WEYL MODULES FOR LIE SUPERALGEBRAS

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Abstract. We define global and local Weyl modules for Lie superalgebras of the form \( g \otimes A \), where \( A \) is an associative commutative unital \( \mathbb{C} \)-algebra and \( g \) is a basic Lie superalgebra or \( \mathfrak{sl}(n, n) \), \( n \geq 2 \). Under some mild assumptions, we prove universality, finite-dimensionality, and tensor product decomposition properties for these modules. These properties are analogues of those of Weyl modules in the non-super setting. We also point out some features that are new in the super case.

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1. INTRODUCTION

Map Lie algebras, also known as generalized current Lie algebras, are Lie algebras of regular maps from a scheme \( X \) to a (generally finite-dimensional) target Lie algebra \( g \). They form a large class of Lie algebras that include the important loop and current algebras as special cases, and their representation theory is an active area of research. We refer the reader to [NS13] for a survey of the field. A vital ingredient in the theory is played by global and local Weyl modules, which are universal objects with respect to certain highest weight properties. The local Weyl modules are finite-dimensional but not, in general, irreducible. They were first defined, in the loop case, in [CP01] and extended to the map case in [FL04].

Replacing the target Lie algebra \( g \) by a Lie superalgebra, we obtain the class of map superalgebras. The study of these algebras is still in its infancy. In the loop case, where \( X \) is a torus, and when \( g \) is a basic Lie superalgebra, the finite-dimensional modules were classified in [ERZ04, ER13]. In the more general setting where the coordinate ring of \( X \) is finitely generated, \( g \) is a basic Lie superalgebra, and where we also consider maps equivariant with respect to a finite abelian group acting freely on the rational points of \( X \), the irreducible finite-dimensional modules were classified in [Sav14]. However, Weyl modules have not been defined in the super setting, except for a quantum analogue in the loop case for \( g = \mathfrak{sl}(m, n) \) considered in [Zha14].

In the current paper, we initiate the study of Weyl modules for Lie superalgebras. In particular, we define global and local Weyl modules for Lie superalgebras of the form \( g \otimes_{\mathbb{C}} A \), where \( A \) is
an associative commutative unital $\mathbb{C}$-algebra and $\mathfrak{g}$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$.

After defining global Weyl modules in the super setting (Definition 3.3), we give a presentation in terms of generators and relations (Proposition 3.4) and prove that these modules are universal highest weight objects in a certain category (Proposition 3.5). We then define local Weyl modules (Definition 4.1), prove that they are finite-dimensional (Theorem 4.12), and that they also satisfy a certain universal property with respect to so-called highest $\text{map-weight}$ modules (Proposition 4.13). Finally, we show that the local Weyl modules satisfy a nice tensor product property (Theorem 4.15).

The above-mentioned results demonstrate that the Weyl modules defined in the current paper satisfy many of the properties that their non-super analogues do. However, there are some important differences. First of all, the Borel subalgebras of basic Lie superalgebras are not all conjugate under the action of the Weyl group, in contrast to the situation for finite-dimensional simple Lie algebras. For this reason, our definitions of Weyl modules depend on a choice of system of simple roots. Second, the category of finite-dimensional modules for a basic Lie superalgebra is not semisimple in general, again in contrast to the non-super setting. For this reason, the so-called Kac modules play an important role in the representation theory. These are maximal finite-dimensional modules of a given highest weight. The Weyl modules defined in the current paper can be viewed as a unification of several types modules in the following sense. If $\mathfrak{g}$ is a simple Lie algebra, then our definitions reduce to the usual ones. Thus, the Weyl modules defined here are generalizations of the Weyl modules in the non-super case. On the other hand, if $A = \mathbb{C}$, then the global and local Weyl modules are equal and coincide with the (generalized) Kac module, which, if $\mathfrak{g}$ is a simple Lie algebra, is the irreducible module (of a given highest weight). These relationships can be summarized in the following diagram:

$$
\begin{array}{ccc}
\text{super} & \longrightarrow & \text{non-super} \\
\text{general } A & \text{global/local Weyl (super)module} & \text{global/local Weyl module} \\
\downarrow & & \downarrow \\
A = \mathbb{C} & \text{generalized Kac module} & \text{irreducible module}
\end{array}
$$

The definition of global and local Weyl modules for Lie superalgebras opens up a number of directions of possible further research. We conclude this introduction by listing some of these.

(a) One should be able to define Weyl modules when $\mathfrak{g}$ is not basic. For example, in [CMS], the finite-dimensional irreducible $\mathfrak{g} \otimes A$-modules have been classified in the case that $\mathfrak{g}$ is the queer Lie superalgebra. The nature of the classification (in terms of evaluation modules) seems to indicate that the theory of Weyl modules should be relatively similar to the case considered in the current paper.

(b) Twisted versions of Weyl modules have been defined and investigated in the non-super setting (see [CFS08, FMS13, FKKS12, FMS15]). One should similarly be able to develop a twisted theory of Weyl modules for equivariant map Lie superalgebras.

(c) A categorical approach to Weyl modules was developed in [CFK10]. It would be interesting to develop this theory in the super setting. In particular, one should be able to define super analogues of Weyl functors.

(d) Recently, in [SVV, BHLW], local Weyl modules for current algebras have appeared as trace decategorifications of categories used to categorify quantum groups. It is natural to ask how the super analogues of Weyl modules defined in the current paper are related to the super analogues, defined in [KKT], of the afore-mentioned categories.
Acknowledgements. We would like to thank S.-J. Cheng and E. Neher for helpful conversations.

Note on the arXiv version. For the interested reader, the tex file of the arXiv version of this paper includes hidden details of some straightforward computations and arguments that are omitted in the pdf file. These details can be displayed by switching the details toggle to true in the tex file and recompiling.

2. Preliminaries

In this section, we review some facts about associative commutative algebras and Lie superalgebras that will be needed in the sequel. We also prove some results about simple root systems for basic Lie superalgebras.

2.1. Commutative algebras. Let $A$ denote an associative commutative unital $\mathbb{C}$-algebra. We define the support of an ideal $I$ of $A$ to be

$$\text{Supp}(I) = \{ m \in \text{MaxSpec } A \mid I \subseteq m \}. $$

Lemma 2.1. Let $I$, $J$ be ideals of $A$.

(a) For all positive integers $N$, we have $\text{Supp}(I) = \text{Supp}(I^N)$.

(b) If $A$ is finitely generated, then the support of $I$ is finite if and only if $I$ has finite codimension in $A$.

(c) Suppose $\text{Supp}(I) \cap \text{Supp}(J) = \emptyset$. Then $I + J = A$ and $IJ = I \cap J$.

Proof. The proofs of parts (b) and (c) can be found in [Sav14, §2.1]. It remains to prove part (a). Fix a positive integer $N$. It is clear that $\text{Supp}(I) \subseteq \text{Supp}(I^N)$. The reverse inclusion follows from the fact that maximal ideals are prime. □

2.2. Lie superalgebras.

Definition 2.2. A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $g = g_{\bar{0}} \oplus g_{\bar{1}}$ with a bilinear multiplication $[\cdot, \cdot]$ satisfying the following axioms:

(a) The multiplication respects the grading: $[g_i, g_j] \subseteq g_{i+j}$ for all $i, j \in \mathbb{Z}_2$.

(b) Skew-supersymmetry: $[a, b] = -(-1)^{|a||b|}[b, a]$, for all homogeneous elements $a, b \in g$.

(c) Super Jacobi Identity: $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$, for all homogeneous elements $a, b, c \in g$.

Observe that $g_{\bar{0}}$ inherits the structure of a Lie algebra and that $g_{\bar{1}}$ inherits the structure of a $g_{\bar{0}}$-module. A Lie superalgebra $g$ is said to be simple if there are no nonzero proper ideals, that is, there are no nonzero proper graded subspaces $i \subseteq g$ such that $[i, g] \subseteq i$. A finite-dimensional simple Lie superalgebra $g = g_{\bar{0}} \oplus g_{\bar{1}}$ is said to be classical if the $g_{\bar{0}}$-module $g_{\bar{1}}$ is completely reducible. Otherwise, it is said to be of Cartan type.

For a classical Lie superalgebra $g$, the $g_{\bar{0}}$-module $g_{\bar{1}}$ is either irreducible or a direct sum of two irreducible representations. In the first case, $g$ is said to be of type II, and in the second case, $g$ is said to be of type I. A classical Lie superalgebra is said to be basic if it admits a nondegenerate invariant bilinear form. Otherwise, it is said to be strange. We will mostly be concerned with basic Lie superalgebras. However, the majority of our results also hold for the Lie superalgebra $\mathfrak{sl}(n, n)$, $n \geq 2$, which is a 1-dimensional central extension of the basic Lie superalgebra $A(n-1, n-1)$. (Throughout the paper we will somewhat abuse terminology by talking of the Lie superalgebras $A(m, n)$, $B(m, n)$, etc., instead of the Lie superalgebras of type $A(m, n)$, $B(m, n)$, etc.)
2.3. Contragredient Lie superalgebras. Let $I = \{1, \ldots, n\}$, let $A = (a_{ij})_{i,j\in I}$ be a complex
matrix, and let $p : I \to \mathbb{Z}_2$ be a set map. Fix an even vector space $\mathfrak{h}$ of dimension $2n - \text{rank} A$ and
linearly independent $a_i \in \mathfrak{h}^*$, $i \in I$, and $H_i \in \mathfrak{h}$, $i \in I$, such that $\alpha_j(H_i) = a_{ij}$, for all $i, j \in I$.
We define $\tilde{\mathfrak{g}}(A)$ to be the Lie superalgebra generated by the even vector space $\mathfrak{h}$ and elements $X_i, Y_i, i \in I$, with the parity of $X_i$ and $Y_i$ equal to $p(i)$, and subject to the relations
$$[X_i, Y_j] = \delta_{ij}H_i, \quad [H, H'] = 0, \quad [H, X_i] = \alpha_i(H)X_i, \quad [H, Y_i] = -\alpha_i(H)Y_i,$$
for $i, j \in I$ and $H, H' \in \mathfrak{h}$.

The contragredient Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ is defined to be the quotient of $\tilde{\mathfrak{g}}(A)$ by the ideal
that is maximal among all the ideals that intersects $\mathfrak{h}$ trivially (see [Mus12, §5.2]). The images of the elements $X_I, Y_I, H_I, i \in I$, in $\mathfrak{g}(A)$ are denoted by the same symbols.

Since the action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable, we have a root space decomposition
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \Delta \subseteq \mathfrak{h}^*,$$
where every root space $\mathfrak{g}_\alpha$ is either purely even or purely odd. A root $\alpha$ is called even (resp. odd)
if $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_0$ (resp. $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_1$). We denote by $\Delta_0$ and $\Delta_1$ the sets of even and odd roots respectively.
A linearly independent subset $\Sigma = \{\beta_1, \ldots, \beta_n\} \subseteq \Delta$ is called a base if we can find $X_{\beta_i} \in \mathfrak{g}_{\beta_i}$ and $Y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$, $i = 1, \ldots, n$, such that $\{X_{\beta_i}, Y_{\beta_i} \mid i = 1, \ldots, n\} \cup \mathfrak{h}$ generates $\mathfrak{g}(A)$, and
$$[X_{\beta_i}, Y_{\beta_j}] = 0 \text{ for } i \neq j.$$

Defining $H_{\beta_i} = [X_{\beta_i}, Y_{\beta_i}]$, it follows that the elements $X_{\beta_i}, Y_{\beta_i}$ and $H_{\beta_i}$ satisfy the following relations:
$$[H_{\beta_j}, X_{\beta_i}] = \beta_i(H_{\beta_j})X_{\beta_i}, \quad [H_{\beta_j}, Y_{\beta_i}] = -\beta_i(H_{\beta_j})Y_{\beta_i}, \quad [X_{\beta_i}, Y_{\beta_j}] = \delta_{ij}H_{\beta_i}, \quad i, j \in \{1, \ldots, n\}.$$  \hfill (2.1)

The matrix $A_\Sigma = (b_{ij})$, where $b_{ij} = \beta_j(H_{\beta_i})$, is called the Cartan matrix with respect to the base $\Sigma$. The original set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is called the standard base. It is clear that $A$ is the Cartan matrix associated to $\Pi$, i.e. $A = A_\Pi$. The relations (2.1) imply that every root is a purely positive or purely negative integer linear combination of elements in $\Sigma$. We call such a root positive or negative, respectively, and we have the decomposition $\Delta = \Delta^+(\Sigma) \cup \Delta^-\Sigma$, where $\Delta^+(\Sigma)$ and $\Delta^-\Sigma$ denote the set of positive and negative roots, respectively. A positive root is called simple if it cannot be written as a sum of two positive roots. It is clear that a root is simple if and only if it lies in $\Sigma$. Thus, $\Sigma$ is a system of simple roots in the usual sense. We define $\Sigma_\pm := \Sigma \cap \Delta_\pm$, and $\Delta^\pm_\Sigma := \Delta_\Sigma \cap \Delta^\pm(\Sigma)$ for $z \in \mathbb{Z}_2$. The triangular decomposition of $\mathfrak{g}$ induced by $\Sigma$ is given by
$$\mathfrak{g} = n^-(\Sigma) \oplus \mathfrak{h} \oplus n^+(\Sigma),$$
where $n^+(\Sigma)$ (resp. $n^-(\Sigma)$) is the subalgebra generated by $X_\beta$ (resp. $Y_\beta$), $\beta \in \Sigma$. The subalgebra $\mathfrak{h}(\Sigma) = \mathfrak{h} \oplus n^+(\Sigma)$ is called the Borel subalgebra corresponding to $\Sigma$. Note that $\Delta_0^+(\Sigma)$ is a system of positive roots for the Lie algebra $\mathfrak{g}_0$. We denote by $\Sigma(\mathfrak{g}_0)$ the set of simple roots of $\mathfrak{g}_0$ with respect to this system.

Suppose that $\mathfrak{g}$ is equal to $A(m, n)$ with $m \neq n$, $\mathfrak{g}(n, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, or $G(3)$. By [Mus12, Theorems 5.3.2, 5.3.3 and 5.3.5], we have that $\mathfrak{g}$ is a contragredient Lie superalgebra. The Lie superalgebra $\mathfrak{sl}(n, n)$ (resp. $A(n, n)$) is isomorphic to $[\mathfrak{g}(n, n), \mathfrak{g}(n, n)]$ (resp. $[\mathfrak{g}(n, n), \mathfrak{g}(n, n)]/C$, where $C$ is a one-dimensional center). The image of $X \in \mathfrak{sl}(n, n)$ in $A(n, n)$ will be denoted by the same symbol. Fixing a base $\Sigma$ of $\mathfrak{g}(n, n)$, the triangular decomposition $\mathfrak{g}(n, n) = n^-(\Sigma) \oplus \mathfrak{h} \oplus n^+(\Sigma)$ induces the triangular decompositions
$$\mathfrak{sl}(n, n) = n^-(\Sigma) \oplus \mathfrak{h}' \oplus n^+(\Sigma) \quad \text{and} \quad A(n, n) = n^-(\Sigma) \oplus (\mathfrak{h}'/C) \oplus n^+(\Sigma),$$
where $\mathfrak{h}'$ is the subspace of $\mathfrak{h}$ generated by $H_\beta$, $\beta \in \Sigma$ (see [Mus12, Lem. 5.2.3]). In particular, any root of $\mathfrak{sl}(n, n)$ or $A(n, n)$ is a purely positive or a purely negative integer linear combination of
elements in $\Sigma$. Therefore $\Delta = \Delta^+(\Sigma) \sqcup \Delta^-(\Sigma)$ is a decomposition of the system of roots of $\mathfrak{sl}(n,n)$ and $A(n,n)$. The matrix $A_\Sigma$ is also called the Cartan matrix of $\mathfrak{sl}(n,n)$ and $A(n,n)$ corresponding to $\Sigma$.

**Remark 2.3.** Assume $\mathfrak{g}$ is a basic Lie superalgebra, $\mathfrak{gl}(n,n)$ with $n \geq 2$, or $\mathfrak{sl}(n,n)$ with $n \geq 3$. Then $[\alpha, \beta, \mathfrak{g}] \neq 0$ if $\alpha, \beta, \alpha + \beta \in \Delta$. In particular, the parity of $\alpha + \beta$ is the sum of the parities of $\alpha$ and $\beta$. Moreover, if $\mathfrak{g} \neq A(1,1)$, then dim $\mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$ (see [Mus12, Ch. 2]).

In Section 4, we will be particularly interested in systems of simple roots $\Sigma$ satisfying the following property:

\[(2.2) \quad \text{For all } \alpha \in \Sigma_1, \text{ there exists } \alpha' \in \Delta^+_r(\Sigma) \text{ such that } \alpha + \alpha' \in \Delta(\Sigma).
\]

Note that such an element $\alpha + \alpha'$ is necessarily an even root. Our next goal is to show that a system of simple roots satisfying $(2.2)$ always exists.

Let $\Sigma$ be a system of simple roots and suppose that $\beta \in \Sigma$ is an odd root with $\beta(H_\beta) = 0$. (Such a root is known as an *isotropic* odd root.) Then define the reflection $r_\beta: \Sigma \to \Delta$ with respect to $\beta$ by

\[
\begin{align*}
    r_\beta(\beta) &= -\beta, \\
    r_\beta(\beta') &= \beta', & \text{for } \beta' \in \Sigma, \beta' \neq \beta, \quad \beta(H_\beta) = \beta'(H_\beta) = 0, \\
    r_\beta(\beta') &= \beta + \beta', & \text{for } \beta' \in \Sigma, \beta' \neq \beta, \quad \beta(H_\beta) \neq 0 \text{ or } \beta'(H_\beta) \neq 0.
\end{align*}
\]

By [CW12, Lem. 1.30], $r_\beta(\Sigma)$ is a system of simple roots, and

\[(2.3) \quad \Delta^+(r_\beta(\Sigma)) \setminus \{ -\beta \} = \Delta^+(\Sigma) \setminus \{ \beta \}.
\]

(We use here the fact that $\mathfrak{gl}(n,n)$ and $\mathfrak{sl}(n,n)$ have the same system of simple roots as $A(n,n)$.)

If $\mathfrak{g}$ is a basic Lie superalgebra, $\mathfrak{gl}(n,n)$, $n \geq 2$, or $\mathfrak{sl}(n,n)$, $n \geq 2$, then $\mathfrak{g}$ admits a system of simple roots with only one odd root (see [Mus12, Tables 3.4.4 and 5.3.1]). Let $\Pi = \{ \gamma_1, \ldots, \gamma_n \}$ denote such a system and let $\gamma_\alpha$ be the unique odd root that lies in $\Pi$. The system $\Pi$ is often called a *distinguished* system of simple roots. When $\mathfrak{g} \neq B(0,n)$, we have that $\gamma_\alpha$ is an odd isotropic regular root. Then we can consider the odd reflection $r_\gamma$ with respect to $\gamma_\alpha$.

**Proposition 2.4.** Let $\Pi$ be a distinguished system of simple roots for $\mathfrak{g}$.

(a) If $\mathfrak{g}$ is a basic Lie superalgebra of type II, then $\Pi$ satisfies condition $(2.2)$.

(b) If $\mathfrak{g}$ is $\mathfrak{gl}(n,n)$, $n \geq 2$, $\mathfrak{sl}(n,n)$, $n \geq 2$, or a basic Lie superalgebra other than $B(0,n)$, then $r_\gamma(\Pi)$ satisfies condition $(2.2)$.

In particular, $\mathfrak{g}$ admits at least one system of simple roots satisfying $(2.2)$.

**Proof.** Part (a) follows from direct examination of the distinguished root systems in type II (see, for example, [FSS00, Tables 3.54, 3.57–3.60]).

Now suppose that $\gamma_\alpha$ is isotropic and let $\Pi' = r_\gamma(\Pi)$. To prove part (b), we will show that $\alpha + r_\gamma(\gamma_\alpha) \in \Delta^+_0(\Pi')$, for all odd roots $\alpha \in \Pi' \setminus \{ r_\gamma(\gamma_\alpha) \}$. First assume that $\mathfrak{g}$ is $\mathfrak{gl}(n,n)$ ($n \geq 2$), $\mathfrak{sl}(n,n)$ ($n \geq 2$), or a basic Lie superalgebra other than $B(0,n)$ or $D(2,1;\alpha)$. One can verify, by looking at each distinguished Cartan matrix, that $\gamma_\alpha(H_{\gamma_\alpha \pm 1}) = -1$ (when $1 \leq s \pm 1 \leq n$) and $\gamma_\alpha(H_{\gamma_\alpha \pm j}) = 0$ when $j \geq 2$ (and $1 \leq s \pm j \leq n$). (See, for example, [FSS00, Tables 3.53–3.58].) The odd root $\gamma_\alpha$ is indicated there by an X on the corresponding node in the Dynkin diagram.) Thus

\[
\begin{align*}
    r_\gamma(\gamma_\alpha) &= -\gamma_\alpha, \\
    r_\gamma(\gamma_\alpha \pm 1) &= \gamma_\alpha + \gamma_\alpha \pm 1 \quad \text{and} \\
    r_\gamma(\gamma_\alpha \pm j) &= \gamma_\alpha \pm j, & \text{for all } j \geq 2.
\end{align*}
\]

Since the only odd root in $\Pi$ is $\gamma_\alpha$, the odd roots of $\Pi'$ are precisely $r_\gamma(\gamma_{\alpha-1}), r_\gamma(\gamma_\alpha), r_\gamma(\gamma_{\alpha+1})$. Now, by (2.3), we have $\Delta^+(\Pi) \setminus \{ \gamma_\alpha \} = \Delta^+(\Pi') \setminus \{ r_\gamma(\gamma_\alpha) \}$, which implies that $\Delta^+_0(\Pi) = \Delta^+_0(\Pi')$. Thus

\[
\begin{align*}
    r_\gamma(\gamma_{\alpha+1}) + r_\gamma(\gamma_\alpha) &= \gamma_{\alpha+1} \in \Delta^+_0(\Pi) = \Delta^+_0(\Pi').
\end{align*}
\]
Finally, assume $g = D(2,1;\alpha)$. Then $\Pi = \{\gamma_1, \gamma_2, \gamma_3\}$, where $s = 1$ and $\gamma_1(H_{\gamma_j}) = -1$, for $j = 2, 3$ (see [FSS00, Table 3.60]). Then every element of $\Pi' = \{r_{\gamma_1}(\gamma_1), r_{\gamma_1}(\gamma_2), r_{\gamma_1}(\gamma_3)\}$ is odd, and again $r_{\gamma_1}(\gamma_j) + r_{\gamma_1}(\gamma_1) = \gamma_j \in \Delta^0_\gamma(\Pi) = \Delta^0_\gamma(\Pi')$, for $j = 2, 3$.

**Remark 2.5.** There exist systems of simple roots that do not satisfy (2.2). For instance, if $g$ is of type I, then a distinguished system $\Pi$ does not satisfy (2.2). This follows from the fact that the induced $\mathbb{Z}$-gradation is of the form $g_{-1} \oplus g_0 \oplus g_1$ (see [Kac78, Prop. 1.6]).

### 2.4. Generalized Kac modules

For the remainder of the paper, we assume that $g$ is a basic Lie superalgebra or $\mathfrak{sl}(n,n)$, $n \geq 2$. We fix a system of simple roots $\Sigma$, define

$$\Delta^+_z = \Delta^+_z(\Sigma) \text{ for all } z \in \mathbb{Z}_2,$$

and let $g = n^- \oplus h \oplus n^+$ be the triangular decomposition induced by $\Sigma$, i.e. $n^\pm = n^\pm(\Sigma)$. In the case that $g$ is $\mathfrak{sl}(n,n)$ or $A(n,n)$, we consider the triangular decomposition induced by $\mathfrak{gl}(n,n)$. Recall that the elements $X_\alpha, Y_\alpha, \alpha \in \Sigma$, generate the subalgebras $n^+$ and $n^-$, respectively.

Since $g_0$ is a reductive Lie algebra, for each even root $\alpha$ we can choose elements $X_\alpha \in g_0, Y_\alpha \in g_{-\alpha}$, and $H_\alpha \in h$, such that the subalgebra generated by these elements is isomorphic to $\mathfrak{sl}(2)$, with these elements satisfying the relations for the standard Chevalley generators. In this case, we say the set $\{X_\alpha, Y_\alpha, H_\alpha\}$ is an $\mathfrak{sl}(2)$-triple.

We denote the irreducible highest weight $g$-module with highest weight $\lambda \in h^*$ by $V(\lambda)$. Define

$$\Lambda^+ = \Lambda^+(\Sigma) = \{\lambda \in h^* \mid \dim V(\lambda) < \infty\}.$$ 

Note that, for $\lambda \in \Lambda^+$, since $V(\lambda)$ is finite dimensional, we have $\lambda(H_\alpha) \in \mathbb{N}$, for all $\alpha \in \Sigma(g_0)$.

**Definition 2.6** (The module $\widetilde{V}(\lambda)$). For $\lambda \in \Lambda^+$, we define $\widetilde{V}(\lambda)$ to be the $g$-module generated by a vector $v_\lambda$ with defining relations

$$n^+_0 v_\lambda = 0, \quad h v_\lambda = \lambda(h)v_\lambda, \quad Y_\alpha^{(0)} v_\lambda = 0, \quad \text{for all } h \in h, \ \alpha \in \Sigma(g_0).$$

**Proposition 2.7.** For all $\lambda \in \Lambda^+$, the module $\widetilde{V}(\lambda)$ is finite-dimensional.

**Proof.** Let $L(\lambda)$ be the irreducible $g_0$-module of highest weight $\lambda$. Since $g_0$ is a reductive Lie algebra and $\lambda(H_\alpha) \in \mathbb{N}$, for all $\alpha \in \Sigma(g_0)$, we have that $L(\lambda)$ is finite dimensional. Moreover, it is well known that $L(\lambda)$ is isomorphic to the $g_0$-module generated a vector $u_\lambda$ with defining relations

$$n^+_0 u_\lambda = 0, \quad h u_\lambda = \lambda(h)u_\lambda, \quad Y_\alpha^{(0)} u_\lambda = 0, \quad \text{for all } h \in h, \ \alpha \in \Sigma(g_0).$$

Let $V' = U(g_0)u_\lambda \subseteq \widetilde{V}(\lambda)$ be the $g_0$-submodule of $\widetilde{V}(\lambda)$ generated by $v_\lambda$. Then the map given by

$$\varphi: L(\lambda) \to V', \quad xu_\lambda \mapsto xv_\lambda, \quad \text{for all } x \in U(g_0),$$

is a well-defined epimorphism of $g_\Lambda$-modules. Thus, $V'$ is finite dimensional. Then it follows from the PBW Theorem for Lie superalgebras (see, for instance, [CW12, Th. 1.36]) that $\widetilde{V}(\lambda)$ is finite dimensional. 

**Lemma 2.8.** Suppose $V$ is a finite-dimensional $g$-module generated by a highest weight vector of weight $\lambda \in \Lambda^+$. Then there exists an unique submodule $W$ of $\widetilde{V}(\lambda)$ such that $\widetilde{V}(\lambda)/W \cong V$ as $g$-modules.

**Proof.** Let $v \in V_\lambda$ be a highest weight vector. Then the first two relations in (2.5) are satisfied by $v$, by the definition of a highest weight vector. The fact that $g_0$ is a reductive Lie algebra and $V$ is finite dimensional implies that $v$ also satisfies the last relation in (2.5). Thus the map $\widetilde{V}(\lambda) \to V$ defined by extending the assignment $v_\lambda \mapsto v$ is a well-defined epimorphism of $g$-modules. Since $\dim V_\lambda = 1 = \dim \widetilde{V}(\lambda)_\lambda$ and homomorphisms between modules preserve weight spaces, this map is unique up to scalar multiple. Thus, the kernel $W$ of this map is unique. 

□
Since every irreducible finite-dimensional \( g \)-module is generated by a highest weight vector of weight \( \lambda \in \Lambda^+ \), Lemma 2.8 applies to irreducible finite-dimensional \( g \)-modules.

**Remark 2.9.** It follows from Lemma 2.8 that \( \bar{V}(\lambda) \) coincides with the *generalized Kac module* defined in [Cou, p. 8]. Thus, when \( \Sigma \) is a distinguished root system, it follows from [Cou, Lem. 3.5] that \( \bar{V}(\lambda) \) is isomorphic to the usual Kac module defined in [Kac78, p. 613].

### 3. Global Weyl modules

Recall that \( g \) is either a basic classical Lie superalgebra or \( \mathfrak{sl}(n, n) \), \( n \geq 2 \). Let \( A \) be an associative commutative unital \( \mathbb{C} \)-algebra. We can then consider the Lie superalgebra \( g \otimes_{\mathbb{C}} A \), where the \( \mathbb{Z}_2 \)-grading is given by \((g \otimes A)_j = g_j \otimes A, j \in \mathbb{Z}_2 \), and the bracket is determined by \([x_1 \otimes a_1, x_2 \otimes a_2] = [x_1, x_2] \otimes a_1 a_2 \) for \( x_i \in g, a_i \in A, i \in \{1, 2\} \). We refer to a superalgebra of this form as a *map Lie superalgebra*, inspired by the case where \( A \) is the ring of regular functions on an algebraic variety. From now on, we consider \( g \subseteq g \otimes A \) as a subalgebra via the natural isomorphism \( g \cong g \otimes \mathbb{C} \).

Let \( \mathcal{I} \) be the full subcategory of the category of \( g_0 \)-modules whose objects are those modules that are isomorphic to direct sums of irreducible finite-dimensional \( g_0 \)-modules. Note that, if \( V \in \mathcal{I} \), then every element of \( V \) lies in a finite-dimensional \( g_0 \)-submodule of \( V \). Let \( \mathcal{I}(g \otimes A, g_0) \) denote the full subcategory of the category of \( g \otimes A \)-modules whose objects are the \( g \otimes A \)-modules whose restriction to \( g_0 \) lies in \( \mathcal{I} \).

If \( V \) is a \( g \)-module, then, by the PBW Theorem, we have an isomorphism of vector spaces

\[
P_A(V) := U(g \otimes A) \otimes_{U(g)} V \cong U(g \otimes A_+) \otimes_{\mathbb{C}} V,
\]

where \( A_+ \) is a vector space complement to \( \mathbb{C} \subseteq A \). We will view \( V \) as a \( g \)-submodule of \( P_A(V) \) via the natural identification \( V \cong \mathbb{C} \otimes \mathbb{C} \otimes V \subseteq P_A(V) \).

**Lemma 3.1.** Let \( V \) be a \( g \)-module whose restriction to \( g_0 \) lies in \( \mathcal{I} \). Then \( P_A(V) \in \mathcal{I}(g \otimes A, g_0) \).

**Proof.** The proof is the same of that in [FMS15, Lem. 3.4], where \( g \) is a finite-dimensional simple Lie algebra. \( \square \)

**Proposition 3.2.** If \( \lambda \in \Lambda^+ \), then \( P_A(\bar{V}(\lambda)) \) is generated, as a \( U(g \otimes A) \)-module, by the element \( v_\lambda \), with defining relations

\[
n^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad Y_\alpha^{(h_\alpha)+1} v_\lambda = 0, \quad \text{for all } h \in \mathfrak{h}, \alpha \in \Sigma(g_0).
\]

**Proof.** It is obvious that the element \( v_\lambda \in P_A(\bar{V}(\lambda)) \) satisfies the relations (3.2). To check that these are all the relations, let \( W \) be the \( g \otimes A \)-module generated by a vector \( w \) with defining relations (3.2). Then we have a surjective homomorphism of \( g \otimes A \)-modules \( \pi_1 : W \to P(\bar{V}(\lambda)) \) which maps \( w \) to \( v_\lambda \). Now, by relations (3.2), \( w \in W \) generates a \( g \)-submodule of \( W \) isomorphic to \( \bar{V}(\lambda) \). Thus, we have an epimorphism

\[
\pi_2 : P(\bar{V}(\lambda)) \to W, \quad u_1 \otimes_{U(g)} u_2 v_\lambda \mapsto u_1 u_2 w, \quad u_1 \in U(g \otimes A), u_2 \in U(g).
\]

Since \( \pi_1 = \pi_2^{-1} \), we have \( W \cong P(\bar{V}(\lambda)) \). \( \square \)

For \( \nu \in \Lambda^+ \) and \( V \in \mathcal{I}(g \otimes A, g_0) \), let \( V^\nu \) be the unique maximal \( g \otimes A \)-module quotient of \( V \) such that the weights of \( V^\nu \) lie in \( \nu - Q^+ \), where \( Q^+ = \sum_{\alpha \in \Sigma} \mathbb{N} \alpha \) is the positive root lattice of \( g \). In other words,

\[
V^\nu = V/ \sum_{\mu \notin \nu - Q^+} U(g \otimes A)V_\mu.
\]

Note that a morphism \( \varphi : V \to W \) of objects in \( \mathcal{I}(g \otimes A, g_0) \) induces a morphism \( \varphi^\nu : V^\nu \to W^\nu \). Let \( \mathcal{I}(g \otimes A, g_0)^\nu \) denote the full subcategory of \( \mathcal{I}(g \otimes A, g_0) \) whose objects are those \( V \in \mathcal{I}(g \otimes A, g_0) \)
such that $V^\nu = V$. Proposition 2.7 and Lemma 3.1 imply that $P_A(\tilde{V}(\lambda)) \in \mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ for all $\lambda \in \Lambda^+$.

**Definition 3.3** (Global Weyl module). We define the **global Weyl module** associated to $\lambda \in \Lambda^+$ to be

$$W(\lambda) := P_A(\tilde{V}(\lambda))^\lambda.$$  

We let $w_\lambda$ denote the image of $v_\lambda$ in $W(\lambda)$.

**Proposition 3.4.** For $\lambda \in \Lambda^+$, the global Weyl module $W(\lambda)$ is generated by $w_\lambda$, with defining relations

$$ (3.3) \quad (n^+ \otimes A)w_\lambda = 0, \quad hw_\lambda = \lambda(h)w_\lambda, \quad Y_\alpha^{1+1}(H_\alpha)w_\lambda = 0, \quad \text{for all } h \in \mathfrak{h}, \alpha \in \Sigma(\mathfrak{g}_0).$$

**Proof.** Since the weights of $W(\lambda)$ lie in $\lambda - Q^+$, it follows that $(n^+ \otimes A)w_\lambda = 0$. The remaining relations are clear since they are already satisfied by $v_\lambda$. To prove that these are the only relations, let $W$ be the module generated by an element $w$ with relations (3.3), so that we have an epimorphism $\pi_1 : W \to W(\lambda)$ sending $w$ to $w_\lambda$. Since the relations (3.3) imply the relations (2.5), the vector $w \in W$ generates a $\mathfrak{g}$-submodule of $W$ isomorphic to a quotient of $\tilde{V}(\lambda)$. Thus we have a surjective homomorphism

$$\pi_2 : P_A(\tilde{V}(\lambda)) \to W, \quad u_1 \otimes_{\mathcal{I}(\mathfrak{g}, \mathfrak{g}_0)} u_2 v_\lambda \mapsto u_1 u_2 w, \quad u_1 \in U(\mathfrak{g} \otimes A), \ u_2 \in U(\mathfrak{g}).$$

Since the $\mathfrak{g}$-weights of $W$ are bounded above by $\lambda$, it follows that $\pi_2$ induces a map $W(\lambda) \to W$ inverse to $\pi_1$. \hfill \Box

In the non-super setting, Proposition 3.4 was proved in [CFK10, Prop. 4].

**Proposition 3.5.** The global Weyl module $W(\lambda)$ is the unique object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$, up to isomorphism, that is generated by a highest weight vector of weight $\lambda$ and admits a surjective homomorphism to any object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ also generated by a highest weight vector of weight $\lambda$.

**Proof.** Let $V \in \mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ be generated by a highest weight vector $v$ of weight $\lambda$. Then

$$ (n^+ \otimes A)v = 0, \quad hv = \lambda(h)v, \quad \text{for all } h \in \mathfrak{h}. $$

Since the $\mathfrak{g}_0$-module generated by $v$ is finite-dimensional, we have that $Y_\alpha^{1+1}(H_\alpha)v = 0$ for all $\alpha \in \Sigma(\mathfrak{g}_0)$. Thus, by Proposition 3.4, we have a surjective homomorphism $W(\lambda) \to V$ such that $w_\lambda \mapsto v$.

Suppose that $W$ is another object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ that is generated by a highest weight vector $w$ of weight $\lambda$ and admits a surjective homomorphism to any object of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ also generated by a highest weight vector of weight $\lambda$. In particular, we have a surjective homomorphism $\pi_1 : W \to W(\lambda)$. It follows from the PBW Theorem that $W(\lambda)_\lambda = U(\mathfrak{h} \otimes A_+) \otimes_{\mathbb{C}} w_\lambda$. The only elements of this weight space that generate $W(\lambda)$ are the $\mathbb{C}$-multiples of $w_\lambda$. Thus, possibly after rescaling, we have $\pi_1(w) = w_\lambda$. Now, as above, $w$ satisfies the relations (3.3). Thus there exists a homomorphism $\pi_2 : W(\lambda) \to W$ sending $w_\lambda$ to $w$. It follows that $\pi_1$ and $\pi_2$ are mutually inverse homomorphisms, and so $W \cong W(\lambda)$. \hfill \Box

Note that, when $A = \mathbb{C}$, the global Weyl module $W(\lambda)$ coincides with the generalized Kac module $\tilde{V}(\lambda)$. In this case, Proposition 3.5 reduces to the universal property given in Lemma 2.8.
4. Local Weyl modules

Recall that $\mathfrak{g}$ is either a basic classical Lie superalgebra or $\mathfrak{sl}(n, n)$, $n \geq 2$, and that $A$ is an associative commutative unital $\mathbb{C}$-algebra. The aim now is to describe, in terms of generators and relations, a universal object in the full subcategory of $\mathcal{I}(\mathfrak{g} \otimes A, \mathfrak{g}_0)$ whose objects are the finite-dimensional modules generated by a highest map-weight vector of a fixed highest map-weight (see Definition 4.2).

**Definition 4.1** (Local Weyl module). Let $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\psi|_\mathfrak{h} \in \Lambda^+$. We define the local Weyl module $W(\psi)$ associated to $\psi$ to be the $\mathfrak{g} \otimes A$-module generated by a vector $w_\psi$ with defining relations

\[(n^+ \otimes A)w_\psi = 0, \quad xw_\psi = \psi(x)w_\psi, \quad Y^{\lambda(H_\alpha) + 1}_\alpha w_\psi = 0, \quad \text{for all } x \in \mathfrak{h} \otimes A, \alpha \in \Sigma(\mathfrak{g}_0).
\]

**Definition 4.2** (Highest map-weight module). A $\mathfrak{g} \otimes A$-module generated by a vector $w_\psi$ satisfying the first and second relations of (4.1) is called a highest map-weight module with highest map-weight $\psi$. The vector $w_\psi$ is called a highest map-weight vector of map-weight $\psi$.

Recall that, for each $\alpha \in \Delta^+_0$, we have an $\mathfrak{sl}(2)$-triple $\{X_\alpha, Y_\alpha, H_\alpha\}$.

**Lemma 4.3.** Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\lambda = \psi|_\mathfrak{h} \in \Lambda^+$. If $\alpha \in \Delta^+_0$, then $Y^{\lambda(H_\alpha) + 1}_\alpha w_\psi = 0$.

**Proof.** The vector $Y^{\lambda(H_\alpha) + 1}_\alpha w_\psi$ has weight $\lambda - (\lambda(H_\alpha) + 1)\alpha$. On the other hand, it follows from Proposition 3.4 that $W(\psi)$ is a quotient of the global Weyl module $W(\lambda)$, and so it is a direct sum of irreducible finite-dimensional $\mathfrak{g}_0$-modules. This implies that the weights of $W(\lambda)$ are invariant under the action of the Weyl group of $\mathfrak{g}_0$. But, if $s_\alpha$ denotes the reflection associated to the root $\alpha$, then $s_\alpha(\lambda - (\lambda(H_\alpha) + 1)\alpha) = \lambda + \alpha$ does not lie below $\lambda$. Therefore, $Y^{\lambda(H_\alpha) + 1}_\alpha w_\psi = 0$. \hfill $\Box$

Let $u$ be an indeterminate and, for $a \in A$, $\alpha \in \Delta^+_0$, define the following power series with coefficients in $U(\mathfrak{h} \otimes A)$:

\[(4.2) \quad p(a, \alpha) = \exp\left(-\sum_{i=1}^{\infty} \frac{H_\alpha \otimes a^i}{i} u^i\right).
\]

For $i \in \mathbb{N}$, let $p(a, \alpha)_i$ denote the coefficient of $u^i$ in $p(a, \alpha)$. In particular, $p(a, \alpha)_0 = 1$.

**Lemma 4.4.** Suppose $m \in \mathbb{N}$, $a \in A$, and $\alpha \in \Delta^+_0$. Then

\[(4.3) \quad (X_\alpha \otimes a)^m (Y_\alpha \otimes 1)^{m+1} - (-1)^m \sum_{i=0}^{m} (Y_\alpha \otimes a^{m-i}) p(a, \alpha)_i \in U(\mathfrak{g} \otimes A)(n^+ \otimes A).
\]

**Proof.** This formula is proved in [CP01, Lem. 1.3(ii)] in the case that $A$ is $\mathbb{C}[t^{\pm 1}]$. However, since the fact that $t$ is an invertible element in $\mathbb{C}[t^{\pm 1}]$ is not used in that proof, the result is still true when $A$ is equal to $\mathbb{C}[t]$. Now, applying the Lie algebra homomorphism

\[\mathfrak{sl}(2) \otimes \mathbb{C}[t] \to \mathfrak{sl}(2) \otimes A, \quad x \otimes t^m \mapsto x \otimes a^m, \quad m \in \mathbb{N}, \quad x \in \mathfrak{sl}(2),\]

gives our result. \hfill $\Box$

For the remainder of the paper we assume that $A$ is finitely generated.

**Proposition 4.5.** Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\lambda = \psi|_\mathfrak{h} \in \Lambda^+$. If $\alpha \in \Delta^+_0$, $a_1, a_2, \ldots, a_t \in A$, and $m_1, \ldots, m_t \in \mathbb{N}$, then

\[(4.4) \quad (Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi \in \operatorname{span}_{\mathbb{C}} \{ (Y_\alpha \otimes a_1^{\ell_1} \cdots a_t^{\ell_t})w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha), \ i = 1, \ldots, t \}.
\]

In particular, $(Y_\alpha \otimes A)w_\psi$ is finite dimensional.
Proof. From the first and third relations in (4.1), together with (4.3), it follows that, for $a \in A$ and $m \geq \lambda(H_\alpha)$, we have

$$0 = (X_\alpha \otimes a)^m(Y_\alpha \otimes 1)^{m+1}w_\psi = \sum_{i=0}^{m}(-1)^{m}(Y_\alpha \otimes a^{m-i})p(a,\alpha_i)w_\psi,$$

for any $a \in A$. Since $p(a,\alpha)_{|\alpha} = 1$, we have

$$(Y_\alpha \otimes a^m)w_\psi \in \text{span}_C\{(Y_\alpha \otimes a^\ell)w_\psi \mid 0 \leq \ell < m\}.$$

This implies, by induction, that

$$(Y_\alpha \otimes a^m)w_\psi \in \text{span}_C\{(Y_\alpha \otimes a^\ell)w_\psi \mid 0 \leq \ell < \lambda(H_\alpha)\}, \quad \text{for all } m \in \mathbb{N}, a \in A.$$

We will now prove (4.4) by induction on $t$. The case $t = 1$ follows immediately from (4.5). Assume that (4.4) holds for some $t \geq 1$. Let $m_1, \ldots, m_{t+1} \in \mathbb{N}$ and choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi = (-\alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}}) + (Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})(a \otimes a_{t+1}^{m_{t+1}}))w_\psi,$$

and so

$$(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi + \alpha(h)(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\psi \in \text{span}_C\{(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi\},$$

since $(h \otimes a_{t+1}^{m_{t+1}})w_\psi \in \mathbb{C}w_\psi$. By the inductive hypothesis, we have

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\psi \in \text{span}_C\left\{(h \otimes a_{t+1}^{m_{t+1}})(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi, (Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$

Then, by (4.6) (with $m_i = \ell_i$ for $i = 1, \ldots, t$), we have

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\psi \in \text{span}_C\left\{(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}a_{t+1}^{m_{t+1}})w_\psi, (Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t})w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$}

Since the above inclusion holds for all $m_1, \ldots, m_{t+1} \in \mathbb{N}$, we can interchange the roles of $m_1$ and $m_{t+1}$ to obtain

$$(Y_\alpha \otimes a_1^{m_1} \cdots a_{t+1}^{m_{t+1}})w_\psi \in \text{span}_C\left\{(Y_\alpha \otimes a_1^{m_1} \cdots a_t^{m_t}a_{t+1}^{m_{t+1}})w_\psi \mid 0 \leq \ell_i < \lambda(H_\alpha)\right\}.$$

This completes the proof of the inductive step. The final statement of the lemma follows from the fact that $A$ is finitely generated. \hfill \square

Let

$$\mathcal{L}(\mathfrak{h} \otimes A) = \{\psi \in (\mathfrak{h} \otimes A)^* \mid \psi(\mathfrak{h} \otimes I) = 0, \text{ for some finite-codimensional ideal } I \text{ of } A\}.$$ 

**Proposition 4.6.** Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ such that $\lambda = \psi|_{\mathfrak{h}} \in \Lambda^+$. If $\psi \not\in \mathcal{L}(\mathfrak{h} \otimes A)$, then $W(\psi) = 0$.

**Proof.** Let $\alpha \in \Delta^+_0$ and let $I_\alpha$ be the kernel of the linear map

$$A \to \text{Hom}_C(W(\psi)_\lambda \otimes \mathfrak{g}_{-\alpha}, (\mathfrak{g}_{-\alpha} \otimes A)w_\psi),$$

$$a \mapsto (v \otimes u \mapsto (u \otimes a)v), \quad a \in A, \quad v \in W(\psi)_\lambda, \quad u \in \mathfrak{g}_{-\alpha}.$$ 

Since $\mathfrak{g}_{-\alpha} = \mathbb{C}Y_\alpha$, Proposition 4.5 implies that $(\mathfrak{g}_{-\alpha} \otimes A)w_\psi$ is finite dimensional. Thus, $I_\alpha$ is a linear subspace of $A$ of finite codimension. We claim that $I_\alpha$ is, in fact, an ideal of $A$. Indeed, since $\alpha \neq 0$, we can choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then, for all $g \in A, \ a \in I_\alpha, \ v \in W(\psi)_\lambda$, and $u \in \mathfrak{g}_{-\alpha}$, we have

$$0 = (h \otimes g)(u \otimes a)v = [h \otimes g, u \otimes a]v + (u \otimes a)(h \otimes g)v = -\alpha(h)(u \otimes ga)v + (u \otimes a)(h \otimes g)v.$$ 

Since $(h \otimes g)v \in W(\psi)_\lambda$, and $a \in I_\alpha$, the last term above is zero. Since we also have $\alpha(h) \neq 0$, this implies that $(u \otimes ga)v = 0$. As this holds for all $v \in W(\psi)_\lambda$ and $u \in \mathfrak{g}_{-\alpha}$, we have $ga \in I_\alpha$. Hence $I_\alpha$ is an ideal of $A$. 


Let $I$ be the intersection of all the $I_{\alpha}, \alpha \in \Delta^+_0$. Since $\mathfrak{g}$ has a finite number of positive roots, this intersection is finite, and thus $I$ is also an ideal of $A$ of finite-codimension. We have
\[(n_0^- \otimes I)W(\psi)_\lambda = 0 \quad \text{and} \quad (n^+ \otimes A)W(\psi)_\lambda = 0.\]
Then, since $\mathfrak{h} \otimes I \subseteq [n^+ \otimes A, n_0^- \otimes I]$, we have $(\mathfrak{h} \otimes I)W(\psi)_\lambda = 0$. In particular, $(\mathfrak{h} \otimes I)w_\psi = 0$.

Assume $\psi \notin \mathcal{L}(\mathfrak{h} \otimes A)$. Then there exists $a \in I$ such that $\psi(h \otimes a) \neq 0$ for some $h \in \mathfrak{h}$, which implies that $w_\psi = 0$, since
\[0 = (h \otimes a)w_\psi = \psi(h \otimes a)w_\psi.\]
Therefore $W(\psi) = 0$.

**Definition 4.7** (The ideal $I_\psi$). For $\psi \in (\mathfrak{h} \otimes A)^*$ with $\psi|_\mathfrak{h} \in \Lambda^+$, let $I_\psi$ be the sum of all ideals $I \subseteq A$ such that $(\mathfrak{h} \otimes I)w_\psi = 0$.

**Remark 4.8.** It follows from the proof of Proposition 4.6 that $I_\psi$ has finite codimension in $A$ and that $(Y_\alpha \otimes I_\psi)w_\psi = 0$ for all $\alpha \in \Delta^+_0$. Furthermore, by Lemma 2.1, parts (a) and (b), since $I_\psi$ has finite codimension and $A$ is finitely generated, we have that $I_\psi^N$ has finite codimension, for all $N \in \mathbb{N}$.

For the remainder of the paper, we assume that
\[\Sigma \text{ is a system of simple roots for } \mathfrak{g} \text{ satisfying (2.2).}\]
Recall that, by Proposition 2.4, such a system always exists.

**Lemma 4.9.** Suppose $\psi \in (\mathfrak{h} \otimes A)^*$ with $\psi|_\mathfrak{h} \in \Lambda^+$. Then there exists $N_\psi \in \mathbb{N}$ such that
\[
\left(n^- \otimes I_{\psi}^{N_\psi}\right)w_\psi = 0.
\]

**Proof.** Recall the set $\{Y_\alpha \mid \alpha \in \Sigma\}$ of generators of $n^-$. We claim that
\[(Y_\alpha \otimes I_\psi)w_\psi = 0, \quad \text{for all } \alpha \in \Sigma.\]
By Remark 4.8, it suffices to consider the case $\alpha \in \Sigma_1$. Fix such an $\alpha$. By (2.2), there exists $\alpha' \in \Delta_1$ such that $\beta := \alpha + \alpha' \in \Delta^+_0$.

First suppose $\mathfrak{g}$ is not $A(1, 1)$ or $\mathfrak{sl}(2, 2)$. Then $\dim \mathfrak{g}_\nu = 1$ for any $\nu \in \Delta$ (see Remark 2.3). Thus, rescaling if necessary,
\[(X_{\alpha'}, Y_\beta) = Y_\alpha.\]
Then,
\[(Y_\alpha \otimes I_\psi)w_\psi = [X_{\alpha'} \otimes A, Y_\beta \otimes I_\psi]w_\psi \subseteq (X_{\alpha'} \otimes A)(Y_\beta \otimes I_\psi)w_\psi + (Y_\beta \otimes I_\psi)(X_{\alpha'} \otimes A)w_\psi = 0,
\]
where the last equality follows from the fact that $(Y_\beta \otimes I_\psi)w_\psi = 0$ by Remark 4.8 and $(X_{\alpha'} \otimes A)w_\psi = 0$ by the first relation in (4.1). This proves (4.7).

To prove (4.7) for $\mathfrak{sl}(2, 2)$ and $A(1, 1)$, we consider $\mathfrak{g} = \mathfrak{sl}(2, 2)$ and we let $\mathfrak{h}$ be the subalgebra of diagonal matrices of $\mathfrak{g}$. Denote by $\{e_i \mid i = 1, \ldots, 4\}$ the basis of $\mathfrak{h}^*$ dual to $\{E_{i,i} \mid i = 1, \ldots, 4\}$. In this case,
\[\Delta_0 = \{\pm(e_1 - e_2), \pm(e_3 - e_4)\}, \quad \Delta_1 = \{\pm(e_1 - e_3), \pm(e_1 - e_4), \pm(e_2 - e_3), \pm(e_2 - e_4)\},\]
and $\mathfrak{g}_{e_i - e_j} = CE_{r,s}$, for $1 \leq r < s \leq 4$. In particular, if we fix $\alpha \in \Sigma_1$ and $\alpha' \in \Delta^+_1$ such that $\beta := \alpha + \alpha' \in \Delta$, then there exist $k, \ell, p, q \in \{1, 2, 3, 4\}$ with $k \neq \ell$ and $p \neq q$, such that $\mathfrak{g}_{\alpha'} = CE_{k,\ell}$ and $\mathfrak{g}_{-\beta} = CE_{p,q}$. Since $\beta \in \Delta^+_0$ and $\alpha' \in \Delta^+_1$, Remark 4.8 and the first relation in (4.1) give us that $([E_{k,\ell}, E_{p,q}] \otimes I_\psi)w_\psi = 0$. Regarding the $\mathfrak{sl}(2, 2)$ case, we choose $Y_\alpha = [E_{k,\ell}, E_{p,q}]$. For the $A(1, 1)$ case, we choose $Y_\alpha$ to be the image of $[E_{k,\ell}, E_{p,q}]$ in $A(1, 1)$. Then $(Y_\alpha \otimes I_\psi)w_\psi = 0$. Since the choice of $\alpha \in \Sigma_1$ was arbitrary, we conclude that $(Y_\alpha \otimes I_\psi)w_\psi = 0$ for all roots $\alpha \in \Sigma_1$. 

Now, for \( \beta = \sum_{\alpha \in \Sigma} m_{\alpha} \alpha \in \Delta^+ \), we define the height of \( \beta \) to be \( \text{ht} \beta := \sum_{\alpha \in \Sigma} m_{\alpha} \). We prove, by induction on the height of \( \beta \), that \((Y_\beta \otimes I_\psi^{\text{ht} \beta})w_\psi = 0\) for all \( \beta \in \Delta^+ \). Since \( \mathfrak{g} \) is finite dimensional, the heights of elements of \( \Delta^+ \) are bounded above, and thus the lemma will follow.

The base case of height one is precisely (4.7). Suppose \( \beta \in \Delta^+ \) with \( \text{ht} \beta > 1 \). Then there exist \( \beta', \beta'' \in \Delta^+ \) with \( \text{ht} \beta', \text{ht} \beta'' < \text{ht} \beta \) such that \( Y_\beta \in \mathbb{C}[Y_{\beta'}, Y_{\beta''}] \). Then

\[
(Y_\beta \otimes I_\psi^{\text{ht} \beta})w_\psi = [Y_{\beta'} \otimes I_\psi^{\text{ht} \beta''}, Y_{\beta''} \otimes I_\psi^{\text{ht} \beta''}])w_\psi = 0. \quad \square
\]

**Corollary 4.10.** Suppose \( \psi \in (\mathfrak{h} \otimes A)^* \) with \( \psi|_h \in \Lambda^+ \), and let \( N_\psi \) be as in Lemma 4.9. Then

\[
(g \otimes I_\psi^{N_\psi})w_\psi = 0.
\]

**Proof.** It follows from the first relation in (4.1) that \((n^+ \otimes I_\psi^{N_\psi})w_\psi = 0\). Since \((\mathfrak{h} \otimes I_\psi)w_\psi = 0\) by the definition of \( I_\psi \), we have \((\mathfrak{h} \otimes I_\psi^{N_\psi})w_\psi = 0\). Finally Lemma 4.9 implies that \((n^- \otimes I_\psi^{N_\psi})w_\psi = 0\). \( \square \)

**Lemma 4.11.** For all \( \lambda \in (\mathfrak{h} \otimes A)^* \) with \( \psi|_h \in \Lambda^+ \), the set of \( g \)-weights (equivalently, \( g_0 \)-weights) of \( W(\psi) \) is finite.

**Proof.** Since the weights of \( W(\lambda) \) are contained in \( \Lambda - Q^+ \), a finite number of weights of \( W(\lambda) \) are dominant integral. Since \( W(\lambda) \) is a direct sum of \( g_0 \)-modules, its weights are invariant under the (finite) Weyl group of \( g_0 \). The result follows. \( \square \)

**Theorem 4.12.** Recall that \( A \) is finitely generated and the system of simple roots \( \Sigma \) satisfies (2.2). The local Weyl module \( W(\psi) \) is finite dimensional for all \( \psi \in (\mathfrak{h} \otimes A)^* \) such that \( \psi|_h \in \Lambda^+ \).

**Proof.** By Definition 4.1, we have \( W(\psi) = U(n^- \otimes A)w_\psi \). By Lemma 4.9, we have \((n^- \otimes I_\psi^{N_\psi})w_\psi = 0\). Thus \( W(\psi) = U(n^- \otimes A/I_\psi^{N_\psi})w_\psi \). By Lemma 4.11, there exists \( N \in \mathbb{N} \) such that

\[
W(\psi) = U_n \left(n^- \otimes A/I_\psi^{N_\psi}\right)w_\psi, \quad \text{for all } n \geq N,
\]

where \( U(a) = \sum_{n=0}^{\infty} U_n(a) \) is the usual filtration on the universal enveloping algebra of a Lie superalgebra \( a \) induced from the natural grading on the tensor algebra. Since the Lie superalgebra \( n^- \otimes A/I_\psi^{N_\psi} \) is finite dimensional (see Remark 4.8), \( W(\psi) \) is also finite dimensional. \( \square \)

In the non-super setting, Theorem 4.12 was proved in [CP01, Th. 1] for \( A = \mathbb{C}[t, t^{-1}] \), and in [FL04, Th. 1] for \( A \) the algebra of functions on a complex affine variety.

**Proposition 4.13.** Let \( \psi \in \mathcal{L}(\mathfrak{h} \otimes A) \) such that \( \psi|_h = \lambda \in \Lambda^+ \). Then the local Weyl module \( W(\psi) \) is the unique (up to isomorphism) finite-dimensional object of \( \mathcal{I}(\mathfrak{g} \otimes A, g_0) \) that is generated by a highest map-weight vector of map-weight \( \psi \) and admits a surjective homomorphism to any finite-dimensional object of \( \mathcal{I}(\mathfrak{g} \otimes A, g_0) \) also generated by a highest map-weight vector of map-weight \( \psi \).

**Proof.** Let \( V \) be a finite-dimensional object of \( \mathcal{I}(\mathfrak{g} \otimes A, g_0) \) that is generated by a highest map-weight vector \( v \) of map-weight \( \psi \). It follows immediately from the definition of a highest map-weight \( \mathfrak{g} \otimes A \)-module that the two first relations in (4.1) are satisfied by \( v \). Since the \( g_0 \)-module generated by \( v \) must be finite dimensional, we have also that \( Y_\alpha^{\lambda(H_\alpha)+1}v = 0 \), for all \( \alpha \in \Sigma(g_0) \). Therefore, there exists a surjective homomorphism \( W(\psi) \to V \) sending \( w_\psi \) to \( v \).

To show that \( W(\psi) \) is the unique representation with the given property, suppose that \( W \) is another module with this property. Then \( W \) is a quotient of \( W(\psi) \) and vice-versa. Since both modules are finite dimensional, it follows that \( W(\psi) \cong W \).

\( \square \)
Corollary 4.14. Let $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ such that $\psi|_{\mathfrak{h}} = \lambda \in \Lambda^+$. Then the local Weyl module $W(\psi)$ is the maximal finite-dimensional quotient of the global Weyl module $W(\lambda)$ that is a highest map-weight module of highest map-weight $\psi$.

By [Sav14, Th. 4.16], any irreducible finite-dimensional $\mathfrak{g} \otimes A$-module is a highest map-weight module, for some $\psi \in \mathcal{L}(\mathfrak{h} \otimes A)$ with $\psi|_{\mathfrak{h}} \in \Lambda^+$. Then, by Proposition 4.13, there exists a surjective homomorphism from the local Weyl module $W(\psi)$ to such an irreducible module. In other words, all irreducible finite-dimensional $\mathfrak{g} \otimes A$-modules are quotients of local Weyl modules.

We conclude by showing that the local Weyl modules possess a tensor product property analogous to the one satisfied in the non-super setting (see, [CP01, Th. 2] and [FL04, Th. 2]).

Theorem 4.15. Recall that $A$ is finitely generated and the system of simple roots $\Sigma$ satisfies (2.2). For $i = 1, 2$, let $\psi_i \in \mathcal{L}(\mathfrak{h} \otimes A)$ with $\lambda_i = \psi_i|_{\mathfrak{h}} \in \Lambda^+$, and suppose that $I_{\psi_1}$ and $I_{\psi_2}$ have disjoint support. Then

$$W(\psi_1 + \psi_2) \cong W(\psi_1) \otimes W(\psi_2)$$

as $\mathfrak{g} \otimes A$-modules.

Proof. By Corollary 4.10, there exist $N_1, N_2 \in \mathbb{N}$ such that $(\mathfrak{g} \otimes I_{\psi_i}^{N_i})w_{\psi_i} = 0$ for $i = 1, 2$. Then the action of $\mathfrak{g} \otimes A$ on $W(\psi_1) \otimes W(\psi_2)$ factors through the composition

$$(4.9) \quad \mathfrak{g} \otimes A \xrightarrow{d} (\mathfrak{g} \otimes A) \oplus (\mathfrak{g} \otimes A) \xrightarrow{\pi} (\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2}),$$

where $d$ is the diagonal embedding. Since $\text{Supp}(I_{\psi_1}) \cap \text{Supp}(I_{\psi_2}) = \emptyset$, Lemma 2.1(a) implies that $\text{Supp}(I_{\psi_1}^{N_1}) \cap \text{Supp}(I_{\psi_2}^{N_2}) = \emptyset$. Then, by Lemma 2.1(c), we have $A = I_{\psi_1}^{N_1} + I_{\psi_2}^{N_2}$ and $I_{\psi_1}^{N_1} \cap I_{\psi_2}^{N_2} = I_{\psi_1}^{N_1}I_{\psi_2}^{N_2}$. Thus, $A/I_{\psi_1}^{N_1}I_{\psi_2}^{N_2} \cong (A/I_{\psi_1}^{N_1}) \oplus (A/I_{\psi_2}^{N_2})$. We therefore have the following commutative diagram:

$$\begin{array}{c}
\mathfrak{g} \otimes A \xrightarrow{d} (\mathfrak{g} \otimes A) \\
\downarrow \\
(\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}I_{\psi_2}^{N_2}) \xrightarrow{\cong} (\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2})
\end{array}$$

It follows that the composition (4.9) is surjective.

Since $W(\psi_1) \otimes W(\psi_2)$ is generated as a $(\mathfrak{g} \otimes A/I_{\psi_1}^{N_1}) \oplus (\mathfrak{g} \otimes A/I_{\psi_2}^{N_2})$-module by the vector $w_{\psi_1} \otimes w_{\psi_2}$, it follows from the above that it is also generated by this vector as a $\mathfrak{g} \otimes A$-module. Moreover, $\mathfrak{h} \otimes A$ acts on $w_{\psi_1} \otimes w_{\psi_2}$ via $\psi := \psi_1 + \psi_2$. Thus $W(\psi_1) \otimes W(\psi_2)$ is a finite-dimensional highest map-weight module of highest map-weight $\psi$. Therefore, by Proposition 4.13, it is a quotient of $W(\psi)$.

To simplify notation, let $I_1 = I_{\psi_1}$, $I_2 = I_{\psi_2}$ and $N = N_{\psi}$. Let $I = I_1I_2 = I_1 \cap I_2$. Then $I \subseteq I_{\psi}$. Therefore, the action of $\mathfrak{b} \otimes A$ on $\mathbb{C}w_{\psi}$ descends to an action of $\mathfrak{b} \otimes A/I^N$ on $\mathbb{C}w_{\psi}$. Consider the induced module

$$M(\psi) := U(\mathfrak{g} \otimes A/I^N) \otimes_{U(\mathfrak{b} \otimes A/I^N)} \mathbb{C}w_{\psi}.$$
It follows from Corollary 4.10 that $W(\psi)$ is a quotient of $M(\psi)$. On the other hand, it is clear that the one-dimensional $\mathfrak{b} \otimes A$-modules $\mathbb{C} w_{\psi_1}$ and $\mathbb{C} w_{\psi_1} \otimes \mathbb{C} w_{\psi_2}$ are isomorphic. Hence,

$$M(\psi) = U(\mathfrak{g} \otimes (A/I_1) \otimes U(\mathfrak{b} \otimes (A/I_2))) \otimes \mathbb{C} w_{\psi_1} \otimes \mathbb{C} w_{\psi_2}$$

and

$$M(\psi) = U(\mathfrak{g} \otimes (A/I_1) \otimes U(\mathfrak{b} \otimes (A/I_2))) \otimes \mathbb{C} w_{\psi_1} \otimes \mathbb{C} w_{\psi_2}$$

So $W(\psi)$ is a quotient of $M(\psi_1) \otimes M(\psi_2)$. Fix a surjection $\theta: M(\psi_1) \otimes M(\psi_2) \to W(\psi)$.

We claim that the image of $M(\psi_1) \otimes M(\psi_2)$ under $\theta$ is zero except for a finite number of weights $\mu$ and $\nu$. By Lemma 4.11, the set $D$ of weights occurring in $W(\psi)$ is finite. Thus, the sets

$$D_1 = (\lambda_1 - Q^+) \cap (-\lambda_2 + D + Q^+) \quad \text{and} \quad D_2 = (\lambda_2 - Q^+) \cap (-\lambda_1 + D + Q^+)$$

are also finite. Since, for $i = 1, 2$, the weights of $M(\psi_i)$ are contained in $\lambda_i - Q^+$, the image of $M(\psi_1) \otimes M(\psi_2)$ under $\theta$ is zero unless $\mu \in \lambda_1 - Q^+$, $\nu \in \lambda_2 - Q^+$, and $\mu + \nu \in D$. Thus it is nonzero only if $\mu \in D_1$ and $\nu \in D_2$, and hence the claim is proved.

For $i = 1, 2$, let $M(\psi_i)^{\prime}$ be the submodule of $M(\psi_i)$ generated by the weight subspaces $M(\psi_i)_{\mu}$ with $\mu \notin D_i$, and let $M(\psi_i) = M(\psi_i)/M(\psi_i)^{\prime}$. Then $W(\psi)$ is a quotient of $M(\psi_1) \otimes M(\psi_2)$. Because $I_i$ has finite codimension and there are only a finite number of weights occurring in the quotient $M(\psi_i)$, this module is a finite-dimensional highest map-weight module of highest map-weight $\psi_i$. Then, by Proposition 4.13, it is a quotient of $W(\psi_i)$. Thus, $M(\psi_1) \otimes M(\psi_2)$ is a quotient of $W(\psi_1) \otimes W(\psi_2)$, which implies that $W(\psi)$ is a quotient of $W(\psi_1) \otimes W(\psi_2)$. Since the modules $W(\psi)$ and $W(\psi_1) \otimes W(\psi_2)$ are both finite dimensional, the fact that one is a quotient of the other implies the isomorphism in the statement of the theorem.  

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