Asymptotic Distribution of Eigenvalues for a Self-Affine String

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We consider a string with fixed endpoints where the mass density and/or the elastic coefficient vary in a self-affine way as function of position. It is demonstrated how the eigenvalues in the asymptotic limit are distributed. Scaling laws for the Weyl term of the asymptotic integrated density of states is established and confirmed numerically.

I. INTRODUCTION

An old problem in mathematical physics is how various irregularities influence the asymptotics of the cumulative eigenvalue distribution for a physical resonator. This problem has many significant physical applications, such as wave scattering from fractal surfaces, liquid flow in porous media, vibrations of cracked bodies or macromolecules (polymers) etc.

In 1910 Lorentz put forward the conjecture that the number of eigenmodes up to some (large) frequency $\omega$, depends on the “volume” and not on the shape of the resonator. This was later, proved by Weyl, under condition of a sufficiently smooth but otherwise arbitrary boundaries [1]. Later Hunt et al. [2] improved this formula by including a correction term which depends on lower powers of the frequency and also on the “surface area” of the resonator’s perimeter.

Berry [3], when working on wave scattering from fractal surfaces, generalized the result of Hunt et al. to fractal boundaries. He conjectured that for fractal boundaries with Hausdorff dimension $D_H$, the first correction term should scale as $\omega^{D_H}$. However, Lapidus has proved [4] that the correct fractal dimension is not the Hausdorff dimension, but another nontrivial dimension known as the Minkowski dimension $D_M$ [21].

Over the last decade or so, there have been renewed interest in this, and related problems, both in the mathematics [1] [11] and physics [13] [14] communities. In this paper we consider a related problem where we study a string which has an irregular (self-affine) mass density and local elastic coefficient. For a physical realization, we could for instance think of a long vibrating (“fussy”) polymer.

This paper is organized as follows. In Section II we introduce some of the general theoretical background for the problem. We then in Section III, make our conjecture for the integrated density of states (IDOS) for our self-affine string based on the results of Section II. In Section IV we discuss the decimation technique, the numerical method used to calculate the IDOS. The numerical results are presented in Section V, and the conclusion of this paper is drawn in Section VI.

II. GENERAL THEORY

In this section we review some results which will prove useful in the later discussion. Let $\Gamma \subset \mathbb{R}^n$ be a bounded open set. Consider the 1D elliptic differential equation

$$\nabla^2 u(x) + a(x)\omega^2 u(x) = 0, \quad x \in \Gamma \quad (2.1)$$

with homogeneous Dirichlet boundary conditions. The “weight function” $a(x)$, will be assumed to be a positive real valued function. This equation has a countable sequence of positive eigenvalues (eigenfrequencies) tending to infinity.

Let $N(\omega, \Gamma)$ denote the integrated density of states (IDOS), i.e. the number of eigenmodes with eigenfre-
frequency below $\omega$. Lapidus and Fleckinger have showed that the asymptotic behavior of $N(\omega, \Gamma)$ is

$$N(\omega, \Gamma) \sim W(\omega, \Gamma) = \frac{B_n \omega^n}{(2\pi)^n} \int_\Gamma \langle a(x) \rangle^H \rho^n \, d^n x, \quad (2.2)$$

as $\omega \to \infty$. Here $B_n$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $\Gamma$ is the resonator domain. This term is usually called the “Weyl term” after Weyl who first proved this asymptotic behavior for a “classical” (i.e. non-fractal) resonator (the Weyl conjecture).

III. CONJECTURES FOR THE SELF-AFFINE STRING

In this paper we will study the situation where the boundary is regular while the weight-function and/or the elastic coefficient is irregular. We will use the following physical model: Consider a freely vibrating string with fixed endpoints at $x = 0$ and $x = L$ (Dirichlet boundary conditions). It has an elastic coefficient $E(x) = E_0 + E_1(x)$ and density profile $\rho(x) = \rho_0 + \rho_1(x)$. Here $E_0 > 0$ and $\rho_0 > 0$ are constants and $E_1(x)$ and $\rho_1(x)$ are strictly positive, self-affine functions. The equation describing the vibrations of this “rough string” is

$$\frac{1}{\rho(x)} \frac{d^2}{dx^2} E(x) \frac{d}{dx} u(x) + \omega^2 u(x) = 0 . \quad (3.1)$$

If $E(x) = E_0$, then $\rho(x) = \rho(x)$ in Eq. (2.1) \[15\]. We will assume that the elastic coefficient $E_1(x)$ and the density $\rho_1(x)$ vary in a self-affine way \[20\]. The concept of self-affinity is a scaling property. A function defined as $h = h(x)$ is said to self-affine if it is statistically invariant under the transformation

$$x \to \lambda x , \quad (3.2a)$$

$$h(x) \to \lambda^H h(\lambda x) , \quad (3.2b)$$

for all positive $\lambda \in \mathbb{R}$, or equivalently

$$h(x) \approx \lambda^{-H} h(\lambda x) , \quad (3.2c)$$

where $\approx$ is used in order to indicate statistical equality. The parameter $H$ is the Hurst exponent (or roughness exponent). When $H > 1$, $h(x)$ is not asymptotically flat and the surface is not statistically invariant under translation. When $H < 0$, the variance of $h(x)$ diverge when the interval over which it is measures goes to zero. $h(x)$ is then referred to as a fractional noise. We will in our analysis assume $0 < H < 1$. Self-affinity is in practice only found within a restricted range of scales. In this work we explicitly introduce a lower cut-off $l$. The upper cut-off is the system size $L$, which is the length of the string.

Asymptotically, on large scale, $E_1(x)$ and $\rho_1(x)$ will dominate the behavior of $E(x)$ and $\rho(x)$ no matter what $E_0$ and $\rho_0$ are. However, we will investigate the system at intermediate scales where the constant terms may or may not dominate over the self-affine terms in $E(x)$ and $\rho(x)$.

We will distinguish four cases:

1. $E_0 \gg \langle E_1(x) \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$,
2. $E_0 \ll \langle E_1(x) \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$,
3. $E_0 \ll \langle E_1(x) \rangle$ and $\rho_0 \gg \langle \rho_1(x) \rangle$,
4. $E_0 \gg \langle E_1(x) \rangle$ and $\rho_0 \gg \langle \rho_1(x) \rangle$,

where $\langle \cdot \rangle$ denotes the averaging operator.

Note that the left hand side of Eq. (3.1), 
\[\omega \int_0^L \sqrt{\rho(x)/E(x)} \, dx\],

involves the derivative of a self-affine function, $dE_1/dx$. This quantity scales as $dE_1/dx = \lambda^{H-1} dE_1/dx$ when $x \to \lambda x$, while $E_1$ scales as $E_1 \to \lambda^H E_1$. Thus, we make the assumption that the term containing $dE_1/dx$ may be neglected in comparison to the term containing $E_1$ in Eq. (3.1).

With this assumption, we expect the IDS of the self-affine string to behave as \(n = 1, B_1 = 2\)

$$N(\omega, L) \sim W(\omega, L) = \frac{\omega}{\pi} \int_0^L \sqrt{\rho(x)/E(x)} \, dx , \quad (3.3)$$

in the asymptotic limit. Note that this expression can be written $W(\omega, L) = \omega L/(\pi \sqrt{\rho(x)/E(x)})$. Thus, the asymptotic behavior of the IDS for a rough string is expected to behave as a classical (non-rough) string with a constant inverse velocity $\langle \sqrt{\rho(x)/E(x)} \rangle$.

We now discuss the four cases in turn. Assume now that we change the system size according to $L \to \lambda L$. Then, for case (1) $\langle E_0 \gg \langle E_1(x) \rangle \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$ after a change of variable, it follows from Eqs. (3.2) and (3.3) that

$$W(\omega, \lambda L) = \frac{\omega}{\pi} \int_0^{\lambda L} \sqrt{\rho(x)/E_0} \, dx$$

$$= \frac{\omega}{\pi} \int_0^L \frac{\rho(\lambda x')}{E_0} \, d(\lambda x')$$

$$\approx \lambda^{1-H} W(\omega, L) . \quad (3.4a)$$

Using Eq. (3.3), this relation may be rewritten

$$W(\omega, \lambda L) \approx \lambda W(\omega, L) . \quad (3.4b)$$

Note that Eqs. (3.4a) and (3.4b) are in principle equivalent, but open up two different possibilities for physical interpretation. In the former case, it is the number of eigenmodes which is scaled, while in the latter it is the frequency.

Through similar arguments, we find for case (2) $(E_0 \ll \langle E_1(x) \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$,)

$$W(\omega, \lambda L) \approx \lambda W(\omega, L) , \quad (3.5a)$$
or equivalently
\[ W(\lambda^{-1}\omega,\lambda L) \simeq W(\omega, L) . \]
(3.5b)
For case (3) \((E_0 \ll \langle E_1(x) \rangle)\) and \(\rho_0 \gg \langle \rho_1(x) \rangle\), we expect
\[ W(\omega, L) \simeq \lambda^{1-H/2} W(\omega, L) , \]
(3.6a)
\[ W(\lambda^{-1+H/2}\omega, \lambda L) \simeq W(\omega, L) . \]
(3.6b)
Case (4) \((E_0 \gg \langle E_1(x) \rangle\) and \(\rho_0 \gg \langle \rho_1(x) \rangle\) leads to the same behavior as Case (2), Eqs. (3.3).

**IV. OUTLINE OF THE NUMERICAL METHOD**

In order to do numerical simulation of our self-affine string we discretize Eq. (3.1) on a lattice of \(N\) sites
\[ A_{ij}^{(0)} U_j(\omega) = \omega^2 U_i(\omega) \quad i, j = 1, 2, \ldots, N. \]
(4.1)
Here \(A^{(0)}\) is the \(N \times N\) matrix representation for the operator \((1/\rho(x))(d/dx)E(d/dx)\). It is tridiagonal.

The method used in this paper to calculate the IDOS for the above equation is the decimation technique of Lambert and Weaire [6]. This method is based on a renormalization philosophy, where successive degrees of freedom are eliminated from the system, see Refs. [16,17] for more details. After removing the sites corresponding to \(k = N - M, \ldots, N\) the system becomes
\[ A_{ij}^{(M)} U_j(\omega) = \omega^2 U_i(\omega), \quad i = 1, \ldots, N-M \]
(4.2)
with
\[ A_{ij}^{(M)} = A_{ij}^{(M-1)} + \frac{A_{i,N+1-M,j}^{(M-1)} A_{N+1-M,j}^{(M-1)}}{D^{(M)}} \]
(4.3)
where the denominator is given by
\[ D^{(M)} = A_{N+1-M,N+1-M}^{(M)} - \omega^2 . \]
(4.4)

At the very heart of the method lies the fact that renormalized system is equivalent to the original one in the sense that the two systems have the same eigenvalues. By repeating this procedure all degrees of freedom can be eliminated. The number of eigenmodes less than \(\omega\), i.e. the IDOS, can now be shown to be equal to the number of negative denominators in the sequence \(\{D^{(i)}\}_{i=1}^N\) [19]. We would like to point out that this algorithm is very efficient.

**V. SIMULATION AND RESULTS**

In order to test the conjecture (3.4), and thus also Eqs. (3.5) to (3.6), we have calculated the IDOS for various system sizes, \(L\), fixing the Hurst exponent \(H\) to the value 0.7. We work in length units where \(l = 1/2^{11}\). Thus, the system size \(N = 2^{11} = 2048\) corresponds to a string of unit length \((L = 1)\). We set \(E_0 = 1\) and \(\langle E_1(x) \rangle = 0.01\) for the cases where \(E_0 \gg \langle E_1(x) \rangle\), and \(E_0 = 0.01\) and \(\langle E_1(x) \rangle = 1\) for the cases where \(E_0 \ll \langle E_1(x) \rangle\). Likewise, we set \(\rho_0 = 1\) and \(\langle \rho_1(x) \rangle = 0.01\) for the cases where \(\rho_0 \gg \langle \rho_1(x) \rangle\), and \(\rho_0 = 0.01\) and \(\langle \rho_1(x) \rangle = 1\) for the cases where \(\rho_0 \ll \langle \rho_1(x) \rangle\). We averaged over 500 samples in case (1), while we used 50 samples in the three other cases.

As can be seen from Figs. [1] the IDOS is a linear function of the frequency \(\omega\), and \(W(\omega,L)\) increases with system size \(L\). This is consistent with the predictions of Eq. (3.4a). A similar linear behavior has been found for case (2) to (4), but these results are not shown explicitly.

We test in Fig. [1] the case (1) scaling relation (3.4a), in Fig. [2] the case (2) scaling relation (3.5a), in Fig. [3] the case (3) scaling relation (3.6a), and in Fig. [4] the case (4) scaling relation (3.5a).

In all cases except case (3), our numerical data are consistent with the theoretical predictions. In case (3), where \(E_0 \ll \langle E_1(x) \rangle\) and \(\rho_0 \gg \langle \rho_1(x) \rangle\), data collapse was obtained by scaling \(W(\omega,L)/\exp(0.46L)\) (see Fig. [3]). This is very different from the predicted behavior, Eq. (3.6a). We do not know the reason for this discrepancy between theory and numerical result.

**VI. SUMMARY AND CONCLUSION**

We have investigated the IDOS for a self-affine string, i.e. a string with self-affine variations in mass density and elastic coefficient. There are four relevant cases to study depending on whether the self-affine variations dominate or not in the mass density and elastic coefficient. We have compared numerical studies with a conjecture based on assuming that a result of Lapidus and Fleckinger [1] is valid for the self-affine string.

We find that our conjecture works for three out of the four cases. However, it fails for the case where \(E_0 \ll \langle E_1(x) \rangle\) and \(\rho_0 \gg \langle \rho_1(x) \rangle\).

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FIG. 1. (a) The asymptotic behavior of the integrated density of states (IDOS) when $E_0 \gg \langle E_1(x) \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$ (Case 1). (b) The scaled IDOS $W(\omega, L)/L^{1+H/2}$ as function of $\omega$.

FIG. 2. The scaled IDOS $W(\omega, L)/L$ as function of $\omega$ for Case (2): $E_0 \ll \langle E_1(x) \rangle$ and $\rho_0 \ll \langle \rho_1(x) \rangle$. 
FIG. 3. The scaled IDOS (a) $W(\omega, L)/L^{1-H/2}$ and (b) $W(\omega, L)/e^{0.46L}$ as function of $\omega$ for Case (3): $E_0 \ll \langle E_1(x) \rangle$ and $\rho_0 \gg \langle \rho_1(x) \rangle$.

FIG. 4. The scaled IDOS $W(\omega, L)/L$ as function of $\omega$ for Case (4): $E_0 \gg \langle E_1(x) \rangle$ and $\rho_0 \gg \langle \rho_1(x) \rangle$. 