Undecidability of the word problem for one-relator inverse monoids

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SandGAL 2019, Cremona, Italy, June 2019

¹Research supported by the EPSRC grant EP/N033353/1 "Special inverse monoids: subgroups, structure, geometry, rewriting systems and the word problem".
Word problem for one-relator groups and monoids

|                | Gp\langle A \mid w = 1 \rangle | Mon\langle A \mid w = 1 \rangle | Inv\langle A \mid w = 1 \rangle |
|----------------|----------------------------------|----------------------------------|----------------------------------|
|                | FG(A)/\langle w \rangle          | A^*/\langle (w, 1) \rangle       | FIM(A)/\langle (w, 1) \rangle     |
| Word problem   | Magnus (1932)                    | Adjan (1966)                     |                                   |
| decidable      | ✓                                | ✓                                |                                   |
|                |                                   |                                   |                                   |

Theorem (Scheiblich (1973) & Munn (1974))
Free inverse monoids have decidable word problem.

Conjecture (Margolis, Meakin, Stephen (1987))
If $M = \text{Inv}\langle A \mid w = 1 \rangle$, then the word problem for $M$ is decidable.
### Word problem for one-relator groups and monoids

|                | \( \text{Gp}(A \mid w = 1) \) | \( \text{Mon}(A \mid w = 1) \) | \( \text{Inv}(A \mid w = 1) \) |
|----------------|-------------------------------|-------------------------------|-------------------------------|
|                | \( \text{FG}(A) / \langle w \rangle \) | \( A^* / \langle (w, 1) \rangle \) | \( \text{FIM}(A) / \langle (w, 1) \rangle \) |
| Word problem   | Magnus (1932)                 | Adjan (1966)                  |                               |
| decidable      | ✓                             | ✓                             | ?                             |

#### Theorem (Scheiblich (1973) & Munn (1974))
Free inverse monoids have decidable word problem.

#### Conjecture (Margolis, Meakin, Stephen (1987))
If \( M = \text{Inv}(A \mid w = 1) \), then the word problem for \( M \) is decidable.

#### Theorem (Ivanov, Margolis, Meakin (2001))
If the word problem is decidable for all inverse monoids of the form \( \text{Inv}(A \mid w = 1) \) then the word problem is also decidable for every one-relator monoid \( \text{Mon}(A \mid u = v) \).
Word problem for one-relator groups and monoids

|                        | Gp\langle A \mid w = 1 \rangle | Mon\langle A \mid w = 1 \rangle | Inv\langle A \mid w = 1 \rangle |
|------------------------|-------------------------------|---------------------------------|---------------------------------|
|                        | FG\langle A \rangle / \langle w \rangle | A^* / \langle (w, 1) \rangle | FIM\langle A \rangle / \langle (w, 1) \rangle |
| Word problem           | Magnus (1932)                 | Adjan (1966)                    |                                  |
| decidable              | ✓                              | ✓                               |                                  |

Theorem (Scheiblich (1973) & Munn (1974))
Free inverse monoids have decidable word problem.

Conjecture (Margolis, Meakin, Stephen (1987))
If \( M = \text{Inv}\langle A \mid w = 1 \rangle \), then the word problem for \( M \) is decidable.

Proved true in many cases e.g. when \( w \) satisfies...
- Idempotent word [Birget, Margolis, Meakin, 1993, 1994]
- \( w \)-strictly positive [Ivanov, Margolis, Meakin, 2001]
- Adjan or Baumslag-Solitar type [Margolis, Meakin, Šunič, 2005]
- Sparse word [Hermiller, Lindblad, Meakin, 2010]
- Certain small cancellation conditions [A. Juhász, 2012, 2014]
Word problem for one-relator groups and monoids

|                       | Gp\langle A \mid w = 1 \rangle | Mon\langle A \mid w = 1 \rangle | Inv\langle A \mid w = 1 \rangle |
|-----------------------|----------------------------------|----------------------------------|----------------------------------|
| FG(A)/\langle w \rangle |                                  | A^*/\langle (w, 1) \rangle       | FIM(A)/\langle (w, 1) \rangle     |
| Word problem          | Magnus (1932)                    | Adjan (1966)                     |                                  |
| decidable             | ✓                                | ✓                                | ✗                                |

Theorem (RDG (2019))
There is a one-relator inverse monoid Inv\langle A \mid w = 1 \rangle with undecidable word problem.
## Word problem for one-relator groups and monoids

|                          | $\text{Gp}(A \mid w = 1)$  | $\text{Mon}(A \mid w = 1)$ | $\text{Inv}(A \mid w = 1)$ |
|--------------------------|-----------------------------|-----------------------------|-----------------------------|
|                          | $\text{FG}(A)/\langle w \rangle$ | $A^*/\langle (w, 1) \rangle$ | $\text{FIM}(A)/\langle (w, 1) \rangle$ |
| Word problem              | Magnus (1932) | Adjan (1966) | ☒ |
| decidable                | ✓                          | ✓                          | ☒                          |

### Theorem (RDG (2019))

There is a one-relator inverse monoid $\text{Inv}(A \mid w = 1)$ with undecidable word problem.

#### Ingredients for the proof:

- Submonoid membership problem for one relator groups.
- HNN-extensions and free products of groups.
- Right-angled Artin groups (RAAGs).
- Right units of special inverse monoids $\text{Inv}(A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1)$
- Stephen’s procedure for constructing Schützenberger graphs.
- Properties of $E$-unitary inverse monoids.
Inverse monoid presentations

An inverse monoid is a monoid $M$ such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \quad (†)$$

$$\text{Inv}\langle A \mid u_i = v_i \ (i \in I) \rangle = \text{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (†) \rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and $x, y$ range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\text{FIM}(A) = \text{Inv}\langle A \mid \rangle$

Munn (1974)

Elements of $\text{FIM}(A)$ can be represented using Munn trees. e.g. in $\text{FIM}(a, b)$ we have $u = w$ where

$$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$$
$$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$$
The word problem

$M$ - a finitely generated monoid with a finite generating set $A$.

$\pi : A^* \rightarrow M$ – the canonical monoid homomorphism.

The monoid $M$ has decidable word problem if there is an algorithm which solves the following decision problem:

**INPUT:** Two words $u, v \in A^*$.

**QUESTION:** $\pi(u) = \pi(v)$? i.e. do $u$ and $v$ represent the same element of the monoid $M$?

For a group or an inverse monoid with generating set $A$ the word problem is defined in the same way except the input is two words $u, v \in (A \cup A^{-1})^*$.

**Example.** The bicyclic monoid $\text{Inv}(a \mid aa^{-1} = 1)$ has decidable word problem.
Proof strategy

\[ M = \text{Inv} \langle A | r = 1 \rangle \quad \text{and} \quad G = \text{Gp} \langle A | r = 1 \rangle \]

- If \( M \) has decidable word problem
- \( \implies \) membership problem for \( U_R \leq M \) is decidable
- since for \( w \in (A \cup A^{-1})^* \)
  \[ w \in U_R \iff ww^{-1} = 1 \]

\( N = \pi(U_R) \)

\( \implies \) membership problem for \( N \leq G \) is decidable
RAAGs induced subgraphs and subgroups

Definition
The right-angled Artin group $A(\Gamma)$ associated with the graph $\Gamma$ is
\[ \text{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma \rangle. \]

Fact: If $\Delta$ is an induced subgraph of $\Gamma$ then the embedding $\Delta \to \Gamma$ induces an embedding $A(\Delta) \to A(\Gamma)$.

Example
\[
\begin{align*}
A(\Delta) &= \text{Gp}\langle a, c, d, e \mid ac = ca, de = ed \rangle \\
A(\Gamma) &= \text{Gp}\langle a, b, c, d, e \mid ac = ca, de = ed, ab = ba, bc = cb, bd = db \rangle
\end{align*}
\]
HNN-extensions of groups

\[ H \cong \text{Gp}(A \mid R), \quad K, L \leq H \text{ with } K \cong L. \]  

Let \( \phi : K \to L \) be an isomorphism. The HNN-extension of \( H \) with respect to \( \phi \) is

\[ G = \text{HNN}(H, \phi) = \text{Gp}(A, t \mid R, t^{-1}kt = \phi(k) \ (k \in K)) \]

**Fact:** \( H \) embeds naturally into the HNN extension \( G = \text{HNN}(H, \phi) \).
**HNN-extensions of RAAGs**

**Definition**

$\Gamma$ - finite graph, $\psi : \Delta_1 \rightarrow \Delta_2$ an isomorphism between finite induced subgraphs.

$A(\Gamma, \psi)$ is defined to be the HNN-extension of $A(\Gamma)$ with respect to the isomorphism $A(\Delta_1) \rightarrow A(\Delta_2)$ induced by $\psi$.

**Fact:** $A(\Gamma)$ embeds naturally into $A(\Gamma, \psi)$. 
HNN-extension of $A(\mathcal{P}_4)$ over $A(\mathcal{P}_3)$

Let $\mathcal{P}_4$ be the graph

\[
\begin{array}{cccc}
  a & b & c & d \\
\end{array}
\]

$A(\mathcal{P}_4) = \langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle$.

$\Delta_1$ - subgraph induced by $\{a, b, c\}$, $\Delta_2$ subgraph induced by $\{b, c, d\}$,

$\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.
HNN-extension of $A(P_4)$ over $A(P_3)$

Let $P_4$ be the graph

```
 a -- b -- c -- d
```

$A(P_4) = \langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle$.

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Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to $\psi$ is

\[
A(P_4, \psi) = \langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle
\]
HNN-extension of $A(P_4)$ over $A(P_3)$

Let $P_4$ be the graph

```
 a --- b --- c --- d
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Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to $\psi$ is

\[
A(P_4, \psi) = \text{Gp}\langle a, t \mid a(tat^{-1}) = (tat^{-1})a, (tat^{-1})(t^2at^{-2}) = (t^2at^{-2})(tat^{-1}), (t^2at^{-2})(t^3at^{-3}) = (t^3at^{-3})(t^2at^{-2}) \rangle.
\]
HNN-extension of $A(P_4)$ over $A(P_3)$

Let $P_4$ be the graph

$$
\begin{array}{cccc}
  a & b & c & d \\
\end{array}
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$A(P_4) = \text{Gp}\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle$.

$\Delta_1$ - subgraph induced by $\{a, b, c\}$, $\Delta_2$ subgraph induced by $\{b, c, d\}$, $\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to $\psi$ is

$$A(P_4, \psi) = \text{Gp}\langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle$$

$$= \text{Gp}\langle a, t \mid a(tat^{-1}) = (tat^{-1})a, (tat^{-1})(t^2at^{-2}) = (t^2at^{-2})(tat^{-1}), (t^2at^{-2})(t^3at^{-3}) = (t^3at^{-3})(t^2at^{-2}) \rangle.$$

$$= \text{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle.$$
HNN-extension of $A(P_4)$ over $A(P_3)$

Let $P_4$ be the graph

```
 a -- b -- c -- d
```

$A(P_4) = \langle a, b, c, d \mid ab = ba, bc = cb, cd = dc \rangle$.

$\Delta_1$ - subgraph induced by $\{a, b, c\}$, $\Delta_2$ subgraph induced by $\{b, c, d\}$,

$\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to $\psi$ is

\[
A(P_4, \psi) = \langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle
\]

\[
= \langle a, t \mid a(tat^{-1}) = (tat^{-1})a, (tat^{-1})(t^{2}at^{-2}) = (t^{2}at^{-2})(tat^{-1}),
(t^{2}at^{-2})(t^{3}at^{-3}) = (t^{3}at^{-3})(t^{2}at^{-2}) \rangle.
\]

\[
= \langle a, t \mid atat^{-1} a^{-1} ta^{-1} t^{-1} = 1 \rangle.
\]

**Conclusion**

$A(P_4)$ embeds into the one-relator group

\[
A(P_4, \psi) = \langle a, t \mid atat^{-1} a^{-1} ta^{-1} t^{-1} = 1 \rangle.
\]
Submonoid membership problem

$G$ - a finitely generated group with a finite group generating set $A$.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

$T$ – a finitely generated submonoid of $G$.

The membership problem for $T$ within $G$ is decidable if there is an algorithm which solves the following decision problem:

**INPUT**: A word $w \in (A \cup A^{-1})^*$.

**QUESTION**: $\pi(w) \in T$?

**Theorem B**

Let $G$ be the one-relator group $Gp\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1 \rangle$. Then there is a fixed finitely generated submonoid $N$ of $G$ such that the membership problem for $N$ within $G$ is undecidable.
Proof of Theorem B

Theorem B
Let $G$ be the one-relator group $\langle a, t \midatat^{-1}a^{-1}t^{-1}t^{-1} = 1 \rangle$. Then there is a fixed finitely generated submonoid $N$ of $G$ such that the membership problem for $N$ within $G$ is undecidable.

Proof. By [Lohrey & Steinberg, 2008] there is a finitely generated submonoid $T$ of $A(P_4)$ such that the membership problem for $T$ within $A(P_4)$ is undecidable. Let $\theta : A(P_4) \rightarrow G$ be an embedding. Then $N = \theta(T)$ is a finitely generated submonoid of $G$ such that the membership problem for $N$ within $G$ is undecidable.

\[
\begin{array}{c}
\text{A}(P_4) \\
\end{array}
\quad \xleftarrow{\theta} \quad
\begin{array}{c}
\text{Gp}\langle a, t \mid aatat^{-1}a^{-1}t^{-1}t^{-1} = 1 \rangle \\
\end{array}
\quad \xrightarrow{\theta} \quad
\begin{array}{c}
\text{N} = \theta(T) \\
\end{array}
\]
Proof strategy

If \( M \) has decidable word problem

\[ \Rightarrow \text{membership problem for } U_R \leq M \text{ is decidable} \]

since \( w \in (A^uA^{-1})^* \) and

\[ w \in U_R \iff ww^{-1} = 1 \]

(sometimes) \( \implies \) membership problem for \( N \leq G \) is decidable.
General observations about inverse monoids

$S$ – an inverse monoid generated by $A$, $E(S)$ – set of idempotents,

$U_R \leq S$ – right units = submonoid if right invertible elements.

- If $e \in E(S)$ and $e \in U_R$ then $e = 1$.
- Two relations for the price of one: If $e$ is an idempotent in $\text{FIM}(A)$ and $r \in (A \cup A^{-1})^*$ then

\[
\text{Inv}\langle A \mid er = 1 \rangle = \text{Inv}\langle A \mid e = 1, r = 1 \rangle.
\]

- $e \in (A \cup A^{-1})^*$ is an idempotent in $\text{FIM}(A)$ if and only if $e$ freely reduces to 1 in the free group $\text{FG}(A)$. e.g.

\[
x^{-1}y^{-1}xx^{-1}yzz^{-1}x \in E(\text{FIM}(x, y, z)).
\]
A general construction

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set $e$ equal to
\[
a_1a_1^{-1} \ldots a_na_n^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1}) \ldots (tw_kt^{-1})(tw_k^{-1}t^{-1})a_n^{-1}a_n \ldots a_1^{-1}a_1
\]
where $t$ is a new symbol. Then

\[
M = \text{Inv}\langle A, t \mid er = 1 \rangle
\]

\[
= \text{Inv}\langle A, t \mid r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A), (tw_it^{-1})(tw_it^{-1})^{-1} = 1 \ (1 \leq i \leq k) \rangle
\]

\[
\cong \text{Gp}\langle A \mid r = 1 \rangle \ast \text{FIM}(t) / \{(tw_it^{-1})(tw_it^{-1})^{-1} = 1 \ (1 \leq i \leq k)\}.
\]

Key claim

Let $T$ be the submonoid of $G = \text{Gp}\langle A \mid r = 1 \rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$. Then for all $u \in (A \cup A^{-1})^*$ we have

\[
u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.
\]
**A general construction**

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, with $A = \{a_1, \ldots, a_n\}$, set $e$ equal to

$$a_1a_1^{-1} \ldots a_na_n^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1})\ldots (tw_kt^{-1})(tw_k^{-1}t^{-1})a_n^{-1}a_n\ldots a_1^{-1}a_1$$

where $t$ is a new symbol. Then

$$M = \text{Inv}(A, t \mid er = 1)$$

$$= \text{Inv}(A, t \mid r = 1, aa^{-1} = 1, a^{-1}a = 1 (a \in A), (tw_it^{-1})(tw_it^{-1})^{-1} = 1 \ (1 \leq i \leq k))$$

$$\cong \text{Gp}(A \mid r = 1) \ast \text{FIM}(t) \ / \ \{(tw_it^{-1})(tw_it^{-1})^{-1} = 1 \ (1 \leq i \leq k)\}.$$

**Key claim**

Let $T$ be the submonoid of $G = \text{Gp}(A \mid r = 1)$ generated by $\{w_1, w_2, \ldots, w_k\}$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.$$

**Theorem**

If $M = \text{Inv}(A, t \mid er = 1)$ has decidable word problem then the membership problem for $T$ within $G = \text{Gp}(A \mid r = 1)$ is decidable.
Proof strategy refined

\[ M = \text{Inv} \langle A, t \mid t^r = 1 \rangle \]

\[ t u t^{-1} = Y \]

\[ U_R = \{ m \in M : m m^{-1} = 1 \} \]

\[ t v t^{-1} \]

If \( M \) has decidable word problem

\[ \implies \text{membership problem for } U_R \leq M \text{ is decidable} \]

\[ \implies \forall u \in (A \cup A^{-1})^* \text{ can decide } t u t^{-1} \in U_R ? \]

(by key claim) can decide

\[ u \in T = \text{Mon} \langle w_1, \ldots, w_k \rangle \leq G \]
Tying things together

**Thoerem A**
There is a one-relator inverse monoid $\text{Inv}\langle A \mid w = 1 \rangle$ with undecidable word problem.

**Proof.**
Let $A = \{a, z\}$ and let $G$ be the one-relator group

$$\text{Gp}\langle a, z \mid azaz^{-1}a^{-1}za^{-1}z^{-1} = 1 \rangle.$$  

Let $W = \{w_1, \ldots, w_k\}$ be a finite subset of $(A \cup A^{-1})^*$ such that the membership problem for $T = \text{Mon}\langle W \rangle$ within $G$ is undecidable. Such a set $W$ exists by Theorem B. Set $e$ to be the idempotent word

$$aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1t^{-1})(tw_2t^{-1})(tw_2t^{-1})\ldots(tw_kt^{-1})(tw_kt^{-1})z^{-1}za^{-1}a.$$  

Then by the above theorem the one-relator inverse monoid

$$\text{Inv}\langle a, z, t \mid eazaz^{-1}a^{-1}za^{-1}z^{-1} = 1 \rangle$$

has undecidable word problem. This completes the proof.  
\[\square\]
Related work

Other negative results

Adjan (1966): Proved the group of units of $\text{Mon}\langle A \mid w = 1 \rangle$ is a one-relator group.

Makanin (1966): Proved that the monoid $\text{Mon}\langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle$ has a finitely presented group of units (with $k$ defining relations), and that $M$ has decidable word problem if and only if its group of units also does.

In recent joint work with Nik Ruškuc we have shown:

- There is a one-relator inverse monoid $\text{Inv}\langle A \mid w = 1 \rangle$ whose group of units is not a one-relator group.
- There is a finitely presented inverse monoid $\text{Inv}\langle A \mid w_1 = 1, \ldots, w_k = 1 \rangle$ whose group of units is not finitely presented.

Some positive results

In recent joint work with Igor Dolinka we have shown the word problem is decidable for some new classes of $\text{Inv}\langle A \mid w = 1 \rangle$ where $w$ is a cyclically reduced and the maximal group image $\text{Gp}\langle A \mid w = 1 \rangle$ is “low down” in the Magnus–Moldovanskii hierarchy.
Open problems

Problem
For which words $w \in (A \cup A^{-1})^*$ does $\text{Inv}(A \mid w = 1)$ have decidable word problem? In particular is the word problem always decidable when $w$ is (a) reduced or (b) cyclically reduced?
Open problems

**Problem**
For which words \( w \in (A \cup A^{-1})^* \) does \( \text{Inv}(A \mid w = 1) \) have decidable word problem? In particular is the word problem always decidable when \( w \) is (a) reduced or (b) cyclically reduced?

**Problem**
Characterise the one-relator groups with decidable submonoid membership problem.

**Problem**
Characterise the one-relator groups with decidable rational subset membership problem.

**Problem**
Is the subgroup membership problem problem decidable for one-relator groups?