Swapping algebra, Virasoro algebra and discrete integrable system

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ABSTRACT

Swapping algebra, introduced by François Labourie, is a commutative Poisson algebra generated by pairs of points of a cyclically ordered set $P$. In this paper, we induce a Poisson structure on the configuration space $M_{N,1}$ of $N$ twisted polygons in $\mathbb{R}P^1$ by the swapping bracket. Then we relate asymptotically the dual of this Poisson structure to the Virasoro algebra. At last, we compare this Poisson structure with another Poisson structure on $M_{N,1}$ induced from the affine $SL(2,\mathbb{R})$ Poisson-Lie group structure, as a result, the two Poisson structure on $M_{N,1}$ are compatible.

1. Introduction

Discrete integrable system of the configuration space $M_{N,n}$ of $N$-twisted polygons in $\mathbb{R}P^n$ is considered by L. Faddeev, A. Yu. Volkov [FV93] for $n=1$, R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT10] for $n=2$, B. Khsein, F. Soloviev [KS13] for $n$ in general and many others. The configuration space $M_{N,n}$ is viewed as the discrete version of the space $L_{n+1}$ of monic $(n+1)$-th order differential operators on the circle. There are two compatible first and second (Lie-Poisson) Adler-Gelfand–Dickey Poisson structures on $L_{n+1}$ induced from Drinfeld-Sokolov [DS85] Hamiltonian reduction of the dual of the affine Lie algebra $\hat{L}gl_{n+1}$. For the discrete version when $n=2$, in [SOT10], they give a cross ratio coordinate system of $M_{N,2}$ and a natural Poisson structure which corresponds to the second Adler-Gelfand-Dickey Poisson structures on $L_3$, they conjecture that there is another Poisson structure on $M_{N,2}$ which corresponds to the first Adler-Gelfand-Dickey Poisson structures on $L_3$, which is compatible with the natural Poisson structure that they defined.

In this paper, we give an answer to the above conjecture when $n=1$. There is a natural cross-ratio coordinate system for $M_{N,1}$ and a natural Poisson structure $\{\cdot,\cdot\}_{S2}$ corresponding to the second Adler-Gelfand-Dickey Poisson structures on $L_2$, we define another Poisson structure $\{\cdot,\cdot\}_{B2}$ on $M_{N,1}$ via the rank 2 swapping algebra [Su14]. By studying the asymptotic behavior of the two Poisson structures, our two main results are the followings.

Theorem 1.1 (Theorem 3.17) We relate asymptotically the dual of the Poisson structure $\{\cdot,\cdot\}_{B2}$ to the Virasoro algebra on $M_{N,1}$.

Theorem 1.2 (Theorem 4.2) We relate asymptotically the dual of the Poisson structure $\{\cdot,\cdot\}_{S2}$ to the Virasoro algebra on $M_{N,1}$.

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We compare these two Poisson structures.

**Theorem 1.3** The Poisson structures \( \{\cdot, \cdot\}_{B_2} \) and \( \{\cdot, \cdot\}_{S_2} \) are compatible on \( M_{N,1} \).

We show that \( \{\cdot, \cdot\}_{B_2} \) and \( \{\cdot, \cdot\}_{S_2} \) are not only asymptotically related to the Poisson-Lie group structure, but also compatible. The relation between the two integration schemes—bihamiltonian one and Poisson-Lie group one is long standing problem in integrable systems theory. We hope this observation will result in a better understanding of integrable systems theory.

We find another compatible Poisson structure on \( M_{N,1} \) via rank 2 swapping algebra. We hope this paper helps to shed light on solving the original conjecture, or even more on the general cases.

### 2. Rank 2 swapping algebra on the configuration space of \( N \)-twisted polygons in \( \mathbb{RP}^1 \)

#### 2.1 The configuration space of \( N \)-twisted polygons in \( \mathbb{RP}^n \)

**Definition 2.1** [Configuration space of \( N \)-twisted polygons in \( \mathbb{RP}^n \)] A \( N \)-twisted polygon in \( \mathbb{RP}^n \) is a map \( f \) from \( \mathbb{Z} \) to \( \mathbb{RP}^n \) such that for any \( k \in \mathbb{Z} \), we have \( f(k+N) = M_f \cdot f(k) \) where \( M_f \) belongs to \( \text{PSL}_{n+1}(R) \). We call \( M_f \) the monodromy of the \( N \)-twisted polygon in \( \mathbb{RP}^n \). We say that \( f \) is in general position if for any \( k \in \mathbb{N} \), the points \( \{f(k+i-1)\}_{i=1}^{n+1} \) are in general position in \( \mathbb{RP}^n \).

The configuration space of \( N \)-twisted polygons in \( \mathbb{RP}^n \), denoted by \( M_{N,n} \), is the space of the \( N \)-twisted polygons in general position in \( \mathbb{RP}^n \) up to projective transformations.

Later on, we only consider the case \( n = 1 \).

It is easy to find a coordinate system of \( M_{N,1} \) by cross ratios.

**Definition 2.2** [A coordinate system of \( M_{N,1} \)] Let \( f(k) := [f_k : 1] \), let
\[
[a, b, c, d] := \frac{a - c}{a - d} \cdot \frac{b - d}{b - c},
\]
then
\[
B_k = [f_{k-1}, f_{k+2}, f_{k+1}, f_k].
\]
We have \( \{B_k\}_{k=1}^N \) is a coordinate system of \( M_{N,1} \).

#### 2.2 Swapping algebra

In this section, we recall briefly some definitions about the swapping algebra introduced by F. Labourie. Our definitions here are based on Section 2 of [L12].

**Definition 2.3** [Linking number] Let \( (r, x, s, y) \) be a quadruple of 4 different points in the interval \( [0, 1] \). Let \( \sigma(\Delta) = -1, 0, 1 \) whenever \( \Delta < 0, \Delta = 0, \Delta > 0 \) respectively. We call \( J(r, x, s, y) \) the linking number of \( (r, x, s, y) \), where
\[
J(r, x, s, y) = \frac{1}{2} \cdot (\sigma(r - x) \cdot \sigma(r - y) \cdot \sigma(y - x) - \sigma(r - x) \cdot \sigma(r - s) \cdot \sigma(s - x)).
\]

If \( (r, x, s, y) \) is a quadruple of 4 points in the oriented circle \( S^1 \), the linking number of 4 points in the interval \( S^1 \setminus o \) for \( o \notin \{r, x, s, y\} \) does not depend on the choice of \( o \). So, \( J(r, x, s, y) \) is defined to be the linking number of 4 points in the circle \( S^1 \). We describe four cases in Figure [1].
Let $\mathcal{P}$ be a finite subset of the circle $S^1$ provided with cyclic order. $\mathbb{K}$ is a field ($\mathbb{C}$ or $\mathbb{R}$). We represent an ordered pair $(r, x)$ of $\mathcal{P}$ by the expression $rx$.

**Definition 2.4** [Swapping ring of $\mathcal{P}$] The swapping ring of $\mathcal{P}$ is the ring

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{x, y \in \mathcal{P}}]/\{xx \mid \forall x \in \mathcal{P}\}$$

over $\mathbb{K}$, where $\{xy\}_{x, y \in \mathcal{P}}$ are variables with values in $\mathbb{K}$.

Notably, $rx = 0$ if $r = x$ in $\mathcal{Z}(\mathcal{P})$. Then we equip $\mathcal{Z}(\mathcal{P})$ with a Poisson bracket defined by F. Labourie in Section 2 of [L12].

**Definition 2.5** [Swapping bracket] The swapping bracket over $\mathcal{Z}(\mathcal{P})$ is defined by extending the following formula to $\mathcal{Z}(\mathcal{P})$ by using Leibniz's rule and additive rule:

$$\{rx, sy\} = J(r, x, s, y) \cdot ry \cdot sx.$$  \hspace{1cm} (4)

(Here is the case for $\alpha = 0$ in Section 2 of [L12].)

Leibniz's rule:

$$\{rx \cdot sy, tz\} = rx\{sy, tz\} + sy\{rx, tz\}$$  \hspace{1cm} (5)

for any $rx, xy, tz$ in $\mathcal{P}$.

Additive rule:

$$\{a + b, c\} = \{a, c\} + \{b, c\}$$  \hspace{1cm} (6)

For any $a, b, c \in \mathcal{Z}(\mathcal{P})$.

**Theorem 2.6** [F. Labourie [L12]] The swapping bracket as above verifies the Jacobi identity. So the swapping bracket defines a Poisson structure on $\mathcal{Z}(\mathcal{P})$.

**Definition 2.7** [Swapping algebra of $\mathcal{P}$] The swapping algebra of $\mathcal{P}$ is $\mathcal{Z}(\mathcal{P})$ equipped with the swapping bracket.

**Definition 2.8** [Swapping fraction algebra of $\mathcal{P}$] The swapping fraction algebra of $\mathcal{P}$ is $\mathcal{Q}(\mathcal{P})$ equipped with the induced swapping bracket.
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**Definition 2.9 [Cross fraction]** Let \( x, y, z, t \) belong to \( \mathcal{P} \) so that \( x \neq t \) and \( y \neq z \). The cross fraction determined by \((x, y, z, t)\) is the element of \( \mathbb{Q}(\mathcal{P}) \):

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{7}
\]

Let \( \mathcal{C} \mathcal{R}(\mathcal{P}) = \{[x, y, z, t] \in \mathbb{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z \} \) be the set of all the cross-fractions in \( \mathbb{Q}(\mathcal{P}) \).

**Definition 2.10 [Swapping Multifraction Algebra of \( \mathcal{P} \)]** Let \( \mathcal{B}(\mathcal{P}) \) be the subring of \( \mathbb{Q}(\mathcal{P}) \) generated by \( \mathcal{C} \mathcal{R}(\mathcal{P}) \). The swapping multifraction algebra of \( \mathcal{P} \) is \( \mathcal{B}(\mathcal{P}) \) equipped with the swapping bracket.

### 2.3 Rank \( n \) Swapping Algebra

We recall some definitions in [Su14].

**Definition 2.11 [The Rank \( n \) Swapping Ring \( \mathcal{Z}_n(\mathcal{P}) \)]** For \( n \geq 2 \), let \( \mathcal{R}_n(\mathcal{P}) \) be the subring of \( \mathcal{Z}(\mathcal{P}) \) generated by

\[
\left\{ D \in \mathcal{Z}_n(\mathcal{P}) \mid D = \det \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_{n+1} \\ \vdots & \ddots & \vdots \\ x_{n+1} y_1 & \cdots & x_{n+1} y_{n+1} \end{pmatrix}, \forall x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in \mathcal{P} \right\}.
\]

Let \( \mathcal{Z}_n(\mathcal{P}) \) be the quotient ring \( \mathcal{Z}(\mathcal{P})/\mathcal{R}_n(\mathcal{P}) \).

**Definition 2.12 [Rank \( n \) Swapping Algebra of \( \mathcal{P} \)]** The rank \( n \) swapping algebra of \( \mathcal{P} \) is the ring \( \mathcal{Z}_n(\mathcal{P}) \) equipped with the swapping bracket.

**Definition 2.13 [Rank \( n \) Swapping Fraction Algebra of \( \mathcal{P} \)]** The rank \( n \) swapping fraction algebra of \( \mathcal{P} \) is the total fraction ring \( \mathcal{Q}_n(\mathcal{P}) \) of \( \mathcal{Z}_n(\mathcal{P}) \) equipped with the swapping bracket.

**Definition 2.14 [Rank \( n \) Multifraction Algebra of \( \mathcal{P} \)]** Let \( x, y, z, t \) belong to \( \mathcal{P} \) so that \( x \neq t \) and \( y \neq z \). The cross fraction determined by \((x, y, z, t)\) is the element of \( \mathcal{Q}_n(\mathcal{P}) \):

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{8}
\]

Let \( \mathcal{C} \mathcal{R}_n(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}_n(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z \} \) be the set of all the cross-fractions in \( \mathcal{Q}_n(\mathcal{P}) \). Let \( \mathcal{B}_n(\mathcal{P}) \) be the subring of \( \mathcal{Q}_n(\mathcal{P}) \) generated by \( \mathcal{C} \mathcal{R}_n(\mathcal{P}) \).

Then, the rank \( n \) multifraction algebra of \( \mathcal{P} \) is the ring \( \mathcal{B}_n(\mathcal{P}) \) equipped with the swapping bracket.

### 2.4 Rank 2 Swapping Poisson Structure on \( \mathcal{M}_{N,1} \)

From now on, let \( N > 3 \).

**Definition 2.15** Let us consider a cyclically ordered set in \( S^1 \)

\[
\mathcal{P} := \{r_k, 1 \leq k \leq N \mid r_1 < r_2 \cdots \cdots r_N < r_1\ldots\}
\]

and its rank 2 multifraction algebra \( \mathcal{B}_2(\mathcal{P}) \).

Recall that \( \mathcal{B}_k = [f_{k-1}, f_{k+2}, f_{k+1}, f_k] \). Let \( \theta' \) be a map from \( \{B_1, \ldots, B_N\} \) to \( \mathcal{B}_2(\mathcal{P}) \) such that:

\[
\theta'(B_k) := \frac{r_k-1r_{k+1}}{r_k-1r_k} \cdot \frac{r_{k+2}r_k}{r_{k+2}r_{k+1}}
\]

for \( k = 1, \ldots, N \) with the convention \( r_k = r_{k+N} \).
Since \( \{B_k\}_{k=1}^N \) is a coordinate system, \( B_1, \ldots, B_N \) are algebraically independent from each other, let \( \mathbb{R}(B_1, \ldots, B_N) \) be the free fraction ring generated by \( \{B_1, \ldots, B_N\} \), we have the following proposition.

**Proposition 2.16** The map \( \theta' \) extends to a ring homomorphism
\[
\theta : \mathbb{R}(B_1, \ldots, B_N) \to B_2(\mathcal{P})
\]

**Proposition 2.17** Let us consider the finite family of elements of \( B_2(\mathcal{P}) \)
\[
C := \left\{ \frac{r_{k-1}r_{k+1} + r_k}{r_{k-1}r_{k+1} + 1}, \frac{r_{k+2}r_{k+1} + r_k}{r_{k+2}r_{k+1} + 1}, \frac{r_{k+3}r_{k+1} + r_k}{r_{k+3}r_{k+1} + 1} \right\}_{k=1}^N
\]
with the convention \( r_k = r_{k+N} \). Let \( \langle C \rangle \) be the subring of \( B_2(\mathcal{P}) \) generated by \( C \). Then for any \( i, j = 1, \ldots, N \), we have \( \frac{\theta(B_i)\theta(B_j)}{\theta(B_i)\theta(B_j)} \) belongs to \( \langle C \rangle \).

**Proof.** Since we have the convention \( r_k = r_{k+N} \), the index of \( B_k \) have the convention \( k = k+N \). By direct calculation, we obtain that

(i) For \( i = k, j = k+1 \), we have
\[
\frac{\theta(B_k)\theta(B_{k+1})}{\theta(B_k)\theta(B_{k+1})} = -1 + \frac{r_{k-1}r_{k+2} + r_k}{r_{k-1}r_{k+2} + 1} + \frac{r_{k+2}r_{k+1} + r_k}{r_{k+2}r_{k+1} + 1} + \frac{r_{k+3}r_{k+1} + r_k}{r_{k+3}r_{k+1} + 1},
\]

(ii) For \( i = k, j = k+1 \), we have
\[
\frac{\theta(B_{k+1})\theta(B_k)}{\theta(B_{k+1})\theta(B_k)} = 1 - \frac{r_{k-1}r_{k+2} + r_k}{r_{k-1}r_{k+2} + 1} + \frac{r_{k+2}r_{k+1} + r_k}{r_{k+2}r_{k+1} + 1} - \frac{r_{k+3}r_{k+1} + r_k}{r_{k+3}r_{k+1} + 1}.
\]

(iii) For \( i = k, j = k+2 \), we have
\[
\frac{\theta(B_k)\theta(B_{k+2})}{\theta(B_k)\theta(B_{k+2})} = \left\{ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_{k+1}}, \frac{r_{k+2}r_{k+1}}{r_{k+2}r_{k+1}}, \frac{r_{k+3}r_{k+1}}{r_{k+3}r_{k+1}} \right\} = -\frac{r_{k+2}r_{k+1} + r_k}{r_{k+2}r_{k+1} + 1} + \frac{r_{k+3}r_{k+1} + r_k}{r_{k+3}r_{k+1} + 1}.
\]

(iv) For \( i = k+2, j = k \), we have
\[
\frac{\theta(B_{k+2})\theta(B_k)}{\theta(B_{k+2})\theta(B_k)} = \frac{r_{k+2}r_{k+1} + r_k}{r_{k+2}r_{k+1} + 1}.
\]

(v) For all the other cases, we have
\[
\frac{\theta(B_i)\theta(B_j)}{\theta(B_i)\theta(B_j)} = 0.
\]

We conclude that \( \frac{\theta(B_i)\theta(B_j)}{\theta(B_i)\theta(B_j)} \) belongs to \( \langle C \rangle \).

**Definition 2.18** Let \( \mathcal{D} = \mathcal{C} \cup \left\{ \frac{r_{k-1}r_{k+1}}{r_{k-1}r_{k+1}}, \frac{r_{k+2}r_{k+1}}{r_{k+2}r_{k+1}} \right\}_{k=1}^N \) with the convention \( r_k = r_{k+N} \). Of course, we have
\[
\theta(\mathbb{R}(B_1, \ldots, B_N)) \subset \langle \mathcal{D} \rangle.
\]
We define a map $\eta' : D \to \mathbb{R}(B_1, \ldots, B_N)$ by:

$$\eta'(r_{k-1}r_{k+1} \quad r_{k+2} \quad r_{k+2}r_{k+1}) = B_k,$$

(14)

$$\eta'(r_{k-1}r_{k+2} \cdot r_kr_{k+1} \quad r_{k-1}r_{k+1} \cdot r_kr_{k+2}) = 1 - \frac{1}{B_k},$$

(15)

$$\eta'(r_{k+2}r_{k+1} \cdot r_{k+3} \quad r_{k+2} \quad r_{k+3} \quad r_{k+1}r_{k+3}) = 1 - \frac{1}{B_{k+1}},$$

(16)

$$\eta'(r_{k+2}r_{k+3} \cdot r_{k+1} \quad r_{k+2} \quad r_{k+1}r_{k+3}) = \frac{1}{B_{k+1}},$$

(17)

for $k = 1, \ldots, N$ with the convention $r_k = r_{k+N}$.

By case by case verification, we have the following proposition.

**Proposition 2.19** We have $\theta \circ \eta' = Id_D$.

**Remark 2.20** The rank 2 swapping algebra condition is used here.

Thus, we have

**Proposition 2.21** The map $\eta'$ extends to a well defined ring homomorphism

$$\eta : \langle D \rangle \to \mathbb{R}(B_1, \ldots, B_N).$$

**Corollary 2.22** We have $\theta \circ \eta = Id_{\langle D \rangle}$.

Then, we define a bracket on $\mathbb{R}(B_1, \ldots, B_N)$ via the rank 2 swapping algebra.

**Definition 2.23** [B2-bracket] We define the B2-bracket $\{ \cdot, \cdot \}_{B2}$ on $\mathbb{R}(B_1, \ldots, B_N)$ by:

$$\{P, Q\}_{B2} := \eta(\{\theta(P), \theta(Q)\})$$

(18)

for any $P, Q \in \mathbb{R}(B_1, \ldots, B_N)$.

**Proposition 2.24** The B2-bracket $\{ \cdot, \cdot \}_{B2}$ on $\mathbb{R}(B_1, \ldots, B_N)$ is Poisson.

**Proof.** We only need to verify the Jacobi identity on three generators $P, Q, R$. Since $\{\theta(P), \theta(Q)\} \in \langle D \rangle$. By Corollary 2.22, we have

$$\{\{P, Q\}_{B2}, R\}_{B2} = \eta(\{\theta \circ \eta(\{\theta(P), \theta(Q)\}), \theta(R)\})$$

$$= \eta(\{\{\theta(P), \theta(Q)\}, \theta(R)\}).$$

(19)

By Theorem 2.6, the swapping bracket verifies the Jacobi identity. Hence, by the above formula, $\{ \cdot, \cdot \}_{B2}$ verifies Jacobi identity. We conclude that $\{ \cdot, \cdot \}_{B2}$ is Poisson.

**Proposition 2.25** [Formulas] We have

$$\{B_k, B_{k+1}\}_{B2} = \left(1 - \frac{1}{B_k} - \frac{1}{B_{k+1}}\right) B_k \cdot B_{k+1},$$

(20)

$$\{B_k, B_{k-1}\}_{B2} = -\left(1 - \frac{1}{B_k} - \frac{1}{B_{k-1}}\right) B_k \cdot B_{k-1},$$

(21)
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\[
\{B_k, B_{k+2}\}_{B2} = -\frac{1}{B_{k+1}} \cdot B_k \cdot B_{k+2},
\]
(22)

\[
\{B_k, B_{k-2}\}_{B2} = \frac{1}{B_{k-1}} \cdot B_k \cdot B_{k-2},
\]
(23)

for the other cases

\[
\{B_i, B_j\}_{B2} = 0.
\]
(24)

with the convention \(k + N = k\), for \(k = 1, \ldots, N\).

Proof. The results follow from Proposition 2.17 and Definition 2.23.

\[\square\]

3. Large N asymptotic relation between the swapping algebra and the Virasoro algebra

3.1 Virasoro algebra and Hill’s operators

This subsection is not original, all the results can be found in Book [KW09]. It serves for the further usages, self-containedness, especially the comparison with its discrete version.

We recall some definitions and propositions related to Virasoro algebra and Hill’s operators. For more informations about this subject, we refer to [Se91] [KW09].

We denote \(\frac{\partial}{\partial \theta}\) by \(\partial_\theta\).

Definition 3.1 [Lie algebra \(Vect(S^1)\)] Let \(Vect(S^1)\) be the space of all the smooth vector fields on \(S^1\). After fixing a coordinate \(\theta\) on the circle \(S^1\), any smooth vector field on \(S^1\) can be written as \(f(\theta)\partial_\theta\), where \(f\) is a smooth function on \(S^1\). Under this identification, the Lie bracket of two elements \(f(\theta)\partial_\theta, g(\theta)\partial_\theta\) in \(Vect(S^1)\) is given by

\[
[f(\theta)\partial_\theta, g(\theta)\partial_\theta] = (f'(\theta)g(\theta) - g'(\theta)f(\theta))\partial_\theta.
\]
(25)

where \(f'(\theta)\) denotes the derivative in \(\theta\) of the function \(f(\theta)\). The Lie algebra \(Vect(S^1)\) is the vector space \(Vect(S^1)\) equipped with the Lie bracket defined above.

Since \(f(\theta)\) is smooth in \(S^1\), we have the Fourier coefficient decomposition

\[
f(\theta) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta},
\]

where \(f_j \in \mathbb{C}\).

Definition 3.2 [Witt algebra] Let \(L_k := ie^{ik\theta}\partial_\theta\). The Witt algebra \(\text{Witt}\) is the Lie subalgebra of \(Vect(S^1) \otimes \mathbb{C}\) generated by \(\{L_k\}_{k=-\infty}^{\infty}\). The restriction of the Lie bracket is the Witt bracket given by

\[
[L_m, L_n] = (m - n)L_{m+n}.
\]
(26)

for any \(m, n \in \mathbb{Z}\).

Definition 3.3 [Virasoro algebra] The map \(\omega_{GF} : \text{Witt} \times \text{Witt} \to \mathbb{R}\) given by

\[
\omega(f(\theta)\partial_\theta, g(\theta)\partial_\theta) = \int_{S^1} f'(\theta)g''(\theta)\,d\theta
\]
(27)

is a nontrivial 2-cocycle on \(\text{Witt}\), called the Gelfand–Fuchs cocycle. The corresponding central extension of \(\text{Witt}\) is called the Virasoro algebra and is denoted by \(\text{vir}\).
Proposition 3.4 [Chapter 2 Proposition 2.3 in [KW09]] Let $\omega$ be a 2-cocycle of $\mathfrak{m}$.t. Let $\delta_{a,b} = 1$ if $a = b$, $\delta_{a,b} = 0$ if $a \neq b$. Then there are two constants $c_1, c_2$ such that

$$\omega(L_n, L_m) = (c_1 \cdot n^3 + c_2 \cdot n) \cdot \delta_{n,-m}. \quad (28)$$

When $c_1 = 0$, $\omega$ is a 2-coboundary.

Definition 3.5 [Hill’s operator and Hill’s equation] Let $a \in \mathbb{R}$. Let $H(t) \in C^2(\mathbb{R}, \mathbb{R})$ with $H(s) = H(s + 1)$ for any $s \in \mathbb{R}$. Then the Hill’s operator defined by $(H(s), a)$ is a map $a \frac{d^2}{ds^2} + H(s)$ from $C^2(\mathbb{R}, \mathbb{R})$ to $C^0(\mathbb{R}, \mathbb{R})$. When

$$a \cdot \frac{d^2 X(s)}{ds^2} + H(s) \cdot X(s) = 0$$

for any $s \in \mathbb{R}$, we say $X(s) \in C^2(\mathbb{R}, \mathbb{R})$ is a solution of the Hill’s equation $a \frac{d^2 X(s)}{ds^2} + H(s) \cdot X(s) = 0$.

We can also consider $H$ as a function belongs to $C^2(S^1, \mathbb{R})$. By definition, the dual space of $\text{Vect}(S^1)$ is the space of the quadratic differential

$$\Omega^{\otimes 2} := \{H(\theta) (d\theta)^2 \mid H(\theta) \in C^2(S^1, \mathbb{R})\}.$$

The dual space of $\text{vir}$ is

$$\text{vir}^* := \{(H(\theta) (d\theta)^2, a) \mid H(\theta) \in C^2(S^1, \mathbb{R}), a \in \mathbb{R}\}.$$

We identify $\text{vir}^*$ with the space of Hill’s operators

$$\left\{a \frac{d^2}{d\theta^2} + H(\theta) \mid H(\theta) \in C^2(S^1, \mathbb{R}), a \in \mathbb{R}\right\}.$$

Later on, we consider the hyperplane $\left\{-\frac{d^2}{d\theta^2} + H(\theta) \mid H(\theta) \in C^2(S^1, \mathbb{R})\right\}$.

Definition 3.6 [Schwarzian derivative] Let $u \in C^2(\mathbb{R}, \mathbb{R})$. Then the Schwarzian derivative of $u$ is

$$S(u) := \frac{u'u'' - \frac{3}{2} (u'')^2}{(u')^2} \quad (29)$$

Proposition 3.7 [Chapter 1 Section 1.2 in [OT05]] Let us fix $x, x_1, x_2, x_3, x_4 \in \mathbb{R}$. Let $[a : b : c : d] := \frac{a-c}{2} : \frac{b-d}{2}$. The Schwarzian derivative $S(u)(x)$ satisfies for all small $\epsilon$

$$[u(x + x_1 \epsilon) : u(x + x_2 \epsilon) : u(x + x_3 \epsilon) : u(x + x_4 \epsilon)] = [x_1 : x_2 : x_3 : x_4] - 2 S(u)(x) \epsilon^2 + O(\epsilon^3). \quad (30)$$

There is a well known result which relates the Hill’s operator to the Schwarzian derivative.

Proposition 3.8 [Chapter 2 Proposition 2.9 in [KW09]] If $f, g$ are two linear independent solutions of the Hill’s equation for $(H(s), -1)$. Let $u = \frac{f}{g}$. Let $S(u)$ be the Schwarzian derivative of $u$. Then, we have

$$H(t) = -\frac{1}{2} \cdot S(u)(t) \quad (31)$$
3.2 The discrete Hill’s operator and the cross-ratios

**Definition 3.9** [Discrete Hill’s equation] Let $N \geq 1$ be an integer. Given a periodic sequence $\{H_k\}_{k=-\infty}^{\infty}$ in $\mathbb{R}$ where $H_{N+k} = H_k$ for any $k \in \mathbb{Z}$. The discrete Hill’s equation is the difference equation in $\{C_k\}_{k=-\infty}^{\infty}$:

$$
\frac{C_{k+2} - C_{k+1}}{N} - \frac{C_{k+1} - C_k}{N} = H_k \cdot C_k,
$$

or equivalently

$$
C_{k+1} = \left( \frac{H_k}{N^2 + 2} \right) C_k - C_{k-1},
$$
for any $k$ belongs to $\mathbb{Z}$.

We define $\{H_k\}_{k=-\infty}^{\infty}$ to be a discrete Hill’s operator, and $\{C_k\}_{k=-\infty}^{\infty}$ as in Equation (33) is the solution to the discrete Hill’s equation.

Given a discrete Hill’s operator, given two initial values $C_0$, $C_1$, by Equation (33) we have the series $\{C_k\}_{k=-\infty}^{\infty}$, since the difference equation is homogeneous, so up to scalar and PSL(2,$\mathbb{R}$) projective transformation, there are exactly two linear independent solutions of a discrete Hill’s operator, which are corresponding to one generic point of $\mathcal{M}_{N,1}$.

Moreover, when $N$ is odd, there is a one to one correspondence between Hill’s operator and one generic point of $\mathcal{M}_{N,1}$. By similar argument of Proposition 4.1 of [SOT10]. We have

**Proposition 3.10** [SOT10] Let $N > 3$ be odd, let $f \in \mathcal{M}_{N,1}$, there exists a unique discrete Hill’s equation such that $\{X_i\}_{i=-\infty}^{\infty}$ and $\{Y_i\}_{i=-\infty}^{\infty}$ are two linear independent solutions of $\{C_k\}$ and $[X_i : Y_i] = [f_i : 1]$.

**Notation 3.11** Let $b_k = \frac{H_k}{N^2} + 2$.

Let $c_1, ..., c_n \in \mathbb{R}$, the continued fraction

$$
[c_1 ; c_2, ..., c_n] := c_1 + \frac{1}{c_2 + \frac{1}{\ddots + \frac{1}{c_n}}},
$$

**Corollary 3.12** Let $N > 3$ be odd, $\{b_k\}_{k=1}^{N}$ and $\{H_k\}_{k=1}^{N}$ are two coordinate systems of $\mathcal{M}_{N,1}$.

The following proposition explains the relation between the discrete Hill's operator and cross ratio coordinate $\{B_k\}_{k=1}^{N}$.

**Proposition 3.13** Let $N > 3$ be odd, $\{X_i\}_{i=-\infty}^{\infty}$ and $\{Y_i\}_{i=-\infty}^{\infty}$ are two linear independent solutions of $\{C_k\}$ and $[X_i : Y_i] = [f_i : 1]$ with the initial condition $X_0 = 0, X_1 = 1, Y_0 = 1, Y_1 = 0$.

(i) for $n \geq 2$, we have

$$
f_n = [-b_1; b_2, -b_3, ..., (-1)^{n-1}b_{n-1}];
$$

(ii) for $k \geq 2$, we have

$$
b_{2k+1} = (f_3 - f_2) \cdot \frac{f_3 - f_2}{f_3 - f_4} \cdot \frac{f_4 - f_5}{f_4 - f_3} \cdots \frac{f_{2k-1} - f_{2k+1}}{f_{2k} - f_{2k-1}} \cdot \frac{f_{2k+2} - f_{2k}}{f_{2k+2} - f_{2k+1}};
$$

(iii) without the initial condition $X_0 = 0, X_1 = 1, Y_0 = 1, Y_1 = 0$, for any $k \geq 1$, we have

$$
b_k \cdot b_{k+1} = [f_{k-1}, f_{k+2}, f_{k+1}, f_k] = B_k.
$$
Proof. (i) We prove Formula \((33)\) by induction on \(n\). By Formula \((33)\) we have

\[ X_{k+1} = b_k X_k - X_{k-1} \]

and

\[ Y_{k+1} = b_k Y_k - Y_{k-1} \]

for any \(k \in \mathbb{Z}\). Since \(X_0 = 0, X_1 = 1, Y_0 = 1, Y_1 = 0\), we have

\[ X_2 = b_1, \ X_3 = b_1 b_2 + 1, \ X_4 = b_1 b_2 b_3 - b_1 - b_3, \ Y_2 = -1, \ Y_3 = -b_2, \ Y_4 = -b_2 b_3 + 1. \]

Thus, we obtain that

\[ f_2 = -b_1, \ f_3 = -b_1 + \frac{1}{b_2}, \ f_4 = [-b_1; b_2, -b_3, b_4]. \]

So Formula \((34)\) is true for \(n = 2, 3, 4\). Suppose that for \(n = k\), the formula is true, then

\[ f_k = \frac{X_k}{Y_k} = \frac{b_k X_k - X_{k-1}}{b_k Y_k - Y_{k-1}} = \frac{b_k X_k - X_{k-1} - \frac{X_{k-1}}{b_k}}{b_k Y_k - Y_{k-1} - \frac{Y_{k-1}}{b_k}} \]

\[ = \frac{(b_k - 1) X_{k-1} - X_{k-2}}{(b_k - 1) Y_{k-1} - Y_{k-2}}. \] (38)

By substitute \(b_{k-1} - \frac{1}{b_k}\) for \(b_{k-1}\) in the continued fraction \(f_k\), we have

\[ f_{k+1} = [-b_1; b_2, -b_3, \ldots, (-1)^{k-2} b_{k-2}, (-1)^{k-1} (b_{k-1} - \frac{1}{b_k})] \]

\[ = [-b_1; b_2, -b_3, \ldots, (-1)^{k-1} b_{k-1}, (-1)^k b_k]. \] (39)

We conclude that the formula \(f_n = [-b_1; b_2, -b_3, \ldots, (-1)^{n-1} b_{n-1}]\) for \(n \geq 2\).

(ii) For any \(k \geq 1\), we have

\[ X_{k+1} Y_k - X_k Y_{k+1} = (b_k X_k - X_{k-1}) \cdot Y_k - X_k \cdot (b_k Y_k - Y_{k-1}) = X_k Y_{k-1} - X_{k-1} Y_k \]

\[ = \ldots = X_1 Y_0 - X_0 Y_1 = 1, \]

\[ f_{k+1} - f_k = \frac{X_{k+1} Y_k - X_k Y_{k+1}}{Y_{k+1} Y_k} = \frac{1}{Y_k Y_{k+1}}, \] (40)

\[ f_{k+2} - f_k = \frac{X_{k+2} Y_k - X_k Y_{k+2}}{Y_{k+2} Y_k} = \frac{X_{k+2} Y_k - X_k Y_{k+2}}{Y_k Y_{k+2}} \]

\[ = \frac{(b_{k+1} X_{k+1} - X_k) \cdot Y_k - X_k \cdot (b_{k+1} Y_{k+1} - Y_k)}{Y_k Y_{k+2}} = \frac{b_{k+1}}{Y_k Y_{k+1}}. \] (42)
By the above formulas, we conclude that for any \( k \geq 2 \), we have

\[
\begin{align*}
(f_3 - f_2) \cdot f_3 - f_2 \cdot f_4 - f_5 \cdots f_{2k} - f_{2k+1} \cdot f_{2k+2} - f_{2k+1} &= \frac{1}{Y_2 Y_3} \cdot \frac{1}{Y_4 Y_5} \cdots \frac{1}{Y_{2k+1} Y_{2k+2}} \cdot \frac{1}{Y_{2k+2} Y_{2k+1}} \\
&= b_{2k+1} Y_2 = b_{2k+1}.
\end{align*}
\]

(iii) Let

\[
h := X_1 Y_0 - X_0 Y_1 \neq 0.
\]

For any \( k \geq 0 \), we have

\[
X_{k+1} Y_k - X_k Y_{k+1} = (b_k X_k - X_{k-1}) \cdot Y_k - X_k \cdot (b_k Y_k - Y_{k-1}) = X_k Y_{k-1} - X_{k-1} Y_k = \ldots = X_1 Y_0 - X_0 Y_1 = h,
\]

\[
f_{k+1} - f_k = \frac{X_{k+1} - X_k}{Y_{k+1} Y_k} = \frac{h}{Y_k Y_{k+1}},
\]

\[
f_{k+2} - f_k = \frac{X_{k+2} - X_k}{Y_{k+2} Y_k} = \frac{h b_{k+1}}{Y_k Y_{k+1}}.
\]

For any \( k \geq 1 \), we have

\[
\frac{f_{k-1} - f_{k+1}}{f_{k-1} - f_k} \cdot \frac{f_{k+2} - f_k}{f_{k+2} - f_{k+1}} = \frac{-h b_k}{h} \frac{h b_{k+1}}{h} = b_k b_{k+1}.
\]

We conclude that \( B_k = b_k \cdot b_{k+1} \).

\[
\text{\[\Box\]}
\]

We give another proof of Proposition 3.8.

Proof. Let \( H(t) \in C^2(\mathbb{R}, \mathbb{R}) \) with \( H(s) = H(s+1) \) for any \( s \in \mathbb{R} \) be a Hill's operator, let \( X(t) \), \( Y(t) \) be two linear independent solutions of \( H(t) \).

Let \( N > 3 \) be odd, let \( H_k = H(k/N) \), let \( X_k, Y_k \) be two linear independent solutions of \( H_k \), \( [X_k : Y_k] = [f_k : 1] \). Suppose that

\[
\frac{X_1}{X_0} \neq \frac{Y_1}{Y_0}.
\]

by Proposition 3.8, we have

\[
[f_{k-1} : f_{k+2} : f_{k+1} : f_k] = b_k b_{k+1}
\]

\[
= (2 + H_{k+1}/N^2)(2 + H_k/N^2)
\]

\[
= [k - 1 : k + 2 : k + 1 : k] + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_k H_{k+1}}{N^4}.
\]

When \( N \) converge to infinity, since \( H(t) \) is continuous, we have

\[
H_{k+1} = H_k + o\left(\frac{1}{N}\right).
\]

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Comparing with the Schwarzian derivative in Proposition 3.7, we have

$$4H_k = -2 \cdot S\left(\frac{X}{Y} \left(\frac{k}{N}\right)\right) + o\left(\frac{1}{N}\right).$$

We conclude that

$$H_k = -\frac{1}{2} \cdot S\left(\frac{X}{Y} \left(\frac{k}{N}\right)\right) + o\left(\frac{1}{N}\right).$$

\[\square\]

### 3.3 Main result

By comparing with Proposition 3.4, we define a discrete version of Virasoro algebra.

**Definition 3.14** \((t_1, t_2, N)\)-Virasoro bracket] Let \(t_1, t_2 \in \mathbb{R}\) and \(N \in \mathbb{N}\), the \((t_1, t_2, N)\)-Virasoro bracket on \(\{I_k\}_{k=-N}^{N}\) is defined to be:

For \(p, q = -\left[\frac{N-1}{2}\right], \ldots, \left[\frac{N}{2}\right]\),

(i) when \(p \neq -q\), we have

\[\{I_p, I_q\}_{N,t_1,t_2} = (p - q) \cdot I_{p+q}\]

with the convention \(I_{k+N} = I_k\);

(ii) when \(p = -q\), we have

\[\{I_p, I_{-p}\}_{N,t_1,t_2} = 2p \cdot I_0 + t_1 \cdot p^3 + t_2 \cdot p.\]

**Remark 3.15** Notice that \((t_1, t_2, N)\)-Virasoro bracket is asymptotic to the Poisson bracket associated to the 2-cocycle with \(c_1 = t_1, c_2 = t_2\) as in Proposition 3.4 when \(N\) converges to infinite, but it is not a Poisson bracket.

Very specific values of \(t_1\) and \(t_2\) correspond to Virasoro algebra. When \(t_1\) is fixed, \(t_2\) varies, they correspond to same element in cohomology group \(H^2(\text{Vect}(S^1), \mathbb{R})\). Different \(t_1\) corresponds to different element in \(H^2(\text{Vect}(S^1), \mathbb{R})\). Virasoro algebra generates all the possible central extension by Proposition 3.4.

**Definition 3.16** [Discrete Fourier transformation] Let \(\{B_k\}_{k=1}^{N}\) be the cross ratio coordinates of \(\mathcal{M}_{N,1}\). Let \(\mathbb{B} = \{B_1, \ldots, B_N\}\). The discrete Fourier transformation \(\mathcal{F}\) of \(\mathbb{B}\) is defined to be

\[\mathcal{F}_{p}\mathbb{B} = \sum_{k=1}^{N} B_k e^{-2\pi i \frac{pk}{N}}.\]

Our main result of this section is

**Theorem 3.17** [Main result] Let \(N > 3\) be odd. For \(k = -\left[\frac{N-1}{2}\right], \ldots, \left[\frac{N}{2}\right]\), let

\[V_k = \frac{\mathcal{F}_{k}\mathbb{B} \cdot N}{8\pi i}.\]

We have

\[\{V_p, V_q\}_{B_2} = \{V_p, V_q\}_{N,\frac{2\pi}{N}, 8N} + o\left(\frac{1}{N^2}\right).\]
Proof. For $p, q = -\left[\frac{N-1}{2}\right], \ldots, \left[\frac{N}{2}\right]$, we have

$$\{F_p, F_q\}_B = 2 \sum_{k=1}^{N} \left( e^{-\frac{2qk\pi i}{N}} \cdot e^{-\frac{2qh(k+1)\pi i}{N}} - e^{-\frac{2q(k+1)\pi i}{N}} \cdot e^{-\frac{2qh\pi i}{N}} \right) \cdot (B_k B_{k+1} - B_k - B_{k+1})$$

$$- \left( e^{-\frac{2qk\pi i}{N}} \cdot e^{-\frac{2q(h+2k)\pi i}{N}} - e^{-\frac{2q\pi i}{N}} \cdot e^{-\frac{2qecessary(k+2)\pi i}{N}} \right) \cdot \frac{B_k B_{k+2}}{B_{k+1}}$$

By

$$B_k = b_k b_{k+1} = 4 + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_k H_{k+1}}{N^4},$$

we have

$$B_k = 4 + \frac{4H_k}{N^2} + o \left( \frac{1}{N^2} \right).$$

We have the above formula equals to

$$\sum_{k=1}^{N} e^{-\frac{2(q+p)k\pi i}{N}} \cdot \left[ \left( 1 + \frac{2q\pi i}{N} + \frac{2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o \left( \frac{1}{N^3} \right) \right) - \left( 1 + \frac{2p\pi i}{N} + \frac{2\pi^2 p^2}{N^2} + \frac{4\pi^3 p^3 i}{3N^3} + o \left( \frac{1}{N^3} \right) \right) \right]$$

$$- \left( 1 + \frac{-4p\pi i}{N} + \frac{-8\pi^2 p^2}{N^2} + \frac{32\pi^3 p^3 i}{3N^3} + o \left( \frac{1}{N^3} \right) \right) \cdot \left( 4 + \frac{4H_k}{N^2} + o \left( \frac{1}{N^2} \right) \right) \right]$$

$$= \sum_{k=1}^{N} e^{-\frac{2(q+p)k\pi i}{N}} \left[ \frac{32H_k(p-q)\pi i}{N^3} - \frac{16\pi^2 (p^2 - q^2)}{N^2} + \frac{32\pi^3 (p^3 - q^3) i}{N^3} + o \left( \frac{1}{N^3} \right) \right].$$

When $p \neq -q$, since $\sum_{k=1}^{N} e^{-\frac{2(q+p)k\pi i}{N}} = 0$, by $B_k = 4 + \frac{4H_k}{N^2} + o(\frac{1}{N^2})$, the above formula equals to

$$\sum_{k=1}^{N} e^{-\frac{2(q+p)k\pi i}{N}} \cdot \frac{8B_k(p-q)\pi i}{N} = \frac{8(p-q)\pi i}{N} \cdot F_{p+q} + o \left( \frac{1}{N^3} \right);$$

When $p = -q$, the above formula equals to

$$\frac{64p\pi i}{N^3} \sum_{k=1}^{N} H_k + \frac{64p^3\pi^3 i}{N^2} + o \left( \frac{1}{N^3} \right)$$

$$= \frac{16p\pi i}{N} \sum_{k=1}^{N} (B_k - 4) + \frac{64p^3\pi^3 i}{N^2} + o \left( \frac{1}{N^3} \right)$$

$$= \frac{16p\pi i}{N} F_0 \cdot B - \frac{64p^3\pi^3 i}{N^2} + o \left( \frac{1}{N^3} \right).$$

Replacing $F_k$ by

$$V_k = \frac{F_k B \cdot N}{8\pi i},$$

we obtain that:

for $p \neq -q$,

$$\{V_p, V_q\}_B = (p-q) \cdot V_{p+q} + o \left( \frac{1}{N^2} \right);$$

for $p = -q$,

$$\{V_p, V_q\}_B = (2p-q) \cdot V_{p+q} + o \left( \frac{1}{N^2} \right).$$
for $p = -q$
\[ \{ V_p, V_{-q} \}_{B_2} = 2p \cdot V_0 + \left( \frac{8\pi^2}{N} \right) \cdot p^3 - 8N \cdot p + o \left( \frac{1}{N^2} \right). \]

We conclude that
\[ \{ V_p, V_q \}_{B_2} = \{ V_p, V_q \}_{N, \frac{8\pi^2}{N}, 8N} + o \left( \frac{1}{N^2} \right). \quad (55) \]

4. Schwartz algebra on $\mathbb{R}(B_1, \ldots, B_N)$ and its relation with swapping algebra

We plan to construct a Poisson structure on coordinates defined by weak cross ratios on $\mathcal{M}_{N,n}$ through the swapping algebra. Then compare this swapping Poisson structure with the Poisson structure considered by R. Schwartz, V. Ovsienko and S. Tabachnikov [SOT10] for $n = 2$ to show that they are compatible, where their continuous limit are the natural Lie-Poisson structure and the freezing structure [SOT10]. This plan is to reply the conjecture mentioned in [SOT10]. But we do not success in finding such a nice swapping Poisson structure. We have only result for $n = 1$ as shown in this section. More general case will be considered later on.

4.1 Schwartz algebra and its asymptotic phenomenon

The Schwartz algebra appears in [SOT10] as a Poisson algebra on the cross ratio coordinate system of $\mathcal{M}_{N,2}$ which is a discrete version of the second Gelfand–Dickey Poisson structure. Here, we consider the case for $\mathbb{R}P^1$ where the bracket is referring to [FV93].

**Definition 4.1 [Schwartz bracket on $\mathbb{R}(B_1, \ldots, B_N)$]** The Schwartz bracket $\{ \cdot, \cdot \}_1$ on $\mathbb{R}(B_1, \ldots, B_N)$ is defined by extending the following formula on generators to the whole ring:
\[ \{ B_i, B_{i+1} \}_{S_2} = \pm B_i \cdot B_{i+1} \quad (56) \]
For the other cases
\[ \{ B_i, B_j \}_{S_2} = 0 \quad (57) \]

We have similar result as Theorem 3.17 for the Schwartz algebra on $\mathbb{R}(B_1, \ldots, B_N)$.

**Theorem 4.2 [Main result]** Let $N > 3$ be odd. For $k = -\left[ \frac{N-1}{2} \right], \ldots, \left[ \frac{N}{2} \right]$, let
\[ W_k = \frac{F_k B \cdot N}{16\pi i}. \]
We have
\[ \{ W_p, W_q \}_{S_2} = \{ W_p, W_q \}_{N, \frac{8\pi^2}{N}, 8N} + o \left( \frac{1}{N^2} \right). \quad (58) \]

**Proof.** For $p, q = -\left[ \frac{N-1}{2} \right], \ldots, \left[ \frac{N}{2} \right]$, we have
\[ \{ F_p B, F_q B \}_{S_2} = \sum_{k=1}^{N} \left( e^{-\frac{2\pi k}{N}} \cdot e^{-\frac{2\pi (k+1)}{N}} - e^{-\frac{2\pi (k+1)}{N}} \cdot e^{-\frac{2\pi k}{N}} \right) \cdot (B_k B_{k+1}) \quad (59) \]
By
\[ B_k = a_k a_{k+1} = 4 + \frac{2(H_k + H_{k+1})}{N^2} + \frac{H_k H_{k+1}}{N^4}, \]

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we have
\[ B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right). \]

We have the above formula equals to
\[
\sum_{k=1}^{N} e^{-2(p+q)k\pi i N} \cdot \left( \left( 1 + \frac{-2q\pi i}{N} + \frac{-2\pi^2 q^2}{N^2} + \frac{4\pi^3 q^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) - \left( 1 + \frac{-2p\pi i}{N} + \frac{-2\pi^2 p^2}{N^2} \right) \right) \\
\frac{4\pi^3 p^3 i}{3N^3} + o\left(\frac{1}{N^3}\right) \right) \\
= \sum_{k=1}^{N} e^{-2(p+q)k\pi i N} \cdot \left[ \frac{32\pi i(p-q)}{N} + \frac{32\pi^2(p^2 - q^2)}{N^2} - \frac{64\pi^3(p^3 - q^3)i}{3N^3} + \frac{64H_k(p-q)\pi i}{N^3} + o\left(\frac{1}{N^3}\right) \right].
\]

When \( p \neq -q \), since \( \sum_{k=1}^{N} e^{-2(p+q)k\pi i N} = 0 \), by \( B_k = 4 + \frac{4H_k}{N^2} + o\left(\frac{1}{N^2}\right) \), the above formula equals to
\[
\sum_{k=1}^{N} e^{-2(p+q)k\pi i N} \cdot \frac{16B_k(p-q)\pi i}{N} = \frac{16(p-q)\pi i}{N} \cdot F_{p+q} B + o\left(\frac{1}{N^3}\right); \tag{60}
\]

When \( p = -q \), since \( \sum_{k=1}^{N} B_k = 0 \), the above formula equals to
\[
\frac{64p\pi i}{N} + \frac{128p\pi i}{N^3} \sum_{k=1}^{N} H_k - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \\
= \frac{64p\pi i}{N} + \frac{32p\pi i}{N} \sum_{k=1}^{N} (B_k - 4) - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right) \tag{62}
\]
\[
= \frac{64p\pi i}{N} + \frac{32p\pi i}{N} F_0 B - 128p\pi i - \frac{128p^3\pi^3 i}{3N^2} + o\left(\frac{1}{N^3}\right). 
\]

Thus we have:

(i) for \( p \neq -q \)
\[ \{W_p, W_q\}_S^2 = (p-q) \cdot W_{p+q} + o\left(\frac{1}{N^2}\right); \]

(ii) for \( p = -q \)
\[ \{W_p, W_{-p}\}_S^2 = 2p \cdot W_0 + (4-8N) \cdot p - \frac{8\pi^2}{3N} \cdot p^3 + o\left(\frac{1}{N^2}\right). \]

We conclude that
\[ \{W_p, W_q\}_S^2 = \{W_p, W_q\}_{N, \frac{8\pi^2}{3N}, 4-8N} + o\left(\frac{1}{N^2}\right). \tag{63} \]

\[ \square \]

4.2 Two Poisson structures are compatible

Let us recall the traditional definition of the bihamiltonian system.

**Definition 4.3** Two Poisson structures \( \{\cdot, \cdot\}_a \) and \( \{\cdot, \cdot\}_b \) on a manifold \( M \) are said to be compatible if and only if for any \( \lambda \), \( \{\cdot, \cdot\}_a + \lambda \{\cdot, \cdot\}_b \) is Poisson on \( M \).
A dynamic system \( \frac{d}{dt} m = \xi(m) \) over \( M \) is bihamiltonian if its vector field \( \xi \) is Hamiltonian with respect to these two Poisson structures \( \{\cdot,\cdot\}_a \) and \( \{\cdot,\cdot\}_b \).

Then we define the compatibility on a ring \( R \) with two Poisson structures.

**Definition 4.4** Two Poisson brackets \( \{\cdot,\cdot\}_a \) and \( \{\cdot,\cdot\}_b \) on a ring \( R \) are said to be compatible if and only if for any \( \lambda, \{\cdot,\cdot\}_a + \lambda\{\cdot,\cdot\}_b \) is Poisson on \( R \).

**Proposition 4.5** \( \{\cdot,\cdot\}_a \) and \( \{\cdot,\cdot\}_b \) are compatible if and only if for any \( x, y, z \in R \), we have
\[
\{\{x, y\}_a, z\}_b + \{\{y, z\}_a, x\}_b + \{\{z, x\}_a, y\}_b + \{\{x, y\}_b, z\}_a + \{\{y, z\}_b, x\}_a + \{\{z, x\}_b, y\}_a = 0
\]

**Proof.** Let the bracket
\[
\{\cdot,\cdot\}_a \lambda b := \{\cdot,\cdot\}_a + \lambda\{\cdot,\cdot\}_b.
\]
The bracket \( \{\cdot,\cdot\}_a \lambda b \) is Poisson if and only if \( \{\cdot,\cdot\}_a \lambda b \) satisfies the Jacobi identity. For any \( x, y, z \in R \), Let \( \sum \) runs over the triplets \((x, y, z), (y, z, x), (z, x, y)\), the Jacobi identity of \( \{\cdot,\cdot\}_a \lambda b \) equals
\[
\sum\{(x, y)_{a \lambda b}, z\}_{a \lambda b}
\]
\[
= \sum\{\{x, y\}_a + \lambda\{x, y\}_b, z\}_{a \lambda b}
\]
\[
= \lambda \sum\{(x, y)_a, z\}_a + \{x, y\}_a, z\}_b + \lambda^2\{x, y\}_b, z\}_b)
\]
\[
= \lambda \sum\{(x, y)_b, z\}_a + \{x, y\}_a, z\}_b
\]
The last equation uses the fact that \( \{\cdot,\cdot\}_a \) and \( \{\cdot,\cdot\}_b \) are Poisson. We conclude that \( \{\cdot,\cdot\}_a \) and \( \{\cdot,\cdot\}_b \) are compatible if and only if
\[
\sum\{(x, y)_b, z\}_a + \{x, y\}_a, z\}_b = 0.
\]

By case by case verification, we have our main result of this subsection.

**Theorem 4.6** [Main Result] For \( N \geq 5 \), \( \{\cdot,\cdot\}_B_2 \) and \( \{\cdot,\cdot\}_S_2 \) are compatible on \( \mathbb{R}(B_1, ..., B_N) \).

**Proof.** Let
\[
K(B_i, B_j, B_k) := \{\{B_i, B_j\}_B_2, B_k\}_{S_2} + \{\{B_j, B_k\}_B_2, B_i\}_{S_2} + \{\{B_k, B_i\}_B_2, B_j\}_{S_2} + \{\{B_i, B_j\}_S_2, B_k\}_B_2 + \{\{B_j, B_k\}_S_2, B_i\}_B_2 + \{\{B_k, B_i\}_S_2, B_j\}_B_2.
\]
By definition, we have to check that
\[
K(B_i, B_j, B_k) = 0
\]
for any \( i, j, k = 1, ..., N \). Since
\[
K(B_i, B_j, B_k) = -K(B_j, B_i, B_k),
\]
when some indexes coincide, for example \( i = j \), we have
\[
K(B_i, B_i, B_k) = 0.
\]
Let $\sigma_s$ be the permutation of the $N$ indexes such that $\sigma(l) = l + s$. The permutation $\sigma_s$ induce a ring automorphism $\chi_s$ of $\mathbb{R}(B_1, ..., B_N)$ such that

$$\chi_s(B_l) = B_{l+s}$$

for $l = 1, ..., N$. Moreover, we have

$$\{\chi_s(B_i), \chi_s(B_j)\}_{S^2} = \chi_s(\{B_i, B_j\}_{S^2})$$

and

$$\{\chi_s(B_i), \chi_s(B_j)\}_{B^2} = \chi_s(\{B_i, B_j\}_{B^2}).$$

Let $\tau$ be the permutation of the $N$ indexes such that

$$\tau(l) = N + 1 - l$$

for $l = 1, ..., N$. The permutation $\tau$ induce a ring automorphism $\nu$ of $\mathbb{R}(B_1, ..., B_N)$ such that

$$\nu(B_l) = B_{N+1-l}.$$ 

Moreover, we have

$$\{\nu(B_i), \nu(B_j)\}_{S^2} = -\nu(\{B_i, B_j\}_{S^2}),$$

$$\{\nu(B_i), \nu(B_j)\}_{B^2} = -\nu(\{B_i, B_j\}_{B^2}).$$

By the above symmetry, we suppose that

$$i = 1$$

and

$$1 < j < k \leq N.$$ 

Let

$$l := \min\{|j-i|, |j-i-N|, |k-j|, |k-j-N|, |i-k|, |i-k-N|\}.$$ 

We suppose that $l = |j-1|$, we have to verify the following cases:

(i) When $1 < j - 1 < k - 2 < N - 1$, we have

$$K(B_1, B_i, B_k)$$

$$K(B_1, B_i, B_k) = \{\{B_1, B_j\}_{B^2}, B_k\}_{S^2} + \{\{B_j, B_k\}_{B^2}, B_1\}_{S^2} + \{\{B_k, B_1\}_{B^2}, B_j\}_{S^2} + \{\{B_1, B_j\}_{B^2}, B_k\}_{B^2} + \{\{B_j, B_k\}_{S^2}, B_1\}_{B^2} + \{\{B_k, B_1\}_{S^2}, B_j\}_{B^2} + \{\{B_1, B_j\}_{B^2}, B_k\}_{S^2} + \{\{B_j, B_k\}_{S^2}, B_1\}_{B^2} + \{\{B_k, B_1\}_{S^2}, B_j\}_{B^2} + \{\{B_1, B_j\}_{B^2}, B_k\}_{S^2} + \{\{B_j, B_k\}_{B^2}, B_1\}_{S^2} + \{\{B_k, B_1\}_{B^2}, B_j\}_{S^2}.$$ 

Since

$$\{B_1, B_j\}_{B^2}$$

is a polynomial of $B_1, ..., B_j$, we have

$$\{\{B_1, B_j\}_{B^2}, B_k\}_{S^2} = 0.$$ 

Similarly, we have

$$\{\{B_j, B_k\}_{B^2}, B_1\}_{S^2} = 0$$

and

$$\{\{B_k, B_1\}_{B^2}, B_j\}_{S^2} = 0.$$ 

We conclude that

$$K(B_1, B_i, B_k) = 0.$$ 


(ii) When $j = 2$, $k = 3$, we have

$$
K(B_1, B_2, B_3) \\
= \{\{B_1, B_2\}_B, B_3\}_S + \{\{B_2, B_3\}_B, B_1\}_S + \{\{B_3, B_1\}_B, B_2\}_S \\
+ \{\{B_1, B_2\}_S, B_3\}_B + \{\{B_2, B_3\}_S, B_1\}_B + \{\{B_3, B_1\}_S, B_2\}_B \\
= \{B_1B_2 - B_1 - B_2, B_3\}_S + \{B_2B_3 - B_2 - B_3, B_1\}_S + \left(\frac{B_3B_1}{B_2}\right)_S B_2 \\
+ \{B_1B_2, B_3\}_B + \{B_2B_3, B_1\}_B \\
= (B_1 - 1)B_2B_3 - (B_3 - 1)B_1B_2 + B_1(B_2B_3 - B_2 - B_3) \\
- \frac{B_1B_3}{B_2}B_2 - (B_1B_2 - B_1 - B_2)B_3 + \frac{B_3B_1}{B_2}B_2 \\
= 0. \\
(67)
$$

(iii) When $j = 2$, $k = 4$ and $N > 5$, we have

$$
K(B_1, B_2, B_4) \\
= \{\{B_1, B_2\}_B, B_4\}_S + \{\{B_2, B_4\}_B, B_1\}_S + \{\{B_4, B_1\}_B, B_2\}_S \\
+ \{\{B_1, B_2\}_S, B_4\}_B + \{\{B_2, B_4\}_S, B_1\}_B + \{\{B_4, B_1\}_S, B_2\}_B \\
= \{B_1B_2 - B_1 - B_2, B_4\}_S + \left(\frac{-B_2B_1}{B_3}\right)_S B_1 + \{B_1B_2, B_4\}_B \\
= \frac{B_1B_2B_4}{B_3} - \frac{B_1B_2B_1}{B_3} \\
= 0. \\
(68)
$$

(iv) When $j = 2$, $k = 4$ and $N = 5$, we have

$$
K(B_1, B_2, B_4) \\
= \{\{B_1, B_2\}_B, B_4\}_S + \{\{B_2, B_4\}_B, B_1\}_S + \{\{B_4, B_1\}_B, B_2\}_S \\
+ \{\{B_1, B_2\}_S, B_4\}_B + \{\{B_2, B_4\}_S, B_1\}_B + \{\{B_4, B_1\}_S, B_2\}_B \\
= \{B_1B_2 - B_1 - B_2, B_4\}_S + \left(\frac{-B_2B_1}{B_3}\right)_S B_1 + \left(\frac{-B_4B_1}{B_5}\right)_S B_2 \\
+ \{B_1B_2, B_4\}_B \\
= \frac{B_1B_2B_4}{B_3} - \frac{B_1B_1B_2}{B_5} - \frac{B_1B_2B_4}{B_3} + \frac{B_1B_4B_2}{B_5} \\
= 0. \\
(69)
$$

We conclude that for $N \geq 5$, $\{\cdot, \cdot\}_B$ and $\{\cdot, \cdot\}_S$ are compatible on $\mathbb{R}(B_1, ..., B_N)$. \hfill \Box

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Swapping algebra, Virasoro algebra and discrete integrable system

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