Uniform local well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation

Mingjuan Chen¹, Boling Guo² & Lijia Han³,*

¹Department of Mathematics, Jinan University, Guangzhou 510632, China;
²Institute of Applied Physics and Computational Mathematics, Beijing 100088, China;
³Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

Email: mjchenhappy@pku.edu.cn, gbl@iapcm.ac.cn, hljmath@ncepu.edu.cn

Received February 5, 2020; accepted November 18, 2020; published online June 11, 2021

Abstract In this paper, we study the Cauchy problem for the Benjamin-Ono-Burgers equation
\[ \partial_t u - \epsilon \partial_x^2 u + \mathcal{H} \partial_x^2 u + uu_x = 0, \]
where \( \mathcal{H} \) denotes the Hilbert transform operator. We obtain that it is uniformly locally well-posed for small data in the refined Sobolev space \( \tilde{H}^\sigma(\mathbb{R}) (\sigma \geq 0) \), which is a subspace of \( L^2(\mathbb{R}) \). It is worth noting that the low-frequency part of \( \tilde{H}^\sigma(\mathbb{R}) \) is scaling critical, and thus the small data is necessary. The high-frequency part of \( \tilde{H}^\sigma(\mathbb{R}) \) is equal to the Sobolev space \( H^\sigma(\mathbb{R}) (\sigma \geq 0) \) and reduces to \( L^2(\mathbb{R}) \). Furthermore, we also obtain its inviscid limit behavior in \( \tilde{H}^\sigma(\mathbb{R}) (\sigma \geq 0) \).

Keywords Benjamin-Ono-Burgers equation, Cauchy problem, inviscid limit behavior

MSC(2020) 35Q53, 35Q55, 35A01

Citation: Chen M J, Guo B L, Han L J. Uniform local well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation. Sci China Math, 2022, 65: 1553–1576, https://doi.org/10.1007/s11425-020-1807-4

1 Introduction

In this paper, we study the Cauchy problem for the Benjamin-Ono-Burgers (BOB) equation on the real line
\[ \begin{aligned}
\partial_t u - \epsilon \partial_x^2 u + \mathcal{H} \partial_x^2 u + uu_x &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
u(x,0) &= \phi(x),
\end{aligned} \tag{1.1} \]

where \( \epsilon \in (0,1] \) is a small parameter, \( u \) is a real-valued function of \( (x,t) \in \mathbb{R} \times \mathbb{R}^+ \), and \( \mathcal{H} \) is the Hilbert transform operator defined as follows:
\[ \mathcal{H}(f)(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy. \]

When \( \epsilon = 0 \), the equation (1.1) reduces to the classical Benjamin-Ono (BO) equation
\[ \partial_t u + \mathcal{H} \partial_x^2 u + uu_x = 0, \quad u(x,0) = \phi(x), \tag{1.2} \]
which was originally derived as a model in the study of one-dimensional long internal gravity waves in deep stratified fluids with great depth (see [2, 21]). The BOB model (1.1) was obtained by Edwin and Roberts [4] in the study of intense magnetic flux tubes of the solar atmosphere. The dissipative effects $-\epsilon \partial_x^2 u$ in that literature are due to weak thermal conduction, where $\epsilon$ is a measure of the importance of thermal conduction and is assumed small.

Recently, there are many researchers who devoted themselves to studying the well-posedness theory and the limit behavior of BO and BOB equations. Until now the best result for the global well-posedness of the BO equation was proved by Ionescu and Kenig [8] in the Sobolev space $H^\sigma$ ($\sigma \geq 0$) (see [14] for another proof). For the BOB equation, thanks to the dissipative effects, there are many results about its well-posedness. Otani [22] derived the global well-posedness in $H^\sigma$ for $\sigma > -1/2$ by using the Picard method. Vento [24] proved that this result is critical in the sense that the mapping data-solution fails to be $C^3$ continuous if $\sigma < -1/2$. For more results, please refer to [1, 9–12, 17–19] and the references therein.

The conjecture that the solutions of the BOB equation converge to those of the BO equation when $\epsilon \to 0$ was put forward by Tao [23]. About the limit behavior for other dispersive equations, please refer to [3, 6, 15, 16, 25]. If we consider the uniform well-posedness and inviscid limit for the solutions of the BOB equation, the dissipative effects which relate to $\epsilon$ cannot be used. Motivated by the work of Ionescu and Kenig [8] in 2007, it seems natural to obtain this limit behavior of the BOB equation in $H^\sigma$ ($\sigma \geq 0$). However, it is highly nontrivial to carry out this problem, which has attracted the attention of many researchers. In 2011, Guo et al. [5] obtained that BOB equations were uniformly globally well-posed in $H^\sigma$ for $\sigma \geq 1$ and the solutions of BOB equations converged to those of BO equations in $C([0, T] : H^\sigma)$ ($\sigma \geq 1$) for any $T > 0$. This result was improved to the energy space $H^{1/2}$ by Molinet [13]. Actually, without using a gauge transformation, Molinet and Vento [20] studied the well-posedness of a large class of dispersive equations, and also obtained the limit behavior of the BOB equation in $H^{1/2}$. To the best of our knowledge, the limit behavior of the BOB equation in $H^\sigma$ ($0 \leq \sigma < 1/2$) is still open. We are devoted to studying this problem. However, due to the difficulty that there is no suitable gauge transformation for the BOB equation, we first consider this problem in the refined Sobolev space.

In this paper, we obtain that the BOB equation is uniformly locally well-posed for small data in the refined Sobolev space $\tilde{H}^\sigma$ ($\sigma \geq 0$), where the low-frequency part is scaling critical, which is the reason why the small initial data is needed, and the high-frequency part is equal to the Sobolev space $H^\sigma$ ($\sigma \geq 0$). In fact, the high-frequency part has already reduced to $L^2$, and only the low-frequency part has some special structure, since the known gauge transformation could not be used to the BOB equation to eliminate the special structure in the low-frequency part.

It is known that considering the well-posedness results in $L^2$ for the BO and BOB equations is very difficult. In order to obtain the global well-posedness results for the BO equation, Ionescu and Kenig [7, 8] assumed that low-frequency functions have some additional structure (see the definitions of $X_0$, $Y_0$ and $B_0$); meanwhile to avoid the logarithmic divergences they worked with high-frequency functions that have two components: a weighted $X^{a,b}$-type component (see $X_k$) and a normalized $L^1_tL^2_x$ component (see $Y_k$) which relate to the smoothing effect. In order to obtain the uniform global well-posedness results for the BOB equation, Guo et al. [5] combined the similar weighted $X^{a,b}$-type space with energy estimates. Some ideas of uniform estimates in [5] and this paper come from [6] which considered the inviscid limit for the KdV-Burgers equation. Different from [5–8, 13], in this paper we construct the uniform homogeneous and inhomogeneous linear estimates for the BOB equation in both the weighted $X^{a,b}$-type space and the smoothing effect space $Y_k$. In addition, in order to avoid some logarithmic divergences, we perform the homogeneous dyadic decomposition to construct the low-frequency space $Y_0$, which is different from the resolution space in [7, 8]. Moreover, it is the first time that the $Y_0$ space is used to a dissipative-dispersive equation. There are some obstacles and difficulties in carrying out these problems.

On the one hand, the decay properties of dissipative effects $-\epsilon \partial_x^2 u$ cannot be used because we want to obtain the uniform bounds which are independent of $\epsilon$. On the other hand, the dissipative structure destroys some symmetries and brings some logarithmic divergences, which will bring several technical difficulties. For example, when treating $1/(\tau - \omega(\xi) - i\epsilon \xi^2)$, we need to conquer the singularity which
occurs in low-frequency low-modulation cases. We lead the readers to Lemma 3.2, where we use some techniques in the harmonic analysis and Poisson kernel. Furthermore, in the proof of Lemma 3.3, we make complex decomposition to avoid the logarithmic divergences according to the types of singularities. We believe that this method can be used in some other problems.

Let $F(F^{-1})$ denote the (inverse) Fourier transform operators on $S'(\mathbb{R} \times \mathbb{R})$. Let $F_{\xi}(F^{-1})$ and $F_{\xi}(F^{-1})$ denote the (inverse) Fourier transform operators with respect to the space variable and the time variable, respectively. We introduce the initial data spaces $\tilde{H}^\sigma(\mathbb{R})$ ($\sigma \geq 0$):

$$\tilde{H}^\sigma(\mathbb{R}) = \left\{ \phi \in L^2(\mathbb{R}) : \| \phi \|_{\tilde{H}^\sigma}^2 := \| \eta_0 \cdot F_x(\phi) \|_{L^2}^2 + \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k \cdot F_x(\phi) \|_{L^2}^2 < \infty \right\},$$

(1.3)

where $\left\{ \eta_k \right\}_{k=0}^{\infty}$ are the symbols of nonhomogeneous dyadic decomposition operators, and the Banach space $B_0(\mathbb{R})$ is defined by

$$B_0(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : f \text{ is supported in } [-2, 2] \right\} \quad \text{and} \quad \| f \|_{B_0} := \inf_{f = g+h} \| F^{-1}_\xi(g) \|_{L^1_{\xi}} + \sum_{k'=-\infty}^{1} 2^{-k'/2} \| \chi_{k'} \cdot h \|_{L^1_{\xi}} < \infty,$$

(1.4)

where $\left\{ \chi_{k'} \right\}_{k'=-\infty}^{1}$ are the symbols of homogeneous dyadic decomposition operators. It is easy to see from the definitions that $\tilde{H}^\sigma \hookrightarrow H^\sigma$ ($\sigma \geq 0$). Moreover, from the scaling point of view, we have

$$\| \phi \|_{\tilde{H}^\sigma} \leq C\| \phi \|_{\tilde{H}^\sigma} \quad \text{for any } \lambda \in (0, 1] \text{ and } \sigma \geq 0,$$

(1.5)

where $\phi(\lambda x) := \lambda \phi(\lambda x)$. In fact, the spaces $\tilde{H}^\sigma$ are scaling critical for the low-frequency part, due to $\| \eta_0 \cdot F_x(\phi) \|_{B_0} \sim \| \eta_0 \cdot F_x(\phi) \|_{B_0}$ for any $\lambda \in (0, 1]$. Because of this, the constant $C$ in the inequality (1.5) is independent of $\lambda$, and thus we cannot expect to obtain the large data results from the small data results.

Let $\tilde{H}^\infty(\mathbb{R}) = \bigcap_{\sigma=0}^{\infty} \tilde{H}^\sigma(\mathbb{R})$ with the induced metric. Let $S^\sigma_\infty : \tilde{H}^\infty(\mathbb{R}) \to C([0, 1] : \tilde{H}^\infty(\mathbb{R}))$ denote the nonlinear mapping of the initial-value problem (1.1) with any initial data $\phi \in \tilde{H}^\infty$. For any Banach space $V$ and $r > 0$, let $B(r, V)$ denote the open ball $\{ v \in V : \| v \|_V < r \}$. Our main theorem states the uniform local well-posedness of the BO initial-value problem (1.1) for small data in $\tilde{H}^\sigma$ ($\sigma \geq 0$).

**Theorem 1.1.** (a) For any $\epsilon \in (0, 1]$, there exists a constant $\delta > 0$ with the property that for any $\phi \in B(\delta, \tilde{H}^0) \cap \tilde{H}^\infty$ there is a unique solution $u' = S^\sigma_\infty(\phi) \in C([0, 1] : \tilde{H}^\infty)$ of the initial-value problem (1.1).

(b) For any $\phi \in B(\delta, \tilde{H}^0)$, the mapping $\phi \to S^\sigma_\infty(\phi)$ extends (uniquely) to a Lipschitz mapping $S^\sigma_0 : B(\delta, \tilde{H}^0) \to C([0, 1] : \tilde{H}^0)$ uniformly on $\epsilon \in (0, 1]$, with the property that $S^\sigma_0(\phi)$ is a solution of the initial-value problem (1.1).

(c) For any $\sigma \in [0, \infty)$ we have the local Lipschitz bound which is independent of $\epsilon$,

$$\sup_{t \in [0, 1]} \| S^\sigma_0(\phi)(t) - S^\sigma_0(\phi')(t) \|_{\tilde{H}^\sigma} \leq C(\sigma, R) \| \phi - \phi' \|_{\tilde{H}^\sigma}$$

for any $R > 0$ and $\phi, \phi' \in B(\delta, \tilde{H}^0) \cap B(\delta, \tilde{H}^\sigma)$. As a consequence, the mapping $S^\sigma_0$ is restricted to a locally Lipschitz mapping $S^\sigma_0 : B(\delta, \tilde{H}^0) \cap \tilde{H}^\sigma \to C([0, 1] : \tilde{H}^\sigma)$ uniformly on $\epsilon \in (0, 1]$.

(d) For any $\sigma \in [0, \infty)$, denote by $\phi \to S^\sigma(\phi)$ the solution mapping of the initial-value problem (1.2). Then we have the limit behavior $\lim_{\epsilon \to 0} \| S^\sigma(\phi) - S^\sigma(\phi) \|_{C([0, 1] : \tilde{H}^\sigma)} = 0$.

Once we have obtained the uniform homogeneous and inhomogeneous linear estimates for the BO equation, the proofs of (a)-(c) in Theorem 1.1 are similar to those in [7], where Ionescu and Kenig got the well-posedness results of the BO equation. Therefore, all the efforts in this paper are devoted to obtaining the uniform estimates, where the bounds are independent of $\epsilon$. 

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The rest of this paper is organized as follows. In Section 2, we present the function spaces and some known properties. Section 3 is the most important part, where we give the uniform homogeneous and inhomogeneous linear estimates. In Section 4, we state the main bilinear estimates. The proof of Theorem 1.1 is completed in Section 5.

Notations. In the sequel, C will denote a universal positive constant which can be different at each appearance. \( x \lesssim y \) (for \( x, y > 0 \)) means that \( x \leq C y \), and \( x \sim y \) stands for \( x \lesssim y \) and \( y \lesssim x \). Let \( \mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty) \). \( \mathcal{F} (\mathcal{F}^{-1}) \) denotes the (inverse) Fourier transform operators on \( \mathcal{S}'(\mathbb{R} \times \mathbb{R}) \). For convenience, let \( F_{\xi}(F_{-\xi}^{-1}) \) and \( F_{\ell}(F_{-\ell}^{-1}) \) denote the (inverse) Fourier transform operators with respect to the space variable and the time variable, respectively. \( \hat{\varphi} \) also denotes the Fourier transform of a distribution \( \varphi \).

## 2 Function spaces and known results

In the beginning, let us recall the dyadic decomposition. Let \( \eta_0 : \mathbb{R} \to [0, 1] \) be an even smooth cut-off function, which is supported in \([-8/5, 8/5] \) and equal to 1 in \([-4/5, 4/5] \). About the homogeneous dyadic decomposition, for \( \ell \in \mathbb{Z} \), let \( \chi_{\ell}(\xi) = \eta_0(\xi/2^\ell) - \eta_0(\xi/2^{\ell+1}) \). Then \( \text{supp} \chi_{\ell} \subset \{ \xi : |\xi| \in [(5/8) \cdot 2^\ell, (8/5) \cdot 2^\ell] \} \). Let

\[
\chi_{[\ell_1, \ell_2]} = \sum_{\ell = \ell_1}^{\ell_2} \chi_{\ell} \quad \text{for any } \ell_1 \leq \ell_2 \in \mathbb{Z}.
\]

About the nonhomogeneous dyadic decomposition, let \( \eta_\ell = \chi_{\ell} \) if \( \ell \geq 1 \) and \( \eta_\ell \equiv 0 \) if \( \ell \leq -1 \). Similarly, for \( \ell_1 \leq \ell_2 \in \mathbb{Z} \), let

\[
\eta_{[\ell_1, \ell_2]} = \sum_{\ell = \ell_1}^{\ell_2} \eta_{\ell} \quad \text{and} \quad \eta_{\leq \ell_2} = \sum_{\ell = -\infty}^{\ell_2} \eta_{\ell}.
\]

Therefore, for any \( \phi \in L^2(\mathbb{R}) \) we can define the dyadic decomposition operator \( P_k \) by

\[
P_k \phi = \mathcal{F}_\xi^{-1} \eta_k(\xi) \mathcal{F}_x(\phi)(\xi) \quad \text{for any } k \in \mathbb{Z}_+.
\]

We also need the operators \( P_k \) on \( L^2(\mathbb{R} \times \mathbb{R}) \) by \( \mathcal{F}(P_k u)(\xi, \tau) = \eta_k(\xi) \mathcal{F}(u)(\xi, \tau) \). For \( \ell \in \mathbb{Z} \), let

\[
I_{\ell} = \{ \xi \in \mathbb{R} : |\xi| \in [2^{\ell-1}, 2^{\ell+1}] \}.
\]

For \( \ell \in \mathbb{Z}_+ \), let \( \widehat{I}_{0} = [-2, 2] \) and \( \widehat{I}_{\ell} = I_{\ell} \) if \( \ell \geq 1 \). For \( k \in \mathbb{Z} \) and \( j \geq 0 \), let

\[
D_{k,j} = \{ (\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in \widehat{I}_j \}, \quad \text{if } k \geq 1,
\]

\[
D_{k,j} = \{ (\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau \in \widehat{I}_j \}, \quad \text{if } k \leq 0.
\]

For \( \xi \in \mathbb{R} \), let \( \omega(\xi) \) denote the dispersive relation of the BO equation, i.e.,

\[
\omega(\xi) = -\xi|\xi|. \quad (2.1)
\]

**Definition 2.1.** We define the weighted Besov-type Banach spaces \( X_k = X_k(\mathbb{R} \times \mathbb{R}) \), \( k \in \mathbb{Z}_+ \). For \( k \geq 1 \), let

\[
X_k = \left\{ f \in L^2 : f \text{ is supported in } I_k \times \mathbb{R} \text{ and } \right. \]

\[
\| f \|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \| \eta_j(\tau - \omega(\xi)) f(\xi, \tau) \|_{L^2_{\xi,\tau}} < \infty \}, \quad (2.2)
\]

where

\[
\beta_{k,j} = 1 + 2^{(j-2k)/2}. \quad (2.3)
\]
For $k = 0$, let

$$X_0 = \left\{ f \in L^2 : f \text{ is supported in } \tilde{I}_0 \times \mathbb{R} \text{ and} \right\}$$

$$\| f \|_{X_0} := \sum_{j=0}^{\infty} \sum_{k'=0}^{1} \| \eta_j(\tau) \chi_{k'}(\xi) f(\xi, \tau) \|_{L^2_{\xi, \tau}} < \infty. \quad (2.4)$$

The choices of the coefficients $\beta_{k,j}$ and the large factor $2^{-k'/2}$ are important in order to obtain the bilinear estimates. Due to various logarithmic divergences involving the modulation variable, we also need the smoothing effect spaces $Y_k = Y_k(\mathbb{R} \times \mathbb{R})$.

**Definition 2.2.** For $k \geq 100$, let

$$Y_k = \left\{ f \in L^2 : f \text{ is supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and} \right\}$$

$$\| f \|_{Y_k} := 2^{-k/2} \| F^{-1}[(\tau - \omega(\xi) + i) f(\xi, \tau)] \|_{L^2_{\xi, \tau}} < \infty, \quad (2.5)$$

where $i$ is the unit imaginary number. For $k = 0$, let

$$Y_0 = \left\{ f \in L^2 : f \text{ is supported in } \tilde{I}_0 \times \mathbb{R} \text{ and} \right\}$$

$$\| f \|_{Y_0} := \sum_{j \geq 1} 2^j \| F^{-1}[\eta_j(\tau) f(\xi, \tau)] \|_{L^2_{\xi, \tau}} + \sum_{j \leq 0} \| F^{-1}[\chi_j(\tau) f(\xi, \tau)] \|_{L^2_{\xi, \tau}} < \infty. \quad (2.6)$$

**Remark 2.3.** The definition of $Y_0$ is different from that in [7,8]. It is easy to see that the space $Y_0$ in this paper is a subspace of the corresponding space (denote it by $Y_0$) in [7,8], whose norm is given by

$$\| f \|_{Y_0} := \sum_{j=0}^{\infty} 2^j \| F^{-1}[\eta_j(\tau) f(\xi, \tau)] \|_{L^2_{\xi, \tau}}. \quad (2.7)$$

There is an essential modification. We use the homogeneous dyadic decomposition and $Y_0$ space to avoid the logarithmic divergences. If we use the space $Y_0$ as in [7,8], there will be some logarithmic divergences in obtaining uniform estimates of the Benjamin-Ono-Burgers equation, especially in Lemma 3.2. $Y_0$ has a finer structure than $Y_0$. The similar idea of using the decomposition to overcome the logarithmic divergences in the uniform estimates of inviscid limit was also used by Guo et al. [5,6].

**Definition 2.4.** The basic Banach spaces $Z_k$ are defined by

$$Z_k := X_k, \quad \text{if } 1 \leq k \leq 99 \quad \text{and} \quad Z_k := X_k + Y_k, \quad \text{if } k \geq 100 \quad \text{or} \quad k = 0. \quad (2.7)$$

In some estimates we will also use the space $Z_0$, which contains the space $Z_0$, i.e., $Z_0 \subseteq Z_0$.

**Definition 2.5.** It holds that

$$Z_0 = \left\{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ is supported in } \tilde{I}_0 \times \mathbb{R} \text{ and} \right\}$$

$$\| f \|_{Z_0} := \sum_{j=0}^{\infty} 2^j \| \eta_j(\tau) f(\xi, \tau) \|_{L^2_{\xi, \tau}} < \infty. \quad (2.8)$$

**Definition 2.6.** For $k \in \mathbb{Z}_+$, let

$$\begin{align*}
A_k(\xi, \tau) = \tau - \omega(\xi) + i, & \quad \text{if } k \geq 1, \\
A_k(\xi, \tau) = \tau + i, & \quad \text{if } k = 0.
\end{align*}$$

Then for $\sigma \geq 0$, we define the Banach spaces $F^\sigma = F^\sigma(\mathbb{R} \times \mathbb{R})$ and $N^\sigma = N^\sigma(\mathbb{R} \times \mathbb{R})$:

$$F^\sigma = \left\{ u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) : \| u \|_{F^\sigma}^2 := \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k(\xi)(I - \partial^2_x) F(u) \|_{Z_k}^2 < \infty \right\} \quad (2.9)$$
and
\[ N^\sigma = \left\{ u \in S'(\mathbb{R} \times \mathbb{R}) : \| u \|_{N^\sigma}^2 := \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k(\xi)A_k(\xi, \tau)^{-1} F(u) \|_{L_k^\infty}^2 < \infty \right\}. \] (2.10)

We establish some basic properties and known estimates which can be found in [8]. If \( k \geq 1 \) and \( f_k \in Z_k \), then \( f_k \) can be written in the following form:
\[ \begin{cases}
    f_k = \sum_{j=0}^{\infty} f_{k,j} + g_k, \\
    \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \| f_{k,j} \|_{L^2} + \| g_k \|_{Y_k} \leq 2 \| f_k \|_{Z_k},
\end{cases} \] (2.11)
such that \( \text{supp } f_{k,j} \subset D_{k,j} \) and \( \text{supp } g_k \subset \bigcup_{j=0}^{k-1} D_{k,j} \) (if \( k \leq 99 \), then \( g_k \equiv 0 \)). If \( f_0 \in Z_0 \), then \( f_0 \) can be written in the following form:
\[ \begin{cases}
    f_0 = \sum_{j=0}^{\infty} \sum_{k'=\infty}^{1} f_{0,k'}^{k'} + \sum_{j=-\infty}^{\infty} g_{0,j}, \\
    \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{1} 2^{-k'/2} \| f_{0,k'}^{k'} \|_{L^2} + \sum_{j=1}^{\infty} 2^j \| F^{-1}(g_{0,j}) \|_{L^1 L^2} + \sum_{j \leq 0} \| F^{-1}(g_{0,j}) \|_{L^1 L^2} \leq 2 \| f_0 \|_{Z_0},
\end{cases} \] (2.12)
such that \( \text{supp } f_{0,k'}^{k'} \subset D_{k',j} \) and \( \text{supp } g_{0,j} \subset \tilde{I}_0 \times I_j \).

**Lemma 2.7.** (a) If \( m, m' : \mathbb{R} \rightarrow \mathbb{C}, k \geq 0 \) and \( f_k \in Z_k \), then
\[ \begin{align*}
    \| m(\xi) f_k(\xi, \tau) \|_{Z_k} &\leq C \| F^{-1}(m) \|_{L^1(\mathbb{R})} \| f_k \|_{Z_k}, \\
    \| m'(\tau) f_k(\xi, \tau) \|_{Z_k} &\leq C \| m' \|_{L^\infty(\mathbb{R})} \| f_k \|_{Z_k}.
\end{align*} \] (2.13)
(b) If \( k \geq 1, j \geq 0 \) and \( f_k \in Z_k \), then
\[ \| \eta_j(\tau - \omega(\xi)) f_k(\xi, \tau) \|_{X_k} \leq C \| f_k \|_{Z_k}. \] (2.14)
(c) If \( k \geq 1, j \in [0, k] \) and \( f_k \) is supported in \( I_k \times \mathbb{R} \), then
\[ \| F^{-1}[\eta_{\leq j}(\tau - \omega(\xi)) f_k(\xi, \tau)] \|_{L^1 L^2} \leq C \| f_k \|_{L^1 L^2}. \] (2.15)

**Lemma 2.8.** If \( k \geq 0, t \in \mathbb{R} \) and \( f_k \in Z_k \), then
\[ \begin{align*}
    \left\| \int_{\mathbb{R}} f_k(\xi, \tau) e^{it\xi} d\tau \right\|_{L_k^2} &\leq C \| f_k \|_{Z_k}, \quad \text{if } k \geq 1, \\
    \left\| \int_{\mathbb{R}} f_0(\xi, \tau) e^{it\xi} d\tau \right\|_{B_0} &\leq C \| f_0 \|_{Z_0}, \quad \text{if } k = 0.
\end{align*} \] (2.16)

As a consequence,
\[ F^\sigma \subseteq C(\mathbb{R} : \tilde{H}^\sigma) \text{ for any } \sigma \geq 0. \] (2.17)

### 3 Uniform linear estimates

In this section, we construct the uniform homogeneous and inhomogeneous linear estimates for the BOB equation. The dissipative structure \(-c \partial_x^2 u\) destroys some symmetries and brings some logarithmic divergences, which will bring several difficulties.

For \( \phi \in L^2(\mathbb{R}) \), we denote the solution of the free BOB equation by
\[ W_c(t)\phi = F^{-1}_x e^{it(c/2)(\xi-x)^2} F_x \phi, \quad t \geq 0, \] (3.1)
where $\omega(\xi)$ is defined in (2.1). To consider the time-space Fourier transform, we extend $W(t)\phi$ to an operator defined on the whole line by $W(t)\phi = F^{-1}(e^{i\omega(t)\xi^2}F_x\phi)$, $t \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \to [0, 1]$ is an even time cut-off function, which is supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. In the following discussions, the implicit constant in the inequality sign $\lesssim$ is independent of $\epsilon$. We first prove a uniform estimate for the free solution.

**Lemma 3.1.** If $\sigma \geq 0$ and $\phi \in \tilde{H}^\sigma$, then for any $\epsilon \in [0, 1]$, $\|\psi(t) \cdot (W(t)\phi)\|_{F^\sigma} \leq C\|\phi\|_{B^\sigma}$, where the constant $C$ is independent of $\epsilon$.

**Proof.** It follows from the definition of $F^\sigma$ that

$$
\|\psi(t) \cdot (W(t)\phi)\|_{F^\sigma}^2 = \sum_{k=0}^{\infty} 2^{2k} \|\eta_k(\xi)(I - \partial^2_x)F(\psi(t)W(t)\phi)\|_{Z_k}^2.
$$

In view of the definition of $\tilde{H}^\sigma$, it suffices to prove that

$$
\|\eta_k(\xi)(I - \partial^2_x)F(\psi(t)W(t)\phi)\|_{Z_0} \lesssim \|\eta_k(\xi)F_x(\phi)(\xi)\|_{B_0}, \quad \text{for any } k \geq 1.
$$

Define $\varphi(t) = (1 + t^2)\psi(t) \in S(\mathbb{R}^+)$. We have

$$
(1 - \partial^2_x)F(\psi(t)W(t)\phi) = F_t((1 + t^2)\psi(t)e^{i\omega(t)\xi^2})F_x(\phi)(\xi)
$$

$$
= (F_t(\varphi(t)e^{-|t|\xi^2}))(\tau - \omega(\xi))F_x(\phi)(\xi).
$$

(1) The proof of (3.2) for $k = 0$. From (3.4), we have

$$
\|\eta_0(\xi)(I - \partial^2_x)F(\psi(t)W(t)\phi)\|_{Z_0} = \|\eta_0(\xi)F_x(\phi)(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi))\|_{Z_0}.
$$

Write $\eta_0 \cdot F_x(\phi) = g(\xi) + \sum_{k \leq 1} h_k$, where $h_k$ is supported in $I_k$. Then

$$
\|F^{-1}(g)\|_{L^1_\tau} + \sum_{k \leq 1} 2^{-k/2}\|h_k\|_{L^2_\tau} \leq 2\|\eta_0 \cdot F_x(\phi)\|_{B_0}.
$$

and (3.5) is controlled by

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi))\|_{Z_0} + \sum_{k \leq 1} \|h_k(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi))\|_{X_0}.
$$

We divide the first term in (3.7) into two parts $I + II$ as follows:

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi))\|_{Y_0} + \|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi)) - F_t(\varphi(t)e^{-|t|\xi^2})(\tau)\|_{X_0}.
$$

For the term $I$, by the definition and Young’s inequality, we know that

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)\|_{Y_0} = \sum_{j \geq 1} 2^j \|\chi_j F^{-1}[g(\xi)\eta_j(\tau)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)]\|_{L^1_\tau L^2_\xi}.
$$

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)\|_{Y_0} \lesssim \sum_{j \geq 1} 2^j \|\chi_j F^{-1}[g(\xi)\eta_j(\tau)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)]\|_{L^1_\tau L^2_\xi}.
$$

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)\|_{Y_0} \lesssim \sum_{j \geq 1} 2^j \|\chi_j F^{-1}[g(\xi)\eta_j(\tau)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)]\|_{L^1_\tau L^2_\xi}.
$$

$$
\|g(\xi)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)\|_{Y_0} \lesssim \sum_{j \geq 1} 2^j \|\chi_j F^{-1}[g(\xi)\eta_j(\tau)F_t(\varphi(t)e^{-|t|\xi^2})(\tau)]\|_{L^1_\tau L^2_\xi}.
$$

(3.8)
It suffices to prove that
\[
\sum_{j \geq 1} 2^j \| \mathcal{F}_\xi^{-1} [\eta(0,1)(\xi) \eta_j(\tau) \mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau)] \|_{L_2^x L_2^t} \lesssim 1 \tag{3.9}
\]
and
\[
\sum_{j' \leq 0} \| \mathcal{F}_\xi^{-1} [\eta(0,1)(\xi) \chi_{j'}(\tau) \mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau)] \|_{L_1^x L_2^t} \lesssim 1. \tag{3.10}
\]

We divide them into two cases where \(|x| \leq C\) and \(|x| > C\); if \(|x| \leq C\), by Hölder’s inequality and Taylor’s expansion we know that
\[
\sum_{j \geq 1} 2^j \| \mathcal{F}_\xi^{-1} [\eta(0,1)(\xi) \eta_j(\tau) \mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau)] \|_{L_2^x L_2^t} \\
= \sum_{j \geq 1} 2^j \left\| \eta(0,1)(\xi) \eta_j(\tau) \mathcal{F}_t \left( \varphi(t) \sum_{n \geq 0} \frac{(-1)^n e^n}{n!} |t|^n \right)(\tau) \right\|_{L_2^x L_2^t} \\
\lesssim \sum_{n \geq 0} \frac{C^n}{n!} \sum_{j \geq 1} 2^j \| \eta_j(\tau) \|_{L_2^x} \| \mathcal{F}_t(\varphi(t)|t|^n)(\tau) \|_{L_2^t} \\
\lesssim \sum_{n \geq 0} \frac{C^n}{n!} \| \varphi(t)|t|^n \|_{L_1^x} \lesssim 1,
\]
where we used the fact that \(\sum_{j' \leq 0} \| \chi_{j'}(\tau) \|_{L_2^x} \lesssim \sum_{j' \leq 0} 2^{j'/2} \lesssim 1\). If \(|x| > C\), then \(|x| \sim \langle x \rangle\). For any fixed \(x\), we have
\[
\sum_{j' \leq 0} 2^{j/2} \| \eta_j(\tau) \mathcal{F}_t |\varphi(t)\mathcal{F}_\xi^{-1}(e^{-|t|\xi^2})|(\tau) \|_{L_2^x} \\
\lesssim \| \varphi(t) \mathcal{F}_\xi^{-1}(e^{-|t|\xi^2}) \|_{H^2} \lesssim \| \varphi(t)(\sqrt{|t|}e)^{-1}e^{-|x|^2/(|t|\xi^2)} \|_{H^2} \lesssim |x|^{-2}
\]
and
\[
\sum_{j' \leq 0} \| \chi_{j'}(\tau) \mathcal{F}_t |\varphi(t)\mathcal{F}_\xi^{-1}(e^{-|t|\xi^2})|(\tau) \|_{L_2^x} \\
\lesssim \| \varphi(t) \mathcal{F}_\xi^{-1}e^{-|t|\xi^2} \|_{L_1^x} \lesssim \| \varphi(t)(\sqrt{|t|}e)^{-1}e^{-|x|^2/(|t|\xi^2)} \|_{L_1^x} \lesssim |x|^{-2}.
\]
Therefore, one can obtain the conclusions (3.9) and (3.10). For the term \(II\), by the definition, the mean value theorem and Taylor’s expansion, for some \(\theta \in [0, 1]\), we have
\[
\| g(\xi) |\mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi)) - \mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau) \|_{L_2^x} \\
\lesssim \sum_{j = 0}^{2j/2} \| \eta_j(\tau) \chi_{k'}(\xi) g(\xi) |\mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau - \theta\omega(\xi)) \|_{L_2^x} \\
\lesssim \sum_{j = 0}^{2j/2} \sup_{|\xi| \leq 2} \| P_j(t\varphi(t)e^{-|t|\xi^2}e^{i\theta\omega(\xi)}) \|_{L_2^x} \sum_{k' \leq 1} 2^{k'} \| g(\xi) \|_{L_\infty^x} \\
\lesssim \sum_{j = 0}^{2j/2} \| P_j(\varphi(t)|t|^{n+1}) \|_{L_2^x} \| \mathcal{F}_\xi^{-1}(g) \|_{L_2^x} \sup_{|\xi| \leq 2} \sum_{n \geq 0} |\xi|^{k'} + i|\theta| |\xi| \| n! \|_{n!}}
\[ \lesssim \sum_{n \geq 0} \frac{C_n}{n!} \| \varphi(t) t^{n+1} \|_{L^2} \| F^{-1}(g) \|_{L^1} \lesssim \| F^{-1}_k(g) \|_{L^1}. \]  

(3.11)

In view of (3.8)–(3.11), we can obtain
\[ \| g(\xi) F_\tau(\varphi(t) e^{-|t|\epsilon \xi^2})(\tau - \omega(\xi)) \|_{L^2} \lesssim \| F^{-1}_k(g) \|_{L^1}. \]  

(3.12)

For the second term in (3.7), recalling that \( h_{k'} \) is supported in \( I_{k'} \), from the definition and Taylor’s expansion, we can obtain that for any fixed \( k' \),
\[
\begin{align*}
&\| h_{k'}(\xi) F_\tau(\varphi(t) e^{-|t|\epsilon \xi^2})(\tau - \omega(\xi)) \|_{L^2} \\
&\lesssim \sum_{j \geq 0} 2^{j-k'/2} \| \eta_j(\tau) h_{k'}(\xi) F_\tau(\varphi(t) e^{-|t|\epsilon \xi^2} e^{-i \tau \xi})(\tau) \|_{L^1} \\
&\lesssim \sum_{j \geq 0} 2^j \sup_{| \xi | \leq 2^j} \| P_j(\varphi(t) e^{-|t|\epsilon \xi^2}) \|_{L^1} \cdot 2^{-k'/2} \| h_{k'}(\xi) \|_{L^1} \\
&\lesssim \sum_{j \geq 0} 2^j \sup_{| \xi | \leq 2^j} \| P_j(\varphi(t) t^n) \|_{L^1} \cdot 2^{-k'/2} \| h_{k'}(\xi) \|_{L^1} \\
&\lesssim \sum_{n \geq 0} C_n \| \varphi(t) t^n \|_{L^1} \cdot 2^{-k'/2} \| h_{k'}(\xi) \|_{L^1} \lesssim 2^{-k'/2} \| h_{k'}(\xi) \|_{L^1}. 
\end{align*}
\]  

(3.13)

Therefore, combining (3.5)–(3.7) and (3.12)–(3.13), we obtain the conclusion (3.2).

(2) The proof of (3.3) for \( k \geq 1 \), by the change of variables and Hölder’s inequality, we obtain
\[
\| \eta_k(\xi)(I - \partial_\xi^2) F_\tau(\varphi(t) W_\tau(t) \phi) \|_{L^1} = \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \| \eta_k(\xi) F_\tau(\phi(\xi)) \|_{L^1} \lesssim \| \eta_k(\xi) F_\tau(\phi(\xi)) \|_{L^1} \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \sup_{| \xi | \sim 2^j} \| P_j(\varphi(t) e^{-|t|\epsilon \xi^2}) \|_{L^1}.
\]

It suffices to show that for any \( k \geq 1 \),
\[
\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \sup_{| \xi | \sim 2^j} \| P_j(\varphi(t) e^{-|t|\epsilon \xi^2}) \|_{L^1} \lesssim 1,
\]  

(3.14)

where the implicit constant is independent of \( \epsilon \) and \( k \). By using Plancherel’s equality and the fact that
\[
F_\tau(e^{-|t|})(\tau) = \frac{C}{1 + |\tau|^2},
\]

which is called the Poisson kernel, we know that if \( |\xi| \sim 2^k \), then for any \( j \geq 0 \),
\[
\| P_j(e^{-|t|\epsilon \xi^2})(t) \|_{L^1} \lesssim \| P_j(e^{-2^k |t|})(t) \|_{L^1}. 
\]  

(3.15)

To prove (3.14) we may assume \( j \geq 100 \) in the summation. Using the para-product homogeneous decomposition, we have
\[
P_j(u_1 u_2) = P_j \left( \sum_{r \geq j-10} (P_{r+1} u_1) (P_{r} u_2) + (P_{r} u_1) (P_{r+1} u_2) \right) =: P_j(I + II). 
\]  

(3.16)

Now we take \( u_1 = e^{-|t|\epsilon \xi^2} \) and \( u_2 = \varphi(t) \). For \( P_j(I) \), it follows from Hölder’s inequality and (3.15) that
\[
\sum_{j \geq 100} 2^{j/2} \beta_{k,j} \sup_{| \xi | \sim 2^j} \| P_j(I) \|_{L^1} \lesssim \sum_{j \geq 100} 2^{j/2} \beta_{k,j} \sum_{r \geq j-10} \| P_{r+1} e^{-|t|\epsilon \xi^2} \|_{L^1} \| P_r \varphi(t) \|_{L^1}.
\]
follows from Bernstein’s estimate, Hölder’s inequality and (3.15) that

\[ \| e^{-|t|} \|_{L^1} \leq \| e^{-|t|} \|_{L^2} \leq C, \]

where we used the facts that \( \| e^{-|t|} \|_{L^1} \sim \lambda^{-1/2} \| e^{-|t|} \|_{L^2} \) and \( e^{-|t|} \in \hat{B}_{2,1}^1 \). For \( P_j(II) \), it follows from Bernstein’s estimate, Hölder’s inequality and (3.15) that

\[ \sum_{j \geq 100} \frac{2^{j/2} \beta_{k,j}}{\epsilon} \sup_{|t| \sim 2^j} \| P_j(II) \|_{L^2} \leq \sum_{j \geq 100} \frac{2^{j/2} \beta_{k,j}}{\epsilon} \sup_{|t| \sim 2^j} \| P_j e^{-|t|^2} \|_{L^2} \leq \sum_{m \geq 1} 2^{m/2} \| P_m e^{-|t|^2} \|_{L^2} \leq \| e^{-|t|} \|_{L^2} \leq \| \hat{f}_0 \|_{L^2} \leq 1. \]

Now we obtain the conclusion (3.14) and then complete the proof of (3.3). \( \square \)

Before giving the inhomogeneous linear estimates, we state an important lemma, which will conquer the singularity when we treat \( 1/(\tau - \omega(\xi) - i\epsilon \xi^2) \). In addition, this lemma will effectively simplify the proof of uniform inhomogeneous estimates.

**Lemma 3.2.** If one of the following two assumptions holds:

1. \( k \geq 100 \) and \( f_k \) is supported in \( \bigcup_{j=1}^{k-1} D_{k,j} \) such that \( f_k \in Y_k \);
2. \( k = 0 \) and \( f_0 \) is supported in \( Y_0 \times \mathbb{R} \) such that \( f_0 \in Y_0 \),

then for any \( \epsilon \in [0, 1] \),

\[ \frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i\epsilon \xi^2} f_k(\xi, \tau) \in Y_k, \ k \geq 100 \ or \ k = 0. \]

In particular, we have

\[ \left\| \frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i\epsilon \xi^2} f_k(\xi, \tau) \right\|_{Y_k} \leq \left\| f_k \right\|_{Y_k}, \ k \geq 100 \ or \ k = 0. \] (3.17)

**Proof.** (1) \( k \geq 100 \). By the definition of \( Y_k \), it suffices to prove that

\[ \left\| F^{-1} \left[ \frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i\epsilon \xi^2} f_k(\xi, \tau) \right] \right\|_{L^1_{\xi} L^2_{\tau}} \leq C. \]

In view of Plancherel’s theorem and the support of \( f_k \), we only need to prove that

\[ \left\| \int_R e^{i\epsilon \xi} \frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i\epsilon \xi^2} \eta_{\leq k} (\tau - \omega(\xi)) \chi_{[k-1, k]}(\xi) d\xi \right\|_{L^1_{\xi} L^2_{\tau}} \leq C. \] (3.18)

The function on the left-hand side of (3.18) is not zero only if \( |\tau| \sim 2^{2k} \). By symmetry, we may assume \( \{ (\xi, \tau) : \xi \in [2^{k-2}, 2^{k+2}], \tau \in [-2^{2k+10}, -2^{2k-10}] \} \). Rewrite

\[ \frac{\tau + \xi^2}{\tau + \xi^2 - i\epsilon \xi^2} = \frac{1}{1 - i\epsilon} \left( 1 + \frac{-i\epsilon \tau}{\tau + \xi^2 - i\epsilon \xi^2} \right) =: I + II. \]

For \( I \), by integration by parts, it is easy to show that

\[ \left\| \int (I - \Delta L) e^{i\epsilon \xi} \eta_{\leq k} (\tau + \xi^2) \chi_{[k-1, k]}(\xi) d\xi \right\| \leq \frac{1}{1 + x^2}, \]

where we used the fact \( \{ \xi \in [2^{k-2}, 2^{k+2}] : |\tau + \xi^2| \leq 2^{k+2} \} \leq C. \)
For $II$, the case $|x| \lesssim 1$ is trivial, and thus we just consider $|x| \geq 1$. Indeed, let

$$a := \sqrt{\frac{\tau}{1 - i\epsilon}} = -i|\tau|^{1/2}(1 + \epsilon^2)^{-1/4}\left(\cos \frac{\arctan \epsilon}{2} + i \sin \frac{\arctan \epsilon}{2}\right), \quad \text{Re} a \sim |\tau|^{1/2}.$$

Then by the properties of the Poisson kernel

$$a \cdot F^{-1}_\xi \left(\frac{-i\epsilon \tau}{\tau + \xi^2 - i \epsilon \xi^2}\right) = F^{-1}_\xi \left(\frac{1}{1 - i\epsilon} \left(\frac{1}{\tau + \xi^2 + i \epsilon \xi^2}\right)\right) = -i\epsilon C a \cdot e^{-a|x|}.$$ 

Therefore,

$$\left|F^{-1}_\xi \left(\frac{-i\epsilon \tau}{\tau + \xi^2 - i \epsilon \xi^2}\right)\right| \leq C e^{2|\tau|^{1/2}e^{-c|\tau|^{1/2}|x|}} \lesssim e^{2k\epsilon e^{-c2^{1/2}|x|}},$$

whose $L^1_x$ norm is bounded, and then we obtain the conclusion (3.18).

(2) $k = 0$. By the definition of $Y_0$, we need to show that for any $j \in \mathbb{Z}$,

$$\left\|F^{-1}_\xi \left(\frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i \epsilon \xi^2} x_j(\tau)f_0(\xi, \tau)\right)\right\|_{L^1_x L^\infty_y} \lesssim \|F^{-1}_\xi x_j(\tau)f_0(\xi, \tau)\|_{L^1_x L^\infty_y}.$$

Combining Plancherel’s theorem with Young’s inequality, we can prove that

$$\left\|F^{-1}_\xi \left(\frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i \epsilon \xi^2} x_j(\tau)\eta_{[0,1]}(\xi)\right)\right\|_{L^1_x L^\infty_y} \lesssim 1. \quad (3.19)$$

Similar to (1), we may assume $\xi > 0$ and rewrite

$$\frac{\tau + \xi^2}{\tau + \xi^2 - i \epsilon \xi^2} = \frac{1}{1 - i\epsilon} \left(1 + \frac{-i\epsilon \tau}{\tau + \xi^2 - i \epsilon \xi^2}\right) = : I + II.$$

Notice that

$$\left|F^{-1}_\xi \left(\frac{-i\epsilon \tau}{\tau + \xi^2 - i \epsilon \xi^2}\right)\right| \leq C e^{2|\tau|^{1/2}e^{-c|\tau|^{1/2}|x|}} \lesssim e^{2^{1/2}e^{-c2^{1/2}|x|}} \in L^1_x.$$

Then we can obtain (3.19) in the same way as in (1). The proof is completed.

For the inhomogeneous linear operator, we have the following uniform estimates.

**Lemma 3.3.** If $\sigma \geq 0$ and $u \in N^\sigma$, then for any $\epsilon \in [0, 1]$,

$$\left\|\psi(t) \cdot \int_0^t W_s(t - s)(u(s))ds\right\|_{L^\infty_\sigma} \leq C \|u\|_{N^\sigma},$$

where the constant $C$ is independent of $\epsilon$.

**Proof.** By the definitions, it suffices to prove that $\forall k \geq 0$,

$$\left\|\eta_k(\xi)(I - \partial^2_x)\mathcal{F} \left[\psi(t) \cdot \int_0^t W_s(t - s)(u(s))ds\right]\right\|_{Z_{k+1}} \lesssim \left\|\eta_k(\xi)A_k(\xi, \tau)^{-1}\mathcal{F}(u)\right\|_{Z_k}. \quad (3.20)$$

From a straightforward calculation, we have

$$(I - \partial^2_x)\mathcal{F} \left[\psi(t) \cdot \int_0^t W_s(t - s)(u(s))ds\right](\xi, \tau)$$
For the sake of briefness, we only consider the case where the denominator is $i(\tau' - \omega(\xi)) + \epsilon \xi^2$ in the above fraction, because it is similar to the case of $i(\tau' - \omega(\xi)) - \epsilon \xi^2$. Let $\varphi(t) := (1 + t^2)\psi(t)$ and $f_k(\xi, \tau) := \eta_k(\xi) A_k(\xi, \tau)^{-1} F(u)(\xi, \tau)$ for $k \in \mathbb{Z}_+$. For $f_k \in Z_k$, let

$$T(f_k)(\xi, \tau) := F_t \left[ \varphi(t) \cdot \int_{\mathbb{R}} \frac{e^{i t(\tau' - \omega(\xi))} - e^{-|t| \epsilon \xi^2}}{i(\tau' - \omega(\xi)) + \epsilon \xi^2} f_k(\xi, \tau') d\tau' \right](\tau - \omega(\xi)).$$

In view of (3.21)–(3.22), to prove (3.20), we only need to prove that

$$\|T\|_{Z_k \to Z_k} \leq C \quad \text{uniformly in } k \in \mathbb{Z}_+ \text{ and } \epsilon \in [0, 1].$$

(1) Case $k \geq 1$. (1-a) Assume first that $f_k \in X_k$. The idea of this part is essential due to [6, 16]. Define $f_k^{\#}(\xi, \mu') = f_k(\xi, \mu' + \omega(\xi))$ and $T(f_k^{\#})(\xi, \mu) = T(f_k)(\xi, \mu + \omega(\xi))$. Then

$$T(f_k^{\#})(\xi, \mu) = F_t \left[ \varphi(t) \cdot \int_{\mathbb{R}} \frac{e^{i t(\tau' - \omega(\xi))} - e^{-|t| \epsilon \xi^2}}{i(\tau' - \omega(\xi)) + \epsilon \xi^2} f_k(\xi, \tau') d\tau' \right](\mu) \quad \text{(3.24)}.$$

It suffices to prove that

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\mu) T(f_k^{\#})(\xi, \mu)\|_{L^2_x} \lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \|\eta_j(\mu) f_k^{\#}(\xi, \mu)\|_{L^2_x}. \quad \text{(3.25)}$$

We divide $T(f_k^{\#})(\xi, \mu)$ into four parts:

$$T(f_k^{\#})(\xi, \mu) = F_t \left[ \varphi(t) \cdot \int_{|\mu'| \leq 1} \frac{e^{i t \mu'} - 1}{i(\mu' + \epsilon \xi^2)} f_k(\xi, \mu') d\mu' \right](\mu) + F_t \left[ \varphi(t) \cdot \int_{|\mu'| \leq 1} \frac{1 - e^{-|t| \epsilon \xi^2}}{i(\mu' + \epsilon \xi^2)} f_k(\xi, \mu') d\mu' \right](\mu) + F_t \left[ \varphi(t) \cdot \int_{|\mu'| \geq 1} \frac{e^{i t \mu'}}{i(\mu' + \epsilon \xi^2)} f_k(\xi, \mu') d\mu' \right](\mu) - F_t \left[ \varphi(t) \cdot \int_{|\mu'| \geq 1} \frac{e^{-|t| \epsilon \xi^2}}{i(\mu' + \epsilon \xi^2)} f_k(\xi, \mu') d\mu' \right](\mu) : = I + II + III - IV.$$

When $|\mu'| \geq 1$, the denominator in the fraction is far from 0, and then $(\mu' + i)/(i\mu' + \epsilon \xi^2)$ is bounded (see the parts III and IV). When $|\mu'| \leq 1$, we could use Taylor’s expansion for the numerator to cancel the denominator (see the parts I and II). We now estimate the contributions of $I$–$IV$. Firstly, we consider the contribution of $IV$, i.e.,

$$\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) F_t(\varphi(t) e^{-|t| \epsilon \xi^2}) (\mu) \int_{|\mu'| \geq 1} \frac{\mu' + i}{i(\mu' + \epsilon \xi^2)} f_k^{\#}(\xi, \mu') d\mu' \right\|_{L^2_x} \lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \sup_{\xi \in I_k} \|P_{\xi}(\varphi(t) e^{-|t| \epsilon \xi^2})(t)\|_{L^2} \cdot \int_{|\mu'| \geq 1} \|f_k^{\#}(\xi, \mu')\|_{L^2} d\mu' \lesssim \int_{|\mu'| \geq 1} \|f_k^{\#}(\xi, \mu')\|_{L^2} d\mu' \lesssim \sum_{j \geq 0} 2^{j/2} \|\eta_j(\mu) f_k^{\#}(\xi, \mu)\|_{L^2_x}.$$
where we used the inequality (3.14). Secondly, we consider the contribution of $III$, i.e.,

\[
\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) \mathcal{F}_t \left[ \varphi(t) \left( \int_{|\mu'| \leq 1} e^{it\mu'} \frac{\mu' + i}{i|\mu' + \epsilon \xi^2 f_k^\#(\xi, \mu')} d\mu' \right) \right](\mu) \right\|_{L^2_{\xi,\mu}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) \mathcal{F}_t \varphi \right\|_{L^2_{\xi,\mu}} + \left\| f_k^\#(\xi, \cdot) \right\|_{L^2_{\xi}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| P_j \mathcal{F}_t^{-1} \left[ \mathcal{F}_t \varphi \right] \cdot \mathcal{F}_t^{-1} f_k^\#(\xi, \mu) \right\|_{L^2_{\xi}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) f_k^\#(\xi, \mu) \right\|_{L^2_{\xi,\mu}},
\]

where we used the facts that $B^1_{2,1}$ and $B^1_{2,1}$ are multiplication algebras and that $\mathcal{F}_t^{-1}(|\mathcal{F}_t \varphi|) \in B^1_{2,1}$ and $\mathcal{F}_t^{-1}(|\mathcal{F}_t \varphi|) \in B^1_{2,1}$. Thirdly, we consider the contribution of $I$. By Taylor’s expansion, we obtain

\[
\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) I I \right\|_{L^2_{\xi,\mu}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \sup_{\xi \in I_k} \left\| P_j (\varphi(t)(1 - e^{-|\xi| \epsilon \xi^2})) \right\|_{L^2_{\xi}} \cdot \int_{|\mu'| \leq 1} \left\| f_k^\#(\xi, \mu') \right\|_{L^2_{\xi}} \left( \mu' \right)
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \left\| \eta_j(\mu) f_k^\#(\xi, \mu) \right\|_{L^2_{\xi,\mu}},
\]

where we used the inequality (3.14) and $\varphi \in B^1_{2,1}$. For $\epsilon \xi^2 \leq 1$, using Taylor’s expansion, we have

\[
\sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) I I \right\|_{L^2_{\xi,\mu}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\mu) \mathcal{F}_t \left[ \varphi(t) \left( \int_{|\mu'| \leq 1} \frac{|\xi|^n (\epsilon \xi^2)^n}{n!} \int_{|\mu'| \leq 1} \frac{\mu' + i}{i|\mu' + \epsilon \xi^2 f_k^\#(\xi, \mu')} d\mu' \right) \right](\mu) \right\|_{L^2_{\xi,\mu}}
\]

\[
\lesssim \sum_{j \geq 0} \frac{\left\| \varphi(t) \right\|_{B^1_{2,1}}}{n!} \left\| \int_{|\mu'| \leq 1} \frac{\epsilon \xi^2}{i|\mu' + \epsilon \xi^2 f_k^\#(\xi, \mu')} d\mu' \right\|_{L^2_{\xi}}
\]

\[
\lesssim \sum_{j \geq 0} 2^{j/2} \left\| \eta_j(\mu) f_k^\#(\xi, \mu) \right\|_{L^2_{\xi,\mu}},
\]

Now we have shown that

\[
\|T\|_{X_k \to X_k} \leq C \quad \text{uniformly in } k \geq 1 \text{ and } \epsilon \in [0,1]. \tag{3.26}
\]

(1-b) Assume now that $k \geq 100$ and $f_k = g_k \in Y_k$. From (2.14), we know that $\|\eta_j(\tau - \omega(\xi)) f_k \|_{X_k} \lesssim \|f_k\|_{Y_k}$, and thus we may assume that $g_k$ is supported in the set $\{ (\xi, \tau) : |\tau - \omega(\xi)| \leq 2^{k-20} \}$. For convenience, we decompose

\[
g_k(\xi, \tau') = \frac{\tau' - \omega(\xi)}{\tau' - \omega(\xi) + i} g_k(\xi, \tau') + \frac{i}{\tau' - \omega(\xi) + i} g_k(\xi, \tau'), \tag{3.27}
\]
and then (3.22) becomes
\[
T(g_k)(\xi, \tau) = \mathcal{F}_t \left[ \varphi(t) \cdot \int_{\mathbb{R}} \frac{\omega(t' - \omega(\xi)) - e^{-|t'|\xi^2}}{i(\tau' - \omega(\xi)) + \xi^2} (\tau' - \omega(\xi))g_k(\xi, \tau')d\tau' \right](\tau - \omega(\xi)) \\
+ iT \left( \frac{1}{\tau' - \omega(\xi) + 1} g_k \right)(\xi, \tau) \\
= \int_{\mathbb{R}} \frac{\tau' - \omega(\xi)}{i(\tau' - \omega(\xi)) + \xi^2} g_k(\xi, \tau')d\tau' \\
- \mathcal{F}_t(\varphi(t)e^{-|t'|\xi^2})(\tau - \omega(\xi)) \int_{\mathbb{R}} \frac{\tau' - \omega(\xi)}{i(\tau' - \omega(\xi)) + \xi^2} g_k(\xi, \tau')d\tau' \\
+ iT \left( \frac{1}{\tau' - \omega(\xi) + 1} g_k \right)(\xi, \tau). 
\]
(3.28)

We can use (3.26) to control the third term in (3.28). Notice that
\[
||\{\xi \in I_k : |\tau - \omega(\xi)| \leq 2^{j+1})|| \lesssim 2^{j-k}. 
\]
(3.29)

Then we have that from (3.26),
\[
\|T((\tau' - \omega(\xi) + i)^{-1}g_k)\|_{X_k} \lesssim \|(\tau' - \omega(\xi) + i)^{-1}g_k\|_{X_k} \\
\lesssim \sum_{0 \leq j \leq k} 2^{j/2} \beta_{k,j} \|\eta_j(\tau' - \omega(\xi))(\tau' - \omega(\xi) + i)^{-1}g_k(\xi, \tau')\|_{L_t^2 L_x^r} \\
\lesssim \sum_{0 \leq j \leq k} 2^{-3j/2}2^{j-k/2}\|\tau' - \omega(\xi) + i\|_{L_t^\infty L_x^r} \|g_k\|_{Y_k}.
\]

For the first and second terms in (3.28), it suffices to prove that
\[
\left\| \int_{\mathbb{R}} \frac{\tau' - \omega(\xi)}{\tau' - \omega(\xi) - i\xi^2} g_k(\xi, \tau')\hat{\varphi}(\tau - \tau')d\tau' \right\|_{Z_k} \lesssim \|g_k\|_{Y_k} 
\]
(3.30)

and
\[
\left\| \mathcal{F}_t(\varphi(t)e^{-|t'|\xi^2})(\tau - \omega(\xi)) \int_{\mathbb{R}} \frac{\tau' - \omega(\xi)}{\tau' - \omega(\xi) - i\xi^2} g_k(\xi, \tau')d\tau' \right\|_{X_k} \lesssim \|g_k\|_{Y_k}. 
\]
(3.31)

Thanks to Lemma 3.2, we know that
\[
\left\| \frac{\tau - \omega(\xi)}{\tau - \omega(\xi) - i\xi^2} f_k(\xi, \tau) \right\|_{Y_k} \lesssim \|f_k\|_{Y_k}.
\]

Then we can make the proof clearer and simpler. To prove (3.30) and (3.31), we just need to prove
\[
\left\| \int_{\mathbb{R}} g_k(\xi, \tau')\hat{\varphi}(\tau - \tau')d\tau' \right\|_{Z_k} \lesssim \|g_k\|_{Y_k} 
\]
(3.32)

and
\[
\left\| \mathcal{F}_t(\varphi(t)e^{-|t'|\xi^2})(\tau - \omega(\xi)) \int_{\mathbb{R}} g_k(\xi, \tau')d\tau' \right\|_{X_k} \lesssim \|g_k\|_{Y_k}. 
\]
(3.33)

The inequality (3.32) has been obtained by Ionescu and Kenig [8]. For the sake of completeness, we give the rigorous proof. For the low modulation part, we divide it into two subparts:
\[
g_k(\xi, \tau') = g_k(\xi, \tau') \left[ \frac{\tau' - \omega(\xi) + i}{\tau - \omega(\xi) + i} \right].
\]

Then the left-hand side of (3.32) is dominated by
\[
\left\| \eta_{[0,k-1]}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} \int_{\mathbb{R}} g_k(\xi, \tau')(\tau' - \omega(\xi) + i)\hat{\varphi}(\tau - \tau')d\tau' \right\|_{Y_k}
\]
\[ + \sum_{j \geq k-1} 2^{j/2} \left\| \eta_j(\tau - \omega(\xi)) (\tau - \omega(\xi) + i)^{-1} \int_{\mathbb{R}} g_k(\xi, \tau') \hat{\varphi}(\tau - \tau') d\tau' \right\|_{L^2_{\xi, \tau}} \]
\[ + \sum_{j \geq k-1} 2^{j/2} \beta_{k,j} \left\| \eta_j(\tau - \omega(\xi)) \int_{\mathbb{R}} g_k(\xi, \tau') \hat{\varphi}(\tau - \tau') d\tau' \right\|_{L^2_{\xi, \tau}} =: I + II + III. \]

For \( I \), we use Lemma 2.7(c) to bound it by
\[ I \lesssim 2^{-k/2} \left\| \mathcal{F}^{-1}_{\eta_0[k-1]}(\tau - \omega(\xi)) \int_{\mathbb{R}} g_k(\xi, \tau') (\tau' - \omega(\xi) + i) \hat{\varphi}(\tau - \tau') d\tau' \right\|_{L^2_{\xi, \tau}} \]
\[ \lesssim 2^{-k/2} \| \varphi(t) \cdot \mathcal{F}^{-1}[(\tau' - \omega(\xi) + i) g_k(\xi, \tau')] \|_{L^1_{\xi} L^2_{\tau}} \lesssim \| g_k \|_{Y_k}, \]

as desired. For \( II \), from (3.29) we can obtain
\[ II \lesssim \sum_{0 \leq j \leq k} 2^{-j/2} \left\| \int_{\mathbb{R}} g_k(\xi, \tau') \hat{\varphi}(\tau - \tau')(\tau - \tau') d\tau' \right\|_{L^2_{\xi, \tau}} \]
\[ \lesssim \| g_k(\xi, \tau) \|_{L^2_{\xi, \tau}} \lesssim \sum_{0 \leq j \leq k-1} 2^{-j} \| \eta_j(\tau - \omega(\xi))(\tau - \omega(\xi) + i) g_k(\xi, \tau) \|_{L^2_{\xi, \tau}} \]
\[ \lesssim \sum_{0 \leq j \leq k-1} 2^{-j/2} \left\| (\tau - \omega(\xi) + i) g_k(\xi, \tau) \right\|_{L^2_{\xi, \tau}} \lesssim \| g_k \|_{Y_k}. \]

For \( III \), let \( g^\#_{k}(\xi, \mu') = g_k(\xi, \mu' + \omega(\xi)). \) Then
\[ III \lesssim \sum_{j \geq k-1} 2^j \left\| \eta_j(\mu) \int_{\mathbb{R}} g^\#_{k}(\xi, \mu') \hat{\varphi}(\mu - \mu') d\mu' \right\|_{L^2_{\xi, \mu}} \]
\[ \lesssim \sum_{j \geq k-1} 2^j \sum_{j' \leq k-20} \left\| \eta_j(\mu) \eta_{j'}(\mu') \int_{\mathbb{R}} g^\#_{k}(\xi, \mu') \hat{\varphi}(\mu - \mu') d\mu' \right\|_{L^2_{\xi, \mu}} \]
\[ \lesssim \sum_{j' \leq k-20} \| \eta_{j'}(\mu) g^\#_{k}(\xi, \mu) \|_{L^2_{\xi, \mu}} \]
\[ \lesssim \sum_{j \geq k-20} 2^{-2j} \| \eta_j(\tau - \omega(\xi))(\tau - \omega(\xi) + i) g_k(\xi, \tau) \|_{L^2_{\xi, \tau}} \lesssim \| g_k \|_{Y_k}. \]

Finally, to prove (3.33), we define the modified Hilbert transform operator
\[ \mathcal{L}_k(g)(\mu) := \int_{\mathbb{R}} (g(\tau)(\tau - \mu + i)^{-1} \eta_{[0,k]}(\tau - \mu) d\tau, \ g \in L^2(\mathbb{R}). \]

Notice that
\[ \mathcal{F}^{-1}_{\tau} \left( \frac{1}{\tau - 1} \right) = \mathcal{F}^{-1}_{\tau} \left( \frac{\tau}{\tau^2 + 1} \right) + \mathcal{F}^{-1}_{\tau} \left( \frac{1}{\tau^2 + 1} \right) = C(-\text{sgn}(t) + i)e^{-ct} \in L^\infty(\mathbb{R}). \]

Hence by Plancherel’s theorem and Hölder’s inequality, we have \( \| \mathcal{L}_k \|_{L^2 \to L^2} \lesssim C \) uniformly in \( k \). We notice that if \( g_k \in Y_k \), then \( g_k \) can be written in the form
\[ g_k(\xi, \tau) = 2^{k/2} \chi_{[k-1,k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1} \eta_{[0,k]}(\tau - \omega(\xi)) \int_{\mathbb{R}} e^{-ix\xi} h(x, \tau) dx, \]
\[ \| g_k \|_{Y_k} = C \| h \|_{L^1_{\xi} L^2_{\tau}}. \]

From (3.14) and a change of variables, the left-hand side of (3.33) is dominated by
\[ \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \left\| \eta_j(\tau - \omega(\xi)) \mathcal{F}_t(\varphi(t)e^{-|t|\xi^2})(\tau - \omega(\xi)) \int_{\mathbb{R}} g_k(\xi, \tau') d\tau' \right\|_{L^2_{\xi, \tau}} \]
\[ \lesssim \sum_{j \geq 0} 2^{j/2} \beta_{k,j} \sup_{\xi \in I_k} \| P_j(\varphi(t)e^{-|t|\xi^2})(t) \|_{L^2_{\tau}} \cdot \left\| \int_{\mathbb{R}} g_k(\xi, \tau') d\tau' \right\|_{L^2_{\xi}} \]
Finally, we consider the contribution of $\eta_{\mu}(\xi,\mu)$. Similar to the previous case, we have

$$
\|f_{0}^{\#}(\xi,\mu)\|_{X_{0}} \leq \|f_{0}^{\#}(\xi,\mu)\|_{X_{0}} \leq \|f_{0}^{\#}(\xi,\mu)\|_{X_{0}} \leq \|f_{0}^{\#}(\xi,\mu)\|_{X_{0}}.
$$

By the similar argument to that for $k \geq 1$, we still divide $T(f_{0}^{\#})(\xi,\mu)$ into four parts:

$$
T(f_{0}^{\#})(\xi,\mu) = I + II + III - IV.
$$

We first consider the contribution of $I$. By the definition of $X_{0}$ and Taylor’s expansion, we obtain

$$
\|I\|_{X_{0}} \leq \sum_{j=0}^{\infty} \sum_{k'=1}^{n} \sum_{n=1}^{2^{j-k/2}/n!} 2^{j-k/2} \|\eta_{j}(\mu)\|_{L_{2}^{\mu}} \|\varphi(t)\|_{L_{2}^{t}} \|\varphi(t)\|_{L_{2}^{t}} \|\chi_{k'}(\xi)\|_{L_{2}^{\xi}} \|f_{0}^{\#}(\xi,\mu)\|_{L_{2}^{\mu}}.
$$

For $II$, we just use Taylor’s expansion for $1 - e^{-|t|\xi^{2}}$ and then use the factor $e^{t|\xi^{2}}$ to eliminate the denominator $i\mu' + \xi^{2}$ and obtain the conclusion similar to $I$. We then consider the contribution of $III$. Due to the algebraic structure of $B_{2,1}^{1}$, we know

$$
\|III\|_{X_{0}} \leq \sum_{j=0}^{\infty} \sum_{k'=1}^{n} 2^{j-k/2} \|\eta_{j}(\mu)\|_{L_{2}^{\mu}} \|\varphi(t)\|_{L_{2}^{t}} \|\varphi(t)\|_{L_{2}^{t}} \|\chi_{k'}(\xi)\|_{L_{2}^{\xi}} \|f_{0}^{\#}(\xi,\mu)\|_{L_{2}^{\mu}}.
$$

Finally, we consider the contribution of $IV$, i.e.,

$$
\|IV\|_{X_{0}} \leq \sum_{j=0}^{\infty} \sum_{k'=1}^{n} 2^{j-k/2} \|\eta_{j}(\mu)\|_{L_{2}^{\mu}} \|\varphi(t)\|_{L_{2}^{t}} \|\varphi(t)\|_{L_{2}^{t}} \|\chi_{k'}(\xi)\|_{L_{2}^{\xi}} \|f_{0}^{\#}(\xi,\mu)\|_{L_{2}^{\mu}}.
$$
Indeed, by the definition of $Y_0$ and Plancherel’s theorem, we have

$$
\|I_1\|_{Y_0} = \sum_{j\geq 1} 2^j \left\| \phi_j(\tau) \int_{\mathbb{R}} \phi(\tau - \tau') F^{-1}_\xi [g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_x L^2_t} + \sum_{j \geq 0} \left\| \chi_j(\tau) \int_{\mathbb{R}} \phi(\tau - \tau') F^{-1}_\xi [g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_x L^2_t} = I_1^h + I_1^l.
$$

For $I_1^h$, if $j \geq j' + C$, we know that $|\tau| \sim |\tau'|$, and thus by using Young’s inequality, we can obtain

$$
I_1^h \lesssim \sum_{j \geq j' + C} 2^j \left\| \phi_j(\tau) |\tau|^{-2} \int_{\mathbb{R}} \phi(\tau - \tau')(\tau - \tau')^2 |F^{-1}_\xi [g_{0,j'}(\xi, \tau')]| d\tau' \right\|_{L^1_x L^2_t} + \sum_{j \leq j' + C} 2^j \left\| \phi(\tau - \tau') F^{-1}_\xi [g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_x L^2_t} \lesssim 2^j \|F^{-1}_\xi [g_{0,j'}(\xi, \tau')]\|_{L^1_x L^2_t} = \|g_{0,j'}\|_{Y_0}.
$$

For $I_1^l$, we could use Hölder’s inequality and Young’s inequality to obtain

$$
I_1^l \lesssim \sum_{j \leq 0} \left\| \chi_j(\tau) \right\|_{L^2_t} \left\| \int_{\mathbb{R}} \phi(\tau - \tau') F^{-1}_\xi [g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_x L^2_t} \lesssim \|F^{-1}_\xi [g_{0,j'}(\xi, \tau')]\|_{L^1_x L^2_t} \lesssim \|g_{0,j'}\|_{Y_0}.
$$
Now the claim (3.34) is obtained, as desired.
For the term $I_2$, from (3.34) we only need to show that
\[
\left\| \frac{\omega(\xi) + i\epsilon \xi^2 + i}{\tau' - \omega(\xi) - i\epsilon \xi^2} g_{0,j'}(\xi, \tau') \right\|_{Y_0} \leq \|g_{0,j'}\|_{Y_0}. \tag{3.35}
\]
By the definition of $Y_0$, Plancherel’s theorem and Hölder’s inequalities, to prove (3.35), it suffices to prove that
\[
\left\| \int \frac{e^{ix\xi} \omega(\xi) + i\epsilon \xi^2 + i}{\tau' - \omega(\xi) - i\epsilon \xi^2} \eta_j(\tau') \chi_{I_0}(\xi) d\xi \right\|_{L^1_{\xi} L^\infty_{\tau'}} \leq C. \tag{3.36}
\]
Using the facts that $|\tau' - \omega(\xi)| \geq 1$ and $|\xi| \leq 2$, from integration by parts, we can easily obtain that
\[
\left| \int \frac{e^{ix\xi} \omega(\xi) + i\epsilon \xi^2 + i}{\tau' - \omega(\xi) - i\epsilon \xi^2} \eta_j(\tau') \chi_{I_0}(\xi) d\xi \right| \leq \frac{1}{(\xi)^2},
\]
which implies (3.36).

The estimates of the term $II$ can be achieved by using the results before. For $II_1$, from (3.12) we see that
\[
\|II_1\|_{Z_0} \lesssim \|F^{-1}_\xi \left[ \int g_{0,j'}(\xi, \tau') d\tau' \right]\|_{L^1_{\xi}} \lesssim 2^{j'/2} \|F^{-1}_\xi g_{0,j'}\|_{L^1_{\xi} L^2_{\tau'}} \lesssim \|g_{0,j'}\|_{Y_0}. \tag{3.37}
\]
Furthermore, (3.35) and (3.37) lead to $\|II_2\|_{Z_0} \lesssim \|g_{0,j'}\|_{Y_0}$.

If $j' \leq 4$, the singularity occurs by the reason that $\int (\tau' - \omega(\xi) + \epsilon \xi^2) is near the origin. We rewrite
\[
T(g_{0,j'})(\xi, \tau) = F_\xi \varphi(t) \cdot \int \frac{e^{it(\tau' - \omega(\xi)) - 1}{i(\tau' - \omega(\xi)) + \epsilon \xi^2} (\tau' + i) g_{0,j'}(\xi, \tau') d\tau'}{(\tau' - \omega(\xi)) + \epsilon \xi^2} (\tau' + i) g_{0,j'}(\xi, \tau') d\tau' + \int \frac{\varphi(\tau - \tau') - \varphi(\tau - \omega(\xi))}{\tau' - \omega(\xi)} (\tau' + i) g_{0,j'}(\xi, \tau') d\tau' + F_\xi |\varphi(t)| (1 - e^{-|t| \epsilon \xi^2}) (\tau - \omega(\xi)) \int \frac{\tau' + i}{i(\tau' - \omega(\xi)) + \epsilon \xi^2} g_{0,j'}(\xi, \tau') d\tau' =: A + B.
\]
Lemma 3.2 yields that for part $A$, we only need to prove
\[
\left\| \int \frac{\varphi(\tau - \tau') - \varphi(\tau - \omega(\xi))}{\tau' - \omega(\xi)} (\tau' + i) g_{0,j'}(\xi, \tau') d\tau' \right\|_{Z_0} \lesssim \|g_{0,j'}\|_{Y_0}. \tag{3.38}
\]
A simple calculation shows that
\[
\frac{\varphi(\tau - \tau') - \varphi(\tau - \omega(\xi))}{\tau' - \omega(\xi)} = c \int_0^1 \varphi'(\tau - \alpha \tau' - (1 - \alpha)\omega(\xi)) d\alpha.
\]
Because of $|\xi| \leq 2$ and $|\tau'| \leq C$, we write
\[
\varphi'(\tau - \alpha \tau' - (1 - \alpha)\omega(\xi))(\tau' + i) = \varphi'(\tau - \alpha \tau')(\tau' + i) + R(\xi, \tau, \tau'),
\]
where
\[
|R(\xi, \tau, \tau')| \leq C \xi^2 (1 + |\tau|)^{-4}.
\]
Therefore, to prove (3.38), we just need to show that for any $\alpha \in [0, 1],
\[
A_1 := \left\| \int \varphi'(\tau - \alpha \tau')(\tau' + i) g_{0,j'}(\xi, \tau') d\tau' \right\|_{Y_0} \lesssim \|g_{0,j'}\|_{Y_0}
\]
and
\[ A_2 := \left\| \xi^2 (1 + |\tau|)^{-4} \int |g_{0,j'}(\xi, \tau')| d\tau' \right\|_{X_0} \lesssim \|g_{0,j'}\|_{Y_0}. \]

In fact, for \(|\tau| \sim 2^j\) and \(|\tau'| \sim 2^{j'}\), if \(j \geq 10\), we have \(|\tau - \alpha \tau'| \sim |\tau|\). Thus, Minkowski’s inequality and Hölder’s inequality give
\[ A_1 = \sum_{j \geq 1} 2^j \left\| \eta_j(\tau) \int \phi(\tau - \alpha \tau') (\tau' + i) F_{\xi}^{-1}[g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_{\xi} L^2_{\tau}} + \sum_{j \geq 0} \left\| \chi_j(\tau) \int \phi(\tau - \alpha \tau') (\tau' + i) F_{\xi}^{-1}[g_{0,j'}(\xi, \tau')] d\tau' \right\|_{L^1_{\xi} L^2_{\tau}} \lesssim \left( \sum_{j \geq 10} 2^j \|\eta_j(\tau)(1 + |\tau|)^{-2}\|_{L^2_{\tau}} + 1 \right) \|F_{\xi}^{-1}[g_{0,j'}(\xi, \tau')]\|_{L^1_{\xi} L^2_{\tau}}, \]
\[ \lesssim 2^{j'/2} \|F_{\xi}^{-1}[g_{0,j'}(\xi, \tau')]\|_{L^1_{\xi} L^2_{\tau}} \lesssim \|g_{0,j'}\|_{Y_0}, \]
where we used the fact that \(\sum_{j' \leq 0} \|\chi_{j'}(\tau)\|_{L^2_{\tau}} \lesssim 1\). In addition, we can easily obtain
\[ A_2 = \sum_{j \geq 0} \sum_{k' \leq 1} 2^{j-k'/2} \left\| \eta_j(\tau) \chi_k(\xi) \xi^2 (1 + |\tau|)^{-4} \int |g_{0,j'}(\xi, \tau')| d\tau' \right\|_{L^2_{\xi} L^1_{\tau}} \lesssim \|g_{0,j'}(\xi, \tau')\|_{L^\infty_{\xi} L^1_{\tau}} \lesssim 2^{j'/2} \|F_{\xi}^{-1}[g_{0,j'}(\xi, \tau')]\|_{L^1_{\xi} L^2_{\tau}} \lesssim \|g_{0,j'}\|_{Y_0}. \]

This completes the proof of (3.38).

For part B, in order to eliminate the singularity, we will divide it into three subparts. Due to Taylor’s expansion, we have
\[ 1 - e^{-|t| \xi^2} = |t| \xi^2 - \sum_{n \geq 2} \frac{(-1)^n |t|^n \xi^{2n}}{n!}. \]

Let
\[ B_1 := (F_{\xi}(\phi(t)(1 - e^{-|t| \xi^2}))(\tau - \omega(\xi)) - F_{\xi}(\phi(t)(1 - e^{-|t| \xi^2}))(\tau)) \int_{\mathbb{R}} \frac{\tau' + i}{\tau' - \omega(\xi) - i \xi^2} g_{0,j'}(\xi, \tau') d\tau', \]
\[ B_2 := \sum_{n \geq 2} \frac{F_{\xi}(\phi(t)|t|^n (\xi^2)^n)(\tau)}{n!} \int_{\mathbb{R}} \frac{\tau' + i}{\tau' - \omega(\xi) - i \xi^2} g_{0,j'}(\xi, \tau') d\tau', \]
and
\[ B_3 := F_{\xi}(\phi(t)|t| \xi^2)(\tau) \int_{\mathbb{R}} \frac{\tau' + i}{\tau' - \omega(\xi) - i \xi^2} g_{0,j'}(\xi, \tau') d\tau'. \]

Next, we will prove that \(B_1, B_2 \in X_0\) and \(B_3 \in Y_0\). For \(B_1\), we use \(\xi^2\), which comes from Taylor’s expansion, to cancel the denominator \(\tau' - \omega(\xi) - i \xi^2\), and use \(\xi^2\), which comes from the mean value theorem, to absorb the big weight \(2^{-k'/2}\) in the definition of \(X_0\). Specifically, by using the mean value theorem and Taylor’s expansion, for some \(\theta \in [0, 1]\), we have
\[ \|B_1\|_{X_0} \lesssim \sum_{j \geq 0} \sum_{k' \leq 1} 2^{j-k'/2} \|\eta_j(\tau) \chi_k(\xi) \xi^2 F_{\xi}(t \phi(t)(1 - e^{-|t| \xi^2}))(\tau - \theta \omega(\xi)) \int_{\mathbb{R}} \frac{\tau' + i}{\tau' - \omega(\xi) - i \xi^2} g_{0,j'}(\xi, \tau') d\tau' \|_{L^1_{\xi} L^2_{\tau}} \lesssim \sum_{n \geq 1} \frac{C_n}{n!} \sum_{j \geq 0} 2^j \|\eta_j(\tau) F_{\xi}(\phi(t)|t|^{n+1})(\tau)\|_{L^2_{\xi}} \int_{\mathbb{R}} \frac{|\xi^2 g_{0,j'}(\xi, \tau')|}{|\tau' - \omega(\xi) - i \xi^2|} d\tau' \|_{L^\infty_{\xi}}. \]
Therefore, we complete the proof of Lemma 3.3.

For $B_2$, there is a small factor $(\epsilon \xi^2)^2$ as $n \geq 2$. We use one $\epsilon \xi^2$ to cancel the denominator $\tau' - \omega(\xi) - i\epsilon \xi^2$, and another $\epsilon \xi^2$ to absorb the big weight $2^{-k'/2}$ in the definition of $X_0$. Thus we can obtain

$$\|B_2\|_{X_0} \lesssim \sum_{n \geq 2} \sum_{k \leq 1} 2^{-k/2} \frac{\|\eta_j(\tau) F_r(\varphi(t) \tau)(\tau)\|_{L^1}}{n!} \cdot 2^{j/2} \|g_{0,j'}(\xi, \tau')\|_{L^\infty},$$

$$\lesssim \sum_{n \geq 2} \sum_{k \leq 1} 2^{-k/2} \frac{\|\eta_j(\tau) F_r(\varphi(t) \tau)(\tau)\|_{L^1}}{n!} \cdot 2^{j/2} \|g_{0,j'}(\xi, \tau')\|_{L^\infty} \lesssim \|g_{0,j'}\|_{Y_0}.$$

For $B_3$, notice that

$$\frac{\epsilon \xi^2}{\tau' - \omega(\xi) - i\epsilon \xi^2} = i \left( 1 - \frac{\tau' - \omega(\xi)}{\tau' - \omega(\xi) - i\epsilon \xi^2} \right).$$

By the proof of Lemma 3.2, we have

$$\left\|F^{-1} \left( \frac{\epsilon \xi^2 g_{0,j'}(\xi, \tau')}{\tau' - \omega(\xi) - i\epsilon \xi^2} \right) \right\|_{L^1 L^2} \lesssim \left\|F^{-1}(g_{0,j'}(\xi, \tau')) \right\|_{L^1 L^2}.$$

Therefore,

$$\|B_3\|_{Y_0} \lesssim \sum_{j \geq 1} 2^j \left\|\eta_j(\tau) F_r(\varphi(t) \tau)(\tau)\right\|_{L^1} \int_{\mathbb{R}} \frac{d\tau'}{\left\|\xi(\tau' + i)^{-1} f_{k_3} * f_0\right\|_{Z_k}} \lesssim \|f_{k_3}\|_{Z_k} \lesssim \|f_{k_3}\|_{Z_k} \|f_0\|_{Z_0}.$$

Now we have proved that

$$\|T\|_{Y_0 \to Z_0} \lesssim C \quad \text{uniformly in } \epsilon \in (0, 1].$$

Therefore, we complete the proof of Lemma 3.3. \hfill \Box

4 Bilinear estimates

In this section, we state the main bilinear estimates. We show first the dyadic bilinear estimates in spaces $Z_k$.

**Lemma 4.1 (High × very low ⇒ high).** Assume $k \geq 20$, $k_2 \in [k - 2, k + 2]$, $f_{k_2} \in Z_{k_2}$ and $f_0 \in Z_0$. Then

$$2^k \left\|\eta_k(\xi) \cdot \tau - \omega(\xi) + i\right\|_{Z_k} \lesssim \|f_{k_2}\|_{Z_{k_2}} \|f_0\|_{Z_0}. \quad (4.1)$$

**Lemma 4.2 (High × low ⇒ high).** Assume $k \geq 20$, $k_2 \in [k - 2, k + 2]$, $f_{k_2} \in Z_{k_2}$ and $f_{k_1} \in Z_{k_1}$ for any $k_1 \in [1, k - 10] \cap \mathbb{Z}$. Then

$$2^k \left\|\eta_k(\xi) \cdot \tau - \omega(\xi) + i\right\|_{Z_k} \lesssim \|f_{k_2}\|_{Z_{k_2}} \sup_{k_1 \in [1, k - 10]} \|f_{k_1}\|_{Z_{k_1}}. \quad (4.2)$$

**Lemma 4.3 (High × high ⇒ low).** Assume $k, k_1, k_2 \in \mathbb{Z}_+$, $k_1, k_2 \geq k + 10$, $|k_1 - k_2| \leq 2$, $f_{k_1} \in Z_{k_1}$ and $f_{k_2} \in Z_{k_2}$. Then

$$\|\tau \cdot \eta_k(\xi) \cdot \tau - \omega(\xi) + i\right\|_{Z_k} \lesssim \|f_{k_1}\|_{Z_{k_1}} \|f_{k_2}\|_{Z_{k_2}}. \quad (4.3)$$
Lemma 4.4 (High × high → high or low × low → low). Assume \(k, k_1, k_2 \in \mathbb{Z}_+\) have the property that \(\max (k, k_1, k_2) \leq \min (k, k_1, k_2) + 30\), \(f_k \in Z_k\) and \(f_{k_2} \in Z_{k_2}\). Then
\[
2^k \|\eta_k (\xi) \cdot A_k (\xi, \tau)\| f_k \| f_{k_2} \| Z_k \lesssim \|f_k\| \|f_{k_2}\| Z_{k_2}.
\]
Moreover, any spaces \(Z_0\) on the right-hand side of (4.4) can be replaced with \(Z_0\).

The main proofs of Lemmas 4.1–4.4 are already given in [8, Sections 7 and 8] and [7, Lemma 3.3]. The same argument of bilinear estimates as in [7,8] works, except the estimates corresponding to \(Y_0\). We only need to consider that \(Y_0\) appears on the left-hand side of bilinear estimates, since the norm of \(Y_0\) in this paper is larger than that in [7,8]. Therefore, we only provide a proof of Lemma 4.4.

Proof of Lemma 4.4. We only consider \(k = 0\) and \(k_1, k_2 \leq 30\). If \(k_1 = 0\) or \(k_2 = 0\), we may replace the spaces \(Z_0\) on the right-hand side of (4.4) with \(Z_0\). A comparison of \(X_k\) (1 \(\leq k \leq 30\)) and \(Z_0\) indicates that the proofs of the cases where \(k_1 = 0\) or \(k_2 = 0\) are identical to the proofs in the corresponding cases where \(k_1 \geq 1\) or \(k_2 \geq 1\). Therefore, we conclude that \(k_1, k_2 \geq 1\), \(f_k = f_{k_1, j_1}\) is supported in \(D_{k_1, j_1}\) and \(f_{k_2} = f_{k_2, j_2}\) is supported in \(D_{k_2, j_2}\), respectively.

We next estimate the term \(I\). From Hölder’s inequality and Plancherel’s theorem, we know that
\[
\|f_k\| \|f_{k_2}\| Z_{k_2} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau}.
\]

We first estimate the term \(I\). If \(j \leq 40\), from Hölder’s inequality and Plancherel’s theorem, we know that
\[
\|f_k\| \|f_{k_2}\| Z_{k_2} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau}.
\]

If \(j \geq 40\), by examining the supports of the functions, we know that \(j \leq \max \{j_1, j_2\} + C\). Therefore, we assume \(j_1 = \max \{j_1, j_2\}\) and \(j \leq j_1 + C\). Then
\[
\|f_k\| \|f_{k_2}\| Z_{k_2} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau}.
\]

We next estimate the term \(II\). From Hölder’s inequality, we achieve
\[
\|f_k\| \|f_{k_2}\| Z_{k_2} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau} \lesssim \|f_k\| \|f_{k_2}\| L^2_{k, \tau} L^2_{k_2, \tau}.
\]

This completes the proof of (4.5). 

With these dyadic bilinear estimates in hand, we can use the para-product to decompose the bilinear product, and divide it into several cases according to the interactions. The idea is similar to that in [8, Section 10], so we omit the details and just state the main bilinear estimates for functions in spaces \(F^\sigma\).

Proposition 4.5. If \(\sigma \geq 0\) and \(u, v \in F^\sigma\), then
\[
\|\partial_u (uv)\|_{\dot{N}^\sigma} \lesssim C_{\sigma} \{\|u\|_{F^\sigma} \|v\|_{\dot{F}^0} + \|u\|_{\dot{F}^0} \|v\|_{F^\sigma}\}.
\]
5 Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. In terms of the uniform estimates Lemmas 3.1 and 3.3 and bilinear estimates Proposition 4.5, the proofs of (a)–(c) in Theorem 1.1 are similar to those in [7], and thus we only give the ideas. For any interval $I \subset [-5/4, 5/4]$ and $\sigma \geq 0$, we define the normed spaces

$$
F^\sigma(I) = \left\{ u \in \mathcal{S}'(\mathbb{R} \times I) : \|u\|_{F^\sigma(I)} := \inf_{\tilde{u} \equiv u \text{ on } \mathbb{R} \times I} \|\tilde{u}\|_{F^\sigma} < \infty \right\},
$$

$$
N^\sigma(I) = \left\{ u \in \mathcal{S}'(\mathbb{R} \times I) : \|u\|_{N^\sigma(I)} := \inf_{\tilde{u} \equiv u \text{ on } \mathbb{R} \times I} \|\tilde{u}\|_{N^\sigma} < \infty \right\}.
$$

With this notation, the uniform estimates in Lemmas 3.1 and 3.3 become

$$
\|W_\epsilon(t - t_0)\|_{F^\sigma(I)} \leq C\|\phi\|_{\tilde{H}_0} \quad \text{(5.1)}
$$

and

$$
\left\| \int_{t_0}^t W_\epsilon(t - s)(u(s))ds \right\|_{F^\sigma(I)} \leq C\|u\|_{N^\sigma(I)} \quad \text{(5.2)}
$$

By combining Proposition 4.5, we obtain

$$
\left\| \int_{t_0}^t W_\epsilon(t - s) (\partial_x(u \cdot v)(s))ds \right\|_{F^\sigma(I)} \leq C_\sigma \|u\|_{F^\sigma(I)} \|v\|_{F^\sigma(I)} + \|u\|_{F^\sigma(I)} \|v\|_{F^\sigma(I)} \quad \text{(5.3)}
$$

for any $u, v \in F^\sigma(I), \sigma \geq 0$. Finally, the estimate (2.17) becomes

$$
\sup_{t \in I} \|u(\cdot, t)\|_{\tilde{H}_0} \leq C_\sigma \|u\|_{F^\sigma(I)} \quad \text{for any } u \in F^\sigma(I). \quad \text{(5.4)}
$$

Given $\phi \in B(\delta, \tilde{H}^0) \cap \tilde{H}^\infty$, we construct a solution of (1.1) by iteration:

$$
\begin{cases}
u_0^\epsilon = W_\epsilon(t_0)\phi, \\
u_{n+1}^\epsilon = W_\epsilon(t)\phi - \frac{1}{2} \int_0^t W_\epsilon(t - s) (\partial_x((u_n^\epsilon)^2))(s)ds \quad \text{for } n \in \mathbb{Z}_+.
\end{cases} \quad \text{(5.5)}
$$

In the following discussion, we assume that $\delta$ is sufficiently small. By using (5.1) and (5.3), we easily obtain that

$$
\|u_n^\epsilon\|_{F^\sigma([0,1])} \leq C\|\phi\|_{\tilde{H}_0} \quad \text{for any } n \in \mathbb{Z}_+, \quad \text{(5.6)}
$$

by induction over $n$. Using (5.3) with $\sigma = 0$, (5.5) and (5.6), we can show that

$$
\|u_n^\epsilon - u_{n-1}^\epsilon\|_{F^\sigma([0,1])} \leq C2^{-n} \cdot \|\phi\|_{\tilde{H}_0} \quad \text{for any } n \in \mathbb{Z}_+, \quad \text{(5.7)}
$$

by induction over $n$. Next, we can obtain that

$$
\|u_n^\sigma\|_{F^\sigma([0,1])} \leq C\|\phi\|_{\tilde{H}_0} \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \sigma \in [0, 2], \quad \text{(5.8)}
$$

$$
\|u_n^\sigma\|_{F^\sigma([0,1])} \leq C(\sigma, \|\phi\|_{\tilde{H}_0}) \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \sigma \in [0, \infty), \quad \text{(5.9)}
$$

and

$$
\|u_n^\sigma - u_{n-1}^\sigma\|_{F^\sigma([0,1])} \leq C(\sigma, \|\phi\|_{\tilde{H}_0}) \cdot 2^{-n} \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \sigma \in [0, \infty). \quad \text{(5.10)}
$$

For $\sigma \in [0, 2]$, the bounds (5.8) and (5.10) follow in the same way as the bounds (5.6) and (5.7), by combining (5.1), (5.3) and induction over $n$. For $\sigma \geq 2$, we write $\sigma = \sigma_0 + \sigma'$, $\sigma' \in \mathbb{Z}_+, \sigma_0 \in [0, 1]$, and argue by induction over $\sigma'$ similar to [7, Section 4] to complete the proofs of (5.8) and (5.10). Therefore, we can use (5.10) and (5.4) to construct $u^\epsilon = \lim_{n \to \infty} u_n^\epsilon \in C([0,1] : \tilde{H}^\infty)$. In view of (5.5),

$$
u^\epsilon = W_\epsilon(t)\phi - \frac{1}{2} \int_0^t W_\epsilon(t - s) (\partial_x((u^\epsilon)^2))(s)ds \quad \text{on } \mathbb{R} \times [0,1], \quad \text{(5.5)}$$
We now prove (5.12), that
\[ S_\epsilon^\infty(\phi) = u' \]
is a solution of the initial-value problem (1.1), which completes the proof of Theorem 1.1(a). For Theorems 1.1(b) and 1.1(c), similar to the above argument, we can easily obtain that for \( \phi, \phi' \in B(\delta, \tilde{H}^\sigma) \cap \tilde{H}^\infty \), then
\[
\sup_{t \in [0, 1]} \| S_\epsilon^\infty(\phi) - S_\epsilon^\infty(\phi') \|_{\tilde{H}^\sigma} \leq C(\sigma, \| \phi \|_{\tilde{H}^\sigma} + \| \phi' \|_{\tilde{H}^\sigma}) \cdot \| \phi - \phi' \|_{\tilde{H}^\sigma},
\]
which implies (b) and (c) in Theorem 1.1.

Finally, we prove Theorem 1.1(d), i.e., the inviscid limit behavior in \( \tilde{H}^\sigma \) \((\sigma \geq 0)\). Assume \( \phi \in B(\delta, \tilde{H}^0) \) \( \cap \tilde{H}^\sigma \). Let \( S_\epsilon^\sigma(\phi) \) and \( S^\sigma(\phi) \) denote the nonlinear solution mappings of the Cauchy problems (1.1) and (1.2) that associate with any initial data \( \phi \). For convenience, we only give the proof of the case \( \sigma = 0 \), since the proof of the case \( \sigma > 0 \) is similar. It suffices to prove
\[
\lim_{\epsilon \to 0} \| S_\epsilon^\sigma(\phi) - S^0(\phi) \|_{C([0,1]; \tilde{H}^\sigma)} = 0.
\]

We know that
\[
\begin{align*}
  u' &= S_\epsilon^0(\phi) = W(t)\phi - \int_0^t W(t - s)(\partial_x((u')^2(s)/2) - c\partial_x^2 u'(s)) ds, \\
  u &= S^0(\phi) = W(t)\phi - \int_0^t W(t - s)(\partial_x(u^2(s)/2)) ds,
\end{align*}
\]
where \( W(t)\phi = F_{\xi}^{-1}\exp(t\xi)F_{\xi}\phi \) is the solution of the free Benjamin-Ono evolution. In terms of (5.2), (5.3), (5.13) and (5.14), we have
\[
\| u' - u \|_{F^0([0,1])} \leq \| S_\epsilon^0(\phi) - S^0(\phi) \|_{F^0([0,1])} \lesssim (\| u' \|_{F^0([0,1])} + \| u \|_{F^0([0,1])})\| u' - u \|_{F^0([0,1])} + \epsilon \| \partial_x^2 u' \|_{N^0([0,1])}.
\]

Similar to (5.6), we have \( \| u' \|_{F^0([0,1])} \leq C\delta \) and \( \| u \|_{F^0([0,1])} \leq C\delta \). By combining them with the definitions of \( N^0 \), \( F^0 \) and (5.8), (5.15) becomes
\[
\| u' - u \|_{F^0([0,1])} \lesssim \epsilon \| \partial_x^2 u' \|_{N^0([0,1])} \lesssim \epsilon \| u' \|_{F^2([0,1])} \lesssim \epsilon \| \phi \|_{\tilde{H}^2}.
\]

In terms of (5.4), we have shown that
\[
\sup_{t \in [0, 1]} \| S_\epsilon^\sigma(\phi) - S^0(\phi) \|_{\tilde{H}^0} \leq C \| S_\epsilon^\sigma(\phi) - S^0(\phi) \|_{F^0([0,1])} \leq C\epsilon \| \phi \|_{\tilde{H}^2}.
\]

We now prove (5.12). \( \forall \eta > 0 \), it follows from the Lipschitz continuity that there exists a \( K > 0 \) such that
\[
\begin{align*}
  &\sup_{t \in [0, 1]} \| S_\epsilon^\sigma(P_{\leq K}\phi) - S^0(P_{\leq K}\phi) \|_{\tilde{H}^0} \leq \eta/4, \quad \forall \epsilon \in (0, 1], \\
  &\sup_{t \in [0, 1]} \| S^0(P_{\leq K}\phi) - S^0(\phi) \|_{\tilde{H}^0} \leq \eta/4.
\end{align*}
\]

Fixing \( K \) and taking \( \epsilon = \epsilon(K) \) sufficiently small, we can obtain from (5.17) that
\[
\sup_{t \in [0, 1]} \| S_\epsilon^\sigma(P_{\leq K}\phi) - S^0(P_{\leq K}\phi) \|_{\tilde{H}^0} \leq C\epsilon K^2 \cdot \| P_{\leq K}\phi \|_{\tilde{H}^0} \leq \eta/4.
\]

Therefore, we have \( \sup_{t \in [0, 1]} \| S_\epsilon^\sigma(\phi) - S^0(\phi) \|_{\tilde{H}^0} < \eta \), which implies (5.12). The proof of Theorem 1.1 is completed.

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant No. 12001236). The second author was supported by National Natural Science Foundation of China (Grant No. 11731014). The third author was supported by National Natural Science Foundation of China (Grant No. 11971166).
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