Comment on “Can disorder really enhance superconductivity?”

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The paper by Mayoh and Garcia-Garcia [arXiv:1412.0029v1] is entitled “Can disorder really enhance superconductivity?”. In our opinion, the answer given by the authors is not satisfactory. We present the alternative picture.

Mean field solution. A basis for description of the spatially inhomogeneous superconductivity is given by the Gor’kov equation for the order parameter $\Delta(r)$

$$\Delta(r) = \int K(r,r')\Delta(r')d^dr'$$

with the kernel $K(r,r')$ satisfying the sum rule [2]

$$\int K(r,r')d^dr' = g\nu_F(r)\ln \frac{1.14\omega_0 T}{\nu_F}$$

where $g$ is the Cooper interaction constant, $\nu_F(r)$ is the local density of states at the Fermi level, $\omega_0$ is a cut-off frequency, and $d$ is a dimension of space.

The Anderson theorem [3] follows from Eq.1 under assumption of a self-averaging order parameter, when $\Delta(r)$ and $K(r,r')$ can be independently averaged over disorder. Since $\langle \Delta(r) \rangle$ does not depend on $r$ due to the spatial uniformity in average, the use of the sum rule (2) gives

$$\langle \Delta \rangle = g\nu_F\ln \frac{1.14\omega_0 T}{\nu_F} \langle \Delta \rangle$$

and the critical temperature $T_c$ is given by the BCS formula, which contains the average density of states $\langle \nu_F \rangle$. If the latter is maintained fixed, $T_c$ is not changed by disorder. However, self-averaging does not hold in the general case, and deviations from the Anderson theorem arise.

Equation (1) can be accurately solved for a small concentration of the point-like impurities [4]. This solution shows possibility of two regimes. The first one corresponds to the moderate variation of $\Delta(r)$ in space, so that it remains more or less of the same order in the whole space. The corresponding $T_c$ is given by the formula

$$\frac{\delta T_c}{T_{c0}} = \frac{1}{\lambda L^d} \int d^dr \nu_0\nu_1(r) + \nu_1(r)^2$$

where $\nu_1(r)$ is a deviation of the local density of states $\nu_F(r)$ from its unperturbed value $\nu_0$, $\lambda = g\nu_0$ is the dimensionless coupling constant, $T_{c0}$ is the transition temperature in the absence of disorder, $L^d$ is a volume for one impurity, and integration is carried out over a vicinity of the single point defect. The linear in $\nu_1(r)$ term exactly corresponds to the Anderson theorem and relates the change in $T_c$ with the change of the average density of states. Generally, $\nu_1(r)$ is comparable with $\nu_0$ and already Eq.4 predicts a possibility of essential violation of the Anderson theorem. It is related with the fact that the initially uniform order parameter is influenced by point defects and can increase or decrease in their vicinity.

More essential deviations from the Anderson theorem arise in the second regime, when the order parameter is mainly localized at the small number of ”resonant” impurities producing the quasi-local states near the Fermi level. The corresponding estimate for $T_c$ [4]

$$T_c \sim g a^{-d} \sim \lambda E_0$$

($a$ is the lattice spacing, $E_0$ is a scale of the order of the Fermi energy or the bandwidth) is valid for small $g$ and saturates by a quantity of the order $\omega_0$, when $g$ increases.

Results (4) and (5), obtained for a small concentration $c$ of impurities, can be qualitatively extrapolated into the $c \sim 1$ region. It allows to give the
adequate answer to the question in the title of the paper:

(a) Disorder can enhance $T_c$ by a trivial reason, due to increase of the average density of states.

(b) If the average density of states remains fixed, then $T_c$ can be changed due to deviations from the Anderson theorem. This effect is always positive in the framework of formula (4).

(c) There is a possibility of the catastrophic increase of $T_c$ due to resonances on the quasi-local levels, though this regime is affected by fluctuations (see below).

**Role of multifractality.** Recently there have been claims [5 6] that a great increase of $T_c$ is possible in the vicinity of the Anderson transition due to multifractality of wave functions. The analysis of [5 6] is based on equation

\[ \Delta(\epsilon) = \frac{\lambda}{2} \int_{-\omega_0}^{\infty} \frac{I(\epsilon, \epsilon') \Delta(\epsilon')}{\sqrt{\epsilon'^2 + \Delta(\epsilon')}} \mathrm{tanh} \left( \frac{\sqrt{\epsilon'^2 + \Delta(\epsilon')}}{2T} \right) \mathrm{d}\epsilon' \]

with the kernel of the form

\[ I(\epsilon, \epsilon') = \frac{E_0}{\epsilon - \epsilon'} \]

motivated by multifractal properties of wave functions. $T_c$ is determined by equation (6) linearized in $\Delta$: accepting $\Delta$ to be a function $\epsilon/T$ and dimensionalizing the integral, one has for infinite $\omega_0$ [5 6]

\[ T_c \sim E_0 \lambda^{1/\gamma}. \]

The singular limit of small $\gamma$ was analyzed in [4] and lead to result

\[ T_c \sim E_0 \left[ \left( \frac{E_0}{\omega_0} \right)^{\gamma} + \gamma \right]^{-1/\gamma} \]

which reproduces (8) in the limit $\omega_0 \to \infty$, the BCS result in the limit $\gamma \to 0$ and describing saturation of (8) by a value $\sim \omega_0$ for large $\gamma$. The claim of [3] that $\omega_0$ is always less than $E_0$ is incorrect: $E_0$ can be small in semimetals and narrow band materials.

The linearized form of equation (6) is not equivalent to (1) (see discussion in [3]) and based on two assumptions: (i) truncation of the Hamiltonian in the BCS spirit, and (ii) averaging of the kernel independently of $\Delta$. Both approximations are uncontrol-

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1 It means a self-averaging assumption, but in the modified form: instead of the usual equation $\langle \Delta \rangle = \langle K \Delta \rangle$ one uses $\langle S \Delta \rangle = \langle SKS^{-1} \rangle \langle S \Delta \rangle$ where $S$ is a certain operator. It can be interpreted in the variational spirit, considering $S$ as a kind of the trial function.

The partial justification of (i) was suggested in [1]: truncation of the Hamiltonian is rigorous for a pure superconductor and should be valid approximately in the case of weak spatial inhomogeneity. One can partially excuse assumption (ii), using some effective exponent in the capacity of $\gamma$. Indeed, the kernel $I(\epsilon, \epsilon')$ is determined by a matrix element $\int d^d r |\psi(\epsilon, r)|^2 |\psi(\epsilon', r)|^2$ whose estimation gives different values of $\gamma$ for the “usual” and “typical” averaging [6]: this uncertainty is aggravated, if averaging is made with the weight $\Delta(r)$.

The first argument explains why the strong localization regime (corresponding to (5)) does not contain in (6) for $I(\epsilon, \epsilon') = const$. The second argument shows that a difference between (5) and (8) arises on the level, which is not controllable in the approach based on Eq.6. In fact, the difference between (5) and (8) is not quantitative, but qualitative. Result (5) is not restricted by a vicinity of the mobility edge and remains approximately the same for any energy inside the band; correspondingly, it has no relation to multifractality. Nevertheless, the real physical mechanism is the same for results (5) and (8) and related with resonances at quasi-local levels [4]. Indeed, if the local density of states $\nu(\epsilon, r)$ is considered as a smooth function of $\epsilon$, then its variation is finite: it corresponds to shifts of the whole band by a value $W$ or $-W$ for the Anderson model with distribution of site energies in the interval $(-W, W)$. Unbounded fluctuations of $\nu(\epsilon, r)$, arising in the context of multifractality, are necessarily related with a partial discretization of the spectrum due to a presence of quasi-local levels. If the usual value of $\gamma$ is exploited in (7), then the $T_c$ value given by (5) is greater than (8); it means that the Cooper instability occurs at configurations, which are governed by individual peaks and not fractal clusters [4].

Weak multifractality considered in [1] is practically actual only for the 2D case in the regime of weak disorder. However, this regime is described by formula (4), which can be obtained in this case making two iterations of the Gor’kov equation and exploiting the sum rule (2). In this case one can integrate along all the system and use its whole volume in the capacity of $L^d$, dividing of disorder into separate “impurities” is not necessary. There is no need to use the approximate equation (6), when the accurate result is available. By the way, in the framework of (4) the order parameter is proportional to $\nu^r(r)$ and the logarithmically normal distribution for $\Delta(r)$ [1] follows trivially in the weak multifrac-
tality regime.

It is clear from this consideration, that multifractality is not a direct cause for the increase of $T_c$ and results of kind (8) are related with the more universal mechanism.

Role of fluctuations. Equation (5) is a result of the mean field theory. The corresponding configuration of the order parameter is a uniform background $\Delta_0$ with abrupt peaks at resonant impurities, whose concentration is of the order $T_c/E_0$. The order parameter can be considered as positive\(^2\), and so its phase is the same in the whole volume. When we come to a fluctuational description, the modulus of the order parameter remains practically unchanged, while the essential phase fluctuations arise. If the uniform background is neglected, then the system is divided into practically independent superconducting "drops", whose phases are fluctuating freely and destroy the macroscopical coherence of the superconducting state. If the uniform contribution $\Delta_0$ is taken into account, the Josephson coupling between drops arises and their phases become correlated. The accurate fluctuational analysis of such a system is nontrivial, but the general character of results is the same as for the granular superconductors [5]. If the ratio $T_c/E_0$ is not too small, then the resonant impurities are close to each other and their Josephson interaction is strong enough for stabilization of the mean-field solution at practically the same $T_c$ value. Contrary, if $T_c/E_0$ is sufficiently small, then the Josephson coupling between drops is weak and fluctuations destroy superconductivity at temperatures close to the mean-field $T_c$ value. However, decreasing of temperature stimulates the growing of tails of the localized solutions [9]; the Josephson coupling between drops increases and stabilizes superconductivity before $T_{c0}$ is reached. Hence, fluctuations suppress $T_c$ in comparison with its mean-field value, but do not eliminate enhancement of $T_c$ completely.

Analogous arguments can be given for configurations corresponding to (8), where a fraction of the superconducting phase is estimated as $\langle T_c/E_0 \rangle^7$ [6]. However, such configurations are not actual, since the Cooper instability corresponds to (5).

The paper [1] suggests another way to deal with fluctuations. Firstly, solution of (6) for $T = 0$ is found, giving the spatially inhomogeneous order parameter $\Delta_0(r)$. Secondly, the field $T_c(r)$ of "local $T_c$" values is introduced, such as $T_c(r) \propto \Delta_0(r)$ [4]. Finally, the global $T_c$ is defined as a percolation threshold in the field $T_c(r)$. Such percolation picture has a sense for certain conditions [10], but it is not the case for weak spatial inhomogeneity.

Indeed, the Gor’kov equation (1) defines the spatially inhomogeneous configuration $\Delta(r)$, which appears at a certain temperature. This temperature, by definition, is a final result for $T_c$: there is no need to consider "local $T_c$" or "percolation". Of course, one should attend for insignificance of fluctuations, but this condition is rather weak. The Ginzburg number is incredibly small for a pure superconductor; it increases gradually with increasing of spatial inhomogeneity, but it is possible to reach rather large values of ratio $\Delta_{\text{max}}/\Delta_{\text{min}}$ before this number will approach unity and fluctuations will become essential. Surely, no percolation is necessary for weak spatial inhomogeneity. The percolation picture becomes reasonable for the Ginzburg number of the order of unity, when the mean-field estimate of $T_c$ is poorly defined and the use of percolation allows to refine it.

Role of interaction. The BCS constant $\lambda$ corresponds to some effective interaction. In a more detailed description it is combined from the electron-phonon coupling and the Coulomb pseudopotential. The latter is known to increase in the presence of disorder and it is the main cause for $T_c$ degradation [11, 12]. This effect was not considered explicitly in [1, 4, 5, 6], but is surely essential in discussion of the experimental situation. It is the main reason why enhancement of $T_c$ is a rare thing in reality.

In paper [7] the result analogous to (8) is obtained in the framework of the Finkelstein renormalization group approach. However, it is completely different from [5, 6] in the initial assumptions and the discussed physical mechanism. The Cooper constant $g$ is kept fixed in [5, 6], but an attempt is made to advance beyond applicability of the Anderson theorem. Contrary, the authors of [7] consider renormalization of $g$ by disorder, while self-averaging is accepted for granted. By the latter reason, the strongly localized regime was not accessible in this approach, while the obtained effect is lesser than (5). The questions also arise, how results of [7] agree with the usual picture\(^3\).

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\(^2\)In the absence of magnetic effects, the kernel $K(r, r')$ is positive, and the Cooper instability corresponds to the nodeless eigenfunction.

\(^3\)In fact, this relation is violated due to the presence of scale $T_{c0}$. 

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of the Coulomb pseudopotential enhancement.

In conclusion, disorder can enhance $T_c$: (a) due to increase of the average density of states; (b) due to deviations from the Anderson theorem; (c) due to resonances on the quasi-local levels. The latter regime is affected by fluctuations, and in some cases can be essentially suppressed. Practically, enhancement of $T_c$ by disorder is a rare thing due to the increase of the Coulomb pseudopotential.

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References

[1] J. Mayoh, A. M. Garcia-Garcia, arXiv:1412.0029v1.

[2] P. G. Gennes, Rev. Mod. Phys., 36, 225 (1964).

[3] P. W. Anderson, J. Phys. Chem. Solids 11, 26 (1959).

[4] A. A. Abrikosov, L. P. Gor’kov, Zh. Eksp. Teor. Fiz. 35, 1158 (1958).

[5] I. M. Suslov, Zh. Eksp. Teor. Fiz. 144, 1184 (2013) [J. Exp. Theor. Phys 117, 1042 (2013)].

[6] M. V. Feigelman, L. B. Ioffe, V. E. Kravtsov, E. A. Yuzbashyan, Phys. Rev. Lett. 98, 027001 (2007).

[7] M. V. Feigelman, L. B. Ioffe, V. E. Kravtsov, E. Cuevas, Annals of Physics 325, 1368 (2010).

[8] I. S. Burmistrov, I.V.Gornyi, A.D.Mirlin, Phys. Rev. Lett. 108, 017002 (2012).

[9] L. B. Ioffe, A. I. Larkin, Zh. Eksp. Teor. Fiz. 81, 707 (1981) [Sov. Phys. JETP 54, 378 (1981)].

[10] L. N. Bulaevskii, M. V. Sadovskii, J. Low-Temp. Phys. 59, 89 (1985).

[11] M. V. Sadovskii, Phys. Reports 282, 225 (1997).

[12] I. M. Suslov, arXiv:1501.05148

[13] J. Mayoh, A. M. Garcia-Garcia, arXiv:1502.06282.

Reply to arXiv:1502.06282

As a reaction to the preceding overview [13], Mayoh and Garcia-Garcia have submitted the comment [14]. This comment has a form of the personal attack and contains the whole series of untrue statements. We give brief remarks, in order to reveal these statements.

1. Our mean field results for a superconductor with a small concentration of the point-like impurities [4] are based on the accurate solution of the Gor’kov equation. These results still persist, even if somebody does not like them. Contrary, the approach of [1] involves uncontrollable approximations [13].

2. It is repeatedly stated in [14] that we predict $T_c \sim 5000 K$ for the superconducting transition temperature, which ”has never, and very likely will never, been found numerically or experimentally”. In fact, it was clearly indicated (see the text after Eq.2 in [4]) that $T_c$ is bounded by the quantity $\omega_0/\pi$ ($\omega_0$ is the cut-off frequency), which for the phonon mechanism corresponds to values already observed for oxide superconductors ($T_c \sim 150 K$, $\omega_0 \sim 400 K$). A possibility to describe the latter in terms of our model was discussed in Sec.7 of [4].

3. The authors of [14] write on ”unjustified use of the mean field approximation”. In fact, insufficiency of the mean field approach was indicated in [4] [13] and the role of fluctuations was extensively discussed. We cannot help, if our arguments are ignored.

4. According to [14], our formula (4) in [13] is related with ”poorly defined” quantities and unclear conditions of applicability. In fact, all explanations were given in [4] [13] and we can repeat those of them that are questioned in [14]: $L^d$ is a volume for one impurity, i.e. the whole volume divided for a number of impurities; the quantity $\nu_1(r)$ is the difference $\nu_F(r)-\nu_0$, where $\nu_F(r)$ is the local density of states, defined in a standard manner (see Eq.8 in [4]), and $\nu_0$ is its unperturbed value. For a small amplitude of disorder ($|\nu_1(r)| \ll \nu_0$) the indicated formula can
be obtained by two iterations of the Gor’kov equation \[13\]. It remains valid for a small concentration of strong impurities, when \(|\nu(r)| \sim \nu_0\) \[4\]. It is not restricted by any assumption on the correlation between impurities.

5. According to \[14\], ”Suslov claims … that we do not discuss phase-fluctuations”. There is no such statement in \[13\]; our comments on the role of interaction are brief and contain no criticism of \[1\].

6. According to \[14\], there is a wrong statement in \[13\]: ”Suslov also states that we claim \(E_0\) is always larger than \(\epsilon_D\)”. One can compare it with the text in \[1\]: ”We believe that this is necessary since \(\epsilon_D \ll E_0\) so it is inconsistent to take the Debye energy to infinity while keeping \(E_0\) finite”. This argument was used in \[1\] to reject the result \(T_c \sim E_0\lambda^{1/7}\) appearing in preceding publications: ”This is an expression that, we are at pains to stress, is not recovered in our formalism” \[14\]. In fact, this result follows from Eq.22 of \[1\] in the limit \(\epsilon_D \to \infty\), independently of the desire of authors.

7. According to \[14\], ”Suslov claims that our results are only valid in small and strictly two dimensional systems”, while they are valid also for thin films and two-dimensional systems with spin-orbit interactions. In fact, it is written in \[13\] that ”weak multifractality considered in \[1\] is practically actual only for the 2D case in the regime of weak disorder”, and there is no rejection of the indicated additional applications.

8. Our main criticism of \[1\] refers to the use of the percolation picture \[11\] beyond the limits of its applicability. Indeed, the case of the ”moderate spatial inhomogeneity” is described by the Gor’kov equation, which has solution \(\Delta(r)\) arising at a certain temperature. This temperature, by definition, is a final result for \(T_c\): there is no need to consider ”local \(T_c\)” or ”percolation”. Such situation persists, till fluctuations are insignificant and the mean field estimation of \(T_c\) is well-defined. The percolation picture becomes reasonable for ”very strong spatial inhomogeneity”, when the Ginzburg number becomes of the order of unity; in this case the mean-field estimate of \(T_c\) is poorly defined and the use of percolation allows to improve it.

The authors of \[1\] try to disprove our arguments basing on the difference between ”weak inhomogeneity” and ”weak multifractality”. This attempt is not successful due to the facts:

(a) The suggested in \[1\] partial justification of Eq.6 in \[13\] refers to ”weak inhomogeneity” and not to ”weak multifractality”.

(b) It is clear from \[13\] that for applicability of percolation, the ratio \(\Delta_{max}/\Delta_{min}\) should be greater than some large parameter; here \(\Delta_{max}\) and \(\Delta_{min}\) are typical (not exclusive) values. The distribution for \(\Delta(r)\) is presented in Fig.4 of \[1\]. This distribution can be cut-off on both sides, since its tails correspond to local perturbations, which have no consequences for global superconductivity. After that the ratio \(\Delta_{max}/\Delta_{min}\) is typically of the order of unity: it corresponds to ”moderate inhomogeneity” and applicability of the Gor’kov equation.

(c) The situation can be discussed constructively. Practically, ”weak multifractality” corresponds to the weakly disordered 2D case, which is described by formula (4) in \[13\]. The corresponding order parameter is either slightly perturbed (for weak impurities), or its perturbations are local (for a small concentration of strong impurities). In both cases, there are no problems with fluctuations, and hence there is no place for percolation.

The authors of \[14\] write correctly that ”according to Suslov our percolation analysis is intended to describe fluctuations”: it is right, in spite of their objections. They are wrong in ascribing to us an idea that ”the value of \(T_c\) resulting from percolation is similar to that obtained by averaging over \(T_c(r)\)”.

9. It is stated in \[14\] that ”in weakly coupled superconductor \(\lambda \ll 1\) so \(T_c/E_0\) is always small and the global critical temperature must be necessary zero”. This statement reveals a complete misunderstanding of our arguments. We say that, for not very small ratio \(T_c/E_0\), the average distance \(a (E_0/T_c)^{1/3}\) between resonant impurities becomes comparable with \(a\) and the background value \(\Delta_0\) of the order parameter becomes comparable with its resonant peaks. Then our ”strongly localized regime” becomes ”moderately localized” and has no problems with fluctuations. In any case, we do not restrict ourselves by weak coupling, and in the worst situation \(T_c\) falls to \(T_{c0}\) and not to zero.