Almost even-Clifford hermitian manifolds with large automorphism group

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Abstract

We study manifolds endowed with an (almost) even Clifford (hermitian) structure and admitting a large automorphism group. We classify them when they are simply connected and the dimension of the automorphism group is maximal, and also prove a gap theorem for the dimension of the automorphism group.

1 Introduction

Recently, there has been some interest in manifolds admitting so-called even Clifford structures [9, 10]. Here, we study such manifolds when they admit a large automorphism group. This type of problem has been studied on Riemannian manifolds [13, 14], almost hermitian manifolds [12], and almost quaternion-hermitian manifolds [11].

It is a classical result [6] that the maximal dimension of the isometry group of a connected $n$-dimensional Riemannian manifold is $\frac{1}{2}n(n+1)$. If the dimension is maximal, the manifold is isometric to either Euclidean space $\mathbb{R}^n$, or the sphere $S^n$, or projective space $\mathbb{RP}^n$, or (simply connected) hyperbolic space. Furthermore, in [13] it was shown that the isometry group contains no $m$-dimensional closed subgroup where

$$\frac{1}{2}n(n-1)+1 < m < \frac{1}{2}n(n+1).$$

In [12], it was shown that the automorphism group of a connected $2n$-dimensional almost-hermitian manifold has dimension at most $n(n+2)$. If the dimension of the automorphism is maximal, the manifold is isometric to either complex Euclidean space $\mathbb{C}^n$, or an open ball with Kähler structure with negative constant holomorphic sectional curvature, or complex projective space $\mathbb{CP}^n$. In this case, however, there is also a (unique) manifold whose automorphism group has dimension one less than the maximal one, namely, euclidean space [4].

In [11], it was shown that the automorphism group of a connected $4n$-dimensional almost quaternion-hermitian manifold has dimension at most $2n^2+5n+3$. If the dimension of the automorphism group is between $2n^2+5n$ and $2n^2+5n+3$, the manifold is isometric to either quaternionic Euclidean space $\mathbb{H}^n$, or quaternionic projective space $\mathbb{HP}^n$, or quaternionic hyperbolic space.

In this paper, we will prove the analogous theorems for almost even-Clifford hermitian manifolds of rank $r \geq 3$. Our terminology differs from that of [9, 10] since we have added the words “almost” and “hermitian” since, in principle, there is no integrability condition on the structure and the compatibility

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with a Riemannian metric is an extra condition. We shall explore integrability conditions in the style of Gray [3] in a future paper.

The note is organized as follows. In Section 2 we recall some preliminaries on Clifford algebras and representations, and almost even-Clifford hermitian manifolds. In Section 3 we give and upper bound in the dimension of the automorphism group (Proposition 3.1), determine the spaces whose automorphism group has dimension equal to this bound (Theorem 3.1), and prove a gap in the dimension of the automorphism group (Proposition 3.2).

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2 Preliminaries

In this section we recall material that can also be consulted in [2, 7]. Let \( Cl_r \) denote the Clifford algebra generated by all the products of the orthonormal vectors \( e_1, e_2, \ldots, e_r \in \mathbb{R}^r \) subject to the relations

\[
e_j e_k + e_k e_j = -2\delta_{j,k}, \quad \text{for } 1 \leq j, k \leq r.
\]

The even Clifford subalgebra \( Cl^0_r \) is defined as the invariant (+1)-subspace of the involution of \( Cl_r \) induced by the map \( -\text{Id}_{\mathbb{R}^r} \). The Spin group \( \text{Spin}(r) \subset Cl_r \) is the subset

\[
\text{Spin}(r) = \{ x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^r, |x_j| = 1, l \in \mathbb{N} \},
\]

endowed with the product of the Clifford algebra. The Lie algebra of \( \text{Spin}(r) \) is

\[
\text{spin}(r) = \text{span}\{e_i e_j \mid 1 \leq i < j \leq r \}.
\]

Now, we summarize some results about real representations of \( Cl^0_r \) in the next table (cf. [7]). Here \( d_r \) denotes the dimension of an irreducible representation of \( Cl^0_r \) and \( v_r \) the number of distinct irreducible representations. Let \( \Delta_r \) denote the irreducible representation of \( Cl^0_r \) for \( r \not\equiv 0 \pmod{4} \) and \( \Delta^\pm_r \) denote the irreducible representations for \( r \equiv 0 \pmod{4} \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
r \pmod{8} & d_r & Cl^0_r & \Delta_r / \Delta^\pm_r & v_r \\
\hline
1 & 2^{\lfloor \frac{r}{2} \rfloor} & \mathbb{R}(d_r) & \mathbb{R}^{d_r} & 1 \\
2 & 2^{\lfloor \frac{r}{2} \rfloor} & \mathbb{C}(d_r/2) & \mathbb{C}^{d_r/2} & 1 \\
3 & 2^{\lfloor \frac{r}{2} \rfloor + 1} & \mathbb{H}(d_r/4) & \mathbb{H}^{d_r/4} & 1 \\
4 & 2^{\lfloor \frac{r}{2} \rfloor} & \mathbb{H}(d_r/4) \oplus \mathbb{H}(d_r/4) & \mathbb{H}^{d_r/4} & 2 \\
5 & 2^{\lfloor \frac{r}{2} \rfloor + 1} & \mathbb{H}(d_r/4) & \mathbb{H}^{d_r/4} & 1 \\
6 & 2^{\lfloor \frac{r}{2} \rfloor} & \mathbb{C}(d_r/2) & \mathbb{C}^{d_r/2} & 1 \\
7 & 2^{\lfloor \frac{r}{2} \rfloor} & \mathbb{R}(d_r) & \mathbb{R}^{d_r} & 1 \\
8 & 2^{\lfloor \frac{r}{2} \rfloor - 1} & \mathbb{R}(d_r) \oplus \mathbb{R}(d_r) & \mathbb{R}^{d_r} & 2 \\
\hline
\end{array}
\]

Table 1

Note that the representations are complex for \( r \equiv 2, 6 \pmod{8} \) and quaternionic for \( r \equiv 3, 4, 5 \pmod{8} \).

Almost even-Clifford hermitian structures

Definition 2.1 A linear even-Clifford hermitian structure of rank \( r \) on \( \mathbb{R}^N, N \in \mathbb{N} \), is a representation

\[
Cl^0_r \rightarrow \text{End}(\mathbb{R}^N)
\]

such that each bivector \( e_i e_j, 1 \leq i < j \leq r \), is mapped to an antisymmetric endomorphism \( J_{ij} \) satisfying

\[
J^2_{ij} = -\text{Id}_{\mathbb{R}^N}.
\]
Notice that the subalgebra spin$(r)$ is mapped injectively into the skew-symmetric endomorphisms \( \text{End}^{-}(\mathbb{R}^{N}) \).

First, let us assume \( r \not\equiv 0 \, (\text{mod} \, 4) \), \( r > 1 \). In this case, \( \mathbb{R}^{N} \) decomposes into a sum of irreducible representations of \( Cl_{r}^{0} \). Since this algebra is simple, such irreducible representations can only be trivial or copies of the standard representation \( \Delta_{r} \) of \( Cl_{r}^{0} \) (cf. [7]). Due to (1), there are no trivial summands in such a decomposition so that

\[
\mathbb{R}^{N} = \Delta_{r} \otimes_{\mathbb{R}} \mathbb{R}^{m}
\]

for some \( m \in \mathbb{N} \). Thus, we see that spin$(r)$ has an isomorphic image

\[
\hat{\text{spin}}(r) := \text{spin}(r) \otimes \left\{ \text{Id}_{m_{x}m_{y}} \right\} \subset \mathfrak{so}(d_{r}m).
\]

Secondly, let us assume \( r \equiv 0 \, (\text{mod} \, 4) \). Recall that if \( \hat{\Delta}_{r} \) is the irreducible representation of \( Cl_{r} \), then by restricting this representation to \( Cl_{0}^{0} \) it splits as the sum of two inequivalent irreducible representations

\[
\hat{\Delta}_{r} = \hat{\Delta}^{+}_{r} \oplus \hat{\Delta}^{-}_{r}.
\]

Since \( \mathbb{R}^{N} \) is a representation of \( Cl_{r}^{0} \) satisfying (1), there are no trivial summands in such a decomposition so that

\[
\mathbb{R}^{N} = \hat{\Delta}^{+}_{r} \otimes \mathbb{R}^{m_{1}} \oplus \hat{\Delta}^{-}_{r} \otimes \mathbb{R}^{m_{2}},
\]

for some \( m_{1}, m_{2} \in \mathbb{N} \). By restricting this representation to \( \text{spin}(r) \subset Cl_{r}^{0} \), consider

\[
\hat{\text{spin}}(r) := \text{spin}(r)^{+} \otimes \left\{ \text{Id}_{m_{x_{1}}m_{x_{1}}} \oplus 0_{m_{y_{2}}m_{y_{2}}} \right\} \oplus \text{spin}(r)^{-} \otimes \left\{ 0_{m_{x_{1}}m_{x_{1}}} \oplus \text{Id}_{m_{y_{2}}m_{y_{2}}} \right\} \subset \mathfrak{so}(d_{r}m_{1} + d_{r}m_{2}),
\]

where \( \text{spin}(r)^{\pm} \) are the images of \( \text{spin}(r) \) in \( \text{End}(\hat{\Delta}^{\pm}_{r}) \) respectively.

**Definition 2.2**

- A rank \( r \) almost even-Clifford hermitian structure, \( r \geq 2 \), on a Riemannian manifold \( M \) is a smoothly varying choice of linear even-Clifford hermitian structure on each tangent space of \( M \). Let \( Q \subset \text{End}^{-}(TM) \) denote the subbundle with fiber \( \text{spin}(r) \).
- A Riemannian manifold carrying such a structure will be called an almost even-Clifford hermitian manifold.
- An almost even-Clifford hermitian structure on a Riemannian manifold \( M \) is called parallel if the bundle \( Q \) is parallel with respect to the Levi-Civita connection on \( M \).

Notice that the definition of (parallel) even-Clifford structure in [7] implies the one we have just given.

### 3 Automorphism group

In this section we derive an upper bound on the dimension of the automorphism group of an almost even-Clifford hermitian manifold and classify the manifolds whose automorphism group’s dimension attains such an upper bound.

The *automorphism group* of an almost even-Clifford hermitian manifold \( M \), denoted by \( \text{Aut}(M) \), is the (sub)group of isometries which preserve the almost even-Clifford hermitian structure. A vector field \( X \) on \( M \) is an *infinitesimal automorphism* if it is a Killing vector field that preserves the structure, i.e. locally

\[
\mathcal{L}_{X} J_{ij} = \sum_{k<l} a_{kl}^{(ij)} J_{kl},
\]

for some (local) functions \( a_{kl}^{(ij)} \), where \( \mathcal{L}_{X} \) denotes the Lie derivative in the direction of \( X \). These vector fields form the Lie algebra \( \text{aut}(M) \) of \( \text{Aut}(M) \).
3.1 Upper bound

Let $X$ be an infinitesimal automorphism of $M$. Consider

$$\mathcal{L}_X(J_{ij}(Y)) = (\mathcal{L}_X J_{ij})(Y) + J_{ij}(\mathcal{L}_X Y),$$

i.e.

$$\nabla_X(J_{ij}(Y)) - \nabla_{J_{ij}(Y)} X = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}(Y) + J_{ij}(\nabla_X Y - \nabla_Y X).$$

Now suppose we are calculations at a point $p$ where $X_p = 0$, so that

$$-\nabla_{J_{ij}(Y)} X = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}(Y) - J_{ij}(\nabla_Y X),$$

i.e.

$$[J_{ij}, \nabla X](Y) = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}(Y).$$

Hence, $(\nabla X)_p$ is a skew-symmetric endomorphism such that

$$[J_{ij}, \nabla X] = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}.$$

i.e. $(\nabla X)_p$ belongs to the normalizer of $\text{spin}(r) = \text{span}(J_{ij})$ in $\text{End}^- (T_p M) = \mathfrak{so}(T_p M)$. Such a normalizer has been calculated in [1] and we list them for $r \geq 3$.

| $r$ (mod 8) | $N$ | $N_{\mathfrak{so}(N)}(\text{spin}(r))$ | $C_{\mathfrak{so}(N)}(\text{spin}(r))$ |
|-------------|-----|-----------------------------------|-------------------------------------|
| 0           | $d_r(m_1 + m_2)$ | $\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2) \oplus \text{spin}(r)$ | $\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2)$ |
| 1           | $d_r m$ | $\mathfrak{so}(m) \oplus \text{spin}(r)$ | $\mathfrak{so}(m)$ |
| 2           | $d_r m$ | $\mathfrak{u}(m) \oplus \text{spin}(r)$ | $\mathfrak{u}(m)$ |
| 3           | $d_r m$ | $\mathfrak{sp}(m) \oplus \text{spin}(r)$ | $\mathfrak{sp}(m)$ |
| 4           | $d_r(m_1 + m_2)$ | $\mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2) \oplus \text{spin}(r)$ | $\mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2)$ |
| 5           | $d_r m$ | $\mathfrak{sp}(m) \oplus \text{spin}(r)$ | $\mathfrak{sp}(m)$ |
| 6           | $d_r m$ | $\mathfrak{u}(m) \oplus \text{spin}(r)$ | $\mathfrak{u}(m)$ |
| 7           | $d_r m$ | $\mathfrak{so}(m) \oplus \text{spin}(r)$ | $\mathfrak{so}(m)$ |

Table 2

**Proposition 3.1** Let $M$ be a $N$-dimensional almost even-Clifford hermitian manifold. Then

| $r$ (mod 8) | $N = \text{dim}(M)$ | upper bound $d_{\text{max}} \geq \text{dim}(\text{Aut}(M))$ |
|-------------|------------------|---------------------------------------------|
| 0           | $d_r (m_1 + m_2)$ | $(\binom{m_1}{2}) + (\binom{m_2}{2}) + (\binom{r}{2}) + d_r (m_1 + m_2)$ |
| 1, 7        | $d_r m$          | $(\binom{m}{2}) + (\binom{r}{2}) + d_r m$ |
| 2, 6        | $d_r m$          | $m^2 + (\binom{r}{2}) + d_r m$ |
| 3, 5        | $d_r m$          | $(\binom{2m+1}{2}) + (\binom{r}{2}) + d_r m$ |
| 4           | $d_r (m_1 + m_2)$ | $(\binom{2m_1+1}{2}) + (\binom{2m_2+1}{2}) + (\binom{r}{2}) + d_r (m_1 + m_2)$ |

Table 3
### 3.2 Large automorphism group

In this subsection, we determine the spaces that support an automorphism group of maximal dimension and prove a gap in the dimension of the automorphism group.

**Proposition 3.2** Let $M$ be a $N$-dimensional, rank $r \geq 3$ almost even-Clifford hermitian manifold and assume that the dimension of its automorphism group is maximal. Then, for any $p \in M$, the isotropy subgroup $A_p$ of $p$ is conjugate to $C_{SO(N)}(Spin(r)) \cdot Spin(r) \subset SO(N)$.

**Proof.** The dimension of the orbit $f p$ under $\text{Aut}(M)$ satisfies
\[
\dim(Aut(M)) - \dim(A_p) \leq N,
\]
so that
\[
\dim(A_p) \geq \dim(Aut(M)) - N = d_{\text{max}} - N = \dim(C_{SO(N)}(Spin(r)) \cdot Spin(r)).
\]
The Lie algebra $a_p$ of $A_p$ maps one-to-one into $C_{\mathfrak{so}(N)}(\mathfrak{spin}(r)) \oplus \mathfrak{spin}(r)$ since a Killing vector field $X$ is determined by its values $X_p$ and $(\nabla X)_p$. Hence,
\[
a_p \cong C_{\mathfrak{so}(N)}(\mathfrak{spin}(r)) \oplus \mathfrak{spin}(r).
\]
\[\blacksquare\]

**Proposition 3.3** Let $M$ be a rank $r \geq 3$ almost even-Clifford hermitian manifold and assume that the dimension of its automorphism group is maximal. Then, $M$ is symmetric and the almost even-Clifford hermitian structure is parallel.

**Proof.** Let $p \in M$ and $A_p$ denote its isotropy group. We know that $A_p = C_{SO(N)}(Spin(r)) \cdot Spin(r)$. Since $C_{SO(N)}(Spin(r))$ contains 1 and $Spin(r)$ contains $-1$, we have $-1 \in A_p$. Thus, there is an element $g \in A_p$ whose derivative $dg_p = -\text{Id}_{T_pM}$ in the isotropy representation of $A_p$ on $T_pM$. In other words, the automorphism $g$ is a (global) symmetry at $p$ and $M$ is symmetric. Since these symmetries generate the translations along geodesics, $M$ has a transitive group of automorphisms, not just isometries.

Proceeding as in [5, p. 264], given a vector field $W$ with $W_p \neq 0$, let $c(t)$ be the geodesic with $\dot{c}(0) = W_p$ and $\tau_t$ be the group of translations along $c$. Then
\[
Z_q := \frac{d}{dt} \tau_t(q)|_{t=0}
\]
is an infinitesimal automorphism since $\tau_t$ are automorphisms. We have that $W_p = Z_p$. For $v \in T_pM$, let $\gamma(s)$ be a curve with $\gamma'(0) = v$. Then
\[
\nabla_v Z_p = \nabla_v \frac{\partial}{\partial t} \tau_t(\gamma(s))|_{s=t=0}
= \nabla_v \frac{\partial}{\partial t} \tau_t(\gamma(s))|_{s=0}
= \nabla_v D\tau_t(v)|_{t=0}
= 0,
\]
since $D\tau_t$ is parallel transport along $c$ and $D\tau_t(v)$ is a parallel vector field along $c$. Hence, for any vector $W_p \in T_pM$, we have an infinitesimal isometry $Z$ such that
\[
Z_p = W_p \quad \text{and} \quad (\nabla Z)_p = 0.
\]
Now, recall that
\[ \nabla_Z(Y) = \mathcal{L}_Z Y + \nabla_Y Z. \]

On the one hand,
\[
(\nabla_W(J_{ij}(Y)))_p = (\nabla_Z(J_{ij}(Y)))_p + ((\nabla_Z J_{ij})(Y))_p + (J_{ij}(\nabla_Z Y))_p,
\]
and, on the other,
\[
(\mathcal{L}_Z(J_{ij}(Y)) + \nabla_{J_{ij}(Y)} Z)_p
= ((\mathcal{L}_Z J_{ij})(Y))_p + (J_{ij}(\mathcal{L}_Z Y))_p
+ (\nabla_{J_{ij}(Y)} Z)_p,
\]
so that
\[
((\nabla_Z J_{ij})(Y))_p
= \left( \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}(Y) \right)_p
+ (J_{ij}(\nabla_Z Y))_p - (J_{ij}(\nabla_Y Z))_p + (\nabla_{J_{ij}(Y)} Z)_p,
\]
i.e.
\[
(\nabla_W J_{ij})_p = \left( \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl} \right)_p.
\]

\[\Box\]

**Theorem 3.1** Let $M$ be a simply connected Riemannian almost even-Clifford hermitian manifold of rank $r \geq 3$ such that the dimension of its group of automorphisms is maximal. Then $M$ is isometric to one of the following spaces:

| $r$  | $M$                                                                 |
|------|----------------------------------------------------------------------|
| 3    | $\Delta^m \oplus (\Delta^+_1)^{m_1} \oplus (\Delta^-_1)^{m_2}$, for some $m, m_1, m_2 \in \mathbb{N}$ |
| 4    | $M_1 \times M_2$, where $M_i = (\Delta_3)^{m_i}$, $Sp(k+1)/(Sp(k) \times Sp(1))$, $Sp(k,1)/(Sp(k) \times Sp(1))$ |
| 5    | $Sp(k+2)/(Sp(k) \times Sp(2))$, $Sp(k,2)/(Sp(k) \times Sp(2))$ |
| 6    | $SU(k+4)/SU(k+4)$, $SU(k,4)^{14}/SU(k,4)$ |
| 8    | $SO(k+8)/(SO(k) \times SO(8))$, $SO(k,8)/(SO(k) \times SO(8))$ |
| 9    | $F_4/Spin(9)$, $E_6^{10}/Spin(9)$ |
| 10   | $E_6/(Spin(10) \cdot U(1))$, $E_6^{14}/(Spin(10) \cdot U(1))$ |
| 12   | $E_7/(Spin(12) \cdot SU(2))$, $E_7^{14}/(Spin(12) \cdot SU(2))$ |
| 16   | $E_8/Spin^{16}$, $E_8^{16}/Spin^{16}$ |

Table 4
Killing vector fields.

and the endomorphisms [9]

Proof.
The flat case is clear and the case \( r = 3 \) was dealt with in [11].

For \( r = 4 \), by [9], \( M \) is a Riemannian product \( M_1 \times M_2 \) of quaternion-Kähler manifolds. We claim that

\[ \dim(\text{Aut}(M)) = \dim(\text{Aut}(M_1)) + \dim(\text{Aut}(M_2)). \]

Indeed, let \( X \in \text{aut}(M) \) an infinitesimal automorphism of \( M \), \( X = X_1 + X_2 \) with \( X_1 \in \Gamma(TM_1) \) and \( X_2 \in \Gamma(TM_2) \). We will prove that \( X_1 \in \text{aut}(M_1) \) and \( X_2 \in \text{aut}(M_2) \). First note that \( X_1 \) and \( X_2 \) are Killing vector fields.

Recall that

\[ \mathcal{L}_X J_{ij} = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl}, \]

and the endomorphisms [9]

\[ J_{12}^\pm = \pm \frac{1}{2}(J_{14} \pm J_{23}), \quad J_{31}^\pm = \pm \frac{1}{2}(J_{13} \mp J_{24}), \quad J_{25}^\pm = \pm \frac{1}{2}(J_{12} \pm J_{34}), \]

where \( J_{kl}^- \) and \( J_{kl}^+ \) vanish on \( M_1 \) and \( M_2 \) respectively. Let \( Z = Z_1 + Z_2 \) with \( Z_1 \in \Gamma(TM_1) \) and \( Z_2 \in \Gamma(TM_2) \),

\[ (\mathcal{L}_{X_1} J_{ij}^+) (Z_1)_p = \mathcal{L}_{X_1} (J_{ij}^+(Z_1))_p - J_{ij}^+(\mathcal{L}_{X_1} Z_1)_p \in T_p M_1, \]
\[ (\mathcal{L}_{X_1} J_{ij}^+) (Z_2)_p = \mathcal{L}_{X_1} (J_{ij}^+(Z_2))_p - J_{ij}^+(\mathcal{L}_{X_1} Z_2)_p = 0, \]
\[ (\mathcal{L}_{X_2} J_{ij}^+) (Z_1)_p = \mathcal{L}_{X_2} (J_{ij}^+(Z_1))_p - J_{ij}^+(\mathcal{L}_{X_2} Z_1)_p = 0, \]
\[ (\mathcal{L}_{X_2} J_{ij}^+) (Z_2)_p = \mathcal{L}_{X_2} (J_{ij}^+(Z_2))_p - J_{ij}^+(\mathcal{L}_{X_2} Z_2)_p = 0. \]

i.e. \( \mathcal{L}_{X_1} J_{ij}^+ = \mathcal{L}_{X} J_{ij}^+ \in \text{End}(TM_1) \). Similarly, \( \mathcal{L}_{X_1} J_{ij}^- = \mathcal{L}_{X_2} J_{ij}^- \in \text{End}(TM_2) \). Now consider, for instance,

\[ \mathcal{L}_{X_1} J_{12}^+ = \frac{1}{2} \mathcal{L}_{X_1} (J_{14} + J_{23}) = \frac{1}{2} \sum_{k<l} \alpha_{kl}^{(14)} J_{kl} + \sum_{k<l} \alpha_{kl}^{(23)} J_{kl} = \frac{1}{2} \sum_{k<l} \beta_{st} J_{st}^+ + \sum_{k<l} \gamma_{st} J_{st}^-, \]

for some functions \( \beta_{st} \) and \( \gamma_{st} \). Since \( \mathcal{L}_{X_1} J_{12}^+ \in \text{End}(TM_1) \), all the coefficients \( \gamma_{st} = 0 \). Therefore

\[ \mathcal{L}_{X_1} J_{12}^+ = \sum \beta_{st} J_{st}^-, \]

By similar calculations \( \mathcal{L}_{X_2} J_{ij}^+ = \sum \beta_{st} J_{st}^+ \) and \( \mathcal{L}_{X_2} J_{ij}^- = \sum \gamma_{st} J_{st}^- \), i.e. \( X_1 \in \text{aut}(M_1) \) and \( X_2 \in \text{aut}(M_2) \). Now let \( m_i = \dim(M_i)/4, i = 1, 2 \). Since

\[ \dim(\text{Aut}(M)) = \left( \frac{2m_1 + 1}{2} \right) + \left( \frac{2m_2 + 1}{2} \right) + 6 + 4m_1 + 4m_2 = \dim(\text{Aut}(M_1)) + \dim(\text{Aut}(M_2)), \]
\[ \dim(\text{Aut}(M_i)) \leq \left( \frac{2m_i + 1}{2} \right) + 3 + 4m_i \]

we must have

\[ \dim(\text{Aut}(M_i)) = \left( \frac{2m_i + 1}{2} \right) + 3 + 4m_i. \]

Therefore, the dimensions of the automorphism groups of \( M_1 \) and \( M_2 \) are maximal.
For $r \geq 5$, the symmetric spaces carrying a parallel even Clifford structure were classified in \[9\] and are listed in Table 4. We claim that the dimension of the automorphism group of each space listed is maximal. Indeed, let $M = G/K$ be one of the spaces in Table 4, where $G$ is the group of isometries of $M$. We need to prove that every Killing vector is an infinitesimal automorphism of $M$. If $X \in \mathfrak{g}$ is a Killing vector field,

\[
(\mathcal{L}_X J_{ij})(Z) = \mathcal{L}_X (J_{ij}(Z)) - J_{ij}(\mathcal{L}_X Z)
\]

\[
= \nabla_X (J_{ij}(Z)) - \nabla_{J_{ij}(Z)} X - J_{ij}(\nabla_X Z) + J_{ij}(\nabla_Z X)
\]

\[
= (\nabla_X J_{ij})(Z) - [\nabla_X, J_{ij}](Z)
\]

\[
= \left( \sum_{k<l} a_{kl}^{(ij)} J_{kl} \right)(Z) - [\nabla_X, J_{ij}](Z)
\]

since the almost even-Clifford hermitian structure is parallel.

Recall that

\[
\mathfrak{g} = \mathfrak{k} + \mathfrak{m},
\]

and at a point $p \in M$,

\[
\mathfrak{m} \cong T_p M,
\]

and from the list of possible spaces

\[
\mathfrak{k} \cong \mathfrak{a}_p \cong C_{\mathfrak{so}(N)}(\hat{\mathfrak{spin}}(r)) \oplus \hat{\mathfrak{spin}}(r).
\]

On the other hand, since $M$ is symmetric,

\[
\mathfrak{g} \cong \mathfrak{b}_p \oplus T_p M,
\]

so that $\mathfrak{b}_p \cong \mathfrak{k} \cong C_{\mathfrak{so}(N)}(\hat{\mathfrak{spin}}(r)) \oplus \hat{\mathfrak{spin}}(r)$ and

\[
[\nabla_X, J_{ij}] = \sum_{k<l} b_{kl}^{(ij)} J_{kl},
\]

i.e.

\[
\mathcal{L}_X J_{ij} = \sum_{k<l} (a_{kl}^{(ij)} + b_{kl}^{(ij)}) J_{kl}.
\]

\[\square\]

**Theorem 3.2** Let $M$ be a $N$-dimensional rank $r \geq 3$, almost even-Clifford hermitian manifold and $p \in M$. Assume the following constraints:

| $r \pmod{8}$ | $N$ constraint | extra constraint |
|-------------|----------------|-----------------|
| 0           | $d_r(m_1 + m_2)$ | $m_1 \geq m_2 > \binom{r}{2} + 1$ or $m_2 \geq m_1 > \binom{r}{2} + 1$ | $m_1 \equiv m_2 \equiv 0 \pmod{2}$ |
| 1, 7        | $d_r m$         | $m > \binom{r}{2} + 1$ | $m \equiv 0 \pmod{2}$ |
| 2, 6        | $d_r m$         | $m > \frac{1}{4} \binom{r}{3} + \frac{1}{4}$ | $m \equiv 0 \pmod{2}$ |
| 3, 5        | $d_r m$         | $m > \frac{1}{4} \binom{r}{3} + 1$ | |
| 4           | $d_r(m_1 + m_2)$ | $m_1 \geq m_2 > \frac{1}{4} \binom{r}{2} + 1$ or $m_2 \geq m_1 > \frac{1}{4} \binom{r}{2} + 1$ | |

Table 5

If $\dim(\text{Aut}(M))$ is not maximal, then

\[
\dim(\text{Aut}(M)) < d_{\text{max}} - \binom{r}{2}.
\]
Proof. Suppose that
\[ d_{\text{max}} - \binom{r}{2} \leq \dim(\text{Aut}(M)) < d_{\text{max}}. \]

At a point \( p \in M \), the isotropy group satisfies
\[
\dim(A_p) \geq \dim(\text{Aut}(M)) - N \\
\geq d_{\text{max}} - \binom{r}{2} - N \\
\geq d_C := \dim(C_{SO(N)}(\text{Spin}(r))).
\]

The Lie algebra \( a_p \) of \( A_p \) maps one-to-one into \( C_{\mathfrak{so}(N)}(\text{spin}(r)) \oplus \text{spin}(r) \) since a Killing vector field \( X \) is determined by its values \( X_p \) and \((\nabla X)_p\). Consider the compositions of this map with the projections to the two factors
\[
\rho_1 : a_p \rightarrow C_{\mathfrak{so}(N)}(\text{spin}(r)), \quad \rho_2 : a_p \rightarrow \text{spin}(r).
\]

The subalgebra \( \rho_1(a_p) \) is either equal to \( C_{\mathfrak{so}(N)}(\text{spin}(r)) \) or is contained in a proper maximal subalgebra of \( C_{\mathfrak{so}(N)}(\text{spin}(r)) \).

If \( r \neq \pm 2 \mod 8 \), the maximal dimension \( d_M \) of a proper maximal subalgebra of \( C_{\mathfrak{so}(N)}(\text{spin}(r)) \) is given in the following table (see [8]),

| \( r \mod 8 \) | \( N \) | \( d_M \) | \( d_C \) |
|----------------|-------|---------|------|
| 0              | \( d_r(m_1 + m_2) \) | max \( \{ (m_1^{2})(2)^{-1}, (m_2^{2})(2)^{-1} \} \) | \( (m_1^{2})(2) \) |
| 1, 7           | \( d_r m \) | \( (m^{2})(2)^{-1} \) | \( (m)(2) \) |
| 3, 5           | \( d_r m \) | \( (m^{2})(2)^{-1} + 3 \) | \( (2m^{2})^{1}(2) \) |
| 4              | \( d_r(m_1 + m_2) \) | max \( \{ (2m_1^{2})(2)^{-1} + 3, (2m_2^{2})(2)^{-1} + 3 \} \) | \( (2m_1^{2})^{1}(2) \) |

Table 6

Thus, due to the constraints on the multiplicities \( m, m_1, m_2 \), if \( \rho_1(a_p) \) is contained in a proper subalgebra of \( C_{\mathfrak{so}(N)}(\text{spin}(r)) \),
\[
d_C > d_M + \binom{r}{2} \geq \dim(\rho_1(a_p)) + \dim(\rho_2(a_p)) \geq \dim(a_p) \geq d_C,
\]

which is a contradiction. Thus
\[
\rho_1(a_p) = C_{\mathfrak{so}(N)}(\text{spin}(r)),
\]

and
\[
a_p \cong C_{\mathfrak{so}(N)}(\text{spin}(r)) \oplus \mathfrak{r} \subset \mathfrak{so}(N),
\]

where
\[
\mathfrak{r} := \ker(\rho_1|_{a_p}) \subset \ker(\rho_1) = \text{spin}(r).
\]

Therefore \( A_p = C_{SO(N)}(\text{Spin}(r)) \cdot K \), where \( K \) is a closed subgroup of \( Spin(r) \). The extra assumptions on \( m, m_1 \) and \( m_2 \) imply \(-1 \in C_{SO(N)}(\text{Spin}(r)) \) and \(-1 \in A_p \). Thus, there is an element \( g \in A_p \) whose derivative \( dg_p = -\text{Id}_{T_p, M} \) in the isotropy representation of \( A_p \) on \( T_p M \). In other words, the automorphism \( g \) is a (global) symmetry at \( p \) and \( M \) is symmetric. Since these symmetries generate the translations along geodesics, \( M \) has a transitive group of automorphisms, not just isometries. As in the proof of Proposition
this implies the almost even-Clifford hermitian structure is parallel. By arguments similar to those in the proof of Theorem 3.1 we have \( \dim(\text{Aut}(M)) = d_{\text{max}} \), which is again a contradiction.

For \( r \equiv \pm 2 \pmod{8} \), \( \dim(A_p) \geq d_c \) will happen if \( \mathfrak{su}(m) \subset \rho_1(\mathfrak{n}_p) \). Hence \( A_p = H \cdot K \), where \( H \) is some subgroup of \( C_{\text{SO}(N)}(\text{Spin}(r)) \) containing \( SU(m) \) and \( K \) is some closed subgroup of \( \text{Spin}(r) \). Since we are assuming \( m \) is even, \(-1 \in SU(m)\) and \(-1 \in A_p\). Therefore \( M \) is symmetric and again, the proofs of Proposition 3.3 and Theorem 3.1 imply \( \dim(\text{Aut}(M)) = d_{\text{max}} \).

\[ \square \]

**Remark.** The constraints in the previous theorem are given in order to ensure that \(-1 \in A_p\). If we were to relax them or change them, a more detailed analysis of the possible subgroups \( K \in \text{Spin}(r) \) would be needed.

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