Algorithm 1024: Spherical Triangle Algorithm: A Fast Oracle for Convex Hull Membership Queries

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The Convex Hull Membership (CHM) tests whether \( p \in \text{conv}(S) \), where \( p \) and the \( n \) points of \( S \) lie in \( \mathbb{R}^m \). CHM finds applications in Linear Programming, Computational Geometry, and Machine Learning. The Triangle Algorithm (TA), previously developed, in \( O(1/\varepsilon^2) \) iterations computes \( p' \in \text{conv}(S) \), either an \( \varepsilon \)-approximate solution, or a witness certifying \( p \notin \text{conv}(S) \). We first prove the equivalence of exact and approximate versions of CHM and Spherical-CHM, where \( p = 0 \) and \( \|v\| = 1 \) for each \( v \) in \( S \). If for some \( M \geq 1 \) every non-witness with \( \|p'\| > \varepsilon \) admits \( v \in S \) satisfying \( \|p' - v\| \geq \sqrt{1 + \varepsilon/M} \), we prove the number of iterations improves to \( O(M/\varepsilon) \) and \( M \leq 1/\varepsilon \) always holds. Equivalence of CHM and Spherical-CHM implies Minimum Enclosing Ball (MEB) algorithms can be modified to solve CHM. However, we prove \((1 + \varepsilon)\)-approximation in MEB is \( \Omega(\sqrt{\varepsilon}) \)-approximation in Spherical-CHM. Thus, even \( O(1/\varepsilon) \) iteration MEB algorithms are not superior to Spherical-TA. Similar weakness is proved for MEB core sets. Spherical-TA also results a variant of the All Vertex Triangle Algorithm (AVTA) for computing all vertices of \( \text{conv}(S) \). Substantial computations on distinct problems demonstrate that TA and Spherical-TA generally achieve superior efficiency over algorithms such as Frank–Wolfe, MEB, and LP-Solver.

CCS Concepts: • Mathematics of computing → Enumeration; Solvers; • Theory of computation → Design and analysis of algorithms; Linear programming; Computational geometry; Convex optimization;

Additional Key Words and Phrases: Machine Learning, Triangle Algorithm, Convex Hull Membership, data reduction, minimum enclosing ball

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1 INTRODUCTION

Given a set \( S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m \) and a distinguished point \( p \in \mathbb{R}^m \), Convex Hull Membership (CHM) is the problem that inquires whether \( p \) lies in \( \text{conv}(S) \), the convex hull of \( S \). CHM is a basic and fundamental problem in linear programming, computational geometry, machine learning, statistics, and more. The homogeneous case of CHM, when \( p = 0 \), arises in some fundamental polynomial time algorithms for linear programming. For instance, Karmarkar’s algorithm \([24]\) deals
with a homogeneous CHM. Another example is Khachiyan’s ellipsoid algorithm [26], which is actually designed to test the feasibility of a strict system of \( n \times m \) inequalities, \( Ax < b \). Using classical LP dualities, it is easy to show that the dual to the strict LP feasibility is the homogeneous CHM corresponding to the equations \( A^Ty = 0, b^Ty + s = 0 \). This implies homogeneous CHM is an inherent dual problem to strict LP feasibility. In fact homogeneous CHM admits a matrix scaling duality that leads to a simple polynomial time interior-point method, see [25]. An important application of CHM in computational geometry and in machine learning is the irredundancy problem, the problem of computing all the vertices of \( \text{conv}(S) \), see e.g., [35].

When the number of points, \( n \), and dimension, \( m \), are large, polynomial time algorithms for CHM are prohibitive. For this reason fully polynomial time approximation schemes for CHM have been studied, see e.g., [2, 13, 18, 22]. These algorithms produce \( \epsilon \)-approximate solutions in time complexities such as \( mn/\epsilon^2 \), see e.g., [13, 22]. There are other criteria for iterative algorithms for large-scale problems, e.g., the representation of an approximate solution and the sparsity of this representation. In CHM, it is preferred that an approximate solution is represented in terms of a small number of points in \( S \). One of the well known algorithms for computing the distance from \( p \) to \( \text{conv}(S) \), sometimes known as the polytope distance problem, is the Frank–Wolfe method [16] and its variations. Letting \( A \) denote the matrix \( [v_1, \ldots, v_n] \) of points in \( S \), \( \epsilon \in \mathbb{R}^n \) the vector of ones, the Frank–Wolfe method considers the convex minimization problem: \( \min_{x} f(x) = \|Ax - p\|^2 : x \in \Sigma \), where \( \Sigma = \{x \in \mathbb{R}^n : e^Tx = 1, x \geq 0\} \), the \((n - 1)\)-dimensional simplex. Given \( x' \in \Sigma \), the Frank–Wolfe algorithm computes an index \( j \) for which the partial derivative \( \partial f(x')/\partial x_j \) is minimized. It then computes \( x'' \) along the line segment connecting \( x' \) and \( e_j \), one of the basis vectors. It replaces \( x' \) with \( x'' \) and repeats. If \( x_n \in \Sigma \) is the optimal solution of the convex minimization, an \( \epsilon \)-approximate solution is an \( x \in \Sigma \) such that \( f(x) - f(x_n) = O(\epsilon) \). The notion of coreset is related both to representation of the approximate solutions, as well as the number of iterations of an algorithm. The Frank–Wolfe algorithm gives an \( \epsilon \)-approximate solution with \( \epsilon \)-coreset of size \( O(1/\epsilon^2) \). Clarkson [13] argues that with a more sophisticated version of the algorithm that uses the Wolfe dual, together with more computation, a coreset of size \( 1/\epsilon \) can be found. Additionally, a popular class of algorithms that has \( O(1/\epsilon) \) number of iterations are the so-called first-order methods, see the fast-gradient method of Nesterov [30].

More generally, for the polytope distance problem, one is interested in computing the distance between two convex hulls. Gilbert’s algorithm [19] for the polytope distance problem coincides with the Frank–Wolfe algorithm, see Gärtner and Jaggi [18]. A related problem is the hard margin support vector machine (SVM): testing if the convex hull of two finite sets of points intersect and if not, computing the optimal pair of supporting hyperplanes separating the convex hulls, see [10].

The Triangle Algorithm (TA), introduced in [22], is a geometrically inspired algorithm designed to solve CHM. When \( p \in \text{conv}(S) \), it works analogously to the Frank–Wolfe algorithm; however, the iterates are not necessarily the same and it offers more flexibility and geometric intuition. When \( p \notin \text{conv}(S) \), the TA computes a witness, a point \( p' \in \text{conv}(S) \), where the orthogonal bisector hyperplane to the line segment \( pp' \) separates \( p \) and \( \text{conv}(S) \). This is an important feature of the TA and has proved to be useful in several applications. As an example in [2], the TA is used efficiently in the All Vertex Triangle Algorithm (AVTA), which is an algorithm for computing the set of all vertices of \( \text{conv}(S) \), or an approximate subset of vertices whose convex hull approximates \( \text{conv}(S) \). The practicality and advantages of the TA over the Frank–Wolfe are supported by large-scale computations in realistic applications. To test if \( p \notin \text{conv}(S) \), there is no need to compute the minimum of \( f(x) \) over \( \Sigma_n \). In fact a witness \( p' \) gives an estimate of the distance from \( p \) to \( \text{conv}(S) \) to within a factor of two. The TA in \( O(1/\epsilon^2) \) iterations computes a point \( p_1 \in \text{conv}(S) \) so that either \( |p - p_1| \leq \epsilon R \), where \( R = \max\{\|p - v_i\| : v_i \in S\} \), or \( p_1 \) is a witness. In each iteration the algorithm uses at most one more of the \( v_i \)'s to represent the current approximation.
\( p' \). It can thus be seen that when \( p \in \text{conv}(S) \), the algorithm produces an \( \varepsilon \)-coreset of size \( O(1/\varepsilon^2) \).

The complexity of the TA improves if \( p \) is contained in a ball of radius \( \rho \), contained in the relative interior of \( \text{conv}(S) \). Specifically, the number of iterations to compute an \( \varepsilon \)-approximate solution \( p_\varepsilon \) is \( O((R^2/\rho^2) \log(1/\varepsilon)) \). The generalization of the TA for computing the distance between two arbitrary compact convex sets is developed in [23]. The algorithm described in [23] either computes an approximate point of intersection, a separating hyperplane, an optimal supporting pair of hyperplanes, or the distance between the sets, whichever is preferred. The complexity of each iteration is dependent on the nature and description of the underlying sets. In the worst case, one needs to solving an LP over one or the other convex set.

There are three major contributions of the current work. First, we propose a variant of the TA called the **Spherical Triangle Algorithm (Spherical-TA)** and provide a novel analysis on its complexity. Second, we list applications of the TA and the Spherical-TA. In particular, we introduce two classes of problems: feasibility problems and the irredunandy problem. Third, we provide substantial computational results to verify the efficiency of the algorithms in both feasibility and irredunandy problems. We view Spherical-TA as a powerful tool for analyzing the complexity of the TA, also making connection with the well known MEB problem. We also show that, as an efficient oracle, the TA, and the Spherical-TA can significantly impact various domains. Indeed we also prove some novel results of independent interest regarding the **Minimum Enclosing Ball** (MEB) problem.

The article is organized as follows: we first review the TA in Section 2. In Section 3, we prove the equivalence of exact and approximate versions of CHM and Spherical-CHM. In Section 4, we describe the Spherical-TA and its complexity analysis. On the one hand, this results in proving the iteration complexity for solving Spherical-CHM to be no worse than \( O(1/\varepsilon^2) \). On the other hand, in Section 5, we combine the analysis of Section 4 with a new analysis to derive an enhanced complexity analysis for Spherical-TA, proving if there is \( M \geq 1 \) such that for every iterate \( p' \in \text{conv}(S) \) that is not a witness and \( \|p'\| > \varepsilon \), there exists \( v \in S \) with \( \|p' - v\| \geq \sqrt{1 + \varepsilon}/M \), the overall number of iterations of Spherical-TA is \( O(M/\varepsilon) \). Since \( M \leq 1/\varepsilon \) is always valid, the number of iterations of Spherical-TA ranges between \( O(1/\varepsilon^2) \) and \( O(1/\varepsilon) \). This geometric assumption is reasonable and suggests a strategy for when it is not satisfied at an iterate. In Section 6, we consider the complexity of solving Spherical-CHM via the MEB algorithms, made plausible in view of our proof of equivalence of CHM and Spherical-CHM. We use the TA distance duality to prove a necessary and sufficient condition for MEB. Moreover, we prove a \((1 + \varepsilon)\)-approximation to the optimal ball radius, in the worst-case, is only an \( \Omega(\sqrt{\varepsilon}) \)-approximate solution to the optimal center, hence to Spherical-CHM. In particular, the state of the art \( O(1/\varepsilon) \) iteration algorithms for MEB to compute an \((1 + \varepsilon)\)-approximation to the optimal ball radius, in the worst-case, runs in \( O(1/\varepsilon^2) \) iterations to compute an \( \varepsilon \)-approximate solution for Spherical-CHM. We also prove that an \( \varepsilon \)-core set for MEB also has the same weakness with respect to CHM. Also, in our computational experimentation, we compare a particular MEB algorithm to Spherical-TA. In Sections 7 and 8, we consider applications of the TA in solving the feasibility problems, i.e., strict LP feasibility and LP feasibility. In Section 9, we introduce the irredunandy problem. In Section 10, we demonstrate our empirical results. Lastly, we conclude with remarks and propose future work.

## 2 A SUMMARY OF TRIANGLE ALGORITHM, DUALITIES, AND COMPLEXITY

The TA described in [22] is an iterative algorithm for solving the CHM problem. Formally, given a set \( S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m \), a distinguished point \( p \in \mathbb{R}^m \), and \( \varepsilon \in (0, 1) \), solving CHM means either computing an \( \varepsilon \)-approximate solution, i.e., \( p_\varepsilon \in \text{conv}(S) \) so that

\[
\|p - p_\varepsilon\| \leq \varepsilon R, \quad R = \max\{\|v_i - p\| : v_i \in S\},
\]
or a hyperplane that separates $p$ from $\text{conv}(S)$. Given an iterate $p' \in \text{conv}(S)$, the TA searches for a pivot to get closer to $p$, where $v \in S$ is a $p$-pivot (or simply pivot) if $\|p' - v\| \geq \|p - v\|$. Equivalently,

$$
(p - p')^T v \geq \frac{1}{2}(\|p\|^2 - \|p'\|^2).
$$

A $p$-witness (or simply witness) is a point $p' \in \text{conv}(S)$, where the orthogonal bisecting hyperplane to the line segment $pp'$ separates $p$ from $\text{conv}(S)$. Equivalently,

$$
\|p' - v_i\| < \|p - v_i\|, \quad \forall i = 1, \ldots, n.
$$

The separating hyperplane $H$ is given as

$$
H = \{x : (p - p')^T x = \frac{1}{2}(\|p\|^2 - \|p'\|^2)\}.
$$

Given an iterate $p' \in \text{conv}(S)$ that is neither an $\varepsilon$-approximate solution nor a witness, the TA finds a $p$-pivot $v \in S$. Then on the line segment $p'v$ it computes the closest point to $p$, denoted by $\text{Nearest}(p; p'v)$. It then replaces $p'$ with $\text{Nearest}(p; p'v)$ and repeats.

**Proposition 1 ([22]).** Suppose $p' \in \text{conv}(S)$ satisfies $\|p' - p\| \leq \min\{\|p - v_i\| : i = 1, \ldots, n\}$, and $v_j$ is a $p$-pivot, then the new iterate is

$$
p'' = \text{Nearest}(p; p'v_j) = (1 - \alpha)p' + \alpha v_j, \quad \alpha = (p - p')^T (v_j - p')/\|v_j - p'\|^2.
$$

If $p' = \sum_{i=1}^n \alpha_i v_i$, a convex combination, $p'' = \sum_{i=1}^n \alpha_i' v_i$, $\alpha_j' = (1 - \alpha)\alpha_j + \alpha$, $\alpha_i' = (1 - \alpha)\alpha_i$, $\forall i \neq j$.

The correctness and complexity of the TA are stated in the following:

**Theorem 1 ([Distance Duality] [22]).** $p \in \text{conv}(S)$ if and only if for each $p' \in \text{conv}(S)$ there exists a pivot $v_j \in S$. Equivalently, $p \notin \text{conv}(S)$ if and only if there exists a witness $p' \in \text{conv}(S)$.

**Theorem 2 ([Complexity Bounds] [22]).** Given $\varepsilon \in (0, 1)$, if the TA starts with $p_0$, the $v_j$ closest to $p$, in $O(1/\varepsilon^2)$ iterations it either computes $p_\varepsilon \in \text{conv}(S)$ with $\|p - p_\varepsilon\| \leq \varepsilon R$, or a witness.

**Definition 1.** Given $p' \in \text{conv}(S)$, $v \in S$ is a strict $p$-pivot (or simply strict pivot) if $\langle p'v, p'v \rangle \geq \pi/2$.

**Theorem 3 ([Strict Distance Duality] [22]).** Assume $p \notin S$. Then $p \in \text{conv}(S)$ if and only if for each $p' \in \text{conv}(S)$ there exists strict $p$-pivot $v \in S$.

**Theorem 4 ([22]).** Suppose $B_{\rho}(p) = \{x : \|x - p\| \leq \rho R\} \subseteq \text{conv}^\circ(S)$, the relative interior of $\text{conv}(S)$. If the TA uses a strict pivot in each iteration, $p_\varepsilon \in \text{conv}(S)$ can be computed in $O(\rho^{-2} \log \frac{1}{\varepsilon})$ iterations.

**Theorem 5 ([2]).** Let $\hat{S} = \{\hat{v}_1, \ldots, \hat{v}_N\}$ be a subset of $S = \{v_1, \ldots, v_n\}$. Given $p \in \mathbb{R}^m$, consider testing if $p \in \text{conv}(\hat{S})$. Given $\varepsilon \in (0, 1)$, the complexity of testing if there exists an $\varepsilon$-approximate solution is

$$
O \left( mN^2 + \frac{N}{\varepsilon^2} \right).
$$

In particular, suppose in testing if $p \in \text{conv}(S), S = \{v_1, \ldots, v_n\}$, the TA computes an $\varepsilon$-approximate solution $p_\varepsilon$ by examining only the elements of a subset $\hat{S} = \{\hat{v}_1, \ldots, \hat{v}_N\}$ of $S$. Then the number of operations to determine if there exists an $\varepsilon$-approximate solution $p_\varepsilon \in \text{conv}(S)$, is as stated in (6).

**Remark 1.** Without any pre-processing, the straightforward iterative complexity of the TA is $O(mN/\varepsilon^2)$. However, with an $O(mN^2)$ pre-processing, the complexity of each iteration is $O(N)$, resulting in the overall complexity in (6).
Spherical Triangle Algorithm: A Fast Oracle for CHM Queries

3 SPHERICAL-CHM AND EQUIVALENCE TO CHM

The Spherical-CHM is the case of CHM, where \( p = 0 \) and each \( v_i \in S \) has unit norm. Given a raw dataset \( S^r = \{v^r_1, \ldots, v^r_n\} \) and \( p^r \), we set \( p = 0 \) and set \( S = \{v_1, \ldots, v_n\} \), where \( v_i = (v^r_i - p^r)/\|v^r_i - p^r\| \). This step scales every point onto a unit sphere. (See Figure 1)

Intuitively, we expect CHM and Spherical-CHM to be equivalent. However, we need to make this precise, that is, we need to convert approximate solutions and separating hyperplanes from one problem to the other. The theorem below shows that given an instance of CHM we can convert it to an instance of Spherical-CHM so that the convex hull of points in CHM contains \( p^r \) if and only if the convex hull of points in Spherical-CHM contains the origin. Next, it proves if we have an \( \varepsilon \)-approximate solution of Spherical-CHM, we can convert it to an \( \varepsilon \)-approximate solution of CHM. Finally, given a separating hyperplane for Spherical-CHM, we can construct a separating hyperplane for the CHM.

**Theorem 6.** Given \( p \in \mathbb{R}^m, S = \{v_i : i = 1, \ldots, n\} \subset \mathbb{R}^m, p \notin S \), let \( R = \max\{\|v_i - p\| : v_i \in S\} \).

\( p \in \text{conv}(S) \) if and only if \( 0 \in \text{conv}(\overline{S}) \) if and only if \( 0 \in \text{conv}(S_0) \).

(i) (Equivalence of Exact Feasibility in CHM and Spherical-CHM)

\( p \in \text{conv}(S) \) if and only if \( 0 \in \text{conv}(\overline{S}) \) if and only if \( 0 \in \text{conv}(S_0) \).

(ii) (Equivalence of Approximate Solutions in CHM and Spherical-CHM)
Given \( \varepsilon \in (0, 1) \), suppose \( p_\varepsilon = \sum_{i=1}^{n} \alpha_i \overline{v}_i/\|\overline{v}_i\|, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0 \) satisfies

\[
\|\overline{p}_\varepsilon\| \leq \varepsilon. \tag{7}
\]

Set

\[
p_\varepsilon = \sum_{i=1}^{n} \beta_i v_i, \quad \beta_i = \frac{\alpha_i/\|\overline{v}_i\|}{\sum_{j=1}^{n} (\alpha_j/\|\overline{v}_j\|)}, \quad i = 1, \ldots, n. \tag{8}
\]

Then

\[
\|p - p_\varepsilon\| \leq \varepsilon R. \tag{9}
\]

(iii) (Equivalence of Separation in CHM and Spherical-CHM)
Assume \( p \not\in \text{conv}(S) \). Without loss of generality assume \( p = 0 \), hence \( R = \max \{ \|v_i\| \; : \; i = 1, \ldots, n \} \). Let \( S_R = \{ v_i / R \; : \; i = 1, \ldots, n \} \). Suppose \( p' \in \text{conv}(S_0) \) is a 0-witness, i.e., the orthogonal bisector hyperplane to the line segment \( 0p' \), say \( H_0 \), separates 0 from \( \text{conv}(S_0) \). Let \( w_i = v_i / R, i = 1, \ldots, n \). Then all \( w_i \)'s lie in the same hemisphere as the one enclosing \( S_0 \). For each \( i \), let \( w'_i \) be the projection of \( w_i \) onto the line segment \( 0p' \). Let the closest of the \( w'_i \) to the origin be denoted by \( \tilde{w}' \). Then the orthogonal bisector hyperplane to the line segment \( 0\tilde{w}' \), say \( H \), separates 0 from \( \text{conv}(S_R) \) (see Figure 2(a)). Equivalently, a scaled version of \( H \) separates 0 from \( \text{conv}(S) \).

**Proof.** (i) Suppose \( p = \sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0 \). Writing \( p = \sum_{i=1}^{n} \alpha_i \bar{v}_i \), we get

\[
0 = \sum_{i=1}^{n} \alpha_i (v_i - p) = \sum_{i=1}^{n} \alpha_i \bar{v}_i \in \text{conv}(\tilde{S}).
\]

Since \( p \neq v_i, \tilde{v}_i \neq 0 \). We can thus rewrite the equation in (10) as

\[
\sum_{i=1}^{n} \alpha_i \|\tilde{v}_i\| \|\tilde{v}_i\| = 0.
\]

Dividing both sides by \( \sum_{j=1}^{n} \alpha_j \|\tilde{v}_j\| \), we get \( 0 \in \text{conv}(S_0) \). We have thus proved one direction of the implications in (i). The other direction follows analogously.

(ii) Multiplying (7) by \( R \) we get

\[
\left\| \sum_{i=1}^{n} \frac{\alpha_i R}{\|\tilde{v}_i\|} \tilde{v}_i \right\| \leq Re.
\]

Dividing each side of (12) by \( \sum_{j=1}^{n} \alpha_j R / \|\tilde{v}_j\| \), and from the definition of the \( \beta_i \)'s in (8) we get,

\[
\left\| \sum_{i=1}^{n} \frac{\beta_i \tilde{v}_i}{\|\tilde{v}_i\|} \right\| = \|p - \sum_{i=1}^{n} \frac{\beta_i \tilde{v}_i}{\|\tilde{v}_i\|}\| \leq eR / \sum_{i=1}^{n} \frac{\alpha_i R}{\|\tilde{v}_i\|}.
\]

From the definition of \( R, R / \|\tilde{v}_i\| \geq 1 \) so that we have

\[
\sum_{j=1}^{n} \frac{\alpha_j R}{\|\tilde{v}_j\|} \geq \sum_{j=1}^{n} \alpha_j = 1.
\]

Using (14) in (13), the proof of (ii) follows.

(iii) Since \( p' \) is a 0-witness, the hyperplane \( H_0 = \{ x : p'^T x = 0.5 \|p'\| \} \) separates 0 from \( \text{conv}(S_0) \). Thus one of the two hemispheres whose base is parallel to \( H_0 \) contains all of \( S_0 \). While \( H_0 \) may not separate 0 from \( \text{conv}(S_R) \), the hemisphere that contains \( S_0 \) must also contain \( S_R \). Thus, the projection of \( w_i = v_i / R \) onto the line segment \( 0p' \) and its extension to a line, strictly lies in the hemisphere containing \( S_R \). Then, the projection \( w_i \) that is closest to the origin gives rise to a separating hyperplane \( H \) (see Figure 2).

\[\Box\]

### 4 SPHERICAL TRIANGLE ALGORITHM AND ITS COMPLEXITY ANALYSIS

Recall that we define the Spherical-TA by converting a CHM into a Spherical-CHM and applying the TA. From now on we consider CHM where \( p = 0 \) and \( S = \{ v_i : i = 1, \ldots, n \} \subset \mathbb{R}^m \), where \( \|v_i\| = 1 \), for all \( i = 1, \ldots, n \), thus a Spherical-CHM. Consider the TA for Spherical-CHM:

#### 4.1 Algorithm Description

In what follows, we will derive the worst-case complexity of Spherical-TA. The worst scenario occurs when in each iteration the iterate is not a witness, and the pivot is orthogonal to the iterate (See Figure 4). Thus, it suffices to analyze the complexity under the worst-case for each iteration. These are formalized next and then used in the next section.
be a point of unit distance, orthogonal to \( v \geq \hat{v} \{ \rangle + H \in \| v \| \) is a right angle. From the similarity of the triangles \( \{ \rangle \) which implies the first inequality in (16).

**Step 2.**

\( \delta \parallel 1 \) (Nearest from \( -\delta \): \( \delta \parallel 1 \)). The lower bound is obvious. The first and last inequalities in (16) follow from (15). The

**Lemma 1.** Given \( p' \in \text{conv}(S) \), let \( v \in S \) be a strict pivot (see Figure 3). Let \( p'' = \text{Nearest}(0, p'v) \), \( \delta = || p' ||, \delta' = || p'' ||, \mu = || p' - p'' ||. \) Let \( \hat{v} \) be a point of unit distance, orthogonal to \( p' \) (drawn for convenience on Figure 4). Let \( \hat{p}'' = \text{Nearest}(0, \hat{p}' \hat{v}) \), \( \delta' = || \hat{p}'' ||, \widehat{\mu} = || p' - \hat{p}'' ||. \) Then we have,

\[
\delta'^2 \leq \frac{\delta^2}{1 + \delta^2}, \quad \widehat{\delta'}^2 = \frac{\delta^2}{1 + \delta^2}, \quad \widehat{\mu}^2 = \frac{\delta^4}{1 + \delta^2} \geq \frac{\delta^4}{2}. \quad (15)
\]

In particular,

\[
\delta' \leq \widehat{\delta'}, \quad \mu \geq \widehat{\mu} \geq \frac{\delta^2}{\sqrt{2}}. \quad (16)
\]

**Proof.** By definition of strict pivot, the angle \( \angle p'0v \) is which implies \( \delta'^2/\delta^2 \leq ||ov||^2/||vp'||^2 \). We have \( ||vp'||^2 \geq ||op'||^2 + ||ov||^2 = ||op'||^2 + 1 \) which implies the first inequality in (15). The equality in (15) holds because \( \triangle p'0\hat{v} \) is a right angle. From the similarity of the triangles \( \triangle p'0\hat{v} \) and \( \triangle 0\hat{p}''\hat{v} \) in Figure 4, we may write \( \widehat{\mu}/\delta = \delta'/1. \) Squaring and substituting for \( \widehat{\delta}^2 \), we get the expression for \( \widehat{\mu}^2 \) in (15). The lower bound is obvious. The first and last inequalities in (16) follow from (15). The
second inequality follows from (15) and from,
\[ \mu^2 = \delta'^2 - \delta''^2 \geq \delta^2 - \frac{\delta^2}{1 + \delta^2} = \hat{\mu}^2. \]

THEOREM 7. For \( k \geq 0 \), let \( \delta_k = \|p_k\| \), where \( p_k \) is the sequence of iterates of the TA, \( p_0 = v_1 \) and none of the iterates is a witness. Let \( \hat{\delta}_0 = \delta_0 \) and define

\[ \hat{\delta}_{k+1} = \frac{\delta_k^2}{1 + \delta_k^2}, \quad k \geq 0. \] (17)

Then for all \( k \geq 1 \),
\[ \delta_k \leq \hat{\delta}_k. \] (18)

PROOF. We prove this by induction on \( k \). From Lemma 1, the inequality is true for \( k = 1 \). Assume true for \( k \). The function \( g(t) = t/(1+t) \) is monotonically increasing on \((0, \infty)\). From the relationship between \( \delta_{k+1} \) and \( \delta_k \) in Lemma 1, together with monotonicity of \( g(t) \), we may write

\[ \delta_{k+1}^2 \leq \frac{\delta_k^2}{1 + \delta_k^2} \leq \frac{\hat{\delta}_k^2}{1 + \hat{\delta}_k^2} = \hat{\delta}_{k+1}. \] (19)

THEOREM 8. Consider Spherical-CHM. The TA terminates in \( O(1/\epsilon^2) \) iterations with \( p_\epsilon \in \text{conv}(S) \), either a witness or \( \|p_\epsilon\| \leq \epsilon \).

PROOF. Let \( p_k, \delta_k \) and \( \hat{\delta}_k \) be as in the previous theorem. We claim for any natural number \( N \),

\[ \hat{\delta}_N^2 = \frac{1}{1 + N}. \] (20)

This is true for \( N = 1 \). By the induction hypothesis and the recursive definition of \( \hat{\delta}_i \), in (17), we have,

\[ \hat{\delta}_{N+1}^2 = \frac{1}{2 + N} = \frac{1}{1 + (N + 1)}. \] (21)
Fig. 4. An iteration of TA at \( p' \) with least reduction if a strict pivot \( \hat{v} \) is orthogonal to \( p' \): \( \hat{p}'' \) projection of 0 on \( p'\hat{v} \), \( \delta = \|p'\| \), \( \hat{\delta}' = \|\hat{p}''\| \), \( \hat{\mu} = \|p' - \hat{p}''\| \).

In particular, if \( N = \lceil 1/\epsilon \rceil \), we get

\[
\hat{\delta}_N = \frac{1}{\sqrt{1+N}} \leq \frac{1}{\sqrt{1+1/\epsilon}} = \frac{\sqrt{\epsilon}}{\sqrt{1+\epsilon}} \leq \sqrt{\epsilon}.
\]

(22)

From Theorem 7 \( \delta_k \leq \hat{\delta}_k \) for all \( k \geq 1 \). From this and (22) if \( 0 \in \text{conv}(S) \), in \( O(1/\epsilon) \) iterations TA computes \( p_k \) such that \( \delta_k \leq \sqrt{\epsilon} \). To complete the proof it suffices to replace \( \sqrt{\epsilon} \) with \( \epsilon \). □

5 IMPROVED COMPLEXITY ANALYSIS FOR SPHERICAL-TA

In this section, we show that rather than using an arbitrary pivot, if TA searches for a better pivot - assuming such pivot exists - an improved iteration complexity can be derived. The following gives a definition of what we mean by a better pivot.

Definition 2. Given an instance of Spherical-CHM, \( \epsilon \in (0,1) \), a number \( M \geq 1 \), we say a point \( p' \in \text{conv}(S) \) that is not a witness and for which \( \|p'\| > \epsilon \) has the \( \epsilon_M \)-property if there exists a pivot \( v \) such that

\[
\|p' - v\| \geq \sqrt{1 + \epsilon/M}.
\]

(23)

As an example, if the ball of radius \( \sqrt{\epsilon} \) is contained in \( \text{conv}(S) \), then Spherical-CHM has the \( \epsilon \)-property everywhere outside of the ball of radius \( \epsilon \). We now establish an improved complexity for Spherical-TA with the \( \epsilon_M \)-property.

Theorem 9. Consider an instance of Spherical-CHM. If every iterate \( p' \in \text{conv}(S) \) of the TA that is not a witness and for which \( \|p'\| > \epsilon \) has the \( \epsilon_M \)-property, then in \( O(M/\epsilon) \) iterations, either TA computes a witness, or \( p_\epsilon \in \text{conv}(S) \) such that \( \|p_\epsilon\| \leq \epsilon \). Moreover, if \( M = 1/\epsilon \), every \( p' \in \text{conv}(S) \) that is not a witness and \( \|p'\| > \epsilon \) satisfies the \( \epsilon_M \)-property.

Proof. The last statement of the theorem about points satisfying \( \epsilon_M \)-property for \( M = 1/\epsilon \) has already been proved in the previous section: In the worst-case \( p' \) and any pivot \( v \) are orthogonal, making the distance between \( p' \) and \( v \) to be \( \sqrt{1+\epsilon^2} = \sqrt{1+\epsilon/M} \).
If $0 \in \text{conv}(S)$, from Theorem 8 in $O(M/\epsilon)$ iterations we get an iterate $p_{k_0}$ such that $\|p_{k_0}\| \leq \sqrt{\epsilon/M}$. If $\|p_{k_0}\| \leq \epsilon$, we are done. Otherwise let $k = k_0$ and we claim that $p_{k+1}$ will decrease the gap sufficiently. More precisely, we claim

$$\delta_{k+1}^2 \leq \frac{\delta_k^2 - \epsilon^2}{2M^2} \leq \frac{\epsilon^2}{M} - \frac{\epsilon^2}{9M^2}. \quad (24)$$

To prove (24), on the one hand we have

$$\delta_{k+1}^2 = \delta_k^2 - \mu_k^2. \quad (25)$$

Since $\delta_k \leq \sqrt{\epsilon/M}$, to prove (24) we need to show $\mu_k^2 \geq \epsilon^2/9M^2$. Consider Figure 5 and assume $p' = p_k$, $\delta = \delta_k$, $\delta' = \delta_{k+1}$, $\mu = \|p' - p''\| = \mu_k$, $\nu = v_k \in S$ satisfying $\|p' - \nu\| \geq \sqrt{1 + \epsilon/M}$. Let $q$ be the point on $vp'$, where $\|\nu - q\| = 1$. Note that $p''$ must be closer to $\nu$ than to $q$. Thus,

$$\mu_k \geq \|p' - q\| \geq \frac{\sqrt{1 + \epsilon}}{M} - \frac{\epsilon}{M} - 1. \quad (26)$$

Multiplying and dividing the right-hand-side of (26) by $\left(\sqrt{1 + \frac{\epsilon}{M}} + 1\right)$ and simplifying we get

$$\mu_k \geq \|p' - q\| \geq \frac{\epsilon}{M} \frac{1}{\sqrt{1 + \frac{\epsilon}{M} + 1}} \geq \frac{\epsilon}{M} \frac{1}{\sqrt{2} + 1} \geq \frac{\epsilon}{3M}. \quad (27)$$

We thus have proved (24). Using (24), after $T$ iterations we get

$$\delta_{k+T}^2 \leq \delta_k^2 - T \frac{\epsilon^2}{9M^2} \leq \frac{\epsilon^2}{M} - T \frac{\epsilon^2}{9M^2}. \quad (28)$$

Since the right-hand-side of (28) is nonnegative, the number of iterations $T$ to get $\delta_{k+T}^2 \leq \epsilon^2$ is $O(M/\epsilon)$. Hence the proof.

5.1 Relationship Between Pivot Property and Cover Set

Let $B^m = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$, $S^{m-1} = \{x \in \mathbb{R}^m : \|x\| = 1\}$, the unit ball and sphere, respectively.

Definition 3. We say a subset $S$ of $S^{m-1}$ is an $r$-cover for $B^m$, if given any $x \in B^m$, there exists $\nu \in S$ such that $\|x - \nu\| \leq r$, i.e., the union of balls of radius $r$ centered at points in $S$, contains the unit ball.

Proposition 2. Let $S \subset S^{m-1}$, $\epsilon \in (0, 1)$, and $M \in [1, 1/\epsilon]$.

(1) If $S$ is a 1-cover of $B^m$, then $0 \in \text{conv}(S)$.

(2) If $S$ is an $r$ cover for $B^m$ with $r \leq \sqrt{2 - \frac{1}{M}} + \epsilon$, then every iterate $p' \in \text{conv}(S)$ of the TA that is not a witness and for which $\|p'\| > \epsilon$ has the $\frac{\epsilon}{M}$-property.

Proof. (1) If $0 \notin \text{conv}(S)$, there exists a witness $p' \in \text{conv}(S)$. Hence the orthogonal bisecting hyperplane to the line segment $0p'$ separates $\text{conv}(S)$ from the origin (see e.g., Figure 2(b), where $S$ consists of the points $v_i/\|v_i\|$, $i = 1, \ldots, 2$). Let $p''$ be the farthest point of $p'$ on the unit sphere. Then, the distance between $p'$ and $p''$ is larger than one and so is the distance from $p''$ to any point of $S$, contradicting that $S$ is a 1-cover.

(2) Consider a point $p' \in \text{conv}(S)$ and assume that $\|p'\| = \epsilon$. Let $p''$ be the farthest point from $p'$ on the sphere, see Figure 6. Now consider a point $\nu'$ that is as the distance of $\sqrt{1 + \epsilon/M}$ from $p'$. After a rotation of the sphere, we can reduce the problem to the Euclidean case shown in Figure 6, where $p' = (0, -\epsilon)$, $p'' = (0, 1)$ and $\nu' = (x, y)$, where $x^2 + (y + \epsilon)^2 = 1 + \epsilon/M$. Since $x^2 + y^2 = 1$ this implies $2xy = \epsilon/M - \epsilon^2$, or $2y = 1/M - \epsilon$. Using this, $\|p'' - \nu'\|^2 = x^2 + (y - 1)^2 = x^2 + y^2 + 1 - 2y = 2 - 2y = 2 - 1/M + \epsilon$. This together with the fact that $S$ is an $r$-cover for the unit ball,
Fig. 5. At iterate $p'$ the pivot $v$ satisfies $\|p' - v\| \geq \sqrt{1 + \varepsilon/M}$. The circle of radius one centered at $v$ intersect $p'v$ at $q$ and $\mu = \|p' - p''\| \geq \|p' - q\| \geq \varepsilon/3\sqrt{M}$.

Fig. 6. Given $p'$ with $\|p'\| = \varepsilon$, if $\|p' - v'\| = \sqrt{1 + \varepsilon/M}$, $\|p'' - v'\| = \sqrt{2 - \frac{1}{M}} + \varepsilon$. Then, $S$ being $r$-cover implies there is $v \in S$, where $\|p'' - v\| \leq \sqrt{2 - \frac{1}{M}} + \varepsilon$. Thus $\|p' - v\| \geq \sqrt{1 + \frac{\varepsilon}{M}}$.

$r \leq \sqrt{2 - 1/M + \varepsilon}$, implies there is a $v \in S$ that is at a distance less than or equal to $\sqrt{2 - 1/M + \varepsilon}$ to $p''$. But this implies $\|p' - v\| \geq \sqrt{1 + \varepsilon/M}$. If $\|p''\| > \varepsilon$ the same property holds. □

Remark 2. If $M$ is a function of $m$ or $n$, $f(m, n)$, then the complexity of TA would be $O(f(m, n)nm/\varepsilon)$. For example, if $M = \sqrt{m}$ or $M = \sqrt{n}$, the overall complexity of Spherical-TA would be $O(m^{1.5}n/\varepsilon)$ or $O(mn^{1.5}/\varepsilon)$, respectively. These would improve $O(mn/\varepsilon^2)$ for small $\varepsilon$.

Remark 3. The intuition behind Proposition 2 is that solving CHM becomes easier if $S$ is dense. Such an observation suggests a heuristic in the implementation of the Spherical-TA. One can
“artificially” generate points using convex combinations of points in \( S \), with the hope that an augmented \( S \) covers the unit ball with smaller \( r \) when the origin is in \( \text{conv}(S) \), thus a better complexity for Spherical-TA. In the experiment section, one can observe that with such a heuristic, the Spherical-TA can potentially serve as a faster version of TA. Spherical-CHM gives deeper insights into the TA, bringing other possibilities and problems into the view. For example, packing and covering problems are among rich classes of subjects that have been studied extensively. Böröczky [9] and Dumer [15] study the problem of covering spheres with spheres. Kochol [27] shows how to construct a set of \( n \) points on the unit sphere in \( \mathbb{R}^m \) so that their convex hull contains a ball centered at the origin with a radius that depends upon \( n \) and \( m \). In particular, this is possible taking \( n = m^2 \), where the ball radius is \( c\sqrt{m^{-1}\log m} \), \( c \) is a constant independent of \( m \). For such an example of points, according to our Theorem 4, Spherical-CHM runs in \( O((m/\log m)\log 1/\varepsilon) \) iterations which is quite good. On the other hand, regarding Spherical-CHM one may ask the following question: Given \( \varepsilon \in (0, 1) \), what is the minimum number of points, or an estimate of this minimum, with centers on the unit sphere that forms a \((1 + \varepsilon)\)-cover of the unit ball? Apparently, for arbitrary small \( \varepsilon \) this number is exponential in \( m \). In summary, Spherical-CHM brings about geometric properties and problems not available to general CHM.

### 5.2 Strategies When Pivot Property Does Not Hold at Iterate

Here, we propose some strategies when we get an iterate \( p_k \) that does not have the \( \varepsilon/M \)-property. Consider Figure 7. At iterate \( p_k \) the pivot property with a number \( M \geq 1 \) is not satisfied, i.e., \( \|p_k - v_k\| < \sqrt{1 + \varepsilon/M} \). We can still compute the next TA iterate \( p_{k+1} \) and for that compute a pivot, \( v_{k+1} \). Next, we can solve the subproblem that tests if \( 0 \in \text{conv}(\{p_k, v_k, v_{k+1}\}) \). If so, we are done. Otherwise, a relative witness is found and we could proceed similarly till we get sufficient reduction at \( p_k \). To formalize this we give a definition.

**Definition 4.** We say an iterate \( p_k \in \text{conv}(S) \) is \( \frac{\varepsilon}{M} \)-reduced at an iterate \( p_t \in \text{conv}(S) \), \( t > k \), if

\[
\|p_t\|^2 \leq \|p_k\|^2 - \frac{\varepsilon^2}{9M^2}. \tag{29}
\]

As proved in Theorem 9, when the \( \varepsilon/M \)-property is valid at \( p_k \) the inequality in definition is satisfied for \( t = k + 1 \). This inequality when satisfied in each iteration resulted in \( O(M/\varepsilon) \) iteration complexity. The strategy we propose when we get an iterate \( p_t \) that does not have the \( \varepsilon/M \)-property and is not a witness, is to compute \( p_t \), if possible, so that it \( \varepsilon/M \)-reduces \( p_k \). For this we try the simplest way possible. Then restart the TA with \( p_t \), checking if it in turn has the \( \varepsilon/M \)-property and so on. Suppose \( v_k \) is a strict pivot for \( p_k \) (see Figure 7). We compute the nearest point to 0 on \( p_k v_k \) to get \( p_{k+1} \). Next, we compute a strict pivot \( v_{k+1} \) for \( p_{k+1} \). Let the restricted Spherical-CHM be the problem of testing if \( 0 \in \text{conv}(\{p_k, v_k, v_{k+1}\}) \). At each iteration in solving the restricted problem we check if the corresponding iterate, say \( p_t \), \( \varepsilon/M \)-reduces \( p_k \). If so, we start from \( p_t \). Otherwise, we obtain a relative witness, say \( p_t \). Next, we compute a strict pivot in \( S \), say \( v_{k+2} \), if such pivot exists (otherwise \( p_t \) is a witness with respect to \( S \)). We then augment the restricted problem to test if \( 0 \in \text{conv}(\{p_k, v_k, v_{k+1}, v_{k+2}\}) \) and repeat the process. This process would stop either with a witness with respect to \( S \), or an iterate \( p_t \) that \( \varepsilon/M \)-reduces \( p_k \) and we return to the TA with \( p_t \) as the current iterate.

The worst-case complexity of such a composite iterate is unknown at this time. However, considering the geometry of the points in \( S \), we would expect that this complexity depends on \( n \) and \( m \) and the relationship between them. For \( n >> m \) and when the pairwise distance between points in \( S \) is not very small, one would expect that the composite iterate will stop after a few iterations so that the overall number of iterations would remain to be \( O(M/\varepsilon) \). However, the complexity of a composite iteration may exceed \( O(n + m) \) or \( O(mn) \), if no preprocessing it done. The theoretical
Fig. 7. At iterate $p_k$ pivot property with $M = 1$ is not satisfied, i.e., $\|p_k - v_k\| < \sqrt{1 + \varepsilon/M}$. The next TA iterate $p_{k+1}$ is computed. If $p_k$ is $\frac{\varepsilon}{M}$-reduced at $p_{k+1}$ then it returns to usual iteration of Spherical-TA. Otherwise, for $p_{k+1}$ a pivot $v_{k+1}$ is found. The subproblem tests if $0 \in \text{conv}(\{p_k, v_k, v_{k+1}\})$. If so, we are done. Otherwise, a relative witness is found in this convex hull (not shown in figure) and treated as $p_{k+2}$ and the process repeated.

analysis of the composite iterate is nevertheless an interesting open problem. In order to investigate this problem, it may be useful to first construct difficult problems for the Spherical-TA with a fixed $m$, where $0 \in \text{conv}(S)$ so that most points are on one semi-sphere, the $\varepsilon/M$-property is not satisfied for a current iterate, and $n$ is as small as possible. When increasing $n$, at some point the $\varepsilon/M$-property will be satisfied in the next iteration. If the pairwise distance between points is $\delta > 0$, then there must be a relationship between $\varepsilon, \delta, m$ and the minimum number $n$ so that the $\varepsilon/M$-property will be satisfied in the next iteration. Understanding such examples are useful in the investigation of the complexity of the composite iterative Spherical-TA. In practice, the $\varepsilon/M$-property is satisfied for a small constant and for the majority of iterates.

6 COMPLEXITY OF SOLVING SPHERICAL-CHM AS MINIMUM ENCLOSING BALL

In this section, we consider the possibility and complexity of solving CHM via algorithms for another geometric problem. Given a set $S$ of $n$ points in $\mathbb{R}^m$, the MEB problem is to compute the ball of minimum radius (volume) enclosing $S$. Specifically, let $B_\rho(c) = \{x \in \mathbb{R}^m : \|x - c\| \leq \rho\}$, the ball of radius $\rho$ centered at $c$. Then MEB is to compute the ball $B_\rho^*(c_*)$ of minimum radius enclosing $S$. In this section, we examine the relevance of MEB with respect to solving CHM and Spherical-CHM.

Like CHM, MEB has many applications such as in machine learning, clustering, data classification and SVM. It has a rich literature, apparently dating back to at least the 19th century, see [33] and also [8] for a detailed account of the earlier history of this problem. Yildrim [37] gives an excellent set of related bibliography, as well as review of both exact and approximation algorithms for MEB. The following definition is relevant to approximation algorithms for MEB.

Definition 5. A ball $B_\rho(c)$ is said to be a $(1 + \varepsilon)$-approximation to MEB if it contains $S$ and $\rho \leq (1 + \varepsilon)\rho_*$.
A subset \( S' \) of \( S \) is said to be an \( \varepsilon \)-core set (or core set) of \( S \) if its MEB has radius \( \tilde{\rho}_s \) satisfying
\[
\tilde{\rho}_s \leq \rho_s \leq (1 + \varepsilon)\tilde{\rho}_s.
\]

The fact that \( \tilde{\rho}_s \leq \rho_s \) is obvious because \( S' \) is a subset of \( S \). The general idea behind the utility of a core set for an optimization problem involving a large input set lies in the satisfiability of the following properties: (1) The core set size is a small fraction of size of the original set; (2) Solving the problem optimally for the core set provides a good approximate solution to the original problem.

Yildirim [37] describes two algorithms for computing \((1 + \varepsilon)\)-approximate MEB the first of which is closely related to the Frank–Wolfe algorithm [16] applied to the dual formulation of the problem. The second algorithm is a modification of the first. Both algorithms take \( O(nm/\varepsilon) \) arithmetic operations. In addition, the algorithms return a core set of size \( O(1/\varepsilon) \), independent of \( n \) and \( m \). Prior to Yildirim’s work another \((1 + \varepsilon)\)-approximate MEB of the same \( O(1/\varepsilon) \) iteration complexity was a geometric algorithm due to Panigrahy [31] which is also very simple to implement. These are the best \( O(1/\varepsilon) \) iterations complexity algorithms known for obtaining a \((1 + \varepsilon)\)-approximate MEB. Prior to the discovery of such fast algorithms many other approximation algorithms were described for MEB, see [37] for details. We now mention some of these algorithms. Badoiu, Har-Peled, and Indyk [5] established the existence of a core set of size \( O(1/\varepsilon^2) \) based on which their algorithm can compute a \((1 + \varepsilon)\)-approximate MEB but with slower complexity than \( O(nm/\varepsilon) \) overall operations. Badoiu and Clarkson [3], and Kumar, Mitchell, and Yildirim [28] independently discovered the existence of an \( \varepsilon \)-core set of size \( O(1/\varepsilon) \). However, this improved core-set does not result in a \( O(1/\varepsilon) \) iteration algorithm for a \((1 + \varepsilon)\)-approximation MEB. Badoiu and Clarkson [4] also show a \( O(1/\varepsilon) \) bound on the size of an \( \varepsilon \)-core set, however, their construction is based on the assumption that \( n \geq 1/\varepsilon \). The algorithms of Panigrahy [31] and Yildirim [37] compute a \((1 + \varepsilon)\)-approximate MEB in \( O(nm/\varepsilon) \) operations with the best known dependence on \( \varepsilon \). Another such algorithm for MEB is due to Clarkson [13] who analyzes the convergence properties of the Frank–Wolfe algorithm [16] for the maximization of a general concave function over the unit simplex. This problem in particular includes the problem of computing for a given point \( b \) in \( \mathbb{R}^m \) the closest point in the convex hull of a set of points \( S \) and hence the CHM problem. Computing a \((1 + \varepsilon)\)-approximate MEB can also be established by its formulation as a convex programming problem and solving it via the ellipsoid method in \( O(m^2n\log(1/\varepsilon)) \) operations [20], or the interior-point algorithms in \( O(m^2n^3.5\log(1/\varepsilon)) \) operations [28].

A reader familiar with the MEB problem and its algorithms may contemplate their application to solve CHM, once the problem is formulated as Spherical-CHM. The reason is due to the known fact that if \( B_\rho = B_{\rho_s}(c_s) \), is the MEB for \( S \), then the center of the ball \( c_s \) must lie in the convex hull of the set \( S \cap \partial B_\rho \), i.e., the set of points of \( S \) that lie on the boundary of \( B_\rho \). In fact, the converse is also true: If \( B_{\rho_s}(c) \) contains \( S \) and \( c \) lies in the convex hull of \( S \cap \partial B_{\rho_s}(c) \), then \( B_{\rho_s}(S) \) is the MEB of \( S \) so that \((c, \rho)\) coincides with \((c_s, \rho_s)\). In this section, we prove several novel results regarding MEB or its connection with Spherical-CHM. First, we give a new definition:

**Definition 6.** A ball \( B_\rho(c) \) containing \( S \) is a \((1 + \varepsilon)\)-strong approximate MEB if
\[
\rho \leq (1 + \varepsilon)\rho_s, \quad \|c - c_s\| \leq O(\varepsilon)\rho_s.
\]

A subset \( S' \) of \( S \) is a \( \varepsilon \)-core set (or strong core set) of \( S \) if its MEB has center \( \tilde{c}_s \) and radius \( \tilde{\rho}_s \) satisfying
\[
\tilde{\rho}_s \leq \rho_s \leq (1 + \varepsilon)\tilde{\rho}_s, \quad \|\tilde{c}_s - c_s\| \leq O(\varepsilon)\rho_s.
\]
The results established in this section are:

(1) We give a proof of necessary and sufficient conditions for a ball to be an MEB of $S$ based on the Distance Duality theorem, Theorem 1 developed for CHM. We also prove its uniqueness.

(2) As a consequence of our reduction of CHM into Spherical-CHM and the equivalence of exact and approximate algorithms for the two problems proved in this article, as well as (1) above, we may conclude that an MEB algorithm can be enhanced to solve the Spherical-CHM. Of course this enhancement is not automatic or immediate and it should also make an MEB algorithm capable of recognizing the case where the center is not in the convex hull of points. We show that $B_{\rho}(c)$ is an $(1 + \varepsilon)$-approximate MEB to $B_{\rho_{*}}(c_{*})$, while $\rho \leq (1 + \varepsilon)\rho_{*}$, in the worst-case $\|c - c_{*}\| \geq \sqrt{\varepsilon}\rho_{*}$. This implies to solve Spherical-CHM to accuracy $\varepsilon$ may require an $(1 + \varepsilon^{2})$-approximate MEB. Thus running an MEB $O(1/\varepsilon)$-time algorithm would become an $O(1/\varepsilon^{2})$-time CHM algorithm.

(3) We show an example of a set $S$ and an $\varepsilon$-core set $S'$ with MEB centered at $\hat{c}_{s}$ of radius $\hat{\rho}_{s}$ such that $\rho_{s} \leq (1 + \varepsilon)\hat{\rho}_{s}$, but $\|\hat{c}_{s} - c\| \geq \sqrt{\varepsilon}$. This shows even if we solve MEB for $S'$ exactly the distance between centers of the balls could be large.

(4) We show a $(1 + \varepsilon)$-approximate MEB computed specifically via Panigrahy algorithm [31] is only a $\sqrt{\varepsilon}$-approximate Spherical-CHM solution.

In summary, a $(1 + \varepsilon)$-approximate MEB may be only a $\varepsilon^{2}$-approximate solution for Spherical-CHM so that in the worst-case an algorithm for MEB is not better than the TA for CHM and certainly not better than the Spherical-TA for Spherical-CHM. It should be mentioned that even the applicability of an MEB algorithm is not immediate by itself and such an algorithm needs to be enhanced to solve the Spherical-CHM. In our computational result section, we actually make a computational comparison of the Panigrahy $(1 + \varepsilon)$-approximation MEB algorithm, once modified to solve the Spherical-CHM, with Spherical-TA. As the computational results show, in practice solving Spherical-CHM via the Panigrahy algorithm is inferior to Spherical-TA.

6.1 Necessary and Sufficient Condition and Uniqueness for MEB

**Theorem 10.** Given $S = \{v_{1}, \ldots, v_{n}\} \subset \mathbb{R}^{m}$, a ball $B_{\rho}(c)$ is a minimum volume enclosing ball for $S$ if and only if $c \in \text{conv}(S \cap \partial B_{\rho}(c))$.

**Proof.** Suppose $c \notin \text{conv}(S \cap \partial B_{\rho}(c))$. Then there exists a witness $c' \in \text{conv}(S \cap \partial B_{\rho}(c))$ (see Distance Duality, Theorem 1). We use the witness to show we can make the containing ball smaller, even if a little. The orthogonal bisecting hyperplane of the line segment $cc'$ separates $c$ from $\text{conv}(S \cap \partial B_{\rho}(c))$. Moving from $c$ toward $c'' = (c + c')/2$, the midpoint of $c, c'$, the distances to all points of $S$ in $\text{conv}(S \cap \partial B_{\rho}(c))$ decrease. Thus there exists some point $\bar{c}$ on the line segment $cc''$ such that if $\rho'$ is the farthest distance from $\bar{c}$ to a point in $\text{conv}(S \cap \partial B_{\rho}(c))$, $B_{\rho'}(\bar{c})$ contains $S$ and $\rho' < \rho$. Hence, $B_{\rho}(c)$ is not optimal.

Conversely, suppose that $c \in \text{conv}(S \cap \partial B_{\rho}(c))$. If $B_{\rho}(c)$ is not optimal, then if $B_{\rho_{*}}(c_{*})$ is optimal we must have $c_{*} \neq c$. By the Distance Duality (Theorem 1) there exists a $v_{l} \in S \cap \partial B_{\rho}(c)$ such that $\|c - v_{l}\| > \|c_{*} - v\|$. But this implies $\rho_{*} > \rho$, a contradiction. \hfill $\square$

**Corollary 1.** MEB of $S$ is unique.

**Proof.** Suppose there are two distinct spheres enclosing $S$, having centers $c_{*}$ and $c'_{*}$. From Theorem 10, $c_{*}$ lies in $\text{conv}(S \cap \partial B_{\rho_{*}}(c_{*}))$. However, by the Strict Distance Duality (Theorem 3) there exists $v \in S \cap \partial B_{\rho_{*}}(c_{*})$ such that $\|c_{*} - v\| > \|c_{*} - v\|$. But this contradicts that the ball centered at $c'_{*}$ is also optimal. \hfill $\square$
6.2 Worst-Case Examples for MEB, Core Set, and Panigrahy Algorithm

The following theorem shows that the center of an \((1 + \varepsilon)\)-approximate MEB could give a poor approximation to the center of MEB.

**Theorem 11.** Given a set of points \(S \subset \mathbb{R}^m\) with \(B_{\rho_s}(c_s)\) as its MEB, let \(B_\rho(c)\) be a \((1 + \varepsilon)\)-approximate MEB. The following inequality may hold,

\[
\|c - c_s\| \geq \sqrt{\varepsilon} \rho_s.
\]

**Proof.** Consider a set of points on the semicircle of radius one centered at \(p\), \(S = \{q, q', q''\}\), see Figure 8. Thus \(p\) lies in \(\text{conv}(S)\). The MEB is the unit circle in this case, \(c_s = p\), \(\rho_s = 1\). Consider the point \(p'\) placed vertically below \(p\) at a distance of \(1 + \varepsilon\) from \(q\). The ball of radius \(1 + \varepsilon\) at \(p'\) contains all the points in \(S\). Hence it is \((1 + \varepsilon)\)-approximate MEB of \(S\). However, \(\|p - p'\| = \sqrt{2\varepsilon} + \varepsilon^2\). \(\square\)

**Remark 4.** This example shows that if we consider solving Spherical-CHM with center in \(\text{conv}(S)\) via an algorithm for MEB, termination of the algorithm with an \((1 + \varepsilon)\)-approximate solution only produces an \(\Omega(\sqrt{\varepsilon})\)-approximate solution to Spherical-CHM. Thus in order to ensure it obtains an \(\varepsilon\)-approximate solution to Spherical-CHM we have to compute an \((1 + \varepsilon^2)\)-approximate solution to MEB.

The next theorem demonstrates that an exact MEB of an \(\varepsilon\)-core set could lead to a poor approximation of the center \(c_s\) of the MEB of the entire set.

**Theorem 12.** Suppose \(S_\varepsilon \subset S\) is an \(\varepsilon\)-core set for \(S\). Let \(B_{\rho_s}(c_s)\) be MEB of \(S\) and \(B_{\hat{\rho}_s}(\hat{c}_s)\) MEB for \(S_\varepsilon\). While \(\rho_s \leq (1 + \varepsilon)\hat{\rho}_s\), \(\|\hat{c}_s - c_s\|\) could satisfy

\[
\|\hat{c}_s - c_s\| \geq \sqrt{\varepsilon}.
\]

**Proof.** Consider the set of points \(S = \{q, q', q'', q''''\}\) in Figure 9. The ball centered at \(p\) of radius one is the MEB for \(S\). Consider the point \(p'\) vertically below \(p\) so that \(\|p - p'\| = \sqrt{\varepsilon}\). Then \(\|p' - q\| = \sqrt{1 + \varepsilon}\). Assume \(c\) is a point on the line segment \(p'q\) so that \(\|c - q\| = 1/(1 + \varepsilon)\). Then \(S_\varepsilon = \{q, q', q''\}\) is an \(\varepsilon\)-core set for \(S\) since \(\rho_s = 1\), \(\hat{\rho}_s = 1/(1 + \varepsilon)\) with the smaller ball as its MEB.
Fig. 9. A core set for $S$ whose MEB center $\hat{c}_s$ gives a poor approximation to $c_s$.

Fig. 10. Panigrahy Algorithm: Given $c_i$ as current center of the ball not yet containing $S$, farthest point $p$ is found and $c_{i+1}$ is computed so that $d(c_{i+1}, p) = 1$.

ALGORITHM 2: Panigrahy Algorithm for MEB with Known Radius($S = \{v_1, \ldots, v_n\}, \varepsilon \in (0, 1)$)

1: Start with a ball of optimal radius (say unit radius).
2: Repeat until every point is within $(1 + \varepsilon)$ of the current center $c$.
   - Find the farthest point $p$ from $c$.
   - Move $c$ towards $p$ till $p$ touches the unit sphere centered at $c$.

Let $w$ be the closest point to $p$ on the line segment $p'q$. Using the similarity of the triangle $p'pw$ and $p'pq$, and that $w$ is the closer to $p$ than $c$, it follows that

$$||p - c|| \geq ||p - w|| = \frac{\sqrt{\varepsilon}}{\sqrt{1 + \varepsilon}} \geq \sqrt{\frac{\varepsilon}{2}}.$$  \[\Box\]

Next, we consider a specific algorithm for MEB, the Panigrahy algorithm [31]. It is a simple algorithm, especially assuming optimal radius is known. Algorithm 2 describes this case. Figure 10 shows how the algorithm works geometrically, assuming the optimal radius is one. Starting with an arbitrary point $c_i$, if not all points in $S$ are within a distance of $1 + \varepsilon$ from $c_i$, the algorithm computes the farthest point $p$ from $c_i$. Then moves $c_i$ towards $p$ until the point $c_{i+1}$ is obtained at the distance of one from $p$ and the process is repeated. The number of iterations is $O(1/\varepsilon)$. When the optimal radius is unknown more steps need to be described. Later in the article, we give the pseudo code for the general case named MEBOPT in Algorithm 4, and also make a computational comparison with Spherical-TA for solving Spherical-CHM.

When the Panigrahy algorithm would take as input set of points $S$ in Figure 8 and start with $p'$ it would terminate with a $(1 + \varepsilon)$-approximate MEB that would only produce $\Omega(\sqrt{\varepsilon})$-approximate
solution to the corresponding Spherical-CHM. The question may arise what if the algorithm when applied to Spherical-CHM starts with a point $c$ on the sphere. But even in this case, the Panigrahy algorithm would perform poorly. We state this as the following theorem.

**Theorem 13.** There are instances of Spherical-CHM such that when the Panigrahy algorithm starts with a point on the sphere it terminates with a $(1 + \varepsilon)$-approximate MEB that is a $\Omega(\sqrt{\varepsilon})$-approximation solution for Spherical-CHM. In particular, to guarantee that the Panigrahy algorithm produces an $\varepsilon$-approximate solution to Spherical-CHM it may have to compute an $(1 + \varepsilon^2)$-approximate MEB.

**Proof.** Consider the case when $S$ consists of the set of points $S = \{p, p', p'', c_1\}$ shown in Figure 11. The circle has radius one. The optimal center is $c_*$. Suppose that the Panigrahy algorithm starts at $c_i$. The farthest point of $S$ from $c_i$ is $p$. $c_i$ is selected so that the point $u$, vertically below $c_*$ on the line segment $c_ip$, is a distance of $\sqrt{\varepsilon + \varepsilon^2}/4$. This means the distance from $u$ to $p$ is $1 + \varepsilon/2$. Thus the next point in the algorithm is $c_{i+1}$, at the distance of $\varepsilon/2$ to $u$. The farthest point of $c_{i+1}$ is $p'$ and since the distance from $u$ to $p'$ is also $1 + \varepsilon/2$, by the triangle inequality the distance from $c_{i+1}$ to $p'$ is less than $1 + \varepsilon/2 + \varepsilon/2 = 1 + \varepsilon$. Thus, the algorithm terminates at $c_{i+1}$. Since the distance from $u$ to $c_*$ is $\Omega(\sqrt{\varepsilon})$, it is easy to argue that the distance between $c_{i+1}$ and $c_*$ is also $\Omega(\sqrt{\varepsilon})$. $\square$

**Remark 5.** We have seen in this section that in comparison of approximation algorithms for MEB and CHM we have to be careful in the sense that a $(1 + \varepsilon)$-approximate solution for MEB could produce only an $\Omega(\sqrt{\varepsilon})$ approximate solution to CHM. In summary, a $(1 + \varepsilon)$-approximate MEB is not a strong $(1 + \varepsilon)$-approximate solution (see Definition 6) and an $\varepsilon$-core set for MEB is not a strong $\varepsilon$-core set. On the other hand, an $\varepsilon$-approximate solution $c$ to $c_*$ trivially gives a $(1 + \varepsilon)$-approximate MEB. In particular, we can apply the Triangl Algorithm to solve MEB, however, this would most likely not give rise to an algorithm as fast as the Panigrahy or Yildirim algorithms. Thus while MEB and CHM have similarities, they are different problems in the sense that they require different exact or approximate algorithms.

**7 SOLVING STRICT LINEAR FEASIBILITY AS SPHERICAL-CHM**

The following lemma connects strict LP feasibility to CHM and is a consequence of Gordan’s Theorem, hence also provable via Farkas Lemma:

**Lemma 2.** Let $A$ be an $n \times m$ real matrix and $b \in \mathbb{R}^n$. Then $Ax < b$ is feasible if and only if there is no feasible solution to the homogeneous CHM: $A^T y = 0, b^T y + s = 0, \sum_{i=1}^n y_i + s = 1$, $y \geq 0$, $s \geq 0$. 
The next theorem shows if we have a witness for the homogenous CHM dual of strict linear feasibility, it solves the strict linear feasibility itself. In particular, the TA can test the solvability of strict linear feasibility.

**Theorem 14.** For \( i = 1, \ldots, n + 1 \), let \( v_i \) be the \( i \)-th column of the \((m + 1) \times (n + 1)\) matrix \( B = \begin{pmatrix} A^T & 0 \\ b^T & 1 \end{pmatrix} \). Suppose \( \text{conv}(\{v_1, \ldots, v_{n+1}\}) \) does not contain the origin. Let \( p' = \begin{pmatrix} x \\ \alpha \end{pmatrix} \in \mathbb{R}^{m+1} \) be a witness. Then \( A(-x/\alpha) < b \).

**Proof.** We will use the Distance Duality (Theorem 1) to prove the theorem. Denote the rows of \( A \) by \( a_i^T \). Then for \( i = 1, \ldots, n \), \( v_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \) and \( v_{n+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), all in \( \mathbb{R}^{m+1} \). Since \( p' \) is a witness,

\[
\|p' - v_i\|^2 < \|v_i\|^2, \quad \forall i = 1, \ldots, n + 1.
\]

From (30) we get

\[
\|x - a_i\|^2 + (\alpha - b_i)^2 < \|a_i\|^2 + b_i^2, \quad \forall i = 1, \ldots, n.
\]

Simplifying (31) we get

\[
-2a_i^T x + \alpha b_i < -\alpha^2, \quad \forall i = 1, \ldots, n.
\]

From (30) for \( i = n + 1 \) we get,

\[
\|x\|^2 + (1 - \alpha)^2 < 1.
\]

From (33) \( \alpha > 0 \). This gives:

\[
-\alpha^2 < 0,
\]

so that from (32)

\[
-a_i^T x < \alpha b_i, \quad \forall i = 1, \ldots, n.
\]

Dividing both sides of (35) by \( \alpha \) implies \(-x/\alpha \) is a feasible solution to the strict linear feasibility problem. \( \square \)

**Remark 6.** Without loss of generality, we may assume that the first \( n \) columns of \( B \) have unit norm. Clearly, the \((n + 1)\)th column has unit norm. Thus, the CHM corresponding to \( B \) can be assumed to be Spherical.

### 8 SPHERICAL-TA FOR LP FEASIBILITY

The LP feasibility problem is to test the feasibility of

\[
Ax = b, \quad x \geq 0.
\]

In other words, to test if \( b \) lies in the cone of columns of \( A \), i.e., \( b \in \text{cone}(A) = \{y | y = \sum_{j=1}^n \alpha_j A_j, \alpha_j \geq 0\} \) where \( A_j, j = 1, \ldots, m \) are columns of \( A \). It is well known that this problem is equivalent to the general Linear Programming problem. Given a bound \( M \) on the feasible solution of (36), it can be converted into the following CHM problem:

\[
\begin{pmatrix}
A & 0 & -b \\
e^T & 1 & -M \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
y
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\frac{1}{M+1}
\end{pmatrix}
\]

\[
e^T \alpha + \beta + y = 1, \quad \alpha, \beta, y \geq 0.
\]

where \( e \in \mathbb{R}^m \) is the vector of ones. It is easy to show that (36) is feasible iff (37) is feasible. This suggests that the TA and the Spherical-TA can be applied to solve the LP feasibility problem.
9 SPHERICAL-TA FOR COMPUTING ALL VERTICES OF A CONVEX HULL

Given a set $S = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^m$, computing all vertices of its convex hull, known as the irredundancy problem [12, 34, 35], is an important problem in computational geometry and machine learning. This problem becomes challenging as $m$ grows, especially for classical algorithms such as Gift Wrapping [14] and QuickHull [6] due to their exponential running times in terms of the dimension. The irredundancy problem can be solved via $O(n)$ membership queries, i.e., for each point, checking if it is an extreme point of the convex hull. One can take LP as an oracle for membership query, however, it is impractical to solve $O(n)$ LPs for large scale problems. Clarkson algorithm [12] for irredundancy determines the vertices of the convex hull of a set of $n$ points in $\mathbb{R}^d$ making use of linear programming but reducing the number of overall linear programming problems. If $LP(n, d)$ is the time it takes to solve an LP with $n$ inequalities in $d$ variables, Clarkson’s algorithm can solve the vertex irredundancy in $O(n \times LP(s, d))$, where $s$ is the number of vertices of the convex hull, i.e., it has the advantage of being output sensitive. It thus must solve many linear programs and exactly. For example, if there are $n = 10^5$ points in dimension $d = 500$ and there are only $s = 1,000$ vertices in the convex hull, it must solve $10^5$ linear programs with 1,000 constraints and 500 variables.

We are not aware of any computational results with Clarkson’s algorithm. Szedlak [34] modifies Clarkson’s algorithm to give a combinatorial algorithm for determining the redundant constraints of a system of inequalities $Ax \leq b$, where $A$ is an $n \times d$ matrix. Indeed the vertex irredundancy and constraint irredundancy problems are reducible to each other via the point-plane duality. We are also not aware of any computational results on Szedlak’s algorithm.

The AVTA has been proposed to tackle the efficiency issue for this class of problems [2]. AVTA has some powerful features, does not require solving linear programs and is also robust. It is capable of giving good approximation to the vertices and the convex hull. In the experiments, we have compared AVTA and AVTA+ with the quickhull algorithm where the software is available. AVTA applies the TA as a membership query oracle and computes all vertices of the convex hull of a set of points under a natural assumption called $\gamma$ robustness [1] (see Definition 7 below). Given a set of points $S = \{v_1, \ldots, v_n\}$, we denote by $T$, $T \subset S$, the set of vertices of $\text{conv}(S)$.

**Definition 7.** The convex hull of $S$ is $\gamma$-robust if the minimum distance from each vertex of the convex hull to the convex hull of the remaining vertices is at least $\gamma$. (See Figure 12(a).)

The intuition behind the $\gamma$ robust assumption is as follows: a vertex is important if it is far away from the convex hull of the remaining vertices. The number of vertices of a $\gamma$ robust convex hull

---

**Fig. 12.** Example of $\gamma$-robust and nonrobust convex hull.
is much smaller compared to the number of vertices of a “non-robust” convex hull. For instance, consider \( S \) as an \( \epsilon \)-Net from a unit sphere \( U \), say \( N_\epsilon \subset U \) so that \( \forall x \in U, \exists v \in N_\epsilon, \) such that \( \|u - v\| \leq \epsilon \). Every point in \( N_\epsilon \) is a vertex of \( \text{conv}(N_\epsilon) \). The size of \( N_\epsilon \) could be exponential in terms of dimension. In such a case, every vertex is of \( O(\epsilon) \) distance to the convex hull of the remaining vertices; thus, no single vertex is important to the geometrical structure of \( \text{conv}(N_\epsilon) \) (See Figure 12(b)). In such a pathological cases, instead of computing all vertices, one will need a “good” subset of vertices to approximate \( \text{conv}(N_\epsilon) \). Indeed, AVTA also works in such approximation schemes. We refer interested readers to [2]. In this article, we only consider the irredundancy problem under the \( \gamma \)-robustness assumption.

The property \( \gamma \)-robustness allows one to test whether a subset \( \widehat{S} \) contains all vertices of \( \text{conv}(S) \). Specifically, a set \( \widehat{S} \subset S \) contains every vertex of \( \text{conv}(S) \) if every point in \( S \) is within a distance less than \( \gamma \) to \( \text{conv}(\widehat{S}) \). As an approximation, the TA can exploit such a property to solve the membership query with precision \( \gamma/2R \) where \( R \) is the diameter of \( \text{conv}(S) \). Indeed, if the query point \( p \notin \text{conv}(\widehat{S}) \), the TA will return a hyperplane \( H \) which separates \( p \) and \( \text{conv}(\widehat{S}) \) (see the Distance Duality proof in [22]). This allows one to find a vertex by the following observation: the set of farthest points along the normal direction of a hyperplane always contains an extreme point. Formally, given a hyperplane defined by its normal direction \( c^t \), maximizers of \( c^t v \) over \( \text{conv}(S) \) includes a vertex. In Figure 13, the set of vertices of \( \text{conv}(S) \) is \( T = \{v_1, \ldots, v_7\} \) and the subset \( \widehat{S} = \{v_1, \ldots, v_5\} \) does not contain all vertices of \( \text{conv}(S) \) as \( v_6, v_7 \) are excluded. Given a query point \( p \notin \text{conv}(\widehat{S}) \), the TA returns a witness \( p' \) and \( H \) the bisecting hyperplane of \( pp' \). In Figure 13(a), the set of farthest points from \( H \) is a single point, the vertex \( v_6 \). In Figure 13(b), the set of farthest points above \( H \) are points on the line segment \( v_6v_7 \) (red line). In such cases, the set of farthest points will be a facet of \( \text{conv}(S) \). One can capture a missing vertex of \( \text{conv}(S) \) by picking any point of \( S \) on the facet and finding its farthest point on this facet.

The above approach is the motivation behind the AVTA: It iteratively adds a new vertex \( v' \) to \( \widehat{S} \) by computing a separating hyperplane, until all points are within distance \( \gamma/2 \) to \( \text{conv}(\widehat{S}) \). Next, we give a detailed description of the AVTA. Given a working subset \( \widehat{S} \) of \( S \), initialized with \( v_1 \in S \) which has the maximum norm, the AVTA randomly selects \( v \in S \setminus \widehat{S} \). It then tests via the TA if \( \text{dist}(v, \text{conv}(\widehat{S})) \leq \gamma/2 \). If so \( v \) cannot be a vertex thus labeled as a redundant point, which

\[ \tau \]

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will not be considered in further computation. In case \( \|v - \text{conv}(\hat{S})\| > \gamma/2 \), AVTA computes a \( v \)-witness \( p' \in \text{conv}(\hat{S}) \). The vector \( c' = v - p' \) leads to a hyperplane separating \( v \) and \( \text{conv}(\hat{S}) \). By maximizing \( c'^T v_i \) where \( v_i \) ranges in \( S \setminus \hat{S} \), one can find a new vertex \( v' \notin \hat{S} \) and the working set \( \hat{S} \) will be updated by including \( v' \). If \( v \) coincides with \( v' \), the AVTA selects a new point in \( S \setminus \hat{S} \). Otherwise, the AVTA continues to test if the same \( v \) (for which a witness was found) is within a distance of \( \gamma/2 \) of the convex hull of the augmented set \( \hat{S} \). Also, as an iterate, the AVTA uses the same witness \( p' \). The algorithm stops when each points in \( S \) is detected either as a redundant point or an extreme point. We next describe AVTA more precisely.

### 9.1 Algorithmic Description of AVTA

Recall that in Section 4, we introduced Spherical-TA, which can be directly applied in AVTA to replace TA as a membership query oracle. Throughout this article, we use AVTA to represent the original version with the TA and AVTA+ if it applies the Spherical-TA.

### 9.2 Applications of AVTA

AVTA has various applications, including Nonnegative Matrix Factorization (NMF) and Topic Modeling which relies on the robustness of the AVTA in recovering vertices of the convex hull of a set of perturbed points. We refer the readers to [2] for details on such classes of problems. In this article, we focus on the Size Reduction problem. Following the goal of irredundancy, the AVTA+ can be applied to reduce an overcomplete dataset i.e., a dataset that can be expressed by a small fraction of itself. In other words, it is applicable when given an \( m \times n \) matrix \( A \) as data, the convex hull of the columns of \( A \), denoted by \( \text{conv}(A) \), has \( K \) vertices, where \( K \ll n \). In certain problems, instead of keeping the full dataset which is of size \( n \), one only needs to focus on the \( K \) vertices. This suggests applying the AVTA or the AVTA+ as a pre-processing algorithm to remove non-extreme points in \( A \). Such problems include, Conditional Gradient, [13], Minimum Volume Enclosing Ellipsoid (MVEE) [17, 29, 32] and Convex Hull Approximation [7]. We briefly introduce MVEE here and demonstrate the improvement of efficiency brought about by AVTA+. The MVEE estimator is based on the smallest volume ellipsoid that covers \( \text{conv}(A) \). The MVEE problem has been studied for decades and has attained interest in broad areas, e.g., outlier detection [36]. Given the MVEE of a dataset, one can identify outliers by picking points on the boundary [36]. Improving the efficiency of algorithms that solve the MVEE problem will impact areas such as robust statistics. Formally, the MVEE is defined as follows:

\[
\min_M \log \det(M^{-1}) \quad \text{s.t.} \quad (v_i - b)^T M (v_i - b) \leq 1, \quad i = 1, \ldots, n
\]

\[
M > 0.
\]
where the optimization computes a vector $b \in \mathbb{R}^m$ and an $m \times m$ symmetric and positive definite matrix $M$ given a set of points $S = \{v_1, \ldots, v_n\}$. Then the resulting ellipsoid $E = \{x | (x - b)^T M (x - b) \leq 1\}$ will be the MVEE centered at $b$ and contains the convex hull of the columns of $A$, the data points. This suggests that one can run AVTA+ before solving MVEE since the number of vertices $K$ is generally much less than the number of columns of of $A$ (See Figure 14).

10 EXPERIMENTS

In this section, we demonstrate the power of the TA and the Spherical-TA in solving CHM and the significance of CHM in solving other problems. In Section 10.1, we compare the efficiency of the Spherical-TA, TA, MEB algorithm, Frank–Wolfe, and LP solver for solving CHM (10.1.1), LP feasibility (10.1.2) and strict LP Feasibility (10.1.3). In Section 10.2.1, we apply the Spherical-TA as a separating hyperplane oracle to find all vertices of a convex hull of a finite set. In Section 10.2.2, we use the AVTA and AVTA+ as preprocessing steps for the MVEE problem.

Implementation Details: We apply our implementations of the TA, the Spherical-TA, the AVTA, and the AVTA+ using MATLAB. In particular, we have a practical implementation of the TA and the Spherical-TA using both the aforementioned strict pivot and also the anti-pivot described in [38]. To describe what is anti-pivot, suppose $p’ = \sum_{i=1}^{n} \alpha_i v_i$, we say $v_i$ is an active component if $\alpha_i > 0$. Let $S_{p’}$ denote the set of active components of $p’$, we say $v_i$ is an anti-pivot if $\angle v_i p’ p$ is obtuse. An anti-pivot can also be used to reduce $\|p’ - p\|$ in a similar way to a pivot by computing the nearest point to $p’$ on $pv_i$. Our implementation of Spherical-TA also incorporates the heuristics described in Remark 3. For the MEB algorithm of Panigrahy, named MEBOPT, we have our own implementation according to Algorithm 4 from [31]. For the Frank–Wolfe Algorithm, we implement according to Algorithm 1 in [21]. We set an early stop criterion for the Frank–Wolfe and MEBOPT algorithm when $\|p - p’\| \leq \varepsilon R$. Note that in the case where $p \notin \text{conv}(S)$, Frank–Wolfe will keep iterating until the number of iterations exceeds the maximum iterate, thus in general, it is less efficient than the TA and the Spherical-TA. For the LP solver, we use the linprog package provided by MATLAB. For the QuickHull solver we use the convhulln package provided by MATLAB. For the MVEE we apply the MinVolEllipse package provided by [29]. All experiments are averaged over 10 repeats.

\footnote{Any advice or opinions posted here are our own, and in no way reflect that of MathWorks.}
ALGORITHM 4: MEOPT ($S = \{v_1, \ldots, v_n\}, \varepsilon \in (0, 1)$)

1. **Step 0.** Set $c = v_i$, $r = \frac{1}{2} \max_{j \neq i} \|v_i - v_j\|$, $\delta = r$
2. **Step 1.** Repeat $O\left(\frac{1}{\varepsilon}\right)$ iterations
   
   Find $p = \arg\max_{c \in S} \|v - c\|
   
   Set $c = \frac{p - c}{\|p - c\|} c + (1 - \frac{\|p - c\|}{\|c - p\|})p$
3. **Step 2.** Set $s = \|p - c\| - r$
4. **Step 3.** If $s \leq \frac{\|p - c\|}{\|c - p\|}$, then $r = r + \frac{\|p - c\|}{\|c - p\|}$; else $r = \frac{\|p - c\|}{\|c - p\|}$
5. **Step 4.** If $\delta \leq \varepsilon$, output $c$, $r$ and Stop; else go to Step 1.

10.1 Feasibility: CHM, LP Feasibility, Strict LP Feasibility

10.1.1 Convex Hull Membership. In our experiments, we generate datasets in two ways. One leverages on the Gaussian distribution, i.e., $v_i \sim N(0, I_m)$, $i = 1, \ldots, K$ and the other on the unit sphere, i.e., vertices of the convex hull are generated by uniformly picking points on a unit sphere. The Gaussian distribution is a natural parametric distribution widely used in statistics. The unit sphere can be viewed as a scaled version of the high dimensional spherical Gaussian. We represent the dataset as a matrix $A \in \mathbb{R}^{m \times n}$, where $m$ is the dimension and $n$ is the number of data points. We compare the efficiency of the following five algorithms for solving the CHM problem: The Simplex method [11], the MEOPT algorithm, the Frank–Wolfe algorithm, the TA [22], and the Spherical-TA. The size of the problems varies from $m = 100$, $n = 500$ to $m = 1,000$, $n = 5,000$ and the value of the precision parameter $\varepsilon$ varies from $10^{-3} \sim 10^{-5}$. The running times of the five algorithms (in log scale) are shown in Figures 15 and 16. One can observe that the TA and the Spherical-TA outperform other iterative algorithms. In addition, they have much better efficiency than the LP solver with a relatively larger value of the precision parameter $\varepsilon$. This is because the number of iterations of the Spherical-TA to obtain an $\varepsilon$-approximate solution increases with smaller value of $\varepsilon$. We also observe that the running time of the TA and the Spherical-TA increases linearly with $m$ and $n$ while the LP solver is more sensitive to large values of $m$ and $n$. We plot the fraction of iterates $p'$ that satisfy $\frac{M}{\varepsilon}$-property with different values of $M$ in Figure 17 with $m = 1,000$, $n = 5,000$ and $\varepsilon = 0.0001$.

10.1.2 LP Feasibility. Here, we compare the efficiency of the TA , Spherical-TA, and an LP solver for the LP feasibility problem introduced in Section 8. We compare the running times of the five algorithms on datasets with different dimension, number of points, precision parameter, and generator for the vertices. We generate the columns of the coefficient matrix $A$ uniformly at random from a unit sphere or an i.i.d Gaussian distribution. In the case $Ax = b$, $x \geq 0$ is feasible, we generate $x \in \mathbb{R}^m$, the solution of the linear system, as an entrywise uniform $(0, 1)$ distributed vector and compute $b$ as $b = Ax$. In the case $Ax = b$, $x \geq 0$ is infeasible, we apply an SVD: $\tilde{A} = U \Sigma V^T$, thresholding half of the singular values to be zeros and obtain a low rank version of $\Sigma$ denoted as $\Sigma'$. The vector is obtained by $b = \tilde{A}x$ where $x$ is an entrywise normal $N(0, 1)$ distributed vector and $\tilde{A} = U \Sigma' V^T$, where $\Sigma$ is $\Sigma'$ perturbed by a Gaussian random matrix. We set an upper bound on $x$, as $M = 1200$ in all cases. The size of the problems varies from $m = 50$, $n = 500$ to $m = 200$, $n = 2,000$ and the value of the precision parameter $\varepsilon$ varies from $10^{-4} \sim 10^{-7}$. The running times of the five algorithms (in log scale) are shown in Figures 18 and 19. In particular, we observe a similar performance between the TA and the Spherical-TA in the number of iterations. Such observations suggest the complexity improvement of the Spherical-TA over the TA is not universal. One can observe that the TA and Spherical-TA outperform other iterative algorithms and the LP solvers.

10.1.3 Strict LP Feasibility. Here, we solve the Strict LP feasibility problem, $Ax < b$, $x \geq 0$, using the TA , the Spherical-TA, and the LP solver. We compare the running times of the
Spherical Triangle Algorithm: A Fast Oracle for CHM Queries

Fig. 15. Running time of different algorithms on CHM problem with Gaussian Vertices.

The Irredundancy Problem

10.2 Finding All Vertices. Here, we apply the AVTA, AVTA+, and Quickhull [6] to solve the irredundancy problem. We compare the efficiency of the three algorithms during the execution of which we control different parameters: (1) the dimension of the problem; (2) the number of points in $S$; (3) the fraction of redundant points in $S$ i.e., fraction of points inside $\text{conv}(S)$; and (4) the distributions used to generate the vertices. We generate vertices according to a Gaussian distribution $\mathcal{N}(0,1)^m$ or uniformly at random from a unit sphere. Having generated the set of vertices, redundant points are generated as convex combination of the vertices. The size of the problems varies from $m = 5, n = 50$ to $m = 200, n = 1,000$ and the fraction of non-vertex points varies from $0 \sim 50\%$. The running times of the three algorithms are shown in Figures 22 and 23.
Fig. 16. Running time of different algorithms on CHM problem with vertices on Unit Sphere.

(a) $p \in \text{conv}(S), \varepsilon = 10^{-3}$

(b) $p \in \text{conv}(S), \varepsilon = 10^{-4}$

(c) $p \in \text{conv}(S), \varepsilon = 10^{-5}$

(d) $p \notin \text{conv}(S)$

Fig. 17. Fraction of times that $p'$ satisfies the $\epsilon_M$ property for different values of $M$. When $M = 1$, the fraction of iterates $p'$ that satisfy the $\epsilon_M$ property is slightly below 50%. When $M \geq 15$, all iterates $p'$ satisfy the $\epsilon_M$ property.
While Quickhull performs better in small size problems, especially for low dimension, it fails to output the vertices with dimension $m > 10$ in any reasonable time, which is due to its exponential dependence on dimension $m$ in the complexity. The AVTA and the AVTA+ demonstrate significantly better efficiency in large size problems since finding a vertex only takes linear time in $m, n$.

### 10.2.2 Minimum Volume Enclosing Ellipsoid

Here, we show that the AVTA and the AVTA+ can handle large scale overcomplete data in the MVEE problem. In our experiments, vertices of the convex hull are generated from a Gaussian distribution. We set the number of vertices $K = 500$. Having generated the vertices, the “redundant” points $d_j$, where $d_j \in \text{conv}(S)$, $j = 1, \ldots, n - K$, are generated using a random convex combination $d_j = \sum_{i=1}^{K} \alpha_i v_i$. The $\alpha_i$’s are scaled so that $\sum_{i=1}^{K} \alpha_i = 1$. The algorithm AVTA+ and MVEE are implemented as follows: First run AVTA+ on $S$ to find all vertices $\hat{S} \subset S$, then run MVEE on $\hat{S}$. The AVTA and MVEE is implemented in a similar manner. The running times of the three algorithms are presented in Figure 24. The results in Figure 24 demonstrate that the AVTA and AVTA+ are an efficient pre-processing steps for data reduction, especially when the number of redundant points dominates the dataset: $n \gg K$. Indeed, the AVTA can reduce the size of dataset from $n$ to $K$ thus the downstream task has much smaller scale problem to solve.
In this article, we considered CHM, a fundamental problem in diverse fields and its special case, called Spherical-CHM, which tests if the origin lies in the convex hull of $n$ points on the unit sphere. This canonical formulation has important features that can be exploited algorithmically. It makes

Fig. 19. Running time of different algorithms on LP feasibility problem (Unit Sphere).

Fig. 20. Running time of different algorithms on Strict LP feasibility problem (Gaussian).

11 CONCLUDING REMARKS

In this article, we considered CHM, a fundamental problem in diverse fields and its special case, called Spherical-CHM, which tests if the origin lies in the convex hull of $n$ points on the unit sphere. This canonical formulation has important features that can be exploited algorithmically. It makes
Fig. 21. Running time of different algorithms on Strict LP feasibility problem (Unit Sphere).

(a) $Ax < b$ feasible
(b) $Ax < b$ infeasible

Fig. 22. Running time of different algorithms on Irredundancy problem (Unit Sphere).

(a) No redundant point
(b) 20% redundant point

(c) 50% redundant point

ACM Transactions on Mathematical Software, Vol. 48, No. 2, Article 23. Publication date: May 2022.
it possible to take a deeper look into the nature of CHM and the theoretical performance of the TA. We first showed that both in the sense of exact and approximate solutions, Spherical-CHM is equivalent to CHM. We then proposed Spherical-TA, which first converts a CHM into a Spherical-CHM and then applies the TA. We derived a novel complexity analysis on the Spherical-TA, showing that under the satisfiability of a verifiable condition at each iteration, called the $\varepsilon_M$-property, the number of iterations is $O(M/\varepsilon)$, where $M \leq 1/\varepsilon$. Thus, the number of iterations ranges between

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig23.png}
\caption{Running time of different algorithms on Irredundancy problem (Gaussian).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig24.png}
\caption{Running time on MVEE problem.}
\end{figure}
O(1/\epsilon) and O(1/\epsilon^2). Indeed in our computational results \( M \) is essentially a small constant. On the other hand, we applied the Spherical-TA to solve a set of distinct problems. Our empirical results demonstrated that the TA and the Spherical-TA achieved impressive performance in solving problems that included, CHM, LP Feasibility, Strict LP Feasibility, and the Irredundancy problem. Substantial computations were carried out on a variety of problems and computational comparison was made between TA and Spherical-TA with several other algorithms, including Frank–Wolfe, MEB, and LP Solver. We also proved that the state-of-the-art algorithm for MEB, when used to solve Spherical-CHM, in the worst case runs in \( O(1/\epsilon^2) \) iterations. Hence is not as efficient as the Spherical-TA. Our computational experimentation also found MEB to be less efficient than both TA and Spherical-TA. In summary, the TA and the Spherical-TA can be used as fast membership query oracles in high dimensional problems. Our computational results strongly support the TA and the Spherical-TA as effective tools in areas such as Linear Programming, Computational Geometry, and Machine Learning. An important advantage of the Spherical-TA over TA lies in the fact that as the iterates progress it gives insights into the overall complexity and even the nature of containment of the center in the convex hull as revealed by the quality of the pivots. Both the TA and Spherical-TA are endowed with the ability to recognize infeasibility and terminate early. Our algorithms are all implemented in MATLAB and available to the readers and researchers.

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