Enhancement of entanglement in one-dimensional disordered systems

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Abstract

The pairwise quantum entanglement of sites in disordered electronic one-dimensional systems (rings) is studied. We focus on the effect of diagonal and off diagonal disorder on the concurrence $C_{ij}$ between electrons on neighbor and non neighbor sites $i,j$ as a function of band filling. In the case of diagonal disorder, increasing the degree of disorder leads to a decrease of the concurrence with respect to the ordered case. However, off-diagonal disorder produces a surprisingly strong enhancement of entanglement. This remarkable effect occurs near half filling, where the concurrence becomes up to 15\% larger than in the ordered system.

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I. INTRODUCTION

It has been shown that entanglement is a necessary requirement for the realization of several algorithms used in quantum computation and information. For example, quantum teleportation, which is a technique for moving a quantum state between two points, requires the use of entangled states \(^1\). On the other hand, there are some evidences that many problems involving strong-coupled many body systems can be characterized and classified by using quantum entanglement. Therefore, this can lead to new and more effective methods for understanding the dynamical behavior of complex quantum systems \(^2\). In particular, it has been conjectured that entanglement can play an important role in quantum phase transitions \(^3,4,5,6\).

Several relations between entanglement and quantum phase transitions have been obtained recently. Osterloh et al., have analyzed the behavior of entanglement near the critical point of the spin 1/2 model XY in a transverse field \(^3\). They have found that in the region close to a quantum phase transition the entropy of entanglement obeys a scaling law. Although the entanglement is not an indicator of that phase transition, they showed that there exists an intimate connection between entanglement, scaling and universality. Moreover, Vidal et al. \(^4\), have established a precise connection between concepts of quantum information, condensed matter physics, and quantum field theory by showing that the behavior of critical entanglement in spin systems is analogous to that of entropy in conformal field theory. In addition, Gu et al. \(^5\) have studied the properties of entanglement in the ground-state of an antiferromagnetic XXZ chain. They have shown that the competition between quantum fluctuation and ordering leads to a maximization of entanglement at the isotropic point.

Interestingly, also in the case of electron systems a quantitative analysis of entanglement can be performed. Recently, Zanardi et al. studied the pairwise entanglement between nearest-neighbor sites in itinerant spinless-fermion systems described by a periodic tight-binding Hamiltonian \(^6\). They calculated the concurrence, which is a quantity which provides the same information as the entropy of entanglement \(^7,8\), and showed that the maximal concurrence between two neighboring sites is obtained at half band filling \(x = n/N_a = 0.5\), being the \(n\) the number of electrons and \(N_a\) the number of atoms (sites). The concurrence is symmetric with respect to the half-filling point.

The impurities or disorder effects have been seen in quantum computation as something
not desired that can induce errors in gating or in the evolution of a quantum state. In addition, if the imperfection strength is increased, new phenomena related with chaotic behavior can occur. On the other hand, it is well known that crystalline systems with impurities can undergo a metal-insulator transition as a function of disorder strength. Therefore, it is interesting to study the effect of disorder on the concurrence and if it is possible to find a relation between both.

In this paper, we present the first study of entanglement in disordered electronic systems. We analyze the effect of disorder on the entanglement between sites in one-dimensional rings. For this purpose, we consider diagonal and off-diagonal disorder and different degrees of disorder. We show that whereas diagonal disorder reduces and suppresses entanglement for all values of the band filling, off-diagonal disorder considerably enhances the entropy of entanglement for near, but not exactly at, half-filling.

The paper is organized as follows. In section II, we discuss the employed model Hamiltonian and how to quantify the entanglement. In Sec. III, the effect of the length of the ring on the entanglement is studied. We analyze the effect of diagonal and off-diagonal disorder on entanglement in Sec. IV A and IV B. Finally, in Sec. V, we summarize our results.

II. THEORY

We consider one-dimensional disordered electronic systems and describe them by an Anderson-Hamiltonian of the form

\[ \hat{H} = \sum_i \varepsilon_i \hat{n}_i + \sum_{\langle ij \rangle} t_{ij} \hat{c}_{i}^{\dagger} \hat{c}_j. \] (1)

For simplicity, we consider spinless electrons. In Eq. (1) \( \hat{c}_{i}^{\dagger} (\hat{c}_i) \) is the usual creation (annihilation) operator of a spinless electron at site \( i \) whereas \( \hat{n}_i = \hat{c}_{i}^{\dagger} \hat{c}_i \) is the number operator, and \( t_{ij} \) is the hopping integral between nearest neighbor (NN) sites \( i \) and \( j \). \( \varepsilon_i \) is the on-site energy for atom \( i \). Diagonal disorder is introduced by assuming that the values of \( \varepsilon_i \) are randomly distributed in the energy interval \([0, W]\) while the NN hopping integrals remain unchanged with respect to the perfectly ordered case, i.e., \( t_{ij} = -1 \). Analogously, the effect of off-diagonal disorder (random hopping) is modeled by considering the hopping integrals \( t_{ij} \) to acquire random values in the interval \([0, W]\) while all on-site energies are equal to zero, like in the ordered lattice.
To calculate the entanglement of formation between pairs of sites we resort to the concept of concurrence\textsuperscript{8,9} and proceed in a similar way as Zanardi and coworkers did for the case of translationally invariant chains.\textsuperscript{7}

Basically, one can consider the ring with \( N_a \) sites and \( n \) electrons as being a bipartite system composed of two subsystems, \( A \) and \( B \), where \( A \) refers to the pair of sites \( i \) and \( j \) and \( B \) to the rest of the ring (see Fig. 1). More precisely, \( A \) contains all 4 possible electronic states in the pair of sites \( i \) and \( j \), i.e., \(|1\rangle_A \equiv |0\rangle_A, |2\rangle_A \equiv c_i^+|0\rangle_A, |3\rangle_A \equiv c_j^+|0\rangle_A, \) and \(|4\rangle_A \equiv c_i^+c_j^+|0\rangle_A \). Analogously, \( B \) contains the states of the rest of the ring. Thus, a particular state \(|\Psi_{AB}\rangle\) of the entire ring can be represented as a direct product of the form

\[
|\Psi_{AB}\rangle = \sum_{\mu\nu} \alpha_{\mu\nu} |\psi_{A\mu}\rangle \otimes |\psi_{B\nu}\rangle,
\]

where \(|\psi_{A\mu}\rangle\) and \(|\psi_{B\nu}\rangle\) are basis states of the subsystems \( A \) and \( B \). Thus, the density matrix of the whole system reads

\[
\hat{\rho} = |\Psi_{AB}\rangle \langle \Psi_{AB}|.
\]

The reduced density matrix \( \rho_A \) for the subsystem \( A \) is obtained by performing a trace on subsystem \( B \) in order to integrate out its degrees of freedom. Therefore

\[
\hat{\rho}_A = Tr_B \{|\Psi_{AB}\rangle \langle \Psi_{AB}|\}.
\]

The entropy of entanglement of a bipartite system is calculated as\textsuperscript{14}

\[
E(|\Psi_{AB}\rangle) = -Tr_A \{\hat{\rho}_A \log \hat{\rho}_A\} = -Tr_B \{\hat{\rho}_B \log \hat{\rho}_B\}
\]

and quantifies the entanglement of formation. It has been shown by Wootters\textsuperscript{8} that the entropy of entanglement can be also characterized by the concurrence \( C \), since \( E(|\Psi_{AB}\rangle) \) is a monotonous function of \( C \), which is defined by

\[
C = max (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0).
\]

The \( \lambda_k \)'s are, in decreasing order from \( k = 1 \) to \( k = 4 \), the eigenvalues of the matrix \( R = \sqrt{\rho_A \rho_A^*} \), where \( \hat{\rho}_A = (\sigma_y \otimes \sigma_y) \rho_A^* (\sigma_y \otimes \sigma_y) \), where \( \rho_A^* \) is the complex conjugated of the matrix \( \rho_A \) and \( \sigma_y \) the “\( y \)” Pauli-matrix.

Thus, in order to calculate \( C \) for disordered 1D systems described by the Hamiltonian of Eq. (1), we should first determine \( \rho_A \) by means of Eqs. (2) and (4). As pointed out in Ref.
the number of electrons $n$ is conserved due to the fact that $[\hat{H}, \hat{n}] = 0$. As a consequence, only the elements $\rho_A(11)$, $\rho_A(22)$, $\rho_A(33)$, $\rho_A(44)$, $\rho_A(23)$ and $\rho_A(32)$ are nonzero. All other elements vanish. This can be seen by looking at the expansion (2). When constructing $|\Psi_{AB}\rangle\langle\Psi_{AB}|$ and performing the trace over $B$ one obtains, for all elements of $\rho_A$ others than the ones mentioned above, scalar products of the type $\langle\psi_{B\nu}|\psi_{B\mu}\rangle$, between states of $B$ having different number of particles and, therefore, being orthogonal. One can take profit of this property of the reduced density matrix and obtain its nonzero elements as quantum expectation values of relevant physical quantities. For this purpose we use the whole wave function $|\Psi\rangle_{AB}$ of the system and write:

$$
\rho_A(11) = \langle\Psi_{AB}|(1 - \hat{n}_i)(1 - \hat{n}_j)|\Psi_{AB}\rangle \tag{7}
$$

$$
\rho_A(44) = \langle\Psi_{AB}|\hat{n}_i\hat{n}_j|\Psi_{AB}\rangle \tag{8}
$$

$$
\rho_A(22) = \langle\Psi_{AB}|\hat{n}_i(1 - \hat{n}_j)|\Psi_{AB}\rangle \tag{9}
$$

$$
\rho_A(33) = \langle\Psi_{AB}|(1 - \hat{n}_i)\hat{n}_j|\Psi_{AB}\rangle \tag{10}
$$

$$
\rho_A(23) = \rho_A^*(32) = \langle\Psi_{AB}|c_j^+c_i|\Psi_{AB}\rangle. \tag{11}
$$

In Eq. (7), the presence of the operator $(1 - \hat{n}_i)(1 - \hat{n}_j)$ automatically projects out all states of the whole system not having 0 electrons on site $i$ and 0 electrons on site $j$. Similarly, the operator $\hat{n}_i\hat{n}_j$ in Eq. (8) ensures that only states having 1 electron on $i$ and 1 electron on $j$ will contribute to the element $\rho_A(44)$. In Eqs. (9) and (10) the corresponding projectors to obtain $\rho_A(22)$ and $\rho_A(33)$ are described. Finally, $\rho_A(23)$ [$\rho_A(32)$] contains only contributions from states having 1 electron on site $i$ ($j$) and no electrons on site $j$ ($i$).

Thus, with the help of the above equations, one can write reduced density matrix $\rho_A$ in the following form

$$
\rho_A = \begin{pmatrix}
v & 0 & 0 & 0 \\
0 & w & z & 0 \\
0 & \bar{z} & u & 0 \\
0 & 0 & 0 & y
\end{pmatrix}, \tag{12}
$$

where the nonvanishing matrix elements are given by

$$
v = 1 - \langle\hat{n}_i\rangle - \langle\hat{n}_j\rangle + \langle\hat{n}_i\hat{n}_j\rangle, \quad y = \langle\hat{n}_i\hat{n}_j\rangle
$$

$$
z = \langle\hat{c}_j^+\hat{c}_i\rangle, \quad w = \langle\hat{n}_i\rangle - y, \quad u = \langle\hat{n}_j\rangle - y. \tag{13}
$$

Eqs. (13) show how the nonzero elements of $\rho_A$, can be calculated as average quantities of the complete ground-state wave function.
Diagonalization of the matrix of Eq. (12) is trivial. Using Eq. (6) one obtains an analytic expression for the concurrence between sites $i$ and $j$ of the form

$$C_{ij} = 2 \max\{0, |z| - \sqrt{vy}\}.$$  \hspace{1cm} (14)

Eq. (14) does not contain $w$ and $u$. Note that $i$ and $j$ do not need be nearest neighbors. In the next section we present calculations of $C_{i,i+\ell}$ with $\ell = 1, 3$.

For every set of parameters, the Hamiltonian can be diagonalized numerically and the concurrence $C$ can be obtained using Eqs. (13) and (14). The average value of the concurrence can be then calculated by using a large number of replicas in order to minimize numerical fluctuations.

To calculate the average concurrence of the tight-binding rings with diagonal and off-diagonal disorder, we have implemented the following procedure: (i) we have generated a set of random parameters (on-site energies or hopping elements) for each of the replicas in order to model the disorder of the finite size rings, (ii) we have numerically diagonalized the Hamiltonian, given by Eq. 1, for obtaining its single-particle eigenvectors, (iii) with the help of these eigenvectors, we have calculated the values $v$, $y$ and $z$ necessary for the calculation of the concurrence and, finally, (iv) we have obtained the average concurrence $\bar{C}_{ij}$ ($\bar{C}_{12}, \bar{C}_{13}, \bar{C}_{14}$) as

$$\bar{C}_{1,1+\ell} = \frac{1}{N_{\text{replicas}}N_a} \sum_{\text{replicas}} \sum_{k} C_{k,k+\ell}.$$  \hspace{1cm} (15)

We have performed these calculations for different ring sizes. We found that for rings consisting of $N_a \approx 200$ sites results achieve convergence. Therefore, in this paper all the results for $\bar{C}_{ij}$ correspond to $N_a = 200$ sites, unless we explicitly specify the ring size. On the other hand, the number of replicas we used was 10000. In order to understand the influence of disorder on the concurrence it is necessary to study first the behavior of the concurrence in ordered systems.

III. ORDERED RINGS: SIZE EFFECTS

The concurrence between $C_{12}$ as a function of $n$ for a perfectly ordered, periodic ring can be easily computed, using the theory presented in the previous section, as a limiting case of $\bar{C}_{ij}$ for $\varepsilon_i = 0 \ \forall i$ and $t_{ij} = -1$). Obviously, the sums of Eq. (15) are not necessary in this case, since they cancel with the factors in the denominator. We have computed $C_{12}$ vs $n$ for
TABLE I: Calculation of the elements of the reduced density matrix $\rho_A$ and the concurrence according to equations 12 to 16, for ring consisting of 4 atoms and $n$ electrons.

| $n$ | $\langle n_1 \rangle$ | $\langle n_2 \rangle$ | $z$ | $\langle n_1 n_2 \rangle$ | $v$ | $y$ | $C_{12}$ |
|-----|------------------|------------------|-----|------------------|-----|-----|---------|
| 1   | 1/4              | 1/4              | 1/4 | 0                | 1/2 | 0   | 1/2    |
| 2   | 3/4              | 1/4              | 1/4 | 1/8              | 1/8 | 1/8 | 1/4    |
| 3   | 3/4              | 3/4              | 1/4 | 1/2              | 1/2 | 1/2 | 1/2    |
| 4   | 1/4              | 1/4              | 0   | 0                | 0   | 0   | 0      |

different sizes. Results are shown in Fig. 2. As an example, we show below the analytical calculation for $N_a = 4$.

The $4 \times 4$ Hamiltonian has the eigenvalues $\epsilon_1 = -2$, $\epsilon_2 = \epsilon_3 = 0$, $\epsilon_4 = 2$, and the eigenvector matrix reads

$$U = \begin{pmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{pmatrix}.$$  

(16)

With this information at hand it is possible to calculate the elements of the reduced density matrix and therefore the concurrence, as summarized in Table I.

Clearly, the concurrence is symmetric with respect to $n = 2$ (half filling). The concurrence for $n = 0$ and $n = 4$ is equal to zero. We observe in Fig. 2(a) that for smaller ring ($N_a = 4, 8$ and 12 sites) it is very difficult to perceive which is the band filling $x$ where $C_{12}$ will have a maximum. In addition, the concurrence as a function of $n$ does not have a monotonous behavior. These size effects can be understood with the help of the correlation functions $z = \langle \hat{c}_1^\dagger \hat{c}_2 \rangle$, $y = \langle \hat{n}_1 \hat{n}_2 \rangle$, and the band filling $x$. It is easy to show that $z = \langle \hat{c}_1^\dagger \hat{c}_2 \rangle$ is related to the ground-state energy $E_{gs}$, i.e. $z = \langle \hat{c}_1^\dagger \hat{c}_2 \rangle = -E_{gs}/2N_a$, while that $v = (x - 1)^2 - z^2$ and $y = x^2 - z^2$ (see Ref. 7). From these expressions, we obtain for systems with band filling $x = 1/N_a$ (only one electron in the lattice) that $C_{12} = 2z = -E_{gs}/N_a$ where $y = \langle \hat{n}_1 \hat{n}_2 \rangle = 0$. This explains the value of $C_{12}$ found for $N_a = 4$ with one electron ($x = 0.25$). In general,
the concurrence of periodic rings can be written as follows:

\[ C_{12} = 2 \max \left\{ 0, \left| \frac{E_{gs}}{2N_a} \right| - \left( \left[ (x - 1)^2 - \left( \frac{E_{gs}}{2N_a} \right)^2 \right] \left[ x^2 - \left( \frac{E_{gs}}{2N_a} \right)^2 \right] \right)^{1/2} \right\} . \]  

(17)

When we begin to increase the ring size, we notice that the finite size effect begins to disappear [see Fig. 2(b)]. Moreover, \( C_{12} \) begins to have a monotonous behavior and we find that the maximum value of NN concurrence occurs at half-band filling. For \( N_a = 64 \) sites the thermodynamic limit has been reached and the values of \( C_{12} \) are almost converged.

IV. DISORDERED RINGS

A. Diagonal disorder

The electronic properties of strongly disorder systems have been investigated by a variety of analytical and numerical methods. In this paper, we study this problem from a very different perspective, the perspective of the concurrence. In Fig. 3 we show the NN concurrence \( \bar{C}_{12} \) for a ring with \( N_a = 200 \) sites where we have introduced only diagonal disorder. We observe that the effect of this diagonal disorder is to decrease \( \bar{C}_{12} \) in a monotonous way.

In Fig 3(a) \( \bar{C}_{12} \) is presented as a function of band filling for some representative values of the disorder strength \( W \). This decrease of the concurrence is related to the localization of the wave function. When disorder is introduced, for example on site \( i \), the wave functions are localized around this site, therefore the correlation function \( z = \langle \hat{c}^+_i \hat{c}_j \rangle \) between two NN sites is decreased. On the other hand, Osterloh et al.\(^3\), in their study of the critical point in the XY model in a transversal magnetic field, showed that the behavior of the concurrence does not have a divergence in the transition point. However, they showed that the derivative of \( C_{12} \) shows this divergence and scaling properties. Therefore, it may be of interest to study the derivative of \( \bar{C}_{12} \) as a function of the disorder strength. Unfortunately, it is known that numerical calculation of derivatives in disorder systems as a function of \( W \) displays errors associated with the numerical division.\(^15\). In order to visualize the behavior of the derivative of \( C_{12} \) as a function of the strength of disorder, in Fig 3(b) we plot \( \bar{C}_{12} \) as a function of the disorder strength \( W \) for some representative values of the band filling \( x \). From the figure, we can deduce that the derivative of \( \bar{C}_{12} \) as a function of disorder strength is almost zero in the range of \( W \in [0, 2.5] \). After that, the derivative of \( \bar{C}_{12} \) as a function of \( W \) is almost constant with a slight negative slope.
A similar analysis can be realized on the concurrence as a function of the distance between sites. In particular, we have focused on next nearest-neighbor concurrence $C_{13}$ and third nearest-neighbor concurrence $C_{14}$. The results are shown as a function of the band filling $x$ for some representative values disorder strength $W$ [Figs. 4(a) and 5(a)] and as a function of the disorder strength $W$ for some values of the band filling $x$ [Figs. 4(b) and 5(b)]. We observe that the values of the concurrence for all band fillings and disorder strengths is much smaller than $C_{12}$. This implies that in systems without disorder and systems with diagonal disorder the concurrence is highly local and strongly decreases as a function of the distance between sites. Notice that the maximum value of the concurrence occurs at $W = 0$ and begins to move towards band fillings smaller than the half band $1/2$ when we increase the distance between sites and disappears in some range of band filling. This result can be explained by using the fact that the correlation function $z = \langle \hat{c}_i^+ \hat{c}_j \rangle$ depends on the overlap between the wave function, which decreases when we increase the distance between sites, whereas $v$ and $y$ fundamentally depend on the band filling. Therefore, it will have a threshold band filling where $\sqrt{vy} > |z|$ and, consequently, $C_{ij} = 0$. On the other hand, as we begin to increase the disorder strength, the concurrence is almost insensitive to the system band fillings. These behaviors can be observed in figures 4(b) and 5(b), where we show the concurrence for systems with different band fillings and $W > 10$ are joined in the same curve. This result is related to the fact that due to the strong disorder $z$, $v$ are $y$ small for all band fillings.

B. Off diagonal disorder

Although for 1D systems with diagonal or off-diagonal disorder the wave function are localized for all disorder strengths, it is known that the physical properties are strongly dependent on the type of disorder. Systems with off-diagonal disorder show an anomalous behavior in the density of states and in the localization length\textsuperscript{16,17}, where it has been found that $E = 0$ state is localized but has an infinite localization length. In this subsection, we show how this property affects the concurrence.

In Fig. 6 we present results for $\bar{C}_{12}$ for a ring with $N_a = 200$ sites and with off-diagonal disorder $t_{NN} \in [0, W]$. In Fig. 6(a) $\bar{C}_{12}$ is showed as a function of band filling $x$ for some representative disorder strengths $W$ whereas in Fig. 6(b) $\bar{C}_{12}$ is presented as a function
of $W$ for some representative values of $x$. From the figures, we observe that, contrary to the diagonal disorder, $C_{12}$ value can increase for some band fillings in comparison with the non disordered case ($W = 0$) when the disorder strength is increased. In particular, we observe that this occurs for band fillings larger than $x \geq 0.25$ and for off-diagonal disorder strength $W \geq 4$. Notice that the maximum value of the concurrence is found for band filling values in the range $0.4 \leq x \leq 0.45$ and for $W \geq 20$. Moreover, the overall form of the concurrence curves does not change anymore for $W > 20$. It is surprising that in the case of system with off-diagonal disorder, the NN concurrence increases in comparison with respect to the non disordered case. This behavior can be related to the anomaly in the density of states found in these kind of systems. To inquire about it, we have calculated the NN concurrence as a function of the band filling for a small system with $N_a = 16$ sites with only one different hopping elements with respect to the others. These results are shown in Fig. 4 where we present the concurrence between the sites with the different hopping integral $(t_{12} = -2, t_{NN} = -1, \text{otherwise}), C_{12}$, and the NN concurrence between sites different from 1 and 2. In the last case, we only show the concurrence between sites close to the impurity. From the figure we observe that $C_{12}$ has much larger values compared to the result show in the previous subsection. This concurrence is almost independent of the band filling and oscillates around 0.72. These high values of $C_{12}$ are related to the fact that the impurity localizes the eigenfunction $|\psi_0\rangle$ with the lowest eigenvalue $\varepsilon_0$ between the sites where the impurity was placed $(t_{12} = -2)$, whereas the other eigenfunctions are almost delocalized. Thus, we have that $z = \langle \psi_0|c_1^\dagger c_2|\psi_0\rangle$ has very large contribution to the concurrence for all band fillings. On the other hand, we know that in the case of one electron, $v y = 0$. Therefore from the figure, we can see that the quasidelocalized eigenfunctions contribute very little to the NN concurrence when we increase the band filling. In the case of $C_{23}$, we observe that the localization effect turns $C_{23}$ into a decreasing function. It is necessary to remark that the localization effect due to one impurity only strongly modifies the NN concurrence between sites very near to this, where the eigenfunction $|\psi_0\rangle$ is localized, whereas the NN concurrence of remote sites from the impurity show a behavior more similar to the case of a periodic ring. Therefore, when the hopping integrals are randomly chosen in the range $t_{NN} \in [0, W]$, the number of hopping integrals having values larger than $W/2$ is $N_a/2$. This implies that in general there are $N_a/2$ eigenfunctions that are localized on the bonds affected by these hopping integrals. These $N_a/2$ bonds will contribute with very high values
to $C_{NN}$, as has been shown in the case of one impurity, whereas the other $N_a/2$ bonds will contribute with much smaller values. As we increase $W$, the localization of the eigenfunction will be increased around these bonds and their $\bar{C}_{NN}$ values will be close to 1 whereas the $\bar{C}_{NN}$ values of the other bonds will be close to zero. These bond-localizations explain the increase in the NN concurrences of systems with off-diagonal disorder in comparison with the non disordered case. In addition, this clearly shows the difference between both types of disorder, in diagonal disorder the one particle eigenfunctions are localized around the sites, which decreases the total value of $\bar{C}_{NN}$ with increasing $W$. On the other hand, in the off-diagonal disorder case the eigenfunctions are localized between the nearest neighbors, which increases $\bar{C}_{NN}$ with increasing $W$.

In order to study the way the off-diagonal disorder affects the concurrence as a function of the distance between sites in this kind of systems, we have calculated the concurrence as a function of the distance. The results are shown in Figs. 8 and 9. In these figures we observe that, as the off-diagonal strength $W$ is increased, the maximum in the concurrence begins to decrease, contrary to the case of the NN concurrence. This confirms that the effect of the increase in the maxima of the NN concurrence is due to the strong localization on the bonds. Contrary to the case of diagonal disorder, the decrease of the maximum values of $\bar{C}_{13}$ and $\bar{C}_{14}$ for large $W$ is relatively small with respect to the non disordered case. We again notice that the concurrence decreases as a function of the distance between sites, which shows that the concurrence is a strongly local property. Moreover in these cases, the curves of $\bar{C}_{13}$ and $\bar{C}_{14}$ depend on the band filling. This behavior can be seen in Figures 8(b) and 9(b), where we show that the curves for different band fillings are completely different. Finally, for a given band filling $x$ and for $W > 10$, the curves are almost insensitive to the strength of the disorder.

V. CONCLUSIONS

In this work, we have studied the effect of diagonal and off diagonal disorder on 1D electronic systems described by the Anderson Hamiltonian. We have found that, in the case of diagonal disorder, as we increase the disorder strength the concurrence between sites decreases in a monotonous way. In addition, we found that the concurrence in these systems is highly local and decays for increasing distances between sites. On the other
hand, in off diagonal disorder case we observe that the increase in the disorder strength results in an increase of the NN concurrence in comparison to finite rings in the band filling interval $0.25 \leq x \leq 0.5$. This behavior of the concurrence for both kinds of disorder can be understood with the help of the one particle wavefunctions. In the diagonal disorder case, the one particle wave functions begin to localize on the sites as we increase the disorder strength. Therefore, this localization decreases the NN concurrence. On the other hand, the increase of the off-diagonal disorder strength localizes the one particle wavefunction on bonds between NN sites, which increases the NN concurrence in a certain interval of the band filling. This result could be useful for the study of entanglement in 3D amorphous solids. Although the metal-insulator transition occurs only in 3D systems, it is known that the off-diagonal disorder in 1D systems produces anomalous properties that should have some effect of the concurrence.

Investigations of the concurrence as an indicator of the insulator-metal phase transition in 3D disordered systems are in progress.

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FIG. 1: (Color online) Schematic illustration of the partition of the disordered rings into two subsystems for the calculation of the entropy of entanglement between sites $i$ and $j$.

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FIG. 2: Nearest neighbor concurrence $C_{12}$ of small periodic ring as a function of band filling $x$. 

- (a) $C_{12}$ for $N_a=4$, $N_a=8$, $N_a=12$.
- (b) $C_{12}$ for $N_a=16$, $N_a=32$, $N_a=64$, $W=0$. 

$C_{12}$ is plotted against $x$ for different values of $N_a$. The plots show how the concurrence changes as a function of band filling.
FIG. 3: (Color online) Nearest neighbor concurrence average $\bar{C}_{12}$ of a ring with $N_a = 200$ sites. In (a) we present $\bar{C}_{12}$ as a function of band filling $x$ for several representative values of the diagonal disorder strength $W$. In (b) $\bar{C}_{12}$ is plotted as a function of disorder strength $W$ for some band filling $x$. 

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FIG. 4: (Color online) Next nearest-neighbor average concurrence $\bar{C}_{13}$ of a ring with $N_a = 200$ sites as a function of band filling $x$ for some representative values of diagonal disorder strength $W$. 

$\varepsilon_i \in [0,W]$
FIG. 5: (Color online) Third neighbor average concurrence $\bar{C}_{14}$ of a ring with $N_a = 200$ sites as a function of band filling $x$ for some representative values of diagonal disorder strength $W$. 

$\varepsilon_i \in [0, W]$

(\textbf{a})

$W=0$

$W=4$

$W=8$

$W=12$

$W=16$

$W=20$

(\textbf{b})
FIG. 6: (Color online) Nearest neighbor average concurrence $\bar{C}_{12}$ of a ring with $N_a = 200$ sites as a function of band filling $n$ for several representative values of the off diagonal disorder strength $W$. 
FIG. 7: Nearest-neighbor concurrence $C_{12}$ for a ring with $N_a = 16$ sites and one impurity. The impurity is localized between the sites 1 and 2 ($t_{12} = -2$). The hopping integral between the other nearest neighbor sites is $t_{NN} = -1.0$. 
FIG. 8: (Color online) Next nearest neighbor average concurrence $C_{13}$ of a ring with $N_a = 200$ sites as a function of band filling $n$ for several representatives values of the off diagonal disorder strength $W$. 
FIG. 9: (Color online) Third neighbor average concurrence $\bar{C}_{14}$ of a ring with $N_a = 200$ sites as a function of band filling $n$ for several representatives values of the off diagonal disorder strength $W$. 
