STABILITY OF SET-VALUED GENERALIZED ADDITIVE CAUCHY FUNCTIONAL EQUATION

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ABSTRACT. The main purpose of this paper is to determine the solution of generalized convex set-valued mappings satisfying certain functional equation. Some conclusions of stability of set-valued functional equations are obtained.

1. Introduction and preliminaries

The goal of this paper is to characterize set-valued solutions of a generalized additive Cauchy functional equation. In 1992, Rassias and Tabor [15] asked whether the functional
\[ f(ax + by + c) = Af(x) + Bf(y) + C \]
with \( abAB \neq 0 \) is stable in the sense of Hyers, Ulam, and Rassias. And the functional equation is called generalized additive Cauchy functional equation.

Badea [1] answered this question of Rassias and Tabor for the case when \( c = C = 0, a = A \) and \( b = B \). As Gajda [2] extended the result of Badea [1] also modified the result of the previous theorem by using a similar method. Jung [4] investigated the stability problem in complex Banach space for a generalized additive Cauchy functional equation
\[ f \left( x_0 + \sum_{j=1}^{m} a_jx_j \right) = \sum_{i=1}^{m} b_i f \left( \sum_{j=1}^{m} a_{ij}x_j \right). \]

An interesting question concerning set-valued mappings that satisfy some functional inclusions is the existence of solutions verifying certain properties. In the stability theory of functional equations, it is important to find solutions of set-valued mappings that are also solutions of certain functional equations. And more results were proved for set-valued mappings satisfying general additive Cauchy of the type
\[ f(ax + by + c) \subseteq Af(x) + Bf(y) + C \quad (1.1) \]
or
\[ Af(x) + Bf(y) \subseteq f(ax + by + c) + C \quad (1.2) \]
where \( f : X \to 2^Y, X \) and \( Y \) are real vector spaces, \( a, b, A, B \in \mathbb{R}, c \in X \) and \( C \in 2^Y \). By \( 2^Y \) we denote the collection of nonempty subsets of \( Y \). Nikodam and Popa [8] considered the general solution of set-valued mappings satisfying linear relation, which.

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can be regarded as a generalization of the additive single-valued functional equation with \(a = A, b = B\) and \(C = c = 0\). Lu et al. \[5, 10\] investigated the approximation of some set-valued functional equations with \(C = c = 0\). Popa \[11\] determined conditions for which a set-valued map that verifies a relation of the form (1.1) admits a unique solution \(f: X \to Y\) that satisfies
\[
f(ax + by + c) \subseteq Af(x) + Bf(y),
\]
for all \(x, y \in X\). The case of equality of form (1.1) was also studied in \[12\]. The similar results \[3, 9\] were obtained in the case of (1.2).

Assume that \(Y\) is a topological vector space satisfying the \(T_0\)-separation axiom. For real numbers \(s, t\) and sets \(A, B \subset Y\) we put \(sA + tB := \{y \in Y; y = sa + tb, a \in A, b \in B\}\). Suppose that the space \(2^Y\) of all subsets of \(Y\) is endowed with the Hausdorff topology (see \[14\]). A set-valued function \(F: X \to 2^Y\) is said to be additive if it satisfies the Cauchy functional equation
\[
F(x_1 + x_2) = F(x_1) + F(x_2), \quad x_1, x_2 \in X.
\]
In this paper, we characterize set-valued solutions of the following functional equation
\[
f(ax + by + c) = AG(x) + BH(y) + C
\] (1.3)
for all \(x, y \in X\), where \(a, b, A\) and \(B\) are positive real numbers, \(c\) and \(C\) are fixed bounded convex sets.

The family of all closed and convex subsets of \(Y\) will be denoted by \(CC(Y)\), and the sets of all real numbers, rational numbers and positive integers are denoted by \(\mathbb{R}, \mathbb{Q}, \mathbb{N}\), respectively.

**Lemma 1.1.** \[6\] Let \(\lambda\) and \(\mu\) be real numbers. If \(A\) and \(B\) are nonempty subsets of a real vector space \(X\), then
\[
\lambda(A + B) = \lambda A + \lambda B,
\]
\[
(\lambda + \mu)A \subseteq \lambda A + \mu B.
\]
Moreover, if \(A\) is a convex set and \(\lambda, \mu \geq 0\), then we have
\[
(\lambda + \mu)A = \lambda A + \mu A.
\]

**Lemma 1.2.** \[13\] Let \(A, B\) be subsets of \(Y\) and assume that \(B\) is closed and convex. If there exists a bounded and nonempty set \(C \subset Y\) such that \(A + C \subset B + C\), then \(A \subset B\).

The following lemmas are rather known and can be easily verified. The proofs of them can be found in \[6, 7\].

**Lemma 1.3.** If \(\{A_n\}_{n \in \mathbb{N}}\) and \(\{B_n\}_{n \in \mathbb{N}}\) are decreasing sequences of compact subsets of \(Y\), then
\[
\bigcap_{n \in \mathbb{N}} (A_n + B_n) = \bigcap_{n \in \mathbb{N}} A_n + \bigcap_{n \in \mathbb{N}} B_n.
\]

**Lemma 1.4.** If \(\{A_n\}_{n \in \mathbb{N}}\) is a decreasing sequence of compact subsets of \(Y\), then \(A_n \to \bigcap_{n \in \mathbb{N}} A_n\).

**Lemma 1.5.** If \(A\) is a bounded subset of \(Y\) and \((\{s_n\}_{n \in \mathbb{N}}\) is a real sequence converging to an \(s \in \mathbb{R}\), then \(s_n A \to sA\).

**Lemma 1.6.** If \(A_n \to A\) and \(B_n \to B\), then \(A_n + B_n \to A + B\).

**Lemma 1.7.** If \(A_n \to A\) and \(A_n \to B\), then \(\text{cl}A = \text{cl}B\).
2. Set-valued solution of the Pexider functional equation

In this section, we give the solution of the following set-valued functional equation

\[ F(ax + by + c) = AG(x) + BH(y) + C \]  

(2.1)

for all \( x, y \in X \).

**Theorem 2.1.** Assume that \((X, +)\) is a vector space and \(Y\) is a \(T_0\) topological vector space. If set-valued functions \( F : X \to CC(Y), G : X \to CC(Y) \) and \( H : X \to CC(Y) \) satisfy the functional equation

\[ F(ax + by + c) = AG(x) + BH(y) + C \]  

(2.2)

for all \( x, y \in X \), where \( a, b, A \) and \( B \) are positive real numbers, \( c \) and \( C \) are fixed bounded convex sets. Then there exist an additive set-valued mapping \( F_0 : X \to CC(Y) \) and sets \( \alpha, \beta \in CC(Y) \) such that

\[ F(x + c) = F_0 \left( \frac{x}{a} \right) + K, \quad AG(x) = F_0(x) + A\alpha \quad \text{and} \quad BH(x) = F_0 \left( \frac{bx}{a} \right) + B\beta \]

for all \( x \in X \).

**Proof.** First, assume that \( 0 \in G(0) \) and \( 0 \in H(0) \). Then, for all \( x, y \in X \), we have

\[
AG(x) + BH(0) + C = F(2ax + c) = F \left( ax + b \frac{ax}{b} + c \right) \\
= AG(x) + BH \left( \frac{ax}{b} \right) + C \\
\subset AG(x) + AG(0) + BH \left( \frac{ax}{b} \right) + C = AG(x) + F(ax + c) \\
= AG(x) + AG(0) + BH(0) + C.
\]

By Lemma 1.2, we get \( G(2x) \subset 2G(x) \), which implies that the sequence \((2^{-n}G(2^n x))_{n \in \mathbb{N}}\) is decreasing. Put \( F_0(x) := \bigcap_{n \in \mathbb{N}} 2^{-n}G(2^n x), x \in X \). It is clear that \( F_0(x) \in CC(Y) \) for all \( x \in X \). Similarly, we get \( H(2x) \subset 2H(x) \) for all \( x \in X \). Then

\[
AG(x) + BH(0) + C = F(ax + c) = AG(0) + BH \left( \frac{ax}{b} \right) + C.
\]

By Lemma 1.3, we obtain that \( AF_0(x) = B \bigcap 2^{-n}H \left( \frac{2^n ax}{b} \right) \) for all \( x \in X \). From (2.2), we get \( F(a2^n x + c) = AG(2^n x) + BH(0) + C, n \in \mathbb{N} \) and so

\[
AF_0(x) = \bigcap_{n \in \mathbb{N}} 2^{-n}F(a2^n x + c)
\]

for all \( x \in X \).
Hence, using once more Lemma 1.3, we get
\[ AF_0(x_1 + x_2) = \bigcap_{n \in \mathbb{N}} 2^{-n} F(a2^n x_1 + a2^n x_2 + c) \]
\[ = \bigcap_{n \in \mathbb{N}} 2^{-n} \left( AG(2^n x_1) + BH \left( \frac{2^n ax_2}{b} \right) + C \right) \]
\[ = \bigcap_{n \in \mathbb{N}} 2^{-n} AG(2^n x_1) + \bigcap_{n \in \mathbb{N}} 2^{-n} BH \left( \frac{2^n ax_2}{b} \right) + \bigcap_{n \in \mathbb{N}} 2^{-n} C \]
\[ = AF_0(x_1) + AF_0(x_2), x_1, x_2 \in X, \]
which means that the set-valued function $F_0$ is additive.

Now observe that
\[ F(nx + c) + (n - 1)AG(0) = F(ax + c) + (n - 1)AG(x) \quad (2.3) \]
for all $x \in X$ and $n \in \mathbb{N}$. Indeed, for $n = 1$ the equality is trivial. Assume that it holds for a natural number $k$. Then, in virtue of (2.2), we obtain
\[ F((k + 1)ax + c) + kAG(0) = AG(x) + BH \left( \frac{akx}{b} \right) + C + kAG(0) \]
\[ = AG(x) + AG(0) + BH \left( \frac{akx}{b} \right) + C + kAG(0) - AG(0) \]
\[ = AG(x) + F(akx + c) + (k - 1)AG(0) \]
\[ = AkG(x) + F(ax + c) \]
which proves that (2.3) holds for $n = k + 1$. Thus, by induction, it holds for all $n \in \mathbb{N}$. In particular, we have
\[ F(2^n ax + c) + (2^n - 1)AG(0) = F(ax + c) + (2^n - 1)AG(x), \]
and so
\[ 2^{-n} F(2^n ax + c) + (1 - 2^{-n})AG(0) = 2^{-n} F(ax + c) + (1 - 2^{-n})AG(x) \]
for all $x \in X$. By Lemma 1.4, $2^{-n} F(2^n ax + c) \to \bigcap_{n \in \mathbb{N}} 2^{-n} F(2^n ax + c) = F_0(x)$.

On the other hand, by Lemma 1.5, $1 - 2^{-n}G(0) \to G(0), 2^{-n}F(ax + c) \to \{0\}$ and $(1 - 2^{-n})G(x) \to G(x)$. Thus, using Lemmas 1.6 and 1.7, we get $cl[F_0(x) + AG(0)] = clAG(x)$, whence $AG(x) = F_0(x) + AG(0)$ for all $x \in X$. Similarly, we can obtain $BH(x) = F_0 \left( \frac{ax}{b} \right) + BH(0), x \in X$. Let $\alpha := G(0)$ and $\beta := H(0)$. Then $AG(x) = F_0(x) + A\alpha$ and $BH(x) = F_0 \left( \frac{ax}{b} \right) + B\beta$ for all $x \in X$. Moreover $F(ax + c) = AG(x) + BH(0) + C = F_0(x) + AG(0) + BH(0) + C = F_0(x) + K, x \in X$, where $K = A\alpha + B\beta + F_0(0) + C$. This finishes our proof in the case that $0 \in G(0)$ and $0 \in H(0)$.

In the opposite case, fix arbitrarily points $a \in G(0)$ and $b \in H(0)$, and consider the set-valued mappings $F_1, G_1, H_1 : X \to CC(Y)$ defined by $G_1(x) := G(x) - a$ and $H_1 := H(x) - b, x \in X$. These set-valued mappings satisfy the equation (2.2) and moreover $0 \in G_1(0)$ and $0 \in H_1(0)$. Therefore, by what we have discussed previously, we can get the same result. This completes the proof. \quad \square
Conclusion

We have determined the solution of generalized convex set-valued mappings satisfying certain functional equation. Some conclusions of stability of set-valued functional equations have been obtained.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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