Alternating direction method of multipliers for convex programming: a lift-and-permute scheme

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Abstract A lift-and-permute scheme of alternating direction method of multipliers (ADMM) is proposed for linearly constrained convex programming. It contains not only the newly developed balanced augmented Lagrangian method and its dual-primal variation, but also the proximal ADMM and Douglas-Rachford splitting algorithm. It helps to propose accelerated algorithms with worst-case $O(1/k^2)$ convergence rates in the case that the objective function to be minimized is strongly convex.

Keywords Convex programming · Augmented Lagrangian method · Alternating direction method of multipliers · Douglas-Rachford splitting

1 Introduction

Consider the convex programming problem with linear equality constraints:

$$\begin{align*}
(P) \quad & \min_x f(x) \\
& \text{s.t. } Ax = b,
\end{align*}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$ is closed, proper, convex, but not necessarily smooth, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

As a fundamental and efficient approach for solving (P), the classical augmented Lagrangian method (ALM), dating back to [8,9], reads as

$$\begin{cases} x^{k+1} \in \arg \min_x \{ f(x) + (\lambda^k)^T (Ax - b) + \frac{\beta}{2} \|Ax - b\|^2 \}, \\ \lambda^{k+1} = \lambda^k + \beta (Ax^{k+1} - b), \end{cases}$$

where $\lambda$ is Lagrange multiplier corresponding to the equality constraints and $\beta > 0$ is a fixed penalty parameter. The update of $x^{k+1}$ may have no closed-form solution due to the coupling of $\|Ax - b\|^2$ and $f(x)$. The following proximal ALM [10] tries to overcome this difficulty by introducing a carefully designed proximity term:

$$\begin{cases} x^{k+1} \in \arg \min_x \{ f(x) + (\lambda^k)^T (Ax - b) + \frac{\beta}{2} \|Ax - b\|^2 + \frac{1}{2\rho} \|x - x^k\|^2 \}, \\ \lambda^{k+1} = \lambda^k + \beta (Ax^{k+1} - b). \end{cases}$$

In fact, by choosing $G = rI_n - \beta A^T A$ with $r > \beta \rho(A^T A)$ and $\rho(\cdot)$ being the spectral norm, we obtain a reduced update of $x^{k+1}$,

$$x^{k+1} \in \arg \min_x \left\{ f(x) + \frac{r}{2} \|x - x^k + \frac{1}{r} A^T [\lambda^k + \beta (Ax^k - b)]\|_G^2 \right\},$$

which is easy to solve if the following proximal mapping of $f(x)$

$$\text{prox}_{\gamma f}(x) := (I + \gamma \partial f)^{-1}(x) := \arg \min_y f(y) + \frac{1}{2\gamma} \|y - x\|^2$$

has a closed-form (or easy-to-compute) solution.

Since $r$ should be set larger than a fixed proportion of $\rho(A^T A)$, the shortcoming of the above proximal ALM is that, for large $\rho(A^T A)$, the iteration sequence $\{x^{k+1}\}$ will get stuck in updating. There is an alternative first-order primal-dual method presented in [1] with the following iteration formula:

$$\begin{cases} x^{k+1} \in \arg \min_x \{ f(x) + \frac{r}{2} \|x - (x^k - \frac{1}{r} A^T \lambda^k)\|^2 \}, \\ \lambda^{k+1} = \lambda^k + \frac{1}{r} [A(2x^{k+1} - x^k) - b], \end{cases}$$

where $r > 0$ and $s > 0$ satisfy that $rs > \rho(A^T A)$. Again, for large $\rho(A^T A)$, either $r$ or $s$ must be large enough. Then either $\|x^{k+1} - x^k\|$ or $\|\lambda^{k+1} - \lambda^k\|$ is small. Recently, He et al. [6] relaxed the requirement to $rs > 0.75 \rho(A^T A)$. 
In order to completely remove the restriction on the step-sizes $r$ and $s$, He and Yuan [7] proposed a simple but effective augmented Lagrangian method, the so-called balanced ALM, which reads as

$$
\begin{align*}
\lambda^{k+1} &= \lambda^k + \left( \frac{1}{r} AA^T + \delta I_m \right)^{-1} \left[ A(2x^{k+1} - x^k) - b \right], \\
x^{k+1} &= \arg \min_x \left\{ f(x) + \frac{r}{2} \| x - (x^k - \frac{1}{r} A^T \lambda^k) \|^2 \right\},
\end{align*}
$$

(1)

where $r > 0$ and $\delta > 0$ are arbitrary parameters. Different from the classical ALM and its proximal variations, balanced ALM (1) has an additional cost in updating $\lambda^{k+1}$ by solving the following linear equations

$$
\left( \frac{1}{r} AA^T + \delta I_m \right) (\lambda - \lambda^k) - (A(2x^{k+1} - x^k) - b) = 0.
$$

(2)

Following this idea, Xu [12] proposed a dual-primal balanced ALM with the same complexity per iteration as balanced ALM. The iteration is given by

$$
\begin{align*}
x^{k+1} &= \arg \min_x \left\{ f(x) + \frac{r}{2} \| x - \left\{ x^k - \frac{1}{r} A^T (2\lambda^k - \lambda^{k-1}) \right\} \|^2 \right\}, \\
\lambda^{k+1} &= \lambda^k + \left( \frac{1}{r} AA^T + \delta I_m \right)^{-1} \left( Ax^{k+1} - b \right),
\end{align*}
$$

(3)

where $r > 0$ and $\delta > 0$ are arbitrary parameters.

(P) can be regarded as a special case of the following two-block problem:

$$
\min_{x,y} \{ f(x) + g(y) : Ax + By = b \},
$$

(4)

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times l}$, $b \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^l \to \mathbb{R}$ are proper and convex, and $g$ is additionally closed. The alternating direction method of multipliers (ADMM) [4,5] is popular for solving (4). The iterative formula reads as

$$
\begin{align*}
x^{k+1} &\in \arg \min_x \mathcal{L}_\beta \left( x, y^k, \lambda^k \right), \\
y^{k+1} &\in \arg \min_y \mathcal{L}_\beta \left( x^{k+1}, y, \lambda^k \right), \\
\lambda^{k+1} &= \lambda^k + \beta (Ax^{k+1} + By^{k+1} - b),
\end{align*}
$$

where $\mathcal{L}_\beta(x, y, \lambda)$ is the augmented Lagrangian function of (4) defined as

$$
\mathcal{L}_\beta(x, y, \lambda) = f(x) + g(y) + \lambda^T (Ax + By - b) + \frac{\beta}{2} \| Ax + By - b \|^2.
$$
Similar to the idea of proximal ALM, proximal ADMM [3] decouples \( \|Ax + By - b\|^2 \) and the objective function by introducing suitable proximity terms. The iterative formula can be written as

\[
\begin{align*}
x^{k+1} &\in \arg \min_x \mathcal{L}_{\beta}(x, y^k, \lambda^k) + \frac{1}{2}\|x - x^k\|^2_C, \\
y^{k+1} &\in \arg \min_y \mathcal{L}_{\beta}(x^{k+1}, y, \lambda^k) + \frac{1}{2}\|x - x^k\|^2_D, \\
\lambda^{k+1} &= \lambda^k + \beta(Ax^{k+1} + By^{k+1} - b),
\end{align*}
\]

where \( C \) and \( D \) are positive semidefinite matrices. As shown above, letting \( C = rI_n - \beta A^T A \) could lead to an easy-to-solve \( x \)-subproblem.

Problem (4) can be alternatively solved by Douglas-Rachford splitting (DRS) algorithm [2]. In general, DRS finds the zero point of the sum of two maximal monotone operators \( A \) and \( B \) via the following iterative formula:

\[
\begin{align*}
w^{k+1} &= (I + \tau A)^{-1}(2z^k - w^k) + w^k - z^k, \\
z^{k+1} &= (I + \tau B)^{-1}(w^{k+1}).
\end{align*}
\]

The equivalence between ADMM and DRS has been established in [2].

In this paper, we propose a lift-and-permute scheme of ADMM for solving problem (P). Each algorithm in our scheme employs a variant ADMM in a permuted order of updating variable to solve the same dual problem of (P) with additional copying variables. Surprisingly, we can show that the above mentioned balanced ALM and its dual-primal variation, proximal ADMM and DRS algorithm all correspond to algorithms in our scheme. With the help of this understanding, we propose in the first time an accelerated balanced ALM and its dual-primal variation with the worst-case \( O(1/k^2) \) convergence rate in the case that \( f(x) \) is strongly convex.

The remainder is organized as follows. Section 2 presents the motivation and the equivalence between DRS and (dual-primal) balanced ALM. Section 3 proposes a lift-and-permute scheme of ADMM and show it includes (dual-primal) balanced ALM and proximal ADMM. Section 4 presents accelerated algorithms and convergence rate analysis. Conclusion and future works are given in Section 5.
2 Why lift

2.1 Solve a lifted version via ADMM

First, we can see that balanced ALM (1) and its dual-primal variation (3) have the same computational complexity as employing the classical ADMM to solve the following reformulation of (P) with additional copied variables:

\[
\min_{x,y} \{ f(x) : Ay = b, x = y \},
\]

which is a case of (4). The corresponding ADMM is rewritten as

\[
\begin{aligned}
x^{k+1} &= \arg \min_x \{ f(x) + \frac{\beta}{2} \| x - (y^k - \lambda^k_2) \|^2 \}, \\
y^{k+1} &= \arg \min_y \{ (\lambda^k_1)^T Ay - (\lambda^k_2)^T y + \frac{\beta}{2} \| Ay - b \|^2 + \frac{\beta}{2} \| x^{k+1} - y^{k+1} \|^2 \}, \\
\lambda^k_1 &= \lambda^k_1 + \beta (Ay^{k+1} - b), \\
\lambda^k_2 &= \lambda^k_2 + \beta (x^{k+1} - y^{k+1}).
\end{aligned}
\]

(5)

Note that the update of \( x^{k+1} \) amounts to the proximal solvers \( \text{prox}_{\frac{1}{\beta} f(y^k - \lambda^k_2)} \).

The update of \( y^{k+1} \) is equivalent to solving the linear equation system

\[
(A^T A + I) z = q^k := \frac{1}{\beta} (\lambda^k_2 + \beta A^T b + \beta x^{k+1} - A^T \lambda^k_1).
\]

When \( m \ll n \), by Sherman-Morrison formula, we have

\[
z = (A^T A + I)^{-1} q^k = \left( I - A^T (I + AA^T)^{-1} A \right) q^k = q^k - A^T \tilde{y},
\]

where \( \tilde{y} \) is a solution of the \( m \)-dimensional system of equations

\[
(I + AA^T) \tilde{y} = Aq^k.
\]

That is, the main computational complexity is the same as (2). Therefore, we have observed that all iterative subproblems in (5) are relatively easy to solve.

Moreover, according to the convergence theory of the classical ADMM, there is no need to assume any restrictive conditions on \( \beta \).

2.2 Equivalence between DRS and (dual-primal) balanced ALM

For a parameter \( \sigma > 0 \), we lift (P) as

\[
\min_{x,y} \{ f(x) : Ax = \sigma y, \sigma y = b \}
\]
and then reformulate it as
\[
\min_{z=(x,y)} \{ f(x) + \phi_{\sigma y=b}(y) + \varphi_{Ax=\sigma y}(z) \},
\]
where \( \phi_{\sigma y=b}(y) \) and \( \varphi_{Ax=\sigma y}(z) \) are indicator functions of \( \sigma y = b \) and \( Ax = \sigma y \), respectively. The proximal mappings of \( \phi_{\sigma y=b}(y) \) and \( \varphi_{Ax=\sigma y}(z) \) are given by
\[
\begin{align*}
\arg \min_{y'} \{ \phi_{\sigma y=b}(y') + \frac{1}{2} \| y' - y' \|^2 \} &= b/\sigma, \\
\arg \min_{z'} \{ \varphi_{Ax=\sigma y}(z') + \frac{1}{2} \| z - z' \|^2 \} &= z + \left( A^T - \sigma I \right)^{-1} (A - \sigma I) z.
\end{align*}
\]
One can easily see that applying DRS algorithm for solving (6) has the same computational complexity as that of balanced ALM (1) and its dual-primal variation (3) for solving (P). Furthermore, we can establish the equivalence between DRS and balanced ALM (1) and then extend the equivalence to dual-primal balanced ALM (3).

**Theorem 1** Let \( F(z) = f(x) + \phi_{\sigma y=b}(y) \). Balanced ALM (1) for solving (P) is equivalent to applying DRS with \( A = \partial \varphi_{Ax=\sigma y} \) and \( B = \partial F \) to solve (6) under the special parametric settings \( \tau r = 1 \) and \( \tau \sigma^2 = \delta \).

**Proof.** Applying DRS with \( A = \partial \varphi_{Ax=\sigma y} \) and \( B = \partial F \) yields
\[
\begin{align*}
w^{k+1} &= \arg \min_{z} \{ \phi_{Ax=\sigma y}(z) + \frac{1}{2} \| z - (2z^k - w^k) \|^2 \} + w^k - z^k, \\
z^{k+1} &= \arg \min_{z} \{ F(z) + \frac{1}{2} \| z - w^{k+1} \|^2 \}.
\end{align*}
\]
Let \( w^k = (\tilde{x}^k, \tilde{y}^k) \), \( z^k = (x^k, y^k) \) and \( H = (AA^T + \sigma^2 I)^{-1} \). According to the optimality conditions, we can explicitly rewrite the above iterative formula as
\[
\begin{align*}
\tilde{x}^{k+1} &= -A^T HA(2x^k - \tilde{x}^k) + \sigma A^T H(2y^k - \tilde{y}^k) + x^k, \\
\tilde{y}^{k+1} &= \sigma HA(2x^k - \tilde{x}^k) - \sigma^2 H(2y^k - \tilde{y}^k) + y^k, \\
0 &\in \partial f(x^{k+1}) + \frac{1}{\tau} (x^{k+1} - \tilde{x}^{k+1}), \\
\sigma y^{k+1} &= b.
\end{align*}
\]
According to (7) and (8), we obtain
\[
\tilde{x}^{k+1} = \frac{1}{\sigma} (-A^T \tilde{y}^{k+1} + \frac{1}{\sigma} A^T b) + x^k.
\]
Substituting (10) into (3) yields that
\[
0 \in \partial f(x^{k+1}) + \frac{1}{\tau} (x^{k+1} - x^k) + \frac{1}{\tau \sigma} A^T (\tilde{y}^{k+1} - \frac{1}{\sigma} b).
\]
By first rewriting (10) as an update of $\tilde{x}^k$ from $\tilde{y}^k$ and $x^{k-1}$ and then taking it into (7), we have

$$\begin{align*}
\tilde{y}^{k+1} &= \sigma HA(2x^k - x^{k-1}) + H(\sigma AA^T + \sigma^2 I)\tilde{y}^k \\
&= \frac{1}{\sigma}H(\sigma AA^T + \sigma^2 I)b - \sigma Hb + \frac{1}{\sigma}b \\
&= \tilde{y}^k + \sigma H[A(2x^k - x^{k-1}) - b].
\end{align*}$$

Let $x^{k+1} := x^{k+1}$, $\lambda^k := \frac{1}{\sigma}(\tilde{y}^{k+1} - \frac{1}{2}b)$, $1/\tau = r$ and $\tau \sigma^2 = \delta$. Then we can recover balanced ALM (11).

Theorem 2 Let $F(z) = f(x) + \phi_{\sigma y=b}(y)$. Dual-primal balanced ALM (3) for solving (P) is equivalent to applying DRS with $A = \partial F$ and $B = \partial \phi_{Ax=\sigma y}$ to solve (6) under the special parametric settings $\tau r = 1$ and $\tau \sigma^2 = \delta$.

Proof. Applying DRS with $B = \partial \phi_{Ax=\sigma y}$ and $A = \partial F$ yields that

$$\begin{align*}
w^{k+1} &= \arg \min \{ F(z) + \frac{1}{2\tau} \| z - (2z^k - w^k) \|^2 + w^k - z^k, \\
x^{k+1} &= \arg \min \{ \phi_{Ax=\sigma y}(z) + \frac{1}{2\tau} \| z - w^{k+1} \|^2 \}.
\end{align*}$$

Let $w^k = (\tilde{x}^k, \tilde{y}^k)$, $z^k = (x^k, y^k)$, $\tilde{x}^{k+1} = \tilde{x}^{k+1} - \tilde{x}^k + x^k$ and $\tilde{H} = (AA^T + \sigma^2 I)^{-1}$. It follows from the optimality conditions that

$$\begin{align*}
0 &\in \partial F(\tilde{z}^{k+1}) + \frac{1}{\tau}(\tilde{x}^{k+1} - 2x^k + \tilde{x}^k), \\
&\tilde{y}^{k+1} - \tilde{y}^k + y^k = \frac{1}{\sigma}b, \\
x^{k+1} &= -A^T \hat{\phi}_{Ax=\sigma y}(\tilde{x}^{k+1} + \sigma A^T H\tilde{y}^{k+1} + \tilde{x}^{k+1}), \\
y^{k+1} &= \sigma HA\tilde{x}^{k+1} - 2\tilde{H} \tilde{y}^{k+1} + \tilde{y}^{k+1}.
\end{align*}$$

By combining (13) with (14), we observe that

$$x^{k+1} = -\frac{1}{\sigma}A^T(y^{k+1} - \tilde{y}^{k+1}) + \tilde{x}^{k+1}. \tag{15}$$

According to the definition of $\tilde{x}^{k+1}$, we have

$$A^T \tilde{y}^{k+1} + \sigma(\tilde{x}^{k+1} - \tilde{x}^{k+1}) \overset{(15)}{=} A^T(y^{k+1} - \tilde{y}^k + y^k) \overset{(12)}{=} \frac{1}{\sigma}A^Tb. \tag{16}$$

We also have

$$\begin{align*}
\frac{1}{\sigma}b + \tilde{y}^{k+1} - \tilde{y}^{k+2} &\overset{(12)}{=} y^{k+1} \overset{(12)}{=} \sigma HA\tilde{x}^{k+1} - 2\tilde{H} \tilde{y}^{k+1} + \tilde{y}^{k+1} \\
&= -H(\sigma AA^T + \sigma^2 I)\tilde{y}^{k+1} + \tilde{y}^{k+1} + \sigma HA\tilde{x}^{k+1} + \frac{1}{\sigma}HA^Tb \\
&= \sigma HA\tilde{x}^{k+1} + \frac{1}{\sigma}HA^Tb.
\end{align*}$$
or equivalently,
\[ \tilde{y}^{k+2} = \tilde{y}^{k+1} - \sigma H(A\tilde{x}^{k+1} - b). \] (17)

Furthermore, we can deduce
\[ -2x^k + \tilde{x}^k = \frac{2}{\sigma}A^T(y^k - \tilde{y}^k) - \tilde{x}^k \]
\[ = \frac{2}{\sigma}A^Ty^k - \frac{1}{\sigma}A^T\tilde{y}^k - \tilde{x}^k - \frac{1}{\sigma^2}A^Tb \]
\[ = A^T\left(-\frac{2}{\sigma}\tilde{y}^{k+1} + \frac{1}{\sigma}\tilde{y}^k + \frac{1}{\sigma^2}b\right) - \tilde{x}^k. \] (18)

By substituting (18) into (11) and then combining it with (17), we recover dual-primal balanced ALM (3) with
\[ x^{k+1} = \hat{x}^{k+1}, \]
\[ \lambda^k := -\frac{1}{\tau}\sigma(A\tilde{y}^k - \hat{x}^k - 1\sigma - \frac{1}{\sigma}b), \]
\[ 1/\tau = r \text{ and } \tau\sigma^2 = \delta. \] □

3 A lift-and-permute scheme of ADMM

Different from the primal lift as in Section 2, we lift the dual problem of (P),
\[ \max_{\lambda} \min_x f(x) + \lambda^T(Ax - b) = \max_{\lambda} \{-f^*(-A^T\lambda) - \lambda^Tb\}, \] (19)

to the following reformulation:
\[ -\min_{u,v,\lambda} \{f^*(u) + \lambda^Tb : -A^Tv = u, v = \lambda\}. \] (20)

The augmented Lagrangian function of problem (20) is given by
\[ L_{\beta_1,\beta_2}(u,v,\lambda,\bar{x},\bar{y}) = f^*(u) + \lambda^Tb + \bar{x}^T(u + A^Tv) + \bar{y}^T(v - \lambda) + \frac{\beta_1}{2}\|u + A^Tv\|^2 + \frac{\beta_2}{2}\|v - \lambda\|^2, \]
where \(\bar{x}, \bar{y}\) are Lagrange multipliers, \(\beta_1 > 0\) and \(\beta_2 > 0\) are two parameters. For convenience, let
\[ L_{\hat{\beta}_1,\hat{\beta}_2}^k(u,v,\lambda,\tilde{x},\tilde{y}) = L_{\beta_1,\beta_2}(u,v,\lambda,\bar{x},\bar{y}) - \frac{1}{2\hat{\beta}_1}\|\tilde{x} - \tilde{x}^k\|^2 - \frac{1}{2\hat{\beta}_2}\|\tilde{y} - \tilde{y}^k\|^2. \]

Now we present the lift-and-permute scheme of ADMM for solving (P).

**Scheme 1**

**Input:** maximum iteration number \(K\) and initial point \(\{u^0, v^0, \lambda^0, \bar{x}^0, \bar{y}^0\}\).
**Output:** \(\{u^{K+1}, v^{K+1}, \lambda^{K+1}, \bar{x}^{K+1}, \bar{y}^{K+1}\}\).
Let \( \{t_1 - t_2 - t_3 - t_4 - t_5\} \) be a permutation of \( \{u, v, \lambda, \bar{x}, \bar{y}\} \).

For \( k = 1, 2, \cdots, K \) do

1. If \( t_1 \in \{u, v, \lambda\} \), then

\[
    t_{1}^{k+1} = \arg\min_{t_1} \mathcal{L}^k_{\beta_1, \beta_2}(t_1, t_2^k, t_3^k, t_4^k, t_5^k),
\]

else

\[
    t_{1}^{k+1} = \arg\max_{t_1} \mathcal{L}^k_{\beta_1, \beta_2}(t_1, t_2^k, t_3^k, t_4^k, t_5^k).
\]

2. If \( t_2 \in \{u, v, \lambda\} \), then

\[
    t_{2}^{k+1} = \arg\min_{t_2} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_2, t_3^k, t_4^k, t_5^k),
\]

else

\[
    t_{2}^{k+1} = \arg\max_{t_2} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_2, t_3^k, t_4^k, t_5^k).
\]

3. If \( t_3 \in \{u, v, \lambda\} \),

\[
    t_{3}^{k+1} = \arg\min_{t_3} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_3, t_4^k, t_5^k),
\]

else

\[
    t_{3}^{k+1} = \arg\max_{t_3} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_3, t_4^k, t_5^k).
\]

4. If \( t_4 \in \{u, v, \lambda\} \),

\[
    t_{4}^{k+1} = \arg\min_{t_4} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_{3}^{k+1}, t_4, t_5^k),
\]

else

\[
    t_{4}^{k+1} = \arg\max_{t_4} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_{3}^{k+1}, t_4, t_5^k).
\]

5. If \( t_5 \in \{u, v, \lambda\} \),

\[
    t_{5}^{k+1} = \arg\min_{t_5} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_{3}^{k+1}, t_{4}^{k+1}, t_5),
\]

else

\[
    t_{5}^{k+1} = \arg\max_{t_5} \mathcal{L}^k_{\beta_1, \beta_2}(t_{1}^{k+1}, t_{2}^{k+1}, t_{3}^{k+1}, t_{4}^{k+1}, t_5).
\]
Scheme 1 contains 5! = 120 algorithms. We can always assume \( t_1 = u \), since otherwise, we can start from a proper initial point and then generate an iterative sequence coinciding with that of Algorithm \( t_1 = u \). So the scheme remains 4! = 24 algorithms.

Up to different initial points, we have the following equivalence,

\[
\{ u - \bar{x} - v - \lambda - \bar{y} \} \iff \{ \bar{y} - u - \bar{x} - v - \lambda \} \iff \{ u - \bar{y} - \bar{x} - v - \lambda \} \iff \{ u - \bar{x} - \bar{y} - v - \lambda \} \quad (21)
\]

\[
\{ u - \bar{x} - v - \lambda - \bar{y} \} \iff \{ \lambda - \bar{y} - u - \bar{x} - v \} \iff \{ u - \lambda - \bar{y} - \bar{x} - v \} \quad (22)
\]

\[
\{ u - \lambda - \bar{x} - \bar{y} - v \} \iff \{ u - \bar{x} - \lambda - \bar{y} - v \} \quad (23)
\]

where (21), the last equivalence in (22), and (23) hold since the update of either \( \bar{y} \) or \( \lambda \) is independent of \( \bar{x} \) and \( u \). In the similar way as above, we can show that all the 24 algorithms (see the first column in Table 1) can be classified into four categories as listed in the second column in Table 1. The equivalence among the remaind four algorithms and (dual-primal) balanced ALM are summarized in Columns II-IV in Table 1. The corresponding proofs are given in the next two subsections, respectively.

3.1 Equivalence between Columns II and III

3.1.1 Equivalence between Algorithm \( \{ u - \bar{x} - v - \lambda - \bar{y} \} \) and balanced ALM

We first write down the algorithm in the scheme corresponding to the order \( \{ u - \bar{x} - v - \lambda - \bar{y} \} \).

**Algorithm 1 (Algorithm \( \{ u - \bar{x} - v - \lambda - \bar{y} \} \))**

\[
u^{k+1} = \arg \min_u \mathcal{L}_{\beta_1, \beta_2}(u, v^k, \lambda^k, \bar{x}^k, \bar{y}^k), \quad (24a)
\]

\[
\bar{x}^{k+1} = \bar{x}^k + \beta_1 (u^{k+1} + \bar{A}^T v^k), \quad (24b)
\]

\[
v^{k+1} = \arg \min_v \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v, \lambda^k, \bar{x}^{k+1}, \bar{y}^k), \quad (24c)
\]

\[
\lambda^{k+1} = \arg \min_\lambda \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v^{k+1}, \lambda, \bar{x}^{k+1}, \bar{y}^k), \quad (24d)
\]

\[
\bar{y}^{k+1} = \bar{y}^k + \beta_2 (v^{k+1} - \lambda^{k+1}), \quad (24e)
\]

Surprisingly, we can show that Algorithm is in fact equivalent to balanced ALM for solving (P).
For Algorithm 1, we have Lemma 1.
Proof. For any $k$ with the special parametric settings $\beta$, balanced ALM Theorem 3 Algorithms in the same row are equivalent to each other.

Table 1

| I            | II | III | IV | V  |
|--------------|----|-----|----|----|
| $u - \bar{x} - \lambda - \bar{y} - v$ | $u - \bar{x} - v - \lambda - \bar{y}$ | balanced ALM |    |    |
| $u - \bar{y} - \bar{x} - v - \lambda$ | $u - \bar{x} - v - \lambda - \bar{y}$ | proximal ADMM | DRS |    |
| $u - \bar{y} - \bar{y} - \bar{x} - \lambda$ | $u - \bar{y} - \bar{x} - \lambda - \bar{x}$ | dual-primal | balanced ALM |    |
| $u - \bar{x} - \bar{y} - \bar{y}$ | $u - \bar{x} - \bar{y} - \lambda$ |    |    |    |
| $u - \bar{x} - \bar{y} - \bar{x}$ | $u - \bar{y} - \bar{x} - \bar{y}$ |    |    |    |
| $u - \bar{y} - \bar{y} - \bar{x}$ | $u - \bar{x} - \bar{y} - \bar{y}$ |    |    |    |
| $u - \bar{y} - \bar{x} - \lambda$ | $u - \bar{x} - \bar{y} - \bar{x}$ |    |    |    |
| $u - \bar{y} - \bar{y} - \bar{x}$ | $u - \bar{x} - \bar{y} - \bar{x}$ |    |    |    |
| $u - \bar{x} - \bar{x} - \bar{x}$ | $u - \bar{x} - \bar{x} - \bar{x}$ |    |    |    |

According to optimality conditions, we first simplify (24a)-(24e) as

\[
0 \in \partial f^* (u^{k+1}) + \bar{x}^k + \beta_1 (u^{k+1} + A^T v^k), \quad (25a)
\]

\[
\bar{x}^{k+1} = \bar{x}^k + \beta_1 (u^{k+1} + A^T v^k), \quad (25b)
\]

\[
0 = A \bar{x}^{k+1} + \bar{y}^k + \beta_1 A u^{k+1} + \beta_1 A A^T v^{k+1} + \beta_2 (v^{k+1} - \lambda^k), \quad (25c)
\]

\[
0 = b - \bar{y}^k - \beta_2 (v^{k+1} - \lambda^{k+1}), \quad (25d)
\]

\[
\bar{y}^{k+1} = \bar{y}^k + \beta_2 (v^{k+1} - \lambda^{k+1}). \quad (25e)
\]

Lemma 1 For Algorithm 1, we have $v^{k+1} \equiv \lambda^{k+1}$ and $\bar{y}^{k+1} \equiv b$ for all $k \geq 0$.

Proof. For any $k \geq 0$, it follows from (25d) and (25c) that

\[
b = \bar{y}^k + \beta_2 (v^{k+1} - \lambda^{k+1}) = \bar{y}^{k+1}. \quad (26)
\]

Substituting (26) into (25d) yields $v^{k+1} = \lambda^{k+1}$. The proof is complete. □

Theorem 3 Balanced ALM (11) for solving (P) is equivalent to Algorithm 1 with the special parametric settings $\beta_1 = 1/r$ and $\beta_2 = \delta$. 


Proof. Define $x^k := -\bar{x}^k$ for all $k \geq 0$. By (25a), we have
\[ x^k + \beta_1 (u^{k+1} + ATv^k) = -x^{k+1}. \] (27)
Substituting (27) into (25a) yields that
\[ 0 \in \partial f^*(u^{k+1}) - x^{k+1} \iff u^{k+1} \in \partial f(x^{k+1}) \]
\[ \iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1} (x^{k+1} - x^k) + ATv^k \]
\[ \iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1} (x^{k+1} - x^k) + AT\lambda^k, \] (28)
where the first equivalence holds since $x \in \partial f^*(u) \iff u \in \partial f(x)$, and equation (28) follows from Lemma 1. With the setting $\beta_1 = 1/r$, (28) is exactly the optimality condition of the $x$-subproblem of balanced ALM (1).

Moreover, it follows from (27) that
\[ u^{k+1} = \frac{1}{\beta_1} x^{k+1} + \frac{1}{\beta_1} x^k - ATv^k. \] (29)
By taking the setting $\beta_2 = \delta$ into (25b), we have
\[ 0 = (\beta_1 AA^T + \beta_2 I)v^{k+1} + Ax^{k+1} + \beta_1 Au^{k+1} - \beta_2 \lambda^k + \bar{y}^k \]
\[ = (\beta_1 AA^T + \beta_2 I)v^{k+1} - 2Ax^{k+1} + Ax^k - \beta_1 AA^T v^k - \beta_2 v^k + \bar{y}^k \]
\[ = (\beta_1 AA^T + \beta_2 I)(v^{k+1} - v^k) - [A(2x^{k+1} - x^k) - b] \]
\[ = \left( \frac{1}{r} AA^T + \delta I \right)(\lambda^{k+1} - \lambda^k) - [A(2x^{k+1} - x^k) - b], \] (30)
where equation (31) is obtained by substituting (29) into (30) and the equation (32) follows from Lemma 1. The proof is complete since the equation (32) corresponds to the $\lambda$-subproblem of balanced ALM (1). \hfill $\Box$

3.1.2 Equivalence between Algorithm \{u - v - \lambda - \bar{x} - \bar{y}\} and dual-primal balanced ALM

By replacing the update order of Algorithm (1) with \{u - v - \lambda - \bar{x} - \bar{y}\}, we obtain the following algorithm, which corresponds to the classical ADMM in solving the three-block convex optimization problem.
Algorithm 2 (Algorithm \{u - v - \lambda - \bar{x} - \bar{y}\})

\[
\begin{aligned}
\left\{
\begin{array}{l}
\bar{u}^{k+1} = \arg\min_u L_{\beta_1, \beta_2}(u, v^k, x^k, y^k), \\
\bar{v}^{k+1} = \arg\min_v L_{\beta_1, \beta_2}(u^{k+1}, v, x^k, y^k), \\
\lambda^{k+1} = \arg\min_{\lambda} L_{\beta_1, \beta_2}(u^{k+1}, v^{k+1}, \lambda, x^k, y^k), \\
\bar{x}^{k+1} = \bar{x}^k + \beta_1 (u^{k+1} + A^T v^{k+1}), \\
\bar{y}^{k+1} = \bar{y}^k + \beta_2 (v^{k+1} - \lambda^{k+1}),
\end{array}
\right.
\end{aligned}
\]

Based on optimality conditions, we can rewrite Algorithm 2 as

\[
\begin{aligned}
0 &= \partial f^* (u^{k+1}) + \bar{x}^k + \beta_1 (u^{k+1} + A^T v^k), \\
0 &= A\bar{x}^k + \bar{y}^k + \beta_1 A u^{k+1} + \beta_2 A A^T v^{k+1} + \beta_2 (v^{k+1} - \lambda^k), \\
0 &= b - \bar{y}^k - \beta_2 (v^{k+1} - \lambda^{k+1}), \\
\bar{x}^{k+1} &= \bar{x}^k + \beta_1 (u^{k+1} + A^T v^{k+1}), \\
\bar{y}^{k+1} &= \bar{y}^k + \beta_2 (v^{k+1} - \lambda^{k+1}).
\end{aligned}
\]

To reveal the equivalence, we first need an observation similar to Lemma 1.

Lemma 2 For Algorithm 2, we have \(v^{k+1} = \lambda^{k+1}\) and \(\bar{y}^{k+1} = b\) for all \(k \geq 0\).

Theorem 4 With the special parametric settings \(\beta_1 = 1/r\) and \(\beta_2 = \delta\), Algorithm 2 is equivalent to dual-primal balanced ALM for solving (P).

Proof. For any \(k \geq 1\), define

\[
x^{k+1} := -\bar{x}^k - \beta_1 u^{k+1} - \beta_1 A^T v^k,
\]

which implies that

\[
u^{k+1} = -\frac{1}{\beta_1} \bar{x}^k - \frac{1}{\beta_1} x^{k+1} - A^T v^k.
\]

Substituting (33a) into (33b) to replace \(u^{k+1}\) yields that

\[
\bar{x}^{k+1} = -\bar{x}^{k+1} + \beta_1 A^T (v^{k+1} - v^k).
\]
Then, we have the following reformulations of (33a):

$$0 \in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^T v^k)$$

$$\iff 0 \in \partial f^*(u^{k+1}) - x^{k+1}$$

(37)

$$\iff 0 \in \partial f(x^{k+1}) - v^{k+1}$$

(38)

$$\iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1} \bar{x}^k + \frac{1}{\beta_1} x^{k+1} + A^T v^k$$

(39)

$$\iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1} (x^{k+1} - x^k) + A^T (2v^k - v^{k-1})$$

(40)

$$\iff 0 \in \partial f(x^{k+1}) + r(x^{k+1} - x^k) + A^T (2\lambda^k - \lambda^{k-1})$$

(41)

where the equation (37) follows from substituting (34) into (33a), the equations (39) and (40) are obtained by substituting (36) and (35) with $k := k - 1$ into (38) and (39), respectively, the equation (41) is due to Lemma 2.

Multiplying both sides of (33a) by $A$ yields that

$$A(\bar{x}^k + \beta_1 u^{k+1}) = A\bar{x}^{k+1} - \beta_1 AA^T v^{k+1}.$$  

(42)

Then, we can simplify (33a) as follows:

$$0 = A\bar{x}^k + \bar{y}^k + \beta_1 A u^{k+1} + \beta_1 A A^T v^{k+1} + \beta_2 (v^{k+1} - \lambda^k)$$

$$\iff 0 = (\beta_1 AA^T + \beta_2 I) u^{k+1} + A(\bar{x}^k + \beta_1 u^{k+1}) + \bar{y}^k - \beta_2 \lambda^k$$

$$\iff 0 = (\beta_1 AA^T + \beta_2 I)(\bar{v}^{k+1} + \bar{v}^k) - A\bar{x}^{k+1} + b$$

(43)

$$\iff 0 = (\bar{v}^{k+1} + \bar{v}^k - A\bar{x}^{k+1} - b)$$

(44)

$$\iff 0 = (\bar{v}^{k+1} + \bar{v}^k - A\bar{x}^{k+1} - b)$$

(45)

where the equation (43) follows from (32), the equation (44) is obtained from substituting (33) into (43), and the equation (45) follows from Lemma 2.

We complete the proof by combining both (41) and (45) with (3).

3.1.3 Equivalence between Algorithm \{ $u - \bar{x} - \lambda - v - \bar{y}$ \} and balanced ALM

Algorithm 3 (Algorithm \{ $u - \bar{x} - \lambda - v - \bar{y}$ \})

$$\begin{align*}
  u^{k+1} &= \arg \min_{u} \mathcal{L}_{\beta_1, \beta_2}(u, v^k, \lambda^k, \bar{x}^k, \bar{y}^k), \\
  \bar{x}^{k+1} &= \bar{x}^k + \beta_1(u^{k+1} + A^T v^k), \\
  \lambda^{k+1} &= \arg \min_{\lambda} \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v^k, \lambda, \bar{x}^{k+1}, \bar{y}^k), \\
  v^{k+1} &= \arg \min_{v} \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v, \lambda^{k+1}, \bar{x}^{k+1}, \bar{y}^k), \\
  \bar{y}^{k+1} &= \bar{y}^k + \beta_2(v^{k+1} + \lambda^{k+1}).
\end{align*}$$
By optimality conditions, we can rewrite Algorithm \textup{3} as follows:

\begin{align}
0 & \in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^T v^k), \quad (46a) \\
\bar{x}^{k+1} & = \bar{x}^k + \beta_1(u^{k+1} + A^T v^k), \quad (46b) \\
0 & = b - \bar{y}^k - \beta_2(v^k - \lambda^{k+1}), \quad (46c) \\
0 & = Ax^{k+1} + \bar{y}^k + \beta_1 Av^{k+1} + \beta_1 AA^T v^{k+1} + \beta_2(v^{k+1} - \lambda^{k+1}), \quad (46d) \\
\bar{y}^{k+1} & = \bar{y}^k + \beta_2(v^{k+1} - \lambda^{k+1}). \quad (46e)
\end{align}

Similarly we can prove that Algorithm \textup{3} is equivalent to balanced ALM.

**Theorem 5** Balanced ALM \textup{(1)} for solving problem \textup{(P)} is equivalent to Algorithm \textup{3} with the special parametric settings \(\beta_1 = 1/r\) and \(\beta_2 = \delta\).

Proof. Define \(x^k := -\bar{x}^k\) for all \(k \geq 0\). By \textup{(46b)}, we obtain

\[x^k + \beta_1(u^{k+1} + A^T v^k) = -x^{k+1}.\]  

(47)

Substituting \textup{(47)} into \textup{(46a)} yields that

\[
0 \in \partial f^*(u^{k+1}) - x^{k+1} \iff u^{k+1} \in \partial f(x^{k+1}) \\
\iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1}(x^{k+1} - x^k) + A^T v^k \\
\iff 0 \in \partial f(x^{k+1}) + r(x^{k+1} - x^k) + A^T v^k.
\]  

(48)

Moreover, we have

\[
u^{k+1} = -\frac{1}{\beta_1}x^{k+1} + \frac{1}{\beta_1}x^k - A^T v^k,
\]  

(49)

\[
\bar{y}^k = \bar{y}^{k+1} - \beta_2(v^{k+1} - \lambda^{k+1}),
\]  

(50)

which follows from \textup{(17)} and \textup{(16c)}, respectively. Substituting \textup{(50)} into \textup{(46c)} yields that

\[
\bar{y}^{k+1} = b + \beta_2(v^{k+1} - v^k).
\]

By substituting \textup{(19)} and \textup{(50)} into \textup{(46d)}, respectively, we can obtain

\[
0 = Ax^{k+1} + [\bar{y}^k + \beta_2(v^{k+1} - \lambda^{k+1})] + \beta_1 Av^{k+1} + \beta_1 AA^T v^{k+1}
\]

\[
= -Ax^{k+1} + \bar{y}^{k+1} + \beta_1 Av^{k+1} + \beta_1 AA^T v^{k+1}
\]

\[
= -Ax^{k+1} + \bar{y}^{k+1} + \beta_2(v^{k+1} - v^k) - Ax^k + \beta_1 AA^T v^k + \beta_1 AA^T v^{k+1}
\]

\[
= (\beta_1 AA^T + \beta_2 I)(v^{k+1} - v^k) - [A(2x^{k+1} - x^k) - b]
\]

\[
= \frac{1}{r}AA^T + \delta I)(v^{k+1} - v^k) - [A(2x^{k+1} - x^k) - b].
\]  

(51)

Notice that \textup{(48)} and \textup{(51)} are optimality conditions of balanced ALM. \hfill \Box
3.1.4 Equivalence between Algorithm \(\{u - \lambda - v - \bar{x} - \bar{y}\}\) and dual-primal balanced ALM

Algorithm 4 (Algorithm \(\{u - \lambda - v - \bar{x} - \bar{y}\}\))

\[
\begin{align*}
u^{k+1} &= \arg\min_u \mathcal{L}_{\beta_1, \beta_2}(u, v^k, \lambda^k, \bar{x}^k, \bar{y}^k), \\
\lambda^{k+1} &= \arg\min_\lambda \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v^k, \lambda, \bar{x}^k, \bar{y}^k), \\
v^{k+1} &= \arg\min_v \mathcal{L}_{\beta_1, \beta_2}(u^{k+1}, v, \lambda^{k+1}, \bar{x}^k, \bar{y}^k), \\
\bar{x}^{k+1} &= \bar{x}^k + \beta_1(u^{k+1} + A^T v^{k+1}), \\
\bar{y}^{k+1} &= \bar{y}^k + \beta_2(v^{k+1} - \lambda^{k+1}).
\end{align*}
\]

Based on optimality conditions, we can rewrite Algorithm 4 as

\[
\begin{align*}
0 &\in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^T v^k), & (52a) \\
0 &= b - \bar{y}^k - \beta_2(v^k - \lambda^{k+1}), & (52b) \\
0 &= A\bar{x}^k + \bar{y}^k + \beta_1 Au^{k+1} + \beta_1 AA^T v^{k+1} + \beta_2(v^{k+1} - \lambda^{k+1}), & (52c) \\
\bar{x}^{k+1} &= \bar{x}^k + \beta_1(u^{k+1} + A^T v^{k+1}), & (52d) \\
\bar{y}^{k+1} &= \bar{y}^k + \beta_2(v^{k+1} - \lambda^{k+1}). & (52e)
\end{align*}
\]

We reveal the equivalence between Algorithm 4 and dual-primal balanced ALM.

**Theorem 6** Algorithm 4 is equivalent to dual-primal balanced ALM for solving \((P)\) with the special parametric settings \(\beta_1 = 1/r\) and \(\beta_2 = \delta\).

Proof. For any \(k \geq 1\), we define

\[
x^{k+1} := -\bar{x}^k - \beta_1 u^{k+1} - \beta_1 A^T v^k,
\]

which implies that

\[
u^{k+1} = -\frac{1}{\beta_1} \bar{x}^k - \frac{1}{\beta_1} x^{k+1} - A^T v^k.
\]

Substituting (54) into (52d) to replace \(u^{k+1}\) yields that

\[
\bar{x}^{k+1} = -x^{k+1} + \beta_1 A^T (v^{k+1} - v^k).
\]
Then, we have the following reformulations of (52a):

\[ 0 \in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^Tv^k) \]
\[ \iff 0 \in \partial f^*(u^{k+1}) - x^{k+1} \]  \hspace{1cm} (56)
\[ \iff 0 \in \partial f(x^{k+1}) + \frac{1}{\beta_1}\bar{x}^k + \frac{1}{\beta_1}x^{k+1} + A^Tv^k \]  \hspace{1cm} (57)
\[ \iff 0 \in \partial f(x^{k+1}) + r(x^{k+1} - x^k) + A^T(2v^k - v^{k-1}) \]  \hspace{1cm} (58)
\[ \iff 0 \in \partial f(x^{k+1}) + \bar{y}^{k+1} + \beta_2(v^{k+1} - v^k) \]  \hspace{1cm} (59)

where the equation (56) follows from substituting (53) into (52a), the equations (58) and (59) are obtained by substituting (54) and (55) (with \( k := k - 1 \)) into (57) and (58), respectively.

Multiplying both sides of (52d) by \( A \) yields that

\[ A(\bar{x}^k + \beta_1u^{k+1}) = A\bar{x}^{k+1} - \beta_1AA^Tv^{k+1}. \]  \hspace{1cm} (60)

By substituting (52c) into (52b), we have

\[ \bar{y}^{k+1} = b + \beta_2(v^{k+1} - v^k). \]  \hspace{1cm} (61)

Then, we can simplify (52c) as follows:

\[ 0 = Ax^k + \bar{y}^k + \beta_1Au^{k+1} + \beta_1AA^Tv^{k+1} + \beta_2(v^{k+1} - \lambda^{k+1}) \]
\[ \iff 0 = A(\bar{x}^k + \beta_1u^{k+1}) + [\bar{y}^k + \beta_2(v^{k+1} - \lambda^{k+1})] + \beta_1AA^Tv^{k+1} \]
\[ \iff 0 = Ax^{k+1} + \bar{y}^{k+1} \]  \hspace{1cm} (62)
\[ \iff 0 = -Ax^{k+1} + \beta_1AA^Tv^{k+1} + b + \beta_2(v^{k+1} - v^k) \]  \hspace{1cm} (63)
\[ \iff (\beta_1AA^T + \beta_2I)(v^{k+1} - v^k) = Ax^{k+1} - b \]
\[ \iff \left( \frac{1}{r}AA^T + \delta I \right)(v^{k+1} - v^k) = Ax^{k+1} - b, \]  \hspace{1cm} (64)

where the equation (62) follows from (60) and (52c), the equation (63) is obtained from substituting (55) and (61) into (62).

We complete the proof by combing both (59) and (64) with (3). \( \square \)

### 3.2 Equivalence between Columns III and IV

Different from the reformulation (20), we rewrite the dual problem (19) as the following compact version:

\[ \min_{u,\lambda} \{ f^*(u) + \lambda^Tb : A^T\lambda + u = 0 \}. \]  \hspace{1cm} (65)
Using Lemma 1, we can reduce the optimality conditions (25a)-(25e) to
\[
\begin{align*}
0 & \in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k}), \\
\bar{x}^{k+1} & = \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k}), \\
0 & = A\bar{x}^{k+1} + b + \beta_1Au^{k+1} + \beta_1AA^T\lambda^{k+1} + \beta_2(\lambda^{k+1} - \lambda^{k}),
\end{align*}
\]
which exactly corresponds to the optimality conditions of the following proximal ADMM for solving the dual problem (65).

Algorithm 5
\[
\begin{align*}
u^{k+1} & = \arg\min_u \{ f^*(u) + (\bar{x}^k)^T u + \frac{\beta_1}{2}\|u + A^T\lambda\|^2 \}, \\
\bar{x}^{k+1} & = \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k}), \\
\lambda^{k+1} & = \arg\min_\lambda \{ b^T\lambda + (A\bar{x}^{k+1})^T\lambda + \frac{\beta_1}{2}\|u^{k+1} + A^T\lambda\|^2 + \frac{\beta_2}{2}\|\lambda - \lambda^{k}\|^2 \}.
\end{align*}
\]
Then, according to Theorem 3, we have the following equivalence result.

**Corollary 1** Balanced ALM (1) for solving (P) is equivalent to the proximal ADMM (Algorithm 5) for solving the dual problem (65) with the special parametric settings \( \beta_1 = 1/r \) and \( \beta_2 = \delta \).

Similarly, we can use Lemma 2 to further simplify the optimality conditions (33a)-(33e) as
\[
\begin{align*}
0 & \in \partial f^*(u^{k+1}) + \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k}), \\
\bar{x}^{k+1} & = \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k+1}), \\
0 & = A\bar{x}^{k+1} + b + \beta_1Au^{k+1} + \beta_1AA^T\lambda^{k+1} + \beta_2(\lambda^{k+1} - \lambda^{k}),
\end{align*}
\]
which exactly corresponds to the optimality conditions of the following proximal ADMM for solving the dual problem (65).

Algorithm 6
\[
\begin{align*}
u^{k+1} & = \arg\min_u \{ f^*(u) + (\bar{x}^k)^T u + \frac{\beta_1}{2}\|u + A^T\lambda\|^2 \}, \\
\lambda^{k+1} & = \arg\min_\lambda \{ b^T\lambda + (A\bar{x}^{k+1})^T\lambda + \frac{\beta_1}{2}\|u^{k+1} + A^T\lambda\|^2 + \frac{\beta_2}{2}\|\lambda - \lambda^{k}\|^2 \}, \\
\bar{x}^{k+1} & = \bar{x}^k + \beta_1(u^{k+1} + A^T\lambda^{k+1}) .
\end{align*}
\]
According to Theorem 4, we have the following conclusion.

**Corollary 2** Dual-primal balanced ALM (3) for solving (P) is equivalent to the proximal ADMM for solving the dual problem (65) with the special parametric settings \( \beta_1 = 1/r \) and \( \beta_2 = \delta \).
4 Acceleration

In this section, we first present accelerated balanced ALM and accelerated dual-primal balanced ALM. Then we provide the convergence rate analysis.

Throughout this section, we assume that $f(x)$ is $\mu$-strongly convex ($\mu \geq 0$).

**Definition 1** $f(x)$ is $\mu$-strongly convex with $\mu \geq 0$ if for all $x, y \in \mathbb{R}^n$,

$$f(x) \geq f(y) + g^T(x - y) + \frac{\mu}{2}||x - y||^2, \quad g \in \partial f(y).$$

**4.1 Accelerated balanced ALM**
We establish convergence rate analysis on the accelerated balanced ALM.

**Algorithm 7 (Accelerated balanced ALM)**

$$\begin{aligned}
    x^{k+1} &= \arg \min_x \{f(x) + \frac{r_k}{2}||x - (x^k - \frac{1}{r_k}A^T\lambda^k)||^2\}, \\
    \lambda^{k+1} &= \lambda^k + \frac{1}{r_k}AA^T + \delta^k I_m \left[ A\tilde{x}^{k+1} - b \right], \\
    \tilde{x}^{k+1} &= x^{k+1} + \theta(x^{k+1} - x^k).
\end{aligned}$$

**Lemma 3** Let $\delta^k = \delta'/r^k + 1$ with $\delta' > 0$ and $H = AA^T + \delta' I_m$. For the sequence $\{(x^k, \lambda^k)\}$ generated by Algorithm 7 and any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, it holds that

$$f(x^{k+1}) + \frac{\lambda^T}{2} A(\tilde{x}^{k+1} - b) - f(x) - (\lambda^k)^T(Ax - b) \leq \frac{r_k}{2}||x^k - x||^2 - \frac{r_k}{2}||x^{k+1} - x||^2 - \frac{1}{2r_k}||x^k - x^{k+1}||^2 + \frac{1}{2r_k} \left[ \frac{\lambda^{k+1} - \lambda^k}{H} \right]^2
- ||\lambda^k - \lambda^k||_H^2 - ||\lambda^{k+1} - \lambda^k||_H^2 + (\lambda^k - \lambda^k)^T A(\tilde{x}^{k+1} - x^{k+1}).$$

(66)

**Proof.** According to the optimality condition of the $x$-subproblem in Algorithm 7, we have

$$0 \in \partial f(x^{k+1}) + A^T \lambda^k + r^k(x^{k+1} - x^k),$$

(67)

Then by using $\mu$-strongly convexity of $f(x)$ and (67), we can obtain

$$f(x^{k+1}) - f(x) \leq (A^T \lambda^k + r^k(x^{k+1} - x^k))^T (x - x^{k+1}) - \frac{\mu}{2}||x^{k+1} - x||^2
= \frac{r_k}{2}||x^k - x||^2 - \frac{r_k}{2}||x^{k+1} - x||^2 - \frac{r_k}{2}||x^k - x^{k+1}||^2 + (\lambda^k)^T A(x - x^{k+1}).$$

(68)
According to the update of $\lambda$ in Algorithm \[ we have
\[
(\lambda^k - \lambda)^T b = (\lambda^k - \lambda)^T \left( A\hat{z}^k - \frac{1}{\gamma} (AA^T + \delta I_m) (\lambda^k - \lambda^{k-1}) \right) \\
= \frac{1}{2r} \left( \|\lambda^{k-1} - \lambda\|_H^2 - \|\lambda^k - \lambda\|_H^2 - \|\lambda^{k-1} - \lambda^k\|_H^2 \right) + (\lambda^k - \lambda)^T A\hat{z}^k.
\] (69)

Putting (68) and (69) together completes the proof. \[\]

We first study the converge rate of Algorithm \[ with the setting $r^k = r$ and $\delta = \delta/r$, which reduces to balanced ALM \[.

**Theorem 7** Suppose $f(x)$ is convex (but not necessarily strongly convex). For any $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, the sequence $\{(x^k, \lambda^k)\}$ generated by balanced ALM \[ satisfies that
\[
f(\hat{x}^K) + \lambda^T (A\hat{x}^K - b) - f(x) - (\hat{\lambda}^K)^T (Ax - b) \\
\leq \frac{1}{K+1} \left( \frac{r}{2} \left( \|x^0 - x\|^2 + \|x^1 - x\|^2 \right) + \frac{1}{2r} \|\lambda^1 - \lambda\|_H^2 \\
+ (\lambda - \lambda)^T A(x^0 - x) \right),
\]
where $\hat{x}^K = (\sum_{k=0}^{K} x^{k+1})/(K+1)$ and $\hat{\lambda}^K = (\sum_{k=0}^{K} \lambda^k)/(K+1)$.

**Proof.** According to Lemma \[ we have
\[
f(x^{k+1}) + \lambda^T (Ax^{k+1} - b) - f(x) - (\lambda^k)^T (Ax - b) \\
\leq \frac{r}{2} \left( \|x^k - x\|^2 - \|x^{k+1} - x\|^2 - \|x^k - x^{k+1}\|^2 \right) + \frac{1}{2r} \left( \|\lambda^{k-1} - \lambda\|_H^2 \\
- \|\lambda^k - \lambda\|^2_H - \|\lambda^{k-1} - \lambda^k\|_H^2 \right) + (\lambda^k - \lambda)^T A(2x^k - x^{k-1} - x^{k+1}) \\
\leq \frac{r}{2} \left( \|x^k - x\|^2 - \|x^{k+1} - x\|^2 + \|x^{k-1} - x\|^2 - \|x^k - x^{k+1}\|^2 \right) \\
+ \frac{1}{2r} \left( \|\lambda^{k-1} - \lambda\|^2_H - \|\lambda^{k-1} - \lambda\|^2_H \right) \\
+ (\lambda^k - \lambda)^T A(x^k - x^{k-1}) - (\lambda^k - \lambda)^T A(x^{k+1} - x^k),
\] (70)

where the last inequality holds as
\[
(\lambda^k - \lambda)^T A(2x^k - x^{k-1} - x^{k+1}) \\
= (\lambda^k - \lambda)^T A(x^k - x^{k+1}) - (\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k) \\
+ (\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k) \\
\leq (\lambda^k - \lambda)^T A(x^k - x^{k+1}) - (\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k) \\
+ \frac{1}{2r} \|\lambda^{k-1} - \lambda\|^2_{AA^T} + \frac{r}{2} \|x^{k-1} - x^k\|^2.
\]
By adding all the inequalities (70) from \( k = 0 \) to \( k = K \) and then dividing both sides by \((K + 1)\), we obtain
\[
\begin{align*}
 & f(\hat{x}^K) + \lambda^T (A\hat{x}^K - b) - f(x) - (\hat{\lambda}^K)^T (Ax - b) \\
\leq & \frac{1}{K + 1} \left( \frac{r}{2} \left( \|x^0 - x\|^2 - \|x^{K+1} - x\|^2 + \|x^{-1} - x^0\|^2 - \|x^K - x^{K+1}\|^2 \right) \\
& + \frac{1}{2r} (\|\lambda^{-1} - \lambda\|^2_H - \|\lambda^K - \lambda\|_H^2) \\
& + (\lambda^{-1} - \lambda)^T A(x^0 - x^{-1}) - (\lambda^K - \lambda)^T A(x^{K+1} - x^K) \right) \\
\leq & \frac{1}{K + 1} \left( \frac{r}{2} \left( \|x^0 - x\|^2 + \|x^{-1} - x^0\|^2 \right) + \frac{1}{2r} \|\lambda^{-1} - \lambda\|^2_H \\
& + (\lambda^{-1} - \lambda)^T A(x^0 - x^{-1}) \right).
\end{align*}
\]

The proof is complete. \(\square\)

He and Yuan [7] established the same \(O(1/K)\) convergence rate based on the convex combination of iteration \(\{x^{k+1}, \lambda^{k+1}\}\). As a contrast, our analysis is based on the new convex combination of iteration \(\{(x^{k+1}, \lambda^k)\}\), which further helps to establish \(O(1/K^2)\) convergence rate for the accelerated balanced ALM (Algorithm 7).

**Theorem 8** Suppose \(f(x)\) is \(\mu\)-strongly convex. For any \(x \in \mathbb{R}^n\) and \(\lambda \in \mathbb{R}^m\), the sequence \(\{(x^k, \lambda^k)\}\) generated by Algorithm 7 with the setting \(\theta^k = r^k/r^{k+1}\) and \((r^k + \mu)\rho^k \geq (r^{k+1})^2\) satisfies that
\[
\begin{align*}
\left( \sum_{k=0}^{K} r^k \right) \left( f(\tilde{x}^K) + \lambda^T (A\tilde{x}^K - b) - f(x) - (\tilde{\lambda}^K)^T (Ax - b) \right) \\
\leq & \frac{(r^0)^2}{2} \|x^0 - x\|^2 + \frac{(r^{-1})^2}{2} \|x^{-1} - x^0\|^2 + \frac{1}{2} \|\lambda^{-1} - \lambda\|_H^2 \\
& + r^{-1} (\lambda^{-1} - \lambda)^T A(x^0 - x^{-1}),
\end{align*}
\]

where \(\tilde{x}^K = (\sum_{k=0}^{K} r^k x^{k+1})/(\sum_{k=0}^{K} r^k)\) and \(\tilde{\lambda}^K = (\sum_{k=0}^{K} r^k \lambda^k)/(\sum_{k=0}^{K} r^k)\).

In particular, with the setting \(r^k = \mu(k + 1)/3\), we have
\[
\begin{align*}
f(\hat{x}^K) + \lambda^T (A\hat{x}^K - b) - f(x) - (\hat{\lambda}^K)^T (Ax - b) \leq O(1/K^2).
\end{align*}
\]
Proof. First we can verify that
\[
(\lambda^k - \lambda)^T A(x^k - x^{k+1})
= (\lambda^k - \lambda)^T A(x^k - x^{k+1}) - \theta^{k-1}(\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k)
+ \theta^{k-1}(\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k)
\leq (\lambda^k - \lambda)^T A(x^k - x^{k+1}) - \theta^{k-1}(\lambda^{k-1} - \lambda)^T A(x^{k-1} - x^k)
+ \frac{1}{2r^k} \|\lambda^{k-1} - \lambda^k\|^2_H + \frac{(r^{k-1})^2}{2r^k} \|x^{k-1} - x^k\|^2.
\]  
(71)
Multiplying both sides of (66) by $r^k$ and using the fact $(r^k + \mu) r^k \geq (r^{k+1})^2$ and (71) yields that
\[
r^k \left( f(x^{k+1}) + \lambda^T(Ax^{k+1} - b) - f(x) - (\lambda^k)^T(Ax - b) \right)
\leq \frac{(r^0)^2}{2} \|x^0 - x\|^2 - \frac{(r^{K+1})^2}{2} \|x^{K+1} - x\|^2 + \frac{1}{2} \left( \|\lambda^{K+1} - \lambda\|^2_H - \|\lambda^k - \lambda\|^2_H \right)
+ \frac{(r^{K-1})^2}{2} \|x^{k-1} - x^k\|^2 - \frac{(r^0)^2}{2} \|x^0 - x\|^2 - \frac{(r^{K-1})^2}{2} \|x^{K-1} - x^K\|^2
- r^{K-1}(\lambda^{K-1} - \lambda)^T A(x^{K-1} - x^K) + r^k(\lambda^k - \lambda)^T A(x^k - x^{k+1}).
\]  
(72)
By adding all the inequalities (72) from $k = 0$ to $k = K$, we obtain
\[
\left( \sum_{k=0}^{K} r^k \right) \left( f(x^K) + \lambda^T(Ax^K - b) - f(x) - (\lambda^k)^T(Ax - b) \right)
\leq \frac{(r^0)^2}{2} \|x^0 - x\|^2 - \frac{(r^{K+1})^2}{2} \|x^{K+1} - x\|^2 + \frac{(r^0)^2}{2} \|x^{K+1} - x^0\|^2
- \frac{(r^{K})^2}{2} \|x^K - x^{K+1}\|^2 + \frac{1}{2} \left( \|\lambda^{K-1} - \lambda\|^2_H - \|\lambda^K - \lambda\|^2_H \right)
+ r^{K-1}(\lambda^{K-1} - \lambda)^T A(x^{K-1} - x^K) - r^K(\lambda^K - \lambda)^T A(x^K - x^{K+1})
\leq \frac{(r^0)^2}{2} \|x^0 - x\|^2 - \frac{(r^{K})^2}{2} \|x^K - x^0\|^2 + \frac{1}{2} \|\lambda^{K-1} - \lambda\|^2_H
+ r^{K-1}(\lambda^{K-1} - \lambda)^T A(x^{K-1} - x^K).
\]
The proof is complete. 
\[\square\]

Remark 1. As shown in Section 3, Algorithm \{u - \bar{x} - \bar{v} - \lambda - \bar{y}\} is equivalent to balanced ALM and proximal ADMM. It is natural to ask whether our accelerated balanced ALM (Algorithm 3) is equivalent to the accelerated proximal ADMM [11].

Let $u^{k+1} := -A^T \lambda^k - r^k(x^{k+1} - x^k)$ and $\bar{x}^k := -x^k$. It holds that
\[
\bar{x}^{k+1} = \bar{x}^k + \frac{1}{r^k}(u^{k+1} + A^T \lambda^k).
\]
According to the optimality condition of $x$-subproblem in Algorithm 7, we have $u^{k+1} \in \partial f(x^{k+1}) \iff x^{k+1} \in \partial f^*(u^{k+1})$. Then it holds that

$$0 \in \partial f^*(u^{k+1}) - x^{k+1} = \partial f^*(u^{k+1}) + \bar{x}^k + \frac{1}{r} (u^{k+1} + A^T \lambda^k).$$

According to the $\lambda$-subproblem in Algorithm 7, we obtain

$$0 = -[A(x^{k+1} + \theta^k(x^{k+1} - x^k)) - b] + \left(\frac{1}{r} \lambda^{k+1} AA^T + \delta^k I_m\right) (\lambda^{k+1} - \lambda^k)$$

$$= Ax^{k+1} + b + \frac{1}{r} (A\bar{x}^{k+1} + AA^T \lambda^{k+1}) + \delta^k (\lambda^{k+1} - \lambda^k).$$

Therefore, Algorithm 7 is equivalent to the following proximal ADMM for (65).

$$\begin{align*}
  u^{k+1} &= \arg\min_u \{ f^*(u) + (\bar{x}^k)^T u + \frac{1}{2r} \| u + A^T \lambda^k \|^2 \}, \\
  \bar{x}^{k+1} &= \bar{x}^k + \frac{1}{r} (u^{k+1} + A^T \lambda^k), \\
  \lambda^{k+1} &= \arg\min_{\lambda} \{ b^T \lambda + (Ax^{k+1})^T \lambda + \frac{1}{2r} \| u^{k+1} + A^T \lambda \|^2 + \frac{1}{2} \| \lambda - \lambda^k \|^2 \}.
\end{align*}$$

With the above observation, one can alternatively establish the $O(1/K^2)$ convergence rate of Algorithm 7 by referring to the analysis on the general accelerated proximal ADMM.

4.2 Accelerated dual-primal balanced ALM

Algorithm 8 (Accelerated dual-primal balanced ALM)

$$\begin{align*}
  x^{k+1} &= \arg\min_x \{ f(x) + \frac{r^k}{2} \| x - (x^k - \bar{x}^k A^T \lambda^k) \|^2 \}, \\
  \lambda^{k+1} &= \lambda^k + (\frac{1}{r} AA^T + \delta^k I_m)^{-1} [Ax^{k+1} - b], \\
  \bar{x}^k &= \lambda^k + \delta^{k-1} (\lambda^k - \lambda^{k-1}).
\end{align*}$$

Lemma 4 Let $\delta^k = \delta'/r^k$ with $\delta' > 0$ and $H = AA^T + \delta' I_m$. For the sequence $\{(x^k, \lambda^k)\}$ generated by Algorithm 8 and any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, it holds that

$$\begin{align*}
  f(x^{k+1}) + \lambda^T (Ax^{k+1} - b) - f(x) - (\lambda^{k+1})^T (Ax - b) \\
  \leq &\frac{r^2}{2} \| x^k - x \|^2 - \frac{r^2 + \mu}{2} \| x^{k+1} - x \|^2 - \frac{r^2}{2} \| x^k - x^{k+1} \|^2 - \frac{1}{2r\kappa} \left( \| \lambda^k - \lambda \|^2_H \\
  &- \| \lambda^{k+1} - \lambda \|^2_H - \| \lambda^k - \lambda^{k+1} \|^2_H \right) + (\bar{x}^k - \lambda^{k+1})^T A(x^{k+1} - x).
\end{align*}$$

(73)
Proof. According to the optimality condition of $x$-subproblem in Algorithm \(8\), we have
\[
0 \in \partial f(x^{k+1}) + AT\hat{x}^k + r^k(x^{k+1} - x^k).
\]
(74)

Then based on $\mu$-strongly convexity of $f(x)$ and (74), we can obtain
\[
f(x^{k+1}) - f(x) \leq (AT\hat{x}^k + r^k(x^{k+1} - x^k))^T(x - x^{k+1}) - \frac{\mu}{2}\|x^{k+1} - x\|^2
= \frac{r}{2}\|x^k - x\|^2 - \frac{r}{2}\|x^{k+1} - x\|^2 - \frac{r}{2}\|x^k - x^{k+1}\|^2 + (\hat{x}^k)^TA(x - x^{k+1}).
\]
(75)

According to the $\lambda$-subproblem in Algorithm \(8\), we have
\[
(\lambda^{k+1} - \lambda)^Tb = (\lambda^{k+1} - \lambda)^T(Ax^{k+1} - \frac{1}{r}(AA^T + \delta' I_m)(\lambda^{k+1} - \lambda^k))
= \frac{1}{2r^k}(\|\lambda^k - \lambda\|^2_H - \|\lambda^{k+1} - \lambda\|^2_H - \|\lambda - \lambda^{k+1}\|^2_H) + (\lambda^{k+1} - \lambda)^TAx^{k+1}.
\]
(76)

Putting (76) and (76) together completes the proof. \(\square\)

**Theorem 9** Suppose $f(x)$ is convex (but not necessarily strongly convex), then the sequence $\{(x^k, \lambda^k)\}$ generated by dual-primal balanced ALM \(8\) satisfies that for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$,
\[
f(\hat{x}^k) + \lambda^TA\hat{x}^k - b = (\lambda^k - \lambda)^T(Ax - b)
\leq \frac{1}{K+1}(\frac{r}{2}\|x^0 - x\|^2 + \frac{1}{2r}(\|\lambda^0 - \lambda\|^2_H - \|\lambda^{-1} - \lambda^0\|^2_H) + (\lambda^{k-1} - \lambda^k)^TA(x^0 - x),
\]
where $\hat{x}^K = (\sum_{k=0}^K x^{k+1})/(K+1)$ and $\hat{\lambda}^K = (\sum_{k=0}^K \lambda^{k+1})/(K+1)$.

Proof. With the setting $r^k = r$ and $\delta = \delta'/r$, Algorithm \(8\) reduces to Algorithm \(8\). Then, according to Lemma \(4\), we have
\[
f(x^{k+1}) + \lambda^T(Ax^{k+1} - b) - f(x) - (\lambda^{k+1})^TAx - b
\leq \frac{r}{2}(\|x^k - x\|^2 - \|x^{k+1} - x\|^2 - \|x^k - x^{k+1}\|^2) + \frac{1}{2r}(\|\lambda^k - \lambda\|^2_H - \|\lambda^{k+1} - \lambda\|^2_H - \|\lambda - \lambda^{k+1}\|^2_H) + (2\lambda^k - \lambda^{k-1} - \lambda^{k+1})^TA(x^{k+1} - x),
\]
(77)
Proof. First we can verify that

\[
(2\lambda^k - \lambda^{k-1} - \lambda^{k+1})T A(x^{k+1} - x) \\
= (\lambda^k - \lambda^{k+1})T A(x^{k+1} - x) - (\lambda^{k-1} - \lambda^k)T A(x^k - x) \\
+ (\lambda^{k-1} - \lambda^k)T A(x^k - x^{k+1}) \\
\leq (\lambda^k - \lambda^{k+1})T A(x^{k+1} - x) - (\lambda^{k-1} - \lambda^k)T A(x^k - x) \\
+ \frac{1}{2\tau} ||\lambda^{k-1} - \lambda^k||_A^2 + \frac{\tau}{2} ||x^k - x^{k+1}||^2 .
\]

Adding all the inequalities (77) from \(k = 0\) to \(k = K\) and then dividing both sides by \((K + 1)\) completes the proof. \(\square\)

Different from the \(O(1/K)\) convergence rate established in [12] based on the convex combination of \(\{(x^k, \lambda^k)\}\), our analysis relies on the new convex combination of iteration \(\{(x^{k+1}, \lambda^{k+1})\}\), which can provide an \(O(1/K^2)\) convergence rate of the accelerated dual-primal balanced ALM (Algorithm 5).

**Theorem 10** Suppose \(f(x)\) is \(\mu\)-strongly convex. For any \(x \in \mathbb{R}^n\) and \(\lambda \in \mathbb{R}^m\), the sequence \(\{x^k, \lambda^k\}\) generated by Algorithm 5 with the special setting \(\theta^k = r^k/r^{k+1}, (r^k + \mu)r^k \geq (r^{k+1})^2, \text{ and } 0 < r^k \leq r^{k+1}\) satisfies that

\[
\left(\sum_{k=0}^{K} r^k\right) \left( f(x^K) + \lambda^T (Ax^K - b) - f(x) - (\hat{\lambda}^K)^T (Ax - b) \right) \\
\leq \frac{(r^0)^2}{2} ||x^0 - x||^2 + \frac{1}{2} ||\lambda^0 - \lambda||_H^2 + \frac{1}{2} ||\lambda^{k-1} - \lambda^0||_H^2 - r^{-1}(\lambda^{k-1} - \lambda^0)^T A(x^0 - x),
\]

where \(\hat{x}^K = (\sum_{k=0}^{K} r^k x^{k+1})/(\sum_{k=0}^{K} r^k)\) and \(\hat{\lambda}^K = (\sum_{k=0}^{K} r^k \lambda^{k+1})/(\sum_{k=0}^{K} r^k).\)

In particular, with the setting \(r^k = \mu(k + 1)/3\), we have

\[
f(\hat{x}^K) + \lambda^T (Ax^K - b) - f(x) - (\hat{\lambda}^K)^T (Ax - b) \leq O(1/K^2)
\]

Proof. First we can verify that

\[
(\hat{\lambda}^k - \lambda^{k+1})T A(x^{k+1} - x) \\
= (\lambda^k - \lambda^{k+1})T A(x^{k+1} - x) - \theta^{k-1}(\lambda^{k-1} - \lambda^k)T A(x^k - x) \\
+ \theta^{k-1}(\lambda^{k-1} - \lambda^k)T A(x^k - x^{k+1}) \\
\leq (\lambda^k - \lambda^{k+1})T A(x^{k+1} - x) - \theta^{k-1}(\lambda^{k-1} - \lambda^k)T A(x^k - x) \\
+ \frac{1}{2\tau} ||\lambda^{k-1} - \lambda^k||_A^2 + \frac{(r^{k-1})^2}{2\tau} ||x^k - x^{k+1}||^2 \\
\leq (\lambda^k - \lambda^{k+1})T A(x^{k+1} - x) - \theta^{k-1}(\lambda^{k-1} - \lambda^k)T A(x^k - x) \\
+ \frac{1}{2\tau} ||\lambda^{k-1} - \lambda^k||_H^2 + \frac{r^k}{2} ||x^k - x^{k+1}||^2.
\]
Multiplying both sides of (73) by \( r^k \) and noting the fact \((r^k + \mu)r^k \geq (r^{k+1})^2\) and (75) yields that

\[
\begin{align*}
r^k \left( f(x^{k+1}) + \lambda^T (Ax^{k+1} - b) - f(x) - (\lambda^{k+1})^T (Ax - b) \right) \\
\leq \frac{(r^k)^2}{2} \|x^k - x\|^2 - \frac{(r^{k+1})^2}{2} \|x^{k+1} - x\|^2 \\
+ \frac{1}{2} \left( \|\lambda^k - \lambda\|_H^2 - \|\lambda^{k+1} - \lambda\|_H^2 + \|\lambda^{k+1} - \lambda^k\|_H^2 - \|\lambda^k - \lambda^{k+1}\|_H^2 \right) \\
- r^{k-1} (\lambda^{k-1} - \lambda^k)^T A (x^k - x) + r^k (\lambda^k - \lambda^{k+1})^T A (x^{k+1} - x).
\end{align*}
\]

By adding all the inequalities (79) from \( k = 0 \) to \( k = K \), we obtain

\[
\left( \sum_{k=0}^{K} r^k \right) \left( f(\bar{x}^K) + \lambda^T (A\bar{x}^K - b) - f(x) - (\bar{\lambda}^K)^T (Ax - b) \right) \\
\leq \frac{(\bar{r}^0)^2}{2} \|x^0 - x\|^2 - \frac{(\bar{r}^{K+1})^2}{2} \|x^{K+1} - x\|^2 \\
+ \frac{1}{2} \left( \|\bar{\lambda}^0 - \lambda\|_H^2 - \|\bar{\lambda}^{K+1} - \lambda\|_H^2 + \|\bar{\lambda}^{K+1} - \lambda^0\|_H^2 - \|\bar{\lambda}^0 - \bar{\lambda}^{K+1}\|_H^2 \right) \\
- \bar{r}^{-1} (\lambda^{-1} - \lambda^0)^T A (x^0 - x) + \bar{r}^K (\lambda^0 - \lambda^{K+1})^T A (x^{K+1} - x).
\]

Then, by further observing that

\[
r^K (\lambda^K - \lambda^{K+1})^T A (x^{K+1} - x) \leq \frac{1}{2} \|\lambda^K - \lambda^{K+1}\|_H^2 + \frac{(r^{K+1})^2}{2} \|x^{K+1} - x\|^2,
\]

we can complete the proof. \( \square \)

\textbf{Remark 2} As shown in Remark 4, Algorithm 7 is equivalent to proximal ADMM for solving the dual problem (65). Analogously, Algorithm 7 is expected to be equivalent to the following proximal ADMM for solving (65):

\[
\begin{cases}
  u^{k+1} = \arg \min_u \left\{ f^*(u) + (\bar{z}^k)^T u + \frac{1}{r^k} \|u + A^T \lambda^k\|^2 \right\}, \\
  \lambda^{k+1} = \arg \min_\lambda \left\{ b^T \lambda + (A\bar{x}^k)^T \lambda + \frac{1}{2r^k} \|u^{k+1} + A^T \lambda\|^2 + \frac{\bar{z}^k}{r^k} \|\lambda - \lambda^{k-1}\|^2 \right\}, \\
  \bar{z}^{k+1} = \bar{z}^k + \frac{1}{r^k} (u^{k+1} + A^T \lambda^{k+1}).
\end{cases}
\]

According to the optimality condition of \( u \)-subproblem, we have

\[
0 \in \partial f^*(u^{k+1}) + \bar{z}^k + \frac{1}{r^k} (u^{k+1} + A^T \lambda^{k+1})
\]

\[
= \partial f^*(u^{k+1}) + \bar{z}^{k+1} + \frac{1}{r^k} A^T (\lambda^k - \lambda^{k+1}).
\]

Let \( \bar{x}^{k+1} = -\bar{z}^{k+1} - \frac{1}{r^k} A^T (\lambda^k - \lambda^{k+1}) \). We obtain

\[
x^{k+1} \in \partial f^*(u^{k+1}) \iff u^{k+1} \in \partial f(x^{k+1}).
\]
Then, it holds that
\[
0 \in \partial f(x^{k+1}) - u^{k+1} = \partial f(x^{k+1}) + r^k(\bar{x} - x^{k+1}) + A^T \lambda^{k+1}
\]
\[
= \partial f(x^{k+1}) + A^T(\lambda^k + \frac{r^k}{\rho} (\lambda^k - \lambda^{k-1})) + r^k(x^{k+1} - x^k).
\]

According to the optimality condition of $\lambda$-subproblem, we have
\[
0 = b + A x^k + \frac{1}{r^k} A(u^{k+1} + A^T \lambda^k) + \delta^k(\lambda^{k+1} - \lambda^k)
\]
\[
= b - A x^{k+1} + (\frac{1}{r^k} A A^T + \delta^k I_m)^{-1}[A x^{k+1} - b].
\]
Hence, Algorithm (81) is equivalent to
\[
\begin{cases}
  x^{k+1} = \text{argmin}\{f(x) + \frac{r^k}{2}\|x - (x^k - \frac{1}{r^k} A^T(\lambda^k + \frac{r^k}{\rho} (\lambda^k - \lambda^{k-1}))\|_2^2} \},
  \lambda^{k+1} = \lambda^k + (\frac{1}{r^k} A A^T + \delta^k I_m)^{-1}[A x^{k+1} - b].
\end{cases}
\]

Algorithm (81) is different from Algorithm (8). The $O(1/K^2)$ convergence rate established in Theorem 10 cannot be extended for Algorithm (81). To the best of our knowledge, it is unknown whether Algorithm (81) enjoys an $O(1/K^2)$ convergence rate if $f(x)$ is $\mu$-strongly convex.

5 Conclusions

We have proposed a lift-and-permute scheme of ADMM for solving convex programming problems with linear equality constraints. We show that not only the recent balanced augmented Lagrangian method and its dual-primal variation, but also the proximal ADMM and Douglas-Rachford splitting algorithm correspond to special algorithms in our scheme. As extensions, we propose accelerated algorithms with worst-case $O(1/k^2)$ convergence rates in the case that $f(x)$ is strongly convex. Our results can be easily generalized to solve more general convex programming problems with additional linear inequality constraints.

We notice that each algorithm in our scheme has a fixed permuted order to update variables. It is interesting to consider algorithms with randomized order for updating variables in each iteration. Future works also include applying our lift-and-permute scheme to the primal lifted problem and then studying their convergence rates and acceleration.
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