TIME-SCALING TRANSFORMATION FOR OPTIMAL CONTROL PROBLEM WITH TIME-VARYING DELAY

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Abstract. This paper focuses on the solution to nonlinear time-delay optimal control problems with time-varying delay subject to canonical equality and inequality constraints. Traditional control parameterization in conjunction with time-scaling transformation could optimize control parameters and switching times at the same time when the time-delays in the dynamic system are taken as constants. The purpose of this paper is to extend this method to solve the dynamic systems with time-varying delay. We introduce a hybrid time-scaling transformation that converts the given time-delay system into an equivalent system defined on a new time horizon with fixed switching times. Meanwhile, we obtain the value of time-delay state utilizing the relationship between the new time scale and the original one. After computing the gradients of the cost and constraints with respect to the control heights and its durations, we could solve the equivalent optimal control problem using gradient based optimization method.

1. Introduction. Optimal control is one of the most important parts of modern control theory, which is concerned with determining control strategies for a dynamic system in the best possible manner such that the system constraints are satisfied and the cost function is optimal. A standard dynamic system can be written as follows:

\[
\frac{d}{dt} x(t) = f(x(t), u(t)), \quad t \in [0, T],
\]

\[
x(0) = x_0.
\]

where \( x(t) \in \mathbb{R}^n \) is called the state vector; \( u(t) \in \mathbb{R}^r \) is called the control vector; \( f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n \) is a given continuously differentiable function. Except for some special cases, it is generally very difficult to find exact solution of optimal control problems and in most cases, we just obtain approximate numerical solutions. Nowadays, there are many effective numerical methods available for solving optimal control problems such as control parameterization technique[20], direct
transcription[2] and methods based on Chebyshev finite difference[19]. Control parameterization is probably the most popular method to solve general optimal control problems with canonical equality/inequality constraints among the existing methods, and has been widely used to solve various types of optimal control problems such as flight level tracking optimal control problem[18], free terminal time optimal control problem[3, 16] and multi-robots system[13, 15]. It works by firstly introducing a partition vector \( \tau = [\tau_0, ..., \tau_p] \) to divide the time interval \([0, T]\) into \(p\) subintervals, \(0 = \tau_0 < \tau_1 < \ldots < \tau_{p-1} < \tau_p = T\). Then, the control function is approximated by a basis function (could be constant, linear or quadratic function) in each subinterval \([\tau_{i-1}, \tau_i)\),

\[
 u(t) \approx \sum_{i=1}^{p} b_i(t) \chi_{[\tau_{i-1}, \tau_i)}(t),
\]

where

\[
\chi_{[\tau_{i-1}, \tau_i)}(t) = \begin{cases} 
1, & \text{if } t \in [\tau_{i-1}, \tau_i), \\
0, & \text{otherwise}.
\end{cases}
\]

One can obtain approximate optimal solution by optimizing parameters of \(b_i(t)\). In traditional control parameterization, switching times \(\tau_i, i = 0, 1, \ldots, p\) are fixed (generally the partition points \(\tau_i, i = 0, 1, \ldots, p\) are evenly distributed on \([0, T]\)). Therefore, to obtain a more accurate solution, the number of subintervals needs to be big enough, which results in a large number of decision variables. To avoid this problem, we regard the switching times \(\tau_i, i = 0, 1, \ldots, p\) as decision variables. In the pioneer work of control parametrization technique by Teo, Goh and Wong[20], the gradients of cost function and constrained functions with respect to switching times are discussed, later in 1997, Teo et al.[10] analyzed the drawbacks of this method and proposed time-scaling transformation to overcome the difficulty caused by switching times.

Time-scaling transformation introduces a new time variable \(s \in [0, p]\) and builds the relationship between \(t\) and \(s\) as follows

\[
\frac{dt(s)}{ds} = \theta_k, \quad s \in [k-1, k), \quad k = 1, \ldots, p, \quad (4)
\]

\[
t(0) = 0, \quad (5)
\]

where \(\theta_k = \tau_k - \tau_{k-1}\) denotes the duration of \(b_k(t)\). It is easy to see that \(t\) is a piecewise linear function of \(s\), and \(\theta_k, \quad k = 1, \ldots, p\) denote the slope of linear function defined on \([k-1, k)\). Moreover, equations (4-5) define a one-to-one correlation between \(t\) and \(s\). In time-scaling transformation, \(\theta_k, \quad k = 1, \ldots, p\) are regarded as decision variables and switching times are fixed, see [14] for more details. Control parameterization in conjunction with time-scaling transformation has been applied successfully in solving different types of optimal control problems, including time-optimal control problem[10], impulsive systems[21], and optimal discrete-valued control problems[11, 25].

However, when solving time-delay optimal control problem, time-scaling transformation faces difficulties. Time-delay system is not only related with the current state and/or control variables but also is influenced by the state and/or control variables at some past time instants, and its mathematical expression can be written
as follows:

\[
\frac{d}{dt} x(t) = f(x(t), x_d, u(t), u_d), \ t \in [0, T],
\]

\[
x(t) = \phi(t), \ t \in [-h, 0],
\]

\[
u(t) = \varphi(t), \ t \in [-h, 0),
\]

where \( x_d = x(t - \tau), 0 \leq \tau \leq h \) represents the trajectory of the state in the past time, \( u_d = u(t - \tau), 0 \leq \tau \leq h \) denotes the control in the past time; \( h > 0 \) is a given constant; \( f : R^n \times R^n \times R^r \times R^r \rightarrow R^n, \ \phi : [-h, 0] \rightarrow R^n \) are all given continuously differentiable functions, and \( \varphi : [-h, 0) \rightarrow R^r \) is a given piecewise continuous function. Applying (4)-(5) to the time-delay system (6),

\[
\frac{d\tilde{x}(s)}{ds} = \frac{d}{ds} \{x(t(s))\} = \frac{dx(t(s))}{dt} \frac{ds}{dt}
\]

\[
= \theta u f(x(t(s)), x_d, u(t(s)), u_d)
\]

\[
= \theta u f(\tilde{x}(s), \tilde{x}(s(\theta)), \tilde{u}(s), u(s(\theta))), \ s \in [0, p].
\]

It is very difficult to find the closed form for \( s(\theta) \) in the new time horizon. Thus, time-scaling transformation is missed for time-delay optimal control problem over the past two decades. In 2016, Yu et al.[26] developed a hybrid time-scaling transformation to solve nonlinear time-delay optimal control problem. By utilizing the relationship between two time horizons, there is no need to find the closed form for \( s(\theta) \). This method is proved to be highly efficient for solving nonlinear time-delay optimal control problem.

For general time-delay optimal control problems, the delay \( \tau \) is normally a constant or a decision variable which needs to be chosen optimally, however, in many real world applications, the delay \( \tau \) may not be fixed, it could be a function of \( t \). In this paper, we first use the control parameterization technique to transform the original problem into a optimal parameter selection problem. Then, to simultaneously optimize the control height as well as the control switching times, we adopt the idea of hybrid time-scaling transformation [26] which is proposed for time-delay optimal control problem with fixed time-delay, and extend this transformation so that it can be used to solve time varying time-delay optimal control problem.

The rest of the paper is organized as follows: in Section 2, we give the mathematical formulation for time-delay optimal control problem with time-varying delay; In Section 3, by applying control parameterization together with hybrid time-scaling transformation, the original problem is transformed into an equivalent problem that can be readily solved. In Section 4, we provide the technical details for calculating the gradients of cost function and constrained functions with respect to the corresponding decision variables, then the problems can be readily solved by sequential quadratic programming method; Finally, we provide two numerical examples to show the effectiveness of this method.

2. Time-delay optimal control problem with time-varying delay. Standard dynamic system with time-varying delay can be described as below:

\[
\frac{d}{dt} y(t) = f(y(t), y_d, u(t), u_d), \ t \in [0, T],
\]

\[
y(t) = \phi(t), \ t \in [h_0, 0],
\]

\[
u(t) = \varphi(t), \ t \in [h_0, 0),
\]
where \( h_0 < 0 \) is a given constant, \( y(t) \in \mathbb{R}^n \) describes the state vector, \( u(t) \in U \) denotes the control vector (\( U \) is a compact and convex subset of \( \mathbb{R}^r \)), Borel measurable function \( u : [h_0, T] \to \mathbb{R}^r \) is called an admissible control if for almost all \( t \in [0, T] \) and \( u(t) = \varphi(t) \) for all \( h_0 \leq t < 0 \). \( y_{d1} = y(t-h(t)) \) and \( u_{d1} = u(t-h(t)) \) denote the state and control vector in the past time, respectively. \( h : [0, T] \to [0, -h_0], f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n \) and \( \varphi : [h_0, 0] \to \mathbb{R}^r \) are given continuously differential functions; \( \varphi : [h_0, 0] \to \mathbb{R}^r \) is a given piecewise continuous function. Let \( \mathcal{U} \) denote the set of all admissible controls and \( y(\cdot | u) \) denote the solution of (9)-(11) for each \( u(t) \in \mathcal{U} \), then canonical optimal control problem can be written as

\[
\begin{align*}
\min_{u} \quad & g_0(u) = \Phi_0(y(T | u)) + \int_0^T \mathcal{L}_0(y(t | u), y_d(t | u), u(t)) dt, \\
\text{s.t.} \quad & g_k(u) = \Phi_k(y(T | u)) + \int_0^T \mathcal{L}_k(y(t | u), y_d(t | u), u(t)) dt \quad k = 1, \ldots, d.
\end{align*}
\]

where \( y_d(t | u) = y(t-h(t) | u); \Phi_k : \mathbb{R}^n \to \mathbb{R}, \quad k = 0,1, \ldots, d \), and \( \mathcal{L}_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}, \quad k = 0,1, \ldots, d \) are all given functions. Let this problem be denoted as Problem \( P \).

To proceed, we assume that following conditions hold throughout the paper:

A1 \( \mathcal{L}_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}, \quad k = 0,1, \ldots, d \) and \( \Phi_k : \mathbb{R}^n \to \mathbb{R}, \quad k = 0,1, \ldots, d \) are continuously differentiable with respect to each of their arguments;

A2 There exists a real number \( L_1 > 0 \) such that \( \| f(e, \nu, \omega, \alpha) \| \leq L_1 (1 + \| e \| + \| \nu \|), (e, \nu, \omega, \alpha) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r \);  

A3 \( f \) is twice continuously differentiable.

3. Problem transformation.

3.1. Control parameterization. Firstly, divide the time interval \([0, T]\) into \( p \) subintervals (\( p \geq 1 \)),

\[
0 = t_0 < t_1 < \ldots < t_{p-1} < t_p = T.
\]

\( t_i, \quad i = 1, \ldots, p-1, \) are switching times, and then control vector can be approximated by

\[
u(t) \approx \sum_{i=1}^{p} \xi_i \chi_{[t_{i-1}, t_i)}(t), \quad t \in [0, T], \tag{12}
\]

where \( \xi_i \) is the value of control vector on the \( i \)th subinterval, \( \chi_{[t_{i-1}, t_i)} \) is the characteristic function:

\[
\chi_{[t_{i-1}, t_i)}(t) = \begin{cases} 1 & \text{if } t \in [t_{i-1}, t_i), \\ 0 & \text{otherwise}. \end{cases}
\]

Applying (12) to (9)-(11), on the subinterval \([t_{i-1}, t_i] \), the original system becomes

\[
\frac{d}{dt} y(t) = f(y(t), y_{d1}, \xi_i, u_{d2}) \tag{13}
\]

\[
y(t) = \varphi(t), \quad t \in [h_0, 0], \tag{14}
\]

where

\[
u_{d2} = \begin{cases} \xi_j, & \text{if } t \in [t_{j-1} + h(t), t_j + h(t)) \\ \varphi(t-h(t)), & \text{if } t < h(t). \end{cases}
\]
Let $\Sigma$ denote the set of all vector $\sigma = [t_1, \ldots, t_{p-1}]^T$. $\Xi$ denote the set of all vector $\xi = [\xi_1^T, \ldots, \xi_p^T]^T$, and $y(\cdot | \sigma, \xi)$ represent the solution of system (13)-(14) when $(\sigma, \xi) \in \Sigma \times \Xi$. Then, Problem $P$ can be converted into Problem $P'$:

$$
\begin{align*}
\min & \quad \Phi_0(y(T | \sigma, \xi)) + \sum_{i=1}^p \int_{t_{i-1}}^{t_i} \mathcal{L}_0(y(t | \sigma, \xi), y_d(t | \sigma, \xi), \xi_i) dt, \\
\text{s.t.} & \quad \Phi_k(y(T | \sigma, \xi)) + \sum_{i=1}^p \int_{t_{i-1}}^{t_i} \mathcal{L}_k(y(t | \sigma, \xi), y_d(t | \sigma, \xi), \xi_i) dt \left\{ \begin{array}{ll} 0, & t \geq h(t), \\
= 1, & t < h(t). \end{array} \right. \\
\end{align*}
$$

(15)

where

$$
\begin{align*}
y_d(t | \sigma, \xi) = \left\{ \begin{array}{ll} y(t - h(t) | \sigma, \xi), & t \geq h(t), \\
\phi(t - h(t) | \sigma, \xi), & t < h(t). \end{array} \right.
\end{align*}
$$

(16)

3.2. Hybrid time-scaling transformation. For Problem $P'$, it is complicated to compute the gradient of the state with respect to switching times. Hybrid time-scaling transformation[26] circumvents this difficulty. It introduces a new time horizon and converts Problem $P'$ into an equivalent optimal control problem defined on the new time horizon, in which the switching times are fixed.

Let $\delta \in \mathbb{R}^p$ satisfy

\begin{align*}
\delta_i = t_i - t_{i-1} \geq \epsilon, & \quad i = 1, \ldots, p, \\
\delta_1 + \ldots + \epsilon = T,
\end{align*}

where $\epsilon$ is a given positive real number, $\Delta$ denotes the set of vector $\delta = [\delta_1, \ldots, \delta_p]^T$, and the relationship between new time variable $\tau$ and $t$ can be described by the following time-scaling function:

$$
t = \mu(\tau | \delta) = \sum_{i=1}^{\lfloor \tau \rfloor} \delta_i + \delta_{\lfloor \tau \rfloor + 1} \{ \tau - \max(\lfloor \tau \rfloor, 0) \}, \quad \tau \in [0, p],
$$

(17)

where $\lfloor \cdot \rfloor$ represents floor function, $\delta_{p+1}$ is arbitrary and $\delta_i = 1$ if $i \leq 0$.

Let

$$
\tilde{y}(\tau) = y(\mu(\tau | \delta)) = y(t),
$$

Applying (17) to (13), for $\mu(\tau | \delta) \in [t_{i-1}, t_i)$, the time-delay system becomes

$$
\frac{d}{d\tau} \tilde{y}(\tau) = \delta, f(\tilde{y}(\tau), y_{d3}, \xi_i, u_{d3})
$$

(18)

where

\begin{align*}
y_{d3} = \left\{ \begin{array}{ll} y(\mu(\tau | \delta) - h(\mu(\tau | \delta))), & \mu(\tau | \delta) - h(\mu(\tau | \delta)) \in [t_i - t_j], \\
\phi(\mu(\tau | \delta) - h(\mu(\tau | \delta))), & \mu(\tau | \delta) - h(\mu(\tau | \delta)) < 0, \end{array} \right.
\end{align*}

\begin{align*}
u_{d3} = \left\{ \begin{array}{ll} \xi_j, & \mu(\tau | \delta) - h(\mu(\tau | \delta)) \in [t_i - t_j], \\
\varphi(\mu(\tau | \delta) - h(\mu(\tau | \delta))), & \mu(\tau | \delta) - h(\mu(\tau | \delta)) < 0. \end{array} \right.
\end{align*}

Note that $\mu(\tau | \delta) - h(\mu(\tau | \delta))$ is related with $\tau$ and $\delta$. Let $\tau_{\text{delay}}$ denote the delayed time point in the new time horizon, namely

$$
\mu(\tau_{\text{delay}} | \delta) = \mu(\tau | \delta) - h(\mu(\tau | \delta)),
$$

(19)
then the dynamic system (18) becomes
\[
\frac{d}{d\tau} \tilde{y}(\tau) = \delta_i f(\tilde{y}(\tau), y_{d4}, \xi_i, u_{d4})
\] (20)
where
\[
y_{d4} = \begin{cases} 
\tilde{y}(\tau_{\text{delay}}), & \text{if } \mu(\tau_{\text{delay}} | \delta) \in [t_{j-1}, t_j) \\
\phi(\mu(\tau_{\text{delay}} | \delta)), & \text{if } \mu(\tau_{\text{delay}} | \delta) < 0,
\end{cases}
\]
\[
u_{d4} = \begin{cases} 
\xi_j, & \text{if } \mu(\tau_{\text{delay}} | \delta) \in [t_{j-1}, t_j) \\
\varphi(\mu(\tau_{\text{delay}} | \delta)), & \text{if } \mu(\tau_{\text{delay}} | \delta) < 0.
\end{cases}
\]

Applying time-scaling function (17),
\[
\sum_{i=1}^{[\tau_{\text{delay}}]} \delta_i[\tau_{\text{delay}}] + 1 \{ \tau_{\text{delay}} - \max([\tau_{\text{delay}}], 0) \}
\]
\[
= \sum_{i=1}^{[\tau]} \delta_i[\tau] + 1 \{ \tau - \max([\tau], 0) \} - h(\mu(\tau | \delta)).
\] (22)

It is easy to see that \( \tau_{\text{delay}} \) is correlated with \( \tau \) and \( \delta \) and for each time point \( \tau \), we calculate the corresponding \( \tau_{\text{delay}} \) according to (22). Thus, it requires at least \( p/\epsilon \) to solve this problem, where \( p \) denotes the terminal time in the new time horizon and \( \epsilon \) denotes the maximum step length which is usually \( 10^{-3} \). Clearly, it would lead to long-time computation.

Considering the relationship between the state defined on two timelines,
\[
\tilde{y}(\tau_{\text{delay}}) = y(\mu(\tau_{\text{delay}} | \delta)) = y(t_{\text{delay}}),
\] (23)
we could utilize it to avoid calculating \( \tau_{\text{delay}} \). Applying (23) to (20), we have
\[
\frac{d}{d\tau} y(\tau) = \delta_i f(\tilde{y}(\tau), y_{d4}, \xi_i, u_{d4})
\] (24)
\[
y(\tau) = \phi(\tau), \quad \tau \in [h_0, 0],
\] (25)

\( t_{\text{delay}} \) therein can be obtained directly according to (21).

Let \( \tilde{y}(\cdot | \delta, \xi) \) denote the solution of time-delay system (24)-(25) and substitute it into (15)-(16) yields Problem Q:
\[
\min \quad \tilde{g}_0(\delta, \xi) = \Phi_0(\tilde{y}(p | \delta, \xi)) + \sum_{i=1}^{p} \int_{i-1}^{i} \delta_i \mathcal{L}_0(\tilde{y}(\tau | \delta, \xi), y_{d4}, \xi_i) d\tau,
\] (26)
\[
s.t. \quad \tilde{g}_k(\delta, \xi) = \Phi_k(\tilde{y}(p | \delta, \xi)) + \sum_{i=1}^{p} \int_{i-1}^{i} \delta_i \mathcal{L}_k(\tilde{y}(\tau | \delta, \xi), y_{d4}, \xi_i) d\tau \begin{cases} = 0, \quad k = 1, \ldots, d. \end{cases}
\] (27)
4. **Solving problem** $Q$. Before using sequential quadratic programming method to solve Problem $Q$, we need to compute the gradients of cost function and constrained functions with respect to $\delta$ and $\xi$ respectively. Firstly, $\tilde{g}_k(\delta, \xi)$, $k = 0, 1, \ldots, d$, are rewritten as the following form:

$$
\tilde{g}_k(\delta, \xi) = \Phi_k(\tilde{y}(p)) + \int_0^p \frac{\partial \mu(\tau | \delta)}{\partial \tau} \hat{L}_k(\tilde{y}(\tau), y_{d1}, \xi) d\tau,
$$

(28)

where

$$
\hat{L}_k(\tilde{y}(\tau), y_{d1}, \xi) = \sum_{i=1}^p L_k(\tilde{y}(\tau), y_{d1}, \xi_i) \chi_{[i-1,i)}(\tau).
$$

Secondly, the gradients of state with respect to $\delta$ and $\xi$ are respectively described in the following lemmas:

**Lemma 4.1.** For each pair $(\delta, \xi) \in \Delta \times \Xi$, we have

$$
\frac{\partial \tilde{y}(\tau | \delta, \xi)}{\partial \delta} = \Lambda(\tau | \delta, \xi), \ \tau \in [0, p].
$$

where $\Lambda(\cdot | \delta, \xi)$ is the solution to system below defined on each subinterval $[i-1, i)$:

$$
\dot{\Lambda}(\tau) = \frac{\partial f^i(\tilde{y}(\tau), y_{d1}, \delta, \xi)}{\partial \tilde{y}} \Lambda(\tau)
$$

$$
+ \frac{\partial f^i(\tilde{y}(\tau), y_{d1}, \delta, \xi)}{\partial y_{d1}} \left[ \Lambda(\tau_{delay}) + \frac{\partial y_{d1}}{\partial \tau_{delay}}\frac{\partial \tau_{delay}}{\partial \delta} \right]
$$

$$
+ \frac{\partial f^i(\tilde{y}(\tau), y_{d1}, \delta, \xi)}{\partial \delta}
$$

(29)

$$
\Lambda(\tau) = 0, \ \tau \leq 0,
$$

(30)

where for $\tau \in [i-1, i)$,

$$
f^i(\tilde{y}(\tau), y_{d1}, \delta, \xi) = \delta f(\tilde{y}(\tau), y_{d1}, \delta, \xi).
$$

**Proof.** The proof of Lemma 4.1 is similar with that in [26], and hence is omitted. \(\square\)

Note that for the computation of $\partial y_{d1}/\partial \tau_{delay}$, we can substitute $\frac{\partial}{\partial t} y(t_{delay})$ for it according to (23); for the computation of $\partial \tau_{delay}/\partial \delta$, calculate the gradients of both sides of (19) with respect to $\delta$

$$
\frac{\partial \mu(\tau_{delay} | \delta)}{\partial \delta} + \frac{\partial \mu(\tau_{delay} | \delta)}{\partial \tau} \cdot \frac{\partial \tau_{delay}}{\partial \delta} = \frac{\partial \mu(\tau | \delta)}{\partial \delta} - \frac{dh(\mu(\tau | \delta))}{\partial \delta} \cdot \frac{\partial \mu(\tau | \delta)}{\partial \delta},
$$

then we have

$$
\frac{\partial \tau_{delay}}{\partial \delta} = \left( \frac{\partial \mu(\tau | \delta)}{\partial \delta} - \frac{dh(\mu(\tau | \delta))}{\partial \delta} \right) \cdot \frac{\partial \mu(\tau | \delta)}{\partial \delta}.
$$

where $\partial \mu(\tau_{delay} | \delta)/\partial \delta$ and $\partial \mu(\tau_{delay} | \delta)/\partial \tau$ can be computed via interpolation.

**Lemma 4.2.** For each pair $(\delta, \xi) \in \Delta \times \Xi$, we have

$$
\frac{\partial \tilde{y}(\tau | \delta, \xi)}{\partial \xi} = \chi(\tau | \delta, \xi), \ \tau \in [0, p].
$$
Numerical examples.

5.1. Problem 1. Consider the following time-delay optimal control problem with time-varying delay:

\[
\begin{align*}
\min g_0(u) &= \frac{3}{2}r(3)^2 + \frac{1}{2} \int_0^3 u(t)^2 dt, \\
\text{s.t. } g_1(u) &= x(3)^2 - 0.03 \geq 0, \\
g_2(u) &= 0.06 - x(3)^2 \geq 0, \\
-3 \leq u(t) \leq 3.
\end{align*}
\]

where \( \Upsilon(\cdot | \delta, \xi) \) is the solution to system below defined on each subinterval \([i-1, i)\):

\[
\dot{\Upsilon} = \frac{\partial f_i(y(\tau), y_{di}, \delta, \xi)}{\partial y} \Upsilon(\tau) + \frac{\partial f_i(y(\tau), y_{di}, \delta, \xi)}{\partial y_{di}} \Upsilon_{\text{delay}}(
\end{align*}
\]

\[
\begin{align*}
\Upsilon(\tau) = 0, \quad \tau \leq 0,
\end{align*}
\]

Proof. The proof of Lemma 4.2 is similar with that of Lemma 4.1, and hence is omitted.

Lastly, Theorem 4.3 and Theorem 4.4 give the gradients of cost function and constrained functions with respect to \( \delta \) and \( \xi \):

**Theorem 4.3.** the gradient of \( \hat{g}_k(\delta, \xi) \), \( k = 0, 1, \ldots, d \), with respect to \( \delta \) can be written as:

\[
\frac{\partial \hat{g}_k(\delta, \xi)}{\partial \delta} = \frac{\partial \Phi_k(\bar{y}(p))}{\partial \delta} \frac{\partial \bar{y}(p)}{\partial \delta} + \frac{\int_{0}^{p} \left\{ \frac{\partial \hat{\lambda}_k(\bar{y}(\tau), y_{di}, \delta)}{\partial \bar{y}} \frac{\partial \bar{y}(\tau)}{\partial \delta} \right\} d\tau}{\partial \delta}
\]

Proof. The proof follows from applying the chain rule to (28).

**Theorem 4.4.** the gradient of \( \hat{g}_k(\delta, \xi) \), \( k = 0, 1, \ldots, d \), with respect to \( \xi \) can be written as:

\[
\frac{\partial \hat{g}_k(\delta, \xi)}{\partial \xi} = \frac{\partial \Phi_k(\bar{y}(p))}{\partial \xi} \frac{\partial \bar{y}(p)}{\partial \xi} + \frac{\int_{0}^{p} \left\{ \frac{\partial \hat{\lambda}_k(\bar{y}(\tau), y_{di}, \delta)}{\partial \bar{y}} \frac{\partial \bar{y}(\tau)}{\partial \xi} \right\} d\tau}{\partial \delta}
\]

Proof. The proof follows from applying the chain rule to (28).

On this basis, we can use sequential quadratic programming method to solve Problem Q, the corresponding programming software packages include FMINCON in MATLAB or NLPQLP in FORTRAN. Next section would demonstrate the effectiveness of this approach through solving two numerical examples.
subject to the dynamic system:
\[
\frac{dx}{dt} = x(t - \sin(t/2) - 1) + u(t), \ t \in [0, 3],
x(t) = 1, \ t \leq 0.
\]

By choosing different numbers of subintervals, we compare the obtained optimal costs by using hybrid time-scaling transformation and traditional control parameterization. More details are listed in Table 1. The results show that for two different methods, the optimal cost decreases as the number of subintervals increases and hybrid time-scaling transformation always achieves a better cost. For the case of \( q = 7 \), optimal control function is shown in Figure 1. and the corresponding optimal trajectories of the state are illustrated in Figure 2 and Figure 3.

\[
\begin{array}{|c|c|}
\hline
\text{(a) using hybrid time-scaling transformation} & \text{(b) using traditional control parameterization} \\
\hline
\text{partition number} & g_0(u^{q,*}) & \text{partition number} & g_0(u^{q,*}) \\
\hline
q = 7 & 2.1275 & q = 7 & 2.1347 \\
q = 5 & 2.1333 & q = 5 & 2.1446 \\
q = 3 & 2.1340 & q = 3 & 2.1812 \\
\hline
\end{array}
\]

Table 1. Optimal cost of Problem 1

![Figure 1. Optimal control for Problem 1 for the case of \( q = 7 \)](image)

5.2. Problem 2. Consider two-dimensional unconstrained time-delay optimal control problem with time-varying delay:
\[
\min g_0(u) = \frac{1}{2} x^\top(t_f)Sx(t_f) + \frac{1}{2} \int_0^{t_f} x^\top Q x + u^\top R u dt,
\]
dynamic system:
\[
\frac{dx}{dt} = A_1(t)x(t) + A_2(t)x(t - e^{-t} - 1) + B(t)u(t),
x(t) = [1, 0]^\top, \ t \leq 0,
\]
Figure 2. Optimal state trajectory for Problem 1 using hybrid time-scaling transformation for the case of \( q = 7 \)

\[
A_1(t) = \begin{bmatrix} 0 & -4\pi^2(a + c\cos 2\pi t) \\ -4\pi^2 c\cos 2\pi t & 0 \end{bmatrix}, \quad A_2(t) = \begin{bmatrix} 0 & 1 \\ -4\pi^2 b\cos 2\pi t & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

parameters of this problem are listed in the following table:

|   | a   | b   | c   | \( t_f \) | Q   | R   | S   |
|---|-----|-----|-----|--------|-----|-----|-----|
|   | 0.2 | 0.5 | 0.2 | 1.5    | \( I_{2\times2} \) | \( I_{2\times2} \) | \( 10^4 I_{2\times2} \) |

Table 2. Parameters in Problem 2

Similarly, by choosing different numbers of subintervals, we could compare the results of hybrid time-scaling transformation and that of traditional control parameterization. More details are given in Table 3. For the case of \( q = 8 \), optimal control function is shown in Figure 4, and the corresponding optimal trajectories of the
state are illustrated in Figure 5 and Figure 6. Again, the optimal costs obtained by using the proposed time scaling transformation are better, as expected.

(a) Using hybrid time-scaling transformation

| partition number | \( g_0(u^{q,+}) \) |
|------------------|---------------------|
| \( q = 8 \)      | 3.8750              |
| \( q = 6 \)      | 4.2319              |
| \( q = 4 \)      | 6.9578              |

(b) Using traditional control parameterization

| partition number | \( g_0(u^{q,+}) \) |
|------------------|---------------------|
| \( q = 8 \)      | 5.3252              |
| \( q = 6 \)      | 6.1942              |
| \( q = 4 \)      | 7.9119              |

Table 3. Optimal cost of Problem 2

Figure 4. Optimal control for Problem 2 for the case of \( q = 8 \)

Figure 5. Optimal state trajectory for Problem 2 using hybrid time-scaling transformation for the case of \( q = 8 \)
6. Conclusion. This paper applies control parameterization in conjunction with hybrid time-scaling transformation to solve time-delay optimal control problem with time-varying delay. We convert the given time-delay system into an equivalent system defined on a new time horizon with fixed switching times. Meanwhile, we obtain the value of time-delay state utilizing the relationship between the new time scale and the original one. Numerical results show that this method is more effective than traditional control parameterization where the switching times are fixed.

REFERENCES

[1] E. B. M. Bashier and K. C. Patidar, Optimal control of an epidemiological model with multiple time delays, *Applied Mathematics and Computation*, 292 (2017), 47–56.
[2] J. T. Betts, S. L. Campbell and K. C. Thompson, Optimal control software for constrained nonlinear systems with delays. in Proceedings, *IEEE Multi Conference on Systems and Control*, (2011), 444–449.
[3] Q. Q. Chai and W. Wang, A computational method for free terminal time optimal control problem governed by nonlinear time delayed systems, *Applied Mathematical Modelling*, 53 (2018), 242–250.
[4] M. Dadkhah, M. H. Farahi and A. Heydari, Optimal control of a class of non-linear time-delay systems via hybrid functions, *IMA Journal of Mathematical Control and Information*, 34 (2017), 255–270.
[5] R. Dehghan and M. Keyanpour, A numerical approximation for delay fractional optimal control problems based on the method of moments, *IMA Journal of Mathematical Control and Information*, 34 (2017), 77–92.
[6] L. Denis-Vidal, C. Jauberthie and G. Joly-Blanchard, Identifiability of a nonlinear delayed-differential aerospace model, *IEEE Trains. Auton. Control*, 51 (2006), 154–158.
[7] L. Göllmann and H. Maurer, Theory and applications of optimal control problems with multiple time-delays, *Journal of Industrial and Management Optimization*, 10 (2014), 413–441.
[8] Z. Gong, C. Liu and Y. Wang, Optimal control of switched systems with multiple time-delays and a cost on changing control, *Journal of Industrial and Management Optimization*, 14 (2018), 183–198.
[9] A. Jajarmi and M. Hajipour, An Efficient finite differencr method for the time-delay optimal control problems with time-varying delay, *Asian Journal of Control*, 19 (2017), 554–563.
[10] H. W. J. Lee, K. L. Teo, V. Rehbock and L. S. Jennings, Control parametrization enhancing technique for time optimal control problems, *Dynamic Systems and Applications*, 6 (1997), 243–261.
[11] H. W. J. Lee, K. L. Teo, V. Rehbock and L. S. Jennings, Control parameterization enhancing technique for optimal discrete-valued control problems, *Automatica*, 35 (1999), 1401–1407.

[12] J. Lei, Optimal vibration control of nonlinear systems with multiple time-delays: An application to vehicle suspension, *Integrated Ferroelectrics*, 170 (2016), 10–32.

[13] G. N. Li, H. L. Xu and Y. Lin, Application of bat algorithm based time optimal control in multi-robots formation reconfiguration, *Journal of Bionic Engineering*, 15 (2018), 126–138.

[14] Q. Lin, R. Loxton and K. L. Teo, The control parameterization method for nonlinear optimal control: A survey, *Journal of Industrial and Management Optimization*, 10 (2014), 275–309.

[15] C. Liu, Z. Gong and K. L. Teo, Robust parameter estimation for nonlinear multistage time-delay systems with noisy measurement data, *Applied Mathematical Modelling*, 53 (2018), 353–368.

[16] C. Liu, R. Loxton and K. L. Teo, A computational method for solving time-delay optimal control problems with free terminal time, *Systems & Control Letters*, 72 (2014), 53–60.

[17] C. Y. Liu, R. Loxton and K. L. Teo, Switching time and parameter optimization in nonlinear switched systems with multiple time-delays, *Journal of Optimization Theory and Applications*, 163 (2014), 957–988.

[18] P. Liu, G. D. Li, X. G. Liu, L. Xiao, Y. L. Wang, C. H. Yang and W. H. Gui, A novel non-uniform control vector parameterization approach with time grid refinement for flight level tracking optimal control problems, *ISA transactions*, 73 (2018), 66–78.

[19] G. R. Marzban and S. M. Hoseini, An efficient discretization scheme for solving nonlinear optimal control problems with multiple time delays, *Optimal Control Applications and Methods*, 37 (2016), 682–707.

[20] K. L. Teo, C. J. Goh and K. H. Wong, *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific and Technical, Essex, 1991.

[21] C. Z. Wu and K. L. Teo, Optimal impulsive control computation, *Journal of Industrial and Management Optimization*, 2 (2006), 435–450.

[22] C. Z. Wu, K. L. Teo, R. Li and Y. Zhao, Optimal control of switched systems with time delay, *Applied Mathematics Letters*, 19 (2006), 1062–1067.

[23] Y. Wu, Z. Yuan and Y. Wu, Optimal tracking control for networked control systems with random time delays and packet dropouts, *Journal of Industrial and Management Optimization*, 11 (2015), 1343–1354.

[24] F. Yang, K. L. Teo, R. Loxton, V. Rehbock, B. Li, C. Yu and L. Jennings, VISUAL MISER: An efficient user-friendly visual program for solving optimal control problems, *Journal of Industrial and Management Optimization*, 12 (2016), 781–810.

[25] C. Yu, B. Li, R. Loxton and K. L. Teo, Optimal discrete-valued control computation, *Journal of Global Optimization*, 56 (2013), 503–518.

[26] C. Yu, Q. Lin, R. Loxton, K. L. Teo and G. Wang, A hybrid time-scaling transformation for time-delay optimal control problems, *Journal of Optimization Theory and Application*, 169 (2016), 876–901.

Received February 2018; revised May 2018.

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