Conformal transformations and the SLE partition function martingale

Michel Bauer\textsuperscript{1} and Denis Bernard\textsuperscript{2}

Service de Physique Théorique de Saclay
CEA/DSM/SPhT, Unité de recherche associée au CNRS
CEA-Saclay, 91191 Gif-sur-Yvette, France

Abstract

We present an implementation in conformal field theory (CFT) of local finite conformal transformations fixing a point. We give explicit constructions when the fixed point is either the origin or the point at infinity. Both cases involve the exponentiation of a Borel subalgebra of the Virasoro algebra. We use this to build coherent state representations and to derive a close analog of Wick’s theorem for the Virasoro algebra. This allows to compute the conformal partition function in non trivial geometries obtained by removal of hulls from the upper half plane. This is then applied to stochastic Loewner evolutions (SLE). We give a rigorous derivation of the equations, obtained previously by the authors, that connect the stochastic Loewner equation to the representation theory of the Virasoro algebra. We give a new proof that this construction enumerates all polynomial SLE martingales. When one of the hulls removed from the upper half plane is the SLE hull, we show that the partition function is a famous local martingale known to probabilists, thereby unravelling its CFT origin.

1 Introduction

Since its very origins, the statistical mechanics of two dimensionnal critical systems has seen a deep interplay between physics and mathematics. This

\textsuperscript{1}Email: bauer@spht.saclay.cea.fr
\textsuperscript{2}Member of the CNRS; email: dbernard@spht.saclay.cea.fr
was already true for Onsager’s solution of the 2d Ising model and the computation of the magnetization by Yang [15]. In the 80’s, the link between physics and mathematics was mainly through representation theory, affine Lie algebras and the Virasoro algebra playing the most central roles. Two dimensional conformal field theories [4] have led to an enormous amount of exact results, including the computation of multipoint correlators and partial classifications. The study of multifractal properties of conformally invariant critical clusters has been less systematic, but nevertheless produced a number of remarkable successes (see eg. refs. [14, 6, 9] and references therein), the famous Cardy formula giving the probability for the existence of a connected cluster percolating between two opposite sides of a rectangle in two dimensional critical percolation [5] being one of the highlights.

More recently, probability theory, stochastic processes to be precise, have started to play an important role, due to a beautiful connection between Brownian motion and critical clusters discovered by Schramm [17]. This connection is via the Loewner evolution equation, which describes locally growing domains $K_t$ (called hulls) in the upper half plane implicitly by prescribing the variation of the normalized uniformizing map for the complement. In this way, the growth of the hull is coded in a real continuous function. Taking this function to be a Brownian sample path leads to stochastic (chordal) Loewner evolutions (SLE) of growing hulls whose properties are those expected for conformally invariant critical clusters. There is a single parameter, denoted $\kappa$, which is the time scale for the Brownian motion. This has led to important probabilistic theorems, among which Brownian intersection exponents [13]. Moreover, this framework made it possible to prove in certain cases that lattice statistical models have a conformally invariant critical behavior. For instance, Cardy’s formula is now a theorem [18].

The link between SLE and standard conformal field theory (CFT) was obscure for several years, but recently we proposed a direct connection [1]. The idea is to couple CFT to SLE via boundary conditions, namely to look at a CFT in the random geometry of the complement of the hull in the upper half plane. The crucial observation is that if one inserts at the origin (where the hull starts to grow) a primary boundary operator (leading to a boundary state $|\omega\rangle$) of appropriate weight in a CFT of appropriate central charge, and then lets the hull grow, the corresponding conformal state is a local martingale in the sense of probability theory, i.e a quantity whose probabilistic average is time independant$^1$. In this way, many quantities

---

$^1$Under certain boundedness conditions: technically, nice linear forms applied to this state are time independant in mean.
computed by probabilistic methods can be shown to be directly related to correlation functions of CFT \[2\].

The purpose of this paper is twofold.

The first is SLE independent. We give a rigorous construction of the CFT operator implementing finite local conformal transformations fixing a point. This amounts to show how to go from certain subalgebras of the Virasoro algebra to a corresponding Lie group via exponentiation.

As a first application, we use coordinates on these groups to build coherent state highest weight representations of the Virasoro algebra. We observe a striking similarity with the representations of the Virasoro algebra that appear in matrix models \[8\]. This is a pedestrian implementation of the geometric ideas à la Borel-Weil presented in \[3\].

Under some global conditions, one can multiply operators corresponding to local conformal transformations fixing different points, leading to an embryonic version of the Virasoro group (which is ill defined in the CFT context: the central extension of the group of diffeomorphisms of the circle is not what is needed). As a byproduct, we give a theorem which does for the Virasoro algebra what Wick’s theorem does for oscillator algebras. This kind of computation could have been made right at the beginning of CFT, in the 80’s. It seems that certain analogous formulæ were derived at that time \[20\], but we have not been able to trace those back in the published literature.

These purely algebraic considerations have applications to SLE. The uniformization of the growing hull \(K_t\) is given, close to the point at infinity, by a suitably normalized local conformal transformation \(k_t\). This leads immediately to a clean definition of the conformal state \(G_{k_t}|\omega\rangle\) describing the growing hull \(K_t\). The invertible operator \(G_{k_t}\) is then shown to satisfy a stochastic differential equation\(^2\) which implies that \(G_{k_t}|\omega\rangle\) is a local martingale.

We give a brief account of the proof, using the above mentioned coherent state representations of the Virasoro algebra, that \(G_{k_t}|\omega\rangle\) is the generating function of all SLE martingales in a precise algebraic sense and that these martingales build a certain highest weight representation of the Virasoro algebra with a non trivial character. This is an elaboration of \[3\].

Finally, we turn to the partition function martingale. If a CFT is coupled via boundary conditions not only to the growing hull \(K_t\) but also to a fixed (deterministic) hull \(A\) disjoint from \(K_t\), the CFT partition function contains a universal contribution corresponding to some kind of interaction between \(A\)

\(^2\)In our previous papers, this equation was used as a heuristic definition of \(G_{k_t}\). We had to leave aside analytical questions of existence of solutions, relying on physical intuition.
and $K_t$. This is by definition a local martingale. We use Wick’s theorem for the Virasoro algebra to give yet another illustration that the SLE quantities computed by probabilists [13] are in fact deeply rooted in CFT. For $\kappa = 8/3$, this martingale computes the probability that $K_t$ never touches $A$.

The previous paragraph is definitely not a claim that mathematicians have rediscovered things that were known to theoretical physicists. Quite the opposite is true: the discoveries of probabilists have motivated us to go back to the foundations of CFT to realize that maybe certain basic construction had not been given enough attention and that some CFT jewels had been left dormant.

**Acknowledgements:** We take this opportunity to warmly thank Wendelin Werner for many illuminating explanations on the probabilistic and geometric intuition motivating SLE constructions and Misha Gromov for his questions on finite conformal transformations in conformal field theory.

Work supported in part by EC contract number HPRN-CT-2002-00325 of the EUCLID research training network.

## 2 (Chordal) stochastic Loewner evolution

The aim of this section is to recall basic properties of stochastic Loewner evolutions (SLE) and its generalizations that we shall need in the following. Most results that we recall can be found in [16, 12, 13]. See [7] for a nice introduction to SLE for physicists and [19] for pedagogical summer school notes.

A hull in the upper half plane $\mathbb{H} = \{ z \in \mathbb{C}, \Im z > 0 \}$ is a bounded simply connected subset $K \subset \mathbb{H}$ (for the usual topology of $\mathbb{C}$) such that $\mathbb{H} \setminus K$ is open, connected and simply connected. The local growth of a family of hulls $K_t$ parametrized by $t \in [0, T]$ with $K_0 = \emptyset$ is related to complex analysis in the following way. The complement of $K_t$ in $\mathbb{H}$ is a domain $\mathbb{H}_t$ which is simply connected by hypothesis, so that by the Riemann mapping theorem $\mathbb{H}_t$ is conformally equivalent to $\mathbb{H}$ via a map $f_t$. This map can be normalized to behave as $f_t(z) = z + 2t/z + O(1/z^2)$: the $PSL_2(\mathbb{R})$ automorphism group of $\mathbb{H}$ allows to impose $f_t(z) = z + O(1/z)$ for large $z$, and then the coefficient of $1/z$ is fixed to be $2t$ by a time reparametrization. The crucial condition of local growth leads to the Loewner differential equation

$$\partial_t f_t(z) = \frac{2}{f_t(z) - \xi_t}, \quad f_{t=0}(z) = z$$

with $\xi_t$ a real function. For fixed $z$, $f_t(z)$ is well-defined up to the time
\( \tau_z \leq +\infty \) for which \( f_{\tau_z}(z) = \xi_{\tau_z} \). Then \( K_t = \{ z \in \mathbb{H} : \tau_z \leq t \} \).

(Chordal) stochastic Loewner evolutions is obtained \([17]\) by choosing \( \xi_t = \sqrt{\kappa} B_t \) with \( B_t \) a normalized Brownian motion and \( \kappa \) a real positive parameter so that \( \mathbb{E}[\xi_t \xi_s] = \kappa \min(t, s) \). Here and in the following, \( \mathbb{E}[\cdots] \) denotes expectation value.

### 3 Connection with conformal field theory

The next section, which also contains basic definitions to which the reader can refer, is devoted to a careful discussion of the implementation of finite local conformal transformations in conformal field theory. In this section, we simply assume that such an implementation is possible, and we derive a direct connection between SLE and CFT.

SLE is defined via an ordinary differential equation, but for our reinterpretation in terms of conformal field theories, it is useful to define \( k_t(z) \equiv f_t(z) - \xi_t \) which satisfies the stochastic differential equation

\[
dk_t = \frac{2dt}{k_t} - d\xi_t.
\]

We observe that the conditions at spatial infinity satisfied by \( k_t \) imply that its germ there, which determines it uniquely, belongs to the group \( N_- \) of germs of holomorphic functions at \( \infty \) of the form \( z + \sum_{m \leq -1} f_m z^{m+1} \), the group law being composition. In this way, the Loewner equations describe trajectories on \( N_- \) in a time dependent left-invariant vector field, whose value at the identity element is \( (2/z - \dot{\xi}_t) \partial_z \).

Due to the fact that \( \xi_t \) is almost surely nowhere differentiable, this observation has to be taken with a grain of salt. We let \( f \in N_- \) act on \( O_\infty \), the space of germs of holomorphic functions at infinity, by composition, \( \gamma_f \cdot F \equiv F \circ f \). Observe that \( \gamma_{gf} = \gamma_f \cdot \gamma_g \) so this is an anti representation. Ito’s formula gives

\[
d\gamma_{k_t} \cdot F = (\gamma_{k_t} \cdot F')(\frac{2dt}{k_t} - d\xi_t) + \frac{\kappa}{2}(\gamma_{k_t} \cdot F'')
\]

from which we derive

\[
\gamma_{k_t}^{-1} \cdot d\gamma_{k_t} = dt\left(\frac{2}{z} \partial_z + \frac{\kappa}{2} \partial_z^2\right) - d\xi_t \partial_z.
\]

The operators \( l_n = -z^{n+1} \partial_z \) are represented in conformal field theories
by operators $L_n$ which satisfy the Virasoro algebra $\text{vir}$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad [c, L_n] = 0.$$  

The representations of $\text{vir}$ are not automatically representations of $N_-$, one of the reasons being that the Lie algebra of $N_-$ contains infinite linear combinations of the $l_n$'s. However, as we shall see in the next section, highest weight representations of $\text{vir}$ can be extended in such a way as to become representations of $N_-$. We take this for granted for the moment and associate to $\gamma_f$ an operator $G_f$ acting on appropriate representations and satisfying $G_g \circ f = G_f G_g$ and

$$G_{k_t}^{-1} dG_{k_t} = dt(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2) + d\xi_t L_{-1}.$$  

The basic observation is the following [1]:

Let $|\omega\rangle$ be the highest weight vector in the irreducible highest weight representation of $\text{vir}$ of central charge $c_\kappa = \frac{(6 - \kappa)(8\kappa - 3)}{2\kappa}$ and conformal weight $h_\kappa = \frac{6 - \kappa}{2\kappa}$. Then $\mathbb{E}[G_{k_t}|\omega\rangle]$ is time independent.

This is a direct consequence of the fact that for this special choice of central charge and weight, the irreducible highest weight representation is degenerate at level 2 and $(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)|\omega\rangle = 0$. Then

$$dG_{k_t}|\omega\rangle = G_{k_t}(dt(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2) + d\xi_t L_{-1})|\omega\rangle = d\xi_t G_{k_t}|\omega\rangle$$

From the definition of Ito integrals, $d\xi_t$ and $G_{k_t}$ are independent, so that $d\mathbb{E}[G_{k_t}|\omega\rangle] = 0$ as announced.

This result can be interpreted as follows. Take a conformal field theory in $\mathbb{H}_t$. The correlation functions in this geometry can be computed by looking at the same theory in $\mathbb{H}$ modulo the insertion of an operator representing the deformation from $\mathbb{H}$ to $\mathbb{H}_t$. This operator is $G_{k_t}$. Suppose that the central charge is $c_\kappa$ and the boundary conditions are such that there is a boundary changing primary operator of weight $h_\kappa$ inserted at the tip of $k_t$ (the existence of this tip is more or less a consequence of the local growth condition). Then in average the correlation functions of the conformal field theory in the fluctuating geometry $\mathbb{H}_t$ are time independent and equal to their value at $t = 0$.

We call $G_{k_t}|\omega\rangle$ a generating function for conserved quantities because for any time-independent bra $\langle v|$, the scalar $\mathbb{E}[\langle v| G_{k_t}|\omega\rangle]$ is a time independent
scalar. We shall see later that in an algebraic sense, all conserved quantities for chordal SLE are of this form.

A word of caution is needed here. Before talking about $\mathbb{E}[\langle v|G_{k_t}|\omega \rangle]$, we should in principle show that $\langle v|G_{k_t}|\omega \rangle$ is an integrable random variable. This is true for instance if $\langle v \rangle$ is a finite excitation of $\langle \omega \rangle$, but this condition is far too restrictive for probability theory and for conformal field theory as well.

In probabilistic terms, a random variable whose Ito derivative contains only a $d\xi_t$ contribution (no $dt$) is called a local martingale. We shall often drop the term local, even if the notion of martingale, though closely related to the notion of local martingale, is more restrictive. In particular, the time independence of expectations is always true for martingales. We refer the interested reader to the mathematical literature [11].

4 Conformal transformations in conformal field theory

A (rather provocative) definition of (boundary) conformal field theory is that it is the representation theory of the Virasoro algebra $\mathfrak{vir}$.

The Virasoro algebra has a subalgebra $\mathfrak{n}_-$, with generators the $L_n$'s $n < 0$, which is closely related to $N_-$, the group of germs of conformal transformations that fix $\infty$. This is crucial for the construction of $G_{k_t}$. Our goal in this section is to show that indeed, $N_-$ acts on sufficiently many physically relevant representations of $\mathfrak{vir}$ to be able to make sense of conformal field theories in the fluctuating geometry $\mathbb{H}_t$.

In the same spirit, the group $N_+$ germs of conformal transformations that fix 0 is closely related to the subalgebra $\mathfrak{n}_+$ of $\mathfrak{vir}$ with generators the $L_n$'s $n > 0$. This group will also play an important role in the forthcoming discussion.

4.1 Background

The theories we shall study will mostly be boundary conformal field theories, and will shall talk of field or operator without making always explicit whether the argument is in the bulk or on the boundary.

The basic principles of conformal field theory state that the fields can be classified according to their behavior under (local) conformal transformations. Then the correlation functions in a region $\mathbb{U}$ are known once they are known in a region $\mathbb{U}_0$ and an explicit conformal map $f$ from $\mathbb{U}$ to $\mathbb{U}_0$ preserving boundary conditions is given. Primary fields have a very simple behavior
under conformal transformations: for a bulk primary field \( \varphi \) of weight \((h, \bar{h})\), 
\[ \varphi(z, \bar{z}) d^h z d^\bar{h} \bar{z} \] is invariant, and for a boundary conformal field \( \psi \) of weight \( \delta \), 
\[ \psi(x) |dx|^\delta \] is invariant. So the statistical averages in \( U \) and \( U_0 \) are related by

\[
\langle \cdots \varphi(z, \bar{z}) \cdots \psi(x) \cdots \rangle_U = \\
\langle \cdots \varphi(f(z), f(\bar{z})) f'(z)^h \bar{f}'(z)^{\bar{h}} \cdots \psi(f(x)) f'(x)^\delta \cdots \rangle_{U_0}.
\]

Such a behavior is described as local conformal covariance.

In a local theory, small deformations are generated by the insertion of a local operator, the stress tensor. Local conformal covariance can then be rephrased: the stress tensor of a conformal field theory is not only conserved and symmetric, but also traceless, so that it has only two independent components, one of which, \( T \), is holomorphic (except for singularities when the argument of \( T \) approaches the argument of other insertions), and the other one, \( \overline{T} \), is antiholomorphic (again except for short distance singularities). The field \( T \) itself is not a primary field in general, but a projective connection:

\[
\langle \cdots T(z) \cdots \rangle_U = \langle \cdots T(f(z)) f'(z)^2 + \frac{c}{12} S f(z) \cdots \rangle_{U_0}.
\]

In this formula, \( c \) is the central charge and \( S f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \) is the Schwartzian derivative of \( f \) at \( z \).

If \( U \) is a non empty simply connected region strictly contained in \( \mathbb{C} \), the Riemann mapping theorem states that \( U_0 \) can be chosen to be unit disk \( \mathbb{D} \) or equivalently the upper-half plane \( \mathbb{H} \) with a point at infinity added, which belongs to the boundary. This second choice will prove most convenient for us in the sequel.

In boundary conformal field theory, \( T \) and \( \overline{T} \) are not independent: they are related by analytic continuation. The relationship is expressed most simply in the upper-half plane. The vectors fields \( z^{n+1} \partial_z \) and \( \overline{z}^{n+1} \partial_{\overline{z}} \) are generators of infinitesimal conformal transformations in \( \mathbb{C} \) but only the combination \( z^{n+1} \partial_z + \overline{z}^{n+1} \partial_{\overline{z}} \equiv -\ell_n \) preserves the boundary of \( \mathbb{H} \), that is, the real axis. Write \( z = x + iy \) and for a while write \( T(x, y) \) for what we usually write \( T(z) \). Choosing boundary conditions such that there is no flow of energy momentum across the boundary \( x = 0 \), \( T(x, y) \) is real along the real axis, and by the Schwartz reflection principle has an analytic extension to the lower half plane as \( T(x, -y) \equiv \overline{T(x, y)} = \overline{T(x, y)} \). Due to this property, most contour integrals involving \( T \) and \( \overline{T} \) in the upper half plane can be seen as contour integrals involving only \( T \) but in the full complex plane.

Using conformal field theory in \( \mathbb{H} \) to express correlators in any simply connected region strictly contained in \( \mathbb{C} \) has another advantage: one can use
the formalism of radial quantization in a straightforward way. The statistical averages are replaced by quantum expectation values:

\[ \langle \cdots T(z) \cdots \varphi(z,\bar{z}) \cdots \psi(x) \cdots \rangle_H \equiv \langle \Omega \mid \cdots \hat{T}(z) \cdots \hat{\varphi}(z,\bar{z}) \cdots \hat{\psi}(x) \cdots \rangle_r \mid \Omega \rangle. \]

In this formula, \( |\Omega\rangle \) is the vacuum and \( r \) denotes radial ordering: the fields are ordered from left to right from the farthest to the closest to the origin. The integral \( \oint dz z^{n+1} \hat{T}(z) \) along any contour of index 1 with respect to 0, defines an operator \( L_n \) (note again that from the point of view of contour integrals in the upper half plane, \( L_n \) involves \( T \) and \( \bar{T} \)). The fact that the stress tensor is the generator of infinitesimal conformal maps implies that

\[
\begin{align*}
[L_n, \hat{\psi}(x)] &= (x^{n+1} \partial_x + \delta(n+1)x^n) \hat{\psi}(x) \\
[L_n, \hat{\varphi}(z,\bar{z})] &= (z^{n+1} \partial_z + h(n+1)z^n + \bar{z}^{n+1} \partial_{\bar{z}} + h(n+1)\bar{z}^n) \hat{\varphi}(z,\bar{z}) \\
[L_n, T(z)] &= (z^{n+1} \partial_z + 2(n+1)z^n) T(z) + \frac{c}{12} (n^3 - n) z^{n-2} \\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}.
\end{align*}
\]

It is no surprise that we recover the commutation relations of \( \mathfrak{vir} \). Except for the anomalous \( c \)-term, the commutation relations of the \( L_n \)'s are those of the \( \ell_n \)'s. Let us take this opportunity to recall that to preserve classical symmetries in quantum mechanics, the crucial point is to have the symmetries act well on operators, i.e. that the adjoint action represents the classical symmetries. This is because the phase of states are not observables. Hence symmetries in quantum mechanics act projectively, and this leaves room for central terms such as \( c \) in \( \mathfrak{vir} \).

The advantage of the operatorial version of conformal field theory is that one can use the powerful methods of representation theory, applied to the Virasoro algebra.

### 4.2 Some representation theory

In the sequel we denote by \( \mathfrak{h} \) the (maximal) abelian subalgebra of \( \mathfrak{vir} \) generated by \( L_0 \) and \( c \), by \( \mathfrak{n}_- \) (resp. \( \mathfrak{n}_+ \)) the nilpotent\(^3\) Lie subalgebra of \( \mathfrak{vir} \) generated by the \( L_n \)'s, \( n < 0 \) (resp. \( n > 0 \)) and by \( \mathfrak{b}_- \) (resp. \( \mathfrak{b}_+ \)) the Borel Lie subalgebra of \( \mathfrak{vir} \) generated by the \( L_n \)'s, \( n \leq 0 \) (resp \( n \geq 0 \)) and \( c \).

If \( \mathfrak{g} \) is any Lie algebra, we denote by \( \mathcal{U}(\mathfrak{g}) \) its universal enveloping algebra. Then a representation of \( \mathfrak{g} \) is the same as a left \( \mathcal{U}(\mathfrak{g}) \)-module.

---

\(^3\)Triangular would be more accurate, but we keep this definition by analogy with finite dimensional Lie algebras.
Let us describe representations of \( \mathfrak{vir} \) by starting with the simplest ones, which we call positive energy representations. These are representations whose underlying space \( M \) splits as a direct sum \( M = \bigoplus_{m \geq 0} M_m \) of finite dimensional subspaces such that \( L_n \) maps \( M_m \) to \( M_{m-n} \) for any \( m, n \in \mathbb{Z} \) (with the convention that \( M_m \equiv \{0\} \) for \( m < 0 \)) and \( L_0 \) is diagonalizable on each \( M_m \).

If \( M \) has a positive energy, we can define the contravariant representation of \( \mathfrak{vir} \) whose underlying space is the little graded dual of \( M \), which we define as \( M^* = \bigoplus_{m \geq 0} M_m^* \), where \( M_m^* \) is the standard algebraic dual of the finite dimensional \( M_m \). Observe that one can view \( L_n \) acting on \( M \) as a collection of linear maps \( L_n : M_m \to M_{m-n} \) indexed by \( m \). For each of these maps, one can take the algebraic transpose \( ^tL_n : M_m^* \to M_m^* \), defined (as usual for finite dimensional spaces) by \( \langle ^tL_n y, x \rangle \equiv \langle y, L_n x \rangle \) for \( (x, y) \in M_m \times M_m^* \). We define \( L_n \) acting on \( M^* \) by the collection \( ^tL_n : M_m^* \to M_m^* \). We decide that \( c \) is the same scalar on \( M^* \) as on \( M \). The representation property is checked by a simple computation. Note that \( M^{**} \) is canonically isomorphic to \( M \) as a \( \mathfrak{vir} \)-module.

The most important examples of positive energy representations are highest weight modules and their contravariants.

A \( \mathfrak{vir} \) highest weight module \( M \) is a representation of the Virasoro algebra which contains a vector \( v \) such that (i) \( \mathbb{C} v \) is a 1-dimensional representation of \( \mathfrak{h} \) and is annihilated by \( \mathfrak{n}_- \) and (ii) the smallest subrepresentation of \( M \) containing \( v \) is \( M \) itself, i.e. all states in \( M \) can be obtained by linear combinations of strings of generators of \( \mathfrak{vir} \) acting on \( v \). Because \( \mathbb{C} v \) is a one dimensional representation of \( \mathfrak{b}_+ \), all states in \( M \) can be obtained by linear combinations of strings of generators of \( \mathfrak{n}_- \) acting on \( v \). On such a representation, the generator \( c \) acts on \( M \) as multiplication by a scalar, which we denote by \( c \) again and call the central charge. The number \( h \) such that \( L_0 v = hv \) is called the conformal weight of the representation. One can write \( M = \bigoplus_{m \geq 0} M_m \) where \( L_0 \) acts on \( M_m \) by multiplication by \( h + m \), \( M_0 = \mathbb{C} v \) and \( M_m \) is finite dimensional with dimension at most \( p(m) \), the number of partitions of \( m \). For convenience, we define \( M_m \equiv \{0\} \) for \( m < 0 \). Then \( L_n \) maps \( M_m \) to \( M_{m-n} \) for any \( m, n \in \mathbb{Z} \). By construction, highest weight cyclic modules have positive energy.

The existence of highest weight modules for given \( c \) and \( h \) is ensured by a universal construction using induced representation. Let \( R(c, h) \) denote the one dimensional representation of \( \mathfrak{h} \), of central charge \( c \) and conformal weight \( h \). View \( R(c, h) \) as a representation of \( \mathfrak{b}_+ \) where \( \mathfrak{n}_+ \) act trivially. This turns \( R(c, h) \) into a left \( \mathcal{U}(\mathfrak{b}_+) \)-module. For any \( \mathfrak{g}, \mathcal{U}(\mathfrak{g}) \) acts on itself on the left and on the right, so by restriction, we can view \( \mathcal{U}(\mathfrak{vir}) \) as a left \( \mathcal{U}(\mathfrak{vir}) \)-module and as a right \( \mathcal{U}(\mathfrak{b}_+) \)-module. Then \( V(c, h) \equiv \mathcal{U}(\mathfrak{vir}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} R(c, h) \)
is a left $\mathcal{U}(\mathfrak{vir})$-module, called the Verma module with parameters $(c, h)$. As a $\mathcal{U}(\mathfrak{n}_-)$-module, $V(c, h)$ is isomorphic to $\mathcal{U}(\mathfrak{n}_-)$ itself, so the number of states in $V(c, h)_n$ is exactly $p(n)$. Any highest weight cyclic module $M$ with parameters $(c, h)$ is a quotient of $V(c, h)$.

The contravariant $M^*$ of a highest weight module is not always highest weight: $\mathcal{U}(\mathfrak{vir})M^*_0$ is always irreducible, hence is a proper submodule of $M^*$ if $M$ is not irreducible.

### 4.3 Completions

In the following, we shall often need to deal with infinite linear combinations of Virasoro generators. For instance, formally $T(z) = \sum_n L_n z^{-n-2}$. So we make some new definitions.

We denote by $\overline{\mathfrak{n}_+}$ the formal completion of $\mathfrak{n}_+$ which is made of arbitrary (not necessarily finite) linear combinations of $L_n$'s, $n > 0$. The Lie algebra structure on $\mathfrak{n}_+$ extends to a Lie algebra structure on $\overline{\mathfrak{n}_+}$ if we define

$$\left[\sum_{m>0} a_m L_m, \sum_{n>0} b_n L_n\right] \equiv \sum_{k>0} \left(\sum_{m>0, n>0, m+n=k} (m-n)a_m b_n\right) L_k.$$ 

As usual with formal power series, this works because for fixed $k$, the sum $\sum_{m>0, n>0, m+n=k}$ is a finite sum.

We can go one step further and define $\overline{\mathfrak{vir}_+}$ as the direct sum $\overline{\mathfrak{n}_+} \oplus \mathfrak{b}_-$, which is still a Lie algebra with the obvious definition.

One can make analogous definitions for $\overline{\mathfrak{vir}_-}$, $\overline{\mathfrak{n}_-}$, $\overline{\mathfrak{b}_-}$, $\overline{\mathfrak{n}_-} \oplus \overline{\mathfrak{vir}_+}$.

All these Lie algebras are contained in $\overline{\mathfrak{n}_-} \oplus \mathfrak{h} \oplus \overline{\mathfrak{vir}_+}$, but we shall not (!) try to put a Lie algebra structure on that space.

Note that $\mathfrak{vir}$, $\mathfrak{n}_-$, $\mathfrak{n}_+$, $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are graded Lie algebras, so their universal enveloping algebras are graded too (the grading should not be confused with the filtration which exists for any Lie algebra). We denote by $\mathcal{U}(\mathfrak{vir})$, $\mathcal{U}(\mathfrak{n}_-)$, $\mathcal{U}(\mathfrak{n}_+)$, $\mathcal{U}(\mathfrak{b}_-)$, $\mathcal{U}(\mathfrak{b}_+)$ the subspace of degree $n$ in each of the corresponding algebras.

Using the grading, one checks that $\overline{\mathcal{U}(\mathfrak{n}_+)} \equiv \prod_{n>0} \mathcal{U}(\mathfrak{n}_+)_n$, the formal completion of $\mathcal{U}(\mathfrak{n}_+)$ has a natural associative algebra structure which extends that of $\mathcal{U}(\mathfrak{n}_+)$. In the same spirit $\overline{\mathcal{U}(\mathfrak{vir})} \equiv \oplus_{n\leq 0} \mathcal{U}(\mathfrak{vir})_n \oplus \prod_{n>0} \mathcal{U}(\mathfrak{vir})_n$ has a natural associative algebra structure which extends that of $\mathcal{U}(\mathfrak{vir})$, and is in fact isomorphic to $\mathcal{U}(\overline{\mathfrak{vir}_+})$.

---

4Following standard practice, if $I$ is a set and $E_i, i \in I$ a family of vector spaces indexed by $I$, $\prod_i E_i$ is the set theoretic product of the $E_i$, whereas $\oplus_i E_i$ is the subspace of $\prod_i E_i$ consisting of families with only a finite number of nonzero components.
Again, one can make analogous remarks for $\mathcal{U}(n_-) \equiv \prod_{n<0} \mathcal{U}(n_-)_n$ and $\mathcal{U}(\text{vir})_- \equiv \oplus_{n \geq 0} \mathcal{U}(\text{vir})_n \oplus \prod_{n<0} \mathcal{U}(\text{vir})_n$.

If $M$ is a finite energy representation, its formal completion $\overline{M} = \prod_m M_m$ is still a $\text{vir}$-module, though not a finite energy one. Any positive energy representation $M$ of $\text{vir}$ is also a representation of $\text{vir}_+ = \mathfrak{p}_+ \oplus \mathfrak{b}_-$ and a $\mathcal{U}(\mathfrak{n}_+)$-module, whereas $\overline{M}$ is a representation of $\text{vir}_- = \mathfrak{p}_- \oplus \mathfrak{b}_+$ and a $\mathcal{U}(\mathfrak{n}_-)$-module.

5 Finite deformations in conformal field theory

Suppose now that $H$ is a domain of the type represented on fig. 1, that is mapped to $\mathbb{H}$ by some conformal transformation $f$.

![Figure 1: A typical hull geometry.](image)

We are going to show that just as an infinitesimal deformation is described by the insertion of an element of the Virasoro algebra, the finite deformation that leads from the conformal field theory on $H$ to that on $\mathbb{H}$ can be represented by an operator $G_f$ implementing the map $f$:

$$\langle \cdots \phi(z, \bar{z}) \cdots \psi(x) \cdots \rangle_H = \langle \Omega | G_f^{-1} \left( \cdots \hat{\phi}(z, \bar{z}) \cdots \hat{\psi}(x) \cdots \right)_r G_f | \Omega \rangle.$$ 

This relates correlation functions in $H$ to correlation functions in $\mathbb{H}$ where the field arguments are taken at the same point (!) but sandwiched inside a conjugation by $G_f$.

5.1 Finite deformations around 0

Let $N_+$ be the space of power series of the form $z + \sum_{m \geq 1} f_m z^{m+1}$ which have a non vanishing radius of convergence. With words, $N_+$ is a subset of the space $O_0$ of germs of holomorphic functions at the origin, consisting of the germs which fix the origin and whose derivative at the origin is 1. In physical applications, we shall only need the case when the coefficients are real. But in certain intermediate constructions, it will be useful to consider
the $f_m$’s as independent commuting indeterminates (so that we forget about convergence and deal with formal power series) : the following statements can be translated in a straightforward way to deal with this more general situation.

As a set, $N_+$ is convex. Moreover, $N_+$ is a group for composition. Our aim is to construct a group (anti)-isomorphism from $N_+$ with composition onto a subset $\mathcal{N}_+ \subset \mathcal{U}(n_+)$ with the associative algebra product. The possibility to do that essentially boils down to the fact that $n_+$ is nilpotent.

We let $N_+$ act on $O_0$ by $\gamma_f \cdot F \equiv F \circ f$ for $f \in N_+$ and $F \in O_0$. This representation is faithful. Because $\gamma_{g \circ f} = \gamma_f \gamma_g$, we see by taking $g = z + \varepsilon v(z)$ for small $\varepsilon$ that $\gamma_{f + \varepsilon v(f)} F = \gamma_f F + \varepsilon \gamma_f \cdot (v \cdot F) + o(\varepsilon)$, where $v \cdot F(z) \equiv v(z) F'(z)$ is the standard action of vector fields on functions. Using the Lagrange inversion formula\(^5\), we compute that for $m \geq 1$

$$z^{m+1} = \sum_{n \geq m} f(z)^{n+1} \int_0^1 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}},$$

so that

$$\frac{\partial \gamma_f}{\partial f_m} = \gamma_f \sum_{n \geq m} \int_0^1 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}} z^n \partial_z.$$

This system of first order partial differential equations makes sense in $\mathcal{U}(n_+)$ if we replace $z^{n+1} \partial_z$ by $-L_n$. We define a connection

$$A_m \equiv \sum_{n \geq m} L_n \int_0^1 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}},$$

which satisfies the zero curvature condition

$$\frac{\partial A_l}{\partial f_k} - \frac{\partial A_k}{\partial f_l} = [A_k, A_l].$$

Hence we may construct $G_f \in \mathcal{U}(n_+)$ for each $f \in N_+$ by solving the system

$$\frac{\partial G_f}{\partial f_m} = -G_f \sum_{n \geq m} L_n \int_0^1 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}} \quad m \geq 1. \quad (2)$$

This system is guarantied to be compatible, because the representation of $N_+$ on $O_0$ is well defined for finite deformations $f$, faithful and solves the

\(^5\)With the convention that $\int_0^1$ is an integration along a small contour of index 1 around the origin, with the prefactor $(2i\pi)^{-1}$ included, or equivalently that $\int_0^1$ is taking the residue at the origin, a purely algebraic operation which can be performed without a real integration.
analogous system. However, as the argument for zero curvature is instructive, we give a direct proof in Appendix A.

Once the compatibility conditions are checked, the existence and unicity of $G_f$, with the initial condition $G_{f|z=0}$ is the identity, is obvious: expansion of $G_f$ using the grading in $U(n_+)$ leads to a recursive system. The group (anti)-homomorphism property is true because it is true infinitesimally and $N_+$ is convex.

As an illustration, 

$$G_f = 1 - f_1 L_1 + \frac{f_2}{2}(L_1^2 + 2L_2) - f_2 L_2 + \cdots$$

Some useful general properties of $G_f$ are collected in Appendix C.

Observe that $N_+$ acts by conjugation on $\text{vir} \equiv n_+ \oplus b_-$. To get orientation, let us consider the action of $f \in N_+$ not on functions but on vector fields. First, we extend the action of $N_+$ on $O_0$ by composition to $Q_0$, the field of fractions of $O_0$. A vector fields $v = v(z)\partial_z$ with coefficient in $Q_0$ (i.e. a derivations of $Q_0$) acts on $Q_0$ too.

Let us consider $(\gamma_f^{-1}.v.\gamma_f)F(z)$. Defining $v_f \equiv \gamma_f^{-1}.v.\gamma_f$, a simple computation shows that $v_f F(z) = (v \circ f^{-1})(z) (f' \circ f^{-1})(z) F'(z)$. So, as expected, $v_f$ is still a derivation, and writing $v_f \equiv v_f(z)\partial_z$, one finds $v_f(z) = (v \circ f^{-1})(z) (f' \circ f^{-1})(z)$. Lagrange inversion shows that

$$v_f(z) = \sum_{n \geq m} z^{n+1} \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n+2}} \quad \text{for} \quad v(z) = z^{m+1}.$$

Because of the correspondence between $-z^{m+1}\partial_z$ and $L_m$, it is not surprising that, for every $m \in \mathbb{Z}$:

$$G^{-1}_f L_m G_f = \frac{c}{12} \int_0^1 dw w^{m+1} S f(w) + \sum_{n \geq m} L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n+2}}$$

$$\equiv L_m(f). \quad (3)$$

The proof of this identity is relegated to appendix B.

One can also check directly and painfully that the $L_m(f)$’s satisfy the Virasoro algebra commutation relation with central term $c$, but this is guaranteed by the fact that $L_m(f)$ is obtained from $L_m$ by a conjugation.

If we define a truncated stress tensor $T_l(z) \equiv \sum_{m \geq l} L_m z^{-m-2}$, which belongs to $\text{vir}_+$, we have that

$$G^{-1}_f T_l(z) G_f = \sum_{m \geq l} L_m(f) z^{-m-2}.$$
Now let us try to let $l \to -\infty$. In the $c$-term the $m$ summation converges to $Sf(z)$ if $z$ belongs to the disk of convergence of $Sf(z)$. In the same way, for fixed $n$, the $m$ summation converges to $f(z)^{-n-2}f'(z)^2$ if $z$ belongs to the disk of convergence of $f(z)^{-n-2}f'(z)^2$. When $n$ varies, this leads only to 2 constraints. So, for $z$ in a non void pointed disk centered at the origin, the infinite summations appearing for fixed vir degree in $G_f^{-1}T(z)G_f$ are absolutely convergent and

$$G_f^{-1}T(z)G_f = T(f(z))f'(z)^2 + \frac{c}{12}Sf(z),$$

(4)

so we have an operatorial version of finite deformations that has all the expected properties. The last equation can then be extended by analytic continuation if $f(z)$ allows it. One important lesson to draw from this computation is that, quite naturally in fact, if the $L_m$’s are the basic objects and $T$ is constructed from them, changes of coordinates act nicely only if some convergence criteria are fulfilled. Similar consideration would apply if we would consider the action of $G_f$ on other local fields.

Now that we have the stress tensor at our disposal, we can rewrite the variations of $G_f$ in a familiar way: if $f$ is changed to $f + \delta f$ with $\delta f = \varepsilon v(f)$, we find that

$$\delta G_f = -\varepsilon G_f \oint T(z)v(z)dz.$$

If v is not just a formal power series at the origin, but a convergent one in a neighborhood of the origin, we can freely deform contours in this formula.

### 5.1.1 Finite deformations around $\infty$

Now, let us look at the holomorphic functions at $\infty$ instead of 0. So let $N_-$ be the space of power series of the form $z + \sum_{m \leq -1} f_m z^{m+1}$ which have a non vanishing radius of convergence. We let it act on $O_\infty$, the space of germs of holomorphic functions at infinity, by $\gamma_f \cdot F \equiv F \circ f$. The adaptation of the previous computations shows that $\frac{\partial \gamma_f}{\partial f_m} = \gamma_f \sum_{n \leq m} \oint dw w^{m+1} \frac{f'(w)}{f(w)^{n+2}} z^{n+1} \partial z$

where $\oint$ is around a small contour of index $-1$ with respect to the point at $-1$. The term $z^{n+1}$ is the monomial at the point $z^-$. We can now rewrite the variations of $G_f$ in a familiar way: if $f$ is changed to $f + \delta f$ with $\delta f = \varepsilon v(f)$, we find that

$$\delta G_f = -\varepsilon G_f \oint T(z)v(z)dz.$$
infinity. We transfer this relation to \( \mathcal{U}(\mathbb{n}) \) to define an (anti)-isomorphism from \( N_- \) to \( N_- \subset \mathcal{U}(\mathbb{n}) \) mapping \( f \) to \( G_f \) such that

\[
\frac{\partial G_f}{\partial f_m} = -G_f \sum_{n \leq m} L_n \int_{\infty}^{\infty} dw w^{m+1} \frac{f'(w)}{f(w)^{n+2}}, \quad m \leq -1.
\]

All the previous considerations could be extended to that case.

5.2 Dilatations and translations

We close this section with a small extensions that, for different reasons, demand to leave the realm of formal power series.

The first one has to do with dilatations. Up to now, we have been dealing with deformations around 0 and \( \infty \) that did not involve dilatation at the fixed point: \( f'(0) \) or \( f'((\infty) \) was unity. Hence the operator \( L_0 \) appears nowhere in the above formulæ. To gain some flexibility in the forthcoming discussion, we decide (this is a convention) to authorize dilatations at the origin. The operator associated to a pure dilatation \( f(z) = f'(0)z \) is \( f'(0)L_0 \). One can view a general \( f \) fixing 0 as the composition \( f(z) = f'(0)(z + \sum_m f_m z^{m+1}) \) of a deformation at 0 with derivative 1 at 0 followed by a dilatation. As before, the operators are multiplied in the opposite order, so that \( G_f = G_{f'}/f'(0) L_0 \). From this formula, one checks that eqs. remain valid even when \( f' \) is not unity. To keep the group composition property, we demand that \( f'(0) \) be real and positive.

The second extension deals with translations. Suppose that \( f(z) = f'(0)(z + \sum_m f_m z^{m+1}) \) is a generic invertible germ of holomorphic function fixing the origin (\( f'(0) \neq 0 \)). If \( a \) is in the interior of the disk of convergence of the power series expansion of \( f \) and \( f'(a) \neq 0 \), we may define a new germ \( f_a(z) \equiv f(a + z) - f(a) \) with the same properties. What is the relationship between \( G_f \) and \( G_{f_a} \)? At the infinitesimal level, we compute \( \frac{df_a}{da}|_{a=0} = vf \). The use of the Lagrange formula yields

\[
v(f) = \sum_{n \geq 0} f^{n+1} \int_{0}^{\infty} dw \frac{f'(w)^2}{f(w)^{n+2}},
\]

which implies

\[
G_{f_a}^{-1} \frac{dG_{f_a}}{da}|_{a=0} = -\sum_{n \geq 0} L_n \int_{0}^{\infty} dw \frac{f'(w)^2}{f(w)^{n+2}} = L_{-1} f'(0) - G_f^{-1} L_{-1} G_f.
\]
The last equality comes eq.(3) for \( m = -1 \). We conclude that for general \( a \),

\[
G_{f_a}^{-1} \frac{dG_{f_a}}{da} = L_{-1} f_a'(0) - G_{f_a}^{-1} L_{-1} G_{f_a}
\]

This differential equation is easy to solve formally:

\[
G_{f_a} = e^{-aL_{-1}} G_{f e^{f(a)L_{-1}}}.
\]

(5)

This formal solution has an analytic meaning at least as long as \( a \) is in the interior of the disk of convergence of the power series expansion of \( f \) and \( f'(a) \neq 0 \) (extensions will require analytic continuation). This is a special case of the yet to come Wick theorem for the Virasoro algebra.

6 An application to representation theory

In this section, we use the above formulæ for finite deformations to make contact with [3]. Our goal is to construct generalized coherent states representations of \( \text{vir} \) that will allow us to understand the structure of SLE martingales.

6.1 Representations associated to deformations near 0

Suppose that \( M \) is a positive energy representation of \( \text{vir} \). Then so is its dual \( M^* \). Let \( f \) be an element of \( N_+ \). For \( (x, y) \in M \times M^* \), consider the expectation value \( \langle G_f y, x \rangle \) or \( \langle G_f^{-1} y, x \rangle \). From eq.(16) in Appendix C these expectations are polynomial in the coefficients of \( f = z + \sum_{m>1} f_m z^{m+1} \).

Take as \( M \) a Verma module \( V(c, h) \) and take \( x \neq 0 \) in the highest weight space of \( M \). Then the space \( \{ \langle G_f y, x \rangle, y \in M^* \} \) or \( \{ \langle G_f^{-1} y, x \rangle, y \in M^* \} \) is the space of all polynomials in the independent variables \( f_1, f_2, \cdots \). Indeed, choose the basis of \( M \) indexed by ordered monomials in the \( L_n \)'s with negative \( n \), acting on the highest weight state \( x \), and the dual basis in \( M^* \). Then eq.(16) shows that when we take for \( y \) successively the elements of the dual basis, the matrix elements \( \langle G_f y, x \rangle \) or \( \langle G_f^{-1} y, x \rangle \) enumerate a basis of the space of polynomials in \( f_1, f_2, \cdots \). So we have two linear isomorphisms from \( M^* \) to \( q[f_1, f_2, \cdots \rangle \) where \( q \) is the preferred field of the reader (\( \mathbb{Q} \) is a minimal choice), and we can use these isomorphism to transport the action of \( \text{vir} \).

6.1.1 The case of \( G_f \)

For \( y \in M^* \), define \( P_y \equiv \langle G_f y, x \rangle \). We are going to give formulæ for \( P_{L_n y} \) as a first order differential operator acting on \( P_y \).
The case when \( n \geq 1 \) is simple. Indeed, using formula (2) for the partial derivatives of \( G_f \), one checks that

\[
- \sum_{m \geq n} \int_0^1 dz \frac{f(z)^{n+1}}{z^{m+2}} \frac{\partial}{\partial f_m} G_f = G_f L_n
\]

So for \( n \geq 1 \),

\[
P_{L_n y} = - \sum_{m \geq n} \int_0^1 dz \frac{f(z)^{n+1}}{z^{m+2}} \frac{\partial P_y}{\partial f_m}.
\]

To deal with \( n < 1 \), we write \( G_f L_n = (G_f L_n G_f^{-1}) G_f \) and use that \( G_f L_n G_f^{-1} \in \overline{\mathfrak{n}}_+ \oplus \mathfrak{b}_- \) to decompose \( G_f L_n G_f^{-1} = (G_f L_n G_f^{-1})_{\overline{\mathfrak{n}}_+} + (G_f L_n G_f^{-1})_{\mathfrak{b}_-} \).

From eq. (3) for the compositional inverse of \( f \), we get after a change of variable

\[
G_f L_n G_f^{-1} = - \frac{c}{12} \int_0^1 dw f(w)^{n+1} \frac{S f(w)}{f'(w)} + \sum_{m \geq n} L_n \int_0^1 dw \frac{f(w)^{n+1}}{w^{m+2} f'(w)} \quad n \in \mathbb{Z}.
\]

The \( \mathfrak{b}_- \) part contains the central charge term and the sum \( n \leq m \leq 0 \). For \( m < 0 \), \( \langle L_m G_f y, x \rangle = \langle G_f y, L_{-m} x \rangle = 0 \) because \( x \) is a highest weight state, and \( \langle L_0 G_f y, x \rangle = h \langle G_f y, x \rangle \) because \( x \) has weight \( h \). So

\[
\langle (G_f L_n G_f^{-1})_{\mathfrak{b}_-} G_f y, x \rangle = \left( - \frac{c}{12} \int_0^1 dw f(w)^{n+1} \frac{S f(w)}{f'(w)} + h \int_0^1 dw f(w)^{n+1} \frac{1}{w^2 f'(w)} \right) P_y.
\]

To deal with the \( \overline{\mathfrak{n}}_+ \) part, we observe that \( G_f^{-1} (G_f L_n G_f^{-1})_{\overline{\mathfrak{n}}_+} G_f \) belongs to \( \overline{\mathfrak{n}}_+ \) but on the other hand \( G_f^{-1} (G_f L_n G_f^{-1})_{\overline{\mathfrak{n}}_+} G_f = L_n - G_f^{-1} (G_f L_n G_f^{-1})_{\mathfrak{b}_-} G_f \).

Hence

\[
G_f^{-1} (G_f L_n G_f^{-1})_{\overline{\mathfrak{n}}_+} G_f = -(G_f^{-1} (G_f L_n G_f^{-1})_{\mathfrak{b}_-} G_f)_{\overline{\mathfrak{n}}_+}, \quad n < 1.
\]

For the second conjugation, we use eq. (3) for \( f \) itself. This leads to

\[
P_{L_n y} + \left( \frac{c}{12} \int_0^1 dw f(w)^{n+1} \frac{S f(w)}{f'(w)} - h \int_0^1 dw \frac{f(w)^{n+1}}{w^2 f'(w)} \right) P_y = \quad (7)
\]

\[
- \sum_{m=n}^0 \int_0^1 dw \frac{f(w)^{n+1}}{w^{m+2} f'(w)} \sum_{l \geq 1} \int_0^1 dz dzz^{l+1} \frac{f'(z)^2}{f(z) l+2} \langle G_f L_l y, x \rangle. \quad (8)
\]

One can express the right hand side of this formula as an explicit differential operator. The details are tedious and best relegated to Appendix.
The final result is that, for \( n < 1 \),

\[
P_{L_n y} + \left( \frac{c}{12} \oint_0 dw f(w)^{n+1} S f(w) \right) - h \oint_0 dw f(w)^{n+1} \left. \frac{f}{f'} \right| w^2 f(w)^n P_y = (9)\]

\[
\sum_{j \geq 1} \sum_{m=n}^0 \oint_0 dw \frac{f(w)^{n+1}}{w^{m+2} f'(w)} \left( f_{j-m} (j - m + 1) - \sum_{k=m}^0 \oint_0 dw \frac{w^{m+1} f'(w)^2}{f(u)^{k+2}} \oint_0 dv \frac{f(v)^{k+1}}{v^{j+2}} \right) \frac{\partial P_y}{\partial f_j}.
\]

Eqs. (9) give the desired representation of the action of the Virasoro algebra on \( V^*(c, h) \) as first order differential operators on the space \( q[f_1, f_2, \cdots] \). To be explicit, we quote the expression for a system of generators of \( \text{vir} \):

\[
L_2 = - \sum_{m \geq 2} \left( \sum_{j+k+l=m-2} f_{j+k+l} \right) \frac{\partial}{\partial f_m}
\]

\[
L_1 = - \sum_{m \geq 1} \left( \sum_{j+k=m-1} f_{j+k} \right) \frac{\partial}{\partial f_m}
\]

\[
L_0 = h + \sum_{m \geq 1} m f_m \frac{\partial}{\partial f_m}
\]

\[
L_{-1} = - 2 f_1 h + \sum_{m \geq 1} ((m + 2) f_{m+1} - 2 f_1 (m + 1) f_m) \frac{\partial}{\partial f_m}
\]

\[
L_{-2} = -(f_2/2 - f_1/12 - f_1^2/3) c - (4f_2 - 7f_1^2) h
\]

+ the differential part

For the positive generators, the convention \( f_0 = 1, \; f_n = 0 \; n < 0 \) is used within the sums.

6.1.2 The case of \( G_f^{-1} \)

For \( y \in M^* \), define \( Q_y \equiv \langle G_f^{-1} y, x \rangle \). We are going to give formulæ for \( Q_{L_n y} \) as a first order differential operator acting on \( Q_y \). Note that \( Q_y \) is nothing but \( P_y \) expressed in terms of the coefficients of the inverse (for composition) of \( f \). So in principle, the two constructions are related by a simple change of variables.

We use eq. (9) to work on \( Q_{L_n y} = \langle (G_f^{-1} L_n G_f) G_f^{-1} y, x \rangle \). Again, we write \( G_f^{-1} L_n G_f = (G_f^{-1} L_n G_f)_{\pi} + (G_f^{-1} L_n G_f)_{b_+} \) and use the definition of contravariant representation on the \( b_+ \) part to keep only the diagonal action of \( \mathfrak{h} \). This leads to

\[
Q_{L_n y} = \frac{c}{12} \oint_0 dw w^{n+1} S f(w) + h \oint_0 dw w^{n+1} \frac{f'(w)^2}{f(w)^2}
\]
\[ + \sum_{m \geq 1} \oint_0 dw \frac{f'(w)^2}{f(w)^{m+2}} \langle L_m G_f^{-1} y, x \rangle. \]

The definition of \( h_m \) in eq. (18) and its characteristic property eq. (19) are in fact valid for every \( m \in \mathbb{Z} \). This allows to rewrite the linear combinations of \( L_m \)'s as linear combinations of partial derivatives as:

\[ Q_{L_n y} = \left( \frac{c}{12} \oint_0 dw \frac{n+1}{2} S f(w) + h \oint_0 dw \frac{n+1}{2} f'(w)^2 f(w)^2 \right) Q_y \]

\[ + \sum_{m \geq \max(1,n)} \left( f_{m-n}(m-n+1) - \sum_{l,n \leq l \leq 0} \oint_0 du \frac{u^{n+1} f'(u)^2}{f(u)^{l+2}} \oint_0 dv \frac{f(v)^{l+1}}{v^{m+2}} \right) \frac{\partial Q_y}{\partial f_m} \]

In particular

\[ L_n = \sum_{m \geq 0} (m+1) f_m \frac{\partial}{\partial f_{n+m}}, \quad n \geq 1 \]

\[ L_0 = h + \sum_{m \geq 1} m f_m \frac{\partial}{\partial f_m} \]

\[ L_{-1} = 2 f_1 h + \sum_{m \geq 1} ((m+2)f_{m+1} - 2f_1 f_m) \frac{\partial}{\partial f_m} \]

\[ L_{-2} = (f_2/2 - f_1/12 - f_1^2/3)c + (4f_2 - f_1^2)h \]

+ the differential part

Let us note that the formula for the action of the positive generators \( L_n \), \( n \geq 1 \) is strikingly similar to the one that arises in matrix models [8].

6.1.3 Representation theoretic remarks

By definition, a (non trivial) highest weight vector \( x \) of a Verma module \( V(c,h) \) generates \( V(c,h) \) when acted on by the Virasoro generators. On the other hand, the dual \( x^* \) of \( x \) in \( V^*(c,h) \) generates the irreducible highest weight representation of weight \((c,h)\) when acted on by the Virasoro generators.

Hence, if \((c,h)\) is generic, i.e. if the Verma module \( V(c,h) \) is irreducible, then so is \( V^*(c,h) \) and they are equivalent as \( \text{vir} \) modules. However, if \((c,h)\) is non generic, \( x^* \) generates only a proper subspace of \( V^*(c,h) \).

For instance, suppose that \( c = \frac{(6-\kappa)(8\kappa-3)}{2\kappa} \) and \( h = \frac{5-\kappa}{2\kappa} \) for some \( \kappa \). Then \( V(c,h) \) is not irreducible, \((2L_{-2} + \frac{5}{2} L_{-1}^2) x \) is a singular vector in \( V(c,h) \).
annihilated by the $L_n$’s, $n \geq 1$, so that it does not couple to any descendant of $x^*$. How does this show up in the two representations on polynomials that we constructed? To keep consistent notations, denote by $\mathcal{P}_n$ (resp. $\mathcal{Q}_n$) the differential operator such that $\langle Gf L_n y, x \rangle = \mathcal{P}_n \langle Gf y, x \rangle$ (resp. $\langle Gf^{-1} L_n y, x \rangle = \mathcal{Q}_n \langle Gf^{-1} y, x \rangle$) for $y \in V^*(c, h)$. If $y$ is a descendant of $x^*$, 

$$\left\langle Gf^{-1} y, (-2L_{-2} + \frac{\kappa}{2}L_{-2}^2) x \right\rangle = 0.$$ 

On the other hand, by copying the argument leading to the formula for $\mathcal{P}_n$, $n \geq 1$, one checks that for $n \geq 1$ $\langle L_n Gf^{-1} y, x \rangle = -\mathcal{P}_n \langle Gf^{-1} y, x \rangle$. We conclude that all the polynomials in $f_1, f_2, \cdots$ obtained by acting repeatedly on the polynomial 1 with the $\mathcal{Q}_m$’s (they build the irreducible representation with highest weight $(c, h)$) are annihilated by $2\mathcal{P}_2 + \frac{\kappa}{2} \mathcal{P}_1^2$. For generic $\kappa$ there is no other singular vector in $V^*(c, h)$, and this leads to a satisfactory description of the irreducible representation of highest weight $(h, c)$: the representation space is given by the kernel of an explicit differential operator acting on $q[f_1, f_2, \cdots]$, and the states are build by repeated action of explicit differential operators on the highest weight state 1. The same argument would apply to general singular vectors.

6.2 Representations associated to deformations near $\infty$

The presentation parallels quite closely the case of deformations around 0 so we shall not give all the details. All arguments can be adapted straightforwardly.

Again, $M$ and its dual $M^*$ are supposed to be positive energy representation of $\mathfrak{vir}$. But now we take $f$ in $N_-$. For $(x, y) \in M \times M^*$, consider the expectation value $\langle y, Gf x \rangle$ or $\langle y, Gf^{-1} x \rangle$. As for the deformations around 0, these expectations are polynomial in the coefficients of $f = z + \sum_{m \leq -1} f_m z^{m+1}$.

As $M$, take a Verma module $V(c, h)$ and take $x \neq 0$ in the highest weight space of $M$. The space $\{\langle y, Gf x \rangle, y \in M^*\}$ or $\{\langle y, Gf^{-1} x \rangle, y \in M^*\}$ is the space of all polynomials in the independent variables $f_{-1}, f_{-2}, \cdots$. So we have two linear isomorphisms from $M^*$ to $q[f_{-1}, f_{-2}, \cdots]$ and we can use these isomorphism to transport the action of $\mathfrak{vir}$.

6Though neither $Gf x$ nor $Gf^{-1} x$ is a finite excitation of $x$ in general, the matrix elements $\langle y, Gf x \rangle$ and $\langle y, Gf^{-1} x \rangle$ are well defined because $y \in M^*$ is by definition a finite excitation.
6.2.1 The case of $G_f$

For $y \in M^*$, define $R_y \equiv \langle y, G_f x \rangle$. We give formul\ae for $R_{L_n y}$ as a first order differential operator acting on $R_y$. We write $\langle L_n y, G_f x \rangle = \langle y, L_{-n} G_f x \rangle$ and conjugate to obtain

$$R_{L_n y} = \frac{c}{12} \oint \infty dzz^{1-n} S f(z) R_y + \sum_{m \leq -n} \oint \infty dzz^{1-n} \frac{f'(z)^2}{f(z)^{m+2}} \langle y, G_f L_m x \rangle,$$

where $\oint \infty$ is around a small contour of index $-1$ with respect to the point at infinity. Using the highest weight property of $x$ we get

$$R_{L_n y} - \left( \frac{c}{12} \oint \infty dzz^{1-n} S f(z) + h \oint \infty dzz^{1-n} \frac{f'(z)^2}{f(z)^2} \right) R_y = \sum_{m \leq -1} \oint \infty dzz^{1-n} \frac{f'(z)^2}{f(z)^{m+2}} \langle y, G_f L_m x \rangle.$$

As in the previous sections, we may express the right hand side as an explicit differential operator. Define, for $n \in \mathbb{Z}$,

$$i_n(z) \equiv z^{1-n} f'(z) - \sum_{m, n \leq m \leq 0} f(z)^{1-m} \oint \infty du \frac{u^{1-n} f'(u)^2}{f(u)^{2-m}},$$

which has the property that $i_n(z) = O(1)$ and

$$\oint \infty dzz^{1-n} \frac{f'(z)^2}{f(z)^{m+2}} = \oint \infty dz \frac{h_n(z) f'(z)}{f(z)^{m+2}} \quad \text{for} \quad m = -1, -2, \cdots.$$

The $z$ expansion reads

$$i_n(z) = \sum_{m \leq -1} z^{m+1} \left( f_{m+n}(m+n+1) - \sum_{l, n \leq l \leq 0} \oint \infty du \frac{u^{1-n} f'(u)^2}{f(u)^{2-l}} \oint \infty dv f(v)^{1-l} \right).$$

This leads to the formula

$$R_{L_n y} = \left( \frac{c}{12} \oint \infty dw w^{1-n} S f(w) + h \oint \infty dw w^{1-n} \frac{f'(w)^2}{f(w)^2} \right) Q_y + \sum_{m \leq \min(-1,-n)} \left( f_{m+n}(m+n+1) - \sum_{l, n \leq l \leq 0} \oint \infty du \frac{u^{1-n} f'(u)^2}{f(u)^{2-l}} \oint \infty dv \frac{f(v)^{1-l}}{v^{m+2}} \right) \frac{\partial R_y}{\partial f_m}.$$
which yields

\[
L_n = - \sum_{m \leq 0} (m + 1) f_m \frac{\partial}{\partial f_{m-n}} \quad n \geq 1
\]

\[
L_0 = h - \sum_{m \leq -1} m f_m \frac{\partial}{\partial f_m}
\]

\[
L_{-1} = -2f_{-1}h - \sum_{m \leq -1} \left( m f_{m-1} - \sum_{k+i=m} f_k f_i + 2f_{-1}f_m \right) \frac{\partial}{\partial f_m}
\]

\[
L_{-2} = -cf_{-2}/2 - h(4f_{-2} - 3f_{-1}^2)
\]

\[
- \sum_{m \leq -1} \left( (m - 1)f_{m-2} - \sum_{j+k+l=m-2} f_j f_k f_l + 3f_{-1} \sum_{k+l=m-1} f_k f_l + (4f_{-2} - 3f_{-1}^2)f_m \right) \frac{\partial}{\partial f_m}
\]

6.2.2 The case of $G_f^{-1}$

For $y \in M^*$, define $S_y \equiv \langle y, G_f^{-1}x \rangle$. We give formulæ for $S_{L_n}y$ as a first order differential operator acting on $S_y$. We write $\langle L_n y, G_f^{-1}x \rangle = \langle y, L_{-n} G_f^{-1}x \rangle$.

The case $n \geq 1$ is easy. From

\[
\sum_{m \leq -n} \int_{-\infty}^{0} dz \frac{f(z)^{1-n}}{z^{m+2}} \frac{\partial}{\partial f_m} G_f^{-1} = L_{-n} G_f^{-1}
\]

we infer that

\[
S_{L_n}y = \sum_{m \leq -n} \int_{-\infty}^{0} dz \frac{f(z)^{1-n}}{z^{m+2}} \frac{\partial S_y}{\partial f_m} \quad n \geq 1.
\]

In particular

\[
S_{L_1}y = \frac{\partial S_y}{\partial f_{-1}}
\]

\[
S_{L_2}y = \sum_{m \leq -2} \int_{-\infty}^{0} dz \frac{1}{f(z)z^{m+2}} \frac{\partial S_y}{\partial f_m}.
\]

The study of the case $n < 1$ follows closely the discussion in section 6.1.1.

As it plays no role in the application to SLE we leave the computation to the reader.
6.2.3 Application to SLE martingales

We assume that $c = \frac{(6-\kappa)(8\kappa-3)}{2\kappa}$ and $h = \frac{6-\kappa}{2\kappa}$ for some $\kappa$. Then $V(c, h)$ is not irreducible, $(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)x$ is a singular vector in $V(c, h)$, annihilated by the $L_n$'s, $n \geq 1$, so that it does not couple to any descendant of $x^*$, the dual of $x$. The descendants of $x^*$ in $V^*(c, h)$ generate the irreducible highest weight representation of weight $(c, h)$. We denote by $R_n$ (resp. $S_n$) the differential operator such that $\langle L_n y, G_f x \rangle = R_n \langle y, G_f x \rangle$ (resp. $\langle L_n y, G_f^{-1} x \rangle = S_n \langle y, G_f^{-1} x \rangle$) for $y \in V^*(c, h)$. Now for $n \geq 1$, $\langle y, G_f L_{-n} x \rangle = -S_n \langle y, G_f x \rangle$. If $y$ is a descendant of $x^*$,

$$\langle y, G_f(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)x \rangle = 0$$

All the polynomials in $f_{-1}, f_{-2}, \cdots$ obtained by acting repeatedly on the polynomial 1 with the $R_m$'s (they build the irreducible representation with highest weight $(c, h)$) are annihilated by $2S_2 + \frac{\kappa}{2}S_1$. For generic $\kappa$ there is no other singular vector in $V(c, h)$, and this leads to a satisfactory description of the irreducible representation of highest weight $(h, c)$: the representation space is given by the kernel of an explicit differential operator acting on $q[f_{-1}, f_{-2}, \cdots]$, and the states are built by repeated action of explicit differential operators (the $R_m$'s) on the highest weight state 1.

We are now in position to rephrase the main results of [3] in the language of this paper. If we take $f = k_t$, the coefficients $f_{-1}, f_{-2}, \cdots$ of $f$ become random functions (for instance $f_{-1}$ is simply a Brownian motion of covariance $\kappa$). One can show (see [3] for details) that for fixed $t$ the coefficients $f_{-1}, f_{-2}, \cdots$ seen as functions over the Wiener sample space are algebraically independent.

So the above computation can be interpreted as follows: the space of polynomials of the coefficients of the expansion of $k_t$ at $\infty$ for SLE$_\kappa$ can be endowed with a Virasoro module structure isomorphic to $V^*(c_\kappa, h_\kappa)$. Within that space, the subspace of martingales is a submodule isomorphic to the irreducible highest weight representation of weight $(c_\kappa, h_\kappa)$.

7 “Wick’s theorem” for the Virasoro algebra

Up to now, we have only dealt with finite deformations close to 0 or $\infty$. These are the most natural points for radial quantization in conformal field theory. However, this is not always convenient. A typical situation is as depicted in fig. [2].

We want to evaluate correlation of operators in a geometry where the natural series at 0 or at $\infty$ either do not exists at all, or do not converge at
7.1 Basic commutative diagram

In this situation, we may obtain a uniformizing map \( f_{A \cup B} \) by first removing \( B \) by \( f_B \), which is regular around \( \infty \) and such that \( f_B(z) = z + O(1) \) at infinity, then \( \tilde{A} \equiv f_B(A) \) by \( f_{\tilde{A}} \) which is regular around \( 0 \) and fixes \( 0 \) (as mentioned before, \( f_{\tilde{A}}'(0) \neq 1 \) is allowed). Suppose that \( B \) is included in an open ball of radius \( r \) and \( \tilde{A} \) is included in the complement of a closed ball of radius \( R \), both centered at the origin. Now choose \( z \) such that \( |z| > r \) but \( |f_B(z)| < R \). For such \( z \)'s, first the composition \( f_{A \cup B}(z) = f_{\tilde{A}} \circ f_B(z) \) can be computed by inserting the series expansions, and second \( G_{f_{\tilde{A}}}^{-1}\left(G_{f_B}^{-1}T(z)G_{f_B}\right)G_{f_{\tilde{A}}} \) is well defined, given by absolutely convergent series, and is equal to \( T(f_{A \cup B}(z)) f_{A \cup B}^{-1}(z)^2 + \frac{c}{12} S f_{A \cup B}(z) \).

As they uniformize the same domain, we know that \( f_{\tilde{A}} \circ f_B \) and \( f_B \circ f_{\tilde{A}} \) differ by a (real) linear fractionnal transformation : there is an \( h \in \text{PSL}_2(\mathbb{R}) \) such that \( f_{\tilde{A}} \circ f_B = h \circ f_B \circ f_{\tilde{A}} \). Suppose that \( f_{\tilde{A}} \) and \( f_B \) are given. There is some freedom in the choice of \( f_{\tilde{A}} \) and \( f_B \) : namely we can replace \( f_{\tilde{A}} \) by \( h_0 \circ f_{\tilde{A}} \) where \( h_0 \) is a linear fractionnal transformation fixing \( 0 \), and \( f_B \) by \( h_\infty \circ f_B \) where \( h_\infty \) is a linear fractionnal transformation such that \( h_\infty(z) = z + O(1) \) at infinity, i.e. a translation. A simple computation shows that unless there is a \( z \) such that \( f_A(z) = \infty \) and \( f_B(z) = 0 \), there is a unique choice of \( f_{\tilde{A}} \) and \( f_B \) such that \( f_{\tilde{A}} \circ f_A = f_B \circ f_{\tilde{A}} \). In the sequel, we shall concentrate on this generic situation. So we deduce that for \( z \)'s in some open set, \( G_{f_{\tilde{A}}}^{-1}\left(G_{f_B}^{-1}T(z)G_{f_B}\right)G_{f_{\tilde{A}}} = G_{f_B}^{-1}\left(G_{f_{\tilde{A}}}^{-1}T(z)G_{f_{\tilde{A}}}\right)G_{f_B} \). As the modes

\[ \text{Figure 2: A typical two hulls geometry.} \]
$L_n$ of $T$ generate all states in a highest weight representation, the operators $G_{f_B}f_A$ and $G_{f_A}f_B$ have to be proportional: they differ at most by a factor involving the central charge $c$. We write

$$G_{f_B}G_{f_A} = Z(A, B) G_{f_A}G_{f_B},$$

or

$$G_{f_A}^{-1}G_{f_B} = Z(A, B) G_{f_B}^{-1}G_{f_A}^{-1}.$$  \(12\)

As implicit in the notation, $Z(A, B)$ depends only on $A$ and $B$ : a simple computation shows that it is invariant if $f_A$ is replaced by $h_0 \circ f_A$ and $f_B$ by $h_\infty \circ f_B$. Formula \(12\) plays for the Virasoro algebra the role that Wick’s theorem plays for collections of harmonic oscillators.

We call $Z(A, B)$ a partition function for the following reason: we can write

$$\langle \Omega \left| G_{f_A}^{-1}G_{f_B}^{-1} \left( \cdots \hat{T}(z) \cdots \right)_r G_{f_B}G_{f_A} \right| \Omega \rangle = \frac{1}{Z(A, B)} \langle \Omega \left| G_{f_B}^{-1}G_{f_A}^{-1} \left( \cdots \hat{T}(z) \cdots \right)_r G_{f_B}G_{f_A} \right| \Omega \rangle$$

But $|\Omega\rangle$ is annihilated by $b_+$ and $\langle \Omega |$ is annihilated by $n_-$ so

$$\langle \cdots T(z) \cdots \rangle_{H_{A\cup B}} = \frac{1}{Z(A, B)} \langle \Omega \left| G_{f_A}^{-1} \left( \cdots \hat{T}(z) \cdots \right)_r G_{f_B} \right| \Omega \rangle,$$

and

$$Z(A, B) = \langle \Omega \left| G_{f_A}^{-1}G_{f_B} \right| \Omega \rangle.$$
7.2 Computation of the partition function

The computation of $Z(A, B)$ goes along the following lines. If one changes $A$ by a small amount, the variation of $f_A$ can be written as $\delta f_A = v_A(f_A)$. In order to keep the initial properties of $A$ and $B$, we impose that $v_A$ is a vector field holomorphic in the full plane but for cuts along the real axis, satisfies the Schwartz reflexion principle ($v_A(\overline{z}) = v_A(z)$), and is such that the open disk of convergence of its power series expansion at 0 contains $\overline{B}$. Similar considerations hold if $B$ is distorted slightly, we write $\delta f_B = v_B(f_B)$ and $v_B$ satisfies corresponding conditions. Then we know that

$$\delta(G_{f_A}^{-1}G_{f_B}) = \oint_0 v_A(u)T(u)duG_{f_A}^{-1}G_{f_B} - G_{f_A}^{-1}G_{f_B} \oint_\infty v_B(v)T(v)dv$$

By hypothesis, we can deform the small contour around 0 to a contour in a region where $v_A$ and $f_B$ have a convergent expansion, and the small contour around $\infty$ to a contour in a region where $v_B$ and $f_A$ have a convergent expansion. Then we may conjugate, with the result

$$\frac{\delta(G_{f_A}^{-1}G_{f_B})}{Z(A, B)} = G_{f_B}(\oint v_A(u)(T(f_B(u)))f_B(u)^2 + \frac{c}{12}Sf_B(u))du$$

Taking the vacuum expectation value yields

$$\delta \log Z(A, B) = \frac{c}{12} \left( \oint v_A(u)Sf_B(u)du - \oint v_B(v)Sf_A(v)dv \right).$$

The explicit value of $\log Z(A, B)$ can be computed by means of several formulæ.

The most symmetrical ones are obtained if $A$ and $B$ are both described by integrating infinitesimal deformations of $\mathbb{H}$. Consider two families of hulls, $A_s$ and $B_t$ that interpolate between the trivial hull and $A$ or $B$ respectively. We arrange that $f_{A_s}$ and $f_{B_t}$ satisfy the genericity condition, so that unique $f_{A_{s,t}}$ and $f_{B_{t,s}}$ exist, which satisfy $f_{B_{t,s}} \circ f_{A_{s,t}} = f_{A_{s,t}} \circ f_{B_t}$.

Define vector fields by $v_{A_s}$ and $v_{B_t}$ by $\frac{\partial f_{A_s}}{\partial s} = v_{A_s}(f_{A_s})$ and $\frac{\partial f_{B_t}}{\partial t} = v_{B_t}(f_{B_t})$. Now set $A_{s,t} = f_{A_s}(A_t)$ and $B_{t,s} = f_{A_s}(B_t)$, and define vector fields $v_{A_{s,t}}$ and $v_{B_{t,s}}$ by $\frac{\partial f_{A_{s,t}}}{\partial s} = v_{A_{s,t}}(f_{A_{s,t}})$ and $\frac{\partial f_{B_{t,s}}}{\partial t} = v_{B_{t,s}}(f_{B_{t,s}})$. Set

$$L(A_s, B_t) \equiv \int_0^s ds \int_0^t dt \oint_{\Gamma_w} dw \oint_{\Gamma_z} dz v_{A_{s,t}}(w) \frac{6}{(z-w)^4} v_{B_{t,s}}(z)$$

27
where the contours $\Gamma_w$ and $\Gamma_z$ are simple contours in $\mathbb{C}$ of index 1 with respect to 0, such that the bounded component of $\mathbb{C}\setminus \Gamma_z$ contains the cuts of $f_{B_{t,s}}^{-1}$, the bounded component of $\mathbb{C}\setminus \Gamma_w$ contains $\Gamma_z$, and the unbounded component contains the cuts of $f_{A_{s,t}}^{-1}$ as described on fig.4. We observe that the kernel is a four order pole, i.e. is proportional to the two-point correlation function for the stress energy tensor in the plane geometry. We claim that

$$Z(A_{\sigma}, B_{\tau}) = \exp \frac{c}{12} L(A_{\sigma}, B_{\tau}).$$

This formula is very symmetrical, but it does not make clear that $\log Z(A_{\sigma}, B_{\tau})$ really depends only on $A_{\sigma}$ and $B_{\tau}$, not on the full trajectories $A_s$, $s \leq \sigma$ and $B_t$, $t \leq \tau$. The following steps are also useful to show that eq.(13) has the correct variationnal derivative.

We start by the change of variable $z = f_{A_{s,t}}(\zeta)$, which is valid for $z$ in a simply connected neighborhood of $\Gamma_w$ containing the origin, hence on $\Gamma_z$. Taking the $t$-derivative of $f_{B_{t,s}} \circ f_{A_{s,t}} = f_{A_{s,t}} \circ f_{B_{t}}$, we obtain

$$v_{B_{t,s}}(f_{A_{s,t}}(\zeta)) = \frac{\partial f_{A_{s,t}}(\zeta)}{\partial t} + f'_{A_{s,t}}(\zeta)v_{B_{t}}(\zeta).$$

But $\frac{\partial f_{A_{s,t}}(\zeta)}{\partial t}$ is a holomorphic function of $\zeta$ in a neighborhood of the origin containing the $\zeta$ integration contour, so in eq.(13) we may replace $v_{B_{t,s}}(z)dz$ by $f'_{A_{s,t}}(\zeta)v_{B_{t}}(\zeta)f'_{A_{s,t}}(\zeta)d\zeta$. Hence

$$L(A_{\sigma}, B_{\tau}) = \int_0^\sigma d s \int_0^t d t \int_{\Gamma_w} dw \int_{f_{A_{s,t}}^{-1}(\Gamma_z)} d\zeta \ v_{A_{s,t}}(w) \frac{6f'_{A_{s,t}}(\zeta)^2}{(f_{A_{s,t}}(\zeta) - w)^4} v_{B_{t}}(\zeta)$$

Figure 4: Integration contours intrication.

$$L(A_{\sigma}, B_{\tau}) = \int_0^\sigma d s \int_0^t d t \int_{\Gamma_w} dw \int_{f_{A_{s,t}}^{-1}(\Gamma_z)} d\zeta \ v_{A_{s,t}}(w) \frac{6f'_{A_{s,t}}(\zeta)^2}{(f_{A_{s,t}}(\zeta) - w)^4} v_{B_{t}}(\zeta)$$

28
\[ = - \int_0^\sigma ds \int_0^\tau dt \oint_{f_{A_{s,t}}^{-1}(\Gamma_z)} d\zeta \, v''_{A_{s,t}}(f_{A_{s,t}}(\zeta)) f'_{A_{s,t}}(\zeta)^2 v_B(\zeta) \] (14)

In the second line, the \( w \) integral has been computed by the residue formula. This is legitimate because, by hypothesis, \( v_{A_{s,t}}(w) \) is holomorphic in the bounded component of \( \mathbb{C} \setminus \Gamma_w \).

We can now make use of a useful identity for the variations of the Schwartzian derivative. From its definition one checks that

\[ S(f + \varepsilon v(f))(f) = \varepsilon v'''(f) + O(\varepsilon^2) \]

Combined with the cocycle property \( S(f + \varepsilon v(f))(z)dz^2 = S(f + \varepsilon v(f))(f)df^2 + S(f)(z)dz^2 \) this yields

\[ \frac{d}{d\varepsilon} S(f + \varepsilon v(f))(z)_{\varepsilon=0} = v'''(f(z))f'(z)^2. \]

Finally

\[
L(A_\sigma, B_\tau) = - \int_0^\sigma ds \int_0^\tau dt \oint_{f_{A_{s,t}}^{-1}(\Gamma_z)} d\zeta \, \frac{d}{ds} Sf_{A_{s,t}}(\zeta)v_B(\zeta)
\]

\[ = - \int_0^\tau dt \oint_{f_{A_{s,t}}^{-1}(\Gamma_z)} d\zeta \, Sf_{A_{s,t}}(\zeta)v_B(\zeta). \] (15)

The roles of \( A_\sigma \) and \( B_\tau \) could be interchanged to remove the \( \Gamma_w \) and \( t \) integrations, leading to

\[ L(A_\sigma, B_\tau) = \int_0^\sigma ds \oint dw \, v_{A_s}(w)Sf_{B_{\tau,s}}(w) \]

\[ = - \int_0^\tau dt \oint dz \, v_B(z)Sf_{A_{\sigma,t}}(z). \]

Using these formulæ, it is apparent that \( L(A_\sigma, B_\tau) \) does not depend on the detailed way the hulls are built: only the final hulls count. It is also clear that setting \( A \equiv A_\sigma, A \cup \delta A = A_{\sigma + d\sigma}, B \equiv B_\tau, B \cup \delta B = B_{\tau + d\tau}, \) the variation of \( \frac{c}{12} L(A_\sigma, B_\tau) \) is exactly the one of \( \log Z(A, B) \). So we have proved

\[ \log Z(A_\sigma, B_\tau) = \frac{c}{12} \int_0^\sigma ds \oint dw \, v_{A_s}(w)Sf_{B_{\tau,s}}(w) \]

\[ = - \frac{c}{12} \int_0^\tau dt \oint dz \, v_B(z)Sf_{A_{\sigma,t}}(z). \]

### 7.3 Two explicit computations

Let \( a \) and \( b \) be real positive numbers
7.3.1 Example 1: two slits

We define the hull $B_b$ to be the segment $\{i0, ib\}$ and $A_a$ the segment $\{ia, i\infty\}$ in $\mathbb{H}$. Assuming that $0 < b < a \leq \infty$ we compute $L(A_a, B_b)$.

We interpolate between the empty hull and $B_b$ (resp. $A_a$, $\alpha \in [a, \infty]$). To uniformize $\mathbb{H} \setminus B_\beta$ we take the map $f_{B_\beta}(z) = (z^2 + \beta^2)^{1/2}$ and for $\mathbb{H} \setminus A_\alpha$ the map $f_{A_\alpha}(z) = (z^{-2} + \alpha^{-2})^{-1/2}$. Observe that $f_{B_\beta}$ maps $A_\alpha$ to $A_\gamma$ where $\gamma = (\alpha^2 - \beta^2)^{1/2}$ while $f_{A_\alpha}$ maps $B_\beta$ to $B_\delta$, where $\delta = \frac{\alpha\beta}{(\alpha^2 - \beta^2)^{1/2}}$. One checks that $f_{B_\delta} \circ f_{A_\alpha} = \frac{1}{1 - b^2/a^2} f_{A_\gamma} \circ f_{B_\beta}$, so we get a commutative diagram by taking $f_{A_\alpha, \beta} = \frac{1}{1 - b^2/a^2} f_{A_\gamma}$ and $f_{B_\beta, \alpha} = f_{B_\beta}$.

Now $S f_{A_\alpha, \beta}(z) = S f_{A_\gamma}(z) = -\frac{3(z^2 + 2 \gamma^2)}{2(z^2 + \gamma^2)^2}$ so $S f_{A_\alpha, \beta}(z) = -\frac{3(z^2 + 2(a^2 - \beta^2))}{2(z^2 + a^2 - \beta^2)^2}$.

On the other hand $\frac{d}{\beta} f_{B_\beta} = \beta f_{B_\beta}$ so $v_{B_\beta}(z) = \frac{\beta}{z}$. To resume,

$$L(A_a, B_b) = -\int_0^b d\beta \int dz v_{B_\beta}(z) S f_{A_\alpha, \beta}(z) = \int_0^b d\beta \int dz \frac{\beta}{z} \frac{3(z^2 + 2(a^2 - \beta^2))}{2(z^2 + a^2 - \beta^2)^2}.$$

The relevant $z$-integral encircles the singularity at 0 and no other, so $L(A_a, B_b) = 3 \int_0^b d\beta \frac{\beta}{z^2 - \beta^2}$. Finally

$$L(A_a, B_b) = -\frac{3}{2} \log(1 - b^2/a^2).$$

7.3.2 Example 2: a slit and a half disc

We keep the definitions above for $A_a$, $A_\alpha$, $\alpha \in [a, \infty]$ and $f_{A_\alpha}$. But now

$B_b$ is the intersection of the disc of center 0 and radius $b$ with $\mathbb{H}$, and we interpolate between the empty hull and $B_b$ we use the half discs $B_\beta$, $\beta \in [0, b]$.

To uniformize $\mathbb{H} \setminus B_\beta$ we choose the map $f_{B_\beta}(z) = z + \beta^2/z$. Observe that $f_{B_\beta}$ maps $A_\alpha$ to $A_\gamma$ where now $\gamma = (\alpha^2 - \beta^2)/\alpha$. The Schwartzian derivative is insensitive to the precise normalization of $f_{A_\alpha, \beta}$, so we can compute it by using $f_{A_\alpha}$: $S f_{A_\alpha, \beta}(z) = -\frac{3(z^2 + 2(a^2 - \beta^2)^2/a^2)}{2(z^2 + (a^2 - \beta^2)^2/a^2)^2}$. On the other hand $\frac{d}{\beta} f_{B_\beta} = \frac{f_{B_\beta} - \sqrt{f_{B_\beta}^2 - 4z^2}}{\beta}$ so $v_{B_\beta}(z) = \frac{z - \sqrt{z^2 - 4\beta^2}}{\beta}$, where the square root is defined to ensure the appropriate properties of $v_{B_\beta}$: this vector field is holomorphic in $\mathbb{H}$ with negative imaginary part, real on the real axis away from the cut and satisfies the Schwarz reflection principle. Hence

$$L(A_a, B_b) = \int_0^b d\beta \int dz \frac{z - \sqrt{z^2 - 4\beta^2}}{\beta} \frac{3(z^2 + 2(a^2 - \beta^2)^2/a^2)}{2(z^2 + (a^2 - \beta^2)^2/a^2)^2}.$$

The relevant $z$-integral encircles the cut $[-2\beta, 2\beta]$ and no other singularity. We may compute it with the help of the residue formula, because
the integrand is meromorphic in the unbounded component of the complement of the integration contour, regular at infinity but with double poles at \( z = \pm i(a^2 - \beta^2)/a \). The index is \(-1\) for both, and the residue is the same as well. This leads to

\[
L(A_b, B_b) = 3 \int_0^b \frac{d\beta \beta^2(\beta^2 + 2a^2)}{\beta^4 - \beta^4}.
\]

Finally

\[
L(A_a, B_b) = \frac{3}{4} \log \frac{1 + b^2/a^2}{(1 - b^2/a^2)^3}.
\]

We observe in these two examples that \( L(A, B) \) becomes singular when \( A \) and \( B \) have a contact. We also observe that \( L(A, B) \) is positive. There is a good reason for that.

### 7.4 Factorisation of unity and Virasoro vertex operators.

Consider a hull \( A \) whose closure does contain neither the origin nor the infinity. There is a one parameter family of maps uniformizing the complement of \( A \) in \( \mathbb{H} \) and which are regular both at the origin and at infinity. Let us pick one of them, which we call \( f_A(z) \). Since \( f_A(z) \) is regular at the origin, we may implement it in conformal field theory by \( G_{A+} f'_A(0)^{-L_0} \) with \( G_{A+} \) in \( \mathcal{N}_+ \). Alternatively, since it is also regular at infinity, we may implement it by \( G_{A-} f'_A(\infty)^{-L_0} \) with \( G_{A-} \in \mathcal{N}_- \). The product

\[
\mathcal{V}_A \equiv G_{A-} f'_A(\infty)^{-L_0} f'_A(0)^{L_0} G_{A+}^{-1}
\]

is the Virasoro analogue of what vertex operators of dual or string models are for the Heisenberg or the affine Kac-Moody algebras. It does not depend on the representative one chooses in the one parameter family. This product is well defined and non trivial in positive energy representation. It may be thought of as the factorization of the identity since the conformal transformation it implements is the composition of two inverse conformal maps.

### 8 The partition function martingale

We now come to the application that has motivated most of our investment in the explicit implementation of conformal transformations. For the convenience of the reader, we start with a quick reminder of \( \mathbb{H} \) phrased in a
more rigorous setting. Remember that \( c_\kappa = \frac{(6-\kappa)(8\kappa-3)}{2\kappa} \) and \( h_\kappa = \frac{6-\kappa}{2\kappa} \). The Verma module \( V(c_\kappa, h_\kappa) \) is not irreducible, and \((-2L_{-2} + \frac{\kappa}{2}L^2_{-1})\) acting on the highest weight state is another highest weight generating a subrepresentation. We quotient \( V(c_\kappa, h_\kappa) \) by this subrepresentation and denote by \(|\omega\rangle\) the highest weight state in the quotient. Then \((-2L_{-2} + \frac{\kappa}{2}L^2_{-1})|\omega\rangle = 0\).

### 8.1 Ito’s formula for \( G_{k_t} \)

The maps \( f_t \) and \( k_t = f_t - \xi_t \) that uniformize the growing hull \( K_t \) fix the point at infinity, so that there are well defined elements \( G_f, G_k \in \mathcal{N}_- \subset \mathcal{U}(\mathfrak{n}_-) \) implementing them in CFT. The maps are related by a change of the constant coefficient in the expansion around \( \infty \), so the operators are related by \( G_k = G_{f_t} e^{\xi_t L^{-1}} \). The map \( f_t \) satisfies the ordinary differential equation \( \partial_t f_t(z) = \frac{2}{f_t(z)-\xi_t} \), the corresponding vector field being \( v(f) = \frac{2}{f-\xi_t} \) whose expansion at infinity reads \( v(f) = 2 \sum_{m \leq -2} f^{m+1} \xi_t^{-m-2} \), so that

\[
G_{f_t}^{-1} dG_{f_t} = -2 dt \sum_{m \leq -2} L_m \xi_t^{-m-2} = -2 e^{\xi_t L^{-1}} L_{-2} e^{-\xi_t L^{-1}} dt.
\]

To get \( G_{k_t}^{-1} dG_{k_t} \) it remains only to compute the Ito derivative of \( e^{\xi_t L^{-1}} \) which reads \( e^{-\xi_t L^{-1}} d e^{\xi_t L^{-1}} = L_{-1} d \xi_t + \frac{\kappa}{2} L^2_{-1} dt \). Finally,

\[
G_{k_t}^{-1} dG_{k_t} = (-2L_{-2} + \frac{\kappa}{2}L^2_{-1}) dt + L_{-1} d \xi_t
\]

as announced in section \( \text{III} \).

In particular, \( dG_{k_t} |\omega\rangle = L_{-1} d \xi_t G_{k_t} |\omega\rangle \), so that \( G_{k_t} |\omega\rangle \) is a (generating function of) local martingale(s).

### 8.2 The partition function martingale

We have given an explicit formula for \( Z(A, B) \), but motivated by the martingale generating function, we shall sandwich \( G_{f_A}^{-1} G_{f_B} \) not with the vacuum \( \Omega \) but with another highest weight state, namely \( |\omega\rangle \). Using the Virasoro Wick theorem, one computes that (remember that \( \langle \omega | \) is annihilated by \( \mathfrak{n}_- \), but \( |\omega\rangle \) is not annihilated by \( \mathfrak{b}_+ \), the \( L_0 \) part contributes)

\[
\langle \omega | G_{f_A}^{-1} G_{f_B} |\omega\rangle = Z(A, B) \langle \omega | G_{f_A}^{-1} |\omega\rangle = Z(A, B) f_A^t(0)^{h_\kappa}.
\]

Observe that while the vacuum expectation value depends only on the hulls, the expectation value in a non conformally invariant state depends on the choices of \( f_A \) and \( f_B \).
We apply the results of section 7 to the case when $B$ is the growing hull $K_t$ and $A$ is another disjoint hull. From the previous computation we know that $\langle \omega | G_f^{-1} G_k \omega \rangle$ is a local martingale.

We start from $f_A$ and $f_t$ to build a commutative diagram as before, with maps denoted by $\tilde{f}_A$ and $\tilde{f}_t$ uniformizing respectively $f_t(A)$ and $f_A(K_t)$, and satisfying $\tilde{f}_t \circ f_A = f_{\tilde{A}_t} \circ f_t$. Now

\[
\langle \omega | G_f^{-1} G_k \omega \rangle = \langle \omega | G_f^{-1} G_f e^{\xi t L^{-1}} \omega \rangle = Z(A, K_t) \langle \omega | G_{f_{\tilde{A}_t}} e^{f_{\tilde{A}_t}(\xi_t) L^{-1}} \omega \rangle = Z(A, K_t) \langle \omega | \left( e^{-\xi t L^{-1}} G_{f_{\tilde{A}_t}} e^{f_{\tilde{A}_t}(\xi_t) L^{-1}} \right)^{-1} \omega \rangle.
\]

From eq. (5) we know that the operator $e^{-\xi t L^{-1}} G_{f_{\tilde{A}_t}} e^{f_{\tilde{A}_t}(\xi_t) L^{-1}}$ corresponds to the map $z \mapsto f_{\tilde{A}_t}(\xi_t + z) - f_{\tilde{A}_t}(\xi_t)$, so that

\[
\langle \omega | \left( e^{-\xi t L^{-1}} G_{f_{\tilde{A}_t}} e^{f_{\tilde{A}_t}(\xi_t) L^{-1}} \right)^{-1} \omega \rangle = \langle \omega | \left( e^{-\xi t L^{-1}} G_{f_{\tilde{A}_t}} e^{f_{\tilde{A}_t}(\xi_t) L^{-1}} \right)^{-1} \omega \rangle = f'_{\tilde{A}_t}(\xi_t) h_\kappa.
\]

From the Loewner equation $v_{K_t}(z) = \frac{2}{z-\xi_t}$ and

\[
L(A, K_t) = -\int_0^t d\tau \oint dz \frac{2}{z - \xi_\tau} S f_{A_\tau}(z)
\]

Finally

\[
\langle \omega | G_f^{-1} G_k \omega \rangle = f'_{\tilde{A}_t}(\xi_t) h_\kappa \exp -\frac{c}{6} \int_0^t d\tau S f_{A_\tau}(\xi_\tau),
\]

were $f_A \circ f_\tau$ uniformizes the two hull geometry corresponding to $A \cup K_\tau$ and $f_{A_\tau}$ is normalized to ensure the commutativity of the uniformization diagram as explained before. It should be noted that the randomness in the above formula is explicit through the appearance of $\xi_t$ but also implicit through $f_{\tilde{A}_t}$ which is a random function.

This local martingale was discovered without any recourse to representation theory by Lawler, Schramm and Werner [13], but we hope to have convinced the reader that it is nevertheless deeply rooted in CFT.

For the sake of completeness, we shall give two illustration of how this machinery is used to compute explicit probabilities. The following discussion does not claim originality, as the derivations merely sketch the ones given in [13].
8.3 Restriction

We already know that

\[ \langle \omega | G_{f_A}^{-1} G_{k_t} | \omega \rangle \]

is a local martingale. One can show that it is a true martingale for \( \kappa \leq 4 \), let us just note that the region \( \kappa \leq 4 \) is also the one for which, almost surely, the SLE hull \( K_t \) is a simple curve that avoids the real axis at all positive times. For the rest of this section assume \( \kappa \leq 4 \).

Suppose that \( A \) is bounded and choose a very large semi circle \( \mathcal{C}_R \) of radius \( R \) in \( \mathbb{H} \) centered at the origin. Let \( \tau_R \) be the first time when \( K_t \) touches either \( A \) or \( \mathcal{C}_R \). Then \( \tau_R \) is a stopping time. It is crucial to normalize \( f_A \) correctly, and one does so by imposing that it fixes 0 (as already done) and that moreover \( f_A(z) = z + O(1) \) close to \( \infty \), which by use of the commutative diagram ensures that ensures that \( \hat{f}_A \) close to \( \infty \) as well. These three conditions fix \( f_A \) completely. Then we claim that \( f_A' (\xi_t |_{\tau_R}) \) is 0 if the SLE hull hits \( A \) at \( \tau_R \) and goes to 1 for large \( R \) if the SLE hull hits \( \mathcal{C}_R \) at \( \tau_R \). Indeed, when the hull approaches \( A \), one or more points on \( \hat{A}_t \) approach \( \xi_t \), and at the hitting time, a bounded connected component is swallowed \( \xi_t \) (this uses the normalization of \( f_A \)) indicating that the derivative has to vanish there. On the other hand, if \( \mathcal{C}_R \) is hit first, then \( \hat{A}_\tau \) is dwarfed so that (this uses again the normalization of \( f_A \)) \( f_{\hat{A}_\tau} \) is close to the identity map away from \( \hat{A}_\tau \) and in particular at the point \( \xi_{\tau_R} \). The behaviour of the other factor in the martingale, \( Z(A, K_t) \), is much harder to control, so we now restrict to \( \kappa = 8/3 \), which is the same as \( c_\kappa = 0 \) because \( \kappa \leq 4 \). So the partition function martingale \( f_{\hat{A}_\tau} (\xi_t)^{\kappa/3} \), at \( t = \tau_R \) is 0 if \( A \) is hit before \( \mathcal{C}_R \) and close to 1 if the opposite is true. But the expectation of a martingale is time independent, so that the probability that \( K_t \) hits \( A \) is \( f_{\hat{A}_\tau} (\xi_t)^{\kappa/3} \bigg|_{t=0} = f_A(0)^{5/8} \).

8.4 Locality

Let us consider again the case when \( B \) is the SLE hull \( K_t \) and \( A \) another disjoint hull. We may apply the Virasoro Wick theorem to \( G_{f_A}^{-1} G_{k_t} \) to get

\[ G_{f_A}^{-1} G_{k_t} = Z(A, K_t) \ G_{k_t} \ G_{f_{\hat{A}_t}}^{-1} \]

Here \( \tilde{k}_t \) is a uniformizing map of the image of the SLE hull by \( f_A \) and it defines the SLE growth in \( \mathbb{H} \setminus A \). Its lift \( \hat{G}_{\tilde{k}_t} \) in \( \mathcal{N}_- \) depending locally on \( k_t \) is random. A simple computation shows that its Ito derivative is

\[ G_{\tilde{k}_t}^{-1} dG_{\tilde{k}_t} = (-2L_{-2} + \frac{\kappa}{2} L_{-1}^2) f_{\hat{A}_t}'(0)^2 dt + \frac{\kappa - 6}{2} L_{-1} f_{\hat{A}_t}''(0) dt + L_{-1} f_{\hat{A}_t}'(0) d\xi_t \]
Hence, for $\kappa = 6$, $G_{k_t}$ is statistically equivalent to $G_{k_t}$ up to a time reparametrisation, $dt \to ds = f_{A_t}'(0)^2 dt$. This expresses the locality property of critical percolation.

A Proof of identity (I)

We start with the proof of eq.(1): the operators $A_m \equiv \sum_{n \geq m} L_n \oint_0 dw^{m+1} \frac{f'(w)}{f(w)^{n+2}}$ satisfy the zero curvature equation

$$\frac{\partial A_l}{\partial f_k} - \frac{\partial A_k}{\partial f_l} = [A_k, A_l].$$

Integration by parts gives

$$\frac{\partial}{\partial f_l} \oint_0 dw^{m+1} \frac{f'(w)}{f(w)^{n+2}} = \frac{m+1}{n+1} \frac{\partial}{\partial f_l} \oint_0 dw^m \frac{1}{f(w)^{n+1}} = -(m+1) \oint_0 dw \frac{w^{l+m+1}}{f(w)^{n+2}}$$

so

$$\frac{\partial A_l}{\partial f_k} - \frac{\partial A_k}{\partial f_l} = (k - l) \sum_j L_j \oint_0 dw \frac{w^{k+l+1}}{f(w)^{j+2}}.$$

On the other hand,

$$[A_k, A_l] = \sum_{m,n} (m - n) L_{m+n} \oint_0 du^k \frac{f'(u)}{f(u)^{m+2}} \oint_0 dv^{l+1} \frac{f'(v)}{f(v)^{n+2}}.$$

Split this sum in two pieces by splitting $m - n = (m + 1) - (n + 1)$. In the sum involving $m + 1$ use

$$(m + 1) \oint_0 du^k \frac{f'(u)}{f(u)^{m+2}} = (k + 1) \oint_0 du \frac{u^k}{f(u)^{m+1}}.$$

In the sum involving $n + 1$ use

$$(n + 1) \oint_0 dv^{l+1} \frac{f'(v)}{f(v)^{n+2}} = (l + 1) \oint_0 dv \frac{v^l}{f(v)^{n+1}},$$

interchange the dummy variables $m$ and $n$, and also $u$ and $v$. This leads to

$$[A_k, A_l] = \sum_{m,n} L_{m+n} \oint_0 du \oint_0 dv \frac{f'(v)((k + 1)u^k v^{l+1} - (l + 1)u^t v^{k+1})}{f(u)^{m+1} f(v)^{n+2}}.$$
Up to now, the contours in the $u$ and $v$ planes were independent. But if they are adjusted in such a way that $|f(v)| < |f(u)|$, we can fix $j = m + n$ and sum over $m$ to obtain

$$[A_k, A_l] = \sum_j L_j \int_0^1 du \int_0^1 dv f'(v) ((k + 1)u^{k+1} - (l + 1)u^{l+1}) (f(u) - f(v))f(v)^{j+2}.$$  

Inside the $u$-plane contour, the singularities of the $u$-integrand consist now in a simple pole at $u = v$, and taking the residue leads to

$$[A_k, A_l] = (k - l) \sum_j L_j \int_0^1 dw w^{k+l+1} \frac{f'(w)^2 f(w)^{n+2}}{f(w)^{j+2}} = \frac{\partial A_l}{\partial f_k} - \frac{\partial A_k}{\partial f_l}.$$  

This concludes the proof.

**B Proof of identity (3)**

We continue with the proof of eq. (3):

$$G_j^{-1} L_m G_f = \frac{c}{12} \int_0^1 dw w^{m+1} S f(w) + \sum_{n \geq m} L_n \int_0^1 dw w^{n+1} \frac{f'(w)^2}{f(w)^{n+2}} \quad m \in \mathbb{Z}.$$  

Observe that if we extend the summation over all $n$'s, the integrals with $n < m$ vanish anyway. Defining $L_m(f)$ to be the right-hand side, one way to prove this identity could be the tedious check that both sides have the same variation when $f$ is changed into $f + \varepsilon z^{k+1}$, i.e.

$$\frac{\partial L_m(f)}{\partial f_k} = \left[ \sum_{l \geq k} L_l \int_0^1 dw w^{k+l+1} \frac{f'(w)^2}{f(w)^{l+2}}, L_m(f) \right] \quad k \geq 1.$$  

This can be done, but it is simpler to consider the variation of $G_f$ and $L_m(f)$ when $f$ is changed to $f + \varepsilon v(f)$. If $v(f) = \sum_{l \geq 1} v_l f^{l+1}$, we know that the variation of $G_f$ is $-G_f \sum_{l \geq 1} v_l L_l$. Now

$$\left[ \sum_{l \geq 1} v_l L_l, \sum_{m \geq n} L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n+2}} \right] = \sum_{l \geq 1} v_l \sum_{n \geq 1} ((l - n)L_{l+n}$$

$$+ \frac{c}{12} \delta_{l+n,0} (l^3 - l) \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n+2}} \right]$$

For the term involving the central charges we sum over $n$, then $l$ and get

$$\varepsilon f_0 dw w^{m+1} f'(w)^2 v''(f(w)).$$

For the remaining terms, for fixed $l$ we replace
the dummy variable $n$ by $n - l$, leading to

$$\sum_{l \geq 1} v_l \sum_n (2l - n) L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n-l+2}},$$

which is the same as

$$\sum_n L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2 (f(w) v'(f(w)) - (n + 2)v(f(w)))}{f(w)^{n+3}}.$$

Finally

$$\left[ \sum_{l \geq 1} v_l L_l, L_m(f) \right] = \sum_n L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2 (f(w) v'(f(w)) - (n + 2)v(f(w)))}{f(w)^{n+3}}$$

$$+ \frac{c}{12} \int_0^1 dw w^{m+1} f'(w)^2 v'''(f(w)).$$

It is easily seen that this is nothing but

$$\frac{d L_m(f + \varepsilon v(f))}{d \varepsilon} \bigg|_{\varepsilon = 0},$$

which shows that

$$G_f^{-1} L_m G_f$$

and

$$\frac{c}{12} \int_0^1 dw w^{m+1} S f(w) + \sum_{n \geq m} L_n \int_0^1 dw w^{m+1} \frac{f'(w)^2}{f(w)^{n+2}},$$

which coincide at $f(z) = z$, have the same tangent map. Convexity ensures that they coincide everywhere.

## C  A few properties of $G_f$

The expansion of $G_f$ in powers of the $f_m$’s has an important property that is already apparent in the expansion above. Let $I = (i_1, i_2, \cdots)$ be a sequence of non negative integers with finitely many nonzero terms. Let $E_m$ be the sequence made of zeroes except for a single 1 in the $m^{th}$ position, so that $I = \sum_m i_m E_m$. We define $|I| = \sum_m i_m$ (which we call degree), $d(I) = \sum_m m i_m$ (which we call grading) $I! = \prod m^{i_m}$, $f_I = \prod f_i$ and $L_I = \prod L_i$ (with the convention that $L_1$ factors are on the utmost right, then $L_2$, and so on).

Then we claim that

$$G_f = \sum_I \left( - \right)^{|I|} I! \cdot f_I (L_I + \text{lower order terms})$$
where “lower order terms” mean $f$-independent linear combinations of $L_j$’s with $|J| < |I|$ but $d(J) = d(I)$. The same statement would be true if we had chosen the opposite convention to order the $L_m$’s in $L_I$.

The statement that $d(J) = d(I)$ is simply that a dilation on $z$ multiplies $L_I$ by $\lambda^I$ but divides $f_I$ by the same factor. Alternatively, one can check that the factor \( \int_0 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}} \) that appears in eq. (2) is a polynomial in the $f_I$’s of grading $n - m$.

The proof that obtained by taking a commuting limit: we set $f_m \equiv \varepsilon \varphi_m$ and $\Lambda_m \equiv \varepsilon L_m$ (think of $\varepsilon$ as $\hbar$). Then in the limit $\varepsilon \to 0$ keeping the $\varphi_m$’s fixed, on the one hand the $\Lambda_m$’s commute, and on the other hand
\[
\int_0 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}} = \delta_{n,m}
\]
so that the differential system defining $G_f$ reduces to
\[
\frac{\partial G_f}{\partial \varphi_m} = -G\Lambda_m,
\]
with solution $G_f = e^{-\sum_m \varphi_m \Lambda_m}$. This implies that in the $\varepsilon$ expansion in terms of $\varphi_m$’s and $\Lambda_m$’s, $G_f = \sum_I (-)^{|I|} \varphi_I (\Lambda_I + O(\varepsilon))$. But expressed in terms of $f_m$’s and $L_m$’s the result is $\varepsilon$-independent. This means that the coefficient of $\varphi_I \varepsilon^k$ involves only $\Lambda_J$’s with $|J| = |I| - k$. This concludes the proof.

An analogous computation would show that
\[
G_f^{-1} = \sum_I \frac{1}{I!} f_I (L_I + \text{lower order terms}).
\]

We can rephrase these results as follows:
\[
G_f = \sum_I \frac{(-)^{|I|}}{I!} L_I (f_I + \text{higher order terms}), \tag{16}
\]
\[
G_f^{-1} = \sum_I \frac{1}{I!} L_I (f_I + \text{higher order terms}), \tag{17}
\]
where “higher order terms” mean $L$-independent linear combinations of $f_J$’s with $|J| > |I|$ but $d(J) = d(I)$. In particular the polynomials in the $f_m$’s that appear as coefficients of the $L_I$’s in the above expansions form a basis of the space of all polynomials in the $f_m$’s.

These observations will be useful for the application to representation theory in section 6.

We can also write down a general recursive formula. We define $P_I$ by $G_f \equiv \sum_I \frac{(-)^{|I|}}{I!} f_I P_I$ and combinatorial coefficients $C_J(m,n)$ by \( \int_0 dww^{m+1} \frac{f'(w)}{f(w)^{n+2}} \equiv \sum_J \frac{(-)^{|J|}}{I!} f_J C_J(m,n) \). The integrand can be written as $w^{m-n} \frac{dw}{w}$ times a function in which each $f_I$ is multiplied by $z^I$: $C_J(m,n) = 0$ unless $d(J) = n - m$. The partial differential equations for $G_f$ lead to difference equations for the
One gets

\[ P_{K+E_m} = \sum_{I+J=K} \frac{K!}{I!J!} C_f(m, m + d(J)) P_I L_{m+d(J)}. \]

One finds \( P_{E_m} = L_m, P_{E_m+E_n} = L_m L_n + (n + 1)L_{m+n}, \cdots. \)

**D Final steps for the proof of (9)**

We start from eq.(7), repeated here for convenience:

\[
\begin{align*}
P_{Lny} + \left( \frac{c}{12} \int_0^1 dw f(w)^{n+1} \frac{Sf(w)}{f'(w)} - h \int_0^1 dw \frac{f(w)^{n+1}}{w^2 f'(w)} \right) P_y &= \\
- \sum_{m=n}^0 \int_0^1 dw \frac{f(w)^{n+1}}{w^{m+2} f'(w)} \sum_{l \geq 1} \int_0^1 dzz^{m+1} \frac{f'(z)^2}{f(z)^{l+2}} \langle G_f L_i y, x \rangle.
\end{align*}
\]

Now, fix \( m \) and concentrate on \( \sum_{l \geq 1} \int_0^1 dzz^{m+1} \frac{f'(z)^2}{f(z)^{l+2}} \langle G_f L_i y, x \rangle. \) From the Lagrange formula, one can expand \( z^{m+1} f'(z) \) in powers of \( f(z) \) as

\[ z^{m+1} f'(z) = \sum_{k \geq m} f(z)^{k+1} \int_0^1 du \frac{u^{m+1} f'(u)^2}{f(u)^{k+2}}. \]

Define

\[ h_m(z) \equiv z^{m+1} f'(z) - \sum_{k, m \leq k \leq 0} f(z)^{k+1} \int_0^1 du \frac{u^{m+1} f'(u)^2}{f(u)^{k+2}}. \tag{18} \]

By definition, \( h_m(z) \) is a \( O(z^2) \) and its \( z \) expansion reads

\[ h_m(z) = \sum_{j \geq 1} z^{j+1} \left( f_{j-m}(j - m + 1) - \sum_{k=m}^0 \int_0^1 du \frac{u^{m+1} f'(u)^2}{f(u)^{k+2}} \int_0^1 dv f(v)^{k+1} \right). \]

On the other hand, by construction, \( h_m(z) \) is such that

\[ \int_0^1 dzz^{m+1} \frac{f'(z)^2}{f(z)^{l+2}} = \int_0^1 dz \frac{h_m(z) f'(z)}{f(z)^{l+2}} \quad \text{for} \quad l = 1, 2, \cdots. \tag{19} \]

so, using again eq(2),

\[ \sum_{l \geq 1} \int_0^1 dzz^{m+1} \frac{f'(z)^2}{f(z)^{l+2}} G_f L_i = \]

\[ - \sum_{j \geq 1} \left( f_{j-m}(j - m + 1) - \sum_{k=m}^0 \int_0^1 du \frac{u^{m+1} f'(u)^2}{f(u)^{k+2}} \int_0^1 dv f(v)^{k+1} \right) \frac{\partial G_f}{\partial f_j}. \]

39
References

[1] M. Bauer, D. Bernard, *SLE growth processes and conformal field theories*, Phys. Lett. **B543** (2002) 135-138.

[2] M. Bauer, D. Bernard, *Conformal field theories of stochastic Loewner evolutions*, arXiv:hep-th/0210015, to appear in Commun. Math. Phys.

[3] M. Bauer, D. Bernard *SLE martingales and the Virasoro algebra*, arXiv:hep-th/0301064, Phys. Lett. **B557** (2003) 309-316.

[4] A. Belavin, A. Polyakov, A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241**, 333-380, (1984).

[5] J. Cardy, *Critical percolation in finite geometry*, J. Phys. **A25**, L201-206, (1992).

[6] J. Cardy, *Conformal invariance and percolation*, arXiv:math-ph/0103018

[7] J. Cardy, *Conformal invariance in percolation, self-avoiding walks and related problems*, arXiv:cond-mat/0209638

[8] F. David, Mod. Phys. Lett. **A5** 1019 (1990),
R. Dijkgraaf, H. Verlinde, E. Verlinde, *Loop equations and Virasoro constraints in nonperturbative 2-d quantum gravity*, Nucl. Phys. **B348** 435 (1991),
V. Kazakov, Mod. Phys. Lett. **A4** 2125 (1989).

[9] B. Duplantier, *Conformally invariant fractals and potential theory*, Phys. Rev. Lett. **84**, 1363-1367, (2000).

[10] B. Duplantier, *Higher conformal multifractality*, J. Stat. Phys. **110** (2003), 691-738.

[11] I. Karatzas, S. E. Shreve *Brownian motion and stochastic calculus*, GTM 113, Springer, (1991).

[12] G. Lawler, *Introduction to the Stochastic Loewner Evolution*, URL [http://www.math.duke.edu/~jose/papers.html](http://www.math.duke.edu/~jose/papers.html) and references therein.
[13] G. Lawler, O. Schramm, W. Werner, *Values of Brownian intersections exponents I: half-plane exponents*, Acta Mathematica 187 (2001) 237-273, arXiv:math.PR/9911084.
G. Lawler, O. Schramm, W. Werner, *Values of Brownian intersections exponents II: plane exponents*, Acta Mathematica 187 (2001) 275-308, arXiv:math.PR/0003156.
G. Lawler, O. Schramm, W. Werner, *Values of Brownian intersections exponents III: two-sided exponents*, Ann. Inst. Henri Poincaré 38 (2002) 109-123, arXiv:math.PR/0005294.
G. Lawler, O. Schramm, W. Werner, *Conformal restriction: the chordal case*, arXiv:math.PR/0209343.

[14] B. Nienhuis, *Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas*, J. Stat. Phys. 34, 731-761, (1983).

[15] L. Onsager, Phys. Rev 65, 117 (1944),
L. Onsager, Nuovo Cimento 6, supplement, 261 (1949),
C. N. Yang Phys. Rev 85, 808 (1952).

[16] S. Rhode, O. Schramm, *Basic properties of SLE*, and references therein, arXiv:math.PR/0106036.

[17] O. Schramm, Israel J. Math., 118, 221-288, (2000);

[18] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits*, C.R. Acad. Sci. Paris, (2001) 333 239-244.

[19] W. Werner, *Lectures notes of the 2002 Saint Flour summer school*

[20] P. A. Wiegmann, private communication.