From $k$-SAT to $k$-CSP: Two Generalized Algorithms

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1 Introduction

Constraint satisfaction problems (CSPs) models many important intractable
$NP$-hard problems such as propositional satisfiability problem (SAT) [1]. Al-
gorithms with non-trivial upper bounds on running time for restricted SAT

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with bounded clause length $k$ ($k$-SAT) can be classified into three styles: DPLL-like, PPSZ-like and Local Search [2], with local search algorithms having already been generalized to CSP with bounded constraint arity $k$ ($k$-CSP) [3]. We generalize a DPLL-like algorithm in its simplest form and a PPSZ-like algorithm [3] from $k$-SAT to $k$-CSP. As far as we know, this is the first attempt to use PPSZ-like strategy to solve $k$-CSP, and before little work has been focused on the DPLL-like or PPSZ-like strategies for $k$-CSP.

For the DPLL-like deterministic $k$-CSP algorithm, a recurrent inequality is tightly solved to get a non-trivial upper bound $O^*((d - \frac{d-1}{d^k})^n)$ on running time, where $n$ is the number of variables and $d$ is the domain size of variables in input. For the PPSZ-like randomized $k$-CSP algorithm, the Satisfiability Coding Lemma [4] is extended to non-Boolean case to show that with probability approaching 1, a satisfying assignment can be found in time $O^*((d^{\sqrt{\frac{d-1}{d^k}}})^n)$. $O^*$ indicates that some polynomial factor in $n$ is ignored in big-$O$ notation.

CSP generalizes SAT in two aspects: each variable can have more than two available values, and each constraint can have more than one falsifying partial assignments. These falsifying partial assignments to some constraint are called nogoods. For example, in graph 3-coloring problem a constraint of two variables $x$ and $y$ with domain $\{0, 1, 2\}$ has tuples of values such as $(x : 0, y : 0)$, $(x : 1, y : 1)$ and $(x : 2, y : 2)$ as its nogoods. If the first $k - 1$ variables in a constraint with arity $k$ have values in agreement with a nogood, then the last variable cannot have the value specified by the nogood, so as not to falsify this constraint. Such a variable is thus called narrowly chosen. Our generalizations rooted from the key observation that nogoods (instead of constraints) in CSP can be treated as clauses in SAT to produce narrowly chosen variables which can be exploited by algorithms to reduce their search efforts.
The remainder of the paper is organized as follows. Section 2 describes and analyzes the generalized DPLL-like deterministic $k$-CSP algorithm. Section 3 extends the original satisfiability coding lemma [4] to non-Boolean case. Section 4 presents the PPSZ-like randomized $k$-CSP algorithm and its analysis.

2 The DPLL-like Deterministic $k$-CSP Algorithm

Our DPLL-like algorithm for $k$-CSP with variables domain size $d$ works as follow: for any nogood $(u_1 : a_1, ..., u_k : a_k)$, branch on $u_1$ to $d - 1$ branches, on each branch a value other than $a_1$ and also different from value assigned on other sister branches is assigned to $u_1$ and then recursively go down the branch. If all these branch fails to find a satisfying assignment, then fix $u_1$ to value $a_1$ and branch on $u_2$ in exactly the same way as on $u_1$ except that this time the number of remaining variables decreased by one. Denote the running time of this algorithm by $T(n)$, then clearly for $d \geq 2$ and $k \geq 2$:

$$T(n) \leq (d - 1)(T(n - 1) + \ldots + T(n - k)) + \text{poly}(n). \quad (1)$$

Note that as usual we can safely ignore the additive $\text{poly}(n)$ term at right hand side and treat the inequality as an equation.

When $d$ is a fixed constant, linear recursion (1) has solution $T(n) = O^*(\lambda^n)$ with $\lambda$ the maximum root in characteristic equation $f(\lambda) = \lambda^k - (d - 1)(\lambda^{k-1} + \ldots + 1) = 0$. Since $\lambda > 1$, our trick is to find the maximum root in equation $g(\lambda) = (\lambda - 1)f(\lambda) = \lambda^{k+1} - d\lambda^k + (d - 1) = 0$. $g(\lambda)$ is strictly increasing when $\lambda > d(1 - \frac{1}{k+1})$. We can find that when $\lambda \geq d - \frac{d-1}{d^k}$, $g(\lambda) > 0$; when $\lambda = d - \frac{1}{d^{k+1}}$, $g(\lambda) < 0$. Hence the tight solution of (1) is $T(n) = O^*((d - \frac{d-1}{d^k})^n)$. 

3
When $d$ is not fixed and varies with $n$, specifically $d = n^\alpha$ with $\alpha$ a constant, this case models some practical problems (e.g. the Latin square problem and the $N$-queen problem) and a random CSP model (called Model RB), which contains many hard instances seemingly quite challenging for various kinds of algorithms, both theoretically [7] and experimentally [6], and a trivial upper bound is $O^*(n^\alpha n)$. Rewrite the recursion (1) as

$$T(n) = (n^\alpha - 1)(T(n-1) + ... + T(n-k)). \quad (2)$$

When $\alpha \leq 1$, for any fixed $\epsilon > 0$, for large $n$ with $n^\alpha - 1 > \frac{1}{\epsilon}$, we have

$$\sum_{i=n-k}^{n-2} T(i) < \epsilon T(n),$$

so for large enough $n$ (actually $n+1$ will be fine for above $n$): $T(n) \leq (n^\alpha - 1)(T(n-1) + \sum_{i=n-k}^{n-2} T(i)) < (n^\alpha - 1)(T(n-1) + \sum_{i=n-k-1}^{n-2} T(i)) < (n^\alpha - 1)(T(n-1) + \epsilon T(n-1)) < n^\alpha(1+\epsilon)T(n-1)$. Substitute $n$ by smaller numbers and combine these inequalities, we have for any fixed number $\epsilon > 0$: $T(n) = O^*((n!)^\alpha(1 + \epsilon)^n) = O^*((\frac{n}{\epsilon})^\alpha n(1 + \epsilon)^n)$.

When $\alpha > 1$, there is some number $\beta$ with $1 < \beta < \alpha$, such that for sufficiently large $n$, $(n-1)^\alpha - 1 > n^\beta$, so

$$T(n) \leq (n^\alpha - 1)(T(n-1) + \sum_{i=n-k}^{n-2} T(i)) < (n^\alpha - 1)(T(n-1) + \sum_{i=n-k-1}^{n-2} T(i)) = (n^\alpha - 1)(T(n-1) + \frac{1}{(n-1)^\beta - 1} T(n-1)) < n^\alpha(1 + \frac{1}{n^\beta})T(n-1).$$

Since $\prod_{n=1}^{\infty} (1 + \frac{1}{n^\beta})$ converges to a finite number, by applying the same analysis as in above paragraph, we have $T(n) = O^*((\frac{n}{\epsilon})^\alpha n)$.

3 A Generalized Satisfiability Coding Lemma

Abbreviation w.r.t. means with respect to. Our key generalization to a definition in [4] about isolated points, critical point and critical variables is:

**Definition 1.** For a $k$-CSP instance $F$ with domain $D$ for its $n$ variables,
call $X = (a_1, ..., a_i, ..., a_n)$ an isolated point w.r.t. a set $S \subseteq D^n$ if there exists a dimension $i \in \{1, 2, ..., n\}$ and an $a'_i \in D - \{a_i\}$ such that $X \in S$ but $X' = (a_1, ..., a'_i, ..., a_n) \notin S$. Call such a dimension $i$ a critical point of $X$ w.r.t. $S$ and the variable $u_i$ at dimension $i$ a critical variable.

We only require that there exist $a'_i \in D - \{a_i\}$ such that $X \in S$ but $X' = (a_1, ..., a'_i, ..., a_n) \notin S$, rather than that for all $a'_i \in D - \{a_i\}$ (which can only work for SAT but not for CSP). This right choice (which works for both SAT and CSP) makes the following two generalized lemmas and the generalized algorithm with analysis in next section straightforward to follow the routine in [4], as follows.

Denote the number of critical points of $X$ w.r.t. $S$ by $J_S(X)$. Call $X$ $j$-isolated w.r.t. $S$ if $X$ is an isolated point in exactly $j$ dimensions w.r.t. $S$. Call an $n$-isolated solution $X$ an isolated solution. When $S$ is the set of all solutions of $F$, we can omit the words w.r.t. $S$. When solution $X = (a_1, ..., a_i, ..., a_n)$ has a critical point $i$, there must be a constraint with a nogood in agreement with $X$ except only in flipping $a_i$ to some $a'_i \in D - \{a_i\}$. Call such a constraint critical. In any value assigning sequence of variables, if a critical variable $u_i$ is assigned value last among all the variables in its critical constraint, and all other variables than $u_i$ are assigned values in agreement with $X$, then the value $a'_i$ should not be assigned to $u_i$ (otherwise the critical constraint will be falsified), thus the domain of $u_i$ is narrowed. Call such a variable $u_i$ narrowly chosen, otherwise fully chosen. For any given partial assignment and any constraint, we can efficiently check if a variable in this constrain is narrowly chosen: it is narrowly chosen iff other variables in this constrain has assigned values in agreement with a nogood for this constraint, and every constraint with arity $k$ can have at most $d^k$ nogoods.
Lemma 1 Let $F$ be a $k$-CSP instance with a $j$-isolated solution $X$. Then over all value assigning sequences of variables with the final value assignment $X$, the average number of narrowly chosen variables is at least $j/k$, thus the average number of fully chosen ones is at most $n - j/k$.

Proof: (As in [4]) For a random value assigning sequence $\sigma$, since no constraint involves more than $k$ variables in a $k$-CSP instance, the probability that a critical variable is assigned last among all the variables in its critical constraint is at least $1/k$. For each critical constraint, if the corresponding critical variable is last assigned, then this variable will be narrowly chosen. The $j$-isolated solution $X$ has exactly $j$ critical points and these $j$ critical variables each has a critical constraint. Thus, the average number of narrowly chosen variables is at least $j/k$ when $X$ is the final assignment. With a total number of variables $n$, the average number of fully chosen variables is no more than $n - j/k$. Q.E.D.

Lemma 2 If a nonempty set $S \subseteq D^n$ with $|D| = d$, then \[ \sum_{x \in S} \left( \frac{1}{d} \right)^{n - J_S(x)} \geq 1. \]

Proof: (By induction on $n$ as in [4].) Case $n = 0$ is trivially true. For $n > 0$, consider a fixed dimension, say $n$. Assume $D = \{a_1, \ldots, a_d\}$ and divide the set $S$ into $d$ subsets $S_1, \ldots, S_d$, such that $S_i = S_i' \times \{a_i\}$ with $S_i'$ the projection of $S_i$ to the first $n - 1$ dimensions. For any $X$ in nonempty $S_i$, denote the image of $X$ in $S_i'$ by $X'$, then induction hypothesis says \[ \sum_{x \in S_i'} \left( \frac{1}{d} \right)^{n - J_{S_i'}(x) - 1} \geq 1. \]

Since $S$ is nonempty, some $S_j$ is nonempty. For any $X \in S_j$, dimension $n$ is surely a critical point of $X$ w.r.t. $S_j$, so $J_{S_j}(X) = J_{S_j'}(X') + 1$. On the other hand, dimension $n$ is a critical point of $X$ w.r.t. $S$ iff some $S_i$ is empty. Say $S_i$ is empty, then dimension $n$ is a critical point of $X$ w.r.t. $S$, so $J_S(X) = J_S(X)$. In this case \[ \sum_{x \in S} \left( \frac{1}{d} \right)^{n - J_S(x)} \geq \sum_{x \in S_j} \left( \frac{1}{d} \right)^{n - J_S(x)} = \sum_{x \in S_j} \left( \frac{1}{d} \right)^{n - J_{S_j'}(x)} = \sum_{x \in S_j'} \left( \frac{1}{d} \right)^{n - J_{S_j'}(x) - 1} \geq 1. \] If no $S_i$ is empty, then dimension $n$ is not a critical
point of $X$ w.r.t. $S$, so $J_S(X) = J_{S_i}(X) - 1$. In this case
\[ \sum_{x \in S} \left( \frac{1}{d} \right)^{n-J_S(x)} = \sum_{i=1}^{d} \sum_{x \in S_i \setminus \{x\}} \left( \frac{1}{d} \right)^{n-J_{S_i}(x) = \sum_{i=1}^{d} \sum_{x \in S_i \setminus \{x\}} \left( \frac{1}{d} \right)^{n-J'_{S_i}(x)} = \frac{1}{d} \sum_{i=1}^{d} \sum_{x \in S_i \setminus \{x\}} \left( \frac{1}{d} \right)^{n-J'_{S_i}(x)-1} \geq \frac{1}{d} \sum_{i=1}^{d} 1 = 1. \]
Q.E.D.

### 4 PPSZ-like Randomized $k$-CSP Algorithm

Our PPSZ-like algorithm for $k$-CSP and its analysis generalize from one for $k$-SAT \cite{4} with the key observation that we can use a partial assignment and nogoods to efficiently produce narrowly chosen variables w.r.t. some value assigning sequence of variables, as explained in introduction and last sections.

**Algorithm A**

repeat $n(n+1)(d^{\frac{b}{d}})^{n}$ times

while there exists an unassigned variable

select an unassigned variable $y$ at random

if $y$ is narrowly chosen

then set $y$ to a random value in the narrowed domain

else set $y$ to a random value in its full domain

if the CSP instance is satisfied, then output the assignment

Now we prove that Algorithm A can find a solution to a satisfiable $k$-CSP instance $F$ in time $O^*((d^{\frac{b}{d}})^n)$ with probability approaching 1. Suppose that $X$ is an $j$-isolated solution of $F$ with $j$ critical points ($1 \leq j \leq n$, since $j = 0$ is a trivial case of tautology input without any nogood). In one iteration of the repeat loop, by lemma 1, the average number of critical variables assigned last among all the variables in its critical constraint is at least $j/k$, over the random value assigning sequences of variables in the while loop. Then by
Markov inequality (on complement event), the probability of the event that for at least \( j/k \) critical constraints, the critical variables occur last among the variables in the critical constraint, is at least \( \frac{1}{n-j/k+1} \). When this event occurs, the number of fully (narrowly) chosen variables is at most \( n-j/k \) (at least \( j/k \)), and each fully (narrowly) chosen variable’s value has probability exactly \( \frac{1}{d} \) (at least \( \frac{1}{d-1} \)) to agree with the corresponding value of \( X \), so the probability of the event that the values assigned to the variables in \( \text{while} \) loop agree with the assignment \( X \) is at least \( \left( \frac{1}{d} \right)^{n-j/k} \left( \frac{1}{d-1} \right)^{j/k} \) conditioned on the above event.

Thus, the probability that a \( j \)-isolated solution \( X \) of \( F \) is output by algorithm \( A \) is at least \( \frac{1}{n-j/k+1} \left( \frac{1}{d} \right)^{n-j/k} \left( \frac{1}{d-1} \right)^{j/k} \). By summing up this probability over set \( S \) of all solutions of \( F \) and by lemma 2, the probability that algorithm \( A \) outputs some solution is at least \( \sum_{x \in S} \frac{1}{n-JS(X)/k+1} \left( \frac{1}{d} \right)^{n-JS(x)/k} \left( \frac{1}{d-1} \right)^{JS(X)/k} \geq \frac{1}{n+1} \left( \frac{1}{d} \right)^{n-k} \left( \sum_{x \in S} \left( \frac{1}{d} \right)^{n-JS(x)} \right)^{1/k} \left( \frac{1}{d-1} \right)^{JS(X)/k} \geq \frac{1}{n+1} \left( \frac{1}{d} \right)^{n-k} \cdot 1 \cdot \left( \frac{1}{d-1} \right)^{n/k} = \frac{1}{n+1} \left( d^{k/d-1} \right)^{-n} \). So by repeating the \( \text{while} \) loop \( n(n+1)(d^{k/d-1})^n = O^\ast \left( (d^{k/d-1})^n \right) \) times, we can find a satisfying assignment with probability approaching 1.

When \( d = n^\alpha \), this upper bound becomes \( O^\ast \left( n^{\alpha n(1-\frac{1}{n^\alpha + 1})} \right) \).

## 5 Conclusion and Future Work

We have generalized two algorithms from \( k \)-SAT to \( k \)-CSP, with running time better than the trivial bound \( O^\ast (d^n) \) when variable domain size \( d \) is fixed.

When \( d \) is unfixed, say \( d = n^\alpha \), the result is only slightly better than the trivial bound \( O^\ast (n^{\alpha n}) \), whether we can reach \( O^\ast (n^{\beta n}) \) (where \( \beta < \alpha \) is a constant) in this case is still open. Our solutions to the recursion (1) and (2) might find other application in the analysis of DPLL-like algorithms. Our randomized algorithm is the first application of PPSZ-like strategy beyond SAT to CSP.
In summary, this paper can be viewed as the first step toward establishing upper bounds for solving $k$-CSP using DPLL-like or PPSZ-like strategies, which leaves much room for further study and improvement, for example, by combining PPSZ-like and local search algorithms as in [3].

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