Conserved quantities in non-abelian monopole fields

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Van Holten’s covariant Hamiltonian framework is used to find conserved quantities for an isospin-carrying particle in a non-Abelian monopole-like field. For a Wu-Yang monopole we find the most general scalar potential such that the combined system admits a conserved Runge-Lenz vector. It generalizes the fine-tuned inverse-square plus Coulomb potential, found before by McIntosh and Cisneros, and by Zwanziger, for a charged particle in the field of a Dirac monopole. Following Fehér, the result is interpreted as describing motion in the asymptotic field of a self-dual Prasad-Sommerfield monopole. In the effective non-Abelian field for nuclear motion in a diatomic molecule due to Moody, Shapere and Wilczek, a conserved angular momentum is constructed, despite the non-conservation of the electric charge. No Runge-Lenz vector has been found.

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I. INTRODUCTION

In a recent paper van Holten [1] outlined an algorithm for deriving conserved quantities for a particle in a given external field, based on using Killing tensors. An example illustrating his method is provided by an isospin-carrying particle [2, 3, 4, 5, 6] in the field of a Wu-Yang monopole [7, 8]. He concluded, in particular, that no Runge-Lenz type vector could exist for such a system. This result reminds one to an earlier result by Fehér [9], who proved that a charged particle in a Dirac monopole field can not have a globally defined Runge-Lenz vector. In the latter case there is a way out, though [9, 10, 11, 12, 13]: adding a fine-tuned inverse square potential removes the obstruction, providing us with a conserved Runge-Lenz vector.

Below we find, using van Holten’s recipe, the most general additional potential which allows for a conserved Runge-Lenz vector in non-Abelian monopole-like fields.

In the Wu-Yang case, the resulting combined system has a remarkable physical interpretation: it describes the motion in the long-distance limit of a self-dual monopole [12].

Similar results hold in the effective field of a diatomic molecule, considered before by Moody, Shapere and Wilczek [14, 15, 16]. Despite the non-conservation of the electric charge, we can construct a conserved angular momentum. No Runge-Lenz vector is found in general, though.

II. CONSERVED QUANTITIES

We start with the equations of motion for an isospin-carrying particle in a static non-Abelian gauge field [28]

\[
\begin{align*}
\dot{\pi}_i &= I^a F^a_{ij} \dot{\pi}^j - D_i V, \\
\dot{\pi}^j &= -\epsilon_{abc} I^b \left( A^c_{ij} \dot{\pi}^j - \frac{\partial V}{\partial \pi^c} \right),
\end{align*}
\]

(1)

where \(\pi_i = \dot{\pi}_i\), and a scalar potential, \(V = V(\vec{r}, \vec{I})\) has also been included for later convenience. Defining the covariant Poisson bracket and Hamiltonian as [1],

\[
\{f, g\} = D_j f \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + I^a F^a_{jk} \frac{\partial f}{\partial \pi_j} \frac{\partial g}{\partial \pi_k} - \epsilon_{abc} I^b \frac{\partial f}{\partial \pi^c} \frac{\partial g}{\partial \pi^b} I^c,
\]

\[
H = \frac{1}{2} \dot{\pi}^2 + V(\vec{r}, \vec{I}),
\]

(2)

(3)

where \(D_j\) is the covariant derivative,

\[
D_j f = \partial_j f - \epsilon_{abc} I^a \frac{\partial f}{\partial I^c} I^c.
\]

(4)

Let us record the commutation relation of the covariant derivatives,

\[
[D_i, D_j] = -\epsilon_{abc} I^a F^b_{ij} \frac{\partial}{\partial \pi^c}.
\]

(5)

The equations (1) can be obtained in a Hamiltonian framework, \(\dot{x}_i = \left\{ x_i, H \right\}, \dot{\pi}_i = \left\{ \pi_i, H \right\}, \dot{I}^a = \left\{ I^a, H \right\}\).

Following van Holten [1], constants of the motion can conveniently be sought for in the form of an expansion into powers of the covariant momentum,

\[
Q = C(\vec{r}, \vec{I}) + C_i(\vec{r}, \vec{I}) \pi_i + \frac{1}{2!} C_{ij}(\vec{r}, \vec{I}) \pi_i \pi_j + \ldots
\]

(6)
Requiring $Q$ to Poisson-commute with the Hamiltonian yields a series of constraints,
\begin{align*}
C_i D_i V &= 0, \quad \text{order 0} \\
D_i C &= I^a F^a_{ij} C + C_{ij} D_j V, \quad \text{order 1} \\
D_i C_j + D_j C_i &= I^a (F^a_{ik} C_{kj} + F^a_{jk} C_{ki}) \\
&\quad + C_{ijk} D_k V, \quad \text{order 2} \\
&\quad \vdots \\
&\quad \vdots \\
&\quad \vdots
\end{align*}
The expansion can be truncated at a finite order, provided the covariant Killing equation is satisfied,
\begin{equation}
D_{(i_1 i_2 \ldots i_n)} = 0, \tag{8}
\end{equation}
when we can set $C_{i_1 \ldots i_n+1} = 0$. For $n = 1$, (8) is a Killing vector. For example, we have, for any unit vector $\vec{n}$, the generator of a rotation around $\vec{n}$,
\begin{equation}
\vec{C} = \vec{n} \times \vec{r}, \tag{9}
\end{equation}
Then van Holten’s recipe yields conserved angular momentum. Similarly, for any unit vector $\vec{n}$,
\begin{equation}
C_{ij} = 2\delta_{ij} \nabla \cdot \vec{r} - (n_i x_j + n_j x_i) \tag{10}
\end{equation}
is a Killing tensor of order 2, associated with the Runge-Lenz vector of planar motion.

### III. WU-YANG MONOPOLE

The prototype of non-Abelian monopoles is the one due to Wu and Yang [7], given by the non-Abelian gauge potential with a “hedgehog” magnetic field,
\begin{equation}
A_i^W Y = \epsilon_{iak} \frac{x_k}{r^2}, \quad F_{ij}^W = \epsilon_{ijk} \frac{x_k x_a}{r^4}. \tag{11}
\end{equation}

Let us now consider an isospin-carrying particle moving in a Wu-Yang monopole field augmented by a scalar potential, and inquire about conserved quantities.

To start, we search for a conserved quantity of order zero, i.e., $C_i = C_{ij} = \cdots = 0$. Then (7) is satisfied for an arbitrary potential if $D_i C = 0$. In particular,
\begin{equation}
Q = \vec{I} \cdot \vec{r} \quad (\vec{r} = \frac{x}{r}), \tag{12}
\end{equation}
is covariantly constant, $\bar{D} Q = 0$, and $Q$ is, therefore, a constant of the motion. An easy calculation shows, furthermore, that (12) is the only such quantity.

Next, we study conserved quantities which are linear in $\pi_i$, $C_{ij} = \cdots = 0$. When the potential is invariant w.r.t. joint rotation of $\vec{r}$ and $\vec{I}$, inserting the Killing vector (10) into equations (7) yields (13),
\begin{equation}
C = -\vec{n} \cdot (Q \cdot \vec{r}). \tag{13}
\end{equation}
and we end up with the angular momentum [1],
\begin{equation}
\vec{J} = \vec{r} \times \vec{r} - Q \cdot \vec{r}. \tag{14}
\end{equation}

Let us now turn to quadratic quantities. We observe that the Killing tensor has the property,
\begin{equation}
C_{ij} x_j = (\vec{n} \cdot \vec{r}) x_i - r^2 n_i = -r^3 \partial_i (\vec{n} \cdot \vec{r}), \tag{15}
\end{equation}
Inserting (10) into (7), from the 2nd-order equation we find, therefore,
\begin{equation}
\vec{C} = \vec{n} \times (Q \vec{r}). \tag{16}
\end{equation}
Restricting ourselves to potentials with fall off at infinity, $V = \sum_{m=0}^{\infty} \alpha_m r^{-m}$, the zeroth order equation allows us to infer that the real coefficients $\alpha_m$ are covariantly constant. The first-order equation requires in turn
\begin{equation}
D_i C = \left[ \frac{Q^2}{r} - \sum_{m=0}^{\infty} m \alpha_m r^{-m+1} \right] r^3 \partial_i (\vec{n} \cdot \vec{r}). \tag{17}
\end{equation}
Our constraint can be solved therefore if $N = 2$ and the leading coefficient in the expansion of $V$ is chosen to cancel the obstruction term $Q^2/r$. Hence,
\begin{equation}
V = \frac{Q^2}{2r^2} + \frac{\alpha}{r} + \beta \quad \text{and} \quad C = \alpha \vec{n} \cdot \vec{r}, \tag{18}
\end{equation}
where $\alpha$ and $\beta$ are arbitrary constants. Note that the coefficient of the inverse-square term is fixed by the requirement of canceling the $Q^2/r^4$ term in the first bracket. Collecting our results yields,
\begin{equation}
\vec{K} = \vec{\pi} \times \vec{J} + \alpha \vec{r}, \tag{19}
\end{equation}
which is indeed a conserved Runge-Lenz vector for an isospin-carrying particle in the Wu-Yang monopole field, combined with the potential [18] [19].

The importance of the conserved quantities $\vec{J}$ and $\vec{K}$ is understood by noting that they determine the trajectory: multiplying (14) by the position, $\vec{r}$, yields
\begin{equation}
\vec{J} \cdot \vec{r} = -Q, \tag{20}
\end{equation}
so that the particle moves, as always in the presence of a monopole, on the surface of a cone of half opening angle $\theta = \arccos(|Q|/|J|)$ ($J = |\vec{J}|$).

On the other hand, the projection of the position onto the vector $\vec{N}$, given by,
\begin{equation}
\vec{N} = \vec{K} + (\alpha/Q) \vec{J}, \quad \vec{N} \cdot \vec{r} = J^2 - Q^2 = \text{const}, \tag{21}
\end{equation}
implicating that the trajectory lies in a plane perpendicular to $\vec{N}$. The motion is, therefore, a conic section. Careful analysis would show that the trajectory is an ellipse, a parabola, or a hyperbola, depending on the energy being smaller, equal or larger as $\beta$ [12]. In particular,
for sufficiently low energies, the nuclear motion remains bounded.

The conserved vectors \( \vec{J} \) and \( \vec{K} \) satisfy, furthermore, the commutation relations
\[
\begin{align*}
\{J_i, J_j\} &= \epsilon_{ijk}J_k, \\
\{J_i, K_j\} &= \epsilon_{ijk}K_k, \\
\{K_i, K_j\} &= -2(H - \beta)\epsilon_{ijk}J_k,
\end{align*}
\] (22)
with Casimir relations,
\[
\vec{J} \cdot \vec{K} = -\alpha Q, \quad K^2 = 2(H - \beta)(J^2 - Q^2) + \alpha^2.
\] (23)
Normalizing \( \vec{K} \) we get, therefore an SO(3)/E(3)/SO(3,1)
dynamical symmetry, depending on the energy being smaller/equal/larger as \( \beta \). [30]

We emphasize that the fine-tuned inverse-square term is necessary to overcome the obstruction in solving the constraint equation; without it, no Runge-Lenz vector would exist.

IV. MOTION IN SELF-DUAL MONOPOLE FIELD

The physical interpretation of the previous result is the following [12]. For large \( r \), dropping exponentially decreasing terms, the field of a self-dual non-Abelian monopole of charge \( m \) is asymptotically that of Wu-Yang, [11], augmented with a “hedgehog” Higgs field,
\[
\Phi^a = \phi^a, \quad \varphi = 1 - \frac{m}{r}, \quad \vec{x}^a = \frac{x^a}{r}.
\] (24)

The equations of motion of an isospin-carrying particle in the combined gauge- plus scalar field can conveniently be found by adding a fictitious fourth spatial dimension, \( x^4 \), and putting \( A_i^4 = \Phi^a \) [12]. Then \( F_{4i} = D_j\Phi^a \), and the equations \( \{1\} \) yield
\[
\begin{align*}
\dot{\pi}_4 &= I^a D_j \phi^a \dot{x}^j, \\
\dot{\pi}_i &= -I^a \left(F_{ij}^a \dot{x}^j + D_j \Phi^a \dot{x}^j\right), \\
\dot{x}^a &= \epsilon_{abc}I^b \left(A_i^c \dot{x}^j + \Phi^c \dot{x}^j\right).
\end{align*}
\] (25)
For the Wu-Yang gauge field, in particular, the direction field, \( \Phi^a \), is covariantly constant, the non-Abelian field strength is parallel to \( \Phi^a \), and the projection of the isospin onto \( \Phi^a \) is the conserved electric charge,
\[
F_{ij}^a = f_{ij}^a \Phi^a, \quad D_j \Phi^a = 0, \quad Q = I^a \Phi^a.
\] (26)
Here \( f_{ij} \) is the (scalar) field strength of a Dirac monopole. The upper equation in \( \{25\} \) is solved therefore by
\[
\dot{x}^4 = \pi_4 = Q\varphi + Q_1
\] (27)
where \( Q_1 \) is another constant of the motion. Inserting \( \dot{x}^4 \) into the second equation in \( \{25\} \) yields, at last, the generalized equations of motion in the self-dual field [12, 18],
\[
\dot{\pi}_i = -Qf_{ij}^a \dot{x}^j - \partial_j V, \quad V = \frac{1}{2}Q^2\varphi^2 - QQ_1\varphi,
\] (28)
(We observe that the third equations in \( \{1\} \) and in \( \{25\} \) are also the same). These equations derive, using the same Poisson bracket structure \( \{2\} \) as above and a Hamiltonian \( \{3\} \), with a potential of the form \( \{18\} \). Our results confirm, hence, the Kepler-type dynamical symmetry in the asymptotic field of a self-dual monopole – or equivalently, in an Abelian monopole field with self-dual scalar potential [12].

V. DIATOMIC MOLECULE

In Ref. [14] Moody, Shapere and Wilczek have shown that, in the Born-Oppenheimer approximation, nuclear motion in a diatomic molecule can be described by the effective non-Abelian gauge field and Hamiltonian,
\[
A_r = 0, \quad A_\theta = \frac{1}{2} \begin{bmatrix} 0 & -\kappa e^{i\phi} \\ -\kappa e^{-i\phi} & 0 \end{bmatrix},
\]
\[
A_\phi = \frac{1}{2} \begin{bmatrix} 1 - \cos \theta & -i\kappa e^{i\phi} \sin \theta \\ i\kappa e^{-i\phi} \sin \theta & -1 - \cos \theta \end{bmatrix},
\]
\[
H = \frac{1}{2} \vec{\pi}^2 + V,
\] (30)
where \( \kappa \) is a real parameter. The field strength resembles that of monopole aligned into the third internal direction, \( F_{\theta\phi} = (1 - \kappa^2) \sin \theta T_3 \), except for the parameter \( \kappa \) being unquantized. The potential \( \{29\} \) is that of a Wu-Yang [i.e., an imbedded Dirac] monopole of unit charge when \( \kappa = 0 \); for other values of \( \kappa \), it is a truly non-Abelian configuration — except for \( \kappa = \pm 1 \), when the field strength vanishes and \( \{29\} \) is a gauge transform of the vacuum.

Our first step is present the field in a more convenient form,
\[
\vec{A}_i^a = (1 - \kappa)\epsilon_{iaj} \frac{x_j}{r^2}, \quad \vec{F}_{ij}^a = (1 - \kappa^2)\epsilon_{ijk} \frac{x_kx_n}{r^4}.
\] (31)
which can be achieved by applying a suitable gauge transformation [13].

Turning to the conserved quantities, we note that, when \( \kappa \neq 0 \), the used-to-be electric charge, \( Q \) in \( \{12\} \), is not more covariantly conserved in general [31],
\[
\{H, Q\} = -\vec{\pi} \cdot \vec{D}Q, \quad D_jQ = \frac{\kappa}{r} \left(F_j^i - Q \frac{x_i}{r}\right).
\] (32)
Nor is \( Q^2 \) conserved, \( \{H, Q^2\} = 2\kappa Q(\vec{\pi} \cdot \vec{D}Q) \). Note for further reference that, unlike \( Q^2 \), the length of isospin, \( I^2 \), is conserved, \( \{H, I^2\} = 0 \).

The gauge field \( \{29\} \) is rotationally symmetric and an isospin-carrying particle moving in it admits a conserved
angular momentum \([14, 15]\). Its form is, however, somewhat unconventional, and we re-derive it, therefore, in detail.

Our starting point is the first-order condition in \([7]\). We take first \(V = 0\); then this is the only condition. Evaluating the r.h.s. with \(F^a\) as given in \([31]\), the equation to be solved is

\[
D_i C = \left(1 - \kappa^2\right) \frac{Q}{r} \left(\hat{n} \cdot \hat{r} \frac{x_i}{r} - n_i\right). \tag{33}
\]

In the Wu-Yang case, \(\kappa = 0\), this equation was solved by \(C = -\hat{n} \cdot \hat{r} \hat{r}\). But for \(\kappa \neq 0\) the electric charge, \(Q\), is not conserved, and using \([32]\), as well as the relations

\[
D_i J^i = \frac{1 - \kappa}{r} \left(Q \delta_{ij} - I^i x_j / r\right),
\]

\[
D_i (Q \hat{n} \cdot \hat{r}) = \frac{Q}{r} \left(n_i + (\hat{n} \cdot \hat{r})(\kappa I_i - (1 + \kappa) \frac{n_i}{r})\right), \tag{34}
\]

\[
I^a F^a_{ij} = (1 - \kappa^2) \frac{Q}{r^3} \epsilon_{ijk} x_k,
\]

we find,

\[
(\kappa - 1) D_i (Q \hat{n} \cdot \hat{r}) = \kappa D_i I^i n_j + (1 - \kappa^2) \left(\hat{n} \cdot \hat{r} \frac{x_i}{r} - n_i\right).
\]

Comparing with \([33]\) allows us to infer that,

\[
C = -\hat{n} \cdot \left( (1 - \kappa) Q \hat{r} + \kappa \hat{I}\right). \tag{35}
\]

The conserved angular momentum is, therefore,

\[
\hat{J} = \hat{r} \times \hat{n} - \hat{\Psi}, \tag{36}
\]

\[
\hat{\Psi} = (1 - \kappa) Q \hat{r} + \kappa \hat{I} = Q \hat{r} + \kappa \left(\hat{r} \times \hat{I}\right), \tag{37}
\]

consistently with the results in \([14, 15, 20]\). Comparison with \([13]\) yields the “replacement rule”

\[
Q \hat{r} \rightarrow \hat{\Psi}. \tag{38}
\]

For \(\kappa = 0\) we recover the Wu-Yang expression, \([14]\).

Eliminating \(\hat{n}\) in favor of \(\hat{p} = \hat{n} + \hat{A}\) allows us to rewrite the total angular momentum as

\[
\hat{J} = \hat{r} \times \hat{p} - \hat{I}, \tag{39}
\]

making manifest the celebrated “spin from isospin term” \([17]\).

Restoring the potential, we see that, again due to the non-conservation of \(Q, D_i V \neq 0\) in general. The zeroth-order condition \(\vec{C} \cdot \vec{D} V = 0\) in \([7]\) is, nevertheless, satisfied if \(V\) is a radial function independent of \(\vec{I}, V = V(\hat{r})\), since then \(\vec{D} V = \nabla V\), which is perpendicular to infinitesimal rotations, \(\vec{C}\). Alternatively, a direct calculation, using the same formulae \([32, 34]\), allows us to confirm that \(\hat{J}\) commutes with the Hamiltonian, \(\{J_i, H\} = 0\).

Multiplying \([37]\) by \(\hat{r}\) yields, once again, the relation \([20]\) i.e. \(\hat{J} \cdot \hat{r} = -Q\), the same as in the Wu-Yang case.

This is, however, less useful as before, since \(Q\) is not a constant of the motion so that the angle between \(\hat{J}\) and the radius vector, \(\hat{r}(t)\), is not more a constant. The components of the angular momentum \([37]\) close, nevertheless, to SO(3), \(\{J_i, J_j\} = \epsilon_{ijk} J_k\).

Turning to quadratic conserved quantities, we have been searching for a Runge-Lenz vector for diatomic molecules associated with the Killing tensor \([10]\). Despite a promising start (summarized in the Appendix) we failed to find such an additional conserved quantity.

We should remark, however, that even if we succeeded to integrate \([A3]\), the resulting potential would break the rotational invariance. The zeroth-order condition in \([7]\) requires in fact the \(D_j V\) be perpendicular to \(\vec{C}\). But the \(\vec{C}\) of angular momentum and the one appropriate for the Runge-Lenz vector, namely the infinitesimal rotation in \([9]\) and \([11]\), respectively, have different orientations, so that the two conditions can not be simultaneously satisfied.

\[\text{VI. RELATION TO THE FORGÁCS-MANTON-JACKIW APPROACH}\]

Moody et al. \([14]\) found the correct expression, \([37]\), for \(\kappa = 0\) but, as they say it, “they are not aware of a canonical derivation when \(\kappa \neq 0\)”\footnote{\textsuperscript{4}}. Our construction here is an alternative to that of Jackiw \([15]\), who obtained it using the method of Ref. \([2]\). In his approach, based on the study of symmetric gauge fields \([21]\), each infinitesimal rotation, \([9]\), is a symmetry of the monopole in the sense that it changes the potential by a surface term. Equivalently,

\[
C_i F_{ij} = D_j \Psi_i \tag{40}
\]

for some Lie algebra valued field \(\Psi\). In this equation, \(\mathcal{D}\) is the gauge-covariant derivative \(\mathcal{D}\Psi = \partial \Psi - [A_j, \Psi]\). Identifying the Lie algebra with \(\mathbb{R}^3\) identifies \(\Psi\) with a vector \(\vec{\Psi}\), the gauge-covariant derivative becoming the covariant derivative of van Holten in Eqn. \([4]\). The rule is simply to replace the generator \(T_a\) with the components of the isospin vector, \(I^a\). Under this transformation, the symmetry condition \([10]\) becomes precisely the first-order condition in \([7]\) that a rotation has to satisfy. Accordingly, in the diatomic case, eqn. \([40]\) is solved by

\[
\Psi = \left( (1 - \kappa)(\hat{n} \cdot \hat{r}) \frac{x_a}{r} + \kappa n_a \right) T_a, \tag{41}
\]

consistently with the indicated correspondence \(\Psi \leftrightarrow \vec{\Psi}\). It represents the response of the (symmetric) gauge field to a space-time transformation and appears as the contribution coming from the gauge field to the associated conserved quantity \([3, 4]\); it is needless to say that the same conserved quantities are obtained in both approaches.

Taken individually, each rotation generator is a symmetry. If we have several generators, an additional con-
sistency condition is required, namely \[3, 21\]

\[ X_i^a X_j^b F_{ij} = \Psi X_n X_m - \left[ \Psi X_n, \Psi X_m \right], \quad (42) \]

where \( X_n \) and \( X_m \) are two infinitesimal rotations around \( \hat{n} \) and \( \hat{m} \), respectively. Writing the l.h.s. as \( r^2 (\hat{n} \mathbf{m} \cdot \mathbf{r}) (\hat{B} \cdot \mathbf{r}) \) where \( \hat{B} = (B_i) \) is the Lie algebra-valued vector \( B^a = \epsilon_{ijk} F^a_{jk} \) allows us to evaluate it, to find, in the diatomic case,

\[ (1 - \kappa^2)(\hat{r} \cdot \hat{n}) \frac{x_a}{r} T_a. \quad (43) \]

On the other hand, we find

\[
\Psi X_n X_m = (\hat{r} \cdot \hat{n}) \frac{x_a}{r} T_a + \kappa [\hat{r} \times (\hat{n} \times \hat{m})] T_a, \quad (44)
\]

\[
- \left[ \Psi X_n, \Psi X_m \right] = -\kappa^2 (\hat{r} \cdot \hat{n} \cdot \hat{m}) \frac{x_a}{r} T_a
\]

\[
-\kappa [\hat{r} \times (\hat{n} \times \hat{m})] T_a \quad (45)
\]

so that \( (42) \) is indeed satisfied.

VII. DISCUSSION

We derived the equations of motion in a Yang-Mills-Higgs background from those in a pure gauge field, by viewing the Higgs field as the fourth component of the Yang-Mills field on an extended space. It is worth mentioning that van Holten’s algorithm could be extended to four dimensions. Then the electric charge could be seen as associated with an internal symmetry generated by the covariantly constant direction field \( \Phi^a \); the second charge, \( Q_2 \) in [10] corresponds in turn to the “vertical” Killing vector which points into the fourth direction. Similarly, the Kepler potential could be viewed as a component of the extended metric, and the Killing tensor \( (10) \) can be lifted to extended space, to yield the Runge-Lenz vector directly \( [23] \).

Our results have a nice interpretation in terms of fiber bundles \( [22] \). The SU(2) gauge field is a connection form defined on a bundle over 3-space. For Wu-Yang, this bundle reduces to an \( U(1) \) bundle, namely that of a Dirac monopole of unit charge. The projection of the isospin onto a direction field, \( \Phi^a, \Phi = 1 \), is a conserved charge if and only if \( \Phi^a \) is covariantly constant. But this amounts precisely to saying that the su(2) connection living on the (trivial) SU(2) bundle reduces to the \( U(1) \) Dirac monopole bundle. This explains why the electric charge \( (12) \) is conserved in the Wu-Yang case: the latter is in fact an imbedded Abelian monopole. The non-Abelian equations of motion \( (1) \) reduce, accordingly, to those of a charged particle in an electromagnetic field.

We remark that although our investigations have been purely classical, there would be no difficulty to extend them to a quantum particle. In the self-dual Wu-Yang case, the SO(4)/SO(3, 1) dynamical symmetry allows, in particular, to derive the bound-state spectrum and the \( S \)-matrix group-theoretically, using the algebraic relations \( [22], [23], [12] \).

The effective field of a diatomic molecule provides us with an interesting generalization. For \( \kappa \neq 0, \pm 1 \), it is truly non-Abelian, i.e., not reducible to one on a \( U(1) \) bundle. No covariantly constant direction field, and, therefore, no conserved electric charge does exist in this case.

The field is nevertheless radially symmetric, but the conserved angular momentum, \( [37] \), has a non-conventional form \( [13] \).

In bundle terms, the action of a symmetry generator can be lifted to the bundle so that it preserves the connection form which represents the potential. But the group structure may not be conserved; this requires another, consistency condition, namely \( (12) \), which may or may not be satisfied. In the diatomic case, it is not satisfied when \( \kappa \neq 0, \pm 1 \).

Is it possible to redefine the “lift” so that the group structure will be preserved? In the Abelian case, the answer can be given in cohomological terms \( [6] \). If this obstruction does not vanish, it is only a central extension that acts on the bundle.

In the truly non-Abelian case, the consistency condition involves the covariant, rather than ordinary derivative and covariantly constant sections only exist in exceptional cases — namely when the bundle is reducible. Thus, only some (non-central extension) acts on the bundle.

It is worth noting that for \( \kappa \neq 0 \) the configuration \( (24) \) does not satisfy the vacuum Yang-Mills equations; it only satisfies them with a suitable (conserved) current \( [15] \).

\[ D_k F_{ik} = j_k, \quad j = \kappa(1 - \kappa^2) \frac{1}{r^4} \mathbf{r} \times \mathbf{T}, \quad (46) \]

Interestingly, this current can also be produced by a hedgehog Higgs field,

\[ j_k = [D_k \Phi, \Phi], \quad \Phi^a = \sqrt{1 - \kappa^2} \frac{x_a}{r}. \quad (47) \]

We have not been able to derive a Runge-Lenz vector for diatomic molecules, except for \( \kappa = 0 \).

Let us emphasize that the derivation of the non-Abelian field configuration \( [31] \) from molecular physics \( [14] \) indicates that our analysis may not be of purely academic interest. The situation could well be analogous to what happened before with the non-Abelian Aharonov-Bohm experiment, first put forward and studied theoretically \( [24], [25] \), but which became recently accessible experimentally, namely by applying laser beams to cold atoms \( [26] \). A similar technique can be used also to create monopole-type fields \( [27] \).
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APPENDIX A: SEARCHING FOR A RUNGE-LENZ VECTOR FOR DIATOMIC MOLECULES

We carried out a promising, but unsuccessful, search for finding a conserved Runge-Lenz vector in the effective gauge field of a diatomic molecule. Inserting the Killing tensor \( T \) into the 2nd-order equations in \( \Sigma \) yields, after a calculation similar to the one sketched for the angular momentum,
\[
\vec{C} = \hat{n} \times \vec{\Psi},
\]
which is, once again, consistent with the “replacement rule” \( 38 \), cf. \( \Sigma \).

Our next step is to try to identify a “good” potential. For this, we observe that the first-order constraint in \( \Sigma \)
can be written as,
\[
D_i C = (1 - \kappa^2) \frac{Q^2}{r^3} \left( (1 - \kappa)(\hat{n} \cdot \hat{r}) x_i - n_i \right) (A2)
\]
\[
+ \kappa(1 - \kappa^2) \frac{Q}{r^3} (\hat{n} \cdot \hat{r}) I^i + C_{ij} D_j V.
\]

In the WY case, \( \kappa = 0 \), our clue has been to remove the first term by a fine-tuned inverse-square term in the potential, \( V = V_0 + V_1, V_0 = Q^2/2r^2 \). We note now that this is once again possible, namely choosing,
\[
\vec{D} V_0 = -(1 - \kappa^2) \frac{Q^2}{r^3} \hat{\Psi},
\] (A3)

once again consistently with our rule \( 38 \). Assuming that such a potential does exist, \( C_{ij} D_j V_0 \) cancels the upper term in \( A2 \) [but contributes others], leaving us with
\[
D_i C = \kappa(1 - \kappa^2) \frac{Q}{r^3} (\hat{n} \times (\hat{r} \times \vec{I}))_i + C_{ij} D_j V_1. \] (A4)

Our remaining task would now be to integrate the equations \( A3 \) and \( A4 \) — that we have not been able to do yet in general. We could do it for \( \kappa = 0 \), though, allowing us to recover the results previously found in Section \( \Sigma \). The electric charge, \( Q \), is now conserved, and eqn. \( A3 \) can now be integrated,
\[
\vec{D} V_0 = -\frac{Q^2}{r^3} \hat{\Psi} \quad \Rightarrow \quad V_0 = \frac{Q^2}{2r^2}.
\] (A5)

Now assuming that \( V_1 \) only depends on \( r \), \( V = V_1(r), \) \( \vec{D}_r V_1 = \vec{\nabla} V_1 \) is radial. On the other hand, using \( \Sigma \) Eqn. \( A4 \) reduces to
\[
D_i C = C_{ij} \partial_j V_1 = -r^2 V'_1 \vec{\nabla} (\hat{n} \cdot \hat{r})
\] (A6)

where \( V'_1 = dV_1/dr \), which can be solved by
\[
V_1 = \frac{\alpha}{r} + \beta, \quad C = \alpha (\hat{n} \cdot \hat{r}),
\] (A7)

consistently with what we found before in \( \Sigma \). Some more calculations show, furthermore, that this is the only possibility.

In the MacIntosh-Zwanziger (and hence the self-dual Wu-Yang) case, the arising of the “fine-tuned” potential can also be understood as follows. Decomposing the momentum into radial and transverse components, \( (\vec{p})^2 = (\hat{r} \cdot \hat{n})^2 + (\hat{\pi} \cdot \hat{n})^2 = \pi^2 + L^2/r^2 \) and using \( \vec{L}^2 = \vec{J}^2 - Q^2 \) allows us to present the Hamiltonian as
\[
H = H_0 + V = \frac{1}{2} \dot{r}^2 + \frac{\vec{J}^2}{2r^2} - \frac{Q^2}{2r^2} + V.
\]

Therefore, choosing the potential as in \( \Sigma \) cancels the centrifugal term \( Q^2/2r^2 \), leaving us with,
\[
H = \frac{1}{2} \dot{r}^2 + \frac{\vec{J}^2}{2r^2} + \frac{\alpha}{r} + \beta,
\] (A8)

which describes an effective Kepler-type problem. For diatoms, the Hamiltonian is, instead
\[
H = \frac{1}{2} \dot{r}^2 + \frac{\vec{J}^2}{2r^2} + \left\{ \frac{-\kappa^2 Q^2 + \kappa^2 I^2 + 2\kappa \vec{I} \cdot \vec{T}}{2r^2} \right\} + V. \] (A9)

One would be tempted to chose the potential so that it cancels the curly bracket leaving us, once again, with a Kepler-type radial Hamiltonian. This does not work, however, since the covariant derivative of such a \( V \) is not perpendicular to the vector \( \vec{C} \) in \( A1 \).

Eliminating the total angular momentum, \( \vec{J} \), in favor of the orbital one, \( \vec{L} = \vec{r} \times \vec{\pi} \), and putting \( Q^2 = I^2 = 1/4 \) would yield the decomposition \# (21) of Jackiw \( \Sigma \). This is, however, only legitimate when \( \kappa = 0 \), since \( Q^2 \) is not conserved for \( \kappa \neq 0 \).
We mention that we can also solve our equations (A3) and (A4) in another particular case, namely when the isospin is radially aligned, \( \mathbf{I} = I_0 \hat{r} \). Then \( Q = I_0 \) is conserved. This condition is very restrictive, however: the equations of motion (A1) imply that if isospin alignment is required as an initial condition at \( t = 0 \), it only remains satisfied for all \( t \) if the space-time motion is radial.

When the isospin is aligned, the curly bracket in (A9) becomes simply \( -Q^2/2r^2 \), and we are back in the Wu-Yang case — but only for radial motions such that the isospin is also radial, \( \mathbf{I} = I_0 \hat{r} \).

For \( \kappa = 0 \), the integrability of the equations (A4) can also be studied as follows. By (5), we must have

\[
-\epsilon^{abc} I^a F^b_{ij} \partial C / \partial F^c = (n_i x_j - n_j x_i) \Delta V_1.
\]  

(A10)

Assuming that \( C \) only depends on \( \mathbf{r} \), the l.h.s. vanishes, and this condition merely requires

\[
\Delta V_1 = 0 \quad \Rightarrow \quad V_1 = \frac{\alpha}{r} + \beta,
\]

(A11)

as we found it before.