Coherent states and global entanglement in an $N$ qubit system

Piroska Dömötör and Mihály G. Benedict*

Department of Theoretical Physics, University of Szeged, Hungary

Abstract

We consider an $N$ qubit system and show that in the symmetric subspace, a state is not globally entangled, iff it is a coherent state. It is also proven that in the orthogonal complement $S_\perp$ all states are globally entangled.

1 Nonentangled pure states

A pure $N$ qubit state is not entangled by definition, if it is a product state. In order to decide if a pure state is entangled or not, we shall use the following formal method used first by Meyer and Wallach [1], who applied this procedure to define a measure of entanglement for $N$ qubit states. The pure state $|\psi\rangle \in \mathbb{C}^{2^N}$, which is expanded in the standard basis $\{|0\rangle, |1\rangle\}^\otimes N$, can be decomposed for each $n = 1, 2 \ldots N$ qubit as

$$|\psi\rangle = |0\rangle_n \otimes |u^n\rangle + |1\rangle_n \otimes |v^n\rangle,$$

where $|u^n\rangle$ and $|v^n\rangle$ are vectors in $\mathbb{C}^{2^{N-1}}$ which are not normalized, in general. Using the above decomposition one can show that $|\psi\rangle$ is a product state if and only if $|u^n\rangle$ is parallel to $|v^n\rangle$ for all possible $n$.

First, assume that $|\psi\rangle$ is a product state and can be written as

$$|\psi\rangle = \bigotimes_{i=1}^N (a_i |1_i\rangle + b_i |0_i\rangle),$$

with $|a_i|^2 + |b_i|^2 = 1$. In this case $|u^n\rangle = b_i \bigotimes_{i=1}^N a_i |1_i\rangle + b_i |0_i\rangle$, while $|v^n\rangle = a_n \bigotimes_{i=1\atop i\neq n}^N (a_i |1_i\rangle + b_i |0_i\rangle)$, and it is obvious that these two vectors are parallel.

*benedict@physx.u-szeged.hu
Second, let \( |u^n\rangle \) be parallel to \( |v^n\rangle \) for all possible \( n \): \( |u^n\rangle = \alpha_n |v^n\rangle \). Then \( |\psi\rangle \) can be written in the following form:

\[
|\psi\rangle = (1 + |\alpha_n|^2)^{-1/2} (|0\rangle_n + \alpha_n |1\rangle_n) \otimes |\tilde{u}^n\rangle \quad \forall \, n,
\]

and the statement can be proven by induction. (Here the \( N - 1 \) qubit state \( |\tilde{u}^n\rangle \) is normalized.) For \( N = 2 \) it is obviously true, because then \( |\psi\rangle = (1 + |\alpha_1|^2)^{-1/2} (|0\rangle_1 + \alpha_1 |1\rangle_1) \otimes |\tilde{u}^1\rangle \) and \( |\tilde{u}^1\rangle \) is a one qubit state.

Suppose now that the statement is true for \( N - 1 \), and let’s prove it for \( N \). We first use the decomposition (3) with respect to the \( i \)-th qubit, where \( |\tilde{u}^i\rangle \) is now an \( N - 1 \) qubit state. Decompose \( |\tilde{u}^i\rangle \) further, with respect to the \( j \)-th qubit:

\[
|\psi\rangle = \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) \otimes |\tilde{u}^i\rangle = \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) \otimes (|0\rangle_j \otimes |u^{ij}\rangle + |1\rangle_j \otimes |v^{ij}\rangle) =
\]

\[
= |0\rangle_j \otimes \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) \otimes |u^{ij}\rangle + |1\rangle_j \otimes \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) \otimes |v^{ij}\rangle,
\]

and compare this with

\[
|\psi\rangle = \left(\frac{|0\rangle_j + \alpha_j |1\rangle_j}{\sqrt{1 + |\alpha_j|^2}}\right) \otimes |\tilde{u}^i\rangle.
\]

As a result we get

\[
\left(\frac{|u^{ij}\rangle}{\sqrt{1 + |\alpha_j|^2}}\right) \otimes \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) = \left(\frac{|0\rangle_i + \alpha_i |1\rangle_i}{\sqrt{1 + |\alpha_i|^2}}\right) \otimes \left(\frac{|0\rangle_j + \alpha_j |1\rangle_j}{\sqrt{1 + |\alpha_j|^2}}\right) \otimes |v^{ij}\rangle
\]

which implies that \( \alpha_j |u^{ij}\rangle = |v^{ij}\rangle \). Then by hypothesis \( |\tilde{u}^i\rangle \) can be written as a product state, and according to \( |\psi\rangle = (1 + |\alpha_i|^2)^{-1/2} (|0\rangle_i + \alpha_i |1\rangle_i) \otimes |\tilde{u}^i\rangle \) the \( N \) qubit state \( |\psi\rangle \) is also a product state.

### 2 Symmetric subspace and atomic coherent states

Now we recall the notion of the symmetric subspace \([2, 3]\). Consider the standard basis, and those vectors of this basis, for which the number of 1-s, \( N_1 \) is fixed, and accordingly the number of 0-s, \( N_0 = N - N_1 \) is also fixed. These vectors span a subspace of dimension \( \binom{N}{N_1} \). The number of such disjoint subspaces is \( N + 1 \), and they obviously exhaust the whole space. In each such subspace there is exactly one state which is symmetric with respect of the permutations of the qubits. One can obtain it by taking any vector \( |\varphi_{N_1}\rangle \) in the given subspace, say \( |\varphi_{N_1}\rangle := |\underbrace{0, \cdots, 0}_{N_0}, 1, \cdots, 1\rangle \) (with obvious simplified notation) and then
applying the symmetrization projector $\mathcal{S}$

$$\mathcal{S} |\varphi_{N_1}\rangle = C \sum_\nu P_\nu |\varphi_{N_1}\rangle ,$$

(7)

where $P_\nu$s are the permutation operators in $\mathbb{C}^{2^N}$ representing the permutation group of the qubits in the natural way, while $C$ is an appropriately chosen normalization coefficient [4]. Taking all these (mutually orthogonal) states with $N_1 = 0, 1 \ldots N$, they span the $N + 1$ dimensional symmetric subspace, $\mathcal{S}$ of the total space.

We shall also need the unnormalized symmetric vectors defined as:

$$|\frac{N}{N_1}\rangle := \left( \begin{array}{c} N \\ N_1 \end{array} \right)^{\frac{1}{2}} \mathcal{S} |\varphi_{N_1}\rangle ,$$

(8)

which will be useful later. For example $|\frac{3}{1}\rangle = |0, 0, 1\rangle + |0, 1, 0\rangle + |1, 0, 0\rangle$.

The state $|\frac{N}{0}\rangle = |0, 0 \ldots 0\rangle$ is obviously symmetric and not entangled. One can consider global rotations of this latter state by introducing the following sums of individual qubit spin operators:

$$J_\alpha = \sum_{n=0}^{N} S^n_\alpha, \quad (\alpha = x, y, z),$$

(9)

with

$$S^n_x = (|1\rangle \langle 0| + |0\rangle \langle 1|)_{n/2}, \quad S^n_y = (|1\rangle \langle 0| - |0\rangle \langle 1|)_{n/2}i, \quad S^n_z = (|1\rangle \langle 1| - |0\rangle \langle 0|)_{n/2}. \quad (10)$$

The symmetric vector $\mathcal{S} |\varphi_{N_1}\rangle$ in (7) is the normalized eigenstate of $J_z$ with the eigenvalue $m = (N_1 - N_0)/2$, and therefore we will denote it by $|m\rangle$ [23], where the possible values of $m$ are $-N/2, -N/2 + 1, \ldots, N/2$.

If $u$ is a unit vector corresponding to a point on the unit sphere in real three dimensional space, making the polar angle $\theta$ with the negative $z$ axis, and the azimuth $\varphi$ with the positive $x$ axis, then

$$R_{\theta\varphi} = e^{-i\theta(J_x \sin \varphi - J_y \cos \varphi)}$$

(11)

is a unitary rotation in $\mathbb{S}$, and the state

$$|\tau_u\rangle = R_{\theta\varphi} |\frac{N}{0}\rangle = R_{\theta\varphi} |m = -N/2\rangle .$$

(12)

is called an atomic coherent state [3]. It can be also shown that $|\tau_u\rangle$ is the normalized eigenstate of $\mathbf{J} \cdot u$ belonging to its highest eigenvalue $N/2$.

$$(\mathbf{J} \cdot u) |\tau_u\rangle = \frac{N}{2} |\tau_u\rangle .$$

(13)

The notation $|\tau_u\rangle$ is related to another parametrization of the state. Following [3] the vector $u$ can be alternatively charcterized by the complex number $\tau = \tan(\theta/2)e^{-i\varphi}$, which is the stereographic projection of the unit vector $u$ to the $x$-$y$ plane. The unit vector $u$ is called sometimes the Bloch (or Poincaré) vector of this state, similarly to the single qubit case. The state $|\frac{N}{0}\rangle = |0, 0 \ldots 0\rangle$, which is obviously symmetric and not entangled, is a coherent state itself, corresponding to $u = -\hat{z}$, i.e., to $\tau = 0$, and it is the eigenstate of $J_z$ with the eigenvalue $m = -N/2 = -j$.  

3
3  A state in $S$ is not entangled iff it is an atomic coherent state

First we point out that the coherent state $|\tau_u\rangle$ is a product state, and thus it is not entangled. This can be seen by expanding it in terms of the symmetrized eigenstates of $J_z$:

$$|\tau_u\rangle = \sum_{m=-j}^{j} \left( \begin{array}{c} j \\ m \end{array} \right) \frac{\tau^{j+m}}{(1+|\tau|^2)^{j/2}} |m\rangle,$$

(14)

where $J_z |m\rangle = m |m\rangle$. Using the unnormalized states given in (8), $|N_k\rangle$ we can write

$$|\tau_u\rangle = \sum_{k=0}^{N} \frac{\tau^k}{(1+|\tau|^2)^{k/2}} |N_k\rangle = \frac{1}{(1+|\tau|^2)^{N/2}} (|0\rangle + \tau |1\rangle)^\otimes N,$$

(15)

where we have used the binomial theorem. The latter form shows that $|\tau_u\rangle$-s are product states, and therefore are not entangled.

Now we prove the reverse statement: in the totally symmetric subspace all the nonentangled states are atomic coherent states. To this end we consider a general linear combination of the symmetric states

$$|\psi\rangle = \sum_{k=0}^{N} \left( \begin{array}{c} N \\ k \end{array} \right)^{1/2} \ c_k \left| m = -\frac{N}{2} + k \right\rangle,$$

(16)

where the square root of the binomial coefficients have been factored out, and the $c_k$-s are arbitrary numbers obeying $\sum_k \left( \begin{array}{c} N \\ k \end{array} \right) |c_k|^2 = 1$, chosen to have $\langle \psi | \psi \rangle = 1$. With the unnormalized states $|N_k\rangle$ we can write

$$|\psi\rangle = \sum_{k=0}^{N} c_k |N_k\rangle, \quad \sum_k \left( \begin{array}{c} N \\ k \end{array} \right) |c_k|^2 = 1.$$

(17)

The $|N_k\rangle$-s have the following property

$$|N_k\rangle = |0\rangle_n \otimes |N_{k-1}\rangle + |1\rangle_n \otimes |N_{k-1}\rangle,$$

(18)

where $|N_{k-1}\rangle = |N_{k-1}\rangle = 0$, by definition. The above decompositions, which correspond to the elementary identity $\left( \begin{array}{c} N \\ k \end{array} \right) = \left( \begin{array}{c} N-1 \\ k \end{array} \right) + \left( \begin{array}{c} N-1 \\ k-1 \end{array} \right)$ are valid for any $n = 1, \ldots N$, as a consequence of the symmetry of the states $|N_k\rangle$ with respect of permutations. Then

$$|\psi\rangle = \sum_{k=0}^{N} c_k |N_k\rangle = \sum_{k=0}^{N} c_k (|0\rangle_n \otimes |N_{k-1}\rangle + |1\rangle_n \otimes |N_{k-1}\rangle).$$

(19)
Therefore we have

\[ l_n(0) |\psi\rangle = \sum_{k=0}^{N-1} c_k |_k^{N-1}\rangle, \]

\[ l_n(1) |\psi\rangle = \sum_{k=1}^{N} c_k |_k^{N-1}\rangle = \sum_{k=0}^{N-1} c_{k+1} |_k^{N-1}\rangle. \]

According to the section 1 \(|\psi\rangle\) is not entangled if these vectors are parallel, requiring \(c_{k+1} = \tau c_k\), and thus

\[ c_k = \tau^k c_0 \]

Then the nonentangled \(|\psi\rangle\) has necessarily the following form:

\[ |\psi\rangle = \sum_{k=0}^{N} \tau^k \cdot c_0 |N\rangle = c_0 \sum_{k=0}^{N} \tau^k |N\rangle = c_0 (|0\rangle + \tau |1\rangle)^{\otimes N}, \]

with \(c_0 = (1 + |\tau|^2)^{-N/2}\) as required by the normalization condition. Comparison with (16) proves the statement: in the symmetric subspace only the coherent states are nonentangled.

4 The vectors orthogonal to \(S\) are all entangled

We shall now consider vectors in \(S_\perp\) the orthogonal complement of the symmetric space. We prove that all vectors in \(S_\perp\) are globally entangled. Assume to the contrary, that there exists a vector \(|\varphi\rangle\) which can be written as a product:

\[ |\varphi\rangle = \bigotimes_{n=1}^{N} (a_n |1\rangle_n + b_n |0\rangle_n) \in S_\perp, \]

with \(|a_n|^2 + |b_n|^2 = 1\) for each \(n\). At least one of the \(a\)-s, say \(a_n\), and one of the \(b\)-s \(b_m\) with different indices \((n \neq m)\) must be zero, otherwise \(|\varphi\rangle\) would have a nonzero projection on \(|m = N/2\rangle = |1,1,\ldots,1\rangle\), and on \(|m = -N/2\rangle = |0,0,\ldots,0\rangle\) that are elements of \(S\), which would contradict to \(|\varphi\rangle\in S_\perp\). Without loss of generality we may assume that the vanishing coefficients are \(a_1 = 0\) and \(b_2 = 0\). Then \(|\varphi\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes_{n=3}^{N} (a_n |1\rangle_n + b_n |0\rangle_n) = |0\rangle_1 \otimes |1\rangle_2 \otimes |\varphi'\rangle\), where \(|\varphi'\rangle\) is the remaining \(N-2\) qubit state.

Now \(|\varphi'\rangle\) must be orthogonal to \(|0\rangle_3 |0\rangle_4 \ldots |0\rangle_{N-1} |0\rangle_N\), as well as to \(|1\rangle_3 |1\rangle_4 \ldots |1\rangle_{N-1} |1\rangle_N\), otherwise \(|\varphi\rangle\) would contain the basis vectors \(|0\rangle_1 |1\rangle_2 |0\rangle_3 |0\rangle_4 \ldots |0\rangle_{N-1} |0\rangle_N\) and \(|0\rangle_1 |1\rangle_2 |1\rangle_3 |1\rangle_4 \ldots |1\rangle_{N-1} |1\rangle_N\) with some nonzero coefficient, which means that \(|\varphi'\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes |\varphi'\rangle\) would have a nonzero projection on \(|m = N/2 - 1\rangle\) and on \(|m = -N/2 + 1\rangle\). Therefore again, at least one of the \(a_n\)-s and the \(b_n\)-s (with different indices) for \(n = 3, 4, \ldots, N\) must be zero, otherwise \(|\varphi\rangle\) would not be in.
S_⊥. We may set \( a_3 = 0 \), \( b_4 = 0 \) and continue this reasoning, until we arrive that \( |\varphi\rangle \) must be of the form

\[
|\varphi\rangle = |0\rangle_1 |1\rangle_2 |0\rangle_3 |1\rangle_4 \ldots |0\rangle_{N-1} |1\rangle_N
\]

if \( N \) is even, \( (25) \)

\[
a_N |0\rangle_1 |1\rangle_2 |0\rangle_3 |1\rangle_4 \ldots |1\rangle_{N-1} |1\rangle_N + b_N |0\rangle_1 |1\rangle_2 |0\rangle_3 |1\rangle_4 \ldots |0\rangle_{N-1} |0\rangle_N
\]

if \( N \) is odd. \( (26) \)

In the first case \( (25) \) has a nonzero projection on the state \( |m = 0\rangle \) in the symmetric subspace, while in the second case \( (26) \) has a nonzero projection on the states \( |m = N/2 \pm 1\rangle \) being also in the symmetric subspace.

We arrived to a contradiction: the nonentangled \( |\varphi\rangle \) cannot be orthogonal to \( S_\perp \), or stated otherwise: all elements of \( S_\perp \) are entangled.

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