Approximations of the tail index estimator of heavy-tailed distributions under random censoring and application

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Abstract

We make use of the empirical process theory to approximate the adapted Hill estimator, for censored data, in terms of Gaussian processes. Then, we derive its asymptotic normality, only under the usual second-order condition of regular variation, with the same variance as that obtained by Einmahl et al. (2008). The newly proposed Gaussian approximation agrees perfectly with the asymptotic representation of the classical Hill estimator in the non censoring framework. Our results will be of great interest to establish the limit distributions of many statistics in extreme value theory under random censoring such as the estimators of tail indices, the actuarial risk measures and the goodness-of-fit functionals for heavy-tailed distributions. As an application, we establish the asymptotic normality of an estimator of the excess-of-loss reinsurance premium.

Keywords: Empirical process; Gaussian approximation; Hill estimator; Limit distribution; Random censoring; Reinsurance premium.

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1. Introduction

For \( n \geq 1 \), let \( X_1, X_2, ..., X_n \) be \( n \) independent copies of a non-negative random variable (rv) \( X \), defined over some probability space \( (\Omega, \mathcal{A}, P) \), with cumulative distribution function (cdf) \( F \). We assume that the distribution tail \( 1 - F \) is regularly varying at infinity, with index \((-1/\gamma_1)\), notation: \( 1 - F \in \mathcal{RV}_{(-1/\gamma_1)} \). That is

\[
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-1/\gamma_1}, \text{ for any } x > 0,
\]

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where $\gamma_1 > 0$, called shape parameter or tail index or extreme value index (EVI), is a very crucial parameter in the analysis of extremes. It governs the thickness of the distribution right tail: the heavier the tail, the larger $\gamma_1$. Its estimation has got a great deal of interest for complete samples, as one might see in the textbook of Beirlant et al. (2004). In this paper, we focus on the most celebrated estimator of $\gamma_1$, that was proposed by Hill (1975):

$$\hat{\gamma}_1^H = \hat{\gamma}_1^H (k) := \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n},$$

where $X_{1,n} \leq ... \leq X_{n,n}$ are the order statistics pertaining to the sample $(X_1, ..., X_n)$ and $k = k_n$ is an integer sequence satisfying

$$1 < k < n, \quad k \to \infty \text{ and } k/n \to 0 \text{ as } n \to \infty. \quad (1.2)$$

The consistency of $\hat{\gamma}_1^H$ was proved by Mason (1982) by only assuming the regular variation condition (1.1) while its asymptotic normality was established under a suitable extra assumption, known as the second-order regular variation condition (see de Haan and Stadtmüller, 1996 and de Haan and Ferreira, 2006, page 117).

In the analysis of lifetime, reliability or insurance data, the observations are usually randomly censored. In other words, in many real situations the variable of interest $X$ is not always available. An appropriate way to model this matter, is to introduce a non-negative rv $Y$, called censoring rv, independent of $X$ and then to consider the rv $Z := \min (X, Y)$ and the indicator variable $\delta := 1 (X \leq Y)$, which determines whether or not $X$ has been observed. The cdf’s of $Y$ and $Z$ will be denoted by $G$ and $H$ respectively. The analysis of extreme values of randomly censored data is a new research topic to which Reiss and Thomas (1997) made a very brief reference, in Section 6.1, as a first step but with no asymptotic results. Considering Hall’s model Hall (1982), Beirlant et al. (2007) proposed estimators for the EVI and high quantiles and discussed their asymptotic properties, when the data are censored by a deterministic threshold. More recently, Einmahl et al. (2008) adapted various EVI estimators to the case where data are censored, by a random threshold, and proposed a unified method to establish their asymptotic normality by imposing some assumptions that are rather unusual to the context of extreme value theory. The obtained estimators are then used in the estimation of extreme quantiles under random censorship. Gomes and Neves (2011) also made a contribution to this field.
by providing a detailed simulation study and applying the estimation procedures on some survival data sets.

We start by a reminder of the definition of the adapted Hill estimator, of the tail index $\gamma_1$, under random censorship. The tail of the censoring distribution is assumed to be regularly varying too, that is $1 - G \in R(V_{-1/\gamma_2})$, for some $\gamma_2 > 0$. By virtue of the independence of $X$ and $Y$, we have $1 - H(x) = (1 - F(x))(1 - G(x))$ and therefore $1 - H \in R(V_{-1/\gamma_1})$, with $\gamma := \gamma_1 \gamma_2 / (\gamma_1 + \gamma_2)$. Let $\{(Z_i, \delta_i), 1 \leq i \leq n\}$ be a sample from the couple of rv’s $(Z, \delta)$ and $Z_{1,n} \leq ... \leq Z_{n,n}$ represent the order statistics pertaining to $(Z_1, ..., Z_n)$. If we denote the concomitant of the $i$th order statistic by $\delta[i:n]$ (i.e. $\delta[i:n] = \delta_j$ if $Z_{i:n} = Z_j$), then the adapted Hill estimator of the tail index $\gamma_1$ is defined by

$$\hat{\gamma}_1^{(H,c)} := \frac{\hat{\gamma}_1^{H}}{\hat{p}},$$

where

$$\hat{\gamma}_1^{H} := \frac{1}{k} \sum_{i=1}^{k} \log Z_{n-i+1:n} - \log Z_{n-k:n}$$

and

$$\hat{p} := \frac{1}{k} \sum_{i=1}^{k} \delta[n-i+1:n],$$

with $k = k_n$ satisfying (1.2). Roughly speaking, the adapted Hill estimator is equal to the quotient of the classical Hill estimator to the proportion of non-censored data.

To derive the asymptotic normality of $\hat{\gamma}_1^{(H,c)}$, we will adopt a new approach which is different from that of Einmahl et al. (2008). We notice that the asymptotic normality of extreme value theory based estimators is achieved in the second-order framework (see de Haan and Stadtmüller, 1996). Thus, it seems quite natural to suppose that cdf’s $F$, $G$ and $H$ satisfy the well-known second-order condition of regular variation. That is, we assume that there exist a constant $\tau_j < 0$ and a function $A_j$, $j = 1, 2$ not changing sign near infinity, such that for any $x > 0$

$$\lim_{t \to \infty} \frac{\overline{F}(tx)/\overline{F}(t) - x^{-1/\gamma_1}}{A_1(t)} = x^{-1/\gamma_1} x^{\tau_1} - 1,$$

$$\lim_{t \to \infty} \frac{\overline{G}(tx)/\overline{G}(t) - x^{-1/\gamma_2}}{A_2(t)} = x^{-1/\gamma_2} x^{\tau_2} - 1,$$

where $\overline{S}(x) := S(\infty) - S(x)$, for any $S$. For convenience, the same condition on cdf $H$ will be expressed in terms of its quantile function $H^{-1}(s) := \inf \{ x : H(x) \geq s \},$
0 < s < 1. There exist a constant \( \tau_3 < 0 \) and a function \( A_3 \) not changing sign near zero, such that for any \( x > 0 \)
\[
\lim_{t \downarrow 0} \frac{H^{-1}(1 - tx)/H^{-1}(1 - t) - x^{-\gamma}}{A_3(t)} = x^{-\gamma}x^{\tau_3} - 1 / \tau_3.
\] (1.7)

Actually what interests us most is the Gaussian approximation to the distribution of the adapted estimator \( \hat{\gamma}_1^{(H,c)} \), similar to that obtained for Hill’s estimator \( \hat{\gamma}_1^H \) in the case of complete data. Indeed, if (1.6) holds for \( F \), then, for an integer sequence \( k \) satisfying (1.2) with \( \sqrt{n/k}A_1(n/k) \to 0 \), we have as \( n \to \infty \),
\[
\sqrt{k}(\hat{\gamma}_1^H - \gamma_1) = \gamma_1\sqrt{n/k} \int_0^1 s^{-1} \tilde{B}_n \left( 1 - \frac{k}{n}s \right) ds - \gamma_1\sqrt{n/k} \tilde{B}_n \left( 1 - \frac{k}{n} \right) + o_p(1),
\]
where \( \{ \tilde{B}_n(s); 0 \leq s \leq 1 \} \) is a sequence of Brownian bridges (see for instance Csörgő and Mason, 1985 and de Haan and Ferreira, 2006, page 163). In other words, \( \sqrt{k}(\hat{\gamma}_1^H - \gamma_1) \) converges in distribution to a centred Gaussian rv with variance \( \gamma_1^2 \).

The Gaussian approximation above enables to solve many problems with regards to the asymptotic behavior of several statistics of heavy-tailed distributions, such as the estimators of: the mean (Peng, 2001 and 2004; Brahimi et al., 2013), the excess-of-loss reinsurance premium (Necir et al., 2007), the distortion risk measures (Necir and Meraghni, 2009 and Brahimi et al., 2011), the Zenga index (Greselin et al., 2013) and the goodness-of-fit functionals as well (Koning and Peng, 2008).

The rest of the paper is organized as follows. In Section 2, we state our main result which consists in a Gaussian approximation to \( \hat{\gamma}_1^{(H,c)} \) only by assuming the second-order conditions of regular variation (1.6) and (1.7). More precisely, we will show that there exists a sequence of Brownian bridges \( \{ B_n(s); 0 \leq s \leq 1 \} \) defined on \((\Omega, \mathcal{A}, \mathbb{P})\), such that as \( n \to \infty \),
\[
\sqrt{k}(\hat{\gamma}_1^{(H,c)} - \gamma_1) = \Psi(B_n) + o_p(1),
\]
for some functional \( \Psi \) to be defined in such a way that \( \Psi(B_n) \) is normal with mean 0 and variance \( p\gamma_1^2 \). Section 3 is devoted to an application of the main result as we derive the asymptotic normality of an excess-of-loss reinsurance premium estimator. The proofs are postponed to Section 4 and some results, that are instrumental to our needs, are gathered in the Appendix.

2. Main result

In addition to the Gaussian approximation of \( \sqrt{k}(\hat{\gamma}_1^{(H,c)} - \gamma_1) \), our main result (stated in Theorem 2.1) consists in the asymptotic representations, with Gaussian
processes, of two other useful statistics, namely \( \sqrt{k} (\hat{p} - p) \) and \( \sqrt{k} \left( \frac{Z_{n-k/n}}{H^{-1}(1-k/n)} - 1 \right) \).

The functions defined below are crucial to our needs

\[
H^0 (z) := \mathbb{P} (Z \leq z, \delta = 0) = \int_0^z F(y) \, dG(y) \tag{2.8}
\]

and

\[
H^1 (z) := \mathbb{P} (Z \leq z, \delta = 1) = \int_0^z G(y) \, dF(y). \tag{2.9}
\]

Throughout the paper, we use the notations

\[
h = h_n := H^{-1} (1 - k/n), \quad \theta := H^1 (\infty) \quad \text{and} \quad p = 1 - q := \gamma / \gamma_1,
\]

and, for two sequences of rv’s, we write \( V_n^{(1)} = o_p \left( V_n^{(2)} \right) \) and \( V_n^{(1)} \approx V_n^{(2)} \) to say that, as \( n \to \infty \), \( V_n^{(1)}/V_n^{(2)} \to 0 \) in probability and \( V_n^{(1)} = V_n^{(2)} (1 + o_p (1)) \) respectively.

**Theorem 2.1.** Assume that the second-order conditions (1.6) and (1.7) hold. Let \( k = k_n \) be an integer sequence satisfying, in addition to (1.2), \( \sqrt{k} A_j (h) \to 0 \), for \( j = 1, 2 \) and \( \sqrt{k} A_3 (k/n) \to \lambda < \infty \) as \( n \to \infty \). Then there exists a sequence of Brownian bridges \( \{ B_n (s) ; 0 \leq s \leq 1 \} \) such that, as \( n \to \infty \),

\[
\sqrt{k} \left( \frac{Z_{n-k/n}}{h} - 1 \right) = \gamma \sqrt{\frac{n}{k} \mathbb{B}_n^* \left( \frac{k}{n} \right)} + o_p (1),
\]

\[
\sqrt{k} (\hat{p} - p) = \sqrt{\frac{n}{k}} \left( q \mathbb{B}_n \left( \frac{k}{n} \right) - p \mathbb{B}_n \left( \frac{k}{n} \right) \right) + o_p (1)
\]

and

\[
\sqrt{k} \left( \bar{\gamma}_1^{(H,c)} - \gamma_1 \right) = \gamma_1 \sqrt{\frac{n}{k}} \int_0^1 s^{-1} \mathbb{B}_n^* \left( \frac{k}{n} s \right) \, ds - \gamma_1 \sqrt{\frac{n}{k}} \mathbb{B}_n \left( \frac{k}{n} \right) + o_p (1),
\]

where

\[
\mathbb{B}_n (s) := B_n (\theta) - B_n (\theta - ps), \quad \tilde{\mathbb{B}}_n (s) := -B_n (1 - qs)
\]

and

\[
\mathbb{B}_n^* (s) := \mathbb{B}_n (s) + \tilde{\mathbb{B}}_n (s), \quad 0 < s < 1,
\]

are sequences of centred Gaussian processes.

**Corollary 2.1.** Under the assumptions of Theorem 2.1, we have

\[
\sqrt{k} \left( \bar{\gamma}_1^{(H,c)} - \gamma_1 \right) \overset{d}{\to} \mathcal{N} (0, p \gamma_1^2), \quad \text{as } n \to \infty.
\]

\( \mathcal{N} (0, \alpha^2) \) designates the centred normal distribution with variance \( \alpha^2 \).

To the best of our knowledge, this is the first time that \( \sqrt{k} \left( \bar{\gamma}_1^{(H,c)} - \gamma_1 \right) \) is expressed in terms of Gaussian processes. This asymptotic representation will be of great usefulness in a lot of applications of extreme value theory under random censoring, as we will see in the following example.
3. Application: Excess-of-loss reinsurance premium estimation

In this section, we apply Theorem 2.1 to derive the asymptotic normality of an estimator of the excess-of-loss reinsurance premium obtained with censored data. The choice of this example is motivated mainly by two reasons. The first one is that the area of reinsurance is by far the most important field of application of extreme value theory. The second is that data sets with censored extreme observations often occur in insurance. The aim of reinsurance, where emphasis lies on modelling extreme events, is to protect an insurance company, called ceding company, against losses caused by excessively large claims and/or a surprisingly high number of moderate claims. Nice discussions on the use of extreme value theory in the actuarial world (especially in the reinsurance industry) can be found, for instance, in Embrechts et al. (1997), a major textbook on the subject, and Beirlant et al. (1994).

Let \(X_1, \ldots, X_n\) \((n \geq 1)\) be \(n\) individual claim amounts of an insured loss \(X\) with finite mean. In the excess-of-loss reinsurance treaty, the ceding company covers claims that do not exceed a (high) number \(R \geq 0\), called retention level, while the reinsurer pays the part \((X_i - R)_+ := \max(0, X_i - R)\) of each claim exceeding \(R\). Applying Wang’s premium calculation principle, with a distortion equal to the identical function (Wang, 1996), to this reinsurance policy yields the following expression for the net premium for the layer from \(R\) to infinity

\[
\Pi(R) := E \left[(X - R)_+\right] = \int_R^\infty F(x) \, dx.
\]

Taking \(h\) as a retention level, we have

\[
\Pi_n = \Pi(h) = hF(h) \int_1^\infty \frac{F(hx)}{F(h)} \, dx.
\]

After noticing that the finite mean assumption yields that \(\gamma_1 < 1\), we use the first-order regular variation condition (1.1) together with Potter’s inequalities, to get

\[
\Pi_n \sim \frac{\gamma_1}{1 - \gamma_1} hF(h), \quad \text{as } n \to \infty, \quad 0 < \gamma_1 < 1.
\]

Let

\[
F_n(x) := 1 - \prod_{Z_{i:n} \leq x} \left[1 - \frac{\delta_{i:n}}{n - i + 1}\right]
\]

be the well-known Kaplan-Meier estimator (Kaplan and Meier, 1958) of cdf \(F\). Then, by replacing \(\gamma_1\), \(h\) and \(F(h)\) by their respective estimates \(\hat{\gamma}_1^{(H,c)}\), \(Z_{n-k:n}\) and

\[
1 - F_n(Z_{n-k:n}) = \prod_{i=1}^{n-k} \left(1 - \delta_{i:n}/(n - i + 1)\right),
\]

be the well-known Kaplan-Meier estimator (Kaplan and Meier, 1958) of cdf \(F\). Then, by replacing \(\gamma_1\), \(h\) and \(F(h)\) by their respective estimates \(\hat{\gamma}_1^{(H,c)}\), \(Z_{n-k:n}\) and
we define our estimator of \( \Pi_n \) as follows

\[
\hat{\Pi}_n := \frac{\hat{\gamma}_{1(H,c)}^{(l)}}{1 - \hat{\gamma}_{1(H,c)}} Z_{n-k} \prod_{i=1}^{n-k} \left( 1 - \frac{\delta_{[i,n]}}{n - i + 1} \right).
\] (3.10)

The asymptotic normality of \( \hat{\Pi}_n \) is established in the following theorem.

**Theorem 3.1.** Assume that the assumptions of Theorem 2.1 hold with \( \gamma_1 < 1 \) and that both cdf’s \( F \) and \( G \) are absolutely continuous, then

\[
\frac{\sqrt{k} (\hat{\Pi}_n - \Pi_n)}{hF(h)} = - \frac{p\gamma_1^2}{1 - \gamma_1} \sqrt{n} B_n^* \left( \frac{k}{n} \right) + \frac{\gamma_1}{(1 - \gamma_1)^2} \left\{ \sqrt{n} \int_0^1 s^{-1} B_n^* \left( \frac{k}{n} s \right) ds - p^{-1} \sqrt{n} B_n^* \left( \frac{k}{n} \right) \right\} + o_p(1),
\]

where \( B_n \) and \( B_n^* \) are those defined in Theorem 2.1.

**Corollary 3.1.** Under the assumptions of Theorem 3.1, we have

\[
\frac{\sqrt{k} (\hat{\Pi}_n - \Pi_n)}{hF(h)} \overset{d}{\to} N \left( 0, \sigma_\Pi^2 \right), \text{ as } n \to \infty,
\]

where

\[
\sigma_\Pi^2 := \frac{p\gamma_1^2}{(1 - \gamma_1)^2} \left[ p\gamma_1^2 + \frac{1}{(1 - \gamma_1)^2} \right], \text{ for } \gamma_1 < 1.
\]

4. **Proofs**

We begin by a brief introduction on some uniform empirical processes under random censoring. The empirical counterparts of \( H_j \) (\( j = 0, 1 \)) are defined, for \( z \geq 0 \), by

\[
H_n^j(z) := \# \{ i : 1 \leq i \leq n, Z_i \leq z, \delta_i = j \} / n, \ j = 0, 1.
\]

In the sequel, we will use the following two empirical processes

\[
\sqrt{n} \left( \overline{H}_n^j(z) - \overline{H}^j(z) \right), \ j = 0, 1; \ z > 0,
\]

which may be represented, almost surely, by a uniform empirical process. Indeed, let us define, for each \( i = 1, ..., n \), the following rv

\[
U_i := \delta_i H^1(Z_i) + (1 - \delta_i) \left( \theta + H^0(Z_i) \right).
\]

From Einmahl and Koning (1992), the rv's \( U_1, ..., U_n \) are iid (0,1)-uniform. The empirical cdf and the uniform empirical process based upon \( U_1, ..., U_n \) are respectively denoted by

\[
\mathbb{U}_n(s) := \# \{ i : 1 \leq i \leq n, U_i \leq s \} / n \text{ and } \alpha_n(s) := \sqrt{n} (\mathbb{U}_n(s) - s), \ 0 \leq s \leq 1.
\]
Deheuvels and Einmahl (1996) state that almost surely
\[ H_n^0(z) = \mathbb{U}_n \left( H^0(z) + \theta \right) - \mathbb{U}_n(\theta), \text{ for } 0 < H^0(z) < 1 - \theta, \]
and
\[ H_n^1(z) = \mathbb{U}_n \left( H^1(z) \right), \text{ for } 0 < H^1(z) < \theta. \]

It is easy to verify that almost surely
\[ \sqrt{n} \left( \overline{\mathbb{P}}^0_n(z) - \overline{\mathbb{P}}^1(z) \right) = \alpha_n(\theta) - \alpha_n \left( \theta - \overline{\mathbb{P}}^1(z) \right), \text{ for } 0 < \overline{\mathbb{P}}^1(z) < \theta, \quad (4.11) \]
and
\[ \sqrt{n} \left( \overline{\mathbb{P}}^0_n(z) - \overline{\mathbb{P}}^1(z) \right) = -\alpha_n \left( 1 - \overline{\mathbb{P}}^0(z) \right), \text{ for } 0 < \overline{\mathbb{P}}^0(z) < 1 - \theta. \quad (4.12) \]

Our methodology strongly relies on the well-known Gaussian approximation given by Csörgő et al. (1986): on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\), there exists a sequence of Brownian bridges \(\{B_n(s); 0 \leq s \leq 1\}\) such that for every \(0 \leq \xi < 1/4\)
\[ \sup_{1/n \leq s \leq 1-1/n} \left| \alpha_n(s) - B_n(s) \right| = O_p(n^{-\xi}), \text{ as } n \to \infty. \quad (4.13) \]

The following processes will be crucial to our needs:
\[ \beta_n(z) := \sqrt{\frac{n}{k}} \left\{ \alpha_n(\theta) - \alpha_n \left( \theta - \overline{\mathbb{P}}^1(zZ_{n-k:n}) \right) \right\}, \text{ for } 0 < \overline{\mathbb{P}}^1(z) < \theta \quad (4.14) \]
and
\[ \tilde{\beta}_n(z) := -\sqrt{\frac{n}{k}} \alpha_n \left( 1 - \overline{\mathbb{P}}^0(zZ_{n-k:n}) \right), \text{ for } 0 < \overline{\mathbb{P}}^0(z) < 1 - \theta. \quad (4.15) \]

### 4.1. Proof of Theorem 2.1
First, observe that
\[ \frac{Z_{n-k:n}}{h} = \frac{H^{-1}(H(Z_{n-k:n}))}{H^{-1}(H_n(Z_{n-k:n}))}. \]
Let \(x_n := \overline{\mathbb{P}}(Z_{n-k:n})/\overline{\mathbb{P}}_n(Z_{n-k:n})\) and \(t_n := \overline{\mathbb{P}}_n(Z_{n-k:n}) = k/n.\) By using the second-order regular variation condition (1.7) we get
\[ \frac{H^{-1}(H(Z_{n-k:n}))}{H^{-1}(H_n(Z_{n-k:n}))} - x_n^{-\gamma} \approx A_3(k/n) x_n^{-\gamma} \frac{x_n^{\tau_3} - 1}{\tau_3}. \]
Since \(x_n \approx 1,\) it follows that \(x_n^{-\gamma} \frac{x_n^{\tau_3} - 1}{\tau_3}\) tends in probability to zero. This means that
\[ \frac{H^{-1}(H(Z_{n-k:n}))}{H^{-1}(H_n(Z_{n-k:n}))} = \left( \frac{\overline{\mathbb{P}}(Z_{n-k:n})}{\overline{\mathbb{P}}_n(Z_{n-k:n})} \right)^{-\gamma} + o_p(A_3(k/n)). \]
Using the mean value theorem, we get
\[ \left( \frac{\overline{\mathbb{P}}(Z_{n-k:n})}{\overline{\mathbb{P}}_n(Z_{n-k:n})} \right)^{-\gamma} - 1 = -\gamma c_n \left( \frac{\overline{\mathbb{P}}(Z_{n-k:n})}{\overline{\mathbb{P}}_n(Z_{n-k:n})} - 1 \right), \]
where \( c_n \) is a sequence of rv’s lying between 1 and \((\overline{\Pi}(Z_{n-k:n})/\overline{\Pi}_n(Z_{n-k:n}))^{-\gamma - 1}\). Since \( c_n \approx 1 \), then
\[
\left( \frac{\overline{H}(Z_{n-k:n})}{\overline{H}_n(Z_{n-k:n})} \right)^{-\gamma} - 1 \approx -\gamma \left( \frac{\overline{H}(Z_{n-k:n})}{\overline{H}_n(Z_{n-k:n})} - 1 \right).
\]

By assumption we have \( \sqrt{k} A_3 (k/n) \to \lambda < \infty \), then
\[
\sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right) = -\gamma \sqrt{k} \left( \frac{\overline{H}(Z_{n-k:n})}{\overline{H}_n(Z_{n-k:n})} - 1 \right) + o_p(1).
\]

We have \( \overline{H}_n(Z_{n-k:n}) = k/n \), then
\[
\sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right) = \gamma \sqrt{k} \frac{n}{k} \left( \overline{\Pi}_n(Z_{n-k:n}) - \overline{\Pi}(Z_{n-k:n}) \right) + o_p(1).
\]

which may be decomposed into
\[
\gamma \sqrt{k} \frac{n}{k} \left( \left( \overline{\Pi}_n(Z_{n-k:n}) - \overline{\Pi}^1(Z_{n-k:n}) \right) + \left( \overline{\Pi}_n(Z_{n-k:n}) - \overline{\Pi}^0(Z_{n-k:n}) \right) \right) + o_p(1).
\]

Using (4.14) and (4.15) with \( z = 1 \), leads to
\[
\sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right) = \gamma \left( \beta_n(1) + \tilde{\beta}_n(1) \right) + o_p(1). \tag{4.16}
\]

Now, we apply assertions (i) and (ii) of Lemma 5.2 to complete the proof of the first result of the theorem.

For the second result of the theorem, observe that
\[
\hat{p} = \frac{n}{k} \overline{\Pi}_n^1(Z_{n-k:n}),
\]
then consider the following decomposition
\[
\hat{p} - p = \frac{n}{k} \left( \overline{\Pi}_n(Z_{n-k:n}) - \overline{\Pi}^1(Z_{n-k:n}) \right) \tag{4.17}
+ \frac{n}{k} \left( \overline{\Pi}_n^1(Z_{n-k:n}) - \overline{\Pi}^1(h) \right) + \left( \frac{n}{k} \overline{\Pi}^1(h) - p \right).
\]

Notice that from (4.14), almost surely, we have
\[
\frac{n}{k} \left( \overline{\Pi}_n(Z_{n-k:n}) - \overline{\Pi}^1(Z_{n-k:n}) \right) = \frac{1}{\sqrt{k}} \beta_n(1). \tag{4.18}
\]

The second term in the right-hand side of (4.17) may be written as
\[
\frac{n}{k} \left( \overline{\Pi}^1(Z_{n-k:n}) - \overline{\Pi}^1(h) \right) = \frac{n}{k} \overline{\Pi}^1(h) \left( \frac{\overline{\Pi}^1(Z_{n-k:n})}{\overline{\Pi}^1(h)} - 1 \right). \tag{4.19}
\]

Making use of Lemma 5.1, with \( z = 1 \) and \( z = Z_{n-k:n}/h \), we respectively get as
\( n \to \infty \)
\[
\frac{n}{k} \overline{\Pi}^1(h) = p + O(A(h)) \quad \text{and} \quad \frac{n}{k} \overline{\Pi}^1(Z_{n-k:n}) = p \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma} + O_p(A(h)), \tag{4.20}
\]
where \( A (h) \), defined later on in Lemma 5.1, is a sequence tending to zero as \( n \to \infty \).

It follows that
\[
\frac{\overline{H}^i (Z_{n-k:n})}{\overline{H}^i (h)} - 1 = \left( \frac{p}{p + O_p (A (h))} \right) \left( (Z_{n-k:n}/h)^{-1/\gamma} - 1 \right) + \frac{O_p (A (h))}{p + O_p (A (h))}.
\]

Putting things in a simple way, we have, since \( A (h) = o (1) \),
\[
\frac{p}{p + O_p (A (h))} = 1 + o_p (1) \quad \text{and} \quad \frac{O_p (A (h))}{p + O_p (A (h))} = O_p (A (h)).
\]

Therefore
\[
\frac{\overline{H}^i (Z_{n-k:n})}{\overline{H}^i (h)} - 1 = (1 + o_p (1)) \left( (Z_{n-k:n}/h)^{-1/\gamma} - 1 \right) + O_p (A (h)).
\]

Recalling (4.19) and using \( \overline{H}^i (h) \) from (4.20), we get
\[
\frac{n}{k} \left( \overline{H}^i (Z_{n-k:n}) - \overline{H}^i (h) \right) = p \left( \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma} - 1 \right) (1 + o (1)) + O_p (A (h))
\]

By applying the mean value theorem and using the fact that \( Z_{n-k:n}/h \approx 1 \), we readily verify that
\[
\left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma} - 1 \approx - \frac{1}{\gamma} \left( \frac{Z_{n-k:n}}{h} - 1 \right).
\]

Hence
\[
\frac{n}{k} \left( \overline{H}^i (Z_{n-k:n}) - \overline{H}^i (h) \right) = - \frac{p}{\gamma} \left( \frac{Z_{n-k:n}}{h} - 1 \right) (1 + o (1)) + O_p (A (h)). \tag{4.21}
\]

From the assumptions on the functions \( A_1 \) and \( A_2 \), we have \( \sqrt{k} A (h) \to 0 \). By combining (4.16) and (4.21), we obtain
\[
\sqrt{k} \frac{n}{k} \left( \overline{H}^i (Z_{n-k:n}) - \overline{H}^i (h) \right) = - \frac{p}{\gamma} \left( \beta_n (1) + \tilde{\beta}_n (1) \right) + o_p (1). \tag{4.22}
\]

For the third term in the right-hand side of (4.17), we use conditions (1.6), as in the proof of Lemma 5.1, to have
\[
\sqrt{k} \left( \frac{n}{k} \overline{H}^i (h) - p \right) \sim \frac{pq}{\gamma_1} \left( \frac{\sqrt{k} A_1 (h)}{1 - pr_1} + \frac{q\sqrt{k} A_2 (h)}{1 - q\gamma_2} \right), \tag{4.23}
\]

which tends to 0 as \( n \to \infty \) because, by assumption, \( \sqrt{k} A_j (h) \) goes to 0, \( j = 1, 2 \).

Substituting results (4.18), (4.22) and (4.23) in decomposition (4.17), yields
\[
\sqrt{k} (\hat{p} - p) = q \beta_n (1) - p \tilde{\beta}_n (1) + o_p (1). \tag{4.24}
\]

The final form of the second result of the theorem is then obtained by applying assertions \((i)\) and \((ii)\) of Lemma 5.2.
Finally, we focus on the third result of the theorem. It is clear that we have the following decomposition
\[ \sqrt{k} \left( \hat{\gamma}_1^{(H,c)} - \gamma_1 \right) = \frac{1}{\hat{p}} \sqrt{k} \left( \hat{\gamma}_1^H - \gamma \right) - \frac{\gamma_1}{\hat{p}} \sqrt{k} (\hat{p} - p). \] (4.25)

Recall that one way to define Hill’s estimator \( \hat{\gamma}_H \) is to use the limit
\[ \gamma = \lim_{t \to \infty} \int_t^\infty z^{-1} H(z)/H(t)dz. \]
Then, by replacing \( H \) by \( H_n \) and letting \( t = Z_{n-k:n} \), we write
\[ \hat{\gamma}_H = \frac{n}{k} \int_{Z_{n-k:n}}^\infty z^{-1} H_n(z)dz. \]
For details, see for instance, (de Haan and Ferreira, 2006, page 69). Let’s consider the following decomposition \( \hat{\gamma}_H - \gamma = T_{n1} + T_{n2} + T_{n3} \), where
\[ T_{n1} := \frac{n}{k} \int_{Z_{n-k:n}}^\infty z^{-1} \left( \overline{H}_n^0(z) - \overline{H}_n^0(z) + \overline{H}_n^1(z) - \overline{H}_n^1(z) \right)dz, \]
\[ T_{n2} := \frac{n}{k} \int_{Z_{n-k:n}}^h z^{-1} \overline{H}(z)dz \text{ and } T_{n3} := \frac{n}{k} \int_h^\infty z^{-1} \overline{H}(z)dz - \gamma. \]
We use the integral convention that \( \int_a^b = \int_{(a,b)} \) as integration is with respect to the measure induced by a right-continuous function. Making a change of variables in the first term \( T_{n1} \) and using the uniform empirical representations of \( \overline{H}_n^0 \) and \( \overline{H}_n^1 \), we get almost surely
\[ \sqrt{k}T_{n1} = \int_1^\infty z^{-1} \left( \beta_n(z) + \tilde{\beta}_n(z) \right)dz. \]
For the second term \( T_{n2} \), we apply the mean value theorem to have
\[ T_{n2} = \frac{\overline{H}(z_n^*)}{z_n^*} \frac{n}{k} (h - Z_{n-k:n}), \]
where \( z_n^* \) is a sequence of rv’s between \( Z_{n-k:n} \) and \( h \). It is obvious that \( z_n^* \approx h \), this implies that \( \overline{H}(z_n^*) \approx k/n \). It follows that the right-hand side of the previous equation is \( \approx -(Z_{n-k:n}/h - 1). \) Hence, from (4.16), we have
\[ \sqrt{k}T_{n2} = -\gamma \left( \beta_n(1) + \tilde{\beta}_n(1) \right) + o_p(1). \]
Finally, for \( T_{n3} \), we use the second-order conditions (1.6) to get
\[ \sqrt{k}T_{n3} \sim p^2 \frac{\sqrt{k}A_1(h)}{1 - pr_1} + q^2 \frac{\sqrt{k}A_2(h)}{1 - qr_2}. \] (4.26)
We achieve the proof of the third result of the theorem by using assertions (i) and (ii) of Lemma 5.2.

4.2. Proof of Corollary 2.1. From the third result of Theorem 2.1, we deduce that \( \sqrt{k} (\hat{\gamma}_1^{(H,c)} - \gamma_1) \) is asymptotically centred Gaussian with variance

\[
\sigma^2 = \gamma_1^2 \lim_{n \to \infty} \mathbb{E} \left[ \sqrt{\frac{n}{k}} \int_0^1 s^{-1} \Phi_n^* (psk/n) ds - \frac{1}{p} \sqrt{\frac{n}{k}} \beta_n (k/n) \right]^2.
\]

We check that the processes \( \mathbb{B}_n (s) \), \( \tilde{\mathbb{B}}_n (s) \) and \( \mathbb{B}_n^* (s) \) satisfy

\[
\frac{p^{-1} \mathbb{E} [\mathbb{B}_n (s) \mathbb{B}_n (t)] - \min (s, t) - \text{pst}}{q^{-1} \mathbb{E} [\tilde{\mathbb{B}}_n (s) \tilde{\mathbb{B}}_n (t)] - \min (s, t) - \text{qst}} = \frac{p^{-1} \mathbb{E} [\mathbb{B}_n^* (s) \mathbb{B}_n^* (t)] - \min (s, t) - \text{st}}{\mathbb{B}_n (s) \mathbb{B}_n (t)} = \frac{p^{-1} \mathbb{E} [\mathbb{B}_n^* (s) \mathbb{B}_n (t)] - \min (s, t) - \text{st}}{\mathbb{B}_n^* (s) \mathbb{B}_n (t)}.
\]

Then, by elementary calculation (we omit details), we get \( \sigma^2 = p \gamma_1^2 \). \( \square \)

4.3. Proof of Theorem 3.1. First, recall that

\[
\Pi_n = \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} \frac{Z_{n-k:n} (1 - F_n (Z_{n-k:n}))}{1 - \hat{\gamma}_1^{(H,c)}}.
\]

Observe that we have the following decomposition

\[
\frac{\hat{\Pi}_n - \Pi_n}{h \mathcal{F} (h)} = \sum_{i=1}^{6} S_{ni},
\]

where

\[
S_{n1} := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} \frac{Z_{n-k:n}}{h} \left\{ \frac{1 - F_n (Z_{n-k:n})}{\mathcal{F} (h)} - \frac{\mathcal{F} (Z_{n-k:n})}{\mathcal{F} (h)} \right\},
\]

\[
S_{n2} := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} \frac{Z_{n-k:n}}{h} \left\{ \frac{\mathcal{F} (Z_{n-k:n})}{\mathcal{F} (h)} - \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} \right\},
\]

\[
S_{n3} := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} \frac{Z_{n-k:n}}{h} \left\{ \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} - 1 \right\},
\]

\[
S_{n4} := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} \frac{Z_{n-k:n}}{h},
\]

\[
S_{n5} := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} - \frac{\gamma_1}{1 - \gamma_1} \text{ and } S_{n6} := \frac{\gamma_1}{1 - \gamma_1} - \frac{\Pi_n}{h \mathcal{F} (h)}.
\]
Since $Z_{n-k:n} \approx h$ and $\hat{\gamma}_1^{(H,c)} \approx \gamma_1$, then

$$S_{n1} \approx -\frac{\gamma_1}{1-\gamma_1} \frac{F_n(Z_{n-k:n}) - F(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})}. \tag{4.29}$$

In view of Proposition 5 in Csörgő (1996), we have for any $x \leq Z_{n-k:n}$,

$$\frac{F_n(x) - F(x)}{\overline{F}(x)} = \frac{H_n^1(x) - H^1(x)}{\overline{H}(x)} - \int_0^x \frac{H_n^1(z) - H^1(z)}{\overline{H}^2(z)} dH(z) \tag{4.30}$$

and recall that from representations (4.11) and (4.12), we have

$$\sqrt{n} \left( \overline{H}_n^1(z) - \overline{H}^1(z) \right) = \alpha_n \left( \theta - \overline{H}^1(z) \right),$$

$$\sqrt{n} \left( \overline{H}_n^0(z) - \overline{H}^0(z) \right) = \alpha_n \left( \theta - \overline{H}^1(z) \right),$$

and

$$\sqrt{n} \left( \overline{H}_n^0(z) - \overline{H}^0(z) \right) = -\alpha_n \left( 1 - \overline{H}^0(z) \right). \tag{4.31}$$

It follows, from (4.31), that

$$\sqrt{n} \left( \overline{H}_n(z) - \overline{H}(z) \right) = \left( \alpha_n \left( \theta - \overline{H}^1(z) \right) \right) - \alpha_n \left( 1 - \overline{H}^0(z) \right).$$

By using the above representations in (4.30), we obtain

$$\sqrt{n} \frac{F_n(x) - F(x)}{\overline{F}(x)} = \frac{\alpha_n \left( \theta - \overline{H}^1(z) \right)}{\overline{H}(x)} - \int_0^x \frac{\alpha_n \left( \theta - \overline{H}^1(z) \right)}{\overline{H}^2(z)} dH(z)$$

$$- \int_0^x \frac{\alpha_n \left( \theta - \overline{H}^1(z) \right) - \alpha_n \left( 1 - \overline{H}^0(z) \right)}{\overline{H}^2(z)} d\overline{H}^1(z) + O_p \left( \frac{\sqrt{n}}{k} \right).$$

By writing

$$\alpha_n \left( \theta - \overline{H}^1(z) \right) = \alpha_n \left( \theta - \left( \alpha_n \left( \theta - \overline{H}^1(z) \right) \right) \right),$$

it is easy to check that

$$\int_0^x \frac{\alpha_n \left( \theta - \overline{H}^1(z) \right)}{\overline{H}^2(z)} dH(z) = \frac{\alpha_n \left( \theta \right)}{\overline{H}(x)} - \int_0^x \frac{\alpha_n \left( \theta - \overline{H}^1(z) \right)}{\overline{H}^2(z)} dH(z),$$
and therefore

\[
\frac{\sqrt{n} F_n(x) - F(x)}{F(x)} = -\frac{\alpha_n(\theta) - \alpha_n(\theta - \bar{H}^1(x))}{\bar{H}(x)} + \int_0^x \frac{\alpha_n(\theta) - \alpha_n(\theta - \bar{H}^1(x))}{\bar{H}^2(z)} dH(z)
\]

\[
- \int_0^x \frac{\alpha_n(\theta) - \alpha_n(\theta - \bar{H}^1(z)) - \alpha_n(1 - \bar{H}^0(z))}{\bar{H}^2(z)} dH^1(z) + O_p\left(\frac{\sqrt{n}}{k}\right).
\]

By multiplying both sides of the previous equation by \(\sqrt{k/n}\), then by using the Gaussian approximation (4.13), in \(x = Z_{n-k:n}\), we get

\[
\frac{\sqrt{k} F_n(Z_{n-k:n}) - F(Z_{n-k:n})}{F(Z_{n-k:n})} = -\sqrt{\frac{n}{k}} B_n(Z_{n-k:n}) + \sqrt{\frac{k}{n}} \int_0^{Z_{n-k:n}} \frac{B_n(z)}{\bar{H}^2(z)} dH(z)
\]

\[
- \sqrt{\frac{k}{n}} \int_0^{Z_{n-k:n}} \frac{B^*_n(z)}{\bar{H}^2(z)} dH^1(z) + O_p\left(\frac{1}{\sqrt{k}}\right),
\]

where \(B_n(z)\) and \(B^*_n(z)\) are two Gaussian processes defined by

\[
B_n(z) := B_n(\theta) - B_n(\theta - \bar{H}^1(z)) \quad \text{and} \quad B^*_n(z) := B_n(z) - B_n(1 - \bar{H}^0(z)).
\]

The assertions of Lemma 5.3 and the fact that \(1/\sqrt{k} \to 0\) yield

\[
\frac{\sqrt{k} F_n(Z_{n-k:n}) - F(Z_{n-k:n})}{F(Z_{n-k:n})} = -\sqrt{\frac{n}{k}} B_n(Z_{n-k:n}) + \sqrt{\frac{k}{n}} \int_0^{h} \frac{B_n(z)}{\bar{H}^2(z)} dH(z) - \sqrt{\frac{k}{n}} \int_0^{h} \frac{B^*_n(z)}{\bar{H}^2(z)} dH^1(z) + o_p(1).
\]

Applying the results of Lemma 5.4 leads to

\[
\frac{\sqrt{k} F_n(Z_{n-k:n}) - F(Z_{n-k:n})}{F(Z_{n-k:n})} = -p \sqrt{\frac{n}{k}} B^*_n\left(\frac{k}{n}\right) + o_p(1),
\]

which in turn implies that

\[
\sqrt{k} S_{n1} \approx \frac{\gamma_1 p}{1 - \gamma_1} \sqrt{\frac{n}{k}} B^*_n\left(\frac{k}{n}\right).
\]

In view of the second-order regular variation condition (1.6) for \(F\), we have \(\sqrt{k} S_{n2} \approx \frac{\gamma_2}{1 - \gamma_1} \sqrt{k} A_1(h)\), which, by assumption tends to 0. As for the term \(S_{n3}\), we use Taylor’s expansion and the fact that \(Z_{n-k:n} \approx h\), to get

\[
\sqrt{k} S_{n3} \approx -\frac{1}{1 - \gamma_1} \sqrt{k} \left(\frac{Z_{n-k:n}}{h} - 1\right).
\]
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By using Theorem 2.1 we get

\[ \sqrt{kS_{n3}} \approx -\frac{\gamma}{1 - \gamma_1} \sqrt{\frac{n}{k}} B_n^* \left( \frac{k}{n} \right). \]

Similar arguments, applied to \( S_{n4} \), yield

\[ \sqrt{kS_{n4}} \approx -\frac{\gamma_1\gamma}{1 - \gamma_1} \sqrt{\frac{n}{k}} B_n^* \left( \frac{k}{n} \right). \]

In view of the consistency of \( \hat{\gamma}_{1(H,c)} \), it is easy to verify that

\[ \sqrt{kS_{n5}} \approx \frac{1}{(1 - \gamma_1)^2} \sqrt{k} \left( \hat{\gamma}_{1(H,c)} - \gamma_1 \right). \]

Once again by using Theorem 2.1, we get

\[ \sqrt{kS_{n6}} \approx \frac{\gamma_1}{(1 - \gamma_1)^2} \left\{ \sqrt{\frac{n}{k}} \int_0^1 s^{-1} B_n^* \left( \frac{k}{n} s \right) ds - \frac{1}{p} \sqrt{\frac{n}{k}} B_n \left( \frac{k}{n} \right) \right\}. \]

For the term \( S_{n6} \), we write

\[ \sqrt{kS_{n6}} = -\sqrt{k} \int_1^\infty \left( \frac{F'(hx)}{F'(h)} - x^{-1/\gamma_1} \right) dx, \]

and we apply the uniform inequality of regularly varying functions (see, e.g., Theorem B.2.18 in de Haan and Ferreira, 2006, page 383) to show that \( \sqrt{kS_{n6}} \approx -\sqrt{k} A_1 (h) \to 0 \), as \( n \to \infty \). Finally, combining the results above, on all six terms \( S_{ni} \), achieves the proof.

4.4. Proof of Corollary 3.1. It is clear that \( \sqrt{k} \left( \hat{\Pi} - \Pi_n \right) / \left( hF'(h) \right) \) is an asymptotically centred Gaussian rv. By using the covariance formulas and after elementary calculation we show that its asymptotic variance equals

\[ \frac{p\gamma_1^2}{(1 - \gamma_1)^2} \left[ \frac{p\gamma_1^2 + \frac{1}{(1 - \gamma_1)^2}}{1} \right]. \]

Concluding notes

The primary object of the present work consists in providing a Gaussian limiting distribution for the estimator of the shape parameter of a heavy-tailed distribution, under random censorship. Our approach is based on the approximation of the uniform empirical process by a sequence of Brownian bridges. The Gaussian representation will be of great use in the statistical inference on quantities related to the tail index in the context of censored data, such as high quantiles, risk measures,...

It is noteworthy that for \( p = 1 \) (which corresponds to the non censoring case), our main result (number three of Theorem 2.1) perfectly agrees with the Gaussian approximation of the classical Hill estimator, given in Section 1. On the other hand,
the variance we obtain in Corollary 2.1 is the same as that given by Einmahl et al. (2008).

5. Appendix

Lemma 5.1. Assume that conditions (1.6) hold and let $k := k_n$ be an integer sequence satisfying (1.2), then for $z \geq 1$, we have

$$\frac{n}{k} \frac{H_i}{H} (zh) = pz^{-1/\gamma} + O (A(h)), \quad n \to \infty,$$

where $A(h) := A_1(h) + A_2(h) + A_1(h) A_2(h)$.

Proof. Let $z \geq 1$ and recall that $\frac{H_i}{H} (z) = - \int_z^\infty G(x) dF(x)$. It is clear that

$$\frac{H_i}{H} (zh) = - \int_z^\infty \frac{G(xh)}{G(h)} d\frac{F(xh)}{F(h)}.$$

Since $\frac{H_i}{H} (zh) = \frac{G(h)}{G(h)} \frac{F(xh)}{F(h)}$, then

$$\frac{H_i}{H} (zh) = - \int_z^\infty \frac{G(xh)}{G(h)} \frac{F(xh)}{F(h)} d \frac{F(xh)}{F(h)}.$$

It is easy to verify that

$$\frac{H_i}{H} (zh) = - \int_z^\infty \left( \frac{G(xh)}{G(h)} - x^{-1/\gamma_2} \right) d \left( \frac{F(xh)}{F(h)} - x^{-1/\gamma_1} \right)$$

$$- \int_z^\infty \left( \frac{G(xh)}{G(h)} - x^{-1/\gamma_2} \right) d x^{-1/\gamma_1} - \int_z^\infty x^{-1/\gamma_2} d \left( \frac{F(xh)}{F(h)} - x^{-1/\gamma_1} \right)$$

$$- \int_z^\infty x^{-1/\gamma_2} d x^{-1/\gamma_1}.$$

For the purpose of using the second-order regular variation conditions (1.6), we write

$$\frac{H_i}{H} (zh) = - A_1(h) A_2(h) \int_z^\infty \frac{G(xh)}{G(h)} - x^{-1/\gamma_2} \frac{F(xh)}{F(h)} - x^{-1/\gamma_1}$$

$$d \left( \frac{F(xh)}{F(h)} - x^{-1/\gamma_1} \right)$$

$$- A_2(h) \int_z^\infty \frac{G(xh)}{G(h)} - x^{-1/\gamma_2} d x^{-1/\gamma_1} - A_1(h) \int_z^\infty x^{-1/\gamma_2} d \frac{F(xh)}{F(h)} - x^{-1/\gamma_1}$$

$$- \int_z^\infty x^{-1/\gamma_2} d x^{-1/\gamma_1}.$$

Next, we apply the uniform inequality of regularly varying functions (see, e.g., Theorem B. 2.18 in de Haan and Ferreira, 2006, page 383). For all $\epsilon, \omega > 0$, there exists $t_1$ such that for $hx \geq t_1$:

$$\left| \frac{F(xh)}{A_1(h)} - x^{-1/\gamma_1} \frac{x^{\tau_1} - 1}{\tau_1} \right| \leq \epsilon x^{-1/\gamma_1} \max \left( x^{\omega}, x^{-\omega} \right).$$
Likewise, there exists $t_2$ such that for $hx \geq t_2$:
\[
\left| \frac{G(xh) - x^{-1/\gamma_2}}{A_2(h)} - x^{-1/\gamma_2}x^{\gamma_2} - \frac{1}{\gamma_2} \right| \leq \varepsilon x^{-1/\gamma_2} \max \left( x^\omega, x^{-\omega} \right).
\]

Making use of the previous two inequalities and noting that $H(h) = k/n$ and
\[
- \int_{z}^{\infty} x^{-1/\gamma_2} dx^{-1/\gamma} = \nu z^{-1/\gamma}
\]
achieve the proof. □

**Lemma 5.2.** In addition to the assumptions of Lemma 5.1, suppose that both cdf’s $F$ and $G$ are absolutely continuous, then for $z \geq 1$, we have

(i) $\beta_n(z) = \sqrt{\frac{n}{k}} B_n \left( \theta \frac{k}{n} z^{-\gamma} \right) + o_p(1)$

(ii) $\tilde{\beta}_n(z) = \sqrt{\frac{n}{k}} \tilde{B}_n \left( \theta \frac{k}{n} z^{-\gamma} \right) + o_p(1)$

(iii) $\int_{1}^{\infty} z^{-1} \left( \beta_n(z) + \tilde{\beta}_n(z) \right) dz = \gamma \sqrt{\frac{n}{k}} \int_{0}^{1} s^{-1} B^*_n \left( \frac{k}{n} s \right) ds + o_p(1)$.

**Proof.** Let’s begin by assertion (i). A straightforward application of the weak approximation (4.13) yields
\[
\beta_n(z) = \sqrt{\frac{n}{k}} \left\{ B_n \left( \theta - \theta_0 \right) - B_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) \right\} + o_p(1).
\]

Then we have to show that
\[
\sqrt{\frac{n}{k}} \left\{ B_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) - B_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) \right\} = o_p(1).
\]
Indeed, let $\{W_n(t) ; 0 \leq t \leq 1\}$ be a sequence of Wiener processes defined on $(\Omega, \mathcal{A}, \mathbb{P})$ so that
\[
\{B_n(t) ; 0 \leq t \leq 1\} \overset{d}{=} \{W_n(t) - tW_n(1) ; 0 \leq t \leq 1\}.
\]

Then without loss of generality, we write
\[
\sqrt{\frac{n}{k}} \left\{ B_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) - B_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) \right\} = \sqrt{\frac{n}{k}} \left\{ W_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) - W_n \left( \theta - \frac{k}{n} \frac{z}{z} \right) \right\}
\]
\[
- \sqrt{\frac{n}{k}} \left( \frac{k}{n} \frac{z}{z} - \frac{k}{n} \frac{z}{z} \right) W_n(1).
\]

Let $z \geq 1$ be fixed and recall that $\theta (z) \approx \frac{z}{k}/n$, then it is easy to verify that the second term of the previous quantity tends to zero (in probability) as
$n \to \infty$. Next we show that the first one also goes to zero in probability. For given $0 < \eta < 1$ and $0 < \varepsilon < 1$ small enough, we have for all large $n$

\[
\mathbb{P}\left( \left\| \frac{\sqrt{1} (z Z_{n-k:n})}{z^{-\gamma} k/n} - 1 \right\| > \eta^2 \frac{\varepsilon^2}{4z^\gamma} \right) < \varepsilon/2.
\]

Observe now that

\[
\mathbb{P}\left( \sqrt{\frac{n}{k}} W_n \left( \theta - \frac{k}{n} z^{-\gamma} \right) > \eta \right) = \mathbb{P}\left( \left| \sqrt{\frac{n}{k}} W_n \left( \frac{k}{n} z^{-\gamma} - \frac{k}{n} z^{-\gamma} \right) \right| > \eta \right)
\]

\[
\leq \mathbb{P}\left( \left| \frac{\sqrt{1} (z Z_{n-k:n})}{z^{-\gamma} k/n} - 1 \right| > \eta^2 \frac{\varepsilon^2}{4z^\gamma} \right) + \mathbb{P}\left( \sup_{0 \leq t \leq \sqrt{\frac{2}{T}} n} W_n (t) > \eta \sqrt{k/n} \right).
\]

It is clear that the first term of the latter expression tends to zero as $n \to \infty$. On the other hand, since $\{W_n (t) ; 0 \leq t \leq 1\}$ is a martingale then by using the classical Doob inequality we have, for any $u > 0$ and $T > 0$

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} W_n (t) > u \right) \leq \mathbb{P}\left( \sup_{0 \leq t \leq T} |W_n (t)| > u \right) \leq \frac{\mathbb{E}|W_n (T)|}{u} \leq \frac{\sqrt{T}}{u}.
\]

Letting $T = \eta^2 \frac{\varepsilon^2}{4z^\gamma} k/n$ and $u = \eta \sqrt{k/n}$, yields

\[
\mathbb{P}\left( \sup_{0 \leq t \leq \eta^2 \frac{\varepsilon^2}{4z^\gamma} n} W_n (t) > \eta \sqrt{k/n} \right) \leq \varepsilon/2,
\]

which completes the proof of assertion (i). The proof of assertion (ii) follows by similar arguments. For assertion (iii), let us write

\[
\int_{1}^{\infty} z^{-1} \left( \beta_n (z) + \tilde{\beta}_n (z) \right) dz
\]

\[
= \sqrt{\frac{n}{k}} \int_{Z_{n-k:n}} z^{-1} \left( \alpha_n (\theta) - \alpha_n \left( \theta - \frac{k}{n} z^{-\gamma} \right) - \alpha_n \left( 1 - \frac{1}{k} \right) \right) dz,
\]

which may be decomposed into $T_{n1}^{(1)} + T_{n1}^{(2)} + T_{n1}^{(3)}$ where

\[
T_{n1}^{(1)} := \sqrt{\frac{n}{k}} \int_{h}^{H^{-1}(1-1/n)} z^{-1} \left( \alpha_n (\theta) - \alpha_n \left( \theta - \frac{k}{n} z^{-\gamma} \right) - \alpha_n \left( 1 - \frac{1}{k} \right) \right) dz,
\]

\[
T_{n1}^{(2)} := \sqrt{\frac{n}{k}} \int_{h}^{H^{-1}(1-1/n)} z^{-1} \left( \alpha_n (\theta) - \alpha_n \left( \theta - \frac{k}{n} z^{-\gamma} \right) - \alpha_n \left( 1 - \frac{1}{k} \right) \right) dz
\]

and

\[
T_{n1}^{(3)} := \sqrt{\frac{n}{k}} \int_{h}^{Z_{n-k:n}} z^{-1} \left( \alpha_n (\theta) - \alpha_n \left( \theta - \frac{k}{n} z^{-\gamma} \right) - \alpha_n \left( 1 - \frac{1}{k} \right) \right) dz.
\]
Once again by using approximation (4.13), we get

$$T_{n1}^{(1)} = \sqrt{\frac{n}{k}} \int_1^{n^{-1(1-1/n)}} z^{-1} \left( B_n(\theta) - B_n(\theta - H^i_1(hz)) - B_n(1 - H^0(hz)) \right) dz + o_p(1).$$

Since $H^{-1}(1 - 1/n)/h \to 0$, then by elementary calculations we show that the latter quantity equals (as $n \to \infty$)

$$\sqrt{\frac{n}{k}} \int_1^0 z^{-1} \left( B_n(\theta) - B_n(\theta - \frac{k}{n}z^{-\gamma}) - B_n(1 - \frac{k}{n}z^{-\gamma}) \right) dz + o_p(1).$$

By a change of variables and inverting the integration limits we end up with

$$\gamma \sqrt{\frac{n}{k}} \int_0^1 s^{-1} \left( B_n(\theta) - B_n(\theta - \frac{k}{n}s) - B_n(1 - \frac{k}{n}s) \right) ds + o_p(1),$$

which equals the right-hand side of equation (i). We have to show that both $T_{n1}^{(2)}$ and $T_{n1}^{(3)}$ tend to zero in probability as $n \to \infty$. Observe that

$$\mathbb{P}\left( \left| T_{n1}^{(2)} \right| > \eta \right) \leq \mathbb{P}(I_n > \eta) + \mathbb{P}\left( \left| \frac{Z_n-k:n}{h} - 1 \right| > \varepsilon \right),$$

where

$$I_n := \sqrt{\frac{n}{k}} \int_h^{(1+\varepsilon)h} z^{-1} \left| \alpha_n(\theta) - \alpha_n(\theta - \mathbf{H}^i(z)) - \alpha_n(1 - \mathbf{H}^0(z)) \right| dz.$$

We already have $\mathbb{P}(\left| Z_n-k:n/h - 1 \right| > \varepsilon) \to 0$, it remains to show that $\mathbb{P}(I_n > \eta) \to 0$ as well. By applying approximation (4.13), we get $I_n = \tilde{I}_n + o_p(1)$, where

$$\tilde{I}_n := \sqrt{\frac{n}{k}} \int_h^{(1+\varepsilon)h} z^{-1} \left| B_n(\theta) - B_n(\theta - \mathbf{H}^i(z)) - B_n(1 - \mathbf{H}^0(z)) \right| dz.$$

Next we show that $\mathbb{P}(\tilde{I}_n > \eta) \to 0$. By letting $B_n^*(z) := B_n(\theta - \mathbf{H}^i(z)) - B_n(1 - \mathbf{H}^0(z))$, we showed that

$$\mathbb{E}\left[ B_n^*(x) B_n^*(y) \right] = \min(\mathbf{H}(x), \mathbf{H}(y)) - \mathbf{H}(x) \mathbf{H}(y),$$

which implies that $\mathbb{E}|B_n^*(z)| \leq \sqrt{\mathbf{H}(z)}$ and since $\mathbf{H}(zh) \sim \frac{k}{n}z^{-\gamma-1}$, then

$$\mathbb{E}\left| \tilde{I}_n \right| \leq \sqrt{\frac{n}{k}} \int_1^{1+\varepsilon} z^{-1} \sqrt{\mathbf{H}(zh)} dz \sim 2\gamma \left( 1 - (1 + \varepsilon)^{-1/2\gamma} \right),$$

which tend to zero as $\varepsilon \downarrow 0$, this means that $\tilde{I}_n \to 0$ in probability. By similar arguments we also show that $T_{n1}^{(3)} \mathbb{P} \to 0$, therefore we omit the details.\(\square\)
Lemma 5.3. Under the assumptions of Lemma 5.1 we have

\[(i) \sqrt{\frac{k}{n}} \int_{h}^{Z_{n-k:n}} \frac{B_n(z)}{H^2(z)} dH(z) = o_p(1),\]

\[(ii) \sqrt{\frac{k}{n}} \int_{h}^{Z_{n-k:n}} \frac{B_n^*(z)}{H^2(z)} dH^1(z) = o_p(1).\]

Proof. We begin by proving the first assertion. To this end let us fix \(\nu > 0\) and write

\[P\left(\left| \sqrt{\frac{k}{n}} \int_{h}^{Z_{n-k:n}} B_n(z) \frac{dH(z)}{H^2(z)} \right| > \nu \right) \leq P\left(\left| \frac{Z_{n-k:n}}{h} - 1 \right| > \nu \right) + P\left(\sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} B_n(z) \frac{dH(z)}{H^2(z)} > \nu \right).

It is clear that the first term of the previous expression tends to zero as \(n \to \infty\). Then we have to show that the second one goes to zero as well. Indeed, observe that

\[E\left| \sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} B_n(z) \frac{dH(z)}{H^2(z)} \right| \leq \sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} E\left| B_n(z) \right| \frac{dH(z)}{H^2(z)}.

Since \(E\left| B_n(z) \right| \leq \sqrt{H^1(z)}\), then the right-hand side of the latter expression is less than or equal to

\[\sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} \sqrt{H^1(z)} \frac{dH(z)}{H^2(z)} \leq \sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} E\left| B_n(z) \right| \frac{dH(z)}{H^2(z)} \leq \sqrt{\frac{k}{n}} \int_{h}^{(1+\nu)h} \frac{1}{H((1+\nu)h)} - \frac{1}{H(h)} \right],

which may be rewritten into

\[\sqrt{\frac{H^1(h)}{H(h)}} \left[ \frac{H(h)}{H((1+\nu)h)} - 1 \right].

Since \(H^1(h) \sim pH(h)\) and \(H \in \mathcal{RV}_{(-\gamma)}\), then the previous quantity tends to

\[p^{1/2} ((1+\nu)^7 - 1)\] as \(n \to \infty\).

Since \(\nu\) is arbitrary then it may be chosen small enough so that the latter quantity tends to zero. By similar arguments we also show assertion \((ii)\), therefore we omit the details. \(\square\)
Lemma 5.4. Under the assumptions of Lemma 5.1 we have, for \( z \geq 1\)

\[
(i) \quad \sqrt{\frac{k}{n}} \int_0^h \frac{B_n(z)}{H^2(z)} \, dH(z) = \sqrt{\frac{n}{k} B_n\left(\frac{k}{n}\right)} + o_p(1),
\]

\[
(ii) \quad \sqrt{\frac{k}{n}} \int_0^h \frac{B_n^*(z)}{H^2(z)} \, dH^1(z) = p \sqrt{\frac{n}{k} B_n^*\left(\frac{k}{n}\right)} + o_p(1),
\]

\[
(iii) \sqrt{\frac{n}{k} B_n(Z_{n-k:n})} = \sqrt{\frac{n}{k} B_n\left(\frac{k}{n}\right)} + o_p(1).
\]

Proof. We only show assertion (i), since (ii) and (iii) follow by similar arguments.

Observe that

\[
\int_0^h \frac{dH(z)}{H(z)} = \frac{1}{H(h)} - 1,
\]

and \( H(h) = k/n \), then

\[
\sqrt{\frac{n}{k} B_n(h)} = \sqrt{\frac{k}{n}} \int_0^h \frac{B_n(h) \, dH(z)}{H^2(z)} + \frac{k}{n}.
\]

Let us write

\[
\sqrt{\frac{k}{n}} \int_0^h \frac{B_n(z) - B_n(h)}{H^2(z)} \, dH(z) = \sqrt{\frac{k}{n}} \int_0^h \frac{B_n(\theta - H^1(h)}{H^2(z)} \, dH(z) + \frac{k}{n}.
\]

We have

\[
\sqrt{\frac{k}{n}} \int_0^h \frac{B_n(z) - B_n(h)}{H^2(z)} \, dH(z) = \sqrt{\frac{k}{n}} \int_0^h \frac{B_n(\theta - H^1(h)}{H^2(z)} \, dH(z).
\]

It is clear that

\[
\sqrt{\frac{k}{n}} \int_0^h \frac{B_n(z) - B_n(h)}{H^2(z)} \, dH(z) = \sqrt{\frac{k}{n}} \int_0^h \frac{B_n(\theta - H^1(h)}{H^2(z)} \, dH(z).
\]

where \( \{W_n(t), 0 \leq t \leq 1\} \) is the sequence of Wiener processes defined in (5.34).
Indeed, it is easy to verify
\[
\mathbb{E} \left| \frac{1}{n} \int_0^h W_n \left( \theta - H^1(h) \right) - W_n \left( \theta - H^1(z) \right) \frac{dH(z)}{H^2(z)} \right|
\]
\[
\leq \sqrt{\frac{k}{n}} \int_0^h \frac{\sqrt{\mathcal{H}^1(z) - \mathcal{H}^1(h)}}{\mathcal{H}^2(z)} dH(z).
\]

It is clear that
\[
\sqrt{\frac{k}{n}} \int_0^h \frac{\sqrt{\mathcal{H}^1(z) - \mathcal{H}^1(h)}}{\mathcal{H}^2(z)} dH(z) = \sqrt{\mathcal{H}(h)} \int_0^h \frac{\sqrt{\mathcal{H}^1(z) - \mathcal{H}^1(h)}}{\mathcal{H}^2(z)} dH(z).
\]

Elementary calculations by using L'Hôpital's rule, we infer that the latter quantity tends to zero as \( n \to \infty \). Likewise, we also show that
\[
\sqrt{\frac{k}{n}} \int_0^h \frac{\sqrt{\mathcal{H}^1(z) - \mathcal{H}^1(h)}}{\mathcal{H}^2(z)} dH(z) \to 0, \quad \text{as} \quad n \to \infty,
\]
which implies that the right side of equation (5.36) goes to zero in probability. It remains to check that
\[
\sqrt{\frac{n}{k}} \mathcal{B}_n(h) = \sqrt{\frac{n}{k}} \mathcal{B}_n \left( \frac{k}{n} \right) + o_p(1).
\]

Recalling
\[
\mathcal{B}_n (h) = B_n (\theta) - B_n \left( \theta - H^1(h) \right) \quad \text{and} \quad \mathcal{B}_n \left( \frac{k}{n} \right) = B_n (\theta) - B_n \left( \theta - \frac{p}{k} \right),
\]
we write
\[
\sqrt{\frac{n}{k}} \left( \mathcal{B}_n (h) - \mathcal{B}_n \left( \frac{k}{n} \right) \right) = \sqrt{\frac{n}{k}} \left( B_n (\theta - pk/n) - B_n \left( \theta - \mathcal{H}^1 (h) \right) \right).
\]

Then, we have to show that this latter tends to zero in probability. By writing \( B_n \) in terms of \( W_n \) as above, it is easy to verify that
\[
\sqrt{\frac{n}{k}} \mathbb{E} \left| B_n (\theta - pk/n) - B_n \left( \theta - H^1(h) \right) \right|
\]
\[
\leq \sqrt{\frac{n}{k}} \mathcal{H}^1(h) - pk/n + \sqrt{\frac{n}{k}} \mathcal{H}^1(h) - pk/n
\]
\[
= \sqrt{\mathcal{H}^1(h) - pk/n} + \sqrt{\frac{k}{n}} \mathcal{H}^1(h) - pk/n,
\]
which converges to zero as \( n \to \infty \), since \( \mathcal{H}^1(h) \approx pk/n \). This achieves the proof. \( \Box \)
References

Beirlant, J., Teugels, J. and Vynckier, P., 1994. Extremes in non-life insurance. In Extreme Value Theory and Applications (ed. J. Galambos), 489-510. Kluwer Academic Publishers.

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J., 2004. Statistics of Extremes - Theory and applications. Wiley.

Beirlant, J., Guillou, A., Dierckx, G., Fils-Villetard, A., 2007. Estimation of the extreme value index and extreme quantiles under random censoring. Extremes. 10, no. 3, 151-174.

Brahimi, B., Meraghni, D., Necir, A. and Zitikis, R., 2011. Estimating the distortion parameter of the proportional-hazard premium for heavy-tailed losses. Insurance Math. Econom. 49, no. 3, 325-334.

Brahimi, B., Meraghni, D., Necir, A. and Yahia, D., 2013. A bias-reduced estimator for the mean of a heavy-tailed distribution with an infinite second moment. J. Statist. Plann. Inference 143, no. 6, 1064-1081.

Csörgő, S. and Mason, D.M., 1985. Central limit theorems for sums of extreme values. Math. Proc. Cambridge Philos. Soc. 98, no. 3, 547-558.

Csörgő, M., Csörgő, S., Horváth, L. and Mason, D.M., 1986. Weighted empirical and quantile processes. Ann. Probab. 14, no. 1, 31-85.

Csörgő, S., 1996. Universal Gaussian approximations under random censorship. Ann. Statist. 24, no. 6, 2744-2778.

Deheuvels, P. and Einmahl, J.H.J., 1996. On the strong limiting behavior of local functionals of empirical processes based upon censored data. Ann. Probab. 24, no. 1, 504-525.

Einmahl, J.H.J., Fils-Villetard, A. and Guillou, A., 2008. Statistics of extremes under random censoring. Bernoulli. 14, no.1, 207-227.

Einmahl, J.H.J. and Koning, A.J., 1992. Limit theorems for a general weighted process under random censoring. Canad. J. Statist. 20, no. 1, 77-89.

Embrechts, P., Klüppelberg, C., Mikosch, T., 1997. Modelling Extremal Events for Insurance and Finance. Springer-Verlag, New York.

Ghosh, S. and Resnick, S., 2010. A discussion on mean excess plots. Stochastic Process. Appl. 120, no 8, 1492-1517.

Gomes, M.I. and Neves, M.M., 2011. Estimation of the extreme value index for randomly censored data. Biometrical Letters. 48, no.1, 1-22.
Greselin, F., Pasquazzi, L. and Zitikis, R., 2013. Heavy tailed capital incomes: Zenga index, statistical inference, and ECHP data analysis. Extremes, DOI 10.1007/s10687-013-0177-2.

de Haan, L. and Stadtmüller, U., 1996. Generalized regular variation of second order. J. Australian Math. Soc. (Series A) 61, 381-395.

de Haan, L. and Peng, L., 1998. Comparison of tail index estimators. Statist. Neerlandica. 52, no. 1. 60-70.

de Haan, L. and Ferreira, A., 2006. Extreme Value Theory: An Introduction. Springer.

Hall, P., 1982. On some simple estimates of an exponent of regular variation. Journal of the Royal Statistical Society. 44. 37-42.

Hill, B.M., 1975. A simple general approach to inference about the tail of a distribution. Ann. Statist. 3, no.5. 1163-1174.

Kaplan, E.L., Meier, P., 1958. Nonparametric estimation from incomplete observations. J. Amer. Statist. Assoc. 53, 457-481.

Koning, A.J. and Peng, L., 2008. Goodness-of-fit tests for a heavy tailed distribution. J. Statist. Plann. Inference. 138, no. 12, 3960-3981.

Mason, D.M., 1982. Laws of large numbers for sums of extreme values. Ann. Probab. 10. 756-764.

Necir, A., Meraghni, D. and Meddi, F., 2007. Statistical estimate of the proportional hazard premium of loss. Scand. Actuar. J., no. 3, 147-161.

Necir, A. and Meraghni, D., 2009. Empirical estimation of the proportional hazard premium for heavy-tailed claim amounts. Insurance Math. Econom. 45, no. 1, 49-58.

Peng, L., 2001. Estimating the mean of a heavy tailed distribution. Statist. Probab. Lett. 52, no. 3, 255-264.

Peng, L., 2004. Empirical-likelihood-based confidence interval for the mean with a heavy-tailed distribution. Ann. Statist. 32, no. 3, 1192-1214.

Reiss, R.D. and Thomas, M., 1997. Statistical Analysis of Extreme Values with Applications to Insurance, Finance, Hydrology and Other Fields. Birkhäuser.

Shorack, G., R. and Wellner, J.A., 1986. Empirical Processes with Applications to Statistics. Wiley.

Stute, W., 1995. The central limit theorem under random censorship. Ann. Statist. 23, 422-439.
Vandewalle, B. and Beirlant, J., 2006. On univariate extreme value statistics and the estimation of reinsurance premiums. *Insurance Math. Econom.* **38**, no. 3, 441-459.

Wang, S.S., 1996. Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin.* **26**, 71-92.