Regenerative multi-type Galton-Watson processes

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August 3, 2018

Abstract

The general Perron-Frobenius theorem describes the growth of powers of irreducible non-negative kernels. In the special case of kernels with an atom this result can be obtained using a regeneration method. If such a kernel is sub-stochastic, then the regeneration method can be intuitively explained in terms of the so-called split-chain. In this paper we give an illuminating probabilistic interpretation of the regeneration method in terms of what we call regenerative Galton-Watson processes. These multi-type branching processes have an intrinsic structure of a single-type Crump-Mode-Jagers process with time inhomogeneous immigration.

1 Introduction

The Galton-Watson processes describe the randomly fluctuating number of independently reproducing particles, see [1]. If each particle has a type, which is an element of a measurable type space \((E, \mathcal{E})\), then the Galton-Watson process \(Z = \{Z_n\}_{n=0}^\infty\) is the sequence of point processes such that \(Z_n(A)\) gives the number of \(n\)-th generation particles whose types lie in the set \(A \in \mathcal{E}\), see [9, Ch 3]. Let \(\Xi\) be the point process describing the types of the offspring produced by a generic particle. We call \(\Xi\) the reproduction measure of the Galton-Watson process, and

\[
M(x, A) = \mathbb{E}_x \Xi(A), \quad x \in E, \quad A \in \mathcal{E},
\]

its reproduction kernel, assumed to be \(\sigma\)-finite. By \(\mathbb{E}_x\) we denote the expectation operator conditioned on the maternal particle having type \(x\). The expected sizes of the consecutive generations

\[
\mathbb{E}Z_n(A) = \int M^n(x, A)\mu_0(dx), \quad \mu_0 = \mathbb{E}Z_0,
\]

are conveniently expressed in terms of the powers of the reproduction kernel

\[
M^0(x, A) = \delta_x(A), \quad M^n(x, A) = \int M^{n-1}(y, A)M(x, dy), \quad n \geq 1,
\]

(2)
where $\delta_x(A) = 1_{\{x \in A\}}$. (Here and elsewhere in this paper, the integrals are taken over the whole type space $E$, unless specified otherwise.)

The asymptotic properties of the multi-type Galton-Watson processes are studied on the basis of the Perron-Frobenius theorem dealing with the limit behavior of the expectation kernels $M^n$ as $n \to \infty$, see [11, Ch 6]. The Perron-Frobenius eigenvalue $R^{-1}$ is used then to distinguish between subcritical ($R > 1$), critical ($R = 1$), and supercritical ($R < 1$) reproduction regimes. This approach was extended to the multi-type Crump-Mode-Jagers processes defined in terms of individuals (rather than particles) belonging to overlapping generations, see [6], and then to the even more general setting of sibling dependence, see [12].

A comprehensive treatment [13] of the Perron-Frobenius theory for the irreducible non-negative kernels is build around the so-called regeneration method. A key step of this method deals with the class of non-negative kernels having an atom, see [13, Ch. 4.3].

**Definition 1**

Let $g$ and $\gamma$ be a measurable non-negative function and a $\sigma$-finite measure. A non-negative kernel $M$ is said to have atom $(g, \gamma)$ if for some non-negative $\sigma$-finite kernel $m$,

$$M(x, A) = m(x, A) + g(x)\gamma(A), \quad x \in E, \quad A \in \mathcal{E}. \tag{3}$$

In the special case, when $M = P$ is a (sub)stochastic kernel with an atom, one can choose $g$ and $\gamma$ such that $0 \leq g(x) \leq 1$ for all $x \in E$ and $\gamma(E) = 1$. This yields a decomposition

$$P(x, A) = g(x)\gamma(A) + (1 - g(x))Q(x, A),$$

which can be interpreted in terms of a "split chain", whose transition from a given state $x$ is governed either by $\gamma(dy)$ or $Q(x, dy)$ depending on the outcome of a $g(x)$-coin tossing [13 Ch. 4.4]. After each $\gamma$-transition step, the future evolution of the chain becomes independent from the past, so the sequence of such regeneration events forms a renewal process with a delay. Finally, it remains to apply the basic renewal theory to establish the Perron-Frobenius theorem for the stochastic kernels.

In this paper we suggest a probabilistic interpretation of the general regeneration method in terms of what we call **regenerative Galton-Watson processes**. Suppose a point process $\Xi$ can be decomposed into the sum

$$\Xi = \xi + \tau_1 + \ldots + \tau_B, \tag{4}$$

of a random number of point processes with conditional expectations

$$E_x\xi(A) = m(x, A), \quad E_xB = g(x), \quad E_x\tau_i(A) = \gamma(A), \quad i \geq 1. \tag{5}$$

Then by the total expectation formula, the kernel (1) satisfies (3).

**Definition 2**

Let (4) hold together with (5). If each component $\tau_i$ in the decomposition (5) is independent of everything else and $\tau_i \overset{d}{=} \tau$, then the multi-type Galton-Watson process $Z$ with the reproduction measure $\Xi$ will be called a regenerative Galton-Watson process. A naturally coupled multi-type Galton-Watson process $z = \{z_n\}_{n=0}^\infty$ with the reproduction measure $\xi$ and $z_0 = Z_0$ will be called a stem Galton-Watson process.
The key property of the regenerative Galton-Watson process is that the sub-process initiated by a block of particles \( \tau_i \) is independent of the other parts of the Galton-Watson process. In Section 2 we show that such a multi-type Galton-Watson process has an intrinsic structure of the single-type Crump-Mode-Jagers process. The main idea is to treat a \( \tau \)-block of particles as a newborn Crump-Mode-Jagers individual (it is somewhat similar to the idea of macro-individuals in the sibling dependence setting of [12]).

In Section 3 we give a proof of the Perron-Frobenious theorem, see Theorem 8, using the renewal property of regenerative Galton-Watson process.

Definition 2 puts no restrictions on the reproduction measure \( \xi \), so that the expectation kernel \( m \) of the stem Galton-Watson process may have or have not an atom of its own. The example from Section 4 presents a case, where the stem Galton-Watson process is reducible over the type space \( E = [0, \infty) \), in that any ordered pair types \( x < y \) communicates in just one direction \( x \to y \).

Remark. Under the following four additional conditions
(i) \( B \) is either 0 or 1,
(ii) \( \xi(E) = B \),
(iii) \( \tau(E) \) has a geometric distribution,
(iv) the types of the particles constituting a \( \tau \)-block are independent,
the regenerative Galton-Watson process becomes a linear-fractional Galton-Watson process with infinitely many types, see [10, 14]. Here, the stem Galton-Watson process is a single lineage Markov chain with the state space \( E \cup \Delta \) supplemented with a graveyard state.

2 Embedded Crump-Mode-Jagers process

By construction, the regenerative Galton-Watson process \( Z \) dominates its stem Galton-Watson process \( z \), and in a typical setting, the size of the \( z \)-population of particles is much smaller than the size of the \( Z \)-population. Let \( X_n \) stand for the number of new \( \tau \)-blocks generated at time \( n \) directly by the process \( z \), so that
\[
E(X_n|Z_0 = \delta_x) = \int g(y)m^{n-1}(x, dy).
\]
Assuming \( Z_0 = \tau \), we will treat the initial \( \tau \)-block of particles as a newborn individual which lives \( L \) units of time, where \( L \) is the time of extinction of the process \( z \) with \( z_0 = \tau \). For such an individual the random variable \( X_n \) becomes the number of births at time \( n \), having the mean value
\[
f_n := E(X_n|Z_0 = \tau) = \int \int g(y)m^{n-1}(x, dy)\gamma(dx).
\]
Thus the random vector \((X_1, \ldots, X_{L+1})\) fully describes the life of the initial individual, see Figure 1 which gives rise to a population of individuals with overlapping generations. This
population forms a single-type Crump-Mode-Jagers process due to a regenerative property of the \( \tau \)-blocks of particles stipulated in Definition 2.

Our standing assumption concerning the regenerative Galton-Watson process:

\[
f(s_0) \in (0, \infty) \text{ for some } s_0 > 0, \text{ where } f(s) = \sum_{n=1}^{\infty} f_n s^n, \tag{6}
\]

requires on one hand, that \( f_n > 0 \) for some \( n \geq 1 \), and on the other hand, that the radius of convergence

\[
r = \inf\{s \geq 0 : f(s) = \infty\}
\]

is positive. The assumption \( r > 0 \) prohibits very fast growing sequences of the type \( f_n = e^{n^2} \).

**Definition 3** Given (6), define a parameter \( R \in (0, \infty) \) as \( R = r \) if \( f(r) < 1 \), and as the unique positive solution of the equation \( f(R) = 1 \) if \( f(r) \geq 1 \).

Since \( f(R) \leq 1 \), the sequence \( (f_n R^n) \) can be viewed as a (possibly defective) distribution on the lattice \( \{1, 2, \ldots\} \). This is the distribution of the waiting time for the renewal process naturally embedded into the Crump-Mode-Jagers process defined above. When this distribution is proper, that is when \( f(r) \geq 1 \), the mean waiting time for the embedded renewal process

\[
\sum_{n=1}^{\infty} n f_n R^n = R f'(R).
\]
gives the average age at childbearing or the mean generation length for the Crump-Mode-Jagers process, see [5].

Focussing on the current waiting time of such a discrete renewal process, we get an irreducible Markov chain with the state space \( \{0, 1, \ldots\} \). The following observation concerning this Markov chain is straightforward.

**Proposition 4** The embedded renewal process is transient if \( f(r) < 1 \), and recurrent if \( f(r) \geq 1 \). If \( f(r) = 1 \), then the embedded renewal process is either positive recurrent or null recurrent depending on whether \( f'(r) < \infty \) or \( f'(r) = \infty \).

Let \( Y_n \) be the number of newborn individuals at time \( n \), or in other words, the total number of \( \tau \)-blocks emerging at time \( n \). Clearly,

\[
F_n := \mathbb{E}(Y_n | Z_0 = \tau) = \int \int g(y) M^{n-1}(x, dy) \gamma(dx).
\]

For the general starting point process \( Z_0 \), putting \( \mu_0 = \mathbb{E}Z_0 \), we get

\[
\tilde{f}_n := \mathbb{E}X_n = \int \int g(y) m^{n-1}(x, dy) \mu_0(dx),
\]

\[
\tilde{F}_n := \mathbb{E}Y_n = \int \int g(y) M^{n-1}(x, dy) \mu_0(dx).
\]

**Proposition 5** The parameter \( R \) from Definition 3 coincides with the radius of convergence of the generating function

\[
F(s) = \sum_{n=1}^{\infty} F_n s^n,
\]

with \( F(R) = \infty \) if and only if \( f(r) < 1 \). Furthermore, the generating functions \( \tilde{f}(s) = \sum_{n=1}^{\infty} \tilde{f}_n s^n \) and \( \tilde{F}(s) = \sum_{n=1}^{\infty} \tilde{F}_n s^n \) are connected by

\[
\tilde{F}(s) = \frac{\tilde{f}(s)}{1 - f(s)} \quad \text{for} \quad s \quad \text{such that} \quad f(s) < 1. \tag{7}
\]

**Proof.** Using the law of total expectation it is easy to justify the following recursion

\[
F_n = f_n + f_{n-1} F_1 + \ldots + f_1 F_{n-1}.
\]

This leads to the equality for generating functions \( F(s) = f(s) + f(s) F(s) \), which yields

\[
F(s) = \frac{f(s)}{1 - f(s)} \quad \text{for} \quad s \quad \text{such that} \quad f(s) < 1. \tag{8}
\]

From here it is obvious that the first statement is valid. To obtain expression (7), observe that

\[
\tilde{F}_n = \tilde{f}_n + \tilde{f}_{n-1} F_1 + \ldots + \tilde{f}_1 F_{n-1},
\]
which gives $\tilde{F}(s) = \tilde{f}(s)(1 + F(s))$, and it remains to apply (8).

Remark. As mentioned above, under the special initial condition $Z_0 = \tau$, the embedded Crump-Mode-Jagers process starts from a single newborn individual. In general, the embedded Crump-Mode-Jagers process has an immigration component characterized by the generating function $\tilde{f}(s)$. The immigration process is the inflow of the newborn Crump-Mode-Jagers individuals representing the $\tau$-blocks of particles generated by the stem Galton-Watson process $z$ with $z_0 = Z_0$.

3 Perron-Frobenius theorem for kernels with atom

Consider a non-negative kernel $M$ with an atom given by Definition 1, and put

$$M_s(x, A) = \sum_{n=1}^{\infty} s^n M^{n-1}(x, A), \quad m_s(x, A) = \sum_{n=1}^{\infty} s^n m^{n-1}(x, A), \quad s \geq 0,$$

so that earlier introduced generating functions have the form

$$F(s) = \int\int g(y)M_s(x, dy)\gamma(dx), \quad f(s) = \int\int g(y)m_s(x, dy)\gamma(dx).$$

Denote

$$h_s(x) = \int g(y)m_s(x, dy), \quad \pi_s(A) = \int m_s(x, A)\gamma(dx),$$

and observe that

$$\int h_s(x)\gamma(dx) = \int g(y)\pi_s(dy) = f(s), \quad \int h_s(y)\pi_s(dy) = s^2 f'(s).$$

The latter equality requires the following argument

$$\int h_s(x)\pi_s(dx) = \int\int g(y)m_s(x, dy)m_s(z, dx)\gamma(dz)$$

$$= \int\int g(y)m^2_s(z, dy)\gamma(dz) = \sum_{n=1}^{\infty} ns^{n+1}f_n = s^2 f'(s),$$

where we used the relation

$$s^{-2}m^2_s(y, A) = \int s^{-1}m_s(x, A)s^{-1}m_s(y, dx) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \int s^n m^n(x, A)s^k m^k(y, dx)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} s^{n+k}m^{n+k}(y, A) = \sum_{j=0}^{\infty} (j + 1)s^j m^j(y, A).$$
Lemma 6 If $s > 0$ is such that $f(s) \leq 1$, then the function $h_s$ and the measure $\pi_s$, defined by (9), are (sub)invariant for the kernel (3), in that

$$\int h_s(y)M(x,dy) = s^{-1}h_s(x) - (1 - f(s))g(x),$$

and

$$\int M(y,A)\pi_s(dy) = s^{-1}\pi_s(A) - (1 - f(s))\gamma(A).$$

Proof. By (3), we have

$$\int m_s(y, A)M(x, dy) = \sum_{n=1}^{\infty} s^n m^n(x, A) + g(x) \int m_s(y, A)\gamma(dy)$$

which implies relation (10):

$$\int h_s(y)M(x, dy) = \int \int g(w)m_s(y, dw)M(x, dy) = s^{-1}h_s(x) - g(x) + g(x)f(s).$$

Similarly, from

$$\int M(y, A)m_s(x, dy) = \sum_{n=1}^{\infty} s^n m^n(x, A) + \gamma(A) \int g(y)m_s(x, dy)$$

we arrive at relation (11). \hfill \Box

Corollary 7 If $f(R) = 1$, then $h = h_R$ and $\pi = \pi_R$ are $R$-invariant function and measure satisfying

$$\int h(x)\gamma(dx) = \int g(y)\pi(dy) = 1, \quad \int h(y)\pi(dy) = R^2 f'(R).$$

If $f'(R) < \infty$, then the non-negative kernel

$$Q(x, A) = \frac{h(x)\pi(A)}{R^2 f'(R)}$$

is such that $Q^n(x, A) = Q(x, A)$ for $n \geq 1$.

Observe that

$$h(x) = \sum_{n=1}^{\infty} R^n \int g(y)m^{n-1}(x, dy)$$

is the expected $R$-discounted number of $\tau$-blocks of particles ever produced by the stem Galton-Watson process $z$ with $z_0 = \delta_x$, in other words,

$$h(x) = E\left( \sum_{n=1}^{\infty} R^n X_n | z_0 = \delta_x \right).$$
From this angle, \( h(x) \) can be interpreted as the reproductive value of type \( x \). On the other hand, \( \pi(A) \) is the expected \( R \)-discounted number of particles whose type belongs to \( A \) and which appear in the stem Galton-Watson process \( z \), having \( z_0 = \tau \):

\[
R^{-1} \pi(A) = \sum_{n=0}^{\infty} R^n \int m^n(x, A) \gamma(dx).
\]

As shown next, see part (iii) of Theorem 8, the measure \( \pi \) can be viewed as an asymptotically stable distribution over the types in the regenerative Galton-Watson process.

**Theorem 8** Consider a non-negative kernel \( M \) with atom \((g, \gamma)\).

(i) If \( s \geq 0 \) is such that \( f(s) < 1 \), then

\[
M_s(x, A) = m_s(x, A) + \frac{h_s(x) \pi_s(A)}{1 - f(s)},
\]

(ii) Let \( f(r) = 1 \) and \( f'(r) = \infty \). If

\[
h(x) < \infty, \quad \pi(A) < \infty, \quad R^n m^n(x, A) \to 0, \quad n \to \infty,
\]

then

\[
R^n M^n(x, A) \to 0, \quad n \to \infty.
\]

(iii) Let \( f(r) \geq 1 \) and \( f'(R) < \infty \). If \( (13) \) holds, then

\[
R^n M^n(x, A) \to Q(x, A), \quad n \to \infty.
\]

**Proof.** By (3), we have the recursion

\[
M^n(x, A) = g(x) \int M^{n-1}(y, A) \gamma(dy) + \int M^{n-1}(y, A) m(x, dy)
\]

\[
= g(x) \int M^{n-1}(y, A) \gamma(dy) + \int g(y) m(x, dy) \int M^{n-2}(z, A) \gamma(dz)
\]

\[
+ \int M^{n-2}(z, A) m^2(x, dz)
\]

\[
= \sum_{i=1}^{n} \int g(y) m^{i-1}(x, dy) \int M^{n-i}(y, A) \gamma(dy) + m^n(x, A),
\]

which in terms of generating functions gives

\[
M_s(x, A) = m_s(x, A) + h_s(x) \int M_s(y, A) \gamma(dy),
\]

and after integration,

\[
\int M_s(x, A) \gamma(dx) = \frac{\pi_s(A)}{1 - f(s)}.
\]
Combining the last two relations we get (12). Observe also that the last formula yields (8).

Turning to the statements (ii) and (iii), suppose \( f(R) = 1 \) and rewrite (12) as

\[
M_{\hat{s}}(x, A) - m_{\hat{s}}(x, A) = \frac{b(s)}{1 - a(s)},
\]

where \( \hat{s} = sR \) and

\[
a(s) = f(sR), \quad b(s) = h(x)\pi_{\hat{s}}(A),
\]

so that \( a'(1) = Rf'(R) \), \( b(1) = h(x)\pi(A) \). Applying Lemma 9, we find that as \( n \to \infty \),

\[
R^n(M^n(x, A) - m^n(x, A)) \to \begin{cases} 
Q(x, A) & \text{if } f'(R) < \infty, \\
0 & \text{if } f'(R) = \infty, h(x)\pi(A) < \infty.
\end{cases}
\]

What remains is to make use of condition (13). \( \square \)

**Lemma 9** Let \( a(s) = \sum_{n=0}^{\infty} a_n s^n \), \( b(s) = \sum_{n=0}^{\infty} b_n s^n \), \( c(s) = \sum_{n=0}^{\infty} c_n s^n \) are generating functions for three non-negative sequences such that \( c(s) = \frac{b(s)}{1 - a(s)} \). If \( a(1) = 1 \), \( a'(1) > 0 \), then

\[
c_n \to \begin{cases} 
\frac{b(1)}{a'(1)} & \text{if } a'(1) < \infty, \\
0 & \text{if } a'(1) = \infty, b(1) < \infty,
\end{cases} \quad n \to \infty.
\]

**Proof.** This is a well-known result from Chapter XIII.4 in [3]. \( \square \)

### 4 Example of a regenerative Galton-Watson process

Here we construct a transparent example of a regenerative Galton-Watson process with the type space \( E = [0, \infty) \). Its positive recurrent expectation kernel is fully specified by just three parameters \( a, c \in (0, \infty) \), and \( b \in (-1, \infty) \):

\[
M(x, dy) = ae^{x-y}1_{\{y \geq x\}}dy + ce^{-bx} \delta_0(dy),
\]

This kernel satisfies (3) with

\[
m(x, dy) = ae^{x-y}1_{\{y \geq x\}}dy, \quad g(x) = ce^{-bx}, \quad \gamma(A) = \delta_0(A), \tag{15}
\]

implying that each \( \tau \)-block consists of a single particle of type 0.

This example is defined in terms of a continuous time Markov branching process modeling the size of a population of Markov particles having the unit life-length mean and offspring mean \( a \). The main idea is to count the Markov particles generation-wise, and to define the type of a Galton-Watson particle as the birth-time of the corresponding Markov particle. Then, \( z_n(A) \) becomes the number of \( n \)-generation Markov particles born in the time period \( A \), so that its conditional mean for \( t > x \), is given by

\[
m^n(x, [0, t]) = a^n \mathbb{P}(x + T_1 + \ldots + T_n \leq t) = a^n \mathbb{P}(N_{t-x} \geq n),
\]

where \( T_i \) are independent exponentials with unit mean and \( \{N_t\}_{t \geq 0} \) is the standard Poisson process.
Proposition 10 Consider a regenerative Galton-Watson process with the triplet \((m, g, \gamma)\) given by (15). Then we have

\[ f(s) = \frac{r cs}{r-s}, \quad r = 1 + \frac{b}{a}, \quad R = \frac{r}{1 + cr}. \]  

(16)

The process is supercritical if \(c > \frac{r-1}{r}\), critical if \(c = \frac{r-1}{r}\), or subcritical if \(c < \frac{r-1}{r}\).

PROOF. Referring to the underlying Poisson process, we find that for \(s \neq 1/a\),

\[ m_s(0, [0, t]) = s \sum_{n=0}^{\infty} s^n a^n \sum_{k=0}^{\infty} \mathbb{P}(N_t = k) = s \sum_{k=0}^{\infty} \mathbb{P}(N_t = k) \frac{1 - (as)^{k+1}}{1 - as} \]

\[ = \frac{s}{1 - as} (1 - as \mathbb{E}(as)^N_t) = \frac{s}{1 - as} (1 - ase^{t(as-1)}). \]

More generally, we have

\[ m_s(x, [0, t]) = m_s(0, [0, t - x]) = \frac{s}{1 - as} (1 - ase^{(t-x)(as-1)})1_{t \geq x} \]

so that

\[ m_s(x, dy) = s\delta_x(dy) + as^2 e^{(as-1)(y-x)}1_{y \geq x}dy. \]

By (9)

\[ h_s(x) = \int g(y)m_s(x, dy) = sce^{-bx} + cas^2 \int_0^{\infty} e^{h(as-1)}e^{-b(x+u)}du = f(s)e^{-bx}, \]

where \(f(s)\) satisfies (16). Since \(f(r) = \infty\), the stated value \(R = \frac{r}{1 + cr}\) is found from the equation \(f(R) = 1\).

Proposition 11 Convergence (14) holds for \(A = [0, t], \ t \in [0, \infty)\), if

\[ Q(x, dy) = e^{-bx}(R\delta_0(dy) + aR^2 e^{(aR-1)y}dy). \]

If \(Ra < 1\), then (14) holds even for \(A = E\) with \(Q(x, E) = \frac{Re^{-bx}}{1-aR}\).

PROOF. Applying once again (9), we find

\[ \pi_s(dy) = \int m_s(x, dy)\gamma(dx) = m_s(0, dy) = s\delta_0(dy) + as^2 e^{(as-1)y}dy. \]

To check this and previously obtained expressions, we verify the general formula for the integral

\[ \int h_s(x)\pi_s(dx) = sf(s) + f(s)as^2 \int e^{-(1+b)x}e^{asx}dx = \frac{rsf(s)}{r-s} = \frac{r^2s^2}{(r-s)^2} = s^2 f'(s). \]
With
\[ h(x) = e^{-bx}, \quad \pi(dx) = R\delta_0(dx) + aR^2e^{(aR-1)x}dx, \]

Theorem 8 (iii) applied to this example says that for \( t \in [0, \infty) \),
\[
R^nM^n(x, [0,t]) \to e^{-bx}(R + aR^2 \int_0^t e^{(aR-1)y}dy)
= e^{-bx}(R + \frac{aR^2}{aR-1}(e^{(aR-1)t} - 1)) = e^{-bx}\frac{aR^2e^{(aR-1)t}}{aR-1} - R, \quad n \to \infty.
\]

If \( aR < 1 \) and \( A = E \), then condition (13) holds since
\[
\pi(E) = R + \frac{aR^2}{1-aR} = \frac{R}{1-aR} < \infty,
\]
and \( R^n\pi^n(x, E) = (Ra)^n \to 0. \)

\[ \square \]

Remark. If we further specialize this example by letting the stem process to be the Yule process, then we have \( a = 2 \). If furthermore, \( b = 2 \) and \( c < \frac{r-1}{r} = \frac{1}{3} \), then the regenerative Galton-Watson process is subcritical, despite the total number of particles in the Yule process is infinite.

References

[1] Athreya, K. AND NEY, P. (1972) Branching processes, John Wiley & Sons, London-New York-Sydney.

[2] Bingham, N. H., Goldie, C. M., and Teugels, J. L. (1987). Regular Variation. Encyclopedia of mathematics and its Applications. Cambridge University Press, Cambridge.

[3] Feller, W. (1959). An introduction to probability theory and its applications, Vol I, 2nd ed. John Wiley & Sons, London-New York-Sydney.

[4] Feller, W. (1971). An introduction to probability theory and its applications, Vol II, 3rd ed. John Wiley & Sons, London-New York-Sydney.

[5] Jagers, P. (1975) Branching processes with biological applications, Wiley, New-York.

[6] Jagers, P. (1989) General branching processes as Markov fields. Stoch. Proc. Appl. 32, 183 - 212.

[7] Jagers, P. AND NERMAN, O. (1996) The asymptotic composition of supercritical multi-type branching populations. Springer Lecture Notes in Mathematics, 1626, 40 - 54.
[8] JAGERS, P. AND Sagitov, S. (2008) General branching processes in discrete time as random trees. *Bernoulli* 14, 949–962.

[9] Harris, T. E. (1963) *The Theory of Branching Processes*, Springer, Berlin.

[10] Lindo, A. and Sagitov, S. (2018) General linear-fractional branching processes with discrete time. *Stochastics* 90 (in press)

[11] Mode, C.J. (1971) Multitype branching processes: theory and applications. Volym 34 av Modern analytic and computational methods in science and mathematics. American Elsevier Pub. Co.,

[12] Olofsson, P. (1996) Branching processes with local dependencies. *Ann. Appl. Probab.* 6, 238–268.

[13] Nummelin, E. (1984) *General Irreducible Markov Chains and Non-negative Operators*, Cambridge University Press, London.

[14] Sagitov, S. (2013) Linear-fractional branching processes with countably many types. *Stoch. Proc. Appl.* 123, 2940–2956.