Using Harmonic Mean to Replace Tsallis’ q-Average

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Abstract— In this paper, a unified mathematical expression for the constraints leading to the equilibrium distributions of both extensive and non-extensive systems is presented. Based on this expression, a recommendation is made to replace Tsallis’ q-average without obvious physical meaning with the statistical harmonic mean for a generalized system.

Key words— Tsallis entropy, q-average, statistical harmonic mean, stimulus-response, power-law, non-extensive, constraint.

I. INTRODUCTION
There is no doubt that Tsallis entropy [1] is an important progress of modern statistical mechanics. However, in the final version, not the original one, of ‘Tsallis’ formalism, there is a weak point. The weak point is that the q-average’s physical meaning is far from being obvious. Recently, H. Hasegawa [2] made a serious comment preferring Tsallis’ original idea of the ordinary average. In previous work of the author [3], a unified mathematical expression about the constraints leading to the equilibrium distributions of non-extensive systems was given as the theorem 2. In this paper a generalized version of that expression is given. The generalized expression is with clear physical meaning. Based on the new expression, the standard Tsallis’ power-law or so-called q-exponential has been found to be associated directly with a new constraint. The new constraint is related with the harmonic mean. Since the statistical harmonic mean is with a relatively clear physical meaning and can completely describe the new constraint, it is recommended to use it to replace Tsallis’ q-average.

II. THE UNIFIED MATHEMATICAL EXPRESSION FOR THE CONSTRAINTS LEADING TO EQUILIBRIUM DISTRIBUTIONS
In the previous work of the author [3], a unified mathematical expression for the constraints leading to the equilibrium distributions of non-extensive systems was given as the theorem 2. After a further and deep study, the author has generalized that unified expression. The generalized expression is described as follows.

Theorem If the dynamic probability distribution of \( p = \{p_1, p_2, \ldots, p_n\} \) of a generalized system is constrained by the following unified expression

\[
\frac{1}{(q-1)} \left[ p_1 (f_1^{(q-1)} - 1) + p_2 (f_2^{(q-1)} - 1) + \ldots + p_n (f_n^{(q-1)} - 1) \right] = \text{const},
\]

(Eq.1)

it is possible for the probability distribution \( p \) to reach the equilibrium state of \( f = \{f_1, f_2, \ldots, f_n\} \). When \( q \to 1 \), the equation 1 becomes

\[
p_1 \ln(f_1) + p_2 \ln(f_2) + \ldots + p_n \ln(f_n) = \text{const},
\]

where, the const is meant by a constant.

Proof
The Tsallis entropy can be expressed as follows

\[
E = \frac{1}{(q-1)} (1 - p_1^q - p_2^q - \ldots - p_n^q),
\]

and there is a natural constraint which can be express as

\[
p_1 + p_2 + \ldots + p_n = 1.
\]

Therefore, the corresponding Lagrangian without taking constants into consideration is

\[
L = \frac{1}{(q-1)} (1 - p_1^q - p_2^q - \ldots - p_n^q)
\]

+\[
\frac{m_1}{(q-1)} [p_1 (f_1^{(q-1)} - 1) + p_2 (f_2^{(q-1)} - 1) + \ldots + p_n (f_n^{(q-1)} - 1)]
\]

+\[
m_2 (p_1 + p_2 + \ldots + p_n),
\]

where \( m_1 \) and \( m_2 \) are Lagrangian multipliers to be determined. In order to make \( L \) take the extreme, one has

\[
\frac{\partial L}{\partial p_i} = 0
\]

Since

\[
\frac{\partial L}{\partial p_i} = -\frac{q}{(q-1)} p_i^{(q-1)} - \frac{m_1}{(q-1)} (f_i^{(q-1)} - 1) + m_2, i = 1, 2, \ldots, n,
\]

one can make \( m_1 = q \) and \( m_2 = \frac{q}{(q-1)} \), a zero \( \frac{\partial L}{\partial p_i} \) will give

\[
p_i = f_i, i = 1, 2, \ldots, n.
\]

The theorem has been proven.
III. THE PHYSICAL MEANING OF THE UNIFIED 
MATHEMATICAL EXPRESSION OF THE 
CONSTRAINTS LEADING TO EQUILIBRIUM 
DISTRIBUTIONS 

There are two types of laws about the stimulus-response 
mechanism of human being. One is Weber–Fechner law 
and the other is Stevens’ power-law [4][5]. For the author, 
equation 1 is related with both of them. Imagine each 
component of the distribution \( f \) is a stimulus to a man or a 
woman. The response as a whole, per Stevens’ power-law, 
will be something like 

\[
\alpha = \{f_1, f_2, ..., f_n\} 
\]

The man or the woman will store this response \( \alpha \) in his or 
her brain. Later on, when he or she “recalls” the distribution 
\( f \), he or she will reconstruct the distribution \( f \) under 
the constraint of that the statistical average of \( \alpha \) will be a 
constant during the whole dynamic process of the 
reconstruction. Let \( \alpha = q - 1 \) and take into consideration the 
natural constraint of 

\[
p_1 + p_2 + ... + p_n = 1, 
\]

one will understand that the constraint leading to the 
reconstructed equilibrium distribution \( f \) is 

\[
p_1 f_1^{(q-1)} + p_2 f_2^{(q-1)} + ... + p_n f_n^{(q-1)} = const, 
\]

or, 

\[
\frac{1}{(q-1)} [p_1 f_1^{(q-1)} - 1] + p_2 f_2^{(q-1)} - 1) + ... 
\]

\[
+ p_n (f_n^{(q-1)} - 1)] = const, 
\]

for a given \( q \). That is exactly what being described in the 
above-mentioned theorem. It is amazing that nature follows 
exactly the same law as human being in the reconstruction of 
a distribution! It should be mentioned that when \( q \rightarrow 1 \), the 
Stevens’ power law can be replaced by Weber–Fechner law 
and the conclusion will remain unchanged.

IV. THE NEW CONSTRAINT LEADING TO THE STANDARD FORM 
OF TSALLIS’ POWER-LAW 

The standard form of Tsallis’ power law is [6] 

\[
f_k \propto [1 + \beta(q-1)E_k]^\frac{1}{q-1} \]

\[
\]

\[
, k = 1,2,...,n, 
\]

where \( \beta \) is a Lagrangian multiplier, \( E_k \) is the energy of the 
\( k \)th component of the generalized system. For this particular 
distribution, per the equation 1, the corresponding constraint 
leading to this distribution is 

\[
p_1[1 + \beta(q - 1)E_1]^{-\frac{1}{q - 1}} + p_2[1 + \beta(q - 1)E_2]^{-\frac{1}{q - 1}} + ... 
\]

\[
+ p_n[1 + \beta(q - 1)E_n]^{-\frac{1}{q - 1}} = const 
\]

(Eq.2) 

For a generalized system, let 

\[
EF_i = 1 + \beta(q - 1)E_i, i = 1,2,...,n, 
\]

one can define \( EF_i \) as the effective energy of the \( i \)th 
component of the generalized system. From equation 2, 

\[
RH = p_1 \left( \frac{1}{EF_1} \right) + p_2 \left( \frac{1}{EF_2} \right) + ... + p_n \left( \frac{1}{EF_n} \right) = const 
\]

(Eq.3) 

One has 

\[
H = \frac{1}{RH} = const 
\]

(Eq.4) 

It is obvious that \( H \) is nothing else but the statistical 
harmonic mean of the effective energies of \( EF_i, i = 1,2,...,n \). 
The meaning of the equation 4 is that a constant statistical 
harmonic mean of the effective energies of a generalized 
system will lead to the standard form of Tsallis’ power law. 
Compared with Tsallis’q-average, the statistical harmonic 
mean’s physical meaning is much clearer. It is recommended 
that the Tsallis’ q-average should be replaced by the statistical 
harmonic mean to make people feel much more comfortable 
when they use the Tsallis entropy for their own purposes.

V. CONCLUSIONS AND DISCUSSION 

A unified mathematical expression for the constraints 
leading to equilibrium distributions of both extensive and non-
extensive systems is presented. The clear physical meaning 
of the expression is discussed. Based on the unified expression, 
It is recommended to use the statistical harmonic mean of 
effective energies to replace Tsallis’ q-average.

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