Generic Torus Orbit Closures in Flag Bott Manifolds

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Received July 31, 2018; revised November 10, 2018; accepted March 16, 2019

Abstract—The generic torus orbit closure in a flag Bott manifold is shown to be a non-singular toric variety, and its fan structure is explicitly calculated.

DOI: 10.1134/S0081543819030088

1. INTRODUCTION

On the full flag manifold $\mathcal{F}(\mathbb{C}^{n+1})$ there is an effective action of the complex torus $(\mathbb{C}^*)^n$. The generic torus orbit closure, which is the closure of a generic $(\mathbb{C}^*)^n$-orbit in the full flag manifold, is well-known to be a non-singular toric variety, called the permutohedral variety. The fan of the permutohedral variety consists of Weyl chambers of a Lie group of $A_n$-type as its maximal cones. Note that the closure of an arbitrary $(\mathbb{C}^*)^n$-orbit is known to be normal and hence is a toric variety (see [4, Proposition 4.8]), but its non-singularity is not determined in general.

Generic torus orbit closures in a generalized flag manifold $G/P$ are studied in [8, 6], and arbitrary orbit closures in Grassmannian manifolds are studied in [9, 10, 3]. Furthermore, generic torus orbit closures in Schubert varieties are studied in [15].

In [14], the notion of a flag Bott manifold is introduced as a generalization of both full flag manifolds and Bott manifolds. In fact, a flag Bott manifold $F_m$ is the total space of an $m$-sequence of iterated fiber bundles whose fibers are full flag manifolds $\mathcal{F}(\mathbb{C}^{n_j+1})$ for $j = 1, \ldots, m$, and there is an effective action of complex torus $\mathbf{H}$ of rank $n = n_1 + \ldots + n_m$. Therefore, it would be interesting to find out when a torus orbit closure of $F_m$ is a non-singular toric variety and to determine its fan structure.

Certain flag Bott manifolds are constructed from generalized Bott manifolds $B_m$, and such manifolds are called the associated flag Bott manifolds to $B_m$. It is shown in [14] that the generic torus orbit closure in the associated flag Bott manifold to $B_m$ is a non-singular toric variety, and such toric variety can be obtained from $B_m$ through a sequence of blow-ups.

In this article, we consider the generic torus orbit closure $X$ of an arbitrary flag Bott manifold $F_m$. The toric variety $X$ is the total space of an $m$-sequence of iterated fiber bundles with permutohedral varieties as its fibers. We calculate the fan of $X$ in Theorem 3.2. As a consequence we can see that $X$ is a non-singular toric variety (see Proposition 3.5).

2. FLAG BOTT MANIFOLDS

In this section we recall flag Bott manifolds from [14] and consider their orbit space construction. Let $M$ be a complex manifold and $E$ an $n$-dimensional holomorphic vector bundle over $M$. Recall from [1, p. 282] that the associated flag bundle $\mathcal{F}(E) \rightarrow M$ is obtained from $E$ by replacing each fiber $E_p$ by the full flag manifold $\mathcal{F}(E_p)$.

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Definition 2.1 [14, Definition 2.1]. Let \( m \) be a positive integer and \( n = (n_1, \ldots, n_m) \) a sequence of positive integers. A flag Bott tower \( F_\bullet(n) = \{ F_j(n) \mid 0 \leq j \leq m \} \) of height \( m \) (or an \( m \)-stage flag Bott tower) is a sequence

\[
F_m(n) \xrightarrow{p_m} F_{m-1}(n) \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_2} F_1(n) \xrightarrow{p_1} F_0(n) = \{ \text{A point} \}
\]

of manifolds \( F_j(n) = \mathcal{F}(\bigoplus_{k=1}^{n_j+1} \mathcal{E}_k^{(j)}) \) where \( \mathcal{E}_k^{(j)} \) is a holomorphic line bundle over \( F_{j-1}(n) \) for all \( 1 \leq k \leq n_j + 1 \) and \( 1 \leq j \leq m \). We call \( F_j(n) \) the \( j \)-stage flag Bott manifold of the flag Bott tower.

For notational convenience, we use \( F_j \) and \( F_j(n) \) instead of \( F_j(n) \) and \( F_0(n) \), respectively, unless there is notational confusion. The full flag manifold \( \mathcal{F}(\mathbb{C}^{n+1}) =: \mathcal{F}(\mathbb{C}) \) is a flag Bott manifold, and the product of flag manifolds \( \mathcal{F}(n_1 + 1) \times \cdots \times \mathcal{F}(n_m + 1) \) is a flag Bott manifold. Also an \( m \)-stage Bott manifold, which is a smooth projective toric variety, is an \( m \)-stage flag Bott manifold (see [11]).

For a sequence \( n = (n_1, \ldots, n_m) \) of positive integers, two flag Bott towers \( F_\bullet(n) \) and \( F'_\bullet(n) \) are said to be isomorphic if there is a collection of holomorphic diffeomorphisms \( \varphi_j : F_j(n) \to F'_j(n) \) that commute with the projections \( p_j : F_j(n) \to F_{j-1}(n) \) and \( p'_j : F'_j(n) \to F'_{j-1}(n) \) for all \( 1 \leq j \leq m \).

Suppose that \( F_\bullet(n) \) is an \( m \)-stage flag Bott tower for a sequence \( n = (n_1, \ldots, n_m) \) of positive integers. To describe the orbit space construction of a flag Bott manifold, we first define the right action \( \Phi^A_j \), which is determined by a set \( \mathcal{A} \) of integer matrices. Let \( B_{\text{GL}(n)} \) be the set of upper triangular matrices in \( \text{GL}(n) := \text{GL}(n, \mathbb{C}) \) for \( n \in \mathbb{Z}_{>0} \). Let also \( H_{\text{GL}(n)} \) be the set of diagonal matrices in \( \text{GL}(n) \). For positive integers \( n \) and \( n' \), let \( A \) be an integer \( (n + 1) \times (n' + 1) \) matrix whose row vectors are \( a_1, \ldots, a_{n+1} \in \mathbb{Z}^{n'+1} \), i.e.,

\[
A = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n+1}
\end{bmatrix}.
\]

Since the character group \( \chi(H_{\text{GL}(n'+1)}) \) is isomorphic to \( \mathbb{Z}^{n'+1} \), the matrix \( A \) defines a homomorphism \( H_{\text{GL}(n'+1)} \to H_{\text{GL}(n+1)} \) given by

\[
h \mapsto \text{diag}(h^{a_1}, \ldots, h^{a_{n+1}}).
\]

Here \( h = \text{diag}(h_1, \ldots, h_{n'+1}) \) is an element of \( H_{\text{GL}(n'+1)} \) and \( h^a := h_1^{a_1} \cdots h_{n'+1}^{a_{n'+1}} \) for \( a = (a(1), \ldots, a(n'+1)) \in \mathbb{Z}^{n'+1} \). The composition of the above homomorphism with the canonical projection \( \Upsilon : B_{\text{GL}(n'+1)} \to H_{\text{GL}(n'+1)} \), which ignores entries not on the diagonals, yields a homomorphism

\[
\Lambda(A) : B_{\text{GL}(n'+1)} \to H_{\text{GL}(n+1)}
\]

induced by the matrix \( A \).

Suppose \( \mathcal{A} = \{ A_{\ell}^{(j)} \in M_{(n_j+1) \times (n_j+1)}(\mathbb{Z}) \mid 1 \leq \ell < j \leq m \} \) is a set of integer matrices. We define a right action \( \Phi^A_j \) of \( B_{\text{GL}(n_1+1)} \times \cdots \times B_{\text{GL}(n_j+1)} \) on the product \( \text{GL}(n_1 + 1) \times \cdots \times \text{GL}(n_j + 1) \) by

\[
\Phi^A_j((g_1, g_2, \ldots, g_j), (b_1, b_2, \ldots, b_j)) := \left( g_1 b_1, \Lambda^2_1(1)^{-1} g_2 b_2, \Lambda^3_1(1)^{-1} \Lambda^2_2(1)^{-1} g_3 b_3, \ldots, \Lambda^j_1(1)^{-1} \Lambda^j_2(1)^{-1} \cdots \Lambda^j_{j-1}(1)^{-1} g_j b_j \right),
\]

(2.1)

where \( \Lambda^j_\ell := \Lambda(A_{\ell}^{(j)}) \) for \( 1 \leq \ell < j \leq m \). Then the action \( \Phi^A_j \) is free and proper (see [14, Lemma 2.7]). Hence we have complex manifolds

\[
(\text{GL}(n_1 + 1) \times \cdots \times \text{GL}(n_j + 1))/\Phi^A_j \quad \text{for} \quad 1 \leq j \leq m.
\]
These manifolds are actually flag Bott manifolds (see [14, Proposition 2.8]), and every flag Bott manifold can be obtained by this construction.

**Proposition 2.2** [14, Proposition 2.11]. Let \( F_\bullet(n) \) be a flag Bott tower of height \( m \) for a sequence \( n = (n_1, \ldots, n_m) \) of positive integers. Then there is a set of integer matrices

\[
\mathcal{A} = \{ A_{\ell}^{(j)} \in M_{(n_{\ell+1}) \times (n_{\ell+1})}(\mathbb{Z}) \mid 1 \leq \ell < j \leq m \}
\]

such that \( F_\bullet(n) \) is isomorphic to

\[
\{(\text{GL}(n_1 + 1) \times \ldots \times \text{GL}(n_j))/\Phi_j^A \mid 0 \leq j \leq m \}
\]

as a flag Bott tower.

**Remark 2.3.** We notice that the integer matrices \( A_{\ell}^{(j)} \in M_{(n_{\ell+1}) \times (n_{\ell+1})}(\mathbb{Z}) \) are associated with the first Chern classes of the line bundles \( \xi_k^{(j)} \) in the construction of a flag Bott manifold. To be more precise, suppose that \( a_{k,j}^{(j)} \) is the \( k \)th row vector of the matrix \( A_{\ell}^{(j)} \). Then the set of vectors \( \{a_{k,1}^{(j)}, \ldots, a_{k,j-1}^{(j)}\} \) determines the first Chern class of the line bundle \( \xi_k^{(j)} \). For more details, see [14, Sect. 2].

**Example 2.4.** Let \( n_1 = 2 \) and \( n_2 = 1 \). Consider \( A_{1}^{(2)} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( \mathcal{A} = \{ A_1^{(2)} \} \). Then the following Bott tower \( \{ F_j \mid 0 \leq j \leq 2 \} \) is isomorphic to \( \{(\text{GL}(n_1 + 1) \times \ldots \times \text{GL}(n_j))/\Phi_j^A \mid 0 \leq j \leq 2 \} \) as a flag Bott tower:

\[
\begin{align*}
\mathcal{F}_\ell(\xi(c_1, c_2, 0) \oplus \mathbb{C}) &\rightarrow \mathcal{F}_\ell(3) \rightarrow \{ \text{A point} \}
\end{align*}
\]

The line bundle \( \xi(c_1, c_2, 0) \) over \( F_1 \) is \( (\text{GL}(3) \times \mathbb{C})/B_{\text{GL}(3)} \), where the right action of \( B_{\text{GL}(3)} \) is defined by

\[
(g, v) \cdot b = (gb, b_1^{-c_1}b_2^{-c_2}v)
\]

for \( g \in \text{GL}(3) \) and \( b = (b_{ij}) \in B_{\text{GL}(3)} \).

3. GENERIC TORUS ORBIT CLOSURES IN FLAG BOTT MANIFOLDS

Let \( n = (n_1, \ldots, n_m) \) be a sequence of positive integers, \( F_m(n) \) an \( m \)-stage flag Bott manifold, and \( \mathbb{H} = H_{\text{GL}(n_1 + 1)} \times \ldots \times H_{\text{GL}(n_m + 1)} \). As introduced in [14, Sect. 3.1], the canonical action of \( \mathbb{H} \) on \( F_j = F_j(n) \) is defined to be

\[
(h_1, \ldots, h_m) \cdot [g_1, \ldots, g_j] := [h_1g_1, \ldots, h_jg_j] \quad \text{for} \quad 1 \leq j \leq m.
\]

Then \( q_j : F_j \rightarrow F_{j-1} \) is an \( \mathbb{H} \)-equivariant fiber bundle. Note that this action is not effective. If we write \( h_j = \text{diag}(h_{j,1}, \ldots, h_{j,n_{j+1}}) \in \text{GL}(n_j + 1) \), then the subtorus

\[
\mathbb{H} := \{(h_1, \ldots, h_m) \in \mathbb{H} \mid h_{1,n_1+1} = \ldots = h_{m,n_m+1} = 1 \} \cong (\mathbb{C}^*)^n
\]

acts effectively on \( F_m \), where

\[
n := n_1 + \ldots + n_m.
\]

In order to consider the closure of a torus orbit with respect to the canonical action, we define a generic element in \( F_m \). Let \( g = (g_{ij}) \) be an element in \( \text{GL}(n+1) \). For an ordered sequence \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n + 1 \), we define the Plücker coordinate

\[
X_{i_1, \ldots, i_k}(g) := \det((g_{i_p,j})_{p=1,\ldots,k}).
\]
The following theorem, whose proof will be given in Section 4.

Definition 3.1 [14, Definition 5.4]. We say that an element \( g \in \text{GL}(n + 1) \) is generic if \( X_{i_1, \ldots, i_k}(g) \) is nonzero for any \( k \in [n + 1] \) and any ordered sequence \( 1 \leq i_1 < \ldots < i_k \leq n + 1 \). For a sequence \( n = (n_1, \ldots, n_m) \) of positive integers, a point \( [g_1, \ldots, g_m] \in F_m(n) \) is generic if \( g_j \in \text{GL}(n_j + 1) \) is generic for all \( 1 \leq j \leq m \). A generic torus orbit in \( F_m(n) \) is the \( \mathbb{H} \)-orbit of a generic point.

The above definition of generic elements and generic torus orbits can be found in [8, 13, 6]. The closure \( X_n \) of a generic torus orbit in the flag manifold \( F\ell(n + 1) \) is a smooth projective toric variety called the permutohedral variety (see [12]). We recall the fan \( \Sigma_X \subset \mathbb{R}^n \) of the permutohedral variety \( X_n \) for the later use. The rays in \( \Sigma_X \) are parametrized by the nonempty proper subsets of \([n + 1]\):

\[
\Sigma_X(1) \overset{\epsilon-1}{\leftrightarrow} \{ S \mid \emptyset \subsetneq S \subseteq [n + 1] \}.
\]

More precisely, for a nonempty proper subset \( S \) of \([n + 1]\), the corresponding ray \( \rho_S \) is generated by

\[
u_S := \begin{cases} 
\sum_{s \in S} \varepsilon_s & \text{if } n + 1 \notin S, \\
- \sum_{s \in [n+1] \setminus S} \varepsilon_s & \text{otherwise},
\end{cases}
\]

where \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) is the standard vector basis of \( \mathbb{R}^n \). Hence there are \( 2^{n+1} - 2 \) rays in \( \Sigma_X \) (Fig. 1a). The maximal cones in \( \Sigma_X \) are indexed by the set of proper chains of \( n \) nonempty proper subsets of \([n + 1]\). For a given proper chain

\[
S_0: \emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq S_n \subsetneq [n + 1]
\]

of nonempty proper subsets, we have the corresponding maximal cone

\[
\text{Cone}(u_{S_1}, u_{S_2}, \ldots, u_{S_n})
\]

(Fig. 1b). Therefore, \(|\Sigma_X(n)| = (n + 1)!\).

Suppose that \( F_0 = F_\bullet(n) \) is an \( m \)-stage flag Bott tower for a sequence \( n = (n_1, \ldots, n_m) \) of positive integers. Considering the effective canonical \( \mathbb{H} \)-action, we see that each fiber of a bundle \( F_j \rightarrow F_{j-1} \) has the restricted \((\mathbb{C}^*)^{n_j}\)-action, and the orbit closure of a generic point is the permutohedral variety \( X_{n_j} \). Hence the closure of a generic torus orbit of the torus \( \mathbb{H} \) in \( F_m \) is an iterated permutohedral-variety bundle. Now we describe the fan of a generic torus orbit closure in \( F_m \) in the following theorem, whose proof will be given in Section 4.
Theorem 3.2. Let $F_m$ be a flag Bott manifold determined by a set of integer matrices $A = \{A^{(j)}_\ell \in M_{(n_{j+1})\times(n_{j+1})} (\mathbb{Z}) \mid 1 \leq \ell < j \leq m\}$. Then the fan $\Sigma \subset \mathbb{R}^n$ of a generic torus orbit closure $X$ in $F_m$ is described as follows:

1. The rays in $\Sigma$ are parametrized by
   \[ \{(\ell,S) \mid \emptyset \subsetneq S \subsetneq [n_\ell + 1], \ 1 \leq \ell \leq m\}. \]

For $(\ell,S)$, the corresponding ray is generated by

\[
 u^\ell_S := \begin{cases} 
  \sum_{s \in S} \varepsilon_{\ell,s} - \sum_{p=\ell+1}^m \sum_{k=1}^{n_p+1} (A^{(p)}_\ell)_{k,d} + \ldots + (A^{(p)}_\ell)_{k,n_\ell+1} \varepsilon_{p,k} & \text{if } n_\ell + 1 \notin S, \\
 - \sum_{s \in [n_\ell+1]\setminus S} \varepsilon_{\ell,s} - \sum_{p=\ell+1}^m \sum_{k=1}^{n_p+1} ((A^{(p)}_\ell)_{k,1} + \ldots + (A^{(p)}_\ell)_{k,d}) \varepsilon_{p,k} & \text{otherwise},
\end{cases}
\]

where $d = |[n_\ell + 1] \setminus S|$ and $\{\varepsilon_{\ell,k}\}_{1 \leq k \leq n_\ell, 1 \leq \ell \leq m}$ is the standard vector basis in $\mathbb{R}^n \cong \text{Lie}(T)$. Here we set $\varepsilon_{1,n_1+1} = \ldots = \varepsilon_{m,n_m+1} = 0$.

2. The maximal cones in $\Sigma$ are indexed by the sequences of proper chains of subsets
   \[ \{(S^1_1, \ldots, S^m_\ell) \mid S^\ell_\ell : \emptyset \subsetneq S^\ell_1 \subsetneq S^\ell_2 \subsetneq \ldots \subsetneq S^\ell_\ell \subsetneq [n_\ell + 1], \ 1 \leq \ell \leq m\}. \]

For $(S^1_1, \ldots, S^m_\ell)$, the corresponding maximal cone is defined to be

\[ \text{Cone}\left( \bigcup_{\ell=1}^m \left\{ u^\ell_{S^\ell_1}, \ldots, u^\ell_{S^\ell_{n_\ell}} \right\} \right). \]

We notice that if $F_m$ is determined by a sequence of integer matrices such that each $A^{(j)}_\ell$ has nonzero values only in the first column, then $F_m$ is an associated flag Bott manifold to a generalized Bott manifold (for more details, see [14, Sect. 4]). The fan of the generic torus orbit closure in an associated flag Bott manifold was computed in [14, Theorem 5.5], and Theorem 3.2 extends this result to arbitrary flag Bott manifolds.

Example 3.3. Let $F_2$ be the two-stage flag Bott manifold given in Example 2.4. Then the fan $\Sigma \subset \mathbb{R}^3$ of the generic torus orbit closure in $F_2$ has eight rays with the ray vectors

\[
 u^1_{(1)} = (1,0,0), \quad u^1_{(2)} = (0,1,0), \quad u^1_{(3)} = (-1,-1,c_1 + c_2), \quad u^2_{(1)} = (0,0,1), \\
 u^1_{(1,2)} = (1,1,-c_2), \quad u^1_{(2,3)} = (-1,0,c_1), \quad u^1_{(1,3)} = (0,-1,c_1), \quad u^2_{(2)} = (0,0,-1).
\]

Moreover, the fan $\Sigma$ has twelve maximal cones

\[
 \text{Cone}(u^1_{[1]}, u^1_{[1,2]}, u^2_{[1]}), \quad \text{Cone}(u^1_{[1]}, u^1_{[1,3]}, u^2_{[1]}), \quad \text{Cone}(u^1_{[2]}, u^1_{[1,2]}, u^2_{[1]}), \quad \text{Cone}(u^1_{[2]}, u^1_{[1,3]}, u^2_{[1]}), \\
 \text{Cone}(u^1_{[3]}, u^1_{[1,3]}, u^2_{[1]}), \quad \text{Cone}(u^1_{[3]}, u^1_{[2,3]}, u^2_{[1]}), \quad \text{Cone}(u^1_{[1]}, u^1_{[1,2]}, u^2_{[2]}), \quad \text{Cone}(u^1_{[1]}, u^1_{[1,3]}, u^2_{[2]}), \\
 \text{Cone}(u^1_{[2]}, u^1_{[1,2]}, u^2_{[2]}), \quad \text{Cone}(u^1_{[2]}, u^1_{[2,3]}, u^2_{[2]}), \quad \text{Cone}(u^1_{[3]}, u^1_{[1,3]}, u^2_{[2]}), \quad \text{Cone}(u^1_{[3]}, u^1_{[2,3]}, u^2_{[2]}).
\]

Example 3.4. Let $n = (3,3,2)$. Let $F_3(n)$ be a three-stage flag Bott manifold determined by

\[
 A^{(2)}_1 = \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{(3)}_1 = \begin{bmatrix} y_1 & y_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{(3)}_2 = \begin{bmatrix} z_1 & z_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Then the fan $\Sigma \subset \mathbb{R}^5$ of the generic torus orbit closure in $F_3(n)$ has 14 rays, which are generated by the following vectors:

\[
\begin{align*}
&u^1_{(1)} = (1, 0, 0, 0, 0), \quad u^1_{(2)} = (0, 1, 0, 0, 0), \quad u^1_{(3)} = (-1, -1, x_{11} + x_{12}, x_{21} + x_{22}, y_1 + y_2), \\
&u^1_{(1,2)} = (1, 1, -x_{12}, x_{22}, -y_2), \quad u^1_{(2,3)} = (-1, 0, x_{11}, x_{21}, y_1), \quad u^1_{(1,3)} = (0, -1, x_{11}, x_{21}, y_1), \\
&u^2_{(1)} = (0, 0, 0, 0, 0), \quad u^2_{(2)} = (0, 0, 0, 1, 0), \quad u^2_{(3)} = (0, 0, -1, -1, z_1 + z_2), \\
&u^2_{(1,2)} = (0, 0, 1, 1, -z_2), \quad u^2_{(2,3)} = (0, 0, -1, 0, z_1), \quad u^2_{(1,3)} = (0, 0, 0, -1, z_1), \\
&u^3_{(1)} = (0, 0, 0, 0, 1), \quad u^3_{(2)} = (0, 0, 0, 0, -1).
\end{align*}
\]

**Proposition 3.5.** All generic torus orbit closures in a flag Bott manifold $F_m$ are isomorphic non-singular toric varieties.

We recall that a toric variety $X_{\Sigma}$ determined by a fan $\Sigma \subset \mathbb{R}^n$ is non-singular if the set of ray generators of each maximal cone in $\Sigma$ forms a basis of $\mathbb{Z}^n$. To give a proof of Proposition 3.5, we use the following lemma.

**Lemma 3.6** [6, Corollary of Theorem 3.3]. The permutohedral variety $X_n$ is smooth; i.e., for a proper chain $S_\bullet: \emptyset \subset S_1 \subset S_2 \subset \ldots \subset S_n \subset [n + 1]$ of nonempty proper subsets of $[n + 1]$, the determinant of the matrix

\[
\begin{bmatrix}
    u_{S_1} & u_{S_2} & \ldots & u_{S_n}
\end{bmatrix}
\]

is $\pm 1$.

**Proof of Proposition 3.5.** Let $\Sigma \subset \mathbb{R}^n$ be the fan of a generic torus orbit closure $X$ in $F_m$ described in Theorem 3.2. To prove the claim, it is enough to show that every maximal cone in $\Sigma$ is smooth. For a maximal cone indexed by $(S_1^*, \ldots, S_m^*)$, consider the matrix whose column vectors are the corresponding ray generators:

\[
\begin{bmatrix}
    u_{S_1}^1 & u_{S_1}^2 & \ldots & u_{S_1}^m \\
    u_{S_2}^1 & u_{S_2}^2 & \ldots & u_{S_2}^m \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{S_m}^1 & u_{S_m}^2 & \ldots & u_{S_m}^m
\end{bmatrix}
\]

Then the matrix in (3.3) is a block lower triangular matrix with blocks of sizes $n_1, \ldots, n_m$. Moreover, the diagonal blocks have the same form as in (3.2). Hence the determinant of the matrix in (3.3) is

\[
\det \left( \begin{bmatrix} u_{S_1}^1 & \cdots & u_{S_1}^m \end{bmatrix} \right) \det \left( \begin{bmatrix} u_{S_2}^1 & \cdots & u_{S_2}^m \end{bmatrix} \right) \cdots \det \left( \begin{bmatrix} u_{S_m}^1 & \cdots & u_{S_m}^m \end{bmatrix} \right) = \pm 1
\]

by Lemma 3.6. Here $\{u_{S_1}^1, \ldots, u_{S_m}^m\}$ is the set of ray generators of the maximal cone in the fan of the permutohedral variety $X_{n_\ell}$ indexed by the proper chain $\emptyset \subset S_1^\ell \subset \ldots \subset S_{n_\ell}^\ell \subset [n_\ell + 1]$ for $1 \leq \ell \leq m$. Hence the non-singularity of $X$ follows.

Since the ray vectors $u_{S}^\ell$ are determined independently of the choice of generic points, the fans of all generic torus orbit closures are identical. Therefore, they are all isomorphic as smooth toric varieties. \(\square\)

**Example 3.7.** Let $F_3(n)$ be the three-stage flag Bott manifold defined in Example 3.4. Consider a maximal cone $\sigma$ indexed by

\[
\{\{2\} \subset \{2, 3\}, \{2\} \subset \{1, 2\}, \{2\}\}.
\]

Then the corresponding ray generators form the following matrix:

\[
\begin{bmatrix}
    u^1_{(2)} & u^1_{(2,3)} & u^2_{(2)} & u^2_{(1,2)} & u^2_{(1)} \\
    0 & -1 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 \\
    0 & x_{11} & 0 & 1 & 0 \\
    0 & x_{21} & 1 & 1 & 0 \\
    0 & y_{1} & 0 & -z_{2} & -1
\end{bmatrix}
\]
We have
\[
\det \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & x_{11} & 0 & 1 & 0 \\
0 & x_{21} & 1 & 1 & 0 \\
0 & y_1 & 0 & -z_2 & -1
\end{bmatrix} = \det \begin{bmatrix}
0 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 1
\end{bmatrix} \det [-1] = 1,
\]
so that the maximal cone \(\sigma\) is smooth.

4. PROOF OF THEOREM 3.2

In this section we give a proof of Theorem 3.2 using the combinatorial structure of the fans of toric-variety bundles and isotropy representations around fixed points. To describe the fan \(\Sigma\) of the torus orbit closure in \(F_m\), we recall a result on an equivariant fiber bundle of toric varieties. Let \(\Sigma\) and \(\Sigma'\) be complete fans in \(N_\mathbb{R} := N \otimes \mathbb{R}\) and \(N''_\mathbb{R} := N' \otimes \mathbb{R}\) for some lattices \(N\) and \(N'\), respectively. Let \(\varphi: N \rightarrow N''\) be compatible with the fans \(\Sigma\) and \(\Sigma'\), and let \(N'' = \ker(\varphi)\). Then we have the corresponding equivariant morphism \(\varphi: X_\Sigma \rightarrow X_{\Sigma'}\) between toric varieties (see [16, Theorem 4.1]). In this setting we have the following proposition.

**Proposition 4.1** [16, Proposition 7.3; 5, § 3.3]. The equivariant morphism \(\varphi: X_\Sigma \rightarrow X_{\Sigma'}\) is an equivariant fiber bundle with the fiber \(X_{\Sigma''}\), where \(\Sigma'' = \{\sigma \in \Sigma \mid \sigma \subset N''_\mathbb{R}\} \subset N''_\mathbb{R}\), if and only if the following conditions are satisfied:

1. \(\varphi: N \rightarrow N''\) is surjective;
2. there exists a lifting \(\tilde{\Sigma}' \subset \Sigma\); i.e., for each \(\sigma' \in \Sigma'\), there exists a unique \(\tilde{\sigma}' \in \tilde{\Sigma}'\) such that \(\varphi\) induces a bijection
   \[\varphi: \tilde{\sigma}' \sim \sigma';\]
3. \(\Sigma = \tilde{\Sigma}' \cdot \Sigma''\); i.e., \(\Sigma\) consists of cones
   \[\sigma = \tilde{\sigma}' + \sigma''\]
   with \(\tilde{\sigma}'\) and \(\sigma''\) running through \(\tilde{\Sigma}'\) and \(\Sigma''\), respectively.

The operation \(\cdot\) is called a join in [7, Ch. III, Sect. 1]. The closure of a generic torus orbit of the torus \(H\) in \(F_m\) is an iterated permutohedral-variety bundle. Hence we have the following lemma.

**Lemma 4.2** (see [14, Lemma 5.11]). Let \(F_m\) be an \(m\)-stage flag Bott manifold. Let \(\Sigma\) be the fan of the closure of a generic torus orbit of the torus \(H\) in \(F_m\). Then there are liftings \(\tilde{\Sigma}_{X_{n_1}}, \ldots, \tilde{\Sigma}_{X_{n_{m-1}}} \cdot \Sigma_{X_{n_m}}\) of fans of permutohedral varieties such that

\[\Sigma = \tilde{\Sigma}_{X_{n_1}} \cdot \ldots \cdot \tilde{\Sigma}_{X_{n_{m-1}}} \cdot \Sigma_{X_{n_m}}\]

where \(\Sigma_{X_{n_m}}\) is the fan of the permutohedral variety \(X_{n_m}\).

The above lemma implies that the combinatorial structure of the fan \(\Sigma\) is indeed as described in assertion 2 of Theorem 3.2. Now it suffices to show that the ray vectors are given by formula (3.1). To complete the proof, we use the weights of the isotropy representation of \(T\) in \(T_vF_m\) for a fixed point \(v\) which is computed in [14, Sect. 3].
in $\mathbb{H}$ and $\mathbf{H}$, respectively. Then, for every fixed point $v := [\hat{v}_1, \ldots, \hat{v}_m]$, the weights of the isotropy representation of $\mathbb{T}$ in $T_vF_m$ are explicitly computed in [14, Proposition 3.5].

Note that there is a one-to-one correspondence between the set $\mathfrak{S}_{n_1+1} \times \cdots \times \mathfrak{S}_{n_m+1}$ and the sequences of chains of subsets \{(S_1^1, \ldots, S_m^m)\}. Namely, for $(v_1, \ldots, v_m) \in \mathfrak{S}_{n_1+1} \times \cdots \times \mathfrak{S}_{n_m+1}$, we define
\[
S_p^\ell := \{v_{\ell}(n_{\ell} + 2 - p), \ldots, v_{\ell}(n_{\ell} + 1)\} \quad \text{for} \quad 1 \leq p \leq n_{\ell}, \quad 1 \leq \ell \leq m. \tag{4.1}
\]
Moreover, for a given maximal cone indexed by $(v_1, \ldots, v_m)$, the adjacent maximal cones are indexed by the permutations
\[
(v_1, \ldots, v_{j-1}, v_j \cdot s_i, v_{j+1}, \ldots, v_m) \tag{4.2}
\]
where $s_i$ is the transposition $(i, i + 1)$ for $1 \leq i \leq n_j$ and $1 \leq j \leq m$.

For a smooth projective toric variety $X_{\Sigma}$ of complex dimension $n$, the weights of the isotropy representations around fixed points and the ray generators are closely related to each other. Let $\rho$ be a ray in $\Sigma(1)$ with $u_{\rho} =: u_1$ as its generating vector. Suppose that $\sigma = \text{Cone}(u_1, \ldots, u_n)$ is a maximal cone containing $\rho$. Let $\tau_1, \ldots, \tau_n$ be the codimension 1 faces of the cone $\sigma$. More precisely, we put
\[
\tau_j = \text{Cone}(u_1, \ldots, \hat{u}_j, \ldots, u_n).
\]
Then $\tau_1$ is the unique codimension 1 face of $\sigma$ which does not contain $\rho$. By the orbit–cone correspondence, these cones correspond to torus-invariant spheres $S_1, \ldots, S_n$ in $X_{\Sigma}$. Then these spheres meet at a point $p \in X_{\Sigma}$, which is exactly the fixed point corresponding to $\sigma$. Suppose that $w_1, \ldots, w_n \in \text{Lie}(T)^*$ are weights of the isotropy representation $T_pX_{\Sigma}$, where $T$ is the compact torus of dimension $n$. Let $H_i$ be the identity component of the kernel of the map
\[
\exp(\sqrt{-1}w_i) : T \to S^1.
\]
Then, by reordering $w_1, \ldots, w_n$ appropriately, one can see that the sphere $S_i$ is the connected component of $X_{\Sigma}(S^2_\mu)$ containing $\rho$. Moreover, we have the following.

**Lemma 4.3** [2, Proposition 7.3.18]. Let $u_1 = u_{\rho}, w_1, \ldots, w_n$ be as above. Then we have the following relation:
\[
\langle w_i, u_{\rho} \rangle = \begin{cases} 
1 & \text{if } i = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

The above lemma implies that the ray generators are completely determined by the weights of isotropy representations around fixed points. Moreover, one can see that the computation of the ray generator $u_{\rho}$ is independent of the choice of a maximal cone containing $\rho$ (see [14, Lemma 5.13]).

We choose $1 \leq \ell \leq m$ and a nonempty proper subset $S$ of $[n_{\ell} + 1]$. To compute the generator $u_{\rho}^\ell_S$ of the ray $\rho_{\Sigma}^\ell$, we consider a specific maximal cone $\sigma_{\Sigma}^\ell$ containing $\rho_{\Sigma}^\ell$. We let $d = \lfloor n_{\ell} + 1 \rfloor \setminus S$ and set $S = \{s_1 < s_2 < \ldots < s_{n_{\ell+1} - 1}\}$ and $[n_{\ell} + 1] \setminus S = \{t_1 < t_2 < \ldots < t_d\}$. Let $v_{\ell,S}$ be a permutation in $\mathfrak{S}_{n_{\ell+1}}$ defined by
\[
v_{\ell,S} = (t_1 \ t_2 \ \ldots \ \ t_d \ s_1 \ s_2 \ \ldots \ s_{n_{\ell+1} - 1}).
\]
Then we choose a maximal cone $\sigma_{\Sigma}^\ell$ indexed by
\[
v = (v_1, \ldots, v_m) := (e, \ldots, e, v_{\ell,S}, e, \ldots, e) \in \mathfrak{S}_{n_1+1} \times \cdots \times \mathfrak{S}_{n_m+1}. \tag{4.3}
\]
so that $\sigma_{\Sigma}^\ell$ contains the ray $\rho_{\Sigma}^\ell$. Here, $e$ is the identity element.

The maximal cones intersecting $\sigma_{\Sigma}^\ell$ are indexed by $(v_1, \ldots, v_{j-1}, v_j \cdot s_i, v_{j+1}, \ldots, v_m)$ for $1 \leq i \leq n_j$ and $1 \leq j \leq m$. Then the weight $w_{\ell}^\ell$ corresponding to the codimension 1 face of $\sigma_{\Sigma}^\ell$ obtained as the intersection of $\sigma_{\Sigma}^\ell$ with the maximal cone indexed by $(v_1, \ldots, v_{j-1}, v_j \cdot s_i, v_{j+1}, \ldots, v_m)$ is computed as follows.
**Proposition 4.4** [14, Proposition 3.5, Theorem 3.11]. For \(1 \leq i \leq n_j\) and \(1 \leq j \leq m\), we have \(w_i^j = r_{i+1}^j - r_i^j \in \text{Lie}(T)^*\), where \(r_i^j\) is the linear combination of the duals
\[
\varepsilon_{1,1}^*, \ldots, \varepsilon_{1,n_1+1}^*, \ldots, \varepsilon_{m,1}^*, \ldots, \varepsilon_{m,n_m+1}^*
\]
of the standard basis vectors with coefficients given by the \(i\)-th row of the matrix
\[
\begin{bmatrix}
X_1^{(j)} & X_2^{(j)} & \ldots & X_{j-1}^{(j)} & B_j & O & \ldots & O
\end{bmatrix}.
\]

Here \(X_k^{(j)}\) is the \((n_j + 1) \times (n_{1} + 1)\) matrix defined by
\[
X_k^{(j)} = \sum_{\ell<k_1<\ldots<k_s<j} (B_{k_1}A_{k_1}^{(j)}) (B_{k_2}A_{k_2}^{(k_1)}) \ldots (B_{k_s}A_{k_s}^{(k_1)}) B_{\ell} + B_j A_{\ell}^{(j)} B_{\ell}
\]
for \(1 \leq \ell < j \leq m\), \(B_j\) is the row permutation matrix corresponding to \(v_j\), i.e., \(B_j = (\hat{v}_j)^T\), and \(O\) is the zero matrix.

We note that the codimension 1 face of the cone \(\sigma^{\ell}_S\) which does not contain the ray \(\rho^{\ell}_S\) corresponds to the weight \(w_j^\ell\). Hence, to complete the proof, it is enough to show that the vector (3.1) in Theorem 3.2 satisfies the relation
\[
\langle w_i^j, u_S^\ell \rangle = \begin{cases}
1 & \text{if } j = \ell \text{ and } i = d, \\
0 & \text{otherwise}. \quad (4.5)
\end{cases}
\]

The weight \(w_i^j\), computed in Proposition 4.4, is an element of \(\text{Lie}(T)^*\). Since we have
\[
T = \{(t_1, \ldots, t_m) \in T \mid t_{1,1} + \ldots = t_{m,n_m+1} = 1 \}
\]
where \(t_j = (t_{j,1}, \ldots, t_{j,n_j+1}) \in (S^1)^{n_j+1}\), we assume that
\[
\varepsilon_{1,1}^* = \ldots = \varepsilon_{m,n_m+1}^* = 0
\]
to compute the pairing (4.5).

To prove the claim, we consider separately the cases \(j < \ell\), \(j = \ell\), and \(j > \ell\).

**Case 1:** \(j < \ell\). By Proposition 4.4, the weight vector \(w_i^j \in \text{Lie}(T)^*\) is a linear combination of \(\varepsilon_{1,1}^*, \ldots, \varepsilon_{1,n_1}^*, \ldots, \varepsilon_{j,1}^*, \ldots, \varepsilon_{j,n_j}^*\). On the other hand, since \(u_S^\ell\) is a linear combination of \(\varepsilon_{\ell,1}, \ldots, \varepsilon_{\ell,n_{\ell}}, \ldots, \varepsilon_{m,1}, \ldots, \varepsilon_{m,n_m}\) and \(j < \ell\), their pairings always vanish.

**Case 2:** \(j = \ell\). By Proposition 4.4, the weight vector \(w_i^\ell \in \text{Lie}(T)^*\) is a linear combination of \(\varepsilon_{1,1}^*, \ldots, \varepsilon_{1,n_1}^*, \ldots, \varepsilon_{1,1}^*, \ldots, \varepsilon_{\ell,1}^*, \ldots, \varepsilon_{\ell,n_\ell}^*\). More precisely, we have
\[
w_i^\ell = \varepsilon_{i,v_S(i)+1}^* - \varepsilon_{i,v_S(i)}^* + \text{other terms},
\]
where the “other terms” are \(\varepsilon_{p,k}^*\) with \(p < \ell\) and \(v_S^\ell\) is a permutation defined in (4.3). Since the vector \(u_S^\ell\) is a linear combination of \(\varepsilon_{\ell,1}, \ldots, \varepsilon_{\ell,n_{\ell}}, \ldots, \varepsilon_{m,1}, \ldots, \varepsilon_{m,n_m}\), we have
\[
\langle w_i^\ell, u_S^\ell \rangle = \langle \varepsilon_{i,v_S(i)+1}^*, \varepsilon_{i,v_S(i)}^* \rangle (u_S^\ell).
\]

Because of the definition of the permutation \(v_S^\ell\), we have \(v_S^\ell(i) \in S\) if and only if \(i \geq d + 1\). Therefore, in the case when \(n_\ell + 1 \notin S\), the value \(\langle \varepsilon_{i,v_S(i)}^*, u_S^\ell \rangle\) is equal to 0 if \(i \leq d\) and to 1 otherwise. In the case when \(n_\ell + 1 \in S\), we find that the pairing \(\langle \varepsilon_{i,v_S(i)}^*, u_S^\ell \rangle\) is equal to 0 if \(i \leq d\) and to 1 otherwise.
By applying (4.6) for \( n_\ell + 1 \notin S \), we have the following:

\[
\langle w_i^\ell, u_S^\ell \rangle = \begin{cases} 
0 - 0 = 0 & \text{for } 1 \leq i \leq d - 1, \\
1 - 0 = 1 & \text{for } i = d, \\
1 - 1 = 0 & \text{for } d + 1 \leq i \leq n_\ell.
\end{cases}
\]

Similarly, when \( n_\ell + 1 \in S \), we get the following:

\[
\langle w_i^\ell, u_S^\ell \rangle = \begin{cases} 
-1 - (-1) = 0 & \text{for } 1 \leq i \leq d - 1, \\
0 - (-1) = 1 & \text{for } i = d, \\
0 - 0 = 0 & \text{for } d + 1 \leq i \leq n_\ell.
\end{cases}
\]

**Case 3: \( j > \ell \).** Since \( v_j = e \) for \( j \neq \ell \), the matrix \( X_\ell^{(j)} \) in Proposition 4.4 (see (4.4)) can be written as

\[
X_\ell^{(j)} = \left( \sum_{\ell < k_1 < \ldots < k_s < j} A_{k_s}^{(k_s)} \ldots A_{\ell}^{(k_1)} + A_{\ell}^{(j)} \right) B_\ell.
\]

Moreover, we have

\[
X_\ell^{(j)} B_\ell^{-1} = \sum_{\ell < k_1 < \ldots < k_s < j} A_{k_s}^{(k_s)} \ldots A_{\ell}^{(k_1)} + A_{\ell}^{(j)}
= X_{j-1}^{(j)} A_{\ell}^{(j-1)} + \ldots + X_{\ell+2}^{(j)} A_{\ell}^{(\ell+2)} + X_{\ell+1}^{(j)} A_{\ell}^{(\ell+1)} + A_{\ell}^{(j)}
= \sum_{p=\ell+1}^{j-1} X_p^{(j)} A_{\ell}^{(p)} + A_{\ell}^{(j)}.
\]

(4.7)

For notational simplicity, we denote the weight determined by the \( i \)th row vector of the matrix \( X_\ell^{(j)} \) by \( x_i^{(j)} \). Then the weight \( r_i^j \) in Proposition 4.4 equals

\[
x_{1,i}^{(j)} + \ldots + x_{j-1,i}^{(j)} + \varepsilon_{j,i}^*
\]

because \( B_j \) is the identity matrix. Hence the pairing

\[
\left\langle r_{i}^{j}, \sum_{p=\ell+1}^{m} \sum_{k=1}^{n_{p+1}} (A_{\ell}^{(p)})_{k,z} \varepsilon_{p,k} \right\rangle = \left\langle x_{1,i}^{(j)} + \ldots + x_{j-1,i}^{(j)} + \varepsilon_{j,i}^*, \sum_{p=\ell+1}^{m} \sum_{k=1}^{n_{p+1}} (A_{\ell}^{(p)})_{k,z} \varepsilon_{p,k} \right\rangle
\]

(4.8)

is the \((i, z)\)-entry of the matrix \( \sum_{p=\ell+1}^{j-1} X_p^{(j)} A_{\ell}^{(p)} + A_{\ell}^{(j)} \) for \( 1 \leq z \leq n_\ell + 1 \). By (4.7), the pairing (4.8) is the same as the \((i, z)\)-entry of the matrix \( X_\ell^{(j)} B_\ell^{-1} \):

\[
\left\langle r_{i}^{j}, \sum_{p=\ell+1}^{m} \sum_{k=1}^{n_{p+1}} (A_{\ell}^{(p)})_{k,z} \varepsilon_{p,k} \right\rangle = (X_\ell^{(j)} B_\ell^{-1})_{i,z}.
\]

(4.9)

**Subcase 1: \( n_\ell + 1 \notin S \).** In this case, we first note that

\[
\left\langle r_{i}^{j}, \sum_{s \in S} \varepsilon_{s}, s \right\rangle = \left\langle x_{1,i}^{(j)}, \sum_{s \in S} \varepsilon_{s}, s \right\rangle = \sum_{s \in S} (X_\ell^{(j)})_{i,s}.
\]

(4.10)

Since \( B_\ell^{-1} \) is the column permutation matrix for \( u_{\ell,A} \), the right-hand side of (4.10) coincides with

\[
(X_\ell^{(j)} B_\ell^{-1})_{i,d+1} + (X_\ell^{(j)} B_\ell^{-1})_{i,d+2} + \ldots + (X_\ell^{(j)} B_\ell^{-1})_{i,n_\ell+1}.
\]

(4.11)
This is the same as the $i$th entry of the sum of the last $n_\ell + 1 - d$ column vectors in $X^{(j)}_\ell B^{-1}_\ell$. Hence we have the following:

$$
\langle r^j_i, u^\ell_S \rangle = \left\langle r^j_i, \sum_{s \in S} \varepsilon_{\ell, s} - \sum_{p=\ell+1}^m \sum_{k=1}^{n_p+1} \left( (A^{(p)}_\ell)_{k,d+1} + \ldots + (A^{(p)}_\ell)_{k,n_\ell+1} \right) \varepsilon_{p,k} \right\rangle
$$

$$
= \sum_{s \in S} \langle X^{(j)}_\ell \rangle_{i,s} - \left( \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,d+1} + \ldots + \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,n_\ell+1} \right) \quad \text{(by (4.9) and (4.10))}
$$

$$
= 0 \quad \text{(by (4.11))}.
$$

Since $w^j_i = r^j_{i+1} - r^j_i$, the pairing $\langle w^j_i, u^\ell_S \rangle$ vanishes.

**Subcase 2:** $n_\ell + 1 \in S$. In this case, we have

$$
\left\langle r^j_i, - \sum_{s \in [n_\ell+1] \setminus S} \varepsilon_{\ell, s} \right\rangle = \left\langle x^{(j)}_{\ell,i} - \sum_{s \in [n_\ell+1] \setminus S} \varepsilon_{\ell, s} \right\rangle = - \sum_{s \in [n_\ell+1] \setminus S} \langle X^{(j)}_\ell \rangle_{i,s}. \quad (4.12)
$$

Since $B^{-1}_\ell$ is the column permutation matrix for $v_{\ell,A}$, the right-hand side of (4.12) coincides with

$$
-\left( \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,1} - \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,2} - \ldots - \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,d} \right). \quad (4.13)
$$

This is the same as the $i$th entry of the sum of the first $d$ column vectors in $-X^{(j)}_\ell B^{-1}_\ell$. Therefore, using an argument similar to that in Subcase 1, one can see that

$$
\langle r^j_i, u^\ell_S \rangle = \left\langle r^j_i, - \sum_{s \in [n_\ell+1] \setminus S} \varepsilon_{\ell, s} + \sum_{p=\ell+1}^m \sum_{k=1}^{n_p+1} \left( (A^{(p)}_\ell)_{k,1} + \ldots + (A^{(p)}_\ell)_{k,d} \right) \varepsilon_{p,k} \right\rangle
$$

$$
= - \sum_{s \in [n_\ell+1] \setminus S} \langle X^{(j)}_\ell \rangle_{i,s} + \left( \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,1} + \ldots + \langle X^{(j)}_\ell B^{-1}_\ell \rangle_{i,d} \right) \quad \text{(by (4.9) and (4.12))}
$$

$$
= 0 \quad \text{(by (4.13))}.
$$

Hence the pairing $\langle w^j_i, u^\ell_S \rangle$ vanishes, since $w^j_i = r^j_{i+1} - r^j_i$.

**FUNDING**

The authors were partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (no. 2016R1A2B4010823). The first author was also partially supported by IBS-R003-D1.

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