The PBW Theorem and simplicity criteria for the Poisson enveloping algebra and the algebra of Poisson differential operators

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Abstract

For an arbitrary Poisson algebra \( P \) over an arbitrary field, an (analogue of) the Poincaré-Birkhoff-Witt Theorem is proven and several presentations/constructions for its Poisson enveloping algebra \( \mathcal{U}(P) \) are given. As a result, explicit sets of generators and defining relations are given for \( \mathcal{U}(P) \) and the algebra \( PD(P) \) of Poisson differential operators on \( P \). Simplicity criteria for the algebras \( \mathcal{U}(P) \) and \( PD(P) \) are given. In the case when the algebra \( P \) is of essentially finite type, a criterion for the algebra \( \mathcal{U}(P) \) to be a domain is presented and a criterion for a natural epimorphism \( \mathcal{U}(P) \to PD(P) \) to be an isomorphism is given. The kernel of the epimorphism is described and for large classes of Poisson algebras an explicit set of generators is given. Explicit formulae for the Gelfand-Kirillov dimension of the algebras \( \mathcal{U}(P) \) and \( PD(P) \) are given. In the case when the Poisson algebra \( P \) is a regular domain of essentially finite type an explicit simplicity criterion for \( P \) is found and a criterion is presented for the algebra \( \mathcal{U}(P) \) to be isomorphic to the algebra \( D(P) \) of differential operators on \( P \).

Key Words: a Poisson algebra, the Poisson enveloping algebra of a Poisson algebra, the algebra of Poisson differential operators, module over a Poisson algebra, the Poisson generalized Weyl algebra, the Poisson simplicity.

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1 Introduction

In this paper, module means a left module, \( K \) is a field, algebra means a \( K \)-algebra (if it is not stated otherwise) and \( K^\times = K \setminus \{0\} \).

An associative commutative algebra \( P \) is called a Poisson algebra if it is a Lie algebra \( (P, \{\cdot, \cdot\}) \) such that \( \{a, xy\} = \{a, x\}y + x\{a, y\} \) for all elements \( a, x, y \in P \).
The Poincaré-Birkhoff-Witt Theorem for Poisson algebras. The (classical) Poincaré-
Birkhoff-Witt Theorem states that for each Lie algebra \( \mathcal{G} \) there is a natural isomorphism of graded algebras,
\[
gr U(\mathcal{G}) \simeq \text{Sym}(\mathcal{G}),
\]
where \( gr U(\mathcal{G}) \) is the associated graded algebra of the universal enveloping algebra \( U(\mathcal{G}) \) of the Lie algebra \( \mathcal{G} \) and \( \text{Sym}(\mathcal{G}) \) is the symmetric algebra of \( \mathcal{G} \). For a smooth Poisson algebra \( \mathcal{P} \), a similar result holds \cite{17} Theorem 3.1:
\[
gr \mathcal{U}(\mathcal{P}) \simeq \text{Sym}_\mathcal{P}(\Omega_\mathcal{P}),
\]
where \( gr \mathcal{U}(\mathcal{P}) \) is the associated graded algebra of the Poisson enveloping algebra \( \mathcal{U}(\mathcal{P}) \) of the Poisson algebra \( \mathcal{P} \) and \( \text{Sym}_\mathcal{P}(\Omega_\mathcal{P}) \) is the symmetric algebra of the \( \mathcal{P} \)-module \( \Omega_\mathcal{P} \) of Kähler differentials of the associative algebra \( \mathcal{P} \). In fact, \cite{17} Theorem 3.1 holds in slightly more general situation, namely, for the universal enveloping algebra of the Lie-Reinhart algebra. The pair \( (\mathcal{P}, \Omega_\mathcal{P}) \) is an example of a Lie-Reinhart algebra and its universal enveloping algebra \( V(\mathcal{P}, \Omega_\mathcal{P}) \) is isomorphic to the Poisson enveloping algebra (PEA) \( \mathcal{U}(\mathcal{P}) \). \cite{17} Section 2, p.197 (see also \cite{6}). Recently, it was proven that the PBW Theorem holds for certain singular Poisson hypersurfaces, \cite{10} Theorem 3.7. One of the main results of this paper, Theorem \cite{16} states that the Poincaré-Birkhoff-Witt Theorem holds for all Poisson algebras (over an arbitrary field).

In \cite{15}, the Poisson enveloping algebra of a Poisson algebra was introduced as a universal object in a certain category and an alternative to Reinhart’s proof of its existence was given. For certain classes of Poisson algebras explicit descriptions of their Poisson enveloping algebras were presented in \cite{16} \cite{19} \cite{20} \cite{10} \cite{11} \cite{9}.

Simplicity criterion for the algebra \( PD(\mathcal{P}) \) of Poisson differential operators on \( \mathcal{P} \). An ideal \( I \) of a Poisson algebra \( \mathcal{P} \) is called a Poisson ideal if \( \{\mathcal{P}, I\} \subseteq I \). A Poisson algebra \( \mathcal{P} \) is called Poisson simple if the ideals 0 and \( \mathcal{P} \) are the only Poisson ideals of the Poisson algebra \( \mathcal{P} \). Let \( \text{Det}_K(\mathcal{P}) \) be the Lie algebra of \( K \)-derivations of the (associative) algebra \( \mathcal{P} \). For each element \( a \in \mathcal{P} \), the derivation \( \text{pad}_a := \{a, \cdot\} \in \text{Det}_K(\mathcal{P}) \) is called the Hamiltonian vector field associated with the element \( a \). Then \( \mathcal{H}_\mathcal{P} := \{\text{pad}_a \mid a \in \mathcal{P}\} \) is a Lie subalgebra of the Lie algebra \( \text{Der}_K(\mathcal{P}) \). The subalgebra \( PD(\mathcal{P}) \) of the \( K \)-endomorphism algebra \( \text{End}_K(\mathcal{P}) \) which generated by \( \mathcal{P} \) and \( \mathcal{H}_\mathcal{P} \) is called the algebra of Poisson differential operators of the Poisson algebra \( \mathcal{P} \). The algebra \( PD(\mathcal{P}) \) is a subalgebra of the algebra \( D(\mathcal{P}) \) of differential operators on \( \mathcal{P} \). In general, \( PD(\mathcal{P}) \neq D(\mathcal{P}) \). Theorem \cite{11} is a simplicity criterion for the algebra \( PD(\mathcal{P}) \) of Poisson differential operators.

**Theorem 1.1** Let \( \mathcal{P} \) be a Poisson algebra over an arbitrary field \( K \). Then the following statements are equivalent:

1. The algebra \( PD(\mathcal{P}) \) is a simple algebra.
2. The Poisson algebra \( \mathcal{P} \) is a Poisson simple algebra.

Simplicity criteria for the Poisson enveloping algebra \( \mathcal{U}(\mathcal{P}) \). There is a natural algebra epimorphism
\[
\pi_\mathcal{P} : \mathcal{U}(\mathcal{P}) \to PD(\mathcal{P}),
\]
see \cite{20}. The algebra \( \mathcal{P} \) is a \( D(\mathcal{P}) \)-module and hence \( PD(\mathcal{P}) \)- and \( \mathcal{U}(\mathcal{P}) \)-module (via \( \pi_\mathcal{P} \)).

**Theorem 1.2** Let \( \mathcal{P} \) be a Poisson algebra over an arbitrary field \( K \). Then the following statements are equivalent:

1. The algebra \( \mathcal{U}(\mathcal{P}) \) is a simple algebra.
2. The algebra \( PD(\mathcal{P}) \) is a simple algebra and \( \mathcal{U}(\mathcal{P}) \simeq PD(\mathcal{P}) \).
3. The Poisson algebra \( \mathcal{P} \) is a Poisson simple algebra and \( \mathcal{P} \) is a faithful left \( \mathcal{U}(\mathcal{P}) \)-module.
If one of the equivalent conditions holds then $U(P) \simeq PD(P)$.

A localization of an affine commutative algebra is called an algebra of essentially finite type. In the case when the Poisson algebra $P = A$ is an algebra of essentially finite type over a field of characteristic zero, Theorem 1.2 can be strengthened, see Theorem 1.3. Let us introduce necessary definitions in order to formulate Theorem 1.3. $P_n = K[x_1, \ldots, x_n]$ is a polynomial algebra over $K$, $I = (f_1, \ldots, f_m)$ is a prime but not a maximal ideal of $P_n$, $A = S^{-1}(P_n/I)$ is a domain of essentially finite type and $Q = Q(A)$ is its field of fractions, $r = r(\frac{\partial f_i}{\partial x_j})$ is the rank (over $Q$) of the Jacobian matrix $\frac{\partial f_i}{\partial x_j}$ of $A$ and $d = d_A = r(C_A)$ is the rank (over $Q$) of the $n \times n$ matrix $C_A = (\{x_i, x_j\}) \in M_n(A)$, GK stands for the Gelfand-Kirillov dimension.

**Theorem 1.3** Let a Poisson algebra $A$ be an algebra of essentially finite type over the field $K$ of characteristic zero. Then the following statements are equivalent:

1. The algebra $U(A)$ is a simple algebra.

2. The algebra $PD(A)$ is a simple algebra and one of the equivalent conditions of Theorem 1.2 holds.

3. The algebra $A$ is Poisson simple and one of the equivalent conditions of Theorem 1.2 holds.

If one of the equivalent conditions holds then the algebra $A = S^{-1}(P_n/I)$ is a regular, Poisson simple domain of essentially finite type over the field $K$ of characteristic zero, the algebra epimorphism $\pi_A : U(A) \to PD(A)$ is an isomorphism (see (26)), $d = n - r$ where $d = r(C_A)$ and $r = r(\frac{\partial f_i}{\partial x_j})$, and the algebra $U(A)$ is a simple Noetherian domain with

$$GK U(A) = GK PD(A) = GK gr U(A) = GK Sym_A(\Omega_A) = 2GK A = 2(n - r).$$

The proofs of Theorem 1.3, Theorem 1.2 and Theorem 1.3 are given in Section 7.

Generators and defining relations for the Poisson enveloping algebra $U(P)$. In Section 2 for each Poisson algebra $P$ explicit sets of generators and defining relations for its Poisson enveloping algebra $U(P)$ are given (Theorem 2.2). Several (expected) results about the Poisson enveloping algebras are proven that are used later in the paper. It is proven that localizations commute with the operation of taking the Poisson enveloping algebra (Theorem 2.10). The symmetric algebra $S(G) = Sym(G)$ of a Lie algebra $G$ admits the canonical Poisson structure that is determined by the Lie structure on $G$. Proposition 2.11 is an explicit description of the algebra $U(S(G))$. It is shown that $U(S(G_1 \times G_2)) \simeq U(S(G_1) \otimes U(S(G_2))$ (Corollary 2.12) where $G_1 \times G_2$ is a direct product of Lie algebras. The structure of the PEA of the Poisson symmetric algebra of a semi-direct product of Lie algebras is described (Corollary 2.13).

The Gelfand-Kirillov dimension of the algebras $U(A)$, $gr U(A)$ and $Sym_A(\Omega_A)$ where $A$ is a domain of essentially finite type.

**Theorem 1.4** Let a Poisson algebra $A = S^{-1}(P_n/I)$ be a domain of essentially finite type over a perfect field $K$ where $I = (f_1, \ldots, f_m)$ is a prime ideal of $P_n$ and $r = r(\frac{\partial f_i}{\partial x_j})$ is the rank of the Jacobian matrix $\frac{\partial f_i}{\partial x_j}$ over the field of fractions of the domain $P_n/I$. Then the algebra $U(A)$ is a Noetherian algebra with

$$GK U(A) = GK gr U(A) = GK Sym_A(\Omega_A) = 2GK A = 2(n - r).$$

The Gelfand-Kirillov dimension of the algebra $PD(A)$ of Poisson differential operators on $A$. Proposition 1.5 gives the exact figure for the Gelfand-Kirillov dimension of the algebra $PD(A)$.
Proposition 1.5 Let a Poisson algebra \( \mathcal{A} \) be a domain of essentially finite type over the field \( K \) of characteristic zero, \( r \) is the rank of Jacobian matrix of \( \mathcal{A} \) and \( d = \nu(C_{\mathcal{A}}) \). Then

\[ \text{GK}(PD(\mathcal{A})) = \text{GK}(\mathcal{A}) + d = n - r + d. \]

The proof of Proposition 1.5 is given in Section 6, see Proposition 6.3.

Criterion for the algebra \( \mathcal{U}(\mathcal{A}) \) to be a domain. Theorem 1.6 (see Theorem 4.3 (3)) and Theorem 1.7 are criteria for the algebra \( \mathcal{U}(\mathcal{A}) \) to be a domain where the Poisson algebra \( \mathcal{A} \) is a domain of essentially finite type. The first one is given in terms of the Jacobian matrix and the Jacobian ideals of \( \mathcal{A} \) and the second one – in terms of the grades of the Jacobian ideals and prime ideals (but for certain class of Poisson algebras of essentially finite type).

For \( i = (i_1, \ldots, i_r) \) such that \( 1 \leq i_1 < \cdots < i_r \leq m \) and \( j = (j_1, \ldots, j_r) \) such that \( 1 \leq j_1 < \cdots < j_r \leq n \),

\[ \Delta(i, j) := \det \left( \frac{\partial f_{j_\nu}}{\partial x_{j_\mu}} \right), \quad \nu, \mu = 1, \ldots, r, \]

denotes the corresponding minor of the Jacobian matrix of the algebra \( \mathcal{A} \) where \( r \) is the rank of the Jacobian matrix of \( \mathcal{A} \). The \( r \)-tuple \( i \) (resp., \( j \)) is called non-singular if \( \Delta(i, j') \neq 0 \) (resp., \( \Delta(i', j) \neq 0 \)) for some \( j' \) (resp., \( i' \)). We denote by \( I_r \) (resp., \( J_r \)) the set of all the non-singular \( r \)-tuples \( i \) (resp., \( j \)). By [1, Lemma 2.1], \( \Delta(i, j) \neq 0 \) iff \( i \in I_r \) and \( j \in J_r \). The Jacobian ideal \( \mathfrak{a}_r \) of the algebra \( \mathcal{A} \) is an ideal of \( \mathcal{A} \) that is generated by all the minors \( \Delta(i, j) \) of the Jacobian matrix of \( \mathcal{A} \).

Theorem 1.6 Let a Poisson algebra \( \mathcal{A} = S^{-1}(I) \) be a domain of essentially finite type over a perfect field \( K \) where \( I = (f_1, \ldots, f_m) \) is a prime but not maximal ideal of \( S \), \( r = r(\partial f_i/\partial x_j) \) is the rank of the Jacobian matrix \( \left( \frac{\partial f}{\partial x_j} \right) \) and \( \mathfrak{a}_r \) is the Jacobian ideal of \( \mathcal{A} \). Then the following statements are equivalent:

1. The algebra \( \mathcal{U}(\mathcal{A}) \) is a domain.
2. The algebra \( \text{gr}\mathcal{U}(\mathcal{A}) \) is a domain.
3. The algebra \( \text{Sym}_{\mathcal{A}}(\Omega_{\mathcal{A}}) \) is a domain.
4. The elements \( \{ \Delta(i, j) \} \) \( i \in I_r, j \in J_r \) are regular in \( \mathcal{U}(\mathcal{A}) \).
5. The element \( \Delta(i, j) \) is a regular element of the algebra \( \mathcal{U}(\mathcal{A}) \) for some \( i \in I_r \) and \( j \in J_r \).
6. \( \text{ann}_{\mathcal{U}(\mathcal{A})}(\mathfrak{a}_r) = \text{r.ann}_{\mathcal{U}(\mathcal{A})}(\mathfrak{a}_r) = 0 \).

In proving the theorem below a result of Huneke is used, see Theorem 4.6. For an \( \mathcal{A} \)-module \( M \), we denote by \( v(M, \mathcal{A}) \) its minimal number of generators.

Theorem 1.7 Let a Poisson algebra \( \mathcal{A} = S^{-1}(P, I) \) be a universally catenarian domain of essentially finite type over a perfect field \( K \) satisfying Serre’s condition \( S_m \) and \( I = (f_1, \ldots, f_m) \).

The following statements are equivalent:

1. The algebra \( \mathcal{U}(\mathcal{A}) \) is a domain.
2. The algebra \( \text{gr}\mathcal{U}(\mathcal{A}) \) is a domain.
3. The algebra \( \text{Sym}_{\mathcal{A}}(\Omega_{\mathcal{A}}) \) is a domain.
4. \( \text{grade}(\mathfrak{a}_t) \geq m + 2 - t \) for \( 1 \leq t \leq m \) where \( \mathfrak{a}_t \) is the ideal of \( \mathcal{A} \) generated by \( t \times t \) minors of the Jacobian matrix \( \left( \frac{\partial f_j}{\partial x_i} \right) \).
5. \( v(\Omega_p, \mathcal{A}_p) \leq n - m + \text{grade}(p) - 1 \) for all nonzero primes \( p \) of \( \mathcal{A} \).
If one of the equivalent conditions holds then the algebra $\text{gr}\, U(\mathcal{A}) \simeq \text{Sym}_A(\Omega)$ is a complete intersection in the polynomial algebra $\mathcal{A}[\delta_1, \ldots, \delta_n]$. In particular, if the algebra $\mathcal{A}$ is Cohen-Macaulay (resp., Gorenstein) then so is the algebra $\text{gr}\, U(\mathcal{A}) \simeq \text{Sym}_A(\Omega)$.

The proofs of Theorem 1.4 (see Theorem 4.3(3)) and Theorem 1.7 are given in Section III.

**Simplicity criterion for the Poisson algebra** $\mathcal{A}$ **of essentially finite type**, i.e. Der$_K(\mathcal{A}) = A\mathcal{H}_A$. For each Poisson algebra $\mathcal{P}$, Der$_K(\mathcal{P}) \geq \mathcal{P}\mathcal{H}_P$. A Poisson algebra $\mathcal{P}$ which is a regular (affine) domain is called a symplectic algebra if Der$_K(\mathcal{P}) = \mathcal{P}\mathcal{H}_P$. Theorem 1.10 is a criterion for Der$_K(\mathcal{A}) = A\mathcal{H}_A$ where $\mathcal{A}$ is a regular domain of essentially finite type. Let $\mathfrak{c}_{A,d}$ be the ideal of the algebra $\mathcal{A}$ which is generated by all the $d \times d$ minors of the matrix $C_A$ where $d := r(C_A)$ is its rank.

**Theorem 1.8** Let a Poisson algebra $\mathcal{A}$ be a regular domain of essentially finite type over a field $K$ of characteristic zero and $d := r(C_A)$. Then the following statements are equivalent:

1. Der$_K(\mathcal{A}) = A\mathcal{H}_A$.
2. $d = n - r$ and $\mathfrak{c}_{A,d} = \mathcal{A}$.
3. For each $i \in \mathbb{I}_r$ and $j \in \mathbb{J}_r$, $\Delta(i, j)^{n-r} \in \mathfrak{m}_j$ where $\mathfrak{m}_j$ is the ideal of $\mathcal{A}$ generated by all the $(n-r) \times (n-r)$ minors of the $n \times (n-r)$ matrix $\mathcal{C}_{A,j}$ (see Proposition 5.1).
4. For each $i \in \mathbb{I}_r$ and $j \in \mathbb{J}_r$, $\Delta(i, j)^{k} \in \mathfrak{c}_{A,n-r}$ for some $k \geq 1$.

Lemma 5.4 and Corollary 5.5 are regularity and symplecticity criteria for certain generalized Weyl Poisson algebras.

**Criteria for** $\ker(\pi_A) = 0$. Recall that there is a natural algebra epimorphism $\pi_{\mathcal{P}} : U(\mathcal{P}) \to PD(\mathcal{P})$, see (28). In the case when the Poisson algebra $\mathcal{P} = \mathcal{A}$ is a regular domain of essentially finite type, Theorem 1.9 is an efficient explicit criterion for $\ker(\pi_A) = 0$, i.e. for the epimorphism $\pi_{\mathcal{A}} : U(\mathcal{A}) \to PD(\mathcal{A})$ to be an isomorphism.

Let $\kappa_{\mathcal{A}}$ be an ideal of the algebra $U(\mathcal{A})$ which is generated by a finite set of explicit elements $\delta_{i,j} \in \Omega_A$ where $i,j \in I_A(d)$, see (28). Then $\kappa_{\mathcal{A}} \subseteq \ker(\pi_{\mathcal{A}})$, see (31). Since Der$_K(\mathcal{A}) \simeq \text{Hom}_{A}(\Omega_A, \mathcal{A})$, there is a pairing of left $\mathcal{A}$-modules (which is an $\mathcal{A}$-bilinear map, see (33)):

$$\text{Der}_K(\mathcal{A}) \times \Omega_A \to \mathcal{A}, \ (\partial, \omega) \mapsto (\partial, \omega) := \partial(\omega).$$

**Theorem 1.9** Let a Poisson algebra $\mathcal{A} = S^{-1}(\mathcal{P}_n/I)$ be a regular domain of essentially finite type over the field $K$ of characteristic zero, $d = r(C_A)$ and $r = r\left(\frac{\partial f}{\partial x_j}\right)$. Then the following statements are equivalent (the derivations $\partial_{i,j}^{\nu}$ of $\mathcal{A}$ are defined in Theorem 4.1):

1. $\ker(\pi_A) = 0$ ($\Leftrightarrow \pi_A : U(\mathcal{A}) \simeq PD(\mathcal{A})$).
2. $\kappa_{\mathcal{A}} = 0$.
3. $d = n - r$ and $\left(\partial_{i,j}^{\nu}, \partial_{i',j'}^{\mu}\right) = 0$ for all elements $i \in \mathbb{I}_r$, $j \in \mathbb{J}_r$, $i' \in \mathbb{I}_A(d)$, $j' \in \mathbb{J}_A(d)$, $\nu = r + 1, \ldots, n$ and $\mu = d + 1, \ldots, n$ where for $j = (j_1, \ldots, j_r)$ and $i' = (i'_1, \ldots, i'_d)$, $\{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\}\setminus\{j_1, \ldots, j_r\}$ and $\{i'_{d+1}, \ldots, i'_n\} = \{1, \ldots, n\}\setminus\{i'_1, \ldots, i'_d\}$.

*Theorem 1.10* is a criterion for $\ker(\pi_A) = 0$ in the general situation, i.e. without the assumption of regularity in the previous theorem.

**Theorem 1.10** Let a Poisson algebra $\mathcal{A} = S^{-1}(\mathcal{P}_n/I)$ be a domain of essentially finite type over a field of characteristic zero, $r = r\left(\frac{\partial f}{\partial x_j}\right)$ and $d = r(C_A)$. Then the following statements are equivalent:

1. $\ker(\pi_A) = 0$ ($\Leftrightarrow \pi_A : U(\mathcal{A}) \simeq PD(\mathcal{A})$).
2. \( \kappa_\mathcal{A} = 0 \) and \( \mathcal{U}(\mathcal{A}) \) is a domain (or any of the equivalent conditions of Theorem 1.10 holds).

3. \( d = n - r, \mathcal{U}(\mathcal{A}) \) is a domain (or any of the equivalent conditions of Theorem 1.10 holds) and \( (\text{Der}_K(\mathcal{A}), \delta, \partial) \equiv 0 \) for all elements \( i' \in \mathcal{I}_\mathcal{A}(d), j' \in \mathcal{J}_\mathcal{A}(d) \) and \( \mu = d + 1, \ldots, n \) where for \( i' = (i_{1}', \ldots, i_{d}'), \{j_{r+1}, \ldots, j_{n}\} = \{1, \ldots, n\}\{j_{1}, \ldots, j_{r}\} \) and \( \{i_{d+1}', \ldots, i_{n}'\} = \{1, \ldots, n\}\{i_{1}', \ldots, i_{d}'\} \).

Criterion for the homomorphism \( \pi_\mathcal{A} : \mathcal{U}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}) \) to be an isomorphism. For a regular domain of essentially finite type \( \mathcal{A} \), Theorem 1.11 is a criterion for the homomorphism \( \pi_\mathcal{A} : \mathcal{U}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}) \) to be an isomorphism, i.e. the algebra \( \mathcal{U}(\mathcal{A}) \) is the algebra of differential operators \( \mathcal{D}(\mathcal{A}) \) on \( \mathcal{A} \).

**Theorem 1.11** Let a Poisson algebra \( \mathcal{A} = S^{-1}(P_\mathcal{A}/I) \) be a regular domain of essentially finite type over the field of characteristic zero, \( r = r(\partial/\delta) \) and \( d = r(C_\mathcal{A}) \). Then the following statements are equivalent:

1. The homomorphism \( \pi_\mathcal{A} : \mathcal{U}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}) \) is an isomorphism.

2. \( \kappa_\mathcal{A} = 0 \) and \( c_\mathcal{A},d = \mathcal{A} \).

If one of the equivalent conditions holds then \( \ker(\pi_\mathcal{A}) = 0 \) and \( \mathcal{U}(\mathcal{A}) = \mathcal{P}\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}) \).

Every Poisson enveloping algebra is isomorphic to its opposite algebra (Theorem 3.22 (2)). So, left and right properties of the Poisson enveloping algebra are identical.

### 2 Generetors and defining relations of the Poisson enveloping algebra of a Poisson algebra

The aim of this section is to prove Theorem 2.2 that for each Poisson algebra \( \mathcal{P} \) given explicit sets of generators and defining relations for the algebra \( \mathcal{U}(\mathcal{P}) \). As a result, for each Poisson factor algebra \( \mathcal{P}' \) of a Poisson algebra \( \mathcal{P} \), explicit sets of generators and defining relations are given for the algebra \( \mathcal{U}(\mathcal{P}') \) (Theorem 2.6). It is shown that \( \mathcal{U}(\mathcal{P}_1 \otimes \mathcal{P}_2) \simeq \mathcal{U}(\mathcal{P}_1) \otimes \mathcal{U}(\mathcal{P}_2) \) (Proposition 2.7) where \( \mathcal{P}_1 \otimes \mathcal{P}_2 \) is a tensor product of Poisson algebras. Proposition 2.9 shows that every endomorphism/automorphism of a Poisson algebra can be lifted to an endomorphism/automorphism of its Poisson enveloping algebra. It is proven that localizations commute with the operation of taking the Poisson enveloping algebra (Theorem 2.10). The symmetric algebra \( S(\mathcal{G}) = \text{Sym}(\mathcal{G}) \) of a Lie algebra \( \mathcal{G} \) admits the canonical Poisson structure that is determined by the Lie structure on \( \mathcal{G} \). Proposition 2.11 is an explicit description of the algebra \( \mathcal{U}(S(\mathcal{G})) \). It is shown that \( \mathcal{U}(S(\mathcal{G}_1 \times \mathcal{G}_2)) \simeq \mathcal{U}(S(\mathcal{G}_1)) \otimes \mathcal{U}(S(\mathcal{G}_2)) \) (Corollary 2.3) where \( \mathcal{G}_1 \times \mathcal{G}_2 \) is a direct product of Lie algebras. The structure of the PEA of the Poisson symmetric algebra of a semi-direct product of Lie algebras is described in Corollary 2.13. A criterion for the PEA of a Poisson algebra to be a commutative algebra is given in Corollary 2.23. Examples of the PEAs are considered. At the beginning of the section we recall some definitions and results about Poisson algebras and their modules.

**Poisson algebras.** An associative commutative algebra \( D \) is called a Poisson algebra if it is a Lie algebra \((D,\{\cdot,\cdot\})\) such that \( \{a,xy\} = \{a,x\}y+x\{a,y\} \) for all elements \( a, x, y \in D \).

For a \( K \)-algebra \( D \), let \( \text{Der}_K(D) \) be the set of its \( K \)-derivations. If, in addition, \((D,\{\cdot,\cdot\})\) is a Poisson algebra then

\[
\text{PDer}_K(D) := \{\delta \in \text{Der}_K(D) | \delta(\{a,b\}) = \{\delta(a),b\} + \{a,\delta(b)\} \text{ for all } a,b \in D\}
\]

is the set of derivations of the Poisson algebra \( D \). The vector space \( \text{Der}_K(D) \) is a Lie algebra, where \( [\delta,\partial] := \delta\partial - \partial\delta \). The set of inner derivations of the Poisson algebra \( D \),

\[
\text{PIDer}_K(D) := \{\text{pad}_a | a \in D\} \text{ (where } \text{pad}_a(b) := \{a,b\}\}
\]
is an ideal of the Lie algebra PDer$_K(D)$ (since $[\delta, \text{pad}_a] = \text{pad}_{i(\delta)}$ for all $\delta \in \text{PDer}_K(D)$ and $a \in D$). By the very definition, the Poisson algebra $D$ is a Lie algebra with respect to the bracket $\{\cdot, \cdot\}$. The map $D \rightarrow \text{PDer}_K(D)$, $a \mapsto \text{pad}_a$, is an epimorphism of Lie algebras with kernel

$$\text{PZ}(D) := \{a \in D \mid \{a, D\} = 0\}$$

which is called the centre of the Poisson algebra (or the Poisson centre of $D$). The Poisson centre $\text{PZ}(D)$ is invariant under the action of $\text{PDer}_K(D)$: Let $z \in \text{PZ}(D)$, $d \in D$ and $\partial \in \text{PDer}_K(D)$; then applying the derivation $\partial$ to the equality $\{z, d\} = 0$ we obtain the equality $\{\partial(z), d\} = 0$, i.e. $\partial(z) \in \text{PZ}(D)$.

**The dual Poisson algebra of a Poisson algebra.** Given an associative algebra $\mathcal{P}$. Then its dual (associative) algebra $\mathcal{P}^{\text{op}}$ coincides with $\mathcal{P}$ as a vector space but the multiplication is given by the rule $a \ast b := ba$. Every left $\mathcal{P}$-module is a right $\mathcal{P}^{\text{op}}$-module, and vice versa.

Similarly, given a Poisson algebra $(\mathcal{P}, \{\cdot, \cdot\})$. Its dual associative algebra $\mathcal{P}^{\text{op}}$ is a Poisson algebra $(\mathcal{P}^{\text{op}}, \{\cdot, \cdot\}^{\text{op}})$, which is called the dual Poisson algebra of $\mathcal{P}$, where $\{a, b\}^{\text{op}} := -\{a, b\}$ for all $a, b \in \mathcal{P}^{\text{op}}$.

**The Poisson structure constants matrix $C_{\mathcal{P}}$ and the ideal $\mathfrak{c}_{\mathcal{P}}$ of a Poisson algebra $\mathcal{P}$.** Let $\mathcal{P}$ be a Poisson algebra and $\{x_i\}_{i \in I}$ be a set of algebra generators of $\mathcal{P}$. The Poisson structure on an associative algebra $\mathcal{P}$ is uniquely determined by the Poisson structure constants $c_{ij} := \{x_i, x_j\}$ where $i, j \in I$. Let $n = \text{card}(I)$ be the cardinality of the set $I$, the case $n = \infty$ is possible. The $n \times n$ matrix

$$C_{\mathcal{P}} := (c_{ij})$$

is called the Poisson structure constants matrix of the Poisson algebra $\mathcal{P}$ and the ideal $\mathfrak{c}_{\mathcal{P}}$ of $\mathcal{P}$, which is generated by all the structure constants $c_{ij}$, is called the Poisson structure constants ideal of the Poisson algebra $\mathcal{P}$.

For all elements $f, g \in \mathcal{P}$,

$$\{f, g\} = f'C_{\mathcal{P}}g^t$$

where $f' := \text{grad}(f) := (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is the gradient of $f$ (a $1 \times n$ matrix with coefficients in $\mathcal{P}$) and $t$ is the transposition of matrix.

If $\{x_i'\}_{i \in I'}$ is another set of algebra generators for $\mathcal{P}$ and $C'_{\mathcal{P}} := (c'_{st})$ is the Poisson matrix constants that correspond to the second set of generators (where $c'_{st} := \{x_s', x_t'\}$). Then (t is the trasposition of matrix)

$$C'_{\mathcal{P}} := J'C_{\mathcal{P}}J$$

where $J := J(x', x) := \frac{\partial x'}{\partial x}$

the Jacobian matrix of the change of the variables from $x = \{x_i\}_{i \in I}$ to $x' = \{x'_i\}_{s \in I'}$, i.e. the $s^{\text{th}}$ row of the $\text{card}(I') \times \text{card}(I)$ matrix $C'_{\mathcal{P}}$ is the gradient $\text{grad}(x'_s) := (\frac{\partial x'_s}{\partial x_i})$ of the function $x'_s = x'_s(x_1, \ldots, x_i, \ldots)$ where $i \in I$.

**Lemma 2.1** Let $\mathcal{P}$ be a Poisson algebra. Then the Poisson structure constants ideal $\mathfrak{c}_{\mathcal{P}}$ is a Poisson ideal of $\mathcal{P}$ (i.e. $[\mathcal{P}, \mathfrak{c}_{\mathcal{P}}] \subseteq \mathfrak{c}_{\mathcal{P}}$) which does not depend on the choice of algebra generators of $\mathcal{P}$.

**Proof.** The lemma follows at once from (2). □

**A module of a Poisson algebra.** Let a commutative associative algebra $(\mathcal{P}, \{\cdot, \cdot\})$ be a Poisson algebra and $M$ be a left $\mathcal{P}$-module ($\mathcal{P} \times M \rightarrow M, (a, m) \mapsto am$). The left $\mathcal{P}$-module $M$ over the associative algebra $\mathcal{P}$ is called a left module over the Poisson algebra or a Poisson left $\mathcal{P}$-module if there is a bilinear map

$$\mathcal{P} \times M \rightarrow M, \ (a, m) \mapsto \delta_a m$$
which is called a Poisson action of \( \mathcal{P} \) on \( M \) such that for all elements \( a, b \in \mathcal{P} \) and \( m \in M \),

\[
\begin{align*}
(\text{PM1}) \, \delta_{\{a, b\}} &= [\delta_a, \delta_b], \\
(\text{PM2}) \, [\delta_a, b] &= \{a, b\}, \text{ and} \\
(\text{PM3}) \, \delta_{ab} &= a\delta_b + b\delta_a.
\end{align*}
\]

Every left Poisson \( \mathcal{P} \)-module \( M \) determines the homomorphism of associative algebras,

\[
\mathcal{P} \to \text{End}_K(M), \quad a \mapsto a_M : M \to M, \quad m \mapsto am
\]
and the homomorphism of Lie algebras,

\[
\mathcal{P} \to \text{End}_K(M), \quad a \mapsto \delta_a : M \to M, \quad m \mapsto \delta_am
\]
such that

\[
[\delta_a, b_M] = \{a, b\}_M \text{ for all } a, b \in \mathcal{P},
\]

\[
\delta_{ab} = a_M\delta_b + b_M\delta_a \text{ for all } a, b \in \mathcal{P},
\]

and vice versa. Indeed, (3) determines a \( \mathcal{P} \)-module structure on \( M \), (4) determines a Lie \( \mathcal{P} \)-module structure on \( M \), and (5) and (6) are equivalent to the properties (PM2) and (PM3), respectively. So, a Poisson \( \mathcal{P} \)-module is a \( \mathcal{P} \)-module over the associative algebra \( \mathcal{P} \) and the Lie algebra \( \mathcal{P} \) and both module structures are related by (5) and (6).

**Example.** Every Poisson algebra \( \mathcal{P} \) is a left Poisson \( \mathcal{P} \)-module where for all \( a \in \mathcal{P} \), \( a_{\mathcal{P}} : \mathcal{P} \to \mathcal{P} \), \( b \mapsto ab \) and \( \delta_a = \{a, \cdot\} : \mathcal{P} \to \mathcal{P} \), \( b \mapsto \{a, b\} \).

**Right modules of a Poisson algebra.** Given a Poisson algebra \( (\mathcal{P}, \{\cdot, \cdot\}) \). A right Poisson module over \( \mathcal{P} \) is, by definition, a left Poisson module over the dual Poisson algebra \( (\mathcal{P}^\text{op}, \{\cdot, \cdot\}^\text{op}) \).

**Example.** Every Poisson algebra \( \mathcal{P} \) is a right Poisson \( \mathcal{P} \)-module where for all \( a \in \mathcal{P} \), \( pa : \mathcal{P} \to \mathcal{P} \), \( b \mapsto ba \) and \( \delta'_a = \{\cdot, a\} : \mathcal{P} \to \mathcal{P} \), \( b \mapsto \{b, a\} \).

The ring \( \mathcal{D}(A) \) of differential operators on an algebra \( A \). Let us recall the definition of the ring of differential operators \( \mathcal{D}(A) \) on a commutative algebra \( A \) over a field \( K \). The ring of (\( K \)-linear) differential operators \( \mathcal{D}(A) \) on \( A \) is defined as a union of \( A \)-modules \( \mathcal{D}(A) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(A) \) where \( \mathcal{D}_0(A) = \text{End}_R(A) \simeq A, \ (x \mapsto ax) \leftrightarrow a \),

\[
\mathcal{D}_i(A) = \{u \in \text{End}_K(A) \mid [a, u] := au - ua \in \mathcal{D}_{i-1}(A) \text{ for all } a \in A\}.
\]
The set of \( A \)-modules \( \{\mathcal{D}_i(A)\} \) is the order filtration for the algebra \( \mathcal{D}(A) \):

\[
\mathcal{D}_0(A) \subseteq \mathcal{D}_1(A) \subseteq \cdots \subseteq \mathcal{D}_i(A) \subseteq \cdots \quad \text{and} \quad \mathcal{D}_i(A)\mathcal{D}_j(A) \subseteq \mathcal{D}_{i+j}(A) \quad \text{for all } i, j \geq 0.
\]
The subalgebra \( \Delta(A) \) of \( \mathcal{D}(A) \) generated by \( A \equiv \text{End}_R(A) \) and the set \( \text{Der}_K(A) \) of all \( K \)-derivations of \( B \) is called the derivation ring of \( A \).

Suppose that \( A \) is a regular affine domain of Krull dimension \( n < \infty \) over a field \( K \) of characteristic zero. In geometric terms, \( A \) is the coordinate ring \( O(X) \) of a smooth irreducible affine algebraic variety \( X \) of dimension \( n \). Then

- \( \text{Der}_K(A) \) is a finitely generated projective \( A \)-module of rank \( n \),
- \( \mathcal{D}(A) = \Delta(A) \),
- \(D(A)\) is a simple (left and right) Noetherian domain of Gelfand-Kirillov dimension \(\text{GK } D(A) = 2n\) \((n = \text{GK } (A) = \text{Kdim}(A))\).

For the proofs of the statements above the reader is referred to [14 Chapter 15]. So, the domain \(D(A)\) is a simple finitely generated infinite dimensional Noetherian algebra, [14 Chapter 15].

The algebra \(PD(P)\) of Poisson differential operators. Let \(P\) be a Poisson algebra. Then \(\mathcal{H}_P := \text{PIDer}_K(P) = \{\text{pad}_a := \{a, \cdot\} | a \in P\}\) is a Lie subalgebra of the Lie algebra \(\text{Der}_K(P)\) (since \([\text{pad}_a, \text{pad}_b] = \text{pad}_{(a,b)}\) and a \(\text{PZ}(P)\)-submodule of \(\text{Der}_K(P)\).

**Definition.** Let \(P\) be a Poisson algebra (not necessarily commutative). The subalgebra \(PD(P)\) of \(\text{End}_K(P)\) which generated by \(L_P\) and \(\mathcal{H}_P\) is called the **algebra of Poisson differential operators**. Clearly, \(PD(P) \subseteq \Delta(P) \subseteq D(P)\).

**Semidirect products of algebras.** Let \(G\) be a Lie algebra and \(U(G)\) be the enveloping algebra of the Lie algebra \(G\) and \(\delta : G \to \text{Der}_K(D), a \mapsto \delta_a\) be a Lie algebra homomorphism \((\delta_{[a,b]} = [\delta_a, \delta_b] \text{ for all } a,b \in G)\). Let \(D \rtimes \delta U(G)\) be the semidirect product of \(D\) and \(U(G)\). It is an associative algebra that is generated by the algebras \(D\) and \(U(G)\) subject to the defining relations: For all elements \(d \in D\) and \(g \in G\), \(gd = dg + \delta_g(d)\). Let \(\{x_i\}_{i \in I}\) be a \(K\)-basis of the Lie algebra \(G\). Then
\[
D \rtimes \delta U(G) = \bigoplus_{\alpha \in \mathbb{N}^{|I|}} Dx^\alpha = \bigoplus_{\alpha \in \mathbb{N}^{|I|}} x^\alpha D
\]
is a free left and right \(D\)-module where \(\mathbb{N}^{|I|}\) is a direct sum of \(I\) copies of the set \(\mathbb{N}\), \(x^\alpha = \prod_{i \in I} x_i^{\alpha_i}\) (in the product the order of multiples is arbitrary and all but finitely many \(\alpha_i\) are equal to zero).

**Example.** Let \(P\) be a Poisson algebra and \(U(P)\) be its universal enveloping algebra as a Lie algebra. Then \(P \rtimes \text{pad } U(P)\) is a semidirect product of \(P\) and \(U(P)\) where \(\text{pad } : P \to \text{PDer}_K(P)\), \(a \mapsto \text{pad}_a := \{a, \cdot\}\).

Given another semidirect product \(D' \rtimes \delta' U(G')\), a homomorphism \(\varphi : D \to D'\) and a Lie homomorphism \(\psi : G \to G'\) such that \([\psi(g), \varphi(d)] = \delta'_{\psi(g)}(\varphi(d))\) for all elements \(d \in D\) and \(g \in G\). Then the map
\[
(\varphi, \psi) : D \rtimes \delta U(G) \to D' \rtimes \delta' U(G'), \ d \mapsto \varphi(d), \ g \mapsto \psi(g)
\]
is a homomorphism.

**Two presentations.** \(P = S^{-1}P_{\Lambda}/I \simeq \overline{S}^{-1}(P_{\Lambda}/I')\). Let \(P_{\Lambda} = K[x_i]_{i \in \Lambda}\) be a polynomial algebra where \(\{x_i\}_{i \in \Lambda}\) is a set of variables (no restriction on the cardinality of \(\Lambda\)), \(S\) be a multiplicative subset of \(P_{\Lambda}\), \(S^{-1}P_{\Lambda}\) is a localization of \(P_{\Lambda}\) at \(S\) and \(P = S^{-1}P_{\Lambda}/I\) be a factor algebra of \(S^{-1}P_{\Lambda}\) modulo an ideal \(I = (f_s)_{s \in \Gamma}\) where \(\{f_s\}_{s \in \Gamma}\) is a set of its generators. The algebra \(P\) can also be written in the form \(\overline{S}^{-1}(P_{\Lambda}/I')\) where \(I'\) is an ideal of the polynomial algebra \(P_{\Lambda}\) and \(\overline{S}\) is a multiplicative subset of regular elements of the factor algebra \(P_{\Lambda}/I'\) (eg, \(\overline{S}\) is the image of the set \(S\) under the epimorphism \(P_{\Lambda} \to P_{\Lambda}/I'\)).

Conversely, given an algebra \(\overline{S}^{-1}(P_{\Lambda}/I')\) where \(I'\) is an ideal of the polynomial algebra \(P_{\Lambda}\) and \(\overline{S}\) is a multiplicative subset of regular elements of the factor algebra \(P_{\Lambda}/I'\). Then the algebra \(\overline{S}^{-1}(P_{\Lambda}/I')\) can be written as \(S^{-1}P_{\Lambda}/I\) for some multiplicative subset \(S_{\Lambda}\) of \(P_{\Lambda}\) and some ideal \(I\) of \(S^{-1}P_{\Lambda}\) (eg, \(S = \varphi^{-1}(\overline{S})\) and \(I = S^{-1}\ker(\varphi)\) where \(\varphi : P_{\Lambda} \to \overline{S}^{-1}(P_{\Lambda}/I')\)). So,
\[
P = S^{-1}P_{\Lambda}/I \simeq \overline{S}^{-1}(P_{\Lambda}/I')
\]
where \(I'\) is an ideal of \(P_{\Lambda}\) and \(\overline{S}\) is a multiplicative subset of regular elements of \(P_{\Lambda}/I'\).

Why we stressed this seemingly obvious fact? In theoretical arguments, the second presentation is slightly more preferable but in dealing with examples and computations, the first one is (as
the number of generators of an ideal under localizations is dropped, as a rule).

**Generators and defining relations of the algebra \( \mathcal{U}(\mathcal{P}) \).**

Definition. Let \(( \mathcal{P}, \{\cdot, \cdot\} )\) be a Poisson algebra over \( K \). A triple \(( \mathcal{U}, \alpha, \beta \) is called a Poisson enveloping algebra \((\text{PEA}, \text{for short})\) of the Poisson algebra \( \mathcal{P} \) if \( \mathcal{U} \) is an (associative) algebra, \( \alpha : \mathcal{P} \rightarrow \mathcal{U} \) is an algebra homomorphism, \( \beta : \mathcal{P} \rightarrow \mathcal{U} \) is a Lie algebra homomorphism such that for all elements \( a, b \in \mathcal{P} \),

\[
[\beta(a), \alpha(b)] = \alpha(\{a, b\}) \quad \text{and} \quad \beta(ab) = \beta(a)\alpha(b) + \alpha(a)\beta(b),
\]

if \(( \mathcal{U}', \alpha', \beta' \) is another triple as above then there is a unique algebra homomorphism \( f : \mathcal{U} \rightarrow \mathcal{U}' \) such that \( \alpha' = f\alpha \) and \( \beta' = f\beta \).

For a Poisson algebra \( \mathcal{P} \) which is defined by generators and defining relations (as an associative algebra), Theorem 2.2 gives explicit sets of generators and defining relations for the Poisson enveloping algebra \( \mathcal{U}(\mathcal{P}) \).

**Theorem 2.2** Let \( \mathcal{P} \) be a Poisson algebra, \( \mathcal{U}(\mathcal{P}) \) be its universal enveloping algebra (as a Lie algebra) and \( \mathcal{U}(\mathcal{P}) \) be its Poisson enveloping algebra. Then

1. \( \mathcal{U}(\mathcal{P}) \simeq \mathcal{P} \rtimes_{\text{pad}} U(\mathcal{P})/\mathcal{I}(\mathcal{P}) \) where \( \mathcal{I}(\mathcal{P}) = (\delta_{ab} - a\delta_{b} - b\delta_{a}, a, b \in \mathcal{P}) \) is the ideal of the algebra \( \mathcal{P} \rtimes_{\text{pad}} U(\mathcal{P}) \) generated by the set \( \{\delta_{ab} - a\delta_{b} - b\delta_{a}, a, b \in \mathcal{P}\} \).

2. If \( \mathcal{P} = S^{-1} K[x_i]_{i \in A}/(f_s)_{s \in \Gamma} \) where \( S \) is a multiplicative subset of the polynomial algebra \( K[x_i]_{i \in A} \) \((\lambda \text{ and } \Gamma \text{ are index sets})\). Then the algebra \( \mathcal{U}(\mathcal{P}) \) is generated by the algebra \( \mathcal{P} \) and the elements \( \delta_i := \delta_{x_i} \mid i \in A \) subject to the defining relations \((a)-(c)\): For all elements \( i, j \in A \) such that \( i \neq j \) and \( s \in \Gamma \),

\[
\begin{align*}
(a) \quad & [\delta_i, \delta_j] = \sum_{k \in A} \frac{\partial(x_i, x_j)}{\partial x_k} \delta_k, \\
(b) \quad & [\delta_i, x_j] = \{x_i, x_j\}, \quad \text{and} \\
(c) \quad & \sum_{i \in A} \frac{\partial f_s}{\partial x_i} \delta_i = 0.
\end{align*}
\]

So, the algebra \( \mathcal{U}(\mathcal{P}) \) is generated by the algebra \( \mathcal{P} \) and the set \( \delta_\mathcal{P} = \{\delta_a \mid a \in \mathcal{P}\} \) subject to the defining relations: For all elements \( a, b \in \mathcal{P} \) and \( \lambda, \mu \in K \),

\[
\begin{align*}
(a) \quad & [\delta_a, \delta_b] = \delta_{\{a, b\}}, \\
(b) \quad & [\delta_a, b] = \{a, b\}, \\
(c) \quad & \delta_{ab} = a\delta_{b} + b\delta_{a}, \\
(d) \quad & \delta_{\lambda a + \mu b} = \lambda\delta_{a} + \mu\delta_{b} \text{ and } \delta_1 = 0.
\end{align*}
\]

3. The map \( \pi_\mathcal{P} : \mathcal{U}(\mathcal{P}) \rightarrow D(\mathcal{P}), a \mapsto a, \delta_b \mapsto \text{pad}_b = \{b, \cdot\} \) is an algebra homomorphism where \( a, b \in \mathcal{P} \) and its image is the algebra \( PD(\mathcal{P}) \) of Poisson differential operators of the Poisson algebra \( \mathcal{P} \).

4. The algebra \( \mathcal{P} \) is a subalgebra of \( \mathcal{U}(\mathcal{P}) \). Furthermore, \( \mathcal{U}(\mathcal{P}) = \mathcal{P} \oplus \text{ann}_{\mathcal{U}(\mathcal{P})}(1) \) is a direct sum of left \( \mathcal{P} \)-modules where \( \text{ann}_{\mathcal{U}(\mathcal{P})}(1) = \sum_{i \in A} \mathcal{U}(\mathcal{P})\delta_i \) is the annihilator of the identity element of the Poisson \( \mathcal{P} \)-module \( \mathcal{P} \). The Poisson \( \mathcal{P} \)-module structure on the Poisson algebra \( \mathcal{P} \) is obtained from the \( D(\mathcal{P}) \)-module structure on \( \mathcal{P} \) by restriction of scalars via \( \pi_\mathcal{P} \).

**Proof.** 1. Consider the triple \(( \mathcal{U}', \alpha, \beta \) where \( \mathcal{U}' := \mathcal{P} \rtimes_{\text{pad}} U(\mathcal{P})/\mathcal{I}(\mathcal{P}) \), \( \alpha : \mathcal{P} \rightarrow \mathcal{U}' \), \( a \mapsto a \) (it is an algebra homomorphism), \( \beta : \mathcal{P} \rightarrow \mathcal{U}' \), \( b \mapsto \delta_b = \text{ad}_b := \{b, \cdot\} \) (it is a Lie algebra homomorphism). Then for all elements \( a, b \in \mathcal{P} \),

\[
[\beta(a), \alpha(b)] = \alpha(\{a, b\}) \quad \text{and} \quad \beta(ab) = \beta(a)\alpha(b) + \alpha(a)\beta(b).
\]
Given an associative algebra $A$, a homomorphism $\alpha' : \mathcal{P} \to A$ and a Lie homomorphism $\beta' : \mathcal{P} \to A$ such that $[\beta'(a), \alpha'(b)] = \alpha'(\{a, b\})$ and $\beta'(ab) = \beta'(a)\alpha'(b) + \alpha'(a)\beta'(b)$ for all elements $a, b \in \mathcal{P}$.

Then by the universal property of the semidirect product there is a unique homomorphism

$$\mathcal{U}' := \mathcal{P} \rtimes_{\text{pad}} U(\mathcal{P})/I(\mathcal{P}) \to A, \ a + I(\mathcal{P}) \mapsto \alpha'(a), \ \delta_0 + I(\mathcal{P}) \mapsto \beta'(b)$$

such that $\alpha' = f\alpha$ and $\beta' = f\beta$. Therefore, $\mathcal{U}(\mathcal{P}) \simeq \mathcal{U}'$.

2. Statement 2 follows at once from statement 1 as the relations (a) – (b) of statement 1 follow from the axioms of Poisson algebra: For all elements $i, j \in A$ such that $i \neq j$ and $s \in \Gamma$,

$$[\delta_i, \delta_j] = \delta_{\{i, j\}} = \sum_{s \in \Lambda} \frac{\partial \delta_{i, j}}{\partial x_s} \delta_k,$$

$$[\delta_i, x_j] = \delta_{\{i, x\}} = \{i, x\},$$

$$0 = \delta_0 = \delta_{\{\cdot, \cdot\}} = \sum_{s \in \Lambda} \frac{\partial \delta_{\{\cdot, \cdot\}}}{\partial x_s} \delta_k = 0.$$

3. Statement 3 follows from statement 1.

4. By the definition of the homomorphism $\pi_\mathcal{P}$, the Poisson $\mathcal{P}$-module structure on the Poisson algebra $\mathcal{P}$ is obtained from the $\mathcal{P}(\mathcal{P})$-module structure on $\mathcal{P}$ by restriction of scalars via $\pi_\mathcal{P}$.

Since $\mathcal{U}(\mathcal{P}) = \mathcal{P} + I$, where $I = \sum_{s \in \Lambda} \mathcal{U}(\mathcal{P})\delta_i$, $I \subseteq \text{ann}_U(\mathcal{P})(1)$ and $\mathcal{P} \cap \text{ann}_U(\mathcal{P})(1)$, we must have $\mathcal{U}(\mathcal{P}) = \mathcal{P} \oplus I$ and $I = \text{ann}_U(\mathcal{P})(1) = 0$, and statement 4 follows. \(\square\)

Corollary 2.3 Every homomorphism of Poisson algebras $f : \mathcal{P} \to \mathcal{P}'$ can be extended to a homomorphism of their Poisson enveloping algebras $f : U(\mathcal{P}) \to U(\mathcal{P}')$ by the rule $f(\delta_i) = \delta_{f(\alpha)}$.

Proof. The corollary follows from Theorem 2.2.(1,2). The map $f$ is well-defined since $f(\delta_{ab}) = f(\delta_{f(a)f(b)}) = f(\delta_{f(a)}f(b)) + f(\delta_{f(b)})f(a)$ for all elements $a, b \in \mathcal{P}$. To finish the proof of the corollary it suffices to show that the relations (a)–(d) of Theorem 2.2.(2) holds. Let us check that the relation (a) holds. The other relations can be verified in a similar way. For all elements $a, b \in \mathcal{P}$,

$$f([\delta_i, \delta_j]) = [f(\delta_{f(a)}), f(\delta_{f(b)})] = f([\delta_i, \delta_j]) = f([\{a, b\}]). \quad \square$$

The PEA of a trivial Poisson algebra. Corollary 2.4 describes the PEA of a trivial Poisson algebra, i.e. $\{\cdot, \cdot\} = 0$.

Corollary 2.4 Suppose that the algebra $\mathcal{P} = S^{-1}K[x_i]_{i \in \Lambda}/(f_s)_{s \in \Gamma}$ is a trivial Poisson algebra (i.e. $\{\cdot, \cdot\} = 0$) where $S$ is a multiplicative subset of regular elements of the polynomial algebra $K[x_i]_{i \in \Lambda}$. Then $U(\mathcal{P}) \simeq U(S^{-1}K[x_i]_{i \in \Lambda}/(\sum_{s \in \Lambda} \delta_{x_s}x_s)_{s \in \Gamma}$.) In particular, $U(K[x_i]_{i \in \Lambda}) = K[x_i, \delta_{x_i}]_{i \in \Lambda} \simeq K[x_i]_{i \in \Lambda} \otimes K[x_i]_{i \in \Lambda}$ is polynomial algebra.

Proof. The corollary follows at once from Theorem 2.2. \(\square\)

Criterion for the algebra $U(\mathcal{P})$ to be a commutative algebra. Corollary 2.5 is such a criterion.

Corollary 2.5 Let $\mathcal{P}$ be a Poisson algebra. Then the algebra $U(\mathcal{P})$ is a commutative algebra iff the Poisson algebra $\mathcal{P}$ is a trivial Poisson algebra.

Proof. $(\Rightarrow)$ Suppose that the algebra $U(\mathcal{P})$ is a commutative algebra. By Theorem 2.2.(4), $\mathcal{P} \subseteq U(\mathcal{P})$. Then, by Theorem 2.2.(2), $0 = \{\delta_i, x_j\} = \{x_i, x_j\} \in \mathcal{P}$, i.e. $\mathcal{P}$ is a trivial Poisson algebra.

$(\Leftarrow)$ This implication follows from Corollary 2.4 or Theorem 2.2.(2). \(\square\)

Generators and defining relations of the PEA of a factor algebra of a Poisson algebra. Proposition 2.6 represents the PEA of a factor algebra of a Poisson algebra as a factor algebra the PEA of the original Poisson algebra.

Proposition 2.6 Let $\mathcal{P}$ be a Poisson algebra, $a$ be a Poisson ideal of $\mathcal{P}$, $\mathcal{P}/a$ be the Poisson factor algebra of $\mathcal{P}$ and $\mathcal{P} \to \mathcal{P}/a$, $a \mapsto \pi = a + a$. Then $U(\mathcal{P}/a) \simeq U(\mathcal{P})/\langle a, \delta_a \rangle$ where $\delta_a = \{\delta_a, a \in a\}$. Furthermore, if $\{a_i\}_{i \in I}$ is a set of generators of the ideal $a$ then $U(\mathcal{P}/a) \simeq U(\mathcal{P})/\langle a, \delta_{a_i} \rangle_{i \in I}$.
Proof. By Theorem 2.2(1), the map $\mathcal{U}(P) \rightarrow \mathcal{U}(\overline{P})$, $a \mapsto \overline{a}$, $\delta_a \mapsto \overline{\delta_a}$ (where $a \in P$) is an algebra epimorphism as the relations (a)–(d) in Theorem 2.2 are respected by the map. Then, by Theorem 2.2(1), the statement follows. $\square$

The PEA of a tensor product of Poisson algebras. Let $P_1$ and $P_2$ be Poisson algebras. Their tensor product $P = P_1 \otimes P_2$ is a Poisson algebra with respect to a unique Poisson bracket that is determined by the Poisson brackets of the tensor components and the condition that $\{P_1, P_2\} = 0$: For all elements $a_1, a_2 \in P_1$ and $b_1, b_2 \in P_2$,

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}.$$  \hfill (9)

Example. The Poisson algebra $P_{2n} = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ where $\{x_i, x_j\} = 0$, $\{y_i, y_j\} = 0$ and $\{y_i, x_j\} = \delta_{ij}$ (the Kronecker delta) is a tensor product $P_{2n} = P_2^\otimes n$ of $n$ copies of the Poisson algebra $P_2 = K[x, y]$ where $\{y, x\} = 1$.

Proposition 2.7 shows that the PEA of a tensor product of Poisson algebras is a tensor product of PEA’s of the tensor components.

Proposition 2.7 Let $P = P_1 \otimes P_2$ be a tensor product of Poisson algebras $P_1$ and $P_2$. Then $\mathcal{U}(P) \simeq \mathcal{U}(P_1) \otimes \mathcal{U}(P_2)$.

Proof. The result follows from Theorem 2.2 $\square$

Proposition 2.8 Let $P_{2n} = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ where $\{x_i, x_j\} = 0$, $\{y_i, y_j\} = 0$ and $\{y_i, x_j\} = \delta_{ij}$ (the Kronecker delta). Then the algebra $\mathcal{U}(P_{2n})$ is isomorphic to the Weyl algebra $A_{2n}$.

Proof. Recall that $P_{2n} \simeq P_2^\otimes n$, see above. By Proposition 2.7, $\mathcal{U}(P_{2n}) \simeq \mathcal{U}(P_2)^\otimes n$. So, it suffices to prove the statement for $n = 1$, i.e. $P_2 = K[x, y]$ with $\{x, y\} = 1$. By Theorem 2.2(2), the algebra $\mathcal{U}(P_2)$ is generated by the elements $x$, $y$, $\delta_x$ and $\delta_y$ subject to the defining relations: $xy = yx$, $[\delta_x, \delta_y] = 0$, $[\delta_x, y] = \{x, y\} = 1$, $[\delta_y, y] = \{y, y\} = 0$, $[\delta_x, y] = \{x, y\} = 1$ and $[\delta_x, x] = \{x, x\} = 0$, i.e. $\mathcal{U}(P_2) \simeq A_2$. $\square$

For an associative algebra $A$, we denote by $\text{End}_K(A)$ (resp., $\text{Aut}_K(A)$) the set of all algebra endomorphisms (resp., automorphisms) of $A$. Clearly, $\text{End}_K(A)$ is a monoid w.r.t. the composition of maps and $\text{Aut}_K(A)$ is its group of units. For a Poisson algebra $P$, we denote by $\text{End}_{P\text{ois}}(P)$ (resp., $\text{Aut}_{P\text{ois}}(P)$) the set of all Poisson algebra endomorphisms (resp., automorphisms) of $A$. Clearly, $\text{End}_{P\text{ois}}(P)$ is a monoid w.r.t. the composition of maps and $\text{Aut}_{P\text{ois}}(P)$ is its group of units.

Proposition 2.9 shows that every endomorphism/automorphism of a Poisson algebra is extended to an endomorphism/automorphism of its Poisson enveloping algebra.

Proposition 2.9 Let $P$ is a Poisson algebra. Then

1. The map $\text{End}_{P\text{ois}}(P) \rightarrow \text{End}_K(\mathcal{U}(P))$, $\sigma \mapsto \sigma : a \mapsto \sigma(a)$, $\delta_a \mapsto \delta_{\sigma(a)}$ ($a \in P$) is a monoid monomorphism.

2. The map $\text{Aut}_{P\text{ois}}(P) \rightarrow \text{Aut}_K(\mathcal{U}(P))$, $\sigma \mapsto \sigma : a \mapsto \sigma(a)$, $\delta_a \mapsto \delta_{\sigma(a)}$ ($a \in P$) is a group monomorphism.

Proof. 1. The proposition follows from Theorem 2.2 $\square$

Localizations commutes with the operation of taking PEA of a Poisson algebra.

Theorem 2.10 Let $P$ be a Poisson algebra and $S$ be a multiplicative subset of $P$, $a = \text{ass}(S) = \{a \in P \mid sa = 0 \text{ for some } s \in S\}$. Then the ideal $a$ is a Poisson ideal, $\overline{P} = P/a$ is a Poisson algebra and $\overline{S} = \{\overline{s} \mid s \in S\}$ is a multiplicative subset of $\overline{P}$ that consists of regular elements of the algebra $\overline{P}$, $S^{-1}P \simeq \overline{S}^{-1}\overline{P}$, $S \in \text{Den}(\mathcal{U}(P), a)$, $\overline{S} \in \text{Den}(\mathcal{U}(\overline{P}), 0)$, $\mathcal{U}(S^{-1}P) \simeq \mathcal{U}(\overline{S}^{-1}\overline{P}) \simeq S^{-1}\mathcal{U}(P) \simeq \mathcal{U}(P)S^{-1} \simeq \overline{S}^{-1}\mathcal{U}(\overline{P}) \simeq \mathcal{U}(\overline{P})\overline{S}^{-1}$.
Proof. (i) \(a\) is a Poisson ideal: By the very definition, \(a\) is an ideal of \(P\). It remains to show that \(\{P,a\} \subseteq a\). Given elements \(a \in a\) and \(b \in P\). Then \(sa = 0\) for some element \(s \in S\). Then \(0 = (b,0) = \{b, s^2 a\} = 2sa\{b, s\} + s^2\{b, a\} = s^2\{b, a\}\), and so \(\{b, a\} \in a\).

By the statement (i), \(\overline{P}\) is a Poisson algebra, the set \(\overline{S}\) consists of regular elements of \(\overline{P}\) and \(S^{-1}P \simeq \overline{S}^{-1}\overline{P}\).

(ii) \(S \in \text{Den}(U(P))\): The statement (ii) follows from Theorem 2.2 and the fact that for all elements \(a \in P\) and \(s \in S\), \(\delta_s a^2 - s^2 \delta_s a = \{a, s^2\} = 2s\{a, s\}\).

(iii) \(U(S^{-1}P) \simeq S^{-1}U(P) \simeq U(P)S^{-1}\).

By the universal property of the PEA, \(U(S^{-1}P) \simeq S^{-1}U(P)\). By the statement (ii), \(S^{-1}U(P) \simeq U(P)S^{-1}\), and the statement (iii) follows.

(iv) \(U(S^{-1}P) \simeq U(\overline{S}^{-1}\overline{P}) \simeq U(P)S^{-1}\): The statement (iv) follows from the statement (iii).

\(\square\)

The PEA of the Poisson symmetric algebra \(S(G)\) of a Lie algebra \(G\). Let \(G\) be a Lie algebra over the field \(K\), \(\{x_i\}_{i \in I}\) be its \(K\)-basis and the lie bracket on \(G\),

\[\{x_i, x_j\} = \sum_{k \in I} c^k_{ij} x_k,\]

is determined by the structure constants \(c^k_{ij} \in K\). The universal enveloping algebra \(U = U(G)\) of \(G\) admits the standard filtration \(\{U_i\}_{i \in \mathbb{N}}\) by the total degree of the generators \(\{x_i\}_{i \in I}\) of \(U\). The associated graded algebra \(S(G) := \text{gr}(U) = \bigoplus_{i \geq 0} U_i / U_{i-1}\) (where \(U_{-1} = 0\)), the so-called, symmetric algebra of \(G\), is a polynomial algebra in the variables \(\{x_i\}_{i \in I}\). Furthermore, the commutative algebra \(S(G)\) is a Poisson algebra where

\[\{x_i, x_j\} = \sum_{k \in I} c^k_{ij} x_k.\] (10)

Proposition 2.11 Let \(G\), \(U(G)\) and \(S(G), \{\ldots, \cdot\}\) be as above. Then \(U(S(G)) \simeq S(G) \ltimes_{\text{pad}} U(G)\) where \(\text{pad} : G \to \text{PDer}_K(S(G)), x \mapsto \text{pad}_x = \{x, \cdot\}\). In more detail, the algebra \(U(S(G))\) is generated by the set \(\{x_i, \delta_{x_i} | i \in I\}\) subject to the defining relations: For all elements \(i, j \in I\), \(x_i x_j = x_j x_i, [\delta_{x_i}, \delta_{x_j}] = \sum_{k \in I} c^k_{ij} \delta_{x_k}\) and \([\delta_{x_i}, x_j] = \{x_i, x_j\}\) (\(= \sum_{k \in I} c^k_{ij} x_k\)). The subalgebra \(K\langle \delta_{x_i} | i \in I\rangle U(S(G))\) of \(U(S(G))\) is isomorphic to the universal enveloping algebra \(U\) of the Lie algebra \(G\) via \(\delta_{x_i} \mapsto x_i\).

\[\square\]

Corollary 2.12 Let \(G = G_1 \times G_2\) be a direct product of Lie algebras. Then \(U(S(G)) \simeq U(S(G_1)) \otimes U(S(G_2))\) is a direct product of algebras.

\[\square\]

Proof. \(S(G) = S(G_1) \otimes S(G_2)\), a tensor product of Poisson algebras. Now, the corollary follows from Proposition 2.7.

\[\square\]

\(U(G_1 \ltimes_{\delta} G_2)\) where \(G_1 \ltimes_{\delta} G_2\) be a semidirect product of Lie algebras. Let \(G := G_1 \ltimes_{\delta} G_2\) be a semidirect product of Lie algebras where \(\delta : G_2 \to \text{Der}_{\text{Lie}}(G_1), x_2 \mapsto \delta_{x_2}\) is a Lie homomorphism \((\delta([x_2, y_2]) = [\delta_{x_2}, \delta_{y_2}]\) for all \(x_2, y_2 \in G_2\), i.e. the lie bracket on \(G\) is given by the rule: For all elements \(x_1, y_1 \in G_1\) and \(x_2, y_2 \in G_2\),

\[\{x_1 + x_2, y_1 + y_2\} = \{x_1, y_1\}_{G_1} + \{x_2, y_2\}_{G_2} + \delta_{x_2}(y_1) - \delta_{y_2}(x_1)\].

Then

\[U(G) \simeq U(G_1) \ltimes_{\delta} U(G_2).\]
and so \( S(\mathcal{G}) \cong (S(\mathcal{G}_1) \times_\delta S(\mathcal{G}_2)) = (S(\mathcal{G}_1) \otimes_S S(\mathcal{G}_2)) \), a tensor product of algebras, the Poisson bracket on the algebra \( S(\mathcal{G}) \) is given by the rule: For all elements \( x_1, y_1 \in \mathcal{G}_1 \) and \( x_2, y_2 \in \mathcal{G}_2 \),
\[
\{ x_1, y_1 \} = [x_1, y_1]_{\mathcal{G}_1}, \quad \{ x_2, y_2 \} = [x_2, y_2]_{\mathcal{G}_2} \quad \text{and} \quad \{ x_2, x_1 \} = \delta_{x_2}(x_1).
\]

**Corollary 2.13** Let \( \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \) be a semidirect product of Lie algebras. Then \( \mathcal{U}(S(\mathcal{G})) \cong (S(\mathcal{G}_1) \times_\delta S(\mathcal{G}_2)) \otimes \mathcal{U}(\mathcal{G}_1) \otimes \mathcal{U}(\mathcal{G}_2)) \).

**Proof.** By Proposition 2.11 \( \mathcal{U}(S(\mathcal{G})) \cong S(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \), and the result follows (since \( S(\mathcal{G}) \cong S(\mathcal{G}_1) \times_\delta S(\mathcal{G}_2) \) and \( \mathcal{U}(\mathcal{G}) \cong \mathcal{U}(\mathcal{G}_1) \otimes \mathcal{U}(\mathcal{G}_2)) \). \( \square \)

### 3 The PBW Theorem for the Poisson enveloping algebras and the module \( \Omega_P \) of Kähler differentials of a Poisson algebra \( P \)

The aim of this section is to prove the PBW Theorem for the Poisson enveloping algebras (Theorem 3.5); to give a projectivity criterion for the algebras \( \mathcal{U}(P) \) and \( \mathcal{U}(\mathcal{P}) \) (Corollary 3.7); to prove that the algebra \( \mathcal{U}(\mathcal{P}) \) is isomorphic to its opposite algebra \( \mathcal{U}(\mathcal{P})^{op} \); a criterion for the algebra \( gr \mathcal{U}(\mathcal{P}) \) to be a left/right Noetherian algebra or a finitely generated algebra is given (Proposition 3.11).

The derivation algebras are introduced at the end of the section. Theorem 3.12 is a criterion for a derivation algebra to have PBW basis. Theorem 3.13 and Theorem 3.14 show that every polynomial Poisson algebra and every Poisson algebra with free module of Kähler differential are derivation algebras with PBW basis, respectively.

**The algebras \( F_b, \ F_{ab} \) and \( \mathcal{U}(\mathcal{P}) \).** Let \( P_\Lambda = K[x_i]_{i \in \Lambda} \) be a polynomial algebra where the cardinality \( |\Lambda| \) of the set \( \Lambda \) can be arbitrary, \( I = (f_s)_{s \in \Gamma} \) be an ideal of \( P_\Lambda \), \( \tilde{P}_\Lambda := P_\Lambda/I \) and \( \mathcal{P} = S^{-1}P_\Lambda \) where \( S \) is a multiplicative subset in \( \tilde{P}_\Lambda \) that consists of regular elements (i.e. non-zero-divisors) of \( \tilde{P}_\Lambda \), eg, \( S = \{1\} \). Suppose that \( \mathcal{P} \) is a Poisson algebra. Let \( F = (\mathcal{P}, \delta_{x_i} = \delta_i)_{i \in \Lambda} \) be a \( K \)-algebra generated freely by the algebra \( \mathcal{P} \) and a set of free generators \( \{\delta_i \mid i \in \Lambda\} \). Let \( \mathcal{R}_a, \mathcal{R}_b \) and \( \mathcal{R}_c \) be ideals of the algebra \( F \) that are generated by the relations (a), (b) and (c) of Theorem 2.2, respectively,

\[
\begin{align*}
\mathcal{R}_a &= \left( \{ \delta_i, \delta_j \} - \sum_{k \in \Lambda} \frac{\partial \{ x_i, x_j \}}{\partial x_k} \delta_k \right)_{i,j \in \Lambda}, \\
\mathcal{R}_b &= \left( \{ \delta_i, x_j \} - \{ x_i, x_j \} \right)_{i,j \in \Lambda}, \\
\mathcal{R}_c &= \left( \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_i \right)_{s \in \Gamma}.\end{align*}
\]

Let \( \mathcal{R}_{ab} = (\mathcal{R}_a, \mathcal{R}_b) \) and \( \mathcal{R} = (\mathcal{R}_a, \mathcal{R}_b, \mathcal{R}_c) \), ideals of \( F \). By Theorem 2.2(2), \( \mathcal{U}(\mathcal{P}) \cong F/\mathcal{R} \). There are algebra epimorphisms

\[
F \to F_b := F/\mathcal{R}_b \to F_{ab} := F/\mathcal{R}_{ab} \to \mathcal{U}(\mathcal{P}) \cong F/\mathcal{R}.
\]

**Proposition 3.1** Let \( \mathcal{P} = S^{-1}(P_\Lambda/I) \) be a Poisson algebra where \( P_\Lambda = K[x_i]_{i \in \Lambda} \) be a polynomial algebra, \( I = (f_s)_{s \in \Gamma} \) be an ideal of \( P_\Lambda \), \( \tilde{P}_\Lambda := P_\Lambda/I \) and \( S \) be a multiplicative subset in \( \tilde{P}_\Lambda \) that consists of regular elements of \( \tilde{P}_\Lambda \). Then

1. \( F_b = \bigoplus_{\delta \in \mathcal{M}(\Lambda)} \mathcal{P} \delta = \bigoplus_{\delta \in \mathcal{M}(\Lambda)} \delta \mathcal{P} \) where \( \mathcal{M}(\Lambda) \) is a free multiplicative monoid on the set \( \{ \delta_i \}_{i \in \Lambda} \).
2. \( F_{ab} = \bigoplus_{\delta \in \mathcal{M}(\Lambda)_{c}} \mathcal{P} \delta = \bigoplus_{\delta \in \mathcal{M}(\Lambda)_{c}} \delta \mathcal{P} \) where \( \mathcal{M}(\Lambda)_{c} \) is a free commutative multiplicative monoid on the set \( \{ \delta_i \}_{i \in \Lambda} \), i.e. \( F_{ab} = \bigoplus_{\alpha \in \mathcal{N}(\Lambda)} \mathcal{P} \delta^\alpha = \bigoplus_{\alpha \in \mathcal{N}(\Lambda)} \delta^\alpha \mathcal{P} \) where \( \delta^\alpha = \prod_{i \in \Lambda} \delta_i^{\alpha_i} \) and \( \alpha = (\alpha_i)_{i \in \Lambda} \). In particular, the algebra \( F_{ab} \) is a free left and right \( \mathcal{P} \)-module. If the algebra \( \mathcal{P} \) is Noetherian and \( |\Lambda| < \infty \) then the algebra \( F_{ab} \) is a Noetherian algebra.
3. The elements \( \{ \delta_s := \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_{x_i} \mid s \in \Gamma \} \subseteq F_{ab} \) belong to the centralizer of the algebra \( \mathcal{P} \) in \( F_{ab} \) (i.e. \( \left[ \delta_{f_s}, \delta_{x_j} \right] = 0 \) for all \( s \in \Gamma \) and \( j \in \Lambda \); \( \left[ \delta_{f_s}, \delta_{x_j} \right] = \sum_{k \in \Lambda} \frac{\partial f_{x_k}}{\partial x_i} \delta_{x_k} ; \mathcal{U}(\mathcal{P}) \simeq F_{ab}/(\delta_{f_s})_{s \in \Gamma} \); and the ideal \( \mathcal{F}_{ab} = (\delta_{f_s})_{s \in \Gamma} \) of the algebra \( F_{ab} \) is equal to the left and right ideals of \( F_{ab} \) that are generated by the elements \( \{ \delta_{f_s} \mid s \in \Gamma \} \).

4. If the algebra \( \mathcal{P} \) is Noetherian and \( |\Lambda| < \infty \) then the algebra \( \mathcal{U}(\mathcal{P}) \) is a Noetherian algebra.

**Proof.**

1. Statement 1 is obvious.

2. Fix a total ordering on the set \( \{ \delta_i := \delta_{x_i} \mid i \in \Lambda \} \). It determines the deg-by-lexicographic ordering on the monoid \( \mathcal{M}(\Lambda) \). In order to prove that the first equality of statement 2 holds (about the direct sum) it suffices to show that the ambiguities \( \{ \delta_i \delta_i a \mid i > j \text{ and } a \in \mathcal{P} \} \) and \( \{ \delta_i \delta_j k \mid i > j > k \} \) are resolved:

\[
\begin{align*}
\delta_i \delta_j a &= \delta_i a \delta_j + \delta_i \{ x_j, a \} = a \delta_i \delta_j + \{ x_i, a \} \delta_j + \{ x_i, \{ x_j, a \} \} \\
&= a \delta_i \delta_j + \sum_{k \in \Lambda} \frac{\partial f_{x_k}}{\partial x_i} \delta_{x_k} + \{ x_i, a \} \delta_j + \{ x_i, \{ x_j, a \} \}
\end{align*}
\]

\[
\begin{align*}
\delta_i \delta_j a &= \delta_j \delta_i a + \sum_{k \in \Lambda} \frac{\partial f_{x_k}}{\partial x_k} \delta_{x_k} a = a \delta_j \delta_i + \{ x_j, a \} \delta_i + \delta_j \{ x_i, a \} + \sum_{k \in \Lambda} \frac{\partial f_{x_k}}{\partial x_k} a \delta_{x_k} + \{ x_k, a \}
\end{align*}
\]

The RHSs of both equalities are equal since \( \{ x_i, \{ x_j, a \} \} = \{ x_j, \{ x_i, a \} \} = \{ x_i, x_j, a \} \). Similarly,

\[
\begin{align*}
\delta_i \delta_j k &= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} = \delta_i \delta_j k + \delta_j \delta_i \{ x_j, x_k \} + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_j \delta_i k + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} k \\
&= \delta_i \delta_j k + \delta_i \{ x_j, x_k \} k
\end{align*}
\]

The RHSs of both equalities are equal since \( \{ x_j, \{ x_i, x_k \} \} = \{ x_k, \{ x_i, x_j \} \} + \{ x_i, \{ x_j, x_k \} \} \). So, \( F_{ab} = \bigoplus_{i \in \mathcal{M}(\Lambda)} \mathcal{P} \delta_i. \) Hence, \( F_{ab} = \bigoplus_{i \in \mathcal{M}(\Lambda)} \mathcal{P} \delta_i. \)

The associated graded algebra \( \text{gr}(F_{ab}) \) of the algebra \( F_{ab} \) w.r.t. the filtration on the algebra \( F_{ab} \) by the total degree of the elements \( \{ \delta_i \mid i \in \Lambda \} \) is a polynomial algebra \( \mathcal{P}[\delta_i]_{i \in \Lambda} \) which is a Noetherian algebra provided the algebra \( \mathcal{P} \) is Noetherian and \( |\Lambda| < \infty \). Hence, under these conditions the algebra \( F_{ab} \) is also Noetherian.

3. For all \( s \in \Gamma \) and \( j \in \Lambda \), we have the following equalities in the algebra \( F_{ab} \):

\[
\left[ \delta_{f_s}, \delta_{x_j} \right] = \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} = \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} = \sum_{i \in \Lambda} \frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j}
\]

\[
\delta_{f_s} \delta_{x_j} - \delta_{x_j} \delta_{f_s} = 0,
\]

\[
\frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} = \sum_{i \in \Lambda} \left( \frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} + \frac{\partial f_s}{\partial x_i} \delta_{x_i} \right)
\]

\[
\frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} = \sum_{i \in \Lambda} \left( \frac{\partial f_s}{\partial x_i} \delta_{x_i} \delta_{x_j} \right)
\]

Clearly, \( \mathcal{U}(\mathcal{P}) \simeq F_{ab}/(\delta_{f_s})_{s \in \Gamma} \), and the rest of the statement 3 follows.
4. The algebra $F_{ab}$ is a Noetherian algebra, by statement 2. Hence, so is its factor algebra $\mathcal{U}(\mathcal{P})$, by statement 3. □

**The Poisson enveloping algebra is isomorphic to its opposite algebra.** Let $A$ be an algebra. The opposite algebra $A^{\text{op}}$ of $A$ is an algebra such that $A^{\text{op}} = A$ (the equality of vector spaces) and the product in $A^{\text{op}}$ is given by the rule: For all elements $a, b \in A^{\text{op}}, a \cdot b = ba$. Theorem 3.2 shows the Poisson enveloping algebra is isomorphic to its opposite algebra.

**Theorem 3.2** We keep the notation of Proposition 3.1.

1. The map $F_{ab} \rightarrow F_{ab}^{\text{op}}, a \mapsto a, \delta_i \mapsto -\delta_i$ is an algebra isomorphism where $a \in \mathcal{P}$ and $i \in \Lambda$.

2. The map $\mathcal{U}(\mathcal{P}) \rightarrow \mathcal{U}(\mathcal{P})^{\text{op}}, a \mapsto a, \delta_i \mapsto -\delta_i$ is an algebra isomorphism where $a \in \mathcal{P}$ and $i \in \Lambda$, i.e. for each Poisson algebra its PEA $\mathcal{U}$ is isomorphic to its opposite algebra $\mathcal{U}^{\text{op}}$ and as a result properties of the algebra $\mathcal{U}$ is left-right symmetric.

3. For all elements $f \in \mathcal{P}$, \(\sum_{j \in \Lambda} \left\{ \frac{\partial f}{\partial x_j}, x_j \right\} = 0\),

Proof. 3. Suppose that $\text{char}(K) \neq 2$. Then

\[
\begin{align*}
\sum_{j \in \Lambda} \left\{ \frac{\partial m}{\partial x_j}, x_j \right\} &= \sum_{j=1}^{l} \left( m_j, x_j \right) = \sum_{i \neq j=1}^{l} m_{ij} \{ x_i, x_j \} = 2 \sum_{i<j=1}^{l} m_{ij} \{ x_i, x_j \} = 0.
\end{align*}
\]

In the case $\text{char}(K) = 2$, it suffices to show that the equality holds for monomials that are products of distinct $x_i$'s. Let $m = x_1 \cdots x_l, m_i$ and $m_{ij} \ (i \neq j)$ be the monomial $m$ where the elements $x_i$ and $x_j$ are deleted. Then

\[
\sum_{j \in \Lambda} \left\{ \frac{\partial m}{\partial x_j}, x_j \right\} = \sum_{j=1}^{l} \left( m_j, x_j \right) = \sum_{i \neq j=1}^{l} m_{ij} \{ x_i, x_j \} = 2 \sum_{i<j=1}^{l} m_{ij} \{ x_i, x_j \} = 0.
\]

1 and 2. The map, say $\theta$, in the statements 1 and 2 respects each of the defining relations $\mathcal{R}_a, \mathcal{R}_b$ and $\mathcal{R}_c$ (by using statement 3):

\[
\begin{align*}
\mathcal{R}_a & : \quad \theta(\delta_i, \delta_j) - \sum_{k \in \Lambda} \delta \frac{x_i}{x_k} \delta_k = \{-\delta_j, -\delta_i\} - \sum_{k \in \Lambda} (-\delta_k) \frac{x_i}{x_k} \partial x_k \\
& = - \left\{ \delta_i, \delta_j \right\} - \sum_{k \in \Lambda} \frac{x_k}{x_k} \partial x_k \delta_k = - \left\{ \delta_i, \delta_j \right\} - \sum_{k \in \Lambda} \frac{x_k}{x_k} \partial x_k \delta_k.
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_b & : \quad \theta(\delta_i, x_j) - \left\{ x_i, x_j \right\} = \left\{ x_j, -\delta_i \right\} - \left\{ x_i, x_j \right\} = \left\{ \delta_i, x_j \right\} - \left\{ x_i, x_j \right\}.
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}_c & : \quad \theta(\delta_{f_s}) = \theta(\sum_{i \in \Lambda} \delta \frac{x_i}{x_i} \delta_i) = \sum_{i \in \Lambda} (-\delta_i) \frac{x_i}{x_i} \partial x_i = - \delta_{f_s} - \sum_{i \in \Lambda} \left\{ x_i, \frac{x_i}{x_i} \partial x_i \right\} = - \delta_{f_s}.
\end{align*}
\]

**The opposite Poisson algebra $\mathcal{P}^{\text{op}}$ of a Poisson algebra $\mathcal{P}$ and $\mathcal{U}(\mathcal{P}^{\text{op}}) \simeq \mathcal{U}(\mathcal{P})^{\text{op}}$.** Let $\mathcal{P}$ be a Poisson algebra. The opposite Poisson algebra of $\mathcal{P}$, denoted by $\mathcal{P}^{\text{op}}$, is a Poisson algebra that coincides with $\mathcal{P}$ as an associative algebra but the Poisson bracket $\{ \cdot, \cdot \}^{\text{op}}$ in $\mathcal{P}^{\text{op}}$ is given by the rule: $\{ a, b \}^{\text{op}} = \{ b, a \}$. We can easily verify that $(\mathcal{P}^{\text{op}}, \{ \cdot, \cdot \}^{\text{op}})$ is a Poisson algebra. Clearly, $(\mathcal{P}^{\text{op}})^{\text{op}} \simeq \mathcal{P}$.

**Proposition 3.3** Let $\mathcal{P}$ be a Poisson algebra. Then $\mathcal{U}(\mathcal{P}^{\text{op}}) \simeq \mathcal{U}(\mathcal{P})^{\text{op}} \simeq \mathcal{U}(\mathcal{P})$. 

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Proof. In view of Theorem 3.2 (2), we have to show that only the first isomorphism of the proposition holds. Let $P$ be as in Theorem 3.2 (2). Then the map

$$
\phi : \mathcal{U}(P^{op}) \rightarrow \mathcal{U}(P)^{op}, \ a \mapsto a, \ \delta_i \mapsto \delta_i \ (a \in P, \ i \in \Lambda)
$$

(11)

is an algebra isomorphism. By the very definition, the map is an algebra isomorphism provided it is well-defined. In order to check this we have to show that the map $\phi$ respects the defining relations of both algebras, i.e. the relations (a)–(c) in Theorem 2.2 (2). This can be easily checked directly by using Theorem 2.2 (2). Furthermore, the map $\phi$ respects each type of the relations (a)–(c). For example, the image under the map $\phi$ of a relation of the type (b) is a relation again of the type (b):

$$
\phi([\delta_i, x_j] - \{x_i, x_j\}^{op}) = \phi(\delta_i) - \{x_i, x_j\} = -\{[\delta_i, x_j] - \{x_i, x_j\}\} = -0 = 0. \ \Box
$$

The module $\Omega_P$ of Kähler differentials of a Poisson algebra $P$ and $\mathcal{U}(P)$. Let $P$ be a Poisson algebra and let $\Omega = \Omega_P$ be the module of Kähler differentials of the (associative) algebra $P$. In more detail, if $P = S^{-1}P_\Lambda/I$ where $I = (f_s)_{s \in \Gamma}$ then

$$
\Omega = \Omega_{S^{-1}P_\Lambda/I} \cong \bigoplus_{i \in \Lambda} Pdx_i / \sum_{s \in \Gamma} Pd\delta_s \text{ where } d\delta_s = \sum_{i \in \Lambda} \frac{\partial \delta_s}{\partial x_i} dx_i.
$$

(12)

Notice that $\mathbb{F} := \bigoplus_{i \in \Lambda} P\delta_i$ is a $P$-submodule of $F_{ab}$ (Proposition 3.1 (2)) that contains the $P$-submodule $\sum_{s \in \Gamma} P\delta_f$. Then the map

$$
\Omega_P \rightarrow \mathbb{F} / \sum_{s \in \Gamma} P\delta_f, \ dx_i \mapsto \delta_i \ (i \in \Gamma)
$$

(13)

is a $P$-modules isomorphism.

Let $\overline{R}_c$ be the image of the ideal $R_c$ of $F$ under the algebra epimorphism $F \rightarrow F_{ab}$. By Proposition 3.1 (3), $\overline{R}_c$ is a left/right/two-sided ideal of the algebra $F_{ab}$ which is generated by the set $\{\delta_f, s \in \Gamma\}$ of the algebra $F_{ab}$. By Proposition 3.1 (3),

$$
\overline{R}_c = \sum_{s \in \Gamma} \delta_f, F_{ab} = M_0F_{ab} = F_{ab}M_0 \text{ where } M_0 := \sum_{s \in \Gamma} P\delta_f = \sum_{s \in \Gamma} \delta_f P \subseteq F_{ab}.
$$

(14)

By Proposition 3.1 (2),

$$
F_{ab} = \bigoplus_{i \geq 0} F_{ab,i} = \bigoplus_{i \geq 0} F'_{ab,i} \text{ where } F_{ab,i} = \bigoplus_{|a|=i} P\delta^a \text{ and } F'_{ab,i} = \bigoplus_{|a|=i} \delta^a P.
$$

So, $F_{ab}$ is a direct sum of left $P$-modules $F_{ab,i}$ and right $P$-modules $F'_{ab,i}$. By (14) and Proposition 3.1 (3),

$$
\overline{R}_c = \bigoplus_{i \geq 0} M_0F_{ab,i} = \bigoplus_{i \geq 0} F'_{ab,i}M_0, \ M_0F_{ab,i} \subseteq F_{ab,i+1} \text{ and } F'_{ab,i}M_0 \subseteq F'_{ab,i+1}, \ i \geq 0.
$$

(15)

By (15), the algebra

$$
\mathcal{U}(P) = F_{ab}/\overline{R}_c = \bigoplus_{i \geq 0} F_{ab,i}/M_0F_{ab,i-1} = \bigoplus_{i \geq 0} F'_{ab,i}/F'_{ab,i-1}M_0
$$

(16)

is a direct sum of left $P$-modules $F_{ab,i}/M_0F_{ab,i-1}$ and right $P$-modules $F'_{ab,i}/F'_{ab,i-1}M_0$. By (13) and (16),

$$
\mathcal{U}(P) = P \oplus \Omega_P \oplus \bigoplus_{i \geq 2} F_{ab,i}/M_0F_{ab,i-1} = P \oplus \Omega'_P \oplus \bigoplus_{i \geq 2} F'_{ab,i}/F'_{ab,i-1}M_0
$$

(17)

where $\Omega'_P := \mathbb{F}'/M_0$ and $\mathbb{F}' := \bigoplus_{i \in \Lambda} \delta_f P$. The first (resp., second) direct sum in (17) is a direct sum of left (resp., right) $P$-modules.
Theorem 3.4  Let $\mathcal{P}$ be a Poisson algebra and $\Omega = \Omega_\mathcal{P}$ be the $\mathcal{P}$-modules of Kähler differentials of $\mathcal{P}$. Then

1. $\Omega$ is a Lie subalgebra of the Lie algebra $(\mathcal{U}(\mathcal{P}), [\cdot, \cdot])$ where $[a, b] = ab - ba$. In particular, for all elements $a_1, a_2, b_1, b_2 \in \mathcal{P}$,
\[
[a_1 da_2, b_1 db_2] = a_1 \{a_2, b_1\} db_2 - b_1 \{b_2, a_1\} da_2 + a_1 b_1 d\{a_2, b_2\}.
\]

2. The left $\mathcal{P}$-module homomorphism $\Omega \to \text{Der}_K(\mathcal{P})$, $adb \mapsto a\{b, \cdot\}$ is a Lie algebra homomorphism.

3. The left $\mathcal{P}$-submodule $\mathcal{P} \oplus \Omega$ of $\mathcal{U}(\mathcal{P})$ is a Lie subalgebra which is a semi-direct product $\mathcal{P} \times \Omega$ of the Lie algebra $\Omega$ by the abelian Lie algebra $\mathcal{P}$ via the Lie homomorphism in statement 2, i.e. $[adb, p] = a\{b, p\}$ for all elements $a, b, p \in \mathcal{P}$. In particular, the Poisson structure on $\mathcal{P}$ can be recovered from the Lie algebra structure on $\mathcal{P} \times \Omega$ by the rule: For all elements $a, b \in \mathcal{P}$, $\{a, b\} = [da, b]$.

4. $\mathcal{U}(\mathcal{P}) \cong \mathcal{P} \times U(\Omega) / R(\Omega)$ where $U(\Omega)$ is the universal enveloping algebra of the Lie $K$-algebra $\Omega$, $\mathcal{P} \times U(\Omega)$ is a skew product of the algebra $U(\Omega)$ and $\mathcal{P}$ and $R(\Omega) = (a \cdot db - adb)_{a, b \in \mathcal{P}}$ (where $a \cdot db$ is a product of the elements $a \in \mathcal{P}$ and $db \in \Omega$ in the algebra $\mathcal{P} \times U(\Omega)$ but $adb \in \Omega$).

Proof. 1-3. Statements 1–3 follow from \cite{[17]}.

4. Statement 4 follows from statement 3 and Theorem \cite{[22]}(2): If we represent the algebra $\mathcal{P}$ as the factor algebra $F_\mathcal{P} / (f_i)_{i \in \Lambda}$ then, in view of \cite{[13]}, the algebra $\mathcal{U}(\mathcal{P}) \cong \mathcal{P} \times U(\Omega) / R(\Omega)$ is generated by the algebra $\mathcal{P}$ and the set $\{\delta_i\}_{i \in \Lambda}$ that satisfies precisely the defining relations (a)–(c) in Theorem \cite{[22]}(2). \hfill $\Box$

The PBW Theorem holds for the Poisson enveloping algebra. By Proposition \cite{[6,1]} \cite{[2]}, the algebra $F_{ab}$ admits a filtration $\{F_{ab, \leq i} = \bigoplus_{|\alpha| \leq i} \mathcal{P}^{\alpha} = \bigoplus_{|\alpha| \leq i} \mathcal{P}^\alpha\}_{i \geq 0}$ by the total degree of the elements $\{\delta_j\}_{j \in \Lambda}$. Its image under the epimorphism $F_{ab} \to \mathcal{U}(\mathcal{P})$ is the filtration $\{\mathcal{U}(\mathcal{P})_{\leq i} = \mathcal{U}(\mathcal{P})\}_{i \geq 0}$ where $\mathcal{U}(\mathcal{P})_{\leq i}$ is the image of $F_{ab, \leq i}$. This is the filtration on the algebra $\mathcal{U}(\mathcal{P})$ by the total degree of the elements $\{\delta_j\}_{j \in \Lambda}$. The associated graded algebra
\begin{equation}
\text{gr} F_{ab} = \bigoplus_{i \geq 0} F_{ab, \leq i} / F_{ab, \leq i-1} \simeq \mathcal{P}[\delta_j]_{j \in \Lambda}
\end{equation}
is a polynomial algebra in the variables $\{\delta_j\}_{j \in \Lambda}$ with coefficients in the algebra $\mathcal{P}$. Theorem \cite{[8,5]} shows that the PBW Theorem holds for the Poisson enveloping algebra.

Theorem 3.5  Let $\mathcal{P}$ be a Poisson algebra and $\text{gr} \mathcal{U}(\mathcal{P}) = \bigoplus_{i \geq 0} \mathcal{U}(\mathcal{P})_{\leq i} / \mathcal{U}(\mathcal{P})_{\leq i-1}$ be the associated graded algebra. Then the algebra $\text{gr} \mathcal{U}(\mathcal{P})$ is isomorphic to the symmetric algebra $\text{Sym}_\mathcal{P}(\Omega)$ of the module of Kähler differentials $\Omega = \Omega_\mathcal{P}$ of the algebra $\mathcal{P}$.

Proof. By \cite{[17]},
\[
\mathcal{U}(\mathcal{P})_{\leq i} / \mathcal{U}(\mathcal{P})_{\leq i-1} = \begin{cases} \mathcal{P} & \text{if } i = 0, \\ \Omega & \text{if } i = 1. \end{cases}
\]

By \cite{[10]} and \cite{[17]}, for all $i \geq 2$, $\mathcal{U}(\mathcal{P})_{\leq i} / \mathcal{U}(\mathcal{P})_{\leq i-1} = F_{ab, i} / M_0 F_{ab, i-1}$. Therefore, by \cite{[13]},
\[
\text{gr} \mathcal{U}(\mathcal{P}) = \mathcal{P}[\delta_j]_{j \in \Lambda} / (M_0) \simeq \text{Sym}_\mathcal{P}(\bigoplus_{j \in \Lambda} \mathcal{P} \delta_j / M_0) \simeq \text{Sym}_\mathcal{P}(\Omega). \hfill \Box
\]

Corollary 3.6  Let $\mathcal{P}$ be a Poisson algebra. Then the algebras $\mathcal{U}(\mathcal{P}) \simeq \text{gr} \mathcal{U}(\mathcal{P}) \simeq \text{Sym}_\mathcal{P}(\Omega)$ are isomorphic as $\mathcal{P}$-filtered left/right $\mathcal{P}$-modules. Furthermore, for all $i \geq 0$ the left (resp., right) $\mathcal{P}$-modules $\mathcal{U}(\mathcal{P})_{\leq i}$ and $\bigoplus_{j=0}^i \mathcal{U}(\mathcal{P})_{\leq j} / \mathcal{U}(\mathcal{P})_{\leq j-1}$ are isomorphic.
The canonical Poisson structure on \( \mathcal{U}(\mathcal{P}) \). Given an \( N \)-filtered algebra \( R = \bigcup_{i \geq 0} R_i \) such that the associated graded algebra \( \text{gr} \, R \) is a commutative algebra, then the algebra \( \text{gr} \, R \) is a Poisson algebra where the Poisson structure is given by the rule: For all elements \( \tau_i = r_i + R_{i-1} \in R_i/R_{i-1} \) and \( \tau_j = r_j + R_{j-1} \in R_j/R_{j-1} \),

\[
\{\tau_i, \tau_j\} := [r_i, r_j] + R_{i+j-2}.
\]

Let \( \mathcal{P} \) be a Poisson algebra. Then the algebra \( \text{gr} \, \mathcal{U}(\mathcal{P}) \simeq \text{Sym}_\mathcal{P}(\Omega_\mathcal{P}) \) is a Poisson algebra where for all elements \( a, b \in \mathcal{P} \), \( \{a, b\} = 0 \), \( \{da, db\} = d\{a, b\} \), and \( \{da, db\} = r \{a, b\} \).

By Proposition 3.9 the algebra \( \mathcal{P} \) is a trivial Poisson subalgebra of \( \text{gr} \, \mathcal{U}(\mathcal{P}) \) (i.e. \( \{\mathcal{P}, \mathcal{P}\} = 0 \)) but the Poisson structure on \( \mathcal{P} \) can be recovered from the Poisson structure on \( \text{gr} \, \mathcal{U}(\mathcal{P}) \) by the rule: For all elements \( a, b \in \mathcal{P} \), \( \{a, b\} = \{da, db\} \).

Proposition 3.10 Let \( \mathcal{P} \) and \( \mathcal{P}' \) be Poisson algebras. The following statements are equivalent:

1. The Poisson algebras \( \mathcal{P} \) and \( \mathcal{P}' \) are isomorphic.
2. The \( N \)-filtered algebras \( \mathcal{U}(\mathcal{P}) \) and \( \mathcal{U}(\mathcal{P}') \) are isomorphic as \( N \)-filtered algebras (where \( \mathcal{U}(\mathcal{P}) = \bigcup_{i \geq 0} \mathcal{U}(\mathcal{P})_{\leq i} \) and \( \mathcal{U}(\mathcal{P}') = \bigcup_{i \geq 0} \mathcal{U}(\mathcal{P}')_{\leq i} \)).
3. The \( N \)-graded Poisson algebras \( \text{gr} \, \mathcal{U}(\mathcal{P}) \simeq \text{Sym}_\mathcal{P}(\Omega_\mathcal{P}) \) and \( \text{gr} \, \mathcal{U}(\mathcal{P}') \simeq \text{Sym}_\mathcal{P}(\Omega_\mathcal{P}') \) are isomorphic as \( N \)-graded algebras.
Proposition 3.11 Let $D$ and $U$ be Poisson algebras. We aim to show that the PEA of a polynomial Poisson algebra is a derivation algebra with PBW basis (Theorem 3.13.1).

Proof. 1. By Theorem 3.5, the algebra $gr\mathcal{H}(\mathcal{P}) \simeq Sym_\mathcal{P}(\Omega_P)$ is a positively graded algebra, and statement 1 follows.

2. (⇒) The algebra $gr\mathcal{H}(\mathcal{P}) \simeq Sym_\mathcal{P}(\Omega_P)$ is a commutative algebra. If it finitely generated then necessarily the algebra $\mathcal{P} \simeq Sym_\mathcal{P}(\Omega_P)/(\Omega_P)$ is also finitely generated, and the algebra $gr\mathcal{H}(\mathcal{P})$ is also Noetherian. Then, by statement 1, the $\mathcal{P}$-module $\Omega_P$ is finitely generated.

(⇐) The implication follows from $gr\mathcal{H}(\mathcal{P}) \simeq Sym_\mathcal{P}(\Omega_P)$. □

The derivation algebras and the polynomial Poisson algebras.

Definition. A polynomial Poisson algebra is a Poisson algebra the associative algebra of it is a polynomial algebra.

We aim to show that the PEA of a polynomial Poisson algebra is a derivation algebra with PBW basis (Theorem 3.13.1).

Definition. Let $D$ be a $K$-algebra, $\delta_1, \ldots, \delta_n$ be $K$-derivations of $D$ and $C = \{c_{ij}^k \in D \mid i, j, k = 1, \ldots, n; i \neq j\}$. A derivation algebra $A = D[t; \delta, C]$ of rank $n$, where $t = (t_1, \ldots, t_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$, is a $K$-algebra which is generated by the algebra $D$ and the elements $t_1, \ldots, t_n$ that satisfy the defining relations: For all $i, j = 1, \ldots, n$ and $i \neq j$,

$$[t_i, d] = \delta_i(d) \quad \text{and} \quad [t_i, t_j] = \sum_{k=1}^n c_{ij}^k t_k. \quad (19)$$

Using (19), the (Jacobi) identity

$$[t_i, [t_j, t_k]] = [[t_i, t_j], t_k] + [t_j, [t_i, t_k]]$$

can be rewritten as follows (we adopt the convention that the summation is assumed over repeated lower and upper indices, i.e. $c_{ij}^k \delta_k$ means $\sum_k c_{ij}^k \delta_k$):

$$\delta_i(c_{jk}^\lambda t_\lambda + c_{jk}^\mu t_\mu) = -\delta_k(c_{ij}^\lambda) t_\lambda + c_{ij}^\lambda c_{jk}^\mu t_\mu + \delta_j(c_{ij}^\lambda) t_\lambda + c_{ik}^\lambda c_{jk}^\mu t_\mu. \quad (20)$$

We say that the derivation algebra $A$ has PBW basis (over $D$) if $A = \bigoplus_{\alpha \in \mathbb{N}^n} Dt^\alpha$ is a free left $D$-module where $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Clearly, if the algebra $A$ has PBW basis then the order of multiples in $t^\alpha$ can be arbitrary and $A = \bigoplus_{\alpha \in \mathbb{N}^n} t^\alpha D$ is also a free right $D$-module.

When the algebra $D$ is a commutative algebra, Theorem 3.12 a criterion for the derivation algebra $A$ to have PBW basis.

Theorem 3.12 Let $D$ be a commutative algebra, $A = D[t; \delta, C]$ be a derivation algebra of rank $n$, $G := \sum_{i=1}^n D t_i$ and $G' := D + G'$. Then the following statements are equivalent:

1. The algebra $A$ has PBW basis.
2. The direct sum of free left $D$-modules of rank 1, $G' = D \oplus \bigoplus_{i=1}^{n} D_{t_{i}}$, is a Lie algebra (w.r.t. $[\cdot, \cdot]$). In particular, $D$ is an abelian Lie ideal of the Lie algebra $G'$ and $[\delta_{i}, \delta_{j}] = \sum_{k} c_{ij}^{k} \delta_{k}$ for all $i$ and $j$.

3. The direct sum of free left $D$-modules of rank 1, $G = \bigoplus_{i=1}^{n} D_{t_{i}}$, is a Lie algebra (w.r.t. $[\cdot, \cdot]$), $G' = D \oplus G$ and the map $G \to \text{Der}_{K}(D)$, $dt_{i} \mapsto [dt_{i}, \cdot]$ is a Lie algebra homomorphism.

If one of the equivalent conditions holds then the algebra $A$ is isomorphic to the factor algebra $U(G')/I$ of the universal enveloping algebra $U(G')$ of the Lie algebra $G'$ modulo the ideal $I$ which is generated by the elements $\{dt_{i} - d \cdot t_{i} | i = 1, \ldots, n; d \in D\}$ (where $dt_{i} \in D_{t_{i}} \subseteq G'$ and $d \cdot t_{i}$ is a product of two elements in $U(G')$).

**Proof.** ($1 \Rightarrow 2$) By statement 1 and the fact that $D$ is a commutative algebra, $G' = D \oplus \bigoplus_{i=1}^{n} D_{t_{i}}$ is a Lie subalgebra of the Lie algebra $(A, [\cdot, \cdot])$ since for all $d, d' \in D$ and $i, j = 1, \ldots, n$:

$$[dt_{i}, d'] = d \delta_{i}(d') \quad \text{and} \quad [dt_{i}, d' t_{j}] = d \delta_{i}(d') t_{j} - d' \delta_{j}(d) t_{i} + d t_{j} \sum_{k=1}^{n} c_{ij}^{k} t_{k}.$$

($2 \Leftrightarrow 3$) Since the algebra $D$ is commutative statements 2 and 3 are equivalent; and $G' = D \times G$ is a semi-direct product of the abelian Lie algebra $D$ and the Lie algebra $G$ that acts on $D$ by the Lie algebra homomorphism $G \to \text{Der}_{K}(D)$, $dt_{i} \mapsto [dt_{i}, \cdot]$.

($2 \Rightarrow 1$) Let $A' := U(G')/I$. Then there is a natural/tautological algebra epimorphism $A' \to A$ (since the relations $I$ hold in the algebra $A$). The epimorphism is an isomorphism since the algebra $A'$ has the same generators and defining relations as the algebra $A$.

Using the ordering $d < t_{i} < t_{j}$ for all $d \in D$ and $i < j$, in order to finish the proof it suffices to show that all the ambiguities can be resolved (and hence the Diamond Lemma guarantees the result). In view of (19), there are two sorts of ambiguities: $t_{j} t_{i} d$ where $i < j$ and $d \in D$; and $t_{k} t_{j} t_{i}$ where $i < j < k$.

- For all elements $d \in D$ and $i < j$,
  $$t_{j}(t_{i}d) = t_{j}(dt_{i} + \delta_{i}(d)) = dt_{i} t_{j} + \delta_{j}(d)t_{i} + \delta_{i}(d)t_{j} + \delta_{i}\delta_{i}(d)$$
  $$= dt_{i} t_{j} + d c_{ij}^{k} t_{k} + \delta_{j}(d)t_{i} + \delta_{i}(d)t_{j} + \delta_{i}\delta_{i}(d),$$
  $$t_{j}(t_{i})d = (t_{i}c_{ij}^{k} t_{k})d = t_{i}(dt_{j} + \delta_{j}(d)) + c_{ij}^{k} dt_{k} + c_{ij}^{k} \delta_{i}(d)$$
  $$= dt_{i} t_{j} + \delta_{i}(d)t_{j} + \delta_{j}(d)t_{i} + \delta_{i}\delta_{i}(d) + c_{ij}^{k} dt_{k} + c_{ij}^{k} \delta_{i}(d).$$

By comparing the quadratic, linear and free terms (w.r.t. the variables $t_{i}$) of both equalities above, we obtained identities (by using the equality $[\delta_{i}, \delta_{j}] = c_{ij}^{k} \delta_{k}$ and the fact that the algebra $D$ is commutative).

- For all $i < j < k$, ...
Kähler differentials of the polynomial algebra

\begin{align*}
t_k(t_j t_i) &= t_{kt}t_j + t_k c_{ij}^k t_i t\lambda = t_{jkt}t_j + c_{ij}^k t j t\mu t_j + c_{ij}^{\lambda} t_k t\lambda + \delta_k(c_{ij}^{\lambda})t\lambda \\
&= t_i t_k t_j + c_{ij}^k t i t\nu + \delta_i(c_{ij}^k) t\nu + \sum c_{ij}^\kappa t\kappa t_j t\mu + \sum c_{ij}^\kappa t\kappa c_{ij}^{\mu} t\lambda + \delta_k(c_{ij}^{\lambda})t\lambda, \\
(t_k t_j) t_i &= t_j t_k t_i + c_{ij}^k t_i t\lambda \\
&= t_j t_k t_i + t_j c_{k}\lambda t_i \\
&= t_j t_k t_i + c_{ij}^\kappa t_i t\lambda t_k + c_{ij}^\mu t\mu t_i + \delta_j(c_{ij}^{\mu}) t\mu + c_{ij}^\lambda t\lambda t_i \\
&= t_i t_j t_k + \sum c_{ij}^\kappa t\kappa t_i t\lambda + \sum c_{ij}^\mu t\mu t_i t\lambda + \sum c_{ij}^\kappa c_{ij}^{\mu} t\lambda t\omega + \\
&+ \sum c_{ij}^\mu t\mu t\lambda t\kappa + \sum c_{ij}^\mu t\mu t\lambda t\kappa + \sum c_{ij}^\mu c_{ij}^{\lambda} t\lambda t\gamma, \\
&+ \sum c_{ij}^\mu t\mu t\lambda t\kappa + \sum c_{ij}^\mu c_{ij}^{\lambda} t\lambda t\gamma.
\end{align*}

By comparing the cubic, quadratic and linear terms (w.r.t. the variables $t_i$) of both equalities above we obtain identities. The equality of quadratic terms is straightforward when we use the equality

\[ c_{ij}^\kappa t_i t\nu = \sum_{i\leq\nu} c_{ij}^\kappa t_i t\nu + \sum_{i>\nu} c_{ij}^\kappa t_i t\nu + \sum_{i>\nu} c_{ij}^\kappa c_{ij}^{\gamma} t_i t\gamma. \]

The equality of the linear terms follows from (20). \hfill \Box

**Theorem 3.13** Let $P = P_n = K[x_1, \ldots, x_n]$ be a Poisson polynomial algebra and $C_{P_n} = (c_{ij})$ where $c_{ij} := \{x_i, x_j\} \in P_n$. Then

1. The Poisson enveloping algebra $U(P_n)$ of the Poisson algebra $P_n$ is isomorphic to the derivation algebra $P_n[t; \delta; C = \{c_{ij}^k\}]$ of rank $n$ that has PBW basis over $P_n$ where $t = (t_1, \ldots, t_n)$, $\delta = (\delta_1 := \{x_1, \} \ldots, \delta_n := \{x_n, \})$ and $c_{ij}^k := \frac{\partial c_{ij}}{\partial x_k}$. In particular, $U(P_n) = \bigoplus_{n \in \mathbb{N}} P_n t^n$ is a free left and right $P_n$-module.

2. As an abstract algebra, the algebra $U(P_n)$ is generated over $K$ by the elements $x_1, \ldots, x_n, t_1, \ldots, t_n$ subject the defining relations:

\[ [x_i, x_j] = 0, \quad [t_i, x_j] = c_{ij} \quad \text{and} \quad [t_i, t_j] = \sum_{i=1}^n \frac{\partial c_{ij}}{\partial x_k} t_k. \quad (21) \]

3. $G' = P_n \oplus \bigoplus_{i=1}^n P_n t_i$ is a Lie subalgebra of $(U(P_n), [\cdot, \cdot])$ and $G' \simeq P_n \rtimes G$ is a semidirect product of the Lie algebras $P_n$ and $G = \bigoplus_{i=1}^n P_n t_i$.

4. The algebra $U(P_n)$ is isomorphic to the factor algebra $U(G')/I$ of the universal enveloping algebra $U(G')$ of the Lie algebra $G'$ modulo the ideal $I$ which is generated by the elements \{ $p t_i - p \cdot t_i$ $|$ $i = 1, \ldots, n; p \in P_n$ \} (where $p t_i \in P_n t_i \subseteq G'$, and $p \cdot t_i$ is a product of two elements in $U(G')$).

**Proof.** 2. Statement 2 follows from Theorem 2.2 (2).

1. Statement 1 follows from Theorem 3.4 (4), Theorem 3.5 and the fact that the module of Kähler differentials of the polynomial algebra $P_n$, $\Omega_{P_n} = \bigoplus_{i=1}^n P_n t_i$, is a free $P_n$-module where $t_i = dx_i$.

3 and 4. Statements 3 and 4 follow from statement 1 and Theorem 3.12. \hfill \Box
Theorem 3.14 Suppose that \( P = S^{-1}K[x_i]_{i \in \Lambda} / (f_s)_{s \in \Gamma} \) where \( S \) is a multiplicative subset of the polynomial algebra \( K[x_i]_{i \in \Lambda} \) (\( \Lambda \) and \( \Gamma \) are index sets) and the algebra \( P \) is a Poisson algebra such that the module of \( \partial x_i \)-differentials \( \Omega P = \bigoplus_{i=1}^n Pdx_i \) is a free left \( P \)-module. Then

1. The Poisson enveloping algebra \( U(P) \) of the Poisson algebra \( P \) is isomorphic to the derivation algebra \( P[t; \delta, C = \{ c^k_{ij} \}] \) of rank \( n \) that has PBW basis over \( P \) where \( t = (dy_1, \ldots, dy_n) \), \( \delta = (\delta_1 := \{ y_1 \}, \ldots, \delta_n := \{ y_n \}) \) and \( dy_i, y_j \} = \sum_{k=1}^n c^k_{ij}dy_k \). In particular, \( U(P) = \bigoplus_{\alpha \in \mathbb{N}} P t^n = \bigoplus_{\alpha \in \mathbb{N}} t^nP \) is a free left and right \( P \)-module.

2. As an abstract algebra, the algebra \( U(P) \) is generated over \( K \) by the algebra \( P \) and the elements \( t_1, \ldots, t_n \) subject the defining relations:

\[
[t_i, x_j] = \{ y_i, x_j \} \quad \text{and} \quad [t_i, t_j] = \sum_{i=1}^n c^k_{ij}t_k. \tag{22}
\]

3. \( G' = P \oplus \bigoplus_{i=1}^n P t_i \) is a Lie subalgebra of \( (U(P), [\cdot, \cdot]) \) and \( G' \cong P \rtimes G \) is a semidirect product of the Lie algebras \( P \) and \( G = \bigoplus_{i=1}^n P t_i \).

4. The algebra \( U(P) \) is isomorphic to the factor algebra \( U(G')/I \) of the universal enveloping algebra \( U(G') \) modulo the ideal \( I \) which is generated by the elements \( \{ pt_i - p \cdot t_i | i = 1, \ldots, n; p \in P \} \) (where \( pt_i \in P t_i \subseteq G' \), and \( p \cdot t_i \) is a product of two elements in \( U(G') \)).

Proof. 1. Statement 1 follows from Theorem 3.12 (4), Theorem 3.13 and the fact that the module of \( \partial x_i \)-differentials for the algebra \( P_n, \Omega_{P_n} = \bigoplus_{i=1}^n P_n t_i \) is a free \( P_n \)-module where \( t_i = dx_i \).

2. Statement 2 follows from statement 1 and Theorem 2.2 (2).

3 and 4. Statements 3 and 4 follow from statement 1 and Theorem 6.12. \( \square \)

Remark. Theorem 3.12, Theorem 3.13 and Theorem 3.14 are true (with the same proofs) for \( n \) of arbitrary cardinality (not necessarily finite).

4 Criterion for the algebra \( U(A) \) to be a domain where a Poisson algebra \( A \) is a domain of essentially finite type

The aim of this section is to prove Theorem 1.13 (3) which is a criterion for the algebra \( U(A) \) to be a domain where a Poisson algebra \( A \) is a domain of essentially finite type. As a corollary we obtain Theorem 3.13 which states that the algebra \( U(A) \) is a domain if, in addition, the algebra \( A \) is regular. Using a result of Huneke (Theorem 4.6) we obtain another criterion for the algebra \( U(A) \) to be a domain (Theorem 1.14) provided the algebra \( A \) satisfies Serre’s Condition \( S_n \).

A localization of an affine commutative algebra is called an algebra of essentially finite type. In this section the following notation will remain fixed if it is not stated otherwise: \( P_n = K[x_1, \ldots, x_n] \) is a polynomial algebra over perfect field \( K \), \( \partial_1 := \frac{\partial}{\partial x_1} \), \( \partial_n := \frac{\partial}{\partial x_n} \in \text{Der}_K(P_n) \), \( I = (f_1, \ldots, f_m) \) is a prime but not a maximal ideal of \( P_n \), \( A = S^{-1}(P_n/I) \) is a domain of essentially finite type and \( Q = Q(A) \) is its field of fractions, \( r = r(\frac{\partial}{\partial x_j}) \) is the rank (over \( Q \)) of the Jacobian matrix \( (\frac{\partial f_i}{\partial x_j}) \) of \( A \), \( a_r \) is the Jacobian ideal of the algebra \( A \) which is (by definition) generated by all the \( r \times r \) minors of the Jacobian matrix \( A \) (the algebra \( A \) is regular iff \( a_0 = A \), it is the Jacobian criterion of regularity, [5] Corollary 16.20), \( \Omega_A \) is the module of \( \partial x_i \)-differentials for the algebra \( A \).

For \( i = (i_1, \ldots, i_r) \) such that \( 1 \leq i_1 < \cdots < i_r \leq m \) and \( j = (j_1, \ldots, j_r) \) such that \( 1 \leq j_1 < \cdots < j_r \leq n \), \( \Delta(i, j) \) denotes the corresponding minor of the Jacobian matrix of \( A \), and the \( i \) (resp., \( j \)) is called non-singular if \( \Delta(i, j') \neq 0 \) (resp., \( \Delta(i', j) \neq 0 \)) for some \( j' \) (resp., \( i' \)). We denote by \( \mathbb{I}_r \) (resp., \( \mathbb{J}_r \)) the set of all the non-singular \( r \)-tuples \( i \) (resp., \( j \)).
Since $r$ is the rank of the Jacobian matrix of $\mathcal{A}$, it is easy to show that $\Delta(i, j) \neq 0$ iff $i \in I$, and $j \in \mathbb{J}_r$. We denote by $\mathbb{J}_{r+1}$ the set of all $(r+1)$-tuples $j = (j_1, \ldots, j_{r+1})$ such that $1 \leq j_1 < \cdots < j_{r+1} \leq n$ and when deleting some element, say $j_v$, we have a non-singular $r$-tuple $(j_1, \ldots, j_v, \ldots, j_{r+1}) \in \mathbb{J}_r$ where the hat over a symbol means that the symbol is omitted from the list. The set $\mathbb{J}_{r+1}$ is called the critical set and any element of it is called a critical singular $(r + 1)$-tuple. $\text{Der}_K(\mathcal{A})$ is the $\mathcal{A}$-module of $K$-derivations of the algebra $\mathcal{A}$. The action of a derivation $\delta$ on an element $a$ is denoted by $\delta(a)$.

The next theorem gives a finite set of generators and a finite set of defining relations for the left $\mathcal{A}$-module $\text{Der}_K(\mathcal{A})$ when $\mathcal{A}$ is a regular algebra.

**Theorem 4.1** ([4], Theorem 4.2] if $\text{char}(K) = 0$; [3], Theorem 1.1] if $\text{char}(K) = p > 0$:) Let the algebra $\mathcal{A}$ be a regular domain of essentially finite type over the perfect field $K$. Then the left $\mathcal{A}$-module $\text{Der}_K(\mathcal{A})$ is generated by the derivations $\partial_{i,j}, i \in \mathbb{I}_r, j \in \mathbb{J}_{r+1}$ where

$$
\partial_{i,j} = \partial_{i_1, \ldots, i_r; j_1, \ldots, j_{r+1}} := \det \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x_{i_1}} & \cdots & \frac{\partial f_1}{\partial x_{i_{r+1}}} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_r}{\partial x_{i_1}} & \cdots & \frac{\partial f_r}{\partial x_{i_{r+1}}} \\
\frac{\partial f_j}{\partial x_{i_1}} & \cdots & \frac{\partial f_j}{\partial x_{i_{r+1}}}
\end{array} \right)
$$

that satisfy the following defining relations (as a left $\mathcal{A}$-module):

$$
\Delta(i, j) \partial_{i', j'} = \sum_{l=1}^{s} (-1)^{r+1+l} \Delta(i, j', \ldots, j_{v'}), j_{v'} = \partial_{i, j_v}
$$

for all $i, j' \in \mathbb{I}_r, j = (j_1, \ldots, j_r) \in \mathbb{J}_r$, and $j' = (j'_1, \ldots, j'_{r+1}) \in \mathbb{J}_{r+1}$ where $\{j'_1, \ldots, j'_{r+1}\} = \{j_1, \ldots, j_r\} \setminus \{j_{v}, \ldots, j_{v'}\}$ and $\partial_{i,j} = \frac{\partial}{\partial x_{i,j}}$.

Remark. Suppose that a Poisson algebra $\mathcal{A} = S^{-1}(P_n/I)$ is of essentially finite type. Then, if necessary, we may assume that $\{x_i, x_j\} \in P_n/I$ for all $i, j = 1, \ldots, n$ and the multiplicative set $S$ consist of regular elements of the algebra $P_n/I$. The second statement follows from the fact that $S^{-1}(P_n/I) \simeq S^{-1}(P_n/J)$ where $\text{ass}_{P_n/I}(S) = J/I$ for some ideal $J$ of $P_n/I$ and $S = \{s+I/s \in S\}$. Let $s$ be the product of all the denominators, say $s_{ij}$, in $\{x_i, x_j\} = s_{ij}^{-1} a_{ij}$ where $a_{ij} \in P_n/I$. Then $P_n/I \simeq P_n[x_{n+1}]/(I, sx_{n+1} - 1)$ and we are done since $s_{ij}^{-1} = s^{-1} \cdot \frac{1}{s_{ij}}$.

The Gelfand-Kirillov dimension of the algebra $\mathcal{U}(\mathcal{A})$. Proposition 4.2 gives a lower bound for the Gelfand-Kirillov dimension of the algebras $\mathcal{U}(\mathcal{A}), \text{gr} \mathcal{U}(\mathcal{A})$, and $\text{Sym}_A(\mathcal{A})$. In some important cases the lower bound is the Gelfand-Kirillov of these algebras (Theorem 4.3 (4) and Theorem 4.4).

**Proposition 4.2** Let a Poisson algebra $\mathcal{A} = S^{-1}(P_n/I)$ be a domain of essentially finite type where $I = (f_1, \ldots, f_m)$ is a prime but not maximal ideal of $P_n$ and $r = r(\frac{\partial f}{\partial x_j})$ is the rank of the Jacobian matrix $(\frac{\partial f}{\partial x_j})$ over the field of fractions of the domain $P_n/I$. Then $\text{GK} \mathcal{U}(\mathcal{A}) \geq \text{GK} \text{gr} \mathcal{U}(\mathcal{A}) = \text{GK} \text{Sym}_A(\Omega_{\mathcal{A}}) \geq 2 \text{GK}(\mathcal{A}) = 2(n - r)$.

**Proof.** By [14] Lemma 8.3.20, $\text{GK} \mathcal{U}(\mathcal{A}) \geq \text{GK} \text{gr} \mathcal{U}(\mathcal{A})$. By Theorem 3.5, $\text{gr} \mathcal{U}(\mathcal{A}) \simeq \text{Sym}_A(\Omega_{\mathcal{A}})$, and so $\text{GK} \text{gr} \mathcal{U}(\mathcal{A}) = \text{GK} \text{Sym}_A(\Omega_{\mathcal{A}})$. By Theorem 4.3 (1), $\text{GK} \text{Sym}_A(\Omega_{\mathcal{A}}) \geq \text{GK} (\mathcal{A}_{\Delta(i,j)}) + n - r = \text{GK}(\mathcal{A}) + n - r = 2(n - r) = 2 \text{GK}(\mathcal{A})$. □

**Criterion for the algebra $\mathcal{U}(\mathcal{A})$ to be a domain.** Theorem 4.3 (3) is a criterion for the algebra $\mathcal{U}(\mathcal{A})$ to be a domain where the Poisson algebra $\mathcal{A}$ is a domain of essentially finite type.

**Theorem 4.3** Let a Poisson algebra $\mathcal{A} = S^{-1}(P_n/I)$ be a domain of essentially finite type where $I = (f_1, \ldots, f_m)$ is a prime but not maximal ideal of $P_n$ and $r(\frac{\partial f}{\partial x_j})$ is the rank of the Jacobian matrix $(\frac{\partial f}{\partial x_j})$ over the field of fractions of the domain $P_n/I$. Then
1. For each \(i \in I_r\) and \(j \in J_r\), the algebra \(U(A)_{\Delta(i,j)} \simeq U(A_{\Delta(i,j)}) \simeq \bigoplus_{a \in \mathbb{N}^n} A_{\Delta(i,j)} \delta_{x_{j+1}}^{a_1} \cdots \delta_{x_j}^{a_n}\) is a Noetherian domain with \(GK U(A_{\Delta(i,j)}) = 2GK(A) = 2(n - r)\) where \(J = (j_1, \ldots, j_r)\) and \(\{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\}\).

2. The kernel of the algebra homomorphism

\[
\theta : U(A) \to \prod_{i \in I_r, j \in J_r} U(A)_{\Delta(i,j)}, \quad u \mapsto (\ldots, u_i, \ldots)
\]

is a finitely generated left and right ideal of the algebra \(U(A)\) which is \(S_{\Delta(i,j)}\)-torsion for all \(i \in I_r\) and \(j \in J_r\). In particular, \(\Delta(i,j)^k \ker(\theta) = 0\) and \(\ker(\theta) \Delta(i,j)^l = 0\) for some natural numbers \(k, l \geq 0\). Furthermore, \(a^* \ker(\theta) = 0\) and \(\ker(\theta) a^*_j = 0\) for some natural numbers \(s, t \geq 0\).

3. The following statements are equivalent:

(a) The algebra \(U(A)\) is a domain.
(b) The algebra \(grU(A)\) is a domain.
(c) The algebra \(\text{Sym}_A(\Omega_A)\) is a domain.
(d) The elements \(\{\Delta(i,j) \mid i \in I_r, j \in J_r\}\) are regular in \(U(A)\).
(e) The element \(\Delta(i,j)\) is a regular element of the algebra \(U(A)\) for some \(i \in I_r, j \in J_r\).
(f) \(\text{ann}_{U(A)}(a_r) = \text{r.ann}_{U(A)}(a_r) = 0\).

4. Suppose that the algebra \(U(A)\) is a domain. Then \(GK U(A) = GK grU(A) = GK \text{Sym}_A(\Omega_A) = 2GK(A) = 2(n - r)\).

Proof. 1. Statement 1 follows from Theorem 3.4 since \(\Omega_{\Delta(i,j)} = \bigoplus_{a \in \mathbb{N}^n} A_{\Delta(i,j)} \delta_{x_{j+1}}^{a_1} \cdots \delta_{x_j}^{a_n}\).

2. Since the algebra \(U(A)\) is a Noetherian algebra statement 2 follows from statement 1 and the fact that the sets \(I_r\) and \(J_r\) are finite.

3. \((b \Rightarrow c)\) By Theorem 3.4 \(grU(A) \simeq \text{Sym}_A(\Omega_A)\), and the equivalence follows.

\((f \Rightarrow e)\) The implication follows from the fact that the ideal \(a_r\) is generated by the elements \(\Delta(i,j)\) where \(i \in I_r, j \in J_r\).

\((e \Rightarrow d)\) The implication is obvious (see statement 2).

\((d \Rightarrow b)\) By Corollary 3.6 the left and right \(A\)-modules \(U(A)\) and \(grU(A)\) are isomorphic. Therefore, the algebra \(grU(A)\) is \(S_{\Delta}\)-torsion free where \(\Delta = \Delta(i,j)\). In particular, the algebra \(grU(A)\) is a subalgebra of its localization at the powers of the element \(\Delta\),

\[
grU(A) \subseteq (grU(A))_\Delta \simeq grU(A)_\Delta \simeq grU(A_\Delta) \simeq A_\Delta[\delta_{x_{j+1}}^{a_1}, \ldots, \delta_{x_j}^{a_n}],
\]

a polynomial algebra over \(A_\Delta\) in the variables \(\delta_{x_{j+1}}^{a_1}, \ldots, \delta_{x_j}^{a_n}\) which is obviously a domain, and so is its subalgebra \(grU(A)\).

4. Since the algebra \(U(A)\) is a domain, \(U(A) \subseteq U(A)_{\Delta} \subseteq U(A)_{\Delta} \simeq U(A)_{\Delta}\). Now, by statement 1, \(GK U(A) \leq GK U(A_{\Delta}) = 2GK(A)\). Now, statement 4 follows from the equalities \(GK(A) = GK(A_{\Delta}) = n - r\). □

The algebra \(U(A)\) is a domain when \(A\) is a regular domain of essentially finite type. As a corollary of Theorem 3.3 (3), we obtain Theorem 4.3 that states that the algebra \(U(A)\) is a domain provided the algebra \(A\) is a regular domain of essentially finite type.

Theorem 4.4 Let a Poisson algebra \(A = S^{-1}(P_n/I)\) be a regular domain of essentially finite type where \(I = (f_1, \ldots, f_m)\) is a prime but not maximal ideal of \(P_n\) and \(r = r(\frac{\partial f}{\partial x})\) is the rank of the Jacobian matrix \((\frac{\partial f}{\partial x})\) over the field of fractions of the domain \(P_n/I\). Then the algebra \(U(A)\) is a Noetherian domain with \(GK U(A) = GK grU(A) = GK \text{Sym}_A(\Omega_A) = 2GK(A) = 2(n - r)\).
Proof. By Proposition 4.3(4), the algebra $U(A)$ is Noetherian. The algebra $A$ is regular, and so $a_r = A$. By Proposition 4.3(3), the algebra $U(A)$ is a domain. Now, the theorem follows from Theorem 4.3(4). □

The $A$-torsion submodules of the algebras $U(P)$, $grU(A)$ and $Sym(Ω_A)$. Theorem 4.5 describes the $A$-torsion submodules of the three algebras above.

Theorem 4.5 Let a Poisson algebra $A = S^{-1}(P/I)$ be a domain of essentially finite type over the field $K$ and $V := grU(A) \simeq Sym(Ω_A)$. Then

1. $tor_{A\setminus \{0\}}(U(A)) = tor_{S_{Δ(i,j)}}(U(A)) = l.ann_{U(A)}(a_i^j) = r.ann_{U(A)}(a_i^j) = l.ann_{U(A)}(Δ(i,j)^i) = r.ann_{U(A)}(Δ(i,j)^j)$ for all $i \gg 1$, $i \in I_r$ and $j \in J_r$.

2. $tor_{A\setminus \{0\}}(V) = tor_{S_{Δ(i,j)}}(V) = l.ann_{V}(a_i^j) = r.ann_{V}(a_i^j) = l.ann_{V}(Δ(i,j)^i) = r.ann_{V}(Δ(i,j)^j)$ for all $i \gg 1$, $i \in I_r$ and $j \in J_r$.

Proof. By Proposition 3.10 the left and right $A$-modules $U(A)$ and $V$ are isomorphic. So, statements 1 and 2 are equivalent. So, it suffices to prove, say, statement 1. The algebras $U(P)$, $grU(A)$ and $Sym(Ω_A)$ are Noetherian finitely generated algebras. The multiplicative sets $A_i \setminus \{0\}$ and $S_i := \{Δ^i \mid i \geq 0\}$ are denominator sets of them where $Δ = Δ(i,j)$, $i \in I_r$ and $j \in J_r$. By Theorem 4.3(1), the algebra $U(A_Δ) \simeq U(A)_{Δ}$ is a Noetherian domain. Hence, $tor_{A\setminus \{0\}}(U(A)) = tor_{S_{Δ(i,j)}}(U(A))$ for all $i \in I_r$ and $j \in J_r$. The other equalities in statement 1 follow at once from the facts that the algebra $U(A)$ is Noetherian, the Jacobian ideal $a_i$ is generated by the finite set $\{Δ(i,j) \mid i \in I_r, j \in J_r\}$ and the sets $A_i \setminus \{0\}$ and $S_i$ are (left and right) denominator sets of $U(A)$. □

The Gelfand-Kirillov dimension of the algebras $U(A)$, $grU(A)$ and $Sym_A(Ω_A)$ where $A$ is a domain of essentially finite type. Theorem 4.4.1 shows that the regularity condition in Theorem 4.3 can be dropped but the result about the Gelfand-Kirillov dimension holds.

Proof of Theorem 4.4 It suffices to show that the theorem holds for a finitely generated algebra $A$.

We prove the theorem by induction on the Gelfand-Kirillov dimension of the algebra $A$. If $GK(A) = 0$, i.e. the algebra $A$ is a finite field extension of the perfect field $K$. In particular, the field $A$ is perfect. Hence, $Ω_A = 0$, and the statement is obvious (Theorem 3.1(4)).

Suppose that $N = GK(A) \geq 1$ and the statement is true for all algebras $A'$ with $GK(A') < N$. By Proposition 3.2 it suffices to show that $GK(U(A)) \leq 2N$.

(i) The algebra $U(A)$ is an almost (left and right) centralizing extension of $A$: The statement (i) follows from Theorem 2.2(2).

(ii) The algebra $U(A)$ is a somewhat commutative algebra: The statement (ii) follows from the statement (i) and [14] Proposition 8.6.9.

(iii) The Gelfand-Kirillov dimension over $U(A)$ is an exact function: The statement (iii) follows from the statement (ii) and [14] Corollary 8.4.9(i).

Fix $Δ = Δ(i,j)$. The Poisson algebra $A_Δ$ is a regular finitely generated domain. Let $U'$ and $J$ be the image and the kernel of the algebra $U(A)$ under the localization map $U(A) \to U(A_Δ) \simeq U(A_Δ, u \mapsto \frac{u}{Δ})$.

(iv) $GK(U') \leq 2N$: By Theorem 4.3 $GK(U') \leq GK(U(A_Δ)) = 2GK(A_Δ) = 2N$.

(v) $GK(J) < 2N$: The algebra $U(A)$ is a finitely generated Noetherian algebra. Let $\min(a_r)$ be the set of minimal primes over the Jacobian ideal $a_r$ of the algebra $A$. We assume that $a_r = A$, otherwise the result follows from Theorem 4.4. By Theorem 4.5 and the statement (iii), there is a natural number $i \geq 1$ such that

$$GK(J) \leq GK(U(A)/(a_i^j)) = \max\{GK(U(A)/(p)) \mid p \in \min(a_r)\} = \max\{2GK(U(A)/(p)) \mid p \in \min(a_r)\} < 2N.$$
(vi) $\text{GK}(U(A)) \leq 2N$: By the statement (iii),

$$\text{GK}(U(A)) = \max\{\text{GK}(U'), \text{GK}(J)\},$$

and the statement (vi) follows from the statements (iv) and (v). □

Let $B$ be a matrix with coefficients in a commutative ring $R$. By $I_i(B)$ we denote the ideal generated by the $s \times s$ minors of $B$. For an $R$-module $M$, let $v(M, R)$ be the least number of its generators. Recall that a commutative ring $R$ satisfies Serre’s property $S_k$ if

$$\text{depth}(R_p) \geq \min\{\dim(R_p), k\}$$

for all prime ideals $p$ of $R$. If $I$ is an ideal of $R$, we denote by $\text{grade}(I)$ the length of a maximal $R$-sequence in $I$.

**Theorem 4.6** (Theorem 1.1) Let $R$ be a universally catenarian Noetherian ring satisfying Serre’s condition $S_m$ and let $M$ be an $R$-module having a finite free resolution,

$$0 \to R^n \xrightarrow{B} R^m \to M \to 0, \quad B = (a_{ij}).$$

The following statements are equivalent:

1. $\text{Sym}_R(M)$ is a domain.
2. $\text{grade}(I_i(B)) \geq m + 2 - t$ for $1 \leq t \leq m$.
3. $v(M_p, R_p) \leq n - m + \text{grade}(p) - 1$ for all nonzero primes $p$ of $R$.

If any (and hence all) of the above conditions hold then $\text{Sym}_R(M)$ is a complete intersection in $R[T_1, \ldots, T_n]$. In particular, if $R$ is Cohen-Macaulay (resp., Gorenstein) then so is $\text{Sym}_R(M)$.

**Proof of Theorem 4.7**  (1 $\iff$ 2 $\iff$ 3) Theorem 4.3(3).

(3 $\iff$ 4 $\iff$ 5) Theorem 4.6 □

5  **Criterion for $\text{Der}_K(A) = \mathcal{A} \mathcal{H}_A$ where $A$ is a regular domain of essentially finite type**

In this section, $K$ is a field of characteristic zero (if it is not stated otherwise) and we keep the assumptions and the notations of Section 4. The aim of this section is to give a criterion for $\text{Der}_K(A) = \mathcal{A} \mathcal{H}_A$ where the Poisson algebra $A$ is a regular domain of essentially finite type (Theorem 4.5). When $A = P_n$ is a polynomial algebra in $n$ variables the criterion looks particularly nice (Corollary 5.3). Examples are considered. Lemma 5.4 and Corollary 5.5 are regularity and symplecticity criteria for certain generalized Weyl Poisson algebras.

**Proposition 5.1** Let a Poisson algebra $A$ be a regular domain of essentially finite type, $Q = Q(A)$ be its field of fractions, $C_A = (c_{ij}) \in M_n(A)$ where $c_{ij} = \{x_i, x_j\}$ and $r(C_A)$ is the rank of the matrix $C_A \in M_n(Q)$, $r = r\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)$ be the rank of the Jacobian matrix $\frac{\partial^2}{\partial x_i \partial x_j}$ of $A$. Then

1. $r(C_A) = \dim_Q(\mathcal{A} \mathcal{H}_A)$.
2. For each $j = (j_1, \ldots, j_r) \in J_r$, $\dim_Q(\mathcal{A} \mathcal{H}_A) = r(C_{A,j})$ where $C_{A,j}$ is an $n \times (n-r)$ matrix which is obtained from the matrix $C_A$ by deleting columns with indices $j_1, \ldots, j_r$, i.e. $C_{A,j} = (c_{ij})$ where $i = 1, \ldots, n$ and $j \in \{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\}\setminus\{j_1, \ldots, j_r\}$. In particular, $r(C_A) \leq \dim_Q(\text{Der}_K(Q)) = n - r$. 

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Corollary 3.5], the derivations $\partial_i$ of the algebra $A$ are uniquely determined by its action on the elements $x_{j+1}, \ldots, x_n$. For each $i = 1, \ldots, n$, $\text{pad}_x = \{x_i, \ldots, x_n\}$, the derivation $\partial_i$ is uniquely determined by the elements $\{x_i, x_{j+1}, \ldots, x_n\}$ of the algebra $A$, and statements 1 and 2 follow. □

The ideals $c_{A,s}$, $s = 1, \ldots, r(C_A)$, and the Poisson ideal $c_{A,1}$. For each integer $s = 1, \ldots, r(C_A)$, let $c_{A,s}$ be the ideal of the algebra $A$ generated by all the $s \times s$ minors of the matrix $C_A$. By [1], the ideals $c_{A,s}$ do not depend on the set of essential generators $x_1, \ldots, x_n$ of the algebra $A$. Clearly,

$$c_{A,1} \supseteq c_{A,2} \supseteq \cdots \supseteq c_{A,r(C_A)}.$$

Lemma 5.2
1. The ideal $c_{A,1}$ is a Poisson ideal of $A$ which is $P\text{Der}_K(A)$-invariant.
2. For all $s = 1, \ldots, r(C_A)$ and $\delta \in P\text{Der}_K(A)$, $\delta(c_{A,i}) \subseteq c_{A,1}c_{A,i-1} \subseteq c_{A,i-1}$. In particular, $\{A, c_{A,i}\} \subseteq c_{A,i-1}$.

Proof. 1. For all $a \in A$ and $i = 1, \ldots, n$, $\{x_i, a\} = \sum_{j=1}^{n} \frac{\partial a}{\partial x_j} \{x_i, x_j\}$, and statement 1 follows.
2. Statement 2 is obvious. □

Criterion for $\text{Der}_K(A) = \mathcal{H}_A$. For each Poisson algebra $P$, $\text{Der}_K(P) \supseteq \mathcal{P}H_P$. A Poisson algebra $P$ which is a regular affine domain is called a symplectic algebra if $\text{Der}_K(P) = \mathcal{P}H_P$. Theorem 1.8 is a criterion for $\text{Der}_K(A) = \mathcal{H}_A$ where $A$ is a regular domain of essentially finite type.

Proof of Theorem 1.8
(1 $\Rightarrow$ 3) Given elements $i \in I_r$ and $j = (j_1, \ldots, j_r) \in \mathcal{J}_r$. Let $\Delta = \Delta(i, j)$ and $\{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\}$. We assume that $j_{r+1} < \cdots < j_n$. By [1] Corollary 3.5], the derivations

$$\Delta^{-1}\partial_{i,j,j+1}, \ldots, \Delta^{-1}\partial_{i,j,j+n}$$

are the partial derivatives $\partial_{j_{r+1}} = \frac{\partial}{\partial x_{j_{r+1}}}, \ldots, \partial_{j_n} = \frac{\partial}{\partial x_{j_n}}$ of the localization $A_{\Delta}$ of the algebra $A$ at the powers of the element $\Delta$, respectively. Notice that $\partial_{i,j,j+1}, \ldots, \partial_{i,j,j+n} \in \text{Der}_K(A)$ and $\text{Der}_K(A) = \mathcal{H}_A$ (by the assumption). Therefore,

$$\left( \begin{array}{c} \partial_{i,j,j+1} \\ \vdots \\ \partial_{i,j,j+n} \end{array} \right) = \Lambda \left( \begin{array}{c} \text{pad}_{x_1} \\ \vdots \\ \text{pad}_{x_n} \end{array} \right) = \mathcal{C}_A \left( \begin{array}{c} \partial_1 \\ \vdots \\ \partial_n \end{array} \right)$$

for some matrix $\Lambda \in M_{n-r,n}(A)$ where $\partial_i = \frac{\partial}{\partial x_i}$. By evaluating this equality of derivations at the elements $x_{j_{r+1}}, \ldots, x_{j_n}$ we obtain the equality of $(n-r) \times (n-r)$ matrices with coefficients in the algebra $A$,

$$\Delta E_{n-r} = \mathcal{C}_{A,j}$$

where $E_{n-r}$ is the $(n-r) \times (n-r)$ identity matrix and $\bar{j} = (x_{j_{r+1}}, \ldots, x_{j_n})$. Let $e_1 = (1, 0, \ldots, 0)$ and $e_{n-r} = (0, \ldots, 0, 1)$ be the standard basis for the free $A$-module $A^{n-r} = \bigoplus_{k=1}^{n-r} A e_k$. Let $R_1, \ldots, R_{n-r} \in A^{n-r}$ be the rows of the matrix $\Delta E_{n-r} = \mathcal{C}_{A,j}$. In the $A$-module $\wedge^{n-r} A^{n-r} = A e_1 \wedge \cdots \wedge e_{n-r}$ (the wedge product), we have the inclusion

$$\Delta^{n-r} e_1 \wedge \cdots \wedge e_{n-r} = R_1 \wedge \cdots \wedge R_{n-r} \subseteq m_j e_1 \wedge \cdots \wedge e_{n-r},$$

and so $\Delta^{n-r} \in m_j$.

(3 $\Rightarrow$ 4) The implication follows from the inclusions $m_j \subseteq c_{A,n-r}$ for all $j \in \mathcal{J}_r$.

(4 $\Rightarrow$ 2) By statement 4, the ideal $c_{A,n-r}$ contains all the elements $\Delta(i, j)^k$ where $i \in I_r$, $j \in \mathcal{J}_r$, and $k = k(i, j)$. The elements $\Delta(i, j)$ are generators of the Jacobian ideal $a_r$ of the algebra $A$. So,

$$a_r^s \subseteq c_{A,n-r}$$

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for some $s \geq 1$. The algebra $A$ is a regular algebra, hence $a_r = A$. Now, $\epsilon_{A,n-r} = A$, and so $d \geq n - r$. By Proposition \ref{prop:regularity}(2), $d \leq n - r$, and so $d = n - r$.

(2 $\Rightarrow$ 1) Let $M_{n-r}$ be the set of nonzero $(n-r) \times (n-r)$ minors of the matrix $C_A$. Then $M_{n-r} \neq \emptyset$ since $d = n - r$ and $\epsilon_{A,d} = A$.

(i) The algebraic extension $A \to \prod_{\mu \in M_{n-r}} A_{\mu}$ is faithfully flat: Since $A = \epsilon_{A,n-r} = (\mu)_{\mu \in M_{n-r}}$, the statement (i) follows.

(ii) For all $\mu \in M_{n-r}$, $\text{Der}_K(A_{\mu}) = A_{\mu} \mathcal{H}_{A_{\mu}}$ (where $A_{\mu}$ is the localization of the algebra $A$ at the powers of the element $\mu$): Fix a minor $\mu \in M_{n-r}$, it is determed by the rows with indices $i_{t+1}, \ldots, i_n$ and columns with indices $j_{t+1}, \ldots, j_n$. Let $j = (j_1, \ldots, j_r)$ where $\{j_1 < \cdots < j_r\} = \{1, \ldots, n\} \setminus \{j_{t+1}, \ldots, j_n\}$. Then

$$
\begin{pmatrix}
\text{pad}_{i_{t+1}} & \ldots & \text{pad}_{i_n} \\
\vdots & \ddots & \vdots \\
\text{pad}_{j_{t+1}} & \ldots & \text{pad}_{j_n}
\end{pmatrix} = \Lambda 
\begin{pmatrix}
\partial_1 \\
\vdots \\
\partial_n
\end{pmatrix}
$$

for some matrix $\Lambda \in M_{n-r,n}(A)$. Let us consider the $(n-r) \times (n-r)$ matrix $M = (\{x_{i_{t+1}}, x_{i_n}\})$ where $s, t = r + 1, \ldots, n$. In particular, $\mu := \text{det}(M)$ and $M^{-1} = \mu^{-1} \tilde{M}$ where $\tilde{M}$ is the adjoint matrix of the matrix $M$. Then

$$
\begin{pmatrix}
\gamma_{r+1} \\
\vdots \\
\gamma_n
\end{pmatrix} = M^{-1} \begin{pmatrix}
\text{pad}_{i_{t+1}} \\
\vdots \\
\text{pad}_{j_{t+1}}
\end{pmatrix} = \mu^{-1} \tilde{M} \begin{pmatrix}
\text{pad}_{i_{t+1}} \\
\vdots \\
\text{pad}_{j_{t+1}}
\end{pmatrix} \in \text{Der}_K(A_{\mu}) \cap A_{\mu} \mathcal{H}_{A_{\mu}}
$$

and $\gamma_s(x_{j_n}) = \delta_{st}$ (the Kronecker delta) where $s, t = r + 1, \ldots, n$. Therefore,

$$
\text{Der}_K(A_{\mu}) = \bigoplus_{s=r+1}^n A_{\mu} \gamma_s \mathcal{H}_{A_{\mu}}
$$

(since $\text{dim}_Q(Q \text{Der}_K(A)) = n - r = \text{tr.deg}_K(Q)$ and the restriction of the derivations $\gamma_{r+1}, \ldots, \gamma_n$ to the subfield $K(x_{j_{t+1}}, \ldots, x_{j_n})$ of rational functions in the variables $x_{j_{t+1}}, \ldots, x_{j_n}$ are equal to the partial derivatives $\partial_{x_{j_{t+1}}}, \ldots, \partial_{x_{j_n}}$, respectively).

(iii) $\text{Der}_K(A) = A \mathcal{H}_A$: Let us consider the left $A$-module $V = \text{Der}_K(A)/A \mathcal{H}_A$. By the statement (ii), $V_\mu = 0$ for all elements $\mu \in M_{n-r}$. Hence $V = 0$, by the statement (i).

**Corollary 5.3** Let $\mathcal{P} = P_n$ be a polynomial Poisson algebra. Then the following statements are equivalent:

1. $\text{Der}_K(P_n) = P_n \mathcal{H}_{P_n}$.
2. $\mathcal{C}_\mathcal{P} \in \text{GL}_n(P_n)$.
3. $\text{det}(\mathcal{C}_\mathcal{P}) \in K^\times$.

**Proof.** (1 \iff 3) Notice that $\text{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ where $\partial_i = \frac{\partial}{\partial x_i}$. By Theorem \ref{thm:regularity}(1,2), $\text{Der}_K(P_n) = P_n \mathcal{H}_{P_n}$ if $d = n$ and $\mathcal{C}_{P,n} = P_n$ if $\text{det}(\mathcal{C}_\mathcal{P}) \notin K^\times$ since $\mathcal{C}_{P,n} = (\text{det}(\mathcal{C}_\mathcal{P}))$.

(2 \iff 3) The equivalence is obvious. □

**Example.** Let $\mathcal{P} = P_2 = K[x_1, x_2]$ be a Poisson algebra and $a = \{x_1, x_2\}$, an arbitrary element of $P_2$. Then $\text{Der}_K(P_2) = P_2 \mathcal{H}_{P_2}$ iff $a \in K^\times$, by Corollary 5.3.

**The generalized Weyl Poisson algebra** $D[X,Y; a, \partial]$. We apply the above results for certain generalized Weyl Poisson algebras. In particular, Lemma \ref{lem:regularity} and Corollary \ref{cor:regularity} are regularity and symplecticity criteria for certain GWPAs.
Definition. Let $D$ be a Poisson algebra, $\partial = (\partial_1, \ldots, \partial_n) \in \text{PDer}_K(D)^n$ be an $n$-tuple of commuting derivations of the Poisson algebra $D$, $a = (a_1, \ldots, a_n) \in \text{PZ}(D)^n$ be such that $\partial_i(a_j) = 0$ for all $i \neq j$. The generalized Weyl algebra

$$A = D[X,Y; (\text{id}_D, \ldots, \text{id}_D), a] = D[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1 Y_1 - a_1, \ldots, X_n Y_n - a_n)$$

admits a Poisson structure which is an extension of the Poisson structure on $D$ and is given by the rule: For all $i, j = 1, \ldots, n$ and $d \in D$,

$$\{Y_i, d\} = \partial_i(d) Y_i, \quad \{X_i, d\} = -\partial_i(d) X_i \quad \text{and} \quad \{Y_i, X_i\} = \partial_i(a_i).$$

(24)

$$\{X_i, X_j\} = \{Y_i, Y_j\} = 0 \quad \text{for all} \quad i \neq j.$$  

(25)

The Poisson algebra is denoted by $A = D[X,Y; a, \partial]$ and is called the generalized Weyl Poisson algebra of rank $n$ (or GWPA, for short) where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$. Lemma 5.4 and Corollary 5.5 are regularity and symplecticity criteria for certain GWPAs.

**Lemma 5.4** Let $A = K[H]|X,Y; a, \partial = b \frac{d}{dH} \}$ be a GWPA of rank 1 where $a, b \in K[H]$ and $a \neq 0$. Then

1. The domain $A = K[H]|X,Y; a, \partial = b \frac{d}{dH} \}$ is regular iff $(a, a') = K[H]$ where $a' = \frac{da}{dH}$, i.e. the polynomials $a$ and $a'$ are co-prime.

2. Suppose that the algebra $A$ is a regular algebra, i.e. $(a, a') = K[H]$. Then

   (a) $\text{Der}_K(A) = AH_A$ iff $b \in K^\times$.

   (b) $C_A = \{\{X,X\}, \{X,Y\}, \{X,H\}, \{Y,X\}, \{Y,Y\}, \{Y,H\}, \{H,X\}, \{H,Y\}, \{H,H\}\}$ and $r(C_A) = \begin{pmatrix} 2 & \text{if} \ b \neq 0, \\ 0 & \text{if} \ b = 0. \end{pmatrix}$

**Proof.** 1. The Jacobian matrix of $A$ is equal to $(Y,X,-a')$. Then the Jacobian ideal $a_1$ of $A$ is equal to $(Y,X,-a')$, the ideal of $A$ which is generated by the elements $Y$, $X$ and $-a'$. Now, $a_1 = A$ iff $K[H] = K[H] \cap (Y,X,-a') = (a, a')$ since $XY = a$ (use the $\mathbb{Z}$-grading of the GWA $A$).

2. (b) The statement (b) is obvious.

   (a) By Theorem 1.2, $\text{Der}_K(A) = AH_A$ iff $r(C_p) = 3 - 1 = 2$ (i.e. $b \neq 0$, by the statement (b)) and $A = c_{A,2} = b^2(a'^2, a'X, a'Y, XY, X^2, Y^2)$ if $b \in K^\times$ and

$$A = (a'^2, a'X, a'Y, a, X^2, Y^2) = (a'^2, a, X, Y),$$

since $(a, a') = 1$ iff $b \in K^\times$ and $K[H] = K[H] \cap (a'^2, a, X, Y) = (a'^2, a)$ iff $b \in K^\times$ since $K[H] = (a, a')$ implies $K[H] = (a, a'^2)$. □

**Corollary 5.5** Let $A = K[H_1, \ldots, H_n]|X,Y; a = (a_1, \ldots, a_n), \partial = (b_1 \frac{d}{dH_1}, \ldots, b_n \frac{d}{dH_n})\}$ be a GWPA of rank $n$ where $a_i, b_i \in K[H_i]$ and $a_i \neq 0$. Then

1. The domain $A = K[H_1, \ldots, H_n]|X,Y; a = (a_1, \ldots, a_n), \partial = (b_1 \frac{d}{dH_1}, \ldots, b_n \frac{d}{dH_n})\}$ is regular iff $(a_i, a'_i) = K[H_i]$ for $i = 1, \ldots, n$ where $a'_i = \frac{da_i}{dH_i}$, i.e. the polynomials $a_i$ and $a'_i$ are co-prime in $K[H_i]$.

2. Suppose that the algebra $A$ is a regular algebra. Then

   (a) $\text{Der}_K(A) = AH_A$ iff $b_i \in K^\times$ for $i = 1, \ldots, n$.

   (b) $r(C_A) = 2n$ iff $b_i \neq 0$ for $i = 1, \ldots, n$.

**Proof.** Notice that $A = \otimes_{i=1}^n A_i$, a tensor product of Poisson algebras where $A_i = K[H_i]|X_i, Y_i; a_i, \partial_i = b_i \frac{d}{dH_i}$ is a GWPA as in Lemma 5.4 for $i = 1, \ldots, n$. Now, the corollary follows from Lemma 5.4.
6 The kernel of the epimorphism $\mathcal{U}(\mathcal{P}) \rightarrow PD(\mathcal{P})$ and the defining relations of the algebra $PD(\mathcal{P})$

In this section we keep the assumptions and the notations of Section 4. In most cases the field $K$ has characteristic zero.

Let $\mathcal{P}$ be a Poisson algebra. The Poisson algebra $\mathcal{P}$ is a Poisson $\mathcal{P}$-module, i.e., a left $\mathcal{U}(\mathcal{P})$-module, and the image of the corresponding algebra homomorphism $\mathcal{U}(\mathcal{P}) \rightarrow \text{End}_K(\mathcal{P})$ is the algebra of Poisson differential operators $PD(\mathcal{P})$ on $\mathcal{P}$. So, the kernel of the algebra epimorphism

$$\pi_\mathcal{P} : \mathcal{U}(\mathcal{P}) \rightarrow PD(\mathcal{P}), \ p \mapsto p, \ \delta_q \mapsto \text{pad}_q := \{q, \cdot\} \ (p, q \in \mathcal{P}) \quad (26)$$

is the annihilator of the left $\mathcal{U}(\mathcal{P})$-module $\mathcal{P}$.

For a domain of essentially finite type $A$ over a field of characteristic zero, Proposition 6.3 determines the exact value for the Gelfand-Kirillov dimension of the algebra $PD(\mathcal{A})$. An explicit ideal $\kappa_\mathcal{A}$ of the algebra $\mathcal{U}(\mathcal{A})$ is introduced such that $\kappa_\mathcal{A} \subseteq \ker(\pi_\mathcal{A})$, see (28) and (29) for the definition of its explicit generators. Corollary 6.8 is a criterion for $\kappa_\mathcal{A} = \ker(\pi_\mathcal{A})$. Proposition 6.4 is about properties of the ideals $\kappa_\mathcal{A}$ and $\ker(\pi_\mathcal{A})$. Proposition 6.4(3) is a sufficient condition for $\kappa_\mathcal{A} = \ker(\pi_\mathcal{A})$. Proposition 6.5 describes the torsion submodule of the module $\Omega_\mathcal{A}$ of Kähler differential of the algebra $\mathcal{A}$. Theorem 6.7 gives explicit descriptions of the kernels $\ker(\pi_\mathcal{A})$ and $\ker(\pi_\mathcal{A})$. Theorem 1.9 and Theorem 1.10 are criteria for $\ker(\pi_\mathcal{A}) = 0$ and their proofs are given in this section. Theorem 1.11 is a criterion for the homomorphism $\pi_\mathcal{A} : \mathcal{U}(\mathcal{A}) \rightarrow D(\mathcal{A})$ to be an isomorphism, its proof is also presented in this section.

In this section, the algebra $\mathcal{P} = \mathcal{A} = S^{-1}(\mathcal{P}/\mathcal{I})$ is a domain of essentially finite type and we keep the assumptions and the notations of Section 4. Let $Q = Q(\mathcal{A})$ be its field of fractions, $d = d_\mathcal{A} = r(\mathcal{C}_\mathcal{A})$ be the rank of the $n \times n$ matrix $\mathcal{C}_\mathcal{A} = (\{x_i, x_j\}) \in M_n(\mathcal{A})$ over the field $Q$. For each $l = 1, \ldots, n$, let

$$\text{ind}_n(l) = \{i = (i_1, \ldots, i_l) \mid 1 \leq i_1 < \cdots < i_l \leq n\}.$$ 

For elements $i = (i_1, \ldots, i_l)$ and $j = (j_1, \ldots, j_l)$ of $\text{ind}_n(l)$, let $\mathcal{C}_\mathcal{A}(i, j) = \{x_{i_\nu}, x_{j_\mu}\}$ be the $l \times l$ submatrix of the matrix $\mathcal{C}_\mathcal{A}$. So, the rows (resp., the columns) of the matrix $\mathcal{C}_\mathcal{A}(i, j)$ are indexed by $i_1, \ldots, i_l$ (resp., $j_1, \ldots, j_l$). The $(i_\nu, j_\mu)$th element of the matrix $\mathcal{C}_\mathcal{A}(i, j)$ is $x_{i_\nu}, x_{j_\mu}$. Let $M_{l,l} = \{\mathcal{C}_\mathcal{A}(i, j) \mid i, j \in \text{ind}_n(l)\}$ be the set of all $l \times l$ submatrices of $\mathcal{C}_\mathcal{A}$ and

$$\mathcal{C}_{l,l} = \{\mu(i, j) := \det(\mathcal{C}_\mathcal{A}(i, j)) \mid i, j \in \text{ind}_n(l)\}$$

be the set of all $l \times l$ minors of $\mathcal{C}_\mathcal{A}$. Let

$$\mathbb{I}(l) = \mathbb{I}_\mathcal{A}(l) = \{i \in \text{ind}_n(l) \mid \mu(i, j) \neq 0 \text{ for some } j \in \text{ind}_n(l)\}$$

$$\mathbb{J}(l) = \mathbb{J}_\mathcal{A}(l) = \{j \in \text{ind}_n(l) \mid \mu(i, j) \neq 0 \text{ for some } i \in \text{ind}_n(l)\}$$

The symmetric group $S_n$ acts on the set of indices $\{1, \ldots, n\}$ by permutation. In particular, the symmetric group $S_l$ acts on the set $\{i_1, \ldots, i_l\}$ by permutation. For each $\sigma \in S_l$, $\text{sign}(\sigma)$ is the sign of $\sigma$.

**Lemma 6.1** 1. For all $i, j \in \text{ind}_n(l)$, $\mathcal{C}_\mathcal{A}(i, j)^t = -\mathcal{C}_\mathcal{A}(j, i)$ where ‘$t$’ stands for the transposition of a matrix.

2. For all $i, j \in \text{ind}_n(l)$, $\mu(i, j) = (-1)^l \mu(j, i)$.

3. $\mathbb{I}_\mathcal{A}(l) = \mathbb{J}_\mathcal{A}(l)$ for all $l = 1, \ldots, n$.

4. For all permutations $\sigma, \tau \in S_l$, $\mu(\sigma(i), \tau(j)) = \text{sign}(\sigma \tau) \mu(i, j)$. 

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Proof. 1. The \((j_1, j_2)^{th}\) element of the matrix \(C_A(i, j)^t\) is \(\{x_i, x_j\} = -\{x_j, x_i\}\), and statement 1 follows.

2. \(\mu(i, j) = \det(C_A(i, j)) = \det((C_A(i, j)^t)^*_{-1}(-1)^{\mu(i, j)} = (-1)^{\mu(i, j)}\).

3. Statement 3 follows from statement 2.

4. Statement 4 is obvious. \(\square\)

In the case \(l = d\), we can say more about the elements \(\mu(i, j)\).

**Proposition 6.2** Let \(K\) be a field of characteristic zero.

1. Given \(i = (i_1, \ldots, i_d) \in \mathbb{I}_A(d)\) where \(d = r(C_A)\). Then \(i \in \mathbb{I}_A(d)\) iff \(Q\mathcal{H}_A = \bigoplus_{s=1}^{d} Q\delta_{s,s}\).

2. For all \(i \in \mathbb{I}_A(d)\) and \(j \in \mathbb{I}_A(d)\), \(\mu(i, j) \neq 0\).

3. Given \(i, j \in \mathbb{I}_A(d)\). Then \(i, j \in \mathbb{I}_A(d)\) iff \(\mu(i, j) \neq 0\).

Proof. 1. Statement 1 follows from the fact that \(\dim_Q(Q\mathcal{H}_A) = d\) and the elements \(\delta_{s,s} = \sum_{s=1}^{d} \{x_i, x_j\}\) are \(Q\)-linearly independent.\(\square\)

2. Since \(\mathbb{I}_A(d)\) and \(\mathbb{I}_A(d)\), \(\mu(i, j') \neq 0\) and \(\mu(i', j') \neq 0\) for some \(i', j' \in \mathbb{I}_A(d)\), \(\mathbb{I}_A(d)\). By statement 1,

\[
Q\mathcal{H}_A = \bigoplus_{s=1}^{d} Q\delta_{s,s} = \bigoplus_{s=1}^{d} Q\delta_{s,s}.
\]

Let \(\delta' = (\delta'_1, \ldots, \delta'_d)\) and \(\delta'' = (\delta''_1, \ldots, \delta''_d)\). Then \(\delta'' = C_A(i', j')C_A(i, j)^{-1}\delta'\) and \(\delta' = C_A(i, j)C_A(i', j')^{-1}\delta''.\) Therefore,

\[
\delta'_i = C_A(i, j)C_A(i', j')^{-1}C_A(i', j')C_A(i, j)^{-1}\delta''_i
\]

and so the product of four matrices is equal to the identity matrix. Hence, their determinants are nonzero. In particular, \(\mu(i, j) = \det(C_A(i, j)) \neq 0\), as required.

3. Statement 3 follows at once from statement 2. \(\square\)

**The ideal \(\kappa_A\) of the algebra \(\mathcal{U}(A)\) and its generators \(\delta_{i_1,i_2}\).** For each pair of elements \(i = (i_1, \ldots, i_d)\) and \(j = (j_1, \ldots, j_d)\) of \(\mathbb{I}_A(d)\) and each element \(i_\nu \in \{i_{d+1}, \ldots, i_n\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_d\}\), let us consider the following elements of the algebra \(\mathcal{U}(A)\) (see Theorem 2.2(2) for the definition of the elements \(\delta_{i}\)),

\[
\delta_{i,j} := \det\left(\begin{array}{ccc}
\{x_{i_1}, x_{j_1}\} & \ldots & \{x_{i_1}, x_{j_d}\} \\
\{x_{i_2}, x_{j_1}\} & \ldots & \{x_{i_2}, x_{j_d}\} \\
\{x_{i_3}, x_{j_1}\} & \ldots & \{x_{i_3}, x_{j_d}\} \\
\vdots & \ddots & \vdots \\
\{x_{i_d}, x_{j_1}\} & \ldots & \{x_{i_d}, x_{j_d}\}
\end{array}\right) = \mu(i, j)\delta_{i_\nu} + \sum_{s=1}^{d} (-1)^{s+d+1}\mu(i_1, \ldots, i_s, \ldots, i_d, i_{i_\nu}; j)\delta_{i_\nu}. \quad (28)
\]

Let \(\{j_{d+1}, \ldots, j_n\} := \{1, \ldots, n\} \setminus \{j_1, \ldots, j_d\}\) and for each \(\nu \in \{d+1, \ldots, n\}\), let us consider the following elements of \(\mathcal{U}(A)\),

\[
\delta'_{i,j} := \det\left(\begin{array}{ccc}
\{x_{i_1}, x_{j_1}\} & \ldots & \{x_{i_1}, x_{j_d}\} \\
\{x_{i_2}, x_{j_1}\} & \ldots & \{x_{i_2}, x_{j_d}\} \\
\{x_{i_3}, x_{j_1}\} & \ldots & \{x_{i_3}, x_{j_d}\} \\
\vdots & \ddots & \vdots \\
\{x_{i_d}, x_{j_1}\} & \ldots & \{x_{i_d}, x_{j_d}\}
\end{array}\right) = -\mu(i, j)\delta_{j_\nu} - \sum_{s=1}^{d} (-1)^{s+d+1}\mu(i; j_1, \ldots, j_s, \ldots, j_d, j_{i_\nu})\delta_{j_\nu}. \quad (29)
\]

The elements \(\delta'_{i,j}\) are ‘dual’ of the elements \(\delta_{i,i,j}\). In fact, they are the same up to sign,

\[
\delta'_{i,j} = (-1)^{d+1}\delta_{j,i,j}. \quad (30)
\]
Hence, $GK(P)$ and the field $\mu_P$ figure for the Gelfand-Kirillov dimension of the algebra of characteristic zero, Proposition 6.3

Let a Poisson algebra $\delta$ be a domain of essentially finite type over the field $K$ of characteristic zero, $r$ is the rank of Jacobian matrix of $A$ and $d = r(\mathcal{C}_A)$. Then

1. $GK(PD(A)) = GK(A) + d = n + r + d.$

2. Let $\mu = \mu(i_1, \ldots, i_d; j_1, \ldots, j_d) = \det((x_{i,s}, x_{j,t}))$ be a nonzero minor of the matrix $\mathcal{C}_A$ where $s, t = 1, \ldots, d$. Then $PD(A)_\mu \simeq PD(A_\mu) = \bigoplus_{\alpha \in \mathbb{N}^d} A_\mu \operatorname{pad}_{x_{i_1}}^{\alpha_1} \cdots \operatorname{pad}_{x_{i_d}}^{\alpha_d}$, and $GK(PD(A)_\mu) = GK(A) + d.$

Proof. 2. By (28) and (31), the localization of the algebra $PD(A)$ at the powers of the element $\mu$ is equal to $PD(A)_\mu = \sum_{\alpha \in \mathbb{N}^d} A_\mu \operatorname{pad}_{x_{i_1}}^{\alpha_1} \cdots \operatorname{pad}_{x_{i_d}}^{\alpha_d}$. Let

$$\begin{pmatrix} \gamma_{i_1} \\ \vdots \\ \gamma_{i_d} \end{pmatrix} := \mathcal{C}_A(i, j)^{-1} \begin{pmatrix} \operatorname{pad}_{x_{i_1}} \\ \vdots \\ \operatorname{pad}_{x_{i_d}} \end{pmatrix}. $$

Then $PD(A)_\mu = \sum_{\alpha \in \mathbb{N}^d} A_\mu \gamma_{i_1}^{\alpha_1} \cdots \gamma_{i_d}^{\alpha_d}$. The sum is a direct sum since

$$\gamma_{i_\nu}(x_{j_\mu}) = \delta_{i_\nu j_\mu} \text{ for all } \nu, \mu = 1, \ldots, d$$

and the field $K$ has characteristic zero where $\delta_{i_\nu j_\mu}$ is the Kronecker delta. Therefore $PD(A)_\mu = \bigoplus_{\alpha \in \mathbb{N}^d} A_\mu \operatorname{pad}_{x_{i_1}}^{\alpha_1} \cdots \operatorname{pad}_{x_{i_d}}^{\alpha_d}$, and so $GK(PD(A)_\mu) = GK(A) + d = n.$

1. By statement 2, $GK(PD(A)) \leq GK(PD(A)_\mu) = GK(A) + d.$ By statement 2,

$$\bigoplus_{\alpha \in \mathbb{N}^d} A \operatorname{pad}_{x_{i_1}}^{\alpha_1} \cdots \operatorname{pad}_{x_{i_d}}^{\alpha_d} \subseteq PD(A).$$

Hence, $GK(PD(A)) \geq GK(A) + d$, and statement 1 follows since $GK(A) = n - r.$

Recall that every left (resp., right) Ore set of a left (resp., right) Noetherian ring is a left (resp., right) denominator set, see [8] Corollary 4.24.

The ring $\mathcal{U}(A)$ is a Noetherian ring and for each nonzero element $s \in A$ the set $S_s = \{s^i \mid i \geq 0\}$ is an Ore set of $\mathcal{U}(A)$. Hence, $S_s$ is a denominator set of $\mathcal{U}(A)$, and so

$$\mathcal{U}(A)_s := S_s^{-1}\mathcal{U}(A) \simeq \mathcal{U}(A)S_s^{-1}$$

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is the localization of the ring \( \mathcal{U}(A) \) at the powers of the element \( s \). By \( \mathcal{A} \) and the inclusion \( \mathcal{A} \subseteq PD(A) \),

\[
\mathcal{A} \cap \kappa_{\mathcal{A}} = 0.
\]

The ring \( \mathcal{U}(A)/\kappa_{\mathcal{A}} \) is a Noetherian ring. Hence, the set \( S_s \) is a denominator set of the ring \( \mathcal{U}(A)/\kappa_{\mathcal{A}} \).

Recall that the algebra \( PD(A) \) is equipped with two filtrations: the order filtration \( \{PD(A)_{s_i} = PD(A) \cap D(A)_{s_i}; \}_{i \geq 0} \) and the filtration \( \{PD(A)_{s_i} = PD(A)_{s_i}; \}_{i \geq 0} \) that is determined by the total degree of the elements \( \delta_{x_1}, \ldots, \delta_{x_n} \). Clearly,

\[
PD(A)_{s_i} \subseteq PD(A)_{s_{i-1}} \quad \text{for all} \quad i \geq 0.
\]

Theorem 6.4 shows that if \( \mathfrak{c}_{A,d} = \mathcal{A} \) then \( \ker(\pi_A) = \kappa_A \) and \( PD(A)_{s_i} = PD(A)_{s_{i-1}} \) for all \( i \geq 0 \).

**Theorem 6.4** Let a Poisson algebra \( \mathcal{A} \) be a domain of essentially finite type over the field \( K \) of characteristic zero and \( d = r (\mathcal{C}_A) \). Then

1. \( \mathfrak{c}^*_{A,d} \mathfrak{c}_\mathcal{A} \) and \( \mathfrak{c}_\mathcal{A} \) are finitely generated, hence Noetherian. Therefore, the left \( \mathcal{A} \)-module \( PD(A)_{s_i} \) is finitely generated and there are natural numbers \( s_i \) and \( t_i \) such that \( \mathfrak{c}^*_{A,d}PD(A)_{s_i} \subseteq PD(A)_{s_{i-1}} \) and \( PD(A)_{t_i} \subseteq PD(A)_{t_{i-1}} \).

2. Suppose that \( \mathfrak{c}_{A,d} = \mathcal{A} \). Then

(a) \( \ker(\pi_A) = \kappa_A \).

(b) For all \( i \geq 0 \), \( PD(A)_{s_i} = PD(A)_{s_{i-1}} \).

**Proof.** 1. (i) For all \( i, j \in \mathbb{N}_d \), \( (\mathcal{U}(A)/\kappa_A)_{\mu, \mathfrak{c}_\mathcal{A}} = \bigoplus_{\alpha \in \mathbb{N}_d} A_{\alpha} \delta^{t_{i,1}} \cdots \delta^{t_{i,d}} \). The statement (i) follows at once from \( \mathcal{U}(A)_{\mu, \mathfrak{c}_\mathcal{A}} = \bigoplus_{\alpha \in \mathbb{N}_d} A_{\alpha} \delta^{t_{i,1}} \cdots \delta^{t_{i,d}} \). The statement (ii) follows at once from \( \mathcal{U}(A)_{\mu, \mathfrak{c}_\mathcal{A}} = \bigoplus_{\alpha \in \mathbb{N}_d} A_{\alpha} \delta^{t_{i,1}} \cdots \delta^{t_{i,d}} \). When we localize the short exact sequence of left \( \mathcal{U}(\mathcal{P}) \)-modules,

\[
0 \to \ker(\pi_A)/\kappa_A \to \mathcal{U}(A)/\kappa_A \to PD(A) \to 0,
\]

at the powers of the element \( \mu = \mu(i, j) \) we obtain the short exact sequence of left \( \mathcal{U}(\mathcal{P})_{\mu} \)-modules,

\[
0 \to \ker(\pi_A)/\kappa_A \to \mathcal{U}(A)/\kappa_A \to PD(A) \to 0.
\]

By the statement (i) and Proposition 6.3 (2), the second map of the short exact sequence above is an isomorphism, and the statement (ii) follows.

The algebra \( \mathcal{U}(A) \) is a Noetherian algebra. Therefore, the ideal \( \kappa_A \) is a finitely generated left and right ideal. The localization at the powers of the element \( \mu \) is a left and right localization. Hence,

\[
\mu^{\mathfrak{c}_\mathcal{A}}(i, j) \ker(\pi_A) \subseteq \kappa_A \quad \text{and} \quad \ker(\pi_A) \mu(t_{i,1}) \subseteq \kappa_A
\]

for some natural numbers \( s(i, j) \) and \( t(i, j) \). The ideal \( \mathfrak{c}_{A,d} \) is generated by the finitely many elements \( \{\mu(i, j) \mid i, j \in \mathbb{N}_d \} \). Now, statement 1 is obvious.

2. By \( \mathcal{U}(A)_{\mu, \mathfrak{c}_\mathcal{A}} = \bigoplus_{\alpha \in \mathbb{N}_d} A_{\alpha} \delta^{t_{i,1}} \cdots \delta^{t_{i,d}} \). The proof of \( \mathcal{U}(A)_{\mu, \mathfrak{c}_\mathcal{A}} = \bigoplus_{\alpha \in \mathbb{N}_d} A_{\alpha} \delta^{t_{i,1}} \cdots \delta^{t_{i,d}} \) works for right \( \mathcal{A} \)-modules, i.e. for each \( i \geq 0 \), the right \( \mathcal{A} \)-modules \( D(A)_{s_i} \) is a finitely generated, hence Noetherian. Therefore, the left and right \( \mathcal{A} \)-submodules \( PD(A)_{s_i} \) of \( D(A)_{s_i} \) is finitely generated, and so is its quotient

\[
V(i) = PD(A)_{s_i}/PD(A)_{s_i-1}.
\]

By Proposition 6.3 (2), \( V(i) = 0 \). Therefore, \( \mathfrak{c}_{A,d} V(i) = 0 \) and \( \mathfrak{c}_{A,d} V(i) = 0 \) for some natural numbers \( s_i \) and \( t_i \), and statement 2 follows.

3. Since \( \mathfrak{c}_{A,d} = \mathcal{A} \), the statements (a) and (b) follow from statements 1 and 2, respectively. \( \square \)
**The torsion \( A \)-submodule** \( T_A \) of \( \Omega_A \).

Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a domain of essentially finite type. Then

\[
T = T_A = \{ \omega \in \Omega_A \mid a \omega = 0 \text{ for some } a \in A \setminus \{0\} \}
\]

is called the torsion \( A \)-submodule of \( \Omega_A \). The set \( A \setminus \{0\} \) of regular elements of \( A \) is a denominator set of \( A \) and \( T \) is the \( A \setminus \{0\} \)-torsion submodule of \( \Omega_A \). For each \( i \in I_r \) and \( j \in J_r \), \( \Delta = \Delta(i,j) \neq 0 \), and the set \( S_\Delta = \{ \Delta^i \mid i \geq 0 \} \) is a denominator set of the algebra \( A \). Let

\[
\text{tor}_{S_\Delta}(\Omega_A) := \{ \omega \in \Omega \mid \Delta^i \omega = 0 \text{ for some } i \geq 0 \},
\]

the \( S_\Delta \)-torsion submodule of the \( A \)-module \( \Omega \).

Proposition [6.5](#) is an explicit description of the torsion submodule \( T \) of \( \Omega \).

**Proposition 6.5** Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a domain of essentially finite type. Then

1. \( T_\Omega = \text{tor}_{(S_\Delta(0,j)}(\Omega_A) \) for all \( i \in I_r \) and \( j \in J_r \).
2. \( T_\Omega = \text{ann}_A(a_r^t) \) for all \( t \gg 0 \) (where \( r \) is the rank of the Jacobian matrix \( \partial f/\partial x_i \) and \( a_r \) is the Jacobian ideal).
3. If, in addition, the algebra \( A \) is a regular algebra then \( T_\Omega = 0 \).

**Proof.**

1. Let \( \Delta = \Delta(i,j) \) where \( i \in I_r \) and \( j \in J_r \). Then \( S_\Delta \subseteq A \setminus \{0\} \), and so \( \text{tor}_{S_\Delta}(\Omega) \subseteq \text{tor}_{A \setminus \{0\}}(\Omega) = T \). The opposite inclusion follows from the fact that the \( A \)-module

\[
S_\Delta^{-1} \Omega = \oplus_{r=1}^{n} A \Delta d_{j_r}
\]

is torsion free where \( \{j_r+1, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\} \).

2. The \( A \)-module \( \Omega \) is finitely generated, hence Noetherian. Then the \( A \)-submodule \( T \) of \( \Omega \) is finitely generated. By statement 1, \( \Delta(i,j)^k T = 0 \) for all \( i \in I_r \), \( j \in J_r \), and some natural number \( k = k(i,j) \geq 0 \). Therefore,

\[
a_r^s T = 0 \text{ for some } s \geq 0
\]

since the ideal \( a_r \) is generated by the finite set \( \{\Delta(i,j) \mid i \in I_r, j \in J_r\} \). Therefore, \( T \subseteq \text{ann}_A(a_r^t) \).

Since \( \Delta(i,j)^l \in a_r^t \) for some \( l \geq 1 \),

\[
\text{ann}_A(a_r^t) \subseteq \text{ann}_A(\Delta(i,j)^l) \subseteq \text{tor}_{S_\Delta(0,j)}(\Omega) = T,
\]

by statement 1, and statement 2 follows.

3. The algebra \( A \) is regular. So, \( a_r = A \), and statement 3 follows from statement 2. □

**The \( A \)-torsion submodule** \( T_{\overline{U}(A)} \) of \( \overline{U}(A) \). Let us consider the factor algebra

\[
\overline{U} := \overline{U}(A) := U(A)/\kappa_A.
\]

There are obvious algebra epimorphisms (since \( \kappa_A \subseteq \ker(\pi_A) \)):

\[
\pi_A : U(A) \xrightarrow{\phi} \overline{U}(A) \xrightarrow{\varphi_A} PD(A) \text{ and } \pi_A = \varphi_A \phi.
\]

Since \( A \subseteq PD(A) \),

\[
A \cap \kappa_A = 0.
\]

The set \( \mathcal{A} \setminus \{0\} \) is an Ore set of the Noetherian algebra \( U(A) \). Hence the set \( \mathcal{A} \setminus \{0\} \) is a denominator set of \( U(A) \). By [34](#), the set \( \mathcal{A} \setminus \{0\} \) is an Ore set of the Noetherian algebra \( \overline{U} \), hence it is a denominator set of \( \overline{U} \). Let

\[
T_{\overline{U}} = T_{\overline{U}(A)} = \text{tor}_{\mathcal{A} \setminus \{0\}}(\overline{U}(A))
\]

be the \( \mathcal{A} \setminus \{0\} \)-torsion submodule of the algebra \( \overline{U} \). The set \( T_{\overline{U}} \) is an ideal of the algebra \( \overline{U} \). Abusing the language we often say ‘\( \mathcal{A} \)-torsion’ meaning ‘\( \mathcal{A} \setminus \{0\} \)-torsion’.

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Proposition 6.6 Let a Poisson algebra $A = S^{-1}(P_n/I)$ be a domain of essentially finite type over the field $K$ of characteristic zero and $d = r(C_A)$. Then

1. $T_{\overline{U}(A)} = \operatorname{tor}_{S_{\mu}(i,j)}(\overline{U}(A))$ for all $i \in I_A(d)$ and $j \in J_A(d)$ where $S_{\mu}(i,j) = \{\mu(i,j)^k \mid k \geq 0\}$ is a denominator set of the algebra $\overline{U}(A)$.

2. $T_{\overline{U}(A)} = r.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^t)$ and $T_{\overline{U}(A)} = \lambda.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^t)$ for all $t \gg 0$.

Proof. 1. Let $\mu = \mu(i,j)$ where $i \in I_A(d)$ and $j \in J_A(d)$. Then the set $S_{\mu} = \{\mu^i \mid i \geq 0\}$ is a denominator set of the Noetherian algebra $\overline{U}(A)$ such that $S_{\mu} \subseteq A\{0\}$. By (34), $S_{\mu} \cap \kappa_A = \emptyset$. Therefore, the set $S_{\mu}$ is a denominator set of the Noetherian algebra $\overline{U} = \overline{U}(A)$. Then the inclusion $S_{\mu} \subseteq A\{0\}$ implies the inclusion

$$\operatorname{tor}_{S_{\mu}}(\overline{U}) \subseteq T_{\overline{U}}.$$ 

The inverse inclusion follows from the fact that (see the statement (i) of the proof of Theorem 6.3 (1))

$$S_{\mu}^{-1}\overline{U} = \bigoplus_{a \in \mathbb{N}^d} A_{\mu} \delta_{x_{i_1}}^a \cdots \delta_{x_{i_d}}^a = \bigoplus_{a \in \mathbb{N}^d} \delta_{x_{i_1}}^a \cdots \delta_{x_{i_d}}^a A_{\mu} = \overline{U} S_{\mu}^{-1}.$$ 

2. The algebra $U = U(A)$ is a Noetherian algebra, hence so is its factor algebra $\overline{U}$. Then the ideal $T_{\overline{U}}$ of $\overline{U}$ is a finitely generated left and right $\overline{U}$-module. By statement 1, $\mu(i,j)^k T_{\overline{U}} = T_{\overline{U}}(i,j)^k = 0$ for all $i \in I_A(d)$, $j \in J_A(d)$ and some natural number $k = k(i,j) \geq 0$. Therefore, $c_{A,d}^s \mu(i,j)^k T_{\overline{U}} = 0$ and $\operatorname{tor}_{\overline{U}}(c_{A,d}^s) = 0$ for some natural number $s \geq 0$ since the ideal $c_{A,d}^s$ is generated by the finite set $\{\mu(i,j) \mid i \in I_A(d), j \in J_A(d)\}$. Therefore,

$$T_{\overline{U}} \subseteq r.\operatorname{ann}_{\overline{U}}(c_{A,d}^s)$$

by statement 1. Similarly, $\lambda.\operatorname{ann}_{\overline{U}}(c_{A,d}^s) \subseteq T_{\overline{U}}$ (by replacing ‘r’ by ‘l’ in the above chain of inclusions), and statement 2 follows. $\square$

Explicit descriptions of $\ker(\pi_A)$ and $\ker(\overline{\pi}_A)$; defining relations of the algebra $PD(A)$. Theorem 6.7 gives explicit descriptions of the kernels $\ker(\pi_A)$ and $\ker(\overline{\pi}_A)$. It gives the set of defining relations of the algebra $PD(A)$ together with Theorem 2.2 (2).

Theorem 6.7 Let a Poisson algebra $A = S^{-1}(P_n/I)$ be a domain of essentially finite type over the field $K$ of characteristic zero, $d = r(C_A)$, $\phi : U(A) \to \overline{U}(A) = U(A)/\kappa_A$, $u \mapsto u + \kappa_A$ and $\pi_A : U(A) \to PD(A)$, $a + \kappa_A \mapsto a$, $\delta_i + \kappa_A \mapsto \{x_i\}$ where $a \in A$. Then

1. $\ker(\pi_A) = \phi^{-1}(T_{\overline{U}(A)}) = \phi^{-1}(\operatorname{tor}_{S_{\mu}(i,j)}(\overline{U}(A))) = \phi^{-1}(r.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^s)) = \phi^{-1}(\lambda.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^s))$ for all $i \in I_A(d)$, $j \in J_A(d)$ and $t \gg 0$.

2. $\ker(\overline{\pi}_A) = T_{\overline{U}(A)} = \operatorname{tor}_{S_{\mu}(i,j)}(\overline{U}(A)) = r.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^s)$ and $\lambda.\operatorname{ann}_{\overline{U}(A)}(c_{A,d}^s)$ for all $i \in I_A(d)$, $j \in J_A(d)$ and $t \gg 0$.

Proof. 1. Since $\kappa_A \subseteq \ker(\pi_A)$ (see [31]), $\ker(\pi_A) = \phi^{-1}(\ker(\overline{\pi}_A))$ and statement 1 follows from statement 2.

2. Notice that $\phi(\ker(\pi_A)) = \ker(\overline{\pi}_A)$ and $\mu = \mu(i,j) \in c_{A,d}$ for all $i \in I_A(d)$ and $j \in J_A(d)$. Now, by Theorem 6.4 (1),

$$\ker(\pi_A) \subseteq \operatorname{tor}_{S_{\mu}(\overline{U})}.$$ 

The epimorphism $\pi_A$ is an $A$-module homomorphism $(\pi_A(a \pi) = a \pi_A(\pi))$ for all elements $a \in A$ and $\pi \in \overline{U}$ and the element $\mu$ is a regular element of the algebra $PD(A)$. Therefore,

$$\ker(\pi_A) \supseteq \operatorname{tor}_{S_{\mu}(\overline{U})},$$

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i.e. \( \ker(\pi_A) = \text{tor}_{S_1}(\mathcal{U}) \) and statement 2 follows from Proposition \[6.6\]. □

**Criterion for** \( \ker(\pi_A) = \kappa_A \Leftrightarrow \ker(\pi_A) = 0 \). The next corollary follows at once from \[6.6\] and Theorem, \[6.7\] (2) and is a criterion for \( \ker(\pi_A) = \kappa_A \Leftrightarrow \ker(\pi_A) = 0 \).

**Corollary 6.8** We keep the assumption of Theorem \[6.7\]. Then \( \ker(\pi_A) = \kappa_A \) iff \( \ker(\pi_A) = 0 \), i.e. \( \mathcal{U}(A) = PD(A) \), iff the nonzero elements of the algebra \( A \) are (left or right) regular in \( \mathcal{U}(A) \) iff all/some of the elements of the set \{\( \mu(\mathbf{i}, \mathbf{j}) \mid \mathbf{i} \in \mathcal{I}_A(d), \mathbf{j} \in \mathcal{I}_A(d) \}\} are (left or right) regular in \( \mathcal{U}(A) \) iff \( r.\text{ann}_{\mathcal{U}(A)}(\mathcal{E}_{A,d}) = 0 \) for all \( t \gg 0 \) iff \( 1.\text{ann}_{\mathcal{U}(A)}(\mathcal{E}_{A,d}) = 0 \) for all \( t \gg 0 \).

The **right kernel** \( \Omega'_{A} \) of the pairing \( \text{Der}_K(A) \times \Omega_A \to A \). Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be an algebra of essentially finite type. Recall \( \Omega = \Omega_A \simeq \bigoplus_{i=1}^{n} A \delta_i/G_A \) where \( G_A := \sum_{j=1}^{n} A \delta_j \) and \( \delta_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \). Let \( \text{Der}_K(A) \simeq \text{Hom}_A(\Omega, A) \), an isomorphism of \( A \)-modules. So, there is a pairing of \( A \)-modules (which is an \( A \)-bilinear map):

\[
\text{Der}_K(A) \times \Omega \to A, \quad (\partial, \omega) \mapsto (\partial, \omega) := \partial(\omega).
\]

If \( \partial = \sum_{i=1}^{n} a_i \partial_i \in \text{Der}_K(A) \) (where \( \partial_i = \frac{\partial}{\partial x_i} \)) and \( \omega = \sum_{i=1}^{n} b_i \delta_i \) where \( a_i, b_i \in A \) then

\[
(\partial, \omega) = \sum_{i=1}^{n} a_i b_i.
\]

In particular, \( (\partial, \delta_i) = a_i := \text{coef}_i(\partial) \), the \( i \)-th coefficient of \( \partial \). The \( A \)-submodule of \( \Omega \)

\[
\Omega' = \Omega'_A := \{ \omega \in \Omega \mid (\text{Der}_K(A), \omega) = 0 \}
\]

is called the **right kernel** of the pairing. In a similar way, the left kernel is defined. Since \( \text{Der}_K(A) \simeq \text{Hom}_A(\Omega, A) \), the left kernel of the pairing is equal to zero.

**Lemma 6.9** Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a domain of essentially finite type and let \( T \) be a multiplicative subset of \( A \). Then \( \Omega_{T^{-1}A} \simeq T^{-1}\Omega'_{A} \).

**Proof.** Since \( \text{Der}_K(T^{-1}A) \simeq \text{Hom}_{T^{-1}A}(\Omega_{T^{-1}A}, T^{-1}A) \simeq T^{-1}\text{Hom}_{T^{-1}A}(T^{-1}\Omega_A, T^{-1}A) \simeq T^{-1}\text{Der}_K(A) \), we must have \( \Omega_{T^{-1}A} \simeq T^{-1}\Omega'_{A} \). □

**Proposition 6.10** Let a Poisson algebra \( A = S^{-1}(P_n/I) \) be a domain of essentially finite type. Then

1. \( (\Omega'_A)_{\Delta(i,j)} = 0 \) for all \( i \in \mathbb{I}_r \) and \( j \in \mathbb{I}_r. \)
2. \( a_r^s \Omega'_A = 0 \) for some natural number \( s \geq 1. \)
3. If, in addition, the algebra \( A \) is a regular algebra than \( \Omega'_A = 0. \)

**Proof.** 1. Let \( \Delta = \Delta(i,j) \), \( j = (j_1, \ldots, j_r) \) and \( \{j_{r+1}, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\}. \) Statement 1 follows at once from the fact that \( (\Omega_A)_{\Delta} = \Omega_{A_\Delta} = \bigoplus_{\nu=1}^{n-1} A \delta_j \) and \( (\Omega'_A)_{\Delta} \simeq \Omega'_A. \)

2. The left \( \mathcal{A} \)-module \( \Omega'_A \) is finitely generated. Then, by statement 1, \( \Delta(i,j)^k \Omega'_A = 0 \) for some \( k = k(i,j). \) Hence, \( a_r \Omega'_A = 0 \) since the ideal \( a_r \) is generated by the finite set \( \{\Delta(i,j)\}. \)

3. Statement 3 follows from statement 2 and the fact that \( a_r = A (\text{since the algebra } A \text{ is a regular algebra}). \) □

**Criteria for** \( \ker(\pi_A) = 0 \). In the case when the Poisson algebra \( A \) is a regular domain of essentially finite type, Theorem \[1.9\] is an efficient explicit criterion for \( \ker(\pi_A) = 0 \), i.e. for the epimorphism \( \pi_A : \mathcal{U}(A) \to PD(A) \) to be an isomorphism.

**Proof of Theorem \[1.9\]** (1 \( \Rightarrow \) 2) This implication is obvious since \( \kappa_A \subseteq \ker(\pi_A) \).
(2 $\Rightarrow$ 1) Since $\kappa_\mathcal{A} = 0$, $\overline{\mathcal{U}}(\mathcal{A}) = \mathcal{U}(\mathcal{A})$, and the implication follows from Theorem 6.7.2\(2\) and the fact that the algebra $\mathcal{U}(\mathcal{A})$ is a domain (Theorem 4.4): $\ker(\pi_\mathcal{A}) = \ker(\overline{\pi}_\mathcal{A}) = T_{\overline{\mathcal{U}}(\mathcal{A})} = 0$.

(2 $\Rightarrow$ 3) If $\kappa_\mathcal{A} = 0$ then $\ker(\pi_\mathcal{A}) = 0$ (since (1 $\Leftrightarrow$ 2)), i.e. $\mathcal{U}(\mathcal{A}) \simeq P\mathcal{D}(\mathcal{A})$. By Theorem 4.4 and Proposition 6.3, we have that

$$2\text{GK}(\mathcal{A}) = \text{GK}(\mathcal{U}(\mathcal{A})) = \text{GK} P\mathcal{D}(\mathcal{A}) = \text{GK}(\mathcal{A}) + d,$$

we have that $d = \text{GK}(\mathcal{A}) = n - r$. Now, the implication follows from Proposition 6.10, $\mathcal{U}(\mathcal{A})$ is a domain and so its subalgebra $\mathcal{U}(\mathcal{A})$.

(2 $\Leftarrow$ 3) Suppose that $d = n - r$. Then $\kappa_\mathcal{A} = 0$ iff all the elements $\delta_{r',i',j'}$ (in statement 3) belong to the $\mathcal{A}$-module $G_\mathcal{A}$, by Corollary 6.8 and Theorem 4.4, we have that $\ker(\pi_\mathcal{A}) = 0$ for all $\delta_{r',i',j'}$ since $\Omega'_\mathcal{A} = 0$, by Proposition 6.10, $\ker(\pi_\mathcal{A}) = 0$ for all elements $\delta_{r',i',j'}$, as in statement 3 since the elements $\partial_{i,j}$ are $\mathcal{A}$-module generators of $\text{Der}_\mathcal{K}(\mathcal{A})$. Therefore, (2 $\Leftrightarrow$ 3). $\square$

Theorem 1.10 is another criterion for $\ker(\pi_\mathcal{A}) = 0$ where we do not assume that the algebra $\mathcal{A}$ is regular. We need the following lemma.

**Lemma 6.11** Let $R$ be a commutative domain of essential finite type over a field of characteristic zero and $\{\mathcal{D}(R),_i\}_{i \geq 0}$ be the order filtration on $\mathcal{D}(R)$. Then

1. Multiplication by a nonzero element of $R$ preserves the order of differential operator.

2. The ring $\mathcal{D}(R)$ is a domain.

**Proof.** 1. We use an induction on the order of differential operators. Given $\delta \in \mathcal{D}(R),_i \setminus \mathcal{D}(R),_{i-1}$. We have to show that $r \delta, r' \delta \in \mathcal{D}(R),_i \setminus \mathcal{D}(R),_{i-1}$ for all $r \in \mathcal{D}(R)^\ast \setminus \{0\}$. This is obvious for $i = 0$ as $\mathcal{D}(R),_0 \simeq R$ is a domain. Suppose that $i > 0$ the result is true for all $i' < i$. Suppose that $r \delta \in \mathcal{D}(R),_{i-1}$ for some $r \not\in \{0\}$, we seek a contradiction. Fix $r'$ such that $\delta' = [r', \delta] \in \mathcal{D}(R),_{i-1} \setminus \mathcal{D}(R),_{i-2}$. Then by induction

$$\mathcal{D}(R),_{i-1} \setminus \mathcal{D}(R),_{i-2} \ni r \delta' = [r', r \delta] \in \mathcal{D}(R),_{i-2},$$

(resp., $\mathcal{D}(R),_{i-1} \setminus \mathcal{D}(R),_{i-2} \ni \delta' r = [\delta' r, r] \in \mathcal{D}(R),_{i-2}$), a contradiction.

2. Take a nonzero element $s$ of the Jacobian ideal $\mathcal{A}_s$ of $\mathcal{A}$. By statement 1, $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})_s \simeq \mathcal{D}(\mathcal{A}_s)$. The algebra $\mathcal{A}_s$ is a regular domain of essentially finite type, hence the algebra $\mathcal{D}(\mathcal{A}_s)$ is a domain and so its subalgebra $\mathcal{D}(\mathcal{A})$. $\square$

**Proof of Theorem 1.10** (1 $\Rightarrow$ 2) If statement 1 holds then $\mathcal{U}(\mathcal{A}) = P\mathcal{D}(\mathcal{A})$. Then the implication follows from the inclusion $\kappa_\mathcal{A} \subseteq \ker(\pi_\mathcal{A})$ and the fact that $\mathcal{U}(\mathcal{A}) = P\mathcal{D}(\mathcal{A})$ is a domain (Lemma 6.11, 2).

(2 $\Rightarrow$ 1) Since $\kappa_\mathcal{A} = 0$, we have that $\overline{\mathcal{U}}(\mathcal{A}) = \mathcal{U}(\mathcal{A})$. Now, the implication follows from Theorem 6.7, 2 and the assumption that the algebra $\mathcal{U}(\mathcal{A}) = \overline{\mathcal{U}}(\mathcal{A})$ is a domain: $\ker(\pi_\mathcal{A}) = \ker(\overline{\pi}_\mathcal{A}) = T_{\overline{\mathcal{U}}(\mathcal{A})} = 0$.

(2 $\Leftrightarrow$ 3) If $\kappa_\mathcal{A} = 0$ and $\mathcal{U}(\mathcal{A})$ is a domain then $\ker(\pi_\mathcal{A}) = 0$ (since (1 $\Leftrightarrow$ 2)), i.e. $\mathcal{U}(\mathcal{A}) \simeq P\mathcal{D}(\mathcal{A})$. By Theorem 1.3 and Proposition 6.3, we have that $d = \text{GK}(\mathcal{A}) = n - r$.

Suppose that $d = n - r$ and $\mathcal{U}(\mathcal{A})$ is a domain. Then $\kappa_\mathcal{A} = 0$ iff all the elements $\delta_{r',i',j'}$ (in statement 3) belong to the $\mathcal{A}$-module $G_\mathcal{A}$, by Theorem 4.4, $\ker(\pi_\mathcal{A}) = 0$ for all $\delta_{r',i',j'}$, by Proposition 6.10, (since $\Omega'_\mathcal{A} = 0$). $\square$

**Criterion for the homomorphism** $\pi_\mathcal{A} : \mathcal{U}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ to be an isomorphism. For a regular domain of essentially finite type $\mathcal{A}$, Theorem 1.11 is a criterion for the homomorphism
\[ \pi_A : \mathcal{U}(A) \to \mathcal{D}(A) \] to be an isomorphism.

**Proof of Theorem 1.11** (1 \( \Rightarrow \) 2) Suppose that the homomorphism \( \pi_A : \mathcal{U}(A) \simeq \mathcal{D}(A) \) is an isomorphism. Then \( PD(A) = D(A) \) and the epimorphism \( \pi_A : \mathcal{U}(A) \to PD(A) \) is an isomorphism. By Theorem 1.3, it suffices to show that \( \text{Der}_K(A) = \mathcal{A}_H(A) \). The equality follows from the Claim.

**Claim.** For all \( i \geq 1 \), \( PD(A)_{\leq i} = PD(A)_i \).

Indeed, for \( i = 1 \), \( PD(A)_{\leq 1} = A \oplus \mathcal{A}_H(A) \) and \( PD(A)_i = A \oplus PD(A) \cap \text{Der}_K(A) = A \oplus D(A) \cap \text{Der}_K(A) = A \oplus \text{Der}_K(A) \), so the equality in the Claim for \( i = 1 \) yields the equality \( \mathcal{A}_H(A) = \text{Der}_K(A) \).

**Proof of the Claim.** Recall that \( PD(A)_{\leq 1} \subseteq PD(A)_i \) for all \( i \geq 1 \).

Since the algebra \( A \) is a regular domain, the algebra \( D(A) = \mathcal{U}(A) = PD(A) \) is a simple Noetherian domain. By Theorem 1.6, the algebra \( \text{gr} \mathcal{U}(A) = \bigoplus_{i \geq 0} PD(A)_{\leq i}/PD(A)_{\leq i-1} \) is a domain. By Theorem 6.4 (2), there are natural numbers \( s_i \) such that

\[ e_{A,d}^i PD(A)_i \subseteq PD(A)_{\leq i} \] for all \( i \geq 0 \).

Since \( D(A) \) is a domain, the degree of a differential operators is preserved by the multiplication by nonzero element of \( A \), and the Claim follows (since \( D(A)_i \cap D(A)_{i-1} \subseteq PD(A)_{\leq i} \cap PD(A)_{\leq i-1} \), by the inclusions above and the fact that the algebra \( \text{gr} \mathcal{U}(A) \) is a domain).

(2 \( \Rightarrow \) 1) By Theorem 6.4 (3), \( \ker(\pi_A) = \kappa_A \), i.e. \( \mathcal{U}(A) = \mathcal{U}(A) \). Since \( d = n - r \) and \( e_{A,d} = A \), \( \ker(\pi_A) = 0 \), by Theorem 6.4 and so

\[ \mathcal{U}(A) \simeq PD(A) \].

Since \( d = n - r \) and \( e_{A,d} = A \), \( \text{Der}_K(A) = \mathcal{A}_H(A) \), by Theorem 1.3. The algebra \( A \) is a regular algebra, hence the algebra \( D(A) \) is generated by \( A \) and \( \text{Der}_K(A) \), and so \( D(A) = PD(A) \). \( \square \)

### 7 Simplicity criteria for the algebras \( \mathcal{U}(\mathcal{P}) \) and \( PD(\mathcal{P}) \)

The aim of this section is to give/prove several simplicity criteria for the algebras \( PD(\mathcal{P}) \) (Theorem 1.11) and \( \mathcal{U}(\mathcal{P}) \) (Theorem 1.12 and Theorem 1.13).

**Subalgebras \( \Lambda \) of \( D(R) \) such that \( R \subset \Lambda \).** Let \( R \) commutative \( K \)-algebra over an arbitrary field \( K \). Let \( \Lambda \) be a subalgebra of the algebra \( D(R) \) of differential operators on \( R \) such that properly contains \( R \) (i.e. \( R \subset \Lambda \) and \( R \neq \Lambda \)). Then \( \Lambda = \bigcup_{i \geq 0} \Lambda_i \) is a filtered algebra where \( \{ \Lambda_i := \Lambda \cap D(R)_{i+1} \} \) is the filtration that is induced by the order filtration of the algebra \( D(R) \). We call this filtration the **order filtration** on \( \Lambda \). The algebra \( \Lambda \) is an \( R \)-bimodule.

**Lemma 7.1** Let \( \Lambda \) and \( D(R) \) be as above. Then \( \Lambda_1 = R \oplus \Gamma \) where \( \Gamma := \Lambda \cap \text{Der}_K(R) \) is a left \( R \)-submodule and a Lie subalgebra of \( \text{Der}_K(R) \).

**Proof.** Since \( R \neq \Lambda \), we must have \( \Lambda_1 \cap \Lambda \neq \emptyset \). Since \( D(R)_1 = R \oplus \text{Der}_K(R) \) and \( R \subseteq \Lambda \),

\[ \Lambda_1 = \Lambda \cap D(R)_1 = \Lambda \cap (R \oplus \text{Der}_K(R)) = R \oplus \Lambda \cap \text{Der}_K(R) = R \oplus \Gamma \]

and \( \Gamma \) is a nonzero left \( R \)-submodule and a Lie subalgebra of \( \text{Der}_K(R) \). \( \square \)

**Definition.** The subalgebra \( \Delta(\Lambda) = \langle R, \Gamma \rangle \) of \( \Lambda \) is called the **derivation subalgebra** of \( \Lambda \) where \( \Gamma = \Lambda \cap \text{Der}_K(R) \).

**Theorem 7.2** Let \( R \) be a commutative \( K \)-algebra, \( K \) be an arbitrary field, \( \Lambda \) be a subalgebra of the algebra \( D(R) \) of differential operators on \( R \) that properly contains \( R \). If \( I \) is a nonzero ideal of \( \Lambda \) then the intersection \( I \cap R \) is a nonzero \( \Gamma \)-stable ideal of \( R \) where \( \Gamma = \Lambda \cap \text{Der}_K(R) \) (i.e. \( \Gamma(I \cap R) \subseteq I \cap R \), the elements of \( \Gamma \) act as derivations) such that \( \Lambda(I \cap R) \Lambda \cap R = I \cap R \).
Proof. Recall that $\Gamma$ is a nonzero left $R$-module and a Lie subalgebra of $\mathrm{Der}_K(R)$ (Lemma 7.1).

(i) $I \cap R \neq 0$: Notice that $I = \bigcup_{t \geq 0} I_t$ where $I_t = \bigcap_{\gamma \in \Gamma} R \cap (\bigcap_{t \geq 0} R)$. Let $s = \min\{t \geq 0 \mid I_t \neq 0\}$. Choose a nonzero element, say $v$, of $I_s$. Then, for all elements $r \in R$, $[a, v] \in I_{s-1} = 0$, i.e. $v \in R$, as required.

(ii) $I \cap R$ is a $\Gamma$-stable ideal of the algebra $R$: For all elements $u \in I$ and $\gamma \in \Gamma$, $I \ni [\gamma, u] = \gamma(u)$, and the statement (ii) follows.

(iii) $\Lambda(I \cap R) \Lambda \cap R = I \cap R$: $I \cap R \subseteq \Lambda(I \cap R) \Lambda \cap R \subseteq I \cap R$ and the statement (iii) follows. □

Definition. Let $\Gamma'$ be a set of $K$-linear maps from $R$ to $R$. We say that the algebra $R$ is $\Gamma'$-simple if $\{0\}$ and $R$ are the only $\Gamma'$-stable ideals of $R$.

Simplicity criterion for the algebra $PD(\mathcal{P})$ of Poisson differential operators on $\mathcal{P}$.

Proof of Theorem 1.1 Without loss of generality we may assume that $\mathcal{H}_\mathcal{P} \neq 0$ (since otherwise the theorem is obvious as $PD(\mathcal{P}) = \mathcal{P}$).

(i) The algebra $PD(\mathcal{P})$ is not simple $\Rightarrow$ the Poisson algebra $\mathcal{P}$ is not simple: The algebra $PD(\mathcal{P})$ is a subalgebra of $D(\mathcal{P})$ that properly contains the algebra $\mathcal{P}$ (since $\mathcal{H}_\mathcal{P} \neq 0$). Suppose that $I$ is a proper ideal of $PD(\mathcal{P})$. Then, by Theorem 7.2 $I \cap \mathcal{P}$ is a proper $\Gamma$-stable ideal of $\mathcal{P}$ where $\Gamma = PD(\mathcal{P}) \cap \mathrm{Der}_K(\mathcal{P}) \supseteq \mathcal{H}_\mathcal{P} = \{\mathrm{pad}_a = \{a\} \mid a \in \mathcal{P}\}$. So, the intersection $I \cap \mathcal{P}$ is a proper $\mathcal{H}_\mathcal{P}$-stable ideal of $\mathcal{P}$, i.e. the Poisson algebra $\mathcal{P}$ is not simple.

(ii) The Poisson algebra $\mathcal{P}$ is not simple $\Rightarrow$ the algebra $PD(\mathcal{P})$ is not a simple algebra: If $J$ is a proper Poisson ideal of $\mathcal{P}$ then $J \cdot PD(\mathcal{P})$ is a proper ideal of $PD(\mathcal{P})$ (since $\mathcal{H}_\mathcal{P}(J) \subseteq J$). So, statements 1 and 2 are equivalent. □

Corollary 7.3 Let $\mathcal{P}$ be a Noetherian Poisson algebra over a field $K$ of characteristic zero. If the algebra $PD(\mathcal{P})$ is a simple algebra (or) the Poisson algebra $\mathcal{P}$ is a Poisson simple algebra, Theorem 7.1 then the algebra $\mathcal{P}$ is a domain.

Proof. Since the minimal primes of a Noetherian algebra over a field of characteristic zero are derivation-stable and the Poisson algebra $\mathcal{P}$ is Poisson simple, the Poisson algebra $\mathcal{P}$ must be a domain. □

Simplicity criteria for the Poisson enveloping algebra $U(\mathcal{P})$.

Proof of Theorem 1.2 (1 $\Rightarrow$ 2) The Poisson algebra $\mathcal{P}$ is a left $U(\mathcal{P})$-module. Let $a$ be the kernel of the algebra epimorphism $U(\mathcal{P}) \rightarrow PD(\mathcal{P})$, $p \mapsto p$, $\delta_q \mapsto \mathrm{pad}_q$ for all $p, q \in \mathcal{P}$. Now, the implication is obvious.

(2 $\Rightarrow$ 1) The implication is obvious.

(2 $\Leftrightarrow$ 3) The equivalence follows from Theorem 1.1. □

In the case when the Poisson algebra $\mathcal{P} = \mathcal{A}$ is an algebra of essentially finite type over a field of characteristic zero, Theorem 1.2 can be strengthened.

Proof of Theorem 1.3 (1 $\Rightarrow$ 2) By Theorem 1.2 the algebra $PD(\mathcal{A})$ is a simple algebra and $U(\mathcal{A}) \cong PD(\mathcal{A})$. The algebra $\mathcal{A}$ is of essentially finite type (hence Noetherian) of characteristic zero. Hence, the Jacobian ideal $\mathcal{a}_r$ and all the minimal primes of the algebra $\mathcal{A}$ are $\mathrm{Der}_K(\mathcal{A})$-stable ideals (Theorem 5) and Theorem 1), respectively. Since the algebra $U(\mathcal{A}) \cong PD(\mathcal{A})$ is simple, the algebra $\mathcal{A}$ is a regular domain. Theorem 1.9 is a criterion for $U(\mathcal{A}) \cong PD(\mathcal{A})$ when the algebra $\mathcal{A}$ is a regular domain, and statement 2 follows.

(1 $\Leftarrow$ 2 $\Leftrightarrow$ 3) The implications follow from Theorem 1.2 and Theorem 1.9. □

In particular, in the case when the algebra $\mathcal{A}$ is a regular domain Theorem 1.3 states that
the algebra $\mathcal{U}(\mathcal{A})$ is a simple algebra iff the Poisson algebra $\mathcal{A}$ is a Poisson simple algebra, $d = n - r$ and $(\partial_{i,j_n}, \partial_{i',j'}; j', j) = 0$ for all elements $i \in I_r, j \in J_r, i' \in I_{\mathcal{A}}(d), j' \in J_{\mathcal{A}}(d)$, $\nu = r + 1, \ldots, n$ and $\mu = d + 1, \ldots, n$ where for $j = \{j_1, \ldots, j_r\}$ and $i' = \{i'_1, \ldots, i'_d\}$ we have that $\{j_{r+1}, \ldots, j_n\} := \{1, \ldots, n\}\backslash\{j_1, \ldots, j_r\}$ and $\{i'_{d+1}, \ldots, i'_n\} := \{1, \ldots, n\}\backslash\{i'_1, \ldots, i'_d\}$.

So, Theorem 1.3 is an efficient tool in proving or disproving simplicity of the algebra $\mathcal{U}(\mathcal{A})$.

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