Statistical auditing and randomness test of lotto $k/N$-type games

H.F. Coronel-Brizio, A.R. Hernández-Montoya,

Facultad de Física e Inteligencia Artificial. Universidad Veracruzana, Sebastián Camacho 5, Xalapa, Veracruz 91000. México

F. Rapallo, E. Scalas

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Via Bellini 25/G, 15100 Alessandria, Italy

Abstract

One of the most popular lottery games worldwide is the so-called “lotto $k/N$”. It considers $N$ numbers $1, 2, \ldots, N$ from which $k$ are drawn randomly, without replacement. A player selects $k$ or more numbers and the first prize is shared amongst those players whose selected numbers match all of the $k$ randomly drawn. Exact rules may vary in different countries.

In this paper, mean values and covariances for the random variables representing the numbers drawn from this kind of game are presented, with the aim of using them to audit statistically the consistency of a given sample of historical results with theoretical values coming from a hypergeometric statistical model. The method can be adapted to test pseudorandom number generators.

Key words: Empirical Randomness Test, Lottery Games, Multivariate Hypothesis Testing, Combinatorial Calculus, Statistical models

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Email addresses: hcoronel@uv.mx (H.F. Coronel-Brizio), alhernandez@uv.mx (A.R. Hernández-Montoya), fabio.rapallo@mfn.unipmn.it (F. Rapallo), enrico.scalas@mfn.unipmn.it (E. Scalas).

URL: www.mfn.unipmn.it/~scalas (E. Scalas).
1 Introduction

The concept of chance has a long history, but, as far as we know, early scientists and mathematicians working in the Hellenistic period did not develop either a theory of probability or statistical methods [1]. Gambling became more and more popular in Europe in the XVIIth century, due to the emergence of a class of people affluent enough to travel along the continent and waste money in such games. Games of chance were at the origin of a new wave of interest on the rules of chance [1] and fostered the first rigorous results in Probability Theory. Among all the games of chance, lotteries have been and still are very popular. They are used by governments to levy indirect taxes on very poor people. It is not clear when the first European lottery games started, but it seems that they could have been already present in the XVth century. Influential names in the history of science, such as D’Alembert, Euler, D. Bernoulli, Huygens, Leibniz, Laplace and many others analyzed lotteries for practical purposes, such as designing them and optimizing governmental collected revenues, but also with theoretical goals in mind, helping to accelerate the development of Statistics and Probability Theory. A very interesting account on the history of lotteries emphasizes the role of Genoa (an Italian Sea Republic of the Middle Ages) in introducing state-run lotteries [2]. That paper includes further interesting references.

Nowadays, analysis, design and simulation of lottery games continue to be an active research area, mainly for statisticians and economists [3,4,5,6], and also studied as a suitable tool for teaching elementary probability theory and Statistics [7,8], but even new interesting theoretical results have been obtained recently [9].

In this work we present a statistical data analysis of randomness of Mexican and Italian lotteries; although, strictly speaking, it is known that there is no way to “prove” the randomness of a sequence of numbers [10], it is always possible to statistically test whether or not historical results exhibit the quantitative properties derived from the probabilistic model assumed to explain the selection mechanism. In this respect, the statistical procedure presented here could be easily used as a test of pseudorandom number generators.
2 Theory

2.1 Probabilities

Readers can find the following references useful to understand the material presented in this section: [11,12,13] for what concerns Probability and combinatorial analysis and [14] for Statistics.

The total number of possible combinations of \( k \) objects chosen from a set of \( N \) objects is given by the combinatorial coefficient "\( N \) choose \( k \)":

\[
\begin{align*}
\binom{N}{k} &= \frac{N!}{k!(N-k)!}.
\end{align*}
\]

We denote by \( X \) the random variable corresponding to the number of matches out of the \( k \) randomly drawn numbers. Here, we use the hypergeometric model and we prefer the technical term “fairness” in place of “equiprobability” as, strictly speaking, all the lottery numbers are equivalent-exchangeable, but the odds of extracting them do not follow the uniform distribution (sampling is without replacement) and, in drawing each of the \( N \) objects, the probability of matching exactly \( i \) numbers, out of \( k \) selected by the player, is given by [15]

\[
P[X = i] = \binom{k}{i} \binom{N-k}{k-i} \binom{N}{k}^{-1}
\] (1)

where \( i = 0, \ldots, k \).

In order to test the hypothesis of fairness, we consider a multivariate test on the mean parameter of the random variable \( \mathbf{Y}' = [Y_{(1)}, \ldots, Y_{(k)}] \), the sorted outcome vector. Here, \( Y_{(i)} \) denotes the random variable corresponding to the number in the \( i \)-th place (recall that the randomly selected numbers are put in ascending order i.e., \( Y_{(1)} < Y_{(2)} < \ldots < Y_{(k)} \)). The probability that the \( i \)-th number corresponds to the value \( r \), is calculated from Eq. (1) with a suitable choice of parameters. In fact, \( Y_{(i)} = r \) if and only if \( i - 1 \) numbers fall between 1 and \( r - 1 \), and \( k - i \) numbers fall between \( r + 1 \) and \( N \). Therefore,

\[
P[Y_{(i)} = r] = \binom{r-1}{i-1} \binom{N-r}{k-i} \binom{N}{k}^{-1}
\] (2)

The joint probability that the \( i \)-th and \( j \)-th numbers have the values \( r \) and \( s \), respectively, is

\[
P[Y_{(i)} = r, Y_{(j)} = s] = \binom{r-i}{i-1} \binom{s-r-1}{j-i-1} \binom{N-s}{k-j} \binom{N}{k}^{-1}
\] (3)
for $i < j$ and $r < s$.

2.2 First and second order moments

The expected value of the $i-$th number in the sorted outcome vector is:

$$
E \left[ Y_{(i)} \right] = \binom{N}{k}^{-1} \sum_{r=i}^{N-k+i} r \left( \frac{r-1}{i-1} \right) \left( \frac{N-r}{k-i} \right)
$$

(4)

and the expected value of its square is:

$$
E \left[ Y_{(i)}^2 \right] = \binom{N}{k}^{-1} \sum_{r=i}^{N-k+i} r^2 \left( \frac{r-1}{i-1} \right) \left( \frac{N-r}{k-i} \right)
$$

(5)

Its variance is then obtained as

$$
\text{Var} \left[ Y_{(i)} \right] = E \left[ Y_{(i)}^2 \right] - \left\{ E \left[ Y_{(i)} \right] \right\}^2
$$

(6)

Finally, the covariance between the values appearing in $i-$th and $j-$th places, can be calculated for $i < j$ as

$$
\text{Cov} \left[ Y_{(i)}, Y_{(j)} \right] = E \left[ Y_{(i)} Y_{(j)} \right] - E \left[ Y_{(i)} \right] E \left[ Y_{(j)} \right]
$$

(7)

where

$$
E \left[ Y_{(i)} Y_{(j)} \right] = \binom{N}{k}^{-1} \sum_{r=i}^{N-k+i} \sum_{s=r+1}^{N-k+j} r s \left( \frac{r-1}{i-1} \right) \left( \frac{s-r-1}{j-i-1} \right) \left( \frac{N-s}{k-j} \right)
$$

(8)

Using the above results, we find that under fairness the $i-$th component of the vector $\mu = E[Y]$ is just

$$
\mu_i = E \left[ Y_{(i)} \right] = \frac{(N+1)i}{(k+1)}, \quad i = 1, \ldots, k.
$$

(9)

On the other hand, the covariance matrix $V = \text{Var}[Y]$ has elements

$$
v_{ij} = v_{ji} = \text{Cov} \left[ Y_{(i)}, Y_{(j)} \right] = \frac{i(k-j+1)(N+1)(N-k)}{(k+1)^2(k+2)}
$$

(10)

for $1 \leq i \leq j \leq k$.

**Remark.** Often, the rules of the game allow for the selection of an additional number, called *bonus number*. In such a case, the formulae above must be slightly modified. For instance, Eq. (1) assumes the following expression:

$$
P'[X = i] = \binom{k}{i} \binom{N-k-1}{k-i-1} \binom{N}{k}^{-1}
$$

(11)
However, in this paper, we do not consider this situation, and in any case, bonus numbers do not affect the distribution of the order statistics.

### 2.3 Examples: lotto 6/51 and 5/90

As an illustration, we present the explicit mean and variance/covariance matrix in two settings: the case $N = 51$ and $k = 6$, as an example of the Mexican game, and the case $N = 90$ and $k = 5$ from the Italian game. Notice that we give the inverse variance/covariance matrices as they are involved in the chi-squared test statistics.

For the 6/51 game, the mean is

$$\mu' = \begin{bmatrix} \frac{52}{7} & \frac{104}{7} & \frac{156}{7} & \frac{208}{7} & \frac{260}{7} & \frac{312}{7} \end{bmatrix}$$

and the inverse variance/covariance matrix is the tri-diagonal matrix

$$V^{-1} = \begin{bmatrix}
\frac{28}{585} & -\frac{14}{585} & 0 & 0 & 0 & 0 \\
-\frac{14}{585} & \frac{28}{585} & -\frac{14}{585} & 0 & 0 & 0 \\
0 & -\frac{14}{585} & \frac{28}{585} & -\frac{14}{585} & 0 & 0 \\
0 & 0 & -\frac{14}{585} & \frac{28}{585} & -\frac{14}{585} & 0 \\
0 & 0 & 0 & -\frac{14}{585} & \frac{28}{585} & -\frac{14}{585} \\
0 & 0 & 0 & 0 & -\frac{14}{585} & \frac{28}{585}
\end{bmatrix}$$

When $k = 5$ and $N = 90$, the mean vector is

$$\mu' = \begin{bmatrix} \frac{91}{6} & \frac{182}{6} & \frac{273}{6} & \frac{364}{6} & \frac{455}{6} \end{bmatrix}$$

and

$$V^{-1} = \begin{bmatrix}
\frac{12}{1105} & -\frac{6}{1105} & 0 & 0 & 0 \\
-\frac{6}{1105} & \frac{12}{1105} & -\frac{6}{1105} & 0 & 0 \\
0 & -\frac{6}{1105} & \frac{12}{1105} & -\frac{6}{1105} & 0 \\
0 & 0 & -\frac{6}{1105} & \frac{12}{1105} & -\frac{6}{1105} \\
0 & 0 & 0 & -\frac{6}{1105} & \frac{12}{1105}
\end{bmatrix}$$
3 Hypothesis testing

Let us denote by \( y_1, \ldots, y_m \) the observed outcome vectors from \( m \) games, and by \( \bar{y} \) the corresponding average.

To test the null hypothesis \( \mathbb{E}[Y] = \mu \), we use both an asymptotic approach and a Monte Carlo one.

With the asymptotic approach, we make use of the multivariate central limit theorem, see [14], Chapter 11. Therefore, under the null hypothesis the quantity

\[
Q = m (\bar{y} - \mu)^\prime V^{-1} (\bar{y} - \mu)
\]

(12)

converges in distribution to a chi-square distribution with \( k \) degrees of freedom, denoted by \( \chi^2_{(k)} \). Should the data exhibit departures from the known mean vector and/or variance/covariance matrix, the quantity \( Q \) will show departures from the \( \chi^2_{(k)} \) distribution. Thus, a test for the parameters can be performed by computing \( Q \), from a sample of \( m \) previous results, and calculating the associated \( p \)-value based on the \( \chi^2_{(k)} \) distribution.

With the Monte Carlo approach, we approximate the distribution of \( Q \) under the null hypothesis through the random generation of 5,000 values of \( Q \), each based on the same sample size as the observed draws.

4 Numerical results

4.1 The Mexican “melate” lotto game

In Mexico, a very popular game is the game known in this country as melate. The historical results are available at \url{www.pronosticos.gob.mx}, the official web-site of “Pronosticos Deportivos para la Asistencia Publica”.

The melate game was available to the Mexican public for the first time on August 19th, 1984, with the scheme of selecting \( k = 6 \) numbers out of \( N = 39 \) until April 4th, 1993, when \( N \) was set to 44. On October 6th, 2002, the game was again modified and \( N \) increased to 47. Another modification to this game was made on December 4th, 2005, raising \( N \) to 51, until December 9, 2007 corresponding to draw number 2088. As of December 12, 2007, \( N \) was raised to 56. For \( N = 51 \) the sample includes 211 results, from December 4, 2005 (draw number 1878) up to December 9, 2007 (draw number 2088). We denote the 4 periods with \( P1, P2, P3, \) and \( P4 \).

Table 1 shows the sample average vectors for each type of game, computed
from the historical results. Using the parameter values found for each case,

| Period | $N$ | $y_{(1)}$ | $y_{(2)}$ | $y_{(3)}$ | $y_{(4)}$ | $y_{(5)}$ | $y_{(6)}$ | Draws |
|--------|-----|-----------|-----------|-----------|-----------|-----------|-----------|-------|
| $P1$   | 39  | 5.634     | 11.679    | 17.195    | 22.859    | 28.699    | 34.153    | 555   |
| $P2$   | 44  | 6.284     | 12.746    | 19.265    | 25.714    | 32.288    | 38.730    | 992   |
| $P3$   | 47  | 6.964     | 13.579    | 20.591    | 27.691    | 34.379    | 41.161    | 330   |
| $P4$   | 51  | 7.739     | 14.104    | 22.038    | 30.227    | 37.564    | 45.635    | 211   |

Table 1
Average results from the Mexican “melate” lotto game. August 19, 1984 to December 30, 2007.

the $Q$–statistic defined in Eq. (12) was calculated and the results are summarized in Table 2, together with the asymptotic and Monte Carlo approximated $p$–values. As it can be seen from Table 2, the historical results for

| Period | $N$ | $Q$     | CLT $p$–value | MC $p$–value |
|--------|-----|---------|---------------|--------------|
| $P1$   | 39  | 6.09    | 0.4127        | 0.3962       |
| $P2$   | 44  | 2.50    | 0.8680        | 0.8746       |
| $P3$   | 47  | 1.76    | 0.9403        | 0.9392       |
| $P4$   | 51  | 18.25   | 0.0056        | 0.0066       |

Table 2
Calculated $Q$–statistic and associated $p$–values for each group of results from the melate lotto game. CLT $p$–value is the Central Limit Theorem-based $p$–value and MC $p$–value is the Monte Carlo approximated $p$–value.

$N = 39, 44, 47$ produce small values of $Q$, with associated $p$–values which show statistical consistency of the sample averages with their corresponding theoretical values.

However, from the 211 available results for $N = 51$, we found $Q = 18.25$ with an associated probability value of $p = 0.0056$, which constitutes strong statistical evidence to conclude that the mechanism that generated the sample is not consistent with the theoretical means and covariances.

Notice that the Monte Carlo $p$–values are close to the asymptotic ones, showing that the Central Limit Theorem is already a good approximation. This feature is due to the use of reasonably large sample sizes in all tests, despite the fact that the order statistics are known to be non-normal.
4.2 The Italian lotto game

In Italy, the lotto game is a 5/90 game and has been available on several wheels at least from 1863. As mentioned in the introduction, the game has a long history, and similar games have been played in Italy at least since 1630. We consider in this paper only one wheel, the Rome wheel, and the same periods of time as for the Mexican lotto. The choice of 4 periods is motivated by the need of reproducing similar sample sizes with respect to the previous analysis on the Mexican data. The historical results from January 7th, 1939 are available at www.lottomatica.it, the official web-site of the game. The results are summarized in Tables 3 and 4. The data are analyzed with the same procedure as discussed in the Mexican case. From table 4, we see that

| Period | Draws |
|--------|-------|
| P1     | 450   |
| P2     | 788   |
| P3     | 359   |
| P4     | 315   |

Table 3
Average results from the Italian lotto game. August 19, 1984 to December 30, 2007.

| Period | Q    | CLT p-value | MC p-value |
|--------|------|-------------|------------|
| P1     | 31.17| < 10^{-5}   | 0          |
| P2     | 2.07 | 0.8387      | 0.8438     |
| P3     | 1.62 | 0.8991      | 0.8962     |
| P4     | 8.05 | 0.1535      | 0.1576     |

Table 4
Calculated $Q$–statistic and associated $p$–values for each group of results from the Italian lotto game. CLT $p$–value is the Central Limit Theorem-based $p$–value and MC $p$–value is the Monte Carlo approximated $p$–value.

the data in the period August 19th, 1984 until April 4th, 1993 produce a $Q$–statistic of 31.17, with a $p$–value near zero. This means that in the decade 1984 – 1993 the data do not agree with the hypothesis of fairness in the draw of the 90 balls.
5 Conclusions

In this paper we have presented an empirical test of randomness applied to historical data samples taken from Mexican and Italian institutional lotteries. The theoretical mean vector and covariance matrix for the random vector representing the outcome in lotto \( k/N \) games for these two sets of data were obtained. Also, and in order to test consistency in our statistical procedure, Monte Carlo data were generated by simulating a lottery game and compared to data. Application of this procedure to computer-generated random numbers is suitable as a test of randomness for the corresponding pseudorandom algorithms.

For certain periods, statistical evidence was found that the observed average vectors of outcomes significantly differ from their theoretical values. The odds associated to the observed difference for the Mexican historical data are less than 1 in 178; roughly speaking, if during the next 356 years, we could apply this test to results corresponding to non-overlapping two-year periods, only in one case would we expect to obtain a difference as large as the one found here. An even worse situation was detected in one period of the Italian 5/90 lottery for the Rome wheel.

The above results are important from the practical point of view, considering that Lotto games are relevant sources of income both for local and national governments in many countries around the world. The regular use of auditing procedures is recommended; monitoring the historical results with the aid of multivariate statistical procedures, will help in improving the quality of the service by detecting possible deviations from the desired ideal behaviour and in strengthening the confidence of the general public in institutional lottery agencies. The cases where the observed results are highly unlikely under fairness assumptions, as those illustrated here, should be further investigated in order to detect the sources of bias.

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