A “quantum spherical model” with transverse magnetic field.

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1 Introduction

The Quantum Ising Model with a transverse magnetic field is well known [1] [2]. In one dimension it has the Hamiltonian

$$\mathcal{H}_N = -J \sum_{n=1}^{N} \sigma_n^x \sigma_{n+1}^x + B \sum_{n=1}^{N} \sigma_n^z + H \sum_{n=1}^{N} \sigma_n^x, \quad (1)$$

where $J > 0$ is the coupling constant and

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. $B$ and $H$ are transverse and longitudinal magnetic fields, respectively. The partition function is

$$Z_N = \text{tr} e^{-\beta \mathcal{H}_N} \quad (2)$$

where $\beta$ is the inverse temperature. In the case where $H = 0$ this model has been exactly solved [1] [3]. The free energy is [2]

$$f(\beta, J, B) = - \lim_{N \to \infty} \frac{1}{\beta N} \log Z_N$$

$$= - \frac{1}{2\pi \beta} \int_0^{2\pi} \log 2 \cosh \beta \Delta(x) \, dx \quad (3)$$

where

$$\Delta(x) = \sqrt{J^2 + B^2 - 2BJ \cos x}. \quad (4)$$
In particular the ground state energy is given by

\[ f_\infty(J, B) = \lim_{\beta \to \infty} f(\beta, J, B) = -\frac{1}{2\pi} \int_0^{2\pi} \Delta(x) \, dx. \]  

(5)

In this limit there is a critical point in \( B \) at \( B = J \). The correlation function

\[ \langle \sigma^x_j \sigma^x_k \rangle = \lim_{N \to \infty} \frac{\text{tr} \, \sigma^x_j \sigma^x_k e^{-\beta H_N}}{Z_N} \]  

(6)

can be written as a Toeplitz determinant of size \(| j - k | \) just as the correlation function of the two dimensional classical Ising model \([4]\), but only in the limit \( \beta \to \infty \). In fact the correlation function \( \lim_{\beta \to \infty} \langle \sigma^x_j \sigma^x_k \rangle \) is the same as the diagonal correlation function \( \langle \sigma_{jj} \sigma_{kk} \rangle \) of the two dimensional classical lattice \( T < T_c \), the ratio \( B/J \) in the one dimensional quantum model corresponding to \((\sinh 2E_1/k_B T \sinh 2E_2/k_B T)^{-1}\) in the two dimensional classical model. (Here \( E_1 \) and \( E_2 \) are the coupling constants in the horizontal and vertical directions, respectively). In particular the limit of infinite separation is given by \([3]\)

\[ \lim_{|j-k| \to \infty} \lim_{\beta \to \infty} \langle \sigma^x_j \sigma^x_k \rangle = \begin{cases} 
{1 - (B/J)^2}^{1/4} & \text{if } B < J, \\
0 & \text{if } B \geq J,
\end{cases} \]  

(7)

which is most easily proved using Szegö’s theorem \([5] [6]\).

2 The quantum spherical model

In analogy with \([1]\) we define a partition function of a \((d\text{-dimensional})\) isotropic quantum spherical model on a lattice \( \Lambda \) as follows:

\[
\begin{align*}
Z_N &= \int_{[0,\infty)^N} \int_{[0,2\pi)^N} \int_{[0,\pi]^N} e^{\sum_{j,k \in \Lambda : \langle jk \rangle} \beta J_{jk} \cos \theta_j \cos \theta_k} \\
& \quad \cdot e^{\sum_{j \in \Lambda} \beta (B_{xj} \sin \theta_j \cos \varphi_j + H_{xj} \cos \theta_j)} \\
& \quad \cdot \prod_{l=1}^N r^2_l \sin \theta_l \, d^N \theta \, d^N \varphi \, \delta \left( \sum_{m=1}^N r^2_m - N \right) \, d^N r \\
& = \int_{\mathbb{R}^{3N}} e^{\sum_{\langle jk \rangle} \beta J_{jk} x_k + \sum_j \beta (B_{xj} + H_{xj})} \delta \left( \sum_{k=1}^N (x_k^2 + y_k^2 + z_k^2) - N \right) d^{3N} \mathbf{x}
\end{align*}
\]  

(8)

Here \( J > 0, \, B \geq 0 \) and \( H > 0 \). \( \delta \) signifies the Dirac distribution. The notation \( \langle jk \rangle \) means that \( j \) and \( k \) are nearest neighbors on \( \Lambda \). Unlike the
Quantum Ising Model with $H = 0$, in this model the critical point is $B = 2Jd$ (in the limit $H \to 0$). In fact, it will be shown that in this limit the ground state free energy $f_{H, \infty} := -\lim_{\beta \to \infty} \lim_{N \to \infty} (N\beta)^{-1} \log Z_N$ is given by

$$f_{0, \infty} = \lim_{H \to 0} f_{H, \infty} = \begin{cases} Jd + B^2/4Jd & \text{if } B \leq 2Jd, \\ B & \text{if } B > 2Jd. \end{cases}$$

We shall now give a proof of (9).

2.1 The case $B > 2Jd$

We use the method of steepest descent to prove this result, following the calculation by Baxter [7]. We let $H = 0$ in (8). Clearly the integrand in (8) may be multiplied by a factor $\exp \left( \sum_{k=1}^{N} (x_k^2 + y_k^2 + z_k^2) - N \right)$ without changing the partition function $Z_N$. Using the identity

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} ds,$$

together with (8) and letting $a > 0$, we get

$$Z_N = \frac{\pi^{N-1}}{2} \int_{\mathbb{R}^N} \int_{-\infty}^{\infty} \left( \frac{1}{a + is} \right)^N \exp \frac{N(\beta B)^2}{4(a + is)} \exp \left[ \sum_{(jk)} \beta J z_j z_k + \sum_j (a + is)(1 - z_j^2) \right] ds d^N z$$

after integrating over $x$ and $y$. Let $V$ be the symmetric matrix such that

$$z^T V z = (a + is) \sum_{j=1}^{N} z_j^2 - \beta J \sum_{(jk)} z_j z_k. \quad (12)$$

In this way (11) can be written as

$$Z_N = \frac{\pi^{N-1}}{2} \int_{\mathbb{R}^N} \int_{-\infty}^{\infty} \left( \frac{1}{a + is} \right)^N \exp \frac{N(\beta B)^2}{4(a + is)} \exp \left[ -z^T V z + N(a + is) \right] ds d^N z. \quad (13)$$

We now choose the constant $a$ so large that all the eigenvalues of $V$ have positive real part. This allows us to change the order of integration, and we may now write (13) as

$$Z_N = \frac{\pi^{3N/2-1}}{2} \int_{-\infty}^{\infty} \left( \frac{1}{a + is} \right)^N (\det V)^{-1/2} \exp \left[ \frac{N(\beta B)^2}{4(a + is)} + N(a + is) \right] ds. \quad (14)$$
We need to calculate the eigenvalues of $V$. Since $V$ is cyclic, this is easily done. We let the lattice be $d$-dimensional hypercubic, so that

$$N = L^d$$  

for some positive integer $L$. It now follows from (12) that the eigenvalues are

$$\lambda(\omega_1, ..., \omega_d) = a + is - \beta J \sum_{j=1}^{d} \cos \omega_j$$  

where each $\omega_j$ takes the values $\{2\pi k/L\}_{k=0}^{L-1}$, and $a > \beta Jd$. The determinant of $V$ is the product of its eigenvalues, so

$$\log \det V = \sum_{\omega_j : 1 \leq j \leq d} \log \lambda(\omega_1, ..., \omega_d).$$  

Clearly

$$Z_N = \frac{\beta J}{2\pi i} \left( \frac{\pi}{\beta J} \right)^{3N/2} \int_{c-i\infty}^{c+i\infty} e^{N\phi(w)} dw,$$  

where

$$\phi(w) = \beta Jw - \frac{1}{2} g(w) + (\beta B)^2 / 4 \beta Jw,$$  

$$c = (a - \beta Jd)/\beta J$$ and

$$g(z) = 2 \log w + \frac{1}{N} \sum_{\omega_j} \log (w - \sum_j \cos \omega_j).$$  

Since $\phi$ approaches $+\infty$ as $w$ approaches 0 or $+\infty$ along the real line, $\phi$ has a minimum at some $w_0$, $0 < w_0 < \infty$. Thus $\Re \phi$ has a maximum at $w_0$ along the line $(w_0 - i\infty, w_0 + i\infty)$. Since $B > 2Jd$, we may choose $c = w_0$. We now use the method of steepest descent (see for instance Murray [5]), by letting $N$ approach infinity. In this way, the free energy is

$$f = -\beta^{-1} \lim_{N \to \infty} N^{-1} \log Z_N$$

$$= -\frac{3}{2\beta} \ln (\pi/\beta J) - \beta^{-1} \phi(w_0).$$  

Now

$$\lim_{\beta \to \infty} w_0 = B/2J,$$  

and thus the ground state energy is

$$\lim_{\beta \to \infty} f = -\lim_{\beta \to \infty} \beta^{-1} \phi(w_0)$$

$$= -B.$$  

2.2 The case \( B \leq 2Jd \)

In this case we let \( H > 0 \), so instead of (13) we have

\[
Z_N = \frac{\pi^{N/2-1}}{2} \int_{\mathbb{R}^N} \int_{-\infty}^{\infty} \left( \frac{1}{a + is} \right)^N \exp \left\{ \frac{N(\beta B)^2}{4(a + is)} \right\} \exp \left[ -z^T V z + h^T z + N(a + is) \right] ds d^N z, \tag{24}
\]

where \( h = \beta H(1, ..., 1) \). We change variables to \( t = z - \frac{1}{2} V^{-1} h \), and rotate the axes in \((t_1, ..., t_N)\) to make \( V \) diagonal. Thus we get

\[
Z_N = \frac{\pi^{N/2-1}}{2} \int_{-\infty}^{\infty} \left( \frac{1}{a + is} \right)^N (\det V)^{-1/2} \exp \left\{ \frac{N(\beta B)^2}{4(a + is)} \right\} \exp \left[ h^T V^{-1} h/4 + N(a + is) \right] ds. \tag{25}
\]

Thus

\[
Z_N = \frac{\pi^{N/2}}{2 \pi i} \int_{c-i\infty}^{c+i\infty} e^{N\phi(w)} \, dw, \tag{26}
\]

where \( a + is - \beta Jd = \beta Jw \) and

\[
\phi(w) = \beta J(w + d) + \frac{(\beta H)^2}{4\beta Jw} + \frac{(\beta B)^2}{4\beta J(w + d)} - \log \beta J(w + d) - \frac{1}{2} \sum_{\omega_j} \log (\beta J(w + d) - \beta J \sum \cos \omega_j). \tag{27}
\]

We proceed in the same way as before, taking the limit \( N \to \infty \) and then \( \beta \to \infty \). In this case \( w_0 \to 0 \) as \( H \to 0 \). The free energy is thus

\[
f = -Jd - \frac{B^2}{4Jd}. \tag{28}
\]

This ends the proof.

3 Discussion

Comparison of (5) and (9) shows that the susceptibilites of the two models at \( B = 0 \) are equal when \( d = 1 \); that is \(-\partial^2 f_\infty/\partial B^2|_{B=0} = 1/2J \) and \(-\partial^2 f_{0,\infty}/\partial B^2|_{B=0} = 1/2Jd \). While the Quantum Ising Model has only been exactly solved in the one dimensional case, the quantum spherical model can be solved in any finite dimension.

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