Double-critical graph conjecture for claw-free graphs

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A graph is **t-colorable** if there exists a function \( c : V(G) \to \{1, \ldots, t\} \) such that \( c(u) \neq c(v) \) for all edges \( uv \in E(G) \).
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A graph \( G \) is \textbf{\textit{t-critical}} if \( \chi(G) = t \), but any proper subgraph of \( G \) is \((t - 1)\)-colorable.
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- A graph $G$ is **t-chromatic** if $\chi(G) = t$.
- A graph $G$ is **t-critical** if $\chi(G) = t$, but any proper subgraph of $G$ is $(t - 1)$-colorable.

- $\omega(G) := \max\{t : K_t \subseteq G\}$. 

*Martin Rolek*

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• A graph is **t-colorable** if there exists a function 
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  of \( G \) is \( (t-1)\)-colorable.

• \( \omega(G) := \max\{t : K_t \subseteq G\} \).

• \( \alpha(G) := \max\{t : \overline{K}_t \subseteq G\} \).
Proposition

If \( \chi(G) = 3t \), then \( G \) contains \( t \) vertex-disjoint odd cycles.
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- $(3t - 2)$-critical graphs may not have $t$ vertex-disjoint odd cycles (Gallai 1968, $t = 2$).
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Question - Erdős (1968)

Is it true for $t \geq 2$ that every $(3t - 1)$-critical graph with sufficiently many vertices contains $t$ vertex-disjoint odd cycles?
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- Is it true that every 5-critical graph with sufficiently many vertices contains two vertex-disjoint odd cycles?
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- Is it true that every 5-critical graph with sufficiently many vertices contains two vertex-disjoint odd cycles?
- If a 5-chromatic graph contains two vertex disjoint odd cycles, then it has two disjoint 3-chromatic subgraphs.
Erdős-Lovász Tihany Conjecture (1968)

Let $G$ be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then $G$ contains two disjoint subgraphs $H_1$ and $H_2$ such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$. 
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- If we fix $s = 2$, the conjecture claims there exists an edge $uv$ such that $\chi(G - u - v) \geq \chi(G) - 1$.
- A connected graph $G$ is **double-critical** if for every edge $uv \in E(G)$, $\chi(G - u - v) = \chi(G) - 2$. 
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Double-Critical Graph Conjecture - Erdős and Lovász (1968)

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- True for $t \leq 5$ (Brown and Jung 1969; Mozhan 1987; Stiebitz 1987)
- Open for $t \geq 6$.
- True for line graphs, quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Kostochka and Stiebitz 2008; Balogh, Kostochka, Prince, and Stiebitz 2009)
A graph is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph.
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If $G$ is a claw-free, double-critical, 6-chromatic graph, then $G = K_6$. 
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**Theorem - R. and Song (2017)**

For $t \in \{6, 7, 8\}$, if $G$ is a claw-free, double-critical, $t$-chromatic graph, then $G = K_t$. 
Proposition - Kawarabayashi, Pedersen, and Toft (2011)

If $G$ is a non-complete, double-critical, $t$-chromatic graph, then

- $\delta(G) \geq t + 1$,
- $\omega(G) \leq t - 2$,
- every edge belongs to at least $t - 2$ triangles, and
- no two vertices of degree $t + 1$ are adjacent.
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Theorem - R. and Song (2017)

If $G$ is a non-complete, double-critical, $t$-chromatic graph, then no vertex of degree $t + 1$ is adjacent to any vertex of degree $\leq t + 3$. 
Proposition - R. and Song (2017)

If $G$ is a non-complete, double-critical, $t$-chromatic, claw-free graph, then $\Delta(G) \leq 2t - 4$. Furthermore, if $d(x) < |V(G)| - 1$, then $d(x) \leq 2t - 6$. 
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Theorem (R. and Song 2017)

If $G$ is a claw-free, double-critical, $t$-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.
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- For $x \in V(G)$, $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$. 
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- For $x \in V(G)$, $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.
- If $t = 7$, then $G$ is 8-regular, contradicting that vertices of degree $t + 1$ are not adjacent.
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- For $x \in V(G)$, $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.
- If $t = 7$, then $G$ is 8-regular, contradicting that vertices of degree $t + 1$ are not adjacent.
- If $t = 8$, then $G$ is 10-regular, and a claw can be found in $N[x]$.
Theorem - Chudnovsky and Seymour (2008)

If $G$ is a claw-free graph, then $G$ is either

- obtained from the icosahedron,
- a circular interval graph,
- antiprismatic,
- covered by three cliques, or
- constructed from certain classes of claw-free “building blocks” in a precise fashion.
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Theorem - Loeb, R., and Yu (2019+)

If $G$ is a non-complete, double-critical, $t$-chromatic, claw-free graph, then $\alpha(G) = 3$. In particular, either $G$ is antiprismatic or $V(G)$ is the union of three cliques.
Each “building block” is a claw free graph
Each block has 1 or two "handles"
- Group together handles
The neighborhoods of each group become cliques
Lemma

If $G$ is a double-critical, claw-free graph constructed in this manner, then there are at most two “handle groups,” and every “handle” belongs to one of these groups.
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If $G$ is a double-critical, claw-free graph constructed in this manner, then there is at most one “building block” which is not a clique.

- Both proofs rely on the property of double-critical graphs that every edge belongs to $t - 2$ triangles.
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A graph is **antiprismatic** if its complement is prismatic.

Equivalently, $G$ is antiprismatic if for every $X \subseteq V(G)$ with $|X| = 4$, there are at least two pairs of vertices in $X$ which are adjacent in $G$. 

Lemma: If $G$ is antiprismatic and double-critical, then $\alpha(G) \leq 3$.

Lemma: If $G$ is the union of three cliques, then $\alpha(G) \leq 3$. 

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**Lemma**

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If $G$ is the union of three cliques, then $\alpha(G) \leq 3$. 
Question

Can it be shown that there are no antiprismatic double-critical graphs or three-cliqued double-critical graphs?
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Can it be shown that there are no double-critical graphs $G$ with $\alpha(G) = 3$?
Thank you!