ARCSINE LAW FOR CORE RANDOM DYNAMICS

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Abstract. In their recent paper [8], G. Hata and the fourth author first gave an example of random iterations of two piecewise linear interval maps without (deterministic) indifferent periodic points for which the arcsine law – a characterization of intermittent dynamics in infinite ergodic theory – holds. The key in the proof of the result is the existence of a Markov partition preserved by each interval maps. In the present paper, we give a class of random iterations of two interval maps without indifferent periodic points but satisfying the arcsine law, by introducing a concept of core random dynamics. As applications, we show that the generalized arcsine law holds for generalized Hata–Yano maps and piecewise linear versions of Gharaei–Homburg maps, both of which do not have a Markov partition in general.

1. Introduction

This paper concerns the generalized arcsine law of random iterations of interval maps with intermittent behavior. Intermittency is the irregular alternation of phases of apparently laminar and chaotic dynamics, and commonly observed in fluid flows near the transition to turbulence. The well-known model of intermittent dynamics is the Pomeau–Manneville map, that is, a piecewise expanding map \( f \) of the interval \([0, 1]\) with two increasing surjective branches and an indifferent fixed point at the boundary point 0, such as

\[
(1.1) \quad f(x) = x + x^{p+1} \mod 1 \quad (p > 0),
\]

named after Pomeau and Manneville since they numerically studied such interval maps in [14]. The existence of the indifferent fixed point 0 makes a typical orbit of \( f \) have long stays around 0 (the laminar phases) while the expanding property of \( f \) makes the orbit have bursts outside the neighborhood of 0 (the chaotic phases). Furthermore, the Pomeau–Manneville map with the indifferent fixed point 0 of order \( p + 1 \), such as (1.1), is known to possess an absolutely continuous invariant measure \( \mu \) whose density is of order \( x^{-p} \) near 0, so \( p \geq 1 \) if and only if \( \mu \) has infinite mass near 0, meaning that the orbit stays most of the time near the indifferent fixed point 0.

Thaler showed in [19] that when \( f \) is a piecewise expanding interval map with two increasing surjective branches and two indifferent fixed points at the boundary...
points 0 and 1 of the same order \( p + 1 \) for \( p \geq 1 \), such as

\[
f(x) = \begin{cases} 
  x + 2^p x^{p+1} & (x \in [0, \frac{1}{2}]) \\
  x - 2^p(1 - x)^{p+1} & (x \in [\frac{1}{2}, 1])
\end{cases}
\]

then \( f \) has a unique absolutely continuous invariant measure \( \mu \) with infinite mass near 0 and 1, and also showed in [17] that the arcsine law holds in the sense that

\[
\lim_{N \to \infty} \mu\left( \frac{S_N^+}{N} \leq a \right) = \frac{2}{\pi} \arcsin \sqrt{a}
\]

with \( S_N^+ = \sum_{n=0}^{N-1} \mathbb{1}_{[\frac{1}{2}, 1]} \circ f^n(x) \) for each \( a \in [0, 1] \) under a condition on wandering rates (see Theorem 2.3 for precise description for the condition). This was recently generalized in [16] to multi-ray settings by T. Sera and the fourth author, by using ideas from excursion theory. See also [2, 3, 15, 18] and reference therein.

Although the existence of indifferent fixed points of maps in the above works is indispensable for the generalized arcsine law of intermittent dynamics, G. Hata and the fourth author recently showed in [8] that random iterations of two piecewise linear interval maps without indifferent periodic points,

\[
(\text{HY}) \quad f_0(x) = \begin{cases} 
  \frac{x}{2} & (x \in [0, \frac{1}{2}]) \\
  2x - 1 & (x \in [\frac{1}{2}, 1])
\end{cases}, \quad f_1(x) = \begin{cases} 
  2x & (x \in [0, \frac{1}{2}]) \\
  \frac{x+1}{2} & (x \in [\frac{1}{2}, 1])
\end{cases}
\]

where each map is chosen with probability \( \frac{1}{2} \) at each step, exhibit the arcsine law. The endpoints 0 and 1 are common fixed points of \( f_0 \) and \( f_1 \), and they are in fact indifferent in average in the sense that

\[
(1.2) \quad \frac{\log |f_0'(0)| + \log |f_1'(0)|}{2} = \log |f_0'(1)| + \log |f_1'(1)| = 1.
\]

Such an indifferent in average fixed point was also considered by Gharaei and Homberg [7]; they showed the “on-off intermittency” (which may be more easily observed than the generalized arcsine law; see Remark 1.6 for details) for random iterations of two maps \( f_0 \) and \( f_1 \) on the interval \([0, 1]\) such that

\[
(\text{GH1}) \quad \text{both } f_0 \text{ and } f_1 \text{ are diffeomorphisms},
\]

\[
(\text{GH2}) \quad f_0(x) < x \text{ and } f_1(x) > x \text{ for all } x \in [0, 1],
\]

\[
(\text{GH3}) \quad f_0(0) = f_1(0) = 0, f_0(1) = f_1(1) = 1 \text{ and } (1.2) \text{ holds.}
\]

See Figure 1. This was generalized to chaotically driven non-autonomous iterations of such a pair of two maps in [11]. See also [4, 9, 10] for recent works for critical intermittency of iterated functions systems.

The key in the proof of the arcsine law in [8] is the existence of a Markov partition preserved by each interval maps. In the present paper, we give a class of random iterations of two interval maps without indifferent periodic points but satisfying the arcsine law, by introducing a concept of core dynamics. As applications, we show that the arcsine law holds for some generalized Hata–Yano (HY) maps and some piecewise linear versions of Gharaei–Homberg (GH) maps, both of which do not have a Markov partition in general (see Figure 2 as examples and refer to Section 1.2 for precise definitions).
1.1. **Main result: core random dynamics.** We consider two interval maps $f_0$, $f_1 : [0, 1] \to [0, 1]$ such that there are a real number $c \in (0, \frac{1}{2}]$ and measurable maps $g_0 : [c, 1 - \frac{c}{2}] \to [\frac{c}{2}, 1 - c]$, $g_1 : [\frac{c}{2}, 1 - c] \to [c, 1 - c]$ satisfying that

\[
(1.3) \quad f_0(x) = \begin{cases} 
\frac{x}{2} & x \in [0, c) \\
g_0(x) & x \in [c, 1 - \frac{c}{2}) \\
2x - 1 & x \in [1 - \frac{c}{2}, 1]
\end{cases}, \quad f_1(x) = \begin{cases} 
2x & x \in [0, \frac{c}{2}) \\
g_1(x) & x \in [\frac{c}{2}, 1 - c) \\
\frac{x + 1}{2} & x \in [1 - c, 1]
\end{cases}.
\]

See Figure 1. Note that given $(f_0, f_1)$ satisfying (1.3) with some $(c, g_0, g_1)$, one may find another pair $(\tilde{c}, \tilde{g}_0, \tilde{g}_1)$ for which (1.3) holds instead of $(c, g_0, g_1)$. Notice also that $(f_0, f_1)$ with $g_0(x) = \frac{x}{2}$ and $g_1(x) = 2x$ converges to the one given in (HY) in the limit $c \to 0$, and $(f_0, f_1)$ satisfies (GH1)-(GH3) when $g_0, g_1$ are diffeomorphisms, $f_0$ is of class $C^1$ near $x = c$ and $x = 1 - \frac{c}{2}$ and $f_1$ is of class $C^1$ near $x = \frac{c}{2}$ and $x = 1 - c$. As one can see below, the slopes 2 and $\frac{1}{2}$ is not essential in our main result: it can be replaced by $\lambda$ and $\lambda^{-1}$ for any $\lambda > 1$. We emphasize that this is contrastive to Hata–Yano map for which the slopes 2 and $\frac{1}{2}$ are important to ensure the existence of a Markov partition. Let $T$ be a random map of $[0, 1]$ such that $T = f_0$ and $f_1$ with equal probabilities.
For notational simplicity, we denote $[\frac{c}{2}, c], [c, 1 - c], [1 - c, 1 - \frac{c}{2}]$ by $I^-_1, I_0, I^+_1$, respectively, and $I^-_1 \cup I_0 \cup I^+_1$ by $Y$. Define two maps $h_0, h_1: Y \to Y$ given by

$$h_0(x) = \begin{cases} g_1(x) & x \in I^-_1 \\ g_0(x) & x \in I_0 \cup I^+_1 \end{cases}, \quad h_1(x) = \begin{cases} g_1(x) & x \in I^-_1 \cup I_0 \\ g_0(x) & x \in I^+_1 \end{cases},$$

and let $T_{\text{core}}$ be a random map of $Y$ such that $T_{\text{core}} = h_0$ and $h_1$ with equal probabilities.

**Definition 1.1.** Let $\{T_n\}_{n=1}^\infty$ be an i.i.d. sequence of random maps whose distribution is the same as that of $T$, and let $T^{(n)} = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for each $n \geq 1$. Similarly, we define $T_{\text{core}}^{(n)}$ for $n \geq 1$ from $T_{\text{core}}$. Then, we call $\{T^{(n)}\}_{n=1}^\infty$ a random dynamics with a core random dynamics $\{T_{\text{core}}^{(n)}\}_{n=1}^\infty$.

Recall that, given a random map $S$ on an interval $I$ with two measurable maps $h_0, h_1$ on $I$, that is, $S = h_0$ and $h_1$ with equal probabilities, a measure $\nu$ on $I$ is called $S$-invariant if $\nu$ is not a zero measure and

$$\frac{(h_0)_*\nu + (h_1)_*\nu}{2} = \nu,$$

where $(h_j)_*\nu = \nu \circ h_j^{-1}$ for $j = 0, 1$. Moreover, an $S$-invariant measure $\nu$ is called annealed metrically transitive if for any two Borel sets $A, B \subset I$ with $\nu(A)\nu(B) > 0$, there exists an integer $n \geq 0$ such that $\nu(S^{(-n)}A \cap B)$ has a positive expectation, where $S^{(-n)}A$ is the inverse image of $A$ by the random composition $S^{(n)}$ of an i.i.d. sequence of $n$ random maps whose distribution is same as the one of $S$.

We can now state our main result.

**Theorem 1.2.** Let $\{T^{(n)}\}_{n=1}^\infty$ be a random dynamics with a core random dynamics $\{T_{\text{core}}^{(n)}\}_{n=1}^\infty$ given in Definition 1.1. Suppose that there exists an annealed metrically transitive $T_{\text{core}}$-invariant probability measure $\nu$ with $\nu(I^-_1)\nu(I^+_1) > 0$. Then, for any random variable $\Theta$ with values in $Y$ which is independent of the random maps $\{T_n\}_{n=1}^\infty$ and whose distribution is absolutely continuous with respect to $\nu$, it holds
that

\[
\frac{1}{N} \sum_{n=0}^{N-1} 1\{T^{(n)}(\Theta) \geq \frac{1}{2}\} \xrightarrow{d_{N \to \infty}} \frac{1}{\pi \sqrt{x(1-x)}} \cdot \frac{b}{b^2x + (1-x)} dx
\]

with

\[
b = \frac{1 - \beta}{\beta}, \quad \beta = \frac{\nu(I^-)}{\nu(I^-) + \nu(I^+)},
\]

where \(d \xrightarrow{N \to \infty}\) means the convergence in distribution.

**Remark 1.3.** When \(\nu(I^-) = \nu(I^+)\), we have \(\beta = \frac{1}{2}\) and \(b = 1\), independently of the value of \(\nu(I_0)\). Thus, it follows from (1.5) that in the case we have

\[
\lim_{N \to \infty} \text{Prob} \left( \frac{1}{N} \sum_{n=0}^{N-1} 1\{T^{(n)}(\Theta) \geq \frac{1}{2}\} \leq a \right) = \int_0^a \frac{1}{\pi \sqrt{x(1-x)}} dx = \frac{2}{\pi} \arcsin \sqrt{a}
\]

for each \(a \in [0, 1]\). Hence, (1.5) is called the generalized arcsine law.

**Remark 1.4.** We show Theorem 1.2 by applying Thaler–Zweimüller’s abstract generalized arcsine law ([18, Theorem 3.2]) to the skew-product induced by \(T\) (which we will recall in Section 2). In the same paper, they also show Darling–Kac law ([18, Theorem 3.1]), so by combining it with the estimates in the proof of Theorem 1.2 for wandering rates, we can show the Darling–Kac law for random dynamics with a core random dynamics: under the assumption of Theorem 1.2, it holds that

\[
\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} 1\{T^{(n)}(\Theta) \in E\} \xrightarrow{d_{N \to \infty}} \frac{2\mu(E)}{\mu(E_0)} |N|,
\]

for any \(E \subset Y\), where \(E_0 := I_0^- \cup I_0^+\) and \(N\) is a random variable with standard normal distribution. See Remark 4.6 for more details.

**Remark 1.5.** As an easy consequence from Theorem 1.2, we finally remark the following pointwise generalized arcsine law: Assume that \(\{T^{(n)}\}_{n=1}^{\infty}\) and \(\nu\) satisfy the conditions in Theorem 1.2. Assume also that \(\nu\) is a discrete measure. Then, for any \(y\) in the support of \(\nu\), it holds that

\[
\frac{1}{N} \sum_{n=0}^{N-1} 1\{T^{(n)}(y) \geq \frac{1}{2}\} \xrightarrow{d_{N \to \infty}} \frac{1}{\pi \sqrt{x(1-x)}} \cdot \frac{b}{b^2x + (1-x)} dx.
\]

(Apply Theorem 1.2 to \(\Theta\) whose distribution is the Dirac measure at \(y\), which is absolutely continuous with respect to \(\nu\) since \(\nu\) is discrete.) This observation may be useful to establish (1.6) for each \(y\) in a large subset of \(Y\) as demonstrated for the example in Section 1.2.2, but (1.6) may hold beyond the above setting. In fact, for the random dynamics in Section 1.2.3, only a finite set of \(Y\) can be supported by an invariant probability measure \(\nu\) but (1.6) holds on a nontrivial interval (i.e. an interval whose interior is nonempty) of \(Y\).

**Remark 1.6.** We recall that Gharaei and Homburg [7] said that the on-off intermittency holds for \(T\) if, for any sufficiently small neighborhood \(U\) of 0 and 1 and any \(y \in (0, 1)\), it almost surely holds that

\[
\lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} 1\{T^{(n)}(y) \in U\} = 1 \quad \text{and} \quad \lim_{N \to \infty} \sum_{n=0}^{N-1} 1\{T^{(n)}(y) \notin U\} = \infty.
\]
From them, we have \( \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{T^n(y) \in U\}} \to 0 \) as \( N \to \infty \). That is, both the scales 1 and \( N \) are not nice to understand the long time behavior of \( \sum_{n=0}^{N-1} 1_{\{T^n(y) \in U\}} \), and the Darling–Kac law in Remark 1.4 tells the appropriate scale \( \sqrt{N} \) together with its limit distribution \( |N| \).

Furthermore, since Theorem 1.2 with "\( \{T^n(\Theta) \in A\} \)" instead of "\( \{T^n(\Theta) \geq \frac{1}{2}\} \)" holds for any interval \( A \) including 1 but not including 0 (as one can see from the proof of Theorem 1.2), by splitting \( U \) into an interval including 1 but not including 0 and its compliment, one would see that the generalized arcsine law is also a (much) stronger limit theorem than the on-off intermittency of Gharaei–Homburg.

1.2. Examples. In this subsection, we give examples that satisfy (or do not satisfy) the hypothesis for the core random dynamics \( T_{\text{core}} \) of Theorem 1.2.

1.2.1. Core deterministic dynamics. The simplest example is a random dynamics with a deterministic core dynamics, that is, a random core dynamics satisfying \( h_0 = h_1 \). Notice that when \( c = \frac{1}{2} \), we automatically have the case. In such a deterministic case, according to classical results for deterministic piecewise smooth interval maps, one can find several type of core dynamics with and without invariant probability measures. For example, when \( h_0 \) is a topologically mixing \( C^2 \) piecewise expanding map on \( Y \), then \( h_0 \) (and thus \( T_{\text{core}} \)) has a unique absolutely continuous ergodic invariant probability measure \( \nu \) whose support is \( Y \) (cf. [13]). Furthermore, the measure \( \nu \) may have different weights on \( I_1^- \) and \( I_1^+ \) (e.g. \( c = \frac{1}{2} \), \( h_0(x) = 2x - \frac{3}{4} \) on \( I_1^- \) and \( h_0(x) = c = \frac{1}{2} \) on \( I_1^+ \)). Moreover, since \( h_0 \) is not continuous, it is even possible that there exists no invariant probability measure for \( h_0 \) (e.g. \( c = \frac{1}{2} \), \( h_0(x) = \frac{1}{3}x + \frac{1}{4} \) on \( I_1^- \) and \( h_0(x) = x - \frac{1}{4} \) on \( I_1^+ \)). Refer also to [5, 6] and references therein for the existence of invariant probability measures of random dynamics.

1.2.2. Generalized Hata–Yano maps. We call the random map \( T \) a generalized Hata–Yano map if, for a constant \( 0 \leq \delta \leq 1/6 \), the maps \( f_0 \) and \( f_1 \) given by

\[
\begin{align*}
 f_0(x) &= \begin{cases} 
 \frac{x}{2} & (x \in [0, \frac{1}{2} + \delta)) \\
 2x - 1 & (x \in [\frac{1}{2} + \delta, 1]) 
\end{cases} , \\
 f_1(x) &= \begin{cases} 
 \frac{x+1}{2} & (x \in [0, \frac{1}{2} - \delta)) \\
 x & (x \in [\frac{1}{2} - \delta, 1]) 
\end{cases} ,
\end{align*}
\]

are randomly chosen with equal probabilities. Note that \( (f_0, f_1) \) is exactly (HY) when \( \delta = 0 \), and that the random map \( T \) admits a core random dynamics \( T_{\text{core}} \) for all \( \delta \neq 0 \) (e.g. \( c = 4\delta \) when \( \delta \leq \frac{1}{8} \), and \( c = \frac{1}{2} \) when \( \delta \geq \frac{1}{8} \)). In fact, when \( \delta \geq \frac{1}{8} \), the core random dynamics is deterministic.

For example, for \( \delta = \frac{1}{8} \) and \( c = \frac{1}{2} \), as depicted in Figure 4, one can easily see that \( h_0 = h_1 \), \( h_0(I) = I \) for \( I := [\frac{1}{4}, \frac{3}{4}] \) and \( h_0(x) = x \) on \( I \), so \( \nu := (\delta_y + \delta_{h_0(y)})/2 \) is a \( T_{\text{core}} \)-invariant probability measure satisfying the conditions of Theorem 1.2 for any \( y \in I \cup h_0(I) \). By virtue of Remark 1.5, pointwise generalized arcsine law (1.6) holds for any \( y \in I \cup h_0(I) \). Similarly, for each \( \delta \geq \frac{1}{8} \), one can see that \( T \) has a core deterministic dynamics with \( h_0(x) = x \) for any \( x \) close to \( \frac{1}{2} \) and (pointwise) generalized arcsine law holds.

1.2.3. Piecewise linear versions of Gharaei–Homburg maps. Finally, we consider piecewise linear versions of Gharaei–Homburg maps. That is, a random map \( T \) given by a random selection with probability \( \frac{1}{2} \) from \( (f_0, f_1) \) satisfying (lGH1) both \( f_0 \) and \( f_1 \) are homeomorphisms, and there are an integer \( N_j \geq 2 \) and real numbers \( c^{(j)}_0 < c^{(j)}_1 < \cdots < c^{(j)}_{N_j} = 1 \) for \( j = 0, 1 \) such that the
restriction of \( f_j \) on \([c_{i-1}^{(j)}, c_i^{(j)}]\) has a constant slope for each \( j = 0, 1 \) and \( i = 1, \ldots, N_j \), and \((\text{GH2}), (\text{GH3})\). It is straightforward to see that \( T \) always has a core random dynamics (e.g. for \( c = \min\{c_1^{(0)}, 1 - c_{N-1}^{(1)}\}\)). For simplicity, as before, in the following we assume that the slope of \( f_0 \) is \( \frac{1}{2} \) near \( x = 0 \) and \( 2 \) near \( x = 1 \).

Notice that if \( N_j = 2 \), then \( f_j \) must have the form

\[
    f_0(x) = \begin{cases} \frac{x}{2} & (x \in [0, \frac{1}{2})), \\ 2x - 1 & (x \in [\frac{1}{2}, 1]) \end{cases}, \quad f_1(x) = \begin{cases} 2x & (x \in [0, \frac{1}{4})), \\ x + \frac{\delta}{2} & (x \in [\frac{1}{4}, \frac{3}{4}]), \\ x + \frac{1}{2} & (x \in [\frac{3}{4}, 1]) \end{cases},
\]

which coincides with the generalized Hata–Yano map for \( \delta = \frac{1}{4} \), and thus the analysis of the case \( N_0 = N_1 = 2 \) is straightforward. Indeed, it satisfies that \( f_0 \circ f_1(x) = f_1 \circ f_0(x) = x \) everywhere, its core random dynamics is deterministic (for \( c = \frac{1}{2} \), \( h_0 \) exchanges \( I^+_1 = [\frac{1}{2}, 1] \) and \( I^-_1 = [\frac{1}{2}, \frac{3}{4}] \), and \( h_0^2(x) = x \) on \( Y = I^-_1 \cup I^+_1 \). Therefore, the simplest nontrivial case may be the case when \( N_0 = 2 \) and \( N_1 = 3 \), such as the family

\[
    f_0(x) = \begin{cases} \frac{x}{2} & (x \in [0, \frac{5}{6})), \\ 2x - 1 & (x \in [\frac{5}{6}, 1]) \end{cases}, \quad f_1(x) = \begin{cases} 2x & (x \in [0, \frac{1}{3})), \\ x + \frac{\delta}{2} & (x \in [\frac{1}{3}, 1 - \delta]), \\ x + \frac{1}{2} & (x \in [1 - \delta, 1]) \end{cases}
\]

with \( 0 < \delta \leq \frac{2}{3} \), see Figure 4. For \( \delta = \frac{1}{3} \), we can take \( c = \frac{1}{2} \) (core deterministic dynamics), \( h_0 \) exchanges \( I^-_1 \) and \( I^+_1 \) and the restriction of \( h_0^2 \) on \( I^-_1 \) has exactly one sink at the left endpoint \( \frac{1}{4} \) whose basin of attraction is \( I^-_1 \). Consequently, there exists a unique \( T_{\text{core}} \)-invariant probability measure \( \nu = (\delta_{\frac{1}{2}} + \delta_{\frac{1}{3}})/2 \), which satisfies the conditions of Theorem 1.2. Furthermore, pointwise generalized arcsine law (1.6) for the attracting periodic point \( p = \frac{1}{2} \) holds by Remark 1.5. However, unlike the generalized Hata–Yano map with \( \delta \geq \frac{1}{6} \), Remark 1.5 does not (directly) tell whether (1.6) holds for \( y \) in a nontrivial interval because there is no \( T_{\text{core}} \)-invariant probability measure on \( Y \) except \( \nu \). Yet, one can easily show (1.6) for all \( y \in I^-_1 \) as follows. Let \( I^-_n = [\frac{1}{2^n - 1}, \frac{1}{2^n}] \) and \( I^+_n = [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}] \) for \( n \in \mathbb{N} \) (see also Section 3). Then, it holds that \( f_{j_k} \circ f_{j_{k-1}} \circ \cdots \circ f_{j_1}(y) \in I^+_n \) if and only if \( f_{j_k} \circ f_{j_{k-1}} \circ \cdots \circ f_{j_1}(p) \in I^+_n \) for any \( k \in \mathbb{N} \), \((j_1, \ldots, j_k) \in \{0, 1\}^k, n \in \mathbb{N} \) and
σ ∈ {+, −} because y, p ∈ I^+_1, y belongs to the basin of attraction of p and f_0, f_1 are monotonically increasing. Hence, noticing that ∪_{n ≥ 1} I^+_n = [1/2, 1), we get (1.6) for y from (1.6) for p.

1.2.4. Problem. Finally, we ask a question on the above examples for further applications of Theorem 1.2.

Problem 1.7. Does there exist a T_{core}-invariant probability measure satisfying the condition of Theorem 1.2 for all generalized Hata–Yano maps (except δ = 0) and piecewise linear versions of Gharaei–Homburg maps?

Notice that in general the maps h_0 and h_1 generating the core random dynamics T_{core} are not continuous, even if both the maps f_0 and f_1 generating T are continuous (see Figure 3). Thus, even the existence of a T_{core}-invariant probability measure for piecewise linear versions of Gharaei–Homburg maps might be a nontrivial problem.

2. The Thaler–Zweimüller Theorem

The proof of Theorem 1.2 is based on the Thaler–Zweimüller theorem, which gives a criterion for generalized arcsine law of abstract infinite ergodic systems. In this subsection we briefly recall the Thaler–Zweimüller theorem.

In this subsection, let τ be a conservative, ergodic and measure preserving transformation over a σ-finite and infinite measure space (X, B, m). Here we mean by conservative that any set W ∈ B with τ^{-k}W ∩ τ^{-l}W = ∅ (mod m) for all distinct k, l ∈ N is a µ-null set and by ergodic that any set E ∈ B with τ^{-1}E = E (mod m) satisfies m(E) = 0 or m(X \ E) = 0. First we recall the definition of the Perron–Frobenius operator of τ (cf. [1]).

Definition 2.1. The Perron–Frobenius operator \( \hat{\tau} \): L^1(X, m) → L^1(X, m) corresponding to τ with respect to m is defined by
\[
\hat{\tau}\phi = \frac{d((\phi \cdot m) \circ \tau^{-1})}{dm} \quad \text{for } \phi \in L^1(X, m)
\]

where \( \phi \cdot m \) is a signed measure given by \( (\phi \cdot m)(A) = \int_A \phi \, dm \) for A ∈ B.

The Perron–Frobenius operator \( \hat{\tau} \) is also characterized by
\[
\int_X \hat{\tau}\phi \cdot \psi \, dm = \int_X \phi \cdot \psi \circ \tau \, dm
\]

for \( \phi \in L^1(X, m) \) and \( \psi \in L^\infty(X, m) \).

We then recall some necessary definitions for stating the Thaler–Zweimüller theorem. In what follows, let \( A ∈ B \) be a set of positive and finite \( m \)-measure. Recall that the first return time for A is a map \( \varphi = \varphi_A : A → \mathbb{N} \cup \{\infty\} \) defined by \( \varphi(x) := \inf \{n ∈ \mathbb{N} : \tau^n x ∈ A \} \) for \( x ∈ A \), where \( \inf \emptyset = \infty \). When τ is conservative and ergodic, \( \varphi \) can be naturally extended to the function defined and finite \( m \)-almost everywhere, which is also written by the same symbol \( \varphi = \varphi_A \) and is referred to be the first return time for A.

Definition 2.2. A measurable function \( H ≥ 0 \) supported on \( A \) is called uniformly sweeping for \( A \) if there is some \( K ∈ \mathbb{N}_0 \) such that
\[
\text{ess inf}_{x ∈ A} \sum_{k=0}^K \hat{\tau}^k H(x) > 0.
\]
For mutually disjoint sets $A^{-}, A, A^{+} \in \mathcal{B}$, we say that a set $A$ dynamically separates $A^{-}$ and $A^{+}$ if $\tau^{k}x \in A^{-}$ and $\tau^{l}x \in A^{+}$ for some $k, l \in \mathbb{N}$ imply $\tau^{n}x \in A$ for some $n \in \mathbb{N}$ with $k < n < l$ or $l < n < k$. For a set $A$, the wandering rate is defined by

$$w_{N}(A) := \sum_{n=0}^{N-1} m(A \cap \{ \varphi > n \}) = \int_{A} \left( \sum_{n=0}^{N-1} \tau^{n}1_{A_{n}} \right) \, dm$$

where $A_{0} := A$ and $A_{n} := A^{c} \cap \{ \varphi = n \} \ (n \in \mathbb{N})$, and when $A$ dynamically separates $A^{-}$ and $A^{+}$ we also define

$$w_{N} (A, A^{\pm}) := \sum_{n=0}^{N-1} m(A \cap \tau^{-1}A^{\pm} \cap \{ \varphi > n \})$$

$$= m(A \cap \tau^{-1}A^{\pm}) + \sum_{n=1}^{N-1} m(A_{n} \cap A^{\pm}).$$

Now we recall the Thaler–Zweimüller theorem. Recall that the strong distributional convergence of random variables $\{L_{N}\}_{N \in \mathbb{N}}$ on a measure space $(X, \mathcal{B}, m)$ to a random variable $R$, denoted by $L_{N} \xrightarrow{d(m)} N \rightarrow \infty R$, means that $\{L_{N}\}_{N \in \mathbb{N}}$ converges to $R$ in distribution with respect to any probability measure $m$, which is absolutely continuous with respect to $m$.

**Theorem 2.3** (18). Let $\tau$ be a conservative, ergodic and measure preserving transformation on a $\sigma$-finite and infinite measure space $(X, \mathcal{B}, m)$. Suppose the following conditions:

(C1) There is a set $A \in \mathcal{B}$ with $0 < m(A) < \infty$ such that

$$\lim_{N \rightarrow \infty} \frac{1}{w_{N}(A)} \sum_{n=0}^{N-1} \tau^{n}1_{A_{n}} = H \quad \text{uniformly on } A$$

for some $H : A \rightarrow [0, \infty)$ uniformly sweeping.

(C2) $\sum_{i}(\cdot) : N \rightarrow \mathbb{R}_{+}$ is regular varying with exponent $1 - \alpha > 0$ i.e., $w_{N}(A) = N^{1-\alpha} \cdot \psi(N)$ for some $\psi : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ such that for any $\lambda > 0$ we have

$$\lim_{x \rightarrow \infty} \frac{\psi(x\lambda)}{\psi(x)} = 1.$$

(C3) There is a partition $X = A^{-} \cup A \cup A^{+}$ with $m(A^{-}) = m(A^{+}) = \infty$ such that $A$ dynamically separates $A^{-}$ and $A^{+}$, and

$$\lim_{N \rightarrow \infty} \frac{1}{w_{N}(A, A^{-})} \sum_{n=0}^{N-1} \tau^{n}1_{A_{n} \cap A^{-}} = H^{-} \quad \text{uniformly on } A$$

for some $H^{-} : A \rightarrow [0, 1)$ uniformly sweeping.

(C4) There exists some $\beta \in (0, 1)$ such that

$$\lim_{N \rightarrow \infty} \frac{w_{N}(A, A^{-})}{w_{N}(A)} = \beta.$$ 

Then, the strong distributional convergence

$$\frac{1}{N} \sum_{n=0}^{N-1} 1_{B} \circ \tau^{n} \xrightarrow{d(m)} N \rightarrow \infty \frac{b \sin \pi \alpha}{\pi} \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{b^{2}2^{2\alpha} + 2bx^{\alpha}(1-x)^{\alpha} \cos \pi \alpha + (1-x)^{2\alpha}} \, dx$$

holds for any $B \in \mathcal{B}$ satisfying $\mu(B \Delta A^{-}) < \infty$, where $b := (1 - \beta)/\beta$. 

---

**Notes:**

1. $\tau^{n}1_{A_{n}}$ denotes the $n$-th iterate of the indicator function of $A_{n}$.
2. $\psi(N)$ is a slowly varying function.
3. $\lim_{x \rightarrow \infty} (\psi(x\lambda)/\psi(x)) = 1$ for $\lambda > 0$. 
4. $\mu(B \Delta A^{-})$ represents the measure of the symmetric difference of $B$ and $A^{-}$. 
5. $b$ is a constant. 

---

**References:**

- Thaler–Zweimüller theorem is often cited in the literature on ergodic theory and dynamical systems.
- The specific form of the convergence condition and the result of the integral are typical in the study of distributional convergence in measure theory.
3. T-invariant measure

In this section, as a preliminary for the proof of Theorem 1.2, we construct a T-invariant measure \( \mu \) from the \( T_{\text{core}} \)-invariant measure \( \nu \) of Theorem 1.2. We first introduce a partition \( \zeta \) of \([0, 1]\) by

\[
\zeta = \{ I_n^-, I_0, I_n^+ : n \in \mathbb{N} \}
\]

where we set, for each \( n \in \mathbb{N} \),

\[
I_n^- = \left( \frac{c}{2^n}, \frac{c}{2^{n-1}} \right), \quad I_0 = [c, 1 - c), \quad I_n^+ = \left[ 1 - \frac{c}{2^n}, 1 - \frac{c}{2^{n-1}} \right).
\]

In order to construct a \( \sigma \)-finite invariant measure for the entire random dynamical system \( T \), we assume that a partial random dynamical system \( T_{\text{core}} \) on \( I_1^- \cup I_0 \cup I_1^+ \) with \( \mathbb{P}(T_{\text{core}} = h_0) = \mathbb{P}(T_{\text{core}} = h_1) = \frac{1}{2} \) admits an invariant probability measure \( \nu \) supported on \( I_1^- \cup I_0 \cup I_1^+ \), that is,

\[
(f_1)_* \nu_- + \frac{1}{2} (f_0)_* \nu_0 + \frac{1}{2} (f_1)_* \nu_0 + (f_0)_* \nu_+ = \nu_- + \nu_0 + \nu_+,
\]

where \( \nu_- (\cdot) = \nu (\cdot \cap I_1^-) \), \( \nu_0 (\cdot) = \nu (\cdot \cap I_0) \) and \( \nu_+ (\cdot) = \nu (\cdot \cap I_1^+) \). If we set a measure \( \mu \) on \([0, 1]\) as

\[
\mu = \begin{cases} 
2(f_0)_n^{-1} \nu_- & \text{on } I_n^- (n \in \mathbb{N}), \\
\nu_0 & \text{on } I_0, \\
2(f_1)_n^{-1} \nu_+ & \text{on } I_n^+ (n \in \mathbb{N}),
\end{cases}
\]

then \( \mu \) is a \( \sigma \)-finite invariant measure for \( T \) which will be shown in the next lemma. Note that \( \nu \) and hence \( \mu \) are possibly singular to the Lebesgue measure on \([0, 1]\).

Lemma 3.1. Let \( \nu \) be an invariant Borel probability measure for \( T_{\text{core}} \). Then \( \mu \) given by the equation (3.2) is a \( \sigma \)-finite \( T \)-invariant Borel measure for which we have \( \mu ([\epsilon, 1 - \epsilon]) < \infty \) for any \( \epsilon \in (0, \frac{1}{2}) \).

Moreover, if \( \nu (I_1^-) \nu (I_1^+) > 0 \) holds then \( \mu ([0, \epsilon)) = \mu ((1 - \epsilon, 1]) = \infty \) for any \( \epsilon \in (0, \frac{1}{2}) \) and hence \( \mu \) is a \( \sigma \)-finite and infinite measure.

Proof. We check the statement by a direct calculation. For any \( A \subset I_n^- \) where \( n \geq 2 \), we have \( f_1^{-1} A \subset I_{n+1}^- \) and \( f_0^{-1} A \subset I_{n-1}^- \). Then from the construction of \( \mu \) (3.2),

\[
\mathbb{E} \left[ \mu \left( T^{-1} A \right) \right] = \left( \frac{1}{2} (f_1)_* \mu + \frac{1}{2} (f_0)_* \mu \right) (A)
\]

\[
= \frac{1}{2} \left( \mu \left( f_1^{-1} A \right) + \mu \left( f_0^{-1} A \right) \right)
\]

\[
= \frac{1}{2} \left( 2 \nu_- \left( f_0^{-(n-1)} \left( f_1^{-1} A \right) \right) + 2 \nu_- \left( f_0^{-(n-2)} \left( f_0^{-1} A \right) \right) \right)
\]

\[
= \nu_- \left( f_0^{-(n-1)} A \right) + \nu_- \left( f_0^{-(n-1)} A \right)
\]

\[
= \mu (A).
\]

In the second last equality, we used the fact that \( f_0^{-1} = f_1 \) on \( I_n^- \) for \( n \geq 2 \). For the case on \( I_n^+ \) \((n \geq 2)\), we can calculate by the same manner and we omit it. On
\(I^- \cup I_0 \cup I^+_1\), we also have for any \(A \subset I^-_1 \cup I_0 \cup I^+_1\),
\[
\mathbb{E}[\mu(T^{-1}A)] = \left(\frac{1}{2}(f_1)_*\mu + \frac{1}{2}(f_0)_*\mu\right)(A)
\]
\[
= \frac{1}{2}\left(\mu(f^{-1}_1A \cap I^-_1) + \mu(f^{-1}_1A \cap I_0) + \mu(f^{-1}_1A \cap I^+_1)\right)
\]
\[
\quad + \mu(f^{-1}_0A \cap I^-_1) + \mu(f^{-1}_0A \cap I_0) + \mu(f^{-1}_0A \cap I^+_1)\right)
\]
\[
= \frac{1}{2}\left(2\nu_-(f^{-1}_1A) + \nu_0(f^{-1}_1A) + 2\nu_-(f^{-1}_0A)\right)
\]
\[
\quad + \nu_0(f^{-1}_0A) + 2\nu_+(f^{-1}_0A)\right).
\]

From the equation (3.1) and that \(f_0\) and \(f_1\) are inverse maps to each other on \(I_n^\pm\) \((n \geq 2)\), we get
\[
\mathbb{E}[\mu(T^{-1}A)] = \nu_-(A) + \nu_0(A) + \nu_+(A) + \nu_-(A) + \nu_+(A)
\]
and hence \(\mu\) is \(T\)-invariant.

By the equation (3.2), we have
\[
\mu(I_{n+1}^-) = 2\nu_-(f_0^{-n}I_{n+1}^-) = 2\nu_-(f_0^{-n}(f_1^{-n}I^-_1)) = 2\nu_-(I^-_1) < \infty
\]
and similarly \(\mu(I_{n+1}^+) = 2\nu_+(I^+_1) < \infty\) for any \(n \in \mathbb{N}\). Then the other claims are valid. \(\square\)

**Remark 3.2.** When a \(T_{\text{core}}\)-invariant probability measure \(\nu\) is Lebesgue-absolutely continuous, we can have the density function of the resulting \(\sigma\)-finite \(T\)-invariant measure \(\mu\): let \(\varphi_-, \varphi_0\) and \(\varphi_+\) be the densities of \(\nu_-, \nu_0\) and \(\nu_+\), respectively. Then the density function of \(\mu\) can be represented as
\[
\varphi(x) := \frac{d\mu}{d\text{Leb}}(x) = \begin{cases} 
2^n\varphi_-(2^{n-1}x) & x \in I^-_n \ (n \in \mathbb{N}), \\
\varphi_0(x) & x \in I_0, \\
2^n\varphi_+(2^{n-1}x - 2^{n-1} + 1) & x \in I^+_n \ (n \in \mathbb{N}).
\end{cases}
\]

4. **Proof of Theorem 1.2**

4.1. **Skew-product representation.** This subsection is devoted to prepare the skew-product transformation corresponding to our random dynamical system. The proof of our main result is based on the application of the Thaler–Zweimüller theorem to the skew-product transformation.

We let \((\Omega, \mathcal{B}(\Omega), \mathbb{P})\) be a probability space with \(\Omega = \{0, 1\}^\mathbb{N}\), the Borel field of \(\Omega\), and
\[
\mathbb{P}(\omega = \{\omega_1, \omega_2, \cdots \} \in \Omega : \omega_1 = i_1, \cdots, \omega_n = i_n) = \frac{1}{2^n}
\]
for any \(i_1, \cdots, i_n \in \{0, 1\}\) and \(n \in \mathbb{N}\). For notational simplicity, we denote \(\bar{B} = B^- = \Omega \times B\) for each \(B \in \mathcal{B}([0, 1])\), and equip \(\{0, 1\}^\sim = \Omega \times [0, 1]\) with the product
For simplicity, we write for every Borel set \( \omega \) \( \in \Omega \) and \( x \in [0,1] \), where \( \theta : \Omega \to \Omega \) is the left shift map \( \theta(\omega_1,\omega_2,\cdots) = (\omega_2,\omega_3,\cdots) \). Note that from the definition of \( \bar{T} \) we have

\[
\bar{T}^n(\omega,x) = (\theta^n \omega, f_\omega^n(x)), \quad f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_1}
\]

for each \( n \in \mathbb{N} \). Given \( n \leq m \) and \( \omega = (\omega_1,\ldots,\omega_m) \in \{0,1\}^m \), we still use the notation \( f_\omega^n \) to denote \( f_{\omega_n} \circ \cdots \circ f_{\omega_1} \), so that the invariance of \( \mu \) for \( T \) implies

\[
\mathbb{E} \left[ \mu(T^{-n}A) \right] = \frac{1}{2^n} \sum_{\omega \in \{0,1\}^n} \mu \left( f_\omega^{-n}A \right) = \mu(A)
\]

for every Borel set \( A \) and \( n \in \mathbb{N} \), where \( f_\omega^{-n}(A) \) is the inverse image of \( A \) by \( f_\omega^n \).

For simplicity, we write \( f_\omega^0 \) for the identity map for each \( \omega \). Similarly, we define \( h_\omega^n \) for \( n \in \mathbb{N} \) and \( \omega \in \{0,1\}^n \), from the measurable maps \( h_0, h_1 \) on \( Y \).

Let \( \tilde{\mu} := \mathbb{P} \otimes \mu \) where \( \mu \) is a \( \sigma \)-finite and infinite \( T \)-invariant measure given in the equation (3.2) under the assumptions of Lemma 3.1 that is, \( \nu \) is a \( T \)-core-invariant probability measure and \( \nu(I_1^-)\nu(I_1^+) > 0 \). Then it is well known (see [12] for example) that \( \tilde{\mu} \) is also an invariant measure for the skew-product transformation \( \bar{T} \) which is \( \sigma \)-finite and infinite. From Lemma 3.1 we also have for each \( \epsilon \in (0,\frac{1}{2}) \),

\[
\tilde{\mu}((\epsilon,1-\epsilon)^c) < \infty \quad \text{and} \quad \tilde{\mu}((0,\epsilon)^c) = \tilde{\mu}((1-\epsilon,1)^c) = \infty.
\]

Furthermore, we have the following lemmas about the annealed metrical transitivity of \( (T,\mu) \) and the ergodicity of \( (\bar{T},\tilde{\mu}) \). Recall that a random dynamical system \( (T,\mu) \) is called ergodic if any \( T \)-invariant set \( A \in \mathcal{B}[0,1] \) in the sense that \( \mathbb{E}[1_A \circ T] = 1_A \) is either \( A = \emptyset \) or \( [0,1] \setminus A = \emptyset \) (mod \( \mu \)).

**Lemma 4.1.** If \( (T,\mu) \) is annealed metrically transitive, then \( (T,\mu) \) is ergodic.

**Proof.** Let \( A \in \mathcal{B} \) be a \( T \)-invariant set. That is, \( A \) satisfies \( 1_A = \mathbb{E}[1_{T^{-1}A}] = \frac{1}{2} \left( f_{I_0}^{-1}A + f_{I_+}^{-1}A \right) \). Then \( 1_A = 1_{f_{I_0}^{-1}A \cap f_{I_+}^{-1}A} + \frac{1}{2} f_{I_0}^{-1}A \Delta f_{I_+}^{-1}A \) and this implies \( f_{I_0}^{-1}A = f_{I_+}^{-1}A = A \) (mod \( \mu \)). From the annealed metrical transitivity of \( (T,\mu) \), for any sets of \( \mu \)-positive measure, say \( B, C \), we can find \( n \geq 1 \) such that \( \mathbb{E}[\mu(T^{-n}B \cap C)] > 0 \). But if we take \( B = A \) and \( C = A^c \) above where \( A \) is a \( T \)-invariant set, then we have \( \mathbb{E}[\mu(T^{-n}B \cap C)] = \mu(A \cap A^c) = 0 \) and hence either \( A = \emptyset \) or \( [0,1] \) (mod \( \mu \)). \( \square \)

**Lemma 4.2.** \( (T,\mu) \) is ergodic if and only if \( (\bar{T},\tilde{\mu}) \) is ergodic.

**Proof.** See Theorem 2.1 in [12] Chapter 1. \( \square \)

Under the help of Lemmas 4.1 and 4.2 we get the following assertion.

**Theorem 4.3.** If \( (T_{\text{core}},\nu) \) is annealed metrically transitive, then \( (\bar{T},\tilde{\nu}) \) is conservative and ergodic.

We prepare a lemma before proving Theorem 4.3

**Lemma 4.4.** Let \( n \geq 2 \). Then for \( \mathbb{P} \)-almost all \( \omega = (\omega_1,\omega_2,\ldots) \in \Omega \), there exists \( N_- \in \mathbb{N} \) (resp. \( N_+ \in \mathbb{N} \)) such that for any \( x \in I_i^- \) (resp. \( I_i^+ \)), we have

\[
f_{\omega}^{(N_-)}(x) \in I_i^- \quad \text{(resp.} \quad f_{\omega}^{(N_+)}(x) \in I_i^+ \).\]

Proof. We fix \( n \geq 2 \). Note that \( \mathbb{P}(T = f_0) = \mathbb{P}(T = f_1) = \frac{1}{2} \) and \( f_0 \circ f_1(x) = f_1 \circ f_0(x) = x \) for any \( x \in I_n^+ \) where \( n \geq 2 \). Then setting for \( \omega = (\omega_1, \omega_2, \ldots) \in \Omega \) and \( k \geq 1 \)

\[
\eta_k(\omega) = \begin{cases} 
-1 & (\omega_k = 0) \\
1 & (\omega_k = 1)
\end{cases}
\]

and

\[
W_k^-(\omega) = -n + \sum_{i=1}^k \eta_i(\omega),
\]

we identify our random dynamics \((T,\mu)\) with a simple random walk on \( \mathbb{Z} \setminus \{-1,0,1\} \) outside \( Y = \bigcup I_n^+ \). Let \( \Omega_{-n} = \{ \omega \in \Omega : \varphi_{-1}(\omega) < \infty \} \) where \( \varphi_{-1}(\omega) = \inf\{ j \geq 1 : W_j^-(\omega) = -1 \} \) stands for the first hitting time of \(-1\) for \( (W_k^-(\omega))_k \). Then from a standard argument for one-dimensional random walk which is recurrent and irreducible, we have \( \mathbb{P}(\Omega_{-n}) = 1 \), i.e., for \( \mathbb{P}\)-almost every \( \omega \in \Omega \), there is \( N = N(\omega) \in \mathbb{N} \) such that \( f_\omega(N)(I^-_n) = I^-_n \). By the symmetry, we conclude the result is true for the positive side \( \bigcup_{n \geq 2} I_n^+ \) and complete the proof. \( \square \)

Proof of Theorem 4.3 We first show that \((\hat{T},\hat{\mu})\) is conservative. To show this, by Maharam’s recurrence theorem (Theorem 1.1.7 in [1]), it is enough to see the existence of a set \( M^- \in \mathcal{B}([0,1]^-) \) such that \( \hat{\mu}(M^-) < \infty \) and \( \bigcup_{n=0}^{\infty} \bigcup_{j=0}^{\infty} \mathbb{Z}^+ \) such that \( \hat{\mu}(M^-') = \{0,1]\} \) (mod \( \hat{\mu} \)). For this, we set \( M^- := \Omega \times Y \) where \( Y = \bigcup I_n^+ \) and \( \omega' = (0 \cdots 0) \) (resp., \( B' = B \cap I_n^+ \) and \( \omega' = (1 \cdots 1) \)). Define \( A' \subset A \), an integer \( n_1 \geq 1 \) and \( \omega \in \{0,1\}^{n_1} \) in a similar manner, except the case \( \mu(A' \cap I_n^+) = \mu(A) \) in which we set \( A' = A \) and \( n_1 = 0 \). Then, observing that \( f_{\omega}^{-j}(I_n^+) = f_{\omega}^{-j}(I_{n-j}^-) = I_{n-j}^- \) and \( f_{\omega}^{-j}(I_n^+) = f_{\omega}^{-j}(I_{n-j}^-) = I_{n-j}^- \) for each \( j = 0, \ldots, n_1 - 1 \) and \( n \geq 1 \) (recall that \( f_0 \circ f_1(x) = f_1 \circ f_0(x) = x \) on \( Y^c \)) and that \( f_0(I_n^+), f_1(I_n^+) \subset Y \), we get

\[
\mu \left( f_{\omega}^{-j}(A') \cap Y \right) > 0, \quad f_{\omega}^{-j}(B') \subset I_{n-j}^- \quad \text{and}
\]

\[
f_{\omega}^{-j}(B') \cap Y = \emptyset \quad \text{for each} \quad j = 0, \ldots, n_2 - 2.
\]

Set \( A'' := f_{\omega}^{-j}(A') \) and \( B'' := f_{\omega}^{-j}(B') \). Then, since \( \nu(E) \geq \frac{\mu(E)}{2} \) for each \( E \in \mathcal{B}(Y) \) and (4.4), \( \nu(A'') \cap B'' > 0 \). Therefore, by the assumption that \((T_{\text{core}}, \nu)\) is annealed metrically transitive, we can find \( n_3 \geq 0 \) such that

\[
\mathbb{E} \left[ \nu \left( T_{\text{core}}^{-n_3} A'' \cap B'' \right) \right] = \frac{1}{2^n} \sum_{\eta \in \{0,1\}^{n_3}} \nu \left( h_{\eta}^{-n_3}(A'' \cap B'') \right) > 0.
\]

In particular, there exists \( \omega'' \in \{0,1\}^{n_3} \) satisfying \( \nu(h_{\omega''}^{-n_3}(A'' \cap B'')) > 0 \). It follows from \( \nu(E) \leq \mu(E) \) for each \( E \in \mathcal{B}(Y) \) that \( \mu(h_{\omega''}^{-n_3}(A'' \cap B'')) > 0 \). Recall that
h_0 = h_1 = f_1 on I_1^-, h_0 = f_0, h_1 = f_1 on I_0, h_0 = h_1 = f_0 on I_1^+. Hence, we have 
\mu(f_{\omega^m}^{(-n_3)} A'' \cap B'' > 0 \text{ for some } \omega'' \in \{0, 1\}^{n_3}. Denote \((\omega'_{n_2 - 1}, \ldots, \omega'_{1 + (mod 1)}) \) by \(\omega'\), then we have \(f_{\omega'}^{(-n_2 - 1)}(C) = f_{\omega'}^{(-n_2 - 1)}(C)\) for any \(C \subset I_1^- \cup I_1^+\) due to \((4.4)\) and \((4.5)\). In particular, \(f_{\omega'}^{(-n_2 - 1)} \circ f_{\omega'}^{(-n_2 - 1)}(B') = B', \) and thus we get

\[
\mu \left( f_{\omega'}^{(-n_2 - 1 + n_3 + n_1)} A \cap B \right) \geq \mu \left( f_{\omega'}^{(-n_2 - 1)} \left( f_{\omega^m}^{(-n_3)} A' \cap f_{\omega'}^{(-n_2 - 1)} B' \right) \right) = \mu \left( f_{\omega'}^{(n_2 - 1)} \left( f_{\omega^m}^{(-n_3)} A'' \cap B'' \right) \right).
\]

On the other hand, for \(D := f_{\omega^m}^{(n_3)} A'' \cap B''\), it follows from \((4.2)\) that

\[
\mu \left( f_{\omega}^{(n_2 - 1)} D \right) = \frac{1}{2^{n_2 - 1}} \sum_{\eta \in \{0, 1\}^{n_2 - 1}} \mu \left( f_{\eta}^{(-n_2 - 1)} \circ f_{\omega}^{(n_2 - 1)} D \right) \geq \frac{1}{2^{n_2 - 1}} \mu \left( f_{\omega}^{(-n_2 - 1)} \circ f_{\omega}^{(n_2 - 1)} D \right) = \frac{1}{2^{n_2 - 1}} \mu \left( f_{\omega^m}^{(-n_3)} A'' \cap B'' \right) > 0.
\]

Therefore, with \(N := n_2 - 1 + n_3 + n_1\), we have

\[
(4.7) \quad \mathbb{E} \left[ \mu \left( T^{(-N)} A \cap B \right) \right] = \frac{1}{2^N} \sum_{\eta \in \{0, 1\}^N} \mu \left( f_{\eta}^{(-N)} A \cap B \right) \geq \mu \left( f_{\omega}^{(-N)} A \cap I_1^- \right) > 0.
\]

This shows the desired result for the first case. In particular, if we take \(B = I_1^-\) above, then we have for any \(A \in B([0, 1])\) with \(\mu(A) > 0\),

\[
(4.8) \quad \mu \left( f_{\omega}^{(-N)} A \cap I_1^- \right) > 0
\]

for some \(\omega \in \Omega\) and \(N \in \mathbb{N}\).

We second consider the case when \(\mu(B \cap I_0) = \mu(I_0)\), implying \(\nu(B \cap I_0) = \mu(I_0) > 0\). From \((4.8)\), we get

\[
\nu \left( f_{\omega}^{(-N)} A \right) \geq \frac{1}{2} \mu \left( f_{\omega}^{(-N)} A \cap I_0^- \right) > 0
\]

for some \(\omega \in \Omega\) and \(N \in \mathbb{N}\). Hence, it follows from the annealed metrical transitivity of \((T_{\text{core}}, \nu)\) that there exists an integer \(m \geq 0\) and \(\omega\) such that \(\nu(f_{\omega}^{(-m)} \circ f_{\omega}^{(-N)} A \cap B) > 0\). Since \(\nu \leq \mu\) on \(Y\), this implies \(\mu(f_{\omega}^{(-m + N)} A \cap B) > 0\), which concludes the annealed metrical transitivity of \((T, \mu)\) by repeating the argument in \((4.7)\).

4.2. The end of the proof of Theorem 1.2

Now we complete the proof of Theorem 1.2 by checking the assumptions (C1)–(C4) in Theorem 2.3 with respect to

\[
(4.9) \quad \tau = \bar{T}, X = [0, 1]^\sim, \ m = \bar{m}, \ A = \bar{Y}, \ A^\perp = \left( \bigcup_{k=2}^{\infty} I_k^\pm \right)^\sim
\]
We set \((\overline{Y})_n = \overline{Y}^c \cap \{\varphi_{\overline{Y}} = n\}\), where \(\varphi_{\overline{Y}}(\omega, x) = \inf\{n \geq 1 : \overline{T}^n(\omega, x) \in \overline{Y}\}\), so that

\[
(\overline{Y})_n = \bigcup_{k=1}^{\infty} \Omega_{n,k}^+ \times I_{k+1}^- \cup \bigcup_{k=1}^{\infty} \Omega_{n,k}^- \times I_{k+1}^+ ,
\]

where

\[
\Omega_{n,k}^- := \left\{ \{\omega_1, \omega_2, \ldots\} \in \Omega : \right. \\
\left. \# \{1 \leq i < l : \omega_i = 1\} - \# \{1 \leq i < l : \omega_i = 0\} < k \text{ for } l < n, \right. \\
\left. \# \{1 \leq i \leq n : \omega_i = 1\} - \# \{1 \leq i \leq n : \omega_i = 0\} = k \right\} 
\]

and

\[
\Omega_{n,k}^+ := \left\{ \{\omega_1, \omega_2, \ldots\} \in \Omega : \right. \\
\left. \# \{1 \leq i < l : \omega_i = 0\} - \# \{1 \leq i < l : \omega_i = 1\} < k \text{ for } l < n, \right. \\
\left. \# \{1 \leq i \leq n : \omega_i = 0\} - \# \{1 \leq i \leq n : \omega_i = 1\} = k \right\} .
\]

Each \(\omega \in \Omega_{n,k}^\pm\) depends only on the first \(n\)-coordinates so that we can write \(\Omega_{n,k}^\pm = \Omega_{n,k}^\pm \mid_n \times \Omega\) for some \(\Omega_{n,k}^\pm \mid_n \subset \{0, 1\}^n\).

In what follows, \((W^n_k)_j = (W^n_k(\omega))_j\) denotes the random walk which starts from \(k \in \mathbb{Z} \setminus \{-1, 0, 1\}\) given in \((4.3)\) and for \(l \in \mathbb{Z}\) with \(lk > 0\), \(\varphi^+_l = \varphi^+_l(\omega) = \inf\{j \geq 1 : W^n_l = l\}\) denotes the first hitting time of \(l\) for the random walk \((W^n_k)_j\). Then we have for \(k \geq 1\)

\[
(4.10) \quad \mathbb{P} (\varphi^+_1 = n) = \mathbb{P}(\varphi^-_{-1} = n) = \mathbb{P}(\Omega_{n,k}^\pm) = \frac{\# \left( \Omega_{n,k}^\pm \mid_n \right)}{2^n}.
\]

For notational simplicity, we also write \(\overline{A}^\pm\) for \((\bigcup_{k=2}^{\infty} I_k^\pm)^\sim\). For these notation, we have the following lemma.

**Lemma 4.5.** For each \(n \geq 1\), it holds that

\[
\widehat{T}_n^1 1_{(\overline{Y})_n \cap \overline{A}^-}(\omega, x) = c_n 1_{I^-_1}(x)
\]

and

\[
\widehat{T}_n^1 1_{(\overline{Y})_n}(\omega, x) = c_n 1_{I^-_1 \cup I^+_1}(x)
\]

where

\[
c_n = \sum_{k=1}^{\infty} \mathbb{P} (\varphi^+_{1} = n).
\]

**Proof.** We prove this lemma in showing

\[
\int_f \psi(\omega, x) \widehat{T}_n^1 1_{(\overline{Y})_n \cap \overline{A}^\pm}(\omega, x) d\overline{\mu} = \int_f \psi(\omega, x) c_n 1_{I^\pm_1}(x) d\overline{\mu}
\]
for \( \psi \in L^\infty(\overline{I}, \overline{\mu}) \) of the form \( \psi(\omega, x) = \psi_1(\omega)\psi_2(x) \). Write
\[
\int_{\overline{I}} \psi(\omega, x) \overline{T}^{\psi_2}_{1(\overline{Y})_{n\cap \overline{\Delta}^-}(\omega, x)} d\overline{\mu} \\
= \int_{\overline{I}} \psi_1(\theta^n \omega) \psi_2 \left(f_{\omega_n} \circ f_{\omega_{n-1}} \cdots \circ f_{\omega_1}x\right) 1(\overline{Y})_{n} (\omega, x) \sum_{k=1}^{\infty} 1_{I_{k+1}^-}(x) d\overline{\mu} \\
= \sum_{k=1}^{\infty} \int_{\Omega} \psi_1(\theta^n \omega) \sum_{(\omega_1, ..., \omega_n) \in \Omega_{k,n} | n} 1_{[\omega_1, ..., \omega_n]}(\omega) d\overline{\mu}(\omega) \int_{I} \psi_2 \left(f^k_{1} x\right) 1_{I_{k+1}^-}(x) d\mu(x)
\]
since \( f^{(n)} = f^k_1 \) for \( (\omega_1, ..., \omega_n) \in \Omega_{k,n} | n \). For \( \hat{\theta} \) the Perron-Frobenius operator of \( \theta \) with respect to \( \overline{P} \), we have \( \hat{\theta}^n 1_{[\omega_1, ..., \omega_n]} = 2^{-n} \cdot 1_{\Omega} \) for each \( (\omega_1, ..., \omega_n) \in \{0, 1\}^n \). We also have \( I_{k+1}^- = f^{-k}_1 I_1^- \) and
\[
\frac{d \left((f^k_1)^* \mu\right)}{d\mu} \bigg|_{I_1^-} = \frac{d \left(2\nu_+ \circ f^{-k}_0 \circ f^{-k}_1\right)}{d\mu} \bigg|_{I_1^-} = \frac{d(2\nu_+)}{d\mu} \bigg|_{I_1^-} = 1
\]
by the construction of \( \mu \) and the fact that \( f^{-k}_0 \circ f^{-k}_1 = \text{id} \) on \( I_1^- \). Hence, by the equation (4.10), it holds that
\[
\int_{\overline{I}} \psi(\omega, x) \overline{T}^{\psi_2}_{1(\overline{Y})_{n\cap \overline{\Delta}^-}(\omega, x)} d\overline{\mu} = \sum_{k=1}^{\infty} \overline{P}(\varphi_{-1}^{-(k+1)} = n) \int_{\Omega} \psi_1 d\overline{P} \int_{I} \psi_2 1_{I_{1}^-} d\mu \\
= \int_{\overline{I}} \psi(\omega, x) \sum_{k=1}^{\infty} 1_{I_{1}^-}(x) \overline{P}(\varphi_{-1}^{-(k+1)} = n) d\overline{\mu}.
\]
For the right half part, the calculation of \( \overline{T}^{\psi_2}_{1(\overline{Y})_{n\cap \overline{\Delta}^+}} \) is similar and is omitted. \( \square \)

From the standard argument of simple symmetric random walks on \( \mathbb{Z} \) (cf. [8]), for \( 0 < s < 1 \) we have
\[
\sum_{n=1}^{\infty} s^n \overline{P}(\varphi_{-1}^{-(k+1)} = n) = \left(1 - \frac{\sqrt{1 - s^2}}{s}\right)^k
\]
for each \( k \in \mathbb{N} \). Thus, we have
\[
\sum_{n=1}^{\infty} c_n s^n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s^n \overline{P}(\varphi_{-1}^{-(k+1)} = n) \\
= \sum_{k=1}^{\infty} \left(1 - \frac{\sqrt{1 - s^2}}{s}\right)^k \\
= \frac{(1 - \sqrt{1 - s^2})}{\sqrt{1 - s^2} - 1 + s} \\
\sim \frac{1}{\sqrt{2(1 - s)}} \text{ as } s \uparrow 1.
\]
Here, \( a_N \sim b_N \) denotes \( \frac{a_N}{b_N} \to 1 \) as \( N \to \infty \). Then by Karamata’s Tauberian Theorem (see Proposition 4.2 in [18]) we conclude
\[
\sum_{n=1}^{N} c_n \sim \frac{1}{\sqrt{2\pi} \left( \frac{3}{2} \right)^{N/2}} N^{1/2} = \frac{1}{\sqrt{2}} N^{1/2}.
\]

From the above computation together with Lemma 4.5, we can calculate asymptotics of the wandering rate:
\[
w_N(\tilde{Y}) = \int_{\tilde{Y}} \sum_{n=0}^{N-1} \hat{T} 1(\tilde{Y})_n d\tilde{\mu}
= \mu(Y) + \left( \mu(I_1^-) + \mu(I_1^+) \right) \sum_{n=1}^{N-1} c_n
\sim \left( \mu(I_1^-) + \mu(I_1^+) \right) \frac{\sqrt{2}}{\pi} N^{1/2}
\]
and hence the condition (C2) is valid with the index \( \frac{1}{2} \). We also have
\[
w_N(\tilde{Y}, \tilde{A}^-) = \mu(\tilde{Y} \cap \tilde{T}^{-1} \tilde{A}^-) + \sum_{n=0}^{N-1} \int_{\tilde{Y}} \hat{T} 1_{\tilde{Y} \cap \tilde{A}^-} d\tilde{\mu}
= \frac{1}{2} \mu(I_1^-) + \mu(I_1^-) \sum_{n=0}^{N-1} c_n.
\]

Therefore, it also follow from Lemma 4.5 that
\[
\lim_{N \to \infty} \frac{1}{w_N(\tilde{Y})} \sum_{n=0}^{N-1} \hat{T} 1(\tilde{Y})_n = \lim_{N \to \infty} \frac{\sum_{n=0}^{N-1} c_n 1_{I_1^- \cup I_1^+}}{w_N(\tilde{Y})} = \frac{1}{\mu(I_1^-) + \mu(I_1^+)} \cdot 1_{I_1^- \cup I_1^+}
\]
uniformly on \( \tilde{Y} \) and
\[
\lim_{N \to \infty} \frac{1}{w_N(\tilde{Y}, \tilde{A}^-)} \sum_{n=0}^{N-1} \hat{T} 1(\tilde{Y})_n \cap \tilde{A}^- = \frac{1}{\mu(I_1^-)} \cdot 1_I^- \]
uniformly on \( \tilde{Y} \) and we have checked the conditions (C1) and (C3).

Finally, for the condition (C4), we have
\[
w_N(\tilde{Y}, \tilde{A}^-) = \mu(I_1^-) \sum_{n=0}^{N-1} c_n + \frac{1}{2} \mu(I_1^-)
\left( \mu(I_1^-) + \mu(I_1^+) \right) \sum_{n=0}^{N-1} c_n
\to \frac{\mu(I_1^-)}{\mu(I_1^-) + \mu(I_1^+)} = \frac{\nu(I_1^-)}{\nu(I_1^-) + \nu(I_1^+)}
\]
as \( N \) tends to infinity.

Now we can apply Theorem 2.3 to the specific \( \tau, X, m, A, A^\pm \) given in (4.9). For any random variable \( \Theta \) with values in \( Y \) which is independent of \( \{T_n\}_{n=1}^{\infty} \) and whose distribution \( m_\Theta \) is absolutely continuous with respect to \( \nu \) (so that \( m_\Theta \) is absolutely continuous with respect to \( \mu \) by construction of \( \mu \)), the probability measure \( P \times m_\Theta \) is absolutely continuous with respect to \( \tilde{\mu} \). Hence, the equation (4.1) and the strong
distributional convergence in Theorem 2.3 with $\alpha = \frac{1}{2}$ and $B = [\frac{1}{2}, 1]$ complete the proof of Theorem 1.2.

**Remark 4.6.** In order to apply Thaler–Zweimüller’s Darling–Kac law ([18, Theorem 3.1]), one needs to compute

$$\hat{a}_N := 4 \pi \cdot \frac{N}{w_N(Y)}$$

for which

$$\frac{1}{\hat{a}_N} \sum_{n=0}^{N-1} 1_{\{T^n(\Theta) \in E\}} \xrightarrow{d} \sqrt{\frac{\pi}{2}} \mu(E) |N|$$

holds for each Borel set $E$. On the other hand, it holds that

$$\hat{a}_N \sim 4 \pi \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{N}} \cdot \frac{N}{\mu(I^-_1) + \mu(I^+_1)} = 2 \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{N}}{\mu(I^-_1) + \mu(I^+_1)}$$

as $N \to \infty$.

Hence, we immediately get the claim in Remark 1.4.

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