Efficient and robust entanglement generation in a many-particle system with resonant dipole-dipole interactions

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We propose and discuss a scheme for robust and efficient generation of many-particle entanglement in an ensemble of Rydberg atoms with resonant dipole-dipole interactions. It is shown that in the limit of complete dipole blocking, the system is isomorphic to a multimode Jaynes-Cummings model. While dark-state population transfer is not capable of creating entanglement, other adiabatic processes are identified that lead to complex, maximally entangled states, such as the $N$-particle analog of the GHZ state [8].

Due to Kramers degeneracy [7] in the underlying non-linear Hamiltonian different unitary operations needed to be applied for even and odd number of particles. Hence the method is also highly sensitive to variations of external and internal parameters and due to the intermediate excitation of decaying Rydberg levels the success probability decreases exponentially with $N$.

Following the proposal of Ref. [2] let us consider an ensemble of $N$ atoms with two lower levels $|a⟩$ and $|b⟩$ both coupled to a Rydberg state $|r⟩$ by coherent laser fields with (real) Rabi-frequencies $Ω_1(t)$ and $Ω_2(t)$ respectively. Let us further assume that there are two additional Rydberg levels above and below $|r⟩$ with equal energy splitting. In such a configuration there is a resonant energy transfer between two atoms in Rydberg levels, leading to a symmetric splitting of the doubly-excited states. If the minimum splitting, given by the atoms of largest separation, exceeds the natural linewidth, resonant laser excitation into doubly- or higher excited states is suppressed (dipole-blockade). In the absence of this dipole-blocking, the Hamiltonian is linear in the total spin of the atoms (SU(2) symmetry) and it is not possible to create entanglement.

On the other hand for perfect dipole-blockade there is never more than one excitation in the Rydberg levels. In this limit the effect of the dipole-dipole interaction can easily be modeled by treating atoms in the Rydberg state as fermions ($σ$, $σ^+$), while representing atoms in levels $|a⟩$ and $|b⟩$ by bosons ($a$, $a^+$), ($b$, $b^+$). The presence of the fermionic component breaks the SU(2) symmetry and the interaction is no longer linear in the total spin but can be described by a multi-mode Jaynes-Cummings Hamiltonian [11]:

$$H(t) = Δ_1a^+a + Δ_2b^+b + (Ω_1a + Ω_2b)σ^+ + h.c.$$  \hspace{1cm} (2)

The detunings $Δ_i$ have to be much smaller than the minimum splitting of the doubly-excited manifold for the blockade-limit to hold.

The isomorphism to the multi-mode Jaynes Cummings
model has a number of interesting consequences. First of all it simplifies the analysis by allowing to employ angular momentum techniques. Secondly many known features of the Jaynes-Cummings dynamics, such as decay and revivals of oscillations [13], squeezed-state generation, and quantum state transfer between different modes [13] can be anticipated in the dipole-blocking system.

The blockade of double and higher excitations results in a chainwise coupling between symmetric collective states as shown in Fig. 1. This coupling with an odd total number of levels suggests the application of dark-state Raman adiabatic passage techniques [13]. To analyze adiabatic passage in such a system it is convenient to introduce dark- and bright-state boson operators

\begin{equation}
D = a \cos \theta - b \sin \theta, \quad B = a \sin \theta + b \cos \theta,
\end{equation}

with \( \tan \theta = \Omega_1/\Omega_2 \). In terms of these variables the interaction Hamiltonian reads

\begin{equation}
H = \frac{\Delta_1 - \Delta_2}{2} (D^+ D + B^+ B) + \frac{\Delta_1 + \Delta_2}{2} (D^+ D - B^+ B) \cos 2\theta
+ \frac{\Delta_1 - \Delta_2}{2} (D^+ B - B^+ D) \sin 2\theta
+ \Omega_0 (B^+ a + a^+ B^+),
\end{equation}

with \( \Omega_0 = \sqrt{\Omega_1^2(t) + \Omega_2^2(t)} \). The first two terms are the free energy of the atoms in the dark and bright states and the third term describes the coupling between dark and bright states. The last term shows that only the bright-state component is coupled to the Rydberg levels.

[Diagram of collective N-atom states in limit of dipole blockade, shown here for \( \Delta_1 = \Delta_2 = 0 \). Individual atoms have two lower states \([a]\) and \([b]\) coupled to Rydberg state \([\beta]\) with Rabi-frequencies \( \Omega_1 \) and \( \Omega_2 \) respectively. \( [a^{N-m}b^m] \) denotes symmetric superposition of \( N-m \) atoms in state \([a]\) and \( m \) atoms in state \([b]\).]

Under conditions of two-photon resonance, i.e. \( \Delta_1 = \Delta_2 \) the dark-state subspace decouples from the remaining system. Its dynamics has however again SU(2) symmetry and factorized states remain factorized. Hence dark-state adiabatic transfer is not suitable for entanglement generation. This result can easily be understood physically. Since the dark state does not contain any excited-state population, the presence of dipole-dipole interactions and the resulting dipole blockade are irrelevant. Nevertheless, as will be shown in the following, adiabatic techniques can be used to create entanglement if other than the zero-eigenvalue state is involved.

To this end we consider here a situation opposite to the two-photon resonance when \( \Delta = \Delta_1 = -\Delta_2 \). We first discuss the case of a substantial delay between the two pulses \( \Omega_1 \) and \( \Omega_2 \) such that the coupling between the dark and bright components, which is proportional to \( \Delta \sin 2\theta \) is negligible. In this approximation the Hamiltonian [13] can be expressed in the simple form

\begin{equation}
H = \frac{\Delta}{2} \sigma_z \cos 2\theta + \Omega_0 \left( B \sigma^+ + B^\dagger \sigma^- \right),
\end{equation}

where the irrelevant constant term \( N\Delta/2 \cos 2\theta \) was omitted. The corresponding Schrödinger equation can be solved analytically in the adiabatic limit, i.e. when the mixing angle \( \theta(t) \) changes sufficiently slowly in time.

To obtain a convenient closed form of the solution we introduce angular momentum operators \( J_1 = (\sigma^+ B + \sigma^- B^\dagger)/2\sqrt{M} \), \( J_2 = i(\sigma^+ B - \sigma^- B^\dagger)/2\sqrt{M} \), and \( J_3 = \sigma_z/2 \), where \( M = B^\dagger B + \sigma^+ \sigma \) is the constant particle number in the bright-state–Rydberg manifold. In terms of these operators the Hamiltonian reads

\[ H = \Omega e^{-i\beta J_2} J_1 e^{i\beta J_2}, \]

where \( \Omega \equiv \sqrt{\Omega_1^2(t) + \Delta^2 \cos^2 2\theta} \), and \( \tan \beta(t) = \Omega_0(t)/\Delta \cos 2\theta \). The corresponding unitary evolution operator then reads

\begin{equation}
U(t) = e^{-i\beta J_2(t) J_1} \exp \left[-i J_3 \int_{-\infty}^{t} \Omega(t') \, dt' \right] e^{i\beta(-\infty) J_2}. \tag{6}
\end{equation}

Let us now consider the case of all \( N \) atoms being initially in \([a]\). If an intuitive pulse sequence is applied, i.e. if \( \Omega_1 \) is switched on and off before \( \Omega_2 \) one has \( \beta : 0 \to \pi \) and the systems starts from a bright state

\begin{equation}
|\Psi_0 \rangle = |a^N \rangle = \frac{1}{\sqrt{N!}} \left( a^\dagger \right)^N |0 \rangle = \frac{1}{\sqrt{N!}} \left( B^\dagger \right)^N |0 \rangle, \tag{7}
\end{equation}

where \( |a^N \rangle \) denotes the collective state with all atoms being in level \([a]\). The unitary evolution operator then reads

\begin{equation}
W = -2i J_2 \exp \left[-i J_3 \int_{-\infty}^{+\infty} \Omega(t) \, dt \right]. \tag{8}
\end{equation}

One-time application of \( W \) generates a symmetric collective state containing a single Rydberg excitation and all other atoms are in \([b]\):

\begin{equation}
|\Psi_1 \rangle = W |\Psi_0 \rangle = \frac{1}{\sqrt{(N-1)!}} \left( B^\dagger \right)^{N-1} \sigma^+ |0 \rangle
= \frac{1}{\sqrt{(N-1)!}} \left( b^\dagger \right)^{N-1} \sigma^+ |0 \rangle, \tag{9}
\end{equation}

FIG. 1. Coupling scheme of collective N-atom states in limit of dipole blockade, shown here for \( \Delta_1 = \Delta_2 = 0 \). Individual atoms have two lower states \([a]\) and \([b]\) coupled to Rydberg state \([\beta]\) with Rabi-frequencies \( \Omega_1 \) and \( \Omega_2 \) respectively. \( [a^{N-m}b^m] \) denotes symmetric superposition of \( N-m \) atoms in state \([a]\) and \( m \) atoms in state \([b]\).
corresponding to
\[ |a^N⟩ \xrightarrow{W} |b^{N-1}⟩. \tag{10} \]
Applying \( W \) twice generates the \( W \)-state of Ref. [15]
\[ |a^N⟩ \xrightarrow{W^2} |a^{N-1}b⟩. \tag{11} \]
On the other hand starting from an initial state with all atoms in \( |b⟩ \) corresponds to a pulse sequence in counter-intuitive order and leads to the transfer
\[ |b^N⟩ \xrightarrow{W} |a^N⟩. \tag{12} \]
Iterative applications of the same operator \( W \) allows one to reach any state in the \( 2N+1 \) dimensional manifold of symmetric many-particle excitations with at most one Rydberg atom. The \( W \)-operation is based on adiabatic evolution and is robust against variations of parameters as long as the condition
\[ γ \int_{-∞}^{∞} dt \frac{\dot{J}^2(t)}{Ω^2(t)} ≪ 1 \tag{13} \]
is fulfilled, with \( γ \) being the decay rate of the Rydberg levels.

Although the application of \( W \) leads to an entangled state whose creation would require many \( π \)-pulses, \( O(N) \) steps are needed for the generation of complex states like the \( N \)-particle analog of the GHZ state [4]. We will now show that a small modification of the \( W \) operation can achieve this goal in very few steps and independent on the number of particles.

For this we assume that the system is initially in an equal superposition of atoms being in \( |a⟩ \) and the symmetric state containing a single Rydberg excitation.
\[ |Ψ_0⟩ = \frac{1}{\sqrt{2}} \left( |a^N⟩ + |a^{N-1}r⟩ \right). \tag{14} \]
\( |Ψ_0⟩ \) can easily be created out of \( |Ψ_0⟩ \) in a robust way e.g. by sweeping the frequency of \( Ω_1 \) through resonance (rapid adiabatic passage) [16]. We now apply the \( W \) operation discussed above, however with a smaller time delay between the two pulses. In this case the dark-bright state coupling in the Hamiltonian [4] proportional to \( \sin 2θ \) needs to be taken into account. Furthermore it is assumed that \( |Ω_0| ≫ |Δ| \). Under these conditions the Schrödinger equation can no longer be solved analytically. However numerically evaluating the equations (for \( N \) up to 20), we found the behavior shown in Fig. 2.

The mechanism can qualitatively be understood as follows: Due to the non-vanishing detuning \( Δ \) and the chosen intuitive pulse order, the state amplitude in \( |a^N⟩ \) undergoes an adiabatic return process [16] and ends up in the same state as it started from. At the same time the chosen pulse order is counter-intuitive for the state \( |a^{N-1}r⟩ \) and hence its amplitude undergoes Raman adiabatic passage to \( |b^{N-1}⟩ \) through a chain of successive \( V \)-type transitions. Since the fields are not in \( N \)-photon resonance, it is essential that \( |Ω_m| ≫ |Δ| \).

FIG. 2. Temporal evolution of population in \( |a^N⟩ \) and \( |b^{N-1}⟩ \) form initial state \( |Ψ_0⟩ \) for \( N = 5 \). The laser pulses are Gaussian \( Ω_{1,2}(t) = Ω_m \exp[-(t ± τ)^2/T^2] \), the delay is \( τ = 0.5T \) the pulse area \( Ω_mT = 125 \), and \( ΔT = 50 \).

The amplitude of the state vector in \( |a^N⟩ \) undergoes some rapid oscillations but returns to the same state for \( t \to ∞ \). At the same time the amplitude in \( |b^{N-1}⟩ \) is transferred to \( |b^{N-1}⟩ \). Applying in a third step the inverse of \( W \), eqs. [14] and [12], eventually leads to the \( N \)-particle GHZ state [4]. This corresponds to the overall 3-step adiabatic process:
\[ |a^N⟩ \to \frac{1}{\sqrt{2}} \left( |a^N⟩ + |a^{N-1}r⟩ \right) \to \frac{1}{\sqrt{2}} \left( |b^N⟩ + |a^N⟩ \right). \tag{15} \]

The transfer is in all parts robust. It does not depend on the exact pulse form of \( Ω_1 \) and \( Ω_2 \), nor does it require an extreme control of the delay time \( τ \) or the pulse length \( T \). Furthermore the mechanism works for even and odd numbers of atoms in the same way. In Fig. 3 we have shown the dependence of the final population in the states \( |a^N⟩ \) and \( |b^{N-1}⟩ \) for \( N = 5 \) as function of pulse delay \( τ \) and pulse area \( Ω_mT \). It can be seen that the mechanism is robust against small variation of the delay time and – above some critical limit – the pulse area. It should be mentioned that for very large values of the pulse area the populations decrease again, since then the term \( Δ \sin 2θ \) in [14] is negligible and there is a transfer \( \sqrt{2} (|a^N⟩ + |a^{N-1}r⟩) \to |b^N⟩ + |b^{N-1}⟩ \).

Since all processes are adiabatic and \( Ω_m \) is limited by the dipole-blockade condition, the question arises how the time \( T \) for generating the GHZ state scales with the number of particles. From our numerical calculations, shown in Fig. 4, we find \( T \sim N^α \) with \( α < 1 \) and decreasing with

\[ \frac{1}{\sqrt{2}} \left( |a^N⟩ + |a^{N-1}r⟩ \right) \to \frac{1}{\sqrt{2}} \left( |b^N⟩ + |b^{N-1}⟩ \right). \]
The numerical calculations for $N = 3\ldots16$ indicate $\alpha \rightarrow 2/3$. Thus in the presence of decay, the success probability decreases less than exponential with $N$. 

![Graph of population as function of delay time $\tau$ for $\Omega_m T = 120$.](image1)

**FIG. 3.** Final population of states $|a^N\rangle$ and $|b^{N-1}r\rangle$ for conditions of Fig. 2. *top:* as function of delay time $\tau$ for $\Omega_m T = 120$, *bottom:* as function of $\Omega_m T$ for $\tau = 0.5T$.

![Graph of minimum pulse area $\Omega_m T_{\text{min}}$ to create GHZ state as function of particle number $N$.](image2)

**FIG. 4.** Minimum pulse area $\Omega_m T_{\text{min}}$ to create GHZ state as function of particle number $N$. Dots represent values from numerical solution of $N$-particle Schrödinger equation for optimized $\tau$.

In conclusion, we have proposed an efficient and robust method to generate complex entanglement structures, such as the $N$-particle GHZ state in a many-particle system with resonant dipole-dipole interactions. The method is robust against variations of parameters since for all steps adiabatic transfer processes are used. Although dark-state adiabatic passage is not suitable for entanglement generation, as it does not involve population of the interacting Rydberg levels, other adiabatic processes are identified that allow e.g. for the generation of the $N$-particle GHZ state in three steps. The suggested method avoids the problem associated with Kramers degeneracy and thus works for even and odd number of particles. Exact knowledge of the number of particles is not required, making the method robust against number fluctuations. As opposed to the proposal of ref. [6] no extreme fine tuning of the interaction time is needed and the minimum interaction time scales only less than linear with the number of particles. Finally it should be mentioned that similar ideas can be applied to other many-particle systems, e.g. ions in a trap.

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