We review on the models of gravity with a constraint by the Lagrange multiplier field. The constraint breaks general covariance or Lorentz symmetry in the ultraviolet region. We report on the $F(R)$ gravity model with the constraint and the proposal of the covariant (power-counting) renormalized gravity model by using the constraint and scalar projectors. We will show that the model admits flat space solution, its gauge-fixing formulation is fully developed, and the only propagating mode is (higher derivative) graviton, while scalar and vector modes do not propagate. The preliminary study of FRW cosmology indicates to the possibility of inflationary universe solution is also given.

Keywords: Spontaneous breakdown of Lorentz symmetry; Quantum gravity

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1. Introduction

Recently the models of gravity with a constraint by the Lagrange multiplier field have been proposed. The constraint breaks general covariance or Lorentz symmetry and given by the following action:

$$S = - \int d^4x \sqrt{-g} \lambda \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + U_0 \right).$$

Here $\lambda$ is the Lagrange multiplier field and $U_0$ is a positive constant. Then the variation of $\lambda$, we obtain the following equation:

$$\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + U_0 = 0,$$

which tells $(\partial_{\mu} \phi)$ is a non-vanishing time-like vector and therefore the general covariance or the Lorentz symmetry is breakdown spontaneously. Locally, one can choose the direction of time to be parallel to $(\partial_{\mu} \phi)$. Then we find

$$\phi = \sqrt{2U_0} t,$$
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In this report, especially we review on the $F(R)$ gravity model with the constraint and the proposal of the covariant (power-counting) renormalized gravity model by using the constraint and scalar projectors.

In 2009, Hořava proposed a candidate of quantum field theory of gravity which is power-counting renormalizable. The model has anisotropy between space and time by the explicit breaking of covariance but it was expected that at long distances, the Lorentz symmetry could be recovered. It was clarified, however, the existence of extra scalar mode violating the Newton law.

After that there were proposals of covariant and power-counting renormalizable model of gravity. It has been shown show that only massless graviton propagates in the model of Ref.

2. Application to $F(R)$ gravity

As an application of the Lagrange multiplier field to the $F(R)$ gravity models, we consider the following action:

$$S = \int d^4x \sqrt{-g} \left\{ F_1(R) - \lambda \left( \frac{1}{2} \partial_\mu R \partial^\mu R + F_2(R) \right) \right\}. \quad (4)$$

By the variation of $\lambda$, we obtain the following constraint

$$\frac{1}{2} \partial_\mu R \partial^\mu R + F_2(R) = 0. \quad (5)$$

On the other hand, by the variation of the metric $g_{\mu\nu}$, we obtain

$$0 = \frac{1}{2} g_{\mu\nu} F_1(R) + \lambda \partial_\mu R \partial_\nu R + (-R_{\mu\nu} + \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) \times (F_1'(R) - \lambda F_2'(R) - \nabla^\nu (\lambda \nabla_\mu R)). \quad (6)$$

If the Ricci curvature is covariantly constant: $R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}$ and therefore $R = R_0$, Eqs. \([5]\) and \([6]\) give $0 = F_2(R_0)$ and $0 = F_1(R_0) - \frac{1}{2} R_0 (F_1'(R_0) - \lambda F_2'(R_0))$ and therefore $\lambda = -\frac{2 F_1(R_0) + R_0 F_1'(R_0)}{R_0 F_2'(R_0)}$. Then if $R_0 > 0$, we obtain a solution describing de Sitter space-time, which may be regarded with the inflation.

For spatially-flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left( dx^i \right)^2. \quad (7)$$

Eqs. \([5]\) and \([6]\) have the following form:

$$0 = -\frac{1}{2} \dot{R}^2 + F_2(R),$$

$$0 = -\frac{1}{2} F_1(R) + 18 \lambda \left( \dot{H} + 4 H \dot{H} \right)^2 + \left\{ 3 \left( \dot{H} + H^2 \right) - 3 H \frac{d \dot{H}}{dt} \right\} \times \left\{ F_1'(R) - \lambda F_2'(R) + \frac{d}{dt} \left( \lambda \frac{dR}{dt} \right) \right\}. \quad (8)$$
By integrating the first equation in (8) when $F_2(R) > 0$, we obtain $$t = \int R \sqrt{2F_2(R)} \, dr$$, which can be solved with respect to $R$ as $R = F_R(t)$. Then since $6\frac{dH}{dt} + 12H^2 = R = F_R(t)$, we obtain the $t$ dependence of $H$ and $\lambda$ as $H = H(t)$ and $\lambda = \lambda(t)$.

Conversely if we know $H(t)$, we can construct $F_2(R)$ to produce the $t$ dependence of $H$ and $\lambda$ as $H = H(t)$ and $\lambda = \lambda(t)$. Conversely if we know $H(t)$, we can construct $F_2(R)$ to produce the $t$ dependence of $H$. As we know $H(t)$, we can find the $t$ dependence of the scalar curvature $R = R(t)$, which could be solved with respect to $t$ as $t = t(R)$. Then the first equation in (8) gives the explicit form of $F_2(R)$ as $F_2(R) = \frac{1}{2} (\frac{dR}{dt})^2 \bigg|_{t=t(R)}$. We should note $F_1(R)$ can be an arbitrary function.

As an example, we may consider $H(t) = \frac{h_0}{t}$. Here $h_0$ is a positive constant. Then we find $F_2(R) = \frac{R^3}{12(-h_0 + 2h_0^2)}$. As a second example, we may consider $R = \frac{R_-}{2} (1 - \tanh \omega t) + \frac{R_+}{2} (1 + \tanh \omega t)$. Here $R_\pm$ and $\omega$ are positive constants. For the curvature $R$ in (9), we find $R \to R_\pm$ when $t \to \pm \infty$, that is, asymptotically de Sitter space-time. Then we may regard $t \to -\infty$ could correspond to inflation, and $t \to +\infty$ to late acceleration. We now find $F_2(R) = \frac{(R_- - R_+)^2 \omega^2}{8} \left(1 - \frac{(R_- + R_+ - 2R)^2}{(R_- - R_+)^2}\right)^2$. (10)

3. Brief review on Hořava gravity

Before we apply the constraint by the Lagrange multiplier field to the covariant power-counting renormalizable gravity, we will briefly review the model in Ref. [3]. We know that Einstein’s general relativity is non-renormalizable as a quantum field theory. If we consider the perturbation from the flat background: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ (Here $\kappa$ is a gravitational coupling), the Einstein-Hilbert action has the following form:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R$$

$$= \int d^4x \left[ -\frac{1}{2} \partial h \partial \bar{h} + \kappa h \partial \bar{h} \partial h + \kappa^2 h^2 \partial \bar{h} \partial h + \cdots + \kappa^n h^n \partial \bar{h} \partial h + \cdots \right].$$ (11)

Since the dimension of $\kappa$ is that of length, Einstein’s general relativity is non-renormalizable. However, if the propagator behaves as $1/p^4$, $1/p^6$, $\cdots$ instead of $1/p^2$ ($p_\mu$: four momentum) in the ultraviolet region, the ultraviolet behavior of the quantum correction could be improved. Usually such a propagator is given by the higher derivative theory, where the unitarity could be broken in general, due to the higher derivative with respect to $t$.

Then Hořava’s idea is to introduce anisotropic treatment between space and time and consider the higher derivative theory with respect to only spacial coordinates. Then in the ultraviolet region, the propagator behaves as $1/p^4$, $1/p^6$, $\cdots$. 

Covariant gravity with Lagrange multiplier constraint
The anisotropy can be expressed by a parameter \( z \) which is given by the scale transformation with a constant \( b \): \( x \rightarrow bx \), \( t \rightarrow b^z t \), \( (z = 2, 3, \cdots) \).

In order to express the action of the model in Ref. 5, we use the well-known ADM decomposition, where the metric is expressed as

\[
ds^2 = -N^2 dt^2 + \sum_{i,j=1,2,3} g^{(3)}_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right) .
\] (12)

Here \( N \) and \( N^i \) are called the lapse variable and the shift variable, respectively. Then we find that the Einstein-Hilbert action has the form of

\[
\int dt d^3 x \sqrt{g^{(3)}} \left( K_{ij} K^{ij} - \lambda K^2 \right) .
\] (13)

Here \( \lambda \) is a parameter. The action is invariant under the spacial diffeomorphism and the temporal diffeomorphism, which are given by

\[
\delta x^i = \zeta^i(t, x) , \quad \delta t = f(t) .
\] (14)

By using \( z \), the dimension of time \( t \) is expressed as \([L^z]\). Here \( L \) is the length. Since \( ds^2 = -N^2 dt^2 + \cdots \), we find the dimension of \( N \) as \([N] = [L^{-1}]\). In perturbation theory, when we fix the gauge of diffeomorphism, we choose \( N = N_0 \) with a constant \( N_0 \). Then as clear from (13), the effective coupling constant is given by \( 1/\kappa_*^2 = N_0/\kappa^2 \), whose dimension is \([\kappa_*^2] = [L^{3-z}]\). Then when \( z = 3 \), \( \kappa_* \) becomes dimensionless and therefore the model becomes power counting renormalizable.

In order to include the “potential”, that is, the terms not including the derivatives with respect to time, the generalized De Witt “metric on the space of metrics”, which is given by

\[
G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl} .
\] Then Hořava has proposed the potential with “detailed balance”, which is given by

\[
S_V = \frac{\kappa^2}{8} \int dt d^3 x \sqrt{g^{(3)}} G_{ijkl} E^{ij} E^{kl} , \quad \sqrt{g} E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}} .
\] (15)

Then for the \( z = 2 \) model, Hořava has chosen \( W = \frac{1}{8\kappa_*^2} \int d^3 x \sqrt{g}(R - 2\Lambda_W) \), which gives

\[
S_V = \frac{\kappa^2}{8\kappa_*^2} \int dt d^3 x \sqrt{g^{(3)}} \left( R^{ij} - \frac{1}{2} R g^{ij} + \Lambda_W g^{ij} \right) G_{ijkl} \left( R^{kl} - \frac{1}{2} R g^{kl} + \Lambda_W g^{kl} \right) .
\] (16)

\(^a\) Here we followed the notation in Ref. 5. Except in this section, \( \lambda \) expresses the Lagrange multiplier field.
For $z = 3$ model, it has proposed as $W = \frac{1}{w^2} \int \omega_3(\Gamma)$. Here $w^2$ is a dimensionless coupling and $\omega_3(\Gamma)$ is gravitational Chern-Simons term given by

$$\omega_3(\Gamma) = \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) \equiv \varepsilon^{ijk} \left( \Gamma^m_{\nu} \partial_j \Gamma^\nu_{km} + \frac{2}{3} \Gamma^m_{\nu} \Gamma^\nu_{jm} \Gamma^m_{kn} \right) d^3x. \quad (17)$$

Then we find

$$S = \int dt d^3x \sqrt{-g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij} C^{ij} \right\} \quad \text{(18)}$$

Here $C^{ij}$ is called the Cotton tensor defined by

$$C^{ij} = \varepsilon^{ik\ell} \nabla_k (R_{j\ell} - \frac{1}{8} \nabla_k R).$$

After the proposal, it has been clarified that there are several problems\textsuperscript{6,7,8} in the Ho\v{r}ava gravity. The model does not have full diffeomorphism symmetry but the direct product of the spacial diffeomorphism and temporal diffeomorphism\textsuperscript{14}. Then in order to impose a gauge condition $N = \text{constant}$, we may assume the projectability condition, that is, $N$ should depend on only time coordinate $N = N(t)$. Although there have been also proposed models which do not satisfy the projectability condition, the degrees of freedom of the Ho\v{r}ava gravity do not coincide with those of the Einstein gravity and therefore the Ho\v{r}ava gravity does not reproduce general relativity even in the low energy region.

### 4. Proposal of covariant and power-counting renormalizable models of gravity

In order to construct models with correct degrees of freedom, we have proposed a model with the covariance (full diffeomorphism invariance), that is, the covariant and power-counting renormalizable model\textsuperscript{9,11} by using the spontaneous breakdown of Lorentz symmetry.

The action of the proposed model is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \alpha \left( \phi^{\mu} \phi^{\nu} \phi \nabla_\mu \nabla_\nu - \partial_\mu \phi \partial^\mu \phi \nabla_\rho \nabla_\rho \right)^n \frac{1}{P_{\alpha}^\mu} P^{\alpha \nu} \right] \times \left( R_{\mu\nu} - \frac{1}{2U_0} \partial_\mu \phi \nabla_\nu \phi \right) \left( \left( \phi^{\mu} \phi^{\nu} \phi \nabla_\mu \nabla_\nu - \partial_\mu \phi \partial^\mu \phi \nabla_\rho \nabla_\rho \right)^n \Delta + \frac{1}{P_{\alpha}^\mu} P^{\alpha \nu} \right) \times \left( R_{\mu\nu} - \frac{1}{2U_0} \partial_\mu \phi \nabla_\nu \phi \right) - \lambda \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_0 \right), \quad (19)$$

Here $\lambda$ is the Lagrange multiplier field, $U_0$ is a constant, and $P^{\alpha \nu}$ is a projection operator defined by $P^{\alpha \nu}_\mu \equiv \delta^{\alpha \nu}_\mu + \frac{\partial^{\alpha} \phi}{2U_\mu}$. For $z = 2n + 2$ model ($n = 0, 1, 2, \cdots$), $\Delta = 0$ and for $z = 2n + 3$ model, $\Delta = 1$. First we should note that the actions admit a flat space vacuum solution. The field equations have the following form:
0 = \frac{1}{16\pi} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + G^\text{higher}_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \left( \frac{1}{2} \partial_\rho \phi \partial^\rho \phi + U_0 \right). \quad \text{Then by assuming the flat vacuum solution, we find} \quad 0 = \lambda \partial_\mu \phi \partial_\nu \phi, \quad \text{which gives} \quad \lambda = 0. \quad \text{Then we obtain the flat space vacuum solution with} \quad \lambda = 0.

5. Perturbation from the flat background

In this section, we consider the perturbation from the flat background and we show that the only propagating mode is higher derivative graviton while scalar and vector modes do not propagate.

We now fix the diffeomorphism invariance with respect to time coordinate by choosing the condition by (3), which is a kind of the unitary gauge condition. Then by the perturbation from flat background: \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), we find

\[
S \rightarrow \int d^4x \left[ -\frac{1}{8\kappa^2} \left\{ -2 h_{tt} (\delta^{ij} \partial_i \partial^j - \partial^i \partial^j) h_{ij} + 2 h_{tij} (\delta^{ij} \partial_k \partial^k - \partial^i \partial^j) h_{ij} \right. \right.
\]

\[
+ h_{ti} \left( 2 \delta^{jk} \partial^i - \delta^{ik} \partial^j - \delta^{ij} \partial^k \right) \partial_i h_{jk} + h_{ij} \left( \left( \delta^{ij} \delta^{kl} - \frac{1}{2} \delta^{ik} \delta^{jl} - \frac{1}{2} \delta^{il} \delta^{jk} \right) \left( -\partial^2_i + \partial_k \partial^k \right) \right.
\]

\[
- \delta^{ij} \partial^i \partial^j - \delta^{ik} \partial^j - \delta^{ij} \partial^k \right) + \frac{1}{2} \left( \delta^{ik} \partial^j \partial^i + \delta^{il} \partial^j \partial^i + \delta^{jl} \partial^i \partial^i + \delta^{il} \partial^j \partial^k \right) h_{kl} \right] \right.
\]

\[
- 2^{2n-2+\Delta} \alpha U_0^{2n+\Delta} \left\{ \left( \partial_\mu \partial^\mu \right)^n \left( h_{k,i} + h_{k,i,k} - \partial_i \partial_j \left( h_{\mu}^\mu \right) \right) \right\} \times \left\{ \left( \partial_\mu \partial^\mu \right)^{n+\Delta} \left( h_{k,i} + h_{k,i,k} - \partial_i \partial_j \left( h_{\mu}^\mu \right) \right) \right\} + U_0 \lambda h_{tt} \right].
\]

By the variation of \( \lambda \), we obtain \( h_{tt} = 0 \). On the other hand by the variation of \( h_{tt} \), we obtain

\[
\lambda = -\frac{1}{4\kappa^2 U_0} \left( \delta^{ij} \partial_i \partial^j - \partial^i \partial^j \right) h_{ij}
\]

\[
+ 2^{2n-1+\Delta} \alpha U_0^{2n+\Delta} \left( \partial_\mu \partial^\mu \right)^{2n+\Delta} \partial^i \partial^j \left( h_{k,i}^k + h_{k,i,k}^k - \partial_i \partial_j \left( h_{\mu}^\mu \right) \right).
\]

The variation of \( \phi \) gives

\[
0 = \partial_t \left\{ \lambda + 2^{2n-1+\Delta} \alpha U_0^{2n-1+\Delta} \left( \partial_\mu \partial^\mu \right)^{2n+\Delta} \partial^i \partial^j \left( h_{k,i}^k + h_{k,i,k}^k - \partial_i \partial_j \left( h_{\mu}^\mu \right) \right) \right\}. \quad \text{(22)}
\]

We now decompose \( h_{ti} \), which corresponds to the shift function \( N_t \), as \( h_{ti} = \partial_i s + v_i \). Here \( v_i \) satisfies the equation \( \partial^i v_i = 0 \) and \( s \) is a spatial scalar. We also linearize the diffeomorphism invariance transformations with respect to the spatial coordinates as \( \delta x^i = \partial^i u + w^i \), where \( u_i \) satisfies the equation \( \partial_i w^i = 0 \). Then we find \( \delta s = \partial_i u \) and \( \delta u_i = \partial_i w_i \). We now choose the gauge fixing condition as \( s = v^i = 0 \), which gives \( h_{tt} = 0 \). The variation of \( h_{ti} \) gives

\[
\partial_t \left( -2 \delta^{ik} \partial^i + \delta^{ik} \partial^j + \delta^{ij} \partial^k \right) h_{jk} = 0,
\]

which is identical with that in the Einstein gravity and does not include higher derivative terms.
We now also decompose \( h_{ij} \) as \( h_{ij} = \delta_{ij} A + \partial_i B_j + \partial_j B_i + C_{ij} + (\partial_i \partial_j - \frac{1}{2} \delta_{ij} \partial_k \partial^k) E \), where \( B_i \) and \( C_{ij} \) satisfy equations \( \partial^i B_i = 0 \), \( \partial^i C_{ij} = \partial^j C_{ij} = 0 \), and \( C_{i,i} = 0 \). Then by the variation of \( \delta h_{ij} \), we obtain

\[
0 = \partial_t \left( -4 \partial_t A + 2 \partial_k \partial^k B_i + \frac{4}{3} \partial_i \partial_k \partial^k E \right) .
\]

(24)

By multiplying (21) with \( \partial^i \), we find \( \partial_t \partial_i \partial^i (-4A + \frac{4}{3} \partial_k \partial^k E) = 0 \), which gives \( A = \frac{1}{3} \partial_k \partial^k E \). Here we have assumed \( A \) and \( E \) vanish at spatial infinity. By using (24), we find \( \partial_t \partial_i \partial^i B_i = 0 \), which gives \( B_i = 0 \) by assuming \( B_i \to 0 \) at spatial infinity. Then Eqs. (21) and (22) can be rewritten as

\[
\lambda = \frac{1}{2\kappa^2 U_0} \partial_k \partial^k \left( -A + \frac{1}{3} \partial_j \partial^j E \right) - 2^{2n+\Delta} \alpha U_0^{2n-1+\Delta} \left( \partial_k \partial^k \right)^{2n+2+\Delta} \left( -A + \frac{1}{3} \partial_j \partial^j E \right) ,
\]

(25)

\[
0 = \partial_t \left\{ \lambda + 2^{2n+\Delta} \alpha U_0^{2n-1+\Delta} \left( \partial_k \partial^k \right)^{2n+2+\Delta} \left( -A + \frac{1}{3} \partial_j \partial^j E \right) \right\} .
\]

(26)

Then we find \( \lambda = 0 \) since \( A = \frac{1}{3} \partial_k \partial^k E \). Therefore we find the scalar modes \( \lambda \) and the vector mode \( B_i \) do not propagate.

By the variation of \( A \) gives

\[
0 = \frac{1}{8\kappa^2} \left\{ -12 \left( -\partial^2 + \partial_k \partial^k \right) A + 8 \partial_k \partial^k A + \frac{4}{3} \left( \partial_k \partial^k \right)^2 E \right\}
\]

\[
- 2^{2n-1+\Delta} \alpha U_0^{2n+\Delta} \left( -\partial_i \partial_j - \delta_{ij} \partial_k \partial^k \right) \left( \partial_k \partial^k \right)^{2n+\Delta} \left( -\partial^i \partial^j A - \delta^{ij} \partial_k \partial^k A + \frac{1}{3} \partial^i \partial^j \partial_k \partial^k E + \frac{1}{3} \delta^{ij} \left( \partial_k \partial^k \right)^2 E \right) \right\} .
\]

(27)

By the variation of \( E \), we obtain

\[
0 = \partial^k \partial_k \left\{ \frac{1}{8\kappa^2} \left\{ \frac{4}{3} \left( -\partial^2 + \partial_k \partial^k \right) \partial_k \partial^k E + \frac{4}{3} \partial_k \partial^k A + \frac{16}{3} \left( \partial_k \partial^k \right)^2 E \right\}
\]

\[
+ \frac{2^{2n-1+\Delta}}{3} \alpha U_0^{2n+\Delta} \left( -\partial_i \partial_j - \delta_{ij} \partial_k \partial^k \right) \left( \partial_k \partial^k \right)^{2n+\Delta} \left( -\partial^i \partial^j A - \delta^{ij} \partial_k \partial^k A + \frac{1}{3} \partial^i \partial^j \partial_k \partial^k E + \frac{1}{3} \delta^{ij} \left( \partial_k \partial^k \right)^2 E \right) \right\} .
\]

(28)

Then we find \( \partial^2 A = 0 \) and therefore by using \( A = \frac{1}{3} \partial_k \partial^k E \), we find \( A = E = 0 \). Therefore we find all the scalar modes \( \phi, \lambda, h_{tt}, s, A, \) and \( E \) and all the vector modes \( v_i \) and \( B_i \) do not propagate. The only propagating mode is massless graviton \( C_{ij} \). This situation should be distinguished from that in the Hořava quantum gravity. The action for the massless graviton \( C_{ij} \) is given by

\[
S = \int d^4x \left\{ \frac{1}{8\kappa^2} \left\{ \left( \partial_k \partial^k \right)^{n+1} C_{ij} \right\} \right\}
\]

\[
- 2^{2n-2+\Delta} \alpha U_0^{2n+\Delta} \left\{ \left( \partial_k \partial^k \right)^{n+1} C_{ij} \right\} \left\{ \left( \partial_k \partial^k \right)^{n+1} C^{ij} \right\} .
\]

(29)
which give the propagator of $C_{ij}$ as follows

$$
\langle h_{ij}(p) h_{kl}(-p) \rangle = \langle C_{ij}(p) C_{kl}(-p) \rangle
$$

$$
= \frac{1}{2^n} \left\{ \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right) \left( \delta_{kl} - \frac{p_k p_l}{p^2} \right) - \left( \delta_{ik} - \frac{p_i p_k}{p^2} \right) \left( \delta_{jl} - \frac{p_j p_l}{p^2} \right) - \left( \delta_{il} - \frac{p_i p_l}{p^2} \right) \left( \delta_{jk} - \frac{p_j p_k}{p^2} \right) \right\} \left\{ \sum_{n=0}^{\infty} \sum_{\Delta=0}^{\max} \alpha_n \left( \left( \partial^{\mu} \phi \partial^{\nu} \phi \nabla^{\alpha} \nabla^{\beta} \nabla^{\gamma} \nabla^{\rho} \phi \right) - \partial_{\mu} \phi \partial^{\mu} \phi \nabla^{\rho} \phi \right)^{n+\Delta} \right\}
$$

Here $p^2 = \sum_{i=1}^{3} (p_i)^2$ and $p^2 = -(p^0)^2 + \mathbf{p}^2$. We should assume $\alpha < 0$ in order to avoid tachyon pole. In the ultraviolet region, the propagator behaves as $\sim 1/|p|^4$ for $z = 2$ ($n = 0$) case and $\sim 1/|p|^6$ for $z = 3$ ($n = 0$) case and therefore the model could be power-counting renormalizable. In the case of $z = 2n + 2$ ($n \geq 1$) or $z = 2n + 3$ ($n \geq 1$) case, the model could be (power-counting) super-renormalizable.

6. FRW cosmology

We now consider the FRW cosmology. We may start with a little bit general action:

$$
S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2k^2} - \sum_{\Delta=0,1} \sum_{n=0}^{\infty} \sum_{\alpha=0}^{\infty} \left( \partial^{\mu} \phi \partial^{\nu} \phi \nabla^{\alpha} \nabla^{\beta} \nabla^{\gamma} \nabla^{\rho} \phi \right)^n \right] P_{\mu} P^{\nu} - \alpha \left( \partial_{\mu} \phi \partial^{\mu} \phi + U_0 \right)
$$

In low energy, in addition to the Einstein-Hilbert term, $\Delta = 0$, $n = 0$ term in (31) could dominate and the action (31) could reduce to

$$
S \sim \int d^4 x \sqrt{-g} \left[ \frac{R}{2k^2} - \alpha_0 P_{\mu} P^{\nu} \left( \partial_{\mu} \phi \partial^{\nu} \phi \nabla^{\alpha} \nabla^{\beta} \nabla^{\gamma} \nabla^{\rho} \phi \right) - \alpha \left( \partial_{\mu} \phi \partial^{\mu} \phi + U_0 \right) \right].
$$

We now assume the FRW metric as $ds^2 = -c^2(t) dt^2 + a(t)^2 \sum_{i=1,2,3} \left( dx_i \right)^2$. Then by the variation of $b$, we obtain the following FRW equation: $\frac{\dot{a}}{a} H^2 = 8 \alpha \left( H^2 + 8 \alpha \right)$, which may correspond to the inflation in the early universe.

7. Superluminal neutrinos

The OPERA experiment results indicate towards the possibility that the neutrino speed might exceed the speed of light [9]. Motivated with the results, in Ref. [10], a
model of superluminal spinor by the spontaneous breakdown due to the Lagrange multiplier field has been proposed. The action we consider is

\[ S = \int d^4x \left[ \bar{\psi} \left\{ \gamma^\mu \partial_\mu + \alpha \left( P^\nu_{\mu} \gamma^\mu \partial_\nu \right)^{2n+1} \right\} \psi - \lambda \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_0 \right) \right]. \quad (33) \]

Here \( \alpha \) is a constant, \( n \) is an integer equal to or greater than 1, and \( P^\nu_{\mu} \) is a projection operator again. By using (3), the equation corresponding to the Dirac equation

\[ 0 = \left\{ \gamma^0 \partial_0 + \gamma^i \partial_i + \alpha \left( \gamma^i \partial_i \right)^{2n+1} \right\} \psi \]

looks as

\[ 0 = \left\{ \gamma^0 \partial_0 + \gamma^i \partial_i + \alpha \left( \gamma^i \partial_i \right)^{2n+1} \right\} \psi. \]

Therefore the dispersion relation for the spinor is given by

\[ \omega = k \sqrt{1 + \alpha^2 k^{4n}}. \]

Here \( \omega \) is the angular frequency corresponding to the energy and \( k \) is the wave number corresponding to the momentum. In the high energy region, the dispersion relation becomes \( \omega \sim |\alpha| k^{2n+1} \) and therefore the phase velocity \( v_p \) and the group velocity \( v_g \) are given, respectively, by

\[ v_p = \frac{\omega}{k} = |\alpha| k^{2n} \quad \text{and} \quad v_g = \frac{d\omega}{dk} = (2n+1) |\alpha| k^{2n}, \]

respectively. When \( k \) becomes larger, both \( v_p \) and \( v_g \) become also larger in an unbounded way and exceed the light speed.

8. Summary

In this report, we investigated the model with the spontaneous breakdown of the Lorentz symmetry by using the Lagrange multiplier field. We considered \( F(R) \) gravity, power-counting renormalizable gravity, and superluminal spinor. We formulated the power-counting renormalizable gravity, and the superluminal spinor by using scalar projectors. For the power-counting renormalizable gravity, we have shown that the theory admits flat space solution and the only propagating mode is (higher derivative) graviton, while scalar and vector modes do not propagate by developing the gauge-fixing formulation. We also gave a preliminary study of FRW cosmology indicates to the possibility of inflationary universe solution. The first FRW equation in the theory turns out to be the first order differential equation which is quite unusual for higher derivative gravity which normally leads to third order differential equation with respect to scale factor.

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References

1. E. A. Lim, I. Sawicki and A. Vikman, JCAP 1005, 012 (2010) [arXiv:1003.5751 [astro-ph.CO]]; C. Gao, Y. Gong, X. Wang and X. Chen, [arXiv:1003.6056 [astro-ph.CO]];
Shin'ichi Nojiri

Y. F. Cai and E. N. Saridakis, Class. Quant. Grav. 28, 035010 (2011) [arXiv:1007.3204 [astro-ph.CO]]; J. Kluson, [arXiv:1101.5880] [hep-th].

2. S. Capozziello, J. Matsumoto, S. Nojiri and S. D. Odintsov, Phys. Lett. B 693 (2010) 198 [arXiv:1004.3691 [hep-th]].

3. S. Nojiri, S. D. Odintsov, Phys. Lett. D81, 043001 (2010), [arXiv:0905.4213 [hep-th]]; Phys. Lett. B691, 60-64 (2010), [arXiv:1004.3613 [hep-th]]; Phys. Rev. D83, 023001 (2011), [arXiv:1007.4856 [hep-th]]; M. Chaichian, M. Oksanen, A. Tureanu, Eur. Phys. J. C71, 1657 (2011). [arXiv:1101.2843 [gr-qc]]; G. Cognola, E. Elizalde, L. Sebastiani and S. Zerbini, Phys. Rev. D 83, 063003 (2011) [arXiv:1007.4676 [hep-th]]; S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011) [arXiv:1101.0544 [gr-qc]].

4. J. Kluson, S. "i. Nojiri, S. D. Odintsov, Phys. Lett. B701, 117-126 (2011). [arXiv:1104.4286 [hep-th]].

5. P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].

6. C. Charmousis, G. Niz, A. Padilla, P. M. Saffin, JHEP 0908, 070 (2009). [arXiv:0905.2579 [hep-th]].

7. M. Li, Y. Pang, JHEP 0908, 015 (2009). [arXiv:0905.2751 [hep-th]].

8. D. Blas, O. Pujolas, S. Sibiryakov, Phys. Rev. Lett. 104, 181302 (2010). [arXiv:0909.3525 [hep-th]].

9. T. Adam et al. [ OPERA Collaboration ], [arXiv:1109.4897] [hep-ex]].

10. S. "i. Nojiri, S. D. Odintsov. [arXiv:1110.0889] [hep-ph]].