Combinatorial Micro-Macro Dynamical Systems

Rafael Díaz and Sergio Villamarín

Abstract

The second law of thermodynamics states that the entropy of an isolated system is almost always increasing. We propose combinatorial formalizations of the second law and explore their conditions of possibilities.

1 Introduction

The second law of thermodynamics is one of the pillars of modern science, enjoying a fundamental status comparable only to that of the law of conservation of energy. The range of applicability of the second law extends well-beyond the confines of its original formulation within thermodynamics and statistical mechanics [19, 30, 31, 44, 48, and for that reason we further refer to it simply as the second law. Despite the efforts of many distinguished researchers a definitive mathematical formulation of the second law, as clear say as the symplectic geometry (Poisson brackets) formulation of the conservation of energy law, has not yet been achieved. With this long term goal in mind we propose combinatorial embodiments of the second law, which give rise to combinatorial problems interesting in their own right.

There are several equivalent formulations of second law, as well as some formulations whose equivalence is not fully understood. The reader will find in [20, 23, 26, 28, 37, 38, 43, 47] mathematically inclined introductions to the subject from quite different viewpoints. Our departure point is the following formulation of the Clausius’ second law due to Boltzmann: “the entropy of an isolated system is almost always increasing.” To make sense of this statement several precisions are in order:

- As formulated the second law is meant to cover transitions between equilibria macrostates, as well as transitions between non-equilibrium macrostates [29]. The word "increasing" is taken in the weak sense, i.e. entropy tends to grow or to remain constant. A system is isolated if it doesn’t interchange neither matter nor heat with its surroundings. We typically think of it as a system enclosed in an insulating box; the universe as whole is an isolated system [5, 45]. Entropy is understood in the Boltzmann’s sense, i.e. as a logarithmic measure of the number of micro-realizations of a macrostate. The Boltzmann approach to the second law has been studied by a number of authors, among them [7, 9, 15, 21, 22, 24, 25, 27, 33, 34, 35, 36]. For
equilibria macrostates Boltzmann and Clausius entropies agree up to an additive constant, in the thermodynamic limit [27, 48].

- The restriction of the second law to transitions between equilibria macrostates is quite well-understood in the thermodynamic limit. A straightforward approach to this case has been developed by Jaynes [27] where the key facts are, first, the equilibrium macrostate can be identified with the probability distribution of maximum Shannon entropy under the available constraints, and, second, Boltzmann and Shannon entropies agree up to a positive multiplicative factor. The second law is thus a consequence of the obvious fact that if constrains are lifted the maximum Shannon entropy increases. Theorem 11 provides a combinatorial analogue for the latter argument within the Boltzmann entropy context. It is thus the increase of entropy for transitions between non-equilibrium macrostates, and in particular the transitions of such kind that arise as a non-equilibrium macrostate gradually approaches the equilibrium, that remains an open problem. The Clausius entropy is defined only for equilibrium macrostates, so the actual problem is to extend the notion of entropy away from the equilibrium. Boltzmann entropy is such an extension, and thus it make sense to ask under what conditions, if any, it possess the properties expected from the second law within a combinatorial setting.

- In contrast with energy, an always conserved quantity, entropy increases almost surely. Strictly decreasing entropy is not ruled out, on the contrary, it is actually predicted by the second law; otherwise the word "almost" should be removed from the law according to Occam’s razor principle. Decreasing entropy is however, according to the second law, an extremely low probability event, turning it into a non-option for most practical purposes. The second law itself does not provide bounds for the probability of decreasing entropy, neither it provides estimates for the rate of entropy production.

We also consider, see Section 8, combinatorial embodiments for stronger versions of the second law, such as the following formulation due to Gibbs in which the equilibrium plays a main role: ”entropy tends to strictly increase until the system reaches the equilibrium, i.e. the state of highest entropy, and then it remains in the equilibrium for a very long period of time.” Indeed we are going to propose several combinatorial properties each covering some aspect of the second law; we are not claiming that any of this properties is the ultimate combinatorial formulation of the second law, but we do claim that understanding these properties, both individually and collectively, provides a deep insight towards grasping the second law as a combinatorial statement.

Although Boltzmann himself was aware of the combinatorial nature of the second law, arising from coarse-graining, it seems that this idea with powerful potential applications hasn’t
had the impact that it deserves. The recent works of Niven [41, 42] may be regarded as fundamental steps towards making the connection combinatorics/second law more explicit. It is our belief that the combinatorial approach to the second law should be pursued in all its depth. In this contribution, we lay down some foundational ideas, discuss the main problems of study, and establish some basic results in the combinatorial approach to the second law. The main advantage of working within a combinatorial context is that we can rigorously define and compute with certain objects whose higher dimensional analogues may be elusive. For example, the set of invertible dynamical systems on a finite set is just the set of permutations on it. Also combinatorial methods often lead to algorithms suitable for numerical computation, allowing hypothesis and conjectures to be probed. Moreover, it is expected that by considering finite sets of large cardinalities the combinatorial models can be used to understand infinite phase spaces. Thus combinatorial models may be useful both as a conceptual guide, and as a computational tool for attacking the more involved cases allowing infinitely many microstates. Along this work we argue that the combinatorial viewpoint leads to a picture of the second law as a subtle balance among six principles:

**Micro/Macro Duality.** Boltzmann entropy relies on the distinction between microstates and macrostates. The micro-macro divide gives rise to dual interpretations. An ontological interpretation where microstates are primordial entities, and macrostates are what the observer measures when the system is in a given microstate. This approach is often referred by phrases such as "subjective or anthropomorphic macrostates" [27, 28, 46]. A phenomenological interpretation where macrostates are primordial, being what is actually accessible to the scientist, and microstates are theoretical constructs whose non-observable individual behaviour is postulated so that it gives rise to the observable behaviour of macrostates. In Boltzmann’s days it was microstates that were regarded as subjective or anthropomorphic, just as today some microstates beyond the standard model are often regarded as lacking an objective basis; in the last few years we have witness the Higgs’ field transition from theoretical construct to experimental fact. It seems that the subjective/objective knowledge qualification correlates weakly with the micro/macro scale division. The choice of interpretation leads to different but ultimately equivalent mathematical models, in their common domain of reference.

**Proportionality.** The idea is that probabilities are proportional to possibilities, the more microstates within a macrostate the higher its probability. Entropy grows simply because there are more microstates with higher entropy than microstates with lower entropy. The proportionality principle may be thought as an application to microstates of the Laplace principle of insufficient reason: a probability is uniform unless we have reasons to claim the contrary. The probabilistic symmetry of microstates arises from the usual methodological division between law of motion and initial conditions, where a theory provides the evolution law for microstates but leaves the
choice of initial microstate to the applied scientist. The proportionality principle is so intuitively appealing that it is tempting to identify it with the second law itself, as some authors seem to do. However the further principles introduced below show the need to complement and restrict the applicability of the proportionality principle in order to understand the second law.

**Large Differences.** In science once a scale is fixed the relevant numbers are often of comparable size. In the realm of the second law however the normal is just the opposite: huge differences in numbers, so pronounced indeed, that they are reminiscent of the mathematical distinction between measure zero and full measure sets. With huge differences low entropy microstates properties likely have a negligible impact on the global properties of a system. However simply disregarding low or decreasing entropy microstates is like disregarding the rational numbers because they have zero Lebesgue measure. In the realm of large differences, small may be huge: suppose a microstate have probability $10^{-23}$ of being non-equilibrium, a probability so low that studying such microstates seems pointless; nevertheless if the total number of microstates is say $10^{10^{23}} + 10^{10^{23}}$, then there are about $10^{10^{23}}$ non-equilibrium microstates leaving plenty of room for interesting behaviour. Assuming large differences the main obstacle towards the "nowhere to go but up" effect are constant entropy microstates. Large differences imply a dominant equilibrium but in general it is a much stronger condition. We talk about large rather than infinite differences, as one of the main aims of the combinatorial approach is to estimate the transition point where differences become dominant.

**Continuity.** Proportionality implies that starting from generic initial conditions the equilibrium will eventually be reached, but against all empirical evidence, it also implies that at any time the most likely move for a microstate is to jump to the equilibrium. Unrestricted proportionality violates the law of gradual changes, a most cherished principle of physics. Continuity places restrictions on proportionality in a couple of ways: it limits the allowed dynamics on microstates, and it demands that macrostates couple to the dynamics in such a way that sudden long jumps in entropy are unlikely, although not completely rule out.

**Microstates Asymmetry.** Reversible systems have as many entropy decreasing as entropy increasing microstates, indeed this is the basic fact behind the Loschmidt’s paradox. Within our combinatorial formalizations of the second law equal increases and decreases in entropy by itself does not give rise to contradictions, but it does point towards a fundamental fact: the second law is a sufficient reason to break the probabilistic symmetry of microstates, a fact materialized with the introduction of not reversion invariant macrostates. As a rule one may expect the equilibrium to be reversion invariant, but it is quite unnatural to demand this property for all macrostates; in particular entropy itself may not be invariant under reversion. The outshot is that any mathematical formalization of the second law must in some way or another break
Localization to Orbits. From Gibbs’ viewpoint properties formalizing the second law should apply orbitwise, allowing a relative small number of microstates to live in badly behaved orbits. Again several more or less related reasonable properties may be proposed. As an example we are going to consider a particularly powerful one: the existence of a reversion invariant equilibrium such that most microstates on each orbit belong to the equilibrium. In such cases the equilibrium reaching time is a strictly decreasing not reversion invariant function on nonequilibrium microstates. Looking at the macrostates associated to this function one obtains, under reasonable hypothesis on the image of the equilibrium down sets, a micro-macro dynamical systems with strictly increasing entropy on non-equilibrium macrostates, i.e. for such systems irreversibility arises naturally from reversibility, the origin of any microstate is a low entropy microstate, and the longer the (past and future) history of a microstate, the lower the entropy of its origin.

Let us describe in details a standard construction given rise to combinatorial models from familiar smooth models through a couple of coarse-graining procedures. Let \((M, \omega)\) be a compact symplectic manifold and \(M \to B\) be a coarse-graining map with \(B\) a finite set. For \(n \in \mathbb{N}_{\geq 1}\) the Hamiltonian map \(H_n : M^n \to \mathbb{R}\) generates the dynamics \(\phi_t : M^n \to M^n\) via the identity \(\omega_n(\dot{\phi}, ) = dH_n\), where \(\omega_n\) is the product symplectic structure on \(M^n\). For \(u \geq 0\) consider the energy shell \(H_n^{-1}(nu) \subseteq M^n\) and its image \(B^n_u\) under the coarse graining map \(M^n \to B^n\). Assume we have a second coarse-graining map \(\text{prop}_B \to A\), where \(\text{prop}_B\) is the space of probability distributions on \(B\), and \(A\) is another finite set. We obtain the chain of maps

\[H_n^{-1}(nu) \to B^n_u \to \text{prop}_B \to A.\]

In this work we focus on the (composition) map \(B^n_u \to A\) since it only involves finite sets; so \(B^n_u\) will be our set of microstates and \(A\) will be our set of macrostates. Under reasonable hypothesis \(B^n_u\) inherits a measure and a stochastic dynamics from the corresponding structures on \(M^n_u\) via the map \(M^n_u \to B^n_u\). In this work however we only consider the case where the induced measure is uniform and the dynamics is deterministic. Although one should really start with a stochastic dynamics on microstates we refrain to do so for several reasons. First, it is worth it to see random processes arising straight out of fully deterministic processes; second, the deterministic case is interesting in itself and deserves its own study; third, studying the deterministic case should be though as preparation for dealing with the more general stochastic case.

In Section 2 we introduce micro-macro dynamical systems and formulate some of the main problems in the combinatorial approach to the second law, e.g. counting the number of strict
decreases in entropy for arbitrary permutations and partitions on finite sets. The partition of microstates into macrostates gives us the notion of Boltzmann entropy, and also a probability distribution and a stochastic dynamics on macrostates. In Section 3 we formalize the notions of reversible micro-macro dynamical systems and global arrow of time. We provide a couple of general construction showing that there are plenty of (invariant, equivariant) reversible micro-macro dynamical systems, and provide formulae for these systems. We show that each (invariant, equivariant) reversible micro-macro dynamical systems can be canonically decomposed into four components, one coming from the constructions just mentioned, and the other ones quite easy to grasp. We also discuss fluctuation theorems [4, 11, 13, 14, 15, 48, 49] for combinatorial micro-macro dynamical systems, and study with a global arrow of time, with emphasis on systems with the same number of strict increases and strict decreases in entropy.

In Section 4 we review some of the structural operations on micro-macro dynamical system such as the product, disjoint union, restriction, coarse-graining, meet and joint; and introduce five general constructions of micro-macro phase spaces. We formulate an analogue of the asymptotic equipartition theorem applicable for micro-macro phase spaces, and provided a couple of interesting examples of coarse-graining. In Section 5 we consider the applicability, within our combinatorial framework, of the second law with the world "almost" removed, i.e. we study invertible micro-macro dynamical systems with no strictly decreasing entropy, and show that a generic system has low probability of having this property. This case is nonetheless interesting because we are able to fully explore for it the dual viewpoints: the partition-based viewpoint where macrostates are fixed and the dynamics vary, and the permutation-based viewpoint where the dynamics is fixed and macrostates vary. As the two viewpoints lead to equivalent results, we obtain an interesting combinatorial identity. In Section 6 we consider invertible micro-macro dynamical systems with the highest possible number of strict decreases in entropy. We introduce a sharp upper bound with a simple combinatorial meaning on the number of such decreases, adopting a partition-based viewpoint, and provide conditions on a partition implying that any permutation coupled to it defines a system satisfying a combinatorial formalization of the second law.

In Section 7 we introduce a pair of new combinatorial formulations of the arrow of time, define the jump of a map from a set provided with a partition to itself, and study combinatorial formulations of the second law for zero jump systems. In Section 8 we adopt Gibbs’ viewpoint and study combinatorial formalizations the second law through properties localized to orbits. We introduce equilibrium bound systems and study the equilibrium reaching time for such systems. In Section 9 we reformulate some of the main problems in the combinatorial approach to the second law in terms of sums over integer points in convex polytopes; this approach allows to fully analyze some simple but revealing cases and opens the door for numerical computations.
In the final Section 10 we consider thermodynamic limits. Although based on the previous sections, readers familiar with maximum entropy methods may feel at home with the techniques and results of this section. At various points throughout this work we use the language of category theory but only basic notions are required [32, 39].

2 Micro-Macro Dynamical Systems

Let set be the category of finite sets and maps, and map be the category of morphisms in set, i.e. objects in map are functions between finite sets. A morphism in map from $f_1 : X_1 \rightarrow A_1$ to $f_2 : X_2 \rightarrow A_2$ is given by maps $k : X_1 \rightarrow X_2$ and $\bar{k} : A_1 \rightarrow A_2$ such that $k f_1 = f_2 \bar{k}$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{k} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
A_1 & \xrightarrow{\bar{k}} & A_2 
\end{array}
$$

Definition 1.

1. A micro-macro phase space is a tuple $(X, A, f)$ where $f : X \rightarrow A$ is a surjective map, $X$ is the set of microstates, and $A$ is the set of macrostates. If $f(i) = a$ we say that the microstate $i$ belongs to the macrostate $a$, and write $i \in a$ instead of $i \in f^{-1}(a)$ whenever $f$ is understood.

2. A micro-macro dynamical system is a tuple $(X, A, f, \alpha)$ where $(X, A, f)$ is a micro-macro phase space, and $\alpha : X \rightarrow X$ is a map defining the dynamics on microstates, i.e. it sends a microstate $i$ to the microstate $\alpha(i)$ in a unit of time. We let mmfds be the category of micro-macro dynamical systems.

3. A micro-macro dynamical system $(X, A, f, \alpha)$ is called invertible if the map $\alpha : X \rightarrow X$ is bijective. We let immds be the full subcategory of mmfds whose objects are micro-macro dynamical systems with invertible dynamics.

The category of micro-macro phase spaces is the full subcategory surj of map whose objects are surjective maps. A morphism $(X_1, A_1, f_1, \alpha_1) \rightarrow (X_2, A_2, f_2, \alpha_2)$ in mmfds is a morphism $k$ of micro-macro phase spaces such that the following diagram commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{k} & X_2 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 \\
X_1 & \xrightarrow{k} & X_2 
\end{array}
$$
Given a finite set \( X \), we let \( \text{Par}_X \) and \( \text{S}_X \) be, respectively, the set of partitions and permutations on \( X \). A partition on \( X \) is a family of non-empty disjoint subsets (called blocks) of \( X \) with union equal to \( X \). A surjective map \( f : X \to A \) defines the partition on \( X \) given by \( \pi = \{ f^{-1}(a) \mid a \in A \} \), which gives rise to the surjective map \( \pi : X \to \pi \) sending \( i \in X \) to the block containing it. The maps \( f : X \to A \) and \( \pi : X \to \pi \) are isomorphic objects in the category \( \text{surj} \), i.e. we have a commutative isomorphism triangle

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow \pi \\
\pi & \xrightarrow{} & \pi \\
\end{array}
\]

where the bottom arrow sends \( a \) to \( f^{-1}(a) \). So, up to isomorphism, a surjective map and a partition on \( X \) define the same structure, thus any micro-macro dynamical system \((X, A, f, \alpha)\) is isomorphic to a micro-macro dynamical system of the form \((X, \pi, \pi, \alpha)\), henceforth denoted by \((X, \pi, \alpha)\). Next we show that \( \text{immds} \) is a coreflective subcategory of \( \text{mmds} \).

**Proposition 2.** The inclusion functor \( i : \text{immds} \to \text{mmds} \) has a right adjoint functor \( i_\ast : \text{mmds} \to \text{immds} \) given on objects by \( i_\ast((X, A, f, \alpha)) = (X, \pi, f^{-1}(\pi), \pi, \alpha) \), where \( X_r \) is the set of recurrent microstates \( \{ i \in X \mid \alpha^n(i) = i \text{ for some } n > 0 \} \), and the restrictions of \( f \) and \( \alpha \) to \( X_r \) are denoted with the same symbols.

**Proof.** We need to show that there is a natural bijection

\[ \text{immds}((X_1, A_1, f_1, \alpha_1), i_\ast((X_2, A_2, f_2, \alpha_2))) \simeq \text{mmds}((X_1, A_1, f_1, \alpha_1), (X_2, A_2, f_2, \alpha_2)). \]

Indeed if \( \alpha_1 \) is invertible, then the image of a morphism in the right-hand set above is necessarily contained in \( (X_2)_r \), and therefore it is also a morphism in the left-hand set above.

**Definition 3.** Let \((X, A, f, \alpha)\) be a micro-macro dynamical system. The following structures arise on \( A \) and \( X \):

1. A measure \( | \cdot | : A \to \mathbb{N} \) given by \( |a| = |f^{-1}(a)| \), inducing the measure \( | \cdot | : X \to \mathbb{N} \) given by \( |i| = |f(i)| \), and the probability measure \( p : A \to [0, 1] \) given by \( p_a = \frac{|a|}{|X|} \).
2. A stochastic map \( T : A \to A \) with \( T_{ab} \in [0, 1] \) giving the probability that a macro-state \( b \) moves to a macro-state \( a \) in a unit of time

\[
T_{ab} = \frac{|\{ i \in b \mid \alpha(i) \in a \}|}{|b|}. \quad \text{Note that } T_{ab} \geq 0 \text{ and } \sum_{a \in A} T_{ab} = 1.
\]

3. The uniform probability on \( A \) and the uniform probabilities on each block \( a \in A \) induce the probability measure \( q \) on \( X \) given by \( q(i) = \frac{1}{|A||\pi|} \).
Definition 4. Let \((X, A, f, \alpha)\) be a micro-macro dynamical system.

1. The Boltzmann entropy on macrostates \(S : A \to \mathbb{R}\) is given by \(S(a) = \ln |a|\).

2. The Boltzmann entropy on microstates \(S : X \to \mathbb{R}\) is given by \(S(i) = \ln |i|\).

3. The Boltzmann entropy of \(A\) is given by \(S(A) = \sum_{a \in A} S(a)p_a = \frac{1}{|X|} \sum_{i \in X} S(i)\).

4. The Shannon entropy of \(p\) is given by \(H(p) = -\sum_{a \in A} \ln(p_a)p_a = \ln |X| - S(A)\).

5. The Shannon entropy of the stochastic map \(T : A \to A\) is given by \(H(T) = \sum_{\bar{b} \in A} H(T_{\bar{a}\bar{b}})p_{\bar{b}} = -\sum_{a, \bar{b} \in A} \ln(T_{\bar{a}\bar{b}})T_{\bar{a}\bar{b}}p_{\bar{b}}\).

Remark 5. Boltzmann’s actual definition of the entropy of a macrostate is \(k \ln |a|\), where \(k\) is the Boltzmann constant. For simplicity we set \(k = 1\), or equivalently, work with the logarithmic function \(\log_{e^{1/k}}(x)\). The entropy of a partition with respect to an automorphism has been studied in ergodic theory [1]. Although related to our constructions, we will not use this notion. Shannon entropy and Boltzmann entropy play complementary roles as \(H(p) + S(A) = \ln |X|\).

Shannon entropy \(H(p)\) measures the mean uncertainty in choosing a macrostate. Boltzmann entropy \(S(A)\) measures the mean uncertainty in choosing a microstate given that a macrostate has already been chosen.

Given a micro-macro phase space \((X, A, f)\) the set of equilibria macrostates \(A^{\text{eq}} \subseteq A\) is the set of macrostates with maximum Boltzmann entropy. In the applications, usually \(A^{\text{eq}}\) has a unique element called the equilibrium. We let \(X^{\text{eq}} \subseteq X\) be the set of microstates in an equilibrium macrostate, and \(X^{\text{neq}} = X \setminus X^{\text{eq}}\) be the set of non-equilibrium microstates. For \(L \subseteq X\) set \(L^{\text{eq}} = L \cap X^{\text{eq}}\) and \(L^{\text{neq}} = L \cap X^{\text{neq}}\).

Let \((X, A, f, \alpha)\) be a micro-macro dynamical system. The sets of microstates where entropy is decreasing, increasing, and constant are respectively given by:

\[D = DX = \{i \in X \mid S(\alpha(i)) < S(i)\}, \quad I = IX = \{i \in X \mid S(\alpha(i)) > S(i)\}, \quad C = CX = \{i \in X \mid S(\alpha(i)) = S(i)\}.\]

More generally, for \(L \subseteq X\) set \(DL = D \cap L\), \(IL = I \cap L\), and \(CL = C \cap L\). We have that

\[
\frac{|DL|}{|L|} + \frac{|IL|}{|L|} + \frac{|CL|}{|L|} = 1.
\]
Throughout this work we use a parameter $\varepsilon$ allowed to be in the interval $[0, 1]$ in order to exhaust all logical possibilities, but meant to be a fairly small positive real number. Next we introduce our first formalization of the second law. We will subsequently provide further formalizations demanding stronger conditions making the systems more closely resemble those likely to be relevant in nature.

**Definition 6.** A micro-macro dynamical system $(X, A, f, \alpha)$ satisfies property $L_1(\varepsilon)$ if and only if $\frac{|D|}{|X|} \leq \varepsilon$, and in this case we write $(X, A, f, \alpha) \in L_1(\varepsilon)$. A sequence $(X_n, A_n, f_n, \alpha_n)$ of micro-macro dynamical systems satisfies property $L_1$, and we write $(X_n, A_n, f_n, \alpha_n) \in L_1$, if for any $\varepsilon > 0$ there exits $N \in \mathbb{N}$ such that $(X_n, A_n, f_n, \alpha_n) \in L_1(\varepsilon)$ for $n \geq N$.

Next result allows to understand property $L_1(\varepsilon)$ in terms of macrostates. Given $b \in A$ we set $A_{<b} = \{a \in A \mid |a| < |b|\}$ and $A_{\leq b} = \{a \in A \mid |a| \leq |b|\}$.

**Proposition 7.** Let $(X, A, f, \alpha)$ be a micro-macro dynamical system. We have that:

1. $\frac{|D|}{|X|} = \sum_{S(a)<S(b)} T_{ab} P_b$, $\frac{|I|}{|X|} = \sum_{S(a)>S(b)} T_{ab} P_b$, and $\frac{|C|}{|X|} = \sum_{S(a)=S(b)} T_{ab} P_b$.

2. $(X, A, f, \alpha) \in L_1(\varepsilon)$ if and only if $\sum_{S(a)<S(b)} T_{ab} P_b \leq \varepsilon$.

3. If $T_{ab} \leq \frac{\varepsilon}{|A_{<b}|}$ for $|a| < |b|$, then $(X, A, f, \alpha) \in L_1(\varepsilon)$.

**Proof.** Item 2 is a direct consequence of item 1. We show the leftmost identity in item 1:

$$\sum_{S(a)<S(b)} T_{ab} P_b = \sum_{S(a)<S(b)} \frac{|\{i \in b \mid \alpha(i) \in a\}| |b|}{|X|} = \frac{1}{|X|} \prod_{S(a)<S(b)} \{i \in b \mid \alpha(i) \in a\} = \frac{|D|}{|X|}.$$  

Item 3 is shown as follows:

$$\frac{|D|}{|X|} = \sum_{S(a)<S(b)} T_{ab} P_b \leq \varepsilon \sum_{S(a)<S(b)} \frac{P_b}{|A_{<b}|} \leq \varepsilon \sum_{b} \frac{|A_{<b}| P_b}{|A_{<b}|} \leq \varepsilon.$$

According to Jaynes [15, 27] a transition on macrostates $b \rightarrow a$ is experimentally reproducible if and only if $T_{ab}$ is nearly equal to 1, meaning that the images under $\alpha$ of almost all microstates in $b$ lie in the macrostate $a$. Accordingly, the stochastic map $T$ is experimentally reproducible if and only if it is nearly deterministic, i.e. if and only if there is a map $t : A \rightarrow A$ such that $T_{t(b)b}$ is nearly equal to 1, say $T_{t(b)b} \geq 1 - \varepsilon$ for $\varepsilon \geq 0$ fairly small.

**Proposition 8.** Let $(X, A, f, \alpha)$ be an invertible micro-macro dynamical system.

1. If the entropy $H(T)$ of the stochastic map $T : A \rightarrow A$ is nearly vanishing, then property $L_1(\varepsilon)$ holds for suitable $\varepsilon > 0$ specified below.
2. If \( H(T) = 0 \), then property \( \text{L}_1(0) \) holds.

**Proof.** Under the hypothesis of item 1, \( \alpha \) induces a map \( \alpha : A \to A \) such that \( T_{\alpha(b)b} \) is nearly equal to 1, say \( T_{\alpha(b)b} \geq 1 - \varepsilon \) with \( \varepsilon > 0 \) fairly small. Then

\[
\frac{|\alpha(b)|}{|b|} \geq \frac{|\{i \in \alpha(b) \mid \alpha^{-1}(i) \in b\}|}{|b|} = \frac{|\{i \in b \mid \alpha(i) \in \alpha(b)\}|}{|b|} = T_{\alpha(b)b} \geq 1 - \varepsilon,
\]

and thus \( S(\alpha(b)) = \ln|\alpha(b)| \geq \ln|b| + \ln(1 - \varepsilon) = S(b) + \ln(1 - \varepsilon) \). Assuming in addition that \( \varepsilon \) is small enough that \( S(a) < S(b) \) implies that \( S(a) < S(b) + \ln(1 - \varepsilon) \), then by Proposition 7 we have that:

\[
\frac{|D|}{|X|} = \sum_{S(a) < S(b)} T_{ab}p_b \leq \varepsilon \sum_b p_b = \varepsilon.
\]

Item 2 follows from item 1, since in this case we can actually set \( \varepsilon = 0 \).

The following result is a direct consequence of the definitions.

**Lemma 9.** Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system. We have that

1) \( \alpha D = I_{\alpha^{-1}}, \) \( \alpha I = D_{\alpha^{-1}}, \) \( \alpha C = C_{\alpha^{-1}}. \)
2) \( |D| = |I_{\alpha^{-1}}|, \) \( |I| = |D_{\alpha^{-1}}|, \) \( |C| = |C_{\alpha^{-1}}|. \)
3) \( |D| + |D_{\alpha^{-1}}| + |C| = |X|. \)

Next we show that a micro-macro dynamical system and its inverse satisfy property \( \text{L}_1(\varepsilon) \) if and only if entropy is nearly constant.

**Proposition 10.** Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system.

1. If \((X, A, f, \alpha) \in \text{L}_1(\varepsilon_1)\) and \((X, A, f, \alpha^{-1}) \in \text{L}_1(\varepsilon_2)\), then \( \frac{|C|}{|X|} \geq 1 - \varepsilon_1 - \varepsilon_2. \)

2. If \( \frac{|C|}{|X|} \geq 1 - \varepsilon \), then \((X, A, f, \alpha) \in \text{L}_1(\varepsilon)\) and \((X, A, f, \alpha^{-1}) \in \text{L}_1(\varepsilon)\).

3. If \( |I| = |D| \), then \((X, A, f, \alpha) \in \text{L}_1(\varepsilon)\) if and only if \( \frac{|C|}{|X|} \geq 1 - 2\varepsilon. \)

**Proof.** Recall that \( |D_{\alpha^{-1}}| = |I| \). The hypothesis of item 1 implies that

\[
\varepsilon_1 + \varepsilon_2 + \frac{|C|}{|X|} \geq \frac{|D|}{|X|} + \frac{|I|}{|X|} + \frac{|C|}{|X|} = 1.
\]

Under the hypothesis of item 2 we have that \( \frac{|D|}{|X|} + \frac{|D_{\alpha^{-1}}|}{|X|} \leq \varepsilon. \) Item 3 follows from the identity \( 2\frac{|D|}{|X|} + \frac{|C|}{|X|} = 1. \)

The set of isomorphism classes of invertible micro-macro dynamical systems on a set \( X \) of micro-states can be identified with the quotient set \((\text{Par}X \times S_X)/S_X\) where:
1. $\beta \in S_X$ acts on $\pi \in \text{Par}X$ by $\beta \pi = \{\beta a \mid a \in \pi\}$.

2. $\beta \in S_X$ acts on $\alpha \in S_X$ by conjugation $\beta(\alpha) = \beta \alpha \beta^{-1}$.

3. $S_X$ acts diagonally on $\text{Par}X \times S_X$.

Isomorphic invertible micro-macro dynamical systems have strictly decreasing entropy sets of the same cardinality, thus we get the map

$$|D| : \text{Par}X \times S_X / S_X \rightarrow [0, d_X]$$

given by $|D|(\pi, \alpha) = |D_{\pi,\alpha}| = \{|i \in X \mid S(\alpha(i)) < S(i)\}$,

where $d_X$ is maximum number of strict decreases for a micro-macro dynamical system on $X$. We show in Section 4 that $d_X = |X| - \min \max \{|i| \mid X| 1 \leq i \leq |X|\}$.

The random variables $D$ and $|l|$ on $(\text{Par}X \times S_X) / S_X$ have the same mean.

**Example 12.** For $X = [3]$ there are three non-uniform partitions $1|23, 2|13, 3|12$, and for each of these partitions there are four permutations with $|D| = 1$. All other choices lead to $D = \emptyset$. Therefore $\frac{|D|}{|X|} = \frac{12}{90} = 0.13$. See Figure 1.

Next we show that, in average, a random invertible micro-macro dynamical system has as many strict increases as strict decreases in entropy.

**Theorem 13.** The random variables $|I|$ and $|D|$ on $(\text{Par}X \times S_X) / S_X$ have the same mean.

**Proof.** Consider the uniform probability on $\text{Par}X \times S_X$. The result follows from Lemma 11 since $|D_{\pi,\alpha}| = |I_{\pi,\alpha^{-1}}|$ for $(\pi, \alpha) \in \text{Par}X \times S_X$. \hfill $\Box$

Next we show that whenever $\pi$ has a dominant equilibrium, then property $L_1(\varepsilon)$ holds for all invertible systems of the form $(X, \pi, \alpha)$. 

12
Theorem 14. Let \((X, \pi)\) be a micro-macro phase space and \(\delta_1 \leq \delta_2 \leq \varepsilon(\delta_1 + 1)\) in \(\mathbb{R}_{\geq 0}\) be such that \(\delta_1 |X^\text{eq}| \leq |X^\text{neq}| \leq \delta_2 |X^\text{eq}|\). Then \((X, \pi, \alpha) \in L_1(\varepsilon)\) for any permutation \(\alpha \in S_X\).

Proof. Under the given hypothesis we have that

\[
\frac{|D|}{|X|} = \frac{|\alpha D|}{|X|} \leq \frac{|X^\text{neq}|}{|X^\text{neq}| + |X^\text{eq}|} \leq \frac{\delta_2 |X^\text{eq}|}{(\delta_1 + 1)|X^\text{eq}|} \leq \varepsilon.
\]

\(\square\)

3 Reversible Systems and the Arrow of Time

In this section we begin to formalize the arrow of time concept \([5, 33, 45]\) within our combinatorial framework; stronger formalizations will be developed subsequently. We also introduce reversible micro-macro dynamical systems and study some of their main properties. Let \((X, A, f, \alpha)\) be a micro-macro dynamical system, \(i \in X\), and \(N, M \in \mathbb{N}\). Entropy defines a \([N, M]\)-arrow of time around \(i\) of time length \(N + M + 1\), if it is strictly increasing at the microstates \(\alpha^n(i)\) for all \(n \in [-N, M]\). In this case the \(\alpha\)-orbit of \(i\) must have at least \(N + M + 2\) elements. As with the second law itself, it is convenient to introduce a less strict condition for the arrow of time, allowing a relative small number of decreases or constant entropy among the microstates \(\alpha^n(i)\).

Definition 15. Entropy defines an \([\varepsilon, N, M]\)-arrow of time around \(i\) of time length \(N + M + 1\) if the following inequality holds

\[
\left| \left\{ n \in [-N, M] \mid S(\alpha^{n+1}(i)) \leq S(\alpha^n(i)) \right\} \right| \leq \frac{N + M + 1}{\varepsilon}.
\]

Definition 15 can be extended for other functions on \(X\) in place of entropy, in particular, one can apply it to negative entropy. The foregoing considerations motivate our next definition.
Proof. We have that
\[ \frac{|DX^{\text{eq}} \sqcup CX^{\text{eq}}|}{|X^{\text{eq}}|} \leq \varepsilon, \] or equivalently
\[ \frac{|IX^{\text{eq}}|}{|X^{\text{eq}}|} \geq 1 - \varepsilon. \]

**Theorem 17.** Let \((X, A, f, \alpha)\) be a micro-macro dynamical system such that \(T_{ab} \leq \frac{\varepsilon}{|A|}\) whenever \(|a| \leq |b|\) and \(b \in A^{\text{eq}}\), then \((X, A, f, \alpha) \in \text{GAT}(\varepsilon)\).

Proof. We have that
\[
\frac{|DX^{\text{eq}} \sqcup CX^{\text{eq}}|}{|X^{\text{eq}}|} = \frac{|X|}{|X^{\text{eq}}|} \sum_{S(a) \leq S(b), b \in A^{\text{eq}}} T_{ab} p_b \leq \frac{\varepsilon |X|}{|X^{\text{eq}}|} \sum_{S(a) \leq S(b), b \in A^{\text{eq}}} \frac{p_b}{|A|} = \varepsilon \frac{|X||X^{\text{eq}}|}{|X^{\text{eq}}|} = \varepsilon.
\]

**Definition 18.** Let \(\varepsilon_1, \varepsilon_2 \in [0, 1]\). A micro-macro dynamical system \((X, A, f, \alpha)\) satisfies property \(L_2(\varepsilon_1, \varepsilon_2)\), and we write \((X, A, f, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)\), if it satisfies \(L_1(\varepsilon_1)\) and \(\text{GAT}(\varepsilon_2)\). A sequence \((X_n, A_n, f_n, \alpha_n)\) of micro-macro dynamical systems satisfies property \(L_2\), and we write \((X_n, A_n, f_n, \alpha_n) \in L_2\), if for any \(\varepsilon_1, \varepsilon_2 > 0\) there exits \(N \in \mathbb{N}\) such that \((X_n, A_n, f_n, \alpha_n) \in L_2(\varepsilon_1, \varepsilon_2)\) for \(n \geq N\).

Let \((X, \pi)\) be micro-macro phase space. The partition \(\pi\) induces another partition \(Z\) of \(X\) into zones. For \(k \in \mathbb{N}_{\geq 1}\) let \(\pi_k \subseteq \pi\) be the subset of \(\pi\) consisting of all blocks of cardinality \(k\), and let the \(k\)-zone \(\hat{\pi}_k \subseteq X\) be given by
\[ \hat{\pi}_k = \bigsqcup_{a \in \pi_k} a. \]

We set \(O_{\pi} = \{k \in \mathbb{N}_{\geq 1} | \hat{\pi}_k \neq \emptyset\}\). A zone of \(X\) is a subset of the form \(\hat{\pi}_k\) for \(k \in O_{\pi}\). Typically we write \(O_{\pi} = \{k_1 < \cdots < k_o\}\) and set \(\hat{\pi}_j = \hat{\pi}_{k_j}\) for \(j \in [o]\).

**Theorem 19.** Let \((X, \pi)\) be a micro-macro phase-space with \(O_{\pi} = \{k_1 < \cdots < k_o\}\) and \(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \in \mathbb{R}_{\geq 0}\) be such that
1. \(\delta_1 |X_{\text{eq}}| \leq |X^{\text{eq}}| \leq \delta_2 |X_{\text{eq}}|\) with \(\delta_2 \leq \varepsilon_1 (\delta_1 + 1)\),
2. \(\delta_3 |\tilde{\pi}_{o-1}| \leq \sum_{i=1}^{o-2} |\hat{\pi}_i| \leq \delta_4 |\tilde{\pi}_{o-1}|\) with \(\frac{\delta_4}{\delta_3 + 1} + \delta_5 \leq \varepsilon_2\),

then \((X, \pi, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)\) for any permutation \(\alpha \in S_X\) with \(|CX^{\text{eq}}| \leq \delta_5 |X^{\text{eq}}|\).
Proof. Theorem 14 implies that \(|D|\leq\varepsilon_1\). The desired result follows from

\[
\frac{|DX^{\text{eq}} \cup CX^{\text{eq}}|}{|X^{\text{eq}}|} \leq \frac{\sum_{i=1}^{a-2} |\tilde{\pi}_i|}{\sum_{i=1}^{a-2} |\tilde{\pi}_i| + |\tilde{\pi}_{a-1}|} + \frac{|CX^{\text{eq}}|}{|X^{\text{eq}}|} \leq \frac{\delta_1 |\tilde{\pi}_{a-1}|}{(\delta_3 + 1) |\tilde{\pi}_{a-1}|} + \delta_5 \leq \varepsilon_2.
\]

\[\square\]

Next we show that if an invertible system has as many strict increases as strict decreases in entropy, and satisfies \(L_2(\varepsilon_1, \varepsilon_2)\), then most microstates are equilibrium microstates with constant entropy. Note that if \(|D| = |I|\), then \(\text{GAT}(\varepsilon)\) implies \(L_1(\varepsilon)\) since \(\frac{|D|}{|X|} \leq \frac{|I|}{|X^{\text{eq}}|} \leq \varepsilon\).

**Theorem 20.** Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system such that \(|D| = |I|\) and \((X, A, f, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)\). We have that:

1. \(\frac{|X^{\text{eq}}|}{|X|} \geq 1 - \frac{\varepsilon_1}{1 - \varepsilon_2}\).
2. \(\frac{|CX^{\text{eq}}|}{|X|} \geq 1 - \frac{\varepsilon_1(2 - \varepsilon_2)}{1 - \varepsilon_2}\).
3. \((1 - 2\varepsilon_2)|X^{\text{eq}}| \leq |DX^{\text{eq}}| \leq |X^{\text{eq}}|\).
4. \((1 - 2\varepsilon_2)|X^{\text{eq}}| \leq |\{i \in X^{\text{eq}} | \alpha(i) \in X^{\text{eq}}\}| \leq |X^{\text{eq}}|\).

**Proof.** From hypothesis we have that \((1 - \varepsilon_2)|X^{\text{eq}}| \leq |D| \leq \varepsilon_1|X|\) which implies item 1. From item 1 and the identity

\[
\frac{|X^{\text{eq}}|}{|X|} = \frac{|DX^{\text{eq}}|}{|X|} + \frac{|CX^{\text{eq}}|}{|X|},
\]

we get that \(1 - \frac{\varepsilon_1}{1 - \varepsilon_2} \leq \frac{|CX^{\text{eq}}|}{|X|} + \varepsilon_1\), which is equivalent to the inequality from item 2. Item 3 follows from

\((1 - \varepsilon_2)|X^{\text{eq}}| \leq |I|X^{\text{eq}}| = |I| = |D| = |DX^{\text{eq}}| + |DX^{\text{eq}}| \leq |DX^{\text{eq}}| + \varepsilon_2|X^{\text{eq}}|\).

Item 4 follows from the identity \(|DX^{\text{eq}}| = |\{i \in X^{\text{eq}} | \alpha(i) \in X^{\text{eq}}\}|\). Indeed we have a bijection \(DX^{\text{eq}} \rightarrow \{i \in X^{\text{eq}} | \alpha(i) \in X^{\text{eq}}\}\) sending \(i \in DX^{\text{eq}}\) to \(\alpha^{e-1}(i)\) where \(e\) is the least positive integer with \(\alpha^{e}(i) \in X^{\text{eq}}\). The inverse map sends \(i\) to \(\alpha^l(i)\), where \(l\) is the largest positive integer such that \(\alpha^l(i) \in X^{\text{eq}}\). \[\square\]

**Theorem 21.** Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system.

1. If \(|X^{\text{eq}}| \geq (1 - \varepsilon_1)|X|\), then \((X, A, f, \alpha) \in L_1(\varepsilon_1)\).
2. If \(|DX^{\text{eq}}| \geq (1 - \varepsilon_2)|X^{\text{eq}}|\), then \((X, A, f, \alpha) \in \text{GAT}(\varepsilon_2)\).
3. If $|X^{eq}| \geq (1 - \varepsilon_1)|X|$ and $|DX^{eq}| \geq (1 - \varepsilon_2)|X^{neq}|$, then $(X, A, f, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$.

Proof. If $|X^{eq}| \geq (1 - \varepsilon_1)|X|$, then $\frac{|D|}{|X|} = \frac{|D\alpha|}{|X|} \leq \frac{|X^{neq}|}{|X|} \leq \varepsilon_1$. Suppose now that $|DX^{eq}| \geq (1 - \varepsilon_2)|X^{neq}|$, then $|IX^{neq}| \geq \{|i \in X^{neq} | \alpha(i) \in X^{eq}\}| = |DX^{eq}| \geq (1 - \varepsilon_2)|X^{neq}|$.

Definition 22. A reversible micro-macro dynamical system is a tuple $(X, A, f, \alpha, r)$ such that $r : X \rightarrow X$ is an involution ($r^2 = 1$), and $r$ conjugates $\alpha$ and $\alpha^{-1}$ ($r\alpha r = \alpha^{-1}$).

We say that $(X, A, f, \alpha, r)$ is invariant if $fr = f$; equivariant if $r$ induces an involution $r : A \rightarrow A$ such that $fr = rf$; entropy preserving if $S(r_i) = S(i)$ for $i \in X$.

Lemma 23. Let $(X, A, f, \alpha, r)$ be a reversible micro-macro dynamical system, $a \in A$, $k \in \mathbb{N}_{\geq 1}$.

- The system is invariant iff $r$ induces bijections $r : a \rightarrow a$; the system is equivariant iff $r$ induces an involution $r : A \rightarrow A$ together with bijections $r : a \rightarrow ra$; the system is entropy preserving iff $r$ induces bijections $r : \hat{\pi}_k \rightarrow \hat{\pi}_k$.

- The reversion map $r : X \rightarrow X$ induces stochastic maps $r_A : A \rightarrow A$ and $r_Z : Z \rightarrow Z$ on macrostates and zones, respectively. We have that $r$ is invariant iff $r_A$ is the identity map; $r$ is equivariant iff $H(r_A) = 0$; and $r$ is entropy preserving iff $H(r_Z) = 0$.

Proposition 24. Let $(X, A, f, \alpha, r)$ be an entropy preserving reversible micro-macro dynamical system, then $|I| = |D|$.

Proof. We show that $r\alpha I = D$, using that $r$ preserves entropy a couple of times. We have that $\alpha I = D_{\alpha^{-1}} = D_{r\alpha r} = D_{\alpha r}$, so the desired result follows from the identity $rD_{\alpha r} = D$. □

Let $\text{rmmds}$ be the category of reversible micro-macro dynamical systems. A morphism $(X_1, A_1, f_1, \alpha_1, r_1) \rightarrow (X_2, A_2, f_2, \alpha_2, r_2)$ in $\text{rmmds}$ is a morphism $k$ of the subjacent micro-macro dynamical systems such that the following diagram commutes.

$$
\begin{array}{c}
X_1 \xrightarrow{k} X_2 \\
\downarrow r_1 \quad \quad \quad \quad \downarrow r_2 \\
X_1 \xrightarrow{k} X_2
\end{array}
$$

Our next constructions show that there are plenty of reversible systems, indeed we associate an invariant reversible micro-macro dynamical system, and an equivariant reversible
Figure 2: Invertible system and its associated invariant reversible system.

micro-macro dynamical system to each invertible micro-macro dynamical system. Let $\text{irmmds}$ and $\text{ermmds}$ be the full subcategories of $\text{rmmds}$ whose objects are, respectively, invariant and equivariant reversible micro-macro dynamical system. We have inclusion functors $\text{irmmds} \rightarrow \text{ermmds} \rightarrow \text{rmmds}$, and the forgetful functor $u : \text{rmmds} \rightarrow \text{immds}$ given by $u(X, A, f, \alpha, r) = (X, A, f, \alpha)$. Set $\mathbb{Z}_2 = \{1, -1\}$.

**Theorem 25.** The forgetful functor $u : \text{irmmds} \rightarrow \text{immds}$ has a left adjoint functor $\mathcal{I}R : \text{immds} \rightarrow \text{irmmds}$ given by $\mathcal{I}R(X, A, f, \alpha) = (X \times \mathbb{Z}_2, A, f \pi_X, \hat{\alpha}, r)$ where:

- $\pi_X : X \times \mathbb{Z}_2 \rightarrow X$ is the projection to $X$;
- $r : X \times \mathbb{Z}_2 \rightarrow X \times \mathbb{Z}_2$ is given by $r(i, s) = (i, -s)$;
- $\hat{\alpha} : X \times \mathbb{Z}_2 \rightarrow X \times \mathbb{Z}_2$ is given by $\hat{\alpha}(i, s) = (\alpha^s(i), s)$.

We have that:

1. $|(f \pi_X)^{-1}(a)| = 2|f^{-1}(a)|$ for $a \in A$, and $S(X \times \mathbb{Z}_2, A, f \pi_X) = S(X, A, f) + \ln(2)$.

2. $|D_{\hat{\alpha}}| = |I_{\hat{\alpha}}| = |D_\alpha| + |I_\alpha|$ and $|C_{\hat{\alpha}}| = 2|C_\alpha|$.

3. $2T_{ab}(\hat{\alpha}) = T_{ab}(\alpha) + T_{ab}(\alpha^{-1})$.

4. $\hat{\alpha} \in L_1(\varepsilon_1)$ iff $|D_\alpha| + |I_\alpha| \leq 2\varepsilon_1|X|$ iff $|C_\alpha| \geq (1 - 2\varepsilon_1)|X|$.

5. $\hat{\alpha} \in \text{GAT}(\varepsilon_2)$ iff $|D_\alpha| + |I_\alpha| \geq 2(1 - \varepsilon_2)|X^{\text{neq}}|$ iff $|C_\alpha| \leq |X^{\text{eq}}| - (1 - 2\varepsilon_2)|X^{\text{neq}}|$.

6. $\hat{\alpha} \in L_2(\varepsilon_1, \varepsilon_2)$ iff $2(1 - \varepsilon_2)|X^{\text{eq}}| \leq |D_\alpha| + |I_\alpha| \leq 2\varepsilon_1|X|$ iff $(1 - 2\varepsilon_1)|X| \leq |C_\alpha| \leq |X^{\text{eq}}| - (1 - 2\varepsilon_2)|X^{\text{neq}}|$.

Figure 2 shows an invertible micro-macro dynamical system and its associated invariant reversible micro-macro dynamical system.
Theorem 26. The forgetful functor $u : \text{ermmds} \to \text{immds}$ has a left adjoint functor $ER : \text{immds} \to \text{ermmds}$ given by $ER(X, A, f, \alpha) = (X \times \mathbb{Z}_2, A \times \mathbb{Z}_2, f \times 1, \tilde{\alpha}, r)$ where:

- $f \times 1 : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2$ is given by $f \times 1(i, s) = (fi, s)$;
- $r : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2$ is given by $r(i, s) = (i, -s)$;
- $\tilde{\alpha} : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2$ is given by $\tilde{\alpha}(i, s) = (\alpha^r(i), s)$.

We have that:

1. $|(f \times 1)^{-1}(a, s)| = |f^{-1}(a)|$ for $a \in A$, and $S(X \times \mathbb{Z}_2, A, f \times 1) = S(X, A, f)$.
2. Properties 2, 4, 5, 6 from Theorem 23 hold.
3. $T_{(a, 1)(b, 1)}(\tilde{\alpha}) = T_{ab}(\alpha)$, $T_{(a, -1)(b, -1)}(\tilde{\alpha}) = T_{ab}(\alpha^{-1})$, and $T_{(a, s)(b, -s)}(\tilde{\alpha}) = 0$.
4. $IR(X, A, f, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$ if and only if $ER(X, A, f, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$.
5. $IR(X, A, f, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)$ if and only if $ER(X, A, f, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)$.

Next result follows from Proposition 24 and Theorems 20, 21, 25, 26.

Proposition 27. Let $(X, \pi)$ be a micro-macro phase space with $|X^\text{eq}| \geq (1 - \varepsilon_1)|X^\text{neq}|$, and consider the (invariant, equivariant) associated reversible system $R(X, \pi, \alpha)$. If $R(X, \pi, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$, then $|D\alpha X^\text{eq}| + |D\alpha^{-1} X^\text{eq}| \geq 2(1 - 2\varepsilon_2)|X^\text{neq}|$. The system $R(X, \pi, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$ for any permutation $\alpha \in S_X$ such that $|D\alpha X^\text{eq}| + |D\alpha^{-1} X^\text{eq}| \geq 2(1 - 2\varepsilon_2)|X^\text{neq}|$.

Below we introduce three methods for constructing invariant reversible micro-macro dynamical systems out of an $A$-colored disjoint union of linearly order sets:

1) Let $(L, A, f)$ be such that $L$ is a non-empty poset obtained as a finite disjoint union $\bigsqcup L_c$ of linearly ordered sets $L_c$ with $c \in C$, and $f : L \to A$ is a map. Let $n(L, A, f, g)$ be the invariant reversible micro-macro dynamical system given by $(L \times \{1, -1\}, A, \tilde{f}, \alpha, r)$ where:

- $\tilde{f} = f\pi_L$ on $L \times \{1, -1\}$, $r(l, 1) = (l, -1)$, and $r(l, -1) = (l, 1)$.
- $\alpha$-orbits with cyclic order $L_c \times \{1\} \sqcup L_c^\text{op} \times \{-1\}$.

2) Consider $(L, A, f, g)$ with $(L, A, f)$ as in item 1 with $L$ allowed to be empty, and $g : C \to A$ another map. Let $o(L, A, f, g)$ be the invariant reversible micro-macro dynamical system given by $(L \times \{1, -1\} \sqcup \{o_c\}_{c \in C}, A, \tilde{f}, \alpha, r)$ where:

- $\tilde{f} = f\pi_L$ on $L \times \{1, -1\}$, $\tilde{f}(o_c) = g(c)$, $r(o_c) = o_c$, $r(l, 1) = (l, -1)$, and $r(l, -1) = (l, 1)$.
- $\alpha$-orbits with cyclic order $o_c \sqcup L_c \times \{1\} \sqcup L_c^\text{op} \times \{-1\}$. 

18
3) Consider \((L, A, f, g, h)\) with \((L, A, f, g)\) as in item 2 and \(h : C \rightarrow A\) another map. Let 
\[t(L, A, f, g, h) = (L \times \{1, -1\} \sqcup \{o_c, t_c\}_{c \in C}, A, \tilde{f}, \alpha, r)\]
be the invariant reversible micro-macro dynamical system given by

- \(\tilde{f} = f \pi_L\) on \(L \times \{1, -1\}\), \(\tilde{f}(o_c) = g(c)\), and \(\tilde{f}(t_c) = h(c)\).
- \(r(o_c) = o_c\), \(r(t_c) = t_c\), \(r(l, 1) = (l, -1)\), and \(r(l, -1) = (l, 1)\).
- \(\alpha\)-orbits with cyclic order \(o_c \sqcup L_c \times \{1\} \sqcup t_c \sqcup L_c^{op} \times \{-1\}\).

The constructions above can be modified to yield equivariant reversible micro-macro dynamical systems out of an \(A\)-colored disjoint union of linearly ordered sets, where now \(A\) is a set provided with an involution map \(r : A \rightarrow A\). One proceeds as in the previous case demanding that macrostates in the image of the maps \(g\) and \(h\) be fixed by \(r\). We denote by \(\tilde{n}, \tilde{o}, \tilde{t}\) the resulting equivariant systems.

**Theorem 28.**

1. An invariant reversible system \((X, A, f, \alpha, r)\) is isomorphic to a system of the form
\[IR(X_1, A, f, \alpha) \sqcup n(X_2, A, f) \sqcup o(X_3, A, f, f) \sqcup t(X_4, A, f, f, f)\].

2. An equivariant reversible system \((X, A, f, \alpha, r)\) is isomorphic to a system of the form
\[ER(X_1, A, f, \alpha) \sqcup \tilde{n}(X_2, A, f) \sqcup \tilde{o}(X_3, A, f, f) \sqcup \tilde{t}(X_4, A, f, f, f)\].

**Proof.** The result follows by an orbitwise analysis. Once the required properties are checked for microstates, the corresponding properties for macrostates follow as well, in both cases. Since \(r\) is an involution its cycles have length one or two, inducing a partition of \(X\) in two blocks \(\hat{r}_1\) and \(\hat{r}_2\), defined as the union of blocks of the respective cardinality. For a cycle \(c\) of \(\alpha\) the following possibilities arise:

1. \(c \subseteq \hat{r}_2\) and \(c \cap r(c) = \emptyset\). The map \(r\) induces an involution without fixed points on the set such cycles. Choose a cycle for each match pair and let \(X_1\) be the reunion of such microstates.

2. \(c \subseteq \hat{r}_2\) and \(c \cap r(c) \neq \emptyset\). In this case necessarily \(c = r(c)\) and \(r\) induces a matching on \(c\). Choose for each such cycle a maximal segment of \(\alpha\)-orbit with unmatched points, and let \(X_2\) be the reunion of such microstates.

3. \(|c \cap \hat{r}_1| = 1\). Away from the fixed point (to be identified with \(o_c\)) \(r\) defines a matching on \(c\). Choose for each such cycle a maximal segment of \(\alpha\)-orbit with unmatched not fixed points, and let \(X_3\) be the union of microstates.
4. \(|c \cap \hat{r}_1| = 2\). Away from the pair of fixed points (to be identified with \(o_c\) and \(t_c\)), \(r\) defines a matching on \(c\). Choose for each such cycle a maximal segment of \(\alpha\)-orbit with unmatched not fixed points, and let \(X_4\) be the union of microstates.

Suppose that an \(\alpha\)-cycle \(c\) has a \(r\)-fixed point \(i\), then \(r\alpha^s(i) = \alpha^{-s}(i)\) for \(s > 0\). It follows that there should be a minimum \(s > 0\) for which either \(\alpha^s(i) = r\alpha^{s-1}(i)\) or \(r\alpha^s(i) = \alpha^s(i)\) (exclusively). If the former condition holds we are in case 3, and if the latter condition holds we are in case.

Statements 1 and 2 of our next result provide the substrate of the Loschmidt’s paradox within the combinatorial framework, note the subtle asymmetry in statements 3 and 4.

**Theorem 29.** Let \((X, A, f, \alpha, r)\) be an entropy preserving reversible micro-macro dynamical system and fix \(N, M \geq 0\).

1. Entropy defines an \((\epsilon, N, M)\)-arrow of time around \(i \in X\) if and only if negative entropy defines an \((\epsilon, M, N)\)-arrow of time around \(r\alpha(i) \in X\).

2. \(|\{i \in X \mid (\epsilon, N, M)\text{-arrow of time around } i\}| = |\{i \in X \mid reversed (\epsilon, M, N)\text{-arrow of time around } i\}|.\)

3. \(|\{i \in X^{neq} \mid (\epsilon, N, M)\text{-arrow of time around } i \text{ and } \alpha^{M+1}(i) \in X^{neq}\}| = |\{i \in X^{neq} \mid reversed (\epsilon, M, N)\text{-arrow of time around } i\}|.\)

4. \(|DX^{neq}| = |\{i \in IX^{neq} \mid \alpha(i) \in X^{neq}\}|.\)

**Proof.** Since entropy is preserved by reversion, it is strictly increasing along the \(\alpha\)-sequence

\[
\alpha^{-N}(i) \rightarrow \cdots \rightarrow \alpha^{-1}(i) \rightarrow i \rightarrow \alpha^1(i) \rightarrow \cdots \rightarrow \alpha^M(i) \rightarrow \alpha^{M+1}(i),
\]

if and only if it is strictly decreasing along the \(\alpha\)-sequence

\[
r\alpha^{M+1}(i) \rightarrow \cdots \rightarrow r\alpha^1(i) \rightarrow ri \rightarrow r\alpha^{-1}(i) \rightarrow \cdots \rightarrow r\alpha^{-N+1}(i) \rightarrow r\alpha^{-N}(i).
\]

Our combinatorial settings provide a straightforward approach to the next results known collectively as fluctuation theorems \([4, 11, 13, 14, 15, 48, 49]\).

**Theorem 30.**

1. Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system, then

\[
T_{a,b} = e^{S(a) - S(b)}T_{b,a}(\alpha^{-1}).
\]
2. Let \((X, A, f, \alpha, r)\) be a reversible micro-macro dynamical system, then
\[
T_{a,b} = e^{S(a) - S(b)} T_{r(b),r(a)}.
\]

Proof. For item 1 set \(l = |\{ i \in b \mid \alpha(i) \in a \}| = |\{ i \in a \mid \alpha^{-1}(i) \in b \}| \). We have that
\[
T_{a,b}(\alpha) = \frac{l}{|b|} = \frac{|a|}{|b|} \frac{l}{|a|} = e^{S(a) - S(b)} T_{b,a}(\alpha^{-1}).
\]
Item 2 follows from item 1 and the identity \( T_{b,a}(\alpha^{-1}) = T_{r(b),r(a)} \), indeed we have that
\[
\left| \{ i \in a \mid \alpha^{-1}(i) \in b \} \right| = \left| \{ i \in a \mid \alpha r(i) \in r(b) \} \right| = \left| \{ i \in r(a) \mid \alpha(i) \in rb \} \right|.
\]
Note that \( r(a) \) need not be a macrostate.

**Definition 31.** Let \((X, A, f, \alpha)\) be a micro-macro dynamical system. The \(n\)-steps entropy production rate is the map \( \sigma_n : X \rightarrow \mathbb{R} \) given by
\[
\sigma_n(i) = \frac{S(\alpha^n i) - S(i)}{n} = \frac{1}{n} \ln \left( \frac{|\alpha^n i|}{|i|} \right).
\]
Let \((X, A, f, \alpha, r)\) be an entropy preserving reversible system. Below we consider three probability distributions on \(X\) : uniform probability \(u\), uniform probability on non-equilibrium microstates, also denoted by \(u\), and the probability \(q\) from Definition 31 given by \( q_i = \frac{1}{|A||i|} \).

Note that \( q_i > \frac{1}{|X|} \) if and only if \( S(i) < \ln(|X|/|A|) \), i.e. \( q \) assigns to low entropy microstates a probability higher than the uniform probability, moreover, the lower the entropy the higher the \(q\)-probability of a microstate. For \( i \in X \) we have that
\[
\sigma_n(r\alpha^n i) = -\sigma_n(i) \quad \text{and} \quad |r\alpha^n i| = e^{n\sigma(i)|i|}.
\]
The density functions \( W_n^u, W_{n,\text{neq}}^u, W_n^q : \mathbb{R} \rightarrow [0,1] \) associated to the uniform probability, the uniform probability on non-equilibrium microstates, and \(q\) via the map \( \sigma_n \) are given by
\[
W_n^u(x) = \frac{|\{ i \mid \sigma_n(i) = x \}|}{|X|}, \quad W_{n,\text{neq}}^u = \frac{|\{ i \in X^{\text{neq}} \mid \sigma_n(i) = x \}|}{|X^{\text{neq}}|}, \quad \text{and} \quad W_n^q(x) = \sum_{\sigma_n (i) = x} \frac{1}{|A||i|}.
\]
The statement and proofs of Theorem 32 and Theorem 33 below are combinatorial renderings of the arguments given by Deward and Maritan [15].

**Theorem 32.** Let \((X, A, f, \alpha, r)\) be an entropy preserving reversible system. For \(x \in \mathbb{R}_{\geq 0}\) we have that: 1) \(W_n^u(x) = W_n^u(-x)\). 2) \(W_{n,\text{neq}}^u(x) \geq W_{n,\text{neq}}^u(-x)\). 3) \(W_n^q(x) = e^{ux}W_n^q(-x)\).

Proof. For item 1 we have that
\[
W_n^u(x) = \frac{|\{ i \mid \sigma_n(i) = x \}|}{|X|} = \frac{|\{ i \mid \sigma_n(r\alpha^n i) = x \}|}{|X|} = \frac{|\{ i \mid \sigma_n(i) = -x \}|}{|X|} = W_n^u(-x).
\]
The item 2 statement is trivial for \( x = 0 \), so we assume \( x > 0 \). We have that

\[
W_{n, \text{neq}}^u(x) = \frac{|\{i \in X^{\text{neq}} \mid \sigma_n(i) = x\}|}{|X|} = \frac{|\{i \in X^{\text{neq}} \mid \sigma_n(i) = x, \ r\alpha^n i \in X^{\text{neq}}\}|}{|X^{\text{neq}}|} = \frac{|\{i \in X^{\text{neq}} \mid \sigma_n(i) = -x\}|}{|X^{\text{neq}}|} + \frac{|\{i \in X^{\text{neq}} \mid \sigma_n(i) = -x\}|}{|X^{\text{neq}}|} = W_{n, \text{neq}}^u(-x) + \frac{|\{i \in X^{\text{neq}} \mid \sigma_n(i) = -x\}|}{|X^{\text{neq}}|} \geq W_{n, \text{neq}}^u(-x).
\]

For item 3 we have that

\[
W_{n, \text{neq}}^q(x) = \sum_{\sigma_n(i) = x} \frac{1}{|A||i|} = \sum_{\sigma_n(r\alpha^n i) = x} \frac{1}{|A||r\alpha^n i|} = \sum_{\sigma_n(i) = -x} \frac{e^{-n\sigma_n(i)}}{|A||i|} = e^{nx} \sum_{\sigma_n(i) = -x} \frac{1}{|A||i|} = e^{nx} W_n(-x).
\]

Below we consider the \( n \)-steps entropy production rate mean value \( \sigma_n^u, \sigma_{n, \text{neq}}^u, \sigma_n^q \) with respect to the uniform probability, the uniform probability over non-equilibrium microstates, and the \( q \) probability on \( X \).

**Theorem 33.** Let \((X, A, f, \alpha)\) be a micro-macro dynamical system, then

\[
n\sigma_n^u = S_{T(\alpha^n)p}(A) - S_p(A)
\]

where \( T(\alpha^n) \) is the stochastic map on \( A \) induced by \( \alpha^n \).

1. If \( \alpha \) is invertible, then \( \sigma_n^u = 0 \).

2. If \( \alpha \) is invertible, then \( \sigma_{n, \text{neq}}^u = \frac{1}{n|X^{\text{neq}}|} \sum_{i \in D_{\alpha^n}(X^{\text{eq}})} [S(i) - S(\alpha^n i)] \geq 0 \), and \( \sigma_{n, \text{neq}}^u = 0 \) if and only if \( X^{\text{eq}} \) is \( \alpha^n \)-invariant.

3. For an entropy preserving reversible system \((X, A, f, \alpha, r)\), we have that \( \sigma_n^q \geq 0 \) and \( \sigma_n^q = 0 \) if and only if \( \alpha^n \) preserves entropy.

**Proof.** Recall that \( p \) and \( T(\alpha^n)p \) are the probability measures on \( A \) given by \( p_a = \frac{|a|}{|X|} \) and \((T(\alpha^n)p)_a = \sum_{b \in A} T_{ab}(\alpha^n p_b)\). We have that

\[
n\sigma_n^u = \sum_{i \in X} \frac{S(\alpha^n i) - S(i)}{|X|} = \sum_{i \in b, \alpha^n(i) \in a} \frac{S(a) - S(b)}{|X|} = \]
\[
\sum_{a,b} [S(a) - S(b)] \frac{|\{i \in b \mid \alpha^n(i) \in a\}|}{|b|} \frac{1}{|X|} = \sum_{a,b} \frac{S(a) - S(b)}{|X|} T_{ab}(\alpha^n) p_b = 0
\]

If \( \alpha \) is invertible, then
\[
\sum_{a,b} S(a) T_{ab}(\alpha^n) p_b - \sum_{a,b} S(b) T_{ab}(\alpha^n) p_b = S_{T(\alpha^n)p}(A) - S_p(A).
\]

Furthermore we have that
\[
\sigma_n^u, \neq = \frac{1}{n|X|} \sum_{i \in X} \left[ S(\alpha^n i) - S(i) \right] = 0.
\]

By the Gibb’s inequality [10] we have that
\[
n \sigma_n^q = \sum_{i \in X} q_i \ln \left( \frac{|\alpha^n i|}{|i|} \right) = \sum_{i \in X} q_i \ln \left( \frac{q_i}{q_{\alpha^n i}} \right) \geq 0.
\]

Items 2 and 3 of Theorem 33 guarantee that with uniform probability on non-equilibrium microstates, and with probability \( q \) the average \( n \)-steps entropy production rate is non-negative. Suppose now that we are given beforehand the mean value \( \alpha_n > 0 \) of the \( n \)-steps entropy production rate, then following the Jaynes’ maximum entropy method is natural to consider the probability \( t \) on \( X \) given on \( i \in a \in \pi \) by
\[
t(i) = \frac{p_a}{|a|} \text{ where } p_a = \frac{e^{\lambda \sigma_n(a)}}{Z(\lambda)}, \quad \sigma_n(a) = \frac{1}{|a|} \sum_{i \in a} S(\alpha^n i) - S(i), \quad \text{and } Z(\lambda) = \sum_{a \in A} e^{\lambda \sigma_n(a)}.
\]

Assuming that \( \alpha_n \in \left[ \min_{a \in A} \sigma_n(a), \max_{a \in A} \sigma_n(a) \right] \), the parameter \( \lambda \) is chosen so that
\[
\sum_{a \in A} \sigma_n(a) \frac{e^{\lambda \sigma_n(a)}}{Z(\lambda)} = \alpha_n.
\]
4 Structural Properties of Micro-Macro Systems

In this section we review some of the structural properties of the category of micro-macro dynamical systems: we introduce the product, disjoint union, restriction, coarse-graining, meet, and joint of micro-macro dynamical systems. We also provide five general construction yielding interesting examples of micro-macro phase spaces.

If \((X, A, f, \alpha)\) satisfies the axioms for a micro-macro dynamical system except that \(f\) may not be surjective, then we have the micro-macro dynamical system \((X, f(X), f, \alpha)\). We use this construction without change of notation, and even without mention. The inversion functor \(\text{inv} : \text{immds} \rightarrow \text{immds}\) is given on objects by \(\text{inv}(X, A, f, \alpha) = (X, A, f, \alpha^{-1})\). The functor \(\text{inv}\) is defined for reversible systems as \(\text{inv}(X, A, f, \alpha, r) = (X, A, f, \alpha^{-1}, r)\).

**Proposition 34.** Let \((X, A, f, \alpha, r)\) be an entropy preserving reversible micro-macro dynamical system. Then \((X, A, f, \alpha, r) \in L_2(\varepsilon_1, \varepsilon_2)\) if and only if \((X, A, f, \alpha^{-1}, r) \in L_2(\varepsilon_1, \varepsilon_2)\).

**Proof.** Follows from \(|D_{\alpha^{-1}}| = |I| = |D|\) and \(|I_{\alpha^{-1}}X| = |I_{\alpha^{-1}}X| = |D| = |I| = |IX|\).

**Definition 35.** Let \((X_1, A_1, f_1, \alpha_1)\) and \((X_2, A_2, f_2, \alpha_2)\) be micro-macro dynamical systems. The product micro-macro dynamical system is given by \((X_1 \times X_2, A_1 \times A_2, f_1 \times f_2, \alpha_1 \times \alpha_2)\). The map \(\times : \text{mmds} \times \text{mmds} \rightarrow \text{mmds}\) is functorial.

The product functor induces a product functor on \(\text{immds}\), and can be compatibly defined on \(\text{rmmds}\) so that \(i : \text{immds} \rightarrow \text{mmds}, i_* : \text{mmds} \rightarrow \text{immds}, u : \text{rmmds} \rightarrow \text{immds}, \text{and} \ inv : \text{immds} \rightarrow \text{immds} \) are product preserving. The following result justifies the presence of the logarithmic function in the Boltzmann entropy from the structural viewpoint.

**Proposition 36.** Consider the system \((X_1 \times X_2, A_1 \times A_2, f_1 \times f_2, \alpha_1 \times \alpha_2)\). We have that \(S(A_1 \times A_2) = S(A_1) + S(A_2), H(p_{A_1 \times A_2}) = H(p_{A_1}) + H(p_{A_2}), H(T_{A_1 \times A_2}) = H(T_{A_1}) + H(T_{A_2})\).

**Proof.** For \((a_1, a_2) \in A_1 \times A_2\) we have that \(p(a_1, a_2) = \frac{|(a_1, a_2)|}{|A_1 \times A_2|} = \frac{|a_1| |a_2|}{|A_1| |A_2|} = p(a_1)p(a_2)\) and \(S(a_1, a_2) = \ln|\{a_1, a_2\}| = \ln|a_1| + \ln|a_2| = S(a_1) + S(a_2)\). Therefore

\[
S(A_1 \times A_2) = \sum_{(a_1, a_2) \in A_1 \times A_2} S(a_1, a_2)p(a_1, a_2) = \sum_{a_1 \in A_1} S(a_1)p(a_1) + \sum_{a_2 \in A_2} S(a_2)p(a_2) = S(A_1) + S(A_2).
\]

Thus \(H(p_{A_1 \times A_2}) = \ln(|A_1||A_2|) - S(A_1) - S(A_2) = H(p_{A_1}) + H(p_{A_2})\). Considering transition maps we have that \(T_{(b_1, b_2),(a_1, a_2)} = T_{(b_1, b_2)}T_{b_1, a_2}\) since

\[
\left|\{(i, j) \in a_1 \times a_2 \mid \{a_1(i), a_2(j)\} \in b_1 \times b_2\}\right| = \left|\{(i) \in a_1 \mid a_1(i) \in b_1\}\right| \left|\{j \in a_2 \mid a_2(j) \in b_2\}\right|.
\]

Therefore

\[
H(T_{A_1 \times A_2}) = -\sum_{(a_1, a_2), (b_1, b_2) \in A_1 \times A_2} \ln|T_{(b_1, b_2),(a_1, a_2)}| T_{(b_1, b_2),(a_1, a_2)}p(a_1, a_2) =
\]
Definition 38. Let \((X, A, f)\) be an invertible micro-macro dynamical system and \(Z\) be a subset of \(X\). The restriction of \((X, A, f, \alpha)\) to \(Z\) is the invertible micro-macro dynamical system \((Z, f(Z), f_Z, \alpha_Z)\) such that \(f_Z : Z \to f(Z)\) is the restriction to \(Z\) of \(f\), and the bijective map \(\alpha_Z : Z \to Z\) is constructed as follows: for \(i \in Z\) find the smallest \(l \in \mathbb{N}_{>0}\) such that \(\alpha^l(i) \in Z\) and set \(\alpha_Z(i) = \alpha^l(i)\).
The restriction construction can be applied to reversible systems \((X, A, f, \alpha, r)\) as follows. Let \(Z\) be a subset of \(X\) closed under \(r\), then \((Z, f(Z), f_Z, \alpha_Z, r_Z)\) is a reversible micro-macro dynamical system. Suppose that \(\alpha_Z(i) = \alpha^l(i) = j\) with \(l > 0\) as small as possible. Note that the identities \(\alpha^l(i) = j\) and \(\alpha^l(rj) = ri\) are equivalent, the former identity implies the latter since \(\alpha^l(rj) = r(rar)^l(j) = r\alpha^{-1}(j) = ri\), the other implication is similar. Therefore we conclude that \(r\alpha^l(rj) = r\alpha^l(ri) = i = \alpha^{-1}_Z(j)\).

**Theorem 41.** Let \((Z, \pi_Z, \alpha_Z)\) be the restriction to \(Z \subseteq X\) of the invertible micro-macro dynamical system \((X, \pi, \alpha)\). The maximum entropy of \((Z, \pi_Z, \alpha_Z)\) is less than or equal to the maximum entropy of \((X, \pi, \alpha)\).

**Proof.** \(\max_{i \in Z} S(i) = \max_{a \in \pi_Z} \ln|a| = \max_{a \in \pi, a \cap Z \neq \emptyset} \ln|a \cap Z| \leq \max_{a \in \pi} \ln|a| = \max_{i \in X} S(i). \)

**Definition 42.** Let \((X, A, f, \alpha)\) be a micro-macro dynamical systems and \(g : A \to B\) be a surjective map. The associated coarse-grained micro-macro dynamical system is given by \((X, B, gf, \alpha)\). Coarse-graining is also naturally defined for reversible systems since \(fr = f \Leftrightarrow (gf)r = gf\), and preserving equivariant reversible systems if \(B\) comes with an involution \(r : B \to B\) such that \(gr = rg\), since in this case \((gf)r = g(rf) = r(gf)\).

**Proposition 43.** Let \((X, B, gf, \alpha)\) be the coarse-grained micro-macro dynamical system obtain from \((X, A, f, \alpha)\) and \(g : A \to B\). We have that \(S(B) \geq S(A)\), \(H(p_B) \leq H(p_A)\), \(H(T_B) - S(B) \leq H(T_A) - S(A)\), and \(H(p_B) + H(T_B) \leq H(p_A) + H(T_A)\).

**Proof.**

\[
S(B) = \sum_{b \in B} \ln|b| p_b = \sum_{b \in B} \sum_{g(a) = b} \ln\left(\sum_{g(a) = b} |a| \frac{|a|}{|X|}\right) \geq \sum_{b \in B} \sum_{g(a) = b} \ln|a| \frac{|a|}{|X|} = \sum_{a \in A} \ln|a| \frac{|a|}{|X|} = \sum_{a \in A} \ln|a| p_a = S(A),
\]

thus \(H(p_B) = \ln|X| - S(B) \leq \ln|X| - S(A) = H(p_A)\). We have that

\[
T_{b_1 b_2} = \left\{\frac{i \in b_1}{b_1} : \alpha(i) \in b_2\right\} = \sum_{f(a_1) = b_1, g(a_2) = b_2} \left\{\frac{i \in a_1}{a_1} : \alpha(i) \in a_2\right\}.
\]

Therefore

\[
H(T_B) \leq -\sum_{b_1, b_2 \in B} \sum_{f(a_1) = b_1, g(a_2) = b_2} \ln\left(\left\{\frac{i \in a_1}{b_1} : \alpha(i) \in a_2\right\}\frac{|i \in a_1, \alpha(i) \in a_2|}{|b_1|}\right)\left\{\frac{i \in a_1}{b_1} : \alpha(i) \in a_2\right\} = \frac{|b_1|}{|X|} = \frac{|b_1|}{|X|} = \frac{|b_1|}{|X|}.
\]

26
Thus we get that $A$ where the amalgamated sum $I \... $X$, macro phase space is given by ($\... $tivities $f$ and $i$. Proposition 45. $S$ $\... $X, A, f, α$ be a micro-macro dynamical system. For $\...$ I, we construct micro-macro phase spaces $(X, A^n, γ_n)$ useful for understanding macrostates transitions in $(X, A, f, α)$. 27
The map $\gamma_n : X \to A^n$ is given by $\gamma_n(i) = (f_i, f\alpha(i), f\alpha^2(i), \ldots, f\alpha^{n-1}(i))$. The entropy $S(a_1, \ldots, a_n)$ of macro-state $(a_1, \ldots, a_n) \in A^n$ is the logarithm of the number of micro-realizations of the transitions $a_1 \to \cdots \to a_n$ through the $\alpha$-dynamics.

II. Given a finite set $B$ (boxes) and $k \in \mathbb{N}$ we let $\mathbb{N}^B_k$ be the set of $\mathbb{N}$-valued measures on $B$ of total measure $k$, that is $\mathbb{N}^B_k = \{w : B \to \mathbb{N} \mid \sum_{b \in B} w(b) = k\}$. Given another finite set $P$ (particles) we let $[P, B]$ be the set of maps from $P$ to $B$ (particles to boxes). We obtain the micro-macro phase space $([P, B], \mathbb{N}^B_{|P|}, c)$ where the surjective map $c : [P, B] \to \mathbb{N}^B_{|P|}$ sends $f$ to the measure $c_f$ given by $c_f(b) = |f^{-1}(b)|$. We study the two-boxes case in Example 52.

III. $S_P$ acts on $[P, B]$ by $(\alpha f)(p) = f(\alpha^{-1}(p)).$ $S_B$ acts respectively on $[P, B]$ and on $\mathbb{N}^B_k$ by $(\beta f)(b) = \beta f(b)$ and $(\beta c)(b) = c(\beta^{-1}(b)).$ We have that

$$c_{\beta f^{-1}} = |(\beta f^{-1})(b)| = |\alpha^{-1}(f^{-1}(\beta^{-1} b))| = |f^{-1}(\beta^{-1} b)| = (\beta c_f)(b).$$

Let $G_P \subseteq S_P$ and $G_B \subseteq S_B$ be subgroups, and $M(P, B) \subseteq [P, B]$ be invariant under the action of $G_P \times G_B$ on $[P, B]$. From the identities above we get the micro-macro phase space $(M(P, B)/G_P \times G_B, \mathbb{N}_k^B/G_B, c)$.

Several instances of this construction, attached to illustrious names, have been study in the literature. Niven [41, 42] considers the following cases:

- Maxwell-Boltzmann statistics: $M(P, B) = [P, B], \ G_P = 1, \ G_B = 1.$
- Lynden-Bell statistics: $M(P, B) = \text{Inj}(P, B), \ G_P = 1, \ G_B = 1.$
- Bose-Einstein statistics: $M(P, B) = [P, B], \ G_P = S_P, \ G_B = 1.$
- $m$-gentile statistics: $M(P, B) = \{f \in [P, B] \mid |f^{-1} b| \leq m \}, \ G_P = S_P, \ G_B = 1.$
- Fermi-Dirac statistics: $M(P, B) = \text{Inj}(P, B), \ G_P = S_P, \ G_B = 1.$
- DI statistics: $M(P, B) = [P, B], \ G_P = 1, \ G_B = S_B.$
- II statistics: $M(P, B) = [P, B], \ G_P = S_P, \ G_B = S_B.$

An unifying aim of these studies has been finding the macrostate of greatest entropy. Other instances of this fairly general construction are yet to be explored.

IV. With the notation of example III let $\text{prob}_B$ be space of probability distributions on $B$, $\mathbb{N}^B_k \to \text{prob}_B$ be the normalization map, and $\mathbb{N}^B_k/G_B \to \text{prob}_B/G_B$ be the induced map.
Let \( \text{prob}_{B} \to A \) be a \( G_{B} \)-invariant map with \( A \) a finite set, and \( \text{prob}_{B}/G_{B} \to A \) be the induced map. Consider the composition map \( f \) obtained from the chain of maps

\[
M(P, B)/G_{P} \times G_{B} \to \mathbb{N}_{k}^{B}/G_{B} \to \text{prob}_{B}/G_{B} \to A.
\]

We have constructed a micro-macro phase space \((M(P, B)/G_{P} \times G_{B}, A, f)\).

V. Our last construction relies on a generalized version of the theory of combinatorial species \([2, 3, 6, 16]\) where \text{surj}, the category of finite sets and surjective maps, plays the role usually reserved for the category of finite sets and bijections. Given functor \( F : \text{surj} \to \text{surj} \), we obtain the map \( \hat{F} : \text{mmds} \to \text{mmds} \) given by \( \hat{F}(X, A, f, \alpha) = (FX, FA, Ff, F\alpha) \), acting functorially on surjective morphisms in \text{mmds}. For example, we have functors \( P, G, L, \text{Par} : \text{surj} \to \text{surj} \) sending a set \( X \) to the set \( PX \) of subsets of \( X \), the set \( GX \) of simple graphs on \( X \), the set \( LX \) of linear orderings on \( X \), and the set \( \text{Par}X \) of partitions on \( X \), respectively. Moreover, given functors \( F, G : \text{surj} \to \text{surj} \) we build new such functors using the following natural operations:

\[
(F + G)(x) = F(x) \sqcup G(x), \quad (F \times G)(x) = F(x) \times G(x),
\]

\[
FG(x) = \bigcup_{a \cup b = x} F(a) \times G(b), \quad F \circ G(x) = F(G(x)),
\]

\[
F(G)(x) = \bigcup_{\pi \in \text{Par}(x)} F(\pi) \times \prod_{a \in \pi} G(a),
\]

in the latter case we set \( F(G)(\emptyset) = F(\emptyset) = G(\emptyset) = \emptyset \) and assume that \( G \) is monoidal, i.e. it comes with functorial (under bijections) maps \( G(a) \times G(b) \to G(a \sqcup b) \) satisfying natural associativity constraints. Note that the functor \( \hat{F} \) can be extended to reversible systems yielding the map \( \hat{F} : \text{rmmds} \to \text{rmmds} \) given by \( \hat{F}(X, A, f, \alpha, r) = (FX, FA, Ff, F\alpha, Fr) \) acting functorially on surjective morphisms in \text{rmmds}.

5 Always Increasing Entropy on Invertible Systems

In this section we consider invertible micro-macro dynamical systems for which entropy is always increasing, i.e. those systems for which property \( L_{1}(0) \) holds. Although we are going to show that this case occurs with low probability, Theorem 48, we develop it in details to illustrate the duality principle described in the introduction, see Theorem 50. We first show a combinatorial analogue of Zermelo’s observation of the tension between recurrence and the second law.

**Proposition 46.** Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system. Entropy is always increasing if and only if entropy is constant on \( \alpha \)-orbits.
Proof. If \( S \) is constant on the orbits of \( \alpha \), then \( S(i) = S(\alpha(i)) \) for all \( i \in X \) and thus entropy is always increasing. Conversely, if entropy is always increasing then
\[
S(i) \leq S(\alpha(i)) \leq \cdots \leq S(\alpha^l(i)) = S(i),
\]
where \( l \) is the cardinality of the \( \alpha \)-orbit of \( i \). Thus the inequalities above are identities.

Corollary 47. Let \((X, A, f, \alpha)\) be an invertible micro-macro dynamical system. The induced stochastic map \( T : A \rightarrow A \) is experimentally reproducible, in Jaynes' sense, if and only if there is a permutation \( t : A \rightarrow A \) such that \( T_{ab} = \delta_{at(b)} \) and \( S(t(a)) = S(a) \) for \( a \in A \).

Theorem 48. Let \((X, \pi)\) be a micro-macro phase space with \( O_\pi = \{k_1 < \cdots < k_o\} \). A random permutation \( \alpha \in S_X \) determines a micro-macro dynamical system \((X, \pi, \alpha)\) with always increasing entropy with probability
\[
\left(\frac{|X|}{|\hat{\pi}_1|, \ldots, |\hat{\pi}_o|}\right)^{-1}.
\]

Proof. It follows from Proposition 46 that such a permutation \( \alpha \) induces and is determined by permutations on the sets \( \hat{\pi}_j \). The induced permutations are arbitrary, so the result follows because a set with \( n \) elements has \( n! \) permutations. The probabilistic statement is then clear assuming uniform probability on \( S_X \). Figure 3 shows a couple of permutations for which entropy is always increasing given the partitioned set.

Suppose now that we are given a set \( X \) together with a permutation \( \alpha \) on it. We want to know how many partitions \( \pi \) are there such that entropy is always increasing on the system \((X, \pi, \alpha)\). Recall that the number of partitions on \( \{1, 2, \ldots, nk\} \) into \( n \) blocks each of cardinality \( k \) is given by \( \frac{(nk)!}{n!k^n} \). Given \( S \subseteq P_X \), a family of subsets \( X \), we let \( S \) be the set of maps \( l : S \rightarrow \mathbb{N}_{\geq 1} \) such that for all \( k \in \mathbb{N}_{\geq 1} \) we have that \( k \) divides \( 7(k) = \sum_{l(A)=k} |A| \).
Given a permutation $\alpha$ on the finite set $X$, we let $\text{Cyc}(\alpha)$ be the partition of $X$ into $\alpha$-cycles. We obtain the micro-macro phase space $(X, \text{Cyc}(\alpha), c)$, where $c$ is the map sending $i \in X$ to the $\alpha$-cycle $c(i)$ generated by $i$.

**Theorem 49.** Let $X$ be a finite set and $\alpha \in S_X$. The number of partitions $\pi \in \text{Par}(X)$ such that entropy is always increasing in $(X, \pi, \alpha)$ is given by

$$
\sum_{l \in \text{Cyc}(\alpha)} \prod_{k \in \text{Im}(l)} \frac{\overline{7(k)!}}{k! \overline{\overline{7(k)!} \overline{k!}^k}}.
$$

**Proof.** Let $\pi$ be a partition on $X$ such that entropy is always increasing in $(X, \pi, \alpha)$ and let $c \in \text{Cyc}(\alpha)$. According to Proposition 46 if $c \cap \hat{\pi}_k \neq \emptyset$, then $c \subseteq \hat{\pi}_k$. Thus we can associate to $\pi$ the map $l_\pi \in \text{Cyc}(\alpha)$ given by $l_\pi(c) = k$ if and only if $c \subseteq \hat{\pi}_k$. Conversely, given $l \in \text{Cyc}(\alpha)$ the partitions $\pi$ with $l_\pi = l$ can be constructed by choosing for each $k \in \text{Im}(l)$ a uniform partition with blocks of cardinality $k$ on the set $\bigcup_{l(c)=k} c \subseteq X$. Entropy is always increasing for such partitions, and there are $\frac{\overline{7(k)!}}{k! \overline{\overline{7(k)!} \overline{k!}^k}}$ of them. Figure 3 displays a couple of examples of this construction. 

We have shown the following instance of the micro/macro duality principle.

**Theorem 50.** Let $X$ be a finite set. The number of invertible micro-macro dynamical systems $(X, \pi, \alpha)$ such that entropy is always increasing is given by

$$
\sum_{\pi \in \text{Par}(X)} |\hat{\pi}_1|! \ldots |\hat{\pi}_0|! = \sum_{\alpha \in S_X} \sum_{l \in \text{Cyc}(\alpha)} \prod_{k \in \text{Im}(l)} \frac{\overline{7(k)!}}{k! \overline{\overline{7(k)!} \overline{k!}^k}}.
$$

**Proof.** Consider the set of pairs $(\pi, \alpha) \in \text{Par}(X) \times S_X$ such that entropy is always increasing on the invertible micro-macro dynamical system $(X, \pi, \alpha)$. Cardinality of this set can be found by fixing $\pi$ and then counting permutations $\alpha$, leading, by Theorem 48, to the left-hand side of the proposed formula. Alternatively, it can be counted by fixing $\alpha$ and then counting partitions $\pi$, leading, by Theorem 49, to the right-hand side of the proposed formula.

### 6 Bounding Entropy Strict Decreases on Invertible Systems

In this section we fix a micro-macro phase space $(X, \pi)$ and find an upper bound for the number $|D(X, \pi, \alpha)|$ of strict decreases in entropy for an arbitrary permutation $\alpha \in S_X$.

**Theorem 51.** Let $(X, \pi)$ be an invertible micro-macro dynamical system. We have that

$$
\max_{\alpha \in S_X} |D(X, \pi, \alpha)| = |X| - |\hat{\pi}_r|,
$$

where $r$ is such that $|\hat{\pi}_r| \geq |\hat{\pi}_k|$ for $k \in \mathbb{N}$. 

31
Figure 4: Partitions on a set with permutation for which entropy is always increasing.

Proof. First we show that \( \max_{\alpha \in S_X} |D(X, \pi, \alpha)| \leq |X| - |\widehat{\pi}_r| \), i.e. we show that \( |D(X, \pi, \alpha)| \leq |X| - |\widehat{\pi}_r| \) for \( \alpha \in S_X \). Consider an \( \alpha \)-orbit containing \( l \geq 1 \) microstates in \( |\widehat{\pi}_r| \); such an orbit can be written as \( a_1*a_2*...*a_l*a_1 \) where the symbol \( * \) stands for the orbit elements (if any) not in \( \widehat{\pi}_r \), and \( a_1, a_2, ..., a_l \) are the orbit elements in \( \widehat{\pi}_r \). Note that in each subsegment of orbit \( a_i* \) there must be at least one microstate with strictly increasing entropy, and thus a total of \( l \) microstates with strictly increasing entropy. Taking all orbits that intersect \( |\widehat{\pi}_r| \) into account we obtain the desired inequality. It remains to show that \( \max_{\alpha \in S_X} |D(X, \pi, \alpha)| \leq |X| - |\widehat{\pi}_r| \), i.e. one has to check that there exists a permutation \( \alpha \) such that \( |D(X, \pi, \alpha)| = |X| - |\widehat{\pi}_r| \). Write the set \( X \) as in Figure 5 with the blocks \( \widehat{\pi}_k \) contained in left justified line and \( \widehat{\pi}_k \) above \( \widehat{\pi}_l \) if \( k > l \). Define the permutation \( \alpha \) by flowing downwards on each vertical column, and sending the bottom element of a column to the highest element in the column. The permutation \( \alpha \) obtained has exactly \( |\widehat{\pi}_r| \), microstates with increasing entropy. 

Figure 5 displays an example of a micro-macro dynamical systems for which the bound from Theorem 51 on the number of strict decreases in entropy is achieved. For a finite set \( X \) set

\[
d_X = \max_{\pi \in \text{Par}_X} \max_{\alpha \in S_X} |D(X, \pi, \alpha)|.
\]

Theorem 52. For any \( X \), we have that \( d_X = |X| - \min_{l \vdash |X|} \max_{1 \leq k \leq |X|} kl_k \), where \( l \) runs over the partitions of \( |X| \), i.e. \( l = (l_1, ..., l_{|X|}) \) and \( \sum_{i=1}^{|X|} kl_k = |X| \).

Proof. Follows from the identities \( d_X = \max_{\pi \in \text{Par}_X} \max_{\alpha \in S_X} |D(X, \pi, \alpha)| = \max_{\pi \in \text{Par}_X} |X| - |\widehat{\pi}_r| = |X| - \min_{\pi \in \text{Par}_X} |\widehat{\pi}_r| = |X| - \min_{l \vdash |X|} \max_{1 \leq k \leq |X|} kl_k. \)

Next we describe various scenarios guaranteeing or not the validity of property \( L_1 \) for arbitrary permutations on combinatorial micro-macro phase spaces under suitable hypothesis on the growth of \( |\widehat{\pi}_k| \). Note that both \( X \) and \( A \) grow to infinity, in subsequence sections we will let \( |X| \) go to infinity but keep the cardinality of \( A \) fixed.
Theorem 53. Let \( \{c_1, \ldots, c_n, \ldots\} \) and \( \{k_1 < \ldots < k_n < \ldots\} \) be a couple of sequences of natural numbers such that \( c_n k_n \leq c_{n+1} k_{n+1} \). For \( n \in \mathbb{N}_{\geq 1} \) consider the micro-macro phase space \((X_n, \pi_n)\) such that

\[
X_n = \bigcup_{s=1}^{n} [k_s]^{c_s},
\]

and \( \pi_n \) is the displayed partition of \( X_n \) with \( c_s \) blocks of cardinality \( k_s \) for \( s \in [n] \).

1. If \( c_n k_n \simeq an^r \) with \( r > 1 \), then there are permutations \( \alpha_n \in S_{X_n} \) such that

\[
\lim_{n \to \infty} \frac{|D(X_n, \pi_n, \alpha_n)|}{|X_n|} = 1.
\]

2. If \( c_n k_n \simeq ar^n \) with \( r > 1 \), then for arbitrary permutations \( \alpha_n \in S_{X_n} \) we have:

\[
\lim_{n \to \infty} \frac{|D(X_n, \pi_n, \alpha_n)|}{|X_n|} \leq \frac{1}{r}.
\]

3. If \( c_n k_n \simeq an^n \), then for arbitrary permutations \( \alpha_n \in S_{X_n} \) we have that:

\[
\lim_{n \to \infty} \frac{|D(X_n, \pi_n, \alpha_n)|}{|X_n|} = 0.
\]

Proof. We study the asymptotic behavior of

\[
\frac{1}{|X_n|} \max_{\alpha_n \in S_{X_n}} |D(X_n, \pi_n, \alpha_n)| \quad \text{as} \quad n \to \infty.
\]

Note that \( |\widehat{\pi}_{n,k_s}| = c_s k_s \) assume its largest value for \( s = n \). By Theorem 51 we have:

\[
\lim_{n \to \infty} \frac{1}{|X_n|} \max_{\alpha_n \in S_{X_n}} |D(X_n, \pi_n, \alpha_n)| = \lim_{n \to \infty} \frac{|X_n| - |\widehat{\pi}_{n,k_n}|}{|X_n|} = \lim_{n \to \infty} \frac{\sum_{s=1}^{n} |\widehat{\pi}_{n,k_s}|}{\sum_{s=1}^{n} |\widehat{\pi}_{n,k_s}|}.
\]
Under the three alternative hypothesis for $c_nk_n$ stated in the Theorem the limit $\lim_{n \to \infty} \frac{c_nk_n}{c_{n+1}k_{n+1}}$ exits. Therefore by the Stolz-Cesaro theorem we have that

$$\lim_{n \to \infty} \frac{1}{|X_n|} \max_{\alpha_n \in S\mathcal{X}_n} |D(X_n, \pi_n, \alpha_n)| = \lim_{n \to \infty} \frac{\sum_{s=1}^{n-1} |\pi_{n,k_s}|}{\sum_{s=1}^{n} |\pi_{n,k_s}|} = \lim_{n \to \infty} \frac{\sum_{s=1}^{n-1} c_{s,k_s}}{\sum_{s=1}^{n} c_{s,k_s}} = \lim_{n \to \infty} \frac{c_nk_n}{c_{n+1}k_{n+1}}.$$ 

The desired result follows since:

1. If $c_nk_n \simeq an^r$ with $r > 1$, then $\lim_{n \to \infty} \frac{c_nk_n}{c_{n+1}k_{n+1}} = 1$.

2. If $c_nk_n \simeq ar^n$ with $r > 1$, then $\lim_{n \to \infty} \frac{c_nk_n}{c_{n+1}k_{n+1}} = \frac{1}{r}$.

3. If $c_nk_n \simeq an^n$, then $\lim_{n \to \infty} \frac{c_nk_n}{c_{n+1}k_{n+1}} = 0$.

So polynomial growth for $c_nk_n$ yields no control on the number of strict decreases in entropy for arbitrary invertible micro-macro dynamical systems on $(X_n, \pi_n)$; exponential growth gives or not a good control on the number of strict decreases in entropy depending on the value of $r$; if $r$ is close to 1 exponential growth gives only a minor improvement over polynomial growth in terms of imposing property $L_1$; if $r$ is quite large, then any invertible micro-macro dynamical systems on $(X_n, \pi_n)$ have a relatively negligible set of strict decreases in entropy, for large $n$; the faster than exponential growth $n^n$ guarantees a vanishing numbers of decreases in entropy for an invertible micro-macro dynamical systems on $(X_n, \pi_n)$, for large $n$.

**Example 54.** For $n \in \mathbb{N}$, let $(P[n], [0, n], | |) = (P[n], \pi_n)$ be the micro-macro phase space with sending $A \subseteq [n]$ to its cardinality $|A|$. We show that for arbitrary permutations one has no control on the number of entropy decreasing microstates. In the odd case $(P[2n+1], \pi_{2n+1})$ we have that

$$O_{\pi_{2n+1}} = \left\{ \binom{2n+1}{k} \mid 0 \leq k \leq n \right\}$$

and

$$|\hat{\pi}_{2n+1}(\binom{2n+1}{k})| = 2 \binom{2n+1}{k}.$$ 

Since $|\hat{\pi}_{2n+1}(\binom{2n+1}{k})| \geq |\hat{\pi}_{2n+1}(\binom{2n+1}{k})|$, by Theorem 51 we have

$$\lim_{n \to \infty} \max_{\alpha \in S(P_{[2n+1]})} |D(P[2n+1], \pi_{2n+1}, \alpha)| = \lim_{n \to \infty} \frac{|D(P[2n+1]) - |\hat{\pi}_{2n+1}(\binom{2n+1}{k})|}{|P[2n+1]|} = 1 - \lim_{n \to \infty} \frac{2(2n+1)}{2n+1} = 1.$$ 

The even case $(P[2n], \pi_{2n})$ case is interesting since the block of larger cardinality does not lie in the zone of largest cardinality for $n \geq 3$. Indeed we have that

$$O_{\pi_{2n}} = \left\{ \binom{2n}{k} \mid 0 \leq k \leq n \right\}, \quad |\hat{\pi}_{2n}(\binom{2n}{k})| = \binom{2n}{k}, \quad |\hat{\pi}_{2n}(\binom{2n}{k})| = 2 \binom{2n}{k} \text{ for } 0 \leq k \leq n - 1.$$
Thus by Theorem 51 we have that
\[
\lim_{n \to \infty} \max_{\alpha \in S_{[2]}^{[n]}} |D(P[2n], \pi_{2n}, \alpha)| = \lim_{n \to \infty} \frac{|P[2n]| - |\hat{\pi}_{2n}(\pi_{2n})|}{|P[2n]|} = 1 - \lim_{n \to \infty} \frac{2(2n-1)}{2^{2n}} = 1.
\]

7 Local Arrow of Time and Zero Jump Permutations

In this section we introduce a couple of further formalizations of the arrow of time, one dealing with zones and the other one dealing with blocks. We introduce jump of a permutation on a micro-macro phase space, and study zero jump micro-macro dynamical systems. We introduce a further "continuity" restriction on permutations by given microstates the structure of a simple graph. Fix a micro-macro phase-space \((X, \pi)\) with \(O_\pi = \{k_1 < \cdots < k_o\}\).

**Definition 55.** Entropy defines a zonal \(\varepsilon\)-arrow of time on \((X, \pi, \alpha)\), written \((X, \pi, \alpha) \in \text{ZAT}(\varepsilon)\), if \(|I_{\hat{\pi}_i}| \geq (1 - \varepsilon)|\hat{\pi}_i|\) for \(i \in [o - 1]\). We say that \((X, \pi, \alpha)\) satisfy the \(L_3(\varepsilon_1, \varepsilon_2)\) property if it satisfies properties \(L_1(\varepsilon_1)\) and \(\text{ZAT}(\varepsilon_2)\).

**Theorem 56.** Let \((X, \pi)\) be a micro-macro phase-space and \(\delta_1, \delta_2, \gamma_i \in \mathbb{R}_{\geq 0}\) for \(i \in [o - 1]\) (\(\gamma_1 = 0\)) be such that

1. \(\delta_1 |X_{\text{eq}}| \leq |X_{\text{neq}}| \leq \delta_2 |X_{\text{eq}}|\) with \(\delta_2 \leq \varepsilon_1(\delta_1 + 1)\),
2. \(\sum_{j=1}^{i-1} |\hat{\pi}_j| \leq \gamma_i|\hat{\pi}_i|\) with \(\gamma_i \leq \varepsilon_2\) for \(2 \leq i \leq o - 1\),

then \((X, \pi, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)\) for \(\alpha \in S_X\) such that \(|C_{\hat{\pi}_i}| \leq (\varepsilon_2 - \gamma_i)|\hat{\pi}_i|\) for \(i \in [o - 1]\).

**Proof.** By Theorem 14 we have that \(\frac{|D|}{|X|} \leq \varepsilon_1\). The desired result holds since

\[
|D_{\hat{\pi}_i} \cup C_{\hat{\pi}_i}| = |D_{\hat{\pi}_i}| + |C_{\hat{\pi}_i}| \leq \sum_{j=1}^{i-1} |\hat{\pi}_j| + |C_{\hat{\pi}_i}| \leq (\gamma_i + \varepsilon_2 - \gamma_i)|\hat{\pi}_i| \leq \varepsilon_2|\hat{\pi}_i|.
\]

Under the hypothesis of Theorem 56 letting \(\lambda = \min_i \gamma_i^{-1}\) we have that \(|\hat{\pi}_i| \geq \lambda (1 + \lambda)^{i-2}|\pi_1|\) for \(2 \leq i \leq o - 1\). The following results are direct consequence of Theorem 56. Our next results illustrate quite well the principle of large differences: zone cardinalities even if relative negligible may actually be approaching infinity.
Corollary 57. Let \((X_n, \pi(n), \alpha_n)\) be a sequence of micro-macro dynamical systems with 
\[ O_{\pi(n)} = \{ k_1(n) < \cdots < k_o(n) \} \] 
and such that:

- \[ \sum_{j=1}^{i-1} \frac{|\pi_j(n)|}{|\pi_i(n)|} \to 0 \text{ as } n \to \infty, \text{ for } 2 \leq i \leq o, \]

- \[ \frac{|C_{\alpha_n} \pi_i(n)|}{|\pi_i(n)|} \to 0 \text{ as } n \to \infty, \text{ for } 2 \leq i \leq o - 1, \]

under these conditions \((X_n, \pi(n), \alpha_n) \in L_3(\varepsilon_1, \varepsilon_2)\).

Corollary 58. Let \((X_n, \pi(n), \alpha_n)\) be a sequence of micro-macro dynamical systems with 
\[ O_{\pi(n)} = \{ k_1(n) < \cdots < k_o(n) \} \] 
satisfying a large deviation principle in the sense that there number \(0 = z_0 < \cdots < z_1\) such that 
\[ |\pi_i(n)| \asymp e^{-n z_i}|X_n| \text{ and } |C_{\alpha_n} \pi_i(n)| \asymp e^{-n c_i}|X_n|. \]
Under these conditions \((X_n, \pi(n), \alpha_n) \in L_3(\varepsilon_1, \varepsilon_2)\).

Let \((X, f, A)\) be a micro-macro phase space, and let \(C \subseteq \text{prob}_A \subseteq \mathbb{R}^{|A|}\) be a non-empty convex subset of the space of probabilities on \(A\). Typically \(C\) is given as the subspace of \(\text{prob}_A\) satisfying linear constrains, i.e. one is given maps \(h_l : A \to \mathbb{R}\) and constants \(u_l \in \mathbb{R}\) for \(l \in [k]\) such that \(q \in C\) if and only if
\[ \sum_{a \in A} q(a) h_l(a) = u_l. \]

Let \(B = \{b_1, \ldots, b_o\}\) be a partition of \(C\) such that \(\text{int}(b_i) = b_i\), and consider a sequence of micro-macro dynamical systems \((X^n_C, L_n, B, \alpha_n)\) constructed as follows:

- Consider the map \(\hat{L}_n\) obtained as the composition of maps \(X^n \to A^n \to \text{prob}_A\), where the map \(A^n \to \text{prob}_A\) sends a tuple \(s \in A^n\) to the empirical probability distribution \(\hat{s}\) on \(A\) given by \(\hat{s}(a) = \frac{1}{n} |\{i \in [n] | s_i = a\}|.\)

- Set \(X_C^n = \hat{L}_n^{-1}C\), and let \(L_n\) be the composition of the of maps \(X^n_C \to C \to B\), where the first map is the restriction to \(X^n_C\) of \(\hat{L}_n\), and the second map is the coarse graining map induced by the partition \(B\).

- \(\alpha_n\) is a permutation on \(X_C^n\).

Relative entropy (Kullback-Leibler divergence) is the map \(D(\cdot | \cdot) : \text{prob}_A \times \text{prob}_A \to [0, \infty]\) given on \(r, q \in \text{prob}_A\) by
\[ D(r|q) = \sum_{a \in A} r(a) \ln \frac{r(a)}{q(a)}. \]

Let \(q_* \in C\) be the probability in \(C\) with minimum relative entropy \(D(q_*|p)\) with respect to the probability \(p\) on \(A\) given by \(p(a) = \frac{|a|}{|X|}\).
Theorem 59. Assume that the systems \((X^n, L_n, B, \alpha_n)\) are such that:

- \(\inf_{q \in b_o} D(q|p) < \inf_{q \in b_{o-1}} D(q|p) < \cdots < \inf_{q \in b_1} D(q|p),\)

- \(\frac{|C_{\alpha_i} b_i|}{|b_i|} \to 0 \text{ as } n \to \infty,\)

then \((X^n, B, L_n, \alpha_n) \in L_3.\)

Proof. Follows from Corollary 57 using Sanov’s theorem [12, 17], which implies for \(E \subseteq C\) with \(\text{int}(E) = E\) that:

\[
p(s \in E | s \in X^n_C) = \frac{p(s \in E, s \in X^n_C)}{p(X^n_C)} \simeq e^{-n \left( \inf_{q \in E} D(q|p) - \inf_{q \in C} D(q|p) \right)} = e^{-n \left( \inf_{q \in C} D(q|p) - D(q*|p) \right)}.
\]

Given a set with a partition on it we have a jump degree on maps from the set to itself. In the applications it is expected that the dynamics is given by a low jump map.

Definition 60. Let \((X, \pi, \alpha)\) be a micro-macro dynamical system. The jump of \(\alpha\) is the cardinality of its set of jumps \(J_\alpha\) given by

\[
J_\alpha = \prod_{i \in X} J_\alpha(i) = \prod_{i \in X} \{ |a| \mid a \in \pi \text{ and } |a| < |i|, |\alpha(i)| > \},
\]

where for \(n, m \in \mathbb{N}\) we set \(< n, m > = (n, m)\) if \(n \leq m\), and \(< n, m > = (m, n)\) if \(n > m\).

We call \(|J_\alpha(i)|\) the jump of \(\alpha\) at \(i \in X\), and let \(S^0_{X,\pi}\) be the set of zero jump permutations of \((X, \pi)\).

Proposition 61. Let \((X, \pi)\) be a micro-macro phase space and \(\alpha \in S^0_{X,\pi}\) then \(|I| = |D|.|\)

Proof. We show that the number of strict increases and the number of strict decreases on each \(\alpha\)-orbit are equal. Let \(i\) be a microstate with lowest entropy among the microstates in an \(\alpha\)-orbit. Assume that there are more strict increases than strict decreases in the \(\alpha\)-orbit of \(i\), a contradiction arises because the orbit can’t return to the microstate \(i\) as it will necessarily end up in a microstate of higher entropy since there are no jumps.

Corollary 62. Let \((X, \pi)\) be a micro-macro phase-space and \(\delta_1, \delta_2, \gamma_i \in \mathbb{R}_{\geq 0}\) for \(i \in [o - 1]\) \((\gamma_1 = 0)\) be such that

1. \(\delta_1|X^{eq}| \leq |X^{neq}| \leq \delta_2|X^{eq}| \) with \(\delta_2 \leq \varepsilon_1(\delta_1 + 1),\)

2. \(|\hat{\pi}_{i-1}| \leq \gamma_i|\hat{\pi}_i| \) with \(\gamma_i \leq \varepsilon_2\) for \(2 \leq i \leq o - 1,\)

then \((X, \pi, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)\) for \(\alpha \in S^0_{X,\pi}\) such that \(|C_{\alpha} \hat{\pi}_i| \leq (\varepsilon_2 - \gamma_i)|\hat{\pi}_i|\) for \(i \in [o - 1].\)
Proof. Follows from Theorem 14 we the inequalities

$$|D\tilde{\pi}_i \cup C\tilde{\pi}_i| = |D\tilde{\pi}_i| + |C\tilde{\pi}_i| \leq |\tilde{\pi}_{i-1}| + |C\tilde{\pi}_i| \leq (\gamma_i + \epsilon - \gamma_i)|\tilde{\pi}_i| \leq \epsilon|\tilde{\pi}_i|. \tag{1}$$

Under the hypothesis of Corollary 62 letting $\lambda = \min_1^{i=0} \gamma_i^{-1}$ we have that $|\tilde{\pi}_i| \geq \lambda^{-1}|\tilde{\pi}_1|$ for $i \in [a-1]$. Next we introduce the arrow of time in block form.

**Definition 63.** Entropy defines a block $\epsilon$-arrow of time on $(X, \pi, \alpha)$, written $(X, \pi, \alpha) \in \text{BAT}(\epsilon)$, if $|Ja| \geq (1-\epsilon)|a|$ for $a \in \pi_{\text{neq}}$. We say that $(X, \pi, \alpha)$ satisfy property $L_4(\epsilon_1, \epsilon_2)$ if it satisfies both $L_1(\epsilon_1)$ and $\text{BAT}(\epsilon_2)$.

Property $\text{BAT}(\epsilon)$ implies property $\text{ZAT}(\epsilon)$ since

$$|I\tilde{\pi}_i| = \sum_{a \in \pi, |a| = k_i} |Ja| \geq \sum_{|a| = k_i} (1-\epsilon)|a| = (1-\epsilon)|\tilde{\pi}_i|.$$

For our next results we assume that the microstates $X$ are the vertices of a simple graph $(X,E)$, i.e. $E$ is a family of subsets of $X$ of cardinality two. The macrostates $\pi$ acquire a simple graph structure $(\pi,E)$, where $\{a,b\} \in E$ if and only if there are microstates $i \in a$ and $j \in b$ with $\{i,j\} \in E$. Given $a \in \pi$ we set

$$Ba = \bigcup_{\{b \in \pi_{\text{neq}} | |b| < |a|, \{a,b\} \in E\}} b \subseteq X.$$ 

A permutation $\alpha \in S_X$ is called $E$-1-Lipschitz continuous if for $\{i,j\} \in E$ we have that either $\alpha(i) = \alpha(j)$ or $\{\alpha(i), \alpha(j)\} \in E$. Let $S_{X,E} \subseteq S_X$ be the set of $E$-1-Lipschitz continuous permutations.

**Theorem 64.** Let $(X, \pi)$ be a micro-macro phase-space and $(X,E)$ a simple graph, and $\delta_1, \delta_2, \gamma_a \in \mathbb{R}_{\geq 0}$ for $a \in \pi_{\text{neq}}$ ($\gamma_a = 0$ if $Ba = \emptyset$) be such that

1. $\delta_1 |X_{\text{neq}}| \leq |X_{\text{neq}}| \leq \delta_2 |X_{\text{neq}}|$ with $\delta_2 \leq \epsilon_1(\delta_1 + 1)$,
2. $|Ba| \leq \gamma_a |a|$ with $\gamma_a \leq \epsilon_2$ for $a \in \pi_{\text{neq}},$

then $(X,f,A,\alpha) \in L_4(\epsilon_1, \epsilon_2)$ for any $E$-1-Lipschitz continuous permutation $\alpha \in S_{X,E}$ such that $|Ca| \leq (\epsilon_2 - \gamma_a)|a|$ for $a \in \pi_{\text{neq}}$.

**Proof.** Follows from Theorem 13 and the inequalities

$$|Da \cup Ca| = |Da| + |Ca| \leq |Ba| + |Ca| \leq (\gamma_a + \epsilon_2 - \gamma_a)|a| \leq \epsilon_2 |a|. \tag{2}$$
Corollary 65. Let \((X, \pi, E)\) be a micro-macro phase-space with \((X, E)\) a simple graph, and \(\delta_1, \delta_2, \gamma_a \in \mathbb{R}_{\geq 0}\) for \(a \in \pi^\text{neq}\) (\(\gamma_a = 0\) if \(B_a = \emptyset\)) be such that

1. \(\delta_1 |\mathcal{X}^\text{eq}| \leq |X^\text{eq}| \leq \delta_2 |\mathcal{X}^\text{eq}|\) with \(\delta_2 \leq \varepsilon_1(\delta_1 + 1)\),
2. \(|F a| \leq \gamma_a|a|\) with \(\gamma_a \leq \varepsilon_2\) for \(a \in A^\text{neq}\), where

\[
F a = \bigcup \{b \in \pi \mid |b| < |a|, \{a, b\} \in \mathcal{E}, \{(b, |a|)\} = \emptyset\}
\]

then \((X, \pi, \alpha) \in L_4(\varepsilon_1, \varepsilon_2)\) for any zero jump \(E\)-1-Lipschitz continuous permutation \(\alpha \in S_{X, \pi}^0\) such that \(|Ca| \leq (\varepsilon_2 - \gamma_a)|a|\) for \(a \in \pi^\text{neq}\).

8 Orbit Properties and the Equilibrium Reaching Time

By design the equilibrium plays a priori no distinguished role in properties \(L_i\). Localizing to orbits suggest further interesting properties inspired by the Gibbs description of the second law for which the equilibrium plays a main role. All definitions and constructions in this section can be weakened by allowing a set of badly behaved orbits not having the required properties, with a small parameter bounding the probability that a microstate be in such orbits. Within this more general framework all arguments given in this section should be thought as applying generically, i.e. to the complement of the bad orbits. We begin by defining, for equilibrium bound systems, a strictly increasing function on non-equilibrium macrostates.

Definition 66. A micro-macro dynamical system \((X, A, f, \alpha)\) is equilibrium bound if each \(\alpha\)-orbit intersects \(X^\text{eq}\). For such systems the equilibrium reaching time map \(e : X \rightarrow \mathbb{N}\) is given by \(e(i) = \text{smallest } k \in \mathbb{N}\) such that \(\alpha^k(i) \in X^\text{eq}\). Set \(E = \max_{i \in X} e(i)\).

Theorem 67. Let \((X, A, f, \alpha)\) be an equilibrium bound micro-macro dynamical system.

1. The map \(e : X \rightarrow [0, E]\) is strictly decreasing on \(X^\text{neq}\) and has value 0 on \(X^\text{eq}\).
2. If \((X, A, f, \alpha) \in L_1(\varepsilon)\), then the probability that \(e\) be strictly decreasing is less than \(\varepsilon\), and the average jump of \(e\) is less than \((|A| - 2)\varepsilon\).
3. Consider the micro-macro dynamical system \(X_e = (X, [0, E], e, \alpha)\) where we assume that \(\alpha\) is invertible, \((X, A, f, \alpha) \in L_1(\varepsilon)\), and \(\alpha X^\text{eq} \cap e^{-1}(k) \neq \emptyset\) for \(k \in [0, E]\). Then \((X, [0, E], e, \alpha) \in L_2(\varepsilon, 0)\).

Proof. For \(i \in X^\text{neq}\) we have that \(e(i) > 0\) and \(\alpha^{e(i) - 1}(\alpha(i)) = \alpha^{e(i)}(i) \in X^\text{eq}\), thus \(e(\alpha(i)) \leq e(i) - 1 < e(i)\). Under the hypothesis of item 2 we have that

\[
\frac{|\{i \in X \mid e(\alpha(i)) > e(i)\}|}{|X|} = \frac{|DX^\text{eq}|}{|X|} \leq \frac{|D|}{|X|} \leq \varepsilon.
\]
Regarding the average jump of $e$ we have that:

$$\frac{1}{|X|} \sum_{i \in X} |J(i)| = \frac{1}{|X|} \sum_{i \in DX^{eq}} |J(i)| \leq \frac{(|A| - 2)|DX^{eq}|}{|X|} \leq \frac{(|A| - 2)|DX|}{|X|} \leq (|A| - 2)e.$$  

We show item 3. Note first that $e^{-1}(k) \neq \emptyset$, since $e^{-1}(E) \neq \emptyset$ by definition, and choosing $i \in e^{-1}(E)$ we have that $\alpha^{E-k}(i) \in e^{-1}(k)$. Note also that $X_{neq}^{eq} = X^{eq}$ and $X_{eq}^{eq} = X^{eq}$. The restriction map $\alpha : e^{-1}(k) \rightarrow e^{-1}(k - 1)$ is injective, thus $|e^{-1}(k)| \leq |e^{-1}(k - 1)|$. Moreover $e^{-1}(k - 1) = \alpha(e^{-1}(k)) \cup \alpha(X^{eq}) \cap e^{-1}(k - 1)$ and thus for $k \in [1, E]$ and $i \in e^{-1}(k)$ we have $S(i) = \ln|e^{-1}(k)| < \ln|e^{-1}(k - 1)| = S(\alpha(i))$. Finally, by item 2 we have that

$$\frac{|DX_e|}{|X|} = \frac{|\{i \in X \mid e(\alpha(i)) > e(i)\}|}{|X|} \leq \varepsilon.$$

\[ \square \]

**Remark 68.** The condition $\alpha X^{eq} \cap e^{-1}(k) \neq \emptyset$ is quite natural for reversible systems since, in this case, the image of entropy decreasing equilibrium microstates nearly covers all non-equilibrium macrostates (Lemma 24, Theorem 20), and thus it is reasonable to expect that $|e^{-1}(k)|$ and $|\alpha X^{eq} \cap e^{-1}(k)|$ be nearly equal.

**Remark 69.** It is worthwhile to analyze Theorem 67 in the light of Zermelo’s critique of the Boltzmann $H$-theorem 50, 51. Zermelo pointed out that a recurring system does not admit a non-constant always decreasing function along orbits (Proposition 16), and thus regardless of further details the main claim of the $H$-theorem can not be correct. Boltzmann accepts the argument but claims that the $H$-theorem remains valid if understood as a probabilistic statement, i.e. allowing the $H$-function to be strictly increasing with low probability. The equilibrium reaching time function $e$ satisfies, under the conditions of Theorem 67 probabilistic properties similar to those expected for the $H$-function, according to Boltzmann, indeed it satisfies stronger properties as it is strictly increasing on non-equilibrium microstates.

**Remark 70.** The Loschmidt’s critique of the $H$-theorem 7, 8, 9, 50 has its combinatorial counterpart in Proposition 29: if the $H$-function is defined on a reversible system (it is not if dynamics is defined via the Boltzmann equation, but it should be if dynamics is defined mechanically), then since it is reversion invariant it must have an equal number of increasing and decreasing microstates, contrary to the claim that it is predominantly decreasing. Boltzmann acknowledges the argument, but points out that the $H$-decreasing orbit segments are the ones that actually show up in nature, i.e. the probabilistic symmetry of microstates is broken. This observation is the origin of the low entropy past hypothesis. The function $e$ can be constructed for equilibrium bound reversible systems as well; as a rule it will not be reversion invariant, indeed if $r$ preserves equilibria, i.e. $r$ restricts to a map $r : X^{eq} \rightarrow X^{eq}$, then $e(ri) = e(i)$ if and only if

smallest $k \in \mathbb{N}$ such that $\alpha^k(i) \in X^{eq} = \text{smallest } k \in \mathbb{N} \text{ such that } \alpha^{-k}(i) \in X^{eq},$
a trivial condition for \( i \in X^\text{eq} \) but fairly restrictive for \( i \in X^\text{neq} \). In fact it holds only for the middle microstate on each maximal \( \alpha \)-orbit segment of odd cardinality in \( X^\text{neq} \).

We proceed to localize to \( \alpha \)-orbits the various properties formalizing the second law previously introduced. Let \( \text{Orb}(\alpha) \) be the set of \( \alpha \)-orbits.

**Definition 71.** Let \((X, \pi, \alpha)\) be a micro-macro dynamical system with \( O_{\pi} = \{k_1 < \cdots < k_o\} \).

1. \((X, \pi, \alpha) \in G_0(\varepsilon)\) if and only if \(|c^\text{eq}| \geq (1 - \varepsilon)|c|\) for \( c \in \text{Orb}(\alpha) \).
2. \((X, \pi, \alpha) \in G_1(\varepsilon)\) if and only if \(|Dc| \leq \varepsilon|c|\) for \( c \in \text{Orb}(\alpha) \).
3. \((X, \pi, \alpha) \in G_2(\varepsilon_1, \varepsilon_2)\) if and only if \(|c^\text{eq}| \geq (1 - \varepsilon_1)|c|\) and \(|Ic| \geq (1 - \varepsilon_2)|c^\text{eq}|\) for \( c \in \text{Orb}(\alpha) \).
4. \((X, \pi, \alpha) \in G_3(\varepsilon_1, \varepsilon_2)\) if and only if \(|c^\text{eq}| \geq (1 - \varepsilon_1)|c|\) and \(|I(\pi_i \cap c)| \geq (1 - \varepsilon)|\pi_i \cap c|\) for \( i \in [o - 1] \) and \( c \in \text{Orb}(\alpha) \).
5. \((X, \pi, \alpha) \in G_4(\varepsilon_1, \varepsilon_2)\) if \(|c^\text{eq}| \geq (1 - \varepsilon_1)|c|\) and \(|I(a \cap c)| \geq (1 - \varepsilon_2)|a \cap c|\) for \( a \in \pi^\text{neq} \) and \( c \in \text{Orb}(\alpha) \).

**Theorem 72.** Let \((X, \pi, \alpha)\) be an invertible equilibrium bound micro-macro dynamical system satisfying property \( G_0(\varepsilon) \), and let \( r, e : X \to \mathbb{N} \) be the first return time and the equilibrium reaching time maps, respectively. We have that \( \frac{T}{r} \leq \varepsilon \).

**Proof.** By definition the maps \( r, e : X \to \mathbb{N} \) are such that \( r(i) = |c| \) if \( i \in c \in \text{Cyc}(\alpha) \), and \( e(i) \) is the smallest \( k \in \mathbb{N} \) with \( \alpha^k(i) \in X^\text{eq} \). Since \( e(i) \leq |c^\text{eq}| \leq \varepsilon|c| \) for \( i \in c \), we have

\[
\frac{T}{r} = \frac{1}{|X|} \sum_{i \in X} e(i) r(i) = \frac{1}{|X|} \sum_{c \in \text{Cyc}} \sum_{i \in c} e(i) r(i) \leq \frac{1}{|X|} \sum_{c \in \text{Cyc}} \varepsilon|c| |c| = \frac{\varepsilon}{|X|} \sum_{c \in \text{Cyc}} |c| = \varepsilon.
\]

\( \blacksquare \)

**Remark 73.** Theorem 72 is consistent with Boltzmann’s response to Zermelo’s critique of his \( H \)-theorem: recurrence, even if it holds for all microstates, occurs long after a microstate have evolved to the equilibrium where it remains for a long period of time, making recurrence of little practical importance. We leave open the problem of determining if an analogue of Theorem 72 holds when \( r \) is replaced by the first block return map, or by the first zone return map.

The proofs of the following results are similar to those of Theorems 20, 21 and 64.

**Theorem 74.** Let \((X, \pi, \alpha)\) be an invertible micro-macro dynamical system.

1. If \((X, \pi, \alpha) \in G_0(\varepsilon)\), then \(|X^\text{eq}| \geq (1 - \varepsilon_1)|X|\) and \((X, \pi, \alpha) \in G_1(\varepsilon)\).
2. If \((X, \pi, \alpha) \in G_1(\varepsilon)\), then \((X, \pi, \alpha) \in L_1(\varepsilon)\).
3. If \((X, \pi, \alpha) \in G_i(\varepsilon_1, \varepsilon_2)\), then \((X, \pi, \alpha) \in L_i(\varepsilon_1, \varepsilon_2)\), for \(i = 2, 3, 4\).

4. If \(\alpha\) is a zero-jump permutation, then \(|Dc^{eq}| \geq (1 - 2\varepsilon_2)|c^{neq}|\).

5. If \((X, \pi, \alpha) \in G_0(\varepsilon_1)\), and \(|Dc^{eq}| \geq (1 - \varepsilon_2)|c^{neq}|\) for all cycles \(c \in \text{Cyc}(\alpha)\), then \((X, \pi, \alpha) \in G_2(\varepsilon_1, \varepsilon_2)\).

6. Let \((X, E)\) be a simple graph and \(\pi\) be another partition on \(X\). Let \(P\) be the set of pairs \((a, s) \in \pi \times \sigma\) such that \(a \cap s \neq \emptyset\), and let \(\mathcal{E}\) be the simple graph on \(P\) such that there is an edge between \((a, s)\) and \((b, t)\) in \(\mathcal{E}\) if and only if \(s = t\) and there are microstates \(i \in a \cap s\) and \(j \in b \cap s\) such that \(i \neq j\) \(\in E\). For \((a, s) \in P\) set

\[M_{as} = \bigcup_{\{(b, s) \in P \mid |b| < |a|, \{(a, b, s)\} \in \mathcal{E}\}} b \cap s \subseteq X.\]

Assume that \((X, \pi, \alpha) \in G_0(\varepsilon_0)\), \(\text{Cyc}(\alpha) = \sigma\), \(\alpha\) is \(E\)-1-Lipschitz continuous, there are constants \(\delta_1, \delta_2, \gamma_{as} \in \mathbb{R}_{\geq 0}\) for \(a \in \pi^{neq}\), \((a, s) \in P\) (with \(\gamma_{as} = 0\) if \(M_{as} = \emptyset\)) such that \(|M_{as}| \leq \gamma_{as}|a \cap s|\) with \(\gamma_{as} \leq \varepsilon_2\), and \(|C(a \cap s)| \leq (\varepsilon_2 - \gamma_{ac})|a \cap s|\), then we have that \((X, \pi, \alpha) \in G_d(\varepsilon_1, \varepsilon_2)\).

9 Second Law and Convex Geometry

In this section we show that several problems arising from the combinatorial formalizations of the second law can be equivalently reformulated as problems in convex geometry and integer programming \([40, 52]\), namely the problem of computing integer sums over lattice points in convex polytopes. This equivalence allows us to analyze a few simple but interesting examples, and provides a pathway towards numerical computations. Given a convex polytope \(P \subseteq \mathbb{R}^d\) we set \(P^\mathbb{Z} = P \cap \mathbb{Z}^d\).

Let \((X, \pi)\) be a micro-macro dynamical system with \(O_\pi = \{k_1 < \ldots < k_o\}\) and consider integers \(0 \leq d, e \leq |X| - |\hat{\pi}_r|\), where \(r\) is such that \(|\hat{\pi}_r| \geq |\hat{\pi}_i|\) for \(i \in [o]\). Let \(\Lambda_d \subseteq \mathbb{R}_{\geq 0}^o\) be the convex polytope given by
\[
\sum_{j=1}^{o} x_{ji} = |\hat{\pi}_i|, \quad \sum_{j=1}^{o} x_{ij} = |\hat{\pi}_i|, \quad \sum_{i< j} x_{ij} = d.
\]

Let \(\Lambda_d^e \subseteq \mathbb{R}_{\geq 0}^o\) be the convex polytope given by
\[
\sum_{j=1}^{o} x_{ji} = |\hat{\pi}_i|, \quad \sum_{j=1}^{o} x_{ij} = |\hat{\pi}_i|, \quad \sum_{i<j} x_{ij} = d, \quad \sum_{i>j} x_{ij} = e.
\]

Let \(\Upsilon_d \subseteq \mathbb{R}_{\geq 0}^{o-1}\) be the convex polytope given by
\[
\sum_{i=1}^{o-1} x_i = d, \quad x_1 \leq |\hat{\pi}_1|, \quad x_{o-1} \leq |\hat{\pi}_o|, \quad x_{i-1} + x_i \leq |\hat{\pi}_i| \text{ for } 2 \leq i \leq o - 1.
\]
Theorem 75. Let $(X, \pi)$ be a micro-macro dynamical system with $O_\pi = \{k_1 < ... < k_o\}$.

1. A random permutation in $S_X$ has $d$ strict decreases in entropy with probability
   \[
   \left(\frac{|X|}{|\hat{\pi}_1| \cdots |\hat{\pi}_o|}\right)^{-1} \sum_{a \in \Lambda^o_d} \prod_{i=1}^o \left(\frac{|\hat{\pi}_i|}{a_{1i}, \ldots, a_{oi}}\right).
   \]

2. A random permutation in $S_X$ has $d$ strict decreases and $e$ strict increases in entropy with probability
   \[
   \left(\frac{|X|}{|\hat{\pi}_1| \cdots |\hat{\pi}_o|}\right)^{-1} \sum_{a \in \Lambda^e_d} \prod_{i=1}^o \left(\frac{|\hat{\pi}_i|}{a_{1i}, \ldots, a_{oi}}\right).
   \]

3. A random permutation in $S^0_{X,\pi}$ has $d$ strict decreases in entropy (and thus $d$ strict increases) with probability
   \[
   \frac{|\hat{\pi}_1|!^2 \cdots |\hat{\pi}_o|!^2}{|S^0_{X,\pi}|} \sum_{a \in \Upsilon^2_d} \prod_{i=1}^o a_{1i}!^2 (\pi_i - a_i - a_{i-1})^{-1},
   \]
   where we set $a_0 = 0$ and $a_o = 0$.

Proof. Items 1 and 2 are similar. For item 1, it is enough to show that the number of permutations $\alpha \in S_X$ such that $(X, \pi, \alpha)$ has $d$ strict decreases in entropy is given by
   \[
   \prod_{i=1}^o |\hat{\pi}_i|! \sum_{a \in \Lambda^o_d} \prod_{i=1}^o \left(\frac{|\hat{\pi}_i|}{a_{1i}, \ldots, a_{oi}}\right).
   \]
A permutation $\alpha \in S_X$ determines the matrix $(a_{ij}) \in \mathbb{N}^{o^2}$ given by
   \[
   a_{ij} = |\{s \in \hat{\pi}_j \mid \alpha(s) \in \hat{\pi}_i\}|.
   \]
The desired result follows from the identities
   \[
   \sum_{j=1}^o a_{ji} = |\hat{\pi}_i|, \quad \sum_{j=1}^o a_{ij} = |\hat{\pi}_i|, \quad |D(X, \pi, \alpha)| = \sum_{i<j} a_{ij}.
   \]
Moreover each matrix $(a_{ij}) \in \mathbb{N}^{o^2}$ satisfying the left and center identities above comes from a permutation $\alpha \in S_X$. Indeed, there are
   \[
   \prod_{i,j=1}^o a_{ij}! \prod_{i=1}^o \left(\frac{|\hat{\pi}_i|}{a_{1i} \cdots a_{oi}}\right)^{a_{1i} \cdots a_{oi}} = \prod_{i=1}^o |\hat{\pi}_i|! \left(\frac{|\hat{\pi}_i|}{a_{1i} \cdots a_{oi}}\right)
   \]
permutations with $(a_{ij})$ as their associated matrix. Item 3 follows from item 2 after setting $x_i = x_{i,i+1}$ for $i \in [o - 1]$, taking into account that $x_{i,i+1} = x_{i+1,i}$, and $x_{ij} = 0$ unless $i = j$, $i = j + 1$, or $i = j - 1$. 

\[\Box\]
Examples 76 and 77 below show that a random permutation is more likely to have exactly one decrease in entropy than being always increasing in entropy.

Example 76. Under the hypothesis of Theorem 75 set $d = 0$. If $(a_{ij}) \in \Lambda^Z_{\pi,0}$, then by definition $a_{ij} = 0$ for $i < j$, which implies that $a_{ij} = |\hat{\pi}_i| \delta_{ij}$. Thus there are

$$
\prod_{i=1}^o |\hat{\pi}_i|! \sum_{a \in \Lambda^Z_0} \left( |\hat{\pi}_1| \overline{a_{11}...a_{ol}} \right) = \prod_{i=1}^o |\hat{\pi}_i|! \left( |\hat{\pi}_i| \overline{0...|\hat{\pi}_i|...0} \right) = \prod_{i=1}^o |\hat{\pi}_i|!
$$

entropy preserving permutations. We have recovered Theorem 48.

Example 77. Under the hypothesis of Theorem 75 set $d = 1$. A matrix $(a_{ij}) \in \Lambda^Z_1$ is uniquely determined by a set of indices $\{i_1 < ... < i_l\} \subseteq [o]$, with $2 \leq l \leq o$, such that $a_{ij} = 0$ for $i \neq j$, except for $a_{i_1i_l} = a_{i_2i_{l-1}} = \cdots = a_{i_{l-1}i_1} = 1$. Thus there are

$$
\prod_{i=1}^o |\hat{\pi}_i|! \sum_{\{i_1 < ... < i_l\} \subseteq [o]} \left( 0...\overline{1_{i_1}}...|\hat{\pi}_{i_l}|...0 \right) \prod_{s=1}^{l-1} \left( 0...|\hat{\pi}_{i_s}|...0 \right) = \prod_{i=1}^o |\hat{\pi}_i|!
$$

permutations with exactly one strict decrease in entropy.

Theorem 78. Let $(X, \pi)$ be a micro-macro phase space with $O_\pi = \{k_1 < k_2\}$ and $|\hat{\pi}_1| < |\hat{\pi}_2|$. Then

1. A random permutation on $X$ has $d$ strict decreases in entropy with probability

$$
\binom{|X|}{|\hat{\pi}_1|}^{-1} \binom{|\hat{\pi}_1|}{d} \binom{|\hat{\pi}_2|}{d}.
$$

2. A random permutation is most likely to have

$$
\frac{|\hat{\pi}_1||\hat{\pi}_2|}{|\hat{\pi}_1| + |\hat{\pi}_2| + 2}
$$

strict decreases in entropy.

3. If $|\hat{\pi}_1|$ is fixed and $|\hat{\pi}_2|$ grows to infinity, then a random permutation is most likely to have a relatively vanishing number of strict decreases in entropy.

4. If $|\hat{\pi}_2| = c|\hat{\pi}_1|$ and $|\hat{\pi}_1|$ grows to infinity, then a random permutation is most likely to have a relative number of strict decreases entropy of

$$
\frac{c}{(1+c)|\hat{\pi}_1|}
$$

5. If $|\hat{\pi}_2| = c|\hat{\pi}_1|^{s}$ with $s > 1$ and $|\hat{\pi}_1|$ grows to infinity, then a random permutation is most likely to have a relatively vanishing number of strict decreases in entropy.

6. If $|\hat{\pi}_2| = ce^{|\hat{\pi}_1|}$ and $|\hat{\pi}_1|$ grows to infinity, then a random permutation is most likely to have a relatively vanishing number of strict decreases in entropy.

44
Proof. A permutation on X with d decreases in entropy gives rise to a matrix \((a_{ij})\) with

\[
a_{12} = d, \quad a_{11} + a_{21} = |\pi_1|, \quad a_{11} + a_{12} = |\pi_1|, \quad a_{12} + a_{22} = |\pi_2|, \quad \text{and} \quad a_{21} + a_{22} = |\pi_2|.
\]

Thus \(0 \leq d \leq |\pi_1|, \quad a_{12} = a_{21} = d, \quad a_{11} = |\pi_1| - d, \quad \) and \(a_{22} = |\pi_2| - d, \quad \) and there are

\[
|\pi_1|!! \frac{|\pi_1|!}{|\pi_2|!} \left( \frac{|\pi_1|}{d} \right) \left( \frac{|\pi_2|}{d} \right)
\]

permutations of \(X\) with \(d\) strict decreases in entropy.

To find out the most likely number of strict decreases in entropy, we should find the integer \(0 \leq d \leq a\) for which the product \(\frac{a!}{(a-k)!k!}\) achieves its maximum. It is not hard to check that

\[
\frac{\binom{a}{k}}{\binom{a+k}{k+1}} \leq 1 \quad \text{is equivalent to} \quad d \leq \frac{a^2 + ak}{2a + k + 2}.
\]

Therefore a random permutation is most likely to have

\[
d \simeq \frac{|\pi_1|^2 + |\pi_1|(|\pi_2| - |\pi_1|)}{2|\pi_1| + |\pi_2| - |\pi_1| + 2} = \frac{|\pi_1||\pi_2|}{|\pi_1| + |\pi_2| + 2}.
\]

If \(|\pi_1|\) is fixed and \(|\pi_2|\) grows to infinity then a permutation is most likely to have \(d \simeq |\pi_1|\) strict increases in entropy, thus \(\frac{|\pi_1|}{|\pi_1| + |\pi_2|} \simeq \frac{|\pi_1|}{|\pi_2|} \simeq 0\). If \(|\pi_2| = c|\pi_1|\) grows to infinity, we have that

\[
d \simeq \frac{c|\pi_1|}{1 + c} \quad \text{and} \quad \frac{d}{|\pi_1| + |\pi_2|} \simeq \frac{c}{(1 + c)^2}.
\]

If \(|\pi_2| = |\pi_1|^s\) with \(s > 1\) and \(|\pi_1|\) growing to infinity, we get that the most likely number of strict decreases in entropy is given by

\[
d \simeq \frac{c|\pi_1|^{s+1}}{|\pi_1| + c|\pi_1|^s + 2} \simeq |\pi_1|, \quad \text{thus} \quad \frac{d}{|\pi_1| + c|\pi_1|^s} \simeq \frac{1}{c|\pi_1|^{s-1}} \simeq 0.
\]

If \(|\pi_2| = e|\pi_1|\) and \(|\pi_1|\) growing to infinity, we get that the most likely number of strict decreases in entropy is given by

\[
d \simeq \frac{c|\pi_1|e|\pi_1|}{|\pi_1| + ce|\pi_1| + 2} \simeq |\pi_1|, \quad \text{thus} \quad \frac{|\pi_1|}{|\pi_1| + ce|\pi_1|} \simeq \frac{|\pi_1|}{c} e^{-|\pi_1|} \simeq 0.
\]

Consider again a micro-macro phase space \((X, \pi)\) with \(O_\pi = \{k_1 < \ldots < k_o\}\), and let\(\Psi_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}^{g^2}_{\geq 0}\) be the convex polytope given by

\[
\sum_{j=1}^o x_{ji} = |\pi_i|, \quad \sum_{j=1}^o x_{ij} = |\pi_i|, \quad \sum_{i<j} x_{ij} \leq \varepsilon_1 |X|, \quad \sum_{i>j} x_{ij} \geq (1 - \varepsilon_2) |X^{\text{neq}}|.
\]
Let $\Theta_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}_{\geq 0}^2$ be the convex polytope given by
\[
\sum_{j=1}^{o} x_{ji} = |\pi_i|, \quad \sum_{j=1}^{o} x_{ij} = |\pi_i|, \quad \sum_{i<j} x_{ij} \leq \varepsilon_1 |X|, \quad \sum_{i>j} x_{ij} \geq (1-\varepsilon_2) |\pi_j|.
\]

Let $\Omega_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}_{\geq 0}^{o-1}$ be the convex polytope given by
\[
\sum_{i=1}^{o-1} x_i \leq \varepsilon_1 |X|, \quad x_1 \leq |\pi_1|, \quad x_{o-1} \leq |\pi_o|, \quad x_i+1 \leq |\pi_i| \quad \text{and} \quad x_i \geq (1-\varepsilon_2) |\pi_i|.
\]

**Theorem 79.** Let $(X, \pi)$ be a micro-macro dynamical system with $O_{\pi} = \{k_1 < \ldots < k_o\}$.

- A random permutation $\alpha \in S_X$ determines a system $(X, \pi, \alpha) \in L_2(\varepsilon_1, \varepsilon_2)$ with probability
  \[
  \left( \frac{|X|}{|\pi_1|, \ldots, |\pi_o|} \right)^{-1} \sum_{a \in \Psi_{\varepsilon_1, \varepsilon_2}} \prod_{i=1}^{o} \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right).
  \]

- A random permutation $\alpha \in S_X$ determines a system $(X, \pi, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)$ with probability
  \[
  \left( \frac{|X|}{|\pi_1|, \ldots, |\pi_o|} \right)^{-1} \sum_{a \in \Theta_{\varepsilon_1, \varepsilon_2}} \prod_{i=1}^{o} \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right).
  \]

- A random permutation in $S_{X, \pi}^0$ determines a system $(X, \pi, \alpha) \in L_3(\varepsilon_1, \varepsilon_2)$ with probability
  \[
  \frac{|\pi_1|!^2 \ldots |\pi_o|!^2}{|S_{X, \pi}^0|} \sum_{a \in \Omega_{\varepsilon_1, \varepsilon_2}} \left( \prod_{i=1}^{o} a_i!^2(\pi_i - a_i - a_{i-1})! \right)^{-1},
  \]

where we set $a_0 = 0$ and $a_o = 0$.

The associated matrix of a reversible system is symmetric
\[
a_{ij} = |\{s \in \pi_j | \alpha(s) \in \pi_i\}| = |\{s \in \pi_i | \alpha(s) \in \pi_j\}| = a_{ji},
\]
since the map $\alpha : \{s \in \pi_j | \alpha(s) \in \pi_i\} \rightarrow \{s \in \pi_i | \alpha(s) \in \pi_j\}$ is a bijection. So it is interesting to consider permutations for which the symmetry condition $a_{ij} = a_{ji}$ holds. We call such systems symmetric and let $\mathcal{S}_{X, \pi}$ be the set of symmetric permutations on $X$. Note that zero jump permutations are symmetric. Let $\Sigma_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}_{\geq 0}^{(o+1)}$ be the convex polytope given on $x_{ij}$ with $1 \leq i \leq j \leq o$ by
\[
\sum_{j=1}^{i} x_{ji} + \sum_{j=i+1}^{o} x_{ij} = |\pi_i|, \quad \sum_{i<j} x_{ij} \geq (1-\varepsilon_2) |\pi_i| \quad \text{for} \quad i \in [o-1], \quad \sum_{i<j} x_{ij} \leq \varepsilon_1 |X|.
\]

46
Theorem 80. Let $(X, \pi)$ be a micro-macro dynamical system with $O_\pi = \{k_1 < \ldots < k_o\}$. A random invertible symmetric system $(X, \pi, \alpha)$ has property $L_2(\varepsilon_1, \varepsilon_2)$ with probability

$$\prod_{i=1}^o |\pi_i|! \sum_{a \in \Sigma_{\varepsilon_1, \varepsilon_2}^o} \prod_{i=1}^o \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right).$$

Let $\Phi_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}^{o^2}_{\geq 0}$ be the convex polytope given by

$$\sum_{j=1}^o x_{ji} + x_{ij} = 2|\pi_i|, \quad (1 - 2\varepsilon_1)|X| \leq \sum_i x_{ii} \leq 2(1 - \varepsilon_2)|X^{eq}| - (1 - 2\varepsilon_2)|X|.$$

The following result follows from Theorem 25.

Theorem 81. Let $(X, \pi)$ be a micro-macro dynamical system with $O_\pi = \{k_1 < \ldots < k_o\}$.

1. A random permutation $\alpha \in S_X$ determines an (invariant, equivariant) reversible system $IR(X, \pi, \alpha)$ in $L_2(\varepsilon_1, \varepsilon_2)$ with probability

$$\left( \frac{|X|}{|\pi_1|, \ldots, |\pi_o|} \right)^{-1} \sum_{a \in \Phi_{\varepsilon_1, \varepsilon_2}^o} \prod_{i=1}^o \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right).$$

2. If $O_\pi = \{k_1 < k_2\}$, then a random permutation $\alpha \in S_X$ determines an (invariant, equivariant) reversible system $IR(X, \pi, \alpha)$ in $L_2(\varepsilon_1, \varepsilon_2)$ with probability

$$\left( \frac{|X|}{|\pi_1|} \right)^{-1} \sum_{(1 - \varepsilon_2)|\pi_1| \leq d \leq \min(|\pi_1|, |\pi_2|, |\pi_3|, |X|)} \left( \frac{|\pi_1|}{|\pi_2|} \right).$$

Let $\Theta_{\varepsilon_1, \varepsilon_2} \subseteq \mathbb{R}^{o^2}_{\geq 0}$ be the convex polytope given by

$$\sum_{j=1}^o x_{ij} + x_{ji} = 2|\pi_i|, \quad \sum_{j=1}^{o-1} x_{ijo} + x_{oj} \geq 2(1 - \varepsilon_2)|X^{neq}|.$$

Next result is a consequence of Theorems 20 and 21.

Theorem 82. Let $(X, \pi)$ be micro-macro phase space with $|X^{eq}| \geq (1 - \varepsilon_1)|X|$. A random permutation $\alpha \in S_X$ determines an (invariant, equivariant) reversible system $R(X, \pi, \alpha)$ in $L_2(\varepsilon_1, \varepsilon_2)$ with probability greater than

$$\left( \frac{|X|}{|\pi_1|, \ldots, |\pi_o|} \right)^{-1} \sum_{a \in \Theta_{\varepsilon_1, \varepsilon_2}^o} \prod_{i=1}^o \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right),$$

and less than

$$\left( \frac{|X|}{|\pi_1|, \ldots, |\pi_o|} \right)^{-1} \sum_{a \in \Theta_{\varepsilon_1, \varepsilon_2}^o} \prod_{i=1}^o \left( \frac{|\pi_i|}{a_{1i}, \ldots, a_{oi}} \right).$$
We close this section describing property $L_4(\varepsilon_1, \varepsilon_2)$ in terms of convex polytopes. Let $(X, \pi, E)$ be a micro-macro phase-space with $(X, E)$ a simple graph. Recall that $(\pi, E)$ denotes the induced simple graph on macrostates. Let $\Gamma_{\varepsilon_1, \varepsilon_2} \subseteq R_{\geq 0}^{A \times A}$ be the convex polytope given by
\[
\sum_{a \in A} x_{ab} = |b|, \quad \sum_{b \in A} x_{ab} = |a|, \quad \sum_{a < b, \{a, b\} \in E} x_{ab} \leq \varepsilon_1 |X|, \quad \sum_{a > b, \{a, b\} \in E} x_{ab} \geq (1 - \varepsilon_2) |b|.
\]

**Theorem 83.** Let $(X, \pi, E)$ be a micro-macro phase-space with $(X, E)$ a simple graph. A random permutation in $S_E^X$ determines a system $(X, \pi, \alpha)$ in $L_4(\varepsilon_1, \varepsilon_2)$ with probability
\[
\frac{\prod_{a \in A} |a|!}{|S_X^E|} \sum_{c \in \Gamma_{\varepsilon_1, \varepsilon_2}} \prod_{b \in A} \left( \frac{|b|}{(c_{ab})_{a \in A}} \right).
\]

### 10 Thermodynamic Limits

Let $f : R_{\geq 0}^o \rightarrow R$ be a map. A thermodynamic (or projective) limit for $f$ is a limit
\[
\lim_{x \rightarrow \infty} f(xp_1, ..., xp_o)
\]
where $(p_1, ..., p_o) \in \Delta^{o-1}$, i.e. $p_i \geq 0$ and $p_1 + \cdots + p_o = 1$. Assuming that $f$ can be written asymptotically as $f(xp_1, ..., xp_o) = x^\alpha g(p_1, ..., p_o) + o(x^\alpha)$, the thermodynamic limits of $f$ are controlled by the map $g : \Delta^{o-1} \rightarrow R$. In our computations below $\ln(f)$ will have such asymptotic behaviour, with $\alpha = 1$, thus $f(xp_1, ..., xp_o) \simeq e^{xg(p_1, ..., p_o)}$.

The zone proportions $p_i$ and transition proportions $\lambda_{ij}$ of a micro-macro dynamical system $(X, \pi, \alpha)$ with $O_\pi = \{k_1, ..., k_o\}$ are given, respectively, by
\[
p_i = \frac{\hat{\pi}_i}{|X|} \quad \text{and} \quad \lambda_{ij} = \frac{|\{\alpha(l) \in \hat{\pi}_i \mid l \in \hat{\pi}_j\}|}{|X|}.
\]

**Definition 84.** A thermodynamic limit of micro-macro dynamical systems with zone proportions $p_i$ and zone transition proportions $\lambda_{ij}$ is a sequence $(X_n, \pi(n), \alpha_n)$ such that:

- $O_{\pi(n)} = \{k_1(n), ..., k_o(n)\}$, and $|X_n| \rightarrow \infty$ as $n \rightarrow \infty$,
- $\lim_{n \rightarrow \infty} p_j(n) = p_j$, and $\lim_{n \rightarrow \infty} \lambda_{ij}(n) = \lambda_{ij}$.

**Remark 85.** We have already study thermodynamics limits with zone proportions $p_o = 1$ and $p_j = 0$ for $j \neq o$ in Corollaries 57 and 58 and Theorem 59.

**Proposition 86.** In a thermodynamic limit with zone proportions $p_i > 0$ for $i \in [o]$ a random permutation has null probability of being always increasing in entropy.
Proof. By Theorem 75 and Stirling’s approximation formula the desired probability is given by the thermodynamic limit

\[
\lim_{n \to \infty} \left( \frac{n}{n p_1, \ldots, n p_o} \right)^{-1} = \lim_{n \to \infty} e^{-nH(p_1, \ldots, p_o)} = 0,
\]

where \( H(p_1, \ldots, p_o) = -\sum_{i=1}^{o} p_i \ln(p_i) > 0 \) is the Shannon entropy of \((p_1, \ldots, p_o) \in \Delta^{o-1} \).

Theorem 87. In a thermodynamic limit with zone proportions \( p_i > 0 \) for \( i \in [o] \) an invertible micro-macro dynamical system \( S \) has null probability of having transition proportions \( \lambda_{ij} \geq 0 \) unless \( \lambda_{ij} = p_i p_j \) which has full probability. If \( 0 < p_1 < p_2 < \ldots < p_o \leq 1 \) we have that:

1. \( S \in L_1(\varepsilon) \) if and only if \( \sum_{i<j} p_i p_j \leq \varepsilon; \) If \( S \in L_1(\varepsilon), \) then \( p_o(1 - p_o) \leq \varepsilon. \)

2. \( S \in GAT(\varepsilon) \) if and only if \( \sum_{i<j<o} p_i p_j \leq \varepsilon; \) \( S \in ZAT(\varepsilon) \) if and only if \( p_o \geq 1 - \varepsilon. \)

3. \( |S^{\text{eq}}| \geq (1 - \varepsilon)|S| \) if and only if \( p_o \geq 1 - \varepsilon; \) \( |DS^{\text{eq}}| \geq (1 - \varepsilon)|S^{\text{eq}}| \) if and only if \( p_o \geq 1 - \varepsilon. \)

4. If \( p_o \geq 1 - \varepsilon, \) then \( S \in L_3(\varepsilon, \varepsilon). \)

5. \( S \) is symmetric and thus \( |D| = |I| \). The proportionality constants of the (invariant or equivariant) reversible system associated to \( S \) agree with those of \( S. \)

6. A micro-state in the zone \( j \) moves to the equilibrium with probability \( p_o. \) The mean jump for such micro-states is greater than \((o - j - 1)p_o.\)

Proof. By Theorem 75 we should consider the thermodynamic limit

\[
\lim_{n \to \infty} \left( \frac{n}{n p_1, \ldots, n p_o} \right)^{-1} \prod_{j=1}^{o} \left( \frac{n p_j}{n p_{j1} p_{j1}, \ldots, n p_{j1} p_{j1}} \right) = \lim_{n \to \infty} e^{n \left[ \sum_{j=1}^{o} p_j \left( \frac{\lambda_{j1}}{p_{j1}}, \ldots, \frac{\lambda_{j1}}{p_{j1}} \right) - H(p_1, \ldots, p_o) \right]},
\]

where \( \sum_{i=1}^{o} p_i = 1, \) \( \sum_{i=1}^{o} \lambda_{ij} = p_j, \) \( \sum_{i=1}^{o} \lambda_{ji} = p_j. \)

Our next goal is to maximize

\[
\sum_{j=1}^{o} p_j H\left( \frac{\lambda_{j1}}{p_{j1}}, \ldots, \frac{\lambda_{j1}}{p_{j1}} \right) - H(p_1, \ldots, p_o) = H(\lambda_{ij}) - 2H(p_1, \ldots, p_o)
\]

with respect to \( \lambda_{ij}. \) Omitting the \( \lambda_{ij}-\)independent summand \( 2H(p_1, \ldots, p_o), \) we maximize \( H(\lambda_{ij}) \) subject to the above constrains. Applying the Jaynes’ max entropy method we get that max entropy is achieved by \( \lambda_{ij} = p_i p_j, \) with entropy \( H(p_i p_j) = 2H(p_1, \ldots, p_o). \) Indeed the maximum entropy distribution is given by

\[
\lambda_{ij} = \frac{e^{-f_i - c_j}}{\sum_{ij} e^{-f_i - c_j}} = \frac{e^{-f_i}}{\sum_i e^{-f_i}} \frac{e^{-c_j}}{\sum_j e^{-c_j}} = p_i p_j,
\]

where \( f_i \) and \( c_j \) are the Lagrangian multipliers associated with the constrains. \( \square \)
Corollary 88. Let \( s(1) > s(2) > \ldots > s(o) > 0 \) be real numbers. If \( \sum_{i=1}^{o} e^{-s(i)} = 1 \) and \( s(o) \leq -\ln(1-\varepsilon) \), then an invertible micro-macro dynamical system with transition proportions \( e^{-s(i)-s(j)} \) belongs to \( L_3(\varepsilon, \varepsilon) \).

Fix a probability \( q \) on a finite set \( X \) and let \( p_i = e^{\ln q_i - \sum c \lambda_c f_c(i) - \ln Z(\lambda)} \) be the probability on \( X \) of minimum relative entropy \( D(p\|q) \) subject to the constraints \( \sum_{i \in X} p_i f_c(i) = a_c \), where \( f_c : X \to \mathbb{R} \), \( a_c \in \mathbb{R} \), and \( Z(\lambda) = \sum e^{\ln q_i - \sum c \lambda_c f_c(i)} \). After reordering, assume that

\[
\sum \lambda_c f_c(1) - \ln q_1 > \cdots > \sum \lambda_c f_c(i) - \ln q_i > \cdots > \sum \lambda_c f_c(o) - \ln q_o.
\]

Corollary 89. Under the above conditions assume that \( \sum_c \lambda_c f_c(o) \geq \ln \left(\frac{(1-\varepsilon)q_o}{Z(\lambda)}\right) \), then an invertible micro-macro dynamical system with transition proportions

\[
e^{\ln(q_i q_j) - \sum \lambda_c (f_c(i)+f_c(j)) - 2\ln Z(\lambda)}
\]

belongs to \( L_3(\varepsilon, \varepsilon) \).

Next we characterize the most likely invertible micro-macro dynamical system in a thermodynamic limit.

Theorem 90. In a thermodynamic limit an invertible micro-macro dynamical system with \( o \) zones most likely have zone and transition proportions \( p_i = \frac{1}{o} \) and \( \lambda_{ij} = \frac{1}{o^2} \). For \( \varepsilon < \frac{1}{4} \), such a system has property \( L_1(\varepsilon) \) if and only if \( o = 1 \).

Proof. We proceed as in Theorem 87 letting both \( p_i > 0 \) and \( \lambda_{ij} > 0 \) vary subject to

\[
\sum_{ij}^o \lambda_{ij} = 1, \quad \sum_{i=1}^o \lambda_{ij} = p_j, \quad \sum_{i=1}^o \lambda_{ji} = p_i.
\]

Maximizing \( \sum_{j=1}^o p_j H(\frac{\lambda_{ij}}{p_j}, \ldots, \frac{\lambda_{oj}}{p_j}) - H(p_1, \ldots, p_o) = H(\lambda_{ij}) - 2H(p_1, \ldots, p_o) \) we get that \( p_i = \frac{1}{o} \), and \( \lambda_{ij} = \frac{1}{o^2} \), i.e. all zones have the same cardinality and transitions between zones are uniformly random. By Theorem 87 we have that such a system has property \( L_1(\varepsilon) \) if and only if \( \frac{1}{o} \geq 1 - 2\varepsilon \). In particular, for \( \varepsilon < \frac{1}{4} \) property \( L_1(\varepsilon) \) holds if and only if \( o = 1 \).

Theorems 87 and 90 show the limitations of the proportionality principle as the only basis of second law. We proceed to supplement it with the continuity principle, making the assumption that only permutations with jump bounded by \( k-1 \geq 0 \) are allowed. Note that setting \( k = o \) we recover the proportionality model discussed above.

Theorem 91. Consider an invertible micro-macro dynamical system \( S \) in a thermodynamic limit with \( k \)-bounded jumps and zone proportions \( p_j \). The system have vanishing relative probability unless its transition probabilities are \( \lambda_{ij} = b_i b_j \) for \( |i-j| \leq k \) and zero otherwise, where \( b_j = b_j + \cdots + b_{j-1} b_j + b_{j+1} b_j + \cdots + b_{j+k} b_j = p_j \), \( b_j > 0 \) for \( j \in [o] \), and \( b_j = 0 \) if \( j \notin [o] \). If \( 0 < b_1 < \ldots < b_o \) we have that:
1. $S \in L_1(\varepsilon)$ if and only if $\sum_{j-k \leq i < j} b_ib_j \leq \varepsilon$.

2. $S \in GAT(\varepsilon)$ if and only if $\sum_{j-k \leq i < j < o} b_ib_j \leq \varepsilon$.

3. $S \in ZAT(\varepsilon)$ if and only if $b_j - k + \cdots + b_j \leq \varepsilon(b_j - k + \cdots + b_j + \cdots + b_{j+k})$ for $j \in [o-1]$.

4. The proportionality constants of the (invariant or equivariant) reversible system associated to $S$ agree with those of $S$.

5. The average jump of system $S$ is bounded by $k - 1$.

Proof. Consider the thermodynamic limit

$$\lim_{n \to \infty} \prod_{j=1}^{o} \left( \frac{np_j}{p_j} \right) = \lim_{n \to \infty} e^{\sum_{j=1}^{o} p_j H\left( \frac{\lambda_{j-k,j}}{p_j}, \ldots, \frac{\lambda_{j+k,j}}{p_j} \right)},$$

with the convention that $\lambda_{ij} = 0$ if $i \notin [o]$. The strict concavity of Shannon’s entropy and the convexity of the constrained domain show that the expression has a unique maximum. Clearly the quotient of any such probability by the probability of the maximum are vanishing as $n$ grows to infinity. Next we maximize

$$\sum_{j=1}^{o} p_j H\left( \frac{\lambda_{j-k,j}}{p_j}, \ldots, \frac{\lambda_{j+k,j}}{p_j} \right) \text{ subject to } \lambda_{j-k,j} + \cdots + \lambda_{jj} + \cdots + \lambda_{j+k,j} = p_j.$$

Associate to each constrain its Lagrangian multiplier $\mu_0$, $\mu_j - 1$ for $1 \leq j \leq o$, respectively. Applying the method of Lagrange multipliers and setting $b_j = \sqrt{p_j e^{-\mu_0} e^{-\mu_j}} > 0$ so $\lambda_{ij} = 0$ for $|i - j| > k$, and for $|i - j| > k$, we get that $b_j = \sqrt{p_j e^{-\mu_0} e^{-\mu_j}} = b_j^2$, $\lambda_{jj} = p_j e^{-\mu_0} e^{-\mu_j} = b_j^2$, $\lambda_{ij} = \sqrt{p_i p_j e^{-\mu_0} e^{-\mu_i} e^{-\mu_j}} = b_j b_{j+1}$.

From the constraints one obtains the desired result. \qed

Note that $b_j^2 = \lambda_{jj}$ measures the proportion of transitions from zone $j$ to itself.

**Theorem 92.** Let $S$ be a micro-macro dynamical system as in Theorem 91 with $b_j = q^{j-1}b_1$ for $j \in [o]$, $o \geq 2$, and $q > 1$. For $q$ large enough that we have that:

1. $b_j^2 \approx \frac{1}{q^{2o-2}}$; $S \in L_1(\varepsilon)$ if and only if $\frac{1}{q} \leq \varepsilon$.

2. $S \in GAT(\varepsilon)$ if and only if $\frac{1}{q^2} \leq \varepsilon$; $S \in ZAT(\varepsilon)$ if and only if $\frac{1}{q^k} \leq \varepsilon$.

3. If $\frac{1}{q} \leq \varepsilon$, then $S \in L_3(\varepsilon, \varepsilon)$.

4. If $k = 1$ or $o = 2$ the average jump is zero. If $k \geq 2$ and $o \geq 3$ the average jump is $\frac{1}{q^2}$.
A subtler approach is to incorporate the principles of proportionality and continuity by fixing beforehand the average jump, and looking for the maximum entropy transitions proportionalties with such jumpiness. For \( i, j \in [o] \) set \( J(i, j) = |i - j| + \delta_{ij} - 1 \). Fix zone proportions \( p_i \) for \( i \in [o] \), and fix \( \delta \in [0, \Delta] \subseteq \mathbb{R}_{\geq 0} \) to be regarded as the average jump of a micro-macro dynamical system, where \( \Delta = \sup_{\lambda} \sum_{ij} J(i, j)\lambda_{ij} \) is the largest average jump for a probability \( \lambda_{ij} \) satisfying the first five constrains below.

**Theorem 93.** The maximum entropy probability \( \lambda_{ij} \) on \([o] \times [o]\) subject to the constrains

\[
\lambda_{ij} = \lambda_{ji} \geq 0, \quad \sum_{ij} \lambda_{ij} = 1, \quad \sum_{i=1}^{o} \lambda_{ij} = p_j, \quad \sum_{i=1}^{o} \lambda_{ji} = p_i, \quad \sum_{ij} J(i, j)\lambda_{ij} = \delta,
\]

exists and it is given setting \( b_j = e^{-f_i} / Z \), \( c = e^{-\lambda} \) by

\[
\lambda_{ij} = \frac{e^{-f_i - f_j - J(i, j)\lambda}}{\sum_{ij} e^{-f_i - f_j - J(i, j)\lambda}} = \frac{e^{-f_i - f_j - J(i, j)\lambda}}{Z} = b_i b_j c^{J(i, j)}.
\]

Assume that \( 0 < b_1 < ... < b_o \) and consider an invertible symmetric micro-macro dynamical system \( S \) with transition proportions \( b_i b_j c^{J(i, j)} \). We have that:

1. \( S \in L_1(\varepsilon) \) if and only if \( \sum_{i < j} b_i b_j c^{J(i, j)} \leq \varepsilon \).
2. \( S \in GAT(\varepsilon) \) if and only if \( \sum_{i, j < o} b_i c^{J(i, j)} \leq \varepsilon \sum_{i, j < o} b_i b_j c^{J(i, j)} \).
3. \( S \in ZAT(\varepsilon) \) if and only if \( \sum_{i < j} b_i c^{J(i, j)} \leq \varepsilon \sum_{i < o} b_i c^{J(i, j)} \) for \( j \in [o - 1] \).
4. \( S \) is symmetric and the proportionality constants of the (invariant or equivariant) reversible system associated to \( S \) agree with those of \( S \).
5. Let \( q > 1 \) and set \( b_j = q^{j-1} b_1 \) for \( j \in [o] \) and \( o \geq 2 \). Properties 1-4 of Theorem 92 hold (setting \( k = 1 \) in property 4).
6. for \( \lambda = 0 \) we recover the proportionality model from Theorem 87 and for \( \lambda \to \infty \) we recover the bounded jump proportionality model from Theorem 91 with \( k = 1 \).

**Theorem 94.** Let \( 0 < c(n) < 1 \) and \( 0 < b_1(n) < ... < b_o(n) \) be \( o + 1 \) sequences of real numbers such that \( b_o(n) \to 1 \) and \( \frac{b_j(n)}{b_{j+1}(n)} \to 0 \) as \( n \to \infty \). A sequence of invertible micro-macro dynamical systems with zone transition proportions \( \lambda_{ij} \) given either as in Theorem 91 or as in Theorem 93 has property \( L_3 \).
Proof. We consider the latter case, the former being similar, by verifying conditions 1 and 3 from Theorem 93. As \( n \to \infty \) we have that

\[
0 \leq \sum_{i<j} b_i(n) b_j(n) c(n)^{J(i,j)} \leq \left( \frac{a}{2} \right) b_{o-1}(n) b_o(n) \to 0, \quad \text{and}
\]

\[
0 \leq \frac{\sum_{i\leq j} b_i(n) c(n)^{J(i,j)}}{\sum_i b_i(n) c(n)^{J(i,j)}} \leq j \frac{b_j(n)}{b_{j+1}(n)} \to 0.
\]

\[ \square \]

References

[1] V. Arnold, A. Avez, Ergodic Problems of Classical Mechanics, Addison-Wesley, Redwood 1989.

[2] F. Bergeron, G. Labelle, P. Leroux, Combinatorial species and tree like structures, University of Cambridge, Cambridge 1998.

[3] H. Blandin, R. Díaz, Rational Combinatorics, Adv. in Appl. Math. 44 (2008) 107-126.

[4] S. Bruers, A discussion on maximum entropy production and information theory, J. Phys. A: Math. Theor. 40 (2007) 7441-7450.

[5] S. Carroll, From Eternity to Here: The Quest for the Ultimate Theory of Time, Penguin, New York 2010.

[6] E. Castillo, R. Diaz, Rota-Baxter Categories, Int. Electron. J. Algebra 5 (2009) 27-57.

[7] C. Cercignani, Ludwig Boltzmann: the man who trusted atoms, Oxford University Press, 2006.

[8] L. Cerino, F. Cecconi, M. Cencini, A. Vulpiani, The role of the number of degrees of freedom and chaos in macroscopic irreversibility, Physica A 442 (2016) 486-497.

[9] S. Chibbaro, L. Rondoni, A. Vulpiani, Reductionism, Emergence and Levels of Reality, Springer, 2014.

[10] T. Cover, J. Thomas, Elements of Information Theory, Wiley-Interscience, Hoboken 2006.

[11] G. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, Phys. Rev. E 60 (1999) 2721-2726.

[12] A. Dembo, O. Zeitouni, Large Deviations Techniques and Applications, Springer-Verlag, New York 1998.
[13] R. Dewar, Maximum entropy production and the fluctuation theorem, J. Phys. A: Math. Gen. 38 (2005) 371-381.

[14] R. Deward, Maximum Entropy Production and Non-equilibrium Statistical Mechanics, in A. Kleidon, R. Lorenz (Eds.), Non-equilibrium Thermodynamics and the Production of Entropy, pp. 41-55, Springer-Verlag, Berlin 2005.

[15] R. Dewar, A. Maritan, The second law, maximum entropy production and Liouville’s theorem, preprint, arXiv:1107.1088.

[16] R. Díaz, E. Pariguan, Super, Quantum and Noncommutative Species, Afric. Diasp. J. Math. 8 (2009) 90-130.

[17] R. Ellis, Entropy, Large Deviations, and Statistical Mechanics, New York 1985.

[18] D. Evans, D. Searle, The Fluctuation Theorem, Adv. Phys. 51 (2002) 1529-1585.

[19] R. Feynman, Statistical Mechanics, Perseus Books, Cambridge 1972.

[20] R. Frigg, C. Werndl, Entropy: A Guide for the Perplexed, in C. Beisbart, S. Hartmann (Eds.), Probabilities in Physics, Oxford Univ. Press, Oxford (1972) 115-142.

[21] R. Frigg, Typicality and the Approach to Equilibrium in Boltzmannian Statistical Mechanics, Philos. Sci. 76 (2009) 997-1008.

[22] R. Frigg, Why Typicality Does Not Explain the Approach to Equilibrium, in M. Suárez (Ed.), Probabilities, Causes and Propensities in Physics, Springer, Berlin 2011.

[23] R. Giles, Mathematical foundations of thermodynamics, Pergamon, Oxford 1964.

[24] S. Goldstein, Boltzmanns Approach to Statistical Mechanics, Lect. Notes Phys. 574 (2001) 39-54.

[25] S. Goldstein, J. Lebowitz, On the Boltzmann Entropy of Nonequilibrium Systems, Physica D 193 (2004) 53-66.

[26] M. Gromov, In a Search for a Structure, Part 1: On Entropy, preprint, http://www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5-2012.pdf

[27] E. Jaynes, Gibbs vs Boltzmann Entropies, Amer. J. Phys. 33 (1965) 391-398.

[28] E. Jaynes, Information Theory and Statistical Mechanics, Phys. Review 106 (1957) 620-630.

[29] E. Jaynes, Macroscopic prediction, in H. Haken (Ed.) Complex systems operational approaches in neurobiology, Springer, Berlin, 1985, pp. 254-269.
[30] E. Jaynes, Probability Theory The Logic of Science, Cambridge Univ. Press, Cambridge 2003.

[31] E. Jaynes, The Evolutions of Carnot’s Principle, in G. Erickson, C. Smith (Eds.), Maximum-Entropy and Bayesian Methods in Science and Engineering, Kluwer Academic Publishers, Dordrecht (1988) 267-282.

[32] W. Lawvere, S. Schanuel, Conceptual Mathematics, Cambridge Univ. Press, Cambridge 1997.

[33] J. Lebowitz, Boltzmanns Entropy and Times Arrow, Physics Today 46 (1993) 32-38.

[34] J. Lebowitz, Macroscopic Laws and Microscopic Dynamics, Ann. New York Aca. Sci. 373 (2006) 220-233.

[35] J. Lebowitz, Macroscopic Laws and Microscopic Dynamics, Times Arrow and Boltzmanns Entropy, Physica A 194 (1993) 1-97.

[36] J. Lebowitz, Microscopic origins of irreversible macroscopic behavior, Physica A 263 (1999) 516-527.

[37] A. Lesne, Shannon entropy: a rigorous mathematical notion at the crossroads between probability, information theory, dynamical systems and statistical physics, Math. Struct. Comp. Sci. 24 (2014) e240311.

[38] E. Lieb, J. Yngvason, A Guide to Entropy and the Second Law of Thermodynamics, Notices Amer. Math. Soc. 45 (1998) 571-581.

[39] S. Mac Lane, Categories for the working mathematician, Springer-Verlag, Berlin 1971.

[40] J. Matoušek, B. Gaertner, Understanding and Using Linear Programming, Springer-Verlag, Berlin 2007.

[41] R. Niven, Combinatorial entropies and statistics, Eur. Phys. J. B 70 (2009) 49-63.

[42] R. Niven, Exact Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistics, Phys. Lett. A 342 (2004) 286-293.

[43] R. Frigg, A field guide to recent work on the foundations of statistical mechanics, in D. Rickles (Ed.), The Ashgate Companion to Contemporary Philosophy of Physics, Ashgate Pub. Limited, Hants (2008) 99-196.

[44] O. Penrose, Foundations of Statistical Mechanics, Pergamon, Elmsford 1970.

[45] R. Penrose, Cycles of Time, Vintage Books, New York 2012.
[46] C. Rovelli, Why do we remember the past and not the future? The 'time oriented coarse graining' hypothesis, preprint, arXiv:1407.3384.

[47] D. Ruelle, Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics, Cambridge Univ. Press, Cambridge 2004.

[48] E. Schrödinger, Statistical Thermodynamics, Cambridge Univ. Press, Cambridge 1946.

[49] E. Sevick, R. Prabhakar, S. Williams, D. Searles, Fluctuation Theorems, Annu. Rev. Phys. Chem. 59 (2008) 603-633.

[50] R. Swendsen, Explaining irreversibility, Amer. J. Phys. 76 (2008) 643-648.

[51] V. Steckline, Zermelo, Boltzmann and the recurrence paradox, Amer. J. Phys. 51 (1983) 894-897.

[52] X. Yang, Introduction to mathematical optimization, Cambridge Int. Science Pub., Cambridge 2008.

ragadiaz@gmail.com
Universidad Nacional de Colombia - Sede Medellín, Facultad de Ciencias, Escuela de Matemáticas, Medellín, Colombia

sergiogyoz@hotmail.com
Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia