A CURVATURE FLOW AND APPLICATIONS TO AN ISOPERIMETRIC INEQUALITY

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Abstract. Long time existence and convergence to a circle is proved for radial graph solutions to a mean curvature type curve flow in warped product surfaces (under a weak assumption on the warp potential of the surface). This curvature flow preserves the area enclosed by the evolving curve, and this fact is used to prove a general isoperimetric inequality applicable to radial graphs in warped product surfaces under weak assumptions on the warp potential.

1. Introduction

Extrinsic geometric curvature flows are concerned with evolving closed hypersurfaces $\Sigma^n(t) \subset N^{n+1}$ in the normal direction with speed equal to some function related to the curvature. A curve flow is the one-dimensional case $n = 1$. A simple example is the curve shortening flow, which evolves closed curves $\gamma \subset \mathbb{R}^2$ in the outward normal direction $\nu$ with speed equal to (minus) the curvature $\kappa$:

$$X_t = -\kappa \nu,$$

where $X$ is the position vector of the curve $\gamma$. In [Gag83] (and [GH86]), Gage (and Gage-Hamilton) prove an isoperimetric inequality related to the curve shortening flow, an idea which was inspiration for the current work relating isoperimetric inequalities with monotonicity properties of a curve flow.

In [GL15], Guan and Li introduce a new type of mean curvature flow for star-shaped hypersurfaces in space forms. This flow is generalized to starshaped hypersurfaces in warped product spaces in [GLW]. For example, if $\Sigma^n$ is a hypersurface in $\mathbb{R}^{n+1}$, then the flow equation is

$$\frac{\partial X}{\partial t} = (n - H u)\nu,$$

where $X$ is the position vector for $\Sigma$, $H$ is its mean curvature, and $u = \langle X, \nu \rangle$ is the support function of the hypersurface. For general warped product manifold $N^{n+1} = (0,R) \times \mathbb{S}^n$ with metric $g^N := dr^2 + \varphi(r)^2 g^\mathbb{S}^n$, the [GLW] flow equation is

$$\frac{\partial X}{\partial t} = (n \varphi'(r) - H u)\nu, \quad u := g^N(\varphi(r) \partial_r, \nu).$$

It is shown in [GL15] that the volume enclosed by the hypersurface is constant along this flow.

When $n \geq 2$, Theorem 1.1 in [GL15] guarantees long-time existence of this flow for smooth initial hypersurfaces, and exponential convergence to a sphere as $t \to \infty$. Furthermore, when $n \geq 2$, Theorem 4.2 in [GLW] (see also Proposition
3.5 in [GL15] guarantees that, under the assumption $\varphi' \varphi' - \varphi \varphi'' \leq 1$, the surface area of the hypersurface is monotonically decreasing along the flow; this argument depends on $n \geq 2$ in a crucial way, since it uses higher elementary symmetric functions of the curvatures: essentially, the key is the Minkowski identity

\begin{equation}
\int_M \varphi' H = \frac{2}{n-1} \int_M u \sigma_2.
\end{equation}

In dimension $n = 1$, this inequality is no longer applicable.

The purpose of this current paper is to extend the long-time existence and monotonicity of surface area results of [GL15] to the case $n = 1$, and to prove an isoperimetric inequality for curves in two-dimensional warped product surfaces with weak assumptions on the warp potential $\varphi$.

The key results of this paper are:

**Theorem 1.1** (long-time existence and convergence to a circle). Let $N$ be a warped-product space with warp potential $\varphi(r)$ satisfying $\varphi' \varphi' - \varphi \varphi'' \geq 0$. If $\gamma_0 \subset N$ is a smooth hypersurface, then there is a unique flow $\gamma(t)$ satisfying (1.2) and $\gamma(0) = \gamma_0$. Furthermore $\min r|_{\gamma_0} \leq r|_{\gamma(t)} \leq \max r|_{\gamma(0)}$, and $r|_{\gamma(t)} \to \text{constant as } t \to \infty$.

**Theorem 1.2** (Isoperimetric inequality). If $\gamma_0 \subset N$ is a piecewise $C^1$ Lipschitz radial graph, and $\varphi' \varphi' - \varphi \varphi'' \in [0, 1]$, then $L(\gamma_0) \geq F(A(\gamma_0))$, where equality holds if and only if $\gamma_0$ is a “circle,” where a “circle” is either

(i) a slice $\{r\} \times S^1$.

(ii) a translated circle contained in a “space-form region” $[r_1, r_2] \times S^1$ where $\varphi' \varphi' - \varphi \varphi'' \equiv 1$.

1.1. **Motivating Example.** In this subsection, we show that, in $\mathbb{R}^2$, the monotonicity of length (e.g. the “surface area” of the hypersurface) of the one-dimensional curve flow (1.1) (with $n = 1$) is intimately connected to the classical isoperimetric ratio in $\mathbb{R}^2$.

Let $\gamma(t)$ be a curve flow which solves (1.1) with initial data $\gamma_0$. Let $L(t)$, $A(t)$, $ds$, $u$, $\kappa$ be the length, area (the “volume”), arc-length one-form, support function, and curvature of $\gamma(t)$, respectively. As we will prove in more generality below, we have the following equations

\begin{equation}
\frac{dA(t)}{dt} = \int_{\gamma(t)} 1 - u \kappa \, ds = 0 \quad \text{and} \quad \frac{dL(t)}{dt} = \int_{\gamma(t)} \kappa - u \kappa^2 \, ds = 2 \pi - \int_{\gamma(t)} u \kappa^2,
\end{equation}

where we have used Gauss-Bonnet in the second equality. Define the isoperimetric difference $\Lambda := L^2 - 4 \pi A$. The classical isoperimetric inequality states that $\Lambda \geq 0$, with equality holding only for circles. We estimate

$$L^2 = \left[ \int u^{1/2} u^{1/2} \kappa \, ds \right]^2 \leq \int u \, ds \int u \kappa^2 \leq 2A \int u \kappa^2,$$

where we have used the well-known fact $\int u \, ds = 2A$, and the fact that $\int 1 - u \kappa \, ds = 0$. From this we can estimate the second side of (1.4) to obtain

$$2L \frac{dL(t)}{dt} \leq \frac{L}{2A} (4\pi A - L^2),$$

which yields

\begin{equation}
\frac{d\Lambda}{dt} \leq -\frac{L}{2A} \Lambda.
\end{equation}
Thus, if $\Lambda > 0$ for our initial curve, then $\Lambda$ decreases as the curve $\gamma(t)$ approaches a circle. By proving that $\Lambda$ is strictly non-negative in a neighborhood of any circle, one can use (1.5) to prove that $\Lambda \neq 0$ for any initial data $\gamma$.

Unfortunately, the argument leading to (1.5) seems to be special to $\mathbb{R}^2$, and the author was unable to reproduce it for more general warped-product spaces. However, a different argument is possible, and one of the main results in this paper is a general isoperimetric inequality applicable to a large class of warped product spaces (subject to a few conditions on the warp potential $\phi$).

2. Geometric Preliminaries

In the remainder of this paper, let $N = (0, R) \times S^1$ be a warped-product surface with metric $g_N := dr^2 + \varphi(r)^2 d\theta^2$, with $\varphi(r) > 0$. Let $\nabla(\cdot, \cdot)$ be the Levi-Civita connection on $N$.

Define $\Phi : N \to (0, \infty)$ by

$$\Phi(r) = \int_0^r \varphi(r') \, dr'.$$

A closed loop $\gamma \subset N$ is called a $C^k$ radial graph if there is a $C^k$ mapping $\rho : S^1 \to (0, R)$ such that $\gamma = \text{im}(\theta \mapsto (\rho(\theta), \theta))$. We denote by $\nu$ and $\partial_s$ the outward normal and unit tangent vector of $\gamma$; we assume $\gamma$ has the orientation such that $g(\partial_r, \partial_r) > 0$ and $g(\partial_\theta, \partial_s) > 0$. The curvature $\kappa$ satisfies $\nabla(\partial_s, \partial_r) = \kappa \partial_s$.

2.1. Isoperimetric Inequality. Let $C_r$ denote the circle $\{r\} \times S^1 \subset N$, and let $L(r)$ and $A(r)$ denote its length and area, respectively. Then, $A(r) = 2\pi \Phi(r)$, and since $\Phi(r) > 0$, we may solve for $r$ as a function of $A$, and consequently, there is some differentiable function $F$ such that

$$L(r)^2 = F(A(r)).$$

The isoperimetric inequality is the statement that, for any curve $\gamma \subset N$,

$$L[\gamma]^2 \geq F(A[\gamma]).$$

Unfortunately, this inequality is not true without some restriction on the warp potential $\varphi$. If $\varphi' \varphi'' - \varphi \varphi''' > 1$, then the inequality will fail. To prove this assertion, we compute the length and area of the radial graph $\gamma(\epsilon)$ with $\rho(\theta) = r_0 + \epsilon g(\theta)$ to second order in $\epsilon$, and show that

$$\frac{L[\gamma(\epsilon)]^2 - F(A[\gamma(\epsilon)])}{\epsilon^2} = 4\pi^2 \left( \frac{1}{2\pi} \int g \, d\theta \right)^2 + \frac{1}{2\pi} \int (g_0)^2 - g^2 \, d\theta \right) + \beta - 1 \left( \frac{1}{2\pi} \int g \, d\theta \right)^2 - \frac{1}{2\pi} \int g^2 \, d\theta \right),$$

where $\beta = \varphi' \varphi' - \varphi \varphi'''$. If $\beta > 1$ and we consider $g = \cos \theta$, for example, we obtain $L^2 - F(A) < 0$, to lowest order in $\epsilon$. We also note that $\varphi' \varphi' - \varphi \varphi''' = 1$ for $\mathbb{R}^2, S^2$ and $\mathbb{H}^2$. 
2.2. Geometric properties of Guan-Li mean-curvature flow. Following [GL15], we prove the following lemma:

**Lemma 2.1.** The vector field $V = \varphi \partial_r$ is a conformal Killing vector field with conformal factor $\varphi'$. More precisely, for all vector fields $X,Y$ on $N$, we have

$$\nabla^2 \Phi(X,Y) = g(\nabla(Y,V),X) = \varphi' g(X,Y).$$

**Proof.** Simply compute:

$$\nabla^2 \Phi(X,Y) = Y \langle d\Phi, X \rangle - \langle d\Phi, \nabla(Y,X) \rangle = g(\nabla(Y,V),X) + g(V,\nabla(Y,X)) - g(V,\nabla(Y,X)).$$

Then

$$\nabla^2 \Phi(\partial_r, \partial_r) = \varphi' \quad \nabla^2 \Phi(\partial_\theta, \partial_\theta) = \varphi' \Gamma^d_{\partial_r} = \varphi' \Gamma^d,$$

as desired. \hfill \Box

**Corollary 2.2.** On a curve $\gamma$, $\Phi_{ss} = \varphi' - u \kappa$.

**Proof.** We have $\Phi_s = g(V, \partial_s)$, and taking a second derivative yields

$$\Phi_{ss} = g(\nabla(\partial_s, V), \partial_s) + g(V, \nabla(\partial_s, \partial_s)) = \varphi' - \kappa g(V, \nu) = \varphi' - u \kappa,$$

as desired. We used the well-known fact that $\nabla(\partial_s, \partial_s) = -\kappa \nu$. \hfill \Box

**Theorem 2.3.** Let $\gamma_0, \gamma(t)$ be closed loops in $N$, and suppose now that $X(\cdot, t) : \gamma_0 \rightarrow \gamma(t)$ parametrizes $\gamma(t)$ in terms of $\gamma_0$. Suppose that $X$ evolves according to $\partial_t X = f \nu$. Then

$$\frac{dL}{dt}(t) = \int_{\gamma(t)} f \kappa\, ds \quad \text{and} \quad \frac{dA}{dt}(t) = \int_{\gamma(t)} f\, ds,$$

where $L$ and $A$ are the length and area of $\gamma(t)$, respectively.

**Proof.** Consider the normal tube $U_\epsilon \simeq \gamma(t_0) \times (-\epsilon, \epsilon)$ around $\gamma(t_0)$ obtained by sending $(p, z)$ to $\text{Exp}_p(z \nu(p))$. For $\epsilon$ small enough, this is a diffeomorphism from $\gamma(t_0) \times (-\epsilon, \epsilon)$ onto an open set $U_\epsilon \subset N$. In the natural coordinates of $U_\epsilon$, we can write $g = A(s, z) \, ds^2 + dz^2$, where $A(s, 0) = g(\partial_s, \partial_s) = g(\nabla(\partial_s, \nu), \partial_s) = \kappa$. Consider $\gamma(t)$ as the graph of $F(\cdot, t) : \gamma(t_0) \rightarrow (-\epsilon, \epsilon)$. Clearly $F(t) = f(s)$. Then

$$L(t) = \int_{\gamma(t_0)} \sqrt{A(s,F(s,t))}^2 + F_s^2\, ds,$$

and so

$$L(t) = \int_{\gamma(t_0)} \kappa(s)f(s)\, ds.$$

Similarly,

$$A(t) = \int_{\gamma(t_0)} \int_{z=0}^{F(s,t)} A(s, z')\, dz'\, ds \quad \Rightarrow \quad A(t) = \int_{\gamma(t_0)} f(s)\, ds.$$

This completes the proof. \hfill \Box

**Theorem 2.4.** Suppose that $\gamma(t)$ evolves with speed function $f = \Phi_{ss}$. Then

$$\frac{dA}{dt}(t) = 0 \quad \text{and} \quad \frac{dL}{dt}(t) = \int_{S^1} (\varphi' \varphi'' - \varphi \varphi') (r_s)^2 - (r_{s\theta})^2 \, d\theta.$$
Proof. The first part of the theorem follows from Theorem 2.3 and fact that \( \int \Phi_{ss} \, ds = 0 \). For the second part of the theorem, we introduce scalar functions \( a, b \) defined by
\[
\partial_r = a \nu + b \partial_s, \quad \text{(note } r_s = b) \]
then it is easy to show that
\[
\partial_s = a \frac{\partial \theta}{\varphi} + b \partial_r, \quad \nu = a \partial_r - b \frac{\partial \theta}{\varphi},
\]
and using this, we may express the curvature \( \kappa \) in terms of \( a \) and \( b \):
\[
\kappa = \frac{\varphi'}{\varphi} \frac{a}{b_s} - \frac{b}{a}.
\]
It is clear that
\[
\Phi_s = \varphi \quad \text{and} \quad \Phi_{ss} = \varphi' b^2 + \varphi b_s,
\]
and thus
\[
\int \kappa \Phi_{ss} \, ds = \int \frac{a \varphi'}{\varphi} \frac{b^2}{a} - \frac{\varphi'}{a} b_s - \frac{\varphi' b^2 b_s}{a} + \varphi' ab_s \, ds.
\]
Integration by parts on the last term yields
\[
\int \kappa \Phi_{ss} \, ds = \int \frac{a \varphi' - \varphi \varphi''}{\varphi} b^2 - \frac{\varphi'}{a} b_s^2 - \frac{\varphi' b^2 b_s}{a} - \varphi' a_s b \, ds,
\]
and thus
\[
\int \kappa \Phi_{ss} \, ds = \int \frac{a (\varphi' - \varphi \varphi'')}{\varphi} b^2 - \frac{\varphi'}{a} b_s^2 - \frac{\varphi' b^2 b_s}{a} - \varphi' a_s b \, ds.
\]
now we use the fact that \( a^2 + b^2 = 1 \) to deduce that \( aa_s = -bb_s \), which makes the last two terms in the integral cancel, and we are left with
\[
\int \kappa \Phi_{ss} \, ds = \int \frac{a (\varphi' - \varphi \varphi'') b^2}{\varphi} - \frac{\varphi'}{a} b_s^2 \, ds
\]
Consider the radial graph parametrization of \( \gamma \), obtained by \( \theta \mapsto (\theta, \rho(\theta)) \). It is straightforward to show that
\[
\frac{a}{\varphi} \, ds = d\theta \quad \text{and} \quad b_s = \frac{a}{\varphi} b_\theta,
\]
whereby we obtain
\[
(2.2) \quad \frac{dL(t)}{dt} = \int \kappa \Phi_{ss} \, ds = \int_{\mathbb{S}^1} (\varphi' \varphi' - \varphi \varphi'')(r_s)^2(\theta) - (r_s \varphi)^2(\theta) \, d\theta,
\]
where we have replaced \( b = r_s \). This completes the proof. \( \square \)

Corollary 2.5. Suppose that \( \varphi' \varphi' - \varphi \varphi'' \leq 1 \). If \( \gamma(t) \) satisfies \( \int_{\mathbb{S}^1} r_s \, d\theta = 0 \), then \( dL(t)/dt \leq 0 \), with equality if and only if \( r_s \equiv 0 \), or \( \varphi' \varphi' - \varphi \varphi'' \equiv 1 \) and \( r_s = a \cos \theta + b \sin \theta \) (\( a, b \) may depend on time).

Proof. Theorem 2.3 guarantees
\[
(2.3) \quad \frac{dL(t)}{dt} \leq \int (r_s)^2 - (r_s \varphi)^2 \, d\theta - 2\pi \int r_s \, d\theta \leq 0,
\]
where we have used the classical Poincaré inequality on the circle. The first inequality is equality only when \( \varphi' \varphi' - \varphi \varphi'' \equiv 1 \) or \( r_s \equiv 0 \), and the second inequality is equality only when \( r_s = a \cos \theta + b \sin \theta \), as desired. \( \square \)
3. PDE Estimates for Guan-Li Curve Flow.

The goal of this section is to prove long-time existence for the curve flow with speed function \( \varphi' - u \kappa \) and convergence to circle as \( t \to \infty \), assuming smooth radial graph \( \gamma_0 \) as initial data.

Following [GL15], we work in the radial graph parametrization: we look for solutions of the form
\[
(3.1) \quad \gamma(t) = \text{im}(\theta \mapsto (\rho(\theta, t), \theta)),
\]
for some \( \rho : \mathbb{S}^1 \times (0, \infty) \to (0, R) \). Parametrizing the flow (3.1) using the radial graph parametrization, we see that
\[
\partial_t X = \rho_t \partial_r = \rho_t \left( \frac{u}{\varphi'} + r_s \partial_s \right),
\]
and thus \( \gamma(t) \) in (3.1) evolves with speed function \( \rho_t u / \varphi' \). Thus, if \( \rho \) satisfies \( \rho_t = \varphi f / u \), (3.1) evolves with speed function \( f \). A straightforward computation yields
\[
\Phi_s = \frac{\varphi \rho_\theta}{\varphi^2 + \rho_\theta^2}, \quad \Phi_{ss} = \frac{\varphi^3 \rho_{\theta\theta} + \varphi' \rho_\theta^5}{(\varphi^2 + \rho_\theta^2)^{3/2}}, \quad \text{(using} \theta_0 = \sqrt{\varphi^2 + \rho_\theta^2} \partial_s),
\]
and \( u = \varphi^2 / \sqrt{\varphi^2 + \rho_\theta^2} \). Therefore, we seek a solution of the following problem
\[
(3.2) \quad \begin{cases}
\rho \in C^\infty(\mathbb{S}^1 \times [0, \infty)) \\
\rho(\theta, 0) = \rho_0(\theta) \quad (\gamma_0 = \text{im}(\theta \mapsto (\rho_0(\theta), \theta))) \\
\rho_t = \frac{\varphi^3 \rho_{\theta\theta} + \varphi' \rho_\theta^5}{\varphi(\varphi^2 + \rho_\theta^2)^{3/2}}
\end{cases}
\]

3.1. \( C^0 \) and Gradient Estimate for Solutions of (3.2). Following [GL15], we first prove solutions of (3.2) satisfy a \( C^0 \) a priori estimate. At critical points of \( \rho \) we have \( \rho_\theta = 0 \) and thus
\[
\rho_t = \frac{1}{\varphi'} \rho_{\theta\theta},
\]
and, by the standard maximum principle, this implies that
\[
(3.3) \quad \min \rho_0 \leq \rho(\theta, t) \leq \max \rho_0.
\]
The gradient estimate requires more work. We begin by showing that \( \omega := \rho_\theta^2 \) satisfies a parabolic evolution equation. We compute
\[
(3.4) \quad \omega_t = 2\rho_\theta \omega_{\theta, t} = \frac{\varphi^3}{\varphi(\varphi^2 + \rho_\theta^2)^{3/2}} \omega_{\theta\theta} - 2 \frac{\varphi^3}{\varphi(\varphi^2 + \rho_\theta^2)^{3/2}} (\rho_{\theta\theta})^2
\]
\[
+ \frac{\partial}{\partial \theta} \left( \frac{\varphi^2}{(\varphi^2 + \rho_\theta^2)^{3/2}} \right) + \frac{2\varphi' \omega}{\varphi(\varphi^2 + \rho_\theta^2)^{3/2}} - \frac{3\varphi \rho_\theta^5}{\varphi(\varphi^2 + \rho_\theta^2)^{5/2}} \omega_\theta
\]
\[
- \frac{2(4\varphi' \varphi' - \varphi'' \varphi) \rho_\theta^6}{(\varphi^2 + \rho_\theta^2)^{5/2}} - \frac{2(\varphi' \varphi' - \varphi'' \varphi) \rho_\theta^8}{(\varphi^2 + \rho_\theta^2)^{5/2}},
\]
and abbreviating yields
\[
(3.5) \quad \omega_t = A(\rho, \omega) \omega_{\theta\theta} - 2A(\rho, \omega) (\rho_{\theta\theta})^2 + B(\rho, \omega, \omega_\theta) \omega_\theta - C_1(\rho, \omega) \omega^3 - C_2(\rho, \omega) \omega^4
\]
We now make the assumption that \( \varphi' \varphi' - \varphi'' \varphi > 0 \), which enables us to conclude that \( C_1 > 0 \) and \( C_2 > 0 \). We remark that this assumption also plays a key role in
proving the convergence to a circle. Supposing that $\omega$ achieves a positive maximum at $(\theta, t)$ for a positive time $t > 0$, we have

$$\omega_t(\theta, t) = A(\rho, \omega)\omega_{\theta \theta}(\theta, t) - C_1(\rho, \omega)\omega^3 - C_2(\rho, \omega)\omega^4,$$

which implies that $\omega_t < 0$, a contradiction. Thus $\omega$ can only attain its maximum on the initial data, so

$$0 \leq \omega(t > 0) < \omega(t = 0).$$

(3.6)

3.2. Long-time existence of solutions of equation (3.2). Appealing to the classical parabolic theory for quasi-linear parabolic equations (see, for instance, [LSU68]), the higher regularity a priori estimates for $\rho$ follow from the uniform $C^0$ and gradient estimates. To be precise, we have:

**Theorem 3.1.** Let $\rho$ solve (3.2), and suppose that $\varphi' \varphi' - \varphi'' > 0$. Then, for any integer $k \geq 0$, and any $t_0 > 0$, there exists a constant $C(t_0, \rho_0, k)$ such that

$$|\rho(\cdot, t)|_{C^k} < C(t_0, \rho_0, k) \quad t > t_0.$$  

(3.7)

These estimates guarantees long-time existence and uniqueness and thus the first part of Theorem 1.1 (see, for instance, the proof of Theorem 8.3 in [Lie96]).

3.3. Convergence to a circle as $t \to \infty$. The goal of this subsection is to prove the second part of Theorem 1.1, that $\kappa_\gamma(\theta(t)) \to \text{constant}$.

Suppose that $\rho$ is a solution to (3.2). As shown in the previous section, $\omega = \rho^2_\theta$ satisfies the following PDE

$$\omega_t = A(\rho, \omega)\omega_{\theta \theta} - 2A(\rho, \omega)(\rho \omega^2) + B(\rho, \omega, \omega_\theta)\omega_\theta - C_1(\rho, \omega)\omega^3 - C_2(\rho, \omega)\omega^4,$$

with constants $a_i, b_i, c_{i,j}$, such that $0 \leq a_1 \leq A \leq a_2 < \infty$, $0 < c_{i,1} \leq C_1 \leq c_{i,2} < \infty$, $i = 1, 2$. Let $v : [0, \infty) \to [0, \infty)$ be the unique solution to

$$v_t = -c_{1,1}v^3 - c_{2,1}v^4 \quad v(0) = \max \omega + \epsilon \quad (\epsilon > 0)$$

It is clear that $v$ is always positive, and that $v = O(t^{-1/2})$.

Letting $u = \omega - v$, which satisfies

$$u_t \leq A(\rho, \omega)u_{\theta \theta} - 2A(\rho, \omega)(u \omega^2) + B(\rho, \omega, \omega_\theta)u_\theta - c_{1,1}u(\omega^2 + v \omega + v^2) - c_{2,1}u(\omega^3 + v \omega^2 + v^2 \omega + v^3).$$

(3.8)

Suppose now that $u$ achieves a positive maximum at $(\theta, t)$. Then evaluating the right hand side of (3.8) yields $u_t(\theta, t) < 0$, which is a contradiction. Thus $u$ cannot achieve a positive maximum, and so $\omega \leq v$, and since $v \to 0$ uniformly as $t \to \infty$, we deduce that $\omega \to 0$ uniformly, and thus $\rho(\cdot, t)$ uniformly approaches a constant function as $t \to \infty$. This proves that the curve flow converges to a circle.

4. Monotonicity of length and the Isoperimetric inequality

Let $\gamma(t)$ be a solution to the flow with speed function $\varphi' - uk$. Unfortunately, it seems difficult to directly prove that $L[\gamma(t)]$ is monotonically nonincreasing - however, we will prove that $L[\gamma(t)]$ is monotonically nonincreasing for bilaterally symmetric curves:
4.1. Bilaterally symmetric radial graphs. For each $\alpha \in S^1$, consider the isometry $\mathcal{R}_\alpha : N \to N$ defined by

$$\mathcal{R}_\alpha : (r, \theta) \mapsto (r, \alpha^2 \theta^{-1}),$$

where multiplication happens in $S^1 \subset \mathbb{C}$. Note that the antipodal points $\pm \alpha$ are fixed, and so we can consider this as a reflection through a line:

It is clear that $\mathcal{R}_\alpha : N \to N$ is an isometry. If $\gamma \subset N$ is a curve such that there is some $\alpha \in S^1$ such that $\gamma$ is fixed under $\mathcal{R}_\alpha$, then we say $\gamma$ has a **bilateral symmetry** with axis $\alpha$.

**Theorem 4.1.** Let $0 \leq \phi' \phi' - \phi'\phi'' \leq 1$. Suppose $\gamma_0$ is a smooth, bilaterally symmetric with axis $\alpha$, radial graph, and suppose $\gamma(t)$ is a solution to the flow with initial data $\gamma(0) = \gamma_0$. Then $\gamma(t)$ is also bilaterally symmetric with axis $\alpha$ and furthermore $L[\gamma(t)]$ is nonincreasing.

**Proof.** The fact that $\gamma(t)$ is also bilaterally symmetric follows from uniqueness of solutions and the fact that $\mathcal{R}_\alpha(\gamma(t))$ also is a solution to the flow.

If $\gamma$ is bilaterally symmetric radial graph, and $p \in \gamma$, then it is an easy computation to show that $r_\gamma(p) = -r_\gamma(\mathcal{R}_\alpha(p))$, and so $\int_{S^1} r_\gamma d\theta = 0$. Then Corollary 2.4 implies that $\frac{d}{dt} L[\gamma(t)] \leq 0$, as desired. \qed

**Corollary 4.2.** Let $0 \leq \phi' \phi' - \phi'\phi'' \leq 1$. If $\gamma_0$ is a smooth bilaterally symmetric radial graph, then $L[\gamma_0] \geq F(A[\gamma_0])$ (cf. subsection 2.1 for definition of $F$).

**Proof.** Using monotonicity of length and the constancy of area, we deduce $L[\gamma_0] \geq L[\gamma(\infty)] = F(A[\gamma(\infty)]) = F(A[\gamma_0])$, as desired. \qed

**Theorem 4.3** (Isoperimetric inequality, without equality case). Let $0 \leq \phi' \phi' - \phi'\phi'' \leq 1$. If $\gamma_0$ is any piecewise $C^1$ and Lipschitz radial graph, then $L[\gamma_0] \geq F[A(\gamma_0)]$.

**Proof.** Let $\rho_0$ be the radial length function of $\gamma_0$, and, for each $\alpha$, define

$$\rho_{\alpha,1}(\theta) = \begin{cases} \rho(\theta) & \text{imag}(\alpha \theta^{-1}) \geq 0 \\ \rho(\alpha^2 \theta^{-1}) & \text{imag}(\alpha \theta^{-1}) \leq 0 \end{cases}$$

and

$$\rho_{\alpha,2}(\theta) = \begin{cases} \rho(\theta) & \text{imag}(\alpha^{-1} \theta) \leq 0 \\ \rho(\alpha^2 \theta^{-1}) & \text{imag}(\alpha^{-1} \theta) \geq 0 \end{cases}$$

Then note that $\text{imag}(\alpha^{-1} \theta) \geq 0$ implies that

$$\text{imag}(\alpha^{-1} \alpha^2 \theta^{-1}) = \text{imag}(\alpha \theta^{-1}) \leq 0,$$

and so $\rho_{\alpha,1}(\alpha^2 \theta^{-1}) = \rho_{\alpha,1}(\theta)$ and $\rho_{\alpha,2}(\alpha^2 \theta^{-1}) = \rho_{\alpha,2}(\theta)$. Let $\gamma_0, \gamma_{\alpha,1}$ and $\gamma_{\alpha,2}$ be the curves defined by $\rho_0, \rho_{\alpha,1}, \rho_{\alpha,2}$. By continuity there is some $\alpha \in S^1$ such that $\gamma_{\alpha,1}$ and $\gamma_{\alpha,2}$ have the same area. Suppose that $L[\gamma_{\alpha,1}] \leq L[\gamma_0] \leq L[\gamma_{\alpha,2}]$. It therefore suffices proving that $L[\gamma_{\alpha,1}] \geq F[A(\gamma_{\alpha,1})] = F[A(\gamma_0)]$, and so we may assume from the outset that $\rho_0$ is piecewise $C^1$, Lipschitz, and bilaterally symmetric.
As in the proof of Theorem 4.3, convolve with smooth mollifiers to obtain \( \rho_n \to \rho_0 \)
be a sequence of smooth, symmetric, functions converging to \( \rho_0 \).
Since \( \rho_0 \) is Lipshitz, its derivative is bounded. Since \( \rho_{n,\theta} \to \rho_{0,\theta} \) uniformly on any compact set \( K \subset S^1 \) such that \( \rho_0 \in C^1(K) \), we deduce that \( L[\gamma_n] \to L[\gamma_0] < \infty \) and \( A[\gamma_n] \to A[\gamma_0] < \infty \). Then \( L[\gamma_0] \geq F(A[\gamma_0]) \) follows, as desired. \( \square \)

5. Equality case for isoperimetric inequality.

The rest of this paper is dedicated to discussing when equality holds in Theorem 4.3. Ideally, \( L[\gamma_0] = F(A[\gamma_0]) \) would imply that \( \gamma_0 \) is a circle \( \{r\} \times S^1 \), but, when \( N \) satisfies \( \phi' \phi' - \phi \phi'' \geq 0 \), there are complications arising from translation isometries of \( N \) (e.g. translated circles in the space forms \( \mathbb{R}^2, S^2 \) and \( H^2 \) are equality cases).

A key tool in our argument is the following existence result:

**Theorem 5.1.** Suppose \( \phi' \phi' - \phi \phi'' \geq 0 \). Let \( \gamma_0 \) be a piecewise \( C^1 \) Lipshitz radial graph. Then there is a smooth solution \( \gamma(t) \) to the flow \( \phi' - \phi \) such that \( \rho_{\gamma(t)}(\cdot,t) \to \rho_{\gamma(0)}(\cdot) \) uniformly, \( L[\gamma(t)] \to L[\gamma_0] \), and \( A[\gamma(t)] \to A[\gamma_0] \) as \( t \searrow 0 \). If \( \gamma_0 \) is bilaterally symmetric, then we may take \( \gamma(t) \) to be bilaterally symmetric as well.

**Proof.** As in the proof of Theorem 4.3, convolve with smooth mollifiers to obtain \( \rho_n \to \rho_0 \) a sequence of smooth radial length functions converging to \( \rho_0 \). If \( \gamma_0 \) (hence \( \rho_0 \)) has a bilateral symmetry, we may take \( \rho_n \) to have the same bilateral symmetry. Using the existence result Theorem 1.1 for smooth initial data, let \( \rho_n(\cdot,t) \) be solutions to (3.2) with initial data \( \rho_n \). For each \( t_0 > 0 \), Theorem 3.1 guarantees that there is a constant \( C(t_0,\rho_0,k) \) such that

\[
|\rho_n(\cdot,t)|_{C^k} \leq C(t_0,\rho_0,k) \quad t \geq t_0.
\]

We can take \( C \) to be independent of \( n \), since it only depends on the \( C_0 \) and \( C_1 \) estimates of the initial data \( \rho_n \), and for \( n \) sufficiently large, the \( C_0 \) and \( C_1 \) estimates of \( \rho_n \) can be estimated from the \( C_0 \) and Lipshitz estimates of \( \rho_0 \).

Now recursively define subsequences of \( (\rho_n : n \in \mathbb{N}) \) by the two properties

(i) \((\rho_{mn} : n \in \mathbb{N})\) converges to a smooth limit function on \([1/m, m] \times S^1 \) satisfying (3.2) (convergence in \( C^k \) for every \( k \)).

(ii) \((\rho_{mn} : n \in \mathbb{N})\) is a subsequence of \((\rho_{m'n} : n \in \mathbb{N})\) if \( m' > m \).

It is always possible to do this because \( \{\rho_n : n \in \mathbb{N}\} \) satisfies the (uniform) estimates (5.1) (here we are using Arzéla-Ascoli, invoking the fact that \( \rho_n : n \in \mathbb{N} \) is bounded in \( C^{k+1}(S^1 \times [1/m, m]) \)).
Then, using the standard diagonal trick, $\rho_{mm}$ converges to a smooth function $\rho$ on $C^k(S^1 \times (0, \infty))$ which satisfies (3.2). By taking more subsequences, we may upgrade this to $C^{\infty}(S^1 \times (0, \infty))$ convergence. It is clear that $\rho(\cdot, t) \to \rho_0(\cdot)$ uniformly and such that the length and area are continuous as $t \to 0$, as desired.

**Theorem 5.2.** Let $\gamma_0 \subset N$ be a piecewise $C^1$ Lipschitz radial graph. Suppose that $\varphi' \varphi'' - \varphi' \varphi'' \equiv 1$ on $\gamma_0$. If $L[\gamma_0] = F(A[\gamma_0])$, then $\gamma_0$ is a circle $\{r \} \times S^1$.

**Proof.** First, using the “cut and reflect” technique in Theorem 4.3, we may cut $\gamma_0$ into two bilaterally symmetric halves $\gamma_1$ and $\gamma_2$ satisfying $A[\gamma_1] = A[\gamma_2] = A[\gamma_0]$. Clearly we have $L[\gamma_1] = F(A[\gamma_1])$. Now apply Theorem 5.1 to deduce a solution to the flow $\gamma_1(t)$. It is clear that $L[\gamma_1(t)]$ is a constant. By Corollary 2.4, we deduce that $r_*(t) = 0$, and so the $\gamma_1(t) = \{r \} \times S^1$ (for all $t > 0$). This is obviously stable as $t \to 0$ by uniform convergence, so $\gamma_1 = \{r \} \times S^1$. Clearly $\gamma_2 = \{r \} \times S^1$ for the same $r$ (by continuity), so $\gamma_0 = \{r \} \times S^1$, as desired.

Now we must consider the case when $\varphi' \varphi' - \varphi' \varphi'' \equiv 1$. First, we note the following result:

**Lemma 5.3.** If $N = (r_1, r_2) \times S^1$ is a warped-product space satisfying $\varphi' \varphi' - \varphi' \varphi'' \equiv 1$, then there is $k > 0$ and $r_0$ such that $\varphi(r) = \sinh(k(r - r_0))$, $\sin(k(r - r_0))$ or $r - r_0$, and $N$ can be isometrically embedded into $k^{-1}H^2$, $k^{-1}S^2$ or $R^2$. (The gauss curvature is $-k^2, k^2, 0$).

**Proof.** We use the well known fact that $k^{-1}H^2 - \{\text{origin}\}$, $k^{-1}S^2 - \{\pm(0, 0, 1)\}$ and $R^2 - \{(0, 0)\}$ are warped product surfaces with potentials $\sinh kr$, $\sin kr$ and $r$, respectively.

Note that $\varphi' \varphi' - \varphi'' \varphi \equiv 1$ implies that the gauss curvature $K = -\varphi'' / \varphi$ is constant, and so
\[
\varphi(r) = \begin{cases} 
A \sinh(k(r - r_0)) & K = -k^2 \\
A \sin(k(r - r_0)) & K = k^2, \\
Ar - r_0 & K = 0
\end{cases}
\]
Then invoking $\varphi' \varphi' - \varphi'' \varphi = 1$, we deduce that $A = 1$ in all cases. Then the map $(r_1, r_2) \times S^1 \to (0, \infty) \times S^1$ defined by $(r, \theta) \to (r - r_0, \theta)$ is an isometric embedding of the tube into $k^{-1}H^2$, $k^{-1}S^2$, or $R^2$, depending on the sign of $K$.

Thanks to this result, if we are given $\gamma_0 \subset N$ satisfying $\varphi' \varphi' - \varphi'' \varphi \equiv 1$ on $\gamma_0$, then we may assume that $\gamma_0 \subset k^{-1}H^2, k^{-1}S^2$ or $R^2$, and then, up to rescaling, $\gamma_0 \subset H^2, S^2$ or $R^2$. If $\gamma_0$ is a radial graph in $N$, then we may assume it is a radial graph in $H^2, S^2$ or $R^2$. Referring to Corollary 2.4, if $r(t)$ is a bilaterally symmetric solution to the curve flow (in one of these three space forms) satisfying $L[\gamma(t)] = \text{constant}$, then $r_\alpha = a \cos \theta + b \sin \theta$ on $\gamma(t)$. We will show that this condition on the radial length function $r$ implies that $\gamma(t)$ is a circle (which may be translated).

We first show that curves satisfying $r_\alpha = a \cos \theta + b \sin \theta$ are unique up to initial point.

**Theorem 5.4.** Suppose that $\gamma_1, \gamma_2 \subset N$ are $C^1$ radial graphs in a warped-product space $N$ and $\gamma_1 \cap \gamma_2 \neq \emptyset$. Suppose that the unit tangent fields $\partial_{\alpha}$ and $\partial_{\beta}$ satisfy $g^N(\partial_\alpha, \partial_\beta) > 0$, $j = 1, 2$. If $r_\alpha = a \cos(\theta + \alpha)$ is true on both $\gamma_1$ and $\gamma_2$ for the same constants $a \in (-1, 1)$ and $\alpha \in R/2\pi\mathbb{Z}$, then $\gamma_1 = \gamma_2$. 
Using the orthogonal decomposition

\[ F_j(0) = p \quad F_j'(s) = \partial_{j,s}, \]

where \( \partial_{j,s} \) is the unit tangent vector to \( \gamma_j \). Let \( F_j = (r_j, \theta_j) \), where \( \theta_j : \mathbb{R} \to \mathbb{R}/2\pi \mathbb{Z} \). Using the orthogonal decomposition

\[ \partial_{j,s} = g^N(\partial_r, \partial_{j,s})\partial_r + g^N(\frac{1}{\varphi}\partial_{\theta}, \partial_{j,s})\partial_{\theta} = r_j'(s)\partial_r + \varphi\theta_j'(s)\partial_{\theta}, \]

we deduce that

\[ \theta_j'(s)^2 = \frac{1 - r_j'(s)^2}{\varphi^2(r_j(s))}, \]

and thus

(5.2)

\[ r_j(0) = r(p) \quad \theta_j(0) = \theta(p) \quad r_j' = a \cos(\theta + \alpha) \quad \theta_j' = \frac{\sqrt{1 - a^2 \cos^2(\theta_j + \alpha)}}{\varphi(r_j)}. \]

We are allowed to choose the positive square root since \( g^N(\partial_{\theta}, \partial_{j,s}) > 0 \) implies that \( \theta_j' > 0 \).

Since solutions to (5.2) are unique, we conclude that \( F_1 = F_2 \), and thus \( \gamma_1 = \gamma_2 \), as desired.

To prove that the only graphs satisfying \( r_s = a \cos \theta \) are circles, we find it convenient to split the argument into three sections depending on the ambient space \( \mathbb{R}^2, \mathbb{S}^2 \) or \( \mathbb{H}^2 \).

**Lemma 5.5.** Given any \( a \in (-1, 1) \) and \( \alpha \in \mathbb{R}/2\pi \mathbb{Z} \), the circle

(5.3) \( \mathbb{C}(a, \alpha, R) := \text{im}(\beta \mapsto (aR \sin \alpha + R \cos \beta, aR \cos \alpha + R \sin \beta)) \)

satisfies \( r_s = a \cos(\theta + \alpha) \).

**Proof.** By rotational symmetry of \( \mathbb{R}^2 \) it suffices to prove the case \( \alpha = 0 \). Then we note that

\[ r_s = a \cos(\theta) \iff rr_s = ax \iff xx_s + yy_s = ax, \]

plugging in \( x(\beta), y(\beta) \), we obtain

\[ xx_s + yy_s = -R \cos \beta \sin \beta + aR \cos \beta + R \cos \beta \sin \beta = aR \cos \beta = ax, \]

and so indeed \( r_s = a \cos(\theta) \) on the circle \( \text{im}(\beta \mapsto (R \cos \beta, aR + R \sin \beta)) \), which completes the proof.

**Theorem 5.6.** If \( \gamma \subset \mathbb{R}^2 \) is a \( C^1 \) radial graph which satisfies \( r_s = a \cos(\theta + \alpha) \), then \( \gamma \) is a circle.

**Proof.** First note that if \( |a| \geq 1 \), then \( |r_s(\theta = -\alpha)| \geq 1 \) which contradicts the fact that \( r_s > 0 \), since \( r_s^2 + r_s' = 1 \). Thus \( a \in (-1, 1) \). This argument uses the fact that \( \gamma \) is a radial graph to deduce that \( r_s > 0 \) and that there exists some point on \( \gamma \) with \( \theta = -\alpha \).

Consider the circles \( \mathbb{C}(a, \alpha, R) \) defined in the previous theorem. It is straightforward to see that the two points

\[ c_+(a, \alpha, R) := ((aR + R) \sin \alpha, (aR + R) \cos \alpha) \]
\[ c_-(a, \alpha, R) := ((aR - R) \sin \alpha, (aR - R) \cos \alpha) \]
both lie on $C(a, \alpha, R)$. Since $|a| < 1$, $R \mapsto c_+(R)$ is surjective onto the ray $\mathbb{R}_+(\sin \alpha, \cos \alpha)$ and $R \mapsto c_-(R)$ is surjective onto the ray $\mathbb{R}_-(\sin \alpha, \cos \alpha)$.

Let $p$ be the point on $\gamma$ where the radius function $r$ is maximized. Then $r_*(p) = 0$, so $\theta(p) + \alpha = \pi/2 + \pi \mathbb{Z}$. It follows that $\gamma$ intersects the line $\mathbb{R}(\sin(\alpha), \cos(\alpha))$, and so there is some $R$ such that either $p = c_+(a, \alpha, R)$ or $p = c_-(a, \alpha, R)$. Theorem 5.4 implies that $\gamma = C(a, \alpha, R)$.

Turning now to the $S^2$ case, we consider $S^2 \subset \mathbb{R}^3$ with $x = \sin r \cos \theta$, $y = \sin r \sin \theta$ and $z = \cos r$.

**Theorem 5.7.** The circle $C_R(p) \subset S^2 - \{(0, 0, 0, \pm 1)\}$, $R \neq 0$, satisfies

$$r_s = \frac{y(p)}{\sin R} \cos \theta - \frac{x(p)}{\sin R} \sin \theta.$$

**Proof.** It suffices to prove that

$$(5.4) \quad \sin r r_s = \frac{y(p)}{\sin R} \sin r \cos \theta - \frac{x(p)}{\sin R} \sin r \sin \theta \iff z_s = \frac{x(p)y - y(p)x}{\sin R},$$

where we have used $x = \sin r \cos \theta$, $y = \sin r \sin \theta$, and $z = \cos r$.

To prove this, we note that, at a point $q$ on the circle $C_R(p)$, the unit tangent vector $\partial_s(q)$ satisfies

$$\partial_s(q) \sin R = p \times q \quad \text{ (vector cross product)},$$

which implies that

$$z_s \sin R = \det \begin{pmatrix} x(p) & y(p) \\ x(q) & y(q) \end{pmatrix},$$

which is equivalent to (5.4). \qed

**Theorem 5.8.** If $\gamma \subset S^2 - \{(0, 0, \pm 1)\}$ is a radial graph which satisfies $r_s = a \cos \theta + b \sin \theta$, then $\gamma$ is a circle.

**Proof.** For simplicity, rotate $\gamma$ so that $r_s = b \sin \theta$, with $b > 0$. As in the proof of Theorem 5.6, we may assume that $b < 1$. There is a unique smooth $f : (0, \pi) \rightarrow (0, \pi)$ such that $b \sin R = \sin f(R)$; clearly $f(R) < R$. Now consider the circle $C_R(p)$ where $p = (\sin f(R), 0, \cos f(R))$. Since $x(p) = b \sin R$, $y(p) = 0$, we conclude

*Figure 1.* $C_R(p)$ shown for various $R$ with $b = 0.75$ and $0.95$, respectively.

from Theorem 5.7 that $C_R(p)$ also satisfies $r_s = b \sin \theta$. It is clear that the points $(\sin(f(R) \pm R), 0, \cos(f(R) \pm R))$ lie on $C_R(p)$. Define

$$g_1(R) = f(R) + R \quad g_2(R) = f(R) - R,$$
Since \( f'(R) = b \cos R / \cos f(R) \), we obtain

\[
f'(R)^2 = \frac{b^2 \cos^2 R}{1 - b^2 + b^2 \cos^2 R} < 1,
\]
so \( g_1(R) \) is increasing, and similarly, \( g_2(R) \) is decreasing. Since \( g_1(\pi) = \pi \) and \( g_2(\pi) = -\pi \), we conclude that \( g_1 \) is surjective onto \((0, \pi)\) and \( g_2 \) is surjective onto \((-\pi, 0)\).

Now let \( \mu \in \gamma \) be the point which maximizes the radial function \( r_\gamma \). Then \( r_\gamma(\mu) = 0 \), so \( \sin \theta(\mu) = 0 \), so \( \mu = (\sin \alpha, 0, \cos \alpha) \) for some \( \alpha \). Since \( \mu \neq (0, 0, \pm 1) \), we can take \( \alpha \in (-\pi, 0) \cup (0, \pi) \). Thus there is some \( R \) such that \( \alpha = g_1(R) \) or \( \alpha = g_2(R) \), and thus \( \mu \in C_R(p) \). Since \( C_R(p) \) also satisfies \( r_\gamma = b \sin \theta \), we may invoke Theorem 5.9 to conclude that \( \gamma = C_R(p) \). \( \square \)

A similar argument to the \( \mathbb{S}^2 \) case works for the \( \mathbb{H}^2 \) case (we found it useful to work in the hyperboloid model \( \mathbb{H}^3 \subset \mathbb{R}^4 \)), and we obtain

**Theorem 5.9.** Let \( N = \mathbb{R}^2, k^{-1}\mathbb{S}^2 \) or \( k^{-1}\mathbb{H}^2 \). If \( \gamma_0 \subset N \) is a \( C^1 \) radial graph satisfying \( r_\gamma = \alpha \cos \theta + b \sin \theta \), then \( \gamma_0 \) is a circle (which may be translated).

**Theorem 5.10.** Let \( N \) be a warped product space, let \( \gamma_0 \subset N \) be a piecewise \( C^1 \) Lipshitz radial graph satisfying \( L[\gamma_0] = F(A[\gamma_0]) \), and suppose \( 0 \leq \varphi' \varphi' - \varphi \varphi'' \leq 1 \). Then, considering \( \gamma_0 \) as lying in a subset of one of the spaceforms, \( \gamma_0 \) is a circle.

**Proof.** We have already proved the case where \( \varphi' \varphi' - \varphi \varphi'' \neq 1 \) in Theorem 5.2 so we assume that \( \varphi' \varphi' - \varphi \varphi'' = 1 \). Using the same “cut and reflect” technique used in Theorem 4.3 and Theorem 5.2, we may cut \( \gamma_0 \) (along an axis \( \alpha \in \mathbb{S}^1 \)) into two bilaterally symmetric halves \( \gamma_1 \) and \( \gamma_2 \). Following Theorem 5.2 the solution \( \gamma_1(t) \) (guaranteed by Theorem 5.1) has constant length, and thus (by Corollary 2.5) \( r_\gamma(t) = a(t) \cos \theta + b(t) \sin \theta \) holds along the flow, and so (by Theorem 5.9) \( \gamma_1(t) \) is a circle for all positive \( t \). Let \( r_1 \) be the (constant) radius of \( \gamma_1(t) \), let \( C_0 \) be the (translated) circle of radius \( r_1 \) fixed under the reflection through \( \alpha \) which contains the points \((\rho_0(\pm \alpha), \pm \alpha)\). The centre of \( C_0 \) is uniquely determined by its axis of symmetry \( \alpha \in \mathbb{S}^1 \), its radius \( r_1 \), and the points \((\rho_0(\pm \alpha), \pm \alpha)\). The centre of \( \gamma_1(t) \) is also determined by its axis of symmetry, its radius, and the point \((\rho_1(\pm \alpha), \alpha)\), and by convergence \( \rho_1(t) \rightarrow \rho_0(\pm \alpha) \), we deduce that \( \gamma_1(t) \) converges to \( C_0 \) uniformly as \( t \rightarrow 0 \). We deduce that \( \gamma_1 = C_0 \), and similarly, \( \gamma_2 = C_0 \), and thus by the construction of \( \gamma_1 \) and \( \gamma_2 \), we conclude \( \gamma_0 = C_0 \), as desired. \( \square \)

Combining Theorems 4.3, 5.2 and 5.10 we conclude the isoperimetric inequality Theorem 1.2 stated in the introduction.

6. Conclusion

It seems to be a general phenomenon that many sophisticated tools used in higher dimensions \( n > 1 \) (such as the Minkowski identity [1.3]) cannot be used when \( n = 1 \). The symmetry argument used in Theorem 4.1 replaced the use of the Minkowski identity, but necessitated a more complicated argument to deal with the non-symmetric case.

As mentioned in section 2.1, the restriction \( \varphi' \varphi' - \varphi \varphi'' \leq 1 \) is sharp - without it the isoperimetric inequality is guaranteed to fail. However, the restriction \( \varphi' \varphi' - \varphi \varphi'' > 0 \) is not sharp (we used this inequality to prove the convergence to a circle). For example, the cylinder \( \mathbb{S}^1 \times [0,1] \) satisfies \( \varphi' \varphi' - \varphi \varphi'' \equiv 0 \), and it is easy to explicitly show that it does satisfy the isoperimetric inequality.
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