GLOBAL WEAK SOLUTIONS TO A GENERAL LIQUID CRYSTALS SYSTEM

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Abstract. We prove the global existence of finite energy weak solutions to the general liquid crystals system. The problem is studied in bounded domain of \( \mathbb{R}^3 \) with Dirichlet boundary conditions and the whole space \( \mathbb{R}^3 \).

1. Introduction. Liquid crystals were discovered in 1888 by F.Reinitzer and O.Lehmann. They are often viewed as intermediate states between the solids and fluids, whose molecular arrangements give rise to preferred directions. As a result, they retain several different features: mechanical, electrical, magnetic and optical properties. According to molecular arrangements, it has been widely supported that liquid crystals can be classified into three types: nematics, cholesterics and smectics. The historical development of liquid crystals confronts two theories, including the swarm theory and the distortion theory. The former theory is well established but only applied to nematics and cholesterics, while the later one, successfully explaining the interactions of nematics with magnetic fields, is not so well-known for us. Based on the predecessors’ work, in the 1960’s, Ericksen and Leslie established the Hydrodynamic theory of nematic liquid crystals system(see [16], [17], [18], [3], [22] and [31]):

\[
\begin{align*}
\rho_t + \text{div} (\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) &= \rho \mathbf{F} + \text{div} \sigma, \\
\rho_1 \frac{d\mathbf{d}}{dt} &= \rho_1 \mathbf{G} + \mathbf{g} + \text{div} \pi,
\end{align*}
\]

where \( \rho \geq 0 \), \( u = (u_1, u_2, u_3) \), \( d = (d_1, d_2, d_3) \) are the fluid density, velocity and molecular direction respectively. \( \mathbf{F} \) denotes the external body force, \( \mathbf{G} \) the external body force for the direction movement, \( g \) the internal body force for the direction movement and \( \rho_1 \frac{d\mathbf{d}}{dt} \) the angular movement per unit time. In low frequency, \( \rho_1 \frac{d\mathbf{d}}{dt} \) is so small that can be neglected. The equations (1.1)-(1.3) represent the conservation...
of mass, linear momentum, and angular momentum respectively. \( \sigma, g \) and \( \pi \) satisfy the following constitutive relations:

\[
\sigma_{ij} = \left(-p - \rho \frac{\partial H}{\partial \rho}\right) \delta_{ij} - \frac{\partial (\rho H)}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij},
\]

\[
\pi_{ij} = \frac{\partial (\rho H)}{\partial d_{i,j}},
\]

\[
g_i = -\frac{\partial (\rho H)}{\partial d_i} + \kappa d_i + \tilde{g}_i,
\]

where \( p = \alpha \rho^\gamma \) denotes the pressure, \( \rho H \) the bulk free energy, and let \( d_{i,j} \) represent \( \frac{\partial d_i}{\partial x_j} \). In equation (1.6), \( \kappa \) is Lagrange multiplier constraint to \( |d| = 1 \). According to Frank’s formula [7] (Chapter 3), we can get the bulk free energy of nematic types.

\[
2\rho H = k_1 (\text{div} d)^2 + k_2 (d \cdot \text{curl} d)^2 + k_3 |d \wedge \text{curl} d|^2 + (k_2 + k_4) (\text{tr}(\nabla d)^2 - (\text{div} d)^2).
\]

Likewise, from [31] we have

\[
\tilde{\sigma} = \mu_1 (d^T A d) d \otimes d + \mu_2 N \otimes d + \mu_3 d \otimes N + \mu_4 A + \mu_5 (A d) \otimes d + \mu_6 d \otimes (A d) + \mu_7 \text{tr}(A) I,
\]

\[
\tilde{g} = \lambda_1 N + \lambda_2 A.
\]

Here \( d \otimes d \) denotes a matrix whose \((i,j)\)-th entry is given by \( d_i d_j \). We use the following notations:

\[
w = \frac{\partial d}{\partial t} + (u \cdot \nabla) d,
\]

\[
N = w - \Omega d, \quad A = \frac{1}{2} \left(\nabla u + (\nabla u)^T\right),
\]

\[
\Omega = \frac{1}{2} (\nabla u - (\nabla u)^T),
\]

here \( N = (N_1, N_2, N_3) \) is used to describe the director movements in satellited coordinates, and \( w = (w_1, w_2, w_3) \) is the material derivatives of \( d \). And \( \lambda_i, \mu_i \) satisfy the following formulas:

\[
\lambda_1 = \mu_2 - \mu_3 < 0,
\]

\[
\lambda_2 = \mu_5 - \mu_6 = -(\mu_2 + \mu_3),
\]

\[
\mu_5 + \mu_6 \geq 0, \quad \mu_1 \geq 0,
\]

\[
\mu_4 \geq 0, \quad \mu_7 \geq 0.
\]

(1.11) is called Parodi’s condition, which is derived from the onager reciprocal relation, see [23], [3], [31]. For simplicity, we assume \( k_4 = 0, k_1 = k_2 = k_3 = 1 \). Observing that

\[
|\nabla d|^2 = \text{tr}(\nabla d)^2 + (\text{curl} d)^2,
\]

\[
(\text{curl} d)^2 = (d \cdot \text{curl} d)^2 + |d \wedge \text{curl} d|^2,
\]

we have the bulk free energy

\[
2\rho H = |\nabla d|^2.
\]
In statics and low frequency, if we set $G = 0$ and neglect $\rho \frac{du}{dt}$, the equation (1.3) reads

$$\Delta d + \kappa d = 0.$$  \hspace{1cm} (1.14)

Indeed equation (1.14) is the Euler equation of the minimum bulk free energy $\rho H$ under the restriction $|d| = 1$. As a consequence, we have $\kappa = |\nabla d|^2$. Because the nonlinear term $|\nabla d|^2 d$ is bad for getting weak solution, we use the Ginzburg-Landau approximate function

$$2\rho H = |\nabla d|^2 + 2F(d),$$  \hspace{1cm} (1.15)

where $F(d) = \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2$ for fixed $\varepsilon > 0$. Thus we use

$$-\Delta d + \frac{1}{\varepsilon^2}(|d|^2 - 1)d = 0$$  \hspace{1cm} (1.16)

instead of (1.14).

In this paper, we set $\rho \frac{du}{dt} = 0$, $F = G = 0$, $\mu_1 = \mu_5 = \mu_6 = \lambda_2 = 0$. Using (1.10)-(1.13), we have $\mu_2 = -\mu_3 < 0$, $\lambda_1 = -2\mu_3 < 0$, $\mu_4 \geq 0$, $\mu_7 \geq 0$.

Finally, the liquid crystals system becomes

$$\rho_t + \text{div}(\rho u) = 0,$$  \hspace{1cm} (1.17)

$$(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \left( p - \frac{1}{2}|\nabla d|^2 - F(d) \right) + \nabla \cdot (\nabla d \otimes \nabla d) = \text{div}\tilde{\sigma},$$  \hspace{1cm} (1.18)

$$\lambda_1 d_t + \lambda_1 u \cdot \nabla d - \lambda_1 \Omega d + \Delta d - f(d) = 0,$$  \hspace{1cm} (1.19)

where $\tilde{\sigma}$ is given by

$$\tilde{\sigma} = \mu_3(N \otimes d - d \otimes N) + \mu_4 A + \mu_7 tr(A) I.$$  \hspace{1cm} (1.20)

After a simple computation, we find $|d|^2 \leq 1$ by the maximum principle.

**The case of bounded domain $D$:**

We are interested in the global weak solutions to the system (1.17)-(1.19) in a bounded domain $D \subset \mathbb{R}^3$ with initial conditions:

$$\left\{ \begin{array}{l}
\rho(x, 0) = \rho_0(x) \geq 0 \ a.e. \ in \ D, \ \rho_0 \in L^\gamma(D), \\
(\rho u)(x, 0) = q(x), \ q(x) = 0 \ a.e. \ on \ \{\rho_0(x) = 0\}, \ \frac{|q|}{\rho_0} \in L^1(D), \\
d(x, 0) = d_0(x), \ |d_0(x)| = 1, \ d_0 \in H^2(D),
\end{array} \right.$$  \hspace{1cm} (1.21)

and the boundary conditions

$$u(x, t) = 0, \ d(x, t) = d_0(x), \ (x, t) \in \partial D \times (0, \infty).$$  \hspace{1cm} (1.22)

In order to give the definition of the weak solutions, we firstly describe energy inequality

$$\frac{d}{dt}E(t) \leq -\int_D \left[ \mu_4 |A|^2 - \lambda_1 |N|^2 + \mu_7 (\text{div}u)^2 \right],$$  \hspace{1cm} (1.23)

where

$$E = \int_D \left[ \frac{1}{2} \rho |u|^2 + \frac{1}{2} |\nabla d|^2 + \frac{1}{\gamma - 1} p + F(d) \right].$$  \hspace{1cm} (1.24)
It reflects the energy dissipation property of the flow of liquid crystals.

Multiplying (1.17) by $B'(ho)$, we formally have

$$
(B(ho))_t + \text{div}(B(ho)u) + b(\rho)\text{div}u = 0,
$$

(1.25)

where $B$ is a smooth function and

$$
b(\rho) = B'(\rho)\rho - B(\rho).
$$

(1.26)

**Definition 1.1.** We call $(\rho, u, d)$ a finite energy weak solution of (1.17)-(1.19) in bounded domain $D \subset \mathbb{R}^3$ with initial and boundary conditions (1.21) and (1.22), if it satisfies the following conditions:

- $\rho \in L^\infty(0,T;L^\gamma(D))$, $\rho \geq 0$ a.e. in $(0,T) \times D$; $d \in L^2(0,T;H^2(D)) \cap L^\infty(0,T;H^1(D)) \cap L^\infty((0,T) \times D)$, and $u \in L^2(0,T;H^0(D))$.
- the energy $E(t)$ is locally integrable on $(0,T)$, and (1.23) holds in $D'(0,T)$;
- $(\rho, u, d)$ satisfies (1.17)-(1.19) in $D'(0,T) \times D$, and (1.17) holds in $D'(0,T) \times \mathbb{R}^3$, provided $(\rho, u)$ is prolonged to be zero on $\mathbb{R}^3 \setminus D$;
- equation (1.17) is satisfied in the sense of renormalized solutions, that is (1.25) holds in $D'(0,T \times D)$ for any $B \in C[0,\infty) \cap C^1(0,\infty)$, $b \in C[0,\infty)$ bounded on $[0,\infty)$, $B(0) = b(0) = 0$.

Then we have the following result:

**Theorem 1.1.** Assume $D \subset \mathbb{R}^3$ is a bounded domain of the class $C^{2+\nu}$, $\nu > 0$, and $\gamma > \frac{3}{2}$. Then the system (1.17)-(1.19) with (1.21)-(1.22) has a finite energy weak solution defined by Definition 1.1 for any $T < \infty$.

**The case of whole space $\mathbb{R}^3$:**

Let Banach space

$$
\mathcal{H}(\mathbb{R}^3) = \{ d \mid d \in L^\infty(\mathbb{R}^3), \ d \in \dot{H}^1(\mathbb{R}^3) \},
$$

and the Orlicz space $L^2_p(\mathbb{R}^3)$ (see [24] pp.288)

$$
L^2_p(\mathbb{R}^3) = \left\{ f \in L^1_{loc}(\mathbb{R}^3), \ f|_{|f| \leq \frac{1}{2}} \in L^2(\mathbb{R}^3), \ f|_{|f| \geq \frac{1}{2}} \in L^p(\mathbb{R}^3) \right\}.
$$

We consider our problem (1.17)-(1.19) in the whole space $\mathbb{R}^3$ with initial conditions:

$$
\begin{align*}
\rho(x,0) &= \rho_0(x) \geq 0 \quad \text{a.e. in } \mathbb{R}^3, \\
(\rho u)(x,0) &= q(x), \ q(x) = 0 \quad \text{a.e. on } \{ \rho_0(x) = 0 \}, \ \frac{|d|^2}{\rho_0} \in L^1(\mathbb{R}^3), \\
d(x,0) &= d_0(x), \ |d_0(x)| = 1, \ d_0 \in \mathcal{H}(\mathbb{R}^3) \cap \dot{H}^2(\mathbb{R}^3),
\end{align*}
$$

(1.27)

where

$$
0 \leq \int_{\mathbb{R}^3} (\rho_0)^\gamma - \gamma (\rho_0 - 1) - 1 \leq C_0.
$$

(1.28)

One can obtain the energy inequality

$$
\frac{d}{dt} E(t) \leq -\int_{\mathbb{R}^3} \left[ \mu_1 |A|^2 - \lambda_1 |N|^2 + \mu_\gamma (\text{div}u)^2 \right],
$$

(1.29)

where

$$
E = \int_{\mathbb{R}^3} \left[ \frac{1}{2} \rho |u|^2 + \frac{1}{2} |\nabla d|^2 + \frac{\rho^\gamma - \gamma (\rho - 1) - 1}{\gamma - 1} + F(d) \right].
$$

We define the weak solution in the sense of following
Definition 1.2. We call \((\rho, u, d)\) is a weak-solution of (1.17)-(1.19) in \(\mathbb{R}^3\) if it satisfies the following conditions:

- \(\rho - 1 \in L^\infty(0, T; L^\gamma(\mathbb{R}^3))\) if \(\gamma \geq 2\) and \(\rho - 1 \in L^\infty(0, T; L^2(\mathbb{R}^3))\) if \(\gamma < 2\), \(\rho \geq 0\) a.e. in \((0, T) \times \mathbb{R}^3\), \(u \in L^2(0, T; H^1(\mathbb{R}^3))\), \(d \in L^\infty(0, T; H(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))\), \(|d| \leq 1\) a.e. in \((0, T) \times \mathbb{R}^3\);
- the energy \(E(t)\) is locally integrable on \((0, T)\), and (1.29) holds in \(D'(0, T)\);
- \((\rho, u, d)\) satisfies (1.17)-(1.19) in \(D'_{\text{loc}}((0, T) \times \mathbb{R}^3)\).

Then we have

Theorem 1.2. Let \(\gamma > \frac{3}{2}\). The system (1.17)-(1.19) with (1.27)-(1.28) has a weak-solution defined by Definition 1.2 for any \(T < \infty\).

There are a lot of results for the incompressible and compressible liquid crystals systems. For the incompressible case with a constant density, F. Lin and C. Liu systematically studied the existence and partial regularity of weak solutions in their papers [9], [10], [11]. For the density is not constant, X. Liu and Z. Zhang [28] proved existence of weak solutions in a bounded domain of \(\mathbb{R}^3\), also to see F. Jiang and Z. Tan [14] for weakening assumption of [28]. For the compressible case, in 2009, X. Liu and J. Qing [29] firstly proved the existence of weak solutions to liquid crystals system (more simpler than one considered in present paper) in a bounded domain of \(\mathbb{R}^3\). Similar results also see D. Wang, Y. Cheng [4]. We also note that papers by S. Ding, C. Wang and H. Wen [26] for one dimension case and by X. Hu, D. Wang [30] for the Besov space case.

In this paper, we consider a more generic(fitting more physical properties) liquid crystals system (1.17)-(1.19) and study its existence of weak solutions in both bounded domain and the whole space. The main difficulties for solving the problem are dealing with pressure term and the nonlinear terms appeared in the stress tensors. Like Navier-Stokes equations, there are phenomenons of losing compactness caused by the higher nonlinear terms of the equations. Fortunately, we overcome those difficulties by using much more technical methods such as compensated compactness used in P. L. Lions’ book [24], also to see E. Feireisl [5], [6].

The remaining part of this paper is organized as follows. Section 2, 3, 4 are devoted to proof Theorem 1.1. Using Theorem 1.1, we prove Theorem 1.2 in Section 5.

2. Approximate solutions. In this section, similar to Eduard Feireisl did on Navier-Stokes equations (see [5], [6]), we firstly construct an approximate problem:

\[
\rho_t + \text{div}(\rho u) = \epsilon \Delta \rho, \quad (2.1)
\]

\[
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla \left( p - \frac{1}{2} |\nabla d|^2 - F(d) \right) + \nabla \cdot (\nabla d \otimes \nabla d)
\]

\[+ \nabla (\delta \rho^2) + \epsilon (\nabla u \cdot \nabla \rho) = \text{div} \sigma, \quad (2.2)\]

\[
\lambda_1 d_t + \lambda_1 u \cdot \nabla d - \lambda_1 \Omega d + \Delta d - f(d) = 0, \quad (2.3)
\]

complemented by the initial conditions:

\[
\rho(x, 0) = \rho_0(x) \in C^2(\overline{D}), \quad 0 < \rho \leq \rho_0 \leq \overline{\rho}, \nabla \rho_0 \cdot \vec{n}|_{\partial \Omega} = 0, \quad (2.4)
\]

\[
(\rho u)(x, 0) = q_0(x), \quad q_0 \in C^2(\overline{D}; \mathbb{R}^3), \quad (2.5)
\]

\[
d(x, 0) = d_0(x), \quad d_0 \in C^2(\overline{D}; \mathbb{R}^3), \quad |d_0(x)| = 1 \quad (2.6)
\]
and the boundary conditions:

\[ \nabla \rho \cdot \vec{n} \big|_{\partial D} = 0, \]  
\[ u \big|_{\partial D} = 0, \]  
\[ d \big|_{\partial D} = d_0, \]  

where \( \vec{n} \) is the out normal vector of \( \partial D \).

For the Neumann problem (2.1), (2.4) and (2.7), by Lemma 2.2 of [29], we have

**Lemma 2.1.** Let \( D \subset \mathbb{R}^3 \), be a bounded domain of \( C^{2+r} \), \( r > 0 \). Then there exists an operator \( \rho = S(u) \) solving the problem (2.1), (2.4) and (2.7),

\[ S : C([0, T]; C^2(D)) \rightarrow C([0, T]; C^{2+r}(D)) \]

having the following properties:

- \( \rho = S(u) \) is the unique classical solution of (2.1), (2.4) and (2.7);
- \( \rho \geq \inf_{x \in \Omega} \rho(0, x) \exp \left( -\int_0^t \| \text{div} u \|_\infty(s) \, ds \right) \), and \( \rho \leq \sup_{x \in \Omega} \rho(0, x) \exp \left( \int_0^t \| \text{div} u \|_\infty(s) \, ds \right) \);
- \( \sup_{t \in [0, T]} \| \rho \|_{L^2(\Omega)} \leq C(T, \rho_0, \| \nabla u \|_{L^\infty(0, T) \times D}) \);
- \( \| S(u^1) - S(u^2) \|_{C([0, T]; L^2(D))} \leq T \frac{C}{2} u^1 - u^2 \|_{L^\infty(0, T; L^2(D))} \).

Next we consider the director movement equation (2.3). Similarly as Lemma 2.3 in [29], we have

**Lemma 2.2.** There exists a mapping \( R = R(u) \),

\[ R : C([0, T]; (C^2(\Omega))^3) \rightarrow C([0, T]; C^\infty(\Omega)) \]

enjoying the following properties:

- \( \delta = R(u) \) is the smooth solution of the problem (2.3), (2.6) and (2.9);
- \( \| R(u) \|_{L^\infty(0, T; H^1(D))} + \| R(u) \|_{L^2(0, T; H^2(D))} \leq C(T, K_0, u) \);
- \( \| R(u^1) - R(u^2) \|_{L^\infty(0, T; H^1(D))} \leq T \frac{C}{2} u^1 - u^2 \|_{L^\infty(0, T; H^1(D))} \).

### 2.1. The Faedo-Galerkin approximations.

In the subsection we solve the equation (2.2) with initial condition (2.5) by Faedo-Galerkin approximate.

Let \( \{ \phi_i \}_{i=1}^\infty \) be the orthogonal basis of \( H^1_0(D) \), which satisfies:

\[ -\Delta \phi_i = a_i \phi_i, \quad \text{in} \; \Omega, \quad \phi_i \big|_{\partial D} = 0. \]

Here \( a_i \), \( (i = 1, 2, \cdots) \) is the eigenvalue of the operator \( -\Delta \), and \( 0 < a_1 \leq a_2 \leq \cdots a_n \leq \cdots, a_n \rightarrow \infty, \; n \rightarrow \infty \). We consider the finite dimensional space \( X_n = \text{span} \{ \phi_i \}_{i=1}^n, \quad n = 1, 2, 3 \cdots \)

And \( X_n \) is Hilbert space equipped with norm given by scalar product of \( L^2 \). We shall look for the approximate solution \( u = u_n \in C([0, T]; X_n) \), satisfying the following integral equation

\[ \int_D \rho u \cdot \eta - \int_D q_{0, \delta} \cdot \eta = \int_0^t \int_D \left( p + \delta \rho^\delta - \frac{1}{2} (|\nabla d|^2 + F(d)) \right) \text{div} \eta - \epsilon (\nabla u \nabla \rho) \eta \]

\[ + \int_0^t \int_D (\rho u \otimes u + \nabla d \otimes \nabla d - \tilde{\sigma} : \nabla \eta), \]

for any \( \eta \in X_n, \; t \in [0, T] \). Define a map \( M[\rho] \).

\[ M[\rho] : X_n \rightarrow X_n^*, \quad \langle M[\rho]u, w \rangle \equiv \int_D \rho u \cdot w dx, \quad u, w \in X_n, \]
where $X_n^*$ is the dual space of $X_n$. Since $\rho$ has a positive lower bound, then $M[\rho]$ is invertible, and satisfies

$$\|M^{-1}[\rho]\|_{L(X_n^*, X_n)} \leq \frac{1}{\text{inf}_D \rho},$$

$$M^{-1}[\rho^1] - M^{-1}[\rho^2] = M^{-1}[\rho^2] (M[\rho^1] - M[\rho^2]) M^{-1}[\rho^1].$$

Obviously from above, we have

$$\|M^{-1}[\rho^1] - M^{-1}[\rho^2]\|_{L(X_n^*, X_n)} \leq C(n, \rho^1, \rho^2)\|\rho^1 - \rho^2\|_{L^\infty(D)}.$$  

The equation (2.10) can be rewritten as

$$u(t) = M^{-1}[\rho(t)] \left( q_0^* + \int_0^t N(S(u)(s), u(s), R(u)(s))ds \right).$$  

(2.11)

Here $q_0^* \in X_n^*$, $N \in X_n^*$ satisfy

$$\langle q_0^*, \psi \rangle = \int_D p_0 \cdot \psi,$$

$$\langle N(\rho(s), u(s), d(s)), \psi \rangle$$

$$= \int_0^t \int_D \left( p + \delta \rho \beta^2 - \frac{1}{2} (|\nabla d|^2 + F(d)) \right) \text{div}\psi$$

$$+ \int_0^t \int_D [-\epsilon (\nabla u \nabla \rho) \psi + (\rho u \otimes u + \nabla d \otimes \nabla d - \tilde{\sigma}) : \nabla \psi].$$

Next we deduce the energy estimates of (2.1)-(2.3) in finite dimensional space $X_n$. Let $(S(u), u, R(u))$ be a solution of (2.1)-(2.3) for $u \in X_n$. We rewrite (2.10) as follows:

$$\int_D (\rho u) \epsilon \cdot \eta = \int_0^t \int_D \left\{ \left( p + \delta \rho \beta^2 - \frac{1}{2} (|\nabla d|^2 + F(d)) \right) \text{div}\eta - \epsilon (\nabla u \nabla \rho) \eta \right\}$$

$$+ \int_0^t \int_D (\rho u \otimes u + \nabla d \otimes \nabla d - \tilde{\sigma}) : \nabla \eta.$$  

(2.12)

Let $\eta = u$ at time $t$. We immediately get

$$\frac{d}{dt} \int_D \frac{1}{2} \rho |u|^2 = \int_D \left[ (p + \delta \rho \beta^2) \text{div} u + f(d)((u \cdot \nabla) d) - \Delta d((u \cdot \nabla) d) - \tilde{\sigma} : \nabla u \right].$$  

(2.13)

Equation (2.1) together with (2.13) yields

$$\frac{d}{dt} \int_D \left[ \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} p + \frac{\delta}{\beta - 1} \rho \beta \right]$$

$$= \int_D [f(d)((u \cdot \nabla) d) - \Delta d((u \cdot \nabla) d) - \tilde{\sigma} : \nabla u]$$

$$+ \int_D (a \gamma \epsilon \Delta \rho \rho \gamma^{-1} + \delta \beta \epsilon \Delta \rho \beta^{-1}).$$
Integrating above equation in time to obtain

\[
\int_D \left[ \frac{1}{2} \rho^2 |u|^2 + \frac{1}{2} |\nabla d|^2 + \frac{1}{\gamma-1} \rho + \frac{\delta}{\beta-1} \rho^\beta + F(d) \right](\tau)
\]

\[
+ \int_0^\tau \int_D \left[ \mu_4 |A|^2 - \lambda_1 |N|^2 + \mu_T (\text{div} u)^2 \right]
\]

\[
+ \int_0^\tau \int_D \epsilon \left( a \gamma |\nabla \rho|^2 + \delta \beta |\nabla \rho|^2 \right)
\]

\[
\leq E_{0,\delta},
\]

where \( E_{0,\delta} \) is

\[
E_{0,\delta} = \int_D \left[ \frac{1}{2} \rho_0^2 + \frac{1}{2} |\nabla d_{0,\delta}|^2 + \frac{1}{\gamma-1} \rho_{0,\delta} + \frac{\delta}{\beta-1} \rho_{0,\delta}^\beta + F(d_{0,\delta}) \right].
\]

From the energy law and (2.14), using the standard method in [28], the integral equation (2.10) can be solve in any interval \([0, T]\).

By energy law again, we have the following lemma describing the information of the approximate solution \((\rho_n, u_n, d_n)\).

**Proposition 2.1.** For any fixed \( n \), and \( T < \infty \), there exists \((\rho, u, d)\) to solve problem (2.1)-(2.9). And we have

\[
\| \sqrt{n} u_n \|_{L^\infty(0,T;L^2(D))} \leq E_{0,\delta}, \quad \| \rho_n \|_{L^\infty(0,T;L^2(D))} \leq E_{0,\delta}, \tag{2.15}
\]

\[
\delta \| \rho_n \|_{L^\infty(0,T;L^2(D))} \leq E_{0,\delta}, \quad \| u_n \|_{L^2(0,T;W^{1,2}(D))} \leq E_{0,\delta}, \tag{2.16}
\]

\[
\| N_n \|_{L^2((0,T) \times D)} \leq E_{0,\delta}, \quad \| d_n \|_{L^2(0,T;W^{1,2}(D))} \leq E_{0,\delta}, \tag{2.17}
\]

\[
\| d_{nn} \|_{L^2(0,T;L^2(D))} \leq C(E_{0,\delta}), \quad \| d_n \|_{L^2(0,T;W^{2,3}(D))} \leq E_{0,\delta}, \tag{2.18}
\]

\[
\epsilon \int_0^T \int_D |\nabla \rho_n|^2 \leq C(T, E_{0,\delta}), \quad \| \rho_n \|_{L^2((0,T) \times D)} \leq C(T, \epsilon, E_{0,\delta}). \tag{2.19}
\]

**Proof.** (2.15)-(2.17) can be directly obtained from (2.14). So we only need to consider (2.18), (2.19). The first term of (2.18) is due to H"{o}lder inequality. By elliptic estimates, we have

\[
\| \nabla d_n \|_{L^2((0,T) \times D)} \leq C(\| \Delta d_n \|_{L^2((0,T) \times D)} + \| d_n \|_{L^2((0,T) \times D)} + \| \nabla d_{0,\delta} \|_{L^2((0,T) \times D)}) \leq \lambda_1 \| N_n \|_{L^2((0,T) \times D)} + \| f(d_n) \|_{L^2((0,T) \times D)} + C \leq C(E_{0,\delta}, |D|). \tag{2.20}
\]

Using equation (2.1), we get

\[
\frac{d}{dt} \int_D \rho_n^2 + 2 \epsilon \int_D |\nabla \rho_n|^2 = \int_D |\rho_n|^2 \text{div} u_n.
\]

and then integrate to obtain

\[
\int_D |\rho_n|^2 + \epsilon \int_0^T \int_D |\nabla \rho_n|^2 \leq \int_D |\rho_{0,\delta}|^2 + \int_0^T \int_D |\rho_n|^2 |\nabla u_n|
\]

\[
\leq \int_D |\rho_{0,\delta}|^2 + \| \rho_n \|^2_{L^\infty(0,T;L^1(D))} \int_0^T \int_D |\nabla u_n|^2
\]

\[
\leq C(T, E_{0,\delta}), \quad \text{if } \beta \geq 4. \tag{2.21}
\]
By inequality (2.14) and $\epsilon\delta\beta\rho_n^\beta - 2|\nabla \rho|^2 = \epsilon\delta|\nabla (\rho_n^\beta)|^2$, we get
$$
\|\nabla \epsilon\delta\beta\rho_n^\beta\|_{L^2(0,T;W^{1,2}(D))} \leq E_0. \delta.
$$

By Sobolev’s inequality, we have
$$
\rho_n^\beta \in L^1(0, T; L^3(D)), \quad N = 3,
$$
$$
\rho_n^\beta \in L^\infty(0, T; L^1(D)).
$$
(2.22) is equivalent to
$$
\rho_n \in L^\beta(0, T; L^3\beta(D)).
$$
Then
$$
\|\rho_n\|_{T^\beta L^\infty(0, T; L^{3\beta}(D))} \leq T^\beta \|\rho_n\|_{L^\infty(0, T; L^3(D))}^\beta \|\rho_n\|_{L^3(0, T; L^{3\beta}(D))}.
$$
(2.25)

Here we use the interpolation of (2.23) and (2.24).

2.2. The process of $n \to \infty$. In this section, we let $n \to \infty$ in the sequence \{(\rho_n, u_n, d_n)\}. The following compactness theorem due to J. Lions (see [19] and [25]).

**Lemma 2.3.** Let $X_0, X, X_1$ be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, $X_0 \hookrightarrow X$ is compact, and $X_0, X_1$ are reflexive. And define $Y = \{v \in L^{\alpha_0}(0, T; X_0); \frac{d}{dt} \in L^\alpha(0, T; X_1)\}$ with norm
$$
\|v\|_Y = \|v\|_{L^{\alpha_0}(0, T; X_0)} + \|v_t\|_{L^\alpha(0, T; X_1)},
$$
$1 < \alpha_i < \infty, i = 1, 2$. Then $Y \hookrightarrow L^{\alpha_0}(0, T; X)$ is compact. Moreover, if $\alpha_0 = \infty$,
$$
Y \hookrightarrow C([0, T]; X) \text{ is compact}.
$$

Using equation (2.1), we have
$$
\int_D \rho_n \varphi = \int_D (\epsilon\Delta \rho_n \varphi - d\rho_n u_n, \varphi) = \int_D (\epsilon\nabla \rho_n \cdot \nabla \varphi + \rho_n u_n \cdot \nabla \varphi),
$$
where $\varphi \in W_*^{1,p}(D), 2 < p' < \infty$. So we obtain
$$
\left|\frac{d}{dt} \rho_n\right|_{H^{-1,p}(D)} \leq \|\rho_n u_n\|_{L^p(D)}^\epsilon + \|\nabla \rho_n\|_{L^p(D)}^\epsilon
\leq \|\nabla \rho_n u_n\|_{L^2(D)} + \|\nabla \rho_n\|_{L^2(D)}, \quad p < 2,
$$
and then
$$
\int_0^T \left|\frac{d}{dt} \rho_n\right|_{H^{-1,p}(D)}^\epsilon \leq C(T, \rho_0, \delta, p, d_0).
$$

By Sobolev embedding theory, we have $H^1(D) \hookrightarrow L^k(D) \hookrightarrow W^{-1,p}(D)$ for $\frac{6}{5} \leq k < 6$, and $H^1(D) \hookrightarrow L^k(D)$ compactly. Using Lemma 2.3, we obtain
$$
\rho_n \to \rho \text{ in } L^\gamma((0, T) \times D), \quad \rho_n \to \rho \text{ in } L^\beta((0, T) \times D), \quad \frac{3}{2} < \gamma, \beta \leq 6.
$$
(2.26)

Moreover, we have
$$
\rho_n^\beta \to \rho^\beta \text{ in } L^1((0, T) \times D), \quad \rho_n^\gamma \to \rho^\gamma \text{ in } L^1((0, T) \times D).
$$
Similarly to ρ, we have
\[ d_n \to d \quad \text{in} \quad L^2(0, T; W^{1,p}(D)), \quad d_n \to d \quad \text{in} \quad C([0, T]; L^p(D)), \quad 1 \leq p < 6. \quad (2.27) \]
Then using \(|d|^2 \leq 1\), we obtain
\[ d_n \to d \quad \text{in} \quad L^{k'}((0, T) \times D) \quad 1 < k' < \infty. \]
It is easy to get
\[ \rho_n u_n \to^* \rho u \quad \text{in} \quad L^\infty(0, T; L^{\frac{2n}{n+1}}(D)). \quad (2.28) \]
As \(d_n\) has good regularities, it is easy to deduce \(\hat{s}_n \to \hat{s}\) in \(D'((0, T) \times D)\) as well as the weak convergence of other terms except for the following two terms
\[ \rho_n u_n \otimes u_n \to \rho u \otimes u, \quad \nabla u_n \nabla \rho_n \to \nabla u \nabla \rho \quad \text{in} \quad D'((0, T) \times D). \]

In order to continue, we need the following lemma (see [6] Lemma 7.7.5).

Lemma 2.4. Suppose \(\rho_n\) is a solution of (2.1) supplement with the boundary conditions (2.7) corresponding to \(u_n\). Then there exist \(r > 1, q > 2\), such that \(\partial_t \rho_n, A_p \rho_n\) are bounded in \(L^r((0, T) \times D)\), \(\nabla \rho_n\) is bounded in \(L^q(0, T; L^2(D))\) independently of \(n\). Accordingly, the limit function \(\rho\) belongs to the same class and satisfies equation (2.1) a.e. on \((0, T) \times D\) together with the boundary conditions (2.7) in the sense of trace.

Since \(\rho_n u_n\) satisfies (2.12), the relation (2.28) can be strengthened as the following term
\[ \rho_n u_n \to \rho u \quad \text{in} \quad L^\infty(0, T; L^{\frac{2n}{n+1}}(D)). \quad (2.29) \]
Observing equation (2.2), we have
\[ \partial_t(\rho_n u_n) \in L^2(0, T; H^{-s}(D)), \quad s \geq \frac{5}{2}. \quad (2.30) \]
Using Lemma 2.3, the above estimates are enough to show
\[ \rho_n u_n \to \rho u \quad \text{in} \quad C([0, T]; W^{-1,2}(D)), \]
and then
\[ \rho_n u_n \otimes u_n \to \rho u \otimes u \quad \text{in} \quad D'((0, T) \times D). \]

Next we consider \(\nabla u \nabla \rho\). By virtue of Lemma 2.4, we have
\[ (\rho - \rho_n)_t - \epsilon \Delta (\rho - \rho_n) = \text{div}(\rho - \rho_n) u + \rho_n (u_n - u), \quad (2.31) \]

Multiplying above equation with \(\rho - \rho_n\), we obtain
\[
\int_D |\rho - \rho_n|^2 + 2\epsilon \int_0^T \int_D |\nabla (\rho - \rho_n)|^2 \\
= \int_0^T \int_D [|\rho - \rho_n|^2 \text{div} u + (\rho - \rho_n) \nabla (\rho - \rho_n) \cdot u] \\
+ \int_0^T \int_D [\rho_n (\rho - \rho_n) \text{div} (u_n - u) + (\rho - \rho_n) \nabla \rho_n \cdot (u_n - u)].
\]
By choosing a suitable \(\beta\) in equation (2.26), the above equation yields
\[ \nabla \rho_n \to \nabla \rho \quad \text{in} \quad L^2((0, T) \times D), \]
\[ \rho_n \to \rho \quad \text{in} \quad L^\infty(0, T; L^2(D)). \]
Proposition 2.2. Suppose $\beta > \max\{\alpha, \gamma\}$, $\Omega \subset \mathbb{R}^3$ is a bounded domain in $C^{2+\alpha}$. Let $\rho_{0, \delta}$, $q_{0, \delta}$, $d_{0, \delta}$ satisfy (2.4)-\(2.9\). Then there exists a weak solution $(\rho, u, d)$ to the problem (2.1)-(2.3), such that
\[
\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(D))} \leq C(E_{0,\delta}), \quad \|\rho\|_{L^\infty(0,T;L^\gamma(D))} \leq C(E_{0,\delta}),
\]
\[
\|d\|_{L^\infty(0,T;H^2(D))} \leq C(E_{0,\delta}), \quad \|u\|_{L^2(0,T;H^1_0(D))} \leq C(E_{0,\delta}),
\]
\[
\|\nabla \rho\|_{L^2((0,T) \times D)} \leq C(E_{0,\delta}), \quad \|\rho\|_{L^{4\alpha}((0,T) \times D)} \leq C(E_{0,\delta}).
\]
and Lemma 2.4 holds. Moreover, we have the following energy law:
\[
\frac{d}{dt} \int_D \left[ \frac{1}{2} |u|^2 + \frac{1}{2} |d|^2 + \frac{1}{\gamma - 1} \rho + \frac{\delta}{\beta - 1} \rho^2 + F(d) \right] = - \int_D \left[ \mu_4 |A|^2 - \lambda_1 |N|^2 + \mu_7 \div u |u|^2 + \epsilon a_\gamma \rho^\gamma - 2 + \delta \beta \rho^2 - 2 |\nabla \rho|^2 \right]. \quad (2.32)
\]

3. Taking limit $\epsilon \to 0$. We introduce an operator $B = [B_1, B_2, B_3]$ corresponding in a certain sense to the inverse of $\div_x$.

\[
\div_x v = g - \frac{1}{|D|} \int_D g \, dx \quad \text{on } D, \quad v|_{\partial D} = 0. \quad (3.1)
\]

It can be shown (see [21] or [15] Theorem 10.3.3) that (3.1) admits an operator $B : g \mapsto v$ enjoying the following properties:

- $B$ is a bounded linear operator from $L^p(D)$ into $W^{1,p}_0(D)$ for any $1 < p < \infty$,
- the function $v = B[g]$ solves the problem (3.1),
- if the function $g \in L^p(D)$ can be written in the form $g = \div_x h$ where $h \in L^r(D)$, $h \cdot n = 0$ on $\partial D$, then
  \[
  \|B[g]\|_{L^r(D)} \leq c(p,r)\|h\|_{L^r(D)}.
  \]

Let $\psi(t) \in C_c^\infty(0,T)$, $0 \leq \psi \leq 1$, $m_\delta = \frac{1}{|D|} \int_D \rho \, dx$. Taking $\varphi = \psi(t)B[\rho_\epsilon - m_\delta]$ as a test function of (2.2), a direct calculation yields
\[
\begin{align*}
&\int_0^T \int_D \left\{ -\rho_\epsilon u_\epsilon \psi B[\rho_\epsilon - m_\delta] - \rho_\epsilon u_\epsilon \psi B[\epsilon \Delta \rho_\epsilon] + \rho_\epsilon u_\epsilon \psi B[\div(\rho_\epsilon u_\epsilon)] \right\} \\
&\quad + \int_0^T \int_D \left\{ -(\rho_\epsilon + \epsilon \rho_\epsilon^2) \div B[\rho_\epsilon - m_\delta] - \psi \rho_\epsilon u_\epsilon \otimes u_\epsilon : \nabla \right\} B[\rho_\epsilon - m_\delta] \\
&\quad + \int_0^T \int_D \left\{ \frac{1}{2} (|\nabla d_\epsilon|^2 + F(d_\epsilon)) \div B[\rho_\epsilon - m_\delta] - (\nabla d_\epsilon \otimes \nabla d_\epsilon) : \nabla B[\rho_\epsilon - m_\delta] \right\} \\
&\quad + \int_0^T \int_D \{ \tilde{\sigma}_\epsilon : \nabla B[\rho_\epsilon - m_\delta] + \epsilon \psi (\nabla u_\epsilon \nabla \rho_\epsilon) B[\rho_\epsilon - m_\delta] \} = 0.
\end{align*}
\]
Then we have

\[ \int_0^T \int_D \psi (a \rho^2 + \delta \rho^3) \]

\[ = \int_0^T \int_D \left\{ \psi m_0 (a \rho^2 + \delta \rho^3) - \psi \rho \rho_{\text{e}} B [\rho_{\text{e}} - m_0] - \psi \rho \rho_{\text{e}} B [\rho_{\text{e}} - m_0] \right\} \]

\[ + \int_0^T \int_D \left\{ \psi \rho \rho_{\text{e}} B [\rho_{\text{e}} - m_0] \right\} \]

\[ + \int_0^T \int_D \left\{ \frac{1}{2} \left( |\nabla \rho_{\text{e}}|^2 + F(d_{\text{e}}) (\rho_{\text{e}} - m_0) - \psi (\nabla d_{\text{e}} \otimes \nabla d_{\text{e}}) : \nabla B [\rho_{\text{e}} - m_0] \right) \right\} \]

\[ + \int_0^T \int_D \left\{ \nabla B [\rho_{\text{e}} - m_0] + \epsilon \psi (\nabla u_{\text{e}} \nabla \rho_{\text{e}}) B [\rho_{\text{e}} - m_0] \right\} \]

\[ = \sum_{i=1}^9 I_i. \]

We estimate each \( I_i \) as follows:

\[ I_1 = \int_0^T \int_D \psi m_0 (a \rho^2 + \delta \rho^3) \leq C(E_{0, \delta}), \]

\[ I_2 = \int_0^T \int_D -\psi \rho_{\text{e}} \rho_{\text{e}} B [\rho_{\text{e}} - m_0] \]

\[ \leq \int_0^T |\psi| \left\| \sqrt{\rho_{\text{e}}} \rho_{\text{e}} \right\|_{L^2(D)} \left\| \sqrt{\rho_{\text{e}}} \right\|_{L^2(D)} \left\| B \right\|_{L^\infty(D)} dt \]

\[ \leq C(E_{0, \delta}) \int_0^T |\psi_t| dt, \]

\[ I_3 = \int_0^T \int_D \psi \rho \rho_{\text{e}} B [\rho_{\text{e}} - m_0] \leq \sqrt{\epsilon} \int_0^T \left\| \psi \sqrt{\rho_{\text{e}}} \right\|_{L^2(D)} \left\| \rho_{\text{e}} \right\|_{L^3(D)} \left\| u_{\text{e}} \right\|_{L^6(D)} dt \]

\[ \leq C(E_{0, \delta}), \]

\[ I_4 = \int_0^T \int_D \psi \rho \rho_{\text{e}} B [\rho_{\text{e}} - m_0] \leq \int_0^T \left\| \psi \rho_{\text{e}} \right\|_{L^2(D)} \left\| \rho_{\text{e}} \rho_{\text{e}} \right\|_{L^2(D)} dt \]

\[ \leq C(E_{0, \delta}), \]

\[ I_5 = \int_0^T \int_D \psi \rho_{\text{e}} \rho_{\text{e}} \otimes u_{\text{e}} : \nabla B [\rho_{\text{e}} - m_0] \]

\[ \leq \int_0^T \left\| \psi \rho_{\text{e}} \right\|_{L^2(D)} \left\| \rho_{\text{e}} \rho_{\text{e}} \right\|^2_{L^6(D)} \left( \left\| \rho_{\text{e}} \right\|_{L^3(D)} + C \right) dt \]

\[ \leq C(E_{0, \delta}), \]

\[ I_6 = \int_0^T \int_D \frac{1}{2} \psi (|\nabla d_{\text{e}}|^2 + F(d_{\text{e}})) (\rho_{\text{e}} - m_0) \leq C(E_{0, \delta}), \]

\[ I_7 = \int_0^T \int_D \psi (\nabla d_{\text{e}} \otimes \nabla d_{\text{e}}) : \nabla B [\rho_{\text{e}} - m_0] \leq C \int_0^T \int_D |\rho_{\text{e}} - m_0| |\nabla d_{\text{e}}|^2 \]

\[ \leq C(E_{0, \delta}), \]

\[ \leq C(E_{0, \delta}). \]
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\[ I_8 = \int_0^T \int_D \psi \epsilon : \nabla B [\rho_\epsilon - m_0] \]
\[ \leq C \int_0^T \int_D (|d_\epsilon| N_\epsilon \| \rho_\epsilon - m_0 \| + |\nabla u_\epsilon| \| \rho_\epsilon - m_0 \|) \]
\[ \leq C \int_0^T \left( \| d_\epsilon \|_{L^3(D)} \| N_\epsilon \|_{L^2(D)} \| \rho_\epsilon - m_0 \|_{L^6(D)} + \| \nabla u_\epsilon \|_{L^2(D)} \| \rho_\epsilon - m_0 \|_{L^2(D)} \right) dt \]
\[ \leq C(E_0, \delta) \]

and

\[ I_9 = \epsilon \int_0^T \int_D \psi (\nabla u_\epsilon \nabla \rho_\epsilon) B [\rho_\epsilon - m_0] \]
\[ \leq \sqrt{\epsilon} \int_0^T \sqrt{\epsilon} \| \nabla \rho_\epsilon \|_{L^2(D)} \| \nabla u_\epsilon \|_{L^2(D)} \| B [\rho_\epsilon - m_0] \|_{L^\infty(D)} dt \]
\[ \leq C(E_0, \delta). \]

Summing up above estimates, we have the following lemma:

**Lemma 3.1.** Let \( \rho_\epsilon, u_\epsilon, d_\epsilon \) be the solution to problem (2.1)-(2.9). Then there exists a constant \( C = C(E_0, \delta) \) which is independent of \( \epsilon \), such that

\[ \| \rho_\epsilon \|_{L^{\gamma+1}((0,T) \times D)} + \| \rho_\epsilon \|_{L^{\gamma+1}((0,T) \times D)} \leq C(E_0, \delta). \] (3.2)

Due to the Proposition 2.2, we have

\[ \int_0^T \int_D \epsilon \varphi \Delta \rho_\epsilon = \int_0^T \int_D -\epsilon \nabla \varphi \nabla \rho_\epsilon \]
\[ \leq \sqrt{\epsilon} (\sqrt{\epsilon} \| \nabla \rho_\epsilon \|_{L^2((0,T) \times D)} \| \nabla \varphi \|_{L^2((0,T) \times D)} \rightarrow 0, \text{ as } \epsilon \rightarrow 0. \] (3.3)

The relation (3.3) yields

\[ \epsilon \Delta \rho_\epsilon \rightarrow 0 \quad \text{in} \ L^2(0,T; H^{-1}(D)). \]

By virtue of Proposition 2.2 and (3.2), we get

\[ \rho_\epsilon \rightarrow \rho \quad \text{in} \ C([0,T]; L^\gamma_{\text{weak}}(D)), \]
\[ \rho_\epsilon \rightarrow \rho \quad \text{in} \ L^{\gamma+1}((0,T) \times D). \]

As we have \( u_\epsilon \rightarrow u \) in \( L^2(0,T; H^1_0(D)) \), the following term holds

\[ \rho_\epsilon u_\epsilon \rightarrow \rho u \quad \text{in} \ C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(D)). \]

Then it is natural to obtain

\[ \rho_\epsilon u_\epsilon \rightarrow \rho u \quad \text{in} \ D'((0,T) \times D). \]

Using Lemma 2.3 and

\[ d_\epsilon \rightarrow d \quad \text{in} \ L^2(0,T; H^2(D)), \]
\[ d_\epsilon \rightarrow d \quad \text{in} \ L^\infty(0,T; H^1(D)), \]

we get

\[ d_\epsilon \rightarrow d \quad \text{in} \ L^2(0,T; W^{1,p}(D)), \]
\[ d_\epsilon \rightarrow d \quad \text{in} \ C([0,T]; L^p(D)), \quad 1 \leq p < 6. \]
Thus one can prove
\[ |\nabla d_c|^2 + F(d_c) \to |\nabla d|^2 + F(d) \text{ in } L^1((0, T) \times D), \]
\[ \nabla d \otimes \nabla d \to \nabla d \otimes \nabla d \text{ in } L^1((0, T) \times D), \]
\[ d_N e \to dN \text{ in } L^1((0, T) \times D). \]
Denoting \( \mathfrak{p} : a \rho \nabla \sigma + \delta \rho \sigma \to \mathfrak{p} \) in \( L^{\frac{2}{p}+1}((0, T) \times D) \), the limit equations read
\[
\rho_t + \text{div}(\rho u) = 0 \tag{3.4}
\]
\[
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla (\mathfrak{p} - \frac{1}{2}|\nabla| - F(d)) + \nabla \cdot (\nabla d \otimes \nabla d) = \text{div} \mathfrak{\tilde{\sigma}}, \tag{3.5}
\]
\[
\lambda_1 d_t + \lambda_1 u \cdot \nabla d - \lambda_1 \Omega d + \Delta d - f(d) = 0, \tag{3.6}
\]
where \( \mathfrak{\tilde{\sigma}} = \mu_3(N \otimes d - d \otimes N) + \mu_4 A + \mu_7 tr(A)I \). The next part will contribute to prove \( \mathfrak{p} = a \rho \nabla + \delta \rho \). 

Let \( 0 \leq \phi_m \leq 1, \phi_m = 1 \text{ in } \{ x | \text{dist}(x, \partial D) \geq \frac{1}{m} \} \), \( \varphi \in C_0^\infty((0, T) \times \mathbb{R}^3) \). Then
\[
\int_0^T \int_{\mathbb{R}^3} \rho \varphi_t = \int_0^T \int_{\mathbb{R}^3} \left[ \rho (\phi_m \varphi) t + \rho (1 - \phi_m) \varphi t \right],
\]
\[
\int_0^T \int_{\mathbb{R}^3} \rho u \nabla \varphi = \int_0^T \int_{\mathbb{R}^3} \left[ \rho u \nabla (\phi_m \varphi) + \rho u (1 - \phi_m) \nabla \varphi - \rho u \nabla \phi_m \right].
\]
Passing to the limit for \( m \to \infty \), we have
\[
\int_0^T \int_{\mathbb{R}^3} \rho (1 - \phi_m) \varphi_t \to 0, \quad \int_0^T \int_{\mathbb{R}^3} \left[ \rho (1 - \phi_m) u \nabla \varphi - \rho u \varphi \nabla \phi_m \right] \to 0. \tag{3.7}
\]
So we conclude
\[
\int_0^T \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \nabla \varphi) = 0. \tag{3.8}
\]
Here we use Hardy’s inequality and Lebesgue’s theorem. Then we get

**Lemma 3.2.** Let \( \rho \in L^2((0, T) \times D), u \in L^2(0, T; H_0^1(D)) \) be a solution of (3.4) in \( D'(0, T) \times D) \). Then prolonging \( \rho, u \) to be zero on \( \mathbb{R}^3 \setminus D \), the equation holds in \( D'(0, T) \times \mathbb{R}^3 \).

We introduce *Riesz integral operator* \( \mathcal{R} \) and *singular integral operator* \( \mathcal{A} \).
\[
\mathcal{R}_i = (-\Delta^{-\frac{1}{2}}) \partial_{x_i},
\]
\[
\mathcal{A}_i = \Delta^{-\frac{1}{2}} \partial_{x_i},
\]
And it holds
\[
\partial_{x_i} \mathcal{A}_j = -\mathcal{R}_i \mathcal{R}_j, \tag{3.9}
\]
These operators have the following properties (see [6] (Lemma 5.5.2) or [2]).

**Lemma 3.3.** The Riesz operator \( \mathcal{R}_i, i = 1, ..., N, \) defined in (3.9) is a bounded linear operator on \( L^p(\mathbb{R}^3) \) for any \( 1 < p < \infty \).

**Lemma 3.4.** Let \( v \in (L^1 \cap L^2)(\mathbb{R}^3), \) then \( \mathcal{A}_i[v] \in (L^1 \oplus L^2)(\mathbb{R}^3), \) and
\[
\| \mathcal{A}_i[v] \|_{(L^1 \oplus L^2)(\mathbb{R}^3)} \leq c \| v \|_{(L^1 \cap L^2)(\mathbb{R}^3)},
\]
\[
\| \partial_{x} \mathcal{A}_i[v] \|_{L^p(\mathbb{R}^3)} \leq c(p) \| v \|_{L^p(\mathbb{R}^3)} \text{ for any } 1 < p < \infty.\]
The next lemma is from [6](Corollary 6.6.1).

**Lemma 3.5.** Let \( \{v_n\}, \{w_n\} \) be the two sequences of vector functions,

\[
\begin{align*}
  v_n &\to v \quad \text{in } L^p(D; \mathbb{R}^3), \\
  w_n &\to w \quad \text{in } L^q(D; \mathbb{R}^3), \\
  B_n &\to B \quad \text{in } L^p(D), \quad \frac{1}{p} + \frac{2}{q} < 1, \quad 1 < p, q < \infty.
\end{align*}
\]

Then the following terms satisfy in distributions:

\[
\begin{align*}
  v_n \cdot (\nabla \Delta^{-1} \text{div})[w_n] - w_n \cdot (\nabla \Delta^{-1} \text{div})[v_n] &\to v \cdot (\nabla \Delta^{-1} \text{div})[w] - w \cdot (\nabla \Delta^{-1} \text{div})[v], \\
  B_n(\nabla \Delta^{-1} \nabla)[B_n] - B_n(\nabla \Delta^{-1} \text{div})[v_n] &\to v(\nabla \Delta^{-1} \nabla)[B] - B(\nabla \Delta^{-1} \text{div})[v].
\end{align*}
\]

**Lemma 3.6.** Let \( \eta \in C_0^\infty(D), \psi \in C_0^\infty(0,T) \) and \( B \) is a bounded measurable function satisfying

\[
\partial_t B + \text{div}(Bu) = h \quad \text{in } D'(0,T \times D), \quad \text{with } h \in L^2(0,T; W^{-1,q}(D)).
\]

Let \( \varphi(t,x) = \psi(t)\eta(x)A[B(t,x)] \) be a test function of (3.5), providing \( B, u \) prolonged to zero outside \( D \). Then we have

\[
\begin{align*}
  \int_0^T \int_{D^2} &\psi \eta (B - S) (\nabla \Delta^{-1} \nabla)[B] \\
  &= \int_0^T \int_D \{ \psi (SV \eta) A[B] - \psi \eta u \otimes u : (\nabla \Delta^{-1} \nabla)[B] - \psi \nabla \eta A[B] \} \\
  &+ \int_0^T \int_D \{ -\psi [(\mu u \otimes u) \nabla \eta] A[B] - \psi \eta (\nabla d \otimes \nabla d) : (\nabla \Delta^{-1} \nabla)[B] \} \\
  &+ \int_0^T \int_D \{ -\psi [(\nabla d \otimes \nabla d) \nabla \eta] A[B] - \frac{1}{2} (|d|^2 + F(d)) \psi \eta B \} \\
  &+ \int_0^T \int_D \{ \frac{1}{2} (|d|^2 + F(d)) \psi \nabla \eta A[B] + \psi \eta (\mu_2 N \otimes d + \mu_3 d \otimes N) : (\nabla \Delta^{-1} \nabla)[B] \} \\
  &+ \int_0^T \int_D \{ \psi (\mu_2 N \otimes d + \mu_3 d \otimes N) \nabla \eta) A[B] - \psi \eta u A[B] \} \\
  &+ \int_0^T \int_D \{ -\psi \eta u A[h] + \psi \eta u A[\text{div}(Bu)] \},
\end{align*}
\]

where \( S = \mu A + \mu \tau \text{tr}(A) I \).

**Proof.** Form the definition of \( A \), we have

\[
\begin{align*}
  \text{div} \varphi &= \psi \nabla \eta A[B] + \psi \eta B, \\
  \nabla \varphi &= \psi \nabla \eta A[B] + \psi \eta (\nabla \Delta^{-1} \nabla)[B], \\
  \varphi_t &= \psi \eta A[B] + \psi \eta A[h] - \psi \eta A[\text{div}(Bu)].
\end{align*}
\]

Taking above relations into (3.5), we conclude this lemma. \( \square \)

**Lemma 3.7.** Let \( \rho, u, d, \) be a solution of (2.1)-(2.9). And \( (\rho, u, d) \) solves (3.4)-(3.6). Then

\[
\begin{align*}
  \int_0^T \int_D &\psi \phi \left( a \rho^2 + \delta \rho^2 - (\mu_4 + \mu_7) \text{div} u_\epsilon \right) \rho \to \int_0^T \int_D \psi \phi \left( \rho - (\mu_4 + \mu_7) \text{div} u \right) \rho \quad \text{as } \epsilon \to 0
\end{align*}
\]

for any \( \psi \in C_0^\infty(0,T), \phi \in C_0^\infty(D) \).
Before proving the lemma, we prolong $\rho_e$ to be zero outside $D$

$$
\partial_t \rho_e = \begin{cases} 
\epsilon \Delta \rho_e - \text{div}(\rho_e u_e) & x \in D \\
0 & x \in \mathbb{R}^3 \setminus D.
\end{cases}
$$

(3.10)

Since $\rho, u$ vanish outside $D$, we have $\text{div}(\rho u) = 0$, $x \in \mathbb{R}^3 \setminus D$ and

$$
\text{div}(1_D \nabla \rho_e) = \begin{cases} 
\Delta \rho_e & x \in D, \\
0 & x \in \mathbb{R}^3 \setminus D.
\end{cases}
$$

(3.11)

**Proof.** Here $\rho_e, \rho$ are extended to zero outside of $D$. Taking $\varphi = \psi \eta A[\rho_e]$ as a test function and using Lemma 3.6, we get

$$
\int_0^T \int_D \left[ \psi \eta \left( (\alpha \rho_e^2 + \delta \rho_e^3) \rho_e - (\mu_4 A_e + \mu_7 tr(A_e) I) : (\nabla \Delta^{-1} \nabla) [\rho_e] \right) \right] dV dt
$$

$$
= \int_0^T \int_D \left\{ \psi (S \nabla \eta) A[\rho_e] - \psi \eta u_e \otimes u_e : (\nabla \Delta^{-1} \nabla) [\rho_e] - \psi (a \rho_e^2 + \delta \rho_e^3) \nabla \eta A[\rho_e] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ -\psi ((\rho_e u_e \otimes u_e) \nabla \eta) A[\rho_e] - \psi \eta (\nabla d_e \otimes \nabla d_e) : (\nabla \Delta^{-1} \nabla) [\rho_e] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ -\psi ((\rho_e u_e \otimes u_e) \nabla \eta) A[\rho_e] + \frac{1}{2} (|\nabla d_e|^2 + F(d_e)) \psi \eta \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ \frac{1}{2} (|\nabla d_e|^2 + F(d_e)) \psi \nabla \eta A[\rho_e] + \psi \eta (\mu_4 A_e + \mu_7 tr(A_e) I) : (\nabla \Delta^{-1} \nabla) [\rho_e] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ -\psi \eta u_e A[\text{div}(1_D \nabla \rho_e)] + \epsilon \psi (\nabla u_e \nabla \rho_e) A[\rho_e] \right\} dV dt
$$

$$
= \sum_{i=1}^{14} I_i.
$$

Similarly, taking $\varphi = \psi \eta A[\rho]$ as a test function of (3.5), we have

$$
\int_0^T \int_D \psi \eta \left\{ \beta \rho - (\mu_4 A + \mu_7 tr(A) I) : (\nabla \Delta^{-1} \nabla) [\rho] \right\} dV dt
$$

$$
= \int_0^T \int_D \left\{ \psi (S \nabla \eta) A[\rho] - \psi \eta u \otimes u : (\nabla \Delta^{-1} \nabla) [\rho] - \psi \eta \nabla \eta A[\rho] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ -\psi ((\rho u \otimes u) \nabla \eta) A[\rho] - \psi \eta (\nabla d \otimes \nabla d) : (\nabla \Delta^{-1} \nabla) [\rho] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ -\psi ((\rho u \otimes u) \nabla \eta) A[\rho] + \frac{1}{2} (|\nabla d|^2 + F(d)) \psi \eta \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ \frac{1}{2} (|\nabla d|^2 + F(d)) \psi \nabla \eta A[\rho] + \psi \eta (\mu_4 A + \mu_7 tr(A) I) : (\nabla \Delta^{-1} \nabla) [\rho] \right\} dV dt
$$

$$
+ \int_0^T \int_D \left\{ \psi (\mu_4 A + \mu_7 tr(A) I) \nabla \eta A[\rho] - \psi \eta \nu A[\rho] + \psi \eta \nu A[\text{div}(\rho u)] \right\} dV dt
$$

$$
= \sum_{i=1}^{12} I_i.
$$
By Proposition 2.2, we have

\[ I_{13} = \int_0^T \int_D -\psi \eta \rho \epsilon u \epsilon A[\epsilon \text{div}(1_D \nabla \rho)] \]
\[ \leq \sqrt{\epsilon} \int_0^T \psi \sqrt{\epsilon} \| \nabla \rho \|_{L^2(D)} \| \rho \|_{L^2(D)} \| u \|_{L^6(D)} \, dt \to 0 \quad \text{as} \ \epsilon \to 0. \]  
\tag{3.12}

and

\[ I_{14} = \int_0^T \int_D \epsilon \psi \eta (\nabla u \nabla \rho \epsilon) A[\rho \epsilon] \]
\[ \leq \sqrt{\epsilon} \int_0^T \psi \sqrt{\epsilon} \| \nabla \rho \|_{L^2(D)} \| A[\rho \epsilon] \|_{L^\infty(D)} \| \nabla u \|_{L^2(D)} \, dt \to 0 \quad \text{as} \ \epsilon \to 0. \]  
\tag{3.13}

Next we show that it still holds \( I_i' \to I_i, \ i = 1, 2 \cdots , 12. \) Using

\[ \rho \epsilon \to \rho \quad \text{in} \ C([0, T]; L^6(D)), \]

we have

\[ A[\rho \epsilon] \to A[\rho] \quad \text{in} \ C([0, T] \times D), \]
\[ R_1 R_2 [\rho \epsilon] \to R_1 R_2 [\rho] \quad \text{in} \ C([0, T]; L^6(D)). \]

And it is easy to get \( I_1' \to I_1, \ I_3' \to I_3, \ I_4' \to I_4, \ I_10' \to I_{10}, \ I_11' \to I_{11}. \) By virtue of \( d \epsilon \to d \) in \( W^{1,k}(D), \ 0 \leq k < 6, \) we have \( I_5' \to I_5, \ I_6' \to I_6, \ I_7' \to I_7, \ I_8' \to I_8. \)

Noting \( \mu_2 = -\mu_3 \), we have \( I_5' = I_9 = 0. \) It only leaves us to consider \( I_2', \ I_12'. \)

\[ I_2' + I_{12}' = \int_0^T \int_D \left\{ \psi \eta \rho \epsilon u \epsilon A[\epsilon \text{div}(\rho \epsilon u)] - \psi \eta \rho \epsilon u \epsilon \otimes u \epsilon : (\nabla \Delta^{-1} \nabla)[\rho \epsilon] \right\} \]
\[ = \int_0^T \int_D \left\{ \psi u \epsilon (\rho \epsilon \Delta^{-1} \text{div}(\eta \rho \epsilon u) - \eta \rho \epsilon (\nabla \Delta^{-1} \nabla)[\rho \epsilon] u \epsilon \right\} \]
\[ \to \int_0^T \int_D \left\{ \psi u (\rho \nabla \Delta^{-1} \text{div}(\eta \rho u) - \eta \rho (\nabla \Delta^{-1} \nabla)[\rho] u \right\} \]
\[ = \int_0^T \int_D \left\{ \psi \eta \rho u A[\epsilon \text{div}(\rho u)] - \psi \eta \rho \epsilon \otimes u : (\nabla \Delta^{-1} \nabla)[\rho] \right\} \]
\[ = I_2 + I_{12}. \]

Here we used Lemma 3.5. It has been deduced that

\[ \int_0^T \int_D \psi \eta \left( (\alpha \rho^2 \epsilon + \delta \rho^3 \epsilon) \rho \epsilon - (\mu_4 A + \mu_7 \text{tr}(A) I) : (\nabla \Delta^{-1} \nabla)[\rho \epsilon] \right) \]
\[ \to \int_0^T \int_D \psi \phi \left( (\rho \epsilon \cdot \nabla \eta - (\mu_4 A + \mu_7 \text{tr}(A) I) : (\nabla \Delta^{-1} \nabla)[\rho] \right). \]  
\tag{3.14}

Straightforward computation yields

\[ \int_0^T \int_D \psi \eta (\mu_4 A + \mu_7 \text{tr}(A) I) : (\nabla \Delta^{-1} \nabla)[\rho \epsilon] \]
\[ = \int_0^T \int_D \left\{ \psi \rho \epsilon (\mu_4 + \mu_7) \text{div}(\eta u \epsilon) - \psi \rho \epsilon \left[ \mu_4 (\nabla \Delta^{-1} \nabla) : (u \epsilon \otimes \nabla \eta) + \mu_7 \rho \epsilon u \epsilon \cdot \nabla \eta \psi \right] \right\} \]
\tag{3.15}
Lemma 3.8. If we rewrite the proof.

Proof. Taking the mollified operator on both sides of \((3.14)\)

\[
\int_0^T \int_D \psi \eta (\mu_4 A + \mu_7 tr(A)I) \cdot (\nabla \Delta^{-1} \nabla) [\rho] \\
= \int_0^T \int_D \left\{ \psi \rho(\mu_4 + \mu_7) \text{div} (\eta u) - \psi \rho \left[ \mu_4 (\nabla \Delta^{-1} \nabla) : (u \otimes \nabla \eta) + \mu_7 \rho u \cdot \nabla \psi \eta \right] \right\},
\]

(3.16)

Also we have

\[
\int_0^T \int_D \psi \rho \left[ \mu_4 (\nabla \Delta^{-1} \nabla) : (u \otimes \nabla \eta) + \mu_7 \rho u \cdot \nabla \eta \right] \\
\rightarrow \int_0^T \int_D \psi \rho \left[ \mu_4 (\nabla \Delta^{-1} \nabla) : (u \otimes \nabla \eta) + \mu_7 \rho u \cdot \nabla \psi \eta \right].
\]

(3.17)

(3.14)-(3.17) lead to our conclusion. \( \square \)

We need the following lemma about renormalized solution (see [6]). For the sake of completeness, we rewrite the proof.

**Lemma 3.8.** Assume \( \rho \in L^1((0,T) \times D), \ u \in L^2(0,T; H_0^1(D)) \) solves \((3.4)\) in the sense of \(D'(0,T) \times \mathbb{R}^3)\). Then \((\rho, u)\) is a renormalized solution of \((3.4)\).

**Proof.** Taking the mollified operator on both sides of \((3.4)\), we have

\[
\frac{\partial}{\partial t} [\rho]_\varepsilon + \nabla [\rho]_\varepsilon u + [\rho]_\varepsilon \text{div} u = [\text{div}(\rho u)]_\varepsilon^\ast - \text{div}([\rho]_\varepsilon^\ast u).
\]

(3.18)

Then we get

\[
\int_0^T \int_D |[\text{div}(\rho u)]_\varepsilon^\ast - \text{div}([\rho]_\varepsilon^\ast u)| \\
\leq \int_0^T \int_D \left\{ \left| \int_{\mathbb{R}^N} \rho(y)(u(x) - u(y)) \nabla \theta^\varepsilon(|x - y|) dy \right| + |[\rho]_\varepsilon^\ast \text{div} u| \right\}.
\]

(3.19)

Noticing

\[
\int_0^T \int_D \int_{\mathbb{R}^N} \rho(y)(u(x) - u(y)) \nabla \theta^\varepsilon(|x - y|) dy \\
= \int_0^T \int_D \int_{\mathbb{R}^N} \rho(x - z) \frac{u(x) - u(x - z)}{|z|} \nabla \theta^\varepsilon(|z|)|z| dz.
\]

(3.20)

and the second term of the right side of \((3.19)\) is bounded, we have

\[
|[\text{div}(\rho u)]_\varepsilon^\ast - \text{div}([\rho]_\varepsilon^\ast u)| \rightarrow 0 \text{ in } L^1((0,T) \times D), \text{ as } \varepsilon \rightarrow 0.
\]

by Lebesgue’s theorem. Multiplying \(B'(\rho)\) on \((3.18)\), the equation reads

\[
\frac{\partial}{\partial t} (B[\rho]_\varepsilon^\ast) + \nabla (B[\rho]_\varepsilon^\ast u) + B([\rho]_\varepsilon^\ast)(\text{div} u) + B'([\rho]_\varepsilon^\ast)\rho u - B([\rho]_\varepsilon^\ast)\text{div}u = B'(\rho) \frac{\partial}{\partial x} s^\ast.
\]

Passing to the limit for \(\varepsilon \rightarrow 0\), we conclude

\[
(B(\rho))_t + \text{div}(B(\rho) u) + (B'(\rho) \rho - B(\rho))\text{div}u = 0.
\]

(3.21)

Taking \(B(z) = z \log z\) in \((3.21)\), we obtain

\[
\int_0^T \int_D \rho \text{div} u = \int_D \rho_0 \log \rho_0 dx - \int_D \rho(T) \log \rho(T) dx.
\]
Similar to lemma 3.8, it holds
\[ B_t(\rho_e) + \text{div}(B(\rho_e)u_e) + (B'(\rho_e)\rho_e - B(\rho_e))\text{div}u_e - \epsilon \Delta B(\rho_e) = -B''(\rho_e)|\nabla \rho_e|^2 \leq 0. \]

The term \( \nabla \rho_e \cdot n = 0 \) leads to \( \int_D \Delta (B(\rho_e)) \, dx = 0 \). Then we have
\[
\int_0^T \int_D \rho_e \text{div}u_e \leq \int_D \rho_0 \log \rho_0 \, dx - \int_D \rho_e(T) \log \rho_e(T) \, dx.
\]

Let \( \psi_m \in C_0^\infty(0, T) \), \( \eta_m \in C_0^\infty(D) \), satisfying \( \psi_m \to 1 \) and \( \eta_m \to 1 \). By virtue of Lemma 3.7, we have
\[
\lim_{\epsilon \to 0^+} \sup_{\epsilon} \int_0^T \int_D \psi_m \eta_m (a \rho_e^\gamma + \delta \rho_e^\beta) \rho_e
\]
\[
= \lim_{\epsilon \to 0^+} \sup_{\epsilon} \int_0^T \int_D \left[ \psi_m \eta_m (a \rho_e^\gamma + \delta \rho_e^\beta - (\mu_4 + \mu_7) \text{div}u_e) \rho_e + \psi_m \eta_m (\mu_4 + \mu_7) \rho_e \text{div}u_e \right]
\]
\[
\leq \int_0^T \int_D \psi_m \eta_m (\bar{\rho} - (\mu_4 + \mu_7) \text{div}u) \rho + (\mu_4 + \mu_7) \lim_{\epsilon \to 0^+} \sup_{\epsilon} \int_0^T \int_D \rho_e \text{div}u_e
\]
\[
+ (\mu_4 + \mu_7) \int_D \rho(T) \log \rho(T) \, dx - \lim \inf_{\epsilon \to 0^+} \int_D \rho_e(T) \log \rho_e(T) \, dx
\]
\[
\leq \int_D \int \psi_m \eta_m \bar{\rho} \rho + o(m^{-1}), \quad (3.22)
\]

The last inequality is due to the fact: \( B(z) \) is convex and globally lipschitz on \( R^+ \).

Thus we have proved
\[
\lim_{\epsilon \to 0^+} \sup_{\epsilon} \int_0^T \int_D \psi_m \eta_m (a \rho_e^\gamma + \delta \rho_e^\beta) \rho_e \leq \int_0^T \int_D \psi_m \eta_m \bar{\rho}, \quad m \gg 1.
\]

Setting \( p(z) = az^\gamma + \delta z^\beta \), it holds
\[
\int_0^T \int_D \psi_m \eta_m (p(\rho_e) - p(v)) (\rho_e - v)
\]
\[
= \int_0^T \int_D \psi_m \eta_m (p(\rho_e) \rho_e - p(\rho_e) v - p(v) \rho_e + p(v) v) \geq 0.
\]

Then
\[
\int_0^T \int_D \psi_m \eta_m \bar{\rho} \rho \, dx \, dt + \int_0^T \int_D \psi_m \eta_m (\bar{\rho} v - p(v) \rho + p(v) v) \geq 0.
\]

Let \( m \to \infty \), we obtain
\[
\int_0^T \int_D (\bar{\rho} - p(v)) (\rho - v) \geq 0.
\]

Choosing \( v = \rho + \zeta \varphi \), for any \( \varphi \), then \( \zeta \to 0 \) yields
\[
\bar{\rho} = a \rho^\gamma + \delta \rho^\beta.
\]

(3.23)
We can rewrite the system (3.4)-(3.6) as
\begin{align}
\rho_t + \text{div}(\rho u) &= 0, \tag{3.24} \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla(p - \frac{1}{2} |\nabla d|^2 - F(d)) + \nabla \cdot (\nabla d \otimes \nabla d) &= \text{div} \tilde{\sigma}, \tag{3.25} \\
\lambda_1 d_t + \lambda_1 u \cdot \nabla d - \lambda_1 \Omega d + \Delta d - f(d) &= 0, \tag{3.26}
\end{align}
where \( \tilde{\sigma} = \mu_3 (N \otimes d - d \otimes N) + \mu_4 + \mu_7 \text{tr}(A)I, \ p = a \rho^\gamma + \delta \rho^\beta. \)

**Proposition 3.1.** Let \( \Omega \) be a bounded domain in class of \( C^{2+\nu} \), and \( \beta > \{4, \frac{6\nu}{2\gamma - 3}\} \). Then there exists a finite energy weak solution \( (\rho, u, d) \) to the problem (3.24)-(3.26), (2.4)-(2.9). And \( \rho, u, d \) satisfy the following estimates:
\begin{align*}
\|\sqrt{\rho}u\|_{L^\infty(0,T; L^2(D))} &\leq CE(\rho_0, q, d_0), \quad \|\rho\|_{L^\infty(0,T; L^q(D))} \leq CE(\rho_0, q, d_0), \\
\delta \|\rho\|^q_{L^\infty(0,T; L^q(D))} &\leq CE(\rho_0, q, d_0), \quad \|\nabla u\|_{L^2(0,T; H^1_0(D))} \leq CE(\rho_0, q, d_0), \\
\|d\|_{L^\infty(0,T; H^1_0(D))} &\leq CE(\rho_0, q, d_0), \quad \|d\|_{L^2(0,T; H^2_0(D))} \leq CE(\rho_0, q, d_0), \\
\|N\|_{L^2_0(0,T; \mathbb{R}^n)} &\leq CE(\rho_0, q, d_0), \quad \|d_t\|_{L^2(0,T; L^2_0(D))} \leq CE(\rho_0, q, d_0).
\end{align*}

4. **Let \( \delta \to 0 \).** We will find \( \rho_{0,\delta}, q_{0,\delta} \) and \( d_{0,\delta} \) which satisfy all the requisite of this paper (for example (2.4)-(2.9), etc). Firstly, one can find \( \rho_{0,\delta} \in C^{2+\alpha}(\overline{D}) \), satisfying
\begin{align*}
0 < \delta \leq \rho_{0,\delta} \leq \delta - \frac{1}{2} \lambda_1, \quad \nabla \rho_{0,\delta} \cdot n|_{\partial D} = 0 \quad \text{and} \quad \int_D |\rho_{0,\delta} - \rho_0|^2 dx \leq \delta^2.
\end{align*}
Indeed one can easily find \( \rho_0 \in C^\infty_0(D), \ |\rho_t - \rho_0|_{L^\gamma} < \delta, \) with \( 0 < \rho_0 < \delta^{-\frac{1}{\gamma}} \). Let \( \rho_{0,\delta} = \rho_0 + \delta, \) with \( \nabla \rho_{0,\delta} \cdot n|_{\partial D} = 0. \) Secondly, Let
\begin{align}
q_0^\delta = \left\{ \begin{array}{ll}
q(x) \sqrt{\frac{\rho_{0,\delta}}{\rho_0}}, & \rho_0 > 0, \\
0, & \rho_0 = 0.
\end{array} \right. \tag{4.1}
\end{align}
(4.1) leads to
\[
\left| q_0^\delta \right|^2_{\rho_{0,\delta}} \text{ is bounded in } L^1(D), \text{ independent of } \delta.
\]
Then we take \( \bar{q}_\delta \in C^2(\overline{D}) \) such that
\[
\| \frac{q_0^\delta}{\sqrt{\rho_{0,\delta}}} - \bar{q}_\delta \|_{L^2(D)} < \delta.
\]
Let \( q_{0,\delta} = \bar{q}_\delta \sqrt{\rho_{0,\delta}} \)
\[
\left| q_{0,\delta} \right|^2_{\rho_{0,\delta}} \text{ is bounded in } L^1(D) \text{ independently of } \delta,
\]
\[
q_{0,\delta} \to q \quad \text{in } L^{\frac{2\gamma}{\gamma + 1}}(D) \quad \text{as } \delta \to 0.
\]
Indeed
\begin{align}
\int_{\{\rho_0(x) > 0\}} \left( \left| q_{0,\delta} - \frac{q_0}{\sqrt{\rho_{0,\delta}}} dx \right|^{\frac{2\gamma}{\gamma + 1}} \right) d^{\frac{\gamma + 1}{\gamma + 3}} &\leq \left( \int_{\{\rho_0(x) > 0\}} \left| \bar{q}_\delta \sqrt{\rho_{0,\delta}} - \sqrt{\rho_0} \bar{q}_\delta \right|^{\frac{2\gamma}{\gamma + 1}} dx \right) \frac{\gamma + 1}{\gamma + 3} \\
&= \left( \int_{\{\rho_0(x) > 0\}} \left| \sqrt{\rho_{0,\delta}}(\bar{q}_\delta - q_0^\delta) \right|^{\frac{2\gamma}{\gamma + 1}} d^{\frac{\gamma + 1}{\gamma + 3}} + \int_{\{\rho_0(x) > 0\}} \left| \frac{\bar{q}_\delta}{\sqrt{\rho_{0,\delta}}} (\sqrt{\rho_0} - \sqrt{\rho_{0,\delta}}) \right|^{\frac{2\gamma}{\gamma + 1}} d^{\frac{\gamma + 1}{\gamma + 3}} \right) \\
&= I_1 + I_2. \tag{4.2}
\end{align}
It is easy to get
\[ I_1 \to 0, \quad \delta \to 0, \]
\[ I_2 < \| \frac{q_\delta}{\sqrt{\rho_0, \delta}} \|_{L^2(\| \rho_0 \|_{L^\gamma} - \| \rho_{0, \delta} \|_{L^\gamma})} \leq C \delta \to 0 \]
and
\[ \left( \int_{\{\rho_0(x) = 0\}} \left| q_{\delta, \alpha} \right|^{\frac{2}{\gamma}} \mathrm{d}x \right)^{\frac{\gamma}{2}} = \left( \int_{\{\rho_0(x) = 0\}} \left| \tilde{q}_\delta \sqrt{\rho_0, \delta} \right|^{\frac{2}{\gamma}} \mathrm{d}x \right)^{\frac{\gamma}{2}} \]
\[ \leq \| \tilde{q}_\delta \|_{L^2(D)} \| \rho_{0, \delta} - \rho_0 \|_{L^\gamma(D)} \to 0 \text{ as } \delta \to 0. \]

Thirdly, we can easily find \( \| d_{0, \delta} - d_0 \|_{H^2(D)} < \delta, \| d_{0, \delta} \| = 1. \) Let \((\rho_\delta, u\delta, d\delta)\) be the approximate solution of the problem \((3.24)-(3.26)\) with the initial data \((\rho_{0, \delta}, d_{0, \delta}, q_{0, \delta})\) of \((2.4)-(2.9)\). Here we still use \((\rho, u, d)\) instead of \((\rho_\delta, u\delta, d\delta)\) for convenience. Noting
\[ \rho \in L^\infty(0, T; L^\gamma(D)) \cap L^\infty(0, T; L^\beta(D)), \quad u \in L^2(0, T; H^1_0(D)), \]
with some \( \beta \geq 2, \) we get that \((\rho, u)\) is a renormalized solution of \((3.24)\) by Lemma \ref{lem:3.8}. Similarly as Lemma \ref{lem:3.1}, Let \( \varphi(t, x) = \psi(t)B[b(\rho) - m_0], b(\rho) = \rho^\theta, \quad m_0 = \int_{D} b(\rho), \) and put \( \theta > 0 \) determined later.

\[
\int_{0}^{T} \int_{D} \left( a\rho^{\gamma+\theta} + \delta \rho^{\beta+\theta} \right) \\
= \int_{0}^{T} \int_{D} \left\{ \psi m_0 (a\rho^{\gamma} + \delta \rho^{\beta}) - \psi_{\gamma} \rho m B[b(\rho) - m_0] + \psi_{\beta} \rho u B[\text{div}(b(\rho) u)] \right\} \\
+ \int_{0}^{T} \int_{D} \left\{ \psi \rho B[(b'(\rho) - b(\rho)) \text{div} u] - \psi \rho u \otimes u : \nabla B[b(\rho) - m_0] \right\} \\
+ \int_{0}^{T} \int_{D} \left\{ \psi \sigma : \nabla B[b(\rho) - m_0] - \psi(\nabla d \otimes \nabla d) : \nabla B[b(\rho) - m_0] \right\} \\
+ \int_{0}^{T} \int_{D} \frac{1}{2} \left( \| \nabla d \|^2 + F(d) \right) [b(\rho) - m_0] \\
= \sum_{i=1}^{8} I_i.
\]

In view of Proposition \ref{prop:3.1}, we estimate \( I_i \) as follows:

\[
I_1 = \int_{0}^{T} \int_{D} \psi m_0 \left( a\rho^{\gamma} + \delta \rho^{\beta} \right) \leq C(E_0), \quad \theta \leq \gamma;
\]
\[
I_2 = \int_{0}^{T} \int_{D} -\psi_{\gamma} \rho u B[b(\rho) - m_0] \\
\leq C(E_0) \| \sqrt{\rho} \|_{L^\infty(0, T; L^{2}(D))} \| \sqrt{\rho} u \|_{L^\infty(0, T; L^{2}(D))} \times \\
\| B[b(\rho) - m_0] \|_{L^\infty(0, T; L^{2}(D))} \int_{0}^{T} |\psi_{\gamma}| \mathrm{d}t \\
\leq C(E_0) \int_{0}^{T} |\psi_{\gamma}| \mathrm{d}t, \quad p > \frac{2\gamma}{\gamma - 1}, \quad \theta \leq \frac{5\gamma}{6} - \frac{1}{2},
\]
\[
I_3 = \int_{0}^{T} \int_{D} \psi \rho u B[\text{div}(b(\rho) u)] \leq C(E_0), \quad \theta \leq \frac{2\gamma}{3} - 1;
\]
Let $(\rho, u, d)$ be a solution to the problem \((3.24)-(3.26)\), then there exists a constant $C = C(\rho_{0,\delta}, p_{0,\delta}, d_{0,\delta})$ which is independent of $\delta$, such that
\[
I_4 = \int_0^T \int_D \psi \rho u B((b'(\rho) \rho - b(\rho)) \text{div} u) \leq C(E_0), \quad \theta \leq \frac{2\gamma}{3} - 1;
\]
\[
I_5 = \int_0^T \int_D -\psi \rho u \otimes u : \nabla B(b(\rho) - m_0) \leq \|u\|_{L^2(0,T;L^6(D))}^2 \|\rho\|_{L^\infty(0,T;L^{2+\theta}(\Omega))}^{1+\theta} \leq C(E_0), \quad \theta \leq \frac{2\gamma}{3} - 1;
\]
\[
I_6 = \int_0^T \int_D \frac{1}{2} \|\nabla d\|^2 + F(d) \, (b(\rho) - m_0) \leq C(E_0);
\]
\[
I_7 = \int_0^T \int_D -\psi (\nabla d \otimes \nabla d) : \nabla B(b(\rho) - m_0) \leq C(E_0);
\]
and
\[
I_8 = \int_0^T \int_D \psi \sigma : \nabla B(b(\rho) - m_0) \leq C(E_0), \quad \theta \leq \frac{\tilde{\gamma}}{2}.
\]
Thus we have proved the following lemma:

**Lemma 4.1.** Let $(\rho, u, d)$ be a solution to the problem \((3.24)-(3.26)\), then there exists a constant $C = C(\rho_{0,\delta}, p_{0,\delta}, d_{0,\delta})$ which is independent of $\delta$, such that
\[
\int_0^T \int_D \psi \left( a \rho^{\gamma+\theta} + \delta \rho^{2+\theta} \right) \leq C, \quad \theta \leq \min \left\{ 1, \frac{2\gamma}{3} - 1, \frac{\tilde{\gamma}}{2} \right\}. \tag{4.3}
\]

By virtue of Proposition 3.1 and (4.3), we have
\[
\rho_\delta \to \rho \quad \text{in} \ C([0,T]; L_{\text{weak}}^\gamma(D)), \quad u_\delta \to u \quad \text{in} \ L^2(0,T; H_0^1(D)),
\]
\[
\rho_{\delta} u_\delta \to \rho u \quad \text{in} \ C([0,T]; L_{\text{weak}}^{\gamma+\theta}(D)), \quad \rho_{\delta} u_\delta \otimes u_\delta \to \rho u \otimes u \quad \text{in} \ D'(0,T \times D),
\]
\[
d_\delta \to d \quad \text{in} \ L^2(0,T; H^2(D)), \quad d_\delta \to^* d \quad \text{in} \ L^\infty(0,T; H^1(D)),
\]
\[
d_{\delta} \to d \quad \text{in} \ L^2(0,T; L^2(D)), \quad \rho_{\delta}^\beta \to \rho^\beta \quad \text{in} \ L_{\text{weak}}^{\beta+\theta}((0,T) \times D),
\]
\[
\delta \rho^\beta \to 0 \quad \text{in} \ L_{\text{weak}}^{\beta+\theta}((0,T) \times D).
\]
Passing to the limit for $\delta \to 0$, $(\rho, u, d)$ satisfies
\[
\rho_t + \text{div}(\rho u) = 0, \tag{4.4}
\]
\[
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla(\tilde{\gamma} - \frac{1}{2} |\nabla d|^2 - F(d)) + \nabla \cdot (\nabla d \otimes \nabla d) = \text{div} \tilde{\sigma}, \tag{4.5}
\]
\[
\lambda_1 d_t + \lambda_1 u \cdot \nabla d - \lambda_1 \Omega d + \Delta d - f(d) = 0, \tag{4.6}
\]
where $\tilde{\sigma} = \mu_3 (\nabla \otimes d - d \otimes N) + \mu_4 A + \mu_7 \text{tr}(A) I, \quad \tilde{\gamma} = a \rho^\gamma$. At last, we want to show $(\rho, u)$ is a renormalized solution and $\tilde{\gamma} = p = a \rho^\gamma$. Here (4.4) still holds in $D'((0,T) \times \mathbb{R}^3)$ provided $\rho, u$ is prolonged to be zero outside $D$. Let us define
\[
T_k = kT(\frac{z}{k}), \quad T(z) = \begin{cases} 
    z & 0 \leq z \leq 1, \\
    \text{concave} & 1 < z < 2, \\
    2 & z \geq 3, \\
    -T(z) & z < 0.
\end{cases} \tag{4.7}
\]
Let $\tilde{\gamma}$ denote the weak convergence limit of $\gamma$ in this paper. We have the following lemma.
Lemma 4.2. Let \((\rho_\delta, u_\delta, d_\delta)\) be a sequence satisfying (3.24)-(3.26), and \((\rho, u, d)\) solve (4.4)-(4.6), then

\[
\int_0^T \int_D \psi\phi(a_{\rho_\delta} - (\mu_4 + \mu_7)\text{div}u_\delta)T_k(\rho_\delta) \rightarrow \\
\int_0^T \int_D \psi\phi(a_{\rho} - (\mu_4 + \mu_7)\text{div}u)\overline{T_k(\rho)} \quad \text{as } \delta \rightarrow 0.
\]

holds for any constant \(k\).

Proof. By virtue of Lemma 3.2, (3.24) still holds. Taking \(\varphi(t, x) = \psi(t)\eta(x)A[T_k(\rho_\delta)], \psi \in D(0, T), \eta \in C_0^\infty(D)\) as a test function for (3.25), with a straightforward computation, we get

\[
\int_0^T \int_D \psi \eta \left\{ (a_{\rho_\delta} + \delta_{\rho_\delta}^2) T_k(\rho_\delta) - (\mu_4 A_\delta + \mu_7 \text{tr}(A_\delta)I) : (\nabla \Delta^{-1}\nabla)[T_k(\rho_\delta)] \right\} \\
= \int_0^T \int_D \left\{ \psi(S_\delta \nabla \eta)A[T_k(\rho_\delta)] - \psi \eta \rho_\delta u_\delta \otimes u_\delta : (\nabla \Delta \nabla)[T_k(\rho_\delta)] \right\} \\
+ \int_0^T \int_D \left\{ -\psi \left( a_{\rho_\delta} + \delta_{\rho_\delta}^2 \right) \nabla \eta A[T_k(\rho_\delta)] + \psi \left( (\rho_\delta u_\delta \otimes u_\delta) \nabla \eta \right) A[T_k(\rho_\delta)] \right\} \\
+ \int_0^T \int_D \left\{ -\psi(\nabla \delta_\delta \otimes \nabla \delta_\delta) : (\nabla \Delta \nabla)[T_k(\rho_\delta)] - \psi[(\nabla \delta_\delta \otimes \nabla \delta_\delta) \nabla \eta] A[T_k(\rho_\delta)] \right\} \\
+ \int_0^T \int_D \left\{ \frac{1}{2} |\nabla \delta_\delta|^2 + F(\delta_\delta) \right\} \psi \eta T_k(\rho_\delta) + \frac{1}{2} \left( |\nabla d_\delta|^2 + F(d_\delta) \right) \psi \nabla \eta A[T_k(\rho_\delta)] \\
+ \int_0^T \int_D \left\{ \mu_3 \psi (N_\epsilon \otimes d_\epsilon - d_\epsilon \otimes N_\epsilon) : (\nabla \Delta \nabla)[T_k(\rho_\delta)] \right\} - \psi \eta \rho_\delta u_\delta A[T_k(\rho_\delta)] \\
+ \int_0^T \int_D \left\{ \psi \left( (N_\epsilon \otimes d_\epsilon - d_\epsilon \otimes N_\epsilon) \nabla \eta \right) A[T_k(\rho_\delta)] + \psi \eta \rho_\delta u_\delta A[\text{div}(T_k(\rho_\delta)u_\delta)] \right\} \\
+ \int_0^T \int_D \psi \eta \rho_\delta u_\delta A[(T_k(\rho_\delta) \rho_\delta - T_k(\rho_\delta)) \text{div}u_\delta] \\
= \sum_{i=1}^{13} I_i.
\]

Using \(\varphi = \psi \eta A[T_k(\rho)]\) as a test function of (4.5), we have

\[
\int_0^T \int_D \psi \eta \left\{ \overline{T_k(\rho)} - (\mu_4 A + \mu_7 \text{tr}(A)I) : (\nabla \Delta^{-1}\nabla)[T_k(\rho)] \right\} \\
= \int_0^T \int_D \left\{ \psi(S \nabla \eta)A[T_k(\rho)] - \psi \eta u \otimes u : (\nabla \Delta \nabla)[T_k(\rho)] \right\} \\
+ \int_0^T \int_D \left\{ -\psi \overline{T_k(\rho)} + \psi \left| (\rho u \otimes u) \nabla \eta \right| A[T_k(\rho)] \right\} \\
+ \int_0^T \int_D \left\{ -\psi(\nabla d \otimes \nabla d) : (\nabla \Delta \nabla)[T_k(\rho)] - \psi[(\nabla d \otimes \nabla d) \nabla \eta] A[T_k(\rho)] \right\}
\]
\[ + \int_0^T \int_D \left\{ \frac{1}{2} (|\nabla d|^2 + F(d)) \psi_\eta \overline{T_k} - \frac{1}{2} \right\} \psi \nabla \eta \mathcal{A} \overline{T_k} \right) \] 
\[ + \int_0^T \int_D \left\{ \psi_\eta (N \otimes d - d \otimes N) : (\nabla \Delta \nabla) [\overline{T_k}] \right\} - \psi_\eta \eta \mu \mathcal{A} [\overline{T_k}] \right) \] 
\[ + \int_0^T \int_D \left\{ \psi ((N \otimes d - d \otimes N) \nabla \eta) \mathcal{A} [\rho] + \psi_\eta \eta \mu \mathcal{A} [\text{div} (\overline{T_k} \eta \mu u)] \right) \] 
\[ + \int_0^T \int_D \psi_\eta \eta \mu \mathcal{A} [\overline{T_k} (\rho - T_k (\rho)) \text{div} u] \] 
\[ = \sum_{i=1}^{13} I_i. \]

One easily observes
\[ \lim_{\delta \to 0} T_k (\rho_\delta) \leq \lim_{\delta \to 0} C \delta \alpha \int_0^{\beta \frac{\pi}{2 \theta}} \| \rho_\delta \|_{L_\alpha (0, T)}^\beta = 0, \]
where \( \theta \) is defined in Lemma 4.1. Noting \( T_k (z) \) is bounded, we can get \( I_i^4 \to I_i \) term by term as in the proof of Lemma 3.7. \( \square \)

Next we define oscillations defect measure \( \text{OSC}_p [\rho_\delta \to \rho] (O) \), for \( O \in ((0, T) \times D) \) (See [6]).
\[ \text{OSC}_p [\rho_\delta \to \rho] (O) = \sup \left\{ \lim_{\kappa \to 0} \frac{\int_0^T \| T_k (\rho_\delta) - T_k (\rho) \|^p }{\int_O \| T_k (\rho_\delta) - T_k (\rho) \|^p } \right\}, \tag{4.9} \]
where \( T_k \) is defined by (4.7).

**Lemma 4.3.** Let \( (\rho_\delta, u_\delta, d_\delta) \), \( (\rho, u, d) \) satisfy the assumption of Lemma 4.2. Then for any bounded set \( O \subset ((0, T) \times D) \), we have
\[ \text{OSC}_{\gamma+1} [\rho_\delta \to \rho] (O) \leq C(O). \]

**Proof.** Using (4.8), we have
\[ \lim_{\delta \to 0} \int_D \int_0^T \left( p(\rho_\delta) T_k (\rho_\delta) - p(\rho) T_k (\rho) \right) \left( \mu_4 + \mu_7 \right) \int_D \int_0^T \text{div} u \overline{T_k} (\rho_\delta) - \text{div} u \overline{T_k} (\rho). \]
Moreover, one easily show,
\[ \lim_{\delta \to 0} \int_D \int_0^T \left[ p(\rho_\delta) T_k (\rho_\delta) - p(\rho) T_k (\rho) \right] = \lim_{\delta \to 0} \int_D \int_0^T \left( p(\rho_\delta) - p(\rho) \right) (T_k (\rho_\delta) - T_k (\rho)) \]
\[ + \int_D \int_0^T \left( \overline{p(\rho_\delta) - p(\rho)} \right) (T_k (\rho) - \overline{T_k}) \]
\[ \geq \lim_{\delta \to 0} \int_D \int_0^T \left( p(\rho_\delta) - p(\rho) \right) (T_k (\rho_\delta) - T_k (\rho)). \tag{4.10} \]
Here we use \( p(z) \) is convex and \( T_k (z) \) is concave. For \( p(z) = az^\gamma \), we have
\[ p(y) - p(z) = \int_z^y p'(s) ds \geq \int_z^y p'(s - z) ds = p(y - z), \quad y \geq z \geq 0. \tag{4.11} \]
Considering the definition of \( T_k \), the inequality (4.11) yields
\[ p(|T_k (y) - T_k (z)|) \leq p(|y - z|). \tag{4.12} \]
Thus we have
\[ a |T_k (\rho_\delta) - T_k (\rho)|^{\gamma+1} \leq p(|T_k (\rho_\delta) - T_k (\rho)|) |T_k (\rho_\delta) - T_k (\rho)| \]
\[ \leq p(|\rho_\delta - \rho|) |T_k (\rho_\delta) - T_k (\rho)| \leq (p(\rho_\delta) - p(\rho))(T_k (\rho_\delta) - T_k (\rho)). \tag{4.13} \]
Using \((4.10)\), we get
\[
\lim_{\delta \to 0} \int_0^T \int_D |T_k(\rho_\delta) - T_k(\rho)|^{\gamma + 1} \leq \lim_{\delta \to 0} \int_0^T \int_D p(\rho_\delta) T_k(\rho_\delta) - p(\rho) T_k(\rho).
\]
And we easily obtain
\[
\lim_{\delta \to 0} \int_0^T \int_D |T_k(\rho_\delta) - T_k(\rho)|^{\gamma + 1} \\
\leq (\mu_4 + \mu_7) \lim_{\delta \to 0} \int_0^T \int_D \left[ \text{div} u_\delta T_k(\rho_\delta) - \text{div} T_k(\rho) \right] \\
= (\mu_4 + \mu_7) \lim_{\delta \to 0} \int_0^T \int_D \left( T_k(\rho_\delta) - T_k(\rho) + T_k(\rho) - T_k(\rho) \right) \text{div} u_\delta \\
\leq 2(\mu_4 + \mu_7) \| \text{div} u_\delta \|_{L^2([0,T] \times D)} \lim_{\delta \to 0} \| T_k(\rho_\delta) - T_k(\rho) \|_{L^2([0,T] \times D)} \\
\leq C \lim_{\delta \to 0} \| T_k(\rho_\delta) - T_k(\rho) \|_{L^{2\gamma+1}(D)} \| D \|^{\frac{\gamma - 1}{\gamma+1}}. \\
\tag{4.14}
\]
Then we complete the proof. \(\square\)

**Lemma 4.4.** Let \(D \in C^{2+\alpha}\) be bounded in \(\mathbb{R}^3\), \(\{ (\rho_\delta, u_\delta, d_\delta) \}\) is a sequence of solutions satisfying \((3.24)-(3.26)\) and \((\rho_\delta, u_\delta)\) is a renormalized solution of \((3.24)\). Assume that \(\rho_\delta \to \rho\) in \(L^\infty(0,T; L^\gamma(D))\) with \(\gamma > \frac{2N}{N+2}\), \(u_\delta \to u\) in \(L^2(0,T; H^1(D))\) and \(\text{OSC}_{\gamma+1}[\rho_\delta \to \rho](O) \leq C(O), O \subset ((0,T) \times D)\). Then \((\rho, u)\) is a renormalized solution of \((4.4)\).

**Proof.** Using \((\rho_\delta, u_\delta)\) is renormalized solution of \((3.24)\), we have
\[
\partial_t T_k(\rho_\delta) + \text{div}(T_k(\rho_\delta) u_\delta) + (T'_k(\rho_\delta) \rho_\delta - T_k(\rho_\delta)) \text{div} u_\delta = 0. \\
\tag{4.15}
\]
Passing to the limit for \(\delta \to 0\), we obtain
\[
\partial_t T_k(\rho) + \text{div}(T_k(\rho) u) + (T'_k(\rho) \rho - T_k(\rho)) \text{div} u = 0, \\
\tag{4.16}
\]
Similarly to Lemma 3.8, we get the following equation,
\[
\partial_t B(T_k(\rho)) + \text{div}(B(T_k(\rho)) u) + B'(T_k(\rho)) T_k(\rho) - B(T_k(\rho)) \text{div} u \\
= B'(T_k(\rho))((T_k(\rho) - T'_k(\rho) \rho) \text{div} u) \quad \text{in } D'((0,T) \times D), \\
\tag{4.17}
\]
where \(B(z)\) satisfies
\[
B \in C^1[0, \infty), \quad B'(z) = 0 \quad \text{for all } z \geq z_B.
\]
Utilizing the weak lower semi-continuity of function’s norm, we deduce
\[
\|T_k(\rho) - \rho\|_{L^1([0,T] \times D)} \leq \liminf_{\delta \to 0} \|T_k(\rho_\delta) - \rho_\delta\|_{L^1((0,T) \times D)} \\
\leq \sup_{\delta} \int_{\rho_\delta \geq k} \rho_\delta \leq k^{1-\gamma} \sup_{\delta} \| \rho_\delta \|_{L^\gamma((0,T) \times D)}.
\]
And passing to the limit for \(k \to 0\), we have
\[
B(T_k(\rho)) \to B(\rho), \quad B'(T_k(\rho)) \to B'(\rho) \quad \text{in the corresponding spaces.}
In order to complete the proof, we should show the right side of (4.17) tends to zero as $k \to \infty$. One easily show
\begin{align}
&\|B'(T_k(\rho))(T_k(\rho) - T_k^\gamma(\rho)\rho)\|_{L^1((0,T) \times D)} \\
&\leq \max_{z \geq 0} \left| B'(z) \right| \int_{T_k(\rho) \leq z_B} \left| (T_k(\rho) - T_k^\gamma(\rho)\rho)\|\right| \, \text{d}v \\
&\leq \max_{z \geq 0} \left| B'(z) \right| \sup_{\delta} \|\text{div}\delta\|_{L^2((0,T) \times D)} \lim_{\delta \to 0} \|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^2(T_k(\rho) \leq z_B)}.
\end{align}
(4.18)

By interpolation, we get
\begin{align}
&\|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^2(T_k(\rho) \leq z_B)} \\
&\leq \|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^1((0,T) \times D)} \|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^{\gamma+1}(T_k(\rho) \leq z_B)}. 
\end{align}
(4.19)

Observing the definition of $T_k$, we have
\begin{align}
&\|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^1((0,T) \times D)} \leq 2\gamma^{-1} \gamma \sup_{\delta} \|\rho_\delta\|_{L^\gamma((0,T) \times D)}.
\end{align}
(4.20)

Using Lemma 4.3 and $T_k^\gamma(z) \leq T_k(z)$, we obtain
\begin{align}
&\lim_{\delta \to 0} \|T_k(\rho) - T_k^\gamma(\rho)\rho_\delta\|_{L^{\gamma+1}(T_k(\rho) \leq z_B)} \\
&\leq 2 \lim_{\delta \to 0} \|T_k(\rho) - T_k(\rho)\|_{L^{\gamma+1}(0,T) \times D} + 2 \|T_k(\rho) - T_k^\gamma(\rho)\|_{L^{\gamma+1}(0,T) \times D} \\
&\quad + 2 \|\overline{T_k(\rho)}\|_{L^{\gamma+1}(T_k(\rho) \leq z_B)} \\
&\leq 4 \text{OSC}(\rho_\delta \to \rho)_{(0,T) \times D} + 2z_B(T \Omega)^{\frac{\gamma-1}{\gamma}}.
\end{align}
(4.21)

Substituting (4.19)-(4.21) into (4.18), and passing to the limit for $k \to \infty$, (4.18) tends to be zero. For general $B$ in Definition 1.1, we can use $z_B = k^{\frac{\gamma-1}{\gamma+1}}$. Then (4.21) still holds.

In the last step, we prove $\rho_\delta \to \rho$ in $L^1((0,T) \times D)$. Let’s define $L_k$:
\begin{align}
L_k(z) = \left\{ \begin{array}{ll}
z \log(z) & 0 \leq z \leq 1, \\
z \int_1^z \frac{T_k(\rho)}{z} \, dz & z > 1.
\end{array} \right.
\end{align}

Observing $(\rho_\delta, u_\delta)$ is a renormalized solution of (3.24) and $(\rho, u)$ is a renormalized solution of (4.4), we have
\begin{align}
&\partial_t L_k(\rho_\delta) + \text{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\|\rho_\delta\|_{L^1} = 0, \\
&\partial_t L_k(\rho) + \text{div}(L_k(\rho)u) + T_k(\rho)\|\rho\|_{L^1} = 0.
\end{align}
(4.22)

Using (4.22), we get
\begin{align}
&L_k(\rho_\delta) \to \overline{L_k(\rho)} \quad \text{in } C([0,T]; L^\gamma_{\text{weak}}(D)), \\
&\rho_\delta \log(\rho_\delta) \to \rho \log(\rho) \quad \text{in } C([0,T]; L^\alpha_{\text{weak}}(D)),
\end{align}
(4.24)
\begin{align}
&1 \leq \alpha < \gamma.
\end{align}
(4.25)

Let $\delta \to 0$ in (4.22),
\begin{align}
&\partial_t \overline{L_k(\rho)} + \text{div}(\overline{L_k(\rho)}u) + \overline{T_k(\rho)\|\rho\|_{L^1}} = 0.
\end{align}
(4.26)

(4.23) and (4.26) yield
\begin{align}
&\partial_t (L_k(\rho) - L_k(\rho)) + \text{div}(L_k(\rho)u - L_k(\rho)u) + T_k(\rho)\|\rho\|_{L^1} - T_k(\rho)\|\rho\|_{L^1} = 0.
\end{align}
(4.27)
For any $\eta_n \in C^\infty_0(D)$, it holds
\[
\lim_{\delta \to 0} \int_D \eta_n [L_k(\rho_{0,\delta}) - L_k(\rho_0)] \, dx = 0.  \tag{4.28}
\]
Then we have
\[
\int_D \eta_n \left[ \frac{L_k(\rho)}{L_k(\rho)} - L_k(\rho) \right] (t) \, dx
= \int_0^T \int_D \left[ \frac{L_k(\rho)}{L_k(\rho)} u - L_k(\rho) u \right] \nabla \eta_n + (T_k(\rho) \nabla \cdot \eta_n) \, \eta_n.
\tag{4.29}
\]
From the definition of $L_k$, we can get $L_k(z) = 2kz - 2k, z \geq 3k$. Let $\{\eta_n\}$ be a sequence such that $\eta_n \to 1$ in $D$. Passing to the limit for $n \to \infty$ in (4.29), we have
\[
\int_D \left[ \frac{L_k(\rho)}{L_k(\rho)} - L_k(\rho) \right] (t) \, dx = \int_0^T \int_D \left[ T_k(\rho) \nabla \cdot \eta_n - \frac{T_k(\rho)}{L_k(\rho)} \right].
\]
Also it holds
\[
\int_0^T \int_D \left[ T_k(\rho) \nabla \cdot \eta_n - \frac{T_k(\rho)}{L_k(\rho)} \right]
= \int_0^T \int_D \left[ \frac{T_k(\rho)}{L_k(\rho)} \nabla \cdot \eta_n - \frac{T_k(\rho)}{L_k(\rho)} \right] + \| \nabla \cdot \eta_n \|_{L^2((0,T)\times D)} \| \frac{T_k(\rho)}{L_k(\rho)} - T_k(\rho) \|_{L^2((0,T)\times D)}.
\tag{4.30}
\]
Using Lemma 4.2 and the weak lower semi-continuity of the function’s norm, one obtains
\[
\lim_{\delta \to 0} \int_0^T \int_D \left[ \nabla \cdot \frac{T_k(\rho)}{L_k(\rho)} - \frac{T_k(\rho)}{L_k(\rho)} \right] \, dx
= \frac{a}{\mu_4 + \mu_7} \lim_{\delta \to 0} \int_0^T \int_D \left[ \frac{\rho^\gamma T_k(\rho)}{L_k(\rho)} - \frac{\rho^\gamma T_k(\rho)}{L_k(\rho)} \right] \leq 0.
\tag{4.31}
\]
For the second term of (4.30), we have
\[
\| \frac{T_k(\rho)}{L_k(\rho)} - T_k(\rho) \|_{L^2((0,T)\times D)} \leq \| \frac{T_k(\rho)}{L_k(\rho)} - T_k(\rho) \|_{L^{2\gamma+1}((0,T)\times D)} + \| T_k(\rho) - \rho \|_{L^{2\gamma+1}((0,T)\times D)}.
\]
Utilizing Lemma 4.3, we have
\[
\| \frac{T_k(\rho)}{L_k(\rho)} - T_k(\rho) \|_{L^{2\gamma+1}((0,T)\times D)} \leq C.
\tag{4.32}
\]
And one easily gets
\[
\| \frac{T_k(\rho)}{L_k(\rho)} - T_k(\rho) \|_{L^1((0,T)\times D)}
\leq \| \frac{T_k(\rho)}{L_k(\rho)} - \rho \|_{L^1((0,T)\times D)} + \| \frac{T_k(\rho)}{L_k(\rho)} - \rho \|_{L^1((0,T)\times D)}
\leq \lim_{\delta \to 0} \| T_k(\rho) - \rho \|_{L^1((0,T)\times D)} + \| T_k(\rho) - \rho \|_{L^1((0,T)\times D)}
\leq k^{1-\gamma} \sup_{\delta} \| \rho \|_{L^{2\gamma+1}((0,T)\times D)} + k^{1-\gamma} \| \rho \|_{L^{2\gamma+1}((0,T)\times D)} \to 0 \quad \text{as } k \to \infty.
\]
Then we have
\[
\rho \log(\rho)(t) \leq \rho \log(\rho)(t).
\]
The convex property of $\rho \log(\rho)$ yields

$$\rho \log(\rho)(t) \geq \rho \log(\rho)(t).$$

Then we get the following result:

$$\rho \log(\rho)(t) = \rho \log(\rho)(t).$$

which leads to $\rho_\delta \to \rho$ in $L^1((0, T) \times D)$. Thus we have proved $\rho^\gamma = \rho^\gamma$. The proof of Theorem 1.1 is completed.

5. The proof of Theorem 1.2. First we consider the system (1.17)-(1.19) in a bounded domain $B_r$ (a bounded ball with radius $r$ and center on the origin). Using the cut off functions and mollified operator, the initial conditions are constructed as follows:

$$\rho_{0,r} = \rho_0|_{B_r}, \quad d_{0,r} = d_0|_{B_r}.$$  

such that

$$\rho_r(x, 0) = \rho_{0,r}(x) \geq 0 \quad \text{a.e. in } B_r,$$

$$(\rho, u_r)(x, 0) = (\rho_0, u_0)(x, 0) = 0 \text{ a.e. on } \{\rho_{0,r}(x) = 0\}, \quad (\rho_{0,r}) \in L^1(B_r),$$

$$d_r(x, 0) = d_{0,r}(x), \quad |d_{0,r}| = 1, \quad d_{0,r} \in H^2(B_r),$$

$$u_r(x, t) = 0, \quad d_r(x, t) = d_{0,r}(x), \quad (x, t) \in \partial B_r \times (0, \infty),$$

and

$$\int_{\mathbb{R}^3} (\rho_{0,r})^\gamma - \gamma(\rho_{0,r} - 1) - 1 \leq C_0.$$  

(5.2)

There exists a constant $E_0$, such that

$$E_{0,r} = \int_D \left[ \frac{|q_{0,r}|^2}{\rho_{0,r}} + \frac{1}{2} |\nabla d_{0,r}|^2 + \frac{(\rho_{0,r})^\gamma - \gamma(\rho_{0,r} - 1) - 1}{\gamma - 1} + F(d_{0,r}) \right] \leq E_0.$$  

The existence of weak solution to system (1.17)-(1.19) with (5.1)-(5.2) can be guaranteed by Theorem 1.1. Using the energy inequality, we have

$$\|\sqrt{\rho} u_r\|_{L^\infty(0, T; L^2(B_r))} \leq E_0, \quad \|\nabla u_r\|_{L^2(0, T \times B_r)} \leq E_0,$$

$$\|N_r\|_{L^2(0, T \times B_r)} \leq E_0, \quad \|\nabla d_r\|_{L^\infty(0, T; L^2(B_r))} \leq E_0,$$

$$\int_{\mathbb{R}^3} (\rho_r)^\gamma - \gamma(\rho_r - 1) - 1 \leq E_0.$$  

(5.5)

Using (5.5) and the following fact:

$$\left\{ \begin{array}{l}
x^\gamma - 1 - \gamma(x - 1) \geq \nu |x - 1|^\gamma, \quad \gamma \geq 2, \\
x^\gamma - 1 - \gamma(x - 1) \geq \nu |x - 1|^2, \quad \gamma < 2, \quad |x - 1| \leq \frac{1}{2}, \\
x^\gamma - 1 - \gamma(x - 1) \geq \nu |x - 1|^\gamma, \quad \gamma < 2, \quad |x - 1| \geq \frac{1}{2}, \end{array} \right.$$  

(5.6)

we have

$$\left\{ \begin{array}{l}
\|\rho_r - 1\|_{L^\infty(0, T; L^\gamma(B_r))} \leq E_0, \quad \text{if } \gamma \geq 2, \\
\|\rho_r - 1\|_{L^\infty(0, T; L^2(B_r))} \leq E_0, \quad \text{if } \gamma < 2. \end{array} \right.$$  

(5.7)

We split $u_r$ as follows:

$$u_r = u_r^1 + u_r^2,$$

$$u_r^1 = u_r|_{|\rho_r - 1| \leq \frac{1}{2}}, \quad u_r^2 = u_r|_{|\rho_r - 1| \geq \frac{1}{2}}.$$
Then we have
\[
\sup_t \int_{B_r} |u_1^1|^2 dx \leq 2 \sup_t \int_{B_r} \rho_r |u_r|^2 dx \leq E_0
\]
\[
\int_{B_r} |u_2^2|^2 dx \leq 2 \int_{B_r} |\rho_r - 1||u_r|^2 dx \leq \|\rho_r - 1\|_{L^2(B_r)} \|u_r^2\|_{L^2(B_r)}^{\gamma} \|\nabla u_r\|_{L^\gamma(B_r)}^{1-\gamma},
\]
which is
\[
\|u_2^2\|_{L^\infty(0,T;L^2(B_r))} \leq E_0, \tag{5.8}
\]
\[
\|u_2^2\|_{L^2(0,T;L^2(B_r))} \leq C(E_0). \tag{5.9}
\]
Thus we have deduced \(u_r \in L^2(0,T;H^1_0(B_r))\) whose bound is independent of \(r\). Like in the bounded case, we have \(|d_r| \leq 1\ a.e.\ in\ B_r\ indenpending\ of\ r\).

Noticing
\[
\|f(d)\|_{L^\infty(0,T);L^2(B_r)} \leq \|F(d)\|_{L^\infty(0,T);L^2(B_r)} \leq C(E_0),
\]
and the bound of \(N_r\) in (5.4), we have
\[
\|\nabla^2 d_r\|_{L^2((0,T)\times B_r)} \leq C(E_0), \text{ independent of } r,
\]
by using interior elliptic estimate on \(\Delta d_r - f(d_r) = \frac{1}{\lambda_1} N_r\). Observing \(N_r = d_{rt} + (u_r \cdot \nabla) d_r - \Omega_r d_r\), we have
\[
\|d_{rt}\|_{L^2((0,T);L^2(B_r))} \leq C(E_0), \text{ independent of } r.
\]
Let us prolong \((\rho_r, u_r, d_r)\) by zero outside \(B_r\). And for the sake of convenience, we still use \((\rho_r, u_r, d_r)\). It exists \((\rho, u, d)\) such that we have the following:
\[
\rho_r - \rho \rightharpoonup 0 \quad \text{in} \quad L^\infty(0,T;L^\gamma(\mathbb{R}^3)), \text{ if } \gamma \geq 2,
\]
\[
\rho_r - \rho \rightharpoonup 0 \quad \text{in} \quad L^\infty(0,T;L^2(\mathbb{R}^3)), \text{ if } \gamma < 2,
\]
\[
u_r \rightharpoonup u \quad \text{in} \quad L^2(0,T;H^1(\mathbb{R}^3)),
\]
\[
 d_r \rightharpoonup d \quad \text{in} \quad L^\infty(0,T;H(\mathbb{R}^3)),
\]
\[
 d_r \rightharpoonup d \quad \text{in} \quad L^2(0,T;H^1(\mathbb{R}^3) \cap \tilde{H}^2(\mathbb{R}^3)).
\]

At last, we only need to show \((\rho, u, d)\) satisfies (1.17)-(1.19) in \(D'_{loc}((0,T) \times \mathbb{R}^3)\).

Observing that it’s nothing but the case of bounded domain(at least, we can use the same process). So we end the prove of Theorem 1.2.

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