CONSTRUCTING SELECTIONS STEPWISE OVER CONES OF SIMPLICIAL COMPLEXES

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Abstract. It is given a simplified proof of a natural generalisation of Uspenskij’s selection characterisation of paracompact $C$-spaces. The method is also applied to give a simplified proof of a similar characterisation of paracompact finite $C$-space. Another application is a characterisation of finite-dimensional paracompact spaces, which generalises both a remark done by Michael and a previous result obtained by the author.

1. Introduction

All spaces in this paper are Hausdorff topological spaces. A space $X$ has property $C$, or is a $C$-space, if for any sequence $\{U_n : n < \omega\}$ of open covers of $X$ there exists a sequence $\{\mathcal{V}_n : n < \omega\}$ of open pairwise-disjoint families in $X$ such that each $\mathcal{V}_n$ refines $U_n$ and $\bigcup_{n<\omega} \mathcal{V}_n$ is a cover of $X$. The $C$-space property was originally defined by W. Haver [10] for compact metric spaces, subsequently Addis and Gresham [1] reformulated Haver’s definition for arbitrary spaces. It should be remarked that a $C$-space $X$ is paracompact if and only if it is countably paracompact and normal, see e.g. [6, Proposition 1.3]. Every finite-dimensional paracompact space, as well as every countable-dimensional metrizable space, is a $C$-space [1], but there exists a compact metric $C$-space which is not countable-dimensional [14].

In what follows, we will use $\Phi : X \rightsquigarrow Y$ to designate that $\Phi$ is a map from $X$ to the nonempty subsets of $Y$, i.e. a set-valued mapping. A map $f : X \rightarrow Y$ is a selection for $\Phi : X \rightsquigarrow Y$ if $f(x) \in \Phi(x)$, for all $x \in X$. A mapping $\Phi : X \rightsquigarrow Y$ is lower locally constant, see [8], if the set $\{x \in X : K \subseteq \Phi(x)\}$ is open in $X$, for every compact subset $K \subseteq Y$. This property appeared in a paper of Uspenskij [15]; later on, it was used by some authors (see, for instance, [3, 16]) under the name “strongly l.s.c.”, while in papers of other authors strongly l.s.c. was already used for a different property of set-valued mappings (see, for instance, [7]). Regarding
our terminology, let us remark that a singleton-valued mapping (i.e. a usual map) is lower locally constant precisely when it is locally constant.

Finally, let us recall that a space $S$ is aspherical if every continuous map of the $k$-sphere ($k \geq 0$) in $S$ can be extended to a continuous map of the $(k+1)$-ball in $S$. The following theorem was obtained by Uspenskij [15, Theorem 1.3].

**Theorem 1.1** ([15]). A paracompact space $X$ is a C-space if and only if for every topological space $Y$, each lower locally constant mapping $\Phi : X \rightrightarrows Y$ with aspherical values, has a continuous selection.

Let $k \geq 0$. For subsets $S, B \subset Y$, we will write that $S \mapsto^k B$ if every continuous map of the $k$-sphere in $S$ can be extended to a continuous map of the $(k+1)$-ball in $B$. Similarly, for mappings $\varphi, \psi : X \rightrightarrows Y$, we will write $\varphi \mapsto^k \psi$ to express that $\varphi(x) \mapsto^k \psi(x)$, for every $x \in X$. In these terms, we shall say that a sequence of mappings $\varphi_n : X \rightrightarrows Y$, $n < \omega$, is aspherical if $\varphi_n \mapsto \varphi_{n+1}$, for every $n < \omega$. Also, to each sequence of mappings $\varphi_n : X \rightrightarrows Y$, $n < \omega$, we will associate its union $\bigcup_{n<\omega} \varphi_n : X \rightrightarrows Y$, defined pointwise by $\left[ \bigcup_{n<\omega} \varphi_n \right](x) = \bigcup_{n<\omega} \varphi_n(x)$, $x \in X$.

In the present paper, we will prove the following generalisation of Theorem 1.1.

**Theorem 1.2.** A paracompact space $X$ is a C-space if and only if for every topological space $Y$, each aspherical sequence $\varphi_n : X \rightrightarrows Y$, $n < \omega$, of lower locally constant mappings admits a continuous selection for its union $\bigcup_{n<\omega} \varphi_n$.

A word should be said about the proper place of Theorem 1.2. By taking $\Phi = \varphi_n = \varphi_{n+1}$, $n < \omega$, the selection property in Theorem 1.2 implies that of Theorem 1.1. Hence, according to Theorem 1.1, the selection property in Theorem 1.2 also implies that $X$ is a C-space. In contrast, the selection property in Theorem 1.1 doesn’t imply that of Theorem 1.2. Here is a simple example.

**Example 1.3.** Let $\mathbb{S} = \{e^{it} : t \in [0, 2\pi]\}$ be the unit circle, and $\{t_n\} \subset [0, 2\pi)$ be an increasing sequence with $2\pi = \lim_{n \to \infty} t_n$. Take any space $X$, and define mappings $\varphi_n : X \rightrightarrows \mathbb{S}$ by $\varphi_n(x) = \{e^{it} : 0 \leq t \leq t_n\}$, $x \in X$. Then each $\varphi_n$, $n < \omega$, is a constant mapping, hence lower locally constant as well. Moreover, each $\varphi_n(x)$ is aspherical being an arc. Thus, $\varphi_n$, $n < \omega$, is an aspherical sequence of lower locally constant mappings. However, $\bigcup_{n<\omega} \varphi_n(x) = \mathbb{S}$ for every $x \in X$. Therefore, the mapping $\Phi = \bigcup_{n<\omega} \varphi_n$ is not aspherical-valued.

Based on the above example, we also have that the inverse implication in Theorem 1.2 — the fact that the selection property implies the property $C$ of the domain, does not cover the same implication of Theorem 1.1. In this regard, let us explicitly remark that in contrast to Theorem 1.1, the proof of Theorem 1.2 is simplified in both directions. Here is briefly the idea of this proof.
In the next section, we provide a simple result about continuous extensions of maps over cones of simplicial complexes, see Proposition 2.1. In Section 3, to each sequence \( \mathcal{F}_n, n < \omega \), of families of subsets of \( X \) whose union forms a cover of \( X \), we associate a special subcomplex \( \Delta(\mathcal{F}_{<\omega}) \) of the nerve \( N(\mathcal{F}_{<\omega}) \) of the disjoint union \( \bigsqcup_{n<\omega} \mathcal{F}_n \), see Example 3.1. Intuitively, \( \Delta(\mathcal{F}_{<\omega}) \) consists of those simplices \( \sigma \in N(\mathcal{F}_{<\omega}) \), which have at most one vertex in each \( \mathcal{F}_n, n < \omega \). One benefit of this subcomplex is that \( \Delta(\mathcal{V}_{<\omega}) = N(\mathcal{V}_{<\omega}) \), whenever \( \mathcal{V}_n, n < \omega \), is a sequence of families as in the defining property of \( C \)-spaces, see Proposition 3.2. Another benefit is discussed in the same section. Namely, for a sequence of covers \( \mathcal{F}_n, n < \omega \), of \( X \), we may associate the sequence of subcomplexes \( \Delta(\mathcal{F}_{\leq n}), n < \omega \), defined as above, where each \( \Delta(\mathcal{F}_{\leq n}) \) corresponds to the indexed cover \( \bigsqcup_{k=0}^n \mathcal{F}_k \) of \( X \). Furthermore, to each \( n < \omega \), we may associate a simplicial-valued mapping \( \Delta[\mathcal{F}_{\leq n}] : X \rightrightarrows \Delta(\mathcal{F}_{\leq n}) \) which assigns to each \( x \in X \) those simplices \( \sigma \subset \bigsqcup_{k=0}^n \mathcal{F}_k \) for which \( x \in \bigcap \sigma \). Finally, we may consider the corresponding geometric realisations \( |\Delta(\mathcal{F}_{\leq n})|, n < \omega \), of these subcomplexes, and the generated set-valued mappings \( \Delta[\mathcal{F}_{\leq n}] : X \rightrightarrows |\Delta(\mathcal{F}_{\leq n})| \subset |\Delta(\mathcal{V}_{<\omega})|, n < \omega \).

In this setting, for a sequence \( \mathcal{F}_n, n < \omega \), of closed locally finite covers of \( X \), the sequence \( \Delta[\mathcal{F}_{\leq n}] : X \rightrightarrows |\Delta(\mathcal{F}_{\leq n})|, n < \omega \), is aspherical and consists of lower locally constant simplicial-valued mappings, Propositions 3.3 and 3.4. Based on this, in Section 4, we deal with the essential part of the proof of Theorem 1.2. Namely, we show that each aspherical sequence \( \varphi_n : X \rightrightarrows Y, n < \omega \), of lower locally constant mappings defined on a paracompact space \( X \), admits a sequence \( \mathcal{F}_n, n < \omega \), of closed locally finite (interior) covers of \( X \) and a continuous map \( f : |\Delta(\mathcal{F}_{<\omega})| \to Y \) such that each composite mapping \( f \circ |\Delta[\mathcal{F}_{\leq n}]| : X \rightrightarrows Y \) is a set-valued selection for \( \varphi_n, n < \omega \), see Theorem 4.1. This theorem is applied in Section 5 to show that for a paracompact space \( X \), the selection problem for aspherical sequences \( \varphi_n : X \rightrightarrows Y, n < \omega \), of lower locally constant mappings is equivalent to that of the simplicial-valued mappings \( |\Delta[\mathcal{F}_{\leq n}]| : X \rightrightarrows |\Delta(\mathcal{F}_{<\omega})|, n < \omega \), associated to closed locally finite (interior) covers \( \mathcal{F}_n, n < \omega \), of \( X \), Proposition 5.1. The step to open covers \( \mathcal{U}_n, n < \omega \), of a paracompact space \( X \) is done in the same section by showing that this selection problem is further equivalent to the existence of canonical maps \( f : X \to |\Delta(\mathcal{U}_{<\omega})| \), Corollary 5.4. Finally, in Section 6, it is shown that in the same setting, the existence of canonical maps \( f : X \to |\Delta(\mathcal{U}_{<\omega})| \) is equivalent to \( C \)-like properties of \( X \), Theorem 6.1. Theorem 1.2 is now obtained as a special case of Theorem 6.1, see Corollary 6.2. Here, let us explicitly remark that two other special cases of Theorem 6.1 are covering two other similar results — Corollary 6.3 (a selection theorem of Valov [16, Theorem 1.1] about finite \( C \)-spaces) and Corollary 6.5 (generalising both a remark done by Michael in [15, Remark 2] and a result obtained by the author in [9, Theorem 3.1]).
2. Extensions of maps over cones of simplicial complexes

By a simplicial complex we mean a collection \( \Sigma \) of nonempty finite subsets of a set \( S \) such that \( \emptyset \neq \tau \subset \sigma \subset \Sigma \). The set \( \bigcup \Sigma \) is the vertex set of \( \Sigma \), while each element of \( \Sigma \) is called a simplex. The \( k \)-skeleton \( \Sigma^k \) of \( \Sigma \) \((k \geq 0)\) is the simplicial complex \( \Sigma^k = \{ \sigma \in \Sigma : \text{card}(\sigma) \leq k + 1 \} \), where card(\( \sigma \)) is the cardinality of \( \sigma \). In the sequel, for simplicity, we will identify the vertex set of \( \Sigma \) with its 0-skeleton \( \Sigma^0 \), and will say that \( \Sigma \) is a \( k \)-dimensional if \( \Sigma = \Sigma^k \).

The vertex set \( \Sigma^0 \) of each simplicial complex \( \Sigma \) can be embedded as a linearly independent subset of some linear normed space. Then to any simplex \( \sigma \in \Sigma \), we may associate the corresponding geometric simplex \( |\sigma| \) which is the convex hull of \( \sigma \). Thus, \( \text{card}(\sigma) = k + 1 \) if and only if \( |\sigma| \) is a \( k \)-dimensional simplex. Finally, we set \( |\Sigma| = \bigcup \{ |\sigma| : \sigma \in \Sigma \} \) which is called the geometric realisation of \( \Sigma \).

As a topological space, we will always consider \( |\Sigma| \) endowed with the Whitehead topology \([17, 18]\). This is the topology in which a subset \( U \subset |\Sigma| \) is open if and only if \( U \cap |\sigma| \) is open in \( |\sigma| \) for every \( \sigma \in \Sigma \).

The cone \( Z * v \) over a space \( Z \) with a vertex \( v \) is the quotient space of \( Z \times [0, 1] \) obtained by identifying all points of \( Z \times \{ 1 \} \) into a single point \( v \). For a simplicial complex \( \Sigma \) and a point \( v \in |\Sigma| \), the cone on \( \Sigma \) with a vertex \( v \), is the simplicial complex defined by

\[
\Sigma * v = \Sigma \cup \{ \sigma \cup \{ v \} : \sigma \in \Sigma \} \cup \{ \{ v \} \}.
\]

Evidently, we have that \( |\Sigma| * v = |\Sigma * v| \).

**Proposition 2.1.** Let \( S_0, \ldots, S_{n+1} \subset Y \) with \( \emptyset \neq S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_{n+1} \), and \( g : |\Sigma| \to S_n \) be a continuous map from an \( n \)-dimensional simplicial complex \( \Sigma \) such that \( g(|\Sigma^k|) \subset S_k \), for every \( k \leq n \). If \( v \notin |\Sigma| \), then \( g \) can be extended to a continuous map \( h : |\Sigma * v| \to S_{n+1} \) such that \( h(|(\Sigma * v)^k|) \subset S_k \), for every \( k \leq n+1 \).

**Proof.** By a finite induction, extend each restriction \( g_k = g \upharpoonright |\Sigma^k| \) to a continuous map \( h_k : |(\Sigma * v)^k| \to S_k \), \( k \leq n \), such that \( h_k \upharpoonright |(\Sigma * v)^{k-1}| = h_{k-1} \), \( k > 0 \). Briefly, define \( h_0 : (\Sigma * v)^0 \to S_0 \) by \( h_0 \upharpoonright |\Sigma^0| = g_0 \) and \( h_0(v) \in S_0 \). Whenever \( u \in \Sigma^0 \) is a vertex of \( \Sigma \), the map \( h_0 : \{ u, v \} \to S_0 \) can be extended to a continuous map \( h_{(1,u)} : \{ u, v \} \to S_1 \) because \( S_0 \not\hookrightarrow S_1 \). Then the map \( h_1 : |(\Sigma * v)^1| \to S_1 \) defined by \( h_1 \upharpoonright |(\Sigma * v)^0 = h_0 \), \( h_1 \upharpoonright |\Sigma^1| = g_1 \) and \( h_1 \upharpoonright \{ u, v \} = h_{(1,u)} \), \( u \in \Sigma^0 \), is a continuous extension of both \( h_0 \) and \( g_1 \). The construction can be carried on by induction to get a continuous extension \( h_n : |(\Sigma * v)^n| \to S_n \) of \( g = g_n \) with the required properties.

Finally, if \( \sigma \in \Sigma * v \) is an \((n+1)\)-dimensional simplex, then \( h_n \) is defined on the boundary \( |\sigma| \cap |(\Sigma * v)^n| \) of \( |\sigma| \) which is homeomorphic to the \( n \)-sphere. Hence, it can be extended to a continuous map \( h_\sigma : |\sigma| \to S_{n+1} \) because \( S_n \not\hookrightarrow S_{n+1} \). The required map \( h : |\Sigma * v| \to S_{n+1} \) is now defined by \( h \upharpoonright |(\Sigma * v)^n| = h_n \) and \( h \upharpoonright |\sigma| = h_\sigma \), for every \((n+1)\)-dimensional simplex \( \sigma \in \Sigma * v \). \[ \square \]
3. Nerves of sequences of covers

The set \( \Sigma_S \) of all nonempty finite subsets of a set \( S \) is a simplicial complex. Another natural example is the nerve of an indexed cover \( \{F_\alpha : \alpha \in A\} \) of a set \( X \), which is the subcomplex of \( \Sigma_A \) defined by

\[
\mathcal{N}(A) = \left\{ \sigma \in \Sigma_A : \bigcap_{\alpha \in \sigma} F_\alpha \neq \emptyset \right\}.
\]

Following Lefschetz [12], the intersection \( \bigcap_{\alpha \in \sigma} F_\alpha \) is called the kernel of \( \sigma \), and is often denoted by \( \ker[\sigma] = \bigcap_{\alpha \in \sigma} F_\alpha \). In case \( F \) is an unindexed cover of \( X \), its nerve is denoted by \( \mathcal{N}(F) \). In this case, \( F \) is indexed by itself, and each simplex \( \sigma \in \mathcal{N}(F) \) is merely a nonempty finite subset of \( F \) with \( \ker[\sigma] = \bigcap_{\sigma} \neq \emptyset \).

Here, an important role will be played by a subcomplex of the nerve of a special indexed cover of \( X \). The prototype of this subcomplex can be found in some of the considerations in the proof of [15, Theorem 2.1].

**Example 3.1.** Whenever \( 0 < \kappa \leq \omega \), let \( F_n, n < \kappa \), be a sequence of families of subsets of \( X \) with \( \bigcup_{n<\kappa} F_n = X \), and \( \bigsqcup_{n<\kappa} F_n \) be the disjoint union of these families (obtained, for instance, by identifying each \( F_n \) with \( F_n \times \{n\}, n < \kappa \)). The nerve of this indexed cover of \( X \) defines a natural simplicial complex

\[
\mathcal{N}(\mathcal{F}_{\leq \kappa}) = \mathcal{N}\left( \bigsqcup_{n<\kappa} F_n \right).
\]

A simplex \( \sigma \in \mathcal{N}(\mathcal{F}_{\leq \kappa}) \) can be described as the disjoint union \( \sigma = \bigsqcup_{i=1}^m \sigma_i \) of finitely many simplices \( \sigma_i \in \mathcal{N}(\mathcal{F}_{n_i}) \), for \( n_1 < \cdots < n_m < \kappa \), such that \( \bigcap_{i=1}^m \ker[\sigma_i] \neq \emptyset \). The simplicial complex \( \mathcal{N}(\mathcal{F}_{\leq \kappa}) \) contains a natural subcomplex \( \Delta(\mathcal{F}_{\leq \kappa}) \), define by

\[
\Delta(\mathcal{F}_{\leq \kappa}) = \{ \sigma \in \mathcal{N}(\mathcal{F}_{\leq \kappa}) : \text{card} (\sigma \cap \mathcal{F}_n \times \{n\}) \leq 1, n < \kappa \}.
\]

In other words, the subcomplex \( \Delta(\mathcal{F}_{\leq \kappa}) \) consists of those simplices \( \sigma \in \mathcal{N}(\mathcal{F}_{\leq \kappa}) \) which are composed of finitely many vertices \( F_i = (F,n_i) \in \mathcal{F}_{n_i} = \mathcal{N}^0(\mathcal{F}_{n_i}), i \leq m \), where \( n_1 < \cdots < n_m < \kappa \). In the special case of \( \kappa = n + 1 < \omega \), we will simply write \( \mathcal{N}(\mathcal{F}_{\leq n}) = \mathcal{N}(\mathcal{F}_{\leq \kappa}) \) and \( \Delta(\mathcal{F}_{\leq n}) = \Delta(\mathcal{F}_{\leq \kappa}) \). □

The subcomplex \( \Delta(\mathcal{F}_{\leq \kappa}) \) in Example 3.1 is naturally related to the definition of \( C \)-spaces. The following proposition is an immediate consequence of (3.3).

**Proposition 3.2.** Let \( 0 < \kappa \leq \omega \) and \( \mathcal{Y}_n, n < \kappa \), be a sequence of pairwise-disjoint families of subsets of \( X \), whose union forms a cover of \( X \). Then

\[
\Delta(\mathcal{Y}_{\leq \kappa}) = \mathcal{N}(\mathcal{Y}_{\leq \kappa}).
\]
For a simplicial complex $\Sigma$, each set-valued mapping $\Omega : X \leadsto \Sigma$ will be called \textit{simplicial-valued}. Such a mapping $\Omega : X \leadsto \Sigma$ generates a mapping $|\Omega| : X \leadsto |\Sigma|$ defined by

$$\tag{3.5} |\Omega|(p) = \bigcup_{\sigma \in \Omega(p)} |\sigma|, \quad p \in X.$$ 

Here is a natural example. Each indexed cover $\{F_\alpha : \alpha \in \mathcal{A}\}$ of $X$ generates a natural simplicial-valued mapping $\Sigma_{\mathcal{A}} : X \leadsto \Sigma_{\mathcal{A}}$, defined by

$$\tag{3.6} \Sigma_{\mathcal{A}}(p) = \bigg\{ \sigma \in \Sigma_{\mathcal{A}} : p \in \bigcap_{\alpha \in \sigma} F_\alpha \bigg\}, \quad p \in X.$$ 

In fact, each $\Sigma_{\mathcal{A}}(p)$ is a subcomplex of $\mathcal{N}(\mathcal{A})$, so $\Sigma_{\mathcal{A}} : X \leadsto \mathcal{N}(\mathcal{A})$.

The benefit of the mapping in (3.6) comes in the setting of the simplicial complex $\Delta(\mathcal{F}_{<\kappa})$ in Example 3.1, associated to a sequence of covers $\mathcal{F}_n$, $n < \kappa$, of $X$ for some $0 < \kappa \leq \omega$. Namely, we may define the corresponding simplicial-valued mapping $\Delta_{[\mathcal{F}_{<\kappa}]} : X \leadsto \Delta(\mathcal{F}_{<\kappa})$ by the same pattern as in (3.6), i.e.

$$\tag{3.7} \Delta_{[\mathcal{F}_{<\kappa}]}(p) = \{ \sigma \in \Delta(\mathcal{F}_{<\kappa}) : p \in \ker[\sigma] \}, \quad p \in X.$$ 

Just like before, we will write $\Delta_{[\mathcal{F}_{\leq \kappa}]} = \Delta_{[\mathcal{F}_{<\kappa}]}$ whenever $\kappa = n + 1 < \omega$.

We now have the following natural relationship with aspherical sequences of lower locally constant mappings.

**Proposition 3.3.** Let $\mathcal{F}_n$, $n < \omega$, be a sequence of covers of a set $X$. Then

$$\tag{3.8} \Delta_{[\mathcal{F}_{\leq \kappa}]}(p) \cdot F \subset \Delta_{[\mathcal{F}_{\leq \kappa + 1}]}(p), \quad \text{whenever } p \in F \in \mathcal{F}_{n+1} \text{ and } n < \omega.$$ 

Accordingly, $|\Delta_{[\mathcal{F}_{\leq \kappa}]}| : X \leadsto |\Delta(\mathcal{F}_{\leq \kappa})| \subset |\Delta(\mathcal{F}_{<\omega})|$, $n < \omega$, is an aspherical sequence of mappings.

**Proof.** The property in (3.8) follows from the fact that $F = (F, n+1) \notin \bigcup_{k=0}^n \mathcal{F}_k$, for every $F \in \mathcal{F}_{n+1}$. Since $|\Delta_{[\mathcal{F}_{\leq \kappa}]}(p) \cdot F| = |\Delta_{[\mathcal{F}_{\leq \kappa}]}(p)| \cdot F$ is contractible, this implies that $|\Delta_{[\mathcal{F}_{\leq \kappa}]}(p)| \xrightarrow{\pi_n} |\Delta_{[\mathcal{F}_{\leq \kappa}]}(p) \cdot F| \subset |\Delta_{[\mathcal{F}_{\leq \kappa + 1}]}(p)|$. \quad \square

**Proposition 3.4.** Let $\mathcal{F}_0, \ldots, \mathcal{F}_n$ be a sequence of locally finite closed covers of a space $X$. Then the mapping $|\Delta_{[\mathcal{F}_{\leq \kappa}]}| : X \leadsto |\Delta(\mathcal{F}_{\leq \kappa})|$ is lower locally constant.

**Proof.** Whenever $p \in X$, the set $V = X \setminus \bigcup \{ F \in \bigcup_{k \leq \kappa} \mathcal{F}_k : p \notin F \}$ is a neighbourhood of $p$. Take a simplex $\sigma \in \Delta_{[\mathcal{F}_{\leq \kappa}]}(p)$ and a point $q \notin \ker[\sigma]$. Then $q \notin F$ for some $F \in \sigma$, and therefore $q \notin V$. Thus, by (3.7), $x \in V$ implies that $\Delta_{[\mathcal{F}_{\leq \kappa}]}(p) \subset \Delta_{[\mathcal{F}_{\leq \kappa}]}(x)$ and, accordingly, $|\Delta_{[\mathcal{F}_{\leq \kappa}]}|$ is lower locally constant. \quad \square

We conclude this section with a remark about the importance of disjoint unions in the definition of the subcomplex in Example 3.1.
Remark 3.5. For a sequence of covers $\mathcal{F}_n$, $n < \kappa$, of $X$, where $0 < \kappa \leq \omega$, one can define the subcomplex $\Delta \mathcal{F}_\kappa \subset \mathcal{N} \mathcal{F}_\kappa$ by considering $\mathcal{N} \mathcal{F}_\kappa$ to be the nerve of the usual unindexed cover $\bigcup_{n<\kappa} \mathcal{F}_n$, rather than the disjoint union $\bigcup_{n<\kappa} \mathcal{F}_n$. However, this will not work to establish a property similar to that in Proposition 3.3, also for the essential results in the next sections (see, for instance, Theorem 4.1 and Lemma 4.2). Namely, suppose that $\mathcal{F}_0$ and $\mathcal{F}_1$ are covers of $X$ which contain elements $F_i \in \mathcal{F}_i$, $i = 0, 1$, with $F_0 \cap F_1 \neq \emptyset$ and $F_i \notin \mathcal{F}_{1-i}$. Then $\sigma = \{F_0, F_1\} \in \Delta \mathcal{F}_{\leq 1}$. However, if $\mathcal{F}_2$ is a cover of $X$ with $F_0, F_1 \in \mathcal{F}_2$, and $\Delta \mathcal{F}_{\leq 2}$ is defined on the basis of unindexed covers, then $\Delta \mathcal{F}_{\leq 1} \not\subset \Delta \mathcal{F}_{\leq 2}$ because $\sigma = \{F_0, F_1\} \notin \Delta \mathcal{F}_{\leq 2}$. □

4. Skeletal selections

For mappings $\varphi, \psi : X \rightsquigarrow Y$, we will write $\varphi \subset \psi$ to express that $\varphi(p) \subset \psi(p)$, for every $p \in X$. In this case, the mapping $\varphi$ is called a set-valued selection, or a multi-selection, for $\psi$. Also, let us recall that a cover $\mathcal{F}$ of a space $X$ is called interior if the collection of the interiors of the elements of $\mathcal{F}$ is a cover of $X$.

The following theorem will be proved in this section.

Theorem 4.1. Let $X$ be a paracompact space and $\varphi_n : X \rightsquigarrow Y$, $n < \omega$, be an aspherical sequence of lower locally constant mappings in a space $Y$. Then there exists a sequence $\mathcal{F}_n$, $n < \omega$, of closed locally finite interior covers of $X$ and a continuous map $f : |\Delta \mathcal{F}_{<\omega}| \rightarrow Y$ such that

(4.1) $f \circ |\Delta_{\mathcal{F}_{\leq n}}| \subset \varphi_n$, for every $n < \omega$.

Let us explicitly remark that, here, $\Delta_{\mathcal{F}_{\leq n}} : X \rightsquigarrow \Delta \mathcal{F}_{\leq n} \subset \Delta \mathcal{F}_{<\omega}$ is the simplicial-valued mapping associated to the covers $\mathcal{F}_k$, $k \leq n$, see (3.7), while $f \circ |\Delta_{\mathcal{F}_{\leq n}}|$ is the composite mapping

$\xymatrix{ |\Delta_{\mathcal{F}_{\leq n}}| \ar[r] & |\Delta \mathcal{F}_{<\omega}| \ar[r] & Y \\
X \ar[u]_{f \circ |\Delta_{\mathcal{F}_{\leq n}}|} & & \ar[l]_{f} }$

According to the definition of $\Delta_{\mathcal{F}_{\leq n}} : X \rightsquigarrow \Delta \mathcal{F}_{\leq n}$, see also (3.5), the property in (4.1) means that $f(|\sigma|) \subset \varphi_n(p)$, for every $\sigma \in \Delta \mathcal{F}_{\leq n}$ and $p \in \ker[\sigma]$.

Turning to the proof of Theorem 4.1, let us remark that the simplicial complex $\Delta \mathcal{F}_{\leq n}$ is $n$-dimensional, see (3.3) of Example 3.1. In what follows, its $k$-skeleton will be denoted by $\Delta^k \mathcal{F}_{\leq n}$. In these terms, following the idea of an $n$-skeletal selection in [9], we shall say that a continuous map $f : |\Delta \mathcal{F}_{\leq n}| \rightarrow Y$ is a skeletal selection for a sequence of mappings $\varphi_0, \ldots, \varphi_n : X \rightsquigarrow Y$ if

(4.2) $f(|\sigma|) \subset \varphi_k(p)$, for every $\sigma \in \Delta^k \mathcal{F}_{\leq n}$, $k \leq n$, and $p \in \ker[\sigma]$. 
Whenever \( k \leq n \), as in (3.7), one can associate the simplicial-valued mapping \( \Delta^k_{|\mathcal{F}_{\leq n}} : X \sim \Delta^k(\mathcal{F}_{\leq n}) \), which assigns to each \( p \in X \) the \( k \)-skeleton \( \Delta^k_{|\mathcal{F}_{\leq n}}(p) \) of the subcomplex \( \Delta_{|\mathcal{F}_{\leq n}}(p) \subset \Delta(\mathcal{F}_{\leq n}) \). Then the property in (4.2) means that the composite mapping \( f \circ |\Delta^k_{|\mathcal{F}_{\leq n}}| : X \sim |\Delta^k(\mathcal{F}_{\leq n})| \) is a set-valued selection for \( \varphi_k \), for every \( k \leq n \).

Another concept that will play a role in the proof of Theorem 4.1 is a simplicial map. A simplicial map \( g : \Sigma_1 \to \Sigma_2 \) is a simplicial isomorphism, we say that \( g \) follows from Proposition (4.3) \( \Delta^p \).

\[
\begin{align*}
\text{Lemma 4.2.} & \quad \text{Let } Y \text{ be a space, } \mathcal{F}_0, \ldots, \mathcal{F}_n \text{ be a sequence of closed locally finite covers of a paracompact space } X, \text{ and } \varphi_0, \ldots, \varphi_{n+1} : X \sim Y \text{ be a sequence of lower locally constant mappings with } \varphi_k \hookrightarrow \varphi_{k+1} \text{ for every } k \leq n. \text{ If } f_n : |\Delta(\mathcal{F}_{\leq n})| \to Y \text{ is a skeletal selection for } \varphi_0, \ldots, \varphi_n, \text{ then there exists a closed locally finite interior cover } \mathcal{F}_{n+1} \text{ of } X \text{ and a continuous extension } f_{n+1} : |\Delta(\mathcal{F}_{\leq n+1})| \to Y \text{ of } f_n \text{ which is a skeletal selection for } \varphi_0, \ldots, \varphi_{n+1}.
\end{align*}
\]

\[
\begin{align*}
\text{Proof.} & \quad \text{Let } \Delta_{|\mathcal{F}_{\leq n}} : X \sim \Delta(\mathcal{F}_{\leq n}) \text{ be the associated simplicial-valued mapping, defined as in (3.7). Whenever } p \in X, \text{ the subcomplex } \\
\quad & \quad \text{(4.3)} \quad \Delta_{|\mathcal{F}_{\leq n}}(p) \text{ is } n\text{-dimensional such that, by (4.2), } f_n(\Delta^k_{|\mathcal{F}_{\leq n}}) \subset \varphi_k(p), \quad k \leq n. \text{ Moreover, by hypothesis, } \varphi_k(p) \hookrightarrow \varphi_{k+1}(p) \text{ for every } k \leq n. \text{ Hence, assuming that } p \notin \Delta_{|\mathcal{F}_{\leq n}} \text{, it follows from Proposition 2.1 that } f_n \upharpoonright \Delta_{|\mathcal{F}_{\leq n}} \text{ can be extended to a continuous map } f_p : \Delta_{|\mathcal{F}_{\leq n}} \to Y \text{ such that } \\
\quad & \quad \text{(4.4)} \quad f_p(\Delta_{|\mathcal{F}_{\leq n}}(p)) \subset \varphi_k(p), \quad \text{for every } 0 \leq k \leq n+1.
\end{align*}
\]

Since all covers are locally finite and closed, the point \( p \in X \) is contained in the open set
\[
\text{(4.5)} \quad O_p = X \setminus \bigcup \{ F \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n : p \notin F \}.
\]

For the same reason, \( \Delta_{|\mathcal{F}_{\leq n}} \) is a finite simplicial complex. Accordingly, each set \( f_p(\Delta_{|\mathcal{F}_{\leq n}}(p)) \), \( k \leq n+1 \), is compact. Hence, by (4.4) and the hypothesis that each mapping \( \varphi_k, k \leq n+1 \), is lower locally constant, we may shrink \( O_p \) to a neighbourhood \( V_p \) of \( p \), defined by
\[
\text{(4.6)} \quad V_p = \{ x \in O_p : f_p(\Delta_{|\mathcal{F}_{\leq n}}(p)) \subset \varphi_k(x), \quad \text{for every } k \leq n+1 \}.
\]
Finally, since $X$ is paracompact, it has an open locally finite cover $\mathcal{U}_{n+1}$ such that \( \{V_p : p \in X\} \) is refined by the associated cover $\mathcal{F}_{n+1} = \{U : U \in \mathcal{U}_{n+1}\}$ of the closures of the elements of $\mathcal{U}_{n+1}$. So, there is a map $p : \mathcal{F}_{n+1} \to X$ such that

\[(4.7) \quad F \subset V_{p(F)} \subset O_{p(F)}, \quad \text{for every } F \in \mathcal{F}_{n+1}.
\]

Having already defined the cover $\mathcal{F}_{n+1}$, we are going to extend $f_n$ to a skeletal selection $f_{n+1} : |\Delta(\mathcal{F}_{\leq n+1})| \to Y$ for the sequence $\varphi_0, \ldots, \varphi_{n+1}$. To this end, take an $F \in \mathcal{F}_{n+1}$, and define the set

$$\Delta_F = \{\tau \in \Delta(\mathcal{F}_{\leq n}) : \tau \cup \{F\} \in \Delta(\mathcal{F}_{\leq n+1})\}.$$  

It is evident that $\Delta_F$ is a subcomplex of $\Delta(\mathcal{F}_{\leq n})$ with $F \notin |\Delta_F|$, hence the cone $\Delta_F * F$ is a subcomplex of $\Delta(\mathcal{F}_{\leq n+1})$. Thus, to extend $f_n$ to a skeletal selection $f_{n+1} : |\Delta(\mathcal{F}_{\leq n+1})| \to Y$ for the sequence $\varphi_0, \ldots, \varphi_{n+1}$, it now suffices to extend each $f_n |_{\Delta_F}$, $F \in \mathcal{F}_{n+1}$, to a continuous map $f_F : |\Delta_F * F| \to Y$ satisfying the condition in (4.2) with respect to the simplices of $\Delta_F * F$. To this end, let us observe that

\[(4.8) \quad \Delta_F \subset \Delta_{p(F)} = \Delta_{[x_{\leq n}]}(p(F)).
\]

Indeed, for $T \in \tau \in \Delta_F$, we have that $\emptyset \neq T \cap F \subset T \cap O_{p(F)}$, see (4.7). Hence, by (4.5), $p(F) \in T$ and according to (3.7) and (4.3), $\tau \in \Delta_{p(F)}$.

We are now ready to define the required maps $f_F : |\Delta_F * F| \to Y$, $F \in \mathcal{F}_{n+1}$. Namely, by (4.8), we can embed $\Delta_F * F$ into the cone $\Delta_{p(F)} * p(F)$ by identifying $p(F)$ with $F$. Let $\ell : \Delta_F * F \to \Delta_{p(F)} * p(F)$ be the corresponding simplicial embedding defined by $\ell | \Delta_0 = \text{id}$, and $\ell | F = p(F)$). Next, define a continuous extension $f_F : |\Delta_F * F| \to Y$ of $f_n |_{\Delta_F}$ by $f_F = f_{p(F)} \circ \ell | F$. Take a simplex $\sigma \in (\Delta_F * F)^k$ for some $k \leq n+1$, and a point $x \in \ker|\sigma|$. If $\sigma \in \Delta_F$, by the properties of $f_n$, see (4.2), $f_F(\sigma) = f_n(\sigma) \subset \varphi_k(x)$. If $F \in \sigma$, then $x \in F \subset V_{p(F)}$ and, by (4.6), we have again that $f_F(\sigma) = f_{p(F)}(\ell(\sigma)) \subset \varphi_k(x)$. The proof is complete. \( \Box \)

Complementary to Lemma 4.2 is the following well-known property, see the proof of [15, Theorem 2.1] and that of [8, Theorem 3.1]. The property itself was stated explicitly in [9, Proposition 3.2], and is an immediate consequence of the defining property of lower locally constant mappings.

**Proposition 4.3.** If $X$ is a paracompact space and $\varphi : X \to Y$ is a lower locally constant mapping, then there exists a closed locally finite interior cover $\mathcal{F}$ of $X$ and a (continuous) map $f : \Delta(\mathcal{F}) = \mathcal{F} \to Y$ such that $f(F) \in \varphi(x)$, for every $x \in F \in \mathcal{F}$.

**Proof of Theorem 4.1.** Inductively, using Proposition 4.3 and Lemma 4.2, there exists a sequence $\mathcal{F}_n$, $n < \omega$, of closed locally finite interior covers of $X$ and
continuous maps \( f_n : |\Delta(\mathcal{F}_{\leq n})| \rightarrow Y, n < \omega \), such that each \( f_n \) is a skeletal selection for the sequence \( \varphi_0, \ldots, \varphi_n \), and each \( f_{n+1} \) is an extension of \( f_n \). Since \( \Delta(\mathcal{F}_{<\omega}) = \bigcup_{n<\omega} \Delta(\mathcal{F}_{\leq n}) \), we may define a map \( f : |\Delta(\mathcal{F}_{<\omega})| \rightarrow Y \) by \( f|_{|\Delta(\mathcal{F}_{\leq n})|} = f_n \), for every \( n < \omega \). Then \( f \) is continuous and clearly has the property in (4.1).

5. Selections and canonical maps

Suppose that \( X \) is a (paracompact) space with the property that for any space \( Y \), each aspherical sequence \( \varphi_n : X \looparrowright Y, n < \omega \), of lower locally constant mappings admits a continuous selection for its union \( \bigcup_{n<\omega} \varphi_n \). As we will see in the next section (Corollaries 6.2, 6.3 and 6.5 and Example 6.4), each one of the following statements determines a different dimension-like property of \( X \).

(5.1) There exists an aspherical sequence \( \varphi_n : X \looparrowright Y, n < \omega \), of lower locally constant mappings such that no \( \varphi_n, n<\omega \), has a continuous selection.

(5.2) For each aspherical sequence \( \varphi_k : X \looparrowright Y, k < \omega \), of lower locally constant mappings there exists an \( n < \omega \) such that \( \varphi_n \) has a continuous selection.

(5.3) There exists an \( n < \omega \) such that for each aspherical sequence \( \varphi_k : X \looparrowright Y, k < \omega \), of lower locally constant mappings, the mapping \( \varphi_n \) has a continuous selection.

Here, we first reduce the above selection problems only to the setting of simplicial-valued mappings associated to closed locally finite (interior) covers of \( X \).

**Proposition 5.1.** For a space \( Y \), a paracompact space \( X \) and \( 0 < \kappa \leq \omega \), the following are equivalent:

(a) For each aspherical sequence \( \varphi_n : X \looparrowright Y, n < \omega \), of lower locally constant mappings, \( \bigcup_{n<\mu} \varphi_n \) has a continuous selection for some \( 0 < \mu \leq \kappa \).

(b) For each sequence \( \mathcal{F}_n, n < \omega \), of closed locally finite interior covers of \( X \), \( |\Delta(\mathcal{F}_{<\mu})| : X \looparrowright |\Delta(\mathcal{F}_{<\mu})| \) has a continuous selection for some \( 0 < \mu \leq \kappa \).

**Proof.** The implication (a) \( \implies \) (b) follows from Propositions 3.3 and 3.4 (for arbitrary locally finite closed covers \( \mathcal{F}_n, n < \omega \), of \( X \)). The converse follows easily from Theorem 4.1. Namely, assume that (b) holds and \( \varphi_n : X \looparrowright Y, n < \omega \), is as in (a). Since \( X \) is paracompact, by Theorem 4.1, there exists a sequence \( \mathcal{F}_n, n < \omega \), of closed locally finite interior covers of \( X \) and a continuous map \( f : |\Delta(\mathcal{F}_{<\omega})| \rightarrow Y \) satisfying (4.1). For this sequence \( \mathcal{F}_n, n < \omega \), by (b), there exists \( 0 < \mu \leq \kappa \) such that \( |\Delta(\mathcal{F}_{<\mu})| : X \looparrowright |\Delta(\mathcal{F}_{<\mu})| \) has a continuous selection \( h : X \rightarrow |\Delta(\mathcal{F}_{<\mu})| \). Evidently, the composite map \( g = f \circ h : X \rightarrow Y \) is a continuous selection for the mapping \( \bigcup_{n<\mu} \varphi_n \).

Complementary to (b) of Proposition 5.1 is the following useful observation extending the property to various covers of \( X \).
Proposition 5.2. Let $\mathcal{U}_n$, $n < \omega$, be a sequence of covers of $X$, $0 < \mu \leq \omega$, and $\mathcal{V}_n$, $n < \mu$, be a sequence of families of subsets of $X$ such that each $\mathcal{V}_n$ refines $\mathcal{U}_n$ and $\bigcup_{n<\mu} \mathcal{V}_n = X$. If $|\Delta(\mathcal{V}_n)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$ has a continuous selection, then so does $|\Delta(\mathcal{U}_n)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$.

Proof. Since each $\mathcal{V}_n$ refines $\mathcal{U}_n$, there are maps $r_n : \mathcal{V}_n \to \mathcal{U}_n$, $n < \mu$, such that $V \subset r_n(V)$, for all $V \in \mathcal{V}_n$. Accordingly, $r = \bigsqcup_{n<\mu} r_n : \Delta(\mathcal{V}_\mu) \to \Delta(\mathcal{U}_\mu)$ is a simplicial map with the property that $\sigma \subset r(\sigma)$, for each simplex $\sigma \in \Delta(\mathcal{V}_\mu)$. In other words, $r \circ \Delta(\mathcal{V}_\mu) \subset \Delta(\mathcal{U}_\mu)$, see (3.7), and therefore $|r| \circ |\Delta(\mathcal{V}_\mu)| \subset |\Delta(\mathcal{U}_\mu)|$.

Thus, if $h : X \to |\Delta(\mathcal{V}_\mu)|$ is a continuous selection for $|\Delta(\mathcal{V}_\mu)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$, then the composite map $f = |r| \circ h : X \to |\Delta(\mathcal{U}_\mu)|$ is a continuous selection for $|\Delta(\mathcal{U}_n)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$. \qed

Suppose that $\mathcal{F}_n$, $n < \omega$, is a sequence of closed (locally finite) interior covers of $X$, and $\mathcal{U}_n$ is the cover composed by the interiors of the elements of $\mathcal{F}_n$. According to Proposition 5.2, if the mapping $|\Delta(\mathcal{U}_n)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$ has a continuous selection for some $0 < \mu \leq \omega$, then so does the mapping $|\Delta(\mathcal{U}_n)| : X \leadsto |\Delta(\mathcal{U}_\mu)|$.

Therefore, the selection problem in (b) of Proposition 5.1 is further reduced to that of a sequence of open covers of the paracompact space $X$. This latter problem is naturally related to the existence of canonical maps for the disjoint union $\bigcup_{n<\mu} \mathcal{U}_n$ of such covers. To this end, let us briefly recall some terminology. For a simplicial complex $\Sigma$ and a simplex $\sigma \in \Sigma$, we use $(\sigma)$ to denote the relative interior of the geometric simplex $|\sigma|$. For a vertex $v \in \Sigma$, the set

$$\text{st}(v) = \bigcup_{v \in \sigma \in \Sigma} (\sigma),$$

is called the open star of the vertex $v \in \Sigma^0$. One can easily see that $\text{st}(v)$ is open in $|\Sigma|$ because $\text{st}(v) = |\Sigma| \setminus \bigcup_{v \not\in \sigma \in \Sigma} |\sigma|$. In these terms, for an indexed cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of a space $X$, a continuous map $f : X \to |\mathcal{N}(\mathcal{A})|$ is called canonical for $\{U_\alpha : \alpha \in \mathcal{A}\}$ if

$$f^{-1}(\text{st}(\alpha)) \subset U_\alpha, \quad \text{for every } \alpha \in \mathcal{A}.$$  

(5.2)

It is well known that each open cover of a paracompact space admits a canonical map, which follows from the fact that such a cover has an index-subordinated partition of unity. The interested reader is refer to [9, Section 2] which contains a brief summary of several facts about canonical maps and partitions of unity. Here, we are interested in a selection interpretation of canonical maps. Namely, in terms of the simplicial-valued mapping $\Sigma_{\mathcal{A}} : X \leadsto \mathcal{N}(\mathcal{A})$ associated to the cover $\{U_\alpha : \alpha \in \mathcal{A}\}$, see (3.6), we have the following characterisation of canonical maps; for unindexed covers it was obtained in [9, Proposition 2.5] (see also Dowker [4]), but the proof for indexed covers is essentially the same.
Proposition 5.3. A map $f : X \to |\mathcal{N}(\mathcal{A})|$ is canonical for a cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of a space $X$ if and only if it is a continuous selection for the simplicial-valued mapping $|\Sigma_{\mathcal{A}}| : X \mapsto |\mathcal{N}(\mathcal{A})|$.

In the special case of a sequence of open covers $\mathcal{U}_n$, $n < \omega$, a canonical map $f : X \to \mathcal{N}(\mathcal{U}_{<\omega})$ for the disjoint union $\bigsqcup_{n<\omega} \mathcal{U}_n$ will be called canonical for the sequence $\mathcal{U}_n$, $n < \omega$. We now have the following further reduction of the selection problem for aspherical sequences of mappings.

Corollary 5.4. For a space $Y$, a paracompact space $X$ and $0 < \kappa \leq \omega$, the following are equivalent:

(a) For each aspherical sequence $\varphi_n : X \mapsto Y$, $n < \omega$, of lower locally constant mappings, $\bigcup_{n<\mu} \varphi_n$ has a continuous selection for some $0 < \mu \leq \kappa$.

(b) Each sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$ admits a canonical map $f : X \to |\Delta(\mathcal{U}_{<\mu})| \subset |\mathcal{N}(\mathcal{U}_{<\omega})|$, for some $0 < \mu \leq \kappa$.

Proof. If $\mathcal{F}_n$ is a closed locally finite interior cover of $X$, then the interiors of the elements of $\mathcal{F}_n$ form a locally finite open cover $\mathcal{U}_n$ of $X$ which refines $\mathcal{F}_n$. If $\mathcal{U}_n$ is an open cover of the paracompact space $X$, then there exists an open locally finite cover $\mathcal{V}_n$ such that $\mathcal{F}_n = \{\overline{V} : V \in \mathcal{V}_n\}$ refines $\mathcal{U}_n$. Accordingly, the equivalence (a) $\iff$ (b) follows from Propositions 5.1, 5.2 and 5.3.

6. Dimension and canonical maps

Here, we finalise the proof of Theorem 1.2 showing that the property $C$ is equivalent to the existence of canonical maps for special covers. To this end, for a sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$ and $0 < \mu \leq \omega$, we shall say that a sequence $\mathcal{V}_n$, $n < \mu$, of families of open subsets $X$ is a $C$-refinement of $\mathcal{U}_n$, $n < \omega$, if each family $\mathcal{V}_n$, $n < \mu$, is pairwise-disjoint and $X = \bigcup_{n<\mu} \mathcal{V}_n$.

Theorem 6.1. For a paracompact space $X$ and $0 < \kappa \leq \omega$, the following are equivalent:

(a) Each sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$ has a $C$-refinement $\mathcal{V}_n$, $n < \mu$, for some $0 < \mu \leq \kappa$.

(b) Each sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$ admits a canonical map $f : X \to |\Delta(\mathcal{U}_{<\mu})|$, for some $0 < \mu \leq \kappa$.

Proof. To see that (a) $\implies$ (b), take a sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$. Then by (a), $\mathcal{U}_n$, $n < \omega$, admits a $C$-refinement $\mathcal{V}_n$, $n < \mu$, for some $0 < \mu \leq \kappa$. Let $\mathcal{N}(\mathcal{V}_{<\mu})$ be the nerve of the disjoint union $\bigsqcup_{n<\mu} \mathcal{V}_n$, see (3.2) of Example 3.1, and $\Sigma_{\mathcal{V}_{<\mu}} : X \mapsto \mathcal{N}(\mathcal{V}_{<\mu})$ be the simplicial-valued mapping associated to this nerve, see (3.6). Since $X$ is paracompact, the indexed cover $\bigsqcup_{n<\mu} \mathcal{V}_n$ has a canonical map. Hence, by Proposition 5.3, the mapping $|\Sigma_{\mathcal{V}_{<\mu}}| : X \mapsto |\mathcal{N}(\mathcal{V}_{<\mu})|$ has a
continuous selection. However, by definition, each family $\mathcal{V}_n$, $n < \mu$, is pairwise-disjoint. Therefore, by Proposition 3.2, see (3.4), $\Delta(\mathcal{V}_{<\mu}) = \mathcal{M}(\mathcal{V}_{<\mu})$ and, consequently, $\Delta[\mathcal{V}_{<\mu}] = \Sigma[\mathcal{V}_{<\mu}]$. Thus, $\left|\Delta[\mathcal{V}_{<\mu}]\right| : X \sim |\Delta(\mathcal{V}_{<\mu})|$ has a continuous selection and, according to Proposition 5.2, the mapping $\left|\Delta[\mathcal{W}_{<\mu}]\right| : X \sim |\Delta(\mathcal{W}_{<\mu})|$ has a continuous selection as well. Finally, by Proposition 5.3, each continuous selection for $|\Delta[\mathcal{W}_{<\mu}]|$ is as required in (b).

Conversely, let $\mathcal{U}_n$, $n < \omega$, and $f : X \rightarrow |\Delta(\mathcal{W}_{<\mu})|$ be as in (b) for some $0 < \mu \leq \kappa$. Define $\mathcal{V}_n = \{f^{-1}(\text{st}(U)) : U \in \mathcal{U}_n\}$, $n < \mu$. Since $f$ is continuous, $\mathcal{V}_n$ is an open family in $X$; moreover, by (5.2), it refines $\mathcal{U}_n$. It is also evident that $\bigcup_{n<\mu} \mathcal{V}_n$ covers $X$, see (5.1). We complete the proof by showing that $\mathcal{V}_n$ is pairwise-disjoint as well. Assume that $U_1, U_2 \in \mathcal{U}_n$ are such that $p \in f^{-1}(\text{st}(U_1)) \cap f^{-1}(\text{st}(U_2))$ for some $p \in X$. Then $f(p) \in \text{st}(U_1) \cap \text{st}(U_2)$, and there are simplices $\sigma_1, \sigma_2 \in \Delta(\mathcal{W}_{<\mu})$ with $f(p) \in \langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle$ and $U_i \in \sigma_i$, $i = 1, 2$, see (5.1). However, the collection $\{\langle \sigma \rangle : \sigma \in \Delta(\mathcal{W}_{<\mu})\}$ forms a partition of $|\Delta(\mathcal{W}_{<\mu})|$, therefore $\sigma_1 = \sigma_2$. According to the defining property of $\Delta(\mathcal{W}_{<\mu})$, see (3.3), this implies that $U_1 = U_2$. Thus, each family $\mathcal{V}_n$, $n < \mu$, is also pairwise-disjoint, and the proof is complete. \hfill $\square$

We finalise the paper with several applications. The first one is the following slight generalisation of Theorem 1.2; it is an immediate consequence of Corollary 5.4 and Theorem 6.1 (in the special case of $\mu = \kappa = \omega$).

**Corollary 6.2.** For a paracompact space $X$, the following are equivalent:

(a) $X$ is a $C$-space.

(b) For every space $Y$, each aspherical sequence $\varphi_n : X \rightarrow Y$, $n < \omega$, of lower locally constant mappings admits a continuous selection for its union $\bigcup_{n<\omega} \varphi_n$.

(c) Each sequence $\mathcal{U}_n$, $n < \omega$, of open covers of $X$ admits a canonical map $f : X \rightarrow |\Delta(\mathcal{W}_{<\omega})|$.

Another consequence is for the case when $0 < \mu < \kappa = \omega$, and deals with the so-called finite $C$-spaces. These spaces were defined by Borst for separable metrizable spaces, see [2]; subsequently the definition was extended by Valov [16] for arbitrary spaces. For simplicity, we will consider these spaces in the realm of normal spaces. In this setting, a (normal) space $X$ is called a *finite $C$-space* if for any sequence $\{\mathcal{U}_k : k < \omega\}$ of finite open covers of $X$ there exists a finite sequence $\{\mathcal{V}_k : k \leq n\}$ of open pairwise-disjoint families in $X$ such that each $\mathcal{V}_k$ refines $\mathcal{U}_k$ and $\bigcup_{k\leq n} \mathcal{V}_k$ is a cover of $X$. It was shown by Valov in [16, Theorem 2.4] that a paracompact space $X$ is a finite $C$-space if and only if each sequence $\{\mathcal{U}_k : k < \omega\}$ of open covers of $X$ admits a finite $C$-refinement, i.e. there exists a finite sequence $\{\mathcal{V}_k : k \leq n\}$ of open pairwise-disjoint families in $X$ such that each $\mathcal{V}_k$ refines $\mathcal{U}_k$.
and $\bigcup_{k \leq n} \mathcal{Y}_k$ is a cover of $X$. Based on this, we have the following consequence of Corollary 5.4 and Theorem 6.1 (in the special case of $\mu < \kappa = \omega$).

**Corollary 6.3.** For a paracompact space $X$, the following are equivalent:

(a) $X$ is a finite $C$-space.

(b) For each aspherical sequence $\varphi_k : X \rightarrow Y$, $k < \omega$, of lower locally constant mappings in a space $Y$, there exists $n < \omega$ such that $\varphi_n$ has a continuous selection.

(c) Each sequence $\mathcal{U}_k$, $k < \omega$, of open covers of $X$ admits a canonical map $f : X \rightarrow |\Delta(\mathcal{U}_{\leq n})|$ for some $n < \omega$.

Let us explicitly remark that the equivalence $(a) \iff (b)$ in Corollary 6.3 was obtained by Valov in [16, Theorem 1.1]. His arguments were following the original arguments for proving Theorem 1.1, accordingly our approach is providing a simplification of these arguments. Regarding the proper place of finite $C$-spaces, it was shown by Valov in [16, Proposition 2.2] that a space $X$ is a finite $C$-space if and only if its Čech-Stone compactification $\beta X$ is a $C$-space. This brings a natural distinction between the selection problems stated in (5.1) and (5.2).

**Example 6.4.** The following example of a $C$-space which is not finite $C$, was given in [11, Remark 3.7]. Let $K_\omega$ be the subspace of the Hilbert cube $[0, 1]^\omega$ consisting of all points which have only finitely many nonzero coordinates. Then $K_\omega$ is a $C$-space being strongly countable-dimensional, but is not a finite $C$-space because every compactification of $K_\omega$ contains a copy of $[0, 1]^\omega$ (as per [5, Example 5.5.(1)]). According to Corollaries 6.2 and 6.3, see also Proposition 5.1, this implies that there exists a space $Y$ and an aspherical sequence $\varphi_k : K_\omega \rightarrow Y$, $k < \omega$, of lower locally constant constant mappings such that $\bigcup_{k < \omega} \varphi_k$ has a continuous selection, but none of the mappings $\varphi_n$, $n < \omega$, has a continuous selection.

Our last application is for the case when $\mu = \kappa = n + 1$, for some $n < \omega$. To this end, following [9], a finite sequence $\varphi_k : X \rightarrow Y$, $0 \leq k \leq n$, of mappings will be called aspherical if $\varphi_k \hookrightarrow \varphi_{k+1}$, for every $k < n$. By letting for $p \in X$ and $k > n$ that $\varphi_k(p) = Y \ast q$ is the cone on $Y$ with a fixed vertex $q$, each finite aspherical sequence $\varphi_k : X \rightarrow Y$, $0 \leq k \leq n$, can be extended to an aspherical sequence $\varphi_k : X \rightarrow Y \ast q$, $k < \omega$. Moreover, let us explicitly remark that in this construction, each resulting new mapping $\varphi_k$, $k > n$, is lower locally constant being a constant set-valued mapping.

Regarding dimension properties of the domain, let us recall a result of Ostrand [13] that for a normal space $X$ with a covering dimension $\dim(X) \leq n$, each open locally finite cover $\mathcal{U}$ of $X$ admits a sequence $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of open pairwise-disjoint families such that each $\mathcal{U}_k$, $k \leq n$, refines $\mathcal{U}_k$ and $\bigcup_{k=0}^n \mathcal{U}_k = X$. The result was refined by Addis and Gresham, see [1, Proposition 2.12], that a paracompact
space $X$ has a covering dimension $\dim(X) \leq n$ if and only if each finite sequence $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of open covers of $X$ has a finite $C$-refinement, i.e. there exists a finite sequence $\mathcal{V}_0, \ldots, \mathcal{V}_n$ of open pairwise-disjoint families of $X$ such that each $\mathcal{V}_k$, $0 \leq k \leq n$, refines $\mathcal{V}_k$, and $\bigcup_{k=0}^{n} \mathcal{V}_k = X$. Just like before, setting $\mathcal{U}_k = \mathcal{U}_n$, $k > n$, the above characterisation of the covering dimension of paracompact spaces remains valid for an infinite sequence $\mathcal{U}_k$, $k < \omega$, of open covers of $X$. Accordingly, we have also the following consequence of Corollary 5.4 and Theorem 6.1 (in the special case of $\mu = \kappa = n + 1 < \omega$).

**Corollary 6.5.** For a paracompact space $X$, the following are equivalent:

(a) $\dim(X) \leq n$.

(b) For each aspherical sequence $\varphi_k : X \twoheadrightarrow Y$, $0 \leq k \leq n$, of lower locally constant mappings in a space $Y$, the mapping $\varphi_n$ admits a continuous selection.

(c) Each sequence $\mathcal{U}_k$, $0 \leq k \leq n$, of open covers of $X$ admits a canonical map $f : X \to \big| \Delta(\mathcal{U}_{\leq n}) \big|$.

A direct proof of the implication (a) $\implies$ (b) in Corollary 6.5 was given in [9, Theorem 3.1]. Let us also remark that in the special case when all mappings $\varphi_k$, $0 \leq k \leq n$, are equal, the equivalence of (a) and (b) of Corollary 6.5 was shown in [15, Remark 2].

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