Poisson-gradient dynamical systems with convex potential

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Abstract

The basic aim is to extend some results and concepts of non-autonomous second order differential systems with convex potentials to the new context of multi-time Poisson-gradient PDE systems with convex potential. In this sense, we prove that minimizers of a suitable action functional are multiple periodical solutions of a Dirichlet problem associated to the Euler-Lagrange equations. Automatically, these are solutions of the associated multi-time Hamiltonian equations.

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1 Poisson-gradient PDEs

There are two methods to study the periodic solutions of boundary problems attached to some partial derivative equations (PDEs):

- the method of Fourier expansions in terms of eigenfunctions of a PDE operator (the method of separation of variables);
- the method of minimizers of suitable action functionals.

Our paper refers to the second method, continuing the ideas in the papers [11], [14], [19]. We start with the set \( T_0 = [0, T^1] \times \ldots \times [0, T^p] \subset \mathbb{R}^p \) determined by the diagonal points \( O = (0, \ldots, 0) \), \( T = (T^1, \ldots, T^p) \), and with the Sobolev space \( W^{1,2}_T \) of the functions \( u \in L^2[T_0, \mathbb{R}^n] \), having weak derivatives \( \frac{\partial u}{\partial t} \in L^2[T_0, \mathbb{R}^n] \). The weak derivatives are defined using the space \( C^{\infty}_T \) of all indefinitely differentiable multiple T-periodic functions from \( \mathbb{R}^p \) into \( \mathbb{R}^n \).
We denote by $H^1_T$ the Hilbert space associated to the Sobolev space $W^{1,2}_T$. The euclidean structure on $H^1_T$ is given by the scalar product

$$
\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i(t) v^j(t) + \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial \psi^\alpha}(t) \frac{\partial v^j}{\partial \psi^\beta}(t) \right) dt^1 \wedge ... \wedge dt^p
$$

and the associated Euclidean norm. These are induced by the scalar product (Riemannian metric)

$$
G = \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \delta^{\alpha\beta} \delta_{ij}
\end{pmatrix}
$$
on $R^{n+np}$ (see the jet space $J^1(T_0, R^n)$).

Let $t = (t^1, ..., t^p)$ be a generic point in $R^p$. Then the opposite faces of the parallelepiped $T_0$ can be described by the equations

$$
S^-_i : t^i = 0, S^+_i : t^i = T^i
$$

for each $i = 1, ..., p$.

Suppose the function $u(t)$ has a weak Laplacian $\Delta u$ and $u \to F(t, u)$ is a convex function. In these hypothesis, we formulate some conditions in which the Dirichlet problem (associated to a Poisson-gradient PDE system)

$$
\begin{align*}
\Delta u(t) &= \nabla F(t, u(t)) \\
\left. u \right|_{S^-_i} &= \left. u \right|_{S^+_i}, \left. \frac{\partial u}{\partial t} \right|_{S^-_i} &= \left. \frac{\partial u}{\partial t} \right|_{S^+_i}, i = 1, ..., p
\end{align*}
$$

has solution. To do that, we denote

$$
\left| \frac{\partial u}{\partial t} \right|^2 = \delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial \psi^\alpha} \frac{\partial u^j}{\partial \psi^\beta}
$$

and we use the Lagrangian

$$
L \left( t, u(t), \frac{\partial u}{\partial t} \right) = \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F(t, u(t))
$$

and the action

$$
\varphi(u) = \int_{T_0} L \left( t, u(t), \frac{\partial u}{\partial t} \right) dt^1 \wedge ... \wedge dt^p
$$
Then, using minimizing sequences, we show that the action $\varphi$ has a minimum point $u$ (extremal, solution of the Poisson-gradient dynamical system (1), satisfying the boundary conditions (2)). Consequently the solution $u$ is multiple periodical, with the reduced period $T = (T^1, \ldots, T^p)$. Our arguments extend those of the book [6], Theorems 1.4, 1.5, 1.7 and 1.8, which are dedicated to single-time problems.

2 Periodic solutions of Poisson-gradient PDEs

Let us show that some conditions upon the potential $F$ ensure periodic solutions for the problem (1)+(2).

**Theorem 1** Let $F : T_0 \times \mathbb{R}^n \to R, (t, x) \to F(t, x)$ and $|x| = \sqrt{\delta_{ij}x^i x^j}$. We consider that $F(t, x)$ is measurable in $t$ for any $x \in \mathbb{R}^n$ and of class $C^1$ in $x$ for any $t \in T_0$.

If there exist $a \in C^1 (\mathbb{R}^+, \mathbb{R}^+)$ with the derivative $a'$ bounded from above and $b \in C (T_0, \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|) b(t), \quad |\nabla x F(t, x)| \leq a(|x|) b(t),$$

for any $x \in \mathbb{R}^n$ and any $t \in T_0$, then the action (4) is of class $C^1$.

**Proof.** The reasons are similar to those in [15, Theorem 3].

**Corollary 2** The same hypothesis as in Theorem 1. If $u \in H^1_T$ is a solution of the equation $\varphi'(u) = 0$ (critical point), then the function $u$ has a weak Laplacian $\Delta u$ (the Jacobian matrix $\frac{\partial u}{\partial t}$ has a weak divergence) and

$$\Delta u = \nabla F(t, u(t))$$
a.e. on $T_0$ and

$$u \big|_{S^-} = u \big|_{S^+}, \quad \frac{\partial u}{\partial t} \big|_{S^-} = \frac{\partial u}{\partial t} \big|_{S^+}.$$ (5)

**Proof.** We build the function

$$\Phi : [-1, 1] \to R,$$

$$\Phi (\lambda) = \varphi (u + \lambda v) =$$
\[ \int_{T_0} \left[ \frac{1}{2} \left\| \frac{\partial}{\partial t} (u(t) + \lambda v(t)) \right\|^2 + F(t, u(t) + \lambda v(t)) \right] \, dt^1 \wedge ... \wedge dt^p, \]

where \( v \in C_T^\infty \). The point \( \lambda = 0 \) is a critical point of \( \Phi \) if and only if the point \( u \) is a critical point of \( \varphi \). Consequently

\[ 0 = \langle \varphi'(u), v \rangle = \int_{T_0} \left[ \delta_{ij} \delta_{\alpha\beta} \frac{\partial u_i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} + \delta_{ij} \nabla^i F(t, u(t)) v^j(t) \right] \, dt^1 \wedge ... \wedge dt^p, \]

for all \( v \in H^1_T \) and hence for all \( v \in C_T^\infty \). Using the definition of the weak divergence,

\[ \int_{T_0} \delta_{ij} \delta_{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} \, dt^1 \wedge ... \wedge dt^p = - \int_{T_0} \delta_{ij} \delta_{\alpha\beta} \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j \, dt^1 \wedge ... \wedge dt^p, \]

the Jacobian matrix \( \frac{\partial u}{\partial t} \) has weak divergence (the function \( u \) has a weak Laplacian) and

\[ \triangle u(t) = \nabla F(t, u(t)) \]

a.e. on \( T_0 \). Also, the existence of the weak derivatives \( \frac{\partial u}{\partial t} \) and weak divergence \( \triangle u \) implies that

\[ u \mid_{s_i^-} = u \mid_{s_i^+}, \quad \frac{\partial u}{\partial t} \mid_{s_i^-} = \frac{\partial u}{\partial t} \mid_{s_i^+}. \]

**Remark.** If the function \( u \) is at least of class \( C^2 \), then the definition of the weak divergence of the Jacobian matrix \( \frac{\partial u}{\partial t} \) (or of the weak Laplacian \( \triangle u \)) coincides with the classical definition. This fact is obvious if we have in mind the formula of integration by parts

\[ \int_{T_0} \delta_{ij} \delta_{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} \, dt^1 \wedge ... \wedge dt^p = \int_{T_0} \delta_{ij} \delta_{\alpha\beta} \left( \frac{\partial u^i}{\partial t^\alpha} v^j \right) \, dt^1 \wedge ... \wedge dt^p - \int_{T_0} \delta_{ij} \delta_{\alpha\beta} \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j \, dt^1 \wedge ... \wedge dt^p. \]

**Corollary 3** The same hypothesis as in Theorem 1. If \( \| x \| \to \infty \) implies

\[ \int_{T_0} F(t, x) \, dt^1 \wedge ... \wedge dt^p \to \infty \] and \( F(t, x) \) is convex in \( x \) for any \( t \in T_0 \),
then there exists a function u that is a solution of the boundary value problem (5).

**Proof.** Let $G : \mathbb{R}^{n} \to \mathbb{R}$, $G(x) = \int_{T_0}^{T} F(t, x) \, dt^1 \land \cdots \land dt^p$. By assumptions, the convex function $G$ has a minimum point $x = \overline{x}$. Consequently, $\nabla G(\overline{x}) = \int_{T_0}^{T} \nabla F(t, \overline{x}) \, dt^1 \land \cdots \land dt^p = 0$.

Let $(u_k)$ be a minimizing sequence for the action (4). We use the decomposition $u_k = \overline{u}_k + \tilde{u}_k$, where $\overline{u}_k = \int_{T_0}^{T} u_k(t) \, dt^1 \land \cdots \land dt^p$. The convexity of $F$ implies

$$F(t, u_k(t)) \geq F(t, \overline{x}) + (\nabla F(t, \overline{x}), u_k(t) - \overline{x}).$$

It follows

$$\varphi(u_k) \geq \frac{1}{2} \int_{T_0}^{T} \left| \frac{\partial u_k}{\partial t} \right|^2 \, dt^1 \land \cdots \land dt^p +$$

$$+ \int_{T_0}^{T} F(t, \overline{x}) \, dt^1 \land \cdots \land dt^p + \int_{T_0}^{T} (\nabla F(t, \overline{x}), u_k(t) - \overline{x}) \, dt^1 \land \cdots \land dt^p$$

$$= \frac{1}{2} \int_{T_0}^{T} \left| \frac{\partial u_k}{\partial t} \right|^2 \, dt^1 \land \cdots \land dt^p +$$

$$+ \int_{T_0}^{T} F(t, \overline{x}) \, dt^1 \land \cdots \land dt^p +$$

$$+ \int_{T_0}^{T} (\nabla F(t, \overline{x}), \tilde{u}_k(t)) \, dt^1 \land \cdots \land dt^p.$$

On the other hand, by Schwartz inequality, we can write

$$(\nabla F(t, \overline{x}), \tilde{u}_k(t)) \leq |\nabla F(t, \overline{x})| |\tilde{u}_k(t)| \leq a(|\overline{x}|) b(t) |\tilde{u}_k(t)|.$$
where \( b_0 = \max_{t \in T_0} b(t) \). Using the Wirtinger inequality for multiple integral, we find

\[
\varphi(u_k) \geq \frac{1}{2} \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p + \int_{T_0} F(t, \bar{x}) dt^1 \wedge \ldots \wedge dt^p - a(|\bar{x}|) b_0 C_1 \left( \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}},
\]

with \( C_1 > 0 \). Thus

\[
\varphi(u_k) \geq \frac{1}{2} \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p + C_2 - C_3 \left( \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}},
\]

and, consequently, the function of degree two in the right hand member must be a decreasing restriction, i.e., there exists \( C_4 > 0 \), such that

\[
\int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p < C_4.
\]

It follows

\[
\int_{T_0} \left| \frac{\partial \tilde{u}_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p < C_4
\]

and so \( \| \tilde{u}_k \| < C_5 \).

Again, the convexity of \( F \) leads to

\[
F \left( t, \frac{u_k}{2} \right) = F \left( t, \frac{1}{2} (u_k(t) - \tilde{u}_k(t)) \right) \leq \frac{1}{2} F(t, u_k(t)) + \frac{1}{2} F(t, -\tilde{u}_k(t)) \leq F(t, u_k(t)) + \frac{1}{2} F(t, -\tilde{u}_k(t)),
\]

\( \forall t \in T_0, \forall k \in N \), so

\[
F(t, u_k(t)) \geq 2 F \left( t, \frac{\tilde{u}_k(t)}{2} \right) - F(t, -\tilde{u}_k(t)).
\]

Consequently

\[
\varphi(u_k) = \frac{1}{2} \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p + \int_{T_0} F(t, u_k(t)) dt^1 \wedge \ldots \wedge dt^p \geq \frac{1}{2} \int_{T_0} \left| \frac{\partial u_k}{\partial t} \right|^2 dt^1 \wedge \ldots \wedge dt^p +,
\]

6
\[ +2 \int_{T_0}^T F \left( t, \frac{\nu_k}{2} \right) \, dt^1 \wedge ... \wedge dt^p - \int_{T_0}^T F \left( t, -\bar{u}_k(t) \right) \, dt^1 \wedge ... \wedge dt^p \]

and hence \( \varphi (u_k) \geq 2 \int_{T_0}^T F \left( t, \frac{\nu_k}{2} \right) \, dt^1 \wedge ... \wedge dt^p - c_0. \)

This means that \( ||\nu_k|| \to \infty. \) So the sequence \((\nu_k)\) is bounded and implicitly the sequence \((u_k)\) is bounded in \(H^1_T.\) The Hilbert space \(H^1_T\) is reflexive. By consequence, the sequence \((u_k)\) (or one of his subsequence) is weakly convergent in \(H^1_T\) with the limit \(u.\) The Mazur’s theorem assure that there exists a sequence \((v_k)\) with the general term \( v_k = \sum_{j=1}^{k} \alpha_{kj} u_j, \sum_{j=1}^{k} \alpha_{kj} = 1, \alpha_{kj} \geq 0, \) which is strongly converges to \(u\) in \(H^1_{\mathcal{S}T}.\)

Now we consider \(c > \lim \varphi (u_k).\) Going if necessary to a subsequence, we can assume that \(c > \varphi (u_k)\) for all \(k \in \mathbb{N}^*.\) Since \(\varphi\) is lower semi-continuous in \(H^1_T\) and \(\varphi\) is convex, we obtain

\[
\varphi (u) \leq \lim \varphi (v_k) \leq \lim \left( \sum_{j=1}^{k} \alpha_{kj} \varphi (u_j) \right) \leq \left( \sum_{j=1}^{k} \alpha_{kj} \right) c = c.
\]

Because \(c > \lim \varphi (u_k)\) is arbitrary, we have \(\varphi (u) \leq \lim \varphi (u_k).\)

Thus, the action \(\varphi (u)\) has a minimum point \(u\) in \(H^1_T,\) and so \(u\) is a solution of the problem (5).

Thanks to the properties of the strictly convex functions, we can reinforce the previous theorem. For that, we recall two equivalent properties of a strictly convex function \(G \in C^1 (R^n, R):\)

1) \(G\) has a critical point \(\bar{x} \in R^n;\)
2) \(G (x) \to \infty \) when \(|x| \to \infty.\)

**Theorem 2.** We consider the problem (1)+(2). Suppose \(F : T_0 \times R^n \to R, (t, x) \to F(t, x)\) has the properties:

1) \(F(t, x)\) is measurable in \(t\) for any \(x \in R^n\) and of class \(C^1\) in \(x\) for any \(t \in T_0.\)
2) There exist \(a \in C^1 (R^+, R^+)\) with the derivative \(a'\) bounded from above and \(b \in C (T_0, R^+)\) so that

\[
|F(t, x)| \leq a(|x|) b(t), |\nabla_x F(t, x)| \leq a(|x|) b(t),
\]

for any \(x \in R^n\) and any \(t \in T_0.\)
3) The function $F(t, \cdot)$ is strictly convex for any $t \in T_0$.
Then, the following statements are equivalent:

1) The problem (1) + (2) has solutions;
2) There exists $\bar{x} \in \mathbb{R}^n$ so that $\int_{T_0} \nabla F(t, \bar{x}) dt^1 \wedge ... \wedge dt^p = 0$;
3) $\int_{T_0} F(t, x) dt^1 \wedge ... \wedge dt^p \to \infty$ as $|x| \to \infty$.

Proof (see single-time case in [6, Theorem 1.8]). Let us prove that 1) implies 2):
We suppose that $u(t)$ is a solution of the problem (1)+(2). By integration we obtain
$$\sum_{i=1}^{p} \int_{T_0} \frac{\partial^2 u^j}{\partial t^i} dt^1 \wedge ... \wedge dt^p = \int_{T_0} \frac{\partial F}{\partial u^j} (t, u(t)) dt^1 \wedge ... \wedge dt^p.$$  
From the boundary conditions it results
$$\int_{T_0} \nabla F(t, u(t)) dt^1 \wedge ... \wedge dt^p = 0. \quad (6)$$

On the other hand, the function $G(x) = \int_{T_0} F(t, x) dt^1 \wedge ... \wedge dt^p$ is strictly convex, because the function $F(t, \cdot)$ is strictly convex.

We suppose $u = \bar{u} + \bar{x}, \bar{x} = \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p,$
$$\tilde{G}(x) = \int_{T_0} F(t, x + u(t)) dt^1 \wedge ... \wedge dt^p.$$  
From (6) we have $\nabla \tilde{G}(\bar{x}) = 0$. From the properties of a strictly convex function, mentioned above, $\tilde{G}(x)$ tends to $\infty$ when $|x|$ tends to $\infty$. Because the function $F(t, \cdot)$ is strictly convex, we obtain:
$$\tilde{G}(x) \leq \frac{1}{2} \int_{T_0} F(t, 2x) dt^1 \wedge ... \wedge dt^p + \frac{1}{2} \int_{T_0} F(t, 2u(t)) dt^1 \wedge ... \wedge dt^p = \frac{1}{2} G(2x) + C.$$  
For $|x| \to \infty$, $\tilde{G}(x) \to \infty$ and consequently $G(2x) \to \infty$ and $G(x) \to \infty$. According to the properties of $G$, there exists $\bar{x}$ so that $\nabla G(\bar{x}) = 0$, i.e.,
$$\int_{T_0} \nabla F(t, \bar{x}) dt^1 \wedge ... \wedge dt^p = 0.$$
Let us show that 2) implies 3):

The properties of $G$ show that if $\pi$ exists so that $\nabla G(\pi) = 0$, then $G(x) \to \infty$ when $|x| \to \infty$, so $\int_{\Gamma_0} F(t, x) \, dt^1 \wedge ... \wedge dt^p \to \infty$ when $|x| \to \infty$.

Now, 3) implies 1). Indeed, the required implication is realized by the Theorem 1.

**Remark.** The previous results can be extended to PDEs produced in [12]-[19].

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