A NOTE ON THE NATURAL DENSITY OF PRODUCT SETS

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ABSTRACT. Given two sets of natural numbers $A$ and $B$ of natural density 1 we prove that their product set $A \cdot B := \{ab : a \in A, b \in B\}$ also has natural density 1. On the other hand, for any $\epsilon > 0$, we show there are sets $A$ of density $> 1 - \epsilon$ for which the product set $A \cdot A$ has density $< \epsilon$. This answers two questions of Hegyvári, Hennecart and Pach.

1. INTRODUCTION

Given two sets of natural numbers $A$ and $B$, let $A \cdot B := \{ab : a \in A, b \in B\}$ be their product set. Also, for any positive integer $k$, let $A^k$ denote the $k$-fold product $A \cdots A$.

The problem of studying the cardinality of product sets has long been of interest in mathematics. The classic multiplication table problem due to Erdős [2, 3] asks for bounds on the cardinality $M_n$ of the $n \times n$ multiplication table, i.e., of the set $\{1, \ldots, n\}^2$. Erdős showed that $M_n = o(n^2)$ and Ford [5], following earlier of Tenenbaum [11], determined the exact order of magnitude of $M_n$. More recently [7], the second author of the present paper provided uniform bounds for $\#(\{1, \ldots, n_1\} \cdots \{1, \ldots, n_s\})$ holding for a wide range of $n_1, \ldots, n_s \in \mathbb{N}$.

For more general sets $A$, the problem of estimating $\#(A \cap [1, x])^2$ was studied by Cilleruelo, Ramana, and Ramaré [1]. For example, they studied this problem when $A$ is the set of shifted primes, the set of sums of two squares, and the set of shifted sums of two squares. Moreover, they computed the (almost sure) asymptotic behavior for $\#A^2$ when $A$ is a random subset of $\{1, \ldots, n\}$ that contains each element of $\{1, \ldots, n\}$ independently with probability $\delta \in (0, 1)$. The third author of the present paper [10] extended this last result to the product of arbitrarily many sets, and Mastrostefano [9] gave a necessary and sufficient condition for having $\#A^2 \sim (\#A)^2 / 2$ almost surely.

Hegyvári, Hennecart and Pach [6] considered the analogous problem for infinite sets of natural numbers. In this context, the role of the cardinality is played by the natural density $d(A)$ of a set $A$, defined as usual by

$$d(A) = \lim_{x \to \infty} \frac{\#A \cap [1, x]}{x}.$$

They asked the following questions (Questions 3 and 2 of [6], respectively):

**Question 1.** If $A$ is a set of natural numbers of density 1, is it true that $A^2$ also has density 1?

**Question 2.** Is it true that $\inf_{A \subseteq \mathbb{N}}: d(A) = \alpha \implies d(A^2) = 0$ for any $\alpha \in [0, 1)$, or at least for $\alpha \in [0, \alpha_0)$ for some $\alpha_0 \in (0, 1)$?

Clearly, Question 1 has an affirmative answer if $1 \in A$, and Hegyvári, Hennecart and Pach showed that it also suffices that $A$ contains an infinite subset of mutually coprime integers $a_1 < a_2 < \cdots$ such that $\sum_{i=1}^{\infty} a_i^{-1} = +\infty$. Here, we show that the answer is “yes” in full generality.

**Theorem 1.** Let $A, B \subseteq \mathbb{N}$. If $d(A) = d(B) = 1$, then $d(A \cdot B) = 1$.

**Corollary.** If $A \subseteq \mathbb{N}$ is such that $d(A) = 1$, then $d(A^k) = 1$ for each $k = 2, 3, \ldots$
Remark. In fact, the case $\mathcal{A} = \mathcal{B}$ of Theorem 1 implies easily the general case. Indeed, if $d(\mathcal{A}) = d(\mathcal{B}) = 1$, then $d(\mathcal{A} \cap \mathcal{B}) = 1$. In addition, if $(\mathcal{A} \cap \mathcal{B})^2$ has density 1, then so does $\mathcal{A} \cdot \mathcal{B}$.

As it will be clear from the proof, the difference in the density of $d(\mathcal{A}^2)$ with respect to Erdős’s multiplication table problem lies in the fact that many elements of $\mathcal{A}^2$ come from very “unbalanced” products, meaning products $ab$ such that the sizes of $a$ and $b$ are completely different.

Let us now turn to Question 2. We will answer it in a strong form that shows, among other things, that the condition that $d(\mathcal{A}) = 1$ in Theorem 1 cannot be relaxed.

Theorem 2. For $\alpha \in [0, 1]$, we have

$$\inf_{\mathcal{A} \subseteq \mathbb{N}} d(\mathcal{A}) = \alpha = \begin{cases} 0 & \text{if } \alpha < 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

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2. Preliminaries

Notation. We employ Landau’s notation $f = O(g)$ and Vinogradov’s notation $f \ll g$ both to mean that $|f| \leq C|g|$ for a some constant $C > 0$. Moreover, we write $f \asymp g$ to mean that $f \ll g$ and $g \ll f$. The notation $f = o(g)$ as $x \to a$ (respectively $f \sim g$ as $x \to a$) means that $\lim_{x \to a} f(x)/g(x) = 0$ (respectively $= 1$). Given an integer $n$, we write $P^-(n)$ and $P^+(n)$ for its smallest and largest prime factors, respectively, with the convention that $P^-(1) = \infty$ and $P^+(1) = 1$. If $P^+(n) \leq y$, we say that $n$ is $y$-smooth, and if $P^-(n) > y$, we say that it is $y$-rough. As usual, we let $\Phi(x, y)$ denote the number of $y$-rough numbers in $[1, x]$. Given any integer $n$, we may write it uniquely as $n = ab$ with $P^+(a) \leq y < P^-(b)$. We then call $a$ and $b$ the $y$-smooth and $y$-rough part of $n$, respectively. Finally, we let $\Omega(n)$ denote the number of prime factors of $n$ counted with multiplicity.

We need some standard lemmas. We give their proofs for the sake of completeness.

Lemma 2.1. For $x \geq y > 1$, we have $\Phi(x, y) \ll x/\log y$.

Proof. This follows from example from Theorem 14.2 in [8] with $f(n) = 1_{P^-(n)>y}$.

Lemma 2.2. Uniformly for example from Theorem 14.2 in [8] with $f(n) = 1_{P^-(n)>y}$.

Proof. Without loss of generality, $u \geq 4$. Let $\mathcal{B}$ denote the set of $n \in \mathbb{Z} \cap [1, x]$ that have a $y^{1/u}$-smooth divisor $d > y$. Given $n \in \mathcal{B}$, let $p_1 \leq p_2 \leq \cdots \leq p_k$ be the sequence of prime factors of $n$ of size $\leq y^{1/u}$ listed in increasing order and according to their multiplicity. By our assumption
on \( n \), we must have \( p_1 \cdots p_k > y \). Let \( j \) be the smallest integer such that \( p_1 \cdots p_j > y \). We must have \( j \geq 5 \) because all factors \( p_i \) are \( \leq y^{1/4} \). We then set \( a = p_1 \cdots p_{j-2}, p = p_{j-1}, \) and \( b = n/(ap) \), so that \( a > y/(p_{j-1}p_j) \geq \sqrt{y}, ap \leq y, \) and \( P^+(a) \leq p \leq P^-(b) \). Consequently,

\[
\#B \leq \sum_{p \leq y^{1/4}} \sum_{P^+(a) \leq p} \sum_{b \leq x/(ap)} \frac{1}{\sqrt{y} < a \leq y/p} \sum_{P^-(b) \geq p} \frac{x}{ap\log p}
\]

by Lemma 2.1. If we let \( \varepsilon_p = \min\{2/3, 2/\log p\} \), then Rankin’s trick implies that

\[
\frac{\#B}{x} \ll \sum_{p \leq y^{1/4}} \sum_{P^+(a) \leq p} \left( \frac{a}{\sqrt{y}} \right)^{\varepsilon_p} \sum_{p \leq y^{1/4}} \frac{y^{\varepsilon_p/2}}{p \log p} \sum_{P^+(a) \leq p} \frac{1}{a^{1-\varepsilon_p}}.
\]

The sum over \( a \) equals \( \prod_{q \leq p} (1 - q^{-1+\varepsilon_p})^{-1} \) with \( q \) denoting a prime number. Since \( q^{\varepsilon_p} = 1 + O(\log q/\log p) \) for \( q \leq y/p \), Mertens’ estimates [8, Theorem 3.4] imply that the sum over \( a \) is \( \ll \log p \). We conclude that

\[
\frac{\#B}{x} \ll y^{-1/3} + \sum_{100 < p \leq y^{1/4}} \frac{e^{-\log y/\log p}}{p} \leq y^{-1/3} + \sum_{j \geq 1} \sum_{y^{1/(u(j+1))} < p \leq y^{1/(uj)}} e^{-ju} p^{-1} \leq y^{-1/3} + \sum_{j \geq 1} e^{-ju} \ll y^{-1/3} + e^{-u}
\]

using Mertens’ estimates once again. This completes the proof. \( \square \)

**Lemma 2.3.** Let \( y \geq 2 \) and \( \lambda \in [0, 1.99] \), and set \( Q(\lambda) = \lambda \log \lambda - \lambda + 1 \) for \( \lambda > 0 \) and \( Q(0) = 0 \). If \( 0 \leq \lambda \leq 1 \), then

\[
\prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \sum_{\Omega(m) \leq y} \frac{1}{m} \ll (\log y)^{-Q(\lambda)},
\]

whereas if \( 1 \leq \lambda \leq 1.99 \), then

\[
\prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \sum_{\Omega(m) \geq \lambda \log \log y} \frac{1}{m} \ll (\log y)^{-Q(\lambda)}.
\]

**Proof.** The result is trivial if \( \lambda = 0 \) by Mertens’ estimates [8, Theorem 3.4], so assume that \( \lambda > 0 \). If \( 0 < \lambda \leq 1 \), then

\[
\sum_{\Omega(m) \leq \lambda \log \log y} \frac{1}{m} \leq \sum_{\Omega(m) \leq \lambda \log \log y} \frac{\lambda^{\Omega(m)-\lambda \log \log y}}{m} = (\log y)^{-\lambda \log \lambda} \prod_{p \leq y} \left( 1 - \frac{\lambda}{p} \right)^{-1} \ll (\log y)^{-Q(\lambda)} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1}
\]

where we used Mertens’ estimates once again. Similarly, if \( 1 \leq \lambda \leq 1.99 \), then

\[
\sum_{\Omega(m) \geq \lambda \log \log y} \frac{1}{m} \leq \sum_{\Omega(m) \geq \lambda \log \log y} \frac{\lambda^{\Omega(m)-\lambda \log \log y}}{m} \ll (\log y)^{-Q(\lambda)} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1}.
\]

This completes the proof. \( \square \)
Lemma 2.4. Let \( \mathcal{P} \) be a set of primes such that \( \sum_{p \in \mathcal{P}} 1/p < \infty \). Then
\[
\mathcal{d}(\{ n \in \mathbb{N} : p|n \Rightarrow p \notin \mathcal{P} \}) = \prod_{p \in \mathcal{P}} \left( 1 - \frac{1}{p} \right).
\]

Proof. The number of integers \( n \leq x \) with a prime divisor \( p > \log x \) from \( \mathcal{P} \) is
\[
\leq \sum_{p > \log x, p \in \mathcal{P}} \frac{x}{p} = o(x)
\]
as \( x \to \infty \), because \( \sum_{p \in \mathcal{P}} 1/p \) converges. Hence, if we write \( \mathcal{P}' = \mathcal{P} \cap [1, \log x] \), then
\[
\#\{ n \leq x : p|n \Rightarrow p \notin \mathcal{P} \} = \#\{ n \leq x : p|n \Rightarrow p \notin \mathcal{P}' \} + o(x) = x \prod_{p \in \mathcal{P}'} \left( 1 - \frac{1}{p} \right) + o(x)
\]
from the inclusion-exclusion principle that has \( \leq 2 \#(\mathcal{P}') \leq 2 \log x = o(x) \) steps (e.g., see [8, Theorem 2.1]). Since \( \prod_{p \in \mathcal{P} \setminus \mathcal{P}'} (1 - 1/p) \sim 1 \) by our assumption that \( \sum_{p \in \mathcal{P}} 1/p < \infty \), the proof is complete. \( \square \)

3. Proof of Theorem 1

Assume \( x \) is sufficiently large and let \( y = y(x) \) and \( u = u(x) \) to be chosen later, with \( y, u \to +\infty \) slowly as \( x \to +\infty \). In particular, \( y \leq \sqrt{x} \). In the following, for the sake of notation, we will often omit the dependence on \( x, y, u \).

With a small abuse of notation, given an integer \( n \), let \( n \) smooth denote its \( y^{1/u} \)-smooth part and let \( n \) rough denote its \( y^{1/u} \)-rough part. We then set
\[
\mathcal{N} = \{ n \leq x : n \text{ smooth} \leq y \}.
\]

By Lemma 2.2, we have \#\( \mathcal{N} \sim x \) as \( x \to \infty \). Therefore, in order to prove Theorem 1, it is enough to show that
\[
\#(\mathcal{C}) = o(x), \quad \text{where} \quad \mathcal{C} := \mathcal{N} \setminus (\mathcal{A} \cdot \mathcal{B}).
\]

Let \( n \in \mathcal{C} \). Since \( n = n_{\text{smooth}} \cdot n_{\text{rough}} \), we must have that either \( n_{\text{smooth}} \notin \mathcal{A} \) or \( n_{\text{rough}} \notin \mathcal{B} \). Consequently,
\[
\#(\mathcal{C}) \leq S_1 + S_2
\]
with
\[
S_1 := \#\{ n \in \mathcal{N} : n_{\text{smooth}} \notin \mathcal{A} \} \quad \text{and} \quad S_2 := \#\{ n \in \mathcal{N} : n_{\text{rough}} \notin \mathcal{B} \}.
\]

Let us first bound \( S_1 \). Letting \( m = n_{\text{smooth}} \), we have
\[
S_1 \leq \sum_{m \leq y, m \notin \mathcal{A}} \Phi(x/m, y^{1/u}) \ll \frac{x}{\log y} \sum_{m \leq y, m \notin \mathcal{A}} \frac{1}{m}
\]
by Lemma 2.1. Since we have assumed that \( \mathcal{d}(\mathcal{A}) = 1 \), we must have that \( \mathcal{d}(\mathbb{N} \setminus \mathcal{A}) = 0 \) and thus
\[
\alpha(t) := \frac{1}{\log t} \sum_{m \leq t, m \notin \mathcal{A}} \frac{1}{m} \to 0 \quad \text{as} \quad t \to \infty.
\]

Hence, setting \( u = u(y) := \alpha(y)^{-1/2} \), we have \( u \to +\infty \) and \( S_1 = o(x) \) as \( x \to +\infty \).

Let us now bound \( S_2 \). Writing \( m' = n_{\text{rough}} \), we have
\[
S_2 \leq \sum_{m \leq y} \#\{ m' \leq x/m : m' \notin \mathcal{B} \}.
\]
By hypothesis, we have \( \mathbf{d}(B) = 1 \), so that \( \mathbf{d}(\mathbb{N} \setminus B) = 0 \). Thus

\[
\beta(t) := \sup_{s \geq t} \frac{\# \left( (\mathbb{N} \setminus B) \cap [1, s] \right)}{s} \to 0 \quad \text{as} \quad t \to \infty.
\]

Hence, setting \( y := \min \left( x^{1/2}, \exp \left( \beta(x^{1/2})^{-1/2} \right) \right) \), we have \( y \to +\infty \) as \( x \to +\infty \) and

\[
S_2 \leq \sum_{d \leq y} \beta(x/d) \cdot \frac{x}{d} \leq x \beta(x/y) \sum_{d \leq y} \frac{1}{d} \ll x \beta(x^{1/2}) \log y \leq x \beta(x^{1/2})^{1/2} = o(x).
\]

In conclusion, \( \# C = o(x) \), as desired. \( \square \)

**Remark.** The proof of Theorem 1 can be made quantitative. For example, if one has \( \# \{ n \leq x : n \notin A \} \leq x (\log x)^{-a} \) for some fixed \( 0 < a < 1 \), then taking \( y = \exp \left( (\log x)^{\frac{a}{1+a}} \right) \) and \( u = \log \log x \) in the above argument yields

\[
\# \{ n \leq x : n \notin A \cdot B \} \ll x e^{-u} + \frac{xyu}{(\log y)^a} + \frac{x \log y}{(\log x)^a} \ll x \log x \frac{a^2}{1+a} + o(1).
\]

An interesting question is to determine the optimal exponent of \( \log x \) in this upper bound.

## 4. Proof of Theorem 2

The case \( \alpha = 1 \) follows from Theorem 1, whereas for the case \( \alpha = 0 \) one can just observe that \( \mathbf{d}(0) = \mathbf{d}(0^2) = 0 \). We may thus assume \( \alpha \in (0, 1) \). Given any \( \varepsilon > 0 \), we need to construct a set \( A \) of density \( \alpha \) such that the density of \( A^2 \) exists and is smaller than \( \varepsilon \).

Let \( k \in \mathbb{N}, y \geq 1 \) and a set of primes \( P \subset (y, +\infty) \) with \( \sum_{p \in P} 1/p < \infty \) to be chosen later. Using the notation \( \Omega_y(n) = \sum_{p^m \mid n, p \leq y} 1 \), let us consider the sets

\[
B_{y,k,p} := \{ n \in \mathbb{N} : \Omega_y(n) \geq k, (n,p) = 1 \forall p \in P \}.
\]

The key property these sets have is that \( B_{y,k,p}^2 = B_{y,2k,p} \).

Now, using Lemma 2.4 twice (once, with \( P_{\text{Lemma 2.4}} = P \cup \{ p \leq y \} \) and once with \( P_{\text{Lemma 2.4}} = \{ p \leq y \} \)), we find that

\[
\mathbf{d}(B_{y,k,p}) = \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \sum_{p^m \mid n, \Omega(m) \geq k} \frac{1}{m} = \mathbf{d}(B_{y,k,0}) \prod_{p \in P} \left( 1 - \frac{1}{p} \right).
\]

Similarly,

\[
\mathbf{d}(B_{y,k,p}^2) = \mathbf{d}(B_{y,2k,p}) = \prod_{p \in P} \left( 1 - \frac{1}{p} \right) \mathbf{d}(B_{y,2k,0}).
\]

Now, take \( y := \exp(\exp(4k/3)) \), so that \( k = \frac{3}{4} \log \log y \). For any fixed \( \varepsilon > 0 \), Lemma 2.3 implies that if \( k \) is sufficiently large in terms of \( \alpha \) and \( \varepsilon \), then \( \mathbf{d}(B_{y,k,0}) > \alpha \) and \( \mathbf{d}(B_{y,2k,0}) < \varepsilon \). Let us fix for the remainder of the proof such a choice of \( k \). We then construct \( P \) in the following way: we take \( p_1 > y \) to be the smallest prime such that \( (1 - 1/p_1) \mathbf{d}(B_{y,k,0}) > \alpha, p_2 > p_1 \) the smallest prime such that \( (1 - 1/p_1)(1 - 1/p_2) \mathbf{d}(B_{y,k,0}) > \alpha \) and so on. Taking \( P := \{ p_1, p_2, \ldots \} \) we clearly have \( \mathbf{d}(B_{y,k,0}) \prod_{p \in P} (1 - 1/p) = \alpha \). Thus, \( \mathbf{d}(B_{y,k,p}) = \alpha \) and \( \mathbf{d}(B_{y,k,p}^2) < \varepsilon \), as desired. \( \square \)
Remark. If $d(A^2)$ in Theorem 2 is replaced by the upper density $\bar{d}(A^2)$, then one could just take $A$ to be any density $\alpha$ subset of $\{ n \in \mathbb{N} : \Omega_y(n) \geq \frac{3}{4} \log \log y \}$ for $y$ large enough. However, in general there is no guarantee that $A^2$ has asymptotic density. For this reason, in order to prove Theorem 2, it is more convenient to construct explicit suitable sets $A$.

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