JACOBI-STIRLING POLYNOMIALS AND P-PARTITIONS

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ABSTRACT. We investigate the diagonal generating function of the Jacobi-Stirling numbers of the second kind $JS(n + k, n; z)$ by generalizing the analogous results for the Stirling and Legendre-Stirling numbers. More precisely, letting $JS(n + k, n; z) = p_k,0(n) + p_{k, 1}(n)z + \cdots + p_{k,k}(n)z^k$, we show that $(1 - t)^{3k-i+1} \sum_{n > 0} p_{k,i}(n)t^n$ is a polynomial in $t$ with nonnegative integral coefficients and provide combinatorial interpretations of the coefficients by using Stanley’s theory of $P$-partitions.

1. Introduction

Let $\ell_{\alpha, \beta}[y](t)$ be the Jacobi differential operator:

$$\ell_{\alpha, \beta}[y](t) = \frac{1}{(1 - t)^{\alpha}(1 + t)^{\beta}} (- (1 - t)^{\alpha+1}(1 + t)^{\beta+1}y'(t))'.$$

It is well known that the Jacobi polynomial $y = P_{n}^{(\alpha, \beta)}(t)$ is an eigenvector for the differential operator $\ell_{\alpha, \beta}$ corresponding to $n(n + \alpha + \beta + 1)$, i.e.,

$$\ell_{\alpha, \beta}[y](t) = n(n + \alpha + \beta + 1)y(t).$$

For each $n \in \mathbb{N}$, the Jacobi-Stirling numbers $JS(n, k; z)$ of the second kind appeared originally as the coefficients in the expansion of the $n$-th composite power of $\ell_{\alpha, \beta}$ (see [7]):

$$(1 - t)^{\alpha}(1 + t)^{\beta}\ell_{\alpha, \beta}^n[y](t) = \sum_{k=0}^{n} (-1)^k JS(n, k; z) \left( (1 - t)^{\alpha+k}(1 + t)^{\beta+k}y^{(k)}(t) \right)^{(k)},$$

where $z = \alpha + \beta + 1$, and can also be defined as the connection coefficients in

$$x^n = \sum_{k=0}^{n} JS(n, k; z) \prod_{i=0}^{k-1} (x - i(z + i)). \quad (1.1)$$

The Jacobi-Stirling numbers $js(n, k; z)$ of the first kind are defined by

$$\prod_{i=0}^{n-1} (x - i(z + i)) = \sum_{k=0}^{n} js(n, k; z)x^k. \quad (1.2)$$

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When \( z = 1 \), the Jacobi-Stirling numbers become the Legendre-Stirling numbers \( 6 \) of the first and second kinds:

\[
\text{ls}(n, k) = \text{js}(n, k; 1), \quad \text{LS}(n, k) = \text{JS}(n, k; 1).
\] (1.3)

Generalizing the work of Andrews and Littlejohn \( 2 \) on Legendre-Stirling numbers, Gelineau and Zeng \( 9 \) studied the combinatorial interpretations of the Jacobi-Stirling numbers and remarked on the connection with Stirling numbers and central factorial numbers. Further properties of the Jacobi-Stirling numbers have been given by Andrews, Egge, Gawronksi, and Littlejohn \( 1 \).

The Stirling numbers of the second and first kinds \( S(n, k) \) and \( s(n, k) \) are defined by

\[
x^n = \sum_{k=0}^{n} S(n, k) \prod_{i=0}^{k-1} (x - i), \quad \prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^{n} s(n, k)x^k.
\] (1.4)

The lesser known central factorial numbers \( 14 \) p. 213–217 \( T(n, k) \) and \( t(n, k) \) are defined by

\[
x^n = \sum_{k=0}^{n} T(n, k) x \prod_{i=1}^{k-1} \left( x + \frac{k}{2} - i \right),
\] (1.5)

and

\[
x \prod_{i=1}^{n-1} \left( x + \frac{n}{2} - i \right) = \sum_{k=0}^{n} t(n, k)x^k.
\] (1.6)

Starting from the fact that for fixed \( k \), the Stirling number \( S(n + k, n) \) can be written as a polynomial in \( n \) of degree \( 2k \) and there exist nonnegative integers \( c_{k,j}, 1 \leq j \leq k \), such that

\[
\sum_{n \geq 0} S(n + k, n)t^n = \sum_{j=1}^{k} c_{k,j}t^j \frac{1}{(1 - t)^{2k+1}},
\] (1.7)

Gessel and Stanley \( 10 \) gave a combinatorial interpretation for the \( c_{k,j} \) in terms of the descents in \textit{Stirling permutations}. Recently, Egge \( 5 \) has given an analogous result for the Legendre-Stirling numbers, and Gelineau \( 8 \) has made a preliminary study of the analogous problem for Jacobi-Stirling numbers. In this paper, we will prove some analogous results for the diagonal generating function of Jacobi-Stirling numbers. As noticed in \( 9 \), the leading coefficient of the polynomial \( \text{JS}(n, k; z) \) is \( S(n, k) \) and the constant term of \( \text{JS}(n, k; z) \) is the central factorial number of the second kind with even indices \( T(2n, 2k) \). Similarly, the leading coefficient of the polynomial \( \text{js}(n, k; z) \) is \( s(n, k) \) and the constant term of \( \text{js}(n, k; z) \) is the central factorial number of the first kind with even indices \( t(2n, 2k) \).
Definition 1. The Jacobi-Stirling polynomial of the second kind is defined by
\[ f_k(n; z) := JS(n + k, n; z). \] (1.8)
The coefficient \( p_{k,i}(n) \) of \( z^i \) in \( f_k(n; z) \) is called the Jacobi-Stirling coefficient of the second kind for \( 0 \leq i \leq k \). Thus
\[ f_k(n; z) = p_{k,0}(n) + p_{k,1}(n)z + \cdots + p_{k,k}(n)z^k. \] (1.9)

The main goal of this paper is to prove Theorems 1 and 2 below.

Theorem 1. For each integer \( k \) and \( i \) such that \( 0 \leq i \leq k \), there is a polynomial \( A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j} t^j \) with positive integer coefficients such that
\[ \sum_{n \geq 0} p_{k,i}(n) t^n = \frac{A_{k,i}(t)}{(1-t)^{2k-i+1}}. \] (1.10)

In order to give a combinatorial interpretation for \( a_{k,i,j} \), we introduce the multiset
\[ M_k := \{1, 1, \bar{1}, 2, 2, \bar{2}, \ldots, k, k, \bar{k}\}, \]
where the elements are ordered by
\[ \bar{1} < 1 < \bar{2} < 2 \ldots < \bar{k} < k. \] (1.11)
Let \([\bar{k}] := \{1, 2, \ldots, k\}\). For any subset \( S \subseteq [\bar{k}] \), we set \( M_{k,S} = M_k \setminus S \).

Definition 2. A permutation \( \pi \) of \( M_{k,S} \) is a Jacobi-Stirling permutation if whenever \( u < v < w \) and \( \pi(u) = \pi(w) \), we have \( \pi(v) > \pi(u) \). We denote by \( JSP_{k,S} \) the set of Jacobi-Stirling permutations of \( M_{k,S} \) and
\[ JSP_{k,i} = \bigcup_{S \subseteq [\bar{k}]} JSP_{k,S}. \]

For example, the Jacobi-Stirling permutations of \( JSP_{2,1} \) are:
\[
\begin{align*}
2211, & \ 2221, \ 2122, \ 2112, \ 21121, \ 12221, \ 12221, \ 12112, \ 22112, \ 12212, \\
11222, & \ 11222, \ 22111, \ 12211, \ 11221, \ 11221, \ 22111, \ 12211, \ 11221, \ 11221, \ 11221.
\end{align*}
\]
Let \( \pi = \pi_1 \pi_2 \ldots \pi_m \) be a word on a totally ordered alphabet. We say that \( \pi \) has a descent at \( l \), where \( 1 \leq l \leq m - 1 \), if \( \pi_l > \pi_{l+1} \). Let \( \text{des} \ \pi \) be the number of descents of \( \pi \). The following is our main interpretation for the coefficients \( a_{k,i,j} \).

Theorem 2. For \( k \geq 1, 0 \leq i \leq k \), and \( 1 \leq j \leq 2k-i \), the coefficient \( a_{k,i,j} \) is the number of Jacobi-Stirling permutations in \( JSP_{k,i} \) with \( j - 1 \) descents.

The rest of this paper is organized as follows. In Section 2, we investigate some elementary properties of the Jacobi-Stirling polynomials and prove Theorem 1. In Section 3 we apply Stanley’s \( P \)-partition theory to derive a first interpretation of the integers \( a_{k,i,j} \) and then reformulate it in terms of descents of Jacobi-Stirling permutations in Section 4.
In Section 5, we construct Legendre-Stirling posets in order to prove a similar result for the Legendre-Stirling numbers, and then to deduce Egge’s result for Legendre-Stirling numbers [3] in terms of descents of Legendre-Stirling permutations. A second proof of Egge’s result is given by making a link to our result for Jacobi-Stirling permutations, namely Theorem 2. We end this paper with a conjecture on the real-rootedness of the polynomials $A_{k,i}(t)$.

2. Jacobi-Stirling Polynomials

Proposition 3. For $0 \leq i \leq k$, the Jacobi-Stirling coefficient $p_{k,i}(n)$ is a polynomial in $n$ of degree $3k - i$. Moreover, the leading coefficient of $p_{k,i}(n)$ is

$$\frac{1}{3^{k-i}i!(k-i)!}$$

for all $0 \leq i \leq k$.

Proof. We proceed by induction on $k \geq 0$. For $k = 0$, we have $p_{0,0}(n) = 1$ since $f_0(n) = \text{JS}(n, n; z) = 1$. Let $k \geq 1$ and suppose that $p_{k-1,i}$ is a polynomial in $n$ of degree $3(k-1) - i$ for $0 \leq i \leq k-1$. From (1.1) we deduce the recurrence relation:

$$\begin{cases} 
\text{JS}(0, 0; z) = 1, \\
\text{JS}(n, k; z) = 0, \text{ if } k \not\in \{1, \ldots, n\}, \\
\text{JS}(n, k; z) = \text{JS}(n-1, k-1; z) + k(k+z)\text{JS}(n-1, k; z), \text{ for } n, k \geq 1. 
\end{cases}$$

Substituting in (1.8) yields

$$f_k(n; z) - f_k(n-1; z) = n(n+z)f_{k-1}(n; z).$$

It follows from (1.9) that for $0 \leq i \leq k$,

$$p_{k,i}(n) - p_{k,i}(n-1) = n^2p_{k-1,i}(n) + np_{k-1,i-1}(n).$$

Applying the induction hypothesis, we see that $p_{k,i}(n) - p_{k,i}(n-1)$ is a polynomial in $n$ of degree at most

$$\max(3(k-1) - i + 2, 3(k-1) - (i - 1) + 1) = 3k - i - 1.$$ 

Hence $p_{k,i}(n)$ is a polynomial in $n$ of degree at most $3k - i$. It remains to determine the coefficient of $n^{3k-i}$, say $\beta_{k,i}$. Extracting the coefficient of $n^{3k-i-1}$ in (2.1) we have

$$\beta_{k,i} = \frac{1}{3k-i}(\beta_{k-1,i} + \beta_{k-1,i-1}).$$

Now it is fairly easy to see that (2.1) satisfies the above recurrence. □

Proposition 4. For all $k \geq 1$ and $0 \leq i \leq k$, we have

$$p_{k,i}(0) = p_{k,i}(-1) = p_{k,i}(-2) = \cdots = p_{k,i}(-k) = 0.$$

(2.5)
Table 1. The first values of $A_{k,i}(t)$

| $k \backslash i$ | 0       | 1       | 2       | 3       |
|------------------|---------|---------|---------|---------|
| 0                | $t + t^2$ | $t$     |         |         |
| 1                | $t + 14t^2 + 21t^3 + 4t^4$ | $2t + 12t^2 + 6t^3$ | $t + 2t^2$ |         |
| 2                | $t + 75t^2 + 603t^3 + 1065t^4 + 460t^5 + 36t^6$ | $3t + 114t^2 + 501t^3 + 436t^4 + 66t^5$ | $3t + 55t^2 + 116t^3 + 36t^4$ | $t + 8t^2 + 6t^3$ |

Proof. We proceed by induction on $k$. By definition, we have

$$f_1(n; z) = JS(n + 1, n; z) = p_{1,0}(n) + p_{1,1}(n)z.$$  

As noticed in [9, Theorem 1], the leading coefficient of the polynomial $JS(n, k; z)$ is $S(n, k)$ and the constant term is $T(2n, 2k)$. We derive from (1.4) and (1.5) that

$$p_{1,1}(n) = S(n + 1, n) = n(n + 1)/2,$$

$$p_{1,0}(n) = T(2n + 2, 2n) = n(n + 1)(2n + 1)/6.$$  

Hence (2.5) is true for $k = 1$. Assume that (2.5) is true for some $k \geq 1$. By (2.4) we have

$$p_{k,i}(n) - p_{k,i}(n - 1) = n^2 p_{k-1,i}(n) + np_{k-1,i-1}(n).$$

Since $JS(0, k; z) = 0$ if $k \geq 2$, we have $p_{k,i}(0) = 0$. The above equation and the induction hypothesis imply successively that

$$p_{k,i}(-1) = 0, \quad p_{k,i}(-2) = 0, \ldots, \quad p_{k,i}(-k + 1) = 0, \quad p_{k,i}(-k) = 0.$$  

The proof is thus complete. □

Lemma 5. For each integer $k$ and $i$ such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = \sum_{j=1}^{2k-i} a_{k,i,j}t^j$ with integer coefficients such that

$$\sum_{n \geq 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1-t)^{3k-i+1}}. \quad (2.6)$$

Proof. By Proposition 3 and standard results concerning rational generating functions (cf. [16, Corollary 4.3.1]), for each integer $k$ and $i$ such that $0 \leq i \leq k$, there is a polynomial $A_{k,i}(t) = a_{k,i,0} + a_{k,i,1}t + \cdots + a_{k,i,3k-i}t^{3k-i}$ satisfying (2.6). Now, by [16, Proposition 4.2.3], we have

$$\sum_{n \geq 1} p_{k,i}(-n)t^n = -\frac{A_{k,i}(1/t)}{(1-1/t)^{2k-i+1}}. \quad (2.7)$$

Applying (2.5) we see that $a_{k,i,2k-i+1} = \cdots = a_{k,i,3k-i} = 0$. □

The first values of $A_{k,i}(t)$ are given in Table 1. The following result gives a recurrence for the coefficients $a_{k,i,j}$. 
Proposition 6. Let $a_{0,0,0} = 1$. For $k, i, j \geq 0$, we have the following recurrence for the integers $a_{k,i,j}$:

$$a_{k,i,j} = j^2 a_{k-1,i,j} + [2(j - 1)(3k - i - j - 1) + (3k - i - 2)]a_{k-1,i,j-1}$$

$$+ (3k - i - j)^2 a_{k-1,i,j-2} + ja_{k-1,i,j} + (3k - i - j)a_{k-1,i,j-1},$$

(2.8)

where $a_{k,i,j} = 0$ if any of the indices $k, i, j$ is negative or if $j \notin \{1, \ldots, 2k - i\}$.

Proof. For $0 \leq i \leq k$, let

$$F_{k,i}(t) = \sum_{n \geq 0} p_{k,i}(n)t^n = \frac{A_{k,i}(t)}{(1 - t)^{3k-i+1}}.$$  

(2.9)

The recurrence relation (2.4) is equivalent to

$$F_{k,i}(t) = (1 - t)^{-1}[t^2 F'_{k-1,i}(t) + tF'_{k-1,i}(t) + tF'_{k-1,i-1}(t)]$$

(2.10)

with $F_{0,0} = (1 - t)^{-1}$. Substituting (2.9) into (2.10) we obtain

$$A_{k,i}(t) = (1 - t)^{3k-i}[t^2(A_{k-1,i}(t)(1 - t)^{-3k-i-2})'$$

$$+ t(A_{k-1,i}(t)(1 - t)^{-3k-i-2})' + t(A_{k-1,i-1}(t)(1 - t)^{-3k-i-1})']$$

$$= [t^2 A''_{k-1,i}(t)(1 - t)^2 + 2(3k - i - 2)t^2 A'_{k-1,i}(t)(1 - t)$$

$$+ (3k - i - 2)(3k - i - 1)t^2 A_{k-1,i}(t)]$$

$$+ [t A'_{k-1,i}(t)(1 - t)^2 + (3k - i - 2)t A_{k-1,i}(t)(1 - t)]$$

$$+ [t A'_{k-1,i-1}(t)(1 - t) + (3k - i - 1)t A_{k-1,i-1}(t)].$$

Taking the coefficient of $t^j$ in both sides of the above equation, we have

$$a_{k,i,j} = j(j - 1)a_{k-1,i,j} - 2(j - 1)(j - 2)a_{k-1,i,j-1} + (j - 2)(j - 3)a_{k-1,i,j-2}$$

$$+ 2(3k - i - 2)(j - 1)a_{k-1,i,j+1} - 2(3k - i - 2)(j - 2)a_{k-1,i,j-2}$$

$$+ (3k - i - 2)(3k - i - 1)a_{k-1,i,j-2} + ja_{k-1,i,j} + (3k - i - 2)a_{k-1,i,j-1}$$

$$+ (j - 2)a_{k-1,i,j-2} + (3k - i - 2)a_{k-1,i,j-1} - (3k - i - 2)a_{k-1,i,j-2}$$

$$+ ja_{k-1,i-1,j} - (j - 1)a_{k-1,i-1,j+1} + (3k - i - 1)a_{k-1,i-1,j-1},$$

which gives (2.8) after simplification. \( \square \)

Corollary 7. For $k \geq 0$ and $0 \leq i \leq k$, the coefficients $a_{k,i,j}$ are positive integers for $1 \leq j \leq 2k - i$.

Proof. This follows from (2.8) by induction on $k$. Clearly, this is true for $k = 0$ and $k = 1$. Suppose that this is true for some $k \geq 1$. As each term in the right-hand side of (2.8) is nonnegative, we only need to show that at least one term on the right-hand side of (2.8) is strictly positive. Indeed, for $k \geq 2$, the induction hypothesis and (2.8) imply that

- if $j = 1$, then $a_{k,i,1} \geq a_{k-1,i-1,1} > 0$;
- if $2 \leq j \leq 2k - i$, then $a_{k,i,j} \geq (3k - i - j)a_{k-1,i,j-1} \geq ka_{k-1,i,j-1} > 0$. 


These two cases cover all possibilities. \[\square\]

Theorem 1 follows then from Lemma 5, Proposition 6 and Corollary 7.

Now, define the Jacobi-Stirling polynomial of the first kind \(g_k(n; z)\) by
\[
g_k(n; z) = js(n, n-k; z). \tag{2.11}
\]

**Proposition 8.** For \(k \geq 1\), we have
\[
g_k(n; z) = f_k(-n; -z). \tag{2.12}
\]

If we write \(g_k(n; z) = q_{k,0}(n) + q_{k,1}(n)z + \cdots + q_{k,k}(n)z^k\), then
\[
\sum_{n \geq 1} q_{k,i}(n)t^n = (-1)^k \frac{\sum_{j=1}^{2k-i} a_{k,i,3k-i+1-j}t^j}{(1-t)^{3k-i+1}}. \tag{2.13}
\]

**Proof.** From (1.2) we deduce
\[
\begin{align*}
js(0, 0; z) &= 1, \\
js(n, k; z) &= 0, \quad \text{if } k \notin \{1, \ldots, n\}, \\
js(n, k; z) &= js(n-1, k-1; z) - (n-1)(n-1+z) js(n-1, k; z), \quad n, k \geq 1.
\end{align*} \tag{2.14}
\]

It follows from the above recurrence and (2.11) that
\[
g_k(n; z) - g_k(n-1; z) = -(n-1)(n-1+z)g_{k-1}(n-1; z).
\]

Comparing with (2.3) we get (2.12), which implies that \(q_{k,i}(n) = (-1)^i p_{k,i}(-n)\). Finally (2.13) follows from (2.11).

**\[\square\]**

3. JACOBI-STIRLING POSETS

We first recall some basic facts about Stanley’s theory of \(P\)-partitions (see [15] and [16, §4.5]). Let \(P\) be a poset, and let \(\omega\) be a labeling of \(P\), i.e., an injection from \(P\) to a totally ordered set (usually a set of integers). A \((P, \omega)\)-partition (or \(P\)-partition if \(\omega\) is understood) is a function \(f\) from \(P\) to the positive integers satisfying
\[
\begin{align*}
(1) & \quad \text{if } x <_P y \text{ then } f(x) \leq f(y) \\
(2) & \quad \text{if } x <_P y \text{ and } \omega(x) > \omega(y) \text{ then } f(x) < f(y).
\end{align*}
\]

A linear extension of a poset \(P\) is an extension of \(P\) to a total order. We will identify a linear extension of \(P\) labeled by \(\omega\) with the permutation obtained by taking the labels of \(P\) in increasing order with respect to the linear extension. For example, the linear extensions of the poset shown in Figure 1 are 2 1 3 and 2 3 1. We write \(\mathcal{L}(P)\) for the set of linear extensions of \(P\) (which also depend on the labeling \(\omega\)).

The order polynomial \(\Omega_P(n)\) of \(P\) is the number of \((P, \omega)\)-partitions with parts in \([n] = \{1, 2, \ldots, n\}\). It is known that \(\Omega_P(n)\) is a polynomial in \(n\) whose degree is the number
of elements of $P$. The following is a fundamental result in the $P$-partition theory [16, Theorem 4.5.14]:

$$
\sum_{n \geq 1} \Omega_P(n) t^n = \frac{\sum_{\pi \in \mathcal{P}(P)} t^{\text{des} \pi + 1}}{(1 - t)^{k+1}},
$$

(3.1)

where $k$ is the number of elements of $P$ and $\text{des} \pi$ is computed according to the natural order of the integers.

For example, the two linear extensions of the poset shown in Figure 1 each have one descent, and the order polynomial for this poset is $2^{(n+1)}_3$. So equation (3.1) reads

$$
\sum_{n \geq 1} 2^{(n+1)_3} t^n = \frac{2t^2}{(1-t)^4}.
$$

By (2.2) the Jacobi-Stirling numbers have the generating function

$$
\sum_{n \geq 0} JS(n, k; z) t^n = \frac{t^k}{(1 - (z + 1)t)(1 - 2(z + 2)t) \cdots (1 - k(z + k)t)},
$$

(3.2)

As $f_k(n; z) = JS(n + k, n; z)$, switching $n$ and $k$ in the last equation yields

$$
\sum_{k \geq 0} f_k(n; z) t^k = \frac{1}{(1 - (z + 1)t)(1 - 2(z + 2)t) \cdots (1 - n(z + n)t)}.
$$

Identifying the coefficients of $t^k$ gives

$$
f_k(n; z) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} j_1(z + j_1) \cdot j_2(z + j_2) \cdots j_k(z + j_k).
$$

(3.3)

For any subset $S$ of $[k]$, we define $\gamma_{S,m}(j)$ by

$$
\gamma_{S,m}(j) = \begin{cases} 
  j & \text{if } m \in S, \\
  j^2 & \text{if } m \notin S,
\end{cases}
$$

and define $p_{k,S}(n)$ by

$$
p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} \gamma_{S,1}(j_1)\gamma_{S,2}(j_2) \cdots \gamma_{S,k}(j_k).
$$

(3.4)
Figure 2. The labeled poset $R_k$.

For example, if $k = 2$ and $S = \{1\}$ then

$$p_{k,S}(n) = \sum_{1 \leq j_1 \leq j_2 \leq n} j_1 j_2^2 = n(n + 1)(n + 2)(12n^2 + 9n - 1)/120.$$  

**Definition 3.** Let $R_k$ be the labeled poset in Figure 2. Let $S$ be a subset of $[k]$. The poset $R_{k,S}$ obtained from $R_k$ by removing the points $3m - 2$ for $m \in S$ is called a Jacobi-Stirling poset.

For example, the posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$ are shown in Figure 3.

Figure 3. The labeled posets $R_{2,\{1\}}$ and $R_{2,\{2\}}$.

**Lemma 9.** For any subset $S \subseteq [k]$, let $A_{k,S}(t)$ be the descent polynomial of $\mathcal{L}(R_{k,S})$, i.e., the coefficient of $t^j$ in $A_{k,S}(t)$ is the number of linear extensions of $R_{k,S}$ with $j - 1$ descents, then

$$\sum_{n \geq 0} p_{k,S}(n)t^n = \frac{A_{k,S}(t)}{(1 - t)^{3k-|S|+1}}. \quad (3.5)$$

**Proof.** It is easy to see that $\Omega_{R_{k,S}}(n) = p_{k,S}(n)$ and the result follows from (3.1). \qed

For $0 \leq i \leq k$, $R_{k,i}$ is defined as the set of $\binom{k}{i}$ posets

$$R_{k,i} = \{ R_{k,S} \mid S \subseteq [k] \text{ with cardinality } i \}.$$
The posets in $R_{2,1}$ are shown in Figure 3. We define $\mathcal{L}(R_{k,i})$ to be the (disjoint) union of $\mathcal{L}(P)$, over all $P \in R_{k,i}$; i.e.,

$$
\mathcal{L}(R_{k,i}) = \bigcup_{S \subseteq [k], |S| = i} \mathcal{L}(R_{k,S}).
$$

(3.6)

Now we are ready to give the first interpretation of the coefficients $a_{k,i,j}$ in the polynomial $A_{k,i}(t)$ defined in (2.6).

**Theorem 10.** We have

$$
A_{k,i}(t) = \sum_{S \subseteq [k], |S| = i} A_{k,S}(t).
$$

(3.7)

In other words, the integer $a_{k,i,j}$ is the number of elements of $\mathcal{L}(R_{k,i})$ with $j - 1$ descents.

**Proof.** Extracting the coefficient of $z^i$ in both sides of (3.3), then applying (1.9) and (3.4), we obtain

$$
p_{k,i}(n) = \sum_{S \subseteq [k], |S| = i} p_{k,S}(n),
$$

so that

$$
\sum_{n \geq 0} p_{k,i}(n)t^n = \sum_{n \geq 0} \sum_{S} p_{k,S}(n)t^n = \sum_{S} \sum_{n \geq 0} p_{k,S}(n)t^n,
$$

where the summations on $S$ are over all subsets of $[k]$ with cardinality $i$. The result follows then by comparing (2.6) and (3.5). □

It is easy to compute $A_{k,S}(1)$ which is equal to $|\mathcal{L}(R_{k,S})|$ and is also $(3k - i)!$ times the leading coefficient of $p_{k,S}(n)$.

**Proposition 11.** Let $S \subseteq [k], |S| = i$ and let $l_j(S) = |\{ s \in S \mid s \leq j \}|$ for $1 \leq j \leq k$. We have

$$
A_{k,S}(1) = \frac{(3k - i)!}{\prod_{j=1}^{k}(3j - l_j(S))}.
$$

(3.8)

**Proof.** We construct a permutation in $\mathcal{L}(R_{k,S})$ by reading the elements of $R_{k,S}$ in increasing order of their labels and inserting each one into the permutation already constructed from the earlier elements. Each element of $R_{k,S}$ will have two natural numbers associated to it: the reading number and the insertion-position number. It is clear that the insertion-position number of $3j$ must be equal to its reading number, which is $3j - l_j(S)$, since it must be inserted to the right of all the previously inserted elements (those with labels less than $3j$). On the other hand, an element not divisible by 3 may be inserted anywhere, so its number of possible insertion positions is equal to its reading number. So the number of possible linear extensions of $R_{k,S}$ is equal to the product of the reading numbers of all elements with labels not divisible by 3. Since the product of all the reading
numbers is $(3j - i)!$, we obtain the result by dividing this number by the product of the reading numbers of the elements with labels $3, 6, \ldots, 3k$. \hfill \qed

From (3.8) we can derive the formula for $A_{k,i}(1)$, which is equivalent to Proposition 3.

**Proposition 12.** We have

$$|\mathcal{L}(R_{k,i})| = A_{k,i}(1) = \frac{(3k - i)!}{3^{k-i}2^i i! (k-i)!}.$$  

**Proof.** By Proposition 11 it is sufficient to prove the identity

$$\sum_{1 \leq s_1 < \cdots < s_i \leq k} \frac{(3k - i)!}{\prod_{j=1}^{k}(3j - l_j(S))} = \frac{(3k - i)!}{3^{k-i}2^i i! (k-i)!},$$  

(3.9)

where $S = \{s_1, \ldots, s_i\}$ and $l_j(S) = |\{s \in S : s \leq j\}|$.

The identity is obvious if $S = \emptyset$, i.e., $i = 0$. When $i = 1$, it is easy to see that (3.9) is equivalent to the $a = 2/3$ case of the indefinite summation

$$\sum_{s=0}^{k-1} \frac{(a)_s}{s!} = \frac{(a+1)_{k-1}}{(k-1)!},$$  

(3.10)

where $(a)_n = a(a+1) \cdots (a+n-1)$ and $(a)_0 = 1$. Since the left-hand side of (3.9) can be written as

$$\sum_{s_i=1}^{k} \frac{(3k - i)!}{\prod_{j=s_i}^{k}(3j - i)} \sum_{1 \leq s_1 < \cdots < s_{i-1} \leq s_i-1} \frac{1}{\prod_{j=1}^{s_i-1}(3j - l_j(S))},$$  

(3.11)

we derive (3.9) from the induction hypothesis and (3.10). \hfill \qed

**Remark 1.** Alternatively, we may prove the formula for $A_{k,i}(1)$ as follows:

$$A_{k,i}(1) = \sum_{S \subseteq [k]} A_{k,S}(1)$$

$$= \sum_{S \subseteq [k]} A_{k,S}(1) + \sum_{S \subseteq [k]} A_{k,S}(1)$$

$$= (3k - i - 1)A_{k-1,i-1}(1) + (3k - i - 1)(3k - i - 2)A_{k-1,i}(1),$$

from which we easily deduce that $A_{k,i}(1) = (3k - i)!/3^{k-i}2^i i! (k-i)!$.

Since both of the above proofs of Proposition 12 use mathematical induction, it is desirable to have a more conceptual proof. Here we give such a proof based on the fact that Proposition 12 is equivalent to

$$|\mathcal{L}(R_{k,i})| = 2^{k-i} \cdot \frac{(3k - i)!}{2^i i! 3^{k-i}(k-i)!}.$$  

(3.12)
A combinatorial proof of Proposition 12. We show that $|\mathcal{L}(R_{k,i})|$ is equal to $2^{k-i}$ times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and $i$ blocks of size 2.

Let $S$ be an $i$-element subset of $[k]$ and let $\pi$ be an element $\mathcal{L}(R_{k,S})$, viewed as a bijection from $[3k-i]$ to $R_{k,S}$. Let $\sigma = \pi^{-1}$. Then $\sigma$ is a natural labeling of $R_{k,S}$, i.e., an order-preserving bijection from the poset $R_{k,S}$ to $[3k-i]$, and conversely, every natural labeling of $R_{k,S}$ is the inverse of an element of $\mathcal{L}(R_{k,S})$.

We will describe a map from the set of natural labelings of elements of $R_{k,i}$ to the set of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and $i$ blocks of size 2. We shall describe all natural labelings $\sigma$ of posets $R_{k,S}$ that correspond to $P$ under the map just defined. First, we list the blocks of $P$ as $B_1, B_2, \ldots, B_k$ in increasing order of their largest elements. Then $\sigma(3m)$ must be the largest element of $B_m$. If $B_m$ has two elements, then the smaller element must be $\sigma(3m-1)$, and $m$ must be an element of $S$. If $B_m$ has three elements then $m \notin S$, and $\sigma(3m-2)$ and $\sigma(3m-1)$ are the two smaller elements of $B_m$, but in either order. Thus $S$ is uniquely determined by $P$, and there are exactly $2^{k-i}$ natural labelings of $R_{k,S}$ in the preimage of $P$. So $|\mathcal{L}(R_{k,i})|$ is $2^i$ times the number of partitions of $[3k-i]$ with $k-i$ blocks of size 3 and $i$ blocks of size 2, and is therefore equal to the right-hand side of (3.12).  

4. Two proofs of Theorem 2

We shall give two proofs of Theorem 2. We first derive Theorem 2 from Theorem 10 by constructing a bijection from the linear extensions of Jacobi-Stirling posets to permutations. The second proof consists of verifying that the cardinality of Jacobi-Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j-1$ descents satisfies the recurrence relation (2.8). Given a word $w = w_1 w_2 \ldots w_m$ of $m$ letters, we define the $j$th slot of $w$ by the pair $(w_j, w_{j+1})$ for $j = 0, \ldots, m$. By convention $w_0 = w_{m+1} = 0$. A slot $(w_j, w_{j+1})$ is called a descent (resp. non-descent) slot if $w_j > w_{j+1}$ (resp. $w_j \leq w_{j+1}$).

4.1. First proof of Theorem 2. For any subset $S = \{s_1, \ldots, s_t\}$ of $[k]$ we define $\bar{S} = \{\bar{s}_1, \ldots, \bar{s}_t\}$, which is a subset of $[\bar{k}]$. Recall that $\mathcal{JSP}_{k,\bar{S}}$ is the set of Jacobi-Stirling permutations of $M_{k,\bar{S}}$. We construct a bijection $\phi : \mathcal{L}(R_{k,S}) \to \mathcal{JSP}_{k,\bar{S}}$ such that des $\phi(\pi) = \text{des } \pi$ for any $\pi \in \mathcal{L}(R_{k,S})$.

If $k = 1$, then $\mathcal{L}(R_{1,0}) = \{123, 213\}$ and $\mathcal{L}(R_{1,1}) = \{23\}$. We define $\phi$ by

$$\phi(123) = \bar{1}11, \ \phi(213) = 11\bar{1}, \ \phi(23) = 11.$$

Suppose that $k \geq 2$ and $\phi : \mathcal{L}(R_{k-1,\bar{S}}) \to \mathcal{JSP}_{k-1,\bar{S}}$ is defined for any $S \subseteq [k-1]$. If $\pi \in \mathcal{L}(R_{k,S})$ with $S \subseteq [k]$, we consider the following two cases:
(i) $k \notin S$, denote by $\pi'$ the word obtained by deleting $3k$ and $3k - 1$ from $\pi$, and $\pi''$ the word obtained by further deleting $3k - 2$ from $\pi'$. As $\pi'' \in \mathcal{L}(R_{k-1,S})$, by induction hypothesis, the permutation $\phi(\pi'') \in \mathcal{JSP}_{k-1,S}$ is well defined. Now, 

a) if $3k - 2$ is in the $r$th descent (or non-descent) slot of $\pi''$, then we insert $\bar{k}$ in the $r$th descent (or non-descent) slot of $\phi(\pi'')$ and obtain a word $\phi_1(\pi'')$;

b) if $3k - 1$ is in the $s$th descent (or non-descent) slot of $\pi'$, we define $\phi(\pi)$ by inserting $kk$ in the $s$th descent (or non-descent) slot of $\phi_1(\pi'')$.

(ii) $k \in S$, denote by $\pi'$ the word obtained from $\pi$ by deleting $3k$ and $3k - 1$. As $\pi' \in \mathcal{L}(R_{k-1,i-1})$, the permutation $\phi(\pi') \in \mathcal{JSP}_{k-1,S}$ is well defined. If $3k - 1$ is in the $r$th descent (or non-descent) slot of $\pi'$, we define $\phi(\pi)$ by inserting $kk$ in the $r$th descent (or non-descent) slot of $\phi(\pi')$.

Clearly this mapping is a bijection and preserves the number of descents. For example, if $k = 3$ and $S = \{2\}$, then $\phi(25137869) = 11232331$. This can be seen by applying the mapping $\phi$ as follows:

\[
213 \rightarrow 25136 \rightarrow 251376 \rightarrow 25137869, \\
11\bar{1} \rightarrow 1122\bar{1} \rightarrow 11232\bar{1} \rightarrow 1123233\bar{1}.
\]

Clearly we have des(25137869) = 2 and des(11232331) = 2.

4.2. Second proof of Theorem 2. Let $\mathcal{JSP}_{k,i,j}$ be the set of Jacobi-Stirling permutations in $\mathcal{JSP}_{k,i}$ with $j - 1$ descents. Let $a'_{0,0,0} = 1$ and $a'_{k,i,j}$ be the cardinality of $\mathcal{JSP}_{k,i,j}$ for $k, i, j \geq 0$. By definition, $a'_{k,i,j} = 0$ if any of the indices $k, i, j < 0$ or $j \notin \{1, \ldots, 2k - i\}$. We show that $a'_{k,i,j}$'s satisfy the same recurrence (2.8) and initial conditions as $a_{k,i,j}$'s.

Any Jacobi-Stirling permutation of $\mathcal{JSP}_{k,i,j}$ can be obtained from one of the following five cases:

(i) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i,j}$, insert $\bar{k}$ and then $kk$ in one of the descent slots (an extra descent at the end of the permutation). Clearly, there are $a'_{k-1,i,j}$ ways to choose the initial permutation, $j$ ways to insert $\bar{k}$, and $j$ ways to insert $kk$.

(ii) Choose a Jacobi-Stirling permutation of $\mathcal{JSP}_{k-1,i,j-1}$,

1) insert $\bar{k}$ in a descent slot and then $kk$ in a non-descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $j - 1$ ways to insert $\bar{k}$, and $3k - i - j - 1$ ways to insert $kk$.

2) insert $\bar{k}$ in a non-descent slot and then $kk$ in a descent slot. In this case, there are $a'_{k-1,i,j-1}$ ways to choose the initial permutation, $3k - i - j - 1$ ways to insert $\bar{k}$, and $j$ ways to insert $kk$.

(iii) Choose a Jacobi-Stirling permutation in $\mathcal{JSP}_{k-1,i,j-2}$, insert $\bar{k}$ and then $kk$ in one of the non-descent slots. In this case, there are $a'_{k-1,i,j-2}$ ways to choose the initial permutation, $3k - i - j$ ways to insert $\bar{k}$, and $3k - i - j$ ways to insert $kk$. 


(iv) Choose a Jacobi-Stirling permutation in $JSP_{k-1,i-1,j}$ and insert $kk$ in one of the descent slots. There are $a'_{k-1,i-1,j}$ ways to choose the initial permutation, and $j$ ways to insert $kk$.

(v) Choose a Jacobi-Stirling permutation in $JSP_{k-1,i-1,j-1}$ and insert $kk$ in one of the non-descent slots. There are $a'_{k-1,i-1,j-1}$ ways to choose the initial permutation, and $3k - i - j$ ways to insert $kk$.

Summarizing all the above five cases, we obtain

$$a'_{k,i,j} = j^2 a'_{k-1,i,j} + [2(j - 1)(3k - i - j - 1) + (3k - i - 2)]a'_{k-1,i,j-1}$$

$$+ (3k - i - j)^2 a'_{k-1,i,j-2} + j a'_{k-1,i-1,j} + (3k - i - j)a'_{k-1,i-1,j-1}.$$ 

Therefore, the numbers $a'_{k,i,j}$ satisfy the same recurrence and initial conditions as the $a_{k,i,j}$, so they are equal.

5. LEGENDRE-STIRLING POSETS

Let $P_k$ be the poset shown in Figure 4 called the Legendre-Stirling poset. The order polynomial of $P_k$ is given by

$$\Omega_{P_k}(n) = \sum_{2 \leq f(2) \leq \ldots \leq f(3k-1) \leq n} \prod_{i=1}^{k} f(3i-1)(f(3i-1) - 1)$$

$$= [x^k] \frac{1}{(1 - 2x)(1 - 6x) \ldots (1 - (n - 1)nx)},$$

which is equal to $JS(n-1+k, n-1; 1)$ by (3.2), and by (1.3) this is equal to $LS(n-1+k, n-1)$. By (3.1), we obtain

$$\sum_{n \geq 0} LS(n + k, n)t^n = \frac{\sum_{\pi \in \mathcal{P}(k)} t^{\text{des} \pi}}{(1 - t)^{3k+1}}.$$

(5.1)

In other words, we have the following theorem.
Theorem 13. Let $b_{k,j}$ be the number of linear extensions of Legendre-Stirling posets $P_k$ with exactly $j$ descents. Then
\[
\sum_{n \geq 0} \text{LS}(n+k,n)t^n = \frac{\sum_{j=1}^{2k-1} b_{k,j} t^j}{(1-t)^{3k+1}}.
\] (5.2)

We now apply the above theorem to deduce a result of Egge \cite[Theorem 4.6]{Egge}.

Definition 4. A Legendre-Stirling permutation of $M_k$ is a Jacobi-Stirling permutation of $M_k$ with respect to the order: $1 < 2 = 2 < \cdots < k = k$.

Here $\bar{1} = 1$ means that neither $1\bar{1}$ nor $\bar{1}1$ counts as a descent. Thus, the Legendre-Stirling permutation $122\bar{1}1\bar{1}$ has one descent at position 4, while as a Jacobi-Stirling permutation, it has three descents, at positions 3, 4 and 5.

Theorem 14 (Egge). The coefficient $b_{k,j}$ equals the number of Legendre-Stirling permutations of $M_k$ with exactly $j - 1$ descents.

First proof. Let $\mathcal{LSP}_k$ be the set of Legendre-Stirling permutations of $M_k$. By Theorem \cite{Egge} it suffices to construct a bijection $\psi : \mathcal{LSP}_k \to \mathcal{L}(P_k)$ such that $\text{des } \psi(\pi) - 1 = \text{des } \pi$ for any $\pi \in \mathcal{LSP}_k$. If $k = 1$, then $\mathcal{LSP}_1 = \{1\bar{1}, \bar{1}1\}$ and $\mathcal{L}(P_1) = \{132, 312\}$. We define $\psi$ by
\[
\psi(1\bar{1}) = 132, \quad \psi(1\bar{1}1) = 312.
\]
Clearly $\text{des } 132 - 1 = \text{des } 1\bar{1}1 = 0$ and $\text{des } 312 - 1 = \text{des } \bar{1}11 = 0$. Suppose that the bijection $\psi : \mathcal{LSP}_{k-1} \to \mathcal{L}(P_{k-1})$ is constructed for some $k \geq 2$. Given $\pi \in \mathcal{LSP}_k$, we denote by $\pi'$ the word obtained by deleting $\bar{k}$ from $\pi$, and by $\pi''$ the word obtained by further deleting $kk$ from $\pi'$. We put $3k - 1$ at the end of $\psi(\pi'')$ and obtain a word $\psi(\pi'')$.

In the following two steps, the slot after $3k - 1$ is excluded, because we cannot insert $3k$ and $3k - 2$ to the right of $3k - 1$.

a) if $\bar{k}$ is in the $r$th descent (or nondescendent) slot of $\pi''$, then we insert $3k$ in the $r$th descent (or nondescendent) slot of $\psi(\pi'')$ and obtain a word $\psi_1(\pi'')$;

b) if $kk$ is in the $s$th descent slot or in the non-descent slot before $\bar{k}$ (in the $j$th non-descent slot other than the non-descent slot before $\bar{k}$) of $\pi''$, we define $\psi(\pi)$ by inserting $3k - 2$ in the $s$th descent slot or in the non-descent slot before $3k$ (in the $j$th non-descent slot other than the non-descent slot before $3k$) of $\psi_2(\pi'')$.

For example, we can compute $\psi(212233\bar{1}1\bar{1}) = 614793258$ by the following procedure:
\[
11\bar{1} \to 211\bar{1} \to 2122\bar{1}\bar{1} \to 21233\bar{1}\bar{1} \to 212233\bar{1}1\bar{1}
\]
\[
132 \to 61325 \to 614325 \to 61493258 \to 614793258.
\]
This construction can be easily reversed and the number of descents is preserved. \hfill \Box

Second proof. By \cite{Egge}, \cite{Egge2}, and \cite{Egge3}, we have
\[
\sum_{n=0}^{\infty} \text{JS}(n+k,n;z)t^n = \sum_{i=0}^{k} z^i \sum_{j=1}^{2k-i} \frac{a_{k,i,j} t^j}{(1-t)^{3k-i+1}}.
\]
Setting $z = 1$ and using (1.3) gives
\[ \sum_{n=0}^{\infty} \text{LS}(n+k,n)t^n = \sum_{i=0}^{k} (1-t)^i \sum_{j=1}^{2k-i} a_{k,i,j} t^j / (1-t)^{3k+1}. \]

Multiplying both sides by $(1-t)^{3k+1}$ and applying (5.2) gives
\[ \sum_{j=1}^{2k-1} b_{k,j} t^j = \sum_{i=0}^{k} (1-t)^i \sum_{j=1}^{2k-i} a_{k,i,j} t^j, \]
so
\[ \sum_{i=0}^{k} \sum_{l=0}^{i} (-1)^l \binom{i}{l} a_{k,i,j-l} = b_{k,j}. \]  

(5.3)

For any $S \subseteq [\bar{k}]$, let $\mathcal{JS}_P{k,S,j}$ be the set of all Jacobi-Stirling permutations of $M_k$ with $j-1$ descents. Let $B_{k,j} = \bigcup_{S \subseteq [\bar{k}]} \mathcal{JS}_P{k,S,j}$ be the set of Jacobi-Stirling permutations with $j-1$ descents. We show that the left-hand side of (5.3) is the number $N_0$ of permutations in $B_{k,j}$ with no pattern $u \bar{u}$.

For any $T \subseteq [\bar{k}]$, let $B_{k,j}(T, \geq)$ be the set of permutations in $B_{k,j}$ containing all the patterns $u \bar{u}$ for $u \in T$. By the principle of inclusion-exclusion [16, Chapter 2],
\[ N_0 = \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} |B_{k,j}(T, \geq)|. \]  

(5.4)

Now, for any subsets $T, S \subseteq [\bar{k}]$ such that $T \subseteq [\bar{k}] \setminus S$, define the mapping
\[ \varphi : \mathcal{JS}_P{k,S,j} \cap B_{k,j}(T, \geq) \to \mathcal{JS}_P{k,S \cup T,j-|T|} \]
by deleting the $\bar{u}$ in every pattern $u \bar{u}$ of $\pi \in \mathcal{JS}_P{k,S,j} \cap B_{k,j}(T, \geq)$. Clearly, this is a bijection. Hence, we can rewrite (5.4) as
\[ N_0 = \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{S, T \subseteq [\bar{k}]} |\mathcal{JS}_P{k,S \cup T,j-|T|}| \]
\[ = \sum_{T \subseteq [\bar{k}]} (-1)^{|T|} \sum_{S : T \cup S = [\bar{k}]} |\mathcal{JS}_P{k,S,j-|T|}|. \]

For any subset $S$ of $[\bar{k}]$ with $|S| = i$, and any $l$ with $0 \leq l \leq i$, there are $\binom{i}{l}$ subsets $T$ of $S$ such that $|T| = l$, and, by definition,
\[ \sum_{S \subseteq [\bar{k}] \atop |S| = i} |\mathcal{JS}_P{k,S,j-|T|}| = a_{k,i,j-l}. \]

This proves that $N_0$ is equal to the left-hand side of (5.3).

Let $\mathcal{LS}_P{k,j}$ be the set of all Legendre-Stirling permutations of $M_k$ with $j-1$ descents. It is easy to identify a permutation $\pi \in B_{k,j}$ with no pattern $u \bar{u}$ with a Legendre-Stirling
permutation $\pi' \in \mathcal{LSP}_{k,j}$ by inserting each missing $\bar{u}$ just to the right of the second $u$. This completes the proof. \hfill \square

Finally, the numerical experiments suggest the following conjecture, which has been verified for $0 \leq i \leq k \leq 9$.

**Conjecture 15.** For $0 \leq i \leq k$, the polynomial $A_{k,i}(t)$ has only real roots.

Note that by a classical result [4, p. 141], the above conjecture would imply that the sequence $a_{k,i,1}, \ldots, a_{k,i,2k-i}$ is unimodal. Let $G_k$ be the multiset $\{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ with $m_i \in \mathbb{N}$. A permutation $\pi$ of $G_k$ is a generalized Stirling permutation (see [3, 12]) if whenever $u < v < w$ and $\pi(u) = \pi(w)$, we have $\pi(v) > \pi(u)$. For any $S \subseteq \{k\}$, the set of generalized Stirling permutations of $M_k \setminus S$ is equal to $\mathcal{JSP}_{k,S}$. By Lemma 9 and Theorem 2 the descent polynomial of $\mathcal{JSP}_{k,S}$ is $A_{k,S}(t)$. It follows from a result of Brenti [3, Theorem 6.6.3] that $A_{k,S}(t)$ has only real roots. By (3.7), this implies, in particular, that the above conjecture is true for $i = 0$ and $i = k$.

One can also use the methods of Haglund and Visontai [11] to show that $A_{k,S}(t)$ has only real roots, though it is not apparent how to use these methods to show that $A_{k,i}(t)$ has only real roots.

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