Erdős space in Julia sets

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Abstract. We prove that the rational Hilbert space \( E \), known as Erdős space, surfaces in complex dynamics via iteration of \( e^z - 1 \).

1. Introduction

The subspace of \( \ell^2 \) consisting of all rational sequences,

\[ E := \{ x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n < \omega \}, \]

was introduced by Paul Erdős in order to show that squaring a topological space of positive dimension does not necessarily raise the dimension. In fact, Erdős space \( E \) is 1-dimensional [7] and homeomorphic to all of its powers, up to and including \( E^\omega \) [5]. Dijkstra and van Mill [5] also found representations of \( E \) in other contexts. For example, they proved that \( E \) is topologically equivalent to the space of homeomorphisms of \( \mathbb{R}^2 \) which map \( \mathbb{Q}^2 \) onto itself. Our goal in this paper is to show that \( E \) surfaces in complex dynamics. Moreover we will show that \( E \) is generated by the transcendental entire function \( f(z) = e^z - 1 \).

The Julia set\(^1\) of \( f \) consists of all complex numbers whose orbits stay in the right half-plane;

\[ J(f) = \bigcap_{n=1}^{\infty} \{ z \in \mathbb{C} : \text{Re}(f^n(z)) \geq 0 \}. \]

The first three sets from this intersection are depicted in Figure 1. Each consists of an infinite collection of domains, which are cut into smaller domains by incrementing \( n \). Every descending sequence of domains accumulates onto a simple curve that connects a point of \( \mathbb{C} \) to the point at infinity. Informally, this shows that \( J(f) \) is a collection of mutually separated curves with endpoints. See Devaney and Krych [3, 4] or Aarts and Oversteegen [1] for more about the structure of \( J(f) \).

Let \( E(f) \) be the set of all finite endpoints of maximal curves in \( J(f) \). Kawamura, Oversteegen and Tymchatyn [8] proved that \( E(f) \) is homeomorphic to complete Erdős space \( E_c := \{ x \in \ell^2 : x_n \in \mathbb{R} \setminus \mathbb{Q} \text{ for all } n < \omega \} \). The main result of this paper is:

\(^{1}\)The Julia set of an entire function \( f \) is traditionally defined to be the set of non-normality for the family \( \{ f^n : n \in \mathbb{N} \} \). For \( f(z) = e^z - 1 \) it is equal to the set \( J(f) \) defined above, owing to the fact that all points in the left half-plane attract to 0.
Theorem 1.1. The imaginary-escaping endpoint set
\[ \hat{E}(f) = \{ z \in E(f) : \text{Im}(f^n(z)) \to \infty \} \]
is homeomorphic to \( \mathcal{E} \).

This provides a partial positive answer to [9, Question 1] insofar as it shows that the escaping endpoint set \( \hat{E}(f) = \{ z \in E(f) : |f^n(z)| \to \infty \} \) contains a dense copy of \( \mathcal{E} \) (although \( \hat{E}(f) \) is homeomorphic to neither \( \mathcal{E} \) nor \( \mathcal{E}_c \) [10]).

**Sketch of proof.** The proof of Theorem 1.1 will be largely based on Rempe [12], Dijkstra and van Mill [5], and Alhabib and Rempe [2]. The topological model of \( J(f) \) in [12] will be used to define a function \( \psi \) whose graph \( G_\infty \psi \) is homeomorphic to \( \hat{E}(f) \). Then we will apply the extrinsic characterization of \( \mathcal{E} \) from [5] to the space \( G_\infty \psi \). In order to show that the characterization fully applies to the model, we will require several results from [2]. The most significant is [2, Theorem 3.6] which is essentially a generalization of the fact that \( E(f) \) is dense in \( J(f) \).

All relevant results from [5] and [12] are presented in Sections 2 and 3. Those in [2] will be cited as needed during the proof, which is in Section 4.

2. Topological models of \( J(f) \) and \( \hat{E}(f) \)

2.1. External addresses. Let \( \mathbb{Z}^\omega \) be the space of integer sequences \( s = \langle s_0, s_1, s_2, \ldots \rangle \). We say that a point \( z \in \mathbb{C} \) has external address \( s \) (in symbols, \( \text{addr}(z) = s \)) if
\[ \text{Im}(f^n(z)) \in [(2s_n - 1)\pi, (2s_n + 1)\pi] \]
for each \( n < \omega \).

If \( s \in \mathbb{Z}^\omega \) and \( |s_n| \) does not increase at a faster-than-exponential rate, then \( s \) is the external address of a unique curve of \( J(f) \); see [4, Section 3] or [13, Proposition 3.2].
Example 2.1. The sequence $\langle -1, 0, 1, 2, 3, \ldots \rangle$ increases at only a linear rate, and is therefore the external address of a curve in $J(f)$. Figure 2 shows the ray at that address, followed by two of its iterates. Note that the endpoint at this address belongs to $\hat{E}(f)$. In particular, this shows that $\hat{E}(f)$ is non-empty.

2.2. Model of the Julia set. Define $F : [0, \infty) \times \mathbb{Z}^\omega \to \mathbb{R} \times \mathbb{Z}^\omega$ by

$$\langle t, \bar{s} \rangle \mapsto (F(t) - 2\pi|s_1|, \sigma(\bar{s})), $$

where $F(t) = e^t - 1$ and $\sigma$ is the shift on $\mathbb{Z}^\omega$ (i.e. $\sigma(\langle s_0, s_1, s_2, \ldots \rangle) = \langle s_1, s_2, s_3, \ldots \rangle$).

Put $T(x) = t$ for each $x = \langle t, \bar{s} \rangle \in [0, \infty) \times \mathbb{Z}^\omega$, and let

$$J(F) = \{ x \in [0, \infty) \times \mathbb{Z}^\omega : T(F^n(x)) \geq 0 \text{ for all } n \geq 0 \}. $$

If $\bar{s} \in \mathbb{Z}^\omega$ and there exists $t \geq 0$ such that $\langle t, \bar{s} \rangle \in J(F)$, then let

$$t_{\bar{s}} = \min \{ t \geq 0 : \langle t, \bar{s} \rangle \in J(F) \}. $$

Otherwise, put $t_{\bar{s}} = \infty$. Observe that

$$J(F) = \bigcup_{\bar{s} \in \mathbb{Z}^\omega} [t_{\bar{s}}, \infty) \times \{ \bar{s} \}. $$

Thus the points $\langle t_{\bar{s}}, \bar{s} \rangle$ with $t_{\bar{s}} < \infty$ are the (finite) endpoints of $J(F)$.

The following is implicit in [12].
Proposition 2.2. There is a homeomorphism $H : J(\mathcal{F}) \to J(f)$ such that
\[
\text{addr}(H(\langle t, \bar{s} \rangle)) = \bar{s}
\]
for every $\langle t, \bar{s} \rangle \in J(\mathcal{F})$.

Proof. Let $H$ be the mapping defined in [12, Theorem 9.1] for the parameter $\kappa = -1$. By construction, $H$ is one-to-one on a set $X$ which contains all non-endpoints of $J(\mathcal{F})$ [12, Observation 3.1]. Since $J(f)$ is a union of disjoint copies of $[0, \infty)$, it follows that all of $H$ is one-to-one. In the proof of [12, Theorem 9.1] it is also noted that $H$ is closed because it extends to a mapping of the one-point compactifications. So $H$ is a homeomorphism.

Now let $\langle t, \bar{s} \rangle \in J(\mathcal{F})$. By the remark after the statement of [12, Theorem 9.1] and the construction of $g$ in [12, Section 4], there exists $t' > t$ such that $H(\langle t', \bar{s} \rangle) = g(\langle t', \bar{s} \rangle)$. By [12, Theorem 4.2], $\text{addr}(g(\langle t', \bar{s} \rangle)) = \bar{s}$. Since $H(\langle t, \bar{s} \rangle)$ and $H(\langle t', \bar{s} \rangle)$ lie on the same curve, and each curve of $J(f)$ is contained in a horizontal strip of the form $\{z \in \mathbb{C} : (2k - 1)\pi < \text{Im}(z) < (2k + 1)\pi\}$, this implies $\text{addr}(H(\langle t, \bar{s} \rangle)) = \bar{s}$. \hfill \square

2.3. Model of the imaginary-escaping endpoints.

Corollary 2.3. $\{\langle t_2, \bar{s} \rangle \in J(\mathcal{F}) : s_n \to \infty\} 
\simeq \bar{E}(f)$.

Proof. Let $H$ be the homeomorphism from Proposition 2.2. For any endpoint $\langle t_2, \bar{s} \rangle \in J(\mathcal{F})$ it is evident that $H(\langle t_2, \bar{s} \rangle) \in \bar{E}(f)$. Note also that if $z \in J(f)$ and $\bar{s} = \text{addr}(z)$, then $\text{Im}(f^n(z)) \to \infty$ if and only if $s_n \to \infty$. Thus $H(\{\langle t_2, \bar{s} \rangle \in J(\mathcal{F}) : s_n \to \infty\}) = \bar{E}(f)$. \hfill \square

3. Characterizations of $\mathcal{E}$

3.1. Sierpiński stratification. The characterizations of $\mathcal{E}$ in [5, Section 7] involve trees of closed subsets that are called Sierpiński stratifications. They are defined as follows.

For any set $A$ we let $A^{<\omega}$ denote the set of all functions $\alpha$ such that $\text{dom}(\alpha) < \omega$ (the domain of $\alpha$ is a finite ordinal), and the range of $\alpha$ is a subset of $A$. Thus if $\alpha \in A^{<\omega}$ and $n = \text{dom}(\alpha)$ then $\alpha$ is an $n$-tuple of elements of $A$; $\alpha = \langle a(0), \ldots, a(n - 1) \rangle$. We shall write $\alpha < \beta$ if $\text{dom}(\alpha) < \text{dom}(\beta)$ and the restriction $\beta \upharpoonright \text{dom}(\alpha)$ is equal to $\alpha$ (that is, $\beta(i) = \alpha(i)$ for all $i < \text{dom}(\alpha)$).

A tree $T$ on an alphabet $A$ is a subset of $A^{<\omega}$ that is closed under initial segments, i.e. if $\beta \in T$ and $\alpha < \beta$ then $\alpha \in T$. An element $\lambda \in A^{\omega}$ is an infinite branch of $T$ provided $\lambda \upharpoonright k \in T$ for every $k < \omega$. We let $[T]$ denote the set of all infinite branches of $T$. If $\alpha, \beta \in T$ are such that $\alpha < \beta$ and $\text{dom}(\beta) = \text{dom}(\alpha) + 1$, then we say that $\beta$ is an immediate successor of $\alpha$ and $\text{succ}(\alpha)$ denotes the set of immediate successors of $\alpha$ in $T$.

Let $X$ be a non-empty separable metrizable space. A system $(X_\alpha)_{\alpha \in T}$ is called a Sierpiński stratification of $X$ if:

1. $T$ is a tree over a countable alphabet,
2. each $X_\alpha$ is a closed subset of $X$,
3. $X_\emptyset = X$ and $X_\alpha = \bigcup\{X_\beta : \beta \in \text{succ}(\alpha)\}$ for each $\alpha \in T$, and
4. if $\lambda \in [T]$ then the sequence $X_{\lambda \upharpoonright 0}, X_{\lambda \upharpoonright 1}, \ldots$ converges to a point in $X$. 
3.2. Graphs of upper-semicontinuous functions homeomorphic to $\mathcal{E}$. A function $\varphi : X \to [0, \infty]$ is upper semi-continuous if $\varphi^{-1}(0, t)$ is open in $X$ for every $t \in \mathbb{R}$.

An upper semi-continuous function $\varphi : X \to [0, 1]$ is called a Lelek function if $X$ is zero-dimensional, $X' = \{x \in X : \varphi(x) > 0\}$ is dense in $X$, and

$$G_0^\varphi := \{(x, \varphi(x)) : \varphi(x) > 0\}$$

is dense in

$$L_0^\varphi := \bigcup_{x \in X'} \{x\} \times [0, \varphi(x)].$$

A Lelek function $\varphi : X \to [0, 1]$ belongs to the Sierpiński-Lelek class $\mathcal{SL}$ if there exists a Sierpiński stratification $(X_\alpha)_{\alpha \in T}$ of $X$ such that:

(a) $\varphi \upharpoonright X_\alpha$ is Lelek for every $\alpha \in T$; and
(b) $G_0^\varphi \upharpoonright X_\alpha$ is nowhere dense in $G_0^\varphi \upharpoonright X_\beta$ for each $\alpha \in T$ and $\beta \in \text{succ}(\alpha)$.

**Proposition 3.1 ([5, Theorem 7.12]).** If $\varphi \in \mathcal{SL}$, then $G_0^\varphi \simeq \mathcal{E}$.

**Example 3.2.** Define $\eta : \mathbb{Q}^\omega \to [0, 1)$ by $\eta(x) = \frac{1}{1 + \|x\|}$, where

$$\|x\| = \sqrt[\infty]{\sum_{n=0}^\infty x_n^2}$$

is the $\ell^2$-norm of $x$, and $1/\infty = 0$. Let $T = \mathbb{Q}^{<\omega}$. For each $\alpha = \langle q_0, \ldots, q_{n-1} \rangle \in T$ define

$$X_\alpha = \{q_0\} \times \cdots \times \{q_{n-1}\} \times \mathbb{Q} \times \mathbb{Q} \times \ldots.$$

The system $(X_\alpha)_{\alpha \in T}$ witnesses that $\eta \in \mathcal{SL}$ (see [5, Proposition 7.11]). Hence $G_0^\eta \simeq \mathcal{E}$.

3.3. Graphs of lower-semicontinuous functions homeomorphic to $\mathcal{E}$. In order to prove Theorem 1.1, we will require a lower semi-continuous version of Proposition 3.1.

A function $\psi : X \to [0, \infty]$ is lower semi-continuous if $\psi^{-1}(t, \infty]$ is open in $X$ for every $t \in \mathbb{R}$. Define

$$G_\infty^\psi := \{\langle \psi(x), x \rangle : \psi(x) < \infty\};$$

and

$$L_\infty^\psi := \bigcup_{x \in X} [\psi(x), \infty] \times \{x\}.$$

Note that the first coordinate is now the output of the function. This is done so that $L_\infty^\psi$ is a collection of horizontal arcs resembling $J(F)$.

**Proposition 3.3.** Let $\psi : X \to [0, \infty]$ be a lower semi-continuous function with zero-dimensional domain $X$. Suppose $(X_\alpha)_{\alpha \in T}$ is a Sierpiński stratification of $X$,

(a) $X'_\alpha := \{x \in X_\alpha : \psi(x) < \infty\}$ is dense in $X_\alpha$ for each $\alpha \in T$;
(b) $G_\infty^\psi \upharpoonright X_\alpha$ is dense in $L_\infty^\psi \upharpoonright X'_\alpha$ for each $\alpha \in T$; and
(c) $G_\infty^\psi \upharpoonright X_\alpha$ is nowhere dense in $G_\infty^\psi \upharpoonright X_\beta$ for each $\alpha \in T$ and $\beta \in \text{succ}(\alpha)$.

Then $G_\infty^\psi \simeq \mathcal{E}$.

**Proof.** Observe that $\varphi := 1/(1 + \psi) \in \mathcal{SL}$ and $G_\infty^\psi \simeq G_0^\varphi$. So the conclusion $G_\infty^\psi \simeq \mathcal{E}$ follows from Proposition 3.1. \qed
Example 3.4. The conditions of Proposition 3.3 are all satisfied if \( (X_n)_{\alpha \in T} \) is the system from Example 3.2, and \( \psi : \mathbb{Q}^\omega \to [0, \infty] \) is defined by \( \psi(x) = \|x\| \).

4. Proof of Theorem 1.1

Let \( X = \{ x \in \mathbb{Z}^\omega : s_n \to \infty \} \). Define \( \psi : X \to [0, \infty] \) by \( \psi(x) = t_x \). The function \( \psi \) is lower semi-continuous [12, Observation 3.1], and \( \bar{E}(f) \simeq G_\infty^\psi \) by Corollary 2.3. It remains to show \( G_\infty^\psi \simeq E \). This will be accomplished by constructing a Sierpiński stratification of \( X \) so that the conditions in Proposition 3.3 are satisfied.

Our tree \( T \) will be a subset of \( (\mathbb{N} \times \mathbb{Z})^\omega \). We may identify each \( \alpha \in (\mathbb{N} \times \mathbb{Z})^\omega \) with an \( n \)-tuple of ordered pairs \( \langle \langle N_0, s_0 \rangle, \langle N_1, s_1 \rangle, \ldots, \langle N_{n-1}, s_{n-1} \rangle \rangle \) where \( n = \text{dom}(\alpha) \) and \( (N_i, s_i) = \alpha(i) \). Given \( \alpha \in (\mathbb{N} \times \mathbb{Z})^\omega \) we define \( N(\alpha(i)) = N_i \) and

\[
\alpha^- \langle N, s \rangle = \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1), (N, s) \rangle.
\]

Let \( X_\emptyset = X \). Supposing \( X_\alpha \) has been defined, for each \( \langle N, s \rangle \in \mathbb{N} \times \mathbb{Z} \) let

\[
X_{\alpha^- \langle N, s \rangle} = \{ x \in X_\alpha : s_{\text{dom}(\alpha)} = s \text{ and } s_n \geq \text{dom}(\alpha) + 1 \text{ for all } n \geq N \}.
\]

In this manner, \( X_\alpha \) is recursively defined for every \( \alpha \in (\mathbb{N} \times \mathbb{Z})^\omega \). Let

\[
T = \{ \alpha \in (\mathbb{N} \times \mathbb{Z})^\omega : X_\alpha \neq \emptyset \text{ and } N(\alpha(i)) \geq i \text{ for each } i < \text{dom}(\alpha) \}.
\]

Claim 4.1. \( (X_\alpha)_{\alpha \in T} \) is a Sierpiński stratification of \( X \).

Proof. We will verify (1) through (4) from the definition in Section 3.1.

Clearly \( T \) is a tree over \( \mathbb{N} \times \mathbb{Z} \), and each \( X_\alpha \) is a closed subset of \( X_\emptyset = X \). Thus (1) and (2) hold.

To see that \( X_\alpha \subset \bigcup \{ X_\beta : \beta \in \text{succ}(\alpha) \} \), let \( \bar{x} \in X_\alpha \). Since \( \bar{x} \in X \), there exists \( N \geq \text{dom}(\alpha) \) such that \( s_n \geq \text{dom}(\alpha) + 1 \) for all \( n \geq N \). Let \( \beta = \alpha^- \langle N, s_{\text{dom}(\alpha)} \rangle \). Then \( \bar{x} \in X_\beta \) and \( \beta \in \text{succ}(\alpha) \). The other inclusion is trivial, so this verifies property (3).

Finally, let \( \lambda = \langle \langle N_0, s_0 \rangle, \langle N_1, s_1 \rangle, \ldots \rangle \in [T] \) be given. The sequence \( X_{\lambda[0]}, X_{\lambda[1]}, \ldots \) clearly converges to \( \bar{x} \). We will prove \( \bar{x} \in X \) by showing that \( s_n \geq k + 1 \) for each \( k < \omega \) and \( n \geq N_k \). To that end, let \( k < \omega \) and suppose \( n \geq N_k \). Since \( \lambda \upharpoonright n + 1 \in T \), there exists \( \bar{x}^0 \in X_{\lambda[n+1]} \). Then \( s_n^0 = s_n \). Further, \( \lambda \upharpoonright k + 1 \in T \) implies \( N_k \geq k \) which gives us \( n \geq k \). Thus \( X_{\lambda[n+1]} \subset X_{\lambda[k+1]} \) and so \( \bar{x}^0 \in X_{\lambda[k+1]} \). Therefore \( s_n^0 \geq k + 1 \). We have \( s_n = s_n^0 \geq k + 1 \) as desired. This completes the proof of (4). \( \square \)

The next three claims will establish (a), (b) and (c) of Proposition 3.3.

Claim 4.2. \( X_\alpha' = \{ \bar{x} \in X_\alpha : t_\bar{x} < \infty \} \) is dense in \( X_\alpha \) for each \( \alpha \in T \).

Proof. Let \( \bar{x} \in X_\alpha \). For each \( n \geq \text{dom}(\alpha) \) define \( \bar{x}^n \in \mathbb{Z}^\omega \) by \( \bar{x}^n \upharpoonright n = \bar{x} \upharpoonright n \) and \( s^n_k = k \) for all \( k \geq n \). Clearly \( \bar{x}^n \to \bar{x} \) and \( \bar{x}^n \in X_\alpha \). Additionally,

\[
\sup_{k \geq 1} F^{-k}(2\pi|s^n_k|) < \infty
\]

because \( F^{-n}(2\pi n) \geq F^{-k}(2\pi k) \) for all \( k \geq n \). Hence \( t_{\bar{x}^n} < \infty \) by [2, Observation 3.7]. Therefore \( \bar{x}^n \in X_\alpha' \). We conclude that \( \bar{x} \) lies in the closure of \( X_\alpha' \), which shows that \( X_\alpha' \) is dense in \( X_\alpha \). \( \square \)

Claim 4.3. \( G_\infty^{\psi |X_\alpha} \) is dense in \( L_\infty^{\psi |X_\alpha} \) for each \( \alpha \in T \).
PROOF. Fix $\mathbf{s}^0 \in X''$. We will show that $[\psi(\mathbf{s}^0), \infty] \times \{\mathbf{s}^0\} \subset \overline{G^{X}_\infty}$.

Let $A_{2^\omega} = \{\mathbf{s} \in \mathbb{Z}^\omega : s_n \geq s_n^0 \text{ for every } n < \omega \} \subset X$. Then $\{(t_2, \mathbf{s}) : \mathbf{s} \in A_{2^\omega}'\}$ is the set of endpoints of

$$\bigcup_{\mathbf{s} \in A_{2^\omega}'} [t_2, \infty] \times \{\mathbf{s}\}.$$ 

The latter set is denoted $X_{2^\omega}(\mathcal{F})$ in [2]. By [2, Theorem 3.6], the endpoints of $X_{2^\omega}(\mathcal{F})$ are dense in $X_{2^\omega}(\mathcal{F})$. Thus we have

$$\overline{G^{\psi|A_{2^\omega}'}_{\infty}} = \{(t_2, \mathbf{s}) : \mathbf{s} \in A_{2^\omega}'\} = \bigcup_{\mathbf{s} \in A_{2^\omega}'} [t_2, \infty] \times \{\mathbf{s}\} = L_{\infty}^{\psi|A_{2^\omega}'}.$$ 

Now let $B_{2^\omega} = \{\mathbf{s} \in \mathbb{Z}^\omega : \mathbf{s} \upharpoonright \text{dom}(\alpha) = \mathbf{s}^0 \upharpoonright \text{dom}(\alpha)\}$, and let $C_{2^\omega} = A_{2^\omega} \cap B_{2^\omega} \subset X_{\alpha}$.

Since $B_{2^\omega}$ is clopen in $\mathbb{Z}^\omega$, the equation above implies that

$$\overline{G^{\psi|C_{2^\omega}}_{\infty}} = L_{\infty}^{\psi|C_{2^\omega}}.$$ 

Therefore $[\psi(\mathbf{s}^0), \infty] \times \{\mathbf{s}^0\} \subset \overline{G^{\psi|C_{2^\omega}}_{\infty}} \subset \overline{G^{\psi|X_{\alpha}}_{\infty}}$. $\square$

CLAIM 4.4. $G^{\psi|X_{\alpha}}_{\infty}$ is nowhere dense in $G^{\psi|X_{\alpha}}_{\infty}$ for each $\alpha \in T$ and $\beta \in \text{succ}(\alpha)$.

PROOF. Since $G^{\psi|X_{\alpha}}_{\infty}$ is a closed subset of $G^{\psi|X_{\alpha}}_{\infty}$, it suffices to show that $G^{\psi|X_{\alpha} \setminus X_\beta}_{\infty}$ is dense in $G^{\psi|X_{\alpha}}_{\infty}$. To that end, let $(t_2, \mathbf{s}) \in G^{\psi|X_{\alpha}}_{\infty}$.

We will demonstrate that a sequence of points in $G^{\psi|X_{\alpha} \setminus X_\beta}_{\infty}$ converges to $(t_2, \mathbf{s})$. For each $n < \omega$ define $s^n \in \mathbb{Z}^\omega$ by $s^n_i = s_i$ for all $i \neq n$, and $s^n_0 = \min\{s_n, \text{dom}(\alpha)\}$. Clearly $|s^n| \leq |s_i|$ for all $i < \omega$. So $t_2^n \leq t_2$ by [2, Observation 3.7]. Then $(t_2^n, \mathbf{s}^n) \to (t_2, \mathbf{s})$ by lower semi-continuity of $\psi$. Note that

$$(t_2^n, \mathbf{s}^n) \in G^{\psi|X_{\alpha} \setminus X_\beta}_{\infty}$$

when $n \geq N_{\text{dom}(\alpha)} := N(\beta(\text{dom}(\alpha)))$. Therefore $(t_2, \mathbf{s}) \in G^{\psi|X_{\alpha} \setminus X_\beta}_{\infty}$. $\square$

From Claims 4.1 through 4.4 and Proposition 3.3, we conclude that $\tilde{E}(f) \simeq \mathcal{E}$.

5. The point at infinity

In proving that $\mathcal{E}$ is 1-dimensional, Erdős showed that $\mathcal{E} \cup \{\infty\}$ is connected, where $\infty$ is the single point needed to compactify $\ell^2$.

We have shown that $\tilde{E}(f)$ is a topological embedding of $\mathcal{E}$ into the complex plane, and we can add that $\tilde{E}(f) \cup \{\infty\}$ is connected, $\infty$ here being the point at infinity on the Riemann sphere. This we will argue using the homeomorphism $\mathcal{H}$ from Proposition 2.2 and the sets defined in Claim 4.3.

THEOREM 5.1. $\tilde{E}(f) \cup \{\infty\}$ is connected.

PROOF. For every $\mathbf{s}^0 \in X'$, $H^{\psi|A_{2^\omega}'}(L_{\infty}^{\psi|A_{2^\omega}'}) \cup \{\infty\}$ is a Lelek fan (a smooth fan with a dense set of endpoints) by [2, Theorem 3.6]. The endpoint set of any Lelek fan becomes connected when the ramification point is added.
to it [2, Section 2]. Thus

\[ H\left( G_{\psi}^{A,0} \right) \cup \{ \infty \} \]

is a connected subset of \( \bar{E}(f) \cup \{ \infty \} \) for every \( \varphi^0 \in X' \). Every point of \( \bar{E}(f) \) is contained in a set of that form. Therefore \( \bar{E}(f) \cup \{ \infty \} \) can be written as a union of connected sets each containing the point \( \infty \).

This makes \( \bar{E}(f) \) a particularly nice embedding of \( \mathcal{E} \). For complete Erdős space \( \mathcal{E}_c \), this type of embedding was discovered by Kawamura, Oversteegen, and Tymchatyn when they proved \( E(f) \simeq \mathcal{E}_c \) [8]. Mayer [11] had already established the connectedness of \( E(f) \cup \{ \infty \} \).

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