MULTIDIMENSIONAL STABILITY OF PLANAR TRAVELING WAVES FOR THE DELAYED NONLOCAL DISPERAL COMPETITIVE LOTKA-VOLTERRA SYSTEM

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Abstract. In this paper, we consider the multidimensional stability of planar traveling waves for the nonlocal dispersal competitive Lotka-Volterra system with time delay in n-dimensional space. More precisely, we prove that all planar traveling waves with speed $c > c^*$ are exponentially stable in $L^\infty(\mathbb{R}^n)$ in the form of $t^{-\frac{n}{2}}\alpha e^{-\varepsilon \tau t}$ for some constants $\sigma > 0$ and $\varepsilon \tau \in (0, 1)$, where $\varepsilon \tau = \varepsilon(\tau)$ is a decreasing function refer to the time delay $\tau > 0$. It is also realized that, the effect of time delay essentially causes the decay rate of the solution slowly down. While, for the planar traveling waves with speed $c = c^*$, we show that they are algebraically stable in the form of $t^{-\frac{n}{2}}\alpha$. The adopted approach of proofs here is Fourier transform and the weighted energy method with a suitably selected weighted function.

1. Introduction. The theory of traveling wave solutions of reaction-diffusion equations has been attached much attention since the seminal works of Fisher [7] and Kolmogorove [16], due to its significant nature in biology, chemistry, epidemiology and physics (see, [3, 7, 8, 29, 32, 34, 38, 39, 40, 41, 43]). Among the basic problems in the theory of traveling wave solutions, the stability of traveling wave solutions, which is one of the central questions in the study of traveling waves, is a very challenge question. Recently, a great interest has been drawn to the study of the multidimensional stability of traveling wave solutions. Xin [36] first considered the following bistable reaction-diffusion equation,

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(x, t) + f(u(x, t)), \quad x \in \mathbb{R}^n, t > 0,$$

where $f(u) = u(1 - u)(u - \theta)$ for some $\theta \in (0, 1/2)$. In fact, he obtained the multidimensional stability of planar traveling waves of (1.1) via an application of linear
semigroup theory. He showed that if the perturbation of a planar traveling wave is small enough in $H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the solution of the initial value problem converges to the planar traveling wave in $H^m(\mathbb{R}^n)$ as $t$ goes to infinity with rate $O(t^{-\frac{n-1}{2}})$. Levermore and Xin [17] further investigated the same problem by using the maximum principle and spectral theory. They proved that the planar traveling waves of (1.1) are stable in $L^2_{loc}(\mathbb{R}^n)$ for $n \geq 2$. Matano et al. [24] obtained that planar traveling waves of (1.1) are asymptotically stable under almost periodic perturbation or under any possibly large initial perturbations which decay at space infinity. Furthermore, they also found a special solution that oscillates permanently between two planar traveling waves, which implies that planar traveling waves are not asymptotically stable under more general perturbations. Matano and Nara [23] extended the results in [24] and obtained that the planar traveling waves are asymptotically stable in $L^\infty(\mathbb{R}^n)$ under spatially ergodic perturbations, which include quasi-periodic and almost periodic ones as special cases. We can study more works of the multidimensional stability of traveling waves by referring to [1, 2, 14, 23, 24, 30, 31, 35, 42] and references therein for more details.

Mei and Wang [26] considered the following Fisher-KPP type reaction-diffusion equation

$$\frac{\partial u(t, x)}{\partial t} = D \Delta u(t, x) - d(u(t, x)) + \int_{\mathbb{R}^n} f_\alpha(y)b(u(t-\tau, x-y))dy, \quad x \in \mathbb{R}^n, t > 0 \quad (1.3)$$

where $D > 0$ denotes the diffusion rate and $d(u), b(u)$ are nonnegative nonlinear functions. They obtained that all noncritical planar traveling waves are exponentially stable and critical planar traveling waves are algebraically stable in the form $t^{-\frac{1}{2}}$ by using weighted energy method and Fourier transform. Huang et al. [10] extended the results in [26] to the nonlocal diffusion equations. Chern et al. [4] studied the stability of non-monotone critical traveling waves for reaction–diffusion equations with time-delay. The adopted method is the technical weighted-energy method with some new flavors to handle the critical oscillatory waves. We can refer to [11, 19, 27] and the references therein for more results on the study of the stability of oscillatory traveling waves.

Very recently, Faye [6] extended the local diffusion equation (1.1) to the following nonlocal diffusion equation and investigated the multidimensional stability of planar traveling waves by using semigroup estimates,

$$\frac{\partial u(t, x)}{\partial t} = \int_{\mathbb{R}^n} J(x-y)u(t, y)dy - u(t, x) + f(u(t, x)), \quad x \in \mathbb{R}^n, t > 0 \quad (1.4)$$

where $J(x)$ is the kernel function and $f$ is a smooth function with bistable type. They showed that if the traveling wave is spectrally stable in one-dimensional space, then it is stable in $n$–dimensional space under some special perturbations of planar traveling waves.

Although the multidimensional stability of planar traveling waves for scalar reaction-diffusion equation has been studied, little attention has been paid to systems especially with time delay in higher dimensional space. In this paper, we study the multidimensional stability of planar traveling waves of the following competitive Lotka-Volterra system with nonlocal diffusion,

$$\begin{cases} \frac{\partial u_1(t, x)}{\partial t} = d_1[J \ast u_1 - u_1](t, x) + r_1 u_1(t, x)[1-u_1(t, x)-b_1 u_2(t-\tau, x)], \\
\frac{\partial u_2(t, x)}{\partial t} = d_2[J \ast u_2 - u_2](t, x) + r_2 u_2(t, x)[1-b_2 u_1(t-\tau, x)-u_2(t, x)], \end{cases} \quad (1.4)$$
for $x \in \mathbb{R}^n, t > 0$, where time delay $\tau > 0$ and

$$ J \ast u_i(t, x) = \int_{\mathbb{R}^n} J(y)u_i(t, x - y)dy, \quad i = 1, 2. $$

In system (1.4), $d_i > 0$ denotes the nonlocal diffusion rate of the species $i$, $r_i > 0$ denotes the intrinsic rate of natural increase of species $i$ and $b_i$ denotes the competitive rate of the inter-species ($i = 1, 2$). The nonlocal kernel function $J$ satisfies

(H1): $J(x) \geq 0, J(x) = J(-x), x \in \mathbb{R}^n, \int_{\mathbb{R}^n} J(x)dx = 1;$

(H2): $\int_{\mathbb{R}^n} J(x)e^{-\lambda x_1}dx < \infty$ for all $\lambda > 0$;

(H3): The Fourier transform $\mathcal{F}[J](\eta) = 1 - K |\eta|^{2\alpha} + o(|\eta|^{2\alpha})$ with $K > 0$ and $\alpha > 0$, where $|\eta| = (\sum_{i=1}^{n} \eta_i^2)^{1/2}$.

The assumption (H1) is natural from a modeling point of view and (H2) is required to ensure the existence of the planar traveling waves to the system (1.4). By assumption (H3), as $\mathcal{F}[J](\eta) \sim -K |\eta|^2$ with $\alpha = 1$ for $\eta \to 0$, the operator $u \mapsto J \ast u - u$ approaches the local operator $K \Delta_{\mathbb{R}^n}$. Thus, we recover the classical local dispersal competitive system. Remark that we have $\mathcal{F}[J](\eta) \sim -1$ for $|\eta| \to +\infty$ such that $u \mapsto J \ast u - u$ is a bounded operator, which is a very different feature from the local operator (Laplacian operator). If we take $J(x) = \frac{1}{(2\pi \rho)^{\frac{n+1}{2}}} e^{-\frac{|x|^2}{2\rho^2}}, x \in \mathbb{R}^n$, the Fourier transform $\mathcal{F}[J](\eta) = e^{-\frac{\alpha^2}{\rho^2} |\eta|^2} \sim 1 - \frac{\alpha^2}{\rho^2} |\eta|^2$ for $\eta \to 0$. Thus, the assumptions (H1)–(H3) can be ensured.

Now, we first examine the competitive Lotka-Volterra system (1.4) without the nonlocal diffusion and time delay response term, which is reduced to

$$
\begin{align*}
\frac{du_1}{dt}(t) &= r_1 u_1(t)(1 - u_1(t) - b_1 u_2(t)), \\
\frac{du_2}{dt}(t) &= r_2 u_2(t)(1 - b_2 u_1(t) - u_2(t)).
\end{align*}
$$

(1.5)

The system (1.5) has four constant equilibria: $(0, 0), (0, 1), (1, 0)$ and coexistence equilibrium $(\frac{1-b_2}{1-b_1 b_2}, \frac{1-b_2}{1-b_1 b_2})$ with the condition $b_1 b_2 \neq 1$. By a phase diagram (see [28] for more details), we list the following asymptotic behavior of the solution as $t \to +\infty$.

(i): $(u_1(t), u_2(t)) \to (1, 0)$ if $0 < b_1 < 1 < b_2$;

(ii): $(u_1(t), u_2(t)) \to (0, 1)$ if $0 < b_2 < 1 < b_1$;

(iii): $(u_1(t), u_2(t)) \to (\frac{1-b_2}{1-b_1 b_2}, \frac{1-b_2}{1-b_1 b_2})$ if $0 < b_1, b_2 < 1$;

(iv): $(u_1(t), u_2(t)) \to (0, 1)$ or $(1, 0)$ (depending on the initial data) if $b_1, b_2 > 1$.

In this paper, we only treat cases (i) and (ii). Since (i) and (ii) are similar by exchanging the role of $u$ and $v$, in the sequel we always assume that $0 < b_1 < 1 < b_2$.

A planar traveling wave of system (1.4) is a special solution in the form of $(u_1, u_2)(t, x) = (\phi_1, \phi_2)(x \cdot \nu + ct)$ (where $\nu \in \mathbb{R}^n$ is a fixed unit vector) connecting the two equilibria of (1.4). In one-dimensional space, existence and stability of traveling waves of system (1.4) have been discussed in [9, 12, 18, 20]. However, to the best of our knowledge, there is no any results for multidimensional stability of planar traveling waves of competitive system (1.4) in high dimensional space. Here, the main purpose of the present paper is to investigate the multidimensional stability of planar traveling waves of system (1.4). We prove that all planar traveling waves with speed $c > c^*$ (here, $c^*$ is not the minimal speed and the definition of $c^*$ can be found in Remark 2.1) are exponentially stable in $L^\infty(\mathbb{R}^n)$ in the form of $t^{-\pi \epsilon} e^{-\epsilon \sigma t}$ by using the weighted energy method and Fourier transform, where
\( \sigma > 0 \) and \( \varepsilon(\tau) \in (0, 1) \) is a decreasing function for \( \tau > 0 \). Furthermore, the effect of time delay can essentially make the decay rate of the solution slow down. While, for \( c = c^* \), the planar traveling waves are algebraically stable like \( t^{-\frac{C}{2}} \) in \( L^\infty(\mathbb{R}^n) \).

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and present the main results of the existence and stability of planar traveling waves. In Section 3, we introduce the time delay ODE system and give the sufficient condition for stability of the trivial solution. In Section 4, we mainly prove the multidimensional stability of planar traveling waves, including the case of \( c = c^* \). In Section 5, we obtain the exact planar traveling waves in some special cases and further give some numerical simulations to illustrate the main results.

2. Preliminaries and main results. First, we introduce some necessary notations throughout this paper. \( C > 0 \) denotes a generic constant and \( C_i(i = 0, 1, 2, \ldots) \) represents a specific constant. Let \( \| \cdot \| \) and \( \| \cdot \|_\infty \) denote 1-norm and \( \infty \)-norm of the matrix (or vector), respectively. Let \( \Omega \) be a domain, typically \( \Omega = \mathbb{R}^n \). Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a multi-index with nonnegative integers \( \alpha_i \geq 0 (i = 1, 2, \ldots, n) \). The derivatives for function \( f(x) \) are denoted as \( \partial^\alpha f(x) = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} f(x) \). \( W^{k,p}(\Omega) (k \geq 1, p \geq 1) \) is the Sobolev space in which the function \( f(x) \) is defined on \( \Omega \) and its weak derivatives \( \partial^\alpha f(x) (|\alpha| \leq k) \) also belong to \( L^p(\Omega) \), and in particular, we denote \( W^{k,2}(\Omega) \) as \( H^k(\Omega) \). Further, \( L^p_w(\Omega) \) denotes the weighted \( L^p \) space with a weighted function \( w(x) > 0 \). Its norm is defined by

\[
\|f\|_{L^p_w(\Omega)} = \left( \int_{\Omega} w(x)|f(x)|^p \, dx \right)^{1/p}.
\]

\( W^{k,p}_w(\Omega) \) is the weighted Sobolev space with the norm

\[
\|f\|_{W^{k,p}_w(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} w(x)|\partial^\alpha f(x)|^p \, dx \right)^{1/p}.
\]

Fourier transform is defined as

\[
\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbb{R}^n} e^{-ix \cdot \eta} f(x) \, dx,
\]

and the inverse Fourier transform is given by

\[
\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \hat{f}(\eta) \, d\eta.
\]

In order to apply the comparison principle, we transform the competitive system (1.4) into a cooperative system by changing the variables \( \tilde{u}_1 = u_1, \tilde{u}_2 = 1 - u_2 \). Thus, we obtain the following system by dropping the tildes for the sake of convenience,

\[
\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= d_1[J * u_1 - u_1](t,x) + r_1 u_1(t,x)[1 - b_1 - u_1(t,x) + b_1 u_2(t, x)], \\
\frac{\partial u_2(t,x)}{\partial t} &= d_2[J * u_2 - u_2](t,x) + r_2 (1 - u_2(t,x))[b_2 u_1(t - \tau, x) - u_2(t,x)],
\end{aligned}
\]

for \( x \in \mathbb{R}^n, t > 0 \), with the initial condition

\[
u_i(s,x) = u_{i0}(s,x), \quad (s,x) \in [-\tau,0] \times \mathbb{R}^n, i = 1, 2.
\]

By the properties of the monotone semiflows [5], we have the following comparison principal.
Lemma 2.1 (Comparison Principle). Assume that (H1)-(H2) hold and \(0 < b_1 < 1 < b_2\). Let \(u^\pm(t,x) = (u_1^\pm, u_2^\pm)(t,x)\) be the solution of (2.1) with the initial data \(u_0^\pm(s,x) = (u_1^\pm, u_2^\pm)(s,x), s \in [-\tau, 0]\), respectively. If
\[
E_- := (0,0) \leq u^-_0(s,x) \leq u^+_0(s,x) \leq (1,1) := E_+,
\]
for \((s,x) \in [-\tau,0] \times \mathbb{R}^n, \) then
\[
(0,0) \leq u^-(t,x) \leq u^+(t,x) \leq (1,1),
\]
for \((t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.\)

If we look for a planar traveling wave \((u_1(t,x), u_2(t,x)) = (\phi_1(\zeta), \phi_2(\zeta))\) \((\zeta = \nu \cdot x + ct, \) where \(\nu \in \mathbb{R}^n\) is a fixed unit vector, here we set \(\nu = e_1 = (1,0,\cdots,0)\) for simplicity) of the system (2.1) connecting \(E_-\) and \(E_+\), then \(\phi_i\) has to satisfy the following system on the line
\[
\begin{cases}
\phi_1''(\zeta) + c_1\phi_1'(\zeta)(1 - b_1\phi_1(\zeta) + b_2 \phi_2(\zeta - c \tau)) = 0, \\
\phi_2''(\zeta) + c_2\phi_2'(\zeta)(1 - b_2\phi_1(\zeta - c \tau) - b_2 \phi_2(\zeta)) - \frac{c_1}{c_2}\phi_1(\zeta - c \tau) - \frac{c_2}{c_1}\phi_2(\zeta) = 0,
\end{cases}
\]
(\(\zeta \in \mathbb{R}\), \(\phi_1(\zeta), \phi_2(\zeta)(-\infty) = (0,0), (\phi_1, \phi_2)(+\infty) = (1,1),\)
\[
\text{(2.3)}
\]
where
\[
\hat{J}(\zeta) = \int_{\mathbb{R}^{n-1}} J(\zeta, x_2, \cdots, x_n)dx_2 \cdots dx_n.
\]
To obtain the existence of planar traveling waves, we consider the following function
\[
\Delta_1(\lambda, c) = d_1 \int_{\mathbb{R}^n} J(x)(e^{-\lambda x_1} - 1)dx - c\lambda + r_1(1 - b_1).
\]

By the properties of function \(\Delta_1(\lambda, c)\) (see [21, Lemma 2.2] for more details), we have the following lemma.

Lemma 2.2. Under the conditions (H1)-(H2) and \(b_1 < 1\), there exist \(\lambda_* > 0\) and \(c_* > 0\) such that
\[
\Delta_1(\lambda_*, c_*) = 0, \quad \frac{\partial \Delta_1(\lambda, c)}{\partial \lambda} \bigg|_{(\lambda_*,c_*)} = 0.
\]
Furthermore,
- if \(0 < c < c_*\), we have \(\Delta_1(\lambda, c) > 0\) for all \(\lambda > 0\);
- if \(c > c_*\), the equation \(\Delta_1(\lambda, c) = 0\) has two positive real roots \(\lambda_1 = \lambda_i(c)(i = 1, 2)\) with \(0 < \lambda_1 < \lambda_* < \lambda_2 < +\infty\), and
\[
\Delta_1(\lambda, c) \begin{cases}
< 0, & \lambda \in (\lambda_1, \lambda_2), \\
> 0, & \lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty).
\end{cases}
\]

To obtain the multidimensional stability of the planar traveling waves, we consider the following function
\[
\Delta_2(\lambda, c) = d_1 \int_{\mathbb{R}^n} J(x)e^{-\lambda x_1}dx - d_1 - c\lambda + r_1(1 + b_1 q),
\]
where \(q = e^{-\lambda \tau} \max\left\{1, \frac{r_2}{r_1 b_1}\right\}.\)

Remark 2.1. For function \(\Delta_2(\lambda, c)\), by the Lemma 2.2, there exist \(\lambda^* > 0\) and \(c^* > 0\) such that \(\Delta_2(\lambda^*, c^*) = 0, \frac{\partial \Delta_2(\lambda, c)}{\partial \lambda} \bigg|_{(\lambda^*,c^*)} = 0\). Due to the properties of function \(\Delta_i(\lambda, c)(i = 1, 2)\) and \(\Delta_2(\lambda, c) - \Delta_1(\lambda, c) = r_1 b_1 (1 + q) > 0\), we can verify that \(c^* > c_*\) by simple analysis.
Li et al. [18] obtained the existence, uniqueness and asymptotic behaviour of the solution of the system (2.3) by using upper-lower solutions, Schauder’s fixed point theorem and the sliding method.

**Theorem 2.1 (Existence).** Assume that (H1)-(H2) hold and \( d_1 \geq d_2, r_1 \geq r_2, 0 < b_1 < 1 < b_2, r_1(1-b_1) \geq r_2(b_2-1) \). For any \( c \geq c_* \), there is a unique solution \( \Phi(\zeta) = (\phi_1, \phi_2)(\zeta) \) of system (2.3) connecting the equilibria \((0,0)\) and \((1,1)\), with the wave profile component \( \phi_i \) increasing. Moreover,
\[
\lim_{\zeta \to -\infty} \frac{\phi_1(\zeta)}{e^{\lambda_1 \zeta}} = 1, \quad \lim_{\zeta \to -\infty} \frac{\phi_1'(\zeta)}{e^{\lambda_1 \zeta}} = \lambda_1. \tag{2.4}
\]

For any \( c < c_* \), system (2.3) has no solution \((\phi_1, \phi_2)(\zeta)\) satisfying (2.4) with the wave speed \( c \) which connects \((0,0)\) and \((1,1)\).

Let \( c \geq c^* \) and \( (\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct)) \) be the planar traveling wave of (2.1) with the speed \( c \) connecting \( E_- \) and \( E_+ \). Now, we define a weighted function as
\[
w(x) = \begin{cases} 
e^{-\lambda^*(x_1-\zeta_0)}, & x_1 \leq \zeta_0, \\ 1, & x_1 > \zeta_0, \end{cases}
\tag{2.5}
\]
where \( \zeta_0 \) is a very large constant and \( \lambda^* \) is defined in Remark 2.1.

Here, we present the main results of this paper.

**Theorem 2.2 (Stability).** Assume that conditions in Theorem 2.1 hold. For any given planar traveling wave \( (\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct)) \) of the system (2.1) with the speed \( c \geq c^* \) connecting \( E_- \) and \( E_+ \), if the initial data satisfies
\[
0 \leq u_{i0}(s, x) \leq 1, \quad (s, x) \in [-\tau, 0] \times \mathbb{R}^n, \quad i = 1, 2, \tag{2.6}
\]
and the initial perturbation
\[
u_{i0} - \phi_i \in C^1([-\tau, 0], W^{1,1}_w(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)), \quad i = 1, 2, \tag{2.7}
\]
then nonnegative solution of the Cauchy problem (2.1) and (2.2) uniquely exists and satisfies
\[
(0, 0) \leq (u_1(t, x), u_2(t, x)) \leq (1, 1), \quad x \in \mathbb{R}^n, \quad t > 0,
\]
and
\[
u_i - \phi_i \in C^1([0, +\infty), W^{1,1}_w(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)), \quad i = 1, 2.
\]
Furthermore, if \( \mathcal{F}(u_{i0} - \phi_i) \in C^1([-\tau, 0], W^{1,1}_w(\mathbb{R}^n)) (i = 1, 2) \) and the assumption (H3) holds, we have

(i): when \( c > c^* \), the solution \((u_1(t, x), u_2(t, x))\) converges to the planar traveling wave \((\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct))\) exponentially in time, i.e.,
\[
\sup_{x \in \mathbb{R}^n} |u_i(t, x) - \phi_i(x \cdot e_1 + ct)| \leq CE_{\varepsilon t} - \frac{\lambda^*}{2} \varepsilon e^{-\varepsilon \tau t}, \quad t > 0, \quad i = 1, 2,
\]
for some constant \( \sigma > 0 \), where \( \varepsilon_\tau = \varepsilon(\tau) \in (0, 1) \) is a decreasing function for \( \tau > 0 \), and
\[
\mathcal{E}_0 = \max_{s \in [-\tau, 0]} \sum_{i=1}^2 \left( \| \partial_s(u_{i0} - \phi_i)(s, \cdot) \|_{L^\infty_w(\mathbb{R}^n)} + \| (u_{i0} - \phi_i)(s, \cdot) \|_{W^{1,1}_w(\mathbb{R}^n)} + \| \partial_s \mathcal{F}(u_{i0} - \phi_i)(s, \cdot) \|_{L^\infty_w(\mathbb{R}^n)} + \| \mathcal{F}(u_{i0} - \phi_i)(s, \cdot) \|_{W^{1,1}_w(\mathbb{R}^n)} \right). \tag{2.8}
\]
(ii): When \( c = c^* \), the solution \((u_1(t, x), u_2(t, x))\) converges to the planar traveling wave \((\phi_1(x \cdot c_1 + c^* t), \phi_2(x \cdot c_1 + c^* t))\) algebraically in time, i.e.,

\[
\sup_{x \in \mathbb{R}^n} |u_i(t, x) - \phi_i(x \cdot c_1 + ct)| \leq C \mathcal{E}_0 t^{-\frac{\sigma}{\alpha}}, \quad t > 0, \quad i = 1, 2.
\]

**Remark 2.2.** The proof of Theorem 2.2 is mainly motivated by [26, 37], but there is some difference. In one-dimensional space, Yü et al. [37] considered the system 1.4 without time delay and proved the stability of traveling waves with speed \( c > c^* \), which will play a key role in the stability proof in Section 4. Moreover, we can obtain the algebraic stability of planar traveling waves with speed \( c = c^* \).

3. **Linearized delay differential system.** In this section, we will derive the solution formulas for the linearized delay differential system and their decay rates, which will play a key role in the stability proof in Section 4.

Now let us consider the following delay differential system,

\[
\begin{align*}
\frac{d}{dt} z(t) &= A z(t) + B z(t - \tau), \quad t > 0, \\
z(s) &= z_0(s), \quad s \in [-\tau, 0],
\end{align*}
\]

where \( A, B \in \mathbb{C}^{N \times N} \) and \( \tau > 0 \) denotes a time delay. In [15], Khusainov and Ivanov presented the solution formula of (3.1) in the case of space dimension \( N = 1 \) and \( A, B \in \mathbb{R} \). Similarly, we can obtain the solution formula of system (3.1) in general case of space dimension \( N \geq 2 \) and \( A, B \in \mathbb{C}^{N \times N} \).

**Lemma 3.1.** If the initial data \( z_0(s) \in C^1([-\tau, 0], \mathbb{C}^N) \), then the solution of system (3.1) can be shown as

\[
z(t) = e^{A(t+\tau)} e^{B_1 t} z_0(-\tau) + \int_{-\tau}^0 e^{A(t-s)} e^{B_1 (t-s-\tau)} [z_0'(s) - A z_0(s)] ds,
\]

where \( B_1 = B e^{-A \tau} \) and \( e^{B_1 t} \) is the so-called delayed exponential function in the form

\[
e^{B_1 t} = \begin{cases} 
0, & -\infty < t < -\tau, \\
1, & -\tau \leq t < 0, \\
1 + B_1 t, & 0 \leq t < \tau, \\
1 + B_1 t + \frac{B_1^2}{2!} (t - \tau)^2, & \tau \leq t < 2\tau, \\
\vdots & \\
1 + B_1 t + \frac{B_1^2}{2!} (t - \tau)^2 + \cdots + \frac{B_1^m}{m!} [t - (m-1)\tau]^m, & (m-1)\tau \leq t < m\tau,
\end{cases}
\]

(3.2)

Now, we are going to give a sufficient condition of the global stability for the trivial solution of the linear delay system (3.1).
Theorem 3.1. Suppose \( \mu(A) := \frac{\mu_1(A) + \mu_{\infty}(A)}{2} < 0 \), where \( \mu_1(A) \) and \( \mu_{\infty}(A) \) denote the matrix measure of \( A \) induced by the matrix \( 1 \)-norm \( \| \cdot \| \) and \( \infty \)-norm \( \| \cdot \|_{\infty} \), respectively. If \( \nu(B) := \frac{\|B\| + \|B\|_{\infty}}{2} \leq -\mu(A) \), then there exists a decreasing function \( \varepsilon_\tau = \varepsilon(\tau) \in (0, 1) \) for \( \tau > 0 \) such that any solution of system (3.1) satisfies
\[
\|z(t)\| \leq C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0,
\]
where \( C_0 \) is a positive constant depending on initial data \( z_0(s), s \in [-\tau, 0] \) and \( \sigma = |\mu(A)| - \nu(B) \). In particular,
\[
\|e^{At}\|_{\infty} \leq C_0 e^{-\varepsilon_\tau \sigma t}, \quad t > 0.
\]
where \( e^{At} \) is defined in (3.2).

Remark 3.1. From Theorem 3.1, the condition \( \mu(A) + \nu(B) < 0 \) implies that the trivial solution of system (3.1) is global exponentially stable whatever the size of delay.

Proof. Let \( z(t) \) be any solution of the linear delay system (3.1) for \( t \geq -\tau \)
\[
W_1(t, z(\cdot)) = \|z(t)\|_2^2 := \sum_{i=1}^N |z_i(t)|^2.
\]
From [33], we have
\[
\mu_1(A) = \lim_{\theta \to 0^+} \frac{\|I + \theta A\| - 1}{\theta} = \max_{1 \leq j \leq N} \left[ \Re(a_{jj}) + \sum_{j \neq i} |a_{ij}| \right],
\]
and
\[
\mu_{\infty}(A) = \lim_{\theta \to 0^+} \frac{\|I + \theta A\|_{\infty} - 1}{\theta} = \max_{1 \leq i \leq N} \left[ \Re(a_{ii}) + \sum_{j \neq i} |a_{ij}| \right].
\]
Then the derivative \( DW_1 \) of \( W_1 \) along the system (3.1),
\[
DW_1(t, z(\cdot))
\]
\[
= \sum_{i=1}^N \left[ \tilde{z}_i(t) \dot{z}_i(t) + z_i(t) \tilde{z}_i(t) \right]
\]
\[
= \sum_{i=1}^N \left\{ \tilde{z}_i(t) \sum_{j=1}^N \left[ a_{ij} z_j(t) + b_{ij} z_j(t-\tau) \right] + z_i(t) \sum_{j=1}^N \left[ \bar{a}_{ij} \tilde{z}_j(t) + \bar{b}_{ij} \tilde{z}_j(t-\tau) \right] \right\}
\]
\[
= \sum_{i=1}^N \left\{ a_{ii} z_i(t) \tilde{z}_i(t) + \sum_{j \neq i} a_{ij} \tilde{z}_i(t) z_j(t) + \sum_{j=1}^N b_{ij} \tilde{z}_i(t) z_j(t-\tau) \right\}
\]
\[
+ \sum_{i=1}^N \left\{ \bar{a}_{ii} z_i(t) \tilde{z}_i(t) + \sum_{j \neq i} \bar{a}_{ij} z_i(t) \tilde{z}_j(t) + \sum_{j=1}^N \bar{b}_{ij} z_i(t) \tilde{z}_j(t-\tau) \right\}
\]
\[
= \sum_{i=1}^N \left[ 2\Re(a_{ii}) |z_i(t)|^2 + 2 \sum_{j \neq i} \Re(a_{ij} \tilde{z}_i(t) z_j(t)) + 2 \sum_{j=1}^N \Re(b_{ij} \tilde{z}_i(t) z_j(t-\tau)) \right]
\]
\[
\leq \sum_{i=1}^N \left[ 2\Re(a_{ii}) |z_i(t)|^2 + \sum_{j \neq i} |a_{ij}| (|z_i(t)|^2 + |z_j(t)|^2) + \sum_{j=1}^N |b_{ij}| (|z_i(t)|^2 + |z_j(t-\tau)|^2) \right]
\]
where

\[ 4. \text{ Stability. (Global Existence and Uniqueness)} \]

Proposition 4.1

Thus, we present the following proposition and omit its proof.

The global existence and uniqueness of the solution to the Cauchy problem (2.1) and (2.2) can be proved via the standard energy method and continuity extension method \([10, 25]\) or the theory of abstract functional differential equation \([22]\). Thus, we present the following proposition and omit its proof.

**Proposition 4.1 (Global Existence and Uniqueness).** Assume that the initial data satisfies

\[ 0 \leq u_{i0}(s, x) \leq 1, \quad s \in [-\tau, 0], x \in \mathbb{R}^n, i = 1, 2. \]
For any given planar traveling wave \((\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct))\) of (2.1) with speed \(c \geq c^*\) connecting the equilibria \(E_-\) and \(E_+\), if the initial perturbation satisfies
\[
u_{i0} - \phi_i \in C^1([\tau, 0], W^{1,1}_w(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)), \quad i = 1, 2,
\]
then there exists a unique global solution \((u_1(t, x), u_2(t, x))\) of the Cauchy problem (2.1) and (2.2) such that
\[
(0, 0) \leq (u_1(t, x), u_2(t, x)) \leq (1, 1), \quad t > 0, x \in \mathbb{R}^n,
\]
and
\[
u_i - \phi_i \in C^1([0, +\infty), W^{1,1}_w(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)), \quad i = 1, 2.
\]
where the function \(w(x)\) is defined by (2.5).

Let \(c \geq c^*\) and initial data \(\nu_{i0}(s, x)\) be such that \(0 \leq \nu_{i0}(x) \leq 1, x \in \mathbb{R}^n\), and define
\[
\bar{u}^+_i = \max \{\nu_{i0}(s, x), \phi_i(x \cdot e_1 + cs)\},
\]
\[
\bar{u}^-_i = \min \{\nu_{i0}(s, x), \phi_i(x \cdot e_1 + cs)\},
\]
for \(x \in \mathbb{R}^n, i = 1, 2\). We can easily obtain
\[
0 \leq \bar{u}^-_i(s, x) \leq \bar{u}^+_i(s, x) \leq u_{i0}(s, x) \leq u_{i0}^+(s, x) \leq 1,
\]
for \(x \in \mathbb{R}^n, i = 1, 2\). Clearly, the initial data \(\bar{u}^{\pm}_i(s, x)(i = 1, 2)\) are piecewise continuous and have a poor regularity, which may also cause the absence of regularity for the corresponding solutions. In order to overcome such a shortcoming, we choose smooth functions \(u^{\pm}_i(s, x)\) instead of these initial data as the new initial data such that
\[
0 \leq u^-_i(s, x) \leq u^+_i(s, x) \leq u_{i0}(s, x) \leq u^{+}_i(s, x) \leq 1,
\]
for \(s \in [-\tau, 0], x \in \mathbb{R}^n, i = 1, 2\).

Let \((u^+_i(t, x), u^-_i(t, x))\) be the corresponding solutions of (2.1) with the initial data \((u^+_i(s, x), u^-_i(s, x))\). Thus, it follows from the comparison principle in Lemma 2.1 that
\[
0 \leq u^-_i(t, x) \leq u^+_i(t, x) \leq u_{i0}(t, x) \leq u^{+}_i(t, x) \leq 1,
\]
for \(x \in \mathbb{R}^n, i = 1, 2\). Letting
\[
U_i(t, \xi) = u^+_i(t, x) - \phi_i(x \cdot e_1 + ct), \quad t > 0,
\]
\[
U_i(s, \xi) = u^+_i(s, x) - \phi_i(x \cdot e_1 + cs), \quad s \in [-\tau, 0],
\]
for \(x \in \mathbb{R}^n, t > 0, i = 1, 2\), where \(\xi = x + ct \cdot e_1 = (x_1 + ct, x_2, \ldots, x_n)(c \geq c^*)\), it follows from (4.2)-(4.4) that
\[
(0, 0) \leq (U_1(s, \xi), U_2(s, \xi)) \leq (1, 1), \quad s \in [-\tau, 0], \xi \in \mathbb{R}^n,
\]
and
\[
(0, 0) \leq (U_1(t, \xi), U_2(t, \xi)) \leq (1, 1), \quad t > 0, \xi \in \mathbb{R}^n.
\]
Clearly, it is easy to see that \((U_1(t, \xi), U_2(t, \xi))\) satisfies
\[
\begin{align*}
U_{1i} + c U_{1i}\xi & = d_1[J * U_1 - U_1] + r_1 U_1 \left[1 - b_1 - (U_1 + \phi_1) + b_1(U_2^+ + \phi_2^+ \right] \\
& + r_1\phi_i \left[b_1 U_2^+ - U_1\right] , \\
U_{2i} + c U_{2i}\xi & = d_2[J * U_1 - U_1] + r_2 U_2 \left[(U_2 + \phi_2) - b_2(U_1^+ + \phi_1^+ \right] \\
& + r_2[1 - \phi_2] \left[b_2 U_1^+ - U_2\right],
\end{align*}
\]
where \(U_i = U_i(t, \xi), U_i^+ = U_i(t - \tau, \xi - c(t - \tau) \cdot e_1), \phi_i = \phi_i(\xi \cdot e_1)\) and \(\phi_i^+ = \phi_i(\xi \cdot e_1 - c\tau)\) for \(t > 0, \xi \in \mathbb{R}^n, i = 1, 2\).
Assume that the conditions in Theorem 2.2 hold throughout this section. The following lemma shows us the decay rate of $U_i$ in $\Omega_- = (-\infty, \zeta_0) \times \mathbb{R}^{n-1}$.

**Lemma 4.1.** There exists a decreasing function $\epsilon_\tau = \epsilon(\tau) \in (0,1)$ such that

$$\sum_{i=1}^{2} \|U_i(t, \cdot)\|_{L^\infty(\Omega_-)} \leq C\epsilon_0 t^{-\frac{\delta}{\delta_0}} e^{-\epsilon_\tau t}, \quad t > 0, \ c > c^*, \quad (4.9)$$

and

$$\sum_{i=1}^{2} \|U_i(t, \cdot)\|_{L^\infty(\Omega_-)} \leq C\epsilon_0 t^{-\frac{\delta}{\delta_0}}, \quad t > 0, \ c = c^*, \quad (4.10)$$

where $\sigma_1 = \sigma_1(c)$ is defined in (4.18) and $\epsilon_0$ is given in (2.8).

**Proof.** Due to the nonnegative of $U_i$ and $U_i + \phi_i = u_i^+ \in (0,1), \ i = 1, 2$, we have

$$r_1 U_1 [1 - b_1 - (U_1 + \phi_1) + b_1 (U_2^+ + \phi_2^+)] + r_1 \phi_1 [b_1 U_2^+ - U_1]$$

$$\leq r_1 U_1 (1 - b_1 + b_1) + r_1 b_1 U_2^+$$

$$= r_1 (U_1 + b_1 U_2^+),$$

and

$$r_2 U_2 [(\phi_2 + U_2) - b_2 (U_1^+ + \phi_1^+)] + r_2 [1 - \phi_2] [b_2 U_1^+ - U_2]$$

$$\leq r_2 (b_2 U_1^+ + U_2).$$

Thus,

\[
\begin{aligned}
&V_{11}(t, \xi) + cV_{11}(t, \xi) \leq d_1 [J * U_1(t, \xi)] + r_1 [U_1(t, \xi) + b_1 U_2(t-\tau, \xi - \epsilon_\tau)] + r_1 \phi_1 [b_1 U_2(t, \xi) - U_1], \\
&V_{22}(t, \xi) + cV_{22}(t, \xi) \leq d_2 [J * U_2(t, \xi)] + r_2 [U_2(t-\tau, \xi - \epsilon_\tau)] + r_2 \phi_2 [b_2 U_1(t, \xi) - U_2],
\end{aligned}
\]

for $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$. Let $v(t, \xi) = (v_1(t, \xi), v_2(t, \xi))^T$ be the solution of the following system with the same initial data $(U_{10}(s, \xi), U_{20}(s, \xi))$,

\[
\begin{aligned}
v_{11}(t, \xi) + cV_{11}(t, \xi) &\leq d_1 [J * v_1(t, \xi)] + r_1 [v_1(t, \xi) + b_1 v_2(t-\tau, \xi - \epsilon_\tau)] + r_1 \phi_1 [b_1 v_2(t, \xi) - v_1], \\
v_{22}(t, \xi) + cV_{22}(t, \xi) &\leq d_2 [J * v_2(t, \xi)] + r_2 [v_2(t-\tau, \xi - \epsilon_\tau)] + r_2 \phi_2 [b_2 v_1(t, \xi) - v_2],
\end{aligned}
\]

for $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, we can obtain

\[
(U_1(t, \xi), U_2(t, \xi)) \leq (v_1(t, \xi), v_2(t, \xi)), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n.
\]

Letting $V_i(t, \xi) = e^{-\lambda^\star (\xi - \epsilon_\tau)} v_i(t, \xi), \ i = 1, 2$. Then, we have

\[
\begin{aligned}
V_{11} + cV_{11} &\leq d_1 [J \ast V_1(t, \xi)] + r_1 (1 - c\lambda^\star) V_1(t, \xi) + r_1 b_1 e^{-\epsilon_\tau t} V_2(t-\tau, \xi - \epsilon_\tau) + r_1 \phi_1 [b_1 V_2(t, \xi) - V_1(t, \xi)], \\
V_{22} + cV_{22} &\leq d_2 [J \ast V_2(t, \xi)] + r_2 (1 - c\lambda^\star) V_2(t, \xi) + r_2 b_2 e^{-\epsilon_\tau t} V_1(t-\tau, \xi - \epsilon_\tau) + r_2 \phi_2 [b_2 V_1(t, \xi) - V_2(t, \xi)],
\end{aligned}
\]

where $J_{\lambda^\star}(y) = J(y) e^{-\lambda^\star y_1}$ and

\[
J_{\lambda^\star} V_i(t, \xi) = \int_{\mathbb{R}^n} J(y) e^{-\lambda^\star y_1} V_i(t, \xi - y) dy, \quad i = 1, 2.
\]

Taking Fourier transform to the system (4.13), we have

\[
\begin{aligned}
\tilde{V}_{11}(t, \eta) &\leq d_1 \tilde{J}_{\lambda^\star}(\eta) - d_1 + r_1 (1 - c\lambda^\star) - ic\eta_1 \tilde{V}_{11}(t, \eta) + r_1 b_1 e^{-\epsilon_\tau (\lambda^\star + ic\eta)} \tilde{V}_2(t-\tau, \eta), \\
\tilde{V}_{22}(t, \eta) &\leq d_2 \tilde{J}_{\lambda^\star}(\eta) - d_1 + r_2 (1 - c\lambda^\star) - ic\eta_1 \tilde{V}_{22}(t, \eta) + r_2 b_2 e^{-\epsilon_\tau (\lambda^\star + ic\eta)} \tilde{V}_1(t-\tau, \eta),
\end{aligned}
\]

(4.14)
where \( \hat{J}_{\lambda^*}(\eta) = \mathcal{F}[J_{\lambda^*}](\eta) \) and

\[
\hat{V}_i(t, \eta) = \mathcal{F}[V](t, \eta) := \int_{\mathbb{R}^n} e^{-i\xi \cdot \eta} V_i(t, \xi) d\xi, \quad i = 1, 2.
\]

Let

\[
A(\eta) = \begin{pmatrix}
  d_1 \hat{J}_{\lambda^*}(\eta) - d_1 + r_1 - c\lambda^* - ic\eta_1 & 0 \\
  0 & d_2 \hat{J}_{\lambda^*}(\eta) - d_2 + r_2 - c\lambda^* - ic\eta_1
\end{pmatrix},
\]

and

\[
B(\eta) = \begin{pmatrix}
  0 & r_1 b_1 e^{-c\tau(\lambda^* + ic\eta_1)} \\
  r_2 b_2 e^{-c\tau(\lambda^* + ic\eta_1)} & 0
\end{pmatrix}.
\]

Then, system (4.14) can be rewritten as

\[
\hat{V}_i(t, \eta) = A(\eta) \hat{V}(t, \eta) + B(\eta) \hat{V}(t - \tau, \eta),
\]

where \( \hat{V}(t, \eta) = (\hat{V}_1(t, \eta), \hat{V}_2(t, \eta))^T \).

By Lemma 3.1, the solution (4.15) can be shown as

\[
\hat{V}(t, \eta) = e^{A(\eta)(t+\tau)} e^{B_1(\eta)t} \hat{V}_0(-\tau, \eta) + \int_{-\tau}^{0} e^{A(\eta)(t-s)} e^{B_1(\eta)(t-s-\tau)} \left[ \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right] ds,
\]

\[
= I_1(t, \eta) + \int_{-\tau}^{0} I_2(t - s, \eta) ds,
\]

where \( B_1(\eta) = B(\eta)e^{A(\eta)\tau} \). Thus, by taking the inverse Fourier transform to (4.16), we have

\[
V(t, \xi) = \mathcal{F}^{-1}[I_1](t, \xi) + \int_{-\tau}^{0} \mathcal{F}^{-1}[I_2](t - s, \xi) ds
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot \eta} e^{A(\eta)(t+\tau)} e^{B_1(\eta)t} \hat{V}_0(-\tau, \eta) d\eta
\]

\[
+ \int_{-\tau}^{0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot \eta} e^{A(\eta)(t-s)} e^{B_1(\eta)(t-s-\tau)} \left[ \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right] d\eta ds.
\]

By Euler’s formula, we have

\[
\text{Re}(\hat{J}_{\lambda^*}(\eta)) = \text{Re} \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} e^{-iy \cdot \eta} dy = \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} \cos(y \cdot \eta) dy
\]

\[
= \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} dy + \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} [\cos(y \cdot \eta) - 1] dy
\]

\[
= \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} dy + \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} + e^{-\lambda^* y_1} [\cos(y \cdot \eta) - 1] dy
\]

\[
\leq \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} dy + \int_{\mathbb{R}^n} J(y)[\cos(y \cdot \eta) - 1] dy
\]

\[
= \int_{\mathbb{R}^n} J(y) e^{-\lambda^* y_1} dy + \hat{J}(\eta) - 1,
\]
since $e^{-\frac{1}{2}x^2} \geq 1$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}^n} J(y) \sin(y \cdot \eta) dy = 0$ (because $J$ is even and $\sin(y \cdot \eta)$ is odd). By assumption (H3), there exist $\eta_0 > 0$ and $\delta \in (0, 1)$ such that

$$\left\{ \begin{array}{l}
1 - \hat{J}(\eta) \geq \frac{K}{2} |\eta|^{2\alpha}, \quad |\eta| \leq \eta_0, \\
1 - \hat{J}(\eta) \geq \delta, \quad |\eta| \geq \eta_0.
\end{array} \right.$$ 

By the definition of $\mu(\cdot)$ and $\nu(\cdot)$, we have

$$|\mu(A(\eta)) - \nu(B(\eta))| = \left| \max \left\{ d_1 \text{Re}(\hat{J}_{\lambda^*}) - d_1 + r_1 - c\lambda^*, d_2 \text{Re}(\hat{J}_{\lambda^*}) - d_2 + r_2 - c\lambda^* \right\} \right|$$

$$\geq c\lambda^* - r_1 - d_1 \int_{\mathbb{R}^n} J(y)(e^{-\lambda^* y_1} - 1) dy - d_1(\hat{J}(\eta) - 1)$$

$$- r_1 b_1 e^{-c\tau \lambda^*} \max \left\{ 1, \frac{r_2 b_2}{r_1 b_1} \right\}$$

$$= - \Delta_2(\lambda^*, c) + d_1(1 - \hat{J}(\eta)),$$

where

$$\Delta_2(\lambda^*, c) = d_1 \int_{\mathbb{R}^n} J(y)(e^{-\lambda^* y_1} - 1) dy - c\lambda^* + r_1 + r_1 b_1 e^{-c\tau \lambda^*} \max \left\{ 1, \frac{r_2 b_2}{r_1 b_1} \right\}.$$ 

It follows from Theorem 3.1 that there exists a decreasing function $\varepsilon_\tau = \varepsilon(\tau) \in (0, 1)$ such that

$$\|e^{A(\eta)t} e^{B_1(\eta)t} - e^{A(\eta)t} e^{B_1(\eta)t} - e^{-\varepsilon_\tau \sigma_1 t} e^{-\varepsilon_\tau d_1(1-J)t}\| \leq C_1,$$

where $C_1$ is a positive constant depending on $\hat{V}_0(s), s \in [-\tau, 0]$ and

$$\sigma_1 := -\Delta_2(\lambda^*, c) \left\{ \begin{array}{ll}
> 0, & c > c^*, \\
= 0, & c = c^*.
\end{array} \right.$$ (4.18)

By the definition of Fourier transform, we have

$$\sup_{\eta \in \mathbb{R}^n} \|\hat{V}_0(-\tau, \eta)\| \leq \int_{\mathbb{R}^n} ||V_0(-\tau, x)|| dx = 2 \sum_{i=1}^2 ||V_0(-\tau, x)||_{L^2(\mathbb{R}^n)}.$$ 

Thus,

$$\sup_{\xi \in \mathbb{R}^n} \|\mathcal{F}^{-1} [I_1](t, \xi)\|$$

$$= \sup_{\xi \in \mathbb{R}^n} \left\{ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \xi \cdot \eta} e^{A(\eta)(t+\tau)} e^{B_1(\eta)t} \hat{V}_0(-\tau, \eta) d\eta \right\}$$

$$\leq C \int_{\mathbb{R}^n} e^{-\varepsilon_\tau \sigma_1 t} e^{-\varepsilon_\tau d_1(1-J)t} \|\hat{V}_0(-\tau, \eta)\| d\eta$$

$$\leq C e^{-\varepsilon_\tau \sigma_1 t} \left( \int_{|\eta| \leq \eta_0} + \int_{|\eta| \geq \eta_0} \right) e^{-\varepsilon_\tau d_1(1-J)t} \|\hat{V}_0(-\tau, \eta)\| d\eta$$

$$\leq C e^{-\varepsilon_\tau \sigma_1 t} \left[ \sup_{\eta \in \mathbb{R}^n} ||V_0(-\tau, \eta)|| \int_{|\eta| \leq \eta_0} e^{-\varepsilon_\tau d_1|\eta|^{2\alpha} t} d\eta + \int_{|\eta| \geq \eta_0} e^{-\varepsilon_\tau d_1|\eta|} d\eta \right]$$

$$\leq C e^{-\varepsilon_\tau \sigma_1 t} \left[ t^{-\frac{\delta}{2\alpha}} \sup_{\eta \in \mathbb{R}^n} \|\hat{V}_0(-\tau, \eta)\| + e^{-\varepsilon_\tau d_1|\eta|} \int_{\mathbb{R}^n} ||\hat{V}_0(-\tau, \eta)|| d\eta \right]$$
\begin{align}
\sup_{\eta \in \mathbb{R}^n} \left| \hat{J}_{\lambda}(\eta) \hat{V}_j(t, \eta) \right| &= \sup_{\eta \in \mathbb{R}^n} \left| \mathcal{F}[J_{\lambda} \ast \hat{V}_j](t, \eta) \right| \\
&= \sup_{\eta \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(y)e^{-\lambda^*y_1} \hat{V}_j(t, \xi - y) e^{-i\xi \cdot \eta} dy d\xi \right| \\
&\leq \int_{\mathbb{R}^n} J(y)e^{-\lambda^*y_1} \| \hat{V}_j(t, \cdot) \|_{L^1(\mathbb{R}^n)},
\end{align}

and

\begin{align}
\sup_{\eta \in \mathbb{R}^n} \left| (i\eta_j) \hat{V}_j(t, \eta) \right| &= \sup_{\eta \in \mathbb{R}^n} \left| \mathcal{F}[\partial_x \hat{V}_j](t, \eta) \right| \\
&\leq \int_{\mathbb{R}^n} |\partial_x \hat{V}_j(t, x)| dx = \| \partial_x \hat{V}_j(t, \cdot) \|_{L^1(\mathbb{R}^n)},
\end{align}

for $j = 1, 2$. Thus,

\begin{align}
\sup_{\eta \in \mathbb{R}^n} \| A(\eta) \hat{V}_0(s, \eta) \| &\leq C \sum_{i=1}^{2} \| V_{0,i}(s, \cdot) \|_{W^{1,1}(\mathbb{R}^n)}.
\end{align}

Similarly, we have

\begin{align}
\sup_{\xi \in \mathbb{R}^n} \left\| \mathcal{F}^{-1}[I_2](t - s, \xi) \right\| \\
&= \sup_{\xi \in \mathbb{R}^n} \left\| \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{i\xi \cdot \eta} e^{A(\eta)(t-s)} e^{B_{\tau}(\eta)(t-s-\tau)} \left[ \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right] d\eta \right\| \\
&\leq C \int_{\mathbb{R}^n} e^{-\varepsilon \sigma_1(t-s)} e^{-\varepsilon d_1(1-J)(t-s)} \left\| \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right\| d\eta \\
&= C e^{-\varepsilon \sigma_1(t-s)} \left( \int_{|\eta| \leq \eta_0} e^{-\varepsilon d_1(1-J)(t-s)} \left\| \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right\| d\eta \\
&+ C e^{-\varepsilon \sigma_1(t-s)} \sup_{\eta \in \mathbb{R}^n} \left\| \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right\| \int_{|\eta| \leq \eta_0} e^{-\frac{\varepsilon}{2} |\eta|^{2n}(t-s)} d\eta \\
&+ C \varepsilon \int_{|\eta| \geq \eta_0} e^{-\varepsilon d_1(1-J)(t-s)} \left\| \partial_s \hat{V}_0(s, \eta) - A(\eta) \hat{V}_0(s, \eta) \right\| d\eta \\
&\leq C \varepsilon_1(s)(t-s) - \frac{\varepsilon}{2} e^{-\varepsilon \sigma_1(t-s)}. 
\end{align}
Because the conditions \( u_{i0} - \phi_i, F(u_{i0} - \phi_i) \in C^1([-\tau, 0], W^{1,1}(\mathbb{R}^n)), i = 1, 2, \) we have
\[
\max\{u_{i0}, \phi_i\} - \phi_i, F(\max\{u_{i0}, \phi_i\} - \phi_i) \in C^1([-\tau, 0], W^{1,1}(\mathbb{R}^n)), \quad i = 1, 2.
\]
Thus, by \( V_{i0}(s, \xi) = e^{-\lambda \tau (\xi - \xi_0)} V_i(s, \xi) \leq w(\xi_1) v_i(s, \xi), s \in [-\tau, 0], \xi \in \mathbb{R}^n \) and (4.1)–(4.5), we have \( \max_{s \in [-\tau, 0]} E_i(s) \leq CE_0. \) Substituting (4.19) and (4.20) to (4.17), we can obtain
\[
\sum_{i=1}^{2} \| V_i(t, \cdot) \|_{L^\infty(\mathbb{R}^n)} \leq CE_0 t^{-\frac{n}{\alpha}} e^{-\gamma t}.
\] (4.21)
Because of \( e^{\lambda \tau (\xi - \xi_0)} \leq 1 \) for \( \xi_1 \leq \xi_0, \) we have
\[
0 \leq U_i(t, \xi) \leq v_i(t, \xi) = e^{\lambda \tau (\xi - \xi_0)} V_i(t, \xi) \leq V_i(t, \xi), \quad i = 1, 2,
\] (4.22)
for \( t > 0, \xi \in \Omega_- \). Thus, (4.9) and (4.10) can be immediately obtained by (4.21) and (4.22).

Next, we will prove the decay rate of \( U_i \) in \( \Omega_+ = [\xi_0, +\infty) \times \mathbb{R}^{n-1} \).

**Lemma 4.2.** It holds that
\[
\sum_{i=1}^{2} \| U_i(t, \cdot) \|_{L^\infty(\Omega_+)} \leq CE_0 t^{-\frac{n}{\alpha}} e^{-\gamma t}, \quad t > 0, c > c^*,
\] (4.23)
and
\[
\sum_{i=1}^{2} \| U_i(t, \cdot) \|_{L^\infty(\Omega_+)} \leq CE_0 t^{-\frac{n}{\alpha}}, \quad t > 0, c = c^*,
\] (4.24)
where \( \gamma \) is a small constant satisfying (4.27).

**Proof.** From (4.7) and Lemma 4.1, it is easy to check that \( (U_1, U_2) \) satisfies, for \( c > c^* \)
\[
\begin{cases}
U_1 + cU_1 \xi_1 \leq d_1 [J * U_1 - U_1] + r_1 (1 - \phi_1) U_1 + r_1 \phi_1 [b_1 U_2^2 - U_1], & t > 0, \xi \in \Omega_+, \\
U_2 + cU_2 \xi_1 \leq d_2 [J * U_2 - U_2] + r_2 (1 - b_2 \phi_1^2) U_2 + r_2 [1 - \phi_2] [b_2 U_1^2 - U_2], & t > 0, \xi \in \Omega_+, \\
U_{i1} \xi_1 = C_2 \xi_0 (1 + t)^{-\frac{n}{\alpha}} e^{-\gamma t}, & t > 0, (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1}, i = 1, 2,
\end{cases}
\]
(4.25)
and for \( c = c^* \),
\[
\begin{cases}
U_1 + cU_1 \xi_1 \leq d_1 [J * U_1 - U_1] + r_1 (1 - \phi_1) U_1 + r_1 \phi_1 [b_1 U_2^2 - U_1], & t > 0, \xi \in \Omega_+, \\
U_2 + cU_2 \xi_1 \leq d_2 [J * U_2 - U_2] + r_2 (1 - b_2 \phi_1^2) U_2 + r_2 [1 - \phi_2] [b_2 U_1^2 - U_2], & t > 0, \xi \in \Omega_+, \\
U_{i1} \xi_1 = C_3 \xi_0 (1 + t)^{-\frac{n}{\alpha}}, & t > 0, (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1}, i = 1, 2,
\end{cases}
\]
(4.26)
Let
\[
0 < \gamma < \min\{\varepsilon, \sigma_1, \delta_1\},
\]
(4.27)
where
\[
\delta_1 = \min\left\{ r_1 (1 - b_1), r_2 (b_2 - 1), \frac{1}{\tau} \frac{1}{b_1} \right\}.
\] (4.28)
Thus, we can choose $\zeta_0$ and $t_*$ large enough and $\gamma$ small enough to ensure that
\[
\begin{align*}
&\left\{\begin{array}{l}
\frac{r_1}{2} \phi_1(\xi_1) \left[1 - b_1 \left(1 + \frac{\tau}{1 + t}\right)^{\frac{1}{\alpha}} e^{\gamma t}\right] + r_1 \phi_1(\xi_1) - 1 - \frac{n}{2\alpha(1 + t + \tau)} - \gamma > 0,
\frac{r_2}{2} [b_2 \phi_1^2(\xi_1) - 1] + r_2 \phi_2(\xi_1) - 1 - \frac{n}{2\alpha(1 + t + \tau)} - \gamma > 0,
\end{array}\right.
\end{align*}
\]
for $\xi_1 > \zeta_0$ and $t > t_*$.  
When $c > c^*$, let
\[
\frac{\partial u_i(t, \xi)}{\partial t} = C_4 \epsilon_0 (1 + t + \tau)^{- \frac{\alpha}{\gamma}} e^{-\gamma t}, \quad t > 0, i = 1, 2,
\]
where $C_4$ large enough such that $U_i(t, \xi) \geq U_i(t, \xi), (t, \xi) \in [0, t_*] \times \mathbb{R}^n$.  
If $\gamma$ is sufficiently small.  By a direct computation, we can verify that
\[
\left\{\begin{array}{l}
U_{1i} + cU_{1i} \geq d_1 [J * U_i - U_i], t > 0, \xi \in \Omega_+, \\
U_{2i} + cU_{2i} \geq d_2 [U_i * U_i - U_i], t > 0, \xi \in \Omega_+,
\end{array}\right.
\]

Thus, for $c > c^*$, we have
\[
0 \leq U_i(t, \xi) \leq \bar{U}_i(t, \xi) = C_4 \epsilon_0 (1 + t + \tau)^{- \frac{\alpha}{\gamma}} e^{-\gamma t}, \quad t > 0, \xi \in \Omega_+, i = 1, 2.  \tag{4.31}
\]

When $c = c^*$, let
\[
\frac{\partial u_i(t, \xi)}{\partial t} = C_5 \epsilon_0 (1 + t + \tau)^{- \frac{\alpha}{\gamma}} e^{-\gamma t}, \quad t > 0, i = 1, 2,
\]
where $C_5$ large enough such that $\bar{U}_i(t, \xi) \geq U_i(t, \xi), t \in [0, t_*] \times \mathbb{R}^n$.  Similarly, we can check that $(\bar{U}_1, \bar{U}_2)$ satisfies
\[
\left\{\begin{array}{l}
\frac{\partial u_i(t, \xi)}{\partial t} \geq d_1 [J * \bar{U}_i - \bar{U}_i], t > 0, \xi \in \Omega_+, \\
\frac{\partial u_i(t, \xi)}{\partial t} \geq d_2 [\bar{U}_i * \bar{U}_i - \bar{U}_i], t > 0, \xi \in \Omega_+.
\end{array}\right.
\]

Thus, for $c = c^*$, we have
\[
0 \leq U_i(t, \xi) \leq \bar{U}_i(t, \xi) = C_5 \epsilon_0 (1 + t + \tau)^{- \frac{\alpha}{\gamma}} e^{-\gamma t}, \quad t > 0, \xi \in \Omega_+, i = 1, 2. \tag{4.32}
\]

Then, (4.23) and (4.24) can be immediately obtained by (4.31) and (4.32).

From Lemmas 4.1 and 4.2, we can obtain the decay rates for $U_i(t, \xi)(i = 1, 2)$ in $L^\infty(\mathbb{R}^n)$.  

**Lemma 4.3.** It holds that
\[
\|U_i(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C \epsilon_0 t^{- \frac{\alpha}{\gamma}} e^{-\gamma t}, \quad c > c^*;
\]
and
\[
\|U_i(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C \epsilon_0 t^{- \frac{\alpha}{\gamma}}, \quad c = c^*,
\]
for $t > 0, i = 1, 2$, where $0 < \sigma < \min\{\sigma_1, \delta_1/\epsilon_\tau\}$, constants $\sigma_1$ and $\delta_1$ are given in (4.18) and (4.28), respectively.
Since \( U_i(t, \xi) = u_i^+(t, x) - \phi_i(x \cdot e_1 + ct), i = 1, 2 \), we have the following convergence for the solution.

**Lemma 4.4.** It holds that
\[
\sup_{x \in \mathbb{R}^n} |u_i^+(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2\sigma}} e^{-\varepsilon \cdot \sigma t}, \quad c > c^*,
\]
and
\[
\sup_{x \in \mathbb{R}^n} |u_i^+(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2}}, \quad c = c^*,
\]
for \( t > 0, i = 1, 2 \), where \( 0 < \sigma < \min\{\sigma_1, \delta_1/\varepsilon_1\} \), constants \( \sigma_1 \) and \( \delta_1 \) are given in (4.18) and (4.28), respectively.

Next, we give the proof of our main result Theorem 2.2.

**Proof of Theorem 2.2.** For \( c \geq c^* \), let \( \xi = x + ct \cdot e_1 \) and
\[
V_i(t, \xi) = \phi_i(x \cdot e_1 + ct) - u_i^+(t, x), \quad V_{i0}(s, \xi) = \phi_i(x \cdot e_1 + cs) - u_{i0}(s, x),
\]
for \( t > 0, s \in [-\tau, 0], x \in \mathbb{R}^n, i = 1, 2 \). Similarly, we can obtain that \( u_i^-(t, x) \) converges to \( \phi_i(x \cdot e_1 + ct) \) as follows,
\[
\sup_{x \in \mathbb{R}^n} |u_i^-(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2\sigma}} e^{-\varepsilon \cdot \sigma t}, \quad c > c^*,
\]
and
\[
\sup_{x \in \mathbb{R}^n} |u_i^-(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2}}, \quad c = c^*,
\]
for \( t > 0, i = 1, 2 \), where \( 0 < \sigma < \min\{\sigma_1, \delta_1/\varepsilon_1\} \). Since
\[
0 \leq u_i^-(t, x) \leq u_i(t, x) \leq u_i^+(t, x) \leq 1,
\]
\[
0 \leq u_i^-(t, x) \leq \phi_i(x \cdot e_1 + ct) \leq u_i^+(t, x) \leq 1,
\]
for \( x \in \mathbb{R}^n, t > 0, i = 1, 2 \), by the squeeze argument, we have
\[
\sup_{x \in \mathbb{R}^n} |u_i(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2\sigma}} e^{-\varepsilon \cdot \sigma t}, \quad c > c^*,
\]
and
\[
\sup_{x \in \mathbb{R}^n} |u_i(t, x) - \phi_i(x \cdot e_1 + ct)| \leq C\varepsilon_0 t^{-\frac{\mu}{2}}, \quad c = c^*,
\]
for \( t > 0, i = 1, 2 \). \( \square \)

5. **Exact planar traveling waves and numerical simulations.** In this section, we first show that exact planar traveling waves of the local dispersal competitive system corresponding to (1.4) can be given explicitly under some certain restrictions. We apply the approach developed in [13] to obtain the exact planar traveling waves. Furthermore, we give some numerical simulations to illustrate our main result.

If we take the nonlocal kernel function \( J(x) = \sum_{i=1}^n \delta_i'(x_i) + \delta_n(x) \) \( (\delta_i(\cdot) \) and \( \delta_n(\cdot) \) are the one-dimensional and \( n \)-dimensional Dirac functions \) and the time delay \( \tau = 0 \), then (1.4) reduces to the following Laplacian diffusion system
\[
\begin{align*}
\frac{\partial u_1(t, x)}{\partial t} &= d_1 \Delta u_1(t, x) + r_1 u_1(t, x)[1 - u_1(t, x) - b_1 u_2(t, x)], \\
\frac{\partial u_2(t, x)}{\partial t} &= d_2 \Delta u_2(t, x) + r_2 u_2(t, x)[1 - b_2 u_1(t, x) - u_2(t, x)],
\end{align*}
\]
(5.1)

**Theorem 5.1.** Let the following hypotheses be fulfilled.

(A1): \( b_1 \in (0, 1), b_2 > 5, r_1 = r > 0 \);
(A2): \( r_2 = \frac{5 - b_1}{b_2 - 5}, d_1 = \frac{1}{24} (1 + b_1)r, d_2 = \frac{1}{24} (1 + b_2)(5 - b_1) \).
Then the system (5.1) has a planar traveling wave \((u_1(x,t), u_2(x,t)) = (\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct))\) in the following form

\[
\begin{align*}
\phi_1(\xi) &= \frac{1}{4} (1 + \tanh(\xi))^2, \\
\phi_2(\xi) &= \frac{1}{4} (1 - \tanh(\xi))^2,
\end{align*}
\]

(5.2)

where \(\xi = x \cdot e_1 + ct\) and \(c = \frac{1}{12} (5 - b_1) r\).

**Proof.** If the system (5.1) has a planar traveling wave \((u_1(x,t), u_2(x,t)) = (\phi_1(x \cdot e_1 + ct), \phi_2(x \cdot e_1 + ct))\), then the system (5.1) can be reduced to

\[
\begin{align*}
cc\phi_1'(\xi) &= d_1\phi_1''(\xi) + r_1 \phi_1(\xi)[1 - \phi_1(\xi) - b_1 \phi_2(\xi)], \\
c\phi_2'(\xi) &= d_2\phi_2''(\xi) + r_2 \phi_2(\xi)[1 - b_2 \phi_1(\xi) - \phi_2(\xi)],
\end{align*}
\]

(5.3)

where \(\xi = x \cdot e_1 + ct\). Now, we are concentrating on finding a planar traveling wave \((\phi_1, \phi_2)\) connecting the equilibria \((0,1)\) and \((1,0)\) in the form

\[
\begin{align*}
\phi_1(\xi) &= a_1(1 + \tanh(\xi))^2, \\
\phi_2(\xi) &= a_2(1 - \tanh(\xi))^2,
\end{align*}
\]

(5.4)

where \(a_1\) and \(a_2\) are constants which will be fixed later. By substituting (5.4) into (5.3), we have

\[
\begin{align*}
2c(1 - \tanh(\xi))\phi_1(\xi) &= d_1 \left[ 6 \tanh^2(\xi) - 8 \tanh(\xi) + 2 \right] \phi_1(\xi) \\
&\quad + r_1 \phi_1(\xi)[1 - a_1(1 + \tanh(\xi))^2 - a_2 b_1(1 - \tanh(\xi))^2],
\end{align*}
\]

and

\[
\begin{align*}
-2c(1 + \tanh(\xi))\phi_2(\xi) &= d_2 \left[ 6 \tanh^2(\xi) + 8 \tanh(\xi) + 2 \right] \phi_2(\xi) \\
&\quad + r_2 \phi_2(\xi)[1 - a_1 b_2(1 + \tanh(\xi))^2 - a_2(1 - \tanh(\xi))^2], \quad \xi \in \mathbb{R}.
\end{align*}
\]

Then,

\[
\begin{align*}
[6d_1 - r_1(a_1 + a_2 b_1)] \tanh^2(\xi) + [2c - 8d_1 + 2r_1(a_2 b_1 - a_1)] \tanh(\xi) + [-2c + 2d_1 + r_1(1 - a_1 - a_1 b_1)] = 0, \quad \xi \in \mathbb{R},
\end{align*}
\]

and

\[
\begin{align*}
[6d_2 - r_2(a_1 b_2 + a_2)] \tanh^2(\xi) + [2c + 8d_1 + 2r_1(a_2 - a_1 b_2)] \tanh(\xi) + [2c + 2d_2 + r_2(1 - a_1 b_2 - a_2)] = 0, \quad \xi \in \mathbb{R}.
\end{align*}
\]

Thus,

\[
\begin{align*}
6d_1 - r_1(a_1 + a_2 b_1) &= 0, \\
2c - 8d_1 + 2r_1(a_2 b_1 - a_1) &= 0, \\
-2c + 2d_1 + r_1(1 - a_1 - a_1 b_1) &= 0,
\end{align*}
\]

(5.5)

and

\[
\begin{align*}
6d_2 - r_2(a_1 b_2 + a_2) &= 0, \\
2c + 8d_1 + 2r_1(a_2 - a_1 b_2) &= 0, \\
2c + 2d_2 + r_2(1 - a_1 b_2 - a_2) &= 0.
\end{align*}
\]

(5.6)

Because of the planar traveling wave \((\phi_1, \phi_2)\) connecting the equilibria \((0,1)\) and \((1,0)\), namely, \((\phi_1, \phi_2)(-\infty) = (0,1)\) and \((\phi_1, \phi_2)(+\infty) = (1,0)\), we can obtain that \(a_1 = a_2 = \frac{1}{4}\). It follows from (5.5) and (5.6) that

\[
\frac{d_1}{r_1} = \frac{1}{24} (1 + b_1), \quad \frac{c}{r_1} = \frac{1}{12} (5 - b_1),
\]

(5.7)
and
\[
\frac{d_2}{r_2} = \frac{1}{24}(1 + b_2), \quad \frac{c}{r_2} = \frac{1}{12}(b_2 - 5). \tag{5.8}
\]
If we set \( r_1 = r > 0, 0 < b_1 < 1 \), then by (5.7)–(5.8) and the nonnegativity of the parameters, we have \( b_2 > 5 \) and
\[
r_2 = \frac{5 - b_1}{b_2 - 5}, d_1 = \frac{1}{24}(1 + b_1)r, d_2 = \frac{1}{24} \left( \frac{(1 + b_2)(5 - b_1)}{b_2 - 5} \right)r.
\]
Thus, the system (5.1) has a planar traveling wave (5.2) under the conditions (A1)–(A2).

Now we present the exact planar traveling wave given in Theorem 5.1. Let \( b_1 = \frac{1}{2}, b_2 = \frac{19}{2}, r_1 = r_2 = r = 16, d_1 = 1, d_2 = 7 \). It is easy to verify that the hypotheses (A1)-(A2) can be ensured. The resulting planar traveling wave is
\[
\begin{align*}
\phi_1(\xi) &= \frac{1}{4}(1 + \tanh(\xi))^2, \\
\phi_2(\xi) &= \frac{1}{4}(1 - \tanh(\xi))^2,
\end{align*}
\]
where \( \xi = x \cdot e_1 + ct \) and \( c = \frac{1}{12}(5 - b_1)r = 6 \).

These profiles are shown in Figure 1.

![Figure 1. Exact planar traveling wave (\( \phi_1, \phi_2 \)) of the system (5.1) with \( b_1 = \frac{1}{2}, b_2 = \frac{19}{2}, r_1 = r_2 = 16, d_1 = 1, d_2 = 7 \).](image)

Next, we consider the system (1.4) with the Neumann boundary conditions and initial data, which are described as
\[
\frac{\partial u_1(t, x)}{\partial n} = \frac{\partial u_2(t, x)}{\partial n} = 0, \quad t \geq 0, x \in \Omega, \tag{5.9}
\]
and
\[
u_1(s, x) = u_{10}(s, x), \quad u_2(s, x) = u_{20}(s, x), \quad s \in [-\tau, 0], x \in \Omega, \tag{5.10}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \frac{\partial}{\partial n} \) denotes the outward normal derivative on \( \partial \Omega \), the homogeneous Neumann boundary conditions imply that the species cannot move across the boundary \( \partial \Omega \).
In system (1.4), we take \( r_1 = r_2 = 1, d_1 = d_2 = 1, b_1 = \frac{1}{2}, b_2 = \frac{3}{2}, \tau = 0.5 \) and \( J(x) = \frac{1}{(4\pi)^{n/2}} e^{-|x|^2}, x \in \mathbb{R}^n \) (here, we assume \( n = 1 \) for simplicity). Then, the system (1.4) with above coefficients has two steady states \((0, 1)\) and \((1, 0)\). From Theorem 2.1, system (1.4) admits a planar traveling wave with speed \( c > c^* = 1.5505 \). Furthermore, if the initial data satisfies (2.6) and the initial perturbation satisfies (2.7), the planar traveling wave with speed \( c > c^* = 2.9955 \) is exponential stable in \( L^\infty(\mathbb{R}^n) \) and the planar traveling wave solution with speed \( c = c^* \) is algebraic stable in \( L^\infty(\mathbb{R}^n) \).

Now, we choose \( \Omega = [-60, 60] \) and the initial data

\[
\begin{align*}
  u_{10}(s, x) &= q(x) \left[ 1 - \frac{1}{4} (1 - \tanh(x))^2 \right], \\
  u_{20}(s, x) &= q(x) \left[ 1 - \frac{1}{4} (1 + \tanh(x))^2 \right],
\end{align*}
\]

\( x \in [-60, 60], s \in [-1, 0], \quad (5.11) \)

where \( q(x) \) is the mollification function of \( p(x) \) and

\[
p(x) = \begin{cases} 
  \frac{15}{16} & x \in [-30, 30], \\
  1 - \frac{1}{31 + \frac{1}{2}x} \cos(x) & x \in \Omega \setminus [-30, 30],
\end{cases}
\]

It is not hard to verify that the initial conditions in Theorem 2.2 can be ensure for such initial data (5.11). With the help of the software MATLAB, we can obtain the numerical solution of (2.1) (see the Figures 2–3). It follows from the Figures 2–3 that the solution of system (1.4) will eventually converge to the equilibrium \((1, 0)\), which implies that the species \( u_1 \) will survive and species \( u_2 \) will die out. After a large time (here, the time \( t \geq 4 \) is enough), the solution of system (1.4) behaves exactly as a stable planar traveling wave in the sense of stability of no change of wave’s shape. From the Figure 2–3, we can obtain that the solution of system (1.4) travels from the positive direction of the axis \( x \) to the negative direction.

An ecological phenomenon related to these results can be explained in the following manner. In the beginning, system (1.4) with species \( u_1 \) being absent only have one native species \( u_2 \). Then the exotic species \( u_1 \) competing with \( u_2 \) invades this system in the way described by system (1.4). Now a natural question arises, namely, can the two species coexist? According to our result, the native species \( u_2 \) will eventually become extinct while the invading species \( u_1 \) will survive. It also can be seen from Theorems 2.2 that this process is stable if the initial perturbation is proper.

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Figure 2. The left picture denotes the solution $u_1$ of system (1.4) with Neumann boundary conditions (5.9) and initial data (5.11). From (a) to (f), the solution $u_1(t,x)$ plots at times $t = 0, 1, 2, 10, 30, 50$ and behaves as a stable monotone increasing traveling wave (no change of the wave's shape after a large time in the sense of stability) and travels from right to left.

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Figure 3. The left picture denotes the solution $u_2$ of system (1.4) with Neumann boundary conditions (5.9) and initial data (5.11). From (a) to (f), the solution $u_2(t, x)$ plots at times $t = 0, 1, 2, 10, 30, 50$ and behaves as a stable monotone increasing traveling wave (no change of the wave’s shape after a large time in the sense of stability) and travels from right to left.

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