OPTICAL ABSORPTION IN THE STRONG COUPLING LIMIT OF ELIASHBERG THEORY

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We calculate the optical conductivity of superconductors in the strong-coupling limit. In this anomalous limit the typical energy scale is set by the coupling energy, and other energy scales such as the energy of the bosons mediating the attraction are negligibly small. We find a universal frequency dependence of the optical absorption which is dominated by bound states and differs significantly from the weak coupling results. A comparison with absorption spectra of superconductors with enhanced electron-phonon coupling shows that typical features of the strong-coupling limit are already present at intermediate coupling.

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I. INTRODUCTION

According to the traditional pairing theory of superconductivity a high transition temperature may have two origins. The BCS-formula

\[ k_B T_c \approx \hbar \omega_0 \exp(-1/N_F V) \]

suggests that the transition temperature is large if either the typical energy of the attractive pairing interaction, $\hbar \omega_0$, or the strength of the pairing interaction, $N_F V$, is large. Both possibilities were taken into consideration as the origin of the high transition temperatures observed in cuprate superconductors. Most theories of high-$T_c$ superconductivity invoke the first mechanism, i.e. a large typical energy of the bosons mediating the pairing interaction and a small or intermediate coupling constant. These systems are well described by the weak-coupling theory of BCS. Alternatively, it has been suggested that the high transition temperatures and other anomalies found in the cuprate superconductors are due to an unusually strong coupling of conduction electrons to the bosons mediating the attractive interaction. Such systems are the subject of Eliashberg’s strong-coupling theory of superconductivity. The weak-coupling BCS theory is obtained from Eliashberg’s theory in the limit $k_B T_c/\hbar \omega_0 \to 0$. The opposite limit, i.e. $k_B T_c/\hbar \omega_0 \to \infty$, is called “strong-coupling limit”, and will be discussed in this article.

The strong-coupling limit of the Eliashberg theory has been of recent interest in the context of the ongoing high-$T_c$ debate. Indeed, Eliashberg’s strong-coupling theory is among the most widely discussed candidates for a proper approach to high-$T_c$ superconductivity. This includes d-wave pairing induced by a coupling of conduction electrons to antiferromagnetic spin fluctuations, as well as s-wave pairing due to a phonon mediated attraction. In both cases, the strong coupling of the electrons to bosonic excitations leads to a very short inelastic lifetime, $\hbar/\tau \approx k_B T_c$, and one of the conditions of the weak-coupling theory ($\hbar/\tau \ll k_B T_c$) is violated. A substantial fraction of the bosons have an energy smaller than $k_B T_c$, and one expects typical features of the strong-coupling limit to be observable in high-$T_c$ superconductors. With this perspective, the investigation of the formal strong-coupling limit, $k_B T_c/\hbar \omega_0 \to \infty$, becomes more than just an interesting intellectual exercise. Indeed, we can hope to gain from this limit some new physical insights into superconductivity in the standard strong-coupling regime, $k_B T_c \lesssim \hbar \omega_0$.

Superconducting properties are not universal in this regime and depend on a number of material parameters. On the other hand, the weak-coupling and the strong-coupling limits are universal, and a thorough understanding of both universal limits will be useful for understanding the anomalies observed in the intermediate cases. Moreover, although no strong-coupling superconductor with $k_B T_c > \hbar \omega_0$ has yet been found, one can envisage that some new physical systems will be discovered in the future which fall in the strong-coupling limit. There are indications that important phonons have energies of the order of $2\pi k_B T_c$ in $C_{60}$-superconductors, and that $C_{60}$ is on the borderline to the very strong coupling regime.
In the following we will assume, for convenience, that the attractive interaction is mediated predominantly by phonons. The universality of the strong-coupling limit implies that details of the phonon spectrum or the electron-phonon coupling do not matter, so that we can let the phonon spectrum collapse to an Einstein spectrum at an average phonon frequency \( \Omega \equiv \langle \Omega^2 \rangle^{1/2} \). The formal strong-coupling limit then amounts to letting \( \Omega \) go to zero. Keeping the electron-phonon coupling parameter, \( \lambda \), finite in this limit would lead to a vanishing \( T_c \), and thus to a non-interesting fixed point. The strong-coupling fixed point of interest is obtained by having the coupling parameter \( \lambda \) go to infinity in a way such that the energy \( \lambda^{1/2} \hbar \Omega \neq 0 \) stays fixed. This limit can also be formulated in an alternative way. One lets the phonon spring constants go to zero (ideal softening of the phonon frequencies), and keeps the ionic mass and the McMillan-Hopfield interaction parameter, \( \eta = N_F \langle I^2 \rangle > 1/4 \), constant (constant electron-ion coupling). There is only one characteristic energy left in the strong-coupling limit, namely

\[
\bar{E} = \hbar \sqrt{\frac{N_F \langle I^2 \rangle}{M}} \equiv \lambda^{1/2} \hbar \Omega ,
\]

where \( N_F \) is the density of states at the Fermi energy, and \( M \) the average mass per site. The BCS formula (1) for the transition temperature of weak-coupling superconductors is replaced in the strong-coupling limit by

\[
k_B T_c = 0.1827 \sqrt{\frac{N_F \langle I^2 \rangle}{M}} .
\]

For convenience we will take \( \bar{E} \) as our energy unit. In these units the limiting critical temperature, \( k_B T_c \), is equal to 0.1827 and the energy gap equal to 1.16\( \bar{E} \).

The basic new feature of the strong-coupling limit is a substantially altered excitation spectrum as compared to the BCS weak-coupling spectrum. At zero temperature, the excitation spectrum is discrete and the density of states is an infinite set of Dirac delta functions. The positions and the weights of these peaks have recently been calculated. Physically they correspond to bound states. The lowest of these excitation energies determines the gap of the superconductor in the strong-coupling limit, and one obtains \( 2 \Delta_0 / k_B T_c = 12.7 \). At a finite temperature the excited states have a finite lifetime due to the interaction with thermally activated phonons, and the energy levels get broadened. The positions of the peak maxima remain essentially unchanged.

These anomalies in the spectrum lead to various anomalies in observable properties. A number of physical properties, in particular thermodynamic properties including the upper critical field and the penetration depth, have already been studied for superconductors in the strong-coupling limit. On the other hand, we know of no calculation of any dynamical properties in this limit. A dynamical quantity of particular interest is the infrared conductivity. The real part of the conductivity of strong-coupling superconductors shows, in general, a quite complicated frequency dependence which reflects the rich variety of different absorption channels. An infrared photon can break a Cooper pair or accelerate a thermally excited electron. As a consequence of the electron-phonon coupling, both processes may be accompanied by the absorption or emission of one or several phonons (Holstein effect). It would be useful to distinguish the more or less universal features in the infrared spectrum (if they exist) from those which depend on material parameters such as the phonon spectrum. The strong-coupling limit is interesting in this respect because it is independent of the phonon spectrum, as was mentioned above. Therefore, the only structures which survive in this limit are the universal strong-coupling features.

We calculate in this paper the infrared absorption for a superconductor in the strong-coupling limit, and discuss the anomalies found in this limit, as well as their relevance for superconductors with a strong electron-phonon coupling. The theoretical background of our calculations is presented in Sec. II. Our results are discussed in Sec. III which also contains the comparison of the strong-coupling limit with numerical results for the infrared absorption in superconductors with a strong electron-phonon coupling. Our conclusions are presented in Sec. IV.

II. CONDUCTIVITY IN THE STRONG-COUPLING LIMIT

We calculate the conductivity in the strong-coupling limit by standard strong-coupling theory, as developed by Nambu, \( \xi \) and \( \eta \) and as used, more recently, in the context of high-\( T_c \) superconductors by several authors, \( R,A,F \). This theory gives the conductivity in the frequency range \( \omega \ll \hbar \bar{E} \), which covers the range of interest, \( \omega \approx \Delta_0 / \hbar \). We use in the following the notation of Lee et al. and Rainer and Sauls. For isotropic electron-phonon scattering one can express the conductivity in terms of the \( \xi \)-integrated Green’s functions and the one-particle self-energies of the superconductor in equilibrium. The retarded (superscript \( R \)) and advanced (superscript \( A \)) self-energies are conveniently written in terms of the renormalized excitation energy, \( \bar{e} R,A(\epsilon) \), and the renormalized gap function, \( \bar{\Delta} R,A(\epsilon) \):
\[ \hat{\sigma}^{R,A}(\epsilon) = (\epsilon - \hat{\epsilon}^{R,A}(\epsilon))\hat{\tau}_3 + \hat{\Delta}^{R,A}(\epsilon)(i\hat{\tau}_2). \]  

(4)

The \(\xi\)-integrated Green’s function is then given by

\[ \hat{g}^{R,A}(\epsilon) = -\pi \frac{\hat{\epsilon}^{R,A}(\epsilon)\hat{\tau}_3 - \hat{\Delta}^{R,A}(\epsilon)(i\hat{\tau}_2)}{\sqrt{\Delta^{R,A}(\epsilon)^2 - \hat{\epsilon}^{R,A}(\epsilon)^2}} \equiv -\pi \frac{\epsilon\hat{\tau}_3 - \Delta^{R,A}(\epsilon)(i\hat{\tau}_2)}{\sqrt{\Delta^{R,A}(\epsilon)^2 - \epsilon^2}}, \]  

(5)

where we followed a standard notation of strong-coupling theory, and introduced the gap function \(\Delta(\epsilon)\),

\[ \Delta^{R,A}(\epsilon) = \Delta^{R,A}(\epsilon) \frac{\epsilon}{\hat{\epsilon}^{R,A}(\epsilon)}. \]  

(6)

The conductivity \(\sigma(\omega)\) is, in the long-wavelength limit \((q = 0)\), given by the following integral:

\[ \sigma(\omega) = \frac{e^2 N_F v_F^2}{2 D n \omega} \times \]

\[ \int_{-\infty}^{+\infty} d\epsilon \left\{ \tanh \frac{\epsilon_-}{2k_BT} \frac{1}{c^R(\epsilon,\omega)} \left[ \frac{\epsilon_- - \epsilon_+ + \Delta^R \Delta^R_{-}}{\sqrt{\Delta^R(\epsilon)^2 - (\epsilon_-)^2}} \right] \right. \]

\[ - \tanh \frac{\epsilon_+}{2k_BT} \frac{1}{c^A(\epsilon,\omega)} \left[ \frac{\epsilon_- - \epsilon_+ + \Delta^A \Delta^A_{-}}{\sqrt{\Delta^A(\epsilon)^2 - (\epsilon_-)^2}} \right] + 1 \]

\[ + \left( \tanh \frac{\epsilon_+}{2k_BT} - \tanh \frac{\epsilon_-}{2k_BT} \right) \frac{1}{c^R(\epsilon,\omega)} \left[ \frac{\epsilon_- - \epsilon_+ + \Delta^R \Delta^A_{-}}{\sqrt{\Delta^A(\epsilon)^2 - (\epsilon_-)^2}} \right] \left[ \frac{\epsilon_- - \epsilon_+ + \Delta^A \Delta^R_{-}}{\sqrt{\Delta^R(\epsilon)^2 - (\epsilon_-)^2}} \right] + 1 \} , \]

where we have used the abbreviations

\[ \Delta^{R,A}_\pm = \Delta^{R,A}(\epsilon \pm \hbar \omega/2) , \]

(8)

\[ \epsilon_\pm = \epsilon \pm \hbar \omega/2, \]

(9)

\[ c^{R,A}(\epsilon,\omega) = \sqrt{\Delta^{R,A}(\epsilon_-)^2 - \hat{\epsilon}^{R,A}(\epsilon_-)^2} + \sqrt{\Delta^{R,A}(\epsilon_+)^2 - \hat{\epsilon}^{R,A}(\epsilon_+)^2} , \]

(10)

and

\[ c^A(\epsilon,\omega) = \sqrt{\Delta^A(\epsilon_-)^2 - \hat{\epsilon}^A(\epsilon_-)^2} + \sqrt{\Delta^A(\epsilon_+)^2 - \hat{\epsilon}^A(\epsilon_+)^2} . \]

(11)

\(N_F\) is the density of states in the normal state, \(v_F\) the Fermi velocity, and \(D\) the dimension of the superconductor.

The self-energies and Green’s functions have been calculated in the strong-coupling limit by Combescot\(^\text{[3]}\). We adopt the notation of this reference and introduce, for convenience, a complex angle \(\varphi(\epsilon)\) defined by

\[ \sin \varphi(\epsilon) = \frac{\epsilon}{\Delta^R(\epsilon)}. \]

(12)

We can now write the conductivity \(\sigma(\epsilon)\) in terms of \(\varphi(\epsilon)\). We first note that \(c^{R,A,a}(\epsilon)\) are constants in the strong-coupling limit, and equal to the inverse lifetime,

\[ e^{R,A,a}(\epsilon) = \frac{\hbar}{\tau} = k_BT \lambda = k_BT \frac{N_F}{\Omega^2} , \]

(13)

where the last equality follows from eq. \(\text{[2]}\). The inverse lifetime is linear in \(T\) and diverges in the strong-coupling limit. In this way the strong-coupling limit is completely analogous to the dirty limit for a superconducting alloy. Hence, we follow the usual procedures of the dirty limit theory, and combine the lifetime with the prefactors of the integral in \(\text{[3]}\) to obtain the conductivity in the normal state,

\[ \sigma_N = \frac{2N_F^2 v_F^2 \tau}{\pi}. \]

(14)
The remaining terms in the integral in (7) can be expressed in terms of \( \varphi \) by using the relations
\[
\frac{\epsilon}{\sqrt{\Delta^R(\epsilon)^2 - \epsilon^2}} = \tan \varphi(\epsilon), \quad \frac{\Delta^R(\epsilon)}{\sqrt{\Delta^R(\epsilon)^2 - \epsilon^2}} = \frac{1}{\cos \varphi(\epsilon)},
\]
and
\[
\frac{\epsilon}{\sqrt{\Delta^A(\epsilon)^2 - \epsilon^2}} = \tan \varphi^*(\epsilon), \quad \frac{\Delta^A(\epsilon)}{\sqrt{\Delta^A(\epsilon)^2 - \epsilon^2}} = \frac{1}{\cos \varphi^*(\epsilon)}.
\]
The relations in (16) for the advanced functions follow from (15) by the fundamental symmetry
\[
\Delta^A(\epsilon) = (\Delta^R(\epsilon))^*.
\]
By making use of (13), (15), (16), (14), and the symmetries
\[
\Delta^{R,A}(\epsilon) = (\Delta^{R,A}(-\epsilon))^*.
\]
one finds the following expression for the conductivity:
\[
\frac{\sigma(\omega)}{\sigma_N} = \frac{1}{\hbar \omega} \int_0^\infty d\epsilon \left[ A(\epsilon, \omega) \tanh \frac{\epsilon - \epsilon_0}{2k_BT} - A^*(\epsilon, \omega) \tanh \frac{\epsilon + \epsilon_0}{2k_BT} + B(\epsilon, \omega) \left( \tanh \frac{\epsilon + \epsilon_0}{2k_BT} - \tanh \frac{\epsilon - \epsilon_0}{2k_BT} \right) \right],
\]
where
\[
2A(\epsilon, \omega) = 1 + \tan \varphi(\epsilon_-) \tan \varphi(\epsilon_+) + \frac{1}{\cos \varphi(\epsilon_-) \cos \varphi(\epsilon_+)},
\]
and
\[
2B(\epsilon, \omega) = 1 + \tan \varphi^*(\epsilon_-) \tan \varphi(\epsilon_+) + \frac{1}{\cos \varphi^*(\epsilon_-) \cos \varphi(\epsilon_+)}.\]

Formula (19) gives us the complex conductivity of a superconductor in the strong-coupling limit, and will be used in the following section. Of special interest here is the real part of \( \sigma \), which determines the contribution of the conduction electrons to the optical absorption. By taking the real part of eq. (19) we find:
\[
\Re \left( \frac{\sigma(\omega)}{\sigma_N} \right) = \frac{1}{\hbar \omega} \int_0^\infty d\epsilon \left[ \tanh \frac{\epsilon + \epsilon_0}{2k_BT} - \tanh \frac{\epsilon - \epsilon_0}{2k_BT} \right] \times \left[ 3m \tan \varphi(\epsilon_-) \Im \tan \varphi(\epsilon_+) \right] + \left( 3m \frac{1}{\cos \varphi(\epsilon_-)} \Im \frac{1}{\cos \varphi(\epsilon_+)} \right) \]
\[
= \frac{1}{\hbar \omega} \int_0^\infty d\epsilon \left[ \tanh \frac{\epsilon + \epsilon_0}{2k_BT} - \tanh \frac{\epsilon - \epsilon_0}{2k_BT} \right] \times \left[ 3m \tan \varphi(\epsilon_-) \Im \tan \varphi(\epsilon_+) \right] + \left( 3m \frac{1}{\cos \varphi(\epsilon_-)} \Im \frac{1}{\cos \varphi(\epsilon_+)} \right)\]
\]
Eq. (22) is the basic formula for the following discussions.

III. INFRARED ABSORPTION

We first calculate the optical absorption at zero temperature from the zero temperature limit of eq. (22). It gives us the absorption in terms of the function \( \varphi(\epsilon) \) defined in eq. (13). It was shown recently that \( \varphi(\epsilon) \) is real, positive, and increases monotonically from zero to infinity at positive energies. This means that \( \tan \varphi \) and \( 1/\cos \varphi \) are real at almost all energies except for a set of isolated points where \( \tan \varphi(\epsilon) \) and \( 1/\cos \varphi(\epsilon) \) have poles, and consequently a
expected value' of $2\Delta_0$.

The effect of the destructive coherence factor is more dramatic. The threshold for optical absorption is no longer at the superconductor is continuous. In the strong-coupling limit, on the other hand, the excitation spectrum is discrete, and which vanishes continuously if the threshold is approached from above. The excitation spectrum of the weak coupling coherence factors compensate for the diverging joint density of states and one finds a finite absorption above threshold.

This result has important consequences for the absorption spectrum, as will be shown below. Eq. (22) can be written at $T = 0$ in the form

$$
\Re \sigma(\omega) = \sigma_N \frac{2}{\hbar \omega} \int_0^{\hbar \omega/2} d\epsilon N(\epsilon + \hbar \omega/2)N(\epsilon - \hbar \omega/2)\times
\left[1 + \frac{1}{\sin \varphi(\epsilon + \hbar \omega/2) \sin \varphi(\epsilon - \hbar \omega/2)}\right].
$$

One can now insert eqs. (24), (26), and one obtains

$$
\Re \sigma(\omega) = \sigma_N \frac{\pi^2}{\hbar \omega} \sum_{m,n} [1 + (-1)^{m+n+1}] P_m P_n \delta(E_m + E_n - \hbar \omega).
$$

The physical interpretation of this result is quite clear. A photon of frequency $\omega$ is absorbed by the superconductor, exciting two quasiparticles of energies $E_m$ and $E_n$. The strength of this absorption is determined by the joint density of states,

$$
N(\epsilon + \hbar \omega/2)N(\epsilon - \hbar \omega/2),
$$

and the coherence factor

$$
1 + \frac{1}{\sin \varphi(\epsilon + \hbar \omega/2) \sin \varphi(\epsilon - \hbar \omega/2)}.
$$

Naturally, the absorption spectrum is discrete, due to the discreteness of the quasiparticle density of states. The interesting new features are stringent selection rules for optical transitions. These rules originate from the coherence factor (20) together with the relation (24), and produce the factor $[1 + (-1)^{m+n+1}]$ in (28). Thus, creation of two excitations with quantum numbers $m$ and $n$ of the same parity ($n + m$ even) has a vanishing coherence factor and is optically forbidden. An interesting example of a forbidden process is the excitation threshold at energy $2\Delta_0$ ($m = n = 1$). Actually, the same phenomenon, i.e. a vanishing coherence factor at threshold, arises in the Mattis-Bardeen theory for the optical absorption of weak-coupling superconductors in the dirty limit. Here, the vanishing coherence factors compensate for the diverging joint density of states and one finds a finite absorption above threshold which vanishes continuously if the threshold is approached from above. The excitation spectrum of the weak coupling superconductor is continuous. In the strong-coupling limit, on the other hand, the excitation spectrum is discrete, and the effect of the destructive coherence factor is more dramatic. The threshold for optical absorption is no longer at the ‘expected value’ of $2\Delta_0 = 2.32$. Instead, it corresponds to the creation of an excitation with energy $E_1 = \Delta_0 = 1.16$.
and one with energy $E_2 = 2.6\Delta_0 = 3.04$. As a result, the threshold for absorption is at $3.6\Delta_0$, which is about a factor 2 higher than what one might otherwise have anticipated.

Since the absorption spectrum is a set of delta functions, it is not convenient to represent it directly on a graph. Rather, we display in Fig.1 the integrated absorption, $\int_0^\infty d\omega \text{Re}\sigma(\omega)$. The positions of the steps give the positions of the absorption peaks, and the height of the steps give the weights of the peaks. The steps in this figure correspond to pair breaking into $E_1 + E_2 = 3.6\Delta_0$, $E_1 + E_4 = 5.6\Delta_0$, $E_2 + E_4 = 6.3\Delta_0$, $E_1 + E_6 = 6.9\Delta_0$, $E_2 + E_5 = 7.9\Delta_0$, $E_1 + E_8 = 8.0\Delta_0$, and $E_3 + E_4 = 8.3\Delta_0$.

We now turn to the case of finite temperatures. The equation for the real part of the conductivity in the strong-coupling limit is given in eq. (22). At finite $T$ the function $\varphi(\epsilon)$ has an imaginary part, and the absorption spectrum becomes continuous. This function can be calculated numerically by various techniques, for example by solving a second order nonlinear differential equation. The results of this calculation are shown on Fig.2 for three temperatures: $T/T_c = 0.3$, 0.6 and 0.9. The absorption for $T/T_c = 0.3$ is easily interpreted with the help of the $T = 0$ result. The main absorption peak at an energy of $\hbar \omega \sim 3.5\Delta_0$ corresponds to the $T = 0$ absorption threshold at $3.6\Delta_0$. Its position is slightly shifted to lower frequencies, in agreement with the small shift of the peaks in the density of states. The next peak corresponds to the breaking of a pair into two excitations at $E_1$ and $E_4$. Its position at $T/T_c = 0.3$ is essentially unchanged from the $T = 0$ position. In addition, we find at $6.3\Delta_0$ the peak corresponding to pair breaking into $E_2 + E_3$. The peak corresponding to $E_1 + E_6$ is barely seen as a shoulder on the side of the preceding two peaks. Finally, the last peak in the figure comprises the $T = 0$ peaks at $7.9\Delta_0$, $8.0\Delta_0$ and $8.3\Delta_0$ which are unresolved.

In addition to these structures, which are in correspondence with the peaks at $T = 0$, the absorption shows for $T/T_c = 0.3$ a small threshold at about the standard absorption threshold $2\Delta_0$. This feature indicates that the coherence factors are no longer strictly zero, because the excitation energies acquire a finite width due to their finite lifetime. At $T/T_c = 0.6$ this $2\Delta_0$ threshold has grown, but the dominant feature of the absorption spectrum is still the main peak at approximately $3.5\Delta_0$ whose frequency is almost unchanged. On the other hand, the structures in the spectrum at higher frequencies have been essentially washed out, except for a small bump at $\approx 6$, corresponding to the fusion of the 5.6 and the 6.3 peaks. At $T/T_c = 0.9$, the gap is filled with states, and there is a substantial amount of absorption down to zero frequency due to the presence of thermal excitations. One clear feature survives in the spectrum at elevated temperatures, which is the peak at $\approx 3.5\Delta_0$. This is the remainder of the $T = 0$ threshold.

![Fig. 1. Integrated conductivity in the strong-coupling limit for $T \to 0$. Frequencies are given in units of $\Delta_0/\hbar$, and the conductivity in units of $\sigma_N(T \to 0)$. The steps indicate the location of $\delta$-peaks in the absorption, and the heights of the steps give the weights of the peaks.](image-url)
We see that the main features of the absorption at $T = 0$ are quite robust, and survive thermal smearing. Thus, it is interesting to check if these features of the strong-coupling limit are still present at finite values of the coupling constant, and even down to realistic couplings. In order to explore this possibility we have solved Eliashberg equations on the real frequency axis for a fixed shape of the spectral function, and several values of the coupling constant $\lambda$. For convenience we have taken a narrow gaussian spectrum (Einstein-type spectrum) of the form:

$$\alpha^2 F(\Omega) = \text{const} \times \exp\left(-\frac{(\Omega - \Omega_0)^2}{\Gamma^2}\right),$$

where $\Omega_0$ is a characteristic phonon frequency, and $\Gamma = \Omega_0/4$. Since, as we have seen, the strong-coupling limit automatically implies the dirty limit, we have also used the dirty limit for our calculations at finite $\lambda$. The reduced temperature is low in all the calculations ($T/T_c = 0.22$ for $\lambda = 1$ and much lower for the higher $\lambda$’s). Figures 3 and 4 show the results for $\Re \sigma(\omega)$ for $\lambda = 1, 2, 3, 4, 10$ and 40, the ratio $\Omega_0/\Delta_0$ being respectively 4.26, 1.83, 1.23, 0.95, 0.46 and 0.183. We note that our program recovers the standard Mattis-Bardeen behavior for $\lambda \to 0$. 

FIG. 2. $\Re \sigma(\omega)$ in the strong-coupling limit as a function of the frequency. Results are shown for $T/T_c = .3$ (solid line), .6 (short dashed line), and .9 (long dashed line). The conductivities are given in units of $\sigma_N(T)$, and the frequencies in units of $\Delta_0/h$. 

$$\alpha^2 F(\Omega)$$
FIG. 3. $\Re \sigma(\omega)$ in the dirty limit for Einstein-type spectra as a function of the frequency. Numerical results at very low temperatures are shown for $\lambda = 40$ ($T/T_c = 0.022$; solid line), and $\lambda = 10$ ($T/T_c = 0.045$; dashed line). The conductivities are given in units of $\sigma_N(T)$, and the frequencies in units of $\Delta_0/\hbar$.

As can be seen in Fig. 3, our numerical calculations for large $\lambda$ ($\lambda = 40$) show a very strong similarity with the strong-coupling limit ($\lambda = \infty$). The numerical results are found to be practically indistinguishable from the strong-coupling limit for $\lambda \geq 100$. Characteristic features of the strong-coupling limit are still visible for smaller coupling...
constants (see Figs. 3 and 4). However, they have smaller amplitudes and are shifted upward in frequency. It is interesting to compare the location of these peaks with the Holstein thresholds located at \( \hbar \omega = 2\Delta_0 + \hbar \Omega_0 \) (see arrows in Fig.4). These are the thresholds for breaking a Cooper pair and, at the same time, emitting a phonon. We see that the positions of the absorption peaks have at intermediate coupling constants (\( \lambda \gtrsim 2 \)) no relation to the Holstein threshold.

IV. CONCLUSION

Our results for the infrared absorption in the limit \( k_B T_c / \hbar \omega_0 \to \infty \) (strong-coupling limit) revealed some surprising features. The absorption threshold of traditional weak and strong-coupling superconductors is at \( 2\Delta_0 \), which is the energy needed to break a pair into two electrons and to put them into the lowest excited quasiparticle state at energy \( \Delta_0 \). In contrast to this, we found that the threshold in the strong-coupling limit is given by a different process. One electron of a broken pair is excited into the lowest energy state at \( \Delta_0 = 6.35 k_B T_c \) and the other one into the next higher state with an energy \( 2.6 \Delta_0 \). Accordingly, the threshold for infrared absorption is found at \( 3.6 \Delta_0 \) instead of \( 2\Delta_0 \) (\( = 12.7 k_B T_c \)). The disappearance of the absorption at \( 2\Delta_0 \) is a consequence of the coherence factors which lead to a vanishing transition amplitude for the \( 2\Delta_0 \) transition. Although the absorption spectrum is smeared out at finite temperatures, a strong absorption peak at \( \hbar \omega \approx 3.6 \Delta_0 \) remains visible in the infrared spectrum for temperatures almost up to the critical temperature. As in all pairing theories of superconductivity, one finds in the strong-coupling limit no absorption gap at a finite temperature. Absorption in the gap is brought about by thermally activated excitations. Absorption near zero frequency becomes important only near \( T_c \). Otherwise, the dominant low-frequency feature at finite temperatures is the appearance of a small absorption at the standard threshold of \( 2\Delta_0 \). This feature grows with increasing temperature and becomes comparable to the main absorption peak near \( T_c \).

Our results for the infrared absorption in the strong-coupling limit are quite different from standard absorption spectra for superconductors with a weak or intermediate coupling. In order to investigate precursor effects of the strong-coupling limit at finite coupling, we have calculated the absorption spectra for dirty strong-coupling superconductors with an Einstein-type spectrum and a series of different coupling strengths. We found that the characteristic structure at \( \hbar \omega \sim 3.6 \Delta_0 \), which corresponds to the main absorption peak in the strong-coupling limit, is clearly visible for \( \lambda \)'s down to \( \sim 3 \). It shows up as a 20% bump over the normal state value. However, the position of this peak shifts upwards progressively when the coupling strength is lowered and, for \( \lambda = 3 \), it is located at about 7.5 times the gap \( \Delta_0 \), which exceeds the \( 3.6 \Delta_0 \) found in the strong-coupling limit.

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