Research Article

On Behavior of the Periodic Orbits of a Hamiltonian System of Bifurcation of Limit Cycles

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In light of the previous recent studies by Jaume Llibre et al. that dealt with the finite cycles of generalized differential Kukles polynomial systems using the first- and second-order mean theorem such as (Nonlinear Anal., 74, 1261–1271, 2011) and (J. Dyn. Control Syst., vol. 21, 189–192, 2015), in this work, we provide upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of Hamiltonian system using the averaging theory of first order.

1. Introduction

Among the many interesting problems in the qualitative theory of planar polynomial differential systems is the study of their limit cycles (see [1, 2]). In particular, concerning Kukles differential system of the form,

\[ \begin{align*}
\dot{x} &= -y, \\
\dot{y} &= f(x, y),
\end{align*} \]

has a long history, where \( f(x, y) \) is a polynomial with real coefficients of degree \( n \). Since it was first introduced in Kukles 1944, many researchers have concentrated on its maximum number of limit cycles and their location. See, for example, [3–5].

In [6], Llibre and Mereu studied the maximum number of limit cycles using the averaging theory as follows:

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - \sum_{k=1}^{\infty} \epsilon^k \left( f_k(x) y + h_k(x) y^2 + g_k(x) y^3 \right),
\end{align*} \]

(2)

where, for every \( k \), the polynomials \( f_k(x) \), \( g_k(x) \), and \( h_k(x) \) have degree \( n_1, n_2, \) and \( n_3 \), respectively. \( \epsilon_0 \neq 0 \) is a real number and \( \epsilon \) is a small parameter.

Also, Makhlouf and Menaceur [7] studied the maximum number for the more generalized polynomial Kukles differential systems in the form

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - \sum_{k=1}^{\infty} \epsilon^k \left( f_k(x) y + h_k(x) y^2 + g_k(x) y^3 \right).
\end{align*} \]

(3)
The number of limit cycles bifurcating from the center \( \dot{x} = -y^{2p-1} \) and \( \dot{y} = x^{2q-1} \), where \( p, q \) are positive integers, for the following two kinds of polynomial differential systems,

\[
\begin{align*}
\dot{x} &= -y^{2p-1}, \\
\dot{y} &= x^{2q-1} - \varepsilon f(x), \\
\dot{x} &= -y^{2p-1} - \varepsilon px f(x, y), \\
\dot{y} &= x^{2q-1} - \varepsilon qy f(x, y),
\end{align*}
\]

(4)

were investigated in the works [8, 9], respectively. In the current study, we discuss the maximum number of limit cycles of the polynomial differential system (5) bifurcating from the periodic orbits of the current study. Theorem 2.

\[
\begin{align*}
\dot{x} &= -y^{2p-1}, \\
\dot{y} &= x^{2q-1} - \varepsilon f(x) + g(x) y^{2n-1} + h(x) y^{2n} + l(x) y^{2n+1},
\end{align*}
\]

(5)

where \( p, q, \) and \( n \) are positive integers, the polynomials \( f(x), g(x), h(x), \) and \( l(x) \) have degree \( n_1, n_2, n_3, \) and \( n_4, \) respectively, and \( \varepsilon \) is a small positive parameter. Clearly, system (5) with \( \varepsilon = 0 \) is an Hamiltonian system with

\[
H(x, y) = \frac{1}{2q} x^{2q} + \frac{1}{2p} y^{2p}. 
\]

(6)

Our main theorems are given as follows.

**Theorem 1.** For the sufficiently small \(|\varepsilon|\), system (5), using averaging theory of first order, has at most

\[
\max \left\{ \left[ \frac{n_1}{2} \right], \left[ \frac{n_2}{2} \right], \left[ \frac{n_3}{2} \right], \left[ \frac{n_4}{2} \right] + p + q \right\}, 
\]

(7)

The limit cycles bifurcating from the periodic orbits of the center are \( \dot{x} = -y^{2p-1} \) and \( \dot{y} = x^{2q-1} \), where \([.]\) denotes the integer part function.

The proof of Theorem 1 is given in Section 3.

**Theorem 2.** Consider system (5) with \( q = lp \), \( l \) is a positive integer, and \(|\varepsilon|\) sufficiently small; let \( H(n_1, l) \) denote the maximum number of limit cycles of the polynomial differential system (5) bifurcating from the periodic orbits of the center \( \dot{x} = -y^{2p-1} \) and \( \dot{y} = x^{2q-1} \) using the averaging theory of first order; then,

\[
\begin{align*}
(a) \quad H(n_1, l) &= \left[ \frac{n_1}{2} \right] + \left[ \frac{n_3}{2} \right] + 1, \quad \text{if} \quad \left[ \frac{n_1}{2} \right] < l, \\
(b) \quad H(n_1, l) &= \left[ \frac{n_1}{2} \right] + l + 1, \quad \text{if} \quad l \leq \left[ \frac{n_3}{2} \right] \leq l + \left[ \frac{n_1}{2} \right], \\
(c) \quad H(n_1, l) &= \left[ \frac{n_1}{2} \right], \quad \text{if} \quad l + \left[ \frac{n_1}{2} \right] < \frac{n_3}{2}. 
\end{align*}
\]

(8)

The proof of Theorem 2 is given Section 4.

2. **First-Order Averaging Method**

The averaging theory is an interesting method to research the limit cycles. Here, some specific function, associated to the initial system, is stated.

**Theorem 3.** The two initial value problems are as follows:

\[
\begin{align*}
\dot{x} &= \varepsilon R(t, x) + \varepsilon^2 G(t, x, \varepsilon), \quad x(0) = x_0, \\
\dot{y} &= \varepsilon f^0(y), \quad y(0) = x_0,
\end{align*}
\]

(9) (10)

where \( x, y \) and \( x_0 \in D \) which is an open domain of \( \mathbb{R} \), \( t \in [0, \infty) \), \( \varepsilon \in (0, \varepsilon_0] \), \( R \) and \( G \) are periodic functions with their period \( T \) with its variable \( t \), and \( f^0(y) \) is the average function of \( R(t, y) \) with respect to \( t \), i.e.,

\[
f^0(y) = \frac{1}{T} \int_0^T R(t, y) dt. \]

(11)

Assume that

(i) \( R, \partial R/\partial x, \partial^2 R/\partial x^2, G, \) and \( \partial G/\partial x \) are well defined, continuous, and bounded by a constant independent by \( \varepsilon \in (0, \varepsilon_0] \) in \( [0, \infty) \times D \).

(ii) \( T \) is a constant independent of \( \varepsilon \).

(iii) \( y(t) \) belongs to \( D \) on the time scale \( 1/\varepsilon \). Then, the following statements hold:

(i) On the time scale \( 1/\varepsilon \), we have

\[
x(t) - y(t) = O(\varepsilon), \quad \varepsilon \to 0. \]

(12)

(ii) If \( p \) is an equilibrium point of the averaged system (10), such that

\[
\frac{\partial f^0}{\partial y} |_{y=p} \neq 0,
\]

(13)

then system (9) has a \( T \)-periodic solution \( \phi(t, \varepsilon) \to p \) as \( \varepsilon \to 0 \).

(iii) If (11) is a negative, therefore, the corresponding periodic solution \( \phi(t, \varepsilon) \) of equation (9) according to \( (t, x) \) is asymptotically stable, for all \( \varepsilon \) sufficiently small; if (11) is a positive, then it is unstable.

For more information about the averaging theory, see [10–12].

3. **Proof of Theorem 1**

Here, we need to transform system (5) to the canonical form from (9). Doing the change of \((p, q)\)-polar coordinates \( x = r^p \cos \theta \) and \( y = r^q \sin \theta \) (see Appendix) and taking \( \theta \) as an independent variable, then system (5) can be written as
According to the notation introduced in Section 2, we have

\[
\begin{align*}
\dot{r} &= -\varepsilon r^{-q+1} (Sn\theta)^{2p+1} + f(r^pC\theta) + g(r^pC\theta)(Sn\theta)^{2p+1} + h(r^pC\theta)(Sn\theta)^{2p+1} + \varepsilon \theta^{-q}C\theta, \\
\dot{\theta} &= r^p \theta^{-q} - \varepsilon pr^{-q}C\theta \\
\end{align*}
\]

If we write

\[
\begin{align*}
 f(x) &= \sum_{k=0}^{n} a_k x^k, \\
g(x) &= \sum_{k=0}^{n} b_k x^k, \\
h(x) &= \sum_{k=0}^{n} c_k x^k, \\
l(x) &= \sum_{k=0}^{n} d_k x^k, \\
\end{align*}
\]

then system (14) becomes

\[
\begin{align*}
\dot{r} &= -\varepsilon r^{-q+1} \left[ \sum_{k=0}^{n} a_k (C\theta)^k (Sn\theta)^{2p-1} r^p + \sum_{k=0}^{n} b_k (C\theta)^k (Sn\theta)^{2p+1} r^{p+q} + \sum_{k=0}^{n} c_k (C\theta)^k (Sn\theta)^{2p+2} r^{p+q} + \sum_{k=0}^{n} d_k (C\theta)^k (Sn\theta)^{2p+3} r^{p+q} \right], \\
\dot{\theta} &= r^p \theta^{-q} - \varepsilon pr^{-q}C\theta \\
\end{align*}
\]

where \( \theta \) is the independent variable we get from system (16).

From

\[
\frac{dr}{d\theta} = \varepsilon R(r, \theta) + O(\varepsilon^2),
\]

where

\[
R(r, \theta) = -r^{-p+1} \left[ \sum_{k=0}^{n} a_k (C\theta)^k (Sn\theta)^{2p-1} r^p + \sum_{k=0}^{n} b_k (C\theta)^k (Sn\theta)^{2p+1} r^{p+q} + \sum_{k=0}^{n} c_k (C\theta)^k (Sn\theta)^{2p+2} r^{p+q} + \sum_{k=0}^{n} d_k (C\theta)^k (Sn\theta)^{2p+3} r^{p+q} \right].
\]
4 Mathematical Problems in Engineering

and we write

\[
f^0(r) = \frac{r^{-pq+p+1}}{T} \left[ \sum_{k=0}^{n_1} a_k r^{pk} + \sum_{k=0}^{n_2} b_k I_{k,2(p+1)} r^{pk+q(2n-1)} \right]
\]

where

\[
I_{i,j} = \int_0^T C_s^j \theta S n^i d\theta.
\]

It is known that

\[
f^0(r) = \frac{r^{q(2n-1)+p+1}}{T} \left[ \sum_{k=k_{\text{even}}}^{n_1} b_k I_{2k,2(p+1)} r^{2ps} + \sum_{k=k_{\text{even}}}^{n_2} d_k I_{2k+1,2(p+1)} r^{2ps+q} \right],
\]

we obtain

\[
f^0(r) = -\frac{r^{q(2n-1)+p+1}}{T} \left[ \sum_{s=0}^{[n/2]} b_{2s} I_{2s,2(p+1)} r^{2ps} + \sum_{s=0}^{[n/2]} d_{2s+1} I_{2s+1,2(p+1)} r^{2ps+q} \right].
\]

For the simplicity of calculation, let \( B_s = b_{2s} I_{2s,2(p+1)} \) and \( D_s = d_{2s+1} I_{2s+1,2(p+1)} \), therefore, (24) can be reduced to

\[
f^0(r) = -\frac{r^{q(2n-1)+p+1}}{T} \left[ \sum_{s=0}^{[n/2]} B_s r^{2ps} + \sum_{s=0}^{[n/2]} D_s r^{2ps+q} \right].
\]

As we all know, the number of positive roots of \( f^0(r) \) is equal to that of

Consider the polynomial differential system (5) with \( q = lp \); from equation (25) we obtain

\[
N(r) = \sum_{s=0}^{[n/2]} B_s r^{2ps} + \sum_{s=0}^{[n/2]} D_s r^{2(p+s+q)}.
\]

Then, to find the real roots of \( N(r) \), we must find the zeros of a polynomial in the variable \( \rho = r^2 \):

\[
M(\rho) = \sum_{s=0}^{[n/2]} B_s \rho^{2ps} + \sum_{s=0}^{[n/2]} D_s \rho^{2(p+s+q)}.
\]

So, the degree of \( M(\rho) \) is bounded by

\[\mu = \max\left[\frac{n}{2} p, \frac{n}{2} p + q\right],\]

we conclude that \( f^0(r) \) has at most \( \mu \) positive roots. Hence, Theorem 1 is proved.

4. Proof of Theorem 2

Consider the polynomial differential system (5) with \( q = lp \);
As we all know, the number of positive roots of \( f^0(r) \) is equal to that of 
\[
G(r) = B_0 + B_1r^{2p} + B_2r^{4p} + \cdots + B_{\lceil n_p/2 \rceil}r^{2p\lceil n_p/2 \rceil} \\
+ D_0r^{2ql} + D_1r^{2q(l+1)} + D_2r^{2q(l+2)} + \cdots \\
+ D_{\lceil n_p/2 \rceil}r^{2q\lceil n_p/2 \rceil}.
\]  
(29)

To find the number of positive roots of polynomials \( G(r) \), we distinguish 3 cases.

\[
f^0(r) = \frac{r^{6q-4}}{T}(B_0 + B_1r^2 + B_2r^4 + D_0r^6 + D_1r^8 + D_2r^{10}).
\]  
(33)

where \( B_s = b_{2s}I_{2s,2} \) and \( D_s = d_{2s}I_{2s,4} \). Using (A.3) of the Appendix, we obtain
\[
\begin{align*}
l_{0,2} &= 2.1033, \\
l_{2,2} &= 0.60460, \\
l_{4,2} &= 0.32339, \\
l_{0,4} &= 0.63098, \\
l_{2,4} &= 0.15115, \\
l_{4,4} &= 6.9298 \times 10^{-2}.
\end{align*}
\]  
(34)

**Case 1.** For \( \lceil n_p/2 \rceil \leq l \), the number terms in polynomial (29) is \( \lceil n_p/2 \rceil + \lceil n_p/2 \rceil + 1 \). Now, we shall apply the Descartes theorem of the Appendix, we can choose the appropriate coefficients \( B_i \) and \( D_j \) so that the simple positive roots’ number of \( G(r) \) is at most \( \lceil n_p/2 \rceil + \lceil n_p/2 \rceil + 1 \). Hence, (a) of Theorem 2 is proved.

**Case 2.** For \( l \leq \lceil n_p/2 \rceil \leq l + \lceil n_p/2 \rceil \), the number terms in polynomial (29) is

\[
\left\lfloor \frac{n_p}{2} \right\rfloor + \left\lfloor \frac{n_p}{2} \right\rfloor + 2 - \left( \left\lfloor \frac{n_p}{2} \right\rfloor - l + 1 \right) = \left\lfloor \frac{n_p}{2} \right\rfloor + l + 1.
\]  
(30)

By Descartes Theorem, we can choose the appropriate coefficients \( B_i \) and \( D_j \) so that the simple positive roots’ number of \( G(r) \) is at most \( \lceil n_p/2 \rceil + l \). Hence, (b) of Theorem 2 is proved.

**Case 3.** For \( l + \lceil n_p/2 \rceil < \lceil n_p/2 \rceil \), the number terms in polynomial (29) is \( \lceil n_p/2 \rceil + 1 \); by Descartes Theorem, we can choose the appropriate coefficients \( B_i \) and \( D_j \) so that the simple positive roots’ number of \( G(r) \) is at most \( \lceil n_p/2 \rceil \). Hence, (c) of Theorem 2 is proved.

**Example 1.** We consider system (5), with \( p = 1, q = 3, n = 1 \), and

\[
f(x) = \sum_{k=0}^2 a_k x^k, \\
g(x) = \sum_{k=0}^2 b_k x^k, \\
h(x) = \sum_{k=0}^2 c_k x^k, \\
l(x) = \sum_{k=0}^4 d_k x^k,
\]
where
\[
\begin{align*}
a_0 &= 1, \\
a_1 &= 2.3, \\
a_2 &= 4.7, \\
b_0 &= -0.5, \\
b_1 &= 1.1, \\
b_2 &= 6.3, \\
b_3 &= 2.5, \\
b_4 &= -15.32, \\
c_0 &= 2.2, \\
c_1 &= -6.4, \\
c_2 &= 7.3, \\
d_0 &= 4.65, \\
d_1 &= 3.4, \\
d_2 &= -5.24, \\
d_3 &= 6.4, \\
d_4 &= 1.13.
\end{align*}
\]  
(32)

In this case, \( C_s \theta \) and \( S_n \theta \) are \( T \)-periodic function with period \( T = 8.4131 \). From equation (28), we obtain

So,
\[
f^0(r) = \frac{r^3}{T} \left[ -1.0517 + 3.8083 r^2 - 4.9553 r^4 \\
+ 2.9320 r^6 - 0.79257 r^8 + 7.8473 \times 10^{-2} r^{10} \right].
\]  
(35)

This polynomial has four positive real roots: \( r_1 = 0.6, r_2 = 0.8, r_3 = 1.1, r_4 = 1.3, \) and \( r_5 = 2 \). According to statement (a) of Theorem 2, the system has exactly 5 limit cycles bifurcating from the periodic orbits of the center \( \dot{x} = -y \) and \( \dot{y} = x^2 \), using the averaging theory of first order.

**5. Conclusion**

In this work, by using averaging theory of the first order, we have proved upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of the Hamiltonian system. In addition, in the next work, a new
condition with a new method will be used to prove our main result in this study.

Appendix

1-p,q)-Polar Coordinates

Following Lyapunov [13], we introduce the (p,q)-trigonometric functions $u(θ) = Csθ$ and $v(θ) = Snθ$ as the solution of the following initial value problem:

\[
\begin{align*}
\dot{u} &= -v^{2p-1}, \\
\dot{v} &= u^{2q-1}, \\
\end{align*}
\]

\[u(0) = \frac{a_1}{\sqrt{p}}, \quad v(0) = 0.\]  
\[\text{(A.1)}\]

Moreover, they satisfy the following properties:

(i) The functions $Csθ$ and $Snθ$ are $T$-periodic with

\[T = 2p^{-\frac{1}{2q}}q^{-\frac{1}{2p}} \Gamma(1/2p)\Gamma(1/2q) / \Gamma((1/2p) + (1/2q)).\]  
\[\text{(A.2)}\]

where $Γ$ is the gamma function.

(ii) For $p = q = 1$, we have $Csθ = \cos θ$ and $Snθ = \sin θ$.

(iii) $pCs^{2p}θ + qSn^{2q}θ = 1$.

(iv) Let $Csθ$ and $Snθ$ be the $(1,q)$-trigonometrical functions, for $i$ and $j$ are both even (see [1]):

\[I_{i,j} = \int_0^T Cs^iθSn^jθdθ = 2q^{-(j+1/2)} \Gamma(i + 1/2q)\Gamma(j + 1/2) / \Gamma((i + 1/2q) + (j + 1/2)).\]  
\[\text{(A.3)}\]

2-Descartes Theorem

The purpose of the Descartes theorem is to provide an insight on how many real roots a polynomial $P(x)$ may have.

Theorem A.1 (see [14]). Consider the real polynomial

\[P(x) = a_1x_1 + a_2x_2 + \cdots + a_kx_k,\]  
\[\text{(A.4)}\]

with $0 < l_1 < l_2 < \ldots < l_k$ and $a_i \neq 0$ real constants, for $i \in \{1, 2, 3, \ldots, k\}$. When $a_i a_{i+1} < 0$, we say that $a_i$ and $a_{i+1}$ have a variation of sign. If the number of variations of signs is $n$, then $P(x)$ has at most $m$ positive real roots. Moreover, it is always possible to choose the coefficients of $P(x)$ in such a way that $P(x)$ has exactly $k - 1$ positive real roots.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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