MODULAR ENVELOPES, OSFT AND NONSYMMETRIC (NON-Σ) MODULAR OPERADS

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Abstract. Our aim is to introduce and advocate non-Σ (non-symmetric) modular operads. While ordinary modular operads were inspired by the structure of the moduli space of stable complex curves, non-Σ modular operads model surfaces with open strings outputs. An immediate application of our theory is a short proof that the modular envelope of the associative operad is the linearization of the terminal operad in the category of non-Σ modular operads. This gives a succinct description of this object that plays an important rôle in open string field theory. We also sketch further perspectives of the presented approach and formulate a principle which we daringly compare to the cobordism hypothesis.

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Introduction

Operads have their non-Σ (non-symmetric) versions obtained by forgetting the symmetric group actions. Likewise, for the non-Σ version of cyclic operads one requires only the actions of the cyclic subgroups of the symmetric groups. In both cases we thus demand less

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1See e.g. definitions II.1.4, II.1.14 and Section II.5.1 of [19].
structure. This straightforward approach however fails for modular operads whose non-Σ versions have been a mystery so far.

There are, fortunately, some clues and inspirations, namely modular envelopes of cyclic operads, introduced under the name modular completions by the author in [15, Definition 2]. The modular envelope Mod(∗C) of the terminal cyclic operad ∗C in the category Set of sets turned out to be the terminal modular operad in Set, see [15, page 382].

The modular envelope Mod(∗C) of the terminal non-Σ cyclic Set-operad, described much later in [4, 6], turned out to be surprisingly complex. Guided by the example of Mod(∗C), one would expect Mod(∗C) to be the (symmetrization of) the terminal Set-operad in the conjectural category of non-Σ modular operads. The description of Mod(∗C) given in [4, Theorem 3.1] and its relation to the moduli space of Riemann surfaces with boundaries [4, Theorem 3.7] therefore gives some feeling what non-Σ modular operads should be.

Another natural requirement is that the conjectural category of non-Σ modular operads should fill the left bottom corner of the diagram

\[
\begin{array}{ccc}
\text{NsCycOp} & \xrightarrow{\text{Sym}} & \text{CycOp} \\
\text{Mod} & \xleftarrow{\square} & \text{Des} \\
\end{array}
\]

in which \(\square: \text{ModOp} \to \text{CycOp}\) is the forgetful functor, Mod : CycOp \to ModOp the modular completion [15], Des : CycOp \to NsCycOp the forgetful functor (the desymmetrization) and the symmetrization Sym : NsCycOp \to CycOp its left adjoint.

The category of (ordinary) modular operads contains the category of cyclic operads as the full subcategory of operads concentrated in genus 0. Requiring the same from the category of non-Σ modular operads leads to the notion of geometricity that does not have analog in the symmetric world.

Our aim is to introduce and advocate the notion of non-Σ (non-symmetric) modular operads. The main definitions are Definitions 8 and 18, and the main result is isomorphism (25a) of Theorem 25. As an immediate application we give a short, elementary proof of the description of the modular envelope Mod(Ass) of the associative operad given in [4, 4].

**Perspectives.** It turns out that the elements of the modular envelope Mod(∗C) of the terminal cyclic Set operad ∗C describe isomorphism classes of oriented surfaces with holes, and likewise the non-Σ modular envelope Mod(∗C) of the terminal non-Σ cyclic Set-operad describe isomorphism classes of oriented surfaces with teethed holes. These geometric objects describe interactions in closed and open string field theory. Very crucially, Theorem 25 asserts

\[2\] We explain why in Example 3.

\[3\] The linearization Span(∗C) of ∗C is the cyclic operad Com for commutative associative algebras.

\[4\] The authors of [4, 6] worked with the linearized version, i.e. with the cyclic operad Ass for associative algebras, but the linear structure is irrelevant here as everything important happens inside Set.

\[5\] A derived version of this modular envelope was studied in [6, 11].

\[6\] Known today as the modular envelope.

\[7\] Not to be mistaken with Batanin’s desymmetrization of ∗C.

[October 12, 2014] nsmod.tex
**Figure 1.** Notation of categories.

that both \( \text{Mod}(\mathbb{C}) \) and \( \text{Mod}(\mathbb{C}) \) are the terminal objects in an appropriate category of modular operads.

In Section 6 we consider the operad \( \mathbb{D} \) describing associative algebras with an involution. It is a cyclic dihedral operad in the sense of [18, Section 3] that equals the Möbiusisation [3, Definition 3.32] of the terminal cyclic operad \( \mathbb{C} \). By [3, Theorem 3.10], its modular envelope \( \text{Mod}(\mathbb{D}) \) consist of isomorphism classes of non-oriented surfaces with teethed holes. We believe that there exist another version of modular operads (say dihedral modular operads) such that \( \text{Mod}(\mathbb{D}) \) is the (symmetrization of the) terminal modular operad of this type.

We believe that similar reasoning applies to other objects, such as the surfaces describing interactions in open-closed string field theory that may include also D-branes, or even higher-dimensional manifolds. In each case the corresponding type of modular operad should reflect how a geometric object is composed from simper pieces, e.g. pair of pants in the case of closed string field theory. We are lead to formulate

**Principle.** For a large class of geometric objects there exists a version of modular operads such that the set of isomorphism classes of these objects is the terminal modular \( \text{Set-operad} \) of a given type.

We venture to compare the above humble principle to the celebrated cobordism hypothesis [4] as they both describe objects of geometric nature by purely categorial means.

**Notations and conventions.** Throughout the paper, \( \mathbb{M} = (\mathbb{M}, \otimes, 1) \) will stand for a complete and cocomplete, possibly enriched, symmetric monoidal category, with the initial object \( 0 \in \mathbb{M} \). Typical examples will be \( \text{k-Mod} \), the category of modules over a commutative unital ring \( \text{k} \), or the cartesian category \( \text{Set} \) of sets. By \( \text{Span} : \text{Set} \to \text{k-Mod} \) we denote the \( \text{k-linear} \) span, i.e. the left adjoint to the forgetful functor \( \text{k-Mod} \to \text{Set} \). The glossary of categories introduced and used throughout the paper is given in Figure 1, various forgetful functors and their adjoints are listed in Figure 2.
An order in this paper will always be a linear order of a finite set. By * we denote a chosen one-point set.

**Assumptions.** We assume certain familiarity with basic notions of operad theory. There exists rich and easily accessible literature, for instance the monograph [19], overview articles [8, 16, 17] or a recent account [14]. For the reader's convenience, we recall a definition of cyclic and modular operads based on finite sets in the Appendix.

1. **Cyclic orders and the first try**

   The notion of a cyclic order is a standard one so we recall it only to fix notation and terminology. Let $S$ be a finite set with $n$ elements. There is an obvious one-to-one correspondence between linear orders of $S$ and isomorphisms $\varphi : \{1, \ldots, n\} \cong S$. We identify the order represented by an isomorphism $\varphi$ with the one represented by the composition $\varphi \alpha_k$, where $k$ is an integer and $\alpha_k : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is the shift by $k$, i.e. it sends $i \in \{1, \ldots, n\}$ to the unique $j \in \{1, \ldots, n\}$ such that $j = i + k \mod n$.

   **Definition 1.** A cyclic order on a finite set $S$ is an equivalence class of total orders on $S$ modulo the above identification. For an ordered set $A$ we denote by $[A]$ the underlying set of $A$ with the cyclic order induced by the order of $A$.

   While linearly ordered sets $A$ are naturally represented by horizontal intervals oriented from the left to the right, or as the combs

   ![Combs](https://example.com/combs.png)

   whose teeth represent elements of $A$, we depict cyclically ordered sets $C$ as circumferences of counterclockwise oriented circles in the plane $\mathbb{R}^2$. We call such pictures *pancakes*:

   ![Pancake](https://example.com/pancake.png)

   (1)

   with the spikes representing elements of $C$.

   We will need to extend the notation of Definition [4] as follows. For finite ordered sets $A_1, \ldots, A_k$ we denote by $A_1 \cdots A_k$, their union with the unique order in which $A_1 < \cdots < A_k$,
and by \([A_1 \cdots A_k]\) the corresponding cyclically ordered set. If e.g. \(A_1 = \{u\}\) we abbreviate \([\{u\} A_2]\) by \([uA_2]\), &c.

**Remark 2.** While \(A_1 A_2 \neq A_2 A_1\) when both \(A_1\) and \(A_2\) are non-empty, \([A_1 A_2]\) always equals \([A_2 A_1]\). Notice that \([A'] = [A'']\) for some finite ordered sets \(A', A''\) if and only if there are ordered sets \(A_1\) and \(A_2\) such that \(A' = A_1 A_2\) and \(A'' = A_2 A_1\).

Let \(\text{Fin}\) (resp. \(\text{Cyc}\)) denote the category of finite (resp. cyclically ordered finite) sets and their isomorphisms. Recall\(^8\) that the pieces \(\mathcal{P}(\langle S \rangle)\) of an (ordinary) cyclic operad \(\mathcal{P}\) are indexed by finite sets \(S \in \text{Fin}\), and their structure operations are

\[
(2) \quad u \circ v : \mathcal{P}(\langle S' \rangle) \otimes \mathcal{P}(\langle S'' \rangle) \to \mathcal{P}(\langle (S' \cup S'') \setminus \{u, v\}\rangle),
\]

where \(S'\) and \(S''\) are disjoint finite sets and \(u \in S', v \in S''\).

A non-\(\Sigma\) cyclic operad \([19, II.5.1]\) \(\mathcal{P}\) has its components \(\mathcal{P}(\langle C \rangle)\) indexed by cyclically ordered sets \(C \in \text{Cyc}\), and structure operations

\[
(3) \quad u \circ v : \mathcal{P}(\langle C' \rangle) \otimes \mathcal{P}(\langle C'' \rangle) \to \mathcal{P}(\langle (C' \cup C'') \setminus \{u, v\}\rangle)
\]

of the same type as \((2)\). The codomain of \((3)\) however does not make sense unless we specify a cyclic order on the set

\[
(4) \quad (C' \cup C'') \setminus \{u, v\} \tag{4}
\]

It is given by the *pancake merging* at \(\{u, v\}\) as follows.

Assume that the cyclic order of \(C'\) is represented by the linear order

\[
a_1 < a_2 < \cdots < a_k < u
\]

and the cyclic order of \(C''\) by

\[
v < b_1 < b_2 < \cdots < b_l.
\]

Then we equip \((4)\) with the cyclic order is represented by

\[
a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_l.
\]

Notice that we allow the case when \(C' = \{u\}\) and \(C'' = \{v\}\), then \((4)\) is an empty cyclically ordered set. In the pancake world, \((4)\) is realized by merging two pancakes into one:

\[
(5)
\]

Modular operads \([10, Section 2]\), \([19, Section II.5.3]\) have, besides \((2)\), also the contractions

\[
(6) \quad \xi_{uv} : \mathcal{P}(\langle S; g \rangle) \to \mathcal{P}(\langle S \setminus \{u, v\}; g + 1\rangle)
\]

where \(S \in \text{Fin}\) and \(u, v \in S\) are distinct elements. It is therefore natural to expect that our conjectural non-\(\Sigma\) modular operad has pieces \(\mathcal{P}(\langle C; g \rangle)\) indexed by \(C \in \text{Cyc}\), \(g \in \mathbb{N}\) and, besides \((3)\), the contractions

\[
(6) \quad \xi_{uv} : \mathcal{P}(\langle C; g \rangle) \to \mathcal{P}(\langle C \setminus \{u, v\}; g + 1\rangle).
\]

---

\(^8\)See the Appendix.

\(^9\)We follow the convention used in \([19]\) and distinguish non-\(\Sigma\) versions of operads by *underlying*.

\(^10\)Here \(C' \cup C''\) is the union of the corresponding underlying sets; we will use this kind of shorthand freely.

\(^11\)Here \(\mathcal{P}\) has an additional grading by the genus \(g \in \mathbb{N}\) which is irrelevant now.
There is only one natural cyclic order on the subset $C \setminus \{u, v\}$ of $C$, namely the restriction of the cyclic order of $C$, so we are forced to equip \([4]\) with this cyclic order. The following example shows that it does not work.

**Example 3.** Consider ordered sets $X$, $Y$ and $Z$, and distinct symbols $u'$, $u''$, $v'$ and $v''$. Let $x \in P([X'Zu'; g'])$ and $y \in P([Y'u''v''; g''])$ be arbitrary elements. According to the definition of the cyclic order of \([4]\) used in \([3]\),

$$(x_{u'\circ_{v''}} y) \in P([Z'XY u''; g' + g'']) \quad \text{and} \quad (x_{u'\circ_{u''}} y) \in P([X'Zv''Y; g' + g'']),$$

thus

$$\xi_{u'u''}(x_{u'\circ_{v''}} y) \in P([XYZ]; g' + g'' + 1) \quad \text{while} \quad \xi_{u'u'}(x_{u'\circ_{u''}} y) \in P([XYZ]; g' + g'' + 1).$$

The standard exchange rule in Definition \([38]\)(iv) between compositions and contractions in a modular operad must of course hold also in the non-$\Sigma$ case, therefore

\begin{equation}
\xi_{u'u''}(x_{u'\circ_{v''}} y) = \xi_{u'u'}(x_{u'\circ_{u''}} y).
\end{equation}

But this is not possible. If $X, Y, Z \neq \emptyset$, the cyclically ordered sets $[XYZ]$ and $[XZY]$ are non-isomorphic, so \([3]\) compares elements of different spaces. This quandary will be resolved by introducing multicyclically ordered sets.

We believe that Figure 3 helps to understand the situation. It shows (from the left to the right the pancake representing the cyclically ordered set $C' = [X'Zu']$, the one representing $C'' = [Y'u''v'']$ and two realizations of the pancakes representing the merging of $C' \cup C''$ at $\{v', v''\}$ resp. the merging $C' \cup C''$ at $\{u', u''\}$. The meaning of the dotted lines will be explained in Example 10.

2. Multicyclic orders

In this section we introduce multicyclically ordered sets as objects appropriately indexing the pieces of non-$\Sigma$ modular operads. The definition is simple:

**Definition 4.** A multicyclic order on a set $S$ is a disjoint decomposition $S = C_1 \cup \cdots \cup C_b$ of $S$ into $b > 0$ possibly empty cyclically ordered sets.
A morphism $\sigma : S' = C'_1 \cup \cdots \cup C'_b \rightarrow S'' = C''_1 \cup \cdots \cup C''_c$ is a map $\sigma : S' \rightarrow S''$ of the underlying sets such that $\sigma(C'_i) \subset C''_{i_\sigma}$ for each $1 \leq i \leq b'$ and some $1 \leq i_\sigma \leq b''$, and that the restriction $f|_{C'_i} : C'_i \rightarrow C''_{i_\sigma}$ preserves the cyclic orders.

We denote by $\text{MultCyc}$ the category of multicyclically ordered sets and their isomorphisms. It contains the full subcategory $\text{Cyc}$ of cyclically ordered finite sets which embedded as multicyclically ordered sets with $b = 1$. Notice that a given set has infinitely many multicyclic orders, but the geometricity that we introduce in Definition 6 below allows only finite number of them.

**Definition 5.** A non-$\Sigma$ modular module is a functor

$$E : \text{MultCyc} \times \mathbb{N} \rightarrow \mathbb{M},$$

where $\mathbb{M}$ is our fixed symmetric monoidal category and the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ are considered as a discrete category.

Explicitly, a non-$\Sigma$ modular module is a rule $(S, g) \mapsto E((S; g))$ that assigns to each multicyclically ordered $S$ and $g \in \mathbb{N}$ an object $E((S; g)) \in \mathbb{M}$, together with a functorial family of maps $E((\sigma)) : E((S'; g)) \rightarrow E((S''; g))$ defined for each isomorphism $\sigma : S' \rightarrow S''$ of multicyclically ordered sets. If $S = C_1 \cup \cdots \cup C_b$, we will sometimes write more explicitly $E((C_1, \ldots, C_b; g))$ instead of $E((S; g))$. We call $g$ the (operadic) genus.

**Definition 6.** We call a couple $(S, g) \in \text{MultCyc} \times \mathbb{N}$ with $S = C_1 \cup \cdots \cup C_b$ geometric if

$$G := \frac{1}{2}(g - b + 1) \in \mathbb{N}.$$  

A non-$\Sigma$ modular module $E$ is geometric, if $E((S; g)) \neq 0$ implies that $(S, g)$ is geometric.

Geometricity therefore means that $g - b + 1$ is an even non-negative integer. We will call $G$ defined in (8) the geometric genus and $b$ the number of boundaries. The reasons for this terminology will be clarified later in Section 6, see also Remark 12.

**Example 7.** For $g \leq 3$, only the following components of a geometric non-$\Sigma$ modular module can be nontrivial: $E((C_1; 0))$, $E((C_1, C_2; 1))$, $E((C_1, C_2, C_3; 2))$ and $E((C_1, C_2, C_3, C_4; 3))$ in the geometric genus 0, and $E((C_1; 2))$, $E((C_1, C_2; 3))$ in geometric genus 1.

Let $C$ be a cyclically ordered set. For $u, v \in C$, the set $C \setminus \{u, v\}$ has an obvious induced cyclic order given by the restriction of the original one. It naturally decomposes as

$$C \setminus \{u, v\} = I_1 \cup I_2$$

where $I_1, I_2$ are the open intervals whose boundary points are $u$ and $v$, considered with the induced cyclic orders. Notice that if $u$ and $v$ are adjacent in the cyclic order, one or both of $I_1, I_2$ may be empty.

If we distribute the elements of $C$ around the circumference of a pancake so that their cyclic order is induced by the (say) counterclockwise orientation of the oven plate, then (9) is realized by cutting the pancake, with the knife running through $u$ and $v$, as in:
Pancake cutting together with pancake merging (5) induce two basic operations on multicyclically ordered sets. The merging starts with two multicyclically ordered sets

\[ S' = C'_1 \cup \cdots \cup C'_{b'} \quad \text{and} \quad S'' = C''_1 \cup \cdots \cup C''_{b''}, \]

whose underlying sets \( S' \) and \( S'' \) are disjoint. For \( u \in C'_i \) and \( v \in C''_j \), let

\[ S' \cup S'' \setminus \{u, v\} \]

be the multicyclically ordered set whose underlying set is \( S' \cup S'' \setminus \{u, v\} \), decomposed as

\[
(10) \quad C'_1 \cup \cdots \cup \widehat{C'_i} \cup \cdots \cup C'_{b'} \cup C''_1 \cup \cdots \cup \widehat{C''_j} \cup \cdots \cup C''_{b''}, \quad \left( C'_i \cup C''_j \setminus \{u, v\} \right),
\]

where \( \widehat{\cdot} \) indicates that the corresponding term has been omitted, and \( \left( C'_i \cup C''_j \setminus \{u, v\} \right) \) is cyclically ordered as in (4).

Let \( S = C_1 \cup \cdots \cup C_b \) is a multicyclically ordered set, \( u \in C_i \), \( v \in C_j \). If \( i \neq j \), we define the cut

\[
(11) \quad S \setminus \{u, v\}
\]

to be the multicyclically ordered set whose underlying set \( S \setminus \{u, v\} \) decomposed as

\[
(12) \quad S \setminus \{u, v\} = C_1 \cup \cdots \cup \widehat{C_i} \cup \cdots \cup C_j \cup \cdots \cup C_b \cup \left( C_i \cup C_j \setminus \{u, v\} \right)
\]

with \( C_i \cup C_j \setminus \{u, v\} \) cyclically ordered as in (4). If \( i = j \), we define (11) as the multicyclically ordered set given by the decomposition

\[
(13) \quad S \setminus \{u, v\} = C_1 \cup \cdots \cup \widehat{C_i} \cup \cdots \cup C_b \cup \left( C_i \setminus \{u, v\} \right),
\]

where \( C_i \setminus \{u, v\} \) is the union of two multicyclically ordered sets as in (4). Notice that the number of cyclically ordered components of (10) is \( b' + b'' - 1 \), of (12) is \( b - 1 \) and of (13) is \( b + 1 \).

3. Biased definition of non-$\Sigma$ modular operads.

We formulate a definition of non-$\Sigma$ modular operads biased towards the bilinear operations \( u \circ_v v \) and contractions \( \xi_{uv} \). Recall that \( \mathbb{M} \) denotes our basic symmetric monoidal category; let \( \tau \) be its commutativity constraint. Regarding multicyclically ordered sets, we use the notation introduced in §2.

**Definition 8.** A non-$\Sigma$ modular operad in \( \mathbb{M} = (\mathbb{M}, \otimes, 1) \) is a non-$\Sigma$ modular module

\[
\mathcal{P} = \{ \mathcal{P}(\langle S; g \rangle) \in \mathbb{M}; (S, g) \in \text{MultCyc} \times \mathbb{N} \}
\]

together with morphisms (compositions)

\[
(14) \quad u \circ_v : \mathcal{P}(\langle S'; g' \rangle) \otimes \mathcal{P}(\langle S''; g'' \rangle) \to \mathcal{P}(\langle S' \cup S'' \setminus \{u, v\}; g' + g'' \rangle)
\]

defined for arbitrary disjoint multicyclically ordered sets \( S' \) and \( S'' \) with elements \( u \in S' \), \( v \in S'' \) of their underlying sets, and contractions

\[
(15) \quad \xi_{uv} = \xi_{vu} : \mathcal{P}(\langle S; g \rangle) \to \mathcal{P}(\langle S \setminus \{u, v\}; g + 1 \rangle)
\]
given for any multicyclically ordered set \( S \) and distinct elements \( u, v \in S \) of its underlying set. These data are required to satisfy the following axioms.
(i) For $S', S''$ and $u, v$ as in (14), one has the equality

$$u \circ_v = v \circ_u \tau$$

of maps $\mathcal{P}((S'; g') \otimes \mathcal{P}((S''; g'')) \to \mathcal{P}((S' \cup S'' \setminus \{u, v\}; g' + g'')).$  

(ii) For mutually disjoint multicyclically ordered sets $S_1, S_2, S_3$, and $a \in S_1, b, c \in S_2, b \neq c, d \in S_3$, one has the equality

$$a \circ_b (id \otimes c \circ_d) = c \circ_d (a \circ_b \otimes id)$$

of maps $\mathcal{P}((S_1; g_1)) \otimes \mathcal{P}((S_2; g_2)) \otimes \mathcal{P}((S_3; g_3)) \to \mathcal{P}((S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\}; g_1 + g_2 + g_3)).$

(iii) For a multicyclically ordered set $S$ and mutually distinct $a, b, c, d \in S$, one has the equality

$$\xi_{ab} \xi_{cd} = \xi_{cd} \xi_{ab}$$

of maps $\mathcal{P}((S; g)) \to \mathcal{P}((S \setminus \{a, b, c, d\}; g + 2)).$

(iv) For multicyclically ordered sets $S', S''$ and distinct $a, c \in S', b, d \in S''$, one has the equality

$$\xi_{ab} c \circ_d = \xi_{cd} a \circ_b$$

of maps $\mathcal{P}((S' \cup S''; g)) \to \mathcal{P}((S' \cup S'' \setminus \{a, b, c, d\}; g + 1)).$

(v) For multicyclically ordered sets $S', S''$ and mutually distinct $a, c, d \in S', b \in S''$, one has the equality

$$a \circ_b (\xi_{cd} \otimes id) = \xi_{cd} a \circ_b$$

of maps $\mathcal{P}((S' \cup S''; g)) \to \mathcal{P}((S' \cup S'' \setminus \{a, b, c, d\}; g + 1)).$

(vi) For arbitrary isomorphisms $\rho : S' \to D'$ and $\sigma : S'' \to D''$ of multicyclically ordered sets and $u, v$ as in (14), one has the equality

$$\mathcal{P}((\rho|_{S' \setminus\{u\}} \cup \sigma|_{S'' \setminus\{v\}})) \circ_v = \rho|_u \circ_{\sigma(v)} (\mathcal{P}(\rho) \otimes \mathcal{P}(\sigma))$$

of maps $\mathcal{P}((S'; g') \otimes \mathcal{P}((S''; g'')) \to \mathcal{P}((D' \cup D'' \setminus \{\rho(u), \sigma(v)\}; g' + g'')).$

(vii) For $S, u, v$ as in (13) and an isomorphism $\rho : S \to D$ of multicyclically ordered sets, one has the equality

$$\mathcal{P}(\rho|_{D \setminus\{\rho(u), \rho(v)\}}) \xi_{ab} = \xi_{\rho(u)\rho(v)} \mathcal{P}(\rho)$$

of maps $\mathcal{P}((S; g)) \to \mathcal{P}((S \setminus \{\rho(u), \rho(v)\}; g + 1)).$

Remark 9. While $\xi_{uv} = \xi_{vu}$, the behavior of the $u \circ_v$-operation under the interchange $u \leftrightarrow v$ is given by axiom (i). Axioms (ii)–(v) are interchange rules between $u \circ_v$'s and the contractions, while the remaining two axioms describe how the structure operations behave under automorphisms. In (vi) and (vii) one sees the restrictions of automorphisms of multicyclically ordered sets. It is clear that they are automorphisms of the corresponding multicyclically ordered subsets.

Example 10. With the definition of non-$\Sigma$ modular operads given above, both sides of (7) belong to the same space, namely to $\mathcal{P}(\{XY, |Z|; g' + g'' + 1\})$. The problem risen in Example 8 is thus resolved by cutting the two rightmost pancakes in Figure 8 along the dotted lines.

Definition 11. A non-$\Sigma$ modular operad $\mathcal{P}$ is geometric, if its underlying non-$\Sigma$ modular module is geometric.
Notice that the $u \circ v$-operations always preserve the geometric genus (8). The contractions $\xi_{uv}$ preserve it if $u, v$ in (13) belong to the same component of the multicyclically ordered set $S$, and raise it by 1 if they belong to different components of $S$. Therefore each non-$\Sigma$ modular operad $\mathcal{P}$ contains a maximal geometric suboperad.

From this point on, we assume that all non-$\Sigma$ modular operads are geometric. With this assumption, the only nontrivial components of $\mathcal{P}$ in (operadic) genus 0 are $\mathcal{P}((C; 0))$, where $C$ is a cyclically ordered set. It is simple to show that the collection

$$\Box \mathcal{P} := \{ \mathcal{P}((C; 0)); \ C \text{ is cyclically ordered} \}$$

together with $u \circ v$ operations (14) is a non-$\Sigma$ cyclic operad. So we have the forgetful functor

(16) $$\Box : \text{NsModOp} \to \text{NsCycOp}. $$

In Section 3 we construct its left adjoint $\text{Mod} : \text{NsCycOp} \to \text{NsModOp}$. One also has the forgetful functor (the desymmetrization) $\text{Des} : \text{ModOp} \to \text{NsModOp}$ given by

$$\text{Des}(\mathcal{P})((S; g)) := \mathcal{P}(\mathcal{P}(S)), $$

where $S$ is the underlying set of the multicyclically ordered set $S$. It has the left adjoint $\text{Sym} : \text{NsModOp} \to \text{ModOp}$ given by

(17) $$\text{Sym}(\mathcal{P})(\mathcal{P}(S; g)) := \bigoplus \mathcal{P}(\mathcal{P}(S; g)), $$

where the coproduct runs over all multicyclically ordered sets whose underlying set equals $S$. Notice that the geometricity guarantees that the coproduct in (17) is finite. We call $\text{Sym}(\mathcal{P})$ the symmetrization of the non-$\Sigma$ modular operad $\mathcal{P}$.

**Remark 12.** Assuming the geometricity, the category of non-$\Sigma$ cyclic operads is isomorphic to the full subcategory of non-$\Sigma$ modular operads $\mathcal{P}$ such that $\mathcal{P}(\mathcal{P}(S; g)) = 0$ for $g \geq 1$. Without the geometricity assumption, this natural property that obviously holds for ordinary modular operads, will not be true. A ‘geometric’ explanation of the geometricity will be given in Remark 33.

**Example 13.** Assume that the basic monoidal category is the cartesian category $\text{Set}$ of sets and let $*$ be a fixed one-point set. Then one has the terminal non-$\Sigma$ modular operad $\ast_M$ with

$$\ast_M((S; g)) := \ast \text{ for each geometric } (S, g) \in \text{MultCyc} \times \mathbb{N},$$

with all structure operations the unique maps $\ast \to \ast$ or $\ast \times \ast \to \ast$.

## 4. UN-BIASED DEFINITION OF NON-$\Sigma$ MODULAR OPERADS.

We give an alternative definition of non-$\Sigma$ modular operads as algebras over a certain monad of decorated graphs representing their pasting schemes, thus extending the table in [14, Figure 14]. This way of defining various types of operads is standard, see e.g. [10, §2.20], [13, II.1.12, II.5.3] or [14, Theorem 40], so we only emphasize the particular features of the non-$\Sigma$ modular case. We start by recalling a definition of graphs suggested by Kontsevich and Manin [13] commonly used in operad theory.\(^{12,13}\)

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\(^{12}\)I.e. a multicyclically ordered set with one component.

\(^{13}\)More refined notions of graphs already exist, see e.g. [3, Part 4], but we will not need them here.
Definition 14. A graph $\Gamma$ is a finite set $\text{Flag}(\Gamma)$ (whose elements are called flags or half-edges) together with an involution $\sigma$ and a partition $\lambda$.

The vertices $\text{Vert}(\Gamma)$ of a graph $\Gamma$ are the blocks of the partition $\lambda$. The edges $\text{edge}(\Gamma)$ are pairs of flags forming a two-cycle of $\sigma$ relative to the decomposition of a permutation into disjoint cycles. The legs $\text{Leg}(\Gamma)$ are the fixed points of $\sigma$.

We denote by $\text{Leg}(v)$ the flags belonging to the block $v$ or, in common speech, half-edges adjacent to the vertex $v$. The cardinality of $\text{Leg}(v)$ is the valency of $v$. We say that two flags $x, y \in \text{Flag}(\Gamma)$ meet if they belong to the same block of the partition $\lambda$. In plain language, this means that they share a common vertex.

Definition 15. A non-$\Sigma$ modular graph $\Gamma$ is a connected graph as above that has, moreover, the following local structure at each vertex $v \in \text{Vert}(\Gamma)$:

(i) a multicyclic order of the set $\text{Leg}(v)$ of half-edges adjacent to $v$ and
(ii) a genus $g_v \in \mathbb{N}$.

We will denote by $\text{Leg}(v)$ the set $\text{Leg}(v)$ with the given multicyclic order and by $b_v$ the number of cyclically ordered subsets in the corresponding decomposition. We say that $\Gamma$ is geometric if at each $v \in \text{Vert}(\Gamma)$,

$$G_v := \frac{1}{2}(g_v + 1 - b_v) \in \mathbb{N}.$$  

The local structure of $\Gamma$ at a vertex $v$ with $\text{Leg}(v) = C_1 \cup \cdots \cup C_b$ can be visualized as a bouquet of sunflowers connected to $v$ by their stalks, the petals of their blossoms representing elements of the individual cyclically ordered sets $C_1, \ldots, C_b$. The flowers may freely move around the common vertex and each may rotate around its stalk, see Figure 4. An alternative representation is an umbel, the pictures that can be found e.g. at en.wikipedia.org/wiki/Umbel clearly convey the idea.

Crucially, the local structure of a non-$\Sigma$ modular graph induces the same kind of structure on its external legs:

Proposition 16. The set of legs of a non-$\Sigma$ modular graph has an induced multicyclic order.

Proof. By an oriented edge cycle in $\Gamma$ we understand a sequence

$$(a_1, b_1), (a_2, b_2), \ldots, (a_s, b_s),$$
where $a_i, b_i$ are half-edges such that $\sigma(a_i) = b_i$ for $1 \leq i \leq s$. So, if $a_i \neq b_i$, $(a_i, b_i)$ is an oriented edge, if $a_i = b_i$ it is a leg of $\Gamma$. We require that $a_i$ is the immediate successor of $b_{i-1}$, $1 < i \leq s$, and that $a_1$ is the immediate successor of $b_s$ in the cyclically ordered set to which these elements belong.

If this cyclically ordered set consists of $b_{i-1}$ resp. $b_s$ only, then of course $a_i = b_{i-1}$ resp. $a_1 = b_s$ so the cycle runs back along the same half-edge. We also assume that each ordered couple $(a_i, b_i)$ occurs exactly once so that the cycle does not run twice along itself. This does not exclude that $(a_i, b_i) = (b_j, a_j)$, i.e. that the cycle runs twice along the same edge, but each time in different direction.

Let $\{X_1, \ldots, X_b\}$ be the set of oriented edge cycles of $\Gamma$ and
\begin{equation}
C_i := \{e \in \text{Leg}(\Gamma); (e, e) \in X_i, 1 \leq i \leq b\}.
\end{equation}
Then $C_1, \ldots, C_b$ is the required disjoint decomposition of $\text{Leg}(\Gamma)$ and each individual $C_i$ is cyclically ordered by the cyclic orientation of the corresponding edge cycle. \hfill \Box

We will denote by $\text{Leg}(\Gamma)$ the set $\text{Leg}(\Gamma)$ with the multicyclic order of Proposition [19] and by $b(\Gamma)$ the number of cyclically ordered sets in the decomposition of $\text{Leg}(\Gamma)$. The (operadic) genus of a non-$\Sigma$ modular graph is defined by the usual formula [19] (II.5.28)
\[ g(\Gamma) := b_1(\Gamma) + \sum_{v \in \text{Vert}(\Gamma)} g(v), \]
where $b_1(\Gamma)$ is the first Betti number of $\Gamma$, i.e. the number of independent circuits of $\Gamma$. We leave as an exercise to prove

**Proposition 17.** If $\Gamma$ is geometric then $(\text{Leg}(\Gamma), g) \in \text{MultCyc} \times \mathbb{N}$ is geometric, too, i.e.
\[ G(\Gamma) := \frac{1}{2}(g(\Gamma) + b(\Gamma) - 1) \in \mathbb{N}. \]

A morphism $f : \Gamma_0 \to \Gamma_1$ of graphs is given by a permutation of vertices, followed by a contraction of some edges of the graph $\Gamma_0$, leaving the legs untouched; a precise definition can be found in [19], II.5.3. Assume that $\Gamma_0$ and $\Gamma_1$ bear a non-$\Sigma$ modular structure. It is simple to see that the non-$\Sigma$ modular structure of $\Gamma_0$ induces via $f$ a non-$\Sigma$ modular structure on $\Gamma_1$. We say that $f : \Gamma_0 \to \Gamma_1$ is a morphism of non-$\Sigma$ modular graphs if this induced structure on $\Gamma_1$ coincides with the given one.

For a multicyclically ordered set $S$, let $\Gamma((S; g))$ be the category whose objects are pairs $(\Gamma, \rho)$ consisting of a non-$\Sigma$ modular graph $\Gamma$ of genus $g$ and an isomorphism $\rho : \text{Leg}(\Gamma) \to S$ of multicyclically ordered sets. Morphisms of $\Gamma((S; g))$ are morphisms as above preserving the labelling of the legs. The category $\Gamma((S; g))$ has a terminal object $*_S.g$, the ‘non-$\Sigma$ modular corolla’ with no edges, one vertex $v$ of genus $g$ and legs labeled by $S$.

For a non-$\Sigma$ modular module $E$ and a non-$\Sigma$ modular graph $\Gamma$, one forms the unordered product [19], Definition II.1.58
\begin{equation}
E(\Gamma) := \bigotimes_{v \in \text{Vert}(\Gamma)} E((\text{Leg}(v); g_v)).
\end{equation}
Let $\text{Iso}\Gamma((S; g))$ denote the subcategory of isomorphisms in $\Gamma((S; g))$. The correspondence $\Gamma \ni E(\Gamma)$ extends to a functor from the category $\text{Iso}\Gamma((S; g))$ to $\mathcal{M}$. We define the endofunctor [October 12, 2014]
$\mathbb{M}: \text{NsModMod} \to \text{NsModMod}$ on the category of non-$\Sigma$ modular modules as the colimit

$$\mathbb{M}(E)((S; g)) := \colim_{\Gamma \in \text{Iso}(S; g)} E(\Gamma).$$

For each non-$\Sigma$ modular graph $\Gamma \in \Gamma((S; g))$ one has the coprojection

$$\iota_\Gamma : E(\Gamma) \to \mathbb{M}(E)((S, g)).$$

In particular, for the corolla $*_{S, g}$, one gets the morphism

$$\iota_{*_{S, g}} : E(S; g) = E(*_{S, s}) \to \mathbb{M}(E)((S; g)).$$

Given a non-$\Sigma$ modular module $E$, the second iterate $(\mathbb{M} \circ \mathbb{M})(E)$ is a colimit of non-$\Sigma$ modular graphs whose vertices are decorated by non-$\Sigma$ modular graphs decorated by $E$, i.e. a colimit of ‘nested’ graphs. Forgetting the nests gives rise to a natural transformation $\mu : \mathbb{M} \circ \mathbb{M} \to \mathbb{M}$ (the multiplication) while morphisms (21) form a natural transformation $\nu : \text{id} \to \mathbb{M}$ (the unit).

Precisely as in [10, §2.17] or in the proof of [19, Theorem II.5.10] one shows that $\mathbb{M} = (\mathbb{M}, \mu, \nu)$ is a monad on the category $\text{NsModMod}$. We have the following theorem/definition.

**Theorem 18.** Non-$\Sigma$ modular operads are algebras for the monad $\mathbb{M} = (\mathbb{M}, \mu, \nu)$.

**Proof.** A straightforward modification of the proof of [19, Theorem II.5.41]. $\square$

A non-$\Sigma$ modular operad is thus a non-$\Sigma$ modular module $\mathcal{P}$ equipped with a morphism

$$(22) \quad \alpha : \mathbb{M}(\mathcal{P}) \to \mathcal{P}$$

of non-$\Sigma$ modular modules having the usual properties [19, Definition II.1.103]. The following claim expresses a standard property of algebras over a monad.

**Proposition 19.** The multiplication $\mu : (\mathbb{M} \circ \mathbb{M})(E) \to \mathbb{M}(E)$ makes $\mathbb{M}(E)$ an algebra for the monad $\mathbb{M}$. It is the free non-$\Sigma$ modular operad on the non-$\Sigma$ modular module $E$.

Thus $\mathbb{M}(-)$ interpreted as a functor $\text{NsModMod} \to \text{NsModOp}$ is the left adjoint to the obvious forgetful functor $\mathcal{F} : \text{NsModOp} \to \text{NsModMod}$.

**Remark 20.** The biased structure operations of the free operad $\mathbb{M}(E)$ are induced by the grafting of the underlying graphs. For graphs $\Gamma', \Gamma''$ with legs $u \in \text{Leg}(\Gamma'), \ v \in \text{Leg}(\Gamma'')$, one has the graph $\Gamma'_u \circ_v \Gamma''$ obtained by grafting the free end of the half-edge $u$ to the free end of $v$. Formally, $\Gamma'_u \circ_v \Gamma''$ is defined by

$$\text{Flag}(\Gamma'_u \circ_v \Gamma'') := \text{Flag}(\Gamma') \cup \text{Flag}(\Gamma''),$$

the partition of $\text{Flag}(\Gamma'_u \circ_v \Gamma'')$ being the union of the partitions of $\text{Flag}(\Gamma')$ and $\text{Flag}(\Gamma'')$, the involution $\sigma$ on $\text{Flag}(\Gamma'_1 \circ_v \Gamma_2)$ agreeing with the involution $\sigma'$ of $\text{Flag}(\Gamma')$ on $\text{Flag}(\Gamma') \setminus \{u\}$, with the involution $\sigma''$ of $\text{Flag}(\Gamma'')$ on $\text{Flag}(\Gamma'') \setminus \{v\}$, and $\sigma(u) := v$. The contraction $\xi_{uv}(\Gamma)$ is, for $u, v \in \text{Flag}(\Gamma)$, defined similarly.
Example 21. Assume that \( E \) is a geometric non-\( \Sigma \) modular module such that \( E((S; g)) = 0 \) for \( g \geq 1 \). In other words, the only nontrivial pieces of \( E \) are \( E((C; 0)) \), where \( C \) is a cyclically ordered set. The elements of the free non-\( \Sigma \) modular operad \( \mathbb{M}(E) \) are the equivalence classes of decorated graphs whose vertices \( v \) are umbels with one blossom, i.e. the pancakes (\( \square \)) with the spikes representing the half-edges in the cyclically ordered set \( \text{Leg}(v) \). Free operads of this form will play a central rôle in our proof of (25b).

A very particular case is the geometric non-\( \Sigma \) modular module \( \ast \) in \( \text{Set} \) defined by

\[
\ast((S; g)) := \begin{cases} * & \text{if } g = 0 \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}
\]

In Section 3 we visualize the generators of \( \mathbb{M}(\ast) \) via toothed wheels (13).

For a non-\( \Sigma \) modular operad \( \mathcal{P} \) and \( \Gamma \in \Gamma((S; g)) \) we will call the composition

\[
\alpha_{\Gamma} : \mathcal{P}(\Gamma) \overset{\iota_{\Gamma}}{\longrightarrow} \mathbb{M}(\mathcal{P})((S; g)) \overset{\alpha}{\longrightarrow} \mathcal{P}((S; g))
\]

of (22) with (21) the contraction along the graph \( \Gamma \). The following analog of [19, Theorem II.5.42] claims that the contractions are part of a functor:

**Theorem 22.** A geometric non-\( \Sigma \) modular module \( \mathcal{P} \) is a non-\( \Sigma \) modular operad if and only if the correspondence \( \Gamma \mapsto \mathcal{P}(\Gamma) \) is, for each geometric \((S, g) \in \text{MultCyc} \times \mathbb{N} \), an object part of a functor \( \alpha : \Gamma((S; g)) \rightarrow \mathbb{M} \) extending (23).

**Proof.** A simple modification of the proof of [19, Theorem II.5.42]. \( \square \)

5. Modular envelopes

Modular envelopes of (ordinary) cyclic operads were introduced under the name modular operadic completions by the author in [15, Definition 2]. The modular envelope functor \( \text{Mod} : \text{CycOp} \rightarrow \text{ModOp} \) is the left adjoint to the obvious forgetful functor \( \square : \text{ModOp} \rightarrow \text{CycOp} \). Analogously we define the modular envelope of a non-\( \Sigma \) cyclic operad via the left adjoint to the non-\( \Sigma \) version \( \square \) of the forgetful functor considered in Section 3. We of course need to prove that this left adjoint exists:

**Proposition 23.** The forgetful functor (14) has a left adjoint \( \text{Mod} : \text{NsCycOp} \rightarrow \text{NsModOp} \).

**Proof.** The proof will be transparent if we assume that the objects of the basic category \( \mathbb{M} \) have elements. Then we take the free non-\( \Sigma \) modular operad \( \mathbb{M}(FP) \) generated by the non-\( \Sigma \) cyclic collection \( FP \) placed in the operadic genus 0 and define \( \text{Mod}(\mathcal{P}) \) as the quotient

\[
\text{Mod}(\mathcal{P}) := \mathbb{M}(FP)/I
\]

of \( \mathbb{M}(FP) \) by the operadic ideal \( I \) generated by

\[
x_u \circ_{\mathcal{P}} y = x_u \circ_{\mathbb{M}} y,
\]

where \( u \circ_{\mathcal{P}} \) resp. \( u \circ_{\mathbb{M}} \) are the \( u \circ_v \) -operations in \( \mathcal{P} \) resp. \( \mathbb{M} \), \( x \in \mathcal{P}(C') \), \( y \in \mathcal{P}(C'') \), \( u \in C' \) and \( v \in C'' \) for some disjoint cyclically ordered sets \( C', C'' \).

If objects of \( \mathbb{M} \) do not have elements, we replace the quotient (24) by an obvious colimit.

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Definition 24. We call $\text{Mod}(\mathcal{P})$ the non-$\Sigma$ modular envelope of the non-$\Sigma$ cyclic operad $\mathcal{P}$.

Informally, $\text{Mod}(\mathcal{P})$ is obtained by adding to $\mathcal{P}$ the results of contractions, splinting the cyclically ordered groups of inputs if the contraction takes place within the same group. This process is nicely visible at Doubek’s construction of the modular envelope of the operad for associative algebras [6].

If the basic category $\mathcal{M}$ is $\text{Set}$, the category of cyclic (resp. non-$\Sigma$ cyclic, resp. modular, resp. non-$\Sigma$ modular) operads has a terminal object $\ast_{\text{C}}$ (resp. $\ast_{\text{C},\Sigma}$, resp. $\ast_{\text{M}}$ resp. $\ast_{\text{M},\Sigma}$) consisting of a chosen one-point set $\ast$ in each arity; the terminal non-$\Sigma$ modular operad $\ast_{\text{M}}$ has already been mentioned in Example 13. The main result of this section is:

Theorem 25. The modular envelope of the terminal operad in the category of cyclic (resp. non-$\Sigma$ cyclic) $\text{Set}$-operads is the terminal modular (resp. the terminal non-$\Sigma$ modular) $\text{Set}$-operad, in formulas:

\begin{align*}
\text{(25a)} & \quad \text{Mod}(\ast_{\text{C}}) \cong \ast_{\text{M}} \quad \text{and} \\
\text{(25b)} & \quad \text{Mod}(\ast_{\text{C},\Sigma}) \cong \ast_{\text{M}}.
\end{align*}

The cyclic operad $\text{Com}$ for commutative associative algebras is the linear span of the terminal cyclic $\text{Set}$-operad, that is $\text{Com} = \text{Span}(\ast_{\text{C}})$. The following immediate corollary of Theorem 25 was stated without proof in [15, page 382].

Theorem 26 ([15]). The modular envelope $\text{Mod}(\text{Com})$ of the cyclic operad for commutative associative algebras is the linear span of the terminal modular set-operad, i.e.

$$\text{Mod}(\text{Com})(\langle S; g \rangle) = k, \quad (S, g) \in \text{Fin} \times \mathbb{N},$$

the maps $\text{Mod}(\text{Com})(\langle \sigma \rangle)$ induced by morphisms in $\text{Fin}$ are the identities, all $u \circ v$-operations are the canonical isomorphisms $k \otimes k \xrightarrow{\cong} k$ and all contractions $\xi_{uv}$ are the identities.

Proof. Both $\text{Span}(\langle \quad \rangle)$ and $\text{Mod}(\langle \quad \rangle)$ are the left adjoints to forgetful functors that commute with each other, so

$$\text{Span}(\text{Mod}(\mathcal{S})) \cong \text{Mod}(\text{Span}(\mathcal{S}))$$

for each cyclic operad $\mathcal{S}$ in $\text{Set}$. \hfill \Box

Since the non-$\Sigma$ cyclic operad $\text{Ass}$ is the linear span of the terminal non-$\Sigma$ cyclic $\text{Set}$-operad $\ast_{\text{C},\Sigma}$, we likewise obtain from Theorem 25:

Theorem 27. The non-$\Sigma$ modular envelope $\text{Mod}(\text{Ass})$ of the non-$\Sigma$ cyclic operad $\text{Ass}$ for associative algebras is the linear span of the terminal non-$\Sigma$ modular operad in the category of sets. Explicitly,

$$\text{Mod}(\text{Ass})(\langle S; g \rangle) = k$$

for each multicyclically ordered set $S$ and $g \in \mathbb{N}$. All structure operations are either the identities of $k$ or the canonical isomorphisms $k \otimes k \xrightarrow{\cong} k$.

Since the symmetrization ([17]) clearly commutes with the non-$\Sigma$ modular envelope functor, Theorem 27 implies the isomorphisms

\begin{equation}
\text{Mod}(\text{Ass}) \cong \text{Sym}(\text{Span}(\ast_{\text{M}}))
\end{equation}

proved in [1, 3] though not expressed in this form there.
Let us start proving the isomorphisms (25a) and (25b) of Theorem 25. From here till the end of this section the basic category will be the category of sets.

**Proof of (25a).** It will be a warm-up for the proof of (25b) given below. The modular envelope \( \Mod(\ast_C) \) is characterized by the adjunction

\[
\ModOpc(\Mod(\ast_C), P) \cong \CycOp(\ast_C, \square P)
\]

that must hold for each modular \( \Set \)-operad \( P \). It is clear that there is a one-to-one correspondence between morphisms in \( \CycOp(\ast_C, \square P) \) and families

\[
\varsigma(S) \in \mathcal{P}((S; 0)), \ S \in \Fin,
\]

such that

\[
\mathcal{P}((\sigma)) (\varsigma(S)) = \varsigma(D) \quad \text{and} \quad \varsigma(S') u \circ v \varsigma(S'') = \varsigma(S' \cup S'' \setminus \{u, v\})
\]

for each \( S', S'', u, v, \sigma \) for which the above expressions make sense.

The theorem will obviously be proved if we exhibit a one-to-one correspondence between morphisms \( \ast_C \to \square P \) represented by (28), and families

\[
\varpi(S; g) \in \mathcal{P}((S; g)), \ S \in \Fin \times \mathbb{N},
\]

such that \( \varpi(S; 0) = \varsigma(S) \) for each \( S \in \Fin \), and

\[
(31a) \quad \mathcal{P}((\sigma)) (\varpi(S; g)) = \varpi(D; g), \ \sigma : S \to D \in \Fin,
\]

\[
(31b) \quad \varpi(S'; g') \circ u \circ v \varpi(S''; g'') = \varpi(S' \cup S'' \setminus \{u, v\}; g' + g''), \quad \text{and}
\]

\[
\xi_{uv} \varpi(S; g) = \varpi(S \setminus \{u, v\}; g - 2),
\]

whenever the above objects are defined. In the light of (27) this is the same as to show that each family (28) uniquely determines a family (30) with \( \varsigma(S; 0) = \varsigma(S) \) for each \( S \in \Fin \).

Let \( \Gamma \) be a connected graph of genus \( g \) with trivial local genera \( g_v \) at each vertex, and \( \text{Leg}(\Gamma) = S \). Decorate the vertices of \( \Gamma \) by (28) and denote the result by

\[
\varsigma(\Gamma) = \prod_{v \in \text{Vert}(\Gamma)} \varsigma(\text{Leg}(v)) \in M(\mathcal{P})(\langle S; g \rangle),
\]

where \( M(\mathcal{P}) \) is the free modular operad [19, Section II.5.3] on the modular module \( F\mathcal{P} \).

The composition (contraction) \( c : M(\mathcal{P})(\langle S; g \rangle) \to \mathcal{P}((S; g)) \) along the graph \( \Gamma \) determines

\[
\varpi_\Gamma := c(\varsigma(\Gamma)) \in \mathcal{P}((S; g)).
\]

Assume we proved, for \( \Gamma, S \) and \( g \) as above, that

\[
\varpi_\Gamma \text{ depends only on the type of } \Gamma, \text{ i.e. on } S \text{ and } g, \text{ not on the concrete } \Gamma.
\]

We claim that then \( \varpi(S; g) := \varpi_\Gamma \) is the requisite extension of (30).

Indeed, to establish (31a), denote by \( \Gamma' \) the graph \( \Gamma \) with the legs relabeled according to \( \sigma \). Then (33) implies that \( \mathcal{P}((\sigma)) \varpi_\Gamma = \varpi_{\Gamma'} \). As for (31b), assume that \( \varpi(S'; g') = \varpi_\Gamma \) and \( \varpi(S''; g'') = \varpi_{\Gamma''} \). Then \( \varpi(S'; g') \circ u \circ v \varpi(S''; g'') \) equals \( \varpi_{\Gamma'\cup \Gamma''} \) with \( \Gamma' \cup \Gamma'' \) the grafting recalled in Remark 20, which in turn equals \( \varpi(S' \cup S'' \setminus \{u, v\}; g' + g'') \). This proves (31b).

Property (31c) can be discussed similarly.

Choose a maximal subtree \( T \) of \( \Gamma \) and denote by \( K := \Gamma / T \) the result of shrinking \( T \subset K \) into a corolla. Notice that \( K \) has only one vertex. By the associativity [19, Theorem II.5.42] 16 Formally correct notation would therefore be \( M(F\mathcal{P}) \) but we want to save space.

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of the contractions, \( \varpi_T = \varpi_K \), it is therefore enough to prove (33) for graphs \( K \) with one vertex. Graphically, such a \( K \) is a non-planar ‘tick’ with \( g \) ‘bellies’ as in

![Graph](image)

where \( g = 3 \). We prove (33) by induction on the genus \( g \).

If \( g = 0 \), \( K \) is the corolla with \( \text{Leg}(K) = S \) whose unique vertex is decorated by \( \varpi_T \) which equals \( \varsigma(S) \), because the contraction in (32) uses only the cyclic part \( \Box \mathcal{P} \) of the operad \( \mathcal{P} \). Therefore \( \varpi_T \) does not depend on the choice of \( T \) and is determined by a given map \( *_C \to \Box \mathcal{P} \).

Assume we have proved (33) for all \( g' < g \). Let \( K' \) be a tick obtained by removing one belly of \( K \). Then clearly \( \text{Leg}(K') = S \cup \{u, v\} \) for some \( u \) and \( v \), and

\[
\varpi_K = \xi_{uv}(\varpi_{K'}).
\]

By the induction assumption, \( \varpi_{K'} \) equals \( \varpi(S \cup \{u, v\}; g - 1) \) and does not depend on the concrete form of \( K' \), so

\[
\varpi_K = \xi_{uv}(\varpi(S \cup \{u, v\}; g - 1))
\]

depends only on the finite set \( S \) and the genus \( g \). \( \square \)

Let us formulate a useful

**Definition 28.** A family (38) satisfying (29) is an internal operad in the cyclic operad \( \Box \mathcal{P} \). Similarly (30) satisfying (31a)–(31c) is an internal operad in the modular operad \( \mathcal{P} \). These notions have obvious non-\( \Sigma \) analogs.

The proof of (25b) occupies the rest of this section. Since it follows the scheme of the proof of Theorem 26, we only emphasize the differences. We must to show that each internal non-\( \Sigma \) cyclic operad

\[
(34) \quad \varsigma(C) \in \mathcal{P}((C; 0)), \quad C \in \text{Cyc},
\]

in \( \Box \mathcal{P} \) uniquely extends to an internal non-\( \Sigma \) modular operad

\[
(35) \quad \varpi(S; g) \in \mathcal{P}((S, g)), \quad S \in \text{MultCyc} \times \mathbb{N}
\]

in \( \mathcal{P} \) such that \( \varpi(C; 0) = \varsigma(C) \) for \( C \in \text{Cyc} \subset \text{MultCyc} \).

As in the proof of Theorem 26 for a non-\( \Sigma \) modular graph \( \Gamma \) with \( \text{Leg}(\Gamma) = S \) whose local genera vanish we define \( \varpi_{\Gamma} \in \mathcal{P}((S; g)) \) using the contractions (23) along \( \Gamma \) in \( \mathcal{P} \). We then need to prove an analog of (33):

\[
(36) \quad \varpi_{\Gamma} \text{ depends only on the type of } \Gamma, \text{ i.e. on the multicyclically ordered set } S = \text{Leg}(\Gamma) \text{ and the genus } g = g(\Gamma).
\]

By choosing a maximal subtree \( T \) of \( \Gamma \) we again reduce (33) to graphs \( K \) with only one vertex. This time, \( K \) is not a ‘tick’ but a pancake

![Graph](image)
with the internal ribs marking the half-edges which have been contracted.

**Example 29.** The only pancake $K$ with the operadic genus $g = 0$ has $b(K) = 0$. It is a circle with the circumference decorated by a cyclically ordered set $C := \text{Leg}(K)$. In this case $\varpi_K = \varsigma(C)$, so (33) is satisfied trivially.

**Example 30.** There is only one type of a pancake with $g = 1$, the left one in Figure 5. When its circumference is labeled by the ordered sets $X$ and $Y$ as in the figure, then, by definition

$$
\varpi_K := \xi_{uv}\varsigma([XuYv]) \in \mathcal{P}([X], [Y]; 1).
$$

We prove that $\varpi_K$ depends only on the induced cyclically ordered sets $[X]$ and $[Y]$, not on the particular orders of $X$ and $Y$.

Let, for instance, $X'$ be an ordered set such that $[X'] = [X]$ and $K'$ be the pancake obtained from $K$ by replacing $X$ by $X'$. We will show that

$$(37) \quad \varpi_K = \varpi_{K'},$$

where

$$
\varpi_{K'} := \xi_{uv}\varsigma([X'vY]) \in \mathcal{P}([X'], [Y]; 1) = \mathcal{P}([X], [Y]; 1).
$$

As we noticed in Remark 2, $[X'] = [X]$ if and only if there are ordered sets $X_1$ and $X_2$ such that $X = X_1X_2$ and $X' = X_2X_1$. By the interchange (iv) of Definition 8,

$$(38) \quad \xi_{v'v''}\{\varsigma([u'v'X_1]) \circ_{u'} \varsigma([X_2v''Yu''])\} = \xi_{u'v''}\{\varsigma([u'v'X_1]) \circ_{v''} \varsigma([X_2v''Yu''])\}.
$$

The corresponding term in the curly bracket in the left hand side equals

$$
\varsigma([u'v'X_1]) \circ_{u'} \varsigma([X_2v''Yu'']) = \varsigma([v'X_1X_2v''Y]) = \varsigma([X_1X_2v''Yv']) = \varsigma([Xv''Yv']),
$$

while the term in the right hand side is

$$
\varsigma([u'v'X_1]) \circ_{v''} \varsigma([X_2v''Yu'']) = \varsigma([X_1u'Yu''X_2]) = \varsigma([X_2X_1u'Yu'']) = \varsigma([X'u'Yu'']),
$$

thus (38) implies

$$
\xi_{v'v''}\varsigma([Xv''Yv']) = \xi_{u'v''}\varsigma([X'u'Yu''])
$$

which is (37). The independence of the particular order of $Y$ can be proved similarly.

**Example 31.** There are two types of pancakes with $g = 2$. The middle one in Figure 5 has $b = 3$ and

$$(39a) \quad \varpi_K = \xi_{u'v'}\xi_{v''v'}\varsigma([Xu'Y_1v'Zv''Y_2u'']) \in \mathcal{P}([X], [Y_1Y_2], [Z]; 3)).$$

The second type in the right hand side of Figure 5 has $b = 1$ and

$$(39b) \quad \varpi_K = \xi_{u'v'}\xi_{v''v'}\varsigma(Xv'Yu''Zv''Uu'') \in \mathcal{P}([UZYX]; 2)).$$
We leave as an exercise on the axioms of non-$\Sigma$ modular operads to prove that the elements $\varpi_K$ in (39a) resp. in (39b) depend only on the cyclically ordered sets $[X]$, $[Y_1Y_2]$ and $[Z]$ resp. $[UZYX]$. 

We prove (35) by induction based on the following simple:

**Lemma 32.** Let $K$ be a pancake with $g > 0$.

(i) If $b(K) = 1$, then there exist a pancake $K'$ obtained by removing one rib of $K$ such that $b(K') = 2$.

(ii) If $b(K) > 1$, then there exist a pancake $K'$ obtained from $K$ by removing one rib that has $b(K') = b(K) - 1$.

**Proof.** The case $b(K) = 1$ may happen only when $g(K)$ is even, by the geometricity (8). Thus by removing an arbitrary rib we obtain a pancake $K'$ with $b(K') = 2$. This proves (i).

To prove (ii) we analyze the pancake (36) representing $K$. The legs of $K$ are irrelevant for the proof so we ignore them. Let us inspect how the ribs enter the circumference of the pancake. The oriented edge cycles (18) are the boundaries of the regions between the ribs:

In this picture, the horizontal line represents a part of the circumference of the pancake and the vertical lines the ribs.

It is simple to see that removing a rib adjacent to two different regions decreases $b(K)$ by one. Since $b(K) > 1$ by assumption, there are at least two different edge cycles, therefore such a rib exists. This finishes the proof. □

Let us finally start the actual inductive proof of (35). The cases when $g(K) \leq 2$ are analyzed in Examples 29–31. Fix $g \geq 3$, assume that we have proved (35) for all $K$'s with $g(K) < g$ and prove it for $K$ with $g(K) = g$. As in Lemma 32 we distinguish two cases.

**Case 1:** $b(K) = 1$. As there are no pancakes with $b(K) = 1$ and $g(K) = 4$, in this case in fact $g(K) \geq 4$. By Lemma 32(i) and the inductive assumption,

$$\varpi_K = \xi_{u''v'} \varpi([X_1u'], [u''X_2]; g - 1),$$

where $X_1$, $X_2$ are ordered sets such that $C := \text{Leg}(K) = [X_1X_2]$. We must show that the right hand side of (40) does not depend on the particular choices of $X_1$ and $X_2$. The choices are represented by a rib of a circle with the circumference decorated by $C$ as in Figure 6 left.

Assume we have two different ribs, $C = [X_1'X_2']$ and $C = [X_1''X_2'']$. They may either be crossing as in the middle of Figure 8, or parallel as in the rightmost picture of Figure 8. The crossing case is parametrized by ordered sets $X, Y, Z, U$ such that

$$X_1' = ZY, \ X_2' = XU, \ X_1'' = YX \ \text{and} \ \ X_2'' = UZ,$$

see Figure 8 again. The element

$$\varpi([v''Yu''Zv'Uu'X]; g - 2) \in \mathcal{P}([v''Yu''Zv'Uu'X]; g - 2))$$

[nsmod.tex]  

[October 12, 2014]
The equalizer of these two choices is
\[ \text{Case 2: } \] 

\[ b \]

Therefore
\[ \xi_{u''u''} \varpi([X'_1 u'']\ \ 2 \ 1) = \xi_{u''u''} \varpi([X''_1 u'],\ [u''X'_2];\ g - 1) \]

as we needed to show. Notice that we need to assume \( g \geq 2 \) in order the equalizer to exist.

The non-crossing case is parametrized by ordered sets \( X, Y, Z, U \) such that
\[ X'_1 = ZYX, \ X'_2 = U, \ X''_1 = XUZ \quad \text{and} \quad X''_2 = Y. \]

The equalizer of these two choices is \( \varpi([Xu'Zv'],\ [v''U];\ g - 2) \) as the reader easily verifies. This finishes the discussion of the \( b(K) = 1 \) case.

**Case 2:** \( b(K) > 1 \). Now \( K \) is of type \( (C_1, \ldots, C_b; g) \) with \( b \geq 2 \). Lemma 32(ii) translates to the formula
\[ \varpi_K = \xi_{u''u''} \varpi([\hat{C}_1, \ldots, \hat{C}_i, \ldots, \hat{C}_j, \ldots, \hat{C}_b, [u'X_i u''X_j];\ g - 1), \]

where \( \hat{ } \) indicates the omission and \( X_i, X_j \) are ordered sets such that \( [X_i] = C_i \) and \( [X_j] = C_j \). As before we must prove that the right hand side of (11) does not depend on the particular choices of \( i, j \) and ordered sets \( X_i, X_j \). So suppose that we have two different choices
\[ [X'_i] = C'_{i'}, \ [X''_j] = C'_{j'}, \quad \text{resp.} \quad [X''_i] = C''_{i''}, \ [X''_j] = C''_{j''} \]

for some ordered sets \( X'_i, X'_j, X''_i, X''_j \) and \( 1 \leq i' < j' \leq b, \ 1 \leq i'' < j'' \leq b \). We distinguish three cases.

**Case 2(i):** \( \{i', j'\} = \{i'', j''\} \). We may clearly assume that \( i' = i'' = 1, \ j' = j'' = 2 \). Since \( C_k \)'s with \( k > 2 \) do not affect calculations we will not explicitly mention them. We therefore have
\[ C_1 = [X'_1] = [X''_1], \ C_2 = [X'_2] = [X''_2], \]

which, as observed in Remark 3, happens if and only if there are ordered sets \( X, Y, Z, U \) such that
\[ X'_1 = XY, \ X'_2 = ZU, \ X''_1 = YX \quad \text{and} \quad X''_2 = UZ. \]

One easily verifies that then \( \varpi([Yv''Yu'], [v''Uv'X]; g - 2) \) is an equalizer of these choices.
Figure 7. Oriented (left) and un-oriented (right) glueing of teethed wheels.

Case 2(ii): the cardinality of \( \{i', j', i'', j''\} \) is 3. We may assume that \( i' = 1, j' = i'' = 2, j'' = 3 \), and neglect \( C_k \)’s with \( k > 3 \). So we have two presentations

\[
\varpi_K = \varpi([u'X_1u'', X_3]; g - 1) \quad \text{and} \quad \varpi_K = \varpi(C, [v'X''_2v'', X_3]; g - 1)
\]

in which \( C_1 = [X_1], C_2 = [X_2] = [X_2''] \) and \( C_3 = [X_3] \). By Remark 2, there are ordered sets \( Y, Z \) such that \( X_2' = YZ \) and \( X_3'' = ZY \). One easily sees that \( \varpi(u'X_1u''Yv'X_3v''Z; g - 2) \) is a equalizer for the two presentations in (42). The last case is

Case 2(iii): the cardinality of \( \{i', j', i'', j''\} \) is 4. This case is simple so we leave its analysis to the reader. This finishes the proof of Theorem 27.

6. Surfaces and Chuang-Lazarev’s approach to \( \text{Mod}(\text{Ass}) \).

In this section we recall the approach which J. Chuang and A. Lazarev used to prove their \cite[Theorem 3.7]{Chuang} that describes the modular envelope \( \text{Mod}(\text{Ass}) \) via the set of isomorphism classes of oriented surfaces with teethed wheels. Our setup nicely conveys their ideas. The authors of \cite{Chuang} of course worked in the category of ordinary operads, but what they did in fact took place within non-\( \Sigma \) modular operads. In the second half we briefly mention a non-oriented modification due to C. Braun \cite{Braun}. Throughout this section, the basic monoidal category will be the cartesian category \( \text{Set} \) of sets.

Let \( \ast \) be the geometric non-\( \Sigma \) modular module of Example 21. Its generators \( \ast((C)) \) will this time be visualized as oriented toothed wheels

\[
(43)
\]

whose teeth are indexed by the cyclically ordered set \( C \). Elements of \( M(\ast) \) are then obtained by gluing these wheels together along the tips of their teeth so that the orientation is preserved, see Figure 7–left. It is an exercise in combinatorial geometry that \( M(\ast)(C_1, \ldots, C_b; g) \) consists of all decompositions of an oriented surface of the genus

\[
G = \frac{1}{2}(g - b + 1)
\]

with \( b \) teethed boundaries whose teeth are labelled by the cyclically ordered sets \( C_1, \ldots, C_b \) shown in Figure 8.

By (24), the modular envelope \( \text{Mod}(\ast) \) is the quotient of \( M(\ast) = M(F(\ast)) \) by the relations that forget how a concrete surface was build from the toothed wheels. As in the proof of \cite[Theorem 3.7]{Chuang} we therefore identify, referring to the results of \cite{Chuang}, \( \text{Mod}(\ast) \) with the set of isomorphism classes of surfaces as in Figure 8. Since there is only one isomorphism class for a given geometric \((C_1, \ldots, C_b; g) \in \text{MultCyc} \times \mathbb{N} \), we get (25b).

\[\text{nsmod.tex} \quad [\text{October 12, 2014}]\]
Despite its conceptual clarity, the above approach relied on a rather deep result of [12] that the classifying space of the category of ribbon graphs of genus $G$ with $b$ boundary components is homeomorphic to the moduli space of Riemann surfaces of the same genus and the same number of boundary components. We therefore still believe that a direct combinatorial description of $\text{Mod}(\mathcal{A}_{\text{ss}})$ given in [4] or here has some merit.

**Remark 33.** Notice that, given $(C_1, \ldots, C_b; g) \in \text{MultCyc} \times \mathbb{N}$, there exists a surface as in Figure 8 if and only if $(C_1, \ldots, C_b; g)$ is geometric. This shall explain our terminology.

A non-oriented variant of the above calculations starts with the cyclic operad $*_{D}$ whose component $*_{D}((S))$ consists of toothed wheels whose teeth are indexed by the finite set $S$ and have their tips decorated by arrows, as

![Figure 8](image.png)

Clearly, if $S$ has $n$ elements, $*_{D}((S))$ has $2^{n-1}(n-1)!$ elements. The structure operations glue the tips of the teeth in such a way that the arrows go in the opposite directions, as in

![Figure 9](image.png)

The operad $*_{D}$ is the Möbiusisation [3, Definition 3.32] of the terminal cyclic operad $*_{C}$, the subscript $D$ referring to the dihedral structure [15, Section 3] that $*_{D}$ carries. Algebras over its linearization $\text{Span}(*_{D})$ are associative algebras with involution [4, Proposition 3.9]. The Chuang-Lazarev approach applies also to this situation, except that the sewing may not preserve the orientations now, see Figure 7–right. Indeed, C. Braun proved\footnote{He of course worked in $k$-$\text{Mod}$ not in $\text{Set}$.}

**Theorem 34 (\cite{Braun}, Theorem 3.10).** The component $\text{Mod}(*_{D})((S; g))$ of the modular envelope $\text{Mod}(*_{D})$ is the set of isomorphism classes of (not necessarily oriented) surfaces with $b$ toothed holes whose teeth are labeled by $S$, with $m$ handles and $u$ crosscaps such that $g = 2m+b+u-1$.

As we theorized in the Introduction, we believe that $\text{Mod}(*_{D})$ is (related to) the terminal operad in a suitable category of dihedral operads.
There are two versions of biased definitions of operads. The skeletal version has natural numbers as the arities, in the non-skeletal the arities are finite sets. We recall, following [3, Section 2], the non-skeletal definitions of classical cyclic and modular operads.

**Definition 35.** A *cyclic module* is a functor $E : \textbf{Fin} \to \mathcal{M}$ from the category of finite sets and their isomorphisms to our fixed symmetric monoidal category $\mathcal{M}$.

**Definition 36.** A *cyclic operad* in $\mathcal{M} = (\mathcal{M}, \otimes, 1)$ is a cyclic module
\[
\mathcal{P} = \{ \mathcal{P}(\{S\}) \in \mathcal{M}; S \in \text{Fin} \}
\]
together with morphisms (compositions)
\[
u \circ v : \mathcal{P}(\{S'\}) \otimes \mathcal{P}(\{S''\}) \to \mathcal{P}(\{(S' \cup S'' \setminus \{u, v\})\})
\]
defined for arbitrary disjoint sets $S'$ and $S''$ with elements $u \in S'$, $v \in S''$ of their underlying sets. These data are required to satisfy the following axioms.

(i) For $S'$, $S''$ and $u$, $v$ as in (45), one has the equality
\[
u \circ v = \nu \circ u \tau
\]
of maps $\mathcal{P}(\{S'\}) \otimes \mathcal{P}(\{S''\}) \to \mathcal{P}(\{(S' \cup S'' \setminus \{u, v\})\})$, where $\tau$ is the symmetry constraint in $\mathcal{M}$.

(ii) For mutually disjoint sets $S_1, S_2, S_3$, and $a \in S_1$, $b, c \in S_2$, $b \neq c$, $d \in S_3$, one has the equality
\[a \circ_b (id \otimes c \circ_d) = c \circ_d (a \circ_b \otimes id)
\]
of maps $\mathcal{P}(\{S_1\}) \otimes \mathcal{P}(\{S_2\}) \otimes \mathcal{P}(\{S_3\}) \to \mathcal{P}(\{(S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\})\})$.

(iii) For arbitrary isomorphisms $\rho : S' \to D'$ and $\sigma : S'' \to D''$ of sets and $u, v$ as in (45), one has the equality
\[
\mathcal{P}(\{\rho_{|S' \setminus \{u\} \cup \sigma_{|S'' \setminus \{v\}}\}) \circ_{\rho(u) \circ \sigma(v)} = \rho_{(\{\rho\}) \otimes \mathcal{P}(\{\sigma\})}
\]
of maps $\mathcal{P}(\{S'\}) \otimes \mathcal{P}(\{S''\}) \to \mathcal{P}(\{(D' \cup D'' \setminus \{\rho(u), \sigma(v)\})\})$.

The category $\textbf{Fin}$ of finite sets is equivalent to its full skeletal subcategory $\textbf{Fin}_{sk}$ whose objects are the sets $[n] := \{1, \ldots, n\}$, $n \geq 0$, with $[0]$ interpreted as the empty set $\emptyset$. The components of the skeletal version of $\mathcal{P}$ are
\[
\mathcal{P}(n) := \mathcal{P}([n+1]), n \geq -1,
\]
with the induced action of the symmetric group $\Sigma_{n+1} = \text{Aut}([n+1])$. The structure operations
\[
i \circ_j : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \ 0 \leq i \leq m, \ 0 \leq j \leq n,
\]
are induced from the equivalence of $\textbf{Fin}$ with $\textbf{Fin}_{sk}$.

Notice that we allow also the component $\mathcal{P}(\{\emptyset\}) = \mathcal{P}(-1)$ and the operation
\[
u \circ v : \mathcal{P}(\{\{u\}\}) \otimes \mathcal{P}(\{\{v\}\}) \to \mathcal{P}(\{\emptyset\}) \text{ resp. } o \circ_0 : \mathcal{P}(0) \otimes \mathcal{P}(0) \to \mathcal{P}(-1),
\]
while the original definition [3, Theorem 2.2] always requires an ‘output,’ i.e. the arities must be non-empty sets (or $n \geq 0$ in the skeletal $\mathcal{P}(n)$). We do not demand operadic units.
Definition 37. A modular module is a functor

\[ E : \text{Fin} \times \mathbb{N} \to \mathbb{M}, \]

where the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) are considered as a discrete category.

Definition 38. A modular operad in \( \mathbb{M} = (\mathbb{M}, \otimes, 1) \) is a modular module

\[ \mathcal{P} = \{ \mathcal{P}(\langle S; g \rangle) \in \mathbb{M}; (S, g) \in \text{Fin} \times \mathbb{N} \} \]

together with morphisms (compositions)

\[ u \circ_v : \mathcal{P}(\langle S'; g' \rangle) \otimes \mathcal{P}(\langle S''; g'' \rangle) \to \mathcal{P}(\langle S' \cup S'' \setminus \{u, v\}; g' + g'' \rangle) \]
defined for arbitrary disjoint sets \( S' \) and \( S'' \) with elements \( u \in S', v \in S'' \) of their underlying sets, and contractions

\[ \xi_{uv} = \xi_{vu} : \mathcal{P}(\langle S; g \rangle) \to \mathcal{P}(\langle S \setminus \{u, v\}; g + 1 \rangle) \]
given for any set \( S \) and distinct elements \( u, v \in S \). These data are required to satisfy the following axioms.

(i) For \( S', S'' \) and \( u, v \) as in (16), one has the equality

\[ u \circ_v = v \circ_u \tau \]
of maps \( \mathcal{P}(\langle S'; g' \rangle) \otimes \mathcal{P}(\langle S''; g'' \rangle) \to \mathcal{P}(\langle S' \cup S'' \setminus \{u, v\}; g' + g'' \rangle) \).

(ii) For mutually disjoint sets \( S_1, S_2, S_3 \), and \( a \in S_1, b, c \in S_2, b \neq c, d \in S_3 \), one has the equality

\[ a \circ_b (id \otimes c \circ_d) = c \circ_d (a \circ_b \otimes id) \]
of maps \( \mathcal{P}(\langle S_1; g_1 \rangle) \otimes \mathcal{P}(\langle S_2; g_2 \rangle) \otimes \mathcal{P}(\langle S_3; g_3 \rangle) \to \mathcal{P}(\langle S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\}; g_1 + g_2 + g_3 \rangle) \).

(iii) For a set \( S \) and mutually distinct \( a, b, c, d \in S \), one has the equality

\[ \xi_{ab} \xi_{cd} = \xi_{cd} \xi_{ab} \]
of maps \( \mathcal{P}(\langle S; g \rangle) \to \mathcal{P}(\langle S \setminus \{a, b, c, d\}; g + 2 \rangle) \).

(iv) For sets \( S', S'' \) and distinct \( a, c \in S', b, d \in S'' \), one has the equality

\[ \xi_{ab} c \circ_d = \xi_{cd} a \circ_b \]
of maps \( \mathcal{P}(\langle S' \cup S''; g \rangle) \to \mathcal{P}(\langle S' \cup S'' \setminus \{a, b, c, d\}; g + 1 \rangle) \).

(v) For sets \( S', S'' \) and mutually distinct \( a, c \in S', b \in S'' \), one has the equality

\[ a \circ_b (\xi_{cd} \otimes id) = \xi_{cd} a \circ_b \]
of maps \( \mathcal{P}(\langle S' \cup S''; g \rangle) \to \mathcal{P}(\langle S' \cup S'' \setminus \{a, b, c, d\}; g + 1 \rangle) \).

(vi) For arbitrary isomorphisms \( \rho : S' \to D' \) and \( \sigma : S'' \to D'' \) of sets and \( u, v \) as in (14), one has the equality

\[ \mathcal{P}(\langle (\rho|_{S' \setminus \{u\}} \cup \sigma|_{S'' \setminus \{v\}}) \rangle) u \circ_v = \rho(u) \circ_{\sigma(v)} (\mathcal{P}(\langle \rho \rangle) \otimes \mathcal{P}(\langle \sigma \rangle)) \]
of maps \( \mathcal{P}(\langle S'; g' \rangle) \otimes \mathcal{P}(\langle S''; g'' \rangle) \to \mathcal{P}(\langle D' \cup D'' \setminus \{\rho(u), \sigma(v)\}; g' + g'' \rangle) \).

(vii) For \( S, u, v \) as in (15) and an isomorphism \( \rho : S \to D \) of sets, one has the equality

\[ \mathcal{P}(\langle (\rho|_{D \setminus \{\rho(u), \rho(v)\}}) \rangle) \xi_{ab} = \xi_{\rho(u) \rho(v)} \mathcal{P}(\langle \rho \rangle) \]
of maps \( \mathcal{P}(\langle S; g \rangle) \to \mathcal{P}(\langle S \setminus \{\rho(u), \rho(v)\}; g + 1 \rangle) \).
Informally, cyclic operads are modular operads without the contractions and the genus grading. In the seminal paper [10] where modular operads were introduced, the stability demanding that

\[ \mathcal{P}((S; g)) = 0 \text{ if } \text{card}(S) < 3 \text{ and } g = 0, \text{ or } \text{card}(S) = 0 \text{ and } g = 1 \]

was assumed, but we do not require this property. As a matter of fact, our main examples of non-\(\Sigma\) modular operads are not stable, though stable versions of our results can easily be formulated and proved.

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