f(R) gravity theories in the Palatini Formalism constrained from strong lensing

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ABSTRACT

f(R) gravity, capable of driving the late-time acceleration of the universe, is emerging as a promising alternative to dark energy. Various f(R) gravity models have been intensively tested against probes of the expansion history, including type Ia supernovae (SNIa), the cosmic microwave background (CMB) and baryon acoustic oscillations (BAO). In this paper we propose to use the statistical lens sample from Sloan Digital Sky Survey Quasar Lens Search Data Release 3 (SQLS DR3) to constrain f(R) gravity models. This sample can probe the expansion history up to z \sim 2.2, higher than what probed by current SNIa and BAO data. We adopt a typical parameterization of the form f(R) = R - \alpha H_0^2(-\frac{R}{M^2})^\beta with \alpha and \beta constants. For \beta = 0 (\Lambda CDM), we obtain the best-fit value of the parameter \alpha = -4.193, for which the 95\% confidence interval that is [-4.633, -3.754]. This best-fit value of \alpha corresponds to the matter density parameter \Omega_{m0} = 0.301, consistent with constraints from other probes. Allowing \beta to be free, the best-fit parameters are (\alpha, \beta) = (-3.777, 0.06195). Consequently, we give \Omega_{m0} = 0.285 and the deceleration parameter q_0 = -0.544. At the 95\% confidence level, \alpha and \beta are constrained to [-4.67, -2.89] and [-0.078, 0.202] respectively. Clearly, given the currently limited sample size, we can only constrain \beta within the accuracy of \Delta \beta \sim 0.1 and thus can not distinguish between \Lambda CDM and f(R) gravity with high significance, and actually, the former lies in the 68\% confidence contour. We expect that the extension of the SQLS DR3 lens sample to the SDSS DR5 and SDSS-II will make constraints on the model more stringent.

Key words: cosmology: theory – cosmological parameters – gravitational lensing – dark matter

1 INTRODUCTION

In modern cosmology, one of the most striking discoveries is that our expanding universe is undergoing a phase of acceleration. The key observational results that support this discovery are: the luminosity-redshift relationship from SNIa surveys (Astier et al. 2006; Perlmutter et al. 1999; Riess et al. 1998, 2004, 2007), the CMB anisotropy spectrum (de Bernardis et al. 2000; Spergel et al. 2003, 2007; Hinshaw et al. 2007), the large scale structure from galaxy redshift surveys (Cole et al. 2005; Seljak et al. 2005; Tegmark et al. 2004a) and BAO (Eisenstein et al. 2005). With the combination of all these observational results, one gets the standard \Lambda CDM cosmology as: \Omega_K = 0, \Omega_M = 0.27, \Omega_\Lambda = 0.73. Amongst the matter content baryonic matter amounts to only 4\%, other 23\% is the so-called dark matter (DM), and the rest component of 73\% dominate the universe that is often referred to as dark energy (DE), which is a negative-pressure ideal fluid smoothly permeated the universe. It is worth noting that, notwithstanding the standard cosmology is very well in agreement with the astrophysical data, the nature of the dark matter and dark energy remains mystery in the modern Cosmology and Physics, so that there are so many theoretical proposals to account for them on the ground. Possible models to explain the acceleration include a classical cosmology constant (Weinberg 1989; Carroll et al. 1992), Chaplygin gas (Kamenshchik et al. 2001; Zhu & Li 2004), a wide variety of scalar-field models such as Quintessence (e.g., Caldwell et al. 1998; Wu & Zhang 2008), K-essence (Armendariz-Picon et al. 2000, 2001), Phantom field (Caldwell 2002; Dabrowski et al. 2003) and so on.

Except above models, there are several popular alternative ideas for the accelerating universe by modifying general relativity rather than resorting to some kinds of exotic fluid. It is a natural consequence to modify the theory of gravity, after the current observations mightly suggesting the failure of General Relativity (GR) as a large cosmological scale gravity theory. Actually, on the right side of Einstein’s field equation, the energy momentum tensor is related to the matter content, and on the left side the Einstein tensor consists of pure geometrical terms, which can be modified to account for DM and DE phenomena (Copeland et al.
Accordingly, both of accelerating behaviour (DE) and dynamical phenomena (DM) can be interpreted as curvature effects. Recently, a lot of people attempt to modify the General Relativity in different ways and propose various modified gravity theories to address the DM or DE phenomenon or both. Amongst them, \( f(R) \) gravity theories, where \( R \) is curvature scalar, have attracted much attention, which are put forward to account for DE by adopting an arbitrary function of \( R \) in Einstein-Hilbert Lagrangian, the so-called \( f(R) \) term, instead of \( R \) in traditional General Relativity. Obviously, an important reason for this interest is that \( f(R) \) term in types of negative as well as positive powers of \( R \) can produce the inflation at early times and now observed acceleration phase at late times, following the well-known gradient, the so-called DE by adopting an arbitrary function of \( R \) in Einstein-Hilbert Lagrangian.

It is worth stressing that how the modifying the lagrangian of the gravitational field affects the standard theory of lensing (built on GR) is not well investigated. Recently, theories with \( f(R) \) gravity models, the gravitational lensing was derived from the Einstein field equation from the Hilbert action. By comparison, the metric approach assumes that the equation has only one variable with respect to metric where the affine connection is the function of metric, while in the Palatini approach the metric and the connection are treated to be independent of each other as two variables. When \( f(R) \) is in form of linear function, both approaches lead to the same results. Once adding any non-linear term to the Hilbert action, however, the two methods would produce enormous differences. The reasons for the Palatini approach seeming appealing compared with another one are that, on one hand, the metric approach leads to the fourth order field equations, which is difficult to solve analytically, whereas the Palatini approach results in the two order field equations. On the other hand, although \( f(R) \) theories via metric approach can give some interesting and successful results, some of them suffer from certain fatal defects. For example, they cannot pass the solar system test, have instabilities, have incorrect Newtonian limit and cannot totally describe all epochs of the universe.

Here we adopt the \( f(R) \) gravity theory within the Palatini approach which can avoid the above mentioned problems. In this framework, the form \( f(R) = R + \alpha (-R)^{\beta} \) is chosen so that it can pass the solar system test and has the correct Newtonian limit (Sotiriou 2006d), and can explain the late accelerating phase in the universe (Fay et al. 2007; Capozziello et al. 2005; Sotiriou 2006a). Furthermore, it is important to go beyond the quantitative studying and test these theories using the observations. Recently constraint from data on above type of \( f(R) \) theory is intensively discussed by many authors. Among them, in these papers (Santos et al. 2008; Fay et al. 2007; Borowiec et al. 2006; Amarzguioui et al. 2006; Sotiriou 2006a), the CMB shift parameter, supernovae Ia surveys data and baryon acoustic oscillations were combined to constrain the parameters and, in particular, they give \( \beta \sim 10^{-1} \). In a previous work, Kovisto (2006) used the matter power spectrum from the SDSS to get further restriction \( \beta \sim 10^{-5} \), which reduced allowed parameter space to a tiny around the \( \Lambda \)CDM cosmology. Recently, Li B. et al. (2007) jointed the WMAP, supernovae Legacy Survey (SNLS) data and Sloan Digital Sky Survey (SDSS) data to tighten the parameter up to \( \beta \sim 10^{-6} \), which made this model hard to distinguish from the standard one, where the parameter \( \beta \) is zero. So far, there has been no attempt to use the strong lensing observation to test the \( f(R) \) gravity theories in the Palatini formalism.

Nowadays, with more and more lens surveys available to enlarge the statistical lensing samples, gravitational lensing has developed into a powerful tool to study a host of important subjects on different scales in astrophysics, from stars to galaxies and clusters, further, to the large structure of the universe. Since the gravitational lensing phenomena involve the source information, two dimensional mass distribution of the lens and the geometry of the universe, it is useful to not only infer the distant source properties far below the resolution limit or sensitivity limit of current observations and offer an ideal way to probe the mass distribution of the universe, but also constrain the parameters of cosmological models (Copeland et al. 2006; Wu 1998). So far the large lens surveys have built both radio (the Cosmic-Lens All Sky Survey, CLASS) and optical (SQLS DR3) lens samples. The CLASS forms a well-defined lens sample containing 13 lenses from 8958 radio sources with image separations of \( 0.3 < \theta < 10'' \) and the \( i \)-band flux ratio \( q \leq 10 \) (bright to faint) (Browne et al. 2002). Meanwhile, the SQLS constructs a statistical sample of 11 lensed quasars from 22683 optical quasars, of which the range of redshift is \( 0.6 < z < 2.2 \) and the apparent magnitude is brighter than \( m < 19.1 \) (Inada et al. 2008). As compared with CLASS, the latter sample has larger image separations (\( 1'' < \theta < 20'' \)) and smaller flux ratio limit, \( q_s = 10^{-0.5} \), faint to bright). Many authors use the strong gravitational lensing statistics to study DE, including the equation of state of DE (Chae et al. 2002; Cherd 2004; Oguri et al. 2008) and the test for the modified gravity theories as alternatives to DE (Zhu & Sereno 2008).

The main goal of this paper is to investigate the constraints of strong lensing observation on the parameters of \( f(R) \) theory in the Palatini approach using the SQLS statistical sample. But there has a big question that how the non-linear terms in \( f(R) \) gravity theories contribute to the light-bending and correct the Einstein deflection angle. If this influence is large enough, it is inevitable that all the models built on GR to compute the lensing statistical probability must be changed. But this question is starting to be solved, here, we still use the same way as that based on GR to calculate the lensing probability because of the following considerations. Firstly, the \( f(R) \) gravity theories are applied to explain the nature of DE, but as far as we know it, DE smoothly fills the space to power the acceleration expansion as the same way in everywhere of the universe and make the universe to be flat. Dave et al. (2002) showed that DE does not cluster on scales less than 100Mpc. Naturally, as an alternative to DE, any modified gravity theory, like \( f(R) \) non-linear term, should not affect the gravitational potential distribution of dark halos as the virialized systems such as galaxies and clusters, and in particular, should have not detectable contribution to the light-bending if they are some successful models. Moreover, it is worth stressing that how the modifying the lagrangian of the gravitational field affects the standard theory of lensing (built on GR) is not well investigated. Recently, theories with \( f(R) \sim R^n \) were been investigated within the metric approach for a point-like lens and results were given that the \( R^n \) modified gravity signatures possibly to be detected through a careful examination of galactic microlensing (Capozziello et al. 2006). And Zhang et al. (2007) derived the complete set of the linearized field equations of the two Newtonian potentials \( \phi \) and \( \psi \) in the metric formalism, and predicted that, for some \( f(R) \) gravity models, the gravitational lensing was virtually identical to that based on GR, under the environments of galaxies and clusters. While, under the Palatini approach, the paper (Ruggiero 2008) is a first step to evaluate some effects of the non-linearity of the gravity Lagrangian on lensing phenomenology. As expected, for a spherically symmetrical point-like model, the estimated results suggest that the effects of \( f(R) \) are confined around a cosmological scale, and hence, they have no effects on smaller
scales such as our Galaxy. For galaxies, the weak-field approximation is valid. The standard lensing theory (based on GR) is on the basis of this assumption. In GR, the deflection angle for a point mass can be easily obtained. For the weak-field assumption, the field equations of GR can be linearized. Therefore, we can first divide the whole galaxy into small elements, and each mass element can be treated as point mass, then the deflection angle of the general galactic mass distribution models can be obtained by the sum (or integration) of the deflections due to the individual mass components (Kochanek et al. 2004). In $f(R)$ gravity, if the gravitational field is weak, we are able to perturb the General Einstein field equations (based on $f(R)$ gravity theories) and simplify them to linear equations. In this case, the superposition principle works, so we can get that the field equations which describe the point-like objects can also be used to describe the ensemble of mass points. That is to say, both of the mass point and the ensemble of mass points have field equations in the same formalism. So we can safely extrapolate the conclusion for point-like lenses, that standard lensing theory is a good approximation in the case of $f(R)$ gravity theories, to galaxy-scale lenses. However, there are no works to prove it through mathematical process, and further theoretical studies are needed to test the $f(R)$ gravity theories in the Palatini formalism. Under these situations, we consider that the standard theory of lensing is still correct in the case of $f(R)$ theory we used.

According to astronomical observations, dynamical analysis and numerical simulations, there are three kinds of popular mass density profile of dark halos as a lens model in the standard lensing theory, namely, the singular isothermal sphere (SIS), the Navarro-Frenk-White (NFW) and the generalized NFW (GNFW) profile. They can reasonably reproduce the results of strong lensing survey (Li & Ostriker 2002, 2003; Sarbu et al. 2001). As been well known, the lensing probability is very sensitive to the density profile of lenses. Moreover, the lensing is determined almost entirely by the fraction of the halo mass that is contained within a fiducial radius that is $\sim 0.4$ of the virial radius. Since in the inner regions the slope of SIS density is steeper than that of NFW density, for the multiple image separations of $\Delta \theta < 5''$, the lensing probability for NFW halos is lower than the corresponding probability for SIS halos by about 3 orders of magnitude (Li & Ostriker 2002, 2003; Keeton & Madau 2001; Wyithe et al. 2001). The cooling mass divided galaxies and clusters for transition from SIS to NFW is $M_c \sim 10^{13} M_\odot$. The SQLS DR3 has 11 multi-imaged lens systems, including ten galaxy-scale lenses and one cluster-scale lens with large image separation (14.62") (Madau et al. 2008). It is the first lens sample that contains both galaxy-scale lenses and cluster-scale lens. In lensing statistics, if we consider lensing halos with all mass scales, from galaxies to clusters, then the effects of substructures (Oguri et al. 2006) and baryon infall (Kochanek & White 2001; Keeton 2001) cannot be neglected, in particular to galaxy groups, for which neither SIS nor NFW can be applied. For early-type galaxies, which dominate the galactic strong lensing, SIS is a good model of lenses, in particular, it results from the original NFW-like halos through the baryon infall effects (Rusin & Kochanek 2003; Koopmans et al. 2006). Moreover, the non-spherical lens profiles do not significantly affect the lensing cross section (Oguri et al. 2005; Huterner et al. 2005; Kochanek 1999). Therefore, in this paper, we use the SIS model to compute the strong lensing probability for galaxies. To avoid the complexity from large mass scales mentioned above, we consider only galaxy-scale lenses in the sample, and thus compute the differential lensing probability rather than the usual integrated lensing probability to match the SQLS DR3 sample (containing 10 galaxy-scale lenses). A more realistic lensing model including SIS and NFW to fit the complete SQLS sample and comparing to the results of this paper is our future work. As comparison, Oguri et al. (2008) adds two additional cuts to select a 7 lensed quasar sub-sample from the DR3 statistical lens sample for constraining the parameter $\omega$ of state of dark energy.

The rest of the paper is organized as follows: In section 2 we outline the $f(R)$ theories in the Palatini approach, including the cosmological dynamical equation and parameters with the FRW universe setting. In section 3 we present the differential lensing probability based on the $f(R)$ gravity theories with SIS model. In section 4, we discuss the specific analytic function $f(R) = R - \alpha H_0^2 \left(\frac{\rho}{\rho_c}\right)^\beta$ used in this paper, give the corresponding cosmological evolution behaviour and investigate the observational constraint on our $f(R)$ theory arising from SQLS DR3 statistical lens sample. Finally, we give discussion and conclusions in section 5.

2 THE $f(R)$ GRAVITY THEORIES AND CORRESPONDING COSMOLOGY PARAMETERS

2.1 The generalized Einstein Equations

The starting point of the $f(R)$ theories in the Palatini approach is Einstein-Hilbert action, whose formalism is:

$$S = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M,$$

where $\kappa = 8\pi G$, light velocity $c = 1$ and $S_M$ is the matter action which is a functional of metric $g_{\mu\nu}$ and the matter fields $\psi_M$. As mentioned previously, the metric and the affine connection are two independent quantities in the Palatini approach, so the Ricci and the scalar curvature the gravity theories constrained from lensing theories in the Palatini approach, including the $f(R)$ gravity theories arising from SQLS DR3 statistical lens sample.

$$R = g^{\mu\nu} \hat{R}_{\mu\nu},$$

in which

$$\hat{R}_{\mu\nu} = \hat{\Gamma}_{\mu\alpha,\nu} - \hat{\Gamma}_{\mu\nu,\alpha} + \hat{\Gamma}_{\alpha\lambda} \hat{\Gamma}^{\lambda}_{\nu,\mu} - \hat{\Gamma}_{\mu\alpha} \hat{\Gamma}^{\lambda}_{\nu,\lambda},$$

where $R$ is negative. From Eq. (1), we derive the generalized Einstein equations. Varying with respect to metric $g_{\mu\nu}$, we get one equation:

$$f'(R) \hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) = -\kappa T_{\mu\nu},$$

where $f'(R) = df/dR$ and the energy momentum tensor $T_{\mu\nu}$ is given as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}.$$ Taking the trace of Eq. (4) gives the so-called structural equation and it controls the solutions of Eq. (4).

$$R f'(R) - 2 f(R) = -\kappa T.$$ Varying with respect to the connection $\hat{\Gamma}^\lambda_{\mu\nu}$ gives another equation

$$\hat{\nabla}_\alpha \left[ f'(R) \sqrt{-g} g^{\mu\nu} \hat{\Gamma}^\nu_{\alpha\mu} \right] = 0,$$

where $\nabla$ denotes the covariant derivative with respect to the affine connection. From this equation it is found that the new metric $h_{\mu\nu}$, which can describe the affine connection as the Levi-Civita connections, is conformal to $g_{\mu\nu}$.

$$h_{\mu\nu} = f'(R) g_{\mu\nu}. $$
where we have chosen $H$ between $D$ the conservation of energy $R$ 

\[ R_{\mu\nu} = R_{\mu\nu} - \frac{3}{2} g_{\mu\nu} \nabla_{\mu} \nabla_{\nu} f' + \frac{1}{f'} \nabla_{\mu} \nabla_{\nu} f' + \frac{1}{g_{\mu\nu}} \nabla_{\mu} \nabla_{\nu} f'. \]  

(9)

Note that the covariant derivative $\nabla$ we refer to is associated with the Levi-Civita connection of metric $g_{\mu\nu}$.

### 2.2 The Background Cosmology

In this subsection, we shall make a detailed study of the cosmological viability of the model based on $f(R)$ theories in Palatini formalism. It is well-known that WMAP’s data is a strong evidence to support a flat universe, so we choose a spatially flat FRW (Friedmann-Robertson-Walker) metric to describe our background universe. The metric takes the standard form:

\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j (i, j = 1, 2, 3), \]

(10)

where $a(t)$ is the scale factor. As usual, we assume a perfect fluid energy-momentum tensor $T_{\mu\nu} = diag(\rho, \rho, \rho, p)$. By contracting Eq. (9) we get the generalized Friedmann equation:

\[ (H + \frac{f'}{2f})^2 = \frac{1}{6} \rho + \frac{3}{2} p \]

(11)

In this paper we consider a matter dominated universe, so the constant equation of state is $p = \omega \rho (\omega = 0)$ and the relation between the matter density and scalar factor is $\rho = \rho_0 R^{-3(\omega + 1)} (\omega = 0)$. Using Eq. (6), we can obtain:

\[ a(R) = (\rho_p R_0)^{1/3} (Rf' - 2f)^{-1}, \]

(12)

where we have chosen $R_0 = 1$. On the other hand, from Eq. (6) and the conservation of energy $T_{\mu\nu,\lambda} = 0$, we have as:

\[ \dot{R} = -\frac{3H \rho M}{Rf''(R) - f'(R)}. \]

(13)

Using Eqs. (5), (11) and (13) we get the Friedmann equation $H(R)$ in the form of $R$,

\[ H^2(R) = \frac{1}{6f'} \left( 1 - \frac{3f'(Rf'' - 2f)}{2f Rf'' - f'f(R)} \right)^2. \]

(14)

Combining Eqs. (12) and (14) we can know the whole expansion history that is determined by $H(a)$, for any specific expression of $f(R)$.

Let us now consider the cosmological distance based on the given model. From the explicit expression for the Hubble parameter $H(R)$ and the relation between $z$ and $a(R)$: $1 + z = a^{-1}(R)$, it is convenient to rewrite the proper distance, luminosity distance, angular diameter distance and the deceleration parameter in terms of $R$. The proper distance and luminosity distance to the object at redshift $z$ are

\[ D^P(z) = \int_0^z \frac{dz}{(1+z)H(z)} = \int_0^{R_0} \frac{dR}{Rf''(R) - 2f} = D^P(R), \]

(15)

\[ D^L(z) = (1+z) \int_0^z \frac{dz}{H(z)} = \frac{1}{3} (k_0)^{-\frac{2}{3}} (Rf'' - 2f)^{\frac{1}{3}} \int_0^{R_0} \frac{dR}{Rf'' - 2f} = D^L(R). \]

(16)

The angular diameter distance from an object at red-shift $z_1$ to an object at red-shift $z_2$ is

\[ D^A(z_1, z_2) = \frac{1}{1 + z_2} \frac{1}{H(z)} \int_{z_1}^{z_2} \frac{dz}{1 + z}, \]

(17)

\[ = \frac{1}{3} (Rf'' - 2f)^{\frac{1}{3}} \int_{z_1}^{z_2} \frac{dR}{(Rf'' - 2f)^{\frac{2}{3}}} = D^A(R_1, R_2). \]

The deceleration parameter is

\[ \frac{2}{3} \frac{\dot{a}(t)}{a(t)} = -\left( 1 + \frac{H^2(R)}{Rf'(R)} \right) = q(R), \]

(18)

when $q > 0$ the universe is in early-time matter-dominated phase and $q < 0$ the universe is in dark energy dominated late-time acceleration phase.

### 3 THE LENSING PROBABILITY

#### 3.1 the SIS lens

As mentioned, in this paper, we assume that the $f(R)$ theories have no effect on the gravitational lensing and still model the early-type galactic lens as SIS as in GR, but these formulae are used in terms of $R$. The density profile is

\[ \rho(r) = \frac{\sigma^2}{2\pi} \frac{1}{r^2}, \]

(19)

where $\sigma$ is the velocity dispersion and $r$ is the distance from the galaxy center. From the geometrical relations between the image position $\theta$, source position $\beta$ and lens, one gets the lensing equation:

\[ \beta = \theta - \frac{D_{ls}^A}{D_S^A} \alpha. \]

(20)

For a SIS lens, the deflection angle is $\alpha = 4\pi(\sigma/c)^2$, which is independent of the impact parameter. The angular Einstein radius is defined as

\[ \theta_E = 4\pi(\frac{\sigma}{c})^2 \frac{D_L^A}{D_S^A}. \]

(21)

The sources are multiply imaged when $\beta < \theta_E$ and a ring-like image occurs when $\beta = 0$. When the multiple images occur, from the lensing equation, there are two images on opposite sides of the lens at angular positions

\[ \theta_{\pm} = \theta_E \pm \beta. \]

(22)

The corresponding magnifications are

\[ \mu_{\pm} = \frac{\theta_{\pm}}{\beta_{\pm}} \frac{d\theta_{\pm}}{d\beta_{\pm}} = \frac{\theta_{E} \pm \beta_{\pm}}{\beta_{\pm}}. \]

(23)

The total magnification of the two images is

\[ \mu_{tot} = \mu_{+} + \mu_{-} = \frac{2\theta_{E}}{\beta}. \]

(24)

The magnification of an image is defined by the ratio between the solid angles of the image and the source, namely, the flux density ratio between the image and the source. Then the flux density ratio $q$ for bright-to-faint is the ratio of the corresponding absolute values of the magnifications.
\[ q = \left[ \frac{\mu_+}{\mu_-} \right] = \frac{\theta_E + \beta}{\theta_E - \beta}. \]  

(25)

Every lens survey has its own spectrum range and selection functions, so there exists an explicit cut flux density ratio, less than which, the survey could not identify the lenses. For example, for SLOLS, the ratio (faint to bright) is larger than \( q_c = 10^{-0.7} \), which means that only sources with \( \beta < \beta_{\text{max}} < \theta_E \) can be detected, where \( \beta_{\text{max}} \) corresponds to \( q_c \). The lensing cross section is

\[ \sigma_{\text{SIS}}(\sigma) = 16\pi^2 \left( \frac{\sigma}{c} \right)^4 \left( \frac{D_A^4}{D_S^4} \right)^2, \]

(26)

and the velocity to produce image separation \( \Delta \theta \) is

\[ \sigma_{\Delta \theta} = 4.392 \times 10^{-8} \left( \frac{\sigma}{c} \right) \left( \frac{D_A^4}{D_S^4} \right)^2 (\Delta \theta)^{2}. \]

(27)

### 3.2 the differential probability

The differential probability for the source quasar at the redshift \( z_s \) (corresponding to \( R_s \)) lensed by foreground dark halos with multiple image separations \( \Delta \theta \) and flux density ratio larger than \( q_c \) is given by

\[ P(\Delta \theta, R_s, > q_c) = \frac{dP(\Delta \theta, R_s, > q_c)}{d\Delta \theta}, \]

(28)

\[ \int_0^{R_s} dR \frac{dP(R)}{dR} [n(\sigma, R_c) d\sigma] \sigma_{\text{real}}(\sigma, R_s, > q_c) \frac{d\sigma}{d\Delta \theta}, \]

\[ \sigma_{\text{real}} = \int d\mu \frac{\phi(L_{\text{min}}/\mu)}{\mu \phi(L_{\text{min}})}, \]

(30)

where \( D^p \) is the proper distance from the observer to the lens, \( n(\sigma, R_c) \) and \( n(\sigma, R_c) d\sigma \) are the comoving number density and the physical number density of galaxies at \( R_c \) (corresponding to redshift \( z \)) with dispersion between \( \sigma \) and \( \sigma + d\sigma \), respectively, the cross section \( \sigma_{\text{real}}(\sigma, R_s, > q_c) \) is dependent on the flux density ratio of multiple images \( (q_c) \) and take into account the magnification bias (see below for details).

In the statistics for gravitational lensing, there are two independent ways to get the mass function of virialized halos. One is the generalized Press-Schechter (PS) theory, and the other is Schechter luminosity function. Since the PS theory in the framework of \( f(R) \) theories does not exist, we adopt the latter. It is established that (Maoz & Rix 1993; Moller et al. 2006; Choi 2007), the early-type galaxies dominant strong lensing, the contribution of the late-type galaxies is well neglected in particular when the image separations are larger than 1”, the reason is that the late-type galaxies have larger rotation velocity while have smaller dispersion velocity, which in turn, lead to smaller mean image separations. Moreover, in our used SLOLS lens sample all lens galaxies are early-type. Many previous studies for lensing statistics applying the no-evolution of the velocity function get appealing results to agree with the galaxy number counts (Im et al. 2002) and the redshift distribution of lens galaxies (Chae & Mao 2003; Ofek et al. 2003). Meanwhile, other studies (Mao & Kochanek 1994; Mitchell et al. 2005; Rix et al. 1994) have investigated the effects of evolution of the velocity function on the lensing statistics, and they concluded that the simple evolution does not significantly affect lensing statistics if all galaxies are early-type. In our paper, we do not take into account the evolution of the velocity function, instead, we use the modified Schechter function (Chae 2007; Chen & Zhao 2004; Oguri et al. 2008; Mitchell et al. 2008; Zhu & Serend 2008)

\[ \phi(\sigma) d\sigma = \phi_*(\frac{\sigma}{\sigma_*})^{\alpha} \exp[-(\frac{\sigma}{\sigma_*})^\beta] \frac{d\sigma}{\Gamma(\alpha/\beta)} \]

(29)

with \( (\phi_*, \sigma_*, \alpha, \beta) = (0.008h^3\text{Mpc}^{-3}, 161\text{hkms}^{-1}, 2.32, 2.67) \), which is derived by (Choi et al. 2007) upon the latest much larger SDSS DR3 data. Then the comoving number density is \( n(\sigma, R_c) = \phi(\sigma) \), and \( \sigma \) is in terms of \( R_c \).

Lensing probabilities must be corrected with the magnification bias, which explains the fact that intrinsically quasar sources with flux below the flux-limited survey can appear by virtue of lensing magnification. The cross section \( \sigma_{\text{real}}(\sigma, R_s, > q_c) \) including the magnification bias parameter \( B \) is defined as (Oguri et al. 2008)

\[ \sigma_{\text{real}} = \int d\mu \frac{\phi(L_{\text{min}}/\mu)}{\mu \phi(L_{\text{min}})}, \]

(30)

here \( d\mu = \frac{2}{1 + \tanh(1.76 - 1.780^\mu)} \)

(32)

By defining \( t = \beta/\theta_E \) and using Eqs. (21), (25), (26) and (30), we can rewrite the cross section as,

\[ \sigma_{\text{real}}(\sigma, R_s, > q_c) = \sigma_{\text{SIS}}(R_s, \Delta \theta) B(R_s, \Delta \theta, q_c), \]

(33)

here the bias \( B \) for the multiply imaged source at scalar \( R_s \) is

\[ B(R_s, \Delta \theta, q_c) = \int_0^{R_s} t \frac{dt}{\mu_t^2 + (1 - \mu_t)^2(\mu_t - 1)} \frac{\phi(L_{\text{min}}/\mu)}{\mu_t \phi(L_{\text{min}})} \]

(34)
where $tc = (1 - q_e)/(1 + q_e)$ and $\phi(L_{\text{min}})$ is the quasar differential luminosity functions at the luminosity limit $L_{\text{min}}$ of the survey, which is described by a typically double power-law in terms of absolute magnitude in $g$-band at $R_e$ by \cite{Hopkins2007, LiG2007}:

$$\phi(M_g) = \frac{\phi_*}{10^{0.4(1+\beta_h)(M_g-M_g^*)} + 10^{0.4(1+\beta_h)(M_g-M_g^*)}},$$

and

$$M_g^*(R_e) = M_g^*(R_0) - 2.5[k_1(a^{-1}(R_e) - 1) + k_2(a^{-1}(R_e) - 1)^2] + k_2(a^{-1}(R_e) - 1)^2].$$

For the low redshift ($z < 2.1$) quasars, the parameters from the 2dF-SDSS Luminous Red Galaxy (LRG) and Quasi-stellar Object (QSO) Survey fitted in the SDSS $g$-band are $(\phi_*, \beta_h, \beta_l, M_g^*(R_0), k_1, k_2) = (1.83 \times 10^{-6} h/0.7)^3\text{Mpc}^{-3}\text{mag}^{-1}, -3.31, -1.45, -21.61 + 5\log(h/0.7), 1.39, -0.29).$ By virtual of the relation between luminosity and absolute magnitude $L \propto 10^{-0.4(M-M^*)}$, we have $\phi(L/\mu)$ in terms of absolute magnitude. The SDSS quasar survey have flux limit in $i$-band and the apparent magnitude limit is $i_{\text{max}} = 19.1$, but the parameters in above luminosity Equations are given in $g$-band, so we first convert the apparent magnitude to the absolute magnitude, then compute the corresponding absolute magnitude in $g$-band using the K-corrections which is given by \cite{Richards2006}:

$$M_g(R_0) = M_i(R_0) + 2.5\alpha_v\log\left(\frac{4670\text{Å}}{7471\text{Å}}\right) - 0.187,$$

where $\alpha_v = -0.5$.

From the differential lensing probability, we calculate the expected number of lensed quasars from the SQLS sample, and compare to the observational results in order to constrain the parameters in $f(R)$ models. The results will be given in the next section.

4 NUMERICAL RESULTS AND OBSERVATIONAL CONSTRAINTS

Here, what we would like to investigate is what extent observations of strong lensing allow deviations from the GR which is corresponding to $f(R) = R$. For our purposes, we adopt the following representation for $f(R)$ \cite{Amarzguioui2006},

$$f(R) = R - \alpha H_0^2\left(-\frac{R}{H_0^2}\right)\beta,$$

which is a possible candidate for the late-time cosmic accelerating expansion found recently. Here, $H_0$ is a constant with dimension Mpc$^{-1}$, which are introduced to make the $\alpha$ and $\beta$ dimensionless. Not all combinations of $\alpha$ and $\beta$ are agreement with a flat universe with matter dominated era flowed by an accelerated expansion today. We now consider the constraint on these parameters. Firstly, at the early time of matter dominated universe, the dark energy is ineffective and the universe is better described by GR. Therefore, with $-R (R$ is negative) being more and more larger, the modified Lagrangian should approach its GR limit, in order to avoid conflict with early-time physics such as Big Bang Nucleosynthesis (BBN) and CMB, and hence it is straightforward to demand that $\beta < 1$. Moreover, in the case of $T = -\rho$, we must demand that the left side of Eq.6 and the right of Eq.14 be always positive. In the special case of $(\alpha, \beta) = (-4.38, 0)$, the $\Lambda$CDM cosmology model is recovered.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure2}
\caption{The background dynamical behaviour based on the $f(R) = R - \alpha(-R)^\beta$ model. The left panel is the hubble parameter $H$ as a function of scale factor $a$. The right panel is the deceleration parameter $q$ with $a$. Different kinds of lines represent a series of values of $\beta = -0.5, -0.1, 0, 0.1, 0.5$ with $\Omega_m = 0.27$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure3}
\caption{The expected number density based on $f(R) = R - \alpha(-R)^\beta$ for different choices of $\beta$. The upper panel is for the cases of $\beta = -0.005, 0, 0.005$ and the lower panel for $\beta = -0.1, -0.05, 0, 0.05, 0.1$ from top to down. In all cases, $\Omega_m = 0.27$. The case $\beta = 0$ corresponds to a $\Lambda$CDM model.}
\end{figure}
4.1 the background evolution

Before using Eqs. (13) and (14) to determine the cosmological evolution behaviour, we must give the initial conditions: \((\rho_{0m}, H_0, R_0)\), it is noting that a subscript 0 denotes evaluation at the present time. By putting \(f(R_0)\) into Eq. (14), we find that the combination \(H_0/R_0\) appears as a single quantity in the equation, in turn, we choose \(H_0 = 1\) to compute \(R_0\). Hence, from \(a_0 = 1\) and the obtained value of \(R_0\), the values of \(\rho_{0m}\) and \(\Omega_{0m} = \kappa \rho_{0m} / (3H_0^2)\) are fixed. In other words, amongst the variables \(\alpha, \beta\) and \(\Omega_{0m}\), two of them are independent. So far, we have obtained all the initial quantities.

Now let us use the equations we derived to examine the evolution history of the universe at late-time. With the present matter component value of \(\Omega_{0m} = 0.27\), the changing of the Hubble parameter and scale factor with the curvature are plotted in figure 3. It is easy to see that for any choice of \(\beta\), the smaller the curvature \(\kappa\), the larger the scale factor, and the smaller the rate of change of \(a (a' (R))\). Contrarily, the Hubble parameter \(H\) drops with decreasing \(\kappa\). Therefore, it is worth noting that the above mentioned constraint conditions for further restricting the parameter \((\alpha, \beta)\) space are satisfied as long as the present curvature \(R_0\) meet these conditions. And curvature \(R_0\) decreases with increasing \(\beta\).

The background dynamical behaviour is well described by lines plotted in figure 2. Obviously, the Hubble parameter drop sharply with the expanding universe, at the same time, the value of deceleration parameter is changed from the positive to negative, representing that the universe evolves from deceleration to acceleration. Different choice of \(\beta\) shows different evolution history. At any given time, the larger value of \(\beta\) harmonizes the larger Hubble parameter and, the larger deceleration parameter after a certain time. It is emphasized that larger value of \(\beta\) corresponds larger acceleration for our accelerated expanding universe at present time. Furthermore, with \(\beta\) increasing, the time for the transition from the gravity-dominated era to the DE-dominated era is gradually late.

4.2 the SQLS sample constraints

In this section, we consider the constraints arising from the strong lensing observations on our gravity model. We have calculated the number density of lensed quasars of the present model using the number density lines based on the bin-method.

Figure 4. The expected number density in the present model for \(\Lambda CDM\), our best-fitted standard model with the value of \((\alpha, \beta) = (-4.193, 0)\) and the \(\Lambda CDM\) in the case of \((z_s) = 1.427\) respectively. The two former number density lines based on the bin-method.

Figure 5. The lensed source quasar number as a function of image separation \(\Delta \theta\). The histogram is the number distribution of the SQLS DR3 statistical lens sample, in which there is a cluster-scale lens with image separation 14.62′. We rule out it because our statistical probability is computed based on the SIS modeled galaxy-scale lenses. The bin-size is 0.5′, so the lines stand for \(N(\Delta \theta) = dN/\Delta \theta = 0.5′\), representing the lens number around \(\Delta \theta\) with width of 0.5′. The solid line shows \(\Lambda CDM\) model, while the dashed line stands for our best-fitting results \((\alpha, \beta) = (-3.777, 0.06195)\). The dotted line at \(\Delta \theta = 1′\) indicates the SQLS resolution limit.

SQLS sample. Specifically, from the differential lensing probability \(P(\Delta \theta, R_s)\), which is multiplying likelihood with respect to two variables, image separation \(\Delta \theta\) and \(R_s\) (determined by the redshift \(z_s\) of the source quasars), if we know the number distribution \(N(z_s)\), then the expected number density is given by

\[
\frac{dN(\Delta \theta)}{d\Delta \theta} = \int \frac{dN(R_s)}{dR_s} P(R_s, \Delta \theta) dR_s, \tag{39}
\]

and the expect lens number with image separations between \(\Delta \theta_1''\) and \(\Delta \theta_2''\) is computed by

\[
N = \int_{\Delta \theta_1}^{\Delta \theta_2} \frac{dN(\Delta \theta)}{d\Delta \theta} d\Delta \theta, \tag{40}
\]

which can be used to compare with 11 lenses in the sample to constrain the parameters of the \(f(R)\) model. Furthermore, it is important to point out that, according to the integral mean value theorem, we can also rewrite the expected lens number with image separation ranging from \(\Delta \theta_1''\) to \(\Delta \theta_2''\) as follows:

\[
N = \frac{dN(\Delta \theta)}{d\Delta \theta} (\Delta \theta_2 - \Delta \theta_1), \tag{41}
\]

where \(\Delta \theta\) is one exist value between \(\Delta \theta_1\) and \(\Delta \theta_2\).

The sample we used has 22683 quasars, and the redshifts range from 0.6 to 2.2. To get \(N(z_s)\), we count the source quasar number for each redshift bin, with bin-size \(\Delta z_s = 0.05\). We investigate how the strong lensing probability depends on the \(f(R)\) model we choose. In Figure 3 we present the lens number density based on SQLS lens sample with a series of choices of \(\beta\). When \(\beta\) is small enough, for example \(\beta = \pm 0.005\), the number density is hard to distinguish from \(\Lambda CDM\) \((\beta = 0)\). Inevitably, we can conclude that the strong lensing statistics restricts the parameter \(\beta\) to not beyond the order of \(10^{-3}\). For any choice of \(\beta\), the number density sharply increases with small image separations, and is slowly down with large image separations. At about \(\Delta \theta = 5''\), the lines drop to zero. Different choices of the parameter \(\beta\) significantly influence the curves near the peak, and \(\beta\) increases while the peak
declines, but they cannot change the position of the peak, which is located approximately at $\Delta \theta = 0.6''$ and is beyond the image separation range of SQLS lens sample.

We derive constraint on the parameter $\alpha$ assuming $\beta = 0$. The best-fit value is $\alpha = -4.193$, so we get $\Omega_{\text{m0}} = 0.301$. From $\Delta \chi^2 = 1$ we can further get the 95% confidence interval for $\alpha$ that is $[-4.633, -3.754]$. Above approach to get the lens number density can be called bin-method. As another interesting approach, we assume that the redshifts of the source quasars follow a Gaussian distribution, which is used to fit the redshift distribution of 22683 quasars. The best fit gives the mean $(\langle z_s \rangle) = 1.427$ and the dispersion $\sigma = 0.519$. In this approach, The predicted number density is computed by $N(\Delta \theta) = 22683 \times P(\Delta \theta, \langle z_s \rangle)$. Which is also plotted in Figure 4 in order to compare the cases of, standard $\Lambda$CDM and our obtained best-fitting standard model based on bin-method.

As mentioned earlier, the typical type of $f(R)$ gravity theories in form of $f(R) = R - \alpha (-R)^{\beta}$ have been extensively studied by using all aspects of observation data. Amarzguioui et al. (2006) combined the SNeIa data from Riess et al. (2004) with the baryon acoustic oscillation length scale data (Eisenstein et al. 2005) and the CMB shift parameter (Spergel et al. 2003) to constrain the parameters of above gravity model, and found that the best-fit model with the value is $(\alpha, \beta) = (-3.6, -0.09)$. Similarly, by using the supernovae data by the supernova Legacy Survey (Astier et al. 2006) together with the baryon acoustic oscillation peak in the SDSS luminosity red galaxy sample (Eisenstein et al. 2005) and the CMB shift parameter (Spergel et al. 2003, 2007), Fay et al. (2007) constrained the parameters to $\alpha = -4.63$ and $\beta = 0.027$ in the best-fit case. Both works found that the $\Lambda$CDM model is well within 68.3% confidence level and the constrained parameter $\beta$ is in $10^{-1}$ magnitude. Koivisto (2006) calculated the matter power spectrum and matched it with the measurements of SDSS (Tegmark et al. 2004b). The result is that the favored values of $\beta$ are in order of $10^{-3}$. Meanwhile, the matter power spectrum is also calculated with TT CMB by Li B. et al. (2007). Where, the combined data from WMAP (Page et al., 2007), Hinshaw et al. (2007), SNLS and SDSS (Tegmark et al. 2004b) is used to restrict the model parameters and it is found that $\beta$ is confined in a even smaller order of $10^{-6}$. From this point, we can see that the matter power spectrum is rather sensitive to the model parameter $\beta$, and the model obtained from the observational constraints on the mater power spectrum is seeming indistinguishable from the standard one with $\beta = 0$. Here, we extend this studying by calculating the strong lensing probability and using the SQLS DR3 lens sample to constrain the parameters. In figure 5 we have plotted $N(\Delta \theta)$ for the $\Lambda$CDM model and our best-fitting model with $\alpha = -3.777$ and $\beta = 0.06195$, as well as the histogram from the SQLS DR3 statistical sample. Except for both small and large image-separation ends, the non-linear gravity model predicts less number of lenses than those predicted by the $\Lambda$CDM model, which is linear in the form of Ricci scale $R$. The cosmological parameters $\Omega_{\text{m0}} = 0.285$ and $q_0 = -0.544$ are derived from the best-fit case, which are broadly in agreement with other measurements. We also present the contours for the joint distribution of $\alpha$ and $\beta$ in Figure 6 we can see that the $\Lambda$CDM model lies in the 68.3% confidence contour, which is consistent with the results of other works. Figure 7 shows the 1$\sigma$ and 2$\sigma$ confidence region of the single parameter $\alpha$ or $\beta$ from the projection of the joint confidence lever estimated from $\Delta \chi^2 = 1.0$ and $\Delta \chi^2 = 4.0$. We have found that the model $f(R) = R - \alpha (-R)^{\beta}$ is compatible with the lensing observational data subject to the parameter constraints $[-4.67, -2.89]$ and $[-0.078, 0.202]$ for $\alpha$ and $\beta$, respectively, at the 95% confidence level. Obviously, the data coming from the SQLS DR3 lens sample can only confine the parameter $\beta$ to order of $10^{-1}$.

5 DISCUSSION AND CONCLUSIONS

To summarize, in this work we have derived new constraints on $f(R) = R - \alpha (-R)^{\beta}$ in the Palatini formalism, using the statistics of strong gravitational lensing and the lens sample come from SQLS DR3. We use $f(R)$ within Palatini approach because of it being free from the instabilities, passing the solar system and having the correct Newtonian limit. This typical type of $f(R)$ theories in form of $f(R) = R - \alpha (-R)^{\beta}$ have been recently put forward to explain the late-time expansion phase dominated by dark energy and have been well studied by using the cosmology measurements
including background universe evolution parameters, matter power spectrum and CMB. In our paper, we assume that the \( f(R) \) gravity theories have hardly detectable effect on the gravitational lensing phenomena of galaxy-scale halos, and then the standard lensing is a realistic approximation of lensing in Palatini \( f(R) \) gravity theories. It is worth emphasizing that in the SQlS lens sample, there is a cluster-scale lens, we use SIS model to calculate the differential lensing probability to fit the other 10 lens data to avoid destroying the well-defined statistical lens sample.

Considering the FRW setting, we have shown the background evolution for the late expansion era of matter-dominated phase based on our \( f(R) \) theory. The expected lens number density arising from the SQLS DR3 sample is not as sensitive as the matter power spectrum to the parameter \( \beta \). The number density distributions for different values of \( \beta \) are indistinguishable to \( 10^{-3} \) in magnitude, while the matter power spectrum changes a lot even \( 10^{-5} \) deviation from the standard can constrain model, any value of separation ranging from \( \theta \) \( \Delta \) \( 0 \) to \( \theta \) \( \Delta \) \( 777 \) and \( \theta \) \( \Delta \) \( 0 \) \( 3 \) \( 0 \) \( 27 \) at the 95\% confidence level, and \( \theta \) \( \Delta \) \( 0 \) \( 6 \) \( 7 \) \( 777 \) and \( \theta \) \( \Delta \) \( 0 \) \( 3 \) \( 8 \) \( 9 \) at the 95\% confidence level, and conclude that the SQLS DR3 statistical lens sample can only constrain the parameter \( \beta \) to \( 10^{-1} \) in magnitude.

In our expected number density distribution (Figure 3), for any value of \( \beta \) with the fixed \( \Omega_{M0} = 0.27 \), the peak is at about \( \Delta \theta = 0.6" \). But the lens sample we used, with the image-separation ranging from 1" to 6.17", provides no data in \( \Delta \theta < 1" \), this significantly restrict the constraint precision of the parameters. The CLASS radio lens sample has the lens image separations of \( 0.3" \leq \Delta \theta < 3" \). Naturally, joining the CLASS and SQLS samples to constrain the model will give better results. Of course, a more larger lens sample from future surveys, (e.g., Square Kilometer Array, Koopmans et al. 2004), will be able to make the constraints on the model more stringent. Furthermore, using the SIS model and NFW model together to constrain the \( f(R) \) theories models arising from the lensing observation data containing both galaxy lens and cluster lens is future work.

Finally, we emphasize that our paper is the first try to test the \( f(R) \) gravity theories using strong lensing statistics. The reason may be that, the possible deviation of gravitational lensing based on \( f(R) \) gravity theories from that built on GR is not well studied practically. Our results hold on the valid hypotheses that this possible deviation is hardly detectable, and they are not contrary to other works. As mentioned in the introduction, if the \( f(R) \) theories are successful models to explain the nature of DE, they should have no important effects which can be detected on the galaxy-scale and cluster-scale dark halos. The reason is that DE is introduced to explain the acceleration of the universe, which is, of course, in the cosmological scale. Namely, DE smoothly permeates the regions of smaller scales, such as galaxies and clusters, then it has no contributions to the inhomogeneity of gravitational potential for such halos. For the dark matter as the dominator of the gravity field, the SIS and NFW lens models are well consistent with the observations.

It is therefore necessary to test the \( f(R) \) theories through verifying that the phenomenology is not contradict those observational results that do agree with the predictions of the standard lensing theory. This critical testing will be presented in future work.

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f(R) theories constrained from lensing
