Ultralocal solutions for quantum integrable nonultralocal models

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Abstract

A challenge in the theory of integrable systems is to show for every nonultralocal quantum integrable model, a possible connection to an ultralocal model. Some of such gauge connections were discovered earlier. We complete the task by identifying the same for the remaining ones along with two new models. We also unveil the underlying algebraic structure for these nonultralocal models.

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1. Introduction

Impressive success has been achieved in the theory of quantum integrable systems during the last twenty years. However, the major progress made and the important results obtained in the subject are limited mostly to a class of models, known as the ultralocal (UL) class. The Liouville model, sine-Gordon (SG) model in laboratory coordinates, relativistic Toda chain, nonlinear Schrödinger equation (NLS) etc. are few well-known examples of this class. The UL models must satisfy the ultralocality condition, i.e. their representative Lax operators at different lattice sites must commute: 

$$L^{ul}_{2k}(\mu)L^{ul}_{1j}(\lambda) = L^{ul}_{1j}(\lambda)L^{ul}_{2k}(\mu) \quad \text{for } j \neq k,$$

which is essential for extending the Yang-Baxter equation (YBE):

$$R_{12}^{ul}(\lambda - \mu)L^{ul}_{1j}(\lambda)L^{ul}_{2j}(\mu) = L^{ul}_{2j}(\mu)L^{ul}_{1j}(\lambda)R_{12}^{ul}(\lambda - \mu),$$

for the monodromy matrix $T^{ul}_a(\lambda) = \prod_{j=1}^N L^{ul}_{aj}(\lambda)$. Note that $a = 1, 2$ labels here the auxiliary or matrix spaces, while $j = 1, 2, \ldots, N$ denotes the associated quantum spaces. For the models with periodic boundary condition, the global YBE leads in turn to the required trace identity $[\tau(\lambda), \tau(\mu)] = 0$, for the transfer matrix $\tau(\lambda) = tr_a(T^{ul}_a(\lambda))$, establishing the quantum integrability of the system by showing mutual commutativity: $[c_n, c_m] = 0$, for the set of conserved operators obtained from $\tau(\lambda) = \sum_n c_n \lambda^n$ or from any other function of it. The eigenvalue problem for all these conserved operators including the Hamiltonian of such models can be solved exactly by the powerful algebraic Bethe ansatz, which effectively exploits the global form of YBE. The ultralocality condition reflects generally the fact that the basic fields involved in the Lax operator of an UL model are of canonical nature, which commute trivially at space-like separated points $x \neq y$.

However, there exists on the other hand another rich class of integrable models which violate the ultralocality condition and make their quantum description through standard YBE formulation difficult. In such nonultralocal (NUL) models the basic fields are either of noncanonical type having nontrivial commutators at $x \neq y$ or their associated Lax operators contain space-derivatives of canonical fields. Nevertheless, there is now a considerable number of NUL systems, namely, nonabelian Toda chain, the quantum mapping, model related to the Coulomb gas picture of CFT, current in WZWN, integrable model on moduli space and the quantum mKdV, for which the quantum integrability is established and the braided extensions of YBE have been formulated. We shall call such NUL models genuine quantum integrable models. However, there are some NUL models like the SG in light-cone coordinates and the nonlinear $\sigma$-model, for which the quantum integrability through braided YBE still remains unsolved.

Nevertheless, for the complete understanding of NUL models and for fully exploiting the powerful machinery of quantum integrable systems developed already for UL models, it is highly desirable to find a possible connection for these NUL systems to UL models. Though in general such a connection is not guaranteed, an UL solution was first discovered for the nonabelian Toda chain using a operator dependent local gauge transformation. Subsequently, similar relation was unveiled also for other quantum NUL systems like current algebra in WZWN model, Chern-Simon theory related integrable model on moduli space and the mKdV model.

It is therefore tempting to conjecture that such a connection to UL model should exist for every
NUL system, at least for the genuine quantum integrable models that respect the braided YBE. However the problem is that, the well known NUL models like the Coulomb gas picture of CFT (CGCFT) and the quantum mapping are not known to have any established UL connection. Moreover for the quantum light-cone SG (LCSG) model, not only its explicit UL relation is unknown, but also, as mentioned already, its quantum integrability through braided YBE could not be formulated. Our aim here is to find the required NUL-UL connection for all these models and thus to resolve the eminent problem for the validity of the above conjecture, at least for the genuine quantum integrable models discovered till today. Along with the above result we find also a new NUL sine-Gordon (NSG) like system and able to identify its connection with a Liouville-like UL model on a lattice. A preliminary announcement on two new models investigated here, e.g. LCSG and NSG, has been made in [13].

2. Nonultralocal models and their ultralocal connections

The NUL models of our concern satisfy the braiding relation

\[ L_{2j+1}^{\text{nul}}(\mu)Z_{12}^{-1}L_{1j}^{\text{nul}}(\lambda) = L_{1j}^{\text{nul}}(\lambda)L_{2j+1}^{\text{nul}}(\mu) \] (1)

involving nearest-neighboring (NN) sites, together with the braided YBE (BYBE) [11]

\[ R_{12}(\lambda - \mu)L_{1j}^{\text{nul}}(\lambda)L_{2j}^{\text{nul}}(\mu)Z_{12}^{-1} = L_{2j}^{\text{nul}}(\mu)L_{1j}^{\text{nul}}(\lambda)Z_{21}^{-1}R_{12}(\lambda - \mu). \] (2)

Note that along with the \( R \)-matrix as in UL models an additional braiding matrix \( Z \) enters into these equations. Remarkably the nonultralocality can be incorporated in the BYBE to obtain its global form for the corresponding monodromy matrix \( T_a^{\text{nul}}(\lambda) = L_{aN}^{\text{nul}}(\lambda) \cdots L_{a1}^{\text{nul}}(\lambda) \) with \( a = 1, 2 \) as

\[ R_{12}(\lambda - \mu)T_1^{\text{nul}}(\lambda)Z_{21}^{-1}T_2^{\text{nul}}(\mu)Z_{12}^{-1} = T_2^{\text{nul}}(\mu)Z_{21}^{-1}T_1^{\text{nul}}(\lambda)Z_{12}^{-1}R_{12}(\lambda - \mu), \] (3)

which leads again to the trace identity establishing the quantum integrability for the NUL models. It may be noticed that unlike the YBE the global form of BYBE differs a bit from its local form. This happens because for the periodic models with \( N + 1 \equiv 1 \), the Lax operators \( L_{aN}^{\text{nul}}(\lambda) \) and \( L_{b1}^{\text{nul}}(\lambda) \), \( a, b = 1, 2 \) become in fact NN operators exhibiting nontrivial commutation relations due to (1). It can be seen easily that by putting \( Z = 1 \) in the above NUL equations, i.e. braiding relation and the BYBEs, one obtains the corresponding equations, i.e. the ultralocality condition and the standard YBEs for the UL models.

The BYBE [3] provides similar Hopf algebra structure to the integrable NUL models, though braiding relation [4] induces now braided algebra property by modifying the multiplication rule [10, 11]. At the same time the trace identity for the NUL models as obtained from [3] is also modified to

\[ tr_{12}(T_1^{\text{nul}}(\lambda)Z_{21}^{-1}T_2^{\text{nul}}(\mu)Z_{12}^{-1}) = tr_{12}(T_2^{\text{nul}}(\mu)Z_{21}^{-1}T_1^{\text{nul}}(\lambda)Z_{12}^{-1}) \] and consequently its crucial factorizability depends heavily on the the structure of the braiding matrix \( Z \), adding another difficulty in tackling such models [11]. Therefore though some progress has been made, as mentioned above, in direct analysis of integrable NUL models, identifying explicit gauge relation of such systems to UL
models would be significantly important for getting insight into their underlying algebraic structures as well as their exact solutions. Following the idea of [3] we aim to find such UL source models for genuine quantum integrable NUL models through a gauge transformation like \( L_{j}^{\text{ul}} = D_{j+1} L_{j}^{\text{ul}} D_{j}^{-1} \), such that \( L_{j}^{\text{ul}} \) satisfies the YBE along with the ultralocality. This in turn is guaranteed, as can be shown by some simple algebraic manipulation, by the condition

\[
D_{a} L_{b}^{\text{ul}} D_{a}^{-1} = L_{b}^{\text{ul}} Z_{ab}, \quad a, b = 1, 2, \ a \neq b,
\]

provided the gauge operator \( D_{j} \) satisfies the YBE together with a related equation as

\[
R_{12}(\lambda - \mu) D_{1j} D_{2j} = D_{2j} D_{1j} R_{12}(\lambda - \mu), \quad R_{12}(\lambda - \mu) D_{2j} D_{1j} = D_{1j} D_{2j} R_{12}(\lambda - \mu).
\]

However in the particular case when \([D_{1j}, D_{2j}] = 0\), both the equations (3) reduce to a single condition \([R_{12}, D_{1j} D_{2j}] = 0\), which actually holds true for gauge operators in all the examples we have considered here. Moreover when \(D_{ab}\) and \(Z_{ab}\) mutually commute (which is true for all our examples except that with the quantum mapping), condition (3) holds also for the gauge related \( L_{j}^{\text{ul}} \). Interestingly the monodromy matrices for these periodic models are linked now as

\[
T^{\text{ul}} = \prod_{j}^{N} L_{j}^{\text{ul}} = D_{N+1} \left( \prod_{j}^{N} L_{j}^{\text{ul}} \right) D_{1}^{-1} = D_{1} T^{\text{ul}} D_{1}^{-1},
\]

and satisfy the relation

\[
D_{a} T_{b}^{\text{ul}} D_{a}^{-1} = T_{b}^{\text{ul}} Z_{ab}^{-1}, \quad a, b = 1, 2, \ a \neq b,
\]

as can be shown starting from (3) with the assumption \([D_{aj+1}, L_{bj}^{\text{ul}}] = 0, \ a \neq b,\), which is fulfilled in our cases. We should mention here that though the global BYBE (3) can be derived from its local form and the braiding relation independent of its conjectured relation with the UL model, by using such relations (3) and (7) one can actually derive (3) starting from the standard YBE for the corresponding \( T^{ul} \). Inspired by the fact that the UL model must have the same \( R \)-matrix as its NUL counterpart related by a gauge transformation, we intend to seek our \( L^{ul} \) solutions using the generating scheme of [4] based on the \( R \)-matrix classification.

Recall that the well known trigonometric \( 4 \times 4 \) \( R(\lambda) \)-matrix is defined by its nontrivial elements

\[
R_{11}^{11} = R_{22}^{22} = \sin(\lambda + \alpha), \quad R_{12}^{12} = R_{21}^{21} = \sin \lambda, \quad R_{21}^{12} = R_{12}^{21} = \sin \alpha,
\]

while the ancestor Lax operator of quantum UL models associated with (8) and satisfying the YBE may be given [4] by

\[
L_{k}^{\text{anc}(\text{ul})}(\lambda) = \left( \begin{array}{ccc}
 c_{1}^{1} \xi q^{S_{k}^{2}} + c_{1}^{1} \xi q^{-S_{k}^{2}} & S_{k}^{+} & q = e^{i\alpha}, \quad \xi = e^{i\lambda} \\
 S_{k}^{-} & c_{2}^{1} \xi q^{-S_{k}^{2}} + c_{2}^{1} \xi q^{S_{k}^{2}} & \end{array} \right),
\]

or multiplying it by \( \sigma^{i}, \ i = 1, 2, 3 \) due to a symmetry of (8) as \([R, \sigma^{i} \otimes \sigma^{i}] = 0\). The basic operators in (8) satisfy a quadratic quantum algebra

\[
[S_{k}^{+}, S_{l}^{-}] = 4 \delta_{kl} \sin \alpha (M^{+} \sin(\alpha 2S^{3}) + M^{-} \cos(\alpha 2S^{3})), \quad [S_{k}^{3}, S_{l}^{-}] = \pm \delta_{kl} S_{k}^{\pm}, \quad [M^{\pm}, \cdot] = 0,
\]
which is a Hopf algebra with well defined coproduct and with \( c_a^\pm, a = 1, 2 \) and \( M^\pm = \pm \frac{1}{2} \sqrt{\pm 1} (c_1^+ c_2^- \pm c_1^- c_2^+) \) being sets of central elements. Reductions of the general \( L \)-operator (9) (modulo its product with \( \sigma^i \)), as shown in [14], generate in a systematic way the Lax operators of integrable UL models having the same \( R \)-matrix (8). At the undeformed \( q \to 1 \) limit the whole procedure is repeated for the rational \( R \)-matrix.

We focus now on the NUL models under consideration for explicit solution of their UL sources. Only the simplest \( SU(2) \) case is considered here; more general group admissible to some models can be treated analogously.

2.1. Coulomb gas picture of CFT

The NUL nature of this CGCFT model [4] follows from the Drinfeld-Sokolov problem

\[
\partial x Q = \mathcal{L}(x) Q,
\]

where \( \mathcal{L}(x) = v(x) \sigma^3 + \sigma^+ \) contains a current-like field \( \{v(x), v(y)\} = -i \alpha \delta'(x-y) \). Quantum and discrete version of this linear operator: \( L_k \) was shown to satisfy the spectral parameter-less analogs of braided equations (11) and (2) with the braiding matrix \( Z \) as presented below and the limit of (8): \( R(\lambda \to \infty) \to R^+_q \) acting as the \( R \)-matrix.

Note that by fixing the central elements in (11) as \( c_1^+ = \Delta \), with \( \Delta \) being the lattice constant and all the rest \( c \)'s as zeroes, the algebra (12) reduces simply to

\[
[S_k^+, S_l^-] = 0 \quad [S^3_k, S^\pm_l] = \pm \delta_{kl} S^\pm_k \]

and we may find a realization for the generators as

\[
S^\mp_k = e^{\pm ip_k}, \quad S^3_k = u_k - \frac{1}{\alpha} p_k
\]

reducing the Lax operator (14) (after multiplying it from right by \( \sigma^1 \)) to

\[
L^{ul}_k(\xi) = e^{i p_k \sigma^3 + \Delta \xi e^{i(\alpha u_k - p_k)} \sigma^+},
\]

in canonical variables \([u_k, p_j] = i \delta_{kj}\). We may conclude that (14) represents a relativistic Toda chain (RTC) like integrable UL model [15] and satisfies the YBE with the \( R \)-matrix (8), as a consequence of the general argument of [14]. We observe that a local gauge matrix \( D_j^{(1)} = e^{-\frac{i}{2} \alpha \sigma^3 u_j} \) along with a change of basic operators:

\[
v^+_k = -\frac{\alpha}{2} (u_{k+1} - u_k) - p_k, \quad v^-_k = \frac{\alpha}{2} (u_{k+1} - u_k) - p_k
\]

can transform (14) to

\[
L^{ul}_k(\xi) = e^{-i v^-_k \sigma^3 + \Delta \xi e^{i v^+_k} \sigma^+},
\]

NUL properties of which are induced by the discrete version of the current algebra for \( v^\pm_k \):

\[
[v^+_k, v^-_l] = \pm i \frac{\alpha}{2} (\delta_{k,l+1} - \delta_{k+1,l}), \quad [v^+_k, v^+_l] = i \frac{\alpha}{2} (\delta_{k+1,l} - 2 \delta_{k,l} + \delta_{k,l+1}).
\]
One may check that (14) as well as (14) satisfy (9) with the braiding matrix \( Z^{(1)} = e^{-\frac{i}{2}a^3 \otimes a^3} \), as found in [4]. This guarantees in turn that the NUL integrability relations (1), (2) and (3) must hold for the Lax operator (13). We observe further that at the continuum limit \( \Delta \to 0, v_k^\pm \to \sqrt{\pm 1} \Delta v(x) \) yield the current-like field and consequently (16) reproduces through \( L_k^{ul}(\xi) = I + \Delta L(x, \xi) + O(\Delta^2) \), \( L(x, \xi) = v(x)\sigma^3 + \xi \sigma^+ \), i.e. the field Lax operator (1) of [4], though generalized here to include spectral parameter \( \xi \) and satisfy the standard YBE, which shows also its novelty. Thus (14) is proved to be the required UL source solution for the NUL integrable CGCFT model.

2.2. NUL sine-Gordon like model

For introducing a new discrete NUL sine-Gordon (NSG) type model, we start from an UL Lax operator

\[
L_k^{ul}(\lambda) = e^{i\lambda \sigma^3 - pk} + e^{i(\alpha u_k - pk)} \sigma^+ + e^{-i(\alpha u_k + pk)} \sigma^-,
\]

(18)

Note that like the previous model it can again be derived from the same general Lax operator (9), though now with a different parameter choice: \( c_1^- = c_2^+ = 0, c_1^+ = c_2^- = 1 \), which reduces the underlying general algebra (10) to an interesting exponentially deformed algebra

\[
[S_k^+, S_l^-] = -2i\delta_{kl} e^{2ia} S_k^3 \sin \alpha, \quad [S_k^3, S_l^\pm] = \pm \delta_{kl} S_k^\pm.
\]

(19)

Note that such reductions, which must represent integrable UL models associated with the trigonometric R-matrix (8) fall in the class of discrete Liouville model [16] and with a realization of (19):

\[
S_k^\pm = e^{\pm i(\alpha u_k \mp pk)}, \quad S_k^3 = -\frac{1}{\alpha} p_k
\]

(20)

we can derive (18) directly from (9).

Remarkably, in spite of the different underlying algebra and its realization from the previous model, (18) permits us to choose the gauge operator again in the same form as the previous one: \( D_j^{(2)} \equiv D_j^{(1)} = e^{-\frac{i}{2}a^3 u_j} \). Performing therefore this gauge transformation on (18) and changing to the current-like operators \( v_k^\pm \) using the map (15) we derive finally the Lax operator of our new NSG model as

\[
L_k^{ul}(\lambda) = \begin{pmatrix}
\xi & W_k^+ \\
W_k^- & \frac{1}{\xi} W_k^-
\end{pmatrix}, \quad W_k^\pm = e^{iv_k^\pm}, \quad \xi = e^{i\lambda}.
\]

(21)

The validity of (1) by both (18) and (21) gives the explicit form of the braiding matrix \( Z_{12}^{(2)} = e^{i\frac{\pi}{2}a^3} \otimes I \), which differs clearly from the previous case. Therefore, as established above, this discrete NSG (21) must be a genuine quantum integrable NUL model, which satisfies (1) and the braided YBE (2). For finding the full set of its mutually commuting conserved operators: \( c_{\pm j}, j = 1, 2, \ldots, N \) we have to factorize the trace identity, which however becomes easy for the present Z matrix yielding \( \tau(\xi) = tr(e^{-i\frac{\pi}{2}a^3}T^{ul}(\xi)) \) as the generating function: \( \tau(\xi) = \sum_{j}^N c_{\pm j} \xi^{\pm j} \). We may define the Hamiltonian of this new integrable model as \( H = \frac{1}{2}(c_{N})^{-1}c_{N-2} + (c_{-N})^{-1}c_{-(N-2)} = \sum_j^N \cos(v_j^+ - v_j^- + \alpha) \) and name it as nonultralocal SG model, due to its resemblance with the standard SG. Note that noncanonical commutation relations (17) involving different sites, which must be used for deriving the dynamical
equations, induce nearest-neighbor interaction in the model. Notice also that by transforming \( \xi \rightarrow \frac{1}{2} \xi, W^+_k \leftrightarrow W^-_k \) in (21) we would get a dual though similar integrable NSG model.

2.3. Quantum light-cone sine-Gordon model

Since quantum formulation through the braided YBE is not available for this well known nonultralocally classical model, our first goal is to discover such a formulation, for which we follow our above strategy and look for its UL source model as a particular realization of the general UL model (3). We observe that the choice of central elements as \( c^-_a = 0, c^+_a = \Delta \) or its complementary set \( c^-_a = \Delta, c^+_a = 0, \) with \( a = 1, 2 \) leads again to the simple algebra (1). Therefore we can use realizations similar to (1):

\[
S^\pm_k = e^{\pm i p_k}, \quad \nabla_k = u_k + \epsilon \frac{1}{\alpha} p_k
\]

with \( \epsilon = \mp \) for the two complementary (left, right) cases and generate from (3) (after multiplying with \( \sigma^\pm \)) the left (right) Lax operator \( L^{(\mp)}_k(\lambda) \) as

\[
L^{(\mp)ul}_{ul}(\lambda) = e^{ip_k \sigma^3} + \Delta \xi^{\pm1} \left( e^{i(\mp\alpha u_k - p_k) \sigma^+} + e^{i(\mp\alpha u_k + p_k) \sigma^-} \right), \quad \xi = e^{i\lambda},
\]

representing again RTC type integrable UL model. Acting now, for example, on the left Lax operator \( L^{(-)}_{ul}(\lambda) \) of (23) by a gauge matrix \( D^{(3)}_j(\alpha) = e^{-i\alpha u_j \sigma^3} \), which may be noticed to be simply the square of that found in the previous cases: \( D^{(2)}_j(\alpha) = (D^{(2)}_j)^2 \), we construct a NUL Lax operator

\[
L^{(-)lcsg}_j(\lambda) = e^{i(p_j - \lambda \nabla u_j) \sigma^3} + \Delta \xi \left( e^{-i(p_j + \alpha u_{j+1}) \sigma^+} + e^{i(p_j + \alpha u_{j+1}) \sigma^-} \right), \quad \nabla u_j \equiv u_{j+1} - u_j.
\]

We may check directly that \( L^{(-)ul}_{ul}(\lambda) \) in (23) as well as \( L^{(-)lcsg}_j(\lambda) \) in (24) respect condition (4) with the braiding matrix \( Z^{(\pm)(3)}_{12} = e^{i\alpha \sigma^3 \otimes \sigma^3} \) and therefore conclude that the NUL Lax-operator (24) together with R-matrix (2) and this \( Z^{(\pm)(3)} \)-matrix satisfies braiding relation (1) and the BYBEs (2), (3). This thus proves the genuine quantum integrability of the associated model which, as we see below, may be considered as the exact lattice version of the quantum light-cone SG (LCSG) model. Therefore, (23) indeed is an UL source model for the LCSG, which is new as a quantum integrable NUL model. At the field limit \( \Delta \rightarrow 0 \), when we have \( p_j \rightarrow \Delta \partial_x u(x), \alpha u_j \rightarrow u(x), \alpha u_{j+1} \rightarrow u(x) + \Delta \partial_x u(x) \), defining \( \partial_t u \pm \partial_x u = \frac{1}{2} \partial_{\pm} u \), it is not difficult to see that (24) reduces to \( L^{(-)lcsg}_j(\lambda) \rightarrow I + \Delta U_-(x) \), where

\[
U_-(x) = \frac{1}{2} \partial_{-} u(x) \sigma^3 + \xi (e^{-iu(x) \sigma^+} + e^{iu(x) \sigma^-})
\]

yields one of the Lax pair: \( \partial_- \Phi = U_- \Phi \) of the well known LCSG field model.

It is important to note that the right operator \( L^{(+)}_k(\lambda) \) in (23) can be obtained formally from the left \( L^{(-)}_k(\lambda) \) through simple mapping \( \xi \rightarrow \frac{1}{2} \xi, \alpha \rightarrow -\alpha \) and therefore the corresponding results for the complementary right LCSG can be recovered easily from that of the left one through the same mapping. Thus we derive the NUL model \( L^{(+)lcsg}_j(\lambda) \) which at the field limit gives similarly the other component of the LCSG Lax pair: \( U_+(x) \). Zero curvature condition: \( \partial_- U_+ - \partial_+ U_- + [U_+, U_-] = 0 \) involving both these Lax operators yields finally the well known form of the sine-Gordon field equation in light-cone coordinates: \( \partial^2_{-} u = 2 \sin 2u \).

It is intriguing to remark that, the UL source models found recently for the quantum left/right mKdV (22) can be given exactly by the same Lax operators (23) discovered here for the light-cone SG
model, though the gauge operator related to the mKdV model coincides rather with $\sqrt{D_j^{(3)}(\alpha)}$.

2.4. Quantum mapping

A rational quantum mapping related to the lattice KdV type equation may be described by a Lax operator

$$U_n = L_{2n}^{nul} L_{2n-1}^{nul}, \text{ where } L_{a,j}^{nul}(\lambda_a) = \frac{1}{2}(1 + \sigma_a^3) v_j + \sigma_a^+ + \lambda_a \sigma_a^-, \ a = 1, 2$$

(25)

which exhibits NUL nature due to the presence of discrete current-like operator $v_j \equiv v_j^+$ [17]. $L_j^{nul}$ and hence $U_n$ in (25) were shown [8] to satisfy both braiding relation and BYBE with the rational $R$-matrix $R_{12}(\lambda_1 - \lambda_2) = 1 + i\frac{\alpha}{2} \frac{\rho_{\lambda_2}}{\lambda_1 - \lambda_2}$, (which may be obtained from [8] at $\alpha \to 0, \lambda \to 0$) and a spectral parameter dependent braiding matrix $Z_{21}^{(4)}(\lambda_2) = 1 - \frac{\alpha}{2\lambda_2} \sigma^- \otimes \sigma^+$ (similarly $Z_{12}(\lambda_1)$).

For finding the UL connection as asserted by the conjecture, we look again for the source model as a realization of the general construction and try to identify a suitable gauge matrix, as done in the above cases. However since the present model is associated with the rational $R$-matrix we have to start now from the ancestor Lax operator of quantum integrable rational UL models [14], which is obtained as a $\alpha \to 0$ (or $q \to 1$) limit of (18) in the form

$$L^{r-anc(ul)}(\lambda) = \left( \begin{array}{cc} c_1^0 (\lambda + s^3) + c_1^1 & s^- \\ s^+ & c_2^0 (\lambda - s^3) - c_2^1 \end{array} \right).$$

(26)

The basic operators satisfy the undeformed quadratic algebra

$$[s^+, s^-] = 2m^+ s^3 + m^-, \quad [s^3, s^\pm] = \pm s^\pm$$

(27)

with $m^+ = c_1^0 c_2^0$, $m^- = c_1^0 c_2^0 + c_1^a c_2^a$ and $c_1^a, a = 1, 2, n = 0, 1$ as central elements. For our construction we choose first these elements as $c_1^0 = i\frac{\alpha}{2}, c_2^0 = 0, c_1^1 = -c_2^1 = 1$ to derive the underlying algebra from (27) in a simple form

$$[s^+, s^-] = -i\frac{\alpha}{2}, \quad [s^3, s^\pm] = \pm s^\pm.$$

(28)

We can therefore realize this algebra simply as

$$s_k^+ = -i\psi_k, \quad s_k^- = \phi_k, \quad s_k^3 = -i\frac{2}{\alpha} \phi_k \psi_k$$

(29)

through operators $[\psi_j, \phi_k] = \frac{\alpha}{2} \delta_{jk}$ to generate from (23) (after multiplying from left by $\sigma^1$) an UL Lax operator

$$L_j^{ul}(\lambda_1) = \left( \begin{array}{cc} -i\psi_j & 1 \\ \lambda_1 - i\phi_j \psi_j & \phi_j \end{array} \right), \lambda_1 = 1 + i\frac{\alpha}{2} \lambda.$$  

(30)

Remarkably this Lax operator is associated with a quantum integrable simple lattice NLS model [17] and satisfies the YBE with rational $R$-matrix. We find next the gauge operator as $D_j^{(4)} = I - \phi_j \sigma^-$, use a simple realization of the current-like operator satisfying (17) as $v_j^+ = -i\psi_j + \phi_{j-1}$ and perform the gauge transformation on our UL model [30] to derive exactly the Lax operator $L_j^{nul}$ of the quantum mapping (25). Note however that since we intend to recover the known form of $L_j^{nul}$ as given in [3], we had to use here a slightly different form of gauge transformation: $D_j L_j^{ul}(\lambda_1) D_j^{-1} = L_j^{nul}(\lambda_1)$,
compared to the rest of our cases. As a result $L^\text{nul}_j$ would satisfy NUL integrable relations, which were used in [3] and are complementary to those presented here and also in place of (6) a similar condition $D_{aj}L^\text{nul}_{bj}D_{aj}^{-1} = Z_{ab}^{-1}L^\text{nul}_{bj}$. This establishes therefore (30) to be a perfect gauge related UL source solution for the well known NUL quantum mapping.

3. Conclusion

We conjecture an ultralocal (UL) connection for every nonultralocal (NUL) quantum integrable model and establish the yet unestablished such connections including some new ones. We find explicitly for the well known quantum NUL systems, e.g. CGCFT [4] and quantum mapping [3], the corresponding gauge related UL source models as the relativistic Toda chain (RTC) and a lattice NLS, which are well studied UL models. At the same time we discover two new quantum integrable NUL models, namely a discrete NUL type SG (NSG) and an exact lattice light-cone SG (LCSG) model and identify their corresponding UL connections as given through a Liouville and a RTC model. Such an explicit UL gauge relation of a NUL model would help in extracting all necessary knowledge of a more complicated NUL system including its Bethe ansatz result through a more tractable and in most cases already known UL integrable model. The NUL-UL connection should become particularly useful, when we are unable to tackle a NUL integrable model directly, due to nonfactorizability of its trace identity or other reasons. In such cases, thanks to the present conjecture, we could switch over to its gauge related UL model and use relation like (6): $\tilde{T}^\text{nul} = T^\text{ul}$ for its UL description. The $\tilde{T}^\text{nul} = L^\text{nul}_{aN} \cdots L^\text{nul}_{a2} L^\text{nul}_{a1}$ however would be an approximation, where one of its boundary terms is modified as $\tilde{L}^\text{nul}_{a1} = D_{1}^{-1} L^\text{nul}_{a1} D_{1}$. It is remarkable that, though we have considered NUL models of completely different nature with no apparent relations between them, their gauge related UL sources could be derived from the same ancestor model (or its $q \to 1$ limit). Such universality of this ancestor is due to its general form, which is capable of generating all possible quantum integrable UL models within the specified class [14]. Therefore we can conjecture further that all other quantum integrable NUL models, not studied here, also should have their gauge related UL models derivable from the same ancestor. Various reductions of this ancestor model, which generate different descendant UL models and serve as the source models for the NUL systems, are due to different choices of the central elements leading to varied underlying algebras and hence various allowed realizations. For generating the corresponding NUL models we need to find further the gauge operators, which depend in general on the structure of individual models as do also the associated braiding matrices. Among the four NUL models we considered, CGCFT and LCSG have the same underlying algebra, while CGCFT and NSG have the coinciding gauge operator. The braiding matrices of all these models however turn out to be different.

Since for our construction we exploit the general algebraic scheme of [14], we also unravel easily the important underlying algebraic structure for each of the NUL models. This algebraic information along with their gauge connections to UL models should make quantum integrable NUL models more interesting and easier objects to study, boosting their development which indeed deserves and needs
more attention.

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