UNBOUNDED TOWERS AND THE MICHAEL LINE TOPOLOGY

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Abstract. A topological space satisfies $(\Omega^\gamma)$ (also known as Gerlits–Nagy’s property $\gamma$) if every open cover of the space such that each finite subset of the space is contained in a member of the cover, contains a point-cofinite cover of the space. A topological space satisfies $(\Omega^\text{ctbl}\Gamma)$ if in the above definition we consider countable covers. We prove that subspaces of the Michael line with a special combinatorial structure have the property $(\Omega^\text{ctbl} \Gamma)$. Then we apply this result to products of sets of reals with the property $(\Omega^\gamma)$. The main method used in the paper is coherent omission of intervals invented by Tsaban.

1. Introduction

By space we mean a topological space. A cover of a space is a family of proper subsets of the space whose union is the entire space. An open cover is a cover whose members are open subsets of the space. An $\omega$-cover is an open cover such that each finite subset of the space is contained in a set from the cover. A $\gamma$-cover is an infinite open cover such that each point of the space belongs to all but finitely many sets from the cover. Given a space, let $\Omega$, $\Omega^\text{ctbl}$, $\Gamma$ be the families of $\omega$-covers, countable $\omega$-covers and $\gamma$-covers, respectively. For families $\mathcal{A}$ and $\mathcal{B}$ of covers of a space, the property that every cover in the family $\mathcal{A}$ has a subcover in the family $\mathcal{B}$ is denoted by $(\mathcal{A} \subseteq \mathcal{B})$. The property $S_1(\mathcal{A}, \mathcal{B})$ means that for each sequence $U_1, U_2, \ldots \in \mathcal{A}$ there are sets $U_1 \subseteq U_2, U_2 \subseteq U_2, \ldots$ such that $\{U_n : n \in \mathbb{N}\} \subseteq \mathcal{B}$.

Let $[\mathbb{N}]^\infty$ be the set of infinite subsets of $\mathbb{N}$ and Fin be the set of finite subsets. For sets $a, b \in [\mathbb{N}]^\infty$ we say that $a$ is an almost subset of $b$, denoted $a \subseteq^* b$, if the set $a \setminus b$ is finite. A pseudointersection of a family of infinite sets is an infinite sets $a$ with $a \subseteq^* b$ for all sets $b$ in the family. A family of infinite sets is centered if the finite intersections of its elements, are infinite. Let $p$ be the minimal cardinality of a subfamily of $[\mathbb{N}]^\infty$ that is centered and has no pseudointersection.

Definition 1.1 ([28, Definition 2.2]). Let $\kappa$ be an uncountable ordinal number. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \kappa$ is a $\kappa$-generalized tower if for each function $a \in [\mathbb{N}]^\infty$, there are sets $b \in [\mathbb{N}]^\infty$ and $S \subseteq X$ with $|S| < \kappa$ such that

$$x \cap \bigcup_{n \in b} [a(n), a(n + 1)) \subseteq \text{Fin}$$

for all sets $x \in X \setminus S$.

Let $\kappa$ be an uncountable ordinal number. A set $X \cup \text{Fin}$ is $\kappa$-generalized tower set if the set $X$ is a $\kappa$-generalized tower.

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The *Michael line* is the set $\mathbb{P}(\mathbb{N})$, with the topology where the points of the set $[\mathbb{N}]^\infty$ are isolated, and the neighborhoods of the points of the set $\text{Fin}$ are those induced by the Cantor space topology on $\mathbb{P}(\mathbb{N})$.

**Lemma 1.2** ([14, Lemma 1.2]). Let $\mathcal{U}$ be a family of open sets in $\mathbb{P}(\mathbb{N})$ such that $\mathcal{U} \in \Omega(\text{Fin})$. There are a function $a \in [\mathbb{N}]^\infty$ and sets $U_1, U_2, \ldots \in \mathcal{U}$ such that for each set $x \in [\mathbb{N}]^\infty$ and all natural numbers $n$:

\[
\text{If } x \cap [a(n), a(n + 1)) = \emptyset, \text{ then } x \in U_n.
\]

For a set $U \subseteq \mathbb{P}(\mathbb{N})$, let $\text{Int}(U)$ be the interior of the set $U$ in the Cantor space topology on $\mathbb{P}(\mathbb{N})$. If $\mathcal{U} \in \Omega(\text{Fin})$ is a family of open sets in $\mathbb{P}(\mathbb{N})$ with the Michael line topology, then $\{ \text{Int}(U) : U \in \mathcal{U} \} \in \Omega(\text{Fin})$. Thus Lemma 1.2 holds for a family of open sets with the Michael line topology.

### 2. Main result

For functions $f, g \in \mathbb{N}^\mathbb{N}$ let $(f \circ g) \in \mathbb{N}^\mathbb{N}$ be a function such that $(f \circ g)(n) := f(g(n))$ for all natural numbers $n$.

**Theorem 2.1.** Let $X \cup \text{Fin}$ be a $p$-generalized tower set with the Michael line topology. The space $X \cup \text{Fin}$ satisfies $(\Omega_{\text{Fin}}^{\text{ctbl}})$.

**Proof.** Let $\mathcal{U} \in \Omega_{\text{ctbl}}(X \cup \text{Fin})$ be a family of open sets in $\mathbb{P}(\mathbb{N})$ with the Michael line topology. Let $S_1 := \text{Fin}$. Fix a natural number $k > 1$, and assume that the set $S_{k-1} \subseteq X \cup \text{Fin}$ with $\text{Fin} \subseteq S_{k-1}$ and $|S_{k-1}| < p$ has been already defined. Since $|S_{k-1}| < p$, there is $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \subseteq \Gamma(S_{k-1})$. By Lemma 1.2 there are a function $a_k \in [\mathbb{N}]^\infty$ and sets $U_1^{(k)}, U_2^{(k)}, \ldots \in \mathcal{V}$ such that for each set $x \in [\mathbb{N}]^\infty$ and all natural numbers $n$:

\[
\text{If } x \cap [a_k(n), a_k(n + 1)) = \emptyset, \text{ then } x \in U_n^{(k)}.
\]

Since the set $X$ is $p$-generalized tower, there are a set $b_k \in [\mathbb{N}]^\infty$ and a set $S_k \subseteq X \cup \text{Fin}$ with $S_{k-1} \subseteq S_k$ and $|S_k| < p$ such that

\[
x \cap \bigcup_{n \in b_k} [a_k(n), a_k(n + 1)) \in \text{Fin}
\]

for all sets $x \in X \setminus S_k$. Then

\[
\{ U_{b_k(j)}^{(k)} : j \in \mathbb{N} \} \in \Gamma((X \setminus S_k) \cup S_{k-1}).
\]

There is a function $a \in [\mathbb{N}]^\infty$ such that for each natural number $k$, we have

\[
|(a_k \circ b_k) \cap [a(n), a(n + 1))| \geq 2,
\]

for all but finitely many natural numbers $n$. Since the set $X$ is $p$-generalized tower, there are a set $b \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|S| < p$ such that

\[
x \cap \bigcup_{n \in b} [a(n), a(n + 1)) \in \text{Fin}
\]

for all sets $x \in X \setminus S$. We may assume that $\bigcup_k S_k \subseteq S$. The sets

\[
c_k := \{ i \in b_k : [a_k(i), a_k(i + 1)) \subseteq \bigcup_{n \in b} [a(n), a(n + 1)) \}
\]
are infinite for all natural numbers $k$. Thus,
\[
\{ U_{ck(j)}^{(k)} : j \in \mathbb{N} \} \in \Gamma((X \setminus S_k) \cup S_{k-1}).
\]
Since the sequence of the sets $S_k$ is increasing, we have $X = \bigcup_k (X \setminus S_k) \cup S_{k-1}$ and each point of $X$ belongs to all but finitely many sets $(X \setminus S_k) \cup S_{k-1}$. For each point $x \in S$, define
\[
g_x(k) := \begin{cases} 0 & x \notin (X \setminus S_k) \cup S_{k-1}, \\ \min\{ j : x \in \bigcap_{i \geq j} U_{ck(i)}^{(k)} \} & x \in (X \setminus S_k) \cup S_{k-1}. \end{cases}
\]
Since $|S| < p$, there is a function $g \in \mathbb{N}^\mathbb{N}$ with $\{ g_x : x \in S \} \leq^* g$ and
\[
a_k(c_k(g(k)+1)) < a_{k+1}(c_{k+1}(g(k+1)))
\]
for all natural numbers $k$. Let
\[
\mathcal{W}_k := \{ U_{ck(j)}^{(k)} : j \geq g(k) \}
\]
for all natural numbers $k$. Then $\mathcal{W}_1, \mathcal{W}_2, \ldots \in \Gamma(S)$. We may assume that families $\mathcal{W}_k$ are pairwise disjoint. Since properties $(\Omega_{\text{ctbl}})$ and $S_1(\Omega_{\text{ctbl}}, \Gamma)$ are equivalent, the set $S$ satisfies $S_1(\Omega_{\text{ctbl}}, \Gamma)$. Then there is a function $h \in \mathbb{N}^\mathbb{N}$ such that $g \leq h$ and
\[
\{ U_{ck(h(k))}^{(k)} : k \in \mathbb{N} \} \in \Gamma(S).
\]
Fix a set $x \in X \setminus S$. By (\ref{2.1.3}), for each natural number $k$, we have
\[
\bigcup_{n \in c_k} [a_k(n), a_k(n+1)] \subseteq \bigcup_{n \in b} [a(n), a(n+1)).
\]
By (\ref{2.1.2}), (\ref{2.1.4}) and the fact that $g \leq h$, the set $x$ omits all but finitely many intervals $[a_k(c_k(h(k))), a_k(c_k(h(k))+1))$.

By (\ref{2.1.1}), we have
\[
\{ U_{ck(h(k))}^{(k)} : k \in \mathbb{N} \} \in \Gamma(X \setminus S).
\]
Then
\[
\{ U_{ck(h(k))}^{(k)} : k \in \mathbb{N} \} \in \Gamma(X \cup \text{Fin}).
\]

3. Applications

For spaces $X$ and $Y$, let $X \sqcup Y$ be the disjoint union of these spaces. Let $X$ be a space satisfying $(\Omega_{\Gamma_{\text{ctbl}}})$. Then the space $X \sqcup X$ satisfies $(\Omega_{\Gamma_{\text{ctbl}}})$. In the realm of sets of reals, the properties $(\Omega_{\Gamma_{\text{ctbl}}})$ and $(\Omega_{\Gamma})$ are equivalent.

**Lemma 3.1** ([14, Proposition 2.3.]). If $X, Y$ are sets of reals then the space $X \times Y$ satisfies $(\Omega_{\Gamma})$ if and only if the space $X \sqcup Y$ satisfies $(\Omega_{\Gamma_{\text{ctbl}}})$.

From our main result we can obtain the following corollary which has originally was proved by Szewczak and Wlodecka [28, Theorem 4.1.(1)].

**Corollary 3.2.** Let $n \in \mathbb{N}$ and $X_1 \cup \text{Fin}, \ldots, X_n \cup \text{Fin}$ be $p$-generalized tower sets with the Cantor topology. Then the space $(X_1 \cup \text{Fin}) \times \cdots \times (X_n \cup \text{Fin})$ satisfies $(\Omega_{\Gamma_{\text{ctbl}}})$.  

Proof. We prove the statement for $n = 2$. The proof for other $n$ is similar. Let $X, Y$ be $p$-generalized towers in $[\mathbb{N}]^{\omega}$. Then $X \cup Y$ is a $p$-generalized tower. By Theorem 2.1, the space $X \cup Y \cup \text{Fin}$ with the Michael line topology satisfies $(\Omega_{\text{ctbl}}^\Gamma)$. Then the space $(X \cup Y \cup \text{Fin}) \cup (X \cup Y \cup \text{Fin})$ satisfies $(\Omega_{\text{ctbl}}^\Gamma)$. Since the property $(\Omega_{\text{ctbl}}^\Gamma)$ is hereditary for closed subset, thus the space $(X \cup \text{Fin}) \cup (Y \cup \text{Fin})$ with the Michael line topology satisfies $(\Omega_{\text{ctbl}}^\Gamma)$. Then $(X \cup \text{Fin}) \cup (Y \cup \text{Fin})$ with the Cantor topology satisfies $(\Omega_{\text{ctbl}}^\Gamma)$, and $(\Omega_{\text{ctbl}}^\Gamma)$, too. By Lemma 3.1, the space $(X \cup \text{Fin}) \times (Y \cup \text{Fin})$ with the Cantor topology satisfies $(\Omega_{\text{ctbl}}^\Gamma)$. □

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