Constant External Fields in Gauge Theory and the Spin $0, \frac{1}{2}, 1$ Path Integrals

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Abstract

We investigate the usefulness of the “string-inspired technique” for gauge theory calculations in a constant external field background. Our approach is based on Strassler’s worldline path integral approach to the Bern-Kosower formalism, and on the construction of worldline (super-) Green’s functions incorporating external fields as well as internal propagators. The worldline path integral representation of the gluon loop is reexamined in detail. We calculate the two-loop effective actions induced for a constant external field by a scalar and spinor loop, and the corresponding one-loop effective action in the gluon loop case.

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1. Introduction

It is by now well-established that techniques from string perturbation theory can be used to improve on calculational efficiency in ordinary quantum field theory. The relevance of string theory for this purpose is based on the fact that many, and perhaps all, amplitudes in quantum field theory can be represented as the infinite string tension limits of appropriately chosen (super-) string amplitudes. This is, of course, an intrinsic and well-known property of string-theory.

It is a more recent discovery, however, that such representations can lead to an interesting alternative to standard Feynman diagram calculations. Following earlier work on the $\beta$ – function for Yang-Mills-theory \cite{1,2,3,4}, Bern and Kosower \cite{5} considered the infinite string tension limit of gauge boson scattering amplitudes, formulated in an appropriate heterotic string model. By a detailed analysis of this limit, they succeeded in deriving a novel type of “Feynman Rules” for the construction of ordinary one-loop gluon (photon) scattering amplitudes. The resulting integral representations are equivalent to the ones originating from Feynman diagrams \cite{6}, but lead to a significant reduction of the number of terms to be computed in gauge theory calculations. This property was then successfully exploited to obtain both five gluon \cite{7} and four graviton amplitudes \cite{8}.

Strassler \cite{9} later showed that, for many cases of interest, the same integral representations can be derived from a first-quantized reformulation of ordinary field theory. In this approach, one starts with writing the one-loop effective action as a particle (super) path-integral, of a type which has been known for many years \cite{10,11,12,13,14,15,16,17,18}. Those path-integrals are then considered as the field-theory analogues of the Polyakov path integral, and evaluated in a way analogous to string perturbation theory (some suggestions along similar lines had already been made \cite{19}). The resulting formalism offers an alternative to standard field theory techniques which circumvents much of the apparatus of quantum field theory. It works equally well for effective action and scattering amplitude calculations, on- and off-shell. It has been applied to a number of calculations in gauge theory \cite{20,21,22,23,24} and generalized to cases where, besides gauge and scalar self-couplings, Yukawa \cite{25,26,27} and axial couplings \cite{28,29,27} are present.

Due to its simplicity, it appears also well-suited to the construction of multiloop generalizations of the Bern-Kosower formalism. Steps in this direction have already been taken by various authors, and along different lines \cite{30,31,32}. In particular, the original Bern-Kosower program becomes very hard to carry through at the two-loop level, due to the complicated structure of genus two string amplitudes. Nevertheless, recently substantial progress has been achieved in this line of work \cite{32,33}.

A multiloop generalization following the spirit of Strassler’s approach has been proposed by two of the present authors, first for scalar field theories in \cite{34}. This generalization uses the concept of worldline Green’s functions on graphs \cite{34,35,33}, and leads to integral representations combining whole classes of graphs. This work was extended to QED in \cite{36}, and its practical viability demonstrated by a recalculation of the two-loop (scalar and spinor) QED $\beta$ – functions. For the case of multiloop amplitudes in scalar QED, a more comprehensive treatment along the same lines was given in \cite{37}. This includes amplitudes involving external scalars.

An important role in quantum electrodynamics is played by calculations involving constant external fields. This subject originates with Euler and Heisenberg’s classic one–loop
calculation of the static limit of photon scattering in spinor quantum electrodynamics. Schwinger's introduction of the proper–time method in 1951 allowed him to reproduce this result, and the analogous one for scalar quantum electrodynamics, with considerably less effort. For calculations of this type, it has turned out generally advantageous to take account of the external field already at the level of the Feynman rules, i.e. to absorb it into the free propagators. Appropriate formalisms have been developed both for QED and QCD.

The purpose of the present paper is threefold. First, we would like to cast the worldline multiloop formalism proposed in into a form suitable for constant external field calculations in quantum electrodynamics. This will be achieved in a way analogous to field theory, namely by modifying the worldline Green's functions so as to take the external field into account. At the one-loop level, similar proposals have already been made by several authors (see also).

Second, we will apply this formalism to a two-loop effective action calculation in QED. In the present work, we will extend this calculation to the two-loop level, and calculate the correction to this effective Lagrangian due to one internal photon, both for scalar and spinor QED. We discuss in how far this calculation improves on previous calculations of the same quantities.

Third, we reconsider the super path integral representation of the spin 1 particle, and use this path integral for calculating the effective action induced for an external constant pseudo–abelian field by a gluon loop in Yang-Mills- theory. While external gluons pose no particular problems in the worldline formalism, apart from forcing path–ordering, internal gluons are a much more delicate matter. While it is not difficult to construct free path integrals describing particles of arbitrary spin, those constructions usually run into consistency problems if one tries to couple a path integral with spin higher than 1/2 to a spin-1 background. A somewhat non–standard path integral, mimicking the superstring, had been proposed by Strassler in his rederivation of the Bern-Kosower rules for the gluon loop case. To the best of our knowledge, it has not yet been proven that this path integral correctly reproduces all the pinch terms implicit in the Bern–Kosower master formula.

We will first give a rigorous derivation of a spin-1 path integral which, while not identical with the one used by Strassler, is easily seen to be equivalent. We then use it for calculating the effective action induced by a gluon loop for a constant pseudo–abelian background gluon field. The result will again be in agreement with the literature, and provides a nontrivial check on the correctness of Strassler's proposal.

The organization of this paper is as follows. In chapter 2 we review the worldline-formalism for one-loop photon scattering in scalar and spinor QED, and its generalization to gluon scattering. We then indicate the changes which are necessary to take a constant external field into account. Chapter 3 extends this analysis to the gluon loop. This so far includes worldline calculations of the one-loop effective actions induced for a (pseudo–abelian) constant external field by a scalar, spinor and gluon loop. For the QED case, we then extend the constant field formalism to the multiloop level in chapter 4. We apply it to calculations of the corresponding two–loop effective actions for scalar QED in chapter
5 and for spinor QED in chapter 6. Chapter 7 contains some remarks on the various ways of calculating the two–loop QED $\beta$–function in this formalism. Our results will be discussed in chapter 8. In appendix A we discuss the path integral representation of the electron propagator in an external field. In appendix B we derive the various worldline Green’s functions used in our calculations, and discuss some of their properties. Appendix C contains a collection of some results concerning the determinants which we encounter in the evaluation of the spin-1 path integral.

2. One–Loop Photon Amplitudes in a Constant Background Field

We begin with shortly reviewing how one–loop photon (gluon) scattering amplitudes are calculated in the worldline formalism [4, 21, 36]. In his rederivation of the Bern–Kosower rules for gauge boson scattering off a spinor loop, Strassler sets out from the following well-known path integral representation for the corresponding one–loop effective action (see e.g. [19, 54]):

$$\Gamma[A] = -2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_D \mathcal{D}x \int_A \mathcal{D}\psi \times \text{tr} \mathcal{P} \exp \left[ - \int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi \dot{\psi} + ieA_\mu \dot{x}_\mu - ie\psi_\mu F^{\mu\nu} \psi_\nu \right) \right]$$

(2.1)

In this formula, $T$ is the usual Schwinger proper–time parameter. The periodic functions $x^\mu(\tau)$ describe the embedding of the circle with circumference $T$ into $D$–dimensional Euclidean spacetime, while the $\psi^\mu(\tau)$'s are antiperiodic Grassmann functions. The periodicity properties are expressed by the subscripts $P,A$ on the path integral. The colour trace $\text{tr}$ and the path ordering $\mathcal{P}$ apply, of course, only to the nonabelian case. We have chosen a constant euclidean worldline metric.

The case of a scalar loop is obtained simply by discarding all Grassmann quantities and the global factor of -2, which takes care of the difference in statistics and degrees of freedom.

Analogous path integral representations exist for the scalar and electron propagators in a background field [4, 11, 53, 56]. The integration is then over a space of paths with fixed boundary conditions. Those are obvious in the scalar case, while in the fermionic case there has been some debate on the correct boundary conditions to be imposed on the Grassmann path integral [53, 54, 19]. An adequate path integral representation for the gluon propagator seems to be missing in the literature, and will be derived in chapter 3. The more familiar cases of the scalar and spin-$\frac{1}{2}$ propagators are included in appendix A for completeness.

While in the present paper we will make use only of the loop path integrals, we expect those propagator path integral representations to play an important role in future extensions of this formalism.

A number of different techniques have been applied to calculating this worldline path integral, and various generalizations thereof (see e.g. [74, 58, 59, 61, 62, 63, 64, 65, 66]).
In the “string–inspired” approach, the first step is to split the coordinate path integral into center of mass and relative coordinates,

\[
\int \mathcal{D}x = \int dx_0 \int \mathcal{D}y \\
x^\mu(\tau) = x_0^\mu + y^\mu(\tau) \\
\int_0^T d\tau y^\mu(\tau) = 0
\]

(2.2)

The path integrals over \( y \) and \( \psi \) are then evaluated by Wick contractions, as in a one-dimensional field theory on the circle. The Green’s functions to be used are

\[
\langle y^\mu(\tau_1)y^\nu(\tau_2) \rangle = -g^{\mu\nu}G_B(\tau_1, \tau_2) = -g^{\mu\nu} \left[ |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \right],
\]

\[
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2) \rangle = \frac{1}{2} g^{\mu\nu}G_F(\tau_1, \tau_2) = \frac{1}{2} g^{\mu\nu}\text{sign}(\tau_1 - \tau_2).
\]

(2.3)

We will often abbreviate \( G_B(\tau_1, \tau_2) =: G_{B12} \) etc. They solve the differential equations

\[
\frac{1}{2} \frac{\partial^2}{\partial \tau_1^2} G_B(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) - \frac{1}{T}
\]

\[
\frac{1}{2} \frac{\partial}{\partial \tau_1} G_F(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2)
\]

(2.4)

with periodic (antiperiodic) boundary conditions for \( G_B(G_F) \). Some freedom exists in the definition of \( G_B \), which has been discussed elsewhere [22, 67]; in particular, a constant added to \( G_B \) would drop out after momentum conservation is applied.

With our conventions, the free path integrals are normalized as

\[
\int_P \mathcal{D}y \exp \left[ - \int_0^T d\tau \frac{1}{4} y^2 \right] = (4\pi T)^{-\frac{D}{2}}
\]

\[
\int_A \mathcal{D}\psi \exp \left[ - \int_0^T d\tau \frac{1}{2} \psi \dot{\psi} \right] = 1
\]

(2.5)

The result of this evaluation is the one-loop effective Lagrangian \( \mathcal{L}(x_0) \). Combined with a covariant Taylor expansion of the external field at \( x_0 \), this yields a new and highly efficient algorithm for calculating higher derivative expansions in gauge theory [21, 22].
One–loop scattering amplitudes are obtained by specializing the background to a finite sum of plane waves of definite polarization. Equivalently, one may define integrated vertex operators

\[ V_A = T^a \int_0^T d\tau \left[ \dot{x}_\mu \varepsilon_\mu - 2i\psi_\mu \gamma_\mu k_\mu \varepsilon_\nu \right] \exp[ikx(\tau)] \]  

(2.6)

for external photons (gluons) of definite momentum and polarization, and calculate multiple insertions of those vertex operators into the free path integral (of course, only the first term has to be taken for the scalar loop). \( T^a \) denotes a generator of the gauge group in the representation of the loop particle. In the nonabelian case, for the spinor loop an additional two-gluon vertex operator is required [9].

The path integrals are performed using the well-known formulas for Wick–contractions involving exponentials, e.g.

\[ \langle \exp[ik_1 \cdot x(\tau_1)] \exp[ik_2 \cdot x(\tau_2)] \rangle = \exp\left[ G_B(\tau_1, \tau_2)k_1 \cdot k_2 \right] \]  

(2.7)

etc. (the factors \( \dot{x}_\mu \varepsilon_\mu \) may be formally exponentiated for convenience).

Explicit execution of the \( \psi \)–path integral would be algebraically equivalent to the calculation of the corresponding Dirac traces in field theory. It can be circumvented by the following remarkable feature of the Bern–Kosower formalism, which may be understood as a consequence of worldsheet or worldline supersymmetry. After evaluation of the \( x \)–path integral, one is left with an integral over the parameters \( T, \tau_1, \ldots, \tau_N \), where \( N \) is the number of external legs. In the nonabelian case, the path–ordering leads to ordered \( \tau \)–integrals, \( \int \prod_{i=1}^{N-1} d\tau_i \theta(\tau_i - \tau_{i+1}) \). The integrand is an expression consisting of an exponential factor

\[ \exp\left[ \sum_{i<j} G_B(\tau_i, \tau_j)k_ik_j \right] \]

multiplied by a polynomial in the first and second derivatives of \( G_B \),

\[ \dot{G}_B(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2) - 2\frac{(\tau_1 - \tau_2)}{T} \]

\[ \ddot{G}_B(\tau_1, \tau_2) = 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \]  

(2.8)

(here and in the following, a “dot” denotes differentiation with respect to the first variable).

All \( \dot{G}_B \)’s can be eliminated by partial integrations on the worldline, leading to an equivalent parameter integral dependent only on \( G_B \) and \( \dot{G}_B \). According to the Bern–Kosower rules, all contributions from fermionic Wick contractions may then be taken into account simply by simultaneously replacing every closed cycle of \( \dot{G}_B \)’s appearing, say \( \dot{G}_{B_1i_2} \dot{G}_{B_2i_3} \cdots \dot{G}_{B_{ni_1}} \), by its “supersymmetrization”

\[ \dot{G}_{B_1i_2} \dot{G}_{B_2i_3} \cdots \dot{G}_{B_{ni_1}} \rightarrow \dot{G}_{B_{1i_2}} \dot{G}_{B_{2i_3}} \cdots \dot{G}_{B_{ni_1}} - G_{Fi_1i_2} G_{Fi_2i_3} \cdots G_{Fi_{ni_1}}. \]  

(2.9)
Note that an expression is considered a cycle already if it can be put into cycle form using
the antisymmetry of $\dot{G}_B$ (e.g. $\dot{G}_{Bab}\dot{G}_{Bab} = -\dot{G}_{Bab}\dot{G}_{Bba}$). Unfortunately the practical
value of this procedure rapidly diminishes with increasing number of external legs ($\geq 5$),
as the number of terms making up the integrand starts to significantly increase in the
partial integration procedure [69].

The worldline supersymmetry (see eq. (A.9)) makes it possible to combine the $x$ – and
$\psi$ – path integrals into the following super path integral [13, 19, 58, 70, 36]

$$\Gamma[A] = -2\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}X \text{tr} P e^{-\int_0^T d\tau \int d\theta \left[ -\frac{1}{4} X D^3 X - ie DX_{\mu} A_\mu(X) \right]}, \quad (2.10)$$

$$X^\mu = x^\mu + \sqrt{2} \theta \psi^\mu = x_0^\mu + Y^\mu$$

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \tau}$$

$$\int d\theta = 1. \quad (2.11)$$

The photon (gluon) vertex operator is then rewritten as

$$- T^a \int_0^T d\tau d\varepsilon_{\mu} D X_{\mu} \text{exp}[ikX], \quad (2.12)$$

and we are left with a single Wick–contraction rule

$$\langle Y^\mu(\tau_1, \theta_1) Y^\nu(\tau_2, \theta_2) \rangle = -g^{\mu\nu} \dot{G}(\tau_1, \theta_1; \tau_2, \theta_2)$$

$$\dot{G}(\tau_1, \theta_1; \tau_2, \theta_2) = G_B(\tau_1, \tau_2) + \theta_1 \theta_2 G_F(\tau_1, \tau_2). \quad (2.13)$$

The fermion loop case can thus be made to look formally identical to the scalar loop case,
and be regarded as its “supersymmetrization”. This analogy has its roots in the fact
that the string-inspired technique corresponds to the use of a second-order formalism for
fermions in field theory (see [71] and ref. therein), instead of the usual first-order ones. In
practical terms it means that the Grassmann Wick contractions are replaced by a number
of Grassmann integrals, which have to be performed at a later stage of the calculation.
Ultimately the superfield formalism leads to the same collection of parameter integrals
to be performed, however we have found it useful for keeping intermediate expressions
compact. In the nonabelian case it has the further advantage that the introduction of an
additional two–gluon vertex operator can be avoided. Instead one introduces a suitable
supersymmetric generalization of the functions $\theta(\tau_i - \tau_{i+1})$ appearing in the ordered $\tau$ –
integrals [70].

Now let us assume that we have, in addition to the background field $A^\mu(x)$ we started
with, a second one, $\bar{A}^\mu(x)$, with constant field strength tensor $\bar{F}_{\mu\nu}$. We will restrict
ourselves to the abelian case for the remainder of this chapter. Using Fock–Schwinger

gauge centered at \( x_0 \), we may take \( \bar{A}^\mu(x) \) to be of the form

\[
\bar{A}_\mu(x) = \frac{1}{2} y_\nu \tilde{F}_{\nu\mu} .
\] (2.14)

The constant field contribution to the worldline lagrangian may then be written as

\[
\Delta \mathcal{L} = \frac{1}{2} i e y_\mu \tilde{F}_{\mu\nu} \dot{y}_\nu - i e \psi_\mu \tilde{F}_{\mu\nu} \psi_\nu
\] (2.15)

in components, or as

\[
\Delta \mathcal{L} = -\frac{1}{2} i e Y_\mu \tilde{F}_{\mu\nu} D Y_\nu
\] (2.16)

in the superfield formalism.

In any case, it is still quadratic in the worldline fields, and therefore need not be
considered part of the interaction lagrangian; we can absorb it into the free worldline
propagator(s). This means that we need to solve, instead of eqs. (2.4) for the worldline
Green’s functions, the modified equations

\[
\frac{1}{2} \left( \frac{\partial^2}{\partial \tau_1^2} - 2 i e \tilde{F} \frac{\partial}{\partial \tau_1} \right) G_B(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) - \frac{1}{T}
\] (2.17)

\[
\frac{1}{2} \left( \frac{\partial}{\partial \tau_1} - 2 i e \tilde{F} \right) G_F(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2)
\] (2.18)

These equations will be solved in appendix B, with the result (deleting the “bar” again)

\[
G_B(\tau_1, \tau_2) = \frac{1}{2(eF)^2} \left( \frac{eF}{\sin(eFT)} e^{-ieFT\hat{G}_{B12}} + ieF \hat{G}_{B12} \right)
\]

\[
G_F(\tau_1, \tau_2) = \frac{G_{F12}}{\cos(eFT)}
\] (2.19)

Equivalent expressions have been given for the pure magnetic field case in [23], and for
the general case in [14]. These expressions should be understood as power series in the field
strength matrix \( F \) (note that eqs. (2.19) do not assume invertibility of \( F \)). Note also that
the generalized Green’s functions are still translation invariant in \( \tau \), and thus functions of
\( \tau_1 - \tau_2 \). They are, in general, nontrivial Lorentz matrices, so that the Wick contraction
rules eq.(2.3) have to be rewritten as

\[
\langle y^\mu(\tau_1) y^\nu(\tau_2) \rangle = -G_B^{\mu\nu}(\tau_1, \tau_2),
\]

\[
\langle \psi^\mu(\tau_1) \psi^\nu(\tau_2) \rangle = \frac{1}{2} \mathcal{G}_F^{\mu\nu}(\tau_1, \tau_2).
\] (2.20)
Again momentum conservation leads to the freedom to subtract from $G_B$ its constant coincidence limit,

$$G_B(\tau, \tau) = \frac{1}{2(eF)^2} \left( eF \cot(eFT) - \frac{1}{T} \right) \quad (2.21)$$

To correctly obtain this and other coincidence limits, one has to apply the rules

$$\dot{G}_B(\tau, \tau) = 0, \quad \ddot{G}_B(\tau, \tau) = 1. \quad (2.22)$$

More generally, coincidence limits should always be taken after derivatives. We also need the generalizations of $\dot{G}_B, \ddot{G}_B$, which turn out to be

$$\dot{G}_B(\tau_1, \tau_2) \equiv 2\langle \tau_1 | (\partial_\tau - 2ieF)^{-1} | \tau_2 \rangle = \frac{i}{eF} \left( eF \sin(eFT) - \frac{i}{eFT} \dot{eF} \right)$$

$$\ddot{G}_B(\tau_1, \tau_2) \equiv 2\langle \tau_1 | (\mathbf{1} - 2ieF\partial_\tau)^{-1} | \tau_2 \rangle = 2\delta_{12} - \frac{2}{eF} \left( eF \sin(eFT) - \frac{i}{eFT} \right)$$

$$\dddot{G}_B(\tau_1, \tau_2) \equiv 2\langle \tau_1 | (\mathbf{1} - 2ieF\partial_\tau)^{-1} | \tau_2 \rangle = \frac{2}{eF} \left( eF \sin(eFT) - \frac{i}{eFT} \right)$$

Let us also give the first few terms of the Taylor expansions in $F$ for those four functions:

$$G_{B12} = \frac{T}{6} - \frac{i}{3} G_{B12} G_{B12} T eF + \left( \frac{T}{3} G_{B12}^2 - \frac{T^3}{90} \right) (eF)^2 + O(F^3)$$

$$\dot{G}_{B12} = \frac{2i(G_{B12} - \frac{T}{6})eF + \frac{2}{3} G_{B12} G_{B12} T (eF)^2 + O(F^3)}{eF}$$

$$\ddot{G}_{B12} = \frac{\dot{G}_{B12}}{eF} - 4(G_{B12} - \frac{T}{6})(eF)^2 + O(F^3)$$

$$G_{F12} = G_{F12} - i G_{F12} G_{B12} T eF + 2 G_{F12} G_{B12} T (eF)^2 + O(F^3)$$

Again $G_B$ and $G_F$ may be assembled into a superpropagator,

$$\hat{G}(\tau_1, \theta_1; \tau_2, \theta_2) \equiv G_B(\tau_1, \tau_2) + \theta_1 \theta_2 G_F(\tau_1, \tau_2). \quad (2.25)$$

At first sight this definition would seem not to accomodate the nonvanishing coincidence limit of $G_F$ (which can not be subtracted). Nevertheless, comparison with the component field formalism shows that the correct expressions are again reproduced if one takes coincidence limits after superderivatives. For instance, the correlator $\langle D_1 X(\tau_1, \theta_1) X(\tau_2, \theta_1) \rangle$ is evaluated by calculating

$$\langle D_1 X(\tau_1, \theta_1) X(\tau_2, \theta_2) \rangle = \theta_1 \theta_2 G_{B12} - \theta_2 G_{F12}, \quad (2.26)$$

and then setting $\tau_2 = \tau_1$.

It is easily seen that the substitution rule eq.(2.9) continues to hold, if one defines the cycle property solely in terms of the $\tau$ – indices, irrespectively of what happens to the Lorentz indices. For example, the expression
\[ \varepsilon_1 \dot{G}_{B12} k_2 \varepsilon_2 \dot{G}_{B23} \varepsilon_3 k_3 \dot{G}_{B31} k_1 \]  \hspace{1cm} (2.27)

would have to be replaced by 

\[ \varepsilon_1 \dot{G}_{B12} k_2 \varepsilon_2 \dot{G}_{B23} \varepsilon_3 k_3 \dot{G}_{B31} k_1 - \varepsilon_1 G_{F12} k_2 \varepsilon_2 G_{F23} \varepsilon_3 k_3 G_{F31} k_1 \]  \hspace{1cm} (2.28)

The only novelty is again due to the fact that, in contrast to \( \dot{G}_B \) and \( G_F \), \( \dot{G}_B \) and \( G_F \) have non-vanishing coincidence limits,

\[ \dot{G}_B(\tau, \tau) = icot(eFT) - \frac{i}{eFT} \]  \hspace{1cm} (2.29)

\[ G_F(\tau, \tau) = -i \tan(eFT) \]  \hspace{1cm} (2.30)

As a consequence, we now have also a substitution rule for one-cycles,

\[ \dot{G}_B(\tau_i, \tau_i) \rightarrow \dot{G}_B(\tau_i, \tau_i) - G_F(\tau_i, \tau_i) = \frac{i}{\sin(eFT) \cos(eFT)} - \frac{i}{eFT} \]  \hspace{1cm} (2.31)

This is almost all we need to know for computing one-loop photon scattering amplitudes, or the corresponding effective action, in a constant overall background field. The only further information required at the one-loop level is the change in the path integral determinants due to the external field, i.e. the vacuum amplitude in the constant field. For spinor QED, this just corresponds to the Euler–Heisenberg Lagrangian, and has, in the present formalism, been calculated in [21]. Let us shortly retrace this calculation (the fact that the Euler-Heisenberg integrand may be represented as a superdeterminant was already noted in [72]). In the scalar QED case, we have to replace

\[ \int \mathcal{D}y \exp \left[ - \int_0^T d\tau \frac{1}{4} y^2 \right] = \text{Det}^{\frac{1}{2}} [-\partial^2] = [4\pi T]^{-\frac{D}{2}} \]  \hspace{1cm} (2.32)

by

\[ \text{Det}^{\frac{1}{2}} [-\partial^2 + 2ieF \partial \tau] = [4\pi T]^{-\frac{D}{2}} \text{Det}^{\frac{1}{2}} \left[ I - 2ieF \partial \tau^{-1} \right] \]  \hspace{1cm} (2.33)

(as usual, the prime denotes the absence of the zero mode in a determinant). Application of the \( \ln \det = tr \ln \) identity yields

\[ \text{Det}^{\frac{1}{2}} \left[ I - 2ieF \partial \tau^{-1} \right] = \exp \left[ \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-2ie)^n \text{tr}[F^n] \text{Tr}[\partial \tau^{-n}] \right] \]

\[ = \exp \left[ \frac{1}{2} \sum_{n=2}^{\infty} \sum_{n \text{ even}} B_n \frac{(2ieT)^n \text{tr}[F^n]}{n!} \right] \]

\[ = \det^{\frac{1}{2}} \left[ \frac{\sin(eFT)}{eFT} \right] \]  \hspace{1cm} (2.34)
where the \( B_n \) are the Bernoulli numbers. In the second step eq.(\ref{eq:2.27}) was used. The analogous calculation for the Grassmann path integral yields a factor
\[
\det_A \left[ I - 2ieF \partial^{-1} \right] = \det^\frac{1}{2} \left[ \cos(eFT) \right].
\] (2.35)
For spinor QED we therefore find a total overall determinant factor of
\[
[4\pi T]^{-\frac{D}{2}} \det^\frac{1}{2} \left[ \frac{\tan(eFT)}{eFT} \right].
\] (2.36)
Expressing \( \text{tr}[F^{2n}] \) in terms of the two invariants of the electromagnetic field,
\[
\text{tr}[F^{2n}] = 2 \left[ (a^2)^n + (-b^2)^n \right],
\]
\[
a^2 = \frac{1}{2} \left[ E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right],
\]
\[
b^2 = \frac{1}{2} \left[ -(E^2 - B^2) + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \right],
\] (2.37)
we obtain the standard Schwinger proper-time representation of the (unsubtracted) Euler-Heisenberg-Schwinger Lagrangians,
\[
\mathcal{L}_{\text{scal}}^{(1)}[F] = \frac{1}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \frac{(eaT)(ebT)}{\sin(eaT) \sinh(ebT)},
\] (3.8)
\[
\mathcal{L}_{\text{spin}}^{(1)}[F] = -\frac{2}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \frac{(eaT)(ebT)}{\tan(eaT) \tanh(ebT)}.
\] (3.9)

3. Gauge Boson Loops in External Fields

In this section we first express the one-loop effective action and also the propagator of spin-1 gauge bosons in an arbitrary external Yang-Mills field in terms of a worldline path integral. We then will evaluate the action in a covariantly constant background.

We employ the background gauge fixing technique so that the effective action \( \Gamma[A^a_\mu] \) becomes a gauge invariant functional of \( A^a_\mu \). The gauge fixed classical action reads, in \( D \) dimensions,
\[
S[a; A] = \frac{1}{4} \int d^D x F^a_{\mu\nu}(A + a) F^a_{\mu\nu}(A + a) + \frac{1}{2\alpha} \int d^D x \left( D^{ab}_\mu[A] a^b_\mu \right)^2.
\] (3.1)
A priori, the background field \( A^a_\mu \) is unrelated to the quantum field \( a^a_\mu \). The kinetic operator of the gauge boson fluctuations one obtains as the second functional derivative of \( S[a, A] \) with respect to \( a^a_\mu \), at fixed \( A^a_\mu \). This leads to the inverse propagator
\[
D^{ab}_{\mu\nu} = -D^{ac}_\rho D^{cb}_\rho \delta_{\mu\nu} - 2tg F^{ab}_{\mu\nu}
\] (3.2)
and the effective action

\[
\Gamma[A] = \frac{1}{2} \ln \det(D)
\]

\[
= -\frac{1}{2} \int_0^\infty \frac{dT}{T} \text{Tr}[e^{-T\mathcal{D}}].
\]  

(3.3)

In writing down eq. (3.2) we adopted the Feynman gauge \( \alpha = 1 \). The covariant derivative \( D_\mu \equiv \partial_\mu + igA_\mu^a T^a \) and the field strength \( F_\mu^a \equiv F^c_\mu (T^c)^{ab} \) are matrices in the adjoint representation of the gauge group, with the generators given by \( (T^a)^{bc} = -if^{abc} \). The full effective action is obtained by adding the contribution of the Faddeev-Popov ghosts to eq. (3.3). The evaluation of the ghost determinant proceeds along the same lines as scalar QED and we shall not discuss it here.

In order to derive a path integral representation of the heat-kernel

\[
\text{Tr}[\exp(-T\mathcal{D})]
\]

(3.4)

we first look at a slightly more general problem. We generalize the operator \( \mathcal{D} \) to

\[
\tilde{\mathcal{D}}_{\mu\nu} = -D^2 \delta_{\mu\nu} + M_{\mu\nu}(x)
\]

(3.5)

where \( M_{\mu\nu}(x) \) is an arbitrary matrix in color space. In particular, we do not assume that the Lorentz trace \( M_{\mu\mu} \) is zero. Given \( M_{\mu\nu} \), we construct the following one-particle Hamilton operator:

\[
\hat{H} = (\hat{p}_\mu + gA_\mu(x))^2 - \hat{\psi}_\mu M_{\mu\nu}(x) \hat{\psi}_\nu.
\]

(3.6)

The system under consideration has a graded phase-space coordinatized by \( x_\mu \) and two sets of anticommuting variables, \( \psi_\mu \) and \( \bar{\psi}_\mu \), which obey canonical anticommutation relations:

\[
\hat{\psi}_\mu \hat{\psi}_\nu + \hat{\psi}_\nu \hat{\psi}_\mu = \delta_{\mu\nu}.
\]

(3.7)

For a reason which will become obvious in a moment we have adopted the “anti-Wick” ordering in (3.6): all \( \bar{\psi} \)'s are on the right of all \( \psi \)'s, e.g.

\[
\hat{\psi}_\alpha \hat{\psi}_\beta : = \hat{\psi}_\alpha \hat{\psi}_\beta
\]

\[
: \hat{\psi}_\beta \hat{\psi}_\alpha : = -\hat{\psi}_\alpha \hat{\psi}_\beta.
\]

(3.8)

We can represent the commutation relations on a space of wave functions \( \Phi(x, \psi) \) depending on \( x_\mu \) and a set of classical Grassmann variables \( \psi_\mu \). The “position” operators \( \hat{x}_\mu = x_\mu \), \( \hat{\psi}_\mu = \psi_\mu \) act multiplicatively on \( \Phi \) and the conjugate momenta as derivatives \( \hat{\bar{\psi}}_\mu = -i\partial_\mu \) and \( \hat{\psi}_\mu = \partial/\partial \psi_\mu \). Thus the Hamiltonian becomes

\[
\hat{H} = -D^2 + \psi_\mu M_{\mu\nu}(x) \frac{\partial}{\partial \psi_\mu}.
\]

(3.9)

The wave functions \( \Phi \) have a decomposition of the form

\[
\Phi(x, \psi) = \sum_{p=0}^D \frac{1}{p!} \phi^{(p)}_{\mu_1...\mu_p}(x) \psi_{\mu_1} \cdots \psi_{\mu_p}.
\]

(3.10)
This suggests the interpretation of $\Phi$ as an inhomogeneous differential form on $\mathbb{R}^D$ with the fermions $\psi_\mu$ playing the role of the differentials $dx^\mu$. The form-degree or, equivalently, the fermion number is measured by the operator

$$\hat{F} = \hat{\psi}_\mu \hat{\psi}_\mu = \psi_\mu \frac{\partial}{\partial \psi_\mu}. \tag{3.11}$$

We are particularly interested in one-forms:

$$\Phi(x, \psi) = \varphi_\mu(x) \psi_\mu. \tag{3.12}$$

The Hamiltonian (3.9) acts on them according to

$$\langle \hat{H} \Phi \rangle(x, \psi) = (\hat{h}_{\mu\nu} \varphi_\nu) \psi_\mu. \tag{3.13}$$

We see that, when restricted to the one-form sector, the quantum system with the Hamiltonian (3.6) is equivalent to the one defined by the bosonic matrix Hamiltonian $\hat{h}_{\mu\nu}$. 

The euclidean proper time evolution of the wave functions $\Phi$ is implemented by the kernel

$$K(x_2, \psi_2, t_2| x_1, \psi_1, t_1) = \langle x_2, \psi_2 | e^{-(t_2-t_1)\hat{H}} | x_1, \psi_1 \rangle \tag{3.14}$$

which obeys the Schrödinger equation

$$\left( \frac{\partial}{\partial T} + \hat{H} \right) K(x, \psi, T| x_0, \psi_0, 0) = 0 \tag{3.15}$$

with the initial condition $K(x, \psi, 0| x_0, \psi_0, 0) = \delta(x-x_0)\delta(\psi-\psi_0)$. It is easy to write down a path integral solution to eq. (3.15). For the trace of $K$ one obtains

$$\text{Tr}[e^{-T\hat{H}_W}] = \int_P \mathcal{D}x(\tau) \int_P \mathcal{D}\psi(\tau) \mathcal{D}\bar{\psi}(\tau) \text{tr} \mathcal{P} e^{-\int_0^T d\tau L} \tag{3.16}$$

with

$$L = \frac{1}{4} \dot{x}^2 + ig A_\mu(x) \dot{x}_\mu + \bar{\psi}_\mu [\partial_\tau \delta_{\mu\nu} - M_{\nu\mu}] \psi_\nu. \tag{3.17}$$

We have again periodic boundary conditions for $x^\mu(\tau)$, and antiperiodic boundary conditions for $\psi^\mu(\tau)$. If we use periodic boundary conditions for the fermions we arrive at a representation of the Witten index rather than the partition function:

$$\text{Tr}[(-1)^F e^{-T\hat{H}_W}] = \int_P \mathcal{D}x(\tau) \int_P \mathcal{D}\psi(\tau) \mathcal{D}\bar{\psi}(\tau) \text{tr} \mathcal{P} e^{-\int_0^T d\tau L}. \tag{3.18}$$

At this point we have to mention a subtlety which is frequently overlooked but will be important later on. If we regard the Hamiltonian (3.6) as a function of the anticommuting $\psi_\mu$ and $\bar{\psi}_\mu$ it is related to the classical Lagrangian (3.17) by a standard Legendre transformation. The information about the operator ordering is implicit in the discretization which is used for the definition of the path-integral. Different operator orderings correspond to different discretizations. Here we shall adopt the midpoint prescription for the discretization, because only in this case the familiar path-integral manipulations
are allowed \[80\]. It is known \[78, 80, 81, 82, 83\] that, at the operatorial level, this is equivalent to using the Weyl ordered Hamiltonian \( \hat{H}_W \). This is the reason why we wrote \( \hat{H}_W \) rather than \( \hat{H} \) on the l.h.s. of eqs. (3.16) and (3.18). In order to arrive at the relation (3.13) we had to assume that the fermion operators in \( \hat{H} \) are “anti-Wick” ordered. Weyl ordering amounts to a symmetrization in \( \tilde{\psi} \) and \( \psi \) so that

\[
\hat{H}_W = \left( \hat{p}_\mu + gA_\mu(\bar{x}) \right)^2 + \frac{1}{2} M_{\nu\mu}(\bar{x})(\hat{\psi}_\nu \hat{\bar{\psi}}_\mu - \hat{\psi}_\mu \hat{\bar{\psi}}_\nu)
\]

\[
= \hat{H} - \frac{1}{2} M_{\mu\mu}(\bar{x}).
\] (3.19)

In the second line of (3.19) we used (3.6) and (3.7). (With respect to \( \hat{\bar{x}}_\mu \) and \( \hat{\bar{p}}_\mu \), Weyl ordering is used throughout.) If we employ (3.19) in (3.16) we obtain the following representation for the partition function of the anti-Wick ordered exponential:

\[
\text{Tr}[e^{-T\hat{H}}] = \int_P \mathcal{D}x(\tau) \int_A \mathcal{D}\psi(\tau) \mathcal{D}\bar{\psi}(\tau)
\]

\[
\times \text{tr}\mathcal{P} \exp \left[ - \int_0^T d\tau \left\{ L(\tau) + \frac{1}{2} M_{\mu\mu}(x(\tau)) \right\} \right].
\] (3.20)

Let us now calculate the partition function \( \text{Tr}[\exp(-T\hat{h})] \) which is a generalization of the heat-kernel needed in eq. (3.3). By virtue of eq. (3.13) we may write

\[
\text{Tr}[e^{-T\hat{h}}] = \text{Tr}_1[e^{-T\hat{H}}]
\] (3.21)

where “\( \text{Tr}_1 \)” denotes the trace in the one-form sector of the theory which contains the worldline fermions. In order to perform the projection on the one-form sector we identify \( M_{\mu\nu} \) with

\[
M_{\mu\nu} = C\delta_{\mu\nu} - 2igF_{\mu\nu}
\] (3.22)

where \( C \) is a real constant. As a consequence,

\[
\hat{H} = \hat{H}_0 + C\hat{F}
\] (3.23)

with

\[
\hat{H}_0 \equiv \left( \hat{p}_\mu + gA_\mu(\bar{x}) \right)^2 - 2igF_{\nu\mu}(\bar{x})\hat{\bar{\psi}}_\nu \hat{\psi}_\mu
\] (3.24)

denoting the Hamiltonian which corresponds to the inverse propagator \( \mathcal{D} \). The fermion number operator \( \hat{F} \equiv \hat{\bar{\psi}}_\mu \hat{\psi}_\mu \) is anti-Wick ordered by definition. Its spectrum consists of the integers \( p = 0, 1, 2, \ldots D \). Note that \( M_{\mu\mu} = DC \), and that because of the antisymmetry of \( F_{\mu\nu} \) the Hamiltonian \( \hat{H}_0 \) has no ordering ambiguity in its fermionic piece. It will prove useful to apply eq. (3.20) not to \( \hat{H} \) directly, but to \( \hat{H} - C = \hat{H}_0 + C(\hat{F} - 1) \). The result reads then

\[
\text{Tr}[e^{-CT(\hat{F} - 1)}e^{-T\hat{H}_0}] = \exp[-CT\left( \frac{D}{2} - 1 \right)] \int_P \mathcal{D}x(\tau) \int_A \mathcal{D}\psi(\tau) \mathcal{D}\bar{\psi}(\tau) \text{tr}\mathcal{P} e^{-\int_0^T d\tau L}
\] (3.25)
with

\[ L = \frac{1}{4} \dot{x}^2 + i g A_\mu(x) \dot{x}_\mu + \bar{\psi}_\mu [(\partial_\tau - C) \delta_{\mu\nu} - 2ig F_{\mu\nu}] \psi_\nu \] (3.26)

After having performed the path integration in (3.25) we shall send \( C \) to infinity. While this has no effect in the one-form sector, it leads to an exponential suppression factor \( \exp[-CT(p - 1)] \) in the sectors with fermion numbers \( p = 2, 3, \ldots D \). Hence only the zero and the one forms survive the limit \( C \to \infty \). In order to eliminate the contribution from the zero forms we insert the projector \( [1 - (-1)^F] / 2 \) into the trace. It projects on the subspace of odd form degrees, and it is easily implemented by combining periodic and antiperiodic boundary conditions for \( \psi_\mu \). In this manner we arrive at a representation of the partition function of \( \hat{H}_0 \) restricted to the one-form sector:

\[
\text{Tr}_1[e^{-T\hat{H}_0}] = \lim_{C \to \infty} \text{Tr} \left[ \frac{1}{2} (1 - (-1)^F) e^{-CT(\hat{F} - 1)} e^{-T\hat{H}_0} \right]
\]

\[
\times \frac{1}{2} \int A - \int P D\psi(\tau) D\bar{\psi}(\tau) \text{Tr} P e^{-\int_0^T d\tau L} (3.27)
\]

Because \( \text{Tr}[\exp(-TD)] = \text{Tr}_1[\exp(-T\hat{H}_0)] \), eq. (3.27) implies for the effective action

\[
\Gamma[A] = -\frac{1}{2} \lim_{C \to \infty} \int_0^\infty \frac{dT}{T} \exp[-CT(D/2 - 1)] \int_P \text{D}x \frac{1}{2} \left( \int_A - \int_P \right)\text{D}\psi\text{D}\bar{\psi}
\]

\[
\times \text{Tr} P \exp \left[ -\int_0^T d\tau \left\{ \frac{1}{4} \dot{x}^2 + i g A_\mu \dot{x}_\mu + \bar{\psi}_\mu [(\partial_\tau - C) \delta_{\mu\nu} - 2ig F_{\mu\nu}] \psi_\nu \right\} \right] (3.28)
\]

Several comments are in order here. The factor \( \exp[-CTD/2] \) in (3.28) is due to the difference between the Weyl and the anti-Wick-ordered Hamiltonian. It is crucial for obtaining a finite result in the limit \( C \to \infty \). In fact, for \( D = 4 \) it converts the prefactor \( \exp[+CT] \) to a decaying exponential \( \exp[-CT] \) [1]. From the point of view of the worldline fermions, \( C \) plays the role of a mass. Their free Green’s function \( G^C(\tau_2 - \tau_1) \delta_{\mu\nu} \) with

\[
G^C(\tau_2 - \tau_1) \equiv \langle \tau_2 | (\partial_\tau - C)^{-1} | \tau_1 \rangle
\] (3.29)

reads for periodic and antiperiodic boundary conditions, respectively,

\[
G_P^C(\tau) = -[\Theta(-\tau) + \Theta(\tau)e^{-CT}] \frac{e^{CT}}{1 - e^{-CT}}
\]

\[
G_A^C(\tau) = -[\Theta(-\tau) - \Theta(\tau)e^{-CT}] \frac{e^{CT}}{1 + e^{-CT}}
\] (3.30)

We observe that for \( C \to \infty \) there is an increasingly strong asymmetry between the forward and backward propagation in the proper time. Further details of the Green’s functions (3.30) can be found in appendix B.

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1In ref. [1] the reordering factor was not taken into account and the change of the sign in \( D = 4 \) was attributed to a difference between Minkowski and Euclidean spacetime which is not correct in our opinion.
We mention in passing that there exists another simple method for the projection on the one-form sector. We can insert a Kronecker-delta \( \delta_1 \) into the partition function (3.20) and exponentiate it in terms of a parameter integral over an angular variable \( \alpha \):

\[
\text{Tr} \left[ e^{-T\hat{H}_0} \right] = \frac{T}{2\pi} \int_0^{2\pi/T} d\alpha \ e^{i\alpha T} \ \text{Tr} \left[ e^{-i\alpha T\hat{F}} e^{-T\hat{H}_0} \right].
\] (3.31)

The r.h.s. of (3.31) can be represented by a path integral which, for the fermions, involves antiperiodic boundary conditions only. The corresponding action is similar to the one used above but with \( C \) replaced by \( i\alpha \). Instead of the limit \( C \to \infty \) one has to perform the \( \alpha \)-integration now. The computational effort is essentially the same in both cases.

The representation (3.28) of the effective action does not coincide with the one used by Strassler [9]. While he uses the same kinetic term in the fermionic worldline Lagrangian, he modifies the interaction term according to

\[
\bar{\psi}_\mu F_{\mu\nu} \psi_\nu \to \frac{1}{2} \chi_\mu F_{\mu\nu} \chi_\nu \equiv \bar{\psi}_\mu F_{\mu\nu} \psi_\nu + \frac{1}{2} F_{\mu\nu} (\bar{\psi}_\mu \psi_\nu + \bar{\psi}_\mu \bar{\psi}_\nu)
\] (3.32)

where

\[
\chi_\mu (\tau) \equiv \psi_\mu (\tau) + \bar{\psi}_\mu (\tau)
\] (3.33)

As a consequence, the modified Hamiltonian \( \hat{H}_0 \) contains terms which change the fermion number by 2 units. This means that, during the proper time evolution, the 1-form representing the gauge boson can evolve into a 3-form. However, in the limit \( C \to \infty \) the substitution (3.32) causes no problems. The reason is that for \( C \to \infty \) the energy gap between 1- and 3-forms becomes infinite, and therefore a 1-form at \( \tau = 0 \) will remain a 1-form for all \( \tau > 0 \). In the modified formalism, Wick contractions of the interaction term (3.3) involve the 2-point function of \( \chi \), i.e., \( \mathcal{G}^{\chi} (\tau) \equiv \mathcal{G}^C (\tau) - \mathcal{G}^C (-\tau) \). From (3.30) we obtain explicitly

\[
\mathcal{G}^{\chi}_F (\tau) = \text{sign}(\tau) \frac{\sinh[C(T/2 - |\tau|)]}{\sinh[CT/2]},
\]

\[
\mathcal{G}^{\chi}_A (\tau) = \text{sign}(\tau) \frac{\cosh[C(T/2 - |\tau|)]}{\cosh[CT/2]}.
\] (3.34)

These Green’s functions do not coincide with the ones given by Strassler [9]; however, they become effectively equivalent in the limit \( C \to \infty \). The substitution (3.32) is motivated by the algebraic simplification which it entails in perturbation theory where one inserts a sum of plane waves for \( A_\mu^a (x) \). We shall not do this in the present paper but rather calculate the path integral exactly for a covariantly constant field strength. In this case the representation (3.28) is more convenient than the one advocated by Strassler.

An important building block for higher-loop calculations is the gluon propagator in an external Yang-Mills field. It is given by the proper-time integral of the evolution kernel (3.14). The latter can be represented by the following path integral with open boundary conditions:

\[
K(x_2, \psi_2, T \mid x_1, \psi_1, 0) = \int_{x(0) = x_1}^{x(T) = x_2} D x(\tau) \int_{\psi(0) = \psi_1}^{\psi(T) = \psi_2} D \psi(\tau) \int D \bar{\psi}(\tau) e^{-\int_0^T d\tau L}. \] (3.35)
The Lagrangian $L$ is given by (3.17) with $M_{\mu\nu} = -2igF_{\mu\nu}$ because we do not need the $C$-term in the case of open boundary conditions. The reason is that the Hamiltonian (3.9) preserves the fermion number. Therefore, the kernel $K$ will map a one-form wave function of the type (3.12) onto another one-form. In fact, in the one-form sector, $K$ is represented by a Lorentz matrix

$$K_{\mu\nu}(x_2, T|x_1, 0) = \langle x_2, \mu|e^{-T\mathcal{D}}|x_1, \nu \rangle$$  (3.36)

which is related to the gluon propagator by

$$\langle x_2, \mu|D^{-1}|x_1, \nu \rangle = \int_0^\infty dT \ K_{\mu\nu}(x_2, T|x_1, 0).$$  (3.37)

(The color indices are kept implicit.) It is easy to express the bosonic matrix $K_{\mu\nu}$ in terms of the kernel $K$ with fermionic arguments. If one writes

$$\Phi(x_2, \psi_2, T) = \int d^Dx_1 d^D\psi_1 K(x_2, \psi_2, T|x_1, \psi_1, 0) \Phi(x_1, \psi_1, 0)$$  (3.38)

and inserts a wave function of the type (3.12) at both $\tau = 0$ and $\tau = T$ one finds that

$$K_{\mu\nu}(x_2, T|x_1, 0) = \frac{\partial}{\partial \psi_{2\mu}} \int d^D\psi_1 K(x_2, \psi_2, T|x_1, \psi_1, 0) \psi_{1\nu}.$$  (3.39)

By combining eq. (3.37) with eqs. (3.38) and (3.35) we obtain the desired path integral representation of the gluon propagator.

In order to get a better understanding of this representation, let us assume that the background $A_\mu$ is either abelian or quasi-abelian, and that no path-ordering must be observed therefore. We may then rewrite (3.35) according to

$$K(x_2, \psi_2, T|x_1, \psi_1, 0) = \int_{x(0)=x_1}^{x(T)=x_2} D\tau D^2 \psi(x_2, T|x_1, 0)$$

$$\times \exp \left[ -\int_0^T d\tau \left\{ \frac{1}{4} \dot{x}_\mu^2 + igA_\mu \dot{x}_\mu \right\} \right].$$  (3.40)

Here the fermionic integral

$$K_F(\psi_2, T|\psi_1, 0) = \int_{\psi(0)=\psi_1}^{\psi(T)=\psi_2} D\psi(\tau) D\bar{\psi}(\tau)$$

$$\times \exp \left[ -\int_0^T d\tau \bar{\psi}_\mu (\partial_\tau \delta_{\mu\nu} - 2igF_{\mu\nu}) \psi_{1\nu} \right]$$  (3.41)

is a functional of the bosonic trajectory $x_\mu(\tau)$. It can be evaluated exactly \cite{75, 76, 77} and has a remarkably simple structure:

$$K_F(\psi_2, T|\psi_1, 0) = \delta(\psi_2 - \mathcal{S}(T)\psi_1).$$  (3.42)
In (3.42) we introduced
\[ S_{\mu\nu}(T) = \mathcal{P} \exp \left[ 2ig \int_{0}^{T} d\tau F(x(\tau)) \right]_{\mu\nu} \] (3.43)

where the path-ordering is necessary because of the Lorentz matrix structure. Alternatively, eq. (3.42) can be established by noting that \( K_F \) solves the Schrödinger equation corresponding to (3.41):
\[ \left[ \partial_{\partial T} - 2ig\dot{\psi}_{\mu} F_{\mu\nu} \partial_{\partial \psi_{\nu}} \right] K_F(\psi, T | \psi_1, 0) = 0. \] (3.44)

Because \( S^T = S^{-1} \), it follows that
\[ \partial_{\partial \psi_{2\mu}} \int d^D \psi_1 K_F(\psi_2, T | \psi_1, 0) \psi_1 = S_{\mu\nu}(T) \] (3.45)

and therefore
\[ K_{\mu\nu}(x_2, T | x_1, 0) = \int_{x(0)=x_1}^{x(T)=x_2} Dx(\tau) D\psi(\tau) \mathcal{P} \exp \left[ 2ig \int_{0}^{T} d\tau F(x(\tau)) \right]_{\mu\nu} \]
\[ \times \exp \left[ -\int_{0}^{T} d\tau \left\{ \frac{1}{4} \dot{x}^2 + igA_{\mu}\dot{x}_{\mu} \right\} \right]. \] (3.46)

Later on we shall use the evolution kernel in the form (3.46). Clearly we could have written down this representation without going through the original path integral (3.35). However, in multiloop calculations it will be advantageous if the gluon loops and the corresponding propagators are represented in a coherent framework. In fact, if we recall that \( \bar{\psi}_{\mu} \) amounts to the derivative \( \partial / \partial \psi_{\mu} \) in the Schrödinger picture, the evolution kernel, for any background, may be rewritten in the following very elegant form:
\[ K_{\mu\nu}(x_2, T | x_1, 0) = \int_{x(0)=x_1}^{x(T)=x_2} Dx(\tau) D\psi(\tau) \int D\bar{\psi}(\tau) \]
\[ \times \delta(\psi(T)) \bar{\psi}_{\mu}(T) \psi_{\nu}(0) e^{-\int_{0}^{T} d\tau L}. \] (3.47)

In eq. (3.47), \( \psi(0) \) and \( \psi(T) \) are integrated independently.

Now we evaluate the path integral (3.28) for the case that the background has a covariantly constant field strength. We assume that the gauge field has the form
\[ A_{\mu}^a(x) = n^a A_{\mu}(x), \quad n^a n^a = 1 \] (3.48)

where \( n^a \) is a constant unit vector in color space. The associated field strength \( F_{\mu\nu}^a = n^a F_{\mu\nu} \) with \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) satisfies \( \mathcal{D}_{\mu}^a F_{\mu\nu}^a = 0 \). Both \( A_{\mu} \) and \( F_{\mu\nu} \) enter eq. (3.28) for the gauge boson loop as matrices in the adjoint representation. Hence the only nontrivial color matrix which enters the path integral is \( n^a T^a \). If we denote the eigenvalues of this

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matrix by \( \nu_l, l = 1, 2, \ldots \), and evaluate the color trace in the diagonal basis, the integral of the path integrand assumes the form

\[
\text{Tr} \mathcal{P} I(gA_\mu) = \sum_l I(g\nu_l A_\mu)
\]

(3.49)

The path ordering has no effect here. We observe that we are effectively dealing with a set of abelian theories whose gauge coupling is given by \( g\nu_1 \).

While the condition \( D^{ab}_\alpha F^{b}_{\mu\nu} = 0 \) does not necessarily imply that \( F_{\mu\nu} \) is constant, we further assume that \( F_{\mu\nu} \) is an \( x \)-independent matrix and that the gauge field is of the form

\[
A_\mu(x) = \frac{1}{2} x_\nu F_{\nu\mu}
\]

(3.50)

In the following we keep the diagonalization of the color matrix implicit and continue to use the notation \( \text{Tr} I(gA_\mu) \) rather than \( \sum_l I(g\nu_l A_\mu) \). We keep in mind, however, that \( A_\mu \) and \( F_{\mu\nu} \) may be treated as pure numbers as far as their color structure is concerned.

For the field (3.48), (3.50) all path integrals in (3.28) are Gaussian. We separate the constant mode from the \( x_\mu \) integration as before and obtain

\[
\Gamma[A] = -\frac{1}{2} \int d^D x_0 \text{tr} \int_0^\infty \frac{dT}{T} \text{Det} \left[ -\frac{1}{2} \partial^2 \delta_{\mu\nu} + 2igF_{\mu\nu}\partial_\tau \right]^{-\frac{1}{2}} \lim_{C \to \infty} Y(C)
\]

(3.51)

with

\[
Y(C) = \frac{1}{2} \exp \left[-CT \left( \frac{D}{2} - 1 \right) \right] \left\{ \text{Det}_A[(\partial_\tau - C)\delta_{\mu\nu} - 2igF_{\mu\nu}] - \text{Det}_P[(\partial_\tau - C)\delta_{\mu\nu} - 2igF_{\mu\nu}] \right\}
\]

(3.52)

We denote the real eigenvalues of \( iF_{\mu\nu} \) by \( f^{(\alpha)}, \alpha = 1, \ldots, D \), and we use the formulas in appendix C in order to express \( Y(C) \) in terms of these eigenvalues:

\[
Y(C) = \frac{1}{2} \exp \left[-CT \left( \frac{D}{2} - 1 \right) \right] \times \left\{ \text{Det}_A[(\partial_\tau - C)\delta_{\mu\nu}] \prod_{\alpha=1}^D \frac{\cosh \left[ T(C + 2gf^{(\alpha)}) \right]}{\cosh \left[ CT/2 \right]} \right. \\
- \left. \text{Det}_P[(\partial_\tau - C)\delta_{\mu\nu}] \prod_{\alpha=1}^D \frac{\sinh \left[ T(C + 2gf^{(\alpha)}) \right]}{\sinh \left[ CT/2 \right]} \right\}
\]

(3.53)

Note that in the case of the fermionic integration with periodic boundary conditions the zero mode of \( \partial_\tau \) was not excluded from the determinant and that eq. (C.6) applies therefore. In (3.53) we have factored out the free determinants because the method of appendix C can yield only ratios of determinants. The normalization factors

\[
\zeta_{A,P} \equiv \text{Det}_{A,P}[(\partial_\tau - C)\delta_{\mu\nu}] = (\text{Det}_{A,P}[\partial_\tau - C])^D
\]

(3.54)
are most easily determined by recalling their operatorial interpretation as the partition function and the Witten index of a free Fermi oscillator, respectively:

\[
\text{Det}_A[\partial_\tau - C] = \text{Tr}[e^{-CT\hat{F}_W}]
\]

\[
\text{Det}_P[\partial_\tau - C] = \text{Tr}[(-1)^\hat{F}e^{-CT\hat{F}_W}]
\]  \hspace{1cm} (3.55)

Here \(\hat{F}_W = (\hat{\psi} \hat{\psi} - \hat{\psi} \hat{\psi})/2 = \hat{F} - 1/2\) is the Weyl-ordered fermion number operator (for \(D = 1\)) with eigenvalues -1/2 and +1/2. As a consequence,

\[
\zeta_A = (2 \cosh[CT/2])^D
\]

\[
\zeta_P = (2 \sinh[CT/2])^D
\]  \hspace{1cm} (3.56)

Taking advantage of \(\sum_\alpha f^{(\alpha)} = 0\), we can rewrite (3.53) as

\[
Y(C) = \frac{1}{2} \eta^{-1} \left[ \prod_\alpha (1 + \eta q_\alpha) - \prod_\alpha (1 - \eta q_\alpha) \right]
\]  \hspace{1cm} (3.57)

with \(\eta \equiv \exp(-CT)\) and \(q_\alpha \equiv \exp[-2gTf^{(\alpha)}]\). In the limit \(C \to \infty\), i.e., \(\eta \to 0\), the leading \(O(1)\) terms cancel among the products in (3.57), and one obtains

\[
\lim_{C \to \infty} Y(C) = \sum_{\alpha=1}^D \exp[-2gTf^{(\alpha)}]
\]

\[
= \text{tr}_L \cos[2gTF]
\]  \hspace{1cm} (3.58)

Here we exploited that if \(f^{(\alpha)}\) is an eigenvalue, so is \(-f^{(\alpha)}\). (\(\text{tr}_L\) denotes the trace with respect to the Lorentz indices.)

In complete analogy with the scalar case, eq.( 2.35), the bosonic determinant in (3.51) gives rise to a factor of

\[
(4\pi T)^{-D/2} \exp \left[ -\frac{1}{2} \text{tr}_L \ln \frac{\sin(gTF)}{(gTF)} \right]
\]  \hspace{1cm} (3.59)

Hence our final result for the gauge boson loop becomes

\[
\Gamma[A] = -\frac{1}{2} \int d^Dx_0 \ (4\pi)^{-D/2} \int_0^\infty \frac{dT}{T} T^{-D/2}
\]

\[
\times \text{tr} \exp \left[ -\frac{1}{2} \text{tr}_L \ln \frac{\sin(gTF)}{(gTF)} \right]
\]

\[
\times \text{tr}_L \cos(2gTF)
\]  \hspace{1cm} (3.60)

Eq. (3.60) coincides with the result which was found with the help of the traditional techniques \[53, 84, 85, 86\].

By a computation similar to the previous one, but with open boundary conditions for the path integral, one can find the gluon propagator in the background (3.48), (3.50). It is given by (3.37) with

\[
K_{\mu\nu}(x_2, T|x_1, 0) = \exp(2igTF)_{\mu\nu} \int_{x(0)=x_1}^{x(T)=x_2} \mathcal{D}x(\tau) e^{-S[x(\cdot)]}.
\]  \hspace{1cm} (3.61)
The path integral which remains to be evaluated is just the one for the scalar propagator. Now in the action functional
\[
S[x(\cdot)] = \frac{1}{4} \int_0^T d\tau \ x_\mu \left\{ -\partial^2_\tau \delta_{\mu\nu} + 2igF_{\mu\nu} \partial_\tau \right\} x_\nu + \frac{1}{4} \left\{ x_\mu(T) \dot{x}_\mu(T) - x_\mu(0) \dot{x}_\mu(0) \right\}
\] (3.62)

one has to be careful about the surface terms which appear after the integration by parts. For open paths they give rise to a non-zero contribution in general. Since \( S \) is quadratic in \( x_\mu \), the saddle point approximation of the path integral (3.61) gives the exact answer. We can solve it by expanding the integration variable about the classical trajectory connecting \( x_1 \) and \( x_2 \):
\[
x_\mu(\tau) = x_\mu^{\text{class}}(\tau) + y_\mu(\tau)
\] (3.63)

Here \( x_\mu^{\text{class}}(\tau) \) obeys
\[
x_\mu^{\text{class}}(\tau) = 2igF_{\mu\nu} \dot{x}_\nu
\] (3.64)

and it satisfies the boundary conditions \( x^{\text{class}}(0) = x_1 \) and \( x^{\text{class}}(T) = x_2 \). The fluctuation \( y(\tau) \) satisfies correspondingly \( y(0) = 0 = y(T) \). Hence the fluctuation determinant is almost the same as in the periodic case, the only difference being that there is no zero-mode integration in the present case:
\[
K_{\mu\nu}(x_2, T|x_1, 0) = \exp(2igTF)_{\mu\nu} \exp(-S[x^{\text{class}}]) \times \det' \left[ -\partial^2_\tau \delta_{\mu\nu} + 2igF_{\mu\nu} \partial_\tau \right]^{-1/2}.
\] (3.65)

The classical trajectory is easily found:
\[
x^{\text{class}}(\tau) = x_1 + \frac{\exp(2igF\tau) - 1}{\exp(2igFT) - 1} (x_2 - x_1).
\] (3.66)

Its action is entirely due to the surface terms in (3.62). In Fock-Schwinger gauge centered at \( x_1 \) it reads
\[
S[x^{\text{class}}] = \frac{1}{4} (x_2 - x_1)gF \cot(gTF)(x_2 - x_1).
\] (3.67)

Putting everything together we obtain the final result for the propagation kernel:
\[
K_{\mu\nu}(x_2, T|x_1, 0) = (4\pi T)^{-D/2} \exp[2igTF]_{\mu\nu} \times \exp \left[ -\frac{1}{4} (x_2 - x_1)gF \cot(gTF)(x_2 - x_1) \right] \times \exp \left[ -\frac{1}{2} \text{tr}_L \ln \frac{\sin(gTF)}{(gTF)} \right].
\] (3.68)
4. A Multiloop Generalization for Quantum Electrodynamics

We return to the case of quantum electrodynamics, and proceed to the multiloop generalization of the formalism developed in chapter 2. This generalization is constructed in strict analogy to the case without an external field [36], and we refer the reader to that publication for some of the details.

We first consider scalar electrodynamics at the two-loop level, i.e. a scalar loop with an internal photon correction. A photon insertion in the worldloop may, in Feynman gauge, be represented in terms of the following current-current interaction term (see e.g. [87, 89, 88]) inserted into the one-loop path integral,

\begin{equation}
\frac{-e^2}{2} \frac{\Gamma(\lambda)}{4\pi^{\lambda+1}} \int_0^T d\tau_a \int_0^T d\tau_b \frac{\dot{x}(\tau_a) \cdot \dot{x}(\tau_b)}{([x(\tau_a) - x(\tau_b)]^2)^{\lambda}}
\end{equation}

(\lambda = \frac{D}{2} - 1). The denominator of this term is again written in the proper-time representation,

\begin{equation}
\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}([x(\tau_a) - x(\tau_b)]^2)^{\lambda}} = \int_0^\infty d\bar{T}(4\pi \bar{T})^{-\frac{D}{2}} \exp \left[-\frac{([x(\tau_a) - x(\tau_b)]^2)}{4\bar{T}}\right].
\end{equation}

It appears then as yet another correction term to the free part of the worldline Lagrangian for the scalar loop path integral. It is convenient to rewrite this term in the form

\begin{equation}
(x(\tau_a) - x(\tau_b))^2 = \int_0^T d\tau_1 \int_0^T d\tau_2 x(\tau_1)B_{ab}(\tau_1, \tau_2)x(\tau_2)
\end{equation}

with

\begin{equation}
B_{ab}(\tau_1, \tau_2) = \left[\delta(\tau_1 - \tau_a) - \delta(\tau_1 - \tau_b)\right]\left[\delta(\tau_2 - \tau_a) - \delta(\tau_2 - \tau_b)\right].
\end{equation}

This allows us to write the new total bosonic kinetic operator as

\begin{equation}
\partial^2 - 2ieF\partial - \frac{B}{T}
\end{equation}

To find the inverse of this operator, we write it as a geometric series,

\begin{equation}
\left(\partial^2 - 2ieF\partial - \frac{B}{T}\right)^{-1} = \left(\partial^2 - 2ieF\partial\right)^{-1} + \left(\partial^2 - 2ieF\partial\right)^{-1}\frac{B_{ab}}{T}\left(\partial^2 - 2ieF\partial\right)^{-1} + \cdots,
\end{equation}

which can be easily summed to yield the following Green's function:

\begin{equation}
G_B^{(1)}(\tau_1, \tau_2) = G_B(\tau_1, \tau_2) + \frac{1}{2} \frac{[G_B(\tau_1, \tau_a) - G_B(\tau_1, \tau_b)][G_B(\tau_a, \tau_2) - G_B(\tau_b, \tau_2)]}{T - \frac{1}{2}C_{ab}}
\end{equation}

with the definition

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\[ C_{ab} \equiv G_B(\tau_a, \tau_a) - G_B(\tau_a, \tau_b) - G_B(\tau_b, \tau_a) + G_B(\tau_b, \tau_b) \]
\[ = \frac{\cos(eFT) - \cos(eFT\hat{G}_{bab})}{(eFT)\sin(eFT)}. \]

(4.8)

Note that this is almost identical with what one would obtain from the ordinary bosonic two-loop Green’s function \( G_B^{(1)} \) by simply replacing all \( G_{Bi} \)'s appearing there by the corresponding \( G_{Bij} \)'s. The more complicated structure of the denominator is due to the fact that the \( G_{Bij} \)'s are not symmetric anymore, rather we have \( G_{Bij} = G_{Bji}^T \), and moreover have nonvanishing coincidence limits. The denominator is now in general a nontrivial Lorentz matrix, and must be interpreted as a matrix inverse (of course, all matrices appearing here commute with each other).

For the free Gaussian path integral, it is again a simple application of the \( \ln \det = \text{tr} \log \) identity to calculate

\[
\text{Det}' \left[ -\partial^2 + 2ieF\partial + \frac{B}{T} \right] = \text{Det}' \left[ -\partial^2 \right] \text{Det}' \left[ I - 2ieF\partial^{-1} \right] \\
\times \text{Det}' \left[ I - \frac{B}{T} \left( \partial^2 - 2ieF\partial \right)^{-1} \right] \\
= [4\pi T]^D \det \left[ \frac{\sin(eFT)}{eFT} \right] \det \left[ I - \frac{1}{2T}C_{ab} \right].
\]

(4.9)

The generalization from one-loop to two-loop photon amplitude calculations in scalar QED thus requires no changes of the formalism itself, but only of the Green’s functions used, and of the global determinant factor. Of course, in the end three more parameter integrations have to be performed.

As in the case without a background field \([36]\), the whole procedure goes through essentially unchanged for the fermion loop, if the superfield formalism is used. As a consequence, the supersymmetrization property carries over to the two-loop level, leading to a close relationship between the parameter integrals for the same amplitude calculated for the scalar and for the fermion loop: They differ only by a replacement of all \( G_B \)'s by \( \hat{G} \)'s, by the additional \( \theta \) – integrations, and by the one-loop Grassmann path integral factor eq.(2.35).

The generalization to an arbitrary fixed number of photon insertions is straightforward, and leads to formulas for the generalized (super-) Green’s functions and (super-) determinants identical with the ones given in \([34, 36]\) up to a replacement of all \( G_B \)'s (\( \hat{G} \)'s) by \( G_{Bij} \)'s (\( \hat{G}_{Bij} \)'s). The only point to be mentioned here is that care must now be taken in writing the indices of the \( G_{Bij} \)'s appearing. For instance, the bosonic three-loop Green’s function must be written
\[ G_B^{(2)}(\tau_1, \tau_2) = \mathcal{G}_B(\tau_1, \tau_2) + \frac{1}{2} \sum_{k,l=1}^{2} \left[ \mathcal{G}_B(\tau_1, \tau_{a_k}) - \mathcal{G}_B(\tau_1, \tau_{b_k}) \right] A^{(2)}_{kl} \left[ \mathcal{G}_B(\tau_{a_l}, \tau_2) - \mathcal{G}_B(\tau_{b_l}, \tau_2) \right]. \] (4.10)

The matrix \( A \) appearing here is the inverse of the matrix

\[
\begin{pmatrix}
T_1 - \frac{1}{2} (\mathcal{G}_{Ba_1a_1} - \mathcal{G}_{Ba_1b_1} - \mathcal{G}_{Bb_1a_1} + \mathcal{G}_{Bb_1b_1}) & -\frac{1}{2} (\mathcal{G}_{Ba_1a_2} - \mathcal{G}_{Ba_1b_2} - \mathcal{G}_{Bb_1a_2} + \mathcal{G}_{Bb_1b_2}) \\
-\frac{1}{2} (\mathcal{G}_{Ba_2a_1} - \mathcal{G}_{Ba_2b_1} - \mathcal{G}_{Bb_2a_1} + \mathcal{G}_{Bb_2b_1}) & T_2 - \frac{1}{2} (\mathcal{G}_{Ba_2a_2} - \mathcal{G}_{Ba_2b_2} - \mathcal{G}_{Bb_2a_2} + \mathcal{G}_{Bb_2b_2})
\end{pmatrix}
\] (4.11)

\( T_1, T_2 \) denote the proper-time lengths of the two inserted propagators.

The discussion of the general case of an arbitrary number of scalar (spinor) loops interconnected by photon propagators requires no new concepts, and will be deferred to a forthcoming review article [90].

While this multiloop construction is done most simply using the Feynman gauge for the propagator insertions, other gauges can be implemented as well (the gauge freedom has also been discussed in [37]). In an arbitrary covariant gauge, the photon insertion term eq. (4.1) would read

\[
-\frac{e^2}{2} \frac{1}{4\pi^2} \int_0^T d\tau_a \int_0^T d\tau_b \left\{ \frac{1 + \alpha}{2} \Gamma \left( \frac{D}{2} - 1 \right) \frac{\dot{x}_a \cdot \dot{x}_b}{\left[ (x_a - x_b)^2 \right]^{\frac{D}{2} - 1}} + (1 - \alpha) \Gamma \left( \frac{D}{2} \right) \frac{\dot{x}_a \cdot (x_a - x_b)(x_a - x_b) \cdot \dot{x}_b}{\left[ (x_a - x_b)^2 \right]^{\frac{D}{2}}} \right\}.
\] (4.12)

Here \( \alpha = 1 \) corresponds to Feynman gauge, \( \alpha = 0 \) to Landau gauge. The integrand may also be written as

\[
\Gamma \left( \frac{D}{2} - 1 \right) \frac{\dot{x}_a \cdot \dot{x}_b}{\left[ (x_a - x_b)^2 \right]^{\frac{D}{2} - 1}} - \frac{1 - \alpha}{4} \Gamma \left( \frac{D}{2} - 2 \right) \frac{\partial}{\partial \tau_a} \frac{\partial}{\partial \tau_b} \left[ (x_a - x_b)^2 \right]^{2 - \frac{D}{2}}. \] (4.13)

This shows that on the worldline gauge transformations correspond to the addition of total derivative terms, which is another fact familiar to string theorists (see e.g. [91]). This form of the photon insertion is also the more practical one for actual calculations. Again it carries over to the fermion loop in the superfield formalism \textit{mutatis mutandis}.
5. The Two-Loop Euler-Heisenberg Lagrangian for Scalar QED

We proceed to the simplest two-loop application of this formalism, which is the two-loop generalization of Schwinger’s formula for the constant field effective Lagrangian due to a scalar loop, eq.(2.38). According to the above, we may write the two-loop correction to this effective Lagrangian in the form

\[
\mathcal{L}^{(2)}_{\text{scal}}[F] = (4\pi)^{-D} \left( -\frac{e^2}{2} \right) \int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \int_0^\infty \frac{d\bar{T}}{\bar{T}} \int_0^T d\tau_a \int_0^T d\tau_b \\
\times \det^{-\frac{1}{2}} \left[ \frac{\sin(eFT)}{eFT} \right] \det^{-\frac{1}{2}} \left[ \bar{T} - \frac{1}{2} C_{ab} \right] \langle \dot{y}_a \cdot \dot{y}_b \rangle .
\]

(5.1)

We have now a fourfold parameter integral, with \( T \) and \( \bar{T} \) representing the scalar and photon proper-times, and \( \tau_{a,b} \) the endpoints of the photon insertion moving around the scalar loop. The first determinant factor is identical with the one-loop Euler-Heisenberg-Schwinger integrand eq.(2.35), and represents the change of the free path integral determinant due to the external field; the second one represents its change due to the photon insertion. A single Wick contraction needs to be performed on the “left over” numerator of the photon insertion, using the modified worldline Green’s function eq.(4.7). This yields

\[
\langle \dot{y}_a \cdot \dot{y}_b \rangle = \text{tr} \left[ \dot{G}_{Bab} + \frac{1}{2} \left( \dot{G}_{Baa} - \dot{G}_{Bab} \right) \left( \dot{G}_{Bab} - \dot{G}_{Bbb} \right) \right] .
\]

(5.2)

Care must be taken again with coincidence limits, as the derivatives should not act on the variables \( \tau_a, \tau_b \) explicitly appearing in the two-loop Green’s function; again the correct rule in calculating \( \langle \dot{y}_a \dot{y}_b \rangle \) is to first differentiate eq.(4.7) with respect to \( \tau_1, \tau_2 \), and put \( \tau_1 = \tau_a, \tau_2 = \tau_b \) afterwards.

After replacing the \( \dot{G}_{Bij} \)’s and \( C_{ab} \) by the explicit expressions given in eqs.(2.23) and eq.(4.8), we have already a parameter integral representation for the bare dimensionally regularized effective Lagrangian.

Alternatively one may, in the spirit of the original Bern-Kosower approach, remove \( \dot{G}_B \) by a partial integration with respect to \( \tau_a \) or \( \tau_b \). Using the formula

\[
d \det(M) = \det(M) \text{tr}(dMM^{-1})
\]

and \( \dot{G}_{Bab} = -\dot{G}_{Bba}^T \), one obtains the equivalent parameter integral

\[
\mathcal{L}^{(2)}_{\text{scal}}[F] = (4\pi)^{-D} \left( -\frac{e^2}{2} \right) \int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \int_0^\infty \frac{d\bar{T}}{\bar{T}} \int_0^T d\tau_a \int_0^T d\tau_b \\
\times \det^{-\frac{1}{2}} \left[ \frac{\sin(eFT)}{eFT} \right] \det^{-\frac{1}{2}} \left[ \bar{T} - \frac{1}{2} C_{ab} \right] \\
\times \frac{1}{2} \left\{ \text{tr} \dot{G}_{Bab} \text{tr} \left[ \dot{G}_{Bab} \right] + \text{tr} \left[ \dot{G}_{Baa} - \dot{G}_{Bab} \right] \dot{G}_{Bab} \dot{G}_{Bbb} \right\} .
\]

(5.4)
For the further evaluation and renormalization of this Lagrangian, we will specialize the constant field $F$ to a pure magnetic and to a pure electric field in turns. To facilitate comparison with previous calculations [46, 47, 48, 49, 50], we will moreover switch from dimensional regularization to proper-time regularization. This means that henceforth we put $D = 4$, and instead introduce a proper-time UV cutoff $T_0$ later on.

We begin with a pure magnetic field. The field is taken along the z-axis, so that $F^{12} = B, F^{21} = -B$ are the only non-vanishing components of the field strength tensor. We also introduce the abbreviations $z = eBT$, and projection matrices

$$
\hat{\mathbf{F}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{I}_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
$$

We may then rewrite the determinant factors eqs.(2.34),(2.36) as

$$
\det -\frac{1}{2} \left[ \sin(eFT) \right] = \frac{z}{\sinh(z)}, \quad \det -\frac{1}{2} \left[ \tan(eFT) \right] = \frac{z}{\tanh(z)}.
$$

The Green’s functions eq.(2.19),(2.23) specialize to

$$
\bar{G}_{B}(\tau_1, \tau_2) = G_{B12}\mathbf{I}_{03} - \frac{T}{2z} \left[ \cosh(z\hat{G}_{B12}) - \cosh(z) \right] \mathbf{I}_{12} + \frac{T}{2z} \left( \frac{\sinh(z\hat{G}_{B12})}{\sinh(z)} - \hat{G}_{B12} \right) i\hat{\mathbf{F}}
$$

$$
\hat{G}_{B}(\tau_1, \tau_2) = \hat{G}_{B12}\mathbf{I}_{03} + \frac{\sinh(z\hat{G}_{B12})}{\sinh(z)} \mathbf{I}_{12} - \left( \frac{\cosh(z\hat{G}_{B12})}{\sinh(z)} - \frac{1}{z} \right) i\hat{\mathbf{F}}
$$

$$
\check{G}_{B}(\tau_1, \tau_2) = \check{G}_{B12}\mathbf{I}_{03} + 2 \left( \delta_{12} - \frac{z\cosh(z\hat{G}_{B12})}{T\sinh(z)} \right) \mathbf{I}_{12} + 2 \frac{z\sinh(z\hat{G}_{B12})}{T\sinh(z)} i\hat{\mathbf{F}}
$$

$$
G_{F}(\tau_1, \tau_2) = G_{F12}\mathbf{I}_{03} + G_{F12} \frac{\cosh(z\hat{G}_{B12})}{\cosh(z)} \mathbf{I}_{12} - G_{F12} \frac{\sinh(z\hat{G}_{B12})}{\cosh(z)} i\hat{\mathbf{F}}.
$$

In writing $G_{B}$ we have already subtracted its coincidence limit, which is indicated by the “bar”. $C_{ab}$ simplifies to
\[ C_{ab} = -2G_{Bab}I_{03} - 2G^z_{Bab}I_{12}, \]  

(5.9)

where we have defined

\[ G^z_{Bab} \equiv \frac{T}{2} \left[ \frac{\cosh(z) - \cosh(z\dot{G}_{ab})}{z \sinh(z)} \right] = G_{Bab} - \frac{1}{3T}G^2_{Bab}z^2 + O(z^4) \]  

(5.10)

We will also use the derivative of this expression,

\[ \dot{G}^z_{Bab} = \frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)} \]  

(5.11)

Similarly we can rewrite

\[ \det \left[ e^{\kappa FT} \left( \frac{\dot{G}_{Bab}}{T - \frac{1}{2}C_{ab}} \right) \right] = \frac{z}{\sinh(z)} \gamma \gamma^z \]

\[ \text{tr} \left[ \dot{G}_{Bab} \right] = 8\delta_{ab} - 4\frac{z \cosh(z\dot{G}_{Bab})}{\sinh(z)} \]

\[ \frac{1}{2} \text{tr} \dot{G}_{Bab} \text{tr} \left[ \frac{\dot{G}_{Bab}}{T - \frac{1}{2}C_{ab}} \right] = 2 \left[ \dot{G}_{Bab} + \frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)} \right] \left[ \dot{G}_{Bab} \gamma + \frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)} \gamma^z \right] \]

\[ \frac{1}{2} \text{tr} \left[ (\dot{G}_{aa} - \dot{G}_{ab})(\dot{G}_{ab} - \dot{G}_{bb}) \right] = -\gamma^z \frac{\sinh^2(z\dot{G}_{Bab}) + \left[ \cosh(z\dot{G}_{Bab}) - \cosh(z) \right]^2}{\sinh^2(z)} \]

\[ -\dot{G}^2_{Bab} \gamma \]  

(5.12)

with the abbreviations

\[ \gamma \equiv (\bar{T} + G_{Bab})^{-1}, \]

\[ \gamma^z \equiv (\bar{T} + G^z_{Bab})^{-1}. \]

We rescale to the unit circle, \( \tau_{a,b} = Tu_{a,b} \), and use translation invariance in \( \tau \) to set \( \tau_b = 0 \). We have then

\[ G_B(\tau_a, \tau_b) = TG_B(\tau_a, 0) = T(u_a - u_a^2), \]

\[ \dot{G}_B(\tau_a, \tau_b) = \dot{G}_B(\tau_a, 0) = 1 - 2u_a. \]

After performance of the \( \bar{T} \) – integration, which is finite and elementary, eq.(5.4) turns into
\[ \mathcal{L}_{\text{scal}}^{(2)}[B] = - (4\pi)^{-1} \frac{e^2}{2} \int_1^\infty \frac{dT}{T^3} e^{-m^2 T} \frac{z}{\sinh(z)} \int_0^1 du_a A(z, u_a), \quad (5.13) \]

with
\[
A = \left\{ A_1 \frac{\ln(G_{Bab}/G_{Bab}^2)}{(G_{Bab} - G_{Bab}^2)^2} + \frac{A_2}{(G_{Bab}^2)(G_{Bab} - G_{Bab}^2)} + \frac{A_3}{(G_{Bab})(G_{Bab} - G_{Bab}^2)} \right\},
\]
\[
A_1 = 4 \left[ G_{Bab}^z \coth(z) - G_{Bab} \right],
\]
\[
A_2 = 1 + 2 \dot{G}_{Bab} \dot{G}_{Bab}^z - 4 G_{Bab}^z \coth(z),
\]
\[
A_3 = -G_{Bab}^2 - 2 \dot{G}_{Bab} \dot{G}_{Bab}^z.
\quad (5.14)
\]

\(G_{Bab}^z\) is now given by eq.(5.10) with \(T = 1\). Here and in the following we often use the identity \(\dot{G}_{Bab}^2 = 1 - 4G_{Bab}\) to eliminate \(\dot{G}_{Bab}\) in favour of \(G_{Bab}\).

Renormalization must now be addressed, and will be performed in close analogy to the discussion in [50]. The integral in eq.(5.13) suffers from two kinds of divergences:

1. An overall divergence of the scalar proper-time integral \(\int_0^\infty dT\) at the lower integration limit.

2. Divergences of the \(\int_0^1 du_a\) parameter integral at the points 0, 1 where the endpoints of the photon propagator become coincident, \(u_a = u_b\).

The first one will be removed by one- and two-loop photon wave function renormalization, the second one by one-loop scalar mass renormalization. As is well known, vertex renormalization and scalar self energy renormalization cancel against each other in this type of calculation, and need not be considered.

By power counting, an overall divergence can exist only for the terms in the effective Lagrangian which are of order at most quadratic in the external field \(B\). Expanding the integrand of eq.(5.13), \(K(z, u_a) = \frac{z}{\sinh(z)} A(z, u_a)\), in the variable \(z\), we find
\[
K(z, u_a) = \left[ 3 - \frac{12}{G_{Bab}^2} - \frac{1}{2} \frac{1}{G_{Bab}^2} + \frac{1}{G_{Bab}} + 2 \right] z^2 + O(z^4) \quad (5.15)
\]
The complicated singularity appearing here at the point \(u_a = u_b\) indicates that this form of the parameter integral is not yet optimized for the purpose of renormalization. In particular, it shows a spurious singularity in the coefficient of the induced Maxwell term \(\sim z^2\). This comes not unexpected as the cancellation of subdivergences implied by the Ward identity has, in a general gauge, no reason to be manifest at the parameter integral level.

We could improve on this either by switching to Landau gauge, or by performing a suitable partial integration on the integrand. The latter procedure is less systematic, but easy enough to implement for the simple case at hand: Inspection of the two versions we have of this parameter integral, the original one eq.(5.1) and the partially integrated
one eq. (5.4), shows that we can optimize the integrand by choosing a certain linear combination of both versions, namely

\[ \mathcal{L}^{(2)}(B) = \frac{3}{4} \times \text{eq. (5.1)} + \frac{1}{4} \times \text{eq. (5.4)} \] (5.16)

After integration over \( \bar{T} \), this leads to another version of eq. (5.13),

\[ \mathcal{L}^{(2)}_{\text{scal}}(B) = -(4\pi)^{-4} \frac{e^2}{2} \int_0^\infty d\bar{T} \frac{\bar{T}^3 e^{-m^2\bar{T}}}{\sinh(z)} \int_0^1 du_a A'(z, u_a), \] (5.17)

with a different integrand

\[
\begin{align*}
A' &= \left\{ A_0 \ln \left( \frac{G_{Bab}/G_{Bab}^2}{(G_{Bab} - G_{Bab}^2)} \right) + A_1 \ln \left( \frac{G_{Bab}/G_{Bab}^2}{(G_{Bab} - G_{Bab}^2)^2} \right) \\
&\quad + \frac{A_2'}{(G_{Bab})^2} + \frac{A_3'}{(G_{Bab})(G_{Bab} - G_{Bab}^2)} \right\}, \\
A_0' &= 3 \left[ 2z^2 G_{Bab}^2 - \frac{z}{\tanh(z)} - 1 \right], \\
A_1' &= A_1 - \frac{3}{2} \left[ \dot{G}_{Bab}^2 - \dot{G}_{Bab}^2 G_{Bab} + \dot{G}_{Bab}^2 G_{Bab} \right], \\
A_2' &= A_2 - \frac{3}{2} \left[ \dot{G}_{Bab}^2 G_{Bab} + \dot{G}_{Bab}^2 G_{Bab} \right], \\
A_3' &= A_3 + \frac{3}{2} \left[ \dot{G}_{Bab}^2 G_{Bab} + \dot{G}_{Bab}^2 G_{Bab} \right].
\end{align*}
\] (5.18)

We have not yet taken into account here the term involving \( \delta_{ab} \), stemming from \( \dot{G}_{Bab} \), which was contained in the integrand of eq. (5.17). This term corresponds, in diagrammatic terms, to a tadpole insertion, and could therefore be safely deleted. However, it will be quite instructive to keep it and check explicitly that it is taken care of by the renormalization procedure. It leads to an integral \( \int_0^\infty d\bar{T} \frac{\bar{T}^2}{T^2} \) which we regulate by introducing an UV cutoff for the photon proper-time,

\[ \int_0^\infty d\bar{T} \frac{\bar{T}^2}{T^2} = \frac{1}{T_0}. \] (5.19)

It gives then a further contribution \( E(\bar{T}_0) \) to \( \mathcal{L}^{(2)}_{\text{scal}}(B) \),

\[ E(\bar{T}_0) = -3(4\pi)^{-4} \frac{e^2}{2} \frac{1}{T_0} \int_0^\infty d\bar{T} \frac{\bar{T}^3 e^{-m^2\bar{T}}}{\sinh(z)} \int_0^1 du_a A'(z, u_a), \] (5.20)

Expanding the new integrand, \( K'(z, u_a) \equiv \frac{1}{\sinh(z)} A'(z, u_a) \), in \( z \), we find the simple result

\[ K'(z, u_a) = -6 \frac{1}{G_{Bab}} + 3z^2 + O(z^4) \] (5.21)

In particular, the absence of a subdivergence for the Maxwell term is now manifest.
We delete the irrelevant constant term, and add and subtract the Maxwell term. Defining
\[ K_{02}(z, u_a) = -6 \frac{1}{G_{Bab}} + 3z^2, \] (5.22)
the Lagrangian then becomes
\[
\mathcal{L}_{\text{scal}}^{(2)}[B] = E(\bar{T}_0) - \frac{\alpha}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} 3z^2 \\
- \frac{\alpha}{2(4\pi)^3} \int_0^1 du_a \left[ K'(z, u_a) - K_{02}(z, u_a) \right].
\] (5.23)
The second term, which we denote by \( F \), is divergent when integrated over the scalar proper-time \( T \). We regulate it by introducing another proper-time cutoff \( T_0 \) for the scalar proper-time integral:
\[
F(T_0) := -\frac{\alpha}{2(4\pi)^3} \int_{2T_0}^\infty \frac{dT}{T^3} e^{-m^2 T} 3z^2
\] (5.24)
(we use \( 2T_0 \) rather than \( T_0 \) for easier comparison with \[ 50 \]). The third term is convergent at \( T = 0 \), but still has a divergence at \( u_a = u_b \), as it contains negative powers of \( G_{Bab} \).

Expanding the integrand in a Laurent series in \( G_{Bab} \), one finds
\[
K'(z, u_a) - K_{02}(z, u_a) = \frac{f(z)}{G_{Bab}} + O(G_{Bab}^0),
\]
\[
f(z) = 3 \left[ 2 - \frac{z}{\sinh(z)} - \frac{z^2 \cosh(z)}{\sinh(z)^2} \right].
\] (5.25)

Again the singular part of this expansion is added and subtracted, yielding
\[
\mathcal{L}_{\text{scal}}^{(2)}[B] = E(\bar{T}_0) + F(T_0) - \frac{\alpha}{2(4\pi)^3} \int_{2T_0}^\infty \frac{dT}{T^3} e^{-m^2 T} \int_0^1 du_a f(z) G_{Bab} \left[ K'(z, u_a) - K_{02}(z, u_a) \right],
\] (5.26)
The last integral is now completely finite. The third term, which we call \( G(T_0) \), is finite at \( T = 0 \), as \( f(z) = O(z^4) \) by construction. Here we have introduced \( T_0 \) for the purpose of regulating the divergence at \( u_a = u_b \). The \( u_a \)– integral for this term is then readily computed and yields, in the limit \( T_0 \to 0 \), a contribution
\[
\int_0^{1-T_0/T} du_a \frac{1}{G_{Bab}} = -2\ln \left( \frac{T_0}{T} \right) = -2\ln(\gamma m^2 T_0) + 2\ln(\gamma m^2 T).
\] (5.27)
We have rewritten this term for reasons which will become apparent in a moment (\(\ln(\gamma)\) denotes the Euler-Mascheroni constant). Next note that we can relate the function \(f(z)\) to the scalar one-loop Euler-Heisenberg Lagrangian, eq. (2.38). If we write this Lagrangian for the pure magnetic field case, and subtract the two divergent terms lowest order in \(z\), we obtain

\[
\bar{\mathcal{L}}_{\text{scal}}^{(1)}[\mathcal{B}] = \frac{1}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \tag{5.28}
\]

On the other hand, we can write

\[
f(z) = 3 \left[ 2 - \frac{z}{\sinh(z)} - \frac{z^2 \cosh(z)}{\sinh(z)^2} \right]
\]

\[
= 3T^3 \frac{d}{dT} \left\{ \frac{1}{T^2} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \right\}
\]  

\[
\tag{5.29}
\]

By a partial integration over \(T\), we can therefore reexpress

\[
\frac{1}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} f(z) = 3 \frac{m^2}{(4\pi)^2} \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right]
\]

\[
= -3m^2 \frac{\partial}{\partial m^2} \bar{\mathcal{L}}_{\text{scal}}^{(1)}[\mathcal{B}] \tag{5.30}
\]

To proceed, we need the value of the one-loop mass displacement in scalar QED, computed in the proper-time regularization. This we borrow from [36]:

\[
\delta m^2 = \frac{3\alpha}{4\pi} m^2 \left[ -\ln(\gamma m^2 T_0) + \frac{7}{6} + \frac{1}{m^2 T_0} \right]. \tag{5.31}
\]

Using this result, we may rewrite

\[
G(T_0) = \left[ \delta m^2 - \frac{7}{2} \frac{\alpha}{4\pi} m^2 - 3 \frac{\alpha}{4\pi T_0} \right] \frac{\partial}{\partial m^2} \bar{\mathcal{L}}_{\text{scal}}^{(1)}[\mathcal{B}]
\]

\[
- \frac{\alpha}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \ln(\gamma m^2 T) f(z).
\]

\[
\tag{5.32}
\]

As expected the \(\frac{1}{T_0}\) term introduced by the one-loop mass renormalization cancels the tadpole term \(E(T_0)\), up to its constant and Maxwell parts. Moreover, the remaining divergence of \(G(T_0)\) for \(T_0 \to 0\) has been absorbed by \(\delta m^2\).

\footnote{Note that this differs by a sign from \(\delta m^2\) as used in [36]. Here this denotes the mass displacement itself, there the corresponding counterterm.}
Putting all pieces together, we can write the complete two-loop approximation to the effective Lagrangian in the following way:

\[
\mathcal{L}^{(\leq 2)} [B_0] = -\frac{1}{2} B_0^2 - \frac{1}{(4\pi)^2} \int_{T_0}^\infty \frac{dT}{T^3} e^{-m_0^2 T} \frac{z^2}{6} + \mathcal{L}^{(1)}_{\text{scal}} [B_0] + \delta m_0^2 \frac{\partial}{\partial m_0^2} \mathcal{L}^{(1)}_{\text{scal}} [B_0] + \frac{7 \alpha_0}{24\pi^2} m_0^2 \frac{\partial}{\partial m_0^2} \mathcal{L}^{(1)}_{\text{scal}} [B_0] - \frac{\alpha_0}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \ln(\gamma m_0^2 T) f(z) \\
- \frac{\alpha_0}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \int_0^1 \frac{du}{u} \left[ K'(z, u) - K_{02}(z, u) - \frac{f(z)}{G_{Bab}} \right] \\
- \frac{\alpha_0}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \frac{z^2}{2} \left[ 1 - \frac{T}{T_0} \right] 
\]  

(5.33)

We have rewritten this Lagrangian in bare quantities, since up to now we have been working in the bare regularized theory. Only mass and photon wave function renormalization are required to make this effective Lagrangian finite:

\[
m^2 = m_0^2 + \delta m_0^2, \\
e = e_0 Z_3^\frac{1}{4}, \\
B = B_0 Z_3^\frac{1}{4}. 
\]

(5.34)

Here \(\delta m_0^2\) has already been introduced in eq. (5.31), while \(Z_3\) is chosen such as to absorb the diverging one- and two-loop Maxwell terms in eq. (5.33). Note that this leaves \(z = e_0 B_0 T\) unaffected. The final answer becomes [1]

\[
\mathcal{L}^{(\leq 2)} [B] = -\frac{1}{2} B^2 + \frac{1}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \\
+ \frac{7 \alpha}{2(4\pi)^3} m_0^2 \int_0^\infty \frac{dT}{T^2} e^{-m_0^2 T} \left[ \frac{z}{\sinh(z)} + \frac{z^2}{6} - 1 \right] \\
- \frac{\alpha_0}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \int_0^1 \frac{du}{u} \left[ K'(z, u) - K_{02}(z, u) - \frac{f(z)}{G_{Bab}} \right] \\
- \frac{\alpha_0}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m_0^2 T} \ln(\gamma m_0^2 T) f(z) 
\]  

(5.35)

\[ ^3 \text{Note added in proof: The constant } \frac{7}{4} \text{ multiplying the third term is incorrect, and should be replaced by } \frac{7}{2}. \text{ This has now been established both by a detailed comparison with [1], and another recalculation using dimensional regularization [2].} \]
This parameter integral representation is of a similar but simpler structure than the one given by Ritus [46].

The corresponding result for the case of a pure electric field is obtained from it by the simple substitution (in Minkowski space)

\[ B \rightarrow -iE. \]  

This makes an important and well-known difference. In the electric field case the \( T \) – integration acquires new divergences due to the appearance of poles. This leads to an imaginary part of the effective action, and to a probability for electron-positron pair creation. At the one-loop level, the first such pole becomes significant at the critical field strength

\[ E_{cr} = \frac{m^2}{e} \]  

(see eq. (2.38)). The two-loop contribution to the effective Lagrangian affects also the imaginary part and the pair creation probability. A detailed investigation of those corrections has been undertaken in [47, 48].

The calculation for the case of a generic constant field would be only moderately more difficult, if one uses the Lorentz frame where the magnetic and electric fields are parallel, as in [46].

### 6. The Two-Loop Euler-Heisenberg Lagrangian for Spinor QED

The corresponding calculation for the spinor loop case proceeds in complete analogy when formulated in the superfield formalism. This allows us to immediately write down the analogue of eqs. (5.1), (5.2):

\[
\mathcal{L}_{\text{spin}}^{(2)}[F] = (-2)(4\pi)^{-D} \left(-\frac{e^2}{2}\right) \int_0^\infty \frac{dT}{T} e^{-m^2 T} T^{-\frac{D}{2}} \int_0^T d\tau_a d\tau_b \int d\theta_a d\theta_b \\
\times \text{det}^{-\frac{1}{2}} \left[ \frac{\tan(eFT)}{eFT} \right] \text{det}^{-\frac{1}{2}} \left[ \frac{1}{2} \hat{C}_{ab} \right] \langle -D_a y_a \cdot D_b y_b \rangle ,
\]

(6.1)

with a superfield Wick contraction

\[
\langle -D_a y_a \cdot D_b y_b \rangle = \text{tr} \left[ D_a D_b \hat{G}_{ab} + \frac{1}{2} \frac{D_a (\hat{G}_{aa} - \hat{G}_{ab}) D_b (\hat{G}_{ab} - \hat{G}_{bb})}{T - \frac{1}{2} \hat{C}_{ab}} \right].
\]

(6.2)

The notations should be obvious.

Performing the Grassmann integrations \( \int d\theta_a \int d\theta_b \), and removing \( \hat{G}_{Bab} \) by partial integration as before, we obtain the equivalent of eq. (5.4),
In writing this formula, we have used the symmetry between \( \tau_a \) and \( \tau_b \) to reduce the number of terms. The same expressions could have been obtained starting from eq.(5.4) and using the generalized one-loop substitution rule.

Specializing to the pure magnetic case and \( D = 4 \), it is then again a matter of simple algebra to calculate the traces and \( \bar{T} \) – integrals. After rescaling and setting \( \tau_b = 0 \), the result can be written as

\[
\mathcal{L}_{\text{spin}}^{(2)}[\mathcal{F}] = \frac{\alpha}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2T - \frac{z}{\tanh(z)}} \int_0^1 d\bar{u}_a B(z, \bar{u}_a),
\]

with

\[
B(z, \bar{u}_a) = \left\{ \begin{array}{c}
B_1 \ln \left( \frac{G_{\bar{B}a b}/G_{\bar{B}a b}^z}{G_{\bar{B}a b} - G_{\bar{B}a b}^z} \right) + \frac{B_2}{G_{\bar{B}a b}^z (G_{Bab} - G_{Bab}^z) + B_3} \\
B_2 = A_2 + 8z \tanh(z) G_{\bar{B}a b}^z - 3 \\
B_3 = A_3 - 4z \tanh(z) G_{\bar{B}a b}^z + 3
\end{array} \right\},
\]

Comparison with an earlier field theory calculation performed by Dittrich and one of the authors [50] shows that the integrand of eq. (6.4) allows for a direct identification with its counterpart there, as given in eqs. (7.21),(7.22). This requires nothing more than a rotation to Minkowskian proper-time, \( T \to \bar{u}_a \), a transformation of variables from \( u_a \) to \( v := G_{\bar{B}a b} \), and the use of trigonometric identities. In particular, our quantities \( G_{Bab}, G_{Bab}^z \) then identify with the quantities \( a, b \) there.

The renormalization of this Lagrangian has, for the spinor-loop case, been carried through in detail in that work. We will therefore not repeat this analysis here, and just give the final result for the renormalized two-loop contribution to the Euler-Heisenberg Lagrangian \([1]\):

\[\text{Note added in proof: The constant } -10 \text{ multiplying the third term is incorrect, and should be replaced by } -18 \text{ [2]} \text{ (compare the footnote before eq.(5.35)).} \]
\[ \mathcal{L}_{\text{spin}}^{(\leq 2)}[B] = -\frac{1}{2}B^2 - \frac{2}{(4\pi)^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[ \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right] \]

\[ -10m^2 \alpha \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \left[ \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right] \]

\[ + \frac{2\alpha}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \ln(\gamma m^2 T) g(z) \]

\[ + \frac{\alpha}{(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \int_0^1 du_a \left[ L(z, u_a) - L_{02}(z, u_a) - \frac{g(z)}{G_{Bab}} \right] \]

(6.6)

with

\[ L(z, u_a) = \frac{z}{\tanh(z)} B(z, u_a), \]

\[ L_{02}(z, u_a) = -\frac{12}{G_{Bab}} + 2z^2, \]

\[ g(z) = -6 \left[ \frac{z^2}{\sinh(z)^2} + z \coth(z) - 2 \right]. \]

(6.7)

For a study of the strong field limit of this Lagrangian see again [50].

The first exact calculation of this two-loop Lagrangian for the fermion loop case is again due to Ritus [49], who also used proper-time methods to arrive at a certain two-parameter integral.

The integral representation given above is equivalent to the one given by Ritus, but simpler. In [50] it was obtained by convoluting a free photon propagator with the polarization tensor of a fermion in a constant magnetic field. The essential part of this calculation consists of deriving a compact integral representation for the polarization tensor. To this end, complicated expressions involving Dirac traces and momentum integrals have to be evaluated. In the string-inspired case, no analogous manipulations are needed, and the computational effort for doing the parameter integrals which it introduces instead is much smaller.

Moreover, when applied to spinor QED, the method of the present paper yields the corresponding result for scalar QED with almost no further effort. This would not be the case for the standard field theory techniques.

Concerning the physical relevance of this type of calculation, let us mention the experiment PVLAS in preparation at Legnaro, Italy, which is an optical experiment designed to yield the first experimental measurement of the Euler-Heisenberg Lagrangian [93, 94]. It is conceivable that the technology used there may even allow for the measurement of the two-loop correction in the near future [94, 95].
7. The 2-Loop QED $\beta$ – Functions Revisited

Finally, let us remark that the method we have employed in this paper for the calculation of the full two-loop Euler-Heisenberg Lagrangians also improves on the calculation of the two-loop QED $\beta$ – functions as it had been presented in [36]. For the extraction of the $\beta$ – function coefficients one needs only to calculate the induced Maxwell terms. Up to the contributions from one-loop mass renormalization, the correct two-loop scalar [96] and spinor [97] QED coefficients can thus be read off from the expansions (see eq.(5.21), eq.(6.7))

$$K'(z,u_a) = -\frac{6}{G_{Bab}} + 3z^2 + O(z^4),$$

$$L(z,u_a) = -\frac{12}{G_{Bab}} + 2z^2 + O(z^4).$$

Comparing with [36] we see that the use of the generalized Green’s functions $G_B, G_F$ has saved us two integrations: The same formulas eq.(B.2) which there had been employed for executing the integrations over the points of interaction $\tau_1, \tau_2$ with the external field, have now entered already at the level of the construction of those Green’s functions. Of course, for the $\beta$ – function calculation all terms of order higher than $O(F^2)$ are irrelevant, so that one could then as well use the truncations of those Green’s functions given in eq.(2.24). Moreover, one would choose an external field with the property $F^2 \sim 1$.

Note that in the fermion loop case a subdivergence-free integrand was obtained proceeding directly from the partially integrated version eq.(5.3). This fact, which had already been noticed in [39], is not accidental, and can be understood by an analysis of the quadratic divergences. In the scalar QED case, there are three possible sources of quadratic divergences for the induced Maxwell term:

1. The contact term containing $\delta_{ab}$.
2. The leading order term $\sim \frac{1}{G_{Bab}}z^2$ in the $\frac{1}{G_{Bab}}$ – expansion of the main term (see e.g. eq.(5.13)).
3. The explicit $1/T_0$ appearing in the one–loop mass displacement eq.(5.31).

The last one should cancel the other two in the renormalization procedure, if those are regulated by the same UV cutoff $T_0$ for the photon proper-time, and this was verified in various versions of this calculation. In the spinor QED case the fermion propagator has no quadratic divergence (this is, of course, manifest in the first order formalism, while in the second order formalism there are various diagrams contributing to the one–loop fermion self energy, and the absence of a quadratic divergence is due to a cancellation among them). The third term is thus missing, and the other two have to cancel among themselves. In particular, the completely partially integrated version of the integrand has no $\delta_{ab}$ – term any more, and consequently the second term must also be absent. However
the $\frac{1}{G_{Bab}}$ – expansion of the main contribution to the Maxwell term is, if one does this
calculation in $D = 4$, always of the form shown in eq.(7.13).

$$\left[ \frac{A}{G_{Bab}^2} + \frac{B}{G_{Bab}} + C \right] \text{tr}(F^2)$$

(7.2)

with coefficients $A, B, C$. In the partially integrated version first consideration of the
quadratic subdivergence allows one to conclude that $A = 0$, and then consideration of the
logarithmic subdivergence that $B = 0$.

Note that this argument does not apply to the scalar QED case, nor does it to spinor
QED in dimensional renormalization, due to the suppression of quadratic divergences by
that scheme. In both cases one would have only one constraint equation for the two
coefficients $A$ and $B$ appearing in the partially integrated integrand, and indeed they
turn out to be nonzero in both cases. In the present formalism, the fermion QED two-
loop $\beta$ – function calculation thus becomes simpler when performed not in dimensional
regularization, but in some four-dimensional scheme such as proper-time or Pauli-Villars
regularization.

The reader may rightfully ask why we have gone to such lengths in analyzing this
2-loop calculation, which is easy to do by modern standards even in field theory. We
find this cancellation mechanism interesting in view of some facts known about the three-
loop fermion QED $\beta$ – function [98, 99, 100]. Apart from the well-known cancellation
of transcendentals occurring between diagrams in the calculation of the quenched (one
fermion loop) contribution to this $\beta$ – function [98, 100], which takes place in any scheme
and gauge, even more spectacular cancellations were found in [99] where this calculation
was performed in four dimensions, Pauli-Villars regularization, and Feynman gauge. In
that calculation all contributions from nonplanar diagrams happened to cancel out exactly.
A recalculation of this coefficient in the present formalism is currently being undertaken
[101].

8. Discussion

In this paper, we have extended previous work of two of the authors on the multiloop
generalization of the string-inspired technique to the case of quantum electrodynamics in a
constant external field. The resulting formalism has been tested on a recalculation of the
two-loop corrections to the Euler-Heisenberg Lagrangian for quantum electrodynamics.
Several advantages of this calculus over standard field theory methods have been pointed
out. In particular, it treats the scalar and spinor loop cases on the same footing, so that
the scalar loop results are always obtained as a byproduct of the corresponding spinor loop
calculations. More technically, our parameter integrals are written in a form convenient
for partial integrations. In particular, the integrands are functions well-defined on the
circle, so that boundary terms do not appear. The usefulness of this property has been
demonstrated in the renormalization of the scalar QED two-loop Lagrangian.

An application to a recalculation of the one-loop QED photon splitting amplitude has
been given separately [102].

We have derived a path integral representation of the gluon loop, and used it for a
recalculation of what is the closest analogon to the one-loop Euler-Heisenberg Lagragian in Yang-Mills theory. More significantly, the analysis of chapter 3 should be viewed as a first step towards an extension of the worldline technique to multiloop calculations in nonablian gauge theory. Our derivation of this path integral is entirely non-heuristic, and thus guaranteed to reproduce the correct one-loop off-shell amplitudes for Yang-Mills theory. From our experience with quantum electrodynamics, this property alone makes us optimistic about the existence of a multiloop generalization of the worldline method for the Yang-Mills case. We hope to have more to say about this in the future.

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Appendix A: Path Integral Representation of the Electron Propagator

Different from the case of the closed loop fermionic worldline Lagrangian eq. (2.4), for the Dirac propagator besides the einbein field gauged to $T$ one has to introduce a gravitino field $\chi$ in the world line action. This can be gauged to a constant, but not to zero. For massive fermions a further field $\psi_5(\tau)$ coupling to mass has to be introduced \[13, 14, 19\] (its supersymmetric partner $x_5$ is not needed for the gauge coupling, but essential for the worldline implementation of Yukawa couplings \[25\]).

Integration over $\chi$ in the path integral leads to a factor $(-\frac{1}{2}\psi_\mu \dot{x}_\mu + m\psi_5)$ corresponding to the numerator of the Dirac propagator. This can be demonstrated nicely for the free propagator \[55, 56, 27\] in the coherent state formalism. This “holomorphic representation” represents operators in terms of their (fermionic) Wick symbols.

In the case with background interaction considered here we prefer a different approach in which fermionic operators are represented by their Weyl symbols. This formalism is manifestly covariant and, contrary to the holomorphic representation, it treats propagators in external fields and one-loop effective actions on the same footing. From a canonical point of view we are dealing with the following algebra of hermitian operators $\hat{\psi}_\mu$ and $\hat{\psi}_5$: \[\hat{\psi}_\mu \hat{\psi}_\nu + \hat{\psi}_\nu \hat{\psi}_\mu = \delta_{\mu\nu}, \quad \hat{\psi}_\mu \hat{\psi}_5 + \hat{\psi}_5 \hat{\psi}_\mu = 0, \quad \psi_5^2 = \frac{1}{2} \] (A.1)

In terms of euclidean Dirac matrices, it can be represented by \[\hat{\psi}_\mu = i \frac{\sqrt{2}}{\sqrt{2}} \gamma_5 \gamma_\mu, \quad \hat{\psi}_5 = \frac{1}{\sqrt{2}} \gamma_5 \] (A.2)

The Weyl symbol map “symb” establishes a linear one-to-one map between operators and functions of the anticommuting c-numbers $\xi_\mu$ and $\xi_5$. In particular, symb($\hat{\psi}_\mu$) $= \xi_\mu$ and symb($\hat{\psi}_5$) $= \xi_5$. The inverse symbol map associates the Weyl-ordered (totally antisymmetrized) operator product to strings of $\xi$’s: symb$^{-1}(\xi_\mu \xi_\nu ... \xi_\rho) = \{ \hat{\psi}_\mu \hat{\psi}_\nu ... \hat{\psi}_\rho \}_\text{Weyl}$ (A.3)

For example, symb$^{-1}(\xi_\mu \xi_\nu) = \frac{1}{2}(\hat{\psi}_\mu \hat{\psi}_\nu - \hat{\psi}_\nu \hat{\psi}_\mu) = \hat{\psi}_\mu \hat{\psi}_\nu - \frac{1}{2} \delta_{\mu\nu}$ (A.4)

and symb($\hat{\psi}_\mu \hat{\psi}_\nu$) $= \xi_\mu \xi_\nu + \frac{1}{2} \delta_{\mu\nu}$. (See ref. [103,104] for further details.)

Let us consider the Dirac propagator in the background of an arbitrary abelian gauge field and let us write down a path integral representation for its kernel (bosonic variables)/symbol (fermionic variables)

\[G^{\text{Dirac}}(x_2, x_1; \xi) = \text{symb}\left[\langle x_2 | (D + m)^{-1} | x_1 \rangle\right](\xi) \] (A.5)

with $D = \gamma_\mu D_\mu = 2i\hat{\psi}_\mu \hat{\psi}_5 D_\mu$. After having integrated out the auxiliary fields $\chi$ and $x_5$ it reads [13,27,104], up to an overall constant:
\[ G^{\text{Dirac}}(x_2, x_1; \xi) \propto \int_0^T dTe^{-m^2T} \]
\[ \times \int_{x(0)=x_1}^{x(T)=x_2} Dx \int_\psi(0)\psi(T)=2\xi D\psi_\mu \int_{\psi_5(0)+\psi_5(T)=2\xi} D\psi_5 \]
\[ \times \frac{1}{T} \int_0^T d\tau \{-\frac{1}{2}\psi_\mu(\tau)\dot{x}_\mu(\tau) + m\psi_5(\tau)\} \exp[-S_B - S_F - S_5] \] (A.6)

The action consists of the following pieces:
\[ S_B = \int_0^T d\tau [\frac{1}{4} \dot{x}_\mu^2 + ieA_\mu(x)\dot{x}_\mu] \]
\[ S_F = \int_0^T d\tau [\frac{1}{2}\psi_\mu\dot{\psi}_\mu - ieF_{\mu\nu}(x)\psi_\mu\psi_\nu] + \frac{1}{2}\psi_\mu(T)\dot{\psi}_\mu(0) \]
\[ S_5 = \int_0^T d\tau \frac{1}{2}\dot{\psi}_5\psi_5 + \frac{1}{2}\psi_5(T)\psi_5(0) \] (A.7)

Note the surface terms in \( S_F \) and \( S_5 \). They are needed in order to correctly reproduce the equations of motion [104]. The factor \( \frac{1}{T} \int_0^T d\tau \{ \ldots \} \) in eq. (A.6) stems from the integration over the world line gravitino field \( \chi \). It is important to realize that the terms inside the curly brackets are actually independent of \( \tau \), and that we may replace
\[ \frac{1}{T} \int_0^T d\tau \{-\frac{1}{2}\psi_\mu(\tau)\dot{x}_\mu(\tau) + m\psi_5(\tau)\} \rightarrow -\frac{1}{2}\psi_\mu(T)\dot{x}_\mu(T) + m\psi_5(T) \] (A.8)

The \( \tau \)-independence of the expectation value of \( \psi_\mu\dot{x}_\mu \) is a consequence of the supersymmetry. In fact, \( S_B + S_F \) is invariant under
\[ \delta x_\mu = -2\eta \psi_\mu \]
\[ \delta \psi_\mu = \eta \dot{x}_\mu \] (A.9)

with a \textit{constant} Grassmann parameter \( \eta \). If, instead, a time-dependent parameter is used in (A.9), the action changes according to
\[ \delta(S_B + S_F) = \int_0^T d\tau \eta(\tau) \frac{d}{d\tau}(\psi_\mu\dot{x}_\mu) \] (A.10)

Obviously \( \psi_\mu\dot{x}_\mu \) is the conserved Noether charge related to the supersymmetry (A.9). If we apply a localized supersymmetry transformation to the path integral \( \int DxD\psi \exp(-S_B - S_F) \) and observe that the measure is invariant we obtain the Ward identity
\[ \frac{d}{d\tau}(\psi_\mu(\tau)\dot{x}_\mu(\tau)) = 0 \] (A.11)

Eq. (A.11) together with a similar argument for \( \psi_5 \) justifies the replacement (A.8).

Using (A.8) in (A.6), the insertion \(-\frac{1}{2}\psi_\mu\dot{x}_\mu + m\psi_5 \) is evaluated at the final point of the trajectory, \( \tau = T \). Hence it may be pulled in front of the path integral, then acting as
a (differential/matrix) operator on the wave function which was time-evolved by the path integral. If we are dealing with a phase-space path integral of the type

\[ \int \mathcal{D}x \mathcal{D}p \exp \{ \int_0^T d\tau (ip\dot{x} - H) \} \]

we know that

\[ \int \mathcal{D}x \mathcal{D}p \ x(T) \ \exp \{ \ldots \} = \int \mathcal{D}x \mathcal{D}p \ \exp \{ \ldots \} \]

\[ \int \mathcal{D}x \mathcal{D}p \ p(T) \ \exp \{ \ldots \} = -i \frac{\partial}{\partial x_2} \int \mathcal{D}x \mathcal{D}p \ \exp \{ \ldots \} \]

By rewriting eq. (A.6) in hamiltonian form, it is easy to see that in the case at hand \( \dot{x}_\mu(T) \) corresponds to the operator \(-2iD_\mu(x_2)\) acting from the left. With an analogous reasoning for the fermions this leads us to the following representation for the Dirac propagator:

\[ G_{\text{Dirac}}(x_2, x_1) \propto [\hat{\psi}_\mu iD_\mu(x_2) + m\hat{\psi}_5] \]

\[ \times \int_0^\infty dTe^{-m^2T} \text{symb}^{-1} \left[ K_{\text{Dirac}}(x_2, T| x_1, 0; \xi_\mu) I_5(\xi_5, T) \right] \]

Here we defined

\[ K_{\text{Dirac}}(x_2, T| x_1, 0; \xi_\mu) \equiv \int_{x(0)=x_1}^{x(T)=x_2} \mathcal{D}x \int_{\psi(0)+\psi(T)=2\xi} \mathcal{D}\psi_\mu \ e^{-S_B-S_F} \]

and

\[ I_5(\xi_5, T) \equiv \int_{\psi_5(0)+\psi_5(T)=2\xi_5} \mathcal{D}\psi_5 \ e^{-S_5} \]

Eq. (A.13) was obtained from (A.6) by applying the inverse symbol map. As for the fermionic degrees of freedom, \( G_{\text{Dirac}}(x_1, x_2) \) is an operator now, i.e., a matrix acting on spinor indices.

Up to this point, no assumption about the gauge field \( A_\mu(x) \) has been made. From now on we consider fields with \( F_{\mu\nu} = \text{const.} \). In this case the path integral (A.16) factorizes:

\[ K_{\text{Dirac}}(x_2, T| x_1, 0; \xi_\mu) = K_B(x_2, T| x_1, 0) \ I_F(\xi_\mu, T) \]

The bosonic piece

\[ K_B(x_2, T| x_1, 0) = \int_{x(0)=x_1}^{x(T)=x_2} \mathcal{D}x \ e^{-S_B} \]

is the same as in the spin-0 or spin-1 case. Its evaluation is described in detail in section 3. The result is given by (3.68) with \( g \) replaced by \( e \), and with the factor \( \exp[2igTF]_{\mu\nu} \) omitted. What remains to be done is to calculate the fermionic contribution

\[ I_F(\xi_\mu, T) = \int_{\psi(0)+\psi(T)=2\xi} \mathcal{D}\psi_\mu \ \exp[-\frac{1}{2} \psi_\mu(T)\psi_\mu(0)] \]

\[ \times \exp \left\{ -\frac{1}{2} \int_0^T d\tau \ \psi_\mu \left[ \partial_\tau \delta_\mu\nu - 2ieF_{\mu\nu} \right] \psi_\nu \right\} \]
Since the $\psi$-integral is Gaussian, the saddle point method will yield its exact value. We decompose the integration variable according to

$$\psi_\mu(\tau) = \psi_\mu^{\text{class}}(\tau) + \varphi_\mu(\tau) \quad (A.21)$$

where $\psi_\mu^{\text{class}}$ is a solution of the classical equation of motion,

$$[\partial_\tau \delta_{\mu\nu} - 2ieF_{\mu\nu}]\psi_\mu^{\text{class}}(\tau) = 0, \quad (A.22)$$

subject to the boundary condition $\psi_\mu^{\text{class}}(0) + \psi_\mu^{\text{class}}(T) = 2\xi_\mu$. The fluctuation field $\varphi_\mu$ satisfies antiperiodic boundary conditions. Using (A.21) and (A.22) in (A.20) we obtain

$$I_F(\xi_\mu, T) = \exp\left[-\frac{1}{2} \psi_\mu^{\text{class}}(T) \psi_\mu^{\text{class}}(0)\right] \times \int_A \mathcal{D}\varphi \exp\left\{-\frac{1}{2} \int_0^T d\tau \varphi_\mu [\partial_\tau \delta_{\mu\nu} - 2ieF_{\mu\nu}] \varphi_\nu\right\} \quad (A.23)$$

In analogy with section 3, the determinant in (A.23) is given by $2^D \det_L[\cos(eFT)]$. The solution to (A.22) which satisfies the correct boundary conditions reads

$$\psi_\mu^{\text{class}}(\tau) = 2 \left(\frac{\exp(2ieF\tau)}{1 + \exp(2ieF\tau)}\right)_{\mu\nu} \xi_\nu \quad (A.24)$$

Inserting this into (A.23) leads us to the final result for $I_F$:

$$I_F(\xi_\mu, T) = 2^{D/2} \det_L[\cos(eFT)]^{1/2} \times \exp[i\xi_\mu \tan(eFT)_{\mu\nu} \xi_\nu] \quad (A.25)$$

Using the same method we can show that $I_5$ equals an unimportant constant which we shall drop. Thus, because

$$\text{symb}^{-1}(K_{\text{Dirac}}) = K_B \text{symb}^{-1}(I_F), \quad (A.26)$$

our last task is to find out which is the operator corresponding to the symbol (A.25). We shall see that

$$\text{symb}[\exp(ieTF_{\mu\nu}\hat{\psi}_\mu\hat{\psi}_\nu)](\xi) = \det_L[\cos(eFT)]^{1/2} \exp[i\xi_\mu \tan(eFT)_{\mu\nu} \xi_\nu] \quad (A.27)$$

In order to prove (A.27), we transform $F_{\mu\nu}$ to block-diagonal form. We assume $D$ even.

$$F_{\mu\nu} = \text{diag} \left[ \begin{pmatrix} 0 & B_1 \\ -B_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B_2 \\ -B_2 & 0 \end{pmatrix}, \ldots \right] \quad (A.28)$$
Since the $\hat{\psi}_\mu$'s pertaining to different blocks are mutually anticommuting, we may prove (A.27) for each block separately. Focusing on the first one, it is convenient to define

$$\hat{\Sigma}_{12} \equiv i(\hat{\psi}_1 \hat{\psi}_2 - \hat{\psi}_2 \hat{\psi}_1)$$

(A.29)

As this operator is Weyl-ordered, we have

$$\text{symb}\left[\hat{\Sigma}_{12}\right] = 2i\xi_1\xi_2$$

(A.30)

Because $(\hat{\Sigma}_{12})^2 = 1$, it follows that

$$\exp[ieTF_{\mu\nu}\hat{\psi}_\mu\hat{\psi}_\nu] = \exp[eB_1 T\hat{\Sigma}_{12}]$$

$$= \cosh(eB_1 T) - \hat{\Sigma}_{12}\sinh(eB_1 T)$$

(A.31)

and therefore

$$\text{symb}\left[\exp(ieTF_{\mu\nu}\hat{\psi}_\mu\hat{\psi}_\nu)(\xi)\right] = \cosh(eB_1 T)[1 - 2i\xi_1\xi_2\tanh(eB_1 T)]$$

$$= \det_L[\cos(eFT)]^{1/2}\exp[ie\xi_\mu\tan(eFT)\xi_\nu]$$

(A.32)

In the last line of (A.32) we used that the eigenvalues of the first block are $\pm iB_1$ and that $\xi_1^2 = 0 = \xi_2^2$. Repeating the same argument for the other blocks establishes eq. (A.27).

Upon inserting (A.26) with (A.25) and (A.27) into (A.15), we obtain the well-known result for the euclidean Dirac propagator in a constant background field [104,105]:

$$G_{\text{Dirac}}(x_1, x_2) = [-\gamma_\mu D_\mu(x_2) + m] \int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2 T}$$

$$\cdot \exp\left[-\frac{1}{4}(x_2 - x_1)eF \cot(eFT)(x_2 - x_1)\right]$$

$$\cdot \exp\left[-\frac{1}{2}\text{tr}_L \ln \frac{\sin(eTF)}{(eTF)}\right]$$

$$\cdot \exp\left[\frac{i}{2}eTF_{\mu\nu}\gamma_\mu\gamma_\nu\right]$$

(A.33)

In (A.33) we used the representation (A.2) for the $\hat{\psi}$'s, and we dropped an overall factor of $\gamma_5$ which is produced by the path integral, but is not included in the standard definition of $G_{\text{Dirac}}$. The expression for the bosonic contribution $K_B$ was taken from section 3. It applies to the Fock-Schwinger gauge centered at $x_1$. In the general case, $K_B$ contains an extra phase factor $\exp(-ie\int_{x_1}^{x_2} dx_\mu A_\mu)$. We also note that the scalar propagator $(-D^2)^{-1}$ is obtained from (A.33) by simply deleting the operator $[-D + m]$ and the last exponential involving the $\gamma$-matrices.

It is remarkable that the above calculation of the propagator is almost identical to the calculation of the one-loop effective action, the only difference being the boundary condition of the fermionic path integral. For the propagator we need $\psi_\mu(T) + \psi_\mu(0) = 2\xi_\mu$, whereby the variables $\xi_\mu$ give rise to its $\gamma$-matrix structure. The effective action, on the other hand, is a scalar quantity, and it is obtained from the same path integral with $\xi_\mu = 0$. Giving a non-zero value to $\xi_\mu$ amounts to creating a fermion line by “opening” a loop.
Appendix B: Derivation of Worldline Green’s Functions

The worldline Green’s functions appearing in this paper are kernels of certain integral operators, acting in the real Hilbert space of periodic or antiperiodic functions defined on an interval of length $T$. We denote by $\bar{H}_P$ the full space of periodic functions, by $H_P$ the same space with the constant mode exempted, and by $H_A$ the space of antiperiodic functions. The ordinary derivative acting on those functions is correspondingly denoted by $\partial_p$, $\bar{\partial}_p$ or $\partial_A$. With those definitions, we can write our Green’s functions as

$$
\mathcal{G}_B(\tau_1, \tau_2) = 2\langle \tau_1 | (\partial_p^2 - 2iF\partial_p)^{-1} | \tau_2 \rangle,
$$

$$
\mathcal{G}_F(\tau_1, \tau_2) = 2\langle \tau_1 | (\partial_A - 2iF)^{-1} | \tau_2 \rangle,
$$

$$
\mathcal{G}_P^C(\tau_1, \tau_2) = \langle \tau_1 | (\bar{\partial}_p - C)^{-1} | \tau_2 \rangle,
$$

$$
\mathcal{G}_A^C(\tau_1, \tau_2) = \langle \tau_1 | (\partial_A - C)^{-1} | \tau_2 \rangle.
$$

(B.1)

(in this appendix we absorb the coupling constant $e$ into the external field $F$). Note that $\mathcal{G}_A^C$ is, up to a conventional factor of 2, formally identical with $\mathcal{G}_F$ under the replacement $C \to 2iF$.

$\mathcal{G}_B$ and $\mathcal{G}_F$ are easy to construct using the following representation of the integral kernels for inverse derivatives on the unit circle [36]

$$
\langle u | \partial_p^{-n} | u' \rangle = -\frac{1}{n!} B_n(|u' - u|) \text{sign}^n(u - u')
$$

$$
\langle u | \partial_A^{-n} | u' \rangle = \frac{1}{2(n-1)!} E_{n-1}(|u' - u|) \text{sign}^n(u - u')
$$

(B.2)

Here $B_n(E_n)$ denotes the $n$-th Bernoulli (Euler) polynomial. Those formulas are valid for $|u - u'| \leq 1$. For instance, the computation of $\mathcal{G}_B$ proceeds as follows:

$$
\mathcal{G}_B(u_1, u_2) = 2\langle u_1 | (\partial_p^2 - 2iF\partial_p)^{-1} | u_2 \rangle
$$

$$
= 2 \sum_{n=0}^{\infty} (2iF)^n \langle u_1 | \partial_p^{-(n+2)} | u_2 \rangle
$$

$$
= -2 \sum_{n=2}^{\infty} \frac{(2iF)^{n-2} \text{sign}^n(u_1 - u_2)}{n!} B_n(|u_1 - u_2|)
$$

$$
= -\frac{1}{iF} \frac{\text{sign}(u_1 - u_2)e^{2iF(u_1 - u_2)} - 1}{e^{2iF\text{sign}(u_1 - u_2)} - 1} + \frac{\text{sign}(u_1 - u_2)}{iF} B_1(|u_1 - u_2|) - \frac{1}{2F^2}
$$

$$
= \frac{1}{2F^2} \left( \frac{F}{\sin F} e^{-iF\mathcal{G}_{B12}} + iF\mathcal{G}_{B12} - 1 \right)
$$

(B.3)
In the next-to-last step we used the generating identity for the Bernoulli polynomials,

\[
\frac{e^{xt} - 1}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\] (B.4)

This is \(G_B\) as given in eq.(2.19) up to a simple rescaling. The computation of \(G_F\) proceeds in a completely analogous way.

This method does not work for the determination of \(G_P^C\), as negative powers of \(\bar{\partial}_P\) are not even well-defined in the presence of the zero mode.

In the following, we will calculate \(G_P^C\), \(A\) in a different, more “physical” way, which corresponds to the usual construction of the Feynman propagator in field theory.

In order to determine \(G_A^C(\tau)\), say, we employ the following set of basis functions over the circle with circumference \(T\):

\[
f_n(\tau) = T^{-1/2} \exp\left[\frac{2\pi}{T} (n + \frac{1}{2}) \tau\right], \quad n \in \mathbb{Z}
\] (B.5)

They satisfy

\[
\int_{\tau}^{\tau + T} d\tau f_n^*(\tau) f_m(\tau) = \delta_{nm},
\]

\[
\sum_{n=-\infty}^{\infty} f_n(\tau_2) f_n^*(\tau_1) = \sum_{m=-\infty}^{\infty} \delta(\tau_2 - \tau_1 - mT)
\] (B.6)

and \(f_n(\tau + T) = -f_n(\tau)\). In this basis, the Green’s function \(G_A^C(\tau)\) becomes

\[
G_A^C(\tau_1 - \tau_2) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{\exp\left[i\frac{2\pi}{T} (n + \frac{1}{2})(\tau_1 - \tau_2)\right]}{i(2\pi/T)(n + \frac{1}{2}) - C} \] (B.7)

By introducing an auxiliary integration in the form \((\tau \equiv \tau_1 - \tau_2)\)

\[
G_A^C(\tau) = \int_{-\infty}^{\infty} d\omega \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}(n + \frac{1}{2})\right) \frac{e^{i\omega\tau}}{i\omega - C}
\] (B.8)

and using Poisson’s resummation formula, the Green’s function assumes the suggestive form \(106\)

\[
G_A^C(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n G_{\infty}^C(\tau + nT)
\] (B.9)

with

\[
G_{\infty}^C(\tau) = \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{2\pi i\omega - C}
\] (B.10)

We verify that

\[
[\partial_\tau - C] G_{\infty}^C(\tau) = \delta(\tau)
\] (B.11)

\[
[\partial_\tau - C] G_A^C(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(\tau + nT)
\] (B.12)
which shows that \( G^C_\infty \) is a Green’s function on the infinitely extended real line, while \( G^C_A \) is defined on the circle. The integral (B.11) yields for \( C > 0 \)

\[
G^C_\infty(\tau) = -\Theta(-\tau)e^{C\tau}
\]

(Hence, from (B.9)

\[
G^C_A(\tau) = -e^{C\tau} \sum_{n=-\infty}^{\infty} (-1)^n \Theta(-\tau-nT)e^{nCT}
\]

For \( \tau \in (0,T) \) only the terms \( n = -\infty, \ldots, -1 \) contribute to the sum in (B.14), while for \( \tau \in (-T,0) \) a nonzero contribution is obtained for \( n = -\infty, \ldots, 0 \). Summing up the geometric series in either case and combining the results we obtain the expression given in eq. (3.30). It is valid for \(-T < \tau < +T\). Using a basis of periodic functions the same arguments lead to \( G^C_P \) as stated in (3.30). Note that in the limit of a large period \( T \)

\[
\lim_{T \to \infty} G^C_{A,P}(\tau) = G^C_\infty(\tau),
\]

as it should be. For \( C \to 0 \), both \( G^C_\infty \) and \( G^C_A \) have a well-defined limit:

\[
G^0_\infty(\tau) = -\Theta(-\tau) \\
G^0_A(\tau) = \frac{1}{2} \text{sign}(\tau)
\]

The periodic Green’s function \( G^C_P \) blows up in this limit because \( \tilde{\partial}_P^{-1} \) does not exist in presence of the constant mode. It is important to keep in mind that \( G^C_P \) is defined in such a way that it includes the zero mode of \( \partial_\tau \).

In the perturbative evaluation of the spin-1 path integral one has to deal with traces over chains of propagators of the form

\[
\sigma^{n}_{A,P}(C) \equiv \text{Tr}_{A,P}[\left(\partial_\tau - C\right)^{-n}]
\]

Because

\[
\sigma^{n}_{A,P}(C) = \frac{1}{(n-1)!} \left( \frac{d}{dC} \right)^{n-1} \sigma^{1}_{A,P}(C)
\]

it is sufficient to know \( \sigma^{1}_{A,P}(C) \). The subtle point which we would like to mention here is that strictly speaking the sum defining \( \sigma^{1}_{A,P} \), say,

\[
\sigma^{1}_{A}(C) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{i(2\pi/T)(n + \frac{1}{2}) - C} \right]
\]

does not converge as it stands, and it is meaningless without a prescription of how to regularize it. The usual strategy is to combine terms for positive and negative values of \( n \), and to replace (B.19) by the convergent series

\[
\sigma^{1}_{A}(C) = -2C \sum_{n=0}^{\infty} \left[ \left( \frac{2\pi}{T} \right)^2 (n + \frac{1}{2})^2 + C^2 \right]^{-1}
\]

\[
= -\frac{T}{2} \tanh \left( \frac{CT}{2} \right).
\]

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It is important to realize that this definition implies a well-defined prescription for the treatment of the $\Theta$ functions in $G_{A,P}^C$ at $\tau = 0$. In fact,

$$\sigma_A^1(C) = \int_0^T d\tau \ G_A^C(\tau - \tau) = T \ G_A^C(0), \quad (B.21)$$

and by combining eqs. (B.20) and (B.21) we deduce that we must set

$$\lim_{\tau \to 0} \Theta(\tau) = \lim_{\tau \to 0} \Theta(-\tau) = \frac{1}{2}. \quad (B.22)$$

With (B.20) we obtain

$$\sigma_A^n(C) = -\frac{1}{(n-1)!} \left( \frac{T}{2} \right)^n \left( \frac{d}{dx} \right)^{n-1} \tanh(x) \bigg|_{x=CT/2}. \quad (B.23)$$

The analogous relation in the periodic case is

$$\sigma_P^n(C) = -\frac{1}{(n-1)!} \left( \frac{T}{2} \right)^n \left( \frac{d}{dx} \right)^{n-1} \coth(x) \bigg|_{x=CT/2} \quad (B.24)$$

if the zero mode of $\partial_\tau$ is included in the trace (B.17), and

$$\sigma_P'^n(C) = -\frac{1}{(n-1)!} \left( \frac{T}{2} \right)^n \left( \frac{d}{dx} \right)^{n-1} \{\coth(x) - x^{-1}\} \bigg|_{x=CT/2} \quad (B.25)$$

if the zero mode is omitted. For $C$ sufficiently small one finds the power series expansions

$$\sigma_A^n(C) = -\frac{1}{(n-1)!} \sum_{k=n/2}^{\infty} \frac{(2^{2k} - 1)B_{2k}}{2k(2k-n)!} T^{2k} C^{2k-n}$$

$$\sigma_P'^n(C) = -\frac{1}{(n-1)!} \sum_{k=n/2}^{\infty} \frac{B_{2k}}{2k(2k-n)!} T^{2k} C^{2k-n} \quad (B.26)$$

$\sigma_A^n$ and $\sigma_P'^n$ have well defined limits for $C \to 0$:

$$\sigma_A^n(0) = -\frac{(2^n - 1)B_n}{n!} T^n = \frac{1}{2} \frac{E_{n-1}}{(n-1)!} T^n \quad (n \text{ even})$$

$$\sigma_P'^n(0) = -\frac{B_n}{n!} T^n \quad (n \text{ even}) \quad (B.27)$$

Those limits vanish for $n$ odd. This brings us, of course, back to eqs. (B.2).
Appendix C: Worldline Determinants

In this appendix we collect a few results about the determinants which arise in the computation of the spin-1 wordline path integral. To start with, we consider the operator $\partial_\tau + \omega$ where $\omega$ is a real constant, and $\partial_\tau$ acts on periodic and antiperiodic functions of period $T$, respectively. Its spectrum reads $i(2\pi/T)n$ in the former and $i(2\pi/T)(n+1/2)$ in the latter case, $n \in \mathbb{Z}$. Ratios of determinants of the form

$$\frac{\text{Det}_A[\partial_\tau + \omega]}{\text{Det}_A[\partial_\tau]} = \prod_{n=-\infty}^{\infty} \frac{i(\frac{2\pi}{T})(n + \frac{1}{2}) + \omega}{i(\frac{2\pi}{T})(n + \frac{1}{2})}$$

are defined by the prescription that terms with positive and negative values of $n$ should be combined so as to obtain the manifestly convergent product

$$\frac{\text{Det}_A[\partial_\tau + \omega]}{\text{Det}_A[\partial_\tau]} = \prod_{n=0}^{\infty} \left[ 1 + \left( \frac{\omega T}{2\pi} \right)^2 \frac{1}{(n+1/2)^2} \right] = \cosh \left( \frac{\omega T}{2} \right)$$

In the periodic case we omit the zero mode from the definition of the determinants and find likewise

$$\frac{\text{Det}'_P[\partial_\tau + \omega]}{\text{Det}'_P[\partial_\tau]} = \frac{\sinh(\omega T/2)}{(\omega T/2)}$$

(compare eqs. (2.35), (2.35)). Next we look at the matrix differential operator

$$(\partial_\tau - C)\delta_{\mu\nu} + \Omega_{\mu\nu}, \quad \mu, \nu = 1, \ldots, D.$$ 

Here $\Omega$ is a constant matrix. We assume that it can be diagonalized and has eigenvalues $\omega$. For antiperiodic boundary conditions we obtain from (C.2)

$$\frac{\text{Det}_A[(\partial_\tau - C)\delta_{\mu\nu} + \Omega_{\mu\nu}]}{\text{Det}_A[(\partial_\tau - C)\delta_{\mu\nu}]} = \prod_{\omega} \frac{\cosh\left[ \frac{T}{2}(C - \omega) \right]}{\cosh[CT/2]}.$$ 

The product extends over the spectrum of $\Omega$. The corresponding formula for periodic boundary conditions, with the zero mode removed, reads

$$\frac{\text{Det}'_P[(\partial_\tau - C)\delta_{\mu\nu} + \Omega_{\mu\nu}]}{\text{Det}'_P[(\partial_\tau - C)\delta_{\mu\nu}]} = \prod_{\omega'} \frac{C}{C - \omega'} \prod_{\omega} \frac{\sinh\left[ \frac{T}{2}(C - \omega) \right]}{\sinh[CT/2]}.$$ 

For $C \neq 0$ we can reinstate the zero mode of $\partial_\tau$. In this case (C.5) is replaced by

$$\frac{\text{Det}_P[(\partial_\tau - C)\delta_{\mu\nu} + \Omega_{\mu\nu}]}{\text{Det}_P[(\partial_\tau - C)\delta_{\mu\nu}]} = \prod_{\omega} \frac{\sinh\left[ \frac{T}{2}(C - \omega) \right]}{\sinh[CT/2]}.$$ 

In this paper we use the above determinants for an exact evaluation of the worldline path integral in the background of a constant field $F_{\mu\nu}$. For more complicated field configurations only a perturbative calculation of the path integral is possible in general. It is based upon the Green’s functions $G_{A,P}^{C}$ which were discussed in appendix B. It can be
checked that the determinants given above are consistent with the perturbative expansion. In perturbation theory, the l.h.s. of eq. (C.4), for instance, is interpreted as a power series in \( \omega \):

\[
\prod_{\omega} \text{Det}_A [1 + \omega G_A^C] = \prod_{\omega} \exp \text{Tr} \ln [1 + \omega G_A^C]
\]

\[
= \prod_{\omega} \exp \left\{ - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \omega^n \sigma_n^A(C) \right\}
\] (C.7)

In the last line of (C.7) we have used (B.17). By virtue of eq. (B.23) one can sum up the perturbation series in closed form:

\[
- \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \omega^n \sigma_n^A(C)
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \frac{\omega T}{2} \right)^n \frac{d}{dx} \tanh(x) \bigg|_{x = CT/2}
\]

\[
= \left[ \exp \left( -\frac{\omega T}{2} \frac{d}{dx} \right) - 1 \right] \ln \cosh(x) \bigg|_{x = CT/2}
\]

\[
= \ln \frac{\cosh(x - \omega T/2)}{\cosh(x)} \bigg|_{x = CT/2}
\] (C.8)

With (C.8) inserted into (C.7) we reproduce precisely the r.h.s. of eq. (C.4).
References

[1] J.A. Minahan, Nucl.Phys. B 298 (1988) 36.

[2] V. Kaplunovsky, Nucl. Phys. B 307 (1988) 145.

[3] R.R. Metsaev, A.A. Tseytlin, Nucl. Phys. B298 (1988) 109.

[4] Z. Bern, D.A. Kosower, Phys. Rev. D 38 (1988) 1888.

[5] Z. Bern, D. A. Kosower, Phys. Rev. Lett. 66 (1991) 1669;
   Z. Bern, D. A. Kosower, Nucl. Phys. B379 (1992) 451.

[6] Z. Bern, D. C. Dunbar, Nucl. Phys. B379 (1992) 562.

[7] Z. Bern, L. Dixon, D. A. Kosower, Phys. Rev. Lett. 70 (1993) 2677 [hep-ph/9302280].

[8] Z. Bern, D. C. Dunbar, T. Shimada, Phys. Lett. B312 (1993) 277 [hep-th/9307001];
   D.C. Dunbar, P. S. Norridge, Nucl. Phys. B433 (1995) 181 [hep-th/9408014].

[9] M. J. Strassler, Nucl. Phys. B385 (1992) 145.

[10] R. P. Feynman, Phys. Rev. 80 (1950) 440.

[11] E. S. Fradkin, Nucl. Phys. 76 (1966) 588.

[12] R. Casalbuoni, J. T. Gomis, G. Longhi, Nuovo Cimento 24 A (1974) 249.

[13] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, P. Howe, Phys.Lett. 64B (1976) 435.

[14] L. Brink, P. Di Vecchia, P. Howe, Nucl.Phys. B118 (1977) 76.

[15] A.P. Balachandran, P. Salomonson, B. Skagerstam, J. Winnberg, Phys. Rev. D15 (1977) 2308.

[16] M. B. Halpern, W. Siegel, Phys. Rev. D16 (1977) 2486.

[17] L. Alvarez-Gaumé, Comm. Math. Phys. 90 (1983) 161.

[18] A.O. Barut and I.H. Duru, Phys. Rep. 172 (1989) 1.

[19] A. M. Polyakov, Gauge Fields and Strings, Harwood 1987.

[20] M. J. Strassler, SLAC-PUB-5978 (1992) (unpublished).

[21] M. G. Schmidt, C. Schubert, Phys. Lett. B318 (1993) 438 [hep-th/9309053].

[22] D. Fliegner, P. Haberl, M. G. Schmidt, C. Schubert, Discourses in Mathematics and its Applications, No. 4, p. 87, Texas A&M 1995 [hep-th/9411177]; New Computing Techniques in Physics Research IV, p. 199, World Scientific 1996 [hep-th/9505077].

[23] D. Cangemi, E. D’Hoker, G. Dunne, Phys. Rev. D51 (1995) 2513 [hep-th/9409113].
[24] V.P. Gusynin, I.A. Shovkovy, UG-9-95 (hep-ph/9509383).
[25] M. Móndragon, L. Nellen, M.G. Schmidt, C. Schubert, Phys. Lett. B351 (1995) 200 (hep-th/9502125).
[26] E. D’Hoker, D. G. Gagné. Nucl. Phys. B467 (1996) 272 (hep-th/9508131).
[27] J.W. van Holten, Nucl.Phys. B457 (1995) 375 (hep-th/9508136).
[28] M. Móndragon, L. Nellen, M.G. Schmidt, C. Schubert, Phys. Lett. B366 (1996) 212 (hep-th/9510036).
[29] E. D’Hoker, D. G. Gagné, Nucl. Phys. B467 (1996) 297 (hep-th/9512080).
[30] C. S. Lam, Phys. Rev. D 48 (1993) 873 (hep-ph/9212296); C. S. Lam, Can. J. Phys. 72 (1994) 415 (hep-ph/9308289).
[31] K. Roland, Phys. Lett. B289 (1992).
[32] P. Di Vecchia, A. Lerda, L. Magnea, R. Marotta, R. Russo, Nucl. Phys. B469 (1996) 235 (hep-th/9601143); DFTT-41-96 (hep-th/9607141).
[33] K. Roland, H. Sato, NBI-HE-96-19 (hep-th/9604152).
[34] M. G. Schmidt, C. Schubert, Phys. Lett. B331 (1994) 69 (hep-th/9403158).
[35] H. Sato, Phys. Lett. B371 (1996) 270 (hep-th/9511158).
[36] M.G. Schmidt, C. Schubert, Phys. Rev. D53 (1996) 2150 (hep-th/9410100).
[37] K. Daikouji, M. Shino, Y. Sumino, Phys. Rev. D53 (1996) 4598 (hep-ph/9508377).
[38] W. Heisenberg, H. Euler, Z. Phys. 38 (1936) 714.
[39] J. Schwinger, Phys. Rev. 82 (1951) 664.
[40] J. Géhéniau, Physica 16 (1950) 822.
[41] W. Y. Tsai, Phys. Rev. D 10 (1974) 1342.
[42] V. N. Baier, V. M. Katkov, and V. M. Strakhovenko, Sov. Phys. JETP 40 (1975) 225; Sov. Phys. JETP 41 (1975) 198.
[43] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, M. A. Shifman, Sov. J. Nucl. Phys. 39 (1984) 77.
[44] R. Shaisultanov, Phys. Lett. B 378 (1996) 354 (hep-th/9512142).
[45] D.G.C. McKeon, T.N. Sherry, Mod. Phys. Lett. A9 (1994) 2167.
[46] V. I. Ritus, Zh. Eksp. Teor. Fiz. 73 (1977) 807 [Sov. Phys. JETP 46 (1977) 423].
[47] S. L. Lebedev, FIAN, No. 254, Moscow 1982 (unpublished).

[48] V. I. Ritus, The Lagrangian Function of an Intense Electromagnetic Field and Quantum Electrodynamics at Short Distances, in Proc. Lebedev Phys. Inst. vol. 168, V. I. Ginzburg ed., Nova Science Publ., NY 1987.

[49] V. I. Ritus, Sov. Phys. JETP 42 (1975) 774.

[50] W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics, Springer 1985.

[51] I.L. Buchbinder, Sh.M. Shvartsman, Int.J.Mod.Phys. A8 (1993) 683.

[52] D.M. Gitman, A.E. Goncalves, I.V. Tyutin, Int.J.Mod.Phys. A10 (1995) 701 (hep-th/9401132).

[53] G. Shore, Ann. Phys. 137 (1981) 262.

[54] J.W. van Holten, in: Proc. Sem. on Math. structures in field theories (1986 – 7), CWI syllabus, Vol. 26 (1990) 109.

[55] F. Bordi and R. Casalbuoni, Phys. Lett. B93 (1980) 308.

[56] J.C. Henty, P.S. Howe and P.K. Townsend, Class. Quant. Grav. 5 (1988) 807.

[57] M.B. Halpern, A. Jevicki, P. Senjanovic, Phys.Rev. D16 (1977) 2476.

[58] S.G. Rajeev, Ann. Phys. 173 (1987) 249.

[59] E.S. Fradkin, D.M. Gitman, S.M. Shvartsman, Quantum Electrodynamics with Unstable Vacuum, Springer 1991;
D.M. Gitman, S.M. Shvartsman, IFUSP-P-1079 (hep-th/9310074).

[60] F. Bastianelli, Nucl. Phys. B376 (1992) 113;
F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. B389 (1993) 53.

[61] J. de Boer, B. Peeters, K. Skenderis, P. van Nieuwenhuizen, Nucl. Phys. B446 (1995) 211 (hep-th/9504087); Nucl.Phys. B459 (1996) 631 (hep-th/9509158).

[62] D.G.C. McKeon, Ann. Phys. 224 (1993) 139.

[63] F.A. Dilkes, D.G.C. McKeon (hep-th/9509003).

[64] A.I. Karanikas and C.N. Ktorides, Phys. Rev. D52 (1995) 5883;
A.I. Karanikas, C.N. Ktorides and N.G. Stefanis, Phys. Rev. D52 (1995) 5898.

[65] D. M. Gitman and S. I. Zlatev, IFUSP-P-1236 (hep-th/9608179).

[66] F. A. Lunev (hep-th/9609166).

[67] D. Fliegner, M.G. Schmidt, C. Schubert, Z. Phys. C 64 (1994) 111 (hep-ph/9401221).
[68] Z. Bern, TASI Lectures, Boulder TASI 92, 471 (hep-ph/9304249).

[69] Private communication by Z. Bern.

[70] O.D. Andreev, A.A. Tseytlin, Phys. Lett. **207B** (1988) 157.

[71] A. Morgan, Phys. Lett. **B351** (1995) 249 (hep-ph/9502230).

[72] F. Cooper, A. Khare, R. Musto and A. Wipf, Ann. Phys. **187** (1988) 1.

[73] L. F. Abbott, Nucl. Phys. **B185** (1981) 189.

[74] W. Dittrich, M. Reuter, *Selected Topics in Gauge Theories*, Springer, Berlin, 1986.

[75] T. Barnes, G.I. Ghandour, Nucl. Phys. **B146** (1978) 483.

[76] E. Gozzi, M. Reuter, W. D. Thacker, Phys. Rev. **D40** (1989) 3363; E. Gozzi, M. Reuter, Phys. Lett. **240B** (1990) 137.

[77] M. Reuter, Int. J. Mod. Phys. **A10** (1995) 65.

[78] B. Sakita, *Quantum Theory of Many-Variable Systems and Fields*, World Scientific, Singapore, 1985.

[79] E. Witten, Nucl. Phys. **B 202** (1982) 253; J. Diff. Geom. **17** (1982) 661.

[80] M. Sato, Prog. Theor. Phys. **58** (1977) 1262.

[81] F. A. Berezin, Theor. Math. Phys. **6** (1971) 141.

[82] M. M. Mizrahi, J. Math. Phys. **16** (1975) 2201.

[83] J.-L. Gervais, A. Jevicki, Nucl. Phys. **B110** (1976) 93.

[84] I. A. Batalin, S. G. Matinyan, G. K. Savvidi, Sov. J. Nucl. Phys. **26** (1977) 214.

[85] W. Dittrich, M. Reuter, Phys. Lett. **B128** (1983) 321; M. Reuter, W. Dittrich, Phys. Lett. **B144** (1984) 99.

[86] S. K. Blau, M. Visser, A. Wipf, Int. J. Mod. Phys. **A6** (1991) 5409.

[87] H.M. Fried, *Functional Methods and Models in Quantum Field Theory*, MIT Press, Cambridge (1972).

[88] H. Dorn, Fortschr. Phys. **34** (1986) 11.

[89] Yu.M. Makeenko and A.A. Migdal, Nucl. Phys. **B 188** (1981) 269.

[90] M.G. Schmidt, C. Schubert, in preparation.

[91] B. Sathiapalan, Mod. Phys. Lett. **A 10** (1995) 4501 (hep-th/9409023).
[92] D. Fliegner, M. Reuter, M.G. Schmidt, C. Schubert, HUB-EP-97-25.

[93] D. Bakalov et al., Nucl. Phys. B35 (Proc. Suppl.) (1994) 180.

[94] D. Bakalov, INFN/AE-94/27 (unpublished).

[95] E. Zavattini, private communication.

[96] Z. Bialynicka-Birula, Bull. Acad. Polon. Sci., Vol. 13, (1965) 369.

[97] R. Jost and J.M. Luttinger, Helv. Phys. Acta 23 (1950) 201.

[98] J. L. Rosner, Ann. Phys. 44 (1967) 11;
   E. de Rafael and J.L. Rosner, Ann. Phys. 82 (1974) 369.

[99] H.E. Brandt, PhD thesis, Univ. of Washington 1970 (unpublished).

[100] D. Broadhurst, R. Delbourgo and D. Kreimer, Phys. Lett. B366 (1996) 421 (hep-ph/9509296).

[101] D. Fliegner, M.G. Schmidt and C. Schubert (work in progress).

[102] S. L. Adler and C. Schubert, Phys. Rev. Lett. 77 (1996) 1695 (hep-th/9605035).

[103] F.A.Berezin, M.S.Marinov, Ann.Phys. 104 (1977) 336.

[104] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, 1992.

[105] C. Itzykson and J. Zuber, Quantum Field Theory, McGraw-Hill 1985.

[106] H. Kleinert, Path Integrals, World Scientific, Singapore, 1993.