Simplicial resolutions and spaces of algebraic maps between real projective spaces

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Abstract

We show that the space $\tilde{A}_d(m, n)$ consisting of all real projective classes of $(n+1)$-tuples of real coefficients homogeneous polynomials of degree $d$ in $(m+1)$ variables, without common real roots except zero, has the same homology as the space $\text{Map}(\mathbb{R}P^m, \mathbb{R}P^n)$ of continuous maps from the $m$-dimensional real projective space $\mathbb{R}P^m$ into the $n$ real dimensional projective space $\mathbb{R}P^n$ up to dimension $(n-m)(d+1)-1$. This considerably improves the main result of [1].

Keywords: Simplicial resolution, truncated simplicial resolution, algebraic map, homotopy equivalence, Vassiliev spectral sequence.

2000 MSC: Primary 55P10, 55R80; Secondly 55P35, 55T99

1. Introduction.

Let $M$ and $N$ be manifolds with some additional structure, e.g holomorphic, symplectic, real algebraic etc. The relation between the topology of the space of continuous maps preserving this structure and that of the space of all continuous maps has long been an object of study in several areas of topology and geometry (e.g. [2], [4], [8], [10], [11], [14], [17]). In [1] we considered the case where the structure is that of a real algebraic variety. The continuous maps that preserve this structure are the rational maps (called also regular

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or real algebraic maps). Using a method analogous to the one invented by Mostovoy \cite{14} to deal with the complex case, we obtained a closely related result, in which the space of rational maps is replaced by the space of tuples of homogeneous polynomials without non-trivial roots. (In fact we believe this space to be equivalent to the space of all rational maps but have not been able to prove this completely). Mostovoy’s original article contained some mistakes that were corrected in \cite{15}. Besides correcting of mistakes, \cite{15} improves the results of \cite{14} by means of a new variant of the main spectral sequence. This new spectral sequence is obtained from a “truncated simplicial resolution” of a discriminant, which replaces the “degenerate simplicial resolution” used in the original argument. It is easy to see that the new spectral sequence can also be applied to our case and makes it possible to obtain better results. The main purpose of this article is to carry out this idea. However, we are able to obtain better results than could be derived by a straight-forward replacement of the degenerate resolutions used in \cite{1} by truncated ones. This is achieved by combining the information obtained by using truncated resolutions (Theorem 4.10 below) with information obtained by using a non-degenerate resolution in the manner of \cite{19}. We conjecture that our main result (Theorem 1.5 below) is optimal.

Notation. Let $m$ and $n$ be positive integers such that $1 \leq m < n$. We choose $e_k = [1 : 0 : \cdots : 0] \in \mathbb{RP}^k$ as the base point of $\mathbb{RP}^k$ ($k = m, n$), and we denote by $\text{Map}^* (\mathbb{RP}^m, \mathbb{RP}^n)$ the space consisting of all based maps $f : (\mathbb{RP}^m, e_m) \to (\mathbb{RP}^n, e_n)$. We also denote by $\text{Map}^*_\epsilon (\mathbb{RP}^m, \mathbb{RP}^n)$ the corresponding path component of $\text{Map}^* (\mathbb{RP}^m, \mathbb{RP}^n)$ for each $\epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_0 (\text{Map}^* (\mathbb{RP}^m, \mathbb{RP}^n))$ (\cite{8}). Similarly, let $\text{Map} (\mathbb{RP}^m, \mathbb{RP}^n)$ denote the space of all free maps $f : \mathbb{RP}^m \to \mathbb{RP}^n$ and $\text{Map}_\epsilon (\mathbb{RP}^m, \mathbb{RP}^n)$ the corresponding path component of $\text{Map} (\mathbb{RP}^m, \mathbb{RP}^n)$.

We shall use the symbols $z_i$ to denote variables of polynomials. A map $f : \mathbb{RP}^m \to \mathbb{RP}^n$ is called an algebraic map of the degree $d$ if it can be represented as a rational map of the form $f = [f_0 : \cdots : f_n]$ such that $f_0, \cdots, f_n \in \mathbb{R}[z_0, \cdots, z_m]$ are homogeneous polynomials of the same degree $d$ with no common real roots except $0_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$. We denote by $\text{Alg}_d (\mathbb{RP}^m, \mathbb{RP}^n)$ (resp. $\text{Alg}_d^* (\mathbb{RP}^m, \mathbb{RP}^n)$) the space consisting of all (resp. based) algebraic maps $f : \mathbb{RP}^m \to \mathbb{RP}^n$ of degree $d$. It is easy to see that there are inclusions $\text{Alg}_d (\mathbb{RP}^m, \mathbb{RP}^n) \subset \text{Map}_{[d]} (\mathbb{RP}^m, \mathbb{RP}^n)$ and $\text{Alg}_d^* (\mathbb{RP}^m, \mathbb{RP}^n) \subset \text{Map}_{[d]}^* (\mathbb{RP}^m, \mathbb{RP}^n)$, where $[d] \in \mathbb{Z}/2 = \{0, 1\}$ denotes the integer $d$ mod 2.
For \( m \geq 2 \) and given an algebraic map \( g \in \text{Alg}_d^*(\mathbb{RP}^m, \mathbb{RP}^n) \), we denote by \( \text{Alg}_d(m, n; g) \) and \( F(m, n; g) \) the subspaces given by

\[
\begin{align*}
\text{Alg}_d(m, n; g) &= \{ f \in \text{Alg}_d^*(\mathbb{RP}^m, \mathbb{RP}^n) : f|\mathbb{RP}^m = g \}, \\
F(m, n; g) &= \{ f \in \text{Map}_{[d]}^*(\mathbb{RP}^m, \mathbb{RP}^n) : f|\mathbb{RP}^m = g \}.
\end{align*}
\]

Note that there is a homotopy equivalence \( F(m, n; g) \simeq \Omega^m \mathbb{RP}^n \simeq \Omega^m S^n \) \((\text{[16]}))\).

**Spaces of polynomials.** Now we turn to spaces of collections of polynomials which represent algebraic maps. These will all be subsets of real affine or real projective spaces.

Let \( \mathcal{H}_{d,m} \subset \mathbb{R}[z_0, \ldots, z_m] \) denote the subspace consisting of all homogeneous polynomials of degree \( d \). Let \( A_d(m, n)(\mathbb{R}) \subset \mathcal{H}_{d,m}^{n+1} \) denote the space of all \((n + 1)\)-tuples \((f_0, \ldots, f_n) \in \mathbb{R}[z_0, \ldots, z_m]^{n+1}\) of homogeneous polynomials of degree \( d \) without non-trivial common real roots (but possibly with non-trivial common non-real roots). These are precisely the collections that represent algebraic maps. Since the algebraic map is invariant under multiplication of all polynomials by a non-zero scalar, it is convenient to define the projectivisation \( \tilde{A}_d(m, n) = A_d(m, n)(\mathbb{R})/\mathbb{R}^* \), which is a subset of the real projective space \( (\mathcal{H}_{d,m}^{n+1} \setminus \{0\})/\mathbb{R}^* \). We have a projection \( \Gamma_d : \tilde{A}_d(m, n) \to \text{Alg}_d^*(\mathbb{RP}^m, \mathbb{RP}^n) \).

To describe base-point preserving maps we need to use the subspace \( A_d(m, n) \subset A_d(m, n)(\mathbb{R}) \), which consists of \((n + 1)\)-tuples \((f_0, \ldots, f_n) \in A_d(m, n)(\mathbb{R}) \) such that the coefficient at \( z_0^d \) is 1 in \( f_0 \) and 0 in the remaining \( f_k \). Note that every algebraic map which preserves the base-point has a representation by such a collection of polynomials, and we get a natural projection map \( \Psi_d : A_d(m, n) \to \text{Alg}_d^*(\mathbb{RP}^m, \mathbb{RP}^n) \).

We define polynomial representations of restricted maps. From now on, let \( m \geq 2 \) and \( g \in \text{Alg}_d^*(\mathbb{RP}^{m-1}, \mathbb{RP}^n) \) be a based algebraic map with some fixed polynomial representation \( g = [g_0 : \cdots : g_n] \) such that \((g_0, \ldots, g_n) \in \tilde{A}_d(m - 1, n)\). Set \( B_k = \{g_k + z_n h : h \in \mathcal{H}_{d-1,m}\} \) \((k = 0, 1, \ldots, n)\) and let

\[
A^*_d := B_0 \times B_1 \times \cdots \times B_n \subset \mathcal{H}_{d,m}^{n+1}.
\]

This space consists of collections of polynomials which restrict to \((g_0, \ldots, g_n)\) when \( z_m = 0 \). Define \( A_d(m, n; g) \subset A^*_d \) to be the subspace of all collections
with no non-trivial common real root:

\[
A_d(m, n; g) = \left\{ (f_0, \cdots, f_n) \in A_d^* \mid f_0, \cdots, f_n \text{ have no common real root}\right. \\
\text{except } 0_{m+1} \in \mathbb{R}^{m+1}\right\}.
\]

Clearly, such a collection determines a map in \( \text{Alg}_d(m, n; g) \) and we again obtain a projection \( \Psi_d : A_d(m, n; g) \to \text{Alg}_d(m, n; g) \). Let

\[
\begin{align*}
  j_d, i_d : &\text{Alg}_d(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{\sim} \text{Map}_{[d]}(\mathbb{R}^m, \mathbb{R}^n) \\
  \quad \quad 
  &\text{Alg}_d^*(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{\sim} \text{Map}_{[d]}^*(\mathbb{R}^m, \mathbb{R}^n) \\
  \quad \quad 
  i'_d : &\text{Alg}_d(m, n; g) \xrightarrow{\sim} F(m, n; g) \simeq \Omega^m \mathbb{R}^n \simeq \Omega^m S^n
\end{align*}
\]

denote the inclusions and let

\[
\begin{align*}
  j_d &\circ \Gamma_d : \tilde{A}_d(m, n) \to \text{Map}_{[d]}^*(\mathbb{R}^m, \mathbb{R}^n) \\
  i_d &\circ \Psi_d : A_d(m, n) \to \text{Map}_{[d]}^*(\mathbb{R}^m, \mathbb{R}^n) \\
  i'_d &\circ \Psi'_d : A_d(m, n; g) \to F(m, n; g) \simeq \Omega^m S^n
\end{align*}
\]

be the natural maps.

The contents of this section can be summarized in the following diagram, for which we assume \( g \) is a based algebraic map of degree \( d \).

The main results. Let \( \lfloor x \rfloor \) denote the integer part of a real number \( x \), and define the integers \( D_*(d; m, n) \) and \( D(d; m, n) \) by

\[
D_*(d; m, n) = (n-m)(\lfloor \frac{d+1}{2} \rfloor+1)-1, \quad D(d; m, n) = (n-m)(d+1)-1.
\]

Now, we recall the following 2 results.
Theorem 1.1 ([10], [13], [20]). If $n \geq 2$ is an integer and $m = 1$, then the natural map $i_d : A_d(1, n) \rightarrow \text{Map}_{[d]}^*(\mathbb{R}P^1, \mathbb{R}P^n) \simeq \Omega^n S^n$ is a homotopy equivalence up to dimension $D(d; 1, n) = (n - 1)(d + 1) - 1$. □

Theorem 1.2 ([1]). If $2 \leq m < n$ are integers and $g \in \text{Alg}_{d}^*(\mathbb{R}P^{m-1}, \mathbb{R}P^n)$ is a fixed based algebraic map of degree $d$, the natural map $i_d' : A_d(m, n; g) \rightarrow F(m, n; g) \simeq \Omega^n S^n$ is a homotopy equivalence through dimension $D_s(d; m, n)$ if $m + 2 \leq n$ and a homology equivalence through dimension $D_s(d; m, n)$ if $m + 1 = n$. □

Theorem 1.3 ([1]). If $2 \leq m < n$ are integers, the natural maps

\[
\begin{cases}
  j_d : \tilde{A}_d(m, n) \rightarrow \text{Map}_{[d]}(\mathbb{R}P^m, \mathbb{R}P^n) \\
i_d : A_d(m, n) \rightarrow \text{Map}_{[d]}(\mathbb{R}P^m, \mathbb{R}P^n)
\end{cases}
\]

are homotopy equivalences through dimension $D_s(d; m, n)$ if $m + 2 \leq n$ and homology equivalences through dimension $D_s(d; m, n)$ if $m + 1 = n$. □

Remark 1.4. A map $f : X \rightarrow Y$ is called a homotopy (resp. a homology) equivalence through dimension $D$ if $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$. Similarly, it is called a homotopy (resp. a homology) equivalence up to dimension $D$ if $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$.

The main purpose of this paper is to improve the above results by replacing $D_s(d; m, n)$ by $D(d; m, n)$. More precisely, we will prove the following.

Theorem 1.5. If $2 \leq m < n$ are integers and $g \in \text{Alg}^*_{d}(\mathbb{R}P^{m-1}, \mathbb{R}P^n)$ is a fixed based algebraic map of degree $d$, the natural map $i_d' : A_d(m, n; g) \rightarrow F(m, n; g) \simeq \Omega^n S^n$ is a homotopy equivalence up to dimension $D(d; m, n)$ if $m + 2 \leq n$ and a homology equivalence up to dimension $D(d; m, n)$ if $m + 1 = n$.

Using the same method as in [1] (cf. [7]), we also obtain the following:

Corollary 1.6. If $2 \leq m < n$ are integers, the natural maps

\[
\begin{cases}
  j_d : \tilde{A}_d(m, n) \rightarrow \text{Map}_{[d]}^*(\mathbb{R}P^m, \mathbb{R}P^n) \\
i_d : A_d(m, n) \rightarrow \text{Map}_{[d]}^*(\mathbb{R}P^m, \mathbb{R}P^n)
\end{cases}
\]

are homotopy equivalences up to dimension $D(d; m, n)$ if $m + 2 \leq n$ and homology equivalences up to dimension $D(d; m, n)$ if $m + 1 = n$. □
Remark 1.7. With the help of Theorem 1.5, it is possible to generalize some results given in [11] concerning the space of algebraic maps from $\mathbb{R}P^m$ to $\mathbb{C}P^n$. We do this in a separate paper [12].

This paper is organized as follows. In section 2, we discuss various kinds of simplicial resolutions and describe a key new idea of Mostovoy [14]. In section 3 we study the spectral sequences induced from the non-degenerate simplicial resolutions of our spaces of discriminants. In section 4, we introduce the spectral sequence induced from the truncated simplicial resolution of our spaces of discriminants and prove the key Theorem (Theorem 4.10). Finally, in section 5, we prove the main result (Theorem 1.5) by using the spectral sequences induced from the non-degenerate simplicial resolutions of stabilized discriminants.

2. Simplicial resolutions.

In this section, we shall recall the definition of non-singular simplicial resolutions and their truncated simplicial resolutions.

Definition 2.1. (i) For a finite set $x = \{x_1, \ldots, x_l\} \subset \mathbb{R}^N$, let $\sigma(x)$ denote the convex hull spanned by $x$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^n$ be an embedding.

Let $\mathcal{X}^\Delta$ and $h^\Delta : \mathcal{X}^\Delta \to Y$ denote the space and the map defined by

$$\mathcal{X}^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \ h^\Delta(y, u) = y.$$

The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called a simplicial resolution of $(h, i)$. In particular, $(\mathcal{X}^\Delta, h^\Delta)$ is called a non-degenerate simplicial resolution if for each $y \in Y$ any $k$ points of $i(h^{-1}(y))$ span $(k - 1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $r \geq 0$, let $\mathcal{X}_r^\Delta \subset \mathcal{X}^\Delta$ be the subspace given by

$$\mathcal{X}_r^\Delta = \{(y, \omega) \in \mathcal{X}^\Delta : \omega \in \sigma(v), v = \{v_1, \ldots, v_l\} \subset i(h^{-1}(y)), l \leq r\}.$$

We make identification $X = \mathcal{X}_1^\Delta$ by identifying $x \in X$ with $(h(x), i(x)) \in \mathcal{X}^\Delta_1$, and we note that there is an increasing filtration

$$\emptyset = \mathcal{X}_0^\Delta \subset X = \mathcal{X}_1^\Delta \subset \mathcal{X}_2^\Delta \subset \cdots \subset \mathcal{X}_r^\Delta \subset \mathcal{X}_{r+1}^\Delta \subset \cdots \subset \bigcup_{r=0}^\infty \mathcal{X}_r^\Delta = \mathcal{X}^\Delta.$$
Lemma 2.2 ([14], [15], [19]). Let $h : X \rightarrow Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \rightarrow \mathbb{R}^N$ be an embedding.

(i) If $X$ and $Y$ are semi-algebraic spaces and the two maps $h$, $i$ are semi-algebraic maps, then $h^{\Delta} : \mathcal{X}^{\Delta} \xrightarrow{\sim} Y$ is a homotopy equivalence.

(ii) There is an embedding $j : X \rightarrow \mathbb{R}^M$ such that the associated simplicial resolution $(\tilde{X}^{\Delta}, \tilde{h}^{\Delta})$ of $(h, j)$ is non-degenerate, and the space $\tilde{X}^{\Delta}$ is uniquely determined up to homeomorphism. Moreover, there is a filtration preserving homotopy equivalence $q^{\Delta} : \tilde{X}^{\Delta} \xrightarrow{\sim} \mathcal{X}^{\Delta}$ such that $q^{\Delta}|X = id_X$. \hfill \Box

Remark 2.3. Even for a surjective map $h : X \rightarrow Y$ which is not finite to one, it is still possible to construct an associated non-degenerate simplicial resolution. In fact, a non-degenerate simplicial resolution may be constructed by choosing a sequence of embeddings $\{\tilde{i}_r : X \rightarrow \mathbb{R}^{N_r}\}_{r \geq 1}$ satisfying the following two conditions for each $r \geq 1$ (cf. [19]).

(2.2)

(i) For any $y \in Y$, any $t$ points of the set $\tilde{i}_k(h^{-1}(y))$ span $(t-1)$-dimensional affine subspace of $\mathbb{R}^{N_r}$ if $t \leq 2r$.

(ii) $N_r \leq N_{r+1}$ and if we identify $\mathbb{R}^{N_r}$ with a subspace of $\mathbb{R}^{N_{r+1}}$, then $\tilde{i}_{r+1} = \hat{i} \circ \tilde{i}_r$, where $\hat{i} : \mathbb{R}^{N_r} \hookrightarrow \mathbb{R}^{N_{r+1}}$ denotes the inclusion.

Let

$\mathcal{X}_r^{\Delta} = \{(y, \omega) \in Y \times \mathbb{R}^{N_r} : \omega \in \sigma(u), u = \{u_1, \ldots, u_l\} \subset \tilde{i}_k(h^{-1}(y)), l \leq r\}$.

Then by identifying naturally $\mathcal{X}_r^{\Delta}$ with a subspace of $\mathcal{X}_{r+1}^{\Delta}$, we now define the non-degenerate simplicial resolution $\mathcal{X}^{\Delta}$ of $h$ as the union $\mathcal{X}^{\Delta} = \bigcup_{r \geq 1} \mathcal{X}_r^{\Delta}$.

Non-degenerate simplicial resolutions have a long been used in algebraic geometry, and play the central role in the work of Vassiliev [19].

In many practical cases the embedding used to construct a simplicial resolution is given by an explicit map which carries geometric information about the corresponding filtration. Typically such an embedding gives rise to a simplicial resolution that is non-degenerate only in low dimensions. In some situations, such a degenerate resolution may provide more information about the homology of the resolved space than the non-degenerate one. This is why a degenerate resolution (defined by a Veronese-like embedding defined in the next section) was used in [14] and [1]. However, in [15] a modification of the non-degenerate resolution, called, the truncated (after a certain term) simplicial resolution was used to obtain results that are (in most dimensions) better than the one derived by means of the degenerate resolution.
Definition 2.4. Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, and $j : X \to \mathbb{R}^N$ be a semi-algebraic embedding. Consider the associated non-degenerate simplicial resolution $(\Delta^X, h^\Delta : \Delta^X \to Y)$.

Let $k$ be a fixed positive integer and let $h^\Delta_k : \Delta^X \to Y$ be the map defined by the restriction $h^\Delta_k := h^\Delta|\Delta^k_k$. The fibres of the map $h^\Delta_k$ are $(k - 1)$-skeleta of the fibres of $h^\Delta$ and, in general, fail to be simplices over the subspace $Y_k = \{y \in Y : h^{-1}(y) \text{ consists of more than } k \text{ points}\}$.

Let $Y(k)$ denote the closure of the subspace $Y_k$. We modify the subspace $\Delta^X_k$ so as to make the all the fibres of $h^\Delta_k$ contractible by adding to each fibre of $Y(k)$ a cone whose base is this fibre. We denote by $\Delta^X(k)$ this resulting space and by $h^\Delta_k : \Delta^X(k) \to Y$ the natural extension of $h^\Delta_k$.

Following Mostovoy \[13\] we call the map $h^\Delta_k : \Delta^X(k) \to Y$ the truncated (after the $k$-th term) simplicial resolution of $Y$. Note that there is a natural filtration

$$0 = \Delta^X_0 \subset \Delta^X_1 \subset \Delta^X_2 \subset \cdots \subset \Delta^X_{k-1} \subset \Delta^X_k \subset \Delta^X_{k+1} = \Delta^X_{k+2} = \cdots = \Delta^X(k),$$

where $\Delta^X_r = \Delta^X_r$ if $r \leq k$ and $\Delta^X_r = \Delta^X(k)$ if $r > k$.

Remark 2.5. Let $\varphi : A \to B$ be a simplicial map between simplicial complexes and let $C_f\varphi : C_fA \to B$ be its fibrewise cone construction. Because $C_f\varphi$ can be constructed in the category of simplicial complexes, it is a quasi-fibration and so it is a homotopy equivalence \[9\]. It is also know that for a semi-algebraic map $f : X \to Y$ between semi-algebraic spaces, there are semi-algebraic triangulations on $X$ and $Y$ such that there is a semi-algebraic trivialization of the map $f$ \[3\, Theorems 9.2.1 and 9.3.2\].

Then using these results, it is not difficult to prove the following.

Lemma 2.6 \[13\]. Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, let $j : X \to \mathbb{R}^n$ be a semi-algebraic embedding with the associated simplicial resolution $(\Delta^X, h^\Delta : \Delta^X \to Y)$, and let $h^\Delta_k : \Delta^X(k) \to Y$ denote the corresponding truncated (after $k$-th term) simplicial resolution of $Y$. Then $h^\Delta_k$ is a quasi-fibration with contractible fibres and hence a homotopy equivalence. \qed
The truncated simplicial resolution has two properties that play a key role in the new, improved argument. First of all, as we saw above, the natural filtration stabilizes above the truncation degree. The second key property is the following:

**Lemma 2.7** ([15], Lemma 2.1). Under the same assumptions as Lemma 2.6

\[ \dim(X_{k+1}^\Delta \setminus X_k^\Delta) = \dim(X_k^\Delta \setminus X_{k-1}^\Delta) + 1. \]

### 3. The spectral sequences.

Because the following notations were already given in [1, section 3], we only explain the important ones.

**Definition 3.1.** Fix a based algebraic map \( g \in \text{Alg}_d^*(\mathbb{R}P^{m-1}, \mathbb{R}P^n) \) of degree \( d \) together with a representation \( (g_0, \ldots, g_n) \in A_d(m-1, n) \). Note that \( A_d(m, n; g) \) is an open subspace of the affine space \( A'_m^* \) as defined in (1).

(i) Let \( N^*_d = \dim A^*_d = (n+1)(m+d-1) \), and define \( \Sigma^*_d \subset A^*_d \) as the discriminant of \( A_d(m, n; g) \) in \( A^*_d \) by

\[ \Sigma^*_d = A^*_d \setminus A_d(m, n; g). \]

In other words, \( \Sigma^*_d \) consists of the \((n+1)\)-tuples of polynomials in \( A^*_d \) which have at least one nontrivial common real root.

(ii) Let \( Z^*_d \subset \Sigma^*_d \times \mathbb{R}^m \) denote the tautological normalization of \( \Sigma^*_d \) consisting of all pairs \((f, x) = ((f_0, \ldots, f_n), (x_0, \ldots, x_{m-1})) \in \Sigma^*_d \times \mathbb{R}^m \) such that the polynomials \( f_0, \ldots, f_n \) have a non-trivial common real root \((x, 1) = (x_0, \ldots, x_{m-1}, 1) \). Projection on the first factor gives a surjective map \( \pi'_d : Z^*_d \to \Sigma^*_d \).

Our goal in this section is to construct, by means of the non-degenerate simplicial resolution of the discriminant, a spectral sequence converging to the homology of the space \( A_d(m, n; g) \).

**Definition 3.2.** Let \( (\mathcal{X}^\Delta_d, \pi^\Delta_d : \mathcal{X}^\Delta_d \to \Sigma^*_d) \) denote the non-degenerate simplicial resolution of the surjective map \( \pi'_d : Z^*_d \to \Sigma_d \) with the natural increasing filtration

\[ \mathcal{X}^\Delta_0 = \emptyset \subset \mathcal{X}^\Delta_1 \subset \mathcal{X}^\Delta_2 \subset \cdots \subset \mathcal{X}^\Delta = \bigcup_{k=0}^{\infty} \mathcal{X}^\Delta_k. \]
By Lemma 2.2 the map \( \pi^\Delta_d : \mathcal{X}^{\Delta,d} \to \Sigma^*_d \) is a homotopy equivalence, and it extends to a homotopy equivalence \( \pi^\Delta_+ : \mathcal{X}^{\Delta,d}_+ \to \Sigma^*_d \), where \( X_+ \) denotes the one-point compactification of a locally compact space \( X \).

Since \( \mathcal{X}^{\Delta,d}_+ / \mathcal{X}^{\Delta,d}_{r-1} \simeq (\mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1})_+ \), we have a spectral sequence

\[
\{ E_t^{r,s}(d), d_t : E_t^{r,s}(d) \to E_t^{r+t,s+1-t}(d) \} \Rightarrow H_c^{r+s}(\Sigma^*_d, \mathbb{Z}),
\]

where \( E_t^{r,s}(d) = H_c^{r+s}(\mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1}, \mathbb{Z}) \) and \( H^k_c(X, \mathbb{Z}) \) denotes the cohomology group with compact supports given by \( H^k_c(X, \mathbb{Z}) := H^k_c(X_+, \mathbb{Z}) \).

By the Alexander duality there is a natural isomorphism

\[
(3.1) \quad H_k(A_d(m, n; g), \mathbb{Z}) \cong H_{c}^{N_d^*(k-1)}(\Sigma^*_d, \mathbb{Z}) \quad \text{for } 1 \leq k \leq N_d^* - 2.
\]

Using (3.1) and reindexing we obtain a spectral sequence

\[
(3.2) \quad \{ \tilde{E}^{t}_{r,s}(d), d^{t} : \tilde{E}^{t}_{r,s}(d) \to \tilde{E}^{t}_{r+s,t-1}(d) \} \Rightarrow H_{s-r}(A_d(m, n; g), \mathbb{Z})
\]

if \( s - r \leq N^*_d - 2 \), where \( \tilde{E}^{1}_{r,s}(d) = H_c^{N_d^*+r-s-1}(\mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1}, \mathbb{Z}) \).

For a connected space \( X \), let \( F(X, r) \) denote the configuration space of distinct \( r \) points in \( X \). The symmetric group \( S_r \) of \( r \) letters acts on \( F(X, r) \) freely by permuting coordinates, and let \( C_r(X) \) be the configuration space of unordered \( r \)-distinct points in \( X \) given by \( C_r(X) = F(X, r) / S_r \).

**Lemma 3.3.** If \( 1 \leq r \leq d + 1 \) and \( m \geq 2 \), \( \mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1} \) is homeomorphic to the total space of a real open disk bundle \( \xi_{d,r} \) over \( C_r(\mathbb{R}^m) \) with rank \( l^*_d,r := N^*_d - nr - 1 \).

**Proof.** The argument is exactly analogous to the one in the proof of [13, Prop. 3.1]. Namely, an element of \( \mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1} \) is represented by \( ((f_0, \ldots, f_n), u) \), where the \( f_i \) are polynomials in \( \Sigma^*_d \) and \( u \) is an element of the span of the images of distinct \( r \) points \( x_1, \ldots, x_n \) at which these polynomials vanish, under a suitable embedding. There is a well defined map \( \pi : \mathcal{X}^{\Delta,d}_r \setminus \mathcal{X}^{\Delta,d}_{r-1} \to C_r(\mathbb{R}^m) \) given by \( ((f_0, \ldots, f_n), u) \mapsto \{x_1, \ldots, x_r\} \). (The points \( x_i \) are uniquely determined by \( u \) by the construction of the non-degenerate simplicial resolution.) Moreover, the condition that the polynomials vanish at \( r \) points gives \( r \) independent linear equations for the coefficients of the polynomials, as long as \( r \leq d + 1 \). Hence, \( \pi \) gives the open disk bundle over \( C_r(\mathbb{R}^m) \) with rank \( l^*_d,r := N^*_d - nr - 1 \). \( \square \)
Remark 3.4. If $m = 1$, the assertion of Lemma 3.3 holds only for $1 \leq r \leq d$. In fact, if $m = 1$ and $r = d + 1$, because an equation in one variable of degree $d$ can have at most $d$ real roots, the condition that the polynomials vanish at $r$ points does not give $r$ independent linear equations for the coefficients of the polynomials.

Lemma 3.5. If $1 \leq r \leq d + 1$, there is a natural isomorphism

$$E_{1,r,s}(d) \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes(n-m)}).$$

Here the meaning of $(\pm \mathbb{Z})^{\otimes(n-m)}$ is the same as in [19].

Proof. Suppose that $1 \leq r \leq d + 1$. By Lemma 3.3 there is a homeomorphism $(X_r^{\Delta,d} \setminus X_{r-1}^{\Delta,d})_+ \cong T(\xi_{d,r})$, where $T(\xi_{d,r})$ denotes the Thom complex of $\xi_{d,r}$. Since $N_d^* + r - s - 1 - l_{d,r} = (n + 1)r - s$ and $rm - \{(n + 1)r - s\} = s - (n - m + 1)r$, by using the Thom isomorphism and Poincaré duality, we obtain a natural isomorphism

$$E_{1,r,s} \cong H^{N_d^* + r - s - 1}(T(\xi_{d,r}), \mathbb{Z}) \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes(n-m)}).$$

4. Spectral sequences induced from truncated resolutions.

In this section, we prove a key result (Theorem 4.10) about the homology stability of “stabilization maps $s_d : A_d(m, n; g) \to A_{d+2}(m, n; g)$.

Definition 4.1. Let $X^\Delta$ denote the (after $(d+1)$-th term) truncated simplicial resolution of $X^{\Delta,d}$ with its natural filtration

$$\emptyset = X^\Delta_0 \subset X^\Delta_1 \subset \cdots \subset X^\Delta_{d+1} \subset X^\Delta_{d+2} = X^\Delta_{d+3} = \cdots = X^\Delta,$$

where $X^\Delta_k = X^{\Delta,d}_k$ if $k \leq d + 1$ and $X^\Delta_k = X^\Delta$ if $k \geq d + 2$.

Remark 4.2. Note that our notation $X^\Delta$ conflicts with that of [15] and Definition 2.3, because usually $X^\Delta$ denotes the non-degenerate simplicial resolution.

By Lemma 2.6 that there is a homotopy equivalence $\pi^\Delta : X^\Delta \xrightarrow{\sim} \Sigma_d$. Hence, analogously to (3.1) and (3.2), we obtain a spectral sequence

$$(4.1) \quad \{E_t^{r,s}, d^t : E_t^{r,s} \to E_{t+r,s+t-1}^{r,s} \} \Rightarrow H_{s-r}(A_d(m, n; g), \mathbb{Z})$$

if $s - r \leq N_d^* - 2$, where $E_{1,r,s}^3 = H_{c}^{N_d^* + r - s - 1}(X^\Delta_r \setminus X^\Delta_{r-1}, \mathbb{Z})$. 

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Lemma 4.3. If $1 \leq r \leq d + 1$, there is a natural isomorphism

$$E_{r,s}^1 \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}).$$

Proof. Since $X^\Delta \setminus X^\Delta_{r-1} = \mathcal{X}^\Delta_r \setminus \mathcal{X}^\Delta_{r-1}$ for $1 \leq r \leq d + 1$, the assertion follows from Lemma 3.5. \hfill \Box

Lemma 4.4. (i) $E^1_{r,s} = 0$ if $r < 0$, or if $r \geq d + 3$, or if $r = 0$ and $s \neq N^*_d - 1$.

(ii) If $1 \leq r \leq d + 1$, $E^1_{r,s} = 0$ for $s \leq (n - m + 1)r - 1$.

(iii) If $r = d + 2$, $E^1_{d+2,s} = 0$ for $s \leq (n - m + 1)(d + 1) - 1$.

Proof. (i) Since $X^\Delta_0 = \emptyset$ and $X^\Delta = X^\Delta_k$ for $k \geq d + 3$, (i) is trivial.

(ii) If $1 \leq r \leq d + 1$, by Lemma 4.3 there is an isomorphism

$$E^1_{r,s} \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}).$$

Because $s - (n - m + 1)r < 0$ if and only if $s \leq (n - m + 1)r - 1$, (ii) follows.

(iii) An easy computation shows that

$$\dim(\mathcal{X}^\Delta_r \setminus \mathcal{X}^\Delta_{r-1}) = N^*_d - r(n + 1) + r - 1 + rm = N^*_d - r(n - m) - 1.$$

By Lemma 2.7 $\dim(X^\Delta_{d+2} \setminus X^\Delta_{d+1}) = \dim(\mathcal{X}^\Delta_{d+1} \setminus \mathcal{X}^\Delta_d) + 1$. So $\dim(X^\Delta_{d+2} \setminus X^\Delta_{d+1}) = N^*_d - (d + 1)(n - m)$. Since $E^1_{d+2,s} = H^c_{N^*_d + d+1-s}(X^\Delta_{d+2} \setminus X^\Delta_{d+1}, \mathbb{Z})$ and $N^*_d + d + 1 - s > N^*_d - (n - m)(d + 1)$ if and only if $s \leq (n - m + 1)(d + 1) - 1$, we see that $E^1_{d+2,s}(d) = 0$ for $s \leq (n - m + 1)(d + 1) - 1$. \hfill \Box

Similarly, let $Y^\Delta$ denote the (after $(d + 1)$-th term) truncated simplicial resolution of $\mathcal{X}^\Delta_{d+2}$ with its natural filtration

$$\emptyset = Y^\Delta_0 \subset Y^\Delta_1 \subset \cdots \subset Y^\Delta_{d+1} \subset Y^\Delta_{d+2} = Y^\Delta_{d+3} = \cdots = Y^\Delta,$$

where $Y^\Delta_k = \mathcal{X}^\Delta_{d+2}$ if $k \leq d + 1$ and $Y^\Delta_k = Y^\Delta$ if $k \geq d + 2$.

By Lemma 2.6 that there is a homotopy equivalence $\pi^\Delta : Y^\Delta \cong \Sigma^s_{d+2}$. Hence, by using the same method as above, we obtain a spectral sequence

$$(4.2) \quad \{ \ell E^t_{r,s}, \ell d^t: E^t_{r,s} \to E^t_{r+t,s+t-1} \} \Rightarrow H_{s-r}(A_{d+2}(m, n; g), \mathbb{Z})$$

if $s - r \leq N^*_d - 2$, where $E^1_{r,s} = H^c_N A_{d+2+r-s-1}(Y^\Delta_r \setminus Y^\Delta_{r-1}, \mathbb{Z})$.

Applying again the same argument we obtain the following two results.
Lemma 4.5. If $1 \leq r \leq d + 1$, there is a natural isomorphism

$$\tilde{E}^1_{r,s} \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^\otimes(n-m)).$$ \hfill $\square$

Lemma 4.6. (i) $\tilde{E}^1_{r,s} = 0$ if $r < 0$, or if $r \geq d + 3$, or if $r = 0$ and $s \neq N_{d+2}^* - 1$.

(ii) If $1 \leq r \leq d + 1$, $\tilde{E}^1_{r,s} = 0$ for $s \leq (n - m + 1)r - 1$.

(iii) If $r = d + 2$, $\tilde{E}^1_{d+2,s} = 0$ for $s \leq (n - m + 1)(d + 1) - 1$. \hfill $\square$

Definition 4.7. Let $g \in \text{Alg}_d^*(\mathbb{RP}^{m-1}, \mathbb{RP}^n)$ be a fixed algebraic map, and let $(g_0, \cdots, g_n) \in A_d(m - 1, n)$ be its fixed representative. If we set $\tilde{g} = \sum_{k=0}^m z_k$, we can see that the tuple $(\tilde{g}g_0, \cdots, \tilde{g}g_n)$ can also be chosen as a representative of the map $g \in \text{Alg}_{d+2}^*(\mathbb{RP}^{m-1}, \mathbb{RP}^n)$. So one can define a stabilization map

$$(4.3) \quad s_d : A_d(m, n; g) \to A_{d+2}(m, n; g)$$

by $s_d(f_0, \cdots, f_n) = (f_0\tilde{g}, \cdots, f_n\tilde{g})$ for $(f_0, \cdots, f_n) \in A_d(m, n; g)$.

Because there is a commutative diagram

$$\begin{array}{ccc}
A_d(m, n; g) & \xrightarrow{s_d} & A_{d+2}(m, n; g) \\
\downarrow{i'_d} & & \downarrow{i'_{d+2}} \\
F(m, n; g) & \cong & F(m, n; g)
\end{array}$$

it induces a map

$$(4.4) \quad s_{d,\infty} = \lim_{k \to \infty} s_{d+2k} : A_{d,\infty}(m, n; g) \to F(m, n; g) \cong \Omega^n S^n,$$

where $A_{d,\infty}(m, n; g)$ denotes the colimit $\lim_{k \to \infty} A_{d+2k}(m, n; g)$ induced from the stabilization maps $s_{d+2k}$'s ($k \geq 0$).

Theorem 4.8. If $2 \leq m < n$, the map $s_{d,\infty} : A_{d,\infty}(m, n; g) \xrightarrow{\cong} \Omega^n S^n$ is a homotopy equivalence if $m+2 \leq n$ and is a homology equivalence if $m+1 = n$.

Proof. This easily follows from Theorem 1.2. \hfill $\square$

Now consider the stabilization map $s_d : A_d(m, n; g) \to A_{d+2}(m, n; g)$. This map naturally extends to the map $\tilde{s}_d : \Sigma_d^* \to \Sigma_{d+2}^*$ by the multiplication by $\tilde{g}$,

$$\tilde{s}_d(f_0, \cdots, f_n) = (f_0\tilde{g}, \cdots, f_n\tilde{g}) \quad \text{for} \quad (f_0, \cdots, f_n) \in \Sigma_d^*.$$
It also naturally extends to the filtration preserving map $\hat{s}_d : X^\Delta \to Y^\Delta$, and induces a homomorphism of spectral sequences

\[
\{\theta^t_{r,s} : E^t_{r,s} \to 'E^t_{r,s}\}. \tag{4.5}
\]

**Lemma 4.9.** If $1 \leq r \leq d+1$, $\theta^1_{r,s} : E^1_{r,s} \to 'E^1_{r,s}$ is an isomorphism for any $s$.

**Proof.** Suppose that $1 \leq r \leq d+1$. Then it follows from the proof of Lemma 3.3 that there is a commutative diagram of open disk bundles

\[
\begin{array}{ccc}
X_r^\Delta \setminus X_{r-1}^\Delta & \xrightarrow{\pi} & C_r(\mathbb{R}^m) \\
\hat{s}_d \downarrow & & \downarrow \\
X_{r-1}^\Delta \setminus X_{r-2}^\Delta & \xrightarrow{\pi} & C_r(\mathbb{R}^m)
\end{array}
\]

Since $X_r^\Delta \setminus X_{r-1}^\Delta \cong X_r^\Delta, d_r \setminus X_{r-1}^\Delta$, and $Y_r^\Delta \setminus Y_{r-1}^\Delta \cong X_r^\Delta, d_r \setminus X_{r-1}^\Delta + 2$, by Lemma 4.5 and the naturality of the Thom isomorphism, we have a commutative diagram

\[
\begin{array}{ccc}
E^1_{r,s} & \xrightarrow{T} & H_{s-r(n-m+1)}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) \\
\theta^1_{r,s} \downarrow & & \downarrow \\
'E^1_{r,s} & \xrightarrow{T} & H_{s-r(n-m+1)}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)})
\end{array}
\]

where $T$ denotes the Thom isomorphism. Hence, $\theta^1_{r,s}$ is an isomorphism. \hfill $\square$

**Theorem 4.10.** If $2 \leq m < n$, $s_d : A_d(m, n; g) \to A_{d+2}(m, n; g)$ is a homology equivalence through dimension $D(d; m, n) - 1$.

**Proof.** Recall that we have two spectral sequences

\[
\begin{align*}
\{E^t_{t,s}, d^t : E^t_{r,s} \to E^t_{r+t,s+t-1}\} & \Rightarrow H_{s-r}(A_d(m, n; g), \mathbb{Z}) \\
\{'E^t_{t,s}, 'd^t : 'E^t_{r,s} \to 'E^t_{r+t,s+t-1}\} & \Rightarrow H_{s-r}(A_{d+2}(m, n; g), \mathbb{Z})
\end{align*}
\]

and a homomorphism $\{\theta^t_{r,s} : E^t_{r,s} \to 'E^t_{r,s}\}$ of spectral sequences.
Now we shall consider the maximal positive integer $D_{\text{max}}$ such that

\[ D_{\text{max}} = \max\{D \in \mathbb{Z} : \theta_{r,s}^\infty \text{ is always an isomorphism as long as } s - r \leq D\} \]

By Lemmas 4.4 and 4.6, we see that $E_{r,s}^1 = 'E_{r,s}^1 = 0$ if $r < 0$, or if $r > d + 2$, or if $r = d + 2$ with $s \leq (n - m + 1)(d + 1) - 1$. Since $(n - m + 1)(d + 1) - (d + 2) = D(d; m, n) - 1$, we easily see that:

$(*)_1$ If $r < 0$ or $r \geq d + 2$, $\theta_{r,s}^\infty$ is an isomorphism for all $(r, s)$ such that $s - r \leq D(d; m, n) - 1$.

Next, we assume that $0 \leq r \leq d + 1$, and investigate the condition that $\theta_{r,s}^\infty$ is an isomorphism. Note that the group $\theta_{r,s}^\infty$ is an isomorphism if $(r, s)$ is an isomorphism for any $(r, s)$.

A similar argument for the differentials $d^1 : E_{r,s}^1 \to E_{r+1,s}^1$ and $'d^1 : 'E_{r,s}^1 \to 'E_{r+1,s}^1$ and applying Lemma 4.9, we see that $\theta_{r,s}^2$ is an isomorphism if $(r, s) \notin S_2$, where

\[
S_2 = \{(r_1, s_1) \in \mathbb{Z}^2 : (r_1 + 1, s_1) \in S_1\}
= \{(d + 1, s_1) \in \mathbb{Z}^2 : s_1 \geq (n - m + 1)(d + 1)\}.
\]

A similar argument for the differentials $d^2$ and $'d^2$ shows that $\theta_{r,s}^3$ is an isomorphism if $(r, s) \notin S_3 = \{(r_1, s_1) \in \mathbb{Z}^2 : (r_1 + 2, s_1 + 1) \in S_1 \cup S_2\}$.

Continuing in the same fashion, considering the differentials $d^k : E_{r,s}^k \to E_{r+t,s+t-1}^k$ and $'d^k : 'E_{r,s}^k \to 'E_{r+t,s+t-1}^k$, and applying Lemma 4.9, we easily see that $\theta_{r,s}^\infty$ is an isomorphism if $(r, s) \notin S := \bigcup_{t \geq 1} S_t = \bigcup_{t \geq 1} A_t$, where $A_t$ denotes the set given by

\[
A_t = \left\{ (r_1, s_1) \in \mathbb{Z}^2 \middle| \begin{array}{l}
\text{There are integers } k_1, k_2, \ldots, k_t \text{ such that,} \\
1 \leq k_1 < k_2 < \cdots < k_t, \ r_1 + \sum_{t=1}^t k_l = d + 2, \\
s_1 + \sum_{t=1}^t (k_l - 1) \geq (n - m + 1)(d + 1)
\end{array} \right\}.
\]

If $A_t \neq \emptyset$, it is easy to see that

\[
a(t) = \min\{s - r : (r, s) \in A_t\} = (n - m + 1)(d + 1) - (d + 2) + t = (n - m)(d + 1) + t + s - (n - m + 1)(d + 1).
\]

Hence, $\min\{a(t) : t \geq 1, A_t \neq \emptyset\} = D(d; m, n) + 1$, and we have the following:

$(*)_2$ If $0 \leq r \leq d + 1$, $\theta_{r,s}^\infty$ is an isomorphism for any $(r, s)$ such that $s - r \leq D(d; m, n)$.
Then, by \((\ast)_1\) and \((\ast)_2\), we see that \(\theta_{r,s}^\infty : E_{r,s}^\infty \xrightarrow{\cong} E_{r,s}^\infty\) is an isomorphism for any \((r, s)\) such that \(s - r \leq D(d; m, n) - 1\). Thus, by using the Comparison Theorem for spectral sequences, \(s_d\) is a homology equivalence through dimension \(D(d; m, n) - 1\).

\[\square\]

5. The proof of the main result.

In this section, we prove our main result (Theorem 1.5). Note that we could simply deduce a weaker result (with \(D(d; m, n) - 1\) in place of \(D(d; m, n)\)) by simply combing Theorem 4.8 with Theorem 4.10. However, we can do better by considering a “stable” non-degenerate resolution in the manner of [19].

**Definition 5.1.**

Let

\[
j'_d : A_d(m, n; g) \to A_{d, \infty}(m, n; g) = \lim_{k \to \infty} A_{d+2k}(m, n; g)
\]

be the natural map.

Recall that \(\mathcal{X}_d^{\Delta,d}\) is a non-degenerate simplicial resolution of \(\pi_d^* : Z_d^* \to \Sigma_d^*\) and it can be defined by using the family of embeddings \(\mathcal{E}_d = \{i_{r,d} : Z_d^r \to \mathbb{R}^{N_r}\}_{r \geq 1}\) satisfying the condition \((2.2)\) as explained in Remark 2.3. The stabilization map \(\tilde{s}_d : \Sigma_d^* \to \Sigma_{d+2}^*\) naturally extends to the map \(\tilde{s}_d : Z_d^* \to Z_{d+2}^*\) by using the multiplication by \(\tilde{g}\). Then it is easy to see that we can choose the families \(\mathcal{E}_d\) of the embeddings satisfying the two conditions \((2.2)\) and

\[
(5.2) \quad \tilde{i}_{r,d+2} \circ \tilde{s}_d = \tilde{i}_{r,d} \quad \text{for each pair of positive integers \((d, r)\).}
\]

Since \(s_{d+2k}\) induces the filtration preserving map \(\hat{s}_{d+2k} : \mathcal{X}_{d+2k}^{\Delta,d} \to \mathcal{X}_{d+2k}^{\Delta,d}\) between non-degenerate simplicial resolutions, it also gives the filtration preserving map \(\hat{s}_{d,\infty} : \mathcal{X}_{d,\infty}^{\Delta,d} \to \mathcal{X}_{d,\infty}^{\Delta,d}\), where \(\mathcal{X}_{d,\infty}^{\Delta,d}\) denotes the colimit \(\lim_{k \to \infty} \mathcal{X}_{d,k}^{\Delta,d}\) induced from the maps \(\hat{s}_{d+2k}\)’s.

It follows from the method due to Vassiliev ([19, §5 of Chap. III], [1, 4.3]), (5.2) and Lemma 3.5 that we may regard \(\mathcal{X}_{d,\infty}^{\Delta,d}\) as a non-degenerate simplicial resolution of the discriminant of \(A_{d,\infty}(m, n; g)\), and that there is a spectral sequence \(\{\hat{E}_{r,s}^t, \hat{d}^t : \hat{E}_{r,s}^t \to \hat{E}_{r+s+t-1}^t\} \Rightarrow H_{s-r}(A_{d,\infty}(m, n; g), \mathbb{Z})\), such that,

\[
(5.3) \quad \hat{E}_{r,s}^1 = H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) \quad \text{for any} \quad r \geq 1.
\]
Since $\hat{s}_{d,\infty}$ is a filtration preserving map between non-degenerate simplicial resolutions, it induces a homomorphism of spectral sequences,

\[(5.4) \quad \{\tilde{\theta}_{r,s}^t : \tilde{E}_{r,s}^t(d) \to \tilde{E}_{r,s}^t\}.
\]

**Lemma 5.2.** $\tilde{E}_{r,s}^1 = \tilde{E}_{r,s}^\infty$ for any $(r, s)$, and $\tilde{d}^t : \tilde{E}_{r,s}^t \to \tilde{E}_{r+t,s+t-1}^t$ is trivial for any $t \geq 1$. Moreover, if $k \geq 1$, the extension problem for the graded group $Gr(H_k(A_{d,\infty}(m, n; g), \mathbb{Z}))$ is trivial and there is an isomorphism

\[H_k(A_{d,\infty}(m, n; g), \mathbb{Z}) \cong \bigoplus_{r=1}^\infty \tilde{E}_{r,r+k}^1 = \bigoplus_{r=1}^\infty H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{(n-m)}).
\]

**Proof.** First, note that by [19] (cf. [5], [18]) there is an isomorphism

\[(5.5) \quad H_k(\Omega^m S^n, \mathbb{Z}) \cong \bigoplus_{r=1}^\infty H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{(n-m)}) \quad \text{for } k \geq 1.
\]

Hence, by Theorem 4.8 for each $k \geq 1$, there is an isomorphism

\[H_k(A_{d,\infty}(m, n; g), \mathbb{Z}) \cong \bigoplus_{r=1}^\infty H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{(n-m)}) = \bigoplus_{r=1}^\infty \tilde{E}_{r,r+k}^1.
\]

So, $\tilde{E}_{r,s}^1 = \tilde{E}_{r,s}^\infty$ if $s - r \geq 1$. If $s - r \leq 0$ then, due to dimensional reasons,

\[\tilde{E}_{r,s}^1 = \tilde{E}_{r,s}^\infty = \begin{cases} 0 & \text{if } (r, s) \neq (0, 0), \\ \mathbb{Z} & \text{if } (r, s) = (0, 0). \end{cases}
\]

Hence, $\tilde{E}_{r,s}^1 = \tilde{E}_{r,s}^\infty$ for any $(r, s)$, so that $\tilde{d}^t = 0$ for any $t \geq 1$. Finally, by from (5.3), we easily see that the extension problem of the graded group is trivial. \hfill $\square$

**Lemma 5.3.**

(i) If $1 \leq r \leq d+1$, $\tilde{\theta}_{r,s}^1 : \tilde{E}_{r,s}^1(d) \to \tilde{E}_{r,s}^1(d)$ is an isomorphism for any $s$.

(ii) If $s - r < D(d; m, n)$, $\tilde{d}^t : \tilde{E}_{r,s}^t(d) \to \tilde{E}_{r+t,s+t-1}^t(d)$ is trivial for any $t \geq 1$.

(iii) If $1 \leq r \leq d$ and $s - r = D(d; m, n)$, $\tilde{\theta}_{r,s}^\infty : \tilde{E}_{r,s}^\infty(d) \to \tilde{E}_{r,s}^\infty(d)$ is an isomorphism.
Proof. (i) The assertion easily follows from the proof of Lemma 4.9.
(ii) By Theorem 4.8 and for dimensional reasons, there is an isomorphism
\[ H_k(A_d(m, n; g), \mathbb{Z}) \cong \bigoplus_{r=1}^d H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) \]  
if \( k < D(d; m, n) \).

Hence, by using the spectral sequence \( \{ E^r_{s,t}, d^r : E^r_{s,t}(d) \to E^r_{s+t,t-1}(d) \} \), we see that \( \tilde{E}^1_{r,s}(d) = \tilde{E}^\infty_{r,s}(d) \) if \( s - r < D(d; m, n) \), and (ii) follows.

(iii) Assume that \( s = r + D(d; m, n) \). Since \( (s + t - 1) - (r + t) = D(d; m, n) - 1 \), by (ii) \( \tilde{E}^1_{r+s,t+s-1}(d) = \tilde{E}^\infty_{r+s,t+s-1}(d) \) for any \( t \geq 1 \). Hence, \( \tilde{d}^t : \tilde{E}^t_{r,s}(d) \to \tilde{E}^t_{r+s,t+s-1}(d) \) is trivial for any \( t \geq 1 \), and there is a natural epimorphism \( \tilde{\pi} : \tilde{E}^1_{r,s}(d) \to \tilde{E}^\infty_{r,s}(d) \). By Lemma 5.2 we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{E}^1_{r,s}(d) & \xrightarrow{\tilde{\theta}^1_{r,s}} & \tilde{E}^1_{r,s} \\
\downarrow{\tilde{\pi}} & & \uparrow{\text{epic.}} \\
\tilde{E}^\infty_{r,s}(d) & \xrightarrow{\tilde{\theta}^\infty_{r,s}} & \tilde{E}^\infty_{r,s}
\end{array}
\]  
(5.6)

Since \( 1 \leq r \leq d \), \( \tilde{\theta}^1_{r,s} \) is an isomorphism by (i). Hence, easy diagram chasing shows that \( \tilde{\theta}^\infty_{r,s} \) is an isomorphism. \( \square \)

Corollary 5.4. If \( 1 \leq r \leq d \) and \( s - r < D(d; m, n) \), \( \tilde{E}^1_{r,s}(d) \) collapses at the \( \tilde{E}^1(d) \) term. Hence, if \( 1 \leq r \leq d \) and \( s - r < D(d; m, n) \), there is an isomorphism \( \tilde{E}^1_{r,s}(d) = \tilde{E}^\infty_{r,s}(d) \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) \).

Proof. Since the proof is analogous, we give the proof only when \( s - r = D(d; m, n) \). If \( s - r = D(d; m, n) \), by (5.6) and Lemma 5.3, \( \tilde{\pi} \) is an isomorphism. Hence, \( \tilde{E}^1_{r,s}(d) = \tilde{E}^\infty_{r,s}(d) \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) \). \( \square \)

Theorem 5.5. If \( 2 \leq m < n \), the map \( j'_d : A_d(m, n; g) \to A_{d,\infty}(m, n; g) \) is a homology equivalence up to dimension \( D(d; m, n) \).

Proof. It follows from Theorem 4.10 that the map \( j'_d \) is a homology equivalence through dimension \( D(d; m, n) - 1 \). So it remains to show that \( j'_d \) induces an epimorphism on the homology \( H_k(\mathbb{Z}) \) for \( k = D(d; m, n) \). However, since \( j'_d \) induces the homomorphism \( \{ \tilde{\theta}^t_{r,s} : \tilde{E}^t_{r,s}(d) \to \tilde{E}^t_{r,s} \} \) of spectral
sequences, it suffices to prove that $\tilde{\theta}_{r,s}^\infty : \tilde{E}_{r,s}^\infty(d) \to \tilde{E}_{r,s}^\infty$ is an epimorphism if $s - r = D(d; m, n)$. If $r \leq 0$, $\tilde{E}_{r,s}^\infty = 0$. So in this case the assertion is trivial. If $r \geq d + 1$, $s - (n - m + 1)r = (n - m)(d + 1 - r) - 1 < 0$. Hence, $\tilde{E}_{r,s}^1 = H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (n-m)}) = 0$ if $r \geq d + 1$. So $\tilde{E}_{r,s}^\infty = 0$ if $r \geq d + 1$ and the assertion is also true in this case. Finally, if $1 \leq r \leq d$, then, by Lemma 5.3, $\tilde{\theta}_{r,s}^\infty$ is an isomorphism and the result follows.

**Proof of Theorem 1.5.** Note that the map $i_d'$ coincides the composite of maps

$$A_d(m, n; g) \xrightarrow{j_d'} A_{d,\infty}(m, n; g) \xrightarrow{s_{d,\infty}} \Omega^m S^n.$$  

It follows from Theorem 4.8 that $s_{d,\infty}$ is a homology equivalence. Since $j_d'$ is a homology equivalence up to dimension $D(d; m, n)$ by Theorem 5.5, the map $i_d'$ is so. If $m + 2 \leq n$, $A_d(m, n; g)$ and $\Omega^m S^n$ are simply connected by [1, Fact 3.2] and $i_d'$ is a homotopy equivalence up to dimension $D(d; m, n)$.

**Acknowledgements.** Both authors should like to take this opportunity to thank Professor Jacob Mostovoy for his valuable suggestions.

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