Transmutations and spectral parameter power series in eigenvalue problems

Vladislav V. Kravchenko and Sergii M. Torba

Abstract. We give an overview of recent developments in Sturm-Liouville theory concerning operators of transmutation (transformation) and spectral parameter power series (SPPS). The possibility to write down the dispersion (characteristic) equations corresponding to a variety of spectral problems related to Sturm-Liouville equations in an analytic form is an attractive feature of the SPPS method. It is based on a computation of certain systems of recursive integrals. Considered as families of functions these systems are complete in the $L_2$-space and result to be the images of the nonnegative integer powers of the independent variable under the action of a corresponding transmutation operator. This recently revealed property of the Delsarte transmutations opens the way to apply the transmutation operator even when its integral kernel is unknown and gives the possibility to obtain further interesting properties concerning the Darboux transformed Schrödinger operators.

We introduce the systems of recursive integrals and the SPPS approach, explain some of its applications to spectral problems with numerical illustrations, give the definition and basic properties of transmutation operators, introduce a parametrized family of transmutation operators, study their mapping properties and construct the transmutation operators for Darboux transformed Schrödinger operators.

Mathematics Subject Classification (2010). Primary 34B24, 34L16, 65L15, 81Q05, 81Q60; Secondary 34L25, 34L40.

Keywords. Sturm-Liouville operator, Sturm-Liouville problem, complex eigenvalue, transmutation operator, transformation operator, Schrödinger operator, spectral parameter power series, Darboux transformation, quantum well, scalar potential.

Research was supported by CONACYT, Mexico. Research of second named author was supported by DFFD, Ukraine (GP/F32/030) and by SNSF, Switzerland (JRP IZ73Z0 of SCOPES 2009–2012).
1. Introduction

Transmutation operators also called operators of transformation are a widely used tool in the theory of linear differential equations (see, e.g., [3], [10], [19], [51], [63] and the recent review [61]). It is well known that under certain quite general conditions the transmutation operator transmuting the operator \( A = -\frac{d^2}{dx^2} + q(x) \) into \( B = -\frac{d^2}{dx^2} \) is a Volterra integral operator with good properties. Its integral kernel can be obtained as a solution of a certain Goursat problem for the Klein-Gordon equation with a variable coefficient. In particular, the elementary solutions of the equation \( Bv = \lambda v \) are transformed into the solutions of the equation \( Au = \lambda u \). If the integral kernel of the transmutation operator is unknown, and usually this is the case, there is no way to apply it to an arbitrary smooth function. This obstacle strongly restricts the application of the transmutation operators confining it to purely theoretical purposes.

Recently, in [9] a relation of the transmutation operators with another fundamental object of the Sturm-Liouville theory was revealed. Sometimes this object is called the \( L \)-basis [24] where \( L \) refers to a corresponding linear ordinary differential operator. The \( L \)-basis is an infinite family of functions \( \{\varphi_k\}_{k=0}^\infty \) such that \( L\varphi_k = 0 \) for \( k = 0, 1 \), \( L\varphi_k = k(k-1)\varphi_{k-2} \), for \( k = 2, 3, \ldots \) and all \( \varphi_k \) satisfy certain prescribed initial conditions. In [41], [42], [45] it was shown that the \( L \)-basis naturally arises in a representation of the solutions of the Sturm-Liouville equation in terms of powers of the spectral parameter. The approach based on such representation is called the spectral parameter power series (SPPS) method. The functions \( \varphi_k \) which constitute the \( L \)-basis appear as the expansion coefficients in the SPPS. In [41], [42] and [45] convenient representations for their practical computation were proposed which converted the SPPS method into an efficient and highly competitive technique for solving a variety of spectral and scattering problems related to Sturm-Liouville equations (see [12], [13], [37], [39], [45], [47]). The above mentioned relation between the transmutation operators and the functions \( \varphi_k \) called in the present paper the recursive integrals consists in the fact established in [9] that for every system \( \{\varphi_k\}_{k=0}^\infty \) there exists a transmutation operator \( T \) such that \( T[a^k] = \varphi_k \), i.e., the functions \( \varphi_k \) are the images of the usual powers of the independent variable. Moreover, it was shown how this operator can be constructed and how it is related to the “canonical” transformation operator considered, e.g., in [51] Chapter 1. This result together with the practical formulas for calculating the functions \( \varphi_k \) makes it possible to apply the transmutation technique even when the integral kernel of the operator is unknown. Indeed, now it is easy to apply the transmutation operator to any function approximated by a polynomial.

Deeper understanding of the mapping properties of the transmutation operators led us in [46] to the explicit construction of the transmutation operator for a Darboux transformed Schrödinger operator by a known transmutation operator for the original Schrödinger operator as well as to several
interesting relations between the two transmutation operators. These relations also allowed us to prove the main theorem on the transmutation operators under a weaker condition than it was known before (not requiring the continuous differentiability of the potential in the Schrödinger operator).

In the present paper we overview the recent results related to the SPPS approach explaining and illustrating its main advantage, the possibility to write down in an analytic form the characteristic equation of the spectral problem. This equation can be approximated in different ways, and its solutions give us the eigenvalues of the problem. In other words the eigenvalue problem reduces to computation of zeros of a certain complex analytic function given by its Taylor series whose coefficients are obtained as simple linear combinations of the values of the functions $\varphi_k$ at a given point. We discuss different applications of the SPPS method and give the results of some comparative numerical calculations.

Following [9] and [46] we introduce a parametrized family of transmutation operators and study their mapping properties, we give an explicit representation for the kernel of the transmutation operator corresponding to the Darboux transformed potential in terms of the transmutation kernel for its superpartner (Theorem 6.2). Moreover, this result leads to interesting commutation relations between the two transmutation operators (Corollary 6.6) which in their turn allow us to obtain a transmutation operator for the one-dimensional Dirac system with a scalar potential as well as to prove the main property of the transmutation operator under less restrictive conditions than it has been proved until now. We give several examples of explicitly constructed kernels of transmutation operators. It is worth mentioning that in the literature there are very few explicit examples and even in the case when $q$ is a constant such kernel was presented recently in [9]. The results discussed in the present paper allow us to enlarge considerably the list of available examples and give a relatively simple tool for constructing Darboux related sequences of the transmutation kernels.

2. Recursive integrals: a question on the completeness

Let $f \in C^2(a,b) \cap C^1[a,b]$ be a complex valued function and $f(x) \neq 0$ for any $x \in [a,b]$. The interval $(a,b)$ is assumed being finite. Let us consider the following functions

$$X^{(0)}(x) \equiv 1, \quad X^{(n)}(x) = n \int_{x_0}^{x} X^{(n-1)}(s) \left( f^2(s) \right)^{-1} ds, \quad x_0 \in [a,b], \quad n = 1, 2, \ldots \quad (2.1)$$

We pose the following questions. Is the family of functions $\{X^{(n)}\}_{n=0}^{\infty}$ complete let us say in $L_2(a,b)$? What about the completeness of $\{X^{(2n)}\}_{n=0}^{\infty}$ or $\{X^{(2n+1)}\}_{n=0}^{\infty}$?
The following example shows that both questions are meaningful and natural.

**Example 2.1.** Let \( f \equiv 1, \ a = 0, \ b = 1 \). Then it is easy to see that choosing \( x_0 = 0 \) we have \( X^{(0)}(x) = 1, \ X^{(1)}(x) = x, \ X^{(2)}(x) = x^2, \ X^{(3)}(x) = x^3, \ldots \). Thus, the family of functions \( \{ X^{(n)} \}_{n=0}^{\infty} \) is complete in \( L^2(0,1) \). Moreover, both \( \{ X^{(2n)} \}_{n=0}^{\infty} \) and \( \{ X^{(2n+1)} \}_{n=0}^{\infty} \) are complete in \( L^2(0,1) \) as well.

If instead of \( a = 0 \) we choose \( a = -1 \) then \( \{ X^{(2n)} \}_{n=0}^{\infty} \) is still complete in \( L^2(-1,1) \) but neither \( \{ X^{(2n)} \}_{n=0}^{\infty} \) nor \( \{ X^{(2n+1)} \}_{n=0}^{\infty} \).

Together with the family of functions \( \{ X^{(n)} \}_{n=0}^{\infty} \) we consider also another similarly defined family of functions \( \{ \tilde{X}^{(n)} \}_{n=0}^{\infty} \),

\[
\tilde{X}^{(0)} \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_{x_0}^{x} \tilde{X}^{(n-1)}(s) \left( f^2(s) \right)^{(1-n)} ds, \quad x_0 \in [a,b], \ n = 1, 2, \ldots \quad (2.2)
\]

**Remark 2.2.** As we show below the introduced families of functions are closely related to the one-dimensional Schrödinger equations of the form \( u'' - qu = \lambda u \) where \( q \) is a complex-valued continuous function. Slightly more general families of functions can be studied in relation to Sturm-Liouville equations of the form \( (py')' + qy = \lambda ry \). Their definition based on a corresponding recursive integration procedure is given in [42], [45], [37].

We introduce the infinite system of functions \( \{ \varphi_k \}_{k=0}^{\infty} \) defined as follows

\[
\varphi_k(x) = \begin{cases} 
  f(x) X^{(k)}(x), & k \text{ odd}, \\
  f(x) \tilde{X}^{(k)}(x), & k \text{ even}.
\end{cases} \quad (2.3)
\]

The system \( \{ \varphi_k \}_{k=0}^{\infty} \) is closely related to the notion of the \( L \)-basis introduced and studied in [24]. Here the letter \( L \) corresponds to a linear ordinary differential operator.

Together with the system of functions \( \{ \varphi_k \}_{k=0}^{\infty} \) we define the functions \( \{ \psi_k \}_{k=0}^{\infty} \) using the “second half” of the recursive integrals (2.1) and (2.2),

\[
\psi_k(x) = \begin{cases} 
  \frac{\tilde{X}^{(k)}(x)}{f(x)}, & k \text{ odd}, \\
  \frac{X^{(k)}(x)}{f(x)}, & k \text{ even}.
\end{cases} \quad (2.4)
\]

The following result obtained in [41] (for additional details and simpler proof see [42] and [45]) establishes the relation of the system of functions \( \{ \varphi_k \}_{k=0}^{\infty} \) and \( \{ \psi_k \}_{k=0}^{\infty} \) to the Sturm-Liouville equation.

**Theorem 2.3 ([41]).** Let \( q \) be a continuous complex valued function of an independent real variable \( x \in [a,b] \) and \( \lambda \) be an arbitrary complex number. Suppose there exists a solution \( f \) of the equation

\[
f'' - qf = 0 \quad (2.5)
\]
on \((a, b)\) such that \(f \in C^2(a, b) \cap C^1[a, b] \) and \(f(x) \neq 0\) for any \(x \in [a, b]\).

Then the general solution \(u \in C^2(a, b) \cap C^1[a, b]\) of the equation

\[
    u'' - qu = \lambda u
\]

on \((a, b)\) has the form

\[
    u = c_1 u_1 + c_2 u_2
\]

where \(c_1\) and \(c_2\) are arbitrary complex constants,

\[
    u_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \varphi_{2k} \quad \text{and} \quad u_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \varphi_{2k+1}
\]

and both series converge uniformly on \([a, b]\) together with the series of the first derivatives which have the form

\[
    u_1' = f' + \sum_{k=1}^{\infty} \frac{\lambda^k}{(2k)!} \left( \frac{f'}{f} \varphi_{2k} + 2k \psi_{2k-1} \right) \quad \text{and}
\]

\[
    u_2' = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \left( \frac{f'}{f} \varphi_{2k+1} + (2k+1) \psi_{2k} \right).
\]

The series of the second derivatives converge uniformly on any segment \([a_1, b_1] \subset (a, b)\).

The representation (2.7) offers the linearly independent solutions of (2.6) in the form of spectral parameter power series (SPPS). The possibility to represent solutions of the Sturm-Liouville equation in such form is by no means a novelty, though it is not a widely used tool (in fact, besides the work reviewed below and in [37] we are able to mention only [4, Sect. 10], [23] and the recent paper [40]) and to our best knowledge for the first time it was applied for solving spectral problems in [45]. The reason of this underuse of the SPPS lies in the form in which the expansion coefficients were sought. Indeed, in previous works the calculation of coefficients was proposed in terms of successive integrals with the kernels in the form of iterated Green functions (see [4, Sect. 10]). This makes any computation based on such representation difficult, less practical and even proofs of the most basic results like, e.g., the uniform convergence of the spectral parameter power series for any value of \(\lambda \in \mathbb{C}\) (established in Theorem 2.3) are not an easy task. For example, in [4, p. 16] the parameter \(\lambda\) is assumed to be small and no proof of convergence is given.

The way of how the expansion coefficients in (2.7) are calculated according to (2.1), (2.2) is relatively simple and straightforward, this is why the estimation of the rate of convergence of the series (2.7) presents no difficulty, see [45]. Moreover, in [7] a discrete analogue of Theorem 2.3 was established and the discrete analogues of the series (2.7) resulted to be finite sums.

Another crucial feature of the introduced representation of the expansion coefficients in (2.7) consists in the fact that not only these coefficients (denoted by \(\varphi_k\) in (2.3)) are required for solving different spectral problems
related to the Sturm-Liouville equation. Indeed, the functions $\tilde{X}^{(2k+1)}$ and $X^{(2k)}$, $k = 0, 1, 2, \ldots$ do not participate explicitly in the representation (2.7). Nevertheless, together with the functions $\varphi_k$ they appear in the representation (2.8) of the derivatives of the solutions and therefore also in characteristic equations corresponding to the spectral problems.

In the present work we also overview another approach developed in [43], [8], [9] and [46] where the formal powers (2.1) and (2.2) were considered as infinite families of functions intimately related to the corresponding Sturm-Liouville operator. As we show this leads to a deeper understanding of the transmutation operators [3], [10] also known as transformation operators [49], [51]. Indeed, the functions $\varphi_k(x)$ also known as transmutation operators (2.5) possess two linearly independent regular solutions $v_1$ and $v_2$ whose zeros alternate. Thus one may choose $f = v_1 + iv_2$. Moreover, for the construction of $v_1$ and $v_2$ in fact the same SPPS method may be used [45].

**Remark 2.4.** It is easy to see that by definition the solutions $u_1$ and $u_2$ from (2.7) satisfy the following initial conditions

$$
\begin{align*}
  u_1(x_0) &= f(x_0), & u'_1(x_0) &= f'(x_0), \\
  u_2(x_0) &= 0, & u'_2(x_0) &= 1/f(x_0).
\end{align*}
$$

**Remark 2.5.** It is worth mentioning that in the regular case the existence and construction of the required $f$ presents no difficulty. Let $q$ be real valued and continuous on $[a, b]$. Then (2.5) possesses two linearly independent regular solutions $v_1$ and $v_2$ whose zeros alternate. Thus one may choose $f = v_1 + iv_2$. Moreover, for the construction of $v_1$ and $v_2$ in fact the same SPPS method may be used [45].

Theorem 2.3 together with the results on the completeness of Sturm-Liouville eigenfunctions and generalized eigenfunctions [51] implies the validity of the following two statements. For their detailed proofs we refer to [43] and [44] respectively.

**Theorem 2.6 ([43]).** Let $(a, b)$ be a finite interval and $f \in C^2(a, b) \cap C^1[a, b]$ be a complex valued function such that $f(x) \neq 0$ for any $x \in [a, b]$.

If $x_0 = a$ (or $x_0 = b$) then each of the four systems of functions

$$
\{X^{(2n)}\}_{n=0}^{\infty} \cup \{X^{(2n+1)}\}_{n=0}^{\infty}, \quad \{\tilde{X}^{(2n)}\}_{n=0}^{\infty} \cup \{\tilde{X}^{(2n+1)}\}_{n=0}^{\infty},
$$

is complete in $L_2(a, b)$.

If $x_0$ is an arbitrary point of the interval $(a, b)$ then each of the following two combined systems of functions

$$
\{X^{(2n)}\}_{n=0}^{\infty} \cup \{X^{(2n+1)}\}_{n=0}^{\infty} \quad \text{and} \quad \{\tilde{X}^{(2n+1)}\}_{n=0}^{\infty} \cup \{\tilde{X}^{(2n)}\}_{n=0}^{\infty}
$$

is complete in $L_2(a, b)$.

**Theorem 2.7 ([44]).** Let $f$ satisfy the conditions of the preceding theorem and $\{\varphi_k\}_{k=0}^{\infty}$ be the system of functions defined by (2.3) with $x_0$ being an arbitrary point of the interval $[a, b]$. Then for any complex valued continuous, piecewise...
continuously differentiable function \( h \) defined on \([a, b]\) and for any \( \varepsilon > 0 \) there exists such \( N \in \mathbb{N} \) and such complex numbers \( \alpha_k, k = 0, 1, \ldots, N \) that

\[
\max_{x \in [a, b]} \left| h(x) - \sum_{k=0}^{N} \alpha_k \varphi_k \right| < \varepsilon.
\]

3. Dispersion relations for spectral problems and approximate solutions

The SPPS representation (2.7) for solutions of the Sturm-Liouville equation (2.6) is very convenient for writing down the dispersion (characteristic) relations in an analytical form. This fact was used in [13], [37], [39], [45], [47] for approximating solutions of different eigenvalue problems. Here in order to explain this we consider two examples: the Sturm-Liouville problem and the quantum-mechanical eigenvalue problem. As the performance of the SPPS method in application to classical (regular and singular) Sturm-Liouville problems was studied in detail in [45] here we consider the Sturm-Liouville problems with boundary conditions which depend on the spectral parameter \( \lambda \). This situation occurs in many physical models (see, e.g., [5, 14, 19, 20, 27, 64] and references therein) and is considerably more difficult from the computational point of view. Moreover, as we show in this section the SPPS method is applicable to models admitting complex eigenvalues - an important advantage in comparison with the best purely numerical techniques all of them being based on the shooting method.

Consider the equation \( u'' - qu = \lambda u \) together with the boundary conditions

\[
\begin{align*}
&u(a) \cos \alpha + u'(a) \sin \alpha = 0, \\
&\beta_1 u(b) - \beta_2 u'(b) = \phi(\lambda)(\beta_1 u(b) - \beta_2 u'(b)),
\end{align*}
\]

where \( \alpha \) is an arbitrary complex number, \( \phi \) is a complex-valued function of the variable \( \lambda \) and \( \beta_1, \beta_2, \beta_1', \beta_2' \) are complex numbers. For some special forms of the function \( \phi \) such as \( \phi(\lambda) = \lambda \) or \( \phi(\lambda) = \lambda^2 + c_1 \lambda + c_2 \), results were obtained [19], [64] concerning the regularity of the problem; we will not dwell upon the details. Notice that the SPPS approach is applicable as well to a more general Sturm-Liouville equation \( (pu')' + qu = \lambda ru \). For the corresponding details we refer to [37] and [45].

For simplicity, let us suppose that \( \alpha = 0 \) and hence the condition (3.1) becomes \( u(a) = 0 \). Then choosing the initial integration point in (2.1) and (2.2) as \( x_0 = a \) and taking into account Remark 2.4 we obtain that if an eigenfunction exists it necessarily coincides with \( u_2 \) up to a multiplicative constant. In this case condition (3.2) becomes equivalent to the equality [45], [37]

\[
(f(b)\phi_1(\lambda) - f'(b)\phi_2(\lambda)) \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k + 1)!} X^{(2k+1)}(b) - \frac{\phi_2(\lambda)}{f(b)} \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} X^{(2k)}(b) = 0,
\]

(3.3)
where $\phi_{1,2}(\lambda) = \beta_{1,2} - \beta'_{1,2}\phi(\lambda)$. This is the characteristic equation of the considered spectral problem. Calculation of eigenvalues given by (3.3) is especially simple in the case of $\phi$ being a polynomial of $\lambda$. Precisely this particular situation was considered in all of the above-mentioned references concerning Sturm-Liouville problems with spectral parameter dependent boundary conditions. In any case the knowledge of an explicit characteristic equation (3.3) for the spectral problem makes possible its accurate and efficient solution. For this the infinite sums in (3.3) are truncated after a certain $N \in \mathbb{N}$. The paper [45] contains several numerical tests corresponding to a variety of computationally difficult problems. All they reveal an excellent performance of the SPPS method. We do not review them here referring the interested reader to [45]. Instead we consider another interesting example from [37], a Sturm-Liouville problem admitting complex eigenvalues.

Example 3.1. Consider the equation (2.6) with $q \equiv 0$ on the interval $(0, \pi)$ with the boundary conditions $u(0) = 0$ and $u(\pi) = -\lambda^2 u(\pi)$. The exact eigenvalues of the problem are $\lambda_n = n^2$ together with the purely imaginary numbers $\lambda_{\pm} = \pm i$. Application of the SPPS method with $N = 100$ and 3000 interpolating points (used for representing the integrands as splines) delivered the following results $\lambda_1 = 1$, $\lambda_2 = 4.00000000000007$, $\lambda_3 = 9.000000000001$, $\lambda_4 = 15.999999999996$, $\lambda_5 = 25.000000002$, $\lambda_6 = 35.99999997$, $\lambda_7 = 49.0000004$, $\lambda_8 = 63.9999994$, $\lambda_9 = 80.9996$, $\lambda_{10} = 100.02$ and $\lambda_{\pm} = \pm i$. Thus, the complex eigenvalues are as easily and accurately detected by the SPPS method as the real eigenvalues. Note that for a better accuracy in calculation of higher eigenvalues of a Sturm-Liouville problem an additional simple shifting procedure described in [45] and based on the representation of solutions not as series in powers of $\lambda$ but in powers of $(\lambda - \lambda_0)$ is helpful. We did not apply it here and hence the accuracy of the calculated value of $\lambda_{10}$ is considerably worse than the accuracy of the first calculated eigenvalues which in general can be improved by means of the mentioned shifting procedure.

Figures 1-3 give us an idea about the stability of the computed eigenvalues when $N$ increases. In Fig. 1 we plot $\lambda_1$ and $\lambda_2$ computed with $N = 14, 16, \ldots, 120$. Figure 2 shows $\lambda_3$ computed with $N = 24, 30, \ldots, 140$ and Figure 3 shows $\lambda_4$ computed with $N = 40, 50, \ldots, 140$ Similar plots can be done for calculated higher eigenvalues. In all cases the computed eigenvalues reveal a remarkable stability when $N$ increases.

An attractive feature of the SPPS method is the possibility to easily plot the characteristic relation. In Fig. 4 we show the absolute value of the expression from the left-hand side of (3.3) as a function of the complex variable $\lambda$ for the considered example. Its zeros coincide with the eigenvalues of the problem. It is important to mention that such plot is obtained in a fraction of a second. This is due to the fact that once the required formal powers $X^{(n)}$ are computed (and this takes several seconds) the calculation of the characteristic relation involves only simple algebraic operations.
Figure 1. The approximate eigenvalues $\lambda_1$ and $\lambda_2$ from Example 3.1 computed using different number $N$ of formal powers.

Figure 2. The approximate values of $\lambda_3$ from Example 3.1 computed using different number $N$ of formal powers.
Figure 3. The approximate values of $\lambda_4$ from Example 3.1 computed using different number $N$ of formal powers.

Figure 4. The absolute value of the expression from the left-hand side of (3.3) as a function of the complex variable $\lambda$ for the considered example. With the arrows we indicate the calculated complex eigenvalues $\lambda_{\pm}$. The other zeros of the graph correspond to the first five real eigenvalues of the problem.
Let us consider the one-dimensional Schrödinger equation

$$Hu(x) = -u''(x) + Q(x)u(x) = \lambda u(x), \quad x \in \mathbb{R}, \quad (3.4)$$

where

$$Q(x) = \begin{cases} 
\alpha_1, & x < 0, \\
q(x), & 0 \leq x \leq h, \\
\alpha_2, & x > h, 
\end{cases} \quad (3.5)$$

\(\alpha_1\) and \(\alpha_2\) are complex constants and \(q\) is a continuous complex-valued function defined on the segment \([0, h]\). Thus, outside a finite segment the potential \(Q\) admits constant values, and at the end points of the segment the potential may have discontinuities. We are looking for such values of the spectral parameter \(\lambda \in \mathbb{C}\) for which the Schrödinger equation possesses a solution \(u\) belonging to the Sobolev space \(H^2(\mathbb{R})\) which in the case of the potential of the form \((3.5)\) means that we are looking for solutions exponentially decreasing at \(\pm \infty\). This eigenvalue problem is one of the central in quantum mechanics for which \(H\) is a self-adjoint operator in \(L^2(\mathbb{R})\) with the domain \(H^2(\mathbb{R})\). It implies that \(Q\) is a real-valued function. In this case the operator \(H\) has a continuous spectrum \([\min \{\alpha_1, \alpha_2\}, +\infty)\) and a discrete spectrum located on the set

$$\left\{ \min_{x \in [0, h]} q(x), \min \{\alpha_1, \alpha_2\} \right\}.$$ 

Computation of energy levels of a quantum well described by the potential \(Q\) is a problem of physics of semiconductor nanostructures (see, e.g., [31]). Other important models which reduce to the spectral problem \((3.4)\) arise in studying the electromagnetic and acoustic wave propagation in inhomogeneous waveguides (see for instance [2], [16], [25], [17], [6], [56], [53]).

A characteristic equation for this spectral problem in terms of spectral parameter power series was obtained in [13] (see also [37]) where a simple numerical algorithm based on the approximation of the characteristic equation was implemented and compared to other known numerical techniques. Here we only give an example from [13].

The usual approach to numerical solution of the considered eigenvalue problem consists in applying the shooting method (see, e.g., [31]) which is known to be unstable, relatively slow and to the difference of the SPPS approach does not offer any explicit equation for determining eigenvalues and eigenfunctions. In [30] another method based on approximation of the potential by square wells was proposed. It is limited to the case of symmetric potentials. The approach based on the SPPS is completely different and does not require any shooting procedure, approximation of the potential or numerical differentiation. Derived from the exact characteristic equation its approximation is considered, and in fact numerically the problem is reduced to finding zeros of a polynomial \(\sum_{k=0}^{N} a_k \mu^k\) in the interval \([\min q(x), 0]\), \((\mu^2 = -\lambda)\).

As an example, consider the potential \(Q\) defined by the expression \(Q(x) = -v \text{sech}^2 x, \ x \in (-\infty, \infty).\) It is not of a finite support, nevertheless its absolute value decreases rapidly when \(x \to \pm \infty\). The original problem is
approximated by a problem with a finite support potential $\hat{Q}$ defined by the equality

$$\hat{Q}(x) = \begin{cases} 
0, & x < -a \\
-v \text{sech}^2 x, & -a \leq x \leq a \\
0, & x > a.
\end{cases}$$

An attractive feature of the potential $Q$ is that its eigenvalues can be calculated explicitly (see, e.g., [26]). In particular, for $v = m(m+1)$ the eigenvalue $\lambda_n$ is given by the formula $\lambda_n = -(m - n)^2$, $n = 0, 1, \ldots$.

The results of application of the SPPS method for $v = 12$ are given in Table 1 in comparison with the exact values and the results from [30].

| $n$ | Exact values | Num.res. from [30] | Num.res. using SPPS ($N = 180$) |
|-----|--------------|---------------------|---------------------------------|
| 0   | $-9$         | $-9.094$            | $-8.999628696$                  |
| 1   | $-4$         | $-4.295$            | $-3.99998053$                   |
| 2   | $-1$         | $-0.885$            | $-0.999927816$                  |

Table 1. Approximations of $\lambda_n$ of the Hamiltonian $H = -D^2 - 12 \text{sech}^2 x$

The results obtained by means of SPPS are considerably more accurate, and as was pointed out above the application of the SPPS method has much less restrictions.

4. Transmutation operators

We slightly modify here the definition given by Levitan [49] adapting it to the purposes of the present work. Let $E$ be a linear topological space and $E_1$ its linear subspace (not necessarily closed). Let $A$ and $B$ be linear operators: $E_1 \to E$.

**Definition 4.1.** A linear invertible operator $T$ defined on the whole $E$ such that $E_1$ is invariant under the action of $T$ is called a transmutation operator for the pair of operators $A$ and $B$ if it fulfills the following two conditions.

1. Both the operator $T$ and its inverse $T^{-1}$ are continuous in $E$;
2. The following operator equality is valid

$$AT = TB$$

or which is the same

$$A = TBT^{-1}.$$ 

Very often in literature the transmutation operators are called the transformation operators. Here we keep ourselves to the original term coined by Delsarte and Lions [23]. Our main interest concerns the situation when $A = -\frac{d^2}{dx^2} + q(x)$, $B = -\frac{d^2}{dx^2}$, and $q$ is a continuous complex-valued function. Hence for our purposes it will be sufficient to consider the functional space
$E = C[a, b]$ with the topology of uniform convergence and its subspace $E_1$ consisting of functions from $C^2[a, b]$. One of the possibilities to introduce a transmutation operator on $E$ was considered by Lions [50] and later on in other references (see, e.g., [51]), and consists in constructing a Volterra integral operator corresponding to a midpoint of the segment of interest. As we begin with this transmutation operator it is convenient to consider a symmetric segment $[-a, a]$ and hence the functional space $E = C[-a, a]$. It is worth mentioning that other well known ways to construct the transmutation operators (see, e.g., [49], [63]) imply imposing initial conditions on the functions and consequently lead to transmutation operators satisfying (4.1) only on subclasses of $E_1$.

Thus, we consider the space $E = C[-a, a]$ and an operator of transmutation for the defined above $A$ and $B$ can be realized in the form (see, e.g., [49] and [51]) of a Volterra integral operator

$$Tu(x) = u(x) + \int_{-x}^{x} K(x, t)u(t)dt \quad (4.2)$$

where $K(x, t) = H\left(\frac{x+t}{2}, \frac{x-t}{2}\right)$ and $H$ is the unique solution of the Goursat problem

$$\frac{\partial^2 H(u, v)}{\partial u \partial v} = q(u + v)H(u, v), \quad (4.3)$$

$$H(u, 0) = \frac{1}{2} \int_{0}^{u} q(s) ds, \quad H(0, v) = 0. \quad (4.4)$$

If the potential $q$ is continuously differentiable, the kernel $K$ itself is the solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right)K(x, t) = \frac{\partial^2}{\partial t^2}K(x, t), \quad (4.5)$$

$$K(x, x) = \frac{1}{2} \int_{0}^{x} q(s) ds, \quad K(x, -x) = 0. \quad (4.6)$$

If the potential $q$ is $n$ times continuously differentiable, the kernel $K(x, t)$ is $n + 1$ times continuously differentiable with respect to both independent variables (see [51]).

An important property of this transmutation operator consists in the way how it maps solutions of the equation

$$v'' + \omega^2 v = 0 \quad (4.5)$$

into solutions of the equation

$$u'' - q(x)u + \omega^2 u = 0 \quad (4.6)$$

where $\omega$ is a complex number. Denote by $e_0(i\omega, x)$ the solution of (4.6) satisfying the initial conditions

$$e_0(i\omega, 0) = 1 \quad \text{and} \quad e'_0(i\omega, 0) = i\omega. \quad (4.7)$$

The subindex “0” indicates that the initial conditions correspond to the point $x = 0$ and the letter “$e$” reminds us that the initial values coincide with the initial values of the function $e^{i\omega x}$. 


The transmutation operator (4.2) maps $e^{i\omega x}$ into $e_0(i\omega, x)$,
\[ e_0(i\omega, x) = T[e^{i\omega x}] \] (4.7)
(see [51, Theorem 1.2.1]).

Following [51] we introduce the following notations
\[ K_c(x, t; h) = h + K(x, t) + K(x, -t) + h \int_t^x \{ K(x, \xi) - K(x, -\xi) \} d\xi \]
where $h$ is a complex number, and
\[ K_s(x, t; \infty) = K(x, t) - K(x, -t). \]

**Theorem 4.2 ([51]).** Solutions $c(\omega, x; h)$ and $s(\omega, x; \infty)$ of equation (4.6) satisfying the initial conditions
\[ c(\omega, 0; h) = 1, \quad c_x'(\omega, 0; h) = h \] (4.8)
\[ s(\omega, 0; \infty) = 0, \quad s_x'(\omega, 0; \infty) = 1 \] (4.9)
can be represented in the form
\[ c(\omega, x; h) = \cos \omega x + \int_0^x K_c(x, t; h) \cos \omega t \, dt \] (4.10)
and
\[ s(\omega, x; \infty) = \frac{\sin \omega x}{\omega} + \int_0^x K_s(x, t; \infty) \frac{\sin \omega t}{\omega} \, dt. \] (4.11)

Denote by
\[ T_c u(x) = u(x) + \int_0^x K_c(x, t; h) u(t) \, dt \] (4.12)
and
\[ T_s u(x) = u(x) + \int_0^x K_s(x, t; \infty) u(t) \, dt \] (4.13)
the corresponding integral operators. As was pointed out in [9], they are not transmutations on the whole subspace $E_1$, they even do not map all solutions of (4.5) into solutions of (4.6). For example, as we show below
\[ \left( -\frac{d^2}{dx^2} + q(x) \right) T_s[1] \neq T_s \left[ -\frac{d^2}{dx^2}(1) \right] = 0 \]
when $q$ is constant.

**Example 4.3.** Transmutation operator for operators $A := \frac{d^2}{dx^2} + c$, $c$ is a constant, and $B := \frac{d^2}{dx^2}$. According to (4.3) and (4.4), finding the kernel of transmutation operator is equivalent to finding the function $H(s, t)$ satisfying the Goursat problem
\[ \frac{\partial^2 H(s, t)}{\partial s \partial t} = -cH(s, t), \quad H(s, 0) = \frac{-cs}{2}, \quad H(0, t) = 0. \]
The solution of this problem is given by [28, (4.85)]
\[ H(s, t) = -\frac{c}{2} \int_0^s J_0(2\sqrt{ct(s-\xi)}) \, d\xi = -\frac{\sqrt{cs} J_1(2\sqrt{cst})}{2t}, \]
where \( J_0 \) and \( J_1 \) are Bessel functions of the first kind, and the formula is valid even if the radicand is negative. Hence,

\[
K(x, y) = H \left( \frac{x + y}{2}, \frac{x - y}{2} \right) = -\frac{1}{2} \sqrt{c(x^2 - y^2)} J_1(\sqrt{c(x^2 - y^2)}) \quad (4.14)
\]

From (4.14) we get the ‘sine’ kernel

\[
K_s(x, t; \infty) = -\frac{t}{2} \sqrt{c(\frac{x^2}{t^2})} J_1(\sqrt{c(\frac{x^2}{t^2})}),
\]

and can check the above statement about the operator \( T_s \),

\[
T_s[1](x) = 1 - \int_0^x \frac{t \sqrt{c(x^2 - t^2)} J_1(\sqrt{c(x^2 - t^2)})}{x^2 - t^2} dt = J_0(x\sqrt{c}),
\]

\[
\left( \frac{d^2}{dx^2} + c \right) T_s[1] = \frac{\sqrt{c} J_1(x\sqrt{c})}{x} \neq 0.
\]

For the rest of this section suppose that \( f \) is a solution of (2.5) fulfilling the condition of Theorem 2.3 on a finite segment \([-a, a]\). We normalize \( f \) in such a way that \( f(0) = 1 \) and let \( f'(0) = h \) where \( h \) is some complex constant. Define the system of functions \( \{ \varphi_k \}_{k=0}^{\infty} \) by this function \( f \) with the use of (2.1), (2.2) and (2.3). The system of functions \( \{ \varphi_k \}_{k=0}^{\infty} \) is related to the transmutation operators \( T_c \) (with the same parameter \( h \) in the kernel) and \( T_s \) in a way that it is the union of functions which are the result of acting of operator \( T_s \) on the odd powers of independent variable and of operator \( T_c \) on the even powers of independent variable. The following theorem holds, see [9] for the details of the proof.

**Theorem 4.4 ([9]).** Let \( q \) be a continuous complex valued function of an independent real variable \( x \in [-a, a] \), and \( f \) be a particular solution of (2.5) such that \( f \in C^2 (-a, a) \), \( f \neq 0 \) on \([-a, a]\) and normalized as \( f(0) = 1 \). Let \( \varphi_k, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) are functions defined by (2.3). Then the following equalities are valid

\[
\varphi_k = T_c[x^k] \quad \text{when } k \in \mathbb{N}_0 \text{ is even or equal to zero}\]

and

\[
\varphi_k = T_s[x^k] \quad \text{when } k \in \mathbb{N} \text{ is odd}.
\]

As for the transmutation operator \( T \), it does not map all powers of the independent variable into the functions \( \varphi_k \). Instead, the following theorem holds.

**Theorem 4.5 ([9]).** Under the conditions of Theorem 4.4 the following equalities are valid

\[
\varphi_k = T[x^k] \quad \text{when } k \text{ is odd} \quad (4.15)
\]

and

\[
\varphi_k - \frac{h}{k+1} \varphi_{k+1} = T[x^k] \quad \text{when } k \in \mathbb{N}_0 \text{ is even or equal to zero} \quad (4.16)
\]

where by \( h \) we denote \( f'(0) \in \mathbb{C} \).
Taking into account the first of former relations the second can be written also as follows

$$\varphi_k = T \left[ x^k + \frac{h}{k+1} x^{k+1} \right]$$

when $k \in \mathbb{N}_0$ is even or equal to zero.

**Remark 4.6.** Let $f$ be the solution of (2.5) satisfying the initial conditions

$$f(0) = 1, \quad f'(0) = 0.$$  \hspace{1cm} (4.17)

If it does not vanish on $[-a,a]$ then from Theorem 4.5 we obtain that $\varphi_k = T[x^k]$ for any $k \in \mathbb{N}_0$. In general, of course there is no guaranty that the solution satisfying (4.17) have no zero on $[-a,a]$. Hence the operator $T$ transmutes the powers of $x$ into $\varphi_k(x)$ whose construction is based on the solution $f$ satisfying (4.17) only in some neighborhood of the origin. In the next section we show how to change the operator $T$ so that the new operator map $x^k$ into $\varphi_k(x)$ on the whole segment $[-a,a]$.

Note that in Theorem 4.5 the operator $T$ does not depend on the function $f$, so the right-hand sides of the equalities (4.15) and (4.16) do not change with the change of $f$. Consider two non-vanishing solutions $f$ and $g$ of (2.5) normalized as $f(0) = g(0) = 1$ and let $\varphi^f_k$ and $\varphi^g_k$ be the functions obtained from $f$ and $g$ respectively by means of (2.1), (2.2) and (2.3). The relation between $\varphi^f_k$ and $\varphi^g_k$ are given by the following proposition which may be easily deduced from equalities (4.15) and (4.16).

**Proposition 4.7.** The following equalities hold

$$\varphi^f_k = \varphi^g_k \quad \text{when } k \in \mathbb{N} \text{ is odd},$$

and

$$\varphi^f_k = \varphi^g_k + \frac{h_f - h_g}{k+1} \varphi^g_{k+1} \quad \text{when } k \in \mathbb{N}_0 \text{ is even},$$

where $h_f = f'(0)$ and $h_g = g'(0)$.

5. **A parametrized family of transmutation operators**

In [9] a parametrized family of operators $T_h$, $h \in \mathbb{C}$ was introduced, given by the integral expression

$$T_h u(x) = u(x) + \int_{-x}^{x} K(x,t;h)u(t)dt$$  \hspace{1cm} (5.1)

where

$$K(x,t;h) = \frac{h}{2} + K(x,t) + \frac{h}{2} \int_{t}^{x} \left( K(x,s) - K(x,-s) \right) ds.$$  \hspace{1cm} (5.2)

They are related to operators $T_s$ and $T_c$ (with the parameter $h$ in the kernel of the latter operator) by

$$T_h = T_c P_e + T_s P_o,$$  \hspace{1cm} (5.3)

where $P_e f(x) = (f(x) + f(-x))/2$ and $P_o f(x) = (f(x) - f(-x))/2$ are projectors onto even and odd functions, respectively. In this section we show
that the operators $T_h$ are transmutations, summarize their properties and in Theorem 5.8 we show how they act on powers of $x$.

Let us notice that $K(x,t;0) = K(x,t)$ and that the expression

$$K(x,t;h) - K(x,-t;h) = K(x,t) - K(x,-t) - \frac{h}{2} \int_{-t}^{t} (K(x,s) - K(x,-s)) \, ds = K(x,t) - K(x,-t)$$

does not depend on $h$. Thus, it is possible to compute $K(x,t;h)$ for any $h$ by a given $K(x,t;h_1)$ for some particular value $h_1$. We formulate this result as the following statement.

**Theorem 5.1 ([9]).** The integral kernels $K(x,t;h)$ and $K(x,t;h_1)$ are related by the expression

$$K(x,t;h) = \frac{h - h_1}{2} + K(x,t;h_1) + \frac{h - h_1}{2} \int_{x}^{t} (K(x,s;h_1) - K(x,-s;h_1)) \, ds.$$  \hspace{1cm} (5.4)

The operator $T_h$ may be expressed in terms of another operator $T_{h_1}$ and in particular, in terms of the operator $T$. The following proposition holds.

**Proposition 5.2.** The operators $T_{h_1}$ and $T_{h_2}$ are related by the expression

$$T_{h_2} u = T_{h_1} \left[ u(x) + \frac{h_2 - h_1}{2} \int_{-x}^{x} u(t) \, dt \right].$$  \hspace{1cm} (5.5)

valid for any $u \in C[-a,a]$. In particular,

$$T_h u = T \left[ u(x) + \frac{h}{2} \int_{-x}^{x} u(t) \, dt \right].$$  \hspace{1cm} (5.6)

**Proof.** Using formulas (5.1) and (5.2) we obtain

$$T_h u = Tu + \frac{h}{2} \int_{-x}^{x} u(t) \, dt + \frac{h}{2} \int_{-x}^{x} u(t) \int_{t}^{x} K(x,s) \, ds \, dt - \frac{h}{2} \int_{-x}^{x} u(t) \int_{-t}^{x} K(x,s) \, ds \, dt,$n

and after changing the order of integration in the last two integrals, we have

$$T_h u = Tu + \frac{h}{2} \int_{-x}^{x} u(t) \, dt + \frac{h}{2} \int_{-x}^{x} K(x,s) \int_{-s}^{s} u(t) \, dt \, ds - \frac{h}{2} \int_{-x}^{x} K(x,s) \int_{-s}^{s} u(t) \, dt \, ds = Tu + \frac{h}{2} \int_{-x}^{x} u(t) \, dt + \frac{h}{2} \int_{-x}^{x} K(x,s) \left[ \int_{-x}^{0} + \int_{0}^{s} u(t) \, dt \right] \, ds - \frac{h}{2} \int_{-x}^{x} K(x,s) \left[ \int_{-x}^{0} - \int_{-s}^{0} u(t) \, dt \right] \, ds = $$
\[ T u + \frac{h}{2} \int_{-x}^{x} u(t) dt + \frac{h}{2} \int_{-x}^{x} K(x, s) \int_{-s}^{s} u(t) dt ds = \mathcal{T} \left[ u(x) + \frac{h}{2} \int_{-x}^{x} u(t) dt \right]. \]

Since \( \int_{-x}^{x} \int_{t}^{x} u(s) ds dt = 0 \) for any function \( u \in C[-a, a] \), we have from (5.6) that
\[ T h_1 \left[ u(x) + \int_{-x}^{x} u(t) dt \right] = \mathcal{T} \left[ u(x) + \frac{h_2 - h_1}{2} \int_{-x}^{x} u(t) dt + \frac{h_1}{2} \int_{-x}^{x} \left( u(t) + \frac{h_2 - h_1}{2} \int_{-t}^{t} u(s) ds \right) dt \right] = \mathcal{T} h_2 u. \]

Using (4.8)–(4.13) and (5.3) it is possible to check how the operators \( \mathcal{T}_h \) act on solutions of (4.5).

**Proposition 5.3 ([46]).** The operator \( \mathcal{T}_h \) maps a solution \( v \) of an equation \( v'' + \omega^2 v = 0 \), where \( \omega \) is a complex number, into a solution \( u \) of the equation \( u'' - q(x)u + \omega^2 u = 0 \) with the following correspondence of the initial values
\[ u(0) = v(0), \quad u'(0) = v'(0) + hv(0). \] \hspace{1cm} (5.7)

**Remark 5.4.** Formulas (5.7) are valid for any function \( v \in C^1[-a, a] \).

We know that the integral kernel of the transmutation operator \( T \) is related to the solution of the Goursat problem (4.3)–(4.4). A similar result holds for the operators \( \mathcal{T}_h \).

**Theorem 5.5 ([46]).** In order for the function \( K(x, t; h) \) to be the kernel of a transmutation operator acting as described in Proposition 5.3, it is necessary and sufficient that \( H(u, v; h) := K(u+v, u-v; h) \) be a solution of the Goursat problem
\[ \frac{\partial^2 H(u, v; h)}{\partial u \partial v} = q(u + v)H(u, v; h), \]
\[ H(u, 0; h) = \frac{h}{2} + \frac{1}{2} \int_{-u}^{u} q(s) ds, \quad H(0, v; h) = \frac{h}{2}. \]

If the potential \( q \) is continuously differentiable, the function \( K(x, t; h) \) itself must be the solution of the Goursat problem
\[ \left( \frac{\partial^2}{\partial x^2} - q(x) \right) K(x, t; h) = \frac{\partial^2}{\partial t^2} K(x, t; h), \] \hspace{1cm} (5.8)
\[ K(x, x; h) = \frac{h}{2} + \frac{1}{2} \int_{0}^{x} q(s) ds, \quad K(x, -x; h) = \frac{h}{2}. \] \hspace{1cm} (5.9)

Under some additional requirements on the potential \( q \) the operators \( \mathcal{T}_h \) are transmutations in the sense of Definition 4.1. The following theorem generalizes the results obtained in [46].

**Theorem 5.6.** Suppose the potential \( q \) satisfies either of the following two conditions.
- \( q \in C^1[-a, a] \);
• \( q \in C[-a,a] \) and there exists a particular complex-valued solution \( g \) of (2.5) non-vanishing on \([-a,a]\).

Then the operator \( T_h \) given by (5.1) satisfies the equality

\[
\left(-\frac{d^2}{dx^2} + q(x)\right) T_h[u] = T_h \left[-\frac{d^2}{dx^2}(u)\right]
\]

(5.10)

for any \( u \in C^2[-a,a] \).

**Proof.** In [46] the theorem was proved for the case \( q \in C^1[-a,a] \) and for the case when the particular solution \( g \) from the statement satisfies conditions \( g(0) = 1 \) and \( g'(0) = 0 \). We may normalize the particular solution \( g \) as \( g(0) = 1 \). Suppose that \( g'(0) = h_1 \). We know already that (5.10) holds for the operator \( T_{h_1} \). To finish the proof, we use (5.5) and obtain

\[
\left(-\frac{d^2}{dx^2} + q(x)\right) T_h[u] = \left(-\frac{d^2}{dx^2} + q(x)\right) T_{h_1} \left[u(x) + \frac{h - h_1}{2} \int_{-x}^{x} u(t) dt\right]
\]

\[
= -T_{h_1} \left[u''(x) + \frac{h - h_1}{2} \frac{d^2}{dx^2} \int_{-x}^{x} u(t) dt\right] = -T_{h_1} \left[u''(x) + \frac{h - h_1}{2} \int_{-x}^{x} u''(t) dt\right] = T_h \left[-\frac{d^2}{dx^2}(u)\right].
\]

□

**Remark 5.7.** As was pointed out in Remark 2.5 in the regular case the non-vanishing solution \( g \) of (2.5) exists due to the alternation of zeroes of two linearly independent solutions. Of course, it would be interesting to prove that the operators \( T_h \) are transmutations in the general case of complex-valued potentials \( q \in C[-a,a] \) without any additional assumption.

Suppose now that a function \( f \) is a solution of (2.5), non-vanishing on \([-a,a] \) and normalized as \( f(0) = 1 \). Let \( h := f'(0) \) be some complex constant. Define as before the system of functions \( \{\varphi_k\}_{k=0}^\infty \) by this function \( f \) and by (2.3). The following theorem states that the operator \( T_h \) transmutes powers of \( x \) into the functions \( \varphi_k \).

**Theorem 5.8 ([9]).** Let \( q \) be a continuous complex valued function of an independent real variable \( x \in [-a,a] \), and \( f \) be a particular solution of (2.5) such that \( f \in C^2(-a,a) \) together with \( 1/f \) are bounded on \([-a,a] \) and normalized as \( f(0) = 1 \), and let \( h := f'(0) \), where \( h \) is a complex number. Then the operator (5.1) with the kernel defined by (5.2) transforms \( x^k \) into \( \varphi_k(x) \) for any \( k \in \mathbb{N}_0 \).

Thus, we clarified that the system of functions \( \{\varphi_k\} \) may be obtained as the result of the Volterra integral operator acting on powers of the independent variable. As was mentioned before, this offers an algorithm for transmuting functions in the situation when \( K(x,t;h) \) is unknown. Moreover, properties of the Volterra integral operator such as boundedness and bounded invertibility in many functional spaces gives us a tool to prove the completeness of the system of function \( \{\varphi_k\} \) in various situations.
Example 5.9. Consider a function \( k(x,t) = \frac{t-1}{2(x+1)} \) (later, in Example 6.8 it is explained how it can be obtained). We have

\[
(\partial_x^2 - \partial_t^2)k(x,t) = \frac{t-1}{(x+1)^3} = \frac{2}{(x+1)^2} \cdot \frac{t-1}{2(x+1)},
\]

\( k(x,-x) = \frac{x-1}{2(x+1)} = -\frac{1}{2} \) and \( k(x,x) = \frac{x-1}{2(x+1)} = -\frac{1}{2} + \frac{1}{2} \int_0^x \frac{2}{(s+1)^2} ds \), thus the function \( k(x,t) \) satisfies the Goursat problem with \( q(x) = 2/(x+1)^2 \) and \( h = -1 \) and by Theorem 5.5 is the kernel of the transmutation operator \( T_{-1} \).

Consider the function \( f = T_{-1}[1] = \frac{1}{x+1} \) as a solution of (2.5) such that \( f(0) = 1 \) and \( f'(0) = h = -1 \), nonvanishing on any \([-a,a] \subset (-1,1)\). The first 3 functions \( \varphi_k \) are given by

\[
\varphi_0 = f = \frac{1}{x+1}, \quad \varphi_1 = \frac{x^3 + 3x^2 + 3x}{3(x+1)}, \quad \varphi_2 = \frac{2x^3 + 3x^2}{3(x+1)}.
\]

It can be easily checked that

\[
T_{-1} x = x + \int_{-x}^x \frac{(t-1)t}{2(x+1)} dt = \frac{x^3 + 3x^2 + 3x}{3(x+1)} = \varphi_1,
\]

\[
T_{-1} x^2 = x^2 + \int_{-x}^x \frac{(t-1)t^2}{2(x+1)} dt = \frac{2x^3 + 3x^2}{3(x+1)} = \varphi_2.
\]

We can calculate the kernel \( K \) of the original operator \( T \) by (5.4), it is given by

\[
K(x,t) = \frac{2x + 2t + x^2 - t^2}{4(x+1)}
\]

and we can check that \( T[x] = \varphi_1 \) and \( T[1] = \frac{x^3 + 3x^2 + 3x + 3}{3(x+1)} = \varphi_0 + \varphi_1 \) in accordance with Theorem 4.5.

6. Transmutation operators and Darboux transformed equations

To construct the system of functions \( \{ \varphi_k \}_{k=0}^\infty \) we use the half of the functions \( \{ X^{(k)}, \tilde{X}^{(k)} \}_{k=0}^\infty \). What about the second half? Note that starting with the function \( 1/f \) we obtain the same system of functions \( \{ X^{(k)}, \tilde{X}^{(k)} \}_{k=0}^\infty \) with the only change that \( X_f^{(k)} \) becomes \( \tilde{X}_f^{(k)} \) and \( \tilde{X}_f^{(k)} \) becomes \( X_f^{(k)} \). Hence the “second half” of the functions \( \{ X^{(k)}, \tilde{X}^{(k)} \}_{k=0}^\infty \) from (2.3) is used. The function \( 1/f \) is continuous complex-valued and non-vanishing, and is a solution of the equation \( u'' - q_2u = 0 \), where \( q_2 = 2 \left( \frac{f'}{f} \right)^2 - q \). The last equation is known as the Darboux transformation of the original equation. The Darboux transformation is closely related to the factorization of the Schrödinger equation, and nowadays it is used in dozens of works, see, e.g., \([18, 29, 52, 58]\) in connection with solitons and integrable systems, e.g., \([11, 32, 55, 57]\) and the review \([59]\) of applications to quantum mechanics.
For the convenience denote the potential of the original equation by $q_1$ and the corresponding Sturm-Liouville operator by $A_1 := \frac{d^2}{dx^2} - q_1(x)$. Suppose a solution $f$ of the equation $A_1 f = 0$ is given such that $f(x) \neq 0$, $x \in [-a, a]$, it is normalized as $f(0) = 1$ and $h := f'(0)$ is some complex number. Denote the Darboux-transformed operator by $A_2 := \frac{d^2}{dx^2} - q_2(x)$, where $q_2(x) = 2\left(\frac{f'(x)}{f(x)}\right)^2 - q_1(x)$.

From the previous section we know that there exists a transmutation operator $T_{1,h}$ for the original equation with the potential $q_1$ and such that

$$T_{1,h} x^k = \varphi_k, \quad k \in \mathbb{N}_0. \quad (6.1)$$

The subindex “1” in the notation $T_{1,h}$ indicates that the transmutation operator corresponds to $A_1$.

Similarly, there exists a transmutation operator $T_{2,-h}$ for the Darboux-transformed operator $A_2$ such that

$$T_{2,-h} x^k = \psi_k, \quad k \in \mathbb{N}_0, \quad (6.2)$$

where the family of functions $\{\psi_k\}_{k=0}^\infty$ is defined by (2.4).

It is interesting to obtain some relations between the operators $T_{1,h}$ and $T_{2,-h}$ and between their integral kernels $K_1$ and $K_2$. In this section we explain how to construct the integral kernel $K_2$ by the known integral kernel $K_1$ and show that the operators $T_{1,h}$ and $T_{2,-h}$ satisfy certain commutation relations with the operator of differentiation.

We remind some well known facts about the Darboux transformation. First, $1/f$ is a solution of $A_2 u = 0$. Second, it is closely related to the factorization of Sturm-Liouville and one-dimensional Schrödinger operators. Namely, we have

$$A_1 = \frac{d^2}{dx^2} - q_1(x) = \left(\partial_x + \frac{f'}{f}\right)\left(\partial_x - \frac{f'}{f}\right) = \frac{1}{f} \partial_x f^2 \partial_x \frac{1}{f}, \quad (6.3)$$

$$A_2 = \frac{d^2}{dx^2} - q_2(x) = \left(\partial_x - \frac{f'}{f}\right)\left(\partial_x + \frac{f'}{f}\right) = f \partial_x \frac{1}{f^2} \partial_x f. \quad (6.4)$$

Suppose that $u$ is a solution of the equation $A_1 u = \omega u$ for some $\omega \in \mathbb{C}$. Then the function $v = (\partial_x - \frac{f'}{f})u = (f \partial_x \frac{1}{f})u$ is a solution of the equation $A_2 v = \omega v$, and vice versa, given a solution $v$ of $A_2 v = \omega v$, the function $u = (\partial_x + \frac{f'}{f})v = (\frac{1}{f} \partial_x f)v$ is a solution of $A_1 u = \omega u$.

Suppose that the operator $T_1 := T_{1,h}$ which transmutes the operator $A_1$ into the operator $B = d^2/dx^2$ and the powers $x^k$ into the functions $\varphi_k$ is known in the sense that its kernel $K_1(x,t;h)$ is given. As before $h = f'(0)$. Then the function $1/f$ is the non-vanishing solution of the equation $A_2 u = 0$ satisfying $1/f(0) = 1$ and $(1/f)'(0) = -h$. Hence we are looking for the operator $T_2 := T_{2,-h}$ transmuting the operator $A_2$ into the operator $B$ and the powers $x^k$ into the functions $\psi_k$.

Let us explain the idea for obtaining the operator $T_2$. We want to find an operator transforming solutions of the equation $Bu + \omega^2 u = 0$ into solutions of the equation $A_2 u + \omega^2 u = 0$, see the first diagram below. Starting with
a solution σ of the equation \((\partial_x^2 + \omega^2)\sigma = 0\), by application of \(T_1\) we get a solution of \((A_1 + \omega^2)u = 0\), and the expression \((f\partial_x\frac{1}{T})T_1\sigma\) is a solution of \((A_2 + \omega^2)v = 0\). But the operator \((f\partial_x\frac{1}{T})T_1\) is unbounded and hence cannot coincide with the operator \(T_2\). In order to find the required bounded operator we may consider the second copy of the equation \((\partial_x^2 + \omega^2)u = 0\), which is a result of the Darboux transformation applied to \((\partial_x^2 + \omega^2)\sigma = 0\) with respect to the particular solution \(g \equiv 1\) and construct the operator \(T_2\) by making the second diagram commutative. In order to obtain a bounded operator \(T_2\), instead of using \(f\partial_x\frac{1}{T}\) for the last step, we will use the inverse of \(\frac{1}{T}\partial_x f\), i.e. \(\frac{1}{f}(\int_0^x f(s) \cdot ds + C)\).

\[
\begin{align*}
\partial_x^2 + \omega^2 &\xrightarrow{T_1} \partial_x^2 - q_1 + \omega^2 \\
&\xrightarrow{T_2} f\partial_x\frac{1}{T} \partial_x^2 - q_2 + \omega^2 \\
\partial_x^2 + \omega^2 &\xrightarrow{T_1} \partial_x^2 - q_1 + \omega^2 \\
&\xrightarrow{T_2} \partial_x^2 + \omega^2 \\
&\xrightarrow{T_2} \partial_x^2 - q_2 + \omega^2 \\
&\xrightarrow{T_2} \partial_x^2 - q_2 + \omega^2
\end{align*}
\]

That explains how to obtain the following theorem.

**Theorem 6.1 ([46]).** The operator \(T_2\), acting on solutions \(u\) of equations \((\partial_x^2 + \omega^2)u = 0\), \(\omega \in \mathbb{C}\) by the rule

\[
T_2[u](x) = \frac{1}{f(x)} \left( \int_0^x f(\eta) T_1[u'](\eta) d\eta + u(0) \right)
\]  

(6.5)

coincides with the transmutation operator \(T_{2,h}\).

Now we show that the operator \(T_2\) can be written as a Volterra integral operator and, as a consequence, extended by continuity to a wider class of functions. To obtain simpler expression for the integral kernel \(K_2(x,t; -h)\) we have to suppose that the original integral kernel \(K_1(x,t; h)\) is known in the larger domain than required by definition (5.1). Namely, suppose that the function \(K_1(x,t; h)\) is known and is continuously differentiable in the domain \(\tilde{\Pi} : -a \leq x \leq a, -a \leq t \leq a\). We refer the reader to [46] for further details.

**Theorem 6.2 ([46]).** The operator \(T_2\) admits a representation as the Volterra integral operator

\[
T_2[u](x) = u(x) + \int_{-x}^x K_2(x,t; -h)u(t) dt,
\]  

(6.6)

with the kernel

\[
K_2(x,t; -h) = -\frac{1}{f(x)} \left( \int_{-t}^x \partial_t K_1(s,t; h)f(s) ds + \frac{h}{2} f(-t) \right).
\]  

(6.7)

Such representation is valid for any function \(u \in C^1[-a,a]\).

By Theorems 6.1 and 6.2 the Volterra operators \(T_2\) and \(T_2\) coincide on the set of finite linear combinations of solutions of the equations \((\partial_x^2 + \omega^2)u = 0\), \(\omega \in \mathbb{C}\). Since this set is dense in \(C[-a,a]\), by continuity of \(T_2\) and \(T_2\) we obtain the following corollaries.
Corollary 6.3 ([46]). The operator $T_2$ given by (6.6) with the kernel (6.7) coincides with $T_2$ on $C[-a,a]$.

Corollary 6.4 ([46]). The operator $T_2$ given by (6.5) coincides with $T_2$ on $C^1[-a,a]$.

Operator $A_1$ is the Darboux transformation of the operator $A_2$ with respect to the solution $1/f$, hence we may obtain another relation between the operators $T_1$ and $T_2$.

Corollary 6.5 ([46]). For any function $u \in C^1[-a,a]$ the equality

$$T_1[u](x) = f(x) \left( \int_0^x \frac{1}{f(\eta)} T_2[u'](\eta) \, d\eta + u(0) \right)$$  \hspace{1cm} (6.8)

is valid.

From the second commutative diagram at the beginning of this subsection we may deduce some commutation relations between the operators $T_1$, $T_2$ and $d/dx$. The proof immediately follows from (6.5) and (6.8).

Corollary 6.6 ([46]). The following operator equalities hold on $C^1[-a,a]$:

$$\partial_x f T_2 = f T_1 \partial_x$$ \hspace{1cm} (6.9)

$$\partial_x \frac{1}{f} T_1 = \frac{1}{f} T_2 \partial_x.$$ \hspace{1cm} (6.10)

In [44] the following notion of generalized derivatives was introduced. Consider a function $g$ assuming that both $f$ and $g$ possess the derivatives of all orders up to the order $n$ on the segment $[-a,a]$. Then in $[-a,a]$ the following generalized derivatives are defined

$$\gamma_0(g)(x) = g(x),$$

$$\gamma_k(g)(x) = (f^2(x))^{(-1)^{k-1}} (\gamma_{k-1}(g))'(x)$$

for $k = 1, 2, \ldots, n$.

Let a function $u$ be defined by the equality

$$g = \frac{1}{f} T_1 u,$$

and assume that $u \in C^n[-a,a]$. Note that below we do not necessarily require that the functions $f$ and $g$ be from $C^n[-a,a]$. With the use of (6.9) and (6.10) we have

$$\gamma_1(g) = f^2 \cdot \left( \frac{1}{f} T_1 u \right)' = f^2 \cdot \frac{1}{f} T_2 u' = f T_2 u',$$

$$\gamma_2(g) = \frac{1}{f^2} \cdot (f T_2 u')' = \frac{1}{f^2} \cdot f T_1 u'' = \frac{1}{f} T_1 u''.$$  

By induction we obtain the following corollary.
Corollary 6.7 \([46]\). Let \(u \in C^n[-a,a]\) and \(g = \frac{1}{f}T_1u\). Then
\[
\gamma_k(g) = f T_2 u^{(k)} \quad \text{if } k \text{ is odd, } k \leq n,
\]
and
\[
\gamma_k(g) = \frac{1}{f} T_1 u^{(k)} \quad \text{if } k \text{ is even, } k \leq n.
\]

Example 6.8. We start with the operator \(A_0 = d^2/dx^2\). We have to pick up such a solution \(f\) of the equation \(A_0 f = 0\) that \(f'/f \neq 0\). This is in order to obtain an operator \(A_1 \neq A_0\) as a result of the Darboux transformation of \(A_0\). For such solution \(f\) consider, e.g., \(f_0(x) = x + 1\). Both \(f_0\) and \(1/f_0\) are bounded on any segment \([-a,a] \subset (-1;1)\) and the Darboux transformed operator has the form \(A_1 = d^2/dx^2 - \frac{2}{(x+1)^2}\).

The transmutation operator \(T\) for \(A_0\) is obviously an identity operator and \(K_0(x,t;0) = 0\). Since \(f_0'(0) = 1\), we look for the parametrized operator \(T_{0;1}\). Its kernel is given by \((5.4)\): \(K_0(x,t;1) = 1/2\). From Theorem 6.2 we obtain the transmutation kernel for the operator \(A_1\)

\[
K_1(x,t;−1) = −\frac{1}{x+1} \left(\frac{1−t}{2(x+1)}\right). \tag{6.11}
\]

The Darboux transformation of the operator \(A_1\) with respect to the solution \(f_1\) is the operator \(A_2 = d^2/dx^2 - \frac{6}{(x+1)^2}\) and by Theorem 6.2 the transmutation operator \(T_{2;−2}\) for \(A_2\) is given by the Volterra integral operator \((5.1)\) with the kernel

\[
K_2(x,t;−2) = −\frac{1}{(x+1)^2}\left(\int_{−t}^{x} \frac{−3t+1}{2(s+1)} (s+1)^2 ds + (1−t)^2\right) = \frac{(3t−1)(x+1)^2 − 3(t−1)^2(t+1)}{4(x+1)^2}.
\]

This procedure may be continued iteratively. Consider the operators

\[
A_n := \frac{d^2}{dx^2} − \frac{n(n+1)}{(x+1)^2}.
\]

The function \(f_n(x) = (x+1)^{n+1}\) is a solution of the equation \(A_n f = 0\). The Darboux transformation of the operator \(A_n\) with respect to the solution \(f_n\) is the operator

\[
\frac{d^2}{dx^2} − 2\left(\frac{f_n'(x)}{f_n(x)}\right)^2 + \frac{n(n+1)}{(x+1)^2} = \frac{d^2}{dx^2} − \frac{(n+1)(n+2)}{(x+1)^2}.
\]
Transmutations and SPPS in eigenvalue problems

i.e., exactly the operator $A_{n+1}$. If we know $K_n(x,t;-n)$ for the operator $A_n$, by (5.4) we compute the kernel $K_n(x,t;n+1)$ corresponding to the solution $f_n(x)$ and by Theorem 6.2 we may calculate the kernel $K_{n+1}(x,t;-n-1)$. Careful analysis shows that we have to integrate only polynomials in all integrals involved, so the described procedure can be performed up to any fixed $n$.

**Example 6.9.** Consider the Schrödinger equation

$$u'' + 2\text{sech}^2(x) u = u. \tag{6.12}$$

This equation appears in soliton theory and as an example of a reflectionless potential in the one-dimensional quantum scattering theory (see, e.g. [48]). Equation (6.12) can be obtained as a result of the Darboux transformation of the equation $u'' = u$ with respect to the solution $f(x) = \cosh x$. The transmutation operator for the operator $A_1 = \partial_x^2 - 1$ was calculated in [9, Example 3]. Its kernel is given by the expression

$$K_1(x,t;0) = -\frac{1}{2} \frac{\sqrt{x^2 - t^2} I_1(\sqrt{x^2 - t^2})}{x - t},$$

where $I_1$ is the modified Bessel function of the first kind. Hence from Theorem 6.2 we obtain the transmutation kernel for the operator $A_2 = \partial_x^2 + 2\text{sech}^2 x - 1$

$$K_2(x,t;0) = \frac{1}{2} \frac{1}{\cosh(x)} \int_{-t}^{x} \left( \frac{I_0(\sqrt{s^2 - t^2})t}{s - t} + \frac{\sqrt{s^2 - t^2} I_1(\sqrt{s^2 - t^2})}{(s - t)^2} \right) \cosh s ds.$$

7. **Transmutation operator for the one-dimensional Dirac equation with a Lorentz scalar potential**

One-dimensional Dirac equations with Lorentz scalar potentials are widely studied (see, for example, [11, 15, 33, 34, 35, 36, 38, 54, 60, 62] and [55] for intertwining techniques for them).

According to [54] the Dirac equation in one space dimension with a Lorentz scalar potential can be written as

$$\left( \partial_x + m + S(x) \right) \Psi_1 = E \Psi_2, \tag{7.1}$$

$$\left( -\partial_x + m + S(x) \right) \Psi_2 = E \Psi_1, \tag{7.2}$$

where $m$ ($m > 0$) is the mass and $S(x)$ is a Lorentz scalar. Denote $\eta = m + S$ and write the system (7.1), (7.2) in a matrix form as

$$\begin{pmatrix} \partial_x + \eta & 0 \\ 0 & \partial_x - \eta \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}.$$ 

In order to apply the results on the transmutation operators and factorizations (6.3), (6.4) we consider a function $f$ such that

$$\frac{f'(x)}{f(x)} = -\eta = -m - S(x).$$

We can take $f(x) = \exp\left( -\int_0^x (m + S(s)) ds \right)$, then $f(0) = 1$ and $f$ does not vanish. Suppose the operators $T_1$ and $T_2$ are transmutations for the
operators \( A_1 = (\partial_x + f') (\partial_x - f') \) and \( A_2 = (\partial_x - f') (\partial_x + f') \) respectively (corresponding to functions \( f \) and \( 1/f \) in the sense of Proposition 5.3). As was shown in [46] with the use of commutation relations (6.9) and (6.10), the operator
\[
T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}
\]
transmutes any solution \( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) of the system
\[
\begin{align*}
u_1' &= Eu_2 \\ 
u_2' &= -Eu_1
\end{align*}
\] (7.3) (7.4)
into the solution \( \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \) of the system (7.1), (7.2) with the initial conditions \( \Psi_1(0) = u_1(0), \Psi_2(0) = u_2(0) \). And vice versa if \( \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \) is a solution of the system (7.1), (7.2), then the operator \( \begin{pmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{pmatrix} \) transmutes it into the solution \( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) of (7.3), (7.4) such that \( u_1(0) = \Psi_1(0), u_2(0) = \Psi_2(0) \).

Consider two systems of functions \( \{\varphi_k\}_{k=0}^{\infty} \) and \( \{\psi_k\}_{k=0}^{\infty} \) constructed from the function \( f \) by (2.3) and (2.4). The general solution of the system (7.3), (7.4) is given by
\[
\begin{align*}
u_1 &= C_1 v_1 + C_2 v_2 \\ 
u_2 &= C_2 v_1 - C_1 v_2,
\end{align*}
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants and
\[
\begin{align*}
v_1(x) &= \cos Ex = \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k}}{(2k)!} x^{2k}, \\
v_2(x) &= \sin Ex = \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k+1}}{(2k+1)!} x^{2k+1}.
\end{align*}
\]
From (6.1) and (6.2), we see that the general solution of the one-dimensional Dirac system (7.1), (7.2) has the form
\[
\begin{align*}
\Psi_1 &= C_1 \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k}}{(2k)!} \varphi_{2k} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k+1}}{(2k+1)!} \varphi_{2k+1}, \\
\Psi_2 &= C_2 \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k}}{(2k)!} \psi_{2k} - C_1 \sum_{k=0}^{\infty} \frac{(-1)^k E^{2k+1}}{(2k+1)!} \psi_{2k+1}.
\end{align*}
\]

Remark 7.1. It is possible to consider the two- or three-dimensional Dirac system and to construct the transmutation operator for it under some conditions on the potential. But the techniques involved, such as bicomplex numbers, pseudoanalytic function theory, Vekua equation and formal powers go well
beyond the scope of the present article. We refer interested readers to the recent paper [8].

References

[1] V. G. Bagrov and B. F. Samsonov, *Darboux transformation, factorization, and supersymmetry in one-dimensional quantum mechanics*. Teoret. Mat. Fiz. **104** (1995), no. 2, 356–367 (in Russian); translation in Theoret. and Math. Phys. **104** (1995), no. 2, 1051–1060.

[2] C. A. Balanis, *Advanced Engineering Electromagnetics*. John Wiley & Sons, 1989.

[3] H. Begehr and R. Gilbert, *Transformations, transmutations and kernel functions, vol. 1–2*. Longman Scientific & Technical, Harlow, 1992.

[4] R. Bellman, *Perturbation techniques in mathematics, engineering and physics*. Dover Publications, 2003.

[5] J. Ben Amara and A. A. Shkalikov, *A Sturm-Liouville problem with physical and spectral parameters in boundary conditions*. Mathematical Notes **66** (1999), no. 2, 127–134.

[6] L. M. Brekhovskikh, *Waves in layered media*. New York, Academic Press, 1960.

[7] H. Campos and V. V. Kravchenko, *A finite-sum representation for solutions for the Jacobi operator*. Journal of Difference Equations and Applications **17** (2011) No. 4, 567–575.

[8] H. Campos, V. V. Kravchenko and L. Mendez, *Complete families of solutions for the Dirac equation: an application of bicomplex pseudoanalytic function theory and transmutation operators*. To appear in the Advances in the Applied Clifford Algebras (2012), available from arxiv.org, arXiv:1111.4198.

[9] H. Campos, V. V. Kravchenko and S. Torba, *Transmutations, L-bases and complete families of solutions of the stationary Schrödinger equation in the plane*. J. Math. Anal. Appl. **389** (2012), No. 2, 1222–1238.

[10] R. W. Carroll, *Transmutation theory and applications*. Mathematics Studies, Vol. 117, North-Holland, 1985.

[11] J. Casahorrán, *Solving simultaneously Dirac and Ricatti equations*. Journal of Nonlinear Mathematical Physics **5** (1985), No. 4, 371–382.

[12] R. Castillo, K. V. Khmelnytskaya, V. V. Kravchenko and H. Oviedo, *Efficient calculation of the reflectance and transmittance of finite inhomogeneous layers*. J. Opt. A: Pure and Applied Optics **11** (2009), 065707.

[13] R. Castillo R, V. V. Kravchenko, H. Oviedo and V. S. Rabinovich, *Dispersion equation and eigenvalues for quantum wells using spectral parameter power series*. J. Math. Phys., **52** (2011), 043522 (10 pp.)

[14] B. Chanane, *Sturm-Liouville problems with parameter dependent potential and boundary conditions*. J. Comput. Appl. Math. **212** (2008), 282–290.

[15] C.-Y. Chen, *Exact solutions of the Dirac equation with scalar and vector Hartmann potentials*. Physics Letters A. **339** (2005), 283–287.

[16] A. H. Cherin, *An introduction to Optical Fibers*. McGraw-Hill, 1983.

[17] W. C. Chew, *Waves and fields in inhomogeneous media*. Van Nostrand Reinhold, New York, 1990.
[18] J. L. Cieński, *Algebraic construction of the Darboux matrix revisited*. J. Phys. A: Math. Theor. **42** (2009), 404003.

[19] W. J. Code and P. J. Browne, *Sturm-Liouville problems with boundary conditions depending quadratically on the eigenparameter*. J. Math. Anal. Appl. **309** (2005), 729–742.

[20] H. Coşkun and N. Bayram, *Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition*. J. Math. Anal. Appl. 306 (2005), no. 2, 548–566.

[21] J. Delsarte, *Sur une extension de la formule de Taylor*. J Math. Pures et Appl. **17** (1938), 213–230.

[22] J. Delsarte, *Sur certaines transformations fonctionnelles relatives aux équations linéaires aux dérivées partielles du second ordre*. C. R. Acad. Sc. **206** (1938), 178–182.

[23] J. Delsarte and J. L. Lions, *Transmutations d’opérateurs différentiels dans le domaine complexe*. Comment. Math. Helv. **32** (1956), 113–128.

[24] M. K. Fage and N. I. Naguibida. *The problem of equivalence of ordinary linear differential operators*. Novosibirsk: Nauka, 1987 (in Russian).

[25] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*. IEEE Press, New York, 1994.

[26] S. Flügge, *Practical Quantum Mechanics*. Berlin: Springer-Verlag, 1994.

[27] Ch. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*. Proc. Roy. Soc. Edinburgh Sect. A **77** (1977), no. 3–4, 293–308.

[28] P. R. Garabedian, *Partial differential equations*. New York–London: John Wiley and Sons, 1964.

[29] C. Gu, H. Hu, and Z. Zhou, *Darboux Transformations in Integrable Systems*, Springer-Verlag, Berlin, 2005.

[30] R. L. Hall, *Square-well representations for potentials in quantum mechanics*. J. Math. Phys. **33** (1992), 3472–3476.

[31] P. Harrison, *Quantum Wells, Wires and Dots: Theoretical and Computational Physics of Semiconductor Nanostructures*. Chichester: Wiley, 2010.

[32] A. D. Hemery and A. P. Veselov, *Whittaker-Hill equation and semifinite-gap Schrödinger operators*. J. Math. Phys. **51** (2010), 072108; doi:10.1063/1.3455367.

[33] J. R. Hiller, *Solution of the one-dimensional Dirac equation with a linear scalar potential*. Am. J. Phys. **70(5)** (2002), 522–524.

[34] C.-L. Ho, *Quasi-exact solvability of Dirac equation with Lorentz scalar potential*. Ann. Physics **321** (2006), No. 9, 2170–2182.

[35] R. Jackiw and S.-Y. Pi, *Persistence of zero modes in a gauged Dirac model for bilayer graphene*. Phys. Rev. B **78** (2008), 132104.

[36] N. Kevlishvili, G. Piranishvili, *Klein paradox in modified Dirac and Salpeter equations*. Fizika **9** (2003), No. 3, 4, 57–61.

[37] K. V. Khmelnyskaya, V. V. Kravchenko and H. C. Rosu, *Eigenvalue problems, spectral parameter power series, and modern applications*. Submitted, available at arXiv:1112.1633.
[38] K. V. Khmelnytskaya and H. C. Rosu, An amplitude-phase (Ermakov–Lewis) approach for the Jackiw–Pi model of bilayer graphene. J. Phys. A: Math. Theor. 42 (2009), 042004.

[39] K. V. Khmelnytskaya and H. C. Rosu, A new series representation for Hill’s discriminant. Annals of Physics 325 (2010), 2512–2521.

[40] A. Kostenko and G. Teschl, On the singular Weyl–Titchmarsh function of perturbed spherical Schrödinger operators. J. Differential Equations 250 (2011), 3701–3739.

[41] V. V. Kravchenko, A representation for solutions of the Sturm-Liouville equation. Complex Variables and Elliptic Equations 53 (2008), 775–789.

[42] V. V. Kravchenko, Applied pseudoanalytic function theory. Basel: Birkhäuser, Series: Frontiers in Mathematics, 2009.

[43] V. V. Kravchenko, On the completeness of systems of recursive integrals. Communications in Mathematical Analysis, Conf. 03 (2011), 172–176.

[44] V. V. Kravchenko, S. Morelos and S. Tremblay, Complete systems of recursive integrals and Taylor series for solutions of Sturm-Liouville equations. Mathematical Methods in the Applied Sciences, Published online, doi: 10.1002/mma.1596.

[45] V. V. Kravchenko and R. M. Porter, Spectral parameter power series for Sturm-Liouville problems. Mathematical Methods in the Applied Sciences 33 (2010), 459–468.

[46] V. V. Kravchenko and S. Torba, Transmutations for Darboux transformed operators with applications. J. Phys. A: Math. Theor. 45 (2012), # 075201 (21 pp.).

[47] V. V. Kravchenko and U. Velasco-García, Dispersion equation and eigenvalues for the Zakharov-Shabat system using spectral parameter power series. J. Math. Phys. 52 (2011), 063517.

[48] G. L. Lamb, Elements of soliton theory. John Wiley & Sons, New York, 1980.

[49] B. M. Levitan, Inverse Sturm-Liouville problems. VSP, Zeist, 1987.

[50] J. L. Lions, Solutions élémentaires de certains opérateurs différentiels à coefficients variables. Journ. de Math. 36 (1957), Fasc 1, 57–64.

[51] V. A. Marchenko, Sturm-Liouville operators and applications. Basel: Birkhäuser, 1986.

[52] V. Matveev and M. Salle, Darboux transformations and solitons. New York, Springer, 1991.

[53] H. Medwin and C. S. Clay, Fundamentals of Oceanic Acoustics. Academic Press, Boston, San Diego, New York, 1997.

[54] Y. Nogami and F. M. Toyama, Supersymmetry aspects of the Dirac equation in one dimension with a Lorentz scalar potential. Physical Review A. 47 (1993), no. 3, 1708–1714.

[55] L. M. Nieto, A. A. Pecheritsin and B. F. Samsonov, Intertwining technique for the one-dimensional stationary Dirac equation, Annals of Physics 305 (2003), 151–189.

[56] O. A. Obrezanova and V. S. Rabinovich, Acoustic field, generated by moving source in stratified waveguides. Wave Motion 27 (1998), 155–167.
[57] A. A. Pecheritsin, A. M. Pupasov and B. F. Samsonov, Singular matrix Darboux transformations in the inverse-scattering method, J. Phys. A: Math. Theor. 44 (2011), 205305.

[58] C. Rogers and W. K. Schief, Backlund and Darboux transformations: geometry and modern applications in soliton theory. Cambridge University Press, 2002.

[59] H. Rosu, Short survey of Darboux transformations, Proceedings of “Symmetries in Quantum Mechanics and Quantum Optics”, Burgos, Spain, 1999, 301–315.

[60] R. K. Roychoudhury and Y. P. Varshni, Shifted 1/N expansion and scalar potential in the Dirac equation. J. Phys. A: Math. Gen. 20 (1987), L1083–L1087.

[61] S. M. Sitnik, Transmutations and applications: a survey. arXiv:1012.3741v1 [math.CA], originally published in the book: “Advances in Modern Analysis and Mathematical Modeling” Editors: Yu.F.Korobeinik, A.G.Kusraev, Vladikavkaz: Vladikavkaz Scientific Center of the Russian Academy of Sciences and Republic of North Ossetia–Alania, 2008, 226–293.

[62] R. Su, Yu Zhong and S. Hu, Solutions of Dirac equation with one-dimensional scalarlike potential. Chinese Phys.Lett. 8 (1991), no.3, 114–117.

[63] K. Trimeche. Transmutation operators and mean-periodic functions associated with differential operators. London: Harwood Academic Publishers, 1988.

[64] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition. Math. Z. 133 (1973), 301–312.

Vladislav V. Kravchenko
Department of Mathematics, CINVESTAV del IPN, Unidad Queretaro, Libramiento Norponiente No. 2000, Fracc. Real de Juriquilla, Queretaro, Qro. C.P. 76230 MEXICO
e-mail: vkravchenko@math.cinvestav.edu.mx

Sergii M. Torba
Department of Mathematics, CINVESTAV del IPN, Unidad Queretaro, Libramiento Norponiente No. 2000, Fracc. Real de Juriquilla, Queretaro, Qro. C.P. 76230 MEXICO
e-mail: storba@math.cinvestav.edu.mx