STRONGLY REAL BEAUVILLE GROUPS

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Abstract. A strongly real Beauville group is a Beauville group that defines a real Beauville surface. Here we discuss efforts to find examples of these groups, emphasizing on the one extreme finite simple groups and on the other abelian and nilpotent groups. We will also discuss the case of characteristically simple groups and almost simple groups. En route we shall discuss several questions, open problems and conjectures as well as giving several new examples of infinite families of strongly real Beauville groups.

1. Introduction

We first issue an apology/assurance. It is the nature of Beauville constructions that this article is likely to be of interest to both geometers and group theorists. The author is painfully aware of this. As a consequence there will be times when we make statements that may seem obvious or elementary to the group theorist but may seem quite surprising to the geometer.

We begin with the usual definitions to establish notation and terminology.

Definition 1. A surface $S$ is a Beauville surface of unmixed type if

- the surface $S$ is isogenous to a higher product, that is, $S \cong (C_1 \times C_2)/G$ where $C_1$ and $C_2$ are algebraic curves of genus at least 2 and $G$ is a finite group acting faithfully on $C_1$ and $C_2$ by holomorphic transformations in such a way that it acts freely on the product $C_1 \times C_2$ and
- each $C_i/G$ is isomorphic to the projective line $\mathbb{P}_1(\mathbb{C})$, and the covering map $C_i \to C_i/G$ is ramified over three points.

What makes these surfaces so easy to work with is the fact that the definition above can be translated into purely group theoretic terms — the following definition imposes equivalent conditions on the group $G$.

Definition 2. Let $G$ be a finite group. Let $x, y \in G$ and let

$$\Sigma(x, y) := \bigcup_{i=1}^{[G]} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An unmixed Beauville structure for $G$ is a pair of generating sets of elements $\{(x_1, y_1), (x_2, y_2)\} \subset G \times G$ such that $(x_1, y_1) = (x_2, y_2) = G$ and

$$(1) \quad \Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}$$

where $e$ denotes the identity element of $G$. If $G$ has a Beauville structure, then we say that $G$ is a Beauville group. Furthermore we say that the structure has type $((o(x_1), o(y_1), o(x_1 y_1)), (o(x_2), o(y_2), o(x_2 y_2)))$.

In the author’s experience, upon seeing the above definition, group theorists often retort “why record just the orders of the elements and not precisely which classes the elements belong to?” Determining precisely which class an element belongs to

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is much harder than determining its order. Furthermore, in practice, when ensuring that a set of elements satisfies condition (†) the easiest way often is to show that \( o(x_1)o(y_1)o(x_1y_1) \) is coprime to \( o(x_2)o(y_2)o(x_2y_2) \). This simple observation has been used to great effect by several authors — see [13, 19, 21, 26] among others. Furthermore, the type alone encodes substantial amounts of geometric information: the Riemann-Hurwitz formula

\[
g(C_i) = 1 + \frac{|G|}{2} \left( 1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_iy_i)} \right)
\]

tells us the genus of each of the curves used to define the surface \( S \). Indeed, whilst some groups have generating pairs that by the above formula define surfaces with the property that \( g(C) \leq 1 \), condition (†) ensures that for each \( i \) we have that \( g(C_i) \geq 2 \). Furthermore, a theorem of Zeuthen-Segre also gives us the Euler number of the surface \( S \) since

\[
e(\mathcal{S}) = 4 \left( g(C_1) - 1 \right) \left( g(C_2) - 1 \right)\frac{|G|}{|G|},
\]

which in turn gives us the holomorphic Euler-Poincaré characteristic of \( S \) from the relation \( 4\chi(\mathcal{S}) = e(\mathcal{S}) \) — see [12, Theorem 3.4].

In light of the above, we make the following non-standard definition which will be of use in what follows.

**Definition 3.** We say that a Beauville structure \( \{(x_1, y_1), (x_2, y_2)\} \) is coprime if \( o(x_1)o(y_1)o(x_1y_1) \) and \( o(x_2)o(y_2)o(x_2y_2) \) are coprime.

Given any complex surface \( S \) it is natural to consider the complex conjugate surface \( \overline{S} \). In particular, it is natural to ask whether the surfaces are biholomorphic.

**Definition 4.** Let \( S \) be a complex surface. We say that \( S \) is real if there exists a biholomorphism \( \sigma : \mathcal{S} \to \overline{\mathcal{S}} \) such that \( \sigma^2 \) is the identity map.

As is often the case with Beauville surfaces, the above geometric condition can be translated into purely group theoretic terms.

**Definition 5.** Let \( G \) be a Beauville group. We say that \( G \) is strongly real if there exists a Beauville structure \( X = \{(x_1, y_1), (x_2, y_2)\} \) such that there exists an automorphism \( \phi \in \text{Aut}(G) \) and elements \( g_i \in G \) for \( i = 1, 2 \) such that

\[ g_i\phi(x_i)g_i^{-1} = x_i^{-1} \quad \text{and} \quad g_i\phi(y_i)g_i^{-1} = y_i^{-1} \]

for \( i = 1, 2 \). In this case we also say that the Beauville structure \( X \) is a strongly real Beauville structure.

In practice we can always replace one generating pair by some conjugate of it and so we can take \( g_1 = g_2 = e \) and often this is what is done in practice.

In [6] Bauer, Catanese and Grunewald show that a Beauville surface is real if, and only if, the corresponding Beauville group and structure are strongly real.

**Example 6.** In [15] Catanese classified the abelian Beauville groups by proving the following.

**Theorem 7.** If \( G \) is an abelian group, then \( G \) is a Beauville group if, and only if, \( G \cong \mathbb{Z}_n \times \mathbb{Z}_m \) where \( \gcd(n,6) = 1 \) and \( \mathbb{Z}_n \) denotes the cyclic group of order \( n > 1 \).

This theorem immediately gives us the following.

**Corollary 8.** Every abelian Beauville group is a strongly real Beauville group making any Beauville structure for these groups strongly real.

**Proof.** If \( H \) is an abelian group, then the map \( H \to H, x \mapsto -x \) is an automorphism. \( \square \)
More recent (and group theoretic) motivation comes from the following. The absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is very poorly understood. Indeed, The Inverse Galois Problem — arguably the hardest open problem in algebra today — forms just one small part of efforts to understand $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (it amounts to showing that every finite group arises as the quotient of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by a topologically closed normal subgroup). When confronted with the task of understanding a group it is natural to consider an action of the group on some set. The group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of Beauville surfaces thanks to Grothendieck’s theory of Dessins d’enfants (“children’s drawings”). See [31, Section 11] for a more detailed discussion of this and related matters.

Henceforth we shall use the standard Atlas notation for group theoretic concepts (aside from occasional deviations to minimise confusion with geometric concepts) as described in some detail in the introductory sections of [16]. In particular, given two groups $A$ and $B$ we use the following notation.

- We write $A \times B$ for the direct product of $A$ and $B$, that is, the group whose members are ordered pairs $(a, b)$ with $a \in A$ and $b \in B$ such that for $(a, b), (a', b') \in A \times B$ we have the multiplication $(a, b)(a', b') = (aa', bb')$. Given a positive integer $k$ we write $A^k$ for the direct product of $k$ copies of $A$.
- We write $A.B$ for the extension of $A$ by $B$, that is, a group with a normal subgroup isomorphic to $A$ whose quotient is $B$ (such groups are not necessarily direct products - for instance $SL_2(5)=2.PSL_2(5)$).
- We write $A : B$ for a semi-direct product of $A$ and $B$, also known as a split extension $A$ and $B$, that is, there is a homomorphism $\phi : B \to Aut(A)$ with elements of this group being ordered pairs $(b, a)$ with $a \in A$ and $b \in B$ such that for $(b, a), (b', a') \in A : B$ we have the multiplication $(b, a)(b, a') = (bb', a\phi(b')a')$.
- We write $A \wr B$ for the wreath product of $A$ and $B$, that is, if $B$ is a permutation group on $n$ points then we have the split extension $A^n : B$ with $B$ acting in a way that permutes the $n$ copies of $A$.

In several places we shall refer to ‘straightforward computations’ or calculations that readers can easily reproduce for themselves. On these occasions either of Magma [10] or GAP [23] can easily be used to do this.

In Section 2 we will discuss the finite simple groups and in particular a conjecture of Bauer, Catanese and Grunewald concerning which of these groups are strongly real Beauville groups. In Sections 3 our attention turns to the characteristically simple groups and in particular the recent work of Jones which we push further in the cases of the symmetric and alternating groups in Section 4. We go on in Section 5 to discuss which of the almost simple groups are strongly real Beauville groups. Finally, in Section 6 we briefly discuss nilpotent groups and $p$-groups.

2. The Finite Simple Groups

Naturally, a necessary condition for being a strongly real Beauville group is being a Beauville group. Furthermore, a necessary condition for being a Beauville group is being 2-generated: we say that a group $G$ is 2-generated if there exist two elements $x, y \in G$ such that $\langle x, y \rangle = G$. It is an easy exercise for the reader to show that the alternating groups $A_n$ for $n \geq 3$ are 2-generated. In [32] Steinberg proved that the simple groups of Lie type are 2-generated and in [11] Aschbacher and Guralnick showed that the sporadic simple groups are 2-generated. We thus have that all of the non-abelian finite simple groups are 2-generated making them natural candidates for Beauville groups. This lead Bauer, Catanese and Grunewald to conjecture that aside from $A_5$, which is easily seen to not be a Beauville group,
every non-abelian finite simple group is a Beauville group - see [6, Conjecture 1] and [7, Conjecture 7.17]. This suspicion was later proved correct [19, 20, 25, 26], indeed the full theorem proved by the author, Magaard and Parker in [19] is actually a more general statement about quasisimple groups (recall that a group $G$ is quasisimple if it is generated by its commutators and the quotient by its center $G/Z(G)$ is a simple group.)

Having found that almost all of the non-abelian finite simple groups are Beauville groups, it is natural to ask which of the non-abelian finite simple groups are strongly real Beauville groups. In [6, Section 5.4] Bauer, Catanese and Grunewald wrote

“There are 18 finite simple nonabelian groups of order $\leq 15000$. By computer calculations we have found strongly [real] Beauville structures on all of them with the exceptions of $A_5$, $PSL_2(7)$, $A_6$, $A_7$, $PSL_3(3)$, $U_3(3)$ and the Mathieu group $M_{11}$.”

On the basis of these computations they conjectured that all but finitely many non-abelian finite simple groups are strongly real Beauville groups. Several authors have worked on this and many special cases are now known to be true.

- In [22] Fuertes and González-Diez showed that the alternating groups $A_n$ ($n \geq 7$) and the symmetric groups $S_n$ ($n \geq 5$) are strongly real Beauville groups by explicitly writing down permutations for their generators and the automorphisms and applying some of the classical theory of permutation groups to show that their elements had the properties they claimed. Subsequently the alternating group $A_6$ was also shown to be a strongly real Beauville group.

- In [21] Fuertes and Jones prove that the simple groups $PSL_2(q)$ for prime powers $q > 5$ and the quasisimple groups $SL_2(q)$ for prime powers $q > 5$ are strongly real Beauville groups. As with the alternating and symmetric groups, these results are proved by writing down explicit generators, this time combined with a celebrated theorem usually (but historically inaccurately) attributed to Dickson for the maximal subgroups of $PSL_2(q)$. General lemmas for lifting Beauville structures from a group to its covering groups are also used.

- Setting the case of the sporadic simple groups makes no impact on the above conjecture, there being only 26 of them. Nonetheless, for reasons we shall return to below, in [18] the author determined which of the sporadic simple groups are strongly real Beauville groups, including the ‘27th sporadic simple group’, the Tits group $^2F_4(2)^\prime$. Of all the sporadic simple groups only the Mathieu groups $M_{11}$ and $M_{23}$ are not strongly real. For all of the other sporadic groups smaller than the Baby Monster group $\mathbb{B}$ explicit words in the ‘standard generators’ [33] for a strongly real Beauville structure are given. (For those unfamiliar with standard generators, we will describe these in Section 5.) For the Baby Monster group $\mathbb{B}$ and Monster group $\mathbb{M}$ character theoretic methods are used.

As we can see from the above bullet points, several of the groups that Bauer, Catanese and Grunewald could not find strongly real Beauville structures for do indeed have strongly real Beauville structures. In particular, we note that the group $PSL_2(9) \equiv A_6$ is in fact strongly real.

Using the results mentioned above, combined with unpublished calculations, the author has pushed Bauer, Catanese and Grunewald’s original computations to every non-abelian finite simple group of order at most $100\,000\,000$ and, as we noted above, several much larger ones in [18]. Many of the smaller groups seemed to require the
use of outer automorphisms to make their Beauville structures strongly real, which explains much of the above difficulty in finding strongly real Beauville structures in certain groups. Slightly larger groups had enough conjugacy classes to be strongly real if they have too few conjugacy classes (as is the case with $A_5$, which we would intuitively expect) or if they have no outer automorphisms — a phenomenon that is extremely rare. We are thus led to the following somewhat stronger conjecture.

**Conjecture 1.** All non-abelian finite simple groups apart from $A_5$, $M_{11}$ and $M_{23}$ are strongly real Beauville groups.

To add further weight to this conjecture we verify this conjecture for the Suzuki groups $2B_2(q^{2n+1})$. Let $q = 2^{2n+1}$.

**Theorem 9.** Each of the groups $2B_2(q)$ has a strongly real Beauville structure of type $((q-1, q-1, q-1), (d_1, d_2, 2))$ where $d_1$ and $d_2$ are odd and coprime to $q-1$.

Throughout the following we shall be using the natural 4-dimensional representation of $2B_2(q)$ over the field of order $q$ as described in some detail in [35, Section 4.2]. To prove Theorem 9 we will use knowledge of the maximal subgroups of the Suzuki groups. The following lemma was proved by Suzuki — see [35, Theorem 4.1]. Here we write $E_q$ for the elementary abelian group of order $q$. Furthermore, by ‘subfield subgroup’ we mean either the subgroup $2B_2(q_{q_0})$ consisting of matrices whose entries come from a subfield of the field $F_q$ of order $F_{q_0}$ where $q_0 > 1$ divides $q$ or one of its conjugates, those appearing in Lemma 10(v) being precisely the maximal subfield subgroups.

**Lemma 10.** If $n > 1$, then the maximal subgroups of $2B_2(q)$ are (up to conjugacy).

(i) $E_q, E_q' : Z_{q-1}$, the subgroup of lower triangular matrices  
(ii) $D_{2(q-1)}$  
(iii) $Z_{q-1} : 4$  
(iv) $Z_{q-1} : 4$  
(v) $2B_2(q_{q_0})$ where $q = q_0^r$, $r$ is prime and $q_0 > 2$.

From the above the following can easily be deduced.

**Lemma 11.**

(a) If $x, y \in 2B_2(q)$ are two elements with the property that  
\[ o(x) = o(y) = o(xy) = q - 1, \]
then $\langle x, y \rangle = Z_{q-1}, E_q, E_q' : Z_{q-1}$ or $2B_2(q)$.

(b) If $x, y \in 2B_2(q)$ are two elements such that $o(x)$ and $o(y)$ have orders dividing $q \pm \sqrt{2q} + 1$ and $o(xy) = 2$, then $\langle x, y \rangle = 2B_2(q)$ or a subfield subgroup.

**Proof of Theorem 9.** For our first generating pair we consider the following elements of $2B_2(q)$ each of which are easily checked to have order 2 by direct calculation.

\[
t_1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad t_2 := \begin{pmatrix} 0 & 0 & 0 & \beta^{-1} \\ 0 & 0 & \beta^{-2^{n+1}+1} & 0 \\ \beta & \beta^{2^{n+1}+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha^{2n+1} & 0 & 1 & 0 \\ \alpha^2 & \alpha^{2n+1} & 0 & 1 \end{pmatrix}
\]
where $\alpha$ and $\beta$ are generators of the multiplicative group $F_q^\times$. The element $x_1 = t_1t_2$ has order $q - 1$. The characteristic polynomial of $y_1 = t_1t_3$ is
\[ p_1(\lambda) = \lambda^4 + \alpha^2\lambda^3 + \alpha^2\lambda^2 + \alpha^2\lambda + 1 \]
and if we set $\gamma := \beta + \beta^{-1} + \beta^{2^{-n+1} - 1} + \beta^{1-2^{-n+1}}$ then the characteristic polynomial of $t_1t_2$ is
\[ p_2(\lambda) = \lambda^4 + \gamma\lambda^3 + (\beta^{2^{-n+1}} + \beta^{-2^{-n+1} + 2} + \beta^{2^{-n+1} - 2} + \beta^{-2^{-n+1}})\lambda^2 + \gamma\lambda + 1 \]
Comparing $p_1$ with $p_2$ we see that the two polynomials are equal if we have $\gamma = \alpha^2$ and
\[ \beta^{2^{-n+1} + 2} + \beta^{2^{-n+1} - 2} + \beta^{2^{-n+1}} = \alpha^{2^{-n+2}} = (\alpha^2)^{2^{-n+1}} = \gamma^{2^{-n+1}}. \]
Since $a \mapsto a^2$ is an automorphism of our underlying field we see that the first of these equalities immediately implies the second if $(\beta + \beta^{-1} + \beta^{2^{-n+1} - 1} + \beta^{1-2^{-n+1}})2^{n+1} = \beta^{2^{n+1}} + \beta^{-2^{n+1} + 2} + \beta^{2^{n+1} - 2} + \beta^{-2^{n+1}}$.

Since $\beta^{(2^{n+1} - 1)2^{n+1}} = (\beta^{2^{n+1}})^2\beta^{-2^{n+1}} = \beta^{-2^{n+1}}$ we can choose $\alpha$ and $\beta$ to satisfy the above condition, so in particular we have that $t_1t_2$ and $t_1t_3$ have the same characteristic polynomial and thus both have order $q - 1$. Furthermore, these are both inverted by conjugation by $t_1$ since $t_1$, $t_2$ and $t_3$ all have order 2. Similarly we find that $t_1t_2t_3$ has characteristic polynomial of the correct form to have order $q - 1$. From Lemma 3(a) we see that these elements generate the group since $x_1$ and $y_1$ are not both contained in a cyclic subgroup (one of them is diagonal) and by direct calculation no one-dimensional subspace in the natural module is preserved by them so there is no proper subgroup containing each of these elements.

For the second triple we consider the matrices
\[ x_2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \delta^4 \\ 1 & \delta^4 & \delta^2 \end{pmatrix}, \quad y_2 := \begin{pmatrix} \epsilon^2 & \epsilon^4 & 0 & 1 \\ \epsilon^4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]
where $\delta, \epsilon \in F_q$ are chosen so that $\delta \neq \epsilon$ and these do not have the correct form for these elements to have order $q - 1$. Direct calculation shows that these elements do not have orders 2 or 4 and that $\alpha(x_2y_2) = 2$. These elements must, therefore, have orders that divide $q(\pm \sqrt{2q} + 1).$ Furthermore their traces are $\epsilon^2$ and $\delta^2$ which can be chosen to be in no proper subfield since $x \mapsto x^2$ is an automorphism of the field $F_q$.

These elements must therefore generate the group by Lemma 3(b). Further direct calculation shows that $x_2^q = x_2^{-1}$ and $y_2^q = y_2^{-1}$. \qed

3. Characteristically Simple Groups

Another class of finite groups that has recently been studied from the viewpoint of Beauville constructions, and seems like fertile ground for providing further examples of strongly real Beauville groups, are the characteristically simple groups that we define as follows (the definition commonly given is somewhat different from that below but in the case finite groups it is equivalent to this).

**Definition 12.** A finite group $G$ is said to be **characteristically simple** if $G$ is isomorphic to some direct product $H^k$ where $H$ is a finite simple groups.

For example, as we saw in Theorem 5 if $p > 3$ is prime then the abelian Beauville groups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ are characteristically simple.

Characteristically simple Beauville groups have recently been investigated by Jones in [29, 30] where the following conjecture is discussed.
Conjecture 2. Let $G$ be a finite non-abelian characteristically simple group. Then $G$ is a Beauville group if and only if it is a 2-generator group not isomorphic to $A_5$.

In particular, the main results of [29, 40] verify this conjecture in the cases where $H$ is any of the alternating groups; the linear groups $PSL_2(q)$ and $PSL_3(q)$; the unitary groups $PSU_3(q)$; the Suzuki groups $^2B_2(2^{2n+1})$; the small Ree groups $^2G_2(3^{2n+1})$ and the sporadic simple groups.

For large values of $k$, the group $H^k$ will not be 2-generated despite the fact that $H$ will be as discussed in Section 2. The values of $k$ for which $H^k$ is 2-generated can be surprisingly large. For example, a special case of the results alluded to in the previous paragraph is the somewhat amusing fact that

$$A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5$$

is a Beauville group, despite the fact that $A_5$ itself is not a Beauville group.

In general, the full automorphism group of $H^k$ will be the wreath product $\text{Aut}(H) \wr S_k$ where $S_k$ is the $k$th symmetric group acting on the product by permuting the groups $H$. This bounteous supply of automorphisms makes it likely that characteristically simple Beauville groups are in general strongly real.

Question 1. Which characteristically simple Beauville groups are strongly real?

As a more specific conjecture on these matters we assert the following.

Conjecture 3. If $H$ is a finite simple group of order greater than 3, then the group $H \times H$ is a strongly real Beauville group.

Note that Corollary 8 tells us that this is true for all abelian characteristically simple Beauville groups. For the nonabelian characteristically simple Beauville groups this conjecture seems rather distant given that, at the time of writing, we have neither a solution to Conjecture 1 nor do we even know if $H \times H$ for a simple group $H$ is even a Beauville group, let alone a strongly real one. Anyone tempted to extend the above conjecture to the products of a larger number of copies of simple groups should see the remarks following Lemma 14, although some hope is provided by the results proven in Section 4.

Theorem 13. Let $G$ be a strongly real Beauville group with coprime strongly real Beauville structure $\{\{x_1, y_1\}, \{x_2, y_2\}\}$. Furthermore, suppose that there exists an automorphism $\phi \in \text{Aut}(G)$ such that

$$\phi(x_1) = x_1^{-1}, \phi(y_1) = y_1^{-1}, \phi(x_2) = x_2^{-1} \text{ and } \phi(y_2) = y_2^{-1}.$$ 

Then the group $G \times G$ is a strongly real Beauville group.

Proof. Consider the following elements of $G \times G$

$$g_1 = (x_1, x_2), \quad h_1 = (y_1, y_2), \quad g_2 = (x_2, x_1) \quad \text{and} \quad h_2 = (y_2, y_1)$$

The pair $\{g_1, h_1\}$ generate the whole of $G \times G$ since the elements $g_1^{o(x_1)}$ and $h_1^{o(y_2)}$ generate the first factor whilst the elements $g_1^{o(x_1)}$ and $h_1^{o(y_1)}$ generate the second factor thanks to our hypothesis that $o(x_1)o(x_2)o(x_1y_1)$ is coprime to $o(x_2)o(y_2)o(x_2y_2)$. Similarly $(g_2, h_2) = G \times G$.

We define an automorphism $\psi \in \text{Aut}(G \times G)$ such that for every $(g, h) \in G \times G$

$$\psi(g, h) = (\phi(g), \phi(h)).$$

This automorphism clearly makes the above Beauville structure for $G \times G$ a strongly real Beauville structure.

Corollary 14. Conjecture 3 is true for each of the following groups.
Proof. For part (a) the results proved in Section 4 provide a strongly real Beauville structure for $A_n \times A_n$ for sufficiently large $n$, the smaller cases being straightforward calculations that are easily performed separately.

For part (b) we note that the strongly real Beauville structures constructed by Fuertes and Jones in [21] for the groups $PSL_2(q)$ satisfy the hypotheses of Theorem 13.

For part (c) we note that the strongly real Beauville structures for $^2B_2(2^{2n+1})$ we constructed in Theorem 9 are coprime and satisfy the hypotheses of Theorem 13.

For part (d) we note that the author’s computations alluded to in Section 2 were performed in such a way that the hypotheses of Theorem 13 are satisfied.

Finally for part (e) we observe that for all the sporadic groups, apart from the Mathieu groups $M_{11}$ and $M_{23}$, the structures given by the author in [18] satisfy the hypotheses of Theorem 13. The groups $M_{11}$ and $M_{23}$ are dealt with separately in Lemma 15 below. □

We remark that the strongly real Beauville structures for the quasisimple groups $SL_2(q)$ where $q > 5$ constructed by Fuertes and Jones in [21] also satisfy the hypotheses of Theorem 13 and so the groups $SL_2(q) \times SL_2(q)$ are also strongly real.

Unfortunately, Theorem 13 cannot be applied to the strongly real Beauville structures constructed by Fuertes and González-Diez in [22] for the symmetric and alternating groups. This is because the types of the Beauville structures in [22] fail to satisfy the coprime hypothesis since their structures use several elements of order 2. We return to this point in Section 4.

Comparing the statement of Conjecture 1 with the statement of Conjecture 3 the reader should immediately be asking “what about the alternating group $A_5$ and the Mathieu groups $M_{11}$ and $M_{23}$?” This concern is immediately addressed by the following further piece of evidence for Conjecture 3.

Lemma 15. The groups $A_5 \times A_5$, $M_{11} \times M_{11}$ and $M_{23} \times M_{23}$ are strongly real Beauville groups.

Proof. This is a straightforward computational calculation. Consider the following permutations.

$$
x_1 := (1,2,3,4,5)(6,7,8,9,10) \quad y_1 := (2,3,4)(7,10)(6,9),
x_2 := (1,4,3,2,5)(7,8,9) \quad y_2 := (1,2)(4,5)(6,9,8,7,10) \quad \text{and}
a := (1,5)(2,4)(6,10)(7,9)
$$

The set $\{x_1, y_1\}$ gives a Beauville structure for the group $A_5 \times A_5$ of type $((5,6,5),(15,10,15))$ acting intransitively on $5+5$ points as a subgroup of the symmetric group $S_{10}$. The automorphism $\alpha$ defined by conjugation by $a$ has the property that $\alpha(x_1) = x_1^{-1}$, $\alpha(y_1) = y_1^{-1}$, $\alpha(x_2) = x_2^{-1}$ and $\alpha(y_2) = y_2^{-1}$ from which we have that this Beauville structure is strongly real.

Next, the group $M_{11} \times M_{11}$. Consider the following permutations.

...
The set \( \{x_1, y_1\}, \{x_2, y_2\} \) gives a Beauville structure for the group \( M_{11} \times M_{11} \) of type \( ((11,11),(8,8,8)) \) acting intransitively on \( 11+11 \) points as a subgroup of the symmetric group \( S_{22} \). The automorphism \( \alpha \) defined by conjugation by \( a \) has the property that
\[
\alpha(x_1) = x_1^{-1}, \quad \alpha(y_1) = y_1^{-1}, \quad \alpha(x_2) = x_2^{-1} \quad \text{and} \quad \alpha(y_2) = y_2^{-1}
\]
from which we have that this Beauville structure is strongly real.

Finally for \( M_{23} \times M_{23} \) we similarly have that the permutations
\[
x_1 := (1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22)
y_1 := (1,6,10,5,2,7,4,9,11,8,3)(12,14,19,16,21,18,13,17,22,20,15)
x_2 := (1,3,9,11,10,7,2,4)(5,8)(12,14,20,22,19,21,16,13)(15,18)
y_2 := (2,6,9,4,8,3,7,5)(10,11)(12,14,20,22,19,21,16,13)(15,18)
a := (1,22)(2,21)(3,20)(4,19)(5,18)(6,17)(7,16)(8,15)(9,14)(10,13)(11,12)
\]
define a strongly real Beauville structure of type \( ((23,23),(11,11,11)) \) for the group \( M_{23} \times M_{23} \) acting intransitively on \( 23+23 \) points as a subgroup of the symmetric group \( S_{46} \).

We remark that in the examples of the above lemma, the automorphisms used are outer automorphisms that interchange the two factors. The lack of automorphisms that stop both \( M_{11} \) and \( M_{23} \) being strongly real will therefore also stop the groups \( M_{11} \times M_{11} \times M_{11} \) and \( M_{23} \times M_{23} \times M_{23} \) being strongly real Beauville groups. Furthermore it is easy to see that the permutations given in the proof of Lemma 15 can be adapted to construct a strongly real Beauville structure of type \( ((88,88),(88,88,88)) \) for the group \( M_{11} \times M_{11} \times M_{11} \times M_{11} \) and to construct a strongly real Beauville structure of type \( ((253,253,253),(253,253,253)) \) for the group \( M_{23} \times M_{23} \times M_{23} \times M_{23} \). Similarly \( A_5 \times A_5 \times A_5 \) is not a strongly real Beauville group. It follows that any extension of Conjecture 3 to products of a larger number of copies of simple groups will necessarily have a much more complicated statement. It is likely that similar remarks apply to \( M_{2k+1}^{k+1}, M_{2k}^k, M_{2k+1}^{k+1}, M_{2k}^{2k+1} A_5^{2k+1} \) and \( A_5^{2k} \) for small values of \( k \).

In light of the above it is natural to ask the following.

**Question 2.** Let \( H \) be a finite simple group, \( n \in \mathbb{Z}^+ \) and \( G = H^n \). When are inner automorphism sufficient to make \( G \) strongly real and when do outer automorphisms interchanging the factors required? Moreover does this have any geometric significance for the corresponding surfaces?

**4. THE SYMMETRIC AND ALTERNATING GROUPS**

In the last section we discussed characteristically simple groups of the form \( H \times H \) for some simple group \( H \). In this section we prove slightly stronger results in the case of the alternating groups and a related result for the symmetric groups. In
each of the below results conjugacy of elements is taken care of by the well known fact that two elements of the symmetric group are conjugate if, and only if, they have the same cycle type. We will use the following recent results of Jones.

**Lemma 16.** Let $H \leq S_n$.

(a) If $H$ is primitive and contains a cycle that fixes at least three points then $H \geq A_n$.

(b) If $H$ is transitive and contains an $m$-cycle where $m > n/2$ and $m$ is coprime to $n$ then $H$ is primitive.

(c) If $H$ is primitive and contains a cycle fixing two points then either $H \geq A_n$ or $PGL_2(q) \leq H \leq PTL_2(q)$ with $n = q + 1$ for some prime power $q$.

**Proof.** See [28] and [29, Section 6]. \hfill \Box

Before proving our main results we recall some facts about generating pairs in simple and characteristically simple groups. Let $H$ be a finite simple group. In [27] Philip Hall showed that the largest $k$ such that the characteristically simple group $H^k$ is 2-generated is equal to the number of orbits of $Aut(H)$ on generating pairs of $H$. (He proved similar results for more general $n$-tuples but we will not be needing these results here.) To show that $H^k$ for some $k$ is generated by a pair of elements it is sufficient to show that each of the ‘coordinates’ of these elements (that is, the parts of each permutation that correspond to each of the factors) are inequivalent under the action of $Aut(H)$. For $n \neq 1, 2, 6$ we have that $Aut(A_n) = S_n$ and in the case $n = 6$ the symmetric group $S_6$ is an index 2 subgroup of $Aut(A_6) \cong PTL_2(9)$.

**Lemma 17.** Let $n \geq 11$ be odd and let $k \leq (n - 6)/2$ be positive integers. Then $A_n^k$ is a strongly real Beauville group.

**Proof.** Since $A_n$ is simple for every $n > 5$ we have that, by the remarks of the previous paragraph, it is sufficient to find $k$ pairs of generating pairs $T_{ij} = \langle x_{ij}, y_{ij} \rangle$ $i = 1, 2, j = 1, \ldots, k$ such that for a fixed $i$ no $T_{ij}$ is an image of $T_{i,j'}$ under the actions of automorphisms of $A_n$ for distinct $j$ and $j'$. For our first pairs we set

$$x_{1j} = (1, \ldots, 2j + 3) \text{ and } y_{1j} = (2j + 3, \ldots, n)$$

for $1 \leq j < (n - 6)/4$. These are cycles of odd length and are thus even permutations. Their product is an $n$-cycle. It is easy to check that $\langle x_{ij}, y_{ij} \rangle$ is primitive and thus equal to $A_n$ since the group contains a cycle with at least three fixed points by Lemma \ref{lemma}(a). These elements are both inverted by the automorphism defined by conjugation by

$$t = (1, 2j)(2, 2j - 1) \cdots (j, j + 1)(2j + 2, n)(2j + 3, n - 1) \cdots ((2j + n + 1)/2, (2j + n + 3)/2)$$

which has only one fixed point, namely $2j + 1$.

For our second pairs we consider the permutations

$$x_{2j} = (1, \ldots, n - 2) \text{ and } y_{2j} = (j + 1, j + 2)(n - j - 1, n - j - 2)((n - 1)/2, n - 1)((n + 1)/2, n)$$

for $1 \leq j < (n - 5)/2$. These are again both even permutations, their product this time being an $(n - 2)$-cycle. To confirm that these elements generate the group we note that $\langle x_{2j}, y_{2j} \rangle$ is clearly transitive and so by Lemma \ref{lemma}(b) must be primitive. It now follows from Lemma \ref{lemma}(c) that $\langle x_{2j}, y_{2j} \rangle = A_n$ since $n > 9$ and the only elements of order 2 in $PTL_2(q)$ do not have the same cycle type as $y_{2j}$. (In the case $n = 9$ these permutations generate $PSL_2(8)$.) These elements are both inverted by the automorphism defined by conjugation by

$$(2, n - 2)(3, n - 3) \cdots ((n - 1)/2, (n + 1)/2)(n - 1, n)$$

which has only one fixed point, namely 1, and thus differs from $t$ solely by an inner automorphism. \hfill \Box
Lemma 18. Let $n \geq 12$ be an even integer and let $k \leq (n-8)/4$. Then $A^n_k$ is a strongly real Beauville group.

**Proof.** Again, we seek a collection of generating pairs that are not mapped to one another by automorphisms of $A_n$.

For our first pairs we consider the following elements.

$$x_{1j} = (1, \ldots, 2j + 5) \text{ and } y_{1j} = (n, \ldots, 2j + 5, 2j + 4)$$

for $1 \leq j \leq (n-8)/4$. These are cycles of odd length and are thus even permutations. Their product is an $(n-1)$-cycle. It is easy to check that $(x_{1j}, y_{1j})$ is primitive and thus equal to $A_n$ since the group contains a cycle with at least three fixed points by Lemma 16(a). These elements are both inverted by the automorphism defined by conjugation by

$$t = (2j+4,2j+5)(1,2j+3) \cdots (j+1,j+3)(2j+6,n) \cdots ((n+2j+4)/2-1, (n+2j+4)/2+1)$$

which has precisely two fixed points namely $j + 2$ and $(n + 2j + 4)/2$.

For our second pair of generators we consider the permutations

$$x_{2j} = (1, \ldots, n-2)(n-1,n) \text{ and } y_{2j} = (n/2, (n-2)/2, n-1)(j+1,j,n,n-j-1,n-j-2)$$

for $1 \leq j < (n-2)/2$. These are both even permutations and their product is an $(n-3)$-cycle that fixes the points $j, n-j-2$ and $(n-2)/2$. The subgroup $\langle x_{2j}, y_{2j} \rangle$ fixes the point $(n-2)/2$ and is transitive on the remaining $n-1$ points. It follows that the group $\langle x_{2j}, y_{2j} \rangle$ is 2-transitive and is therefore primitive. Since the group $\langle x_{2j}, y_{2j} \rangle$ also contains the 3-cycle $y_{2j}^3$ (and the 5-cycle $y_{2j}^5$) it is equal to $A_n$ by Lemma 16(a). These elements are both inverted by the automorphism defined by conjugation by

$$t = (1, n-2)(2, n-3) \cdots ((n-2)/2, n/2)$$

which has precisely two fixed points, namely $n-1$ and $n$ and thus differs from $t$ solely in an inner automorphism.

When considering the alternating groups it is natural to seek similar results for the symmetric groups. Unfortunately, here we are somewhat limited: if $k > 2$ then $S^n_k$ is not 2-generated since its abelianisation is $Z^2_k$ and this is not 2-generated. It follows that we can only find analogous results for $k \leq 2$ and since $k = 1$ comes straight from the work of Fuertes and González-Diez we are left only to consider the case $k = 2$. Note that in this case the outer automorphism of $S_n \times S_n$ that interchanges the two factors is useless since the only permutations that are inverted by this automorphism are even.

**Lemma 19.** For $n \geq 5$ the group $S_n \times S_n$ is strongly real.

**Proof.** We will explicitly construct our Beauville structure in the case of $n$ even and then describe the differences in the case of $n$ odd.

For our first pair we consider the following elements.

$$x_1 = (1, \ldots, n-1)(2n-1, 2n) \text{ and } y_1 = (n, n-1)(n+1, \ldots, 2n-1)$$

The product of these permutations is a pair of $n$-cycles. It is easy to check that $\langle x_1^n, y_1^n \rangle$ generates the first of the two factors whilst $\langle x_1^{n-1}, y_1^n \rangle$ generates the second and so $\langle x_1, y_1 \rangle$ is the whole group. These elements are both inverted by the automorphism defined by conjugation by

$$t = (1, n-2)(2, n-3) \cdots (n/2, n/2 - 1)(n+1, 2n-2) \cdots (3n/2, 3n/2 - 1)$$

which has precisely four fixed points, namely $n-1$, $n$, $2n-1$ and $2n$. 

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For our second pair of generators we consider the permutations
\[ x_2 = (1, 2, 3)(4, \ldots, n)(n+4, n+3, n+2, n+1) \]
and \[ y_2 = (4, 3, 2, 1)(n+1, n+2, n+3)(n+4, \ldots, 2n) \]
The product of these permutations is a pair of \( n-2 \) cycles. It is easy to check that \( \{x_1^t, y_1^{3(n-3)}\} \) generates the first factor (this group is easily seen to be 2-transitive, and thus primitive, by considering conjugates of \( y_1^{3(n-1)} \) under powers of \( x_1^t \) and since this group contains a 4-cycle it contains the whole of \( S_n \) by [28 Corollary 1.3]). Similarly \( \{x_1^{3(n-1)}, y_1^{t}\} \) generates the second factor. These elements are both inverted by the automorphism defined by conjugation by
\[ (1,3)(5,n) \cdots (n/2+2, n/2+3)(n+1, n+3)(n+5, 2n) \cdots (3n/2+2, n/2+3) \]
which has precisely four fixed points, namely 2, 4, n + 2 and n + 4 and thus differs from \( t \) solely in an inner automorphism. (The case \( n = 6 \) requires a little care — using
\[ x_2 = (1, 2, 3, 4)(10, 11, 12), y_2 = (4, 5, 6)(7, 8, 9, 10), \]
and the automorphism defined by \( (1,3)(5,6)(7,9)(11,12) \) avoids being trapped inside copies of \( S_5 \) acting transitively on six points.)

If \( n \) is odd then for the first pair we need only replace the \( (n-1) \)-cycles with \( n \)-cycles to ensure that the elements have odd parity and for the automorphism instead use
\[ t = (n, n-1)(n-1,1) \cdots ((n-1)/2-1, (n-1)/2+1) \cdots \]
\[ (2n, 2n-1)(2n-2, n+1) \cdots ((3n-1)/2-1, (3n-1)/2+1). \]
For the second generating pair we must now replace the 4-cycles with some longer \( p \)-cycle whose length is coprime to \( 3(n-3) \) (if 5 fails then \( n \geq 2 \times 5 + 3 \) and we can try 7; if both 5 and 7 fail then \( n \geq 2 \times 5 \times 7 + 3 \) and we can try 11 etc.). The permutation of order 2 for the automorphism needs to be adjusted in the obvious manner (i.e. \( (1,3)(4,p)(5,p-1) \cdots (p+1,n)(p+2, n-1) \cdots \)). Again, the smallest case needs separate attention but it is easily checked that if
\[ x_1 = (1,4)(2,5)(6,10)(7,8,9) \text{ and } y_1 = (1,5)(2,3,4)(6,9)(7,10) \]
and
\[ x_2 = (1,2,3,4,5)(6,7,9,10) \text{ and } y_2 = (5,4,2,1)(10,9,8,7,6) \]
then \( \{x_1, y_1\}, \{x_2, y_2\} \) is a strongly real Beauville structure whose elements are inverted by conjugation by the element \( (1,5)(2,4)(6,10)(7,9). \)

5. Almost Simple Groups

Let \( G \) be a group. Recall that we say \( G \) is almost simple if there exists a simple group \( S \) such that \( S \leq G \leq \text{Aut}(S) \). For example, any simple group is almost simple, as are the symmetric groups. Given our earlier remarks on the finite simple groups it is natural to ask the following.

Question 3. Which of the almost simple groups are strongly real?

This is particularly pertinent in light of Fuertes and González-Diez proof that the symmetric groups \( S_n \) for \( n > 5 \) are strongly real. Unfortunately, the general picture here is much more complicated with many almost simple groups not even being Beauville groups, let alone strongly real Beauville groups. Worse, infinitely many of the almost simple groups are not even 2-generated: the smallest example is \( PSL_4(9) \) whose outer automorphism group is \( Z_2 \times D_8 \) (and more generally, if \( p \) is an odd prime and \( r \) is an even positive integer then \( \text{Aut}(PSL_4(p^r)) \) is not 2-generated). We can at least add the following to the list.
Theorem 20. The non-simple almost simple sporadic groups are strongly real Beauville groups.

Before proceeding to the proof of Theorem 20 we make the following remarks for those who are unfamiliar with standard generators of finite groups (those who are familiar with them may skip to the proof in the next paragraph). Any given group will have many generating sets and in particular if \( x, y \in G \) are such that \( \langle x, y \rangle = G \) then \( \langle x^g, y^g \rangle = G \) for any \( g \in G \). To provide some standardisation to computational group theory, Wilson [33] introduced the notion of ‘standard generators’. These are generators for a group that are unusually easy to find and are specified in terms of which conjugacy classes that they belong to (which can often be determined solely from their orders) and which classes some word(s) in these elements belong to. Representatives for many of the finite simple groups and various other groups closely related to them may be found in explicit permutations and/or matrices for many of their most useful representations on the web-based Atlas of Group Representations [34].

To construct our Beauville structures that prove Theorem 20 we proceed as follows. We first recall some well-known facts about the sporadic simple groups. If \( G \) is one of the 27 sporadic simple groups (including the Tits group \( ^2\text{F}_4(2)' \)) then the outer automorphism group of \( G \) has order at most 2 and that in all cases in which there exists a non-trivial outer automorphism \( \text{Aut}(G) \) is a non-split extension, apart from the Tits group \( ^2\text{F}_4(2)' \) and may thus be written \( G : 2 \) in ATLAS notation (see Section 1). Let \( G \) be a simple group such that \( \text{Aut}(G) = G : 2 \). Let \( t, t' \in G : 2 \) have order 2 such that one of the elements lies in \( G \) and the other lies in \( G : 2 \setminus G \). For \( i = 1, 2 \) we define the elements \( x_i = tt^{g_i} \) for some \( g_i \in G : 2 \). If for \( i = 1, 2 \), \( u(i) \) is coprime to \( o(t) \) and \( o(g_i) \), then we can further define the elements \( y_i = (x_i^{(1)})^{u(i)} \) for some positive integers \( j(i) \). Note that since \( u(i) \) commutes with \( t \) the automorphism defined by conjugation by \( t \) inverts both \( x_i \) and \( y_i \). Using knowledge of the subgroup structure of \( G : 2 \) it is often possible to choose the elements \( g_1, g_2, u(1) \) and \( u(2) \) in such a way that \( \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G : 2 \). Unfortunately, in the case of almost simple groups we must have that the orders of \( x_i \) and \( y_i \) all have even order and so verifying the conjugacy condition (1) of Definition 2 is more difficult than simply showing that \( o(x_1)o(y_1)o(x_1y_1) \) is coprime to \( o(x_2)o(y_2)o(x_2y_2) \). For some of the larger groups verifying that they generate the whole group can also be difficult. In these cases, generation is verified by finding words in our elements with the property that no proper subgroup can contain them (in many cases the maximal subgroups for these groups may be found in [16]). The words defining our Beauville structures are given in Table 1 and their types are given in Table 2.

We remark that this construction will not work in cases that are non-split extensions. This includes the almost simple ‘sporadic’ Tits group \( ^2\text{F}_4(2) \). Straightforward computations verify that this group is not a strongly real Beauville group.

In light of the above we make the following tentative conjecture.

Conjecture 4. A split extension of a simple group is a Beauville group if, and only if, it is a strongly real Beauville group.

We remark that some (unpublished) progress on this conjecture has been made by the author’s PhD student, Emilio Pierro, whilst the question of which of the groups \( \text{PGL}_2(q) \) are Beauville is discussed by Garion in [24].

6. Nilpotent Groups

It is immediate that the direct product of two Beauville groups of coprime order is again a Beauville group (though slightly more is true — see [2] Lemma 1.3). Recall that a finite group is nilpotent if and only if it is the direct product of
of the rest only two fail to be strongly constructions. We saw in Section 2 that among the non-abelian finite simple groups only one fails to be a Beauville group and of the rest only two fail to be strongly real. This immediately raises the following question.

| $G$  | $t_1$          | $t_2$          | $x_1$          | $x_2$          |
|------|----------------|----------------|----------------|----------------|
| $M_{12}$ : 2 | $c$            | $(cd)^2$        | $t_1t_2$       | $t_1t_2^2$     |
| $M_{23}$ : 2 | $((cd)^2d)^3$  | $d^2$           | $t_1t_2$       | $t_1t_2^2$     |
| $J_2$ : 2    | $c$            | $(cd)^2(cd)^2$  | $t_1t_2$       | $t_1t_2^4$     |
| HS : 2       | $c$            | $(cd)^3cd^2$    | $t_1t_2$       | $t_1t_2^2$     |
| $J_3$ : 2    | $c$            | $(cd)^2$        | $t_1t_2$       | $t_1t_2^2$     |
| McL : 2      | $c$            | $(cd)^2(cd^2)^2$| $t_1t_2$       | $t_1t_2^2$     |
| He : 2       | $c$            | $d^3$           | $t_1t_2$       | $t_1t_2^2$     |
| Suz : 2      | $c$            | $(cd)^{14}$     | $t_1t_2$       | $t_1t_2^2$     |
| O’N : 2      | $c$            | $d^2$           | $t_1t_2$       | $t_1t_2^2$     |
| Fi$_{22}$ : 2| $(cd^1)^{10}$  | $(cd^3)^{15}$   | $t_1t_2^{cd^6}$| $t_1t_2^{cd}$  |
| HN : 2       | $c$            | $(cd^3(cd^2)^2)^{12}$ | $t_1t_2^{cd^3d^2}$ | $t_1t_2^{cd^3d^2}$ |
| Fi$_{24}$   | $d^4$         | $(cd^3d^3)^{33}$ | $t_1t_2^{t_1^2d^3c}$ | $t_1t_2^{t_1^2d^3d^2c}$ |

| $G$  | $u_1$          | $u_2$          | $j(1)$         | $j(2)$         |
|------|----------------|----------------|----------------|----------------|
| $M_{12}$ : 2 | $(c,(dc)^2d^2)^4$ | $(dc)^2 [c,(dc)^2d^2]^2$ | 1 | 1 |
| $M_{23}$ : 2 | $cd^2cd^2[cd,t_1,cd^2cd]^5$ | $[t_1, c]^2$ | 5 | 9 |
| $J_2$ : 2    | $d[c, d]^3$   | $d[c, d]^3$    | 1 | 9 |
| HS : 2       | $d[c, d]$    | $d[c, d]$      | 1 | 1 |
| $J_3$ : 2    | $d[c, d]^4$  | $d[c, d]^4$    | 21 | 1 |
| McL : 2      | $d^2[c, d]^7$ | $dcd[c, cd]^7$ | 1 | 7 |
| He : 2       | $d[c, d]^7$  | $d[c, d]^7$    | 15 | 19 |
| Suz : 2      | $d[c, d]^3$  | $d[c, d]^3$    | 9 | 3 |
| O’N : 2      | $d[c, d]^5$  | $d[c, d]^5$    | 7 | 1 |
| Fi$_{22}$ : 2| $[t_1, d]^3$  | $[t_1, d]^3$   | 3 | 1 |
| HN : 2       | $d^2[cd]^3$  | $d[c, d]^4$    | 1 | 39 |
| Fi$_{24}$   | $c[t_1, c]$   | $c[t_1, c]$    | 7 | 25 |

Table 1. Words in the standard generators providing strongly real Beauville structures for each of the non-simple almost simple sporadic groups.

| $G$  | type            | $G$  | type            |
|------|-----------------|------|-----------------|
| $M_{12}$ : 2 | $((4,4,5),(6,6,3))$ | He : 2 | $((16,16,7),(30,30,5))$ |
| $M_{23}$ : 2 | $((12,12,4),(10,10,5))$ | Suz : 2 | $((10,10,3),(8,8,13))$ |
| $J_2$ : 2    | $((24,24,15),(14,14,7))$ | O’N : 2 | $((38,38,19),(56,56,28))$ |
| HS : 2       | $((8,8,8),(6,6,15))$ | Fi$_{22}$ : 2 | $((10,10,11),(12,12,4))$ |
| $J_3$ : 2    | $((34,34,17),(24,24,4))$ | HN : 2 | $((18,18,25),(44,44,22))$ |
| McL : 2      | $((8,8,3),(10,10,5))$ | Fi$_{24}$ | $((86,66,33),(84,84,26))$ |

Table 2. The types of the Beauville structures specified by the words in Table 1.

its Sylow subgroups. Since Sylow subgroups for different primes will have coprime orders this observation reduces the study of nilpotent Beauville groups to that of Beauville $p$-groups.

There is another motivation for wanting to study Beauville $p$-groups and that is to study how finite groups in general behave from the point of view of Beauville constructions. We saw in Section 2 that among the non-abelian finite simple groups only one fails to be a Beauville group and of the rest only two fail to be strongly real. This immediately raises the following question.
Question 4. Are most Beauville groups strongly real Beauville groups?

Finite simple groups are rare gems in the rough — for every positive integer \( n \) there are at most two finite simple groups of order \( n \) and for most values of \( n \) there are none at all. Taking our lead from their behaviour is therefore somewhat dangerous.

Few mathematicians outside finite group theory seem to realise that in some sense most finite groups are \( p \)-groups, indeed most finite groups are 2-groups. There are 49,910,529,484 groups of order at most 2,000. Of these 49,487,365,422 have order precisely 1,024 — that’s more than 99.1% of the total! When we throw in the other 2-groups of order at most 1,024 and the other \( p \)-groups of order at most 2,000 we have almost all of them. Determining which of the Beauville \( p \)-groups are strongly real Beauville \( p \)-groups thus goes a long way to answering the above question for groups in general. (For details of these extraordinary computational feats and a historical discussion of the problem of enumerating groups of small order, which has been worked on for almost a century and a half, see the work of Besche, Eick and O’Brian in [8, 9].)

Theorem 7 and Corollary 8 tell us that if \( p \geq 5 \) is prime then there are infinitely many strongly real Beauville \( p \)-groups - just let \( n \) be any power of \( p \). These results are, however, useless for the primes 2 and 3. As far as the author is aware there are no known examples.

Problem 1. Find strongly real Beauville 2-groups and 3-groups.

The only known infinite family of Beauville 2-groups are those recently constructed by Barker, Boston, Peyerimhoff and Vdovina in [3]. One of the main results of [3] is that the groups constructed there are not strongly real. Furthermore there remain only finitely many known examples of Beauville 3-groups.

In general, \( p \)-groups have large outer automorphism groups [12, 13], so it seems likely that most Beauville \( p \)-groups are in fact strongly real. Again, as far as the author is aware, this matter remains largely uninvestigated.

Problem 2. Find non-abelian strongly real Beauville \( p \)-groups.

The best general discussion of work on Beauville \( p \)-groups is Boston’s contribution to these proceedings [11]. The work of the Barker, Boston and the author in [2] and the work of Barker, Boston, Peyerimhoff and Vdovina in [3, 4, 5] are also worth consulting.

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