Large-System Analysis of Correlated MIMO Multiple Access Channels with Arbitrary Signaling in the Presence of Interference

Maksym A. Girnyk, Mikko Vehkaperä, Lars K. Rasmussen

Abstract—Presence of multiple antennas on both sides of a communication channel promises significant improvements in system throughput and power efficiency. In effect, a new class of large multiple-input multiple-output (MIMO) communication systems has recently emerged and attracted both scientific and industrial attention. To analyze these systems in realistic scenarios, one has to include such aspects as co-channel interference, multiple access and spatial correlation. In this paper, we study the properties of correlated MIMO multiple-access channels in the presence of external interference. Using the replica method from statistical physics, we derive the ergodic sum-rate of the communication for arbitrary signal constellations when the numbers of antennas at both ends of the channel grow large. Based on these asymptotic expressions, we also address the problem of sum-rate maximization using statistical channel information and linear precoding. The numerical results demonstrate that when the interfering terminals use discrete constellations, the resulting interference becomes easier to handle compared to Gaussian signals. Thus, it may be possible to accommodate more interfering transmitter-receiver pairs within the same area as compared to the case of Gaussian signals. We also design precoding matrices for both Gaussian and QPSK signaling, and demonstrate numerically that they improve the achievable rates significantly at low-to-mid signal to noise ratios.

I. INTRODUCTION

During the last decade, multi-antenna communications has received an increased interest both from academia and industry. Pioneering research by Foschini, Gans and Telatar [1], [2] on the topic suggested that the new class of multiple-input multiple-output (MIMO) systems allowed the transmission rate to be increased roughly linearly in the number of antennas available at the transmitter and receiver. Measurements both indoors [3] and outdoors [4] have also confirmed the throughput gains of the multi-antenna transmission. Very recently, large MIMO communications or massive MIMO has become a new emerging topic [5]–[8]. Typically this concept entails a multiuser system where a single base station equipped with a very large antenna array is used to serve a small number of terminals simultaneously. Apart from the aforementioned throughput gains, such systems allow for a significant reduction of the transmit power.

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The main price to pay for the benefits offered by multi-antenna transmission is the hardware and signal processing complexity at both the transmitter and receiver. It is therefore of great importance to analyze the potential performance gains of MIMO processing in realistic scenarios before employing the techniques in practice. For example, in the uplink of a cellular system, the effects of co-channel interference emerging from other cells need to be taken into account. As observed in [2], such interference has also a surprising influence on the optimal power allocation strategy at the transmitter. In addition to co-channel interference, spatial correlation [10] and the type of channel inputs [11] have a great impact on the achievable rate of the channel. However, analysis of realistic scenarios tend to be mathematically challenging and numerical simulations are time consuming, especially if discrete signaling is employed at the transmitter. Some simplifying assumptions are therefore needed to make the problem tractable.

Asymptotic approaches developed within the field of random matrix theory greatly facilitate the analysis of achievable ergodic rates (mutual information averaged over channel realizations) in MIMO systems. Such methods were used already in the early works [1], [2] to assess the capacity of multi-antenna transmission. At the same time, several approaches using random matrix theory for the analysis of the spectral efficiency of large code division multiple access (CDMA)
systems [12]–[15] were reported. The multi-antenna results were later extended to the case of spatial correlation in [16] and then to MIMO multiple-access channel (MIMO-MAC) in [17]. Some analysis of MIMO systems in the case of co-channel interference have also been carried out under the assumption of Gaussian channel inputs. The first analytical results using random matrix theory were obtained by Lozano and Tulino [18] assuming uncorrelated channels and interferers. Some later efforts [20]–[22] have extended this analysis to different assumptions about correlations and numbers of antennas present in the system.

The aforementioned studies regarding MIMO systems with co-channel interference have all concentrated on the special case, where Gaussian signals are transmitted both by the desired user and the interfering terminals. This is in contrast to real-world systems, where discrete constellations such as QPSK and QAM are used. These realistic cases are, however, out-of-bounds for random matrix theory, except for setups where sub-optimal linear detection and per-stream decoding is considered. To investigate the performance bounds of generic systems with non-Gaussian channel inputs, a tool borrowed from the field of statistical physics, namely the replica method, has been recently used.

The replica method was invented by Kac [23], and is widely known due to its early applications to spin glasses [24], [25]. The replica framework provides a powerful set of mathematical tools for computing average quantities within large many-body systems and has since been applied to various problems in science and engineering. In the context of information theory, it was used to assess the spectral efficiency of large CDMA systems with antipodal signaling by Tanaka in [26], [27]. Later, Guo and Verdú generalized the approach to CDMA with arbitrary signaling [28]. Meanwhile, Müller in [11] applied the method to spatially correlated MIMO channels with binary inputs. This work was further generalized in [29] to the analysis of the sum-rate of a MIMO-MAC.

A somewhat different approach was taken by Moustakas et al. [30], where the replica method was used to analyze the moments of mutual information of a MIMO system with co-channel interference. These results were obtained, however, under the assumption of Gaussian signaling at all terminals.

In the present paper, we extend our previous work in [31] and investigate the performance and sum-rate maximization of a correlated MIMO-MAC in the presence of correlated non-Gaussian interferers. To summarize, the following contributions are reported:

- We derive an expression for the asymptotic sum-rate of the MIMO-MAC with spatial correlation and in the presence of spatially correlated multi-antenna interferers. The analysis is valid for arbitrary channel inputs at all terminals and carried out in the large-system limit (LSL), where the numbers of antennas at both ends of each MIMO channel grow without bound at a constant rate. As expected, several prior results [18], [29], [30], [32] are obtained as special cases of the analysis.
- We address the precoder optimization problem for both Gaussian and finite-alphabet signaling schemes under the assumptions of full channel state information (CSI) at the receiver and statistical CSI at the transmitter. By using the asymptotic sum-rate as a utility function for the corresponding optimization problem, we obtain the precoding matrices for each user.

The remainder of the paper is organized as follows. In the following section, we describe the system model and formulate the main problem. In addition, we discuss the necessary details regarding the MIMO channels with perfect CSI at the transmitter. Next, in Section III we present the main result of the paper, that is, the asymptotic sum-rate of a MIMO-MAC in the presence of interference. Section IV then addresses the precoder optimization problem, followed by Section VI where we present numerical results and discussion. Finally, in Section VII we conclude the paper. The proofs are relegated to the appendices.

**Notation:** Throughout this paper we will use upper case bold-faced letters to denote matrices, e.g., $A$, with elements denoted by $(A)_{i,j}$, lower case bold-faced letters to denote column vectors, e.g., $a$, with elements $a_i$, and lower case light-faced letters to denote scalar variables, e.g., $a$. Superscripts $^T$ and $^H$ denote transpose and Hermitian adjoint operators, respectively. Meanwhile, $A^{1/2}$, $\text{tr}(A)$ and $\text{det}(A)$ denote the principal square root, the trace and the determinant of matrix $A$. We differentiate between operators $\text{diag}(a)$, which denotes a diagonal matrix containing the coefficients of vector $a$ on its main diagonal, and $\text{Diag}(A)$, which denotes a column vector containing the diagonal entries of matrix $A$. Also, $I$, $0$ and $I$ denote the identity matrix, the zero matrix and the all-ones vector of appropriate sizes. Operator $E\{\cdot\}$ denotes the expectation, $\otimes$ represents the Kronecker product, and $A \succeq 0$ implies that the matrix $A$ is positive semidefinite. Finally, $\Re\{\cdot\}$ and $\Im\{\cdot\}$ stand for the real and imaginary parts of the argument.

## II. Preliminaries

### A. System Model

Consider the scenario where $K$ multi-antenna terminals communicate to a single multi-antenna receiver in the presence of $L$ multi-antenna interferers. An uplink cellular communication system in the presence of inter-cell interference can be regarded as an example of such a scenario. The numbers of antennas at transmitter $k$, interferer $\ell$ and the receiver, are denoted by $M_{k}, M_{\ell}$, and $N$, respectively. The discrete-time received vector is given by

$$y = \sum_{k=1}^{K} H_{s,k} x_{s,k} + \sum_{\ell=1}^{L} H_{i,\ell} x_{i,\ell} + n,$$

where $x_{s,k} \in \mathbb{C}^{M_{k}}$ is the zero-mean transmitted signal vector of the $k$th user with covariance matrix $E\{x_{s,k}x_{s,k}^H\} = P_{s,k}$ and $x_{i,\ell} \in \mathbb{C}^{M_{\ell}}$ is the transmitted signal vector of the $\ell$th interferer having zero-mean and covariance $E\{x_{i,\ell}x_{i,\ell}^H\} = P_{i,\ell}$. To satisfy long-term power constraints at the transmitters, we require that $\text{tr}(P_{s,k}) \leq M_{k}$ and $\text{tr}(P_{i,\ell}) \leq M_{\ell}$. The noise vector $n \in \mathbb{C}^{N}$ has independent zero-mean circularly symmetric complex Gaussian (ZMCSGC) entries with unit variance. Matrices $H_{s,k} \in \mathbb{C}^{N \times M_{k}}$ and $H_{i,\ell} \in \mathbb{C}^{N \times M_{\ell}}$
denote the MIMO channels between user $k$ and the receiver and between interferer $\ell$ and the receiver, respectively. The channels are assumed to be flat-fading and are modeled via the Kronecker model \([33]\), that is
\[
H_{s,k} = \sqrt{\frac{\rho_{s,k}}{M_s}} R_{s,k}^{1/2} W_{s,k} T_{s,k}^{1/2}, \tag{2a}
\]
\[
H_{i,\ell} = \frac{\rho_{i,\ell}}{M_{i,\ell}} P_{i,\ell}^{1/2} W_{i,\ell} T_{i,\ell}^{1/2}, \tag{2b}
\]
where $\rho_{s,k}$ and $\rho_{i,\ell}$ represent average signal-to-noise ratios (SNRs) of the corresponding links and matrices $W_{s,k}$ and $W_{i,\ell}$ have i.i.d. ZMCG entries of unit variance. The correlation matrices at the receive end are denoted by $R_{s,k}$ and $R_{i,\ell}$, while $T_{s,k}$ and $T_{i,\ell}$ represent the correlation matrices at the transmit end of the corresponding channels. To ensure that the correlation matrices do not influence the average path gains, they are normalized as
\[
\text{tr}\{R_{s,k}\} = N_s, \quad \text{tr}\{T_{s,k}\} = M_s, \tag{3a}
\]
\[
\text{tr}\{R_{i,\ell}\} = N_i, \quad \text{tr}\{T_{i,\ell}\} = M_i. \tag{3b}
\]

For later convenience, we write the input covariance matrices in terms of two precoder matrices, that is, $P_{s,k} = G_{s,k} G_{s,k}^H$ and $P_{i,\ell} = G_{i,\ell} G_{i,\ell}^H$, and let them depend on the statistical CSI, i.e., the knowledge of $\{\rho_{s,k}, T_{s,k}, R_{s,k}\}$ and $\{\rho_{i,\ell}, T_{i,\ell}, R_{i,\ell}\}$, respectively. Thus, denoting $s_{s,k}$ and $s_{i,\ell}$ for independent vectors with i.i.d. zero-mean unit variance entries, we may write $x_{i,k} = G_{s,k} s_{s,k}$ and $x_{i,\ell} = G_{i,\ell} s_{i,\ell}$ without loss of generality. This formulation is especially useful when we consider the optimization of the input covariance for discrete signals. For notational simplicity, we also denote $M_s = \sum_{k=1}^K M_{s,k}$ and $M_i = \sum_{\ell=1}^L M_{i,\ell}$, and rewrite the input-output relation of the resulting MIMO channel as
\[
y = H_s x_s + H_i x_i + n, \tag{4}
\]
where $H_s \triangleq [H_{s,1}, \ldots, H_{s,K}] \in \mathbb{C}^{N \times M_s}$, $H_i \triangleq [H_{i,1}, \ldots, H_{i,L}] \in \mathbb{C}^{N \times M_i}$, $x_{s,k} \triangleq [x_{s,k,1}, \ldots, x_{s,k,K}]^T \in \mathbb{C}^{M_s}$, and $x_{i,\ell} \triangleq [x_{i,\ell,1}, \ldots, x_{i,\ell,L}]^T \in \mathbb{C}^{M_i}$.

B. Problem Statement

Define the instantaneous CSI at the receiver as $H \triangleq \{H_s, H_i\}$. Given that all channels are ergodic and the receiver knows both $H$ and the distribution of $x_s$, we can write down the average mutual information
\[
I(y; x_s) = h(y|H) - h(y|x_s, H), \tag{5}
\]
where the differential entropy terms are given by
\[
h(y|H) = -E_{y,H} \ln E_{x_s,H} p(y|x_s, x_i, H), \tag{6a}
\]
\[
h(y|x_s, H) = -E_{y|x_s,H} \ln E_{x_i,H} p(y|x_s, x_i, H), \tag{6b}
\]
and the conditional distribution of the channel \([33]\) reads
\[
p(y|x_s, x_i, H) = \frac{1}{\pi^N} e^{-\|y - H x_s - H_i x_i\|^2}. \tag{7}
\]
The mutual information in \([3]\) represents an achievable sum-rate of the MIMO-MAC \([1]\) when the receiver does not decode the interference signal $x_i$. Given statistical knowledge of the channels, the sum-rate can then in principle be maximized by designing the precoder matrices $G_{s,k}, \forall k$ and $G_{i,\ell}, \forall \ell$. Unfortunately, the explicit expression for $G_{s,k}$ is not known in general. Moreover, its numerical evaluation is computationally expensive due to averaging of \([6b]\) over the channel realizations. Even more serious difficulty arises when the data symbols are non-Gaussian. In this case, one needs to compute two sums over an exponential number (w.r.t. the numbers of transmit antennas and bits in the constellation) of terms for every realization of the channel.

The aim of the present paper is to find a computationally feasible expression for the ergodic mutual information \([5]\) given arbitrary channel inputs. To make the analysis tractable, we consider the asymptotic regime where the system size grows large and use the replica method to compute the individual entropy terms. These expressions \((vide Section \(III))\) are then used to optimize the covariance matrices so that the mutual information is maximized \((vide Section \(IV))\). Finally, the large system result is used to give an approximation for the original quantity \([5]\) when the system size is finite.

C. Mutual Information and the MMSE of a Fixed MIMO Channel

To finish this section, we discuss the problem of finding the mutual information and the minimum mean squared error (MMSE) of a fixed MIMO channel. These results are used later in the paper to evaluate the asymptotic mutual information obtained via the replica method.

1) General Case: Consider the following multi-antenna communication channel
\[
z = Ax + w, \tag{8}
\]
where $A$ is a fixed $N \times M$ channel matrix and $w \in \mathbb{C}^N$ has i.i.d. ZGC entries of unit variance. The channel inputs $x = Gs \in \mathbb{C}^M$ are a combination of a precoder matrix $G$ and vector $s$ that has i.i.d. zero-mean unit variance entries, with constraint $\text{tr}\{P\} \leq M$ where $E[x x^H] = P$. The conditional distribution of this fixed single-user (su) channel \([4]\) is given by
\[
p_{su}(z|x, A) = \frac{1}{\pi^N} e^{-\|z - A x\|^2}. \tag{9}
\]
The posterior mean estimate of $x$ is denoted $\langle x \rangle \triangleq E\{x|z, A\}$, where the expectation is taken over the posterior density $p_{su}(x|z, A)$ obtained from the prior distribution $p(x)$ and \((9)\) via Bayes’ theorem. For future convenience, the conditional MMSE matrix is defined here through parametrization
\[
\text{mmse}(x, A; p_{su}) \triangleq E_{z,x} \left\{ (x - \langle x \rangle)(x - \langle x \rangle)^H \right\} \in \mathbb{C}^{M \times M}, \tag{10}
\]
where the expectation is w.r.t. the joint distribution $p_{su}(z, x|A)$. Similarly, the mutual information reads
\[
I(z; x|A, p_{su}) = E_{z,x} \ln p_{su}(y|x, A) - E_{z} \ln E_{x} p_{su}(y|x, A), \tag{11}
\]
where the expectations are again w.r.t. $p_{su}(z, x|A)$.

1In this and the following sections, the probabilities that are connected to the transition probabilities of the channel \([3]\) are denoted $p_{su}$. This is to make a clear separation to the probabilities related to the original channel \([4]\).
Below we present two important special cases and provide the corresponding expressions for the mutual information and the MMSE.

**Example 1 (Gaussian inputs).** The MMSE detector becomes linear if the channel input \( x \) is a ZMCSCG vector given \( G \). In this case, the output of the MMSE detector reads

\[
\langle x \rangle = \left( P^{-1} + A^H A \right)^{-1} A^H z,
\]

and the MMSE matrix is given by

\[
\text{mmse}(x, A; p_{su}) = \left( P^{-1} + A^H A \right)^{-1}.
\]

The mutual information reduces also to the well-known formula

\[
I(z; x|A) = \ln \det \left( I_N + AP A^H \right).
\]

**Example 2 (Discrete channel inputs).** Let the entries of \( s \) be independently drawn from a discrete constellation (e.g., QPSK, QAM) of cardinality \( C \), so that \( s \) is uniformly distributed over the set \( \{s_1, \ldots, s_{C^d}\} \). For fixed \( G \) we may then treat also \( x \) as being uniformly drawn from a set \( \{x_1, \ldots, x_{CM} \} \) of possible input vectors. Denoting

\[
p_{su}(z|A) = \frac{1}{C^M} \sum_{i=1}^{C^M} \frac{1}{\pi^M} e^{-\|z-Ax_i\|^2},
\]

the MMSE estimate of \( x \) is by definition given as

\[
\langle x \rangle = \sum_{i=1}^{C^M} x_i p_{su}(z|x_i, A)
\]

The MMSE matrix is thus obtained from

\[
\text{mmse}(x, A; p_{su}) = P - \int \langle x \rangle \langle x \rangle^H p_{su}(z|A)dz,
\]

while the mutual information between \( z \) and \( x \) reads

\[
I(z; x|A, p_{su}) = N + M \ln C
\]

subject to

\[
- \frac{1}{C^M} \sum_{i=1}^{C^M} E_m \left\{ \ln \sum_{j=1}^{C^M} e^{-\|A(x_i - x_j) + w\|^2} \right\}.
\]

2) Parallel Gaussian Channels: Assume that the channel matrix \( A \) is diagonal with real-valued entries \( a_1, \ldots, a_M \). Let \( G \) also be a real diagonal matrix formed by \( g_1, \ldots, g_M \) so that the MIMO channel \( (8) \) decouples into a bank of parallel channels

\[
z_m = a_m x_m + w_m.
\]

The MMSE estimate of \( x_m \) for the \( m \)th channel is then

\[
\langle x_m \rangle = E \{x_m z_m, a_m\},
\]

while the MMSE matrix is diagonal with \( E_{z_m, x_m} \{[x_m - \langle x_m \rangle]^2|a_m\} \) as its \((m, m)\)th element.

The mutual information \([11] \) reduces in this case to a form

\[
I(z; x|A, p_{su}) = \sum_{m=1}^{M} I(z_m; x_m|a_m, p_{su}).
\]

In the following examples we discuss three analytically tractable special cases, which will be useful later in Section [V].

**Example 3 (Gaussian inputs).** In this scenario, we have \( x_m \sim \mathcal{CN}(0, g_m^2) \) so that the MMSE estimate of \( x_m \) becomes

\[
\langle x_m \rangle = \frac{g_m^2 a_m z_m}{1 + g_m^2 a_m^2},
\]

leading to

\[
E_{z_m, x_m} \{[x_m - \langle x_m \rangle]^2|a_m\} = \frac{g_m^2}{1 + g_m^2 a_m^2}.
\]

The mutual information between the input and output of \([19] \), on the other hand, is quantified as

\[
I(z_m; x_m|a_m, p_{su}) = \ln \left( 1 + g_m^2 a_m^2 \right),
\]

so that using \([20] \) we obtain the total achievable sum-rate \([11] \) of a single-user MIMO system with fixed diagonal channel.

**Example 4 (QPSK inputs).** When the prior distribution of the elements of \( s \) is given by \( p(s) = 1/4 \delta(s \pm 1/\sqrt{2} \pm j/\sqrt{2}) \), we have

\[
\langle x_m \rangle = \frac{g_m}{\sqrt{2}} \tanh \left( \sqrt{2} g_m a_m \Re\{z_m\} \right) + j \frac{g_m}{\sqrt{2}} \tanh \left( \sqrt{2} g_m a_m \Im\{z_m\} \right).
\]

Furthermore, the \((m, m)\)th element of the MMSE matrix is given by

\[
E_{z_m, x_m} \{[x_m - \langle x_m \rangle]^2|a_m\} = g_m^2 - \frac{g_m^2}{2\pi} \int_R \tanh \left( g_m^2 a_m^2 - g_m a_m s \right) e^{-2\pi} ds,
\]

and the per-stream mutual information is evaluated as

\[
I(z_m; x_m|a_m, p_{su}) = 2 g_m^2 a_m^2 - \sqrt{\frac{2}{\pi}} \int_R \ln \cosh \left( g_m^2 a_m^2 - g_m a_m s \right) e^{-\frac{\pi^2}{2}} ds.
\]

**Example 5 (16-QAM inputs).** When the elements of \( s \) are uniformly drawn from the standard 16-QAM constellation, the diagonal terms of the MMSE matrix are evaluated as in \([27] \) on the top of the page (the minor typo in \([34] \) \((27) - (28) \) is corrected there). The mutual information is then obtained numerically through the I-MMSE relation \([35] \).

### III. ASYMPTOTIC ACHIEVABLE SUM-RATE

In this section, we present the main findings of the paper, namely, the asymptotic sum-rate of reliable communication
over a multi-access MIMO channel in the presence of interferers. The expression is derived in the large-system limit (LSL), where the numbers of antennas at each terminal grow without bounds at constant ratios, i.e.,

\[
\begin{align*}
\beta_{s,k} M_{s,k} &= N \rightarrow \infty, & \beta_{s,k} &= \text{const}, \\
\beta_{i,\ell} M_{i,\ell} &= N \rightarrow \infty, & \beta_{i,\ell} &= \text{const}.
\end{align*}
\]  

(28a)

(28b)

In the remainder of the section, we first present the asymptotic result for a general (correlated) MIMO channel, and then specialize to the uncorrelated case where the expression are much simpler.

A. General Case

The main result of the paper is given in the following proposition.

** Proposition 1.** Given the input distributions and spatial covariance matrices, the differential entropy \(h(y|\mathcal{H})\) defined in (2b) reads in the LSL

\[
h(y|\mathcal{H}) = \sum_{k=1}^{K} I \left( z_{s,k}; x_{s,k} | A_{s,k}, p_{su} \right) - \sum_{k=1}^{K} M_{s,k} \epsilon_{s,k} \epsilon_{s,k} + \sum_{\ell=1}^{L} I \left( z_{i,\ell}; x_{i,\ell} | A_{i,\ell}, p_{su} \right) - \sum_{\ell=1}^{L} M_{i,\ell} \xi_{i,\ell} \xi_{i,\ell} + \ln \det \left( I_N + \sum_{k=1}^{K} \epsilon_{s,k} R_{s,k} + \sum_{\ell=1}^{L} \epsilon_{i,\ell} R_{i,\ell} \right).
\]

(29)

The parameters \(\epsilon_{s,k}, \xi_{s,k}, \epsilon_{i,\ell}, \text{ and } \xi_{i,\ell}\) satisfy the following set of fixed-point equation:

\[
\begin{align*}
\epsilon_{s,k} &= \frac{1}{M_{s,k}} \text{tr} \left\{ R_{s,k} \left[ I_N + \sum_{k=1}^{K} \epsilon_{s,k} R_{s,k} + \sum_{\ell=1}^{L} \epsilon_{i,\ell} R_{i,\ell} \right]^{-1} \right\}, \\
\epsilon_{i,\ell} &= \frac{1}{M_{i,\ell}} \text{tr} \left\{ R_{i,\ell} \left[ I_N + \sum_{k=1}^{K} \epsilon_{s,k} R_{s,k} + \sum_{\ell=1}^{L} \epsilon_{i,\ell} R_{i,\ell} \right]^{-1} \right\},
\end{align*}
\]

(30a)

(30b)

\[
\begin{align*}
\xi_{s,k} &= \frac{\rho_{s,k}}{M_{s,k}} \text{tr} \{ \text{mmse} (x_{s,k}, A_{s,k}, p_{su}) T_{s,k} \}, \\
\xi_{i,\ell} &= \frac{\rho_{i,\ell}}{M_{i,\ell}} \text{tr} \{ \text{mmse} (x_{i,\ell}, A_{i,\ell}, p_{su}) T_{i,\ell} \},
\end{align*}
\]

(30c)

(30d)

where \(A_{s,k} = \sqrt{\rho_{s,k} \xi_{s,k}} T_{s,k}^{1/2}, A_{i,\ell} = \sqrt{\rho_{i,\ell} \xi_{i,\ell}} T_{i,\ell}^{1/2}\) and the MMSE matrices are obtained using (10).

**Proof:** The proof is given in Appendix 4.

Here the mutual information terms in (29), as well as the terms \(30\) and \(30\), are associated with two different fixed channels given by \(3\) with corresponding channel matrices \(A_{s,k}\) and \(A_{i,\ell}\). Hence, the terms \(\epsilon_{s,k}\) and \(\epsilon_{i,\ell}\) include the MMSE matrix of a fixed single-user channel discussed in

2Since some of the steps in the replica method are still lacking rigorous proof, we refer to the key results of the paper as propositions.

3In general, the fixed-point equations may have more than one solutions. Among these, the one minimizing \(29\) corresponds to entropy \(h(y|\mathcal{H})\). In physics, this phenomenon is referred to as phase coexistence \(30\).

Subsection 11.1 transmit covariances of the original multiuser-MIMO channel \(2\). Despite looking a bit cumbersome, (29) has a simple interpretation; it represents the contributions of both users and interferers to the sum-rate of the MIMO-MAC presented in (4).

**Proposition 2.** Given the input distributions and spatial covariance matrices, the conditional differential entropy \(h(y|x, \mathcal{H})\), given in (6b), in the LSL given by

\[
h(y|x, \mathcal{H}) = \sum_{\ell=1}^{L} I \left( z_{i,\ell}; x_{i,\ell} | A_{i,\ell}, p_{su} \right) + \ln \det \left( I_N + \sum_{\ell=1}^{L} \varpi_{i,\ell} R_{i,\ell} \right) - \sum_{\ell=1}^{L} M_{i,\ell} \xi_{i,\ell} \xi_{i,\ell},
\]

(31)

where parameters \(\varpi_{i,\ell}\) and \(\xi_{i,\ell}\) satisfy the following set of fixed-point equation:

\[
\begin{align*}
\varpi_{i,\ell} &= \frac{1}{M_{i,\ell}} \text{tr} \left\{ R_{i,\ell} \left[ I_N + \sum_{\ell=1}^{L} \varpi_{i,\ell} R_{i,\ell} \right]^{-1} \right\}, \\
\xi_{i,\ell} &= \frac{\rho_{i,\ell}}{M_{i,\ell}} \text{tr} \{ \text{mmse} (x_{i,\ell}, A_{i,\ell}, p_{su}) T_{i,\ell} \},
\end{align*}
\]

(32a)

(32b)

and we denoted \(A_{i,\ell} = \sqrt{\rho_{i,\ell} \xi_{i,\ell} T_{i,\ell}^{1/2}}\).

**Proof:** The proof is given in Appendix 3.

The mutual information term in (31), as well as the MMSE-like term (32), are again associated with a single-user MIMO system (8) whose channel matrix is fixed and given as \(A_{i,\ell}\). The above entropy describes the amount of information discarded at the receiver due to noise and interference removal.

Here we emphasize the difference from the MAC system studied in (29), where interferers were absent and white noise was the only source of disturbance. In contrast, our results, given in Propositions 1 and 2 describe the sum-rate of the MAC in the presence of interference. In the case of a single user and one interferer, both using Gaussian signals, the above result directly reduces to the mean mutual information reported in (30), where it was obtained in a slightly different way. Note that in contrast to (30), our result is not restricted to Gaussian channel inputs, while the authors there computed also higher moments of mutual information.

B. Uncorrelated Channels

The next result provides the sum-rate for the special case where \(T_{s,k}, T_{i,\ell}, R_{s,k}, R_{i,\ell}, G_{s,k}\) and \(G_{i,\ell}\) are all identities.

**Corollary 1.** In the LSL, the asymptotic average sum-rate of an uncorrelated multi-access MIMO channel as given in (4) in the presence of interference is given by (33) on the top of the next page, where parameters \(\epsilon_{s,k}, \xi_{s,k}, \epsilon_{i,\ell}, \xi_{i,\ell}, \varpi_{i,\ell}, \xi_{i,\ell}\) are all identities.
I(y; x) = \sum_{k=1}^{K} M_{s,k} I(z_{s,k}; x_{s,k} | \sqrt{p_{s,k}} \xi_{s,k}, p_{su}) + \sum_{\ell=1}^{L} M_{t,\ell} I(z_{t,\ell}; x_{t,\ell} | \sqrt{p_{t,\ell}} \xi_{t,\ell}, p_{su}) + N \ln \left(1 + \sum_{k=1}^{K} \varepsilon_{s,k} + \sum_{\ell=1}^{L} \varepsilon_{t,\ell}\right) - N \ln \left(1 + \sum_{k=1}^{K} \varepsilon_{s,k} + \sum_{\ell=1}^{L} M_{s,k} \xi_{s,k} \varepsilon_{s,k} \right), (33)

satisfy the following set of fixed-point equations

\begin{align}
\xi_{s,k} &= \beta_{s,k} \left(1 + \sum_{k=1}^{K} \varepsilon_{s,k} + \sum_{\ell=1}^{L} \varepsilon_{t,\ell}\right)^{-1}, \tag{34a}

\xi_{t,\ell} &= \beta_{t,\ell} \left(1 + \sum_{k=1}^{K} \varepsilon_{s,k} + \sum_{\ell=1}^{L} \varepsilon_{t,\ell}\right)^{-1}, \tag{34b}

\varepsilon_{s,k} &= \frac{\rho_{s,k}}{M_{s,k}} \text{tr}\left\{\text{mmse}(x_{s,k}, \sqrt{p_{s,k}} \xi_{s,k} I_{M_{s,k}}; p_{su})\right\}, \tag{34c}

\varepsilon_{t,\ell} &= \frac{\rho_{t,\ell}}{M_{t,\ell}} \text{tr}\left\{\text{mmse}(x_{t,\ell}, \sqrt{p_{t,\ell}} \xi_{t,\ell} I_{M_{t,\ell}}; p_{su})\right\}, \tag{34d}

\tilde{\xi}_{t,\ell} &= \beta_{t,\ell} \left(1 + \sum_{\ell=1}^{L} \varepsilon_{t,\ell}\right)^{-1}, \tag{34e}

\tilde{\varepsilon}_{t,\ell} &= \frac{\rho_{t,\ell}}{M_{t,\ell}} \text{tr}\left\{\text{mmse}(x_{t,\ell}, \sqrt{p_{t,\ell}} \xi_{t,\ell} I_{M_{t,\ell}}; p_{su})\right\}, \tag{34f}
\end{align}

and the mutual information terms are obtained using (20).

Proof: The proof follows directly from Propositions 1 and 2. The same result was also reported in our previous work [31].

IV. PRECODER OPTIMIZATION

As each transmitter has statistical CSI, by carefully choosing the precoder matrix, the transmitters could, in principle, maximize the sum mutual information between the inputs and outputs of their channels. The corresponding optimization problem is described as

\begin{align}
\max_{G_{s,k}} &\quad I(z_{s,k}; x_{s,k} | A_{s,k}, p_{su}) \\
\text{s.t.} &\quad \text{tr}\{G_{s,k}G_{s,k}^H\} \leq M_{s,k}, \quad k \in \mathcal{K} \tag{35}
\end{align}

a set of individual per-transmitter optimization problems

\begin{align}
\max_{G_{s,k}} &\quad I(z_{s,k}; x_{s,k} | A_{s,k}, p_{su}) \\
\text{s.t.} &\quad \text{tr}\{G_{s,k}G_{s,k}^H\} \leq M_{s,k}, \quad k \in \mathcal{K} \tag{36}
\end{align}

where \( A_{s,k} = \sqrt{\rho_{s,k}} \xi_{s,k} T_{s,k}^{1/2} \). Namely, each transmitter \( k \) adjusts its own precoder matrix \( G_{s,k} \) according to its own transmit correlation matrix \( T_{s,k} \), which is available by the statistical-CSI assumption.

Note that, in principle, parameters \( \varepsilon_{s,k}, \xi_{s,k}, \varepsilon_{t,\ell}, \xi_{t,\ell}, \tilde{\varepsilon}_{t,\ell}, \tilde{\xi}_{t,\ell} \) depend on the precoder matrices in \( G_{s,k} \). To obtain a feasible point satisfying the KKT conditions [37] the derivatives of the fixed-point parameters w.r.t. the precoder matrices have to be zero. However, due to the particular way the analysis in Appendix A was carried out, these parameters constitute a saddle-point for a min-max problem with a similar objective as in (36) (vide (64) in the Appendix). Hence, the corresponding derivatives w.r.t. the parameters in question are zero and we may consider them as being independent of \( G_{s,k} \).

Albeit the above optimization problem is, in general, non-convex, it can still be efficiently solved for the following (most practically relevant) special cases.

A. Gaussian Inputs

In the case of Gaussian channel inputs, it is convenient to work with covariance matrices instead of precoders since the objective function of the optimization problem (36) reduces to

\begin{align}
I(z_{s,k}; x_{s,k} | A_{s,k}, p_{su}) = \ln \det \left(I_{M_{s,k}} + A_{s,k} P_{s,k} A_{s,k}^H\right). \tag{37}
\end{align}

Let the singular-value decomposition (SVD) of the effective fixed channel be given by \( A_{s,k} = U_{A_{s,k}} \Sigma_{A_{s,k}} V_{A_{s,k}}^H \), where \( U_{A_{s,k}} \) and \( V_{A_{s,k}} \) are orthonormal matrices and \( \Sigma_{A_{s,k}} = \text{diag}(\{\sigma_1(A_{s,k}), \ldots, \sigma_{M_{s,k}}(A_{s,k})\}) \) is the matrix with the singular values on the diagonal. Given the solution to the fixed-point equations (\( \xi_{s,k} \) and \( \varepsilon_{s,k} \)), the optimal input covariance matrix is then given by the water-filling solution [38]

\begin{align}
P_{s,k}^* = V_{A_{s,k}} \Sigma_{P_{s,k}} V_{A_{s,k}}^H, \tag{38}
\end{align}

where \( \Sigma_{P_{s,k}} \) is a diagonal matrix whose non-zero entries are

\begin{align}
[\Sigma_{P_{s,k}}]_{m,m} = \left[\frac{1}{\nu} - \frac{1}{\xi_{s,k} \sigma_m(A_{s,k})}\right]^+ , \tag{39}
\end{align}

where \( \nu \) is chosen so that the power constraint \( \text{tr}\{P_{s,k}\} = M_{s,k} \) is satisfied.

We remark that, as pointed out in [18], to obtain the optimal transmit covariance matrix one has to iterate the solution to
the fixed-point equations with the above statistical water-filling until the stopping criterion is reached.

B. Discrete Inputs

Unlike the previous case, finding the optimal precoder for discrete constellations is a difficult task. For such cases [40], there is no longer a convex optimization problem. It has been shown in [39], [40] that the mutual information is a concave function in the quadratic form $F_{s,k} \triangleq A_{s,k}G_{s,k}G_{s,k}^H A_{s,k}^H$; yet, due to the power constraint, $\text{tr}(G_{s,k}G_{s,k}^H) = M_{s,k}$, one cannot directly apply convex optimization tools for solving the problem. For instance, when using the gradient ascent method for solving (36), one updates $F_{s,k}$ iteratively as

$$F_{s,k}^{(l+1)} = F_{s,k}^{(l)} + \mu_F \Delta F_{s,k},$$

with $\mu_F$ being the step size and $\Delta F_{s,k}$ being the gradient of the mutual information $I(z_{s,k};x_{s,k}|A_{s,k},p_{su})$ w.r.t. $F_{s,k}$. It is shown in [41] that the gradient of the mutual information in the single-user setup is

$$\nabla_F I(z_{s,k};x_{s,k}|A_{s,k},p_{su}) = E_{s,k},$$

where we denoted for notational simplicity $E_{s,k} \triangleq \text{mmse}(x_{s,k},A_{s,k};p_{su})$ for the MMSE matrix defined in (10). However, in practice, the precoder matrix $G_{s,k}$ is updated subject to a power constraint, which limits its feasible region and complicates the problem.

Here we apply an algorithm similar to that proposed in [39], [40], based on the alternating optimization between the following two subproblems.

1) Per-Eigenmode Power Allocation: Let the SVD of the precoder matrix be given by $G_{s,k} = U_{G_{s,k}} \Sigma_{G_{s,k}} V_{G_{s,k}}^H$. For fixed $V_{G_{s,k}}$ the first subproblem is

$$\max_{\Sigma_{G_{s,k}}} I(z_{s,k};x_{s,k}|A_{s,k},p_{su})$$

s.t. $\text{tr}(\Sigma_{G_{s,k}}^2) \leq M_{s,k}$

$$\Sigma_{G_{s,k}}^2 = 0_{M_{s,k}}.$$  

According to [42], this problem is convex, provided that the precoder has optimal structure.

Since the matrix of interest, $\Sigma_{G_{s,k}}^2$, is a diagonal matrix, we introduce for notational convenience a vector $g_k$, such that $\Sigma_{G_{s,k}}^2 = \text{diag}(g_k)$. We then choose an initial value for $g_k$, e.g., $g_k^{(1)} = 1/M_{s,k}$, and perform the gradient update $g_k^{(l+1)} = g_k^{(l)} + \mu g_{l} \left( \text{Diag}(\Sigma_{G_{s,k}}^2)^{-1} F_{s,k} V_{G_{s,k}} V_{G_{s,k}}^H - \gamma 1_{M_{s,k}} \right)$, with $\gamma = 1/M_{s,k}$, $1_{M_{s,k}}$ as the identity matrix, $\text{Diag}(\Sigma_{G_{s,k}}^2)^{-1}$ being an appropriately chosen step size, e.g., obtained by the backtracking line search algorithm [37]. If $g_k^{(l+1)}$ has negative entries, one sets those to zero and renormalizes $g_k^{(l+1)}$ so that the power constraint is satisfied and then sets $\Sigma_{G_{s,k}}^2 = \text{diag}(g_k^{(l+1)})$.

2) Optimization of the Eigenvectors of $F_{s,k}$: In this subproblem, for fixed $\Sigma_{G_{s,k}}^2$ we optimize the eigenvectors of the quadratic form $F_{s,k} = A_{s,k}G_{s,k}G_{s,k}^H A_{s,k}^H$. Let $A_{F_{s,k}}$ be the diagonal matrix, whose entries are eigenvalues of $F_{s,k}$. The second subproblem is then formulated as

$$\max_{F_{s,k}} I(z_{s,k};x_{s,k}|A_{s,k},p_{su})$$

s.t. $A_{F_{s,k}} = \Sigma_{G_{s,k}}^2 \Sigma_{G_{s,k}}^2$.

We remind the reader that $\text{Diag}(A)$ denotes a column vector containing the diagonal entries of matrix $A$, whereas $\text{diag}(a)$ denotes a diagonal matrix containing the coefficients of vector $a$. 

Fig. 2. Average mutual information per dimension vs. SNR for the single-user single-interferer scenario. Both, the user and interferer, have the same type of signaling. The terminals are equipped with $M = N = 4$ antennas. Solid curves denote analytic results, markers denote the results of Monte-Carlo simulation.

Fig. 3. Average mutual information per dimension vs. the inverse of the number of antennas $M = N$ ($M, N \in \{4, \ldots, 11\}$) at terminals for both Gaussian and QPSK signaling schemes at $\rho = 10$ dB. The star markers at $1/M = 0$ denote the predictions obtained by the replica analysis.
The gradient of the mutual information is given by (41), and hence the gradient update for $F_{s,k}$ becomes

$$F_{s,k}^{(l+1)} = F_{s,k}^{(l)} + \mu F_{s,k}.$$  \hfill (45)

The obtained update has to be further projected into a matrix with the prescribed eigenvalues, which is as close to $F_{s,k}^{(l+1)}$ as possible [39], [40].

V. NUMERICAL RESULTS

In this section, we provide numerical results alongside with some discussion. For the simulations, the spatial correlation at the transmitter side is assumed to be generated by a uniform linear antenna array with Gaussian power azimuth spectrum [30]. Hence, correlation matrices $(T_{s,k}$ and $T_{i,l})$ consist of entries given by

$$(T)_{a,b} = \frac{1}{2\pi\delta^2} \int_{-\pi}^{\pi} e^{2\pi i d_{\lambda}(a-b)\sin(\phi) - \frac{(a-b)^2}{2\delta^2}} d\phi,$$ \hfill (46)

where $d_{\lambda}$ is the nearest neighbor antenna spacing (in wavelengths $\lambda$), $\theta$ is the mean angle and $\delta^2$ is the mean-square angle spread. For the sake of simplicity, we assume that there is no correlation at the receiver side, that is, $R_{s,k} = I_{M_s,k}, \forall k$ and $R_{i,l} = I_{M_i,l}, \forall l$.

A. Uncorrelated Channels

To begin, we complement the obtained expression [33] for the uncorrelated case with Monte-Carlo simulation [43]. We consider the setup, where a single user transmits its signal towards the receiver in the presence of a single interferer. All terminals have equal numbers of antennas, that is, $N = M_s = M_i = M$. Both the user and interferer utilize the same type of signals (Gaussian or QPSK), and have the same total transmit power, that is, $\rho_s = \rho_i = \rho$. In Fig. 2 we plot the average mutual information per transmit antenna as a function of SNR. Both the asymptotic results obtained via the replica method and Monte Carlo simulations for $M = 4$ antennas are shown. For QPSK, the simulations and asymptotic results are the farthest apart at SNRs around $\rho = 10$ dB due to the phase transition phenomenon. Namely, in this region the system instantly switches from one state to another, mimicking the “water-ice” transition in physics [11]. For Gaussian inputs, the plotted curve does not experience a phase transition and the asymptotic results are accurate already for small numbers of antennas.

To illustrate how the small scale simulations converge to the asymptotic result obtained using the replica method, Fig. 3 plots the simulated values of the mutual information [4] vs. $1/M$ for $M \in \{4, \ldots, 11\}$ at $\rho = 10$ dB. The markers at $1/M = 0$ represent the results obtained using Corollary [1] and quadratic curves are fitted to the simulated data using non-linear least-squares regression. From the extrapolation we observe that the simulated per-antenna mutual information approaches close to the value predicted by the replica analysis also in the region nearby the phase transition.

Next, we consider the effect of signal constellations on the achievable sum-rate. Fig. 4 depicts the mutual information of the desired user when the Gaussian and/or QPSK signaling are used by the terminals. The interference-free case is also drawn for comparison. We directly see that for the desired user it is always best to employ Gaussian signaling. On the other hand, Gaussian signaling, when used by the interferer, creates more disturbance. Hence, in a cellular system where inter-cell interference is present, the network might be able to handle more users if some of them are assigned discrete constellations. This is due to the fact that the most severe (unoptimized) interference is in fact Gaussian [45].

Fig. 5 presents the average mutual information per transmit antenna of a single user in the presence of $L = 1, 2, 3$ interferers using different signaling schemes (Gaussian, QPSK).

$^4$Note that here we do not consider the optimization of the interferer’s signal constellation aiming to jam the user. In the latter case, Gaussian signaling would not cause the worst-case interference, whereas an optimized discrete signaling would degrade the user’s performance the most [44].
interference. The terminals have L interferers with QPSK create smaller performance degradation than L interferers with 16-QAM. Again, we see that Gaussian signaling causes the worst-case degradation in the desired user’s performance.

B. Correlated Channels

In this section, we study the behavior of the system under spatial correlation and quantify the gains of precoding. Fig. 6 depicts the normalized ergodic mutual information as a function of SNR of a single-user (no interference) MIMO channel with N = M = 3 antennas under various conditions. Namely, we consider the cases of correlated and uncorrelated channels with and without precoding at the transmitter. The transmit side correlation parameters are set to all the terminals as follows. The antenna spacing is set to d = 1, the mean angle is set to θ = 0° and the root-mean-square angle spread is chosen to be δ = 5°. The receive side correlation matrix is set to identity, i.e., R = I_N.

Fig. 6(a) plots the case of Gaussian inputs. As expected, at high SNR transmit correlation decreases the achievable rate regardless of precoding. However, quite remarkably, at low SNR transmit correlation together with precoding, based on the statistical water-filling (38), is beneficial in terms of the mutual information. From Fig. 6(b) we observe somewhat similar behavior for the case of QPSK signals. At low SNR a precoder in combination with transmitter-side correlation allows for improving the performance of the system. However, since the per-stream mutual information saturates at 2 bits/cu, transmit correlation does not affect the rates too much at high SNR. To optimize the precoder matrix for the case of QPSK signals, we use the algorithm described in Subsection IV-B.

For the aforementioned set of the transmit and receive side correlation parameters, the obtained optimal precoder reads

\[
G^*_s = \begin{bmatrix}
0.2251 & -0.8723 & 0.4342 \\
0.2251 & -0.8723 & 0.4342 \\
0.2251 & -0.8723 & 0.4342
\end{bmatrix}
\]

Next, we investigate the performance of a MIMO-MAC with correlated channels. The rate region of a generic K-user MIMO-MAC using Gaussian signaling is given by

\[
\mathcal{C}_{\text{MAC}} = \bigcup_{\left\{ \text{tr}(P_{s,k}) \leq M_k, P_{s,k} \succeq 0_{M_k, M_k} \right\} \forall k \in \mathcal{K}} \left\{ \{R_k\}, \forall k \in \mathcal{K} : \sum_{i \in \mathcal{S}} \ln \det \left( I_N + \sum_{i \in \mathcal{S}} H_{s,i} P_{s,i} H_{s,i}^H \right), \forall \mathcal{S} \subseteq \mathcal{K} \right\}.
\]
Note that the corresponding large-system ergodic mutual information terms can be directly obtained from Proposition 1. To illustrate this region, we consider a symmetric setup with two users who both have $M = M_{u,1} = M_{u,2} = 3$ antennas. We further fix the available transmit powers $\rho = \rho_{u,1} = \rho_{u,2}$ to $\rho \in \{0, 20\}$ dB and evaluate the achievable rate regions for the given 2-user MAC. The result is depicted in Fig. 7 where both uncorrelated and correlated channels with and without precoding are present. It is clear that using precoding at both terminals is beneficial when transmit correlation is present. As expected though, at high SNR the rate region is largest for the uncorrelated MAC. On the contrary, at low SNR the rate region is achieved in the presence of correlation and optimal precoding.

To finish this section, we return to the case of one desired user and an interfering interfering terminal that carries out its own precoding. We add an interferer with $N_{i} = 3$ antennas and having the same transmit power $\rho_i = \rho$ and correlation $\rho_{u}=\rho_i$ and correlation $\rho_{i}$. Fig. 8 depicts the average mutual information as a function of SNR for this scenario under Gaussian and QPSK signaling schemes. Both the user and interferer either do or do not realize precoding. Note that the scenario is symmetric and hence the precoders used by the terminals are the same. Moreover, the terminals adapt to their own correlation matrices aiming to increase their own rates. From Fig. 8(a) we see that, quite expectedly, for the case of Gaussian signals, precoding at the user increases its own ergodic rate. At the same time, we also observe that utilizing the optimal precoder at the interferer results in higher rate at the user’s terminal at high SNR, degrading the performance of the latter in the other SNR region only slightly. Similar behavior is observed for the case of QPSK inputs (vide Fig. 8(b)), apart from expected saturation at 2 bits/cu at high SNR.

VI. CONCLUSIONS

In this paper, we derived an explicit expression for the asymptotic achievable sum-rate of the MIMO multiple-access channel in the presence of interference. The result accounts for the spatial correlation at the terminals and, in contrast to the previous results, is not restricted to Gaussian signals. Although derived in the large system limit, it approximates relatively well the achievable sum-rate of small systems. We have also studied the impact of the number of interferers, their signaling scheme and spatial correlation structure on the system performance. For instance, Gaussian signaling is seen to create the worst-case (unoptimized) interference. Thus, the system may handle more interferers if they use discrete signal constellation, as compared to the case of Gaussian interferers. The obtained large-system approximation has been further used to find precoder matrices for maximizing the sum-rate for both Gaussian and finite-alphabet signaling schemes. It has been demonstrated that properly optimized precoder significantly increases the achievable rates. More interestingly, in the low-SNR region the presence of spatial correlation, in combination with an optimal precoder, is beneficial and can in fact improve the system performance as compared to the case of uncorrelated channels. The proposed approach is general and degenerates to many well-known results as special cases.

APPENDIX

In general, direct computation of (6a) and (6b) is very difficult if we allow arbitrary channel input distributions. To overcome this obstacle we use the replica trick to compute the entropies in the large system limit by considering them as two separate systems. To stay coherent with the existing work, we partially keep the statistical physics terminology but avoid unnecessary jargon whenever possible.

Let us concentrate on the first term (6a), and define the partition function related to it as

$$Z(y, H) \triangleq \mathbb{E}_{x, x, y} \left\{ \frac{1}{\pi N} e^{-\|y - H x, x - H x, x\|^2} \right\} . \quad (49)$$

In statistical physics, virtually all interesting macroscopic quantities can be derived from the partition function of the system. Often, however, it is more convenient to work with the logarithm of the partition function, or free energy instead. If we further average the free energy w.r.t. the remaining randomness in the MIMO setup (49), we get

$$F \triangleq -\mathbb{E}_{y, H} \ln Z(y, H) ,$$

that is just the entropy $h(y|H)$. Then, we take the first step towards making the evaluation of $F$ solvable by writing

$$F = -\lim_{u \to 0^+} \frac{\partial}{\partial u} \ln \mathbb{E}_{y, H} \{Z^u(y, H)\} , \quad (51)$$

and implicitly assume that the system size also grows large as discussed in Section III. This identity is exact when $u$ is a real number but on its own does not solve the problem. Thus, we invoke the replica trick and write the under-log term as

$$\mathbb{E}_{y, H} \{Z^u(y, H)\} = \mathbb{E}_{x, x, y, u} \left\{ \int \frac{1}{\pi N} \prod_{a=0}^u e^{-\|y - H x(s), x(s) - H x_i, x\|_2^2} \, dy \right\} , \quad (52)$$

where $x(s)$ and $x_i(s)$ are the $a$th replica vectors of the signals transmitted by the users and interferers. The distributions $p(x(s))$ and $p(x_i(s))$ are identical to $p(x)$ and $p(x_i)$, respectively, and conditionally independent for $a = 0, 1, \ldots, u$ given $y$ and $H$. For ease of exposition, we also defined the following vectors containing the replicated vectors of desired users $X_s \triangleq [x_s(0)^T, \ldots, x_s(u)^T]^T \in \mathbb{C}^{KM, (u+1)}$ and interferers $X_i \triangleq [x_i(0)^T, \ldots, x_i(u)^T]^T \in \mathbb{C}^{LM, (u+1)}$.

After applying the replica trick, the problem of finding the free energy reduces to evaluating $\mathbb{E}_{y, H} \{Z^u(y, H)\}$ for integer-valued $u$ using techniques from large deviations theory and then assuming that the result generalizes to real positive values, at least in the vicinity of zero.\footnote{Note that mathematical rigor of this step is still an open problem. However, some results obtained by the replica method are confirmed to match the ones derived via systematic approaches (e.g. [47, 60]). Moreover, the results can be furthermore verified via Monte-Carlo simulations, as we saw in Section V. Therefore, we consider the replica analysis as a valid mathematical tool.} Completely analogous formulation can be made for the second entropy term (6b).

The details of how to compute these terms are given in the following two sections.
Fig. 8. Average mutual information per dimension vs. SNR for a single-user correlated MIMO channel with / without precoding in the presence of a single interferer with / without precoding. The terminals have \( M = N = 3 \) antennas.

A. Proof of Proposition 7

Let us concentrate on the first term, \( h(y|\mathcal{H}) \), and define the following set of random vectors

\[
\begin{align*}
\psi^{(a)}_{i,k} &\triangleq \sqrt{\frac{\rho_{i,k}}{M_{i,k}^2}} H_{i,k} x^{(a)}_{i,k} \in \mathbb{C}^N, \\
v^{(a)}_{i,\ell} &\triangleq \frac{\rho_{i,\ell}}{M_{i,\ell}^2} H_{i,\ell} x^{(a)}_{i,\ell} \in \mathbb{C}^N.
\end{align*}
\]

Denote also \( v^{(a)}_s \triangleq \sum_{k=1}^K \psi^{(a)}_{s,k} \) and \( v^{(a)}_v \triangleq \sum_{\ell=1}^L v^{(a)}_{v,\ell} \), and group them into a big vector

\[
V \triangleq [v^{(0)}_s, v^{(0)}_v, \ldots, v^{(u)}_s, v^{(u)}_v]^T \in \mathbb{C}^{(u+1)N}.
\]

Conditioned on the channel inputs \( X_s \) and \( X_v \), we know by the central limit theorem that as the dimensions of the channel matrices \( H_{s,k} \) and \( H_{v,\ell} \) grow large, \( V \) converges to a zero-mean Gaussian random vector with conditional covariance

\[
Q = \sum_{k=1}^K (Q_{s,k} \otimes R_{s,k}) + \sum_{\ell=1}^L (Q_{v,\ell} \otimes R_{v,\ell}),
\]

whose corresponding entries are given by

\[
\begin{align*}
[Q_{s,k}]_{a,b} &= \frac{\rho_{s,k}}{M_{s,k}^2} x^{(b)H}_{s,k} T_{s,k} x^{(a)}_{s,k}, \\
[Q_{v,\ell}]_{a,b} &= \frac{\rho_{v,\ell}}{M_{v,\ell}^2} x^{(b)H}_{v,\ell} T_{v,\ell} x^{(a)}_{v,\ell},
\end{align*}
\]

for \( a, b \in \{0, 1, \ldots, u\} \). Note that we used (52) and the assumption that \( W_{s,k} \) and \( W_{v,\ell} \) have i.i.d. ZMCSCG entries of unit variance to derive the result.

Let us define \( Q \triangleq \{(Q_{s,k}, Q_{v,\ell}) : \forall k, \ell\} \), so that the expectation over the replicated vectors \( X_s \) and \( X_v \) may be rewritten as an integral over a probability measure of \( Q \). Treating \( V \) as a Gaussian random vector, it can be shown via the Edgeworth expansion that in the LSL (27)

\[
\mathbb{E}_{y,\mathcal{H}} \{ Z^u(y, \mathcal{H}) \} = \int e^{G^u(Q)} d\mu^u(Q),
\]

where we have omitted constants terms and \( \mu^u(Q) \) reads

\[
\begin{align*}
\mu^u(Q) &= \mathbf{E} \left\{ \prod_{a,b=0}^u \prod_{k=1}^K \delta\left( \rho_{s,k} x^{(b)H}_{s,k} T_{s,k} x^{(a)}_{s,k} - M_{s,k} Q_{s,k} a,b \right) \right. \\
&\quad \times \prod_{\ell=1}^L \delta\left( \rho_{v,\ell} x^{(b)H}_{v,\ell} T_{v,\ell} x^{(a)}_{v,\ell} - M_{v,\ell} Q_{v,\ell} a,b \right) \right\}. \quad (58)
\end{align*}
\]

The above expectation is w.r.t. \( \{X_s, X_v, \mathcal{H}\} \) and \( \delta(\cdot) \) denotes the Dirac delta function. If we plug \( V \) into (52) and assess the expectations w.r.t. \( V \) and \( y \) using Gaussian integration, we get

\[
G^u(Q) = -Nu \ln \pi - N \ln(u+1) - \ln \det \left( I_{N(u+1)} + \sum_{k=1}^K Q_{s,k} \otimes R_{s,k} + \sum_{\ell=1}^L Q_{v,\ell} \otimes R_{v,\ell} \right),
\]

where \( \Sigma \triangleq I_{u+1} - \frac{1}{u+1} I_{u+1} I_{u+1}^T \in \mathbb{R}^{(u+1) \times (u+1)} \).

To compute the integral in (57), we note that since both \( Q_{s,k} \) and \( Q_{v,\ell} \) are formed by summing independent random variables, the measure (58) satisfies the large deviations property and by Varadhan’s theorem [48]

\[
\ln \mathbb{E}_{y,\mathcal{H}} \{ Z^u(y, \mathcal{H}) \} \rightarrow \max_{\mathcal{Q}} \left\{ G^u(Q) - I^u(Q) \right\},
\]

in the LSL. The second term inside the maximization is called the rate function and can be obtained via Cramér’s theorem [48]

\[
I^u(Q) = \max_{\mathcal{Q}} \left\{ M_{s} \sum_{k=1}^K \text{tr}\{Q_{s,k} Q_{s,k}\} \right. \right.
\]

\[
+ \left. M_{v} \sum_{\ell=1}^L \text{tr}\{Q_{v,\ell} Q_{v,\ell}\} - \ln M^u(Q) \right\}, \quad (61)
\]

where \( \mu^u(Q) \) reads

\[
\begin{align*}
\mu^u(Q) &= \mathbf{E} \left\{ \prod_{a,b=0}^u \prod_{k=1}^K \delta\left( \rho_{s,k} x^{(b)H}_{s,k} T_{s,k} x^{(a)}_{s,k} - M_{s,k} Q_{s,k} a,b \right) \right. \\
&\quad \times \prod_{\ell=1}^L \delta\left( \rho_{v,\ell} x^{(b)H}_{v,\ell} T_{v,\ell} x^{(a)}_{v,\ell} - M_{v,\ell} Q_{v,\ell} a,b \right) \right\}. \quad (58)
\end{align*}
\]
where the moment-generating function of $\mu^{(u)}(Q_s, Q)$ reads
\[
M^{(u)}(\tilde{Q}) = E_{X_s, \hat{X}} \left\{ \prod_{k=1}^{K} e^{\rho_k X_{s,k}^u(Q_{s,k} \otimes T_{s,k})X_{s,k}} \times \prod_{\ell=1}^{L} e^{\rho_\ell \hat{X}_{\ell}^u(\tilde{Q}_{\ell} \otimes T_{\ell})X_{\ell}} \right\}, \tag{62}
\]
and we denoted $X_{s,k} \triangleq [x_{s,k}^{(0)T}, \ldots, x_{s,k}^{(u)T}]^T \in \mathbb{C}^{M(u+1)}$, $X_{\ell} \triangleq [\bar{x}_{\ell}^{(0)T}, \ldots, \bar{x}_{\ell}^{(u)T}]^T \in \mathbb{C}^{M(u+1)}$. As before, we group the auxiliary “Q-matrices” as $\tilde{Q} \triangleq \{(Q_{s,k}, \tilde{Q}_{\ell}); \forall k, \ell\}$.

To make the optimization problems in (60) and (61) tractable, we next assume that the saddle-point solutions are invariant under the permutation of the replica indices. This is known as the replica symmetric (RS) ansatz and here implies that we can write the members of $Q$ and $\tilde{Q}$ as
\[
Q_{s,k} = \bar{q}_{s,k} I_{s+1} + (p_{s,k} - \bar{q}_{s,k}) I_{s+1}, \tag{63a}
\]
\[
\tilde{Q}_{s,k} = \tilde{q}_{s,k} I_{s+1} + (\tilde{p}_{s,k} - \tilde{q}_{s,k}) I_{s+1}, \tag{63b}
\]
\[
Q_{\ell} = \bar{q}_{\ell} I_{s+1} + (\tilde{p}_{\ell} - \bar{q}_{\ell}) I_{s+1}, \tag{63c}
\]
\[
\tilde{Q}_{\ell} = \tilde{q}_{\ell} I_{s+1} + (\tilde{p}_{\ell} - \tilde{q}_{\ell}) I_{s+1}. \tag{63d}
\]
Under the RS assumption, the free energy becomes in the LSL
\[
F = N(1 + \ln \pi) + \lim_{u \to 0^+} \frac{\partial}{\partial u} \min_{\tilde{Q}} \sum_{i=1}^{3} T_{3}^{(u)}(Q, \tilde{Q}), \tag{64}
\]
where the term of $T_{3}^{(u)}(Q, \tilde{Q})$ are given at the top of the next page where we denoted $A_{s,k} \triangleq \sqrt{\rho_k \bar{q}_{s,k} T_{s,k}^{1/2}}$, $B_{\ell} \triangleq \sqrt{\tilde{p}_{\ell} \tilde{q}_{\ell} T_{\ell}^{1/2}}$, $B_{s,k} \triangleq \rho_k \bar{q}_{s,k} T_{s,k}$ and $B_{\ell} \triangleq \rho_{\ell} \tilde{q}_{\ell} T_{\ell}$. We also assumed there that all terminals have independent channel inputs. Using the Hubbard-Stratonovich transform [51, 52] on (65c) to decouple the quadratic terms gives
\[
T_{3}^{(u)}(Q, \tilde{Q}) = \frac{1}{\pi M} \int_{\mathbb{C}^M} E_{x_{s,k}} \left\{ e^{-\|z_{s,k} - A_{s,k} x_{s,k}\|^2} e^{x_{s,k}^u B_{s,k} x_{s,k}} \right\} \times \left[ E_{z_{s,k}} \left\{ e^{2Re(z_{s,k}^u A_{s,k} x_{s,k}) x_{s,k}^u (A_{s,k}^2 - B_{s,k}) x_{s,k}} \right\} \right] u d z_{s,k} \nonumber \times \left[ E_{z_{s,k}} \left\{ e^{-\|z_{s,k} - A_{s,k} x_{s,k}\|^2} e^{x_{s,k}^u B_{s,k} x_{s,k}} \right\} \right] u d z_{s,k} \nonumber \times \left[ E_{z_{s,k}} \left\{ e^{2Re(z_{s,k}^u A_{s,k} x_{s,k}) x_{s,k}^u (A_{s,k}^2 - B_{s,k}) x_{s,k}} \right\} \right] u d z_{s,k}, \tag{66}
\]
where $z_{s,k} \in \mathbb{C}^M$ and $z_{s,k} \in \mathbb{C}^M$ are auxiliary variables.

To solve (64), we find the conditions under which the derivatives of the argument w.r.t. all RS parameters vanish. After taking $u \to 0$, we get
\[
\bar{p}_{s,k} = 0, \quad \forall k \in \{1, \ldots, K\}, \tag{67a}
\]
\[
\bar{p}_{\ell} = 0, \quad \forall \ell \in \{1, \ldots, L\}, \tag{67b}
\]
\[
\tilde{q}_{s,k} = \frac{\rho_k}{\tilde{M}_{s,k}} \text{tr} \{ S^{-1} R_{s,k} \}, \tag{67c}
\]
where $\rho_k \triangleq \sqrt{\rho_k \bar{q}_{s,k} T_{s,k}^{1/2}}$, $\tilde{M}_{s,k} \triangleq \sqrt{\tilde{p}_{\ell} \tilde{q}_{s,k} T_{s,k}^{1/2}}$ and $\tilde{M}_{\ell} \triangleq \rho_{\ell} \tilde{q}_{\ell} T_{\ell}^{1/2}$. Clearly, the Hubbard-Stratonovich trans-

\[\text{This assumption has been widely accepted in the field of statistical physics [50] and information theory [11, 23, 29]. However, the cases of replica-symmetry breaking are known in the literature [59, 50].}\]
form is then performed only on the second term, giving

\[
\hat{T}_3(Q, \hat{Q}) = \frac{1}{\pi M} \int \left\{ e^{-\|\bar{x}_{i,t} - A_{i,t}x_{i,t}\|_2^2} e^{jx_{i,t}^H B_{i,t}x_{i,t}} \right\}^u d\bar{x}_{i,t}. 
\]

Setting the derivatives of the argument in (72) w.r.t all the RS parameters to zero and defining \( \bar{x}_{i,t} \triangleq \hat{x}_{i,t} - \hat{\bar{x}}_{i,t} \) provides the set of fixed-point equations (72). Finally, differentiating w.r.t. \( u \) and letting \( u \to 0 \) gives (31).

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