Covariant isolation from an Abelian gauge field of its nondynamical potential, which, when fed back, can transform into a “confining Yukawa”

Steven Kenneth Kauffmann
American Physical Society Senior Life Member

43 Bedok Road
#01-11
Country Park Condominium
Singapore 469564
Handphone: +65 9370 6583

and

Unit 802, Reflection on the Sea
120 Marine Parade
Coolangatta QLD 4225
Australia
Tel/FAX: +61 7 5536 7235
Mobile: +61 4 0567 9058

Email: SKKauffmann@gmail.com

Abstract

For Abelian gauge theory a properly relativistic gauge is developed by supplementing the Lorentz condition with causal determination of the time component of the four-vector potential by retarded Coulomb transformation of the charge density. This causal Lorentz gauge agrees with the Coulomb gauge for static charge densities, but allows the four-vector potential to have a longitudinal component that is determined by the time derivative of the four-vector potential’s time component. Just as in Coulomb gauge, the two transverse components of the four-vector potential are its sole dynamical part. The four-vector potential in this gauge covariantly separates into a dynamical transverse four-vector potential and a nondynamical timelike/longitudinal four-vector potential, where each of these two satisfies the Lorentz condition. In fact, analogous partition of the conserved four-current shows each to satisfy a Lorentz-condition Maxwell-equation system with its own conserved four-current. Because of this complete separation, either of these four-vector potentials can be tinkered with without affecting its counterpart. Since it satisfies the Lorentz condition, the nondynamical four-vector potential times a constant with dimension of inverse length squared is itself a conserved four-current, and so can be fed back into its own source current, which transforms its time component into an extended Yukawa, with both exponentially decaying and exponentially growing components. The latter might be the mechanism of quark-gluon confinement: in non-Abelian color gauge theory the Yukawa mixture ratio ought to be tied to color, with palpable consequences for “colorful” hot quark-gluon plasmas.
Introduction

Gauge theories have a quintessential dual nature. They simultaneously encompass dynamical transverse waves that travel at the speed of light, which phenomenon can be independently quantized, and nondynamical potential/force fields whose source is appropriately "charged" matter, and which, in turn, affect the behavior of such matter. The nondynamical pure potentials/forces, which are part of what gauge theories encompass, are not subject to independent quantization: they merely conveniently isolate and abstract a certain intermediate mathematical aspect of physical behavior that is inherent to the "charged" matter itself—useful mathematical abstractions, however suggestive or extremely convenient, are still not physical degrees of freedom that can be independently quantized; any quantum characteristics of such nondynamical pure potential/force fields are merely the secondary consequences of the quantum character of their "charged" matter source.

In view of the dual nature of gauge theories, their typically seamless-appearing mathematical formulations present a conundrum and challenge to the theoretical physicist. Part of the formal smoothness of gauge theories is attributable to the physics which they describe: the selfsame "charged" matter which inherently interacts with itself in a manner that can be conveniently mathematically analyzed using the intermediate constructs of source-determined pure potential/force fields, also emits, absorbs and scatters the dynamical transverse waves/radiation/massless quanta—whose interactions with that "charged" matter are made part and parcel of the physics treated by that same gauge theory. This naturally promotes the formal similarity of fields that do have dynamical content to fields which are merely extremely convenient intermediate mathematical constructs for the analysis of "charged" matter’s intrinsic interaction. Another technical aspect of gauge theories which can contribute to such a lack of formal distinction between actual dynamical fields and nondynamical “intermediate mathematical construct” pure potential fields is the gauge invariance ambiguity of those theories: the physically nonexistent degree of freedom often serves to enhance misleading surface manifestation of physically nonexistent symmetry.

There are a number of reasons to seek to, in a formally neat and relativistically covariant fashion, “pull apart” gauge theory into two physically natural parts, one encompassing the purely dynamical transverse gauge fields and the other the purely nondynamical gauge potential fields. Obviously, pushing the nondynamical potential fields out of the way in a covariant fashion might conceivably be a boon to covariant quantization of the dynamical transverse gauge fields. More intriguingly, having a clean such separation in hand raises the theoretical possibility of tinkering with one of these parts without affecting the other, or perhaps even trying to discard one of the parts altogether. Certainly the issue of quark confinement lends itself to thoughts of somehow relativistically drastically reshaping “gluon” gauge potential fields without upending altogether the notion of the gluon as a gauge particle. The last half of this paper tentatively explores just this matter, albeit only in the theoretically insufficient context of purely Abelian gauge theory.

As a first step along this route, this paper deals only with the simplest case, namely Abelian gauge theory. Hopefully there will be other researchers who will be inspired to look at the more involved non-Abelian cases. The approach to carrying out covariant cleavage of Abelian gauge theory into two natural constituents will lean heavily on the identification of a sensible potential field. Since the potential field is an intermediate mathematical construct for facilitating the understanding of the intrinsic interaction of charged matter, it should be entirely the creature of its charged matter source; in other words, with the charged matter source in hand, the potential field should follow uniquely, and it should, of course, vanish altogether in the absence of the charged matter source. Since the equations of Abelian gauge theory are linear, it is clear that the potential field must be a homogeneous functional of its source. Furthermore, the kernel of the appropriate homogeneous functional is a Green’s function of the linear equation that relates the potential field to its charged matter source. To ensure that a relativistically sensible Green’s function is available for this purpose, we must take care that our choice of gauge for the Abelian gauge theory does not impose as the equation which links the potential field to its charged matter source one that is incompatible with special relativity. In other words, success of our envisioned project hinges on a relativistically compatible choice of gauge. It is to be noted that the physics of the dynamical gauge waves/radiation/massless particles in contrast does not enslave their transverse fields to a charged matter source; those fields can have a nonvanishing existence even in the complete
absence of charged matter, albeit they interact with charged matter if it is present. But their fields are not determined by the charged matter, whereas the pure potential field is entirely so determined, even in a fully quantum context.

At first blush it might appear that selection of the Coulomb gauge of itself accomplishes the project just set out: at one stroke not only is gauge ambiguity abolished, but also potential effects are transparently assigned to the time component of the four-vector potential, dynamical effects are equally transparently assigned to the four-vector potential's spatial transverse components, and its spatial longitudinal component is set to zero by fiat. Everything, apparently, is now clear-cut and simple. The shock comes when examining the full consequences for that time-component potential: it responds instantaneously throughout all of space to any time variation of its charge-density source. Thus the penny drops: the marvelously simple and efficacious consequences for that time-component potential: it responds instantaneously throughout all of space to any time variation of its charge-density source. Thus the penny drops: the marvelously simple and efficacious abolition of the longitudinal component of the spatial vector potential, which is the Coulomb gauge condition, spared no thought or concern for special relativity! Still, the Coulomb gauge’s treatment of the static limit of the charge density is absolute bedrock; any proposed alternative gauge won’t be worth its salt if it deviates from that.

The obvious alternative to the Coulomb gauge is the celebrated Lorentz condition. No slighting of relativity there, and an immediate payoff with a marvelous automatic formal simplification of the Maxwell equation system. But hold on, doesn’t that equation system now look too symmetrical in the components of the four-vector potential? Is there the slightest indication of just where to seek the nondynamical potential versus the dynamical radiation fields? And what about the troubling restricted gauge invariance ambiguity which still remains?

In fact, it happens that the Lorentz condition itself, with no reference to the Maxwell equations, completely determines the spatial longitudinal component of the four-vector potential in terms of the time derivative of its time component. Thus although the Lorentz condition does not flatly abolish the longitudinal component of the vector potential as the Coulomb gauge does, it does completely subordinate it to the time derivative of the four-potential’s time component, which has a similar effect. A puzzling feature of this consequence of the Lorentz condition is that it is far from obvious how it could be demonstrated by manifestly covariant operations on four-vectors. The obvious approach is to note the precise mathematical analogy between the Lorentz condition and the Coulomb-law Maxwell equation, and then to recall that the latter determines the longitudinal component of the electric field in terms of the charge density. The determination of the longitudinal component of the spatial vector potential in terms of the time derivative of its time component is then worked out in strict analogy.

Though very helpful and enlightening, this consequence of the Lorentz condition leaves a “loose screw” in its wake; the restricted gauge invariance ambiguity that the Lorentz condition permits must still be addressed. Since the Lorentz condition ties the longitudinal space component of the four-vector potential to its time component, it is apparent that if the relation of the time component of the four-vector potential to the charge density could have all the slack removed from it (without offending special relativity, of course), then that time component would be the full-fledged potential field, and therefore the dynamical field would be compelled to occupy only the two transverse spatial components of the four-vector potential, being that the four-vector potential’s longitudinal spatial component has already been fully determined by the Lorentz condition to be entirely the creature of the time derivative of what is now the potential field.

The key to removing the slack from the relation of the time component of the four-vector potential to the charge density—which slack is a manifestation of the restricted gauge invariance ambiguity—of course lies with the Green’s functions of the light-speed wave equation which connects that time component to the charge density after imposition of the Lorentz condition. The physically most appealing such light-speed wave-equation Green’s function to select is clearly the celebrated traditional causal retarded one, which makes perfect intuitive relativistic sense. The result bears a considerable resemblance to the consequences of using Coulomb gauge, except that the potential field is now tied to the charge density in a relativistically causal retarded fashion rather than in a relativistically offensive instantaneous spatially uniform fashion, and the time derivative of that potential field now fully determines the longitudinal spatial component of the four-vector potential, instead of its being decreed to vanish under all circumstances. The upshot for the dynamical gauge field is
exactly the same as it is in Coulomb gauge: that field is described by the two remaining transverse spatial components of the four-vector potential. In the charge density’s static limit, the causal retarded potential field as well comes out to be static, and therefore the longitudinal spatial component of the four-vector potential vanishes identically. This last is, of course, the Coulomb gauge condition, so that in the charge density’s static limit, there is no difference from the Coulomb gauge result. Obviously this selection of the causal retarded Green’s function to specify a completely determined relation of the time component of the four-vector potential to the charge density, in addition to the imposition of the Lorentz condition, has also determined a gauge, which we dub the causal Lorentz gauge. This gauge is as close to the Coulomb gauge as one can get without contravening the tenets of special relativity.

The causal Lorentz gauge permits the full four-vector potential to be written as the sum of two physically distinct four-vector potentials, the first one consists of only the time and spatial longitudinal components of that full four-vector potential in this gauge, while the second one consists of just the remaining two spatial transverse components of that full four-vector potential in this gauge. This separation is covariant because both of these “pulled apart” four-vector potentials satisfy the Lorentz condition. Obviously the first, purely timelike/longitudinal four-vector potential in this causal Lorentz gauge is totally determined by the charge density, and is thus entirely in the nature of a collection of potential fields, whereas the second, remaining purely transverse four-vector potential in this causal Lorentz gauge is purely dynamical.

Now the four-current conservation constraint condition is highly analogous to the Lorentz condition, and permits the four-current itself to be similarly partitioned into the sum of two conserved four currents, the first one encompassing the time component of the total four-current (i.e., the charge density) as well as the spatial longitudinal component of the total four-current, which turns out to be completely determined by the charge density’s time derivative. It is readily shown that this first timelike/longitudinal four-current is explicitly conserved. The second four-current consists of just the two remaining purely transverse components of the total four-current, and is readily shown to be conserved as well. With this split up of the four-current into a conserved timelike/longitudinal part plus a conserved purely transverse part complementing the corresponding split up of the four-vector potential into two highly analogous parts which each satisfy the Lorentz condition, it turns out that Lorentz-condition light-speed wave-equation versions of the Maxwell equations neatly apply to both of the split-up four-vector potentials in causal Lorentz gauge.

We thus are now in a position to deal entirely separately with the pure potential and the pure dynamical parts of the Abelian gauge theory in causal Lorentz gauge. Being that adherence to the Lorentz condition is a pervasive feature of this gauge, the four-vector potentials themselves times a constant with the dimensions of inverse length squared qualify as conserved currents. In other words, in this gauge one can envision feeding back the four-potentials as partial contributors to their own source current. To do this to the purely transverse dynamical part of the four-vector potential seems tantamount to giving the gauge particles mass, which is physically questionable in view of their transverse character: such particles with mass are supposed to have integer spin, which is incompatible with the two transverse spin states of such particles when they are massless.

Feeding back only the timelike/longitudinal purely potential part of the four-vector potential in this causal Lorentz gauge does not affect the properties of the transverse dynamical massless gauge particles but does transform the potential field to one having Yukawa character. In fact, there is no reason to believe that such feedback would result in only the traditional exponentially decaying Yukawa potential; exponentially growing “Yukawa” potential components should make their presence felt as well. One can at least speculate that such exponentially growing feedback Yukawas might have a role in quark confinement. It turns out that the causal Lorentz gauge appears to be compatible with every possible ratio of exponentially growing Yukawa potential component to exponentially decaying Yukawa potential component. This result undoubtedly has to do with the limitations of considering Abelian gauge theory only. One might entertain the hope that these ratios of exponentially growing Yukawa potential component to exponentially decaying Yukawa potential component might eventually become tied to non-Abelian gauge color in such a way that color singlet states experience only the traditional exponentially decaying Yukawa potential. If the confinement of non-color-singlet quark/gluon configurations is indeed effected by exponentially growing Yukawa potentials, it seems rather obvious that quark-gluon “plasmas” which are too “hot” (thermally disturbed) to readily reorganize themselves into color
singlets (“hot” enough to be “colorful”) will not behave at all like a free gas.

We now proceed to the mathematical detail in Abelian gauge theory of the results we have been outlining above in words. We begin by considering the consequences of the imposition of the Lorentz condition on Abelian gauge theory, and continue to the development of the full causal Lorentz gauge and the consequence it has of allowing the four-vector potential to be covariantly cleaved into a timelike/longitudinal four-vector potential of purely potential field character and a remaining transverse four-vector potential of purely dynamical field character.

Isolation of the nondynamical four-vector potential in causal Lorentz gauge

The Lorentz-covariant four-vector Abelian gauge field $A^\mu$, 

$$A^\mu(r, t) = (\phi(r, t), A(r, t)),$$  

(1a)

whose source is the Lorentz-covariant four-vector current density $j^\mu$,

$$j^\mu(r, t) = (c\rho(r, t), j(r, t)),$$  

(1b)

is governed by a “mixed bag” of constraint and dynamical equations [1] which are expressed in Lorentz-covariant notation as,

$$\partial_\lambda \partial^\lambda A^\mu - \partial^\mu (\partial_\nu A^\nu) = j^\mu/c,$$  

(1c)

where the Lorentz-covariant four-vector first-derivative operators $\partial_\mu$ and $\partial^\mu$ are given by,

$$\partial_\mu = (c^{-1}\partial/\partial t, \nabla_r),$$  

(1d)

and,

$$\partial^\mu = (c^{-1}\partial/\partial t, -\nabla_r),$$  

(1e)

which imply that the Lorentz-scalar contracted second-derivative operator $\partial_\lambda \partial^\lambda$ comes out to be,

$$\partial_\lambda \partial^\lambda = c^{-2}\partial^2/\partial t^2 - \nabla_r^2.$$  

(1f)

Upon contracting both sides of Eq. (1c) with $\partial_\mu$, we obtain the current conservation source constraint,

$$\partial_\mu j^\mu = 0.$$  

(1g)

It turns out that Eq. (1c) fails to uniquely determine $A^\mu$. It is readily verified that if the four-gradient of an arbitrary Lorentz-scalar function $f(r, t)$ is added to $A^\mu$, i.e., if,

$$A^\mu \rightarrow A^\mu + \partial^\mu f,$$  

(1h)

then Eq. (1c) continues to be satisfied. We now take advantage of this gauge-invariance ambiguity [2] of Eq. (1c) to formally simplify it by imposing on it the Lorentz-invariant Lorentz condition [3],

$$\partial_\nu A^\nu = 0,$$  

(2a)

which results in its becoming just,
\[ \partial_\lambda \partial^\lambda A^\mu = j^\mu /c. \quad (2b) \]

The Lorentz condition has not, however, fully removed the gauge-invariance ambiguity, since we readily see that Eq. (2b) will still be satisfied after the gauge transformation of Eq. (1h) provided that the scalar gauge function \( f(r, t) \) satisfies,

\[ \partial_\lambda \partial^\lambda f = 0, \quad (2c) \]

which is restricted gauge-invariance ambiguity. We note that the current conservation source constraint of Eq. (1g) follows from Eq. (2b) upon taking account of the Lorentz condition of Eq. (2a).

Experience with electrostatics suggests that the pure potential effects [4] which arise from \( A^\mu = (\phi, \mathbf{A}) \) are primarily associated with \( \phi \), whereas it is almost universally agreed that the dynamical, radiative effects that arise from \( A^\mu \) are not at all formally apparent at this stage; in fact they are missing altogether from Eq. (2b). It turns out, however, that the Lorentz condition of Eq. (2a) is able to nail down a unique longitudinal part \( A_L \) of \( \mathbf{A} \) with the property that \( A_L \) is a pure homogeneous functional of the time derivative of \( \phi \), and, as a wonderful bonus, that the four-vector \((\phi, A_L)\) also satisfies the Lorentz condition!

Upon writing out the Lorentz condition of Eq. (2a) as,

\[ \nabla \cdot \mathbf{A} = -\dot{\phi}/c, \quad (3a) \]

we realize its extremely close formal similarity to the Coulomb law \( \nabla \cdot \mathbf{E} = \rho \) of Maxwell’s equations [1]. The Coulomb law of course uniquely determines the longitudinal part of \( \mathbf{E} \) in terms of the charge density \( \rho \). Here we proceed to determine the longitudinal part of \( \mathbf{A} \) in terms of the time derivative of \( \phi \) in exactly the same way, and obtain,

\[ \mathbf{A} = \mathbf{A}_L + \mathbf{A}_T, \quad (3b) \]

where,

\[ \mathbf{A}_L(r, t) = c^{-1} \nabla ((4\pi)^{-1} \int d^3r' \dot{\phi}(r', t)/|\mathbf{r} - \mathbf{r}'|), \quad (3c) \]

and \( \mathbf{A}_T \) must be transverse, i.e.,

\[ \nabla \cdot \mathbf{A}_T = 0. \quad (3d) \]

That Eqs. (3b) through (3d) satisfy Eq. (3a) is readily verified by use of the Coulomb potential Green’s function identity,

\[ \nabla^2_r (1/|\mathbf{r} - \mathbf{r}'|) = -4\pi \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (3e) \]

What is thus in fact verified is that,

\[ \nabla \cdot \mathbf{A}_L = -\dot{\phi}/c, \quad (3f) \]

which implies that the four-vector field,

\[ A^\mu_{(0L)} \overset{\text{def}}{=} (\phi, A_L), \quad (3g) \]
also satisfies the Lorentz condition,

$$\partial_\mu A_\mu^{(0L)} = 0,$$

(3h)

and this, together with the fact that $A_\mu$ satisfies the the Lorentz condition (by Eq. (2a)), implies that,

$$A_\mu^{(T)} \overset{\text{def}}{=} A_\mu - A_\mu^{(0L)},$$

(3i)

as well satisfies the Lorentz condition,

$$\partial_\mu A_\mu^{(T)} = 0.$$

(3j)

From Eqs. (3g), (3i) and (3b), we see that,

$$A_\mu^{(T)} = (0, A^T),$$

(3k)

which shows that $A_\mu^{(T)}$ is completely transverse. We now further note that any four-vector field which satisfies the Lorentz condition is necessarily Lorentz-covariant. That is because such a four-vector field contracted with the manifestly Lorentz-covariant four-vector differential operator $\partial_\mu$ yields zero identically, which is a manifest Lorentz scalar. Therefore $A_\mu^{(0L)}$ and $A_\mu^{(T)}$ are Lorentz-covariant four-vector fields.

The Lorentz condition has thus enabled us to covariantly separate $A_\mu$ into $A_\mu^{(T)}$, which is purely transverse and $A_\mu^{(0L)}$, which is purely timelike and longitudinal, with its time component being $\phi$ itself, while its longitudinal part is a pure homogeneous functional of $\dot{\phi}$. Now the current density four-vector $j_\mu = (c\rho, j)$ satisfies the current conservation source constraint given by Eq. (1g), which can be reexpressed as,

$$\nabla \cdot j = -\dot{\rho},$$

(4a)

in extremely close analogy with Eq. (3a) for $A_\mu = (\phi, A)$. Therefore we have for the current density four-vector $j_\mu$ extremely close analogs of all the results given by Eqs. (3) for the four-vector gauge field $A_\mu$. We therefore simply list the most important of these results with a minimum of comment,

$$j = j_L + j_T,$$

(4b)

where,

$$j_L(r, t) = \nabla((4\pi)^{-1} \int d^3r' \dot{\rho}(r', t)/|r - r'|),$$

(4c)

and $j_T$ must be transverse, i.e.,

$$\nabla \cdot j_T = 0.$$  

(4d)

In fact,

$$\nabla \cdot j_L = -\dot{\rho},$$

(4e)

which implies that the four-vector field,

$$j_\mu^{(0L)} \overset{\text{def}}{=} (c\rho, j_L),$$

(4f)
also satisfies the current conservation constraint,
\[ \partial_\mu j^\mu_{(0L)} = 0. \quad (4g) \]
We of course have that,
\[ j^\mu_{(T)} \overset{\text{def}}{=} j^\mu - j^\mu_{(0L)}, \quad (4h) \]
\text{as well satisfies the current conservation constraint,}
\[ \partial_\mu j^\mu_{(T)} = 0. \quad (4i) \]
We note that,
\[ j^\mu_{(T)} = (0, j_T), \quad (4j) \]
which shows that \( j^\mu_{(T)} \) is completely transverse. Since \( j^\mu_{(0L)} \) and \( j^\mu_{(T)} \) both satisfy the current conservation constraint, they are therefore both Lorentz-covariant four-vector fields.

It is interesting to use Eq. (2b) and Eqs. (3) to determine the equations that are satisfied \( A^\mu_{(0L)} \) and \( A^\mu_{(T)} \). From the time component of Eq. (2b), we, of course obtain that,
\[ \partial_\lambda \partial^\lambda \phi = \rho, \quad (5a) \]
or,
\[ \ddot{\phi}/c^2 - \nabla^2 \phi = \rho. \quad (5b) \]
Thus we see that the operators which comprise \( \partial_\lambda \partial^\lambda \) are \( c^{-2} \partial^2/\partial t^2 \) and \( -\nabla^2 \). Now from Eqs. (3c), (3e) and (4c) we readily work out that we can write \( A_L \) and \( j_L \) in the very convenient schematic operator forms,
\[ A_L = c^{-1} \nabla(-\nabla^2)^{-1} \dot{\phi}, \quad (5c) \]
and,
\[ j_L = \nabla(-\nabla^2)^{-1} \dot{\rho}. \quad (5d) \]
We clearly see that the two operators which comprise the operator \( \partial_\lambda \partial^\lambda \) both commute with all of the three operators that appear in front of \( \phi \) in Eq. (5c) for \( A_L \). Now the action of \( \partial_\lambda \partial^\lambda \) on \( \phi \) is given by Eq. (5a). With that and the form of Eq. (5d), we conclude that,
\[ \partial_\lambda \partial^\lambda A_L = j_L/c. \quad (5e) \]
We can now combine our result of Eq. (5e) with the forms for \( A^\mu_{(0L)} \) and \( j^\mu_{(0L)} \) given by Eqs. (3g) and (4f) respectively, plus Eq. (5a), to obtain that,
\[ \partial_\lambda \partial^\lambda A^\mu_{(0L)} = j^\mu_{(0L)}/c. \quad (5f) \]
If we combine the definitions of \( A^\mu_{(T)} \) and \( j^\mu_{(T)} \) that are given in Eqs. (3i) and (4h) with the results of Eqs. (5f) and (2b), we also obtain,
\[ \partial_\Lambda \partial^\Lambda A^\mu_{(TL)} = j^\mu_{(T)} / c. \] (5g)

The four-vector potential \( A^\mu_{(0L)} \) is completely determined by \( \phi \) because, as we see from Eqs. (3g) and (5c), its time component is \( \phi \) itself and its remaining longitudinal part is a purely homogeneous functional of \( \phi \). The relation of \( \phi \) to \( \rho \), however, is given by Eq. (5a), which leaves open the possibility that the relation of \( \phi \) to \( \rho \) is inhomogeneous and not fully determined by just \( \rho \) itself. What we face here is simply an aspect of the restricted gauge-invariance ambiguity in the face of the imposition of only the Lorentz condition, as was pointed out in the discussion centered on Eq. (2c). Indeed it is entirely clear from Eq. (5a) that any contribution to \( \phi \) which is inhomogeneous in \( \rho \) must satisfy precisely the relation pointed out in Eq. (2c). To jettison this annoying remnant of gauge-invariance ambiguity, we need to supplement the Lorentz condition with a restriction on the solution space of Eq. (5a) for \( \phi \) that rejects any elements which are inhomogeneous in \( \rho \). The intuitively/physically most appealing way to achieve this is to simply assert that the relation of \( \phi \) to \( \rho \) is a totally causal one in both space and time. This uniquely pins down the following extremely well-known and justly celebrated homogeneous causal retard ed Coulomb transformation of \( \rho \) as the desired solution of Eq. (5a) for \( \phi \), i.e.,

\[ \phi(\mathbf{r}, t) = (4\pi)^{-1} \int d^3 \mathbf{r}' \rho(\mathbf{r}', t - c^{-1}|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'|. \] (6)

With the use of Eq. (6), a particular gauge has at long last been precisely determined. This gauge is as close to the Coulomb gauge as it is possible to get without clashing with the tenets of special relativity. Recall that the Coulomb-gauge version of \( \phi \) simply omits the time retardation of the functional on the right-hand side of Eq. (6), and thereby manifests an instantaneous response of \( \phi \) throughout the whole of space to any change in \( \rho \), which is relativistically problematic. The two gauges agree perfectly, however, when \( \rho \) is time-independent (static). In that case the \( \phi \) of Eq. (6) is also time-independent, and therefore, from Eq. (5c), the longitudinal part \( A_L \) of \( A \) vanishes identically, which is what the Coulomb gauge mandates under all circumstances by relativistically dubious fiat.

In the gauge determined by the Lorentz condition and Eq. (6), which we dub the causal Lorentz gauge, there is nothing remotely dynamical about the timelike/longitudinal four-potential \( A^\mu_{(0L)} = (\phi, A_L) \), because Eq. (6) completely ties \( \phi \) to \( \rho \) without the least trace of dynamical freedom, and, of course, Eq. (5c) just as completely ties \( A_L \) to \( \phi \). Therefore, in causal Lorentz gauge, the timelike/longitudinal four-potential \( A^\mu_{(0L)} \) isolates the nondynamical sector of the Abelian gauge theory, and it does so in relativistically compliant fashion, satisfying its own covariant Lorentz condition, Eq. (3h), and its own covariant “equation of motion”, Eq. (5f) (to which only the completely causal solution that is set out in Eqs. (6) and (5c) is selected), a very far cry indeed from the utter disregard for relativity inherent in the Coulomb gauge.

Thus devoid in causal Lorentz gauge of any dynamical content, the timelike/longitudinal four-potential \( A^\mu_{(0L)} \) cannot be independently quantized; in causal Lorentz gauge the independently quantizable part of the gauge theory resides entirely in the relativistically compliant transverse, dynamical four-potential \( A^\mu_{(T)} \), which satisfies its own covariant Lorentz condition, Eq. (3j), and its own covariant transverse equation of motion, Eq. (5g). With the dynamical transverse and the nondynamical timelike/longitudinal sectors of the gauge theory thus cleanly and covariantly separated in causal Lorentz gauge, the possibility of discarding one of these sectors may be entertained. Whether that could be called for is an empirical issue: some parton-style analyses of empirical hadronic data have suggested that quarks alone are inadequate to account for that data, that strong participation by gluons (independently-quantized, transverse-spin dynamical gauge particles) is in fact required.
Feeding back the nondynamical four-vector potential in causal Lorentz gauge

A fascinating aspect of the Lorentz condition is that it precisely parallels the four-current conservation constraint. Thus a gauge field which adheres to the Lorentz condition could conceivably be made to contribute to its own input four-current, but it would need to be multiplied by a factor which has dimensions of inverse length squared in order to be four-current compatible. For the transverse dynamical four-potential in causal Lorentz gauge, such a maneuver would, at least naively, appear to endow the independently-quantized transverse gauge particle with mass. That seems uncomfortable from a physics standpoint, however, since the inherently transverse quantized gauge particle only has two spin degrees of freedom, not the three that a spin 1 particle with mass evidently requires.

In causal Lorentz gauge, the dynamical transverse and nondynamical timelike/longitudinal four-potentials, however, each individually adheres to the Lorentz condition, and each also has its own particular individually conserved input four-current, so it is possible for the nondynamical timelike/longitudinal four-potential in causal Lorentz gauge to be made to contribute to itself only, and to therefore not give rise to problematic gauge-particle mass, but to nevertheless transform its time component $\phi$ into an entity with Yukawa-like properties. Such a fed-back $\phi$ in causal Lorentz gauge would very likely be endowed with an exponentially growing Yukawa-type component in addition to a traditional exponentially decaying Yukawa component—it obviously requires a near-miracle for an exponentially growing Yukawa component to not develop in consequence of feedback. It is naturally very tempting to speculate that such an exponentially growing fed-back $\phi$ might be the mechanism of permanent quark-gluon confinement.

A technical/mathematical difficulty with exponentially growing Yukawa-type potentials is that since they cannot be spatially Fourier transformed, neither can they ever be the result of any approach which entails spatial Fourier analysis for obtaining the feedback potential response occasioned by an external charge density. Convenient handling of such potentials might conceivably entail unusual techniques such as use of the Laplace transformation. In what follows, we cope with this issue by first writing down a standard inverse space-time Fourier-transformation expression which applies to the causal retarded exponentially-decaying Yukawa feedback homogeneous potential response to a given external charge density, on which we carry out closed-form evaluation of the spatial part of this inverse Fourier transformation in order to obtain the explicit decaying Yukawa potential response directly in configuration space, with only its time still Fourier-transformed, and then demonstrate that this result can be straightforwardly extended to a sizable class of causal retarded exponentially-growing Yukawa-type feedback homogeneous potential responses to the very same external charge density.

Some physically-based criterion for choosing amongst the many different types of causal retarded exponentially-growing Yukawa feedback homogeneous potential responses to an external charge density will eventually need to be developed: perhaps in the more realistic non-Abelian color-gauge context, that choice can somehow be tied to color in such a way that color singlet states uniquely encounter the exponentially-decaying Yukawa feedback potential, whilst any nonsinglet color configuration encounters an exponentially-growing version of the Yukawa feedback retarded homogeneous potential which serves to effectively confine that nonsinglet color configuration.

Without feedback, we recall that in causal Lorentz gauge the nondynamical timelike/longitudinal four-potential consists of $(\phi, A_L)$, where the longitudinal three-potential $A_L$ is, in fact, the homogeneous functional of $\phi$ that is given by Eq. (5c) in such a way that the Lorentz condition $\nabla \cdot A_L + \dot{\phi}/c = 0$ is identically satisfied. Similarly, in causal Lorentz gauge, this four-potential’s conserved four-current source $(c \rho , j_L)$ has its longitudinal part $j_L$ given as a homogeneous functional of $\dot{\rho}$ by Eq. (5d) in such a way that the current conservation constraint $\nabla \cdot j_L + \dot{\rho}$ is identically satisfied. The equation satisfied by $(\phi, A_L)$ in terms of its source four-current $(c \rho, j_L)$ is,

$$\partial_t \partial^\lambda (\phi, A_L) = (c \rho, j_L)/c.$$  

Now given a nonnegative constant $\kappa$ whose dimension is inverse length, we proceed to make the nondynamical timelike/longitudinal $(\phi, A_L)$ four-potential contribute to its own source by adding to its above purely external causal Lorentz gauge source current the conserved “nondynamical timelike/logitudinal four-potential self-current” $(-c^2 \phi, -c^2 A_L)$ to produce the new net four-current source, $(c(\rho - \kappa^2 \phi), j_L - c\kappa^2 A_L)$, which, in...
view of the Lorentz condition, clearly also identically satisfies the current conservation constraint. If we now replace the original current source \((cρ, j_L)\) by this new feed-back net current source, the equation satisfied by \((\phi, A_L)\) becomes,

\[
\partial_\lambda \partial^\lambda (\phi, A_L) = (\rho - \kappa^2 \phi, j_L/c - \kappa^2 A_L),
\]

or,

\[
(\partial_\lambda \partial^\lambda + \kappa^2)(\phi, A_L) = (\rho, j_L/c).
\]

We in fact only need solve the time-component feedback equation,

\[
(\partial_\lambda \partial^\lambda + \kappa^2)\phi = \rho, \tag{7a}
\]

for \(\phi\), because, in view of the Lorentz condition, \(A_L\) is still the homogeneous functional of \(\dot{\phi}\) which is given by Eq. (5c). If we restrict ourselves to merely solving for a Green’s function \(G(r, t; \kappa)\) which satisfies,

\[
(\partial_\lambda \partial^\lambda + \kappa^2)G(r, t; \kappa) = \delta(t)\delta^{(3)}(r), \tag{7b}
\]

then a solution of Eq. (7a) which is homogeneous in the charge density \(\rho\), a key property that we insist on, will be given by,

\[
\phi(r, t) = \int dt' d^3r' \rho(r', t') G(r - r', t - t', \kappa). \tag{7c}
\]

Now a causal retarded Green’s function that satisfies,

\[
G(r, t; \kappa) = 0 \text{ when } t < 0, \tag{7d}
\]

and whose static reduction is a purely exponentially decaying Yukawa potential, is given by the inverse space-time Fourier transformation expression [2],

\[
G_{-1}(r, t; \kappa) = (2\pi)^{-4} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \int d^3k e^{i\mathbf{k}\cdot\mathbf{r}} \left[-((\omega/c) + i\epsilon)^2 + |\mathbf{k}|^2 + \kappa^2\right]^{-1}. \tag{7e}
\]

Because,

\[
(\partial_\lambda \partial^\lambda + \kappa^2) = c^{-2} \partial^2/\partial t^2 - \nabla^2 + \kappa^2, \tag{7f}
\]

it is easily seen \(G_{-1}(r, t; \kappa)\) satisfies the basic Green’s function requirement of Eq. (7b). Since both poles in the \(\omega\)-dependence of the integrand of \(G_{-1}\) occur in the lower-half \(\omega\)-plane, and because, for \(t < 0\), the \(\omega\) contour must be closed in the upper-half \(\omega\)-plane, we see that for \(t < 0\), \(G_{-1}\) vanishes, which makes it a causal retarded Green’s function. With the aid of a contour integration, the inverse spatial \(d^3\mathbf{k}\)-integration of Eq. (7e) for \(G_{-1}\) can be analytically carried out in closed form. We have proceeded to straightforwardly extend the particular expression which thereby results for \(G_{-1}\) to a class of objects whose members we denote as \(G_\eta(r, t; \kappa)\). We will now demonstrate that each such object \(G_\eta\) satisfies the basic Green’s function requirement of Eq. (7b) irrespective of the value of \(\eta\), while still retaining the causal retarded nature of \(G_{-1}\). The expression for \(G_\eta\) is in the form of an inverse time Fourier transformation times the factor \((2\pi)\),

\[
G_\eta(r, t; \kappa) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t}\left[\theta(1 - (\omega/(c\kappa))^2)g_\eta(r, \kappa(1 - (\omega/(c\kappa))^2)^{1/2}) + \theta(1 - (c\kappa/\omega)^2)h(r, \omega(1 - (c\kappa/\omega)^2)^{1/2})\right], \tag{8a}
\]

where \(r \overset{\text{def}}{=} |\mathbf{r}|\), \(\theta\) is the standard Heaviside unit step function, which equals zero for negative argument and
unity for positive argument, and
\[ g_\eta(r, \kappa') \overset{\text{def}}{=} (8\pi^2(r + \epsilon))^{-1}[\cosh(\kappa'r) + \eta \sinh(\kappa'r)], \tag{8b} \]

\[ h(r, \omega') \overset{\text{def}}{=} (8\pi^2(r + \epsilon))^{-1}e^{i\omega' r/\epsilon}, \tag{8c} \]

and \( \epsilon \) is a positive infinitesimal length. The expression in square brackets in the integrand of Eq. (8a) equals \((2\pi)^{-1}\) times \( G_\eta(r, \omega; \kappa) \), which is the time Fourier transformation of \( G_\eta(r, t; \kappa) \). From that expression and Eq. (8c) we can see that for asymptotically large values of \( |\omega| \) (i.e., asymptotically high Fourier frequencies), \( G_\eta(r, \omega; \kappa) \) behaves as \((4\pi(r + \epsilon))^{-1}e^{i\omega r/\epsilon} \), which is an outgoing spherical wave in light of the time-dependence \( e^{-i\omega t} \) of the integrand of Eq. (8a). This purely outgoing spherical-wave high-Fourier-frequency asymptotic behavior of the integrand of the inverse time Fourier transformation of \( G_\eta(r, t; \kappa) \) shows that \( G_\eta(r, t; \kappa) \) is a causal retarded Green’s function. We also note from Eqs. (8a) through (8c) that at the somewhat confusing “critical points”, \( \omega = \pm \kappa \), \( G_\eta(r, \omega; \kappa) \) is continuous as a function of \( \omega \) and assumes the value \((4\pi(r + \epsilon))^{-1}\) irrespective of the value of \( \eta \), which ensures, for any value of \( \eta \), that \( G_\eta(r, \omega; \kappa) \) is a sensibly continuous well-defined function of \( \omega \). It turns out that the \( G_\eta(r, t; \kappa) \) of Eq. (8a) can be expressed in terms of Bessel functions [2], but it doesn’t appear to be necessary or worthwhile to write them as such here. The positive infinitesimal length \( \epsilon \) in Eqs. (8b) and (8c) can usually be dropped, but the key exception is that, if one bears in mind the identity,

\[ \nabla^2 f(r) = r^{-1}d^2(rf(r))/dr^2, \tag{8d} \]

then meticulously careful calculation yields,

\[ -\nabla^2((8\pi^2(r + \epsilon))^{-1}) = (4\pi^2r)^{-1}e/(r + \epsilon)^3 \xrightarrow{\epsilon \to 0^+} (2\pi)^{-1}\delta^{(3)}(r). \tag{8e} \]

From Eqs. (8b) through (8c), it is readily shown that,

\[ -\nabla^2 g_\eta(r, \kappa') = (2\pi)^{-1}\delta^{(3)}(r) - (\kappa')^2 g_\eta(r, \kappa'), \tag{8f} \]

and,

\[ -\nabla^2 h(r, \omega') = (2\pi)^{-1}\delta^{(3)}(r) + (\omega'/\epsilon)^2 h(r, \omega'). \tag{8g} \]

From Eqs. (8f) and (8g), together with Eq. (7f) we obtain,

\[ (\partial_\lambda \partial^\lambda + \kappa^2)(e^{-i\omega t}g_\eta(r, \kappa(r(1 - (\omega/(\kappa \omega))^2)\frac{1}{\epsilon^2}))) = (2\pi)^{-1}e^{-i\omega t}\delta^{(3)}(r), \tag{8h} \]

and,

\[ (\partial_\lambda \partial^\lambda + \kappa^2)(e^{-i\omega t}h(r, \omega(r(1 - (\kappa^2/\omega))^2)\frac{1}{\epsilon^2}))) = (2\pi)^{-1}e^{-i\omega t}\delta^{(3)}(r), \tag{8i} \]

and therefore,

\[ (\partial_\lambda \partial^\lambda + \kappa^2)G_\eta(r, t; \kappa) = (2\pi)^{-1}\delta^{(3)}(r) \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} = \delta(t)\delta^{(3)}(r), \tag{8j} \]

which shows that all \( G_\eta(r, t; \kappa) \), regardless of the value of \( \eta \), are Green’s functions of \((\partial_\lambda \partial^\lambda + \kappa^2)\). In the limit that we do not feed back, i.e., that \( \kappa \to 0^+ \), we obtain,
\[ G_\eta(r, t; \kappa = 0) = \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} h(r, \omega) = (8\pi^2 r)^{-1} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-r/c)} = (4\pi r)^{-1} \delta(t - r/c), \]  

which is the standard causal \textit{retarded} Green’s function of the Abelian gauge theory—together with Eq. (7c) it yields the celebrated Eq. (6). Note that in this no-feedback limit where \( \kappa \to 0^+ \), there is no dependence whatsoever on \( \eta \).

Finally, in the interesting case that the source charge density \( \rho(r, t) \) has no time dependence, i.e., is static, we readily see from Eq. (7c) that the corresponding static Green’s function, \( G_\eta(r; \kappa) \), is simply given by the integral over all time of the dynamical Green’s function, \( G_\eta(r, t; \kappa) \). Since,

\[ \int_{-\infty}^{+\infty} dt e^{-i\omega t} = (2\pi)\delta(\omega), \]  

we obtain from this and Eqs. (8a) and (8b) that,

\[ G_\eta(r; \kappa) = (2\pi)g_\eta(r, \kappa) = (4\pi r)^{-1}[\cosh(\kappa r) + \eta \sinh(\kappa r)], \]

which properly reduces to the point-charge static Coulomb potential \((4\pi r)^{-1}\) in the limit of no feedback, i.e., \( \kappa \to 0^+ \). For \( \eta = -1 \), it is the classic purely exponentially \textit{decaying} point-charge static Yukawa potential,

\[ (4\pi r)^{-1} \exp(-\kappa r), \]  

which is the static reduction of the classic retarded causal Fourier-transformation Green’s function that is given by Eq. (7e). In the case of non-vanishing feedback, we began with this particular inverse space-time Fourier-transformation Green’s function of Eq. (7e), which we denoted as \( G_{-1}(r, t; \kappa) \). We then \textit{extended} it’s inverse \textit{purely time} Fourier transformation to a whole additional class of extremely similar inverse \textit{purely time} Fourier transformations for which, however, the corresponding inverse \textit{spatial} Fourier transformations simply \textit{fail to exist because those Green’s functions feature growing exponential components in configuration space}. We have denoted the members of this class as \( G_\eta(r, t; \kappa) \) for arbitrary values of \( \eta \), and their static reductions, which \textit{prominently display} those \textit{growing} exponential components (\textit{except} for the case \( \eta = -1 \)), are given by Eq. (10b) above.

In diametrical opposition to the classic purely exponentially \textit{decaying} static Yukawa potential associated with \( \eta = -1 \), there is the purely exponentially \textit{growing} static “contra-Yukawa” potential associated with \( \eta = +1 \), namely \((4\pi r)^{-1} \exp(+\kappa r)\), which although patently unavailable via inverse spatial Fourier transformation, is just as much a legitimate consequence of the potential feedback idea as is the classic purely exponentially \textit{decaying} Yukawa case. Indeed the range of \( \eta \)-values which have \textit{some admixture} of exponentially growing static Yukawa potential component \textit{utterly swamps} the single \( \eta = -1 \) value for which that component is precariously just canceled out. But because of the \textit{immense bias} introduced by the almost universally employed inverse spatial Fourier transformation \cite{Yukawa}, the \textit{overwhelming prevalence} of feedback-Yukawa static potentials which \textit{grow exponentially rather than decay exponentially} has been effectively \textit{entirely wiped off the radar screen}!

**Conclusion**

One can at least entertain the \textit{hope} that the feedback-Yukawa exponentially \textit{growing} static potential components provide a vital clue as to the mechanism of quark-gluon confinement. Of course this idea, here treated only in a bare-bones Abelian gauge theory, needs to be properly implemented in the non-Abelian Yang-Mills color gauge theory, where the whole \( \eta \)-range of mixed-component exponentially growing and exponentially decaying static Yukawa potentials can hopefully be \textit{linked to color} in such a way that \textit{only the traditional} \( \eta = -1 \) \textit{purely exponentially decaying Yukawa potential operates for color singlet entities}. The apparently exponentially \textit{growing} nature of the confinement potential also provides some feeling for why a quark-gluon “plasma” that is too “hot” (strongly thermally disorganized) to be able to readily reorganize itself into \textit{unconfined color singlets}—i.e., “hot” enough to be “colorful”—can \textit{never} have characteristics that are \textit{at all akin} to those of a free gas.

It is worthwhile to point out once again that the timelike/longitudinal \textit{potential phenomena} that we have
treated here in causal Lorentz gauge are *strictly non-dynamical* in nature, and thus are *absolutely not subject to independent quantization*: the independent dynamical gauge *quanta* are all associated to the *transverse* components of the gauge field, which in causal Lorentz gauge can be cleanly and covariantly *separated* from the timelike/longitudinal nondynamical potential phenomena discussed here.

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