Contributions to the kinematics of double harmonic transmission

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Abstract. The construction of modern industrial robots has imposed the assurance of a higher kinematic precision of orientation and positioning, which led to the appearance of better mechanical transmissions. The paper presents the construction and operating of such a new mechanical transmission, named the double harmonic transmission (DHT). Through the kinematic analysis of the double harmonic transmission, it was developed an original analytical calculation method of the transmission ratio for the cases of two constructive variants of these transmissions. The detailed study of the harmonic engagement processes in the two stages of double harmonic transmission has highlighted the relative movements of the teeth of the conjugated wheels.

1. Introduction

The continuous automation of technological processes, as well as the widespread deployment of industrial robots and manipulators in current technique has led to the use of more advanced mechanical transmissions.

In this category are also included harmonic gear transmissions, which have imposed in almost all areas of top technique, due to the many advantages they presents: compact and modular constructions, very low gauges, large and very high transmission ratios, high cinematic precision, silent operation, high mechanical efficiency, etc. [1-5].

The uses of harmonic gear transmissions are the most varied, these being found in the construction of the: industrial robots, spacecraft and rockets, airplanes and helicopters, nuclear reactors, radar antennas, naval mechanisms, cranes, servomotors, machine tools, precision dividing heads, actuators in hermetic spaces from chemical and petroleum industry, etc. [6-19].

The presence of the flexible element in the construction of the harmonic gear transmissions determines a series of specific functional and constructive particularities in comparison to those of the classical gears (cylindrical, conical and worm).

The principle of operation of the harmonic gear transmissions is based on the deformability of an element of its own, called the flexible toothed wheel, which allows the propagation with a certain frequency of elastic deformations (so-called waves) on its periphery, usually after a law harmonic [1, 3, 4, 5, 15, 20].

In the new category of harmonic gear transmissions, can also be included the double harmonic transmission, which has the peculiarity that the flexible toothed wheel has a short annular form and is provided with two toothing (one outer and the other inside) at its two extremities [20-24].
The paper presents the construction and the functioning of this double harmonic transmission variant, as well as a kinematic analysis used for the study of the relative motion between the outer teeth of the flexible wheel and the inner teeth of rigid wheel, respectively the inner teeth of the flexible wheel and the outer teeth of the mobile rigid wheel.

2. Construction and functioning of the double harmonic transmission

The constructive particularity of this double harmonic transmission consists of the annular form of the flexible wheel, which is provided with two tooth crowns located on opposite cylindrical surfaces (one on the outside and the other on the inside), by one at each extremity.

Figure 1 shows the double harmonic transmission scheme which is composed of the following four elements: 1 - the wave generator is the driving element, 2 - the flexible toothed wheel is the intermediate element having an outer toothing ($z_2$) and the other inside ($z'_2$), 3 - the rigid wheel with internal teeth ($z_3$) is the fixed element and 4 - the mobile rigid wheel with external teeth ($z_4$) is the driven element [20].

The double harmonic transmission was based on the same operating principle as the simple radial harmonic transmission, but has in addition to this a second harmonic engagement stage.

Thus, the first harmonic stage engages the flexible toothed wheel (2) with the fixed rigid wheel (3) on the direction of the large symmetry axis of the wave generator in the front plane (I-I) of the flexible wheel. In the second harmonic stage engages the flexible wheel (2) with the mobile rigid wheel (4) on a direction from the front plane (II-II) of the flexible wheel, which is parallel to the direction of the small symmetry axis of the wave generator from plane (I-I).

In order to ensure the optimal conditions of harmonic engaging between the flexible and the fixed rigid wheel, respectively between the mobile rigid wheel and the flexible wheel, it is necessary that the teeth numbers of the conjugated wheels satisfy the following relations:

$$z_3 - z_2 = k \cdot n_u; \quad z'_3 - z'_4 = k \cdot n_u$$

where: $n_u$ - is number of waves deformation, $z$ - number of the teeth of wheels; $k = (1, 2, ...)$.

In the case of the studied double harmonic transmission (where: $k = 1$ and $n_u = 2$), the conditions imposed by the relation (1) can also be written in the following form:

$$z_3 - z_2 = z'_3 - z'_4 = n_u = 2 \iff z_3 = z_2 + 2; \quad z'_3 = z'_4 + 2$$

When assembling the transmission, the wave generator is mounted forced inside the short flexible wheel, and causes its elastic deformation. Thus, in cross-section, the flexible wheel will be elliptical and will have four harmonic equidistant engagement zones: two opposite with fixed rigid wheel (in vertical direction), and two with rigid mobile wheel (in horizontal direction). The two pairs of opposite harmonic engagement areas are positioned at an angle of 90°.
For the case of the analysed double harmonic transmission it is observed that when rotating (clockwise) a wave generator with a 180° angle, the flexible wheel will rotate with one tooth in a trigonometric sense relative to fixed rigid wheel. In turn, the mobile rigid wheel will rotate in a trigonometric sense with two teeth relative to fixed rigid wheel, respectively with a tooth relative to flexible wheel.

At a 360° rotation of the wave generator, the flexible wheel will rotate with two teeth relative to the fixed rigid wheel (in a trigonometric sense), and the mobile rigid wheel will also rotate (in a trigonometric sense) with four teeth in relation to the fixed rigid wheel.

3. Determination of the transmission ratio of the double harmonic transmission

The transmission ratio of the double harmonic transmission, corresponding to the scheme shown in Figure 1, was determined analytically according to the way it is defined and by using the motion inversion method proposed by R. Willis [2]:

\[ i_{14}^3 = \frac{\omega_1}{\omega_4} ; i_{34}^1 = \frac{\omega_3 - \omega_1}{\omega_4 - \omega_1} = \frac{z_2}{z_2} \Rightarrow \omega_4 = \frac{i_{34}^1 - 1}{i_{14}^3} \cdot \omega_1 \]  

\[ i_{14}^3 = \frac{\omega_1}{\omega_4} = \frac{i_{34}^1 - 1}{i_{34}^1} \frac{z_2 \cdot z_4}{z_2' \cdot z_4 - z_3 \cdot z_2'} \]  

where: \( \omega_1 \) - is the angular speed of the wave generator; \( \omega_3 \) - angular speed of the fixed rigid wheel; \( \omega_4 \) - angular speed of the mobile wheel; \( z_2, z_2', z_3 \) and \( z_4 \) - number of teeth of the conjugated wheels.

The same result was obtained by considering the double harmonic transmission as consisting of two harmonic stages (I and II respectively), characterized by the transmission ratios [15]:

\[ i_J = \frac{z_2}{z_2 - z_2} ; i_{II} = \frac{z_4}{z_4 - z_2'} \]  

\[ i_{14}^3 = \frac{i_J \cdot i_{II}}{i_J - i_{II} + 1} = \frac{z_2 \cdot z_4}{z_2' \cdot z_4 - z_3 \cdot z_2'} \]  

where: \( i_J \) and \( i_{II} \) - are transmission ratios from the two harmonic stages.

For the case when the second harmonic stage of the double harmonic transmission operates as a toothed coupling (\( z_2' = z_4 \)), then the transmission ratio becomes:

\[ i_{14}^3 = \frac{z_2}{z_2 - z_3} = -\frac{z_2}{2} \]  

The transmission ratio \( i_{14}^3 \) of the studied double harmonic transmission (\( z_2 \neq z_2' \)) can also be determined on the basis of a geometric reasoning, which takes into consideration the dynamics of the flexible toothed wheel.

In Figure 2, b was represented the kinematic model of the double harmonic transmission by highlighting the medium dynamic curves of the conjugated wheels. This kinematic model was obtained by fictional transposition of the two pairs of harmonic engagement zones (from stage I and stage II respectively), which are offset to 90°, in a median cross section (A-A) of the transmission (Figure 2, a). It is admitted that the length of the medium dynamic curve of the deformed flexible wheel is kept constant in any cross section thereof.

For the kinematic analysis of the double harmonic transmission, the following reference systems were adopted: \( S_1(XOY) \) - the mobile system fixed by the wave generator, \( S_2(xOy) \) - the mobile system fixed to the flexible toothed wheel, \( S_3(xOy) \) - the immobile system connected to the rigid fixed wheel and \( S_4(x'Oy') \) - the mobile system fixed to the mobile rigid wheel.
Before assembling the double harmonic transmission, the rigid fixed wheel (3), the rigid mobile wheel (4) and the undeformed flexible toothed wheel (2), have the medium dynamic curves in the form of concentric circles. After mounting the wave generator, the flexible wheel deforms and its medium dynamic curve takes an elliptical form.

**Figure 2.** The kinematic model of the double harmonic transmission.

The lengths of the medium dynamic curves of the wheels of a double harmonic transmission will be proportional to the teeth numbers of the respective wheels \((z_2, z_3\text{ and } z_4)\), i.e:

\[
L_2 = \pi \cdot d_2 = \pi \cdot m \cdot z_2; L_3 = \pi \cdot d_3 = \pi \cdot m \cdot z_3; L_4 = \pi \cdot d_4 = \pi \cdot m \cdot z_4
\]  

(8)

By rotating (clockwise) of the wave generator with the \(\varphi_1\) angle, the medium dynamic curve of the flexible wheel (2) will roll without sliding (in the trigonometric sense) on the medium dynamic curve of the rigid fixed wheel (3), thus that point M will describe the trajectory materialized by the geometric arc M_0M.

Simultaneously, the median dynamic curve of the flexible wheel will roll without sliding on the medium dynamic curve of the mobile rigid wheel (4) in areas at 90° relative to the large symmetry axis of ellipse so that the point M’ will describe the trajectory M’_0M’ in the fixed coordinate system S_3.

The transmission ratio achieved in the first harmonic stage of the double harmonic transmission can be expressed according to the geometric rolling arcs and the angles of rotation of the coupled wheels in the following form:

\[
i_{12}^3 = \frac{\omega_1}{\omega_2} = \frac{\varphi_1}{\varphi_1} = \frac{-AM_0/r_1}{AM/r_2 - AM_0/r_3} = \frac{-s/r_5}{r_3 - r_2} = \frac{z_2}{z_3 - z_2}
\]  

(9)

The transmission ratio achieved in the second harmonic stage of the double harmonic transmission, expressed by the angles of rotation of the conjugated wheels, will be:

\[
i_{24}^3 = \frac{\omega_2 - \omega_1}{\omega_4 - \omega_1} = \frac{\varphi_1 - \varphi_1}{\psi - \varphi_1} = \frac{z_4}{z_2} \Rightarrow \psi - \varphi_1 = \frac{z_4}{z_2} \left( \varphi_1 - \varphi_1 \right)
\]  

(10)

Using the relations (9) and (10), the total transmission ratio of the double harmonic transmission can be determined:
From the relations (4), (6) and (11) it is noticed that, regardless of the reasoning applied, the same formula for calculating the transmission ratio of the double harmonic transmission was found.

The transmission ratio of the double harmonic transmission depends on the teeth number of the component wheels. Larger transmission ratios can be obtained by conveniently choosing the teeth numbers of the wheels, but in close correlation with the loading and the gauge imposed to the double harmonic transmission.

4. The relative motion of the teeth in the double harmonic transmission
The study of the relative motion of the teeth of the conjugated wheels in the two stages of the double harmonic transmission, is of a particular importance for the appreciation of the engagement conditions, in order to avoid teeth interference as well as for choosing the basic geometric parameters of the respective gears (the angle profile ($\alpha$), the tooth height ($h$), the clearance ($c_0$), etc.).

The determination of the relative successive positions of the teeth of conjugated wheels from the two stages of the double harmonic transmission was performed according to the methodology presented in the reference [20].

The graphical highlighting of the relative movement of the conjugated teeth was accomplished by drawing the motion trajectory of two points located on the axis of symmetry of the mobile tooth.

In Figure 3 it was represented (using continuous line) the relative position of the two teeth from first harmonic stage of the double harmonic transmission (one from the fixed rigid wheel and the other from the flexible wheel) along the direction of the small axis of the wave generator, at the initial moment, for the deformed state of the flexible wheel. With broken line, it was represented the position of the same tooth of the flexible wheel, when it is not deformed.

![Figure 3](image_url)

**Figure 3.** The initial positions of the teeth in first stage (I-I).

In the axis system $S_2$, the initial position of the tooth of fixed rigid wheel is determined by the coordinates of two points M and N, located on the symmetry axis of the tooth, at its intersection with the addendum circle (index $a$ - for point M), respectively with the dedendum circle (index $f$ - for point N) of the fixed wheel. The relations (12) and (13) were used for the point M, and the relations (14) and (15) for the point N:
\[ x_M^r = v_{a3} = -r_{a3} \cdot (2\pi / z_3) \cdot \psi \]  \hspace{1cm} (12) \\
\[ y_M^r = w_{a3} = r_{a3} \cdot \cos[(2\pi / z_3) \cdot \psi] - r_{02} \]  \hspace{1cm} (13) \\
\[ x_N^r = v_{f3} = -r_{f3} \cdot (2\pi / z_3) \cdot \psi \]  \hspace{1cm} (14) \\
\[ y_N^r = w_{f3} = r_{f3} \cdot \cos[(2\pi / z_3) \cdot \psi] - r_{02} \]  \hspace{1cm} (15)

Similarly, the position of the tooth of the flexible wheel (in the mobile system S_2) is determined by the coordinates of the points P and R, located at the intersection of the axis of symmetry of the tooth with the addendum circle (index a - for point P), respectively with the dedendum circle (index f - for R point). The relations (16) and (17) were used for the point P, and the relations (18) and (19) for the point R:

\[ x_P^r = v_{a2} = v + (r_{a2} - r_{02}) \cdot \theta - (r_{a2} + w) \cdot \varphi_3 - v_3 \]  \hspace{1cm} (16) \\
\[ y_P^r = w_{a2} = (r_{a2} + w) \cdot \cos \varphi_3 - r_{02} - w_3 \]  \hspace{1cm} (17) \\
\[ x_R^r = v_{f2} = v + (r_{f2} - r_{02}) \cdot \theta - (r_{f2} + w) \cdot \varphi_3 - v_3 \]  \hspace{1cm} (18) \\
\[ y_R^r = w_{f2} = (r_{f2} + w) \cdot \cos \varphi_3 - r_{02} - w_3 \]  \hspace{1cm} (19)

where: \( r_{02} \) - is the radius of the medium dynamic curve of the flexible wheel; \( w, v \) - the radial respectively tangential displacements; \( \theta \) - the rotation of a section of the deformed flexible element; \( \varphi_3 \) - the angle of relative rotation of the wheels; \( w_3 = 0; v_3 = 0; \psi = 0 \).

In Figure 4 it was represented the graphical construction of successive positions of the flexible wheel tooth in relation to the fixed rigid wheel tooth, for the case of the double harmonic transmission characterized by the following kinematics and geometric parameters: transmission ratio, \( i_{14} = 50 \); teeth modulus, \( m = 0.3 \) mm; radius of the medium dynamic curve, \( r_{02} = 29.3 \) mm; number of the teeth, \( z_2 = 200; z_3 = 202; z_2' = 192; z_4 = 190 \); maximum radial deformation in the front sections of the flexible wheel, \( w_{01} = 0.3 \) mm; \( w_{02} = 0.27 \) mm.

![Figure 4](image-url)

**Figure 4.** The successive positions of the teeth in first stage of the double harmonic transmission.

For the second stage of the double harmonic transmission proceeding analogously as in the first stage, were generated the successive positions of the two teeth (one tooth of the deformed flexible wheel, and the other of the rigid mobile wheel).
In Figure 5 it was represented (using continuous line) the relative position of the two teeth in the second stage, which at the initial moment, are in the direction of the large axis of the wave generator.

With broken line it was represented the position of the same tooth of the flexible wheel, when it is not deformed.

The relative positions of the two teeth in the mobile system $S_2$ (linked to the flexible toothed wheel) are defined by the coordinates of two points located on the axes of symmetry of the teeth, at the intersection with the addendum circles (index a - for points $M'$ and $P'$), respectively with the dedendum circles (index f - for points $N'$ and $R'$).

Thus, the coordinates of the points $M'$ and $N'$ on the axis of symmetry of the tooth of the mobile rigid wheel, are determined by means of the relations:

$$x_{M'} = v_{a4} = -r_{a4} \cdot [(2\pi / z_4) \cdot \psi]$$  \hspace{1cm} (20)

$$y_{M'} = w_{a4} = r_{a4} \cdot \cos [(2\pi / z_4) \cdot \psi] - r_{02}$$ \hspace{1cm} (21)

$$x_{N'} = v_{f4} = -r_{f4} \cdot [(2\pi / z_4) \cdot \psi]$$ \hspace{1cm} (22)

$$y_{N'} = w_{f4} = r_{f4} \cdot \cos [(2\pi / z_4) \cdot \psi] - r_{02}$$ \hspace{1cm} (23)

The coordinates of the points $P'$ and $R'$ defining the relative position of the flexible wheel tooth in relation to the tooth of the rigid mobile wheel considered fixed, will be expressed through the following relationships:

$$x_{P'} = v_{a2} = v - (r_{a2} - r_{02}) \cdot \theta + (r_{a2}' + w) \cdot (\varphi_1 - \varphi_2)$$ \hspace{1cm} (24)

$$y_{P'} = w_{a2} = (r_{a2}' + w) \cdot \cos (\varphi_1 - \varphi_2) - r_{02}$$ \hspace{1cm} (25)

$$x_{R'} = v_{f2} = v - (r_{f2} - r_{02}) \cdot \theta + (r_{f2}' + w) \cdot (\varphi_1 - \varphi_2)$$ \hspace{1cm} (26)

$$y_{R'} = w_{f2} = (r_{f2}' + w) \cdot \cos (\varphi_1 - \varphi_2) - r_{02}$$ \hspace{1cm} (27)

Figure 6 shows the graphical construction of successive positions of tooth of the flexible wheel in relation to the tooth of the mobile wheel, in the case of double harmonic transmission studied.
Figure 6. The successive positions of the teeth in second stage of the double harmonic transmission.

5. Conclusions
In this paper were presented the constructive and functional aspects of the double harmonic transmission, as well as a kinematics study of this transmission.

The presentation of the constructive and functional peculiarities, manufacture and testing of this double harmonic transmission, have demonstrated viability and cinematic performance of very high orientation and positioning.

The study of the kinematics of the double harmonic transmission allowed the determination of the formula for calculating the transmission ratio for the cases of two constructive variants of these transmissions ($z_2 \neq z'_2$, and $z_4 = z'_2$ respectively).

From the comparative analysis of the calculus relations of the transmission ratio of double harmonic transmission, it was found that it depends on the number of teeth of the conjugated wheels, being limited in value (upper - max. 150, and lower - min. 40). This limitation of the transmission ratio is caused by limited technological possibilities of execution and precision, referring to the maximum number of teeth of the wheels and the minimum modulus of the teeth that can be manufactured.

The mathematical modelling of the harmonic engagement processes in the two stages of the double harmonic transmission, has led to the graphical construction of the relative successive positions of the engaging teeth which can be easily used to define the basic geometric parameters of the transmission wheels so as to avoid tooth interference.

References
[1] Ghinzburg E G 1977 Harmonic Gear Transmission, Mechanical Engineering
[2] Ianici S and Ianici D 2010 Design of Mechanical Systems, Eftimie Murgu
[3] Ivanov M N 1981 Harmonic Gear Transmission, Graduate School
[4] Musser C W 1955. Strain Wave Gearing, US Patent 2.906.143
[5] Miloiu G, Dudita F and Diaconescu D V 1980 Modern Mechanical Drives, Technics
[6] Afronie E M, Manescu T S and Ianici S 2012 Calculus of Metallic and Steel-Concrete Mixed Structures at Fire Actions, 7th International Symposium on Machine and Industrial Design in Mechanical Engineering KOD, Balatonfured, Hungary, May 24-26, pp. 515-518
[7] Coman L, Hamat C, Pittner A and Ianici S 2009 Experimental Research Regarding the Hydraulic Forming of Revolution Shape Parts, 20th International Danube Adria Association for Automation and Manufacturing Symposium DAAAM, Vienna, Austria, November 25-28, pp. 141-142
[8] Dong H, Zhu Z, Zhou W and Chen Z 2012 Dynamic Simulation of Harmonic Gear Drives
Considering Tooth Profile Parameters Optimization, *Journal of Computers* 7(6) 1429-1436

[9] Folega P and Siwiec G 2012 Numerical Analysis of Selected Materials for Flexsplines, *Archives of metallurgy and materials* 57(1) 185-191

[10] Harachová D and Tóth T 2014 Determining Opposite Profile to the Flexible Wheel the Harmonic Gear After Deformation, *GRANT Journal* 3(2) 83-86

[11] Ianici S and Ianici D 2015 Constructive Design and Dynamic Testing of the Double Harmonic Gear Transmission, *Annals of Eftimie Murgu University* 22(1) 231-238

[12] Ianici D and Ianici S 2015 Dynamic Research of the Flexible Wheel of a Double Harmonic Gear Transmission, *Annals of Eftimie Murgu University* 22(2) 223-230

[13] Ianici S and Ianici D 2014 *Numerical Simulation of Stress And Strain State of the Flexible Wheel of the Double Harmonic Gear Transmission*, 8th International Symposium on Machine and Industrial Design in Mechanical Engineering KOD, Balatonfured, Hungary, June 12-15, pp. 135-138

[14] Ianici D and Ianici S 2014 Simulation of Dynamic Behavior of the Flexible Wheel of the Double Harmonic Gear Transmission, *Robotica & Management* 19(1) 17-20

[15] Ianici S 1998 *Contributions to the Synthesis of the Transmission with Deformable Elements*, Polytechnic University of Timisoara, PhD Thesis

[16] Kuzmanovic S, Ianici S and Rackov M 2010 *Analysis of Typical Method of Connection of Electric Motor-Gear Unit in the Frame of Universal Motor Gear Reducer*, International Conference Machine Design, Novi Sad, Serbia, May 18, pp. 141-146

[17] Nedelcu D, Ianici D, Nedeloni M D, Daia D, Pop F M and Avasiloaie R C 2011 *The Aerodynamic Force Calculus for a Plate Immersed in a Uniform Air Stream using SolidWorks Flow Simulation Module*, 4th WSEAS International Conference on Finite Differences – Finite Elements – Finite Volumes – Boundary Elements, Paris, France, April 28-30, pp. 98-103

[18] Routh B, Maiti R and Ray A K 2015 An Investigation on Secondary Force Contacts of Tooth Pairs in Conventional Harmonic Drives with Involute Toothed Gear Set, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science* 230 622-638

[19] Sahoo V and Maiti R 2016 *State of Stress in Strain Wave Gear Flexspline Cup on Insertion of Drive Cam-Experiment And Analysis*, International Conference of Mechanical Engineering ICME, London, UK, April 29 - July 1, pp. 98-103

[20] Ianici D 2012 *Contributions to the Constructive-Functional Improvement of the Double Gear Harmonic Transmission*, Eftimie Murgu University of Resita, PhD Thesis

[21] Ianici S 1998 *Contributions to the Synthesis of the Transmission with Deformable Elements*, Polytechnic University of Timisoara

[22] Ianici D, Nedelcu D, Ianici S and Coman L 2010 *Dynamic Analysis of the Double Harmonic Transmission*, 6th International Symposium about Forming and Design in Mechanical Engineering KOD, Palić, Serbia, September 29-30, pp. 155-158

[23] Ianici S, Ianici D and Potoceanu N 2008 *Design of Double Harmonic Transmission*, 6th International Conference of DAAAM Baltic Industrial Engineering, Tallinn, Estonia, April 24-26, pp. 95-100

[24] Ianici S, Vela D, Ianici D and Miclosina C 2004 *Double Harmonic Transmission used in Industrial Robotic Drives*, 2nd International Conference on Robotics, Timisoara, Romania, October 14-16, pp. 89-90