New methods to find patches of invisible integer lattice points

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It is a surprising fact that the proportion of integer lattice points visible from the origin is exactly \(6/\pi^2\), or approximately 60%. Hence, approximately 40% of the integer lattice is hidden from the origin. Since 1971, many have studied a variety of problems involving lattice-point visibility, in particular, searching for patterns in that 40% of the lattice composed of invisible points. One such pattern is a square patch, an \(n \times n\) grid of \(n^2\) invisible points, which we call a hidden forest.

It is known that there exist arbitrarily large hidden forests in the integer lattice. However, the methods up to now involve the Chinese remainder theorem (CRT) on the rows and columns of matrices with prime number entries, and they have only been able to locate hidden forests very far from the origin. For example, using this method the closest known \(4 \times 4\) hidden forest is over 3 quintillion, or \(3 \times 10^{18}\), units away from the origin. We introduce the concept of quasiprime matrices and utilize a variety of computational and theoretical techniques to find some of the closest known hidden forests to date. Using these new techniques, we find a \(4 \times 4\) hidden forest that is merely 184 million units away from the origin. We conjecture that every hidden forest can be found via the CRT-algorithm on a quasiprime matrix.

1. Introduction

Imagine the plane \(\mathbb{R}^2\) as a forest in which each nonorigin lattice point in \(\mathbb{Z}^2\) is a tree and each tree is infinitely thin yet also opaque. In this scenario, we say that a tree is hidden if some other tree lies in our line of sight from the origin.

Consider the four lines of sight denoted by the dashed line segments emanating from the origin in Figure 1, left. In these four lines of sight in the first quadrant, exactly four trees are visible — one per each line of sight. These visible trees are located at the black bullet points. Obscured by them are three other trees at the white bullet points, which are not visible from the origin. The tree at (2, 6) is

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obscured by the visible tree at (1, 3), while the tree at (6, 3) is obscured by the tree at (4, 2), which in turn is obscured by the visible tree at (2, 1). The question of the visibility (or invisibility) of a lattice point from the origin can be recast in a number-theoretic setting, where it turns out that the only visible points are the points $(x, y)$ such that $\gcd(x, y) = 1$. A proof for this visibility criterion is given in Proposition 2.1.

It is well known that approximately 60% of the integer lattice is visible from the origin (see Proposition 2.4). So a natural question to ask about the approximately 40% of the integer lattice which is hidden from view is the following:

**Are there arbitrarily large square patches of invisible lattice points?**

The answer to this question is yes, and in this paper we focus on invisible $n \times n$ square patches, which we call *hidden forests*. An example of a $2 \times 2$ hidden forest is given in Figure 1, right. In this figure we note the specific four visible trees that obscure this hidden forest.

Lattice-point visibility is a well-studied subject that arises in a variety of areas such as number theory, integer optimization, and even theoretical physics (see [Brass et al. 2005, Chapter 10.4] for a brief survey). Herzog and Stewart [1971] studied patterns of both visible and invisible lattice points; one such invisible pattern they explored is the one we call a hidden forest. Schumer [1990] also examined hidden forests. He used the Chinese remainder theorem (in a form similar to our Theorem 3.4) and gave an example of a $3 \times 3$ hidden forest very far from the origin and questioned whether a closer one could be found. He then noted that finding a $4 \times 4$ forest would require solving systems of linear congruence equations modulo the product of the first 16 primes, the so-called 16th primorial which is approximately 32 quintillion, and declared, “Such a project is beyond the courage of this author!” In this paper we not only undertake this task of finding closer $4 \times 4$ hidden forests, but also introduce a variety of theoretical and computational
techniques to aid in finding the closest known $n \times n$ hidden forests for $n \geq 4$, a task which has not yet been done to date. The paper is broken down as follows:

In Section 2, we give a brief overview of lattice-point visibility and provide a detailed proof of the well-known result that the probability of two randomly selected integers being relatively prime is $6/\pi^2$.

In Section 3, we give the known method of finding hidden forests in Section 3A. Given $n \in \mathbb{N}$ and a prime matrix $P_n$, there exists an $n \times n$ hidden forest $H_n^{(x,y)}$ in the first quadrant with bottom-left corner $(x, y)$ that is found by applying the Chinese remainder theorem to the rows and columns of $P_n$. We denote this process by the term CRT-algorithm. In Section 3B, we elaborate on the relationship between a prime matrix $P_n$ and its hidden forest $H_n^{(x,y)}$ with the introduction of an object called a gcd-matrix. In Section 3C, we apply this method to find hidden forests $H_n^{(x,y)}$ for $n = 2, 3, 4$.

In Section 4, we introduce the concept of a quasiprime matrix $QP_M$ and the QP-algorithm in Section 4A and define the method of strings of strongly composite integers in Section 4B. In Section 4C, we explore the notion of an optimal gcd-matrix by considering the minimal number of prime factors required in a quasiprime matrix to produce an $n \times n$ hidden forest. With these three tools and some computational programming techniques, we then use the CRT-algorithm on quasiprime matrices to find $n \times n$ hidden forests. The resulting hidden forests turn out to be much closer to the origin than those found by the traditional method (given in Section 3).

In Section 5, we combine the techniques detailed in the previous section to find the closest known (to date) $5 \times 5$ hidden forest.

In Section 6, we give a selection of open problems.

2. Density of visible lattice points in $\mathbb{Z}^2$

As mentioned in the introduction, a criterion for the visibility of an integer lattice point can be recast in the number-theoretic setting as the following proposition shows.

**Proposition 2.1.** Let $(x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Then $(x, y)$ is visible if and only if $\gcd(x, y) = 1$.

**Proof.** Let $(x, y)$ be a nonorigin point in $\mathbb{Z}^2$. Suppose $d = \gcd(x, y)$. If $d > 1$ then $(x/d, y/d)$ lies strictly between the points $(0, 0)$ and $(x, y)$, and hence $(x, y)$ is not visible. Thus $(x, y)$ visible implies that $\gcd(x, y) = 1$.

Conversely, assume that $d = 1$ and suppose by way of contradiction that $(x, y)$ is not visible from the origin. Then there is a point $(x_0, y_0) \in \mathbb{Z}^2$ such that we have $(x, y) = (cx_0, cy_0)$ for some integer $c > 1$. That is, $c$ divides both $x$ and $y$. But $d = 1$ is the greatest common divisor of $x$ and $y$, contradicting that $c > 1$. Thus if $\gcd(x, y) = 1$ then $(x, y)$ is visible. 

\(\square\)
Now that we have a simple criterion for an integer lattice point’s visibility, it is natural to inquire what fraction of integer lattice points are visible from the origin. That is, we ask:

**What is the density of visible lattice points in \( \mathbb{Z}^2 \)?**

Let \( T(n) \) equal the total number of integer lattice points in an \( n \times n \) square centered at the origin, and let \( V(n) \) equal the number of these points visible from the origin. Then it suffices to compute the limit of \( V(n)/T(n) \) as \( n \) approaches infinity. It turns out this limit is \( 6/\pi^2 \). Proofs of this famous result are well known with the earliest proofs given in the late 19th century (see references in Remark 2.2). Many modern solutions involve the Möbius inversion formula and Euler’s totient function. In Proposition 2.4, we provide an alternative proof that is essentially an application of Euler’s famous product formula and utilizes the number-theoretic criterion for the visibility of a lattice point given in Proposition 2.1.

**Remark 2.2** (historical background to the problem). The historical record of the original authorship of the result in Proposition 2.4 is inaccurately described on a number of occasions in the literature. Originally, the question on the probability of two random integers being coprime was raised by Cesàro [1881]. Two years later, Cesáro [1883] and Sylvester [1883] independently proved the result. Earlier Dirichlet [1849] proved a slightly weaker form of the result. The generalization to \( k \) coprime integers with \( k > 2 \) was presented again by Cesàro [1884]. This result was also proven independently by Lehmer [1900].

**Remark 2.3.** Since there is no uniform distribution on the natural numbers, it is somewhat imprecise to speak about the probability that two integers chosen at random are relatively prime. However, if we consider the uniform distribution on the set \( \{1, 2, \ldots, n\} \) and take the limit as \( n \) approaches infinity, then it is within this context that we make any probability statements in Proposition 2.4.

**Proposition 2.4.** The density of integer lattice points that are visible from the origin is \( 6/\pi^2 \), or approximately 60%.

**Proof.** It suffices to show that a lattice point chosen at random has a probability of \( 6/\pi^2 \) of being visible from the origin. Let \( x \) and \( y \) be randomly selected integers. Recall that \((x, y)\) is visible if and only if \( \gcd(x, y) = 1 \) by Proposition 2.1. Hence it suffices to compute the probability that no prime divides both \( x \) and \( y \). The probability that \( x \) is divisible by the prime \( p \) is \( 1/p \). Similarly \( y \) is divisible by \( p \) with probability \( 1/p \). By mutual independence, the probability that both \( x \) and \( y \) are divisible by \( p \) is \( 1/p^2 \). Hence, the probability that both integers \( x \) and \( y \) are not divisible by \( p \) is \( 1 - 1/p^2 \). For distinct primes, these divisibility events are mutually
independent; thus the probability that no prime divides both $x$ and $y$ is

$$\prod_p \left( 1 - \frac{1}{p^2} \right).$$

To calculate this infinite product, it is helpful to consider the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

for $s > 1$. A result of Euler connects this infinite sum with an infinite product of infinite sums over the primes. The essence of Euler’s proof is his use of the fundamental theorem of arithmetic to observe that the sum $\zeta(s)$ can be written as

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right).$$

(1)

To prove (1), Euler observed that since each $n$ in the denominator on the left-hand side is of the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for some $k$ by the fundamental theorem of arithmetic, by multiplying out the product on the right-hand side, each term $1/n^s$ on the left-hand side appears exactly once, as a product of the appropriate powers of the primes in $n$. And since each multiplicand on the right-hand side is a geometric series of the form $1/(1 - 1/p^s)$, Equation (1) becomes

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - 1/p^s}.$$

Setting $s = 2$ and taking reciprocals, we get

$$\zeta(2)^{-1} = \frac{1}{\sum_{n \geq 1} 1/n^2} = \prod_p \left( 1 - \frac{1}{p^2} \right),$$

where the right-hand side is the probability value we seek, and the left-hand side is the reciprocal of the well-known evaluation of the Riemann zeta function at $s = 2$, namely $\zeta(2) = \pi^2/6$ (the solution to the famous Basel problem).\footnote{The Basel problem asks for the exact sum of the reciprocals of the squares of the positive integers. There are a variety of proofs of this result. The details of 14 different proofs are given in [Chapman 2003], and [Moreno 2015] compiles a comprehensive list of 85 references from Euler to the present that address the Basel problem.} Hence the fraction of lattice points $(x, y)$ visible from the origin is $6/\pi^2$ (or approximately 0%), as desired. □
3. The traditional method to find hidden forests

In the previous section we showed that approximately 60% of the integer lattice is visible, and hence approximately 40% lies hidden from view. In this section, we find arbitrarily large patches of hidden square regions in $\mathbb{Z}^2$ using the known method. This technique is what we call the CRT-algorithm, since it is mainly an application of the Chinese remainder theorem (CRT). The strategy is to find two sets of $n$ consecutive integers

$$X = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_n\}$$

such that $X \cap Y = \emptyset$ and $\gcd(x_i, y_j) > 1$ for all $1 \leq i, j \leq n$. Then it is clear that the $n^2$ points in the set $\{(x_i, y_j) \mid 1 \leq i, j \leq n\}$ yield the desired hidden square region. To this end, we first establish some necessary preliminary definitions.

**Definition 3.1** (hidden forest). An $n \times n$ hidden forest in $\mathbb{Z}^2$ is a square patch of $n^2$ adjacent invisible integer lattice points. We denote this hidden forest by the symbol $H_{(x,y)}^n$, where $(x, y)$ is the closest corner lattice point of the square to the origin. By the remark below, this closest corner point is well-defined.

**Remark 3.2.** Observe that the points $(x, \pm 1)$ and $(\pm 1, y)$ are visible for all $x, y \in \mathbb{Z}$ by Proposition 2.1. Hence no nontrivial (that is, $n > 1$) hidden forest $H_{(x,y)}^n$ will contain any points on the $x$- or $y$-axes. Hence we conclude that any $H_{(x,y)}^n$ for $n > 1$ is completely contained in the interior of one of the four quadrants.

**Definition 3.3** (prime matrix). Let $\{p_1, p_2, \ldots, p_{n^2}\}$ be the set of the first $n^2$ primes. Construct an $n \times n$ matrix with these primes by filling row $i$ with $p_{(i-1)n+1}$ through $p_{(i-1)n+n}$ for each $1 \leq i \leq n$ to yield

$$
\begin{pmatrix}
  p_1 & p_2 & \cdots & p_j & \cdots & p_n \\
p_{n+1} & p_{n+2} & \cdots & p_{n+j} & \cdots & p_{2n} \\
p_{2n+1} & p_{2n+2} & \cdots & p_{2n+j} & \cdots & p_{3n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{(i-1)n+1} & p_{(i-1)n+2} & \cdots & p_{(i-1)n+j} & \cdots & p_{(i-1)n+n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{(n-1)n+1} & p_{(n-1)n+2} & \cdots & p_{(n-1)n+j} & \cdots & p_{n^2}
\end{pmatrix}.
$$

Note that the prime $p_{(i-1)n+j}$, boxed for visual ease, is located in row $i$ and column $j$ of the matrix. We call this $n \times n$ matrix a prime matrix and denote it by $P_n$.

**3A. The CRT-algorithm.** The following theorem is the primary tool used in the CRT-algorithm to find hidden forests of arbitrary size.

**Theorem 3.4.** For each $n \in \mathbb{N}$, there exist two sets of $n$ consecutive numbers $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that $X \cap Y = \emptyset$ and $\gcd(x_i, y_j) > 1$ for all $1 \leq i, j \leq n$. 
Proof. Fix \( n \in \mathbb{N} \). Consider the prime matrix \( P_n \). Let \( R_i \) and \( C_j \) be the product of the entries in row \( i \) and column \( j \), respectively, so we have

\[
R_i = \prod_{k=1}^{n} p_{(i-1)n+k} \quad \text{and} \quad C_j = \prod_{k=0}^{n-1} p_{kn+j}.
\]

Since they share no primes in common, the row products \( R_1, R_2, \ldots, R_n \) are pairwise relatively prime. Similarly, the column products \( C_1, C_2, \ldots, C_n \) are pairwise relatively prime. Consider the pair of systems of linear congruences

\[
\begin{align*}
x + 1 &\equiv 0 \pmod{R_1}, \\
x + 2 &\equiv 0 \pmod{R_2}, \\
&\vdots \\
x + n &\equiv 0 \pmod{R_n},
\end{align*}
\]

\[
\begin{align*}
y + 1 &\equiv 0 \pmod{C_1}, \\
y + 2 &\equiv 0 \pmod{C_2}, \\
&\vdots \\
y + n &\equiv 0 \pmod{C_n}.
\end{align*}
\]

Observe that \( R_1 \cdot R_2 \cdots R_n = C_1 \cdot C_2 \cdots C_n = \prod_{i=1}^{n^2} p_i \), which we denote by \( M \).

By the Chinese remainder theorem, there exist solutions \( x_0 \) and \( y_0 \) to the left and right systems, respectively, such that \( x_0 \) and \( y_0 \) are unique modulo \( M \). Define the set \( X = \{x_0 + 1, x_0 + 2, \ldots, x_0 + n\} \) and the set \( Y = \{y_0 + 1, y_0 + 2, \ldots, y_0 + n\} \).

We claim that none of the integers in \( X \) are pairwise relatively prime to any of the integers in \( Y \). For an arbitrary \( x_0 + i \in X \) and \( y_0 + j \in Y \), these two elements by construction are multiples of \( R_i \) and \( C_j \), respectively. Hence the prime that lies in the intersection of row \( i \) and column \( j \) in the matrix, namely \( p_{(i-1)n+j} \), divides \( \gcd(x_0 + i, y_0 + j) \). Thus \( \gcd(x_0 + i, y_0 + j) > 1 \) as desired.

Observe that for \( n \geq 2 \) the sets \( X \) and \( Y \) are necessarily disjoint. Otherwise if \( X \cap Y \neq \emptyset \) then some element \( a \in X \) is relatively prime to some element \( a \pm 1 \in Y \) since \( \gcd(a, a \pm 1) = 1 \), contradicting \( \gcd(x_0 + i, y_0 + j) > 1 \) for all \( 1 \leq i, j \leq n \).

For the trivial case when \( n = 1 \), the algorithm above yields \( X = Y = \{2\} \). So set \( Y = \{4\} \), and hence \( X \cap Y = \emptyset \). \( \square \)

Algorithm (CRT-algorithm to construct a hidden forest \( H^n_{(x,y)} \)).

1. Fix a value \( n \in \mathbb{N} \).
2. Construct the prime matrix \( P_n \).
3. Apply Theorem 3.4 to \( P_n \) to yield sets \( X \) and \( Y \).
4. Construct the hidden forest \( H^n_{(x,y)} \) from \( X \) and \( Y \).

3B. The gcd-grid and gcd-matrix yielded by prime matrices. By Theorem 3.4, the prime matrix \( P_n \) yields the hidden forest \( H^n_{(x,y)} \) composed of the \( n^2 \) points \((x_i, y_j)\), where \( x_i = x_0 + i \in X \) and \( y_j = y_0 + j \in Y \) for \( 1 \leq i, j \leq n \). The forest \( H^n_{(x,y)} \) is shown in Figure 2. For each \( H^n_{(x,y)} \) we can write a corresponding \( n \times n \) array of numbers called the gcd-grid where \( g_{i,j} = \gcd(x_i, y_j) \) for \( 1 \leq i, j \leq n \). The gcd-grid is shown in Figure 2.
We call this matrix arising from the gcd-grid the \textit{gcd}-matrix. 

\textbf{Remark 3.5.} The \((i, j)\)-entry of \(P_n\) is \(p_{i,j}\). However, the \((i, j)\)-entry of \(\text{Gcd}_{P_n}\) is not \(g_{i,j}\). In fact, the entry \(g_{i,j}\) is in row \(n - (j - 1)\) and column \(i\) of \(\text{Gcd}_{P_n}\).

Comparing the locations of the entries \(g_{i,j}\) and \(p_{i,j}\) of the matrices \(\text{Gcd}_{P_n}\) and \(P_n\), respectively, as the values \(i\) and \(j\) vary, we observe that the subscripts of the entries in one matrix are a rotation of the subscripts of the entries in the other. In particular, the following proposition describes the relationship between the matrices \(\text{Gcd}_{P_n}\) and \(P_n\) via a third matrix which we call \(\text{Gcd}_{P_n}\).

\textbf{Proposition 3.6.} Let \(P_n\) be a prime matrix. A rotation by 90° counterclockwise of the entries in \(P_n\) gives a corresponding matrix, which we denote by \(\text{Gcd}_{P_n}\), and the \((i, j)\)-entry in \(\text{Gcd}_{P_n}\) divides the \((i, j)\)-entry in the \textit{gcd}-matrix \(\text{Gcd}_{P_n}\).
Proof. The rotational relationship between \( P_n \) and \( \widetilde{\text{gcd}}_{P_n} \) is given by a simple matrix calculation. If we let \( \text{AD}_n \) be the antidiagonal matrix — that is, a matrix with ones in the antidiagonal and zeroes elsewhere, then \( \widetilde{\text{gcd}}_{P_n} = (P_n \cdot \text{AD}_n)^T \), where \( T \) denotes the transpose of a matrix. In particular, multiplying \( P_n \) on the right by \( \text{AD}_n \) reverses the columns of \( P_n \), and then transposing this result yields \( \widetilde{\text{gcd}}_{P_n} \) as desired. After this rotation on \( P_n \) is performed, the entry \( p_{i,j} \) of \( \widetilde{\text{gcd}}_{P_n} \) is then located in row \( n - (j - 1) \) and column \( i \). In this same location in \( \text{gcd}_{P_n} \) is \( g_{i,j} \). Below we give an illustration of this process:

\[
P_n = \begin{pmatrix}
p_{1,1} & p_{1,2} & \cdots & p_{1,j} & \cdots & p_{1,n} \\
p_{2,1} & p_{2,2} & \cdots & p_{2,j} & \cdots & p_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{i,1} & p_{i,2} & \cdots & p_{i,j} & \cdots & p_{i,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{n,1} & p_{n,2} & \cdots & p_{n,j} & \cdots & p_{n,n}
\end{pmatrix}
\]

\( \xrightarrow{\circlearrowleft 90^\circ \text{left}} \)

\[
\widetilde{\text{gcd}}_{P_n} = \begin{pmatrix}
p_{1,n} & p_{2,n} & \cdots & p_{i,n} & \cdots & p_{n,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{1,j} & p_{2,j} & \cdots & p_{i,j} & \cdots & p_{n,j} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{1,2} & p_{2,2} & \cdots & p_{i,2} & \cdots & p_{n,2} \\
p_{1,1} & p_{2,1} & \cdots & p_{i,1} & \cdots & p_{n,1}
\end{pmatrix}.
\]

In the proof of Theorem 3.4, we observed that by construction the prime \( p_{i,j} \) divides the value \( \gcd(x_0 + i, y_0 + j) = g_{i,j} \). Hence, the \( (i, j) \)-entry in \( \widetilde{\text{gcd}}_{P_n} \) divides the \( (i, j) \)-entry in \( \text{gcd}_{P_n} \).

\( \square \)

Remark 3.7. The rotational relationship between \( P_n \) and \( \widetilde{\text{gcd}}_{P_n} \) proves to be very important in Section 4 when we perform the reverse rotation. Starting from a gcd-matrix, a clockwise rotation will help us produce a quasiprime matrix, crucial for finding closer hidden forests.

3C. An application: the \( n = 2, 3, 4 \) cases.

Example 3.8. In the \( 2 \times 2 \) case, using Theorem 3.4, we set \( n = 2 \) and the prime matrix is

\[
P_2 = \begin{pmatrix}2 & 3 \\ 5 & 7\end{pmatrix}.
\]

The row products are \( R_1 = 6 \) and \( R_2 = 35 \), while the column products are \( C_1 = 10 \) and \( C_2 = 21 \). Hence the corresponding linear congruences we need to solve are

\[
x + 1 \equiv 0 \pmod{6}, \quad y + 1 \equiv 0 \pmod{10},
\]
\[
x + 2 \equiv 0 \pmod{35}, \quad y + 2 \equiv 0 \pmod{21}.
\]

By the CRT-algorithm, the left and right systems have the unique solutions \( x_0 = 173 \pmod{210} \) and \( y_0 = 19 \pmod{210} \), respectively. Set \( X = \{174, 175\} \) and \( Y = \{20, 21\} \). Then \( X \cap Y = \emptyset \) and \( \gcd(x_i, y_j) > 1 \) for all \( 1 \leq i, j \leq 2 \). Thus
there is a hidden forest \( H^2_{(174,20)} \) of four trees at \((174, 20)\), \((174, 21)\), \((175, 20)\), and \((175, 21)\). Below we give the hidden forest on the left and its corresponding gcd-grid on the right:

\[
\begin{array}{c c c c}
(174,21) & (175,21) & \text{gcd}(174, 21) = 3 & 7 = \text{gcd}(175, 21) \\
(174,20) & (175,20) & \text{gcd}(174, 20) = 2 & 5 = \text{gcd}(175, 20)
\end{array}
\]

Then by Proposition 3.6, we have the following map from \( P_2 \) to \( \widetilde{\text{gcd}}_{P_2} \):

\[
P_2 = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \xrightarrow{90^\circ \text{ left}} \widetilde{\text{gcd}}_{P_2} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}.
\]

In this example, the \( \widetilde{\text{gcd}}_{P_2} \) coincides with the gcd-matrix \( \text{gcd}_{P_2} \), and so the \((i, j)\)-entry of \( \widetilde{\text{gcd}}_{P_2} \) divides the \((i, j)\)-entry of \( \text{gcd}_{P_2} \) as Proposition 3.6 guarantees. This is an effect of the so-called law of small numbers, since we see in the cases of larger \( n \) to follow that this coincidence does not occur.

**Example 3.9.** In the \( 3 \times 3 \) case, using Theorem 3.4, we set \( n = 3 \) and the prime matrix is

\[
P_3 = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix}.
\]

The CRT-algorithm gives the solutions \( x_0 = 119,740,619 \) and \( y_0 = 121,379,047 \), which are both unique modulo 223,092,870. Hence the nine coordinates \((x_i, y_j)\) of \( H^3_{(119740620,121379048)} \) have the following values and respective prime factorizations:

\[
x_1 = 119,740,620 = 2^2 \cdot 3 \cdot 5 \cdot 1,995,677, \quad y_1 = 121,379,048 = 2^3 \cdot 7 \cdot 17 \cdot 59 \cdot 2161, \\
x_2 = 119,740,621 = 7 \cdot 11 \cdot 13 \cdot 37 \cdot 53 \cdot 61, \quad y_2 = 121,379,049 = 3^2 \cdot 11 \cdot 19 \cdot 173 \cdot 373, \\
x_3 = 119,740,622 = 2 \cdot 17 \cdot 19 \cdot 23 \cdot 8059, \quad y_3 = 121,379,050 = 2 \cdot 5^2 \cdot 13 \cdot 23^2 \cdot 353.
\]

It is readily verified that the corresponding \( 3 \times 3 \) hidden forest has the following gcd-grid:

\[
\begin{array}{c c c c}
2 \cdot 5 & 13 & 2 \cdot 23 \\
3 & 11 & 19 \\
2^2 & 7 & 2 \cdot 17
\end{array}
\]

For example, the top-right node corresponds to the \((x_3, y_3)\)-coordinate, and it is labeled by the value \( \text{gcd}(x_3, y_3) = 2 \cdot 23 \) since \( x_3 = 2 \cdot 17 \cdot 19 \cdot 23 \cdot 8059 \) and \( y_3 = 2 \cdot 5^2 \cdot 13 \cdot 23^2 \cdot 353 \). And by Proposition 3.6, we have the following map from
\( P_3 \) to \( \widetilde{\text{Gcd}}_{P_3} \):

\[
P_3 = \begin{pmatrix}
2 & 3 & 5 \\
7 & 11 & 13 \\
17 & 19 & 23
\end{pmatrix}
\]

\[
\begin{array}{c}
\text{left} \\
\begin{pmatrix}
2 & 3 & 5 \\
7 & 11 & 13 \\
17 & 19 & 23
\end{pmatrix}
\end{array} \rightarrow
\]

\[
\begin{pmatrix}
5 & 13 & 23 \\
3 & 11 & 19 \\
2 & 7 & 17
\end{pmatrix}
\]

Observe that, as expected, the \((i, j)\)-entry of \( \widetilde{\text{Gcd}}_P \) divides the \((i, j)\)-entry of the gcd-matrix

\[
\text{Gcd}_{P_3} = \begin{pmatrix}
2 \cdot 5 & 13 & 2 \cdot 23 \\
3 & 11 & 19 \\
2^2 & 7 & 2 \cdot 17
\end{pmatrix}
\]

**Example 3.10.** In the 4×4 case, the CRT-algorithm on the prime matrix gives the solution

\[
x_0 = 2,847,617,195,518,191,809,
\]

\[
y_0 = 1,160,906,121,308,397,397.
\]

The absurdly large solution values in Examples 3.9 and 3.10 reveal that the CRT-algorithm applied to prime matrices is hardly useful for finding \( n \times n \) hidden forests which are close to the origin for cases even as small as \( n = 3 \) and \( n = 4 \). For instance, we prove later that the closest \( H^3_{(x,y)} \) is at \( x = 1274 \) and \( y = 1308 \). Furthermore, we reveal that there is an \( H^4_{(x,y)} \) at \( x = 134,043 \) and \( y = 184,785,885 \). The \( x \)-value of the \( H^4_{(x,y)} \) which the CRT-algorithm on a prime matrix yields is \( 2.12441 \times 10^{13} \) times larger than this \( x \)-value, 134,043, of the \( H^4_{(x,y)} \) which we found.

**Remark 3.11.** It turns out that the number 134,043 is a very interesting integer; it is the smallest positive integer \( n \) such that the numbers in the set \( \{n, n+1, n+2, n+3\} \) have exactly four prime factors each.\(^2\) We later use this value to calculate the closest known \( H^4_{(x,y)} \), bearing the smallest \( x \)-value, in Example 4.8.

### 4. New methods to find closer hidden forests

The previous section detailed the well-known method of using the CRT-algorithm on prime matrices to find arbitrarily large hidden forests. The main problem with that method is that for \( n \geq 3 \) the locations of these \( H^n_{(x,y)} \) are substantially further away from the origin and thus progressively harder to compute. The aim of this section is to introduce two concepts, namely quasiprime matrices and strings of strongly composite integers, to help find substantially closer \( H^n_{(x,y)} \). In this section we give the closest \( H^n_{(x,y)} \) for \( n = 2, 3 \) and the closest known (to date) \( H^n_{(x,y)} \) for \( n = 4 \).

\(^2\)This is known as Problem 47 on the website [https://projecteuler.net/about](https://projecteuler.net/about) started in 2001 by Colin Hughes [Somers 2011]. Project Euler gives a series of challenging computational problems that require more than just mathematical insights to solve. Nayuki Minase’s very simple solution via Mathematica to Problem 47 is given in Section 4B.
Definition 4.1 (the closest $n \times n$). A hidden forest $H_{(x,y)}^n$ is said to have distance $d$ from the origin where $d$ is given by

$$d(x, y) = \sqrt{x^2 + y^2}.$$ 

We say that $H_{(x,y)}^n$ is the closest $n \times n$ hidden forest if it has the minimum distance $d$ of all hidden $n \times n$ forests.

Note that Remark 3.2 guarantees that the closest corner point $(x, y)$ of $H_{(x,y)}^n$ is well-defined up to quadrant selection.

Convention 4.2. In searching for the closest hidden forest it suffices to search only half of quadrant I. As seen in Figure 3, any $H_{(x,y)}^n$ has seven copies that are reflectional symmetries, which are the same distance from the origin. We choose to focus only on $H_{(a,b)}^n$ in quadrant I such that $(a, b)$ lies above the diagonal $y = x$ (that is, $a < b$). Note that $H_{(a,a)}^n$ can never exist if $n > 1$ since gcd$(a, a + 1) = 1$ and hence $(a, a + 1)$ is a visible point.

4A. Quasiprime matrices and the QP-algorithm. In Proposition 3.6, we begin with a prime matrix $P_n$ and observe that a $90^\circ$ counterclockwise rotation of $P_n$ yields $\widehat{\text{gcd}}_P$, which relates very closely to the gcd-matrix $\text{gcd}_P$ of the corresponding $H_{(x,y)}^n$ (in particular, recall that the $(i, j)$-entry of $\text{gcd}_P$ divides the $(i, j)$-entry of $\text{gcd}_P$). Now suppose instead that we start with an $H_{(x,y)}^n$ and its associated gcd-matrix, which we denote by $\text{gcd}_M$. If we rotate this matrix $90^\circ$ clockwise, then we get some matrix $M$ that is not necessarily a prime matrix. Furthermore, applying
the CRT-algorithm on $M$ may not even be possible (see Example 4.5). But from $M$, can we find a matrix $\tilde{M}$ such that the $(i, j)$-entry of $\tilde{M}$ divides the $(i, j)$-entry of $M$ and applying the CRT-algorithm on $\tilde{M}$ gives the original $H^n_{(x,y)}$ from which we started? Based on much computational evidence, the answer appears to be yes. The matrix $\tilde{M}$ is what we call a quasiprime matrix $QP_M$ to be defined in Definition 4.6, but a formal proof still awaits. For now, we proceed to give very substantial support that this conjecture holds for all $H^n_{(x,y)}$ (see Question 1).

To find $M$, we use the matrix equality in Proposition 3.6 and solve for $M$:

\[
Gcd_M = (M \cdot AD_n)^T \implies (Gcd_M)^T = M \cdot AD_n
\]

\[
\implies M = (Gcd_M)^T \cdot AD_n.
\]

(2)

where (2) follows since an antidiagonal matrix with all ones in its nonzero entries is its own inverse. Below we give an illustration of this process:

\[
Gcd_M = \begin{pmatrix}
g_{1,n} & g_{2,n} & \cdots & g_{i,n} & \cdots & g_{n,n} \\
\vdots & \vdots & & \vdots & & \vdots \\
g_{1,j} & g_{2,j} & \cdots & g_{i,j} & \cdots & g_{n,j} \\
\vdots & \vdots & & \vdots & & \vdots \\
g_{1,2} & g_{2,2} & \cdots & g_{i,2} & \cdots & g_{n,2} \\
g_{1,1} & g_{2,1} & \cdots & g_{i,1} & \cdots & g_{n,1}
\end{pmatrix}
\]

\[
\overset{90^\circ \text{ right}}{\rightarrow} M = \begin{pmatrix}
g_{1,1} & g_{1,2} & \cdots & g_{1,j} & \cdots & g_{1,n} \\
g_{2,1} & g_{2,2} & \cdots & g_{2,j} & \cdots & g_{2,n} \\
\vdots & \vdots & & \vdots & & \vdots \\
g_{i,1} & g_{i,2} & \cdots & g_{i,j} & \cdots & g_{i,n} \\
\vdots & \vdots & & \vdots & & \vdots \\
g_{n,1} & g_{n,2} & \cdots & g_{n,j} & \cdots & g_{n,n}
\end{pmatrix}.
\]

In the case of $n = 2$, we see that $M$ is a prime matrix (see Example 4.3). However in the case of $n = 3$, the matrix $M$ can have repeated prime number entries and hence is not a prime matrix (see Example 4.5). And in the case of $n \geq 4$, the matrix can have both repeated primes and composite number entries, and hence is not a prime matrix (see Example 4.8). In these $n \geq 3$ cases, we construct a quasiprime version of $M$ which we denote by $QP_M$. And an application of the CRT-algorithm on $QP_M$ yields the $H^n_{(x,y)}$ that has the original gcd-matrix corresponding to $H^n_{(x,y)}$. Before we present an algorithm on how to produce $QP_M$ from $M$, we give two motivating examples in the $n = 2$ and $n = 3$ cases.

Example 4.3 (the closest 2×2). By examining a small grid of points in quadrant I of $\mathbb{Z}^2$, it is easy to notice that the closest $H^2_{(x,y)}$ occurs at $x = 14$ and $y = 20$. Below we give $H^2_{(14,20)}$ and to its right the gcd-grid corresponding to the four nodes:
By (2), we can retrieve a matrix $M$ from the gcd-grid above as follows:

$$
\text{Gcd}_M = \begin{pmatrix} 7 & 3 \\ 2 & 5 \end{pmatrix} \xrightarrow{90^\circ \text{ right}} M = \begin{pmatrix} 2 & 7 \\ 5 & 3 \end{pmatrix}.
$$

Applying the CRT-algorithm to this matrix $M$, we get $x_0 = 13$ and $y_0 = 19$ as desired. Hence at the distance of $d \approx 24.4131$ we have the closest hidden forest $H_{(14,20)}^2$.

**Remark 4.4.** Alternate permutations of the same primes in the gcd-grid, and consequently the matrix $M$, may produce different solutions under the CRT-algorithm. This means that for each unique set of $n^2$ primes in the matrix $P_n$, the CRT-algorithm can yield up to $n^2$ factorial (not necessarily distinct) $H_n^{n^2}(x,y)$. Compare the previous example with Example 3.8. Both examples use the same set of primes $\{2, 3, 5, 7\}$ but yield drastically different $H_n^{n^2}(x,y)$.

**Example 4.5** (the closest $3 \times 3$). At the distance of $d \approx 1825.91$ we find the closest hidden forest $H_3^{1274,1308}$. Though others have cited $H_3^{1274,1308}$ as a hidden forest [Herzog and Stewart 1971; Weisstein], none of these sources have asserted that it is the closest. For the $n = 3$ case, the problem of finding the closest hidden forest is computationally tractable via exhaustive means. In fact, we wrote Java code,\(^3\) which exhaustively checked the square region with lower-left endpoint $(0, 0)$ and upper-right endpoint $(1308, 1308)$, finally confirming that this is the closest $3 \times 3$ hidden forest. Below we give the hidden forest $H_3^{1274,1308}$ and its corresponding gcd-grid:

\[
(1274, 1310) \quad (1275, 1310) \quad (1276, 1310) \\
(1274, 1309) \quad (1275, 1309) \quad (1276, 1309) \\
(1274, 1308) \quad (1275, 1308) \quad (1276, 1308)
\]

\[
\begin{array}{ccc}
2 & 5 & 2 \\
7 & 17 & 11 \\
2 & 3 & 2^2
\end{array}
\]

Incidentally, the $M$ given here is very similar to the one given in [Herzog and Stewart 1971], but neither of these matrices can possibly produce the correct $H_3^{1274,1308}$ because the Chinese remainder theorem simply cannot work on such matrices. For example, since the products of row 1 and row 3 of $M$ each have a factor of 4, any solution $x_0$ to the three row equations would also have to satisfy $x_0 + 1 \equiv 0 \pmod{4}$.

\(^3\)See the online supplement for the Java code.
and \( x + 3 \equiv 0 \pmod{4} \), but the existence of such an \( x_0 \) is absurd. However, this problem is resolved by introducing the concept of a quasiprime matrix.

**Definition 4.6.** Given a matrix \( M \) arising from a \( \text{Gcd}_M \) via (2), we produce a *quasiprime* matrix \( QP_M \) defined by the QP-algorithm given below.

**Algorithm (QP-algorithm to construct a quasiprime matrix \( QP_M \)).**

1. Construct matrix \( M \) arising from a \( \text{Gcd}_M \) via (2).

2. Let \( \{ p_i \}_{i=1}^s \) be the union of the sets of all primes appearing in the prime factorizations of each entry of \( M \).

3. For a fixed \( p_i \) with \( 1 \leq i \leq s \), locate the entry in \( M \) which contains \( p_i^k \) for \( k \geq 1 \) such that \( k \) is largest. If there is more than one entry which contains \( p_i^k \), then choose exactly one.

4. Place the selected \( p_i^k \) in \( QP_M \) in the same location where it appears in \( M \). Place the value 1 in \( QP_M \) in every location where a \( p_j^i \) appears in \( M \) for each \( j \leq k \).

5. Repeat the previous two steps for each \( p_i \) with \( 1 \leq i \leq s \).

**Example 4.7** *(the closest \( 3 \times 3 \) via a quasiprime matrix).* From the matrix \( M \) in **Example 4.5**, we can produce the quasiprime matrix as follows using the QP-algorithm:

\[
M = \begin{pmatrix} 2 & 7 & 2 \\ 3 & 17 & 5 \\ 2^2 & 11 & 2 \end{pmatrix} \quad \xrightarrow{\text{QP-algorithm}} \quad QP_M = \begin{pmatrix} 1 & 7 & 1 \\ 3 & 17 & 5 \\ 2^2 & 11 & 1 \end{pmatrix}.
\]

By use of the CRT-algorithm on \( QP_M \), we solve the system of linear congruences

\[
\begin{align*}
    x + 1 & \equiv 0 \pmod{7}, & y + 1 & \equiv 0 \pmod{2^2 \cdot 3}, \\
    x + 2 & \equiv 0 \pmod{3 \cdot 5 \cdot 17}, & y + 2 & \equiv 0 \pmod{7 \cdot 11 \cdot 17}, \\
    x + 3 & \equiv 0 \pmod{2^2 \cdot 11}, & y + 3 & \equiv 0 \pmod{5},
\end{align*}
\]

which has solutions \( x_0 = 1273 \) and \( y_0 = 1307 \). Hence the \( QP_M \) yields the closest \( 3 \times 3 \) hidden forest \( H^3_{(1274,1308)} \).

**4B. Computer-heavy approach: strings of strongly composite integers.** Another technique we implement that proves very powerful in finding hidden forests involves using strings of consecutive integers each with several prime factors. In **Section 5**, we find that combining the technique below with a clever computational use of quasiprime matrices yields the closest known \( n \times n \) hidden forests for \( n \geq 4 \).

Schumer [1990] proved that there exist strings of \( n \) consecutive integers each divisible by at least \( k \) distinct primes, which he calls strings of strongly composite integers. The proof uses the Chinese remainder theorem, and hence, like **Theorem 3.4**, it produces very large numbers for \( n \geq 3 \). However, there is an efficient way to find
the smallest set of $n$ consecutive integers each with at least $k$ prime factors each (ignoring multiplicity). We can then use these values as our $x$-values in our hunt for closer hidden forests for $n \geq 4$.

The following Mathematica code, which is Nayuki Minase’s solution to Project Euler Problem 47, easily produces the very first number $n$ in a sequence of four consecutive integers each with four prime factors (ignoring multiplicity):

```mathematica
Has4PrimeFactors[n_] := Length[FactorInteger[n]] == 4
i = 2;
While[! (Has4PrimeFactors[i] && Has4PrimeFactors[i + 1] && Has4PrimeFactors[i + 2] && Has4PrimeFactors[i + 3]), i++]
i
```

The value that it yields is 134,043. This number and the next three consecutive integers have the prime factorizations

\[
134,043 = 3 \cdot 7 \cdot 13 \cdot 491, \\
134,044 = 2^2 \cdot 23 \cdot 31 \cdot 47, \\
134,045 = 5 \cdot 17 \cdot 19 \cdot 83, \\
134,046 = 2 \cdot 3^2 \cdot 11 \cdot 677.
\]

**Example 4.8** (the closest $4 \times 4$ to date with the smallest $x$-value). Using the four values 134,043 to 134,046 as the $x$-values of $4 \times 4$ hidden forest we seek, we exhaustively searched for the very first set of four consecutive integers such that all four values share at least one prime factor with each of the four values 134,043 to 134,046. After running for only two minutes,\(^4\) the Java program which we wrote outputs the value 184,785,885. This number and the next three consecutive integers have the prime factorizations

\[
184,785,885 = 3^2 \cdot 5 \cdot 31^2 \cdot 4273, \\
184,785,886 = 2 \cdot 17 \cdot 491 \cdot 11,069, \\
184,785,887 = 11 \cdot 13 \cdot 19 \cdot 23 \cdot 2957, \\
184,785,888 = 2^5 \cdot 3 \cdot 7 \cdot 83 \cdot 3313.
\]

Using these four numbers as the $y$-values of our $H_4(x,y)$ and the four values 134,043 to 134,046 as the $x$-values, we get a hidden forest $H_4^{(134043,184785885)}$ with the following gcd-grid:

\[\]
By (2), we retrieve a matrix $M$ from the gcd-grid above as follows:

$$\text{Gcd}_M = \begin{pmatrix} 3 \cdot 7 & 2^2 & 83 & 2 \cdot 3 \\ 13 & 23 & 19 & 11 \\ 491 & 2 & 17 & 2 \\ 3 & 31 & 5 & 3^2 \end{pmatrix} \xrightarrow{90^\circ \text{ right}} M = \begin{pmatrix} 3 & 491 & 13 & 3 \cdot 7 \\ 31 & 2 & 23 & 2^2 \\ 5 & 17 & 19 & 83 \\ 3^2 & 2 & 11 & 2 \cdot 3 \end{pmatrix}.$$

Applying the QP-algorithm to $M$, we get the quasiprime matrix

$$QP_M = \begin{pmatrix} 1 & 491 & 13 & 7 \\ 31 & 1 & 23 & 2^2 \\ 5 & 17 & 19 & 83 \\ 3^2 & 1 & 11 & 1 \end{pmatrix}.$$

Then applying the CRT-algorithm to $QP_M$ indeed yields the forest $H^4_{(134043,184785885)}$ as desired.

**Remark 4.9.** The hidden forest $H^4_{(134043,184785885)}$ is distance $d \approx 1.84786 \times 10^8$ from the origin. In comparison recall Example 3.10, where we used the only known method to date in the literature (that is, the prime matrix $P_4$ and Theorem 3.4). Using that traditional method we found the hidden forest $H^4_{(x_1, y_1)}$ with

$$x_1 = 2,847,617,195,518,191,810,$$
$$y_1 = 1,160,906,121,308,397,398$$

at a distance $d \approx 3.07516 \times 10^{18}$, which is $1.66418 \times 10^{10}$ times farther than the hidden forest which we found in Example 4.8! The matrix $P_4$ and its associated gcd-grid of our closer hidden forest $H^4_{(134043,184785885)}$ are as follows:

$$P_4 = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 11 & 13 & 17 & 19 \\ 23 & 29 & 31 & 37 \\ 41 & 43 & 47 & 53 \end{pmatrix} \xrightarrow{\text{Theorem 3.4}} \xrightarrow{\text{CRT-algorithm}} \begin{pmatrix} 2 \cdot 5 & 17 & 2^2 \cdot 31 & 47 \\ 3 & 13 & 29 & 3 \cdot 43 \\ 7 & 19 & 37 & 53 \\ 2 & 11 & 2 \cdot 23 & 41 \end{pmatrix}$$
Pighizzini and Shallit 2002 2.30574 \times 10^8 no
Baake and Grimm 2013 1.90265 \times 10^7 no
Goodrich, Mbirika, and Nielsen 2014 1.84786 \times 10^8 yes

Table 1. Distances of the three closest known 4\times4 hidden forests in the literature. (We discovered this forest in 2014; however, it is in this paper in which we give its existence and proof.)

Hence the gcd-matrix of this 4\times4 hidden forest is

\[
\text{Gcd}_M = \begin{pmatrix}
7 & 19 & 37 & 53 \\
2 \cdot 5 & 17 & 2^2 \cdot 31 & 47 \\
3 & 13 & 29 & 3 \cdot 43 \\
2 & 11 & 2 \cdot 23 & 41
\end{pmatrix}
\]

Remark 4.10. Other researchers have found 4\times4 hidden forests of distances relatively close to the one shown in Remark 4.9. Pighizzini and Shallit [2002] addressed the issue of finding the closest \(n\times n\) hidden forests. For a positive integer \(n\), they define a function \(S(n)\), which is the least positive integer \(r\) such that there exists \(m \in \{0, 1, \ldots, r\}\) with gcd\((r - i, m - j) > 1\) for \(0 \leq i, j < n\). This is equivalent to finding the closest \(n \times n\) hidden forest. They were only successful in finding this value for \(n = 1, 2, 3\), but for \(n = 4\) they were able to give bounds \(450000 < S(4) \leq 172379781\) by finding a hidden forest \(H^4_{(x,y)}\) with \(x = 172,379,778\) and \(y = 153,132,342\). An even closer 4\times4 hidden forest was later revealed in [Baake and Grimm 2013, p. 422]. The forest they find has bottom left corner \(x = 13,458,288\) and \(y = 13,449,225\); however, no proof or justification of how this was found is given. Moreover, they give no assertion regarding whether this is the closest known 4\times4 hidden forest. Table 1 gives the distances of the three closest known 4\times4 hidden forests in the literature.

4C. Computer-free approach: minimum prime factors in an optimal gcd-matrix. The concept of an “optimal” gcd-matrix for a hidden \(n \times n\) forest \(H^n_{(x,y)}\) depends on \(n\) and is based on the minimal number of prime factors required in the gcd-grid of \(H^n_{(x,y)}\). We find that minimizing the number of primes used in the gcd-matrix while simultaneously maximizing the number of locations in the gcd-grid where a prime can be used again leads to a closer \(H^n_{(x,y)}\) than the traditional method given in Section 3.

Observe that the gcd-matrix of the \(H^4_{x,y}\) in Remark 4.9 is hardly optimal in the sense that if the corner entries were all multiples of 3, then we immediately get the four corners “hidden for free”, as in the forest in Example 4.8 — that is,
NEW METHODS TO FIND PATCHES OF INVISIBLE INTEGER LATTICE POINTS

the values $x_1, x_4, y_1$ and $y_4$ would all be divisible by 3 and hence none of the
four points $(x_1, y_1), (x_4, y_1), (x_1, y_4), \text{ or } (x_4, y_4)$ would be visible. An optimal
situation is to have one corner, for example, the bottom-left coordinate $(x_1, y_1)$, be
divisible by both 2 and 3. Then we would have a forest where the gcd of each of
the 16 coordinates $(x_i, y_j)$ with $1 \leq i, j \leq 4$ are divisible by 2, 3, and the nine other
primes $p_1, \ldots, p_9$ as in Figure 4.

This leads one to consider a different type of gcd-matrix that does not give the
exact gcd $g_{i,j}$ (recall Figure 2, right) for each coordinate $(x_i, x_j)$ (recall Figure 2,
left) of $H_n^{(x,y)}$. But on the other hand, this new matrix would simply give the
smallest prime divisor of the gcd for each coordinate. We make this more precise in
Definition 4.12. But first we need to recall the following number-theoretic function.

Definition 4.11. The prime counting function $\pi : \mathbb{R} \rightarrow \mathbb{N}$ counts the number of
primes less than or equal to a given real number.

Definition 4.12. Construct an optimal gcd-matrix as follows. Let one of the four
corner entries of the $n \times n$ matrix contain the product of the first $k_n := \pi(n-1)$ primes
(where $\pi$ is the prime counting function) if $n \geq 3$, and set $k_2 = 1$. Without loss of
generality, choose the bottom-left corner for this value. Denote these first $k_n$ primes
as $q_1, q_2, \ldots, q_{k_n}$. For each $q_i$ with $1 \leq i \leq k_n$, any entry in the matrix that is a mul-
tiple of $q_i$ rows to the right of the bottom-left corner and/or a multiple of $q_i$ columns
above the bottom-left corner must be filled with the value $q_i$. If more than one prime
fits this criteria for a specific matrix entry, then simply multiply the primes in that
every entry together. In the remaining unfilled entries, place one prime in each entry from
the set of the next smallest primes larger than the prime $q_{k_n}$. Denote this set of primes
by $\{p_1, p_2, \ldots\}$. We denote this optimal gcd-matrix by the symbol opt-Gcd$_M$.

Example 4.13. An optimal $4 \times 4$ gcd-matrix is

$$ \text{opt-Gcd}_M = \begin{pmatrix}
3 & p_9 & p_8 & 3 \\
2 & p_5 & 2 & p_7 \\
p_3 & p_2 & p_4 & p_6 \\
2 \times 3 & p_1 & 2 & 3
\end{pmatrix}, $$

where the entries $p_1, \ldots, p_9$ are the nine smallest prime numbers other than 2 or 3.
Observe that the locations of the primes 2 and 3 correspond exactly to their location
Figure 5. An optimal gcd-grid with bottom left corner having the product of $k_n$ primes.

in the gcd-grid in Figure 4. The manner in which the $p_1$ through $p_9$ are distributed in this particular matrix is the $n = 4$ case that arises from the grid in Figure 5.

The grid in Figure 5 shows us the minimum number of primes and their relative locations in a candidate for an optimal gcd-matrix for an $n \times n$ hidden forest. In this grid, we choose the bottom-left corner (denoted with the symbol •) to contain the product of powers of the first $k_n$ primes where $k_n$ is the value given in Definition 4.12. In the far-left column, each entry refers to the size $n$ of the corresponding $n \times n$ grid. In the bottom row, in each box we give the number of additional primes that are needed to go from an $n \times n$ grid to an $(n+1) \times (n+1)$ grid. For example, for $n = 5$, we need a minimum $k_5 + 3 + 2 + 4 + 4 = 15$ distinct primes in the optimal gcd-matrix for an $H^5_{(x,y)}$. Indeed in Section 5, we see that this minimum is achieved.

Example 4.14. If $n = 4$ then $k_4 = 2$, and hence we place $2 \times 3$ in a corner location. This is because the $4 \times 4$ portion of the grid in Figure 5 says that we need a minimum of nine primes, not counting the primes 2 and 3, which are placed in the locations where they appear in the grid. Hence an optimal gcd-matrix might be

$$opt-Gcd_M = \begin{pmatrix}
3 & 29 & 31 & 3 \\
2 & 19 & 2 & 23 \\
7 & 11 & 13 & 17 \\
2 \times 3 & 5 & 2 & 3
\end{pmatrix}.$$ 

In the boxed entries in the matrix above, we place the nine smallest primes larger than 3 where the grid in Figure 5 places $p_1, \ldots, p_9$.

Observe that the forest $H^4_{(134043,184785885)}$ found in Example 4.8 is the closest known $4 \times 4$ forest and is attained by cleverly using the method of strings of strongly
composite integers, that is, using computer computation to find the smallest four consecutive values \(x_1, \ldots, x_4\) which each have at least four primes factors each, and then using computer computation again to compute the next set of four values \(y_1, \ldots, y_4\), each of which is not relatively prime to all four \(x\)-values. However, the \(QP_M\) associated to this closest forest is not optimal in the sense that it uses ten primes (not including 2 and 3), whereas an optimal \(QP_M\) uses at most nine primes (not including 2 and 3).

Note that much computer assistance was required to generate \(H_4^{(134043,184785885)}\); however, no computer assistance whatsoever is required to create the optimal gcd-matrix, \(\text{opt-Gcd}_M\). From the matrix \(\text{opt-Gcd}_M\) in this example, we produce the quasiprime matrix as follows using the QP-algorithm:

\[
\text{opt-Gcd}_M = \begin{pmatrix}
  3 & 29 & 31 & 3 \\
  2 & 19 & 2 & 23 \\
  7 & 11 & 13 & 17 \\
  2 \times 3 & 5 & 2 & 3
\end{pmatrix}
\xrightarrow{\text{QP algorithm}}
\begin{pmatrix}
  1 & 29 & 31 & 1 \\
  1 & 19 & 1 & 23 \\
  7 & 11 & 13 & 17 \\
  6 & 5 & 1 & 1
\end{pmatrix}.
\]

Applying the CRT-algorithm on \(QP_M\), we then get the forest \(H_4^{(x,y)}\) with

\[
x = 153,630,616,137, \quad y = 116,380,988,514
\]

and the following prime factorizations of the 16 coordinates \((x_i, y_j)\) for all \(1 \leq i, j \leq 4\):

\[
x_1 = 153,630,616,137 = 3 \cdot 29 \cdot 31 \cdot 229 \cdot 248,749,
y_1 = 116,380,988,514 = 2 \cdot 3^3 \cdot 7 \cdot 37 \cdot 8,321,249,
x_2 = 153,630,616,138 = 2 \cdot 19 \cdot 23 \cdot 1723 \cdot 102,019,
y_2 = 116,380,988,515 = 5 \cdot 11 \cdot 19 \cdot 29 \cdot 47 \cdot 101 \cdot 809,
x_3 = 153,630,616,139 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 9,028,067,
y_3 = 116,380,988,516 = 2^2 \cdot 13 \cdot 31 \cdot 72,196,643,
x_4 = 153,630,616,140 = 2^2 \cdot 3^3 \cdot 5 \cdot 103 \cdot 2,762,147,
y_4 = 116,380,988,517 = 3 \cdot 17 \cdot 23 \cdot 883 \cdot 112,363.
\]

In Table 2 we summarize the distances of the 4×4 hidden forests found by the traditional method versus the two new methods given in this paper.

So we can easily see that the two new methods produce substantially closer hidden forests than the traditional methods. However, we find that merging the method of strings of composite integers with the method of the optimal matrix is an even better idea. And that is precisely what we do in the 5×5 case in the following section.
### Table 2. Distances of the 4×4 hidden forests found by the traditional method versus the two new methods given in this paper.

| method                                      | distance       | location     |
|---------------------------------------------|----------------|--------------|
| traditional approach CRT-algorithm          | 3.07516×10^{18} | Example 3.10 |
| computer-heavy approach with QP-algorithm   | 1.84786×10^{8}  | Example 4.8  |
| computer-free approach with QP-algorithm    | 1.92735×10^{11} | Example 4.14 |

#### 5. An application: the closest known 5×5 hidden forest

We employ a combination of the techniques from both strings of strongly composite integers and an optimal quasiprime matrix to find the closest known 5×5 hidden forest to date. We first calculate a length-5 analogue of the Project Euler Problem 47 (recall the footnote given in Remark 3.11). By slightly altering Minase’s solution (see Section 4B), we find the smallest set of five consecutive integers each with at least five prime factors. Mathematica completed this computation in 36 minutes. These five integers and their prime factorizations are

\[
\begin{align*}
x_1 &= 129,963,314 = 2 \cdot 13 \cdot 37 \cdot 53 \cdot 2549, \\
x_2 &= 129,963,315 = 3 \cdot 5 \cdot 31 \cdot 269 \cdot 1039, \\
x_3 &= 129,963,316 = 2^2 \cdot 7 \cdot 97 \cdot 109 \cdot 439, \\
x_4 &= 129,963,317 = 11^2 \cdot 17 \cdot 23 \cdot 41 \cdot 67, \\
x_5 &= 129,963,318 = 2 \cdot 3 \cdot 89 \cdot 199 \cdot 1223.
\end{align*}
\]

In Example 4.8, it took the Java code only 2 minutes to find the smallest four consecutive values which are each not relatively prime to the four values 134,043 through 134,046. However in this \( n = 5 \) case, it is not as simple. After the Java code ran continuously for four days,\(^5\) it had checked up to the \( y \)-value of 500 billion and still did not find an \( H^5_{(x,y)} \) with the \( x \)-values \( x_1, \ldots, x_5 \) given earlier. So we approached this problem from a more theoretical perspective instead.

Consider the list of five consecutive integers \( x_1, \ldots, x_5 \). Observe that \( x_1, x_3, \) and \( x_5 \) are divisible by 2 and that \( x_2 \) and \( x_5 \) are divisible by 3. Hence a hidden 5×5 forest bearing these \( x \)-values would be “optimal” if the corresponding five \( y \)-values (which we denote \( y_1, \ldots, y_5 \)) have the property that \( y_1, y_3, \) and \( y_5 \) are divisible by 2 and that \( y_2 \) and \( y_5 \) are divisible by 3. The benefit of this optimal situation is that 12 of the 25 coordinates will automatically have \( \gcd(x_i, y_j) > 1 \) and hence these 12 points are hidden. In the following matrices, we represent each of these 12 points with the symbol • in the gcd-matrix \( \text{Gcd}_M \) on the left, and to its right we give the 90° clockwise rotation matrix \( M \) from which we construct a quasiprime matrix:

\(^5\)Recall the footnote regarding computation times in Example 4.8.
Since we know that \( x_5 \) and \( y_5 \) are both divisible by 2 and 3 in this optimal case, we place a 6 in this entry, and the \( QP_M \) matrix has the abstract form

\[
QP_M = \begin{pmatrix}
1 & a_1 & 1 & a_2 & 1 \\
1 & b_1 & b_2 & b_3 & 1 \\
1 & c_1 & 1 & c_2 & 1 \\
d_1 & d_2 & d_3 & d_4 & d_5 \\
1 & 1 & 1 & e_1 & 6
\end{pmatrix},
\]

where

\[
x_1 = 2 \cdot 13 \cdot 37 \cdot 53 \cdot 2549 \quad \Rightarrow \quad a_1, a_2 \in \{13, 37, 53, 2549\},
\]
\[
x_2 = 3 \cdot 5 \cdot 31 \cdot 269 \cdot 1039 \quad \Rightarrow \quad b_1, b_2, b_3 \in \{5, 31, 269, 1039\},
\]
\[
x_3 = 2^2 \cdot 7 \cdot 97 \cdot 109 \cdot 439 \quad \Rightarrow \quad c_1, c_2 \in \{7, 97, 109, 439\},
\]
\[
x_4 = 11^2 \cdot 17 \cdot 23 \cdot 41 \cdot 67 \quad \Rightarrow \quad d_1, d_2, d_3, d_4, d_5 \in \{11, 17, 23, 41, 67\},
\]
\[
x_5 = 2 \cdot 3 \cdot 89 \cdot 199 \cdot 1223 \quad \Rightarrow \quad e_1 \in \{89, 199, 1223\}.
\]

Observation 5.1. Consider the \( QP_M \) matrix in (3). Then the following hold:

(1) There are 1,244,160 distinct ways to produce a quasiprime matrix \( QP_M \).

(2) Applying the CRT-algorithm to any of the \( QP_M \) yields the same solution values \( x_1, \ldots, x_5 \) as the \( x \)-values of the hidden forest \( H_{(x_1, y_1)}^5 \). In particular, this unique \( x_1 \)-value is 129963314.

(3) The \( y \)-value solutions have the property that \( y_1, y_3, y_5 \in 2\mathbb{Z} \) and \( y_2, y_5 \in 3\mathbb{Z} \).

Proof. (1) There are \( P(4, 2) = 4!/(4 - 2)! = 12 \) possible 2-permutations of a 4-element set. So since \( a_1 \) and \( a_2 \) must be distinct elements of \( \{13, 37, 53, 2549\} \), the ordered tuple \( (a_i)_{i=1}^2 \) can be chosen in 12 ways. Applying a similar argument to count the possible \( (b_i)_{i=1}^3 \), \( (c_i)_{i=1}^2 \), \( (d_i)_{i=1}^5 \), and the \( (e_1) \), we see that the ordered tuple \( (b_i)_{i=1}^3 \) can be chosen in 12 ways, the \( (c_i)_{i=1}^2 \) in 12 ways, the \( (d_i)_{i=1}^5 \) in 120 ways, and \( (e_1) \) in 3 ways. Thus there are 1,244,160 distinct ways to produce a quasiprime matrix \( QP_M \), which proves (1).
(2) Unfortunately, we only proved this by computational exhaustion using Mathematica. See part (a) of Question 1.

(3) Consider an arbitrary \(QP_M\). Suppose \(y_0\) is a solution to the five linear congruences \(y + k \equiv 0 \pmod{C_k}\), where \(C_k\) equals the product of the column entries of \(QP_M\) for \(1 \leq k \leq 5\). Setting \(y_k = y_0 + k\), we observe that \(y_5 \equiv 0 \pmod{6 \cdot d_5}\), and thus \(y_5 \equiv 0 \pmod{2}\) and \(y_5 \equiv 0 \pmod{3}\). Hence \(y_5 \in 2\mathbb{Z} \cap 3\mathbb{Z}\). Setting \(y_{5k} = y_0 + k\), we observe that \(y_5 \equiv 0 \pmod{6 \cdot d_5}\), and thus \(y_5 \equiv 0 \pmod{2}\) and \(y_5 \equiv 0 \pmod{3}\). Hence \(y_5 \in 2\mathbb{Z} \cap 3\mathbb{Z}\). Since \(y_5 \in 2\mathbb{Z}\), it follows that \(y_3 = y_5 - 2\) implies \(y_3 \in 2\mathbb{Z}\), and \(y_1 = y_5 - 4\) implies \(y_1 \in 2\mathbb{Z}\). Moreover since \(y_5 \in 3\mathbb{Z}\), it follows that \(y_2 = y_5 - 3\) implies \(y_2 \in 3\mathbb{Z}\). Hence (3) holds. □

We wrote a program in Mathematica which applies the CRT-algorithm to each of the possible \(1,244,160\) matrices. Four minutes later, the program yields that the smallest \(y\)-value solution is given by the quasiprime matrix

\[
QPM = \begin{pmatrix} 1 & 37 & 1 & 13 & 1 \\ 31 & 1 & 5 & 269 & 1 \\ 1 & 109 & 1 & 7 & 1 \\ 67 & 17 & 41 & 23 & 11 \\ 1 & 1 & 1 & 89 & 6 \end{pmatrix}.
\]

(4) This \(y\)-value and the next four consecutive integers have the following prime factorizations (with commas omitted in the factorizations for readability):

\[
\begin{align*}
y_1 &= 2,546,641,254,872,348 = 2^2 \cdot 31 \cdot 67 \cdot 461 \cdot 664921471, \\
y_2 &= 2,546,641,254,872,349 = 3^2 \cdot 17 \cdot 37 \cdot 109 \cdot 8681 \cdot 475421, \\
y_3 &= 2,546,641,254,872,350 = 2 \cdot 5^2 \cdot 41 \cdot 11113 \cdot 111784759, \\
y_4 &= 2,546,641,254,872,351 = 7^2 \cdot 13 \cdot 23 \cdot 73 \cdot 89 \cdot 269 \cdot 271 \cdot 367, \\
y_5 &= 2,546,641,254,872,352 = 2^5 \cdot 3 \cdot 11 \cdot 2411592097417.
\end{align*}
\]

Comparing the \(x_1, \ldots, x_5\) with the \(y_1, \ldots, y_5\) we see that \(\gcd(x_i, y_j) > 1\) for all indices \(1 \leq i, j \leq 5\) and in fact \(H_5^{(x_1, y_1)}\) has the following \(\text{gcd}\)-matrix \(\text{Gcd}_M\) and corresponding matrix \(M\):

\[
\text{Gcd}_M = \begin{pmatrix} 2 & 3 & 2^2 & 11 & 2 \cdot 3 \\ 13 & 269 & 7 & 23 & 89 \\ 2 & 5 & 2 & 41 & 2 \\ 37 & 3 & 109 & 17 & 3 \\ 2 & 31 & 2^2 & 67 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 37 & 2 & 13 & 2 \\ 31 & 3 & 5 & 269 & 3 \\ 2^2 & 109 & 2 & 7 & 2^2 \\ 67 & 17 & 41 & 23 & 11 \\ 2 & 3 & 2 & 89 & 2 \cdot 3 \end{pmatrix}.
\]

**Remark 5.2.** If we apply the \(QP\)-algorithm to the \(M\) above, then we are forced to place a \(2^2\) in either the \((3, 1)\)- or \((3, 5)\)-entry of \(QP_M\), and consequently the 6 in the \((5, 5)\) entry becomes a 3. Hence this new \(QP_M\) differs from the quasiprime matrix in (4). However, applying the CRT-algorithm to this new \(QP_M\) gives the same hidden forest as expected.
Remark 5.3. The forest $H_{(x,y)}^5$ with
\[ x = 129,963,314, \quad y = 2,546,641,254,872,348 \]
is at a distance $d \approx 2.54664 \times 10^{15}$ from the origin. Using the only known method until now (that is, Theorem 3.4), we get a forest $H_{(x,y)}^5$ with
\[ x = 251,080,644,933,696,940,130,615,676,720,763,950, \]
\[ y = 108,580,359,501,475,197,963,484,708,875,960,338. \]
This forest is at a distance $d \approx 2.73553 \times 10^{35}$ from the origin, and hence is $1.07417 \times 10^{20}$ times farther than the forest we reveal in this paper! We have not found a computationally tractable method to find the closest $5 \times 5$ hidden forest, nor do we believe that anyone else has. So for the time being, the $H_{(x,y)}^5$ we present in this paper is the closest $5 \times 5$ hidden forest to date.

6. Open problems

There are many avenues for further research motivated from the work in this present paper. In this section, we give some open problems identified during our research process.

**Question 1.** Is it true that for every hidden forest $H_{(x,y)}^n$, there exists a quasiprime matrix $QP_M$ in $\text{Mat}_n(\mathbb{Z})$ such that the CRT-algorithm applied to $QP_M$ yields $H_{(x,y)}^n$? Related to this question are the following subquestions:

- (a) Why do all 1,244,160 distinct quasiprime matrices in matrix (3) yield exactly the same $x$-value solution under the CRT-algorithm?
- (b) Do all distinct quasiprime matrices produce unique solutions?
- (c) Can one code a computationally efficient method to search for the closest $H_{(x,y)}^n$ for $n \geq 4$?

**Question 2.** Higher-dimensional analogues of patches of invisible points can be found. Observe that our proof of Proposition 2.4 can easily be extended to higher dimensions by setting the value $s$ (in the proof) to the appropriate dimension. That is, the probability that $(x_1, x_2, \ldots, x_s)$ is visible in $\mathbb{Z}^s$ is $1/\xi(s)$. In Example 6.1, we find a hidden $2 \times 2 \times 2$ forest using a 3-dimensional analogue of the CRT-algorithm, and we see that the forest found by this method is very far from the origin. Can we generalize the quasiprime matrix to these higher-dimensional settings and find closer hidden $n$-dimensional forests?

**Example 6.1.** In Figure 6, we give an example of a hidden $2 \times 2 \times 2$ forest with corner point $(x_1, y_1, z_1)$ at $x_1 = 9,126,194$, $y_1 = 8,286,564$, and $z_1 = 8,822,099$. 
To find this 3-dimensional hidden forest, we considered a 3-dimensional version of the prime matrix as a cube whose corners contain the first eight prime numbers. Then to each face of the cube, we multiplied the four numbers in each corner as the following image illustrates:

![Image of a 2x2x2 hidden forest]

Solving the three systems of congruences
\[
\begin{align*}
    x+1 &\equiv 0 \pmod{\text{back}}, \\
    y+1 &\equiv 0 \pmod{\text{left}}, \\
    z+1 &\equiv 0 \pmod{\text{bottom}}, \\
    x+2 &\equiv 0 \pmod{\text{front}}, \\
    y+2 &\equiv 0 \pmod{\text{right}}, \\
    z+2 &\equiv 0 \pmod{\text{top}}
\end{align*}
\]
yields the three simultaneous solutions \(x_0 = 9,126,193\), \(y_0 = 8,286,563\), and \(z_0 = 8,822,098\). Then the following values \(x_1, y_1, z_1, x_2, y_2, \) and \(z_2\) have the prime factorizations
\[
\begin{align*}
    x_1 &= 2 \cdot 7 \cdot 11 \cdot 19 \cdot 3119, \\
    y_1 &= 2^2 \cdot 3 \cdot 11^2 \cdot 13 \cdot 439, \\
    z_1 &= 11 \cdot 13 \cdot 17 \cdot 19 \cdot 191, \\
    x_2 &= 3 \cdot 5 \cdot 13 \cdot 17 \cdot 2753, \\
    y_2 &= 5 \cdot 7 \cdot 17 \cdot 19 \cdot 733, \\
    z_2 &= 2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 4201.
\end{align*}
\]
It is readily verified from these factorizations that each of the eight tuples of coordinates \((x_i, y_j, z_k)\) for \(1 \leq i, j, k \leq 2\) have the property \(\gcd(x_i, y_j, z_k) > 1\). Hence this 3-dimensional forest is indeed hidden from the origin.

**Question 3.** What can be said about hidden forests in the \(\mathbb{Z}[i] \times \mathbb{Z}[i]\) lattice? What is meant by the coordinate values \((x, y) \in \mathbb{Z}[i] \times \mathbb{Z}[i]\) being relatively prime? Recall that if \(R\) is a Euclidean domain (as is the case for the ring \(\mathbb{Z}[i]\) of Gaussian integers), then greatest common divisors can be computed using the Euclidean algorithm. Can we apply methods in this paper to the visibility of points in the lattice \(\mathbb{Z}[i] \times \mathbb{Z}[i]\)?
**Question 4.** Recently, the notion of lattice-point visibility was generalized to include all curves through the origin given by power functions of the form $f(x) = ax^b$, where $a \in \mathbb{Q}$ and $b \in \mathbb{N}$, by Goins, Harris, Kubik, and Mbirika [Goins et al. 2018]. In this generalized setting, forests invisible from the origin are called $b$-invisible forests. Can we apply the techniques detailed in Section 4 to find the closest $b$-invisible $n \times n$ forests?

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**References**

[Baake and Grimm 2013] M. Baake and U. Grimm, *Aperiodic order, vol. 1: A mathematical invitation*, Encyclopedia of Mathematics and its Applications 149, Cambridge University Press, 2013. MR Zbl

[Brass et al. 2005] P. Brass, W. Moser, and J. Pach, *Research problems in discrete geometry*, Springer, 2005. MR Zbl

[Cesàro 1881] E. Cesàro, “Question proposée 75”, *Mathesis* 1 (1881), 184.

[Cesàro 1883] E. Cesàro, “Question 75 (solution)”, *Mathesis* 3 (1883), 224–225.

[Cesàro 1884] E. Cesàro, “Probabilité de certains faits arithmétiques”, *Mathesis* 4 (1884), 50–151.

[Chapman 2003] R. Chapman, “Evaluating $\zeta(2)$”, preprint, 2003, http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf.

[Dirichlet 1849] P. G. L. Dirichlet, “Über die Bestimmung der mittleren Werte in der Zahlentheorie”, *Abhandl. Kgl. Preuss. Akad. Wiss. Berlin* 1849 (1849), 63–83.

[Goins et al. 2018] E. H. Goins, P. E. Harris, B. Kubik, and A. Mbirika, “Lattice point visibility on generalized lines of sight”, *Amer. Math. Monthly* 125:7 (2018), 593–601. MR Zbl

[Herzog and Stewart 1971] F. Herzog and B. M. Stewart, “Patterns of visible and nonvisible lattice points”, *Amer. Math. Monthly* 78 (1971), 487–496. MR Zbl

[Lehmer 1900] D. N. Lehmer, “Asymptotic Evaluation of Certain Totient Sums”, *Amer. J. Math.* 22:4 (1900), 293–335. MR Zbl

[Moreno 2015] S. G. Moreno, “A one-sentence and truly elementary proof of the basel problem”, preprint, 2015. arXiv
[Pighizzini and Shallit 2002] G. Pighizzini and J. Shallit, “Unary language operations, state complexity and Jacobsthal’s function”, Internat. J. Found. Comput. Sci. 13:1 (2002), 145–159. MR Zbl

[Schumer 1990] P. Schumer, “Strings of strongly composite integers and invisible lattice points”, College Math. J. 21:1 (1990), 37–40. MR Zbl

[Somers 2011] J. Somers, “How I failed, failed, and finally succeeded at learning how to code”, The Atlantic (June 3 2011).

[Sylvester 1883] J. Sylvester, “Sur le nombre de fractions ordinaires inégales qu’on peut exprimer en se servant de chiffres qui n’excède pas un nombre donné”, C. R. Acad. Sci. Paris 96 (1883), 409–413. Zbl

[Weisstein] E. W. Weisstein, “Visible point”, http://mathworld.wolfram.com/VisiblePoint.html. From MathWorld.

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