Infinite families of linear codes supporting more $t$-designs*

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Abstract

Tang and Ding [IEEE IT 67 (2021) 244-254] studied the class of narrow-sense BCH codes $C(q,q+1,4,1)$ and their dual codes with $q = 2^m$ and established that the codewords of the minimum (or the second minimum) weight in these codes support infinite families of 4-designs or 3-designs. Motivated by this, we further investigate the codewords of the next adjacent weight in such codes and discover more infinite classes of $t$-designs with $t = 3, 4$. In particular, we prove that the codewords of weight 7 in $C(q,q+1,4,1)$ support 4-designs when $m \geq 5$ is odd and 3-designs when $m \geq 4$ is even, which provide infinite classes of simple $t$-designs with new parameters. Another significant class of $t$-designs we produce in this paper has supplementary designs with parameters 4-(2$^{2s+1}+1,5,5$); these designs have the smallest index among all the known simple 4-(q + 1, 5, λ) designs derived from codes for prime powers $q$; and they are further proved to be isomorphic to the 4-designs admitting the projective general linear group PGL(2, 2$^{2s+1}$) as the automorphism group constructed by Alltop in 1969.

2000 MSC: 05B05, 51E10, 94B15

Keywords: BCH code, trace code, $t$-design, elementary symmetric polynomial, isomorphic.

1 Introduction

Let $P$ be a $v$-element set and let $B$ be a collection of $k$-subsets of $P$. If every $t$-subset of $P$ is contained in exactly $λ$ elements of $B$, then the incidence structure $D = (P, B)$ is called a $t$-(v, k, λ) design, or simply $t$-design. The elements of $P$ and $B$ are called points and blocks, and the parameters $t$ and $λ$ are called strength and index respectively. Let $\binom{P}{k}$ denote the set of all $k$-subsets of $P$. Obviously, a $t$-design with $t = k$ or $k = v$ trivially exists; such $t$-designs are called complete designs. A $t$-design $(P, B)$ is said to be simple if $B$ does not contain any repeated blocks. In this paper, we will mainly consider simple $t$-designs and hence the expression “$t$-design” will mean “simple $t$-design”, unless otherwise stated. A Steiner system, denoted by $S(t, k, v)$, is a $t$-(v, k, λ) design where $t \geq 2$ and $λ = 1$. It is clear that a $t$-(v, k, λ) design also forms an $s$-(v, k, λ$_s$) design with

$$λ_s = λ\left(\frac{v-s}{t-s}\right)\left(\frac{k-s}{t-s}\right)$$

(1)

*The research was supported by NSFC under Grant No. 11971053.
for any $s$ with $1 \leq s \leq t$.

Let $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design. By [18], $\text{supp}(\mathbb{D}) := (\mathcal{P}, \{P \setminus B : B \in \mathcal{B}\})$ is defined to be the supplementary design of $\mathbb{D}$, which is a $t$-$(v, v-k, \lambda')$ design with

$$\lambda' = \lambda \binom{v-t}{k}/\binom{v-t}{k-t};$$

and $\text{comp}(\mathbb{D}) := (\mathcal{P}, (P \setminus k \setminus B)$ is defined to be the complementary design of $\mathbb{D}$, which is a $t$-$(v, k, \lambda'')$ design with

$$\lambda'' = \binom{v-t}{k-t} - \lambda.$$

In what follows, the block set $(P \setminus k \setminus B$ is also denoted by $\mathcal{B}$ for short.

Let $\mathcal{C}$ be an $[n, k, d]$ linear code over the finite field $\text{GF}(q)$. For $0 \leq w \leq n$, denote by $A_w(\mathcal{C})$ or $A_w$ the number of codewords with Hamming weight $w$ in $\mathcal{C}$. The sequence $(A_0, A_1, \ldots, A_n)$ is called the weight distribution of $\mathcal{C}$, and the polynomial $\sum_{w=0}^{n} A_w x^w$ is the weight enumerator of $\mathcal{C}$. For convenience we index the coordinates of all codewords in $\mathcal{C}$ by considering codes admitting $t$-designs from linear codes (see, for example, [8, 21]). Tang et al. [20] generalized Assmus-Mattson Theorem [2] is very useful in producing $t$-designs. The Assmus-Mattson Theorem [2] is very useful in producing $t$-designs from linear codes (see, for example, [8, 21]). Tang et al. [20] generalized Assmus-Mattson Theorem and their new criterion outperforms the original Assmus-Mattson Theorem in some special cases. Additionally, we have another fundamental method to produce $t$-designs by considering codes admitting $t$-homogeneous or $t$-transitive automorphism groups (see, for example, [4, 12, 16]).

An $[n, k, d]$ linear code $\mathcal{C}$ over $\text{GF}(q)$ is cyclic if any cyclic shift of a codeword is again a codeword, i.e., $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ implies $(c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}$. We usually treat codewords in a cyclic code as polynomials in $\text{GF}(q)[x]$. This means if $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ then we associate the polynomial $c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in \text{GF}(q)[x]/(x^n - 1)$. In such a way any cyclic code of length $n$ over $\text{GF}(q)$ corresponds to a subset of the residue class ring $\text{GF}(q)[x]/(x^n - 1)$. It is obvious that a linear code $\mathcal{C}$ is cyclic if and only if the corresponding subset in $\text{GF}(q)[x]/(x^n - 1)$ is an ideal of the $\text{GF}(q)[x]/(x^n - 1)$.

Let $\mathcal{C}$ be a cyclic code of length $n$ over $\text{GF}(q)$. Because $\text{GF}(q)[x]/(x^n - 1)$ is a principal ideal ring, there is a unique monic polynomial $g(x) \in \text{GF}(q)[x]$ of lowest degree such that $\mathcal{C} = \langle g(x) \rangle$. This polynomial $g(x)$ (as a divisor of $x^n - 1$) is called the generator polynomial of $\mathcal{C}$.

Let $m = \text{ord}_p(q)$ and let $\alpha$ be a generator of $\text{GF}(q^m)^* := \text{GF}(q^m) \setminus \{0\}$. Let $\beta = \alpha^{(q^m - 1)/m}$. Then clearly $\beta$ is a primitive $n$-th root of unity in $\text{GF}(q^m)$. The minimal polynomial $M_{\beta^s}(x)$ of $\beta^s$ over $\text{GF}(q)$ is the monic polynomial of lowest degree over $\text{GF}(q)$ with $\beta^s$ as a zero. Let $\delta, h$ be positive integers with $2 \leq \delta \leq n$. A $\text{BCH code}$ over $\text{GF}(q)$ with length $n$ and designed distance $\delta$, denoted by $\mathcal{C}_{(q, n, \delta, h)}$, is a cyclic code of length $n$ whose generator polynomial is

$$g(x) = \text{lcm}(M_{\beta^h}(x), M_{\beta^h+1}(x), \ldots, M_{\beta^{h+\delta-2}}(x))$$

with the least common multiple calculated over $\text{GF}(q)$. When $h = 1$, the code $\mathcal{C}_{(q, n, \delta, h)}$ is called a narrow-sense $\text{BCH code}$. As is known to us all, $\text{BCH codes}$ are an important subclass of cyclic codes with many attractive properties.
A code $C$ is called a maximum distance separable code (or MDS code), if the minimum distance $d$ meets $d = n - k + 1$. The Singleton defect of an $[n, k, d]$ code $C$ is defined to be $\text{def}(C) = n - k + 1 - d$ and we say that $C$ is an $A^s$MDS code if $s = \text{def}(C)$. Thus, $A^0$MDS code is the same as an MDS code and an $A^1$MDS code is also called an almost MDS code (AMDS code). Hence, AMDS codes have parameters $[n, k, n - k]$. A code $C$ is said to be a near MDS code (NMDS code) if both $C$ and $C^\perp$ are AMDS codes.

MDS codes do hold $t$-designs with large $t$, but these designs are all trivial unfortunately. Ding and Tang [9] presented an infinite family of NMDS codes over $\text{GF}(3^m)$ holding 3-designs and an infinite family of NMDS codes over $\text{GF}(2^{2^e})$ holding 2-designs. The first NMDS code supporting a 4-design was the $[11, 6, 5]$ ternary Golay code which holds a Steiner system $S(4, 5, 11)$ discovered in 1949 by Golay [15]. Recently Tang and Ding [19] studied the narrow-sense BCH, Ding and Tang [9] presented an infinite family of NMDS codes over $\text{GF}(3^m)$ with odd and they support 3-designs when $m = 2^e$. It was shown that the codewords of the minimum (or the second minimum) weight in $C(q,q+1,4,1)$ support 4-designs when $m \geq 5$ is odd and they support 3-designs when $m \geq 4$ is even. In this paper, we will further investigate the codewords of the next adjacent weight (of weight 7) and prove that $B_7(C(q,q+1,4,1))$ also supports a 4-design when $m \geq 5$ is odd and it supports a 3-design when $m \geq 4$ is even. For $q = 2^m$ with odd $m \geq 5$, the minimum weight codewords in $C(q,q+1,4,1)$ were shown to support $4(q + 1, q - 5, \lambda)$ designs in [19] and in this paper we will show the second minimum weight codewords (of weight $q - 4$) support again 4-designs. It is very interesting and significant that the supplementary designs of these 4-designs have parameters 4-$\{q + 1, 5, 5\}$, because on one hand they achieve the smallest index among all the known simple 4-$\{q + 1, 5, 5\}$ designs (with $q$ prime powers) derived from codes and on the other hand they are proved to be isomorphic to the 4-designs admitting the projective general linear group $\text{PGL}(2,q)$ as the automorphism group constructed by Alltop [1] in 1969. This paper is not only a sequel of [19], but also we define generalized combinatorial objects, adopt new approaches and produce more infinite families of $t$-designs with new parameters.

The rest of the paper is organized as follows. In Section 2, we introduce the definition of elementary symmetric polynomials (ESPs) and develop block sets produced from three types of variants of ESPs. We also document several useful results produced in [19]. In Section 3, we present several new infinite families of $t$-designs with $t = 3, 4$ by exploring the block sets generated from the variants of ESPs. In Section 4, we consider the narrow-sense BCH codes $C(q,q+1,4,1)$ with $q = 2^m$ and prove that the codewords of weight 7 in $C(q,q+1,4,1)$ support 4-designs when $m \geq 5$ is odd and they support 3-designs when $m \geq 4$ is even. In Section 5, we represent the dual codes $C^\perp(q,q+1,4,1)$ ($q = 2^m, m \geq 5$ odd) in terms of trace codes in order to study the automorphism groups of their support sets. Then we prove that the codewords of weight $q - 4$ support 4-designs, whose supplementary designs are isomorphic to the 4-$\{q + 1, 5, 5\}$ designs constructed by Alltop [1]. In Section 6 we summarize our main results and conclude the paper with some remarks.

## 2 Preliminaries

A polynomial $f$ is symmetric if the polynomial is invariant under all permutations of its variables. The elementary symmetric polynomial (briefly by ESP) of degree $l$ in $k$ variables $u_1, u_2, \ldots, u_k$,
is defined by
\[ \sigma_{k,l}(u_1, u_2, \ldots, u_k) = \sum_{I \subseteq [k]} \prod_{j \in I} u_j. \]

For simplicity, \( \sigma_{k,l}(u_1, u_2, \ldots, u_k) \) is also denoted by \( \sigma_{k,l}(B) \) where \( B = \{ u_1, u_2, \ldots, u_k \} \) or briefly denoted \( \sigma_{k,l} \) if the context is clear.

Throughout the paper we always let \( q \) be a prime power and let \( U_{q+1} \) be the set of all \((q+1)\)-th roots of unity in \( \text{GF}(q^2) \), that is,
\[ U_{q+1} = \{ u \in \text{GF}(q^2) : u^{q+1} = 1 \}. \]

Let \( f \) be a symmetric polynomial in \( k \) variables whose coefficients are taken in \( \text{GF}(q^2) \). We define
\[ B_{f,q+1} = \left\{ \{ u_1, u_2, \ldots, u_k \} \in \binom{U_{q+1}}{k} : f(u_1, u_2, \ldots, u_k) = 0 \right\}. \]

This produces an incidence structure \( D = (U_{q+1}, B_{f,q+1}) \). If \( D \) forms a \( t-(q+1, k, \lambda) \) design, then we say that \( f \) supports a \( t-(q+1, k, \lambda) \) design. In particular, for any positive integer \( k \leq q+1 \), the block set \( B_{\sigma_{k,l},q+1} \) produced from the ESP \( \sigma_{k,l} \) is defined by
\[ B_{\sigma_{k,l},q+1} = \left\{ B \in \binom{U_{q+1}}{k} : \sigma_{k,l}(B) = 0 \right\}. \]

In [19], Tang and Ding presented several infinite families of linear codes supporting \( t \)-designs, whose block sets are isomorphic to \( B_{\sigma_{k,l},q+1} \) where \((k, l, q) \in \{(4, 2, 2^s), (4, 2, 3^m), (5, 2, 2^s), (6, 3, 2^m)\}\). In this paper we also concentrate on the topic of linear codes supporting \( t \)-designs by handling the incidence structures produced primarily from three types of variants of ESPs.

We define a block set \( B_{u,\sigma_{k,l}, q+1} \) by
\[ B_{u,\sigma_{k,l}, q+1} = \left\{ B \in \binom{U_{q+1}}{k} : \sigma_{k,l}(B - a) = 0 \text{ for some } a \in U_{q+1} \right\}, \]
where \( B - a := \{ b - a : b \in B \} \). If \((U_{q+1}, B_{u,\sigma_{k,l}, q+1})\) forms a \( t-(q+1, k, \lambda) \) design, we say that the \( u \)-variant of the ESP \( \sigma_{k,l} \) supports a \( t-(q+1, k, \lambda) \) design.

More restrictively, we define the block sets \( B_{b,\sigma_{k,l}, q+1} \) and \( B_{\bar{b},\sigma_{k,l}, q+1} \) generated from the \( b \)-variant and \( \bar{b} \)-variant of ESPs \( \sigma_{k,l} \) respectively by
\[ B_{b,\sigma_{k,l}, q+1} = \left\{ B \in \binom{U_{q+1}}{k} : \sigma_{k,l}(B - a) = 0 \text{ for some } a \in B \right\}, \]
and
\[ B_{\bar{b},\sigma_{k,l}, q+1} = B_{u,\sigma_{k,l}, q+1} \setminus B_{b,\sigma_{k,l}, q+1}. \]

Let \( B = \{ u_1, u_2, \ldots, u_k \} \in \binom{U_{q+1}}{k} \). One gets
\[ \sigma_{k,l}(B - a) = \sum_{I \subseteq [k]} \prod_{j \in I} (u_j - a) = \sum_{i=0}^{l} (-a)^{l-i} \binom{k-i}{l-i} \sigma_{k,i}(B). \]
Then $B \in \mathcal{B}_{q,4}^{u}$ if and only if $\sum_{i=0}^{t} (-a)^{t-i}(k-i)\sigma_{k,i}(B) = 0$ for some $a \in U_{q+1}$.

In this paper we will present new infinite families of $t$-designs with $t = 3, 4$ by exploring the block sets generated from the $u$-variant, $b$-variant, or $\mathcal{B}$-variant of ESPs. The effect is twofold. From the perspective of coding theory, the three types of variants usually provide succinct descriptions for the designs held by the codewords of a fixed weight in a linear code. On the other hand, from the aspect of design theory, we produce several infinite families of $t$-designs. Comparing with \cite{18}, we find that most of these parameters are new and an attractive feature of these designs lies in the simplicity of the designs although the indices $\lambda$ are somewhat large.

Now we record several conclusions in \cite{19} and derive some corollaries for later use.

**Lemma 2.1.** \cite{19} Lemmas 20, 22, 24-26] Let $q = 2^m$ and $m \geq 4$. Let $\{u_1, u_2, u_3, u_4\} \in \binom{U_{q+1}}{4}$ such that $\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5) \neq 0$ for any $u_5 \in U_{q+1} \setminus \{u_1, u_2, u_3, u_4\}$. Define $S_1$ and $S$ by

$$S_1 = \left\{ \frac{\sigma_{4,3} + u_i \sigma_{4,2}}{\sigma_{4,2} + u_i \sigma_{4,1}} : i = 1, 2, 3, 4 \right\} \bigcup \left\{ \sqrt{\frac{\sigma_{4,3}}{\sigma_{4,1}}} \right\}$$

(10)

and

$$S = S_1 \cup \{u_i : i = 1, 2, 3, 4\}.$$  

(11)

Then we have the following.

1. $S \subseteq U_{q+1}$, $|S_1| = 5$ and $|S| = 9$.

2. If $m$ is even and $\{\alpha, \beta\} \in \binom{U_{q+1}}{2}$ such that $\sigma_{5,2}(u_1, u_2, u_3, \alpha, \beta) = 0$, then $\alpha, \beta \notin S$.

3. Let $u = \frac{\sigma_{5,3}(u_1, u_2, u_3, u_4, u_5)}{\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5)}$. Then $u \in \{u_1, u_2, u_3, u_4, u_5\}$ if and only if $u_5 \in S_1$.

**Lemma 2.2.** \cite{19} Lemma 17] For $q = 2^m$ with odd $m \geq 5$ and $B \in \binom{U_{q+1}}{5}$, one has $\sigma_{5,2}(B) \neq 0$.

**Lemma 2.3.** \cite{19} Theorem 3] For $q = 2^m$ where $m \geq 4$ is even, $(U_{q+1}, \mathcal{B}_{5,2,q+1})$ forms a Steiner system $S(3, 5, q + 1)$.

**Lemma 2.4.** \cite{19} Theorem 2] For $q = 2^m$ where $m \geq 5$ is odd, $(U_{q+1}, \mathcal{B}_{6,3,q+1})$ forms a $4-(q + 1, 6, \frac{q-8}{2})$ design.

**Proof.** Let $\{u_1, u_2, u_3, u_4\}$ be a fixed 4-subset of $U_{q+1}$. For any $u_5 \in U_{q+1} \setminus \{u_1, u_2, u_3, u_4\}$, $\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5) \neq 0$ from Lemma 2.2. Define

$$\mathcal{T} = \left\{ \{u_1, u_2, u_3, u_4, u_5\} \in \binom{U_{q+1}}{6} : u_5 \in U_{q+1} \setminus S, u_6 = \frac{\sigma_{5,3}(u_1, u_2, u_3, u_4, u_5)}{\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5)} \right\}.$$  

where $S$ is given by Eq. (11). Note that $\sigma_{5,3} = \sigma_{5,5} \sigma_{5,2}$. Then $\frac{\sigma_{5,3}}{\sigma_{5,2}} = \frac{\sigma_{5,3}^q}{\sigma_{5,2}} = (\frac{\sigma_{5,3}}{\sigma_{5,2}})^{q+1} = 1$. This shows that $\sigma_{5,3}^q = \sigma_{5,2}^q$. Then $\frac{\sigma_{5,3}^{q+1}}{\sigma_{5,2}} = \frac{\sigma_{5,3}^{q+1}}{\sigma_{5,2}^{q+1}} = 1$. It is easily checked that $E \subseteq B$ and $B \in \mathcal{B}_{6,3,q+1}$ if and only if $B \in \mathcal{T}$. From Lemma 2.1 (3), $\frac{\sigma_{5,3}}{\sigma_{5,2}} \notin S$ if $u_5 \notin S$. Noting the symmetry of $u_5$ and $u_6$, we have $|\mathcal{T}| = \frac{q+1-9}{2}$ by Lemma 2.1 (1). As a result, $(U_{q+1}, \mathcal{B}_{6,3,q+1})$ is a $4-(q + 1, 6, \frac{q-8}{2})$ design.

**Lemma 2.5.** \cite{19} Theorem 4] For $q = 2^m$ with even $m \geq 4$, $(U_{q+1}, \mathcal{B}_{6,3,q+1})$ gives a $3-(q + 1, 6, \frac{(q-4)^2}{6})$ design.
Proof. Let $E = \{u_1, u_2, u_3\}$ be a fixed 3-subset of $U_{q+1}$. By Lemma 2.3 there is a unique block $A \in \binom{U_{q+1}}{3}$ such that $E \subseteq A$ and $\sigma_{5,2}(A) = 0$. Set
\[
\mathcal{T}_1 = \{ A \cup \{u_i\} : u_i \in U_{q+1} \setminus A \},
\]
and
\[
\mathcal{T}_2 = \left\{ \{u_1, u_2, u_3, u_4, u_5, u_6\} \in \binom{U_{q+1}}{6} : \begin{array}{l}
u_4 \in U_{q+1} \setminus A, u_5 \in U_{q+1} \setminus (A \cup S), \\u_6 = \frac{\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5)}{\sigma_{5,2}(u_1, u_2, u_3, u_4, u_5)} \end{array} \right\},
\]
where $S$ is given by (11). Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. Obviously $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint. It is easily checked that $E \subseteq B$ and $B \in \mathcal{B}_{\sigma_{6,3},q+1}$ if and only if $B \in \mathcal{T}$. Note that $|\mathcal{T}_1| = q + 1 - 5 = q - 4$ and $|\mathcal{T}_2| = \frac{(q+1-|A|)(q+1-|A\cup S|)}{6}$. By Lemma 2.1, $|A \cup S| = 11$. Then $|\mathcal{T}_2| = \frac{(q-4)(q-10)}{6}$ and $(U_{q+1}, B_{\sigma_{6,3},q+1})$ is a 3-$(q+1, 6, \frac{(q-4)^2}{6})$ design.

The following corollary follows from the proof of Lemma 2.3.

Corollary 2.6. Let $q = 2^m$ with even $m \geq 4$. Let $E$ be a fixed 3-subset of $U_{q+1}$.

1. Set $\mathcal{T}_1 = \{ B \in \mathcal{B}_{\sigma_{6,3},q+1} : E \subseteq B, B \setminus \{u\} \in \mathcal{B}_{\sigma_{5,2},q+1} \text{ for some } u \in B \setminus E \}$. Then $|\mathcal{T}_1| = q - 4$.

2. Set $\mathcal{T}_2 = \{ B \in \mathcal{B}_{\sigma_{6,3},q+1} : E \subseteq B, B \setminus \{u\} \notin \mathcal{B}_{\sigma_{5,2},q+1} \text{ for any } u \in B \setminus E \}$. Then $|\mathcal{T}_2| = \frac{(q-4)(q-10)}{6}$.

Combining Lemma 2.3 with the proofs of Lemmas 2.4 and 2.5 gives the following corollary, which is very useful in this paper.

Corollary 2.7. (1) Let $q = 2^m$ with even $m \geq 4$. For any $B_1, B_2 \in \mathcal{B}_{\sigma_{5,2},q+1}$ with $B_1 \neq B_2$, we have $|B_1 \cap B_2| \leq 2$.

(2) Let $q = 2^m$ with even $m \geq 4$. For any $B_1, B_2 \in \mathcal{B}_{\sigma_{6,3},q+1}$ with $B_1 \neq B_2$, we have $|B_1 \cap B_2| \leq 5$; equality occurs only if $\sigma_{5,2}(B_1 \cap B_2) = 0$.

(3) Let $q = 2^m$ with odd $m \geq 5$. For any $B_1, B_2 \in \mathcal{B}_{\sigma_{6,3},q+1}$ with $B_1 \neq B_2$, we have $|B_1 \cap B_2| \leq 4$.

Lemma 2.8. [19] Theorem 4] Let $q = 2^m$ with even $m \geq 4$. Then $(U_{q+1}, \mathcal{B}_{\sigma_{6,3},q+1}^0)$ forms a 3-$(q+1, 6, 2(q-4))$ design and $(U_{q+1}, \mathcal{B}_{\sigma_{6,3},q+1} \setminus \mathcal{B}_{\sigma_{6,3},q+1}^0)$ gives a 3-$(q+1, 6, \frac{(q-4)(q-16)}{6})$ design, where the block set $\mathcal{B}_{\sigma_{6,3},q+1}^0$ is defined by
\[
\mathcal{B}_{\sigma_{6,3},q+1}^0 = \left\{ B \in \binom{U_{q+1}}{6} : \sigma_{5,2}(B \setminus \{u\}) = 0 \text{ for some } u \in B \right\}. \tag{12}
\]

To conclude this section we give two theorems showing several cases when the ESP $\sigma_{k,l}$ and the $u$-variant of $\sigma_{k,l}$ support the same designs.

Theorem 2.9. For $q = 2^m$ with even $m \geq 4$, $(U_{q+1}, \mathcal{B}_{\sigma_{5,2},q+1}^u)$ forms a Steiner system $S(3, 5, q+1)$. 
Theorem 2.10. (1) For \( q = 2^m \) with even \( m \geq 4 \), \((U_{q+1}, B_{\sigma_{6,3,q+1}}^u) \) forms a \( 3-(q+1, 6, \frac{(q-4)^2}{6}) \) design.

(2) Let \( q = 2^m \) with odd \( m \geq 5 \), \((U_{q+1}, B_{\sigma_{6,3,q+1}}^u) \) forms a \( 4-(q + 1, 6, \frac{q-8}{2}) \) design.

Proof. Let \( B \in (U_{q+1})_6 \). By Eq. (9), \( B \in B_{\sigma_{5,2,q+1}}^u \) if and only if \( \sigma_{5,2}(B-a) = 0 \) for some \( a \in U_{q+1} \). From Eq. (9), \( \sigma_{5,2}(B-a) = \sigma_{5,2} - 4a\sigma_{5,1} + 10a^2 \). Since \( q = 2^m \), \( \sigma_{5,2}(B-a) = 0 \) is the same as \( \sigma_{5,2}(B) = 0 \). This shows that \( B_{\sigma_{5,2,q+1}}^u = B_{\sigma_{5,2,q+1}} \). So the conclusion follows by Lemma 2.3 \( \square \)

Theorem 3.1. For \( q = 2^m \) with even \( m \geq 4 \), \((U_{q+1}, B_{\sigma_{4,1,q+1}}^u) \) forms a \( 3-(q+1, 4, 2) \) design.

Proof. By Eq. (9), \( B \in B_{\sigma_{4,2,q+1}}^u \) if and only if \( \sigma_{4,2}(B-a) = 0 \) for some \( a \in U_{q+1} \). Let \( B = \{u_1, u_2, u_3, u_4\} \) be any block of \( B_{\sigma_{4,2,q+1}}^u \). From Eq. (9) one has

\[
\sigma_{4,2}(B-a) = \sigma_{4,2}(B) - 3a\sigma_{4,1}(B) + 6a^2 = \sigma_{4,2}(B) + a\sigma_{4,1}(B).
\]

Then \( \sigma_{4,2}(B-a) = 0 \) is equivalent to \( \sigma_{5,2}(B \oplus \{a\}) = 0 \). Next we prove \( a \notin B \). Otherwise let \( a = u_4 \) and \( \sigma_{4,2} + u_4\sigma_{4,1} = 0 \). Thus

\[
\sigma_{3,2} + u_4^2 = (\sigma_{3,2} + u_4\sigma_{3,1}) + u_4(\sigma_{3,1} + u_4) = \sigma_{4,2} + u_4\sigma_{4,1} = 0.
\]

Hence, we have

\[
u_4^2\sigma_{3,1} + \sigma_{3,3} = u_4^2(\sigma_{3,3}\sigma_{3,2}) + \sigma_{3,3} = u_4^2\sigma_{3,3}(\sigma_{3,2} + u_4^2) = u_4^2\sigma_{3,3}(\sigma_{3,2} + u_4^2) = 0.
\]

Multiplying both sides of \( \sigma_{3,2} + u_4^2 = 0 \) by \( u_4 \) and then adding \( u_4^2\sigma_{3,1} + \sigma_{3,3} \) yields

\[
\sigma_{3,3} + u_4\sigma_{3,2} + u_4^2\sigma_{3,1} + u_4 = 0.
\]

Then

\[
(u_4 + u_1)(u_4 + u_2)(u_4 + u_3) = 0,
\]
which is contrary to the assumption \( \{u_1, u_2, u_3, u_4\} \in (U_{q+1}^{\frac{5}{q+1}}) \). It follows that \( a \not\in B \) and thus

\[
B_{a,2,q+1}^u = \left\{ B \in \left( U_{q+1}^{\frac{5}{q+1}} \right) : \sigma_{5,2}(B \cup \{a\}) = 0 \text{ for some } a \in U_{q+1} \setminus B \right\}.
\]

Let \( E = \{u_1, u_2, u_3\} \) be a fixed 3-subset of \( U_{q+1} \). From Lemma 2.23 \( (U_{q+1}, B_{5,2,q+1}) \) is a 3-\((q+1, 5, 1)\) design. Then there is a unique pair \((\alpha, \beta)\) \( \in (U_{q+1}^{\frac{5}{q+1}}) \) such that \( E \cup \{\alpha, \beta\} \in B_{5,2,q+1} \) and so \( E \cup \{\alpha\}, E \cup \{\beta\} \in B_{a,2,q+1}^u \). It readily yields that \( (U_{q+1}, B_{a,2,q+1}^u) \) forms a 3-\((q+1, 4, 2)\) design.

Lemma 3.2. Let \( q = 2^m \) with \( m \geq 4 \). Let \( B \in \left( U_{q+1}^{\frac{5}{q+1}} \right) \) such that \( \sigma_{5,2}(B) = 0 \). Then \( \sigma_{6,3}(B \uplus \{u\}) = 0 \) for any \( u \in U_{q+1} \).

Proof. Since \( \sigma_{5,3}(B) = \sigma_{5,5}(B)\sigma_{3,2}(B) \), then \( \sigma_{5,3}(B) = 0 \) if and only if \( \sigma_{5,2}(B) = 0 \) by noting \( \sigma_{5,5}(B) \neq 0 \). Note that

\[
\sigma_{6,3}(B \uplus \{u\}) = \sigma_{5,3}(B) + u\sigma_{5,2}(B).
\]

Then \( \sigma_{6,3}(B \uplus \{u\}) = 0 \) if \( \sigma_{5,2}(B) = 0 \).

Theorem 3.3. For \( q = 2^m \) with odd \( m \geq 5 \), \( (U_{q+1}, B_{a,5,q+1}^u) \) is a complete 4-design. In particular we have the following.

1. \( (U_{q+1}, B_{5,5,q+1}^a) \) forms a 4-\((q + 1, 5, 5)\) design.

2. \( (U_{q+1}, B_{5,5,q+1}^b) \) forms a 4-\((q + 1, 5, q - 8)\) design.

Proof. Let \( B \in \left( U_{q+1}^{\frac{5}{q+1}} \right) \). By Eq. 31., \( B \in B_{a,5,q+1}^u \) if and only if \( \sigma_{5,3}(B - a) = 0 \) for some \( a \in U_{q+1} \). From Eq. 32. one obtains

\[
\sigma_{5,3}(B - a) = \sigma_{5,3}(B) - 3a\sigma_{5,2}(B) + 6a^2\sigma_{5,1}(B) - 10a^3 = \sigma_{5,3}(B) + a\sigma_{5,2}(B).
\]

Then \( \sigma_{5,3}(B - a) = 0 \) is equivalent to \( \sigma_{6,3}(B \uplus \{a\}) = 0 \). Let \( E = \{u_1, u_2, u_3, u_4\} \) be a fixed 4-subset of \( U_{q+1} \) and \( B = \{u_1, u_2, u_3, u_4, u_5\} \) be any block of \( B_{a,5,q+1}^u \) containing \( E \). Then we consider the following two cases.

1. \( B \in B_{a,5,q+1}^u \). Then there is \( u_i \in B \) such that \( \sigma_{6,3}(B \uplus \{u_i\}) = 0 \) by 33. From Lemma 2.2, \( \sigma_{5,2}(B) \neq 0 \). Note that

\[
\sigma_{6,3}(B \uplus \{u_i\}) = \sigma_{5,3}(B) + u_i\sigma_{5,2}(B).
\]

Then \( u_i = \frac{\sigma_{5,3}(B)}{\sigma_{5,2}(B)} \). From Lemma 2.1 (3), \( \frac{\sigma_{5,3}(B)}{\sigma_{5,2}(B)} \in B \) if and only if \( u_5 \in S_1 \), where \( S_1 \) is given by Eq. 10. Hence, \( |\{B \in B_{a,5,q+1}^u : E \subseteq B\}| = |S_1| = 5 \) by Lemma 2.1 (1). So \( E \) is contained in five distinct blocks of \( B_{a,5,q+1}^u \) and the incidence structure \( (U_{q+1}, B_{a,5,q+1}^u) \) is a 4-\((q + 1, 5, 5)\) design.

2. \( B \in B_{a,5,q+1}^b \). Then there is \( a \in U_{q+1} \setminus B \) such that \( \sigma_{6,3}(B \uplus \{a\}) = 0 \) by 35. From Lemma 2.4, \( (U_{q+1}, B_{a,6,q+1}) \) is a 4-\((q + 1, 6, \frac{q - 8}{2})\) design. Then there are \( \frac{q - 8}{2} \) blocks of \( B_{a,6,q+1} \) containing \( E \). Noticing each block of \( B_{6,q+1} \) containing \( E \) provides two blocks of \( B_{a,6,q+1} \) containing \( E \) and applying Corollary 2.7, we know that \( (U_{q+1}, B_{a,5,q+1}^u) \) is a 4-\((q + 1, 5, q - 8)\) design.
Combining (1) and (2) yields that \((U_{q+1}, B^u_{\sigma_{5,3,q+1}})\) is a \((q+1, 5, q-3)\) design. This is a complete 4-design. \(\square\)

**Remark 3.4.** Let \(q = 2^m\) and \(m \geq 4\) be even. If we take each block \(B \in B^b_{\sigma_{5,3,q+1}}\) such that \(\sigma_{5,2}(B) = 0\) five times and each of the other blocks of \(B^b_{\sigma_{5,3,q+1}}\) once, then we produce a non-simple \((q+1, 5, q)\) design by applying similar arguments to the proof of Theorem 3.3 (1).

**Theorem 3.5.** For \(q = 2^m\) with even \(m \geq 4\), \((U_{q+1}, B^b_{\sigma_{5,3,q+1}})\) forms a \((3(q+1), 5, \frac{q^2-10q+26}{2})\) design.

**Proof.** Let \(B \in (U_{q+1})\). Analogous to the proof of Theorem 3.3, we have \(B \in B^b_{\sigma_{5,3,q+1}}\) if and only if \(\sigma_{6,3}(B \cup \{a\}) = 0\) for some \(a \in U_{q+1} \setminus B\) by noting Eq. (3). Let \(E = \{u_1, u_2, u_3\}\) be a fixed 3-subset of \(U_{q+1}\) and let \(B\) be any block of \(B^b_{\sigma_{5,3,q+1}}\) containing \(E\). So by Corollary 2.6 there is \(A \in (U_{q+1})\) such that \(B \subseteq A\) and \(\sigma_{6,3}(A) = 0\). Thus \(A \in T_1 \cup T_2\), where \(T_1\) and \(T_2\) are the same defined as in Corollary 2.6. Let

\[
B_i = \left\{ B \in \binom{U_{q+1} \setminus B}{5} \mid \text{there is } A \in T_i \text{ such that } E \subseteq B \subseteq A \right\},
\]

where \(i = 1, 2\). Clearly \(B_1\) and \(B_2\) are disjoint.

For any \(A \in T_1\), there exists \(u \in A \setminus E\) such that \(\sigma_{5,2}(A \setminus \{u\}) = 0\). From Lemma 2.3 there is a unique \(B_0 \in \binom{U_{q+1}}{5}\) such that \(E \subseteq B_0\) and \(\sigma_{5,2}(B_0) = 0\). Hence \(A \setminus \{u\} = B_0\). Let \(B_0 = \{u_1, u_2, u_3, u_4, u_5\}\). By Lemma 3.2 clearly \(B_0 \in B_1\) and also \(\{u_1, u_2, u_3, u_i, \alpha\} \in B_1\) for any \(i = 4, 5\) and \(\alpha \in U_{q+1} \setminus B_0\). As a consequence, each \(A \in T_1\) provides \(1 + 2(q - 4)\) distinct blocks of \(B_1\) and hence \(|B_1| = 1 + 2(q - 4)|E|\).

For any \(A \in T_2\), obviously \(A \setminus \{u\} \in B_2\) for any \(u \in A \setminus E\). From Corollary 2.6 (2), \(|T_2| = \frac{(q-4)(q-10)}{6}\). Applying Corollary 2.7 (2) yields that each \(A \in T_2\) provides three distinct blocks of \(B_2\). Hence \(|B_2| = 3|T_2| = \frac{(q-4)(q-10)}{2}\).

Now we conclude that \(|B_1| + |B_2| = \frac{q^2-10q+26}{2}\) and \((U_{q+1}, B^b_{\sigma_{5,3,q+1}})\) is a \((3(q+1), 5, \frac{q^2-10q+26}{2})\) design. \(\square\)

**Theorem 3.6.** For \(q = 2^m\) with even \(m \geq 4\), \((U_{q+1}, B^b_{\sigma_{6,2,q+1}})\) forms a \((3(q+1), 6, 2q-8)\) design.

**Proof.** Let \(B \in (U_{q+1})\). By Eq. (6), \(B \in B^b_{\sigma_{6,2,q+1}}\) if and only if \(\sigma_{6,2}(B - a) = 0\) for some \(a \in B\). From Eq. (9) one has

\[
\sigma_{6,2}(B - a) = \sigma_{6,2}(B) - 5a\sigma_{6,1}(B) + 15a^2 = \sigma_{6,2}(B) + a\sigma_{6,1}(B) + a^2.
\]

It is clear by noting \(a \in B\) that

\[
\sigma_{6,2}(B) + a\sigma_{6,1}(B) + a^2
= \sigma_{5,2}(B \setminus \{a\}) + a\sigma_{5,1}(B \setminus \{a\}) + a(\sigma_{5,1}(B \setminus \{a\}) + a) + a^2
= \sigma_{5,2}(B \setminus \{a\}).
\]

So \(\sigma_{6,2}(B - a) = 0\) for \(a \in B\) is the same as \(\sigma_{5,2}(B \setminus \{a\}) = 0\). Let \(E = \{u_1, u_2, u_3\}\) be a fixed 3-subset of \(U_{q+1}\) and let \(B\) be any block of \(B^b_{\sigma_{6,2,q+1}}\) containing \(E\). Then we have two possibilities for \(B\).
Case 1. \(\sigma_{5,2}(B \setminus \{a\}) = 0\) for some \(a \in B \setminus E\). Denote the collection of such \(B\) by \(B_1\). From Lemma 2.3, \((U_{q+1}, B_{\sigma_{5,2},q+1})\) is an S(3, 5, \(q+1\)). Then \(\{|A \in B_{\sigma_{5,2},q+1} : E \subseteq A\| = 1\). From Corollary 2.7 (1), \(|A_1 \cap A_2| \leq 2\) for any \(A_1, A_2 \in B_{\sigma_{5,2},q+1}\) with \(A_1 \neq A_2\). Then \(B_1\) is simple and \(|B_1| = \{|A \cup \{u\} : A \in B_{\sigma_{5,2},q+1}, E \subseteq A, u \in U_{q+1} \setminus A\| = q + 1 - 5 = q - 4\).

Case 2. \(\sigma_{5,2}(B \setminus \{a\}) = 0\) for some \(a \in E\). Denote the collection of such \(B\) by \(B_2\). Also note that \((U_{q+1}, B_{\sigma_{5,2},q+1})\) is an S(3, 5, \(q+1\)). Then there exists a unique block of \(B_{\sigma_{5,2},q+1}\) containing \(E\). Applying Eq. (1), there are \(\lambda_2 = \frac{q+1-2}{3}\) blocks of \(B_{\sigma_{5,2},q+1}\) containing any fixed pair of \(E\). Excluding the unique block containing \(E\) and applying Corollary 2.7 (1), we have that \(B_2\) is simple and \(|B_2| = 3(\frac{q+1}{3} - 1) = q - 4\).

Now we prove \(B_1 \cap B_2 = \emptyset\). Otherwise, there exists \(B \in B_1 \cap B_2\). Then \(\sigma_{5,2}(B \setminus \{u_i\}) = 0\) for some \(u_i \in B \setminus E\) and \(\sigma_{5,2}(B \setminus \{u_j\}) = 0\) for some \(u_j \in E\). Note that \(|(B \setminus \{u_i\}) \cap (B \setminus \{u_j\})| = 4\), contradicting Corollary 2.7 (1). Then \(B_1\) and \(B_2\) are disjoint. So \(E\) is contained in \(|B_1| + |B_2| = 2(q - 4)\) blocks of \(B_{\sigma_{5,2},q+1}\) and \((U_{q+1}, B_{\sigma_{5,2},q+1})\) is a \(3\((q+1,6,2q-8)\) design. \(\square\)

Lemma 3.7. For \(q = 2^m\), one has

\[
B_{\sigma_{5,3},q+1}^u = \left\{ B \in \left(U_{q+1}\right)^7 : \sigma_{6,3}(B \setminus \{a\}) = 0 \text{ for some } a \in B \right\}.
\]

Proof. Let \(B = \{u_1, u_2, \ldots, u_7\}\) be any block of \(B_{\sigma_{5,3},q+1}^u\). By Eq. (4), \(\sigma_{7,3}(B - a) = 0\) for some \(a \in U_{q+1}\). From Eq. (9) one obtains

\[
\sigma_{7,3}(B - a) = \sigma_{7,3}(B) - 5a\sigma_{7,2}(B) + 15a^2\sigma_{7,1}(B) - 35a^3
= \sigma_{7,3}(B) + a\sigma_{7,2}(B) + a^2\sigma_{7,1}(B) + a^3 = 0.
\]

Next we prove \(a \in B\). Note that

\[
a^3\sigma_{7,4} + a^2\sigma_{7,5} + a\sigma_{7,6} + \sigma_{7,7}
= a^3\sigma_{7,2}\sigma_{7,3}^q + a^2\sigma_{7,2}\sigma_{7,3}^q + a\sigma_{7,2}\sigma_{7,3}^q + \sigma_{7,7}
= a^3\sigma_{7,2}(\sigma_{7,3}^q + a^{-1}\sigma_{7,2} + a^{-2}\sigma_{7,2} + a^{-3})
= a^3\sigma_{7,2}(\sigma_{7,3}^q + a^q\sigma_{7,2} + a^{2q}\sigma_{7,2} + a^{3q})
= a^3\sigma_{7,2}(\sigma_{7,3}^q + a^{2}\sigma_{7,2} + a^{3}\sigma_{7,2}) = 0.
\]

Multiplying both sides of \(\sigma_{7,3} + a\sigma_{7,2} + a^2\sigma_{7,1} + a^3 = 0\) by \(a^4\) and then adding \(a^3\sigma_{7,4} + a^2\sigma_{7,5} + a\sigma_{7,6} + \sigma_{7,7}\) yields

\[
\sigma_{7,7} + a\sigma_{7,6} + a^2\sigma_{7,5} + a^3\sigma_{7,4} + a^4\sigma_{7,3} + a^5\sigma_{7,2} + a^6\sigma_{7,1} + a^7 = 0.
\]

So

\[
(a + u_1)(a + u_2)(a + u_3)(a + u_4)(a + u_5)(a + u_6)(a + u_7) = 0.
\]

Thus \(a = u_i\) for some \(u_i \in B\) and

\[
\sigma_{7,3}(B - u_i)
= \sigma_{7,3} + u_i\sigma_{7,2} + u_i^2\sigma_{7,1} + u_i^3
= \sigma_{6,3}(B \setminus \{u_i\}) + u_i\sigma_{6,2}(B \setminus \{u_i\}) + u_i(\sigma_{6,1}(B \setminus \{u_i\}) + u_i + u_i^3)
= \sigma_{6,3}(B \setminus \{u_i\}) + 2(u_i\sigma_{6,2}(B \setminus \{u_i\}) + u_i(\sigma_{6,1}(B \setminus \{u_i\}) + u_i^2)
= \sigma_{6,3}(B \setminus \{u_i\}).
\]

Hence the conclusion follows. \(\square\)
Theorem 3.8. For \( q = 2^m \) with odd \( m \geq 5 \), \((U_{q+1}, B^u_{\sigma, q+1})\) forms a \((q + 1, 7, 7(q-8)(q-5)/6)\) design.

Proof. By Lemma 3.7, \( B^u_{\sigma, q+1} = \{B \in (U_{q+1}^7) : \sigma(B \backslash \{a\}) = 0 \text{ for some } a \in B\} \). Let \( E = \{u_1, u_2, u_3, u_4\} \) be a fixed 4-subset of \( U_{q+1} \) and let \( B \) be any block of \( B^u_{\sigma, q+1} \) containing \( E \). Then we have two possibilities for \( B \).

Case 1. \( \sigma(B \backslash \{a\}) = 0 \text{ for some } a \in B \backslash E \). Denote the collection of such \( B \) by \( B_1 \). From Lemma 2.4, \((U_{q+1}, B_{\sigma, q+1})\) is a \((q+1, 4, q^2/2)\) design. Then \(|\{A \in B_{\sigma, q+1} : E \subseteq A\}| = q^2/2\). From Corollary 2.7 (3), \(|A_1 \cap A_2| \leq 4\) for any \( A_1, A_2 \in B_{\sigma, q+1} \) with \( A_1 \neq A_2 \). Then \( B_1 \) is simple and \(|B_1| = |\{A \cup \{u\} : A \in B_{\sigma, q+1}, E \subseteq A, u \in U_{q+1} \backslash A\}| = q^2/2 \). As a result, \(|B_1| = \frac{(q-5)(q-5)}{2}\).

Case 2. \( \sigma(B \backslash \{a\}) = 0 \text{ for some } a \in E \). Denote the collection of such \( B \) by \( B_2 \). By Lemma 2.4, there are \( q^2/2 \) blocks of \( B_{\sigma, q+1} \) containing \( E \). Applying Eq. (1), there are \( \lambda_3 = q^2/2 \cdot q^3 \) blocks of \( B_{\sigma, q+1} \) containing any fixed 3-subset of \( E \). Excluding \( q^2/2 \) blocks containing \( E \) and applying Corollary 2.7 (3) yields that \( B_2 \) is simple and \(|B_2| = 4(\lambda_3 - q^2/2) = 2(q-8)(q-5)/3\).

Now we prove \( B_1 \cap B_2 = \emptyset \). Otherwise, there exists \( B \in B_1 \cap B_2 \). Then \( \sigma(B \backslash \{u_i\}) = 0 \) for some \( u_i \in B \backslash E \) and \( \sigma(B \backslash \{u_j\}) = 0 \) for some \( u_j \in E \). Note that \(|(B \backslash \{u_i\}) \cap (B \backslash \{u_j\})| = 5\), contradicting Corollary 2.7 (3). Hence \( B_1 \) and \( B_2 \) are disjoint. So \( E \) is contained in \(|B_1| + |B_2| = \frac{7(q-8)(q-5)}{6}\) blocks of \( B^u_{\sigma, q+1} \) and thus \((U_{q+1}, B^u_{\sigma, q+1})\) is a \((q + 1, 7, 7(q-8)(q-5)/6)\) design. \( \square \)

Theorem 3.9. For \( q = 2^m \) with even \( m \geq 4 \), \((U_{q+1}, B^u_{\sigma, q+1})\) is a \((3(q + 1, 7, 7(q-4)(q-5)(q-10)/24)\) design.

Proof. Let \( B \in (U_{q+1}^7) \). By Lemma 3.7, \( B \in B^u_{\sigma, q+1} \) if and only if \( \sigma(B \backslash \{a\}) = 0 \) for some \( a \in B \). Let \( E = \{u_1, u_2, u_3\} \) be a fixed 3-subset of \( U_{q+1} \). Let \( B = \{u_1, u_2, \ldots, u_7\} \) be any block of \( B^u_{\sigma, q+1} \) containing \( E \). We treat four possibilities of \( B \) according to Corollary 2.6. Clearly \( B \in \bigcup_{i=1}^4 B_i \), where

\[
B_1 = \left\{ B \in (U_{q+1}^7) : \begin{array}{l}
E \subseteq B, \sigma(B \backslash \{u_i\}) = 0 \text{ for some } u_i \in B \backslash E, \\
\sigma_5(B \backslash \{u_i, u_j\}) = 0 \text{ for some } u_j \in B \backslash (E \cup \{u_i\})
\end{array} \right\},
\]

\[
B_2 = \left\{ B \in (U_{q+1}^7) : \begin{array}{l}
E \subseteq B, \sigma(B \backslash \{u_k\}) = 0 \text{ for some } u_k \in E, \\
\sigma_5(B \backslash \{u_k, u_j\}) = 0 \text{ for some } u_j \in B \backslash E
\end{array} \right\},
\]

\[
B_3 = \left\{ B \in (U_{q+1}^7) : \begin{array}{l}
E \subseteq B, \sigma(B \backslash \{u_i\}) = 0 \text{ for some } u_i \in B \backslash E, \\
\sigma_5(B \backslash \{u_i, u_j\}) \neq 0 \text{ for any } u_j \in B \backslash (E \cup \{u_i\})
\end{array} \right\},
\]

\[
B_4 = \left\{ B \in (U_{q+1}^7) : \begin{array}{l}
E \subseteq B, \sigma(B \backslash \{u_k\}) = 0 \text{ for some } u_k \in E, \\
\sigma_5(B \backslash \{u_k, u_j\}) \neq 0 \text{ for any } u_j \in B \backslash E
\end{array} \right\}.
\]

Next we prove that \( B_2 \subseteq B_3 \) and that \( B_1, B_3 \) and \( B_4 \) are mutually disjoint.

(i) For any \( B \in B_2 \), w.l.o.g. we assume \( \sigma(B \backslash \{u_1\}) = 0 \) and \( \sigma_5(B \backslash \{u_1, u_7\}) = 0 \). Then we have \( \sigma_5(B \backslash \{u_7\}) = 0 \) by Lemma 3.2. Note that \(|\{B \backslash \{u_1, u_7\}\} \cap (B \backslash \{u_7, u_j\})| = 4\) for any \( u_j \in B \backslash E \). Then from Corollary 2.7 (1), \( \sigma_5(B \backslash \{u_7, u_j\}) \neq 0 \) as \( \sigma_5(B \backslash \{u_1, u_7\}) = 0 \). This shows that \( B \in B_3 \) by taking \( u_i = u_7 \). As a result, \( B_2 \subseteq B_3 \).
(ii) For any $B \in B_1 \cap B_3$, w.l.o.g., we assume $\sigma_{6,3}(B \setminus \{u_7\}) = 0$, $\sigma_{5,2}(B \setminus \{u_6,u_7\}) = 0$, $\sigma_{6,3}(B \setminus \{u_1\}) = 0$ for some $u_i \in B \setminus E$ and $\sigma_{5,2}(B \setminus \{u_i,u_j\}) \neq 0$ for any $u_j \in B \setminus (E \cup \{u_i\})$. It is obvious that $u_i \not\in \{u_6,u_7\}$ and $|B \setminus \{u_i\}| \cap (B \setminus \{u_7\}) = 5$. So we have $\sigma_{5,2}(B \setminus \{u_i,u_j\}) = 0$ from Corollary 2.7 (2), which contradicts $B \in B_3$. Hence $B_1 \cap B_3 = \emptyset$.

Similarly, we can prove that $B_1 \cap B_4 = \emptyset$ and $B_3 \cap B_4 = \emptyset$. To prove the final conclusion we only need to show that $|B_1| + |B_3| + |B_4| = \frac{(q-4)(q-5)}{24}$. Next we calculate the cardinalities of $B_1, B_3$ and $B_4$.

By Lemma 2.3 there is a unique block $A \in \binom{U_{q+1}}{5}$ such that $E \subseteq A$ and $\sigma_{5,2}(A) = 0$. From Lemma 3.2, $\sigma_{6,3}(A \cup \{u\}) = 0$ for any $u \in U_{q+1} \setminus A$. This implies $B \in B_1$ if and only if $B = A \cup \{u_6,u_7\}$ where $\{u_6,u_7\} \in \binom{U_{q+1}}{2}$. It then follows that

$$|B_1| = \frac{(q-4)(q-5)}{2}.$$  

(13)

It is clear that $B \in B_3$ if and only if $B = B' \cup \{u\}$ for some $B' \in B'$ and $u \in U_{q+1} \setminus B'$, where

$$B' = \left\{ B' \in \binom{U_{q+1}}{6} : E \subseteq B', \sigma_{6,3}(B') = 0, \sigma_{5,2}(B' \setminus \{u\}) \neq 0 \right\}.$$  

From Corollary 2.6 (2), we have $|B'| = \frac{(q-4)(q-10)}{6}$. And from Corollary 2.7 (2), $|B_1' \cap B_2'| \leq 4$ for any $B_1', B_2' \in B_{\sigma_{6,3},q+1}$ with $B_1' \neq B_2'$. It then follows that

$$|B_3| = \frac{(q-4)(q-10)}{6} \cdot (q + 1 - 6) = \frac{(q-4)(q-10)(q-5)}{6}.$$  

(14)

In order to calculate the size of $B_4$, we define $T_{k,1}$ and $T_{k,2}$ for any fixed $u_k \in E$ by

$$T_{k,1} = \left\{ B \in \binom{U_{q+1}}{7} : E \subseteq B, \sigma_{6,3}(B \setminus \{u_k\}) = 0 \right\},$$

and

$$T_{k,2} = \left\{ B \in \binom{U_{q+1}}{7} : E \subseteq B, \sigma_{5,2}(B \setminus \{u_k,u_j\}) = 0 \text{ for some } u_j \in B \setminus E \right\}.$$  

Then $T_{k,2} \subseteq T_{k,1}$ by Lemma 3.2. Let $T_k = T_{k,1} \setminus T_{k,2}$. Clearly, $B_4 = T_1 \cup T_2 \cup T_3$.

It is obvious that $B \in T_{k,1}$ if and only if $B = B' \cup \{u_k\}$ for some $B' \in B_{\sigma_{6,3},q+1}$ such that $E \cap B' = E \setminus \{u_k\}$. From Lemma 2.3 and Eq. (11), $E$ is contained in $\frac{(q-4)^2}{6}$ blocks of $B_{\sigma_{6,3},q+1}$ and $E \setminus \{u_k\}$ is contained in $\lambda_2 = \frac{(q-4)^2 \cdot q-1}{4} = \frac{(q-4)^2(q-1)}{24}$ blocks of $B_{\sigma_{6,3},q+1}$. It follows that

$$|T_{k,1}| = \lambda_2 - \frac{(q-4)^2}{24}$$

as $u_k \not\in B'$.

Clearly, $B \in T_{k,2}$ if and only if $B = B' \cup \{u_k,u_j\}$ for some $B' \in B_{\sigma_{5,2},q+1}$ with $E \cap B' = E \setminus \{u_k\}$ and $u_j \in U_{q+1} \setminus (B' \cup \{u_k\})$. From Lemma 2.3 and (11), we have $\frac{q^3-1-2}{3} - \frac{q-4}{3}$ choices of $B'$ and $|T_{k,2}| = \frac{(q-4)(q-5)}{48}$ by noting Corollary 2.7 (1). Hence,

$$|T_k| = |T_{k,1}| - |T_{k,2}| = \frac{(q-4)(q-5)(q-12)}{24}.$$  

(15)

Let $\{u_i,u_j\} \subseteq E$ be fixed and define

$$A_{ij} = \left\{ B \in \binom{U_{q+1}}{7} : E \subseteq B, \sigma_{5,2}(B \setminus \{u_i,u_j\}) = 0 \right\}.$$
Now we prove $\mathcal{T}_i \cap \mathcal{T}_j = A_{ij}$. For any $B \in A_{ij}$, we have $\sigma_{5,2}(B \setminus \{u_i, u_j\}) = 0$. From Lemma 3.2, $\sigma_{6,3}(B \setminus \{u_i\}) = 0$ and $\sigma_{6,3}(B \setminus \{u_j\}) = 0$. Since $|(B \setminus \{u_i, u_j\}) \cap (B \setminus \{u_i, u_k\})| = 4$ for any $u_k \in U_{q+1} \setminus E$, we have $\sigma_{5,2}(B \setminus \{u_i, u_k\}) \neq 0$ by Corollary 2.7 (1) as $\sigma_{5,2}(B \setminus \{u_i, u_j\}) = 0$. Thus $B \in \mathcal{T}_i$. Similarly, we also have $B \in \mathcal{T}_j$ and thus $A_{ij} \subseteq \mathcal{T}_i \cap \mathcal{T}_j$. On the other hand, for any $B \in \mathcal{T}_i \cap \mathcal{T}_j$, we have $\sigma_{6,3}(B \setminus \{u_i\}) = 0$, $\sigma_{6,3}(B \setminus \{u_j\}) = 0$ and $|(B \setminus \{u_i\}) \cap (B \setminus \{u_j\})| = 5$. So $\sigma_{5,2}(B \setminus \{u_i, u_j\}) = 0$ from Corollary 2.7 (2). Then $B \in A_{ij}$. Clearly $\mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{T}_3 = \emptyset$.

Now we calculate the size of $A_{ij}$ ($1 \leq i < j \leq 3$). From Lemma 2.3, there is a unique block of $B_{\sigma_{5,2}, q+1}$ containing $E$. By Lemma 2.3 and Eq. (1), there are $\lambda_1 = \frac{q+1-1}{(q-1)(q-3)} = \frac{q-1}{12}$ blocks of $B_{\sigma_{5,2}, q+1}$ containing any fixed point of $E$ and $\lambda_2 = \frac{q-1}{3}$ blocks of $B_{\sigma_{5,2}, q+1}$ containing any fixed pair of $E$. Applying Corollary 2.7 (1) and the principle of inclusion-exclusion,

$$|A_{ij}| = \lambda_1 - 2\lambda_2 + 1 = \frac{(q-4)(q-5)}{12}. \quad (16)$$

Since $B_4 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, $\mathcal{T}_i \cap \mathcal{T}_j = A_{ij}$, and $\mathcal{T}_i \cap \mathcal{T}_2 \cap \mathcal{T}_3 = \emptyset$, using (15) and (16) gives

$$|B_4| = \sum_{k=1}^{3} |\mathcal{T}_k| - \sum_{1 \leq i < j \leq 3} |A_{ij}| = \frac{(q-4)(q-5)(q-14)}{8}. \quad (17)$$

Combining (13), (14) and (17) yields that $E$ is contained in $|\mathcal{B}_1| + |\mathcal{B}_3| + |\mathcal{B}_4| = \frac{7(q-4)(q-5)(q-10)}{24}$ blocks of $\mathcal{B}_{\sigma_{7,3}, q+1}$. Consequently, $(U_{q+1}, \mathcal{B}_{\sigma_{7,3}, q+1}^u)$ is a 3-$(q + 1, 7, \frac{7(q-4)(q-5)(q-10)}{24})$ design. \hfill $\square$

From Theorems 3.8 and 3.9 one gets

$$|\mathcal{B}_{\sigma_{7,3}, q+1}^u| = \begin{cases} \frac{(q-4)(q-5)(q-10)}{3} \binom{q+1}{3}, & \text{if } q = 2^{2s}, \\ \frac{(q-5)(q-8)}{30} \frac{(q+1)}{4}, & \text{if } q = 2^{2s+1}, \end{cases} \quad (18)$$

In general, it is difficult to determine $|\mathcal{B}_{\sigma_{k,l}, q+1}^u|$. It would be interesting to settle the following problem.

**Problem 3.10.** Determine the cardinality of $\mathcal{B}_{\sigma_{k,l}, q+1}^u$ for $(k, l) \neq (4, 2), (5, 2), (5, 3), (6, 3), (7, 3)$.

**Theorem 3.11.** Let $q = 2^m$ and $m \geq 4$ be even. Then $(U_{q+1}, \mathcal{B}_{\sigma_{7,3}, q+1}^0)$ forms a 3-$(q + 1, 7, \frac{7(q-5)(q-4)}{4})$ design, where the block set $\mathcal{B}_{\sigma_{7,3}, q+1}^0$ is given by

$$\mathcal{B}_{\sigma_{7,3}, q+1}^0 = \left\{ B \in \left( U_{q+1} \right)^2 : \sigma_{5,2}(B \setminus \{u_i, u_j\}) = 0 \text{ for some } u_i \neq u_j \in B \right\}.$$

**Proof.** Let $E = \{u_1, u_2, u_3\}$ be a fixed 3-subset of $U_{q+1}$ and let $B$ be any block of $\mathcal{B}_{\sigma_{7,3}, q+1}^0$ containing $E$. Then we have three possibilities for $B$.

Case 1. $\sigma_{5,2}(B \setminus \{u_i, u_j\}) = 0$ for some $u_i, u_j \in B \setminus E$ ($u_i \neq u_j$). Denote the collection of such $B$ by $\mathcal{B}_1$. Clearly $B \in \mathcal{B}_1$ if and only if $B = A \cup \{u_i, u_j\}$ for $A \in \mathcal{B}_{\sigma_{5,2}, q+1}$ with $E \subseteq A$ and $\{u_i, u_j\} \subseteq (U_{q+1} \setminus A)$. By Lemma 2.3 and Corollary 2.7 (1), we have that $\mathcal{B}_1$ is simple and $|\mathcal{B}_1| = \frac{(q-4)(q-5)}{2}$.

Case 2. $\sigma_{5,2}(B \setminus \{u_k, u_j\}) = 0$ for some $u_k \in E$ and $u_j \in B \setminus E$. Denote the collection of such $B$ by $\mathcal{B}_2$. Clearly $B \in \mathcal{B}_2$ if and only if $B = A \cup \{u_k, u_j\}$ for $A \in \mathcal{B}_{\sigma_{5,2}, q+1}$ with $E \cap A = E \setminus \{u_k\}$.
and $u_j \in U_{q+1} \setminus (A \cup \{u_k\})$. By Lemma 2.3 and Eq. (1), we have $\lambda_2 - 1 = \frac{q+1}{3} - 1 = \frac{q-4}{3}$ choices of $A$ as $u_k \notin A$. Then $B_2$ is simple and $|B_2| = 3 \cdot \frac{q-4}{3} \cdot (q+1-6) = (q-4)(q-5)$ from Corollary 2.7 (1).

Case 3. $\sigma_5,2(B \setminus \{u_k, u_l\}) = 0$ for some $u_k, u_l \in E$ ($u_k \neq u_l$). Denote the collection of such $B$ by $B_3$. Clearly $B \in B_3$ if and only if $B = A \cup \{u_k, u_l\}$ for $A \in B_{\sigma_5,2,q+1}$ with $E \cap A = E \setminus \{u_k, u_l\}$.

By Lemma 2.3 and Eq. (1), we have $\lambda_1 - 2\lambda_2 + 1 = \frac{(q+1-1)}{3} - 2 \cdot \frac{q-1}{3} + 1 = \frac{(q-4)(q-5)}{12}$ choices of $A$ as $\{u_k, u_l\} \subseteq A$. Then $B_3$ is simple and $|B_3| = 3 \cdot \frac{(q-4)(q-5)}{12} = \frac{(q-4)(q-5)}{4}$ from Corollary 2.7 (1).

Now we prove $B_1 \cap B_2 = 0$. Otherwise, there is $B \in B_1 \cap B_2$. Then $\sigma_5,2(B \setminus \{u_i, u_j\}) = 0$ and $\sigma_5,2(B \setminus \{u_k, u_l\}) = 0$ for $u_i, u_j, u_k, u_l \in B \setminus E$ and $u_k \in E$. Note that $|\{B \setminus \{u_i, u_j\} \cap (B \setminus \{u_k, u_l\})| = 3$ or 4, which contradicts Corollary 2.7 (1). Then $B_1$ and $B_2$ are disjoint. Similarly, $B_1, B_2$ and $B_3$ are pairwise disjoint. So $E$ is contained in $|B_1| + |B_2| + |B_3| = \frac{7(q-4)(q-5)}{4}$ blocks of $B_{\sigma_5,3,q+1}$ and $U_{q+1}, B_{\sigma_5,3,q+1}$ is a $3-(q+1, 7, \frac{7(q-4)(q-5)}{4})$ design.

\section{BCH codes supporting $t$-designs}

In this section, we consider the codewords of weight 7 in the narrow-sense BCH codes $C_{(q,q+1,4,1)}$ over $\text{GF}(q)$ where $q = 2^m$. We will prove that $B_5(C_{(q,q+1,4,1)})$ supports a 4-design when $m \geq 5$ odd and it supports a 3-design when $m \geq 4$ is even.

\textbf{Lemma 4.1.} [19, Theorems 34, 35] Let $q = 2^m$ and $m \geq 4$. Then $C_{(q,q+1,4,1)}$ has parameters $[q+1,q-5,d]$ with $d = 6$ if $m$ is odd and $d = 5$ if $m$ is even. The dual $C_{(q,q+1,4,1)}^\perp$ has parameters $[q+1, 6, q-5]$. In particular, $C_{(q,q+1,4,1)}$ is an NMDS code if $m$ is odd and $A^2$MDS code if $m$ is even.

We display here the connections between $B_k(C_{(q,q+1,4,1)})$ and $B_{\sigma_{k,1,q+1}}$ for $k \in \{5, 6, 7\}$.

\textbf{Lemma 4.2.} Let $q = 2^m$ with $m \geq 4$ and denote $C := C_{(q,q+1,4,1)}$. If we index the coordinates of the codewords(245,605),(933,885) with the elements in $U_{q+1}$, then we may have the following.

(i) For odd $m$, $B_6(C) = B_{\sigma_6,3,q+1}$.

(ii) For even $m$, $B_5(C) = B_{\sigma_5,2,q+1}$ and $B_6(C) = B_{\sigma_6,3,q+1} \setminus B_0(\sigma_6,3,q+1)$ (see (12)).

(iii) $B_7(C) \subseteq \overline{B}_{\sigma_7,3,q+1}$, where $\overline{B}_{\sigma_7,3,q+1} = (U_{q+1}) \setminus B_{\sigma_7,3,q+1}$.

\textbf{Proof.} The first two assertions follow immediately from [19, Theorems 36, 39] and their proofs. Now we prove (iii). We denote by $\text{wt}(c)$ and $\text{Supp}(c)$ the weight and the support of a codeword $c \in C$, respectively.

For any $c \in C$ with $\text{wt}(c) = 7$, denote $\text{Supp}(c) = B = \{u_1, u_2, \ldots, u_7\}$. Then $B \in B_7(C)$. We will prove that $B \in \overline{B}_{\sigma_7,3,q+1}$ by contradiction. Suppose on the contrary that $B \in B_{\sigma_7,3,q+1}$. Then there is $u_i \in B$ such that $\sigma_{6,3}(B \setminus \{u_i\}) = 0$ from Lemma 3.7. W.l.o.g. let $\sigma_{6,3}(B \setminus \{u_7\}) = 0$. Contradictions will be derived by considering the following two cases.

Case 1. Let $m$ be odd. The assertion (i) shows that there is $c_1 \in C$ such that $\text{Supp}(c_1) = B \setminus \{u_7\}$ since $B_6(C) = B_{\sigma_6,3,q+1}$. Doing a linear combination of $c$ and $c_1$ gives a codeword
\(c_2 \in C\) such that \(\text{Supp}(c_2) = B \setminus \{u_1\}\) and \(\text{wt}(c_2) = 6\) as the minimum weight of \(C\) is 6 by Lemma 4.1. Note that \(\sigma_{6,3}(B \setminus \{u_1\}) = \sigma_{6,3}(B \setminus \{u_7\}) = 0\) and \(|(B \setminus \{u_1\}) \cap (B \setminus \{u_7\})| = 5\), contradicting Corollary 2.7 (3).

Case 2. Let \(m\) be even. We begin with a claim.

**Claim:** There does not exist \(c_1 \in C\) such that \(\text{Supp}(c_1) = B \setminus \{u_7\}\).

If the claim is not true, then let \(c_1 \in C\) and \(\text{Supp}(c_1) = B \setminus \{u_7\}\). A linear combination of \(c\) and \(c_1\) gives a codeword \(c_2 \in C\) such that \(u_7 \in \text{Supp}(c_2) \subseteq B \setminus \{u_1\}\) and \(\text{wt}(c_2) = 5, 6\) as the minimum weight of \(C\) is 5 by Lemma 4.1. If \(\text{wt}(c_2) = 6\), then \(\text{Supp}(c_2) = B \setminus \{u_1\}\) (similarly to Case 1) and \(\sigma_{6,3}(B \setminus \{u_1\}) = \sigma_{6,3}(B \setminus \{u_7\}) = 0\) and thus \(\sigma_{5,2}(B \setminus \{u_1, u_7\}) = 0\) by Corollary 2.7 (2), contradicting \(c_2 \in B_0(C) = B_{\sigma_{6,3,q+1}} \setminus B_{\sigma_{6,3,q+1}}^0\) by (ii). So we must have \(\text{wt}(c_2) = 5\). Suppose \(\text{Supp}(c_2) = B \setminus \{u_1, u_2\}\). So \(\sigma_{5,2}(B \setminus \{u_1, u_2\}) = 0\) by (ii). It is immediate that \(\sigma_{6,3}(B \setminus \{u_1\}) = \sigma_{6,3}(B \setminus \{u_2\}) = 0\) by Lemma 3.2. Since we also have \(\sigma_{6,3}(B \setminus \{u_7\}) = 0\), combining together and also noting \((B \setminus \{u_i\}) \cap (B \setminus \{u_j\}) = 5\) \((i \neq j \in \{1, 2, 7\})\) yields from Corollary 2.7 (2) that \(\sigma_{5,2}(B \setminus \{u_1, u_7\}) = \sigma_{5,2}(B \setminus \{u_2, u_7\}) = 0\), but this contradicts Corollary 2.7 (1). This completes the proof of the claim.

Applying the claim we know that \(B \setminus \{u_7\}\) does not support any codeword of \(C\). Since \(\sigma_{6,3}(B \setminus \{u_7\}) = 0\), there exists a 5-subset \(A \subseteq B \setminus \{u_7\}\) such that \(\sigma_{5,2}(A) = 0\) by (ii), say, \(A = \{u_1, u_2, u_3, u_4, u_5\}\). Again from (ii), there is \(c_1 \in C\) such that \(\text{Supp}(c_1) = A\). A linear combination of \(c\) and \(c_1\) gives a codeword \(c_2 \in C\) such that \(\{u_6, u_7\} \subseteq \text{Supp}(c_2) \subseteq B \setminus \{u_1\}\) and \(\text{wt}(c_2) = 5, 6\) as the minimum weight of \(C\) is 5 by Lemma 4.1. We have \(\text{wt}(c_2) = 6\) because if \(\text{wt}(c_2) = 5\) then \(\sigma_{5,2}(\text{Supp}(c_1)) = \sigma_{5,2}(\text{Supp}(c_2)) = 0\) by (ii) and \(|\text{Supp}(c_1) \cap \text{Supp}(c_2)| = 3\), contradicting Corollary 2.7 (1). Thus \(|\text{Supp}(c_2)| = 6\) and \(\text{Supp}(c_2) = B \setminus \{u_1\}\). By assertion (ii), \(\sigma_{6,3}(B \setminus \{u_1\}) = 0\). So we have \(\sigma_{5,2}(B \setminus \{u_1, u_7\}) = 0\) by Corollary 2.7 (2) as \(\sigma_{6,3}(B \setminus \{u_7\}) = 0\). However, this contradicts that \(c_2 \in B_0(C) = B_{\sigma_{6,3,q+1}} \setminus B_{\sigma_{6,3,q+1}}^0\) by (ii).

**Lemma 4.3.** Let \(q = 2^m\) with \(m \geq 4\) and \(C\) be the narrow-sense BCH code \(C_{(q,q+1,4,1)}\) over \(\text{GF}(q)\) with the minimum weight \(d\). Then \(|B_i(C)| = \frac{A_i}{d^i}\) for \(d \leq i \leq 7\), where \(A_i\) denotes the number of codewords with weight \(i\) in \(C\).

**Proof.** From Lemma 4.1, we have \(d = 5\) if \(m\) is even and \(d = 6\) if \(m\) is odd. We only need to prove that if \(c \in C\) and \(c_1 \in C\) are two codewords of weight \(i\) \((d \leq i \leq 7)\) with \(\text{Supp}(c) = \text{Supp}(c_1)\), then \(c = \alpha c_1\) for some nonzero \(\alpha \in \text{GF}(q)\). The proof will proceed based on the notation and three assertions of Lemma 4.2.

Assume that \(c\) and \(c_1\) are two different codewords of \(C\) such that \(\text{wt}(c) = \text{wt}(c_1) = i\) \((d \leq i \leq 7)\), \(\text{Supp}(c) = \text{Supp}(c_1)\), but \(c\) is not a multiple of \(c_1\). Then we can find a nonzero element \(\beta \in \text{GF}(q)\) such that \(c_2 = c - \beta c_1 \in C, \text{Supp}(c_2) \subseteq \text{Supp}(c)\) and \(\text{wt}(c_2) \leq i - 1\). This clearly derives a contradiction if \(i = d\). So we let \(i \geq d + 1\) next.

If \(i = 6\) then we only need to let \(m\) be even by Lemma 4.1. Clearly \(\text{wt}(c_2) = 5\) as the minimum distance \(d = 5\). By Lemma 4.2 (ii), \(\text{Supp}(c_2) \subseteq B_{\sigma_{5,2,q+1}}\). Since \(\text{Supp}(c_2) \subseteq \text{Supp}(c)\), one has \(\text{Supp}(c) \in B_{\sigma_{6,3,q+1}}^0\). But this contradicts \(B_0(C) = B_{\sigma_{6,3,q+1}} \setminus B_{\sigma_{6,3,q+1}}^0\).

Now we deal with the only remaining case of \(i = 7\). One has \(\text{wt}(c_2) = 5, 6\) and if \(\text{wt}(c_2) = 5\) then \(m\) must be even by Lemma 4.1. If \(\text{wt}(c_2) = 6\), then from Lemma 4.2 (i) and (ii), \(\sigma_{6,3}(\text{Supp}(c_2)) = 0\). If \(\text{wt}(c_2) = 5\) and \(m\) is even, then \(\sigma_{5,2}(\text{Supp}(c_2)) = 0\) by Lemma 4.2 (ii). Thus \(\sigma_{6,3}(B) = 0\) for any 6-subset \(B\) with \(\text{Supp}(c_2) \subseteq B \subseteq \text{Supp}(c)\) by Lemma 3.2. As a result, in either case, according to Lemma 2.7 one has \(\text{Supp}(c) \in B_{\sigma_{6,3,q+1}}^0\), which is contrary
to Lemma 4.2 (iii). This completes the proof.

We have the weight distribution formula for $A^s$-MDS codes as follows.

**Lemma 4.4.** [11] Theorem 9 Let $C$ be an $[n, k, d]$ $A^s$-MDS code over $GF(q)$ with $s \geq 1$ and let the dual code $C^\perp$ be an $[n, k, d^\perp]$ $A^s$-MDS code. Then the weight distribution $(A_0, A_1, \ldots, A_n)$ of $C$ satisfies

$$A_{n-d^\perp+r} = \sum_{j=d^\perp}^{n-d} \binom{j}{d^\perp-r} \left( \sum_{i=d^\perp}^{j} (-1)^{i-d^\perp+r} \binom{j-i}{j-i} \right) A_{n-j}$$

\begin{equation}
+ \left( \sum_{i=0}^{n-d^\perp-r} (-1)^{n-d^\perp+r-i} \binom{n}{i} A_{n-i} \right) (q^{k-d^\perp+r-i-1})
\end{equation}

for $r = 1, 2, \ldots, d^\perp$. In particular, $A_{d_1}, \ldots, A_{n-d^\perp}$ determine the weight distribution of $C$ completely.

In particular, we have the following weight distribution formula for near MDS codes.

**Lemma 4.5.** [11] Theorem 4.1 Let $C$ be an $[n, k, n-k]$ near MDS code over $GF(q)$. Then the weight distribution $(A_0, A_1, \ldots, A_n)$ of $C$ is given by

$$A_{n-k+s} = \binom{n}{k-s} \sum_{j=0}^{s-1} (-1)^j \binom{n-k+s}{j} \binom{q^s-j-1}{s} + (-1)^s \binom{k}{s} A_{n-k}$$

\begin{equation}
(20)
\end{equation}

for $s \in \{1, 2, \ldots, k\}$.

**Lemma 4.6.** For $q = 2^m$ with odd $m \geq 5$, the incidence structure

$$(\mathcal{P}(C_{(q,q+1,4,1)}), \mathcal{B}_7(C_{(q,q+1,4,1)}))$$

is isomorphic to the complementary design of $(U_{q+1}, B^u_{\sigma_{7,3},q+1})$ with block set $\overline{B}^u_{\sigma_{7,3},q+1}$.

**Proof.** From Lemma 4.2 if we index the coordinates of the codewords in $C := C_{(q,q+1,4,1)}$ with the elements in $U_{q+1}$, then we may let $B_0(C) = B_{\sigma_{6,3},q+1}$ and $B_7(C) \subseteq \overline{B}^u_{\sigma_{7,3},q+1}$. So we only need to prove that

$$|B_7(C)| = |\overline{B}^u_{\sigma_{7,3},q+1}|.$$

Since $C_{(q,q+1,4,1)}$ is a near MDS code with parameters $[q+1, q-5, 6]$ if $m$ is odd by Lemma 4.1 according to Eq. (20) one has

$$A_7 = -(q-5)A_6 + \left( \frac{q+1}{7} \right) (q-1).$$

From Lemmas 4.3 and 2.4

$$A_6 = |B_0(C)|(q-1) = \frac{(q-8)(q-1)(q+1)}{30}.$$ 

Then

$$\left( \frac{q+1}{7} \right) - |B_7(C)| = \left( \frac{q+1}{7} \right) - \frac{A_7}{q-1} = \frac{(q-5)(q-8)(q+1)}{30},$$

which is the same as $|\overline{B}^u_{\sigma_{7,3},q+1}|$ for odd $m$ from (18). This completes the proof.
Theorem 4.7. For $q = 2^m$ with odd $m \geq 5$, the codewords of weight 7 in $C_{(q,q+1,4,1)}$ support a 4-$(q+1, 7, \lambda)$ design where
\[
\lambda = \left( \frac{q - 3}{3} \right) - \frac{7(q - 5)(q - 8)}{6}.
\]

Proof. The desired conclusion follows from Lemma 4.6 and Eq. (3).

Lemma 4.8. For $q = 2^m$ with even $m \geq 4$, the incidence structure
\[
(P(C_{(q,q+1,4,1)}), B_7(C_{(q,q+1,4,1)}))
\]
is isomorphic to the complementary design of $(U_{q+1}, B_{q,\sigma_{7,3},q+1})$ with block set $\overline{B}_{\sigma_{7,3},q+1}$.

Proof. Similarly to the proof of Lemma 4.6, we only need to prove that $|B_7(C_{(q,q+1,4,1)}))| = |\overline{B}_{\sigma_{7,3},q+1}|$ by Lemma 4.12. Since $C_{(q,q+1,4,1)}$ is an A$^2$ MDS code with parameters $[q+1, q-5, 5]$ if $m$ is even by Lemma 4.11, then according to Eq. (19) one has
\[
A_7 = \binom{q + 1}{7}(q - 1) - (q - 5)A_6 - \frac{(q - 4)(q - 5)}{2}A_5.
\]
From Lemmas 4.3, 2.3 and 2.8, one has
\[
\begin{align*}
A_6 &= |B_6(C_{(q,q+1,4,1)}))|(q - 1) = \frac{(q - 4)(q - 6)(q - 1)}{120}(q + 1), \\
A_5 &= |B_5(C_{(q,q+1,4,1)}))|(q - 1) = \frac{(q - 1)}{10}(q + 1)^3.
\end{align*}
\]
Then
\[
\left( \frac{q + 1}{7} \right) - |B_7(C_{(q,q+1,4,1)}))| = \left( \frac{q + 1}{7} \right) - \frac{A_7}{q - 1} = \frac{(q - 4)(q - 5)(q - 10)}{120} \left( \frac{q + 1}{3} \right),
\]
which is the same as $|\overline{B}_{\sigma_{7,3},q+1}|$ for even $m$ from (18). This completes the proof.

Theorem 4.9. For $q = 2^m$ with even $m \geq 4$, the codewords of weight 7 in $C_{(q,q+1,4,1)}$ support a 3-$(q+1, 7, \lambda)$ design where
\[
\lambda = \left( \frac{q - 2}{4} \right) - \frac{7(q - 4)(q - 5)(q - 10)}{24}.
\]

Proof. The desired conclusion follows from Lemma 4.8 and Eq. (3).

Denote by $C$ the narrow-sense BCH code $C_{(q,q+1,4,1)}$ over GF$(q)$. From what has been investigated up to now, we have Tables 1 and 2 for $t$-designs supported by the codewords of a fixed weight in $C$.

| Block sets | Designs |
|------------|---------|
| $B_5(C)$ $\cong B_{\sigma_{5,2},q+1}$ | 3-$(q + 1, 5, 1)$ |
| $B_6(C)$ $\cong B_{\sigma_{6,3},q+1}$ $\setminus B_{\sigma_{6,3},q+1}^0$ | 3-$(q + 1, 6, \frac{(q - 4)(q - 16)}{6})$ |
| $B_7(C)$ $\cong \binom{U_{q+1}}{7}$ $\setminus B_{\sigma_{7,3},q+1}^0$ | 3-$(q + 1, 7, \frac{(q - 2)^4}{24}, \frac{7(q - 4)(q - 5)(q - 10)}{24})$ |

Table 1: Known designs supported by $C_{(q,q+1,4,1)}$ with $q = 2^{2s}$

According to Magma [5] experiments, we have the following two examples.
This section is divided into three subsections. In subsection 5.1 we define trace codes and use $\text{Tr}_{q^2/q}(\mathcal{C}_{(1,2,3)})$ to represent the dual code $\mathcal{C}_{(q,q+1,4,1)}^\perp$. We show that the trace code $\text{Tr}_{q^2/q}(\mathcal{C}_{(1,2,3)})$ supports the supplementary design of $(U_{q+1}, B_{24,3,q+1}^6)$ with parameters $4-(q + 1, 5, 5)$ where $q = 2^{2s+1}$. In subsection 5.2 we prove that the set of supports of a fixed weight in $\text{Tr}_{q^2/q}(\mathcal{C}_{(1,2,3)})$ is invariant under the action of a 3-transitive group, which is isomorphic to $\text{PGL}(2,q)$. In the last subsection we employ the known information on the permutation character of $\text{PGL}(2,q)$ to prove that the $4-(q+1,5,5)$ designs produced from $\text{Tr}_{q^2/q}(\mathcal{C}_{(1,2,3)})$ are isomorphic to the designs with the same parameters constructed by Alltop [1] in 1969.

### 5.1 $\text{Tr}_{q^2/q}(\mathcal{C}_{(1,2,3)})$ and $4-(q+1,5,5)$ design

Let $\mathcal{C}$ be a code of length $n$ over $\text{GF}(q^m)$. Then we call $\mathcal{C} \cap \text{GF}(q)^n$ the subfield subcode over $\text{GF}(q)$ and usually denote it by $\mathcal{C}|_{\text{GF}(q)}$. The trace code of $\mathcal{C}$ is defined by

$$\text{Tr}_{q^m/q}(\mathcal{C}) = \{(\text{Tr}_{q^m/q}(c_0), \text{Tr}_{q^m/q}(c_1), \ldots, \text{Tr}_{q^m/q}(c_{n-1})) : (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}\},$$

where $\text{Tr}_{q^m/q}$ represents the trace function from $\text{GF}(q^m)$ to $\text{GF}(q)$. Delsarte [3] stated that

$$(\text{Tr}_{q^m/q}(\mathcal{C}))^\perp = \mathcal{C}^\perp|_{\text{GF}(q)}.$$
Ding et al. [10] gave a cyclic code over GF(q²) of length q + 1 by defining
\[ C_{3,5} = \{(a_3u^3 + a_2u^2 + a_1u + a_0) | u \in U_{q+1} : a_3, a_2, a_1, a_0 \in GF(q^2)\}. \]

Thus the trace code of \( C_{3,5} \) is given by
\[ \text{Tr}_{q^2/q}(C_{3,5}) = \{(\text{Tr}_{q^2/q}(au^3 + bu^5)) | u \in U_{q+1} : a, b \in GF(q^2)\}. \]

Similarly, we define another cyclic code and its trace code by
\[ C_{1,2,3} = \left\{ (a_1u + a_2u^2 + a_3u^3) | u \in U_{q+1} : a_1, a_2, a_3 \in GF(q^2) \right\}, \]
\[ \text{Tr}_{q^2/q}(C_{1,2,3}) = \{(\text{Tr}_{q^2/q}(au + bu^2 + cu^3)) | u \in U_{q+1} : a, b, c \in GF(q^2)\}. \]

It is immediate that if \((c_u)_{u \in U_{q+1}} \in C_{1,2,3}\), then \((c_u^q)_{u \in U_{q+1}} \in C_{1,2,3}\). From [13] Lemma 7 we derive that
\[ \text{Tr}_{q^2/q}(C_{1,2,3}) = C_{1,2,3} |_{GF(q)} \]
and
\[ \text{Tr}_{q^2/q}(C_{1,2,3}^\perp) = C_{1,2,3}^\perp |_{GF(q)}. \]

Recall the proof of [19] Theorem 34, where the trace expression of \( C_{q,q+1,4,1} \) was given. Next combining (21) and (22) yields that we may identify the codes \( C_{q,q+1,4,1} \), \( \text{Tr}_{q^2/q}(C_{1,2,3}) \) and \( C_{1,2,3} |_{GF(q)} \) and they have parameters \([q + 1, 6, q - 5]\) by Lemma 3.1

Lemma 5.1. Let \( f(u) = \text{Tr}_{q^2/q}(au + bu^2 + cu^3) \) where \((a, b, c) \in GF(q^2)^3 \setminus \{0\}\). Define \( \text{zero}(f) = \{u \in U_{q+1} : f(u) = 0\} \). Then \( |\text{zero}(f)| \leq 6 \) and we have the following.

1. \( |\text{zero}(f)| = 6 \) if and only if \( a = \frac{\tau_{\sigma_{5,2}(B)}}{\sqrt{\sigma_{6,6}(B)}}, b = \frac{\tau_{\sigma_{6,1}(B)}}{\sqrt{\sigma_{6,6}(B)}} \) and \( c = \frac{\tau}{\sqrt{\sigma_{6,6}(B)}} \), where \( B \in B_{\sigma_{6,3},q+1} \) and \( \tau \in GF(q)^* \).

2. \( |\text{zero}(f)| = 5 \) if and only if there is \( B \in B_{\sigma_{5,3},q+1} \) and \( u_i \in B \) such that \( a = \frac{\tau_{(\sigma_{5,2}(B)+u_i\sigma_{5,1}(B))}}{\sqrt{u_i\sigma_{5,5}(B)}}, b = \frac{\tau_{\sigma_{5,1}(B)+u_i}}{\sqrt{u_i\sigma_{5,5}(B)}}, c = \frac{\tau}{\sqrt{u_i\sigma_{5,5}(B)}} \), where \( \tau \in GF(q)^* \).

Proof. When \( u \in U_{q+1} \) one has
\[ f(u) = au + bu^2 + cu^3 + a^2u^{-1} + b^2u^{-2} + c^2u^{-3}. \]
If \( c = 0 \), then \( f(u) = au + bu^2 + a^2u^{-1} + b^2u^{-2} = \frac{1}{u^2}(bu^4 + au^3 + a^2u + b^3) \). This shows that \( \text{zero}(f) \leq 4 \). Next let \( c \neq 0 \). Then,
\[ f(u) = \frac{1}{u^3}(cu^6 + bu^5 + au^4 + a^2u^2 + b^2u + c^3). \]
Hence, \( |\text{zero}(f)| \leq 6 \).

(1) The proof can be found in [19] Lemma 33.

(2) Assume that \( |\text{zero}(f)| = 5 \) and let \( B = \{u_1, u_2, \ldots, u_5\} \in \binom{U_{q+1}}{5} \) be the set of the five roots of \( f \) in \( U_{q+1} \). Then we have another root \( \alpha \) in the splitting field of \( f \). So \( f(u) = \frac{c(u+\alpha)\prod_{i=1}^{5}(u+u_i)}{u^3} \). By Vieta’s formula, \( c\sigma_{5,5}(B) = c^3, c(\sigma_{5,5}(B) + \alpha\sigma_{5,4}(B)) = b^3, \)
\[\alpha\sigma_{5,3}(B) = \alpha^q, \quad c(\sigma_{5,3}(B) + \alpha\sigma_{5,2}(B)) = 0, \quad c(\sigma_{5,2}(B) + \alpha\sigma_{5,1}(B)) = a, \quad c(\sigma_{5,1}(B) + \alpha) = b.\]

One obtains \(\alpha = \frac{\sigma_{5,1}(B)}{\sigma_{5,5}(B)}\) from \(\alpha\sigma_{5,5}(B) = \alpha^q - 1\). Note that \(\alpha^{q+1} = \frac{\sigma_{5,5}(B)^{q+1}}{\sigma_{5,5}(B)^q} = 1\) as \(\sigma_{5,5}(B) \in U_{q+1}\).

Then \(\alpha \in U_{q+1}\). This shows that \(\alpha = u_i\) is a double root of \(f\) in \(U_{q+1}\) for some \(u_i \in B\). One also obtains \(B \in B_{\sigma_{5,3},q+1}^b\) from \(\sigma_{5,3}(B)_+ u_i\sigma_{5,2}(B) = 0\) and \(c = \frac{\tau_\gamma}{u_i\sigma_{5,5}(B)}\) from \(u_i\sigma_{5,5}(B) = \alpha^q - 1\), where \(\tau \in GF(q)^*\). Then \(a = \frac{\tau(\sigma_{5,2}(B) + u_i\sigma_{5,1}(B))}{u_i\sigma_{5,5}(B)}\) and \(b = \frac{\tau(\sigma_{5,1}(B) + u_i\sigma_{5,5}(B))}{u_i\sigma_{5,5}(B)}\).

Conversely, assume that \(B = \{u_1, u_2, \ldots, u_5\} \in B_{\sigma_{5,3},q+1}^b\) and there is \(u_i \in B\) such that \(a = \frac{\tau(\sigma_{5,2}(B) + u_i\sigma_{5,1}(B))}{u_i\sigma_{5,5}(B)}\), \(b = \frac{\tau(\sigma_{5,1}(B) + u_i)}{u_i\sigma_{5,5}(B)}\) and \(c = \frac{\tau}{u_i\sigma_{5,5}(B)}\), where \(\tau \in GF(q)^*\). Then \(f(u) = \frac{c(u + u_i)}{\prod_{u \in B} (u + u_i)}\). Thus, \(\text{zero}(f) = B\) and \(|\text{zero}(f)| = 5\).

**Theorem 5.2.** Let \(q = 2^m\) with odd \(m \geq 5\). Then the incidence structure

\[\Delta = (U_{q+1}, B_q - 4(\text{Tr}_{q^2/q}(C_{1,2,3})))\]

forms a 4-\((q+1,q-4,\binom{q-4}{4})\) design and its supplementary design is isomorphic to \((U_{q+1}, B_{\sigma_{5,3},q+1}^b)\), which forms a 4-\((q+1,5,5)\) design.

**Proof.** Lemma \([5,1](2)\) shows that the incidence structure \((U_{q+1}, B_q - 4(\text{Tr}_{q^2/q}(C_{1,2,3})))\) is isomorphic to the supplementary design of \((U_{q+1}, B_{\sigma_{5,3},q+1}^b)\), which is a 4-\((q+1,5,5)\) design from Theorem \([5,3](1)\). According to Eq. \([2]\), \((U_{q+1}, B_q - 4(\text{Tr}_{q^2/q}(C_{1,2,3})))\) forms a 4-\((q+1,q-4,\lambda)\) design, where

\[\lambda = \frac{\binom{q+1}{5} \binom{q-4}{4}}{\binom{q+4}{5}} = \frac{q-4}{4}.\]

This completes the proof.

According to Magma \([5]\) experiments, we have the following example.

**Example 5.3.** Let \(q = 2^5\). Then the trace code \(\text{Tr}_{q^2/q}(C_{1,2,3})\) has parameters \([33,6,27]\) and its weight distribution equals

\[1 + 1014816z^{27} + 1268520z^{28} + 20296320z^{29} + 64609952z^{30} + 210132384z^{31} + 399584823z^{32} + 376835008z^{33}.\]

The codewords of weight 27 support a 4-\((33,27,14040)\) design, and the codewords of weight 28 support a 4-\((33,28,20475)\) design.

**Lemma 5.4.** For \(q = 2^m\) with even \(m \geq 4\), one has \(B_{\sigma_{4,2},q+1}^u = B_{\sigma_{4,2}^2 + \sigma_{4,4}}\) (see \([4]\) for definition of \(B_{f,q+1}\)).

**Proof.** From the proof of Theorem \([5,1]\) \(B_{\sigma_{4,2},q+1}^u = \left\{B \in \binom{U_{q+1}}{4} : \sigma_{5,2}(B \cup \{a\}) = 0 \text{ for some } a \in U_{q+1} \setminus B\right\}\). Combine \([19\text{ Lemmas 12, 18}]\) and \([10\text{ Lemma 18}]\) to yield that \(B_{\sigma_{4,2},q+1}^u = B_{\sigma_{4,2}^2 + \sigma_{4,4}}\).

**Corollary 5.5.** Let \(q = 2^m\) and \(m \geq 4\) be even. Then

\[(U_{q+1}, B_4((\text{Tr}_{q^m/q}(C_{2,3,5}))^\perp))\]

from the minimum weight codewords in \((\text{Tr}_{q^m/q}(C_{2,3,5}))^\perp\) is isomorphic to \((U_{q+1}, B_{\sigma_{4,2},q+1}^u)\), which is a 3-\((q+1,4,2)\) design.
Hence Supp(π) where a, b, d Supp(π) by the translation where (c) of weight is said to be invariant under PGL(2, q) if Supp((c(x))x∈PG(1,q)) ∈ Bk(C) for every permutation π ∈ PGL(2, q) and any codeword (c(x))x∈PG(1,q) of weight k in C.

Let StabU_{q+1} = \{g ∈ PGL(2, q^2) : g(U_{q+1}) = U_{q+1}\} be the setwise stabilizer of U_{q+1} under the action of PGL(2, q^2) on PG(1, q^2). Then the following lemma presents the specific structure of StabU_{q+1}.

**Lemma 5.6.** [10 Corollary 6] Let q = 2^m. Then StabU_{q+1} is generated by three types of linear fractional transformations as follows.

1. (1) u ↦ u0u, where u0 ∈ U_{q+1};
2. (2) u ↦ u^−1;
3. (3) u ↦ u+c^e/du+1, where c ∈ GF(q^2)* \ U_{q+1}.

**Proposition 5.7.** [10 Proposition 7] Let q = 2^m and StabU_{q+1} the setwise stabilizer of U_{q+1}. Then StabU_{q+1} is conjugate in PGL(2, q^2) to the group PGL(2, q) and its action on U_{q+1} is equivalent to the action of PGL(2, q) on PG(1, q).

We will establish the invariance of the support set of any fixed weight in Tr_{q^2/q}(C_{1,2,3}) under the action of StabU_{q+1} in the following theorem.

**Theorem 5.8.** Let q = 2^m with m ≥ 4. Let k be a positive integer with k ≤ q + 1 and A_k(Tr_{q^2/q}(C_{1,2,3})) > 0. Then B_k(Tr_{q^2/q}(C_{1,2,3})) is invariant under the action of StabU_{q+1} and hence it forms the block set of a 3-(q + 1, k, λ) design for some positive integer λ when 3 < k < q + 1.

**Proof.** The proof is similar to that of [10 Theorem 23]. In view of Proposition 5.7, we only need to prove the former part of the conclusion because it is well-known that PGL(2, q) acts on PG(1, q) 3-transitively. As a result, it suffices to show that if c ∈ Tr_{q^2/q}(C_{1,2,3}) and π is a transformation in Lemma 5.6 then there exists a c_1 ∈ Tr_{q^2/q}(C_{1,2,3}) such that Supp(π(c)) = Supp(c_1). Let c(a, b, d) denote the codeword (Tr_{q^2/q}(au + bu^2 + du^3))_{u∈U_{q+1}} of Tr_{q^2/q}(C_{1,2,3}), where a, b, d ∈ GF(q^2). The following three cases for π will be handled.

If π : u ↦ u0u for some u0 ∈ U_{q+1}, then it is obvious that π(c(a, b, d)) = c(au0, bu_0^2, du_0^3). Hence Supp(π(c(a, b, d))) = Supp(c(au0, bu_0^2, du_0^3)).
If \( \pi : u \mapsto u^{-1} \), then obviously \( \pi(c(a, b, d)) = c(a^q, b^q, d^q) \). Thus \( \text{Supp}(\pi(c(a, b, d))) = \text{Supp}(c(a^q, b^q, d^q)) \).

Finally let \( \pi : u \mapsto \frac{u + c^q}{cu + 1} \) where \( c \in GF(q^2) \setminus U_{q+1} \). Let \( f(u) = \text{Tr}_{q^2/q}(au + bu^2 + du^3) \) and \( A = cu + 1 \). Then \( u + c^q = uA^q \). Simple computation gives

\[
\begin{align*}
  f \left( \frac{u + c^q}{cu + 1} \right) &= \text{Tr}_{q^2/q}(a' \frac{u + c^q}{cu + 1} + b' \frac{(u + c^q)^2}{cu + 1} + d' \frac{(u + c^q)^3}{cu + 1}) \\
  &= \text{Tr}_{q^2/q}(a(u + c^q)(cu + 1)^2 + b(u + c^q)^2(cu + 1) + d(u + c^q)^3) \\
  &= \text{Tr}_{q^2/q}(a \frac{uA^{q+2} + bu^2A^q + du^3}{cu + 1}) \\
  &= \frac{1}{A^3A^{q^2}} \text{Tr}_{q^2/q}(A^{q+2} au + A^{q+1} bu^2 + A^q du^3),
\end{align*}
\]

and

\[
\begin{align*}
  au^{A^q+2} &= auA^{4q} \\
  &= au \cdot (cu + 1)^{4q} \\
  &= au \cdot c^{4q}u^{q+1} + 1 \cdot (c^2u^2 + 1) \\
  &= a(u + c^2u^2 + c^{q+2}u^{-1} + c^{q}u^{-3}).
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
  bu^2A^{5q+1} &= bu^2A^{4q}a^q \\
  &= bu^2(cu + 1)^{4q} \\
  &= bu^2 \left( c^{4q}u^{q+1} + 1 \right) \cdot (cu + 1) \\
  &= b(cu^3 + (c^{q+1} + 1)u^2 + c^2u + c^{q+1}u^{-1} + (c^{4q} + c^{5q+1})u^{-2} + c^{5q}u^{-3}),
\end{align*}
\]

and

\[
\begin{align*}
  du^3A^{6q} &= du^3A^{4q}a^q \\
  &= du^3 \cdot (cu + 1)^{4q} \cdot (cu + 1)^{2q} \\
  &= du^3 \left( c^{4q}u^{q+1} + 1 \right) \cdot (c^{2q}u^{2q} + 1) \\
  &= d(u^3 + c^{2q}u + c^{4q}u^{-1} + c^{5q}u^{-3}).
\end{align*}
\]

Combining Eq. (25)–(27) gives

\[
\text{Tr}_{q^2/q}(auA^{q+2} + bu^2A^{5q+1} + du^3A^{6q}) = \text{Tr}_{q^2/q}(a'u + b'u^2 + c'u^3),
\]

where

\[
\begin{align*}
  &\left\{ \begin{array}{l}
  a' = a + be^q + dc^{2q} + a^qe^{2q+4} + b'^2e^{q+4} + d'^2e^4, \\
  b' = b(c^{q+1} + 1) + b'^2(c^{q} + e^{q+5}), \\
  d' = ac^{2} + be + d + a^qe^{4} + b'^2e^{5} + a^qe^{6}.
\end{array} \right.
\end{align*}
\]

Plugging (28) into (24) yields

\[
\begin{align*}
  f \left( \frac{u + c^q}{cu + 1} \right) &= \frac{1}{A^3A^{q^2}} \text{Tr}_{q^2/q}(a'u + b'u^2 + d'u^3).
\end{align*}
\]

So we have \( \text{Supp}(\pi(c(a, b, d))) = \text{Supp}(c(a', b', d')) \). The conclusion then follows. \( \square \)
Corollary 5.9. Let $q = 2^m$ and $m \geq 4$ be odd. Then $B^b_{5,3,q+1}$ is invariant under the action of $\text{Stab}_{U_{q+1}}$.

Proof. Theorem 5.8 implies that $B_{q-4}((q^2/q)(C_{1,2,3}))$ is invariant under the action of $\text{Stab}_{U_{q+1}}$ by noting $A_{q-4}(C_{1,2,3})) > 0$ from Eq. (19). According to Theorem 5.2, we have

\[ B^b_{5,3,q+1} = \left\{ B \in \left( \frac{U_{q+1}}{5} \right) : (U_{q+1} \setminus B) \in B_{q-4}((q^2/q)(C_{1,2,3})) \right\}. \]

Then the conclusion follows. \hfill \Box

5.3 Relationship with Alltop’s family

The subgroup structure of $\text{PGL}(2,q)$ is known \cite{7} and the permutation character $\chi$ for the action of $\text{PGL}(2,q)$ on $\text{PG}(1,q)$ is given in Table 3, where $\varphi(x)$ denotes Euler’s totient.

| Order of $g$ | 1 | 2 | $d|(q - 1), d \neq 1$ | $d|(q + 1), d \neq 1$ |
|-------------|---|---|----------------------|----------------------|
| Order of the centralizer of $g$ | $q^3 - q$ | $q$ | $q - 1$ | $q + 1$ |
| Number of conjugacy classes | 1 | 1 | $\varphi(d)/2$ | $\varphi(d)/2$ |
| Number of fixed points | $q + 1$ | 1 | 2 | 0 |

Table 3: Permutation character of $\text{PGL}(2,q)$, $q = 2^m$

Keranen and Kreher \cite{17} gave complete information on the action of $\text{PGL}(2,q)$ on all quadruples and quintuples of the projective line $\text{PG}(1,q)$. In particular, whenever $q = 2^m$ with $m$ being odd, under the action of $\text{PGL}(2,q)$, all quintuples form exactly $\frac{q^2 - 2}{6}$ short orbits with stabilizer size 4 and exactly $\frac{(q-2)(q+8)}{120}$ orbits with trivial stabilizer, see \cite{17} Theorem 3.3. Alltop \cite{1} proved that the union of all those short orbits forms a $(q + 1, 5, 5)$ design. In subsection 5.2 we showed that the support set of all codewords with a given weight in $(q^2/q)(C_{1,2,3})$ is invariant under the action of $\text{Stab}_{U_{q+1}}$ on $U_{q+1}$, which is equivalent to the action of $\text{PGL}(2,q)$ on $\text{PG}(1,q)$.

Let $\text{supp}(\mathbb{D})$ be the supplementary design of the 4-design $\mathbb{D}$ from Theorem 5.2 which is isomorphic to the 4-$(q + 1, 5, 5)$ design corresponding to $(U_{q+1}, B^b_{5,3,q+1})$. According to Magma \cite{5} experiments, $\text{supp}(\mathbb{D})$ is isomorphic to the Alltop’s design with the same parameters when $q \in \{2^5, 2^7\}$. In order to prove that $\text{supp}(\mathbb{D})$ is isomorphic to the Alltop’s design in general, we need the following lemmas.

Lemma 5.10. Let $q = 2^m$ and $m$ be odd. Suppose that $B$ is a 5-subset of $\text{PG}(1,q)$ which is fixed by a non-identity element $f \in \text{PGL}(2,q)$. Then we have the following.

1. $f$ has order 2; $f$ has exactly one fixed point $\gamma$ and $\frac{q}{2}$ 2-cycles $(\alpha_i, f(\alpha_i)), 1 \leq i \leq \frac{q}{2}$.
2. $B = \{\alpha_i, f(\alpha_i), \alpha_j, f(\alpha_j), \gamma\}$ for some $1 \leq i < j \leq \frac{q}{2}$.

Proof. Let $\text{ord}(f) = a$ and $B \in \binom{\text{PG}(1,q)}{5}$. If $f(B) = B$, then it is not difficult to show that $B$ consists of $b \leq 2$ fixed points of $f$ and $c = \frac{5 - b}{a}$ $a$-cycles of $f$. By Table 3, $b \leq 2$. Because $3 \nmid q - 1$, $5 \nmid q + 1$ when $m$ is odd, we must have $b = 1$, $a = 2$ and $c = 2$. Then the conclusion follows. \hfill \Box
Lemma 5.11. Let $q = 2^m$ and $m$ be odd. Let $B \in \binom{U_{q+1}}{5}$ be a 5-subset which is fixed by a non-identity element $f \in \text{Stab}_{U_{q+1}}$. Then there exists $h \in \text{Stab}_{U_{q+1}}$ and $\alpha \neq \beta \in U_{q+1}$ such that $B = h(\{\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, 1\})$.

Proof. We apply Proposition 5.7, Table 3 and Lemma 5.10. Let $g : u \mapsto \frac{1}{u}, u \in U_{q+1}$. Then $g$ is an element of $\text{Stab}_{U_{q+1}}$ of order 2. So $g$ has one fixed point 1 and $\frac{1}{2}$ 2-cycles $(\alpha_i, g(\alpha_i)), 1 \leq i \leq \frac{q}{2}$ by Lemma 5.10 (1). Assume $B \in \binom{U_{q+1}}{5}$ is fixed by a non-identity element $f \in \text{Stab}_{U_{q+1}}$. According to Lemma 5.10 (1), $\text{ord}(f) = 2$. By Table 3, there is only one conjugacy of order 2 in $\text{Stab}_{U_{q+1}}$. Then there exists $h \in \text{Stab}_{U_{q+1}}$ such that $f = hgh^{-1}$. Since $B = f(B)$, we have $B = hgh^{-1}(B)$. This shows that $f^{-1}(B) = g(h^{-1}(B))$, i.e., $h^{-1}(B)$ is fixed by $g$. From Lemma 5.10 (2), $h^{-1}(B) = \{\alpha, g(\alpha), \beta, g(\beta), 1\}$ where $(\alpha, g(\alpha))$ and $(\beta, g(\beta))$ are two 2-cycles in $g$. As a result, $B = h(\{\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, 1\})$.

Lemma 5.12. For $q = 2^m$ with odd $m \geq 5$, $(U_{q+1}, \mathcal{B}_{\sigma_5,3,q+1})$ is isomorphic to the 4-$(q+1,5,5)$ design constructed by Alltop [1].

Proof. Consider the action of $\text{Stab}_{U_{q+1}}$ on all 5-subsets of $U_{q+1}$, which is equivalent to the action of $\text{PGL}(2,q)$ on 5-subsets of $\text{PG}(1,q)$ by Proposition 5.7. From Lemma 5.11, any short orbit (under the action of $\text{Stab}_{U_{q+1}}$) must have a quintuple representative $A = \{\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, 1\}$ for some $\alpha \neq \beta \in U_{q+1} \setminus \{1\}$. Clearly $\sigma_5,3(A) + \sigma_5,2(A) = 0$, yielding $A \in \mathcal{B}_{\sigma_5,3,q+1}$. By Corollary 5.9, $\mathcal{B}_{\sigma_5,3,q+1}$ is invariant under the action of $\text{Stab}_{U_{q+1}}$. As a consequence, we have $A \subseteq \mathcal{B}_{\sigma_5,3,q+1}$, where $A$ is the set of all 5-subsets in all short orbits. From [17] Theorem 3.3, we have $\frac{4q-4}{6}$ short orbits whose stabilizers are all of size 4. Hence $|A| = \frac{q^2 - 2 \cdot (q+1)(q-1)}{4} = \frac{(q+1)}{4}$, which is the same as $|\mathcal{B}_{\sigma_5,3,q+1}|$. So $\mathcal{B}_{\sigma_5,3,q+1}$ equals the union of all short orbits and thus $(U_{q+1}, \mathcal{B}_{\sigma_5,3,q+1})$ is isomorphic to the 4-$(q+1,5,5)$ design constructed by Alltop [1].

Theorem 5.13. Let $q = 2^m$ with odd $m \geq 5$. Then the incidence structure

$$(U_{q+1}, \mathcal{B}_{q-4}(\text{Tr}_{q^2/q}(\mathcal{C}_{1,2,3})))$$

forms a 4-$(q+1, q-4, (q-4)^2)$ design and its supplementary design is isomorphic to the 4-$(q+1,5,5)$ design constructed by Alltop [1].

Proof. The conclusion follows by combining Theorem 5.2 with Lemma 5.12.

Baartmans et al. [3] constructed a class of 4-$(2s^2 + 1, 5, 2)$ designs by extending the 3-designs formed by the minimum weight codewords in the Preparata code of length $n = 2^{2s}$. Unfortunately, this class of 4-designs is not simple. In this paper we provide an infinite family of linear codes giving rise to an infinite family of simple 4-$(2s^{2s} + 1, 5, 5)$ designs; so far this class of designs has the smallest index among all known simple 4-$(q + 1, 5, \lambda)$ designs derived from codes for prime powers $q$.

6 Summary and concluding remarks

Coding theory and design theory interact with each other intimately, with results and methods from one area being applied to the other. In this paper, we mainly investigated the topic of linear codes supporting $t = 3, 4$ designs by handling the incidence structures produced from some variants of ESPs. Specifically, the main contributions of this paper are the following:
The codewords of weight 7 in the BCH code $C_{(q,q+1,4,1)}$ for $q = 2^m$ support the complementary design of a 4-$(q + 1, 7, \lambda_1)$ design when $m \geq 5$ is odd and they support the complementary design of a 3-$(q + 1, 7, \lambda_2)$ design when $m \geq 4$ is even where

$$\lambda_1 = \frac{7(q - 5)(q - 8)}{6}; \quad \lambda_2 = \frac{7(q - 4)(q - 5)(q - 10)}{24}.$$

The codewords of weight $q - 4$ in the trace code $\text{Tr}_{q^2/q}(C_{(1,2,3)})$ for $q = 2^{2^s+1}$ support the supplementary design of $(U_{q+1}, B_{\sigma,3,q+1}^u)$ which is isomorphic to the 4-$(q + 1, 5, 5)$ design constructed by Alltop [1].

We produced infinite families of simple $t$-designs with new parameters (in comparison with [18] and [19]), which are summarized in Table 4 (complementary designs and supplementary designs not included).

| Block sets          | Designs                                      | Conditions                                      | Ref.     |
|---------------------|----------------------------------------------|-------------------------------------------------|----------|
| $B_{\sigma,3,q+1}^b$ | $3-((q + 1, 5, \frac{q^2-10q+26}{2})$      | $q = 2^{2s}$                                   | Theorem 3.5 |
| $B_{\sigma,3,q+1}^u$ | $4-((q + 1, 7, \frac{7(q-5)(q-8)}{6})$      | $q = 2^{2s+1}$                                 | Theorem 3.8 |
| $B_{\sigma,3,q+1}^u$ | $3-((q + 1, 7, \frac{7(q-4)(q-5)(q-10)}{24})$ | $q = 2^{2s}$                                   | Theorem 3.9 |
| $B_{\sigma,3,q+1}^0$ | $3-((q + 1, 7, \frac{7(q-4)(q-5)}{4})$      | $q = 2^{2s}$                                   | Theorem 3.11 |

Table 4: $t$-Designs with new parameters

An interesting open problem is to construct infinite families of $t$-designs with new parameters from elementary symmetric polynomials $\sigma_{k,l}$ and their variants, such as $(k, l) = (8, 3), (9, 3)$ and $(10, 3)$. Another open problem is whether there exist linear codes holding $t$-designs isomorphic to those produced from ESPs and their variants.

References

[1] W. O. Alltop, An infinite class of 4-designs, Journal of Combinatorial Theory, 6 (1969) 320–322.

[2] E. F. Assmus Jr. and H. F. Mattson Jr., New 5-designs, Journal of Combinatorial Theory, 6 (1969) 122–151.

[3] A. H. Baartmans, I. Bluskov, and V. D. Tonchev, The Preparata codes and a class of 4-designs, Journal of Combinatorial Designs, 2 (1994) 167–170.

[4] T. P. Berger and P. Charpin, The automorphism groups of BCH codes and of some affine-invariant codes over extension fields, Designs Codes and Cryptography, 18 (1999) 29–53.

[5] J. Cannon and W. Bosma, Handbook of Magma Functions, Version 2.12, University of Sydney, Sydney, 2005.

[6] P. Delsarte, On subfield subcodes of modified Reed-Solomon codes, IEEE Transactions on Information Theory, 21 (1975) 575–576.
[7] L. E. Dickson, *Linear groups with an exposition of the Galois field theory*, Dover Publications Incorporated, New York, 1958.

[8] C. Ding, *Designs from Linear Codes*, World Scientific, Singapore, 2018.

[9] C. Ding and C. Tang, *Infinite families of near MDS codes holding t-designs*, IEEE Transactions on Information Theory, 66 (2020) 5419–5428.

[10] C. Ding, C. Tang, and V. D. Tonchev, *The projective general linear group PGL(2, 2^m) and linear codes of length 2^m + 1*, Designs Codes and Cryptography, 89 (2021) 1713–1734.

[11] S. Dodunekov and I. Landgev, *On near-MDS codes*, Journal of Geometry, 54 (1995) 30–43.

[12] X. Du, R. Wang, and C. Fan, *Infinite families of 2-designs from a class of cyclic codes*, Journal of Combinatorial Designs, 28 (2020) 157–170.

[13] A. Faldum and W. Willems, *Codes of small defect*, Designs Codes and Cryptography, 10 (1997) 341–350.

[14] M. Giorgetti and A. Previtali, *Galois invariance, trace codes and subfield subcodes*, Finite Fields and Their Applications, 16 (2010) 96–99.

[15] M. J. E. Golay, *Notes on digital coding*, Proceedings of the IEEE, 37 (1949) 657.

[16] W. C. Huffman and V. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge University Press, Cambridge, 2003.

[17] M. S. Keranen and D. L. Kreher, *3-designs of PSL(2, 2^n) with block sizes 4 and 5*, Journal of Combinatorial Designs, 12 (2004) 103–111.

[18] G. B. Khosrovshahi and H. Lane, *t-Designs with t ≥ 3*, in C. J. Colbourn and J. H. Dinitz (Eds.), Handbook of Combinatorial Designs, CRC Press, (2007) 79–101.

[19] C. Tang and C. Ding, *An infinite family of linear codes supporting 4-designs*, IEEE Transactions on Information Theory, 67 (2021) 244–254.

[20] C. Tang, C. Ding, and M. Xiong, *Codes, differentially δ-uniform functions, and t-designs*, IEEE Transactions on Information Theory, 66 (2020) 3691–3703.

[21] V. D. Tonchev, *Codes*, in C. J. Colbourn and J. H. Dinitz (Eds.), Handbook of Combinatorial Designs, CRC Press, (2007) 677–701.