Quantum many particle systems in ring-shaped optical lattices

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Abstract

In the present work we demonstrate how to realize 1d-optical closed lattice experimentally, including a \textit{tunable} boundary phase-twist. The latter may induce “persistent currents”, visible by studying the atoms’ momentum distribution. We show how important phenomena in 1d-physics can be studied by physical realization of systems of trapped atoms in ring-shaped optical lattices. A mixture of bosonic and/or fermionic atoms can be loaded into the lattice, realizing a generic quantum system of many interacting particles.
Studies of one dimensional systems constitute an intense research activity both in experimental and theoretical physics. They are particularly interesting mainly because quantum effects are strongest at low dimensionality and peculiar phenomena emerge. Prominent examples are the spin-charge separation in Luttinger liquids[1], one dimensional persistent currents in mesoscopic rings[2], and transmutation of quantum statistics[3]. Most of the approximate schemes working in higher dimensions break down in 1-d. Only for a restricted class of model Hamiltonians, physical properties can be obtained analytically resorting to powerful techniques as Bethe ansatz[4] or conformal field theory[5]. For more generic 1d systems, numerical analysis is the standard route to extract physical information. Degenerate atoms in optical lattices could constitute a further tool for the investigations[6], thus rediscovering Feynman’s ideas[7], suggesting that an ideal system with a “quantum logic” can be used to study open problems in quantum physics. Precise knowledge of the model Hamiltonian, manipulation of its coupling constants, possibility of working with controllable disorder are some of the great advantages of atomic systems in optical lattices compared with solid state devices to experimentally realize Feynman’s ideas. The upsurge of interest of the scientific community has been remarkable, and some perspectives disclosed by trapped–atom “labs” have been already explored: the observation of the superfluid–Mott insulator quantum phase transition[8], the analysis of the Tonks-Girardeau regime in strongly interacting bosons[9], and the physical realization of a 1d-chain of Josephson junctions[10] were relevant achievements for condensed matter physics. The two most widely used methods to trap and manipulate atoms are based on the conservative interaction of atoms with either magnetic fields or with far off–resonant laser beams. For our purposes the magnetic trapping potential has a parabolic symmetry. Laser light interacts with the atomic induced dipoles creating attractive or repulsive potentials depending on the sign of the detuning $\Delta$ from resonance[11]. This can be used to create different potentials for different atoms, but with a single tunable laser beam. Notice that no light absorption occurs in creating the potential; therefore the medium can be considered transparent to the laser.

So far open optical lattices have been studied. This constitutes a limitation of optical apparata since a variety of studies for finite 1d lattices with Periodic Boundary Conditions (PBC) exists in the literature, that cannot be accessed with them. In the same way as Gaussian laser beams are useful to produce open optical lattices, we shall take advantage of the rotational symmetry of Laguerre–Gauss (LG) laser modes to produce
FIG. 1: The optical potential resulting from the interference of a plane wave with an LG mode with $L = 14$, $p = 0$. For $p \neq 0$ the potential is virtually unaltered.

Closed optical lattices. LG beams, obtained experimentally making use of computer generated holograms [12], have already been used in the field of ultra-cold atoms [13]. A LG mode with frequency $\omega$, wave-vector $k$ and amplitude $E_0$ propagating along the $z$ axis can be written in cylindrical coordinates $(r, \varphi, z)$ as [14] $E(r, \varphi) = E_0 f_{pl}(r) e^{i \varphi} e^{i (\omega t - k z)}$, $f_{pl}(r) = (-1)^p \sqrt{\frac{2p!}{\pi (p + |l|)!}} \xi^{|l|} L_p^{|l|} (\xi^2) e^{-\xi^2}$, $\xi = \sqrt{2} r/r_0$, where $r_0$ is the waist of the beam. 

and $L_p^{|l|}$ are associate Laguerre polynomials $L_m^n(x) := (-)^m d^m/dx^m[L_{n+m}(x)]$, $L_{n+m}(x)$ being the Laguerre polynomials themselves. The numbers $p$ and $l$ label the radial and azimuthal quantum-coordinates, respectively. The lattice modulation is obtained by interference of a LG beam with a plane wave $E_0 e^{i (\omega t - k z)}$: in the far field, the interferogram is periodic in $\varphi$ with $l$ wells. For even $l$ a perfect 1$d$-ring with $L = l$ lattice sites is obtained. By reflecting the combined beam (LG beam plus plane wave) back on itself one achieves confinement also along $z$. Indeed a series of disk shaped traps are obtained. We point out that tunneling between the disks $t_z$ can be made much weaker than the corresponding tunneling within each ring $t_\phi$ adjusting $r_0/\lambda$ (i. e. focusing the LG beam). Such a parameter depends monotonously only on $L$; for $L \gtrsim 15$, $t_z/t_\phi \lesssim 10^{-2}$ can be achieved with $r_0/\lambda \sim 100$. The resulting lattice potential (see Fig.(1)) is described by

$$V_{latt} = 4E_0^2 \left[1 + f_{pl}^2 + 2f_{pl} \cos(l \varphi)\right] \cos(kz)^2$$  (1)

Note that, contrary to what was done in [13], here we need the laser frequency to be tuned below the atomic resonance since we want to trap atoms into the ring. For example with a laser intensity of $I = 5W/cm^2$ and $\Delta = -10^6 MHz$ the potential wells would be separated by a barrier of $\sim 5\mu K$ much larger than the chemical potential of a standard condensate (whose temperature can reach few $nK$); with these parameters the scattering rate is $\ll$
1 photon/sec. It is worth noting that, because of the relation:

\[ L_m^{n-m}(r) \leftrightarrow H_n[(x-y)/\sqrt{2}]H_m[(x+y)/\sqrt{2}], \]

LG modes can be realized also from Hermite–Gauss modes (modulo a \( \pi/2 \) phase change). Such a “mode-converter”, realized experimentally in [15], can switch from an open to a closed lattice potential with the same periodicity and \( L \). As we shall discuss further, this device might be useful in the experiments.

We have just illustrated how to realize an optical lattice with PBC. Now we show how to twist them. The task can be achieved by applying an external, cone-shaped magnetic field \( B = B_\varphi e_\varphi + B_z e_z \). In this way the atomic magnetic dipoles \( \mu_{m_F} \) experience a field varying along the ring, eventually equipping the periodic lattice by a twist factor: \( \Psi \rightarrow e^{i\phi_{m_F}}\Psi \) at each winding, \( \Psi \) being a generic wave function. The phase factor \( \phi_{m_F} = m_F \pi \cos \theta \), with \( \tan \theta = B_\varphi/B_z \), is the analog of the Berry phase [16] of the two state system corresponding to the Zeeman splitting of the hyperfine atomic ground states; the role of time is played by the angle \( \varphi \). We can adjust \( \phi_{m_F} \) using an additional laser beam (with a suitable frequency), relying on the AC-Stark shift: \( A_E(m_F) \), where the function \( A_E \) depends on the intensity of the laser and on the Clebsch-Gordan coefficients corresponding to the matrix element of the electric dipole interaction energy [20]. The resulting phase twist is \( \Phi_\sigma = A_E(m_F) + m_F \pi \cos \theta \) where \( \sigma = m_F \). Whereas boundary twists induced by a magnetic field piercing the ring are “symmetric”, \( \Phi_+ \equiv \Phi_- \), our protocol realizes 1d-models with a tunable \( \Phi_\sigma \), thus opening the way to novel investigations discussed below.

For OBC, \( \Phi_\sigma \) can be “gauged away” completely from the system. In contrast, the boundary phase cannot be eliminated for closed loops and alters the phase diagram of the system [17]. Infact \( \Phi_\sigma \) emerges from the sum of site dependent phases causing an increase of the velocity field (\( \propto \) to the tight binding amplitude \( t \)) that, in absence of dissipation, may set a persistent current. Therefore different regions in the phase diagram are identified depending on the dynamical response of the system by perturbing \( \Phi_\sigma \). The effect is reflected in the curvature of the \( N \)-particle energy levels \( E_n \) respect to the phase twist:

\[ \rho_\sigma = L^2 \sum_n p_n [E_n(\Phi_\sigma) - E_n(0)]/(Nt\Phi_\sigma^2), \]

where \( p_n = e^{-\beta E_n}/Z \) are the Boltzmann weights. For (spinless) bosons \( \rho_+ = \rho_- = \rho \) is proportional to the superfluid fraction. Persistent currents are studied analyzing the charge stiffness \( D_c \propto \rho_+ + \rho_- \) (for electrons, it is the zero frequency conductivity or Drude weight); a non vanishing \( D_c \) sets a persistent current, visible by releasing the condensate for a time much longer than the typical atomic oscillation.
FIG. 2: Interference pattern for condensates released by the lattice (1), obtained resorting to the analog of light diffraction from a circular grating [18]. The figures show the square of the order parameter $|\psi(k_x, k_y)|^2 = |\psi_0(k_x, k_y)\sum_{j=0}^{L-1}\cos[i(k_x \cos(2\pi j/L) + k_y \sin(2\pi j/L) + \phi_j)]|^2$. On the left $\phi_j = 0$ for all the condensates; on the right the atoms move along the ring with velocity $\propto \nabla \phi$; the interference pattern reflects a loss of matter at the trap center caused by centrifugal effects.

period in the lattice wells. Then the spatial distribution of the condensates $|\Psi(\mathbf{r} = k\mathbf{t}, t)|$ is indicative of the initial atomic momentum distribution $|\Psi(k_0)|$ [18]; in particular the phase difference between atoms trapped in different sites, produces characteristic interference patterns in the released condensates. In Fig. (2) we show such a pattern for condensates released from the potential of Fig. (1) in mean-field approximation (see also Fig. (3)). Supercurrent/superfluid fractions can be studied looking at the response of the system under imprinting of a dynamical phase $\alpha_d(j, \sigma)\delta \tau$ to the atomic wave functions, flashing the atoms with an additional Gaussian laser beam (can be much closer to resonance than those creating the potential) with a waist larger than the LG mode and with $\varphi$–dependent intensity. The time $\delta \tau$ must be too short to induce atomic motion by absorption during the pulse).

The case $\Phi_+ = -\Phi_-$ is useful to study the spin stiffness $D_s \propto \rho_+ - \rho_-$ indicating long range spin correlations in the system (for charged particles $D_s$ would be proportional to the inverse bulk spin susceptibility [17]). Generic values of $\Phi_+ \neq \pm \Phi_-$ can be seen also as a result of certain correlated–hopping processes (on the untwisted models) [19] and correspond to more exotic cases that, as far as we know, have not been realized yet in physical systems. To be specific we consider $\mathcal{N}$ fermions described by the Hubbard model with particle-density
FIG. 3: The zero temp. momentum distribution for fermions with Hubbard dynamics is presented: $|\Psi(k_x, k_y)|^2 \propto |w(k_x, k_y)|^2 \sum_{i,j} e^{i(k_x-x_j)} \sum_{k_\phi} e^{ik_\phi(\phi_i-\phi_j)} \langle n_{k_\phi} \rangle$; $N/L = 32/16$ ($N/L = 1$ is the less advantageous case to discern the effects of $\Phi$ at finite size, since the metal-insulator transition strongly suppresses $D$, $[23]$); $\langle n_{k_\phi} \rangle$ is calculated perturbatively, at second order in $U/t$. For $\Phi = 0$ (left), $k_\phi = \{-\pi(N-1)/L \ldots \pi(N-1)/L\}$. For $\Phi \neq 0$, $k_\phi = \{-\pi(N-1)/L + \Phi/L \ldots \pi(N-1)/L + \Phi/L\}$; the asymmetry in $|\Psi(k_x, k_y)|^2$ is due to the offset of $\langle n_{k_\phi} \rangle$ caused by $\Phi$.

modulated kinetic energy

$$H_{Hub} = - \sum_{j,\sigma} \mu_{j,\sigma} N_{j,\sigma} - \sum_{j,\sigma} (\bar{t}_j(\sigma) c_{j+1,\sigma}^\dagger c_{j,\sigma} + h.c.) + U \sum_j N_{j,+}^2 N_{j,-}$$

(2)

$$\bar{t}_j(\sigma) = t \exp \left[ i \gamma_j(\sigma) + i \sum_{l} (\alpha_{j,l}(\sigma) N_{l,-} + A_{j,l}(\sigma) N_{l,+}) \right]$$

(3)

where $c_{j,\sigma}$'s are fermionic operators, and $N_{i,\sigma} := c_{i,\sigma}^\dagger c_{i,\sigma}$. $U = \pi b_s \int dx |w(x)|^4/m$ and $t = \int dx w(x) [-\frac{1}{2m} \nabla^2 + V_{lat}] |w(x+a)$, $(b_s, a$, and $w(x)$ indicates the scattering length, the lattice spacing, and Wannier functions respectively) play the role of the Coulomb and hopping amplitudes respectively; $\mu_{j,\sigma}$ is of the order of the Bloch band separation$[21]$; the site dependence can be achieved by tuning the magnetic confinement out of the symmetry axis of the optical ring. For the model (2) in a closed lattice, (3) can be gauged away everywhere but at the boundary; therefore (2), (3) is equivalent to the ordinary Hubbard model, but with twisted BC$[19]$. The phase twist is $\Phi_\sigma := \phi(\sigma) + \phi_{1-}(\sigma) N_{-\sigma} + \phi_{1+}(\sigma) (N_{\sigma} - 1)$, where $\phi_{1-}(\sigma) = \sum_{j=1}^{L} \alpha_{j,m}(\sigma)$, $\phi(\sigma) = \sum_{j=1}^{L} (\gamma_j(\sigma) + A_{j,j}(\sigma))$, $\phi_{1+}(\sigma) = \sum_{j \neq m-1,m} A_{j,m}(\sigma) + A_{m,m-1}(\sigma) + A_{m-1,m+1}(\sigma)$. Hence, loading the Hubbard model into the twisted ring effectively leads to the physical realization of the model (2), (3). To point out the effects of $U$ (smearing of the Fermi distribution with algebraic singularity at $k_F$) in the persistent current, $|\Psi(k)|^2$ is calculated for the Hubbard ground state at small $U/t$, and with $\Phi_\sigma = \phi(\sigma)$, $\phi(+) = \phi(-) = \phi$ (see Fig. (3)).
The proposed setups could be used to study several issues in one dimensional systems.

I. The concept of conformal invariance plays a central role in 1 + 1 dimensional critical phenomena: universality is characterized by a single parameter, the conformal anomaly $c$. The physical meaning of $c$ resides in the concept of Casimir energy, namely the variation of the vacuum energy density to a change in the BC. For PBC it was shown $^5$ that the finite size correction to the bulk ground state energy is related to $c$: $\mathcal{E}_{PBC} - \mathcal{E}_{bulk} = - \pi c v / 6 L$; resorting the modular invariance this correction should be visible in the specific heat of the system, at low temperature: $C(T) = \pi c L T / 3 v$, for each collective mode of the system; the speed of sound $v$ can be extracted from the dispersion curve, at small $k$: $v = \Delta \mathcal{E} / \Delta k$, for sufficiently large $L$ (for the XXZ model, numerical analysis suggests that $L \gtrsim 15$ $^2$). Except for integrable models, it is hard to measure or even have numerical estimates of $c$ in solid state systems $^3$, $^2$ With the presented setups for highly controllable loaded models these measurements can be done with unique accuracy. Both $C(T)$ and $\Delta \mathcal{E} / \Delta k$ can be measured following the techniques employed by Cornell et al. $^1$. To discern finite size effects in $C(T)$ the PBC to OBC converter, discussed above, could be a valid tool. Indeed, the finite size correction to $\mathcal{E}_{bulk}$ for OBC is also proportional to $c$, but with a different coefficient $^5$. Then: $cv = \frac{8L}{\pi}(\mathcal{E}_{PBC} - \mathcal{E}_{OBC}) + \mathcal{F}_S$, where $\mathcal{F}_S$ is the bulk limit of the surface energy that, being non-universal, can be fixed by performing the measurements for different $L$. (mimicking a “finite size scaling analysis”). Remarkably, both the energies $\mathcal{E}_{PBC}$ and $\mathcal{E}_{OBC}$ might be accessible measuring the second moment of the velocity of the released condensate $^2$.

II. A general model we can engineer in the ring shaped lattice is

$$H = H_{BH} + H_{Hub} + H_I$$

(4)

where $H_{BH}$ is the Bose-Hubbard Hamiltonian $^8$ and $H_I$ describes a density-density, fermion-boson interaction $^6$. By tuning $\Delta$ within the fine structure of the fermionic atoms, a spin dependence can be inserted in the hopping amplitude of the Hubbard model: $t \rightarrow t_\sigma$. At $\mathcal{N}/L = 1$ and $t_\sigma \ll U$ the Hubbard ring effectively accounts for the physical realization of the twisted XXZ model with anisotropy $\gamma = (t_+^2 + t_-^2) / (2t_+ t_-)$ and external field $h = 4 \sum_{\sigma} \sigma t_\sigma^2 / \mu_\sigma$ $^1$. Loading quantum systems described by Hamiltonians of the type (4) in lattices with twisted BC could serve to study charge and spin stiffness in physical systems with tunable interaction and/or disorder. For example a mixture of $^{87}$Rb and $^{40}$K atoms constitutes an ideal system to check the recent experimental evidence suggesting that the
supersolid order would be effectively favoured by the insertion of fermionic degrees of freedom into homogenous bosonic systems. The off-diagonal long range order manifests in superfluid currents. Jumps between non-vanishing supercurrents should reveal the existence of the supersolid phase. This should be accompanied by a macroscopic occupation in the condensate at a non vanishing wave vector ($\sim \pi/(na)$, $n \geq 2$) signalling the charge-density-wave instability. The two condensates should be traced in the interference fringes.

It was proved that exactly solvable twisted Hubbard/XXZ rings are equivalent to untwisted models for particles with intermediate statistics; this results in modifications of the exponents of the (low energy) correlation functions. The spatial profile of the latter might be detected by photoassociation techniques, as suggested in [34].

III Another interesting issue we can study is the conjecture that Poisson or Wigner-Dyson level-statistics manifest in that the thermal Drude weights have qualitatively different slopes for integrable (smooth algebraic temperature-decrease, universal behaviour of $D(T)/D(0)$) or non-integrable (sharp, non-universal suppression of $D(T)/D(0)$) systems. Due to the precise knowledge of the model-Hamiltonian under analysis we can address the problem directly in a physical system. For example, we could consider $^{40}$K pure-XXZ rings with twisted BC, for different $L$’s; using the Feshbach resonance one could tune $b_s \sim 2a$; the resulting XXZ model with next-nearest neighbor density-density interaction is non-integrable (another way is to destroy the integrability introducing disorder into the ring by site-dependent $h_j$). In short: integrability can be switched on and off by tuning the Feshbach resonance (or adjusting the energy offsets $h_j$). The presence of persistent currents can be detected along the lines described above (see Fig.(2), (3)). Numerical investigations for the XXZ model suggest that the effect should be visible for $T/L \gtrsim 0.1\gamma$.[36]

In summary we have suggested a number of protocols to realize closed rings of many quantum particles, by optical means. This is possible by employing slight variations and combinations of techniques already developed within the current experimental activity in atomic physics. We have discussed how several open problems in condensed matter physics can be enlightened by such a setup. We finally observe that the clockwise/anticlockwise currents in a few-wells-ring constitutes a controllable two state-system analogous to the flux-qubit realized by a SQUID. As the current is neutral, the corresponding decoherence-rate is much lower compared to solid state devices (charged currents). Information transfer could be mediated by an induced-dipole–dipole atomic interaction.
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