Disentanglement and decoherence in two-spin and three-spin systems under dephasing

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We compare disentanglement and decoherence rates within two-spin and three-spin entangled systems subjected to all possible combinations of local and collective pure dephasing noise combinations. In all cases, the bipartite entanglement decay rate is found to be greater than or equal to the dephasing-decoherence rates and often significantly greater. This sharpens previous results for two-spin systems [Phys. Rev. B 56, 165322 (2003)] and extends them to the three-spin context.

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I. INTRODUCTION

One goal of nanoscience is to understand the behavior of quantum entanglement in states of multiparticle systems, such as spin arrays. In such contexts (e.g. [1]), spins are often called qubits, as we will refer to them here. Entanglement can be viewed as a resource that allows uniquely quantum phenomena, such as quantum state teleportation [2] using quantum memories, to be exhibited [3, 4]. Entanglement can be affected by a number of noise influences, including both vacuum noise [5] and classical noise [6]. One important effect is that of the loss of entanglement, which arises in both these cases. Recently, for example, it has been noted that in initially entangled two-qubit systems a sudden loss of entanglement can occur, an effect known as "entanglement sudden death" [5, 6]. It is also interesting to note that in certain situations the irreversible process of spontaneous emission may lead to the revival of entanglement [7]. An experimental method for the probing of disentanglement in cavity QED has also been introduced [8]. Recent analyses have analytically examined the relationship between dephasing decoherence and the reduction of bipartite entanglement in two-qubit systems under such noise [5, 9, 10, 11, 12], as well as in pairs of three-level systems, i.e., two-qutrit systems [13]. Here, we extend the analytic consideration of this relationship to the case of three-qubit systems.

Decoherence results from unwanted interactions of a quantum system with its environment. When such decoherence occurs, it is exhibited by the decay of off-diagonal elements of the density matrix describing the system [14]. This may be due to either local or collective interactions, acting either at the scale of subsystems or of the composite system; that is, environmental noise may act on some or all system constituents similarly or differently and on some not at all [9, 10]. Examples of environments capable of inducing such interactions include electromagnetic fields and thermal baths [1, 9, 10]. Here, we consider quantum state decoherence in systems of two or three qubits due to pure-dephasing noise. In particular, we address the relationship between degree of entanglement (concurrence) and state coherence under environmental noise by determining characteristic decay timescales at the single-qubit, double-qubit, and triple-qubit scales. We find decoherence timescales to bound corresponding disentanglement timescales from above in all cases.

We begin by reviewing and sharpening the analysis begun in an earlier study of two-qubit systems, and then extend that approach to the case of three-qubit systems, in which more complex combinations of dephasing noise are relevant. This paper is accordingly organized as follows. In Sec. II, we revisit the analysis of two-qubit systems begun by Yu and Eberly in [11] that explicitly treats disentanglement in the operator-sum formalism under local dephasing-noise environments and explicitly address disentanglement under collective dephasing noise, and sharpen the conclusion of the analysis by showing that decoherence times bound disentanglement times from above in all cases. In Sec. III, we
extend the analysis of the effects of dephasing on coherence and bipartite entanglement to three-qubit systems in the more complex dephasing noise environments available there. We conclude with a summary of results in Section IV.

II. TWO-QUBIT SYSTEM

Before considering the effects of dephasing on three-qubit systems, we complete a previous analysis of two-qubit systems by Yu and Eberly \[10\]. There, a number of dephasing scenarios involving characteristic classes of qubit-pair states were discussed; in one of these, the situation in which collective dephasing, that is, dephasing between two-qubit-basis states occurs under a non-zero global magnetic field alone, disentanglement was not explicitly treated. Here, we discuss disentanglement in this case. The general conclusion in \[10\] was that bipartite disentanglement can be much more rapid than local decoherence under local dephasing. Here, we demonstrate a more general and precise result that accords with that conclusion: the decoherence timescale bounds the disentanglement timescale from above for both local and collective phase-damping noise environments for the two classes of states introduced there.

A. MODEL AND MEASURES

There exist many related two-component few-state models corresponding to physical systems the coherence of which may be influenced. For example, \[10\] modeled two-qubit systems such as pairs of atoms in a dephasing-noise environment, whereas \[13\] modeled two-qutrit systems corresponding to two three-level atoms whose two uncoupled excited levels undergo spontaneous emission to the ground state. In this section, we examine the model first introduced in \[10\], namely, that of a two-qubit system subject to classical dephasing noise.

Consider a two-qubit system coupled to external sources of noise that can act both on single qubits individually and on the joint two-qubit system as a whole. There are many physical situations that may be described in this way, for example, two spin-1/2 fermions on a solid-state matrix subject to random external electromagnetic fields. In this section, we specifically consider situations wherein the interaction between this compound system and the environment is described by the following Hamiltonian.

\[
H (t) = \frac{1}{2 \mu} \left[ B (t) \left( \sigma^A_z + \sigma^B_z \right) + b_A (t) \sigma^A_z + b_B (t) \sigma^B_z \right],
\]

where, for example, \( \mu \) is taken to be the gyromagnetic ratio of the particle and \( B (t), b_A (t), \) and \( b_B (t) \) represent stochastic environmental noise from electromagnetic fields, in particular, with \( B(t) \neq 0 \); we take \( \hbar = 1 \) and \( \sigma_z \) to be the well-known Pauli matrix; superscripts \( A(B) \) indicate the particle Hilbert spaces \( \mathcal{H}_{A(B)} \) in which the operators act. The noise terms \( B (t), b_A (t), \) and \( b_B (t) \) are taken to be statistically independent classical Markov processes satisfying

\[
\langle B (t) \rangle = 0,
\]

\[
\langle B (t) B (t') \rangle = \frac{\Gamma_{AB}}{\mu^2} \delta (t - t'),
\]

\[
\langle b_X (t) \rangle = 0,
\]

\[
\langle b_X (t) b_X (t') \rangle = \frac{\Gamma_X}{\mu^2} \delta (t - t'), \quad X = A, B,
\]

where \( \langle \cdots \rangle \) is the ensemble time-average; \( \Gamma_X \) denotes the phase-damping rates associated with \( b_X (t) \) \((X = A, B)\) and \( \Gamma_{AB} \) is that associated with \( B(t)\). The two-qubit standard-basis eigenstates here are

\[
|1\rangle_{AB} = | + \rangle_{AB}, |2\rangle_{AB} = | - \rangle_{AB},
\]

\[
|3\rangle_{AB} = | + \rangle_{AB}, |4\rangle_{AB} = | - \rangle_{AB}.
\]

The Hamiltonian of Eq. 1, the same one assumed in \[10\], is representative of the class of interactions giving rise to pure dephasing decoherence. The corresponding dynamical process is described here as a “quantum channel,” in accord with the standard terminology of qubit evolution as described by linear maps \[3, 4\], namely, a phase-damping channel, using the operator-sum representation which we now introduce.

The time-dependent density matrix for the two-qubit system can be obtained by taking ensemble averages over the three noise fields, \( B (t), b_A (t), \) and \( b_B (t) \), that is,

\[
\rho (t) = \langle \langle \rho_{st} (t) \rangle \rangle,
\]
where the statistical density operator $\rho_{st}(t)$ and the unitary operator $U(t)$ associated with $H(t)$ are

$$\rho_{st}(t) = U(t) \rho(0) U^\dagger(t) \quad \text{and} \quad U(t) = \exp \left[-i \int_0^t dt' H(t') \right],$$

respectively. It is helpful to consider the dynamical evolution of $\rho(t)$ as a completely positive trace preserving (CPTP) linear map $E(\rho)$, that is, a combination of local and collective quantum channels, any of which can be turned off in particular cases, taking an input state $\rho(0)$ to the output state $\rho(t)$ given by the operator sum

$$\rho(t) = E(\rho(0)) = \sum_{\mu=1}^N E^\dagger_\mu(t) \rho(0) E_\mu(t),$$

where $E_\mu$ are decomposition operators that satisfy the completeness relation

$$\sum_\mu E^\dagger_\mu E_\mu = I.$$

In each of the various cases considered here, the internal structure of the $E_\mu$ will be accord with the Hamiltonian; various terms may or may not nontrivially contribute in a given case. The most general solution of Eq. 10, assuming that the system is not initially correlated with any of the three environments, is

$$\rho(t) = \sum_{i,j=1}^2 \sum_{k=1}^3 (D^B_i E^A_j E^B_k) \rho(0) (E^A_i E^B_j D_k),$$

where the terms describing the interaction with the local magnetic fields $b_A(t)$ and $b_B(t)$ involve the decomposition operators

$$E^A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_A(t) \end{pmatrix} \otimes I, \quad E^A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A(t) \end{pmatrix} \otimes I,$$

$$E^B_1 = I \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_B(t) \end{pmatrix}, \quad E^B_2 = I \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B(t) \end{pmatrix}.$$

By contrast, the operators describing the collective interaction with a non-zero global magnetic field, $B(t)$, are

$$D_1 = \begin{pmatrix} \gamma(t) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma(t) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \omega_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2(t) \\ 0 & 0 & \omega_2(t) & 0 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_3(t) \end{pmatrix}.$$

The time-dependent parameters appearing in Eqs. 13–17 above are

$$\gamma_A(t) = e^{-t/2T_A}, \quad \gamma_B(t) = e^{-t/2T_B},$$

$$\omega_A(t) = \sqrt{1 - \gamma_A^2}, \quad \omega_B(t) = \sqrt{1 - \gamma_B^2},$$

$$\gamma(t) = e^{-t/2T_{AB}},$$

$$\omega_1(t) = \sqrt{1 - \gamma^2},$$

$$\omega_2(t) = -\gamma \sqrt{1 - \gamma^2},$$

$$\omega_3(t) = \sqrt{(1 - \gamma^2)(1 - \gamma^4)}.$$
where $T_X = \frac{1}{\Gamma_X}$ ($X = A, B, AB$) are the phase-relaxation times introduced in \cite{10}, $\Gamma_X$ being the rate parameters appearing on the right-hand sides of Eqs. 3 and 5; from here on, for tractability of notation, time does not explicitly appear as an argument for these quantities but is implied.

The two pertinent subclasses, distinguished by the large-time behavior of their coherence under collective dephasing noise, a normalized state-vector with

$$\rho = \sum_{i=1}^{4} C_i |i\rangle \langle i|$$

where $C_i$ are the eigenvalues of the matrix

$$\rho = \sum_{i=1}^{4} |\phi_i\rangle \langle \phi_i|$$

indexed according to decreasing magnitude, $\rho_{AB}$ denotes the complex conjugate of $\rho_{AB}$, and $\sigma_y^A(B)$ is the standard Pauli matrix acting in the space of qubit A(or B) \cite{15}. Concurrence is related to the canonical measure of entanglement, “$E_f$”, by

$$E_f (\rho) = h \left( 1 + \sqrt{1 - C^2 (\rho)} \right),$$

where $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$, of which it is, therefore, a monotonic function.

### B. DECOHERENCE AND DISENTANGLEMENT

In the standard-basis representation of Eq. 6, the generic pure state of the two-qubit system is

$$|\Psi\rangle = a|1\rangle_{AB} + b|2\rangle_{AB} + c|3\rangle_{AB} + d|4\rangle_{AB},$$

a normalized state-vector with $a, b, c, d \in \mathbb{C}$. The generic class of two-qubit pure states represented by $|\Psi\rangle$ contains two pertinent subclasses, distinguished by the large-time behavior of their coherence under collective dephasing noise, that is, dephasing in which each qubit interacts with the same global noise field $B(t)$: one class is fragile, whereas the other is robust \cite{10}.

(i) The fragile class has the forms

$$|\phi_1\rangle = a|1\rangle_{AB} + b|2\rangle_{AB} + d|4\rangle_{AB},$$

$$|\phi_2\rangle = a|1\rangle_{AB} + c|3\rangle_{AB} + d|4\rangle_{AB}.$$  

(ii) The robust class has the forms

$$|\psi_1\rangle = a|1\rangle_{AB} + b|2\rangle_{AB} + c|3\rangle_{AB},$$

$$|\psi_2\rangle = b|2\rangle_{AB} + c|3\rangle_{AB} + d|4\rangle_{AB}.$$  

Without loss of generality, we consider the first of the two forms in each case, for the purposes of precisely characterizing the decoherence and disentanglement behavior of these classes.

The fragile states have been shown to decohere under collective dephasing noise in such a way that all off-diagonal matrix elements go to zero exponentially as a function of time. A well known example of the fragile class is that with $a = d = 1/\sqrt{2}, b = c = 0$, namely, the spin-triplet Bell state $|\Phi^+\rangle$. By contrast, the robust states remain partially coherent under collective noise, some off-diagonal elements being constants \cite{17}. A well known example of the robust class is that with $a = d = 0, b = c = 1/\sqrt{2}$, the spin-singlet Bell state $|\Psi^-\rangle$, which is known to be decoherence-free under collective dephasing \cite{10}. In fact, each class of states contains two Bell states.

In \cite{10}, the evolution of state entanglement under local dephasing noise for both the fragile and robust classes was characterized; it was shown for both classes, for both the case of single-qubit local and the case of multi-local dephasing noise, that the disentanglement rate can be much faster than the decoherence rate when either (and so both) occur. We now generalize and sharpen this result, by pointing out that the decoherence timescale bounds the disentanglement timescale from above and explicitly showing that such a bounding relation also holds under collective dephasing noise, i.e., when $B(t) \neq 0$, a case in which disentanglement was not explicitly investigated in \cite{10} and only to a limited extent and in a somewhat different manner and context in the preceding study \cite{9}. Here, we consider all possible cases in an identical way in a common formalism (that in subsequent sections is extended to the three-qubit
and robust classes. One finds identical, it suffices to consider the reduced density matrix of just one qubit (here, we choose A) for both the fragile of off-diagonal elements of the single-qubit reduced density matrices. Because the action of \( P_{AB}(t = 0) = P(|\phi_1\rangle) = \left( \begin{array}{cccc} |a|^2 & a b^* & 0 & a d^* \\ b a^* & |b|^2 & 0 & b d^* \\ 0 & 0 & 0 & 0 \\ d a^* & d b^* & 0 & |d|^2 \end{array} \right) \) (32), whereas for the robust class one has

\[
\rho_{AB}^R(t = 0) = P(|\psi_1\rangle) = \left( \begin{array}{cccc} |a|^2 & a b^* & a c^* & 0 \\ b a^* & |b|^2 & b c^* & 0 \\ c a^* & c b^* & |c|^2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),
\]

which exhibits two pertinent two-qubit decoherence timescales, parameterized by \( \gamma^4 \) (fast) and \( \gamma \) (slow), and becomes diagonal in the limit of large times. By comparison with the result in the independent multi-local noise case (see Eq. 64 of \[10\]) wherein two timescales of decoherence exist that are determined by \( \gamma_A|B\rangle \) and \( \gamma_A\gamma_B \), the decay of some off-diagonal terms in this case is more rapid, being determined by \( \gamma^4 \).

The initial-time density matrix for the representative fragile-state class is

\[
\rho_{AB}^F(t = 0) = P(|\psi_1\rangle) = \left( \begin{array}{cccc} |a|^2 & a b^* & 0 & a d^* \\ b a^* & |b|^2 & 0 & b d^* \\ 0 & 0 & 0 & 0 \\ d a^* & d b^* & 0 & |d|^2 \end{array} \right),
\]

whereas for the robust class one has

\[
\rho_{AB}^R(t = 0) = P(|\psi_1\rangle) = \left( \begin{array}{cccc} |a|^2 & a b^* & a c^* & 0 \\ b a^* & |b|^2 & b c^* & 0 \\ c a^* & c b^* & |c|^2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),
\]

wherein two timescales of decoherence exist that are determined by \( \gamma_A|B\rangle \) and \( \gamma_A\gamma_B \), the decay of some off-diagonal terms in this case is more rapid, being determined by \( \gamma^4 \).

The robust states \( \rho_{AB}^R(0) \) under collective dephasing noise become

\[
\rho_{AB}^R(t) = \left( \begin{array}{cccc} |a|^2 & a b^* \gamma & a c^* \gamma & 0 \\ b a^* \gamma & |b|^2 & b c^* \gamma & 0 \\ c a^* \gamma & c b^* & |c|^2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),
\]

and, by contrast, do not become diagonal in the large-time limit. Furthermore, also in contrast to the case of the fragile class, we see that the off-diagonal elements that do decay do so according to only a single, slow timescale, namely, that given by a single factor of \( \gamma \). The robustness of this class is due to the fact that the off-diagonal elements \( \rho_{23} \) and \( \rho_{32} \) (corresponding to the symmetric spin-singlet component \[10\]) do not decay; the robust states remain partially coherent in the limit of large time. By contrast, under multi-local dephasing (see Eq. 67 of \[10\]), density matrix elements \( \rho_{23} \) and \( \rho_{32} \) do decay to zero, in the timescale determined by \( \gamma \). Under collective dephasing noise, the two-qubit system coherence-decay timescales determined by \( \gamma \) factors in the above density matrices, are thus

\[
\tau_{2-\text{dec}}^{F(\text{slow})} = 2 \left( \frac{1}{\Gamma_{AB}} \right),
\]

\[
\tau_{2-\text{dec}}^{F(\text{fast})} = 1 \left( \frac{1}{\Gamma_{AB}} \right),
\]

\[
\tau_{2-\text{dec}}^{R} = 2 \left( \frac{1}{\Gamma_{AB}} \right) \quad \text{(for decaying elements)}.
\]

Now consider single-qubit decoherence under collective dephasing noise, corresponding to the reduction of magnitude of off-diagonal elements of the single-qubit reduced density matrices. Because the action of \( B(t) \) on each qubit is identical, it suffices to consider the reduced density matrix of just one qubit (here, we choose A) for both the fragile and robust classes. One finds

\[
\rho_A^F = \text{tr}_B \rho_{AB}^F(t) = \left( \begin{array}{cc} |a|^2 + |b|^2 & b d^\gamma \\ d b^\gamma & |d|^2 \end{array} \right),
\]

\[
\rho_A^R = \text{tr}_B \rho_{AB}^R(t) = \left( \begin{array}{cc} |a|^2 + |b|^2 & a c^\gamma \\ c a^\gamma & |c|^2 \end{array} \right).
\]
For the two classes, single-qubit coherence is entirely lost as parameterized by \( \gamma \), namely, in the slow decoherence time-scale given above, so the single-qubit decoherence timescales are the same:

\[
\tau_{1-\text{dec}}^F = \frac{1}{\Gamma}, \quad \text{(41)}
\]
\[
\tau_{1-\text{dec}}^R = \frac{1}{\Gamma}. \quad \text{(42)}
\]

Quantum state disentanglement is described by a reduction of concurrence, \( C(\rho) \). For fragile states, one finds

\[
C(\rho_{AB}(t)) = 2\gamma^4 |a||d|. \quad \text{(43)}
\]

Disentanglement of these states is therefore characterized by the time

\[
\tau_{\text{dis}} = \frac{1}{2} \left( \frac{1}{\Gamma_{AB}} \right), \quad \text{(44)}
\]

that is, the fast time-scale for two-qubit decoherence above, cf. Eq. 37. A factor of \( \gamma^4 \) appears here, by contrast to the case of multi-local dephasing (see Eq. 66 of [10]) where disentanglement takes place via the factor \( \gamma_A \gamma_B \). Under collective dephasing, disentanglement thus occurs more quickly than in multi-local dephasing due to an additional factor of \( \gamma^2 \) in off-diagonal terms (assuming \( \gamma_{AB}, \gamma_A \) and \( \gamma_B \) to be comparable in value, which enables a meaningful comparison of effects). For the robust class of states, the entanglement is

\[
C(\rho_{AB}(t)) = 2|b||c|; \quad \text{(45)}
\]

that is, there is no disentanglement for the robust class, which also does not decohere. We therefore see that this class of state can be identified on the basis of the robustness of its entanglement, as well as that of its coherence. In the following section, we find a similar behavior distinguishing the two major entanglement classes of three-qubit states.

Thus, the relation between the timescales of decoherence and disentanglement described in [10] continues to hold under the collective dephasing channel, in which \( B(t) \neq 0 \) and \( b_X(t) = 0 \). With the above results, together with those of [10], one now can consider the behavior of entanglement under dephasing noise to be fully characterized for both these complementary classes of two-qubit system states, because both local-noise and collective-noise cases have been fully analyzed. In the following section, we consider three-qubit composite systems under local and collective dephasing noise, which involves a greater number of scenarios involving combinations of these noise types.

III. THREE-QUBIT SYSTEM

Now consider the effect of similar dephasing noise on states of three qubits. In three-qubit systems (ABC) there are known to be two qualitatively different pure-state entanglement classes,

\[
|W^g\rangle = \tilde{a}_1|001\rangle + \tilde{a}_2|010\rangle + \tilde{a}_4|100\rangle,
\]
\[
|\text{GHZ}^g\rangle = \tilde{a}_0|000\rangle + \tilde{a}_7|111\rangle,
\]

with coefficients \( \tilde{a}_i \in \mathbb{C} \) such that \( |\tilde{a}_1|^2 + |\tilde{a}_2|^2 + |\tilde{a}_4|^2 = 1 \), and \( |\tilde{a}_0|^2 + |\tilde{a}_7|^2 = 1 \), defined by interconvertibility under stochastic local operations and classical communication (SLOCC) operations [18]. We will accordingly consider each class individually under the available combinations of local and collective dephasing-noise that this system might encounter, assuming that a single noise type affects each qubit. In the above classification of (here, initial) states, the distinguishing property of the W class is, equivalently, the robustness of its internal entanglement under the loss of any one qubit of the system, in the sense of retaining the entanglement of the remaining two qubits. By contrast, the GHZ class has maximum three-qubit entanglement [10] but no two-qubit entanglement after the loss of any qubit.

A. MODEL AND MEASURES

We now consider the three-qubit system ABC in dephasing-noise environments that may act at the level of each of the three individual single-qubit subsystems (\( X = A, B, C \)) and/or the three two-qubit subsystems (\( XY = AB, AC, \text{and } BC \)), or the full three-qubit system (ABC) alone. Thus, the noise terms considered here are those due to the local-dephasing environment which acts on a single qubit, \( B^{(1)} \), a two-qubit collective-dephasing environment, \( B^{(2)} \),
and a three-qubit collective-dephasing environment, $B^{(3)}$. We consider only cases in which a given qubit is influenced by a single type of noise, in order to clearly distinguish the influence of combinations of dephasing noise on three-qubit systems.

The corresponding three-qubit interaction Hamiltonian is

$$
H(t) = -\frac{1}{2\mu} \left[ B_A^{(1)}(t) \sigma_z^A + B_B^{(1)}(t) \sigma_z^B + B_C^{(1)}(t) \sigma_z^C 
+ B_{AB}^{(2)}(t) (\sigma_z^A + \sigma_z^B) 
+ B_{BC}^{(2)}(t) (\sigma_z^B + \sigma_z^C) 
+ B_{AC}^{(2)}(t) (\sigma_z^A + \sigma_z^C) 
+ B^{(3)}(t) (\sigma_z^A + \sigma_z^B + \sigma_z^C) \right],
$$

(47)

where $\mu$ is taken to be unity, and for example, $\mu$ is the gyromagnetic ratio and the $B^{(i)} (i = 1, 2, 3)$ are the imposed stochastic magnetic fields at each of the three available scales, with subscripts indicating the region of influence in cases in which only subsystems are affected; Pauli matrices are also labeled by the subsystem on which they act. The standard three-qubit Hilbert space basis assumed here is

$$
\begin{align*}
|1\rangle_{ABC} &= |000\rangle_{ABC}, \\
|2\rangle_{ABC} &= |001\rangle_{ABC}, \\
|3\rangle_{ABC} &= |010\rangle_{ABC}, \\
|4\rangle_{ABC} &= |011\rangle_{ABC}, \\
|5\rangle_{ABC} &= |100\rangle_{ABC}, \\
|6\rangle_{ABC} &= |101\rangle_{ABC}, \\
|7\rangle_{ABC} &= |110\rangle_{ABC}, \\
|8\rangle_{ABC} &= |111\rangle_{ABC},
\end{align*}
$$

(48)

where $|ijk\rangle_{ABC} \equiv |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C$ for $i,j,k = 0, 1$ denote the eigenstates of the product operator $\sigma_z^A \otimes \sigma_z^B \otimes \sigma_z^C$. For simplicity, much as before, we assume that the stochastic fields are taken to be classical and characterized as statistically independent Markov processes satisfying

$$
\begin{align*}
\langle B_X^{(1)} \rangle &= 0, \\
\langle B_X^{(1)}(t) B_X^{(1)}(t') \rangle &= \frac{\Gamma_1}{\mu^2} \delta(t-t'), \\
\langle B_{XY}^{(2)} \rangle &= 0, \\
\langle B_{XY}^{(2)}(t) B_{XY}^{(2)}(t') \rangle &= \frac{\Gamma_2}{\mu^2} \delta(t-t'), \\
\langle B^{(3)} \rangle &= 0, \\
\langle B^{(3)}(t) B^{(3)}(t') \rangle &= \frac{\Gamma_3}{\mu^2} \delta(t-t'),
\end{align*}
$$

(49)-(54)

where $\langle \cdots \rangle$ stands for ensemble average. Here $\Gamma_i$ is the damping rate due, for example, to the respective interaction with magnetic fields localized to $i = 1, 2, 3$ qubits at a time. The imposed white-noise conditions on the three-qubit system and any subsystems ensure a Markovian time-evolution. They also require interactions at a given scale to be of the same strength, which allows one to make an objective assessment of the importance of interactions by scale, as is done below.

The compound-system density matrix can be obtained by taking the ensemble averages over the three noise sources, which are again given by decomposition-operators. For the single-qubit dephasing channel, in this section labeled $D$, we have the following decomposition-operator matrices for qubit A at the one-qubit level.

$$
D_A^A = \left( \begin{array}{cc} 1 & 0 \\ 0 & \gamma_A(t) \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
$$

$$
D_A^I = \left( \begin{array}{cc} 0 & 0 \\ 0 & \omega_A(t) \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
$$

(55)-(56)
(For the dephasing of qubits B and C, the matrices are analogous. Note also that the choice of decomposition operator labels in this section differs from that of the previous section, which there was made to agree with the notation of [10].) As before, the time parameter in $\gamma$s and $\omega$s will be implicit from here on.

For the collective dephasing channel that acts only on two-qubit subsystems, in this section labeled $\mathcal{E}$, we have the following decomposition-operator matrices for the AB qubit system at the two-qubit level, which are just those that are collective at the two-qubit level. (For collective-dephasing-noise contributions acting the two-qubit systems BC and AC, the decomposition operator matrices are analogous.)

\[
E_{1}^{AB} = \begin{pmatrix}
\gamma_{AB} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \gamma_{AB}
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

\[
E_{2}^{AB} = \begin{pmatrix}
\omega_{AB1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_{AB2}
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

\[
E_{3}^{AB} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_{AB3}
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

Finally, for the three-qubit collective dephasing channel, we have the following decomposition-operator matrices acting on the whole three-qubit system.

\[
F_{1}^{ABC} = \begin{pmatrix}
\gamma_{ABC} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{ABC}
\end{pmatrix}.
\]

\[
F_{2}^{ABC} = \begin{pmatrix}
\omega_{ABC1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{ABC2}
\end{pmatrix}.
\]

\[
F_{3}^{ABC} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{ABC3}
\end{pmatrix}.
\]

The parameters appearing in the previous matrices are:

\[
\gamma_{i} = e^{-t/2T_{i}},
\]

\[
\omega_{i} = \sqrt{1 - e^{-t/2T_{i}}} = \sqrt{1 - \gamma_{i}^{2}},
\]

\[
\omega_{i1} = \sqrt{1 - \gamma_{i}^{2}},
\]

\[
\omega_{i2} = -\gamma_{i}^{2} \sqrt{1 - \gamma_{i}^{2}},
\]

\[
\omega_{i3} = \sqrt{(1 - \gamma_{i}^{2})(1 - \gamma_{i}^{4})}.
\]
Here, as before, $T_i = 1/\Gamma_i$ is the phase-relaxation time associated with the corresponding interaction. Below, the numeric labels $i$ may be replaced by the labels of the pertinent qubits A, AB, ABC, and so on, for specificity.

The explicit form of the time-evolved density matrix subject to the available one-qubit, two-qubit, and three-qubit dephasing-noise environments is given by

$$\rho(t) = \sum_{i,j,k=1}^{2} \sum_{l,m,n,p=1}^{3} \left( F_{p}^{AB} E_{n}^{AC} E_{m}^{BC} + F_{i}^{AB} D_{j}^{C} D_{k}^{A} D_{l}^{B} \right) \rho(0) \left( D_{l}^{A} D_{k}^{B} D_{j}^{C} E_{m}^{AC} E_{n}^{BC} F_{p}^{AB} \right).$$

For each sort of dephasing-noise environment considered, the relevant off-diagonal elements from the single-qubit-reduced, two-qubit-reduced, and three-qubit density matrices exhibit any one-qubit, two-qubit, and three-qubit decoherence effects, respectively, and bipartite disentanglement occurs when there is a reduction of $C_{XY}^2$ defined in Eq. [24] where $X, Y = A, B, C$. (See the note [20] for a discussion of the grounds for this choice of entanglement measure.)

### B. BEHAVIOR OF W-CLASS STATES

Here, the environmental noise channel inducing single-qubit local-dephasing on qubit A is given explicitly; the single-qubit local dephasing noise acting on qubit B and qubit C, respectively, are of the same general form. In the case of subsystem-collective dephasing noise, considered in Subsection 2 below, where the collective-noise channel acting on qubits A and B is discussed, a similar statement applies for the channels acting on B and C only and A and C only. The initial generalized W-class state density matrix

$$\rho_{ABC}^W(0) = P(|W^X\rangle) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\bar{a}_1|^2 & \bar{a}_1 \bar{a}_2 & 0 & \bar{a}_1 \bar{a}_4^* & 0 & 0 & 0 \\ 0 & \bar{a}_2 \bar{a}_1^* & |\bar{a}_2|^2 & 0 & \bar{a}_2 \bar{a}_4^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_4 \bar{a}_1^* & \bar{a}_4 \bar{a}_2^* & 0 & |\bar{a}_4|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$\hspace{1cm}(69)

We now examine the behavior of the time-evolved state $\rho(t)$, given by Eq. (68) above, in detail.

#### 1. ONE-QUBIT LOCAL DEPHASING CHANNEL: $\mathcal{D}$

Under local dephasing noise,

$$\rho_{ABC}^W(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\bar{a}_1|^2 & \bar{a}_1 \bar{a}_2 & 0 & \bar{a}_1 \bar{a}_4^* & 0 & 0 & 0 \\ 0 & \bar{a}_2 \bar{a}_1^* & |\bar{a}_2|^2 & 0 & \bar{a}_2 \bar{a}_4^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}_4 \bar{a}_1^* & \bar{a}_4 \bar{a}_2^* & 0 & |\bar{a}_4|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$\hspace{1cm}(70)

is the time-evolved three-qubit density matrix obtained when beginning at $t = 0$ with a pure W-class state. One sees that several but not all off-diagonal elements decay under local dephasing noise, as determined by $\gamma_A$. The decoherence timescale for these terms is thus

$$\tau_{3-dec, \mathcal{D}}^W = 2\frac{1}{\Gamma_1},$$

where the notation

$$\tau_{i-dec, \mathcal{C}}^{c(r)},$$

$$\tau_{i-dis, \mathcal{C}}^{c(r)}$$

(72)
represent the decoherence and disentanglement timescales, respectively, where \( i \) is the number of qubits affected, the pertinent dephasing noise channel is \( C \) and the state class is \( c \); further special cases may be designated by relative decay rate, \( r (= \text{fast or slow}) \), and is used from here on. The two-qubit reduced density matrices are then

\[
\rho_{AB}^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_2|^2 & \bar{a}_2\bar{a}_4^\gamma & 0 \\
0 & \bar{a}_4\bar{a}_2^\gamma & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(73)

\[
\rho_{AC}^W(t) = \begin{pmatrix}
|\bar{a}_2|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_4^\gamma & 0 \\
0 & \bar{a}_4\bar{a}_1^\gamma & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(74)

\[
\rho_{BC}^W(t) = \begin{pmatrix}
|\bar{a}_4|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^\gamma & 0 \\
0 & \bar{a}_2\bar{a}_1^\gamma & |\bar{a}_2|^2 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(75)

Thus, when part of a triad beginning in a \( W \)-class state, qubit pairs that include the influenced qubit decohere fully, as exhibited by the presence of the factors \( \gamma_A \) in the expected off-diagonal terms of the first two of the above density matrices, whereas subsystem BC maintains its coherence; no such factors appear in their joint reduced density matrix because in this case the dephasing channel only acts on qubit A. The two-qubit system decoherence timescale determined by the factors in the above density matrix is thus

\[
\tau_{2-\text{dec.}D}^W = 2 \left( \frac{1}{\Gamma_1} \right).
\]

(76)

Finally, the single-qubit reduced density matrices are

\[
\rho_A^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 + |\bar{a}_2|^2 & 0 & 0 \\
0 & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

(77)

\[
\rho_B^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 + |\bar{a}_4|^2 & 0 & 0 \\
0 & |\bar{a}_2|^2 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

(78)

\[
\rho_C^W(t) = \begin{pmatrix}
|\bar{a}_2|^2 + |\bar{a}_4|^2 & 0 & 0 \\
0 & |\bar{a}_1|^2 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

(79)

because the individual-qubit matrices are already diagonal at time \( t = 0 \) and so cannot be dephased further.

Consider now the degrees of entanglement of these two-qubit subsystems as quantified by the concurrence.

\[
C_{AB}^2 = 4 |\bar{a}_2|^2 |\bar{a}_4|^2 \gamma^2,
\]

(80)

\[
C_{AC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_4|^2 \gamma^2,
\]

(81)

\[
C_{BC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_2|^2.
\]

(82)

The disentanglement timescale determined by the gamma factor \( \gamma_A \) for the subsystems AB and AC that decohere is thus

\[
\tau_{\text{dis}D}^W = \left( \frac{1}{\Gamma_1} \right).
\]

(83)

One sees that disentanglement in subsystems containing a qubit on which local dephasing noise acts is faster than both the two-qubit and three-qubit decoherence:

\[
\tau_{\text{dis}D}^W < \tau_{3-\text{dec.}D}^W,
\]

\[
\tau_{\text{dis}D}^W < \tau_{2-\text{dec.}D}^W.
\]

(84)

Neither decoherence nor disentanglement occurs in subsystems not containing qubit A.
2. TWO-QUBIT COLLECTIVE DEPHASING CHANNEL: $\mathcal{E}$

In the presence of two-qubit collective dephasing noise, acting on the two-qubit subsystem $AB$, the time-evolved full three-qubit density matrix is

$$\rho_{ABC}^W(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^{\gamma_{AB}} & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{a}_1\bar{a}_2^{\gamma_{AB}} & |\bar{a}_2|^2 & \bar{a}_2\bar{a}_4^* & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{a}_2\bar{a}_4^* & |\bar{a}_4|^2 & 0 & 0 & 0 & 0 \\
0 & \bar{a}_2\bar{a}_4^* & \bar{a}_2\bar{a}_4^* & |\bar{a}_4|^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (85)$$

There is a decay of some but not all off-diagonal density matrix elements under the local dephasing noise, as determined by $\gamma_{AB}$. A different set of terms now decay compared to the case of local dephasing noise exhibited in Eq. 70, although in both cases the terms $\rho_{25}$ and $\rho_{52}$ decay. The timescale for the decoherence that occurs in this three-qubit system, determined by the two-qubit gamma factor $\gamma_{AB}$ in the above density matrix, is therefore

$$\tau_{3-\text{dec.},W}^W = 2 \left( \frac{1}{\Gamma_2} \right). \quad (86)$$

The two-qubit reduced density matrices obtained from Eq. 70 are

$$\rho_{AB}^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_2|^2 & \bar{a}_2\bar{a}_4^* & 0 \\
0 & \bar{a}_2\bar{a}_4^* & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (87)$$

$$\rho_{AC}^W(t) = \begin{pmatrix}
|\bar{a}_2|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_4^* \gamma_{AB} & 0 \\
0 & \bar{a}_1\bar{a}_4^* \gamma_{AB} & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad (88)$$

$$\rho_{BC}^W(t) = \begin{pmatrix}
|\bar{a}_4|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_1|^2 & \bar{a}_1\bar{a}_2^{\gamma_{AB}} & 0 \\
0 & \bar{a}_1\bar{a}_2^{\gamma_{AB}} & |\bar{a}_2|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (89)$$

It is significant that the off-diagonal elements of $\rho_{AB}^W$ do not decay, whereas the those of $\rho_{AC}^W$ and $\rho_{BC}^W$ do. By comparing these expressions with those of Eqs. 73-75, we also see that collective noise acting on subsystems including only one affected qubit has an effect analogous to that of the local noise environment on that channel. This behavior also appears below in the case of total-system collective dephasing of initially W-class states, where a group of qubits subjected to the fully collective noise also do not lose decoherence as a result of that collective noise. The timescale of two-qubit system decoherence determined by the factors in the above density matrices, when it occurs, is thus

$$\tau_{2-\text{dec.},W}^W = 2 \left( \frac{1}{\Gamma_2} \right). \quad (90)$$

Again single-qubit reduced density matrices are diagonal from the outset, cf. Eqs. 77-79, and so cannot decohere further.

The degrees of entanglement of two-qubit subsystems are

$$C_{AB}^2 = 4 |\bar{a}_2|^2 |\bar{a}_4|^2, \quad (91)$$

$$C_{AC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_4|^2 \gamma_{AB}^2, \quad (92)$$

$$C_{BC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_2|^2 \gamma_{AB}^2. \quad (93)$$

Comparing these expressions with the local-dephasing noise scenario expressions in Eqs. 80-82, one again sees similar behavior when only one qubit of a subsystem is subject to collective noise. The timescale of disentanglement determined by the factors in the above density matrix, when it occurs, is thus

$$\tau_{\text{dis.},W}^W = \left( \frac{1}{\Gamma_2} \right). \quad (94)$$
In this case, one again sees that disentanglement, which occurs only when decoherence occurs, is always faster than
three-qubit and two-qubit decoherence in all instances in which these effects occur:

\[ \tau_{\text{dis,}E}^{W} < \tau_{3-\text{dec,}E}^{W}, \]
\[ \tau_{\text{dis,}E}^{W} < \tau_{2-\text{dec,}E}^{W}. \] (95)

3. THREE-QUBIT COLLECTIVE DEPHASING CHANNEL: \( F \)

The time-evolved three-qubit density matrix, however, shows no effect from the three-qubit collective dephasing
channel:

\[
\rho_{ABC}^{W}(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{a}_{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{a}_{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{a}_{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{a}_{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{a}_{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{a}_{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} .
\] (96)

Hence, the W-class states can be identified as a class that is entirely robust under fully collective noise, in the sense
being “decoherence-free,” which can be attributed to the permutation symmetry introduced with the initial state
\(|W\rangle\), analogously to what occurs in two-qubit systems under collective dephasing (cf. [16]). For the GHZ-class
states, however, one finds decoherence does occur, as shown in Section C, below.

4. THREE-QUBIT MULTI-LOCAL DEPHASING: \( D^A D^B D^C \)

Under multi-local dephasing noise, wherein each qubit is individually subject to the same sort of noise, one finds
the time-evolved density matrix arising from initially W-class states to be

\[
\rho_{ABC}^{W}(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{a}_{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{a}_{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{a}_{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{a}_{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{a}_{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{a}_{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
\] (97)

wherein all off-diagonal elements decay according to a product of two differing \( \gamma \) factors. The three-qubit system
decoherence timescale determined by those factors (recalling that \( \gamma_A = \gamma_B = \gamma_C \)) in the above density matrices is thus

\[ \tau_{3-\text{dec,}D^A D^B D^C}^{W} \leq \left( \frac{1}{\Gamma_1} \right). \] (98)
The two-qubit reduced density matrices are

$$\rho_{AB}^W(t) = \begin{pmatrix} |\tilde{a}_1|^2 & 0 & 0 & 0 \\ 0 & |\tilde{a}_2|^2 & \tilde{a}_2\tilde{a}_4^*\gamma_A\gamma_B & 0 \\ 0 & \tilde{a}_2\tilde{a}_4^*\gamma_A\gamma_B & |\tilde{a}_4|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  

(99)

$$\rho_{AC}^W(t) = \begin{pmatrix} |\tilde{a}_2|^2 & 0 & 0 & 0 \\ 0 & |\tilde{a}_1|^2 & \tilde{a}_1\tilde{a}_4^*\gamma_A\gamma_C & 0 \\ 0 & \tilde{a}_1\tilde{a}_4^*\gamma_A\gamma_C & |\tilde{a}_4|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  

(100)

$$\rho_{BC}^W(t) = \begin{pmatrix} |\tilde{a}_4|^2 & 0 & 0 & 0 \\ 0 & |\tilde{a}_1|^2 & \tilde{a}_1\tilde{a}_2^*\gamma_B\gamma_C & 0 \\ 0 & \tilde{a}_1\tilde{a}_2^*\gamma_B\gamma_C & |\tilde{a}_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  

(101)

similar in form to those in Eqs. 73-75 but having a $\gamma_X$ factor in off-diagonal elements for each corresponding local (to qubit X) noise source. One sees that all off-diagonal elements decay, as determined by two pertinent $\gamma$ factors. The two-qubit system decoherence timescale determined by the factors in the above density matrices is similarly thus

$$\tau_{2-\text{dec},D^A\cdot D^B\cdot D^C}^W = \left(\frac{1}{\Gamma_1}\right).$$  

(102)

The single-qubit reduced density matrices are again diagonal, as shown in Eqs. 77-79, and so are not able to dephase further.

The degrees of entanglement of the two-qubit subsystems given by concurrence are

$$C_{AB}^2 = 4|\tilde{a}_2|^2|\tilde{a}_4|^2\gamma_A\gamma_B^2,$$  

(103)

$$C_{AC}^2 = 4|\tilde{a}_1|^2|\tilde{a}_4|^2\gamma_A\gamma_C^2,$$  

(104)

$$C_{BC}^2 = 4|\tilde{a}_1|^2|\tilde{a}_2|^2\gamma_B\gamma_C^2,$$  

(105)

with the particular $\gamma$ changes reflecting the dephasing channel. The disentanglement timescale is thus

$$\tau_{2-\text{dis},D^A\cdot D^B\cdot D^C}^W = \frac{1}{2}\left(\frac{1}{\Gamma_1}\right).$$  

(106)

We thus find again that disentanglement is always faster than three-qubit decoherence and two-qubit decoherence

$$\tau_{\text{dis},D^A\cdot D^B\cdot D^C}^W < \tau_{3-\text{dec},D^A\cdot D^B\cdot D^C}^W,$$

$$\tau_{\text{dis},D^A\cdot D^B\cdot D^C}^W < \tau_{2-\text{dec},D^A\cdot D^B\cdot D^C}^W,$$  

(107)

for multi-local dephasing noise.

5. ONE-QUBIT LOCAL, TWO-QUBIT COLLECTIVE DEPHASING DECOHERENCE

For three-qubit systems, the possibility of combinations of local and subsystem-collective dephasing noise exists, which we now consider. Under such noise, where qubit A, for example, is affected by local noise and the remaining subsystem BC is subject to collective noise (i.e. noise collective at the scale of BC), the time-evolved state is

$$\rho_{ABC}^W(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\tilde{a}_1|^2 & \tilde{a}_1\tilde{c}_2 & 0 & \tilde{a}_1\tilde{c}_4\gamma_A\gamma_{BC} & 0 & 0 & 0 \\ 0 & \tilde{a}_1\tilde{c}_2 & |\tilde{a}_2|^2 & 0 & \tilde{a}_2\tilde{c}_4\gamma_A\gamma_{BC} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{a}_4\tilde{c}_1\gamma_A\gamma_{BC} & \tilde{a}_4\tilde{c}_2\gamma_A\gamma_{BC} & 0 & |\tilde{a}_4|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$  

(108)
The effective three-qubit system decoherence timescale determined by the factors in the above density matrices is thus
\[
\tau_{3-\text{dec, DE}}^W = 2 \left( \frac{1}{\Gamma_1} \right) + 2 \left( \frac{1}{\Gamma_2} \right). \tag{109}
\]
In this case, one finds slightly different decay behavior than in cases above: the total decay rate is composed of a product of \(\gamma\) factors. In this case and in subsequent similar cases, we will assume, for clarity of exposition, that either one of the two \(\gamma\) factors, one corresponding to local noise and the other to collective noise, is much larger or that the two are of comparable strength.

The two-qubit reduced density matrices in this case are
\[
\rho_{AB}^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_2|^2 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & 0 \\
0 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{110}
\]
\[
\rho_{AC}^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_2|^2 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & 0 \\
0 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{111}
\]
\[
\rho_{BC}^W(t) = \begin{pmatrix}
|\bar{a}_1|^2 & 0 & 0 & 0 \\
0 & |\bar{a}_2|^2 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & 0 \\
0 & \bar{a}_2 c_2 \gamma_A \gamma_{BC} & |\bar{a}_4|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{112}
\]
The off-diagonal elements also decay due to the appearance of a combination of the two pertinent \(\gamma\) factors in the first two subsystems above. Because the two qubits of subsystem BC are subjected to collective dephasing noise \textit{and} because W-class states are symmetrized, this subsystem does not decohere, differing from the case of two-qubit subsystem-collective dephasing only and the three-qubit collective dephasing cases. The two-qubit system decoherence timescale, in subsystems AB and AC, determined by these factors is thus
\[
\tau_{2-\text{dec, DE}}^W = 2 \left( \frac{1}{\Gamma_1} \right) + 2 \left( \frac{1}{\Gamma_2} \right), \tag{113}
\]
the same combination of rates as for three-qubit decoherence.

Recall again that the single-qubit reduced density matrices, given by Eqs. 77-79, are fully dephased from the outset.

The entanglement of subsystems given by the concurrences
\[
C_{AB}^2 = 4 |\bar{a}_2|^2 |\bar{a}_4|^2 \gamma_A^2 \gamma_{BC}^2, \tag{114}
\]
\[
C_{AC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_4|^2 \gamma_A^2 \gamma_{BC}^2, \tag{115}
\]
\[
C_{BC}^2 = 4 |\bar{a}_1|^2 |\bar{a}_2|^2, \tag{116}
\]
has behavior determined by the factors in the above density matrices, so that for subsystems AB and AC
\[
\tau_{\text{dis, DE}}^W = \left( \frac{1}{\Gamma_1} \right) + \left( \frac{1}{\Gamma_2} \right). \tag{117}
\]
Disentanglement is therefore always faster than two-qubit and three-qubit decoherence under this noise,
\[
\tau_{\text{dis, DE}}^W < \tau_{3-\text{dec, DE}}^W, \tag{118}
\]
when it occurs.

6. DISCUSSION

From the above results for the W class of states, we have shown that the disentanglement timescale is always faster than or equal to the decoherence rate as described by the off-diagonal terms in all five dephasing-noise environments, namely, one-qubit local (\(D\)), two-qubit collective (\(E\)), three-qubit collective (\(F\)), three-qubit multi-local (\(DDE\)), and one-qubit local plus two-qubit collective (\(DE\)) environments. Thus, the decoherence rate bounds the disentanglement rate from below for systems beginning out in W-class states.
C. BEHAVIOR OF GHZ-CLASS STATES

The analysis of the GHZ-class of states will proceed in a similar fashion as that of the W-class but can be explicated more briefly. The initial density matrix of the GHZ-class state of Eq. 46 is

$$\rho_{ABC}^{GHZ}(0) = P([\text{GHZ}^6]) \equiv \begin{pmatrix} |\bar{a}_0|^2 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{a}_0 \bar{a}_7^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{a}_7 \bar{a}_0^* & 0 & 0 & 0 & 0 & 0 & |\bar{a}_0|^2 \end{pmatrix}.$$  (119)

1. DEPHASING CHANNELS

The effect of all dephasing noise types upon the GHZ-class density matrix is similar, differing only by the representative decay factor $\tilde{\gamma}$ of the decay channel, in the five previously considered noise-scenarios:

- $\mathcal{D}^A$: $\tilde{\gamma} = \gamma_A$ 1 - qubit local,
- $\mathcal{E}^{AB}$: $\tilde{\gamma} = \gamma_{AB}^4$ 2 - qubit local,
- $\mathcal{F}$: $\tilde{\gamma} = \gamma_{ABC}^4$ 3 - qubit collective,
- $\mathcal{D}^A \mathcal{D}^B \mathcal{D}^C$: $\tilde{\gamma} = \gamma_{\alpha \beta \gamma \gamma \gamma \gamma \gamma}$ multi-local,
- $\mathcal{D}^A \mathcal{E}^{BC}$: $\tilde{\gamma} = \gamma_A \gamma_{BC}^4$ 1 - qubit local and two-qubit collective.

which appear in all of the off-diagonal elements in the density matrices in the following time-evolved density matrix, so that

$$\rho_{ABC}^{GHZ}(t) = \begin{pmatrix} |\bar{a}_0|^2 & 0 & 0 & 0 & 0 & 0 & \bar{a}_0 \bar{a}_7^* \tilde{\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{a}_7 \bar{a}_0^* \tilde{\gamma} & 0 & 0 & 0 & 0 & 0 & |\bar{a}_0|^2 \end{pmatrix},$$  (121)

which decays to a diagonal matrix as determined by the particular form of $\tilde{\gamma}$ under the influence of the noise models introduced in the previous section. The three-qubit-system decoherence timescales are thus

$$\tau_{3-dec, \mathcal{D}^A}^{GHZ} = 2 \left( \frac{1}{\Gamma_1} \right),$$  (122)
$$\tau_{3-dec, \mathcal{E}^{AB}}^{GHZ} = \frac{1}{2} \left( \frac{1}{\Gamma_2} \right),$$  (123)
$$\tau_{3-dec, \mathcal{F}_{ABC}}^{GHZ} = \frac{1}{2} \left( \frac{1}{\Gamma_3} \right),$$  (124)
$$\tau_{3-dec, \mathcal{D}^A \mathcal{D}^B \mathcal{D}^C}^{GHZ} = \frac{2}{3} \left( \frac{1}{\Gamma_1} \right),$$  (125)
$$\tau_{3-dec, \mathcal{D}^A \mathcal{E}^{BC}}^{GHZ} = 2 \left( \frac{1}{\Gamma_1} \right) + \frac{1}{2} \left( \frac{1}{\Gamma_2} \right).$$  (126)
The two-qubit reduced density matrices are, unlike the case of the W class, seen to be diagonal from the outset and so cannot decohere or dephase further:

\[
\rho_{\text{GHZ}}^{AB}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & |\bar{a}_\tau|^2
\end{pmatrix},
\]
\[\text{(127)}\]

\[
\rho_{\text{GHZ}}^{AC}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & |\bar{a}_\tau|^2
\end{pmatrix},
\]
\[\text{(128)}\]

\[
\rho_{\text{GHZ}}^{BC}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & |\bar{a}_\tau|^2
\end{pmatrix}.
\]
\[\text{(129)}\]

The single-qubit reduced density matrices are similarly always diagonal:

\[
\rho_{\text{GHZ}}^{A}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 \\
0 & |\bar{a}_\tau|^2
\end{pmatrix},
\]
\[\text{(130)}\]

\[
\rho_{\text{GHZ}}^{B}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 \\
0 & |\bar{a}_\tau|^2
\end{pmatrix},
\]
\[\text{(131)}\]

\[
\rho_{\text{GHZ}}^{C}(t) = \begin{pmatrix}
\bar{a}_0^2 & 0 \\
0 & |\bar{a}_\tau|^2
\end{pmatrix}.
\]
\[\text{(132)}\]

For two-qubit-subsystem entanglement, one finds

\[
C_{AB}^2 = 0,
\]
\[\text{(133)}\]

\[
C_{AC}^2 = 0,
\]
\[\text{(134)}\]

\[
C_{BC}^2 = 0.
\]
\[\text{(135)}\]

The reduced states of all proper subsystems in states initially of the GHZ-class are neither coherent nor entangled at any time, when subject to all the above forms of dephasing noise; all entanglement appears only at the triple-qubit level, for which there is no analytic mixed-state entanglement measure of which we are aware. Nonetheless, one sees that, at the two-qubit level, the behavior of this class of states still formally agrees with our general conclusion regarding the W-class, with the difference that the two-qubit (lack of) decoherence trivially bounds the trivial (lack of) bipartite entanglement. Unfortunately, as in the case of the W-class states but more significantly in this case, the lack of a well defined analytic mixed-state three-qubit entanglement measure precludes one from performing a detailed comparison of three-qubit disentanglement versus decoherence at the three-qubit level, which would be useful.

IV. CONCLUSIONS

We have shown for the fundamental three-qubit entanglement classes that there are different timescales for two-qubit disentanglement and decoherence under dephasing noise. In particular, we have shown for three qubits initially in an entangled state subject to an array of combinations of dephasing-noise types, disentanglement, when it occurs, does so on a timescale shorter than or equal to that of decoherence. In the course of obtaining these results, we determined the precise dephasing and bipartite disentanglement behavior for each of the various pertinent classes of two-qubit and three-qubit states under all dephasing noise scenarios available at their respective scales, and have clearly identified dephasing-free and disentanglement-free classes. The coherence and bipartite entanglement behavior for these classes has thereby been provided for future investigators, who can now select and/or engineer states for practical applications, such as quantum metrology, communication and information processing, according to the coherence and entanglement characteristics these tasks require.
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[20] Although often useful, the commonly used three-qubit residual-tangle multipartite entanglement measure ("three-tangle"

is applicable only to three-qubit pure states. Therefore, here, one may only make use of this measure for the classification of initial three-qubit states, which are pure at t=0, but unfortunately cannot do so thereafter, when the influence of dephasing noise has begun, because these states then become mixed. It is an open problem to find an analytic expression quantifying entanglement for mixed states of more than two qubits. As a result, we are currently restricted to considering only bipartite entanglement in mixed three-qubit states.