Bell-Wigner inequalities: their logical relationship and satisfaction by quantum mechanics

Louis Sica\textsuperscript{1,2}

\textsuperscript{1}Institute for Quantum Studies, Chapman University, Orange, CA & Burtonsville, MD, USA

\textsuperscript{2}Inspire Institute Inc., Alexandria, VA, USA

Email: lousica@jhu.edu

The Bell and Wigner inequalities are commonly derived using logically separate procedures. It is not generally appreciated that they are closely related. Their relationship follows from the fact that the Bell inequality describes a constraint on the correlations of random variable pairs and leads to a constraint on the probabilities from which they are computed. In the case of the Bell inequality, the logic of the constraint is further clarified when it is found that the inequality that Bell derived for correlation functions must be identically satisfied by the data sets of $\pm 1$'s used to compute the correlations. This data set inequality is independent of the assumptions used by Bell in the course of derivation of the correlation inequality. Thus, the Bell inequality in its most fundamental form cannot be violated by the number of data sets used in deriving it regardless of their individual characteristics. When the Bell inequality is applied to three predicted correlations using properties based on perfect entanglement, the resulting symmetries allow correlations to be replaced by their probabilities in the inequality. The Wigner inequality follows. The two related inequalities are satisfied by correlations and probabilities respectively computed using quantum mechanical principles.

Keywords: Bell theorem, Bell inequality, Wigner inequality, cross-correlations, entanglement

I. Introduction

The Bell theorem based on Bell inequality violation [1] has had a dominating influence on the understanding of quantum mechanics (QM) for more than fifty years [2]. The inequality in correlations that Bell derived using the concepts and notation of a random process has been applied to three (and four) quantum mechanical correlations associated with entanglement of two particles, photons being the most easily realized. However, due to a failure to cross-correlate data variables as carried out in the derivation, the inequality has been violated, while the cause has been attributed to violation of the specific statistical assumptions used by Bell in its derivation. It will be shown below in the three variable case, that the inequality is identically satisfied by data sets independently of their random characteristics [3,4], and therefore it holds independently of Bell’s sufficient but not necessary assumptions.

Not recognizing that the inequality derived statistically was identically satisfied by any three data sets, random quantum mechanical or deterministic, Bell made an unstated assumption in applying his statistically derived correlation inequality to QM experiments. He assumed that all three correlations among the quantum mechanical measurements on two particles had the same mathematical form, though the third measurement was a non-commutative predicted alternative measurement not carried out. For two of the three variable pairs, the correlation was measured or predicted, but the third correlation used data already acquired in the first two pairs. Had Bell computed the third correlation from such predicted measurement data using quantum mechanical probabilities, he would have found that the inequality was satisfied.
Inserting the probabilities corresponding to the correlations in the Bell inequality produces the Wigner inequality [5]. When the correct probabilities are applied, both the Wigner and Bell inequalities are satisfied.

While the above discussion has been tailored to the three variable inequality, similar logic applies to the four variable inequality [3] that is identically satisfied by four cross-correlated data sets. How does one apply such purely mathematical results to quantum mechanical measurements on two particles? The task is more complicated for four than three measurements. Due to a failure to recognize the purely algebraic bases of the Bell inequalities, six and eight uncorrelated data sets, rather than the three and four cross-correlated sets used in the derivation of the inequalities, have been inserted into them. Unsurprisingly, the inequalities are violated.

One further point needs to be made in this summary of basic mathematical facts. The random measurements at alternative settings used in the inequalities may appear in classical as well as quantum contexts. They would be dealt with by repeating experiments with different combinations of settings keeping one the same, and comparing the results at different values for the others. This will be described in detail below for the Bell quantum case.

II. Bell data and correlation inequalities

The Bell inequality for laboratory data will now be derived [3,4]. It applies the same algebraic steps to finite data sets as Bell applied to infinite data sets assumed to have converged to correlations. In this regard, a pivotal fact must be noted: in laboratory Bell experiments, correlations are not measured – finite data sets are measured from which correlations are computed. The relevance of the result to be derived below follows from that fact.

Assume that three data sets, random or deterministic, labeled \(a, b, b'\) have been obtained so that they can be written down. The data set items are denoted by \(a_i, b_i,\) and \(b'_i\) with \(N\) items in each set. Each datum equals \(\pm 1\). One may form the equation

\[ a_i b_i - a_i b'_i = a_i (b_i - b'_i), \]

and sum this equation over the \(N\) data triplets of the data sets. After dividing by \(N\), one obtains

\[
\frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i b'_i}{N} = \frac{\sum_i^N a_i (b_i - b'_i)}{N}.
\]

Taking absolute values of both sides,

\[
\left| \frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i b'_i}{N} \right| = \frac{\sum_i^N a_i (b_i - b'_i)}{N} = \left| \frac{\sum_i^N a_i b_i (1 - b'_i)}{N} \right| \leq \sum_i^N (1 - b'_i),
\]

or
The sums in (2.4) have the form of correlation estimates although the data may in fact be deterministic. In the case where they are random, the data may exhibit correlations due to a variety of circumstances, e.g., the correlations may result from correlation to other variables not indicated or known. The data may also represent nonsense or be severely corrupted due to electronic interference between detectors. In the case where the data are random and the estimates statistically converge to correlations as $N \to \infty$ one has

$$|C_1(a,b) - C_2(a,b')| \leq 1 - C_3(b,b'),$$

(2.5)

where the correlation arguments $a, b, \text{etc.}$, may refer to instrument settings that label the data sets. The correlations such as $C_i(a,b)$ will in general have different functional forms as indicated by their subscripts. Since (2.5) is to be applied to Bell experiments, the arguments of the correlation functions will refer to angular settings of detectors. (When subscripted they refer to the data acquired at those settings.) Inequality (2.5) has the form of the inequality derived by Bell except for the minus sign before the right-most correlation. As will be discussed below, Bell ultimately replaced the final $b'$ on the right by a negative value corresponding to a measurement on the opposite side of the apparatus with the result:

$$|C_1(a,b) - C_2(a,b')| \leq 1 + C_3(b,b').$$

(2.6)

III. The Wigner inequality from the Bell inequality

Two of the three variables in (2.4) have been given labels $b_i$ and $b'_i$ since they are to represent values of measurements on the right-hand $B$-side of a Bell experiment apparatus (Figure 1). The final term in (2.4) may then be written

$$\sum_i^N b_i (-1)^i (-1)^{b'_i} = - \sum_i^N b_i (-1) b'_i = - \sum_i^N b_i a'_i,$$

(3.1)

where $a'_i = -b'_i$, and $a' = b'$ denoting that the settings $a'$ and $b'$ are the same but are on opposite sides of the apparatus. (Given the properties of entanglement, measurements at the same angular settings on opposite sides of a Bell apparatus have opposite signs, i.e., $+1 \Rightarrow -1$ and $-1 \Rightarrow +1$.) Thus, using Bell measurements in (2.4) for which each pair of variables in a product term occurs on opposite sides of the apparatus, one has

$$\left| \sum_i^N a_i b_i - \sum_i^N a_i b'_i \right| \leq 1 + \sum_i^N b_i a'_i,$$

(3.2)

or

$$|C(a,b) - C(a,b')| \leq 1 + C_3(b,a').$$

(3.3)

assuming convergence of the correlation estimates as $N \to \infty$. 

3
Given that probabilities exist that determine the correlations between each of the variable pairs in (3.3) the correlations may be written in terms of those probabilities. The notation to be used for the probabilities is $P_x(a,b)$ where $x$ and $y$ each equal $+1$ or $-1$ and $a$ and $b$ indicate instrument setting angles. In the quantum mechanical case after assuming that (perfectly) entangled particle pairs produce the measurements, the probabilities are symmetrical so that [6,7]

$$P_+(a,b) = P_-(a,b) \text{ and } P_-(a,b) = P_+(a,b)$$

(3.4a)

with normalization condition

$$P_+(a,b) + P_-(a,b) + P_+(a,b) + P_-(a,b) = 1.$$  \hspace{1cm} (3.4b)

Then

$$P_+(a,b) = 1/2 - P_-(a,b)$$ \hspace{1cm} (3.4c)

so that

$$C(a,b) = 2P_+(a,b) - 2P_-(a,b) = 4P_+(a,b) - 1.$$ \hspace{1cm} (3.4d)

Using (3.4d) with the appropriate substituted variables corresponding to each of the correlations in (3.3) yields

$$4P_+(a,b) - 1 - (4P_+(a,b') - 1) \leq 1 + (4P_+(b,a') - 1),$$

or the Wigner inequality

$$P_+(a,b) - P_+(a,b') \leq P_+(b,a')$$ \hspace{1cm} (3.5)

in which variable pairs for each probability are on opposite sides of a Bell measurement apparatus. However, it is important to note again that $a' = b'$ on the $A$-side and $B$-side, respectively, so that their data values have opposite signs.

This inequality is usually derived from purely probability assumptions [6] based on entanglement. However, since it is used to interpret results of Bell experiments, it is necessary to examine the physical conditions of experiments under which the data and probabilities obtain. From the above, the Wigner inequality is consistent with and follows from the Bell inequality and this fact is important in its interpretation.

IIIa. Quantum mechanical probabilities for Bell experiments

First one must identify the probabilities that produce the correlations of (2.5) in the case of Bell experiments. The probabilities for instrument setting pairs $(a,b)$ and $(a,b')$ are well known from the properties of entanglement that result in the Bell cosine correlation [6,7]:

$$P_+(a,b) = P_-(a,b) = 1/2 \sin^2 \frac{b-a}{2}; \quad P_+(a,b) = P_+(a,b) = 1/2 \cos^2 \frac{b-a}{2}.$$ \hspace{1cm} (3.6)

where the angular setting difference divided by 2 is appropriate for Bell’s original application to entangled spins. (In the optical version that corresponds to most Bell experiments, the 2 does not occur, with the result that a factor of 2 occurs in the final correlation.) The basic conception of a Bell experiment involves two detectors at settings $(a, b)$ that register counts recorded as $\pm 1$, of two entangled particles. The probabilities determined from quantum mechanics for obtaining such data, are given by (3.6). However, the probabilities corresponding to setting coordinates $(b, b')$ require additional
careful consideration. To obtain a constraint on the data correlations of the statistical analysis, Bell required three correlations as used in (2.5) and (2.6). How were these to be obtained from measurements on two particles such as photons, for which a measurement destroys the particle? This question is best answered by quoting Bell [8]: “We can suppose, for instance, that \( A' \) is obtained after \( A \), and \( B' \) after \( B \). But by no means. We are not at all concerned with sequences of measurements on a given particle, or of pairs of measurements on a given pair of particles. We are concerned with experiments in which for each pair the ‘spin’ of each particle is measured once only. The quantities \( A(\hat{a}',\lambda), B(\hat{b}',\lambda) \) are just the same functions \( A(\hat{a},\lambda), B(\hat{b},\lambda) \) with different arguments.” (Bell used “hats” on the angular setting parameters, and \( A \) and \( B \) indicate measurement readouts equal to \( \pm 1 \).)

There would appear to be only one way to realize data corresponding to Bell’s prescription. Perform two experiments using the same setting \( a \) and correlate the values at \( b \) and \( b' \) randomly obtained for each of the outcomes \( \pm 1 \) at \( a \). The quantum mechanically predicted results of such an experiment will be described below.

This situation is analogous to that which occurs in drug efficacy testing. A group of volunteers is divided into two equal groups with one subgroup being given the drug and the other a placebo. (A drug and then placebo, or placebo and then drug, are not given to each individual.) Various symptoms between the two subgroups may then be correlated.

IV. Predicted quantum mechanical correlations satisfy the Bell inequality

The correlations \( C(a,b) \) and \( C(a,b') \) defined by Bell (correlation subscripts are dropped since the functions have the same form) are given by correlations that equal the negative cosine of the difference of angular settings. Fifty years of controversy over the implications of the Bell theorem result from Bell’s assumption that \( C(b,b') \), the function resulting from his conceptual construction quoted above, is also of this form. From the derivation of (2.4) however, it is clear that this cannot be the case due to the resulting violation of (2.5) that would occur from large converging data sets that are ultimately finite. It should be clear from the derivation of (2.4) that the same data collected (or predicted) used to compute the left side correlations immediately determines the right side correlation as well, e.g., \( a_\lambda b_\lambda a_\lambda b'_\lambda = b_\lambda b'_\lambda \). This has been obscured due to the fact that in Bell’s derivation, the operations resulting in the inequality are applied to already formed correlation functions. Thus, it is not obvious that the inequality follows simply as a result of the algebra of cross-correlations applied to data sets, regardless of their statistical origin or whether they are even random. As follows from Sec 2, no three existing data sets can violate the Bell inequality in data correlations.

There remain two separate questions. One is how to measure data to allow computation of \( C(b,b') \) consistent with (2.4-2.5); the other is how to predict \( C(b,b') \) from quantum mechanical principles to determine whether the results are internally consistent with (3.3) and (3.5). The necessary methodology is implied by Bell in the quote above. One cannot undo an experimental result at setting \( b \) to obtain one at \( b' \) on the same
particle. However, consistent with the probability prescriptions for the random variables in the experiment, one may realize the required probabilities using multiple particle pairs i.e., correlate outcomes at $b$ and $b'$ for each of the +1 and −1 outcomes at setting $a$ using different particle pairs under the same random conditions and QM probabilities.

From quantum mechanics it follows that measurements at two alternative settings $b$ and $b'$ on one spin or photon do not commute [9,10]. (Measurements at $a$ and $b$ commute because they are on two different particles.) The probabilities for these counts are conditional on those that occur at $a$ that have equal probability $\frac{1}{2}$ for outcomes ±1. The conditional probabilities may be immediately obtained from (3.6) since the preceding factors of $\frac{1}{2}$ are the probabilities for $P_+(a)$. These conditional probabilities evidently may be used to determine those indicated in the Bell quotation above. Thus, $C_3(b,b')$ given by

$$C_3(b,b') = 2P_+(b,b') - 2P_-(b,b')$$

may be obtained from

$$P_+(b,b') = P_{++}(b,b'|a)P_+(a) + P_{+-}(b,b'|a)P_-(a)$$

$$= \frac{1}{2}[P_{++}(b|a)P_+(b'|a) + P_{+-}(b|a)P_+(b'|a)],$$

$$= \frac{1}{2}[\sin^2 \frac{b-a}{2} \sin^2 \frac{b'-a}{2} + \cos^2 \frac{b-a}{2} \cos^2 \frac{b'-a}{2}]$$

(4.2)

where subscript $++$ indicates positive counts at $b$ and $b'$ conditional on a positive count at $a$ etc. Similarly

$$P_-(b,b') = \frac{1}{2}[\sin^2 \frac{b-a}{2} \cos^2 \frac{b'-a}{2} + \cos^2 \frac{b-a}{2} \sin^2 \frac{b'-a}{2}].$$

(4.3)

Inserting (4.3) and (4.2) into (4.1) yields [11]

$$C(b,b') = \cos(b-a) \cos(b'-a).$$

(4.4)

Using (4.4) in (2.5) along with Bell correlations may be shown to satisfy the Bell inequality [3].

V. How quantum mechanical probabilities satisfy the Wigner inequality

The Wigner inequality is derived using variable pairs on opposite sides of a Bell apparatus. This implies that in (3.5) a +1 count at setting $a'$ corresponds to a −1 count at setting $b'$ since setting $a'$ equals setting $b'$. Thus the probability of $(+,+)$ at settings $b$ and $a'$ in (3.5) is given by $P_+(b,b')$ of (4.3). Inserting this and the other probabilities into (3.5) one obtains

$$\frac{1}{2} \sin^2 [(a-b)/2] \leq \frac{1}{2} \sin^2 [(a-b)/2] + \frac{1}{2} \left[ \sin^2 [(b-a)/2] \cos^2 [(b'-a)/2] \right] + \cos^2 [(b-a)/2] \sin^2 [(b'-a)/2].$$

(5.1)
That is equivalent to
\[
0 \leq 2 \sin^2\left(\frac{(a - b')}{2}\right) - 2 \sin^2\left(\frac{(a - b)}{2}\right) \sin^2\left(\frac{(a - b')}{2}\right)
\]
\[
0 \leq 2 \sin^2\left(\frac{(a - b')}{2}\right) \cos^2\left(\frac{(a - b)}{2}\right).
\]

(5.2)

Thus, the Bell and Wigner inequalities are both satisfied by quantum mechanical predictions.

Probabilities follow from the details of experiments [12]. In the common treatment of Wigner inequalities in which it is claimed that they are violated by QM, appropriate joint probabilities from (3.6) are used for each pair of measurements. But the measurements at \((a', b)\) are already determined by the two alternatives prescribed by Bell for \((a, b)\) and \((a, b')\). The probabilities for results at \((b, b')\) given \(a\) are determined by the conditional probabilities (4.2-4.3) used above. In this case, the probabilities are now consistent physically and mathematically.

**VI. Conclusion**

From sections IV and V above, the quantum mechanical correlations predicted from entanglement and their corresponding probabilities satisfy the Bell and Wigner inequalities, respectively. It has been shown above that the Wigner inequality follows from the Bell inequality when both the correlation and probability variable pairs are appropriate to opposite sides of a Bell measurement apparatus.

However, a more detailed outline of the logical links between Bell and Wigner inequalities is perhaps in order. The linkages specified assume the entanglement-based symmetry of the probabilities used in the above analysis: \(P_+ = P_-\) and \(P_+ = P_-\) at a given pair of angular settings. From this condition and probability normalization, it follows that a correlation may be expressed in terms of one probability and vice versa. From this, it follows that the Bell inequality in correlations results in the Wigner inequality in probabilities. However, neither the three correlations nor corresponding probabilities are in general equal, since the Bell correlation inequality is the same as that identically satisfied by any three data sets having equal or unequal cross-correlations. If the data set correlation estimates converge, the same inequality constraints result as derived by Bell, but is independent of Bell’s assumptions. Since laboratory procedures measure correlations only by recording data sets, the Bell inequality in data sets becomes logically central to the analysis. It follows that if the first two data set pairs for measurements and outcomes at \(a, b,\) and \(b'\) at fixed \(a\) are specified, the third correlation is determined by the data already in hand. The third correlation must be computed from that data to be consistent with the Bell inequality. When quantum mechanical probabilities are used in the computation, the third correlation has a different mathematical form from the first two and is consistent with quantum mechanical principles of non-commutation. There is thus no conflict between quantum mechanical probability predictions and basic mathematics, or quantum mechanical measurements and algebra.
Finding that the Bell theorem is fatally flawed does not constitute a proof that local hidden variables exist for Bell correlations. However, since a physical superposition of entangled waves no longer exists in Bell experiments at the separated spatial locations where measurements occur, it must initially be concluded that Bell correlations arise due to source initial conditions, followed by local conditions at detection. Since Bell correlations also follow theoretically on the assumption of measurements of an entanglement superposition, it is implied that the production of a Bell correlation is not mathematically unique.

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