Landis-Oleinik Conjecture in the Exterior Domain

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Abstract

In 1974, Landis and Oleinik conjectured that if a bounded solution of a parabolic equation decays fast at a time, then the solution must vanish identically before that time, provided the coefficients of the equation satisfy appropriate conditions at infinity. We prove this conjecture under some reasonable assumptions on the coefficients which improved the earlier results.

Keywords: Carleman estimates; Unique continuation; Backward uniqueness; Landis and Oleinik; Parabolic equation.

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1 Introduction

The behavior of solutions of heat equations arose many interests in last few decades. In 1974, Landis and Oleinik [1] proposed the following conjecture:

If \( u(x,t) \) is a bounded solution of a uniformly parabolic equation

\[
\sum_{i,j} \partial_i (a^{ij}(x) \partial_j u) - \partial_t u + b(x) \cdot \nabla u + c(x) u = 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T],
\]

and the condition

\[
u(x,T) \leq Ne^{-|x|^{2+\epsilon}}, \quad x \in \mathbb{R}^n,
\]

holds for some positive constants \( N \) and \( \epsilon \), then \( u(x,t) \equiv 0 \) in \( \mathbb{R}^n \times [0,T] \), provided that the coefficients of the equation satisfy appropriate conditions at infinity.

The original conjecture only assumes that the coefficients are time-independent and does not mention the precise conditions, however, the Lipschitz continuous assumption with some decay at infinity on \( a^{ij}(x) \) seems reasonable and we may also consider the space-time dependent case.

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Landis-Oleinik conjecture is closely related to many important problems. In particular, if \( u(x,T) = 0 \), the conjecture is reduced to the backward uniqueness problem for parabolic equations. The backward uniqueness problem has a natural background in the control theory for PDEs, and it also appeared in the regularity theory of parabolic equations, such as the Navier-Stokes equations \([7]\), semi-linear heat equations \([8]\), heat flow of harmonic maps \([9]\).

This conjecture has an elliptic version, where probably the problem originated, the Landis conjecture, namely, if a solution of an elliptic equation decays faster than a given rate at infinity, then it is identically zero. The complex case of Landis conjecture is solved by Meshkov \([10]\), and a quantitative result is proved by Bourgain and Kenig \([11]\), while the real case remains open.

Now we denote the backward parabolic operator

\[
P = \partial_t + \sum_{i,j} \partial_i (a^{ij}(x,t) \partial_j) = \partial_t + \nabla \cdot (A \nabla),
\]

where \( A(x,t) = (a^{ij}(x,t))_{i,j=1}^n \) is a real symmetric matrix such that for some \( \Lambda \geq \lambda > 0 \),

\[
\lambda |\xi|^2 \leq \sum_{i,j} a^{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.
\]  

(2)

In the following we always assume that the lower coefficients \( b \) and \( c \) are space-time dependent and bounded, and rather than \((1)\), we assume a weaker condition

\[
|u(x,0)| \leq C_k e^{-k|x|^2}, \quad \forall \ k > 0.
\]  

(3)

There are some earlier results about Landis-Oleinik conjecture. In the constant coefficients case, i.e., the heat equation, this conjecture was solved by Escauriaza, Kenig, Ponce and Vega \([2]\). They introduced some interesting Carleman estimates and proved both qualitative and quantitative results for the heat equation with bounded space-time dependent coefficients in whole space and half space.

For the general case, the first result is obtained by Nguyen \([3]\) where both qualitative and quantitative results are proved for the conjecture in whole space and half space under the following assumptions

\[
|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \leq M,
\]

(4)

\[
|\nabla_x a^{ij}(x,t)| \leq M \langle x \rangle^{-1-\epsilon},
\]

(5)

\[
|a^{ij}(x,t) - a^{ij}(x,s)| \leq M \langle x \rangle^{-1} |t-s|^{1/2},
\]

(6)

where \( \langle x \rangle = \sqrt{1 + |x|^2} \) and \( \epsilon > 0 \). We remark that condition \((4)\), the Lipschitz regularity assumption is reasonable, as shown in \([12, 13]\), and some decay assumptions seems necessary. However, condition \((5)\) is not scaling invariant and we wonder if condition \((6)\) is necessary.

Another related result is the backward uniqueness result for general parabolic equations in half space proved by the authors \([14]\) under condition \((1)\) and the decay at infinity condition:

\[
|\nabla_x a^{ij}(x,t)| \leq \frac{E}{|x|}, \quad \text{where} \quad E \leq E_0(n, \Lambda, \lambda).
\]  

(7)
Note that condition (7) is scaling invariant. In [13], the authors also constructed examples to show that both condition (4) and (7) are almost optimal.

All these results suggest that if Landis-Oleinik conjecture is true, then certain regularity and decay at infinity assumptions on the coefficients should be required, and assumptions (4) and (7) seem to be optimal.

Now in the exterior domain, under assumptions (4) and (7), we shall prove the Landis-Oleinik conjecture. Our main result is the following.

**Theorem 1.1** Suppose \( \{a^{ij}\} \) satisfy (2), and for some constants \( E, N > 0 \),

\[
|\nabla_x a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \leq M, \quad \forall (x,t) \in \mathbb{R}^n \setminus B_1 \times [0, 1],
\]

and

\[
|\nabla_x a^{ij}(x,t)| \leq \frac{E}{|x|}, \quad \forall (x,t) \in \mathbb{R}^n \setminus B_1 \times [0, 1].
\]

Assume that \( u \) satisfies

\[
\begin{cases}
|Pu| \leq N(|u| + |\nabla u|) & \text{in } \mathbb{R}^n \setminus B_1 \times [0, 1], \\
|u(x,t)| \leq Ne^{-N|x|^2} & \text{in } \mathbb{R}^n \setminus B_1 \times [0, 1], \\
|u(x,0)| \leq Cke^{-k|x|^2}, & \forall k > 0 \text{ in } \mathbb{R}^n \setminus B_1.
\end{cases}
\]

Then there exists a constant \( E_0 = E_0(n, \Lambda, \lambda) \), such that when \( E < E_0 \), we have \( u(x,t) \equiv 0 \) in \( \mathbb{R}^n \setminus B_1 \times [0, 1] \).

By the unique continuation (see [5, 6]) result, we immediately have the following corollary.

**Corollary 1.2** Theorem 1.1 is still valid if we replace \( \mathbb{R}^n \setminus B_1 \) by \( \mathbb{R}^n \).

Theorem 1.1 can be obtained immediately by the following upper bound and lower bound estimates.

**Proposition 1.3 (Upper Bound)** Suppose \( \{a^{ij}\} \) and \( u \) are the same as above. Then there exists a constant \( E_0 = E_0(n, \Lambda, \lambda) \), such that when \( E < E_0 \), we have

\[
|u(x,t)| + |\nabla u(x,t)| \leq e^{-k|x|^2}, \quad \forall k > 0,
\]

when \( |x| \geq R_1(n, \Lambda, \lambda, M, E, N, k) \) and \( 0 \leq t \leq T_1(\Lambda, N) \).

**Proposition 1.4 (Lower Bound)** Suppose \( \{a^{ij}\} \) are the same as above, \( u \) satisfies the first two conditions of (10), and \( u(x,0) \neq 0 \). Then there exists a positive constant \( E_0 = E_0(n, \Lambda, \lambda) \), such that when \( E < E_0 \), there exists a constant \( C_* = C_*(n, \Lambda, \lambda, M, E, N) \), such that the following estimate

\[
\frac{1}{T} \int_{T/8}^{7T/8} \int_{R-1 \leq |x| \leq R} (u^2 + |\nabla u|^2)dxdt \geq e^{-C_* \frac{T^2}{8}}.
\]

holds when

\[
R \geq R_2(n, \Lambda, \lambda, M, E, N, ||u(\cdot,0)||_{L^2(B(10e_1, \frac{1}{2}))})
\]

and

\[
0 < T \leq T_2(n, \Lambda, \lambda, M, N, ||u(\cdot,0)||_{L^2(B(10e_1, \frac{1}{2}))}),
\]

where \( e_1 = (1, 0, \ldots, 0) \).
Combining these two estimates together, we must have $u(x,0) = 0$, then by the backward uniqueness (see [14]) result, we have $u(x,t) \equiv 0$. Thus we proved Theorem 1.1.

**Remark 1.5** This lower bound of the integration form is optimal, which can be seen from the solution of the backward heat equation $\partial_t \Gamma + \Delta \Gamma = 0$ that

$$\Gamma(x,t) = (T-t)^{-n/2} e^{-\frac{|x|^2}{4(T-t)}}.$$

The upper bound can be obtained by the following Carleman inequality.

**Proposition 1.6** Suppose $\{a^{ij}\}$ are the same as above. Let

$$Q = \mathbb{R}^n \setminus B_1 \times [0,1], \quad f(t) = (t+1)^{-\beta} - 2^{-\beta}.$$

There exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, for any function $v \in C_0^\infty(Q)$ and any $\gamma > 0$, we have

$$\int_Q e^{2\gamma f|v|^2} dx dt \leq \int_Q e^{2\gamma f|x|^{2}-\frac{2b|x|^2}{1+\lambda}} |Pv|^2 dx dt + c \int_{\mathbb{R}^n} |x|^2 e^{-b|x|^2} |v(x,1)|^2 dx + c(1 + \gamma)^2 \int_{\mathbb{R}^n} |x|^2 e^{-b|x|^2} |v(x,0)|^2 dx,$$

where $b = \frac{1}{16\Lambda}$, $\beta = \beta(n, \Lambda, \lambda, M, E) \geq 1$, and $c$ is an absolute constant.

The lower bound can be proved mainly by the following Carleman inequality. First, let $\psi(t)$ be a cut-off function satisfying

$$\psi(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]; \\ 2, & \text{if } t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

**Proposition 1.7** Suppose $\{a^{ij}\}$ are the same as above. Let

$$Q_R = \{(x,t) | \ 1 < |x| < R, t \in (\frac{1}{8}, \frac{7}{8})\},$$

$$\Psi = \gamma(1-t)R^{2/3}|x|^{4/3} + \psi(t)R^2.$$ Then there exists a positive constant $E_0 = E_0(n, \Lambda, \lambda)$, such that when $E < E_0$, for any function $v \in C_0^\infty (Q_R)$ and any $\gamma \geq \gamma_0(n, \Lambda, \lambda, M, E)$, we have

$$c\lambda^2 \int_{Q_R} e^{2\Psi (\gamma^3 R^2 |v|^2 + \gamma |\nabla v|^2)} dx dt \leq \int_{Q_R} e^{2\Psi |Pv|^2} dx dt.$$

The paper organized as follows. We first use the two Carleman inequalities to prove the upper and lower bound, then we prove the two Carleman inequalities.
2 Proof of Upper Bound and Lower Bound

In this section, we prove the upper bound and lower bound by assuming Proposition 1.6 and Proposition 1.7 first, and we postpone the proof of the two Carleman inequalities to the next section.

2.1 Upper Bound

Proof of Proposition 1.3. We use Carleman inequalities (12) to prove the upper bound for the solution.

Step 1. By the regularity theory for solutions of parabolic equations, we have

\[ |u(x,t)| + |\nabla u(x,t)| \leq C(n, \Lambda, \lambda, M, N)e^{2N|x|^2} \]  \hspace{1cm} (14)

for \((x,t) \in \mathbb{R}^n \setminus B_2 \times [0, \frac{1}{2}]\). Let

\[ \tau = \min \{ \frac{1}{2}, \frac{1}{2N}, \frac{b}{8} \}, \]  \hspace{1cm} (15)

where \(b\) is the one in Proposition 1.6. Define

\[ \tilde{u}(x,t) = u(\tau x, \tau^2 t), \]

and

\[ \tilde{a}^{ij}(x,t) = a^{ij}(\tau x, \tau^2 t) \]

for \((x,t) \in \mathbb{R}^n \setminus B_{\tau} \times [0,1]\). Then it is easy to see

\[ |\nabla_x \tilde{a}^{ij}| + |\partial_t \tilde{a}^{ij}| \leq \tau M \leq M, \quad |\nabla_x \tilde{a}^{ij}| \leq \frac{E}{|x|}. \]

We denote

\[ \tilde{P}\tilde{u} = \partial_t \tilde{u} + \sum_{ij} \partial_i (\tilde{a}^{ij} \partial_j \tilde{u}), \]

then by (10) we have

\[ |\tilde{P}\tilde{u}| \leq \tau N(|\tilde{u}| + |\nabla \tilde{u}|) \leq \frac{1}{2}(|\tilde{u}| + |\nabla \tilde{u}|). \]  \hspace{1cm} (16)

By (14) and (15), we have

\[ |\tilde{u}(x,t)| + |\nabla \tilde{u}(x,t)| \leq C(n, \Lambda, \lambda, M, N)e^{2N\tau^2|x|^2} \leq C(n, \Lambda, \lambda, M, N)e^{\frac{b}{8}|x|^2}. \]  \hspace{1cm} (17)

We keep in mind that

\[ |u(x,0)| \leq C_k e^{-k|x|^2}, \quad \forall k > 0, \]  \hspace{1cm} (18)

and we always take \(k\) large enough.
Step 2. In order to apply Carleman inequality (12), we define a cut-off function $\theta$ satisfying

$$
\theta(|x|) = \begin{cases} 
0, & \text{if } |x| < R \text{ or } |x| > k^2R + 1; \\
1, & \text{if } R + 1 \leq |x| \leq k^2R,
\end{cases}
$$

where $R > \frac{3}{2}$. Let $v = \theta \tilde{u}$, then by (16) we have

$$
|\tilde{P}v| = |\tilde{P}\theta \tilde{u} + \theta \tilde{P}\tilde{u} + 2\tilde{a}^i \partial_i \theta \partial_j \tilde{u}|
\leq \frac{1}{2} \theta(|\tilde{u}| + |\nabla \tilde{u}|) + C(n, \Lambda, M) \chi(|\tilde{u}| + |\nabla \tilde{u}|)(|\nabla \theta| + |\nabla^2 \theta|) 
\leq \frac{1}{2} (|v| + |\nabla v|) + C(n, \Lambda, M) \chi(|\tilde{u}| + |\nabla \tilde{u}|),
$$

(19)

where $\chi$ is the characteristic function and

$$
\Omega = \{0 < \theta < 1, t \in [0, 1]\}
= \{R < |x| < R + 1, t \in [0, 1]\} \cup \{k^2R < |x| < k^2R + 1, t \in [0, 1]\}.
$$

Step 3. We apply Carleman inequality (12) for $v$, then

$$
J \equiv \int_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2 + \beta}{c+1}} (|v|^2 + |\nabla v|^2) dxdt
\leq \int_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2 + \beta}{c+1}} |\tilde{P}v|^2 dxdt
+ c \int_{\mathbb{R}^n} |x|^2 e^{-\frac{b|x|^2}{2}} (|v(x, 1)|^2 + |\nabla v(x, 1)|^2) dx
+ c(1 + \gamma)^2 \int_{\mathbb{R}^n} |x|^2 e^{2\gamma |x|^{3/2} - \beta |x|^2} |v(x, 0)|^2 dx.
$$

By (19) we have

$$
J \leq \frac{3}{4} J + C(n, \Lambda, M) \int_Q e^{2\gamma f|x|^{3/2} - \frac{b|x|^2 + \beta}{c+1}} \chi(|\tilde{u}| + |\nabla \tilde{u}|)^2 dxdt
+ c \int_{\mathbb{R}^n} |x|^2 e^{-\frac{b|x|^2}{2}} (|v(x, 1)|^2 + |\nabla v(x, 1)|^2) dx
+ c(1 + \gamma)^2 \int_{\mathbb{R}^n} |x|^2 e^{2\gamma |x|^{3/2} - \beta |x|^2} |v(x, 0)|^2 dx,
$$

thus

$$
J \leq C(n, \Lambda, M) \int_\Omega e^{2\gamma f|x|^{3/2} - \frac{b|x|^2 + \beta}{c+1}} (|\tilde{u}| + |\nabla \tilde{u}|)^2 dxdt
+ c \int_{|x| \geq R} |x|^2 e^{-\frac{b|x|^2}{2}} (|\tilde{u}(x, 1)|^2 + |\nabla \tilde{u}(x, 1)|^2) dx
+ c(1 + \gamma)^2 \int_{|x| \geq R} |x|^2 e^{2\gamma |x|^{3/2} - \beta |x|^2} |\tilde{u}(x, 0)|^2 dx
\equiv I_1 + I_2 + I_3.
$$
**Step 4.** Now we estimate both sides of the above inequality. We estimate $I_2$ and $I_3$ first, then $I_1$, at last $J$.

**Estimate of $I_2$.**
By (17),

$$I_2 \leq C(n, \Lambda, \lambda, M, N) \int_{|x| \geq R} |x|^2 e^{-\frac{b|x|^2}{4}} dx \leq C(n, \Lambda, \lambda, M, N)$$

(20)

**Estimate of $I_3$.**
Recall (18), then

$$|\bar{u}(x, 0)| = |u(\tau x, 0)| \leq C(\frac{bk}{\tau^2}) e^{-\frac{bk}{\tau^2}|\tau x|^2} = C(\Lambda, N, k) e^{-\frac{bk}{\tau^2}|x|^2},$$

and thus

$$I_3 \leq C(\Lambda, N, k)(1 + \gamma)^2 \int_{|x| \geq R} |x|^2 e^{-\frac{bk}{\tau^2}|x|^2 + 2\gamma|x|^{3/2} - \frac{b}{\tau^2}|x|^2} dx.$$ 

Now we choose

$$\gamma = \frac{bk}{16} R^{1/2}. \quad (21)$$

In the region $\{|x| \geq R\}$,

$$2\gamma |x|^{3/2} = \frac{bk}{8} R^{1/2} |x|^{3/2} \leq \frac{bk}{8} |x|^2,$$

then

$$I_3 \leq C(\Lambda, N, k) k^2 R \int_{|x| \geq R} |x|^2 e^{-\frac{bk}{\tau^2}|x|^2 - \frac{b}{\tau^2}|x|^2} dx$$

$$\leq C(\Lambda, N, k) k^2 R e^{-\frac{bk}{\tau^2} R^2} \int_{|x| \geq R} |x|^2 e^{-\frac{b}{\tau^2}|x|^2} dx$$

(22)

$$\leq C(n, \Lambda, N, k) k^2 R e^{-\frac{b}{\tau^2} R^2} \leq 1,$$

if $R \geq R_0(n, \Lambda, N, k)$ large enough.

**Estimate of $I_1$.**

$$I_1 \leq C(n, \Lambda, M) \int_{\Omega} e^{2\gamma|x|^{3/2} - \frac{b|x|^2}{4}} (|\bar{u}| + |\nabla \bar{u}|)^2 dx dt.$$ 

Use (17) again, we obtain

$$I_1 \leq C(n, \Lambda, \lambda, M, N) \int_{\Omega} e^{2\gamma|x|^{3/2} - \frac{b|x|^2}{4}} dx dt$$

$$\leq C(n, \Lambda, \lambda, M, N) (\int_{k^2 R \leq |x| \leq k^2 R + 1} + \int_{R \leq |x| \leq R + 1} ) e^{2\gamma|x|^{3/2} - \frac{b|x|^2}{4}} dx$$

$$= I_{1,1} + I_{1,2}.$$
In the region \( \{ k^2 R \leq |x| \leq k^2 R + 1 \} \),
\[
2 \gamma |x|^{3/2} = \frac{bk}{8} R^{1/2} |x|^{3/2} \leq \frac{b}{8} |x|^2,
\]
then
\[
I_{1,1} \leq C(n, \Lambda, \lambda, M, N) \int_{k^2 R \leq |x| \leq k^2 R + 1} e^{-\frac{|x|^2}{8}} \, dx \leq C(n, \Lambda, \lambda, M, N).
\]
In \( \{ R \leq |x| \leq R + 1 \} \),
\[
2 \gamma |x|^{3/2} = \frac{bk}{8} R^{1/2} |x|^{3/2} \leq \frac{bk}{8} |x|^2,
\]
then
\[
I_{1,2} \leq C(n, \Lambda, \lambda, M, N) \int_{R \leq |x| \leq R + 1} e^{\frac{bk}{8} |x|^2 - \frac{b|x|^4}{8}} \, dx
\leq C(n, \Lambda, \lambda, M, N) e^{\frac{bk}{8} (R+1)^2} \int_{R \leq |x| \leq R + 1} e^{-\frac{|x|^2}{8}} \, dx
\leq C(n, \Lambda, \lambda, M, N) e^{\frac{bk}{8} R^2}.
\]
Thus we have
\[
I_1 \leq C(n, \Lambda, \lambda, M, N) e^{\frac{bk}{8} R^2}. \tag{23}
\]
Combining (20), (22) and (23), we have that when \( R \geq R_0(n, \Lambda, N, k) \),
\[
J \leq C(n, \Lambda, \lambda, M, N) e^{\frac{bk}{8} R^2}. \tag{24}
\]
Next we estimate a lower bound for \( J \).

**Estimate of \( J \).**
If \( k \geq 4^{\beta+5} \), then \( \{ 4^{\beta+2} R \leq |x| \leq 4^{\beta+3} R \} \subset \{ \theta = 1 \} \), and thus
\[
J \geq \int_0^{1/2} \int_{4^{\beta+2} R \leq |x| \leq 4^{\beta+3} R} e^{2\gamma f(|x|^{3/2} - \frac{b|x|^2 + b}{c+1} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx/dt.}
\]
Notice that when \( t \in [0, \frac{1}{2}] \), \( f(t) \geq f(\frac{1}{2}) \geq 2^{-\beta-2} \), then
\[
J \geq e^{-\beta} \int_0^{1/2} \int_{4^{\beta+2} R \leq |x| \leq 4^{\beta+3} R} e^{2^{-\beta-1} \gamma |x|^{3/2} - b|x|^2 (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx/dt.}
\]
In the region \( \{ 4^{\beta+2} R \leq |x| \leq 4^{\beta+3} R \} \),
\[
2^{-\beta-1} \gamma |x|^{3/2} = 2^{-\beta-5} bk R^{1/2} |x|^{3/2} \geq 2^{-\beta-5} bk (4^{-\beta-3} |x|)^{1/2} |x|^{3/2} = 4^{-\beta-4} bk |x|^2,
\]
then
\[
2^{-\beta-1} \gamma |x|^{3/2} - b|x|^2 \geq (4^{-\beta-4} k - 1) b|x|^2.
\]
Notice that $k \geq 4^{\beta+5}$, then
$$4^{-\beta-4}k - 1 \geq 4^{-\beta-5}k,$$
and
$$2^{-\beta-1}x^{3/2} - b|x|^2 \geq 4^{-\beta-5}bk|x|^2 \geq 4^{-\beta-5}bk(4^{\beta+2}R)^2 = 4^{\beta-1}bkR^2 \geq bkR^2.$$ 

Thus
$$J \geq e^{-\beta}e^{bkR^2} \int_0^{1/2} \int_{A^{\beta+2}R \leq |x| \leq A^{\beta+3}R} (|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt$$
$$\geq \tau^{-n}e^{-\beta}e^{bkR^2} \int_0^{\tau^{x/2}} \int_{A^{\beta+2}R \leq |x| \leq A^{\beta+3}R} (|u|^2 + |\nabla u|^2) dx dt. \quad (25)$$

Combining (24) and (25) together, we have
$$\int_{0}^{\tau^{x/2}} \int_{A^{\beta+2}R \leq |x| \leq A^{\beta+3}R} (|u|^2 + |\nabla u|^2) dx dt \leq C(n, \Lambda, \lambda, M, E, N)e^{-\frac{bkR^2}{4}} \leq e^{-\frac{bkR^2}{4}}$$
when $R \geq R_0(n, \Lambda, \lambda, M, E, N, k)$.

We replace $\tau A^{\beta+2}R$ by $R$, and let
$$T_1 = \frac{\tau^{x/2}}{4} = \frac{1}{16} \min\{1, \frac{b^2}{N^2}, \frac{b^2}{16}\},$$
then we obtain
$$\int_0^{T_1} \int_{R \leq |x| \leq 4R} (|u|^2 + |\nabla u|^2) dx dt \leq e^{-ckR^2}.$$ 

Finally, by the regularity theory for solutions of parabolic equations, we obtained our upper bound estimate.

### 2.2 Lower Bound

The lower bound can be proved by the following two lemmas. The first one is due to Escauriaza, Fernandez and Vessella (see [4]), and we copy it here.

**Lemma 2.1** There is a constant $C = C(n, \Lambda, \lambda, M, N)$ such that the inequalities

$$C \log (C \Theta_\rho) \geq 2 \quad \text{and} \quad C \int_{B_2} u^2(x,t)dx \geq \int_{B_0} u^2(x,0)dx \quad (26)$$

hold when $0 < t \leq \rho^2/C \log (C \Theta_\rho)$ and $0 < \rho \leq 1$. Here

$$\Theta_\rho = \frac{\int_0^{1} \int_{B_\rho} u^2(x,t)dx dt}{\rho^2 \int_{B_\rho} u^2(x,0)dx}.$$ 

The second one is derived from Carleman inequality (13).
Lemma 2.2 Suppose \( \{a^{ij}\} \) are the same as above, \( u \) satisfies the first two conditions of (10). Then there exists a positive constant \( E_0 = E_0(n, \Lambda, \lambda) \), such that when \( E < E_0 \), there exists \( C_* = C_*(n, \Lambda, \lambda, M, E, N) \), such that the following estimate

\[
e^{C_* \frac{R^2}{T^{3/2}}} \int_0^{T/2} \int_{|x| < 12} |u|^2 dx dt \leq 1 + e^{C_* \frac{R^2}{T^{5/8}}} \int_{R^{-1} \leq |x| \leq R} (|u|^2 + |\nabla u|^2) dx dt,
\]

holds when \( R \geq R_3(n, N) \) and \( 0 < T \leq 1 \).

In the following, we prove Lemma 2.2 first, then we use the two lemmas to prove the lower bound.

Proof of Lemma 2.2 We use Carleman inequality (13) to prove Lemma 2.2. We again divided the proof into several steps.

**Step 1.** For any \( 0 < T \leq 1 \), we define

\[
\tilde{u}(x, t) = u(\sqrt{T}x, Tt),
\]

\[
\tilde{a}^{ij}(x, t) = a^{ij}(\sqrt{T}x, Tt),
\]

\[
\tilde{P} \tilde{u} = \partial_t \tilde{u} + \sum_{ij} \partial_i (\tilde{a}^{ij} \partial_j \tilde{u}),
\]

for \((x, t) \in (x, t) \in \mathbb{R}^n \setminus B_{\sqrt{T}} \times [0, 1] \). Similarly, we have

\[
|\tilde{u}(x, t)| + |\nabla \tilde{u}(x, t)| \leq C(n, \Lambda, \lambda, M, N)e^{2NT|x|^2},
\]

(28)

and

\[
|\tilde{P} \tilde{u}| \leq \sqrt{T} N (|\tilde{u}| + |\nabla \tilde{u}|) \leq N(|\tilde{u}| + |\nabla \tilde{u}|).
\]

(29)

**Step 2.** In order to apply Carleman inequality (13), we choose two smooth cut-off functions. Let

\[
\eta_1(|x|) = \begin{cases} 
0, & \text{if } |x| \leq \frac{2}{\sqrt{T}} \text{ or } |x| \geq \gamma^{-3/4} R; \\
1, & \text{if } \frac{3}{\sqrt{T}} \leq |x| \leq \gamma^{-3/4} R - \frac{1}{\sqrt{T}},
\end{cases}
\]

where \( \gamma \) and \( R \) are the parameters in Carleman inequality (13), and

\[
\gamma^{-3/4} R \geq \frac{20}{\sqrt{T}}.
\]

(30)

We always take both \( \gamma \) and \( R \) large enough. Let

\[
\eta_2(t) = \begin{cases} 
0, & \text{if } t \in [0, \frac{1}{10}] \cup [\frac{7}{8}, 1]; \\
1, & \text{if } t \in [\frac{1}{4}, \frac{3}{4}].
\end{cases}
\]
Let $\eta(x, t) = \eta_1(|x|)\eta_2(t)$ and $v = \eta \tilde{u}$. Then $\text{supp} \eta \subset Q_R$ and so $\text{supp} w \subset Q_R$.

By (29) we have

$$
|\hat{P}v| = |\eta \hat{P} \tilde{u} + \tilde{u} \hat{P} \eta | + 2 \hat{a} \partial_\gamma \eta \partial_t \tilde{u} | \leq N \eta (|\tilde{u}| + |\nabla \tilde{u}|) + C(n, \Lambda, M)(|\tilde{u}| + |\nabla \eta | + |\nabla \eta | + |\nabla^2 \eta |)
$$

(31)

\[ \leq N (|v| + |\nabla v|) + C(n, \Lambda, M)(|\tilde{u}| + |\nabla \tilde{u}|) \chi_{\{0 < \eta < 1\}}, \]

**Step 3.** We apply Calman inequality (13) for $v$, then we get

$$
c \lambda^2 \int_{Q_R} e^{2\psi}(\gamma^3 R^2|v|^2 + \gamma |\nabla v|^2) dx dt \leq \int_{Q_R} e^{2\psi}|\hat{P}v|^2 dx dt.
$$

By (31), we have

$$
c \lambda^2 \int_{Q_R} e^{2\psi}(\gamma^3 R^2|v|^2 + \gamma |\nabla v|^2) dx dt \leq 4N^2 \int_{Q_R} e^{2\psi}(|v|^2 + |\nabla v|^2) dx dt
$$

$$
+ C \int_{\{0 < \eta < 1\}} e^{2\psi}(|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt.
$$

In the above inequality, if we take $\gamma = \gamma(n, \Lambda, \lambda, M, E, N)$ large enough, then the first term of the right hand side can be absorbed by the term of the left hand side, thus we obtain

$$
\int_{Q_R} e^{2\psi}(|v|^2 + |\nabla v|^2) dx dt \leq C \gamma^{-1} \int_{\{0 < \eta < 1\}} e^{2\psi}(|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt.
$$

Denote that

$$
\Omega_1 = \{(x, t) \mid \frac{9}{\sqrt{T}} \leq |x| \leq \frac{11}{\sqrt{T}}, \ t \in \left[\frac{1}{3}, \frac{2}{3}\right]\},
$$

then $\Omega_1 \subset \{\eta = 1\}$ and thus

$$
\int_{\Omega_1} e^{2\psi}(|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt \leq C \gamma^{-1} \int_{\{0 < \eta < 1\}} e^{2\psi}(|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt.
$$

(32)

We divide the set $\{0 < \eta < 1\}$ into three parts:

$$
\{0 < \eta < 1\} \subset \Omega_2 \cup \Omega_3 \cup \Omega_4,
$$

where

$$
\Omega_2 = \{(x, t) \mid \frac{2}{\sqrt{T}} < |x| < \frac{3}{\sqrt{T}}, \ t \in \left[\frac{1}{8}, \frac{7}{8}\right]\},
$$

$$
\Omega_3 = \{(x, t) \mid \frac{3}{\sqrt{T}} < |x| < \gamma^{-3/4}R - \frac{1}{\sqrt{T}}, \ t \in \left[\frac{1}{8}, \frac{1}{4}\right] \cup \left[\frac{3}{4}, \frac{7}{8}\right]\},
$$

(33)

$$
\Omega_4 = \{(x, t) \mid \gamma^{-3/4}R - \frac{1}{\sqrt{T}} < |x| < \gamma^{-3/4}R, \ t \in \left[\frac{1}{8}, \frac{7}{8}\right]\}.
$$

If we denote that

$$
J_i = \int_{\Omega_i} e^{2\psi}(|\tilde{u}|^2 + |\nabla \tilde{u}|^2) dx dt, \quad i = 1, 2, 3, 4,
$$

then we get
then we rewrite (32) as
\( J_1 \leq C \gamma^{-1}(J_2 + J_3 + J_4). \)  

(34)

**Step 4.** We estimate them respectively.

**Estimate of \( J_1. \)**
In \( \Omega_1, \psi(t) = 2, \) and
\[
\Psi \geq \frac{\gamma}{3} R^{2/3} \left( \frac{9}{\sqrt{T}} \right)^{4/3} + 2R^2 \geq 6\gamma (\frac{R}{T})^{2/3} + 2R^2,
\]
then
\[
J_1 \geq \exp\{12\gamma (\frac{R}{T})^{2/3} + 4R^2\} \int_{\Omega_1} |\tilde{u}|^2 \, dx \, dt
\]
\[
= T^{-\frac{n}{2}} \exp\{12\gamma (\frac{R}{T})^{2/3} + 4R^2\} \int_{T/3}^{2T/3} \int_{|x| \leq 11} |u|^2 \, dx \, dt.
\]

(35)

**Estimate of \( J_2. \)**
In \( \Omega_2, \)
\[
\Psi \leq \gamma R^{2/3} \left( \frac{3}{\sqrt{T}} \right)^{4/3} + 2R^2 \leq 5\gamma (\frac{R}{T})^{2/3} + 2R^2,
\]
and by (28),
\[
|\tilde{u}| + |\nabla \tilde{u}| \leq C(n, \Lambda, \lambda, M, N)e^{18N} \leq C,
\]
thus
\[
J_2 \leq CT^{-\frac{n}{2}} \exp\{10\gamma (\frac{R}{T})^{2/3} + 4R^2\}.
\]

(36)

**Estimate of \( J_3. \)** In \( \Omega_3, \psi(t) = 0, \)
\[
\Psi \leq \gamma R^{2/3} (\gamma^{-3/4} R)^{4/3} = R^2,
\]
and by (28),
\[
|\tilde{u}| + |\nabla \tilde{u}| \leq C \exp\{2NT(\gamma^{-3/4} R)^2\} \leq C \exp\{2N(\gamma^{-3/4} R)^2\},
\]
then we have
\[
J_3 \leq C(\gamma^{-3/4} R)^n \exp\{2R^2 + 4N(\gamma^{-3/4} R)^2\}.
\]

Notice that if \( \gamma^{-3/4} R > C(n, N) \), then
\[
(\gamma^{-3/4} R)^n \leq \exp\{N(\gamma^{-3/4} R)^2\},
\]

hence
\[
J_3 \leq C \exp\{2R^2 + 5N(\gamma^{-3/4} R)^2\} \leq C \exp\{3R^2\}.
\]

(37)

**Estimate of \( J_4. \)**
In \( \Omega_4, \)
\[
\Psi \leq \gamma R^{2/3} (\gamma^{-3/4} R)^{4/3} + 2R^2 = 3R^2,
\]
In the above inequality, we divide both sides by \( \exp \{ \gamma \} \) and have
\[
\gamma = \frac{\gamma}{\Lambda}
\]
provided \( \gamma \) large enough, then when \( \gamma \leq n, \lambda, M, E, N \)

\[
\exp \{ 12 \gamma (\frac{R}{T})^{2/3} + 4R^2 \} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9 \leq |x| \leq 11} |u|^2 dx dt
\]

\[
\leq C \gamma^{-1} \exp \{ 10 \gamma (\frac{R}{T})^{2/3} + 4R^2 \}
\]

\[
+ \exp \{ 6R^2 \} \frac{1}{T} \int_{T/8}^{T/8} \int_{\gamma^{-3/4} \sqrt{T} - 1 < |x| < \gamma^{-3/4} \sqrt{T}} (|u|^2 + |\nabla u|^2) dx dt.
\]

Now we combine (34), (35), (36), (37) and (38) together, then we have
\[
\exp \{ 2 \gamma (\frac{R}{T})^{2/3} \} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9 \leq |x| \leq 11} |u|^2 dx dt
\]

\[
\leq 1 + \exp \{ 2R^2 \} \frac{1}{T} \int_{T/8}^{T/8} \int_{\gamma^{-3/4} \sqrt{T} - 1 < |x| < \gamma^{-3/4} \sqrt{T}} (|u|^2 + |\nabla u|^2) dx dt,
\]

In the above inequality, we divide both sides by \( \exp \{ 10 \gamma (\frac{R}{T})^{2/3} + 4R^2 \} \), and take \( \gamma = \gamma(n, \Lambda, \lambda, M, E, N) \) large enough, then when \( \gamma^{-3/4} R \leq \frac{C(n, N)}{\sqrt{T}} \), we have
\[
\exp \{ 2 \gamma (\frac{R}{T})^{2/3} \} \frac{1}{T} \int_{T/3}^{2T/3} \int_{9 \leq |x| \leq 11} |u|^2 dx dt
\]

\[
\leq 1 + \exp \{ 2 \gamma R^2 \} \frac{1}{T} \int_{T/8}^{T/8} \int_{R^{-1} < |x| < R} (|u|^2 + |\nabla u|^2) dx dt,
\]

provided \( \gamma = \gamma(n, \Lambda, \lambda, M, E, N) \) large enough, and \( R \geq C(n, N) \). Thus we proved Lemma 2.2

Proof of Proposition 1.4 Since \( u(x, 0) \neq 0 \), then by the unique continuation (see [5, 6]), we must have \( u(x, 0) \neq 0 \) in \( B(10e1, \frac{1}{2}) \), and thus \( ||u(\cdot, 0)||_{L^2(B(10e1, \frac{1}{2}))} > 0 \).

Now we apply Lemma 2.1 for \( \rho = \frac{1}{2} \) and the ball \( B(10e1, \frac{1}{2}) \), then when
\[
0 < t \leq 1/C \log(C \Theta_{1/2}),
\]

we have
\[
C \int_{B(10e1, 1)} u^2(x, t) dx \geq \int_{B(10e1, \frac{1}{2})} u^2(x, 0) dx.
\]

(39)

Notice that
\[
\Theta_{1/2} \leq C(N)/||u(\cdot, 0)||_{L^2(B(10e1, \frac{1}{2}))}^2.
\]
and
\[
1/C \log(C \Theta_{1/2}) \geq C(n, \Lambda, \lambda, M, N, ||u(\cdot, 0)||_{L^2(B(10e_1, \frac{1}{2}))}) \equiv T_2,
\]
then when $0 < t \leq T_2$, we have (39).

For $0 < T \leq T_2$, we apply Lemma 2.2, then when $R \geq R_3(n, N)$, we have
\[
e^{C \cdot \frac{R^2}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{|x| \leq 11} u^2 \, dx \, dt \leq 1 + e^{C \cdot \frac{R^2}{T}} \frac{1}{T} \int_{T/8}^{T/4} \int_{|x| \leq R} (u^2 + ||u||^2_{L^2(B(10e_1, \frac{1}{2}))}) \, dx \, dt,
\]
(40)

Notice that the left hand side of (40)
\[
e^{C \cdot \frac{R^2}{T}} \frac{1}{T} \int_{T/3}^{2T/3} \int_{|x| \leq 11} u^2 \, dx \, dt \geq e^{C \cdot \frac{R^2}{2T}} \geq 2.
\]
(41)

By (40) and (41), we have
\[
\frac{1}{T} \int_{T/8}^{T/4} \int_{|x| \leq R} (u^2 + ||u||^2_{L^2(B(10e_1, \frac{1}{2}))}) \, dx \, dt \geq e^{C \cdot \frac{R^2}{2T}}.
\]

Thus we proved the lower bound estimate.

3 Proof of Carleman Inequalities

In this section, we shall prove the two Carleman Inequalities. The main idea is to choose a proper weighted functions $G$. We denote
\[
\tilde{\Delta} v = \partial_i (a^{ij} \partial_j v).
\]

Here and in the following argument, we use the summation convention on the repeated indices. We shall make use of the following lemma which is due to Escauriaza and Fernández in [5] (see also [3]).
Lemma 3.1 Suppose $\sigma(t) : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function, $F$ is differentiable, $G$ is twice differentiable and $G > 0$. Then the following identity holds for any $v \in C_0^\infty(\mathbb{R}^n \times [0, T])$ and any $\alpha \in \mathbb{R}$:

$$
2 \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} |Lv|^2 G dx dt + \frac{1}{2} \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} v^2 MG dx dt
$$

$$
+ \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} \langle A \nabla v, \nabla v \rangle [(\log \frac{\sigma}{\sigma'})' + \frac{\partial_t G - \tilde{G}}{G} - F] G dx dt
$$

$$
+ 2 \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} \langle D_G \nabla v, \nabla v \rangle G dx dt - \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} v \langle A \nabla v, \nabla F \rangle G dx dt
$$

$$
= 2 \int_{\mathbb{R}^n \times [0, T]} \frac{\sigma^{1-\alpha}}{\sigma'} LvP v G dx dt + \int_{\mathbb{R}^n} \frac{\sigma^{1-\alpha}}{\sigma'} \langle A \nabla v, \nabla v \rangle G dx |^T_0
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^n} \frac{\sigma^{1-\alpha}}{\sigma'} v^2 (F - \frac{\alpha \sigma'}{\sigma}) G dx |^T_0
$$

where

$$Lv = \partial_t v - \langle A \nabla \log G, \nabla v \rangle + \frac{1}{2} (F - \frac{\alpha \sigma'}{\sigma}) v,$$

$$M = (\log \frac{\sigma}{\sigma'})' F + \partial_t F + (F - \frac{\alpha \sigma'}{\sigma}) (\partial_t G - \tilde{G} \frac{\partial_t G - \tilde{G}}{G} - F) - \langle A \nabla F, \nabla \log G \rangle,$$

and

$$D_G^{ij} = a^{ik} \partial_{kl} (\log G) a^{lj} + \frac{\partial_t (\log G)}{2} (a^{ki} \partial_k a^{lj} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij}) + \frac{1}{2} \partial_t a^{ij}.$$ 

We first give a modification of this lemma which will be used in our proof. Let $\alpha = 0$ and $\sigma(t) = e^t$ in Lemma 3.1 then we obtain the following identity for $v \in C_0^\infty(\mathbb{R}^n \times [0, T])$

$$
\frac{1}{2} \int_{\mathbb{R}^n \times [0, T]} v^2 MG dx dt + \int_{\mathbb{R}^n \times [0, T]} \langle [2D_G + (\partial_t G - \tilde{G}) \frac{F - \tilde{G}}{G} - F] A \nabla v, \nabla v \rangle G dx dt
$$

$$- \int_{\mathbb{R}^n \times [0, T]} v \langle A \nabla v, \nabla F \rangle G dx dt = 2 \int_{\mathbb{R}^n \times [0, T]} Lv (P v - L v) G dx dt
$$

$$+ \int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle G dx |^T_0 + \frac{1}{2} \int_{\mathbb{R}^n} v^2 F G dx |^T_0.$$

If $\nabla F$ is differentiable, we can integrate by parts to obtain that

$$- \int_{\mathbb{R}^n \times [0, T]} v \langle A \nabla v, \nabla F \rangle G dx dt
$$

$$= \frac{1}{2} \int_{\mathbb{R}^n \times [0, T]} v^2 \tilde{F} G dx dt + \frac{1}{2} \int_{\mathbb{R}^n \times [0, T]} v^2 \langle A \nabla F, \nabla \log G \rangle G dx dt.$$

The function $\nabla F$ may not be differentiable, so we approximate $F$ by some twice differentiable function $F_0$ and use the above identity with $F_0$ in place of $F$, following Nguyen’s idea in [3]. Thus a direct corollary follows.
Corollary 3.2 Suppose $F$ is differentiable, $F_0$ and $G$ is twice differentiable and $G > 0$. Then the following identity holds for any $v \in C_0^\infty(\mathbb{R}^n \times [0, T])$:

$$
\frac{1}{2} \int_{\mathbb{R}^n \times [0, T]} v^2 M_0 G dx dt + \int_{\mathbb{R}^n \times [0, T]} \langle [2D_G + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) A] \nabla v, \nabla v \rangle G dx dt
$$

$$
- \int_{\mathbb{R}^n \times [0, T]} v \langle A \nabla v, \nabla (F - F_0) \rangle G dx dt = 2 \int_{\mathbb{R}^n \times [0, T]} L v (P v - L v) G dx dt
$$

$$
+ \int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle G |_{x=0}^T + \frac{1}{2} \int_{\mathbb{R}^n} v^2 F G |_{x=0}^T,
$$

where

$$
L u = \partial_t u - \langle A \nabla u, \nabla \log G \rangle + \frac{F u}{2},
$$

$$
M_0 = \partial_t F + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) + \tilde{\Delta} F_0 - \langle A \nabla (F - F_0), \nabla \log G \rangle,
$$

and

$$
D_G^{ij} = a^{ik} \partial_{kl} (\log G) a^{lj} + \frac{\partial_t (\log G)}{2} \left( a^{ki} \partial_k a^{lj} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij} \right) + \frac{1}{2} \partial_t a^{ij}.
$$

Before we prove our Carleman inequalities, we need to prove a result which can be viewed as another version of Corollary 3.2.

In (43), we let $G = e^{2\Phi}$, $w = e^\Phi v$, and we denote

$$
B = 2D_G + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) A.
$$

Then the third term of the left hand side of (43) is

$$
- \int_{Q} v \langle A \nabla (F - F_0), \nabla v \rangle e^{2\Phi} dx dt
$$

$$
= - \int_{Q} w \langle A \nabla (F - F_0), \nabla w - \nabla \Phi w \rangle dx dt
$$

$$
= - \int_{Q} w \langle A \nabla (F - F_0), \nabla w \rangle dx dt + \int_{Q} \langle A \nabla (F - F_0), \nabla \Phi \rangle w^2 dx dt.
$$

We use the above identity and rewrite (43) as

$$
\frac{1}{2} \int_{Q} M_1 w^2 dx dt + \int_{Q} \langle B \nabla v, \nabla v \rangle e^{2\Phi} dx dt - \int_{Q} w \langle A \nabla (F - F_0), \nabla w \rangle dx dt
$$

$$
= 2 \int_{Q} L v (P v - L v) e^{2\Phi} dx dt + \int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle e^{2\Phi} |_{x=0}^T + \frac{1}{2} \int_{\mathbb{R}^n} v^2 F e^{2\Phi} |_{x=0}^T
$$

where

$$
M_1 = \partial_t F + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) + \tilde{\Delta} F_0,
$$

and

$$
D_G^{ij} = a^{ik} \partial_{kl} (\log G) a^{lj} + \frac{\partial_t (\log G)}{2} \left( a^{ki} \partial_k a^{lj} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij} \right) + \frac{1}{2} \partial_t a^{ij}.
$$
\[ \mathbf{B} = 4 \mathbf{A} D^2 \Phi \mathbf{A} + 2 \partial_t \Phi (a^{ki} \partial_k a^{ij} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij}) + \partial_t a^{ij} + \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) \mathbf{A}. \]  

By direct calculations we have

\[ \frac{\partial_t G - \tilde{\Delta} G}{G} = 2 \partial_t \Phi - 2 a^{ij} \partial_{ij} \Phi - 2 \partial_t a^{ij} \partial_j \Phi - 4 \langle \mathbf{A} \nabla \Phi, \nabla \Phi \rangle. \]  

Let

\[ F = 2 \partial_t \Phi - 2 a^{ij} \partial_{ij} \Phi - 4 \langle \mathbf{A} \nabla \Phi, \nabla \Phi \rangle - H, \]  

where \( H \) is a smooth function to be determined. We choose

\[ F_0 = 2 \partial_t \Phi - 2 a^{ij} \epsilon \partial_{ij} \Phi - 4 a^{ij} \epsilon \partial_t \Phi \partial_j \Phi - H, \]  

where

\[ a^{ij}_\epsilon (x, t) = \int_{\mathbb{R}^n} a^{ij}(x - y, t) \phi_\epsilon(y) dy, \]  

\( \phi \) is a mollifier, and \( \epsilon = \frac{1}{2} \).

By (45)-(47), we have

\[ \mathbf{B} = 4 \mathbf{A} D^2 \Phi \mathbf{A} + 2 \partial_t \Phi (a^{ki} \partial_k a^{ij} + a^{kj} \partial_k a^{li} - a^{kl} \partial_k a^{ij} - a^{ij} \partial_k a^{kl}) + \partial_t a^{ij} + H \mathbf{A}. \]  

Now we begin to prove our Carleman inequalities.

### 3.1 Proof of Proposition 1.6.

Note that Carleman inequality (12) is very similar to the second Carleman inequality in [14], and their proofs are also similar.

In this part, we let

\[ \Phi = \gamma f(t)|x|^{3/2} - \frac{b|x|^2 + \beta}{2(t + 1)}, \]  

where \( b = \frac{1}{16 \Lambda} \) and \( \beta = \beta(n, \Lambda, \lambda, M, E) \) large enough.

**Step 1.** Estimate matrix \( \mathbf{B} \).

We need to estimate the lower bounds of the matrices in the right side of (48).

First we estimate \( D^2 \Phi \). Denote that

\[ h = \gamma f|x|^{-1/2}. \]

By direct calculations we have

\[ D^2 \Phi = \frac{3}{2} h \left( I_n - \frac{x \cdot x^T}{2|x|^2} \right) - \frac{b}{t + 1} I_n \geq \left( \frac{3}{4} h - \frac{b}{t + 1} \right) I_n, \]

and hence

\[ 4 \mathbf{A} D^2 \Phi \mathbf{A} \geq \left( 3 \lambda^2 h - \frac{C}{t + 1} \right) I_n. \]
Second, we estimate matrix $\partial \Phi_{a^k a^{ij}}$ and $\partial_t a^{ij}$. For any $\xi \in \mathbb{R}^n$,

$$|\partial \Phi_{a^k a^{ij}} \xi | \leq n^2 \Lambda \frac{E}{|x|} |\nabla \Phi| \sum_{i,j} |\xi_i| |\xi_j| \leq \frac{n^3 \Lambda E}{|x|} |\nabla \Phi||\xi|^2.$$  

Since

$$\nabla \Phi = \left( \frac{3}{2}h - \frac{b}{t+1} \right)x,$$

then

$$|\partial \Phi_{a^k a^{ij}} \xi | \leq n^3 \Lambda E \left( \frac{3}{2}h + \frac{b}{t+1} \right),$$

and thus

$$\partial \Phi_{a^k a^{ij}} \geq -n^3 \Lambda E \left( \frac{3}{2}h + \frac{b}{t+1} \right) I_n.$$  

Similarly,

$$\partial_t a^{ij} \geq -nMI_n.$$  

Consequently,

$$B \geq (3\lambda^2 - 12n^3 \Lambda E)hI_n - \frac{C}{t+1} I_n - nMI_n + \lambda HI_n$$

$$\geq 2\lambda^2 hI_n + (\lambda - \frac{C}{t+1})I_n,$$

if we take $E < E_0(n, \Lambda, \lambda)$. Now in this part, we choose

$$H = \frac{d}{t+1},$$

where $d = d(n, \Lambda, \lambda, M, E)$ large enough, then we have

$$B \geq 2\lambda^2 (h + \frac{1}{t+1})I_n + I_n. \quad (50)$$

**Step 2.** Prove the Carleman inequality.

By (50), we can estimate the second term of the left hand side of (44),

$$\int_Q (B \nabla v, \nabla v) e^{2\Phi} dx dt$$

$$\geq \int_Q e^{2\Phi} |\nabla v|^2 dx dt + 2\lambda^2 \int_Q (h + \frac{1}{t+1}) e^{2\Phi} |\nabla v|^2 dx dt$$

$$= \int_Q e^{2\Phi} |\nabla v|^2 dx dt + 2\lambda^2 \int_Q (h + \frac{1}{t+1}) |\nabla w|^2 dx dt$$

$$+ 2\lambda^2 \int_Q [(h + \frac{1}{t+1}) |\nabla \Phi|^2 + \nabla h \cdot \nabla \Phi + (h + \frac{1}{t+1}) \Delta \Phi] w^2 dx dt. \quad (51)$$

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By (44), (51) and the Cauchy inequality, we have

\[
\int_Q e^{2\Phi} |\nabla v|^2 \, dx \, dt + 2\lambda^2 \int_Q (h + \frac{1}{t+1}) |\nabla w|^2 \, dx \, dt + \int_Q M_2 w^2 \, dx \, dt + 2\lambda^2 \int_Q \left(h + 1 \atop t+1 \right) \left| \nabla w \right|^2 \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle e^{2\Phi} \, dx \bigg|_0^1 + \frac{1}{2} \int_{\mathbb{R}^n} F e^{2\Phi} v^2 \, dx \bigg|_0^1,
\]

where

\[
M_2 = 2\lambda^2 [(h + \frac{1}{t+1}) |\nabla \Phi|^2 + \nabla h \cdot \nabla \Phi + (h + \frac{1}{t+1}) \Delta \Phi] + \frac{1}{2} \partial_t F + \frac{1}{2} F \left( \frac{\partial G - \tilde{\Delta} G}{G} - F \right) + \frac{1}{2} \tilde{\Delta} F_0.
\]

We use inequality (52) to prove Proposition 1.6. We need some estimates which we list in the following lemma.

**Lemma 3.3** Set \( b = \frac{1}{16 \lambda} \), \( \beta = 20 \frac{1}{\lambda} d \) and \( d = d(n, \Lambda, \lambda, M, E) \) large enough. There exists \( E_0 = E_0(n, \Lambda, \lambda) \), such that when \( E < E_0 \), for any \( \gamma > 0 \), we have

\[
M_2 \geq \lambda^2 h^3 |x|^2 + \frac{db}{8} \frac{|x|^2}{(t+1)^3};
\]

\[
|\nabla (F - F_0)| \leq C(n) E [h^2 + \frac{1}{(t+1)^2}] |x|;
\]

\[
F(x, 0) \geq -2\beta |x|^2 (1 + \gamma)^2;
\]

\[
F(x, 1) \leq \frac{\beta}{2} |x|^2.
\]

We shall prove this lemma later.

By applying Lemma 3.3, in particular by (55), we have

\[
| \int_Q w \langle A \nabla (F - F_0), \nabla w \rangle \, dx \, dt | \leq \Lambda \int_Q |\nabla (F - F_0)||w||\nabla w| \, dx \, dt
\]

\[
\leq C(n) \Lambda E \int_Q [h^2 + \frac{1}{(t+1)^2}] |x| |w||\nabla w| \, dx \, dt.
\]

Using the Cauchy inequality, we have

\[
| \int_Q w \langle A \nabla (F - F_0), \nabla w \rangle \, dx \, dt | \leq C(n) \Lambda E \int_Q (h^3 |x|^2 + \frac{|x|^2}{(t+1)^2}) w^2 \, dx \, dt
\]

\[
+ C(n) \Lambda E \int_Q (h + \frac{1}{t+1}) |\nabla w|^2 \, dx \, dt.
\]
When \( E < E_0(n, \Lambda, \lambda) \), we have
\[
| \int_Q w \langle A \nabla(F - F_0), \nabla w \rangle dx \, dt | \leq \lambda^2 \int_Q (h^3 |x|^2 + \frac{|x|^2}{(t + 1)^2}) w^2 dx \, dt \\
+ \lambda^2 \int_Q (h + \frac{1}{t + 1}) |\nabla w|^2 dx \, dt. \tag{58}
\]

Because of (52), (58) and (54), we have
\[
\int_Q e^{2\Phi} |\nabla v|^2 dx \, dt + \left( db_8 - C \right) \int_Q |x|^2 (t + 1)^3 w^2 dx \, dt \leq \int_Q e^{2\Phi} |Pv|^2 dx \, dt \tag{59}
\]

Now we estimate the second term of the right hand side of (59).
\[
\int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle e^{2\Phi} dx \mid_{t=1} \leq \lambda \int_{\mathbb{R}^n} |\nabla v(x, 1)|^2 e^{-\frac{b |x|^2 + \beta}{2}} dx \, dt \tag{60}
\]

Notice that
\[
\Lambda e^{-\frac{\alpha}{2}} \leq \beta e^{-\frac{\alpha}{2}} \leq c,
\]

then
\[
\int_{\mathbb{R}^n} \langle A \nabla v, \nabla v \rangle e^{2\Phi} dx \mid_0 \leq c \int_{\mathbb{R}^n} |\nabla v(x, 1)|^2 e^{-\frac{b |x|^2 + \beta}{2}} dx \, dt
\]

Finally we estimate the third term of the right hand side of (59).
\[
\frac{1}{2} \int_{\mathbb{R}^n} Fe^{2\Phi} v^2 dx \mid_0 = \frac{1}{2} \int_{\mathbb{R}^n} F(x, 1) e^{-\frac{b |x|^2 + \beta}{2}} v^2(x, 1) dx \\
- \frac{1}{2} \int_{\mathbb{R}^n} F(x, 0) e^{2\gamma(1-2^{-\beta}) |x|^{3/2} - b |x|^2 - \beta} v^2(x, 0) dx
\]

By (56) and (57), we have
\[
\frac{1}{2} \int_{\mathbb{R}^n} Fe^{2\Phi} v^2 dx \mid_0 \leq \frac{\beta}{4} \int_{\mathbb{R}^n} |x|^2 e^{-\frac{b |x|^2 + \beta}{2}} v^2(x, 1) dx \\
+ \beta (1 + \gamma)^2 \int_{\mathbb{R}^n} |x|^2 e^{2\gamma(1-2^{-\beta}) |x|^{3/2} - b |x|^2 - \beta} v^2(x, 0) dx \\
\leq c \int_{\mathbb{R}^n} |x|^2 e^{-\frac{b |x|^2}{2}} v^2(x, 1) dx \\
+ c(1 + \gamma)^2 \int_{\mathbb{R}^n} |x|^2 e^{2\gamma |x|^{3/2} - b |x|^2} v^2(x, 0) dx. \tag{61}
\]

We combine (59), (60) and (61), and take \( d \) large enough, then we proved Carleman inequality (12).
It remains to prove Lemma 3.3.

**Step 3.** Prove Lemma 3.3

**Estimate of $M_2$.**

We estimate the terms of $M_2$ respectively.

**Estimate of the first three terms.**

By (49), we have

\[(h + \frac{1}{t + 1})|\nabla \Phi|^2 \geq h|\nabla \Phi|^2 = h(\frac{3}{2} h - \frac{b}{t + 1})^2 |x|^2 \]

\[\geq h|x|^2[\frac{9}{8} h^2 - \frac{b^2}{(t + 1)^2}] \]

\[= \frac{9}{8} h^3 - \frac{C}{(t + 1)^2} h|x|^2;\]

\[\nabla h \cdot \nabla \Phi = -\frac{1}{2} h(\frac{3}{2} h - \frac{b}{t + 1}) \geq -\frac{3}{4} h^2;\]

\[(h + \frac{1}{t + 1})\Delta \Phi \geq -\frac{nb}{t + 1} (h + \frac{1}{t + 1}) \geq -\frac{C}{t + 1} h - \frac{C}{(t + 1)^2}.\]

Combining them together, we obtain

\[2\lambda^2(h|\nabla \Phi|^2 + \nabla h \cdot \nabla \Phi + h\Delta \Phi) \geq \frac{9}{4} \lambda^2 h^3 - Ch^2 - \frac{C}{(t + 1)^2} h|x|^2 - \frac{C}{(t + 1)^3}. \tag{62}\]

**Estimate of $\frac{1}{2} \partial_t F$.**

Recall (47), then

\[\frac{1}{2} \partial_t F = \partial_t^2 \Phi - \partial_t a^{ij} \partial_{ij} \Phi - a^{ij} \partial_{ijt} \Phi - 2\partial_t \langle A \nabla \Phi, \nabla \Phi \rangle - \frac{1}{2} \partial_t H.\]

We estimate them one by one. Keep in mind that $f' < 0$.

\[\partial_t^2 \Phi = \gamma f'' |x|^{3/2} - \frac{b|x|^2 + \beta}{(t + 1)^3} = \frac{f''}{f} h|x|^2 - \frac{b|x|^2 + \beta}{(t + 1)^3};\]

\[-\partial_t a^{ij} \partial_{ij} \Phi = -\frac{3}{2} h(\partial_t a^{ii} - \frac{\partial_t a^{ij} x_i x_j}{2 |x|^2}) + \frac{b \partial_t a^{ii}}{t + 1} \geq -Ch - \frac{C}{t + 1};\]

\[-a^{ij} \partial_{ijt} \Phi = -\frac{3 f'}{2f} h(a^{ii} - \frac{a^{ij} x_i x_j}{2 |x|^2}) - \frac{ba^{ii}}{(t + 1)^2} \geq C \frac{f'}{f} h - \frac{C}{(t + 1)^2}.\]
\[-2\partial_t \langle A \nabla \Phi, \nabla \Phi \rangle \]
\[= [-9 \frac{f'}{f} h^2 + 6b \frac{f'}{(t+1)f} - \frac{1}{(t+1)^2} h + \frac{4b^2}{(t+1)^3}] a^{ij} x_i x_j \]
\[-2\left( \frac{3}{2} h - \frac{b}{t+1} \right)^2 \partial_t a^{ij} x_i x_j \]
\[\geq [-9 \frac{f'}{f} h^2 + C \frac{f'}{(t+1)f} - \frac{1}{(t+1)^2} h] |x|^2 - C[h^2 + \frac{1}{(t+1)^2}] |x|^2 \]
\[\geq [(-9 \frac{f'}{f} - C)h^2 + C \frac{f'}{(t+1)f} - \frac{1}{(t+1)^2} h] |x|^2 - C|x|^2 \frac{1}{(t+1)^2}; \]
\[\leq -\frac{1}{2} \partial_t H = \frac{d}{2(t+1)^2}. \]

Combining them together, we have

\[\frac{1}{2} \partial_t F \geq \left[ (-9 \frac{f'}{f} - C)h^2 + \left( \frac{f''}{f} + \frac{C f'}{(t+1)f} - \frac{C}{(t+1)^2} \right) h \right] |x|^2 - \frac{C}{(t+1)^2} |x|^2 + \beta; \quad (63) \]

Estimate of $\frac{1}{2} F \left( \frac{\partial G - \tilde{\Delta} G}{G} - F \right)$.
First we have

\[\frac{1}{2} F \left( \frac{\partial G - \tilde{\Delta} G}{G} - F \right) = (\partial_t \Phi - 2 \langle A \nabla \Phi, \nabla \Phi \rangle - a^{ij} \partial_t \Phi - \frac{1}{2} H)(H - 2 \partial_t a^{ij} \partial_j \Phi) \equiv J_1 - J_2 - J_3, \]

where

\[J_1 = \partial_t \Phi (H - 2 \partial_t a^{ij} \partial_j \Phi) \]
\[J_2 = 2 \langle A \nabla \Phi, \nabla \Phi \rangle (H - 2 \partial_t a^{ij} \partial_j \Phi) \]
\[J_3 = (a^{ij} \partial_t \Phi + \frac{1}{2} H)(H - 2 \partial_t a^{ij} \partial_j \Phi). \]

Before we estimate $J_1$, $J_2$ and $J_3$, we estimate $2 \partial_t a^{ij} \partial_j \Phi$ first.

\[|2 \partial_t a^{ij} \partial_j \Phi| \leq \frac{2n^2 E}{|x|} |\nabla \Phi| \leq 2n^2 E \left( \frac{3}{2} h + \frac{b}{t+1} \right) \leq 3n^2 E h + \frac{C}{t+1}, \]

then

\[-3n^2 E h + \frac{d - C}{t+1} \leq H - 2 \partial_t a^{ij} \partial_j \Phi \leq 3n^2 E h + \frac{d + C}{t+1}. \]

Now we estimate $J_1$, $J_2$ and $J_3$ respectively.

\[J_1 = \frac{f''}{f} h |x|^2 (H - 2 \partial_t a^{ij} \partial_j \Phi) + \frac{b |x|^2 + \beta}{2(t+1)^2} (H - 2 \partial_t a^{ij} \partial_j \Phi) \]
\[\geq \frac{f''}{f} h |x|^2 (3n^2 E h + \frac{d + C}{t+1}) + \frac{b |x|^2 + \beta}{2(t+1)^2} (-3n^2 E h + \frac{d - C}{t+1}) \]
\[\geq \left[ 3n^2 E \frac{f''}{f} h^2 + \left( \frac{(d + C)f'}{(t+1)f} - \frac{C + \beta + C}{(t+1)^2} \right) h \right] |x|^2 + \left( \frac{d}{2} - C \right) \frac{(b |x|^2 + \beta)}{(t+1)^3}. \]

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Combining $J_1$, $J_2$ and $J_3$ together, we obtain
\[
\frac{1}{2} F(\partial_G - \tilde{\Delta}G - F) \geq \left\{-27n^2\Lambda Eh^3 + (3n^2E\frac{f'}{f} - 9\Lambda d + C)h^2 + \left(\frac{d + C}{t + 1}\right)f' - C\beta + C\right\}(3n^2Eh + \frac{d + C}{t + 1}) \leq \left(Ch + \frac{d}{t + 1}\right)^2 \leq Ch^2 + \frac{2d^2}{(t + 1)^2} \leq Ch^2|x|^2 + \frac{4d^2}{(t + 1)^3}.\]

Estimate of $\frac{1}{2}\tilde{\Delta} F_0$.

In order to estimate $\tilde{\Delta} F_0$ and $|\nabla (F - F_0)|$, we need some estimates about $\{a_i^{ij}\}$ which we will prove in the appendix.

In fact, $\{a_i^{ij}\}$ satisfy the following properties:

\[i)\ \lambda |\xi|^2 \leq a_i^{ij}(x, t)\xi_i\xi_j \leq \Lambda |\xi|^2, \ \forall \xi \in \mathbb{R}^n;\]

\[ii)\ |\nabla a_i^{ij}(x, t)| \leq M; \ \ |\nabla a_i^{ij}(x, t)| \leq \frac{2E}{|x|} \text{ when } |x| \geq 1;\]

\[iii)\ |a_i^{ij}(x, t) - a_i^{ij}(x, t)| \leq 2\Lambda; \ \ |a_i^{ij}(x, t) - a_i^{ij}(x, t)| \leq \frac{E}{|x|} \text{ when } |x| \geq 1;\]

\[iv)\ |\partial_{kl} a_i^{ij}(x, t)| \leq c(n)M; \ \ |\partial_{kl} a_i^{ij}(x, t)| \leq \frac{c(n)E}{|x|} \text{ when } |x| \geq 1.\]

Recall that
\[F_0 = 2\partial_t \Phi - 2a_i^{ij}\partial_j \Phi - 4a_i^{ij}\partial_i \Phi \partial_j \Phi - H\]
\[= 2\gamma f'|x|^{3/2} + \frac{b|x|^2 + \beta}{(t + 1)^2} - \frac{d}{t + 1} - 2a_i^{ij}(\partial_{ij} \Phi + 2\partial_i \Phi \partial_j \Phi),\]

then
\[
\frac{1}{2} \tilde{\Delta} F_0 = \gamma f' \tilde{\Delta}(|x|^{3/2}) + \frac{b}{2(t + 1)^2} \tilde{\Delta}(|x|^2) - \tilde{\Delta}[a_i^{ij}(\partial_{ij} \Phi + 2\partial_i \Phi \partial_j \Phi)] \geq C \gamma f'|x|^{-1/2} - \frac{C}{(t + 1)^2} - \tilde{\Delta}[a_i^{ij}(\partial_{ij} \Phi + 2\partial_i \Phi \partial_j \Phi)] \]
\[= C \frac{f'}{f} - \frac{C}{(t + 1)^2} - \tilde{\Delta}[a_i^{ij}(\partial_{ij} \Phi + 2\partial_i \Phi \partial_j \Phi)],\]
and thus it remains to estimate $\tilde{\Delta}[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)]$

By (65) we have that $|a^ij|$, $|\nabla a^ij|$ and $|\nabla^2 a^ij|$ are all bounded, and it is easy to verify that

$$|\nabla^k\Phi| \leq C(n)(h + \frac{1}{t+1})|x|^{2-k}, \quad k = 1, 2, 3, 4.$$ (67)

Direct calculations give us

$$|\tilde{\Delta}[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)]|$$

$$= |\partial_k a^{kl}\partial_l[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)] + a^{kl}\partial_l[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)]|$$

$$\leq C(|\nabla^2\Phi| + |\nabla^3\Phi| + |\nabla^4\Phi| + |\nabla\Phi|\nabla^2\Phi + |\nabla\Phi|\nabla^3\Phi + |\nabla^2\Phi|^2).$$

By the Cauchy inequality, we have

$$|\tilde{\Delta}[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)]|$$

$$\leq C(|\nabla^2\Phi| + |\nabla^3\Phi| + |\nabla^4\Phi| + |\nabla\Phi|^2 + |\nabla^2\Phi|^2 + |\nabla^3\Phi|^2)$$ (68)

then by (67),

$$|\tilde{\Delta}[a^ij(\partial_j\Phi + 2\partial_i\Phi \partial_j\Phi)]| \leq C(h + \frac{1}{t+1})^2|x|^2 \leq C[h^2 + \frac{1}{(t+1)^2}]|x|^2.$$ (69)

We combine (66) and (69) and obtain that

$$\frac{1}{2} \tilde{\Delta} F_0 \geq \left( \frac{Cf'}{f} h - C h^2 \right) |x|^2 - \frac{C|x|^2}{(t+1)^3}. \quad \text{(70)}$$

At last, combining (62), (63), (64) and (70), we have

$$M_2 \geq \left[ \frac{9}{4} \lambda^2 - 27n^2 \Lambda E \right] h^3 + (9\lambda - 3n^2 E) \left( \frac{f'}{f} - \frac{9\Lambda d + C}{t+1} \right) h^2 |x|^2$$

$$+ \left( \frac{f''}{f} + \frac{(d + C)f'}{(t+1)f} - \frac{C\beta + C}{(t+1)^2} \right) h|x|^2$$

$$+ \left( \frac{db}{2} - 4d\Lambda b^2 - C \right) \frac{|x|^2}{(t+1)^3} + \frac{(d/2 - C)\beta - 4d^2 - C}{(t+1)^3}. $$

Now we choose

$$b = \frac{1}{16\Lambda}, \quad \beta = 20\frac{\Lambda}{\lambda} d,$$

and we take $d$ large enough, then

$$\frac{db}{2} - 4d\Lambda b^2 - C = \frac{db}{4} - C \geq \frac{db}{8};$$

$$\left( \frac{d}{2} - C \right) \beta - 4d^2 - C \geq \frac{d}{4} \beta - 5d^2 \geq 0;$$

and thus when $E < E_0(n, \Lambda, \lambda)$, we have

$$M_2 \geq \left[ \lambda^2 h^3 + \left( \frac{-9\lambda f'}{2f} - \frac{18\Lambda d}{t+1} \right) h^2 \right] |x|^2$$

$$+ \left( \frac{f''}{f} + \frac{2df'}{(t+1)f} - \frac{C\beta}{(t+1)^2} \right) h|x|^2 + \frac{db}{8} \frac{|x|^2}{(t+1)^3}. $$
We take into account that 
\[ f(t) = (t + 1)^{-\beta} - 2^{-\beta}, \]
then we have
\[
\frac{9\lambda f'}{2f} - \frac{18\Lambda d}{t + 1} = \frac{9\lambda \beta}{2(t + 1)[1 - \left(\frac{t + 1}{2}\right)^\beta]} - \frac{18\Lambda d}{t + 1} \\
\geq \frac{9\lambda \beta}{2(t + 1)} - \frac{18\Lambda d}{t + 1} = \frac{9(\lambda \beta - 4\Lambda d)}{2(t + 1)} \geq 0,
\]
and
\[
\frac{f''}{f} + \frac{2df'}{f(t + 1)} = \frac{\beta(\beta + 1 - 2d)}{(t + 1)^2[1 - \left(\frac{t + 1}{2}\right)^\beta]} - \frac{C\beta}{(t + 1)^2} \\
\geq \frac{\beta(\beta + 1 - 2d)}{(t + 1)^2} - \frac{d\beta}{(t + 1)^2} \\
= \frac{\beta(\beta + 1 - 3d)}{(t + 1)^2} \geq 0,
\]
thus
\[ M_2 \geq \lambda^2 h^3 |x|^2 + \frac{db}{8} \frac{|x|^2}{(t + 1)^3}. \]  \(71\)

**Estimate of \(\nabla(F - F_0)\).**

Since
\[ F - F_0 = 2(a^i_j - a^i_j)(\partial_i \Phi + 2\partial_i \Phi \partial_j \Phi), \]
then
\[
|\nabla(F - F_0)| = 2|\nabla a^i_j - \nabla a^i_j| (\partial_i \Phi + 2\partial_i \Phi \partial_j \Phi) \\
+ |a^i_j - a^i_j| (4\partial_i \Phi \partial_j \Phi)| \\
\leq 2|\nabla a^i_j - \nabla a^i_j| (|\partial_i \Phi| + |\partial_i \Phi|^2 + |\partial_j \Phi|^2) \\
+ 2|a^i_j - a^i_j| (|\nabla \partial_i \Phi| + 2|\partial_i \Phi|^2 + 2|\nabla \partial_j \Phi|^2).
\]

By (65) we have
\[ |\nabla a^i_j| \leq \frac{2E}{|x|}, \quad |a^i_j - a^i_j| \leq \frac{E}{|x|}, \]
then
\[
|\nabla(F - F_0)| \leq \frac{6E}{|x|} (n|\nabla^2 \Phi| + 2n|\nabla \Phi|^2) + \frac{2E}{|x|} (n|\nabla^3 \Phi| + 2n|\nabla \Phi|^2 + 2n|\nabla^2 \Phi|^2) \\
\leq \frac{nE}{|x|} (6|\nabla^2 \Phi| + 2|\nabla^3 \Phi| + 16|\nabla \Phi|^2 + 4|\nabla^2 \Phi|^2). \]  \(72\)

By (67) we have
\[ |\nabla(F - F_0)| \leq C(n)E[h^2 + \frac{1}{(t + 1)^2}]|x|. \]

**Estimate of \(F(x, 0)\) and \(F(x, 1)\).**
By \((47)\) and direct calculations, we have
\[
F = 2\partial_t \Phi - 2a^{ij} \partial_j \Phi - 4\langle A \nabla \Phi, \nabla \Phi \rangle - \frac{d}{t+1}
\]
\[
= -2\beta \gamma (t+1)^{-\beta-1}|x|^{3/2} - 9h^2 a^{ij} x_i x_j + 3h[\langle \frac{1}{2|x|^2} + \frac{4b}{t+1} \rangle a^{ij} x_i x_j - a^{ii}]
\]
\[
+ \frac{b|x|^2 - 4b^2 a^{ij} x_i x_j}{(t+1)^2} + \frac{\beta}{(t+1)^2} + \frac{2ba^{ii} - d}{t+1},
\]
then
\[
F(x, 1) = -2\beta \gamma 2^{-\beta-1}|x|^{3/2} + \frac{b|x|^2 - 4b^2 a^{ij}(x, 1) x_i x_j}{4} + \frac{\beta}{4} + \frac{2ba^{ii}(x, 1) - d}{2}
\]
\[
\leq \frac{b|x|^2 + \beta}{4} \leq \frac{\beta}{4} (|x|^2 + 1) \leq \frac{\beta}{2} |x|^2,
\]
and
\[
F(x, 0) = -2\beta \gamma |x|^{3/2} - 9\gamma^2 (1 - 2^{-\beta})^2 |x|^{-1} a^{ij}(x, 0) x_i x_j
\]
\[
+ 3\gamma (1 - 2^{-\beta}) |x|^{-1/2} [\langle \frac{1}{2|x|^2} + 4b \rangle a^{ij}(x, 0) x_i x_j - a^{ii}(x, 0)]
\]
\[
+ b|x|^2 - 4b^2 a^{ij}(x, 0) x_i x_j + \beta + 2ba^{ii} - d
\]
\[
\geq -2\beta \gamma |x|^{3/2} - 9\gamma^2 \Lambda |x| - 3\gamma \Lambda |x|^{-1/2} + (b - 4b^2 \Lambda) |x|^2
\]
\[
\geq -2\beta \gamma |x|^{3/2} (1 + \gamma + 1) \geq -2\beta |x|^2 (1 + \gamma)^2.
\]
Thus we complete the proof of Lemma 3.3.

### 3.2 Proof of Proposition 1.7

In this part, we let
\[
\Phi = \Psi = \gamma (1 - t) R^{2/3} |x|^{4/3} + \psi(t) R^2,
\]
and we denote by \(c\) absolute constants and \(C = C(n, \Lambda, \lambda, M, E)\). We keep in mind that
\[
\frac{|x|}{R} \leq 1, \quad \frac{1}{8} \leq t \leq \frac{7}{8} \quad \text{in} \quad Q_R.
\]

**Step 1.** Estimate matrix \(B\).

First we estimate the Hessian matrix \(D^2 \Phi\). Denote
\[
g = \gamma \left( \frac{|x|}{R} \right)^{-2/3}.
\]
By direct calculations, we have
\[
D^2 \Phi = \frac{4}{3}(1-t)g(I_n - \frac{2x \cdot x^T}{3|x|^2}) \geq \frac{4}{9}(1-t)gI_n \geq cgI_n,
\]
and hence
\[
4AD^2 \Phi A \geq \epsilon \lambda^2 gI_n.
\]
Then we estimate $\partial_t \Phi a^{ki} \partial_k a^{lj}$ and $\partial_t a^{ij}$.

For any $\xi \in \mathbb{R}^n$,

$$|\partial_t \Phi a^{ki} \partial_k a^{lj} \xi_i \xi_j| \leq n^2 \Lambda \frac{E}{|x|} |\nabla \Phi| \sum_{i,j} |\xi_i| |\xi_j| \leq \frac{n^2 \Lambda E}{|x|} |\nabla \Phi||\xi|^2,$$

Since

$$\nabla \Phi = \frac{4}{3} (1 - t) g x,$$

then

$$\frac{1}{6} \gamma R^{2/3} |x|^{1/3} \leq |\nabla \Phi| \leq \frac{4}{3} \gamma R^{2/3} |x|^{1/3}, \quad (73)$$

and

$$|\partial_t \Phi a^{ki} \partial_k a^{lj} \xi_i \xi_j| \leq cn^3 \Lambda Eg |\xi|^2,$$

thus

$$\partial_t \Phi a^{ki} \partial_k a^{lj} \geq -cn^3 \Lambda Eg I_n.$$

Similarly,

$$\partial_t a^{ij} \geq -nMI_n.$$

Consequently,

$$B \geq c(\lambda^2 - c_1 n^3 \Lambda E) g I_n - CI_n + HA.$$

Now we take

$$H = 4n^2 \varphi(t) Eg,$$

where $\varphi(t)$ is a smooth decreasing function on $[0, 1]$ satisfying

$$\varphi(t) = 1 \text{ in } [0, \frac{1}{3}], \quad \varphi(t) = -1 \text{ in } [\frac{2}{3}, 1],$$

$$\varphi(t) > 0 \text{ in } [0, \frac{1}{2}], \quad \varphi(t) < 0 \text{ in } (\frac{1}{2}, 1].$$

Then

$$B \geq c(\lambda^2 - c_1 n^3 \Lambda E) g I_n - CI_n - 4n^2 \Lambda Eg I_n$$

$$\geq c(\lambda^2 - c_2 n^3 \Lambda E) g I_n - CI_n,$$

When $E < E_0(n, \Lambda, \lambda)$, and we take $\gamma(n, \Lambda, \lambda, M, E)$ large enough, then

$$B \geq 2c\lambda^2 g I_n. \quad (74)$$

**Step 2.** Prove the Carleman inequality.

By (74), we have the estimates of the second term of the left hand side of (44), in fact

$$\int_{Q_R} \langle B \nabla v, \nabla v \rangle e^{2\Phi} dx dt \geq 2c\lambda^2 \int_{Q_R} g e^{2\Phi} |\nabla v|^2 dx dt$$

$$= c\lambda^2 \int_{Q_R} g e^{2\Phi} |\nabla v|^2 dx dt + c\lambda^2 \int_{Q_R} g |\nabla w|^2 dx dt \quad (75)$$

$$+ c\lambda^2 \int_{Q_R} [g |\nabla \Phi|^2 + \nabla g \cdot \nabla \Phi + g \Delta \Phi] w^2 dx dt.$$
By (44), (75) and the Cauchy inequality, we have
\[ cλ^2 \int_{Q_R} g e^{2Φ} |∇v|^2 dxdt + cλ^2 \int_{Q_R} g |∇w|^2 dxdt + \int_{Q_R} M_2 w^2 dxdt - \int_{Q_R} w \langle A \nabla(F - F_0), ∇w \rangle dxdt \leq \int_{Q_R} e^{2Φ} |Pv|^2 dxdt, \] (76)
where
\[ M_2 = cλ^2 (g |∇Φ|^2 + ∇g \cdot ∇Φ + g \DeltaΦ) + \frac{1}{2} \partial_t F + \frac{1}{2} F (\partial_t G - \tilde{∆}G G - F) + \frac{1}{2} \tilde{∆}F_0. \]

We use inequality (76) to prove Proposition 1.7. We also need some estimates which we list in the following lemma.

**Lemma 3.4** There exists a constant \( E_0(n, Λ, λ) \), such that when \( E < E_0 \), for any \( γ ≥ γ_0(n, Λ, λ, M, E) \), we have
\[ M_2 ≥ cλ^2 γ^3 R^2; \] (77)
\[ |∇(F - F_0)| ≤ cE γ^2 R^{4/3} |x|^{-1/3}. \] (78)

We will prove this lemma later.

Then by (78), we have
\[ \int_{Q_R} w \langle A \nabla(F - F_0), ∇w \rangle dxdt \leq Λ \int_{Q_R} |∇(F - F_0)| |w| |∇w| dxdt \leq cΛE \int_{Q_R} γ^2 R^{4/3} |x|^{-1/3} |w| |∇w| dxdt. \]

Using the Cauchy inequality,
\[ \int_{Q_R} w \langle A \nabla(F - F_0), ∇w \rangle dxdt \leq cΛE [\int_{Q_R} γ^3 R^2 w^2 dxdt + \int_{Q_R} γ (\frac{|x|}{R})^{-2/3} |∇w|^2 dxdt]. \]

When \( E < E_0(n, Λ, λ) \), we have
\[ \int_{Q_R} w \langle A \nabla(F - F_0), ∇w \rangle dxdt \leq \frac{1}{2} cλ^2 [\int_{Q_R} γ^3 R^2 w^2 dxdt + \int_{Q_R} γ (\frac{|x|}{R})^{-2/3} |∇w|^2 dxdt] \geq \frac{1}{2} \int_{Q_R} M_2 w^2 dxdt + cλ^2 \int_{Q_R} g |∇w|^2 dxdt. \] (79)

Because of (76) and (79), we have
\[ \int_{Q_R} e^{2Φ} |Pv|^2 dxdt ≥ cλ^2 \int_{Q_R} g e^{2Φ} |∇v|^2 dxdt + \frac{1}{2} \int_{Q_R} M_2 w^2 dxdt \geq cλ^2 \int_{Q_R} e^{2Φ} (γ^3 R^2 v^2 + γ |∇v|^2) dxdt. \]
Thus we proved Carleman inequality (13).
It remains to prove Lemma 3.4.

**Step 3.** Prove Lemma 3.4.

**Estimate of $M_2$.**

We estimate the terms of $M_2$ respectively. The leading term of $M_2$ is $h|\nabla \Phi|^2$ and we need pay attention to two quantities, $\partial_t^2 \Phi$ and $\partial_t \Phi(H - 2\partial_i a^{ij} \partial_j \Phi)$.

**Estimate of the first three terms.**

By (73), we have

$$g|\nabla \Phi|^2 \geq c\gamma^3 R^2,$$

$$|\nabla g \cdot \nabla \Phi| \leq |\nabla g||\nabla \Phi| \leq c\gamma^2 \left(\frac{|x|}{R}\right)^{-4/3},$$

$$g\Delta \Phi \geq 0,$$

then

$$c\lambda^2 (g|\nabla \Phi|^2 + \nabla g \cdot \nabla \Phi + g\Delta \Phi) \geq c\lambda^2 \gamma^3 R^2 - c\gamma^2 \left(\frac{|x|}{R}\right)^{-4/3}. \quad (80)$$

**Estimate of $\frac{1}{2} \partial_t F$.**

Recall (47), then

$$\frac{1}{2} \partial_t F = \partial_t^2 \Phi - \partial_t a^{ij} \partial_j \Phi - a^{ij} \partial_t \Phi - 2\partial_t \langle A\nabla \Phi, \nabla \Phi \rangle - \frac{1}{2} \partial_t H.$$

We estimate them one by one.

$$\partial_t^2 \Phi = \psi'' R^2 \geq -cR^2;$$

$$-\partial_t a^{ij} \partial_j \Phi \geq -C|\nabla^2 \Phi| \geq -Cg;$$

$$-a^{ij} \partial_t \Phi = \frac{4}{3} g(a^{ii} - \frac{2a^{ij} x_i x_j}{3|x|^2}) \geq -Cg;$$

$$-2\partial_t \langle A\nabla \Phi, \nabla \Phi \rangle = -2\partial_t a^{ij} \partial_i \Phi \partial_j \Phi - 4a^{ij} \partial_t \Phi \partial_t \Phi \geq -C|\nabla \Phi|^2 + \frac{64}{9} \gamma^2 (1 - t) \left(\frac{|x|}{R}\right)^{-4/3} a^{ij} x_i x_j$$

$$\geq -C|\nabla \Phi|^2 \geq -C\gamma^2 R^{4/3} |x|^{2/3};$$

$$-\frac{1}{2} \partial_t H = -2n^2 \varphi'(t) Eg \geq 0.$$

Combining them together, we have

$$\frac{1}{2} \partial_t F \geq -cR^2 - Cg - C\gamma^2 R^{4/3} |x|^{2/3}\quad (81)$$

**Estimate of $\frac{1}{2} F (\frac{\partial G - \tilde{\Delta} G}{G} - F)$.**
First we have
\[
\frac{1}{2} F \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) = (\partial_t \Phi - 2 \langle A \nabla \Phi, \nabla \Phi \rangle - a^{ij} \partial_j \Phi - \frac{1}{2} H)(H - 2 \partial_i a^{ij} \partial_j \Phi).
\]
Since
\[
\partial_t \Phi = -\gamma R^{2/3} |x|^{4/3} + \psi' R^2,
\]
then
\[
\frac{1}{2} F \left( \frac{\partial_t G - \tilde{\Delta} G}{G} - F \right) = \psi' R^2 (H - 2 \partial_i a^{ij} \partial_j \Phi)
\]
\[
- [\gamma R^{2/3} |x|^{4/3} + 2 \langle A \nabla \Phi, \nabla \Phi \rangle + a^{ij} \partial_j \Phi + \frac{1}{2} H](H - 2 \partial_i a^{ij} \partial_j \Phi)
\]
\[
\equiv J_1 - J_2.
\]
Before we estimate $J_1$ and $J_2$, we estimate $2 \partial_i a^{ij} \partial_j \Phi$ first.
\[
|2 \partial_i a^{ij} \partial_j \Phi| \leq \frac{2n^2 E}{|x|} |\nabla \Phi|,
\]
and by (73), we have
\[
|2 \partial_i a^{ij} \partial_j \Phi| \leq \frac{8}{3} n^2 E g.
\]
For $J_1$, we notice that
\[
\psi'(t) = 0 \text{ in } [0, \frac{1}{4}] \cup \left[\frac{1}{3}, \frac{2}{3}\right] \cup \left[\frac{3}{4}, 1\right],
\]
so we just need to consider the case when $t \in \left[\frac{1}{4}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{4}\right]$.
When $t \in \left[\frac{1}{4}, \frac{1}{3}\right]$, $\psi' \geq 0$, $\varphi(t) = 1$, $H = 4n^2 E g$, then
\[
H - 2 \partial_i a^{ij} \partial_j \Phi \geq 0,
\]
and thus $J_1 \geq 0$.
When $t \in \left[\frac{2}{3}, \frac{3}{4}\right]$, $\psi' \leq 0$, $\varphi(t) = -1$, $H = -4n^2 E g$, then
\[
H - 2 \partial_i a^{ij} \partial_j \Phi \leq 0,
\]
and thus $J_1 \geq 0$.
Above all, we have
\[
J_1 \geq 0.
\]
For $J_2$,
\[
J_2 \leq [\gamma R^{2/3} |x|^{4/3} + 2 \Lambda |\nabla \Phi|^2 + C |\nabla^2 \Phi| + cn^2 E g] \cdot cn^2 E g
\]
\[
\leq [\gamma R^{2/3} |x|^{4/3} + c \Lambda \gamma R^{4/3} |x|^{2/3} + C g] \cdot cn^2 E g
\]
\[
\leq [c \Lambda \gamma R^{4/3} |x|^{2/3} + C \gamma R^{2/3} |x|^{4/3}] \cdot cn^2 E g
\]
\[
= cn^2 \Lambda E \gamma R^2 + C \gamma R^{4/3} |x|^{2/3}.
\]
Combining $J_1$ and $J_2$ together, we obtain
\[
\frac{1}{2} F (\frac{\partial G - \Delta G}{G} - F) \geq -cn^2 \Lambda E \gamma^3 R^2 - C \gamma^2 R^{4/3} |x|^{2/3}. \tag{82}
\]

**Estimate of $\frac{1}{2} \Delta F_0$.**
Recall that
\[
F_0 = 2 \partial_t \Phi - 2a^{ij}_{\epsilon} \partial_i \Phi - 4a^{ij}_{\epsilon} \partial_i \partial_j \Phi - H = -2 \gamma R^2/3 |x|^{4/3} + 2 \phi' R^2 - 4n^2 \phi(t) E \gamma R^2/3 |x|^{-2/3} - 2a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi),
\]
then by (84),
\[
\frac{1}{2} \Delta F_0 = -\gamma R^2/3 \Delta (|x|^{4/3}) - 2n^2 \phi(t) E \gamma R^2/3 \Delta (|x|^{-2/3}) - \Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)]
\]
\[
\geq -C \gamma R^2/3 |x|^{-2/3} - C \gamma R^2/3 |x|^{-8/3} - \Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)]
\]
\[
\geq -C \gamma - \Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)],
\]
and thus it remains to estimate $\Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)]$.

By (65) we have that $|a^{ij}_{\epsilon}|$, $|\nabla a^{ij}_{\epsilon}|$ and $|\nabla^2 a^{ij}_{\epsilon}|$ are all bounded, and it is easy to verify that
\[
|\nabla^k \Phi| \leq C \gamma R^2/3 |x|^{4/3-k}, \quad k = 1, 2, 3, 4. \tag{84}
\]
Similarly to (68),
\[
|\Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)]| \leq C (|\nabla^2 \Phi| + |\nabla^3 \Phi| + |\nabla^4 \Phi| + |\nabla \Phi|^2 + |\nabla^2 \Phi|^2 + |\nabla^3 \Phi|^2),
\]
then by (84),
\[
|\Delta [a^{ij}_{\epsilon} (\partial_i \Phi + 2 \partial_i \partial_j \Phi)]| \leq C |\nabla \Phi|^2 \leq C \gamma^2 R^{4/3} |x|^{2/3}. \tag{85}
\]
We combine (83) and (85) and obtain that
\[
\frac{1}{2} \Delta F_0 \geq -C \gamma - C \gamma^2 R^{4/3} |x|^{2/3} \geq -C \gamma^2 R^{4/3} |x|^{2/3}. \tag{86}
\]

At last, combining (80), (81), (82) and (86), we have
\[
M_2 \geq (c \lambda^2 - c \Lambda n^2 \Lambda E) \gamma^3 R^2 - C \gamma^2 R^{4/3} |x|^{2/3} - cR^2
\]
\[
\geq (c \lambda^2 - c \Lambda n^2 \Lambda E) \gamma^3 R^2 - C \gamma^2 R^2,
\]
When $E < E_0(n, \Lambda, \lambda)$, we have
\[
M_2 \geq (c \lambda^2 \gamma^3 - C \gamma^2) R^2 \geq c \lambda^2 \gamma^3 R^2.
\]
if $\gamma \geq \gamma_0(n, \Lambda, \lambda, M, E)$ large enough.

**Estimate of $|\nabla (F - F_0)|$.**

Similarly to (72),
\[
|\nabla (F - F_0)| \leq \frac{n E}{|x|} (6 |\nabla^2 \Phi| + 2 |\nabla^3 \Phi| + 16 |\nabla \Phi|^2 + 4 |\nabla^2 \Phi|^2),
\]
then by (84) we have
\[
|\nabla (F - F_0)| \leq \frac{c n E}{|x|} |\nabla \Phi|^2 \leq c n E \gamma^2 R^{4/3} |x|^{-1/3}.
\]
Thus we complete the proof of Lemma 3.4.
4 Appendix

The properties of \( \{a_{ij}^\epsilon\} \).

\[ a_{ij}^\epsilon(x, t) = \int_{\mathbb{R}^n} a^{ij}(x - y, t)\phi_\epsilon(y)dy, \]
where \( \phi \) is a mollifier and \( \epsilon = \frac{1}{2} \), then \( \{a_{ij}^\epsilon\} \) satisfy:

i) \( \lambda |\xi|^2 \leq a_{ij}^\epsilon(x, t)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n; \)

ii) \( |\nabla a_{ij}^\epsilon(x, t)| \leq M; \quad |\nabla a_{ij}^\epsilon(x, t)| \leq \frac{2E}{|x|} \text{ when } |x| \geq 1; \)

iii) \( |a^\epsilon_{ij}(x, t) - a^{ij}(x, t)| \leq 2\Lambda; \quad |a^\epsilon_{ij}(x, t) - a^{ij}(x, t)| \leq \frac{E}{|x|} \text{ when } |x| \geq 1; \)

iv) \( |\partial_{kl}a_{ij}^\epsilon(x, t)| \leq c(n)M; \quad |\partial_{kl}a_{ij}^\epsilon(x, t)| \leq \frac{c(n)E}{|x|} \text{ when } |x| \geq 1. \)

Proof.

i) It is obvious.

ii) \[
|\nabla a_{ij}^\epsilon(x, t)| \leq \int_{\mathbb{R}^n} |\nabla a^{ij}(x - y, t)|\phi_\epsilon(y)dy \leq M \int_{\mathbb{R}^n} \phi_\epsilon(y)dy = M,
\]
and when \( |x| \geq 1, \)

\[
|\nabla a_{ij}^\epsilon(x, t)| \leq \int_{\mathbb{R}^n} |\nabla a^{ij}(x - y, t)|\phi_\epsilon(y)dy \leq \int_{\mathbb{R}^n} \frac{E}{|x - y|} \phi_\epsilon(y)dy \leq \int_{\mathbb{R}^n} \frac{E}{|x| - \frac{1}{2}} \phi_\epsilon(y)dy \leq \frac{2E}{|x|}.
\]

iii) The first part is obvious. We only need to prove the second one.

\[
|a^\epsilon_{ij}(x, t) - a^{ij}(x, t)| \leq \int_{\mathbb{R}^n} |a^{ij}(x - y, t) - a^{ij}(x, t)|\phi_\epsilon(y)dy \leq \int_{\mathbb{R}^n} |\nabla a^{ij}(x - \theta y, t)||y|\phi_\epsilon(y)dy, \quad (0 < \theta < 1)
\]
and when \( |x| \geq 1, \)

\[
|a^\epsilon_{ij}(x, t) - a^{ij}(x, t)| \leq \int_{\mathbb{R}^n} \frac{E}{2|x - \theta y|} \phi_\epsilon(y)dy \leq \int_{\mathbb{R}^n} \frac{E}{2(|x| - \frac{1}{2})} \phi_\epsilon(y)dy \leq \frac{E}{|x|}.
\]

iv) \[
|\partial_{kl}a_{ij}^\epsilon(x, t)| \leq \int_{\mathbb{R}^n} |\partial_{kl}a^{ij}(x - y, t)||\partial_{p}\phi_\epsilon(y)|dy \leq \epsilon^{-n-1} \int_{\mathbb{R}^n} |\partial_{kl}a^{ij}(x - y, t)||\partial_{p}\phi(y/\epsilon)|dy \leq \frac{M}{\epsilon} ||\partial_{p}\phi||_{L^1} \leq 2M||\nabla\phi||_{L^1},
\]
and when $|x| \geq 1$,

$$
|\partial_{kl}a^i_j(x, t)| \leq \epsilon^{-n-1} \int_{\mathbb{R}^n} |\partial_k a^i_j(x - y, t)||\partial_l \phi\left(\frac{y}{\epsilon}\right)|dy
$$

$$
\leq \epsilon^{-n-1} \int_{\mathbb{R}^n} E \left|\frac{y}{x - y}\right||\partial_l \phi\left(\frac{y}{\epsilon}\right)|dy
$$

$$
\leq \frac{2E}{\epsilon|x|} ||\partial_l \phi||_{L^1} \leq \frac{4E||\nabla \phi||_{L^1}}{|x|}.
$$

Then we finished the proof.

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