Spatial Intermittency in Electron Magnetohydrodynamic Turbulence

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Abstract

Fractal and multi-fractal aspects of spatial intermittency in the energy cascade of electron magnetohydrodynamic (EMHD) turbulence is considered. Fractal and multi-fractal models for the energy dissipation field are used to determine intermittency corrections to the scaling behavior in the high-wavenumber (electron hydrodynamic limit) and low-wavenumber (magnetization limit) asymptotic regimes of the inertial range. Extrapolation of the multi-fractal scaling down to the dissipative microscales confirms in these asymptotic regimes a dissipative anomaly previously indicated by the numerical simulations of EMHD turbulence. Several basic features of the EMHD turbulent system which are universal and transcend the existence of the characteristic length scale $d_e$ (which is the electron skin depth) in the EMHD problem are highlighted.

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1. Introduction

The high-temperature plasmas in space (e.g. solar flares and magnetospheric substorms) and laboratory (tokamak discharges) have been found to be collisionless. An important aspect of a collisionless plasma is the enhancement by an order of magnitude of the magnetic reconnection rate (Yamada [1]). In situations where the spatial scales are shorter than the ion-inertial length $d_i$ and time scales are shorter than the ion-cyclotron period, the ions do not have time to respond and merely provide a neutralizing background, and the dynamics are entirely controlled by electrons. A fluid description for the electrons then leads to the electron magnetohydrodynamic (EMHD) model (Kingsep et al. [3], Gordeev et al. [4]). The strongly sheared electron flows in the current sheets in EMHD undergo Kelvin-Helmholtz instability and lead to turbulence in EMHD (which is to be contrasted with turbulence generation/intensification via the tearing mode instability of current sheets in MHD). The energy cascade in EMHD turbulence proceeds directly even in two dimensions (2D), as in MHD turbulence, thanks to the Lorentz force on the electrons. Biskamp et al. [5], [6] did high resolution numerical simulation of decaying 2D isotropic homogeneous EMHD turbulence and found that the energy spectrum follows the Kolmogorov spectrum in the electron hydrodynamic limit ($d_e/\ell \gg 1$, $d_e$ being the electron inertial length) in spite of the fact that the whistler waves (which are generic to EMHD) would be expected to mediate the energy cascade. (A whistler-like relation implying an equipartition of energy between the poloidal and axial components of the magnetic field was however found to hold.) Celani et al. [8] further showed that a Kolmogorov 4/5th law type result also holds for the energy cascade in 2D EMHD turbulence. Numerical simulations of Boffetta [9] revealed the presence of spatial intermittency in EMHD turbulence - the energy dissipation field was found not to be uniformly distributed in space and the dissipative structures were of filament shape. Numerical simulations of Germaschewski and Grauer [10] showed deviations from a Kolmogorov-type linear law of the characteristic scaling exponent of higher order structure functions further validating this aspect. Numerical simulations of Biskamp et al. [5] and [6] also showed that the energy dissipation rate in EMHD turbulence was apparently independent of the dissipation coefficients suggesting the possibility of a dissipative anomaly in the direct energy cascade in EMHD.

In this paper, we consider fractal (Frisch et al. [11]) and multi-fractal (Frisch and Parisi [12]) models to describe the effects of spatial intermittency in 2D fully-developed EMHD turbulence. We will then extrapolate multi-fractal scaling in the inertial range down to the dissipative microscale and provide analytical evidence for a dissipative anomaly in the high-wavenumber (electron hydrodynamic limit) and low wavenumber (magnetization limit) asymptotic regimes. Several basic features of the EMHD turbulent system which are universal and transcend the existence of the characteristic length scale $d_e$ in the EMHD problem are highlighted.

\[ \text{In situ measurements in the solar wind have provided evidence of magnetic field fluctuations characterized by such spatial and time scales (Alexandrova et al. [2]).} \]

\[ \text{3The extent of the whistlerization in EMHD turbulence was numerically investigated by Dastgeer et al. [7].} \]

\[ \text{4The finiteness of the energy dissipation even in the limit the dissipation coefficients vanish constitutes a dissipative anomaly (persistence of symmetry breaking even in the limit the symmetry breaking factors vanish). There is experimental support (Sreenivasan [13]) for this in 3D hydrodynamic turbulence.} \]
2. Governing Equations of EMHD

The 2D EMHD system of equations can be written in terms of two scalar potentials - the magnetic flux function $A$ describing the magnetic field in the plane $\mathbf{B} = \nabla \times A \hat{z}$ and the stream function $\psi$ describing the electron flow velocity in the plane $\mathbf{v}_e = \nabla \times \psi \hat{z}$, which is proportional to the in-plane current density (so $\psi$ also represents the out-of-plane magnetic field):

- the equation of generalized vorticity:
  \[
  \frac{\partial}{\partial t} \left( \omega + \frac{\psi}{d_e^2} \right) + (\mathbf{v}_e \cdot \nabla) \omega - \frac{1}{m_e n_e c} (\mathbf{B} \cdot \nabla) J = \frac{\nu}{d_e^2} \nabla^2 \omega \tag{1}
  \]

- the generalized Ohm’s law:
  \[
  \frac{\partial}{\partial t} \left( A + \frac{d_e^2}{c} J \right) + (\mathbf{v}_e \cdot \nabla) \left( A + \frac{d_e^2}{c} J \right) = n \nabla^2 A \tag{2}
  \]

where,
\[
\frac{1}{c} J = -\nabla^2 A, \quad \omega = -\nabla^2 \psi. \tag{3}
\]

The number density $n_e$ is constant, in accordance with the incompressibility of the electron flow $\nabla \cdot \mathbf{v}_e = 0$ which implies $\nabla \cdot \mathbf{J} = 0$ - this presupposes that the displacement current $\partial \mathbf{E}/\partial t$ is negligible.

In the ideal limit ($\nu$ and $n \to 0$), equations (1) and (2) have the Hamiltonian integral invariant (upon appropriately non-dimensionalizing the various quantities (Biskamp et al. [5] and [6])) -
\[
H = \frac{1}{2} \int_S \left[ (\nabla A)^2 + \psi^2 + d_e^2 \{ J^2 + (\nabla \psi)^2 \} \right] dS \tag{4}
\]

$S$ being the area occupied by the plasma. (4) shows that the magnetization effects introduce a characteristic length scale, namely $d_e$ in the EMHD problem. As a result, the latter exhibits some departures from the properties of MHD turbulence. One such feature is a decrease of the energy flux, leading to energy pileup of scales $\ell_n \sim d_e$ in the energy cascade. This could lead to an ordered quasi-crystalline phase signifying the appearance of long-range order in the system (similar to the case with geostrophic turbulence (Kukharin et al. [14])).

(4) implies, on noting a whistler-like relation $\psi \sim A/\ell$ holds between the poloidal and axial components of the magnetic field (Biskamp et al. [5] and [6]), that the energy per unit mass at length scale $\ell$ is given by
\[
E \sim \psi^2 \left( 1 + \frac{d_e^2}{\ell^2} \right) \tag{5}
\]

\footnote{This relation also implies an equipartition in the energy contents of the in-plane magnetic field and velocity fluctuations (Alexandrova et al. [2]).}
which, in the magnetization \((d_e/\ell \ll 1)\) and the electron hydrodynamic \((d_e/\ell \gg 1)\) asymptotic regimes, leads to

\[ E \sim \begin{cases} \psi^2, & d_e/\ell \ll 1 \\ (d_e/\ell^2)\psi^2, & d_e/\ell \gg 1 \end{cases} \]  

(6a, b)

It is of interest to note that EMHD turbulence also exhibits some basic features which transcend the existence of the characteristic length scale \(d_e\) in the EMHD problem. One such feature becomes apparent on applying the equilibrium statistical mechanics approach to the EMHD problem.

3. Equilibrium Statistical Mechanics

Consider an EMHD turbulence within a square which can be expanded into an infinite series of discrete wave vectors \(k_n\) with stream function amplitudes \(\Psi(k_n)\) related to each other via equations (1) and (2). The Fourier analysis of this system allows a formulation in terms of many degrees of freedom and hence leads to a consideration of this problem from the viewpoint of statistical mechanics.

Application of equilibrium statistical mechanics to this system (Burgers [15], Hopf [16], Lee [17] and Kraichnan [18]) requires the latter to be considered ideal\(^6\). This, in turn, requires a truncation in the Fourier space by retaining the Fourier modes lower than a cut-off wavenumber \(k_{\text{max}}\). This set of N wavenumbers conserves the energy (according to (5)), which is a quadratic rugged invariant, and survives the spectral truncation (because it is conserved by an interacting triad)

\[
\frac{1}{2} \sum_{k_n} (1 + k_n^2d_e^2) |\Psi(k_n)|^2 = \text{const.}
\]  

(7)

If \(y_{n1}(k_n)\) and \(y_{n2}(k_n)\) are the real and imaginary parts of each mode \(\Psi(k_n)\), the system can be represented by a point of \(m = 2N\) coordinates in a phase space and evolves ergodically in this phase space on the energy sphere,

\[
\frac{1}{2} \sum_{\alpha=1}^{m} (1 + k_\alpha^2d_e^2) y_\alpha^2 = \text{const.}
\]

(8)

Consider now a collection of such systems which is represented at each instant of time by a cluster of points in the phase space of density \(\rho(y_1, \ldots, y_m, t)\). Since the total number of such systems and hence the volumes occupied by their representative points in the phase space are preserved, we have the Liouville Theorem:

\[
\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{m} \frac{dy_\alpha}{dt} \frac{\partial \rho}{\partial y_\alpha} = 0.
\]

(9)

\(^6\)Formally, equilibrium statistical mechanics does not seem to be applicable to turbulence which, being dissipational, is in a non-equilibrium state. However, a turbulent system is believed to relax via nonlinear interactions toward equilibrium (which is confirmed by the numerical calculations of Orszag and Patterson [19]). Indeed, one may interpret the energy cascade to small length scales as a consequence of this tendency (Novikov [20]).
Statistical mechanics seeks to explain the statistical behavior of a system in terms of its structural properties, such as the conservation of energy. This enables the equilibrium spectrum of EMHD turbulence to be predicted from the viewpoint of canonical ensemble averages.

The elementary Gibbsian methods of statistical mechanics allow construction of equilibrium solutions of Liouville’s equation (9) as functions of the conserved quantities, such as the energy (8), via the Boltzmann-type distribution

\[ P(y_1, \ldots, y_m) = \frac{1}{Z} e^{-\frac{1}{2} \sum_{\alpha=1}^{m} \sigma(1+k_{\alpha}^2 d^2_e) y_{\alpha}^2} \]  

where \( \sigma \) is a constant (interpretable as “inverse temperature”) and \( Z \) is the partition function of the system,

\[ Z \equiv \int \cdots \int e^{-\frac{1}{2} \sum_{\alpha=1}^{m} \sigma(1+k_{\alpha}^2 d^2_e) y_{\alpha}^2} dy_1 \ldots dy_m. \]  

The canonical ensemble average \( \langle \rho(y_1, \ldots, y_m, t) \rangle \) of an ensemble of the given system \( \rho(y_1, \ldots, y_m, t) \) is stipulated to relax eventually toward this equilibrium distribution over the energy sphere (8) in the phase space.

The mean variance of the mode \( \alpha \) is given by

\[ \langle y_{\alpha}^2 \rangle = \frac{1}{Z} \int \cdots \int e^{-\frac{1}{2} \sum_{\alpha=1}^{m} \sigma(1+k_{\alpha}^2 d^2_e) y_{\alpha}^2} dy_1 \ldots dy_m \]  

or

\[ \langle y_{\alpha}^2 \rangle = \frac{1}{1 + k_{\alpha}^2 d^2_e}. \]  

The energy spectrum is then given by

\[ E(k) \sim \pi k \left(1 + k^2 d^2_e\right) \langle |\Psi(k)|^2 \rangle \sim \pi k, \forall k. \]  

(13) shows that EMHD turbulence, like the infrared regime of 2D hydrodynamic turbulence, exhibits the equipartition spectrum \( E(k) \sim k \), for small wavenumbers. This result signifies basic characteristic aspects of EMHD turbulence which transcend the existence of the characteristic length scale \( d_e \) in the EMHD problem, as is also apparent in the following developments.

4. **Inertial-Range Scaling Laws**

One may consider for the energy cascade in EMHD turbulence an inertial range of Kolmogorov type which is in a state of statistical equilibrium and the energy is assumed to cascade smoothly through nonlinear processes in a stationary state.

Consider a discrete sequence of scales,

\[ \ell_n \sim \ell_0 \cdot 2^{-n} ; \quad n = 0, 1, 2, \ldots. \]  

\( ^7 \)The number of modes in 2D is proportional to \( 2\pi k \).
Let us assume that we have a statistically stationary EMHD turbulence, where energy is introduced into the plasma at scales $\sim \ell_0$, and is then transferred successively to scales $\sim \ell_1, \ell_2, \cdots$ until some scale $\ell_d$ is reached where dissipative effects are able to compete with nonlinear transfer.

The energy per unit mass in the $n$th scale, according to (5), is given by

$$E_n \sim \Psi_n^2 \left( 1 + \frac{d_e^2}{\ell_n^2} \right). \tag{15}$$

The rate of energy transfer per unit mass from the $n$th scale to the $(n+1)$th scale is given by

$$\epsilon_n \sim \frac{E_n}{t_n} \sim \frac{\Psi_n^3 d_e}{\ell_n^2} \left( 1 + \frac{d_e^2}{\ell_n^2} \right) \tag{16}$$

where $t_n$ is a characteristic time of the $n$th scale,

$$t_n \sim \frac{\ell_n^2}{d_e \Psi_n}. \tag{17}$$

In the inertial range, we assume a stationary process in which the energy transfer rate is constant,

$$\epsilon_n = \text{const} = \epsilon, \quad \ell_d \leq \ell_n \leq \ell_0. \tag{18}$$

Using (18), (17) leads to

$$\Psi_n \sim \epsilon^{1/3} \ell_n^{2/3} \frac{d_e^{1/3}}{\ell_n^{1/3}} \left( 1 + \frac{d_e^2}{\ell_n^2} \right)^{-1/3}. \tag{19}$$

Using (19), (16) gives

$$E_n \sim \epsilon^{2/3} \ell_n^{4/3} \frac{d_e^{2/3}}{\ell_n^{2/3}} \left( 1 + \frac{d_e^2}{\ell_n^2} \right)^{1/3} \tag{20}$$

from which we have

$$E_n \sim \begin{cases} \epsilon^{2/3} d_e^{-2/3} \ell_n^{-1/3}, & d_e/\ell_n \ll 1, \\ \epsilon^{2/3} \ell_n^{2/3}, & d_e/\ell_n \gg 1. \end{cases} \tag{21a, b}$$

(21) leads to the following energy spectra (Biskamp et al. [5] and [6]),

$$E_k \sim \begin{cases} \epsilon^{2/3} d_e^{-2/3} k^{-7/3}, & k d_e \ll 1, \\ \epsilon^{2/3} k^{-5/3}, & k d_e \gg 1. \end{cases} \tag{22a, b}$$

The electron hydrodynamic limit corresponds to $kd_e \gg 1$ while the magnetization limit corresponds to $kd_e \ll 1$.

5. Spatial Intermittency

The inertial range theory discussed in Section 4 does not take into account the spatial intermittency in EMHD turbulence that was revealed by the numerical simulations (Boffetta et al. [9] and Geramshewski and Grauer [10]). Spatial intermittency effects would cause systematic departures from the scaling laws (22) which use mean transfer rates. One may follow
Mandelbrot [21] and argue that the spatial intermittency effects in EMHD turbulence are related to the fractal aspects connected with the strongly convoluted dissipative structures (like the current sheets revealed in the numerical simulations [10]). This may be simulated in a first approximation by representing the dissipative structures via a homogeneous fractal with non-integer Hausdorff dimension $D_0$. This amounts to assuming the energy flux to be transferred to only a fixed fraction $\beta$ of the eddies downstream in the cascade (Frisch et al. [11]).

5.1 Homogeneous Fractal Model

Consider a discrete sequence of scales as in (15), but now assume that at the nth step, only a fraction $\beta_n$ of the total space has an appreciable excitation with a fractal dimension $D_0$.

The energy per unit mass in the nth scale is given by

$$E_n \sim \beta^n \Psi^2_n \left(1 + \frac{d^2}{\ell_n^2}\right)$$

where,

$$\beta^n \sim \left(\frac{\ell_n}{\ell_0}\right)^{2-D_0}.$$  (24)

The energy transfer rate per unit mass from the nth scale to the $(n + 1)$th scale is given by

$$\epsilon_n \sim \frac{E_n}{t_n} \sim \beta^n \Psi^3_n d_e \left(1 + \frac{d^2}{\ell^2_n}\right).$$

(25)

In the inertial range, the energy transfer rate is constant for a stationary process, so on using (24) and (25), and assuming the scaling behavior,

$$\Psi_n \sim \ell_n^\alpha$$

we have, from (18),

$$3\alpha + 2 - D_{0(1)} - 2 = 0, \quad d_e/\ell_n \ll 1$$

$$3\alpha + 2 - D_{0(2)} - 4 = 0, \quad d_e/\ell_n \gg 1$$

(27a, b)

from which,

$$\alpha = \begin{cases} \frac{D_{0(1)}}{3}, & d_e/\ell_n \ll 1 \\ \frac{D_{0(2)} + 2}{3}, & d_e/\ell_n \gg 1. \end{cases}$$

(28a, b)

Using (26) and (28), we have from (23),

$$E(\ell) \sim \begin{cases} c^2/3 d_e^{-2/3} \ell_n^{4/3 + 1/3(2 - D_{0(1)})}, & d_e/\ell_n \ll 1 \\ c^2/3 \ell_n^{2/3 + 1/3(2 - D_{0(2)})}, & d_e/\ell_n \gg 1. \end{cases}$$

(29a, b)

(29) leads to the following energy spectra,

$$E(k) \sim \begin{cases} c^2/3 d_e^{-2/3} k^{-7/3 - 1/3(2 - D_{0(1)})}, & kd_e \ll 1 \\ c^2/3 k^{-5/3 - 1/3(2 - D_{0(2)})}, & kd_e \gg 1. \end{cases}$$

(30a, b)
Observe that the intermittency corrections \( D_{0(1,2)} < 2 \) make the spectra steeper, as expected.

Noting that in the electron hydrodynamic limit \( (kd_e \gg 1) \) the dissipative structures are typically vortex-filament like \( (D_{0(2)} = 0) \), and in the magnetization limit \( (kd_e \ll 1) \) they are typically current-sheet like \( (D_{0(1)} = 1) \) \( [10] \), (30a, b) would lead to

\[
E(k) \sim \begin{cases} 
\epsilon^{2/3} d_e^{-2/3} k^{-\frac{8}{3}}, & k d_e \ll 1 \\
\epsilon^{2/3} k^{-\frac{2}{3}}, & k d_e \gg 1.
\end{cases}
\]  

(31a, b)

On the other hand, noting that the structure function of \( S_p(\ell) \), order \( p \), for the EMHD turbulence problem is defined in terms of the magnetic field in the magnetization limit \( (d_e/\ell \ll 1) \) and the electron flow velocity in the electron hydrodynamic limit \( (d_e/\ell \gg 1) \),

\[
S_p(\ell) \sim \begin{cases} 
\langle |\delta \psi(\ell)|^p \rangle, & d_e/\ell \ll 1 \\
\langle |\delta (\partial \psi/\partial \ell)(\ell)|^p \rangle, & d_e/\ell \gg 1.
\end{cases}
\]  

(32a, b)

Using (26), and noting that the probability to belong to this fractal at scale \( \ell \) goes like \( \ell^{2-D_0} \), (32) leads to

\[
S_p(\ell) \sim \ell^{\zeta_p} \sim \begin{cases} 
\ell^{\alpha p + 2 - D_{0(1)}}, & d_e/\ell \ll 1 \\
\ell^{(\alpha - 1)p + 2 - D_{0(2)}}, & d_e/\ell \gg 1.
\end{cases}
\]  

(33a, b)

So, the characteristic exponent \( \zeta_p \) is given by

\[
\zeta_p = \begin{cases} 
\alpha p + 2 - D_{0(1)}, & d_e/\ell \ll 1 \\
(\alpha - 1)p + 2 - D_{0(2)}, & d_e/\ell \gg 1.
\end{cases}
\]  

(34a, b)

Using (28), (34) becomes

\[
\zeta_p = \begin{cases} 
\frac{2p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{0(1)}\right), & d_e/\ell \ll 1 \\
\frac{p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{0(2)}\right), & d_e/\ell \gg 1.
\end{cases}
\]  

(35a, b)

which does not show a nonlinear dependence on \( p \) as is required of the characteristic exponent for large \( p \). It is therefore necessary to consider the multi-fractal model \[12\] to address this issue.

5.2 Multi-fractal Model

Let us assume that the energy flux (or dissipation) is concentrated on a multi-fractal object \( (12) \) which is characterized by a continuous spectrum of scaling exponents \( \alpha \), \( \alpha \in I \equiv [\alpha_{\text{min}}, \alpha_{\text{max}}] \). Each \( \alpha \in I \) has the support set \( S(\alpha) \subset \mathbb{R}^3 \) of fractal dimension \( f(\alpha) \) such that, as \( \ell \Rightarrow 0 \), the stream function increment has the scaling behavior -

\[
|\delta \psi(\ell)| \sim \ell^\alpha.
\]  

(36)
The sets $S(\alpha)$ are nested so that $S(\alpha') \subset S(\alpha)$, for $\alpha' < \alpha$. The fractal dimension $f(\alpha)$ is obtained via a Legendre transformation of the scaling exponent of the $p$th order structure function of the electron flow velocity (or magnetic field),

$$S_p(\ell) \sim \begin{cases} \int d\mu(\alpha) \ell^{\alpha p + 2 - f(\alpha)} \sim \ell^{\zeta_p(1)}, & d_e/\ell \ll 1 \\ \int d\mu(\alpha) \ell^{(\alpha-1)p + 2 - f(\alpha)} \sim \ell^{\zeta_p(2)}, & d_e/\ell \gg 1 \end{cases} (37a, b)$$

where the measure $d\mu(\alpha)$ gives the weight of different scaling exponents $\alpha$, and $\ell^{2-f(\alpha)}$ represents the probability of encountering the set $S(\alpha)$ within a 2D circle of radius $\ell$. (37a, b) reflect the asymptotic scalings exhibited by (6a, b).

One may use the method of steepest descent to extract the dominant terms in the integrals in (9), in the limit of very small $\ell$. This gives

$$\zeta_p = \begin{cases} \alpha^* p + 2 - f(\alpha^*), & d_e/\ell \ll 1 \\ (\alpha^* - 1) p + 2 - f(\alpha^*), & d_e/\ell \gg 1 \end{cases} \quad (38a, b)$$

where,

$$f'(\alpha^*) = p. \quad (38c)$$

Next, in order to relate the singularity spectrum $f(\alpha)$ to the generalized fractal dimension (GFD) of the energy dissipation field, note that the energy transfer rate per unit mass at length scale $\ell$ is given by

$$\dot{\epsilon}(\ell) \sim \frac{E}{\ell} \sim \begin{cases} (d_e/\ell^2) \psi^3, & d_e/\ell \ll 1 \\ (d_e^3/\ell^4) \psi^3, & d_e/\ell \gg 1 \end{cases} \quad (39a, b)$$

If the energy dissipation field is assumed to be a multi-fractal, the sums of the moments of the total energy dissipation $U(\ell) \sim \epsilon(\ell)\ell^2$ occurring in $N(\ell)$ squares of size $\ell$ covering the support of the measure $\epsilon$ exhibit the following asymptotic scaling behavior (Halsey et al. [22])

$$\sum_{i=1}^{N(\ell)} [U_i(\ell)]^q \sim \ell^{(q-1)D_q} \sim \begin{cases} \int d\mu(\alpha) \ell^{3\alpha q - f(\alpha)} \sim \ell^{\zeta_p(1)}, & d_e/\ell \ll 1 \\ \int d\mu(\alpha) \ell^{(3\alpha-2)q - f(\alpha)} \sim \ell^{\zeta_p(2)}, & d_e/\ell \gg 1 \end{cases} \quad (40a, b)$$

where $D_q$ is the GFD of the $\epsilon$-field (Hentschel and Proccacia [23]), and we have assumed that the number of iso-$\alpha$ squares for which $\alpha$ takes on values between $\alpha$ and $\alpha + d\alpha$ is proportional to $d\mu(\alpha)\ell^{-f(\alpha)}$. (40a, b) again reflect the asymptotic scalings exhibited by (6a, b). The dominant terms in the integrals in (40) may again be extracted, in the limit $\ell \Rightarrow 0$, using the method of steepest descent, to give

$$(q - 1)D_q = \begin{cases} 3\alpha^* q - f(\alpha^*), & d_e/\ell \ll 1 \\ (3\alpha^* - 2)q - f(\alpha^*), & d_e/\ell \gg 1 \end{cases} \quad (41a, b)$$

where,

$$f'(\alpha^*) = 3q. \quad (41c)$$

The coincidence of the values of $\alpha^*$ given by (38) and (41), for which the integrands in (37a, b) and (40a, b) become extremum, is insured by assuming a Kolmogorov refined similarity type hypothesis (Meneveau and Sreenivasan [24]) in the dissipative microscale regime.
Eliminating $f(\alpha)$ from (38) and (41), and putting $q = p/3$, we obtain

$$
\zeta_p = \begin{cases} 
\frac{2p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{p/3}\right), & d_e/\ell \ll 1 \\
\frac{p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{p/3}\right), & d_e/\ell \gg 1.
\end{cases} \quad (42a, b)
$$

For a fractally homogeneous EMHD turbulence,

$$
D_{p/3} = \begin{cases} 
D_{0(1)}, & d_e/\ell \ll 1 \\
D_{0(2)}, & d_e/\ell \gg 1
\end{cases} \quad (43a, b)
$$

(42a, b) reduce to

$$
\zeta_p = \begin{cases} 
\frac{2p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{0(1)}\right), & d_e/\ell \ll 1 \\
\frac{p}{3} - \left(\frac{p}{3} - 1\right) \left(2 - D_{0(2)}\right), & d_e/\ell \gg 1
\end{cases} \quad (44a, b)
$$
in agreement with (35a, b). The energy per unit mass then shows the following scaling behavior,

$$
E(\ell) \sim \begin{cases} 
\epsilon^{2/3} d_e^{-2/3} \ell^{4/3} + 1/3(2-D_{0(1)}), & d_e/\ell \ll 1
\end{cases} \quad (45a, b)
$$

and the energy spectra are,

$$
E(k) \sim \begin{cases} 
\epsilon^{2/3} d_e^{-2/3} k^{-7/3} - 1/3(2-D_{0(1)}), & k d_e \ll 1
\end{cases} \quad (46a, b)
$$
in agreement with (30a, b).

### 5.3 Multi-fractal Scaling at the Dissipative Microscale

We now consider extrapolation of the multi-fractal scaling in the inertial range discussed in Section 5.2 down to the dissipative microscale by assuming that an inertial behavior persists at scales smaller than $d_e$ - this assumption may be justifiable for tenuous plasmas like those in space ($d_e \approx 10$ km for the magnetospheric plasma).

On taking into account the spatial intermittent character of the energy dissipation field, the dissipative microscales $\xi_{D(1),(2)},$

$$
\xi_{D(1)} \sim \frac{\eta^{3/2} d_e^{-1}}{\epsilon^{1/2}}, \quad d_e/\ell \ll 1 \quad (47a)
$$

$$
\xi_{D(2)} \sim \frac{\nu^{3/4}}{\epsilon^{1/4}}, \quad d_e/\ell \gg 1 \quad (47b)
$$

(along the lines of the development of Paladin and Vulpiani [25] and Nelkin [26] for the hydrodynamic case), are found to exhibit the scaling behavior,

$$
\xi_{D(1)} \sim R_m^{-1/\alpha}, \quad d_e/\ell \ll 1 \quad (48a)
$$
\[ \xi_{D(2)} \sim \bar{R}_h^{-1/\alpha}, \quad d_e/\ell \gg 1 \quad (48b) \]

where \( \bar{R}_m \) and \( \bar{R}_h \) are, respectively, mean magnetic and hydrodynamic Reynolds numbers,

\[ \bar{R}_m \sim \frac{(\bar{\epsilon}\ell^5/d_e)^{1/3}}{\eta}, \quad \bar{R}_h \sim \frac{(\bar{\epsilon}\ell^7/d_e)^{1/3}}{\nu} \quad (49) \]

\( \bar{\epsilon} \) is the mean energy dissipation rate, and \( \eta \) is the resistivity and \( \nu \) is the kinematic viscosity of the plasma. (Observe that the mean magnetic and hydrodynamic Reynolds numbers are both dependent on the electron skin depth \( d_e \).) The identity of the scaling exponents in the two opposite asymptotic regimes is symptomatic of certain universal features in these regimes, as seen further in the following.

The moments of the electron-flow velocity (or magnetic field)-gradient distribution,

\[ A_p \equiv \begin{cases} \langle |\partial \psi / \partial x|^p \rangle, & d_e/\ell \ll 1 \\ \langle |\partial^2 \psi / \partial x^2|^p \rangle, & d_e/\ell \gg 1 \end{cases} \quad (50a, b) \]

are then given by

\[ A_p \sim \begin{cases} \int d\mu(\alpha) \left( \bar{R}_m \right)^{-1/\frac{1}{\alpha}(\alpha - 1)p + 2 - f(\alpha)}, & d_e/\ell \ll 1 \\ \int d\mu(\alpha) \left( \bar{R}_h \right)^{-1/\frac{1}{\alpha}(\alpha - 2)p + 2 - f(\alpha)}, & d_e/\ell \gg 1 \end{cases} \quad (51) \]

In the limit of large \( \bar{R}_m \) and \( \bar{R}_h \), the dominant exponents in (51) correspond to

\[ \alpha^* [p - f(\alpha^*)] = (\alpha^* - 1)p + 2 - f(\alpha^*), \quad d_e/\ell \ll 1 \quad (52a) \]

\[ \alpha^* [p - f(\alpha^*)] = (\alpha^* - 2)p + 2 - f(\alpha^*), \quad d_e/\ell \gg 1 \quad (52b) \]

The coincidence of the values of \( \alpha^* \) given by (41) and (52) for which the integrands in (40a, b) and (51a, b) become extremum, is again insured by assuming the Kolmogorov refined similarity type hypothesis ([24]) in the dissipative microscale regime. (52a, b), in conjunction with (41a, b), lead to

\[ A_p \sim \begin{cases} \frac{-D_Q(p - 3) - 3p + 6}{(\bar{R}_m)}, & \text{where} \quad Q = \frac{D_Q + p - 2}{D_Q}, \quad d_e/\ell \ll 1 \\ \frac{-D_Q(p - 3) - 6p + 6}{(\bar{R}_h)}, & \text{where} \quad Q = \frac{D_Q + 2p - 2}{D_Q + 2}, \quad d_e/\ell \gg 1 \end{cases} \quad (53a, b) \]

from which,

\[ A_2 \sim \begin{cases} (\bar{R}_m)^1, & d_e/\ell \ll 1 \\ (\bar{R}_h)^1, & d_e/\ell \gg 1 \end{cases} \quad (54a, b) \]

So, the mean energy dissipation has the following scaling behavior,

\[ \eta A_2 \sim (\bar{R}_m)^0, \quad d_e/\ell \ll 1 \quad (55a) \]

\[ \nu A_2 \sim (\bar{R}_h)^0, \quad d_e/\ell \gg 1 \quad (55b) \]

(55a, b) implies an inviscid dissipation of energy in the electron hydrodynamic limit and a non-resistive dissipation of energy in the magnetization limit and hence a dissipative anomaly.
in high- and low-wavenumber asymptotic regimes of EMHD turbulence in conformity with DNS ([5], [6]). Note further from (53a, b) that the energy dissipation field in these asymptotic regimes has the GFD $D_Q$ equal to the information entropy dimension $D_1$. The dissipative anomaly signifies another basic characteristic aspect of EMHD turbulence which transcends the existence of the characteristic length $d_e$ in the EMHD problem.

6. Discussion

One may view the energy dissipation rate $\epsilon$ to be the order parameter à la Landau [27] for the EMHD turbulence problem because it appears to indicate the degree of broken symmetry and exhibits fluctuations in the presence of spatial intermittency. Further, noting that the critical point for EMHD turbulence corresponds to the limit $\tilde{R}_m$ and $\tilde{R}_h \to \infty$, the non-zero limiting value of $\epsilon$, as the critical point is approached appears to validate this view. Indeed, one may define the critical exponent $\sigma$ (Shivamoggi [28]) for this problem by

$$\epsilon \sim \begin{cases} (\tilde{R}_m)^\sigma, & \tilde{R}_m \to \infty, \ d_e/\ell \ll 1 \\ (\tilde{R}_h)^\sigma, & \tilde{R}_h \to \infty, \ d_e/\ell \gg 1 \end{cases}$$

(56)

where, as per (53a, b),

$$\sigma = 3(Q - 1), \ \forall d_e/\ell.$$  

(57)

Comparison of the above results (see Table 1) with the corresponding results for various FDT systems in fluid and plasma dynamics (Shivamoggi [33]-[35]) indicates that the energy (or enstrophy in 2D hydrodynamic FDT) dissipation rate $\epsilon$ is the right choice for the order parameter for the FDT problem with an apparently universal form for the critical exponent

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8This perspective therefore allows (Shivamoggi [28]) the Kolmogorov type inertial range theory described in Section 4, which assumes $\epsilon$ to be uniform, to be appropriately regarded as a mean field theory à la Landau [27].

9One of the goals of critical phenomena formulation of the turbulence problem (Nelkin [29], Yakhot and Orszag [30], Eyink and Goldenfeld [31], Esser and Grossmann [32], Shivamoggi [28]) has been to determine the critical exponents that are intrinsic features of the turbulence dynamics and are not artifacts of the large-scale stirring mechanisms.

10It may be mentioned that different choices (Nelkin [29], Rose and Sulem [36]) have been considered for the order parameter for the FDT problem; the present choice seems to be appealing because it agrees with all the implications posited in Landau’s order parameter concept (see also footnote 8, as well as remark above equation (56)).
The variations in the amplitude $a$ reflect the residual effect of variant cascade physics in the diverse FDT systems. Observe in Table 1 that the energy (or enstrophy) dissipation fields in the various FDT systems have the GFD $D_Q$ equal to the information entropy dimension $D_1$ (because, corresponding to $p = 2$, the GFD index $Q$ turns out to be unity for all these FDT cases).

Further insight can be gained into this aspect by looking at the probability distribution function (PDF) of the electron-flow velocity (or magnetic field) gradient. In order to derive the PDF of the electron-flow velocity (or magnetic field) gradient, note that the scaling behavior of the dissipative microscales, on using (47a, b), is given by

$$
\xi_{D(1)} \sim \left( \frac{\eta}{\psi_0} \right)^{1/\alpha}, \quad d_e/\ell \ll 1
$$

$$
\xi_{D(2)} \sim \left( \frac{\nu}{\psi_0} \right)^{1/\alpha}, \quad d_e/\ell \gg 1
$$

where $\psi_0$ is the stream function increment on a macroscopic length $L$.

The scaling behavior of the electron-flow velocity (or magnetic field) gradient is then

$$
s \sim \begin{cases} 
\frac{\psi_0}{\xi_{D(1)}} & d_e/\ell \ll 1 \\
\frac{d_e \psi_0}{\xi_{D(2)}} & d_e/\ell \gg 1 
\end{cases}
$$

The PDF of the electron-flow velocity (or magnetic field) gradient may then be determined in terms of that for the characteristic stream function increment $\psi_0$ for large scales as follows,

$$
P(s; \alpha) = P(\psi_0) \frac{d\psi_0}{ds}.
$$

---

11 The critical exponent $\sigma$ may be connected with the critical exponents $\gamma$, $\nu$, and $\eta$ introduced by Rose and Sulem according to

$$
\frac{k_d}{\sqrt{2}} \xi_D^{-1} \sim R^\nu \sim \bar{R}^{1/\alpha} \quad \text{so} \quad \nu = 1/\alpha
$$

as follows,

$$
\sigma = 3 (\gamma - 1)
$$

with

$$
Q = \gamma = \nu (2 - \eta).
$$

Noting, from (28) and (29), that

$$
\nu = \begin{cases} 
3/2, \quad d_e/\ell \ll 1 \\
3/4, \quad d_e/\ell \gg 1 
\end{cases}
$$

and

$$
\eta = \begin{cases} 
4/3, \quad d_e/\ell \ll 1 \\
2/3, \quad d_e/\ell \gg 1 
\end{cases}
$$

we obtain

$$
\begin{align*}
Q &= \gamma = 1 \\
\sigma &= 0
\end{align*}
$$

as required.

12 This is totally in accord with the idea of universality which implies that near a critical point all systems can be grouped into a relatively small number of classes (depending on the specific dynamics) with identical critical exponents within each class (Hohenberg and Halperin).
### Table 1: Critical exponents for various FDT cases.

| FDT case                        | Critical Exponent $\sigma$ | Generalized Fractal Dimension Index                                                                 |
|---------------------------------|----------------------------|-----------------------------------------------------------------------------------------------------|
| 3D incompressible FDT           | $3(Q - 1)$                 | $Q = \frac{D_Q + 2p - 3}{D_Q + 1}$                                                                  |
| 2D incompressible FDT-entröphy cascade | $3(Q - 1)$                 | $Q = \frac{D_Q + 3p - 2}{D_Q + 4}$                                                                  |
| 3D compressible FDT              | $\left(\frac{3\gamma - 1}{\gamma + 1}\right)(Q - 1)$ | $Q = \frac{D_Q + \frac{2\gamma}{\gamma + 1}p - 3}{D_Q + \frac{4\gamma}{\gamma + 1} - 3}$          |
| 3D MHD FDT                      | $2(Q - 1)$                 | $Q = \frac{2D_Q + 3p - 6}{2D_Q}$                                                                    |
| 2D EMHD FDT                     | $3(Q - 1)$                 | $Q = \begin{cases} \frac{D_Q + p - 2}{D_Q}, & d_e/\ell \ll 1 \\ \frac{D_Q + 2p - 2}{D_Q + 2}, & d_e/\ell \gg 1 \end{cases}$ |
Taking $P(\psi_0)$ to be Gaussian,
\[ P(\psi_0) \sim e^{-\psi_0^2/2<\psi_0^2>} \]  \hspace{1cm} (62)
and using (60a, b), (61) leads to
\[ P(s; \alpha) \sim \begin{cases} 
\left( \frac{\eta}{|s|} \right)^{1-\alpha} e^{-\left[ \frac{\eta^{2(1-\alpha)}|s|^{2\alpha}}{2<\psi_0^2>} \right]}, \, d_e/\ell \ll 1 \\
\left( \frac{\nu}{|s|^{1/2}} \right)^{2-\alpha} e^{-\left[ \frac{\nu^{2(2-\alpha)}|s|^{\alpha}}{2<\psi_0^2>} \right]}, \, d_e/\ell \gg 1 .
\end{cases} \]  \hspace{1cm} (63a, b)

For EMHD turbulence, on noting from (26) and (28),
\[ \alpha = \begin{cases} 
2/3, \, d_e/\ell \ll 1 \\
4/3, \, d_e/\ell \gg 1 .
\end{cases} \]  \hspace{1cm} (64a, b)
(63a, b) become
\[ P(s) \sim \begin{cases} 
\left( \frac{\eta}{|s|} \right)^{1/3} e^{-\left[ \frac{\eta^{4/3}|s|^{4/3}}{2<\psi_0^2>} \right]}, \, d_e/\ell \ll 1 \\
\left( \frac{\nu}{|s|} \right)^{1/3} e^{-\left[ \frac{\nu^{4/3}|s|^{4/3}}{2<\psi_0^2>} \right]}, \, d_e/\ell \gg 1 .
\end{cases} \]  \hspace{1cm} (65a, b)

The identity of the $|s|$-dependence exhibited by $P(s)$, as per (65a, b), (which is also the same as the PDF for the velocity gradient for 3D hydrodynamic turbulence given by Frisch and She \cite{38}), appears to be consistent with the demonstration of dissipative anomaly, as per (55a, b), in the asymptotic regimes (this is validated further by the critical exponent (57) for EMHD). The stretched exponential decay of the PDF exhibited by (65) has also been indicated by the numerical simulations (\cite{9} and \cite{10}).

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