FULLY COVARIANT VAN DAM-VELTMAN-ZAKHAROV DISCONTINUITY, AND ABSENCE THEREOF

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Abstract

In both old and recent literature, it has been argued that the celebrated van Dam-Veltman-Zakharov (vDVZ) discontinuity of massive gravity is an artifact due to linearization of the true equations of motion. In this letter, we investigate that claim. First, we exhibit an explicit –albeit somewhat arbitrary– fully covariant set of equations of motion that, upon linearization, reduce to the standard Pauli-Fierz equations. We show that the vDVZ discontinuity still persists in that non-linear, covariant theory. Then, we restrict our attention to a particular system that consistently incorporates massive gravity: the Dvali-Gabadadze-Porrati (DGP) model. DGP is fully covariant and does not share the arbitrariness and imperfections of our previous covariantization, and its linearization exhibits a vDVZ discontinuity. Nevertheless, we explicitly show that the discontinuity does disappear in the fully covariant theory, and we explain the reason for this phenomenon.
1 Introduction

In a famous paper, van Dam and Veltman [1] (see also Zakharov [2]) studied a massive spin-2 field that couples to matter as the graviton, namely as \( h^{\mu\nu} T_{\mu\nu} \) (\( T_{\mu\nu} \) is the conserved stress-energy tensor). They showed that, at distances much smaller than the Compton wavelength of the massive graviton, one recovers Newton’s law by an appropriate choice of the spin-2 coupling constant. On the other hand, in the small-mass limit, the bending angle of light by a massive body approaches \( \frac{3}{4} \) of the Einstein result. This is the vDVZ discontinuity. A physical explanation of this phenomenon is that a massive spin-2 field carries 5 polarizations, whereas a massless one carries only two. In the limit \( m \to 0 \), therefore, a massive spin-2 field decomposes into massless fields of spin 2, 1, and 0. The spin-0 field couples to the trace of the stress-energy tensor, so that in the limit \( m \to 0 \) one does not recover Einstein’s gravity but rather a scalar-tensor theory.

This result seems to rule out any modification of Einstein’s gravity in which the principle of equivalence still holds, but the graviton acquires a mass, no matter how tiny.

In the presence of a negative cosmological constant \( \Lambda \), on an Anti de Sitter background, the one-graviton amplitude between conserved sources is continuous in the limit \( m^2 / \Lambda \to 0 \) [3, 4], so that one cannot rule a massive graviton with a Compton wavelength of the order of the Hubble scale. In refs. [5, 6], it was shown from various viewpoints that the AdS graviton may indeed become massive, when standard gravity is coupled to conformal matter.

On a de Sitter background, a massive spin-2 field is unitary only if \( m^2 \geq 2\Lambda/3 \) [7].

All of this makes perfect sense, yet, the very fact that experiments at a scale of roughly an astronomical unit can tell that the mass of the graviton is smaller than the inverse Hubble radius is baffling to some. After all, the latter scale is \( 10^{16} \) times smaller than the former!

In fact, several old [8] and recent [9, 10, 11] papers have claimed or argued that the vDVZ discontinuity is an artifact of the Pauli-Fierz Lagrangian, i.e. of the linearization of the true, covariant, non-linear equations of massive gravity.

Most of the renewed attempts to go beyond the linear approximation to massive gravity have exploited the DGP model [12], which has both a massive-graviton spectrum and four-dimensional general covariance.

In this letter, we would like to study the existence of a general covariant vDVZ discontinuity from two points of view.

First of all, in Section 2, we exhibit explicitly covariant, fully nonlinear equations of motion for a massive spin-2 field coupled to matter that, after linearization, reduce to the Pauli-Fierz system studied in [1, 2]. We show that the discontinuity found in the linearized equations persists at the non-linear, fully covariant level by finding a covariant constraint not present in standard, massless gravity. This definitely settles in the affirmative the
question of whether a general-covariant extension of the vDVZ result exists.

In Section 3, we study the DGP model, for two reasons. The first is that it is a promising candidate for a brane-world realization of gravity, which may even shed light on the cosmological constant problem. The second is that the covariantization studied in the first part of the paper is far from being unquestionable. Besides being somewhat arbitrary, so that it does not rule out the possibility of other discontinuity-free covariantizations, it is also non-local. Non-locality signals the presence of other light, possibly unphysical degrees of freedom coupled to ordinary matter (negative-norm ghosts, for instance). The DGP model, instead, is a consistent model that exhibits a vDVZ discontinuity at linear order. We show that there, as argued in [9, 10, 11], the discontinuity does indeed disappear when the DGP is studied beyond its linear approximation. To prove that, we relate the breakdown of the linear approximation to the fact that the brane can bend in the fifth dimension, so that its induced curvature may be large even when the source on the brane is weak.

2 Covariantization of the Pauli-Fierz Action

A long time ago, Pauli and Fierz [13] found a local, covariant action describing a free, massive spin-2 field. The action is unique up to field redefinitions and it reads

$$S = S_L[h_{\mu\nu}] + \int d^4x \left[ \frac{M^2}{64\pi G} (h_{\mu\nu}^2 - h^2) - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right].$$

(1)

Here, $T_{\mu\nu}$ is an external source, that we assume to be conserved and identify with the stress-energy tensor of matter. $S_L[h_{\mu\nu}]$ is the Einstein action expanded to quadratic order in the metric fluctuations around flat space:

$$S_L[h_{\mu\nu}] = \frac{1}{64\pi G} \int d^4x h^{\mu\nu} L_{\mu\nu,\rho\sigma} h^{\rho\sigma},$$

$$S_E[\eta_{\mu\nu} + h_{\mu\nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-\det(\eta + h)} R(\eta + h) = S_L[h_{\mu\nu}] + O(h^3).$$

(2)

At $M = 0$, the action in Eq. (1) is obviously invariant under linearized diffeomorphisms

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu,$$

(3)

but the mass term explicitly breaks this invariance. A gauge-invariant form of the Pauli-Fierz action is achieved by using the St"uckelberg mechanism, i.e. by adding a vector field that transforms linearly under local diffeomorphisms:

$$A_\mu \rightarrow A_\mu - \xi_\mu.$$

(4)
Substituting \( h_{\mu \nu} \rightarrow h_{\mu \nu} + 2 \partial_{(\mu} A_{\nu)} \) in Eq. (1) we find the manifestly gauge-invariant \textquotedblleft St"uckelberg-Pauli-Fierz\textquotedblright{} (SPF) action

\[
S_{SPF}[h] = S_L[h] + \int d^4x \left\{ \frac{M^2}{64\pi G} \left[ (h_{\mu \nu} + 2 \partial_{(\mu} A_{\nu)}^2 - (h + 2 \partial \cdot A)^2 \right] - \frac{1}{2} h_{\mu \nu} T^{\mu \nu} \right\}. \tag{5}
\]

The gauge-invariant equations of motion are

\[
L_{\mu \nu, \rho \sigma} h^{\rho \sigma} + M^2[h_{\mu \nu} + 2 \partial_{(\mu} A_{\nu)} - \eta_{\mu \nu}(h + 2 \partial \cdot A)] = 16\pi GT_{\mu \nu}, \tag{6}
\]

\[
\partial^\nu F_{\nu \mu} + J_\mu = 0, \quad J_\mu = \partial^\nu h_{\mu \nu} - \partial_\mu h. \tag{7}
\]

Of course, \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). Notice that the Pauli-Fierz mass term is precisely the combination that gives a gauge-invariant equation of motion for \( A_\mu \). Equation (7) is easily solved by

\[
A_\mu = -\Box^{-1} J_\mu + \partial_\mu \phi. \tag{8}
\]

\( \phi \) is an arbitrary function since Eq. (7) is invariant under the gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\mu \chi \). We can then select a particular solution to Eq. (7) by choosing \( \phi = -\Box^{-1} h \):

\[
A_\mu = -\Box^{-1} I_\mu, \quad I = \partial^\nu h_{\mu \nu} - \frac{1}{2} \partial_\mu h. \tag{9}
\]

Substituting this \( A_\mu \) into Eq. (3) we arrive at a particularly interesting form of the equations of motion:

\[
L_{\mu \nu, \rho \sigma} h^{\rho \sigma} + M^2[h_{\mu \nu} + h_{\mu \nu} + -2 \partial_{(\mu} \Box^{-1} I_{\nu)} - \eta_{\mu \nu}(h - 2 \Box^{-1} \partial \cdot I)] = 16\pi GT_{\mu \nu}. \tag{10}
\]

Recalling the definition of \( I_\mu \), and noticing that \( L_{\mu \nu, \rho \sigma} h^{\rho \sigma} \) is by construction proportional to the linearized Einstein tensor, \( L_{\mu \nu, \rho \sigma} h^{\rho \sigma} = 2G^L_{\mu \nu} = 2R_{\mu \nu}^L - \eta_{\mu \nu} R^L \), we can be recast Eq. (10) into the suggestive form

\[
G^L_{\mu \nu} - M^2 \Box^{-1} (R^L_{\mu \nu} - \eta_{\mu \nu} R^L) = 8\pi GT_{\mu \nu}. \tag{11}
\]

It is now obvious how to promote the Pauli-Fierz equations into a fully covariant form. First, we notice that any symmetric tensor \( S_{\mu \nu} \) can be decomposed as \( S_{\mu \nu} = S^T_{\mu \nu} + D_{(\mu} S_{\nu)} \), \( D^\mu S^T_{\mu \nu} = 0 \). Then, we replace all linearized tensors in Eq. (11) with their exact form

\[
G_{\mu \nu} - M^2 \left( \Box^{-1} G_{\mu \nu} \right)^T + \frac{1}{2} M^2 g_{\mu \nu} \Box^{-1} R = 8\pi GT_{\mu \nu}, \tag{12}
\]

where \( T_{\mu \nu} \) obeys the covariant conservation equation \( D^\mu T_{\mu \nu} = 0 \). We can also find the covariant form of the vDVZ discontinuity. By taking the double divergence of Eq. (12), we get a new constraint on the metric, not present in Einstein’s gravity:

\[
D^\mu D^\nu \left[ G_{\mu \nu} - M^2 \left( \Box^{-1} G_{\mu \nu} \right)^T + \frac{1}{2} M^2 g_{\mu \nu} \Box^{-1} R \right] = -\frac{M^2}{2} R = 0. \tag{13}
\]
Clearly this constraint, implying that the scalar curvature is zero everywhere, cannot be satisfied by a metric obeying Einstein’s equations up to a small deformation $o(M)$. Notice also that we would have missed the existence of the discontinuity if we only studied the Einstein vacuum equations, $R_{\mu\nu} = 0$. In the covariantization studied here, the discontinuity appears only when comparing Eq. (13) with the Einstein equations in matter, where $R = -8\pi G T \neq 0$.

Eq. (12) is fully covariant and it reduces to the PFS equations to linear order, but it is far from satisfactory. The first problem is that it is by no means the only covariantization of Eq. (11), so that we cannot exclude a priori that other covariantizations exist, in which the vDVZ discontinuity disappears. Secondly, Eq. (12) cannot be derived from a covariant action, since if that were the case its covariant divergence would automatically vanish, instead of giving the constraint Eq. (13). One could hope that a “good” covariantization, where the divergence of the equations of motions vanishes identically, may also cure the discontinuity.

A third, more serious problem, is that Eq. (12) is nonlocal and it may, therefore, describe the propagation of other light, possibly unphysical degrees of freedom.

We address the first and third problems in the next Section, when discussing a consistent embedding of massive gravity into a ghost-free theory: the DGP model.

The second problem is addressed here, by showing that another covariantization of Eq. (11) exists, with the desired property that the covariant divergence vanishes identically, but in which the vDVZ discontinuity is still present.

First of all, recall that Eq. (8) depends on an arbitrary scalar function. We can then write, generically,

$$ A_{\mu} = -\Box^{-1} I_{\mu} + \partial_{\mu}\varphi. $$

(14)

We can also introduce another scalar, $\psi$, and redefine the linearized metric as

$$ h_{\mu\nu} \rightarrow h_{\mu\nu} + \eta_{\mu\nu}\psi. $$

(15)

This redefinition changes the (linearized) Einstein tensor and the scalar curvature as follows

$$ G_{\mu\nu} \rightarrow G_{\mu\nu} + \eta_{\mu\nu}\Box \psi - \partial_{\mu}\partial_{\nu}\psi, \quad R^L \rightarrow R^L - 3\Box \psi. $$

(16)

Thanks to Eqs. (14,16) we can re-write the PFS equations as

$$ G_{\mu\nu}^L - M^2\Box^{-1}\left[ G_{\mu\nu}^L - \frac{1}{2}\eta_{\mu\nu}\left( R^L - 3\Box \psi \right) \right] + (1 - M^2\Box^{-1}) (\eta_{\mu\nu}\Box \psi - \partial_{\mu}\partial_{\nu}\psi) - M^2 (\eta_{\mu\nu}\Box \varphi - \partial_{\mu}\partial_{\nu}\varphi) = 8\pi G T_{\mu\nu}. $$

(17)

This equation can be simplified by setting $3\Box \psi = R^L$, and $(1 - M^2\Box^{-1})\psi = M^2 \varphi$:

$$ (1 - M^2\Box^{-1}) G_{\mu\nu}^L = 8\pi G T_{\mu\nu}. $$

(18)
By taking the trace of Eq. (18), we find $(M^2 \Box^{-1} - 1)R^L = 8\pi G T$, so that we can re-write the equation that defines $\psi$ in a more instructive form

$$(\Box - M^2)\psi = -\frac{8\pi}{3} GT. \quad (19)$$

The covariantization of Eqs. (18,19) is now obvious

$$G_{\mu\nu} - M^2 \left( \Box^{-1} G_{\mu\nu} \right)^T = 8\pi G T_{\mu\nu}, \quad (\Box - M^2)\psi = -\frac{8\pi}{3} GT. \quad (20)$$

Notice that the divergence of the tensor equation is automatically zero thanks to the covariant conservation of the stress-energy tensor and a standard Bianchi identity of general relativity. Notice also that the vDVZ discontinuity is still present, as Eqs. (20) describe a scalar-tensor theory, in which the massive scalar $\psi$ couples with gravitational strength to the trace of the stress-energy tensor.

### 3 Absence of vDVZ Discontinuity in the DGP Model

The results of the previous Section seem to indicate that even a consistent theory of massive gravity may suffer from a vDVZ discontinuity besides the linear order. Nevertheless, we will show that this is not the case in the DGP model, as already argued in [9, 10, 11] (see also [14]).

The DGP model describes a 4-d brane moving in a 5-d space with vanishing cosmological constant. In five dimensions, the Einstein action is

$$S_5 = \int_{\Sigma} d^5x \frac{1}{16\pi\hat{G}} \sqrt{-\hat{g}} \hat{R}(\hat{g}) + S_{GH}. \quad (21)$$

Here, hatted quantities are five-dimensional, while un-hatted ones are four-dimensional. The integral is performed over a space $\Sigma$ that is, topologically, the direct product of the real half-line $R^+$ and the 4-d Minkowski space $M_4$. We parametrize this space with four-dimensional coordinates $x^\mu$, $\mu = 0, 1, 2, 3$ and a fifth coordinate $y \equiv x^4$. $S_{GH}$ is the Gibbons-Hawking boundary term [15], whose explicit form we will not need.

The model is specified by the Einstein equations inside $\Sigma$ and by boundary conditions at the brane’s position, i.e. at the $\Sigma$ boundary $\partial \Sigma = M_4$:

$$\frac{1}{16\pi\hat{G}} K_{\mu\nu} \equiv -\frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_5[g]}{\delta g^{\mu\nu}} = \frac{1}{16\pi\hat{G}} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{1}{2} T_{\mu\nu}. \quad (22)$$

As evident from this equation, the brane has a 4-d nonzero Newton’s constant $G$. $T_{\mu\nu}$ is the stress-energy tensor of the matter living on the brane. The 4-d cosmological constant is assumed to be negligible. This corresponds to a limit in which the brane is almost
tensionless, and possesses only a bending energy, proportional to the extrinsic curvature $K_{\mu\nu}$. It is convenient to work in Gaussian coordinates where the brane is located at $y = 0$, and where

$$g_{\mu 4}(x, y)|_{y=0} = 0.$$  \hfill (23)

In these coordinates,

$$K_{\mu\nu} = \frac{1}{2} \sqrt{g_{44}} (\dot{g}_{\mu\nu} - g_{\mu\nu} g^{\alpha\beta} \dot{g}_{\alpha\beta}).$$  \hfill (24)

The dot denotes the derivative w.r.t. $y$. The linearization of Eq. (22) has been given in [9, 11, 16]. It is most conveniently performed in the 5-d harmonic gauge:

$$\partial_a h^a_b - \frac{1}{2} \partial_b h = 0, \quad \dot{g}_{ab} = \eta_{ab} + h_{ab}, \quad a, b = 0, ..., 4.$$  \hfill (25)

This gauge choice is compatible with Eqs. (23); indeed, it is compatible with setting $g_{\mu 4} = 0$ everywhere in $\Sigma$. After this last gauge choice, the linearized equations assume the simple form

$$\Box h_{ab}(z) + \ddot{h}_{ab}(z) = 0, \quad h_{\mu 4}(z) = 0, \quad z \in \Sigma; \quad \frac{1}{L} [\dot{h}_{\mu\nu}(x) - \eta_{\mu\nu} \dot{h}(x)] = \Box h_{\mu\nu}(x) - \partial_{\mu} \partial_{\nu} h(x) + 16\pi G T_{\mu\nu}(x), \quad x \in \partial \Sigma.$$  \hfill (26)

The ratio $L \equiv \hat{G}/G$ plays a fundamental role in the DGP model, since it is the transition length beyond which 4-d gravity turns into 5-d gravity.

Eqs. (26,27) are easily solved by Fourier transforming the 4-d coordinates $x^\mu$:

$$\tilde{h}_{\mu\nu}(p) = \tilde{\hat{h}}_{\mu\nu}(p) + p_{\mu} p_{\nu} \frac{16\pi G L}{p^2 + p^2/L} \tilde{T}(p) \exp(-py), \quad \frac{16\pi G}{p^2 + p/L} \left[ \tilde{T}_{\mu\nu}(p) - \frac{1}{3} \eta_{\mu\nu} \tilde{T}(p) \right] \exp(-py).$$  \hfill (28)

These equations contain a term proportional to $L$, that diverges in the decoupling limit $L \to \infty$. To linear order, this divergence is an artifact of our gauge choice, in which the brane lies at $y = 0$. It can be canceled by transforming into new coordinates, $\bar{y}$, $\bar{x}^\mu$, in which the brane lies at $\bar{y} = \epsilon_4(x, 0)$.

$$\bar{x}^\mu = x^\mu + \epsilon^\mu(x, y), \quad \bar{y} = y + \epsilon_4(x, y), \quad \bar{\epsilon}_4(p, y) = \frac{8\pi G L}{p^2 + p/L} \tilde{T}(p) \exp(-py), \quad \bar{\epsilon}_\mu(p, y) = -i \frac{p_{\mu}}{p} \bar{\epsilon}_4(p, y).$$  \hfill (30)

The new coordinate system still obeys $\tilde{h}_{\mu 4} = 0$ everywhere in $\Sigma$, since $\epsilon_\mu$ obeys

$$\partial_\mu \epsilon_4 + \dot{\epsilon}_\mu = 0.$$  \hfill (32)
The metric fluctuation in the new coordinate system is given by Eq. (29); it is finite in the limit $L \to \infty$. Moreover, $\bar{h}_{\mu \nu}$ is linear in the source $T_{\mu \nu}$, and small everywhere in $\Sigma$, if the energy density of the source is well below the black-hole limit. This is in prefect analogy with standard Einstein's gravity.

The story does not end here, though, as it can be seen by closer inspection of Eqs. (30,31).

Consider for simplicity a static, spherically-symmetric distribution of matter on the brane, with total mass $M$. Outside matter, in the region $GM \ll r \equiv |\vec{x}| \ll L$, the position of the brane in the new coordinate system is

$$\epsilon_4(r, 0) = \frac{2GM}{r}. \quad \text{(33)}$$

This function can be large even in the region $r \gg GM$, so that the limit of validity of the linear approximation must be re-examined. Recall that the metric transforms as follows under the reparametrization $\bar{x}^\mu = x^\mu + \epsilon^\mu(x, y)$, $\bar{y} = y + \epsilon_4(x, y)$:

$$g_{\mu \nu}(x, y) = \frac{\partial \bar{z}^a}{\partial x^\mu} \frac{\partial \bar{z}^b}{\partial x^\nu} \bar{g}_{ab}(\bar{z}), \quad \bar{z}^a = \bar{x}^\mu, \bar{y}. \quad \text{(34)}$$

If we expand to linear order in $\bar{h}_{\mu \nu}$, and to quadratic order in $\epsilon_a$, we find, at $y = 0$,

$$h_{\mu \nu}(x, 0) = \bar{h}_{\mu \nu}(x) + \partial_\mu \epsilon_\nu(x, 0) + \partial_\nu \epsilon_\mu(x, 0) + \partial_\mu \epsilon^a(x, 0) \partial_\nu \epsilon_a(x, 0) + \partial_\mu \epsilon^4(x, 0) \partial_\nu \epsilon_4(x, 0). \quad \text{(35)}$$

Notice that the last term in this expansion is not a 4-d reparametrization and that when $\epsilon_4$ is given by Eq. (33) it is $O(G^2M^2L^2/r^4)$.

Clearly, when we assume that matter is so diluted that $|\bar{h}_{\mu \nu}| \ll 1$ everywhere on the brane, the linear approximation for $\bar{h}_{\mu \nu}$ is justified by assumption, but the linear approximation for $\epsilon_a$ breaks down when $\partial_\mu \epsilon^4 \partial_\nu \epsilon_4 \gg \bar{h}_{\mu \nu}$. This happens when $G^2M^2L^2/r^4 \gg GM/r$, i.e. when

$$r^3 \ll GM^2L^2. \quad \text{(36)}$$

This is exactly the condition found in ref. [11].

The breakdown of the linear approximation for $\epsilon_a$ means that, in the region $r \leq (GM^2L^2)^{1/3}$, the position of the brane is still $\bar{y} = \epsilon^4(x, 0)$, but $\epsilon^4(x, 0)$ is no longer given by Eq. (33). To study the brane inside that region, we choose $\epsilon^4$ and $\epsilon^\mu$ by demanding only that $g_{\mu 5} = 0$ and that $\epsilon^4$ is still given by Eq. (33) at large distances:

$$\epsilon_4(r, 0) \approx \frac{2GM}{r}, \quad \text{for } r \geq (GM^2L^2)^{1/3}. \quad \text{(37)}$$

\[\text{\scriptsize \footnote{\text{To linear order } r = |\vec{x}|.}}\]
We can always set $\epsilon^\mu(x,0) = 0$ with a 4-d coordinate transformation. The metric fluctuation is then

$$h_{\mu\nu}(x,0) = \bar{h}_{\mu\nu}(x) + \partial_\mu \epsilon^4(x,0) \partial_\nu \epsilon_4(x,0).$$

(38)

To find the metric on the brane, we begin by making the following ansatz:

$$\bar{h}_{00}(r) = \frac{2F(r)M}{r}, \quad \bar{h}_{ii}(r) = \frac{F(r)M_i}{r}.$$  

(39)

The asymptotic behavior of the function $F(r)$ is:

$$F(r) = G \quad \text{for} \quad r \ll (GML^2)^{1/3}, \quad F(r) = \frac{4}{3} G \quad \text{for} \quad r \gg (GML^2)^{1/3};$$  

(40)

otherwise, $F(r)$ is arbitrary.

The linearized scalar curvature of the ansatz vanishes identically everywhere. At large distances, $r \gg (GML^2)^{1/3}$, it approximates the metric of the linearized DGP equations [9, 11, 16] [see also Eq. (29)].

Suppose now that a 5-d diffeomorphism $\epsilon_4$ exists, such that a) it obeys Eq. (37), b) the metric $h_{\mu\nu}$, given by Eq. (38), solves the linearized 4-d Einstein equations in the region $r \ll (GML^2)^{1/3}$. With these assumptions, we can write the solution to Eq. (22) as

$$g_{\mu\nu}(x,0) = \eta_{\mu\nu} + h_{\mu\nu}(x,0) + \Delta_{\mu\nu}(x,0),$$

(41)

with $|\Delta_{\mu\nu}(x,0)| \ll |h_{\mu\nu}(x,0)|$ everywhere on the brane. The last statement can be proven by approximating Eq. (22) as

$$L_{\mu\nu,\rho\sigma} \Delta^{\rho\sigma} = J_{\mu\nu},$$

(42)

$$J_{\mu\nu} = \frac{2}{L} K_{\mu\nu} + 16\pi G T_{\mu\nu} - L_{\mu\nu,\rho\sigma} h^{\rho\sigma}. $$

(43)

The source $J_{\mu\nu}$ is conserved, and everywhere smaller than $L_{\mu\nu,\rho\sigma} h^{\rho\sigma}$, because $h_{\mu\nu}$ solves by assumption the Einstein equations, for $r^3 \ll GML^2$, and, for $r^3 \gg GML^2$, it solves by construction the linearized DGP equations.

Conservation of the source ensures that Eq. (42) can be solved, while $|J_{\mu\nu}| \ll |L_{\mu\nu,\rho\sigma} h^{\rho\sigma}|$ guarantees that $|\Delta_{\mu\nu}(x,0)| \ll |h_{\mu\nu}(x,0)|$. Extending $h_{\mu\nu}$ and $\Delta_{\mu\nu}$ to the interior of $\Sigma$ is straightforward because $h_{\mu\nu} + \Delta_{\mu\nu}$ obeys Eq. (28).

At this point, we are left only with the task of finding a shift $\epsilon^4$ which satisfies our assumption. In the spherically symmetric case, we notice that the Schwarzschild metric at $r \gg GM$ is $h_{00}(r) = 2GM/r$, $h_{ii} = 2GM/r$, $i = 1, 2, 3$, so that the diffeomorphism we need is

$$\epsilon_4(r,0) = 2\sqrt{GM}r.$$  

(44)

Let us conclude with a few remarks.

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2We have assumed again that matter is diluted, i.e. that $|h_{\mu\nu}| \ll 1$ everywhere on the brane. This assumption has been made for clarity’s sake and can be relaxed.
• The limit of validity of Eq. (44) can be found by demanding that the contribution to the extrinsic curvature due to the brane bending in Eq. (44) is smaller than that given by Eq. (33). Since the curvature due to bending is \(\sim |\frac{d^2 \epsilon_4}{dr^2}|\) we find \(r \ll (GML^2)^{1/3}\). Therefore, the domain of validity of Eq. (44) is complementary to that of Eq. (33).

• The fact that quadratic corrections to the linear approximation cure the vDVZ discontinuity is at the heart of Refs. [8, 10, 11]. In this paper, we spelled out that it is the linear approximation for the fluctuations of the brane that fails at \(r \ll (GML^2)^{1/3}\), not the linearization of the 5-d metric (see also [11]).

• The previous observation makes the breakdown of linearity at such large distances more palatable, since the brane is almost tensionless, and can, therefore, bend significantly even over macroscopic (astronomical) distances.

• When the position of the brane is given by Eq. (44), the sub-leading correction to the induced metric, \(\Delta_{\mu\nu}\), is proportional to \(\sqrt{GM}\) [see Eqs. (42,43)]. It is tantalizing to conjecture that this correction may give rise to interesting modifications of Newtonian dynamics at some macroscopic length scale.

• Absence of a vDVZ discontinuity is only a qualified good news for the DGP theory. Indeed, the very breakdown of the linear approximation at the macroscopic length scale \(r = (GML^2)^{1/3}\) may signal that the 4-d scalar \(\epsilon_4(x,0)\) interacts strongly with the stress-energy tensor at the quadratic level—for instance through an interaction term \(\sim L\sqrt{G(\epsilon_4)^2T}\).

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