A Diagrammatic Calculus for Algebraic Effects

Preliminary Report

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Abstract

We introduce a new diagrammatic notation for representing the result of (algebraic) effectful computations. Our notation explicitly separates the effects produced during a computation from the possible values returned, this way simplifying the extension of definitions and results on pure computations to an effectful setting. Additionally, we show a number of algebraic and order-theoretic laws on diagrams, this way laying the foundations for a diagrammatic calculus of algebraic effects. We give a formal foundation for such a calculus in terms of Lawvere theories and generic effects.

Keywords algebraic effect, generic effect, diagrammatic notation, graphical calculus, monad, Lawvere theory

1 Introduction

In this note we are concerned with the problem of finding a convenient presentation of effectful (sequential) computations when effects are produced by algebraic operations [25?]. Concretely, what we have in mind are computations generated by sorts of $\lambda$-calculi enriched with algebraic operations in the spirit of Plotkin and Power [23]. Say we have fixed such a $\lambda$-calculus: how do we represent the result $[t]$ of the evaluation of a term $t$? Two standard answers to this question are the following:

1. $[t]$ is a possibly infinite tree whose nodes are labeled with algebraic operations, and whose leaves are either values or the bottom symbol $\bot$ denoting pure divergence. These structures are known as computation trees [6] and are used to give operational semantics to calculi with algebraic effects in, e.g., [11, 23, 28].
2. $[t]$ is an element in $T(V)$, where $T$ is a suitable monad and $V$ is the set of values. This is the approach followed in, e.g., [3, 13].

Both these approaches have their drawback. The first one is essentially syntactical, and does not give any semantical information on the effects produced. The second, on the contrary, is purely semantical and does not allow one to separate terms/values from effects. This is rather unsatisfactory as effects are performed by operations, and thus we would expect $[t]$ to be made of an ‘effect part’ and a ‘value part’, rather than being a monolithic object.

We work with (strong) monads on $\mathbf{Set}$, the category of set and functions, which we present as Kleisli triples [18]. The latter are triples $(T, \eta, \mu)$, where $f^\dagger : TX \to TY$ is the Kleisli extension of $f : X \to Y$ and $\eta : X \to TX$ is the unit of $T$, satisfying the following laws, for $f, g$ having appropriate (co)domains.

$$\eta_X = id_{T(X)}; \quad f^\dagger \circ \eta_X = f; \quad g^\dagger \circ f^\dagger = (g^\dagger \circ f)^\dagger.$$ 

Since we work in $\mathbf{Set}$, we oftentimes use the bind notation $\gg$ for monadic sequencing. That is, given $\mu \in T(X)$ and $f : X \to T(Y)$, we write $\mu \gg f$ in place of $f^\dagger(\mu)$.

We assume $\eta$ to be an injection, meaning that $T$ is non-trivial [19]. This is the case for all monads on $\mathbf{Set}$, except for the monad with $T(X) = 1$ for every set $X$, and the one with $T(\emptyset) = \emptyset$ and $T(X) = 1$, for $X \neq \emptyset$. We denote by $\mathbf{Kl}(T)$ the Kleisli category of $T$.

Example 2.1. The target monads we have in mind are those modeling notions of computations [20, 21, 30]. Among those are:

1. The maybe monad $M(X) \triangleq X + \{\top\}$ modeling divergence.

In this note we show that we can rely on the correspondence between algebraic effects and generic operations [25] to express $[t]$ as a pair $(\Gamma, \bar{v})$, where $\Gamma$ is a (generic) effect and $\bar{v}$ is a list of values. In order to facilitate calculations with generic effects we present the pair $(\Gamma, \bar{v})$ diagrammatically, this way obtaining a lightweight diagrammatic calculus for effectful sequential computations.

Advertisement This note is a work in progress. The authors plan to systematically review and update it—especially with examples and informal explanations—in the next weeks. The authors have also noticed that different browsers have different rendering of diagrams, whereas ‘desktop’ pdf viewers tend to have a better rendering.

2 Preliminaries: Monads and Algebraic Operations

In this section, we give some background notions on monads and algebraic operations. The current version of this work still lacks a proper introduction to monads and algebraic effects, and thus assume the reader to be familiar with basic category theory [18] and domain theory [1].

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Example 2.1. The target monads we have in mind are those modeling notions of computations [20, 21, 30]. Among those are:

1. The maybe monad $M(X) \triangleq X + \{\top\}$ modeling divergence.
2. The exception monad $E(X) \triangleq X + E$ modeling computations raising exceptions in a given set $E$ from a given $X$.
3. The powerset monad $\mathcal{P}$ modeling pure nondeterminism.
4. The (discrete) subdistribution monad $\mathcal{D}$ modeling probabilistic nondeterminism.
5. The global state monad $G(X) \triangleq M(X \times S)^S$, modeling imperative computations over a store $S$ (e.g. given a set $L$ of locations, take $S \triangleq \{0,1\}^L$, meaning that a store assigns to each location a boolean value).
6. The output monad $O(X) \triangleq C^\infty \times M(X)$, where $C^\infty$ is the set of finite and infinite strings over a fixed alphabet $C$.
7. Combinations thereof, such as monads of the form $T(M(X))$ or $T(X \times S)^S$ (see [8]).

Monads alone can structure effects and (sequentially) compose them, but do not have the structure to actually produce them (with the irrelevant exception of the trivial effect). For this reason we consider monads coming with effect-triggering operations. Following Plotkin and Power [23, 24], we require such operations to be algebraic.

**Definition 2.2.** An $n$-ary (set-indexed family of) operation $\sigma_X : T^n X \rightarrow TX$ is algebraic, if for all sets $X, Y, f : X \rightarrow T(Y)$, and $\mu_1, \ldots, \mu_n \in T(X)$, we have:

$$(\sigma_X(\mu_1, \ldots, \mu_n)) \gg f = \sigma_Y(\mu_1 \gg f, \ldots, \mu_n \gg f).$$

**Example 2.3.** Referring to the monads in Example 2.1, the following operations are all algebraic.
1. The maybe monad comes with no operation.
2. The exception monad comes with a set of zero-ary operations $\text{raise}_e$, for $e \in E$, that simply corresponds to $\text{inr} (e)$.
3. The powerset monad $\mathcal{P}$ comes with set-theoretic union $\cup$ modeling binary nondeterministic choice.
4. The subdistribution monad $\mathcal{D}$ comes with binary fair nondeterministic choice $\oplus$ (recall that $(\mu \oplus \nu)(x) = \frac{1}{2} \mu(x) + \frac{1}{2} \nu(x)$).
5. The global state monad $G(X) \triangleq M(X \times S)^S$, comes with a set of binary operations $\text{read}_\ell$ for reading locations, and a set of unary operations $\text{write}_\ell$ for writing locations (see [24, 27]).
6. The output monad comes with a set of unary operations $\text{print}_\ell$ indexed over elements of $C$ for printing. That is, $\text{print}_t.(w, x) = (cw, x)$.

As highlighted by Example 2.3 several computational effects can be modeled using monads and algebraic operations. Notable exceptions are continuations and exception handling. In this paper we will consider computational effects modeled as a monad $T$ together with a set $\Sigma$ of algebraic operations on $T$. In those cases, we say that $T$ is $\Sigma$-algebraic.

### 3 A $\lambda$-Calculus with Algebraic Effects

At this point of the work we have to make some design choices, as well as introduce some (minor) restrictions on the collection of monads and effects studied. In order to motivate such choices and restrictions, it is convenient to have a concrete computational calculus with algebraic effects. We take the calculus of [3].

**Definition 3.1.** Let $\Sigma$ be a set of operations on a given monad $T$. The calculus $\Lambda_\Sigma$ has terms and values defined by the following grammar, where $x$ ranges over a countable set of variables, and $\sigma \in \Sigma$.

$$t, s ::= x \mid \lambda x.t \mid ts \mid \sigma(t, \ldots, t)$$

$$v, w ::= x \mid \lambda x.t$$

Notice that $\Lambda_\Sigma$ is parametric with respect to a set $\Sigma$ of operations and to a $\Sigma$-algebraic monad $T$. Although the monad $T$ plays no role in Definition 3.1 and calculi are usually defined relying on uninterpreted operation symbols (which are then interpreted as algebraic operations once giving semantics to the calculus), for the sake of the economy of the work we chose the 'semantic-oriented' presentation of Definition 3.1.

**Example 3.2.** 1. Taking the maybe monad (and no operation) we obtain the pure $\lambda$-calculus.
2. Taking the exception monad with the raising exception operation(s) of Example 2.3 we obtain a $\lambda$-calculus with exceptions.
3. Taking the powerset with set-theoretic union (as in Example 2.3), we obtain the nondeterminism $\lambda$-calculus [5, 14, 22].
4. Taking the subdistribution monad $\mathcal{D}$ and fair probabilistic choice (as in Example 2.3) we obtain the probabilistic $\lambda$-calculus of [4].
5. Taking the global state monad and the operation for reading and writing stores of Example 2.3 we obtain the imperative $\lambda$-calculus.
6. Taking the output monad and the printing operation(s) of Example 2.3 we obtain the $\lambda$-calculus with output [7, 29].

We follow standard notational conventions, as in [2]. In particular, we denote by $\Lambda$ and $\mathcal{V}$ the collection of closed terms (programs) and values. Additionally, we write $\text{let } x = t \text{ in } s$ for $(\lambda x.t)s$.

Next, we want to give, say call-by-value, semantics to $\Lambda_\Sigma$. To do so, we follow [3] and give a monadic semantics to the calculus. That is, to any closed term $t$ is associated an element $[t]$ in $T(\mathcal{V})$ (we call such elements monadic values). Intuitively, the map $[-] : \Lambda \rightarrow T(\mathcal{V})$ should be defined as follows.
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4 Countable Monads and Generic Effects

In previous section we defined the (result of the) evaluation of a program \( t \) as a monadic value \( [t] \in T(V) \). Working with monadic values has the major advantage of providing semantical information on the effects performed. However, it also has a major drawback: \( [t] \) being a monolithic object, it does not allow a clear distinction between the effects produced by \( t \) and the possible results (values) obtained.

Intuitively, since during the evaluation of \( t \) effects can only be produced by (algebraic) operations, we would expect \( [t] \) to be a pair of the form \( (\Gamma, \langle \nu_i \rangle_{i \in I}) \), where \( \Gamma \) is a mathematical object describing the effects produced by \( t \), and \( \langle \nu_i \rangle_{i \in I} \) is the list of possible values returned.

Notice that since the calculus may be nondeterministic, \( I \) has cardinality bigger than one in general, and that due to recursion \( I \) may be infinite (but countable). To see that, consider the nondeterministic program \( \mathbf{null} \), where \( \mathbf{null} \) stands for the (Church) numeral of 0, \( \text{suc} \) is the successor function, and \( t \) is recursively defined by the equation \( t = \lambda x. (x \cup t(\text{suc} x)) \).

The purpose of this section is to prove that the aforementioned composition is indeed always possible.

4.1 Countable Monads

First, we have to show that any monadic value has indeed a ‘value component’. Mimicking the terminology employed for subdistributions, we refer to such a component as support. As the set of terms is countable, we expect any monadic value to have countable support. In general, not all monads come with a notion of support, and even those that have such a notion may have objects with uncountable support (e.g., the continuation monad \( \mathbb{K} \)). As a consequence, we need to isolate the class of monads whose elements have countable support.

First, we need monad to preserves injections (the reason why we need such monads will become clear soon). That is, if \( A \rightarrow X \) is the subset inclusion map, then \( T(A) \rightarrow T(X) \) is an injection (i.e., a mono), which we regard as monadic inclusion. Notice that if \( T \) preserves weak pullbacks, then it also preserves monos. This condition is met by all the monads in Example 2.1 (see, e.g., \( \mathbb{P} \)).

Given a monadic object \( \mu \in T(X) \), the support of \( \mu \) is the smallest set \( A \rightarrow X \) such that \( \mu \in T(A) \). We denote such a set by \( \text{supp}(\mu) \). Of course, in general the support of \( \mu \) need not exist and thus we restrict our analysis to monads coming with a notion of countable support.

Definition 4.1. We say that a monad is countable if for any set \( X \) and any element \( \mu \in T(X) \), there exists a smallest countable set \( Y \rightarrow X \), denoted by \( \text{supp}(\mu) \), such that \( \mu \in T(Y) \) (i.e., there exists \( v \in T(Y) \) such that \( \mu = (Tv)(v) \)).

Example 4.2. All the monads in Example 2.1 are countable, with the exception of the powerset monad. Nonetheless, we can regard \( \mathbb{P} \) as countable (by taking its countable restriction), the collection of \( \lambda \)-terms being countable itself.

Remark 1. The notion of support is sometimes formalized throughout the notion of an accessible functor (monad, in our case). Accordingly, a monad \( T \) is \( \kappa \)-accessible, for a cardinal \( \kappa \), if for any element \( \mu \in T(X) \) there exists \( A \subseteq X \) with cardinality strictly smaller than \( \kappa \) such that \( \mu \in T(A) \). If \( \kappa = \aleph_0 \), then \( T \) is said to be finitary as, intuitively, any element in \( T(X) \) has a finite support. Moreover, if \( T \) preserves weak pullbacks, then it also preserves finite intersection, and one can define for \( \mu \in TX \), the support of \( \mu \) as:

\[
\text{supp}(\mu) \triangleq \bigcap \{ A \subseteq X \mid \mu \in TA \}.
\]

Since we deal with elements with countable support, we need to shift from finitary monads to \( \aleph_0 \)-accessible monad. As a consequence, preservation of weak pullbacks is not enough to guarantee the existence of \( \cap \{ A \subseteq X \mid \mu \in TA \} \), as we now need \( T \) to preserve countable intersections.

4.2 Generic Effects

Working with countable monads we can think of the support of a monadic object as its ‘value component’. What
about its ‘effect component’? Here we show that such a component can be formally described using the notion of a generic effects [25]. Achieving such a goal, however, requires to introduce some mathematical abstractions. As a consequence, we first study a concrete example which will then generalise to arbitrary (countable) monads.

**Example 4.3** (Probabilistic computations). When working with (discrete) subdistributions, hence with the monad $D$, it is oftentimes convenient to represent subdistributions as syntactic objects called formal sums. A formal sum (over a set $X$) is an expression of the form $\sum_{i \in I} p_i x_i$, where $I$ is a countable set, $p_i \in [0, 1]$ and $x_i \in X$, and $\sum p_i \leq 1$.

The notation $\sum_{i \in I} p_i x_i$ is meant to recall the semantic counterpart of formal sums, namely subdistributions. However, we should keep in mind that formal sums are purely syntactical expressions. For instance, abusing a bit the notation

$\sum_{i=1}^{\infty} \frac{1}{2^i} x_0 + \frac{1}{2} x_0$ and $1; x_0$ are two distinct formal sums, although they both denote the Dirac distribution on $x_0$. More generally, there is an interpretation function $I$ mapping each formal sum $\sum_{i \in I} p_i x_i$ to a subdistribution $\mu$ on $X$ defined as $\mu(x) \triangleq \sum_{i \in x} p_i$. Additionally, the map $I$ is a surjection, meaning that any subdistribution can be represented as a (non-unique) formal sum.

Examining a bit more carefully a formal sum $\sum_{i \in I} p_i x_i$, we see that the latter consists of an $I$-indexed sequence $\langle p_i \rangle_{i \in I}$ of elements in $[0, 1]$ and an $I$-indexed sequence $\langle x_i \rangle_{i \in I}$ of elements in $X$. Therefore, a formal sum is just a pair of sequences $\langle p_i \rangle_{i \in I}, \langle x_i \rangle_{i \in I} \in [0, 1]^I \times X^I$ such that $\sum_i p_i \leq 1$. But the latter requirement means precisely that $\langle p_i \rangle_{i \in I}$ is actually a subdistribution on $I$ (the one mapping $I$ to $p_i$). Therefore, we see that formal sums are just elements in $D(I) \times X^I$.

Putting all these observations together, we see that for any $\mu \in D(X)$, there exists a countable set $I$ and an element $F \in D(I) \times X^I$ such that $I(F) = \mu$. As a consequence, stipulating two formal sums $F_1, F_2 \in D(I) \times X^I$ to be equal

$\langle F_1 \rangle \equiv \langle F_2 \rangle$ if $I(F_1) = I(F_2)$, then we see that $D(X)$ is isomorphic to the quotient set $(\bigcup_I D(I) \times X^I)/\equiv_I$, where $I$ ranges over countable sets.

Summing up, Example 4.3 shows that any subdistribution $\mu \in D(V)$ can be decomposed as a pair $\langle (p_i)_{i \in I}, (v_i)_{i \in I} \rangle$, for some (countable) set $I$, where $\langle p_i \rangle_{i \in I}$ is the ‘effect’ part of $\mu$, and $\langle v_i \rangle_{i \in I}$ is the value part of $\mu$. Even if such a decomposition is not unique, we can give definitions and prove results on such decompositions and extend them to subdistributions by showing invariance with respect to $\equiv_I$, which is usually trivial.

Can we generalize Example 4.3 to arbitrary (countable) monads?

First, let us observe that since the set $I$ in Example 4.3 is countable, we can replace it with an enumeration of its elements. That is, we replace $I$ with sets $\mathbf{n}$, where $n \in \mathbb{N}^\omega \triangleq \mathbb{N} \cup \{\omega\}$ and $1^n \triangleq \{1, \ldots, n\}$ if $n \neq \omega$, and $1^n \triangleq \mathbb{N}$, if $n = \omega$.

Formally, we should work $\mathbb{N}_1$, the skeleton of the category of countable sets and all functions between them, instead of $\mathbb{N}^{\omega}$ (notice that $\mathbb{N}_1$ and $\mathbb{N}^\omega$ have the same objects, but the former has ‘more arrows’, so to speak). Nonetheless, thinking of $\mathbb{N}^{\omega}$ in place of $\mathbb{N}_1$ is perfectly fine for building intuitions. Abusing the notation, we write $n \in \mathbb{N}_1$ to state that $n$ is an object of $\mathbb{N}_1$.

**Theorem 4.4** ([9, 16, 26]). For any set $X$ we have the isomorphism

$$ TX \cong \int_{n \in \mathbb{N}_1} T(n) \times X^n. $$

Let us decode Theorem 4.4. First, $\int_{n \in \mathbb{N}_1} T(n) \times X^n$ is a co-end [17], an abstract notion that is not needed to achieve our goals. For us, $\bigcup_{n \in \mathbb{N}_1} T(n) \times X^n$ is the quotient of $\prod_{n \in \mathbb{N}_1} T(n) \times X^n$ for a suitable equivalence relation.

More precisely, we can translate Theorem 4.4 as follows:

**Theorem 4.5.** For any set $X$, all elements in $T(X)$ can be (non-uniquely) presented as elements in

$$ \bigcup_{n \in \mathbb{N}_1} T(n) \times X^n $$

Moreover, there is a surjective map $I$ such that for any $\mu \in T(X)$ there exists $n \in \mathbb{N}^{\omega}$ such that $\mu$ can be uniquely represented as an equivalence class modulo $\equiv_I$, the kernel of $I$, of an element in $T(\mathbf{n}) \times X^n$.

Replacing $\bigcup_{n \in \mathbb{N}_1} T(n) \times X^n$ with the more correct (yet morally equivalent) $\prod_{n \in \mathbb{N}_1} T(n) \times X^n$, we achieve the correspondence

$$ T(X) \cong \int_{n \in \mathbb{N}_1} T(n) \times X^n. $$

Let us see how to prove Theorem 4.5. In the following, we will oftentimes regard a sequence in $X^n$ as a function in $n \to X$.

**Proof of Theorem 4.5.** First, let us define the map $I$. Given $\Gamma \in T(n)$ and $s : n \to X$ (where $n \in \mathbb{N}_1$), define $I(\Gamma, s) \triangleq T(s)(\Gamma)$. Next we prove Surjectivity of $I$. Let $\mu \in T(X)$. Since $T$ is countable, $\text{supp}(\mu)$ is countable, and thus isomorphic to $\mathbf{n}$, for some object $n$ of $\mathbb{N}_1$. Any $\mu \in T(\mathbf{n})$ has a countable support. Let $f : \text{supp}(\mu) \to n$ be such a bijection. Then we present $\mu$ as $(T(f) \mu, f^{-1})$. Since $I(T(f)(\mu), f^{-1}) = T(f^{-1})(T(f)(\mu)) = \mu$ we are done.

We call elements in $\prod_{n \in \mathbb{N}_1} T(n) \times X^n$ formal presentations. Before giving examples of formal presentations we introduce a diagrammatic notation for them.

\footnote{Counting from one rather than from zero simplifies the notation. We have also implicitly used this convention writing $\sigma(t_1, \ldots, t_n)$ for $n$-ary algebraic operations.}
5 A Diagrammatic Notation for Generic Effects

Representing monadic elements as formal presentations has the major drawback of introducing bureaucracy in the treatment of indexes. Consider, for instance, a program of the form \( \text{let } x = t \text{ in } s \). We try to define \( \llbracket \text{let } x = t \text{ in } s \rrbracket \) using formal presentation. First, we evaluate \( t \) obtaining \( (\Gamma, \langle x_i \rangle_{i \in \mathbb{N}}) \in T(\mathbb{N}) \times \mathcal{V}^n \), for some \( n \in \mathbb{N} \). For any \( i \in n \), we then evaluate \( s[x := x_i] \), obtaining \( (\Lambda_i, (w_j)_{j \in \mathbb{N}_i}) \), for some \( m_i \in \mathbb{N} \). As a consequence, evaluating \( \llbracket \text{let } x = t \text{ in } s \rrbracket \) as a diagram of the form

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\Gamma} \\
\xrightarrow{x_i}
\end{array}
\end{array}
\]

for \( i \) ranges over elements in \( n \) and to each \( i \) it is associated the corresponding \( x_i \). That is, the horizontal bar with subscript \( i \) and target \( x_i \) stands for the function \( i \mapsto x_i \).

If \( n \) is finite, we can modify Definition 5.1 by extensionally listing all elements \( x_1, \ldots, x_n \). For instance,

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\Gamma} \\
\xrightarrow{x_{i_1}} \\
\ldots \\
\xrightarrow{x_{i_n}}
\end{array}
\end{array}
\]

is the extensional version of \( \begin{array}{c}
\begin{array}{c}
\xrightarrow{\Gamma} \\
\xrightarrow{x_i}
\end{array}
\end{array} \) for \( n \in \mathbb{N} \).

**Example 5.2.** 1. Consider the maybe monad \( M \). We present an object \( \mu \in M(\mathbb{X}) \) as a pair in \( M(\mathbb{N}) \times \mathcal{X}^n \), for some \( n \in \mathbb{N} \). Since

\[
M(\mathbb{N}) \times \mathcal{X}^n = (\mathbb{N} + \{\|\}) \times \mathcal{X}^n \equiv (\mathbb{N} \times \Lambda^n) + (\Lambda^n \times \{\|\}),
\]

\( \mu \) (presented as) either a pair \((k, \langle x_i \rangle_{i \in \mathbb{N}})\) or a pair \((\|, \langle x_i \rangle_{i \in \mathbb{N}})\). The former corresponds to the case of convergence to \( x_k \), whereas the latter to divergence. In particular, if \( \mu \) is the result of evaluating a \( \lambda \)-term, then we will actually have \( n = 1 \) (if the term converges) or \( n = 0 \) (if the term diverges).

• If \( n = 1 \), we obtain pairs of the form \((1, \langle x \rangle)\), which we write as \( \xrightarrow{\Gamma} x \).

• If \( n = 0 \), then we can only have the pair \((\|, \langle \rangle)\), where \( \langle \rangle \) is the empty sequence. We write such a pair as \( \xrightarrow{\Gamma} \| \).

2. Consider the output monad \( O \). We present an object \( \mu \in O(\mathbb{X}) \) as a pair in

\[
O(\mathbb{N}) \times \mathcal{X}^n = \mathcal{O}^\infty \times (\mathbb{N} + \{\|\}) \times \mathcal{X}^n,
\]

for some \( n \in \mathbb{N} \). Therefore, \( \mu \) is presented as either a triple \((w, \|, \langle x_i \rangle_{i \in \mathbb{N}})\), or as a triple \((w, k, \langle x_i \rangle_{i \in \mathbb{N}})\). The former case means that we have divergence, and that the string \( w \) is outputted, whereas the latter case means that we converge to \( x_k \), and that the string \( w \) is outputted. As before, if \( \mu \) is the result of evaluating a term, we will have either \( n = 1 \) (if the term converges) or \( n = 0 \) (if the term diverges).

1. If \( n = 1 \), we have triples of the form \((w, \|, x)\) which we write as \( \xrightarrow{\Gamma} w \rightarrow x \).

2. If \( n = 0 \), the we can only have triples of the form \((w, \|, \langle \rangle)\), which we write as \( \xrightarrow{\Gamma} w \).

3. The case for the subdistribution monad goes exactly as in Example 4.3. We can represent a formal sum \( (\langle p_i \rangle_{i \in \mathbb{N}}, \langle x_i \rangle_{i \in \mathbb{N}}) \) as \( \xrightarrow{\Gamma} \sum_{i \in \mathbb{N}} p_i \cdot \xrightarrow{\Delta_i} x_i \).

4. Replacing \((\{0, 1\} \cup \{0, 1\}) \) with \((\{0, 1\}, \vee)\) in Example 4.3 we obtain formal presentations for elements in \( \mathcal{P}(\mathbb{X}) \), which we may write as diagrams \( \xrightarrow{\Gamma} \sum_{i \in \mathbb{N}} p_i \cdot \xrightarrow{\Delta_i} x_i \).

5.1 A Calculus of Diagrams: Sequential Composition

One of the strengths of monads (at least concerning their usage in the semantics of programming languages) is that they naturally support the sequential composition of effects. Do diagrams do the same?

Let us consider again \( \llbracket \text{let } x = t \text{ in } s \rrbracket \). Suppose:

1. \( s[t] = \langle x_i \rangle_{i \in \mathbb{N}} \)
2. For any \( i \in \mathbb{N} \), \( s[x := x_i] = \sum_{j \in \mathbb{N}} m_i (w_j) \).

Can we find a diagram for \( \llbracket \text{let } x = t \text{ in } s \rrbracket \)? Since \( \llbracket \text{let } x = t \text{ in } s \rrbracket \) in \( s \) is nothing but a sequential composition, a natural proposal is to write

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\Gamma} \\
\xrightarrow{\Delta_i} \\
\xrightarrow{m_i} \\
\xrightarrow{w_j}
\end{array}
\end{array}
\]

for \( \llbracket \text{let } x = t \text{ in } s \rrbracket \). But: is this figure meaningful? That is, do we have a notion of composition for diagrams? We are going to answer such questions in the affirmative.

5.1.1 Sequential Composition

There are at least two ways to define the sequential composition of two diagrams/formal presentations. The first one relies on the correspondence between generic effects and algebraic operations [25].

**Proposition 5.3.** Any generic effect \( \Gamma \in T(\mathbb{N}) \) corresponds to a \( n \)-ary algebraic operation \( \gamma \), and any \( n \)-ary algebraic operation \( \gamma \) corresponds to a generic effect \( \Gamma \in T(\mathbb{N}) \). Moreover, if \( \langle \Gamma, (x_i)_{i \in \mathbb{N}} \rangle \) is a formal presentation of \( \mu \), then \( \mu = \gamma(\ldots, \eta(x_i), \ldots) \).

**Proof.** The bijection between algebraic operations and generic effects is given in [25]. Given \( \Gamma \in T(\mathbb{N}) \) we define \( \gamma_X : T(\mathbb{N}) \rightarrow T(\mathbb{X}) \) as \( \gamma_X(\tilde{x}) \triangleq (\eta_X \circ \tilde{x})(\Gamma) \), where \( \tilde{x} : \mathbb{N} \rightarrow \mathbb{X} \). Vice versa, given \( \gamma_X : T(\mathbb{N}) \rightarrow T(\mathbb{X}) \) we define \( \Gamma \triangleq \gamma_X(\eta_X) \). Suppose now \( \langle \Gamma, (x_i)_{i \in \mathbb{N}} \rangle \) is a formal presentation of \( \mu \). We write \( \tilde{x} \) for the sequence \( \langle x_i \rangle_{i \in \mathbb{N}} \) regarded as a function \( n \rightarrow \mathbb{X} \). We
have:

\[ \mu = I(\Gamma, (x_i)_{i \in \mathbb{N}}) = (\eta_X \circ \xi)^\dagger(\Gamma) = (\eta_X \circ \xi)^\dagger(\eta_n(\eta_n)) = \gamma_n((\eta_X \circ \xi)^\dagger \circ \eta_n) = \gamma_n(\eta_X \circ \xi) \]

where the penultimate equality follows by the defining identity law of algebraic operations (see Definition 2.2).

**Example 5.4.** 1. The generic effect corresponding to operation(s) print \( : O(X) \rightarrow O(X) \) is \((c, 1)\).

2. The generic effect corresponding to fair probabilistic choice \( \oplus \) is the distribution \((0.5, 0.5)\) mapping \( i \in \{1, 2\} \) to 0.5.

**Corollary 5.5.** There exists a notion of sequential composition for generic effects: given a generic effect \( \Gamma \in T(n) \) and a \( n \)-indexed family of generic effects \( \Delta_i \in T(m_i) \), there exists their sequential composition which is a generic effect in \( T(I) \), for \( l \triangleq \sum_{i \in \mathbb{N}} m_i \).

**Proof.** Let us consider \( \Gamma \in T(n) \) and \( \Delta_i \in T(m_i) \), for any \( i \in \mathbb{N} \). The former gives the algebraic operation \( \gamma : T(X)^n \rightarrow T(X) \). Similarly, the latter gives a family of algebraic operations \( \delta_i : (TX)^m_i \rightarrow T(X) \). Taking products, we obtain

\[ T(X)^\sum_{i \in \mathbb{N}} m_i \cong \prod_{i \in \mathbb{N}} T(X)^{m_i} \cong T(X)^\sum_{i \in \mathbb{N}} \delta_i \cong T(X)^n \]

We then take advantage of Proposition 5.3 and define the composition of \( \Gamma \) and \( \Delta_i \)'s as the generic effect associated to the algebraic operation \( \gamma \circ \langle \delta_i \rangle_{i \in \mathbb{N}} \). \( \square \)

The drawback of Corollary 5.5 is that we would like to define the composition of generic effects without going through their associated algebraic operations. To achieve such a goal, we look at the countable Lawvere theory [9, 15] induced by generic effects.

First, observe that \( \mathbb{N}_1 \) has countable coproducts and, up-to equivalence, it is the free category with countable coproducts on 1, where coproducts are given by standard sums (thus, for instance, for \( n \in \mathbb{N} \) we have \( n = 1 + \cdots + 1 \)). By duality, \( \mathbb{N}_1^{op} \) has countable products.

**Definition 5.6** ([26]). A countable Lawvere theory is a small category \( \mathcal{L} \) with countable coproducts and a strict countable-coproduct preserving identity-on-objects functor \( I : \mathbb{N}_1^{op} \rightarrow \mathcal{L} \).

In particular, objects of \( \mathcal{L} \) are exactly those of \( \mathbb{N}_1 \), and every function between such objects gives a map in \( \mathcal{L} \). A map in \( \mathcal{L}(n, 1) \) represents an operation of arity \( n \) (meaning that we also consider operations with arity 0), whereas a map in \( \mathcal{L}(n, m) \) represent \( m \) operations of arity \( n \). In particular, given operations \( \sigma \in \mathcal{L}(n, m) \) and \( \tau \in \mathcal{L}(m, l) \), their composition \( \tau \circ \sigma \in \mathcal{L}(n, l) \) gives \( l \) operations of arity \( n \).

**Proposition 5.7.** Generic effects form a Lawvere theory, and thus we can define the composition of generic effects as composition on their Lawvere theory.

**Proof.** Let us begin showing that generic effects form a Lawvere theory. Let \( K(T)^{op} \) be the opposite of \( K(T) \) restricted to objects of \( \mathbb{N}_1 \). Since \( K(T) \) has countable coproducts, and the canonical functor \( I : \text{Set} \rightarrow K(T) \) preserves them, restricting \( I \) to \( \mathbb{N}_1 \) we obtain a countable-coproduct preserving identity-on-objects functor \( I : \mathbb{N}_1 \rightarrow K(T)_{\mathbb{N}_1} \). As a consequence, \( K(T)_{\mathbb{N}_1} \) is the opposite of a Lawvere theory, and thus \( K(T)_{\mathbb{N}_1}^{op} \) is a (countable) Lawvere theory. Notice that a generic effect \( \Gamma \in T(n) \) corresponds to a map in \( K(T)_{\mathbb{N}_1}^{op}(n, 1) \), i.e. as an operation in the Lawvere theory \( K(T)_{\mathbb{N}_1}^{op} \). We now define composition of generic effects. Given generic a effect \( \Gamma \in T(n) \) and an \( n \)-indexed family of generic effects \( \Delta_i \in T(m_i) \) (for any \( i \in \mathbb{N} \)), i.e. arrows \( \Gamma \in K(T)_{\mathbb{N}_1}^{op}(n, 1) \) and \( \Delta_i \in K(T)_{\mathbb{N}_1}^{op}(m_i, 1) \), we can define its composition relying on composition in \( K(T)_{\mathbb{N}_1}^{op}(n, 1) \) as follows. Since \( \Delta_i \in K(T)_{\mathbb{N}_1}^{op}(m_i, 1) \) and \( K(T)_{\mathbb{N}_1} \) has products, we have the map \( \langle \Delta_i \rangle_{i \in \mathbb{N}} \in K(T)_{\mathbb{N}_1}^{op}(\sum_i m_i, 1) \), and thus \( \Gamma \circ \langle \Delta_i \rangle_{i \in \mathbb{N}} \in K(T)_{\mathbb{N}_1}^{op}(\sum_i m_i, 1) \) is the desired generic effect. \( \square \)

Coming back to diagrams, given \( \begin{array}{c} \Gamma \\ n \end{array} \xrightarrow{i} x_i \) and \( \begin{array}{c} \Delta_j \\ m_j \end{array} \xrightarrow{j} y_j \), for any \( i \in \mathbb{N} \), we write their composition as

\[
\begin{array}{c} \Gamma \\ n \end{array} \xrightarrow{i} \begin{array}{c} \Delta_j \\ m_j \end{array} \xrightarrow{j} \begin{array}{c} y_j \\ \end{array}
\]

Such a notation gives us several advantages, as we are going to see. However, we first look at some concrete examples.

**Example 5.8.** 1. Consider the maybe monad and consider representations that can be obtained as the result of computations. Then sequential composition is defined by the following laws:

\[
\begin{array}{c} w \\ \end{array} \xrightarrow{\xi} \begin{array}{c} w \\ \end{array} \xrightarrow{\xi} \begin{array}{c} w \\ \end{array}
\]

2. Consider the output monad. As before, we consider representations that can be obtained as the result of computations. Sequential composition is thus defined:

\[
\begin{array}{c} w \\ \end{array} \xrightarrow{\xi} \begin{array}{c} (w, \xi) \\ \end{array} \xrightarrow{\xi} \begin{array}{c} (w, \xi) \\ \end{array}
\]

3. Consider the distribution monad, and denote by \( p \xi \) scalar multiplication of \( p \) with \( \xi \). We then define sequential composition as:

\[
\begin{array}{c} \xi \\ \end{array} \xrightarrow{\xi} \begin{array}{c} \xi \\ \end{array} \xrightarrow{\xi} \begin{array}{c} \xi \\ \end{array}
\]

\[
\begin{array}{c} \xi \\ \end{array} \xrightarrow{\xi} \begin{array}{c} \xi \\ \end{array} \xrightarrow{\xi} \begin{array}{c} \xi \\ \end{array}
\]
We immediately notice that composition is associative. This is a notion of equality for diagrams. In fact, what we are doing is de facto building an algebra of diagrams. Without much of a surprise, we say that two diagrams \( \xi, \rho \) are equal (notation \( \xi = \rho \)) if and only if \( \xi =_f \rho \) is the map of Theorem 4.5.

It goes by itself that working with \( =_f \) directly may be quite heavy. For this reason, we now prove a collection of algebraic results that dispense us from working with \( =_f \). Such results concern sequential composition, monadic binding, and sequential composition.

### 5.2 Algebra of Sequential Composition

We immediately notice that composition is associative. This property is actually built-in the diagrammatic notation, as evident when writing the composition of three diagrams:

\[
\begin{array}{c}
\Gamma^m \overset{\eta}{\xrightarrow{i}} \Delta^j \overset{\xi}{\xrightarrow{j}} \Delta^k \\
\end{array}
\]

Linguistically, this notation describes associativity of sequencing, usually expressed by program equivalences of the following form:

\[
\text{let } x = t \text{ in } \text{(let } y = s \text{ in } r) \equiv \text{(let } x = t \text{ in } s) \text{ in } r.
\]

Diagram (1) also highlights another important feature of diagrams, namely the way they manage index dependencies. The ‘geography’ of (1) shows that \( i \in n, j \in m_i, \) and \( k \in l_j \). Moreover, by reading from the right to the left we recover index dependencies: we see that since \( k \) (being in \( l_j \)) depends on \( j \) which depends (being in \( m_i \)) on \( i \), \( k \) depends on \( i \) as well.

Second, there is a trivial generic effect \( H \in T(1) \) (read capital \( \eta \)) which behaves as a neutral element for composition. That is, for any \( x \in X \), the diagram \( H \xrightarrow{\eta} x \) represent the computation trivially returning \( x \) performing no effect (i.e. \( \eta(x) \)). We have the following laws:

\[
\begin{array}{c}
H \xrightarrow{\xi} = \xi \\
\Gamma^m \overset{\eta}{\xrightarrow{i}} H \xrightarrow{\xi} = \Gamma^m \overset{\eta}{\xrightarrow{i}} x_i
\end{array}
\]

Linguistically, these laws correspond to the following program equivalences:

\[
\begin{align*}
\text{let } x = u \text{ in } t & \equiv t[x := u] \\
\text{let } x = t \text{ in } x & \equiv t
\end{align*}
\]

\[\text{Example 5.9.} \] The object \( \Gamma \xrightarrow{\eta} x \) is \( \Gamma \xrightarrow{\eta} x \) in the maybe monad \( M \), \( \Gamma \xrightarrow{\eta} x \) in the output monad \( O \), and \( \Gamma \xrightarrow{\eta} x \) in the subdistribution (resp. powerset) monad \( D \) (resp. \( P \)).

Additionally, diagram equality is preserved by diagrams.

\[\text{Example 5.11.} \] Let us consider monadic terms, i.e. elements in \( T(\Lambda) \). Recall that we have defined an evaluation map \( [-] : \Lambda \to T(V) \). Define the monadic extension of \( [-]^T \) by:

\[
\left[ \Gamma \overset{\xi}{\xrightarrow{i}} \right]^T \equiv \Gamma \overset{\xi}{\xrightarrow{i}} [t_i]
\]

\[\text{Example 5.11.} \] Let us consider monadic terms, i.e. elements in \( T(\Lambda) \). Recall that we have defined an evaluation map \( [-] : \Lambda \to T(V) \). Define the monadic extension of \( [-]^T \) by:

\[\left[ \Gamma \overset{\xi}{\xrightarrow{i}} \right]^T \equiv \Gamma \overset{\xi}{\xrightarrow{i}} [t_i]\]
A Diagrammatic Calculus for Algebraic Effects,

Then \([-\,]^T\) is automatically well-defined, as by Theorem 5.10 \([-\,]^T\) is invariant under diagram equality, and thus it is independent of the choice of representatives. Notice also how the diagrammatic notation makes explicit how definitions and results ‘distribute’ over effects. This is the very essence of algebraic effects, as stated in the defining identity of algebraic operations (Definition 2.2). The diagrammatic notation extends this identity to any monadic expressions, as the latter are ultimately built using generic effects, which bijectively correspond to algebraic operations.

5.2.2 Algebraic Operations

Thanks to Proposition 5.3, diagrams are also a natural way to write algebraic operations. In general, if we present objects \(\mu_i \in T(X)\) as \(\xi_i\), then we write

\[
\sigma \xrightarrow{\xi_1} \xi_n \text{ or } \sigma \xrightarrow{n} f(\xi_i),
\]

for the presentation of \(\sigma(\mu_1, \ldots, \mu_n)\) (that is, we use the notation \(\sigma\) both for the algebraic operation and the generic effects associated to it). Notice that thanks to Theorem 5.10 both \(\sigma(\mu_1, \ldots, \mu_n) \Rightarrow f\) and \(\sigma(\mu_1 \Rightarrow f, \ldots, \mu_n \Rightarrow f)\) are presented as

\[
\sigma \xrightarrow{n} f(\xi_i),
\]

where \(\xi_i\) is a presentation of \(\mu_i\), this way encoding the defining identity of algebraic operations (Definition 2.2) in diagrams. Linguistically, this corresponds to the program transformation

\[
\text{let } x = \sigma(t_1, \ldots, t_n) \text{ in } s \equiv \sigma(\ldots, \text{let } x = t_i \text{ in } s, \ldots).
\]

Having seen the basic algebra of diagrams, we move to the study of their order-theoretic properties.

5.3 A Calculus of Diagrams: Order

So far we have focused on algebraic properties of diagrams. However, working with \(\Sigma\)-continuous monads, we can extend the order \(\subseteq\) on diagrams by stipulating \(\xi \subseteq \rho\) if and only if \(I(\xi) \subseteq I(\rho)\). This way, we see that diagrams enjoy pleasant order-theoretic properties. Before studying such properties, however, it is useful to relate the elements \(\langle x_i \rangle_{i \in n}\) in a diagram \(\Gamma \xrightarrow{n} \xi_i\) presenting \(\mu\) with the support of \(\mu\).

On Support Given an element \(\mu \in T(X)\) presented as a diagram \(\Gamma \xrightarrow{n} \xi_i\), we see that the set \(\{x_i \mid i \in n\}\) is a superset of \(\text{supp}(\mu)\). For instance, the Dirac distribution on \(x\) (i.e. \(\eta(x)\)) is presented as \(H \xrightarrow{n} x\) (i.e. as the formal sum \(1; x\)). However, we can also represent it as the formal sum \(1; x + 0; y\).

More generally, given \(\Gamma \in T(n)\) and \(n \xrightarrow{i} m\), we also have the injection \(T(n) \xrightarrow{T(i)} T(m)\). For instance, taking \(T = \mathcal{D}\) and \((\rho_i)_{i \in n} \in \mathcal{D}(n)\), the subdistribution \(\mathcal{D}(\langle i \rangle)(\rho_i)_{i \in n}\) maps \(i\) to \(\rho_i\) if \(i \in n\), and to 0 otherwise (i.e. if \(i \in m \setminus n\)).

Formally, given a countable set \(X\), any formal presentation \((\Gamma, \vec{x}) \in T(s) \times X^n\) can be extended to a formal presentation \((\Delta, \vec{y})\) in \(T(m) \times X^n\), for any \(n \xrightarrow{i} m\).

**Lemma 5.12.** Let \(X\) be a countable set. Then any pair \((\Gamma, \vec{x}) \in T(n) \times X^n\) can be extended to a formal presentation \((\Delta, \vec{y})\) in \(T(m) \times X^n\), for any \(n \xrightarrow{i} m\).

**Proof.** Since \(n \xrightarrow{i} m\), then \((n) \xrightarrow{T(i)} T(m)\). Take \(\Delta \triangleq T(i)(\Gamma)\) and \(\vec{y}\) to be any map extending \(\vec{x}\) according to the following diagram (such maps indeed exists):

\[
\begin{array}{ccc}
 m & \xrightarrow{\vec{y}} & X \\
 \downarrow & & \downarrow \\
 n & \xrightarrow{\vec{x}} & \n 
\end{array}
\]

Indeed \((\Gamma, \vec{x}) = (\Delta, \vec{y})\).

For instance, we have seen that the trivial generic effect \(H \in T(1)\) corresponds to the map \(\eta_1 : 1 \rightarrow T(1)\). However, for any \(n\), there is a map \(\eta_n : n \rightarrow T(n)\) such that, for any \(i \in n\), \(\eta(i) = T(i)(H)\), for \(1 \xrightarrow{i} n\) sending \(1 \in 1\) to \(i\). As a consequence, we see that we can indeed think of \(H\) as the ‘real’ trivial effect, and regard the others as its extension to larger supports. For instance, we can regard \(1; x\) as the presentation of the Dirac distribution on \(x\), and formal sums such as \(1; x + 0; y + 0; z\) as extensions of \(1; x\).

**Order-theoretic Properties** We now analyze the order-theoretic properties of diagrams. For technical reason, we need to require \(T(1)\) to have at least two elements\(^6\), i.e. \(\eta(1) \neq 1\). First, we observe that there is a bottom effect \(\bot \in T(0)\), which we write as \(\bot\). It is easy to see that such element is \(\bot\) in the maybe monad, \([\varepsilon, T]\) (where \(\varepsilon\) is the empty string) in the output monad, and \([\emptyset]\) in the subdistribution monad.

Actually, for any \(n\) there is a bottom effect \(\bot_n \in T(n)\) such that \(\bot_n \xrightarrow{n} ^\bot x_i \equiv \bot m \xrightarrow{n} ^\bot y\). This is a bit unsatisfactory, as we might expect the ‘real’ bottom effect to be \(\bot\). That is actually the case.

In fact, since \(T\) is \(\Sigma\)-continuous, \(T(f)\) is strict, for any \(f : X \rightarrow Y\). Therefore, for any \(n\), the map \(T(0) \xrightarrow{i} n\) is strict, meaning that \(\bot_n = T(i)(\bot)\). That means that we essentially have a unique bottom effect, viz. \(\bot\) and we can regard any \(\bot_n\) as its extension to larger supports. In fact, we have the equality

\[
\bot_n \xrightarrow{n} ^\bot x_i = \bot
\]

as well as the inequality

\[
\bot \subseteq ^\bot
\]

\(^6\) This property ensures the possibility to embed the Boolean algebra (which is isomorphic to) \(2\) to \(1\). Notice also that if \(\bot = \eta(1)\), then \(T(1) \simeq 1\). It is a straightforward exercise to verify that all monads mentioned so far satisfy this condition.
For this reason we write $\bot$ in place of $\bot_n$.

We may also ask whether the ‘dual’ equality holds, i.e.

$$\Gamma_n \bot = \bot$$

This is not the case. In fact, there are algebraic operations such as $\text{print}$, that are not strict. However, if effects are commutative, then such an equality holds.

**Definition 5.13.** We say that a monad is commutative if

$$\Gamma_n \Delta_j \xi_{ij} = \Delta_j \Gamma_n \xi_{ij}$$

Notice that $m$ is independent of $i$, and $n$ of $j$.

Please observe that Definition 5.13 is rather different from standard definitions of commutative monads one meets in the literature. Indeed, one of the advantages of our notation is to allow for a simple, operational definition of commutativity of monads. Accordingly, we easily notice that the commutativity equation in Definition 5.13 is just the semantical counterpart of the following program equivalence:

$$\text{let } x = t \text{ in } (\text{let } y = s \text{ in } r) \equiv \text{let } y = s \text{ in } (\text{let } x = t \text{ in } r)$$

where $x \in \text{FV}(s)$ and $y \notin \text{FV}(t)$.

**Proposition 5.14.** If $T$ is commutative, then

$$\Gamma_n \bot = \bot$$

**Proof.** Calculate

$$\Gamma_n \bot = \Gamma_n \bot \bot_n \equiv \bot_n \Gamma_n \bot = \bot$$

Having clarified the role of the bottom effect, let us now move to monotonicity laws.

**Proposition 5.15.** The following monotonicity laws hold:

$$\Gamma_n f(x_i) \sqsubseteq \Delta_j \Gamma_n f(y_j)$$

$$f, g : X \rightarrow T(Y) \quad \Rightarrow \quad f \sqsubseteq g$$

**Proof.** We prove the first rule (the second is similar). Let $\xi \triangleq \Gamma_n f(x_i)$, $\rho \triangleq \Delta_j \Gamma_n f(y_j)$, $\gamma$ the algebraic operation corresponding to $\Gamma$, and $\delta$ the one corresponding to $\Delta$. Notice that by hypothesis we know $\gamma(\ldots, \eta(x_i), \ldots) \sqsubseteq \delta(\ldots, \eta(x_i), \ldots)$, and recall that $f^\gamma$ is monotone. We have $I(\xi) \sqsubseteq \gamma I(\rho)$. For:

$$I(\xi) = f^\gamma(\gamma(\ldots, \eta(x_i), \ldots))$$

$$f^\gamma(\delta(\ldots, \eta(x_i), \ldots)) = I(\rho).$$

**Corollary 5.16.** The following monotonicity law hold:

$$\forall i \in n. \xi_i \sqsubseteq \rho_i$$

**Proof.** Define $f, g : n \rightarrow T(x)$ by $f(i) \triangleq \xi_i$ and $g(i) \triangleq \rho_i$. Clearly, we have $f \sqsubseteq g$. Moreover, since $\Gamma_n \xi_i = \Gamma_n f(i)$ and $\Gamma_n \rho_i \sqsubseteq \Gamma_n g(i)$ we conclude the thesis by Proposition 5.15. □

6 Conclusion

We have introduced a diagrammatic notation and calculus for countable monads with algebraic operations. We have shown by means of examples and general results some of advantages of our notation. Additionally, the authors are currently using this notation to prove new, nontrivial theorems on calculi with algebraic effects whose proofs turned out to be extremely heavy using the standard, linear notation. We leave as a future work the investigation of further applications of such a notation.

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