Hadronic and radiative three–body decays of $J/\psi$
involving the scalars $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$

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We study the role of the scalar resonances $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ in the strong and radiative three–body decays of $J/\psi$ with $J/\psi \to V + PP(\gamma\gamma)$ and $J/\psi \to \gamma + PP(VV)$, where $P(V)$ denotes a pseudoscalar (vector) meson. We assume that the scalars result from a glueball–quarkonium mixing scheme while the dynamics of the transition process is described in an effective chiral Lagrangian approach. Present data on $J/\psi \to V + PP$ are well reproduced, predictions for the radiative processes serve as further tests of this scenario.

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I. INTRODUCTION

There have been suggestions that mass, production and decay properties of some of the isoscalar scalar mesons are consistent with a situation where a glueball should have mixing with quarkonia states in a mass range roughly below 2 GeV. The idea that a glueball configuration is present in the scalar meson spectrum was suggested in [1] and later extensively used as for example in Refs. [2–6]. The discussion on possible evidence for the scalar ground state glueball has dominantly centered on the scalar-isoscalar resonances $f_0(1500)$ and $f_0(1710)$. The claim that at least partial glueball nature can be attributed to these states is to some part based on the mass predictions of Lattice QCD. In the quenched approximation the lightest glueball state is predicted [7] to be a scalar with a mass of about 1650 MeV and uncertainty of around 100 MeV. But note that first results in an unquenched calculation could indicate a strong downward mass shift towards 1 GeV [8]. For reviews on the experimental situation of scalar mesons and their possible glueball nature can be attributed to these states is to some part based on the mass predictions of Lattice QCD. In particular, they indicated the branching ratios for $f_0(1370)$ and $f_0(1500)$ and $f_0(1710)$.

An important object for studying the nature of the scalar resonances above 1 GeV has been the strong decay patterns observed for these states. Thereby several analyses conclude that the three scalar states $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ result from a mixing of the glueball with the $n\bar{n} = \sqrt{1/2}(u\bar{u} + d\bar{d})$ and $s\bar{s}$ states of the scalar $3P_0$ quarkonium nonet. Furthermore, the nonappearance or the strength of the appearance of these scalar states in $\gamma\gamma$ or in central $pp$ collisions have been argued to give further signals for a possible quantification of the glueball component residing in the scalars [2–6]. A study of the strong and radiative three–body decays of $J/\psi$ with $J/\psi \to V + PP(\gamma\gamma)$ and $J/\psi \to \gamma + PP(VV)$, where $V$ and $P$ are vector and pseudoscalar mesons, opens a further opportunity to test the nature of the scalar mesons $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$. The strong three-body $J/\psi$ decays with $PP = \pi^+\pi^-$ and $K^+K^-$ in the final state have been studied by the Mark III [10], DM2 [11] and BES II [12] collaborations. Particularly the BES II [13] experiment with a much larger statistics indicated a clear signal for $f_0(1370) \to \pi^+\pi^-$ in the $\phi\pi^+\pi^-$ data. An enhancement in the $\phi KK$ data can be fitted by interference between the $f_0(1500)$ and the $f_0(1710)$. In [14] BES II also reported on partial wave analyses on radiative decays of the type $J/\psi \to \gamma PP$ and $J/\psi \to \gamma VV$. In particular, they indicated the branching ratios $B(J/\psi \to \gamma f_0(1710) \to \gamma KK)$ [14], $B(J/\psi \to \gamma f_0(1710) \to \gamma \pi^+\pi^-)$ , $B(J/\psi \to \gamma f_0(1500) \to \gamma \pi^+\pi^-)$ [15] and $B(J/\psi \to \gamma f_0(1710) \to \gamma \omega\omega)$ [16].

Theoretical studies of $J/\psi$ decay into a vector meson and two pseudoscalars and opens a further opportunity to study the role of the scalar resonances $f_0(1370)$ and $f_0(1710)$ in $J/\psi$ radiative decays has been performed in Ref. [17] [18].

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It was shown in Refs. [22, 23] that the dominant contribution to the Okubo-Zweig-Iizuka (OZI) suppressed radiative decays of the $J/\psi$ comes from the photon emission from the initial state. This means that the strong $J/\psi$ decays involving isoscalar vector mesons (like $\omega$ and $\phi$) are not related to the radiative decays of the $J/\psi$ via the vector meson dominance (VMD) mechanism. For this reason the branching ratios of strong and radiative $J/\psi$ decays are compatible. The three-body $J/\psi$ decays can be factorized in terms of the two-body rates. Therefore, the analysis of the $J/\psi$ three-body decays can shed light on the two-body transitions of the $J/\psi$ and $f_i$ states: $J/\psi \to f_i V$, $J/\psi \to f_i \gamma$, $f_i \to V \gamma$ and $f_i \to \gamma \gamma$. The electromagnetic decays of the scalar mesons $f_i \to V \gamma$ and $f_i \to \gamma \gamma$ have been studied in detail using chiral approaches, nonrelativistic and light-front quark models (see e.g. Refs. [6, 24–28]).

The purpose of the present paper is to analyze the role of the scalar mesons $f_i$ in the strong and radiative three-body decays of $J/\psi$ using a chiral Lagrangian approach suggested and developed in [6, 29, 30]. In these works we originally studied the strong and electromagnetic decay properties of scalar mesons above 1 GeV. The isoscalar scalars were treated as mixed states of glueball and quarkonia configurations. Later on we extended the phenomenology to the study of decay properties of excited tensor, vector and pseudoscalar mesons. Present evaluation is meant as a continuation of the full decay analysis presented in [6] but now we include the additional constraints set by $J/\psi$ decays involving these scalars. At this stage we do not consider explicitly the mixing of the scalar resonances induced by the continuum decay channels, we also neglect final state interaction in the decay modes (note that inclusion of final-state interaction in $J/\psi$ three-body decays has been done in [19])—both interaction mechanisms are higher-order effects in the framework of our perturbative considerations. We are interested to give predictions for scalar resonance contributions to $J/\psi$ decay modes where the physical nature of the scalar resonances $f_i = f_0(1370), f_0(1500)$ and $f_0(1710)$ can possibly be tested.

In the present paper we proceed as follows. In Sec. II, we present the effective Lagrangian which will be used for the calculation of the resonance contributions of scalar mesons $f_i$ to the matrix elements of strong and radiative three-body decays of the $J/\psi$. In Sec. III we present our results for the resonance contributions of the scalar mesons $f_i$ to the decay widths of the $J/\psi$. A short summary is given in Sec. IV.

II. APPROACH

In Refs. [6, 29, 30] we presented the lowest-order chiral Lagrangian describing strong and radiative decays of pseudoscalar, scalar, vector and tensor mesons. This Lagrangian was motivated by chiral perturbation theory [31–34]. Here we extend the formalism by including the additional couplings $J/\psi f_i V$, $J/\psi f_i \gamma$, $f_i V V$ and $f_i V \gamma$. These interaction terms are necessary for elaborating the contributions of the scalar mesons $f_i$ to the strong and radiative three-body $J/\psi$ decays. The full Lagrangian relevant for the $J/\psi$ meson decays involves the $J/\psi$ meson, the scalar glueball $G$, the nonets of pseudoscalar $P$, scalar $S$ and vector $V$ mesons with

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}(D_{\mu}U D^{\mu}U^\dagger + \chi_{+}) + \frac{1}{2}(D_{\mu}S D^{\mu}S - M_S^2 S^2) + \frac{1}{2}(\partial_{\mu}G \partial^{\mu}G - M_G^2 G^2)$$
$$- \frac{1}{2}\left(\nabla_{\mu}W^{\mu\nu} \nabla^{\rho}W_{\rho\nu} - \frac{1}{2}M_W^2 W_{\mu\nu}W^{\mu\nu}\right) - \frac{1}{2}(\partial_{\mu}J^{\mu\nu} \partial^{\rho}J_{\rho\nu} - \frac{1}{2}M_J^2 J_{\mu\nu} J^{\mu\nu})$$
$$+ c_g^s (S u \mu u^\mu) + c_m^s (S \chi_{+}) + \frac{c_g^S}{\sqrt{3}} G(u \mu u^\mu) + \frac{c_{\gamma}^G}{\sqrt{3}} G(\chi_{+}) + \mathcal{L}^P + \mathcal{L}^{S}$$
$$+ c'_{\epsilon} (S F_{\mu\nu} \epsilon^{\mu\nu}) + \frac{c_{\gamma}^F}{\sqrt{3}} G(F^{\mu\nu} F^{\mu\nu}) + \frac{c_f^F}{2} \langle S W^{\mu\nu} W^{\mu\nu} \rangle + \frac{c^{\gamma}_{\epsilon}}{\sqrt{3}} G(W^{\mu\nu} F^{\mu\nu})$$
$$+ c'_\epsilon (S W_{\mu\nu} W^{\mu\nu}) + \frac{c_{\gamma}^W}{\sqrt{3}} G(W_{\mu\nu} W^{\mu\nu}) + c_{\epsilon}^W J^{\mu\nu} \langle S W_{\mu\nu} \rangle + c_{\epsilon}^W J^{\mu\nu} \langle S \rangle \langle W_{\mu\nu} \rangle$$
$$+ \frac{c_{\gamma}^G}{\sqrt{3}} J^{\mu\nu} G(W_{\mu\nu}) + J^{\mu\nu} F_{\mu\nu} (c_{\epsilon}^N N + c_{\epsilon}^S S + c_{\epsilon}^G G).$$

The symbols $\langle \rangle$ and \{\} occurring in Eq. (1) denote the trace over flavor matrices and the anticommutator, respectively. $|N\rangle = |(u u + \bar{d} d)/\sqrt{2}\rangle$, $|S\rangle = |s s\rangle$, $|G\rangle$ are the nonstrange, strange quarkonia states and the glueball, respectively [6]. The constants $c_g^s$ describe the coupling of scalar fields and of the glueball to pseudoscalar, vector mesons, $J/\psi$ and to photons. For the vector mesons we use the tensorial representation in terms of antisymmetric tensor fields $W_{\mu\nu}$ (SU(3) nonet of vector mesons) and $J_{\mu\nu}$ ($J/\psi$ meson) [see details in 34, 32]. The couplings of the scalars to vector mesons and photons ($c'_\epsilon, c'_F, c_f^F, c_{\epsilon}^W, c_{\epsilon}^N$ and $c_{\epsilon}^G$) can be constrained by vector meson dominance.
(VMD) and SU(3) flavor symmetry as:

\[
c^s_i = 2\sqrt{2}g_{\rho\gamma}c^s_i, \quad c^\rho_i = 2\sqrt{2}g_{\rho\gamma}c^\rho_i, \\
c^e = \frac{g_{\rho\gamma}^2}{2}c^e, \quad c^f = \frac{g_{\rho\gamma}^2}{\sqrt{6}}c^e.
\]

(2)

As we already stressed in the introduction, the couplings of \(J/\psi\) to scalars and photons contained in \(c^s_i (i = n, s, g)\) are not constrained by VMD and, therefore, are independent on the couplings \(c^s_i\) and \(c^\rho_i\). In the derivation of the expressions in Eq. (2) we also used SU(3) relations for the vector–meson to photon couplings:

\[
g_{\rho\gamma} = 3g_{\omega\gamma} = \frac{3}{\sqrt{2}}g_{\phi\gamma}
\]

(3)

with \(g_{\rho\gamma} = 0.2\) fixed from data on \(\Gamma(\rho^0 \rightarrow e^+e^-)\) [36]. The other notations are standard for the basic blocks of the ChPT Lagrangian [31][34]: \(U = u^2 = \exp(iP\sqrt{2}/F)\) is the chiral field collecting pseudoscalar fields in the exponential parametrization, \(F\) is the pseudoscalar meson decay constant, \(D_u\) and \(\nabla\) denote the chiral and gauge-invariant derivatives acting on the chiral fields and other mesons. Furthermore, \(\chi_{\pm} = u^\dagger \chi u^\dagger \pm u^\dagger \chi u\) with \(\chi = 2B(s+u)\), \(s = 3/3\) and \(F_{\mu\nu}^\rho = u^\dagger F_{\rho\mu}Qu + u F_{\mu\nu}Q_u\), where \(F_{\mu\nu}\) is the stress tensor of the electromagnetic field. The charge and the current quark mass matrices are represented by \(Q = e \operatorname{diag}(2/3, -1/3, -1/3)\) and \(M = \operatorname{diag}(\hat{m}, \hat{m}, m_s)\) where \(\hat{m} = (m_u + m_d)/2\) and \(m_s\) are the nonstrange and strange quark masses. In the following we restrict to the isospin symmetry limit \(m_u = m_d\). The terms \(\mathcal{L}_{\text{mix}}^P\) and \(\mathcal{L}_{\text{mix}}^S\) give rise to flavor singlet-octet mixing in the pseudoscalar and scalar sector, while for the scalar case quarkonia-glueball mixing is included in addition [see details in [3]]. In particular, the mass matrix involving nonstrange \(|N\rangle\) and strange \(|S\rangle = |ss\rangle\) quarkonia states and the glueball \(|G\rangle\) has the form [6]

\[
M^2_{\text{bare}} = \begin{pmatrix}
M^2_N & f\sqrt{2} & \varepsilon \\
\varepsilon & f/\sqrt{2} & M^2_G \\
M^2_G & \varepsilon & M^2_G
\end{pmatrix},
\]

(4)

where \(f\) and \(\varepsilon\) are free parameters controlling the quarkonia–glueball mixing scheme. The orthogonal physical states are obtained by diagonalization of \(M^2_{\text{bare}}\) with the transformation matrix \(B\) as

\[
BM^2_{\text{bare}}B^T = M^2_{\text{phys}} = \operatorname{diag}(M^2_{f_1}, M^2_{f_2}, M^2_{f_3}).
\]

(5)

The eigenvalues of \(M^2_{\text{phys}}\) represent the masses of the physical states \(f_1 \equiv f_0(1370), f_2 \equiv f_0(1500)\) and \(f_3 \equiv f_0(1710)\). These diagonal states \(|i\rangle\), with \(i = f_1, f_2, f_3\), are then given in terms of the bare states as

\[
\begin{pmatrix}
|f_1\rangle \\
|f_2\rangle \\
|f_3\rangle
\end{pmatrix} = B
\begin{pmatrix}
|N\rangle \\
|G\rangle \\
|S\rangle
\end{pmatrix}.
\]

(6)

The \(J/\psi\) three-body decays which proceed through the scalar resonances are described by the diagrams shown in Figs.1–4. The diagram in Fig.1 contributes to the strong decay \(J/\psi \rightarrow Vf_i \rightarrow VPP\). There are two graphs in Fig.2 relevant for the radiative decay \(J/\psi \rightarrow V\gamma\): one described by the two–step process \(J/\psi \rightarrow f_iV \rightarrow V\gamma\) [Fig.2(a)] and the other one with \(J/\psi \rightarrow f_iV \rightarrow V\gamma\) [Fig.2(b)]. Fig.3 contains the graph responsible for the radiative decay \(J/\psi \rightarrow f_i\gamma \rightarrow \gamma PP\) involving two pseudoscalars in the final state. Finally, the diagrams of Figs.4(a) and 4(b), representing the two–step transitions \(J/\psi \rightarrow f_i\gamma \rightarrow \gammaVV\) and \(J/\psi \rightarrow f_iV \rightarrow \gammaVV\), contribute to the decay \(J/\psi \rightarrow \gammaVV\) with two vector mesons in the decay products. The corresponding matrix elements and decay widths resulting from these specific graphs are given in full detail in the Appendix. Note, that the three-body decay widths of \(J/\psi\) are factorized in terms of two-body decays of the subprocesses \(J/\psi \rightarrow f_iV\), \(J/\psi \rightarrow f_i\gamma\), \(f_i \rightarrow PP\), \(f_i \rightarrow VV\), \(f_i \rightarrow V\gamma\) and \(f_i \rightarrow \gamma\gamma\).

III. RESULTS

A. Mixing schemes of the scalar mesons \(f_0(1370), f_0(1500)\) and \(f_0(1710)\)

In Ref. [6] we already presented a detailed analysis of strong and radiative decays of the scalar mesons \(f_i = f_0(1370), f_0(1500)\) and \(f_0(1710)\) e.g. considering different scenarios of quarkonia–glueball mixing. Bare masses,
mixing parameters and decay couplings of the quarkonia and glueball configurations were extracted from a fit to experimental masses and well-established decay rates into two pseudoscalar mesons. In the present discussion we also indicate a general fit of these parameters, especially for the decay constants \( c_d \) and \( c_m \) of the direct glueball decay. In Ref.~[6] we constrained these parameters by either studying the flavor-symmetry limit \( c_m = 0 \) or by matching these constants to first lattice results for the glueball decay. In the following fits no constraints are set on these decay parameters, but previously deduced mixing schemes will approximately result again from this procedure. The quantities entering in the fitting procedure and the results obtained from both fits are displayed in Table I. In the fit here we also include the additional decay mode \( f_0(1500) \to \eta \eta' \). As in Ref.~[6] \( \Gamma_{f_0}^{(1500)} \) is the sum of partial decay widths into two pseudoscalar mesons approximated by the total width. By releasing the constraints on the direct glueball decay we obtain two solutions where the quality of fits in terms of \( \chi^2_{\text{tot}} \) is somewhat improved compared to the ones of Ref.~[6]. We consider two options resulting from local minima in \( \chi^2_{\text{tot}} \): Scenario I, the mass of the bare glueball \( G \) is on the lower end of the unquenched lattice predictions with \( m_G \sim 1.5 \text{ GeV} \); Scenario II, the mass of the bare glueball \( G \) is larger with \( m_G \sim 1.7 \text{ GeV} \). The fit parameters for these two schemes are fixed as follows:

**Scenario I:**

\[
M_N = 1.485 \text{ GeV , } M_G = 1.482 \text{ GeV , } M_S = 1.698 \text{ GeV} \\
f = 0.068 \text{ GeV}^2 , \quad \epsilon = 0.236 \text{ GeV}^2 , \quad c_d^I = 8.8 \text{ MeV} \\
c_m^I = 2.2 \text{ MeV} , \quad c_d^g = 1.8 \text{ MeV} , \quad c_m^g = 27.7 \text{ MeV} 
\]

**Scenario II:**

\[
M_N = 1.360 \text{ GeV , } M_G = 1.686 \text{ GeV , } M_S = 1.439 \text{ GeV} \\
f = 0.23 \text{ GeV}^2 , \quad \epsilon = 0.30 \text{ GeV}^2 , \quad c_d^I = 6.5 \text{ MeV} \\
c_m^I = 5.5 \text{ MeV} , \quad c_d^g = -2.0 \text{ MeV} , \quad c_m^g = 48.3 \text{ MeV} 
\]

The corresponding mixing matrices \( B_I \) (Scenario I) and \( B_{II} \) (Scenario II) are:

\[
B_I = \begin{pmatrix} 0.75 & 0.6 & 0.26 \\
-0.59 & 0.8 & -0.14 \\
-0.29 & -0.05 & 0.95 \end{pmatrix}, 
\]

\[
B_{II} = \begin{pmatrix} 0.79 & 0.29 & 0.55 \\
0.57 & \sim 0 & -0.82 \\
0.23 & -0.96 & 0.17 \end{pmatrix}. 
\]

For solution I the glueball component dominantly resides in the \( f_0(1500) \) (with \( \chi^2_{\text{tot}} \simeq 20 \)), while for the second one (with \( \chi^2_{\text{tot}} \simeq 15 \)) \( f_0(1710) \) is identified with the glueball. Both level schemes of the bare states and mixing scenarios contained in \( B_{II} \) are very similar to the original solutions of Ref.~[6], where the direct glueball decay is included (denoted in [6] as third and fourth solution). Releasing the constraint on the direct glueball decay parameters allows a further fine-tuning of the results. The two solutions also represent the situation discussed in the literature, when studying the mixing of the glueball with quarkonia in the \( f_0(1370) \), \( f_0(1500) \) and \( f_0(1710) \) sector – the glueball resides dominantly in the \( f_0(1500) \) or the \( f_0(1710) \). Qualitatively both solutions are in acceptable agreement with the experimental data, except for the underestimate of \( \Gamma_{f_0}^{(1340)} \to K \pi \), which as in the previous work \( \text{[6]} \) remains unresolved. In the context of the effective chiral theory the second solution is slightly favored. We present additional theoretical ratios not contained in Table I, compared for example to experimental data from the WA102 Collaboration~[37] (see Table II). Note that in contrast to the result of \( \Gamma_{f_0}^{(1500)} \to K K / \Gamma_{f_0}^{(1500)} \to \pi \pi = 0.46 \pm 0.19 \) of the WA102 Collaboration~[37] the analysis of the OBELIX Collaboration~[38] results in \( 0.91 \pm 0.20 \).

**B. Hadronic and radiative \( J/\psi \) decays**

In this subsection we analyze the role of the scalar resonances \( f_i \) in the hadronic \( J/\psi \) decays \( J/\psi \to V f_i \to VPP \). Details of the calculations are presented in the Appendix: analytical formulas for the matrix elements and the decay rates. Our strategy for determining the parameters involved in these processes is the following: using data (central values) on the three-body decay modes \( J/\psi \to (\phi, \omega) f_0(1710) \to (\phi, \omega) K K \) \[36\] (see Table III) as well as the predictions of our approach for the two-body decay modes of the scalars (see Table I) we fix the couplings \( c_h^I, c_w^I, c_h^g \) in the effective chiral Lagrangian \([1]\) as

\[
c_h^I = 1.092 \times 10^{-3} \text{ GeV}^{-1} , \quad c_w^I = -0.626 \times 10^{-3} \text{ GeV}^{-1} , \quad c_h^g = 10^{-4} \text{ GeV}^{-1}, \\
c_h^{I,II} = 1.340 \times 10^{-3} \text{ GeV}^{-1} , \quad c_w^{I,II} = -0.924 \times 10^{-3} \text{ GeV}^{-1} , \quad c_h^{g,II} = 0.722 \times 10^{-3} \text{ GeV}^{-1}. 
\]
The indices I, II refer to the respective mixing scenarios. With these couplings we determine the strong couplings $g_{JfV}$ [they are linear combinations of $c_{fi}^h$ and $c_{fi}^g$ (10)] and the electromagnetic couplings $g_{Jf\gamma}$ of the physical $f_i$ states. In Table IV we list the results for the effective couplings $g_{JfV}$ and $g_{Jf\gamma}$ and the corresponding two-body decay widths.

In Table III we give our final results for the hadronic three-body decays of $J/\psi$ for both scenarios (I and II). Results are compared to data [36] and the predictions of Ref. [3], where a similar mixing scheme [1] has been used to determine $B$. Especially in scenario I we obtain a reasonable description of present data on the hadronic $J/\psi$ decays, while in version II some discrepancies occur. In particular, one can see that in scenario I we can reproduce the central value of the ratio

$$R = \frac{\text{Br}(J/\psi \to f_3\phi \to \phi K K)}{\text{Br}(J/\psi \to f_3\omega \to \omega K K)} = \frac{4}{3}$$

(12)

while in scenario II this quantity is close to 2. The explanation for this quantitative difference between the two mixing schemes is quite simple. The ratio $R$ is approximately equal to the ratio of the corresponding strong two-body decay modes of $J/\psi$, which is expressed in terms of the elements of the mixing matrix $B$ as:

$$R \approx \frac{\text{Br}(J/\psi \to f_3\phi)}{\text{Br}(J/\psi \to f_3\omega)} = 2 \left[ 1 + \frac{c_{33}^k (B_{33}/\sqrt{2} - B_{33})}{c_{33}^k B_{33} + c_{33}^g B_{32}/\sqrt{3} + c_{33}^e (B_{31} \sqrt{2} + B_{33})} \right]^2.$$  

(13)

The ratio depends crucially on the value of the combination of mixing matrix elements with $\Delta = B_{31}/\sqrt{2} - B_{33}$. In scenario I we have $\Delta = -1.16$ and therefore it is possible to generate the observed value for the ratio $R$ with the appropriate choice of free parameters $c_{33}^h, c_{33}^g$ and $c_{33}^e$ (as we did in our fitting procedure). In case of scenario II this quantity has the value $\Delta = -0.007 \approx 0$, and now the ratio $R$ is always close to 2 independent on the choice of the relevant parameters $c_{33}^h, c_{33}^g$ and $c_{33}^e$. After the mixing matrix $B$ is constrained by a fit to the properties of the scalars (as compiled in Table I) scenario II is less preferable when analysing the hadronic $J/\psi$ decays.

Analysis of the radiative $J/\psi$ decays $J/\psi \to \gamma + PP(VV)$ and $J/\psi \to P + \gamma\gamma$ can produce further important information about the electromagnetic structure of the scalar resonances involved. The relevant matrix elements involve further unknown couplings $(c_{33}^h, c_{33}^g, c_{33}^e, c_{33}^i, c_{33}^j, c_{33}^k, c_{33}^l, c_{33}^m)$ from the effective Lagrangian (11). Note, only 5 parameters from this set are independent, because the parameters $c_{33}^i, c_{33}^e$ and $c_{33}^j, c_{33}^l, c_{33}^m$ are expressed through the parameters $c_{33}^h, c_{33}^g$ via VMD constraints (2).

The other parameters involved were already fixed from the previous analysis of the strong $J/\psi$ decays. We find that acceptable results for the radiative $J/\psi$ decays can be achieved with the following choice of the parameters $c_{33}^h, c_{33}^g, c_{33}^e, c_{33}^i, c_{33}^j, c_{33}^k, c_{33}^l$:

$$c_{33}^h = 2.815 \text{ GeV}^{-1}, \quad c_{33}^g = 0.138 \text{ GeV}^{-1},$$

$$c_{33}^e = 0.241 \times 10^{-2} \text{ GeV}^{-1}, \quad c_{33}^i = -0.313 \times 10^{-2} \text{ GeV}^{-1}, \quad c_{33}^j = -0.271 \times 10^{-3} \text{ GeV}^{-1},$$

$$c_{33}^{III} = 3.036 \text{ GeV}^{-1}, \quad c_{33}^{III} = 1.898 \text{ GeV}^{-1},$$

$$c_{33}^{III} = 0.132 \times 10^{-2} \text{ GeV}^{-1}, \quad c_{33}^{III} = 0.187 \times 10^{-2} \text{ GeV}^{-1}, \quad c_{33}^{III} = -0.352 \times 10^{-2} \text{ GeV}^{-1}.$$  

(14)

Knowledge of the couplings $c_{33}^h$ and $c_{33}^g$ gives access to the effective couplings of scalar mesons to vector mesons and photons. They are related by the VMD relations (2) and (3). With the use of Eq. (14) and the VMD relations we determine the effective couplings of the scalar $f_i$ mesons to photons and vector mesons as listed in Table V.

Our predictions for the radiative decays of $J/\psi$ and the scalar mesons $f_i$ are given in Tables VI and VII. Note that we reproduce all known data for radiative decays of the $J/\psi$ involving the $f_i$ states [36]:

$$\text{Br}(J/\psi \to f_2\gamma) = (1.01 \pm 0.32) \times 10^{-4},$$

$$\text{Br}(J/\psi \to f_3\gamma \to \gamma \pi\pi) = (4.0 \pm 1.0) \times 10^{-4},$$

$$\text{Br}(J/\psi \to f_3\gamma \to \gamma K \bar{K}) = (8.5^{+1.2}_{-0.8}) \times 10^{-4},$$

$$\text{Br}(J/\psi \to f_3\gamma \to \gamma \omega\omega) = (3.1 \pm 1.0) \times 10^{-4}.$$  

(15)

In case of the scalar mesons we set our results in comparison to the ones of other approaches and with available data. In particular, the results of Ref. [24] and [25] are given in the form $(L, M, H)$, where $L=$Light, $M=$Medium and $H=$Heavy correspond to the three possibilities for the bare glueball mass: lighter than the bare $\bar{n}n$ mass, between the $\bar{n}n$ and $s\bar{s}$ masses, heavier than the $s\bar{s}$ mass. The results of Ref. [26] are listed in the form $(K, P)$, where $K = K\bar{K}$ and $P = \pi\pi$ indicate the contributions of the intermediate $K\bar{K}$ and $\pi\pi$ loop, respectively.
The matrix elements describing the two- and three-body decays of $J/\psi$ and the $f_i$ mesons satisfy the following approximate ratios:

$$
R_{i1} = \frac{\text{Br}(J/\psi \to f_i \gamma \to \pi \pi \gamma)}{\text{Br}(J/\psi \to f_i \gamma \to KK\gamma)} \simeq R_{i2} = \frac{\text{Br}(f_i \to \pi \pi)}{\text{Br}(f_i \to KK)},
$$

$$
R_{i33}^{VPP} = \frac{\text{Br}(J/\psi \to f_i V \to PPV)}{\text{Br}(J/\psi \to f_i V)} \simeq R_{i44}^{PP} = \frac{\text{Br}(J/\psi \to f_i V \to PP\gamma)}{\text{Br}(J/\psi \to f_i \gamma)} \simeq \text{Br}(f_i \to PP). \quad (16)
$$

We want to illustrate these approximate constraints in case of the scenario I. The set of the radiative three-body $J/\psi$ branching ratios $R_{11} = 1.04$, $R_{21} = 3.80$, $R_{31} = 0.56$ is similar to the corresponding set of the strong two-body $f_i$ branching ratios $R_{12} = 0.94$, $R_{22} = 3.61$, $R_{32} = 0.43$. Also, the ratios $R_{i33}^{VPP}$ and $R_{i44}^{PP}$ are approximately equal to the corresponding branchings of the strong two-body decays of the scalar mesons $f_i$. E.g. in case of $f_3 = f_0(1710)$ and $KK$ in the final state we have

$$
R_{33}^{KK} = 0.45 \simeq R_{33}^{KK} = 0.44 \simeq R_{34}^{KK} = 0.45 \simeq \text{Br}(f_3 \to K\bar{K}) = 0.49. \quad (17)
$$

Similar consistency is found in case of the scenario II with:

$$
R_{11} = 0.51 \simeq R_{12} = 0.44, \quad R_{21} = 0.41 \simeq R_{21} = 0.39, \quad R_{31} = 0.64 \simeq R_{31} = 0.49, \quad (18)
$$

$$
R_{33}^{KK} \simeq R_{33}^{KK} \simeq R_{34}^{KK} = 0.38 \simeq \text{Br}(f_3 \to K\bar{K}) = 0.42. \quad (19)
$$

**IV. CONCLUSIONS**

In conclusion, we present an analysis of the role of the scalar resonances $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ in the strong and radiative $J/\psi$ three-body decays. This analysis can shed light on the nature of the scalar resonances $f_i$ and the possibility that these states arise from glueball-quarkonia mixing. We tested two cases for the glueball-quarkonia mixing schemes: scenario I, the bare glueball dominantly resides in the $f_0(1500)$; scenario II, the scalar $f_0(1710)$ contains the largest glueball component. We found that the first scenario is more consistent with present data, especially for the case of the strong three-body decays of $J/\psi$. The results are also consistent with available data on radiative decays. The detailed set of predictions for the radiative decays can serve to further distinguish between the different structure assumptions of the scalars. It would therefore be extremely useful to have more data on the radiative processes.

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**Appendix A: Matrix elements and decay widths of the $J/\psi$ three-body decays**

For the calculation of the three-body decays of the $J/\psi$ we introduce several notations — $p$ is the momentum of the $J/\psi$ and $q$, $q_1$, $q_2$, are the momenta of the final state particles: $V(q), P(q_1), P(q_2)$ [for the decay $J/\psi \to Vf_i \to VPP$, $V(q), \gamma(q_1), \gamma(q_2)$ [decay $J/\psi \to f_iV(\gamma) \to \gamma V\gamma$, $\gamma(q), P(q_1), P(q_2)$ [decay $J/\psi \to f_i \gamma \to \gamma PP$] and $\gamma(q), V(q_1), V(q_2)$ [decay $J/\psi \to f_iV(\gamma) \to V\gamma V$].

We also define the invariant variables $s_i (i = 1, 2, 3)$:

$$
p = q + q_1 + q_2,
$$

$$
s_1 = (q + q_1)^2 = (p - q_2)^2,
$$

$$
s_2 = (q_1 + q_2)^2 = (p - q)^2,
$$

$$
s_3 = (q + q_2)^2 = (p - q_1)^2. \quad (A1)
$$

The matrix elements describing the $J/\psi$ three-body strong and radiative decays proceeding through a scalar resonance $f_i$ are denoted by $M^{a(b)}_{i(II,III,IV)}$ with super- and subscripts referring to the graphs of Figs.1-4 in an obvious notation.
The explicit expressions for these amplitudes read as

\[ M_I \equiv M(J/\psi \to f_i V \to V P P) = c_{ij}^f(p) c_{ij}^v(q) g_{Jf,V} g_{f_i,PP} (g_{\mu\nu} p - p_{\mu} q_{\nu}) \left( \frac{1}{M_{f_i}^2 - s_2 - iM_{f_i} \Gamma_{f_i}} \right), \]  

(A2)

\[ M_{II} \equiv M(J/\psi \to f_i V(\gamma) \to V\gamma\gamma) = M_{II}^{(a)} + M_{II}^{(b)}, \]  

(A3)

\[ M_{II}^{(a)} \equiv M(J/\psi \to f_i \gamma \to V\gamma\gamma) = c_j^f(p) c_{ij}^v(q) c_{ij}^{\gamma\alpha}(q_1) c_{ij}^{\gamma\beta}(q_2) g_{Jf,V} g_{f_i,\gamma\gamma} \times \left( g_{\mu\nu} p - p_{\mu} q_{\nu} \right) (g_{\alpha\beta} q_1 q_2 - q_{\beta} q_{2\alpha}) \left( \frac{1}{M_{f_i}^2 - s_2 - iM_{f_i} \Gamma_{f_i}} \right), \]  

\[ M_{II}^{(b)} \equiv M(J/\psi \to f_i V \to V\gamma\gamma) = c_j^f(p) c_{ij}^v(q) c_{ij}^{\gamma\alpha}(q_1) c_{ij}^{\gamma\beta}(q_2) g_{Jf,V} g_{f_i,\gamma\gamma} \times \left[ \left( g_{\mu\alpha} q_1 p - p_{\mu} q_{\alpha} \right) (g_{\nu\beta} q_2 - q_{\beta} q_{2\nu}) \left( \frac{1}{M_{f_i}^2 - s_3 - iM_{f_i} \Gamma_{f_i}} \right) \right], \]  

(A4)

\[ M_{III} \equiv M(J/\psi \to \gamma f_i \to \gamma PP) = c_j^f(p) c_{ij}^v(q) g_{Jf,V} g_{f_i,PP} (g_{\mu\nu} p - p_{\mu} q_{\nu}) \left( \frac{1}{M_{f_i}^2 - s_2 - iM_{f_i} \Gamma_{f_i}} \right), \]  

(A5)

\[ M_{IV} \equiv M(J/\psi \to f_i V(\gamma) \to \gamma VV) = M_{IV}^{(a)} + M_{IV}^{(b)}, \]  

\[ M_{IV}^{(a)} \equiv M(J/\psi \to f_i \gamma \to \gamma VV) = c_j^f(p) c_{ij}^v(q) c_{ij}^{\gamma\alpha}(q_1) c_{ij}^{\gamma\beta}(q_2) g_{Jf,V} g_{f_i,\gamma VV} \times \left( g_{\mu\nu} p - p_{\mu} q_{\nu} \right) (g_{\alpha\beta} q_1 q_2 - q_{\beta} q_{2\alpha}) \left( \frac{1}{M_{f_i}^2 - s_2 - iM_{f_i} \Gamma_{f_i}} \right), \]  

\[ M_{IV}^{(b)} \equiv M(J/\psi \to f_i V \to \gamma VV) = c_j^f(p) c_{ij}^v(q) c_{ij}^{\gamma\alpha}(q_1) c_{ij}^{\gamma\beta}(q_2) g_{Jf,V} g_{f_i,\gamma VV} \times \left[ \left( g_{\mu\alpha} q_1 p - p_{\mu} q_{\alpha} \right) (g_{\nu\beta} q_2 - q_{\beta} q_{2\nu}) \left( \frac{1}{M_{f_i}^2 - s_3 - iM_{f_i} \Gamma_{f_i}} \right) \right], \]  

where \( M_{f_i} \) and \( \Gamma_{f_i} \) are the mass and width of the scalar meson \( f_i \). Here, the coefficients \( g_{Jf,V}, g_{f_i,PP}, g_{f_i,VV}, \) \( g_{f_i,\gamma}, g_{f_i,\gamma\gamma} \) and \( g_{f_i,\gamma\gamma} \) are the couplings involving mixed scalars, which are related to the couplings of the effective Lagrangian (1) involving unmixed states as:

\[ g_{Jf,\omega} = 2B_{1i} c_{h}^q + 2\sqrt{\frac{2}{3}} B_{1i} c_{h}^q + 2\sqrt{2}(B_{1i} \sqrt{2} + B_{i3}) c_{e}^q, \quad g_{Jf,\phi} = -2B_{1i} c_{h}^q - 2\sqrt{\frac{1}{3}} B_{1i} c_{h}^q - 2(B_{1i} \sqrt{2} + B_{i3}) c_{e}^q, \]  

(A6)

\[ g_{f_i,\pi\pi} = -\frac{2B_{i1}}{F^2 \sqrt{2}} \left( (M_{f_i}^2 - 2M_{\pi}^2) c_{d}^q + 2M_{\pi}^2 c_{e}^q \right) - \frac{2B_{i1} c_{d}^q}{F^2 \sqrt{3}} \left( (M_{f_i}^2 - 2M_{\pi}^2)^2 c_{d}^q + 2M_{\pi}^2 c_{e}^q \right), \]  

(A7)

\[ g_{f_i,KK} = -\frac{B_{i1} + \sqrt{2} B_{i3}}{F^2 \sqrt{2}} \left( (M_{f_i}^2 - 2M_{K}^2) c_{d}^q + 2M_{K}^2 c_{e}^q \right) - \frac{2B_{i1}}{F^2 \sqrt{3}} \left( (M_{f_i}^2 - 2M_{K}^2)^2 c_{d}^q + 2M_{K}^2 c_{e}^q \right), \]  

(A8)

\[ g_{f_i,\rho\rho} = g_{f_i,\omega\omega} = \frac{4}{\sqrt{2}} c_{e}^q B_{1i} + \frac{4}{\sqrt{3}} c_{e}^q B_{i2}, \quad g_{f_i,\phi\phi} = 4c_{e}^q B_{1i} + \frac{4}{\sqrt{3}} c_{e}^q B_{i2}, \]  

(A9)

\[ g_{f_i,\gamma} = 2(B_{1i} c_{e}^q + B_{1i} c_{e}^q + B_{3i} c_{e}^q), \quad g_{f_i,\gamma\gamma} = \frac{16}{9} \left( \frac{5}{\sqrt{2}} B_{1i} + B_{i3} \right) c_{e}^q + \frac{32}{3\sqrt{3}} B_{i2} c_{e}^q, \]  

(A10)

\[ g_{f_i,\rho\gamma} = 3g_{f_i,\omega\gamma} = B_{1i} c_{e}^q + \sqrt{\frac{2}{3}} B_{i2} c_{e}^q, \quad g_{f_i,\phi\gamma} = \frac{2}{3} B_{1i} c_{e}^q + \frac{2}{3\sqrt{3}} B_{i2} c_{e}^q. \]  

(A11)

In above expressions the \( B_{ij} \) are the elements of the matrix \( B \) relating the mixed \((f_1, f_2, f_3)\) and unmixed \((N, G, S)\) scalar states as in Eq. (6). If the process is kinematically allowed, the couplings \( g_{Jf,V}, g_{f_i,\gamma}, g_{f_i,\gamma\gamma}, g_{f_i,\gamma\gamma}, \) and
\[ g_{f,\gamma\gamma} \] define the corresponding two-body decay widths of \( \Jpsi \) and the physical \( f_i \) states as:

\[
\Gamma(J/\psi \to f_i V) = \frac{g_{J,V,i}^2}{16\pi M_f^2} M_f^2 \lambda^{1/2}(M_f^2, M_{f_i}^2, M_V^2) \left( 1 + \frac{\lambda(M_f^2, M_{f_i}^2, M_V^2)}{6M_f^2 M_V^2} \right),
\]

\[
\Gamma(f_i \to PP) = \frac{g_{f,PP,i}^2}{16\pi M_f^2} N_{PP} \sqrt{1 - \frac{4M_f^2}{M_{f_i}^2}},
\]

\[
\Gamma(f_i \to VV) = \frac{g_{f,VV,i}^2}{64\pi} M_{f_i}^3 \sqrt{1 - \frac{4M_{f_i}^2}{M_{f_i}^2} \left( 1 - \frac{4M_f^2}{M_{f_i}^2} + \frac{6M_f^2}{M_{f_i}^2} \right)},
\]

\[
\Gamma(J/\psi \to f_i \gamma) = \frac{\alpha}{24} g_{f,i,\gamma}^2 M_f^2 \left( 1 - \frac{M_f^2}{M_{f_i}^2} \right)^3,
\]

\[
\Gamma(f_i \to V\gamma) = \frac{\alpha}{8} g_{f,V,\gamma}^2 M_{f_i}^2 \left( 1 - \frac{M_f^2}{M_{f_i}^2} \right)^3,
\]

\[
\Gamma(f_i \to \gamma\gamma) = \frac{\pi}{4} \alpha^2 g_{f,\gamma\gamma}^2 M_{f_i}^2.
\]

where \( \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \) is the Källen function. The factor

\[
N_{PP} = \begin{cases} 
\frac{9}{2}, & PP = \pi\pi \\
\frac{1}{2}, & PP = K\bar{K} 
\end{cases}
\]

takes into account the sum over charged modes:

\[
\Gamma(f_i \to \pi\pi) = \Gamma(f_i \to \pi^+\pi^-) + \Gamma(f_i \to \pi^0\pi^0),
\]

\[
\Gamma(f_i \to K\bar{K}) = \Gamma(f_i \to K^+\bar{K}^-) + \Gamma(f_i \to K^0\bar{K}^0).
\]

Note, the couplings \( g_{f,V,V}, g_{f,V,\gamma}, g_{f,\gamma\gamma} \) and \( g_{f,f,V}, g_{f,f,\gamma} \) are constrained by VMD and SU(3) flavor symmetry relations (see also Eqs. \ref{eq:2} and \ref{eq:3}):

\[
g_{f,\gamma\gamma} = g_{\rho\gamma} \left( g_{f,\rho\gamma} + \frac{1}{3} g_{f,\omega\gamma} + \frac{\sqrt{2}}{3} g_{f,\phi\gamma} \right) = g_{\rho\gamma}^2 \left( g_{f,\rho\gamma} + \frac{1}{9} g_{f,\rho\omega} + \frac{2}{9} g_{f,\rho\phi} \right),
\]

\[
g_{f,\rho\gamma} = g_{\rho\gamma} g_{f,\rho\rho}, \quad g_{f,\omega\gamma} = \frac{1}{3} g_{\rho\gamma} g_{f,\omega\omega}, \quad g_{f,\phi\gamma} = \frac{\sqrt{2}}{3} g_{\rho\gamma} g_{f,\phi\phi},
\]

\[
g_{f,f,\gamma} = g_{\rho\gamma} \left( \frac{1}{3} g_{f,\rho\omega} + \frac{\sqrt{2}}{3} g_{f,\rho\phi} \right)
\]
with \( g_{\rho\gamma} = 0.2 \).

Next we write down the expressions for the \( J/\psi \) three-body decay widths. The decay width \( \Gamma_1 \equiv \Gamma(J/\psi \to f_i V \to VPP) \) is given by:

\[
\Gamma_1 = \frac{N_{PP}}{768\pi^3 M_f^2} \int_{4M_f^2}^{(M_f-M_V)^2} ds_2 \int_{s_1}^{s_1^+} ds_1 \sum_{pol} |M_i|^2 \frac{g_{J,V,i}^2 g_{f,PP,i}^2}{1536\pi^3 M_f^4} N_{PP} R,
\]

where the sum is performed over the polarizations and where

\[
R = \int_{4M_f^2}^{(M_f-M_V)^2} ds_2 \lambda^{1/2}(M_f^2, M_{f_i}^2, s_2) \sqrt{1 - \frac{4M_f^2}{s_2} \left( \lambda(M_f^2, M_{f_i}^2, s_2) + 6M_f^2 M_{f_i}^2 \right)},
\]

\[
s_{1,1}^+ = M_f^2 + \frac{1}{2} \left( M_f^2 + M_{f_i}^2 - s_2 \pm \lambda^{1/2}(s_2, M_f^2, M_{f_i}^2) \sqrt{1 - \frac{4M_f^2}{s_2}} \right).
\]

Using Eqs. \ref{eq:A12} and \ref{eq:A13} we express \( \Gamma_1 \) through the two-body decay widths \( \Gamma(J/\psi \to f_i V) \) and \( \Gamma(f_i \to PP) \):

\[
\Gamma_1 = \frac{\Gamma(J/\psi \to f_i V) \Gamma(f_i \to PP)}{\pi \lambda^{1/2}(M_f^2, M_{f_i}^2, M_{f_i}^2) \sqrt{M_{f_i}^2 - 4M_f^2}} \lambda(M_f^2, M_{f_i}^2, M_V^2) + 6M_f^2 M_{f_i}^2 R.
\]
The decay width \( \Gamma_{\Pi} \equiv \Gamma(J/\psi \rightarrow f_i V(\gamma) \rightarrow V\gamma\gamma) \) is calculated according to the expression:

\[
\Gamma_{\Pi} = \frac{1}{1536\pi^3 M_f^3} \left( \frac{(M_f - M_\gamma)^2}{s_{\Pi}} \right) \int ds_2 \int ds_1 |M_{\Pi}|^2,
\]

\( (A26) \)

where

\[
s_{\Pi}^\pm = \frac{M_f^2 + M_V^2 - s_2}{2} \pm \frac{\lambda^{1/2}(s_2, M_f^2, M_V^2)}{2}.
\]

\( (A27) \)

The sum over the polarizations is rewritten as

\[
\sum_{\text{pol}} |M_{\Pi}|^2 = 4\pi^2 \alpha^2 \left( g_{Jf,V}^2 g_{f_1,\gamma\gamma}^2 R_1 + g_{Jf,V}^2 g_{f_1,\gamma\gamma}^2 R_2 + g_{Jf,V} g_{f_1,\gamma\gamma} g_{f_1,\gamma\gamma} R_3 \right)
\]

\( (A28) \)

where

\[
R_1 = \frac{s_2^2}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2} \left( \lambda(M_f^2, M_V^2, s_2) + 6M_f^2 M_V^2 \right),
\]

\( (A29) \)

\[
R_2 = \frac{(M_f^2 - s_2)^2}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2} \left( \lambda(M_f^2, M_V^2, s_2) + 6M_f^2 M_V^2 \right)
\]

\( (A30) \)

\[
R_3 = \frac{s_2^2}{(M_f^2 - s_2)^3 + M_f^2 \Gamma_f^2} \left( \lambda(M_f^2, M_V^2, s_2) + 6M_f^2 M_V^2 \right)
\]

\( (A31) \)

The decay width \( \Gamma_{\Pi} \equiv \Gamma(J/\psi \rightarrow f_i \gamma \rightarrow \gamma PP) \) is given by:

\[
\Gamma_{\Pi} = \frac{N_{PP}}{768\pi^3 M_f^3} \int ds_2 \int ds_1 |M_{\Pi}|^2 = \frac{\alpha g_{Jf,V}^2 g_{f_1,PP}^2 N_{PP}}{384\pi^2 M_f^3} \int ds_2 \sqrt{1 - \frac{4M_f^2}{s_2} \frac{(M_f^2 - s_2)^3}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2}},
\]

\( (A32) \)

where

\[
s_{\Pi}^{\pm} = M_f^2 + \frac{M_f^2 - s_2}{2} \left( 1 \pm \sqrt{1 - \frac{4M_f^2}{s_2}} \right).
\]

\( (A33) \)

Using Eqs. \( (A13) \) and \( (A15) \) we again express \( \Gamma_{\Pi} \) through the two-body decay widths \( \Gamma(J/\psi \rightarrow f_i \gamma) \) and \( \Gamma(f_i \rightarrow PP) \) as:

\[
\Gamma_{\Pi} = \frac{\Gamma(J/\psi \rightarrow f_i \gamma) \Gamma(f_i \rightarrow PP) M_f^2}{\pi (M_f^2 - M_f^2)^3 \sqrt{M_f^2 - 4M_f^2}} \int ds_2 \sqrt{1 - \frac{4M_f^2}{s_2} \frac{(M_f^2 - s_2)^3}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2}},
\]

\( (A34) \)

The decay width \( \Gamma_{IV} \equiv \Gamma(J/\psi \rightarrow f_i V(\gamma) \rightarrow \gamma VV) \) is determined according to:

\[
\Gamma_{IV} = \frac{1}{1536\pi^3 M_f^3} \int ds_2 \int ds_1 |M_{IV}|^2
\]

\( (A35) \)
where

\[ s_{1,IV}^\pm = M_\nu^2 + \frac{M_f^2 - s_2}{2} \left( 1 \pm \sqrt{1 - \frac{4M_\nu^2}{s_2}} \right). \quad (A36) \]

Again, the sum can be expressed as

\[ \sum_{pol} |M_{1V}|^2 = \pi \alpha \left( g_{f,\gamma}^2 g_{f,VV}^2 Q_1 + g_{f,V}^2 g_{f,VV}^2 Q_2 + g_{f,V}^2 g_{f,VV}^2 g_{f,V\gamma} Q_3 \right) \quad (A37) \]

with

\[
Q_1 = \frac{1}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2}(s_1^2 - 2s_3^2(M_f^2 + 2M_\nu^2) + s_2^2(M_f^4 + 8M_f^2 M_\nu^2 + 6M_\nu^4) \\
- 4s_2 M_f^2 M_\nu^2 (M_f^2 + 3M_\nu^2) + 6M_f^4 M_\nu^4), \quad (A38)
\]

\[
Q_2 = \frac{1}{(M_f^2 - s_1)^2 + M_f^2 \Gamma_f^2}(s_1^2 - 2s_3^2(M_f^2 + 2M_\nu^2) + s_2^2(M_f^4 + 8M_f^2 M_\nu^2 + 6M_\nu^4) \\
- 2s_1 M_\nu^2 (M_f^4 + 5M_f^2 M_\nu^2 + 2M_\nu^4) + M_\nu^4 (M_f^4 + 4M_f^2 M_\nu^2 + M_\nu^4) \\
+ \frac{(M_f^2 - s_3)(M_f^2 - s_1) + M_f^2 \Gamma_f^2}{(M_f^2 - s_1)^2 + M_f^2 \Gamma_f^2((M_f^2 - s_3)^2 + M_f^2 \Gamma_f^2)}(s_1^2 s_3^2 + (s_1^2 + s_3^2)M_\nu^2 (M_f^2 + M_\nu^2) \\
- 2(s_1 + s_3)M_\nu^4 (M_f^2 + 2M_\nu^2) + 2M_\nu^6 (2M_f^2 + 5M_\nu^2)), \quad (A39)
\]

\[
Q_3 = \frac{1}{(M_f^2 - s_2)^2 + M_f^2 \Gamma_f^2}(\frac{(M_f^2 - s_2)(M_f^2 - s_1) + M_f^2 \Gamma_f^2}{(M_f^2 - s_1)^2 + M_f^2 \Gamma_f^2}) (s_1^2 s_3^2 + 2s_1^2 M_f^2 M_\nu^2 + s_2^2 M_\nu^4 (M_f^2 + M_\nu^2) \\
- 2(s_1 + s_2)M_\nu^2 (M_f^2 + 2M_\nu^2) + M_f^2 M_\nu^4 (M_f^2 + 4M_f^2 M_\nu^2 + 2M_\nu^4) \\
+ \frac{(M_f^2 - s_3)(M_f^2 - s_2) + M_f^2 \Gamma_f^2}{(M_f^2 - s_3)^2 + M_f^2 \Gamma_f^2} (s_3^2 s_2^2 + 2s_3^2 M_f^2 M_\nu^2 + s_2^2 M_\nu^4 (M_f^2 + M_\nu^2) \\
- 2(s_3 + s_2)M_\nu^2 (M_f^2 + 2M_\nu^2) + M_f^2 M_\nu^4 (M_f^2 + 4M_f^2 M_\nu^2 + 2M_\nu^4) \right). \quad (A40)
\]
FIG. 1: Contribution of the scalar resonances $f_i = f_0(1370), f_0(1500), f_0(1710)$ to the decay $J/\psi \rightarrow f_i V \rightarrow VPP$

FIG. 2: Contribution of the scalar resonances $f_i = f_0(1370), f_0(1500), f_0(1710)$ to the decay $J/\psi \rightarrow f_i (\gamma) \rightarrow V \gamma \gamma$

FIG. 3: Contribution of the scalar resonances $f_i = f_0(1370), f_0(1500), f_0(1710)$ to the decay $J/\psi \rightarrow f_i \gamma \rightarrow \gamma PP$

FIG. 4: Contribution of the scalar resonances $f_i = f_0(1370), f_0(1500), f_0(1710)$ to the decay $J/\psi \rightarrow f_i (\gamma) \rightarrow \gamma VV$
TABLE I: Masses and decay properties of the scalar mesons \( f_i \).

| Quantity | Data \[36\] | Fit I | \( \chi^2 \) | Fit II | \( \chi^2 \) |
|----------|-------------|-------|------|-------|------|
| \( M_{f_1} \) (MeV) | 1350 ± 200 | 1432 | 0.16 | 1231 | 0.36 |
| \( M_{f_2} \) (MeV) | 1505 ± 6 | 1510 | 0.72 | 1510 | 0.70 |
| \( M_{f_3} \) (MeV) | 1720 ± 6 | 1720 | ~ 0 | 1720 | ~ 0 |
| \( \Gamma_{f_2 \to \eta' \eta} \) (MeV) | 2.07 ± 0.87 | 1.2 | 0.97 | 1.02 | 1.45 |
| \( \Gamma_{f_2 \to \eta \eta} \) (MeV) | 5.56 ± 0.98 | 2.8 | 7.98 | 4.70 | 0.76 |
| \( \Gamma_{f_2 \to \bar{K}K} \) (MeV) | 9.37 ± 1.09 | 10.4 | 0.96 | 8.5 | 0.19 |
| \( \Gamma_{f_2 \to \pi \pi} \) (MeV) | 38.04 ± 2.51 | 37.7 | 0.02 | 38.38 | 0.02 |
| \( \Gamma_{f_3 \to \eta \eta} \) (MeV) | 5.56 ± 0.98 | 2.8 | 7.98 | 4.70 | 0.76 |
| \( \Gamma_{f_3 \to \bar{K}K} \) (MeV) | 9.37 ± 1.09 | 10.4 | 0.96 | 8.5 | 0.19 |
| \( \Gamma_{f_3 \to \pi \pi} \) (MeV) | 38.04 ± 2.51 | 37.7 | 0.02 | 38.38 | 0.02 |

TABLE II: Comparison of strong scalar decays with WA102 data \[37\].

| Quantity | WA102 data \[37\] | I | II |
|----------|----------------|---|----|
| \( \Gamma_{f_1 \to \bar{K}K} / \Gamma_{f_1 \to \eta \eta} \) | 0.46 ± 0.19 | 1.07 | 2.27 |
| \( \Gamma_{f_1 \to \eta \eta} / \Gamma_{f_1 \to \bar{K}K} \) | 0.16 ± 0.07 | 0.22 | 0.4 |
| \( \Gamma_{f_3 \to 2P} \) (MeV) | small | 314 | 131 |
| \( \Gamma_{a_0 \to \pi \eta} / \Gamma_{a_0 \to \eta \eta} \) | 0.35 ± 0.16 | 0.29 | 0.15 | 0.29 | 0.13 |
| \( \Gamma_{a_0 \to \bar{K}K} / \Gamma_{a_0 \to \pi \eta} \) | 0.88 ± 0.23 | 0.8 | 1.06 | 0.64 |
| \( \Gamma_{K_{(1430)} \to K \pi} \) (MeV) | 251 ± 74.0 | 64.7 | 6.27 | 41.64 | 7.92 |

\( \chi^2_{tot} \) | 19.82 | 14.99 |

TABLE III: Branchings of the \( J/\psi \) hadronic decays in units \( 10^{-4} \).

| Meson | \( \phi \bar{K}K \) Data \[36\] | \( \omega \bar{K}K \) Data \[36\] | \( \phi \pi \pi \) Data \[36\] | \( \omega \pi \pi \) Data \[36\] |
|-------|----------------|----------------|----------------|----------------|
| \( f_3 \) | 3.6 | 2.5 | 3.6 ± 0.4 | 4.8 | 5.2 | 4.8 | 4.8 ± 1.1 | 1.7 | 1.3 | 0.40 | - | 2.2 | 2.7 | 0.53 | - |
| \( f_2 \) | 0.5 | 2.3 | 0.8 ± 0.5 | 0.2 | 1.1 | 1.38 | - | 2.0 | 9.4 | 0.77 | 1.7 ± 0.8 | 0.6 | 4.6 | 5.60 | - |
| \( f_1 \) | 4.1 | 4.7 | 0.3 ± 0.3 | 1.3 | 8.7 | 0.08 | - | 4.3 | 2.4 | 0.01 | 4.3 ± 1.1 | 1.4 | 4.4 | 0.83 | - |
TABLE IV: The branchings of the $J/\psi$ two-body decays in $10^{-4}$ and effective couplings in $10^{-3} \times \text{GeV}^{-1}$.

| Quantity                  | I     | II    | Data [36] |
|---------------------------|-------|-------|-----------|
| $g_{Jf_1 \omega}$        | -0.60 | -1.91 |
| $g_{Jf_2 \omega}$        | 0.57  | 2.2   |
| $g_{Jf_3 \omega}$        | -1.60 | -1.80 |
| $g_{Jf_1 \phi}$          | 1.02  | 1.35  |
| $g_{Jf_2 \phi}$          | -1.00 | 2.18  |
| $g_{Jf_3 \phi}$          | -1.40 | 1.27  |
| $g_{Jf_1 \gamma}$        | 2.34  | 2.12  |
| $g_{Jf_2 \gamma}$        | -1.53 | -1.53 |
| $g_{Jf_3 \gamma}$        | -7.42 | 7.98  |
| $\text{Br}(J/\psi \rightarrow f_1 \omega)$ | 2.2   | 27.5 |
| $\text{Br}(J/\psi \rightarrow f_2 \omega)$ | 1.8   | 13.6 |
| $\text{Br}(J/\psi \rightarrow f_3 \omega)$ | 10.7  | 13.5 |
| $\text{Br}(J/\psi \rightarrow f_1 \phi)$ | 6.8   | 14.9 |
| $\text{Br}(J/\psi \rightarrow f_2 \phi)$ | 5.9   | 28.0 |
| $\text{Br}(J/\psi \rightarrow f_3 \phi)$ | 8.1   | 6.6  |
| $\text{Br}(J/\psi \rightarrow f_1 \gamma)$ | 2.6   | 2.6  |
| $\text{Br}(J/\psi \rightarrow f_2 \gamma)$ | 1.0   | 1.0  |
| $\text{Br}(J/\psi \rightarrow f_3 \gamma)$ | 17.7  | 20.5 |

TABLE V: Effective couplings of $f_i$ in units $\text{GeV}^{-1}$.

| Coupling   | I     | II    | Data [36] |
|------------|-------|-------|-----------|
| $g_{f_1 \gamma \gamma}$ | 0.30  | 0.41  |
| $g_{f_2 \gamma \gamma}$ | -0.21 | 0.13  |
| $g_{f_3 \gamma \gamma}$ | -0.01 | -0.08 |
| $g_{f_1 \rho \gamma}$   | 1.24  | 1.60  |
| $g_{f_2 \rho \gamma}$   | 0.29  | 0.75  |
| $g_{f_3 \rho \gamma}$   | -0.90 | -0.98 |
| $g_{f_1 \phi \gamma}$   | -0.13 | -0.94 |
| $g_{f_2 \phi \gamma}$   | -0.47 | -0.44 |
| $g_{f_3 \phi \gamma}$   | 0.10  | -0.20 |
| $g_{f_1 \rho \rho}$     | 6.19  | 8.01  |
| $g_{f_2 \rho \rho}$     | 3.09  | 8.00  |
| $g_{f_3 \rho \rho}$     | -4.44 | 4.91  |
| $g_{f_1 \phi \phi}$     | -1.33 | -9.97 |
| $g_{f_2 \phi \phi}$     | -2.35 | -2.19 |
| $g_{f_3 \phi \phi}$     | 10.74 | 2.16  |
TABLE VI: Branchings of the $J/\psi$ radiative decays in units $10^{-4}$.

| Mode                  | I         | II         | Data [36] |
|-----------------------|-----------|------------|-----------|
| $J/\psi \rightarrow f_1\gamma \rightarrow \gamma\pi\pi$ | 1.65      | 0.42       |           |
| $J/\psi \rightarrow f_2\gamma \rightarrow \gamma\pi\pi$ | 0.34      | 0.34       |           |
| $J/\psi \rightarrow f_3\gamma \rightarrow \gamma\pi\pi$ | 4.48      | 5.02       | 4.0 ± 1.0 |
| $J/\psi \rightarrow f_4\gamma \rightarrow \gamma K\bar{K}$ | 1.58      | 0.82       |           |
| $J/\psi \rightarrow f_5\gamma \rightarrow \gamma K\bar{K}$ | 0.09      | 0.08       |           |
| $J/\psi \rightarrow f_6\gamma \rightarrow \gamma K\bar{K}$ | 8.00      | 7.85       | 8.5$^{+1.2}_{-0.9}$ |
| $J/\psi \rightarrow f_1\gamma \rightarrow \gamma\omega\omega$ | 0.33      | 0.22       |           |
| $J/\psi \rightarrow f_2\gamma \rightarrow \gamma\omega\omega$ | 0.13      | 0.16       |           |
| $J/\psi \rightarrow f_3\gamma \rightarrow \gamma\omega\omega$ | 3.09      | 3.07       | 3.1 ± 1.0 |
| $J/\psi \rightarrow f_1\gamma \rightarrow \rho\gamma\gamma$ | 2.0$\times10^{-2}$ | 1.7$\times10^{-2}$ |           |
| $J/\psi \rightarrow f_1\omega(\gamma) \rightarrow \omega\gamma\gamma$ | 2.2$\times10^{-3}$ | 3.3$\times10^{-3}$ |           |
| $J/\psi \rightarrow f_1\phi(\gamma) \rightarrow \phi\gamma\gamma\gamma$ | 0.8$\times10^{-3}$ | 2.5$\times10^{-3}$ |           |
| $J/\psi \rightarrow f_2\gamma \rightarrow \rho\gamma\gamma$ | 0.9$\times10^{-2}$ | 1.1$\times10^{-2}$ |           |
| $J/\psi \rightarrow f_2\omega(\gamma) \rightarrow \omega\gamma\gamma\gamma$ | 1.1$\times10^{-3}$ | 1.5$\times10^{-3}$ |           |
| $J/\psi \rightarrow f_2\phi(\gamma) \rightarrow \phi\gamma\gamma\gamma$ | 0.4$\times10^{-4}$ | 0.5$\times10^{-2}$ |           |
| $J/\psi \rightarrow f_3\gamma \rightarrow \rho\gamma\gamma$ | 6.0$\times10^{-2}$ | 6.5$\times10^{-2}$ |           |
| $J/\psi \rightarrow f_3\omega(\gamma) \rightarrow \omega\gamma\gamma\gamma$ | 0.7$\times10^{-2}$ | 0.7$\times10^{-2}$ |           |
| $J/\psi \rightarrow f_3\phi(\gamma) \rightarrow \phi\gamma\gamma\gamma$ | 0.16      | 0.8$\times10^{-2}$ |           |

TABLE VII: Decay widths of $f_i \rightarrow V\gamma$ and $f_i \rightarrow \gamma\gamma$ transitions (in keV) in comparison with other theoretical predictions.

| Mode                  | Ref. [24]                  | Ref. [25]                  | Ref. [26]                  | Ref. [28]                     | Ref. [6]                      | Our (I, II) in keV |
|-----------------------|----------------------------|----------------------------|----------------------------|-------------------------------|-------------------------------|--------------------|
| $f_1 \rightarrow \gamma\gamma$ | $(1.6, 3.9^{+0.8}_{-0.7}, 5.6^{+1.4}_{-1.3})$ | $(1.6, 3.9^{+0.8}_{-0.7}, 5.6^{+1.4}_{-1.3})$ | 0.35                       | 1.31                          | (11.1, 13.4)           |
| $f_2 \rightarrow \gamma\gamma$ | $(0.92, 1.3^{+0.2}_{-0.2}, 3.0^{+1.4}_{-1.2})$ | $(0.92, 1.3^{+0.2}_{-0.2}, 3.0^{+1.4}_{-1.2})$ | 0.019                      | 0.05                          | (0.02, 1.2)            |
| $f_3 \rightarrow \gamma\gamma$ | $(443, 1121, 1540)$         | $(443, 1121, 1540)$         | $(79 \pm 40, 125 \pm 80)$  | 726                           | (1441, 957)            |
| $f_1 \rightarrow \omega\gamma$ | $(8, 9, 32)$               | $(8, 9, 32)$               | $(7 \pm 3, 128 \pm 80)$    | 0.04                          | (156, 102)             |
| $f_1 \rightarrow \phi\gamma$ | $(0.98, 0.83^{+0.27}_{-0.23}, 4.5^{+4.5}_{-3.0})$ | $(0.98, 0.83^{+0.27}_{-0.23}, 4.5^{+4.5}_{-3.0})$ | $(11 \pm 6, -)$            | 0.01                          | (27, 30)            |
| $f_2 \rightarrow \rho\gamma$ |                           |                           |                           |                               | (990, 1209)          |
| $f_2 \rightarrow \omega\gamma$ |                           |                           |                           |                               | (108, 131)            |
| $f_2 \rightarrow \phi\gamma$ |                           |                           |                           |                               | (8, 447)             |
| $f_3 \rightarrow \rho\gamma$ | $(42, 94, 705)$            | $(42, 94, 705)$            | $(100 \pm 40, -)$          | 24                            | (519, 450)            |
| $f_3 \rightarrow \omega\gamma$ |                           |                           | $(3.3 \pm 1.2, -)$         | 82                            | (57, 49)              |
| $f_3 \rightarrow \phi\gamma$ |                           |                           | $(15 \pm 5, -)$            | 94                            | (1300, 53)            |