Generalized Dyck paths of bounded height

Axel Bacher

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Abstract

Generalized Dyck paths (or discrete excursions) are one-dimensional paths that take their steps in a given finite set $S$, start and end at height 0, and remain at a non-negative height. Bousquet-Mélou showed that the generating function $E_k$ of excursions of height at most $k$ is of the form $F_k/F_{k+1}$, where the $F_k$ are polynomials satisfying a linear recurrence relation. We give a combinatorial interpretation of the polynomials $F_k$ and of their recurrence relation using a transfer matrix method. We then extend our method to enumerate discrete meanders (or paths that start at 0 and remain at a non-negative height, but may end anywhere). Finally, we study the particular case where the set $S$ is symmetric and show that several simplifications occur.

1 Introduction and notations

A Dyck path is a one-dimensional path taking its steps in $\{-1, 1\}$, starting and ending at 0 and visiting only non-negative points (Figure 1 left). It is well-known that the number of Dyck paths of length $2n$ is the $n$th Catalan number $C_n$ [10, Chapter 6]:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

Moreover, let $E = E(t)$ be the generating function of Dyck paths. This generating function satisfies the following algebraic equation:

$$1 - E + t^2E^2 = 0.$$ 

A generalized Dyck path (or discrete excursion) takes its steps in a given finite set $S$ instead of $\{-1, 1\}$. A discrete meander is a slightly more general path: it takes its steps in $S$, starts at 0 and visits only non-negative points, but may end anywhere (figure 1 right).

Figure 1: Left: a Dyck path of height 3. Right: a discrete meander with set of steps $S = \{\pm 1, \pm 2\}$, of height 4 and with final height 3.
A large number of papers enumerate generalized Dyck paths with varying amounts of generality. Methods used include some properties of Laurent series [6], grammars [7, 8, 5, 1], or the kernel method [2, 3]. Assume that all steps $s$ in $S$ have a weight $\omega_s$ taken in some field of characteristic 0 (typically a field of fractions with one or several variables over $\mathbb{Q}$). Let $E$ and $M$ be the generating functions of excursions and meanders, respectively, according to the weights $\omega_s$.

Banderier and Flajolet [2] showed that both these generating functions are algebraic. More precisely, let $a = \max S$ and $b = -\min S$; one may compute polynomials of degree $(a+b) - a - b$ canceling $E$ and $M$.

In this paper, we consider excursions and meanders with bounded height, that is, that never go above a certain level, say $k$. We denote by $E_k$ the generating function of excursions of height at most $k$ and by $E_{k,\ell}$ the generating function of meanders of height at most $k$ with final height $\ell$. We also denote by $M_k$ the generating function of meanders of height at most $k$ regardless of final height.

Bousquet-Mélou [3] proved that the generating function $E_k$ is of the form:

$$E_k = \frac{F_k}{F_{k+1}},$$

where the $F_k$ are polynomials in the weights $\omega_s$. She also proved, with the use of symmetric functions, that the polynomials $F_k$ satisfy a linear recurrence relation of order $(a+b)$. In other words, we have

$$\sum_{k \geq 0} F_k z^k = \frac{N(z)}{D(z)},$$

where the polynomial $D(z)$ has degree $(a+b)$ (the polynomial $N(z)$ has degree $(a+b) - a - b)$. Moreover, it can be seen that the polynomial $D(z)$ cancels the generating function of excursions $E$.

To our knowledge, the only cases of the generating function $E_{k,\ell}$ that were studied before with general steps are $\ell = 0$, that corresponds to excursions, and $\ell = k$, that corresponds to culminating paths [4]. In this paper, we use a transfer matrix method to compute both generating functions $E_k$ and $E_{k,\ell}$.

We now set some notations. As argued in [3], excursions of height at most $k$ are walks in a finite graph, with vertices $\{0, \ldots, k\}$ and an arc from $i$ to $j$ if $j - i$ is in $S$. We denote by $A_k$ the adjacency matrix of this graph. If $s$ is in $\mathbb{Z}$, let $\beta_s$ be the quantity:

$$\beta_s = \delta_{s,0} - \begin{cases} \omega_s & \text{if } s \in S, \\ 0 & \text{otherwise}. \end{cases}$$

In this way, the $(i, j)$ entry of the matrix $1 - A_k$ is $\beta_{j-i}$.

If $m$ and $n$ are integers, we use the notation $\lbrack m, n \rbrack$ to denote the set of integers $i$ such that $m \leq i \leq n$. We also use the notation $\mathbb{N}_{\geq n}$ to denote the set of integers greater than or equal to $n$. Finally, if $X$ is a set and $n$ an integer, we call $n$-subset of $X$ a subset of $X$ of cardinality $n$.

The paper is organized as follows. Section 2 deals with bounded excursions, re-proving Bousquet-Mélou’s results using a simple transfer matrix method. This method is expanded in Section 3 to cover bounded meanders as well. In Section 4, we discuss the case where the set $S$ is symmetric and show that several simplifications occur.

2
2 Bounded excursions

Our first step to enumerate excursions of height at most $k$ is the same as in [3]. As these excursions are walks that go from 0 to 0 in the graph described by the matrix $A_k$, the generating function $E_k$ is the $(0,0)$ entry of the matrix $(1 - A_k)^{-1}$ (see [9, Chapter 4]). Therefore, we have:

$$E_k = \frac{F_k}{F_{k+1}},$$

(1)

where $F_k$ is the determinant of the matrix $1 - A_{k-1}$, with the convention $F_0 = 1$.

As the entry $(i, j)$ of the matrix $1 - A_{k-1}$ is $\beta_{j-i}$, we have, by the definition of the determinant:

$$F_k = \sum_{\sigma \in S_k} \varepsilon(\sigma) \prod_{i=0}^{k-1} \beta_{\sigma(i)-i}.$$

In the following, we use this expression to compute $F_k$. The permutations $\sigma$ that it involves are, of course, bijections from $[0, k-1]$ to itself; however, we find it more convenient to regard them as bijections from $\mathbb{N}$ to itself that fix all points in $\mathbb{N} \geq k$.

**Definition 1.** Let $I$ be an $a$-subset of the set $[-b, a-1]$. We call $I$-permutation of order $k$ a bijection from $\mathbb{N}$ to $I \cup \mathbb{N} \geq a$ that fixes all points in $\mathbb{N} \geq k$.

Note that for a $I$-permutation to exist when $k < a$, all points in $[k, a-1]$ must be in the set $I$ since they are fixed points. Also note that a $[0, a-1]$-permutation of order $k$ is the same as a standard permutation of order $k$; for this reason, we set $I_0 = [0, a-1]$.

We denote by $S_k^I$ the set of $I$-permutations of order $k$. Let $\sigma$ be a $I$-permutation. We define the number of inversions of $\sigma$, denoted by $\text{inv}(\sigma)$, in the same manner as a regular permutation:

$$\text{inv}(\sigma) = \# \{ (i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j) \}.$$  

We define the signature of $\sigma$, denoted by $\varepsilon(\sigma)$, to be the number $(-1)^{\text{inv}(\sigma)}$. We also define the quantity $\beta(\sigma)$ to be:

$$\beta(\sigma) = \varepsilon(\sigma) \prod_{i=0}^{k-1} \beta_{\sigma(i)-i}.$$

Finally, we call head of $\sigma$ the value $\sigma(0)$; we call tail of $\sigma$ the mapping $\tau$ defined for all $n \in \mathbb{N}$ by:

$$\tau(n) = \sigma(n+1) - 1.$$

We now denote by $F_k$ the vector indexed by the $a$-subsets of $[-b, a-1]$ and the entries of which are:

$$F_k[I] = \sum_{\sigma \in S_k^I} \beta(\sigma).$$

The above remark means that the entry $I_0$ of this vector coincides with $F_k$.

We compute the vector $F_k$ using a simple transfer matrix method, in a manner similar to [9, Example 4.7.7 and Proposition 4.7.8a]. If $I$ is a subset of
[-b, a - 1] and s an integer, we denote by \( \varepsilon_s(I) \) the number \(-1\) to the power of the number of elements of I lower than s:

\[
\varepsilon_j(I) = (-1)^{\# \{i \in I \mid i < s\}}.
\]

We denote by \( T \) the matrix whose rows and columns are indexed by the \( a \)-subsets of \([-b, a - 1]\) and the entries of which are:

\[
T[I, J] = \begin{cases} 
\varepsilon_s(I)\beta_s & \text{if } I \cup \{a\} = (J + 1) \cup \{s\}, \\
0 & \text{otherwise}.
\end{cases}
\] (2)

Figure 2 shows two instances of this matrix. The first corresponds to the paths with set of steps \( S = \{0, \pm 1\} \) (Motzkin paths); the second corresponds to \( S = \{0, \pm 1, \pm 2\} \) (this is known as the basketball problem, and studied in [1]). In both cases, we prefer to represent the graph \( G \), the vertices of which are the \( a \)-subsets of \([-b, a - 1]\) and the adjacency matrix of which is \( T \).

![Graph G](image)

Figure 2: The graph \( G \) for the sets of steps \( S = \{0, \pm 1\} \) (above) and \( S = \{0, \pm 1, \pm 2\} \) (below). In each case, the subset \( I \) of \([-b, a - 1]\) corresponding to each vertex is represented by a sequence of \( \bullet \) and \( \circ \) (e.g. \( \bullet\bullet\bullet \) corresponds to the subset \( \{-1, 1\} \) of \([-2, 1]\)). The vertex corresponding to the set \( I_0 \) is colored gray.

To help the reader familiarise himself with this definition, we start by stating two basic properties of the matrix \( T \). We make use of them later.

**Lemma 2.** Let \( I \) and \( J \) be \( a \)-subsets of \([-b, a - 1]\) and assume that \( T[I, J] \neq 0 \). The two following implications hold.

1. If \(-b\) is in \( I \), then \( J \) is such that \( I \cup \{a\} = (J + 1) \cup \{-b\} \). In this case, we have \( T[I, J] = \beta_{-b} \).

2. If \( a - 1 \) is not in \( J \), then \( I \) is equal to \( J + 1 \). In this case, we have \( T[I, J] = (-1)^s \beta_a \).
We can check this lemma by looking at Figure 2 all vertices with a label starting with $a$ have only one outgoing arc, with label $\beta_{-b}$; all vertices with a label ending with $a$ have only one ingoing arc, with label $(−1)^a\beta_a$.

**Proof.** By the definition of the matrix $T$, if $I$ and $J$ are such that $T[I, J] \neq 0$, we have $I \cup \{a\} = (J + 1) \cup \{s\}$ for some $s$ in $[−b, a − 1]$.

Let us prove the first implication. As $J$ is a subset of $[−b, a − 1]$, the point $−b$ cannot be in $J + 1$. Therefore, if $−b$ is in $I$, we have $s = −b$. To prove the second implication, we note that the point $a$ is obviously in $I \cup \{a\}$ and therefore in $(J + 1) \cup \{s\}$. Therefore, if $a − 1$ is not in $J$, we have $s = a$. 

We now define some particular vertices of the graph $G$, defined as cyclic permutations of $I_0$. Specifically, if $m$ is such that $−b \leq m < a$, let $I_m$ be the vertex of $G$ defined as the set $[m, m + a − 1]$ modulo $a + b$, where the values modulo $a + b$ are taken in $[−b, a − 1]$. Since $I_{−b} = I_a$, these vertices number $a + b$.

**Lemma 3.** If $0 < m \leq a$, any walk in the graph $G$ going from $I_m$ visits $I_0$. If $−b \leq m < 0$, any walk in the graph $G$ going backwards from $I_m$ visits $I_0$.

**Proof.** The first result stems from the fact that $−b$ is in $I_m$ if $0 < m \leq a$. Lemma 2 entails that the only arc of $G$ going from $I_m$ goes to $I_{m−1}$. Thus, any walk going from $I_m$ reaches $I_0$ in $m$ steps. Likewise, the second result stems from the fact that $a − 1$ is not in $I_m$ if $−b \leq m < 0$. Lemma 2 entails that the only arc of $G$ going to $I_m$ goes from $I_{m+1}$. This means that any walk going backwards from $I_m$ reaches $I_0$.

**Proposition 4.** The vectors $F_k$ satisfy for $k \geq 0$:

$$F_{k+1} = TF_k.$$

**Proof.** To prove the proposition, we show that for every $a$-subset $I$ of $[−b, a − 1]$, we have:

$$F_{k+1}[I] = \sum J T[I, J]F_k[J].$$

To do this, we let $σ$ be a $I$-permutation of order $k + 1$ such that $β(σ) \neq 0$. Let $s = σ(0)$ be the head of $σ$ and $τ$ be its tail, defined above. Since the number of inversions created by 0 in the permutation $σ$ is equal to the number of elements of $I$ lower than $s$, we have:

$$β(σ) = \varepsilon_s(τ) β_s β(τ).$$

It remains to show that $τ$ is a $J$-permutation, where $I \cup \{a\} = (J + 1) \cup \{s\}$. Since $β_s \neq 0$, we have $s \in [−b, a]$. Thus, we may write:

$$σ : \mathbb{N} → \{s\} \cup (I \cup \{a\} \setminus \{s\}) \cup \mathbb{N}_{\geq a+1}.$$  

From this, we deduce:

$$τ : \mathbb{N} → J \cup \mathbb{N}_{\geq a}.$$  

From its definition, the set $J$ is a priori an $a$-subset of $[−b − 1, a − 1]$; however, since $β(τ) \neq 0$, the point $−b − 1$ cannot be in $J$. This finishes the proof.

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From Proposition 4 we derive the following theorem, which already appears in [3] but is found using completely different methods.

**Theorem 5.** The generating function of the polynomials $F_k$ is:

$$
\sum_{k \geq 0} F_k z^k = \frac{N(z)}{D(z)},
$$

where $D(z)$ is the determinant of $1 - zT$ and $N(z)$ is the $(I_0, I_0)$ cofactor of the same matrix.

Moreover, the degree of the polynomial $D(z)$ is $(a+b)^2$; the degree of $N(z)$ is $(a+b) - a - b$.

**Proof.** The only possible $I$-permutation of order 0 is the identity, which is an $I_0$-permutation. This implies that the polynomial $F_0^I$ is 1 if $I = I_0$ and 0 otherwise. With Proposition 4 we deduce that the polynomial $F_k$ is equal to the entry $(I_0, I_0)$ in the matrix $T^k$.

The generating function $\sum_{k \geq 0} F_k z^k$ is therefore equal to the entry $(I_0, I_0)$ in the matrix $(1 - zT)^{-1}$. The announced form follows from Cramer’s rule.

To compute the degree of the polynomial $D(z)$, we let $d = (a+b)$. Since $T$ is a $d \times d$ matrix, the polynomial $D(z)$ has degree at most $d$; the coefficient of $z^d$ in this polynomial is (up to a sign) $\det(T)$. Denoting by $S_d$ the set of permutations of the set of $a$-subsets of $[\{0, a - 1\}]$, we have:

$$
\det(T) = \sum_{\pi \in S_d} \varepsilon(\pi) \prod_{I} T[I, \pi(I)].
$$

Let $\pi$ be a permutation with a nonzero contribution in this sum. Lemma 2 asserts that:

1. if $-b$ is in $I$, then $I \cup \{a\} = (\pi(I) + 1) \cup \{-b\}$;
2. if $a - 1$ is not in $\pi(I)$, then $I = \pi(I) + 1$.

Condition 1 determines $\pi(I)$ if $-b$ is in $I$; Condition 2 determines $\pi(I)$ if $-b$ is not in $I$. The permutation $\pi$ is thus uniquely determined. Moreover, the values of $T[I, \pi(I)]$ are given by Lemma 2. This gives, up to a sign, the value of $\det(T)$:

$$
\det(T) = \pm \beta_{-b} \beta_a^{a+b-1} \beta_a^{a+b-1},
$$

which is nonzero. Therefore, the polynomial $D(z)$ has degree $d$.

To compute the degree of $N(z)$, we adopt a more combinatorial point of view: we regard the determinant $D(z)$ as the generating function of configurations of cycles of the graph $\mathcal{G}$, counted up to a sign, where $z$ accounts for the number of vertices visited by the configuration. Let $\pi_0$ be the unique permutation, defined above, that contributes to the dominant term of $D(z)$; the permutation $\pi_0$ can be interpreted as the only configuration of cycles that visits all vertices of $\mathcal{G}$.

The cofactor $N(z)$ is the generating function of the configurations of cycles that avoid the vertex $I_0$. By Lemma 3 such a configuration cannot visit any of the vertices $I_m$ (since if it would, it would also visit $I_0$). This means that $N(z)$ has degree at most $d - a - b$. Moreover, we easily check that the vertices $I_m$ form a cycle of the configuration $\pi_0$; by removing this cycle, we thus obtain a configuration of cycles visiting $d - a - b$ vertices and not visiting $I_0$. Therefore, the degree of $N(z)$ is exactly $d - a - b$. 

\[\square\]
As examples, let us compute the polynomials $D(z)$ and $N(z)$ corresponding to the two examples of Figure 2. In the case of Motzkin paths $S = \{0, \pm 1\}$, let us take the weights $\omega_0 = 0$ and $\omega_1 = \omega_{-1} = t$ (i.e., the case of standard Dyck paths). Theorem 5 shows that the polynomials $F_k$ satisfy:

$$\sum_{k \geq 0} F_k z^k = \frac{1}{1 - z + t^2 z^2},$$

which is equivalent the recurrence relation:

$$F_0 = F_1 = 1, \quad F_k = F_{k-1} - t^2 F_{k-2} \quad \text{if } k \geq 2.$$

These polynomials are commonly known as the Fibonacci polynomials, due to the similarities with the recurrence relation of the Fibonacci numbers.

In the basketball case $S = \{0, \pm 1, \pm 2\}$, let us take the weights $\omega_0 = 0$, $\omega_1 = \omega_{-1} = t_1$ and $\omega_2 = \omega_{-2} = t_2$. In this case, the polynomials $D(z)$ and $N(z)$ are:

$$D(z) = (1 + t_2 z)^2(1 - z - 2t_2 z + t_1^2 z^2 + 2t_2 z^2 + 2t_2^2 z^2 - t_2^2 z^3 - 2t_2^3 z^3 + t_4^2 z^4);$$

$$N(z) = (1 + t_2 z)(1 - t_2 z).$$

As we can see, a simplification by a factor of $1 + t_2 z$ occurs in the computation of the fraction $N(z)/D(z)$. This implies that the polynomials $F_k$ follow a linear recurrence relation of order 5 instead of 6, as shown in [1]. The factorisation of the polynomial $D(z)$ is also predicted in [3]; it is linked to the fact that the set $S$ and the weights $\beta_s$ are symmetric.

3 Bounded meanders

We enumerate bounded meanders in the same manner as bounded excursions. Specifically, the generating function $E_{k,\ell}$ counts walks from 0 to $\ell$ in the graph described by the matrix $A_k$. Therefore, we have

$$E_{k,\ell} = \frac{F_{k,\ell}}{F_{k+1}},$$

where $F_{k,\ell}$ is the $(\ell, 0)$ cofactor of the matrix $1 - A_k$. Obviously, we have $F_{k,0} = F_k$.

We again compute the cofactors $F_{k,\ell}$ using a transfer matrix method. We start by writing $F_{k,\ell}$ in terms of the determinant of the matrix $1 - A_k$ with the $\ell$th row and the 0th column cut. This reads:

$$F_{k,\ell} = (-1)^{\ell} \det \begin{pmatrix} \beta_{j-i+1} & \text{if } i < \ell \\ \beta_{j-i} & \text{if } i \geq \ell \end{pmatrix}_{0 \leq i,j \leq k-1}$$

$$= (-1)^{\ell} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=0}^{\ell-1} \beta_{\sigma(i)-i+1} \prod_{i=\ell}^{k-1} \beta_{\sigma(i)-i}.$$
For every integer $s$, we set $\tilde{\beta}_s = -\beta_{s+1}$. We also define, for every permutation $\sigma$ of order $k$, the quantity:

$$\beta_\ell(\sigma) = \varepsilon(\sigma) \prod_{i=0}^{\ell-1} \tilde{\beta}_{i-\sigma(i)} \prod_{i=\ell}^{k-1} \beta_{i-\sigma(i)}.$$ 

We rewrite the above formula as:

$$F_{k,\ell} = \sum_{\sigma \in \tilde{S}_k} \beta_\ell(\sigma).$$

Let now $I$ be an $a-1$-subset of the set $[-b - 1, a - 2]$. Let $\tilde{S}_k(I)$ be the set of $I$-permutations of order $k$ with respect to the set $\tilde{S} = S - 1$. We define the following polynomial:

$$F_k(I, \ell) = \sum_{\sigma \in \tilde{S}_k(I)} \beta_\ell(\sigma).$$

We also denote by $F_{k,\ell}$ the vector whose $I$-component is $F_k(I, \ell)$.

Let $T$ be the transfer matrix defined in Section 2 corresponding to the set $\tilde{S}$ and the weights $\tilde{\beta}_s$. Let $U$ be the matrix whose rows are indexed by the $a-1$-subsets of $[-b - 1, a - 2]$, whose columns are indexed by the $a$-subsets of $[-b, a - 1]$ and whose entries are:

$$U[I, J] = \begin{cases} 1 & \text{if } I \cup \{a-1\} = J, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We also let $\tilde{G}$ be the graph with adjacency matrix $\tilde{T}$. Let $H$ be the graph $G \cup \tilde{G}$, with additional arcs between the vertices of $G$ and $\tilde{G}$ coded by the matrix $U$. Examples are shown in Figure 3.

**Proposition 6.** The vectors $F_{k,\ell}$ satisfy for $k \geq \ell \geq 0$:

$$F_{k+1,\ell+1} = \tilde{T}F_{k,\ell};$$

$$F_{k,0} = UF_k.$$ 

**Proof.** The proof of the first identity follows the same lines as that of Proposition 4. Let $I$ be an $a-1$-subset of $[-b - 1, a - 2]$ and $\sigma$ be in $\tilde{S}_k(I)$. Let $s$ be the head and $\tau$ be the tail of $\sigma$. We have:

$$\beta_{\ell+1}(\sigma) = \varepsilon_s(I)\tilde{\beta}_s\beta_\ell(\tau).$$

The rest of the proof is identical to that of Proposition 4.

To prove the second identity, we let $I$ be an $a-1$-subset of $[-b - 1, a - 2]$ and $\sigma$ be in $\tilde{S}_k(I)$. By definition, we have:

$$\beta_0(\sigma) = \beta(\sigma).$$

Moreover, assume that $\beta_0(\sigma) = \beta(\sigma) \neq 0$. This means that $-b - 1$ cannot be in $I$. Therefore, the set $J = I \cup \{a - 1\}$ is an $a$-subset of $[-b, a - 1]$. Again by definition, the mapping $\sigma$ is in $\tilde{S}_k(I)$. This completes the proof. \(\Box\)
Theorem 7. The bivariate generating function of the polynomials $F_{k,\ell}$ satisfies:

$$
\sum_{k \geq \ell \geq 0} F_{k,\ell} u^k z^k = \sum_{k \geq 0} F_k(u) z^k = \frac{\tilde{N}(u, z)}{D(u) D(z)},
$$

where $D(z)$ is the determinant of $1 - zT$, $\tilde{D}(z)$ is the determinant of $1 - z\tilde{T}$, and $\tilde{N}(u, z)$ may be computed as:

$$
\tilde{N}(u, z) = \sum_{U[i, J] = 1} \text{Cof}[I, \tilde{I}_0](1 - uz\tilde{T}) \text{Cof}[I_0, J](1 - zT).
$$

The polynomial $D(z)$ has degree $(a+b)$, the polynomial $\tilde{D}(z)$ has degree $(a+b)$, and the polynomial $\tilde{N}(u, z)$ has a dominant term in

$$
z^{(a+b+1)} - a - b - 1 (a+b) - a.
$$

Proof. As the only $I_0$-permutation of order 0 is the identity, Proposition 6 entails that the polynomial $F_{k,\ell} z^k$ is equal to the entry $(\tilde{I}_0, I_0)$ in the matrix $T^\ell U T^{k-\ell}$. In other terms, $F_{k,\ell}$ is the generating function of walks from $I_0$ to $I_0$ in the graph $H$ taking $\ell$ steps in the graph $\tilde{G}$ and $k - \ell$ steps in the graph $G$. The generating function $\sum_{k,\ell} F_{k,\ell} u^k z^k$ is thus equal to the entry $(\tilde{I}_0, I_0)$ in the matrix $(1 - uzT)^{-1}U(1 - zT)^{-1}$. This yields the announced form.

Let us now compute the degrees; let $\tilde{d} = (a+b)$. Theorem 6 shows that the polynomial $\tilde{D}(z)$ has degree $\tilde{d}$.

To compute the dominant term of the polynomial $N(u, z)$, we first remark that if $I$ and $J$ are such that $U[I, J] = 1$, we have $-b-1 \not\in I$ and $a-1 \in J$. This implies that Lemma 5 is still valid in the graph $H$. Let $I_m$, for $-b-1 \leq m \leq a-1$, be the vertices of $\tilde{G}$ defined in the same way as the vertices $I_m$. Since Lemma 2 is valid, Lemma 3 is also valid regarding both the vertices $I_m$ and $\tilde{I}_m$.

We now consider the polynomial $N(u, z)$. This polynomial is the generating function of configurations in the graph $H$ composed of elementary cycles and a self-avoiding walk going from $I_0$ to $I_0$; the variable $u$ takes into account the number of arcs of $\tilde{G}$ in the configuration. By Lemma 3 such a configuration cannot visit a vertex $I_m$ with $0 \leq m \leq a-1$ (since it would contain an arc going into $I_0$), nor can it visit a vertex $I_m$ with $-b \leq m < 0$ (since it would contain an arc going from $I_0$). This proves that the dominant term is at most in

$$
z^{\tilde{d}-a+d-b-1}(u)^{d-a}.
$$

Let $\bar{\pi}_0$ and $\pi_0$ be the only configurations of cycles visiting all vertices of the graphs $\tilde{G}$ and $G$, respectively. Consider the configuration consisting of:

- all cycles of $\bar{\pi}_0$ except the one containing the vertices $\tilde{I}_m$;
- all cycles of $\pi_0$ except the one containing the vertices $I_m$;
- the self-avoiding walk $\tilde{I}_0 \rightarrow \cdots \rightarrow \tilde{I}_{-b} \rightarrow I_a \rightarrow \cdots \rightarrow I_0$.

We check that this configuration contains $\tilde{d} - a$ arcs of $G$ and $d - b - 1$ arcs of $\tilde{G}$. We thus derive the dominant term of the polynomial $N(u, z)$.

\[ \square \]
We now consider the generating function $M_k$ of all meanders of height at most $k$ regardless of final height. From (3), we find
\[ M_k = \frac{F_k(1)}{F_{k+1}}, \]
where the polynomial $G_k$ is the sum of $F_{k,\ell}$ for all $\ell$ between 0 and $k$. The generating function of the polynomials $G_k$ is found by setting $w = 1$ in the expression of Theorem 7 this proves that the polynomials $G_k$ follow a linear recurrence relation of order $(a_n + b_n + 1)$.

Let us now take the two examples detailed in Section 2 The case of Dyck paths (Figure 3 left) is very simple since the graph $\tilde{G}$ has only one vertex. If we set $\omega_0 = 0$ and $\omega_1 = \omega_{-1} = t$, the generating function of the polynomials $F_{k,\ell}$ is:
\[ \sum_{k \geq \ell \geq 0} F_{k,\ell} u^\ell z^k = \frac{1}{(1 - tz)(1 - z + t^2 z^2)}. \]
In other words, we have:
\[ F_{k,\ell} = t^\ell F_{k-\ell}. \]

Let us now examine the case where $S = \{0, \pm 1, \pm 2\}$, $\omega_0 = 0$, $\omega_1 = \omega_{-1} = t_1$ and $\omega_2 = \omega_{-2} = t_2$ (Figure 3 right). The polynomials $\tilde{D}(z)$ and $\tilde{N}(u, z)$ are:
\[ \tilde{D}(z) = 1 - t_1 z - t_2 z^2 - t_1 t_2 z^3 + t_2^2 z^4; \]
\[ \tilde{N}(u, z) = (1 + t_2 z)(1 - t_2 z + t_1 t_2 u z^2 - t_2^3 u^2 z^3 + t_2^4 u^2 z^4). \]
Once again, a simplification by a factor of $1 + t_2 z$ occurs in the computation of the generating function $\sum_{k,\ell} F_{k,\ell} u^\ell z^k$.

4 Symmetric set of steps

We now consider the special case where the set of steps $S$ is symmetric, that is, where $-S = S$ and $\omega_i = \omega_j$ for all $i, j \in S$. Bousquet-Mélou already considers this case and shows (3) that the generating function $E(t)$ is canceled by a polynomial of degree $2^n$ instead of $(\binom{2n}{n})$.

While we were not able to recover Bousquet-Mélou’s results with our methods, we show that another phenomenon occurs: namely, the polynomials $F_k$ factor into two parts, and simplifications occur in the computation of the generating functions of meanders $E_{k,\ell}$ and $M_k$. These simplifications are basically due to the fact that as $S$ is symmetric, the entry $(i, j)$ of the matrix $A_k$ is identical to the entry $(k - i, k - j)$. Define the following two matrices:
\[ A_k^+ = \begin{pmatrix} \omega_{j-i} + \begin{cases} \omega_{k-j-i} & \text{if } j < k/2 \\ 0 & \text{if } j = k/2 \end{cases} & \end{pmatrix}_{0 \leq i, j \leq k/2}, \]
\[ A_k^- = \begin{pmatrix} \omega_{j-i} - \omega_{k-j-i} \end{pmatrix}_{0 \leq i, j < k/2} \]
(all values of $\omega_s$ are taken to be 0 if $s$ is not in $S$). If $k$ is an odd number, both matrices have the same dimension and the condition $j = k/2$ never occurs; if $k$ is an even number, the dimension of $A_k^+$ is one more than that of $A_k^-$. In
Figure 3: The graphs $\tilde{G}$ for the sets of steps $S = \{0, \pm 1\}$ (above) and $S = \{0, \pm 1, \pm 2\}$ (below). Graphical conventions are identical to those of Figure 2 with the vertex corresponding to $\tilde{I}_0$ colored gray. Arcs with weight 1, coded by the matrix $U$ and leading to vertices of the graph $\mathcal{G}$ shown in Figure 2 are also shown.

Figure 4: The graphs corresponding to the three matrices $A_3$, $A_3^+$ and $A_3^-$ with $S = \{\pm 1\}$ and $\omega_1 = \omega_{-1} = t$. They differ only by the vertex 3.
both cases, the sum of the two dimensions is \( k + 1 \). The graphs with adjacency matrices \( A_k^+ \) and \( A_k^- \) are illustrated in Figure 4.

We denote by \( F_k^+ \) and \( F_k^- \) the determinants of the matrices \( 1 - A_k^+ \) and \( 1 - A_{k-1}^- \), respectively. We also denote by \( F_{k,\ell}^+ \) and \( F_{k,\ell}^- \) the \((\ell,0)\) cofactors of the matrices \( 1 - A_k^+ \) and \( 1 - A_k^- \), respectively.

**Theorem 8.** The polynomial \( F_k \) satisfies for all integers \( k \):

\[
F_k = F_k^+ F_k^-.
\]

Moreover, the generating functions \( E_{k,\ell} \) satisfy, for all \( \ell \) such that \( 0 \leq \ell < k/2 \):

\[
E_{k,\ell} + E_{k,k-\ell} = \frac{F_{k,\ell}^+}{F_{k+1}^+}; \quad E_{k,\ell} - E_{k,k-\ell} = \frac{F_{k,\ell}^-}{F_{k+1}^-}.
\]

Finally, if \( k \) is even and \( \ell = k/2 \), we have:

\[
E_{k,\ell} = \frac{F_{k,\ell}^+}{F_{k+1}^-}.
\]

**Proof.** Let \( B = (b_0, \ldots, b_k) \) be the canonical basis of the underlying vector space of the matrix \( A_k \). We denote by \( B_0, B_1 \) and \( B_2 \) the following collections of vectors:

\[
B_0 = (b_i)_{0 \leq i < k/2}; \quad B_1 = (b_{k/2}) \text{ if } k \text{ is even and } \emptyset \text{ otherwise}; \quad B_2 = (b_{k-i})_{0 \leq i < k/2}.
\]

Since the entry \((i, j)\) of the matrix \( A_k \) is equal to the entry \((k-i, k-j)\), the matrix \( A_k \) written as a block matrix with respect to the basis \((B_0, B_1, B_2)\) looks like:

\[
A_k = \begin{pmatrix} B & U & C \\ V & x & V \\ C & U & B \end{pmatrix}.
\]

Let \( P \) be the following passage matrix:

\[
P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix},
\]

where \( 1 \) denotes the identity matrix of the appropriate dimension. We compute:

\[
P^{-1}A_kP = \begin{pmatrix} B + C & U & 0 \\ 2V & x & 0 \\ 0 & 0 & B - C \end{pmatrix} = \begin{pmatrix} A_k^+ & 0 \\ 0 & A_k^- \end{pmatrix}.
\]

All identities of the theorem are readily derived from this form.
This result shows that the generating functions $E_{k,\ell}$ can be computed using smaller polynomials than expected, since the matrices $A_{k}^{+}$ and $A_{k}^{-}$ (and thus their determinants and cofactors) are about twice as small as $A_{k}$. Moreover, we can derive from the proposition a simplified expression for the generating function of meanders $M_{k}$. By writing $M_{k}$ as the sum of $E_{k,\ell}$ for all $0 \leq \ell \leq k$ and grouping the terms by pairs, we find:

$$M_{k} = \frac{F_{k}^{+}(1)}{F_{k+1}}.$$  \hspace{1cm} (6)

This expression involves smaller polynomials that \[5\] (a simplification occurs by a factor of $F_{k+1}$).

**Theorem 9.** The generating functions of the polynomials $F_{k}^{+}$ and $F_{k}^{-}$ are of the form:

$$\sum_{k \geq 0} F_{k}^{+} z^{k} = \frac{N^{+}(z)}{D(z^{2})}; \quad \sum_{k \geq 0} F_{k}^{-} z^{k} = \frac{N^{-}(z)}{D(z^{2})},$$

where $D(z)$ is the polynomial defined in Theorem \[2\] and $N^{+}(z)$ and $N^{-}(z)$ are polynomials. Moreover, the bivariate generating functions of the polynomials $F_{k,\ell}^{+}$ and $F_{k,\ell}^{-}$ are of the form:

$$\sum_{k,\ell \geq 0} F_{k,\ell}^{+} u^k z^{k} = \frac{\tilde{N}^{+}(u,z)}{D(uz^2)D(z^2)}; \quad \sum_{k,\ell \geq 0} F_{k,\ell}^{-} u^k z^{k} = \frac{\tilde{N}^{-}(u,z)}{D(uz^2)D(z^2)},$$

where $\tilde{D}(z)$ is the polynomial defined in Theorem \[4\] and $\tilde{N}^{+}(u,z)$ and $\tilde{N}^{-}(u,z)$ are polynomials.

**Proof.** This theorem is a consequence of a fact hinted at in Figure \[4\]; the matrices $A_{k}^{+}$ and $A_{k}^{-}$ are nearly identical to the matrices $A_{k+1}$ and $A_{k-1}$, respectively, where $k^{+} = \left\lfloor \frac{k}{2} \right\rfloor$ and $k^{-} = \left\lfloor \frac{k}{2} - 1 \right\rfloor$. More precisely, the term $\omega_{k-j-i}$ is zero if $k-j-i > a$; since $j \leq k/2$, this is true whenever $i < k/2 - a$.

For simplicity, we only prove the results associated to the matrix $A_{k}^{-}$, but the case of $A_{k}^{+}$ is identical. The proof follows the same techniques used in the previous sections. If $I$ is a $a$-subset of $[-a, a-1]$, we denote by $F_{k+1}^{-}$ the vector whose $I$-coefficient is:

$$F_{k}^{-}[I] = \sum_{\sigma \in \mathcal{S}_{k}^{a}} \varepsilon(\sigma) \prod_{i=0}^{k-1} (\beta_{\sigma(i)-1} - \omega_{k-1-\sigma(i)-1}).$$

Assume now that $k-1 > 2a$ and examine the first term of the product: the above remark entails that $\omega_{k-\sigma(i)-1} = 0$, which means that the first term is equal to $\beta_{\sigma(i)-1}$. This allows us to repeat the proof of Proposition \[4\]. As the matrix $A_{k}^{-}$ minus its first row and first column is equal to $A_{k-2}^{-}$, we find:

$$F_{k}^{-} = T F_{k-2}^{-}.$$  

In the same way, we define the vectors $F_{k,\ell}^{-}$; by repeating the proof of Proposition \[6\] we find, if $k$ is sufficiently large:

$$F_{k+2,\ell+2}^{-} = \hat{T} F_{k,\ell}^{-}; \quad F_{k,0}^{-} = UF_{k}^{-}.$$  

All the identities of the theorem are derived from these recurrence relations. \hspace{1cm} $\blacksquare$
As an example, we take the Fibonacci polynomials, corresponding to the set $S = \{\pm 1\}$ (see Section 2). The values of the polynomials $F_k^+$ and $F_k^-$ are given by:

\[
\begin{align*}
F_{2k}^+ &= F_k - tF_{k-1}; & F_{2k+1}^+ &= F_{k+1} - t^2F_{k-1}; \\
F_{2k}^- &= F_k + tF_{k-1}; & F_{2k+1}^- &= F_k.
\end{align*}
\]

Obviously, these four sequences of polynomials follow the same recurrence relation as the polynomials $F_k$.

References

[1] Arvind Ayyer and Doron Zeilberger. The number of [old-time] basketball games with final score $n$ where the home team was never losing but also never ahead by more than $w$ points. *Electron. J. Combin.*, 14(1):Research Paper 19, 8 pp. (electronic), 2007.

[2] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002. Selected papers in honour of Maurice Nivat.

[3] Mireille Bousquet-Mélou. Discrete excursions. *Sém. Lothar. Combin.*, 57:Art. B57d, 23, 2006/08.

[4] Mireille Bousquet-Mélou and Yann Ponty. Culminating paths. *Discrete Math. Theor. Comput. Sci.*, 10(2):125–152, 2008.

[5] Philippe Duchon. On the enumeration and generation of generalized Dyck words. *Discrete Math.*, 225(1-3):121–135, 2000. Formal power series and algebraic combinatorics (Toronto, ON, 1998).

[6] Ira M. Gessel. A factorization for formal Laurent series and lattice path enumeration. *J. Combin. Theory Ser. A*, 28(3):321–337, 1980.

[7] Jacques Labelle and Yeong Nan Yeh. Generalized Dyck paths. *Discrete Math.*, 82(1):1–6, 1990.

[8] Donatella Merlini, D. G. Rogers, Renzo Sprugnoli, and M. Cecilia Verri. Underdiagonal lattice paths with unrestricted steps. *Discrete Appl. Math.*, 91(1-3):197–213, 1999.

[9] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[10] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.