ON SOME PARTITIONS OF A FLAG MANIFOLD

G. LUSZTIG

1. Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic exponent $p \geq 1$. We shall assume that $p$ is 1 or a good prime for $G$. Let $B$ be the variety of Borel subgroups of $G$ and let $W$ be the Weyl group of $G$. Note that $W$ naturally indexes ($w \mapsto O_w$) the orbits of $G$ acting on $B \times B$ by simultaneous conjugation on the two factors. For $g \in G$ we set $B_g = \{B \in B; g \in B\}$. The varieties $B_g$ play an important role in representation theory and their geometry has been studied extensively. More generally for $g \in G$ and $w \in W$ we set $B_w^g = \{B \in B; (B, gBg^{-1}) \in O_w\}$. Note that $B_1^g = B_g$ and that for fixed $g$, $(B_w^g)_{w \in W}$ form a partition of the flag manifold $B$.

For fixed $w$, the varieties $B_w^g$ ($g \in G$) appear as fibres of a map to $G$ which was introduced in [L3] as part of the definition of character sheaves. Earlier, the varieties $B_g^w$ for $g$ regular semisimple appeared in [L1] (a precursor of [L3]) where it was shown that from their topology (for $k = \mathbb{C}$) one can extract nontrivial information about the character table of the corresponding group over a finite field.

In this paper we present some experimental evidence for our belief that from the topology of $B_w^g$ for $g$ unipotent (with $k = \mathbb{C}$) one can extract information closely connected with the map [KL, 9.1] from unipotent classes in $G$ to conjugacy classes in $W$.

I thank David Vogan for some useful discussions.

2. We fix a prime number $l$ invertible in $k$. Let $g \in G$ and $w \in W$. For $i, j \in \mathbb{Z}$ let $H^i_c(B_w^g, \mathbb{Q}_l)_j$ be the subquotient of pure weight $j$ of the $l$-adic cohomology space $H^i_c(B_g^w, \mathbb{Q}_l)$. The centralizer $Z(g)$ of $g$ in $G$ acts on $B_w^g$ by conjugation and this induces an action of the group of components $\tilde{Z}(g)$ on $H^i_c(B_g^w, \mathbb{Q}_l)$ and on each $H^i_c(B_g^w, \mathbb{Q}_l)_j$. For $z \in \tilde{Z}(g)$ we set

$$\Xi_{g, z}^w = \sum_{i, j \in \mathbb{Z}} (-1)^i \text{tr}(z, H^i_c(B_g^w, \mathbb{Q}_l)_j) v^j \in \mathbb{Z}[v]$$

Supported in part by the National Science Foundation
where $v$ is an indeterminate; the fact that this belongs to $\mathbb{Z}[v]$ and is independent of the choice of $l$ is proved by an argument similar to that in the proof of [DL, 3.3].

Let $l : W \to \mathbb{N}$ be the standard length function. The simple reflections $s \in W$ (that is the elements of length 1 of $W$) are numbered as $s_1, s_2, \ldots$. Let $w_0$ be the element of maximal length in $W$. Let $\mathcal{W}$ be the set of conjugacy classes in $W$.

Let $H$ be the Iwahori-Hecke algebra of $W$ with parameter $v^2$ (see [GP, 4.4.1]; in the definition in loc. cit. we take $A = \mathbb{Z}[v, v^{-1}], a_s = b_s = v^2$). Let $(T_w)_{w \in W}$ be the standard basis of $H$ (see [GP, 4.4.3, 4.4.6]). For $w \in W$ let $\hat{T} = v^{-2l(w)}T_w$. If $s_1s_2 \ldots s_t$ is a reduced expression for $w \in W$ we write also $\hat{T}_w = \hat{T}_{s_1s_2 \ldots s_t}$.

For any $g \in G, z \in \mathcal{Z}(g)$ we set

$$
\Pi_{g,z} = \sum_{w \in \mathcal{W}} \Xi_{g,z}^w \hat{T}_w \in H.
$$

The following result can be proved along the lines of the proof of [DL, Theorem 1.6] (we replace the Frobenius map in that proof by conjugation by $g$); alternatively, for $g$ unipotent, we may use 6(a).

(a) $\Pi_{g,z}$ belongs to the centre of the algebra $H$.

According to [GP, 8.2.6, 7.1.7], an element $c = \sum_{w \in \mathcal{W}} c_w \hat{T}_w$ ($c_w \in \mathbb{Z}[v, v^{-1}]$) in the centre of $H$ is uniquely determined by the coefficients $c_w$ ($w \in \mathcal{W}_{\text{min}}$) and we have $c_w = c_{w'}$ if $w, w' \in \mathcal{W}_{\text{min}}$ are conjugate in $W$; here $\mathcal{W}_{\text{min}}$ is the set of elements of $W$ which have minimal length in their conjugacy class. This applies in particular to $c = \Pi_{g,z}$, see (a). For any $C \in \mathcal{W}$ we set $\Xi_{g,z}^C = \Xi_{g,z}^w$ where $w$ is any element of $C \cap \mathcal{W}_{\text{min}}$.

Note that if $g = 1$ then $\Pi_{g,1} = (\sum_w v^{2l(w)}).1$. If $g$ is regular unipotent then $\Pi_{g,1} = \sum_{w \in \mathcal{W}} v^{2l(w)} \hat{T}_w$. If $G = \text{PGL}_3(k)$ and $g \in G$ is regular semisimple then $\Pi_{g,1} = 6 + 3(v^2 - 1)(\hat{T}_1 + \hat{T}_2) + (v^2 - 1)^2(\hat{T}_{12} + \hat{T}_{21}) + (v^6 - 1)\hat{T}_{121};$ if $g \in G$ is a transvection then $\Pi_{g,1} = (2v^2 + 1) + v^4(\hat{T}_1 + \hat{T}_2) + v^6\hat{T}_{121}$.

For $g \in G$ let $\text{cl}(g)$ be the $G$-conjugacy class of $g$; let $\overline{\text{cl}(g)}$ be the closure of $\text{cl}(g)$. Let $S_g$ be the set of all $C \in \mathcal{W}$ such that $\Xi_{g,1}^C \neq 0$ and $\Xi_{g',1}^C = 0$ for any $g' \in \overline{\text{cl}(g)} - \text{cl}(g)$. If $C$ is a conjugacy class in $G$ we shall also write $S_C$ instead of $S_g$ where $g \in C$.

We describe the set $S_g$ and the values $\Xi_{g,1}^C$ for $C \in S_g$ for various $G$ of low rank and various unipotent elements $g$ in $G$. We denote by $u_n$ a unipotent element of $G$ such that $\dim B_{u_n} = n$. The conjugacy class of $w \in \mathcal{W}$ is denoted by $(w)$.

$G$ of type $A_1$.

$$
S_{u_1} = (1), S_{u_0} = (s_1); \Xi_{u_1,1}^1 = 1 + v^2, \Xi_{u_0,1}^1 = v^2.
$$

$G$ of type $A_2$.

$$
S_{u_3} = (1), S_{u_1} = (s_1), S_{u_0} = (s_1s_2).
$$
\[ \Xi_{u_{3,1}}^1 = 1 + 2v^2 + 2v^4 + v^6, \Xi_{u_{1,1}}^{(s_1)} = v^4, \Xi_{u_{0,1}}^{(s_1 s_2)} = v^4. \]

G of type \( B_2 \). (The simple reflection corresponding to the long root is denoted by \( s_1 \).)
\[ S_{u_1} = (1), S_{u_2} = (s_1), S_{u_3} = \{(s_2), (s_1 s_2 s_1 s_2)\}, S_{u_4} = (s_1 s_2). \]
\[ \Xi_{u_{1,1}}^{(s_1)} = v^4, \Xi_{u_{0,1}}^{(s_1 s_2)} = 2v^4, \Xi_{u_{3,1}}^{(s_1 s_2)} = v^6(v^2 - 1), \Xi_{u_{0,1}}^{(s_1 s_2)} = v^4. \]

G of type \( G_2 \). (The simple reflection corresponding to the long root is denoted by \( s_2 \).)
\[ S_{u_0} = (1), S_{u_1} = (s_2), S_{u_2} = \{(s_1), (s_1 s_2 s_1 s_2)\}, S_{u_3} = (s_1 s_2), S_{u_4} = (s_1 s_2). \]
\[ \Xi_{u_{0,1}}^{(s_2)} = v^8(v^4 - 1), \Xi_{u_{1,1}}^{(s_1 s_2)} = 2v^8, \Xi_{u_{0,1}}^{(s_1 s_2)} = v^4. \]

G of type \( B_3 \). (The simple reflection corresponding to the short root is denoted by \( s_3 \) and \( (s_1 s_3)^2 = 1 \).)
\[ S_{u_0} = (1), S_{u_1} = (s_3), S_{u_2} = \{(s_1), (s_2 s_3 s_2 s_3)\}, S_{u_3} = \{(s_1 s_3), (w_0)\}, S_{u_4} = (s_2 s_3). \]
\[ \Xi_{u_{0,1}}^{(s_3)} = v^8(1 + v^2)^2, \Xi_{u_{1,1}}^{(s_1 s_3)} = 2v^6(1 + v^2)^2, \Xi_{u_{2,1,1}}^{(s_1 s_3)} = 2v^8, \Xi_{u_{0,1}}^{(s_1 s_2 s_3)} = v^6. \]

G of type \( C_3 \). (The simple reflection corresponding to the long root is denoted by \( s_3 \) and \( (s_1 s_3)^2 = 1 \); \( u''_2 \) denotes a unipotent element which is regular inside a Levi subgroup of type \( C_2 \); \( u''_2 \) denotes a unipotent element with \( \dim B_{u''_2} = 2 \) which is not conjugate to \( u''_2 \).)
\[ S_{u_0} = (1), S_{u_1} = (s_3), S_{u_2} = \{(s_1), (s_2 s_3 s_2 s_3)\}, S_{u_3} = \{(s_1 s_3), (w_0)\}, S_{u_4} = (s_1 s_2). \]
\[ \Xi_{u_{0,1}}^{(s_3)} = v^{10}(v^4 - 1), \Xi_{u_{1,1}}^{(s_1)} = 2v^8(1 + v^2)^2, \Xi_{u_{2,1,1}}^{(s_1 s_3)} = v^{14}(v^4 - 1), \Xi_{u_{0,1}}^{(s_1 s_2 s_3)} = v^6. \]
3. Consider the following properties of $G$:

(a) $W = \sqcup_u S_u$

$u$ runs over a set of representatives for the unipotent classes in $G$;

(b) $c_u \in S_u$ where for any unipotent element $u \in G$, $c_u$ denotes the conjugacy class in $W$ associated to $u$ in [KL, 9.1]. Note that (a),(b) hold for $G$ of rank $\leq 3$ (see §2).

We have used the computations of the map in [KL, 9.1] given in [KL, §9], [S1], [S2]. We will show elsewhere that (a), (b) hold for $G$ of type $A_n$ or $C_n$. We expect that a proof similar to that in type $C_n$ would yield (a),(b) in type $B_n$ and $D_n$. We expect that (a),(b) hold also for $G$ simple of exceptional type; (a) should follow by computing the product of some known (large) matrices using 6(a). The equality $W = \sqcup_u S_u$ is clear since for a regular unipotent $u$ and any $w$ we have $\Xi^w_{u,1} = v^{2i(w)}$.

4. Assume that $G = Sp_{2n}(k)$. The Weyl group $W$ can be identified in the standard way with the subgroup of the symmetric group $S_{2n}$ consisting of all permutations of $\{1,2,\ldots,n,n',\ldots,2',1'\}$ which commute with the involution $1 \leftrightarrow 1', 2 \leftrightarrow 2', \ldots, n \leftrightarrow n'$. We say that two elements of $W$ are equivalent if they are contained in the same conjugacy class of $S_{2n}$. The set of equivalence classes in $W$ is in bijection with the set of partitions of $2n$ in which every odd part appears an even number of times (to $C \in W$ we attach the partition which has a part $j$ for every $j$-cycle of an element of $C$ viewed as a permutation of $\{1,2,\ldots,n,n',\ldots,2',1'\}$).

The same set of partitions of $2n$ indexes the set of unipotent classes of $G$. Thus we obtain a bijection between the set of equivalence classes in $W$ and the set of unipotent classes of $G$. In other words we obtain a surjective map $\phi$ from $W$ to the set of unipotent classes of $G$ whose fibres are the equivalence classes in $W$. One can show that for any unipotent class $C$ in $G$ we have $\phi^{-1}(C) = S_u$ where $u \in C$.

5. Recall that the set of unipotent elements in $G$ can be partitioned into "special pieces" (see [L5]) where each special piece is a union of unipotent classes exactly one of which is "special". Thus the special pieces can be indexed by the set of isomorphism classes of special representations of $W$ which depends only on $W$ as a Coxeter group (not on the underlying root system). For each special piece $\sigma$ of $G$ we consider the subset $S_\sigma := \sqcup_{C \in \sigma} S_C$ of $W$ (here $C$ runs over the unipotent classes contained in $\sigma$). We expect that each such subset $S_\sigma$ depends only on the Coxeter group structure of $W$ (not on the underlying root system). As evidence for this we note that the subsets $S_\sigma$ for $G$ of type $B_3$ are the same as the subsets $S_\sigma$ for $G$ of type $C_3$. These subsets are as follows:

$$\{1\}, \{(s_1), (s_3), (s_2s_3s_2s_3)\}, \{(s_1s_3), (w_0)\}, \{(s_1s_2)\}, \{(s_2s_3)\}, \{(s_2s_3s_1s_2s_3)\}, \{(s_1s_2s_3)\}.$$
ON SOME PARTITIONS OF A FLAG MANIFOLD 5

6. Let \( g \in G \) be a unipotent element and let \( z \in \bar{Z}(g) \), \( w \in W \). We show how the polynomial \( \Xi^{w}_{g,z} \) can be computed using information from representation theory. We may assume that \( p > 1 \) and that \( k \) is the algebraic closure of the finite field \( \mathbb{F}_p \). We choose an \( \mathbb{F}_p \) split rational structure on \( G \) with Frobenius map \( F_0 : G \to G \). We may assume that \( g \in G^{F_0} \). Let \( q = p^m \) where \( m \geq 1 \) is sufficiently divisible. In particular \( F := F_0^m \) acts trivially on \( \bar{Z}(g) \) hence \( cl(g)^F \) is a union of \( G^F \)-conjugacy classes naturally indexed by the conjugacy classes in \( \bar{Z}(g) \); in particular the \( G^F \)-conjugacy class of \( g \) corresponds to \( 1 \in \bar{Z}(g) \). Let \( g_z \) be an element of the \( G^F \)-conjugacy class in \( cl(g)^F \) corresponding to the \( \bar{Z}(g) \)-conjugacy class of \( z \in \bar{Z}(g) \). The set \( B^w_{g_z} \) is \( F \)-stable. We first compute the number of fixed points \( |(B^w_{g_z})^F| \).

Let \( H_q = \tilde{Q}_l \otimes_{\mathbb{Z}_q} \mathbb{Z}[v,v^{-1}] \mathcal{H} \) where \( \tilde{Q}_l \) is regarded as a \( \mathbb{Z}_q[v,v^{-1}] \)-algebra with \( v \) acting as multiplication by \( \sqrt{q} \). We write \( T_w \) instead of \( \otimes T_w \). Let \( \text{Irr} \mathcal{W} \) be a set of representatives for the isomorphism classes of irreducible \( \mathcal{W} \)-modules over \( \tilde{Q}_l \). For any \( E \in \text{Irr} \mathcal{W} \) let \( E_q \) be the irreducible \( H_q \)-module corresponding naturally to \( E \). Let \( \mathcal{F} \) be the vector space of functions \( \mathcal{B}^F \to \tilde{Q}_l \). We regard \( \mathcal{F} \) as a \( G^F \)-module by \( \gamma : f \mapsto f' \), \( f'(B) = f(\gamma^{-1}B\gamma) \) for all \( B \in \mathcal{B}^F \). We identify \( H_q \) with the algebra of all endomorphisms of \( \mathcal{F} \) which commute with the \( G^F \)-action, by identifying \( T_w \) with the endomorphism \( f \mapsto f' \) where \( f'(B) = \sum_{B' \in \mathcal{B}^F} (B,B') \otimes H_w f(B) \) for all \( B \in \mathcal{B}^F \). As a module over \( \tilde{Q}_l[G^F] \otimes \mathcal{H}_q \) we have canonically \( \mathcal{F} = \bigoplus_{E \in \text{Irr} \mathcal{W}} \rho_E \otimes E_q \), where \( \rho_E \) is an irreducible \( G^F \)-module. Hence if \( \gamma \in G^F \) and \( w \in \mathcal{W} \) we have \( \text{tr}(T_w,F) = \sum_{E \in \text{Irr} \mathcal{W}} \text{tr}(\gamma,T_w,F) = \sum_{E \in \text{Irr} \mathcal{W}} \text{tr}(\gamma,\rho_E) \text{tr}(T_w,E_q) \).

From the definition we have \( \text{tr}(T_w,F) = |\{ B \in \mathcal{B}^F ; (B,\gamma B\gamma^{-1}) \in \mathcal{O}_w \}| = |(B^w_{g_z})^F| \).

Taking \( \gamma = g_z \) we obtain

\[
|B^w_{g_z}| = \sum_{E \in \text{Irr} \mathcal{W}} \text{tr}(g_z,\rho_E) \text{tr}(T_w,E_q).
\]

The quantity \( \text{tr}(g_z,\rho_E) \) can be computed explicitly, by the method of [L4], in terms of generalized Green functions and of the entries of the non-abelian Fourier transform matrices [L2]; in particular it is a polynomial with rational coefficients in \( \sqrt{q} \). The quantity \( \text{tr}(T_w,E_q) \) can be also computed explicitly (see [GP], Ch.10,11); it is a polynomial with integer coefficients in \( \sqrt{q} \). Thus \( |B^w_{g_z}| \) is an explicitly computable polynomial with rational coefficients in \( \sqrt{q} \). Substituting here \( \sqrt{q} \) by \( v \) we obtain the polynomial \( \Xi^{w}_{g,z} \). This argument shows also that \( \Xi^{w}_{g,z} \) is independent of \( p \) (note that the pairs \((g,z)\) up to conjugacy may be parametrized by a set independent of \( p \)).

This is how the various \( \Xi^{w}_{g,z} \) in §2 were computed, except in type \( A_1, A_2, B_2 \) where they were computed directly from the definitions. (For type \( B_3, C_3 \) we have used the computation of Green functions in [Sh]; for type \( G_2 \) we have used directly [CR] for the character of \( \rho_E \) at unipotent elements.)

7. In this section we assume that \( G \) is simply connected. Let \( \tilde{G} = G(k((\epsilon))) \) where \( \epsilon \) is an indeterminate. Let \( \mathcal{B} \) be the set of Iwahori subgroups of \( \tilde{G} \). Let \( \mathcal{W} \) the
affine Weyl group attached to $\tilde{G}$. Note that $\tilde{W}$ naturally indexes $(w \mapsto \mathcal{O}_w)$ the orbits of $\tilde{G}$ acting on $\tilde{B} \times \tilde{B}$ by simultaneous conjugation on the two factors. For $g \in \tilde{G}$ and $w \in \tilde{W}$ we set

$$\tilde{B}_g^w = \{ B \in \tilde{B} ; (B, gBg^{-1}) \in \mathcal{O}_w \}.$$ 

By analogy with [KL, §3] we expect that when $g$ is regular semisimple, $\tilde{B}_g^w$ has a natural structure of a locally finite union of algebraic varieties over $k$ of bounded dimension and that, moreover, if $g$ is also elliptic, then $\tilde{B}_g^w$ has a natural structure of algebraic variety over $k$. It would follow that for $g$ elliptic and $w \in \tilde{W}$,

$$\Xi_{g,w}^1 = \sum_{i,j \in \mathbb{Z}} (-1)^i \dim H^i_c(\tilde{B}_w^g, \mathcal{O}) \in \mathbb{Z}[v]$$

is well defined; one can then show that the formal sum $\sum_{w \in \mathbb{W}} \Xi_{g,w}^1 \tilde{T}_w$ is central in the completion of the affine Hecke algebra consisting of all formal sums $\sum_{w \in \mathbb{W}} a_w \tilde{T}_w$ ($a_w \in \mathbb{Q}(v)$) that is, it commutes with any $\tilde{T}_w$. (Here $\tilde{T}_w$ is defined as in §2 and the completion of the affine Hecke algebra is regarded as a bimodule over the actual affine Hecke algebra in the natural way.)

8. In this and the next subsection we assume that $G$ is adjoint. For $g \in G, z \in \tilde{Z}(g), w \in \mathbb{W}$ we set

$$\xi_{g,z}^w = \Xi_{g,z}^w |_{v=1} = \sum_{i \in \mathbb{Z}} (-1)^i \tr(z, H^i_c(\tilde{B}_w^g, \mathcal{O})) \in \mathbb{Z}.$$ 

This integer is independent of $l$. For any $g \in G, z \in \tilde{Z}(g)$ we set

$$\pi_{g,z} = \sum_{w \in \mathbb{W}} \xi_{g,z}^w w \in \mathbb{Z}[W].$$

This is the specialization of $\Pi_{g,z}$ for $v = 1$. Hence from 2(a) we see that $\pi_{g,z}$ is in the centre of the ring $\mathbb{Z}[W]$. Thus for any $C \in \mathbb{W}$ we can set $\xi_{g,z}^C = \xi_{g,z}^w$ where $w$ is any element of $C$. For $g \in G$ let $s_g$ be the set of all $C \in \mathbb{W}$ such that $\xi_{g,z}^C \neq 0$ for some $z \in \tilde{Z}(g)$ and $\xi_{g',z'}^C = 0$ for any $g' \in \overline{\mathcal{C}(g)} - \mathcal{C}(g)$ and any $z' \in \tilde{Z}(g')$. We describe the set $s_g$ and the values $\xi_{g,z}^C = 0$ for $C \in s_g, z \in \tilde{Z}(g)$, for various $G$ of low rank and various unipotent elements $g$ in $G$. We use the notation in §2. Moreover in the case where $\tilde{Z}(g) \neq \{1\}$ we denote by $z_n$ an element of order $n$ in $\tilde{Z}(g)$.

$G$ of type $A_1$.

$$s_{u_1} = (1), s_{u_0} = (s_1); \xi_{u_1,1}^1 = 2, \xi_{u_0,1}^1 = 1.$$ 

$G$ of type $A_2$.

$$s_{u_3} = (1), s_{u_1} = (s_1), s_{u_0} = (s_1 s_2).$$
\[ \xi_{u_4,1}^1 = 6, \xi_{u_1,1}^{(s_1)} = 1, \xi_{u_0,1}^{(s_1s_2)} = 1. \]

**G of type B₂.**

\[ s_{u_4} = (1), s_{u_2} = (s_1), s_{u_4} = \{(s_2), (s_1s_2s_1s_2)\}, s_{u_0} = (s_1s_2). \]

\[ \xi_{u_4,1}^1 = 8, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_1s_2s_1s_2)} = 0, \xi_{u_0,1}^{(s_1s_2)} = 2. \]

**G of type G₂.**

\[ s_{u_6} = (1), s_{u_4} = (s_2), s_{u_2} = (s_1), s_{u_4} = \{(s_1s_2s_1s_2s_1s_2), (s_1s_2s_1s_2s_1s_2)\}, s_{u_0} = (s_1s_2). \]

\[ \xi_{u_6,1}^1 = 12, \xi_{u_3,1}^{(s_2)} = 2, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_1s_2s_1s_2s_1s_2)} = -3, \xi_{u_1,2}^{(s_1s_2s_1s_2s_1s_2)} = 3, \]
\[ \xi_{u_1,2}^{(s_1s_2)} = 3, \xi_{u_1,1}^{(s_1s_2s_1s_2)} = 0, \xi_{u_1,2}^{(s_1s_2s_1s_2)} = 2, \xi_{u_1,3}^{(s_1s_2s_1s_2)} = 2, \xi_{u_0,1}^{(s_1s_2)} = 1. \]

**G of type B₃.**

\[ s_{u_9} = (1), s_{u_5} = (s_1), s_{u_4} = \{(s_3), (s_2s_3s_2s_3)\}, s_{u_3} = (s_1s_3), \]
\[ s_{u_2} = \{(s_1s_2), (w_0)\}, s_{u_0} = \{(s_2s_3), (s_2s_3s_1s_2s_3)\}. \]

\[ \xi_{u_9,1}^1 = 48, \xi_{u_5,1}^{(s_1)} = 4, \xi_{u_4,2}^{(s_2s_3s_2s_3)} = 0, \xi_{u_4,2}^{(s_2s_3s_2s_2s_3)} = 4, \xi_{u_4,1}^{(s_3)} = 8, \]
\[ \xi_{u_4,1}^{(s_3)} = 0, \xi_{u_3,1}^{(s_3)} = 2, \xi_{u_2,1}^{(w_0)} = 0, \xi_{u_2,2}^{(w_0)} = 6, \]
\[ \xi_{u_2,1}^{(s_1s_2)} = 2, \xi_{u_2,2}^{(s_2s_3)} = 2, \xi_{u_2,1}^{(s_2s_3)} = 0, \xi_{u_1,1}^{(s_1s_2s_3)} = 2, \xi_{u_1,2}^{(s_1s_2s_3)} = 2. \]

**G of type C₃.**

\[ s_{u_9} = (1), s_{u_6} = (s_3), s_{u_4} = \{(s_1), (s_2s_3s_2s_3)\}, s_{u_3} = (s_1s_3), \]
\[ s_{u_2} = (s_1s_2), s_{u_2'} = (s_2s_3), s_{u_0} = \{(s_2s_3s_1s_2s_3), (w_0)\}. \]

\[ \xi_{u_9,1}^1 = 48, \xi_{u_6,1}^{(s_3)} = 8, \xi_{u_4,1}^{(s_2s_3s_2s_3)} = 0, \xi_{u_4,2}^{(s_2s_3s_2s_3)} = 4, \]
\[ \xi_{u_4,1}^{(s_1)} = 4, \xi_{u_4,1}^{(s_3)} = 0, \xi_{u_3,1}^{(s_1s_3)} = 2, \xi_{u_2,1}^{(s_1s_3)} = 2, \xi_{u_2,1}^{(s_2s_3)} = 2, \xi_{u_1,1}^{(s_2s_3)} = 1, \xi_{u_1,1}^{(s_2s_3s_1s_2s_3)} = 1, \xi_{u_1,1}^{(w_0)} = -3, \xi_{u_1,2}^{(w_0)} = 3, \xi_{u_0,1}^{(s_1s_2s_3)} = 1. \]
9. For any unipotent element $u \in G$ let $n_u$ be the number of isomorphism classes of irreducible representations of $\bar{Z}(u)$ which appear in the Springer correspondence for $G$. Consider the following properties of $G$:

(a) $W = \bigsqcup_u s_u$

($u$ runs over a set of representatives for the unipotent classes in $G$); for any unipotent element $u \in G$,

(b) $|s_u| = n_u$.

By the results in §8, (a),(b) hold if $G$ has rank $\leq 3$. We will show elsewhere that (a),(b) hold if $G$ is of type $A$. We expect that (a),(b) hold in general. The equality $W = \bigsqcup u s_u$ is clear since for a regular unipotent $u$ and any $w$ we have $\xi_{w,1} = 1$.

Consider also the following property of $G$: for any $g \in G$, $w \in W$,

$$\xi_w g, 1$$ is equal to the trace of $w$ on the Springer representation of $W$ on $\bigoplus_i H^2(B_g, \bar{Q}_l)$.

(c) $\xi_{g,1}$ is equal to the trace of $w$ on the Springer representation of $W$ on $\bigoplus_i H^2(B_g, \bar{Q}_l)$.

Again (c) holds if $G$ is of type $A$ and in the examples in §8; we expect that it holds in general. Note that in (c) one can ask whether for any $z$, $\xi_{w,z}$ is equal to the trace of $wz$ on the Springer representation of $W \times \bar{Z}(g)$ on $\bigoplus_i H^2(B_g, \bar{Q}_l)$; but such an equality is not true in general for $z \neq 1$ (for example for $G$ of type $B_2$).

10. In this subsection we assume that $k = \mathbb{C}$. We show how the method of [KL, 9.1] extends to the case of symmetric spaces.

Let $K$ be a subgroup of $G$ which is the fixed point set of an involution $\theta : G \to G$. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of $G, K$. Let $\mathfrak{p}$ be the $(-1)$-eigenspace of $\theta : \mathfrak{g} \to \mathfrak{g}$. Let $\mathfrak{p}_{nil}$ be the set of elements $x \in \mathfrak{p}$ which are nilpotent in $\mathfrak{g}$. Note that $K$ acts naturally on $\mathfrak{p}_{nil}$ with finitely many orbits. Let $\mathcal{T}$ be the variety of all tori $T$ in $G$ such that $\theta(t) = t^{-1}$ for all $t \in T$ and such that $T$ has maximum possible dimension. It is known that $K$ acts transitively on $\mathcal{T}$ by conjugation. For $T \in \mathcal{T}$ let $\mathcal{W}_T$ be the normalizer of $T$ in $K$ modulo the centralizer of $T$ in $K$. Let $\mathcal{W}$ be the set of conjugacy classes in the finite group $\mathcal{W}_T$; this is independent of the choice of $T$.

Let $\Phi = \mathbb{C}((\epsilon))$, $A = \mathbb{C}[[\epsilon]]$ where $\epsilon$ is an indeterminate. Let $\Phi'$ be an algebraic closure of $\Phi$. Then the groups $G(\Phi), K(\Phi)$ are well defined and the set $\mathcal{T}(\Phi)$ of $\Phi$-points of $T$ is well defined. Let $\mathfrak{p}_\Phi = \Phi \otimes \mathfrak{p}$, $\mathfrak{p}_A = A \otimes \mathfrak{p}$.

The group $K(\Phi)$ acts naturally by conjugation on $\mathcal{T}(\Phi)$; as in [KL, §1, Lemma 2] we see that the set of $K(\Phi)$-orbits on $\mathcal{T}(\Phi)$ is naturally in $1-1$ correspondence with the set $\mathcal{W}$. For $\gamma \in \mathcal{W}$ let $\mathcal{O}_\gamma$ be the $K(\Phi)$-orbit on $\mathcal{T}(\Phi)$ corresponding to $\gamma$.

An element $\xi \in \mathfrak{p}_\Phi$ is said to be "regular semisimple" if there is a unique $T' \in \mathcal{T}(\Phi')$ such that $\xi \in \text{Lie}(T')$; we then set $T'_\xi = T'$ and we have necessarily $T'_\xi \in \mathcal{T}(\Phi)$.
Let $N \in \mathfrak{p}_{\text{nil}}$. We consider the subset $N + \epsilon \mathfrak{p}_A$ of $\mathfrak{p}_\Phi$. As in [KL, 9.1] there is a unique element $\gamma \in \mathcal{W}$ such that the following holds: there exists a "Zariski open dense" subset $V$ of $N + \epsilon \mathfrak{p}_A$ such that for any $\xi \in V$, $\xi$ is "regular semisimple" and $T'_\xi \in \mathcal{O}_\gamma$. Note that $N \mapsto \gamma$ is constant on $K$-orbits hence it defines a map $\Psi$ from the set of $K$-orbits on $\mathfrak{p}_{\text{nil}}$ to $\mathcal{W}$. (In the case where $G = H \times H$ where $H$ is a connected reductive group, $K$ is the diagonal in $H \times H$ and $\theta(a,b) = (b,a)$, the map $\Psi$ reduces to the map defined in [KL, 9.1].)

One can show that if $(G, K) = (GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$, then $\Psi$ is a bijection. Note that for general $(G, K)$, $\Psi$ is neither injective nor surjective.

\textbf{References}

[CR] B.Chang and R.Ree, \textit{The characters of $G_2(q)$}, Istituto Naz.di Alta Mat. Symposia Math. \textbf{XIII} (1974), 395-413.

[DL] P.Deligne and G.Lusztig, \textit{Representations of reductive groups over finite fields}, Ann.Math. \textbf{103} (1976), 103-161.

[GP] M.Geck and G.Pfeiffer, \textit{Characters of finite Coxeter groups and Iwahori-Hecke algebras}, Clarendon Press Oxford, 2000.

[KL] D.Kazhdan and G.Lusztig, \textit{Fixed point varieties on affine flag manifolds}, Isr.J.Math. \textbf{62} (1988), 129-168.

[L1] G.Lusztig, \textit{On the reflection representation of a finite Chevalley group}, Representation theory of Lie groups, LMS Lect-notes Ser.34, Cambridge U.Press, 1979, pp. 325-337.

[L2] G.Lusztig, \textit{Unipotent representations of a finite Chevalley group of type $E_8$}, Quart.J.Math. \textbf{30} (1979), 315-338.

[L3] G.Lusztig, \textit{Character sheaves, I}, Adv.in Math. \textbf{56} (1985), 193-237.

[L4] G.Lusztig, \textit{On the character values of finite Chevalley groups at unipotent elements}, J.Alg. \textbf{104} (1986), 146-194.

[L5] G.Lusztig, \textit{Notes on unipotent classes}, Asian J.Math. \textbf{1} (1997), 194-207.

[Sh] T.Shoji, \textit{On the Green polynomials of Chevalley groups of type $F_4$}, Comm.in Alg. \textbf{10} (1982), 505-543.

[S1] N.Spaltenstein, \textit{Polynomials over local fields, nilpotent orbits and conjugacy classes in Weyl groups}, Astérisque \textbf{168} (1988), 191-217.

[S2] N.Spaltenstein, \textit{On the Kazhdan-Lusztig map for exceptional Lie algebras}, Adv.Math \textbf{83} (1990), 48-74.