THE LOGARITHMIC SOBOLEV INEQUALITY FOR A SUBMANIFOLD IN MANIFOLD WITH NONNEGATIVE SECTIONAL CURVATURE

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Abstract

We prove a sharp logarithmic Sobolev inequality which holds for compact submanifolds without boundary in Riemannian manifold with nonnegative sectional curvature of arbitrary dimension and codimension, while the ambient manifold needs to have a specific Euclid-like property. Like the Michael-Simon Sobolev inequality, this inequality includes a term involving the mean curvature. This extends a recent result of S. Brendle with Euclidean setting.

1 Introduction

In 2019, S. Brendle [1] proved a Sobolev inequality which holds on submanifolds in Euclidean space of arbitrary dimension and codimension. The inequality is sharp if the codimension is at most 2. Soon, he [2] proved a sharp logarithmic Sobolev inequality which holds on submanifolds in Euclidean space of arbitrary dimension and codimension at the same year. In 2020, he [3] extended the result of the Sobolev inequality to Riemannian manifolds with nonnegative curvature which gives the asymptotic volume ratio due to the Bishop-Gromov volume comparison theorem. Inspired by [3], we extend the result of the logarithmic Sobolev inequality to ambient Riemannian manifolds with nonnegative sectional curvature under an assumption.

Let $M$ be a complete noncompact Riemannian manifold of dimension $k$. We say $M$ satisfies the condition (P), if there is a point $p \in M$ such that the following limit exists and is positive.

$$\lim_{r \to \infty} \left( (4\pi)^{-\frac{k}{2}} r^{-k} \int_M e^{-\frac{4\pi r^2}{4\pi} dvol(x)} \right).$$

We denote the limit by $\theta$. Note that the limit is equal to 1 when $M = \mathbb{R}^k$. So we call it a specific Euclid-like property. We have the following result

Theorem 1.1. Let $M$ be a complete noncompact Riemannian manifold of dimension $n+m$ with nonnegative sectional curvature and satisfies the condition (P). Let $\Sigma$ be a compact $n$-dimension submanifold of $M$ without boundary, and let $f$ be a positive smooth function on $\Sigma$. Then

$$\int_{\Sigma} f \left( \log f + \frac{n}{2} \log(4\pi) + \log \theta \right) dvol - \int_{\Sigma} \left| \nabla^\Sigma f \right|^2 dvol - \int_{\Sigma} f |H|^2 dvol \leq \int_{\Sigma} \log \left( \int_{\Sigma} f dvol \right).$$

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where $H$ denotes the mean curvature vector of $\Sigma$.

Recall the paraboloid of revolution $\Gamma : \mathbb{R}^2 (u, v) = (u \cos v, u \sin v, \frac{1}{2}au^2)$ with $u \in [0, \infty)$ and $v \in [0, 2\pi)$, where $a$ is a positive constant. We find that it also satisfies the condition (P). Moreover $\theta = 2$ which is independent on $a$. It’s amazing! And we have the result

**Corollary 1.2.** Let $\gamma$ be a smooth closed curve on $\Gamma$, and let $f$ be a positive smooth function on $\gamma$. Then

$$
\int_{\gamma} f \left( \log f + 1 + \frac{1}{2} \log(4\pi) + \log 2 \right) d\text{vol} - \int_{\gamma} \frac{|\nabla f|^2}{f} d\text{vol} - \int_{\gamma} f |\kappa_n|^2 d\text{vol}
$$

where $\kappa_n$ denotes the geodesic normal curvature of $\gamma$ with respect to $\Gamma$.

However, cylinder $S^1 \times \mathbb{R}$ doesn’t satisfy the condition (P) in spite of $\theta = 0$.

The logarithmic Sobolev inequality has been studied by numerous authors (see e.g. [6–10]). Our proof of theorem 1.1 is in the spirit of ABP-techniques in [2]. ABP-techniques have been applied to various classes of linear and nonlinear elliptic equations in the Euclidean space for a long time. Due to some difficulties, it was not until 1997 that Cabré [4] developed them to Riemannian manifolds.

## 2 Preliminaries

Let’s talk about the condition (P) first.

**Proposition 2.1.** Let $M$ be a complete noncompact Riemannian manifold of dimension $k$, then the following are equivalent:

(a) There exists a point $p \in M$ such that the limit

$$
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_M e^{-\frac{d(x,p)^2}{4r^2}} d\text{vol}(x)
$$

exists.

(b) The limit

$$
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_M e^{-\frac{d(x,p)^2}{4r^2}} d\text{vol}(x)
$$

exists for every $p \in M$.

(c) For any compact subset $K \subset M$ and for any Borel map $p : M \to K$, the limit

$$
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_M e^{-\frac{d(x,p(x))^2}{4r^2}} d\text{vol}(x)
$$

exists.

Moreover, both of these limits are the same one if exist.

**Proof.** Clearly, (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are trivial. It remains to show (a) $\Rightarrow$ (c). We assume that there exists a point $p_0 \in M$ such that the limit

$$
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_M e^{-\frac{d(x,p_0)^2}{4r^2}} d\text{vol}(x)
$$
exists and equals to \( \theta \). Given a compact subset \( K \subset M \) and a Borel map \( p : M \to K \). We define a positive constant

\[
C := \sup \{ d(p_0, p(x)) : x \in M \}.
\]

For any fixed \( \varepsilon > 0 \) sufficiently small, note that

\[
\frac{d(x, p(x))^2}{4\pi^2} = -\frac{d(x, p_0)^2}{4\pi^2} + \frac{d(x, p(x))^2}{4\pi^2} \geq \frac{d(x, p_0)^2}{4\pi^2} \left( 1 + \frac{d(p_0, p(x))}{d(x, p_0)} \right)^2 \geq \frac{d(x, p_0)^2}{4\pi^2} (1 + C\varepsilon)^2
\]

for all \( x \in M \setminus B_{\varepsilon^{-1}}(p_0) \). Similarly, we have

\[
\frac{d(x, p(x))^2}{4\pi^2} \leq \frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2
\]

for all \( x \in M \setminus B_{\varepsilon^{-1}}(p_0) \). Thus,

\[
\frac{1}{r^k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2}} (1 + C\varepsilon)^2 \, d\text{vol}(x) \leq \frac{1}{r^k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2}} \, d\text{vol}(x) \leq \frac{1}{r^k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2} \, d\text{vol}(x).
\]

It’s easy to see that

\[
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_{B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2}} \, d\text{vol}(x) = 0
\]

and

\[
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2} \, d\text{vol}(x) = 0.
\]

From the assumption, we have

\[
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 + C\varepsilon)^2} \, d\text{vol}(x) = (1 + C\varepsilon)^{-k} \theta
\]

and

\[
\lim_{r \to \infty} \left( 4\pi \right)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2} \, d\text{vol}(x) = (1 - C\varepsilon)^{-k} \theta.
\]

Combining with

\[
(4\pi)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 + C\varepsilon)^2} \, d\text{vol}(x) + (4\pi)^{-\frac{k}{2}} r^{-k} \int_{B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2} \, d\text{vol}(x)
\]

\[
\leq (4\pi)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2}} \, d\text{vol}(x)
\]

\[
\leq (4\pi)^{-\frac{k}{2}} r^{-k} \int_{M \setminus B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2}} \, d\text{vol}(x) + (4\pi)^{-\frac{k}{2}} r^{-k} \int_{B_{\varepsilon^{-1}}(p_0)} e^{-\frac{d(x, p_0)^2}{4\pi^2} (1 - C\varepsilon)^2} \, d\text{vol}(x),
\]

(c) follows from a standard \( \varepsilon - \delta \) discussion. \( \Box \)
3 Proof of Theorem 1.1

Recall the definition of the second fundamental form $II$ of $\Sigma$ with respect to $M$:

$$\langle II(X,Y), V \rangle = \langle \bar{D} X Y, V \rangle = -\langle \bar{D} X V, Y \rangle,$$

where $X,Y$ are tangent vector fields, $V$ is a normal vector field and $\bar{D}$ denotes the connection on $M$. Moreover, the mean curvature vector $H$ is defined as the trace of the second fundamental form $II$.

We now give the proof of Theorem 1.1. We first consider the special case that $\Sigma$ is connected. By scaling, we may assume that

$$\int_\Sigma f \log f \, d\text{vol} - \int_\Sigma |\nabla \Sigma f|^2 \, f \, d\text{vol} - \int_\Sigma f |H|^2 \, d\text{vol} = 0.$$

From functional analysis and standard elliptic theory, we can find a smooth function $u : \Sigma \to \mathbb{R}$ such that

$$\text{div}_\Sigma (f \nabla^\Sigma u) = f \log f - |\nabla \Sigma f|^2 - |H|^2.$$

In the following, we fix a positive number $r$. We denote the contact set

$$A = \{(\bar{x}, \bar{y}) \in T^\perp \Sigma : ru(\bar{x}) + \frac{1}{2} d(\bar{x}, \exp_{\bar{x}}(r \nabla^\Sigma u(\bar{x}) + r \bar{y}))^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2), \forall x \in \Sigma\}.$$

Moreover, we define a map $\Phi : T^\perp \Sigma \to M$ by

$$\Phi(x, y) = \exp_x (r \nabla^\Sigma u(x) + ry)$$

for all $(x, y) \in T^\perp \Sigma$.

**Lemma 3.1.** Suppose that $(\bar{x}, \bar{y}) \in A$, then

$$d(\bar{x}, \Phi(\bar{x}, \bar{y}))^2 = r^2 (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2).$$

**Proof.** Let $\bar{\gamma}(t) := \exp_x (rt \nabla^\Sigma u(\bar{x}) + r\bar{y})$ for $t \in [0, 1]$. From the definition of $A$, we have

$$ru(\bar{x}) + \frac{1}{2} d(\bar{x}, \exp_x (r \nabla^\Sigma u(\bar{x}) + r\bar{y}))^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2).$$

Thus, $d(\bar{x}, \Phi(\bar{x}, \bar{y}))^2 \geq r^2 (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2)$. On the other hand,

$$r^2 (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2) = |\bar{\gamma}'(0)|^2 = \left(\int_0^1 |\bar{\gamma}'(t)| \, dt\right)^2 \geq d(\bar{x}, \Phi(\bar{x}, \bar{y}))^2.$$

Then, the lemma follows.

**Lemma 3.2.** $\Phi(A) = M$.

**Proof.** Fix a point $p \in M$. Since $\Sigma$ is compact without boundary, the function $x \mapsto ru(x) + \frac{1}{2} d(x, p)^2$ must attain its minimum at some point denoted by $\bar{x}$ on $\Sigma$. Moreover, we can find
a minimizing geodesic \( \tilde{\gamma} : [0, 1] \to M \) such that \( \tilde{\gamma}(0) = \bar{x} \) and \( \tilde{\gamma}(1) = p \). For every path \( \gamma : [0, 1] \to M \) satisfying \( \gamma(0) \in \Sigma \) and \( \gamma(1) = p \), we obtain

\[
\begin{align*}
ru(\gamma(0)) + E(\gamma) & \geq ru(\gamma(0)) + \frac{1}{2}d(\gamma(0), p)^2 \\
& \geq ru(\bar{x}) + \frac{1}{2}d(\bar{x}, p)^2 \\
& = ru(\bar{\gamma}(0)) + \frac{1}{2}|\gamma'(0)|^2 \\
& = ru(\bar{\gamma}(0)) + E(\bar{\gamma}),
\end{align*}
\]

where \( E(\gamma) \) denotes the energy of \( \gamma \). In other words, the path \( \gamma \) minimizes the functional \( ru(\gamma(0)) + E(\gamma) \) among all paths \( \bar{\gamma} : [0, 1] \to M \) satisfying \( \bar{\gamma}(0) \in \Sigma \) and \( \bar{\gamma}(1) = p \). Hence, the formula for the first variation implies

\[
\bar{\gamma}'(0) - r\nabla^\Sigma u(\bar{x}) \in T_x^\perp \Sigma.
\]

Consequently, we can find a vector \( \bar{y} \in T_x^\perp \Sigma \) such that

\[
\bar{\gamma}'(0) = r\nabla^\Sigma u(\bar{x}) + r\bar{y}.
\]

It remains to show \( (\bar{x}, \bar{y}) \in A \). For each point \( x \in \Sigma \), we have

\[
ru(x) + \frac{1}{2}d\left(x, \exp_x\left(r\nabla^\Sigma u(\bar{x}) + r\bar{y}\right)\right)^2 = ru(x) + \frac{1}{2}d(x, p)^2
\]

\[
\geq ru(\bar{x}) + \frac{1}{2}d(\bar{x}, p)^2
\]

\[
= ru(\bar{\gamma}(0)) + \frac{1}{2}|\gamma'(0)|^2
\]

\[
= ru(\bar{\gamma}(0)) + E(\bar{\gamma}),
\]

\[
\Box
\]

**Lemma 3.3.** Suppose that \( (\bar{x}, \bar{y}) \in A \), and let \( \tilde{\gamma}(t) := \exp_x\left(rt\nabla^\Sigma u(\bar{x}) + rt\bar{y}\right) \) for \( t \in [0, 1] \). If \( Z \) is a vector field along \( \tilde{\gamma} \) satisfying \( Z(0) \in T_{A} \Sigma \) and \( Z(1) = 0 \), then

\[
\begin{align*}
ru(D_2^2 u(Z(0), Z(0))) - r\langle \nabla^\Sigma u(\bar{x}), \bar{y}\rangle
+ \int_0^1 \left( |D_t Z(t)|^2 - R(\tilde{\gamma}'(t), Z(t), \tilde{\gamma}'(t), Z(t)) \right) dt & \geq 0.
\end{align*}
\]

**Lemma 3.4.** Suppose that \( (\bar{x}, \bar{y}) \in A \). Then \( g + rD_2^2 u(\bar{x}) - r\langle \nabla^\Sigma u(\bar{x}), \bar{y}\rangle \geq 0 \).

**Lemma 3.5.** Suppose that \( (\bar{x}, \bar{y}) \in A \), and let \( \tilde{\gamma}(t) := \exp_x\left(rt\nabla^\Sigma u(\bar{x}) + rt\bar{y}\right) \) for \( t \in [0, 1] \). Moreover, let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_{A} \Sigma \). Suppose that \( W \) is a Jacobi field along \( \tilde{\gamma} \) satisfying \( W(0) \in T_{A} \Sigma \) and \( \langle D_t W(0), e_j\rangle = r\langle D_2^2 u(W(0), e_j) - r\langle \nabla^\Sigma u(\bar{x}), \bar{y}\rangle \rangle \) for each \( 1 \leq j \leq n \). If \( W(\tau) = 0 \) for some \( 0 < \tau < 1 \), then \( W \) vanishes identically.

**Lemma 3.6.** The Jacobian determinant of \( \Phi \) satisfies

\[
|\det \Phi (x, y)| \leq r^m \det (g + rD_2^2 u(x) - r\langle \nabla^\Sigma u(\bar{x}), \bar{y}\rangle)
\]

for all \( (x, y) \in A \).

The proofs of Lemma 3.3-3.6 are identical to Lemma 2.1-2.3 and Lemma 2.5 in [3] respectively. We omit them.

**Lemma 3.7.** The Jacobian determinant of \( \Phi \) satisfies

\[
e^{-\frac{(x-x_0)(y-y_0)^2}{4r^2}}|\det \Phi (x, y)| \leq r^{n+m} f(x) e^{-n - \frac{(x-x_0)^2 + (y-y_0)^2}{4}}
\]

for all \( (x, y) \in A \).
Proof. Given a point \((x, y) \in A\). Using the identity \( \div \big( f\nabla^\Sigma u \big) = f \log f - \frac{\| \nabla^\Sigma u \|^2}{f} - |H|^2 \), we have
\[
\Delta^\Sigma u(x) - \langle H(x), y \rangle = E \log f(x) - E \frac{\| \nabla^\Sigma u(x) \|^2}{f(x)} - |H(x)|^2
- \frac{\| \nabla^\Sigma u(x) \|^2}{2f(x)} - \langle H(x), y \rangle
\]
\[
= \log f(x) + \frac{\| \nabla^\Sigma u(x) \|^2}{4f(x)} - \frac{\| \nabla^\Sigma u(x) \|^2}{4f(x)} - |2H(x)+y|^2.
\]
Using Lemma 3.4, Lemma 3.6 and the elementary inequality \( \lambda \leq e^{\lambda - 1} \), we have
\[
|\det D\Phi(x, y)| \leq r^{n} \log (g + r D^2 \Phi(x) - rH(x, y))
\]
\[
= r^{n+m} \det (\frac{\partial}{\partial y} + D^2 \Phi(x) - (n+1)(x, y))
\]
\[
\leq r^{n+m} e^{\log f(x) + \| \nabla^\Sigma u(x) \|^2/2} - |2H(x)+y|^2
\]
\[
= r^{n+m} f(x) e^{-n} e^{-|2H(x)+y|^2/4} e^{\log f(x)}.
\]
The lemma follows. \(\square\)

By Lemma 3.2, for any fixed \( p \in M \), we choose some point \((x(p), y(p)) \in A\) arbitrarily such that \( \Phi(x(p), y(p)) = p \). Using Lemma 3.2, Lemma 3.7 and area formula [5], we have
\[
\int_M e^{-\frac{d(x, y)^2}{4f}} d\vol(p) \leq \int_M \left( \int_{\Phi^{-1}(p)} e^{-\frac{d(x, y)^2}{4f}} d\mathcal{H}^0 \right) d\vol(p)
\]
\[
= \int_{\Sigma} \left( \int_{\Sigma_1} e^{-\frac{d(x, y)^2}{4f}} |\det D\Phi(x, y)| d\vol(x, y) dy \right) d\vol(x)
\]
\[
\leq \int_{\Sigma} \left( \int_{\Sigma_1} r^{n+m} f(x) e^{-n} e^{-|2H(x)+y|^2/4} d\vol(x, y) dy \right) d\vol(x)
\]
\[
= \int_{\Sigma} \left( \int_{\Sigma_1} r^{n+m} f(x) e^{-n} e^{-|2H(x)+y|^2/4} dy \right) d\vol(x)
\]
where \( \mathcal{H}^0 \) denotes the counting measure. Using Proposition 2.1, we can divide by \( r^{n+m} \) and send \( r \to \infty \) since \( M \) satisfies the condition (P). This gives
\[
(4\pi)^{n+m} \theta \leq e^{-n} \int_{\Sigma} f(x) d\vol(x).
\]
Consequently,
\[
n + \frac{n}{2} \log (4\pi) + \log \theta \leq \log \left( \int_{\Sigma} f d\vol \right).
\]
Combining this inequality with the normalization
\[
\int_{\Sigma} f \log f d\vol - \int_{\Sigma} \frac{\| \nabla^\Sigma f \|^2}{f} d\vol - \int_{\Sigma} f |H|^2 d\vol = 0
\]
gives
\[
\int_{\Sigma} \left( f \log f + \frac{n}{2} \log(4\pi) + \log \theta \right) d\vol - \int_{\Sigma} \frac{\| \nabla^\Sigma f \|^2}{f} d\vol - \int_{\Sigma} f |H|^2 d\vol
\]
\[
= \left( \int_{\Sigma} f d\vol \right) \log \left( \int_{\Sigma} f d\vol \right).
\]
It remains to consider the case when $\Sigma$ is disconnected. For completeness, we list Brendle’s proof. In that case, we apply the inequality to each individual connected component of $\Sigma$, and sum over all connected components. Since
\[
a \log a + b \log b < a \log (a+b) + b \log (a+b) = (a+b) \log (a+b)
\]
for $a, b > 0$, we conclude that
\[
\int_{\Sigma} f \left( \log f + n + \frac{n}{2} \log(4\pi) + \log \theta \right) d\text{vol} - \int_{\Sigma} |\nabla f|^2 d\text{vol} - \int_{\Sigma} f |H|^2 d\text{vol}
\]
for $\Sigma$ disconnected. This completes the proof of Theorem 1.1.

4 Proof of Corollary 1.2

By computing, the volume form $d\text{vol}(u,v)$ is equal to $u \sqrt{1 + a^2 u^2} du dv$, and the intrinsic distance from the Origin to the point $p(u,v)$ satisfies
\[
d(O, p(u,v)) = \int_0^u \sqrt{1 + a^2 t^2} dt = \frac{u}{2} \sqrt{1 + a^2 u^2} + \frac{1}{2a} \ln (au + \sqrt{1 + a^2 u^2}).
\]
So we have
\[
\int_{\Gamma} e^{-\frac{d(O, p(u,v))^2}{4r^2}} d\text{vol}(u,v) = 2\pi \int_0^\infty e^{-\frac{A(u)^2}{4r^2}} u \sqrt{1 + a^2 u^2} du,
\]
where $A(u) = d(O, p(u,v))$. Since $u \leq 2A(u)$ for all $u \geq 0$, we have
\[
\int_0^\infty e^{-\frac{A(u)^2}{4r^2}} u \sqrt{1 + a^2 u^2} du \leq \int_0^\infty e^{-\frac{A(u)^2}{4r^2}} 2A(u) \sqrt{1 + a^2 u^2} du = \int_0^\infty e^{-\frac{A(u)}{r}} du = 4r^2.
\]
Note that $\lim_{u \to 0^+} A(u) = \frac{1}{2} u$. Thus, for any $\varepsilon > 0$, we can find a positive number $\delta = \delta(\varepsilon)$ such that $u \geq \frac{2}{1+\varepsilon} A(u)$ for all $u \in [0, \delta]$. So we have
\[
\int_0^\infty e^{-\frac{A(u)^2}{4r^2}} u \sqrt{1 + a^2 u^2} du \geq \int_0^{\delta} e^{-\frac{A(u)^2}{4r^2}} u \sqrt{1 + a^2 u^2} du \geq \frac{2}{1+\varepsilon} \int_0^{\delta} e^{-\frac{A(u)^2}{4r^2}} A(u) \sqrt{1 + a^2 u^2} du = \frac{4r^2}{1+\varepsilon} \int_0^{\delta} e^{-t} dt = \frac{4r^2}{1+\varepsilon} (1 - e^{-\frac{\delta}{r}}).
\]
And we can find a positive number $N = N(\varepsilon)$ such that $e^{-\frac{A(u)^2}{4r^2}} < \varepsilon$ for all $r > N$. From standard $\varepsilon - \delta$ language, we can conclude that $\theta = 2$. Using Theorem 1.1, the corollary follows.

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