YANG–BAXTER SYMMETRY IN INTEGRABLE MODELS:
NEW LIGHT FROM THE BETHE ANSATZ SOLUTION

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ABSTRACT

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We show how any integrable 2D QFT enjoys the existence of infinitely many non–abelian conserved charges satisfying a Yang–Baxter symmetry algebra. These charges are generated by quantum monodromy operators and provide a representation of $q$–deformed affine Lie algebras. We review and generalize the work of de Vega, Eichenherr and Maillet on the bootstrap construction of the quantum monodromy operators to the sine–Gordon (or massive Thirring) model, where such operators do not possess a classical analogue. Within the light–cone approach to the mT model, we explicitly compute the eigenvalues of the six–vertex alternating transfer matrix $\tau(\lambda)$ on a generic physical state, through algebraic Bethe ansatz. In the thermodynamic limit $\tau(\lambda)$ turns out to be a two–valued periodic function. One determination generates the local abelian charges, including energy and momentum, while the other yields the abelian subalgebra of the (non–local) YB algebra. In particular, the bootstrap results coincide with the ratio between the two determinations of the lattice transfer matrix.

1. Introduction

When a physical model is integrable it always possess extra conserved quantities not related to manifest symmetries but presumably with hidden dynamical symmetries. For 2D lattice models and 2D quantum field theory (QFT), integrability is a consequence of the Yang-Baxter equation (YBE).

In lattice vertex models, using the $R$–matrix elements to define the local statistical weights, the monodromy matrix $T_{ab}(\lambda)$ for a lattice line obeys the YB algebra:

$$R(\lambda - \mu) [T(\lambda) \otimes T(\mu)] = [T(\mu) \otimes T(\lambda)] R(\lambda - \mu)$$

(1.1)

where $\lambda$ stands for the spectral parameter. Its trace, $t(\lambda) \equiv \sum_a T_{aa}(\lambda)$, provides a commuting family of operators,

$$[ t(\lambda), t(\mu) ] = 0$$

for any lattice size. In general the transfer matrix $t(\lambda)$ is conserved, since, among all
commuting charges, it generates also the Hamiltonian. It is now clear from eq.(1.1), that the \( T_{ab}(\lambda) \) do not commute with \( t(\lambda) \) and are generally not conserved. That is, for finite lattice size we have only an infinite abelian symmetry generated by \( t(\lambda) \) (or by its series expansion coefficients).

In ref.[1], a bootstrap construction of monodromy matrices \( T_{ab}(u) \) was proposed in a class of integrable QFT. These \( T_{ab}(u) \) are conserved and obey a YB algebra analogous to eq.(1.1). Hence, this class of QFT enjoy an infinite YB non-abelian symmetry generated by the \( T_{ab}(u) \). (This construction is valid in the infinite space).

A classically conserved limit of \( T_{ab}(u) \) exists in the class of models considered in ref. [1] where the \( R \)–matrix is a rational function of \( u \).

In the present paper we first show that the bootstrap construction of conserved \( T_{ab}(u) \) generalizes to integrable models with trigonometric \( R \)–matrices such as the sine-Gordon or massive Thirring model. In such cases the classical limit is abelian, as shown explicitly in sec.3.

The main aim of this work is then to investigate and clarify, from a microscopic point of view, the problem of unveiling the existence of the infinite YB symmetry of the sG–mT model. In other words, since lattice models provide regularized version of QFT, we seek an explicit connection between the lattice and the bootstrap YB algebras. For this purpose we adopt the so–called light-cone approach, which is a general method to precisely derive QFT’s as scaling limits of integrable lattice models [2,3]. One starts from a diagonal-to-diagonal lattice with lines at angles \( 2\Theta \). The light-cone evolution operators \( U_R \) and \( U_L \) are introduced (eq.(5.2)) which define the lattice hamiltonian and momentum (eq.(5.3)). They can be expressed in terms of the values at \( \lambda = \pm \Theta \) of the alternating transfer matrix \( t(\lambda, \Theta) \) (eqs. (5.5)–(5.6)). Through algebraic Bethe Ansatz the ground state and excited states are constructed in the thermodynamic limit. When the ground state is antiferromagnetic, it corresponds to a Dirac sea of interacting pseudoparticles. Excited states around it describe particle-like physical excitations. In this way, the sG–mT model is obtained from the six-vertex model (with anisotropy \( \gamma \))[2]. QFT like mul-
ticomponent Thirring models, sigma models and others follow from various vertex models [3].

In order to investigate the operators present in such QFT, it is important to learn how the monodromy operators \( T_{ab}(\lambda, \Theta) \) act on physical states. In the present paper, we explicitly compute the eigenvalues of the alternating six–vertex transfer matrix \( t(\lambda, \Theta) \equiv \sum_a T_{aa}(\lambda, \Theta) \), on a generic \( n \)--particle state, in the thermodynamic limit. The explicit formulae are given by eqs.(6.32),(6.37) and (6.38). The eigenvalues of \( t(\lambda, \Theta) \) turn out to be \( i\pi \)--periodic and multi–valued functions of \( \lambda \), each determination of \( t(\lambda, \Theta) \) being a meromorphic function of \( \lambda \). We call \( t^{II}(\lambda, \Theta) \) and \( t^{I}(\lambda, \Theta) \) the determinations associated with the periodicity strips closer to the real axis (see fig. 4). The ground–state contribution \( \exp[-iG(\lambda)V] \) is exponential on the lattice size, as expected, whereas the excited states contributions are finite and express always in terms of hyperbolic functions [see sec. 6].

We then compare these Bethe Ansatz eigenvalues with the eigenvalues of the bootstrap transfer matrix \( \tau(u) \equiv \sum_a T_{aa}(u) \). Remarkably enough, we find the following simple relation between the two results, for \( 0 < \gamma < \pi/2 \) (repulsive regime),

\[
\tau(u) = t^{II}\left(\frac{\gamma}{\pi}u - i\frac{\gamma}{2}, \Theta\right) t^{I}\left(\frac{\gamma}{\pi}u - i\frac{\gamma}{2}, \Theta\right)^{-1}
\]

(1.2)

where \( t^{II}(\lambda, \Theta) \) and \( t^{I}(\lambda, \Theta) \) have been normalized to one on the ground state. Thus, we succeed in connecting the bootstrap transfer matrix \( \tau(u) \) of the sG-mT model with the alternating transfer matrix \( t(\lambda, \Theta) \) of the six vertex model. In the thermodynamic limit \( \tau(u) \) coincide with the jump between the two main determinations of \( t(\lambda, \Theta) \). Notice the renormalization of the rapidity by \( \gamma/\pi \) and the precise overall shift by \( i\gamma/2 \) in the argument in order the equality to hold.

We find in addition that \( t(\lambda, \Theta) \), for \( 0 < \text{Im} \lambda < \gamma/2 \), generates the hamiltonian and momentum together with an infinite number of higher–dimension and higher–spin conserved abelian charges, through expansion in powers of \( e^{\pm \pi \lambda/\gamma} \). We see therefore that the same bare operator generates two kinds of conserved quantities.
Energy and momentum as well the higher-spin abelian charges are local in the basic fields which interpolate physical particles, whereas the infinite set of charges obtained from the jump from $t^{II}(\lambda, \Theta)$ to $t^{I}(\lambda, \Theta)$ are nonlocal in the same fields. The fact that local and nonlocal charges come from different sides of a natural boundary, clearly shows that they carry independent information. That is, one cannot produce the nonlocal charges from the sole knowledge of the local charges. We also recall that the monodromy matrix $T(\lambda, \Theta)$ can be written in terms of the lattice fermi fields of the mT model [2], so that local and nonlocal charges do admit explicit expressions in terms of local field operators.

As a byproduct of this analysis, we find an explicit relation for the light-cone lattice hamiltonian and momentum $P_{\pm}$ in terms of the continuum hamiltonian and momentum $p_{\pm}$ plus an infinite series of higher conserved continuum charges $I_{j}\pm$, playing the role of irrelevant operators,

$$P_{\pm} = (P_{\pm})_{V} + p_{\pm} + \frac{m}{4} \sum_{j=1}^{\infty} \left(\frac{ma}{4}\right)^{2j} I_{j}\pm$$

(1.3)

where $(P_{\pm})_{V}$ stands for the ground state contribution.

We expect eqs.(1.2) - (1.3), and the discussion in-between, to be valid for many other integrable models provided the appropriate rapidity renormalization and imaginary shift are introduced.

The next natural step after finding the connection (1.2) between transfer matrices would be to relate the bare and renormalized monodromies $T_{ab}(\lambda, \Theta)$ and $T_{ab}(u)$ . This is necessarily more involved. For $\gamma \neq 0$ they obey the same YB algebra but with different anisotropy parameters $\gamma$ and $\hat{\gamma} \equiv \frac{2}{1-\gamma/\pi}$, respectively. The rational case ($\gamma = 0$) is evidently simpler and it is the only case where classically conserved monodromies are present.

The quantum monodromy operators $T_{ab}(u)$ generate a Fock representation of the $q$–deformed affine Lie algebra $U_{q}(\hat{G})$ corresponding to the given $R$–matrix. More precisely, by expanding $T_{ab}(u)$ in powers of $z = e^{u}$ around $z = 0$ and $z = \infty$,
one obtains non–abelian non–local conserved charges representing the algebra $U_q(\hat{G})$ on the Fock space of in– and out–particles. This connects our approach based on the YB symmetry, to the $q$–deformed algebraic approach of ref.[4]. $U_q(\hat{G})$ is a Hopf algebra endowed with an universal $R$–matrix, which reduces to the $R$– explicitly entering the YB algebra, upon projection to the finite–dimensional vector space spanned by the indexes of $\mathcal{T}_{ab}(u)$ [5]. In particular, the two expansions around $z = 0$ and $z = \infty$ generate the two Borel subalgebras of $U_q(\hat{G})$. A single monodromy matrix $\mathcal{T}(u)$ is sufficient for this purpose, since this field–theoretic representation has level zero [6]. This fact receives a new explanation in the light–cone approach, since $U_q(\hat{G})$ emerges as true symmetry only in the infinite–volume limit above the antiferromagnetic ground state (with no need to take the continuum limit), but its action is uniquely defined already on finite lattices, and all finite–dimensional representations have level zero.

Besides the conserved operators $\mathcal{T}_{ab}(u)$, Zamolodchikov-Faddeev non-conserved operators $Z_\alpha(\theta)$ act by creating particles on physical states. Their algebra with the $\mathcal{T}_{ab}(u)$ is determined by the two body S-matrix:

$$\mathcal{T}_{ab}(u)Z_\beta(\theta) = \sum_{cc'} Z_\alpha(\theta)\mathcal{T}_{ac}(u)S^{cc'}_{b\beta}(u + \theta)$$

(1.4)

The ZF operators provide a representation of the dynamical symmetry of $q$–deformed vertex operators in the sense of [7].
2. Bootstrap construction of quantum monodromy operators.

We briefly review in this section the work of refs. [1] where the exact (renormalized) matrix elements of a quantum monodromy matrix $T_{ab}(u)$ ($u$ is the generally complex spectral parameter) were derived using a bootstrap–like approach for a class of integrable local QFT’s. In such theories there is no particle production and the $S$–matrix factorizes. The two–body $S$–matrix then satisfies the Yang–Baxter (YB) equations. Moreover, in the models considered in refs.[1] (the O(N) nonlinear sigma model, the SU(N) Thirring model and the 0(2N) Gross–Neveu model), thanks to scale invariance there exist classically conserved monodromy matrices. In general, the quantum $T_{ab}(u)$ can be constructed by fixing its action on the Fock space of physical in and out many–particle states. The starting point are the following three general principles:

a) $T_{ab}(u)$, $a,b=1,2,\ldots,n$, exist as quantum operators and are conserved.

b) $T_{ab}(u)$ fulfil a quantum factorization principle.

c) $T_{ab}(u)$ is invariant under P, T and the internal symmetries of the theory.

The quantum factorization principle referred above under b) is nowadays called the ”coproduct rule”. This means that there exists the following relation between the action of $T_{ab}(u)$ on $k$–particles states and its action on one–particle states

\begin{equation}
T_{ab}(u) |\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{in} = \sum_{a_1 a_2 \ldots a_{k-1}} T_{aa_1}(u) |\theta_1 \alpha_1\rangle T_{a_1 a_2}(u) |\theta_2 \alpha_2\rangle \cdots T_{a_{k-1}b}(u) |\theta_k \alpha_k\rangle
\end{equation}

(2.1)

\begin{equation}
T_{ab}(u) |\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{out} = \sum_{a_1 a_2 \ldots a_{k-1}} T_{a_1 b}(u) |\theta_1 \alpha_1\rangle T_{a_2 a_1}(u) |\theta_2 \alpha_2\rangle \cdots T_{a_{k-1}a_k}(u) |\theta_k \alpha_k\rangle
\end{equation}

(2.2)

where $\theta_j$ and $\alpha_j$ ($1 \leq j \leq k$) label the rapidities and the internal quantum numbers of the particles, respectively, in the asymptotic in and out states. Hence it is understood that $\theta_i > \theta_j$ for $i > j$.

Although $T_{ab}(u)$ acts differently on in and out states, the assumption of conservation is nonetheless consistent. All the eigenvalues of a maximal commuting
subset of \{T_{ab}(u), a, b = 1, 2, \ldots, n, u \in \mathbb{C}\} are identical for in and out states with given rapidities. Indeed the two in and out forms of the action on the internal quantum numbers are related by the unitary permutation \(|\alpha_1, \alpha_2, \ldots, \alpha_k\rangle \rightarrow |\alpha_k, \alpha_{k-1}, \ldots, \alpha_1\rangle\).

Furthermore, principles (a) and (c) imply that \(T_{ab}(u)\) acts in a trivial way on the physical vacuum state \(|0\rangle\):

\[
T_{ab}(u) |0\rangle = \delta_{ab} |0\rangle
\]

This also fixes the normalization of \(T_{ab}(u)\) in agreement with the classical limit [1].

An immediate consequence of point (b) is that when \(T_{ab}(u)\) is expanded in powers of the spectral parameter \(u\), it generates an infinite set of noncommuting and nonlocal conserved charges. This is the clue to the matching of the quantum monodromy matrix with its classical counterpart which is written nonlocally in terms of the local fields.

The main result in refs.[1] was to derive from (a), (b) and (c) the explicit matrix elements of \(T_{ab}(u)\) on one–particle states. This result can be written as

\[
\langle \theta \alpha | T_{ab}(u) | \theta' \beta \rangle = \delta(\theta - \theta') S_{b \beta}^{a \alpha}(\kappa(u) + \theta)
\]

where \(S_{b \beta}^{a \alpha}(\theta - \theta')\) stands for the \(S\)–matrix of two–body scattering

\[
|\theta b, \theta' \beta\rangle_{in} = \sum_{aa} |\theta a, \theta' \alpha\rangle_{out} S_{b \beta}^{a \alpha}(\theta - \theta')
\]

and \(\kappa(u)\) is an odd function of \(u\). Notice that this requires the presence in the model of particles with indices \(a, b, \ldots\) as internal state labels. In the simplest situation these new labels coincide with those of the original particles. The appearance of a nontrivial “renormalization” \(u \rightarrow \kappa(u)\) is to be expected when there exist a definition of the spectral parameter outside the bootstrap itself. This is the case
of the models of refs.\[1\], which posses Lax pairs and auxiliary problems which fix the definition of $u$. Here we shall adopt the purely bootstrap viewpoint and fix the definition of $u$ so that $\kappa(u) = u$. In principle, an extra $u-$ and $\theta-$dependent phase factor may appear in the r.h.s. of eq.(2.4). However, no phase showed up in the specific models of refs.\[1\], when nonperturbative checks were performed using the operator product expansion. Eq.(2.4) can be written in a more suggestive way as

$$T_{ab}(u) |\theta\beta\rangle = \sum_\alpha |\theta\alpha\rangle S_{b\beta}^{a\alpha}(u + \theta)$$

(2.6)

This equation, when combined with eqs.(2.1) and (2.2), completely defines the quantum monodromy operators in the Fock space. From the YB equations satisfied by the $S-$matrix it then follows that $T_{ab}(u)$ fulfils the YB algebra

$$\hat{R}(u - v) [T(u) \otimes T(v)] = [T(u) \otimes T(v)] \hat{R}(u - v)$$

(2.7)

where $\hat{R}_{b\beta}^{a\alpha}(u) = S_{b\beta}^{a\alpha}(u)$. It should be stressed that the conservation of $T_{ab}(u)$ implies that this YB algebra is a true non–abelian infinite symmetry algebra of the relativistic local QFT. On the contrary the rôle of the YB algebra in integrable vertex and face models on finite lattices or in nonrelativistic quantum models is that of a dynamical symmetry underlying the Quantum Inverse Scattering Method. In these latter cases, only the transfer matrix, namely

$$\tau(u) = \sum_a T_{aa}(u)$$

(2.8)

is conserved. Since $[\tau(u), \tau(v)] = 0$, the transfer matrix just generates an abelian symmetry.

The dynamical symmetry underlying the integrable QFT includes in addition non–conserved operators $Z_{\alpha}(\theta)$ which create the particle eigenstates out of the
vacuum. In the bootstrap framework they can be introduced à la Zamolodchikov–Faddeev, by setting

$$|\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{\text{in}} = Z_{\alpha_k}(\theta_k) Z_{\alpha_{k-1}}(\theta_{k-1}) \ldots Z_{\alpha_1}(\theta_1) |0\rangle$$

$$|\theta_1 \alpha_1, \theta_2 \alpha_2, \ldots, \theta_k \alpha_k\rangle_{\text{out}} = Z_{\alpha_1}(\theta_1) Z_{\alpha_2}(\theta_2) \ldots Z_{\alpha_k}(\theta_k) |0\rangle$$

with the fundamental commutation rules

$$Z_{\alpha_2}(\theta_2) Z_{\alpha_1}(\theta_1) = \sum_{\beta_1 \beta_2} S_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} (\theta_1 - \theta_2) Z_{\beta_1}(\theta_1) Z_{\beta_2}(\theta_2)$$

Combining now eqs. (2.1), (2.2), (2.6) and (2.10), we obtain the algebraic relation between monodromy and Zamolodchikov–Faddeev operators:

$$\mathcal{T}_{ab}(u) Z_\beta(\theta) = \sum_{c \alpha} Z_\alpha(\theta) \mathcal{T}_{ac}(u) S_{b \beta}^{c \alpha}(u + \theta)$$

Together with eqs. (2.7) and (2.10), these relations close the complete dynamical algebra of an integrable QFT. For the XXZ spin chain in the regime $|q| < 1$, the ZF operators have been identified in ref. [7] with special vertex operators (or representation intertwiners of the relevant $q$–deformed affine Lie algebra). They are uniquely characterized by being solutions of the $q$–deformed Knizhnik–Zamolodchikov equation and by their normalization [5].

3. Yang-Baxter symmetry in the sine-Gordon model

In refs.[1] the infinite YB symmetry was explicitly considered and exhibited for classically scale–invariant models like the $O(N)$ nonlinear sigma, the $SU(N)$ Thirring and the $0(2N)$ Gross–Neveu models. Indeed, it is this scale–invariance which guarantees the conservation also of the nondiagonal elements of the classical monodromy matrix. An important example of integrable QFT which does not belong to this category is the sine–Gordon (sG) model. The presence of mass at
the classical level implies that only its transfer matrix is conserved. Then the classical symmetry algebra is commuting (as Poisson brackets) and admits a basis of local conserved charges. It is well known that these charges survive quantization, leading to factorization of the scattering with corresponding YB equations satisfied by the two-body $S-$matrix [8].

It appears therefore natural to apply the general bootstrap methods of the previous section also to the sG model. The quantum $T_{ab}(u)$ (with $a, b = \pm$) defined in this way is assumed to be conserved from the outset, and does not reduce to its classical counterpart in the classical limit. Let us consider first soliton and antisoliton states. They have mass $M$ and a charge $\alpha = +1 (-1)$ for solitons (antisolitons). Let us recall that the solitons (antisolitons) are the fermions (antifermions) of the massive Thirring model, which is equivalent to the sG model as QFT in 2D Minkowski space.

Eq.(2.6) in the one-particle soliton/antisoliton sector takes now the form

\begin{align}
T_{\pm\pm}(u) |\theta\pm\rangle &= S(u + \theta) |\theta\pm\rangle \\
T_{\pm\mp}(u) |\theta\mp\rangle &= S_T(u + \theta) |\theta\mp\rangle \\
T_{\pm\mp}(u) |\theta\pm\rangle &= S_R(u + \theta) |\theta\mp\rangle \\
T_{\pm\mp}(u) |\theta\mp\rangle &= 0
\end{align}

where $S(\theta)$ is the soliton/soliton scattering amplitude, while $S_T(\theta)$ and $S_R(\theta)$ are, respectively the transmission and reflection amplitudes of the soliton/antisoliton scattering. Explicitly they read [8]:

\begin{align}
S(\theta) &= \exp i \int_0^\infty \frac{dk}{k} \frac{\sinh(\pi/\hat{\gamma} - 1)k/2 \sin k\theta/\pi}{\sinh(\pi k/2\hat{\gamma}) \cosh k/2} \\
S_T(\theta) &= S(i\pi - \theta) \\
S_R(\theta) &= S(\theta) \frac{\sin \hat{\gamma}}{\sin \hat{\gamma}(1 - \theta/i\pi)}
\end{align}
where $\hat{\gamma}$ is related to the usual sG coupling constant $\beta$ by

$$\frac{\hat{\gamma}}{\pi} = \frac{8\pi}{\beta^2} - 1 \quad (3.3)$$

The action of $\mathcal{T}_{ab}(u)$ on multiparticle soliton/antisoliton states is obtained by simply inserting eqs. (3.1) into eqs. (2.1) and (2.2). Notice that $S(\theta)$, $S_T(\theta)$ and $S_R(\theta)$ possess essential singularities at $\beta^2 = 0$. That is,

$$S(\theta) \xrightarrow{\beta \to 0} \exp -i \frac{16\pi}{\beta^2} \int_0^\infty dk \frac{k}{\sinh(k/2)} \sin \frac{k\theta}{\pi} \equiv S_c(\theta) \quad (3.4)$$

Hence the quantum monodromy matrix is singular in the free boson limit $\beta = 0$. Of course it is regular, although trivial ($\mathcal{T}_{ab}(u) = \delta_{ab}$), in the free fermion limit $\hat{\gamma} = \pi$.

In the classical limit instead, we must replace the scattering amplitudes with the analogous quantities computed for soliton field configurations in the classical sG model, namely

$$S(\theta) = S_T(i\pi - \theta) = S_c(\theta), \quad S_R(\theta) = 0$$

(there is no soliton reflection at the classical level). Hence $\mathcal{T}_{ab}(u)$ becomes diagonal and the YB algebra becomes abelian. This is consistent with the fact that the integrals of motion of the classical sG equation are all in involution.

Besides solitons and antisolitons states, the sG-MTM model possess breathers states for $\hat{\gamma} > \pi$. These particles are labeled by and index $n$ running from 1 to $[\hat{\gamma}/\pi] - 1$ (where $[.]$ stands for integer part) and have masses

$$m_n = 2M \sin\left(\frac{n\pi^2}{2\hat{\gamma}}\right) \quad (3.5)$$

[$n = 1$ corresponds to the fundamental particle associated to the sG-field $\theta$]. Applying the general formula (2.4) to these particle states yields

$$\langle \theta_n | \mathcal{T}_{ab}(u) | \theta'm \rangle = \delta(\theta - \theta')\delta_{nm}\delta_{ab}S_n(u - \theta) \quad (3.6)$$

where $|\theta_n\rangle$ stands for a $n$th breather state with rapidity $\theta$ and $S_n(\theta - \theta')$ is the
soliton-breather S-matrix. That is [8],

\[ S_n(\theta) = \frac{\sinh \theta + i \cos \frac{n\pi^2}{2\gamma} \prod_{l=1}^{n-1} \sin^2\left(\frac{n}{2} - l\right) \frac{\pi^2}{2\gamma} - \frac{\pi}{4} + i\theta}{\sinh \theta - i \cos \frac{n\pi^2}{2\gamma} \prod_{l=1}^{n-1} \sin^2\left(\frac{n}{2} - l\right) \frac{\pi^2}{2\gamma} - \frac{\pi}{4} - i\theta} \]  

(3.7)

In particular,

\[ S_1(\theta) = \frac{\sinh \theta + i \cos \frac{\pi^2}{2\gamma}}{\sinh \theta - i \cos \frac{\pi^2}{2\gamma}} \]  

(3.8)

We conclude that \( T_{ab}(u) \) has a rather trivial action on breather states

\[ T_{ab}(u) = \delta_{ab}S_B(u) \]  

(3.9)

where \( S_B(u) \) is a diagonal operator with eigenvalues \( S_n(u-\theta) \) on the \( n \)th breather state. In conclusion, we have uncovered the infinite YB symmetry of the sG-mT model providing the explicit form of its conserved operators on all the asymptotic states.

It is instructive to study the \( u \to \infty \) limit of the YB operator \( T_{ab}(u) \). We find from eqs. (3.1)-(3.2) for \( u \to \infty \),

\[ T_{ab}(u) = \exp\left(ia \frac{4\pi^2}{\beta^2} \sigma_3\right) \delta_{ab} + 2ie^{i\frac{8\pi u}{\beta^2}} \exp\left(-\frac{8\pi u}{\beta^2}\right) Q_b(1 - \delta_{ab}) + O(e^{-\frac{16\pi u}{\beta^2}}) \]  

(3.10)

Here \( \beta \) is the usual sine-Gordon coupling constant and \( Q_a \) acts on one-particle soliton/antisoliton states as

\[ Q_a = e^{-\frac{2\theta}{\pi}} \sigma_a \]  

(3.11)

This is a \( SU(2)_q \) generator for the spin 1/2 representation. (For spin 1/2, SU(2) and \( SU(2)_q \) generators coincide). Using eq.(3.10) and the coproduct relations (2.1) and (2.2), we find that eq.(3.10) holds as it stands on two (or more) particle states.
but now with
\[ \sigma_3 = \sigma_3^{(1)} + \sigma_3^{(2)} \quad , \quad Q_a = e^{-ia \frac{b}{\beta^2} \sigma_3^{(1)}} Q_a^{(2)} + Q_a^{(1)} e^{ia \frac{b}{\beta^2} \sigma_3^{(2)}} \]  

(3.12)

Analogous relations hold for multiparticle states. This tells us that \( Q_a \) and \( \sigma_3 \) are related to \( SU(2)_q \) generators with
\[ q = e^{\frac{8i\pi}{\beta}} \]  

(3.13)
as
\[ J_+ = Q_+ \quad , \quad J_- = Q_-^\dagger \quad , \quad J_z = \sigma_3 \]  

(3.14)

Alternatively, we can make the identification \( q = e^{-\frac{8i\pi}{\beta}} \) with:
\[ J_+ = Q_+^\dagger \quad , \quad J_- = Q_- \quad , \quad J_z = \sigma_3 \]  

(3.15)

A nonlocal charge equivalent to \( Q_a \) studied in ref.[9]. The fact that YB generators for \( u = \infty \) yield \( SU(2)_q \) generators in this way is typical of periodic boundary conditions [10]. For fixed boundary conditions (that is scattering of particles between two walls) the connection is much cleaner [11].

4. Bethe Ansatz at the bootstrap level

The maximal abelian subalgebra of the YB algebra (2.7) is generated by the transfer matrix \( \tau(u) \) (eq.(2.8)). With respect to this subalgebra, the remaining elements of \( \mathcal{T}_{ab}(u) \) act as generalized raising and lowering operators. This observation provides the basis for the so-called Algebraic Bethe Ansatz, which is a purely algebraic method to construct the eigenvectors and the eigenvalues of \( \tau(u) \) [12]. The crucial starting point is the identification of the highest weight states annihilated by the raising operators. Since particles are conserved in an integrable QFT model, one can restrict the problem to states with a fixed number, say \( k \), of particles.
In the case of the sG model the highest weight states are the ferromagnetic states containing only solitons, that is the states $|\theta_1^+, \theta_2^+, \ldots, \theta_k^+\rangle$ with the highest possible value $J_z = k/2$ of the $z$–projection of the $SU(2)_q$ spin in the sector with $k$ particles. On such states the monodromy matrix $T_{ab}(u)$ is indeed upper triangular (compare eqs. (2.1) and (2.2) with eqs. (3.1)). The rapidities $\theta_n$ of the solitons are arbitrary and act as fixed parameters in the problem, since they are left unchanged by the action of $T_{ab}(u)$. Then the BA in–eigenstates of $\tau(u) = T_{++}(u) + T_{--}(u)$ with $k - m$ solitons and $m$ antisolitons can be written

$$B(u_1)B(u_2)\ldots B(u_m)|\theta_1^+, \theta_2^+, \ldots, \theta_k^+\rangle_{in}$$

where $B(u) \equiv T_{+-}(u + i\pi/2)$ act as lowering operators of $J_z$ and the distinct numbers $u_1, u_2, \ldots, u_m$ must satisfy the BA equations

$$\prod_{n=1}^{k} \frac{\sinh \hat{\gamma}[i/2 + (u_j + \theta_n)/\pi]}{\sinh \hat{\gamma}[i/2 - (u_j + \theta_n)/\pi]} = -\prod_{r=1}^{m} \frac{\sinh \hat{\gamma}[+i + (u_j - u_r)/\pi]}{\sinh \hat{\gamma}[-i + (u_j - u_r)/\pi]} \quad (4.1)$$

The eigenvalues $\xi(u)$ of $\tau(u)$ on the BA states read

$$\xi(u) = \left\{ \prod_{n=1}^{k} S(u + \theta_n) \right\} \left[ \xi_+(u) + \xi_-(u) \right]$$

$$\xi_+(u) = \prod_{j=1}^{m} \frac{\sinh \hat{\gamma}[i/2 + (u - u_j)/\pi]}{\sinh \hat{\gamma}[i/2 - (u - u_j)/\pi]} \quad (4.2)$$

$$\xi_-(u) = \left[ \prod_{n=1}^{k} \frac{\sinh \hat{\gamma}(u + \theta_n)/\pi}{\sinh \hat{\gamma}[-i/2 + (u + \theta_n)/\pi]} \right] \prod_{j=1}^{m} \frac{\sinh \hat{\gamma}[3i/2 + (u - u_j)/\pi]}{\sinh \hat{\gamma}[-i/2 + (u - u_j)/\pi]}$$

Up to the common factor $\prod_{n=1}^{k} S(u + \theta_n)$, $\xi_{\pm}(u)$ is just the contribution coming from $T_{\pm\pm}(u)$. It is clear, moreover, that the presence of breathers introduce no further complications, due to the diagonal action (3.9) of the monodromy matrix on breather states.
Eqs. (4.1) and (4.2) follow directly from the YB algebra (2.7) satisfied by
construction by the bootstrap monodromy matrix. This algebraic Bethe Ansatz
can be generalized to a whole class of integrable field theories where the bootstrap
construction of sec. 2 applies. Furthermore, let us observe that the diagonaliza-
tion of the bootstrap transfer matrix represents the basic step in the so-called
Thermodynamic BA, which is a way to obtain off–shell exact results on the inte-
grable relativistic QFT at hand. In fact, the transfer matrix \( \tau(u) \), as trace of the
monodromy matrix (eq.(2.8)), is directly related to the multiscattering amplitudes
suffered by each particle in a system of \( k \) particles confined on a ring, namely

\[
\tau(-\theta_j) = S_{jk} \ldots S_{j\,j+1} S_{j\,j-1} \ldots S_{j1}
\]

(4.3)

where \( j = 1, 2, \ldots, k \) and the two–body matrices \( S_{ij} \) are defined by

\[
S_{ij} \langle \theta_1 \alpha_1, \ldots, \theta_k \alpha_k \rangle = \sum_{\beta_i, \beta_j} \left( \prod_{n \neq i,j} \delta_{\beta_n}^{\beta_{n'}} \right) S_{\beta_i \beta_j}^{\alpha_j \alpha_k} (\theta_i - \theta_j) \langle \theta_1 \beta_1, \ldots, \theta_k \beta_k \rangle
\]

By periodicity, eqs. (4.3) and (4.2) determine the quantization of the momentum
of each particle in the standard way

\[
\xi(-\theta_j) \exp \left( i m_j L \sinh \theta_j \right) = 1
\]

(4.4)

where \( L \) is the length of the ring. Together with the BA equations (4.1) for the roots
\( u_1, u_2, \ldots, u_m \) (the so–called magnon parameters), these new equations provide the
basis for the TBA [15].
5. Light-cone lattice regularization.

In order to obtain a first-principles, microscopic understanding of the bootstrap picture presented above, we now consider the integrability-preserving lattice regularization of an integrable relativistic QFT defined by the so-called light-cone approach [2,3] to vertex models.

In this approach one starts from the discretized Minkowski 2D space-time formed by a regular diagonal lattice of right-oriented and left-oriented straight lines (see fig. 1). These represent true world-lines of “bare” objects (pseudo-particles) which are thus naturally divided in left- and right-movers. The right-movers have all the same positive rapidity $\Theta$, while the left-movers have rapidity $-\Theta$. One can regard $\Theta$ as a cut-off rapidity, which will be appropriately taken to infinity in the continuum limit. Furthermore, we shall denote by $\mathcal{V}$ the Hilbert space of states of a pseudo-particle (we restrict here to the case in which $\mathcal{V}$ is the same for both left- and right-movers and has finite dimension $n$, although more general situations can be considered).

The dynamics of the model is fixed by the microscopic transition amplitudes attached to each intersection of a left- and a right-mover, that is to each vertex of the lattice. This amplitudes can be collected into linear operators $R_{ij}$, the local $R-$matrices, acting non-trivially only on the space $\mathcal{V}_i \otimes \mathcal{V}_j$ of $i$th and $j$th pseudo-particles. $R_{ij}$ thus represent the relativistic scatterings of left-movers on right-movers and depend on the rapidity difference $\Theta - (\Theta) = 2\Theta$, which is constant throughout the lattice. Moreover, by space-time translation invariance any other parametric dependence of $R_{ij}$ must be the same for all vertices. We see therefore that attached to each vertex there is a matrix $R(2\Theta)_{cd}^{ab}$, where $a, b, c, d$ are labels for the states of the pseudo-particles on the four links stemming out of the vertex, and take therefore $n$ distinct values (see fig. 1). This is the general framework of a vertex model. The difference with the standard statistical interpretation is that the Boltzmann weights are in general complex, since we should require the unitarity of the matrix $R$. In any case, the integrability of the model is guaranteed
whenever $R(\lambda)^{cd}_{ab}$ satisfy the Yang–Baxter equations

$$R_{ij}(\lambda)R_{jk}(\lambda + \mu)R_{ij}(\mu) = R_{jk}(\mu)R_{ij}(\lambda + \mu)R_{jk}(\lambda)$$ (5.1)

For periodic boundary conditions, the one–step light–cone evolution operators $U_L(\Theta)$ and $U_R(\Theta)$, which act on the ”bare” space of states $\mathcal{H}_N = (\otimes V)^{2N}$, ($N$ is the number of sites on a row of the lattice, that is the number of diagonal lines), are built from the local $R$–matrices $R_{ij}$ as [2]

$$U_R(\Theta) = U(\Theta)V, \quad U_L(\Theta) = U(\Theta)V^{-1}$$

$$U(\Theta) = R_{12}R_{34}\ldots R_{2N-1\ 2N}$$ (5.2)

where $V$ is the one-step space translation to the right. $U_R$ ($U_L$) evolves states by one step in right (left) light–cone direction. $U_R$ and $U_L$ commute and their product $U = U_RU_L$ is the unit time evolution operator. The graphical representation of $U$ is given by the section of the diagonal lattice with fat lines in fig. 1. If $a$ stands for the lattice spacing, the lattice hamiltonian $H$ and total momentum $P$ are naturally defined through

$$U = e^{-iaH}, \quad U_RU_L^{-1} = e^{iaP}$$ (5.3)

The action of other fundamental operators is naturally defined on the same Hilbert space $\mathcal{H}_N$. These are the $n^2$ Yang-Baxter operators for $2N$ sites, which are conventionally grouped into the $n \times n$ monodromy matrix $T(\lambda) = \{T_{ab}(\lambda), \ a, b = 1, \ldots, n\}$. One usually regards the indices $a, b$ of $T_{ab}$ as horizontal indices fixing the out– and in–states of a reference pseudo–particle. Then $T(\lambda)$ is defined as horizontal coproduct of order $2N$ of the local vertex operators $L_j(\lambda) = R_{0j}(\lambda)P_{0j}$, where $0$ label the reference space and $P_{ij}$ is the transposition in $V_i \otimes V_j$. Explicitly

$$T(\lambda) = L_1(\lambda)L_2(\lambda)\ldots L_{2N}(\lambda)$$

The inhomogeneous generalization $T(\lambda, \vec{\omega})$ then reads

$$T(\lambda, \vec{\omega}) = L_1(\lambda + \omega_1)L_2(\lambda + \omega_2)\ldots L_{2N}(\lambda + \omega_{2N})$$

and has the graphical representation of fig. 2. The formal structure of this ex-
pression is identical to that of eq.(2.1). In fact \( L_j(\lambda + \omega_j) \) can be regarded as the scattering of the \( j \)th pseudo–particle carrying formal rapidity \( \omega_j \) with the reference pseudo–particle carrying formal rapidity \(-\lambda\). In the same way, thanks to eq.(2.6), the single particle terms in eq.(2.1) represent the scattering of the corresponding particle on a reference particle carrying physical rapidity \(-u\). In the case of our diagonal lattice of right– and left–moving pseudoparticles, there exists a specific, physically relevant choice of the inhomogeneities, namely

\[
\omega_k = (-1)^k \Theta, \quad k = 1, 2, \ldots 2N
\]  

(5.4)

leading to the definition of the alternating monodromy matrix

\[
T(\lambda, \Theta) \equiv T(\lambda, \{\omega_k = (-1)^k \Theta\})
\]  

(5.5)

In fact, the evolution operators \( U_L(\Theta) \) and \( U_R(\Theta) \) can be expressed in terms of the alternating transfer matrix \( t(\lambda, \Theta) = \text{tr}_0 T(\lambda, \Theta) \) as [3]

\[
U_R(\Theta) = t(\Theta, \Theta), \quad U_L(\Theta) = t(-\Theta, \Theta)^{-1}
\]  

(5.6)

At any rate, no matter how the \( \omega_k \) are chosen, the monodromy matrix \( T(\lambda, \vec{\omega}) \) fulfills the YB algebra

\[
R(\lambda - \mu) \left[ T(\lambda, \vec{\omega}) \otimes T(\mu, \vec{\omega}) \right] = \left[ T(\mu, \vec{\omega}) \otimes T(\lambda, \vec{\omega}) \right] R(\lambda - \mu)
\]  

(5.7)

just as the quantum \( T(u) \) satisfies the YB algebra (2.7). We see that the “bare” YB algebra involves the finite–dimensional operators \( T_{ab}(\lambda, \vec{\omega}) \) and, correspondingly, the “bare” \( R \)–matrix \( R(\lambda) \) defines it.

Notice that \( T(\lambda, \Theta) \) fails to be conserved on the lattice only because of boundary effects. Indeed from fig. 3, which graphically represents the insertion of \( T(\lambda, \Theta) \) in the lattice time evolution, one readily sees that \( U \) and \( T(\lambda, \Theta) \) fail to commute
only because of the free ends of the horizontal line. For all vertices in the bulk, the graphical interpretation of the YB equations (5.1), namely that lines can be freely pulled through vertices, allows to move \( T(\lambda, \Theta) \) up or down, that is to freely commute it with the time evolution. The problem lays at the boundary: if periodic boundary conditions are assumed, then the free horizontal ends of \( T(\lambda, \Theta) \) cannot be dragged along with the bulk, unless they are tied up, to form the transfer matrix \( t(\lambda, \Theta) \). After all, for p.b.c., the boundary is actually equivalent to any point of the bulk and thus \( t(\lambda, \Theta) \) commutes with \( U \), as obvious also from eqs. (5.6) and the general fact that \([t(\lambda, \Theta), t(\mu, \Theta)] = 0\). One might think that the thermodynamic limit \( N \to \infty \), by removing infinitely far away the troublesome free ends of \( T(\lambda, \Theta) \), will allow for its conservation and thus for the existence of an exact YB symmetry with bare \( R \)-matrix. The situation however is not so simple: first of all one must fix the Fock sector of the \( N \to \infty \) non–separable Hilbert space in which to take the thermodynamic limit. Different choices leads to different phases with dramatically different dynamics. Then the non–local structure of \( T(\lambda, \Theta) \) must be taken into account. It is evident, for instance, that in the spin–wave Fock sector above ferromagnetic reference states \( T(\lambda, \Theta) \) can never be conserved. Indeed, the working itself of the Quantum Inverse Scattering Method, where energy eigenstates are built applying non–diagonal elements of \( T(\lambda, \Theta) \) on a specific ferromagnetic reference state, of course depends on \( T(\lambda, \Theta) \) not commuting with the hamiltonian!

From the field–theoretic point of view, the most interesting phase is the antiferromagnetic one, in which the ground state plays the rôle of densely filled interacting Dirac sea (this holds for all known integrable lattice vertex models [2,3,12]). The corresponding Fock sector is formed by particle–like excitations which become relativistic massive particles within the scaling limit proper of the light–cone approach [3]. This consists in letting \( a \to 0 \) and \( \Theta \to \infty \) in such a way that the physical mass scale

\[
\mu = a^{-1} e^{-\kappa \Theta}
\]  

(5.8)
stays fixed. Here $\kappa$ is a model–dependent parameter which for the so–called rational class of integrable model (to this class belong the models considered in ref. [1]) takes the general form [13]

$$\kappa = \frac{2\pi t}{h s}$$

(5.9)

where $h$ is the dual Coxeter number of the underlying Lie algebra, $s$ equals 1, 2 or 3 for simply, doubly and triply laced algebras, respectively, and $t = 1$ ($t = 2$) for non–twisted (twisted) algebras. For the class of model characterized by a trigonometric $R$–matrix (with anisotropy parameter $\gamma$) the expression (5.9) for $\kappa$ is to be divided by $\gamma$ [13].

The ground state or (physical vacuum) and the particle–like excitations of this antiferromagnetic phase are extremely more complicated than those of the ferromagnetic phase. It is therefore very hard to control, in the limit $N \to \infty$, the action of the alternating monodromy matrix $T(\lambda, \Theta)$ on the particle–like BA eigenstates of the alternating transfer matrix $t(\lambda, \Theta)$. On the other hand, since these particles enjoy a factorized scattering, one can proceed according to the general tenets of the bootstrap approach described in sec. 2. In this way one constructs the bootstrap monodromy matrix $\mathcal{T}(u)$ and it is natural to search for an explicit connection between $\mathcal{T}(u)$ and $T(\lambda, \Theta)$. It is a connection like this that would provide the microscopic interpretation of the bootstrap results.

In order to study the infinite volume limit of $T(\lambda, \Theta)$ on the physical Fock space (that is, finite energy excitations around the antiferromagnetic vacuum), one needs to compute scalar products of Bethe Ansatz states to derive relations like (2.4) or (2.6) with $T(\lambda, \Theta)$ instead of $\mathcal{T}(\lambda, \Theta)$ in the l.h.s. Since this kind of calculations are indeed possible but rather involved, we start by computing the eigenvalues of $t(\lambda, \Theta)$ on a generic state of the physical Fock space. Then, we shall compare these eigenvalues with those of $\tau(u)$. This will tell us whether the bare and the renormalized YB algebras have a common abelian subalgebra. Notice that this fact alone would provide a microscopic basis for the TBA, which originally relies solely on the bootstrap.
We shall consider once more the sG model as example, although the same result would apply to any integrable QFT admitting a light–cone lattice regularization. This class of models contains also the O(N) nonlinear sigma model and the $SU(N)$ Thirring model considered from the bootstrap viewpoint in refs.[1].

The integrable light–cone lattice regularization of the sG–mT model is provided the six-vertex model [2]. Therefore, the space $V$ is two–dimensional and the unitarized local $R$–matrices can be written

$$R_{jk}(\lambda) = \frac{1 + c}{2} + \frac{1 - c}{2} \sigma^x_j \sigma^x_k + \frac{b}{2} (\sigma^z_j \sigma^z_k + \sigma^y_j \sigma^y_k)$$

$$b = \frac{\sinh \lambda}{\sinh(i\gamma - \lambda)}, \quad c = \frac{\sinh i\gamma}{\sinh(i\gamma - \lambda)}$$

(5.10)

where $\gamma$ is commonly known as anisotropy parameter.

The standard Algebrized BA can be applied to the diagonalization of the alternating transfer matrix $t(\lambda, \Theta)$ with the following results [2,3,10]. The BA states are written

$$\Psi(\vec{\lambda}) = B(\lambda_1)\ldots B(\lambda_M)\Omega$$

(5.11)

where $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_M)$, $B(\lambda_i) = T_{+-}(\lambda_i + i\gamma/2, \Theta)$ and $\Omega$ is the ferromagnetic ground-state (all spins up). They are eigenvectors of $t(\lambda, \Theta)$

$$t(\lambda, \Theta)\Psi(\vec{\lambda}) = \Lambda(\lambda; \vec{\lambda})\Psi(\vec{\lambda})$$

(5.12)

provided the $\lambda_i$ are all distinct roots of the “bare” BA equations

$$\left( \frac{\sinh[i\gamma/2 + \lambda_j - \Theta]}{\sinh[i\gamma/2 - \lambda_j + \Theta]} \right)^N = -\prod_{k=1}^M \frac{\sinh[i\gamma + \lambda_j - \lambda_k]}{\sinh[-i\gamma + \lambda_j - \lambda_k]}$$

(5.13)

The eigenvalues $\Lambda(\lambda; \vec{\lambda})$ are the sum of a contribution coming from $A(\lambda) = T_{++}(\lambda, \Theta)$.
and one coming from $D(\lambda) = T_{--}(\lambda, \Theta)$,

$$\Lambda(\lambda; \vec{\lambda}) = \Lambda_A(\lambda; \vec{\lambda}) + \Lambda_D(\lambda; \vec{\lambda}) \quad (5.14)$$

Here

$$\begin{align*}
\Lambda_A(\lambda; \vec{\lambda}) &= \exp \left[ -iG(\lambda, \vec{\lambda}) \right] \\
\Lambda_D(\lambda; \vec{\lambda}) &= e^{-iN[\phi(\lambda-i\gamma/2-\Theta, \gamma/2)+\phi(\lambda-i\gamma/2+\Theta, \gamma/2)]} \exp [iG(\lambda-i\gamma, \vec{\lambda})]
\end{align*} \quad (5.15)$$

and

$$G(\lambda, \vec{\lambda}) \equiv \sum_{j=1}^{M} \phi(\lambda - \lambda_j, \gamma/2), \quad \phi(\lambda, \gamma) \equiv i \log \frac{\sinh(i\gamma + \lambda)}{\sinh(i\gamma - \lambda)} \quad (5.16)$$

$G(\lambda, \vec{\lambda})$ is manifestly a periodic function of $\lambda$ with period $i\pi$. Notice also that $\Lambda_D(\pm \Theta, \vec{\lambda}) = 0$. That is, only $\Lambda_A(\pm \Theta, \vec{\lambda})$ contributes to the energy and momentum eigenvalues:

$$\begin{align*}
E(\Theta) &= a^{-1} \sum_{j=1}^{M} [\phi(\Theta + \lambda_j, \gamma/2) + \phi(\Theta - \lambda_j, \gamma/2) - 2\pi] \\
P(\Theta) &= a^{-1} \sum_{j=1}^{M} [\phi(\Theta + \lambda_j, \gamma/2) - \phi(\Theta - \lambda_j, \gamma/2)]
\end{align*} \quad (5.17)$$

The ground state and the particle–like excitations of the light–cone six–vertex model are well known [2,12]: the ground state corresponds to the unique solution of the BAE with $N/2$ consecutive real roots (notice that the energy in eq.(5.17) is negative definite, so that the ground state is obtained by filling the interacting Dirac sea). In the limit $N \to \infty$ this yields the antiferromagnetic vacuum. Holes in the sea appear as physical particles. A hole located at $\varphi$ carries energy and
momentum, relative to the vacuum,

\[ e(\varphi) = 2a^{-1} \arctan \left( \frac{\cosh \pi \varphi/\gamma}{\sinh \pi \Theta/\gamma} \right), \quad p(\varphi) = -2a^{-1} \arctan \left( \frac{\sinh \pi \varphi/\gamma}{\cosh \pi \Theta/\gamma} \right) \] (5.18)

In the scaling limit \( a \to 0, \Theta \to \infty \) with \( e(0) \) held fixed, we then obtain \( (e,p) = m(\cosh \theta, \sinh \theta) \) with

\[ m \equiv 4a^{-1} \exp \left( -\pi \Theta/\gamma \right), \quad \theta \equiv -\pi \varphi/\gamma \] (5.19)

identified, respectively, as physical mass and physical rapidity of a sG soliton (mT fermion) or antisoliton (antifermion). Complex roots of the BAE are also possible. They correspond to magnons, that is to different polarization states of several sG solitons (mT fermions), or to breather states (in the attractive regime \( \gamma > \pi/2 \)). In the rest of this paper, we shall restrict our attention to the repulsive case \( \gamma < \pi/2 \), where the complex roots corresponding to the breathers are absent.

6. Thermodynamic limit of the transfer matrix

We proceed now to evaluate the function \( G(\lambda, \vec{x}) \) in the infinite volume limit \( (N \to \infty \) at fixed lattice spacing) for the antiferroelectric ground state (the physical vacuum) and for the excited states, in the repulsive regime \( \gamma < \pi/2 \).

For the vacuum, the density of roots \( \vec{x}_V \) results to be [10]

\[ \rho(\lambda)_V = N \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik\lambda} \frac{\cos k\Theta}{\cosh k\gamma/2} \] (6.1)

Using the integral representation

\[ \phi(\lambda, \gamma/2) = \text{P} \int_{-\infty}^{+\infty} \frac{dk}{ik} e^{ik\lambda} \frac{\sinh k[\pi/2 - \gamma]}{\sinh k\pi/2} \] (6.2)
which is valid for $|\text{Im}\lambda| < \frac{\gamma}{2}$, and eq.(6.1), we obtain for $G(\lambda)_V \equiv G(\lambda, \vec{\lambda}_V)$

$$G(\lambda)_V = -iN \text{P} \int_{-\infty}^{+\infty} \frac{dk}{k} e^{ik\lambda} \cos k\Theta \frac{\sinh k(\pi - \gamma)/2}{\cosh k\gamma/2 \sinh k\pi/2} , \quad |\text{Im}\lambda| < \gamma/2 \quad (6.3)$$

When $\pi - \gamma/2 > |\text{Im}\lambda| > \frac{\gamma}{2}$ we need another integral representation for $\phi(\lambda, \gamma/2)$,

$$\phi(\lambda, \gamma/2) = -\text{P} \int_{-\infty}^{+\infty} \frac{dk}{ik} e^{ik\lambda + (\pi k/2)\text{sign}(\text{Im}\lambda)} \frac{\sinh k\gamma/2}{\sinh k\pi/2} \quad (6.4)$$

We then find using eqs.(5.16),(6.1) and (6.4),

$$G(\lambda)_V = iN \text{P} \int_{-\infty}^{+\infty} \frac{dk}{k} e^{ik\lambda + (\pi k/2)\text{sign}(\text{Im}\lambda)} \frac{\cos k\Theta \sinh k\gamma/2}{\cosh k\gamma/2 \sinh k\pi/2} \quad (6.5)$$

when $\pi - \gamma/2 > |\text{Im}\lambda| > \gamma/2$. That is, the function $G(\lambda)_V$ is discontinuous on the lines $\text{Im}\lambda = \pm \gamma/2$. As we shall see this fact holds true also for all excited states. On the other hand $G(\lambda, \vec{\lambda})$ is periodic with period $i\pi$, so that there exist two main analytic determinations of its infinite volume limit $G(\lambda)$, that we shall call henceforth $G^I(\lambda)$ and $G^{II}(\lambda)$.

$$G(\lambda) = G^I(\lambda) , \quad \text{for the strip I :} \quad -\gamma/2 < \text{Im}\lambda < \gamma/2 \quad (6.6)$$

$$G(\lambda) = G^{II}(\lambda) , \quad \text{for the strip II :} \quad -\pi + \gamma/2 < \text{Im}\lambda < -\gamma/2$$

$G^I(\lambda)_V$ and $G^{II}(\lambda)_V$ have, respectively, the integral representations (6.3) and (6.5). The functions $G^I(\lambda)_V$ and $G^{II}(\lambda)_V$ analytically continued in $\lambda$ are meromorphic functions. Of course, they do not coincide with $G(\lambda)$ except for the strips indicated in eq. (6.6). For $\text{Im}\lambda$ outside these two strips, $G(\lambda)$ can be expressed in terms of
\[ G^I(\lambda)_V \text{ or } G^{II}(\lambda)_V \] using the \( i\pi \) periodicity as follows:

\begin{align*}
G(\lambda) &= G^I(\lambda - in\pi) \quad \text{for} \quad n\pi - \gamma/2 < \text{Im} \lambda < n\pi + \gamma/2 \\
G(\lambda) &= G^{II}(\lambda - in\pi) \quad \text{for} \quad (n-1)\pi + \gamma/2 < \text{Im} \lambda < n\pi - \gamma/2
\end{align*}

(6.7)

where \( n \in \mathbb{Z} \). The reflection principle also holds here:

\[ G(\lambda) = \bar{G}(\bar{\lambda}) \]

We find from eqs.(6.3) and (6.5) the following expression for the difference between the meromorphic functions \( G^I(\lambda)_V \) and \( G^{II}(\lambda)_V \):

\[ G^{II}(\lambda)_V - G^I(\lambda)_V = -2iN \text{Arg} \tanh \left( \frac{\cosh (\pi\Theta/\gamma)}{\cosh (\pi\lambda/\gamma)} \right) \]

(6.8)

The discontinuities of \( G(\lambda) \) through the other cuts follow by \( i\pi \) periodicity and the reflection principle.

In addition, when \( \lambda \) and \( \lambda - i\gamma \) lay both in strip II (which is indeed possible for \( \gamma < \pi/2 \)), we can relate the functions \( G(\lambda)_V \) and \( G(\lambda - i\gamma)_V \) as follows:

\[ G^{II}(\lambda)_V + G^{II}(\lambda - i\gamma)_V = -iN \text{P} \int_{-\infty}^{+\infty} \frac{dk}{k} e^{ik\lambda - \pi k/2} (1 + e^{k\gamma}) \frac{\cos k\Theta \sinh k\gamma/2}{\cosh k\gamma/2 \sinh k\pi/2} \]

\[ = N \left[ \phi(\lambda - i\gamma/2 - \Theta, \gamma/2) + \phi(\lambda - i\gamma/2 + \Theta, \gamma/2) \right] \]

(6.9)

We find an analogous relation when \( \lambda \) lays in strip I and \( \lambda - i\gamma \) in strip II:

\[ G^I(\lambda)_V + G^{II}(\lambda - i\gamma)_V = N \left[ \phi(\lambda - i\gamma/2 - \Theta, \gamma/2) + \phi(\lambda - i\gamma/2 + \Theta, \gamma/2) \right] \]

\[ + iN \log \frac{\cosh \frac{\pi\Theta}{\gamma} - i \sinh \frac{\pi\lambda}{\gamma}}{\cosh \frac{\pi\Theta}{\gamma} + i \sinh \frac{\pi\lambda}{\gamma}} \]

(6.10)

Let us now consider excited states. We start with a two hole state (the number of
holes is always even when $N$ is even). The density of roots is then [10]

$$
\rho(\lambda) = \rho(\lambda)_{V} + \rho(\lambda - \varphi_{1})_{h} + \rho(\lambda - \varphi_{2})_{h} - \delta(\lambda - \varphi_{1}) - \delta(\lambda - \varphi_{2})
$$

(6.11)

where $\varphi_{1}$ and $\varphi_{2}$ are the hole positions and

$$
\rho(\lambda)_{h} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{ik\lambda} \sinh[k(\pi - 2\gamma)]}{2\pi \sinh[k\pi/2 + \sinh[k(\pi - 2\gamma)]}
$$

(6.12)

The function $G(\lambda)$ takes then the form

$$
G(\lambda) = G(\lambda)_{V} + G(\lambda - \varphi_{1})_{h} + G(\lambda - \varphi_{2})_{h}
$$

(6.13)

We find from eqs.(5.16), (6.2), (6.11), and (6.12)

$$
G^{I}(\lambda)_{h} = i P \int_{-\infty}^{+\infty} \frac{dk}{2k} \frac{e^{ik\lambda} \sinh[k(\pi - 2\gamma)]}{2 \cosh[k\gamma/2 + \sinh[k(\pi - 2\gamma)]}
$$

$$
= -2 \arctan \left( \tanh \frac{\pi\lambda}{2\gamma} \right)
$$

(6.14)

$$
G^{II}(\lambda)_{h} = -i P \int_{-\infty}^{+\infty} \frac{dk}{2k} \frac{e^{ik\lambda+k(\pi/2)\text{sign}(\text{Im}\lambda)} \sinh[k\gamma/2]}{\cosh[k\gamma/2 \sinh[k(\pi - \gamma)]}
$$

$$
= -2 \arctan \left( \tanh \frac{\pi\lambda}{2\gamma} \right) + i \log S(\pi\lambda/\gamma - \text{sign}(\text{Im}\lambda)i\pi/2)
$$

where $S(\theta)$ is recognized as the soliton–soliton scattering amplitude (3.2) upon the identification

$$
\hat{\gamma} \equiv \frac{\gamma}{1 - \frac{\gamma}{\pi}}
$$

(6.15)

The function $S(\theta)$ enjoys the crossing property

$$
S(i\pi - \theta) = \hat{b}(\theta) S(\theta)
$$

(6.16)

where

$$
\hat{b}(\theta) = \frac{\sinh(\frac{\pi\theta}{\gamma})}{\sinh(\frac{i\pi}{\gamma} \mp i\frac{\pi}{2})}
$$

(6.17)

Notice that $\hat{b}(\theta) \hat{b}(i\pi - \theta) = 1$. We see from eq.(6.14) that $G(\lambda)_{h}$ has cuts on the lines $\text{Im}\lambda = \pm\gamma/2$, with discontinuity $i \log S(\frac{\pi\lambda}{\gamma} \mp i\frac{\pi}{2})$. 

27
Next consider states containing complex roots. There are four kinds of complex roots [14] associated to excited states close to the $N \to \infty$ antiferromagnetic vacuum, in the regime $\gamma < \pi/2$:

(a) Close roots with $|\text{Im } \lambda| < \gamma$. They appear as quartets: $\lambda = (\sigma \pm i\eta, \sigma \pm i[\gamma - \eta])$, where $0 < \eta < \gamma$, or as two strings: $\lambda = \sigma \pm i\gamma/2$.

(b) Wide roots with $|\text{Im } \lambda| > \gamma$. They appear in pairs $\lambda = \sigma \pm i\eta, \gamma/2 < \eta < \pi/2$, or as self–conjugate single roots with $|\text{Im } \lambda| = \pi/2$.

The presence of such complex roots produces a backflow in the real roots density. For a close pair we have [11]

$$
\rho_{\eta}(\lambda)_c = -\frac{1}{2\pi} [p(\lambda - \sigma - i\eta) + p(\lambda - \sigma + i\eta)], \quad \eta < \gamma < \pi/2 \quad (6.18)
$$

while for a wide pair [11]

$$
\rho_{\eta}(\lambda)_w = -\frac{1}{2\pi} \frac{d}{d\lambda} [\phi_\gamma(\lambda - \sigma, \eta - \gamma) - \phi_\gamma(\lambda - \sigma, \eta)], \quad \eta > \gamma < \pi/2 \quad (6.19)
$$

where

$$
\phi_\gamma(\lambda, \eta) \equiv \phi\left(\frac{\lambda}{1 - \gamma/\pi}, \frac{\eta}{1 - \gamma/\pi}\right) \quad (6.20)
$$

A self–conjugate root at $\sigma + i\pi/2$ gives instead

$$
\rho(\lambda)_{sc} = \frac{1}{2}\rho_{\pi/2}(\lambda)_w \quad (6.21)
$$

Let us denote by $G_{\eta}(\lambda)_c$ and $G_{\eta}(\lambda)_w$ the contribution of a closed pair and of a wide pair to the function $G(\lambda)$, respectively. For self–conjugate roots one simply has $G(\lambda)_{sc} = \frac{1}{2}G_{\pi/2}(\lambda)_w$. We find from eqs. (5.16) and (6.18):

$$
G_{\eta}(\lambda)_c = \int_{-\infty}^{+\infty} d\mu \phi(\lambda - \mu, \gamma/2)\rho_{\eta}(\mu)_c + \phi(\lambda - \sigma - i\eta, \gamma/2) + \phi(\lambda - \sigma + i\eta, \gamma/2) \quad (6.22)
$$
Then the integral representations \((6.2)\) for \(\phi(\lambda, \gamma/2)\) and the density \((6.18)\) yield

\[
G^I_\eta(\lambda) = 2 \arctan \left( \tanh \frac{\pi}{2\gamma} \left[ \lambda - \sigma - i\eta \right] \right) + 2 \arctan \left( \tanh \frac{\pi}{2\gamma} \left[ \lambda - \sigma + i\eta \right] \right) \tag{6.23}
\]

It is easy to see that the total contribution for a quartet vanishes when \(|\text{Im} \lambda| < \gamma/2\) and that the two–string contributions equal \(\pm \pi\) in this region:

\[
G^I_\eta(\lambda) + G^I_{\gamma-\eta}(\lambda) = 0 \mod 2\pi, \quad G^I_{\gamma/2}(\lambda) = \text{i} \log(-1), \quad |\text{Im} \lambda| < \gamma/2 \quad (6.24)
\]

Hence, quartets and two–strings do not contribute to the energy and momentum.

Let us now consider the more interesting strips of type II. There, using the integral representation \((6.4)\) for \(\phi(\lambda, \gamma/2)\), we obtain

\[
G^{II}_\eta(\lambda) = 2 \arctan \left( \tanh \frac{\pi}{2\gamma} \left[ \lambda - \sigma - i\eta \right] \right) + 2 \arctan \left( \tanh \frac{\pi}{2\gamma} \left[ \lambda - \sigma + i\eta \right] \right) - \text{i} \log S \left( \frac{\pi}{\gamma} \left[ \lambda - \sigma - i\eta \right] - \frac{i\pi}{2} \right) - \text{i} \log S \left( \frac{\pi}{\gamma} \left[ \lambda - \sigma + i\eta \right] - \frac{i\pi}{2} \right) \tag{6.25}
\]

Then, the total contribution for quartets and two strings results in

\[
G^{II}_\eta(\lambda) + G^{II}_{\gamma-\eta}(\lambda) = \text{i} \log \left[ \hat{b} \left( \frac{\pi}{\gamma} \left[ \lambda - \sigma + i\eta \right] - \frac{i\pi}{2} \right) \hat{b} \left( \frac{\pi}{\gamma} \left[ \lambda - \sigma - i\eta \right] - \frac{i\pi}{2} \right) \right], \quad G^{II}_{\gamma/2}(\lambda) = \text{i} \log \left[ -\hat{b} \left( \frac{\pi}{\gamma} \left[ \lambda - \sigma \right] \right) \right] \tag{6.26}
\]

where we used eq.(6.16).

Let us finally consider the wide pairs. Their contribution is given by

\[
G_\eta(\lambda)_w = \int_{-\infty}^{+\infty} d\mu \phi(\lambda-\mu, \gamma/2)\rho_\eta(\mu)_w + \phi(\lambda-\sigma-i\eta, \gamma/2) + \phi(\lambda-\sigma+i\eta, \gamma/2) \tag{6.27}
\]

Use of eqs. \((6.2)\), \((6.4)\) and \((6.19)\) now yields

\[
G^I_\eta(\lambda)_w = 0 \mod 2\pi, \quad G^{II}_\eta(\lambda)_w = \phi_\gamma(\lambda - \sigma - i\gamma/2, \gamma - \eta) + \phi_\gamma(\lambda - \sigma - i\gamma/2, \eta) \tag{6.28}
\]

We see from eq.(6.28) that wide pairs do not contribute to the energy and momentum.
We are now in position to analyze the excitation spectrum of the infinite-volume transfer matrix \( t(\lambda, \Theta) \). Let us begin with \( \lambda \) lying in strip I. From eqs. (5.15) and (6.10), we find for the vacuum

\[
\Lambda_A(\lambda)_V = \exp \left[ -iG^I(\lambda)_V \right]
\]
\[
\Lambda_D(\lambda)_V = \exp \left[ -iG^I(\lambda)_V \right] \left( \frac{\cosh \pi \Theta/\gamma + i \sinh \pi \lambda/\gamma}{\cosh \pi \Theta/\gamma - i \sinh \pi \lambda/\gamma} \right)^N \tag{6.29}
\]

The extra factor in \( \Lambda_D(\lambda) \) tends to zero (infinity) for \( \text{Im} \lambda \) positive (negative) when \( N \to \infty \). Since this vacuum contribution is present in any physical particle-like excitation, we see that the \( \Lambda_D(\lambda) \) will always behave like \( \Lambda_D(\lambda)_V \). Let us recall that \( \Lambda_D(\pm \Theta) = 0 \) for any finite \( N \), giving no contribution to energy and momentum.

Hence, choosing \( 0 < \text{Im} \lambda < \gamma/2 \), we are guaranteed that the two limits \( \lambda \to \pm \Theta \) and \( N \to \infty \) commute. The reduced strip \( 0 < \text{Im} \lambda < \gamma/2 \) is therefore the most natural one to define the renormalized type I transfer matrix

\[
t^I(\lambda) = \lim_{N \to \infty} t(\lambda, \Theta) \exp \left[ iG^I(\lambda)_V \right] (-)^{J_z - N/2} \tag{6.30}
\]

where \( J_z = N/2 - M \) is to be identified with the soliton (or fermion) charge of the continuum sG-mT model. The last sign factor in eq.(6.30) corresponds to square-root branch choice suitable to obtain the relation

\[
t^I(\pm \Theta) = \exp \left\{ -ia [P_\pm - (P_\pm)_V] \right\} \tag{6.31}
\]

where \( P_\pm \equiv (H \pm P)/2 \) (see eqs.(5.3), (5.6), (5.17)) and \( (P_\pm)_V \) stands for the vacuum contribution. Notice that the \( \Theta \)-dependence of \( t^I(\lambda) \) has been completely canceled out, since it is present only in the vacuum contribution. In fact, from eqs.(6.13), (6.14), (6.24) and (6.28), we read the eigenvalue \( \Lambda^I(\lambda) \) of \( t^I(\lambda) \) on a generic particle state:

\[
\Lambda^I(\lambda) = \exp \left[ -2i \sum_{n=1}^{k} \arctan \left( e^{\pi \lambda/\gamma + \theta_n} \right) \right] = \prod_{n=1}^{k} \coth \left( \frac{\pi \lambda}{2\gamma} + \frac{\theta_n}{2} + \frac{i\pi}{4} \right) \tag{6.32}
\]

where \( \theta_n \equiv -\pi \varphi_n/\gamma \) are the physical particle rapidities. Suppose now we expand
log $\Lambda^I(\lambda)$ in powers of $z = e^{-\pi|\lambda|/\gamma}$ around $\lambda = \pm \infty$, 

$$
\pm i \log \Lambda^I(\lambda) = \sum_{j=0}^{\infty} z^{2j+1} \frac{(-1)^j}{j + 1/2} \sum_{n=1}^{k} e^{\pm(2j+1)\theta_n} 
$$

(6.33)

One has to regard the coefficients of the expansion parameter $z$ as the eigenvalues of the conserved abelian charges generated by the transfer matrix. The additivity of the eigenvalues implies the locality of the charges. In terms of operators we can write, around $\lambda = \pm \infty$, 

$$
\pm i \log t^I(\lambda) = \sum_{j=0}^{\infty} \left[ \frac{4z}{m} \right]^{2j+1} I^\pm
$$

(6.34)

where $I^\pm = p^\pm$ is the continuum light–cone energy–momentum and the $I^\pm_j, j \geq 1$, are local conserved charges with dimension $2j + 1$ and Lorentz spin $\pm(2j + 1)$. Their eigenvalues 

$$
\frac{(-1)^j}{j + 1/2} \sum_{n=1}^{k} \left[ \frac{m}{4} e^{\pm\theta_n} \right]^{2j+1}
$$

coincide with the values on multisoliton solutions of the higher integrals of motion of the sG equation [16]. It is remarkable that these eigenvalues are free of quantum corrections although the corresponding operators in terms of local fields certainly need renormalization. Let us stress that explicit expressions for these conserved charges can be obtained by writing the local $R$–matrices in terms of fermi operators, as in ref.[2]. Notice also that, combining eqs.(6.31) with (6.34), and recalling the scaling law (5.19), we can write 

$$
P^\pm - (P_0)^V = p^\pm + \frac{m}{4} \sum_{j=1}^{\infty} \left( \frac{ma}{4} \right)^{2j} I^\pm_j
$$

(6.35)

That is, the light-cone lattice hamiltonian and momentum can be expressed in a precise way as the continuum hamiltonian and momentum plus an infinite series of continuum higher conserved charges, playing the rôle of irrelevant operators.
We now come back to the problem of comparing the light–cone results with the bootstrap predictions. As we have just seen, there is no chance to match the bootstrap predictions for $\lambda$ in strip I, since $\Lambda_A(\lambda)$ and $\Lambda_D(\lambda)$ cannot be renormalized by a common factor (see eq.(6.29)). Indeed, the structure of the sum $\Lambda_A(\lambda) + \Lambda_D(\lambda)$, that is of the eigenvalue $\Lambda(\lambda)$ of $t(\lambda, \Theta)$, will never match that of the eigenvalue $\xi(u)$ of the bootstrap transfer matrix $\tau(u)$ (eq.(4.2)). The situation is more favourable when both $\lambda$ and $\lambda - i\gamma$ lay in strip II. In this case eq.(6.9) applies and we find that

$$\Lambda_A(\lambda)V = \Lambda_D(\lambda)V = \exp\left[-iG^{II}(\lambda)V\right]$$

(6.36)

In order to consider all other excited states, it is important to recall that in the infinite volume limit the complex roots and the holes are coupled by equations with the BA structure [14]. These “higher–level” BAE follow from the original BAE, eq.(5.13), by summing up the Dirac sea of real roots in much the same way as we have done here for the function $G(\lambda)$. The result can be cast in the most symmetrical form by parametrizing the complex roots as follows [14]:

- a) $\sigma = \frac{2}{\pi}u$, for two–strings
- b) $\sigma + i\eta = \frac{2}{\pi}(u + i\pi/2)$ and $\sigma - i\eta = \frac{2}{\pi}(\bar{u} - i\pi/2)$, for quartets and wide pairs.
- c) $\sigma + i\pi/2 = \frac{2}{\pi}(u + i\pi/2)$ for self–conjugate roots.

Then the equations satisfied by the new complex root parameters $\{u_j, j = 1, 2, \ldots, m\}$ exactly coincide with the bootstrap BAE (5.13), upon the natural identification of $-\pi\varphi_n/\gamma$ with the physical rapidity $\theta_n$ of the $n$th hole (or particle) where $1 \leq n \leq k$. By construction, the number $m$ of higher–level roots is equal to the number of two–strings and self–conjugated roots plus twice the number of quartets and wide pairs. Notice that a self–conjugate root in the bare BAE is also self–conjugate in the higher–level BAE.

Then, combining eqs.(6.26), (6.28), (6.36) and using the new $u$–parametrization for the complex roots, we obtain the general form of the $A$ and $D$ contributions to
the eigenvalue of $t(\lambda, \Theta)$ on the $N \to \infty$ limit of the BA states for $\lambda$ in strip II:

$$\Lambda_A(\lambda) = -e^{-iG^{II}(\lambda)V} \left\{ \prod_{n=1}^{k} S(x_n) \coth \frac{x_n}{2} \right\} \prod_{j=1}^{m} \frac{\sinh \frac{\hat{\gamma}}{2} + \left( \frac{\pi}{\gamma}(\lambda + i\gamma/2) - u_j \right)/\pi}{\sinh \frac{\hat{\gamma}}{2} - \left( \frac{\pi}{\gamma}(\lambda + i\gamma/2) + u_j \right)/\pi}$$

(6.37)

and

$$\Lambda_D(\lambda) = -e^{-iG^{II}(\lambda)V} \left\{ \prod_{n=1}^{k} S(x_n) \hat{b}(x_n) \coth \frac{x_n}{2} \right\} \prod_{j=1}^{m} \frac{\sinh \frac{\hat{\gamma}}{2} + \left( \frac{\pi}{\gamma}(\lambda + i\gamma/2) - u_j \right)/\pi}{\sinh \frac{\hat{\gamma}}{2} + \left( \frac{\pi}{\gamma}(\lambda + i\gamma/2) - u_j \right)/\pi}$$

(6.38)

where for definiteness we chose the strip II, $-\pi + \gamma/2 < \text{Im} \lambda < -\gamma/2$ and set $x_n = \frac{\pi}{\gamma}(\lambda + i\gamma/2) + \theta_n$. These last two expressions can be connected with that for the eigenvalues of the bootstrap transfer matrix $\tau(u)$, eq.(4.2), provided we identify $u$ with $\frac{\pi}{\gamma}(\lambda + i\gamma/2)$. We find indeed from eqs.(4.2), (5.14), (6.37), (6.38):

$$\Lambda(\lambda) = -e^{-iG^{II}(\lambda)V} \xi(u) \prod_{n=1}^{k} \coth \left( \frac{u + \theta_n}{2} \right)$$

(6.39)

with $\lambda$ in strip II. In analogy with eq.(6.30), we now define the type II renormalized transfer matrix

$$t^{II}(\lambda) = \lim_{N \to \infty} t(\lambda, \Theta) \exp \left[ iG^{II}(\lambda)V \right] (-J_s^{-N/2})$$

(6.40)

Then, taking into account eq.(6.32), eq.(6.39) can be rewritten

$$\xi(u) = \frac{\Lambda^{II}(\frac{\pi}{2}u - i\frac{\gamma}{2})}{\Lambda^{I}(\frac{\pi}{2}u - i\frac{\gamma}{2})}$$

(6.41)

Notice that the dependence on the cutoff rapidity $\Theta$ has completely disappeared from the r.h.s. of eq.(6.41). This holds true both for the explicit dependence in the vacuum function $G(\lambda)V$ and for the implicit dependence through the bare BAE, which are now replaced by the $\Theta$–independent higher–level ones. In other
words, the eigenvalues of the bootstrap transfer matrix can be recovered from the light–cone regularization already on the infinite diagonal lattice, with no need to take the continuum limit. This should cause no surprise, since after all a factorized scattering can be defined also on the infinite lattice, with physical rapidities replaced by lattice rapidities (see eq.(5.18)). The bootstrap construction of the quantum monodromy operators $T_{ab}(u)$ then proceeds just like on the continuum. In this case, some $q_0$–deformation of the two dimensional Lorentz algebra should act as a symmetry on the physical states. This $q_0$ becomes unit when $\Theta \to \infty$.

7. Final remarks

In the previous section we have established the precise relation (6.41) between the BA eigenvalues of the bootstrap and microscopic lattice transfer matrices sG–mT–6V model, when $|\text{Im } u| < \pi/2$ and $\gamma < \pi/2$. With the implicit understanding that the thermodynamic limit $N \to \infty$ is taken in the ground state representation, such a relation extends to the operators themselves:

$$\tau(u) = t^{II}(\frac{\gamma}{\pi} u - i\frac{\gamma}{2}) t^{I}(\frac{\gamma}{\pi} u - i\frac{\gamma}{2})^{-1}$$ (7.1)

where $\tau(u)$ is the bootstrap operator (2.8). The relation (7.1) between $\tau(u)$ and $t(\lambda, \Theta)$ is remarkably simple, specially taking into account the long chain of steps involved in their totally independent constructions. For $t(\lambda, \Theta)$ we have:

1. Defined the light-cone lattice with the alternating parameter $\Theta$.

2. Found the antiferroelectric ground state.

3. Considered general finite energy excitations around it.

4. Let the volume $N$ become infinity.

In the other hand, $\tau(u)$ follows solely from the bootstrap principles (a)–(c) of sec. 2.
Notice that the bootstrap construction by itself does not provide any relationship between $T_{ab}(u)$ and the local fundamental fields entering the lagrangian which supposedly corresponds to the given factorized scattering model. On the other hand $T_{ab}(\lambda, \Theta)$ can be explicitly written in terms of the bare fermi field of the mT–model [2], so that eq.(7.1) represents a relevant piece of information for the search of such a relationship. It is clear, however, that a direct extension of eq.(7.1) to the full monodromy matrix would not work: indeed, suppose that operators $\tilde{T}_{ab}(u)$ are consistently defined by the relation

$$\tilde{T}_{ab}(u) = T_{ab}^{II}(\frac{u}{2} - \frac{i\gamma}{2}, \Theta) t^{I}(\frac{u}{2} - \frac{i\gamma}{2})^{-1}$$ (7.2)

then certainly the trace $\sum_a \tilde{T}_{aa}(u)$ coincides with $\tau(u)$, due to eq.(7.1), but $\tilde{T}_{ab}(u)$ cannot be identified with $T_{ab}(u)$ because it still satisfies a bare YB algebra, with anisotropy $\gamma$ rather than $\hat{\gamma}$. [In the YB algebra (2.7) the R-matrix elements, as given by eq.(3.2), depend on $\hat{\gamma}$]. It is presumable therefore that eq.(7.2) does not provide a consistent renormalization for the complete monodromy matrix. It should be noted, in this respect, that all the models considered in refs.[1], where the existence of a classssical analogue of $T_{ab}(u)$ allows to relate it to local curvature–free divergenceless non–abelian currents, correspond to rational forms of the $R$–matrix. But then one would find no finite renormalization like $\gamma \to \hat{\gamma}$ when taking the $N \to \infty$ limit in the light–cone lattice regularization of these models. Since both bare and bootstrap $R$–matrices are rational and depend non–trivially only on the spectral parameter, it is always possible to rescale the latter so that bare and boootstrap YB algebras coincide. In other words, in these rational models, there exist a thermodynamic limit in which the microscopically defined lattice monodromy matrix is conserved. Notice that this lattice monodromy matrix can be written in term of lattice non–abelian currents [3] in a way which represents an integrable regularization of the classical monodromy matrix. The picture is therefore fully consistent for the rational models.

Evidently, the situation appears to be more subtle in a trigonometric integrable model like the sG–mT–6V model considered here in detail. At the microscopic
level the model enjoys a dynamical YB symmetry characterized by the anisotropy \( \gamma \), which underlies the BA solution based on the ferromagnetic reference state \( \Omega \).

At the “renormalized” level, when the reference state is the physical antiferromagnetic ground state of the infinite lattice (and still in the presence of the UV cutoff provided by the lattice spacing), the model acquires a true YB symmetry characterized by the anisotropy \( \hat{\gamma} \). Eq.(7.1) shows that the Cartan subalgebras of these two YB algebras are essentially identical, strongly supporting both the bootstrap and the light–cone lattice constructions. It would be very interesting to relate the complete monodromy matrices, that is to find general YB–algebraic arguments to provide a microscopic interpretation for the bootstrap monodromy. The recent work reported in refs. [7], which relies on the \( q \)–deformed affine algebra approach to the YB symmetry, seems very promising in this respect, although it is restricted to the regime \( |q| < 1 \) (while \( |q| = 1 \) in the sG–mT–6V model).

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**FIGURE CAPTIONS**

*Fig.1.* Light–cone lattice representing a discretized portion of Minkowski space–time. An $R$–matrix of probability amplitudes is attached to each vertex. The bold lines correspond to the action, at a given time, of the one–step evolution operator $U$.

*Fig.2.* Graphical representation of the inhomogeneous monodromy matrix. The angles between the horizontal and the vertical lines are site–dependent in an arbitrary way.

*Fig.3.* Insertion of the alternating monodromy matrix in the light–cone lattice.

*Fig.4.* The two main determinations, $G^I(\lambda)$ and $G^{II}(\lambda)$ are defined by $G(\lambda)$ with $\lambda$ in strips I and II, respectively.