COUNTING RATIONAL POINTS ON CUBIC HYPERSURFACES

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Abstract. Let $X \subset \mathbb{P}^N$ be a geometrically integral cubic hypersurface defined over $\mathbb{Q}$, with singular locus of dimension $\leq \dim X - 4$. Then the main result in this paper is a proof of the fact that $X(\mathbb{Q})$ contains $O_{\epsilon,X}(B^\dim X + \epsilon)$ points of height at most $B$.

1. Introduction

Let $C \in \mathbb{Z}[x_1, \ldots, x_n]$ be an absolutely irreducible cubic form, defining a hypersurface $X_C \subset \mathbb{P}^{n-1}$. The primary goal of this paper is to investigate the density of rational points on $X_C$. Given any rational point $x = [\mathbf{x}] \in \mathbb{P}^{n-1}(\mathbb{Q})$, with $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and $\gcd(x_1, \ldots, x_n) = 1$, we write $H(x) := \max_{1 \leq i \leq n} |x_i|$. Let

$$N_{X_F}(P) := \#\{x \in X_F \cap \mathbb{P}^{n-1}(\mathbb{Q}) : H(x) \leq P\},$$

for any $P \geq 1$ and any hypersurface $X_F \subset \mathbb{P}^{n-1}$ defined by a form $F \in \mathbb{Z}[\mathbf{x}]$ in $n$ variables. This paper is motivated by the following basic conjecture, due to Heath-Brown [8, Conjecture 2].

Conjecture. Let $\epsilon > 0$ and suppose that $F \in \mathbb{Z}[x_1, \ldots, x_n]$ is an absolutely irreducible form of degree $d \geq 2$. Then we have

$$N_{X_F}(P) = O_{d,\epsilon,n}(P^{n-2+\epsilon}).$$

We will henceforth refer to this conjecture as the “uniform dimension growth conjecture”. There is a version of the conjecture in which one allows an arbitrary dependence on the coefficients of the form under consideration. We will refer to this as the “dimension growth conjecture”. These conjectures are essentially best possible, as examples of the shape

$$F(\mathbf{x}) = x_1^d + x_2(x_3^{d-1} + \cdots + x_n^{d-1})$$

show. Here, one obtains $N_{X_F}(P) \gg P^{n-2}$ by considering rational points of the shape $[0, 0, x_3, \ldots, x_n]$. The dimension growth conjectures have received considerable attention in recent years, to the extent that the uniform dimension growth conjecture is known to hold when $F$ is non-singular, when $d = 2$ or $n \leq 5$, and when $d \geq 4$. This is one of the major outcomes of the body of work [1, 2, 8, 11]. Thus the single outstanding case concerns singular absolutely irreducible cubic forms $C \in \mathbb{Z}[x_1, \ldots, x_n]$, with $n \geq 6$. In this setting, the best general result available is due to Salberger [11], who has shown that

$$N_{X_C}(P) \ll_{\epsilon,n} P^{n-3+2/\sqrt{3}+\epsilon},$$

(1.1)

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for any $\varepsilon > 0$. Note that $-1 + 2/\sqrt{3} = 0.1547 \ldots$

Let $s(C) \in \mathbb{Z} \cap [-1, n-2]$ denote the projective dimension of the singular locus of the hypersurface $X_C$. When $s(C) = -1$ we have already seen that the uniform dimension growth conjecture holds. One might therefore hope to improve on (1.1) when $s(C)$ is not too large compared with $n$. This is the point of view adopted by Salberger [12], who has recently established the validity of the conjecture for $n \geq 8 + s(C)$. Salberger’s result is based on refining an argument involving the $q$-analogue of van der Corput’s method for exponential sums, developed by Heath-Brown [7]. As observed independently by the author and Salberger, the weaker bound $n \geq 11 + s(C)$ is readily derived from Heath-Brown’s work. At the expense of uniformity in the implied constant we will be able to improve on all of these results. Let $\|C\|$ denote the maximum modulus of the coefficients of the underlying cubic form $C$. Then the following is our main result.

**Theorem.** Let $X_C \subset \mathbb{P}^{n-1}$ be a geometrically integral cubic hypersurface, and let $\varepsilon > 0$. Assume that $n \geq 6 + s(C)$. Then there exists a positive number $\theta = O_n(1)$ such that

$$N_{X_C}(P) \ll_{\varepsilon, n} \|C\|^\theta P^{n-2+\varepsilon}.$$ 

In particular, the dimension growth conjecture holds for $n \geq 6 + s(C)$.

With more work, the value of $\theta$ could be made explicit in the statement of the theorem. We have not attempted to do so here, however, being content to show that there is at worst polynomial dependence on $\|C\|$. An inspection of the proof of the theorem reveals that we obtain a small improvement on the exponent $n - 2$ as soon as $n > 6 + s(C)$.

We will actually establish a version of the theorem for the number of integral points in certain expanding regions that lie on arbitrary affine cubic hypersurfaces $g = 0$. The proof of this estimate will be established by induction on the dimension $s(g_0)$ of the singular locus of the projective hypersurface $g_0 = 0$, where $g_0$ denotes the cubic part of the polynomial $g$. The idea is to use hyperplane sections to reduce consideration to a family of hypersurfaces in $\mathbb{A}^5_{\mathbb{Q}}$, for which the dimension of the relevant singular locus is reduced by 1. The main work comes in having to handle the inductive base $s(g_0) = -1$, and this will be dealt with by an application of the Hardy–Littlewood circle method.

**Notation.** Throughout our work $\mathbb{N}$ will denote the set of positive integers. For any $\alpha \in \mathbb{R}$, we will follow common convention and write $e(\alpha) := e^{2\pi i\alpha}$ and $e_q(\alpha) := e^{2\pi i\alpha/q}$. We will allow both the small positive quantity $\varepsilon$ and the constant $\theta$ to vary from time to time, so that we may write $P^\varepsilon \log P \ll_{\varepsilon} P^\varepsilon$ and $H^{2\theta} \ll H^{\theta}$, for example. All of the implied constants in our work are allowed to depend implicitly on $\varepsilon$ and $n$, and $\theta$ will always be bounded in terms of $n$ alone. Finally, we will write $|z| := \max_{1 \leq i \leq n} |z_i|$ for the norm of any vector $z \in \mathbb{R}^n$, and we will use the notation $A \asymp B$ to denote that $A \ll B \ll A$. 
2. Preliminaries

It will be convenient to work with infinitely differentiable weight functions \( w : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \), with compact support. Given such a function \( w \), we set \( R(w) \) to be the smallest \( R \) such that \( w \) is supported in the hypercube \([-R, R]^n\), and we let

\[
R_j(w) := \max \left\{ \left| \frac{\partial^{j_1} \cdots \partial^{j_n} w(x)}{\partial^{j_1} x_1 \cdots \partial^{j_n} x_n} \right| : x \in \mathbb{R}^n, \ j_1 + \cdots + j_n = j \right\},
\]

for each integer \( j \geq 0 \). Let constants \( c_n \) and \( c_{n,j} \) be given, and define \( W_n \) to be the set of infinitely differentiable functions \( w : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) of compact support, such that \( R(w) \leq c_n \) and \( R_j(w) \leq c_{n,j} \) for all \( j \geq 0 \). In future all our order constants will be allowed to depend on \( c_n \) and the \( c_{n,j} \), without further comment.

Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial and let \( w \in W_n \). All our efforts are centred upon determining the asymptotic behaviour of the quantity

\[
N_w(g; P) := \sum_{x \in \mathbb{Z}^n, g(x) = 0} w(x/P),
\]
as \( P \to \infty \). For this we will employ the form of the Hardy–Littlewood circle method developed by Heath-Brown [6], which incorporates a single Kloosterman refinement. Define the cubic exponential sum

\[
T(\alpha) = T_P(\alpha; g, w) := \sum_{x \in \mathbb{Z}^n} w(x/P)e(\alpha g(x)),
\]
for \( P \geq 1 \). Then \( T(\alpha) \) converges absolutely, and for any \( Q \geq 1 \) we have

\[
N_w(g; P) = \int_0^1 T(\alpha) \, d\alpha = \int_{-1/1+Q}^{1-1/1+Q} T(\alpha) \, d\alpha.
\]

In [6] Heath-Brown proceeds to break the interval \([-1/1+Q, 1-1/1+Q]\) according to the Farey dissection of order \( Q \). This ultimately yields

\[
N_w(g; P) = \sum_{q \leq Q} \int_{-1/1+Q}^{1/1+Q} S_0(q; z) \, dz + O\left(Q^{-2} E_w(g; P, Q)\right),
\]
for any \( Q \geq 1 \), where

\[
E_w(g; P, Q) := \sum_{q \leq Q} \sum_{|u| \leq \frac{Q}{2}} \frac{\max_{\frac{1}{2} \leq |z| \leq 1} \left| S_u(q; z) \right|}{1 + |u|},
\]
and

\[
S_u(q; z) := \sum_{q=1}^{\infty} \sum_{\gcd(a,q)=1} c_q(\overline{a}u) T(a/q + z).
\]
This is [6, Lemma 7]. We will find that our work is optimised by taking \( Q = P^{3/2} \) in (2.2).
Recall the notation \( \|g\| \) for the maximum modulus of the coefficients of \( g \). We will follow the convention that \( g_0 \) denotes the homogeneous cubic part of \( g \). Given \( P \geq 1 \), we will need to work with the function

\[
\|g\|_P := \|P^{-3}g(Px_1, \ldots, Px_n)\|. \tag{2.4}
\]

It is clear that

\[
\|g_0\| \leq \|g\|_P \leq \|g\|
\]

with equality if \( g \) is homogeneous. The bulk of this paper will be spent establishing the following result, from which the proof of the theorem will flow rather swiftly.

**Proposition 1.** Suppose we are given \( w \in \mathcal{W}_n, \varepsilon > 0 \) and \( H \geq 1 \). Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial for which \( g_0 \) is non-singular. Assume that \( n \geq 5 \). Then there exists a positive number \( \theta \) such that

\[
N_w(g; P) \ll H^\theta P^{n-2+\varepsilon}, \tag{2.5}
\]

for any \( H \geq \|g\|_P \).

The proof of this result relies upon (2.2) in a crucial way, and will involve two basic estimates for \( S_u(g; z) \). The first uses repeated Weyl differencing, and is based on the approach taken by Davenport in [5]. This will be the subject of §3. The second estimate is based on an application of the Poisson summation formula, and in particular, the treatment of cubic exponential sums appearing in the author’s recent joint work with Heath-Brown [3]. This will be the focus of §4. Finally, in §5 we will stitch all of this together in order to complete the proof of Proposition 1.

We end this section by showing how the theorem follows from Proposition 1. Let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial with cubic part \( g_0 \). Let \( w \in \mathcal{W}_n \) and \( \varepsilon > 0 \) be given. We will show that there exists \( \theta > 0 \) such that (2.5) holds, provided that \( H \geq \|g\|_P \) and

\[
n \geq 6 + s(g_0). \tag{2.6}
\]

Before establishing this claim, let us indicate how this suffices for the statement of the theorem. Define

\[
w_1(x) := \prod_{i=1}^n \gamma(x_i),
\]

where \( \gamma : \mathbb{R} \to \mathbb{R}_{\geq 0} \) is given by

\[
\gamma(x) = \begin{cases} 
  e^{-1/(1-x^2)}, & \text{if } |x| < 1, \\
  0, & \text{if } |x| \geq 1.
\end{cases}
\]

It is clear that \( w_1 \in \mathcal{W}_n \). Let \( C \subset \mathbb{Z}[x] \) be a cubic form defining a cubic hypersurface \( X_C \subset \mathbb{P}^{n-1} \). Then it follows from the above claim that

\[
N_{X_C}(P) \ll N_{w_1}(C; P) \ll H^\theta P^{n-2+\varepsilon},
\]

provided that \( n \geq 6 + s(C_0) = 6 + s(C) \) and \( H \geq \|C\|_P = \|C\| \). This therefore establishes the theorem subject to the claim.

To confirm the claim we will argue by induction on \( s(g_0) \), the base case \( s(g_0) = -1 \) being taken care of by Proposition 1. To handle the inductive step we will use a simpler version of the argument based on hyperplane sections developed in [3, §4]. Let \( w \in \mathcal{W}_n \) and let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a
cubic polynomial such that \( s(g_0) \geq 0 \). Let \( P \geq 1 \) and let \( H \) be such that \( \|g\|_P \leq H \). Our plan will be to use hyperplane sections, in order to reduce the problem to a consideration of cubic polynomials in only \( n - 1 \) variables, whose cubic part defines a hypersurface with singular locus of dimension \( s(g_0) - 1 \). According to [3, Lemma 5] in the special case \( d = 3 \) and \( r = 0 \), there exists a primitive vector \( m \in \mathbb{Z}^n \), with \( |m| \ll 1 \), such that

\[
\dim \text{sing}(X_{g_0} \cap H_m) = s(g_0) - 1. \tag{2.7}
\]

Here \( X_{g_0} \) is the hypersurface defined by \( g_0 = 0 \) and \( H_m \) is the hyperplane \( m \cdot x = 0 \). In order to apply the induction hypothesis we will sum over affine hyperplane sections \( m \cdot x = k \), for integers \( k \ll P \). This gives

\[
N_w(g; P) = \sum_{k \ll P} \sum_{\substack{x \in \mathbb{Z}^n \\ m \cdot x = k \\ g(x) = 0}} w(x/P) = \sum_{k \ll P} N_k, \tag{2.8}
\]

say. Now \( N_k \) is zero unless there exists a vector \( t \in \mathbb{Z}^n \) such that \( m \cdot t = k \) and \( |t| \ll P \). Let us fix such a choice of vector, and write \( x = t + y \) in \( N_k \). Then clearly \( m \cdot x = k \) if and only if \( m \cdot y = 0 \). This condition defines a lattice \( \Lambda \subseteq \mathbb{Z}^n \) of rank \( n - 1 \) and determinant \( |m| \), by part (i) of [8, Lemma 1]. We now choose a basis \( e_1, \ldots, e_{n-1} \) for \( \Lambda \). Then \( 1 \ll |e_i| \ll 1 \), for \( 1 \leq i \leq n - 1 \). Moreover, any of the vectors \( y \) we are interested in can be written as \( y = \sum_{i=1}^{n-1} \lambda_i e_i \), for \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1} \) such that \( \lambda \ll P \).

Putting all of this together, we conclude that

\[
N_k = \sum_{\substack{\lambda \in \mathbb{Z}^{n-1} \\ h(\lambda) = 0}} w_0(\lambda/P) = N_{w_0}(h; P), \tag{2.9}
\]

where

\[
h(u) := g\left(t + \sum_{i=1}^{n-1} u_i e_i\right), \quad w_0(u) := w\left(P^{-1}t + \sum_{i=1}^{n-1} u_i e_i\right),
\]

and \( u = (u_1, \ldots, u_{n-1}) \). Note that \( h \) is a cubic polynomial in only \( n - 1 \) variables. We need to show that the induction hypothesis can be applied to estimate \( N_{w_0}(h; P) \). Now it is trivial to see that \( w_0 \in \mathcal{W}_{n-1} \) for our choice of \( m \) and \( t \), and furthermore, that \( \|h\|_P \ll H \). Finally, the argument in [3, §4] ensures that \( s(h_0) = s(g_0) - 1 \), under the assumption that (2.7) holds. On applying the induction hypothesis in (2.9), and combining it with (2.8), we therefore deduce that

\[
N_w(g; P) \ll H^\theta \sum_{k \ll P} P^{n-3+\varepsilon} \ll H^\theta P^{n-2+\varepsilon}.
\]

This completes the proof of (2.5) subject to (2.6).

3. Estimating \( S_u(g; z) \): Weyl differencing

In this section we will establish an estimate for the exponential sum (2.3) by arguing along the lines of Davenport [5]. The fact that we are working with possibly non-homogeneous polynomials makes no difference to the opening steps of the argument. It will be convenient to draw upon
Heath-Brown’s recent reworking of Davenport’s approach [9], where possible. Throughout this section we will suppose that \( s(g_0) = -1 \), so that the cubic part \( g_0 \) of \( g \) is non-singular.

We will sum trivially over the numerator in (2.3), giving

\[
|S_u(q; z)| \leq q \max_{1 \leq a \leq q} |T(a/q + z)|. \tag{3.1}
\]

Our interest lies with values of \( q \leq Q \) and \( |z| \leq (qQ)^{-1} \). As indicated in §2, the final analysis will be optimised by taking \( Q = P^{3/2} \). In particular, we may henceforth assume that

\[
|z| \leq q^{-1}P^{-3/2}. \tag{3.2}
\]

The purpose of this section is to establish the following result.

**Proposition 2.** Let \( q \in \mathbb{N} \) such that \( q \leq P^{3/2} \), and let \( z \in \mathbb{R} \) such that (3.2) holds. Let \( w \in W_n \), \( \epsilon > 0 \) and let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial such that \( g_0 \) is non-singular. Then we have

\[
S_u(q; z) \ll \|g_0\|^{n/8}q^{1-n/8}P^{n+\epsilon} \min \{1, (|z|P^3)^{-n/8}\}.
\]

Recall the definition (2.1) of \( T(\alpha) = T_\alpha(\alpha; g, w) \), for \( w \in W_n \). The central idea in Davenport’s approach is an application of Weyl differencing. The first step in this process produces the bound

\[
|T(\alpha)|^2 \ll \sum_{w \leq P} \left| \sum_{x} w((x + w)/P)w(x/P)e(\alpha(g(x + w) - g(x))) \right|.
\]

An application of Cauchy’s inequality now yields

\[
|T(\alpha)|^4 \ll P^n \sum_{w, x \leq P} \left| \sum_{y \in \mathbb{Z}^n} w_{w, x}(y)e(\alpha G(w, x; y)) \right|, \tag{3.3}
\]

where

\[
G(w, x; y) := g(w + x + y) - g(w + y) - g(x + y) + g(y), \tag{3.4}
\]

and

\[
w_{w, x}(y) = w((w + x + y)/P)w((w + y)/P)w((x + y)/P)w(y/P).
\]

Recall our notation \( g_0 \) for the homogeneous cubic part of the cubic polynomial \( g \). Suppose that

\[
g_0(x_1, \ldots, x_n) = \sum_{i,j,k=1}^n c_{ijk}x_ix_jx_k,
\]

in which the coefficients \( c_{ijk} \) are symmetric in the indices \( i, j, k \). On replacing \( g \) by \( 6g \), we may assume that the \( c_{ijk} \) are all integral. If we now define the bilinear forms

\[
B_i(w; x) := \sum_{j,k=1}^n c_{ijk}w_jx_k,
\]

for \( 1 \leq i \leq n \), then we find that

\[
G(w, x; y) = 6 \sum_{i=1}^n y_iB_i(w; x) + \Gamma(w, x),
\]
where $G$ is given by (3.4) and $\Gamma(w, x)$ is independent of $y$. It therefore follows from (3.3) that

$$|T(\alpha)|^4 \ll P^n \sum_{w, x \in P} \left| \sum_{y \in \mathbb{Z}^n} w_{w, x}(y) e\left(6\alpha \sum_{i=1}^{n} y_i B_i(w; x)\right) \right|.$$ 

On combining the standard estimate for linear exponential sums with partial summation, we deduce that

$$|T(\alpha)|^4 \ll P^n \sum_{w, x \in P} \prod_{i=1}^{n} \min\{P, \|6\alpha B_i(w; x)\|^{-1}\}.$$ 

The aim is now to establish a link between this bound and the density of integer solutions to the system of simultaneous bilinear equations

$$B_i(x; y) = 0, \quad (1 \leq i \leq n). \quad (3.5)$$

This is described in detail by Heath-Brown [9, §2]. Following this more or less verbatim we obtain

$$|T(a/q + z)|^4 \ll \frac{P^{2n}(\log P)^n}{Z^{2n}} \#\{(w, x) \in \mathbb{Z}^{2n}: (w, x) \ll ZP, \text{ (3.5) holds}\},$$

for any $Z \in \mathbb{R}$ such that

$$0 < Z < 1, \quad Z^2 < (12cq|z|P^2)^{-1}, \quad Z^2 < P/(2q),$$

and

$$Z^2 < \max\left\{\frac{q}{6cP^2}, qP|z|\right\}.$$ 

Here, $c = \sum |c_{ijk}|$, where $c_{ijk}$ are the coefficients of $g_0$. In particular, we clearly have $\|g_0\| \ll c \ll \|g_0\|$.

Now it is not hard to see that the system of equations (3.5) is just the system $H_{g_0}(x)y = 0$, where

$$H_F(x) := \left\{ \frac{\partial^2 F}{\partial x_i \partial x_j} \right\}_{1 \leq i, j \leq n}$$

is the Hessian matrix formed from the second order partial derivatives of any form $F \in \mathbb{Z}[x]$. It follows from [6, Lemma 1] that the variety cut out by (3.5) has dimension $n$ in $\mathbb{A}^{2n}$. An application of [1, Eq. (2.3)] now yields

$$|T(\alpha)|^4 \ll \frac{P^{2n}(\log P)^n}{Z^{2n}} (ZP)^n \ll Z^{-n} P^{3n}(\log P)^n,$$

provided that $Z \geq P^{-1}$. This bound clearly holds trivially when $Z < P^{-1}$. We will need to choose $Z$ as large as possible, given the constraints above. The choice

$$Z = \frac{1}{2} \min\left\{1, \frac{1}{12cq|z|P^2}, \frac{P}{2q}, \max\left\{\frac{q}{6cP^2}, q|z|P\right\}\right\}^{1/2},$$

is clearly satisfactory. On taking this value, we therefore deduce that

$$|T(\alpha)|^4 \ll \|g_0\|^{n/2} P^{3n+\varepsilon} \left(1 + q|z|P^2 + qP^{-1} + \frac{1}{q} \min\left\{P^2, \frac{1}{|z|P}\right\}\right)^{n/2},$$

$$= \|g_0\|^{n/2} P^{4n+\varepsilon} \left(P^{-2} + q|z| + qP^{-3} + \frac{1}{q} \min\left\{1, \frac{1}{|z|P^3}\right\}\right)^{n/2},$$

with 

$$\varepsilon = \frac{1}{2} \min\left\{\frac{1}{\log P}, 1\right\}.$$
for any $\varepsilon > 0$. We complete the proof of Proposition 2 by substituting this into (3.1).

4. Estimating $S_u(q; z)$: Poisson summation

The goal of this section is to give an alternative treatment of the cubic exponential sum (2.3), for cubic polynomials $g \in \mathbb{Z}[x_1, \ldots, x_n]$ such that the cubic part $g_0$ is non-singular. Our treatment is based on drawing together ideas already present in the author’s joint work with Heath-Brown [3], and the Kloosterman refinement carried out by Heath-Brown [6] in the context of non-singular cubic forms.

We will continue to assume that $\alpha = a/q + z$, with $q \leq P^{3/2}$ and $z$ satisfying (3.2). We will write $q = bc^2d$, where

$$b_1 := \prod_{p|q} p, \quad b_2 := \prod_{p^2|q} p, \quad d := \prod_{e \geq 3, 2|e} p$$

and $b = b_1b_2$. It is not hard to see that $d$ divides $c$, and that there exist a divisor $d_0$ of $d$ such that $d_0^{-1}d^{-1}c$ is a square-full integer. Moreover, $\gcd(b, c^2d) = 1$. Finally, we recall the definition of the function (2.4) for $P \geq 1$. The following is our main estimate for $S_u(q; z)$.

**Proposition 3.** Let $w \in W_n$, let $\varepsilon > 0$ and let $g \in \mathbb{Z}[x_1, \ldots, x_n]$ be a cubic polynomial such that $g_0$ is non-singular and $\|g\|_P \leq H$, for some $H \leq P$. Let $q = bc^2d$, in the notation of (4.1). Define

$$V := qP^{-1} \max\{1, \sqrt{|z|P^3}\},$$

and

$$W := V + (c^2d)^{1/3}. \quad (4.3)$$

Then there exists a positive number $\theta$ such that

$$S_u(q; z) \ll H^\theta q^{-n/2+1} P^{n+\varepsilon} (W^n M_1 + \min\{W^n, M_2, M_3\}),$$

where

$$M_1 := \max_{0 < N \ll (HP)^{\theta}} \frac{\gcd(b_1, N)^{1/2}}{b_1^{1/2}},$$

and

$$M_2 := c^n \left(1 + \frac{V}{c}\right)^{n-3/2}, \quad M_3 := V^n \left(1 + \frac{c^2d}{V^3}\right)^{n/2}.$$

Note that by taking $M_1 \leq 1$ and $\min\{W^n, M_2, M_3\} \leq W^n$ in this estimate we retrieve what is given by summing trivially over $a$ in [3, Proposition 1]. Not surprisingly, our sharpening of this estimate is based on taking advantage of cancellation in the summation over $a$.

The first step in the proof of Proposition 3 involves introducing complete exponential sums modulo $q$ via an application of Poisson summation. Thus it follows from summing over $a$ in [3, Lemma 8] that

$$S_u(q; z) = q^{-n} \sum_{v \in \mathbb{Z}^n} S_u(q; v) I(z; q^{-1}v),$$
where
\[ S_u(q; v) := \sum_{a=1}^{q} e_q(\bar{a}q) \sum_{y \mod q} e_q(ay + v \cdot y) \]  
and
\[ I(z; \beta) := \int \frac{w(x/P)e(zg(x) - \beta \cdot x)}{x} \, dx. \]
Before discussing the complete exponential sums (4.4), we first dispatch the integral \( I(z; q^{-1}v) \) that appears in our formula for \( S_u(q; z) \). For this we may apply [3, Lemma 9]. This leads to the conclusion that
\[ S_u(q; z) \ll P^{-N} + q^{-n} \int_{|x - qz\nabla g(x)| \leq P^2 \epsilon \sqrt{V}} |S_u(q; v)| \, dx \]
\[ \ll P^{-N} + q^{-n} P^\epsilon \max_{v_0 \leq HP} \sum_{|v - v_0| \leq P^2 \epsilon \sqrt{V}} |S_u(q; v)|. \]
Here \( N \in \mathbb{N} \) is arbitrary, \( V \) is given by (4.2) and we have used the fact that
\[ qz\nabla g(x) \ll P^{-1} ||\nabla g(Px)|| = P||P^{-2}\nabla g(Px)|| \ll HP, \]
for any \( x \ll P \) and \( |z| \leq q^{-1}P^{-3/2} \leq (qP)^{-1} \).

The bound in (4.5) constitutes a key difference between our current approach and that used in Heath-Brown’s work on cubic forms [6]. In the latter one uses instead the bound
\[ S_u(q; z) \ll \frac{1}{P^{\epsilon n}} + \frac{1}{q^n} \sum_{v_0 \leq HP^{1+\epsilon}} |S_u(q; v)| \meas\{x \ll P : |v_0(x) - v| \leq P^2 \epsilon \sqrt{V}\}, \]
for \( v_0(x) = qz\nabla g(x) \). In fact this difference is already capitalised upon in [3]. The point is that the approach in [6] requires information about the size of the Hessian \( \det H_{g_0}(x) \), and this is achieved by working with weight functions that detect points lying very close to a fixed point \( x_0 \in \mathbb{R}^n \) satisfying \( g_0(x_0) = 0 \), but which does not vanish in the Hessian. In our present investigation we seek an upper bound for integer solutions \( x \in \mathbb{Z}^n \) to the equation \( g = 0 \), that have modulus at most \( P \), and not just those that lie sufficiently close to \( x_0 \). Thus we have found it convenient to adopt the device pioneered in [3].

It remains to study the average order of \( S_u(q; v) \), as \( v \) ranges over an interval of length \( P^2 \epsilon \sqrt{V} \), centred upon a point \( v_0 \). Our investigation of this topic will adhere to the basic approach in [3]. Beginning with the multiplicativity property of the sums \( S_u(q; v) \), we have the following result.

**Lemma 1.** We have
\[ S_u(rs, v) = S_{ur^2}(s, rv)S_{us^2}(r, sv), \]
provided that \( r, s \) are coprime and \( rv, s \) are any integers such that \( rv + sv = 1 \).

**Proof.** Now it follows from [3, Lemma 10] that
\[ T(a, rs; v) = T(a\tilde{s}, r; s\tilde{v})T(a\tilde{r}, s; rv), \]
where
\[ T(a, q; v) := \sum_{y \mod q} e_q(ay + v \cdot y). \]
As $\alpha$ (resp. $\beta$) ranges over integers coprime to $r$ modulo $r$ (resp. coprime to $s$ modulo $s$), so $a = \alpha s\bar{s} + \beta r\bar{r}$ ranges over a set of residues modulo $q$ that are coprime to $q$. Hence it follows that

$$S_u(rs, v) = \sum_{\alpha \mod r} \sum_{\beta \mod s} e_{rs}(u(\alpha s\bar{s} + \beta r\bar{r}))T(\alpha s, r; \bar{s}v)T(\beta r, s; \bar{r}v)$$

$$= S_{u^2}(s, \bar{r}v)S_{u^2}(r, \bar{s}v),$$

as required for the statement of the lemma.

Lemma 1 allows us to concentrate on the value of $S_u(q, v)$ at prime power values of $q$. We begin by considering the sums $S_u(p, v)$ for primes $p$ such that $g_0$ remains non-singular modulo $p$. It follows from [4, Lemma 4] that

$$S_u(p; v) \ll p^{(n+1)/2}, \quad (4.7)$$

provided that $p \nmid u$. For the case in which $p | u$ the situation is more complicated. Let $g^* \in \mathbb{Z}[x]$ denote the dual form to $g_0$. Since $g_0$ is non-singular, so it follows that $g^*$ is absolutely irreducible and has degree $D$, with $2 \leq D \ll 1$. It now follows from [4, Lemma 5] that

$$S_0(p; v) \ll p^{(n+1)/2}(p, g^*(v))^{1/2}. \quad (4.8)$$

When $p$ is a prime such that $g_0$ is singular modulo $p$, so that $p$ divides the discriminant $\Delta_{g_0}$ of $g_0$, it will suffice to employ the trivial bound

$$|S_u(p^j; v)| \leq p^{j(n+1)} \leq \gcd(p, \Delta_{g_0})^j \ll \|g_0\|^j \leq H^j, \quad (j = 1, 2). \quad (4.9)$$

To handle moduli involving $p^2$ when $p \nmid \Delta_{g_0}$ we employ [3, Lemma 7], giving

$$S_u(p^2; v) \ll p^{n+2}. \quad (4.10)$$

Recall the decomposition $q = bc^2d = b_1b_2^2c^2d$ in (4.1), with $b_1, b_2, c^2d$ pairwise coprime. Drawing together (4.7), (4.8) and (4.9), we conclude from Lemma 1 that

$$S_u(b_1; v) \leq H^\theta A^{\omega(b_1)}b_1^{(n+1)/2} \gcd(b_1, u, g^*(v))^{1/2},$$

for some absolute constant $A \ll 1$. Similarly, it follows from (4.9) and (4.10) that

$$S_u(b_2; v) \leq H^\theta A^{\omega(b_2)}b_2^{n+2}.$$
Observing that
\[ |S_{ab^2}(c_2d; \tilde{b}v)| \leq \sum_{a \mod c_2d \atop \gcd(a, c_2d) = 1} |T(a, c_2d; \tilde{b}v)|, \]
where \(T(a, c_2d; \tilde{b}v)\) is given by (4.6), we may now combine [3, Lemmas 11, 15 and 16] in the manner indicated at the close of [3, §5], in order to conclude that
\[ S_1 \ll \mathcal{H}^\theta q^{n/2+1+\varepsilon} W^n \max_{0 < N \ll (HP)^\theta} \frac{\gcd(b_1, N)^{1/2}}{b_1^{1/2}}. \tag{4.11} \]
Here \(W\) is given by (4.3), and we have taken \(\max_{1 \leq i \leq n} r_i \ll \mathcal{H}^\theta\) in [3].

In order to estimate \(S_2\), in which case \(u = 0\), we begin as above with the observation that
\[ S_2 \ll \mathcal{H}^\theta q^{n/2+1+\varepsilon} W^n. \tag{4.12} \]
Alternatively, we apply [3, Lemma 11] to deduce that
\[ S_2 \ll \mathcal{H}^\theta q^{n/2+1+\varepsilon} b^{n/2+1} (c_2d)^{n/2} \sum_{a \mod c_2d \atop \gcd(a, c_2d) = 1} \sum_{\nu - \nu_0 \leq \rho^2 v} \sum_{a \mod c \atop \gcd(a, c) = 1} \sum_{\nu \leq \rho^2 v} \sum_{\nu - \nu_0 \leq \rho^2 v} \sum_{a \mod c \atop \gcd(a, c) = 1} \sum_{\nu \leq \rho^2 v} M_d(a)^{1/2}, \]
where
\[ M_d(x) := \# \{ y \mod d : \nabla^2 g(x)y \equiv 0 \pmod{d} \}. \]
We proceed to employ the following result to estimate the number of available \(v\).

**Lemma 2.** We have
\[ \# \{ v \in \mathbb{Z}^n : |v - \nu_0| \leq X, g^*(v) = 0, v \equiv a \mod q \} \ll 1 + \left( \frac{X}{q} \right)^{-3/2+\varepsilon}, \]
uniformly in \(v_0, a\) and the coefficients of \(g^*\).

**Proof.** Let \(N(q; X)\) denote the quantity that is to be estimated. Dropping the condition \(g^*(v) = 0\), we trivially have \(N(q; X) = O(1)\) if \(q > X\). Assume henceforth that \(q \leq X\), and write \(v = a + qw\) for \(w \in \mathbb{Z}^n\). Since \(|v - \nu_0| \leq X\), so it follows that \(|w - w_0| \ll X/q\), where the components of \(w_0\) are obtained by taking the integer part of the components of \(q^{-1}(v_0 - a)\). On writing \(z = w - w_0\), we therefore deduce that
\[ N(q; X) \ll \# \{ z \in \mathbb{Z}^n : z \ll X/q, f(z) = 0 \}, \]
where \(f(z) = g^*(a + qw_0 + qz)\).

In view of the fact \(g^*\) is an absolutely irreducible form of degree at least 2, so it follows that the shifted polynomial \(f\) must be absolutely irreducible...
with degree at least 2. Our new polynomial \( f \) need not be homogeneous, and has coefficients that depend on \( a, v_0 \) and \( q \). We now appeal to a very general uniform bound due to Pila [10, Theorem A], which implies that the affine hypersurface \( f = 0 \) contains 
\[
\ll Y^{n-2+1/\deg f + \varepsilon}
\]
integer points of height at most \( Y \), for any \( Y \geq 1 \). Furthermore, the implied constant in this estimate is uniform in the coefficients of \( f \). Once inserted into our bound for \( N(q; X) \), this therefore completes the proof of the lemma.

Employing Lemma \ref{lem:uniform-bound} in our bound for \( \mathcal{S}_2 \) we deduce that
\[
\mathcal{S}_2 \ll H^\theta q^{n/2+1+\varepsilon} \max_{a \mod c^2 d} \sum_{\gcd(a, c^2 d) = 1} M_d(a)^{1/2} \sum_{\substack{|v - v_0| \leq P^\varepsilon V \\ g^*(v) = 0}} \frac{1}{c|(a \nabla g(a) + v)} \ll H^\theta q^{n/2+1+\varepsilon} P^\varepsilon \bigg( 1 + \frac{V}{c} \bigg)^{n-3/2} U(q),
\]
where
\[
U(q) := \sum_{a \mod c} M_d(a)^{1/2}.
\]

At this point it is worth comparing our investigation with the corresponding argument in [4] and [6]. There a refinement of [3, Lemma 11] is used in order to obtain a better estimate for \( S_0(c^2 d; \mathbf{v}) \) when \( g^*(\mathbf{v}) = 0 \). This ultimately leads to a version of the above bound for \( \mathcal{S}_2 \) with the additional constraint that \( c | g(a) \) in the definition of \( U(q) \). If we denote this quantity by \( U^*(q) \), then in the setting of homogeneous polynomials Heath-Brown shows that
\[
U^*(q) \ll c^{n-1+\varepsilon} d^{1/2},
\]
provided that \( n \geq 10 \). A slightly weaker estimate is achieved in [4], but which is valid for cubic polynomials that are not necessarily homogeneous. It turns out that neither of these estimates is readily extended to smaller values of \( n \), but fortunately we have found it sufficient to work with \( U(q) \) instead. We can then combine [3, Lemma 14] with Cauchy’s inequality to deduce that
\[
U(q) \ll c^{n/2} \left( \sum_{a \mod c} M_d(a) \right)^{1/2} \ll c^{n/2} \left( \frac{c^n}{d^n} \sum_{a \mod d} M_d(a) \right)^{1/2} \ll H^\theta c^{n+\varepsilon}.
\]

Once inserted into the preceding bound for \( \mathcal{S}_2 \), we deduce that
\[
\mathcal{S}_2 \ll H^\theta q^{n/2+1+\varepsilon} P^\varepsilon M_2, \quad (4.13)
\]
in the notation of Proposition \ref{prop:main}.

To obtain a final estimate for \( \mathcal{S}_2 \), we simply drop the condition that \( g^*(\mathbf{v}) = 0 \) in the summation over \( \mathbf{v} \). In this way [3, Lemma 16] easily leads us to the conclusion that
\[
\mathcal{S}_2 \ll H^\theta q^{n/2+1+\varepsilon} P^\varepsilon M_3, \quad (4.14)
\]
in the notation of Proposition \ref{prop:main}.

Drawing together (4.11), (4.12), (4.13), and (4.14) in (4.5), we therefore complete the proof of Proposition \ref{prop:main}. 
5. Proof of Proposition 1

In this section we establish Proposition 1. Let \( w \in \mathcal{W}_n \) and let \( g \in \mathbb{Z}[x] \) be a cubic polynomial with \( s(g_0) = -1 \) and \( n \geq 5 \). Taking (2.2) as our starting point, with the choice \( Q = P^{3/2} \), we have

\[
N_w(g; P) = T_1 + O(T_2),
\]

where

\[
T_1 := \sum_{q \leq Q} \int_{\frac{1}{q}}^{\frac{1}{1-q}} S_0(q; z) dz, \quad T_2 := \frac{1}{P^3} \sum_{q \leq Q} \sum_{|u| \leq \frac{1}{2}} \frac{\max_{1 \leq qQ|z| \leq 1} |S_u(q; z)|}{1 + |u|}.
\]

Let \( P \geq 1 \) and \( H \geq \|g\|_P \). Throughout this section we may assume that \( P \geq H \), since the alternative hypothesis simply contributes \( O(H^n) \) to \( N_w(g; P) \), which is satisfactory for Proposition 1.

We will consider the contribution to \( T_1, T_2 \) from \( q \) restricted to lie in certain intervals. Write \( q = b_1 b_2 c^2 d \), where \( b_1, b_2, d \) are given by (4.1). Let \( R, R_0, \ldots, R_3 \geq 1/2 \) and \( t > 0 \). Then we will write \( \Sigma_i(R, \mathbf{R}; t) \) for the overall contribution to \( T_i \), for \( i = 1, 2 \), from those \( q, z \) for which

\[
R < q \leq 2R, \quad R_0 < b_1 \leq 2R_0, \quad R_1 < b_2 \leq 2R_1, \quad R_2 < c \leq 2R_2, \quad R_3 < d \leq 2R_3,
\]

and

\[
t < |z| \leq 2t.
\]

Our plan will be to show that

\[
\Sigma_i(R, \mathbf{R}; t) \ll H^9 P^{n-2+\epsilon},
\]

for \( i = 1, 2 \), under the assumption that \( n \geq 5 \) and \( s(g_0) = -1 \). Once summed over \( O((\log P)^6) \) dyadic intervals for \( R, \mathbf{R} \) and \( t \), this will clearly suffice to complete the proof of Proposition 1.

Recall that \( d \mid c \). Thus \( \Sigma_i(R, \mathbf{R}; t) = 0 \) for \( i = 1, 2 \), unless

\[
R_3 \ll R_2, \quad R \ll R_0 R_1^2 R_2^2 R_3 \ll R \leq P^{3/2}.
\]

Similarly, it is clear that \( \Sigma_i(R, \mathbf{R}; t) = 0 \) unless

\[
(RP^{3/2})^{-1} \geq t \gg \begin{cases} 0, & \text{if } i = 1, \\ (RP^{3/2})^{-1}, & \text{if } i = 2. \end{cases}
\]

The following simple result will be useful in our work.

**Lemma 3.** We have

\[
\#\{q = b_1 b_2 c^2 d : (5.1) \text{ holds}\} \ll R_0 R_1^{1/2} R_2^{1/2} R_3^{1/2}.
\]

**Proof.** It is clear that we have to count the number of quadruples \((b_1, b_2, c, d)\) for which \( d \mid c \) and (5.1) holds. The number of choices for \( b_1 \) and \( b_2 \) is \( O(R_0 R_1) \). To count the possible pairs \( c, d \) recall from (4.1) that there exist a positive integer \( d_0 \) such that \( d_0 \mid d \) and \( d_0^{-1} d^{-1} c \) is square-full. Hence, for fixed values of \( d \), the number of available choices for \( c \) is

\[
\ll \sum_{d \mid d_0} \left( \frac{R_2}{d_0 R_3} \right)^{1/2}.
\]
On summing over values of \( d \), we deduce that the overall number of choices for \( c, d \) is

\[
\ll R_2^{1/2} R_3^{-1/2} \sum_{d \in R_3} \sum_{d_0 | d} \frac{1}{d_0^{1/2}} \ll R_2^{1/2} R_3^{1/2}.
\]

This suffices for the proof of Lemma 3.

Our main tool in bounding \( \Sigma_1(R, R; t) \) and \( \Sigma_2(R, R; t) \) will be Proposition 3, but this will be supplemented with Proposition 2 to handle certain awkward ranges of \( R \).

5.1. Estimating \( \Sigma_2(R, R; t) \). We begin with our treatment of \( \Sigma_2(R, R; t) \). Since the size of \( t \) is effectively determined by (5.4), so it will be convenient to write \( \Sigma_2(R, R) = \Sigma_2(R, R; t) \) throughout this section.

It follows from Proposition 3 that

\[
\Sigma_2(R, R) \ll H^\theta P^{n-3+\epsilon} \sum_q R_{1}^{1-n/2} \max_{|z| \asymp (RQ)^{-1}} (W^{n} M_1 + \min\{M_2, M_3\}),
\]

where \( W, M_1, M_2, M_3 \) are as in the statement of the proposition and the summation over \( q \) is over all \( q = b_1 b_2 c d \) such that \( q, b_1, b_2, c, d \) are constrained to lie in the dyadic ranges (5.1). Let us write \( \Sigma_{2,a} \) for the overall contribution to the right hand side from the term involving \( W^n M_1 \), and \( \Sigma_{2,b} \) for the corresponding contribution from the term involving \( \min\{M_2, M_3\} \). Thus we have

\[
\Sigma_2(R, R) \ll \Sigma_{2,a} + \Sigma_{2,b}.
\] (5.5)

We begin by estimating \( \Sigma_{2,a} \), for which it is convenient to note that

\[
V \ll R^{1/2} P^{-1/4}, \quad W \ll R^{1/2} P^{-1/4} + (R_2^2 R_3)^{1/3},
\] (5.6)

for any \( z \) such that \( |z| \asymp (RQ)^{-1} \), with \( Q = P^{3/2} \). Now it is trivial to see that

\[
\sum_{b \leq B} \gcd(b, N) \ll \tau(N) B \ll N^\epsilon B,
\]

for any \( N \in \mathbb{N} \) and \( B \geq 1 \). By adjusting the proof of Lemma 3 slightly it therefore follows that

\[
\sum_{q} \gcd(b_1, N)^{1/2} \ll N^\epsilon R_0 R_1 R_2^{1/2} R_3^{1/2},
\]

for any \( N \in \mathbb{N} \). Bringing this all together we conclude that

\[
\Sigma_{2,a} \ll H^\theta \frac{P^{n-3+\epsilon}}{(R_0^{1/2} R_3^{n/2-1})^{0 < N < (HP)^\rho}} \sum_q \gcd(b_1, N)^{1/2} \max_{z} W^n
\]

\[
\ll H^\theta \frac{P^{n-3+\epsilon}}{(R_0^{1/2} R_3^{n/2-1})^{0 < N < (HP)^\rho}} \frac{R_1^{1/2} R_2^{1/2} R_3^{1/2}}{R_{1/2}^{n/2-1}} \max_{z} W^n
\]

\[
\ll H^\theta \frac{P^{n-3+\epsilon}}{(R_0^{1/2} R_3^{n/2-1})^{0 < N < (HP)^\rho}} \frac{(R_1^{1/2} P^{-1/4} + (R_2^2 R_3)^{1/3})^n},
\]

since \( R_0 \ll R/(R_1^2 R_2^2 R_3) \) by (5.3). Our aim is to show that

\[
\Sigma_{2,a} \ll H^\theta P^{n-2+\epsilon},
\] (5.7)
provided that \( n \geq 5 \). We have two terms to consider in our estimate for \( \Sigma_{2,a} \). Beginning with the term involving \( R^{1/2} P^{-1/4} \), we obtain the contribution
\[
\ll H^\theta P^{3n/4 - 3 + \varepsilon} R^{3/2} \ll H^\theta P^{3n/4 - 3/4 + \varepsilon},
\]
since \( R \leq P^{3/2} \). This is satisfactory for \( n \geq 5 \). Finally, the term involving \((R_2^2 R_3)^{1/3}\) contributes
\[
\ll H^\theta \frac{P^{n-3+\varepsilon}(R_2^2 R_3)^{n/3}}{R^{n/2 - 3/2}} \ll H^\theta P^{n-3+\varepsilon} R^{3/2 - n/6},
\]
since \( R_2^2 R_3 \ll R \). When \( n \geq 9 \) the exponent of \( R \) is non-positive, which clearly yields a satisfactory contribution. When \( 5 \leq n \leq 8 \), we obtain the contribution \( O(H^\theta P^{3n/4 - 3/4 + \varepsilon}) \) by taking \( R \leq P^{3/2} \). This completes the proof of (5.7).

We now turn to the task of estimating \( \Sigma_{2,b} \), for which we want to show that
\[
\Sigma_{2,b} \ll H^\theta P^{n-2+\varepsilon}, \quad (5.8)
\]
provided that \( n \geq 5 \). Once combined with (5.7) in (5.5) this will be enough to establish (5.2) in the case \( i = 2 \). We will need to supplement our estimate with Proposition 2. A little thought reveals that
\[
\Sigma_{2,b} \ll H^\theta P^{n-3+\varepsilon} R \sum_q \min \left\{ P^{-3n/16}, R^{-n/2} \max_{z} \min\{M_2, M_3\} \right\},
\]
where \( M_2, M_3 \) are as in the statement of Proposition 3, and the maximum is over \( z \) such that \( |z| \asymp (RQ)^{-1} \). In particular (5.6) holds in the definitions of \( M_2, M_3 \).

Suppose first that \( V \geq R_2 \). Then we take \( \min\{M_2, M_3\} \leq M_2 \), in order to conclude from (5.3) and Lemma 3 that
\[
\Sigma_{2,b} \ll H^\theta P^{n-3+\varepsilon} R^{1-n/2} R_0 R_1 R_2^{n+1/2} R_3^{1/2} \left( \frac{V}{R_2} \right)^{n-3/2}
\ll H^\theta P^{n-3+\varepsilon} R^{2-n/2} V^{n-3/2}
\ll H^\theta P^{3n/4 - 21/8 + \varepsilon} R^{5/4}
\ll H^\theta P^{3n/4 - 3/4 + \varepsilon}.
\]
This is satisfactory for \( n \geq 5 \).

Suppose now that \((R_2^2 R_3)^{1/3} \leq V < R_2 \). Then we may take
\[
\min\{M_2, M_3\} \leq M_2^{3/10} M_3^{7/10} \ll R_2^{3n/10} \left( \frac{R_1^{1/2}}{P^{1/4}} \right)^{7n/10}
\]
in the above. Lemma 3 and (5.3) together reveal that
\[
\Sigma_{2,b} \ll H^\theta P^{n-3+\varepsilon} \frac{R_2^{2-n/2}}{R_2^{3/2} R_3^{1/2}} \min\{M_2, M_3\} = H^\theta P^{n-2+\varepsilon} E_n,
\]
where
\[
E_n = P^{-1-7n/40} R^{2-3n/20} R_2^{3n/10-3/2}.
\]
We wish to show that $E_n \ll 1$ for $n \geq 5$. But clearly $3n/10 - 3/2 \geq 0$ for $n$ in this range, whence we may take $R_2 \ll R^{1/2}$ in this estimate. It follows that

$$E_n \ll P^{-1-7n/40} R_5^{5/4} \ll P^{7/8-7n/40} \ll 1,$$

for $n \geq 5$, as required.

Turning to the case in which $V < (R_2^2 R_3)^{1/3}$, we note that

$$M_3 \ll \left( \frac{R_2^2 R_3}{V} \right)^{n/2} \ll \frac{P^{n/8}(R_2^2 R_3)^{n/2}}{R^{n/4}}$$

in the statement of Proposition 3. Hence

$$
\Sigma_{2,b} \ll H^n \frac{P^{n-3+\varepsilon} R^2}{R_2^{3/2} R_3^{1/2}} \min \left\{ \frac{1}{P^{3n/16}}, \frac{P^{n/8}(R_2^2 R_3)^{n/2}}{R^{3n/4}} \right\}
$$

$$\ll H^n \frac{P^{n-3+\varepsilon}}{P^{3n/16}} \min \left\{ \frac{R^2}{R^{3n/16}}, \frac{P^{n/8}}{R^{n/4-3/4}} \right\},$$

since $R_2^{3/2} R_3^{1/2} \gg (R_2^2 R_3)^{2/3}$. When $n = 5$, so that $n/4 - 4/3 < 0$, we take $R \ll P^{1/2}$ to deduce that

$$\Sigma_{2,b} \ll H^n P^{21/8+\varepsilon} (P^{3/2})^{1/12} \ll H^n P^{11/4+\varepsilon} \ll H^n P^3,$$

which is satisfactory. When $n \geq 6$ we apply the bound coming from Weyl differencing when $R < P$, and the bound coming from Poisson summation when $R \geq P$. This yields

$$\Sigma_{2,b} \ll H^n P^{\varepsilon} (P^{13n/16-1} + P^{7n/8-5/3}),$$

which is satisfactory for $n \geq 6$. This completes the proof of (5.8).

5.2. **Estimating $\Sigma_1(R, \mathbf{R}; t)$**. It follows from Proposition 3 and the argument in Lemma 3 that

$$\Sigma_1(R, \mathbf{R}; t) \ll H^n P^{n+\varepsilon} t \left( \frac{R_2^{3/2-n/2} W_n}{R_2^{1/2}} + \frac{R_2^{2-n/2}}{R_1 R_2^{3/2} R_3^{1/2}} \min \{ M_2, M_3 \} \right),$$

where $W, M_2, M_3$ are as in the statement of the proposition, but with individual variables replaced by appropriate lower or upper bounds corresponding to the interval that the variable is assumed to lie in. Let us write $\Sigma_{1,a}$ for the overall contribution to the right hand side from the first term, and $\Sigma_{1,b}$ for the corresponding contribution from the second term. In order to establish (5.2) with $i = 1$, it will suffice to show that

$$\max \{ \Sigma_{1,a}, \Sigma_{1,b} \} \ll H^n P^{n-2+\varepsilon},$$

for $n \geq 5$.

Let us begin by estimating $\Sigma_{1,a}$, for which we have

$$V \asymp \begin{cases} \frac{R}{P}, & \text{if } t < P^{-3}, \\ \frac{R t^{1/2} P^{1/2}}{P^{3/2}}, & \text{if } t \geq P^{-3}, \end{cases}$$

and $W \ll V + (R_2^2 R_3)^{1/3}$. When $t \geq P^{-3}$ the term involving $V$ makes the contribution

$$\ll H^n P^{n+\varepsilon} t R^{3/2-n/2} (R t^{1/2} P^{1/2})^n \ll H^n P^{3n/2+\varepsilon} t^{1+n/2} R^{3/2+n/2} \ll H^n P^{3n/4-3/4+\varepsilon}$$
to $\Sigma_{1,a}$, since $t \leq (RP^{3/2})^{-1}$. This is satisfactory for $n \geq 5$. Likewise, when $t < P^{-3}$, one obtains a satisfactory contribution. To handle the contribution from the term involving $(R_2^2 R_3)^{1/3}$ we will need to supplement our estimate with an application of Proposition 2, in addition to differentiating according to the size of $t$. Suppose first that $t \geq P^{-3}$. Then we have the overall contribution

$$\ll H^\theta P^{n+\varepsilon} \min \left\{ \frac{R^{3/2-n/2} t (R_2^2 R_3)^{n/3}}{R_2^{1/2}}, \frac{R^{2-n/8} t^{1-n/8}}{R_2^{3/2} R_3^{1/2} P^{3n/8}} \right\},$$

$$\ll H^\theta P^{n+\varepsilon} \min \left\{ \frac{R^{3/2-n/2} t (R_2^2 R_3)^{3-1/6}}{R_2^{1/2}}, \frac{R^{2-n/8} t^{1-n/8}}{(R_2^2 R_3)^{2/3} P^{3n/8}} \right\},$$

since $R_3 \ll R_2$. We apply the basic inequality $\min \{A, B\} \leq A^{1/3} B^{2/3}$ to derive the overall contribution $O(H^\theta P^{n-2+\varepsilon} E_n)$, with

$$E_n = P^{2-n/4} t^{1-n/12} R^{11/6-n/4} (R_2^2 R_3)^{n/9-1/2}.$$ 

Suppose first that $5 \leq n \leq 12$. Then we may take $t \leq (RP^{3/2})^{-1}$ to deduce that

$$E_n \leq P^{1/2-n/8} R^{5/6-n/6} (R_2^2 R_3)^{n/9-1/2} \ll P^{1/2-n/8} R^{1/3-n/18},$$

whence $E_n \ll 1$. Alternatively, when $n \geq 13$ we have

$$E_n \leq P^{-1} R^{11/6-n/4} (R_2^2 R_3)^{n/9-1/2} \ll P^{-1} R^{4/3-5n/36} \ll 1,$$

since $t \geq P^{-3}$.

So far we have established a satisfactory bound for $\Sigma_{1,a}$ under the assumption that $t \geq P^{-3}$. When $t < P^{-3}$, we easily obtain the overall contribution

$$\ll H^\theta P^{n+\varepsilon} \frac{R^{3/2-n/2} t (R_2^2 R_3)^{n/3}}{R_2^{1/2}} \ll H^\theta P^{n-3+\varepsilon} R^{3/2-n/2} (R_2^2 R_3)^{n/3-1/6} \ll H^\theta P^{n-3+\varepsilon} R^{4/3-n/6}.$$ 

The exponent of $R$ is non-positive when $n \geq 8$, in which case the bound is clearly satisfactory. When $5 \leq n \leq 7$, we take $R \leq P^{3/2}$ to obtain the satisfactory contribution $O(H^\theta P^{3n/4-1+\varepsilon})$. This establishes the bound for $\Sigma_{1,a}$ recorded in (5.9), for $n \geq 5$.

We now turn to the task of estimating $\Sigma_{1,b}$. We have

$$\Sigma_{1,b} \ll H^\theta P^{n+\varepsilon} \frac{R^{2-n/2}}{R_2^{3/2} R_3^{1/2}} \min \{M_2, M_3, M_4\}, \quad (5.11)$$

with

$$M_2 = R_2^n \left( 1 + \frac{V}{R_2} \right)^{-n/2}, \quad M_3 = V^n \left( 1 + \frac{R_3^2}{V^2} \right)^{n/2},$$

and

$$M_4 = R^{3n/8} \min \{1, (tP^3)^{-n/8}\}. \quad (5.13)$$

Here $M_4$ arises from an application of Proposition 2 and $V$ satisfies (5.10). Let us begin by handling the case in which $t \geq P^{-3}$, so that $V \asymp R t^{1/2} P^{1/2}$. 


Suppose first that \( V \geq R_2 \). Then we take \( \min\{M_2, M_3, M_4\} \leq M_2 \), in order to conclude that
\[
\Sigma_{1,b} \ll H^\theta P^{n+\varepsilon} t R^2 - n/2 R_2^{n-3/2} \left( \frac{R^1/2 P^1/2}{R_2} \right)^{n-3/2} \\
\ll H^\theta P^{3n/2 - 3/4 + \varepsilon} p^n/2 + 1/4 R^{n/2 + 1/2} \\
\ll H^\theta P^{3n/4 - 9/8 + \varepsilon} R^{1/4} \\
\ll H^\theta P^{3n/4 - 3/4 + \varepsilon}.
\]
This is satisfactory for \( n \geq 5 \).

Suppose now that \( (R_2^2 R_3)^{1/3} \leq V < R_2 \). We take \( \min\{M_2, M_3, M_4\} \leq M_2^{3/10} M_3^{7/10} \) in (5.11). This gives \( \Sigma_{1,b} \ll H^\theta P^{n-2+\varepsilon} E_n \), where
\[
E_n \ll P^{1/2 - 7n/20} R^{1/4} \ll P^{7/8 - 7n/20} \ll 1
\]
for \( n \geq 5 \), since \( t \leq (RP^{3/2})^{-1} \). Finally we consider the case \( V < (R_2^2 R_3)^{1/3} \).

In this setting we have \( M_2 \ll R_n^2 \) and \( M_3 \ll (R_2^2 R_3/V)^{n/2} \) in (5.12), and \( M_4 \ll R_3^{n/8} (tP^3)^{-n/8} \) in (5.13). Taking \( \min\{A, B, C\} \ll A^{1/10} B^{1/5} C^{7/10} \) in (5.11), we therefore deduce that \( \Sigma_{1,b} \ll H^\theta P^{n-2+\varepsilon} E_n \), with
\[
E_n = P^{2 - 5n/16} R^{1 - 11n/80} R_2^{n - 3/2} R_3^{n/10 - 1/2}.
\]

Suppose first that \( n \leq 7 \), so that \( 1 - 11n/80 > 0 \). Then the upper bound \( t \leq (RP^{3/2})^{-1} \) gives
\[
E_n \ll P^{1/2 - 17n/160} R_1^{n-1/5} R_2^{n - 3/2} R_3^{n/10 - 1/2} \\
\ll P^{1/2 - 17n/160} R_1^{1/4 - n/20} \\
\ll 1,
\]

since \( R_2^{n - 3/2} R_3^{n/10 - 1/2} \ll R_3^{n/20 - 3/4} \) for \( n \geq 5 \). When \( n \geq 8 \) we instead take \( t \geq P^{-3} \) in the above, obtaining \( E_n \ll P^{1 - n/10} R_1^{5/4 - 3n/16} \ll 1 \), when \( n \leq 10 \). Finally, when \( n \geq 11 \), we instead take \( \min\{A, B, C\} \ll C \) in the above to get a satisfactory contribution.

In order to complete the proof of (5.9) it remains to show that \( \Sigma_{1,b} \ll H^\theta P^{n-2+\varepsilon} \) when \( n \geq 5 \) and \( t < P^{-3} \). In particular we have \( V \asymp R/P \), by (5.10), and it now follows from (5.11) that
\[
\Sigma_{1,b} \ll H^\theta P^{n-3+\varepsilon} R_2^{2 - n/2} R_3^{1/2} \min\{M_2, M_3, M_4\},
\]

with \( M_2, M_3, M_4 \) being given by (5.12) and (5.13). When \( V \geq R_2 \), we take \( M_2 \) in the minimum, giving
\[
\Sigma_{1,b} \ll H^\theta P^{n-3+\varepsilon} R_2^{n/2 + 1/2} / P^{n-3/2} \ll H^\theta P^{3n/4 - 3/4 + \varepsilon}.
\]
This is satisfactory for \( n \geq 5 \). When \( (R_2^2 R_3)^{1/3} \leq V < R_2 \) we take \( \min\{M_2, M_3, M_4\} \ll M_2^{3/10} M_3^{7/10} \) to deduce that \( \Sigma_{1,b} \ll H^\theta P^{n-2+\varepsilon} E_n \), with
\[
E_n \ll \frac{R_2^{n/5 + 3/2} R_3^{n/10 - 3/2}}{P^{n/10 + 1}} \ll \frac{R_2^{7n/20 + 5/4}}{P^{n/10 + 1}} \ll P^{7/8 - 7n/40} \ll 1,
\]
for \( n \geq 5 \).
Finally we must deal with the case $V < (R_2^2 R_3)_{1/3}$, in which setting
\[
\Sigma_{1,b} \ll H^{\theta} \frac{P^{n-3+\varepsilon} R^{2-n/2}}{R_2^{3/2} R_3^{1/2}} \min \left\{ R_2^n, \left( \frac{P R_2^2 R_3}{R} \right)^{n/2}, R_3^{3n/8} \right\}.
\]
We use the inequality $\min\{A, B, C\} \leq A^{1/10} B^{11/30} C^{8/15}$ to deduce that $\Sigma_{1,b} \ll H^{\theta} P^{n-2+\varepsilon} E_n$, with
\[
E_n = P^{11n/60-1} R_2^{2-29n/60} R_2^{7n/15-3/2} R_3^{11n/60-1/2} \\
\leq P^{11n/60-1} R_2^{4/2-3n/10} R_3^{n/10-1/2}.
\]
In particular we have $E_5 \ll P^{-1/12} \ll 1$. When $n \geq 6$ we have
\[
E_n \ll P^{11n/60-1} R_2^{5/4-n/4}.
\]
This is clearly $O(1)$ when $R \geq P^{7/10}$ and $6 \leq n \leq 15$. Assume now that $n \geq 16$, or else $6 \leq n \leq 15$ and $R < P^{7/10}$. Then we take $\min\{A, B, C\} \leq C$ in the above estimate instead, obtaining $\Sigma_{1,b} \ll H^{\theta} P^{n-2+\varepsilon} E_n$, but this time with
\[
E_n = P^{-1} R_2^{2-n/8}.
\]
If $6 \leq n \leq 15$ and $R < P^{7/10}$ then clearly $E_n \leq P^{2/5-7n/80} \ll 1$. Alternatively, if $n \geq 16$ then $E_n \ll P^{-1} \ll 1$. This completes the proof of Proposition 1.

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