A combinatorial and field theoretic path to quantum gravity: 
the new challenges of group field theory

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Group field theories are a new type of field theories over group manifolds and a generalization of matrix models, that have recently attracted much interest in quantum gravity research. They represent a development of and a possible link between different approaches such as loop quantum gravity and simplicial quantum gravity. After a brief introduction to the GFT formalism we put forward a long but still far from exhaustive list of open issues that this line of research faces, and that could represent interesting challenges for mathematicians and mathematical physicists alike.

I. INTRODUCTION

This article (as our talk at the workshop) has one main goal: to draw attention on one recent approach to quantum gravity, named group field theory, and on some of the many outstanding open issues of this approach, in particular those that can be (in our humble opinion) both mathematically attractive for those mathematicians and mathematical physicists interested in combinatorics and quantum field theory, as well as physically crucial from a quantum gravity perspective. The hope and the expectation (as stressed during the workshop) are therefore that more mathematicians and mathematical physicists will join quantum gravity theorists in the analysis and development of the group field theory formalism, for mutual benefit and amusement, as well as for speeding up progress towards a complete theory of quantum gravity.

Already at this point one reason of interest in GFTs becomes apparent: GFTs can potentially represent a common framework for different current approaches to quantum gravity, in particular canonical loop quantum gravity and simplicial quantum gravity formalisms, namely quantum Regge calculus and (causal) dynamical triangulations, because the same mathematical structures that characterize these approaches also enter necessarily and in very similar fashion in the GFT framework. Let us be slightly more explicit here, even though all of this will become clear later on when the GFT formalism will be described in more detail. The connection with loop quantum gravity arises first of all because GFT boundary states are given by (open or closed) spin network states, i.e. graphs labelled by group representations or group elements, which are indeed the kinematical quantum states of gravity as discovered by loop quantum gravity. Also, when the GFT partition function is expanded in Feynman diagrams, they turn out to be given by spin foams, i.e. labelled 2-complexes that first arose in the loop quantum gravity context to represent the histories of spin network states, and the Feynman amplitudes that are associated to them are nothing else than spin foam models, i.e. combinatorial and algebraic sum over histories first introduced to encode the dynamics of loop quantum gravity states. The same Feynman diagrams, as we shall see, identify simplicial complexes to which

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the GFT assigns geometric data, weighted by amplitudes that can be derived from or related to path integrals for simplicial gravity on the given complex. Thus, for given Feynman diagram, the GFT provides a model quantization of gravity in the spirit and language of quantum Regge calculus, while, for given assignment of field degrees of freedom (i.e. fixing the geometric data), the same GFT provides a definition of the dynamics of quantum geometry via a sum over triangulations (the perturbative expansion of the same GFT) in the same spirit of the dynamical triangulations approach.

We refer the reader to the literature (especially [1]) for a more extensive discussion on this. Here we stress only that, this set of relations between different approaches to quantum gravity, within the GFT formalism, suggests that any future development and improved understanding of any aspect of GFTs is likely to have implications and an impact in all of them. In this sense, at the very least, GFTs may play a crucial role in current quantum gravity research, in our opinion.

II. THE GFT FORMALISM

We now proceed to introduce the GFT formalism, in its main features. Our treatment is going to be rather sketchy, and we refer once more to the literature, in particular the reviews [1 2 3], for a more complete and detailed treatment and a more extensive list of references.

A. Kinematics

We start from a field taken to be a \( C \)-valued function of \( D \) group elements, for a generic group \( G \), one for each of the \( D \) boundary (D-2)-faces of the (D-1)-simplex that the field \( \phi \) represents:

\[
\phi(g_1, g_2, ..., g_D) : G^\times D \to C.
\]

We can identify the ordering of the arguments of the field with a choice of orientation for the (D-1)-simplex it represents, and we then encode the orientation properties of the corresponding (D-1)-simplex in the complex structure by requiring invariance of the field under even permutations \( \sigma \) of its arguments (that do not change the orientation) and trading odd permutations of them with complex conjugation of the field. Given this symmetry, then, a more precise definition of the field is: \( \phi(g_1, ..., g_D) = \sum_\sigma \phi(g_{\sigma(1)}, ..., g_{\sigma(D)}) \). Other symmetry properties can also be considered.

An additional symmetry that is usually imposed on the field (but this is again model-dependent, of course) is the invariance under diagonal action of the group \( G \) on the \( D \) arguments of the field: \( \phi(g_1, ..., g_D) = \phi(g_1 g, ..., g_D g) \). This is the simplicial counterpart of the Lorentz gauge invariance of continuum and discrete first order gravity actions, and it has also the geometric interpretation, at the simplicial level, of requiring the \( D \) faces of a (D-1)-simplex to close to form the close \( S^{D-2} \) surface representing its boundary. This reduces the number of degrees of freedom from those represented by the group elements associated to the \( (D-2) \) faces, but at the same time intertwines them in a rather non-trivial way, as it is clear by the resulting more complicated geometry and topology of configuration space.

As in any other field theory, a momentum representation for the field and its dynamics is also available and is obtained by harmonic analysis on the group manifold \( G \). The field can be expanded in modes as:

\[
\phi(g_i) = \sum_{J, \Lambda, k_i} \phi_{J, \Lambda}^{k_i} \left( \prod_i D_{k_i}^{J_i}(g_i) \right) C_{i_1, ..., i_D}^{J_1, ..., J_D \Lambda},
\]

with the \( J \)'s labelling representations of \( G \), the \( k \)'s vector indices in the representation spaces, and the \( C \)'s being intertwiners of the group \( G \). We have labelled an orthonormal basis of intertwiners by an extra parameter \( \Lambda \) (depending on the group chosen and on the dimension \( D \), this may actually be a shorthand notation for a set of parameters).

That this decomposition is possible is not guaranteed in general (as the harmonic analysis of non-compact groups, for example, may not be under control), but it is in fact true for all the known quantum gravity GFT models, which are based on the Lorentz group or on extensions of it. A geometric interpretation of the field variables is obtained either looking at the Feynman amplitudes for the GFT at hand, in turn usually obtained from a discretization of some continuum gravity action, or from the direct (1st) quantization of simplicial structures, e.g. by geometric quantization methods. The group variables are then seen to represent parallel transport of a (gravity) connection along elementary paths dual to the (D-2)-faces, and the representations \( J \) are usually put in correspondence with the volumes of the same (D-2)-faces, the details of this correspondence depending, however, on the specific model.

Just as one identifies a single field (or the corresponding 1st quantized wave function) with a single (D-1)-simplex, a simplicial space built out of \( N \) such (D-1)-simplices is described by the tensor product of \( N \) such wave functions, with
FIG. 1: For the $D = 3$ case, the association of a field with a 2-simplex, a triangle, or equivalently its dual vertex, and of its arguments with the edges (1-faces) of the triangle, or equivalently with the links incident to the vertex, together with the consequent labelling of the by group-theoretic variables.

FIG. 2: A ‘2-particle state’(again, in the $D=3$ example)

suitable constraints implementing the fact that some of their (D-2)-faces are identified. For example, a state describing two (D-1)-simplices glued along one common (D-2)-face would be represented by: $\phi^{J_1 J_2 \ldots J_D, \Lambda}$$\phi^{J_1 J_2 \ldots J_D, \Lambda}$, where the gluing is along the face labelled by the representation $J_2$, and effected by the contraction of the corresponding vector indices (of course, states corresponding to disjoint (D-1)-simplices are also allowed).

We see that states of the theory are then labelled, in momentum space, by spin networks of the group $G$. The corresponding second quantized theory is a GFT. Spin networks are also relevant for the construction of GFT observables. These are given by gauge invariant functionals of the GFT field, and can be constructed in momentum space using spin networks according to the formula:

$$O_{\Psi^{(\gamma, j_e, i_v)}}(\phi) = \left( \prod_{(ij)} \int dg_{ij} dg_{ji} \right) \Psi_{(\gamma, j_e, i_v)}(g_{ij} g_{ji}^{-1}) \prod_i \phi(g_{ij}),$$

where $\Psi_{(\gamma, j_e, i_v)}(g)$ identifies a spin network functional for the spin network labelled by a graph $\gamma$ with representations $j_e$ associated to its edges and intertwiners $i_v$ associated to its vertices, and $g_{ij}$ are group elements associated to the edges $(ij)$ of $\gamma$ that meet at the vertex $i$.

### B. Dynamics

On the basis of the above kinematical structure, one aims at defining a field theory for describing the interaction of fundamental building blocks of space ((D-1)-simplices or spin network vertices), and in which a typical interaction process will be characterized by a D-dimensional simplicial complex. In the dual picture, the same will be represented as a spin foam (labelled 2-complex). This is the straightforward generalization of the way in which 2d discretized surfaces emerge from the interaction of matrices (graphically, segments), or ordinary Feynman graphs emerge from the interaction of point particles. A discrete spacetime emerge then from the theory as a virtual construct, a possible interaction process among the quanta of the theory.

In order for this to be realized, the classical field action in group field theories has to be chosen appropriately. In this choice lies the main peculiarity of GFTs with respect to ordinary field theories. This action, in configuration space, has the general QFT structure:

$$S_D(\phi, \lambda) = \frac{1}{2} \left( \prod_{i=1}^{D} \int dg_i d\tilde{g}_i \right) \phi(g_i) \mathcal{K}(g_i g_i^{-1}) \phi(\tilde{g}_i) + \frac{\lambda}{(D+1)!} \left( \prod_{i \neq j=1}^{D+1} \int dg_{ij} \right) \phi(g_{ij}) \ldots \phi(g_{D+1, j}) \mathcal{V}(g_{ij} g_{ji}^{-1}), \quad (1)$$
and it is of course the choice of kinetic and interaction functions $K$ and $V$ that define the specific model considered. Obviously, the same action can be written in momentum space after harmonic decomposition on the group manifold.

The mentioned peculiarity is in the combinatorial structure of the pairing of field arguments in the kinetic and vertex terms, as well as from their degree as polynomials in the field. The interaction term describes the interaction of D+1 (D-1)-simplices (the fundamental 'atoms of space') to form a D-simplex ('the fundamental virtual 'atom of spacetime') by gluing along their (D-2)-faces (arguments of the fields), that are pairwise linked by the interaction vertex. The nature of this interaction is specified by the choice of function $V$. The (quadratic) kinetic term involves two fields each representing a given (D-1)-simplex seen from one of the two D-simplices (interaction vertices) sharing it, so that the choice of kinetic functions $K$ specifies how the information and therefore the geometric degrees of freedom corresponding to their D (D-2)-faces are propagated from one vertex of interaction (fundamental spacetime event) to another. Let us mention here that one can consider generalizations of the above combinatorial structure, for example defining vertex functions whose combinatorics corresponds to the gluing of (D-1)-simplices to form different sorts of D-dimensional complexes (e.g. hypercubes etc). We are not going to elaborate further on this.

Because of the peculiar way in which field arguments are paired in the interaction term, we may consider GFTs as \textit{combinatorially non-local field theories} (as opposed to theories in which the nonlocality is originated by higher derivatives terms, for example). Before detailing more the interesting structure of Feynman diagrams resulting from this combinatorial nonlocality, let us stress once more that it is basically in it that lies the peculiarity of GFTs as field theories. Indeed, as for the rest, we have an almost ordinary field theory, in that we can rely on a fixed background metric structure, given by the invariant Killing-Cartan metric on the group manifold, a fixed topology, given again by the topology of the group manifold, the usual splitting between kinetic (quadratic) and interaction (higher order) term in the action, that will later allow for a straightforward perturbative expansion, and the usual conjugate pictures of configuration and momentum space. This allows us to use all usual QFT techniques and language in the analysis of GFTs, and thus of quantum gravity, even though we remain in a background independent (in the physical sense of 'spacetime independent') context. The importance of this, in a non-perturbative quantum gravity framework, should not be underestimated, we think.

Let us now turn to the quantum dynamics. Most of the research in this area has concerned the perturbative aspects of this dynamics around the no-particle state, the complete vacuum, and the main guide for model building have been, up to now, only the properties of the resulting Feynman amplitudes. This relevant Feynman expansion is:

$$Z = \int D\phi e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^{N_v(\Gamma)}}{\text{sym}[\Gamma]} Z(\Gamma),$$

where $N_v$ is the number of interaction vertices $v$ in the Feynman diagram $\Gamma$, $\text{sym}[\Gamma]$ is the number of automorphisms of $\Gamma$ and $Z(\Gamma)$ the corresponding Feynman amplitude. Each edge of the Feynman graph is made of $D$ strands, one for each argument of the field\(^1\) and each one is then re-routed at the interaction vertex, with the combinatorial structure of an $D$-simplex, following the pairing of field arguments in the vertex operator. Diagrammatically:

Each strand in an edge of the Feynman diagram goes through several vertices, coming back where it started, for closed Feynman diagrams, and therefore identifies a 2-cell (for open graphs, it may end up on the boundary, identifying then an open 2-cell). Each Feynman diagram $\Gamma$ is then a collection of 2-cells (faces), edges and vertices, i.e. a 2-complex, that, because of the chosen combinatorics for the arguments of the field in the action, is topologically dual

\(^1\) One could write explicitly down and keep track of each of the spacetime coordinates on which the field depends also in ordinary QFT in Minkowski space, but it would be a pedantic and rather useless complication. The need to keep track of each argument of the field, in a GFT context, comes from the mentioned nonlocality of the interaction term, which is at the origin of the combinatorially nontrivial structure of the resulting Feynman diagrams.
to a D-dimensional simplicial complex. Notice that the resulting 2-cells can be glued (i.e. can share edges) in all sorts of ways, forming for example “bubbles”, i.e. closed 3-cells.

No restriction on the topology of the resulting diagram/complex is imposed, a priori, in their construction, so the resulting complexes/triangulations can have arbitrary topology. Each resulting 2-complex or triangulation corresponds to a particular scattering process of the fundamental building blocks of space, i.e. (D-1)-simplices. Each line of propagation, made as we said out of D strands, is labelled, on top of the group/representation data, by a permutation of (1,...,D), representing the labelling of the field variables, and all these data are summed over in the construction of the Feynman expansion. The sum over permutations affects directly the combinatorics of the allowed gluings of vertices with propagators. The above choice of permutation symmetry for the field (with the orientation encoded in the complex structure) implies that only even permutation appear as labellings of propagation lines. In turn, this ensures that only orientable complexes are generated in the Feynman expansion of the field theory (see [8] for a more detailed treatment). The more restrictive choice of invariance of the field under any permutation of its arguments results, for example, in the presence of non-orientable complexes as well in the Feynman expansion.

FIG. 3: The gluing of vertices of interaction through propagators, again in the D=3 example. The rectangles represent the additional integrations imposing gauge invariance under the action of $G$, while the ellipses represent the implicit sum over permutations of the (labels of the) strands to be glued.

FIG. 4: An example, in D=3, of a closed GFT Feynman diagram, with 4 vertices and 8 propagators.

As said, each strand in a propagation line carries a field variable, i.e. a group element in configuration space or a representation label in momentum space. After the closure of the strand to form a 2-cell in a closed diagram, the same representation label ends up being associated to this 2-cell. Therefore in momentum space each Feynman graph is given by a spin foam (a 2-complex with faces labelled by representation variables), and each Feynman amplitude (a complex function of the representation labels, obtained by contracting vertex amplitudes with propagator functions) by a so-called spin foam model [9]:

$$Z(\Gamma) = \sum_{J_f} \prod_f A_f(J_f) \prod_e A_e(J_e) \prod_v A_v(J_v),$$

where we have highlighted the fact that the amplitudes can be factori

Given the mentioned geometric interpretation of the representation variables (edge lengths, areas, etc) (see [1, 9]), each of these Feynman amplitudes corresponds...
to a definition of a sum-over-histories for discrete quantum gravity on the specific triangulation dual to the Feynman graph, although the quantum amplitudes for each geometric configuration are not necessarily given by the exponential of a discrete gravity action.

One can show that the inverse is also true: any local spin foam model can be obtained from a GFT perturbative expansion \[3, 11\] (even though this does not imply that the reconstruction of the underlying GFT action from the knowledge of the spin foam amplitudes is immediate nor easy). This implies on the one hand that the GFT approach incorporates the spin foam in its perturbative aspects, and on the other hand that it goes potentially far beyond it, since there is of course much more in a QFT than its perturbative expansion. The sum over Feynman graphs gives then a sum over spin foams, and equivalently a sum over triangulations, augmented by a sum over algebraic data (group elements or representations) with a geometric interpretation, assigned to each triangulation. This perturbative expansion of the partition function also allows for a perturbative evaluation of expectation values of GFT observables, as in ordinary QFT. In particular, the transition amplitude (probability amplitude for a certain scattering process) between certain boundary data represented by two spin networks, of arbitrary combinatorial complexity, can be expressed as the expectation value of the field operators having the same combinatorial structure of the two spin networks \[1, 3\].

\[
\langle \Psi_1 | \Psi_2 \rangle = \int D\phi O_{\Psi_1} O_{\Psi_2} e^{-S(\phi)} = \sum_{\Gamma/\partial \Gamma = \gamma_{\Psi_1} \cup \gamma_{\Psi_2}} \frac{\lambda^N}{\text{sym}[\Gamma]} Z(\Gamma)
\]

where the sum involves only 2-complexes (spin foams) with boundary given by the two spin networks chosen.

The above perturbative expansion involves thus two types of sums: one is the sum over geometric data (group elements or representations of \(G\)) entering the definition of the Feynman amplitudes as the GFT analogue of the integral over momenta or positions of usual QFT; the other is the overall sum over Feynman diagrams. We stress again that, in absence of additional restrictions being imposed on the GFT, the last sum includes a sum over all triangulations for a given topology and a sum over all topologies\(^2\).

C. Examples

We now give a few examples of specific GFT models, again referring to the literature for more details.

1. \(D=2\) and matrix models

The easiest example is a straightforward generalization of matrix models for 2d quantum gravity to a GFT, obtained by adding group structure to them, but keeping the same combinatorics, and it is given:

\[
S[\phi] = \int_G dg_1 dg_2 \phi(g_1, g_2) \phi(g_1, g_2) + \frac{\lambda}{3} \int dg_1 dg_2 dg_3 \phi(g_1, g_2) \phi(g_1, g_3) \phi(g_2, g_3)
\]

where \(G\) is a generic compact group, say \(SU(2)\), and the symmetries mentioned above are imposed on the field \(\phi\) implying, in this case: \(\phi(g_1, g_2) = \phi(g_1, g_2^{-1})\). The relation with matrix models is apparent in momentum space, expanding the field in representations \(j\) of \(G\) to give:

\[
S[\tilde{\phi}] = \sum_j \text{dim}(j) \left( \frac{1}{2} \text{tr}(\tilde{\phi}_j^2) + \frac{\lambda}{3} \text{tr}(\tilde{\phi}_j^3) \right)
\]

where the field modes \(\tilde{\phi}_j\) are indeed matrices with dimension \(\text{dim}(j)\). Thus, one gets a sum of matrix models actions of increasing dimensions. Alternatively, one can see the above as the action for a single matrix model in which the dimension of the matrices has been turned from a parameter into a dynamical variable.

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\(^2\) There is no algorithmic procedure that allows to distinguish topologies in \(D \geq 3\), so we cannot partition this sum into the two sub-sums mentioned. What ensures us that all topologies are present in the full sum is, however, that any simplicial complex is obtained by an appropriate gluing and face identification of fundamental simplicial building blocks, and that all possible such gluings and identifications are included in the GFT Feynman expansion.
2. \(D=3\) and topological BF theory

Another much studied example (actually the best understood one) is the group field theory, first proposed by D. Boulalov, for topological BF theory in 3d, which in turn is closely related to 3-dimensional quantum gravity in first order formalism.

Kinetic and vertex terms are then chosen as follows:

\[
\mathcal{K}(g_i, \tilde{g}_i) = \int_G dg \prod_i \delta(g_i \tilde{g}_i^{-1} g), \quad \mathcal{V}(g_{ij}, g_{ji}) = \prod_i \int_G dg_i \prod_{i<j} \delta(g_i g_{ij}^{-1} g_{ji}^{-1}),
\]

where the integrals impose the gauge invariance under the action of \(G\). The choice of \(G = SO(3)\) or \(G = SO(2,1)\) provides a quantization of 3d gravity in the Euclidean and Minkowskian signatures, respectively, and the Feynman amplitudes are given by the so-called Ponzano-Regge spin foam model. The choice of the quantum group \(SU(2)_q\) gives the Turaev-Viro topological invariant, conjectured to correspond, when \(q\) is a root of unity, to 3d gravity with positive cosmological constant. For the \(SU(2)\) case, and for real field, the action is then:

\[
S[\phi] = \prod_i \int_{SU(2)} \phi(g_1,g_2,g_3) \phi(g_1,g_2,g_3) + \frac{\lambda}{4!} \prod_{i=1}^6 \int_{SU(2)} dg_i \phi(g_1,g_2,g_3) \phi(g_3,g_4,g_5) \phi(g_5,g_2,g_6) \phi(g_6,g_1,g_1),
\]

and the Feynman amplitudes are:

\[
Z(\Gamma) = \left( \prod_{e \in \Gamma} \int dg_{e*} \right) \prod_{f*} \delta(e*) \prod_{e \in \partial f*} g_{e*}
\]

where \(e*\) are dual edges of the 2-complex and \(f*\) its 2-cells. This is the same quantity one obtains from a path integral quantization of discretized BF theory on the triangulation dual to \(\Gamma\), confirming the above-given interpretation.

Lots is known about the last model, including the appropriate gauge fixing procedure of its Feynman amplitudes, the coupling of matter fields, etc. and we refer once more to the literature for more details, starting from [10].

3. Models with additional structure

One may then consider models with additional structure and more involved kinetic and vertex operators. One example are the generalized models of [12], based, in the \(D=3\) case, on a complex field on \((G \times \mathbb{R})^3\), but with a \(G\)-only diagonal invariance, with \(G = SU(2)\) or \(G = SL(2,\mathbb{R})\) in the Riemannian and Lorentzian cases respectively. The kinetic and vertex operators are then:

\[
\mathcal{K}(g_i, s_i, \tilde{g}_i, \tilde{s}_i) = \sum_{\sigma} \int dg \prod_i (i\partial + \Box) \delta(g_i \tilde{g}_i^{-1}) \delta(s_i - \tilde{s}_i^{-1}) \quad \mathcal{V}(g_{ij}, s_{ij}) = \prod_{i \neq j} \delta(g_{ij}^{-1}) \delta(s_{ij} - s_{ji})
\]

where \(g_i \in G, s_i \in \mathbb{R}\) and \(\Box\) is the Laplace-Beltrami operator on \(G\).

One more example of a similar type, a sort of ‘relativistic’upgrading of the previous one, and that is currently being studied and developed [13], as the possible explicit common ground for loop quantum gravity and simplicial quantum gravity, uses instead a complex field over \((G \times \mathbb{R}^3)^3\) and the kinetic and vertex terms:

\[
\mathcal{K}(g_i, x_i, \tilde{g}_i, \tilde{x}_i) = \sum_{\sigma} \int dg \prod_i (\triangle + \Box) \delta(g_i \tilde{g}_i^{-1}) \delta(x_i - \tilde{x}_i^{-1}) \quad \mathcal{V}(g_{ij}, x_{ij}) = \prod_{i \neq j} \delta(g_{ij}^{-1}) \delta(x_{ij} - x_{ji})
\]

where \(g_i \in G, x_i \in \mathbb{R}^3, \triangle\) is now the Laplace-Beltrami on \(\mathbb{R}^3\) and \(\Box\) is again the Laplace-Beltrami on \(G\). The interest in this last model lies in the fact that the corresponding Feynman amplitudes have exactly the form of path integrals for simplicial quantum gravity, i.e. quantum Regge calculus, in 1st order form. Similar models exist in higher dimension (\(D \geq 3\)) [13].
III. A (FAR FROM COMPREHENSIVE) LIST OF OPEN ISSUES

We now outline, in a rather schematic way, a list of open issues within the group field theory approach, that are in our opinion of considerable mathematical interest as well as crucial for a better physical understanding of this class of models. We stress once more that this list is, by all means, far from exhaustive, even from a purely mathematical perspective. At the same time, we also stress that it is only for the stated aims of this article that we do not review or discuss more what, instead, is known about GFTs, and the many results already obtained in this approach. For an account of them, we refer to the cited literature [1, 2, 3].

A. Classical solutions and their relevance

GFTs were historically developed from the perspective of either loop quantum gravity and spin foam models or dynamical triangulations, and thus the focus of attention has always been their perturbative structure and the properties of their Feynman amplitudes. However, changing slightly the perspective on them, and seeing them just as field theories (of a peculiar type), the first thing one would think of analyzing is their classical structure. In particular, one would start from their classical equations of motion. The importance of these equations from the GFT point of view are obvious: they define the classical dynamics of the field theory, they identify the classical background configurations around which to expand in a semi-classical perturbative definition of the path integral, etc. Their relevance from a quantum gravity perspective is easily understood if one recalls the interpretation of GFTs as second quantized theories of simplicial geometry or of spin network states. Then the GFT classical action should encode the full 1st quantized dynamics, and the classical equations of motion should correspond to the full 1st quantized wave function equations. In the specific case of gravity, then, solving the GFT classical equations means identifying non-trivial quantum gravity wave functions satisfying all the quantum gravity constraints, an important and still unachieved goal of canonical quantum gravity, even in the modern loop quantized formalism.

The classical equations of motion following from the general form of the action (considering for simplicity a real field) are:

\[
\prod_j \int d\tilde{g}_j K(g_1\tilde{g}_1^{-1})\phi(\tilde{g}_j) + \frac{\lambda}{D!} \prod_{j=1}^{D+1} \int dg_1...dg_{D+1} \phi(g_2)...\phi(g_{D+1}) V(\{g_i\}) = 0.
\]

For a generic differential operator \( K \), they are then rather complicated non-linear equations of integro-differential type, with the main complications coming once more from the particular pairing of variables in the interaction term. No detailed analysis of these equations in any specific GFT model nor of their solutions has been carried out to date.

Solving these equations in complete generality, obtaining a complete classification of their exact solutions, is probably beyond reach, even in low D. However, two more modest goals are probably within reach, and both would be of importance.

1) Identify at least some exact solutions of the full GFT equations, for some currently studied GFT model, and analyze their mathematical structure and physical meaning. Most likely, this has to be done first in symmetry reduced cases, and, on top of identifying some particularly simple solution of a given GFT model, it would be of considerable interest to develop a general theory of symmetry reduction for GFTs, to be then applied to the various models.

2) Develop an exact formalism for obtaining approximate solutions to the above field equations, e.g. in perturbation expansion in the coupling parameter \( \lambda \), or in some other parameter that could allow to take into account both the non-linearity of the equations and the non-local nature of the field coupling in a perturbative way.

Indeed, the classical solutions of the (local and linear) free field theory equations

\[
\left( \prod_j \int d\tilde{g}_j \right) K(g_j\tilde{g}_j^{-1})\phi(\tilde{g}_j) = 0
\]

are rather easily obtained, but at the same time are not very interesting physically, for most known models.

Let us now given one example of a GFT equation of motion, to clarify some of the above issues. Take the, arguably simplest, GFT model the corresponding equations are:

\[
\int d\tilde{g} \phi(g_1\tilde{g}, g_2\tilde{g}, g_3\tilde{g}) + \lambda \prod_{i=1}^{3} \int d\tilde{g}_i \prod_{j=4}^{6} \int dg_j \phi(g_3\tilde{g}_1, g_4\tilde{g}_1, g_5\tilde{g}_1) \times \phi(g_5\tilde{g}_2, g_6\tilde{g}_2, g_2\tilde{g}_2, g_6\tilde{g}_3, g_4\tilde{g}_3, g_1\tilde{g}_3) = 0
\]
where we have included additional integrals implementing the gauge invariance. We do not know the general solution of this equation, nor any exact method for obtaining it or any exact procedure for approximating it. We do know some exact simplified (families of) solutions \[14\], though, which are the following:

\[
\phi_f(g_1, g_2, g_3) = \sqrt{3!/3!} \int dg \delta(g_1 g) f(g_2 g) \delta(g_3 g),
\]

parametrized by a real function \( f \) on \( SU(2) \). More solutions of the same type can be constructed, even in higher dimensions.

However, there is a lot yet to be clarified concerning the physical meaning of these simple known solutions, and how their mathematical structure implements it. We refer in particular to the issue of topology change. Let us clarify. As discussed above, the GFT \[3\] provides a quantization of topological BF theory in 3d, which takes into account topology changing configurations, as it is clear in perturbative expansion. Accordingly, \( any \) classical solution should be interpreted as the wave function for a \textit{flat} geometry and some prescribed dynamics for the \textit{topological} degrees of freedom. How a flat geometry is implemented in the above solutions, and which prescription for the dynamics for space topology is encoded in them, as well as how the two ingredients are intertwined, is mathematically yet to be understood. Same applies to other known simple solutions.

The same issues, from the purely mathematical ones to those concerning the physical interpretation, are of course even more complex and interesting at the same time in the \( D = 4 \) case, for non-topological models, where geometry itself has a highly non-trivial dynamics. For example, the model \[7\] which presents already a set of non-trivial free field solutions (e.g. (the group analogue of) plane waves on both \( \mathbb{R}^3 \) and \( G \)).

**B. Classical and quantum Hamiltonian analysis**

Still sticking to a purely formal field theoretic perspective on GFTs, i.e. keeping aside momentarily the quantum gravity interpretation of the same, one more basic aspect of them is both undeveloped, and mathematically challenging: their canonical/Hamiltonian formulation. A general formalism for the canonical analysis of GFTs at both classical and quantum levels has been recently proposed \[16\], and applied to the model \[6\] but it represents, especially at the mathematical level, only a first step.

What are the main difficulties involved in this analysis for GFTs?

Once more, they stem from the peculiar combinatorial structure of the GFT action, but also from the need, resulting from the quantum gravity interpretation, to deal with all of the field arguments on equal footing, since they propagate almost independently from each other (due again to the combinatorics in the vertex).

Let us be more explicit. Consider the action with kinetic and vertex term \[6\] and restrict the attention to the kinetic term only (determining the symplectic structure of the theory and thus providing the basis for the hamiltonian analysis):

\[
S = \left( \prod_i \int_G dg_i \int_\mathbb{R} ds_i \right) \phi_{\text{kin}}(g_1, s_1; \ldots; g_D, s_D) \prod_i (i \partial_{s_i} + \Box_i) \phi(g_1, s_1; \ldots; g_D, s_D) + \text{h.c.}
\]

for generic group \( G \) (Riemannian or Lorentzian).

The kinetic term has the structure of a product of differential operators, each acting independently on one of the \( D \) (sets of) arguments of the field. Each of them is a Schroedinger-like operator with “Hamiltonian”\( \Box \). This suggests that one should consider the variables \( s_i \) as “time”variables, to be used in a GFT generalization of the usual time+space splitting of the configuration space coordinates, with the group elements treated instead as “space”. This is reasonable (and indeed the correct thing to do), but it implies we have a field theory with \( D \) “times”, all to be treated on equal footing. In turn this immediately implies that the naive phase space has coordinates \( (\phi, \pi_i = \frac{\delta L}{\delta \dot{\phi}_i}) \), with one field and \( D \) conjugate momenta. Clearly, a generalization of usual Hamiltonian mechanics is needed. One approach, and the one chosen in \[16\], is to use the DeDonder-Weyl generalized Hamiltonian mechanics, as developed at both the classical and quantum level by Kanatchikov \[13\], as a starting point and to adapt it to the peculiar GFT setting. The framework chosen is thus that of \textit{polysymplectic (or polymomentum) mechanics} \[13\], and we refer to the literature for more mathematical details on this beautiful formalism.

The general idea of how to adapt this general formalism to the GFT case is the following. One starts from a “covariant”definition of momenta, hamiltonian density, Poisson brackets, etc treating all “time variables”on equal footing at first, i.e. when defining densities. Then one defines the ‘scalar’ quantities referring to each ‘time direction’ (to
be turned into operators at the quantum level), including a set of D Hamiltonians, by integration over appropriate hypersurfaces in \((G \times \mathbb{R})^D\), so that each Hamiltonian refers to a single time direction, but at the same time all time directions are treated equally but independently, corresponding to an equal but independent evolution (propagation) of the corresponding degrees of freedom. A similar procedure is adopted for other canonical quantities, e.g. Poisson brackets, scalar products etc.

Let us sketch one example of such procedure, for the case \(D = 2\) and the above choice of GFT action, outlining only the definition of the Hamiltonians, in order to convey at least the flavour of the formalism. We refer to \([10]\) for the full treatment of this case as well as for a full, but still in many ways preliminary, description of the GFT Hamiltonian analysis, while we refer to \([15]\) for the complete exposition of the state of the art in polymomentum Hamiltonian mechanics.

We start from the naive phase space \((\phi, \phi^\dagger, \pi_\phi, \pi^\dagger_\phi)\) (we call it ‘naive’ simply because in the polymomentum formalism phase space variables are actually differential forms on spacetime, while the \(\pi\) above are not), with the product structure of the kinetic term resulting in a peculiar expression for the momenta, e.g. \(\pi^1_\phi = (-i\partial_2 + \Box_2)\phi^\dagger\), and define the DeDonder-Weyl Hamiltonian density (summation over repeated indices understood):

\[
\mathcal{H}_{DW} = \pi_\phi^1 \delta \partial s_i \phi + \pi^\dagger_\phi^1 \delta \partial s_i \phi^\dagger - L = 2\pi_\phi^1 \pi^\dagger_\phi + i\pi_\phi^1 \Box_1 \phi + i\pi^\dagger_\phi \Box_2 \phi + h.c.. 
\]

One then proceeds to re-write it as a sum of two contributions, each uniquely associated to a single time parameter: \(\mathcal{H}_{DW} = \mathcal{H}_1 + \mathcal{H}_2\), with \(\mathcal{H}_i = \pi_\phi^1 \pi^\dagger_\phi + i\pi_\phi^1 \Box_1 \phi + i\pi^\dagger_\phi \Box_2 \phi + h.c\..\)

The Hamiltonians governing the ‘time evolution’ with respect to the different time directions identified by each variable \(s_i\) are then defined by integration over independent hypersurfaces, each orthogonal to a different time direction, e.g. \(H_1 = \int d\sigma_d s_i d\sigma \mathcal{H}_1\). Each \(H_1\) results in being independent of time \(s_i\), as one would expect.

One can then proceed, after suitable decomposition in modes of fields and momenta, the definition of (a GFT-adapted version of) the covariant Poisson brackets, etc, to the canonical quantization of the theory, with the definition of a Fock structure on the space of states. In this way, one is able to make precise the intuition of GFTs being a dynamical theory of creation and annihilation of fundamental quanta of space. We refer once more to \([10]\) for the results of this analysis, among which we mention only the interesting interplay between statistics and group structure in GFTs, that causes fields on a Riemannian group \(G\) to be quantized as bosons, and fields on a Lorentzian group \(G\) to be quantized, necessarily, as fermions, in order to preserve positivity of the Hamiltonians \(H_i\). However, this result may depend rather crucially on the specific choice of kinetic term, and so be model-dependent, but we believe that this shows how the issue of field statistics in GFTs deserves to be further analyzed in more precise and general terms.

This is only one of the many features of the polysymplectic formalism for field theories, in general, and of its GFT incarnation, in particular, that need to be developed and clarified at both the mathematical and physical level.

They include, at the classical level, and for the general polysymplectic formalism: the formulation of a covariant Hamilton-Jacobi theory corresponding to it, and the study of its implications for quantum field theory; an analysis of the relationship between the DeDonder-Weyl polysymplectic field theory and other known covariant extension of ordinary Hamiltonian field theory (see again \([15]\) for a discussion); a complete analysis of the exact reduction of formalism and results of polymomentum Hamiltonian field theory to the usual single-time Hamiltonian formalism; more technically, there many unsolved issues concerning the algebraic structure induced on the space of (horizontal) differential forms constituting the generalized phase space by Kanatchikov’s definition of the (graded) Poisson bracket for them (that seems to be a generalized type of Gerstenhaber algebra); also, the general theory of conservation laws in polymomentum mechanics, with the corresponding definition of conserved currents and charges, the generalization of Noether theorem, etc, is to be developed in more detail and its physical consequences for know theories (and here we refer to ordinary field theories as well as to GFTs) have to be analyzed, also given the great physical importance attached to them.

At the quantum level, the state of the art is even more full of open issues, given that work on this has really just started: for the quantization of field theories based on the full DeDonder-Weyl-Kanatchikov Hamiltonian theory we refer to \([15]\), and subsequent work by the same author, for the first steps; for the GFT-adapted formalism, the same consideration apply, so we refer to \([10]\). We only mention three open problems specifically related to the GFT approach. The first is the general issue of conserved quantities and symmetries for GFTs; this is crucial for a better understanding of the appropriate scalar product in the space of fields, or equivalently for the GFT definition of the kinematical inner product for canonical 1st quantized quantum gravity wave functions from the perspective of GFTs, and at the same time would provide the basis for the analysis of symmetries at the quantum level, and thus the consequent gauge fixing of Feynman amplitudes, i.e. spin foam models; we will discuss this issue more in the following. The second is related to the first: what is the GFT analogue of the notion of anti-particles? Being strictly intertwined with the complex structure of the field, one would expect it to be linked with the orientation of the simplices corresponding to the fundamental quanta of the GFT, and this would match some insights coming from
recent developments in spin foam models; however, the whole issue is far from clear, and probably is best addressed within an Hamiltonian context, like the one outlined above. The third is the canonical derivation of GFT quantum propagators, i.e. 2-point functions; in particular, one would like to put on more solid grounds the choice of propagator made in the construction of the spin foam models/Feynman amplitudes corresponding to the models 7 and 6 clearly, in light of the above discussion, this means providing, among other things, a suitable definition, in an Hamiltonian setting, of a “multi-time-ordering” of field operators.

C. Symmetries

We have mentioned above the open issue of symmetries in group field theories. We now expand on it, trying to clarify what is known and what is not known.

All the past work on symmetries in group field theories has proceeded in a rather awkward way, from a field theoretic perspective. Consider the model 4 which is the only one of which we understand reasonably well the symmetries and their gauge fixing at the level of Feynman amplitudes. As said, the same Feynman amplitudes can be derived from a discretization of the continuum action and path integral of topological BF theory in 3 dimensions. One can identify the discrete analogue of the continuum symmetries of BF theory: the translational and local Lorentz symmetry, at the level of each GFT Feynman diagram/simplicial complex, then devise the appropriate gauge fixing procedure of the spin foam model/GFT Feynman amplitude, to get rid of the redundant gauge degrees of freedom, and the corresponding Faddeev-Popov determinant, to obtain in the end fully gauge-fixed and finite GFT Feynman amplitude.

Now, first of all, this has been done, to date, -only- for this specific GFT model, and not much is known about the relevant symmetries of other models. Second, even in this case, from a GFT perspective there is still much to be understood. In particular, while the Lorentz symmetry has a clear GFT origin in the invariance of the field under the diagonal action of the group G on its 3 arguments, the GFT origin of the translation symmetry remains mysterious.

On top of this, as said, it is rather obvious that this is a cumbersome way of proceeding, from a purely field theory perspective: one would study directly the symmetries of the GFT action that originates the Feynman amplitudes under consideration, and derive the relevant perturbative identities between Feynman amplitudes and n-point function, rather than try to guess what these symmetries are from an analysis of the individual Feynman amplitudes, which are effectively path integrals for the corresponding single and multi-particle theories.

This is indeed what needs to be done and understood in all mathematical and physical details. As mentioned, the natural starting point would be the Hamiltonian analysis of symmetries for the various GFTs, and the derivation of the corresponding conserved currents and charges. But one can as well remain at the Lagrangian level and be concerned only with the perturbative expansion of GFTs in terms of spin foam models; in this case, one should identify the various symmetry transformations of the fields in the GFT action, for any specific model considered, and derive from them the corresponding Ward identities for the n-point functions. These would be exactly the identities between spin foam amplitudes that can (and were) in some cases discovered by direct analysis of the classical discrete theory from which the same amplitudes, can be derived. Both the general formalism for deriving Ward identities and explicit examples of symmetry analysis for specific GFT models are yet to be developed.

Notice that this is far from a trivial task, even at the classical level, as one needs to work out a suitable generalization of Noether theorem, adapted to GFTs, that overcomes the difficulties posed by presence of several time variables (leading necessarily to a polymomentum formalism), and by the product form of the general kinetic operator, as well as by the higher-order in the “spacetime” derivatives that results from it, even for simple choices of the kinetic term.

One more reason why this would be of interest is that, as it was stressed forcefully by P. Cvitanovic in his lectures at the workshop, the presence of symmetries and of the consequent Ward identities in a field theory affects greatly, among other things, the growth rate and convergence properties of the corresponding perturbative series, and thus in the GFT case it can tell us a lot about the mathematical structure and properties of the sum over topologies, as well as over geometries, implicit in their expansion in Feynman diagrams.

3 And a better understanding of this translation symmetry in GFTs would be of great relevance from a quantum gravity point of view since to has been shown [10] that it is strictly related to the discrete Bianchi identities on the simplicial complex dual to the GFT Feynman diagram.
D. Combinatorial structure of the Feynman diagrams

Let us now move on, indeed, to the discussion of the open problems concerning the Feynman diagrams themselves. These have to do with their combinatorial structure mainly, and can be motivated by the need, and at the same time the chance that group field theories offer, to re-phrase quantum gravity questions in purely (quantum) field theoretic terms, and to tackle them with (quantum) field theoretic tools.

To start with there is a urgent need to clarify the general combinatorial structure of GFT Feynman diagrams, by developing the basic concepts of ordinary quantum field theory in this new and rather peculiar context. What are straightforward questions in QFT, become a bit less straightforward ones in GFTs. For example: what are the 1-particle irreducible diagrams, now that lines of propagation are actually formed by several strands (equivalently, what is the generalization of 1PI to 2-complexes)? and what are their properties? what is the (approximate) form of the 1-loop effective action for the various GFTs? what is the exact combinatorial content of the Dyson-Schwinger equations in this setting?

If these are still rather simple questions (although may involve technical or formal complications), as soon as we try to unravel in more explicit term the combinatorial structure of GFT Feynman diagrams with respect to their dual picture as simplicial complexes, things start to complicate considerably.

Even if every 2-complex arising as a GFT Feynman diagram can be understood as topologically dual to a D-dimensional simplicial complex, this would not be, in general, a simplicial manifold. In fact, the data attached to GFT Feynman diagrams, nor the feynmanological rules for their construction, do not constrain the neighbourhoods of simplices of dimensions from (D-3) downwards to be spheres. This implies that in the general case, the resulting simplicial complex, obtained by gluing D-simplices along their (D-1)-faces, would correspond to a pseudo-manifold, i.e. to a manifold with conical singularities [8]. The issue does not arise in D=2, where all GFT Feynman diagrams (combinatorially the same as those of simple matrix models) are dual to simplicial manifolds, if the orientation condition is satisfied. Neglecting for the moment the issue of whether this is a problem from a physical perspective4, or whether on the contrary it is possible to give some physical meaning to these singularities, one remains with the task of analyzing these configurations from a purely mathematical point of view. A precise set of conditions under which the GFT Feynman diagrams correspond to manifolds is identified and discussed at length in [8], both at the level of simplicial complexes and of the corresponding dual 2-complexes, in D=2,3,4. All the relevant conditions can be checked algorithmically on any given Feynman graph. However, we feel it would be good to build upon the analysis of [8], in two main directions: 1) try to identify a suitable reformulation of the found conditions, or an alternative but equivalent set of conditions, that would make the quantum field theoretic interpretation and role of the pseudo-manifold configurations more transparent; 2) if and once this can be done, find how to impose these conditions at the GFT level (a sort of superselection rules?) or construct suitably constrained GFT, that would generate only manifold-like complexes in their Feynman expansion. This last task makes sense, of course, only if the first does not prove that, because of their field theoretic interpretation, pseudo-manifold configurations are in fact needed for consistency at the quantum level (in the same sense, for example, as loop diagrams are in ordinary field theory). It may also turn out that non-manifold-like configurations can not be removed but are instead suppressed in certain sectors of the theory, in specific models, as for example happens in some 3d tensor models [17].

Another wide landscape of interesting questions opens up when considering the topological structure of the GFT Feynman diagrams. As discussed above, the sum over GFT Feynman diagrams includes a sum over all simplicial topologies as well as over all simplicial decompositions of the same topology. In D=2 different topologies are weighted by a single topological invariant, the Euler characteristics, and one can then try to express the GFT Feynman amplitudes as a function of this topological parameter, and then identify the sector of the theory or of the parameter space in which, say, the trivial topology dominates. In matrix models, indeed, where the Feynman amplitudes have a simpler form (as discussed above, they correspond to the truncation of the simple GFT to a fixed representation J), one can easily show that non-trivial topologies are suppressed in the limit of infinite matrix dimension (large J). No analogue of this result in the full D=2 GFTs is known. In D ≥ 3 topologies are not classified so the situation is much more intricate. A first guess at how non-trivial topologies enter the GFT Feynman expansion is that they only appear beyond tree level, i.e. in the quantum regime; in fact, in matrix models the expansion in the matrix dimension can also be understood as a loop expansion and confirms the above expectation. This was confirmed also for generic GFTs in [8] by analyzing the Dyson-Schwinger equations: at tree level only diagrams of trivial topology appear, and the order of loops can then be related to the number of handles in the simplicial complex dual to the Feynman diagram. Moreover, by suitable re-scaling of the field, the number of loops could be related to (a power of)

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4 As pointed out by one of the participants during the workshop, after all, we do not know whether or not spacetime really possesses conical singularities in the microscopic regime....
the GFT coupling constant, that thus acquires a possible interpretation as the parameter governing topology change; we refer to [3] for more details.

The number of handles is, however, just one of the quantities that can characterize simplicial topology, and a more in depth analysis of the relation between the topological properties of the simplicial complexes appearing in the Feynman expansion and their field theoretic interpretation would be very much welcome. And more generally, one would also like to be able to identify more clearly the dependence of the GFT Feynman amplitudes (for generic dimension, and for non-trivial models) on the topology of the underlying diagram. This would help greatly in the attempt to gain control over the sum over topologies implicit in the GFT perturbative expansion.

For example, just as one can relate handles and quantum loops, is there a QFT interpretation of other known topological invariants or for other topological properties of simplicial complexes?

The model [4] for example, generates Feynman amplitudes that are topological invariants themselves [10], meaning that they evaluate to the same number (after appropriate gauge fixing and regularization) on any element of equivalence class of simplicial complexes related by Pachner moves, while they still provide a different amplitude for topologically inequivalent simplicial complexes. Can one identify the field theoretic feature of this model (and of its higher dimensional equivalents, whose amplitudes share the same property) that can be seen as the origin of this property of its amplitudes? If this can be done, can we use this new insight to construct new topological invariants of simplicial complexes by field theoretic means?

Even remaining at the level of trivial topology, e.g. say we just consider diagrams with the cylindrical topology $\Sigma \times \mathbb{R}$, there are further structures that one would like to identify in the diagrams and characterize in field-theoretic terms. In the modern version of the dynamical triangulations approach [6], namely causal dynamical triangulations, one finds that imposing additional combinatorial restrictions to the simplicial complexes summed over in the definition of the quantum gravity path integral, the continuum properties of the same are drastically improved and one can recover many of the wanted properties of a continuum spacetime from its purely combinatorial quantum definition, e.g. its spectral dimension [6]. These additional restrictions include the presence of a fixed foliation of the triangulations and the absence of branching (baby universe) configurations with respect to this foliation. Given that the GFT partition function can be re-expressed, in perturbative expansion, as a sum over triangulations, weighted by Feynman amplitudes, what is the field theory interpretation of these combinatorial restriction? or, can one identify specific GFT models or general properties of GFT Feynman amplitudes that would lead to a strong suppression, if not the absence, of configurations not satisfying these restrictions?

E. Renormalization

We now come to the important issue of renormalization of group field theories, the importance of which we certainly do not need to stress, also given that all known GFT models, at present, are defined by their perturbative expansion.

Once more, this is almost completely unexplored territory. The whole question of perturbative renormalization of the various GFT models has not been analyzed in any detail, nor any general scheme for performing the perturbative renormalization of GFTs has been developed. Once more, it is the peculiar combinatorial structure of the GFT Feynman diagrams, and the non-local pairing of variables in the vertex term in the action, that makes the whole issue of divergences and renormalization at the same time very intricate and challenging. For any given Feynman diagram, and after gauge fixing, the sum over geometric data has two potential sources of divergences: depending on the kinetic term (propagator) one chooses, one can have a potential divergence for each loop, i.e. each 2-cell, but also one has a potential divergence for every ‘bubble’of the GFT Feynman diagram, i.e. for every closed surface of it identified by a collection of 2-cells glued together along common edges. This is, in a sense, the true GFT analogue of loop divergences of usual QFT. In addition to this ‘double-layer’structure of potential divergences, there are other combinatorial peculiarities of GFTs that make the conventional wisdom (including, for example, simple counting of the degree of divergence of a diagram) less easily applicable, and a brand new scheme of analysis badly needed. One example is the fact that a closer look at the way degrees of freedom are propagated within a Feynman diagram reveals a sort of ‘propagation-only’structure, in the sense that also within vertices of interaction the field theory degrees of freedom are simply re-routed, i.e. propagated, and there is no coincidence at the same “point”of more than two field variables. This may mean that a regularization of the GFT propagators may suffice to cure loop divergences. However, also this needs a more careful study. Certainly, due to this combinatorial peculiarities, usual results on the

\footnote{Much remains to be understood concerning this correspondence, anyway, as one can easily find high order (in $\lambda$) diagrams which still correspond to trivial topology.}
non-renormalizability of \( \phi^n \) theories for high enough \( n \) have to be at least reconsidered, to check that they still apply in the GFT framework.

In fact, whether the GFT amplitudes are divergent or not depends on the specific model, even for the same polynomial order in the interaction term. For example, the model \( \mathbf{4} \) turns out to be finite after gauge fixing, while the most natural definition of the group field theory for the Barrett-Crane spin foam model for 4d gravity \( \mathbf{9} \), for example, presents indeed bubble divergences, and the Perez-Rovelli modification of it \( \mathbf{9} \), producing a different version of the same model based on a GFT of the same order and same combinatorics, but different symmetries, possesses again finite Feynman amplitudes, i.e. it is perturbatively finite without the need for any regularization, even before any gauge fixing.

Once more, one has to develop thus a general theory of perturbative renormalization for GFTs, that would allow to unravel first the combinatorial structure of GFT divergences, and then to regularize them away. On the one hand, conventional regularization and renormalization techniques would be the first thing to try as they look like the most efficient way of performing explicit computations and extract physical results from a field theory; indeed, the development and use of their GFT-adapted analogues would be of great value. On the other hand, the Hopf algebra approach to renormalization \( \mathbf{18} \) have proven to be especially suitable for capturing and elucidating the combinatorial structure of Feynman diagrams and of their divergences; therefore it is a natural guess and hope that one can apply similar techniques to GFTs, where usual tools may be less powerful exactly because of the combinatorial intricacies of the GFT Feynman diagrams (some of which have been highlighted in the previous section).

That the Hopf algebra techniques developed by Kreimer and Connes, among many others, can be the correct language to tackle the issue of GFT renormalization is suggested also by recent work on the renormalization of spin foam models, in particular by the work of \( \mathbf{19} \). Here the conceptual setup and the perspective on spin foam models were rather different from the one presented in the present article, and group field theories were really not part of the picture. Instead, spin foam models were studied as background independent discrete quantum gravity path integrals (or statistical mechanical state sum models), and the task was to develop a coarse graining and renormalization procedure, a background independent analogue of the usual one used in lattice statistical mechanics and statistical field theory, that would bypass the difficulties coming from the absence of a fixed lattice geometry (e.g. fixed lattice spacing) and could be used for lattices of arbitrary combinatorial structure, as spin foams. The most natural language was found to be, in fact, that of Hopf algebra renormalization and a Hopf algebra of spin foams (more precisely, a Hopf algebra of partitioned spin foams) was defined, together with a Hopf algebra of (parenthesized) spin foam weights (amplitudes), based on the identification of appropriate subfoams. As in the Connes-Kreimer approach, it is the antipode of the two algebras that plays a crucial role in the definition of renormalization group transformations. Indeed, Markopoulou went on defining exact and approximate block transformations on spin foams, based on the discovered Hopf algebra structure, and in particular on its antipode, and suggested that this could be the correct starting point for the analysis of renormalization of known spin foam models.

This approach acquires a new light and, in a sense, a further justification from a group field theory perspective.\(^6\). Recall that spin foams are nothing more than the Feynman diagrams of group field theories, and spin foam models are their Feynman amplitudes. The definition of a renormalization procedure for spin foam models would therefore amount to a definite prescription for perturbative renormalization of group field theories. From this standpoint, then, the fact that the Kreimer-Connes Hopf algebra approach to renormalization, originally developed for perturbative renormalization of QFTs, can be adapted to spin foam models seems only natural. The tasks however are: to identify the Kreimer-Connes algebra of Feynman diagrams for GFTs and compare it with the algebra of spin foams as defined by Markopoulou (as a first step, this involves comparing GFT 1PIs to the subfoam structure proposed in \( \mathbf{19} \)); go on identifying the corresponding Hopf algebra of GFT perturbative renormalization, and again compare it with the one of \( \mathbf{19} \); clarify the role of GFT gauge symmetries within it; apply it to specific GFT models. Given the large number of interesting mathematical results coming out of recent work on QFT Hopf algebra renormalization, we expect further work along these lines in the GFT framework to be particularly exciting and rewarding, also considering on the one hand the dual simplicial spacetime interpretation of GFT diagrams, and on the other hand the relevance of any result so obtained for quantum gravity.\(^7\)

In the context of renormalization, let us also mention that it would be of great interest to go beyond the perturbative level and develop an exact Wilsonian renormalization group analysis of group field theories. On top of providing important information on the renormalizability of GFTs, it would represent an even more (with respect to the perturbative renormalization of their Feynman diagrams/spin foam models) powerful tool for the study of their

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\(^6\) This was indeed noted already in \( \mathbf{19} \).

\(^7\) In fact such an analysis would represent one direct way of tackling the issue of the continuum limit of spin foam models and of their relation with continuum Einstein’s gravity.
continuum limit and relation with classical General Relativity.

Another related issue, concerning the perturbative expansion of GFTs in Feynman diagrams/spin foam models, is that of summability of the perturbative series. Of course, even though, from the interpretation of it as a discretized version of a quantum gravity path integral, one may hope it to be finite, this is not such a reasonable expectation from a purely field theory perspective, which is the one we would like to advocate here. However, summability is a more limited but perfectly reasonable hope for a QFT perturbative series. Notice that such a property would still be rather remarkable, even though only if achieved in a physically meaningful way, from a quantum gravity perspective, given that it amounts to the possibility of gaining (non-perturbative) control over a sum over all simplicial geometries and all simplicial topologies. Also this issue has been largely unexplored, both in general terms and for specific GFT models. The only (important) result in this direction obtained so far concerned the model 4 and is the following: a simple modification of the GFT action gives a model whose perturbative expansion is Borel summable. The modification amounts to adding another vertex term of the same order to the original one, to give:

\[
\lambda \sum_{i=1}^{6} \int_{SU(2)} dg_i \left[ \phi(g_1, g_2, g_3) \phi(g_3, g_4, g_5) \phi(g_5, g_2, g_6) \phi(g_6, g_4, g_1) + \delta \phi(g_1, g_2, g_3) \phi(g_3, g_4, g_5) \phi(g_4, g_2, g_6) \phi(g_6, g_5, g_1) \right],
\]

with \(| \delta | < 1\).

The new term corresponds simply to a slightly different recoupling of the group/representation variables at each vertex of interaction, and geometrically to the only other possible way of gluing 4 triangles to form a closed surface, i.e. to form a “pillow” instead of a regular tetrahedron. In turn, this “pillow” configurations are equivalent to two tetrahedra glued together along -two- common triangles, instead of one. Even if the above modification of the Boulatov GFT model has no clear physical interpretation yet from the quantum gravity point of view, e.g. in terms of gravity coupled to some sort of matter, it is indeed a very mild modification, and most importantly one that one would expect to be forced upon us by renormalization group-type of argument, that usually require us to include in the action of our field theory all possible terms that are compatible with the symmetries. What the other terms would be and what their effect on the perturbative series is another interesting open issue, together with the (more physically important) possibility of obtaining a similar result for any of the known GFT models of quantum gravity in \(D = 4\).

IV. CONCLUSIONS

In this article, we have introduced the group field theory formalism, trying to clarify its relevance for quantum gravity research, as well as key aspects and peculiarities of GFT models, defining combinatorially non-local field theories on the one hand, and a generalization of matrix models and field theories of random surfaces on the other. Most importantly, we have presented and discussed many outstanding open issues and unanswered questions that we thought could be of special interest to mathematical physicists and mathematicians working on quantum field theories in general, and in particular those interested in the related perturbative combinatorics and renormalization. Our expectation is that GFTs can represent for mathematicians and mathematical physicists a very valuable playground and toolbox, and an important source of insights and, obviously, amusement, also in light of the immense corpus of mathematical results, tools and applications that has already developed from work on matrix models and random surfaces (and which is by all means still growing). Adding to this the mentioned relevance of all of these open issues for the construction of a satisfactory theory of quantum gravity, the urge to join the current efforts of many theoretical physicists working in this area should be, we hope, unrestrained.....

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8 It would be interesting to understand in more detail the field theoretic interpretation of these configurations.
9 The selection, of course, has been operated from the particular (and certainly limited) perspective of a theoretical physicist working on quantum gravity, and this has affected also the style of the discussion. Being unavoidable, we do not feel the need to apologize for this.
Acknowledgements

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