On hitting times for simple random walk on dense Erdös-Rényi random graphs

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Abstract

Let \((V,E)\) be a realization of the Erdös-Rényi random graph model \(G(N,p)\) and \((X_n)_{n \in \mathbb{N}}\) be a simple random walk on it. We study the size of \(\sum_{i \in V} \pi_i h_{ij}\) where \(\pi_i = \frac{d_i}{2|E|}\) for \(d_i\) the number of neighbors of node \(i\) and \(h_{ij}\) the hitting time for \((X_n)_{n \in \mathbb{N}}\) between nodes \(i\) and \(j\). We always consider a regime of \(p = p(N)\) such that realizations of \(G(N,p)\) are asymptotically almost surely connected as \(N \to \infty\). Our main result is that \(\sum_{i \in V} \pi_i h_{ij}\) is almost surely of order \(N(1 + o(1))\) as \(N \to \infty\). This coincides with previous non-rigorous results in the physics literature [SRBA04]. Our techniques are based on large deviations bounds on the number of neighbors of a typical node and the number of edges in \(G(N,p)\) [CL06] together with bounds on the spectrum of the (random) adjacency matrix of \(G(N,p)\) [EKYY11].

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1 Introduction

Random walks have been used since a couple of years to investigate properties of finite and infinite graphs e.g. [DS84, Lov93, Woe00], partially as part of a larger program for developing a theory of probability on finite and infinite graphs that accounts for their intrinsic geometrical structure, e.g. [AF00, Gri10, LPW09, LP09], partially as an independent and exciting research area with its own rights. Meanwhile, the study of random graph models has received great attention not only within the probability community e.g. [Bol01, CL06, Dur06, JLR03, Kol09, VDH09], but also within the physics, biology, engineering, computer sciences and social sciences communities, among others. In this paper, we give a small contribution to that program by studying some properties of hitting times of a random walk on Erdös-Rényi random graphs.

Let \(\mathcal{G} := (\mathcal{G}, \mathcal{F}, P_p)\) denote the probability space of the Erdös-Rényi random graph model \(G(N,p)\) on \(N\) vertices. More precisely, \(\mathcal{G}\) is the set of all graphs on \(N\) vertices, \(\mathcal{F}\) is its powerset, and \(P_p\) is the probability measure for which every edge is created, independently one from each other, with probability \(p \in [0,1]\). We will call a realization of such a graph \((V,E)\). We say that an event \(A_N \subset \mathcal{G}\) happens asymptotically almost surely (abbreviated by a.a.s.) if \(P_p(A_N) \to 1\) as \(N \to \infty\). Given \((V,E) \in \mathcal{G}\), let \((X_n)_{n \in \mathbb{N}}\) be a simple random walk on \((V,E)\), i.e. \((X_n)_{n \in \mathbb{N}}\) is the discrete time Markov Chain with state space \(V\) and transition

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probabilities given by
\[ p^n_{ij} := P(X_{n+1} = j | X_n = i) = \begin{cases} 1/d_i & \text{if } ij \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases} \]
Here \( d_i \) is the degree of vertex \( i \in V \). If \((V, \mathcal{E})\) is connected, it is well known that:
  1. \((X_n)_{n \in \mathbb{N}}\) has a unique stationary distribution defined by the vector
     \[ \pi = (\pi_1, \ldots, \pi_N)^T \quad \text{where } \pi_i := d_i/(2|\mathcal{E}|). \]
  2. For \( i, j \in V \), \( h_{ij} \) the expected number of steps \((X_n)_{n \in \mathbb{N}}\) takes to visit \( j \) when starting from \( i \) is (a.s.) finite.

The quantity \( h_{ij} \) is called hitting (or access) time for \((X_n)_{n \in \mathbb{N}}\) between \( i \) and \( j \), see (2.1) below. Note that \( h_{ij} \) is a random variable on \( G \) and is itself an expectation with respect to the law of \((X_n)_{n \in \mathbb{N}}\) as well. In the present note, we would like to estimate the so called random target time, defined by
\[ H_j := \sum_{i \in V} \pi_i h_{ij}, \quad (1.1) \]
for certain regime of \( p = p(N) \) such that \((V, \mathcal{E})\) is a.a.s. connected. In [LPW09a, chapter 10] connections between \( H_j \) and some other relevant quantities (e.g. mixing time for random walks) are shown. Our main motivation is to give a rigorous proof of \( H_j = N + o(N) \) for dense Erdős-Rényi graphs, as proposed in the physics work of SRBA04. In principle, \((V, \mathcal{E})\) is a.a.s. connected for \( p = p_c := \kappa \log N/N \) for \( \kappa > 1 \) constant (cf. [Dur06], section 2.8 and references therein). In this note, it will be necessary to take \( p(N) = \Omega((\log N)^{2\gamma-1}) \) (therefore much larger than \( p_c \)), because we use results on the spectrum of \( A \) only available in this regime [EKYY11] (cf. proof of Proposition 5.6 in our Section 3). However, we conjecture that our results are true already for \( p \geq p_c \).

The rest of this note is organized as follows. In Section 2 we give the definitions and our main results on the order of magnitude of \( H_j \), together with some applications estimating the order of magnitude of the so called random starting time, see (2.5), and commenting on the order of magnitude of the commute time for \((X_n)_{n \in \mathbb{N}}\), see (2.7). Section 3 explains how to take advantage of an spectral decomposition for \( h_{ij} \), for which every term will be bounded depending on the regime of \( p = p(N) \) and finally the a.a.s. order of magnitude will be determined. The bounds on the degree of nodes and on the number of edges are valid for every regime of \( p = p(N) \), though they are sharper in the regime where \((V, \mathcal{E})\) becomes a.a.s. connected (and therefore also in the regime where our main theorem is stated for). Our technique could eventually be applied to the largest component in other regimes of \( G(N, p) \), provided corresponding bounds on the spectrum of its (random) adjacency matrix.

### 2 Definitions and main results

Let \((V, \mathcal{E})\) be an a.a.s. connected random graph in \( G \) and \((X_n)_{n \in \mathbb{N}}\) be a simple random walk taking values on \( V \) as defined above. Its law is denote by \( \mathbb{P} \) and the corresponding expectation by \( \mathbb{E} \), and as usual, for \( i \in V \), let \( \mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | X_0 = i) \) and \( \mathbb{E}_i(\cdot) = \mathbb{E}(\cdot | X_0 = i) \). As mentioned in the introduction, our aim is to estimate the order of magnitude of quantities involving hitting times of \((X_n)_{n \in \mathbb{N}}\). Let \( \tau_j \) be the first time \((X_n)_{n \in \mathbb{N}}\) is at \( j \in V \), i.e.
\[ \tau_j := \inf\{m > 0 : X_m = j\}. \quad (2.1) \]

The hitting time \( h_{ij} \) is then given by \( h_{ij} := \mathbb{E}_i(\tau_j) \). We will represent \( h_{ij} \) in terms of eigenvalues and eigenvectors of the adjacency matrix of \((V, \mathcal{E})\). Let \( D \) be the diagonal matrix
with entries \((D)_{ij} = 1/d_i\) and \(A\) be the adjacency matrix of \((V, \mathcal{E})\). Consider the symmetric (random) matrix
\[
B := D^{1/2} A D^{1/2}
\]
and its (random) eigenvalues \(1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\). The corresponding orthonormal (random) eigenvectors are denoted by \(v_1, \ldots, v_N\), and there components are \(v_k = (v_{kj})_{j=1, \ldots, N}\). Note that the positive (random) vector \(w := (\sqrt{d_1}, \ldots, \sqrt{d_N})\) satisfies \(B \cdot w = \mathbf{1}^T w\), therefore by the Perron-Frobenius Theorem \(v_1 = \sqrt{d_j} / |\mathcal{E}|\). Here \(|\mathcal{E}|\) is the number of edges in \((V, \mathcal{E})\).

For later use, remark that:

- For every \(k = 2, \ldots, N\),
  
  \[
  0 = v_k \cdot v_1 = \sum_{j=1}^N v_{kj} v_{1j} = \frac{1}{\sqrt{2|\mathcal{E}|}} \sum_{j=1}^N v_{kj} \sqrt{d_j} \Rightarrow \sum_{j=1}^N v_{kj} \sqrt{d_j} = 0 \quad (2.2)
  \]

- The matrix \(V := (v_{kj})_{k,j=1,\ldots,N}\) is unitary, hence its rows and its columns form an orthonormal set, therefore
  
  \[
  1 = \sum_{j=1}^N v_{kj}^2 = \sum_{k=1}^N v_{kj}^2 \quad (2.3)
  \]

The following theorem gives a spectral decomposition of the hitting times:

**Theorem 2.1.** [Lov93]

\[
\begin{align*}
  h_{ij} &= 2|\mathcal{E}| \sum_{k=2}^N \frac{1}{1 - \lambda_k} \left( \frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{d_i d_j} \right), \\
  &\text{a.a.s. for any } i, j \in V, \text{ where } |\mathcal{E}| \text{ is the number of edges in } G(N,p).
\end{align*}
\]

Our main results requires a quantity \(\xi\) that stems from the estimation of the spectral gap of \(A\): In what follows \(a_0 > 0\) and \(\tilde{a}_0 \geq 10\) will always be constants and \(\xi = \xi_N\) will be a parameter such that

\[
1 + a_0 \leq \xi \leq \tilde{a}_0 \log \log N \quad (2.4)
\]

Our main result is:

**Theorem 2.2.** Let \(\xi\) be as in (2.4). Then, there exists a constant \(C > 0\) not depending on \(N\) such that in the regime of \(p = p(N)\) where \((\log N)^{2C\xi} / Np \to 0\) as \(N \to \infty\), we have a.a.s.

\[
H_j = N(1 + o(1)).
\]

Next, let us analyze quantities related to \(H_j\). Given \(i \in V\), let \(H^i\) be the random starting time defined by

\[
H^i := \sum_{j \in V} \pi_j h_{ij}. \quad (2.5)
\]

Note that in general \(h_{ij} \neq h_{ji}\) for \(i, j \in V\), so therefore in general \(H^i \neq H_i\). However, if the graph has a vertex-transitive automorphism group then \(h_{ij} = h_{ji}\) for every \(i, j \in V\) [Lov93, corollary 2.6]. From Theorem 2.1 we can deduce that

\[
H^i = \sum_{j=1}^N \pi_j h_{ij} = \frac{1}{1 - \lambda_k} \left( \sum_{j=1}^N v_{kj}^2 - \sum_{j=1}^N v_{kj} \sqrt{d_j} \right)
\]

\[
= \frac{1}{1 - \lambda_k} \left( \sum_{j=1}^N v_{kj}^2 - \sum_{j=1}^N v_{ki} \sqrt{d_j} \right) \sum_{j=1}^N v_{kj} \sqrt{d_j}
\]

\[
= \frac{1}{1 - \lambda_k}
\]
where the last equality follows from (2.2). So we have for every \( i \in V \),

\[
H_i = \sum_{k=2}^{N} \frac{1}{1 - \lambda_k}. \tag{2.6}
\]

We therefore obtain the following order of magnitude for \( H_i \):

**Theorem 2.3.** Let \( \xi \) be as in (2.4). Then, there exists a constant \( C > 0 \) not depending on \( N \) such that in the regime of \( p = p(N) \) where \((\log N)^{2C\xi}/Np \to 0 \) as \( N \to \infty \), we have a.a.s.

\[
H_i = N(1 + o(1))
\]

for each \( i \in V \).

Another related quantity is the commute time. Let \( \kappa_{ij} \) be the commute time between \( i \) and \( j \), i.e. the expected number of steps that \((X_n)_{n \in \mathbb{N}}, \) starting at \( i \in V \), needs for coming back to \( i \) visiting \( j \in V \) before,

\[
\kappa(i, j) := h_{ij} + h_{ji}. \tag{2.7}
\]

Due to the definition of the commute time in terms of hitting times and the spectral decomposition in Theorem 2.1, we have \([Lov93\text{, corollary 3.2}]\)

\[
\kappa(i, j) = 2|\mathcal{E}| \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} \left( \frac{v_{kj}}{d_j} - \frac{v_{ki}}{d_i} \right)^2
\]

and therefore we have \([Lov93\text{, corollary 3.3}]\)

\[
|\mathcal{E}| \left( \frac{1}{d_i} + \frac{1}{d_j} \right) \leq \kappa(i, j) \leq \frac{2|\mathcal{E}|}{1 - \lambda_2} \left( \frac{1}{d_i} + \frac{1}{d_j} \right). \tag{2.8}
\]

Remember that these results are also valid for random graphs though the graphs in \([Lov93]\) are deterministic. Therefore, combining Corollary 3.4, Proposition 3.6 and inequality (2.8) we have:

**Corollary 2.4.** Let \( \xi \) be as in (2.4). Then, there exists a constant \( C > 0 \) not depending on \( N \) such that in the regime of \( p = p(N) \) where \((\log N)^{2C\xi}/Np \to 0 \) as \( N \to \infty \), we have a.a.s.

\[
N(1 + o(1)) \leq \kappa(i, j) \leq N(2 + o(1))
\]

for every \( i, j \in V \).

### 3 Intermediate results and proofs

#### 3.1 Spectral decomposition.

For the readers’ convenience, we give a brief summary over spectral decomposition of hitting times, to be found in \([Lov93]\), which will be our starting point. Following the proof of Theorem 2.10 b) in \([Lov93]\), we have

\[
H_j = \sum_{i=1}^{N} \pi_i h_{ij} = \sum_{i=1}^{N} \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} \left( v_{ji} d_i - v_{ki} v_{kj} \sqrt{d_i d_j} \right)
\]

\[
= \left( \frac{1}{d_j} \sum_{i=1}^{N} d_i \right) \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2 - \sum_{k=2}^{N} v_{kj} \frac{1}{d_j} \sum_{i=1}^{N} v_{ki} \sqrt{d_i}
\]

\[
= \left( \frac{1}{d_j} \sum_{i=1}^{N} d_i \right) \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2
\]
where the last inequality follows because of (2.2). At last,

\[ H_j = \frac{2|E|}{d_j} \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2. \]  

(3.1)

Now, note that by using (2.3) we can re-write

\[ \sum_{k=2}^{N} v_{kj}^2 = \sum_{k=1}^{N} v_{kj}^2 - \pi_j = 1 - \pi_j \]  

(3.2)

Therefore, from the inequality between the arithmetic and harmonic means (considering \( v_{kj}^2 \) as weights) which can be written as

\[ \frac{1}{\sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2} \geq \frac{\sum_{k=2}^{N} v_{kj}^2}{\sum_{k=2}^{N} (1 - \lambda_k) v_{kj}^2} \]

and by using (3.2), (3.3) and (3.1) we have that

\[ H_j = \frac{1}{\pi_j} \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2 \geq \frac{1}{\pi_j} (1 - \pi_j)^2 = \frac{1}{\pi_j} - 2 + \pi_j \geq \frac{1}{\pi_j} - 2 = \frac{2|E|}{d_j} - 2, \]

which is a lower bound for \( H_j \) only in terms of \(|E|\) and \( d_j \), namely

\[ H_j \geq \frac{2|E|}{d_j} - 2. \]  

(3.4)

An upper bound for \( H_j \) can be obtained by starting from (3.1) as follows:

\[ H_j = \frac{2|E|}{d_j} \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} v_{kj}^2 \leq \frac{2|E|}{d_j} \cdot \frac{1}{1 - |\lambda_2|} \sum_{k=2}^{N} v_{kj}^2 \]

\[ = \frac{2|E|}{d_j} \cdot \frac{1}{1 - |\lambda_2|} \left( \sum_{k=1}^{N} v_{kj}^2 - \pi_j \right) = \frac{2|E|}{d_j} \cdot \frac{1}{1 - |\lambda_2|} \left( \frac{d_j}{2|E|} \right) \]

which can be written as

\[ H_j \leq \left( \frac{2|E|}{d_j} - 1 \right) \left( \frac{1}{1 - |\lambda_2|} \right). \]  

(3.5)

Hence, (3.4) and (3.5) show that:

**Proposition 3.1.** For every \( j \in V \), we have

\[ \frac{2|E|}{d_j} - 2 \leq H_j \leq \left( \frac{2|E|}{d_j} - 1 \right) \left( \frac{1}{1 - |\lambda_2|} \right). \]

The next step is now obvious: We will estimate the almost sure order of magnitude of \( 2|E|/d_j \) and \((1 - |\lambda_2|)^{-1}\) (the inverse of the spectral gap) as \( N \to \infty \).

### 3.2 Order of the stationary distribution.

The following two results are valid for all regimes of \( p = p(N) \), and even hold for a more general model of weighted random graphs. The first one, is about large deviations for the random variable \( d_j \):

\[ \frac{2|E|}{d_j} - 2 \leq H_j \leq \left( \frac{2|E|}{d_j} - 1 \right) \left( \frac{1}{1 - |\lambda_2|} \right). \]
Lemma 3.2. \cite{CL06} For \((V, E)\), with probability \(1 - e^{-c^2/2}\), the degree \(d_j\) satisfies
\[
d_j > Np - c\sqrt{Np},
\]
and with probability \(1 - e^{-c^2/2}\) the degree \(d_j\) satisfies
\[
d_j < Np + c\sqrt{Np}.
\]

Lemma 3.2 provides useful inequalities in the regime \(Np > c' \log N\) with \(c' > 1\) constant. The second result is about large deviations for the number of edges \(|E|\):

Lemma 3.3. \cite{CL06} With probability \(1 - 2e^{-c^2/6}\), the number \(|E|\) of edges in \((V, E)\) satisfies
\[
|2|E| - N^2p| < c\sqrt{N^2p}
\]
for every \(0 < c \leq \sqrt{N^2p}\).

As a consequence, we can deduce:

Corollary 3.4. In the regime of \(p = p(N)\) such that \(\frac{\log N}{Np} \to 0\) as \(N \to \infty\), we have a.a.s.
\[
\frac{2|E|}{d_j} = N(1 + o(1))
\]
for every \(j \in V\).

Proof. Take \(c = \sqrt{\log N}\) in (3.6) and in (3.8). Then, we have that a.a.s.
\[
\frac{2|E|}{d_j} \leq N^2p + \sqrt{N^2p \log N} \leq N \left(1 + \frac{\log N}{Np} \frac{1}{\sqrt{Np}} \right) = N(1 + o(1)).
\]
For the other inequality, again taking \(c = \sqrt{\log N}\), now in (3.7) and in (3.8), we have a.a.s.
\[
\frac{2|E|}{d_j} \geq N^2p - \sqrt{N^2p \log N} \geq N \left(1 - \frac{\log N}{Np} \frac{1}{\sqrt{Np}} \right) = N(1 + o(1)). \]

3.3 Order of the spectral gap.

The idea here is to use the a recent analysis on eigenvalues statistics of the adjacency matrix of dense Erdős-Rényi graphs. This was obtained in \cite{EKYY11} to estimate the order of \((1 - |\lambda_2|)^{-1}\). Let us start by deriving an a.a.s. relation between the eigenvalues of the matrix \(A = (a_{ij})_{i,j=1,\ldots,N}\) and the eigenvalues of the matrix \(B = D^{1/2}AD^{1/2} = (b_{ij})_{i,j=1,\ldots,N}\). Note that
\[
b_{ij} = \frac{a_{ij}}{\sqrt{d_i d_j}} = \frac{1}{Np} a_{ij} + \left(\frac{Np - \sqrt{d_i d_j}}{Np \sqrt{d_i d_j}}\right) a_{ij}.
\]
Therefore, \(B = \frac{1}{Np} A + R\) where the matrix \(R = (r_{ij})_{i,j=1,\ldots,N}\) is defined by
\[
r_{ij} = \left(\frac{Np - \sqrt{d_i d_j}}{Np \sqrt{d_i d_j}}\right) a_{ij}.
\]
Recall that if \(M = (m_{ij})\) is an \(N \times N\)-matrix with real entries, the spectral radius of \(M\) given by
\[
\rho(M) := \max\{|\lambda| : \lambda 	ext{ eigenvalue of } M\}
\]
and it holds \(\rho(M) \leq ||M||_{\infty} := \max_{1 \leq i \leq N} \sum_{j=1}^{N} m_{ij}\). Let \(\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N\) be the eigenvalues of \(A\), and \(w_1, w_2, \ldots, w_N\) be their corresponding normalized eigenvectors. We
are interested in the case where \((V, E)\) is (a.a.s.) not the complete graph, such that \(A\) will not have \((1/N, \ldots, 1/N)\) as normalized eigenvector. Now, it is easy to see that

\[
w_2^T B w_2 = \frac{\nu_2}{Np} + w_2^T R w_2
\]

therefore

\[
|\lambda_2| = \max_{w_2: w_2^T w_2 = 1} |w_2^T B w_2| \leq \frac{|\nu_2|}{Np} + \max_{w_2: w_2^T w_2 = 1} |w_2^T R w_2|
\]

\[
= \frac{|\nu_2|}{Np} + \rho(R) \leq \frac{|\nu_2|}{Np} + ||R||_{\infty}
\]

which means that

\[
|\lambda_2| \leq \frac{|\nu_2|}{Np} + \max_{i=1, \ldots, N} \sum_{j=1}^{N} \frac{|Np - \sqrt{d_i d_j}|}{Np \sqrt{d_i d_j}} a_{ij}.
\]

(3.9)

**Lemma 3.5.** In the regime of \(p = p(N)\) such that \(\log \frac{N}{Np} \to 0\) as \(N \to \infty\), we have that a.a.s. \(\lambda_2\) and \(\nu_2\) satisfy

\[
|\lambda_2| \leq \frac{|\nu_2|}{Np} + o(1).
\]

**Proof.** Because of inequality (3.9), it is enough to show that a.a.s. we have

\[
\max_{i=1, \ldots, N} \sum_{j=1}^{N} \frac{|Np - \sqrt{d_i d_j}|}{Np \sqrt{d_i d_j}} a_{ij} = o(1)
\]

for any \(j \in V\). Indeed, setting \(c := \sqrt{\log N}\), from Lemma 3.2 we get that

\[
|Np - \sqrt{d_i d_j}| \leq \sqrt{Np \log N}
\]

with probability \((1 - \frac{1}{\sqrt{N}}) \cdot (1 - 2 \exp(\frac{-1}{2} (\frac{1}{\sqrt{Np \log N}}) - 1))) which tends to 1 as \(N \to \infty\). Moreover, we obtain

\[
d_i d_j \geq (Np - \sqrt{Np \log N})^2
\]

with probability \((1 - \frac{1}{\sqrt{N}})^2\) which tends to 1 as \(N \to \infty\). By combining these two inequalities, given any \(\varepsilon > 0\), there exists \(N_0 = N_0(\varepsilon) > 0\) such that

\[
\sum_{j=1}^{N} \frac{|Np - \sqrt{d_i d_j}|}{Np \sqrt{d_i d_j}} a_{ij} \leq \frac{\sqrt{Np \log N}}{Np(Np - \sqrt{Np \log N})} \sum_{j=1}^{N} a_{ij}
\]

\[
= \frac{\sqrt{Np \log N}}{Np \sqrt{Np \log N}} \frac{d_i}{d_i}
\]

\[
\leq \frac{\sqrt{Np \log N} (Np + \sqrt{Np \log N})}{Np(Np - \sqrt{Np \log N})}
\]

\[
= \sqrt{\frac{\log N}{Np}} \left( \frac{1 + \frac{\sqrt{Np \log N}}{Np}}{1 - \frac{\sqrt{Np \log N}}{Np}} \right) < \varepsilon
\]

a.s. for every \(N > N_0\). 

The next result establishes a bound for \((1 - |\lambda_2|)^{-1}:

**Proposition 3.6.** In the regime of \(p = p(N)\) such that \((\log N)^{2C\xi} / Np \to 0\) as \(N \to \infty\), we have that a.s.s.

\[
\frac{1}{1 - |\lambda_2|} \leq 1 + o(1).
\]
Proof. Let $\tilde{A} := (Np(1 - p))^{-1/2} A$. Let $\mu_1 \leq \cdots \leq \mu_N$ be the eigenvalues of $\tilde{A}$. One checks that the matrix $\tilde{A}$ satisfies Definition 2.2 in [EKYY1] with $f = \sqrt{Np/(1 - p)}$. On the other hand, it is clear that

$$\tilde{A} w_2 = (Np(1 - p))^{-1/2} A w_2 = (Np(1 - p))^{-1/2} v_2 w_2$$

therefore $\mu_{N-1} = (Np(1 - p))^{-1/2} v_2$. Moreover, inequality (3.19) in [EKYY1] tells us that there exist constants $\theta, C > 0$ (both not depending on $N$) such that

$$|\mu_{N-1}| \leq 2 + (\log N)^{C\xi} \left( \frac{1}{Np} + \frac{1}{N^{2/3}} \right)$$

with probability at least $1 - e^{-\theta(\log N)^{\xi}}$, which tends to one as $N \to \infty$. Combining this with (3.10) and Lemma 3.3, we have that

$$|\lambda_2| \leq \sqrt{\frac{1 - p}{Np}} \left[ 2 + (\log N)^{C\xi} \left( \frac{1}{Np} + \frac{1}{N^{2/3}} \right) + o(1) \right] \leq \sqrt{\frac{4}{Np}} + \sqrt{\frac{(\log N)^{2C\xi}}{Np}} \left( \frac{1}{Np} + \frac{1}{N^{2/3}} \right) + o(1) = o(1)$$

a.s.s. where the last $o(1)$ is due to the hypothesis $(\log N)^{2C\xi}/Np \to 0$ as $N \to \infty$. So, a.a.s. we have that

$$1 - |\lambda_2| \geq 1 - o(1) \Rightarrow \frac{1}{1 - |\lambda_2|} \leq \frac{1}{1 - o(1)} = 1 + \left( \frac{1}{1 - o(1)} - 1 \right) = 1 + o(1). \quad \square$$

3.4 Proof of the main results.

Proof of Theorem 2.2. Take $j \in V$. The regime in Proposition 3.0 also satisfies the condition $\frac{\log N}{Np} \to 0$ as $N \to \infty$, so a.a.s. we have that

$$H_j \leq \left( \frac{2|\xi|}{d_j} - 1 \right) \left( \frac{1}{1 - |\lambda_2|} \right) \leq [N(1 + o(1)) - 1](1 + o(1)) = N(1 + o(1))$$

which follows from the upper bound in Proposition 3.7 and Proposition 3.0, and

$$N(1 + o(1)) = N(1 + o(1)) - 2 = \frac{2|\xi|}{d_j} - 2 \leq H_j$$

which follows from the lower bound in Proposition 3.7 and Corollary 3.2. \square

Proof of Theorem 2.3. In this regime, we use Proposition 3.6 and 2.6 to deduce a.a.s. the upper bound

$$\sum_{k=2}^{N} \frac{1}{1 - \lambda_k} \leq \frac{1}{1 - |\lambda_2|} (N - 1) = (1 + o(1))(N - 1) = N(1 + o(1)).$$

For the lower bound, note that $\frac{1}{1 - \lambda_k} 1_{\{\lambda_k \geq 0\}}$ is positive, and under $1_{\{\lambda_k \geq 0\}}$ we are in the case $0 \leq \lambda_k < 1$ for $k = 2, \ldots, N$. Therefore $1 \geq 1 - \lambda_k > 0$ and so it follows $\frac{1}{1 - \lambda_k} 1_{\{\lambda_k \geq 0\}} \geq 1$, which implies

$$\sum_{k=2}^{N} \frac{1}{1 - \lambda_k} \geq \sum_{k=2}^{N} \frac{1}{1 - \lambda_k} 1_{\{\lambda_k \geq 0\}} \geq (N - 1) = N(1 + o(1)). \quad \square$$

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