Group averaging for de Sitter free fields in terms of hyperspherical functions

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Abstract

We study the convergence of inner products of free fields over the homogeneous spaces of the de Sitter group and show that the convergence of inner products in the of $N$-particle states is defined by asymptotic behavior of the hypergeometric functions. We calculate the inner product for two-particle states on the four-dimensional hyperboloid in detail.

Keywords: de Sitter group, group averaging, inner product, hyperspherical function, homogeneous space.

1 Introduction

As is known, the procedure introduced by Dirac [1] for quantizing constrained systems is currently intensively studied in theoretical physics. In the Dirac approach, constraints are considered operators acting on the vectors of some Hilbert space and also conditions selecting “physical” states, from which a physical Hilbert space is then formed. But this quantization procedure contains a number of unsolved problems related to the structure of the Hilbert space of physical states and to the definition of inner products for these states. Some progress in this area has been achieved in such approaches as the BRST-method [2], the method of geometric quantization [3], coherent state quantization [4], $C^*$-algebra methods [5], algebraic quantization [6], and refined algebraic quantization (RAQ) [7, 8], related with the Rieffel induction [9, 10] and other works (see e.g., [12]). In the framework of RAQ, the inner product of states is defined using the technique of group averaging. Group averaging uses the integral

$$\int_G \langle \phi_1 | U(g) | \phi_2 \rangle dg$$

over the gauge group $G$, where $dg$ is a so-called symmetric Haar measure on $G$, $U(g)$ is a representation of $G$, and $\phi_1$ and $\phi_2$ are state vectors from an auxiliary Hilbert space $\mathcal{H}_{aux}$. Convergent group averaging gives an algorithm for construction of a complete set of observables of a quantum system [11, 13, 14, 15].
Here, we study inner products of free fields of any spin over homogeneous spaces of the de Sitter group $\text{SO}_0(1,4)$. The key point in studying the convergence of group averaging is the method for defining matrix elements of representations $U(g)$ of $\text{SO}_0(1,4)$ using an additional theorem for generalized spherical functions previously developed \textsuperscript{16} \textsuperscript{17} \textsuperscript{18} \textsuperscript{19}. The main advantage of this way of defining the matrix elements is their explicit factorization (according to the Cartan decomposition) with respect to the subgroups of the original group. Hence, for the group $\text{SO}_0(1,4)$, the matrix elements can be factorized with respect to both $\text{SO}(4)$ (a maximal compact subgroup) and $\text{SO}_0(1,3)$ (Lorentz group). This factorization allows segregating the variables in the integral that defines the group averaging for the inner product, i.e., calculating the integrals over compact and noncompact subgroups separately. As an example, we calculate the inner product in the two-particle case on the four-dimensional hyperboloid in detail. We show that the convergence of inner products is defined by the asymptotic behavior of the hypergeometric functions.

2 Group averaging and inner products over the de Sitter group

The main idea in the Dirac approach is to impose the additional conditions

$$\hat{\Lambda}_a^+ | \Psi \rangle = 0, \quad a = 1, M,$$

on the wave function $| \Psi \rangle$, where $\hat{\Lambda}_a^+$ are quantum analogues of the constraints. The commutation relations

$$[\hat{\Lambda}_a^+, \hat{\Lambda}_b^+] = i(U^c_{ab})^\dagger \hat{\Lambda}_c^+$$

then hold, where operators $U^c_{ab}$ usually called structure functions. The most difficult problem of the Dirac approach is to construct the inner product because $\Psi(q)$ are probability distributions and not square-integrable functions.

An alternative method in quantum theory is the RAQ. The essence of this method is the introduction of arbitrary functions $\Phi(q) \in \mathcal{H}_{aux}$ called auxiliary state vectors, whose inner product is given by the formula $(\Phi, \eta \Phi)$, where the map $\eta$ is such that $| \Psi \rangle = \eta | \Phi \rangle$. The Dirac states $| \Psi \rangle$ form the physical Hilbert space $\mathcal{H}_{phys}$. In the case of a general closed algebra, the inner product is expressed via the integral over the gauge group $G$ using the group averaging formula

$$\int d_R g (\det \text{Ad} \{ g \})^{-1/2} (\Phi | \hat{T}(g) | \Phi),$$

where $d_R g$ is a right-invariant Haar measure on the group $G$ and $\hat{T}(g)$ is a representation of $G$. In the case of $\text{SO}_0(1,4)$, we have the inner product

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{g \in G} dg \langle \Phi_1 | U(g) | \Phi_2 \rangle.$$  

(1)
The convergence of this integral is completely defined by the matrix elements of irreducible unitary representations of SO\(_0(1,4)\).

It was shown in [16] that the matrix elements and spherical functions of irreducible representations of the de Sitter group SO\(_0(1,4)\) form a universal covering Spin\(_4(1,4)\) ∼ Sp(1,1) of SO\(_0(1,4)\). Spherical functions on SO\(_0(1,4)\) are understood as functions of class-1 representations realized on homogeneous spaces of SO\(_0(1,4)\). A list of homogeneous spaces of SO\(_0(1,4)\) including symmetric Riemannian and non-Riemannian spaces was given in [16]. Matrix elements realized on the SO\(_0(1,4)\) group manifold \(\mathcal{S}_10\) have the form

\[ \mathcal{M}^l_{mn}(q) = e^{-im\varphi^q} \mathfrak{J}^l_{mn}(\cos \theta^q) e^{-in\psi^q}, \]

where \(l = 0,1/2,1,\ldots\), and \(-l \leq m,n \leq l\). The hyperspherical function \(\mathfrak{J}^l_{mn}(\cos \theta^q)\) is expressed as a series in products of three hypergeometric functions

\[ \mathfrak{J}^l_{mn}(\cos \theta^q) = \frac{\Gamma(l + m + 1)\Gamma(l - n + 1)}{\Gamma(l - m + 1)\Gamma(l + n + 1)} \cos^l \frac{\theta^q}{2} \cos^m \frac{\phi^q}{2} \cosh^l \frac{\tau}{2} \times \]

\[ \sum_{k=-l}^{l} \sum_{t=-l}^{l} i^{m-k} \tan^{m-k} \frac{\theta^q}{2} \tan^{t-k} \frac{\phi^q}{2} \tanh^{k-n} \frac{\tau}{2} \mathbf{F}_1 \left( \begin{array}{c} m-l,-t-l \\ m-t+1 \end{array} \left| -\tan^2 \frac{\theta^q}{2} \right) \right) \]

\[ \mathbf{F}_1 \left( \begin{array}{c} t-l,-k-l \\ t-k+1 \end{array} \left| -\tan^2 \frac{\phi^q}{2} \right) \right) \mathbf{F}_1 \left( \begin{array}{c} k-l,-n-l \\ k-n+1 \end{array} \left| \tanh^2 \frac{\tau}{2} \right) \right) \]

for \(m \geq t, t \geq k, k \geq n\). There also exist seven expressions of the hypergeometric type for the functions \(\mathfrak{J}^l_{mn}(\cos \theta^q)\) with the index values \(m \geq t, k \geq t, k \geq n, t \geq m, t \geq k, n \geq k, t \geq m, k \geq t, k \geq n, t \geq m, t \geq k, n \geq k, m \geq t, t \geq k, n \geq k;\) and \(m \geq t, k \geq t, n \geq k\).

### 3 Convergence of inner products

Returning to the group-averaging formula, we see that the convergence of the inner product

\[ \langle \Psi_1 | \Psi_2 \rangle := \int_{g \in G} dg(\phi_1 | U(g) | \phi_2) \]

depends on the matrix elements of irreducible representations \(U(g)\) of the de Sitter group SO\(_0(1,4)\). It was previously shown in [16] that such matrix elements are defined via the hyperspherical function of the form

\[ \mathcal{M}^l_{mn}(q) = e^{-i(m\varphi^q+n\psi^q)} \sum_{k=-l}^{l} \sum_{t=-l}^{l} P^l_{mn}(\cos \phi) P^l_{kt}(\cos \theta) \mathfrak{P}^l_{tn}(\cosh \tau), \]

where \(P^l_{mn}(\cos \phi)\), \(P^l_{kt}(\cos \theta)\) are spherical functions on the subgroup SU(2) and \(\mathfrak{P}^l_{tn}(\cosh \tau)\) is a spherical function on the subgroup SU(1,1). This expression
follows directly from the Cartan decomposition $U(g) = A^q K^q A^q$, where $K^q = \text{SU}(2) \otimes \text{SU}(2)$ and $A^q$ are maximal compact and commutative subgroups of $\text{Sp}(1,1)$. With this expression taken into account, group averaging inner product on the group manifold $S_{10}$ becomes

$$\langle \Psi_1 | \Psi_2 \rangle := \int_0^\infty d\tau d\epsilon d\omega \sin \theta^q e^{-m\epsilon-n(\epsilon+\omega)} \langle \psi_1 | P_0 K^q P_0 | \psi_2 \rangle,$$

where $P_0$ are projectors on $\text{SU}(2)$-invariant states. Because $\text{SU}(2)$ is a compact group, the operators $P_0$ do not affect the convergence properties of the group-averaging inner product. For $N$-particle states in which each particle occupies a definite mode, we have

$$\langle \Psi_1 | \Psi_2 \rangle := \int_0^\infty d\tau d\epsilon d\omega \sin \theta^q e^{-m\epsilon-n(\epsilon+\omega)} \mathcal{P}^{l}_{m_1,n_1}(\cosh \tau) \cdots \mathcal{P}^{l}_{m_N,n_N}(\cosh \tau).$$

The convergence of this integral follows from asymptotic behavior of the hypergeometric type functions

$$\mathcal{P}^{l}_{mn}(\cosh \tau) = \frac{1}{\Gamma(l-m-n+1)\Gamma(l+m+1)} \times \cosh^{m+n} \tau \sinh^{m-n} \frac{\tau}{2} 2^{l-m-n} \left( \begin{array}{c} l + m + 1, m - l \\ m - n + 1 \end{array} \right) - \sinh^2 \frac{\tau}{2} F_1 \left( \begin{array}{c} l + m + 1, m - l \\ m - n + 1 \end{array} \right) - \sinh^2 \frac{\tau}{2}. \quad (2)$$

As an example, we consider the inner product of two-particle states on the homogeneous space $\mathcal{M}_4 = \text{SO}_0(1,4)/\text{SO}(4)$. This space is homeomorphic to a two-sheeted four-dimensional hyperboloid $H^4$. We note that a four-dimensional Lobatchevski space $\mathcal{L}^4$, also called a de Sitter space, is realized on the hyperboloid $H^4$. Spherical functions defined on the homogeneous space $\mathcal{M}_4 = H^4 \sim \text{SO}_0(1,4)/\text{SO}(4)$, i.e., on the upper sheet of the hyperboloid $x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = 1$, have the form [16]

$$\mathcal{M}^{l}_{mn}(\epsilon, \tau, \omega) = e^{-m\epsilon} \mathcal{P}^{l}_{mn}(\cosh \tau) e^{-n(\epsilon+\omega)}.$$

Hence, in the two-particle case, we have the inner product

$$\langle \Psi_1 | \Psi_2 \rangle = \int_0^\infty d\tau d\epsilon d\omega \sinh \epsilon e^{-m\epsilon-n(\epsilon+\omega)} \mathcal{P}^{l}_{m_1,n_1}(\cosh \tau) \mathcal{P}^{l}_{m_2,n_2}(\cosh \tau).$$

It is obvious that the integral

$$I_1 = \int_0^\infty d\epsilon d\omega e^{-m\epsilon-n(\epsilon+\omega)}$$

is finite.
converges. We calculate the integral

$$I_2 = \int_0^\infty \sinh \tau \mathcal{P}_{m_1 n_1}^l (\cosh \tau) \mathcal{P}_{m_2 n_2}^l (\cosh \tau) d\tau. \quad (3)$$

Using (2), we express the functions \( \mathcal{P}_{m_1 n_1}^l (\cosh \tau) \) and \( \mathcal{P}_{m_2 n_2}^l (\cosh \tau) \) in terms of the hypergeometric functions. Then

$$I_2 = \frac{1}{\Gamma(m_1 - n_1 + 1) \Gamma(m_2 - n_2 + 1)} \times$$

$$\frac{\Gamma(l - n_1 + 1) \Gamma(l + m_1 + 1) \Gamma(l - n_2 + 1) \Gamma(l + m_2 + 1) \Gamma(l - m_1 + 1) \Gamma(l + n_1 + 1) \Gamma(l - m_2 + 1) \Gamma(l + n_2 + 1)}{\sqrt{\Gamma(l - m_1 + 1) \Gamma(l + m_1 + 1) \Gamma(l - n_2 + 1) \Gamma(l + n_2 + 1)}} \times$$

$$\int_0^\infty \cosh^{m_1 + m_2 + n_1 + n_2} \tau \sinh^{m_1 + m_2 - n_1 - n_2} \tau \times$$

$$\frac{\Gamma(l + m_1 + 1, m_1 - l)}{\Gamma(m_1 - n_1 + 1)} \times$$

$$\sum_{s=0}^{l-m_1} (-1)^s \frac{\Gamma(l + m_1 + s + 1) \sinh^2 \tau \Gamma(s + 1) \Gamma(m_1 - n_1 + s + 1) \Gamma(l - m_1 - s + 1)}{\Gamma(s + 1) \Gamma(m_1 - n_1 + s + 1) \Gamma(l - m_1 - s + 1)}. \quad (4)$$

The first function \( \mathcal{P}_{m_1 n_1}^l (\cosh \tau) \) can be written as

$$\mathcal{P}_{m_1 n_1}^l (\cosh \tau) = \sqrt{\frac{\Gamma(l - m_1 + 1) \Gamma(l - n_1 + 1)}{\Gamma(l + m_1 + 1) \Gamma(l + n_1 + 1)}} \cosh^{m_1 + n_1} \frac{\tau}{2} \times$$

$$\sinh^{m_1 - n_1} \frac{\tau}{2} \sum_{s=0}^{l-m_1} \frac{(-1)^s \Gamma(l + m_1 + s + 1) \sinh^2 \tau}{\Gamma(s + 1) \Gamma(m_1 - n_1 + s + 1) \Gamma(l - m_1 - s + 1)}. \quad (5)$$

Taking (5) into account, we rewrite the integral (4) as

$$I_2 = \frac{1}{\Gamma(m_2 - n_2 + 1)} \times$$

$$\frac{\Gamma(l - m_1 + 1) \Gamma(l - n_1 + 1) \Gamma(l - n_2 + 1) \Gamma(l + m_2 + 1) \Gamma(l + m_1 + 1) \Gamma(l + n_1 + 1) \Gamma(l - m_2 + 1) \Gamma(l + n_2 + 1)}{\sqrt{\Gamma(l - m_1 + 1) \Gamma(l + m_1 + 1) \Gamma(l - n_2 + 1) \Gamma(l + n_2 + 1)}} \times$$

$$\sum_{s=0}^{l-m_1} \frac{(-1)^s \Gamma(l + m_1 + s + 1)}{\Gamma(s + 1) \Gamma(m_1 - n_1 + s + 1) \Gamma(l - m_1 - s + 1)}. \quad (4)$$
\[
\int_0^\infty \cosh^{m_1+m_2+n_1+n_2} \tau \sinh^{m_1+m_2-n_1-n_2+2s} \tau \times \\
 F_1 \left( \begin{array}{c} l + m_2 + 1, m_2 - l \\ m_2 - n_2 + 1 \end{array} \bigg| -\sinh^2 \frac{\tau}{2} \right) \sinh \tau d\tau. \tag{6}
\]

Substituting \( z = \cosh \tau \) in the integral

\[
I_3 = \int_0^\infty \cosh^{m_1+m_2+n_1+n_2} \tau \times \\
\sinh^{m_1+m_2-n_1-n_2+2s} \tau F_1 \left( \begin{array}{c} l + m_2 + 1, m_2 - l \\ m_2 - n_2 + 1 \end{array} \bigg| -\sinh^2 \frac{\tau}{2} \right) \sinh \tau d\tau
\]

we obtain

\[
I_3 = \int_1^\infty \left( \frac{z^2-1}{4} \right)^{m_1+m_2} \left( \frac{z+1}{z-1} \right)^{n_1+n_2} \\
\left( \frac{z-1}{2} \right)^s F_1 \left( \begin{array}{c} l + m_2 + 1, m_2 - l \\ m_2 - n_2 + 1 \end{array} \bigg| -\frac{z-1}{2} \right) dz.
\]

Further, introducing the new variable \( t = -(z-1)/2 \), we obtain

\[
I_3 = (-1)^{m_1+m_2+n_1+n_2+s+1} \\
\int_0^\infty (1-t)^{m_1+m_2+n_1+n_2} t^{m_1+m_2+s-n_1-n_2} F_1 \left( \begin{array}{c} l + m_2 + 1, m_2 - l \\ m_2 - n_2 + 1 \end{array} \bigg| t \right) dt.
\]

Decomposing \((1-t)^{m_1+m_2+n_1+n_2}\) according to the Newton binomial formula, we obtain

\[
I_3 = \sum_{p=0}^{m_1+m_2+n_1+n_2} (-1)^{m_1+m_2+n_1+n_2+s+p+1} \\
\frac{(m_1+m_2+n_1+n_2)!}{p!(m_1+m_2+n_1+n_2-p)!} \\
\int_0^\infty t^{m_1+m_2+s+p-n_1-n_2} F_1 \left( \begin{array}{c} l + m_2 + 1, m_2 - l \\ m_2 - n_2 + 1 \end{array} \bigg| t \right) dt.
\]
To calculate this integral, we use the formula [21]:

\[
I_4 = \int t^{n-2} F_1 \left( \begin{array}{c} a, b \\ c \end{array} \mid t \right) dt = n! \sum_{k=1}^{n+1} (1)^{k+1} \frac{(c-k) t^{n-k+1}}{(a-k+1)!(c-k)(b-k)} F_1 \left( \begin{array}{c} a-k, b-k \\ c-k \end{array} \mid t \right).
\]

Then

\[
I_3 = \sum_{p=0}^{m_1+m_2+n_1+n_2} \sum_{p=1}^{m_1+m_2+s+p-n_1-n_2+1} (-1)^{m_1+m_2+n_1+n_2+s+p+k} \times (m_1+m_2+s+p-n_1-n_2)! \times
\]

\[
\frac{(m_1+m_2+n_1+n_2)!(m_2-n_2-k+1)_k}{(l+m_1-k+2)!(l+m_2-n_2-k+1)_k} \times
\]

\[
l^{m_1+m_2+s+p-n_1-n_2-k+1} F_1 \left( \begin{array}{c} l+m_2-k+1, m_2-l-k \\ m_2-n_2-k+1 \end{array} \mid t \right).
\]

Taking (6) into account, we obtain the expression

\[
\langle \Psi_1 | \Psi_2 \rangle = \frac{1}{\Gamma(m_2-n_2+1)} \times
\]

\[
\frac{\Gamma(l-m_1+1)\Gamma(l-n_1+1)\Gamma(l-n_2+1)\Gamma(l+m_2+1)}{\sqrt{\Gamma(l+m_1+1)\Gamma(l+n_1+1)\Gamma(l-m_2+1)\Gamma(l+n_2+1)}} \times
\]

\[
\sum_{s=0}^{l-m_1} \sum_{p=0}^{m_1+m_2+n_1+n_2} \sum_{k=1}^{m_1+m_2+s+p-n_1-n_2+1} (-1)^{m_1+m_2+n_1+n_2+s+p+k} \times
\]

\[
\frac{\Gamma(l+m_1+s+1)(m_1+m_2+n_1+n_2)!}{\Gamma(s+1)\Gamma(m_1-n_1+1)\Gamma(l-m_1+s+1)\Gamma(l+m_2-k+2)!} \times
\]

\[
\frac{(m_2-n_2-k+1)_k}{(l+m_2-k+1)_k(m_2-l-k)_k} \times
\]

\[
l^{m_1+m_2+s+p-n_1-n_2-k+1} F_1 \left( \begin{array}{c} l+m_2-k+1, m_2-l-k \\ m_2-n_2-k+1 \end{array} \mid t \right)
\]

for the two-particle inner product. To investigate the convergence of \(\langle \Psi_1 | \Psi_2 \rangle\),
we apply the asymptotic expansion for the hypergeometric function \[20\]

\[
_2F_1\left(\frac{a, b}{c} \mid t\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-t)^{-a}_2F_1\left(\frac{a, 1-c+a}{1-b+a} \mid \frac{1}{t}\right) + \\
+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-t)^{-b}_2F_1\left(\frac{b, 1-c+b}{1-a+b} \mid \frac{1}{t}\right).
\]

Therefore,

\[
\langle \Psi_1 \mid \Psi_2 \rangle = \frac{1}{\Gamma(m_2 - n_2 + 1)} \times \\
\sqrt{\frac{\Gamma(l - m_1 + 1)\Gamma(l - n_1 + 1)\Gamma(l - n_2 + 1)\Gamma(l + m_2 + 1)}{\Gamma(l + m_1 + 1)\Gamma(l + n_1 + 1)\Gamma(l - m_2 + 1)\Gamma(l + n_2 + 1)}} \times \\
\sum_{s=0}^{l-m_1} \sum_{p=0}^{m_1+m_1+n_1+n_2} \sum_{k=1}^{m_1+m_2+s+p-n_1-n_2+1} (-1)^{m_1+m_2+n_1+n_2+p+k} \times \\
\frac{\Gamma(l + m_1 + s + 1)(m_1 + m_2 + n_1 + n_2)!}{\Gamma(s + 1)\Gamma(m_1 - n_1 + s + 1)\Gamma(l - m_1 - s + 1)(l + m_2 - k + 2)!} \times \\
\frac{(m_2 - n_2 - k + 1)k}{(l + m_2 - k + 1)k(m_2 - l - k)k} \times \\
\left[(-1)^{l+m-1} \frac{\Gamma(m_2 - n_2 - k - 1)\Gamma(-2l - 1)}{\Gamma(m_2 - l - k)\Gamma(-n_2 - l)} \times \\
t^{m_1+s+p-n_1-n_2-l}_2F_1\left(l + m_2 - k + 1, l + n_2 + 1 \mid \frac{1}{t}\right) + \\
(-1)^{m_2-l-k} \frac{\Gamma(m_2 - n_2 - k + 1)\Gamma(2l + 1)}{\Gamma(l + m_2 - k + 1)\Gamma(-n_2 - l)} \times \\
t^{m_1+l+s+p-n_1-n_2+l}_2F_1\left(m_2 - l - k, n_2 - l \mid \frac{1}{t}\right) \right].
\]

In this expression the hypergeometric function \(_2F_1\) can be written as a power series in \(1/t\). It hence follows that \(\langle \Psi_1 \mid \Psi_2 \rangle \sim t^{m_1+l+s+p-M-n_1+n_2+1}\), and because \(M \to \infty\), \(\langle \Psi_1 \mid \Psi_2 \rangle\) converges for \(M > m_1 + l + s + p - n_1 - n_2 + 1\).

### 4 Summary

We have presented an extended group-averaging method by determining the integrals giving the inner products of free fields on homogeneous spaces of the de
Sitter group $\text{SO}_0(1, 4)$. We considered $N$-particle case on the four-dimensional hyperboloid $H^4$. It would be interesting to consider inner products and also their convergence on other homogeneous spaces of $\text{SO}_0(1, 4)$ (both symmetric Riemannian and non-Riemannian), such as the three-dimensional real sphere $S^3$, the two-dimensional quaternion sphere $S^q_2$, and the group manifold $\mathcal{S}_{10}$ of $\text{SO}_0(1, 4)$. Our next paper will be devoted to these questions.

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