On some soliton equations in 2+1 dimensions and their 1+1 and/or
2+0 dimensional integrable reductions

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Abstract
Some soliton equations in 2+1 dimensions and their 1+1 and/or 2+0 dimensional
integrable reductions are considered.
1 Introduction

Consider the Myrzakulov IX (M-IX) equation

\[ iS_t + \frac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0 \]  \hspace{1cm} (1a)
\[ M_2u = \frac{\alpha^2}{2\iota} tr(S[S_x, S_y]) \]  \hfill (1b)

where \(\alpha, b, a = \text{consts and} \)

\[ S = \begin{pmatrix} S^+_3 & rS^- \\ rS^+ & -S^-_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2 \quad S^2 = EI, \quad r^2 = \pm 1, \]

\[ M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b-a)\frac{\partial^2}{\partial x\partial y} + 4(a^2 - 2ab - b)\frac{\partial^2}{\partial x^2}, \]

\[ M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a+1)\frac{\partial^2}{\partial x\partial y} + 4a(a+1)\frac{\partial^2}{\partial x^2}, \]

\[ A_1 = i\{(\alpha(2b+1)u_y - 2(2ab + a + b)u_x \}, \]

\[ A_2 = i\{4\alpha^{-1}(2a^2b + a^2 + 2ab + b)u_x - 2(2ab + a + b)u_y \}. \]

This equation was introduced in [1] and arises from the compatibility condition of the following linear equations

\[ \alpha \Phi_y = \frac{1}{2} [S + (2a+1)I] \Phi_x \quad (2a) \]

\[ \Phi_t = 2i[S + (2b+1)I] \Phi_{xx} + W \Phi_x \quad (2b) \]

with

\[ W = 2i\{(2b+1)(F^+ + F^- S) + (F^+ S + F^-) + (2b-a + \frac{1}{2})SS_x + \frac{1}{2}S_x + \alpha SS_y \}, \]

\[ F^\pm = A \pm D, \quad A = i[u_y - \frac{2a}{\alpha}u_x], \quad D = i\{\frac{2(a+1)}{\alpha}u_x - u_y \}. \]

It is well known that equation (1) is gauge and Lakshmanan equivalent to the following Zakharov equation (ZE)[2]

\[ iq_t + M_1q + vq = 0, \quad (3a) \]

\[ ip_t - M_1p - vp = 0, \quad (3b) \]

\[ M_2v = -2M_1(pq), \quad (3c) \]

The Lax representation of this equation has the form[2]

\[ \alpha \Psi_y = B_1 \Psi_x + B_0 \Psi, \quad (4a) \]

\[ \Psi_t = iC_2 \Psi_{xx} + C_1 \Psi_x + C_0 \Psi, \quad (4b) \]

with

\[ B_1 = \begin{pmatrix} a + 1 & 0 \\ 0 & a \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \]

\[ C_2 = \begin{pmatrix} b + 1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]
Here \( c_{jj} \) is the solution of the following equations

\[
(a + 1)c_{11}x - \alpha c_{11}y = i[(2b - a + 1)(pq)x + \alpha(pq)y],
\]

\[
ac_{22}x - \alpha c_{22}y = i[(a - 2b)(pq)x - \alpha(pq)y].
\]  

Below we consider the some reductions of the M-IX equation (1) and their gauge equivalent counterparts.

2 Some \( \sigma \)-models with potentials and the Klein-Gordon-type equations

First, let us consider the some \( \sigma \)-models with potentials, which are the some reductions of equation (1) in 1+1 or 2+0 dimensions.

2.1 The Myrzakulov XXXII (M-XXXII) equation

Suppose that now \( \nu = t \) is the some ”hidden” parameter, \( S = S(x, y), \ u = u(x, y), \ q = q(x, y), \ p = p(x, y), \ v = v(x, y) \) and at the same time \( \Phi = \Phi(x, y, \nu), \ \Psi = \Psi(x, y, \nu). \) Then equation (1) reduces to the following \( \sigma \)-model with potential

\[
M_1S + \{k_1S_x^2 + k_2S_xS_y + k_3S_y^2\}S + A_2SS_x + A_1SS_y = 0 \quad (6a)
\]

\[
M_2u = \frac{\alpha^2}{2i}tr(S[S_x, S_y]) \quad (6b)
\]

where \( M_1 \) we write in the form

\[
M_1 = k_3 \frac{\partial^2}{\partial y^2} + k_2 \frac{\partial^2}{\partial x \partial y} + k_1 \frac{\partial^2}{\partial x^2},
\]

which is the compatibility condition \( \Phi_{y\nu} = \Phi_{\nu y} \) of the set (2) and called the M-XXXII equation [1]. The G-equivalent and L-equivalent counterpart of this equation is given by

\[
M_1q + vq = 0, \quad M_1p + vp = 0, \quad M_2v = -2M_1(pq). \quad (7)
\]

This is the some modified complex Klein-Gordon equation (mKGE).

2.2 The Myrzakulov XV (M-XV) equation

Let us consider the case: \( a = b \). Then we obtain the M-XV equation [1]

\[
\alpha^2S_{yy} - a(a + 1)S_{xx} + \{\alpha^2S_y^2 - a(a + 1)S_x^2\}S + + A_2''SS_x + A_1''SS_y \quad (8a)
\]
\[ M_2 u = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \]  

(8b)

where \( A''_j = A_j \) as \( a = b \). The corresponding mKGE has the form

\[ \alpha^2 q_{yy} - a(a + 1)q_{xx} + vq = 0, \quad M_2 v = -2[\alpha^2(| q |^2)_{yy} - a(a + 1)(| q |^2)_{xx}] \]  

(9)

The Lax representations of these equations we obtain from (2) and (4) respectively as \( a = b \).

### 2.3 The Myrzakulov XIV (M-XIV) equation

Now we consider the reduction: \( a = -\frac{1}{2} \). Then the equation (1) reduces to the M-XIV equation[1]

\[ S_{xx} + 2\alpha(2b+1)S_{xy} + \alpha^2 S_{yy} + \{ S_{xx} + 2\alpha(2b+1)S_{xy} + \alpha^2 S_{yy} \} S + A'_2 SS_x + A'_1 SS_y = 0 \]  

(10a)

\[ \alpha^2 u_{yy} - u_{xx} = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \]  

(10b)

where \( A'_j = A_j \) as \( a = -\frac{1}{2} \). The corresponding gauge equivalent equation is obtained from (3) and looks like

\[ q_{xx} + 2\alpha(2b+1)q_{xy} + \alpha^2 q_{yy} + vq = 0 \]  

(11a)

\[ \alpha^2 v_{yy} - v_{xx} = -2\{\alpha^2(pq)_{yy} + 2\alpha(2b+1)(pq)_{xy} + (pq)_{xx}\} \]  

(11b)

From (2) and (4) we obtain the Lax representations of (10) and (11) respectively as \( a = -\frac{1}{2} \).

### 2.4 The Myrzakulov XIII (M-XIII) equation

Now let us consider the case: \( a = b = -\frac{1}{2} \). In this case the equations (1) reduce to the \( \sigma \)-model

\[ S_{xx} + \alpha^2 S_{yy} + \{ S^2_x + \alpha^2 S^2_y \} S + iu_y SS_x + iu_x SS_y = 0 \]  

(12a)

\[ \alpha^2 u_{yy} - u_{xx} = \frac{\alpha^2}{2i} tr(S[S_y, S_x]) \]  

(12b)

which is the M-XIII equation[1]. The equivalent counterpart of the equation(12) is the following equation

\[ q_{xx} + \alpha^2 q_{yy} + vq = 0 \]  

(13a)

\[ \alpha^2 v_{yy} - v_{xx} = -2\{\alpha^2(pq)_{yy} + (pq)_{xx}\} \]  

(13b)

that follows from the mKGE(7).
2.5 The Myrzakulov XII (M-XII) equation

Now let us consider the case: $a = b = -1$. In this case the equations (6) reduce to the $\sigma$-model - the M-XII equation[1]

$$S_{YY} + S_Y^2 S + iwSS_Y = 0$$  \hspace{1cm} (14a)

$$w_Y + w_X + \frac{1}{4i} tr(S[S_X, S_Y]) = 0$$  \hspace{1cm} (14b)

where $X = 2x$, $Y = \alpha y$, $w = -\frac{wY}{\alpha}$. The equivalent counterpart of the equation (14) is the mKGE

$$q_{YY} + vq = 0$$  \hspace{1cm} (15a)

$$v_X + v_Y + 2(pq)_Y = 0.$$  \hspace{1cm} (15b)

that follows from the (7).

2.6 The Myrzakulov XXXI (M-XXXI) equation

This $\sigma$-model equation is read as[1]

$$bS_{\eta\eta} - (b + 1)S_{\xi\xi} + \{bS_{\eta\eta} - (b + 1)S_{\xi\xi}\} S + i(b + 1)w_{\eta}SS_{\eta} + ibw_{\xi}SS_{\xi} = 0$$  \hspace{1cm} (16a)

$$w_{\eta\eta} = -\frac{1}{4i} tr(S[S_{\eta}, S_{\xi}])$$  \hspace{1cm} (16b)

which is the M-XXXI equation, where $w = -\alpha^{-1}u$. The equivalent mKGE looks like

$$(1 + b)q_{\xi\xi} - bq_{\eta\eta} + vq = 0$$  \hspace{1cm} (17a)

$$v_{\eta\eta} = -2\{((1 + b)(pq)_{\xi\xi} - b(pq)_{\eta\eta}\}$$  \hspace{1cm} (17b)

So we have found the G-equivalent counterparts of the $\sigma$-models with potentials.

3 The (2+1)-dimensional integrable spin systems and NLSE type equations

The equation (1) admits some interesting (2+1)-dimensional integrable reductions. Let us now consider these particular integrable cases.
3.1 The M-VIII equation

Let $b = 0$. Then the equations (1) take the form

$$iS_t = \frac{1}{2}[S_{\xi\xi}, S] + iwS_{\xi}$$

(18a)

$$w_{\eta} = \frac{1}{4i} tr(S[S_{\eta}, S_{\xi}])$$

(18b)

where

$$\xi = x + \frac{a + 1}{\alpha} y, \quad \eta = -x - \frac{a}{\alpha} y, \quad w = -\frac{1}{\alpha} u_{\xi}$$

which is the M-VIII equation[1]. The corresponding Lax representation has the form

$$\Phi_{Z^+} = S\Phi_{Z^-}$$

(19a)

$$\Phi_t = 2i[S + I]\Phi_{Z^-} - iK\Phi_{Z^-}$$

(19b)

where $Z^\pm = \xi \pm \eta$ and

$$K = E_0 + E_0S + S_{Z^-} + 2SS_{Z^+} + SS_{Z^+}, \quad E_0 = \frac{i}{\alpha}(u_{Z^+} + u_{Z^-}).$$

The gauge equivalent counterpart of the equation(18) we obtain from(3) as $b = 0$

$$iq_t + q_{\xi\xi} + vq = 0,$$

(20a)

$$v_{\eta} = -2r^2(\bar{q}q)_{\xi},$$

(20b)

which is the other Zakharov equation[2].

3.2 The Ishimori equation

Now let us consider the case: $a = b = -\frac{1}{2}$. In this case the equations(1) reduce to the well-known Ishimori equation

$$iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} + \alpha^2 S_{yy})] + iu_yS_x + iu_xS_y = 0$$

(21a)

$$\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i} tr(S[S_y, S_x]),$$

(21b)

The gauge equivalent counterpart of the equation(21) is the Davey-Stewartson equation

$$iq_t + \frac{1}{4}q_{xx} + \alpha^2 q_{yy} + vq = 0$$

(22a)

$$\alpha^2 v_{yy} - \frac{1}{4}v_{xx} = -2(\alpha^2 pq)_{yy} + \frac{1}{4}(pq)_{xx}$$

(22b)

that follows from the ZE(3). This fact was for first time established in [3]. The Lax representations of (21) and (22) we can get from (2) and (4) respectively as $a = b = -\frac{1}{2}$. 
3.3 The M-XVIII equation

Now we consider the reduction: \( a = -\frac{1}{2} \). Then the equation (1) reduces to the M-XVIII equation\[1\]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, \left( \frac{1}{4} S_{xx} - \alpha (2b + 1) S_{xy} + \alpha^2 S_{yy} \right)] + A'_2 S_x + A'_1 S_y &= 0 \tag{23a} \\
    \alpha^2 u_{yy} - \frac{1}{4} u_{xx} &= \frac{\alpha^2}{4i} tr(S[S_y, S_x]) \tag{23b}
\end{align*}
\]

where \( A'_j = A_j \) as \( a = -\frac{1}{2} \). The corresponding gauge equivalent equation is obtained from (3) and looks like

\[
\begin{align*}
    iq_t + \frac{1}{4} q_{xx} - \alpha (2b + 1) q_{xy} + \alpha^2 q_{yy} + v q &= 0 \tag{24a} \\
    \alpha^2 v_{yy} - \frac{1}{4} v_{xx} &= -2\{\alpha^2 (pq)_{yy} - \alpha (2b + 1)(pq)_{xy} + \frac{1}{4}(pq)_{xx}\} \tag{24b}
\end{align*}
\]

From (2) and (4) we obtain the Lax representations of (23) and (24) respectively as \( a = -\frac{1}{2} \).

3.4 The M-XIX equation

Let us consider the case: \( a = b \). Then we obtain the M-XIX equation \[1\]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, \alpha^2 S_{yy} - a(a + 1) S_{xx}] + A''_2 S_x + A''_1 S_y &= 0 \tag{25a} \\
    M_2 u &= -\frac{\alpha^2}{4i} tr(S[S_x, S_y]) \tag{25b}
\end{align*}
\]

where \( A''_j = A_j \) as \( a = b \). The corresponding NLSE has the form

\[
\begin{align*}
    iq_t + \alpha^2 q_{yy} - a(a + 1) q_{xx} + v q &= 0, \tag{26a} \\
    M_2 v &= -2[\alpha^2 (|q|^2)_{yy} - a(a + 1)(|q|^2)_{xx}] \tag{26b}
\end{align*}
\]

The Lax representations of these equations we obtain from (2) and (4) respectively as \( a = b \).

3.5 The M-XX equation

This equation is read as\[1\]

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, b S_{yy} - (b + 1) S_{xx}] + w_\eta S_\eta + w_\xi S_\xi &= 0 \tag{27a} \\
    w_{\eta \eta} &= -\frac{1}{4i} tr(S[S_\eta, S_\xi]) \tag{27b}
\end{align*}
\]
where \( w = -\alpha^{-1} u \). The associated linear problem is given by
\[
\Phi_{Z^+} = S \Phi_{Z^-} \tag{28a}
\]
\[
\Phi_t = 2i[S + (2b + 1)I] \Phi_{Z^-} + iW_0 \Phi_{Z^-} \tag{28b}
\]
where \( Z^\pm = \xi \pm \eta \) and
\[
W_0 = (2b + 1)(E + FS) + F + ES + \frac{1}{2}S_{Z^-} + 2(2b + 1)SS_{Z^-} + SS_{Z^+}
\]
\[
E = \frac{i}{\alpha} u_{Z^-}, \quad F = \frac{i}{\alpha} u_{Z^+}.
\]
The gauge equivalent equation looks like
\[
iq_t + (1 + b)q_{\xi \xi} - bq_{\eta \eta} + vq = 0 \tag{29a}
\]
\[
v_{\xi \eta} = -2\{(1 + b)(pq)_{\xi \xi} - b(pq)_{\eta \eta}\} \tag{29b}
\]
This equation is integrated by the linear problem
\[
f_{Z^+} = \sigma_3 f_{Z^-} + B_0 f \tag{30a}
\]
\[
f_t = 4iC_2 f_{Z^-} + 2C_1 f_{Z^-} + C_0 f. \tag{30b}
\]
where \( B_0, \ C_j \) are given as in (4).

Thus, we have presented the some reductions of the equation (1). All of these reductions are integrable in the sense that they admit the Lax representations.

4 The (2+1)-dimensional integrable spin system with anisotropy

4.1 Gauge equivalent counterpart of the anisotropic spin system

As integrable one, the anisotropic LLE can admits the several integrable (2+1)-dimensional extensions\[1\]. One of the such integrable (2+1)-dimensional extension of the LLE as \( J_1 = J_2 = 0, J_3 = \triangle \) is the following M-I equation with one-ion anisotropy
\[
S_t = (S \wedge S_y + uS)_x + vS \wedge n, \tag{31a}
\]
\[
u_x = -S \cdot (S_x \wedge S_y), \tag{31b}
\]
\[
v_x = \triangle(S_y \cdot n) \tag{31c}
\]
where \( u \) and \( v \) are scalar functions, \( n = (0,0,1) \), and \( \triangle < 0 \) and \( \triangle > 0 \) correspond respectively to the system with an easy plane and to that with an easy axis. Note that if the symmetry \( \partial_x = \partial_y \) is imposed then the M-I equation (31) reduces to the well-known LLE with single-site anisotropy
\[
S_t = S \wedge (S_{xx} + \triangle(S \cdot n)n) \tag{32}
\]
which is the particular case of the anisotropic LLE as $J_1 = J_2 = 0, J_3 = \triangle$. On the other hand, in the case when $\triangle = 0$, the equation (31) becomes the isotropic M-I equation. It is known that the equation (32) is gauge equivalent to the NLSE [4–7]. In this subsection we construct the NLSE which is gauge equivalent to the equation (31) with the easy-axis anisotropy ($\triangle > 0$).

The Lax representation of the equation (31) may be given by [6]

$$
\psi_x = L_1 \psi, \quad \psi_t = 2\lambda \psi_y + M_1 \psi
$$

where

$$
L_1 = i\lambda S + \mu [\sigma_3, S], \quad M_1 = 2\lambda A + 2i\mu [A, \sigma_3] + 4i\mu^2 \{\sigma_3, V\} \sigma_3
$$

with

$$
S = \sum_{k=1}^{3} S_k \sigma_k, \quad A = \frac{1}{4} ([S, S_y] + 2iuS), \quad \mu = \sqrt{\frac{\triangle}{4}}, \quad \triangle > 0.
$$

and

$$
V = \triangle \int_{-\infty}^{x} S_y dx.
$$

Here $\sigma_k$ is Pauli matrices, $[,]$ ($\{,\}$) denoting commutator (anticommutator), and $\lambda$ is a spectral parameter. The matrix $S$ has the following properties: $S^2 = I, \quad S^* = S, \quad tr S = 0$. The compatibility condition of system (33) $\psi_{xt} = \psi_{tx}$ gives the equation (31). Let us now consider the gauge transformation induced by $g(x,y,t) : \psi = g^{-1} \phi$, where $g^* = g^{-1} \in SU(2)$. It follows from the properties of the matrix $S$ that it can be represented in the form $S = g^{-1} \sigma_3 g$. The new gauge equivalent operators $L_2, M_2$ are given by

$$
L_2 = gL_1 g^{-1} + g_x g^{-1}, \quad M_2 = gM_1 g^{-1} + g_t g^{-1} - 2\lambda g_y g^{-1}
$$

and satisfy the following system of equations

$$
\phi_x = L_2 \phi, \quad \phi_t = 2\lambda \phi_y + M_2 \phi.
$$

Now choosing

$$
g_x g^{-1} + \mu [\sigma_3, S] g^{-1} = U_0, \quad gS g^{-1} = \sigma_3
$$

$$
g_t g^{-1} + 2i\mu [A, \sigma_3] g^{-1} + 4i\mu^2 g \{\sigma_3, V\} \sigma_3 g^{-1} = V_0
$$

with

$$
U_0 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad V_0 = i\sigma_3 (\partial_x |q|^2 - U_{0y})
$$

where $q(x,y,t)$ the new complex valued fields. Hence we finally obtain

$$
L_2 = i\lambda \sigma_3 + U_0, \quad M_2 = V_0
$$


The compatibility condition $\phi_{xt} = \phi_{tx}$ of the system (37) with the operators $L_2, M_2$ (39) leads to the (2+1)-dimensional NLSE\cite{2,8}

$$iq_t = q_{xy} + wq, \quad w_x = 2(|q|^2)_y.$$  \hspace{1cm} (40)

This equation under the reduction $\partial_y = \partial_x$ equation(40) becomes the well known (1+1)-dimensional NLSE. Thus we have shown that the M-I equation with single-site anisotropy(31) is gauge equivalent to the (2+1)-dimensional NLSE - the Zakharov equation(40).

4.2 The isotropic and anisotropic spin systems: gauge equivalence

It is known that the equations (40) is gauge and geometrical equivalent to the isotropic M-I equation

$$iS'_t = \frac{1}{2}([S', S'_y] + 2iu'S'_x)$$  \hspace{1cm} (41a)

$$u'_x = -\frac{1}{4i} tr(S'[S'_x, S'_y])$$  \hspace{1cm} (41b)

which was introduced in\cite{1} and arises from the compatibility condition of the linear problem

$$f_x = L'_1 f, \quad f_t = 2\lambda f_y + \lambda M'_1 f$$  \hspace{1cm} (42)

where

$$L'_1 = i\lambda S', \quad M'_1 = \frac{1}{2}([S', S'_y] + 2iu'S').$$  \hspace{1cm} (43)

Now we show that between the isotropic(41) and anisotropic(31) versions of the M-I equation the gauge equivalence takes place. Indeed the Lax representations, (33) and (42), which reproduce equations (31) and (41), respectively can be obtained from each other by the $\lambda$-independent gauge transformation

$$L'_1 = hL_1 h^{-1} + h_x h^{-1}, \quad M'_1 = hM_1 h^{-1} + h_t h^{-1}.$$  \hspace{1cm} (44)

with $h(x, y, t) = \psi^{-1}|_{\lambda=\lambda_0}$, where $\lambda_0$ is some fixed value of the spectral parameter $\lambda$.

So, the solutions of equations(31) and (41) are connected with each other by formulas $S = h^{-1}S'h$. Now we present the important relations between the field variables $q$ and $S$:

$$|q|^2 = \frac{1}{2}[S_x^2 - 8\mu S_{3x} + 16\mu^2(1 - S_2^2)]$$  \hspace{1cm} (45a)

$$\bar{q}_x q - \bar{q} q_x = \frac{i}{4}S \cdot [S_{xx} + 16\mu^2(S \cdot n)n] + 4\mu S \cdot (S_{xx} \wedge n)$$  \hspace{1cm} (45b)

These relations coincide with the corresponding connections between $q$ and $S$ from the one-dimensional case.
Note that the equation (31) with \(\triangle < 0\) easy plane single-site anisotropy is gauge equivalent to the following general (2+1)-dimensional NLSE[2,8]

\[
\begin{align*}
    iq_t &= q_{xy} + wq \quad (46a) \\
    ip_t &= -p_{xy} - wp \quad (46b) \\
    w_x &= 2(pq)_y. \quad (46c)
\end{align*}
\]

Besides, there can be shown that the equation (31), when \(S \in SU(1,1)/U(1)\), i.e. the non-compact group case, is gauge equivalent to the NLSE (46) with the repulsive interaction, \(p = -q\).

Finally, we note that the M-I equation (31) is the particular case of the M-III equation

\[
\begin{align*}
    \frac{d}{dt}(S \wedge \text{V}) &= (S \wedge S_y + uS)_x + 2b(cb + d)S_y - 4cvS_x + S \wedge \text{V} \quad (47a) \\
    u_x &= -S(S_x \wedge S_y), \quad (47b) \\
    v_x &= \frac{1}{4(2bc + d)^2}(S_x^2)_y \quad (47c) \\
    \text{V}_x &= JS_y. \quad (47d)
\end{align*}
\]

Note these equations admit the some integrable reductions: a) the isotropic M-I equation as \(c = J = 0\); b) the anisotropic M-I equation as \(c = J_1 = J_2 = 0\); c) the M-II equation as \(d = J_3 = 0\); d) the isotropic M-III equation as \(J = 0\) and so on[1].

## 5 Integrable spin-phonon systems and the Yajima-Oikawa and Ma equations

Let us now we consider the reduction of the M-IX equation (1) as \(a = b = -1\). We have

\[
\begin{align*}
    iS_t + \frac{1}{2}[S, S_{YY}] + iwS_Y &= 0 \quad (48a) \\
    w_X + w_Y + \frac{1}{4t}\text{tr}(S[S_X, S_Y]) &= 0 \quad (48b)
\end{align*}
\]

This equation is the Myrzakulov VIII (M-VIII) equation[1]. The G-equivalent and L-equivalent counterpart of equation (48) is given by

\[
\begin{align*}
    iq_t + q_{XY} + vq &= 0, \quad (49a) \\
    ip_t - p_{YY} - vp &= 0, \quad (49b) \\
    v_X + v_Y + 2(pq)_Y &= 0. \quad (49c)
\end{align*}
\]
Now let us take the case when $X = t$. Then the M-VIII equation (48) pass to the following M-XXXIV equation

$$
i S_t + \frac{1}{2} [S, S_{YY}] + iw S_Y = 0 \quad (50a)$$

$$w_t + w_Y + \frac{1}{4} \{ tr(S^2_Y) \}_Y = 0 \quad (50b)$$

The M-XXXIV equation (50) was proposed in [1] to describe a nonlinear dynamics of the compressible magnets (see, Appendix). It is integrable and has the different soliton solutions[1].

In our case equation (49) becomes

$$iq_t + q_{YY} + vq = 0, \quad (51a)$$

$$ip_t - p_{YY} - vp = 0, \quad (51b)$$

$$v_t + v_Y + 2(pq)_Y = 0. \quad (51c)$$

that is the Yajima-Oikawa equation(YOE)[9]. So we have proved that the M-XXXIV equation (50) and the YOE (51) is gauge equivalent to each other. The Lax representations of (50) and (51) we can get from (2) and (4) respectively as $a = b = -1$ (see, for example, the ref.[1]). Note that our Lax representation for the YOE (51) is different than that which was presented in [9].

Also we would like note that the M-VIII equation (48) we usually write in the following form

$$i S_t = \frac{1}{2} [S_{\xi\xi}, S] + iw S_{\xi} \quad (52a)$$

$$w_\eta = \frac{1}{4i} tr(S[S_\eta, S_\xi]) \quad (52b)$$

The gauge equivalent counterpart of this equation is the following ZE[2]

$$iq_t + q_{\xi\xi} + vq = 0, \quad (53a)$$

$$v_\eta = -2r^2(\bar{q}q)_\xi. \quad (53b)$$

As $\eta = t$ equation (52) take the other form of the M-XXXIV equation

$$i S_t = \frac{1}{2} [S_{\xi\xi}, S] + iw S_{\xi} \quad (54a)$$

$$w_t = \frac{1}{4i} \{ tr(S^2_\xi) \}_\xi \quad (54b)$$

Similarly, equation (53) becomes

$$iq_t + q_{\xi\xi} + vq = 0, \quad (55a)$$

$$v_t = -2r^2(\bar{q}q)_\xi, \quad (55b)$$

which is called the Ma equation and was considered in [10].
6 Conclusion

The some integrable reductions of the Myrzakulov IX equation are considered. We have established the gauge equivalence between the (1+1)-, or (2+0)- dimensional σ-models and the Klein-Gordon type equations. Also we have shown that the Myrzakulov XXXIV equation which describe nonlinear dynamics of compressible magnets and the Yajima-Oikawa-Ma equations are gauge equivalent to each other[11-13].

Finally, we note that between the some above mentioned equations take place the Lakshmanan equivalence (see, for example, the refs.[14-16]).

7 Appendix: On some soliton equations of compressible magnets

Solitons in magnetically ordered crystals have been widely investigated from both theoretical and experimental points of view. In particular, the existence of coupled magnetoelectric solitons in the Heisenberg compressible spin chain has been extensively demonstrated. In [1] were presented a new classes of integrable and nonintegrable soliton equations of spin systems. Below we present the some of these nonlinear models of magnets - the some of the Myrzakulov equations(ME), which describe the nonlinear dynamics of compressible magnets.

7.1 The 0-class of spin-phonon systems

The Myrzakulov equations with the potentials have the form:

the Myrzakulov LVII(M-LVII) [the $M_{10}^{10}$ - equation]:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$

the Myrzakulov LVI(M-LVI) [the $M_{00}^{10}$ - equation]:

$$2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$

the Myrzakulov LV(M-LV) [the $M_{00}^{10}$ - equation]:

$$2iS_t = \{ (\mu S^2_x - u + m)[S, S_x] \}_x + h[S, \sigma_3]$$

the Myrzakulov LIV(M-LIV) [the $M_{00}^{20}$ - equation]:

$$2iS_t = n[S, S_{xxxx}] + 2\{(\mu S^2_x - u + m)[S, S_x] \}_x + h[S, \sigma_3]$$

the Myrzakulov LIII(M-LIII) [the $M_{00}^{30}$ - equation]:

$$2iS_t = [S, S_{xx}] + 2iuS_x$$
where $v_0, \mu, \lambda, n, m, a, b, \alpha, \beta, \rho, h$ are constants, $u$ is a scalar function (potential), subscripts denote partial differentiations, $[,] (\{,\})$ is commutator (anticommutator),

\[
S = \begin{pmatrix}
S_3 & rS^-
nS^+ & -S_3
\end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad r^2 = \pm 1 \quad S^2 = I.
\]

7.2 The 1-class of spin-phonon systems

The Myrzakulov LII(M-LII) [the $M_{10}^{11}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^3)_{xx}
\]

the Myrzakulov LI(M-LI) [the $M_{10}^{12}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^3)_{xx}
\]

the Myrzakulov L(M-L) [the $M_{10}^{13}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]
\]

\[
u_t + u_x + \lambda(S^3)_x = 0
\]

the Myrzakulov XXXXIX(M-XXXXIX) [the $M_{10}^{14}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]
\]

\[
u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(S^3)_x = 0
\]

7.3 The 2-class of spin-phonon systems

The Myrzakulov XXXXVIII(M-XXXXXVIII) [the $M_{10}^{21}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^2_3)_{xx}
\]

the Myrzakulov XXXXVII(M-XXXXXVII) [the $M_{10}^{22}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]
\]

\[
\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_3)_{xx}
\]

the Myrzakulov XXXXVI(M-XXXXXVI) [the $M_{10}^{23}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]
\]

\[
u_t + u_x + \lambda(S^2_3)_x = 0
\]

the Myrzakulov XXXXV(M-XXXXXV) [the $M_{10}^{24}$ - equation]:

\[
2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]
\]

\[
u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(S^2_3)_x = 0
\]
7.4 The 3-class of spin-phonon systems

The Myrzakulov XXXXIV (M-XXXXIV) [the $M^{31}_{00}$ - equation]:

$$2iS_t = \{ (\mu S^2_x - u + m) [S, S_x] \}_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda (S^2_x)_{xx}$$

the Myrzakulov XXXXIII (M-XXXXIII) [the $M^{32}_{00}$ - equation]:

$$2iS_t = \{ (\mu S^2_x - u + m) [S, S_x] \}_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_x + \beta u_{xxxx} + \lambda (S^2_x)_{xx}$$

the Myrzakulov XXXXII (M-XXXXII) [the $M^{33}_{00}$ - equation]:

$$2iS_t = \{ (\mu S^2_x - u + m) [S, S_x] \}_x$$
$$u_t + u_x + \lambda (S^2_x)_x = 0$$

the Myrzakulov XXXXI (M-XXXI) [the $M^{34}_{00}$ - equation]:

$$2iS_t = \{ (\mu S^2_x - u + m) [S, S_x] \}_x$$
$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda (S^2_x)_x = 0$$

7.5 The 4-class of spin-phonon systems

The Myrzakulov XXXX (M-XXXX) [the $M^{41}_{00}$ - equation]:

$$2iS_t = [S, S_{xxxx}] + 2\{((1 + \mu)S^2_x - u + m) [S, S_x] \}_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda (S^2_x)_{xx}$$

the Myrzakulov XXXIX (M-XXXIX) [the $M^{42}_{00}$ - equation]:

$$2iS_t = [S, S_{xxxx}] + 2\{((1 + \mu)S^2_x - u + m) [S, S_x] \}_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_x + \beta u_{xxxx} + \lambda (S^2_x)_{xx}$$

the Myrzakulov XXXVIII (M-XXXVIII) [the $M^{43}_{00}$ - equation]:

$$2iS_t = [S, S_{xxxx}] + 2\{((1 + \mu)S^2_x - u + m) [S, S_x] \}_x$$
$$u_t + u_x + \lambda (S^2_x)_x = 0$$

the Myrzakulov XXXVII (M-XXXVII) [the $M^{44}_{00}$ - equation]:

$$2iS_t = [S, S_{xxxx}] + 2\{((1 + \mu)S^2_x - u + m) [S, S_x] \}_x$$
$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda (S^2_x)_x = 0$$
7.6 The 5-class of spin-phonon systems

The Myrzakulov XXXVI(M-XXXVI) [the $M_{00}^{51}$ - equation]:

$$2iS_t = [S, S_{xx}] + 2iuS_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(f)_{xx}$$

the Myrzakulov XXXV(M-XXXV) [the $M_{00}^{52}$ - equation]:

$$2iS_t = [S, S_{xx}] + 2iuS_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxx} + \lambda(f)_{xx}$$

the Myrzakulov XXXIV(M-XXXIV) [the $M_{00}^{53}$ - equation]:

$$2iS_t = [S, S_{xx}] + 2iuS_x$$

$$u_t + u_x + \lambda(f)_{x} = 0$$

the Myrzakulov XXXIII(M-XXXIII) [the $M_{00}^{54}$ - equation]:

$$2iS_t = [S, S_{xx}] + 2iuS_x$$

$$u_t + u_x + \alpha(u^2)_{x} + \beta u_{xxx} + \lambda(f)_{x} = 0$$

Here $f = \frac{1}{4} tr(S_x^2)$, $\lambda = 1$.

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