Intuitionistic fuzzy semigroup

M. Elomari, S. Melliani\textsuperscript{*}, R. Ettoussi and L. S. Chadli

Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco
\textsuperscript{*} Corresponding author
e-mail: s.melliani@yahoo.fr

Abstract: In this work, we generalize the definition of an intuitionistic fuzzy strongly continuous semi-group and its generator. We establish some of their properties and some results about the existence and uniqueness of solutions for intuitionistic fuzzy equation.

Keywords: Intuitionistic fuzzy semigroup, Intuitionistic fuzzy strongly continuous semi-group.

AMS Classification: 03E72.

1 Introduction

The initial value crisp problem

\[
\begin{align*}
  x(t) &= Ax(t) + f(t, x(t)) & t \in [0, T] \\
  x(0) &= x_0
\end{align*}
\]  

(1.1)

has a unique mild solution under assumption some conditions, if \( A \) is the generator of a \( C_0 \)-semigroup, \((S(t))_{t \geq 0}\) on a Banach space \( X \), the system (1.1) has a unique mild solution \( x \in \mathcal{C}([0, T]) \). In [7], C. G. Gal and S. G. Gal studied, with more details, fuzzy linear and semi-linear (additive and positive homogeneous) operators theory, introduced semigroups of operators of fuzzy-number-valued functions, and gave various applications to fuzzy differential equation.

In this work, we study the existence and uniqueness of mild solution for fractional differential equation with intuitionistic fuzzy data of the following form:

\[
\begin{align*}
  x(t) &= Ax(t) + f(t, x(t)) & t \in [0, T] \\
  x(0) &= x_0 \in IF_1
\end{align*}
\]  

(1.2)
2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $X = \mathbb{R}$ and we denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

**Definition 2.1.** We denote
\[
\text{IF}_1 = \left\{ (u, v) : \mathbb{R} \to [0, 1]^2 \mid \forall x \in \mathbb{R} / 0 \leq u(x) + v(x) \leq 1 \right\}
\]
where
1. $(u, v)$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
2. $u$ is fuzzy convex and $v$ is fuzzy concave.
3. $u$ is upper semicontinuous and $v$ is lower semicontinuous
4. $\text{supp}(u, v) = \text{cl}(\{ x \in \mathbb{R} : v(x) < 1\})$ is bounded.

For $\alpha \in [0, 1]$ and $(u, v) \in \text{IF}_1$, we define
\[
\left[ (u, v) \right]^\alpha = \{ x \in \mathbb{R} \mid v(x) \leq 1 - \alpha \} \quad \text{and} \quad \left[ (u, v) \right]_{\alpha} = \{ x \in \mathbb{R} \mid u(x) \geq \alpha \}
\]

**Remark 2.1.** We can consider $\left[ (u, v) \right]^\alpha$ as $[u]^\alpha$ and $\left[ (u, v) \right]_{\alpha}$ as $[1 - v]^\alpha$ in the fuzzy case.

**Definition 2.2.** The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by
\[
0_{(1,0)}(x) = \begin{cases} 
(1,0) \quad &\text{if } x = 0 \\
(0,1) \quad &\text{if } x \neq 0
\end{cases}
\]

**Definition 2.3.** Let $(u, v), (u', v') \in \text{IF}_1$ and $\lambda \in \mathbb{R}$, we define the addition by:
\[
\left( (u,v) \oplus (u',v') \right)(z) = \left( \sup_{z = x+y} \min \left( u(x), u'(y) \right) ; \inf_{z = x+y} \max \left( v(x), v'(y) \right) \right)
\]
\[
\lambda(u, v) = \begin{cases} 
(\lambda u, \lambda v) \quad &\text{if } \lambda \neq 0 \\
0_{(0,1)} \quad &\text{if } \lambda = 0
\end{cases}
\]

According to Zadeh’s extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $\text{IF}_1$ as follows:
\[
\left[ (u, v) \oplus (z, w) \right]_{\alpha} = \left[ (u, v) \right]_{\alpha} \oplus \left[ (z, w) \right]_{\alpha}, \quad \left[ \lambda(u, v) \right]_{\alpha} = \lambda \left[ (u, v) \right]_{\alpha}
\]

where $(u, v), (z, w) \in \text{IF}_1$ and $\lambda \in \mathbb{R}$.

We denote
\[
\left[ (u, v) \right]_l^- (\alpha) = \inf \left\{ x \in \mathbb{R} \mid u(x) \geq \alpha \right\}, \quad \left[ (u, v) \right]_r^+ (\alpha) = \sup \left\{ x \in \mathbb{R} \mid u(x) \geq \alpha \right\}
\]
\[
\left[ (u, v) \right]_l^- (\alpha) = \inf \left\{ x \in \mathbb{R} \mid v(x) \leq 1 - \alpha \right\}, \quad \left[ (u, v) \right]_r^- (\alpha) = \sup \left\{ x \in \mathbb{R} \mid v(x) \leq 1 - \alpha \right\}
\]
**Theorem 2.1.** [5] Let \( M = \{M_\alpha, M^\alpha : \alpha \in [0, 1]\} \) be a family of subsets in \( \mathbb{R} \) satisfying Conditions (i) – (iv)

i) \( \alpha \leq \beta \Rightarrow M_\beta \subset M_\alpha \) and \( M^\beta \subset M^\alpha \)

ii) \( M_\alpha \) and \( M^\alpha \) are nonempty compact convex sets in \( \mathbb{R} \) for each \( \alpha \in [0, 1] \).

iii) for any nondecreasing sequence \( \alpha_i \to \alpha \) on \([0, 1]\), we have \( M_\alpha = \bigcap_i M_{\alpha_i} \) and \( M^\alpha = \bigcap_i M^{\alpha_i} \).

iv) For each \( \alpha \in [0, 1], M_\alpha \subset M^\alpha \) and define \( u \) and \( v \), by

\[
\begin{align*}
u(x) &= \begin{cases} 
0 & \text{if } x \notin M_0 \\
\sup \{ \alpha \in [0, 1] : x \in M_\alpha \} & \text{if } x \in M_0
\end{cases} \\
v(x) &= \begin{cases} 
1 & \text{if } x \notin M^0 \\
1 - \sup \{ \alpha \in [0, 1] : x \in M^\alpha \} & \text{if } x \in M^0
\end{cases}
\end{align*}
\]

Then \((u, v) \in IF_1\).

The space \( IF_1 \) is metrizable by the distance of the following form:

For \( p \in [1, \infty) \)

\[
d_p((u, v), (z, w)) = \left\{ \frac{1}{4} \int_0^1 \left| [(u, v)]_i^+(\alpha) - [(z, w)]_i^+(\alpha) \right|^p d\alpha + \frac{1}{4} \int_0^1 \left| [(u, v)]_r^+(\alpha) - [(z, w)]_r^+(\alpha) \right|^p d\alpha + \frac{1}{4} \int_0^1 \left| [(u, v)]_i^-\beta(\alpha) - [(z, w)]_i^-\beta(\alpha) \right|^p d\alpha + \frac{1}{4} \int_0^1 \left| [(u, v)]_r^-\beta(\alpha) - [(z, w)]_r^-\beta(\alpha) \right|^p d\alpha \right)^{\frac{1}{p}}
\]

and

\[
d_\infty((u, v), (z, w)) = \frac{1}{4} \sup_{0 < \alpha, \beta \leq 1} \left| (u, v)]_i^+(\alpha) - [(z, w)]_i^+(\alpha) \right| + \frac{1}{4} \sup_{0 < \alpha, \beta \leq 1} \left| (u, v)]_r^+(\alpha) - [(z, w)]_r^+(\alpha) \right| + \frac{1}{4} \sup_{0 < \alpha, \beta \leq 1} \left| (u, v)]_i^-\beta(\alpha) - [(z, w)]_i^-\beta(\alpha) \right| + \frac{1}{4} \sup_{0 < \alpha, \beta \leq 1} \left| (u, v)]_r^-\beta(\alpha) - [(z, w)]_r^-\beta(\alpha) \right|
\]

**Theorem 2.2.** [5] For \( p \in [1, \infty) \)

\( (IF_1, d_p) \) is a complete and separable metric space, but \( (IF_1, d_\infty) \) is a complete and not separable metric space.

**Remark 2.3.** We can see that \( d_1 \leq d_\infty \).
Definition 2.4. we say that a mapping $F : [a, b] \rightarrow IF_1$ is strongly measurable if for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : [a, b] \rightarrow P_k(\mathbb{R})$ defined by $F_\alpha (t) = [F(t)]^\alpha$ and $F^\alpha : [a, b] \rightarrow P_k(\mathbb{R})$ defined by $F^\alpha (t) = [F(t)]^\alpha$ are (Lebesgue) measurable, when $P_k(\mathbb{R})$ is endowed with the topology generated the Hausdorff metric $d_H$.

Lemma 2.1. [5] Let $F : [a, b] \rightarrow IF_1$ be strongly measurable and denote $F_\alpha (t) = [\lambda_\alpha (t), \lambda^\alpha (t)]$, $F^\alpha (t) = [\mu_\alpha (t), \mu^\alpha (t)]$ for $\alpha \in [0, 1]$. Then $\lambda_\alpha$, $\lambda^\alpha$, $\mu_\alpha$, $\mu^\alpha$ are measurable.

Definition 2.5. Suppose $A = [a, b]$, $F : [a, b] \rightarrow IF_1$ is integrably bounded and strongly measurable for each $\alpha \in (0, 1)$,

$$\left[\int_A F(t)dt\right]_\alpha = \int_A \left[F(t)\right]_\alpha dt = \left\{\int_A f dt : f : A \rightarrow \mathbb{R} \text{ is a measurable selection for } F_\alpha \right\}$$

$$\left[\int_A F(t)dt\right]^\alpha = \int_A \left[F(t)\right]^\alpha dt = \left\{\int_A f dt : f : A \rightarrow \mathbb{R} \text{ is a measurable selection for } F^\alpha \right\}.$$

if there exist $(u, v) \in IF_1$ such that $\left[(u, v)\right]^\alpha = \left[\int_A F(t)dt\right]^\alpha$ and $\left[(u, v)\right]^\alpha = \left[\int_A F(t)dt\right]^\alpha$ for all $\alpha \in (0, 1)$.

Then $F$ is called integrable on $A$, write $(u, v) = \int_A F(t)dt$.

Remark 2.4. 1. If $F(t) = (u_t, v_t)$ is integrable, with theorem 2.1 we have $\int (u_t, v_t) = \left(\int u_t, \int v_t\right)$

2. If $F : [a, b] \rightarrow IF_1$ is integrable then in view of Lemma (2.1) $\int A F$ is obtained by integrating the $\alpha$-level curves, that is

$$\left[\int F\right]_\alpha = \left[\int \lambda_\alpha, \int \lambda^\alpha\right] \text{ and } \left[\int F\right]^\alpha = \left[\int \mu_\alpha, \int \mu^\alpha\right], \alpha \in [0, 1]$$

$$F_\alpha (t) = \left[F(t)\right]_\alpha \text{ and } F^\alpha (t) = \left[F(t)\right]^\alpha.$$
\[ \lambda(u,v)(x) = \begin{cases} 
\left( u(x/\lambda), v(x/\lambda) \right) & \text{if } \lambda \neq 0 \\
0 & \text{if } \lambda = 0 \text{ and } x \neq 0 \\
\left( \sup_{y \in X} u(y), 1 - \sup_{y \in X} v(y) \right) & \text{if } \lambda = 0 \text{ and } x = 0
\end{cases} \]

**Theorem 3.1.** There exists a normed space \( \mathcal{X} \) and a function \( j : IF_1 \to \mathcal{X} \) with properties

1. \( j \) is an isometry i.e. \( \| j(u,v) - j(u',v') \| = d_1((u,v);(u',v')) \leq d_\infty((u,v);(u',v')) \)
2. \( j((u,v) \oplus (u',v')) = j(u,v) + j(u',v') \)
3. \( j(\lambda(u,v)) = \lambda j(u,v) \), \( \lambda \geq 0 \)

**Proof.** Define an equivalence relation in \( IF_1 \times IF_1 \) by

\[ ((u,v);(u',v')) R (z,w);(z',w')) \iff (u,v) \oplus (z,w) = (u',v') \oplus (z',w') \]

The space of equivalence class \( [(u,v);(u',v')] \) of pairs \( (u,v);(u',v') \) is denoted by \( \mathcal{X} \). The norm in \( \mathcal{X} \) is defined by \( \| (u,v);(u',v') \| = d_1((u,v);(u',v')) \). It is easy to check that \( j : IF_1 \to \mathcal{X} \) defined by

\[ j((u,v)) = (u,v);(O_{0,1}) \]

is an isometry, and properties (2), (3) follow the definition. \( \square \)

**Proposition 3.1.** \[8\] \( \mathcal{X} \) is a Banach real space.

### 4 Intuitionistic fuzzy strongly continuous semi-group

In this section we give the approach of the concept of intuitionistic fuzzy Semi-group.

**Definition 4.1.** We called an intuitionistic fuzzy \( C^0 \)-Semi-group (one parameter, strongly continuous, nonlinear) the whole family \( \{ T(t), t \geq 0 \} \) of operators from \( IF_1 \) into itself satisfying the following conditions

- \( T(0) = i \), the identity mapping on \( IF_1 \).
- \( T(t+s) = T(t)T(s), \forall t, s \geq 0. \)
- The function \( g : [0, +\infty[ \to IF_1 \), defined by \( g(t) = T(t)(u,v) \) is continuous at \( t = 0 \) for all \((u,v) \in IF_1 \) i.e
  \[
  \lim_{t \to 0^+} T(t)(u,v) = (u,v)
  \]
There exist two constants $M > 0$ and $\omega \in \mathbb{R}$ such that
\[
d_\infty \left(T(t)(u, v), T(t)(u', v')\right) \leq M e^{\omega t} d_\infty \left((u, v), (u', v')\right),
\]
for $t \geq 0$, $(u, v), (u', v') \in IF_1^2$

In particular, if $M = 1$ and $\omega = 0$, we say that $\{T(t); t \geq 0\}$ is a contraction intuitionistic fuzzy semigroup.

**Remark 4.1.** The continuity of $g$ at 0, implies the continuity of $g$ at $t_0 \geq 0$.

**Definition 4.2.** Let $\{T(t), t \geq 0\}$ be an intuitionistic fuzzy $C^0$-semigroup on $IF_1$ and $(u, v) \in IF_1$, if for $h > 0$ sufficiently small, the Hukuhara difference $T(h)(u, v) - H(u, v)$ exists, we define
\[
\lim_{h \to 0} d_\infty \left(T(h)(u, v) - H(u, v), A((u, v))\right) = 0
\]
whenever this limit exists in the metric space $\left(IF_1, d_\infty\right)$. Then the operator $A : (u, v) \to A(u, v)$ defined on
\[
D(A) = \left\{(u, v) \in IF_1, \lim_{h \to 0} \left(T(h)(u, v) - H(u, v)\right) \text{ exists}\right\}
\]
is called the infinitesimal generator of the intuitionistic fuzzy semigroup $\{T(t), t \geq 0\}$.

**Proposition 4.1.** Let $A : IF_1 \to IF_1$ and $A_1 = jAj^{-1} : \mathcal{X} \to \mathcal{X}$ two operators. $A$ is the infinitesimal generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on $IF_1$ if and only if $A_1$ is the infinitesimal generator of the semigroup $\{T_1(t); t \geq 0\}$ defined on $\mathcal{X}$ by $T_1(t) = jT(t)j^{-1}$ for $t \geq 0$.

**Proof.** We assume that $A$ is the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on $IF_1$, then we have for all $(u, v) \in j^{-1} (D(A))$
\[
\lim_{h \to 0^+} \frac{T_1(h)(u, v) - (u, v)}{h} = \lim_{h \to 0^+} \frac{jT(h)j^{-1}(u, v) - jj^{-1}(u, v)}{h} = j \lim_{h \to 0^+} \frac{T(h)j^{-1}(u, v) - Hj^{-1}(u, v)}{h} = jAj^{-1}(u, v) = A_1(u, v)
\]
Conversely, if $A_1$ is the generator of an semigroup $\{T_1(t); t \geq 0\}$ on $\mathcal{X}$, then for all $(u, v) \in D(A)$
\[
\lim_{h \to 0^+} \frac{T(h)(u, v) - H(u, v)}{h} = \lim_{h \to 0^+} \frac{j^{-1}T_1(h)j(u, v) - Hj^{-1}(u, v)}{h} = j^{-1} \lim_{h \to 0^+} \frac{T_1(h)(u, v) - j(u, v)}{h} = j^{-1}A_1j(u, v) = A(u, v)
\]

\[\square\]
**Remark 4.2.** Since the infinitesimal generator $A_1$ of $\{T_1(t); t \geq 0\}$ is unique, we deduce that the infinitesimal generator $A$ of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ is also unique.

**Lemma 4.1.** Let $A$ be the generator of an intuitionistic fuzzy semigroup $\{T(t); t \geq 0\}$ on $IF_1$, then for all $(u, v) \in IF_1$ such that $T(t)(u, v) \in D(A)$ for all $t \geq 0$, the mapping $t \rightarrow g(t) = T(t)(u, v)$ is differentiable and

$$g'(t) = AT(t)(u, v)$$

**Proof.** Let $(u, v) \in IF_1$, for $t, h \geq 0$ we have

$$T(t + h)(u, v) = T(h)T(t)(u, v)$$

Since $T(t)(u, v) \in D(A)$ then

$$\lim_{h \rightarrow 0^+} \frac{g(t + h) - g(t)}{h} = \lim_{h \rightarrow 0^+} \frac{T(t + h)(u, v) - T(t)(u, v)}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{T(h)T(t)(u, v) - T(t)(u, v)}{h}$$

$$= AT(t)(u, v)$$

Denote $A_h = \frac{T(h) - i}{h}$, for $h > 0$. Using the continuity of $g$ and the definition of $A$, we have

$$\lim_{h \rightarrow 0^+} \frac{g(t - h) - g(t)}{-h} = \lim_{h \rightarrow 0^+} \frac{T(t - h)(u, v) - T(t)(u, v)}{-h}$$

$$= \lim_{h \rightarrow 0^+} \frac{T(h)T(t - h)(u, v) - T(t - h)(u, v)}{h}$$

$$= \lim_{h \rightarrow 0^+} A_hT(t - h)(u, v)$$

$$= AT(t)(u, v)$$

Hence, $g$ is differentiable and $g'(t) = AT(t)(u, v)$, for all $t \geq 0$. 

**References**

[1] Atanassov, K. (1986) Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20, 87–96.

[2] Atanassov, K. (1999) Intuitionistic Fuzzy Sets : Theory and Applications, Springer Physica-Verlag, Heidelberg.

[3] Atanassov, K. T., Vassilev, P. M., & Tsvetkov, R. T. (2013) Intuitionistic Fuzzy Sets, Measures and integrals Bulgarian Academic Monographs (12), Professor Marin Drinov Academic Publishing House, Sofia.

[4] Bede, B., & Gal, S. G. (2005) Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, 151, 581–599.
[5] Melliani, S., Elomari, M., Chadli, L. S. & Ettoussi, R. (2015) Intuitionistic fuzzy metric space, *Notes on Intuitionistic Fuzzy Sets*, 21(1), 43–53.

[6] Melliani, S., Ettoussi, R., Elomari, M. & Chadli, L. S. Solution of intuitionistic fuzzy differential equations (submitted in *Notes on Intuitionistic Fuzzy Sets*).

[7] Gal, C. G., & Gal, S. G. (2013) Semigroups of Operators on Spaces of Fuzzy-Number-Valued Functions with Applications to Fuzzy Differential Equations, arXiv:1306.3928v1, 17 June 2013.

[8] Klement, E. P., Puri, M. L., & Ralescu, D. A. (1986) Limit Theorems for Fuzzy Random Variables, *Proc. R. Soc. Lond. A* 407, 171–182.