Model Companions of $T_\sigma$ for stable $T$

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Let $T$ be a complete first order theory in a countable relational language $L$. We assume relation symbols have been added to make each formula equivalent to a predicate. Adjoin a new unary function symbol $\sigma$ to obtain the language $L_\sigma$; $T_\sigma$ is obtained by adding axioms asserting that $\sigma$ is an $L$-automorphism.

The modern study of the model companion of theories with an automorphism has two aspects. One line, stemming from Lascar \cite{Lascar}, deals with ‘generic’ automorphisms of arbitrary structures. A second, beginning with Chatzidakis and Hrushovski \cite{ChatzidakisHrushovski} and questions of Macintyre about the Frobenius automorphism is more concerned with specific algebraic theories. This paper is more in the first tradition: we find general necessary and sufficient conditions for a stable first order theory with automorphism to have a model companion.

Kikyo investigates the existence of model companions of $T_\sigma$ when $T$ is unstable in \cite{Kikyo}. He also includes an argument of Kudaibergenov showing that if $T$ is stable with the finite cover property then $T_\sigma$ has no model companion.

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This argument was implicit in [9] and is a rediscovery of a theorem of Winkler in the 70’s. We provide necessary and sufficient conditions for $T_\sigma$ to have a model companion when $T$ is stable. Namely, we introduce a new condition: $T_\sigma$ admits obstructions, and show that $T_\sigma$ has a model companion iff and only if $T_\sigma$ does not admit obstructions. This condition is weakening of the finite cover property: if a stable theory $T$ has the finite cover property then $T_\sigma$ admits obstructions.

Kikyo also proved that if $T$ is an unstable theory without the independence property, $T_\sigma$ does not have a model-companion. Kikyo and Shelah have improved this by weakening the hypothesis to, $T$ has the strict order property.

For $p$ a type over $A$ and $\sigma$ an automorphism with $A \subset \text{dom } p$, $\sigma(p)$ denotes $\{\phi(x, \sigma(a)) : \phi(x, a) \in p\}$. References of the form II.4.13. are to [8].

1 Example

In the following example we examine exactly why a particular $T_\sigma$ does not have a model companion. Eventually, we will show that the obstruction illustrated here represents the reason $T_\sigma$ (for stable $T$) can fail to have a model companion. Let $L$ contain two binary relation symbols $E$ and $R$ and unary predicates $P_i$ for $i < \omega$. The theory $T$ asserts that $E$ is an equivalence relation with infinitely many infinite classes, which are refined by $R$ into two-element classes. Moreover, each $P_i$ holds only of elements from one $E$-class and contains exactly one element from each $R$-class of that $E$-class.

Now, $T_\sigma$ does not have a model companion. To see this, let $\psi(x, y, z)$ be the formula: $E(x, z) \land E(y, z) \land R(x, y) \land x \neq y$. Let $\Gamma$ be the $L_\sigma$-type in the variables $\{z\} \cup \langle x_i y_i : i < \omega \rangle$ which asserts for each $i$, $\{\psi(x_i, y_i, z)\}$, the sequence $\langle x_i y_i : i < \omega \rangle$ is $L$-indiscernible and for every $\phi(x, w) \in L(T)$:

$$(\forall w) \bigwedge_{i \in U} \phi(x_i, w) \leftrightarrow \phi(y_i, \sigma(w)) : U \subseteq \lg(w) + 3, |U| > (\lg(w) + 3)/2.$$ 

Thus if $\langle b_i c_i : i < \omega \rangle a$ realize $\Gamma$ in a model $M$,

$$\sigma(\text{avg}(\langle b_i : i < \omega \rangle / M)) = \text{avg}(\langle c_i : i < \omega \rangle / M).$$

For any finite $\Delta \subset L(T)$, let $\chi_{\Delta, k}(x, y, z)$ be the conjunction of the $\Delta$-formulas satisfied by $\langle b_i c_i : i < k \rangle a$ where $\langle b_i c_i : i < k \rangle a$ are an initial
segment of a realization of $\Gamma$. Let $\theta_{\Delta,k}$ be the sentence

$$(\forall x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}, z) \chi_{\Delta,k}(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}, z) \rightarrow

(\exists x_0, y_0, x_1, y_1)[\psi(x_0, y_0, z) \land \psi(x_1, y_1, z) \land \sigma(x_1) = y_1].$$

We now claim that if $T_\sigma$ has a model companion $T_\sigma^*$, then for some $k$ and $\Delta$,

$$T_\sigma^* \models \theta_{\Delta,k}.$$  

For this, let $M \models T_\sigma^*$ such that $\langle b, c : i < k \rangle$ satisfy $\Gamma$ in $M$. Suppose $M \models L \prec N$ and $N$ is an $|M|^+$-saturated model of $T$. In $N$ we can find $b, c$ realizing the average of $\langle b : i < \omega \rangle$ and $\langle c : i < \omega \rangle$ over $M$ respectively. Then

$$\sigma(\text{avg}((b : i < \omega)/M)) = \text{avg}((c : i < \omega)/M)$$

and so there is an automorphism $\sigma^*$ of $N$ extending $\sigma$ and taking $b$ to $c$. Since $(M, \sigma)$ is existentially closed ($T_\sigma^*$ is model complete), we can pull $b, c$ down to $M$. By compactness, some finite subset $\Gamma_0$ of $\Gamma$ suffices and letting $\Delta$ be the formulas mentioned in $\Gamma_0$ and $k$ the number of $x_i, y_i$ appearing in $\Gamma_0$ we have the claim.

But now we show that if $(M, \sigma)$ is any model of $T_\sigma$, then for any finite $\Delta$ and any $k$, $(M, \sigma) \models \neg \theta_{\Delta,k}$. For this, choose $b_i, c_i$ for $i < k$ which are $E$-equivalent to each other and to an element $a$ in a class $P_j$ where $P_j$ does not occur in $\Delta$ and with $R(b_i, c_i)$ and $b_i \neq c_i$. Then $b, c, a$ satisfy $\chi_{\Delta,k}$ but there are no $b, c$ and automorphism $\sigma$ which makes $\theta_{\Delta,k}$ true. For, for each $j$,

$$T \models (\forall x, y, z)(\psi(x, y, z) \land P_j(z) \rightarrow [P_j(x) \leftrightarrow \neg P_j(y)]).$$

To put this situation in a more general framework, recall some notation from [3]. $\Delta$ will note a finite set of formulas: $\{\phi_i(x, y) : \text{lg}(x) = m, i < |\Delta|\}; p$ is a $\Delta$-$m$-type over $A$ if $p$ is a set of formulas $\phi_i(x, a)$ where $x = \langle x_1, \ldots, x_{m-1} \rangle$ (these specific variables) and $a$ from $A$ is substituted for $y_i$. Thus, if $A$ is finite there are only finitely many $\Delta$-$m$-types over $A$.

Now let $\Delta_1$ contain Boolean combinations of $x = y, R(x, y), E(x, y)$. Let $\Delta_2$ expand $\Delta_1$ by adding a finite number of the $P_j(z)$ and let $\Delta_3$ contain $P_j(x)$ where $P_j$ does not occur in $\Delta_2$.

Now we have the following situation: there exists a set $X = \{b_0, b_1, c_0, c_1, a\}$, $P_j(a)$ holds, all 5 are $E$-equivalent and $R(b_i, c_i)$ for $i = 0, 1$ such that:

1. $\langle b, c_i : i \leq 2 \rangle$ is $\Delta_2$-indiscernible over $a$.  

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2. \( \langle b_0c_0, b_1c_1 \rangle \) can be extended to an infinite set of indiscernibles \( \overline{bc} \) which satisfy the following.

3. \( \psi(b_i, c_i, a) \).

4. \( \sigma(\text{avg}_{\Delta_2}(\overline{b}/M)) = \text{avg}_{\Delta_2}(\overline{c}/M). \)

5. \( \text{tp}_{\Delta_1}(b_2c_2/X) \vdash \sigma(\text{tp}_{\Delta_3}(b_2/X)) \neq \text{tp}_{\Delta_3}(c_2/X). \)

We call a sequence like \( \langle b_i c_i : i \leq 2 \rangle \) a \( (\sigma, \Delta_1, \Delta_2, \Delta_3, n) \)-obstruction over the emptyset. In order to ‘finitize’ the notions we will give below more technical formulations of the last two conditions: we will have to discuss obstructions over a finite set \( A \). In the example, the identity was the only automorphism of the prime model. We will have to introduce a third sequence \( b' \) to deal with arbitrary \( \sigma \). But this example demonstrates the key aspects of obstruction which are the second reason for \( T_\sigma \) to lack a model companion.

## 2 Preliminaries

In order to express the notions described in the example, we need several notions from basic stability theory. By working with finite sets of formulas in a stable theory without the finite cover property we are able to refine arguments about infinite sets of indiscernibles to arguments about sufficiently long finite sequences. Let \( \Delta \) be a finite set of formulas which we will assume to be closed under permutations of variables and negation; \( \neg \neg \phi \) is identified with \( \phi \). Recall that an ordered sequence \( E = \langle a_i : i \in I \rangle \) is said to be \( (\Delta, p) \) indiscernible over \( A \) if any two properly ordered \( p \)-element subsequences of \( I \) realize the same \( \Delta \)-type over \( A \). It is \( \Delta \)-indiscernible if it is \( (\Delta, p) \)-indiscernible for all \( p \), or equivalently for all \( p' \) with \( p' \) at most the maximum number of variables in a formula in \( \Delta \). For any sequence \( E = \langle a_i : i \in I \rangle \) and \( j \in I \) we write \( E_j \) for \( \langle a_i : i < j \rangle \).

We will rely on the following facts/definitions from [8] to introduce two crucial functions for this paper: \( F(\Delta, n) \) and \( f(\Delta, n) \).

**Fact 2.1** Recall that if \( T \) is stable, then for every finite \( \Delta \subset L(T) \) and \( n < \omega \) there is a finite \( \Delta' = F(\Delta, n) \) with \( \Delta \subseteq \Delta' \subset L(T) \) and a \( k^* = f(\Delta, n) \) such that
1. A sequence $\langle e_i : i \in I \rangle$ of $n$-tuples such that for $i < j$ and a finite set $A$, $tp_{F(\Delta,n)}(e_i/E_iA) = tp_{F(\Delta,n)}(e_j/E_jA)$ and $R(F(\Delta,n),2)(e_j/E_jA) = R(F(\Delta,n),2)(e_i/E_iA)$, (whence, $tp_{F(\Delta,n)}(e_j/E_iA)$ is definable over $A$) is a sequence of $\Delta$-indiscernibles (II.2.17).

2. For any set of $\Delta'$-indiscernibles over the empty set, $E = \langle e_i : i < k \rangle$ with $\lg(e_i) = n$ and $k \geq k^*$ for any $\theta(u,v) \in \Delta$ and any $d$ with $\lg(d) = \lg(v) = m$ either $\{e_i : \phi(e_i, d)\}$ or $\{e_i : \neg\phi(e_i, d)\}$ has strictly less than $k^*/2$ elements. (II.4.13., II.2.20)

3. This implies that, for appropriate choice of $k^*$,

   (a) there is an integer $m = m(\Delta, n) \geq n$ such that for any set of $\Delta'$-indiscernibles $\langle e_i : i < k \rangle$ over $A$ with $\lg(e_i) = n$ and $k \geq k^*$ and any $a$ with $\lg(a) \leq m$ there is a $U \subseteq k$ with $|U| < k^*/2$ such that $\langle e_i : i \in k - U \rangle$ is $\Delta$-indiscernible over $Aa$;

   (b) moreover if $k \geq k^*$, for any set $A$, $\text{avg}_{\Delta}(\langle e_i : i < k \rangle/A)$ is well-defined. Namely, $\text{avg}_{\Delta}(\langle e_i : i < k \rangle/A) = \{\phi(x,a) : |\{e_i : i < k, \phi(e_i,a)\} \geq \frac{k^*}{10}, a \in A, \phi(x,y) \in \Delta\}$.

In a), $m$ is the least $k \geq n$ such all $\phi \in \Delta$ have at most $k$ free variables. But $\text{avg}_{\Delta}(\langle e_i : i < k \rangle/A)$ need not be consistent. (Let $E$ be all the members of one finite class in the standard fcp example and let $A = E$.) However,

**Fact 2.2** If, in addition to Fact 2.1, $T$ does not have the finite cover property, we can further demand

1. If $E = \langle e_i : i < k^* \rangle$ is a set of $\Delta'$-indiscernibles over the empty set, for any $A$, $\text{avg}_{\Delta}(E/A)$ is a consistent complete $\Delta$-type over $A$.

2. Moreover $k^*$ can be chosen so that any set of $\Delta'$-indiscernibles (of $n$-tuples) with length at least $k^*$ can be extended to one of infinite length (II.4.6).

3. For any pair of $F(F(\Delta,n))$-indiscernible sequences $E^1 = \langle e^1_i : i < k \rangle$ and $E^2 = \langle e^2_i : i < k \rangle$ over $a$ with $\lg(e_i) = n$ and $k \geq k^*$ such that

   $\text{avg}_{F(\Delta)}(E^1/aE^1E^2) = \text{avg}_{F(\Delta)}(E^2/aE^1E^2),$

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there exist \( J = \langle e_j : k < j < \omega \rangle \) such that both \( E^1 J \) and \( E^2 J \) are \( F(\Delta) \)-indiscernible over \( a \). We express the displayed condition on \( \bar{e}^1, \bar{e}^2 \) by the formula: \( \lambda_\Delta(\bar{e}^1, \bar{e}^2, a) \).

4. If \( E_1 \) and \( E_2 \) contained in a model \( M \) are \( F(\Delta, n) \)-indiscernible over \( a \in M \) and each have length at least \( k^* \), there is a formula \( \lambda_\Delta(x^1, x^2, z) \) such that \( M \models \lambda_\Delta(e^1, e^2, a) \) if and only if \( \text{avg}_\Delta(e^1/M) = \text{avg}_\Delta(e^2/M) \).

Proof. For 1, make sure that \( k^* \) is large enough that every \( \Delta \)-type which is \( k^* \)-consistent is consistent (II.4.4 3)). Now 3) follows by extending the common \( F(\Delta) \)-average of \( E^1 \) and \( E^2 \) over \( a \in M \) by 2). Finally, condition 4 holds by adapting the argument for III.1.8 from the set of all \( L \)-formulas to \( \Delta \); \( \lambda_\Delta \) is the formula from 3).

Note that both \( F \) and \( f \) can be chosen increasing in \( \Delta \) and \( n \).

3 Obstructions

In this section we introduce the main new notion of this paper: obstruction.

We are concerned with a formula \( \psi(x, y, z) \) where \( \log(x) = \log(y) = n \) and \( \log(z) = m \). We will apply Facts 2.1 and 2.2 with \( e_i = b_i b'_i c_i \) where each of \( b_i, b'_i, \) and \( c_i \) has length \( n \). Thus, our exposition will depend on functions \( F(\Delta, 3n), f(\Delta, 3n) \). In several cases, we apply Fact 2.1 with \( \phi(u_1, u_2, u_3, v) \) as \( \theta(u_2, v) \leftrightarrow \theta(u_3, v) \) for various \( \theta \).

The following notation is crucial to state the definition.

**Notation 3.1** If \( \bar{d} = \langle d_i : i < r \rangle \) is a sequence of \( 3n \)-tuples, which is \( \Delta \)-indiscernible over a finite sequence \( f \), and \( r \geq k^* = f(\Delta, 3n) \), then \( \tau_\Delta(z, \bar{d}f) \) is the formula with free variable \( z \) and parameters \( \bar{d}f \) which asserts that there is a subsequence \( \bar{d} \) of \( \bar{d} \) with length \( f(\Delta, 3n) \) so that \( \bar{d}z \) forms a sequence of \( \Delta \)-indiscernibles over \( f \).

The following result follows easily from Fact 2.2 (3) and the definition of \( f(\Delta, n) \).

**Lemma 3.2** Let \( p \geq k = f(\Delta, n) \) and suppose \( \langle e_i : i < p \rangle \) is a sequence of \( 3n \)-tuples which is \( F(\Delta, 3n) \)-indiscernible over \( f \). If \( S_1, S_2 \subseteq p \) with \( |S_1|, |S_2| \geq k^*/2 \) then \( \tau_\Delta(z, \bar{d}|S_1f) \) and \( \tau_\Delta(z, \bar{d}|S_2f) \) are equivalent formulas.
Now we come to the main notion. Intuitively, \( \langle b_i, b'_i, c_i : i \leq k \rangle a \) is a \((\Delta_1, \Delta_2, \Delta_3, n)\) obstruction over \( A \) if \( \langle b_i, b'_i, c_i : i \leq k \rangle a \) is an indefinitely extendible sequence of \( \Delta_2 \) indiscernibles over \( a \) such that the \( b_i \)'s, \( b'_i \)'s and \( c_i \)'s each have length \( n \) and the \( \Delta_2 \)-average of the \( b'_i \) and \( c_i \) is the same (over any set) but any realizations of the \( \Delta_1 \)-type of the \( b'_i \) and the \( \Delta_1 \)-type of the \( c_i \) over \( a \) and the sequence have different \( \Delta_3 \)-types over \( A \). More formally, we define:

**Definition 3.3** For finite \( \Delta_1 \subseteq \Delta_2 \subseteq L(T) \) and \( \Delta_3 \subseteq L(T) \), finite \( a \subseteq A \subseteq M \models T \) with \( \lg(a) \leq m(\Delta, n) \) (as in Fact 2.1), \( \sigma \) an automorphism of \( M \), and a natural number \( n \), \( \langle b_i \sigma(b_i)c_i : i \leq k \rangle a \) is a \((\sigma, \Delta_1, \Delta_2, \Delta_3, n)\) obstruction over \( A \) if the following conditions hold.

1. \( \langle b_i \sigma(b_i)c_i : i \leq k \rangle \) is \( F(\Delta_2, 3n) \)-indiscernible over \( a \).
2. \( k \geq f(\Delta_2, 3n) \).
3. \( \text{avg}_{\Delta_2}(\overline{e}^1/M) = \text{avg}_{\Delta_2}(\overline{e}^2/M) \) where \( \overline{e}^1 = \langle \sigma(b_i) : i < k \rangle \) and \( \overline{e}^2 = \langle c_i : i < k \rangle \).
4. Writing \( \tau_{\Delta_1}(\langle b_i \sigma(b_i)c_i : i < k \rangle a) \) with free variables \( x, x', y, \) we have

\[
M \models (\forall x, x', y)[\tau_{\Delta_1}(x, x', y, \langle b_i \sigma(b_i)c_i : i < k \rangle a) \rightarrow \bigvee \{\phi(x', f) \land \neg\phi(y, f) : f \in A \cup \bigcup_{i<k} \langle b_i \sigma(b_i)c_i \rangle \cup a, \phi \in \Delta_3\}]
\]

By Fact 2.2, Condition 3) is expressed by a formula of \( e^1, e^2 \) and \( a \). Crucially, the hypothesis of the fourth condition in Definition 3.3 is an \( L \)-formula with parameters \( \langle b_i \sigma(b_i)c_i : i < k \rangle a \); the conclusion is an \( L \)-formula with parameters from \( A \) as well. The disjunction in the conclusion of condition 4) is nonempty since each \( b_i, \sigma(b_i) \) is in the domain if \( \Delta_3 \) is nontrivial. \( \Delta_1 \) and \( \Delta_2 \) have \( 3n \) type-variables; \( \Delta_3 \) has \( n \) type-variables.

**Fact 3.4** Note that if \( \langle b_i, c_i : i \leq k \rangle a \) is a \((\sigma, \Delta_1, \Delta_2, \Delta_3, n)\) obstruction over \( A \) and \( \Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2 \), then \( \langle b_i, c_i : i \leq k \rangle a \) is a \((\sigma, \Delta_1, \Delta'_2, \Delta_3, n)\) obstruction over \( A \). Further, if \( \langle b_i, c_i : i \leq k \rangle a \) is a \((\sigma, \Delta_1, \Delta_2, \Delta_3, n)\) obstruction over \( A \) and \( A \subseteq A' \), where \( A' \) is finite, \( \Delta_1 \subseteq \Delta'_2 \subseteq \Delta_2 \), then \( \langle b_i, c_i : i \leq k \rangle a \) is a \((\sigma, \Delta_1, \Delta'_2, \Delta_3, n)\) obstruction over \( A' \).
Definition 3.5 1. We say \((M, \sigma) \models T\) has no \(\sigma\)-obstructions when there is a function \(G(\Delta_1, n)\) with \(F(\Delta_1, 3n) \subseteq G(\Delta_1, n)\) such that if \(\Delta_1\) is a finite subset of \(L(T)\) and \(G(\Delta_1, n)\) is contained in the finite \(\Delta_3 \subseteq L(T)\), then for every finite subset \(A\) of \(M\), there is no \((\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)\) obstruction over \(A\).

2. We say \(T\) has no \(\sigma\)-obstructions when there is a function \(G(\Delta_1, n)\) (which does not depend on \((M, \sigma) \models T\)) such that for each \((M, \sigma) \models T\), if \(\Delta_1\) is a finite subset of \(L(T)\), \(A\) is finite subset of \(M\), and \(\Delta_3\) is a finite subset of \(L(T)\), there is no \((\sigma, \Delta_1, G(\Delta_1, n), \Delta_3, n)\) obstruction over \(A\).

Definition 3.6 A simple obstruction is an obstruction where the automorphism \(\sigma\) is the identity. The notions of a theory or model having a simple obstruction are the obvious modifications of the previous definition.

Lemma 3.7 \(T\) has obstructions if and only if \(T\) has simple obstructions.

Proof. Suppose \(T\) has obstructions; we must find simple obstructions. So, suppose for some \(\Delta_1\), and \(n\), and for every finite \(\Delta_2 \supseteq F(\Delta_1, 3n)\), there is a finite \(\Delta_3\) and a tuple \((\Delta_2^M, \sigma^A, k^A)\) such that: \((\Delta_2^M, \sigma^A) \models T\), \(\Delta_2^M\) is a finite subset of \(M\) and \(b^\Delta, \sigma(b^\Delta), c^\Delta, a^\Delta\) contained in \(M\) are a \((\sigma^A, \Delta_1, \Delta_2, \Delta_3, n)\) obstruction of length \(k^A\) over \(\Delta_2^M\). Without loss of generality \(\log(a) = m = m(\Delta_1, 3n)\) and we can write \(\Delta_3 = \Delta_3(\Delta_2)\). Now, define a family of simple obstruction by replacing each component of the given sequence of obstructions by an appropriate object with left prefix sim.

\[
\begin{align*}
sim A^\Delta_2 &= A^\Delta_2 \cup \{b^\Delta, \sigma(b^\Delta), c^\Delta\} \\
\sim(b^\Delta) &= \sigma(b^\Delta) \\
\sim c^\Delta &= c^\Delta.
\end{align*}
\]

We use the same sequence of formulas for the \(\Delta_2\) and \(\Delta_3(\Delta_2)\). It is routine to check that we now have an obstruction with respect to the identity.

Lemma 3.8 If \(T\) is a stable theory with the finite cover property then \(T\) has obstructions.
Proof. By II.4.1.14 of [8], there is a formula $E(x, y, z)$ such that for each $d$, $E(x, y, z)$ is an equivalence relation and for each $n$ there is a $d_n$ such that $E(x, y, d_n)$ has finitely many classes but more than $n$. Let $\Delta_1$ be \{ $E(x, y, z), \neg E(x, y, z)$ \} and consider any $\Delta_2$. Fix $\lg(x) = \lg(y) = r$. There are arbitrarily large sequences $b_n = \langle b^n_j : j < n \rangle$, $c_n = \langle c^n_j : j < n \rangle$ such that for some $d_{n_1}, d_{n_2}$, $b_n$ is a set of representatives for distinct classes of $E(x, y, d_{n_1})$ while $c_n$ is a set of representatives of distinct classes for $E(x, y, d_{n_2})$ and $n_1 < n_2$. So by compactness and Ramsey, for any $\Delta_2$ we can find such $b_k, c_k$ where $k = f(\Delta_2, 2r)$ and $b_k, c_k$ are a sequence of length $k$ of $\Delta_2$ indiscernibles. Now, if $\Delta_3$ contains formulas which express that the number of equivalence classes of $E(x, y, z)$ is greater than $n_1$ and $A$ contains $d_{n_1}, d_{n_2}$, we have an $(\text{identity}, \Delta_1, \Delta_2, \Delta_3, r)$ obstruction over $A$. (Let $b = b' = b_k$, $c = c_k$.)

4 Model Companions of $T_\sigma$

In this section we establish necessary and sufficient conditions on stable $T$ for $T_\sigma$ to have a model companion. First, we notice when the model companion, if it exists, is complete.

Note that $\text{acl}(\emptyset) = \text{dcl}(\emptyset)$ in $C_{eq}^T$ means every finite equivalence relation $E(x, y)$ of $T$ is defined by a finite conjunction: $\bigwedge_{i<n} \phi_i(x) \leftrightarrow \phi_i(y)$.

Fact 4.1 1. If $T$ is stable, $T_\sigma$ has the amalgamation property.

2. If, in addition, $\text{acl}(\emptyset) = \text{dcl}(\emptyset)$ in $C_{eq}^T$ then $T_\sigma$ has the joint embedding property.

Proof. The first part of this Lemma was proved by (Theorem 3.3 of [3]) using the definability of types. For the second part, the hypothesis implies that types over the empty set are stationary and the result follows by similar arguments.

Lemma 4.2 Suppose $T$ is stable and $T_\sigma$ has a model companion $T^*_\sigma$.

1. Then $T^*_\sigma$ is complete if and only if $\text{acl}(\emptyset) = \text{dcl}(\emptyset)$ in $C_{eq}^T$.

2. If $(M, \sigma) \models T_\sigma$ then the union of the complete diagram of $M$ (in $L$) with the diagram of $(M, \sigma)$ and $T^*_\sigma$ is complete.
Proof. 1) We have just seen that if acl(∅) = dcl(∅) in $C^eq_T$, then $T_σ$ has the joint embedding property; this implies in general that the model companion is complete. If acl(∅) \neq dcl(∅) in $C^eq_T$, let $E(x, y)$ be a finite equivalence relation witnessing acl(∅) \neq dcl(∅). Because $E$ is a finite equivalence relation,

$$T_1 = T_σ \cup \{(∀x)E(x, σ(x))\}$$

is a consistent extension of $T_σ$. But since

$$T_σ \cup \{¬E(x, y)\} \cup \{ϕ(x) ↔ ϕ(y) : ϕ ∈ L(T)\}$$

is consistent so is

$$T_2 = T_σ \cup \{(∃x)¬E(x, σ(x))\}.$$  

But $T_1$ and $T_2$ are contradictory, so $T^*_σ$ is not complete.  

2) Since we have joint embedding (from amalgamation over any model) the result follows as in Fact [4.1].

We now prove the equivalence of three conditions: the first is a condition on a pair of models. The second is given by an infinite set of $L_σ$ sentences (take the union over all finite $Δ_2$) and the average requires names for all elements of $M$. The third is expressed by a single first order sentence in $L_σ$. The equivalence of the first and third suffices (Theorem [4.7]) to show the existence of a model companion. In fact 1 implies 2 implies 3 requires only stability; the nfcp is used to prove 3 implies 1.

**Lemma 4.3** Suppose $T$ is stable without the fcp. Let $(M, σ) \models T_σ$, $a \in M$ and suppose that $(M, σ)$ has no $σ$-obstructions. Fix $ψ(x, y, z)$ with $lg(x) = lg(y) = n$ and $lg(z) = lg(a) = m$.

The following three assertions are equivalent:

1. There exists $(N, σ)$, $(M, σ) ⊆ (N, σ) \models T_σ$ and $N \models (∃xy)[ψ(x, y, a) ∧ σ(x) = y]$.

2. Fix $Δ_1 = \{ψ(x, y, z)\}$ and without loss of generality $lg(z) ≤ m(Δ_1, n)$. For $k ≥ 5 \cdot f(Δ_1, 3n)$ any finite $Δ_2 \supseteq F(Δ_1, 3n)$ (Fact [2.4]), there are $b_i, σ(b_i)c_i ∈ ^{3n}M$ for $i < k$ such that

   (a) $\langle b_iσ(b_i)c_i : i < k \rangle$ is $F(Δ_2, 3n)$-indiscernible over $a$,

   (b) for each $i < k$, $ψ(b_i, c_i, a)$ holds,
(c) For every $d \in {}^m M$ and $\phi(u, v) \in \Delta_2$ we have
\[ |\{i < k : \phi(\sigma(b_i), d) \leftrightarrow \phi(c_i, d)\}| \geq f(\Delta_2, 3n)/2.\]

3. Let $\Delta_2 = G(\Delta_1, n)$. Then there are $b_i, \sigma(b_i)c_i \in {}^{3n} M$ for $i < k = 5 \cdot f(\Delta_2, 3n)$ such that :

(a) $\langle b_i, \sigma(b_i)c_i : i < k \rangle$ is $F(G(\Delta_1, n), 3n)$-indiscernible over $a$,
(b) for each $i < k$, $\psi(b_i, c_i, a)$,
(c) $\lambda_{\Delta_2}(\langle \sigma(b_i) : i < k \rangle, (c_i : i < k), a)$.

Proof. First we show 1) implies 2). Fix $b, c \in N$ with $N \models \psi(b, c, a) \land \sigma(b) = c$. For $\Delta_2$, let $\Delta_2^+ = F(F(\Delta_2, 3n), 3n)$. For each $\Delta_2$, choose a finite $p \subseteq tp_{L(T)}(b, c/M)$ with the same $(\Delta_2^+, 2)$ rank as $tp_{\Delta_2}(b, c/M)$ (so $tp_{\Delta_2}(b, c/M)$ is definable over dom $p$). Now inductively construct an $F(\Delta_2, 3n)$-indiscernible sequence (by Fact 2.1 1)) $\langle b_i, c_i : i < \omega \rangle$ by choosing $b_i, c_i$ in $M$ realizing the restriction of $tp_{\Delta_2^+}(b, c/M)$ to dom $p$ along with the points already chosen. Let $b'_i = \sigma(b_i)$. By Ramsey’s Theorem for some infinite $U \subseteq \omega$, $\langle b_i, b'_i, c_i : i \in U \rangle$ is $F(\Delta_2, 3n)$-indiscernible over $a$; renumbering let $U = \omega$. Now conditions a) and b) of assertion 2) are clear. For clause c),
\[
\text{avg}_{\Delta_2}(\langle c_i : i < \omega \rangle/M) = tp_{\Delta_2}(c, M)
\]
\[
= \sigma(tp_{\Delta_2}(b, M)) = \sigma(\text{avg}_{\Delta_2}(\langle b_i : i < \omega \rangle/M))
\]
since $\sigma(b) = c$. So, for each $\phi \in \Delta_2$ and each $d \in M$ of appropriate length,
\[
\phi(x, d) \in \text{avg}_{\Delta_2}(\langle c_i : i < \omega \rangle/M)
\]
if and only if
\[
\phi(x, \sigma^{-1}(d)) \in \text{avg}_{\Delta_2}(\langle b_i : i < \omega \rangle/M).
\]
So for some $S_1, S_2 \subset \omega$ with $|S_1|, |S_2| < f(\Delta_2, 3n)/2$, we have for all $i \in \omega - (S_1 \cup S_2)$, $\phi(c_i, d)$ if and only if $\phi(b_i, \sigma^{-1}(d))$. Since $\sigma$ is an automorphism of $M$ this implies for $i \in \omega - (S_1 \cup S_2)$, $\phi(c_i, d)$ if and only if $\phi(\sigma(b_i), d)$ which gives condition c) by using the first $k$ elements of $\langle b_i, \sigma(b_i)c_i : i < \omega - S_1 \cup S_2 \rangle$.

3) is a special case of 2). To see this, note that 3c) is easily implied by the form analogous to 2c): For every $m \leq m(\Delta_1, n)$ and $d \in {}^m M$ and $\phi(u, v) \in G(\Delta_1, n)$ we have
\[ |\{i < k : \phi(\sigma(b_i), d) \leftrightarrow \phi(c_i, d)\}| \geq f(\Delta_2, 3n)/2.\]
If \( T \) does not have f.c.p., the converse holds and we use that fact implicitly in the following argument. It remains only to show that 3) implies 1) with \( \Delta_1 = \{ \psi \} \) and \( \Delta_2 = G(\Delta_1, n) \). Without loss of generality we may assume \( N \) is \( \aleph_1 \)-saturated. We claim the type

\[
\Gamma = \{ \psi(x, y, a) \} \cup \{ \phi(x, d) \leftrightarrow \phi(y, \sigma(d)) : d \in M, \phi \in L(T) \} \cup \text{diag}(M)
\]

is consistent. This clearly suffices.

Let \( k = f(\Delta_2, 3n) \). Suppose \( \langle b_i \sigma(b_i) c_i : i < k \rangle a \) satisfy 3). Let \( \Gamma_0 \) be a finite subset of \( \Gamma \) and suppose only formulas from the finite set \( \Delta_3 \) and only parameters from the finite set \( A \) appear in \( \Gamma_0 \). Write \( b_i' \) for \( \sigma(b_i) \).

Now \( \langle b, b' c_i : i \leq f(\Delta_2, 3n) \rangle a \) easily satisfy the first two conditions of Definition 3.3 for being a \( (\Delta_1, \Delta_2, \Delta_3, n) \)-obstruction over \( A \) and, in view of Fact 2.2 (3), 4), the third is given by condition 3c). Since there is no obstruction the 4th condition must fail. So there exist \( b^*, (b^*)', c^* \) so that

\[
M \models \tau_{\Delta_1}(b^*, (b^*)', c^*, \langle b_i, b_i', c_i : i < k \rangle a).
\]

and \( tp_{\Delta_1}(b^*/A) = tp_{\Delta_1}(c^*/A) \) so \( \Gamma_0 \) is satisfiable.

As we’ll note in Theorem 4.7, we have established a sufficient condition for \( T_\sigma \) to have a model companion. The next argument shows it is also necessary.

**Lemma 4.4** Suppose \( T \) is stable; if \( T \) has an obstruction then \( T_\sigma \) does not have a model companion.

More precisely, suppose for some \( \Delta_1 \) and \( n \), and for every finite \( \Delta_2 \supseteq F(\Delta_1, 3n) \), there is a finite \( \Delta_3 \) and a tuple \( (M^{\Delta_2}, \sigma^{\Delta_2}, A^{\Delta_2}, k^{\Delta_2}) \) such that: \( (M^{\Delta_2}, \sigma^{\Delta_2}) \models T_\sigma \), \( A^{\Delta_2} \) is a finite subset of \( M^{\Delta_2} \), \( b^{\Delta_2}, \sigma(b^{\Delta_2}), c^{\Delta_2}, a^{\Delta_2} \) contained in \( M^{\Delta_2} \) are a \( (\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n) \) obstruction of length \( k^{\Delta_2} \) over \( A^{\Delta_2} \). Without loss of generality \( \lg(a) = m = m(\Delta_1, 3n) \) and we can write \( \Delta_3 = \Delta_3(\Delta_2) \).

Then the collection \( K_\sigma \) of existentially closed models of \( T_\sigma \) is not an elementary class.

Proof. We may assume \( T \) does not have f.c.p., since if it does we know by Winkler and Kudaibergerov that \( T_\sigma \) does not have a model companion. By the usual coding we may assume \( \Delta_1 = \{ \psi(x, y, z) \} \) with \( \lg(x) = \lg(y) = n, \lg(z) = m, k = f(\Delta_1, 3n) \). Without loss of generality each \( M^{\Delta_2} \) is existentially closed. Let \( D \) be a nonprincipal ultrafilter on \( Y = \{ \Delta_2 : F(\Delta_1, 3n) \subseteq \)
Claim 4.5

1. \( \lg(b_i) = \lg(\sigma(b_i)) = \lg(c) = n \); \( \lg(a) = m \) and \( \sigma^*(a_i) = b_i' \).

2. \( (b_i, b'_i, c_i : i \in I) \) is a sequence of \( L(T) \)-indiscernibles over \( a^* \).

3. For each finite \( \Delta_2 \subseteq L(T) \) with \( \Delta_2 \subseteq F(\Delta_1, 3n) \) and each finite subsequence from \( (b_i, b'_i, c_i : i \in I) \) indexed by \( J \) of length at least \( k = f(\Delta_2, 3n) \) the \( \Delta_2 \)-type of \( (b_i, b'_i, c_i : i \in J) \) \( a \) is the \( \Delta_2 \)-type of some \( (\sigma^{\Delta_2}, \Delta_1, \Delta_2, \Delta_3, n) \)-obstruction \( (b_{i_1}^{\Delta_2}, \sigma(b_i^{\Delta_2})c_i^{\Delta_2})a^{\Delta_2} \) in \( M^{\Delta_2} \) over the empty set.

4. \( \text{avg}_{L}(\lg(b_i : i \in I)/N) = \text{avg}_{L}(\lg(c : i \in I)/N) \).

This claim follows directly from the properties of ultraproducts. (For item 3, apply Fact 3.4 and the definition of the ultrafilter \( D \).)

Let \( \Gamma \) be the \( L \)-type in the variables \( (x, x'_i, y_i : i \in I) \cup \{z\} \) over the empty set of \( (b_i, b'_i, c_i : i \in I) a^* \). For any finite \( \Delta \subseteq L(T) \), let \( \chi_{\Delta,k}(x, x', y, z) \) be the \( \Delta \)-type over the empty set of a subsequence of \( k \) elements from \( (b_i, b'_i, c_i : i \in I) \) and \( a^* \) from a realization of \( \Gamma \) with \( z \) for \( a^* \).

Recall the definition of \( \tau_{\Delta_1} \) from Notation 3.1. Let \( r = f(\Delta_1, n) \) and let \( \theta_{\Delta_1}(x_0, \ldots, x_{r-1}, x'_0, \ldots, x'_{r-1}, y_0, \ldots, y_{r-1}, z) \) be the formula:

\[
(\exists x, x', y)[\chi_{\Delta_1,r}(x_0, \ldots, x_{r-1}, x'_0, \ldots, x'_{r-1}, y_0, \ldots, y_{r-1}, z) \\
\land \tau_{\Delta_1}(x, x', y, x_0, \ldots, x_{r-1}, x'_0, \ldots, x'_{r-1}, y_0, \ldots, y_{r-1}, z) \land \sigma(x') = y].
\]

Without loss of generality we assume \( 0, 1, \ldots, r-1 \) index disjoint sequences.

Now we claim:

Claim 4.6 If \( K_\sigma^* \), the family of existentially closed models of \( T_\sigma^* \), is axiomatized by \( T_\sigma^* \),

\[
T_\sigma^* \cup \Gamma \cup \{\sigma(x_i) = x'_i : i \in I\} \models \theta_{\Delta_1}(x_0, \ldots, x_{r-1}, x'_0, \ldots, x'_{r-1}, y_0, \ldots, y_{r-1}, z).
\]
and so that:

By the definition of an obstruction, \( \sigma \) holds since for each \( M \) holds. Thus, \( M' \) is an \(|M'|^+\)-saturated model of \( T \). In \( M' \) we can find \( \mathbf{b}', \mathbf{c} \) realizing the average of \( \langle \mathbf{b}, \mathbf{b}', \mathbf{c} : i \in I \rangle \) over \( M' \). Then

\[
\sigma'(\text{tp}(\mathbf{b}/M')) = \sigma'(\text{avg}(\langle \mathbf{b}_i : i \in I \rangle/M')) = \text{avg}(\langle \mathbf{b}'_i : i \in I \rangle/M') = \text{tp}(\mathbf{c}/M')
\]

(The first and last equalities are by the choice of \( \mathbf{b}, \mathbf{b}', \mathbf{c} \); the second holds since for each \( i \), \( \sigma'((\mathbf{c})_i) = (\mathbf{b}')_i \), the third follows from clause 4 in the description of the ultraproduct.) Now since \( M'' \) is \(|M'|^+\)-saturated there is an automorphism \( \sigma'' \) of \( N \) extending \( \sigma' \) and taking \( \mathbf{b} \) to \( \mathbf{c} \).

As \( (M', \sigma') \models T^*_\sigma \), it is existentially closed. So we can pull \( \mathbf{b}, \mathbf{b}' \mathbf{c} \) down to \( M' \). Thus, \( (M', \sigma') \models \theta_{\Delta_1}(\mathbf{b}_0, \ldots, \mathbf{b}_{r-1}, \mathbf{b}'_0, \ldots, \mathbf{b}'_{r-1}, \mathbf{c}_0, \ldots, \mathbf{c}_{r-1}, \mathbf{a}) \). But \( (M', \sigma') \) was an arbitrary model of \( T^*_\sigma \cup \Gamma \cup \{ \sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I \} \); so

\[ T^*_\sigma \cup \Gamma \cup \{ \sigma(\mathbf{x}_i) = \mathbf{x}'_i : i \in I \} \models \theta_{\Delta_1}(\mathbf{x}_0, \ldots, \mathbf{x}_{r-1}, \mathbf{x}'_0, \ldots, \mathbf{x}'_{r-1}, \mathbf{y}_0, \ldots, \mathbf{y}_{r-1}, \mathbf{z}). \]

By compactness, some finite subset \( \Gamma_0 \) of \( \Gamma \) and a finite number of the specifications of \( \sigma \) suffice; let \( \Delta^*_0 \) be the formulas mentioned in \( \Gamma_0 \) along with those in \( F(\Delta_1, 3n) \) and \( k \) the number of \( x_i, y_i \) appearing in \( \Gamma_0 \) and let \( \Delta_2 = F(\Delta^*_0, n) \). Without loss of generality, \( k \geq f(\Delta_1, 3n) \). Then, \( T^*_\sigma \models \)

\[ (\forall \mathbf{x}_0 \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0 \ldots \mathbf{y}_{k-1})[(\chi_{\Delta_2,k}(\mathbf{x}_0 \ldots \mathbf{x}_{k-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{k-1}, \mathbf{y}_0 \ldots \mathbf{y}_{k-1}, \mathbf{z}) \wedge \bigwedge_{i<k} \sigma(\mathbf{x}_i) = \mathbf{x}'_i \rightarrow \theta_{\Delta_1}(\mathbf{x}_0 \ldots \mathbf{x}_{r-1}, \mathbf{x}'_0 \ldots \mathbf{x}'_{r-1}, \mathbf{y}_0 \ldots \mathbf{y}_{r-1}, \mathbf{z})]. \]

By item 3) in Claim \[ \ref{claim:obstruction} \] fix a \( \Delta_2 \) and \( \Delta_3 = \Delta_3(\Delta_2, m) \) containing \( \Delta_2 \) and \( \langle \mathbf{b}_i^{\Delta_2}, \mathbf{b}_i^{\Delta_2}, \mathbf{c}_i^{\Delta_2} : i < k \rangle \mathbf{a}^{\Delta_2} \) which form a \( (\sigma, \Delta_1, \Delta_2, \Delta_3, n) \)-obstruction over \( \mathbf{a}^{\Delta_2} \) and so that:

\[ M^{\Delta_2} \models \chi_{\Delta_2,k}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b}_i^{\Delta_2}, \mathbf{c}_i^{\Delta_2} : i < k \rangle \mathbf{a}^{\Delta_2}). \]

By the definition of an obstruction, \( \sigma(\mathbf{b}_i^{\Delta_2}) = \mathbf{b}_i^{\Delta_2} \). So by the choice of \( \Gamma_0 \),

\[ (M^{\Delta_2}, \sigma^{\Delta_2}) \models \theta_{\Delta_1}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b}_i^{\Delta_2}, \mathbf{c}_i^{\Delta_2} : i < r \rangle \mathbf{a}^{\Delta_2}). \]

Now, let \( \mathbf{b}, \mathbf{b}' \mathbf{c} \in M^{\Delta_2} \) with \( \sigma^{\Delta_2}(\mathbf{b}') = \mathbf{c} \) witness this sentence. Then

\[ (M^{\Delta_2}, \sigma^{\Delta_2}) \models \tau_{\Delta_1}(\langle \mathbf{b}_i^{\Delta_2} \mathbf{b}_i^{\Delta_2}, \mathbf{c}_i^{\Delta_2} : i < k \rangle \mathbf{a}^{\Delta_2}) \]

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By the definition of obstruction,
\[
\sigma^2(\text{tp}_{\Delta_3}(b'/\Delta_2 \cup \bigcup_{i<k} (b_i^2 \sigma(b_i^2) c_i) \cup a^2)) \neq \text{tp}_{\Delta_3}(c/A^2 \cup \bigcup_{i<k} (b_i^2 \sigma(b_i^2) c_i^2) \cup a^2).
\]

This contradicts that \(\sigma^2\) is an automorphism and we finish.

Finally we have the main result.

**Theorem 4.7** If \(T\) is a stable theory, \(T_\sigma\) has a model companion if and only if \(T_\sigma\) admits no \(\sigma\)-obstructions.

Proof. We showed in Lemma 4.4 that if \(T_\sigma\) has a model companion then there is no obstruction. If there is no obstruction, Lemma 3.8 implies \(T\) does not have the finite cover property. By Lemma 4.3 for every formula \(\psi(x,y,z)\) there is an \(L\)-formula \(\theta_\psi(z)\) (write out condition 3 of Lemma 4.3) which for any \((M,\sigma)\) holds of any \(a\) in \(M\) if and only if there exists \((N,\sigma), (M,\sigma) \subseteq (N,\sigma) \models T_\sigma\) and

\[
(N,\sigma) \models (\exists xy)\psi(x,y,a) \land \sigma(x) = y.
\]

Thus, the class of existentially closed models of \(T_\sigma\) is axiomatized by the sentences: \((\forall z)\theta_\psi(z) \rightarrow (\exists xy)\psi(x,y,a) \land \sigma(x) = y\). (We can restrict to formulas of the form \(\psi(x,\sigma(x),a)\) by the standard trick ([4, 1]).

Kikyo and Pillay [4] note that if a strongly minimal theory has the definable multiplicity property then \(T_\sigma\) has a model companion. In view of Theorem 4.7, this implies that if \(T\) has the definable multiplicity property, then \(T_\sigma\) admits no \(\sigma\)-obstructions. Kikyo and Pillay conjecture that for a strongly minimal set, the converse holds: if \(T_\sigma\) has a model companion then \(T\) has the definable multiplicity property. They prove this result if \(T\) is a finite cover of a theory with the finite multiplicity property. It would follow in general from a positive answer to the following question.

**Question 4.8** If the strongly minimal theory \(T\) with finite rank does not have the definable multiplicity property, must it omit obstructions?

Pillay has given a direct proof that if a strongly minimal \(T\) has the definable multiplicity property, then \(T_\sigma\) admits no \(\sigma\)-obstructions. Pillay has provided an insightful reworking of the ideas here in a note which is available on his website [7].

Here is a final question:

**Question 4.9** Can \(T_\sigma\) for an \(\aleph_0\)-categorical \(T\) admit obstructions?
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