\delta_s}(\Sigma_i') \cap \widehat{\Sigma}_i = \emptyset$ for any $s \in \{1, \ldots, m\}$ and any $i \in \{2, \ldots, n(l)\}$.

So, by the condition 2) of Lemma 5.5 and (17) we have that

\begin{align*}
\Lambda &= \Lambda \cap \bigcup_{j = 1}^{l} \phi^{-(\gamma, \gamma)}(\text{int } \Sigma_j) \subset \Lambda \cap \left(\bigcup_{j = 2}^{l} \phi^{-(\gamma, \gamma)}(\text{int } \Sigma_j) \cup \phi^{-(\gamma, \gamma)}(\text{int } \phi^{\delta}(\Sigma_1))\right) \\
&= \Lambda \cap \left(\bigcup_{j \in B_1} \phi^{-(\gamma, \gamma)}(\text{int } \Sigma_j) \cup \bigcup_{j \in T_1 \setminus \{1\}} \phi^{-(\gamma, \gamma)}(\text{int } \Sigma_j) \cup \phi^{-(\gamma, \gamma)}(\text{int } \phi^{\delta}(\Sigma_1))\right) \\
&\subset \Lambda \cap \left(\bigcup_{i = 2}^{n(l)} \phi^{(-2\gamma, 2\gamma)}(\text{int } \widehat{\Sigma}_i) \cup \bigcup_{s = 1}^{m_0} \phi^{(-2\gamma, 2\gamma)}(\text{int } \phi^{\delta}(\Sigma_i'))\right).
\end{align*}

Therefore, the last inclusion concludes our proof, since $m = n(l) - 1 + m_0$. \hfill \Box
Proof of Hyperbolicity of Poincaré Map

Our main goal of this section is to prove Lemma 3.6, which has [4] as its main reference. We recall some information. Let $\Xi = \bigcup_{i=1}^{m} \Sigma_i$ be a finite union of cross-sections to the flow $\phi^t$ given by Remark 7, which are pairwise disjoint. Sometimes, abusing of notation, we consider $\Xi = \{\Sigma_1, \ldots, \Sigma_m\}$. Let $\mathcal{R}: \Xi \to \Xi$ be a Poincaré map, that is, the map of first return to $\Xi$, $\mathcal{R}(y) = \phi^{t_+(y)}(y)$, where $t_+(y)$ corresponds to the first time that the positive orbits of $y \in \Xi$ encounter $\Xi$.

The splitting $E_s^x \oplus \Phi \oplus E_u^x$ over a neighborhood $U_0$ of $\Lambda$ defines a continuous splitting $E_{\Sigma}^s \oplus E_{\Sigma}^u$ of the tangent bundle $T\Sigma$ with $\Sigma \in \Xi$ given by

$$E_{\Sigma}^s(y) = E_{\Sigma}^{cs} \cap T_{\Sigma} \Sigma \text{ and } E_{\Sigma}^u(y) = E_{\Sigma}^{cu} \cap T_{\Sigma} \Sigma, \tag{18}$$

where $E_{\Sigma}^{cs} = E_y^s \oplus \langle \Phi(y) \rangle$ and $E_{\Sigma}^{cu} = E_y^u \oplus \langle \Phi(y) \rangle$.

We will show that for a sufficiently large iterated of $\mathcal{R}$, $\mathcal{R}^n$, the splitting (18) defines a hyperbolic splitting for transformation $\mathcal{R}^n$ on the cross-sections, at least restricted to $\Lambda \cap \Xi$ (cf. [4, chap. 6]). To achieve this goal, we will take into consideration the following:

Remark 24

(1) In what follows, we use $K \geq 1$ as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction. Also depending on upper and lower bounds for the norm of the vector field on the cross-sections.

(2) Let us consider unit vectors, $e^s_x \in E^s_x$ and $\hat{e}^s_x \in E^s_x(x)$, and write

$$e^s_x = a_x \hat{e}^s_x + b_x \frac{\Phi(x)}{\|\Phi(x)\|}. \tag{19}$$

Since the angle between $E^s_x$ and $\Phi(x)$, $\angle(E^s_x, \Phi(x))$, is greater than or equal to the angle between $E^s_x$ and $E^cu_x$, $\angle(E^s_x, E^cu_x)$, due to the fact $\Phi(x) \in E^cu_x$. The latter is uniformly bounded from zero, we have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow.

Let $0 < \lambda < 1$ be, then there is $t_1 > 0$ such that $\lambda^{t_1} < \frac{K}{K^3}$ and $\lambda^{t_1} < \frac{\lambda}{K^3}$, take $n$, such that $t_n(x) := \sum_{i=1}^{n} t_i(x) > t_1$ for all $x \in \Lambda \cap \Xi$, where $t_i(x) = t_+((\mathcal{R}^i)^{-1}(x))$. So, we have the following proposition:

Proposition 3 Let $\mathcal{R}: \Xi \to \Xi$ be a Poincaré map and $n$ as before. Then $D\mathcal{R}^n(\mathcal{R}^n(x)) = E^s_{\Sigma}((\mathcal{R}^n(x)))$ at every $x \in \Sigma \in \{\Sigma_1\}_{i}$ and $D\mathcal{R}^n(\mathcal{R}^n(\mathcal{x})) = E^u_{\Sigma}(\mathcal{R}^n(\mathcal{x}))$ at every $x \in \Lambda \cap \Sigma$ where $\mathcal{R}^n(\mathcal{x}) \in \Sigma' \in \{\Sigma_1\}_{i}$.

Moreover, we have that

$$\left\|D\mathcal{R}^n|_{E^s_{\Sigma}(x)}\right\| < \lambda \text{ and } \left\|D\mathcal{R}^n|_{E^u_{\Sigma}(x)}\right\| > \frac{1}{\lambda}$$

at every $x \in \Sigma \in \Xi$.

Proof The differential of the map $\mathcal{R}^n$ at any point $x \in \Sigma$ is given by

$$D\mathcal{R}^n(x) = \mathcal{P}_{\mathcal{R}^n(x)} \circ D\phi^{t_n(x)}|_{T_x \Sigma}.$$

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where $P_{\mathcal{R}^n(x)}$ is the projection onto $T_{\mathcal{R}^n(x)} \Sigma'$ along the direction of $\Phi(\mathcal{R}^n(x))$.

Note that $E^c_{\Sigma}$ is tangent to $\Sigma \cap W^{cs}$. Since the center stable manifold $W^{cs}(x)$ is invariant, we have that the stable bundle is invariant:

$$D\mathcal{R}^n(x)(E^s_{\Sigma}(x)) = E^s_{\Sigma}(\mathcal{R}^n(x)).$$

Moreover, for all $x \in \Sigma$ we have

$$D\phi^n(x)(E^u_{\Sigma}(x)) \subset D\phi^n(x)(E^u_x) = E^u_{\mathcal{R}^n(x)},$$

since $P_{\mathcal{R}^n(x)}$ is the projection along the vector field, it sends $E^u_{\mathcal{R}^n(x)}$ to $E^u_{\Sigma}(\mathcal{R}^n(x))$.

This proves that the unstable bundle is invariant restricted to $\Lambda$, that is, $D\mathcal{R}^n(x)(E^u_{\Sigma}(x)) = E^u_{\mathcal{R}^n(x)}$, because they have the same dimension 1.

Next, we prove the expansion and contraction statements. We start by noting that

$$\| P_{\mathcal{R}^n(x)} \| \leq K, \text{ with } K \geq 1.$$

Then we consider the basis $\left\{ \frac{\Phi(x)}{\| \Phi(x) \|}, e^u_x \right\}$ of $E^u_x$, where $e^u_x$ is a unit vector in the direction of $E^u_{\Sigma}(x)$ and $\Phi(x)$ is the direction of flow. Since the direction of the flow is invariant by $D\phi^t$, then the matrix of $D\phi^t|_{E^u_x}$ relative to this basis is upper triangular:

$$D\phi^n(x)|_{E^u_x} = \begin{bmatrix} \frac{\Phi(\mathcal{R}^n(x))}{\| \Phi(x) \|} & * \\ 0 & a \end{bmatrix}$$

since $D\phi^n(x)(\phi(x)) = \Phi(\phi^n(x)) = \Phi(\mathcal{R}^n(x))$.

Then,

$$\| D\mathcal{R}^n(x)e^u_x \| = \| P_{\mathcal{R}^n(x)}(D\phi^n(x)(x))e^u_x \| = \| ae^u_{\mathcal{R}^n(x)} \| = |a| \geq \frac{1}{K} \| \Phi(x) \| | det(D\phi^n(x)|_{E^u_x}) \| \geq \frac{1}{K^3 \lambda^{-t_n(x)}} \geq K^{-3} \lambda^{-t_1} > \frac{1}{\lambda}. $$

To prove that $\| D\mathcal{R}^n|_{E^s_{\Sigma}(x)} \| < \lambda$, let us consider unit vectors, $e^s_x \in E^s_x$ and $\hat{e}^s_x \in E^s_{\Sigma}(x)$, and write as in (19)

$$e^s_x = a_x \hat{e}^s_x + b_x \frac{\Phi(x)}{\| \Phi(x) \|},$$

with $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow.

Then, since $P_{\mathcal{R}^n(x)}\left( \frac{\Phi(\mathcal{R}^n(x))}{\| \Phi(x) \|} \right) = 0$ we have that

$$\| D\mathcal{R}^n(x)\hat{e}^s_x \| = \| P_{\mathcal{R}^n(x)}(D\phi^n(x)(x))\hat{e}^s_x \| = \frac{1}{|a_x|} \| P_{\mathcal{R}^n(x)}(D\phi^n(x)(x)) \left[ \frac{1}{a_x} \left[ e^s_x - b_x \frac{\Phi(x)}{\| \Phi(x) \|} \right] \right] \| = \frac{1}{|a_x|} \| P_{\mathcal{R}^n(x)}(D\phi^n(x)(x)) \left[ e^s_x - b_x \frac{\Phi(x)}{\| \Phi(x) \|} \right] \|.$$
We assume, without loss of generality, that \( R \) defined by

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Lemma 5.7

\[ \lim_{n \to \infty} \frac{1}{|a_n|} \left\| P_{R^n(x)}(D\phi_{a_n}^n(x))(e_x^n) - b_x P_{R^n(x)} \left( \frac{\Phi(R^n(x))}{\|\Phi(x)\|} \right) \right\| \]

\[ \leq \frac{K}{\kappa} \left\| D\phi_{a_n}^n(x)(e_x^n) \right\| \leq \frac{K}{\kappa} \lambda_{a_n} \leq \frac{K}{\kappa} \lambda \leq \lambda. \tag{20} \]

\[ \square \]

The next step is to prove that there exists \( n \) such that \( R^n \) is defined for every point of \( \Lambda \cap \Xi \) and consequently, by Proposition 3, it is a hyperbolic set for \( R^n \). Moreover, it should be a hyperbolic set for \( R \), since \( \Lambda \cap \Xi \) is invariant by \( R \).

For every \( x \in \Sigma \subset \Xi \), we define \( W^s(x, \Sigma) \) to be the connected component of \( W^{cs}(x) \cap \Sigma \) that contains \( x \). Given \( \Sigma, \Sigma' \subset \Xi \) we set \( \Sigma(\Sigma')_n = \{ x \in \Sigma : R^n(x) \subset \Sigma \} \) the domain of the map \( R^n \) from \( \Sigma \) to \( \Sigma' \). Remembering relation (20), the tangent direction to each \( W^s(x, \Sigma) \) is contracted at an exponential rate \( \left\| D\Phi(R^n(x))e_x^n \right\| \leq C e^{-\beta t_n(x)} \), with \( C = \frac{K}{\kappa} \) and \( \beta = -\log \lambda > 0 \). Since the cross-section of \( \Xi \) are GCS and satisfies (1) for some \( \delta > 0 \), then we can take \( n \) such that \( t_n(x) > t_1 \) as in Proposition 3 with \( t_1 \) satisfying

\[ C e^{-\beta t_1} \sup \{ l(W^s(x, \Sigma)) : x \in \Sigma \} < \delta \quad \text{and} \quad C e^{-\beta t_1} < \frac{1}{2}, \tag{21} \]

where \( l(W^s(x, \Sigma)) \) is the length of \( W^s(x, \Sigma) \). Under these conditions, we have

**Lemma 5.7** Let \( n \) be satisfying conditions from Proposition 3. If \( R^n : \Sigma(\Sigma')_n \to \Sigma' \) defined by \( R^n(x) = \phi_{a_n}^n(x) \). Then,

1. \( R^n(W^s(x, \Sigma)) \subset W^s(R^n(x), \Sigma') \) for every \( x \in \Sigma(\Sigma')_n \).
2. \( d(R^n(y), R^n(z)) \leq \frac{1}{2} d(y, z) \) for every \( y, z \in W^s(x, \Sigma) \) and \( x \in \Sigma(\Sigma')_n \).

We let \( \{ U_{\Sigma_i} : i = 1, \ldots, m \} \) be a finite cover of \( \Lambda \), as in the Lemma 3.5 where the \( \Sigma_i \) is a GCS for each \( i \), and we set \( T_3 \) to be an upper bound for the time it takes any point \( z \in U_{\Sigma_i} \) to leave this tubular neighborhood under the flow, for any \( i = 1, \ldots, l \). We assume, without loss of generality, that \( t_1 \) in Proposition 3 and (21) is bigger than \( T_3 \) and we consider \( n \) of Lemma 5.7. If the point \( z \) never returns to one of the cross-sections, then the map \( R \) is not defined at \( z \). Moreover, by the Lemma 5.7, if \( R^n \) is defined for \( x \in \Sigma \) for some \( \Sigma \subset \Xi \), then \( R^n \) is defined for every point in \( W^s(x, \Sigma) \). Hence, the domain of \( R^n \Sigma \) consists of strips of \( \Sigma \). The smoothness of \( (t, x) \to \phi^t(x) \) ensure that the strips

\[ \Sigma(\Sigma')_n = \{ x \in \Sigma : R^n(x) \subset \Sigma' \} \]

have non-empty interior in \( \Sigma \) for every \( \Sigma, \Sigma' \subset \Xi \). Note that by the Tubular Flow Theorem and the smoothness of the flow, the map \( R \) is locally smooth for all points \( x \in \text{int} \Sigma \) such that \( R(x) \in \text{int} \Xi \), where \( \text{int} \Xi = \{ \text{int} \Sigma_i \}_{i=1}^m \). We will denote \( \partial^j \Xi = \{ \partial^j \Sigma_i \}_{i=1}^m \) for \( j = s, u \).

**Lemma 5.8** The set of discontinuities of \( R \) in \( \Xi \setminus (\partial^s \Xi \cup \partial^u \Xi) \) is contained in the set of point \( x \in \Xi \setminus (\partial^s \Xi \cup \partial^u \Xi) \) such that, \( R(x) \) is defined and belongs to \( (\partial^s \Xi \cup \partial^u \Xi) \).
**Proof** Let \( x \) be a point in \( \Sigma \setminus (\partial^s \Sigma \cup \partial^u \Sigma) \) for some \( \Sigma \in \Xi \), not satisfying the condition. Then \( \mathcal{R}(x) \) is defined and \( \mathcal{R}(x) \) belongs to the interior of some cross-section \( \Sigma' \). By the smoothness of the flow, we have that \( \mathcal{R} \) is smooth in a neighborhood of \( x \) in \( \Sigma \). Hence, any discontinuity point for \( \mathcal{R} \) must be in the condition of the Lemma. \( \square \)

Let \( D_j \subset \Sigma_j \) be the set of points sent by \( \mathcal{R}^n \) into stable boundary points of some GCS of \( \Xi \), if we define the set

\[
L_j = \{ W^s(x, \Sigma_j) : x \in D_j \},
\]

then the Lemma 5.7 implies that \( L_j = D_j \). Let \( B_j \subset \Sigma_j \) be the set of points sent by \( \mathcal{R}^n \) into unstable boundary points of some GCS of \( \Xi \). Denote

\[
\Gamma_j = \bigcup_{x \in D_j} W^s(x, \Sigma_j) \cup B_j \quad \text{and} \quad \Gamma = \bigcup \Gamma_j \cup (\partial^s \Xi \cup \partial^u \Xi).
\]

Then, \( \mathcal{R}^n \) is smooth in the complement \( \Xi \setminus \Gamma \) of \( \Gamma \). Observe that if \( x \in D_j \) for some \( j \in \{1, \ldots, l\} \), then

\[
\mathcal{R}^n(W^s(x, \Sigma_j)) \subset \partial^s \Sigma' \quad \text{for some } \Sigma' \in \Xi.
\]

We know that \( \partial^s \Sigma \cap \Lambda = \emptyset \), then \( \mathcal{R}^n(W^s(x, \Sigma_j)) \cap \Lambda = \emptyset \) for all \( x \in D_j \), which implies that \( W^s(x, \Sigma_j) \cap \Lambda = \emptyset \) for all \( x \in D_j \). However, if \( x \in B_j \), then \( \mathcal{R}^n(x) \in \partial^u \Sigma' \) for some \( \Sigma' \in \Xi \) and again we know that \( \partial^u \Sigma \cap \Lambda = \emptyset \), this implies that \( B_j \cap \Lambda = \emptyset \). Therefore, \( \Gamma_j \cap \Lambda = \emptyset \) for all \( j \in \{1, \ldots, l\} \), so \( \Gamma \cap \Lambda = \emptyset \). The latter arguments proved the following:

**Lemma 5.9** If \( x \in \Lambda \cap \Xi \), then \( \mathcal{R}^n(x) \) is defined and \( \mathcal{R}^n(x) \in \text{int} \Xi \).

**Proof of Lemma 3.6** Note simply that by Lemma 5.9 the set \( \Lambda \cap \Xi \) is an invariant set for \( \mathcal{R}^n \) and by Proposition 3, \( \Lambda \cap \Xi \) is hyperbolic set for \( \mathcal{R}^n \) and since \( \Lambda \cap \Xi \) is invariant for \( \mathcal{R} \), then \( \Lambda \cap \Xi \) is hyperbolic for \( \mathcal{R} \). \( \square \)

**Conservative Poincaré Map**

Recall that \( \phi = (\phi^t)_{t \in \mathbb{R}} \) preserves a volume form \( \omega \) on \( M \). We define the 2-form on \( \Xi \), \( \omega_\Xi : T \Xi \times T \Xi \rightarrow \mathbb{R} \), defined by \( \omega_\Xi)_\theta(u, v) = \omega(u, v, \Phi(\theta)) \), where \( u, v \in T_\theta \Xi \) and \( T \Xi = \bigcup_\theta T_\theta \Xi \) is the tangent bundle of \( T \Xi \).

**Lemma 5.10** The 2-form \( \omega_\Xi \) is a volume form on \( \Xi \) which is invariant by \( \mathcal{R} \).

**Proof** Note simply that

\[
(\omega_\Xi)_{\mathcal{R}(\theta)}(D\mathcal{R}(u), D\mathcal{R}(v)) = \omega_{\Phi^t(\theta)}(D\mathcal{R}(u), D\mathcal{R}(v), \Phi(\phi^t(\theta)))
\]

\[
= \omega(P_{\mathcal{R}(\theta)}(D\mathcal{R}(u)), P_{\mathcal{R}(\theta)}(D\mathcal{R}(v)), \Phi(\phi^t(\theta))),
\]

where \( P_{\mathcal{R}(\theta)} \) is the projection onto \( T_{\mathcal{R}(\theta)} \Xi \) along the direction of \( \Phi(\mathcal{R}^n(x)) \) (see proof of Proposition 3). Note that \( D\mathcal{R}(\phi^t(\theta))u = P_{\mathcal{R}(\theta)}(D\mathcal{R}(\phi^t(\theta))u + A \Phi(\phi^t(\theta))) \) and
\[ D\phi^{t+}(\theta)(v) = P_{R(\theta)}(D\phi^{t+}(\theta)(v)) + B \Phi(\phi^{t+}(\theta)), \] for some constants \( A \) and \( B \). Since \( \omega \) is a volume form, then
\[
\omega(P_{R(\theta)}(D\phi^{t+}(\theta)(u)), P_{R(\theta)}(D\phi^{t+}(\theta)(v)), \Phi(\phi^{t+}(\theta))) = \omega(u, v, \Phi(\theta)) = (\omega_{\Xi})_{\theta}(u, v).
\]
Therefore, the 2-form \( \omega_{\Xi} \) is an invariant form for \( R \). As \( \Xi \) is transverse to the flow and \( \omega \) is a volume form, then it is easy to see that \( \omega_{\Xi} \) is also a volume form for \( \Xi \). \( \square \)

**Regular Cantor Sets**

Let \( A \) be a finite alphabet, \( B \) a subset of \( A^2 \), and \( \Sigma_B \) the subshift of finite type of \( A^\mathbb{Z} \) with allowed transitions \( B \). We will always assume that \( \Sigma_B \) is topologically mixing and that every letter in \( A \) occurs in \( \Sigma_B \).

An **expansive map of type** \( \Sigma_B \) is a map \( g \) with the following properties:

(i) the domain of \( g \) is a disjoint union \( \bigcup_B I(a, b) \). Where for each \( (a, b) \), \( I(a, b) \)
    is a compact subinterval of \( I(a) := [0, 1] \times \{a\} \);
(ii) for each \( (a, b) \in B \), the restriction of \( g \) to \( I(a, b) \) is a smooth diffeomorphism
    onto \( I(b) \) satisfying \( |Dg(t)| > 1 \) for all \( t \).

The **regular Cantor set** associated to \( g \) is the maximal invariant set
\[
K = \bigcap_{n \geq 0} g^{-n}\left( \bigcup_B I(a, b) \right).
\]

Let \( \Sigma_B^+ \) be the unilateral subshift associated to \( \Sigma_B \). There exists a unique homeomorphism \( h: \Sigma_B^+ \to K \) such that
\[
 h(a) \in I(a_n), \text{ for } a = (a_0, a_1, \ldots) \in \Sigma_B^+ \text{ and } h \circ \sigma = g \circ h,
\]
where \( \sigma^+: \Sigma_B^+ \to \Sigma_B^+ \), is defined as follows \( \sigma^+((a_n)_{n \geq 0}) = (a_{n+1})_{n \geq 0} \).

**Expanding Maps Associated to a Horseshoe**

Let \( \Lambda \) be a horseshoe associated to \( C^2 \)-diffeomorphism \( \varphi \) on a surface \( M \) and consider a finite collection \( (R_a)_{a \in \Lambda} \) of disjoint rectangles of \( M \), which are a Markov partition of \( \Lambda \). Define the sets
\[
W^s(\Lambda, R) = \bigcap_{n \geq 0} \varphi^{-n}\left( \bigcup_{a \in \Lambda} R_a \right),
\]
\[
W^u(\Lambda, R) = \bigcap_{n \leq 0} \varphi^{-n}\left( \bigcup_{a \in \Lambda} R_a \right).
\]
There is a \( r > 1 \) and a collection of \( C^r \)-submersions \( (\pi_a : R_a \to I(a))_{a \in \mathcal{A}} \), satisfying the following property:

If \( z, z' \in R_{a_0} \cap \varphi^{-1}(R_{a_1}) \) and \( \pi_{a_0}(z) = \pi_{a_0}(z') \), then we have

\[
\pi_{a_1}(\varphi(z)) = \pi_{a_1}(\varphi(z')).
\]

In particular, the connected components of \( W^s(\Lambda, R) \cap R_a \) are the level lines of \( \pi_a \).

Then we define a mapping \( g^u \) of class \( C^r \) (expansive of type \( \Sigma_{\mathbb{B}} \)) by the formula

\[
g^u(\pi_{a_0}(z)) = \pi_{a_1}(\varphi(z))
\]

for \( (a_0, a_1) \in \mathbb{B}, z \in R_{a_0} \cap \varphi^{-1}(R_{a_1}) \). The regular Cantor set \( K^u \) defined by \( g^u \), describes the geometry transverse of the stable foliation \( W^s(\Lambda, R) \). Analogously, we can describe the geometry transverse of the unstable foliation \( W^u(\Lambda, R) \) using a regular Cantor set \( K^s \) defined by a mapping \( g^s \) of class \( C^r \) (expansive of type \( \Sigma_{\mathbb{B}} \)).

Also, the horseshoe \( \Lambda \) is locally the product of two regular Cantor sets \( K^s \) and \( K^u \). So, the Hausdorff dimension of \( \Lambda \), \( HD(\Lambda) \) is equal to \( HD(K^s \times K^u) \), but for regular Cantor sets, we have that \( HD(K^s \times K^u) = HD(K^s) + HD(K^u) \). Thus \( HD(\Lambda) = HD(K^s) + HD(K^u) \) (cf. [24, chap 4]).

**Intersections of Regular Cantor Sets and Property \( V \)**

Let \( r \) be a real number \( > 1 \), or \( r = +\infty \). The space of \( C^r \) expansive maps of type \( \Sigma \) (cf. Subsection 3), endowed with the \( C^r \) topology, will be denoted by \( \Omega^r_{\Sigma} \). The union \( \Omega_{\Sigma} = \bigcup_{r > 1} \Omega^r_{\Sigma} \) is endowed with the inductive limit topology.

Let \( \Sigma^− = \{ (\theta_n)_{n \leq 0}, (\theta_i, \theta_{i+1}) \in \mathbb{B} \text{ for } i < 0 \} \). We equip \( \Sigma^− \) with the following ultrametric distance: for \( \theta \not= \tilde{\theta} \in \Sigma^− \), set

\[
d(\theta, \tilde{\theta}) = \begin{cases} 1 & \text{if } \theta_0 \not= \tilde{\theta}_0, \\ |I(\theta \land \tilde{\theta})| & \text{otherwise}, \end{cases}
\]

where \( \theta \land \tilde{\theta} = (\theta_{-n}, \ldots, \theta_0) \) if \( \tilde{\theta}_j = \theta_{-j} \) for \( 0 \leq j \leq n \) and \( \tilde{\theta}_{-n-1} \not= \theta_{-n-1} \).

Now, let \( \theta \in \Sigma^− \); for \( n > 0 \), let \( \theta^n = (\theta_{-n}, \ldots, \theta_0) \), and let \( B(\theta^n) \) be the affine map from \( I(\theta^n) \) onto \( I(\theta_0) \) such that the diffeomorphism \( k^n_\theta = B(\theta^n) \circ f^{\theta^n}_\theta \) is orientation preserving.

We have the following well-known result (cf. [25]):

**Proposition** Let \( r \in (1, +\infty) \), \( g \in \Omega^r_{\Sigma} \).

1. For any \( \theta \in \Sigma^− \), there is a diffeomorphism \( k^\theta_\pi \in \text{Diff}^r_+(I(\theta_0)) \) such that \( k^\theta_\pi \) converges to \( k^\theta_\pi \) in \( \text{Diff}^r_+(I(\theta_0)) \), for any \( r' < r \). The convergence is also uniform in a neighborhood of \( g \in \Omega^r_{\Sigma} \).
2. If $r$ is an integer or $r = + \infty$, $k^n_\theta$ converge to $k^\varnothing$ in $\text{Diff}_+^r(I(\theta_0))$. More precisely, for every $0 \leq j \leq r - 1$, there is a constant $C_j$ (independent on $\theta$) such that

$$\left| D^j \log D \left[ k^n_\theta \circ (k^\varnothing)^{-1} \right](x) \right| \leq C_j |I(\varnothing^n)|.$$

It follows that $\theta \rightarrow k^\varnothing$ is Lipschitz in the following sense: for $\theta_0 = \tilde{\theta}_0$, we have

$$\left| D^j \log D \left[ k^n_{\tilde{\theta}} \circ (k^\varnothing)^{-1} \right](x) \right| \leq C_j d(\theta, \tilde{\theta}).$$

Let $r \in (1, +\infty]$. For $a \in A$, we denote by $P^r(a)$ the space of $C^r$-embeddings of $I(a)$ into $\mathbb{R}$, endowed with the $C^r$ topology. The affine group $\text{Aff}(\mathbb{R})$ acts by composition on the left on $P^r(a)$, the quotient space being denoted by $\overline{P}^r(a)$. We also consider $P(a) = \bigcup_{r \geq 1} P^r(a)$ and $\overline{P}(a) = \bigcup_{r \geq 1} \overline{P}^r(a)$, endowed with the inductive limit topologies.

**Remark 25** In [20] was considered $P^r(a)$ for $r \in (1, +\infty]$, but all the definitions and results involving $P^r(a)$ can be obtained considering $r \in [1, +\infty]$.

Let $A = (\theta, A)$, where $\theta \in \Sigma^-$ and $A$ is now an affine embedding of $I(\theta_0)$ into $\mathbb{R}$. We have a canonical map

$$A \rightarrow P^r = \bigcup_A P^r(a)$$

$$(\theta, A) \mapsto A \circ k^\varnothing \quad (\in P^r(\theta_0)).$$

Now assume we are given two sets of data $(A, B, \Sigma, g)$, $(A', B', \Sigma', g')$ defining regular Cantor sets $K$, $K'$. We define as in the previous the spaces $P = \bigcup_A P(a)$ and $P' = \bigcup_{A'} P(a')$. A pair $(h, h')$, $(h \in P(a), h' \in P'(a'))$ is called a smooth configuration for $K(a) = K \cap I(a)$, $K'(a') = K' \cap I(a')$. Actually, rather than working in the product $P \times P'$, it is better to go to the quotient $Q$ by the diagonal action of the affine group $\text{Aff}(\mathbb{R})$. Elements of $Q$ are called smooth relative configurations for $K(a)$, $K'(a')$. We say that a smooth configuration $(h, h') \in P(a) \times P(a')$ is
As in previous, we can introduce the spaces $A$ making sense for smooth relative configurations. All these definitions are invariant under the action of the affine group and, therefore, there is an open and dense set $U$ allowing to define linked, intersecting, and stably intersecting configurations at the level of $A$. Let $K_s, K_u$ are the stable and unstable Cantor sets respectively. Suppose that $\lambda^1 \neq 0$ and $\lambda + \lambda^{-1}$ is symplectic) and $0$ a hyperbolic fixed point with eigenvalues $\lambda$ and $\lambda^{-1}$, then the Birkhoff normal form (cf. [18]) says that there is an area-preserving change of coordinates $\eta$ such that $\eta^{-1} \circ f \circ \eta = N$, where $N(x, y) = (U(xy)x, U^{-1}(xy)y)$ and

- **linked** if $h(I(a)) \cap h'(I(a')) \neq \emptyset$;
- **intersecting** if $h(K(a)) \cap h'(K(a')) \neq \emptyset$, where $K(a) = K \cap I(a)$ and $K(a') = K \cap I(a')$;
- **stably intersecting** if it is still intersecting when we perturb it in $P \times P'$, and we perturb $(g, g')$ in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

All these definitions are invariant under the action of the affine group and, therefore, make sense for smooth relative configurations.

As in previous, we can introduce the spaces $A, A'$ associated to the limit geometries of $g, g'$, respectively. We denote by $C$ the quotient of $A \times A'$ by the diagonal action on the left of the affine group. An element of $C$, represented by $(\Theta, A) \in A$, $(\Theta', A') \in A'$, is called a relative configuration of the limit geometries determined by $\Theta, \Theta'$. We have canonical maps

$$A \times A' \to P \times P'$$

$$C \to Q$$

allowing to define linked, intersecting, and stably intersecting configurations at the level of $A \times A'$ or $C$.

**Remark** For a configuration $((\Theta, A), (\Theta', A'))$ of limit geometries, one could also consider the weaker notion of stable intersection obtained by considering perturbations of $g, g'$ in $\Omega_\Sigma \times \Omega_{\Sigma'}$ and perturbations of $(\Theta, A), (\Theta', A')$ in $A \times A'$. We do not know of any example of expansive maps $g, g'$, and configurations $(\Theta, A), (\Theta', A')$ which are stably intersecting in the weaker sense, but not in the stronger sense.

We consider the following subset $V$ of $\Omega_\Sigma \times \Omega_{\Sigma'}$. A pair $(g, g')$ belongs to $V$ if for any $[(\Theta, A), (\Theta', A')] \in A \times A'$ there is a translation $R_t$ (in $\mathbb{R}$) such that $(R_t \circ A \circ k_\Theta, A' \circ k_{\Theta'}^\tau)$ is a stably intersecting configuration.

**Definition 10** We say that a pair $(\psi, \Lambda)$, where $\Lambda$ is a horseshoe for $\psi$, has the property $V$ if the stable and unstable cantor sets have the property $V$ in the above sense.

The more important result in this setting is

**Theorem 5.1** (Moreira-Yoccoz [21]) Let $\varphi$ be a $C^\infty$ diffeomorphism with a horseshoe $\Lambda$. Let $K^s, K^u$ are the stable and unstable Cantor sets respectively. Suppose that $HD(K^s) + HD(K^u) > 1$. If $U$ is sufficiently small neighborhood $\varphi$ in $Diff^\infty(M)$, there is an open and dense set $U^* \subset U$ such that, for every $\psi \in U^*$ the pair $(\psi, \Lambda_\psi)$ has the property $V$.

**The Birkhoff Invariant**

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a germ of diffeomorphism area-preserving (in dimension two is symplectic) and 0 a hyperbolic fixed point with eigenvalues $\lambda$ and $\lambda^{-1}$, then the Birkhoff normal form (cf. [18]) says that there is an area-preserving change of coordinates $\eta$ such that $\eta^{-1} \circ f \circ \eta = N$, where $N(x, y) = (U(xy)x, U^{-1}(xy)y)$ and
U(xy) is a power series \( \lambda + U_2xy + \cdots \) convergent in a neighborhood of \( x = y = 0 \). In other words, in this coordinates \( f \) can be written by

\[
f(x, y) = (\lambda x(1 + axy + O(\|(x, y)\|^4)), \lambda^{-1}y(1 - axy + O(\|(x, y)\|^4)))
\] (22)

and the number \( a \) is called the Birkhoff Invariant of \( f \).

**Lemma 5.11** The Birkhoff invariant for area-preserving diffeomorphisms in \((\mathbb{R}^2, 0)\) only depends on 3-jets in 0, \( J^3(0) \). Moreover, the set of diffeomorphism area-preserving in \((\mathbb{R}^2, 0)\) such that the Birkhoff invariant is non-zero is open, dense, and invariant in \( J^3(0) \).

**Proof** For the proof of [18, Theorem 1 and 2], we have the first part and opening. For density, suppose that for some \( f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \), the Birkhoff invariant is zero, then for \( \epsilon > 0 \) we consider the function \( N_\epsilon(x, y) := (\lambda x(1 + O(\|(x, y)\|^4)), \lambda^{-1}y(1 + O(\|(x, y)\|^4))) + \epsilon(x^2y, -xy^2) \), then the function \( f_\epsilon = \eta \circ N_\epsilon \circ \eta^{-1} \) is area-preserving diffeomorphism close to \( f \) with the Birkhoff invariant \( \epsilon \).

Let \( f \), \( g \) be as above and suppose that the Birkhoff invariant for \( f \) is non-zero, then \( g^{-1} \circ f \circ g \) has the Birkhoff invariant non zero. Indeed, by the Birkhoff Normal Form [18, Theorem 1], there is an area-preserving change of coordinates \( \eta \) such that \( \eta^{-1} \circ g^{-1} \circ f \circ g \circ \eta \) has the form (22), then \( (g \circ \eta)^{-1} \circ f \circ (g \circ \eta) \) has the form (22). In other words, there is another area-preserving change of coordinates \( g \circ \eta \) such that \( f \) has the form (22), but by the unicity of the Birkhoff normal form (see [18, page 674]), we have that the Birkhoff invariant of \( g^{-1} \circ f \circ g \) is equal to the Birkhoff invariant of \( f \), therefore non-zero. \( \square \)

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