Anderson Localization in 1D Systems with Correlated Disorder

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Abstract. Anderson localization has been a subject of intense studies for many years. In this context, we study numerically the influence of long-range correlated disorder on the localization behavior in one dimensional systems. We investigate the localization length and the density of states and compare our numerical results with analytical predictions. Specifically, we find two distinct characteristic behaviors in the vicinity of the band center and at the unperturbed band edge, respectively. Furthermore we address the effect of the intrinsic short-range correlations.

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1 Introduction

Over the past fifty years the Anderson model of localization has become a paradigm for investigations of electronic transport in the presence of static disorder [1, 2]. Depending on the strength of the disorder the wave functions of non-interacting electrons vary from delocalized to exponentially localized. This so called Anderson localization was demonstrated recently in experiments with matter waves of Bose-Einstein condensates in random or quasi-periodic optical potentials [3, 4]. Due to significant electron-electron and electron-phonon interactions in realistic materials, e.g. imperfect crystals, the direct observation of Anderson localization is rather difficult. Therefore the success of the model is driven to a large extent by the universal properties of the resulting phenomena.

The characteristics of the disorder potential have a prominent influence on Anderson localization [2]. For one dimensional (1D) systems with a spatially uncorrelated potential it has been shown that all electronic states are exponentially localized [6]. Later it was realized that by introducing correlations, one can partly suppress the localization, at least for weak disorder [6]. Moreover, there have been discussions of a delocalization-localization transition even in 1D for long-range correlated disorder potentials [7–9]. However, it has early been recognized that the apparent transition results from a rescaling of the disorder potential by normalizing the variance, which becomes system-size dependent for the considered self-affine potential landscapes [10].

In the present work, we focus on long-range power law correlated potentials with a correlation function \( C(\ell) \propto \ell^{-\alpha} \) and correlation exponents \( \alpha > 0 \). The variance of the resulting potential is independent of the system size. This so called scale-free disorder is by no means artificial. It appears naturally in a large variety of physical systems [11–13].

The outline of this article is as follows. In the next section we briefly review the main results for Anderson localization in 1D and in the presence of correlated disorder. We also discuss the numerical methods used in our calculations. In Sec. 3 we present and discuss the obtained results. Finally, the article is summarized in Sec. 4.

2 Model and Numerical Techniques

2.1 Anderson Model of Localization with Long-Range Correlated Disorder

The Anderson model [12] is widely used to investigate the phenomenon of localization in disordered materials. It is based upon a tight-binding Hamiltonian in site representation,

\[
H = \sum_i \varepsilon_i \langle i | i \rangle - \sum_{\langle i,j \rangle} | i \rangle \langle j | ,
\]

where \( | i \rangle \) is a localized state at lattice site \( i \) and \( \langle i,j \rangle \) denotes a restriction of the sum to \( i \) and \( j \) being nearest neighbors. The hopping parameter \( t = \hbar^2 / (2m^*a^2) \) is defined in terms of the lattice spacing \( a \) and the effective mass \( m^* \). In the following, we set \( t \equiv 1 \) and thus fix the unit of energy; lengths are measured in units of \( a \).

The on-site potentials \( \varepsilon_i \) are random numbers, chosen according to some probability distribution \( P(\varepsilon) \) characterized by the mean \( \langle \varepsilon_i \rangle \) and the correlation function \( \langle \varepsilon_i \varepsilon_{i+\ell} \rangle = C(\ell) \). However, usually the site energies are taken to be statistically independent. For example, a common choice of \( P(\varepsilon) \) is a Gaussian white noise distribution or a box distribution of width \( W \), both with \( \langle \varepsilon_i \rangle = 0 \) and \( C(\ell) = \frac{W^2}{12} \delta_{\ell,0} \).
Other distributions have also been considered \cite{1,4,16}. In the present work we are interested in the influence of long-range correlated disorder potentials on the Anderson model, i.e., using a correlation function of the form

\[ C(\ell) \equiv \langle \varepsilon_i \varepsilon_{i+\ell} \rangle \propto |\ell|^{-\alpha} \quad (2) \]

where \( \alpha \) is the correlation exponent which determines the strength of the correlations. The associated spectral density is given by

\[ S(q) = \sum_\ell C(\ell) e^{-iq\ell} \propto |q|^{\alpha-1}. \quad (3) \]

In order to generate physically reasonable long-ranged correlated potentials, the correlation function (2) should decay with distance \( \ell \) and therefore \( \alpha > 0 \). For 1D systems the correlations are considered to be long ranged for \( 0 < \alpha < 1 \). The value \( \alpha = 1 \) corresponds to the case of uncorrelated disorder. Notice that the delocalization-localization transition observed in Refs. [7–9] relies on power-law correlations with \( \alpha < 0 \). These correlations increase with distance \( \ell \) and lead to a system-size dependence of the disorder potential. Thus a rescaling of the results is required which yields the correct localization behavior without transition \cite{10}.

In general, it is extremely complicated to obtain analytical results of transport properties for the Anderson model of localization. For example, only in the 1D case rigorous proofs of strong localization for all energies and disorder strengths have been given \cite{5}. Moreover, the explicit energy and disorder dependence of the localization length \( \Lambda \) for weak disorder has been derived \cite{17,18}. There are also some results for 1D systems with long-range correlated disorder. For energies close to the band center \((|E| \ll 2)\) a weak disorder expansion for \( \Lambda \) yields \cite{6}.

\[ \Lambda(q, W) = \left[ \frac{1}{8 \sin^2(q)} \frac{W^2}{12} S(2q) \right]^{-1}, \quad (4) \]

where \( E(q) = -2 \cos(q) \). It is important to notice that even for the correlated case the localization length remains proportional to \( W^{-2} \). Thus, for small disorder strength and in the vicinity of the band center, the presence of correlations leads to an effective disorder strength \( W_{\text{eff}} = W \sqrt{S(2q)} \), which depends on energy via the spectral density \( S(q) \).

On the other hand, close to the unperturbed band edge \((|E| \approx 2)\) a different universal behavior can be found \cite{19,20}. Here, the density of states (DOS) and the localization length are characterized by units of energy \( \epsilon \) and length \( \lambda \),

\[ \rho_\alpha(E) = \lambda^{-1} \epsilon^{-1} f_\alpha(E/\epsilon), \quad (5a) \]

\[ \Lambda_\alpha(E) = \lambda g_\alpha(E/\epsilon), \quad (5b) \]

where \( f_\alpha \) and \( g_\alpha \) are universal functions. For uncorrelated potentials, i.e. \( \alpha = 1 \), there is a closed analytic form available for both functions \cite{21}. The units \( \epsilon \) and \( \lambda \) can be expressed in terms of the hopping parameter \( t \) and the disorder strength \( w = W/\sqrt{12} \),

\[ \epsilon = w^{4/(4-\alpha)} \quad (6a) \]

\[ \lambda = w^{-2/(4-\alpha)} \quad (6b) \]

In particular, it follows that at the unperturbed band edge \((|E| = 2)\),

\[ A_\alpha(|E| = 2, W) \propto W^{-y} \quad (7a) \]

\[ \rho_\alpha(|E| = 2, W) \propto W^{-y} \quad (7b) \]

with \( y = 2/3 \) for \( \alpha = 1 \) and \( y = 2/(4-\alpha) \) for \( \alpha \leq 1 \) \cite{21,22}. Most interestingly, within the so called white-noise model (WNM) scaling expressions identical to Eqs. (5) are obtained where \( \alpha \) is replaced by the dimension \( d \) of the system. We would like to emphasize that the WNM does not consider long-range correlated disorder, but the similarity suggests that the influence of power-law correlations at the band edge may possibly be interpreted in terms of an effectively reduced dimensionality.

\subsection{2.2 Numerical Methods}

In order to generate random numbers with power-law correlations \cite{2} we use the modified Fourier filtering method (FFM) \cite{25} with one additional step. Hereby, the correlation function is given by

\[ C(\ell) = (1 + \ell^2)^{-\alpha/2} \propto |\ell|^{-\alpha} \quad (\ell \gg 1), \quad (8) \]

which avoids the singularity at \( \ell = 0 \) and therefore improves the filtering. For large distances \( \ell \gg 1 \) this form resembles the desired power-law behavior \cite{2}. An additional benefit of Eq. (8) consists in avoiding aliasing effects, which may obscure the long-range character of the correlations. The spectral density \( S(q) \) can be calculated analytically and is given in terms of a modified Bessel function \cite{25}. Overall, the special choice of the correlation function (8) leads to a modified behavior with respect to the correlation exponent \( \alpha \). While for the pure power-law case the localization behavior is obtained for \( \alpha = 1 \) (see also Eq. (5a)), the modified FFM generates uncorrelated random numbers only in the limit \( \alpha \to \infty \). Therefore, it is expected to find an influence on the localization behavior even for \( \alpha > 1 \).

Additionally, we shift and normalize the obtained sequence of correlated random numbers such that the mean vanishes and the variance is \( W^2/12 \) \cite{2}. Thereby, only the strength of the disorder is adjusted while the correlations of the potential are preserved. Then we calculate the localization length using a standard transfer-matrix method (TMM) \cite{2} with a new seed of the random number generator for each parameter combination \((E, W, \alpha)\). The DOS is calculated by diagonalizing the Hamiltonian \cite{1} and counting the eigenvalues in finite energy bins. In each case 100 realizations are taken into account for averaging.

Due to the symmetry of the Hamiltonian \cite{1} and the chosen symmetric disorder distribution \( P(\epsilon) \) the results...
should depend on the absolute value of $E$ only. This is confirmed by our calculations. Therefore we use the arithmetic average of values for $-E$ and $+E$ to improve the accuracy of our results.

In the following, we consider chains of fixed length $L = 2^{19}$ for the TMM calculations and $L = 2^{13}$ for the DOS computation. The correlation exponents are $\alpha = \infty, 2.0, 1.0$ and 0.5.

3 Results and Discussion

In Fig. 1 we show the energy dependence of $\Lambda$ and the DOS for fixed disorder strength $W = 0.5$ and different correlation strengths $\alpha$. For energies inside the unperturbed band ($|E| < 2$) the localization length is increased by long-range correlations. In this region one observes an overall good agreement with the weak-disorder result given by Eq. (4). On the other hand, near the band edge ($|E| \approx 2$) $\Lambda$ is decreased by correlations, which will be discussed in more detail below. Another important quantity for transport studies in general is the DOS $\rho(E)$. For an ordered system the DOS can be calculated analytically. In this case it develops van Hove singularities at the band edges $|E| = 2$, which is typical for 1D systems [24] and indicates the presence of long-range order [2]. In case of uncorrelated on-site energies the van Hove singularities are smeared out while maintaining peaks at $|E| = 2$. Inside the unperturbed band the DOS is only weakly changed compared to the DOS of the ordered system. For $|E| > 2$ there is an exponential band tail [2]. For correlated on-site energies Fig. 2 illustrates that the van Hove peak is even less pronounced and shifted towards the band center, which results in a decreased DOS at $|E| = 2$ with decreasing $\alpha$. On the other hand, in the band tails the DOS is getting larger for decreasing $\alpha$.

In the following we discuss the two regions separately and in more detail. Firstly, we focus specifically on the behavior in the center of the band. Figure 3 shows the disorder strength dependence of the localization length at $E = 0$. One can see that introducing correlations leads to a systematically larger localization length. Only for very strong disorder the influence of the correlations vanishes and the localization length for finite $\alpha$ approaches the uncorrelated localization length. Also shown is the localization length obtained from Eq. (4) using the same parameters as in the TMM calculation. It is important to notice, that there is no fitting procedure involved at this point. Therefore, Eq. (4) allows for an independent check of the numerics and in particular of the correct behavior of the generated correlations. Figure 3 shows a good agreement of $\Lambda$ and the weak disorder expansion result (4) for $W < 1$. In contrast to $\Lambda$ the DOS in the band center has no particular dependence on the correlation strength for weak disorder as demonstrated in Fig. 2. Therefore the influ-
enence of the long-ranged correlations can be interpreted in accordance with Eq. (4) as an effective disorder of the potential. Hereby, the effective disorder strength becomes smaller with decreasing $\alpha$.

A qualitatively different behavior of the localization length is observed for energies close to the unperturbed band edge ($\vert E \vert = 2$). This is shown in Fig. 4. Here the localization is enhanced in case of a correlated potential and $\Lambda$ is consequently smaller compared to the uncorrelated case. However, for weak disorder $W < 1$ one can still observe a power-law decay according to Eq. (7), but with an exponent $y < 2/3$. The results of least-squares fits of Eq. (7) to the numerical data are summarized in Tab. 1 and also shown in Fig. 4. The obtained exponents for the uncorrelated potential as well as for $\alpha = 2.0$ and 0.5 agree with the values predicted by Eqs. (7). Only for $\alpha = 1.0$ we observe a distinct deviation from the expected value $y = 2/3$ which could be a consequence of the modified FFM [23]. From Eq. (5) one expects the FFM to generate an uncorrelated potential for $\alpha = 1$ since $S(q)$ becomes independent of $q$ in this case. However, due to the short-range cutoff introduced in Eq. (5), the spectral density is given in terms of a modified Bessel function [23], which is independent of $q$ only in the limit $\alpha \to \infty$. Therefore, the behavior of the localization length and the DOS at the band edge is sensitive to the specific choice of the short-range cutoff for sufficiently weak disorder. On the other hand, for very strong disorder the localization lengths for the correlated cases coincide with the uncorrelated case. Figure 5 shows the DOS at the band edge for small disorder strengths. Similar to the localization length one observes a power-law decay of the DOS, which is also in accordance with the scaling law (5). The fitted exponents are listed in Tab. 1. Their values agree with the results obtained from the localization length but are less accurate.

4 Summary and Conclusions

In summary, we have numerically investigated the role of long-range correlated disorder on Anderson localization in 1D systems. Specifically, we have studied the behavior of the localization length and the DOS as functions of energy and disorder strength for different correlation exponents. Hereby, we found two regions in the electronic band, where universal behavior can be observed. In the vicinity of the band center and for weak disorder the localization

![Fig. 3. (Color online) Localization length $\Lambda_\alpha$ vs disorder strength $W$ at the band center $E = 0$ for a 1D chain of length $L = 2^{19}$ obtained from TMM calculations. Different symbols denote various correlation parameters $\alpha$. Dashed lines show $\Lambda_\alpha$ given by Eq. (4).](image1)

![Fig. 4. (Color online) Localization length $\Lambda_\alpha$ vs disorder strength $W$ at the unperturbed band edge $E = 2$ for a 1D chain of length $L = 2^{19}$. Different symbols indicate results obtained from TMM calculations. Error bars are smaller than the symbol size. The full line shows the exact result for uncorrelated disorder [21]. Different colors denote various correlation parameters $\alpha$. Dashed lines show results of a least squares fit of $\Lambda_\alpha \propto W^{-\gamma}$ to the data.](image2)
length is determined by an effective disorder strength. The localization length remains proportional to $W^{-2}$ in this regime. The explicit expression, which has been derived in Ref. [6], is given by Eq. (4). A qualitatively different behavior is found at the band edge. Here, the localization length and DOS are given by Eqs. (5), which implies a power-law behavior for both quantities as functions of disorder strength. A comparison with the WNM suggests that in this region the long-range correlations lead to an effectively reduced dimension.

Overall, we find good agreement of our numerical results with the respective analytical expressions. However, we also find that the particular realization of the long-range correlations plays an important role. In the present case the FFM [23] makes use of a modified correlation function given by Eq. (8), which contains the desired long-range but also short-range contributions. Our numerical results indicate that the modification leads to observing an unexpected influence of the correlations even for correlation exponents $\alpha > 1$. This behavior should be taken into account for investigations involving realistic disorder potentials with long-range correlations.

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