A LOCAL FORM OF THE AUTOMORPHISMS OF THE SPECTRAL BALL

LUKASZ KOSIŃSKI

Abstract. We show that the group generated by shears and overshears is dense in the group of automorphisms of the spectral ball $\Omega_2$. Moreover, it is shown that any automorphism of $\Omega_2$ is a local holomorphic conjugation.

1. Introduction and statement of the result

Let $\Omega_n$ denote the spectral ball in $\mathbb{C}^{n^2}$, that is a domain composed of $n \times n$ complex matrices whose spectral radius is less than 1.

The natural question that arises is to classify its group of automorphisms. It may be easily checked that among them there are the following three forms:

(i) Transposition: $\tau : x \mapsto x^t$,
(ii) Möbius maps: $m_{\alpha,\gamma} : x \mapsto \gamma(x-\alpha)(1-\alpha x)^{-1}$, where $\alpha$ lies in the unit disc and $|\gamma|=1$,
(iii) Conjugations: $J_u : x \mapsto u(x)^{-1}xu(x)$, where $u : \Omega \to M_n^{-1}$ is a conjugate invariant holomorphic map, i.e. $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and $q \in M_n^{-1}$

(throughout the paper $M_n^{-1}$ denotes the group of invertible $n \times n$ complex matrices).

Ransford and White have asked in [Ran-Whi] whether the compositions of the three above forms generate the whole Aut($\Omega$) - the group of automorphisms of the spectral ball. In [Kos] we have shown that the answer to this question is negative. Nevertheless the question about the description of Aut($\Omega$) remains open. In this note we deal with that problem.

Let us introduce some notation. $\mathcal{M}_n$ denotes the algebra of $n \times n$ matrices with complex coefficients. Moreover, $\mathcal{T}_n$ is the set of cyclic matrices lying in $\Omega_n$. In the case $n = 2$ we shall simply write $\Omega = \Omega_2$, $\mathcal{M} = M_2$ and $\mathcal{T} = T_2$.

Definition 1. Let $U$ be an open subset of the spectral ball $\Omega$. We shall say that a mapping $\varphi : U \to M_n$ is a holomorphic conjugation if there is a holomorphic mapping $p : U \mapsto M_n^{-1}$ such that $\varphi(x) = p(x)xp(x)^{-1}$, $x \in U$.

Moreover, we shall say that $\varphi$ is a local holomorphic conjugation if any $x \in U$ has a neighborhood $V$ such that $\varphi$ restricted to it is a holomorphic conjugation.

The paper is organized as follows. Recall that we presented in [Kos] two counterexamples to the question on the description of the group of automorphisms of the spectral ball. We focus on them in the sense that we classify all automorphisms of the spectral ball of the form

$\Omega \ni x \mapsto u(x)xu(x)^{-1} \in \Omega$.

Key words and phrases. Spectral unit ball, Danielewski surface, automorphisms.
where \( u \in \mathcal{O}(\Omega, M^{-1}) \) is such that \( u(x) \) is either diagonal or triangular, \( x \in \Omega \) (we shall call the diagonal and triangular conjugations respectively).

It is known that any automorphism of the spectral ball fixing the origin preserves the spectrum. Therefore, trying to describe the group \( \text{Aut}(\Omega) \) it is natural to investigate the behavior of automorphisms restricted to the fibers of \( \Omega \), i.e. sets of the form \( \mathcal{F}_{(\lambda_1, \lambda_2)} := \{ x \in \Omega : \sigma(x) = \{ \lambda_1, \lambda_2 \} \} \), \( \lambda_1, \lambda_2 \in \mathbb{D} \), where \( \sigma(x) \) denotes the spectrum of \( x \in M \). If \( \lambda_1 \neq \lambda_2 \), the fiber \( \mathcal{F}_{(\lambda_1, \lambda_2)} \) forms a submanifold known as the Danielewski surfaces. Recall that the Danielewski surface associated with a polynomial \( p \in \mathbb{C}[z] \) is given by

\[
D_p := \{(x, y, z) \in \mathbb{C}^3 : xy = p(x)\}
\]

and its complex structure is naturally induced from \( \mathbb{C}^3 \). Algebraic properties of Danielewski surfaces have been intensively studied in the literature. Anyway a little is known about their holomorphic automorphisms. It was lastly shown (see [Kal-Kut] and [Kut-Lin]) that the group generated by shears and overshears (for the definition see e.g. [Lin]) is dense in the group of holomorphic automorphisms. It was a little surprise to us that shear are just the restriction to the fiber of diagonal and triangular conjugations.

Following the idea idea from [Kut-Lin] (we recall all details for the convenience of the reader) we shall show that the spectral ball satisfies the property obtained by Andersén and Lempert for holomorphic automorphisms of \( \mathbb{C}^n \) (see [And], [And-Lem] and [For-Ros]). To be more precise we shall show that the group generated by triangular and diagonal conjugations is dense in \( \text{Aut}(\Omega) \) (in the local-uniform topology).

Finally, we shall show that the uniform limit of conjugations is a local conjugation in a neighborhood of \( T \). This, together with the density of the the group generated by triangular and diagonal conjugations and the results of P.J. Thomas and J. Rostand (see [Pas] and [Ros]) imply that any automorphism of the spectral ball is a local conjugation.

## 2. Diagonal and triangular conjugations

Let us focus on conjugations of the following form:

\[
(1) \quad \tilde{D}_a x \mapsto \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix} x \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix}^{-1}
\]

and

\[
(2) \quad T_b : x \mapsto \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix}^{-1}
\]

Our aim is to describe \( a \) and \( b \) such that the above mappings are automorphisms of the spectral unit ball.

The case (1) is easy. First note that the simply-connectedness of \( \Omega \) imply that there is \( \tilde{a} \in \mathcal{O}(\Omega) \) such that \( a = \exp(\tilde{a}/2) \). Moreover, for fixed \( x_{11} \) and \( x_{22} \) the mapping \((x_{12}, x_{21}) \mapsto (\exp(\tilde{a}(x)) x_{12}, \exp(-\tilde{a}(x)) x_{21})\) is injective on its domain. In particular, the mapping

\[
z \mapsto \exp(\tilde{a}(x_{11}, z, t/z, x_{22})) z
\]
is an automorphism of $\mathbb{C}_s$ for $t$ sufficiently small. Therefore $z \mapsto \tilde{a}(x_{11}, z, t/z, x_{22})$ is constant, whence $\tilde{a}$ depends only on $x_{11}$, $x_{22}$ and $x_{12}x_{21}$.

Now we focus our attention on (2). We want to find $b$ such that

$$x \mapsto \begin{pmatrix} x_{11} - b(x)x_{12} \\ b(x)x_{11} + x_{21} - b^2(x)x_{12} + b(x)x_{22} \\ b(x)x_{12} + x_{22} \end{pmatrix}$$

is an automorphism of $\Omega$. Putting $(s, p) := (\text{tr}x, \text{det}x) \in \mathbb{G}_2$ (throughout the paper $\mathbb{G}_2$ denotes the symmetrized bidisc - see e.g. [Edi-Zwo]) and looking at the automorphism restricted to the fibers (it is obvious that conjugations preserve fibers) we get that the mapping

$$(x_{11}, x_{12}) \mapsto (x_{11} - x_{12}b\left(\begin{array}{cc} x_{11} & x_{12} \\ (x_{11}(s - x_{11}) - p)x_{12} & s - x_{11} \end{array}\right)\), x_{12})$$

is an automorphism of $\mathbb{C} \times \mathbb{C}_s$. It is quite easy to observe that if $(x, y) \mapsto (x - f(x, y), y)$ is an automorphism of $\mathbb{C} \times \mathbb{C}_s$, iff $f(x, y) = x(1 - c(y)) - y(g(y), x \in \mathbb{C}, y \in \mathbb{C}_s$, where $c \in \mathcal{O}(\mathbb{C}_s, \mathbb{C}_s)$ and $g \in \mathcal{O}(\mathbb{C}_s, \mathbb{C})$.

Applying this reasoning to $b$ we simple find that there are $c \in \mathcal{O}(\mathbb{C}_s \times \mathbb{G}_2, \mathbb{C}_s)$ and $g \in \mathcal{O}(\mathbb{C}_s \times \mathbb{G}_2)$ such that

$$b(x) = x_{11}\frac{1 - c(x_{12}, \text{tr}x, \text{det}x)}{x_{12}} + \frac{\gamma(x_{12}, \text{tr}x, \text{det}x)}{x_{12}}.$$

Putting $x_{11} = 0$ we see that $\gamma(x_{12}, s, p) = x_{12}\beta(x_{12}, s, p)$, where $\beta \in \mathcal{O}(\mathbb{C} \times \mathbb{G}_2)$. Using this we see that $c$ may be extend holomorphically through $x_{12} = 0$. Moreover, one may easily check that $c(x_{12}, s, p) = e^{x_{12}\alpha(x_{12}, \text{tr}x, \text{det}x)}$, where $\alpha \in \mathcal{O}(\mathbb{C} \times \mathbb{G}_2)$. Thus

$$(3) \quad b(x) = x_{11}\frac{1 - \exp(x_{12}\alpha(x_{12}, \text{tr}x, \text{det}x))}{x_{12}} + \beta(x_{12}, \text{tr}x, \text{det}x).$$

**Remark 1.** Note that $T_b$ is generated by $T_\beta$ and $T_\gamma$, $\gamma(x) = x_{11}\frac{1 - \exp(x_{12}\alpha(x_{12}, \text{tr}x, \text{det}x))}{x_{12}}$, where $\alpha$ and $\beta$ satisfy (3).

### 3. Vector fields generated by triangular and diagonal conjugations and relations between them

To describe the vector fields generated by (1) and (2) it is convenient to introduce the following notation:

- $D_\alpha := \tilde{D}_{\exp(\alpha/2)}$.
- $T'_\alpha := T_b$, where $b(x) = x_{11}\frac{1 - \exp(x_{12}\alpha(x_{12}, \text{tr}x, \text{det}x))}{x_{12}}$.

Note that the mappings $D_\alpha$, $T_\beta$ and $T'_\alpha$ generate the following vector fields (i.e. $\frac{d}{dt}\Phi_t(\Phi^{-1}_t(x))|_{t=t_0}$, where $\Phi_t \in \{D_{t\alpha}, T_{t\beta}, T'_{t\alpha}\}$):
\[ HD_a = a x_{12} \frac{\partial}{\partial x_{12}} - a x_{21} \frac{\partial}{\partial x_{21}}, \]

\[ HT_\beta = -\beta x_{12} \frac{\partial}{\partial x_{11}} + (\beta x_{11} - \beta^2 x_{12} + \beta x_{22}) \frac{\partial}{\partial x_{21}} + \beta x_{12} \frac{\partial}{\partial x_{22}}, \]

\[ HT'_\alpha = x_{11} x_{12} \alpha \frac{\partial}{\partial x_{11}} + (x_{22} - x_{11}) x_{11} \alpha \frac{\partial}{\partial x_{21}} - x_{11} x_{22} \alpha \frac{\partial}{\partial x_{22}}. \]

Additionally \( \tau \circ T_\beta \circ \tau \) and \( \tau \circ T'_\alpha \circ \tau \) (where \( \tau(x) = x^t \) is a transposition) generate \[ \tilde{HT}_\beta = -\beta x_{21} \frac{\partial}{\partial x_{11}} + (b x_{11} - b^2 x_{21} + b x_{22}) \frac{\partial}{\partial x_{12}} + b x_{21} \frac{\partial}{\partial x_{22}}, \]

\[ \tilde{HT}'_\alpha = x_{11} x_{12} \alpha \frac{\partial}{\partial x_{11}} + (x_{22} - x_{11}) x_{11} \alpha \frac{\partial}{\partial x_{12}} - x_{11} x_{22} \alpha \frac{\partial}{\partial x_{22}}. \]

Let us write the above vector fields in ”spectral” coordinates

\[ (x, y, s, p) = (x_{11}, x_{12}, \text{tr} x, \det x). \]

Observe that for any vector field \( V \) on \( \Omega \) orthogonal to \( \det x \) and \( \text{tr} x \) its divergence \( \text{div} V \) (in Euclidean coordinates) is equal to \( \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} - \frac{v_3}{y} \), where \( V = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} \). We get:

\[ (4) \quad HD_a = a y \frac{\partial}{\partial y}, \quad HT_\beta = -\beta y \frac{\partial}{\partial x} \quad \text{and} \quad HT'_\beta = xy \beta \frac{\partial}{\partial x}. \]

A straightforward calculation leads to:

\[ [HD_a, HT_b] = ya(yb)' \frac{\partial}{\partial y} - y^2 a'_x b \frac{\partial}{\partial y}, \quad \text{div}[HD_a, HT_b] = 0, \]

\[ [HD_a, HT'_b] = xy a(yb)' \frac{\partial}{\partial y} - xy^2 a'_x b \frac{\partial}{\partial y}, \quad \text{div}[HD_a, HT'_b] = ay(yb)'. \]

\[ \tilde{HT}_b = x^2(s-x)^2 y b \frac{\partial}{\partial x} + (s-2x)x b \frac{\partial}{\partial y}. \]

Putting \( b = 1 \) we see moreover that \([HD_a, HT_1], \tilde{HT}'_b] = -a(p-x(s-x)) \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \) for some \( v \).

4. DENSITY OF TRIANGULAR AND DIAGONAL CONJUGATIONS THE SPECTRAL UNIT BALL IN \( \mathbb{C}^{2 \times 2} \)

**Proposition 2.** The group generated by finite compositions of the transposition, M"obius maps, \( T_b \) and \( D_a \), where \( a, b \) are described above is dense in the group of holomorphic automorphisms of the spectral unit ball.

**Proof.** Let \( \varphi \) be an automorphism of \( \Omega \). Composing it, if necessary, with the M"obius map one may assume that \( \varphi(0) = 0 \) (see [Ran-Whi] and [Edi-Zwo]). Then \( \varphi'(0) \) is also an automorphism of the spectral ball. By [Ran-Whi] there is an invertible matrix \( a \) such that either \( \varphi'(0)(x) = axa^{-1}, x \in \Omega \) or \( \varphi'(0)(x) = ax^t a^{-1}, x \in \Omega \). Losing no generality assume that the first possibility holds. Since any invertible matrix may be represented as a finite product of triangular and diagonal matrices \( \varphi'(0) \) satisfies trivially the assertion.
Note that \( t \mapsto \Phi_t := t^{-1} \varphi(t \cdot) \) is a well defined mapping from \( \mathbb{R} \) into \( \text{Aut}(\Omega) \). Since any automorphism of \( \Omega \) fixing the origin preserves the spectrum we see that \( \sigma(\Phi_t(x)) = \sigma(x) \), \( t \in \mathbb{R}, \ x \in \Omega \) (\( \sigma(x) \) denotes the spectrum of a matrix \( x \)).

Consider the following vector field:

\[
X_{t_0}(x) := \frac{d}{dt} (\Phi_t(\Phi_{t_0}^{-1}(x)))|_{t=t_0}, \quad x \in \Omega.
\]

It is clear that \( \Phi_{t_0} \circ \Phi_{t_0}^{-1} \) is obtained by integrating \( X_t \) from 0 to \( t_0 \). Since \( \Phi_t \) preserves the spectrum it follows that \( X_{t_0}(\det x) = 0 \) and \( X_{t_0}(\tr x) = 0 \).

We proceed as in [Lin]. \( X - t \) may be approximated on compact sets by integrating the time dependent vector field \( X_{k t_0/N} \) from time \( k t_0/N \) to \( (k + 1)t_0/N \). It is clear that every \( X_{k t_0/N} \) may approximated by polynomial vector fields \( X' \) such that \( X'(\det x) = 0 \) and \( X'(\tr x) = 0 \). We shall show that \( X' \) is a Lie combination of polynomial vector fields whose flows are conjugations. Then the standard argument (called sometimes the Euler’s method) would imply that we could approximate \( \Phi_1 \circ \Phi_{t_0}^{-1} \) by compositions of conjugations, thus we would be able to approximate \( \Phi_1 \) as well.

Let us denote

\[
X' = v_1 \frac{\partial}{\partial x_{11}} + v_2 \frac{\partial}{\partial x_{12}} + v_3 \frac{\partial}{\partial x_{21}}.
\]

Assumptions on \( X_{t_0} \) imply that \( v_4 = -v_1 \) and \( v_1(x_{22} - x_{11}) = v_2 x_{21} + v_3 x_{12} \). Note that \( v_1 \) may be written as

\[
(5) \quad v_1 = \sum_{j=1}^{n} x_{12}^j f_j(x_{11}, x_{22}, x_{12} x_{21}) + \sum_{j=1}^{n} x_{21}^j g_j(x_{11}, x_{22}, x_{12} x_{21}) + \varphi(x_{11}, x_{22}, x_{12} x_{21})
\]

for some polynomials \( f_j, g_j, \varphi \in \mathbb{C}[x_{11}, x_{22}, x_{12} x_{21}] \).

It is easily seen that \( \varphi(x_{11}, x_{22}, 0) = 0 \), so \( \varphi(x_{11}, x_{22}, x_{12} x_{21}) = x_{12} x_{21} \alpha(x, s, p) \). Adding to \( X' \), if necessary, \( [[HD_α, HT^0], HT^0] \) with suitable chosen \( α \) we may assume that \( \varphi = 0 \).

Now adding vector fields of the form \([HD_α, HT^0] \) and \([HD_α, HT^0] \) we may assume that \( \text{div} X' = \psi(x_{12} x_{21}, x_{11}, x_{22}) \) for some polynomial \( \psi \). Again, adding to \( X' \), if necessary, vector fields of the forms \([HD_α, HT^0] \) and \([HD_α, HT^0] \) we may assume that \( v_1 = 0 \).

Thus, up to adding Lie combinations of the vector fields generated by triangular and diagonal conjugations we may assume that

\[
X' = v_2 \frac{\partial}{\partial x_{12}} + v_3 \frac{\partial}{\partial x_{21}}
\]

and \( X'(\det x) = 0 \) and \( \text{div} X' = \psi(x_{12} x_{21}, x_{11}, x_{22}) \). The second condition means that \( x_{21} v_2 + x_{12} v_3 = 0 \) so \( v_2 = x_{12} w, \ v_3 = -x_{21} w \) for some polynomial \( w \). In particular, \( \text{div} X' = x_{12} \frac{\partial w}{\partial x_{12}} - x_{21} \frac{\partial w}{\partial x_{21}} \). Let us write \( w = \sum x_{12}^j x_{21}^k \zeta_{j,k}(x_{11}, x_{22}) \). Then it is straightforward to see that \( \sum x_{12}^j x_{21}^k (j - k) \zeta_{j,k}(x_{11}, x_{22}) = \psi(x_{12} x_{21}, x_{11}, x_{22}) \). Therefore \( \zeta_{j,k} = 0 \) whenever \( j \neq k \) and \( w = w(x_{12} x_{21}, x_{11}, x_{22}) = w(x, s, p) \). In particular \( X' \) is of the form \( HD_α \). \( \square \)
Remark 2. Note that the holomorphic mapping \( p \) occurring in Definition 1 is defined up to a multiplication with \( x \mapsto a(x) + b(x)x \). More precisely, \( p(x)xp(x)^{-1} = q(x)xq(x)^{-1}, \ x \in U \), where \( p, q \in \mathcal{O}(U, \mathcal{M}^{-1}) \) if and only if there are holomorphic functions \( a, b : U \to \mathbb{C} \) such that \( p(x) = (a(x) + b(x)x)q(x) \) and \( \det(a(x) + b(x)x) \neq 0, \ x \in U \).

Lemma 3. Let \( p_n \in \mathcal{O}(W, \mathcal{M}^{-1}), \ \mathcal{T} \subset W \subset \Omega, \ det p_n = 1 \) be a sequence of holomorphic mappings such that \( \varphi_n(x) := p_n(x)xp_n(x)^{-1}, \ x \in W \) is convergent locally uniformly to \( \psi \in \mathcal{O}(W, \Omega) \). Then there is a neighborhood \( U \) of \( \mathcal{T} \) and diagonal mappings \( a_n, b_n \in \mathcal{O}(U, \mathcal{M}) \) such that \( p_n(x)(a_n(x) + xb_n(x)) \) is locally uniformly convergent on \( U \) to \( u \in \mathcal{O}(U, \mathcal{M}^{-1}) \) and \( \det(a_n(x) + xb_n(x)) = 1 \).

The proof presented below is elementary and relies upon purely analytic methods. We do not know whether the lemma would follow from more general algebraic properties. Note that the main difficulty lies in the fact that numerical methods do not work for cyclic matrices.

Proof. To simplify the notation we will omit subscript \( n \). Composing \( \varphi \) with Möbius maps from both sides (more precisely taking \( m_{-a} \circ \varphi \circ m_a, \ \text{where} \ m_a = m_{a,1} \)) we easily see that \( q(a, x) := p(m_a(x))xp(m_a(x)) \) is convergent locally uniformly with respect to \( (a, x) \) in a neighborhood of \( \mathbb{D} \times \mathcal{T} \). Therefore \( \frac{\partial \varphi}{\partial x}(a, x)(h) \) converges locally uniformly in an open neighborhood of \( \mathbb{D} \times \mathcal{T} \) for any \( h \). Putting \( x = 0 \) we find that

\[
\frac{\partial \varphi}{\partial x}(a, 0)(h) = p(a)hp(a)^{-1},
\]

whence \( p(a) \) is convergent locally uniformly with respect to \( a \in \mathbb{D} \). Therefore, replacing \( \varphi \) with

\[
x \mapsto p^{-1}\left( \begin{array}{cc} \text{tr} x/2 & 0 \\ 0 & \text{tr} x/2 \end{array} \right) \varphi(x)p^{-1}\left( \begin{array}{cc} \text{tr} x/2 & 0 \\ 0 & \text{tr} x/2 \end{array} \right)
\]

we may assume that \( p(x) = 1 \) for all cyclic matrices \( x \in \Omega \).

Put \( \Omega' := \{ \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{11} \end{pmatrix} : \ x_{11} \in \mathbb{D}, x_{21} \in \mathbb{C} \} \). First we show that there are diagonal mappings \( a', b' \) defined on a \( U' \) neighborhood of \( \mathcal{T} \) in \( \Omega' \) such that \( p(x)(a'(x) + xb'(x)) \) is convergent and \( \det(a'(x) + xb'(x)) = 1 \) for \( x \in \Omega' \) lying in a neighborhood of \( \mathcal{T} \). Multiplying \( p(x)xp(x)^{-1} \) out we get

\[
\left( \begin{array}{cc} x_{11} + p_{12}p_{22}x_{21} & \frac{p_{22}^2x_{21}}{p_{12}^2x_{21}} \\ \frac{p_{12}^2x_{21}}{p_{22}^2x_{21}} & x_{11} - p_{12}p_{22}x_{21} \end{array} \right), \ x \in \Omega'.
\]

Using the fact that \( p(x)xp(x)^{-1} \) converges locally uniformly on \( \Omega' \), we get that \( p_{21} \) and \( p_{12} \) converge uniformly on compact subsets of \( \Omega' \). Since \( p_{21} \equiv 0 \) on \( \mathcal{T} \) we get that there is \( q \in \mathcal{O}(\Omega') \) such that \( p_{21}(x) = x_{21}q(x) \). Moreover, \( p_{22} \equiv 0 \) on \( \mathcal{T} \), so there is a neighborhood \( U_0 \) of \( \mathcal{T} \) in \( \Omega' \) (uniform with respect to \( n \)) on which \( p_{22} \) does not vanish. Put \( b'(x) := -q(x)/p_{22}(x), \ a'(x) := (p_{22}(x) - b'(x)x_{11})/p_{22}(x), \ x \in U \). Direct calculations show that \( a' \) and \( b' \) satisfy the desired claim.
Now we shall show that there are diagonal mappings \( a'' \), \( b'' \) on a neighborhood \( U'' \) of \( T \) in \( \Omega'' := \{ \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} : x_{11}, x_{22} \in \mathbb{D}, x_{21} \in \mathbb{C} \} \) such that \( p(x)(a'(x) + x b'(x)) \) is convergent locally uniformly and \( \det(a'(x) + x b'(x)) = 1 \) for \( x \in \Omega'' \) in a small neighborhood of \( T \). Let us consider the following projection

\[
j_1 : \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} (x_{11} + x_{22})/2 & 0 \\ x_{21} & (x_{11} + x_{22})/2 \end{pmatrix}
\]

and note that \( V'' := j_1^{-1}(U'') \) is a neighborhood of \( T \) in \( \Omega'' \).

It follows from the previous step that \( u(x) := p(j_1(x))(a'(j_1(x)) + b'(j_1(x))j_1(x)), \quad x \in V'' \)
is convergent locally uniformly on \( V'' \) and \( \det u = 1 \) there. Therefore, replacing \( \varphi \) with \( x \mapsto u^{-1}(x)\varphi(x)u(x) \) we may assume that \( \varphi(x) = x \) for \( x \in V'' \cap \Omega' = U' \). Multiplying \( p(x)xp(x)^{-1} \) out we get

\[
\begin{pmatrix}
(p_{11}p_{22}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} & -p_{11}p_{12}(x_{11} - x_{22}) - p_{12}^2x_{21} \\
p_{21}p_{22}(x_{11} - x_{22}) + p_{22}^2x_{21} & x_{22} - p_{12}p_{21}(x_{11} - x_{22}) - p_{12}p_{22}x_{21}
\end{pmatrix},
\]

\( x \in \Omega'' \). Since \( p(x)xp(x)^{-1} = x \) on \( U' \) we deduce from the formula above that \( p_{12} = 0 \) and \( p_{22}^2 = 1 \) and thus \( p_{11} = p_{22} = 1 \) on \( U' \).

In particular, there is a holomorphic function \( q_{12} \) on \( V'' \) such that \( p_{12}(x) = (x_{11} - x_{22})q_{12}(x) \). Similarly, \( \tilde{b}(x) := (p_{22}(x) - p_{11}(x))(x_{11} - x_{22})^{-1} \), \( x \in V' \) is a well-defined holomorphic function on \( V' \). Let us put \( \tilde{a}(x) := p_{11}(x) + q_{12}(x)x_{21} - \tilde{b}(x)x_{22} \). It is quite elementary to verify that \( p(x)(\tilde{a}(x) + \tilde{b}(x)x) \) is convergent in \( V'' \) (actually, to check it observe that \( p(x)(\tilde{a}(x) + \tilde{b}(x)x)(x_{11} - x_{22}) \) converges locally uniformly). Moreover, \( \det(a(x) + b(x)x) = (p_{11}(x) + q_{12}(x)x_{21})(p_{22}(x) + q_{12}(x)x_{21}), \) so it is equal to 1 when \( x \in T \). Therefore there is a simply-connected neighborhood \( U'' \) of \( T \) in \( \Omega'' \) (uniform with respect to subscripts \( n \), \( U'' \subset V'' \) such that \( \det(\tilde{a}(x) + x \tilde{b}(x)) \) does not vanish there. Let us take the branch of the square root \( s(x) := \det(\tilde{a}(x) + x \tilde{b}(x))^{1/2} \) preserving 1. Observe that \( a'' := \tilde{a}/s, \quad b'' := \tilde{b}/s \) satisfy the claim.

Now we prove the existence of \( a \) and \( b \) satisfying the assertion of the lemma. Put

\[
j_2 : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix}.
\]

Note that \( V := \Omega \cap j_2^{-1}(U'') \) is a neighborhood of \( T \) in \( \Omega \) so we may repeat the previous reasoning: since \( v := p \circ j_2 \circ (a'' \circ j_2 + j_2 \cdot b'' \circ j_2) \) is convergent on \( V \) and \( \det v = 1 \) there, replacing \( \varphi \) with \( v^{-1} \varphi v \) we may assume that \( \varphi(x) = x \) for \( x \in U'' \). Comparing the coefficients of \( p(x)xp(x)^{-1} \) for \( x \in U'' \):

\[
\begin{pmatrix}
p_{11}p_{22}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} + x_{22} & -p_{11}p_{12}(x_{11} - x_{22}) - p_{12}^2x_{21} \\
p_{21}p_{22}(x_{11} - x_{22}) + p_{22}^2x_{21} & -p_{11}p_{21}(x_{11} - x_{22}) - p_{12}p_{22}x_{21} + x_{21}
\end{pmatrix}
\]

and the ones of \( x \) we get that \( p_{12}(p_{11}(x_{11} - x_{22}) + p_{12}x_{21}) = 0 \), so by the identity principle either \( p_{12} = 0 \) or \( p_{11}(x_{11} - x_{22}) + p_{12}x_{21} = 0 \). If the second possibility would hold, then
comparing the elements lying in the first column and the first row we would get that $x_{11} = p_{11}p_{22}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} + x_{22} = x_{22}$, a contradiction. Therefore $p_{12} = 0$. Then, in particular, $p_{12} = 0$ on $U''$. Thus $p_{12}(x) = x_{12}q(x), x \in V$, for some holomorphic function $q$. Put $b := -q$ and $a := p_{11} - bx_{22}$. Then $p(x)(a(x) + b(x)x)$ is convergent locally uniformly in $V$. In particular, $x \mapsto \det(a(x) + b(x)x)$ is convergent. Moreover $\det(a(x) + b(x)x) = 1$ on $\mathcal{T}$, therefore shrinking, if necessary, $V$ and dividing $a$ and $b$ by a proper non-vanishing holomorphic mapping we finish the proof.

\[ \square \]

### 6. Local Form of the Automorphisms of the Spectral Unit Ball

**Corollary 4.** Let $\varphi$ be an automorphism of $\Omega$. Then for any $x \in \Omega$ there is $u \in \mathcal{O}(U, M_{2 \times 2})$ defined in an open neighborhood $U$ of $x$ such that $\varphi(x) = u(x)xu(x)^{-1}$ on $U$.

**Proof.** It follows from Proposition 2 that there is a sequence $(p_n) \subset \mathcal{O}(\Omega, M^{-1})$ such that $p_n(x)xp_n(x)^{-1}$ converges locally uniformly to $\varphi(x)$. It follows from Lemma 3 that there is a neighborhood $U$ of $T$ and $u \in \mathcal{O}(U, M^{-1})$ such that $\varphi(x) = u(x)xu(x)^{-1}, x \in U$.

On the other hand it is well known (see [Pas] and [Ros]) that $\varphi$ is a local conjugation on $\Omega \setminus T$.

It is very natural to ask whether a local conjugation on a domain $U$ satisfying ”nice” topological properties (for example $H^1(U, \mathcal{O}) = H^2(U, \mathbb{Z}) = 0$) a local holomorphic conjugation is a holomorphic conjugation. Note, that this is equivalent to finding a solution of the following problem which may be viewed as a counterpart of the meromorphic Cousin problem:

given a covering $\{\Omega_\alpha\}$ and $a_{\alpha \beta}, b_{\alpha \beta} \in \mathcal{O}(\Omega_\alpha \cap U_\beta), \det(a_{\alpha \beta}(x) + xb_{\alpha \beta}(x)) \neq 0, x \in \Omega_\alpha \cap \Omega_\beta$, such that

$$(a_{\alpha \beta}(x) + xb_{\alpha \beta}(x))(a_{\beta \gamma}(x) + xb_{\beta \gamma}(x)) = a_{\alpha \gamma}(x) + xb_{\alpha \gamma}(x), \ x \in \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma$$

and

$$(a_{\alpha \beta}(x) + xb_{\alpha \beta}(x))(a_{\beta \alpha}(x) + xb_{\beta \alpha}(x)) = 1, \ x \in \Omega_\alpha \cap \Omega_\beta$$

find $a_{\alpha}, b_{\alpha} \in \mathcal{O}(\Omega_\alpha)$ such that $a_{\alpha \beta}(x) + xb_{\alpha \beta}(x) = (a_{\alpha}(x) + xb_{\alpha}(x))(a_{\beta}(x) + xb_{\beta}(x))^{-1}$ on $\Omega_\alpha \cap \Omega_\beta$ and $\det(a_{\alpha}(x) + xb_{\alpha}(x))$ does not vanish on $\Omega_\alpha$.

Actually, assume that the problem stated above has a solution. A local conjugation $\varphi$ gives a data for the above problem in the following way: We may locally expand $\varphi$ a a holomorphic conjugation, i.e. there are $u_{\alpha}$ and $U_{\alpha}$ such that $\varphi(x) = u_{\alpha}(x)xu_{\alpha}(x)^{-1}$, where $u_{\alpha} \in \mathcal{O}(U_\alpha, M^{-1})$ and $\{U_\alpha\}$ is an open covering of $\Omega_\alpha$. Then, it follows from Remark 2 that $u_{\alpha \beta} := u_{\alpha}^{-1}u_{\beta}$ are data for the problem stated above. Solving it we find that $u_{\alpha}(x)u_{\beta}(x)^{-1} = (a_{\alpha}(x) + xb_{\alpha}(x))(a_{\beta}(x) + xb_{\beta}(x))^{-1}$ on $\Omega_\alpha \cap \Omega_\beta$. Putting $w(x) := u_{\alpha}(x)(a_{\alpha}(x) + xb_{\alpha}(x))$, $x \in \Omega_\alpha$ we get a well defined holomorphic mapping on $U$ such that $\varphi(x) = w(x)xw(x)^{-1}$.

On the other hand, suppose that $\{\Omega_{\alpha \beta}, a_{\alpha \beta}, b_{\alpha \beta}\}$ are data for the above problem. Then, solving the second Cousin problem for matrices we get $u_{\alpha} \in \mathcal{O}(\Omega_\alpha, M^{-1})$ such that $a_{\alpha \beta}(x) + xb_{\alpha \beta}(x) = u_{\alpha}(x)u_{\beta}(x)$ on $\Omega_\alpha \cap \Omega_\beta$. Putting $\varphi(x) := u_{\alpha}(x)xu_{\alpha}(x)^{-1}$ we get a local holomorphic conjugation on $U$. If it were a holomorphic conjugation, we would get $u \in \mathcal{O}(U, M^{-1})$ such that $\varphi(x) = u(x)xu(x)^{-1}$. Making use of Remark 2 again we get $a_{\alpha}, b_{\beta}$ such that
\[ u_\alpha(x) = (a_\alpha(x) + xb_\alpha(x))u(x), \ x \in \Omega_\alpha. \] Then it is a direct to observe that \( a_\alpha, b_\alpha \) solve the above problem.

**References**

[And] E. Andersén, *Volume-preserving automorphisms of \( \mathbb{C}^n \)*, Complex Variables Theory Appl. 14 (1990), no. 1-4, 223-235.

[And-Lem] E. Andersén and L. Lempert, *On the group of holomorphic automorphisms of \( \mathbb{C}^n \)*, Invent. Math. 110 (1992), no. 2, 371-388.

[Edi-Zwo] A. Edigarian and W. Zwonek, *Geometry of the symmetrized polydisc*, Arch. Math. (Basel) 84 (2005), no. 4, 364-374.

[For-Ros] F. Forstnerič, J. P. Rosay, *Approximations of biholomorphic mappings by automorphisms of \( \mathbb{C}^n \)*, Invent. Math. 112 (1993), no. 2, 323-349.

[Kal-Kut] S. Kaliman and F. Kutzschebauch, *Density property for hypersurfaces \( UV = P(X) \)*, Math. Z. 258 (2008), no. 1, 115-131.

[Kos] Kosiński, *The group of automorphisms of the spectral ball*, to appear in Proc. AMS.

[Kut-Lin] F. Kutzschebauch and A. Lind, *Holomorphic automorphisms of Danielewski surfaces I: density of the group of overshears*, Proc. AMS, 139 (2011), no. 11, 3915-3927.

[Lin] A. Lind, *Holomorphic automorphisms of Danielewski surfaces*, Phd-thesis, 2009.

[Pas] P. J. Thomas, *A local form for the automorphisms of the spectral unit ball*, Collect. Math. 59 (2008), no. 3, 321-324.

[Ran-Whi] T.J. Ransford and M.C. White, *Holomorphic self-maps of the spectral unit ball*, Bull. London Math. Soc. 23 (1991), 256-262.

[Ros] J. Rostand, *On the automorphisms of the spectral unit ball*, Studia Math. 155 (2003), 207-230

Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. prof. St. Lojasiewicza 6, 30-348 Kraków,

*E-mail address: lukasz.kosinski@gazeta.pl*