Quantum Theory of Measurement

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Abstract

We describe a measurement in quantum mechanics as a variational principle including a simple interaction between the system under measurement and the measurement apparatus. Augmenting the action with a nonlocal term (a double integration over the duration of the measurement interaction) results in a theory capable of describing both the measurement process (agreement between system state and the pointer state of the measurement apparatus) and the collapse of both systems into a single eigenstate (or superposition of degenerate eigenstates) of the operator corresponding to the measured variable. In the absence of the measurement interaction, a superposition of states is stable, and the theory agrees with the predictions of standard quantum theory. Because the theory is nonlocal, the resulting wave equation is an integrodifferential equation (IDE). We demonstrate these ideas using a simple Lagrangian for both systems, as proof of principle. The variational principle is time–symmetric and retrocausal, so the solution for the measurement process is determined by boundary conditions at both initial and final times; the initial condition is determined by the experimental preparation and the final condition is the natural boundary condition of variational calculus. We hypothesize that one or more hidden variables (not ruled out by Bell’s Theorem, due both to the retrocausality and the nonlocality of the theory) influence the outcome of the measurement, and that distributions of the hidden variables that arise plausibly in a typical ensemble of experimental realizations give rise to outcome frequencies consistent with Born’s rule. We outline steps in a theoretical validation of the hypothesis. We discuss the role of both initial and final conditions to determine a solution at intermediate times, the mechanism by which a system responds to measurement, time symmetry of the new theory, causality concerns, and issues surrounding solution of the IDE.

I. INTRODUCTION

A. Motivation and philosophical stance

Quantum theory in general, and its description of measurement in particular, seems to violate several reasonable expectations about the the characteristics of a correct physical theory. Ordinarily, to be accepted as correct and complete, a theory must predict future
phenomena, given a complete set of the relevant initial conditions. Quantum theory fails to do this in the case of a measurement; in fact, it is understood that the mathematical description (wave equation) describing system evolution in the absence of a measurement does not apply to a measurement. In effect, two different theories are required, for the measurement and non–measurement cases. While it may be acceptable for a theory to treat different cases in different ways, quantum theory lacks an unambiguous definition of a measurement, with the result that measurement and non–measurement configurations may be arbitrarily similar physically, and the bipartite theoretical description is implausible.

In addition, the theory of quantum measurement (as distinguished from the wave equation) as usually interpreted (e.g. by the Copenhagen interpretation) has multiple features that are unknown in any other generally accepted fundamental theory. One is intrinsic randomness, the idea that nature samples from a random distribution, and no prediction can be made about the result of sampling that goes beyond a description of the distribution function. Another is temporal asymmetry; after the measurement, but not before, the system is understood to be “collapsed” into an eigenstate or set of degenerate eigenstates of the operator corresponding to the measured quantity.

A third feature unique to the quantum measurement process is dependence on the eigenstate structure of the problem. The observed behavior that a measurement always finds the system in a single eigenstate (or a superposition of degenerate eigenstates) of the operator requires a nonlocal theory. As we will discuss in subsection II E, the information (e.g., potential $V(x_0)$) available at a single point $x_0$ is insufficient to determine whether a particular solution at that point (values of the wavefunction $\psi(x_0)$ and its derivative(s) at $x_0$) is consistent with a single eigenfunction $\psi$ (the function defined for all allowed values of $x$). Nature cannot reliably make that determination at $x_0$ without using information at points $x \neq x_0$.

In addition, we call attention to quantum phenomena that seem to violate causality. One is correlations between spacelike separated measurements in ways that violate special–relativity–based expectations (“EPR correlations,” for short) but have been verified in a long sequence of increasingly more sophisticated experiments. Another is delayed–choice experiments, in which the path of a particle (through one or two slits, for example) has been observed to be apparently determined by a choice made after the particle is committed to a particular path.
In this paper, we propose that a quantum theory can be constructed so as to either avoid or explain most of these objectionable or unique features. To be specific, we will exhibit a wave equation that applies even when a measurement is being done, in which case it describes evolution ("collapse," although not instantaneous) of the wavefunction to a state or states with a single eigenvalue. The theory is time–symmetric. Instead of relying on intrinsic randomness to explain differing results of identically prepared measurements, it proposes that some hidden variable(s), presumably uncontrolled or overlooked by the experimenter, determine(s) the outcome. The Born’s–rule distribution of outcomes attributed to randomness by standard quantum theory presumably appears instead as a result of a naturally–arising distribution of values of the hidden variable(s)—although the complete proof of that result must await further investigation.

On the other hand, the theory we describe relies on some unusual assumptions; we do not expect to replace conventional quantum theory with one that completely resembles other physical theories. One such assumption is retrocausality, roughly speaking, the idea that effects may precede their causes in time. (To be more precise, in a retrocausal theory the solution at \( t \) is found as a function of variables at \( t' > t \).) The theory is also nonlocal, as needed to produce the needed dependence on eigenstate structure; for this reason the wave equation is a integrodifferential equation (IDE). Finally, as mentioned earlier, we posit the existence of hidden variables. Bell’s Theorem and its experimental tests are generally understood to rule out local hidden–variable theories, but that does not restrict our nonlocal theory. In addition, it has been pointed out that the proof of Bell’s theorem relies on an assumption violated by retrocausality, so for that reason also, hidden variables are not off limits in this case.

We point out here that because retrocausality can allow information to propagate backward in time, it trivially explains EPR correlations and delayed–choice experiments. Since those two issues are already disposed of, we will focus our attention on the remaining ones.
B. Elements of the theory

We consider that a legitimate measurement is understood to require a duration $T$ limited by the time-energy uncertainty relation

$$T \Delta E \geq \frac{\hbar}{2} \quad (1)$$

where $\Delta E$ is the smallest energy difference between states that must be distinguished by the measurement. Typical experiments are designed with $T \Delta E \gg \hbar/2$. In our analysis, we will suppose that the system is prepared and the experiment begun at time $t_i$, and the measurement is determined or read at $t_f = t_i + T$.

We desire the theory to be time–symmetric, and it is appealing to do so by couching it as a variational principle. In this case the state $\psi$ of the system is found to be a critical point of the action

$$I[\psi] \equiv \int_{t_i}^{t_f} L[\psi, \dot{\psi}, t] \quad (2)$$

where $L$ is the system Lagrangian, typically the spatial integral of a Lagrangian density. Critical points are choices of the function $\psi$ where $I$ is stationary with respect to infinitesimal variations of $\psi$. For functionals that depend smoothly on their arguments, maxima and minima are critical points, so the search for critical points is often described as finding extrema. Schwinger [9] developed quantum field theory as a variational principle based on the action. For our purposes, we note that the action has the same value regardless of the direction of time, so the resulting theory is time–reversal invariant.

Our exposition will be nonrelativistic, but the variational principle is inherently compatible with special relativity [9], and we expect that it can readily be expressed in a relativistically covariant formulation. Relativistic Lagrangians routinely appear in quantum field theory, [10] and the four-dimensional integration of the Lagrangian density to produce the action is of course a relativistically appropriate operation, invariant under change of reference frames.

In its simplest form, a variational principle leads to a differential equation, the Euler equation [11]. In order to introduce nonlocality, we will employ a more complex form (a double rather than a single integral in time) that will result in an integro-differential equation (IDE); see Appendix for the mathematical details. The IDE involves an integral from $t_i$ to $t_f$, so nonlocality in time is evident. Note that the conventionally–understood
unitary evolution of the system (as described by the wave equation) would be predicted by the conventionally–derived action, without our modification. We expect that the modified action will predict, via the variational principle, the combination of the non–measurement evolution of the system and the effect of the measurement.

This mathematical form apparently requires solving for $\psi$ simultaneously for all times in $[t_i, t_f]$. This contrasts with a typical physical theory in which variables and their time derivatives at $t$ depend on other variables and derivatives at $t$, or in some cases on $t$ and its past. Wharton [12] has designated these two approaches as Lagrangian and Newtonian respectively, and argued persuasively that the former may be appropriate for physical theories. Note that this picture is definitely retrocausal, because the wavefunction at time $t$ may depend on conditions or the wavefunction at times $> t$, and in particular at $t_f$.

The most obvious way to solve such a mathematical problem is with specified initial and final conditions $\psi(t_i)$ and $\psi(t_f)$. Consider a typical measurement problem in which the system to be measured is prepared at $t_i$ in a given quantum state, defined as an eigenstate or a specified superposition of eigenstates of a given operator. Then a measurement concluding (“read out”) at $t_f$ determines in which of the eigenstates of that operator the system is found at that time. In this case the initial condition is fixed by the specified experimental preparation, but the final condition appears to be missing. Calculus of variations [11] supplies the missing constraint, namely, a “natural boundary condition” (NBC) that inevitably applies at a boundary where the value of the unknown function is not specified by the problem definition.

As a familiar example, consider a vibrating string of length $L$. It is described by a simple wave equation (a differential equation) for the displacement $y(x,t)$, but we could equally well cast the problem as a variational principle and deduce the wave equation from the system Lagrangian. Now if the string is fixed at both ends, the variational principle (and hence the wave equation) must satisfy ordinary BCs at both ends: $y(0,t) = y(L,t) = 0$. But if at $x = 0$ the string is not fixed but rather free to slide frictionlessly along a rod perpendicular to the string, the BC at that point is the NBC $\partial y/\partial x|_{x=0} = 0$. Note that that condition is caused not by the rod, which does not constrain the string’s position or slope, but by the Lagrangian, by which the condition follows from the requirement of stationarity of the action.

We conclude that the solution of the IDE must be constrained by the specified prepa-
ration at $t_i$ (an initial condition) and the NBC at $t_f$ (a final condition). However, the empirical fact that different outcomes may result from identically-prepared repetitions of the same measurement proves that those two conditions underconstrain the problem. At $t_i$, the system is prepared in a given quantum state or superposition of states (e.g., the ground state of a square–well potential), but that description falls short of a specification of every possible variable (including e.g. both position and momentum), as it must by quantum complementarity [13]. The full specification of the initial (ontological) state consists of the given quantum state, plus additional “hidden variables” unknown to or uncontrolled by the experimenter. Similarly, the measurement of a quantum state at $t_f$ does not determine the ontological state at that moment; in fact, the measurement readout at $t_f$ is a weaker constraint than the preparation at $t_i$, because it determines only the variable (operator) measured but not its value (eigenvalue).[14] This indeterminacy provides the opportunity for hidden variables to participate in determining the result of the measurement.

We will see that our theory predicts the collapse (not necessarily instantaneous; perhaps a better term is decay) of the wavefunction to a single eigenvalue at $t_f$. We expect that the BCs together with the hidden variable(s) determine which final state results from the collapse. Ultimately, the frequencies of the different outcomes possible from a single experimental definition must reflect the distribution of hidden variable values in a large number of realizations of the experiment. The observed fact that those frequencies may be described by a simple law (Born’s rule) presumably reflects a likelihood of an approximately universal distribution of the hidden variable values in experiments that are likely to be conducted (without knowledge or control of the hidden variable(s)). For instance, suppose the experimental result depends on a high–frequency sinusoidal function of some experimental time. If in an ensemble of experimental realizations that time is naturally distributed over a range large compared to the period of oscillation, it is an excellent approximation to say that that time has a uniform distribution over a single period. In this way, it is reasonable to expect that naturally–occurring ensembles of experiments may be found reliably to give outcome frequencies satisfying Born’s rule.
C. Model of the measurement problem

As the principal issues motivating our theoretical development have to do with quantum measurement, we will consider an idealized model of such a measurement. Suppose that the system is prepared in a known superposition \( \sum_j C_j |\psi_j \rangle \) of eigenstates of the operator \( \sigma_{op} \) at time \( t_i \); that is, the eigenstates are well-defined and the coefficients \( C_j(t_i) \) are known. This superposition is known to be initially stable; \( \dot{C}_j(t_i) = 0 \) for all \( j \). The eigenstates themselves must be stable, so \( \sigma_{op} \) commutes with the Hamiltonian. Finally, the stability of an (unperturbed, unmeasured) superposition implies that the system is linear when in isolation, that is, it satisfies a wave equation linear in the wavefunction. This consideration will be seen to constrain the form of possible Lagrangians for the system. We will develop our ideas using a particular simple form, as proof of principle.

During all or part of the interval \([t_i, t_f]\), a measurement apparatus (which we will call system 2) interacts with the measured system (system 1). A requirement for generality of the theory—validity of the properties of “quantum measurements” across all types of measurements—excludes all but the most general description of the measurement apparatus and its interaction with the measured system. We therefore use a minimal description, that the apparatus has a “pointer state” variable \( \sigma^2 \), and that it is coupled to the measured variable \( \sigma^1 \) of the system. Without loss of generality, we define \( \sigma^2 \) so that its value in a successful measurement equals the value of \( \sigma^1 \). Then the composite (system + apparatus) Lagrangian must include an interaction term that depends on both measured and pointer state variables, and attains an extreme (or stationary) value when they are equal. The simplest such term is quadratic in the corresponding operators, that is, proportional to \( (\sigma^1_{op} - \sigma^2_{op})^2 \).

Note that good experimental design dictates that the combined system (1 and 2) be well isolated in spacetime. Spatial isolation is accomplished by physical isolation or other control of the boundaries of the domain, and temporal isolation by system preparation at \( t_i \) and measurement readout at \( t_f \). This blocks influences from outside the spacetime region, which is important so that the spacetime integrals in this nonlocal theory can legitimately be limited to the experimental domain.

What is known experimentally is that if there is no measurement, system 1 remains indefinitely in the same superposition of states in which it was prepared. If there is a
measurement, it is found (measured) to be in a single eigenstate. (Actually, this may be a superposition of degenerate eigenstates—states with a single eigenvalue.) Finally, in an ensemble of identically prepared measurements, measured eigenstates occur in proportion to their weight $|C_j(t_i)|^2$ in the initial superposition (Born’s rule). We seek a theory that predicts these empirical facts.

This description of the “measurement problem” is understood to be very well established by a large body of experimental evidence. On the other hand, that body of evidence is silent on the outcomes of measurements violating (1), because such experiments would be understood to be ineligible to invalidate any of the above points. In other words, we may consider Born’s rule to be a summary of observations about experiments conforming to (1), since nonconforming experiments would not have been considered proper measurements.

### D. Outline

In the next section, we will develop the theory based on a variational principle, generalized so as to result in a nonlocal equation. The subsequent section will discuss the predictions of that equation and compare them to the properties that we have argued must appear in a successful theory. In some cases the agreement will be clear, although it will remain for the future to describe the details of approach to the solution, and to prove that the solution is unique. For Born’s rule, we will show how hidden variables may arise and the way in which the expected output frequencies may follow from their distribution; however, analytic proof or numerical demonstration that our theory yields frequencies consistent with Born’s rule remains to be done. In the last section we will summarize what we have done, discuss new perspectives required by retrocausality and nonlocality, and list some of the next steps to be taken to continue developing these ideas. A mathematical appendix derives an extension of variational calculus used in our analysis of the nonlocal variational principle.

### II. THEORETICAL DEVELOPMENT

#### A. Variational approach

An isolated system (system 1 or system 2, in our case, when they are not interacting), is described by a Lagrangian $L[\psi, \dot{\psi}, t]$—a functional of the wavefunction $\psi(t)$—which is
typically the spatial integral of a Lagrangian density \( \mathcal{L}[\psi(t, \mathbf{x})] \). The variational principle (to be specific, the Hamiltonian principle) says that the action \( S \equiv \int dt L \) is stationary with respect to variations of \( \psi \), a condition that may be denoted \( \delta S = 0 \). (This is the principle that was employed by Schwinger\footnote{9} as the foundation for quantum field theory.) A choice of \( \psi \) for which \( S \) is stationary is said to be a critical point of \( S \). For functionals that depend smoothly on their arguments, maxima and minima are critical points, so the search for critical points is often described as finding extrema. A necessary condition is given by the Euler equation

\[
0 = \frac{\partial L}{\partial \psi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}}
\]

(Equation 3)

Evidently the requirement that \( \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} \) yield a linear wave equation implies that the Lagrangian must be quadratic in \( \psi \) and its time derivative.

B. Normal–mode expansion—Single system

Since the point of the measurement problem is to describe the evolution of a superposition of eigenstates of a given operator to a single eigenstate, it will simplify matters to define a basis set of such eigenstates. This expansion will be specific to a given inertial reference frame—the frame in which the measurement is performed and described by the above characteristics—because that will simplify the analysis and its comparison to those points. However, as explained above, we expect that the general theory (the form of the action, without dependence on the normal–mode expansion we will use here) will be relativistically appropriate and can be expressed in covariant form.

We will describe each system \( \ell (=1 \text{ or } 2) \) by a wavefunction \( \psi^\ell(t, \mathbf{x}) \), normalized in the usual way in terms of the spatial integral or otherwise–defined inner product

\[
\langle \psi^\ell | \psi^\ell \rangle = 1
\]

(Equation 4)

At any given time \( t \), let \( |\psi^\ell_j(t)\rangle \) be for system \( \ell = 1 \text{ or } 2 \) an eigenstate of a Hermitian operator \( \sigma^\ell_{\text{op}}, \)

\[
\sigma^\ell_{\text{op}} |\psi^\ell_j(t)\rangle = \sigma^\ell_j |\psi^\ell_j(t)\rangle
\]

(Equation 5)

satisfying the applicable spatial BCs, and let those eigenstates form an orthonormal basis for states of system \( \ell \):
\[ \langle \psi^\ell_j(t) | \psi^\ell_k(t) \rangle = \delta_{jk} \quad (\ell = 1, 2; \forall t) \] (6)

Since external fields acting on the system may change during the course of the measurement (perhaps due to the measurement process itself), the eigenvalues and eigenstates are in general functions of time. In many interesting cases they are slowly varying functions of time, and for simplicity we will confine ourselves to the case in which the eigenvalues \( \sigma^\ell_j \) are constant. We expect that the analysis presented below can be readily generalized to the time-dependent case, for sufficiently slow variation.

We will also require each normal mode \( |\psi^\ell_j\rangle \) to satisfy the variational principle based on its single-system Lagrangian \( L^\ell \). This is possible because as stated above, the operator corresponding to the measured variable commutes with the Hamiltonian. The basis states will be taken to be simultaneous eigenstates of both operators, and eigenstates of the Hamiltonian satisfy the variational principle. Since a basis vector \( |\psi^\ell_j(t)\rangle \) was defined to be an eigenstate of the Hamiltonian, it has an energy \( E_j^\ell \) and a time derivative

\[ \frac{d}{dt} |\psi^\ell_j(t)\rangle = -\frac{i}{\hbar} E_j^\ell |\psi^\ell_j(t)\rangle \] (7)

(Schödinger picture). We will also take the energies \( E_j^\ell \) to be constant; then it follows that

\[ \langle \psi^\ell_j(t_1) | \psi^\ell_k(t_2) \rangle = \delta_{jk} e^{-\frac{i}{\hbar} E_j^\ell (t_2 - t_1)} \] (8)

Now if system \( \ell = 1 \) (measured system) or 2 (measurement apparatus) is isolated, its wavefunction can be expanded

\[ |\psi^\ell(t)\rangle = \sum_j C^\ell_j(t) |\psi^\ell_j\rangle \] (9)

and the normalization condition (4) implies

\[ \sum_j |C^\ell_j(t)|^2 = 1 \] (10)

At present we expect this condition to hold for any \( t \), but in subsection III we will argue for removing this constraint.
The action is

$$S^\ell \equiv \int_{t_i}^{t_f} dt \, L^\ell(t)$$

$$= \int_{t_i}^{t_f} dt \, \langle \psi^\ell(t) \vert L^\ell_{op} \vert \psi^\ell(t) \rangle$$

$$= \sum_{j,k} \int_{t_i}^{t_f} dt \, \langle \psi_j^\ell(t) \vert C^\ell_j(t) L^\ell_{op} C^\ell_k(t) \vert \psi_k^\ell(t) \rangle$$

(11)

Since the complete wavefunction $\psi^\ell$ is completely determined by the set of coefficients $C^\ell_j(t)$, the condition of stationarity of the action reduces to the problem of finding those coefficients, which must satisfy

$$0 = \frac{\partial L^\ell}{\partial C^\ell_j} - \frac{d}{dt} \frac{\partial L^\ell}{\partial \dot{C}^\ell_j} \quad \forall j$$

(12)

This formulation of the problem replaces (3).

It is traditional in quantum field theory to perform the variational calculus analysis by varying (differentiating with respect to) the physically significant canonical fields and momenta, and that approach is extremely useful in producing intuitively appealing and useful evolution equations. However, the stationarity of the action is a mathematical condition, and as long as our formulation spans the space of its allowed variations, the mathematics does not dictate our choice of the functions in terms of which those variations are expressed. Because we are interested in the eigenstate content of the wavefunction, the corresponding coefficients are particularly useful to us, and we use them to analyze the variational principle.

C. Combined systems

Now we can use the normalization condition (4) to write from (11)

$$S^1 + S^2 = \int_{t_i}^{t_f} dt \, \langle \psi^1(t) \vert \langle \psi^2(t) \vert (L^1_{op} + L^2_{op}) \vert \psi^1(t) \rangle \vert \psi^2(t) \rangle$$

(13)

if there is no interaction or entanglement between the two systems, that is, the combined state factors as $\vert \psi \rangle \equiv \vert \psi^1 \rangle \vert \psi^2 \rangle$.

To allow the two subsystems to be entangled, we replace the product of single-system states $\vert \psi^1 \rangle$ and $\vert \psi^2 \rangle$ by the joint state

$$\vert \psi(t) \rangle = \sum_{j,k} C_{jk}(t) \vert \psi^1_j(t) \rangle \vert \psi^2_k(t) \rangle$$

(14)
whereupon the normalization condition becomes

$$\sum_{j,k} |C_{jk}(t)|^2 = 1 \quad \forall t \quad (15)$$

Then

$$S^1 + S^2 = \sum_{j,k,\ell,m} \int_{t_i}^{t_f} dt \left\langle \psi^1_j(t) | C^*_{jk}(t)(L^1_{op} + L^2_{op}) C_{\ell m}(t) | \psi^1_\ell(t) \right\rangle \left\langle \psi^2_m(t) \right\rangle \quad (16)$$

and

To simplify the single-system terms, suppose $L^1_{op}$ and $L^2_{op}$ are of the form

$$L^\ell_{op} = A^\ell - B^\ell \frac{d^2}{dt^2}$$

so $L^1$ and $L^2$ take the form

$$L^\ell \equiv \left\langle \psi^\ell | L^\ell_{op} | \psi^\ell \right\rangle = A^\ell \left\langle \psi^\ell | \psi^\ell \right\rangle + B^\ell \left\langle \dot{\psi}^\ell | \dot{\psi}^\ell \right\rangle \quad (18)$$

with real constants $A^\ell$ and $B^\ell$. Then the fact that $|\psi^\ell_j\rangle$ is an eigenstate of the Hamiltonian means that it satisfies the Euler equation (3), which we can write as

$$0 = \frac{\partial L^\ell}{\partial \left\langle \psi^\ell_j \right\rangle} - \frac{d}{dt} \frac{\partial L^\ell}{\partial \left\langle \dot{\psi}^\ell_j \right\rangle} = A^\ell |\psi^\ell_j\rangle - B^\ell |\ddot{\psi}^\ell_j\rangle \quad (19)$$

At this point we observe that the functional in question is a physical action and therefore real, so it is unchanged if we drop any imaginary part of the integrand. This has a simplifying advantage. When we use variational calculus to find a stationary state with respect to variations of a complex quantity ($|\psi^\ell_j\rangle$ or $C^*_{jk}$), we may treat the real and imaginary parts of that quantity independently, with an Euler equation for each of them. Alternatively, we may treat the quantity and its complex conjugate ($\langle \psi^\ell_j |$ or $C^*_{jk}$) as the two functions to be varied. In our case, with a real integrand, doing so has the convenient feature that the two resulting Euler equations are complex conjugates of each other and we only need to solve one of them. Here in (19) we choose to vary the bra vector.
Substituting (17) into (16) and using property (19) of the eigenvectors, we find (introducing the shorthand notation $B \equiv B^1 + B^2$) that

$$S^1 + S^2 = \sum_{j,k,\ell,m} \int_{t_i}^{t_f} dt \left\langle \psi_j^1(t) \right| \left( \psi_k^2(t) \right| C^*_{jk}(t) \left[ \left( L^1 + L^2 \right) C_{\ell m}(t) \right] \left| \psi_\ell^1(t) \right\rangle \left| \psi_m^2(t) \right\rangle$$

$$= -B \sum_{j,k,\ell,m} \int_{t_i}^{t_f} dt \left\langle \psi_j^1 \right| \left( \psi_k^2 \right| C^*_{jk} \left( \dot{C}_{\ell m} + 2 \dot{C}_{\ell m} \frac{d}{dt} \right) \left| \psi_\ell^1 \right\rangle \left| \psi_m^2 \right\rangle$$

(20)

Then

$$S^1 + S^2 = -B \sum_{j,k,\ell,m} \int_{t_i}^{t_f} dt \left\langle \psi_j^1 \right| \left( \psi_k^2 \right| C^*_{jk} \left( \dot{C}_{\ell m} - \frac{2i}{\hbar} E_{\ell m} \dot{C}_{\ell m} \right) \left| \psi_\ell^1 \right\rangle \left| \psi_m^2 \right\rangle$$

$$= -B \sum_{j,k} \int_{t_i}^{t_f} dt \left\langle C_{jk} \left( \dot{C}_{jk} - \frac{2i}{\hbar} E_{jk} \dot{C}_{jk} \right) \left| \psi_\ell^1 \right\rangle \left| \psi_m^2 \right\rangle$$

$$= B \sum_{j,k} \int_{t_i}^{t_f} dt \left( \dot{C}_{jk}^2 + \frac{2i}{\hbar} E_{jk} C_{jk}^* \dot{C}_{jk} \right)$$

(21)

where in the last step we rely on the hypothesis that $\dot{C}_{jk}$ vanishes at $t_i$ as a condition imposed by the experimental preparation, and at $t_f$ since that is implied by the NBC. Finally, as intended, we discard the imaginary part of the integrand:

$$S^1 + S^2 = B \sum_{j,k} \int_{t_i}^{t_f} dt \left( \left| \dot{C}_{jk} \right|^2 + \frac{2i}{\hbar} E_{jk} C_{jk}^* \dot{C}_{jk} \right)$$

(22)

D. Interaction term

As argued above, we must account for interaction by including in the action a term proportional to $(\sigma^1_{op} - \sigma^2_{op})^2$. A simple form for such an interaction term is

$$S^I = \mu \int_{t_i}^{t_f} dt \left\langle \psi(t) \right| (\sigma^1_{op} - \sigma^2_{op})^2 \left| \psi(t) \right\rangle$$

(23)

for some constant $\mu$. Then, defining another shorthand notation $\Delta_{jk} \equiv \sigma^1_{jk} - \sigma^2_{jk}$,

$$S^I = \mu \sum_{j,k,\ell,m} \int_{t_i}^{t_f} dt \left\langle \psi_j^1 \right| \left( \psi_k^2 \right| C^*_{jk}(t) (\sigma^1_{op} - \sigma^2_{op})^2 C_{\ell m}(t) \left| \psi_\ell^1 \right\rangle \left| \psi_m^2 \right\rangle$$

$$= \mu \sum_{j,k} \int_{t_i}^{t_f} dt \left\langle \psi_\ell^1 \right| \left( \psi_m^2 \right| C_{jk}(t)^2 \right\rangle$$

(24)

Then we might expect the complete action to be

$$S = S^1 + S^2 + S^I$$

(25)
E. Necessity of a nonlocal theory

However, as we indicated earlier, to reproduce the observed behavior that a measurement always finds the system in a single eigenstate (or a superposition of degenerate eigenstates) of the operator corresponding to the measured quantity, the theory must be nonlocal. For a simple example of this, consider a system described by the one-dimensional Schrödinger equation

\[
-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - V(x) \psi(x) = E \psi(x)
\]

(26)

with potential function \( V(x) \) and boundary conditions at positions \( x_1, x_2 \) (which may be \( \pm \infty \)). It is customary to require the solution to be normalized according to

\[
\int |\psi|^2 dx = 1
\]

(27)

although for our purposes it suffices to require that integral to be finite. Consider the case in which the potential is attractive and the spectrum of energy eigenvalues \( E \) is discrete, with (for simplicity) no degenerate eigenstates.

Now suppose that the values \( \psi(x_0) \) and \( \psi'(x_0) \) are proposed as solutions at some point \( x = x_0 \), and we ask whether they belong to a solution that is a single eigenstate. In a conventional interpretation of quantum mechanics, this is the question Nature must answer when those values have developed due to the operation of the wave equation and then a measurement is made, requiring a single eigenvalue as its result. (We are here dealing with the case in which the measured quantity is energy, but that case is enough to prove our point.) Nature must decide whether to accept the proposed values of \( \psi(x_0) \) and \( \psi'(x_0) \) as given or “collapse” to different values consistent with a single eigenstate.

In a local theory, that question must be answered on the basis of local information alone, that is, \( V(x_0) \). That information is insufficient. With nonlocal information, namely, the entire function \( V(x) \), it would be possible, given \( E \), to find the solution \( \psi(x) \) of (26) by integrating the differential equation twice. However, for most values of \( E \), either the integrated solution violates the boundary conditions or the normalization integral diverges (or both). We conclude that determining whether \( \psi(x_0) \) and \( \psi'(x_0) \) are consistent with a single eigenstate requires the use of information \( V(x) \) at all \( x \) to integrate the solution and test boundary conditions and normalizability. A local theory cannot make that determination.

Therefore, since a measurement always finds an eigenstate of the relevant operator, we
conclude that its complete mathematical description must be nonlocal in space. But a
description that is nonlocal in space in one inertial reference frame is nonlocal in both space
and time in any other frame, so in general the description of a measurement must be nonlocal
in time as well.

A differential equation (aside from the specification of BCs) is local, depending on a
function and its derivatives at a single point. By contrast, a nonlocal relationship is naturally
expressed as an integral equation. Calculus of variations shows us that the stationary states
of an integral expression like $\int du F(g(u), g(\dot{u}), u)$ satisfy a differential equation (the Euler
equation) for $g(u)$, so such a description corresponds to a strictly local process. In order to
obtain an integral equation as the simplest description of a measurement process, we need
the action to involve at least two integrations of some function of the quantum state.

F. Nonlocal interaction term

Since the phenomenon that requires nonlocality (measurement–induced collapse of the
wavefunction) is due to the interaction between systems 1 and 2, we suppose that it is the
interaction term $S^I$ that must be made nonlocal. We propose to add to it a nonlocal piece
involving two integrations on time. We start with an expression resembling $S^I$ in (23) but
with two integrations on time:

$$\nu \left[ \int_{t_1}^{t_f} dt \, \langle \psi(t)| (\sigma_{op}^1 - \sigma_{op}^2) |\psi(t)\rangle \right]^2$$

$$= \nu \int_{t_1}^{t_f} dt_1 \int_{t_i}^{t_1} dt_2 \, \langle \psi(t_1)| \langle \psi(t_2)| (\sigma_{op}^1 - \sigma_{op}^2) (\sigma_{op}^1 - \sigma_{op}^2) |\psi(t_1)\rangle |\psi(t_2)\rangle \rangle \quad (28)$$

Here $\nu$ is a real constant, and the primed $\sigma_{op}^\ell$ operators combine with the primed bra and
ket vectors in an inner product, as do the unprimed operators and bra and ket vectors. Now
we make changes so as to couple the $t_1$ and the $t_2$ integrals. We move one of the primes in
the operator kernel, changing it from $(\sigma_{op}^1 - \sigma_{op}^2) (\sigma_{op}^1 - \sigma_{op}^2)$ to $(\sigma_{op}^1 - \sigma_{op}^2) (\sigma_{op}^1 - \sigma_{op}^2)$. We also
move the prime from one ket vector to the other. Finally, we observe that in this form the
interaction between the state at $t_1$ and at that at $t_2$ is independent of the time difference. It
may be that that effect weakens with temporal separation, so a dimensionless non-negative
real function $f(t_1 - t_2)$ should be included in the integrand. By symmetry, $f$ must be an
even function, and we expect it to be a monotonically decreasing function of the absolute
value of its argument. For later convenience, let us suppose that there is a real constant \( \tau \) such that \( f(t_1 - t_2) = 0 \) whenever \( |t_1 - t_2| \geq \tau \). These changes result in the term

\[
R^I \equiv \nu \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \, f(t_1 - t_2) \langle \psi(t_1) | \langle \psi(t_2) | (\sigma^1_{op} - \sigma^2_{op}) (\sigma^1_{op} - \sigma^2_{op}) | \psi(t_1) \rangle | \psi(t_2) \rangle
\]

Physically this expresses an interaction or “auto-entanglement” between the state \(|\psi\rangle\) at time \(t_1\) and the same state at \(t_2\); this is an expression of retrocausality in the sense that the state at the later time interacts with its earlier value. A more speculative interpretation, based on the time symmetry of the variational principle, is that this term describes interaction between “forwards” and “backwards” histories. This sounds very much like the “transaction” in Cramer’s transactional interpretation,\[15\] but it is not quite the same; Cramer proposed a two-way interaction between lightlike separated events, whereas our form allows for the possibility of timelike, lightlike and spacelike interactions. (We may of course restrict those options as we gain future understanding.)

We point out that for the extreme choice of \( f \)

\[
f(t_1 - t_2) = \delta(t_1 - t_2)
\]

the integrand takes a more intuitive form in terms of quantum expectation values \( \langle O \rangle \equiv \langle \psi | O | \psi \rangle \):

\[
\langle \psi(t) | \langle \psi(t) | (\sigma^1_{op} - \sigma^2_{op}) (\sigma^1_{op} - \sigma^2_{op}) | \psi(t) \rangle | \psi(t) \rangle = \langle \sigma^1 \rangle^2 - 2 \langle \sigma^1 \sigma^2 \rangle + \langle \sigma^2 \rangle^2
\]

\[
= \langle (\sigma^1 - \sigma^2)^2 \rangle - \langle (\Delta \sigma^1)^2 \rangle - \langle (\Delta \sigma^2)^2 \rangle
\]

in which

\[
\Delta \sigma^\ell \equiv \sigma^\ell - \langle \sigma^\ell \rangle \quad \ell = 1, 2
\]

This suggests that minimizing the term \( \langle (\sigma^1 - \sigma^2)^2 \rangle \) drives the action of measurement (system and apparatus evolve to states with the same eigenvalue) and the other two terms drive wavefunction collapse (until each system ultimately has only a single eigenvalue \( \sigma^\ell = \langle \sigma^\ell \rangle \)). However, we will find that the \( \delta \)-function form of \( f \) is unsuitable for our objectives, so the physical interpretation of \( R^I \) is more subtle.

Next we expand in normal modes according to \[14\] and use the eigenvalue relation \[5\]:

\[
R^I = \nu \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \sum_{j,k,\ell,m,\nu,p,q,r} \langle \psi^j_1(t_1) | \langle \psi^k_2(t_1) | C^*_{\ell m}(t_2) \Delta_{\nu p} \Delta_{\nu r} C_{\nu p}(t_1) \langle \psi^r_1(t_1) | \langle \psi^\ell_2(t_2) | \psi^k_2(t_2) \rangle | \psi^j_1(t_2) \rangle | \psi^r_2(t_2) \rangle
\]

\[
C^*_{\ell m}(t_2) \Delta_{\nu p} \Delta_{\nu r} C_{\nu p}(t_1) \langle \psi^r_1(t_1) | \langle \psi^\ell_2(t_2) | \psi^k_2(t_2) \rangle | \psi^j_1(t_2) \rangle | \psi^r_2(t_2) \rangle
\]

\[
(33)
\]

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Then, using (8) and defining $E_{jk} \equiv E_{j}^{1} + E_{k}^{2}$, 

$$R^{I} = \int_{t_{i}}^{t_{f}} dt_{1} \int_{t_{i}}^{t_{f}} dt_{2} r^{I}(t_{1}, t_{2})$$

$$= \frac{1}{2} \int_{t_{i}}^{t_{f}} dt_{1} \int_{t_{i}}^{t_{f}} dt_{2} \left[ r^{I}(t_{1}, t_{2}) + r^{I}(t_{2}, t_{1}) \right]$$

(34)

where

$$r^{I}(t_{1}, t_{2}) \equiv \nu f(t_{1} - t_{2}) \sum_{j,k,l,m} \Delta_{jm} \Delta_{lk} C_{jk}^{*}(t_{1}) C_{lm}^{*}(t_{2}) C_{tm}(t_{1}) C_{jk}(t_{2}) e^{-\frac{i}{\hbar}(E_{jk} - E_{lm})(t_{2} - t_{1})}$$

(35)

In the second line of (34) we have replaced the integrand by its real part, for the reasons discussed above, utilizing the property

$$[r^{I}(t_{1}, t_{2})]^* = r^{I}(t_{2}, t_{1})$$

(36)

G. Complete action and variational analysis

Then the full action is

$$S = S^{I} + S^{I} + S^{I} + R^{I}$$

$$= \int_{t_{i}}^{t_{f}} dt \left[ s^{12}(t) + s^{I}(t) \right] + \frac{1}{2} \int_{t_{i}}^{t_{f}} dt_{1} \int_{t_{i}}^{t_{f}} dt_{2} \left[ r^{I}(t_{1}, t_{2}) + r^{I}(t_{2}, t_{1}) \right]$$

$$= \int_{t_{i}}^{t_{f}} dt_{1} \int_{t_{i}}^{t_{f}} dt_{2} \left\{ \frac{1}{2T} \left[ s^{12}(t_{1}) + s^{I}(t_{1}) + s^{12}(t_{2}) + s^{I}(t_{2}) \right] + \frac{1}{2} \left[ r^{I}(t_{1}, t_{2}) + r^{I}(t_{2}, t_{1}) \right] \right\}$$

(37)

where $s^{12}$ and $s^{I}$ are the integrands (including prefactors) in $(S^{1} + S^{2})$ and $S^{I}$, as given in (22) and (24):

$$s^{12} = B \sum_{j,k} \left[ |\dot{C}_{jk}|^{2} + \frac{i}{\hbar} E_{jk} \left( C_{jk}^{*} \dot{C}_{jk}^{*} - \dot{C}_{jk}^{*} C_{jk} \right) \right]$$

(38)

$$s^{I} = \mu \sum_{j,k} \Delta^{2}_{jk} |C_{jk}|^{2}$$

(39)

We observe that in this form, the integrand of $S$ is real and symmetric in $t_{1}$ and $t_{2}$. It depends on the coefficients $\{C_{pq}\}$ at two times. We need to find a critical point of the action subject to the constraint (15). In the Appendix we outline the analysis of such a problem, including the use of a Lagrange multiplier $\lambda(t)$ to enforce the constraint, leading to integral equation (A.22). Varying $C_{jk}^{*}$ by that procedure and defining the operator

$$W \equiv \frac{\partial}{\partial C_{jk}^{*}(t_{1})} - \frac{\partial}{\partial t_{2}} \left| \frac{\partial}{\partial \dot{C}_{jk}^{*}(t_{1})} \right|$$

(40)
we find
\[ 0 = \frac{1}{2} W s^2(t_1) + \frac{1}{2} W s'(t_1) + \frac{1}{2} \int_{t_i}^{t_f} dt_2 \ W \left[ r^I(t_1, t_2) + r^I(t_2, t_1) \right] + T \lambda(t_1) \frac{\partial}{\partial C_{jk}^*(t_1)} \left[ \sum_{j,k} |C_{jk}(t_1)|^2 - 1 \right] \]  
(41)

This becomes
\[ \ddot{C}_{jk}(t) = \frac{2i}{\hbar} E_{jk} \dot{C}_{jk}(t) + \frac{1}{B} \left[ \mu \Delta_{jk}^2 + 2T \lambda(t) \right] C_{jk}(t) + \frac{\nu}{B} \tilde{C}_{jk}(t) \]  
(42)
in which we define the function
\[ \tilde{C}_{jk}(t) \equiv \sum_{\ell,m} \Delta_{jm} \Delta_{\ell k} C_{\ell m}(t) \int_{t_i}^{t_f} dt' C_{\ell m}^*(t') C_{jk}(t') f(t - t') e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \]  
(43)

It can be seen by varying the action by \( C_{jk} \) instead of \( C_{jk}^* \) that \( \lambda(t) \) must be real. To find it, we note that the second derivative of the normalization condition (15) is
\[ 2 \sum_{j,k} \left( |\dot{C}_{jk}|^2 + \text{Re} \left\{ C_{jk}^* \ddot{C}_{jk} \right\} \right) = 0 \]  
(44)

We eliminate \( \ddot{C}_{jk} \) between (42) and (44) and then solve for (a constant times) \( \lambda(t) \):
\[ \frac{2T \lambda}{B} = -\sum_{j,k} \left( |\dot{C}_{jk}|^2 + \frac{\mu}{B} \Delta_{jk}^2 |C_{jk}|^2 + \text{Re} \left\{ \frac{2i}{\hbar} E_{jk} C_{jk}^* \dot{C}_{jk} + \frac{\nu}{B} C_{jk}^* \tilde{C}_{jk} \right\} \right) \]  
(45)

Substituting that expression into (42),
\[ \ddot{C}_{jk} = \frac{2i}{\hbar} E_{jk} \dot{C}_{jk} + \frac{\mu}{B} C_{jk} \left( \Delta_{jk}^2 - \sum_{\ell,m} \Delta_{\ell m}^2 |C_{\ell m}|^2 \right) \]  
\[ + \frac{\nu}{B} \left( \dot{C}_{jk} - C_{jk} \text{Re} \left\{ \sum_{\ell,m} C_{\ell m}^* \dot{C}_{\ell m} \right\} \right) - C_{jk} \sum_{\ell,m} \left( |\dot{C}_{\ell m}|^2 + \text{Re} \left\{ \frac{2i}{\hbar} E_{\ell m} C_{\ell m}^* \dot{C}_{\ell m} \right\} \right) \]  
(46)

This is the equation that we expect describes the evolution of the complete system (that is, system + apparatus), as described by the coefficients \( \{C_{jk}(t)\} \) in the normal–mode expansion (9).

The BCs to be applied with (46) are specified values of \( \{C_{jk}(t_i)\} \) from the initial preparation, and the NBC at \( t_f \), which takes the form
\[ 0 = \int_a^b dt_2 \left( \frac{\partial L}{\partial \ddot{C}_{jk}^*(t_1)} \right) \bigg|_{t_1 = t_f} \]  
(47)
in which \( L \) is the integrand in the full action on the last line of (37). This form of the NBC is derived in the appendix.
H. Alternative treatment of the normalization condition

Comparison of (42) with (46) shows that rigorous enforcement of the normalization condition (14) or (15) has complicated the mathematics. Since we hope to show that experimental results of great simplicity and generality (e.g. Born’s rule) follow from this theory, we are suspicious of the additional complexity and wonder whether it is absolutely necessary to satisfy the stated normalization condition at every instant $t$.

Our skepticism about that requirement is also based on a thought experiment described by Renninger[16], which is equivalent to the following description. An excited atom at the origin is known to emit a photon at $t = 0$, but the direction is unknown, so the photon’s wavefunction satisfies $|\psi|^2 = \delta(r - ct)/(4\pi r^2)$. A perfectly collecting hemispherical detector screen occupies the upper half of the sphere $r = 1$ light-second. Therefore, if the photon’s emission direction is within $\theta < \pi/2$, it is collected and extinguished at $t = 1$ second. Otherwise, it is not registered by the detector screen, and its wavefunction changes to satisfy $|\psi|^2 = \delta(r - ct)H(\theta - \pi/2)/(2\pi r^2)$, where $H$ is the Heaviside function. The instantaneous change in the denominator from $4\pi r^2$ to $2\pi r^2$ at $t = 1$ is not due to any measurement, for there is none, nor to any physical change in the photon; it arises entirely from the normalization requirement. This seems unphysical, and our suspicion deepens when we consider that this description depends on choice of reference frame; for instance, in any other frame the detector screen would not be (hemi)spherical but spheroidal, and so the resulting change in magnitude of the uncollected wavefunction would happen over a nonzero interval of time.

A more physically sound description would be that a photon intercepted by the detector screen does not simply vanish; it interacts with (a) particle(s) of the screen to produce some physical effect, for instance dislodging a photoelectron. A more complete description of the experiment would include that effect. Since half of the outgoing spherical photon wavefunction participates in that effect, it is unreasonable for the uncollected half to double its weight to satisfy a normalization condition. We argue instead that the outgoing uncollected photon wavefunction after $t = 1$ should be normalized to integrate to $1/2$, and with that change we see that a discontinuous and unphysical change is no longer needed in that uncollected part at $t = 1$.

Armed with our reasoning that the normalization condition (14) is not absolute, we pro-
pose to relax it for the experiment that is the subject of this paper. Although for many experiments we do not expect to lose any of the wavefunction weight in mid-experiment, we point out that the total weight of the wavefunction (unity, meaning one particle of whatever type is being described) is known only at $t_i$ and $t_f$. There is not, nor can there be, any experimental evidence for a unity (or any other) value of the weight at intermediate times. Therefore we propose that (4) is a constraint only at $t_i$ and $t_f$. This is easily handled mathematically; we simply stipulate that (15) is part of the initial and final conditions. Then we can dispense with the Lagrange multiplier altogether, so the IDE to be satisfied is

$$\ddot{C}_{jk}(t) = \frac{2i}{\hbar} E_{jk} \dot{C}_{jk}(t) + \frac{\mu}{B} \Delta_{jk}^2 C_{jk}(t) + \frac{\nu}{B} \tilde{C}_{jk}(t) \quad (48)$$

Since we regard the simplicity of this equation in comparison to (46) as an argument for its plausibility, we will adopt it rather than the latter in the remaining sections of the paper; nevertheless, much of the following reasoning can be applied to (46) as well at the cost of more algebra.

III. COMPARISON TO DESIRED PROPERTIES

A. Stability of a superposition in the absence of a measurement

We observe at this point that (48) predicts the stability of an unperturbed superposition, as it should. When there is no interaction between the system and the measurement apparatus, $\mu = \nu = 0$. The resulting equation

$$\dot{C}_{jk} = \frac{2i}{\hbar} E_{jk} \dot{C}_{jk} \quad (49)$$

has the solution $\dot{C}_{jk} = 0 \forall j, k$, that is, stability of the superposition. Furthermore, since (for each subsystem $\ell = 1$ or 2) the modes in the expansion (14) were defined as solutions of the no-measurement wave equation, the stable solution resulting from our analysis here agrees with the solution of the ordinary wave equation for each isolated system.

B. Collapse to a single eigenstate with $\sigma^1_j = \sigma^2_k$

This includes three events we expect in a measurement: system 1 must collapse to a single eigenstate of $\sigma^1_{op}$, or a superposition of eigenstates with the same eigenvalue; system
2 must similarly collapse; and the eigenvalues of the two systems must agree. The third condition (measurement) requires that for any \( j, k \),

\[
\Delta_{jk} = 0 \quad \text{or} \quad C_{jk} = 0
\]  

(50)

Although we will not analyze the differential equation (48) to describe the approach to these three conditions, we will show that it is consistent with their satisfaction in the steady state, when all time derivatives of \( \{C_{pq}\} \) vanish. Thus it is plausible for the combined system to reach such a state, and having done so, to remain in that state.

We see that condition (50) together with the steady-state condition cause every term in (48) to vanish except possibly the last. To understand those terms, consider that after the system attains a steady state, we can replace all the factors \( C_{pq} \) or \( C_{pq}^* \) on the RHS of (43) by their final values, which satisfy (50). Then at times \( t \) greater than \( \tau \) after the full system reaches its steady state, any nonzero terms \( \ell, m \) on the RHS must have

\[
\Delta_{jk} = \Delta_{\ell m} = 0
\]  

(51)

If either of systems 1 and 2 has collapsed to a single state (or a set of states with a single eigenvalue), then by (50) the other system has also collapsed, and it is easy to see that (51) implies that \( \Delta_{jm} = \Delta_{\ell k} = 0 \), so the only possible nonzero term in \( \tilde{C}_{jk} \) is zero after all. Therefore the last term in (48) vanishes, so the equation is consistent with the supposed late-time steady state. On the other hand, if systems 1 and 2 have not collapsed, there are terms in (43) that do not trivially vanish. We conclude that the evolution equation predicts that a late-time steady state is only possible if both the measurement condition is satisfied (the apparatus state corresponds to the state of the system being measured) and both systems have collapsed to a single eigenvalue.

We would prefer to have a more rigorous analysis, both disposing of the possibility that the combined system never reaches a steady state and describing the approach to the steady state. This analysis must await future work, possibly including numerical studies. Our objective in this paper is to show the possibility that a variational principle of the type we have developed can explain the measurement problem.
C. Consistency with Born’s rule

The well-known experimental observation is that in an ensemble of identically-prepared measurements of some property (eigenvalue), beginning with a system in a superposition of modes with different values of the eigenvalue, the expected proportion of outcomes equal to a particular value will be the weight of that value in the superposition. (At this point we take it as given that the system will collapse to a single value of the eigenvalue.) In our case, where the system being measured is denoted $\ell = 1$, the weight corresponding to eigenvalue $\sigma_j^1$ is

$$P_j = \sum_k |C_{jk}(t_i)|^2$$

(52)

(More generally, it is $\sum_{j,k} |C_{jk}(t_i)|^2$, where the sum on $j$ is over all modes with a single value of the eigenvalue. For simplicity, we will consider only the non-degenerate case, but the extension to the more general case should be straightforward.)

It will be convenient to denote averages over an ensemble of identically prepared experimental realizations by an overbar. Then, if it is taken as given that the collapse to a single eigenvalue is complete by $t_f$, we can see that the relation

$$\overline{P_j(t_i)} = \overline{P_j(t_f)} \quad \forall j$$

(53)

is equivalent to Born’s rule. This equivalence holds because at the initial time $t_i$, by the requirement of identical preparation, every member of the ensemble contributes the same value $P_j(t_i)$ to the ensemble average. At $t_f$, $P_j = 1$ in a fraction $P_j(t_i)$ of the realizations in the ensemble, and 0 in the others. So (53) is the relation that should be predicted by a successful theory.

We would like to be able prove that Born’s rule (53) follows from our nonlocal wave equation (46). The theoretical proof has eluded us so far; we may ultimately have to rely on numerical studies. However, we sketch out here some of the ideas that may contribute to the theoretical analysis.

By differentiating (52) twice, we see that (supposing that by the system preparation $\dot{P}_j(t_i) = 0$)

$$\dot{P}_j(t) = 2 \int_{t_i}^t dt' \sum_k \left( |\dot{C}_{jk}|^2 + \text{Re} \left\{ C_{jk}^* \dot{\bar{C}}_{jk} \right\} \right)$$

$$= 2 \int_{t_i}^t dt' \sum_k \left( |\dot{C}_{jk}|^2 + \frac{\hbar}{B} \Delta_{jk}^2 |C_{jk}|^2 + \text{Re} \left\{ \frac{2i}{\hbar} E_{jk} C_{jk}^* \dot{\bar{C}}_{jk} + \frac{\nu}{B} C_{jk}^* \dot{\bar{C}}_{jk} \right\} \right)$$

(54)
In the term in the integrand involving $E_{jk}$, let $C'_{jk} = X e^{i\phi}$ for real $X$ and $\phi$. Then
\[
\text{Re} \left\{ \frac{2i}{\hbar} E_{jk} C'^*_{jk} \dot{C}'_{jk} \right\} = \text{Re} \left\{ \frac{2i}{\hbar} E_{jk} X (X + iX \dot{\phi}) \right\} = -\frac{2}{\hbar} E_{jk} X^2 \dot{\phi}
\]
so
\[\text{Re} \left\{ \frac{2i}{\hbar} E_{jk} C'^*_{jk} \dot{C}'_{jk} \right\} = 0 \quad (55)\]
by symmetry, since the phase $\phi$ is equally likely to increase or decrease.

To deal with the term $\text{Re} \left\{ \frac{\nu}{\hbar} B_{jk} C'^*_{jk} \tilde{C}_{jk} \right\}$, we note that
\[
C'^*_{jk}(t) \tilde{C}_{jk}(t) = \Delta_{jk}^2 \langle \langle C^2_{jk}(t) \rangle \rangle |C_{jk}(t)|^2 
+ C'^*_{jk}(t) \sum_{\ell,m}' \Delta_{jm} \Delta_{\ell k} C_{\ell m}(t) \int_{t_i}^{t_f} dt' \ C^*_{\ell m}(t') C_{jk}(t') f(t - t') e^{-\frac{1}{\hbar} (E_{jk} - E_{\ell m})(t - t')} \quad (57)
\]
in which we define the “moving average”
\[
\langle \langle C^2_{jk}(t) \rangle \rangle \equiv \int_{t_i}^{t_f} dt' f(t - t') |C_{jk}(t')|^2
\]
and the primed sum denotes the sum over all $\ell, m$ except the single term $\ell = j, m = k$.

We have hypothesized that the solution $\{C_{jk}(t) \forall j, k, t\}$ of the variational principle is constrained by the initial (preparation) condition at $t_i$ and the final (NBC) condition at $t_f$. We now venture a little further and suppose that the desired solution, in order to extremize the action, uses the entire interval from $t_i$ to $t_f$ to evolve from initial to final values of $\{C_{jk}\}$; this is plausible due to the term in the action $|\langle \psi^\ell|\dot{\psi}^\ell \rangle|$ in (18) or $\int dt|\dot{C}_{jk}|^2$ in (22) that penalizes rapid transitions. Therefore $|\dot{C}_{jk}| \sim 1/T$. But a measurement adequate to resolve two states $j, k$ and $\ell, m$ with $E_{jk} \neq E_{\ell m}$ is conventionally understood to require a duration
\[
T \gg \frac{\hbar}{|E_{jk} - E_{\ell m}|} \quad (59)
\]
We conclude therefore that there is an $\epsilon$ such that
\[
\frac{\hbar |\dot{C}_{pq}|}{|E_{jk} - E_{\ell m}|} < \epsilon \ll 1 \quad (60)
\]
for any $p, q$ and for any choice of $j, k, \ell, m$ for which $E_{jk} \neq E_{\ell m}$. We may also require the function $f$ to be slowly varying in the sense that
\[
\frac{\hbar |\dot{f}|}{|E_{jk} - E_{\ell m}| f_{max}} < \epsilon \ll 1 \quad (61)
\]
where \( f_{\text{max}} \) is the maximum value taken by \( f \). Consequently, with the additional assumption that \( E_{jk} = E_{\ell m} \) only if \( j = \ell \) and \( k = m \), the integral in the second term of \( \text{(57)} \) can be integrated by parts twice:

\[
\int_{t_i}^{t_f} dt' \, C_{\ell m}^*(t') \, C_{jk}(t') \, f(t - t') \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \nonumber
\]

\[
= \frac{i\hbar}{E_{jk} - E_{\ell m}} \left[ C_{\ell m}^*(t') \, C_{jk}(t') \, f(t - t') \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \right]_{t' = t_i}^{t_f} \nonumber
\]

\[
+ \frac{\hbar^2}{(E_{jk} - E_{\ell m})^2} \int_{t_i}^{t_f} dt' \, \frac{d^2}{dt'^2} \left[ C_{\ell m}^*(t') \, C_{jk}(t') \, f(t - t') \right] \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \nonumber
\]

\[
+ \frac{\hbar^2}{(E_{jk} - E_{\ell m})^2} \int_{t_i}^{t_f} dt' \, \frac{d^2}{dt'^2} \left[ C_{\ell m}^*(t') \, C_{jk}(t') \, f(t - t') \right] e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \nonumber
\]

\[
= \frac{i\hbar}{E_{jk} - E_{\ell m}} \left[ C_{\ell m}^*(t') \, C_{jk}(t') \, f(t - t') \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t' - t)} \right]_{t' = t_i}^{t_f} [1 + O(\epsilon)] \quad (62) \nonumber
\]

Our hypothesis to explain the apparent randomness of quantum mechanical measurements is that some “hidden variable” is not sufficiently well controlled in typical practice to determine a single outcome. Here the hidden variable appears to be the stop time \( t_f \) or equivalently the duration \( T \) of the experiment. If the uncertainty in \( t_f \) is \( \gg 1/\Delta E \) for the smallest energy difference \( \Delta E \), the realization average of the complex exponential factor \( \exp[-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t_f - t)] \) is zero. We would like to infer from that, neglecting \( O(\epsilon) \), that the realization average of \( \text{(62)} \) vanishes, but there are two problems. We cannot factor the realization average

\[
\frac{C_{\ell m}^*(t_f) \, C_{jk}(t_f) \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t_f - t)}}{C_{\ell m}^*(t_f) \, C_{jk}(t_f) \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t_f - t)}} \neq \frac{C_{\ell m}^*(t_f) \, C_{jk}(t_f) \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t_f - t)}}{C_{\ell m}^*(t_f) \, C_{jk}(t_f) \, e^{-\frac{i}{\hbar}(E_{jk} - E_{\ell m})(t_f - t)}} \quad (63) \nonumber
\]

because the final values of the coefficients \( C_{\ell m}^* \) and \( C_{jk} \) are correlated with the complex exponential factor. Also, the \( t' = t_i \) term in \( \text{(62)} \) will not average to zero; since the initial conditions are imposed at the start time, uncertainty is \( t_i \) is presumably not a source of variation in the outcome.

From the surviving terms in the realization average of \( \text{(54)} \) we see that

\[
\langle P_j(t_f) \rangle - \langle P_j(t_i) \rangle = \int_{t_i}^{t_f} dt' \, \langle P_j(t') \rangle \nonumber
\]

\[
= 2 \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} dt'' \, p(t'') \nonumber
\]

\[
= 2 \int_{t_i}^{t_f} dt'' \, (t_f - t'') \, p(t'') \quad (64) \nonumber
\]
and therefore
\[ |P_j(t_f) - P_j(t_i)| < 2T \int_{t_i}^{t_f} dt'' |p(t'')| \] (65)

with
\[ p \equiv \sum_k |\dot{C}_{jk}|^2 + \frac{\mu}{B} \Delta^2_{jk}|C_{jk}|^2 + \frac{\nu}{B} \Delta^2_{jk} \langle \langle C^2_{jk} \rangle \rangle |C_{jk}|^2 \] (66)

If the previously identified issues in the proof of Born’s rule are resolved, it remains to show that the LHS of (65) vanishes, at least in the limit at \( T \to \infty \). (As noted earlier, experimental results at variance with Born’s rule are likely to be rejected as invalid if \( T \) is too small.) To do that, we must show that \( p(t) \) decays fast enough that the integral in (65) decreases faster than \( 1/T \).

IV. DISCUSSION

A. Sensitivity of the system evolution to a measurement

Traditional discussions of quantum mechanics maintain that making a measurement changes the evolution of a quantum system from its unitary evolution, as described by the wave equation, to a collapsed state, as described by the measurement side of the bipartite theory. Thus the unitary evolution cannot be observed without interrupting it. This remarkable sensitivity to observation is not explained except as the inevitable corollary of the special treatment of measurement in the theory.

We also find this sensitivity to observation in our picture, but can give more of an explanation for it. The act of measuring a system involves causing it to physically interact with a measurement apparatus, and the variational principle describes the evolution of the combined system. The readout of the measurement at \( t_f \) defines the end of the domain of integration of the variational principle. Of course, the theory continues to apply after \( t_f \), but the observation at \( t_f \), like its preparation at \( t_i \) and its spatial boundary conditions, imposes a leakproof barrier to influences from outside the problem domain, so that a solution may be found within that domain without reference to the rest of the universe.

Now if the measurement apparatus were read at some intermediate time \( t_m \), the structure of the problem would be different. Instead of applying between \( t_i \) and \( t_f \), the variational principle would apply twice, from \( t_i \) to \( t_m \) and from \( t_m \) to \( t_f \). The appearance of a constraint at \( t_m \) as a final condition on the first interval and an initial condition on the second would
make this a different problem than the original one from \( t_i \) to \( t_f \). (As we have explained, the intervention at \( t_m \) results in the appearance of an NBC on the solution between \( t_i \) and \( t_m \), even though it does not dictate the result of the reading at \( t_m \).) Consequently, the act of observing the system at \( t_m \) changes it, just as in conventional interpretations.

The reader may object that we have not removed the mystery but moved it to a different concept. Instead of declaring by fiat that a measurement changes the system, we have declared that the domain of integration of the variational principle must end at the time (and place) at which the measurement apparatus is read. We haven’t explained what is special about the events at \( t_f \) that allow us to end the domain there.

The criticism is valid, but we point out that we have pushed back the mystery, or made it less mysterious, by relating it to considerations of BCs. Certainly the description of a measurement in terms of an action integral bounded at \( t_i \) and \( t_f \) must be an approximation to a more complete theory that includes a greater time interval before and after \([t_i, t_f]\) and a fuller description of the measurement process. On the other hand, the empirical fact that broad statements of great generality apply to measurements, regardless of the system under study or the mechanism of the process, strongly suggests that a simple description is possible, particularly regarding a time before the measurement \((t_i)\) and a time after its completion \((t_f)\). The validity of the simple description is not necessarily a surprise; it may be that the interactions that can be so described have been adopted as measurement procedures precisely because of their ability to give repeatable quantitative results.

If the simple description proposed in this paper turns out to be successful in description and prediction at some level of approximation, that will be evidence of its usefulness, without denying the possibility of a more complete theory. Eventually such an improved theory may show that collapse/decay to a single eigenvalue occurs at \( t_f \) in a physically justifiable way, based on the role of the apparatus in the action, and so it is appropriate to simplify the problem as we have done by terminating the integral at \( t_f \) and accepting the NBC there.

An extended analysis of that type would also be appropriate to explore another aspect of the new theory. We have argued that we can solve the variational principle between \( t_i \) and \( t_f \), which would presumably enable a prediction of the experimental outcome at \( t_f \) (based on (a) fixed value(s) of hidden variable(s), of course). We have asserted that the final condition at \( t_f \) provides a leakproof barrier to influences from outside that problem domain. But the theory must apply under reversal of the direction of time, so it should also
be possible to apply an experimental preparation (initial condition) at $t_{f2} \equiv t_f + T$ and a
measurement readout (NBC as a final condition) at $t_f$ to predict an outcome at $t_f$ based on
physics between $t_f$ and $t_{f2}$. We suspect that the theory retains sufficient flexibility to allow
the two solutions (for $t_i \leq t \leq t_f$ and $t_f \leq t \leq t_{f2}$) to agree at $t_f$. It probably helps that
we expect (in both cases) to apply natural BCs at $t_f$, so we are not actually constraining
the value of the measured variable. Also, continuity constraints on fields, wavefunctions
and derivatives appearing in the action may help to avoid contradictions. Since these two
predictions must agree, the barrier at $t_f$ is not completely leakproof. It is rather a partially
permeable membrane, as suggested by the applicability of an NBC that constrains some but
not all properties of the system at $t_f$. This type of study may give insight into the nature
of the constraint imposed by the measurement readout.

B. Causality and time–ordering issues

Retrocausality—the dependence of phenomena at a given time on phenomena in their
future—conflicts with the usual notion of causality—the concept that causes precede their
effects in time. However, multiple authors[15, 18, 19] have pointed out that such a notion
of causality is not necessary to avoid contradictions. If event $A \Rightarrow B$, then $B \Rightarrow \neg A$
would produce a contradiction. But if we are somehow prevented from declaring that $B \Rightarrow \neg A$
(or an equivalent combination of statements), then in principle $A \Rightarrow B$ is possible even if $B$
occurs earlier than $A$.

To apply this to our use of retrocausality in the variational principle, we are asserting
that the NBC at $t_f$ (which applies because a measurement is made at that time, even though
the result of the measurement is unconstrained) is an event $A$ that constrains the solution
between $t_i$ and $t_f$, so that solution at some intermediate time $t_m$ can be considered as event
$B$. But the event $B$ thus chosen is by definition consistent with $A$, since it is a point along
the solution based on $A$. It is not possible to claim that $B \Rightarrow \neg A$, so no contradiction is
possible.

Of course, the usual objection to this is that one could intervene at $t_m$ to change the
trajectory of events and produce $\neg A$ at $t_f$ (going back in time and shooting one’s grandpar-
ent, in the usual cliche). But doing this changes the problem, as described above; now the
variational principle applies from $t_i$ to $t_m$ and from $t_m$ to $t_f$, with the intervention imposing
new BCs at \( t_m \). Since this is a different problem than the original one, the original solution does not apply and no claim of a contradiction can be made.

C. Choice of the function \( f \)

We have relied on a supposed interaction between wavefunctions at \( t_1 \) and \( t_2 \), as expressed in the nonlocal action term (29). The interaction is a physical process with a temporal range described by the function \( f \). It will be important to determine the form of \( f \); this may be explored numerically, but additional physical insight could be very useful.

Our earlier hypotheses that \( f \) is a decreasing function of the absolute value of its argument and that it has a finite range \( \tau \) are intuitively appealing, but they are not the only possibility. In fact, we cannot rule out the opposite extreme, that \( f(t) \equiv 1 \). This would mean that the nonlocal interaction has infinite range, but in practice for a given measurement it would be limited to the interval \( [t_i, t_f] \). (Without the finite–range limit \( \tau \), our analysis in section III B would have to be revisited.)

D. Solving the integrodifferential equation

As mentioned above, it will be important to solve, or otherwise study, the IDE (48). That effort may be made theoretically, or numerically if need be. We would like to understand under what conditions the system reaches the collapsed state described in section III B, how fast that late–time state is approached, and which of the possible collapsed states is reached, as a function of the hidden variable(s). It will also be important to test whether the equation produces outcome frequencies consistent with Born’s rule, possibly following ideas in section III C.

One question is whether, given a choice of initial conditions and hidden variable(s), the solution to the IDE is unique (and even whether a solution exists). If there is always a unique solution, the theory may be completely deterministic (although it remains to be seen what that means for a retrocausal theory), so we may be able to dispense completely with the idea that quantum mechanical processes depend on intrinsically random variables. Such a discovery might have far–reaching ramifications in quantum information technologies that rely on (supposed) randomness.
If this understanding enables us to make predictions based on the theory, we will look for experimentally testable predictions. Although we have argued that the new theory will agree with many features of conventional theory, it is certainly possible that it could differ in some ways. One possibility is that results that have historically been seen to vary, supposedly due to intrinsic randomness, may vary less or not at all if a hidden (that is, historically uncontrolled) variable is controlled in new experiments (guided by new predictions about how well or to what values it must be controlled).

Of course, it is possible that the particular choice of action we have made, and the IDE resulting from it, do not correspond to nature. Even in that case, our exposition here shows that a variational principle of this type, including our assumptions of retrocausality, nonlocality, and one or more hidden variables, can lead to a plausible theory that avoids, resolves or explains problematic features of conventional quantum theory. If the theory presented here is not borne out, a similarly–constructed theory with a different form of the action may be more successful.

V. ACKNOWLEDGMENTS

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Appendix: Calculus of Variations: Two–time Variant

A basic problem in the calculus of variations [11] is to find the function \( \phi(t) \) for which the integral

\[
S[\phi] = \int_a^b dt \, F(t, \phi(t), \dot{\phi}(t))
\]

is stationary with respect to infinitesimal changes in the function \( \phi \). Here \( F \) is a given function with continuous first partial derivatives and piecewise continuous second derivatives. The function \( \phi(t) \) is required to be continuous with piecewise continuous first derivative, and must satisfy

\[
\phi(a) = A \quad \phi(b) = B
\]
for given $A$ and $B$. Under these conditions a necessary condition for (A.1) is the Euler equation

$$0 = \frac{\partial F}{\partial \phi} - \frac{d}{dt} \frac{\partial F}{\partial \dot{\phi}}$$

(A.3)

1. Two–time variant

In our case the integrated function $F$ depends on the unknown function $\phi$ at two times, both of which are integrated over:

$$S[\phi] = \int_a^b \int_a^b dt_1 \int_a^b dt_2 F(t_1, t_2, \phi(t_1), \dot{\phi}(t_1), \phi(t_2), \dot{\phi}(t_2))$$

(A.4)

As in the standard derivation, we find a necessary condition by defining

$$\theta(t, \epsilon) = \phi(t) + \epsilon \eta(t)$$

(A.5)

and requiring that

$$\frac{dS[\theta]}{d\epsilon} \bigg|_{\epsilon=0} = 0$$

(A.6)

for any continuous function $\eta(t)$ with piecewise continuous derivative and

$$\eta(a) = \eta(b) = 0$$

(A.7)

Condition (A.5) becomes

$$0 = \int_a^b \int_a^b dt_1 \int_a^b dt_2 \left[ \eta(t_1) \frac{\partial F}{\partial \phi(t_1)} + \eta(t_1) \frac{\partial F}{\partial \phi(t_1)} + \eta(t_2) \frac{\partial F}{\partial \phi(t_2)} + \eta(t_2) \frac{\partial F}{\partial \phi(t_2)} \right]$$

$$= \int_a^b \int_a^b dt_1 \int_a^b dt_2 \left[ \eta(t_1) \left( \frac{\partial F}{\partial \phi(t_1)} - \frac{\partial}{\partial t_1} \bigg|_{t_2} \frac{\partial F}{\partial \phi(t_1)} \right) + \eta(t_2) \left( \frac{\partial F}{\partial \phi(t_2)} - \frac{\partial}{\partial t_2} \bigg|_{t_1} \frac{\partial F}{\partial \phi(t_2)} \right) \right]$$

(A.8)

Since $\eta(t)$ is arbitrary (subject to the restrictions already stated), this requires that

$$0 = \int_a^b dt_2 \left( \frac{\partial F}{\partial \phi(t_1)} - \frac{\partial}{\partial t_1} \bigg|_{t_2} \frac{\partial F}{\partial \phi(t_1)} \right)$$

(A.9)

and

$$0 = \int_a^b dt_1 \left( \frac{\partial F}{\partial \phi(t_2)} - \frac{\partial}{\partial t_2} \bigg|_{t_1} \frac{\partial F}{\partial \phi(t_2)} \right)$$

(A.10)

as necessary conditions for the stationarity of $S[\phi]$. 31
2. Special cases

A special case of interest is when $F$ factors into $t_1$–dependent and $t_2$–dependent factors:

$$F(t_1, t_2, \phi(t_1), \dot{\phi}(t_1), \phi(t_2), \dot{\phi}(t_2)) = G(t_1, \phi(t_1), \dot{\phi}(t_1))H(t_2, \phi(t_2), \dot{\phi}(t_2))$$  \hspace{1cm} (A.11)

so that

$$S[\phi] = \int_a^b dt_1 G(t_1, \phi(t_1), \dot{\phi}(t_1)) \int_a^b dt_2 H(t_2, \phi(t_2), \dot{\phi}(t_2))$$  \hspace{1cm} (A.12)

and the necessary conditions (A.9) and (A.10) become

$$0 = \frac{\partial G}{\partial \phi(t_1)} - \frac{d}{dt_1} \frac{\partial G}{\partial \dot{\phi}(t_1)}$$  \hspace{1cm} (A.13)

and

$$0 = \frac{\partial H}{\partial \phi(t_2)} - \frac{d}{dt_2} \frac{\partial H}{\partial \dot{\phi}(t_2)}$$  \hspace{1cm} (A.14)

if we exclude the possibility that either of the integrals in (A.12) vanishes. These relations are of course the stationarity conditions for those two integrals if they were considered independently. We observe that the special case in which $F$ factors as in (A.11) is significantly different than the general case, in that the solution of the former can be expressed as differential equations but the latter requires integral equations.

In this paper our concern is limited to functions $F$ that are symmetric in $t_1$ and $t_2$, that is, invariant under their interchange. For this special case, equations (A.9) and (A.10) are equivalent, as are (A.13) and (A.14).

3. Natural boundary condition

Consider the case in which the boundary conditions (A.2) are replaced by

$$\phi(a) = A$$  \hspace{1cm} (A.15)

that is, the solution is not constrained at $t = b$ (except, as will be shown, by the NBC). Then condition (A.7) is replaced by

$$\eta(a) = 0$$  \hspace{1cm} (A.16)
(no constraint on $\eta(b)$) and so the second line of (A.8) becomes
\[
0 = \int_a^b dt_1 \int_a^b dt_2 \left[ \eta(t_1) \left( \frac{\partial F}{\partial \dot{\phi}(t_1)} - \frac{\partial}{\partial t_1} \left|_{t_2} \right. \frac{\partial F}{\partial \dot{\phi}(t_1)} \right) + \eta(t_2) \left( \frac{\partial F}{\partial \dot{\phi}(t_2)} - \frac{\partial}{\partial t_2} \left|_{t_1} \right. \frac{\partial F}{\partial \dot{\phi}(t_2)} \right) \right] \\
+ \int_a^b dt_2 \eta(t_1) \left( \frac{\partial F}{\partial \dot{\phi}(t_1)} \right) \Big|_{t_1=b}^b
\]
(A.17)

But since the functions $\eta(t)$ satisfying (A.7) are among the set of functions allowed by (A.16), $F$ must satisfy (A.9) and (A.10), so the last equation becomes simply
\[
0 = \int_a^b dt_2 \eta(t_1) \left( \frac{\partial F}{\partial \dot{\phi}(t_1)} \right) \Big|_{t_1=a}^b \\
= -\eta(b) \int_a^b dt_2 \left( \frac{\partial F}{\partial \dot{\phi}(t_1)} \right) \Big|_{t_1=b}^b
\]
(A.18)

so we find that the NBC is
\[
0 = \int_a^b dt_2 \left( \frac{\partial F}{\partial \dot{\phi}(t_1)} \right) \Big|_{t_1=b}^b
\]
(A.19)
and of course by symmetry
\[
0 = \int_a^b dt_1 \left( \frac{\partial F}{\partial \dot{\phi}(t_2)} \right) \Big|_{t_2=b}^b
\]
(A.20)

4. Lagrange multipliers

A related problem is to find a stationary point of $S[\phi]$, as given by (A.4), subject to a constraint
\[
K(t, \phi(t)) = 0 \quad \forall t
\]
(A.21)
This can be addressed by the method of Lagrange multipliers, in a straightforward extension of the derivation given in reference [11]. For the special case of symmetric $F$, that analysis shows that we can introduce a Lagrange multiplier $\lambda(t)$ and replace condition (A.9) by
\[
0 = \int_a^b dt_2 \left( \frac{\partial F}{\partial \phi(t_1)} - \frac{\partial}{\partial t_1} \left|_{t_2} \right. \frac{\partial F}{\partial \phi(t_1)} + \lambda(t_1) \frac{\partial K}{\partial \dot{\phi}(t_1)} \right) \\
= (b-a) \lambda(t_1) \left. \frac{\partial K}{\partial \phi} \right|_{t=t_1} + \int_a^b dt_2 \left( \frac{\partial F}{\partial \phi(t_1)} - \frac{\partial}{\partial t_1} \left|_{t_2} \right. \frac{\partial F}{\partial \phi(t_1)} \right)
\]
(A.22)
The solution of this differential equation is \( \phi \) as a function of \( t \) and the entire function \( \lambda \). Finally, \( \lambda(t) \) is determined by requiring the satisfaction of (A.21).

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Note of course that if the experiment being described is the Renninger experiment, or some other experiment with sources or sinks of the wavefunction (the quantum field), then the normalization values at $t_i$ and $t_f$ will be modified in the ways just described, or in more complicated ways. For instance, if the Renninger experiment were augmented with a lower–hemisphere detector at $r = 2$, then there would be one final condition at $t = 1$ and $r = 1$, $\theta \leq \pi/2$ and another at $t = 2$ and $r = 2$, $\theta \geq \pi/2$, with a normalization value of 1 applied to the union of both collector surfaces at their respective collecting times.

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We expect that it will differ in the normalization factor applied to the wavefunction in experiments like that of Renninger, as discussed above, but that is a difference in how a physical state is described mathematically, not a difference in the state itself, and so not experimentally testable.