Stressed state “boundary layer” in a round plate of variable thickness according to refined theory

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Abstract. Based on the refined theory, the edge stress state of an isotropic round plate of variable thickness under the influence of local load was considered. In constructing the mathematical model of the plate, three-dimensional equations of the theory of elasticity and the variation Lagrange principle were used. Displacements were represented in the form of polynomials along with the coordinate normal to the middle surface, which was two degrees higher than the classical theory of the Kirchhoff–Love type. The resolving system of equations includes eleven ordinary differential equations with variable coefficients. The solution of the formulated boundary-value problem was carried out by finite difference methods and matrix sweeps. The deformations and tangential stresses of the plate were determined from the corresponding geometric and physical equations of the elasticity theory. This article has been focused on identifying the stress state of the boundary layer type near rigidly and elastically fixed edges by the round plate where the destruction of thin-walled structural elements in machinery, including aviation and space technology, takes place.

1. Introduction

Analysis or solution of plates and shells means the determination of displacement, moment, stress, etc. at various points in space of it. This analysis is very important to ensure the stability of the plate to resist the design load. A number of research works were performed on the solution of plate problems by various classical, numerical and engineering methods in the past years. Recently, there has been increasing interest in constructing refined theories of plates and shells. This is due to the need of study on the stress-deformed state (SDS) when designing elements of modern structures, including aircraft and rocketry products. When constructing an approximate theory of plates that is free of Kirchhoff–Love hypotheses, the method of direct asymptotic integration of differential equations of the three-dimensional theory of elasticity became widespread in [1]. The problem of an additional “boundary layer” type in relation to the classical theory of SDS near the clamped edge of rectangular and round plates was solved in a variation formulation by the Vlasov–Kantorovich method using a specially constructed polynomial approximating function. Another approach [2] was based on the representation of displacements in polynomial series along with the coordinate normal to the median plane and relates to the number of terms in these expansions in tangential and transverse directions.

The methods for calculating SDS based on analytical models including structures such as plates and shells with significantly varying stiffness characteristics that are more effective than numerical methods
are still relevant [3-5]. It should also be noted that the works [6-9], in which other methods of refined calculation of plates and shells are considered. Rajasekaran Sundaramoorthy and Wilson Antony [10] studied the determination of exact buckling loads and vibration frequencies of variable thickness isotropic plates using well known finite difference technique. The plates are subjected to uni, biaxial compression and shear loadings with various combinations of boundary conditions. In most of the cases, researchers carried out their analysis using the refined theory and then compared their results with other available known solutions to check validity of their analysis.

Therefore, this article aims to build a solution to calculate the SDS of the round plates of isotropic material with a thickness symmetrically changing in the radial direction. Here, we present a refined theory for determining SDS with the same accuracy in all areas of the plate, including the narrow boundary zones near the rigidly fixed edge. Comparison of the results obtained by the refined theory with the data of the classical theory showed that in the edge zone of the plate, the normal tangential stresses are substantially refined and the transverse normal stresses, which are neglected in the classical theory, are of the same order as the maximum values of the main bending stress.

2. Materials and methods
Let us consider the elastic equilibrium of a round isotropic plate with a thickness symmetric with respect to the median plane, loaded with a transverse local load \( q(r, \varphi) \), referred to a cylindrical coordinate system \((r, \varphi, z)\) (figure 1). Denoted by \(a\) and \(b\) are the external and internal radii of the plate, and by \(2h(r)\) its variable thickness. The outer and inner edges of the plate \(r = a, r = b\) are rigidly pinched.

![Figure 1. A round plate of variable thickness.](image)

In accordance to [2], the desired elastic displacements \( U_1, U_2, U_3 \) are represented in the form

\[
U_1 (r, \varphi, z) = u_0 (r, \varphi) + u_1 (r, \varphi) z + u_2 (r, \varphi) \frac{z^2}{2!} + u_3 (r, \varphi) \frac{z^3}{3!},
\]

\[
U_2 (r, \varphi, z) = v_0 (r, \varphi) + v_1 (r, \varphi) z + v_2 (r, \varphi) \frac{z^2}{2!} + v_3 (r, \varphi) \frac{z^3}{3!},
\]

\[
U_3 (r, \varphi, z) = w_0 (r, \varphi) + w_1 (r, \varphi) z + w_2 (r, \varphi) \frac{z^2}{2!},
\]

where the indices 1, 2, 3 correspond to the axes \(x, y\) and \(z\). Decomposition (1) corresponds to a two-degree increase in polynomials approximating the desired displacements.

The geometric equations of the three-dimensional elasticity theory have the form
\[ \varepsilon_r = \frac{\partial U_1}{\partial r}, \quad \varepsilon_\phi = \frac{1}{r} \left( \frac{\partial U_1}{\partial \phi} + U_1 \right), \quad \varepsilon_z = \frac{\partial U_3}{\partial z}, \]

\[ \gamma_{r\phi} = r \frac{\partial U_1}{\partial r} + \frac{1}{r} \frac{\partial U_1}{\partial \phi}, \quad \gamma_{rz} = \frac{\partial U_3}{\partial r} + \frac{\partial U_1}{\partial z}, \quad \gamma_{\phi z} = \frac{1}{r} \frac{\partial U_3}{\partial \phi} + \frac{\partial U_2}{\partial z}. \]  

(2)

Putting formulas (1) in (2), we find

\[ \varepsilon_r = \sum_{i=0}^{3} \frac{\partial u_i}{\partial r} \frac{z^i}{i!}, \quad \varepsilon_\phi = \frac{1}{r} \left( \sum_{i=0}^{3} \frac{\partial v_i}{\partial \phi} \frac{z^i}{i!} + \sum_{i=0}^{3} u_i \frac{z^i}{i!} \right), \quad \varepsilon_z = w_1 + w_2 z, \]

\[ \gamma_{r\phi} = \frac{1}{r} \sum_{i=0}^{3} \left( \frac{\partial u_i}{\partial \phi} + \frac{\partial v_i}{\partial r} \right) \frac{z^i}{i!}, \]

\[ \gamma_{rz} = \left( u_1 + \frac{\partial w_0}{\partial r} \right) + \left( u_2 + \frac{\partial w_1}{\partial r} \right) z + \left( u_3 + \frac{\partial w_2}{\partial r} \right) \frac{z^2}{2!}, \]

\[ \gamma_{\phi z} = \left( v_1 + \frac{1}{r} \frac{\partial w_0}{\partial \phi} \right) + \left( v_2 + \frac{1}{r} \frac{\partial w_1}{\partial \phi} \right) z + \left( v_3 + \frac{1}{r} \frac{\partial w_2}{\partial \phi} \right) \frac{z^2}{2!}. \]  

(3)

The physical equations of the three-dimensional elasticity theory for the plate are written as following

\[ \sigma_r = A_{11} \varepsilon_r + A_{12} \varepsilon_\phi + A_{13} \varepsilon_z, \quad \sigma_\phi = A_{21} \varepsilon_r + A_{22} \varepsilon_\phi + A_{23} \varepsilon_z, \quad \sigma_z = A_{33} \varepsilon_z, \quad \tau_{r\phi} = A_{44} \gamma_{r\phi}, \quad \tau_{rz} = A_{55} \gamma_{rz}, \quad \tau_{\phi z} = A_{60} \gamma_{\phi z}, \]  

(4)

where the coefficients \( A_{ij} \) (\( i = 1, 6, j = 1, 6 \)) are the elastic constants of the isotropic material of the plate.

We find differential equilibrium equations and natural boundary conditions for plates on the basis of the Lagrange variation principle \cite{2}

\[ \int \int \left( \sigma_r \delta \varepsilon_r + \sigma_\phi \delta \varepsilon_\phi + \sigma_z \delta \varepsilon_z + \tau_{r\phi} \delta \gamma_{r\phi} + \tau_{rz} \delta \gamma_{rz} + \tau_{\phi z} \delta \gamma_{\phi z} \right) dr d\phi dz - \int \int q(r, \phi) \delta \left( w_1 (r, \phi) + w_2 (r, \phi) \frac{h^2}{2} \right) dr d\phi = 0. \]  

(5)

First we represent the external load, strain (3) and stress (4) in the form of trigonometric series along the circumferential coordinate. Substituting the obtained decompositions into equality (5), we obtain the system of equations of equilibrium in displacements:

\[ \left( K_{1}^{u00} + K_{1}^{u01} \frac{d}{dr} + K_{1}^{u011} \frac{d^2}{dr^2} - K_{1}^{u022} \cdot m^2 \right) U_{0w} (r) + \]

\[ + \left( K_{1}^{w20} + K_{1}^{w21} \frac{d}{dr} + K_{1}^{w211} \frac{d^2}{dr^2} - K_{1}^{w222} \cdot m^2 \right) U_{2w} (r) - \]

\[ - m(K_{1}^{v00} + K_{1}^{v012} \frac{d}{dr}) W_{0w} (r) - m(K_{1}^{v12} + K_{1}^{v121} \frac{d}{dr}) W_{2w} (r) + \]

\[ + (K_{1}^{w00} + K_{1}^{w11} \frac{d}{dr}) W_{1w} (r) = 0, \]

\[ \left( K_{2}^{u00} + K_{2}^{u01} \frac{d}{dr} + K_{2}^{u011} \frac{d^2}{dr^2} - K_{2}^{u022} \cdot m^2 \right) U_{1w} (r) + \]

\[ + \left( K_{2}^{v00} + K_{2}^{v01} \frac{d}{dr} + K_{2}^{v011} \frac{d^2}{dr^2} - K_{2}^{v022} \cdot m^2 \right) U_{3w} (r) - \]
\[-m(K_2^{112} + K_2^{v112}) \frac{d}{dr} V_{1n}(r) - m(K_2^{v32} + K_2^{v312}) \frac{d}{dr} W_{3n}(r) +

+ (K_2^{v210} + K_2^{v212}) \frac{d}{dr} W_{2m}(r) + K_2^{v01} \frac{d}{dr} W_{0m}(r) = 0,

\left(K_3^{v100} + K_3^{v101} \frac{d}{dr} + K_3^{v011} \frac{d^2}{dr^2} - K_3^{v022} \cdot m^2 \right) U_{0m}(r) +

\left(K_3^{v201} + K_3^{v21} \frac{d}{dr} + K_3^{v211} \frac{d^2}{dr^2} - K_3^{v222} \cdot m^2 \right) U_{2m}(r) -

-m(K_3^{v102} + K_3^{v012}) \frac{d}{dr} W_{1m}(r) - m(K_3^{v32} + K_3^{v312}) \frac{d}{dr} W_{3m}(r) +

+ (K_3^{v10} + K_3^{v111} \frac{d}{dr}) W_{1m}(r) = 0,

\left(K_4^{v10} + K_4^{v111} \frac{d}{dr} + K_4^{v111} \frac{d^2}{dr^2} - K_4^{v212} \cdot m^2 \right) U_{1m}(r) +

\left(K_4^{v310} + K_4^{v311} \frac{d}{dr} + K_4^{v3111} \frac{d^2}{dr^2} - K_4^{v322} \cdot m^2 \right) U_{3m}(r) -

-m(K_4^{v12} + K_4^{v112}) \frac{d}{dr} W_{1m}(r) - m(K_4^{v32} + K_4^{v312}) \frac{d}{dr} W_{3m}(r) +

+ (K_4^{v20} + K_4^{v211} \frac{d}{dr}) W_{2m}(r) + K_4^{v01} \frac{d}{dr} W_{0m}(r) = 0,

m \left(K_5^{v012} + K_5^{v010} \frac{d}{dr} \right) U_{0m}(r) + m \left(K_5^{v22} + K_5^{v212} \frac{d}{dr} \right) U_{2m}(r) +

+ (K_5^{v01} \frac{d^2}{dr^2} - m^2 K_5^{v022}) V_{0m}(r) + (K_5^{v211} \frac{d^2}{dr^2} - m^2 K_5^{v222}) V_{2m}(r)

+ m \cdot K_5^{v312} W_{1m}(r) = 0,

m \left(K_6^{v11} \frac{d^2}{dr^2} - m^2 K_6^{v112} + K_6^{v10} \right) V_{1m}(r) + (K_6^{v311} \frac{d^2}{dr^2} - m^2 K_6^{v322} + K_6^{v310}) V_{3m}(r) +

+ m \cdot K_6^{v02} W_{0m}(r) + m \cdot K_6^{v22} W_{2m}(r) = 0,

m \left(K_7^{v012} + K_7^{v010} \frac{d}{dr} \right) U_{0m}(r) + m \left(K_7^{v22} + K_7^{v212} \frac{d}{dr} \right) U_{2m}(r) +

+ (K_7^{v011} \frac{d^2}{dr^2} - m^2 K_7^{v022}) V_{0m}(r) + (K_7^{v211} \frac{d^2}{dr^2} - m^2 K_7^{v222} + K_7^{v20}) V_{2m}(r) +

+ m \cdot K_7^{v312} W_{1m}(r) = 0,

m \left(K_8^{v112} + K_8^{v110} \frac{d}{dr} \right) U_{1m}(r) + m \left(K_8^{v32} + K_8^{v312} \frac{d}{dr} \right) U_{3m}(r) +

+ (K_8^{v111} \frac{d^2}{dr^2} - m^2 K_8^{v112} + K_8^{v10}) V_{1m}(r) + (K_8^{v311} \frac{d^2}{dr^2} - m^2 K_8^{v322} + K_8^{v30}) V_{3m}(r) +
\[ + m \cdot K_8^{w22} W_{0m}(r) + m \cdot K_8^{w22} W_{2m}(r) = 0, \]
\[ K_9^{w11} \frac{d^2 U_{1m}}{dr^2}(r) + K_9^{w31} \frac{d U_{3m}}{dr}(r) - m \cdot K_9^{w12} V_{1m}(r) - m \cdot K_9^{w32} V_{3m}(r) + \]
\[ + (K_9^{w01}) \frac{d^2}{dr^2} - m^2 K_9^{w22} \right) W_{0m}(r) + (K_9^{w21}) \frac{d^2}{dr^2} - m^2 K_9^{w22} \right) W_{2m}(r) + \]
\[ + K_9^{Q33} Q33_m(r) = 0, \]
\[ (K_9^{w11}) \frac{d}{dr} U_{1m}(r) + (K_9^{w31}) \frac{d U_{3m}}{dr}(r) - m \cdot K_9^{w12} V_{1m}(r) - \]
\[ - m \cdot K_9^{w22} V_{2m}(r) + (K_9^{w21}) \frac{d^2}{dr^2} - m^2 K_9^{w22} \right) W_{0m}(r) = 0, \]
\[ (K_9^{w11}) \frac{d}{dr} U_{1m}(r) + (K_9^{w31}) \frac{d U_{3m}}{dr}(r) - m \cdot K_9^{w12} V_{1m}(r) - \]
\[ - m \cdot K_9^{Q33} Q33_m(r) = 0. \]

On rigidly clamped edges, the boundary conditions have the form
\[ r = a: \quad U_{1m} = V_{1m} = W_{jm} = 0, \quad i = 0,1,2,3, j = 0,1,2, \]
\[ r = b: \quad U_{1m} = V_{1m} = W_{jm} = 0, \quad i = 1,2,3, j = 0,1,2. \]

In equations (6), the coefficients \( K \) with alphabetic and numerical indices are variable coefficients depending on the geometric parameters and elastic constants of the plate material. To solve system (6), we use the finite-difference method [11-12]. Approximating the first and second order derivatives by the central differences of the second order of accuracy [13-14], we obtain a system of linear algebraic equations, the eleventh order matrix of which is solved by the matrix sweep method using a computer program Maple v.18.

The deformations and tangential stresses of the plate are found from the geometric and physical relations of the theory of elasticity. Transverse stresses are obtained by direct integration of the equilibrium equations of the three-dimensional elasticity theory
\[ \tau_{rz} = - \int_{-h}^{z} \left( \frac{\partial \sigma_z}{\partial r} + \frac{1}{r} \frac{\partial \tau_{rp}}{\partial \phi} \right) dz, \quad \tau_{pq} = - \int_{-h}^{z} \left( \frac{1}{r} \frac{\partial \sigma_p}{\partial \phi} + \frac{\partial \tau_{rp}}{\partial r} \right) dz, \]
\[ \sigma_z = - \int_{-h}^{z} \left( \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{pq}}{\partial \phi} - \frac{1}{r} \sigma_q \right) dz. \]

3. Results and discussion
Consider a steel round plate, rigidly clamped at two edges \( r = a, r = b \) and under the influence of local load
\[ q(r, \phi) = \begin{cases} 0, & \text{if } b \leq r < r_i, \\ Q_0 \sin(\phi), & \text{if } r_i \leq r \leq r_2, \quad \text{where } Q_0 = \text{const}, \\ 0, & \text{if } r_2 < r \leq a \end{cases} \]
\[ r_i = (a + 19b)/20, \quad r_2 = (19a + b)/20. \quad \text{Poisson's ratio } \mu = 0.3, \quad \text{elastic modulus } E = 2 \cdot 10^{11} \text{Pa}. \quad \text{Plate dimensions: outer and inner radii } a = 0.5(m), \quad b = 0.15(m), \quad h = 0.05(m), \quad h_0 = 0.015(m); \quad \text{linearly} \]
varying thickness is determined by the ratio \( h(r) = h_m - \tan(\alpha) \cdot r \), where \( \tan(\alpha) = (h_m - h_0) / (a - b) \). The results of calculating the SDS of a round plate are presented in figures 2-5. Here the abbreviations “Class” and “Non-class” correspond to the results of calculations according to classical and refined theories. In figure 2 shows the variation in the additional stresses of the “boundary layer” in thickness according to refined theories at the rigidly clamped edge of the plate.

An analysis of the results shows that when moving away from the edge, the stresses obtained from the refined and classical theories practically coincide, which confirms the reliability of the refined theories; maximum mismatch of the calculation results takes place (figure 3) in determining normal stresses \( \sigma_r \) and is about 10%.

![Figure 2](image1.png)  
**Figure 2.** Change of stresses \( \sigma_r, \sigma_\phi, \sigma_z, \tau_{rz} \) along the thickness at the edge \( r = b \).

![Figure 3](image2.png)  
**Figure 3.** Change \( \sigma_r \) in radius.

![Figure 4](image3.png)  
**Figure 4.** Change \( \sigma_\phi \) in radius.

![Figure 5](image4.png)  
**Figure 5.** Change \( \sigma_z \) in radius.
It can be noted that the stresses in the edge zone are substantially refined: normal tangential stresses $\sigma_t$ – by 21.4% (figure 3) and $\sigma_\theta$ – by 30.9% (figure 4). In addition, the normal transverse stresses $\sigma_z$, negligible in the classical theory, according to the revised theory, account for about 28% of the maximum bending stresses $\sigma_r$ (figures 2, 3, 5).

Compared with the classical theory, the refined theory of round plates considers quickly decaying additional boundary SDSs of the “boundary layer” type that make a significant contribution to the total SDS of the plate. When moving away from the edge of the plate - a zone of the distorted stressed state – the results obtained according to the refined and classical theories almost coincide, thus confirming the validity of the proposed theory.

It can be noted that the additional stresses of the border layer type determined in rectangular plates by other methods [1] (asymptotic integration of differential equations of three-dimensional problem of the elasticity theory) nearly coincide in damping magnitude and nature with the results of this work. For cylindrical shell by this method [2] of solving three-dimensional elasticity theory problems, obtained the same result. Using the modified semi-inverse Saint-Venant method, an approximate solution of the spatial problems of the theory of elasticity, in the work [3] also obtained such results for thin shells. In the future, the method presented in this paper can be used to solve SDS problems for spheres, cones, and other shapes. In the next research process, experiments will be conducted to verified results of this research.

4. Conclusion

Based on the refined theory, a boundary-value problem is constructed for a circular plate of variable thickness under the influence of local load. The results of calculations of the stress state of the plate according to the classical and refined theories are presented and their comparison with each other is given. It was established that in the study of SDS in the marginal zones of the plate, the proposed mathematical model should be used, since the maximum stresses are substantially refined. It is shown that the transverse normal stresses turn out to be of the same order with the maximum values of the main bending stress. This result is important because it allows to reliably assess the strength and crack resistance of the power hulls of aircraft.

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