Universality of the Threshold for Complete Consensus for the Opinion Dynamics of Deffuant et al.

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Abstract

In the compromise model of Deffuant et al., opinions are real numbers between 0 and 1 and two agents are compatible if the difference of their opinions is smaller than the confidence bound parameter $\epsilon$. The opinions of a randomly chosen pair of compatible agents get closer to each other. We provide strong numerical evidence that the threshold value of $\epsilon$ above which all agents share the same opinion in the final configuration is $1/2$, independently of the underlying social topology.

Keywords: Sociophysics, Monte Carlo simulations.

The last few years witnessed many attempts to describe society as a physical system [1], with people playing the role of atoms or classical spins undergoing elementary interactions. There are meanwhile several models to explain how hierarchies [2] and consensus [3]–[8] may originate in a society.

In this paper we focus on a consensus model, that of Deffuant et al. [4]. It is a model with binary interactions, i.e. where the opinions of the agents are modified pairwise, according to a compromise strategy. One starts from a graph with $N$ vertices, which are the agents of the society. The edges of the graph represent the relationships between the agents and interactions can take place only between neighbouring agents. Next, opinions are randomly distributed among the agents; the opinions, usually indicated with $s$, are real numbers in the range $[0, 1]$. Furthermore two parameters are introduced, the confidence bound $\epsilon$ and the convergence parameter $\mu$. They are both real numbers, which take values in $[0, 1]$ and $[0, 1/2]$, respectively. The dynamics is very simple: one chooses a pair $\{i, j\}$ of neighbouring agents and checks whether $|s_i - s_j| < \epsilon$. If this is not true we do nothing. Otherwise the agents get the new opinions $s'_i = s_i + \mu (s_j - s_i)$ and $s'_j = s_j + \mu (s_i - s_j)$. This means
that the opinions of the agents shift towards each other, by a relative amount \( \mu \). In the particular case \( \mu = 1/2 \), the two opinions jump to their mean. By repeating the procedure one sees that the agents group in opinion clusters until, at some stage, a stable configuration is reached. Stable configurations can only be superpositions of \( \delta \)-functions in the opinion space, such that the opinion of any \( \delta \) is farther than \( \epsilon \) from all others. In this case, in fact, if we take a pair of agents, their opinions are either identical, because the agents belong to the same cluster, or they differ more than \( \epsilon \): in both cases nothing happens. The number of clusters in the final configuration depends on the value of \( \epsilon \). In particular, there is a special \( \epsilon_c \) such that, for \( \epsilon > \epsilon_c \), all agents belong to a single cluster. We will show here that \( \epsilon_c = 1/2 \), no matter what kind of graph one takes to represent the relationships among the agents. We study the question numerically by means of Monte Carlo simulations. We analyzed four different graph structures:

- a complete graph, where everybody talks to everybody [4];
- a square lattice;
- a random graph a la Erdös and Rényi [9];
- a scale free graph a la Barabási-Albert [10].

We update the opinions of the agents in ordered sweeps over the population. For any agent we randomly select one of its neighbours as partner of the interaction. The program stops if no agent changed opinion after an iteration; since opinions are double precision real numbers, our criterion is to check whether any opinion varied by less than \( 10^{-9} \) after a sweep. The results do not depend on the value of the convergence parameter \( \mu \), so we have always set \( \mu = 1/2 \). Our method is quite simple. For a given population \( N \) and confidence bound \( \epsilon \) we collected a number of samples ranging from 500 to 1000. Once the system has reached its final configuration, we check whether all agents belong to the same cluster or not. The fraction of samples with a single final opinion cluster is the probability \( P_c \) to have complete consensus, that we study as a function of \( \epsilon \).

For a society where all agents interact with each other, a beautiful analysis was carried on by Ben-Naim et al. [11]. In contrast to us they fixed the confidence bound to 1 and introduced a parameter \( \Delta \) to mark the bounds of the opinion space, which goes from \(-\Delta\) to \(\Delta\), but the results can be simply
translated in our conventions. They derived a rate equation for the dynamics and solved it numerically, finding that for $\Delta > 1$, which corresponds to $\epsilon > 1/2$, all agents share the same opinion. Because of that, we did not study this case in detail, but we performed some simple simulations to test our method. Fig. 1 shows the behaviour of $P_c$ with $\epsilon$ for a society with 10000 agents. We observe that $P_c$ is basically zero up to $\epsilon \sim 0.46$, it rises rapidly for $0.46 < \epsilon < 1/2$ and it saturates to one for $\epsilon > 1/2$, which is compatible with the result of [11]. Fig. 1 suggests that $P_c$ may converge to a step function $\theta$ in the limit $N \to \infty$. This can be best verified by using several values of $N$ and analyzing what happens when $N$ increases, and this is the strategy we adopted for the other social topologies.

Let us now examine the situation for an ordered structure like a lattice. We took a square lattice with periodic boundary conditions, so each agent has exactly four neighbours. Early studies of the Deffuant dynamics on a lattice were carried on in [12]. In Fig. 2 we again plot $P_c$ as a function
of $\epsilon$, but this time we repeated the procedure for three different population sizes, $N = 2500$, 10000 and 40000. The convergence to a step function with threshold $1/2$ is manifest.

In a realistic model of a society, it is neither true that everybody knows everybody else (unless one considers small communities), nor that every person has the same number of friends. A plausible model is given by a random graph (or random network). We considered two types of random graphs, the classical model of Erdős and Rényi [9] and the scale free model proposed by Barabási and Albert [10], which has attracted an exceptional attention in the last years [13].

The random graph of Erdős and Rényi is characterized by a parameter $p$, which is the connection probability of the nodes. One assumes that each of the $N$ nodes of the graph has probability $p$ to be linked to any other node. In this way, the total number of edges $m$ of the graph is $m = pN(N-1)/2$ and
the average degree of the graph, i.e. the average number of neighbours of a node, is \( k = p(N - 1) \) which can be well approximated by \( pN \) when \( N \to \infty \).

In order to have a finite degree, the product \( pN \) must then be finite. In our simulations we built graphs with the same average degree \( k = pN = 400 \), and number of nodes \( N = 10000, 40000 \) and \( 100000 \). Fig. 3 shows the behaviour of \( P_c \) with \( \epsilon \) for this special topology: \( P_c \) equals one for \( \epsilon > 1/2 \). The fact that for any \( \epsilon < 1/2 \) \( P_c \) decreases with \( N \) confirms the impression that, in the limit \( N \to \infty \), \( P_c = 0 \) for \( \epsilon < 1/2 \).

Finally, we examined our problem for agents sitting on the nodes of a scale free network. This topology was adopted for the Deffuant model in [14] and [15] as well. To build the network we must specify the outdegree \( m \) of the nodes, i.e. the number of edges which originate from a node. The procedure is dynamic; one starts from \( m \) nodes which are all connected to each other and adds further \( N - m \) nodes one at a time. When a new node is added, it selects \( m \) of the preexisting nodes as neighbours, so that the probability to get linked to a node is proportional to the number of its
neighbours. In all networks created in this way the number of agents with degree \( k \) is proportional to \( 1/k^3 \) for \( k \) large, independently of \( m \); here we chose \( m = 3 \). The results are illustrated in Fig. 4. Once more, we get the same pattern observed in the previous cases.

We have then discovered a general feature of the opinion dynamics of Deffuant et al.: no matter how society is structured, for \( \epsilon > 1/2 \) there is always complete consensus, whereas for \( \epsilon < 1/2 \) there are at least two different opinion clusters. When we are slightly below 1/2 one cluster includes all agents except just a few (or even a single agent!). There is a simple argument to convince ourselves why this happens. If \( \epsilon > 1/2 \) there cannot be more than one large opinion cluster. As a matter of fact, because of the symmetry of the model, the disposition of the clusters in the opinion space is also symmetric with respect to the center opinion \( s = 1/2 \). Let us concentrate on the two clusters which lie close to the extremes of the opinion interval. The agents whose opinions lie close to 0, for instance, can interact with agents with opinions in the range \([0, \epsilon] \). Due to the symmetry of the dynamics, the...
center $\epsilon_l$ of the peak will approximately coincide with the middle point of the latter range, so that $\epsilon_l \sim \epsilon/2$. Analogously, the center of the rightmost peak $\epsilon_r \sim 1 - \epsilon/2$. In this way, the distance between the rightmost and the leftmost cluster $\epsilon_r - \epsilon_l \sim 1 - \epsilon$, which is smaller than $\epsilon$ if $\epsilon > 1/2$. The two clusters will then fuse at some stage to a unique cluster at $s = 1/2$. The agents of this big central cluster are compatible with all remaining agents, as the maximal difference between $1/2$ and all possible opinions is $1/2 < \epsilon$. At the end of the day, all agents which are not already in the major cluster will be sooner or later attracted by it. The argument we exposed is obviously independent of the way the agents are connected to each other, but it only relies on the dynamics. However, it is not a rigorous proof, and this is the reason why we recurred to the numerical simulations we have presented.

We believe that the result holds as well for other versions of the Deffuant model. For example, one could assign different values of $\mu$ to all agents, like in [16]. As long as the $\mu$'s are distributed independently of the opinions of the agents the result should still be valid, for any distribution. We performed some tests, by using a uniform and an exponential distribution for $\mu$, and they confirm our expectation.

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