Linear source invertible bimodules and Green correspondence

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Abstract

We show that the Green correspondence induces an injective group homomorphism from the linear source Picard group \( \mathcal{L}(B) \) of a block \( B \) of a finite group algebra to the linear source Picard group \( \mathcal{L}(C) \), where \( C \) is the Brauer correspondent of \( B \). This homomorphism maps the trivial source Picard group \( \mathcal{T}(B) \) to the trivial source Picard group \( \mathcal{T}(C) \). We show further that the endopermutation source Picard group \( \mathcal{E}(B) \) is bounded in terms of the defect groups of \( B \) and that when \( B \) has a normal defect group \( \mathcal{E}(B) = \mathcal{L}(B) \). Finally we prove that the rank of any invertible \( B \)-bimodule is bounded by that of \( B \).

1 Introduction

Let \( p \) be a prime and \( k \) a perfect field of characteristic \( p \). We denote by \( \mathcal{O} \) either a complete discrete valuation ring with maximal ideal \( J(\mathcal{O}) = \pi \mathcal{O} \) for some \( \pi \in \mathcal{O} \), with residue field \( k \) and field of fractions \( \mathcal{K} \) of characteristic zero, or \( \mathcal{O} = k \). We make the blanket assumption that \( k \) and \( \mathcal{K} \) are large enough for the finite groups and their subgroups in the statements below.

Let \( A \) be an \( \mathcal{O} \)-algebra. An \( A \)-\( A \)-bimodule \( M \) is called invertible if \( M \) is finitely generated projective as a left \( A \)-module, as a right \( A \)-module, and if there exists an \( A \)-\( A \)-bimodule \( N \) which is finitely generated projective as a left and right \( A \)-module such that \( M \otimes_A N \cong A \cong N \otimes_A M \) as \( A \)-bimodules. The set of isomorphism classes of invertible \( A \)-\( A \)-bimodules is a group, denoted \( \text{Pic}(A) \) and called the Picard group of \( A \), where the product is induced by the tensor product over \( A \).

Given a finite group \( G \), a block of \( \mathcal{O}G \) is an indecomposable direct factor \( B \) of \( \mathcal{O}G \) as an algebra. Any such block \( B \) determines a \( p \)-subgroup \( P \) of \( G \), called a defect group of \( B \), uniquely up to conjugation. Moreover, \( B \) determines a block \( C \) of \( \mathcal{O}N_G(P) \) with \( P \) as a defect group, called the Brauer correspondent of \( B \). When regarded as an \( \mathcal{O}(G \times G) \)-module, \( B \) is a trivial source module with vertices the \( G \times G \)-conjugates of the diagonal subgroup \( \Delta P = \{ (u, u) \mid u \in P \} \) of \( P \times P \). The Brauer correspondent \( C \) is the Green correspondent of \( B \) with respect to the subgroup \( N_G(P) \times N_G(P) \) of \( G \times G \). We denote by \( \mathcal{L}(B) \) the subgroup of \( \text{Pic}(B) \) of isomorphism classes of invertible \( B \)-\( B \)-bimodules \( X \) having a linear source (that is, a source of \( \mathcal{O} \)-rank 1) for some (and hence any) vertex. We denote by \( \mathcal{T}(B) \) the subgroup of \( \mathcal{L}(B) \) of isomorphism classes of invertible \( B \)-\( B \)-bimodules \( X \) having a trivial source for some vertex. Note that if \( \mathcal{O} = k \), then \( \mathcal{L}(B) = \mathcal{T}(B) \).

In general, the canonical surjection \( B \to k \otimes_\mathcal{O} B \) induces an isomorphism \( \mathcal{T}(B) \cong \mathcal{T}(k \otimes_\mathcal{O} B) \) which extends to a surjective group homomorphism \( \mathcal{L}(B) \to \mathcal{T}(k \otimes_\mathcal{O} B) \). If \( \text{char}(\mathcal{O}) = 0 \), then the
kernel of this homomorphism is canonically isomorphic to \( \text{Hom}(P/\mathfrak{soc}(F), O^X) \); see \cite{2} Theorem 1.1, Remark 1.2.(d),(e) for more details.

**Theorem 1.1.** Let \( G \) be a finite group, \( B \) a block of \( OG \), and \( P \) a defect group of \( B \). Set \( N = N_G(P) \) and denote by \( C \) the block of \( ON \) which has \( P \) as a defect group and which is the Brauer correspondent of \( B \). Let \( X \) be a linear source invertible \( B \)-\( B \)-bimodule, and let \( Q \) be a vertex of \( X \) contained in \( P \times P \). Then \( N \times N \) contains the normaliser in \( G \times G \) of the vertex \( Q \) of \( X \). Denote by \( Y \) the \( N \times N \)-Green correspondent of \( X \) with respect to \( Q \). Then \( Y \) is a linear source invertible \( C \)-\( C \)-bimodule whose isomorphism class does not depend on the choice of \( Q \) in \( P \times P \). Moreover, if \( X \) has a trivial source, so does \( Y \). The map \( X \to Y \) induces an injective group homomorphism

\[
\mathcal{L}(B) \to \mathcal{L}(C)
\]

which restricts to an injective group homomorphism

\[
\mathcal{T}(B) \to \mathcal{T}(C).
\]

The strategy is to translate this to a statement on the source algebras of \( B \) and \( C \), and then play this back to block algebras via the canonical Morita equivalences between blocks and their source algebras. By a result of Puig \cite{16} 14.6, a source algebra \( A \) of \( B \) contains canonically a source algebra \( L \) of \( C \).

As a first step we observe that if \( \alpha \) is an automorphism of \( A \) which preserves \( L \), then the \( B \)-\( B \)-bimodule corresponding to the invertible \( A \)-\( A \)-bimodule \( A_\alpha \) is the Green correspondent of the \( C \)-\( C \)-bimodule corresponding to the invertible \( L \)-\( L \)-bimodule \( L_\beta \), where \( \beta \) is the automorphism of \( L \) obtained from restricting \( \alpha \). See Proposition \cite{31} for a precise statement and a proof, as well as the beginning of Section \cite{2} for the notation.

The second step is to observe that an element in \( \mathcal{L}(B) \), given by an invertible \( B \)-\( B \)-bimodule \( X \), corresponds via the canonical Morita equivalence, to an invertible \( A \)-\( A \)-bimodule of the form \( A_\alpha \) for some algebra automorphism \( \alpha \) of \( A \) which preserves the image of \( OP \) in \( A \).

The third and key step is to show that \( \alpha \) can be chosen in such a way that \( \alpha \) preserves in addition the subalgebra \( L \) of \( A \). Such an \( \alpha \) restricts therefore to an automorphism \( \beta \) of \( L \), and yields an invertible \( L \)-\( L \)-bimodule \( L_\beta \). By step one, the corresponding invertible \( C \)-\( C \)-bimodule \( Y \) is then the Green correspondent of \( X \), and the map \( X \to Y \) induces the group homomorphism \( \mathcal{L}(B) \to \mathcal{L}(C) \) as stated in the theorem. This third step proceeds in two stages - first for the subgroup \( \mathcal{T}(B) \), and then for \( \mathcal{L}(B) \). This part of the proof relies significantly on the two papers \cite{12} and \cite{2}.

**Example 1.2.** Consider the special case in Theorem \cite{11} where \( X \) is induced by a group automorphism \( \alpha \) of \( G \) which stabilises \( B \). We use the same letter \( \alpha \) for the extension of \( \alpha \) to an algebra automorphism of \( OG \). Note that \( OG_\alpha \) is a permutation \( O(G \times G) \)-module. Suppose that \( \alpha \) stabilises \( B \). Then the indecomposable direct summand \( B_\alpha \) of \( OG_\alpha \) is a trivial source \( O(G \times G) \)-module. If \( (P, e) \) is a maximal \( B \)-Brauer pair, then \( (\alpha(P), \alpha(e)) \) is a maximal \( B \)-Brauer pair as well, hence \( G \)-conjugate to \( (P, e) \). After possibly composing \( \alpha \) by a suitable chosen inner automorphism of \( G \), we may assume that \( \alpha \) stabilises \( (P, e) \). Then \( \alpha \) restricts to a group automorphism \( \beta \) of \( N = N_G(P) \) which stabilises the Brauer correspondent \( C \) of \( B \). The bimodule \( B_\alpha \) represents an element in \( \mathcal{T}(B) \). Its \( N \times N \)-Green correspondent is the \( C \)-\( C \)-bimodule \( C_\beta \), and this bimodule represents the image in \( \mathcal{T}(C) \) under the homomorphism \( \mathcal{T}(B) \to \mathcal{T}(C) \) in Theorem \cite{11}.
By a result of Eisele [5], if \( \mathcal{O} \) has characteristic zero, then \( \text{Pic}(B) \) is a finite group. We do not know, however, whether in that case the order of \( \text{Pic}(B) \) is determined ‘locally’, that is, in terms of the defect groups of \( B \). We show that there is a local bound, without any assumption on the characteristic of \( \mathcal{O} \), for the order of the subgroup \( \mathcal{E}(B) \) of isomorphism classes of invertible \( B \)-\( B \)-bimodules \( X \) having an endopermutation module as a source, for some vertex.

**Theorem 1.3.** Let \( G \) be a finite group and \( B \) a block algebra of \( \mathcal{O}G \). Let \( P \) be a defect group of \( B \). Then the order of \( \mathcal{E}(B) \) is bounded in terms of a bound which depends only on \( P \).

This will be proved as a consequence of [2, Theorem 1.1].

**Theorem 1.4.** Let \( A \) be an \( \mathcal{O} \)-algebra which is free of finite rank as an \( \mathcal{O} \)-module. Suppose that \( k \otimes_\mathcal{O} A \) is split and has a symmetric positive definite Cartan matrix. Then for every invertible \( A \)-\( A \)-bimodule \( M \) we have \( \text{rk}_\mathcal{O}(M) \leq \text{rk}_\mathcal{O}(A) \). Moreover, we have \( \text{rk}_\mathcal{O}(M) = \text{rk}_\mathcal{O}(A) \) if and only if \( M \cong A_\alpha \) for some automorphism \( \alpha \) of \( A \).

This result applies in particular to any algebra \( A \) which is Morita equivalent to a block algebra of a finite group algebra over \( \mathcal{O} \) or \( k \). We use this in the proof of the next result which is in turn used to show, in Example 4.2, that Theorem 1.1 does not hold with \( \mathcal{L} \) replaced by \( \mathcal{E} \).

**Theorem 1.5.** Let \( G \) be a finite group and \( B \) a block of \( \mathcal{O}G \) with a normal defect group. Then \( \mathcal{E}(B) = \mathcal{L}(B) \).

## 2 Background

Let \( A, B \) be \( \mathcal{O} \)-algebras, and let \( \alpha : A \to B \) be an algebra homomorphism. For any \( B \)-module \( V \) we denote by \( \alpha_* V \) the \( A \)-module which is equal to \( V \) as an \( \mathcal{O} \)-module, and on which \( \alpha \in A \) acts as \( \alpha(a) \). We use the analogous notation for right modules and bimodules.

Any \( A \)-\( A \)-bimodule of the form \( A_\alpha \) for some \( \alpha \in \text{Aut}(A) \) is invertible, and we have \( A \cong A_\alpha \) as bimodules if and only of \( \alpha \) is inner. The map \( \alpha \mapsto A_\alpha \) induces an injective group homomorphism \( \text{Out}(A) \to \text{Pic}(A) \). This group homomorphism need not be surjective. An invertible \( A \)-\( A \)-bimodule \( M \) is of the form \( A_\alpha \) for some \( \alpha \in \text{Aut}(M) \) if and only if \( M \cong A \) as left \( A \)-modules, which is also equivalent to \( M \cong A \) as right \( A \)-modules. See e.g. [3, §55 A] or [10, Proposition 2.8.16] for proofs and more details.

**Lemma 2.1** (cf. [12, Lemma 2.4]). Let \( A \) be an \( \mathcal{O} \)-algebra and \( L \) a subalgebra of \( A \). Let \( \alpha \in \text{Aut}(A) \) and let \( \beta : L \to A \) be an \( \mathcal{O} \)-algebra homomorphism. The following are equivalent.

(i) There is an automorphism \( \alpha' \) of \( A \) which extends the map \( \beta \) such that \( \alpha \) and \( \alpha' \) have the same image in \( \text{Out}(A) \).

(ii) There is an isomorphism of \( A \)-\( L \)-bimodules \( A_\beta \cong A_\alpha \).

(iii) There is an isomorphism of \( L \)-\( A \)-bimodules \( \beta A \cong \alpha A \).

**Remark 2.2.** For \( G, H \) finite groups, we switch without further comment between \( \mathcal{O}(G \times H) \)-modules and \( \mathcal{O}G-\mathcal{O}H \)-bimodules as follows: given an \( \mathcal{O}G-\mathcal{O}H \)-bimodule \( M \), we regard \( M \) as an \( \mathcal{O}(G \times H) \)-module (and vice versa) via \((x, y) \cdot m = x \alpha y^{-1} \), where \( x \in G, y \in H \), and \( m \in M \). If \( M \)
is indecomposable as an $OG$-$OH$-bimodule, then $M$ is indecomposable as an $O(G \times H)$-module, hence has a vertex (in $G \times H$) and a source. If $Q$ is a subgroup of $G$, $R$ a subgroup of $H$, and $W$ an $O(Q \times R)$-module, then with these identifications, we have an isomorphism of $O(G \times H)$-modules (or equivalently of $OG$-$OH$-bimodules)

$$\text{Ind}_{Q \times R}^{G \times H}(W) \cong OG \otimes_{OQ} W \otimes_{OR} OH$$

sending $(x, y) \otimes w$ to $x \otimes w \otimes y^{-1}$, where $x \in G$, $y \in H$, and $w \in W$. Thus if $M$ is a relatively $(Q \times R)$-projective $O(G \times H)$-module, then, as an $OG$-$OH$-bimodule, $M$ is isomorphic to a direct summand of

$$OG \otimes_{OQ} W \otimes_{OR} OH$$

for some $OQ$-$OR$-bimodule $W$. Note further that if $B$ is a block of $OG$ with defect group $P$ and $C$ a block of $OH$ with defect group $Q$, then $B \otimes_{O} C^{op}$ is a block of $O(G \times H)$ with defect group $P \times Q$, via the canonical algebra isomorphisms $O(G \times H) \cong OG \otimes_{O} OH \cong OG \otimes_{O} (OH)^{op}$. By [1] Lemma 2.3, if $A$ and $L$ are source algebras of $B$ and $C$, respectively, then $A \otimes_{O} L^{op}$ is a source algebra of $B \otimes_{O} C^{op}$.

Let $G$ be a finite group and $B$ a block algebra of $OG$ with a defect group $P$. Recall our standing assumption that $K$ and $k$ are splitting fields for the subgroups of $G$. Choose a block idempotent $e$ of $kC_{G}(P)$ such that $(P, e)$ is a maximal $B$-Brauer pair and a source idempotent $i \in B^{P}$ associated with $e$; that is, $i$ is a primitive idempotent in $B^{P}$ such that $\text{Br}_{P}(i)e \neq 0$. Since $k$ is assumed to be large enough, it follows that the choice of $(P, e)$ determines a (saturated) fusion system $\mathcal{F}$ on $P$. In particular, the group $\text{Out}(\mathcal{F})(P) \cong N_{G}(P, e)/PC_{G}(P)$ is a $p'$-group, and hence lifts uniquely up to conjugation by an element in $\text{Inn}(P)$ to a $p'$-subgroup $E$ of $\text{Aut}(\mathcal{F})(P) \cong N_{G}(P, e)/C_{G}(P)$. The group $E$ is called the inertial quotient of $B$ (and depends on the choices as just described). As in [1] 1.13, we denote by $\text{Aut}(P, \mathcal{F})$ the subgroup of $\text{Aut}(P)$ consisting of all automorphisms of $P$ which stabilise $\mathcal{F}$. In particular, the automorphisms in $\text{Aut}(P, \mathcal{F})$ normalise the subgroup $\text{Aut}(\mathcal{F})(P)$ of $\text{Aut}(P)$, and we set

$$\text{Out}(P, \mathcal{F}) = \text{Aut}(P, \mathcal{F})/\text{Aut}(\mathcal{F})(P).$$

The algebra $A = iBi$ is called a source algebra of $B$. If no confusion arises, we identify $P$ with its image $iP = Pi$ in $A^{x}$. Following [2], we set $\text{Aut}_{P}(A)$ to be the group of algebra automorphisms of $A$ which fix $P$ elementwise, and by $\text{Out}_{P}(A)$ the quotient of $\text{Aut}_{P}(A)$ by the subgroup of inner automorphisms induced by conjugation with elements in $(A^{P})^{x}$.

**Remark 2.3.** We will make use of the following standard facts on source algebras. With the notation above, by [13] 3.5 the $B$-$A$-bimodule $Bi$ and the $A$-$B$-bimodule $iB$ induce a Morita equivalence between $A$ and $B$. More precisely, the equivalence $iB \otimes_{B} - : \text{mod}(B) \to \text{mod}(A)$ is isomorphic to the functor sending a $B$-module $U$ to the $A$-module $iU$ and a $B$-module homomorphism $\varphi : U \to U'$ to the induced $A$-module homomorphism $iU \to iU'$ obtained from restricting $\varphi$ to $iU$.

Following [9] 6.3, this Morita equivalence between $A$ and $B$ keeps track of vertices and sources in the following sense: if $U$ is an indecomposable $B$-module, then there exists a vertex-source pair $(Q, W)$ of $U$ such that $Q \leq P$ and such that $W$ is isomorphic to a direct summand of $\text{Res}_{Q}(iU)$. In particular, a finitely generated $B$-module $U$ is a $p$-permutation $B$-module if and only if $iU$ is a
$P$-permutation module. Since a $p$-permutation $k \otimes B$-module lifts uniquely, up to isomorphism, to a $p$-permutation $B$-module, it follows that a $P$-permutation $k \otimes O$ $A$-module lifts uniquely, up to isomorphism, to a $P$-permutation $A$-module.

This applies to bimodules over block algebras via the Remark 2.2. Morita equivalences are compatible with tensor products of algebras. In particular, there is a Morita equivalence between $B \otimes_O B^{op}$ and $A \otimes_O A^{op}$ sending a $B$-$B$-bimodule $X$ to the $A$-$A$-bimodule $iXi$. If $X$ is an invertible $B$-$B$-bimodule, then $iXi$ is an invertible $A$-$A$-bimodule, and the map $X \mapsto iXi$ induces a canonical group isomorphism

$$\text{Pic}(B) \cong \text{Pic}(A).$$

By the above remarks on vertices and sources applied to the block $B$ of $O$ (cf. [8, Theorem A], [16, Propositions 14.6, 14.9], [6, Proposition 4.10]), in the following Proposition the construction of this embedding and some of its properties.

Following the notation in [2], we denote by $\mathcal{E}(B)$, $\mathcal{L}(B)$, $\mathcal{T}(B)$ the subgroups of $\text{Pic}(B)$ of isomorphism classes of invertible $B$-$B$-bimodules with endopermutation, linear, trivial sources, respectively. We denote by $\mathcal{E}(A)$, $\mathcal{L}(A)$, $\mathcal{T}(A)$ their respective images in $\text{Pic}(A)$ under the canonical isomorphism $\text{Pic}(B) \cong \text{Pic}(A)$. Again by the above remarks on vertices and sources, this creates no conflict of notation if $A$ is a itself isomorphic as an interior $P$-algebra to a block algebra of some (other) finite group with defect group $P$. See [11, Section 6.4] for an expository account on source algebras which includes the statements in this Remark.

The Brauer correspondent $C$ of $B$ has a source algebra $L$ of the form $L = O_C(P \times E)$ as an interior $P$-algebra, for some $\tau \in H^2(E, k^\times)$, where as above $E \cong \text{Out}_F(P)$ is the inertial quotient of $B$ and of $C$ determined by the choice of the maximal $B$-Brauer pair $(P,e)$ (which is also a maximal $C$-Brauer pair), and where we identify $k^\times$ with its canonical inverse image in $O_C^\times$.

The fusion system of $C$ determined by the choice of $(P,e)$ is $N_F(P) = \mathcal{F}_P(P \times E)$. Since $\text{Aut}(P,\mathcal{F})$ is a subgroup of $\text{Aut}(P,N_F(P))$ and since $\text{Aut}_{N_F(P)}(P) = \text{Aut}_F(P)$, it follows that $\text{Out}(P,\mathcal{F})$ is a subgroup of $\text{Out}(P,N_F(P))$.

By a result of Puig, there is a canonical embedding of interior $P$-algebras $L \to A$. We review in the following Proposition the construction of this embedding and some of its properties.

**Proposition 2.4** (cf. [8 Theorem A], [16 Propositions 14.6, 14.9], [6 Proposition 4.10]). Let $G$ be a finite group, $B$ a block of $O_G$, and $(P,e)$ a maximal $B$-Brauer pair, with associated inertial quotient $E$. Let $C$ be the Brauer correspondent of $B$. Denote by $\hat{e}$ the unique block idempotent of $O_C(P)$ which lifts $e$. The following hold.

(i) Let $j$ be a primitive idempotent of $O_C(P)\hat{e}$. Then $j$ remains primitive in $C^P$, and $j$ is a source idempotent both for the block $C$ of $O_NG(P)$ as well as for $O_NG(P,e)\hat{e}$. More precisely, the algebra $L = jO_NG(P)j$ is a source algebra of $C$, and we have $L = jO_NG(P,e)j$. There is $\tau \in H^2(E, k^\times)$, inflated to $P \times E$, such that

$$L \cong O_{\tau}(P \times E)$$

as interior $P$-algebras.

(ii) Let $f$ be a primitive idempotent in $B^{N_G(P,e)}$ satisfying $\text{Br}_P(f)e \neq 0$. Then $i = jf$ is a source idempotent in $B^P$ satisfying $\text{Br}_P(i)e \neq 0$. Set $A = iBi$. The idempotent $f$ commutes with
L, and multiplication by $f$ induces an injective homomorphism of interior $P$-algebras

$L \to A$

which is split injective as an $L$-$L$-bimodule homomorphism. Moreover, every indecomposable direct $L$-$L$-bimodule summand of $A$ in a complement of $L$ is relatively projective, as an $O(P \times P)$-module, with respect to a twisted diagonal subgroup of $P \times P$ of order strictly smaller than $|P|$.

(iii) As an $A$-$A$-bimodule, $A$ is isomorphic to a direct summand of $A \otimes _L A$, and every other indecomposable direct $A$-$A$-bimodule summand of $A \otimes _L A$ is relatively projective, as an $O(P \times P)$-module, with respect to a twisted diagonal subgroup of $P \times P$ of order strictly smaller than $|P|$.

(iv) The map sending $\zeta \in \text{Hom}(E,k^\times)$ to the linear endomorphism of $L$ given by the assignment $uy \mapsto \zeta(y)uy$, where $u \in P$ and $y \in E$, and where we identify $L = O_r(P \times E)$ induces a group isomorphism

$$\text{Hom}(E,k^\times) \cong \text{Out}_P(L)$$

(v) The map sending an automorphism $\alpha$ of $A$ which fixes $P$ elementwise and stabilises $L$ to the restriction of $\alpha$ to $L$ induces an injective group homomorphism

$$\text{Out}_P(A) \to \text{Out}_P(L) \cong \text{Hom}(E,k^\times).$$

Proofs of the statements in Proposition 2.4 can be found in the expository account of this material in [11, Theorem 6.14.1, Theorem 6.7.4, Theorem 6.15.1, and Lemma 6.16.2]. We record the following elementary group theoretic observation.

**Lemma 2.5.** Let $G$ be a finite group and $P$ a subgroup. Let $Q$ be a subgroup of $P \times P$. Suppose that the two canonical projections $P \times P \to P$ both map $Q$ onto $P$. The following hold.

(i) If $(x,y) \in G \times G$ such that $(x,y)Q \leq P \times P$, then $(x,y) \in N_G(P) \times N_G(P)$.

(ii) We have $N_{G \times G}(Q) \leq N_G(P) \times N_G(P) = N_{G \times G}(P \times P)$.

**Proof.** Let $(x,y) \in G \times G$ such that $(x,y)Q \leq P \times P$. Let $u \in P$. Since the first projection $P \times P \to P$ maps $Q$ onto $P$, it follows that there is $v \in P$ such that $(u,v) \in Q$. Then $(x,y)(u,v) \in P \times P$. In particular, $xu \in P$. Thus $x \in N_G(P)$. The same argument yields $y \in N_G(P)$, and hence $(x,y) \in N_G(P) \times N_G(P)$. This shows (i), and (ii) follows immediately from (i).

**Remark 2.6.** Let $G$ be a finite group, $P$ a $p$-subgroup, and $X$ an indecomposable $O(G \times G)$-module with a vertex $Q$ contained in $P \times P$ such that the two canonical projections $P \times P \to P$ map $Q$ onto $P$. By Lemma 2.5(ii), the Green correspondence yields, up to isomorphism, a unique indecomposable direct summand $f(X)$ of the $O(N_G(P) \times N_G(P))$-module $\text{Res}^{G \times G}_{N_G(P) \times N_G(P)}(X)$ with vertex $Q$ and a source which remains a source of $X$. Since any two vertices of $X$ are $G \times G$-conjugate, it follows from Lemma 2.5(i) that the isomorphism class of $f(X)$ does not depend on the choice of a vertex $Q$ of $X$ in $P \times P$. 

6
Lemma 2.7. Let $A$ be a source algebra of a block $B$ of a finite group algebra $OG$ with defect group $P$. Let $M$ be an invertible $A$-$A$-bimodule, and let $X$ be an invertible $B$-$B$-bimodule. The following hold.

(i) $M$ remains indecomposable as an $A$-$OP$-bimodule and as an $OP$-$A$-bimodule.

(ii) As an $O(P \times P)$-module, $M$ has an indecomposable direct summand with a vertex $Q$ such that both canonical projections $P \times P \to P$ map $Q$ onto $P$. In particular, $Q$ has order at least $|P|$.

(iii) As an $O(G \times G)$-module, $X$ has a vertex $Q$ contained in $P \times P$, and any such vertex $Q$ has the property that both canonical projections $P \times P \to P$ map $Q$ onto $P$. In particular, the vertices of $X$ have order at least $|P|$.

Proof. Since $M$ is an invertible bimodule, by Morita’s theorem, we have an algebra isomorphism $A \cong \text{End}_{A^{op}}(M)$ sending $c \in A$ to the right $A$-endomorphism of $M$ given by left multiplication by $c$ on $M$. This restricts to an algebra isomorphism $A^P \cong \text{End}_{OP \otimes A^{op}}(M)$. Now $1_A = i$ is primitive in $B^P$, hence $A^P$ is local, and thus so is $\text{End}_{OP \otimes A^{op}}(M)$, implying the statement (i). We prove next (iii). Suppose that the first projection $P \times P \to P$ maps $Q$ onto the subgroup $R$ of $P$. Then $Q$ is contained in $R \times P$, hence in $R \times G$. It follows that $X$ is relatively $(R \times G)$-projective. Thus $X$ is isomorphic to a direct summand of $OG \otimes_{OR} X$ as a bimodule. Let $U$ be a $B$-module. Then $U \cong X \otimes_B V$ for some $B$-module $V$ because $X$ is an invertible bimodule. Thus $U \cong X \otimes_B V$ is isomorphic to a direct summand of $OG \otimes_{OR} X \otimes_B V \cong \text{Ind}_Q^U(U)$. This shows that every $B$-module is relatively $R$-projective, which forces $R = P$. A similar argument, using right $B$-modules, shows that the second projection $P \times P \to P$ maps $Q$ onto $P$. This implies (iii). Statement (ii) follows from (iii) by the Remark 2.3.

Remark 2.8. The statements (ii), (iii) in Lemma 2.7 hold more generally if $M, X$ induce a stable equivalence of Morita type; see [18, Section 6].

Lemma 2.9 (cf. [10, Corollary 2.4.5]). Let $P$ be a finite group and $\varphi \in \text{Aut}(P)$. Set $Q = \{(\varphi(u), u) \mid u \in P\}$. We have an isomorphism of $O(P \times P)$-modules

$$\text{Ind}_Q^{P \times P}(O) \cong (OP)_{\varphi}$$

sending $(x, y) \otimes 1$ to $x\varphi(y^{-1})$, for all $x, y \in P$.

Proof. This is easily verified directly. Note that this is the special case of [10, Corollary 2.4.5] applied to $G = P \times \langle \varphi \rangle$, $H = L = P$ and $x = \varphi$.

Lemma 2.10. Let $\alpha$ be an algebra automorphism of $A$ which preserves $L$. Denote by $\beta$ the algebra automorphism of $L$ obtained from restricting $\alpha$ to $L$.

(i) The class of $\beta$ in $\text{Out}(L)$ is uniquely determined by the class of $\alpha$ in $\text{Out}(A)$.

(ii) If $\alpha$ is an inner automorphism of $A$, then $\beta$ is an inner automorphism of $L$.

(iii) If $c \in A^\times$ satisfies $cLc^{-1} = L$, then there exists $d \in L^\times$ such that $cd^{-1}$ centralises $L$.
Proof. We first prove (iii). Let \( c \in A^\times \) such that \( cLc^{-1} = L \). Then \( Lc \) is an \( L-L \)-bimodule summand of \( A \). Note that \( L \) and \( Lc \) have the same \( O \)-rank \( r \), and that \( \frac{|E|}{|P|} = \text{max} \) is prime to \( p \). It follows from Proposition [2.4](ii) that \( L \) is up to isomorphism the unique \( L-L \)-bimodule summand of \( A \) with this property, and hence \( L \cong Lc \) as \( L-L \)-bimodules. This implies that conjugation by \( c \) on \( L \) induces an inner automorphism of \( L \), given by an element \( d \in L^\times \). Then \( cd^{-1} \) acts as the identity on \( L \). This proves (iii). The statements (ii) and (iii) are clearly equivalent. In order to show (i), let \( \alpha, \alpha' \) two automorphisms which preserve \( L \) and which represent the same class in \( \text{Out}(A) \). Thus there exists \( c \in A^\times \) such that \( \alpha'(a) = co(a)c^{-1} \) for all \( a \in A \). Since \( \alpha, \alpha' \) preserve \( L \), it follows that conjugation by \( c \) preserves \( L \). By (ii), conjugation by \( c \) induces an inner automorphism of \( L \) and hence the restrictions to \( L \) of \( \alpha, \alpha' \) belong to the same class in \( \text{Out}(L) \). The result follows. \( \square \)

Lemma 2.11. Let \( A \) be a source algebra of a block with defect group \( P \) and fusion system \( \mathcal{F} \) on \( P \). Let \( \psi \in \text{Aut}(P) \). There is an isomorphism of \( A-OP \)-bimodules \( A \cong A_\psi \) if and only if \( \psi \in \text{Aut}_\mathcal{F}(P) \).

Proof. This is the special case of the equivalence of the statements (i) and (iii) in [11] Theorem 8.7.4, applied to \( Q = R = P \) and \( m = n = 1_A \). \( \square \)

For further results detecting fusion in source algebras see [15], or also [11] Section 8.7.

3 Source algebra automorphisms and Green correspondence

We use the notation and facts reviewed in Proposition 2.4. In particular, \( A = iBi \) and \( L = jCj \) are source algebras of the block \( B \) of \( OG \) and its Brauer correspondent \( C \), respectively, both associated with a maximal Brauer pair \( (P,e) \), and chosen such that multiplication by a primitive idempotent \( f \) in \( B_{N_G(P,e)} \) satisfying \( Br_p(f)e \neq 0 \) induces an embedding \( L \to A \) as interior \( P \)-algebras. In particular, \( i = jf \). This embedding is split as a homomorphism of \( L-L \)-bimodules. We set \( N = N_G(P) \). We keep this notation throughout this section.

The following Proposition describes the Green correspondence at the source algebra level for certain invertible bimodules induced by automorphisms.

Proposition 3.1. Let \( \beta \) be an algebra automorphism of \( L \) which extends to an algebra automorphism \( \alpha \) of \( A \) through the canonical embedding \( L \to A \). Then the \( B-B \)-bimodule

\[
X = OGi_\alpha \otimes_A iOG
\]

is invertible. As an \( O(G \times G) \)-module, \( X \) has a vertex \( Q \) contained in \( P \times P \). The \( O(N \times N) \)-Green correspondent of \( X \) with respect to \( Q \) is isomorphic to

\[
Y = ONj_\beta \otimes_L jON.
\]

Proof. The \( A-A \)-bimodule \( A_\alpha \) is obviously invertible, and hence so is the \( B-B \)-bimodule \( X \), since \( X \) is the image of \( A_\alpha \) under the canonical Morita equivalence between \( A \otimes_O A^{\text{op}} \) and \( B \otimes_O B^{\text{op}} \). Similarly, \( L_\beta \) and \( Y \) are invertible bimodules. By Lemma 2.4, \( Y \) has a vertex \( Q \) contained in \( P \times P \) such that both canonical projections \( P \times P \to P \) map \( Q \) onto \( P \). By Lemma 2.5 we have \( N_{G \times G}(Q) \leq N \times N \), so \( Y \) has a well-defined Green correspondent. In order to show that
$X$ is this Green correspondent, we start by showing that $X$ is isomorphic to a direct summand of $\text{Ind}_{N \times N}^{G \times G}(Y)$. Rewrite
\[
\text{Ind}_{N \times N}^{G \times G}(Y) = OGj_\beta \otimes_L jOG.
\]
Decompose $j = i + (j - i)$; since $i = jf$, this is an orthogonal decomposition of $j$ into two idempotents both of which commute with $L$; that is, $OGi_\beta \otimes_L iOG$ is isomorphic to a direct summand of $OGj_\beta \otimes_L jOG$. We show that $X$ is isomorphic to a direct summand of $OGi_\beta \otimes_L iOG$. Multiplying both sides by $i$, this is equivalent to showing that $A_\alpha$ is isomorphic to a direct summand of $A_\beta \otimes_L A$. Now $A$ is isomorphic to a direct summand of $A \otimes_L A$ (cf. Proposition 2.16). Tensoring on the left with $A_\alpha$ shows that $A_\alpha$ is isomorphic to a direct summand of $A_\alpha \otimes_L A \equiv A_\beta \otimes_L A$, where the last equality holds since $\alpha$ extends $\beta$. This shows that $X$ is indeed isomorphic to a direct summand of $\text{Ind}_{N \times N}^{G \times G}(Y)$, and therefore $X$ has a subgroup of $Q$ as a vertex. In order to show that $X$ is the Green correspondent of $Y$, we need to show that $Q$ is a vertex of $X$. It suffices to show that $Y$ is isomorphic to a direct summand of $\text{Res}_{N \times N}^{G \times G}(X) = OGi_\alpha \otimes_A iOG$. Thus it suffices to show that $L_\beta$ is isomorphic to a direct summand of $\text{Res}_{N \times N}^{G \times G}(X) = OGi_\alpha \otimes_A iOG$. Using as before the decomposition $j = i + (j - i)$, it suffices to show that $L_\beta$ is isomorphic to a direct summand of $A_\alpha$, as an $L$-$L$-bimodule. By Proposition 2.16 (ii), $L$ is isomorphic to a direct summand of $A$. The claim now follows by tensoring on the right with $L_\beta$ and noting that $A \otimes_L L_\beta \equiv A_\alpha$ as $L$-$L$-bimodules. 

As before, we denote by $E(B)$, $L(B)$, $T(B)$ the subgroups of $\text{Pic}(B)$ represented by invertible bimodules whose sources are endopermutation, linear, or trivial, respectively. We denote by $E(A)$, $L(A)$, $T(A)$ the subgroups of $\text{Pic}(A)$ which correspond to $E(B)$, $L(B)$, $T(B)$, respectively, under the canonical group isomorphism $\text{Pic}(B) \cong \text{Pic}(A)$ (cf. Remark 2.3). Summarising special cases of results in [18] 6.7, [2] Section 2, the bimodules representing elements in $T(A)$ can be described as follows.

**Proposition 3.2.** An invertible $A$-$A$-bimodule $M$ represents an element in $T(A)$ if and only if $M$ is isomorphic to a direct summand of an $A$-$A$-bimodule of the form
\[
A_\varphi \otimes_{OP} A
\]
for some $\varphi \in \text{Aut}(P,F)$. Moreover, the class of $\varphi$ in $\text{Out}(P,F)$ is uniquely determined by the isomorphism class of $M$, and the map $M \mapsto \varphi$ induces a group homomorphism $T(A) \to \text{Out}(P,F)$. This group homomorphism corresponds to the group homomorphism $T(B) \to \text{Out}(P,F)$ in [2] Theorem 1.1. (ii) through the canonical isomorphism $T(B) \cong T(A)$.

**Proof.** The fact that the elements in $T(A)$ are represented by bimodules as stated follows from the reformulation [2] Theorem 2.4] of results of Puig [18] 7.6 together with the canonical Morita equivalence between $B$ and $A$. The statement on the uniqueness of the class of $\varphi$ in $\text{Out}(P,F)$ follows from [2] Lemma 2.7. The fact that this yields a group homomorphism $T(A) \to \text{Out}(P,F)$ follows from [2] Lemma 2.6. By construction, this yields the group homomorphism $T(B) \to \text{Out}(P,F)$ in [2] Theorem 1.1. (ii)] when precomposed with the canonical isomorphism $T(B) \cong T(A)$.

The following characterisation of $A$-$A$-bimodules in Proposition 3.2 (i) below representing elements in $T(A)$ is essentially a reformulation of work of L. L. Scott [20] and L. Puig [18], where it is shown that Morita equivalences between block algebras given by $p$-permutation bimodules are induced by source algebra isomorphisms. As in Proposition 3.2, the homomorphism
\[ T(A) \rightarrow \text{Out}(P, F) \] in Proposition 3.3 (ii) corresponds to the one at the bottom of the diagram in 2 Theorem 1.1.

**Proposition 3.3.** With the notation above, the following hold.

(i) An invertible \( A\)-\( A\)-bimodule \( M \) represents an element in \( T(A) \) if and only if \( M \cong A_\alpha \) for some \( O\)-algebra automorphism \( \alpha \) of \( A \) which preserves the image of \( P \) in \( A^\times \). In particular, \( T(A) \) is a subgroup of the image of \( \text{Out}(A) \) in \( \text{Pic}(A) \).

(ii) Let \( \varphi \in \text{Aut}(P) \) and let \( \alpha \) be an \( O\)-algebra automorphism of \( A \) which extends \( \varphi \). Then \( \varphi \in \text{Aut}(P, F) \), and the map \( \alpha \mapsto \varphi \) induces a group homomorphism

\[ T(A) \rightarrow \text{Out}(P, F) \]

with kernel \( \text{Out}_P(A) \).

**Proof.** Note that \( A \) is a permutation \( OP\)-\( OP\)-bimodule. Thus if \( \alpha \in \text{Aut}(A) \) preserves the image of \( P \) in \( A^\times \), then \( A_\alpha \) is again a permutation \( OP\)-\( OP\)-bimodule. Therefore \( A_\alpha \) represents in that case an element in \( T(A) \). For the converse, let \( M \) be an invertible \( A\)-\( A\)-bimodule which represents an element in \( T(A) \). By Proposition 3.2 there is \( \varphi \in \text{Aut}(P, F) \) such that \( M \) is isomorphic to a direct summand of \( A_\varphi \otimes_{OP} A \). By Lemma 2.7 the restriction of \( M \) as an \( A\)-\( OP\)-bimodule remains indecomposable. Thus, using Krull-Schmidt, as an \( A\)-\( OP\)-bimodule, \( M \) is isomorphic to a direct summand of \( A_\varphi \otimes_{OP} W \) for some indecomposable direct summand \( W \) of \( A \) as an \( OP\)-\( OP\)-bimodule.

By [11 Theorem 8.7.1], we have \( W \cong OP_\tau \otimes_{OQ} OP \) for some subgroup \( Q \) of \( P \) and some \( \tau \in \text{Hom}_F(Q, P) \). By Lemma 2.4 (ii), we have \( Q = P \), and hence \( \tau \in \text{Aut}_F(P) \) and \( W = OP_\tau \). Thus \( M \) is isomorphic to a direct summand of \( A_\varphi \otimes_{OP} OP_\tau \cong A_{\varphi \tau} \cong A_{\varphi \tau \varphi} \cong A_\varphi \), where the last isomorphism uses Lemma 2.11 and the fact that \( \varphi \tau \in \text{Aut}_F(P) \). But then \( M \), as an \( A\)-\( OP\)-module, is isomorphic to \( A_\varphi \), since, by Lemma 2.7 (i), this module is indecomposable. In particular, \( M \cong A \) as a left \( A \)-module. Thus \( M \cong A_\alpha \) for some \( \alpha \in \text{Aut}(A) \). By Lemma 2.1 we can choose \( \alpha \) to extend \( \varphi \).

The fact that we have a group homomorphism \( T(A) \rightarrow \text{Out}(P, F) \) follows from Proposition 3.2 and that its kernel is \( \text{Out}_P(A) \) from identifying it with the corresponding homomorphism in 2 Theorem 1.1.

The next result shows that in the situation of Proposition 3.3 (ii) it is possible to choose \( \alpha \) in such a way that it preserves the subalgebra \( L = O_\tau(P \rtimes E) \).

**Proposition 3.4.** With the notation above, let \( \varphi \in \text{Aut}(P) \) such that \( \varphi \) extends to an \( O\)-algebra automorphism \( \alpha \) of \( A \). Then \( \varphi \) extends to an \( O\)-algebra automorphism \( \alpha' \) of \( A \) such that the images of \( \alpha \) and \( \alpha' \) in \( \text{Out}(A) \) are equal and such that \( \alpha' \) preserves the subalgebra \( L \). The correspondence \( \alpha \mapsto \alpha'|_L \) induces an injective group homomorphism \( \rho : T(A) \rightarrow T(L) \), and we have a commutative diagram of finite groups with exact rows of the form.

\[
\begin{array}{c}
1 \longrightarrow \text{Out}_P(A) \longrightarrow T(A) \longrightarrow \text{Out}(P, F) \\
\downarrow & & \downarrow \rho \\
1 \longrightarrow \text{Hom}(E, k^\times) \longrightarrow T(L) \longrightarrow \text{Out}(P, N_F(P))
\end{array}
\]
where the leftmost vertical map is from Proposition 2.4 (v), after identifying Hom\(E, k^\times\) with Out\(P(L)\) via Proposition 2.4 (iv), the rightmost horizontal arrows are those from Proposition 3.3, and the right vertical map is the inclusion.

Proof. In order to prove the first statement, we need to show that \(\alpha(L)\) is conjugate to \(L\) via an element \(w\) in \((A^P)^\times\). This proof is based on a ‘Maschke type’ argument, constructing \(w\) explicitly. This is a well-known strategy; see e.g. [7, Remark 4.4], [13, Proposition 4].

Note that any inner automorphism of \(P\) extends trivially to an algebra automorphism of \(A\). Since \(\varphi\) extends to an algebra automorphism \(\alpha\) of \(A\), it follows that any \(\varphi' \in \text{Aut}(P)\) representing the same class as \(\varphi\) in Out\((P)\) extends to an algebra automorphism of \(A\) representing the same class as \(\alpha\) in Out\((A)\). Therefore, in order to prove Proposition 3.3 we may replace \(\varphi\) by any automorphism of \(P\) representing the same class as \(\varphi\) in Out\((P)\).

We identify \(P \rtimes E\) as a subset of \(L = O_{\tau}(P \rtimes E)\), hence of \(A\). Note that this is a subset of \(A^\times\), but not a subgroup, because of the twist of the multiplication by \(\tau\). In particular, the inverse \(x^{-1}\) in the group \(P \rtimes E\) of an element \(x \in P \rtimes E\) is in general different from the inverse of \(x\) in the algebra \(L\). More precisely, the inverses of \(x\) in the group \(P \rtimes E\) and in the algebra \(L\) differ by a scalar.

For group elements \(x, y \in P \rtimes E\), we denote by \(xy\) the product in the group \(P \rtimes E\), and by \(x \cdot y\) the product in the algebra \(L\); that is, we have

\[
x \cdot y = \tau(x, y)xy.
\]

We denote by \(y x\) the conjugate of \(x\) by \(y\) in the group \(P \rtimes E\). By the above, this differs by a scalar from the conjugate of \(x\) by \(y\) in \(A^\times\).

Let \(\varphi \in \text{Aut}(P)\) and \(\alpha \in \text{Aut}(A)\) such that \(\alpha\) extends \(\varphi\). By Proposition 3.3 we have \(\varphi \in \text{Aut}(P, F)\). In particular, \(\varphi\) normalises the group \(\text{Aut}_{\tau}(P) = \text{Inn}(P) \cdot E\). Then \(\varphi \circ E \circ \varphi^{-1}\) is a complement of \(\text{Inn}(P)\) in \(\text{Inn}(P) \cdot E\), so conjugate to \(E\) by an element in \(\text{Inn}(P)\) by the Schur-Zassenhaus theorem. That is, after possibly replacing \(\varphi\) by another representative in \(\text{Aut}(P)\) of the class of \(\varphi\) in \(\text{Out}(P)\), we may assume that \(\varphi\) normalises the subgroup \(E\) of \(\text{Aut}(P)\).

Let \(y \in E\) (regarded as an automorphism of \(P\)). Since \(\varphi\) normalises \(E\), there is an element \(\psi(y) \in E\) such that

\[
\varphi \circ y \circ \varphi^{-1} = \psi(y).
\]

That is, \(\psi\) is the group automorphism of \(E\) induced by conjugation with \(\varphi\) in \(\text{Aut}(P)\).

In what follows we denote by \(\psi(y)^{-1}\) the inverse of \(\psi(y)\) in the subalgebra \(L\) of \(A\); by the above, this may differ from the group theoretic inverse of \(\psi(y)\) in \(E\) by a scalar in \(O^\times\). The elements \(\alpha(y)\) and \(\psi(y)\) in \(A^\times\) act in the same way on the image of \(P\) in \(A\) up to scalars in \(O^\times\). That is, conjugation by \(\alpha(y)\psi(y)^{-1}\) in \(A^\times\) sends \(u \in P\) to \(\zeta(u)u\) for some scalar \(\zeta(u) \in O^\times\). The map \(u \mapsto \zeta(u)\) is then a group homomorphism from \(P\) to \(O^\times\). It follows from [12, Lemma 3.9] that \(\zeta(u) = 1\) for all \(u \in P\). This shows that \(\alpha(y)\psi(y)^{-1}\) belongs to \((A^P)^\times\). Since conjugation by the elements \(\alpha(y), \psi(y)\) in \(A^\times\) preserves \(OP\), these conjugations also preserve the centraliser \(A^P\) of \(OP\) in \(A\). In other words, \(\alpha(y)\) and \(\psi(y)\) normalise the subgroups \((A^P)^\times\) and \(1 + J(A^P)\) of \(A^\times\).

Since \(k\) is perfect, we have a canonical group isomorphism \(O^\times \cong k^\times \times (1 + J(O))\). Now \(A^P\) is a local algebra, so \((A^P)^\times = k^\times (1 + J(A^P))\), or equivalently, every element in \((A^P)^\times\) can be written uniquely in the form \(\lambda \cdot 1_A + r\) for some \(\lambda \in k^\times\) (with \(k^\times\) identified to its canonical preimage in \(O^\times\)) and some \(r \in J(A^P)\). Thus

\[
\alpha(y)\psi(y)^{-1} = \lambda y + r_y.
\]
for a uniquely determined $\lambda_y \in k^\times$ and $r_y \in J(A^P)$. It follows that $\lambda_y^{-1} \alpha(y)\psi(y)^{-1} \in 1 + J(A^P)$. Set
\[ w = \frac{1}{|E|} \sum_{y \in E} \lambda_y^{-1} \alpha(y)\psi(y)^{-1}. \]
This is well defined since $|E|$ is prime to $p$. By construction, we have $w \in 1 + J(A^P)$, so in particular, $w$ is invertible in $A^P$, and conjugation by $w$ fixes the elements of $P$, hence preserves $OP$. Therefore, in order to show that $\alpha(L) = wLw^{-1}$, it suffices to show that for any $y \in E$, the element $\alpha(y)$ is a scalar multiple of the conjugate $w\psi(y)w^{-1}$. More precisely, we are going to show that
\[ \alpha(y)w = \lambda_y w\psi(y). \]
For any further element $x \in E$, applying $\alpha$ to the equation $y \cdot x = \tau(y, x)yx$ yields
\[ \alpha(y)\alpha(x) = \tau(y, x)\alpha(yx). \]
Similarly, we have
\[ \psi(y) \cdot \psi(x) = \tau(\psi(y), \psi(x))\psi(yx). \]
We show next that the 2-cocycles $\tau$ and $\tau(\psi(-), \psi(-))$ in $Z^2(E, k^\times)$ represent the same class, via the 1-cochain $y \mapsto \lambda_y$. By construction, $\alpha(y)$ and $\lambda_y\psi(y)$ differ by an element in $1 + J(A^P)$. Calculating modulo $1 + J(A^P)$ in the two previous equations yields
\[ \tau(\psi(y), \psi(x)) = \lambda_y^{-1} \lambda_x^{-1} \lambda_{yx} \tau(y, x). \]
In other words, the class of $\tau$ is stable under $\psi$.

Using these equations, we have
\[ \alpha(y)w = \frac{1}{|E|} \sum_{x \in E} \lambda_x^{-1} \alpha(y)\alpha(x)\psi(x)^{-1} = \frac{1}{|E|} \sum_{x \in E} \lambda_x^{-1} \tau(y, x)\alpha(y)\alpha(yx)\tau(\psi(y), \psi(x))^{-1}\psi(yx)^{-1}\psi(y) = \frac{1}{|E|} \sum_{x \in E} \lambda_x^{-1} \tau(y, x)\alpha(y)\lambda_x \lambda_y \lambda_{yx}^{-1} \tau((y, x)^{-1}\psi(yx)^{-1}\psi(y) = \frac{1}{|E|} \sum_{x \in E} \lambda_y \lambda_{yx}^{-1} \alpha(y)\psi(yx)^{-1}\psi(y) = \lambda_y w\psi(y). \]
This shows that $\alpha(L) = wLw^{-1}$. Thus setting
\[ \alpha' = c_{w^{-1}} \circ \alpha, \]
where here $c_{w^{-1}}$ is conjugation by $w^{-1}$ in $A^\times$, yields an automorphism $\alpha'$ of $A$ in the same class as $\alpha$ which extends $\varphi$ and stabilises $L$. If $\alpha$ fixes $P$, so does $\alpha'$, and hence its restriction to $L$ fixes $P$. Together with Lemma 2.10 this shows that the map sending $\alpha$ to the restriction of $\alpha'$ to $L$ induces a group homomorphism $T(A) \to T(L)$ mapping the image of $\text{Out}_P(A)$ in $T(A)$ to the image of $\text{Out}_P(L)$ in $T(L)$, and by Proposition 2.4 (iv) we have $\text{Out}_P(L) \cong \text{Hom}(E, k^\times)$.

For the injectivity of this group homomorphism, suppose that $\alpha$ stabilises $L$ and restricts to an inner automorphism of $L$. By Proposition 2.4 (iii), the $A$-$A$-bimodule $A_\alpha$ is isomorphic to a direct
summand of $A_\alpha \otimes_L A$. Since the restriction of $\alpha$ to $L$ is inner, we have $A_\alpha \cong A$ as $A$-$L$-bimodules. Thus $A_\alpha$ is isomorphic to a direct summand of $A \otimes_L A$. But then Proposition 2.4 (iii) implies that $A_\alpha \cong A$ as $A$-$A$-bimodules, and hence $\alpha$ is an inner automorphism of $A$. This concludes the proof.

**Proposition 3.5.** Let $\gamma : L \to A$ be an algebra homomorphism such that $\gamma(u) = u$ for all $u \in P$ and such that the induced map $k \otimes_O L \to k \otimes_O A$ is the canonical inclusion. Then there is an element $c \in 1 + \pi A^P$ such that $\gamma(y) = yc$ for all $y \in L$.

**Proof.** The hypotheses imply that $A$ and $A_\gamma$ are permutation $P \times P$-modules such that $k \otimes_O A \cong k \otimes_O A_\gamma$ as $A$-$L$-bimodules, with the isomorphism given by $1 \otimes a \mapsto 1 \otimes a$. Since $p$-permutation modules over finite group algebras lift uniquely, up to isomorphism from $k$ to $O$, and since homomorphisms between $p$-permutation modules lift from $k$ to $O$ (see e.g. [10, Theorem 5.11.2]) it follows that there is an $A$-$L$-bimodule isomorphism $A \cong A_\gamma$, lifting the identity map on $k \otimes O A$. (Note we must temporarily pass to the block algebras to apply the results of [10].) Consequently this bimodule isomorphism is induced by right multiplication on $A$ with an element $c$ in $1 + \pi A$. Since right multiplication by $c$ is also an isomorphism of right $L$-modules, it follows that $\gamma \gamma^{-1}(y) = yc$ for all $y \in L$. That is, composing $\gamma$ with the automorphism given by conjugating with $c$ gives the inclusion map $L \to A$. Since $\gamma$ fixes $P$, it follows that $c \in A^P$, hence $c \in 1 + \pi A^P$, whence the result.

## 4 Proofs

**Proof of Theorem 1.1** We use the notation from Proposition 2.4 as briefly reviewed at the beginning of Section 3. Let $X$ be an invertible $B$-$B$-bimodule $X$ representing an element in $\mathcal{L}(B)$. We will show that $X$ corresponds (via the standard Morita equivalence) to an invertible $A$-$A$-bimodule of the form $A_\alpha$, for some algebra automorphism $\alpha$ of $A$ which preserves $L$, or equivalently, which restricts to an algebra automorphism $\beta$ of $L$. Together with Proposition 3.1 this implies that $L_\beta$ corresponds to the Green correspondent $Y$ of $X$. Before getting into details, we show how this completes the proof of Theorem 1.1. Since the Green correspondence is a bijection on the isomorphism classes of the bimodules under consideration, this shows that the class of $\beta$ in $\text{Out}(L)$ is uniquely determined by the class of $\alpha$ in $\text{Out}(A)$ (a fact which follows also directly from Lemma 2.10), and hence that the map $\mathcal{L}(B) \to \mathcal{L}(C)$ induced by the map $A_\alpha \to L_\beta$ is an injective map. This is a group homomorphism because for any two algebra automorphisms $\alpha$, $\alpha'$ of $A$ we have a bimodule isomorphism $A_\alpha \otimes_A A_{\alpha'} \cong A_{\alpha \circ \alpha'}$.

We turn now to what remains to be proved, namely that an invertible $B$-$B$-bimodule $X$ representing an element in $\mathcal{L}(B)$ corresponds to an invertible $A$-$A$-bimodule of the form $A_\alpha$, for some algebra automorphism $\alpha$ of $A$ which preserves $L$.

As pointed out in [2] Remark 1.2.(e), it follows from [12, Theorem 1.1, Lemma 3.15] that we have canonical isomorphisms

$$\mathcal{L}(B) \cong \text{Hom}(P/\mathfrak{i}\mathfrak{o}(F), O^x) \rtimes T(B),$$

$$\mathcal{L}(C) \cong \text{Hom}(P/[P,P \rtimes E], O^x) \rtimes T(C),$$

where in the second isomorphism we use the fact that the fusion system of $L$ is $N_F(P)$, which is the same as the fusion system of the group $P \rtimes E$ on $P$, and hence its focal subgroup is $[P,P \rtimes E]$.
This is a subgroup of \( \text{fo}(\mathcal{F}) \), and therefore we may identify \( \text{Hom}(P/\text{fo}(\mathcal{F}), \mathcal{O}^\times) \) with a subgroup of \( \text{Hom}(P/[P,P \times E], \mathcal{O}^\times) \).

It suffices to show separately for \( X \) representing an element in \( \mathcal{T}(B) \) and in \( \text{Hom}(P/\text{fo}(\mathcal{F}), \mathcal{O}^\times) \) that \( X \) corresponds to an invertible \( A-A \)-bimodule of the form \( A_\alpha \) as above. As far as \( \mathcal{T}(B) \) is concerned, this holds by the Propositions \([5, 3]\) and \([5, 4]\) Note that if \( \mathcal{O} = k \), then \( \text{Hom}(P/\text{fo}(\mathcal{F}), \mathcal{O}^\times) \) is trivial, so this concludes the proof of Theorem \([1, 1]\) in that case.

We assume now that \( \mathcal{O} \) has characteristic zero (and enough roots of unity, by our initial blanket assumption). Suppose that \( X \) represents an element in the canonical image of \( \text{Hom}(P/\text{fo}(\mathcal{F}), \mathcal{O}^\times) \) in \( \mathcal{L}(B) \). We need to show that then \( X \) corresponds to an invertible \( A-A \)-bimodule of the form \( A_\alpha \) as above. Note that the image of the group \( \text{Hom}(P/\text{fo}(\mathcal{F}), \mathcal{O}^\times) \) in \( \mathcal{L}(B) \) is equal to the kernel of the canonical map \( \mathcal{L}(B) \to \mathcal{T}(k \otimes \mathcal{O} B) \); this follows from \([2\text{ Remark 1.2.(d)}]\). This kernel consists only of isomorphism classes of linear source invertible \( B-B \)-bimodules with diagonal vertex \( \Delta P = \{(u, u) \mid u \in P\} \). More precisely, by \([2\text{ Lemmas 2.3, 2.7}]\), if \( X \) is an invertible \( B-B \)-bimodule with a linear source, then there is a unique group homomorphism \( \zeta : P \to \mathcal{O}^\times \) such that \( \text{fo}(\mathcal{F}) \leq \ker(\zeta) \), and such that \( X \) is isomorphic to a direct summand of \( O\text{Gi}_\eta \otimes_{\mathcal{O}_P} i\mathcal{O}_G \), where \( \eta \) extends to an algebra automorphism of \( \mathcal{O} \mathcal{P} \) given by \( \eta(u) = \zeta(u)u \) for all \( u \in P \).

By the results in \([12\text{ Section 3}]\), \( \eta \) extends to an algebra automorphism \( \alpha \) of \( \mathcal{A} \) which induces the identity on \( k \otimes \mathcal{O} A \), and through the canonical Morita equivalence, \( X \) corresponds to the \( A-A \)-bimodule \( A_\alpha \). By Proposition \([5, 5]\) applied to \( \alpha^{-1} \) restricted to \( L \), we may choose \( \alpha \) such that it stabilises \( L \). Thus the restriction of \( \alpha \) to \( L \) yields an element \( L_\alpha \) whose isomorphism class belongs to the image of \( \text{Hom}(P/[P,P \times E], \mathcal{O}^\times) \) in \( \mathcal{L}(L) \).

**Proof of Theorem \([1, 3]\)** We use the notation introduced in Section \([2\text{ Remark 1.2.(d)})\]. The key ingredient is the exact sequence of groups

\[
1 \longrightarrow \text{Out}_P(A) \longrightarrow \mathcal{E}(B) \longrightarrow D(P, \mathcal{F}) \times \text{Out}(P, \mathcal{F}) \cong D(P, \mathcal{F}) \times \text{Out}(P, \mathcal{F})
\]

from \([2\text{ Theorem 1.1}]\) (we write here \( D(P, \mathcal{F}) \) instead of \( D_\mathcal{O}(P, \mathcal{F}) \)). Denote by \( \mathcal{E}^\Delta(B) \) the subgroup of all elements in \( \mathcal{E}(B) \) whose image in \( D(P, \mathcal{F}) \times \text{Out}(P, \mathcal{F}) \) is of the form \((V, \text{Id}_P)\), for some element \( V \) in \( D(P, \mathcal{F}) \). That is, \( \mathcal{E}^\Delta(B) \) is the inverse image in \( \mathcal{E}(B) \) under the map \( \Phi \) of the normal subgroup \( D(P, \mathcal{F}) \) of \( D(P, \mathcal{F}) \times \text{Out}(P, \mathcal{F}) \). Thus \( \mathcal{E}^\Delta(B) \) is a normal subgroup of \( \mathcal{E}(B) \) such that the quotient \( \mathcal{E}(B)/\mathcal{E}^\Delta(B) \) is isomorphic to a subgroup of \( \text{Out}(P, \mathcal{F}) \). In particular, the order of this quotient is determined in terms of the defect group \( P \). Now \( \mathcal{E}(B) \) is a finite group (by \([2\text{ Theorem 1.1 (iii)]}\)), and hence \( \mathcal{E}^\Delta(B) \) is finite. Thus the image of \( \mathcal{E}^\Delta(B) \) under \( \Phi \) in \( D(P, \mathcal{F}) \) is finite, hence contained in the torsion subgroup of the Dade group \( D(P) \). By \([17\text{ Corollary 2.4}]\), the torsion subgroup of \( D(P) \) is finite, hence a finite invariant of \( P \), and so the image of \( \mathcal{E}^\Delta(B) \) in \( D(P) \) is bounded in terms of \( P \). By \([16\text{ Proposition 14.9}]\), the kernel \( \text{Out}_P(A) \) of \( \Phi \) is isomorphic to a subgroup of \( \text{Hom}(E, \mathcal{O}^\times) \), where \( E \cong \text{Out}(P) \) is the inertial quotient. Thus \( \ker(\Phi) \) is also bounded in terms of \( P \). The result follows.

For the proof of Theorem \([1, 4]\) we need the following observations; we use the well-known fact that the Cartan matrix of a split finite-dimensional \( k \)-algebra \( A \) is of the form \( \left( \dim_k(iA_j) \right) \), where \( i, j \) run over a set of representatives of the conjugacy classes of primitive idempotents in \( A \) (see e.g. \([10\text{ Theorem 4.10.2}]\)).


Lemma 4.1. Let $A$ be a split finite-dimensional $k$-algebra. Let $I$ be a set of representatives of the conjugacy classes of primitive idempotents in $A$. For $i \in I$ set $S_i = Ai/J(A)i$, and for $i, j \in I$ set $c_{ij} = \dim_k(iA_j)$. Let $M$ be an invertible $A$-$A$-bimodule. Denote by $\pi$ the unique permutation of $I$ satisfying $S_{\pi(i)} \cong M \otimes_A S_i$ for all $i \in I$. We have
\[
\dim_k(M) = \sum_{i,j \in I} c_{ij} \dim_k(S_{\pi(i)}) \dim_k(S_j) .
\]
Moreover, for any $i, j \in I$, we have $c_{\pi(i)\pi(j)} = c_{ij}$.

Proof. Since $A$ is split, for any $i \in I$ we have $i S_i \cong k$, and for any two different $i, j \in I$ we have $j S_i = 0$. As a right $A$-module, $M$ is a progenitor, and hence we have an isomorphism of right $A$-modules $M \cong \oplus_{i \in I} (iA)^{m_i}$ for some positive integers $m_i$. Thus we have an isomorphism of vector spaces $M \otimes_A S_i \cong \oplus_{j \in I} (j A \otimes_A S_i)^{m_j}$. By the above, the terms with $j \neq i$ are zero while $i A \otimes_A S_i$ is one-dimensional, and hence $\dim_k(S_{\pi(i)}) = \dim_k(M \otimes_A S_i) = m_i$. Note that $\dim_k(iA) = \sum_{j \in I} c_{ij} \dim_k(S_j)$. Thus
\[
\dim_k(M) = \sum_{i \in I} \dim_k(iA) \cdot m_i = \sum_{i,j \in I} c_{ij} \dim_k(S_{\pi(i)}) \dim_k(S_j)
\]
as stated. Since the functor $M \otimes_A -$ is an equivalence sending $S_i$ to a module isomorphic to $S_{\pi(i)}$, it follows that this functor sends $Ai$ to a module isomorphic to $A\pi(i)$ and induces isomorphisms $\text{Hom}_A(Ai, A j) \cong \text{Hom}_A(A\pi(i), A\pi(j))$, hence $i A j \cong A\pi(i)A\pi(j)$. The equality $c_{\pi(i)\pi(j)} = c_{ij}$ follows. 

Proof of Theorem 4.4. In order to prove Theorem 4.4 we may assume that $O = k$. We use the notation as in Lemma 4.1. By the assumptions, the Cartan matrix $C = (c_{ij})_{i,j \in I}$ of $A$ is symmetric and positive definite. Thus the map $(x, y) \to x^T C y$ from $\mathbb{R}^{|I|} \times \mathbb{R}^{|I|}$ to $\mathbb{R}$ is an inner product. The Cauchy-Schwarz inequality yields $|x^T C y|^2 \leq |x^T C x| \cdot |y^T C y|$. We are going to apply this to the dimension vectors $x = (\dim_k(S_i))_{i \in I}$ and $y = (\dim_k(S_{\pi(i)}))_{i \in I}$. By Lemma 4.1, we have $\dim_k(M) = x^T C y$. Applied to $M = A$ (and $\pi = 1d$) we also have that $\dim_k(A) = x^T C x$. The last statement in Lemma 4.1 implies that $x^T C x = y^T C y$. Thus the Cauchy-Schwarz inequality yields $\dim_k(M) = x^T C y \leq x^T C x = \dim_k(A)$ as stated.

The Cauchy-Schwarz inequality is an equality if and only if the dimension vectors $x$ and $y$ are linearly dependent. Since both vectors consist of the same positive integers (in possibly different orders) this is the case if and only if $x = y$, or equivalently, if and only if $\dim_k(S_{\pi(i)}) = \dim_k(S_i)$ for all $i \in I$. By [10] Proposition 4.7.18, this holds if and only if there is an $A$-$A$-bimodule isomorphism $M \cong A_\alpha$ for some $\alpha \in \text{Aut}(A)$. This completes the proof. 

Proof of Theorem 4.5. It is clearly enough to prove the theorem for $O = k$. Let $G$ be a finite group and $B = kGb$ a block of $kG$ with normal defect group $P$. We first claim that we may assume $B$ is isomorphic to its own source algebra. More specifically we assume that $G = P \rtimes H$, for a $p'$-group $H$ and $Z = C_H(P)$ a cyclic subgroup such that $b \in kZ$. Indeed $B$ is certainly source algebra equivalent (or equivalent as an interior $P$-algebra) to such a block (see e.g. [11] Theorem 6.14.1). Since, by [2] Lemma 2.8(ii)], source algebra equivalences preserve $E(B)$ and $L(B)$, we may assume that $B$ is of the desired form.
Let $M$ be an invertible $B$-$B$-bimodule with endopermutation source. Let $Q$ be a vertex of $M$ which, by [2, Lemma 1.1], is necessarily of the form $\Delta \varphi = \{ (\varphi(u), u) \mid u \in P \} \leq P \times P$, for some $\varphi \in \text{Aut}(P)$. Let $V$ be a source for $M$ with respect to the vertex $Q$. In particular, $V$ is absolutely indecomposable (see e.g. [11, Proposition 7.3.10]). It follows from Green’s Indecomposability Theorem that $U = \text{Ind}_Q^{P \times P}(V)$ is indecomposable. We now consider $I = \text{Stab}_{G \times G}(U)$, the stabiliser of $U$ in $G \times G$. If $h \in H$, then the $\mathcal{O}(P \times P)$-module $(h, 1)U$ has vertex $\Delta(c_h \circ \varphi)$, where $c_h$ denotes conjugation by $h$. Therefore, if $(h, 1) \in I$, then

$$\Delta(c_h \circ \varphi) = (x, y)(\Delta \varphi)(x, y)^{-1} = \Delta(c_x \circ \varphi \circ c_y^{-1}),$$

for some $x, y \in P$. In other words,

$$c_h = c_x \circ \varphi \circ c_y^{-1} \circ \varphi^{-1} = c_x \circ c_{\varphi(y)^{-1}}.$$

In particular, $c_h$ is an inner automorphism of $P$. However, since $H/Z$ is a $p'$-group, $c_h$ is an inner automorphism of $P$ if and only if $h \in Z$. So we have an injective map between sets $H/Z \to (G \times G)/I$, giving that $[G \times G : I] \geq [H : Z]$. Therefore, since $M \cong \text{Ind}_I^{G \times G}(W)$, for some direct summand $W$ of $\text{Ind}_{P \times P}(U)$,

$$\dim_k(M) = [G \times G : I] \cdot \dim_k(W) \geq [H : Z] \cdot \dim_k(W) \geq [H : Z] \cdot \dim_k(U)$$

$$= [H : Z] \cdot [P \times P : Q] \cdot \dim_k(V) = [H : Z] \cdot [P : Q] \cdot \dim_k(V) = [G : Z] \cdot \dim_k(V) = \dim_k(B) \cdot \dim_k(V).$$

By Theorem [14] we have $\dim_k(B) \geq \dim_k(M)$. This yields that $\dim_k(V) = 1$ as desired. 

**Example 4.2.** Let $p = 3$ and $G = Q_8 \times P$, where the action of $P \cong C_3$ on $Q_8$ is non-trivial. Set $B = \mathcal{O}b$, where $b = (1 - z)/2$ and $z$ is the unique non-trivial central element in $Q_8$. Now $\mathcal{O}Q_b \cong M_2(\mathcal{O})$ and conjugation by a non-trivial element of $P$ induces an non-trivial automorphism of $\mathcal{O}Q_b$. We temporarily assume $\mathcal{O} = k$. Since every $\mathcal{O}$-algebra automorphism of $M_2(\mathcal{O})$ is inner and every non-trivial element of order three in $M_2(\mathcal{O})$ is conjugate to

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

it follows that $(\mathcal{O}Q_b)^P \cong \mathcal{O} \cdot 1 \oplus \mathcal{O} \cdot x \subset M_2(\mathcal{O})$. In particular, $(\mathcal{O}Q_b)^P$ is local and so $B^P \cong (\mathcal{O}Q_b)^P \otimes_\mathcal{O} \mathcal{O}P$ is also local meaning $B$ is its own source algebra. This holds even with the assumption that $\mathcal{O} = k$ dropped.

As $B$ is nilpotent, we can apply [2, Example 7.2] and construct an element of $\mathcal{E}(B \otimes_\mathcal{O} \mathcal{O}P)$ not in $\mathcal{L}(B \otimes_\mathcal{O} \mathcal{O}P)$. With Theorem [15] in mind, we have shown that Theorem [14] does not hold with $\mathcal{L}$ replaced by $\mathcal{E}$.

**Remark 4.3.** In view of the notational conventions in [2, Proposition 2.6], the group homomorphism $\Phi$ in [2, Theorem 1.1] should send $X$ to $(V, \varphi^{-1})$ instead of $(V, \varphi)$.

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