Strange Quark Stars in 4D Einstein-Gauss-Bonnet Gravity

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\textbf{Abstract:} The existence of strange matter in compact stars may pose striking sequels of the various physical phenomena. Alternative to neutron stars, a new class of compact stars called strange stars should exist if the strange matter hypothesis is true. In the present article, we investigate the possible construction of the strange stars in quark matter phases based on the MIT bag model. We consider scenarios in which strange stars have no crusts. Then we apply two types of the equations of state to quantify the mass-radius diagram for static strange star models performing the numerical calculation to the modified Tolman-Oppenheimer-Volkoff (TOV) equations in the context of 4D Einstein-Gauss-Bonnet gravity. It is worth noting that the GB term gives rise to a non-trivial contribution to the gravitational dynamics in the limit $D \to 4$. We also study the effects of coupling constant $\alpha$ on the physical properties of the constructed strange stars including the energy conditions, velocity of sound, and the adiabatic stability. Finally, we compare our results to those obtained from the standard GR.

\textbf{Keywords:} 4D EGB gravity; Strange matter; Strange stars
1 Introduction

In modern gravity theories, higher derivative gravity (HDG) theories have attracted considerable attention, as an alternative theories beyond GR. Among many impressive outcomes, HDG shows quite different aspects from that in four dimensions, and Einstein-Gauss-Bonnet (EGB) theory [1] is one of them. The EGB theory is a natural extension of GR to higher dimensions, which emerges as a low energy effective action of heterotic string theory [2–4]. As the string theory yields additional higher order curvature correction terms to the Einstein action [5]. Interestingly, the EGB Lagrangian is a linear combination of Euler densities continued from lower dimensions, has been widely studied from astrophysics to cosmology. The EGB theory which contains quadratic powers of the curvature is a special case of Lovelocks’ theory of gravitation (LG) [6, 7] and is free of ghost. In 4D spacetime EGB and GR are equivalent, as the Gauss-Bonnet (GB) term does not give any contribution to the dynamical equations.

According to the recent theoretical developments, Glavan and Lin [8] proposed a 4-dimensional EGB gravity theory by rescaling the coupling constant $\alpha \rightarrow \alpha/(D - 4)$, and then taking the limit $D \rightarrow 4$, a non-trivial black hole solution was found. It was suggested that one can bypass the Lovelock’s theorem and the GB term gives rise to a non-trivial contribution to the gravitational dynamics. However, it seems that regularization procedure was originally be traced back to Tomozawa [9] with finite one-loop quantum corrections to Einstein gravity. One can say that this interesting proposal has opened up a new window
for several novel predictions, though the validity of this theory is at present under debate and doubts. The spherically symmetric black hole solutions and their physical properties have been discussed [8] that claims to differ from the standard vacuum-GR Schwarzschild BH. Furthermore, rotating and non-rotating black hole solutions and their physical properties have been discussed, see [10–18]. Beside that geodesics motion and shadow [19], the strong/weak gravitational lensing by black hole [20–23], spinning test particle [24], thermodynamics AdS black hole [25], Hawking radiation [26, 27], quasinormal modes [28–30], and wormhole solutions [31, 32], were extensively analyzed. It has attracted a great deal of recent attention, see [33–36] for more. More recently, the study of the possible existence of thermal phase transition between AdS to dS asymptotic geometries in vacuum in the context of novel 4D Einstein-Gauss-Bonnet (EGB) gravity has been proposed in Ref.[37].

Hence, the 4D EGB gravity witnessed significant attention that includes finding astrophysical solutions and investigating their properties. In particular, the mass-radius relations are obtained for realistic hadronic and for strange quark star EoS [38]. Precisely speaking, we are interested to investigate the behaviour of compact star namely strange quark stars in regularized 4D EGB gravity. Matter at densities exceeding that of nuclear matter will have to be discussed in terms of quarks. As mentioned in Ref. [39] that for quark matter models massive neutron stars may exist in the form of strange quark stars. Usually the quark matter phase is modeled in the context of the MIT bag model as a Fermi gas of u, d, and s quarks. At finite densities and zero or small temperature, quark matter can exhibit substantial rich phase structures resulting from different pairing mechanisms due to the coupling of color, flavor and spin degrees of freedom, see e.g. Refs. [40–42]. In addition, a variety of different condensates underlying fundamental descriptions may be plausible.

An expectation is that quark matter might play an important role in cosmology and in astrophysics, see e.g. [43, 44]. On the one hand, in cosmology, it may provide an explanation of a source of density fluctuation and as a consequence of how galaxies form generated by the quark-hadron transition. On the other hand, in astrophysics, quark matter is an interplay between general relativistic effects and the equation of state of nuclear particle physics. These objects are present in the form of the stellar equilibrium including neutron stars with a quark core, super massive stars, white dwarfs and even strange quark stars. Nevertheless, in all possible applications of quark matter from cosmology and astrophysics, our lack of knowledge of the exact equation posses the main source of uncertainties in describing stars. In order to study the stable/unstable configurations and even other physical properties of stars, the realistic equations of state (EoS) have to be proposed. The color-flavor locked phase appearing in three flavor (up, down, strange) matter posses the importance of condensates [45–48] and is shown to be the asymptotic ground state of quark matter at low temperature [49]. For instant, the authors of Ref.[50] studied a class of static and spherically symmetric compact objects made of strange matter in the color flavor locked (CFL) phase in 4D EGB gravity.

The structure of the present work is as follows: after the introduction in Sec.1, we quickly review how to derive the field equations in the context of 4D EGB gravity and show that it makes a nontrivial contribution to gravitational dynamics in 4D in Sec.2.
In Sec. 3 we discuss a class of static and spherically symmetric compact objects invoking
the equation of state parameters in quark matter phases invoking massless quark and cold
star approximations. In Sec. 4, we discuss the numerical procedure used to solve the field
equations. In the same section, we report the general properties of the spheres in terms
of the massless quark and cold star approximations. We analyzed the energy conditions
as well as other properties of the spheres, such as sound velocity and adiabatic stability.
Finally, we conclude our findings in the last section.

2 Basic equations of EGB gravity

Let us start from the general action of EGB gravity in
$D$-dimensions and also derive the
equations of motion. The action takes the form
$$I_G = \frac{c^4}{16\pi G} \int d^D x \sqrt{-g} \left[ R + \frac{\alpha}{D-4} L_{GB} \right] + S_{\text{matter}}, \quad (2.1)$$
where $g$ denotes the determinant of the metric $g_{\mu\nu}$ and $\alpha$ is the Gauss-Bonnet coupling
constant. The Ricci scalar $R$ provides the general relativistic part of the action. The
Einstein-Gauss-Bonnet Lagrangian $L_{GB}$ is given by
$$L_{GB} = R_{\mu\nu\sigma\tau} R_{\mu\nu\sigma\tau} - 4 R_{\mu\nu} R_{\mu\nu} + R^2. \quad (2.2)$$

We add also the matter action $S_{\text{matter}}$ which induces the energy momentum tensor $T_{\mu\nu}$.
If the above action, Eq. (2.1), is varied with respect to $g_{\mu\nu}$, one obtains the field equations
$$G_{\mu\nu} + \frac{\alpha}{D-4} H_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.3)$$
with $G_{\mu\nu}$ is the Einstein tensor and $H_{\mu\nu}$ is the contribution of the Gauss-Bonnet term with
the following expression
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad H_{\mu\nu} = 2 \left( R R_{\mu\nu} - 2 R_{\mu\rho} R_{\nu}^{\rho} - 2 R_{\mu\nu\rho\sigma} R_{\rho\sigma} - R_{\mu\nu\rho\delta} R_{\rho\delta}^{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} L_{GB}, \quad (2.4)$$
where $R_{\mu\nu}$ is the Ricci tensor, $R$ and $R_{\mu\rho\nu\sigma}$ are the Ricci scalar and the Riemann tensor,
respectively. Additionally, GB terms is total derivative in 4D space-time, and hence do
not contribute to the field equations. However, it was proposed that re-scaling the coupling constant as $\alpha/(D-4)$, maximally symmetric spacetimes with curvature scale $K$, the
variation of the Gauss-Bonnet [51], yield
$$\frac{g_{\mu\nu} \delta L_{GB}}{\sqrt{-g} \delta g_{\nu\sigma}} = \frac{\alpha (D-2)(D-3)}{2(D-1)} K^2 \delta_{\mu\nu}, \quad (2.5)$$
with this re-scaled coupling constant [8], the Eq. (2.5) does not vanish in $D = 4$ [8].
For solution describing stellar objects, we use the regularization process (see Ref. [8, 52],
where obtained spherically symmetric solutions are also exactly same as of other regularised
theories [35, 53–55].
Here, we consider static spherically symmetric $D$-dimensional metric ansatz with two independent functions of radial coordinate, which is:

$$ds^2_D = -e^{2\Phi(r)}c^2dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\Omega^2_{D-2},$$  

where $d\Omega^2_{D-2}$ is the metric on the unit $(D - 2)$-dimensional sphere and $\Phi(r)$ and $\Lambda(r)$ are functions of $r$, only. The energy momentum tensor $T_{\mu\nu}$ is a perfect fluid matter source and describe the interior of a star, which in this study is written as

$$T_{\mu\nu} = (\epsilon + P)u_\mu u_\nu + P\delta_{\mu\nu},$$  

where $P = P(r)$ is the pressure, $\epsilon = \epsilon(r)$ is the energy density of matter, and $u_\mu$ is the contravariant $D$-velocity. On using the metric (2.6) with stress tensor (2.7), in the limit $D \to 4$, the $tt$, $rr$ and hydrostatic continuity equations (2.3) read:

$$\frac{2}{r} \frac{d\Phi}{dr} = e^{2\Lambda} \left[ \frac{8\pi G}{c^4} \frac{\epsilon - \frac{1 - e^{-2\Lambda}}{r^2}}{1 - \frac{\alpha(1 - e^{-2\Lambda})}{r^2}} \right] \left[ 1 + \frac{2\alpha(1 - e^{-2\Lambda})}{r^2} \right]^{-1},$$  \hspace{1cm} (2.8)

$$\frac{2}{r} \frac{d\Phi}{dr} = e^{2\Lambda} \left[ \frac{8\pi G}{c^4} \frac{P + \frac{1 - e^{-2\Lambda}}{r^2}}{1 - \frac{\alpha(1 - e^{-2\Lambda})}{r^2}} \right] \left[ 1 + \frac{2\alpha(1 - e^{-2\Lambda})}{r^2} \right]^{-1},$$  \hspace{1cm} (2.9)

$$\frac{dP}{dr} = -(\epsilon + P) \frac{d\Phi}{dr}. \hspace{1cm} (2.10)$$

As usual, the asymptotic flatness imposes $\Phi(\infty) = \Lambda(\infty) = 0$ while the regularity at the center requires $\Lambda(0) = 0$. It is advantageous to define the gravitational mass within the sphere of radius $r$, such that $e^{-2\Lambda} = 1 - \frac{2GM(r)}{c^2r}$. Now, we are ready to write the Tolman-Oppenheimer-Volkoff (TOV) equations in a form we want to use. So, using (2.9-2.10), we obtain the modified TOV as

$$\frac{dP}{dr} = -\frac{Ge(r)m(r)}{c^2r^2} \left[ 1 + \frac{P(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4\pi^3P(r)/4\alpha Grm(r)}{c^2r^4} \right] \left[ 1 + \frac{4\alpha Grm(r)}{c^2r^4} \right] \left[ 1 - \frac{2\alpha m(r)}{c^2r^4} \right].$$  \hspace{1cm} (2.11)

If we take the $\alpha \to 0$ limit, the above equation reduces to the standard TOV equation of GR. Replacing the last equality in Eq. (2.8), we obtain the gravitational mass:

$$m'(r) = \frac{6\alpha Grm(r)^2 + 4\pi r^6\epsilon(r)}{4\alpha Grm(r) + c^2r^4},$$  \hspace{1cm} (2.12)

using the initial condition $m(0) = 0$. Then we use the dimensionless variables $P(r) = \epsilon_0 \bar{P}(r)$ and $\epsilon(r) = \epsilon_0 \bar{\epsilon}(r)$ and $m(r) = M_\odot \bar{M}(r)$, with $\epsilon_0 = 1 \text{ MeV/fm}^3$. As a result, the above two equations become

$$\frac{d\bar{P}(r)}{dr} = -\frac{Ge(r)M_\odot \bar{M}(r)}{c^2r^2} \left[ 1 + \frac{\bar{P}(r)}{\bar{\epsilon}(r)} \right] \left[ 1 + \frac{4\pi^3\epsilon_0 \bar{P}(r)/c^2M_\odot \bar{M}(r)}{c^2r^4} \right] \left[ 1 + \frac{4\alpha M_\odot \bar{M}(r)}{c^2r^4} \right] \left[ 1 - \frac{2\alpha M_\odot \bar{M}(r)}{c^2r^4} \right] \left[ 1 + \frac{4\epsilon_0 \bar{M}(r)}{r^4} \right], \hspace{1cm} (2.13)$$

$$\frac{d\bar{\epsilon}(r)}{dr} = \frac{c_1 \bar{\epsilon}(r) \bar{M}(r)}{r^2} \left[ 1 + \frac{\bar{P}(r)}{\bar{\epsilon}(r)} \right] \left[ 1 + \frac{c_2 \pi^3 \bar{P}(r)/M(r)}{r^4} \right] \left[ 1 + \frac{4c_1 \bar{M}(r)}{r^4} \right] \left[ 1 - \frac{2\bar{M}(r)}{r^4} \right], \hspace{1cm} (2.14)$$
where \( c_1 \equiv \frac{GM_\odot}{c^2} = 1.474 \text{ km} \) and \( c_2 \equiv \frac{4\pi\varepsilon_0 M_\odot c^2}{c^2} = 1.125 \times 10^{-5} \text{ km}^{-3} \). The relationship between mass \( M \) and radius \( R \) can be straightforwardly illuminated using Eq. (2.14) with a given EoS. Therefore, the final two Eqs. (2.13) and (2.14) can be numerically solved for a given EoS \( P = P(\epsilon) \). In the next section, we will discuss the strange matter hypothesis.

3 Equation of state and numerical techniques

To understand, what kind of matter compact stars may be built up from, assuming an EoS is the most important step, which encompasses all the information regarding the stellar inner structure. Here, we solve the hydrostatic equilibrium Eq. (2.13)- (2.14) numerically for a specific EoS, \( \epsilon = f(P) \), where \( \epsilon \) is the energy density and \( p \) is the pressure. Since each possible EoS, there is a unique family of stars, parametrized by, say, the central density and the central pressure. The standard procedure is to derive the expressions \( P = f(\rho) \) and \( \epsilon = g(\rho) \), with \( \rho \) being the baryon density, and then obtain an \( \epsilon - P \) pair for every value of \( \rho \). Fitting a curve to this data results in the EoS.

The bag model is a simple tool when we work with quark stars and is invoked in this present work. It is worth noting that in a simple model of free quarks in a bag, analytical expressions of pressure, energy density, and other relevant parameters can be derived. For a Fermi gas of quarks, we can separate into two cases: (1) \( T \neq 0, m = 0 \) and (2) \( T = 0, m \neq 0 \). The pressure, energy density, and baryon number density are given by [56]

\[
P = \sum_f \gamma_f \frac{1}{3} \frac{\gamma_f}{2\pi^2} \int_0^\infty \frac{\partial E_f(k)}{\partial k} [n(k, \mu_f) + n(k, -\mu_f)] k^2 dk - B, \tag{3.1}
\]

\[
\epsilon = \sum_f \gamma_f \frac{1}{2\pi^2} \int_0^\infty E_f(k) [n(k, \mu_f) + n(k, -\mu_f)] k^2 dk + B, \tag{3.2}
\]

\[
\rho = \sum_f \frac{1}{3} \frac{\gamma_f}{2\pi^2} \int_0^\infty [n(k, \mu_f) - n(k, -\mu_f)] k^2 dk, \tag{3.3}
\]

where \( E_f(k) = \left( m_f^2 + k^2 \right)^{1/2} \) is the is the quark kinetic energy,

\[
n(k, \mu_f) = (\exp [E_f(k) - \mu_f] / T + 1)^{-1},
\]

is the Fermi distribution function for temperature \( T \), and the quark degeneracy for each flavor is \( \gamma_f = 2^{\text{spin}} \times 3^{\text{spin}} \), with \( B \) is the MIT Bag constant that represents the positive energy shift per unit volume in the deconfined vacuum relative to the confined vacuum. The factor 1/3 is due to there are three quarks per baryon. Note that the electric charge density can also be computed.
3.1 Massless quark approximation

Here, we consider the first of the limiting cases outlined above. As mentioned in Ref. [56], one can obtain the analytical expressions if quarks are massless while the temperature is finite. Invoking the standard integrals and applying some algebraic computation, we end up with

\[
P = \sum_f \left( \frac{7}{60} \pi^2 T^4 + \frac{1}{2} T^2 \mu_f^2 + \frac{1}{4\pi^2} \mu_f^4 \right) - B, \tag{3.4}
\]

\[
\epsilon = 3p + 4B, \tag{3.5}
\]

\[
\rho = \sum_f \frac{1}{3} \left( T^2 \mu_f + \frac{\mu_f^3}{\pi^2} \right). \tag{3.6}
\]

Interestingly we obtain a simple form of an EoS given in Eq. (3.5). What we have to do next is to solve three equations with four unknown functions, which are \(m(r), \Phi(r), P(r)\) and \(\epsilon(r)\). Notice that the EoS for the massless quark approximations explicitly depends on the bag constant \(B\) and the pressure \(P(r)\). Due to the long range effects of confinement of quarks, the stability of strange quark stars is represented by the bag constant, \(B\). We then consider the re-scaled TOV equations Eq. (2.13) and mass function Eq. (2.14). Therefore, the mass is measured in the solar mass unit \((M_\odot)\), radius in km, while energy density and pressure are in MeV/fm\(^3\). The bag constant \(B\) is also in MeV/fm\(^3\). In the present analysis, we treat the values of \(B\) and \(\alpha\) as free constant parameters. Since, the parameter \(B\) can vary from 57 to 94 MeV/fm\(^3\) [57]. For the study of quark matter with massless strange quark, we consider \(B = 70\) MeV/fm\(^3\).

![Figure 1](image_url)

**Figure 1.** Variation of pressure (left panel) and the energy density (right panel) with radius for the strange quark stars using the massless quark approximation for different values of \(\alpha = 0, \pm 2.5, \pm 5\), where we set \(P(r_0) = 800.00\) MeV/fm\(^3\), \(B = 70.00\) MeV/fm\(^3\), respectively.

Given the set of differential equations (2.13) and (2.14) together with the EoS (3.5), we apply numerical approach for integrating and calculate the maximum mass and other properties of the strange quark matter star. To do so, one can consider the boundary conditions \(P(r_0) = P_c\) and \(M(R) = M\), and integrates Eq. (2.13) outwards to a radius \(r = R\) in which fluid pressure \(P\) vanishes for \(P(R) = 0\). This leads to the strange star
Figure 2. The mass-radius diagram using the massless quark approximation for different values of $\alpha = 0, \pm 2.5, \text{and } \pm 5$, where we set $P(r_0) = 800.00 \text{MeV/fm}^3, B = 70.00 \text{MeV/fm}^3$, respectively. The curve corresponds to $\alpha = 0$ is representing GR case (solid black lines).

radius $R$ and mass $M = m(R)$. The initial radius $r_0 = 10^{-5}$ and mass $m(r_0) = 10^{-30}$ are set to very small numbers rather than zero to avoid discontinuities, as they appear in denominators within the equations.

We start from the center of the star for a certain value of central pressure, $P(r_0) = 800 \text{MeV/fm}^3$ and the radius of the star is identified when the pressure vanishes or drops to a very small value. For such a choice, we plot pressure and density versus distance from the center of strange star (see Fig. 1). At that point we recorded the mass-radius relation of the star in Fig. 2. As one can see, the mass-radius ($M - R$) relation depends on the choice of the value of coupling constant $\alpha$. For $\alpha > 0$ the mass of star for given radius increases with fixed value of $B$. In all the presented cases, we can note that there is significantly different for positive and negative values of $\alpha$, but $\alpha = 0$ case is equivalent to pure general relativity. Moreover, it can be seen from Table 1 and comparing the results to GR, one may obtain maximum mass for strange stars with positive $\alpha$. Therefore, we argue that a confirmed determination of a compact star with $2M_\odot$, which are actually very close to the ones of realistic neutron star models [58].

3.2 Cold star approximation

This section contains a discussion of the zero temperature ($T = 0$) and $m \neq 0$. Then the expressions given in Eqs. (3.1-3.3) can be further simplified. To be more accuracy, we add the electrons to the system with their statistical weights (= 2) due to the spin. Performing
the standard calculations, we obtain [56]

\[
P = -B + \sum_f \left[ \frac{1}{4\pi^2} \left( \mu_f k_f \left( \frac{\mu^2_f - 5}{2m^2_f} \right) + \frac{3}{2} m^4_f \ln \left( \frac{\mu_f + k_f}{m_f} \right) \right) \right]
\]

\[
+ \frac{1}{12\pi^2} \left[ \mu_e k_e \left( \frac{\mu^2_e - 5}{2m^2_e} \right) + \frac{3}{2} m^4_e \ln \left( \frac{\mu_e + k_e}{m_e} \right) \right],
\]

(3.7)

\[
\epsilon = B + \sum_f \left[ \frac{3}{4\pi^2} \left( \mu_f k_f \left( \frac{\mu^2_f - 1}{2m^2_f} \right) + \frac{1}{2} m^4_f \ln \left( \frac{\mu_f + k_f}{m_f} \right) \right) \right]
\]

\[
+ \frac{1}{4\pi^2} \left[ \mu_e k_e \left( \frac{\mu^2_e - 1}{2m^2_e} \right) + \frac{1}{2} m^4_e \ln \left( \frac{\mu_e + k_e}{m_e} \right) \right],
\]

(3.8)

\[
\rho = \sum_f \frac{k^3_f}{3\pi^2},
\]

(3.9)

where \( k_f \) is the Fermi momentum for flavor \( f \) with \( k_f = \left( \frac{\mu^2_f - m^2_f}{2} \right)^{1/2} \) and \( k_e = \left( \frac{\mu^2_e - m^2_e}{2} \right)^{1/2} \). Notice that there are four independent variables appeared in the above equations, i.e. \( \mu_u, \mu_d, \mu_s \) and \( \mu_e \). Strange stars are composed of \( uds \) quarks. Hence, we constrain the chemical potentials of the quarks to a single independent variable \( \mu \) such that \( \mu_d = \mu_s = \mu \) and \( \mu_u + \mu_e = \mu \). Thus, the two independent variables \( \mu \) and \( \mu_e \), two equations are necessary to produce a set of chemical potentials and solve the system for a pair of values for \( \epsilon \) and \( P \).

It seems like we end up with the higher order/nonlinear EoS and the TOV equations are very hard to be solved in this particular situation. This is because the quark chemical potentials increase when we increase the baryon number density, while the electron chemical potential is neglectable. Therefore, from Eqs. (3.7-3.8) we only focus a simple linear relationships of \( \epsilon \) and \( P \), which can be simplified to \( \epsilon = 3.05P(r) + 368.00 \), see Appendix A for more details.

![Figure 3](image)

**Figure 3.** Variation of pressure (left panel) and the energy density (right panel) with radius for the strange quark stars using the cold star approximation for different values of \( \alpha = 0, \pm 2.5, \) and \( \pm 5 \), where we set \( P(r_0) = 800.00 \text{ MeV/fm}^3, B = 70.00 \text{ MeV/fm}^3 \), respectively.
Figure 4. Unification diagram for the variation of mass with radius (left panel) and the relations between the maximum mass and radius of the strange quark stars using the cold star approximation with $\alpha = 0, \pm 2.5, \pm 5$ and parameters $P(r_0) = 800.00 \text{ MeV}/\text{fm}^3, B = 70 \text{ MeV}/\text{fm}^3$.

Figure 5. Unification diagram for mass versus central energy density with variation of $\alpha = 0, \pm 2.5$ and $\pm 5$. The left panel is for the massless quark star while the right panel is for the cold star case.

The input data for the numerical calculation are similar to aforementioned. The pressure and density versus radial distance from the center of cold star i.e. quark matter at zero temperature are represented in Fig. 3. All curves in Fig. 3, note that the pressure and density are maximum at the center and decrease monotonically towards the boundary. In turn, to study the mass-radius relation and the mass vs. central density for cold quark matter EoS are given for 5 representative values of $\alpha$ in Figs. 4 and 5, respectively. For a given central density, the star mass grows with increasing $\alpha$. The maximum mass increases with increasing value of $\alpha$ and we find that, for $\alpha = 5$, the maximum mass becomes $M_{\text{max}} = 1.80 M_\odot$. At that point we recorded the mass of the star, $1.52 M_\odot$ when $\alpha = 0$ in GR. For more clarity, the properties of stars with maximal mass are tabulated in Tables 1 and are compared to GR ($\alpha = 0$). Finally, we compare between the two mass-radius relationships (see Fig. 6). Interestingly, all values of $M_{\text{max}}$ for massless quarks are higher than Chandrasekhar limit, which is about $1.4 M_\odot$. 

-- 9 --
Massless Quark | Cold Star
---|---
| \( \alpha \) | \( M_{\text{max}} \) (\( M_\odot \)) | \( R \) (km) | \( \epsilon_c \) (10\(^{15}\) g/cm\(^3\)) | \( M_{\text{max}} \) (\( M_\odot \)) | \( R \) (km) | \( \epsilon_c \) (10\(^{15}\) g/cm\(^3\))
-5.0 | 1.50 | 8.98 | 4.78 | 1.24 | 7.78 | 5.02
-2.5 | 1.61 | 9.04 | 4.78 | 1.38 | 7.92 | 5.02
0 | 1.74 | 9.14 | 4.78 | 1.52 | 8.09 | 5.02
2.5 | 1.87 | 9.25 | 4.78 | 1.66 | 8.27 | 5.02
5.0 | 1.99 | 9.37 | 4.78 | 1.80 | 8.44 | 5.02

Table 1. We summarize the parameters of the strange quark stars using various values of the 4D EGB coupling constant, \( \alpha \). We show the maximum mass of the stars \( M \) in a unit of the solar mass \( M_\odot \) with their radius \( R \) in km and the central energy density \( \epsilon_c \) for both the massless quark model and the cold star approximation.

Figure 6. Figures display the variation of mass (M) with star’s radius (R) of the two types of EoS: massless quark (Mlq) and cold stars (Cs) with \( \alpha = 0, \pm 2.5, \) and \( \pm 5 \).

4 Structural properties of strange stars

For completeness, we would also like to explore the physical properties in the interior of the fluid sphere.

4.1 Energy conditions

In this section, we will discuss the energy conditions (ECs). The ECs are local inequalities, depending on \( T_{\mu\nu} \), that capture the idea of energy should be positive for strange star. Most notable weak energy condition (WEC), i.e. \( T_{\mu\nu}U^\mu U^\nu \), requires that

\[
\epsilon(r) \geq 0 \quad \text{and} \quad \epsilon(r) + P(r) \geq 0,
\]
where $U^{\mu}$ is a timelike vector and follows that if WEC is satisfied then NEC also satisfied. NEC is the assertion that for any null vector $k^{\mu}$, we should have $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$. In addition to its role in constraining the set of physical spacetimes, the NEC demands $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for null vector $k^{\mu}$. Since, the NEC is the simplest energy condition to deal with algebraically.

Whereas, the strong energy condition (SEC) asserts that $(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) U^{\mu}U^{\nu} \geq 0$ for any timelike vector $U^{\mu}$. The SEC asserts that gravity is attractive,

$$\epsilon(r) + \sum P_i(r) \geq 0, \quad \Rightarrow \quad \epsilon(r) + 3P(r) \geq 0.$$ (4.2)

Note that the SEC does not imply the WEC, but it follows that any violation of the NEC also violates the SEC and WEC. Finally, using the inequalities (4.1-4.2), we plot Fig. 7 for different values of $\alpha$. We see from Fig. 7 that all energy conditions are satisfied, and our assumed EoS is suitable for modelling viable strange stars.

![Figure 7](image)

**Figure 7.** Plots for energy conditions, WEC and SEC. The results from the massless quark approximation are displayed in the upper panels, while those from the cold star approximation are illustrated in the lower panels. Here, we have varied the values of the 4D EGB coupling $\alpha$ such that $\alpha = 0, \pm 1, \pm 3,$ and $\pm 5$, where the central pressure is $P(r_0) = 800\text{MeV/fm}^3$. We have considered the bag parameter for the massless quark approximation and the cold star approximation $B = 70.00\text{MeV/fm}^3$.

### 4.2 Speed of sound and Le Chatelier’s principle

In a perfect fluid described by an EoS of the form $P = P(\epsilon)$, we are going to consider the speed of sound propagation, $c^2 = dP/d\rho$. In order to preserve the causality, one expects that the velocity of sound ($v$) should be less than the lights velocity ($c$). Thus one can put a constraint using the following expression $0 \leq v^2 = dP/d\rho \leq c^2$, i.e $dP/d\epsilon \leq 1$. 

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Now, we are going to use Eqs. (3.5) and (3.8) for obtaining the speed of sound inside the fluid sphere. It is seen that for massless case $dP/d\epsilon \approx 0.333$, whereas for cold star we obtain $dP/d\epsilon \approx 0.328$. We can, then, say that the speed of sound is constant everywhere within the strange star and less than unity.

In [59], it was suggested that matter of star satisfies $dP/d\epsilon \geq 0$ which is a necessary condition that a body is stable both as a whole, and also with respect to the non-equilibrium elementary regions with spontaneous contraction or expansion (Le Chatelier’s principle). The results of such calculations for the given parameters $P$ and $\epsilon$ are shown in Fig. 8. Thus, Le Chatelier’s principle is established.

4.3 The stability criterion and the adiabatic indices

Now we will discuss the adiabatic index ($\gamma$) based on our theoretical quark matter models. The adiabatic index is related to the thermodynamical quantity. In order to solve the instability problem, Chandrasekhar [60] introduced a criterion for dynamical stability based on the variational method. To be more specific, the final expression for the adiabatic index reads

$$\gamma \equiv \left(1 + \frac{\epsilon}{P}\right) \left(\frac{dP}{d\epsilon}\right)_S,$$

where $dP/d\epsilon$ is the speed of sound in units of speed of light and the subscript $S$ indicates the derivation at constant entropy. The Eq. (4.3) is a dimensionless quantity measuring the stiffness of the EoS.

Therefore, we need to study the effects of $\gamma$ corresponding to the respective EoSs. In particular, the EoS related to neutron star matter, this value lies between 2 to 4 [61]. In [62], authors have shown that $\gamma$ should exceed 4/3 in a stable polytropic star by an amount that depends on that ratio $\epsilon/P$ at the centre of the star. In resent studies it was shown that for dynamical stability $\gamma$ should be more than 4/3 (i.e $\gamma > 1.33$). In Table 1, we present a set of parameters used in the present work, and Fig. 9 shows that stellar model is stable against the radial adiabatic infinitesimal perturbations. Additionally, increasing values of $\gamma$ mean the growth of pressure for a given increase in energy density, i.e. a stiffer
EoS. In view of the above, we conclude that massless quark matter gives more stiffer EoS compare to cold and dense quark matter.

![Figure 9](image)

Figure 9. Figures show the adiabatic index, $\gamma$, of the strange quark stars using the massless quark approximation (left panel) and the cold star approximation (right panel) with various values of $\alpha = 0, \pm 2.5$ and $\pm 5$. The green horizontal line represents a value of $\gamma$ at $\gamma = 4/3$.

5 Conclusions and astrophysical implications

In this paper, we investigate the features of 4D Einstein-Gauss-Bonnet gravity in an extreme circumstances such as those arising within highly compact static spherically symmetric bodies. We considered the self-bound strange matter hypothesis. The interesting part of this theory is that the resulting regularized 4D EGB gravity has nontrivial dynamics and free from the Ostrogradsky instability.

There exist considerable evidences that the possible existence of compact stars are partially or totally made up of quark matter. But the existence of quark stars is still controversial and its EoS is also uncertain. Here, we first considered the static spherically symmetric $D$-dimensional metric and derived corresponding field equations taking a limit of $D \to 4$ at the level of field equations. We then numerically solved field equations for strange matter hypothesis. To clarify the astrophysical implications of our work, we discuss two important scenarios. Firstly, we considered quark matter phases consisting of massless quarks, and secondly quark matter at zero temperature.

To gain better understanding of the physical properties, we quantified the maximal mass from the central density and mass-radius relation of the stellar structure. The mass-radius results are graphically shown which strictly depends on the values of the coupling constant and the chosen EoS. Then, we showed that for $\alpha \to 0$ limit, the obtained TOV in 4D EGB gravity reduces to the standard Einstein theory, and solutions are compared in Table 1. The main difference between the two classes is that the maximum stable mass of the massless quark star is almost the same as that for a cold quark star for increasing value of $\alpha$. Since negative $\alpha$ reduces the maximum mass of a compact star for a given EoS. Furthermore, we obtained the interesting results of their physical properties such as velocity of sound, energy conditions and adiabatic stability. It is worth noting that the obtained solutions are in favour of the physical requirements (see from Figs. 7-9).
Finally, it is notable that the investigation for other compact objects such as neutron star and white dwarf using the same context and its modified TOV equation are interesting subjects. However, we will leave these interesting topics for our future work.

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A Linear approximation of EoS for the cold star approximation

In order to study the effect of the non-vanishing strange quark mass on the EoS for uncharged quark matter with two massless flavors (up & down quarks) and one massive flavor (strange quark), we have to include electrons with the chemical potential \( \mu_e \). We impose the chemical equilibrium of three quarks which maintain the following interactions [63]:

\[
\begin{align*}
  d &\rightarrow u + e + \bar{\nu}_e, \\
  u + e &\rightarrow d + \nu_e, \\
  s &\rightarrow u + e + \nu_e, \\
  u + e &\rightarrow s + \nu_e, \\
  s + u &\rightarrow d + u.
\end{align*}
\]  

(A.1)

As mentioned in Ref. [63], neutrinos are rapidly lost, so one may set their chemical potential to zero. As a result, we assume the chemical potentials of the above system as a single independent variable, \( \mu \):

\[
\mu_d = \mu_s = \mu, \quad \mu_u + \mu_e = \mu,
\]  

(A.2)

where \( \mu_d, \mu_s \) and \( \mu_u \) are the up, down and strange quark chemical potentials, respectively. The EoS describes the relation between the energy density \( \epsilon \) and pressure \( P \) for the above system which have been already given in Eqs. (3.7-3.8). Basically, the pressure and the energy density are solved in terms of the baryon density, \( \rho \). For the cold star approximation, these three variables, \( \epsilon, P, \rho \), are written in terms of the quark mass \( m_f \) and chemical potential \( \mu_f \). The charge neutrality of quarks taking account of electrons must satisfy the following condition:

\[
\tilde{q} = \sum_f \left( \tilde{q}_f \frac{k_f^3}{\pi^2} \right) - \frac{k_e^3}{3\pi^2} = 0.
\]  

(A.3)

In order to obtain the EoS, we have to solve the system of equations; Eqs. (3.7-3.9) and Eq. (A.3). What we have to do first is to quantify the relations between the chemical potentials \( \mu_u, \mu_d, \mu_s, \mu_e \) and the baryon density \( \rho \). In this step, we use the numerical calculation considering the baryon density and the conservation of charge of quarks; Eq. (3.7) and Eq. (A.3), respectively.
It is worth noting that the effects of the finite strange quarks mass on the energy density ($\epsilon$) and the pressure ($P$) for neutral quark matter including electrons have been already examined in Ref. [63]. The results showed that there was a sizable difference in the energy density and the pressure between zero strange quark mass and non-vanishing strange quark mass. However, the EoS in this case basically shows a non-linear behavior between $\epsilon$ and $P$, and as a result this non-linearity is very hard to be solved.

Fortunately, it has been also noticed from Ref. [63] that the resulting EoS $\epsilon = \epsilon(P)$ can be approximated by a non-ideal bag model which is written in the following form:

$$\epsilon = aB + bP, \quad (A.4)$$

with $a$ and $b$ being arbitrary constants. In our case, we find that $\epsilon = 3.05P + 368$ which we have used $B = 70$ MeV/fm$^3$. Using the numerical calculations, the chemical potentials and the number density $\rho$ are simultaneously obtained. After substituting the results into Eq. (3.7) and Eq. (3.8), we finally end up with the EoS displaying a relationship between the energy density and pressure. However, the linear behavior is maintained by the relation given in Eq.A.4.

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