Rudiments of Holography

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Abstract

An elementary introduction to Maldacena’s AdS/CFT correspondence is given, with some emphasis in the Fefferman-Graham construction. This is based on lectures given by one of us (E.A.) at the Universidad Autónoma de Madrid.
1 Introduction

The following is an introduction to the AdS/CFT correspondence proposed by Maldacena in [37].

The style is informal, and only some computations towards the end are done in detail. The topic is already large, and a comprehensive (up to the date it was written) review is available ([1]). References are only given to material which has actually been used in the original lectures, and are not complete in any sense.

Although strings almost do not appear at the level of approximation we shall work, they lurk in the horizon. Standard introductions to strings are [19], [44] (cf. also [3] for a quick overview).

We shall use the Landau-Lifshitz Timelike Conventions for General Relativity; that is, metric signature $s = -2$, the Riemann tensor defined as $R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \ldots$, and the Ricci tensor defined by $R_{\alpha\beta} \equiv R^\gamma{}_{\alpha\gamma\beta}$. The flat line element with $p$ times and $q$ spaces will be denoted by

$$d\vec{x}^2_{(p,q)} \equiv \sum_1^p (dx^i)^2 - \sum_1^q (dx^i)^2$$  \hspace{1cm} (1)

Volume elements will be similarly shortened:

$$dx_d \equiv \sqrt{g} dx^1 \wedge \ldots \wedge dx^d$$  \hspace{1cm} (2)
2 The Holographic Principle

The line of reasoning that led G. ’t Hooft to propose the holographic principle (⁵¹) stems from noticing that if we want a piece of matter of given mass $M$, and contained in a given volume $V \equiv L^3$ to be observable from far away, we have to assume that the volume considered is bigger than the Schwarzschild scale:

$$R_s \leq L \quad (3)$$

where the Schwarzschild radius is given by:

$$R_s \equiv \frac{2GM}{c^2} \quad (4)$$

because otherwise the system as a whole would be the interior of a black hole and, as such, could not be observed from the outside. This means that if we assume the system to be at equilibrium, then, in first approximation at high enough temperature, the energy is given by

$$E \sim VT^4 \quad (5)$$

and the corresponding entropy is

$$S \sim VT^3 \quad (6)$$

and the bound just espoused is equivalent to

$$S \leq \frac{V^{1/2}}{G^{3/4}} \sim (\frac{A}{G})^{3/4} \quad (7)$$

which is still quite small compared with the black hole entropy

$$S = \frac{1}{4}A \quad (8)$$

It seems strange at first sight that the entropy is not proportional to the volume

$$S \sim V \quad (9)$$
as would have been predicted by ordinary quantum field theory. One has to conclude that most of the would be quantum field states lie inside their own Schwarzschild radius. For consistency one is then led following this line of reasoning to the postulate that the quantum theory of gravity should be described by a sort of topological quantum theory, in the sense that all degrees of freedom could be projected onto the boundary.
3 The Maldacena conjecture

3.1 Physics on the world volume of a D-brane versus the brane as a source of spacetime curvature

In the last few years following the seminal work of Polchinski ([44]) it has become increasingly clear that some topological defects, spanning a given number $p$ of space dimensions, generically called branes, play an essential rôle in the formulation of string theory. The simplest ones are those which can be defined as the locus of open string endpoints, which are called $D$-brane--branes.

There is a fascinating duality between two different aspects of the physics of D-branes ([31]): the physics on the brane, described by the Dirac-Born-Infeld (DBI) action, and the spacetime physics stemming from the brane as a source of energy-momentum.

We indeed know ([36]) that the effective action of a D-brane, (from which the Weyl anomaly coefficients are derived by a variational principle) is the DBI action, which in the simplest case reads:

$$S_p \equiv -T_p \int d^{p+1} \xi e^{-\Phi} \sqrt{\det(g_{ab} + b_{ab} + 2\pi\alpha' F_{ab})}$$  \hspace{1cm} (10)

where $T_p$ is the brane tension, of mass dimension $p+1$, and $g_{ab} \equiv g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu$ is the metric induced on the brane by the imbedding from the world volume of the brane, $\Sigma$, into the external spacetime, $M$,

$$\xi^a \in \Sigma_{p+1} \rightarrow x^\mu \in M_{10}$$  \hspace{1cm} (11)

(and similarly for $b_{ab}$). The gauge field strength is not pulled back from spacetime; gauge degrees of freedom live on the brane only.

On the other hand, it can be argued ([41]) that the gravitational field produced by a black D-brane of $p$ spacelike dimensions is given (in the string frame) by the universal formula $^2$:

$$ds^2 = H^{-1/2}(r)[W(r)dt^2 - \delta_{ij}dx^idx^j]$$

$^2$When there are no extra translational isometries
\[-H^{1/2}(r)[W^{-1}(r)dr^2 + r^2d\Omega^2_{8-p}]\]  
(12)

with \(i, j, \ldots = 1, \ldots p\) and a dilaton field given by

\[e^{\phi - \phi_0} = H^{\frac{3+p}{4}}(r)\]  
(13)

the functions \(H\) and \(W\) are given by:

\[H(r) \equiv 1 + \frac{l^{7-p}}{r^{7-p}}\]  
(14)

and

\[W(r) \equiv 1 - \frac{r_0^{7-p}}{r^{7-p}}\]  
(15)

and \(d\Omega^2_{8-p}\) is the line element on the sphere \(S^{8-p}\). The source is at \(r = r_0\), and the extremal limit is \(r_0 = 0\), that is, \(W = 1\).

There is a RR background as well, given by the \(p+1\)-form:

\[A_{p+1} \equiv \alpha e^{\phi_0}(H^{-1} - 1)H^{-1/4}W^{1/2}dt \wedge dx^1 \wedge \ldots dx^p\]  
(16)

and a relationship between the constants, namely:

\[r_0^{7-p} = l^{7-p}(1 - \alpha^2)\]  
(17)

Let us now concentrate, for the rest of this work, in the case \(p = 3\), in which the world volume is four-dimensional and, besides, the dilaton is constant.

The self-dual RR five-form field strength is actually given by

\[F_5 \equiv 4l^4(\epsilon_5 + \star \epsilon_5)\]  
(18)

\(\epsilon_n\) being the volume element on \(S^n\).

The normalization is done as follows \[43\]. We first define the charge of the brane,

\[\mu_3 \equiv \frac{1}{\sqrt{2\kappa_{10}^2}} \int_{S^5} \star F_5\]  
(19)
which yields $\mu_3 = \frac{4l^4\Omega_2}{\sqrt{2}\kappa_{10}}$. The BPS condition (in the case in which we have $N$ coincident branes; that is, a BPS system of charge $N$), is the equivalent to the following relationship with the string tension:

$$\mu_3 = N\sqrt{2\kappa_{10}^2\tau_3}$$  \hspace{1cm} (20)

which leads to (after plugging the values $\tau_3 = \frac{1}{g_s(2\pi)^3\alpha'}$ and $\kappa_{10}^2 = 2^6\pi^7\alpha'^4g_s^2$ taken from [44])

$$l^4 = 4\pi g_sNl_s^4$$  \hspace{1cm} (21)

where $\alpha' \equiv l_s^2$.

### 3.2 Absorption Cross Sections

The fate of scalar particles when approaching a D-brane can be computed from the two different viewpoints alluded above.

Indeed I. Klebanov ([31]) realized that absorption cross sections could be calculated both from the D-brane (DBI) point of view, or from the gravitational field of the D-brane itself, with identical results.

In the simplest case of a dilaton incident at right angles with frequency $\omega$ the relevant DBI coupling (obtained from an low energy, weak field expansion of the full DBI action) is:

$$S_{int} = \frac{\pi^{1/2}}{\kappa_{10}} \int d^4x \frac{1}{4} \Phi tr F_{\mu\nu}^2$$  \hspace{1cm} (22)

This implies that the cross section for decaying into a pair of gluons with vanishing spacial momentum, is given by:

$$\sigma_{DBI} = \frac{\kappa_{10}^2\omega^3N^2}{32\pi}$$  \hspace{1cm} (23)

(where the factor $N^2$ comes from the degeneracy of the final state).

On the other hand, the radial part of the equation of motion for a dilaton $\Phi(x) \equiv \phi(r)e^{i\omega t}$ is just

$$\left[\frac{1}{r^5} \frac{d}{dr} r^5 \frac{d}{dr} + \omega^2(1 + \frac{l^4}{r^4})\right] \phi(r) = 0$$  \hspace{1cm} (24)
Using the convenient variable in the inner region,

\[ z \equiv \frac{\omega l^2}{r} \]  

(25)

and substituting

\[ \phi(r) = z^{3/2} f(z) \]  

(26)

yields

\[ \left( \frac{d^2}{dz^2} - \frac{15}{4z^2} + 1 + \frac{(\omega l)^4}{z^4} \right) f = 0 \]  

(27)

This implies an absorption cross section:

\[ \sigma_{\text{sugra}} = \frac{\pi^4}{8} \omega^3 l^8 \]  

(28)

Both cross sections can be shown to coincide by using the relationship \(^{\text{3}}\)

\[ l^4 = \frac{\kappa_{10}}{2\pi^{5/2}} N \sim g_s l_s^4 N \]  

(29)

The range of validity of the supergravity calculation is

\[ l >> l_s \iff N g_s >> 1 \]  

(30)

The condition that the incident energy is small is

\[ \omega l_s << 1 \]  

(31)

On the gauge theory side, this corresponds (because \( g_{YM}^2 \sim g_s \)) to large ’t Hooft coupling, \( g_{YM}^2 N \to 0 \). If we want to suppress string loop corrections, we need in addition \( g_s \to 0 \), implying that \( N >> 1 \).

\(^{3}\text{This is nothing more than the consistency condition equating the mass of the extremal D-brane solution with RR charge with } N \text{ times the mass of a single brane}\)
3.3 Maldacena’s near horizon limit

Motivated by the preceding results, Maldacena ([38]) realized that when we are simultaneously interested in the near horizon solution (which means \( r \to 0 \)) and low energies (that is, \( \alpha' \to 0 \)), there is a natural variable that can be introduced (a natural blow up, from the mathematical point of view), namely:

\[
    u \equiv \frac{r}{\alpha'}
\]

This variable has dimensions of energy, and, in spite of the fact that the starting point is the near horizon limit, the variable clearly is a continuous one, and it can reach arbitrary real values.

It has indeed been suggested that this variable has to do with the renormalization group scale of the gauge theory living on the stack of branes, namely \( \mathcal{N} = 4 \) SUSY Yang Mills.

Performing the limit in the supergravity solution, and using \( l^4 = 4\pi g_s N \alpha'^2 \) yields

\[
    ds^2 = \alpha'[\frac{u^2}{\sqrt{4\pi g_s N}} d\mathbf{x}_R^2 + \sqrt{4\pi g_s N} \frac{du^2}{u^2} + \sqrt{4\pi g_s N} d\Omega_5^2] \tag{33}
\]

which happens to be the metric of Anti de Sitter spacetime of radius \( l^2 = l_s^2 \sqrt{4\pi g_s N} \) cross a five sphere of the same radius, \( AdS_5 \times S_5 \) ([17]).

And the physics on the brane itself is described by a \( d = 4 \) conformal field theory (CFT), namely \( \mathcal{N} = 4 \) SUSY Yang-Mills, with gauge group \( SU(N) \) and coupling constant \( g = g_s^{1/2} \). The results of the last section lead us to expect that there is a close relationship between these two descriptions of the D-brane stack.

The lagrangian of \( \mathcal{N} = 4 \) supersymmetric Yang-Mills in 4 dimensions is given by

\[
    L = -\frac{1}{4} Tr (F_{\mu \nu} F^{\mu \nu}) + i \lambda_i \sigma^\mu D_\mu \bar{\lambda}^i + \frac{1}{2} D_\mu \Phi_{ij} D^\mu \Phi^{ij} \\
    + i \lambda_i [\lambda_j, \Phi^{ij}] + i \bar{\lambda}^i [\bar{\lambda}^j, \Phi_{ij}] + \frac{1}{4} [\Phi_{ij}, \Phi_{kl}] [\Phi^{ij}, \Phi^{kl}] \tag{34}
\]

where the gauginos are represented by four Weyl spinors \( \lambda_i \), transforming in the 4 of \( SO(6) \), and the six scalar fields \( \Phi_{ij} \) obey \((\Phi_{ij})^\dagger \equiv \Phi^{ij} = \frac{1}{2} \epsilon^{ijkl} \Phi_{kl} \). Sometimes we shall represent the six scalars as \( \Phi^I \).
The ten-dimensional Newton’s constant is given by:

\[ \kappa_{10}^2 \sim l_p^8 = g_s^2 l_s^8 \sim \frac{l_s^8}{N^2} \]  

The effective string tension is just

\[ T_{eff} \equiv \frac{l_s^2}{l_s^2} \sim \sqrt{g_s N} \sim \lambda^{1/2} \]  

where we have introduced the ’t Hooft coupling \( \lambda \equiv g^2 N \). The ’t Hooft limit is precisely

\[ g \to 0 \]
\[ N \to \infty \]
\[ \lambda \equiv g^2 N \to constant \]  

and corresponds to quasi-free strings. There is a slightly different, more holographic limit, to wit

\[ g \to constant \]  
\[ N \to \infty \]
\[ l/s \to \infty \]  

In this limit the effective string tension grows large, so that it is to be expected that classical supergravity is a good approximation.

Actually, from this starting point a whole mapping (52) between strings on one side and CFT on the other has been slowly inferred. Indeed, from this point on, one can forget about the way the conjecture was first posited, and consider \( AdS_5 \times S_5 \) as a new string background by itself. These considerations do allow the calculation (in the aforementioned large \( N \), large ’t Hooft coupling) of gauge invariant correlators in the gauge theory side, using tree level supergravity computations; that is, computing the action of supergravity of certain bulk fields \( \Phi_i \) with given values in the (conformal) boundary, \( \Phi_i|_\partial = \phi_i \). The
mapping itself is the association \( \phi_i \rightarrow O_i \), obtained through an expansion of the DBI action.

The whole setup is summarized in the generating functional (\[52\])

\[
\langle e^{\int \sum O_i \phi_i} \rangle = e^{-S_{\text{sugra}}[\Phi_i]} |_{\Phi_i |_{\partial} = \phi_i}
\]

(40)

It is plain that if the operators \( O_i \) have conformal weight \( \Delta_i \), then the associated fields \( \phi_1 \) (which in practice behave as currents on the boundary) will have conformal dimension \( 4 - \Delta_i \).

Maldacena actually proposed that for any value of the coupling there was an exact quantum equivalence, between \( IIB \) string theory in \( AdS_5 \times S_5 \) and \( \mathcal{N} = 4 \) SUSY Yang-Mills, for any value of \( G = SU(N) \). From the string theory side, the value of \( N \) appears as the Ramond-Ramond flux on the sphere \( S^5 \).

The global symmetries of both theories are the same, namely \( SO(4,2) \times SO(6) \), which includes the four dimensional conformal group which appears as an isometry group from the string side, and as a \( R \)-symmetry \( SU(4) \sim SO(6) \) in the CFT side. When fermions are considered, both groups appear as the bosonic part of the supergroup \( SU(2,2|4) \). Besides a non perturbative S-duality group \( SL(2,\mathbb{Z}) \) is conjectured to exist on both sides.

### 3.4 The Infrared/Ultraviolet Connection

Anti de-Sitter space is non-compact; its volume \( Vol(AdS_p) \) diverges. An infrared (IR) regulator in the bulk (such as making believe that the boundary is at \( r = \epsilon \) instead of \( r = 1 \) in the form \([59]\) of the AdS metric to be introduced momentarily) is equivalent to an ordinary ultraviolet (UV) cutoff in the CFT living on the boundary. Giving the fact that in the gauge theory there are in the large \( N \) limit approximately \( N^2 \) degrees of freedom per point, the number of degrees of freedom per unit of three-dimensional volume in the cutoff theory will be:

\[
N_{d.o.f.} = \frac{N^2}{\epsilon^3}
\]

(41)
Now the regularized area of the eight-dimensional spatial boundary at constant time is (taking into account an $l^5$ factor from the sphere $S^5$)

$$A = \frac{l^8}{\epsilon^3}\quad (42)$$

in such a way that the number of degrees of freedom per unit boundary area is

$$\frac{N_{d.o.f.}}{A} = \frac{N^2}{l^8} = \frac{1}{G}\quad (43)$$

in accordance with the holographic principle.
4 Structure of the Anti de Sitter Geometry.

Given the basic importance of Anti de Sitter metric in the whole description of the space-time region close to the brane, we have collected here some geometric facts, relevant for the discussion of boundary conditions in the main text, specially in connection with the generating functional formerly introduced in the equation (40).

Anti de Sitter space in $p$ dimensions ($AdS_p$) is the symmetric space

$$AdS_p \equiv SO(2, p - 1)/SO(1, p - 1)$$

Indeed, all real forms of $SO(p+1)/SO(p)$ are closely related through Weyl’s unitary trick. This suggests the definition of an *euclidean* version of AdS:

$$EAdS_p \equiv SO(1, p)/SO(p)$$

$AdS_p$ could also be explicitly defined ([16]) as the induced metric on the hyperboloid

$$(X^0)^2 + (X^p)^2 - \delta_{ij}X^iX^j = l^2;$$

$(i, j = 1 \ldots p - 1)$ embedded in an ambient space $\mathbb{R}_{2,p-1}$ (that is, $\mathbb{R}_{p+1}$ endowed with a Minkowskian metric with two times,

$$ds^2 = (dX^0)^2 + (dX^p)^2 - \delta_{ij}dX^idX^j.$$  

Defined in that way, it clearly has topology $S^1 \times \mathbb{R}^{p-1}$ (as well as closed timelike curves). The universal covering space ($CAAdS_p$) has topology $\mathbb{R}^p$.

$AdS$ is an Einstein space; its Ricci tensor is proportional to the metric:

$$R_{\mu\nu} = \frac{p-1}{l^2}g_{\mu\nu}$$

which corresponds to a *positive* $^4$ cosmological constant,

$$\lambda = \frac{(p - 1)(p - 2)}{2l^2}$$

$^4$The unconventional sign for the $AdS$ cosmological constant is due to our choice of signature.
This definition makes it manifest the underlying $O(2, p - 1)$ symmetry. The $p(p+1)/2$ Killing vectors are given by
\begin{equation}
    k_{ab} \equiv X^a \partial_b - X^b \partial_a
\end{equation}
(for $0 < a, b < p$). As it is well known, there is a $2 - 1$ correspondence between $O(2, p - 1)$ and the conformal group of Minkowski space in $p - 1$ dimensions, $C(1, p - 2)$.

A first, provisional, definition of the boundary at infinity $\partial AdS$ can be defined as the region where all $X^\mu$ are rescaled by an infinite amount, $X^\mu \to \xi X^\mu$, where $\xi \to \infty$. In that way, the boundary is characterized by the relationship $(X^0)^2 + (X^p)^2 - \delta_{ij} X^i X^j = 0$, which is nothing but the well-known $O(2, p - 1)$ null-cone compactification of Minkowski space, $M^C$. The way it works is that to any regular point of Minkowski space, $x^\mu \in M$, there corresponds another point in $M^C$, namely
\begin{equation}
\begin{cases}
    X^0 &= x^0, \\
    X^i &= x^i, \\
    X^{p-1} &= \frac{1+x^2}{2}, \\
    X^p &= \frac{1-x^2}{2}.
\end{cases}
\end{equation}

The points in $M^C$ which are not in $M$ correspond to $X^p + X^{p-1} = 0$. This means that this compactification amounts to add an extra null cone at infinity.

The $AdS$ metric can be easily put in the globally static form by means of the ansatz (in which we introduce two closely related sets of coordinates simultaneously)
\[
\begin{align*}
X^0 &= l \cos \tau \cosh \chi = l \frac{\cos \tau}{\cos \rho}, \\
X^p &= l \sin \tau \cosh \chi = l \frac{\sin \tau}{\cos \rho}, \\
X^i &= l n^i \sinh \chi = l n^i \tan \rho,
\end{align*}
\]
(52)

where \( \delta_{ij} n^in^j = 1, (i, j = 1, \ldots, p - 1) \). The result, in terms of the first set of coordinates, is

\[
ds^2 = l^2 [(\cosh \chi)^2 d\tau^2 - (d\chi)^2 - (\sinh \chi)^2 d\Omega^2_{p-2}].
\]
(53)

AdS corresponds to \( 0 \leq \tau \leq 2\pi \), and CAdS to \( 0 \leq \tau \leq \infty \). The boundary lies at \( \chi = \infty \).

The antipodal map \( J : X \to -X \), corresponds in this coordinates simply to \( (\tau, \chi, \vec{n}) \to (\tau + \pi, \chi, -\vec{n}) \).

The second set of equalities gives the form conformal to Einstein’s static universe as used in [4],

\[
ds^2 = \frac{l^2}{\cos^2 \rho} [d\tau^2 - d\rho^2 - \sin \rho^2 d\Omega^2_{p-2}]
\]
(54)

where \( 0 \leq \rho < \pi/2, 0 \leq \theta < \pi \) and \( 0 \leq \phi < 2\pi \). The boundary is now located at \( \rho = \pi/2 \).

This clearly shows that the boundary is timelike, because the normal vector is spacelike.

This fact is the root of much of the peculiar behavior of this space.

Hawking and Page ([25]) parametrize this as

\[
r \equiv l \tan \rho
\]
(55)

and

\[
T \equiv l \tau \sim T + 2\pi l
\]
(56)

in such a way that the metric reads

\[
ds^2 = (1 + \frac{r^2}{l^2})dT^2 - \frac{dr^2}{(1 + \frac{r^2}{l^2})} - r^2 d\Omega^2_{p-2}
\]
(57)
These coordinates are actually the best adapted to write down the would-be newtonian potential (this is only a somewhat formal concept here in the sense that although the space can be considered static with Killing vector $\frac{\partial}{\partial T}$, it is not asymptotically flat).

$$V(r) = \frac{r^2}{l^2}$$

which clearly puts into evidence the confining character of the AdS space.

In spite of the fact that we have already introduced coordinates which fully cover the space, in some physical applications the AdS space appears linked to some specific set of coordinates. Let us quickly list some of the most important ones.

A different, but closely related set of coordinates is the one used by Susskind and Witten in [48] (see also [16]). The metric has the form

$$ds^2 = \frac{l^2}{(1-r^2)^2}(-4\sum_{i=1}^{p-1}(dx^i)^2 + (1+r^2)d\tau^2).$$

They are easily obtained from the globally static form by

$$\sinh \chi = \frac{2r}{1-r^2}.$$  (60)

$CAdS$ itself corresponds to the ball $r < 1$, and the boundary sits on the sphere $r = 1$.

Another interesting set of coordinates (common to all constant curvature spaces) is Riemann’s, in which the metric reads

$$ds^2 = \frac{\eta_{\mu\nu}dy^\mu dy^\nu}{(1 - \frac{r^2}{l^2})^2},$$  (61)

where $\mu, \nu = 0, \ldots p - 1$ and $\eta_{\mu\nu}$ is the ordinary Minkowski metric, and $r^2 \equiv \eta_{\mu\nu}y^\mu y^\nu$.

In order to understand them, it is useful to introduce first another canonical set of coordinates, valid for any constant curvature space as well (cf. [50]). Let us start from the fact that the geodesic deviation between neighboring geodesics grows as

$$\eta = l \left( \frac{d\eta}{ds} \right)_{s=0} \sinh \frac{s}{l},$$  (62)
(easily obtained from the geodesic deviation equation). Now, we use the fact that the angle between the tangents to such neighboring geodesics is precisely the volume element on the unit Minkowskian sphere, which can be easily obtained in terms of the ordinary volume on the unit Euclidean sphere: 

\[ d\Omega^2_{p-1} (\text{hyperbolic}) \equiv -d\xi^2 - \sinh^2 \xi d\Omega^2_{p-2}. \]

In that way we can use Pithagoras’ theorem of the triangle with hypotenuse \( ds \) and other sides \( dr \) and \( d\eta \), getting easily for the volume element:

\[ ds^2 = dr^2 - l^2 \sinh^2 \frac{r}{l} (d\xi^2 + \sinh^2 \xi d\Omega^2_{p-2}). \] (63)

Then, Riemann’s coordinates can be constructed \([50]\) by

\[ y^\mu \equiv 2l u^\mu \tanh \frac{r}{2l}, \] (64)

where \( u^\mu \) is the unit tangent vector to the geodesic going to the point \( P \) from a fiducial point \( P_0 \); and \( r \) is the geodesic distance from \( P_0 \) to \( P \). Plugging equation (64) into (63) easily yields (61).

The Poincaré coordinates (also called horospheric coordinates in the old british literature) \([21]\) are defined as

\[
\begin{cases}
  X^0 = t/z, \\
  X^a = x^a/z, \\
  X^p - X^{p-1} = \frac{l}{z},
\end{cases}
\] (65)

(where \( a = 1, \ldots, p-2 \)), and the coordinate \( z \) is dimensionless.

The metric reads

\[ ds^2 = \frac{1}{z^2} (dt^2 - dx^2_{p-2} - l^2 dz^2), \] (66)

where \( dx^2_{p-2} \) is the Euclidean line element in \( \mathbb{R}_{p-2} \).
The metric above enjoys a manifest $O(1,p-2)$ symmetry. Besides, it is invariant under dilatations $x^\mu \to \lambda x^\mu$ ($x^\mu = (t, \vec{x}, z)$) and inversions $x^\mu \to \frac{x^\mu}{x^2}$. Of course those transformations just convey an action of $O(2,p-1)$ on the horospheres.\(^5\)

Poincaré coordinates break down at $z = \infty$ ($u = 0$), (which we shall call the horizon); which in terms of the embedding is just $X^p = X^{p-1}$. In terms of the global static coordinates of (54), this equation has solution for a given $\tau$ for all $\chi < \chi(\tau) \equiv \sinh^{-1}|\tan \tau|$.

In terms of the coordinates in (54), this means

$$n^{p-1} = \frac{\sin \tau}{\sin \rho}$$  \hspace{1cm} (71)

(which has a physical solution as long as $\sin \tau \leq \sin \rho$).

This region can be easily parametrized, using $(X^0)^2 - \delta_{ij}X^iX^j = 1$ ($i = 1 \ldots p - 2$) by $X^0 = \cosh z; X^i = n^i \sinh z$, and the induced metric on the horizon is:

$$ds^2 = -dz^2 - \sinh^2 zd\Omega^2_{p-3}.$$

The boundary of $AdS$ can be identified with the surface $z = 0$, which in terms of

\(^5\)The coordinates used by Maldacena in [37] are essentially $v \equiv \frac{l}{z}$:

$$ds^2 = -\frac{l^2}{v^2}dv^2 + \frac{v^2}{l^2}dx^2_\parallel,$$

(where $dx^2_\parallel$ stands for the ordinary Minkowski metric in $M_{p-1}$)

Actually what Maldacena did is to work with the adimensional radius, $\bar{l}$, (which just happens to be equal to $\bar{l} \equiv (4\pi gN)^{1/4}$),

$$\ell^2 \equiv \bar{l}^2 \alpha'$$  \hspace{1cm} (68)

where $\alpha' \equiv \frac{1}{\ell^2}$ is the string tension, and define a coordinate with mass dimension one by

$$u \equiv \frac{v}{\alpha'}$$  \hspace{1cm} (69)

This yields:

$$ds^2 = \alpha'[\frac{\ell^2}{u^2}du^2 + \frac{u^2}{\ell^2}dx^2_\parallel].$$  \hspace{1cm} (70)
the embedding coordinates is equivalent to $X^p - X^{p-1} = \infty$. The normal vector to the boundary is the spacelike vector $n = \frac{\rho^2 \partial}{\partial z}$, which is precisely the reason why we say that the boundary is a timelike surface. In the coordinates of equation (54) it corresponds to $\rho = \pi/2$, unless $\sin \tau = \sin \rho n^{p-1}$ (that is, the point is in the horizon), in which case a $0/0$ ambiguity is encountered, and further expansion is needed.

A glance at equation (66) clearly shows that the induced metric on the boundary is conformal (with a singular factor) to the Minkowski metric.
5 Some Comments on De Sitter space

The De Sitter space $dS_p$ is in many senses an analytic continuation of $AdS_p$. There are, however, two important differences: on the one hand, it does not have a timelike infinity; on the other, any observer in it has got a horizon.

It can be globally defined \[^{[46]}\) as the hypersurface

$$X_0^2 - \delta_{ij}X^iX^j = -l^2\tag{73}$$

($i, j, \ldots = 1 \ldots p$), in a convenient Minkowski space $\mathbb{R}_{1,p}$

$$ds^2 = dX_0^2 - \delta_{ij}dX^idX^j\tag{74}$$

Global coordinates can be defined through

$$X^0 = l\sinh \tau$$

$$X^i = l n^i \cosh \tau\tag{75}$$

and the metric reads

$$ds^2 = l^2(d\tau^2 - \cosh^2 \tau d\Omega_{p-1}^2)\tag{76}$$

The Ricci tensor corresponds to a constant curvature space,

$$R_{\mu\nu} = -\frac{p-1}{l^2}g_{\mu\nu}\tag{77}$$

If we perform the change

$$\cosh \tau = \sec T\tag{78}$$

(where $-\pi/2 \leq T \leq \pi/2$) then the metric reads

$$ds^2 = l^2 \sec^2 T(dT^2 - d\Omega_{p-1}^2)\tag{79}$$

This clearly shows that the only natural definition of infinity in $dS_p$ is at $\tau = \infty$, that is $T = \pm \pi/2$, (a spacelike surface) with induced metric corresponding to the unit sphere
This, in turn, does imply that there are horizons associated to a given observer. The future event horizon is the boundary between events which can eventually be detected by the observer and those that can not. The past horizon is the boundary between those events which can detect the observer, and those that can not.

The final set of coordinates we will introduce is the so called static ones, in which

\[
X^0 = l \sqrt{1 - r^2} \sinh t \\
X^a = l r n^a \\
X^p = l \sqrt{1 - r^2} \cosh t
\]  

(80)

(where \(a = 1 \ldots p - 1\)). The metric reads

\[
ds^2 = l^2 \left( (1 - r^2) dt^2 - \frac{dr^2}{1 - r^2} - r^2 d\Omega_{p-2}^2 \right)
\]

(81)

(which is the analytic continuation to imaginary radius of the \(AdS_p\) metric in (57)).

By redefining coordinates

\[
\tilde{r} \equiv lr \\
\tilde{t} \equiv lt
\]

(82)

we get directly the corresponding newtonian potential,

\[
V(\tilde{r}) = -\frac{\tilde{r}^2}{\tilde{l}^2}
\]

(83)

which is a repulsive one.

The Killing vector defining staticity, namely

\[
k = \frac{\partial}{\partial \tilde{t}}
\]

(84)

is not globally timelike; actually it is so only in a wedge covering a quarter of \(dS_p\), namely \(r < 1\).
If we perform a Wick rotation in the global coordinates, and we want that the euclidean manifold is an sphere $S^p$, the timelike angular coordinate must be periodic,

$$\tau \sim \tau + 2\pi$$ (85)

which means (cf. [8]) that the period in the proper length is

$$l_0 \sim l_0 + 2\pi l$$ (86)

signalling the presence of a temperature associated to the horizon,

$$\beta = 2\pi l$$ (87)
6 Penrose’s Conformal Infinity

In \( AdS_p \) the infinity with the explicit coordinates introduced in (54) in this sense is located at \( \rho = \pi/2 \), so that the metric at the boundary is given by

\[
ds^2 = d\tau^2 - d\Omega_{p-2}^2
\]

(88)

conveying a topological \( \mathbb{R} \times S^{p-2} \) structure.

Penrose’s construction of conformal infinity (cf. [42]) was originally designed as a general procedure to address the physics of the asymptotic structure of the space time in General Relativity. The main idea used to bring infinity back to a finite distance is to Weyl rescale the physical metric, looking for a new manifold \( \tilde{\mathcal{M}} \) with boundary, such that the interior of \( \tilde{\mathcal{M}} \) coincides with our original spacetime \( \mathcal{M} \), endowed with the metric \( \tilde{g}_{\mu\nu} \) to be defined in a moment. In the physics literature it is customary to denote the part of \( \partial\mathcal{M} \) corresponding to end-points of null geodesics by \( \mathcal{J} \).

Based on the previous construction we will say that a spacetime \( (\mathcal{M}, g_{\mu\nu}) \) is asymptotically Einstein if there exists a smooth manifold \( \tilde{\mathcal{M}} \), with metric \( \tilde{g}_{\mu\nu} \), and a smooth scalar field \( \Omega(x) \) defined in \( \mathcal{M} \) such that:

i) \( \mathcal{M} \) is the interior of \( \tilde{\mathcal{M}} \).

ii) \( \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \)

iii) \( \Omega(x) = 0 \) if \( x \in \mathcal{J} \); but \( N_{\mu} \equiv -\nabla_{\mu}\Omega \) is nonsingular in \( \mathcal{J} \).

iv) Every null geodesic in \( \mathcal{M} \) has two endpoints in \( \mathcal{J} \).

and, finally, the field equation:

v) \( R_{\mu\nu} \sim \lambda g_{\mu\nu} \) (cf. [12][1][21]).

(where we have followed Penrose’s notation \( a \sim b \) to indicate two things that are equal on \( \mathcal{J} \) only).

For example, \( AdS_p \) in the coordinates used in equation (54) is conformal to the metric of Einstein’s static universe (ESU). We see that in this example

\[
\Omega \equiv \cos \rho
\]

(89)
and the infinite $J$ is located in these coordinates at the finite distance $\rho = \pi/2$.

One of the simplest ways of characterizing the behavior of the conformal factor $\Omega$ is to study null geodesics in its vicinity (cf. [42]). We shall choose an affine parameter $\hat{u}$ on them, (that is, $l^\mu \hat{\nabla}_\mu \hat{u} \sim 1$; where $l^\mu$ is the tangent (null) vector to the geodesic) and we further fix the origin of the affine parameter by $\hat{u} \sim 0$.

There is a corresponding parameter $u$ associated to the metric $g_{\mu\nu}$, defined by

$$\frac{d\hat{u}}{du} \equiv l^\mu \nabla_\mu \hat{u} \equiv \Omega^2 l^\mu \hat{\nabla}_\mu \hat{u} = \Omega^2$$

(90)

The conformal factor will have, by analiticity, an expansion of the type

$$\Omega(\hat{u}) = -\sum_{n=1}^{\infty} A_n \hat{u}^n$$

(91)

If $\mathcal{M}$ is an Einstein space with scalar curvature given by

$$R = \frac{2n\lambda}{n - 2}$$

(92)

it is not difficult to show that the vector $N_\mu \equiv -\nabla_\mu \Omega$ obeys

$$\hat{N}^2 \sim \frac{2\lambda}{(n - 1)(n - 2)}$$

(93)

conveying the fact that the sign of the cosmological constant is related to the spacetime properties of the $J$ boundary (and, in particular, in the ordinary $(1, 3)$ case, we see that $J$ will be timelike when the cosmological constant is negative only).

It is also possible to show that

$$\hat{\nabla}_\nu \hat{\nabla}_\mu \Omega = \frac{1}{n} \hat{\Delta} \Omega \hat{g}_{\mu\nu}$$

(94)

which in turn, (using the fact that $l^2 = 0$), enforce $A_2 = 0$ in the preceding expansion (91) (42).

Let us remark that we are here associating to a metric in $\mathcal{M}$, a whole conformal class in $J$. (That is, all the construction above is invariant under $\Omega \rightarrow t \Omega$, with $t \in \mathbb{R}^+$).
7 Witten’s Holography

7.1 The bulk action expressed in terms of boundary values.

The special properties of $AdS_p$ allow for uniqueness of a solution of a wave equation, given data on the boundary. Precise mathematical theorems can be found in [18]. Let us concentrate in the euclidean case, where $\partial E AdS_p = S_{p-1}$.

Using horospheric coordinates (66) it can be easily shown that the appropriate Green function is:

$$K(z, \vec{x}, \vec{x}') \equiv l^{p-1} \frac{\Gamma(p-1)}{\pi^{(p-1)/2}\Gamma((p-1)/2)} \frac{z^{p-1}}{|l^2z^2 + (\vec{x} - \vec{x}')^2|^{p-1}}$$

which obeys the equation

$$\Delta K(z, \vec{x}, \vec{x}') = 0$$

as well as

$$\lim_{z \to 0} K(z, \vec{x}, \vec{x}') = \delta^{p-1}(\vec{x} - \vec{x}')$$

Using that, it is plain that the solution for the bulk field $\Phi(z, \vec{x})$, with action

$$S \equiv \frac{1}{2} \int_{AdS_p} d(vol) \partial_\mu \Phi \partial^\mu \Phi$$

and fixed value at the boundary,

$$\Phi(z = 0, \vec{x}) = \phi(\vec{x})$$

is given by:

$$\Phi(z, \vec{x}) = \int_{S_{p-1}} d(vol)_{\vec{x}} K(z, \vec{x}, \vec{x}') \phi(\vec{x}')$$

In order to compute the bulk action corresponding to fixed boundary values, it is convenient to first regularize the boundary to $z = \epsilon$ and only at the end make $\epsilon = 0$. In that way one gets:

$$S(\phi) \sim -\frac{p-1}{2} \int d(vol)_{\vec{x}} d(vol)_{\vec{x}'} l^{p-2} \frac{\phi(\vec{x})\phi(\vec{x}')}{|\vec{x} - \vec{x}'|^2(p-1)}$$

26
which is equivalent to the fact, on the CFT side

\[ < \mathcal{O}(\vec{x}) \mathcal{O}(\vec{x}') > \sim \frac{1}{(\vec{x} - \vec{x}')^{2(p-1)}} \quad (102) \]

In the massive case, we have to add to the action an extra term:

\[ \Delta S \equiv \frac{1}{2} \int d(\text{vol}) m^2 \Phi^2 \quad (103) \]

The wave equation is now best analyzed in the euclidean version of the coordinates (63), namely

\[ ds^2 = dy^2 + l^2 \sinh^2(y/l) \ d\Omega^2_{p-1} \quad (104) \]

For large values of the coordinate y, the laplacian on the sphere should be negligible, and the equation reduces to:

\[ e^{-(p-1)y/l} \frac{d}{dy}(e^{(p-1)y/l} \frac{d}{dy} \Phi) = m^2 \Phi \quad (105) \]

which admits an exponential behaviour \( e^{\lambda \pm y/l} \), with \( \lambda_{\pm}(\lambda_{\pm} + p - 1) = l^2 m^2 \). This means that in the massive case it is not possible to extend to the bulk an arbitrary function \( \phi \) on the boundary. Massive field should tend to boundary fields coupling to operators \( \mathcal{O}_\Delta \) with scale dimension \( (p - 1) + \lambda_+ \), because a boundary Weyl transformation \( y \to y + w \) can be compensated by the transformation \( \phi \to e^{-\frac{w}{l} \lambda_+} \phi \), which means that the boundary fields have got, in this sense, scale dimension \(-\lambda_+\).

The Green’s function is now:

\[ K(z, \vec{x}, \vec{x}') \equiv l^{p-1+2\lambda_+} \frac{\Gamma(p - 1 + \lambda_+)}{\pi^{(p-1)/2} \Gamma((p - 1)/2 + \lambda_+)} \frac{z^{(p-1)+\lambda_+}}{(l^2 z^2 + (\vec{x} - \vec{x}')^2)^{(p-1)+\lambda_+}} \quad (106) \]

leading easily to the bulk action:

\[ S \sim \int d(\text{vol})_z d(\text{vol})_{\vec{x}} \ l^{p-2+2\lambda_+} \frac{\phi(\vec{x}) \phi(\vec{x}')}{(\vec{x} - \vec{x}')^{2(p-1)+\lambda_+}} \quad (107) \]

Given the fact that \( \lambda_{\pm} \equiv -\frac{p-1}{2} \pm \frac{1}{2} \sqrt{(p - 1)^2 + 4l^2 m^2} \), the scaling dimensions of the operators which can be represented in this way are necessarily \( \Delta > \frac{p-1}{2} \).
It can be further shown (cf. [32], [40]) that the use of the \textit{irregular boundary conditions} \((\lambda_-)\) allows to obtain correlators for CFT operators with scaling dimensions \((p-1)/2-1 < \Delta < (p-1)/2\).

7.2 Operator Mapping

Let us examine the massless fields of the IIB string theory living in \(AdS_5 \times S^5\). Besides the field corresponding to the graviton, \(g_{\mu\nu}\), which will be expanded around the background as \(\bar{g}_{\mu\nu} + h_{\mu\nu}\), there is a complex scalar \(B\), a complex two-form, \(A^{(2)}_{\mu\nu}\) and a real self-dual four form, \(A^{(+)}_{\mu\nu\rho\sigma}\). The fermionic sector consists in a complex gravitino \(\psi_\mu\), as well as a complex fermion \(\lambda\).

All these fields are expanded (cf. [30]) in terms of spherical harmonics corresponding to the sphere \(S^5\), so that for example in the scalar case

\[
\bar{g}^{ab} h_{ab} = \sum \pi^{I_1} Y_{I_1} \tag{108}
\]

Antisymmetric tensors need more terms in the expansion:

\[
A_{\mu\nu} = \sum \alpha^{I_{10}} Y^{I_{10}}_{[\mu\nu]} + \ldots \tag{109}
\]

On the other hand, the different fields of \(\mathcal{N} = 4\) SYM can be packed in several ways (cf. [33]). For example, in terms of \(\mathcal{N} = 1\) superfields, they span three chiral superfields \(\phi^I\) (transforming on the adjoint of the gauge group) as well as a vector superfield \(V\), transforming also in the adjoint. The three scalar superfields give three complex scalars and three Weyl fermions. The vector superfield give another Weyl fermion, namely the gaugino, and a real vector. In this language, only a \(SU(3) \times U(1)\) subgroup of the full \(SU(4)\) symmetry is manifest. In terms of \(\mathcal{N} = 2\) superfields, everything can be packed into a vector plus a hypermultiplet. In the vector there is a complex scalar (which we shall call \(a\), the vector field an a couple of fermions; and in the hypermultiplet there are four real scalars and another two fermions.
Witten ([52]) gave the first entries of a dictionary relating fields on the two sides of the
correspondence. Let us consider, for example, the $\mathcal{N} = 1$ superfield $T^{I_1...I_n} \equiv tr(\phi^{I_1}...\phi^{I_n})$, which has got dimension $n$, which corresponds to a conformal weight $\lambda = n - 4$. Looking at
the mass formula, this means that the corresponding bulk field has a mass $m^2 = n(n + 4)$. This corresponds in the IIB side to the expansion of the graviton trace.

Another field is $V^{I_1...I_n} \equiv tr(W_a W^a \phi^{I_1}...\phi^{I_n})$ where $W_a$ is the superfield streng, and
the term $W_a W^a$ contains a gluino bilinear. Its total dimension is $n + 3$, so that the mass
of the bulk field is $m^2 = (n + 3)(n - 1)$. This corresponds to the expansion of the two-form
in the IIB theory.

Finally, there is the field $Q^n \sim tr(a^{n-2} F_{\mu\nu} F^{\mu\nu} + ...) (where a is the particular scalar included in the $\mathcal{N} = 2$ vector multiplet). Its dimension is $n + 2$, so that the mass of the corresponding bulk field is $m^2 = (n + 2)(n + 6)$. This corresponds to the expansion of the traceless graviton in the IIB theory.

All these operators enjoy special properties that guarantee protection of their dimen-
sions from quantum corrections.

### 7.3 Finite Temperature

Following the holographic philosophy, if we want to represent a conformal theory at finite
temperature (which in the euclidean case means that we are working in a manifold $M_n =
S_{n-1} \times S_1$ or $M_n = \mathbb{R}_{n-1} \times S_1$, the first thing we have to do is to look for a negative
curvature manifold $B_{n+1}$ such that $\partial B_{n+1} = M_n$. It so happens that Hawking and Page
in [25] studied this very problem, and discovered that there are two such manifolds. The
first one is essentially EAdS with time running in a circle:

\[
ds^2 = (1 + \frac{r^2}{l^2})dT^2 + \frac{dr^2}{(1 + \frac{r^2}{l^2})} + r^2 d\Omega^2_{p-2}
\]

with $T = T + \beta'$, where $\beta'$ is in principle arbitrary.
The second manifold is Schwarzschild Anti-de Sitter, which we will call SAdS. Its metric is:

$$ds^2 = (1 + \frac{r^2}{l^2} - \frac{c_n M}{r^{n-2}})dT^2 + \frac{dr^2}{1 + \frac{r^2}{l^2} - \frac{c_n M}{r^{n-2}}} + r^2 d\Omega_{p-2}^2$$  \hspace{1cm} (111)$$

where the constant $c_n = \frac{16\pi G\Gamma(n/2)}{(n-1)2^n\pi^n}$. The horizon is defined by

$$(1 + \frac{r^2}{l^2} - \frac{c_n M}{r^{n-2}})|_{r=r_+} = 0$$  \hspace{1cm} (112)$$

When we compactify the euclidean time on a circle there will in general appear a conic singularity, unless the temperature happens to have the particular value:

$$\beta_0 \equiv \frac{4\pi r_+ l^2}{nr_+^2 + (n - 2)l^2}$$  \hspace{1cm} (113)$$

The topology of SAdS is $\mathbb{R}^2 \times S_{n-1}$. It is possible to define a rescaling such that the topology is $\mathbb{R}^2 \times \mathbb{R}^{n-1}$ namely,

$$\rho^n \equiv \frac{b^{n-2}}{c_n M}r^n$$
$$t^n \equiv \frac{b^{n-2}}{c_n M}\tau^n$$  \hspace{1cm} (114)$$

Then, in the high mass limit, $M \to \infty$ we can neglect the 1 in the metric coefficients, getting

$$ds^2 = (\frac{b^2}{b^n} - \frac{b^{n-2}}{\rho^{n-2}})dT^2 + \frac{d\rho^2}{(\frac{\rho^2}{b^n} - \frac{b^{n-2}}{\rho^{n-2}})} + \rho^2 d\vec{x}^2$$  \hspace{1cm} (115)$$

The radius of the sphere $S^{n-1}$ is now of the order $M^{1/n}$, so that in the limit the topology is as stated, $\mathbb{R}^2 \times \mathbb{R}^{n-1}$. This solution had been previously considered by Horowitz and Ross in [28].

It is natural to assume that when there are several manifolds $B_{n+1}$ bounding the same $M_n$, one should consider a superposition of the two. In a given physical situation, the dominant contribution will be provided by the solution with least action. In our case the action is given by

$$I \equiv -\frac{1}{2\kappa^2} \int \sqrt{g}d^{n+1}x(l - 2\lambda) = \frac{n}{\kappa^2 l^2}V_{n+1}$$  \hspace{1cm} (116)$$
York’s boundary term vanishes, and the second equality stems from the fact that the scalar curvature is $R = 2 \frac{n+1}{n-1} \lambda$. We have represented by $V_{n+1}$ the total volume of the space, which diverges. If we reguarize through a cutoff $\epsilon^{-1}$, then

$$Vol(AdS) = \int_0^{\beta'} dt \int_0^{\epsilon^{-1}} drr^{n-1} \int_{S^{n-1}} d\Omega$$

$$Vol(SAdS) = \int_0^{\beta_0} dt \int_{r_+}^{\epsilon^{-1}} drr^{n-1} \int_{S^{n-1}} d\Omega$$  \hspace{1cm} (117)

We can determine $\beta'$ by demanding that the geometry of the hypersurface $r = \epsilon^{-1}$ is the same both in AdS and in SAdS. This means that:

$$\beta' \sqrt{1 + \frac{r^2}{l^2}} = \beta_0 \sqrt{1 + \frac{r^2}{l^2} - \frac{c_n M}{r^{n-2}}} \Big|_{r=\epsilon^{-1}}$$ \hspace{1cm} (118)

This gives

$$\beta' = \beta_0 \left(1 - \frac{1}{2} \frac{c_n M l^2 \epsilon^n}{\epsilon^{n+1}} \right) \hspace{1cm} (119)$$

yielding for the difference in action the value:

$$\Delta I \equiv \lim_{\epsilon \to 0} \left( I(SAdS) - I(AdS) \right) = \frac{Vol(S_{n-1})}{\kappa^2} \frac{\pi r_+^{n-1} l^2}{n r_+^2 + (n-2) l^2} (l^2 - r_+^2)$$ \hspace{1cm} (120)

The average value of the energy is given by:

$$<E> = \frac{\partial I}{\partial \beta_0} = \frac{(n-1)r_+^{n-2}Vol(S_{n-1})}{2\kappa^2} (r_+^2 + l^2) \hspace{1cm} (121)$$

In such a way that the canonical entropy reads:

$$S \equiv \beta_0 <E> - \Delta I = \frac{Vol(2\pi l^2 r_+^{n-1} S_{n-1})}{\kappa^2} \sim A$$ \hspace{1cm} (122)

where $A$ is the area of the horizon.

All this is consistent with the AdS/CFT conjecture. When $\beta_0 \to 0$ one expects the high temperature limit on the CFT side, which means that the entropy density should behave as

$$S \sim T^{n-1}$$ \hspace{1cm} (123)
On the gravity side,

$$r_+ = \frac{n - 2\beta_0}{4\pi^2}$$  \hspace{2cm} (124)

which we discard, or else

$$r_+ = \frac{4\pi l^2}{n\beta_0}$$  \hspace{2cm} (125)

This last possibility gives

$$S_{AdS} \sim \beta_0^{-(n-1)}$$  \hspace{2cm} (126)
8 Wilson Loops

It is natural, following Maldacena, to make the ansatz that the value of the Wilson loop $C$ (supposedly lying on the four dimensional submanifold $z = 0$) is given in the leading approximation by the area of the minimal area surface $D$ (such that $C = \partial D$) extending on the five-dimensional bulk manifold. Before proceeding, it is easy to show that in AdS, conformal invariance prevents an area law, because if we start from the expression

$$W(C) = e^{-A(D)}$$

(127)

and rescale by $\lambda$, then

$$W(\lambda C) = e^{-A(\lambda D)} = e^{-A(D)}$$

(128)

(by dilatation invariance of AdS). This implies that

$$W(\lambda C) = W(C)$$

(129)

This clearly leads to

$$A \sim \frac{T}{L}$$

(130)

and to an expression for the static potential

$$V \sim \frac{1}{L}$$

(131)

It is worth remarking that this argument ceases to apply in the SAdS case. Using the blowup

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)} + \rho^2 d\vec{x}^2$$

(132)

(where now the loop itself is placed at $\rho = \infty$) we see that

$$\rho \geq b$$

(133)

so that the presence of the horizon breaks conformal invariance and allows the possibility of confining behavior.
Let us see in some detail how the calculation proceeds in the conformal situation.

To begin with, there are no quarks in the fundamental in $\mathcal{N}=4$ SYM. What we can do is to start with a stack of $N+1$ D-branes, and pull one of them apart from the others. The long strings stretched between the pack and the isolated brane reproduce the $W$ boson with a very large mass, which then behaves in some respects as a particle in the fundamental.

Furthermore, the natural operator to consider (in the sense that this is the one that comes from dimensional reduction of $\mathcal{N}=1$ SYM in ten dimensions) is:

$$W \equiv \frac{1}{N} tr \, P e^{\int ds (iA_\mu \dot{x}^\mu + \Phi_a \dot{z}^a)}$$

(134)

(the reason for the funny $i$ is explained in [13]) where $x^\mu = x^\mu(s)$ is a parametrized loop in ordinary Minkowski space, and $z^a = z^a(s)$ another loop in the complementary six dimensional space. It can be argued that $\dot{x}^2 = \dot{z}^2$. In the CFT side, this is the condition for the absence of a linear divergence. On the string side, there is another reason dealing with boundary conditions. The total ten dimensional metric can be written in horospheric coordinates as:

$$ds^2 = \frac{1}{z^2} (dx_{1,3}^2 - l^2 \delta_{ab} dz^a dz^b)$$

(135)

where $z^a \equiv zn^a$ and $\vec{n}$ is a unitary vector living on $S^5$, $\vec{n}^2 = 1$. In [13] the following boundary conditions have been proposed on the imbeddings of the string in the space-time:

$$x^\mu (\sigma_1, 0) = x^\mu (\sigma_1)$$

(136)

and

$$J^a_1 \partial_\sigma z^a (\sigma_1, 0) = \dot{z}^a (\sigma_1)$$

(137)

(where $J$ is the two dimensional complex structure on the worldsheet of the string) and we have parametrized the boundary of the worldsheet as $\sigma_2 = 0$. The additional condition that the minimal surface terminates at the boundary of AdS, $z^a = 0$, is only compatible with the above boundary conditions precisely when $\dot{x}^2 = \dot{z}^2$. In this case the boundary
conditions can be written as Dirichlet boundary conditions on $S^5$:
\[ n^a(\sigma_1, 0) = \frac{z^a}{|z|} \]  
(138)

All this means that we are really computing:
\[ W(C) \equiv \frac{1}{N} tr P e^{\int ds A_\mu(\xi) \dot{\xi}^\mu + \theta I(s) X_\mu(\xi)\sqrt{\epsilon^2}} \]  
(139)

The one-dimensional loop is imbedded into $AdS_5 \times S_5$ through
\[ s \in S^1 \rightarrow ((\xi^\mu(s), u(s)), \theta I(s)) \]  
(140)

Let us assume that we place the loop at the boundary, $u(s) = \infty$ and that besides we map the loop to a fixed point on the sphere, $\theta I(s) = \theta I_0$. The simplest way to proceed (138) is to consider a rectangular static loop, extending from $x = -L/2$ to $x = L/2$, and in the temporal direction from 0 to $T$. In that way, in the large $T$ limit we can easily extract the static potential,
\[ W \sim e^{-TV(L)} \]  
(141)

We can furthermore parametrize the two-dimensional surface bounded by the loop by $\sigma = x^1 \equiv x$ and $\tau = x^0 \equiv t$. Assuming that the surface itself extends only in the holographic direction, and owing to invariance under time translations, it is uniquely characterized by only one function
\[ u(x) \]  
(142)

The induced metric on the two-surface is
\[ ds^2 = h_{ab} d\sigma^a d\sigma^b = \frac{u^2}{l^2} dt^2 + \left( \frac{u^2}{l^2} + \left( \frac{l^2}{u^2} \frac{\partial u}{\partial x} \right)^2 \right) dx^2 \]  
(143)

so that its area is given by:
\[ A \equiv \int dt dx \sqrt{det h_{ab}} \]  
(144)

that is
\[ A = T \int_{-L/2}^{L/2} dx \sqrt{\frac{u^4}{l^4} + (u_x)^2} \]  
(145)
The problem of finding the minimal area surface bounded by the loop \( C \) is equivalent to minimizing the above expression in terms of the function \( u(x) \). This can be easily done by using the expression for the first integral coming from the fact that \( x \) itself is an ignorable coordinate. The result is

\[
x = \frac{l^2}{u_0} \int_{1}^{u/u_0} \frac{dz}{z^2 \sqrt{z^4 - 1}} \tag{146}
\]

Where \( u_0 \) is the unknown value of the minimum of the function \( u(x) \). Its numerical value can be determined by enforcing the boundary condition:

\[
\frac{L}{2} = \frac{l^2}{u_0} \int_{1}^{\infty} \frac{dz}{z^2 \sqrt{z^4 - 1}} \tag{147}
\]

This gives

\[
A = Tu_0 \int dy \frac{y^2}{\sqrt{y^4 - 1}} \tag{148}
\]

Which, although goes as \( T/L \) (because \( u_0 \sim 1/L \)) as it should by conformal invariance, actually diverges. This divergence can be eliminated by substracting the free loop corresponding to the cuboid extending all the way to \( u = 0 \):

\[
A_{\text{ren}} = Tu_0 \int dy \left( \frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) \tag{149}
\]

The fact that the effective string tension \( T_{\text{eff}} \sim \lambda^{1/2} \) implies then that the static potential behaves in this case as:

\[
V(L) \sim \frac{\lambda^{1/2}}{L} \tag{150}
\]
9 The Fefferman-Graham construction

In mathematical terms the problem associated with the holographic projection is that of finding the conformal invariants of a given manifold $M_n$, with generic signature $(p, q) \equiv ((1)^p, (-1)^q)$ and dimension $n = p + q$, in terms of the Riemannian invariants of some other manifold $\tilde{M}$ in which $M_n$ is contained in some precise sense. In order to employ a more physical terminology, we will refer to $M_n$ as the space-time and to $\tilde{M}$ as the bulk (in case it has one dimension more) or ambient space (in the case it has got two more dimensions).

Following [15] we will work out this geometrical problem from two different points of view, depending on the dimension and signature of the bulk space. In the so-called Lorentzian approach the ambient space $A_{n+2}$ will have signature $(p + 1, q + 1)$ while in the second approach, based on Penrose’s definition of conformal infinity the bulk space $B_{n+1}$ will have either $(p, q + 1)$ or $(p + 1, q)$ signature, leading to two different kinematical types of geometric holography.

9.1 Lorentz Holography

Conformal invariant tensors, considered as functionals of the metric tensor, $g$, $P(g)$ are defined by the transformation law:

$$P(\lambda g) = \lambda^{-\Delta} P(g)$$

where $\Delta$ is the conformal weight; they are thus associated with a given conformal class of metrics, $[g]$.

We shall represent the extra two coordinates of the ambient space by $\rho$ and $t$. Remarkably enough, Fefferman and Graham were able to prove that diff invariant expressions on $A_{n+2}$ give rise to conformal invariants provided that the metric on $A_{n+2}$, $d\tilde{s}^2$ is such that:

\begin{align*}
  i) & \quad d\tilde{s}^2(\rho = 0) = t^2 ds_n^2 \\
  ii) & \quad d\tilde{s}^2(x, \lambda t, \rho) = \lambda^2 d\tilde{s}^2(x, t, \rho)
\end{align*}
$$iii) \quad R_{\mu\nu}(\bar{g}) = 0 \quad (152)$$

where $ds^2_n$ is a convenient reference metric chosen in $M_n$.

In the odd case, $n \in 2\mathbb{Z} + 1$ there is a perturbative solution to this mathematical problem as a formal series in the variable $\rho$. Moreover, it so happens that the conditions just stated force the metric in $B$ to be of the form

$$ds^2 = t^2 ds^2(x, \rho) - 2 \rho dt^2 - 2td\rho dt \quad (153)$$

where $ds^2(x, \rho)$ is such that

$$ds^2(x, \rho = 0) = ds^2_n(x) \quad (154)$$

which is the fiducial line element in $M_n$. The generator of the dilatations $t \to \lambda t$ is

$$T \equiv t \frac{\partial}{\partial t} \quad (155)$$

On the region of $A_{n+2}$ defined by $\rho = 0$ the following relationships are true:

$$\tilde{R}_{abct} = 0$$
$$\tilde{R}_{abcd} = t^2 W_{abcd}$$
$$\tilde{R}_{abcp} = t^2 C_{abc}$$
$$\tilde{R}_{\rho abp} = \frac{t^2}{n - 4} B_{ab} \quad (156)$$

where the latin indices $a, b, c \ldots \in (1 \ldots n)$ and we have explicitly indicated the extra indices $t$ and $\rho$. The symbols $W$, $C$ and $B$ stand for the Weyl, Cotton and Bach tensors, defined (in any dimension) by means of the tensor

$$A_{\alpha\beta} \equiv \frac{1}{n - 2} \left( R_{\alpha\beta} - \frac{R}{2(n-1)} g_{\alpha\beta} \right) \quad (157)$$

as:

$$W_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} - (A_{\beta\mu} g_{\alpha\nu} + A_{\alpha\nu} g_{\beta\mu} - A_{\beta\nu} g_{\alpha\mu} - A_{\alpha\mu} g_{\beta\nu}) \quad (158)$$
(this definition implies that the Weyl tensor vanishes identically when \( n = 2 \) or \( n = 3 \)). The Weyl tensor is conformal invariant of weight \( \Delta = -1 \). When \( n > 3 \) the space is conformally flat iff \( W = 0 \).

On the other hand, the Cotton tensor, \( C_{\mu \nu \rho} \), is defined by:

\[
C_{\alpha \beta \gamma} \equiv \nabla_\alpha A_{\beta \gamma} - \nabla_\beta A_{\alpha \gamma}
\]  
(159)

For dimension \( n = 3 \) the Cotton tensor is a conformal invariant of weight zero. For bigger dimension, it is not conformal invariant. In \( n = 3 \) the spacetime is conformally flat iff the Cotton tensor vanishes

and

\[
B_{\mu \nu} \equiv \nabla^\rho C_{\rho \mu \nu} + A^{\alpha \beta} W_{\alpha \beta \mu \nu}
\]  
(160)

It is to be stressed that the Bach tensor is conformally invariant (of weight \( \Delta = 1 \)) when \( n = 4 \) only.

Let us consider the simplest example in order to visualize this result. We shall take \( M = S^1 \). The defining equation

\[
x_1^2 + x_2^2 = 1
\]  
(161)

can equally well be written in projective coordinates \( x^i \equiv \xi^i (i = 1, 2) \) as the cone \( C \):

\[
\xi_0^2 - \xi_1^2 - \xi_2^2 = 0.
\]  
(162)

In this picture the ambient space \( A_3 \) is then defined in terms of those (projective) coordinates \( \xi^0, \xi^1, \xi^2 \).

The dilatation generator on the cone \( C \), that is, the vector

\[
T \equiv \xi^\mu \frac{\partial}{\partial \xi^\mu}
\]  
(163)

is forced to be a null vector with respect to the ambient metric \( \tilde{g} \) (restricted to \( C \)). All this is obviously equivalent to identify the cone \( C \) with the light cone of the three-dimensional ambient space with signature (1, 2) (where the coordinate \( \xi_0 \) is a time); that is,

\[
ds^2 = d\xi_0^2 - d\xi_1^2 - d\xi_2^2
\]  
(164)
In terms of the variables \((\xi_0, x_1, x_2)\) this reads

\[
d\tilde{s}^2 = (1 - x_1^2 - x_2^2)d\xi_0^2 - \xi_0^2(dx_1^2 + dx_2^2) + 2\xi_0d\xi_0(x_1dx_1 + x_2dx_2)
\]  

so that on \(C\)

\[
d\tilde{s}^2|_C = -\xi_0^2(dx_1^2 + dx_2^2)
\]  

Canonical ambient coordinates can be introduced by first passing to polar coordinates \((\xi_0, r, \theta)\), with

\[
x_1 = r \cos \theta \\
x_2 = r \sin \theta
\]  

in such a way that

\[
d\tilde{s}^2 = d\xi_0^2 - dr^2 - r^2 d\theta^2
\]  

and then defining the holographic coordinates

\[
t \equiv 2(\xi_0 + r) \\
\rho \equiv \frac{1}{2} r - \xi_0
\]  

yielding the metric in the canonical form:

\[
d\tilde{s}^2 = -t^2(\rho + 2)^2d\theta^2 - 2\rho dt^2 - 2tdtd\rho
\]  

We clearly see that we have changed the signature from \((0, 1)\) in \(M = S^1\) to \((1, 2)\) in the ambient space.

Generically, one goes from signature \((p, q)\) in \(M\), to signature \((p + 1, q + 1)\) for the Lorentzian ambient space; that is, of the two extra coordinates, one is spacelike and the other timelike.

In the even case, \(n \in 2\mathbb{Z}\), there is an obstruction to the perturbative solution, the Fefferman-Graham tensor, \(F_{\mu\nu}\), which is nothing other than the Bach tensor when \(n = 4\), but in general dimension is a new tensor, a conformal invariant of weight \((n - 2)/2\). As we shall see later on, this obstruction is related to the conformal anomaly.
9.2 Penrose holography

There is another, mathematically equivalent construction, based in a bulk space, $B_{n+1}$ such that $M_n = \partial B_{n+1}$ in Penrose’s sense. If we extend the coordinates of $M$, $x^a$ by a new holographic coordinate, $r$, such that $r = 0$ is precisely the boundary, then the construction is such that the metric in the bulk space obeys:

$$ds_\gamma^2 = \frac{1}{r^2} (-dr^2 + g_{ab}(x, r)dx^adx^b)$$

(171)

(we shall see the reason for the $-$ label in a moment). The bulk space is a constant curvature space, that is

$$R_{ab} = n g_{ab}$$

(172)

(in our previous conventions, this is equivalent to normalize the total radius of AdS to unity, $l = 1$). The signature of this metric, as advertised, is $(p, q + 1)$.

There is a canonical way of constructing the bulk metric from the ambient metric. If we consider the hypersurface in the ambient space defined by constant values of the modulus of the dilatation generator,

$$T^2 = -1$$

(173)

that is, $2\rho t^2 = 1$. This space of codimension one, which we shall identify with the bulk space, $B_{n+1}$, contains $M_n$ as its conformal boundary. The bulk metric as induced by the imbedding, is given by

$$ds_{n+1}^{(-)} = \frac{1}{2\rho} ds^2(x, \rho) - \frac{1}{4\rho^2} d\rho^2$$

(174)

(that is, the extra coordinate is spacelike). This can be put precisely of the form in (171)

$$ds_{n+1}^{(-)} = \frac{1}{r^2} (-dr^2 + ds^2(x, \rho = \frac{r^2}{2}))$$

(175)

through $r^2 \equiv 2\rho$.

It should be clear by the reasoning above that when $M$ is of signature $(p, q)$, $B_{n+1}$ enjoys signature $(p, q + 1)$.
We could repeat the previous construction for

\[ T^2 = 1 \]  

(176)

with the the result

\[ (ds^{(+)}_{n+1}) = -\frac{1}{2\rho}dS^2(x, \rho) - \frac{1}{4\rho^2}d\rho^2 \]  

(177)

(This is actually the promised reason for the subscript ± on the metric). For example, in our simplest \( M = S^1 \) case, this gives

\[ ds^{(+)}_{n+1} = \frac{t^2(\rho + 2)^2}{8\rho}d\theta^2 - \frac{1}{4\rho^2}d\rho^2 \]  

(178)

the extra coordinate is timelike now. The metric can be formally put into the canonical form through \( \rho = -\frac{r^2}{2} \):

\[ ds^{(+)}_{n+1} = \frac{1}{r^2}(-dr^2 + ds^2(x, \rho = -\frac{r^2}{2})) \]  

(179)

In general this would mean that the holographic coordinate is in this case timelike, and therefore in order to get a boundary with the desired signature \( (p, q) \) we should consider a bulk space of signature \( (p + 1, q) \) (instead of \( (p, q + 1) \)).

The preceding remark is potentially interesting for a boundary physical spacetime of Minkowskian signature \((1, 3)\), since in this case we could try to perform a holographic projection with positive cosmological constant, but on a bulk spacetime with signature \((2, 3)\). This could presumably be the mathematical basis for the de Sitter/CFT duality proposed by Strominger in [47].

9.2.1 Appendix

There is a useful generalization of the usual horospheric coordinates which gives the metric induced on pseudospheres by the imbedding on a flat space of arbitrary signature. Actually, for arbitrary ± signs, denoted by \( \epsilon_i = \pm 1 \), the metric induced on the surface

\[ \sum_{i=1}^{n} \epsilon_i x_i^2 = 1 \]  

(180)
by the imbedding on the flat space with metric

$$ds^2 = \sum_{i=1}^{n} \epsilon_i dx_i^2$$  \hspace{1cm} (181)

can easily be reduced to a generalization of Poincaré’s metric for the half-plane by introducing the coordinates

$$z \equiv x^-$$

$$y^\mu \equiv z \, x^\mu$$  \hspace{1cm} (182)

where we have chosen the two last coordinates, $x^{n-1}$ and $x^n$ in such a way that their contribution to the metric is $dx_{n-1}^2 - dx_n^2$ (this is always possible if we have at least one timelike coordinate); and we define $x^- \equiv x^n - x^{n-1}$. $\mu \in (1, \ldots n - 2)$. The generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum \epsilon_\mu dy_\mu^2 - dz^2}{z^2}$$  \hspace{1cm} (183)

### 9.3 The News Function and the Holographic Map

The geometric holography just described strongly depends on the existence of a unique solution for a Cauchy problem defined in terms of Einstein’s equations on the bulk and with initial conditions fixed by the conformal class of the physical spacetime metric at the boundary. A necessary condition for the uniqueness of the solution is the vanishing of the (gravitational) Bondi-Sachs news function $N$ through $\mathcal{J}$ (cf. (2)). The precise definition of the complex quantity $N$ is:

$$N \equiv -\frac{1}{2} R_\mu^\nu \bar{m}^\mu \bar{m}^\nu$$  \hspace{1cm} (184)

where $m^\nu$ is one of the elements of a complex null Newman-Penrose tetrad, which is formed by four null vectors, two of them real, $l^2 = 0$ and $n^2 = 0$, and two complex conjugate of one another, $m^2 = 0$ and $\bar{m}^2 = 0$. They are normalized in such a way that $l.n = -m.\bar{m} = 1$.

The physical meaning of this is that the spacetime boundary $\mathcal{J}$ is opaque to gravitational radiation; in the four-dimensional case, with topology $S^3 \times \mathbb{R}$ absence of Bondi-Sachs news
requires that the Bach tensor vanishes on $\mathcal{J}$. To be specific, (cf. [42]) the variation of Bondi mass is given by an integral of two terms. The first one is a convenient projection of the energy-momentum tensor of the matter, whereas the second one is proportional to the modulus of the news function:

$$\delta M = \int_{\Sigma} A^2 T_{\mu\nu} n^\mu n^\nu + \frac{N\tilde{N}}{4\pi G}$$

(185)

where $A$ is a scalar field defined asymptotically in terms of the formerly introduced vector $N_\mu \equiv -\nabla_\mu \Omega$ by

$$\hat{N}^\mu = A n^\mu$$

(186)

and $\Sigma$ is a surface comprised between two cuts of the conformal boundary, $\mathcal{J}$.

When $n=3$ (with topology $S^2 \times \mathbb{R}$), the Bach tensor vanishes, conveying the fact that there are no gravitational news in this case ([4]). This is the simplest instance of the much alluded to general theorem proved in [15] stating that for a spacetime $M$ of even dimension there is no obstruction for the existence of a formal power series solution to the Cauchy problem with initial data on the boundary.

In the case of a $n=3$ boundary spacetime $S^2 \times \mathbb{R}$, however, Bondi news exist in general for matter fields if the Cotton tensor does not vanish. This means that for four dimensional bulk spaces there is the possibility of having a well defined Cauchy problem in the Fefferman-Graham sense, and yet, Bondi news for fields with spin different from 2. It is obvious that this is problematic from the holographic point of view (except in the case of pure gravity).

Geometrically, the vanishing of the Cotton tensor in the three-dimensional case is the necessary and sufficient condition for the existence of conformal Killing spinors. (In the four dimensional case, the equivalent condition (implying that the space is conformally Einstein) is the vanishing of the Bach tensor (cf.[34]). Only in this case the conformal symmetry is realized asymptotically in such a way that one can define asymptotically conserved charges associated to the $O(3, 2)$ conformal group ($O(4, 2)$ in the four-dimensional case).
This reduction of the asymptotic symmetry group to AdS is similar to the reduction from the asymptotic Bondi-Metzner-Sachs group in the asymptotically flat case, towards Poincaré, as has been pointed out in [42]. In this case, however, the condition $B_{\mu\nu} \sim 0$ is too strong, and, in particular, it is not stable against gravitational perturbations. There is then a curious discontinuity in the limit $\lambda \to 0$.

It then would seem that the vanishing of the conformal anomaly in the three dimensional case in the holographic setting [26] does not need the vanishing of the Cotton tensor.

Another fact worth stressing is that gravitational Bondi news are generically non-vanishing when $\mathcal{J}$ is spacelike, or even null. It is plain that the interplay between holography and Bondi news is related to the existence of a Cauchy surface for asymptotically anti de Sitter spacetimes (i.e. $\mathcal{J}$ timelike). The simplest example is, obviously, AdS itself, where in order to define a Cauchy surface one is forced to impose reflective boundary conditions on $\mathcal{J}$, enforcing the desired absence of Bondi news for matter fields.

Remarkably enough, Hawking [24] has proved that the physics of this set of boundary conditions is equivalent to assuming that the gravitational fields tend to AdS at infinity fast enough. Physically, absence of Bondi news on $\mathcal{J}$ is necessary in order that a CFT living on $\mathcal{J}$ could propagate holographically to the bulk in a unique way.
10 Holography and the Conformal Anomaly

As we have just seen, in the framework of the geometric approach to holography in its Poincaré form (that is, when the holographic image $M_d$ is represented as Penrose’s conformal infinity of another $B_{d+1}$ manifold), there is a privileged system of coordinates such that the boundary $\partial B_{d+1} \sim M_d$ is located at $\rho = 0$, namely

$$ds^2 = \frac{l^2 \rho^2}{4 \rho^2} + \frac{1}{\rho} h_{ij}(x, \rho) dx^i dx^j$$  \hspace{1cm} (187)

(This coordinate is related to the canonical one in (171) by $\rho = r^2$; the normalization corresponds to a cosmological constant $\lambda \equiv -\frac{d(d-1)}{2l^2}$ when $h_{ij} = \delta_{ij}$). Physically, the boundary condition is

$$h_{ij}(x, \rho = 0) = g_{ij}(x)$$  \hspace{1cm} (188)

where $g_{ij}$ is an appropriate metric on $M_d$.

Those coordinates are in conformal backgrounds essentially our old friends the horospheric coordinates: $z \equiv \rho^2$.)

The Ricci tensor for the metric (187) can be expressed as

$$R_{\rho \rho} = -\frac{d}{4 \rho^2} + \frac{1}{4} tr(h^{-1}h')^2 - \frac{1}{2} tr(h^{-1}h'')$$

$$R_{\rho i} = \frac{1}{2} \nabla_j(h^{-1}h')^j - \frac{1}{2} \nabla_i(tr h^{-1}h')$$

$$R_{ij} = R_{ij}[h] - \frac{2 - d}{l^2} h_{ij} - \frac{2 \rho}{l^2} h''_{ij} - \frac{d}{\rho l^2} h_{ij} + \frac{1}{l^2} tr(h^{-1}h')h_{ij}$$

$$-\frac{\rho}{l^2} tr(h^{-1}h')h''_{ij} + \frac{2 \rho}{l^2} (h'h^{-1}h')_{ij}$$  \hspace{1cm} (189)

where a prime means $\frac{d}{d \rho}$, and $\nabla_i$ is the covariant derivative of the Levi-Civita connection of the metric $h_{ij}$.

Einstein’s equations

$$R_{\mu \nu} = -\frac{d}{l^2} g_{\mu \nu}$$  \hspace{1cm} (190)

can be then rewritten for the metric (187) as:

$$\rho[2h''_{ij} - 2h'_{il}h'^{lm}h'_{mj} + h'^{ki}h'^{lj}h'^{km} - l^2 R_{ij} - (d - 2)h'_{ij} - h'^{kl}h'^{m}h_{ij}] = 0$$
\[(h^{-1})^{jk}(\nabla_i h'_{jk} - \nabla_k h'_{ij}) = 0\]
\[(h^{jk} h''_{kj}) - \frac{1}{2}(h^{id} h''_{in} h^{mn} h''_{ni}) = 0 \quad (191)\]

There is a natural scale symmetry associated with the preceding metric, namely

\[\rho \rightarrow \lambda \rho\]
\[h_{ij} \rightarrow \lambda h_{ij} \quad (192)\]

The famous Fefferman-Graham obstruction implies, however, that when \(d \in 2\mathbb{Z}\), there appear logarithmic terms in the expansion of the preceding metric around \(\rho = 0\), which begin at \(\rho^{d/2}\), and spoil a consistent power solution. (Although they are absent if one uses dimensional regularization, as in [29], [39]). As has been shown by Henningson and Skenderis in [26] (following a suggestion of E. Witten in [52]), these terms are responsible for the conformal anomaly. Let us sketch their argument. In the basic work by Fefferman and Graham it is proved that there is a formal power series solution to Einstein’s equations with negative cosmological constant, up to \(\rho^{d/2}\). This means that in \(d = 4\), for example, a consistent expansion exists of the form

\[h_{ij} = g_{ij} + h^{(1)}_{ij} \rho + h^{(2)}_{ij} \rho^2 + \tilde{h}^{(2)}_{ij} \rho^2 \log \rho + o(\rho^3) \quad (193)\]

Even the term \(h^{(d/2)}_{ij}\) is not completely determined; Einstein’s equations only give its trace as well as its covariant derivative, (cf. [23]).

It is, on the other hand, obvious that the Einstein-Hilbert action is divergent. To be specific,

\[S \equiv \frac{1}{2\kappa^2_{d+1}} \left[ \int_{M_{d+1}} d(vol)_{d+1}(R_{d+1} - 2\lambda_{d+1}) + \int_{M_d} dx_d 2K \right] \quad (194)\]

where \(K \equiv h^{ij}_{(ind)} \nabla_i n_j\) is the trace of the second fundamental form, \(n^i\) being the normal to the boundary and \(h^{ij}_{(ind)} \equiv h^{ij} - n^i n^j\) the induced metric on the boundary.
Explicit calculation shows that $R - 2\lambda = -\frac{4}{d - 2}\lambda = -\frac{2d}{l^2}$

$$L_{\text{bulk}} = -\frac{2d}{l^2} \int \sqrt{\frac{l^2}{4\rho^2} \rho^{-d} h^{1/2}} d\rho$$

$$= -\frac{d}{l} \int \rho^{-1-d/2} h^{1/2} d\rho. \quad (195)$$

This integral is, as advertised, divergent, a reflect of the fact that AdS is a non compact space. Actually, it diverges in both limits, both $\rho = \infty$ (which should correspond to the infrared (IR) region in the CFT, according to the IR/UV connection) and in $\rho = 0$, which is the UV region of the CFT. The divergence at $\rho = \infty$ would appear only at order $\rho^{d/2+1}$ or higher in the expansion of the metric, which is higher that the order that can be determined unambiguously.

Let us concentrate in the UV divergences. A way to regularize them is to cut-off the integral over $\rho$ with a $\theta(\rho - \epsilon)$. This leads to an inverse power series in $\epsilon$

$$S(\epsilon) \sim \sum_{n=d/2}^{0} \epsilon^{-n} S^{(n)} + \hat{S} \log \epsilon + S_{\text{ren}} \quad (196)$$

The logarithmically divergent term comes from the integral of the $\rho^{d/2}$ term in the expansion of the $M_d$ volume element, combined with the pre-factor $\rho^{-d/2-1}$. This explains why it only appears for even $d$. It is remarkable that this term has a priori nothing to do with the logarithmic ambiguity of the expansion noticed above, (although cf. later on) and is a purely bulk effect.

Actually, if we write the regularized action in the form,

$$S(\epsilon) \equiv \frac{1}{2\kappa_{d+1}^2} \int \sqrt{g} dx dL_{\epsilon} \quad (197)$$

then

$$L_{\epsilon} = a_0 \epsilon^{-d/2} + a_1 \epsilon^{-d/2+1} + \ldots + \epsilon^{-1} a_{d-1} - \log \epsilon a_d + L_{\text{finite}} \quad (198)$$

Incidentally, the two logarithmically divergent terms in equations (193) and (196) are related in the sense that, as has been proved in [23],

$$\tilde{h}^{d/2}_{ij} = -\frac{4}{d\sqrt{\tilde{g}}} \frac{\delta}{\delta g^{ij}} \int dx \sqrt{\tilde{g}} a_{(d)} \quad (199)$$
Now, under the scale invariance mentioned above, all powers are invariant by themselves (cf. [26]), meaning that the variation of the logarithmically divergent term has to be cancelled with an anomalous variation of the finite part: if \( \lambda = 1 + 2\delta\sigma \), then \( \delta h_{ij} = 2h_{ij}\delta\sigma \) and \( \delta(\log \epsilon) = 2\delta\sigma \)

\[-\delta S_{\text{ren}} \equiv \int_{M_d} \sqrt{g}dx_d\delta\sigma \mathcal{A} \quad (200)\]

Where the anomaly, \( \mathcal{A} \) is given by:

\[\mathcal{A} = -\frac{a_d}{\kappa_{d+1}^2} \quad (201)\]

General theorems [10] ensure that the anomaly can always be written as:

\[a_d = d\, l^{d-1}(E_d + I_d + D[g]J_{d-1}) \quad (202)\]

where \( E_d \) is proportional to Euler’s density in \( d \) dimensions, \( I_d \) is a conformal invariant, and the total derivative can be cancelled by a (finite covariant) counterterm.

In [26] this property has been used to compute the Weyl anomaly in several interesting cases, by just expanding carefully \( h^{1/2} \), and finding complete agreement for the leading term when \( N \to \infty \).

For example, in the physically important case of \( d = 4 \), the logarithmically divergent terms read:

\[S = \frac{2}{l^2\kappa_{d+1}^2} \int d^4x \sqrt{g} \log \epsilon \left[ 1/2(g^{ij}h_{(2)ij}) - 1/4(g^{il}h_{(1)lm}g^{mn}h_{(1)ni}) + 1/8(g^{ij}h_{(1)ji})(g^{kl}h_{(1)kl}) \right] \quad (203)\]

The \( \rho^0 \) term of the first of Einstein’s equations [101] gives:

\[R_{ij} = -\frac{1}{l^2}(2h_{(1)ij} + g^{kl}h_{(1)kl}g_{ij}) \quad (204)\]

so that

\[R = -\frac{6}{l^2} g^{ij}h_{(1)ji} \quad (205)\]

and (using \( h^{ij} = g^{ij} - \rho g^{il}h_{(1)lm}g^{mj} + \rho^2(g^{il}h_{(1)lm}g^{mn}h_{(1)np}g^{pj} - g^{il}h_{(2)lm}g^{mj}) \)),

\[R_{ij}R_{ij} = \frac{1}{l^4} (4g^{il}h_{(1)lm}g^{mn}h_{(1)ni} + 8(g^{il}h_{(2)li})^2) \quad (206)\]
whereas the third yields:

\[ g^{ij} h_{(2)ji} = \frac{1}{4} g^{ij} h_{(1)lm} g^{mn} h_{(1)ni} \]  

(207)

This altogether leads to:

\[ a_4 = \frac{l^3}{8} (-R^{ij} R_{ij} + \frac{1}{3} R^2) \]  

(208)

The four dimensional invariants are:

\[ E_4 \equiv \frac{1}{64} (R^{ijkl} R_{ijkl} - 4 R^{ij} R_{ij} + R^2) \]  

(209)

and

\[ I_4 = -\frac{1}{64} (R^{ijkl} R_{ijkl} - 2 R^{ij} R_{ij} + \frac{1}{3} R^2) \equiv W^{ijkl} W_{ijkl} \]  

(210)

(where \( W_{ijkl} \) is the Weyl tensor).

The anomaly is then given by

\[ \mathcal{A} = \frac{1}{2\kappa_5^2} (-2a_4) = -\frac{N^2}{\pi^2} (E_4 + I_4) \]  

(211)

where we have used the fact that

\[ \frac{1}{\kappa_5^2} = \frac{\text{Vol}(S_5)}{\kappa_{10}^2} = \frac{l^5 \pi^3}{64 \pi^7 g_s^2 l_s^8} \]  

(212)

and that

\[ l = (4\pi g_s N)^{1/4} l_s \]  

(213)

This reproduces the leading term in the large \( N \) limit of the four dimensional conformal anomaly, which is given in full by:

\[ \mathcal{A} = -N^2 \frac{1 - N^{-2}}{\pi^2} (E_4 + I_4) \]  

(214)

In (9) non-leading (in \( N \)) contributions to the Weyl anomaly were computed, with only partial success.

Incidentally, for any Ricci-flat metric on \( M_d \), Einstein’s equations for \( M_{d+1} \) are obeyed with

\[ h_{ij}(x, \rho) = g_{ij}(x) \]  

(215)
### 10.1 PBH Diffeomorphisms

The Penrose-Brown-Henneaux (PBH),\(^{12}\[11\]29) diffeomorphisms were introduced in \(^{29}\) as particular bulk diffs which include conformal transformations on the boundary. If we impose in the canonical form of the bulk metric we employed in equation \([187]\) that the diff is such that

\[
\delta g_{d+1,d+1} = \delta g_{d+1,i} = 0
\]  \hspace{1cm} (216)

(with \(x^{d+1} \equiv \rho\)), then we get that the diff must be generated by a vector such that

\[
\xi^{n+1} = -2\rho \sigma(x)
\]
\[
\xi^i = a^i(x, \rho)
\]  \hspace{1cm} (217)

and, besides,

\[
\partial_\rho a^i = -l^2 h^{ij} \partial_j \sigma
\]  \hspace{1cm} (218)

This implies, in particular, that

\[
\delta h_{ij} = (h)\nabla_i \xi_j + (h)\nabla_j \xi_i - 2\sigma h_{ij} - 2\sigma \rho \partial_\rho h_{ij}
\]  \hspace{1cm} (219)

We assume that there is an analytic expansion

\[
a^i = \sum_{n=1} a_{(n)}^i \rho^n
\]  \hspace{1cm} (220)

which implies that, to the lowest order in the holographic coordinate, the diff is a pure Weyl transformation on the boundary metric defined on \(M, h_{ij}^{(0)} \equiv g_{ij}\)

\[
\delta g_{ij} = -2\sigma g_{ij}
\]  \hspace{1cm} (221)

(while as in the last paragraph, we assume an expansion \(h_{ij} = \sum_q h_{ij}^{(q)} \rho^q\)).

The variation of the other terms in the expansion are easily obtained from \(219\). For example, the next one is:

\[
\delta h_{ij}^{(1)} = (0)\nabla_i a_j^{(1)} + (0)\nabla_j a_i^{(1)}
\]  \hspace{1cm} (222)
The basic differential equation just written down in eq. (218) determines the different terms in the expansion of the PBH diffs in terms of the coefficients in the expansion of the bulk metric. For example, the first one is:

\[ a_{(1)}^i = \frac{l^2}{2} g^{ij} \partial_j \sigma \]  

(223)

Imbimbo et al first noticed in [29] the remarkable fact that from these variations it is easy to get expressions for the coefficients in the expansion of the bulk metric, for example,

\[ h_{ij}^{(1)} = \frac{l^2}{d-2} \left( R_{ij} - \frac{1}{2(d-1)} R g_{ij} \right) \]  

(224)

This formula fails if the spacetime dimension is \( d = 2 \); this illustrates the claims made in the last paragraph on \( h_{ij}^{(1)} \). In this case, for example,

\[ h_{ij}^{(1)} = \frac{1}{2} (R g_{ij} + t_{ij}) \]  

(225)

with \( \nabla_i t^{ij} = 0 \) and \( g^{ij} t_{ij} = -R \).

Sometimes there are terms which appear with arbitrary coefficients; this phenomenon starts at second order in which

\[ c_1 l^4 W_{klmn} W^{klmn} g_{ij} + c_2 W_{iklm} W_{j}^{klm} \]  

(226)

can be added to the expression of \( h_{ij}^{(2)} \) for any \( c_1 \) and \( c_2 \).

Were not for these constants, this procedure would allow to determine the bulk metric in terms of boundary data; that is, to decode the hologram.

What is perhaps even more remarkable is that the whole approach can be used to recover the conformal anomaly in any dimension.

In order to achieve this goal, we shall consider an arbitrary gravitational action in the bulk space. We shall only assume that the Dirichlet problem for the metric has a unique solution. If we chose to write the action in the form (29)

\[ S = \frac{1}{2\kappa_{d+1}^2} \frac{l}{2} \int d^d x d\rho^{-\left(1+d/2\right)} \sqrt{g(x)} b(x, \rho) \]  

(227)
(where we assume that $b$ is a functional of $g$ on shell), then by expanding on a power series in the holographic coordinate

$$b(x, \rho) \equiv \sum_n b_n(x) \rho^n$$

and performing the integration over $d\rho$, one gets

$$S = \frac{1}{\kappa_{d+1}^2} \sum_{p \neq d/2} \frac{1}{2 p - d} \int d^d x \sqrt{g} \delta_{d}(x).$$

(229)

There is a pole in the expansion for any even dimension. Actually, the coefficient $b_p$ represents a trace anomaly in dimension $d = 2p$.

Then, using the fact that the total variation of the integrand $I$ under any diff generated by the vector $\xi$, must be

$$\delta I = \nabla_{\alpha}(\xi^\alpha I)$$

(230)

as well as the curious property that for PBH diffs (where $\xi = a^i \partial_i - 2 \rho \sigma \partial_{\rho}$),

$$\delta \sqrt{h} = \nabla_{\mu} \xi^\mu = (h)^{a^i} \nabla_i a^i + d\sigma - \rho \sigma h^{-1} \partial_{\rho} h$$

(231)

and also that PBH act as Weyl on the boundary,

$$\delta \sqrt{g} = d\sigma \sqrt{g}$$

(232)

then, the PBH variation of the construct $b$ is easily found to be:

$$\delta b = -2\sigma \rho \partial_{\rho} b + (0) \nabla_i (ba^i)$$

(233)

which can be easily translated in corresponding formulas for the modes $b_p$, for example

$$\delta b_0 = 0$$

(234)

$$\delta b_1 = -2\sigma b_1 + \frac{l^2}{2} b_0 \Box \sigma$$

(235)

These formulas start having arbitrary parameters in $\delta b_3$, reflecting the corresponding arbitrariness in $h^ {(2)}_{ij}$. The authors of [29] have argued that from here, local expressions for the modes can always be found. The first two are:

$$b_0 = constant$$

(236)
Starting with $b_2$ there is an increasing number of arbitrary parameters in the solution. It is however possible to use this information to find the form of the Euler density contribution to the conformal anomaly valid for any gravitational action with the characteristics indicated \cite{29}.

10.2 The Holographic Energy-momentum tensor

The expectation value of the boundary energy momentum tensor (cf. \cite{3}, \cite{14}, \cite{35}, \cite{12}) is given by the variation of the gravitational action with respect to the metric on the boundary (cf. \cite{11} for a comprehensive treatment of related matters).

The starting point is the regularization of the gravitational action we made earlier in (196):

$$S(\epsilon) = \frac{1}{2\kappa^2} \int d^d x \sqrt{g} \left[ \epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-2)} - \log \epsilon a_d \right] + S_{\text{ren}}$$

where $S_{\text{ren}}$ defines the renormalized action.

The expectation value of the energy momentum tensor is:

$$<T_{ij}> = \frac{2}{\sqrt{g} \delta g^{ij}} S_{\text{ren}} = \lim_{\epsilon \to 0} \epsilon^{1-d/2} T_{ij}[\gamma]$$

where $T_{ij}[\gamma]$ is the energy-momentum tensor corresponding to the regulated theory with respect to the induced metric on the boundary, $\gamma_{ij} \equiv \frac{1}{\epsilon} h_{ij}(x, \epsilon)$ (cf. \cite{23}).

This energy momentum can, in turn, be separated in two different contributions, coming from the regulated action, and from the counterterms:

$$T_{ij}[\gamma] = T_{ij}^\epsilon[\gamma] + T_{ij}^{\text{counterterms}}[\gamma]$$

Haro et al have given in ref. \cite{23} explicit expressions for this energy momentum tensor in different dimensions. In the simplest of all cases, $d = 2$, one gets

$$<T_{ij}> = \frac{l}{2\kappa^2} t_{ij} = \frac{l}{\kappa^2} (h_{ij}^{(2)} - g_{ij} g^{lm} h_{lm}^{(2)})$$
And indeed we recover in that way the standard two-dimensional conformal anomaly:

\[ < T^i_i > = -\frac{l}{2\kappa^2} R \]  \hfill (242)

Let us remark, finally, that the explicit transformation rules under PBH diffs, combined with those formulas, allow to determine the explicit Weyl variations of the holographic energy momentum tensor. For example, in the much discussed two-dimensional example:

\[ \delta < T_{ij} > = \frac{l}{2\kappa^2} (\nabla_i \nabla_j \sigma - g_{ij} \nabla^2 \sigma) \]  \hfill (243)
11 Conclusions

In a sense, mathematical holography is pure kinematics. Is this all there is to it?

In spite of much effort devoted to it, the extension of the above ideas to non conformal situations is still unclear.

One topic which seems worth exploring is to clarify the rôle of the cutoff (cf. [22]) and its suggested relationship with the Randall-Sundrum approach ([45]).

A curious, but well-known fact is that all black hole solutions of the holographic type dominate the path integral (that is, enjoy lower action) only in the region in which the specific heat is positive, $c_V > 0$. This leaves open the question as to whether holography is possible at all for systems such as the Schwarzschild black hole, for which the specific heat is always negative.

It has been recently suggested that some of these ideas could be extended to the constant positive curvature spaces (de Sitter, dS) ([17]), by analyzing the asymptotic diffeomorphisms in the Brown and Henneaux sense (cf. [11]). It remains to explore in detail its physical meaning as well as how these ideas fit in the Fefferman-Graham framework.
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