Open superstring partition function in constant gauge field background at finite temperature

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Abstract

We find the general expression for the open superstring partition function on the annulus in a constant abelian gauge field background and at finite temperature. We use the approach based on Green-Schwarz string path integral in the light-cone gauge and compare it with NSR approach. We discuss the super Yang-Mills theory limit of the string free energy and mention some D-brane applications.

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1. INTRODUCTION

Leading-order interactions between BPS states of D-branes \([1]\) admit both supergravity and super Yang-Mills descriptions which give equivalent results for interaction potentials (see, e.g., \([2,3,4,5,6]\) and references there). This equivalence can be understood as being, in particular, a consequence of the universality of the leading \(F^4\) term in the open superstring partition function on the annulus \([3,4]\).

An important question is whether this correspondence between the two descriptions applies also to non-BPS (excited, or non-extremal) states of D-branes. As was found in \([8]\), starting with the one-loop \(F^4\)-term in the SYM effective action and assuming certain averaging over SYM backgrounds which have the right energy and charges to describe near-extremal branes on supergravity side one obtains the expressions which have the same structure as supergravity interaction potentials between extremal and near-extremal branes. The precise coefficients do not seem to match however.

From supergravity point of view, the non-extremal branes can be assigned certain temperature and entropy. It is thus natural to expect that the SYM description of non-extremal RR branes should, in fact, be based on thermal gauge theory states. In particular, one may try to interpret the Hawking radiation of a certain class of near-extremal black holes with RR charges in terms of emission of closed string modes by a D-brane configuration in an excited state \([10,11]\). The entropy of near-extremal D3-brane can be reproduced as the entropy of finite-temperature ensemble of states of the \(N = 4\) four-dimensional SYM theory \([12]\). As discussed in \([13]\), there are two kinds of non-extremal states of D-branes: one can be thought of as a D-brane with a small number of long (massive) strings, and another as a D-brane with a large number of short (massless) open strings. For large deviation from extremality (or large temperature, \(\beta < \sqrt{\alpha'}\)) long string state has greater entropy, while for small excess energy the gas of light open strings (or SYM modes) is the relevant description.

This suggests that to describe the potential between non-extremal D-branes one should perform finite-temperature analogs of computations in \([2,3,5,14]\), i.e. determine the corresponding terms in the finite-temperature open string partition function or finite-temperature effective action of SYM theory.

This is one of motivations behind the formal discussion of the present paper. In more general context, it is of interest to study the combined effect of the temperature and magnetic and electric background fields on the behaviour of open string ensemble. Here we shall compute the finite temperature superstring partition function on the annulus in a constant gauge field background and obtain the corresponding SYM free energy in the \(\alpha' \to 0\) limit. Possible applications will be mentioned only briefly.

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1 The agreement for potentials between non-BPS branes found in \([1]\) seems to apply only to a class of configurations which are spherically symmetric in \(D = 11\) sense.
We shall start in Section 2 with a detailed discussion of the zero-temperature case. The string partition function in external fields is directly related to the string effective action. This relation is particularly simple in the open string theory case. The computation of the one-loop (annulus) superstring partition function in a constant gauge field background $Z(F)$ is straightforward in the light-cone gauge Green-Schwarz (GS) formulation and was originally considered in [19]. In section 2.1 we shall put $Z(F)$ found in [19] in a more explicit form and then in section 2.2 compare it with related results found using NSR path integral and GS boundary state approaches.

For constant $F$ the GS path integral becomes gaussian in both bosonic ($x$) and fermionic ($S$) coordinates. The approach used in [19] was to compute the corresponding determinants with the free-theory boundary conditions $\partial_n x = 0$, $S_1 = S_2$. This is in direct correspondence with the standard definition of the string scattering amplitudes as correlators of l.c. gauge GS vertex operators $[23,24,25]$. Somewhat surprisingly, we find that the resulting $Z(F)$ is different from the result of the NSR approach of [2] (which is equivalent to the result of the GS boundary state approach of [20]). The two partition

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2 Low-energy effective action in string theory can be defined as a ‘superposition’ of string scattering amplitudes with massless tree-level poles subtracted. The scattering amplitudes are given by correlators of vertex operators in flat background. In the Polyakov path integral approach it is possible to represent the string effective action in terms of the (renormalized) string sigma model partition function in background fields. Indeed, the latter is the generating functional for string amplitudes (average of the exponential of vertex operators multiplied by external fields) and renormalisation of logarithmic 2d divergences effectively subtracts the massless poles $[15,16]$.

3 At the tree (disc) level the divergences associated with the Möbius group volume are absent in the superstring case $[17]$ and can be easily renormalised away in the bosonic case $[18,19]$. The problem of Möbius infinities does not appear at one and higher loop level where in computing $Z$ one needs only to subtract logarithmic divergences associated with massless poles in the amplitudes. The local part of these divergences is absent in the case of the constant abelian vector field strength background so that $Z(F)$ is finite (apart from modular divergences that may or may not cancel depending on a particular problem and theory under consideration).

4 One of our aims is to clarify the structure of the light-cone GS path integral approach to computation of the string partition function with a hope that it can be applied to the problems of determining derivative $O(\partial F)$ corrections to the one-loop $Z(F)$ and computing the two-loop contribution to the string effective action (in particular, in order to check the conjecture $[6,20]$ that, like $F^4$ term in one-loop $Z$ $[3]$, the $F^6$ term in two-loop $Z$ is ‘universal’, i.e. has trivial dependence on $\alpha'$ and thus on space-time IR cutoff or distance between branes). Such computations in the NSR formalism where one needs to sum over spin structures appear to be very complicated. Related examples of the utility of the l.c. GS approach to computing the one-loop partition function in closed string theory can be found in $[21,22]$. 

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functions agree in the $\alpha' \to 0$ (massless state) limit but differ in the contributions of massive states.

Similar gaussian path integral in the NSR approach can be computed either directly, by evaluating determinants in external field $F$ using free-theory boundary conditions, or, as in [27], by first solving the boundary conformal field theory in terms of free oscillators (imposing $F$-dependent boundary conditions [28,29]) and then obtaining the partition function using the standard operator formalism relation (the two procedures give, of course, equivalent expressions for $Z(F)$). The same result can be found in the GS approach if instead of computing path integral with the free-theory boundary conditions as in [19] one first solves the 2d conformal theory using appropriate ('supersymmetric') boundary condition $S_1^a = \hat{M}^{ab}(F) S_2^b$ [26]. This boundary condition is the one that is consistent with space-time supersymmetry [29,26] (though it is not the one implied by the form of the $F$-dependent l.c. gauge GS action).

While the approach of [19] is in direct correspondence with the standard l.c. gauge GS vertex operator definition of the string amplitudes, it may not manifestly preserve supersymmetry.\footnote{Indeed, here one uses the constant $F$ truncation, i.e. keeps only (momentum)$^n$ term in the $n$-point vector amplitude while the symmetries of the theory are expected to be preserved provided one defines the amplitudes by an analytic continuation in momenta.} This suggests that the difference between the expressions of [19] and [27,26] may be interpreted as being due to certain ‘contact terms’\footnote{A lesson seems to be that one may get inequivalent results for a superstring partition function depending on whether one reconstructs it from an expansion near free-theory point or defines it in terms of a non-trivial conformal field theory which uses appropriate boundary conditions.} (which for constant $F$ may have the same structure as the ‘main’ terms). The two expressions may be related by a redefinition of the field strength $F \to F + O(\alpha' F^2)$ containing all powers of $F$.\footnote{The finite temperature case will be the subject of section 3. In section 3.1 we shall find the finite-temperature analogue of the one-loop effective action (free energy) in SYM theory in a magnetic background. In section 3.2 we shall use the GS approach to generalise the string partition function obtained in section 2.2 to the finite-temperature case. The}
resulting expression $Z(\beta, F)$ will have the SYM free energy as its $\alpha' \to 0$ limit. In section 3.3 we shall study the dependence of $Z(\beta, F)$ on the background field and temperature. In particular, we shall show that the value of the Hagedorn temperature is not modified by the magnetic field and that the infra-red magnetic instability of zero-temperature partition function (present in both string theory and SYM theory) remains also at finite temperature. The finite-temperature case with an electric background will be discussed in section 3.4. The presence of an electric field is known to modify the value of the Hagedorn temperature of the neutral open bosonic string gas [33] and we find that the same is true for the open superstring gas.

2. OPEN SUPERSTRING PARTITION FUNCTION ON THE ANNULUS IN CONSTANT BACKGROUND FIELD (ZERO TEMPERATURE CASE)

2.1. Green-Schwarz path integral with free-theory boundary conditions

Our starting point is the open superstring partition function in the constant abelian gauge field background given by light-cone gauge GS path integral on the annulus

$$Z(F^{(1)}, F^{(2)}) = \int [dq][dx][dS] \ exp[i(I_0 + I_{int}^{(1)} + I_{int}^{(2)})] ,$$

where (the indices $i,j$ and $a,b$ run from 1 to 8)

$$I_0 = -\frac{1}{4\pi\alpha'} \int d^2\sigma \ [\partial_+ x^i \partial_- x^i - \frac{i}{2}\alpha' (S_1^a \partial_+ S_1^a + S_2^a \partial_- S_2^a)] ,$$

$$I_{int} = \int dt \ [\dot{x}^i A_i(x) - \frac{i}{2}\alpha' S^a S^b \tilde{F}_{ab}] ,$$

$$\tilde{F}_{ab} = \frac{1}{4} \gamma_{ij} F_{ij} .$$

For the constant field strength,

$$A_i = -\frac{1}{2} F_{ij} x^j , \quad F_{ij} = \text{const} ,$$

the path integral is gaussian [18][19]. To be able to use the l.c. gauge GS formalism, we assume that the electric field components are vanishing. At zero temperature the final result for $Z(F)$ should admit a Lorentz-covariant $SO(1,9)$ generalisation, so that the

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7 We shall be interested in the case of oriented open superstrings which is relevant for the description of D-branes and discuss only the annulus diagram. All considerations below can be straightforwardly repeated for the case of the Möbius strip diagram of type I theory.
dependence on electric components may be deduced by analytic continuation in magnetic components.

We shall assume the standard \[25\] boundary conditions

\[
\partial_\sigma x^i = 0, \quad S_1^a = S_2^a \equiv S^a,
\]

with both \(x\) and \(S\) being periodic in the angular coordinate of the annulus. For generic vector fields \(Z\) is the generating functional for the scattering amplitudes (\(I_{int}\) corresponds to the standard l.c. gauge form of the vector field vertex operator [25]). The form of the interaction action (2.2),(2.3) follows also upon fixing the l.c. gauge in the covariant action for a GS superstring coupled to the on-shell \(D = 10\) SYM superfield background [15].

In the present case of a gaussian theory one may compute the path integral either by first directly expanding the fields subject to the free-theory boundary conditions (2.5) in modes and then integrating them out, or by first solving the classical equations of the theory using field-dependent boundary conditions implied by the action (2.2), i.e.

\[
\partial_\sigma x^i + F^{ij} \partial_\tau x^j = 0, \quad \text{or} \quad \partial_+ x^i = M^{ij} \partial_- x^j, \quad M = \frac{1 + \hat{F}}{1 - \hat{F}},
\]

\[
S_1^a - S_2^a = \hat{F}^{ab}(S_1^b + S_2^b), \quad \text{or} \quad S_1^a = \tilde{M}^{ab} S_2^b, \quad \tilde{M} = \frac{1 + \hat{F}}{1 - \hat{F}}.
\]

The results for \(Z(F)\) found using these two approaches are, of course, equivalent.

One may, in principle, consider a possibility of replacing \(\hat{F}_{ab}\) in (2.3) with \(\hat{F}'_{ab}\) given by a local power series in \(F\), \(\hat{F}_{ab} = \frac{1}{4} \gamma_{ab} F_{ij} + O(F^2)\), as this would modify the corresponding scattering amplitudes only by certain contact terms (which may be necessary to add to maintain Lorentz symmetry or space-time supersymmetry [30]). Equivalently, one may consider replacing (2.7) by a different field-dependent boundary condition (implied by the action (2.2) with \(\hat{F}\) given not by (2.3) but by \(\hat{F}_{ab}\))

\[
S_1^a = \tilde{M}^{ab} S_2^b, \quad \tilde{M} = \hat{M}(F).
\]

This again would change the scattering amplitudes only by contact terms. The choice of such modified boundary condition with \(\tilde{M}^{ab}\) related to \(M\) in (2.6) as a spinor rotation is related to a vector rotation, i.e. by

\[
M^{ij} \gamma^i = \tilde{M}^{-1} \gamma^i \tilde{M},
\]

follows from the condition that the boundary conformal field theory defined by (2.2),(2.4) should preserve space-time supersymmetry [29,28], As we shall find below, such modification is, indeed, required in order to obtain the same expression for \(Z\) as in manifestly Lorentz-covariant NSR path integral approach (in the special case of \(F\) having only one
non-vanishing component this was shown in [26] where the NSR result of [27] was reproduced using (2.9)).

In this subsection we shall follow the approach of [19] based on (2.3),(2.5) but many of equations below will not depend on the explicit form of the relation between $\hat{F}$ and $F$ so that the modification required for restoring the equivalence with the NSR approach will be straightforward to implement later. We shall often absorb $2\pi\alpha'$ in the two magnetic field $8\times8$ matrices $F_{ij}^{(r)}$ ($r = 1, 2$) representing the interactions at the boundaries of the annulus. The gaussian path integral over the bosons and fermions gives [19]

$$Z = c_0 \Lambda(F^{(1)} + F^{(2)}) \int_0^1 \frac{dq}{q} \prod_{I=1}^4 Z_I(F^{(1)}, F^{(2)}; q) , \quad (2.10)$$

where $c_0 \sim (2\pi\alpha')^{-5}V_{10}$, $\Lambda(F)$ is the fermionic zero mode factor given below and

$$Z_I \equiv \left[ \frac{\det(1 - \hat{K}_I \cdot \hat{K}_I)}{\det(1 - K_I \cdot K_I)} \right]^{1/2} \quad (2.11)$$

is the contribution of the non-zero modes. The advantage of the GS formalism is that the treatment of non-zero modes of bosons and fermions is parallel as both $x$ and $S$ are periodic functions of the boundary angle $\psi$ (which is the euclidean analogue of the open-string time). The bosonic function $K_I = K \cdot F_I$ ($I = 1, 2, 3, 4$) is the product of the derivative of the boundary values $K_{rs}$ of the Green function of the Laplace operator on the annulus ($r, s = 1, 2$)

$$K_{rs} = \partial_\psi G_{rs} = -\frac{1}{\pi} \sum_{n=1}^\infty G_{nrs} \sin n\psi_{rs} , \quad \psi_{rs} = \psi_r - \psi'_s , \quad (2.12)$$

$$G_n \equiv \begin{pmatrix} A_n & B_n \\ B_n & A_n \end{pmatrix} , \quad A_n = \frac{1 + q^{2n}}{1 - q^{2n}} , \quad A_n^2 - B_n^2 = 1 ,$$

with the matrix $F_I = \text{diag}(f_1^{(1)}, f_1^{(2)})$, where $f_1^{(r)}$ are the non-vanishing entries in $F_{ij}^{(r)}$ taken in the block-diagonal form

$$F_{ij}^{(r)} = \text{diag} \begin{pmatrix} \begin{pmatrix} 0 & f_1^{(r)} \\ -f_1^{(r)} & 0 \end{pmatrix} , \ldots , \begin{pmatrix} 0 & f_4^{(r)} \\ -f_4^{(r)} & 0 \end{pmatrix} \end{pmatrix} . \quad (2.13)$$

8 In the case of the open string with charges $e_1, e_2$ in an external magnetic field $F_{ij}$ one has $F_{ij}^{(r)} = e_r F_{ij}$ (the neutral string case corresponds to $e_1 + e_2 = 0$ or $F_{ij}^{(1)} + F_{ij}^{(2)} = 0$). We shall, however, keep $F_{ij}^{(1)}$ and $F_{ij}^{(2)}$ independent which is useful in view of D-brane applications.

9 We assume that both $F_{ij}^{(1)}$ and $F_{ij}^{(2)}$ can be simultaneously put in such form.
The identity operator in (2.11) is \(1 = \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n \psi_{rs}\). The fermionic function \(\hat{K}_I\) has a similar definition \(\hat{K}_I = K \cdot \hat{F}_I\) in terms of \(K\) (which is also equal to the \(2 \times 2\) matrix of the boundary values of the Dirac operator Green function) and \(\hat{F}_I = \text{diag}(\hat{f}_I^{(1)}, \hat{f}_I^{(2)})\), where \(\hat{f}_I^{(r)}\) are the eigen-values of the matrix \(\hat{F}_I^{(r)}\).

In the case of \(\hat{F}_{ab}\) given by (2.3) one may use that \(i\gamma_{12}, i\gamma_{34}, i\gamma_{56}, i\gamma_{78}\) are projectors to show that the non-vanishing elements \(f_I\) and \(\hat{f}_I\) in the block-diagonal forms of \(F_{ij}\) and \(\hat{F}_{ab}\) are related by

\[
\hat{f}_1 = \frac{1}{2}(-f_1 + f_2 + f_3 + f_4), \quad \hat{f}_2 = \frac{1}{2}(f_1 - f_2 + f_3 + f_4), \quad \hat{f}_3 = \frac{1}{2}(f_1 + f_2 - f_3 + f_4), \quad \hat{f}_4 = \frac{1}{2}(f_1 + f_2 + f_3 - f_4),
\]

i.e.

\[
\hat{f}_I = P_{IJ} f_J, \quad P_{IJ} \equiv \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}, \quad P^{-1} = P, \quad \det P = -1 .
\] (2.14)

As follows from (2.14),

\[
\sum_{I=1}^{4} \hat{f}_I = \sum_{I=1}^{4} f_I, \quad \sum_{I=1}^{4} \hat{f}_I^2 = \sum_{I=1}^{4} f_I^2 .
\] (2.15)

Note that it is not necessary to assume that \(\hat{F}_{ab}\) in (2.3) has block-diagonal form: using the projector property of \(i\gamma_{ij}\) one can compute the fermionic determinant with the only assumption being that \(F_{ij}\) has block-diagonal form. The result is then expressed in terms of \(f_I\) according to (2.11) where \(\hat{K}_I = K \cdot \text{diag}(\hat{f}_I^{(1)}, \hat{f}_I^{(2)})\), with \(\hat{f}_I\) defined by (2.13).

The fermionic zero-mode factor in (2.10) is \([19,34]\)

\[
\Lambda(F) = -4 \prod_{I=1}^{4} \hat{f}_I = \sqrt{\det \hat{F}_{ab}} = \frac{1}{3 \cdot 2^8 t_8 FFFF} = -\frac{1}{2} \sqrt{\det F_{ij}} + \frac{1}{16} [F^4 - \frac{1}{4} (F^2)^2] .
\] (2.16)

Since the matrices \(K_I\) and \(\hat{K}_I\) are obtained from \(2 \times 2\) matrix \(K\) (2.12) of the first derivative of the bosonic Green function on the annulus with legs on any of the two boundaries

\footnote{This is the same as the transformation in the weight space of \(SO(8)\) that rotates vectors into spinors, see, e.g., \([25]\). The choice of signs of \(\hat{f}_I\) is not important as the determinant depends only on their squares (we, in fact, invert the sign of \(\hat{f}_1\) as compared to \(\hat{F}_{12}\) so that \(\prod_{I=1}^{4} \hat{f}_I = -\sqrt{\det \hat{F}_{ab}}\).}
by multiplying it by \( \text{diag}(f_I^{(1)}, f_I^{(2)}) \) and \( \text{diag}(\hat{f}_I^{(1)}, \hat{f}_I^{(2)}) \), the ratio of the functional determinants (2.11) reduces to the infinite product of the ratios of determinants of \( 2 \times 2 \) matrices

\[
Z_I = \left[ \frac{\det(1 - \hat{K}_I \cdot K_I) - \hat{f}_I^{(1)} \hat{f}_I^{(2)}}{\det(1 - K_I \cdot K_I)} \right]^{1/2} = \prod_{n=1}^{\infty} \frac{\det \hat{\Omega}_{n,I}}{\det \Omega_{n,I}}, \tag{2.17}
\]

where

\[
\Omega_{n,I} = 1 + (K_{n,I})^2, \quad K_{n,I} \equiv \left( \begin{array}{cc} f_I^{(1)} A_n & f_I^{(2)} B_n \\ f_I^{(1)} B_n & f_I^{(2)} A_n \end{array} \right), \quad \hat{\Omega} = \Omega(f_I^{(r)} \to \hat{f}_I^{(r)}).
\]

In [19] \( f_I^{(2)} \) were set equal to zero. Keeping the non-zero background fields at both boundaries of the annulus we get

\[
Z_I = \prod_{n=1}^{\infty} \frac{\det \hat{\Omega}_{n,I}}{\det \Omega_{n,I}} = \prod_{n=1}^{\infty} \frac{(1 - \hat{f}_I^{(1)} \hat{f}_I^{(2)} + (\hat{f}_I^{(1)} + \hat{f}_I^{(2)})^2 A_n^2}{(1 - f_I^{(1)} f_I^{(2)} + (f_I^{(1)} + f_I^{(2)})^2 A_n^2}, \tag{2.18}
\]

or, equivalently,

\[
Z_I = \left[ \frac{1 + (f_I^{(1)})^2[1 + (f_I^{(2)})^2]}{1 + (\hat{f}_I^{(1)})^2[1 + (\hat{f}_I^{(2)})^2]} \right]^{1/2} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos 2\pi \hat{\varphi}_I + q^{4n}}{1 - 2q^{2n} \cos 2\pi \varphi_I + q^{4n}}, \tag{2.19}
\]

where we have used that in the \( \zeta \)-function regularisation \( \prod_{n=1}^{\infty} c = c^{-1/2} \). We defined the parameters \( \varphi_I \) and \( \hat{\varphi}_I \) related to \( f_I^{(r)} \) and \( \hat{f}_I^{(r)} \) by

\[
\varphi_I = \varphi_I^{(1)} + \varphi_I^{(2)}, \quad \tan \pi \varphi^{(r)} = f_I^{(r)}, \tag{2.20}
\]

\[
\hat{\varphi}_I = \hat{\varphi}_I^{(1)} + \hat{\varphi}_I^{(2)}, \quad \tan \pi \hat{\varphi}_I^{(r)} = \hat{f}_I^{(r)}, \tag{2.21}
\]

so that

\[
\tan \pi \varphi_I = \frac{f_I^{(1)} + f_I^{(2)}}{1 - f_I^{(1)} f_I^{(2)}}, \quad \cos 2\pi \varphi_I = \frac{1 - \tan^2 \pi \varphi_I}{1 + \tan^2 \pi \varphi_I}, \tag{2.22}
\]

\[
\sin \pi \varphi_I = \frac{f_I^{(1)} + f_I^{(2)}}{\sqrt{1 + (f_I^{(1)})^2[1 + (f_I^{(2)})^2]}}. \tag{2.22}
\]

We reserve the notation \( \hat{\varphi}_I \) for the linear combinations of \( \varphi_I \) defined as in (2.14),

\[
\hat{\varphi}_I = P_{IJ} \varphi_J, \quad \text{i.e.} \quad \hat{\varphi}_1 = \frac{1}{2}(-\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4), \ldots . \tag{2.23}
\]

\[\text{11} \] There is a misprint in eq.(34) of [19] (the equations that follow (34) are correct): the power of the determinant should be -1 (as in eq.(4.9) in [33]).
Using the expression for the Jacobi $\vartheta_1$-function

$$\vartheta_1(\varphi | i\tau) = 2q^{1/4} \sin \pi \varphi \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2\pi \varphi + q^{4n}) \ , \quad q = e^{-\pi \tau} \ , \quad (2.24)$$

we finish with

$$Z_I = \frac{f_I}{\tilde{f}_I} \frac{\vartheta_1(\tilde{\varphi}_I | i\tau)}{\vartheta_1(\varphi_I | i\tau)} \ , \quad (2.25)$$

where we defined

$$f_I \equiv f^{(1)}_I + f^{(2)}_I \ , \quad \tilde{f}_I \equiv \tilde{f}^{(1)}_I + \tilde{f}^{(2)}_I \ . \quad (2.26)$$

The final result for the partition function (2.10) is thus remarkably simple ($c_1 = -\pi c_0$)\footnote{Note that the fermionic zero mode factor $\Lambda(F^{(1)} + F^{(2)})$ or $\prod_{I=1}^{4} \hat{f}_I$ effectively got ‘replaced’ by its bosonic ‘analogue’ $\prod_{I=1}^{4} f_I$ after we expressed the infinite product in terms of the $\vartheta$-functions (the weak-field expansion of $Z$ is of course still proportional to $\prod_{I=1}^{4} \hat{f}_I$).}

$$Z = c_1 \int_0^\infty d\tau \prod_{I=1}^{4} f_I \frac{\vartheta_1(\tilde{\varphi}_I | i\tau)}{\vartheta_1(\varphi_I | i\tau)} \ . \quad (2.27)$$

2.2. Green-Schwarz path integral with supersymmetric boundary condition and equivalence with NSR path integral

It is easy to show that if one replaces the fermionic boundary condition (2.7) by the ‘supersymmetric’ one (2.8),(2.9) or, equivalently, makes the corresponding replacement of $\hat{F}$ in (2.3) by $\hat{F}'_{ab}$ such that

$$\hat{M}(F) = \frac{1 + \hat{F}'}{1 - \hat{F}'} \ ,$$

and repeats the above computation of $Z$ using (2.5) and $\hat{F}'$ in place of $\hat{F}$, one finishes with the same expression (2.25),(2.27) but with

$$\varphi_I \to \tilde{\varphi}_I \ , \quad (2.28)$$

where $\tilde{\varphi}_I$ is defined by (2.23), i.e.

$$Z = c_1 \int_0^\infty d\tau \prod_{I=1}^{4} f_I \frac{\vartheta_1(\tilde{\varphi}_I | i\tau)}{\vartheta_1(\varphi_I | i\tau)} \ . \quad (2.29)$$

To see the reason for the replacement (2.28) note first that the matrix in the bosonic boundary condition (2.6) $M = \frac{1 + F}{1 - F}$ has block-diagonal form with four $2 \times 2$ entries, $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ where

$$M_I = e^{2\pi \varphi_I J} = 1 \cos 2\pi \varphi_I + J \sin 2\pi \varphi_I = \begin{pmatrix} \frac{1 - f_I^2}{1 + f_I^2} & \frac{2f_I^2}{1 + f_I^2} \\ -\frac{2f_I^2}{1 + f_I^2} & \frac{1 - f_I^2}{1 + f_I^2} \end{pmatrix} \ , \quad J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad (2.30)$$
where \( \varphi_I = \frac{1}{\pi} \arctan f_I \). Its 8 eigenvalues are thus \( e^{\pm 2\pi i \varphi_I} \). Since (2.6) and (2.7) are related by \( F \to \hat{F} \), i.e. \( f_I \to \hat{f}_I = P_{IJ} f_J \), we find that the eigenvalues of the ‘naive’ GS fermion rotation matrix \( \hat{M} \) in (2.7) are

\[
\hat{M} = \{ e^{\pm 2\pi i \tilde{\varphi}_I} \} \quad .
\]

(2.31)

The matrix \( \hat{M} \) in (2.8),(2.9) can be represented as

\[
\hat{M} = e^{\frac{1}{2} \pi \gamma^I \varphi_I}, \quad \text{where } \varphi_{ij} \text{ is the matrix whose 8 eigenvalues are } \pm i \varphi_I, \text{ with } \tan \pi \varphi_I = f_I.
\]

Using that \( (\gamma_{2I-1,2I})^2 = -1, [\gamma_{2I-1,2I}, \gamma_{2J-1,2J}] = 0, \gamma_{2I-1,2I} = \gamma_{2I-1} \gamma_{2I} \) one finds [26]

\[
\hat{M} = e^{\pi \sum_{I=1}^{4} \varphi_I \gamma_{2I-1,2I}} = \prod_{I=1}^{4} \left( \cos \pi \varphi_I + \gamma_{2I-1,2I} \sin \pi \varphi_I \right) = \prod_{I=1}^{4} \frac{1 + \gamma_{2I-1,2I} f_I}{\sqrt{1 + f_I^2}} \quad .
\]

(2.32)

If we diagonalise \( \hat{M} \) (which is just a spinor rotation matrix with angles \( \varphi_I \)) then its 8 eigenvalues will be (cf. (2.3),(2.13),(2.14))

\[
\hat{M} = \{ e^{\pm 2\pi i \tilde{\varphi}_I} \} \quad , \quad \tilde{\varphi}_I = P_{IJ} \varphi_J \quad .
\]

(2.33)

Comparing (2.31) and (2.33) we conclude that replacing (2.7) by (2.8), i.e. \( \hat{M} \to \hat{M} \) corresponds to the replacement \( \tilde{\varphi}_I \to \tilde{\varphi}_I \) in the fermionic GS sector. The equivalent transformation \( \hat{F} \to \hat{F}' \) that relates \( \hat{M} = \frac{\hat{F} + \hat{F}'}{1 - \hat{F}} \) and \( \hat{M} = \frac{\hat{F} + \hat{F}'}{1 - \hat{F}'} \) is determined by the following non-linear transformation of the eigenvalues

\[
\hat{f}_I = P_{IJ} f_J \to \hat{f}'_I = \tan(P_{IJ} \arctan f_J) = P_{IJ} f_J + O(f^2) \quad .
\]

(2.34)

It would be interesting to find an independent argument of why this non-linear redefinition is required.

The equivalent ‘proper-time’ form of (2.29) is found by performing the Jacobi transformation,

\[
\vartheta_1 \left( \frac{i \varphi}{\tau} \right) = i \sqrt{\tau} \left( e^{\frac{\pi \varphi^2}{\tau}} \right) \vartheta_1 \left( \varphi | i \tau \right) \quad ,
\]

and noting that the exponential factors coming from the bosonic and fermionic \( \vartheta \)-functions cancel out because of the relation between squares of \( \varphi_I \) and \( \tilde{\varphi}_I \) similar to the one in (2.13).\(^{14}\) We finish with

\[
Z = c_1 \int_0^\infty \frac{dt}{t^2} \prod_{I=1}^{4} f_I \left( \frac{\vartheta_1 (i t \tilde{\varphi}_I | i t)}{\vartheta_1 (i t \varphi_I | i t)} \right) , \quad t \equiv \frac{1}{\tau} \quad .
\]

(2.35)

\(^{13}\) Equivalent form is \([29] \hat{M} = [\sqrt{\det(1 + F)}]^{-1/2} \exp(2\hat{F}) \) where \( \hat{F} \) is defined in (2.3) and the expansion of the exponent is understood in the following sense: \( \gamma_i \) are treated as Grassmann, i.e. they all anticommute and have zero squares.

\(^{14}\) Such relation is not true for \( \tilde{\varphi}_I \) and \( \varphi_I \) and thus the form of (2.27) is not ‘modular-invariant’.  

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In the special case of one non-vanishing field component this expression was obtained in \cite{20} using GS boundary state approach with the boundary condition (2.8).

The expressions (2.29), (2.35) can be put also into ‘NSR form’ by using the Riemann identity \cite{36}

\[
\prod_{I=1}^{4} \vartheta_{1}(\tilde{y}_{I}|it) = \frac{1}{2} \prod_{I=1}^{4} \vartheta_{1}(y_{I}|it) + \frac{1}{2} \sum_{k=2}^{4} (-1)^{k} \prod_{I=1}^{4} \vartheta_{k}(y_{I}|it), \tag{2.36}
\]

where four \( \tilde{y}_{I} \) are related to \( y_{I} \) as in (2.14), i.e. \( \tilde{y}_{I} = P_{I,J} y_{J} \). Then from (2.29) we get

\[
Z = \frac{1}{2} c_{1} \int_{0}^{\infty} \cdots \prod_{I=1}^{4} \frac{f_{I}}{\vartheta_{1}(\varphi_{I}|it)} \left[ \prod_{J=1}^{4} \vartheta_{2}(it\varphi_{J}|it) - \prod_{J=1}^{4} \vartheta_{3}(it\varphi_{J}|it) + \prod_{J=1}^{4} \vartheta_{4}(it\varphi_{J}|it) \right] \tag{2.37}
\]

and from (2.35) \cite{37}

\[
Z = \frac{1}{2} c_{1} \int_{0}^{\infty} \cdots \prod_{I=1}^{4} \frac{f_{I}}{\vartheta_{1}(it\varphi_{I}|it)} \left[ \prod_{J=1}^{4} \vartheta_{2}(it\varphi_{J}|it) - \prod_{J=1}^{4} \vartheta_{3}(it\varphi_{J}|it) + \prod_{J=1}^{4} \vartheta_{4}(it\varphi_{J}|it) \right] + \frac{1}{2} c_{1} \int_{0}^{\infty} \cdots \prod_{I=1}^{4} f_{I}. \tag{2.38}
\]

Apart from the last term, this is, indeed, the result which one finds by doing a similar calculation in the NSR approach.

The last term (i.e. \( \sim \sqrt{\det F_{ij}} \) which does not have a \( D = 10 \) Lorentz-covariant extension) does not appear in the standard D9-brane NSR partition function in magnetic background (the periodic sector contribution vanishes because of the remaining fermionic zero modes \( \psi_{0,\psi_{g}} \) ) but is present in the closely related D8-brane expression \cite{31,37,38}, obtained by assuming the Dirichlet boundary condition along the 9-th direction. The reason why it appears in the GS approach is that because of the choice of the l.c. gauge here one treats the 0,9 directions as the Dirichlet ones \cite{39,26}, i.e. the above expression (2.35) effectively corresponds to D8-instanton, with time being one of the two orthogonal directions.

In NSR approach the fermionic terms in the action in (2.1), (2.2) are replaced by \( (\mu, \nu = 0, 1, \ldots, 8, 9) \)

\[
\frac{i}{8\pi} \int d^{2}\sigma (\psi_{R}^{\mu} \partial_{+} \psi_{R}^{\mu} + \psi_{L}^{\mu} \partial_{-} \psi_{L}^{\mu}) - \frac{i}{2} \alpha' \int dt \ F_{\mu\nu} \psi^{\mu} \psi^{\nu}. \tag{2.39}
\]

\[\text{Note that } \frac{\vartheta_{2}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{3}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{1}(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{4}'(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \]

\[\text{Note that } \frac{\vartheta_{2}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{3}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{1}(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{4}'(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \]

\[\text{Note that } \frac{\vartheta_{2}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{3}(v/z) - 1/z}{\vartheta_{1}(v/z) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{1}(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \quad \frac{\vartheta_{4}(v/z)}{\vartheta_{4}'(0) - 1/z} = \frac{i}{2} \vartheta_{4}(v/z), \]

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with the boundary conditions $\psi^\mu_R = \psi^\mu_L = \psi^\mu$ at $\sigma = 0$ and $\psi^\mu_R = \mp \psi^\mu_L = \psi^\mu$ at $\sigma = \pi$. The calculation of the fermionic determinants is analogous to the one discussed above but now we are to sum over the contributions of different sectors. Since here we have the same $F_{\mu\nu}$ matrix appearing in the bosonic and fermionic determinants, the arguments of the $\vartheta$-functions are also the same. For $F_{\mu\nu}$ having only magnetic $F_{ij}$ ($i,j = 1,\ldots,8$) components one finds that the final expression for the partition function is given by (2.38) (without the last term).

The NSR approach allows, in principle, to obtain the expression for $Z$ for the general $D = 10$ choice of (euclidean) $F_{\mu\nu}$ having all five ‘eigenvalues’ $f_0, f_1, \ldots, f_4$ being non-vanishing ($f_0 = F_{09} = iE$ is the electric field component). The natural generalization of (2.38) to the case of $f_0^{(1)}, f_0^{(2)} \neq 0$ is

$$Z = \frac{1}{2} c_1 \int_0^\infty d\tau \prod_{M=0}^4 f_M \left[ \sum_{k=2}^4 (-1)^k \frac{\vartheta_1(0)|it\tau\rangle}{\vartheta_k(0)|it\tau\rangle} \prod_{N=0}^4 \vartheta_k(\varphi_N|i\tau) + \prod_{N=0}^4 \vartheta_1(\varphi_N|i\tau) \right]$$

or, equivalently, after the modular transformation ($t = 1/\tau$)

$$Z = \frac{1}{2} c_1 \int_0^\infty dt \prod_{M=0}^4 f_M \left[ it \sum_{k=2}^4 (-1)^k \frac{\vartheta_1(0)|it\rangle}{\vartheta_k(0)|it\rangle} \prod_{N=0}^4 \vartheta_k(it\varphi_N|it) + \prod_{N=0}^4 \vartheta_1(it\varphi_N|it) \right]$$

$$= \frac{1}{2} i c_1 \int_0^\infty dt \prod_{M=0}^4 f_M \left[ \frac{\vartheta_1(0)|it\rangle}{\vartheta_1(0)|it\rangle} \prod_{N=0}^4 \frac{\vartheta_2(it\varphi_N|it)}{\vartheta_1(it\varphi_N|it)} - \frac{\vartheta_1(0)|it\rangle}{\vartheta_1(0)|it\rangle} \prod_{N=0}^4 \frac{\vartheta_3(it\varphi_N|it)}{\vartheta_1(it\varphi_N|it)} + \frac{\vartheta_1(0)|it\rangle}{\vartheta_1(0)|it\rangle} \prod_{N=0}^4 \frac{\vartheta_4(it\varphi_N|it)}{\vartheta_1(it\varphi_N|it)} \right] + \frac{1}{2} c_1 \int_0^\infty dt \prod_{M=0}^4 f_M . \quad (2.41)$$

The $D = 10$ expressions (2.40) and (2.41) which are symmetric in all five field strength eigenvalues reduce to (2.37) and (2.38) in the limit $f_0^{(1,2)} \to 0$, $\varphi_0 \equiv \frac{1}{\pi} (\arctan f_0^{(1)} + \arctan f_0^{(2)}) \to 0$. Some of the properties of the partition function (2.41) will be studied in sections 2.3 and 2.4 below. In particular, it has the correct $D = 10$ SYM ($\alpha' \to 0$) limit.

\[16\] In the case of Minkowski signature there is another irreducible form of constant $F_{\mu\nu}$ which is the direct superposition of the block-diagonal magnetic field strength $F_{pq}$ ($p,q = 5,\ldots,9$) and the 4-dimensional ‘plane-wave’ field $F_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3$) with orthogonal electric and magnetic components, e.g., $F_{+3} = 0$, i.e. $F_{93} = F_{13} = \text{const}$. The open bosonic string spectrum in such background was studied in [40]. Since the non-trivial 4-dimensional part of this background preserves supersymmetry, the corresponding superstring partition function vanishes. The bosonic string partition function is also trivial since all Lorentz-invariant scalars vanish when evaluated on the plane-wave background $F_{\alpha\beta}$. 
The special case of the superstring partition function (2.38), (2.41) when only one (electric, \( f_0 = iE \)) gauge field component is non-vanishing was first obtained in [27] (instead of computing the determinants directly with field-independent boundary conditions, the authors of [27] followed the equivalent procedure of solving the string equations with field-dependent boundary conditions [28, 29, 31], i.e. (2.6) and \( \psi_R^\mu - \psi_L^\mu = F_{\mu\nu}(\psi_R^\nu + \psi_L^\nu) \) at \( \sigma = 0 \), and \( \psi_R^\mu \pm \psi_L^\mu = F_{\mu\nu}(\psi_R^\nu \pm \psi_L^\nu) \) at \( \sigma = \pi \) and explicitly determining the corresponding string spectrum). The resulting expression was applied in D-brane context in [3] and was extended to the case of four or less non-vanishing eigenvalues of \( F_{\mu\nu} \) in [31] (ref. [31] gave also partial \( f_0 << f_I \) result for the general five-eigenvalue case). It is easy to check that the expressions in [27, 2, 5, 31] are indeed the appropriate special cases of (2.38).

### 2.3. D-brane applications and some properties of \( Z \)

The partition function (2.33), (2.37), (2.41) has several applications. When the boundary background fields are equal \( F^{(1)} = F^{(2)} = F \) it may be interpreted as a one-loop contribution to the tension of D9-brane with some background field distribution \( F \). Similar expressions are found for other Dp-branes by setting some field components to zero and adding extra factor \( s^{(9-p)/2} \) in the integration measure (reflecting the Dirichlet nature of \( 9-p \) transverse directions). If instead one introduces the open string mass factor \( \exp(-M^2 t) \), with \( M^2 = \frac{v^2}{2\pi\sigma} \) representing the separation \( r \) between parallel D-branes [3], one finds the potential between two Dp-branes with generic backgrounds \( F^{(1)} \) and \( F^{(2)} \) on each brane (keeping the transverse electric field component non-vanishing allows also to describe the potential between moving branes [3]). Various other cases are obtained as special ones by sending some of the field components \( f_I^{(r)} \) to zero or infinity.

For example, for two parallel D8-branes one finds

\[
Z = c_2 \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{r^2}{4\pi\sigma t}} \prod_{I=1}^4 f_I \frac{\vartheta_1(it\varphi_I|it)}{\vartheta_1(it\varphi_I|it)} ,
\]

where \( c_2 = \frac{2V_6}{(2\pi^2\sigma')^{1/2}(2\pi\alpha')^4} \) and \( f_I, \varphi_I, \varphi_I \) are defined in (2.26), (2.27), (2.28). If \( f_I^{(2)} = 0 \) while \( f_I^{(1)} \) is generic we get the potential between ‘pure’ D8-brane and a non-marginal bound state \( 8+6+...+0 \) of branes. It is clear from (2.1), (2.2) or from the form of the boundary conditions (2.6), that sending the boundary components \( f_I^{(2)} \) to infinity is equivalent to changing from Neumann to Dirichlet conditions in the two directions \( x^{2I-1}, x^{2I} \) on the second brane. This then gives the potential between D8-brane and D6-brane (or between D8-brane and D0-brane if \( f_I^{(2)} \to \infty \) for all \( I = 1, 2, 3, 4 \)) [3]. The limit \( f_I^{(r)} \to \infty \) on both boundaries corresponds to performing T-duality in the \( 2I - 1, 2I \) directions and

---

\[17\] For \( f_I^{(2)} = 0 \) one has \( \varphi_I = \varphi_I^{(1)} \), while for \( f_I^{(2)} \to \infty \) (2.22) implies that \( \tan \pi \varphi = -1/f_I^{(1)} \), i.e. \( \varphi_I = \varphi_I^{(1)} - \frac{1}{2} \), so that one can use \( \vartheta_1(\varphi - \frac{1}{2}|i\tau) = -\vartheta_2(\varphi|i\tau), \vartheta_3(\varphi - \frac{1}{2}|i\tau) = \vartheta_4(\varphi|i\tau). \)
thus gives potential between two D6-branes (or D7-branes if the starting point is D9-brane expression), etc.

The last zero-mode term in (2.41) proportional to \( \prod_{M=0}^{4} f_M = \sqrt{\det F_{\mu \nu}} \) reduces to the last term in (euclidean) D7-brane analogue of (2.38) if one sends \( f_1 (f_0) \) to infinity which is equivalent to performing T-duality transformation along the two orthogonal directions.\(^{18}\) The analogs of the last term in (2.41) (which is absent in the SYM theory limit) appear also in the case of lower-dimensional D-branes and may be related to the top ‘Chern-Simons’ terms which multiply the RR fields in D-brane actions \(^{23,32}\).

The superstring partition function (2.10),(2.35),(2.41) vanishes not only in the zero-field limit but also in the ‘neutral string’ \( F \equiv F^{(1)} + F^{(2)} = 0 \) limit (\( \hat{t}_I = 0 \) implies \( \varphi_I = 0 \) and \( \hat{\varphi}_I = 0 \), see (2.20),(2.23))\(^{19}\). In D-brane context this corresponds to the vanishing of the potential between two parallel Dp-branes with the same field backgrounds (i.e. the same extra RR charge distributions) or the vanishing of potential between two parallel Dp-branes moving with the same velocities (equal to electric fields after T-duality).

As follows from (2.10),(2.16), the ‘magnetic’ partition function (2.35) vanishes when any of the parameters \( \hat{\varphi}_I \) is zero, i.e. for the gauge field backgrounds which have residual supersymmetry from string-theory point of view, cf. (2.8),(2.33)\(^{26,29}\) (see also, e.g., \(^{12,14}\)). Note that, in general, the condition that some \( \hat{\varphi}_I = \frac{1}{\pi} \text{sgn} f^{(1)}_I + \text{sgn} f^{(2)}_I = 0 \) is not equivalent to the vanishing of some of \( \hat{t}_I = P_{I,J} f_{IJ} \), i.e. to the SYM theory supersymmetry condition (when \( \hat{t}_I = 0 \) the corresponding matrix \( \hat{F}_{ab} = \frac{1}{4} \gamma^{ij} F_{ij} \) has zero modes).\(^{22}\) For \( F^{(1)} = F^{(2)} \) this implies, in particular, the vanishing of the one-loop contribution to the tension of a supersymmetric bound state of Dp-branes, while for generic \( F^{(1)} \) and \( F^{(2)} \) this implies the vanishing of the potential for a supersymmetric configuration of Dp-branes.

\(^{18}\) In this limit one should fix the product of the field component with the volume in the orthogonal directions so that the finite coefficient in front of the (euclidean) D7-brane analogue of (2.38) is proportional to \( V_8 \). Analogous considerations can be used to relate D9-brane action to D8-brane one. Similar term (which is the contribution of \( R^{(−1)} \) sector) is present in the interaction potential between D0-brane and D8-brane \(^{13,31,37,38}\) and the related case of interaction between D4-branes at angles \(^{32}\).

\(^{19}\) This is in contrast to what happens in the neutral bosonic string case where \( Z(F) = \det (\delta_{ij} + F_{ij}) Z(0) \)\(^{28}\). The one-loop \( Z \) in the open bosonic string has non-trivial dependence on the fields if \( F^{(1)} \neq - F^{(2)} \). \( Z \) with \( F^{(1)} = F^{(2)} \) was found in \(^{13}\); the case of \( F^{(2)} = 0 \) and its relation to YM effective action in the \( \alpha' \rightarrow 0 \) limit was studied in \(^{15}\) (some related computations of bosonic string partition function in external electromagnetic field and at finite temperature appeared in \(^{13,14}\)).

\(^{20}\) The two conditions are equivalent, however, in the most interesting cases, e.g., for the 4d instanton configuration (\( f_1 = \pm f_2, f_3 = f_4 = 0 \) implies \( \varphi_1 = \varphi_2, \varphi_3 = \varphi_4 \), and thus \( \hat{\varphi}_1 = \hat{\varphi}_2 = 0 \).
Let us now discuss the $t \to 0$ (open string channel) and $t \to \infty$ (closed string channel) behaviour of the integral in $Z$ (2.41) for arbitrary background fields. Let us first consider the ‘magnetic’ case when $Z$ is given by (2.35), (2.38). In the $t \to 0$ ($\tau \to \infty$) region it is clear from the representation (2.29) and (2.24) that the integral in (2.35) reduces to

$$Z \to \int_{t=0} dt \prod_{I=1}^{4} \hat{f}_I \frac{\sin \pi \hat{\varphi}_I}{\sin \pi \varphi_I}.$$  \tag{2.43}$$

The weak-field expansion\footnote{In general, the the weak-field limit $Z$ (2.23) is proportional to $\prod_{I=1}^{4} \hat{f}_I$, i.e. to $\sqrt{\det \hat{F}_{ab}}$.} of this expression starts with $\prod_{I=1}^{4} \hat{f}_I$, i.e. $Z$ contains $O(F^4)$ (quadratic) UV divergence (cf. (2.10)) which in type I string theory is canceled as in [24] after adding the Möbius strip contribution [19]. In Dp-brane context, the measure gets extra factor of $s^{1/2}$ for each of the $9-p$ Dirichlet directions so that the integral becomes convergent at $t \to 0$ for $p<7$.

For large $t$ (2.35) and (2.24) lead to a similar expression but now with $\varphi_I \to i t \varphi_I$:

$$Z \to \int_{t=0}^{t \to \infty} dt \prod_{I=1}^{4} \hat{f}_I \frac{\sinh \pi t \varphi_I}{\sinh \pi t \varphi_I}.$$  \tag{2.44}$$

This is convergent when, e.g., all $f_I^{(r)}$ are such that all $\varphi_I$ and $\hat{\varphi}_I$ have the same sign: the divergence cancels out because of $\sum_{I=1}^{4} (\hat{\varphi}_I - \varphi_I) = 0$ (cf. (2.13)). If only one (e.g., the first) component of $f_I$ and thus of $\varphi_I$ is non-vanishing, one finds that the integral is divergent at large $t$. Introducing the IR cutoff factor $e^{-M^2 t}$ we get

$$Z \to f_1 \int_{t=0}^{t \to \infty} dt \frac{e^{-M^2 t}}{t^5} \frac{\sinh \frac{1}{2} \pi t \varphi_1}{\sinh \pi t \varphi_1} \sim f_1 \int_{t=0}^{t \to \infty} dt \frac{e^{-M^2 t}}{t^5} e^{\pi |\varphi_1| s} + \ldots .$$  \tag{2.45}$$

The integral is convergent if $M^2 > \pi |\varphi_1|$ (similar remark was made in D-brane context in [26]). For $M = 0$ the resulting divergence is the string analogue of the well-known IR instability [16] of the YM theory in magnetic backgrounds (which remains also in the super YM theory [17]).

In general, starting with the full $D=10$ partition function (2.41) we find in the $t \to \infty$ limit\footnote{Note that the fact that the leading $F^4/M^{7-p}$ term (in the integral with factor $e^{-M^2 t}$ included) originates from the zero-mode factor explains its ‘universality’, i.e. its trivial scale-factor dependence on $\alpha'$ or $r$ (cf. [6,3]).}:

$$Z \to \frac{1}{2c_1} \int_{t=0}^{t \to \infty} dt \frac{e^{-\frac{M^2}{4}} \prod_{M=0}^{4} f_M}{16 \prod_{M=0}^{4} e^{-\frac{M^2}{4}} \sinh \pi t \varphi_M} \left[ 16 e^{\frac{\pi t}{4}} \prod_{N=0}^{4} e^{-\frac{\pi t}{4}} \cosh \pi t \varphi_N \right].$$

Note that for $t \to \infty$

$\vartheta_1(iz|it) \to 2ie^{-\frac{\pi t}{4}} \sinh \pi z$, $\vartheta_2(iz|it) \to 2e^{-\frac{\pi t}{4}} \cosh \pi z$, $\vartheta_3(iz|it) \to 1 + 2e^{-\pi t} \cosh 2\pi z$, $\vartheta_4(iz|it) \to 1 - 2e^{-\pi t} \cosh 2\pi z$.\footnote{We use that for $t \to \infty$ $\vartheta_1(iz|it) \to 2ie^{-\frac{\pi t}{4}} \sinh \pi z$,
$\vartheta_2(iz|it) \to 2e^{-\frac{\pi t}{4}} \cosh \pi z$, $\vartheta_3(iz|it) \to 1 + 2e^{-\pi t} \cosh 2\pi z$, $\vartheta_4(iz|it) \to 1 - 2e^{-\pi t} \cosh 2\pi z$.}
\[
- \prod_{N=0}^{4} \left( \frac{1 + 2e^{-\pi t} \cosh 2\pi t \varphi_N}{1 + 2e^{-\pi t}} \right) + \prod_{N=0}^{4} \left( \frac{1 - 2e^{-\pi t} \cosh 2\pi t \varphi_N}{1 - 2e^{-\pi t}} \right) + \ldots
\]
\[
\to \frac{1}{8} c_1 \int_{t \to \infty} dt \frac{\prod_{M=0}^{4} f_{M}}{t \prod_{M=0}^{4} \sinh \pi t \varphi_M} \left[ 4 \prod_{N=0}^{4} \cosh \pi t \varphi_N - \sum_{N=0}^{4} \cosh 2\pi t \varphi_N + 1 \right] + \ldots, \quad (2.46)
\]
were we kept only the leading term in each \( \vartheta \)-function factor and dropped the last term in (2.41) which is convergent at \( t \to \infty \). In the euclidean case with all \( \varphi_M \) being real we find that the condition of convergence at \( t \to \infty \) is determined by the term \( \prod_{M=0}^{4} \sinh \pi t \varphi_M \). Since in this case \( \tanh \pi \varphi_{(r)} = E^{(r)} \), there is the restriction \( E^{(r)} \leq 1 \) (at the critical value of the electric field \( \varphi_{E}^{(r)} \) goes to infinity and the production rate diverges [27]).

2.4. Super Yang-Mills theory limit

Let us now show that the \( \alpha' \to 0 \) limit of the string theory partition function (2.27), (2.29), (2.41) reproduces the \( D = 10 \) super Yang-Mills one-loop effective action in constant abelian background. As discussed in [33], the \( D = 10 \) field-theory limit corresponds to \( \alpha' \to 0 \) for fixed UV cutoff at \( t = 0 \), i.e. to \( t \to \infty \) (or \( \tau \to 0 \)). We shall restore the factors \( 2\pi \alpha' \) in \( F_{ij} \), i.e. \( f_M \to 2\pi \alpha' f_M \), and define \( t = 2\pi \alpha' \) which has dimension (length)\(^2\) so that \( 2\pi \alpha' f_M t = f_M t \) is dimensionless. The parameter \( t \) is the (open string) field-theory proper-time parameter, \( \epsilon^2 \leq t < \infty \), where \( \epsilon \) is the field-theory UV cutoff. Taking \( \alpha' \to 0 \) for fixed \( t \) and \( \epsilon \) we find that (see (2.21)) \( \varphi_M \to 2\alpha' f_M \), \( \hat{\varphi}_M \to 2\alpha' f_M \), so that the limiting form of \( Z \) depends only on the sum of the two boundary fields \( f_M^{(1)} + f_M^{(2)} = f_M \).

Since in this limit \( t = \frac{1}{2\pi \alpha'} \to \infty \) while \( 2\pi \alpha' f_M t = f_M t \to \pi \varphi_M t \) is fixed, the resulting limiting form of \( Z \) (2.41) is essentially given by (2.46) (with \( \pi \varphi_M \to f_M \) and extra overall factor of \( (2\pi \alpha')^5 \) coming out of \( \prod_{M=0}^{4} f_M \)). This is to be compared with the \( D = 10 \) SYM analogue of the one-loop background field effective action. In the case of the \( SU(2) \) theory with (euclidean) \( U(1) \) \( D = 10 \) background \( F_{\mu
u} = \frac{1}{2} \sigma_3 F_{\mu
u} \), where \( F_{\mu\nu} \) has block-diagonal form with 5 non-zero entries \( f_M \), one finds for the one-loop SYM effective action [14, 40]

\[
\Gamma(F) = -\frac{2V_{10}}{(4\pi)^5} \int_{\epsilon^2}^{\infty} dt \prod_{M=0}^{4} \frac{f_M}{\sinh f_M t} \left( \sum_{N=0}^{4} \cosh 2f_N t \right. - 1 - 4 \prod_{N=0}^{4} \cosh f_N t \right). \quad (2.47)
\]

\footnote{In general, one should also fix \( g_{\varphi M}^2 = g_0 \alpha'(D-4)/2 \) in the limit. There is no cutoff dependence in \( D < 8 \) [49].}

\footnote{Similar expressions in \( D < 10 \) SYM theories are found after one corrects the power of \( t \) in the measure \( \left( \frac{1}{t} \to \frac{1}{t + \frac{1}{2} D} \right) \) to take into account the effect of dimensional reduction (or, equivalently, sets the components \( f_M \) corresponding to ‘extra’ \( D - 10 \) dimensions to zero).}
This is in precise agreement with the expression that follows from (2.41), (2.46) with the normalisation of the string-theory result fixed as

$$c_1 = \frac{V_{10}}{2(2\pi)^5(2\pi\alpha')^5}.$$ (2.48)

Note that the last $\sqrt{\text{det} F_{\mu\nu}}$ term in (2.41) has no counterpart in the SYM theory: it disappears in the $\alpha' \to 0$ limit (it gets one extra power of $\alpha'$ after $t \to \frac{t}{2\pi\alpha'}$ and $f_M \to 2\pi\alpha' f_M$).

Let us consider explicitly the purely magnetic case ($f_0 = 0$). Both expressions (2.27) and (2.29) for the ‘magnetic’ GS partition function have the same field-theory limit (both $\tilde{\phi}_I$ (2.21) and $\hat{\phi}_I$ (2.23) are equal to $2\alpha' \hat{f}_I$ in the $\alpha' \to 0$ limit). Since $\vartheta_1(t|\frac{it}{2\pi\alpha'})_{\alpha' \to 0} \to 2i e^{-\frac{t}{2\pi\alpha'}} \sinh f_I t$, we get that in the limit $\alpha' \to 0$ $Z$ becomes

$$Z = c_0 \int_{e^2}^{\infty} \frac{dt}{t^2} \prod_{I=1}^{4} f_I \left( \prod_{J=1}^{4} \frac{\sinh \hat{f}_J t}{\sinh f_J t} - \frac{1}{2} \right), \quad c_0 = (2\pi\alpha')^5 c_1.$$ (2.49)

The term $-\frac{1}{2}$ in the bracket is introduced to subtract the last $\sqrt{\text{det} F_{ij}}$ term in (2.38) which has a non-vanishing $\alpha' \to 0$ limit but is not reproduced by the SYM theory (as was mentioned above, it is absent in the $f_0 \to 0$ limit of the covariant D9-brane NSR expression (2.41) but appears in related euclidean D8-brane expression). Like (2.47), this integral has the well-known UV divergence of SYM theory (quadratic in $D = 10$ and logarithmic in $D = 8$) which is proportional to (cf. (2.16)) $F^4 - \frac{1}{4}(F^2)^2$ [7,35]. Eq. (2.49) is indeed equivalent to the $f_0 = 0$ limit of (2.47)

$$\Gamma(F) = -2V_{10} \int_{e^2}^{\infty} \frac{dt}{t^2} \prod_{I=1}^{4} \frac{f_I}{\sinh f_I t} \left( \sum_{J=1}^{4} \cosh 2f_J t - 4 \prod_{J=1}^{4} \cosh f_J t \right),$$ (2.50)

since, as follows from (2.13),

$$8 \prod_{I=1}^{4} \sinh \hat{f}_I t - 4 \prod_{I=1}^{4} \sinh f_I t = - \sum_{I=1}^{4} \cosh 2f_I t + 4 \prod_{I=1}^{4} \cosh f_I t.$$ (2.51)

We shall later use a similar identity (obtained from (2.51) by $f_k \to f_k + \frac{i\pi}{2}$)

$$8 \prod_{I=1}^{4} \cosh \hat{f}_I t - 4 \prod_{I=1}^{4} \sinh f_I t = \sum_{I=1}^{4} \cosh 2f_I t + 4 \prod_{I=1}^{4} \cosh f_I t.$$ (2.52)

Eq. (2.49) gives a useful expression for the 1-loop effective action in $D = 10$ SYM theory in purely magnetic (or ‘8-dimensional’) background. In particular, it is clear from it that the effective action becomes simply proportional to $\prod_{I=1}^{4} f_I$ for all supersymmetric
abelian gauge field backgrounds for which $\hat{F}_{ab} = \frac{1}{4} \gamma_{ij}^{ab} F_{ij}$ has zero modes, i.e. for which some of $\hat{f}_I$ vanish (in particular, (2.49) vanishes for the $D = 4$ instanton background, $f_1 = \pm f_2$, but, in contrast to (2.35), (2.38), is non-vanishing for its ‘$D = 8$ generalisation’, $f_1 = \pm f_2$, $f_3 = \mp f_4$).

The special case of (2.49) with $f_3 = f_4 = 0$, $f_2 \to f_0$ (i.e. $\hat{f}_1 = -\hat{f}_2 = \frac{1}{2} (f_0 - f_1)$, $\hat{f}_3 = \hat{f}_4 = \frac{1}{2} (f_0 + f_1)$) is directly related to the effective action of the four-dimensional $N = 4$ $SU(2)$ SYM theory in constant $U(1)$ background [17]

$$\Gamma(F) = -\frac{V_4}{\pi^2} \int_0^\infty \frac{dt}{t} f_0 f_1 \frac{\sinh^2 \frac{f_0 - f_1}{2} t}{\sinh f_0 t} \frac{\sinh^2 \frac{f_0 + f_1}{2} t}{\sinh f_1 t}.$$  (2.53)

Here $f_0$ stands for the euclidean analogue of the $D = 4$ electric field component. This integral is convergent at $t \to 0$ but has the same magnetic IR instability at $t \to \infty$ as in YM theory.

3. FINITE TEMPERATURE CASE

3.1. Free energy of Super Yang-Mills theory

It is useful first to recall that in a $D = p + 1$ dimensional field theory with bosonic and fermionic degrees of freedom with mass operators $\hat{M}_B$ and $\hat{M}_F$ (which may depend on a background field) the proper-time representation for the free energy $F$ has the form

$$Z = \beta F = -\frac{V_p \beta}{2(4\pi)^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} \left[ \vartheta_3(0) \frac{i \beta^2}{4\pi t} \left( \text{Tr} e^{-t \hat{M}_B^2} - \text{Tr} e^{-t \hat{M}_F^2} \right) \right].$$  (3.1)

The inverse temperature $\beta$ is the period of the euclidean time direction. Using that

$$\vartheta_3(0|iz) \pm \vartheta_4(0|iz) = \sum_{n=-\infty}^\infty [1 \pm (-1)^n] e^{-\pi z n^2} = 2 \vartheta_{3,2}(0|4iz),$$  (3.2)

we can represent (3.1) in the form

$$Z = -\frac{V_p \beta}{2(4\pi)^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{\frac{p+3}{2}}} \left[ \vartheta_3(0) \frac{i \beta^2}{\pi t} \left( \text{Tr} e^{-t \hat{M}_B^2} - \text{Tr} e^{-t \hat{M}_F^2} \right) \right]$$

$$+ \vartheta_2(0) \frac{i \beta^2}{\pi t} \left( \text{Tr} e^{-t \hat{M}_B^2} + \text{Tr} e^{-t \hat{M}_F^2} \right),$$

or, equivalently, as

$$Z = -\frac{V_p}{4(4\pi)^{p/2}} \int_0^\infty \frac{dt}{t^\frac{p+1}{2}} \left[ \vartheta_3(0) \frac{i \pi t}{\beta^2} \left( \text{Tr} e^{-t \hat{M}_B^2} - \text{Tr} e^{-t \hat{M}_F^2} \right) \right].$$  (3.4)
In the zero-temperature limit \( \beta \to \infty \) only the first term in (3.4) survives \( (\vartheta_2(0|i\beta^2/t) \to 0, \vartheta_{3,4}(0|i\beta^2/t) \to 1) \), i.e. (3.4) reduces to the standard integral of \( \text{Tr} e^{-t\hat{M}_B^2} - \text{Tr} e^{-t\hat{M}_F^2} \). In the special case of free supersymmetric theory with equal masses of bosons and fermions \( \hat{M}_B = \hat{M}_F = \hat{M} \) this becomes

\[
Z = -\frac{V_{9\beta}}{(4\pi)^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{p+2}} \vartheta_2(0|i\beta^2/t) \text{Tr} e^{-t\hat{M}^2}.
\]  

(3.5)

If the fields are massless with total \( N \) of bosonic degrees of freedom (\( N = 8N^2 \) for the \( U(N) \) SYM theory)

\[
Z(\beta) = -N\kappa_p V_{9\beta}^{-p}, \quad \kappa_p = [1 + (1 - 2^{-p})](2\pi)^{-p}\omega_{p-1}(p-1)! \zeta(p+1),
\]

(3.6)

where \( \omega_{p-1} = \frac{2\pi^2}{\Gamma(\frac{p}{2})} \).

The one-loop \( SU(2) \) SYM effective action (2.47),(2.50) is given by the sum of contributions of bosonic and fermionic determinants. Observing that the fermionic contribution in (2.50),(2.47) is represented by the term proportional to \( \prod_J \cosh f_J t \), we find the following finite-temperature generalisation of (2.50), i.e. the one-loop free energy of the \( D = 10 \) SYM theory in a magnetic background (cf. (3.1))

\[
Z(\beta, F) = \beta F(\beta, F) = -\frac{2V_{9\beta}}{(4\pi)^5} \int_{e^2}^\infty \frac{dt}{t^2} \prod_{I=1}^{4} \frac{f_I}{\sinh f_I t} \times \left[ \vartheta_3(0|i\beta^2/4\pi t) \sum_{J=1}^{4} \cosh 2f_J t - \vartheta_4(0|i\beta^2/4\pi t) \sum_{J=1}^{4} \cosh f_J t \right].
\]

(3.7)

Using (2.11),(2.12),(3.2) we can rewrite (3.7) in the form of (3.4) (cf. (2.49))

\[
Z(\beta, F) = \frac{V_{9\beta}}{2(2\pi)^5} \int_{e^2}^\infty \frac{dt}{t^2} \prod_{I=1}^{4} \frac{f_I}{\sinh f_I t} \left[ \vartheta_3(0|i\beta^2/4\pi t) \left( \prod_{J=1}^{4} \frac{\sinh \hat{f}_J t}{\sinh f_J t} - \frac{1}{2} \right) \right. \\
- \left. \vartheta_2(0|i\beta^2/\pi t) \left( \prod_{J=1}^{4} \frac{\cosh \hat{f}_J t}{\sinh f_J t} - \frac{1}{2} \right) \right],
\]

(3.8)

or

\[
Z(\beta, F) = \frac{V_{9\beta}}{2(2\pi)^5} \int_{e^2}^\infty \frac{dt}{t^2} \left[ \vartheta_3(0|i\beta^2/\pi t) \prod_{I=1}^{4} \frac{\sinh \hat{f}_I t}{\sinh f_I t} - \vartheta_2(0|i\beta^2/\pi t) \prod_{I=1}^{4} \frac{\cosh \hat{f}_I t}{\sinh f_I t} \\
- \left. \vartheta_2(0|i\beta^2/\pi t) \prod_{I=1}^{4} \frac{\cosh \hat{f}_I t}{\sinh f_I t} \right].
\]

(3.8)
\[-\frac{1}{2} \vartheta_4(0) \frac{i \beta^2}{4 \pi t} \prod_{l=1}^{4} f_I \] (3.9)

In the zero-temperature limit $\beta \to \infty$ the second term vanishes and we get back to (2.49), (2.50). In the zero-field limit $f_I \to 0$ it is the second $\vartheta_2$-term that gives a non-vanishing contribution which is equal to the free energy of the massless $D = 10$ SYM modes (cf. (3.6))

$$Z(\beta, 0) = -\frac{V_9 \beta}{2(2\pi)^5} \int_{0}^{\infty} \frac{dt}{t^6} \vartheta_2(0) \frac{i \beta^2}{\pi t} = -32\kappa_0 V_9 \beta^{-9} \, .$$ (3.10)

The presence of the second $\vartheta_2$-term in (3.8) with $\cosh \hat{f}_I t$ factors instead of $\sinh \hat{f}_I t$ in the first $\vartheta_3$-term can be related to antiperiodicity of the fermionic fields in euclidean time direction. That finite temperature explicitly breaks supersymmetry is reflected in the fact that in contrast to (2.49), the finite-temperature expression (3.7) is non-trivial on supersymmetric (e.g., self-dual) configurations with $\hat{f}_I = 0$.

In contrast to (2.49) and the integral of the first $\vartheta_3$-term in (3.7), the integral of the second $\vartheta_2$-term in (3.7) is convergent for $t \to 0$ (the finite temperature provides an effective UV cutoff in this term since $\vartheta_2(0) \frac{i \beta^2}{\pi t} \to 2e^{-\frac{a^2}{4\pi}}$). Rescaling $t$, we can represent (3.8) as

$$Z(\beta, F) = \beta^{-p} H(\beta^2 F) \, ,$$ (3.11)

where we have added the factor $t^{(9-p)/2}$ to the measure in (3.8) to describe the case of SYM theory in $D = p + 1$ dimensions and defined (cf. (2.16))

$$G_3(tF) \equiv \prod_{I=1}^{4} f_I \frac{\sinh \hat{f}_I t}{\sinh f_I t} = \frac{1}{16} t^4 \left[ -8 \prod_{I=1}^{4} f_I + 2 \sum_{I=1}^{4} f_I^4 - \left( \prod_{I=1}^{4} f_I^2 \right)^2 \right] + O(t^6 f^6) \, ,$$

$$G_2(tF) \equiv \prod_{I=1}^{4} f_I \frac{\cosh \hat{f}_I t}{\sinh f_I t} = 1 + \frac{1}{3} t^2 \sum_{I=1}^{4} f_I^2 + \frac{1}{720} t^4 \left[ 39 \sum_{I=1}^{4} f_I^4 - 10 \left( \sum_{I=1}^{4} f_I^2 \right)^2 \right] + O(t^6 f^6) \, ,$$

$$G_4(tF) \equiv \frac{1}{2} t^4 \prod_{I=1}^{4} f_I \, .$$

The weak-field expansion of $Z$ thus has the following structure

$$Z \sim b_1 \beta^{-p} + b_2 \beta^{-p+4} F^2 + (b_0 + b_3 \beta^{-p+8}) F^4 + \ldots \, ,$$ (3.12)

where $b_0 F^4$ stands for the $F^4 - \frac{1}{4}(F^2)^2$ terms in the zero-temperature SYM effective action ($b_0 \sim e^{8-p}$).

For $t \to \infty$ in (3.8) one finds the same IR singularity as in the zero-temperature case. It is known that finite temperature does not eliminate the magnetic instability of the YM theory [51]. Since this instability has its origin in the vector-field sector, it is not affected by the presence of fermions (irrespective of the choice of their boundary conditions).
3.2. String theory partition function in magnetic background

Below we shall consider an ensemble of open superstrings at finite temperature in an external magnetic field. Our aim will be to find the finite temperature analogs of the partition functions (2.27),(2.29),(2.38). Taking the $\alpha' \to 0$ limit, we will reproduce the free energy (3.8) of the $N = 4$, $D = 4$ SYM theory in a constant abelian magnetic field.

As in the zero background field case (cf. [52,53]) we shall obtain the finite temperature string partition function

$$Z(\beta, F) = \beta \mathcal{F}(\beta, F) = -\ln \hat{Z}(\beta, F),$$

where $\beta$ is the inverse temperature, $\mathcal{F}$ is the free energy and $\hat{Z}$ is the canonical partition function of string field theory, using the l.c. GS path integral formalism. Compared to the zero-temperature case of section 2 now the euclidean time coordinate has period $\beta$ and thus includes winding modes in the angular coordinate $\psi$ of the annulus. The fermionic coordinate $S$ in (2.2) may be either periodic or antiperiodic in $\psi$ (which effectively plays the role of the euclidean time coordinate). In fact, both sectors should be included in the GS partition function with appropriate temperature factors (this corresponds to taking into account different statistics in the cases of space-time bosonic or fermionic states propagating in the loop).

The periodic sector contribution is the obvious generalisation of (2.35)

$$Z^+(\beta, F) = a_1 \beta \int_0^\infty \frac{dt}{t^2} \theta_3(0|\frac{i\beta^2}{2\pi^2\alpha'}t) \prod_{I=1}^{4} \frac{\partial_1(it\varphi_I|it)}{\vartheta_1(it|it)}.$$

(3.13)

It vanishes in the absence of the external field but provides correspondence with the zero-temperature expression for the string partition function (2.33) in the limit $\beta \to \infty$.

In the antiperiodic sector there is no fermionic zero-mode factor present in (2.10),(3.13), and the space-time supersymmetry is explicitly broken by the temperature. The contribution $Z^-$ of the antiperiodic sector can be written as (cf. (2.10))

$$Z^-(\beta, F) = -a_1 \int_0^1 \frac{dq}{q} Z(\beta, q) \prod_{I=1}^{4} Z_I^-(F^{(1)}, F^{(2)}; q),$$

(3.14)

where $Z_I^-$ has the same form as in (2.11) (and is equal to 1 for $F^{(r)} = 0$), while the temperature-dependent factor $Z(\beta, q)$ is the same as in the absence of the magnetic field [53,54]

$$Z(\beta, q) = \pi^4 \beta \partial_2(0|\frac{i\beta^2}{2\pi^2\alpha'}\tau) \left[\frac{\partial_1(0|i\tau)}{\vartheta_1'(0|i\tau)}\right]^4, \quad q = e^{-\pi \tau},$$

(3.15)
where \( \vartheta'_0(\imath \tau) = 2\pi \eta^3(\imath \tau) \). Equivalent forms of \( Z(\beta, q) \) are found using that

\[
\vartheta_2\left(\frac{i v}{\tau}, \frac{1}{\tau}\right) = \sqrt{\tau} \ e^{\pi i z^2} \ \vartheta_4(v|\imath \tau), \quad \frac{\vartheta_2\left(\frac{i v}{\tau}, \frac{1}{\tau}\right)}{\vartheta_1\left(\frac{i v}{\tau}, \frac{1}{\tau}\right)} = -i \frac{\vartheta_4(v|\imath \tau)}{\vartheta_1(v|\imath \tau)}.
\]

To compute \( Z^\pm \) we note that the fermionic Green function \( \hat{K} \) in \( Z_I^- \) has the same form as (2.12) but now with the sum going over half-integers \( r = 1/2, 3/2, \ldots \), or, equivalently, over integers \( n = r + \frac{1}{2} \), but with \( A_n \to \hat{A}_n = \frac{1+q^{2n-1}}{1-q^{2n-1}} \). As a result, \( Z_I^- \) becomes (cf. (2.18))

\[
Z_I^- = \prod_{n=1}^{\infty} \frac{(1 - \hat{f}_I^{(1)}(1) \hat{f}_I^{(2)}(1))^2 + (\hat{f}_I^{(1)}(1) + \hat{f}_I^{(2)}(1))^2 \hat{A}_n^2}{(1 - f_I^{(1)}(1)^2 + (f_I^{(1)}(1) + f_I^{(2)}(1))^2 A_n^2},
\]

i.e. (cf. (2.19))

\[
Z_I^- = \left(1 + (f_I^{(1)}(1)^2)[1 + (f_I^{(2)}(1)^2)]\right)^{1/2} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})^2} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n-1} \cos 2\pi \varphi_I + q^{4n-2}}{1 - 2q^{2n} \cos 2\pi \varphi_I + q^{4n}}.
\]

Here we have used the fact that taking a constant out of the product over half-integers does not produce an overall factor, \( \prod_{r=1}^{\infty} c = c^{2\pi} \). Since

\[
\vartheta_4(\varphi|\imath \tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi N t^2 + 2\pi n \varphi} = \prod_{n=1}^{\infty} (1 - q^{2n}) (1 - 2q^{2n-1} \cos 2\pi \varphi + q^{4n-2}),
\]

we finally get (for the definitions of parameters see (2.20),(2.21),(2.26))

\[
Z_I^- = \frac{f_I}{\pi} \ \vartheta_4\left(0|\imath \tau\right) \ \vartheta_4\left(\varphi_I|\imath \tau\right) \ \vartheta_4\left(\varphi_I|\imath \tau\right) \ \vartheta_4\left(\varphi_I|\imath \tau\right).
\]

Combining this with (3.15) we find that \( Z^- \) (3.14) takes the following simple form (cf. (2.27))

\[
Z^- = -a_1 \beta \int_0^{\infty} d\tau \ \vartheta_2\left(0|\frac{\imath \beta^2 \tau}{2\pi^2 a'}\right) \prod_{I=1}^{4} f_I \ \vartheta_4\left(\varphi_I|\imath \tau\right) \ \vartheta_4\left(\varphi_I|\imath \tau\right).
\]

As in the zero-temperature case, to have the agreement with the NSR result one should make the replacement of \( \varphi_I \) (2.21) by \( \varphi \) (2.23) in the fermionic \( \vartheta_4 \)-contribution (cf. (2.29))

\[
Z^- = -a_1 \beta \int_0^{\infty} d\tau \ \vartheta_2\left(0|\frac{\imath \beta^2 \tau}{2\pi^2 a'}\right) \prod_{I=1}^{4} f_I \ \vartheta_4\left(\varphi_I|\imath \tau\right) \ \vartheta_4\left(\varphi_I|\imath \tau\right).
\]

\[\text{The infinite products are to be regularised with the generalised \( \zeta \)-function} \ (n+a)^{-z} \text{ for which} \ \zeta(0,0) = -\frac{1}{12}, \text{ but} \ \zeta(0, \frac{1}{2}) = 0. \text{ The latter relation is also the reason why the fermionic contribution (of the only possible antiperiodic spinor on the disc) does not change the bosonic Born-Infeld expression for the tree-level string effective action} \ (19,53).\]
This can be put into the form of an integral over the open-string proper time variable \( t = 1/\tau \) as in (2.35)\(^{27}\):

\[
Z^- = -a_1 \beta \int_0^\infty \frac{dt}{t^2} \frac{i\beta^2}{2\pi^2 \alpha' t} \prod_{I=1}^4 f_I \frac{\vartheta_2(it\hat{\phi}_I|it)}{\vartheta_1(it\varphi_I|it)}.
\] (3.23)

\( Z^\prime \) vanishes in the zero-temperature limit and reduces to the free string partition function \([53]\) in the zero-field limit.

The final expression for the partition function which is the finite-temperature analogue of (2.35) is thus

\[
Z = Z^+ + Z^-
\]

\[
= a_1 \beta \int_0^\infty \frac{dt}{t^2} \left[ \vartheta_3(0) \frac{i\beta^2}{2\pi^2 \alpha' t} \prod_{I=1}^4 f_I \frac{\vartheta_1(it\hat{\phi}_I|it)}{\vartheta_1(it\varphi_I|it)} - \vartheta_2(0) \frac{i\beta^2}{2\pi^2 \alpha' t} \prod_{I=1}^4 f_I \frac{\vartheta_2(it\hat{\phi}_I|it)}{\vartheta_1(it\varphi_I|it)} \right].
\] (3.24)

The equivalent form of (3.24) is

\[
Z(\beta, F) = a_2 \int_0^\infty \frac{dt}{t^{3/2}} \left[ \vartheta_3(0) \frac{2\pi^2 \alpha' it}{\beta^2} \prod_{I=1}^4 f_I \frac{\vartheta_2(it\hat{\phi}_I|it)}{\vartheta_1(it\varphi_I|it)} - \vartheta_4(0) \frac{2\pi^2 \alpha' it}{\beta^2} \prod_{I=1}^4 f_I \frac{\vartheta_2(it\hat{\phi}_I|it)}{\vartheta_1(it\varphi_I|it)} \right],
\] (3.25)

where (cf. (2.48))

\[
a_2 = (2\pi^2 \alpha')^{1/2} a_1,
\]

\[
a_1 = \frac{N^2 V_9}{2(2\pi)^5 (2\pi \alpha')^5},
\] (3.26)

and \( N \) is the Chan-Paton number (number of D9-branes). Using a version of the Riemann identity (2.36) \( Z \) can be written also in the NSR form similar to (2.37),(2.38) (cf. \[56\]).

### 3.3. Some properties of finite temperature partition function

Starting with the string-theory partition function (3.24) and repeating the same steps as in section 2.4, i.e. restoring the \( 2\pi \alpha' \) factors in \( f_I \), defining \( t = 2\pi \alpha' t \) and taking the \( \alpha' \to 0 \) limit, one finds indeed the SYM partition function (3.9) (apart from the last \( \prod_{I=1}^4 f_I \) term the role of which in the zero-temperature field-theory expression is to cancel a similar product term coming out of the expansion of the first term in (3.9), see section 2).

\(^{27}\) The exponential factors appearing in the Jacobi transformation again cancel out since \( \sum_{I=1}^4 (\hat{\phi}_I^2 - \varphi_I^2) = 0 \).
In the case of the ‘neutral’ background ($f_I = f_I^{(1)} + f_I^{(2)} = 0$) the partition function $Z(\beta, F)$ is equal to $Z^−$ and reduces (as in the bosonic string theory) simply to its zero-field value $Z(\beta, 0) = −a_1 \int \frac{dq}{q} Z(\beta, q)$ multiplied by the overall factor $\det(\delta_{ij} + F_{ij}) = \prod_{I=1}^{4}(1 + f_I^2)$ (cf. (3.18)).

The behaviour of the integral (3.23) in the open-string UV region $t \to 0$ is determined by the $\tau \to \infty$ region of (3.22): since $\vartheta_4(\varphi_I | i\tau) \to 1$, $\vartheta_1(\varphi_I | i\tau) \to 2e^{-\frac{\pi i}{4\tau}} \sin \pi \varphi_I$, $\vartheta_2(0 | \frac{i\beta^2}{2\pi c_3}) \to 2e^{-\frac{\pi^2}{8\pi c_3}}$, we conclude that the integral is convergent at $t \to 0$ provided $\beta > \beta_c = 2\pi \sqrt{2}\alpha'$. Thus the presence of the magnetic field does not change the value of the Hagedorn temperature of the free open superstring gas.

The open-string IR behaviour $t \to \infty$ of the integral (3.23) is similar to that in the zero-temperature case (2.35),(2.44) discussed in sect. 2.2 (the temperature-dependent factors in (3.25) become trivial in this limit, $\vartheta_{3,4}(0 | \frac{2\pi^2 \alpha' \alpha}{\beta^4})$), so that the zero-temperature case (2.35),(2.44) discussed in section 2.3. For example, the finite-temperature analogue of the D8-brane corresponds to the gas of open strings with both ends attached to the same Dp-brane, so that $Z$ may be interpreted as determining the thermal self-energy of the Dp-brane (or correction to the tension of a Dp-brane in a thermal state). The temperature should correspond to the Hawking temperature of a non-extremal brane in the supergravity description, while the magnetic field may be used to represent bound states of D-branes.

Keeping $f_I^{(1)}$ and $f_I^{(2)}$ general and adding the factor $e^{-\frac{\pi^2}{8\pi c_3}t}$ one finds the thermal partition function of open strings stretched between two Dp-branes. The zero magnetic field limit of the Dp-brane partition function is [57,58]

$$Z(\beta, 0) = -\frac{8\pi^4 N^2 V_p}{2(8\pi^2 \alpha')^{p/2}} \int_0^\infty \frac{dt}{t^{1+\frac{p}{2}}} e^{-\frac{\pi^2}{8\pi c_3}t} \vartheta_4(0 | \frac{2\pi^2 i\alpha' t}{\beta^2}) \left[ \vartheta_2(0 | it) \vartheta_1(0 | it) \right]^4,$$

where $8\pi^4 \left[ \vartheta_2(0 | it) \vartheta_1(0 | it) \right]^4 = 8 \prod_{n=1}^{\infty} \left( \frac{1 + e^{-\frac{8\pi t}{\alpha' n}}}{1 - e^{-\frac{8\pi t}{\alpha' n}}} \right)^8$. The $p = 9$ case of (3.28) corresponds to the $F^{(r)} = 0$ limit of (3.23). For example, the finite-temperature analogue of the D8-brane
expression (2.42) is given by the obvious modification of (3.25) which reduces to (3.28) for $F = 0$.

Having in mind possible applications to the study of potentials between extremal and non-extremal branes one would be interested in expanding $Z(\beta, F)$ for small field and temperature. In contrast with the zero-temperature case where the expansion of $Z$ in powers of $F$ starts with the universal $F^4$ term, here one finds also lower powers of $F$ with temperature-dependent coefficients, and there is no a priori reason for the ‘universality’ of such terms. The leading terms in the expansion of the partition function (3.25) in powers of the field will have the form (cf. (3.12))

$$Z \sim k_1 \beta^{-9} + k_2 \beta^{-5} F^2 + (k_0 + k_3 \beta^{-1}) F^4 + ...,$$

where $k_i$ with $i > 0$ will be non-trivial functions of $r$ and $\alpha'$.

### 3.4. Neutral superstrings in electric background

In general, one does not expect an equilibrium distribution for charged strings in an electric field, so it is natural to consider the neutral string case, $f_0^{(1)} = -f_0^{(2)} = iE$, $E = 2\pi \alpha' \mathcal{E}$. The corresponding partition function $Z(\beta, E)$ can be computed using either real-time or imaginary-time formalism. For the bosonic string $Z(\beta, E)$ was found in [33]. In the real-time approach one should take into account that according to [41] the oscillation part of the mass spectrum of a bosonic neutral string in electric and magnetic fields gets rescaled by $1 - E^2$ factor

$$M^2 = -\sum_{l=1}^{D-2} \frac{E^2 + f_l^2}{1 + f_l^2} P_l^2 + \frac{1}{\alpha'} (1 - E^2) \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} n \tilde{a}_n^i a_n^i, \quad (3.29)$$

where the oscillators are canonically normalised, $[a_n^i, \tilde{a}_m^j] = i \delta^{ij} \delta_{nm}$.[28] Using the open string theory proper-time representation one then finds that the factor $1 - E^2$ appears multiplying the integration variable $t$ in the arguments of $\vartheta$-functions, or multiplying the temperature term $\beta^2 / t$ after a redefinition of $t$. In the imaginary-time approach one arrives at the same expression after taking into account that the winding modes of the imaginary time coordinate couple to the electric field term in the string action. As a result, the critical temperature of the bosonic string gets rescaled by the factor $\sqrt{1 - E^2}$ [33].

Similar conclusion is reached in the superstring case (for simplicity we shall set the magnetic components to zero since the generalisation to the ‘mixed-field’ case is obvious).

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[28] The special role of the electric field compared to the magnetic one is related to the fact that it couples to the time coordinate and thus contributes to the momentum. The definition of momentum becomes important at non-zero temperature since the partition function is defined by a phase-space integral.
In the case of the neutral string in the background field with just one electric component we get

\[ Z = -a_1 \pi^4 [1 - (2\pi \alpha' \mathcal{E})^2] \beta \int_0^\infty \frac{dt}{t^6} \frac{\vartheta_2(0) \left[1 - (2\pi \alpha' \mathcal{E})^2\right] \beta^2}{2\pi^2 \alpha' t} \left[\frac{\vartheta_2(0|it)}{\vartheta_1'(0|it)}\right]^4. \tag{3.30} \]

As follows from (3.30), the critical inverse temperature \( \beta > \beta_c \) becomes dependent on the electric field:

\[ \beta_c = \frac{2\pi \sqrt{2\alpha'}}{\sqrt{1 - (2\pi \alpha' \mathcal{E})^2}}. \tag{3.31} \]

The field-dependent rescaling factor may be interpreted as a modification \( (T_0 \to T_{\text{eff}}) \) of the string tension which enters the expression for the critical temperature:

\[ T_{\text{eff}} = T_0 [1 - (T_0^{-1} \mathcal{E})^2], \quad T_0 = \frac{1}{2\pi \alpha'}. \tag{3.32} \]

Note that the SYM theory limit of (3.30) is trivial as all dependence on the electric field disappears for \( \alpha' \to 0 \)

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\[ \text{\textsuperscript{29}} \text{This rescaling of the open-string tension is similar to the one discussed in D-string context in [59].} \]

\[ \text{\textsuperscript{30}} \text{For discussions of related (electric field, finite temperature) problems in field theory context see, e.g., [60], where the one-loop effective action of Dirac fermions in an approximately constant electro-magnetic field at finite temperature was computed, and [61], where a weak-field expansion of the one-loop finite temperature YM effective action in generic background was considered.} \]
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