A Characterization Theorem for the Distribution of a Continuous Local Martingale and Related Limit Theorems

Andriy Yurachkivsky

e-mail: yap@univ.kiev.ua

Abstract: The main result of the article reads: the distribution of a continuous starting from zero local martingale whose quadratic characteristic is almost surely absolutely continuous with respect to some non-random increasing continuous function is determined by the distribution of the quadratic characteristic. Functional limit theorem based on this characterization are proved.

AMS 2000 subject classifications: Primary 60G44; secondary 60F17.
Keywords and phrases: martingale, quadratic characteristic, distribution, convergence.

Introduction

The famous Lévy theorem (see, e.g., [1], Theorem 3.2.2) asserts that every continuous local martingale with zero initial value and non-random absolutely continuous quadratic characteristic is a (non-homogeneous) Wiener process whose drift coefficient is zero and diffusion coefficient is the derivative of the quadratic characteristic. Thus the distribution (this is the abbreviation for "the system of finite-dimensional distributions") of a starting from zero continuous local martingale is determined (explicitly!) by its quadratic characteristic provided the latter is non-random (and, in the original formulation, absolutely continuous, but this demand can be easily waived). The main goal of the present article is to extend this result to the case when the characteristic may be random (and an analogue of the absolute continuity condition is retained). It is accomplished in Section 1. The refusal from non-randomness of the quadratic characteristic deprives us of the possibility to find the distribution of the martingale explicitly, which in Lévy's case exhausts the proof. Thus, our approach involves another techniques.

In Section 3, this characterization is used to derive new functional limit theorems for sequences of martingales. The interstitial Section 2 contains preparatory results.

All vectors are thought of as columns. All random variables and processes are, unless otherwise stated, assumed \( \mathbb{R}^d \)-valued. The tensor square \( xx^T \) of \( x \in \mathbb{R}^d \) will be otherwise denoted \( x \otimes 2 \). The integral \( \int_0^t \varphi(s) dX(s) \) will be written shortly (following [2] and [3]) as \( \varphi \circ X(t) \) if this integral is pathwise (i.e. \( X \) is a process of locally bounded variation) or \( \varphi : X(t) \) if it is stochastic. We
use properties of stochastic integral and other basic facts of stochastic analysis without explanations, relegating the reader to [1 – 3].

1. The characterization

In this section, all the processes under consideration are implied to be given on a common probability space \((\Omega, \mathcal{F}, P)\). If \(\mathcal{F} = (\mathcal{F}(t), t \in \mathbb{R}_+)\) is a flow (in the other terminology – filtration) on this space and \(\xi\) is an \(\mathcal{F}\)-adapted random process such that for any \(t > s \geq 0\) the increment \(\xi(t) - \xi(s)\) does not depend on \(\mathcal{F}(s)\), then we say that \(\xi\) is a process with \(\mathcal{F}\)-independent increments (so, it is clear what is a Wiener process w. r. t. \(\mathcal{F}\)). We assume throughout that \(\mathcal{F}\) satisfies the usual conditions [1 – 3] and use the following notation:

\[ \mathcal{F}_\xi(t) = \bigcap_{\varepsilon > 0} \sigma(\xi(s), s \leq t + \varepsilon), \]

\[ \mathcal{F}_\xi = (\mathcal{F}_\xi(t), t \in \mathbb{R}_+); \]

\[ I_A\] – the indicator of a set \(A\);

\[ \langle M \rangle\] – the quadratic characteristic (if defined) of \(M\);

\[ \mathcal{S}\] – the class of all symmetric square matrices of a fixed size (in our case – \(d\)) with real entries, \(\mathcal{S}_+\) – its subclass of nonnegative (in the spectral sense) matrices.

Theorem 1.1. Let \(Y\) be a continuous local martingale w. r. t. \(\mathcal{F}\). Suppose that there exist an \(\mathcal{S}_+\)-valued \(\mathcal{F}Y\)-progressive random process \(\Phi\) and an \(\mathbb{R}\)-valued increasing continuous non-random function \(\Lambda\) such that

\[ \langle Y \rangle = \Phi \circ \Lambda. \]  

Then the joint distribution of \(Y\) and \(\langle Y \rangle\) is determined by \(\Lambda\) and the joint distribution of \(Y(0)\) and \(\Phi\).

We shall deduce this statement from two lemmas.

Lemma 1.1. Let \(S\) be a continuous process with \(\mathcal{F}\)-independent increments such that for all \(t\)

\[ \mathbb{E}S(t) = 0, \quad \mathbb{E}S(t)^{\otimes 2} = T(t)1, \]

where \(T\) is a non-random \(\mathbb{R}\)-valued continuous function on \(\mathbb{R}_+\). Let, further, \(\alpha\) be an \(\mathcal{F}(0)\)-measurable random variable and \(R\) be an \(\mathcal{S}\)-valued \(\mathcal{F}\)-adapted left-continuous random process such that for all \(t\)

\[ \int_0^t \|R(s)\|^2dT(s) < \infty \quad \text{a. s.} \]

Then the distribution of the triple \((\alpha, R \cdot S, R)\) is determined by \(T\) and the distribution of the pair \((\alpha, R)\).

Proof. Denote \(\Delta_{ni} = [(i - 1)/n, i/n[\), \(\overline{\Delta}_{ni} = [(i - 1)/n, i/n]\),

\[ R_n(t) = \sum_{i=1}^\infty R \left( \frac{i - 1}{n} \right) I_{\Delta_{ni}}(t), \]

\[ Z = R \cdot S, \quad Z_n = R_n \cdot S. \] By construction

\[ Z_n(t) = Z_n \left( \frac{i - 1}{n} \right) + R \left( \frac{i - 1}{n} \right) \left( S(t) - S \left( \frac{i - 1}{n} \right) \right) \quad \text{as} \quad t \in \overline{\Delta}_{ni}. \]
Hence and from the assumptions about $S$ and $R$ we deduce by induction in $k$ that the distribution of $(\alpha, Z_n, R)$ in the segment $\Delta_{nk+1}$ is determined by the joint distribution of

$$\alpha, Z_n \left(\frac{1}{n}\right), \ldots, Z_n \left(\frac{k}{n}\right), R(0), R \left(\frac{1}{n}\right), \ldots, R \left(\frac{k}{n}\right)$$

and the latter is in turn determined by $T$ and the joint distribution of $(\alpha, Z_n, R)$ is determined by that of $(\alpha, R)$.

Denote $\xi_n = Z_n - Z = (R_n - R) \cdot S$. The assumptions about $S$ and $R$ imply

$$E \xi_n^{\otimes 2} = E(R_n - R)^2 \circ T.$$  

Equality (3) where $R$ is, by assumption, left-continuous, shows us that $R_n(s) \to R(s)$ for all $s$. Consequently, if there exists a non-random constant $C$ such that

$$\sup_s \|R(s)\| \leq C,$$  

then by the dominated convergence theorem $E\xi_n(t)^{\otimes 2} \to 0$ for all $t$. Hence, noting that for any $x \in \mathbb{R}^d$ $|x|^2 = \operatorname{tr} x \otimes 2$, we get $E|\xi_n(t)|^2 \to 0$, which proves the lemma under the extra assumption (4).

Denote, for $m \in \mathbb{N}$,

$$R[m] = m R \vee \|R\|, \quad Z[m] = R[m] \cdot S.$$  

The random process $R[m]$ is left-continuous and $\mathbb{F}$-adapted since so is $R$. Furthermore, $\|R[m]\| \leq m$ by construction. So, by what has been proved, given $T$, the distribution of $(\alpha, Z[m], R)$ is determined by that of $(\alpha, R[m])$ and all the more by the distribution of $(\alpha, R)$.

Obviously, $\|R[m]\| \leq \|R\|$ and $\lim_{m \to \infty} R[m](s) = R(s)$ for all $s$. Hence and from (2) we get by the dominated convergence theorem

$$E \left(\frac{R[m] - R}{T(t)} \right)^2 \to 0 \quad \text{as} \quad m \to \infty.$$  

But the left hand side of this relation equals $E \left(\frac{Z(m)(t) - Z(t)}{T(t)} \right)^2$. Consequently, $E |Z[m](t) - Z(t)|^2 \to 0$ for all $t$.

The following statement is a modification of Lemma 1.1, with stronger assumptions about $T$ and $S$ but (and this turns out more important) without the demand that $R$ be left-continuous.

**Lemma 1.2.** Let $w$ be the standard Wiener process w. r. t. $\mathbb{F}$, $\alpha$ be an $\mathcal{F}(0)$-measurable random variable and $H$ be an $\mathbb{F}$-progressive $\mathcal{G}$-valued random process such that for any $t$

$$\int_0^t \|H(s)\|^2 \, ds < \infty \quad \text{a. s.}$$  

(5)
Then the distribution of the triple \((\alpha, H \cdot w, H)\) is determined by that of the pair \((\alpha, H)\).

Proof. Let us take an arbitrary nonnegative continuous function \(g\) on \(\mathbb{R}_+\) such that \(g(0) = 0,\) \(\text{supp} \, g \subset [0, 1]\) and \(\int_0^1 g(s) \, ds = 1.\) Denote \(g_n(t) = ng(nt),\)

\[
H_n(t) = \int_0^t H(t - s)g_n(s) \, ds = \int_0^t H(s)g_n(t - s) \, ds, \tag{6}
\]

\[
X = H \cdot w, \, X_n = H_n \cdot w \tag{7}
\]

and assume for a while that

\[
\sup_s \norm{H(s)} \leq C \tag{8}
\]

for some non-random \(C.\) Then

\[
\sup_{n,s} \norm{H_n(s)} \leq C. \tag{9}
\]

Furthermore, each \(H_n\) is, by construction, continuous and \(\mathbb{F}\)-adapted. So Lemma 1.1 asserts that the distribution of \((\alpha, X_n, H_n)\) is determined by that of \((\alpha, H_n)\). And the latter is determined by the distribution of \((\alpha, H),\) since the function \(g_n\) in (6) is non-random.

Equality (6) together with the definition of \(g_n\) yields for \(\tau \geq 1/n\)

\[
H_n(\tau) - H(\tau) = \int_0^{1/n} (H(\tau - s) - H(\tau)) \, g_n(s) \, ds,
\]

whence with account of (8) and (9) we have for \(t \geq 1/n\)

\[
\int_0^t \norm{H_n(\tau) - H(\tau)} \, d\tau \leq \frac{2C}{n} + \int_{1/n}^t \, d\tau \int_0^{1/n} \norm{H(\tau - s) - H(\tau)} \, g_n(s) \, ds.
\]

Recalling the definition of \(g_n\) and denoting

\[
f_n(t,s) = \int_{1/n}^t \norm{H \left( \tau - \frac{s}{n} \right) - H(\tau)} \, d\tau, \tag{10}
\]

we can rewrite the last inequality in the form

\[
\int_0^t \norm{H_n(\tau) - H(\tau)} \, d\tau \leq \frac{2C}{n} + \int_0^t f_n(t,s) g(s) \, ds \tag{11}
\]

(this was derived for \(t \geq 1/n\) but is valid, due to (8) and (9), for \(t < 1/n,\) too).

The process \(H,\) being progressive, has measurable paths, which together with (8) implies that all its paths on \([0, 1]\) belong to \(L_1([0,1],d\tau).\) Then from (10) we have by the M. Riesz theorem

\[
f_n(t,s) \to 0 \quad \text{as} \quad n \to \infty.
\]
(This is the central point of our rationale. Only now it becomes clear why we could not consider the more general setting of Lemma 1.1 and were forced to choose \( T(t) = t \).

Formulae (10) and (8) imply also that \( f_n(t, s) \leq 2Ct \). From the last two relations we have by the dominated convergence theorem

\[
\int_0^1 f_n(t, s)g(s)ds \to 0
\]

for all \( t \), which together with (11) yields

\[
\int_0^t \|H_n(\tau) - H(\tau)\|d\tau \to 0.
\]

Hence we get with account of (8) and (9)

\[
\mathbb{E} \int_0^t (H_n(\tau) - H(\tau))^2 d\tau \to 0.
\]

But the left hand side of this relation equals, as is seen from (7), \( \mathbb{E} (X_n(t) - X(t))^{\otimes 2} \).

Thus we have proved, under the extra assumption (8), the relation

\[
\mathbb{E} |X_n(t) - X(t)|^2 \to 0
\]

and therefore, in the light of the first paragraph of the proof, the whole lemma.

Condition (5) allows to waive assumption (8) exactly like (4) was waived in the proof of Lemma 1.1.

**Proof of Theorem 1.1.** Let us make two provisional assumptions: 1) for any \( s \) the matrix \( \Phi(s) \) is non-degenerate; 2) there exists a non-random number \( C \) such that \( \|\Phi(s)\| \leq C \) for all \( s \).

Denote \( \Psi(s) = \Phi(s)^{-1/2} \), \( \Lambda^\dagger(t) = \sup \{ s : \Lambda(s) \leq t \} \) (note that if \( \Lambda \) increases strictly, then \( \Lambda^\dagger \) is simply the inverse function \( \Lambda^{-1} \)), \( \Theta(s) = \sqrt{\Phi(s)} \equiv \Psi(s)^{-1} \), \( W = \Psi \cdot Y \), \( w(t) = W (\Lambda^\dagger(t)) \),

\[
X(t) = Y (\Lambda^\dagger(t)) - Y(0).
\]

(12)

Then:

\[
\Lambda (\Lambda^\dagger(t)) = t
\]

(13)

(but not certainly \( \Lambda^\dagger (\Lambda(t)) = t \)); \( W \), \( X \) and \( w \) are starting from zero martingales;

\[
Y = Y(0) + \Theta \cdot W;
\]

(14)

\[
\langle W \rangle = \Psi^2 \circ \langle Y \rangle \overset{(1)}{=} (\Psi^2 \Phi) \circ \Lambda, \text{ i. e.}
\]

\[
\langle W \rangle = \Lambda \mathbf{1};
\]

(15)

\[
\langle w \rangle(t) = \langle W \rangle (\Lambda^\dagger(t)), \text{ which together with (15) and (13) yields}
\]

\[
\langle w \rangle(t) = t \mathbf{1}.
\]

(16)
Notwithstanding possible discontinuities of \( \Lambda^l \) the processes \( X \) and \( w \) are continuous. Indeed, if \( \Lambda^l(t-)=a<b=\Lambda^l(t) \), then \( \Lambda(a)=\Lambda(b) \), whence in view of (1) \( \langle Y \rangle (a)=\langle Y \rangle (b) \) and therefore \( Y(s)=Y(a), \, W(s)=W(a) \) for all \( s \in [a,b] \). One may say that \( Y \) and \( W \) "skip" the discontinuities of \( \Lambda \).

Thus \( X \) and \( w \) are continuous martingales starting from zero. Then equality (16) implies, by Lévy’s theorem, that \( w \) is the standard Wiener process.

Writing on the basis of (12) and (14)
\[
X(t) = \int_0^{\Lambda^l(t)} \Theta(s)dW(s),
\]
making the change of variables \( s = \Lambda^l(\tau) \) (possibly discontinuous but yet correct due to "skipping"), denoting
\[
H(s) = \Theta \left( \Lambda^l(s) \right)
\]
and recalling the definition of \( w \), we get \( X = H \cdot w \).

Since for any \( A \in \mathcal{G} \) \( ||A^2|| = ||A||^2 \), the second provisional assumption implies that \( \sup_s||\Theta(s)|| \leq \sqrt{C} \) and therefore \( H \) satisfies condition (5). Then Lemma 1.2 asserts that the distribution of \( \langle Y(0), X, H \rangle \) is determined by that of \( \langle Y(0), H \rangle \). Obviously, for any \( l \in \mathbb{N} \) and \( t_1, \ldots, t_l \in \mathbb{R}_+ \) there exist \( s_1, \ldots, s_l \in \mathbb{R}_+ \) such that \( Y(t_1) = Y(s_1), \ldots, Y(t_l) = Y(s_l) \) and \( \Lambda^l(\Lambda(s_j)) = s_j \), \( j = 1, \ldots, l \). Then because of (12) \( Y(t_j) = Y(0) + X(\Lambda(s_j)) \). So the distribution of \( \langle Y(0), H \rangle \) determines, together with \( \Lambda \), the distribution of \( \langle Y, H \rangle \). And the process \( H \) is, in turn, determined (pathwise!) by \( \Phi \) and \( \Lambda \) by virtue of formula (17). Thus we have proved the theorem under two extra assumptions, of which the second can be waived exactly like the similar assumption (4) was.

To get rid of the first provisional assumption we take (extending, if necessary, the probability space) an independent of \( \mathcal{F} \) (and therefore of \( Y \)) process \( W_0 \) with independent increments such that for all \( t \) \( EW_0(t) = 0, \, EW_0(t)^2 = \Lambda(t) \) and put
\[
Y_\varepsilon = Y + \varepsilon W_0 \quad (\varepsilon > 0).
\]
Obviously, \( Y_\varepsilon(0) = Y(0) \) and \( \langle Y_\varepsilon \rangle = \Phi_\varepsilon \circ \Lambda \), where \( \Phi_\varepsilon = \Phi + \varepsilon^2 1 \). So the distribution of \( \langle Y_\varepsilon, H \rangle \) is determined by \( \Lambda \) and the distribution of \( \langle Y(0), \Phi \rangle \). It remains to let \( \varepsilon \to 0 \).

**Lemma 1.3.** Let \( \zeta \) be an \( \mathbb{R} \)-valued random process whose trajectories are w. p. 1 absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with distribution function \( \Lambda \). Then there exists an \( \mathcal{F} \)-progressive random process \( \varphi \) such that \( \zeta = \varphi \circ \Lambda \) a. s.

**Proof.** Let us introduce the notation: \( s_{ni} = 2^{-ni} \), \( I_{ni} \) the indicator of \( [s_{ni-1}, s_{ni}] \), \( \mu \) the random signed measure on \( \mathcal{B}(\mathbb{R}_+) \) with distribution function \( \zeta \); \( \nu \) the measure on \( \mathcal{B}(\mathbb{R}_+) \) with distribution function \( \Lambda \);
\[
\alpha_{ni} = \text{sgn} \left( \Lambda(s_{ni}) - \Lambda(s_{ni-1}) \right) \frac{\zeta(s_{ni}) - \zeta(s_{ni-1})}{\Lambda(s_{ni}) - \Lambda(s_{ni-1})},
\]
(18)
\[ \psi_n(\omega, s) = \sum_{i=1}^{\infty} \alpha_{ni}(\omega)I_{ni}(s), \]

\[ \Omega_0 = \{ \omega \in \Omega : \mu(\omega, \cdot) \text{ is absolutely continuous w. r. t. } \nu \}, \quad \Gamma = \Omega_0 \times \mathbb{R}_+, \]

\[ F = \{ (\omega, s) \in \Gamma : \text{the sequence } (\psi_n(\omega, s)) \text{ converges} \}, \quad (19) \]

\[ \varphi_n = I_F \psi_n, \quad \varphi = \lim_{n \to \infty} \varphi_n, \quad \mathbb{F}_n^\varphi = (\mathbb{F}_n^\varphi(s + 2^n), s \in \mathbb{R}_+). \quad (20) \]

Rewriting the definition of \( \psi_n \) in the form \( \psi_n(s) = \alpha_n[2^n s] \), we deduce from (18) – (20) the following: \( \psi_n \) is \( \mathbb{F}_n^\varphi \)-progressive; \( I_F \) and \( \varphi \) are \( \mathbb{F}_n^\varphi \)-progressive. The remaining part of the proof is standard.

We continue the list of notation: \( \mathcal{B}_n \) – the \( \sigma \)-algebra in \( \mathbb{R}_+ \) generated by the sets \( \{0\}, [s_{ni-1}, s_{ni}], i \in \mathbb{N}; \mu_n \) and \( \nu_n \) – the restrictions of \( \mu \) and \( \nu \) respectively to \( \mathcal{B}_n \). Obviously, \( \psi_n = d\mu_n / d\nu_n \). Then a well-known theorem of measure theory (see, e. g., Proposition 48.1 [4]) asserts that for each \( \omega \in \Omega_0 \)

\[ \nu \left\{ s : \lim_{n \to \infty} \psi_n(\omega, s) \text{ does not exist} \right\} = 0. \]

So \( \varphi_n \circ \Lambda = \psi_n \circ \Lambda \) for all \( (\omega, t) \in \Gamma \). By the same theorem \( \varphi_n \circ \Lambda \rightarrow \varphi \circ \Lambda \) for all \( (\omega, t) \in \Gamma \). Recall also that, by assumption, \( \mathbb{P}(\Omega_0) = 1 \). Now, to finalize the proof, it remains to show that \( \psi_n \circ \Lambda \rightarrow \zeta \) for all \( (\omega, t) \in \Gamma \). To this end we write, by the definition of \( \psi_n \), the relations

\[ \psi_n \circ \Lambda(s_{n_j}) = \zeta(s_{n_j}), \quad \left| \psi_n \circ \Lambda(t) - \psi_n \circ \Lambda(s_{n_i[2^n t]}) \right| \leq \left| \zeta(t) - \zeta(s_{n_i[2^n t]}) \right|, \]

implying, obviously, that \( \left| \psi_n \circ \Lambda(t) - \zeta(t) \right| \leq 2 \left| \zeta(t) - \zeta(s_{n_i[2^n t]}) \right| \).

**Lemma 1.4.** Let \( K \) be an \( \mathcal{S}_+ \)-valued increasing random process such that almost all trajectories of \( \text{tr } K \) are absolutely continuous w. r. t. some non-random measure. Then for any \( a, b \in \mathbb{R}^d \), \( a^\top K b \) is also absolutely continuous w. r. t. this measure.

**Proof.** For any \( A \in \mathcal{S}_+ \)

\[ a^\top A a \leq \| A \| \cdot |a|^2 \] and \( \| A \| \leq \text{tr } A \). Hence, taking \( t > s \) and putting \( A = K(t) - K(s) \), we deduce absolute continuity of \( a^\top K a \). It remains to write \( 2a^\top A b = (a + b)^\top A(a + b) - a^\top A a - b^\top A b \).

Lemmas 1.3 and 1.4 yield

**Corollary 1.1.** Let \( K \) be an \( \mathcal{S}_+ \)-valued increasing random process such that almost all trajectories of \( \text{tr } K \) are absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with continuous distribution function \( \Lambda \). Then there exists an \( \mathcal{S}_+ \)-valued \( \mathbb{F}^\varphi \)-progressive random process \( \Phi \) such that \( K = \Phi \circ \Lambda \).

Hence and from Theorem 1.1 we obtain

**Corollary 1.2.** Let \( Y \) be a continuous local martingale such that almost all trajectories of \( \text{tr } (Y) \) are absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with continuous distribution function. Then the joint distribution of \( Y \) and \( (Y) \) is determined by that of \( Y(0) \) and \( (Y) \).
2. Some technical results

Some statements of this ancillary section are almost trivial, some other are easy consequences of well-known facts. But Lemma 2.3 conveys technical novelty and Lemma 2.2 appearing here as a quotation is by far not trivial.

We consider henceforth sequences of random processes given, maybe, on different probability spaces. So, for the $n$th member of a sequence, $P$ and $E$ should be understood as $P_n$ and $E_n$.

Let $X, X_1, X_2, \ldots$ be random processes with trajectories in the Skorokhod space $D (= \text{càdlàg processes on } \mathbb{R}_+)$.

We write $X_n \xrightarrow{D} X$ if the induced by the processes $X_n$ measures on the Borel $\sigma$-algebra in $D$ weakly converge to the measure induced by $X$. If herein $X$ is continuous, then we write $X_n \xrightarrow{C} X$.

We say that a sequence $(X_n)$ is relatively compact (r.c.) in $D$ (in $C$) if each its subsequence contains, in turn, a subsequence converging in the respective sense.

The weak convergence of finite-dimensional distributions of random processes, in particular the convergence in distribution of random variables, will be denoted $d \xrightarrow{}$. Likewise $d = \xrightarrow{}$ signifies equality of distributions.

Denote $\Pi(t, r) = \{(u, v) \in \mathbb{R}^2 : (v-r)_+ \leq u \leq v \leq t\}$,

$$\Delta_d(f; t, r) = \sup_{(u, v) \in \Pi(t, r)} |f(v) - f(u)| \quad (f \in D, \ t > 0, \ r > 0).$$

Proposition VI.3.26 (items (i), (ii)) [3] together with VI.3.9 [3] asserts that a sequence $(\xi_n)$ of càdlàg random processes is r.c. in $C$ iff for all positive $t$ and $\varepsilon$

$$\lim_{N \to \infty} \lim_{n \to \infty} P \left\{ \sup_{s \leq t} |\xi_n(s)| > N \right\} = 0$$

and

$$\lim_{r \to 0} \lim_{n \to \infty} P \{ \Delta_d(\xi_n; t, r) > \varepsilon \} = 0.$$

Hence two consequences are immediate.

**Corollary 2.1.** Let $(\xi_n)$ and $(\Xi_n)$ be sequences of $\mathbb{R}^m$-valued càdlàg processes such that $(\Xi_n)$ is r.c. in $C$, $|\xi_n(0)| \leq |\Xi_n(0)|$ and for any $v > u \geq 0$

$$|\xi_n(v) - \xi_n(u)| \leq |\Xi_n(v) - \Xi_n(u)|.$$

Then the sequence $(\xi_n)$ is also r.c. in $C$.

**Corollary 2.2.** Let $(\xi_n)$ and $(\zeta_n)$ be r.c. in $C$ sequences of càdlàg processes taking values in $\mathbb{R}^d$ and $\mathbb{R}^p$ respectively. Suppose also that for each $n \xi_n$ and $\zeta_n$ are given on a common probability space. Then the sequence of $\mathbb{R}^{d+p}$-valued processes $(\xi_n, \zeta_n)$ is also r.c. in $C$.

For a function $f \in D$ we denote $\Delta f(t) = f(t) - f(t-)$. The quadratic variation (see the definition in § 2.3 [1] or Definition I.4.45 together with Theorem I.4.47 in [3]) of a random process $\xi$ will be denoted $[\xi]$. We shall use the conditions:

**RC.** The sequence $(\text{tr } (Y_n))$ is r.c. in $C$.

**UI.** For any $t$ the sequence $(|Y_n(t) - Y_n(0)|^2)$ is uniformly integrable.
Lemma 2.1. Let \((Y_n)\) be a sequence of local square integrable martingales satisfying the conditions: \(\text{RC}\),

\[
\lim_{L \to \infty} \lim_{n \to \infty} P\{Y_n(0) > L\} = 0
\]

and, for each \(t > 0\), the condition

\[
\max_{s \leq t} |\Delta Y_n(s)| \xrightarrow{P} 0.
\]

Then \((Y_n)\) is r. c. in \(C\).

Proof. It follows from \(\text{RC}\) and (21) by Rebolledo’s theorem [3, VI.4.13] that \((Y_n)\) is r. c. in \(D\). Hereon the desired conclusion follows from Proposition VI.3.26 (items (i) and (iii)) [3] with account of VI.3.9 [3]. □

Lemma 2.2. Let \((Y_n)\) be a sequence of martingales satisfying conditions \(\text{RC}, \text{UI}\) and (22). Then for any \(t\)

\[
[Y_n](t) - (Y_n)(t) \xrightarrow{P} 0.
\]

Proof. For the dimension \(d = 1\), this was proved in [5] under the extra assumption that the \(Y_n\)’s are quasicontinuous which was waived in [6]. If \(d > 1\), then, in view of the familiar expression \(\|A\| = \sup_{\|x\| = 1} |x^\top Ax|\) for the norm of a Hermitian operator \(A\), it suffices to show that for any \(t > 0\) and sequence \((x_n)\) in the unit sphere

\[
[x_n^\top Y_n](t) - \langle x_n^\top Y_n \rangle(t) \xrightarrow{P} 0.
\]

And this emerges from the statement for the one-dimensional case, since the numeral processes \(x_n^\top Y_n\) satisfy, evidently, the conditions of the lemma. □

Lemma 2.3. Let \((\eta^l_n, l, n \in \mathbb{N}), (\eta^l), (\eta_n)\) be sequences of càdlàg random processes such that: for any positive \(t\) and \(\varepsilon\)

\[
\lim_{l \to \infty} \lim_{n \to \infty} P\left\{\sup_{s \leq t} |\eta^l_n(s) - \eta_n(s)| > \varepsilon\right\} = 0;
\]

for each \(l\)

\[
\eta^l_n \xrightarrow{D} \eta^l \quad \text{as} \quad n \to \infty;
\]

the sequence \((\eta^l)\) is r. c. in \(D\). Then there exists a random process \(\eta\) such that \(\eta^l \xrightarrow{D} \eta\).

Proof. Let \(\rho\) be a bounded metric in \(D\) metrizing Skorokhod’s \(J\)-convergence (see, for example, [3, VI.1.26]). Then condition (23) with arbitrary \(t\) and \(\varepsilon\) implies that

\[
\lim_{l \to \infty} \lim_{n \to \infty} E\rho(\eta^l_n, \eta_n) = 0.
\]

Hence by the triangle inequality

\[
\lim_{l \to \infty} \lim_{n \to \infty} E\rho(\eta^m_n, \eta^k_n) = 0.
\]
Let $F$ be a uniformly continuous w. r. t. $\rho$ bounded functional on $D$. Denote $A = \sup_{x \in D} |F(x)|$, $\vartheta(r) = \sup_{x,y \in D; \rho(x,y) < r} |F(x) - F(y)|$. Then $\vartheta(0+) = 0$ and for any $r > 0$

$$
\mathbb{E} |F(\eta^m_n) - F(\eta^k_n)| \leq AP \{ \rho(\eta^m_n, \eta^k_n) > r \} + \vartheta(r),
$$

which together with (26) yields

$$
\lim_{n \to \infty} \lim_{m \to \infty} |EF(\eta^m_n) - EF(\eta^k_n)| = 0.
$$

(27)

By condition (24)

$$
\lim_{n \to \infty} |EF(\eta^m_n) - EF(\eta^k_n)| = |EF(\eta^m_n) - EF(\eta^k_n)|,
$$

which jointly with (27) proves fundamentality and therefore convergence of the sequence $(EF(\eta^l), l \in \mathbb{N})$. Now, the desired conclusion emerges from relative compactness of $(\eta^l)$ in $D$. \qed

**Corollary 2.3.** Let the conditions of Lemma 2.3 be fulfilled. Then $\eta_n \xrightarrow{D} \eta$, where $\eta$ is the existing by Lemma 2.3 random process such that $\eta^l \xrightarrow{D} \eta$.

**Proof.** Repeating the derivation of (27) from (26), we derive from (25) the relation

$$
\lim_{l \to \infty} \lim_{n \to \infty} |EF(\eta^l_n) - EF(\eta_n)| = 0.
$$

It remains to write $|EF(\eta_n) - EF(\eta)| \leq |EF(\eta_n) - EF(\eta^l_n)| + |EF(\eta^l_n) - EF(\eta^l)| + |EF(\eta^l) - EF(\eta)|$.

\qed

**Corollary 2.4.** Let $(\eta^l_n)$, $(\eta^l)$, $(\eta_n)$ be sequences of càdlàg random processes such that: for any $t \in \mathbb{R}_+$ and $\varepsilon > 0$ equality (23) holds; for each $l \in \mathbb{N}$ relation (24) is valid; the sequence $(\eta^l)$ is r. c. in $C$. Then there exists a random process $\eta$ such that $\eta^l \xrightarrow{C} \eta$ and $\eta_n \xrightarrow{C} \eta$.

Below, $U$ is the symbol of the locally uniform (i. e. uniform in each segment) convergence.

**Lemma 2.4.** Let $X, X_1, X_2 \ldots$ be càdlàg random processes such that $X_n \xrightarrow{C} X$. Then $F(X_n) \xrightarrow{d} F(X)$ for any $U$-continuous functional $F$ on $D$.

**Proof.** Lemma VI.1.33 and Corollary VI.1.43 in [3] assert completeness and separability of the metric space $(D, \rho)$, where $\rho$ is the metric used in the proof of Lemma 2.3. Then it follows from the assumptions of the lemma by Skorokhod’s theorem [7] that there exist given on a common probability space random processes $X', X'_1, X'_2 \ldots$ such that $X' \xrightarrow{d} X$ (so that $X'$ may be considered continuous), $X_n' \xrightarrow{d} X_n$ and $\rho(X_n', X') \to 0$ a. s. By the choice of
\( \rho \) the last relation is tantamount to \( X'_n \xrightarrow{d} X' \) a. s. Hence and from continuity of \( X' \) we get by Proposition VI.1.17 [3] \( X'_n \xrightarrow{d} X' \) a. s. and therefore, by the choice of \( F \), \( F(X'_n) \rightarrow F(X') \) a. s. It remains to note that \( F(X'_n) \equiv F(X_n), \ F(X') \equiv F(X) \).

The next statement is obvious.

**Lemma 2.5.** Let \( (M^l) \) be a sequence of martingales such that

\[
M^l \xrightarrow{d} M
\]

and for any \( t \) the sequence \( |M^l(t)| \) is uniformly integrable. Then \( M \) is a martingale.

**Lemma 2.6.** Let \( (M^l) \) be a sequence of local square integrable martingales such that \( (28) \) holds and

\[
\sup_{t \in \mathbb{N}} \mathbb{E} \left[ \text{tr}(M^l)(t) \right] < \infty.
\]

Then \( \sup_t \mathbb{E}|M(t)|^2 \leq \infty \). 

**Proof.** By condition \( (29) \) and the definition of quadratic characteristic there exists a constant \( C \) such that \( \mathbb{E}|M^l(t)|^2 \leq C \) for all \( t \) and \( l \). Hence and from \( (28) \) we have by Fatou’s theorem (applicable due to the above-mentioned Skorokhod’s principle of common probability space) \( \mathbb{E}|M(t)|^2 \leq C \). □

**Corollary 2.5.** Let a sequence \( (M^l) \) of square integrable martingales satisfy conditions \( (28) \) and \( (29) \). Then \( M \) is a uniformly integrable martingale.

**Lemma 2.7.** Let \( Y \) be a local martingale and \( K \) be an \( \mathcal{G} \)-valued random process. Suppose that they are given on a common probability space and \( (Y, K) \xrightarrow{d} (Y, [Y]) \). Then for any \( t \) \( K(t) = [Y](t) \) a. s.

**Proof.** By assumption

\[
\sum_{i=1}^{n} (Y(t_i) - Y(t_{i-1}))^\otimes 2 - K(t) \xrightarrow{d} \sum_{i=1}^{n} (Y(t_i) - Y(t_{i-1}))^\otimes 2 - [Y](t)
\]

for any \( n, t \) and \( t_0 < t_1 < \ldots t_n \). Hence, recalling the definition of quadratic variation, we get \( [Y](t) - K(t) \equiv 0 \) a. s. □

Theorem 2.3.5 [1] asserts that

\[
[Y] = \langle Y \rangle
\]

for a a continuous local martingale \( Y \).

**Corollary 2.6.** Let \( Y \) be a continuous local martingale and \( K \) be a continuous \( \mathcal{G} \)-valued random process. Suppose that they are given on a common probability space and \( (Y, K) \xrightarrow{d} (Y, [Y]) \). Then w. p. 1 \( K(t) = \langle Y \rangle(t) \) for all \( t \).

**Proof.** Lemma 2.7 and formula \( (30) \) yield \( \mathbb{P}\{\forall t \in \mathbb{Q}_+ \ K(t) = \langle Y \rangle(t) \} = 1 \). Continuity of both processes enables us to substitute \( \mathbb{Q}_+ \) by \( \mathbb{R}_+ \). □
3. Functional limit theorems for martingales

**Theorem 3.1.** Let \( (Y_n) \) be a sequence of martingales satisfying condition \( \text{UI} \). Suppose that there exists an \( \mathbb{R}^d \times \mathcal{S}_+ \)-valued random process \( (Y, K) \) such that

\[
(Y_n, \langle Y_n \rangle) \xrightarrow{C} (Y, K).
\]

Then: 1) the relation

\[
[Y_n] - \langle Y_n \rangle \xrightarrow{C} 0
\]

holds; 2) \( Y \) is a continuous martingale and \( (Y, K) \overset{d}{=} (Y, \langle Y \rangle) \).

**Proof.** Recalling Doob’s inequality

\[
E \sup_{s \leq t} |M(s)|^2 \leq 4E|M(t)|^2
\]

for a square integrable martingale \( M \), we deduce from \( \text{UI} \) that for any \( t \)

\[
\sup_n E \max_{s \leq t} |\Delta Y_n(s)| < \infty.
\]

Also, \( \text{UI} \) together with (31) implies, by Lemma 2.5, that \( Y \) is a martingale. Then from (33) and (34) we get by Corollary VI.6.7 [3] \( (Y_n, \langle Y_n \rangle) \xrightarrow{C} (Y, \langle Y \rangle) \). This together with (31) (entailing both \( \text{RC} \) and (22)) results, by Lemma 2.2, in (32) which together with the previous relation yields \( (Y_n, \langle Y_n \rangle) \xrightarrow{C} (Y, \langle Y \rangle) \). And this is, in view of (30), tantamount to

\[
(Y_n, \langle Y_n \rangle) \xrightarrow{C} (Y, \langle Y \rangle).
\]

**Theorem 3.2.** Let \( (Y_n) \) be a sequence of martingales satisfying conditions \( \text{RC}, \text{UI}, \) (22) and

\[
(Y_n(0), \langle Y_n \rangle) \overset{d}{\longrightarrow} \left( \hat{Y}, K \right),
\]

where \( \hat{Y} \) is a random variable and \( K \) is an \( \mathcal{S}_+ \)-valued random process such that almost all trajectories of \( \text{tr} K \) are absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with continuous distribution function. Then relation (31) holds, where \( \hat{Y} \) is a continuous martingale with quadratic characteristic \( K \) and initial value \( Y \).

**Proof.** Obviously, (36) entails (21). Conditions \( \text{RC}, \) (21) and (22) imply by Lemma 2.1 and Corollary 2.2 that the sequence of compound processes \( (Y_n, \langle Y_n \rangle) \) is r. c. in \( C \). Consequently, for any infinite set \( J_0 \subset \mathbb{N} \) there exist an infinite set \( J \subset J_0 \) and an \( \mathbb{R}^d \times \mathcal{S}_+ \)-valued random process \( (Y^J, K^J) \) such that

\[
(Y_n, \langle Y_n \rangle) \xrightarrow{C} (Y^J, K^J) \quad \text{as} \quad n \to \infty, \; n \in J.
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]

\[
\text{C}\ - \rightarrow
\]
Then Theorem 3.1 asserts that \( Y^J \) is a continuous martingale and \((Y^J, K^J) \overset{d}{=} (Y^J, \langle Y^J \rangle)\). Also, (37) and (36) yield \((Y^J(0), K^J) \overset{d}{=} (\bar{Y}, K)\). Juxtaposing these two relations, we get \((Y^J(0), \langle Y^J \rangle) \overset{d}{=} (\bar{Y}, K)\). Then Corollary 1.2 asserts that the distribution of \((Y^J, \langle Y^J \rangle)\) (or, the same, of \((Y^J, K^J)\)) does not depend on \( J \). And this means, since a set \( J_0 \) was taken arbitrarily, that (37) holds for \( J = \mathbb{N} \), too.

\[ \text{Proof.} \]

Then Theorem 3.3.

Let \((Y_n)\) be a sequence of local square integrable martingales satisfying conditions RC, (21) and, for each \( t \), the condition

\[ \mathbb{E} \max_{s \leq t} |\triangle Y_n(s)|^2 \to 0. \]  

Then for any infinite set \( J_0 \subset \mathbb{N} \) there exist an infinite set \( J \subset J_0 \) and a continuous local martingale \( Y^J \) such that

\[ (Y_n, \langle Y_n \rangle) \overset{C}{\longrightarrow} (Y^J, \langle Y^J \rangle) \quad \text{as} \quad n \to \infty, \quad n \in J. \]

\[ \text{Proof.} \]

1. Denote \( \tau_n^l = \inf \{ t : |Y_n(t)| \geq l \} \), \( Y_n^l(t) = Y_n(t \wedge \tau_n^l) \), \( K_n = \langle Y_n \rangle \), \( K_n^l = \langle Y_n^l \rangle \) (so that \( K_n^l(t) = K_n(t \wedge \tau_n^l) \)),

\[ \eta_n = (Y_n, K_n), \quad \eta_n^l = (Y_n^l, K_n^l), \quad \text{for each} \quad n \in J. \]

regarding \( \eta_n \) and \( \eta_n^l \) as \( \mathbb{R}^{d+d^2} \)-valued processes.

Conditions RC, (21) and (38) imply (see the proof of Theorem 3.2) that the sequence \((\eta_n)\) is r. c. in \( C \). Then by Corollaries 2.1 and 2.2 for any \( l \in \mathbb{N} \) the sequence of compound processes \((\eta_n^1, \ldots, \eta_n^l, K_n)\) is r. c. in \( C \), too. Hence, using the diagonal method, we deduce that for any infinite set \( J_0 \subset \mathbb{N} \) there exist an infinite set \( J \subset J_0 \) and random processes \( Y^1, K^1, Y^2, K^2 \ldots \) such that for all \( l \in \mathbb{N} \)

\[ (\eta_n^1, \ldots, \eta_n^l, K_n) \overset{C}{\longrightarrow} (\eta^1, \ldots, \eta^l, K) \quad \text{as} \quad n \to \infty, \quad n \in J, \]

where

\[ \eta^l = (Y^i, K^i). \]  

The distribution of the right hand side of (41) may depend on \( J \), so the minute notation would be something like \((\eta^{i,1}, \ldots, \eta^{i,l}, K^i)\) (cf. with the proof of Theorem 3.2). For technical reasons, we suppress the superscript \( J \), keeping, however, it in mind.

2. By the definition of \( Y_n^l \)

\[ |Y_n^l(t)| \leq l + \max_{s \leq t} |\triangle Y_n(s)|, \]

which together with (38) shows that for any \( l \) and \( t \) the sequence \( \left( |Y_n^l(t)|^2, n \in \mathbb{N} \right) \) is uniformly integrable. Then it follows from (40) – (42) by Theorem 3.1 and Corollary 2.6 that \( Y^l \) is a continuous martingale and

\[ K^l = \langle Y^l \rangle. \]
3°. Writing
\[
\left\{ \sup_{s \leq t} |\eta^i_n(s) - \eta_n(s)| > 0 \right\} \subset \{ \tau_n^l < t \} \subset \left\{ \sup_{s \leq t} |Y_n(s)| \right\}
\]
and recalling that \((Y_n)\) is r. c. in \(C\), we arrive at (23).

4°. Note that the processes \(\eta^1, \eta^2, \ldots\) are given, in view of (41), on a common probability space. Let us show that
\[
\lim_{l \to \infty} \sup_{i > l} E r(\eta^i, \eta^l) = 0,
\]
where \(r\) is the metric in \(D\) defined by
\[
r(f, q) = \sum_{m=1}^{\infty} 2^{-m} \left( 1 \wedge \sup_{s \leq m} |f(s) - g(s)| \right).
\]

From (41) we have by Lemma 2.4
\[
\sup_{s \leq m} |\eta^i_n(s) - \eta^l_n(s)| \xrightarrow{d} \sup_{s \leq m} |\eta^i(s) - \eta^l(s)| \quad \text{as} \quad n \to \infty, \ n \in J
\]
for all natural \(m, i\) and \(l\). Then Alexandrov’s theorem asserts that for any \(\varepsilon > 0\)
\[
P \left\{ \sup_{s \leq m} |\eta^i(s) - \eta^l(s)| > \varepsilon \right\} \leq \lim_{n \to \infty, n \in J} \lim_{i \to \infty, i \in \mathbb{N}} \sup_{s \leq m} |\eta^i_n(s) - \eta^l_n(s)| > \varepsilon
\]
which together with the definitions of \(\eta^k_n\), \(\lim\) and \(\lim\) yields, for \(i > l\),
\[
P \left\{ \sup_{s \leq m} |\eta^i(s) - \eta^l(s)| > \varepsilon \right\} \leq \lim_{n \to \infty, n \in J} \sup_{s \leq m} |Y_n(s)| > l.
\]
Hence and from the evident inequality
\[
E(1 \wedge \gamma) \leq \varepsilon + P\{\gamma > \varepsilon\},
\]
where \(\gamma\) is an arbitrary non-negative random variable, we get for \(i > l\)
\[
E \left( 1 \wedge \sup_{s \leq m} |\eta^i(s) - \eta^l(s)| \right) \leq \varepsilon + \lim_{n \to \infty, n \in J} \sup_{s \leq m} |Y_n(s)| \geq l
\]
By the Lenglart – Rebolledo inequality
\[
P \left\{ \sup_{s \leq m} |Y_n(s)| \geq l \right\} \leq \frac{a}{l^2} + P \{ tr K_n(m) \geq a \}
\]
for any \(a > 0\). Relation (41) implies, by Alexandrov’s theorem, that
\[
\lim_{n \to \infty, n \in J} P \{ tr K_n(m) \geq a \} \leq P \{ tr K(m) \geq a \},
\]
which together with (45) – (47) yields

$$\sup_{i>l} E r (\eta_i, \eta_l) \leq \varepsilon + \frac{a}{l^2} + \sum_{m=1}^{\infty} 2^{-m} P \{ \text{tr } K(m) \geq a \}.$$ 

Hence, letting $l \to \infty$, then $a \to \infty$ and finally $\varepsilon \to 0$, we obtain (44).

5°. Obviously, $r$ metrizes the $U$-convergence and the metric space $(C, r)$ is complete. Relation (44) means that the sequence $(\eta^l)$ of C-valued random elements is fundamental in probability. Then by the Riesz theorem each its subsequence contains a subsequence converging w. p. 1. The limits of every two convergent subsequences coincide w. p. 1 because of (44). So there exists a C-valued random element (= continuous random process) $\eta$ such that

$$\lim_{l \to \infty} E r (\eta^l, \eta) = 0. \quad (48)$$

And this is a fortified form of the relation

$$\eta^l \xrightarrow{C} \eta. \quad (49)$$

In particular, the sequence $(\eta^l)$ is r. c. in C (which can be proved directly, but such proof does not guarantee that partial limits are given on the same probability space that the pre-limit processes are).

6°. Relation (41) together with the conclusions of items 3° and 5° shows that all the conditions of Corollary 2.4 (with the range of $n$ restricted to $J$) are fulfilled (and even overfulfilled: relation (49) proved above without recourse to Corollary 2.4 contains both an assumption and a conclusion of the latter). So Corollary 2.4 asserts, in addition to (49), that

$$\eta_n \xrightarrow{C} \eta \quad \text{as} \quad n \to \infty, \ n \in J.$$ 

This pair of relations can be rewritten, in view of (40) and (42), in the form

$$(Y^l, K^l) \xrightarrow{C} (Y, K), \quad (Y_n, K_n) \xrightarrow{C} (Y, K) \quad \text{as} \quad n \to \infty, \ n \in J, \quad (50)$$

where $(Y, K)$ is a synonym of $\eta$. We wish to stress again that, firstly, all the processes in (50) are given on a common probability space and, secondly, they depend on the choice of $J$.

7°. Let us show that $Y$ is a local martingale.

Denote $\sigma_m = \inf \{ t : \text{tr } K(t) \geq m \}$, $M^l_m(t) = Y \uparrow \sigma^l_m$, $M^l_n = Y^l \uparrow \sigma^l_m$. Equalities (48), (45) and (42) yield

$$\lim_{l \to \infty} E r (M^l_m, M_m) = 0,$$

whence

$$M^l_m \xrightarrow{d} M_m \quad \text{as} \quad l \to \infty. \quad (52)$$
On the strength of (43)
\[ \langle M^1_m \rangle (t) = K^1 (t \wedge \sigma_m). \]  

(53)

By the construction of the processes \( Y^l_n \) and \( K^l_n \) for any \( s \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \) the sequence \( (\text{tr} K^l_n, l \in \mathbb{N}) \) increases. Then due to (41) so does \( (\text{tr} K^l(s), l \in \mathbb{N}) \). Hence we have with account of (48), (45) and (42)
\[
\text{tr} K^l(s) \leq \text{tr} K(s)
\]

for all \( s \) and \( l \). Comparing this with (53), we see that
\[
\mathbb{E} \text{tr} \langle M^1_m \rangle (t) \leq \mathbb{E} \text{tr} K (t \wedge \sigma_m).
\]

(54)

But \( \text{tr} K \) is a continuous increasing process, so \( \text{tr} K(\sigma_m) = m, \text{tr} K(t \wedge \sigma_m) \leq m \). Now it follows from (52) and (54) by Corollary 2.5 that \( M_m \) is a uniformly integrable martingale. Thus the sequence \( (\sigma_m) \) localizes \( Y \).

8°. Relation \( Y^l \xrightarrow{C} Y \) (a part of (50)) where the pre-limit processes are, according to item 2°, continuous martingales implies by Corollary VI.6.7 [3]
\[
(\langle Y^l \rangle, [Y^l]) \xrightarrow{C} (\langle Y \rangle, [Y]).
\]

Comparing this with (50), we get with account of (30) \( (Y, K) \equiv (\langle Y \rangle), \) hereupon Corollary 2.6 asserts that \( K = \langle Y \rangle \).

\[ \square \]

**Corollary 3.1.** Let \( (Y_n) \) be a sequence of local square integrable martingales satisfying conditions RC, (34) and (38). Then relation (32) holds.

**Proof.** It was shown in items 1° and 2° of the proof of Theorem 3.3 that, for each \( l \), every subsequence of \( (Y^l_n, n \in \mathbb{N}) \) contains, in turn, a subsequence satisfying the conditions of Theorem 3.1 which therefore asserts that \( \langle Y^l_n \rangle - \langle Y^l_n \rangle \xrightarrow{C} 0 \) as \( n \to \infty \). It remains to recall (see item 3° of the proof of Theorem 3.3) that for any \( t \lim_{l \to \infty} \lim_{n \to \infty} \mathbb{P} \{ \tau^l_n < t \} = 0 \).

\[ \square \]

**Corollary 3.2.** Let \( (Y_n) \) be a sequence of local square integrable martingales satisfying conditions RC, (34) and (38). Then \( Y \) is a continuous local martingale and relation (35) holds.

**Proof.** Let \( J_0 \) be an arbitrary infinite set of natural numbers. Then Theorem 3.3 whose condition (21) is covered by (34) asserts existence of an infinite set \( J \subset J_0 \) and a continuous local martingale \( Y^J \) such that (39) holds. By assumption the distribution of \( Y^J \) and, consequently, of \( (Y^J, \langle Y^J \rangle) \) does not depend on \( J \), which allows to delete the superscript in (39). Hence, taking to account arbitrariness of \( J_0 \), we conclude that (39) holds for \( J = \mathbb{N} \).

\[ \square \]

**Corollary 3.3.** Let \( (Y_n) \) be a sequence of local square integrable martingales satisfying conditions RC, (36) and (38). Suppose that almost all trajectories of \( \text{tr} K \) are absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with continuous distribution function. Then there exists a continuous local martingale \( Y \) such that \( (Y(0), \langle Y \rangle) \equiv \left( \hat{Y}, K \right) \) and (35) holds.
Proof. Let $J_0$ be an arbitrary infinite set of natural numbers. Then Theorem 3.3 whose condition (21) is covered by (36) asserts existence of an infinite set $J \subset J_0$ and a continuous local martingale $Y^J$ such that (39) holds. This relation jointly with (36) yields $(Y^J(0), \langle Y^J \rangle) \overset{d}{=} \left( \tilde{Y}, K \right)$. Then Corollary 1.2 asserts that the distribution of $(Y^J, \langle Y^J \rangle)$ does not depend on $J$. And this means, since a set $J_0$ was taken arbitrarily, that (39) holds for $J = \mathbb{N}$, too.

Theorem 3.4. Let for each $n \in \mathbb{N}$ $X_n, X^1_n, X^2_n \ldots$ be local square integrable martingales given on a common probability space which may depend on $n$. Suppose that for all $m \in \mathbb{N}, t > 0$ and $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbf{E} \max_{s \leq t} |\Delta X^n_m(s)|^2 = 0, \quad (55)$$

$$\lim_{L \to \infty} \sup_{t} \lim_{n \to \infty} \mathbf{P} \left\{ |X^n_t(0)| > L \right\} = 0, \quad (56)$$

$$\lim_{L \to \infty} \sup_{t} \lim_{n \to \infty} \mathbf{P} \left\{ \text{tr} \langle X^n_t \rangle (t) > L \right\} = 0, \quad (57)$$

$$\lim_{l \to 0} \lim_{n \to \infty} \mathbf{P} \left\{ \sup_{(t_1, t_2) \in \Pi(t, r)} \left( \text{tr} \langle X^n_t \rangle (t_2) - \text{tr} \langle X^n_t \rangle (t_1) \right) > \varepsilon \right\} = 0, \quad (58)$$

$$\lim_{l \to 0} \sup_{n \to \infty} \mathbf{P} \left\{ |X^n_t(0) - X_n(0)| > \varepsilon \right\} = 0, \quad (59)$$

$$\lim_{l \to 0} \sup_{n \to \infty} \mathbf{P} \left\{ \text{tr} \langle X^n_t \rangle (t) > \varepsilon \right\} = 0. \quad (60)$$

Then for any infinite set $J_0 \subset \mathbb{N}$ there exist an infinite set $J \subset J_0$ and a continuous local martingale $X$ such that

$$(X_n, \langle X_n \rangle) \overset{C}{\to} (X, \langle X \rangle) \quad \text{as} \quad n \to \infty, \; n \in J. \quad (61)$$

Note that relation (61) is, up to notation, a duplicate of (39). So the superscript $J$ on the right hand side is tacitly implied (but suppressed because the conditions of the theorem contain another superscript).

Proof. Conditions (57) and (58) imply that for each $m$ the sequence $(\langle X^m_n \rangle, n \in \mathbb{N})$ is r. c. in $C$. Then it follows from (55) and (56) by Lemma 2.1 that the sequence $(X^m_n, n \in \mathbb{N})$ is r. c. in $C$. So there exist an infinite set $J_m \subset J_{m-1}$ and a random process $X^m$ such that

$$X^m_n \overset{C}{\to} X^m \quad \text{as} \quad n \to \infty, \; n \in J_m.$$

Consequently, if we denote by $J$ the set whose $m$th member is that of $J_m$, then for each $m$

$$X^m_n \overset{C}{\to} X^m \quad \text{as} \quad n \to \infty, \; n \in J.$$

And this together with (55) and relative compactness of $(\langle X^m_n \rangle, n \in \mathbb{N})$ implies by Corollary 3.2 that $X^m$ is a continuous local martingale and

$$\eta^m_n \equiv (X^m_n, \langle X^m_n \rangle) \overset{C}{\to} \eta^m \equiv (X^m, \langle X^m \rangle) \quad \text{as} \quad n \to \infty, \; n \in J. \quad (62)$$
Then it follows from (56) – (58) that
\[
\lim_{L \to \infty} \sup_l \mathbb{P} \{ |X^{(0)}_l| > L \} = 0, \quad \lim_{L \to \infty} \sup_l \mathbb{P} \{ \text{tr} \langle X^{(t)}_l \rangle > L \} = 0,
\]
\[
\lim_{r \to 0} \sup_l \mathbb{P} \{ \sup_{(t_1, t_2) \in \Pi(t, r)} (\text{tr} \langle X^{(t)}_l \rangle (t_2) - \text{tr} \langle X^{(t)}_l \rangle (t_1)) > \varepsilon \} = 0
\]
and therefore the sequences \((\langle X^{(t)}_l \rangle)\), \((X^{(t)}_l)\) and \((\eta^{(t)}_l)\) are r. c. in \(C\).

Conditions (59) and (60) imply by the Lenglart – Rebolledo inequality that for all positive \(t\) and \(\varepsilon\)
\[
\lim_{l \to \infty} \lim_{n \to \infty} \mathbb{P} \{ \sup_{s \leq t} |X^{(t)}_n(s) - X_n(s)| > \varepsilon \} = 0,
\]
which together with (60) yields (23). Then Corollary 2.4 asserts existence of a random process \(\eta \equiv (X, H)\) such that
\[
(X^{(t)}, \langle X^{(t)} \rangle) \xrightarrow{C} (X, H)
\]
and
\[
(X_n, \langle X_n \rangle) \xrightarrow{C} (X, H) \quad \text{as} \quad n \to \infty, \; n \in J.
\]
The ensuing relation \(X^{(t)} \xrightarrow{C} X\), continuity (due to (62)) of all \(X^{(t)}\) and relative compactness of \((\langle X^{(t)} \rangle)\) imply by Corollary 3.2 that \(X\) is a continuous local martingale and \((X^{(t)}, \langle X^{(t)} \rangle) \xrightarrow{C} (X, \langle X \rangle)\). Comparing this with (63), we get \((X, H) \cong (X, \langle X \rangle)\), which converts (64) to (62).

From this theorem, arguing as in the proof of Corollary 3.3, we deduce

**Corollary 3.4.** Let for each \(n \in \mathbb{N}\) \(X_n, X_{n_1}^{(t)}, X_{n_2}^{(t)} \ldots\) be local square integrable martingales given on a common probability space. Suppose that conditions (55) – (60) are fulfilled and there exist given on a common probability space a random variable \(\hat{X}\) and an \(\mathcal{S}_+\)-valued random process \(H\) such that
\[
(X_n(0), \langle X_n \rangle) \xrightarrow{d} \left(\hat{X}, H\right)
\]
and almost all trajectories of \(H\) are absolutely continuous w. r. t. some non-random measure on \(B(\mathbb{R}_+)\) with continuous distribution function. Then there exists a continuous local martingale \(X\) such that
\[
(X_n, \langle X_n \rangle) \xrightarrow{C} (X, \langle X \rangle) \cong (X, H).
\]

Let us denote, for \(x \in \mathbb{R}^d\) and \(N > 0\),
\[
x^{[N]} = \frac{N x}{N \vee |x|}, \quad f_N(x) = \left| x - x^{[N]} \right|^2.
\]
**Corollary 3.5.** Let for each \( n \in \mathbb{N} \) the process \( X_n \) be defined by

\[
X_n = \bar{X}_n + \varphi_n \cdot M_n,
\]

where \( M_n \) is a local square integrable martingale w. r. t. some flow \( \mathcal{F}_n \), \( \bar{X}_n \) is an \( \mathcal{F}_n(0) \)-measurable random variable and \( \varphi_n \) is an \( \mathcal{F}_n \)-predictable random \( \mathbb{R} \)-valued process such that for all \( t \) \( \varphi_n^2 \circ \text{tr}(M_n)(t) < \infty \). Suppose that: for all positive \( t \) and \( \varepsilon \)

\[
\mathbb{E} \max_{s \leq t} |\Delta M_n(s)|^2 \to 0, \quad (65)
\]

\[
\lim_{L \to \infty} \lim_{n \to \infty} P \left\{ |\bar{X}_n| > L \right\} = 0, \quad (66)
\]

\[
\lim_{l \to \infty} \lim_{n \to \infty} P \left\{ \int_0^t f_l(\varphi_n(s)) \, d \text{tr} (M_n)(s) > \varepsilon \right\} = 0; \quad (67)
\]

the sequence \( \varphi_n^2 \circ \text{tr}(M_n) \) is r. c. in \( C \). Then for any infinite set \( J_0 \subset \mathbb{N} \) there exist an infinite set \( J \subset J_0 \) and a continuous local martingale \( X \) such that (61) holds.

**Proof.** Denote \( X^m_n = \bar{X}_n + \varphi^{[m]}_n \cdot M_n \). By construction and due to the assumptions about \( M_n, \varphi_n \) and \( \bar{X}_n \) the processes \( X_n, X^1_n, X^2_n \ldots \) are local square integrable martingales with common initial value \( X_n \) (so that condition (59) becomes trivial and condition (56) turns to (66)) and quadratic characteristics

\[
\langle X_n \rangle = \varphi_n^2 \circ \text{tr}(M_n), \quad \langle X^m_n \rangle = \left( \varphi^{[m]}_n \right)^2 \circ \text{tr}(M_n), \quad (68)
\]

so that the sequence \( (\text{tr}(X_n)) \) is relative compact in \( C \). Also by construction of \( X_n \) and the definition of \( f_N \)

\[
\langle X^m_n - X_n \rangle = f_m(\varphi_n) \circ \text{tr}(M_n),
\]

so that condition (67) entails (59). Equalities (68) show that for any \( l \in \mathbb{N} \) and \( t \geq s \geq 0 \)

\[
\text{tr} \langle X^l_n \rangle(t) - \text{tr} \langle X^l_n \rangle(s) \leq \text{tr} \langle X_n \rangle(t) - \text{tr} \langle X_n \rangle(s)
\]

and therefore relations (57) and (58) follow from relative compactness of \( (\varphi_n^2 \circ \text{tr}(M_n)) \). The evident inequality \(|\Delta X^m_n| \leq |\Delta X_n| \) shows that condition (55) is also fulfilled. \( \square \)

**Corollary 3.6.** Let the conditions of Corollary 3.5 be fulfilled and let, furthermore, there exist given on a common probability space a random variable \( \bar{X} \) and an \( \mathcal{G}_+ \)-valued random process \( H \) such that

\[
\left( \bar{X}_n, \varphi_n^2 \circ \text{tr}(M_n) \right) \overset{d}{\to} \left( \bar{X}, H \right)
\]

and almost all trajectories of \( H \) are absolutely continuous w. r. t. some non-random measure on \( \mathcal{B}(\mathbb{R}_+) \) with continuous distribution function. Then \( X_n \overset{C}{\to} X \), where \( X \) is a continuous local martingale with initial value \( \bar{X} \) and quadratic characteristic \( H \).
References

[1] Gikhman, I. I. and Skorokhod, A. V. (1982). *Stochastic Differential Equations and Their Applications*, Naukova Dumka, Kyiv. (Russian)

[2] Liptser, R. Sh. and Shiryaev, A. N. (1989). *Theory of Martingales*. Kluwer, Dordrecht.

[3] Jacod, J. and Shiryaev, A. N. (1987). *Limit Theorems for Stochastic Processes*, Springer, Berlin.

[4] Parthasarathy, K. R. (1980) *Introduction to Probability and Measure*, M, New Delhi.

[5] Yurachkivsky, A. P. (2003). *A functional central limit theorem for the measure of a domain covered by a flow of random sets*. Th. Prob. Math. Stat. No. 67, 151–160.

[6] Yurachkivsky, A. P. (2003). *Conditions for convergence of a sequence of martingales in terms of their quadratic characteristics*. Dopovidi Natsionahoi Akademii Nauk Ukrainy [Rep. Nat. Acad. Sci. Ukr.], No. 1, 33–36.

[7] Skorokhod, A.V. (1956). *Limit theorems for stochastic processes*. Th. Prob. and Appl. 1, 261–290.