The sudden birth and sudden death of thermal fidelity in a two-qubit system

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(Dated: May 2, 2011)

We study the energy level crossings of the states and thermal fidelity for a two-qubit system in the presence of a transverse and inhomogeneous magnetic field. It is shown clearly the effects of the anisotropic factor of the magnetic field through the contour figures of energy level crossing in two subspaces, the isotropy subspace and anisotropy subspace. We calculate the quantum fidelity between the ground state and the state of the system at temperature $T$, and the results show the strong effect of the anisotropic factor again. In addition, by making use of the transition of Yangian generators in the tensor product space, we study the evolution of the thermal fidelity after the transition. The potential applications of Yangian algebra, as a switch to turn on or off the fidelity, are proposed.

PACS numbers: 75.10.Pq, 71.10.Hf, 32.80.Xx, 03.67.-a.

Keywords: qubits XY model, the ground state, energy level crossing, quantum fidelity

I. INTRODUCTION

Energy levels and eigenvalues of a Hamiltonian, especially the ground state, play an important role in determining the properties of a quantum system \cite{1}. When the energy gap between the ground and first excited levels tends to zero, the energy level crossing will appear \cite{2}. It is well known that the degeneracy and the quantum phase transition lead to various peculiar phenomena \cite{1,3}. It is very important to investigate level crossings to comprehend the quantum phase transition. Bhattacharya and Raman presented an effective algebraic method for finding level crossings \cite{4}. It is still important to explore a way of studying level crossings and understand how eigenvalues of the Hamiltonian change at crossing.

In order to detect phase transitions, the fidelity, as a new concept originated from quantum information theory, has been put forward \cite{5,7}. Recently, the fidelity as a useful probe has attracted much attention \cite{8,13}. In addition, as a geometric measure, the fidelity may represent an effective approach to identify quantum phase transitions \cite{14,16}. Moreover, it could be a good indicator to energy level crossings \cite{14,18}. At present most efforts have been devoted to the ground-state fidelity corresponding to slightly different values of the controlling parameters. In this work, we study the fidelity between the ground state and the state of the system at temperature $T$ to explore the cases of the system in the ground state.

In this paper, we study the two distinguishing qubits system in a transverse and inhomogeneous magnetic field, and calculate the energy level crossings and the quantum fidelity. For XY spin chain model, the fidelity has been studied in the homogeneous magnetic field \cite{3,10}. However, we will study the fidelity in the non-homogeneous magnetic field, which acts on two different qubits. By taking advantage of the nonuniform characteristic, we study its effect to the energy level crossings and thermal fidelity, and find new interesting physical phenomena.

In addition, due to the particular characteristic of Yangian algebra to deal with physical models: symmetry and transition, Yangian algebra method has been widely studied \cite{14,21} in recent years. People have found the Yangian symmetry in many physical models \cite{22,24}. It is also important and helpful for exploring physical systems in terms of the transition characteristic of Yangian. In this paper, by making use of the transition effect of Yangian operators in the tensor product space, we obtain the better regulation and control to the thermal fidelity.

The paper is organized as follows: In Sec. II, we introduce a two-qubit XY spin model Hamiltonian and divide the system into the two subspaces: the isotropy and the anisotropy spaces. As an important physical phenomenon, the energy level crossings are studied for some models in detail in Sec. III. In Sec. IV, the thermal fidelity between the ground state and the state of the system at temperature $T$ is studied in the parametric space. The effects of the anisotropic factor of the magnetic field to the fidelity are explored. Sec. V presents the effects of Yangian operators on the thermal fidelity. Finally, our conclusions are given.

II. THE MODEL HAMILTONIAN

The Hamiltonian of a two-qubit XY spin model with an inhomogeneous magnetic field in the $Z$ direction can be expressed as

$$H = -\frac{1}{2} \gamma_1 \sigma_1^x \sigma_2^x - \frac{1}{2} \gamma_2 \sigma_1^y \sigma_2^y - B(\sigma_1^z + \lambda \sigma_2^z), \quad (1)$$

where $\sigma_\alpha^\beta$ are the Pauli matrices of the $i$-th qubit with $\alpha = x, y, z$ and $B$ is the strength of the external magnetic

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field. \( \lambda \) is an anisotropic factor of the magnetic field acting on the second qubit and denotes the non-uniformity degree of magnetic field. \( \gamma \) is an anisotropic factor in the interaction.

The eigenstates and corresponding eigenvalues of the Hamiltonian can be obtained as

\[
H|\psi_1\rangle = E_1|\psi_1\rangle = -\sqrt{\xi^2 + 1}|\psi_1\rangle, \\
H|\psi_2\rangle = E_2|\psi_2\rangle = \sqrt{\xi^2 + 1}|\psi_2\rangle, \\
H|\psi_3\rangle = E_3|\psi_3\rangle = \sqrt{\eta^2 + \gamma^2}|\psi_3\rangle, \\
H|\psi_4\rangle = E_4|\psi_4\rangle = -\sqrt{\eta^2 + \gamma^2}|\psi_4\rangle,
\]

(2)

where

\[
|\psi_1\rangle = \frac{1}{N_1}(|10\rangle + a_1|01\rangle), \\
|\psi_2\rangle = \frac{1}{N_2}(|10\rangle + a_2|01\rangle), \\
|\psi_3\rangle = \frac{1}{N_3}(|11\rangle + a_3|00\rangle), \\
|\psi_4\rangle = \frac{1}{N_4}(|11\rangle + a_4|00\rangle).
\]

(3)

Here \( a_1 = \xi + \sqrt{\xi^2 + 1}, a_2 = \xi - \sqrt{\xi^2 + 1}, a_3 = \eta - \sqrt{\eta^2 + \gamma^2} \) and \( a_4 = \eta + \sqrt{\eta^2 + \gamma^2} \), where \( \xi = B(1 - \lambda), \eta = B(1 + \lambda) \), while \(|0\rangle\) stands for spin down and \(|1\rangle\) stands for spin up. \( N_i \) is the normalization coefficient of \(|\psi_i\rangle \) \( (i = 1, 2, 3, 4) \).

The Hamiltonian \( H \) is acquired a more effective form by rewriting Eq. 1 in the matrix form 23, 26,

\[
H = H_e \oplus H_o,
\]

(4)

where \( H_e = -\begin{pmatrix} \eta & \gamma \\ \gamma & -\eta \end{pmatrix} \) and \( H_o = -\begin{pmatrix} \xi & 1 \\ 1 & -\xi \end{pmatrix} \), acting in an isotropy subspace spanned by \{|\uparrow\rangle, |\downarrow\rangle\} and in an anisotropy subspace by \{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}, respectively. From Eqs. 2 and 4, we can know \( E_{3,4} \) and \(|\psi_{3,4}\rangle\) are the eigenvalues and eigenvectors of Hamiltonian \( H_e \) respectively, and \(|\psi_4\rangle\) is the ground states in the isotropy subspace. Similarly, the eigenvalues \( E_{1,2} \) and the eigenvectors \(|\psi_{1,2}\rangle\) correspond to the eigenvalues and eigenvectors of the Hamiltonian \( H_o \) with the ground state \(|\psi_1\rangle\) in the anisotropy subspace. Based on the two subspaces, which are also closely related to Yangian algebra, the energy level crossings and the thermal fidelity will be studied in detail as follows.

### III. THE ENERGY LEVEL CROSSINGS OF THE GROUND STATE

From Eq. 2, we learn that the ground state can only be \(|\psi_1\rangle\) or \(|\psi_4\rangle\). However, by changing the parameters, the rules of the ground state may be exchanged. When the energy gap \( \Delta E = E_1 - E_4 \) is less than zero, it indicates that \(|\psi_1\rangle\) is the ground state; while \( \Delta E = E_1 - E_4 > 0 \), it shows that \(|\psi_4\rangle\) is the ground state. However, the ground states are the degenerate states of \(|\psi_1\rangle\) and \(|\psi_4\rangle\) for \( \Delta E = E_1 - E_4 = 0 \). In order to express clearly the evolution of the energy gap \( \Delta E \) in the parameter space, we are going to investigate them graphically.

Figure 1 shows the contour plots of the energy gap as a function of the magnetic field strength \( B \) and the anisotropic factor \( \lambda \) or \( \gamma \) in the different cases. The energy level crossing of the system occurs at the zero-contour lines, on which \( E_1 = E_4 \), namely, the energy level of the ground state happens to mutate at these points. It is clear that the ground state changes abruptly at \( \Delta E = 0 \) in Fig. 1.

For the different magnetic fields: nonhomogeneous for \( \lambda = -1 \) or homogeneous for \( \lambda = 1 \), the contours of \( \Delta E \) versus \( B \) and \( \gamma \) have obvious differentiations. In the Fig. 1(A), the contours of \( \Delta E < 0 \) are to be in the upper and under polylines, which indicate that the ground state is \(|\psi_1\rangle\), while those of the left and right sides are larger or equal to 0, which show the ground state is \(|\psi_4\rangle\) or the degenerate state. In Fig. 1(B), the contours are a series of concentric ellipses and the energy gap becomes bigger as the radius increases. Especially, \( \Delta E < 0 \) is within the zero contour, which indicate \(|\psi_1\rangle\) is the ground state. Without the zero contour, \( \Delta E > 0 \) indicates that the ground state is \(|\psi_4\rangle\).
By comparing, we can find that the probability of $|\psi_1\rangle$ as the ground state, the areas of $\Delta E < 0$ in the ($\gamma$, $B$) plan, is comparatively large for the inhomogeneous magnetic field, and almost disappears completely in the case of the homogeneous magnetic field.

Without loss of generality, for $\gamma = 0.5$, the system is the XY model, and only has the coupling strength in the X or Y direction for $\gamma = \pm 1$, that is to say, the XY model becomes the transverse-field Ising mode. Figs. 1(C) and (D) show the contour graphs of $\Delta E$ in the models. In Figs. 1(C) and (D), the positive and negative of $\Delta E$ have the same indications with Figs. 1(A) and (B). In the case of $\gamma = 0.5$, the probability of $|\psi_1\rangle$ as the ground state is bigger than $|\psi_4\rangle$. However, for $\gamma = \pm 1$, the zero contour is a straight line at $\lambda = 0$ and the contour plots are bilateral. The probability of $|\psi_1\rangle$ as the ground state is equal to the case of $|\psi_4\rangle$. It is worth noting that $|\psi_4\rangle$ is in the isotropy subspace and $|\psi_1\rangle$ is in the anisotropy subspace. If the ground state of the system, $|\psi_1\rangle$ or $|\psi_4\rangle$ is identified, one can confirm the system is in the isotropy or anisotropy subspace at zero temperature. From Fig. 1, one can obtain the different ground state by adjusting the parameters.

IV. EVOLUTION OF THE FIDELITY

Let us now turn our attention to probe the probability of the system in the ground state of the anisotropy subspace. As a usable method, the fidelity is defined by

$$F = \langle \psi | \rho | \psi \rangle,$$

(5)

where $|\psi\rangle$ is the pure state and $\rho$ is the density matrix of the system or state. In our problem, we will study the fidelity between the ground state of the anisotropy subspace and the state of the system at temperature $T$. The density matrix $\rho(T)$ of the model in the thermal-equilibrium state at temperature $T$ is given by

$$\rho = \sum_{i=1}^{4} p_i |\psi_i\rangle\langle\psi_i|,$$

(6)

where $p_i = \exp(-E_i/kT)/Z$ are the probability distributions and the partition function $Z = Tr[\exp(-H/kT)]$. For simplicity, we set $k = 1$ in all following equations. Based on $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, $\rho$ of the system is a standard $X$ state given by

$$\rho(T) = \frac{1}{Z} \begin{pmatrix} v_1 & 0 & 0 & u \\ 0 & w_1 & y & 0 \\ 0 & y & w_2 & 0 \\ u & 0 & 0 & v_2 \end{pmatrix}.$$  

(7)

Here $v_1 = (b_3 + b_4)^2$, $v_2 = b_3 a_3^2 + b_4 a_4^2$, $u = (b_3 a_3 + b_4 a_4)\gamma$, $y = b_1 a_1 + b_2 a_2$, $w_1 = b_1 + b_2$ and $w_2 = b_1 a_1^2 + b_2 a_2^2$.

In Fig. 2, we show the evolution behavior of the thermal fidelity vs $T$ with three different magnetic fields in the XY model ($\gamma = 0.2$) and Ising model ($\gamma = 1$). Under the conditions of Fig. 2, the ground state is $|\psi_1\rangle$ for the inhomogeneous magnetic field and $|\psi_4\rangle$ for the homogeneous magnetic field. At zero temperature, the system...
will be in the ground state. If the system is in $|\psi_1\rangle$, the overlap between the state of the system and $|\psi_1\rangle$ is maximum, so one can obtain $F = 1$ for $\lambda = -1$. On the contrary, $F = 0$ for $\lambda = 1$ because $|\psi_4\rangle$ and $|\psi_1\rangle$ are orthogonal. In the cases of $\lambda = 0$, one get $F = 1$ in Fig. 2(A), because $|\psi_1\rangle$ is still the ground state for $\gamma = 0.2$; however, $F = 0.5$ in Fig. 2(B), because the ground states become the degeneracy states of $|\psi_1\rangle$ and $|\psi_4\rangle$ for $\gamma = 1$. As $T$ increases, the ground state mixes with the excited states, the change trends of the fidelity are to be shown in the illustration.

Figure. 3 displays the fidelity contours in $(B, \lambda)$ space for $\gamma = 0.2$ and $\gamma = 1$ at $T = 0.2$ respectively. The fidelities are very susceptible to $\lambda$ and vary dramatically jumping from zero to maximal around the critical points. The high thermal fidelities mainly appear in the third and the fourth quadrants of the $(B, \lambda)$ space. Especially for the reversed inhomogeneous magnetic field $\lambda < 0$ and near $B = 0$, the fidelity is enhanced as shown in Fig. 3(A). When the XY model reduces to the Ising model, by comparing Fig. 3(A) with Fig. 3(B), one can find the fidelity declines around $B = 0$ because of the degeneracy as displayed in Fig. 3(B).

In order to illustrate the evolution of the thermal fidelity around the critical points, we plot the graphs of $F$ vs $\lambda$ for $\gamma = 0.2$ in Fig. 4(A) and for $\gamma = 1$ in Fig. 4(B) at $T = 0.2$. At low temperature, the system is primarily covered the ground state. According to the expressions of energy levels in Eq. (2), the ground state is $|\psi_1\rangle$ for $|\gamma| \leq 1$ and $\lambda < 0$, and $|\psi_4\rangle$ is the ground state in the other cases. We have $F = 1$ for $|\psi_1\rangle$ as the ground state, $F = 0$ for $|\psi_4\rangle$ as the ground state, and $0 < F < 1$ for the mixed states of $|\psi_1\rangle$ and $|\psi_4\rangle$ as the ground state in Fig. 4. In Fig. 4(A), as $\lambda$ increases, $F$ quickly decays from 1 to 0 around the critical point ($\lambda = 0$). It is worth noting that the probabilities of $|\psi_1\rangle$ and $|\psi_4\rangle$ in the ground state are respectively 50% for $\gamma = \pm 1$ and $\lambda = 0$ with the random $B$, so the three curves cross at a point of $F = 0.5$ and $\lambda = 0$, as shown in Fig. 4(B). In addition, especially $F = 0$, the fidelity is rapid death because $|\psi_1\rangle$ and $|\psi_4\rangle$ are orthogonal. Due to the sharp change of the fidelity as a function of $\lambda$, $\lambda$ can be used as a switch to turn on or off the fidelity. For these properties, possible applications are expected to quantum logical gate.

V. EVOLUTION OF THE FIDELITY WITH THE TRANSITION EFFECT OF YANGIAN GENERATORS

Since Yangian algebra was presented by Drinfeld in 1985, it has attracted much attention. Not only can Yangian algebra describe the symmetry of quantum integrable models, but also can present the transitions of the states between different weights beyond the Lie algebra. For a bi-spin system, the realization of Yangian $Y(sl(2))$ is taken the form of [29, 30]

$$I = S = S_1 + S_2,$$
$$J = \mu S_1 + \nu S_2 + \lambda S_1 \times S_2,$$  \hspace{1cm} (9)

where $S_1$, $S_2$ are the spin-$\frac{1}{2}$ operators and $\mu$, $\nu$ and $\lambda$ are arbitrary parameters. $I_\pm = I_1 \pm iI_2$ and $J_\pm = J_1 \pm iJ_2$. $I$ is the total spin operator satisfying $[I_i^a, I_j^b] = i\varepsilon_{abc}I^c \delta_{ij}$, $(i, j = 1, 2)$, namely, the generators of the Lie algebra. By making use of the transition, $I$ can only transit between the states with the same weight. However, $J$ can transit mutually to different sites or particles and jump between states with different weights [21].

Fig. 5 sketches the transition characteristic of the Yangian algebra in the two subspaces. $J_3$ only makes the transitions of states inside the subspace $H_e$ and $H_o$ respectively, however $J_\pm$ can transit from a state of one subspace to a state of another subspace. This important link between the two subspaces and Yangian algebra is suitable for any bi-spin systems, not only the XY model. The effect of Yangian operators will contribute to the boost of the high fidelity.

To illustrate the effect of Yangian algebra on the fidelity, we discuss its as follows. By the transition characteristic of the Yangian algebra, first let us act the transition operators $J_\pm$ on the initial state $|\psi_1\rangle$, the corresponding final states are given by...
by taking advantage of the operator \( J_\pm \) of by the operator \( J_+ \) and to the first quadrant by \( J_- \) in \( (B, \lambda) \) space. For \( \gamma = 0.2 \), the ground state is \( |\psi_1\rangle \), and there is not degeneracy in the case of \( B = 0 \). The initial state is orthogonal to the final states \( |\psi_1'\rangle \) and \( |\psi_4'\rangle \), namely, the overlap between two final states and the ground state passes out. Hence \( F' = F'' = 0 \), and there is not a long tail as shown in Fig. 6. However, the degeneracy of the ground states will appear for \( \gamma = 1 \), namely, the ground state is made up of \( |\psi_1\rangle \) and \( |\psi_4\rangle \). The overlap between two final states and the ground state is not zero. So one can see a long tail around \( B = 0 \) in Fig. 7. Whatever, the high fidelities are concentrated more on the one quadrant of a plan and are almost sudden death in the others through the transition effect of Yangian generators in the tensor product space. These results show that Yangian has the better hand, as a switch, to turn on or off the fidelity. To the better properties, possible applications are expected.

VI. CONCLUSIONS

In summary, we have investigated the properties of the energy level crossings of the ground states in the subspaces and the evolution of the thermal fidelity in the two-qubit system with a transverse non-homogeneous magnetic field respectively. We also discussed the effects of the quantum algebra on the states and high fidelity in the isotropy subspace and anisotropy subspace. For important parameters, we plotted the contour figures of energy gap, with the help of which the ground state can be distinguished and the level crossings are easily recognized. The thermal fidelity is completely calculated in both the XY model (\( \gamma = 0.2 \)) and Ising model (\( \gamma = 1 \)). In the two models the evolution tendencies of thermal fidelity versus the temperature are uniform when \( \lambda = \pm 1 \); but when \( \lambda = 0 \) at \( T = 0 \), the value of the \( F \) in XY model is two times than one in Ising model, because the energy level crossing appears. At \( T = 0.2 \) the contours of fidelity versus the parameters \( \lambda \) and \( B \) show \( F \) rapid death around \( \lambda = 0 \) besides minimal region like \( \delta \) function. Therefore the anisotropic factor of the magnetic field \( \lambda \), one of the focus in this paper, can be modulated to control the thermal fidelity. In addition, the thermal fidelity figures between the two final states and the state of the system at temperature \( T \) are obtained by the transition effect of Yangian generators, and show that the high fidelity only appears in the one branch (\( \lambda > 0 \)) of the equilateral hyperbola in the \( (B, \lambda) \) plane. Moreover, Yangian algebra plays important roles in enhancing the high \( F \) and making it more concentrate on the one quarter plane. So to speak, we present a more optimal proposal to turn on or off the fidelity as a switch.

These results show that Yangian algebra shed new light on fidelity controlling and the effect of the Yangian algebra is tremendous to deal with some physical models.
Acknowledgments

This work is partly supported by the NSF of China (Grant No. 11075101), Shanghai Leading Academic Discipline Project (Project No. S30105), and Shanghai Research Foundation (Grant No. 07d222020).

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