1. Lecture 1. Non-commutative Probability Spaces

Let $\mathcal{A}$ be an algebra of operators that act on a Hilbert space. We will assume that $\mathcal{A}$ contains the identity operator (such algebras are called unital) and that it is closed under the operation of taking adjoints, that is, if $X \in \mathcal{A}$, then $X^* \in \mathcal{A}$.

It is often convenient to assume further that $\mathcal{A}$ is closed either with respect to uniform operator norm ($\|X\| = \sup_{\|v\| = 1} \|Xv\|$), or with respect to weak topology ($X_i \to X$ if and only if $\langle u, X_i v \rangle \to \langle u, Xv \rangle$ for all $u$ and $v$).
In the first case, the algebra is called a $C^*$-algebra, and in the second case, it is called a $W^*$-algebra or a von Neumann algebra.

A state on the algebra $\mathcal{A}$ is a linear functional $E : \mathcal{A} \to \mathbb{C}$, which has the following positivity property: for all operators $X$,

$$E(X^*X) \geq 0.$$  

A typical example of a state is $E(X) = \langle u, Xu \rangle$ where $u$ is a unit vector. Typically, a state is denoted by letters $\varphi$ or $\tau$ but we will use letter $E$ to emphasize the parallel with expectation functional in the classical probability theory.

The name “state” is due to a relation of operator algebras to quantum mechanics. Let a state of a quantum system be described by a unit vector $u$ in a Hilbert space $H$. In the standard formalism of quantum mechanics, measurable quantity corresponds to a self-adjoint operator $X$, and the expected result of a measurement equals $\langle u, Xu \rangle$. Hence, there is a correspondence between states $u$ and the corresponding functionals $E$. Irving Segal in [18] suggested to call all such functionals “states.”

We will use the words “state” and “expectation” interchangeably.

States may have some additional properties. If $E(A^*A) = 0$ implies that $A = 0$, then the state is called faithful. If $X_n \to X$ weakly implies that $E(X_n) \to E(X)$ then the state is called normal. If $E(XY) = E(YX)$ then the state is called tracial, or simply trace.

**Definition 1.1.** A non-commutative probability space $(\mathcal{A}, E)$ is a pair of a unital $C^*$-operator algebra $\mathcal{A}$ and a state $E$ with an additional property $E(I) = 1$.

If the state $E$ is tracial, then we call $(\mathcal{A}, E)$ a tracial non-commutative probability space. Also, if $\mathcal{A}$ is a von Neumann algebra and $E$ is normal, then we call $(\mathcal{A}, E)$ a $W^*$-probability space.

Here are several examples.

**1. A classical probability space**

Let $(\Omega, \mathcal{F}, \mu)$ be a classical probability space with the space of events $\Omega$, sigma-algebra of subsets $\mathcal{F}$, and probability measure $\mu$.

In this case we take the Hilbert space $H = L^2(\Omega, \mu)$ and let $\mathcal{A}$ be the algebra of bounded measurable functions on $\Omega$ that act on $H$ by multiplication. The expectation is the usual probability expectation:

$$Ef = \int_{\Omega} f \mu \, (d\omega).$$

**2. The algebra of $N$-by-$N$ matrices**

In this case, we can use the normalized trace as the expectation:

$$EX = \text{tr}(X) := \frac{1}{N} \sum_{i=1}^{N} X_{ii}.$$  

**3. The algebra of random $N$-by-$N$ matrices**

The joint probability distribution of the entries of these matrices is such that all joint moments of these entries are finite. We define the functional $E$ as the
expectation of the trace:
$$EX = \langle \text{tr} (X) \rangle .$$
Here we used a convenient notation from the physical literature: \( \langle a \rangle \) denotes the average of \( a \) over the statistical ensemble, that is, the expectation of the random variable \( a \).

4. The group algebra of a finitely generated group

Let \( G \) be a finitely generated group and \( \mathbb{C} G \) be its group algebra over the field of complex numbers. This algebra is an infinite-dimensional vector space over \( \mathbb{C} \), and we can introduce a scalar product by linearity and the formula
$$\langle a, b \rangle = \delta_{ab},$$
where \( a \) and \( b \) are elements of the group \( G \).

A completion of \( \mathbb{C} G \) with respect to this scalar product defines a Hilbert space \( L^2 (G) \), and the elements of the group algebra \( \mathbb{C} G \) act on \( L^2 (G) \) by left multiplication. In this way, we see that \( \mathbb{C} G \) is an algebra of operators.

The expectation is given by the formula:
$$EX = \langle e, X e \rangle ,$$
where \( e \) denotes the identity element of \( G \). Informally speaking, \( X \) is a linear combination of the elements of \( G \), and the expectation is the coefficient before the identity element.

It turns out that this expectation is faithful and tracial.

Exercise 1.2. Show that the expectation in this example is faithful and tracial.

5. The full Fock space

Let \( H \) be a Hilbert space and define the corresponding Fock space \( F (H) \) as follows. Let us define a direct sum of tensor products of \( H \):
$$\tilde{F}(H) = \mathbb{C} \oplus H \oplus H \otimes H \oplus H \otimes H \otimes H \oplus \ldots$$

An example of a vector in \( \tilde{F}(H) \) is
$$v = 3 \Omega + e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_1 \otimes e_2 .$$
Here \( \Omega \) is the number 1, considered as a unit basis vector of the vector space of complex numbers, \( \mathbb{C} \).

It is easy to define the scalar product in \( \tilde{F}(H) \) by extending the scalar product of \( H \). For example,
$$\langle e_1 \otimes e_2 \otimes e_3, f_1 \otimes f_2 \otimes f_3 \rangle = \langle e_1, f_1 \rangle \langle e_2, f_2 \rangle \langle e_3, f_3 \rangle .$$

We define the full Fock space \( F (H) \) as the completion of the vector space \( \tilde{F}(H) \) with respect to this scalar product.

For each vector \( v \) in \( H \) we can define the left creation and annihilation operators, which we denote \( a^* (v) \) and \( a (v) \), respectively. They are defined by linearity and by the following formulas:
$$a^* (v) f = v \otimes f ,$$
$$a (v) \Omega = 0 .$$
and

\[ a(v) e_1 \otimes e_2 \otimes \ldots \otimes e_n = \langle v, e_1 \rangle e_2 \otimes \ldots \otimes e_n. \]

Note that \( a(v) \) and \( a^*(v) \) are adjoint with respect to the scalar product that we just defined.

We define the algebra \( \mathcal{A}(F(H)) \) as the algebra generated by all left creation and annihilation operators. The expectation is given by the formula:

\[ E(X) = \langle \Omega, X\Omega \rangle. \]

This expectation is neither faithful nor tracial. Indeed,

\[ E(a^*(v)a(v)) = 0, \]

hence the expectation is not faithful.

In addition,

\[ E(a(v)a^*(v)) = 1, \]

which means that the expectation is not tracial.
2. Lecture 2. Distributions

Suppose that $X_1, X_2, \ldots, X_n$ be elements of a non-commutative probability space $(A, E)$. We will call them random variables. Their distribution is the linear map from the algebra of polynomials in non-commuting variables $x_1, \ldots, x_n$ to $\mathbb{C}$,

$$f (x_1, \ldots, x_n) \rightarrow E [f (X_1, \ldots, X_n)] .$$

The $*$-distribution is a similar map for polynomials in non-commutative variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, which is given by the formula:

$$f (x_1, \ldots, x_n, y_1, \ldots, y_n) \rightarrow E [f (X_1, \ldots, X_n, X_1^*, \ldots, X_n^*)]$$

In other words, the distribution of a family of random variables is a collection of their joint moments.

We will write $X \cong Y$ and we will say that $X$ is equivalent to $Y$ if these two variables have the same $*$-distributions.

**Definition 2.1.** The sequence of variables $(X_1^{(i)}, \ldots, X_n^{(i)})$, $i = 1, 2, \ldots$, converges in distribution to $(X_1, \ldots, X_n)$ if for every non-commutative polynomial $f$ in $n$ variables it is true that

$$E [f (X_1^{(i)}, \ldots, X_n^{(i)})] \rightarrow E [f (X_1, \ldots, X_n)]$$

as $i \rightarrow \infty$.

Convergence in $*$-distribution is defined similarly.

Note that this definition makes sense even if all $(X_1^{(i)}, \ldots, X_n^{(i)})$ and the limit $(X_1, \ldots, X_n)$ belong to different probability spaces.

**Proposition 2.2.** Suppose that $X$ is a bounded self-adjoint element of a $W^*$-probability space $(A, E)$. Then there exists a probability measure $\mu$ on $\mathbb{R}$ such that

$$E (X^k) = \int_{\mathbb{R}} x^k \mu (dx) .$$

**Proof:** By using the spectral theorem, we can write

$$X = \int_{\mathbb{R}} x P (dx),$$

where $P$ is the projection-valued measure. We define $\mu (A) := E (P (A))$. It is easy to see that $\mu$ is a probability measure. Then,

$$X^k = \int_{\mathbb{R}} x^k P (dx)$$

and by taking expectations on both sides, we obtain

$$E (X^k) = \int_{\mathbb{R}} x^k \mu (dx) .$$

QED.
This proposition can also be extended to normal $X$, that is, to operators $X$, that satisfy the condition $X^*X = XX^*$. In this case the measure $\mu$ is a measure on $\mathbb{C}$, and for every polynomial in two variables, we have

$$E\left(P(X,X^*)\right) = \int_{\mathbb{R}} P(z,\overline{z}) \mu\left(dz\right).$$

Measure $\mu$ defined by Proposition 2.2 is called the spectral probability distribution of $X$.

Examples.
1. If $X$ is a Hermitian matrix, then

$$\mu = \frac{1}{n} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where $\lambda_i$ are eigenvalues of $X$, counted with multiplicity

**Exercise 2.3.** Suppose that $X$ is a random Hermitian matrix considered as an element of non-commutative probability space from Example 3 on page 2. What is its spectral probability distribution of $X$?

2. Consider the probability space of a group algebra, $\mathbb{C}G$ which we defined in Example 4 on page 3. In particular, let $G$ be isomorphic to $\mathbb{Z}$ and let $g$ be the generator of $G$. Then, $g^* = g^{-1}$ and therefore $g$ is unitary as an operator. It is easy to see that

$$E\left(g^k\right) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \quad (1)$$

Every unitary random variable with the moments as in (1) is called the Haar unitary.

The spectral probability distribution of $g$ is the uniform (“Haar”) measure on the unit circle. Indeed,

$$E\left(g^k (g^*)^l\right) = \delta_{kl}$$

and

$$\int_{\mathbb{C}} z^k \overline{z}^l \mu\left(dz\right) = \frac{1}{2\pi} \int_0^{2\pi} \theta^{k-l} d\theta = \delta_{kl}.$$

3. Now consider the same probability space as in the previous example but let $X = g + g^{-1}$. This is a self-adjoint operator and its moments are given by the following formula:

$$E\left(X^k\right) = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \left(\frac{k}{k/2}\right), & \text{if } k \text{ is even}. \end{cases} \quad (2)$$

What is its spectral probability distribution? In order to figure this out, let us define the Cauchy transform of a probability distribution $\mu$ by the formula

$$G(z) = \int_{\mathbb{R}} \frac{\mu\left(dt\right)}{z - t}.$$
Here, \( z \) is the complex variable that takes values in the upper half-plane \( \mathbb{C}^+ = \{ z | \text{Im} z > 0 \} \). Another frequently used name for this object (or sometimes for its negation) is the \textit{Stieltjes transform} of \( \mu \). We will use these names as synonyms.

If \( \mu \) is the spectral probability distribution of a random variable \( X \), then
\[
G(z) = E \left( \frac{1}{z - X} \right). \tag{3}
\]
Therefore, we will call function \( G(z) \) defined by formula (3) the \textit{Cauchy transform} of a random variable \( X \). Since \( X \) is assumed to be bounded, we can expand \((z - X)^{-1}\) as series in \( z^{-1} \),
\[
G(z) = z^{-1} + \sum_{k=1}^{\infty} E \left( X^k \right) z^{-k-1},
\]
and these series are convergent if \( |z| \geq \|X\| \).

If the Cauchy transform is known, then it can be used to recover the probability distribution by the so-called \textit{Stieltjes inversion formula} . Namely,
\[
\mu(B) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_B \text{Im}G(x + i\varepsilon) \, dx, \tag{4}
\]
provided that \( B \) is Borel and \( \mu(\partial B) = 0 \).

![Figure 1. The density of the arcsine distribution.](image)

In particular, if \( G(z) \) can be analytically continued to a point \( x \) on the real axis, then \( \mu \) is absolutely continuous at this point and its density is equal to \(-\pi^{-1} \text{Im}G(x)\).

**Exercise 2.4.** Check that if \( X = g + g^{-1} \), then
\[
G(z) = \frac{1}{\sqrt{z^2 - 4}}.
\]
Conclude that the spectral distribution of \( X \) has the density
\[
p(x) = \begin{cases} \frac{1}{\pi \sqrt{4 - x^2}}, & \text{if } |x| \leq 2, \\ 0, & \text{if } |x| > 2. \end{cases}
\]
This distribution is called the *arcsine distribution*. The plot of its density is presented in Figure 1.

4. Now consider the full Fock space $F(H)$ where $H$ is one-dimensional with a unit basis vector $e$. For shortness, let us write $a$ for the creation operator $a(e)$, and $a^*$ for the annihilation operator $a^*(e)$. Also, let

$$e_n := \begin{cases} e \otimes \ldots \otimes e, & \text{if } n \geq 1, \\ \Omega, & \text{if } n = 0. \end{cases}$$

The algebra generated by $a$ and $a^*$ is called the Toeplitz algebra. Recall that we defined the expectation by the formula $E(X) = \langle e_0, X e_0 \rangle$. That is, the expectation of $X$ is the upper-left most entry in the matrix of $X$ in the basis $\{e_n\}$.

What is the spectral distribution of the operator $a + a^*$ with respect to this expectation?

As a first step, we want to calculate $E((a + a^*)^n)$. For this purpose, let us code the terms in the expansion of $(a + a^*)^n$ by certain paths on the $\mathbb{Z}^2$ lattice. We start with the origin (i.e., $(0,0)$ point). At each step we go one step to the right and either up or down, depending on whether we got $a$ or $a^*$. Also, we read the terms in the expansion from right to left. For example, the term $aa^*a^*$ in the expansion of $(a + a^*)^3$ corresponds to the picture in Figure 2.

![Figure 2. Path of $aa^*a^*$.](image)

And the term $a^*a^*aaa^*a$ in the expansion of $(a + a^*)^6$ corresponds to the path in Figure 3.
Note that a particular term in the expansion of \((a + a^*)^n\) has zero expectation if and only if the corresponding path either goes below the horizontal axis, or does not return to zero at the end, or both. The paths that are always on or above the horizontal axis and that return to zero at the end are sometimes called the Dyck paths. Hence, we need to enumerate the Dyck paths.

Obviously, the number of Dyck paths with odd \(n\) is zero. It turns out that the number of Dyck paths with odd \(n\) is given by the so-called Catalan numbers. Namely, let \(n = 2k\). Then the number of Dyck paths is given by

\[
C_k = \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{k!(k+1)!}.
\]

Indeed, the number of all paths that start at the origin and end at the point \((2k, 0)\) is equal to \(\binom{2k}{k}\), which is simply the number of ways in which we can distribute \(k\) upward steps among the total of \(2k\) steps. Let us count those of them that go below the horizontal axis.

Suppose that such a path reached the line \(y = -1\) and let us reflect the remainder of the path relative to the line \(y = -1\). As a result, we obtain a path that starts at \((0, 0)\) and ends at \((2k, -2)\). It is easy to see that in this way we can construct a bijection between the paths that go from the origin to \((2k, 0)\) and drop below the horizontal axis, and the paths that go from the origin to \((2k, -2)\). The number of the latter paths is easy to calculate as \(\binom{2k}{k-1}\). For we have \(k - 1\) upward steps that have to be distributed among the total of \(2k\) steps.

Hence the number of Dyck paths is

\[
\binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1} \binom{2k}{k} = C_k.
\]

The first Catalan numbers are given in the following Table:

| \(k\) | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| \(C_k\) | 1  | 1  | 2  | 5  | 14 | 42 | 132|

A useful recursive formula for Catalan numbers is as follows:

\[
C_n = \sum_{k=0}^{n-1} C_{n-1-k} C_k.
\]

OK. We have shown that the moments of \(a + a^*\) are given by the Catalan numbers. This allows us to recover its spectral probability distribution.

**Exercise 2.5.** Check that the Cauchy transform of \(a + a^*\) is given by the formula

\[
G(z) = \frac{1}{2} \left( z - \sqrt{z^2 - 4} \right).
\]

Conclude that the spectral distribution of \(a + a^*\) has the density

\[
p(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2}, & \text{if } |x| \leq 2, \\ 0, & \text{if } |x| > 2. \end{cases}
\]
This distribution is called the *semicircle distribution*. The plot of its density is presented in Figure 4.

![Figure 4. The density of the semicircle distribution.](image)

To conclude this lecture, note that Catalan numbers occur very often in enumerative problems (see Stanley). We mention two of them which are closely related to the subject of large random matrices and free probability. First of all, Catalan numbers enumerate the oriented rooted trees (sometimes called fat trees).

![Figure 5. Oriented rooted trees with 3 edges.](image)

An oriented tree is a tree (i.e., a graph without loops) embedded in a plane. A rooted tree has a special edge ("root"), which is oriented, that is, this edge has a start and end vertices. One oriented rooted tree is isomorphic to another if there is a homeomorphism of the corresponding planes which sends one tree to another in such a way that the root goes to the root and the orientations of the root and the plane are preserved. The 5 non-isomorphic oriented rooted trees with 3 edges are shown in Figure 5.

![Figure 6. Two non-isomorphic oriented rooted trees.](image)
Figure 6 shows an example of two trees that are equivalent as rooted trees, but not equivalent as oriented rooted trees. (There is no isomorphism of planes that sends the first tree to the second one.)

Figure 7. A fat tree

The oriented rooted tree can be thought of as a “fat tree”. Namely, start with the initial vertex of the root and travel near the root edge so that the edge is on the right of you. Continue the travel along the tree so that the edges will always remain on the right of you. Finish the travel when you arrive at the initial point of the root. Note that we will travel on both sides of each edge of the tree. Note also that the sequence of the vertices visited during the travel determines the plane embedding of the tree. If we cut the plane along the path of our travel we will get the fat tree.

Figure 7 shows an example of a fat tree.

The claim is that there is a bijection between Dyck paths of length $2n$ and the non-isomorphic oriented rooted trees with $n$ edges. This bijection works as follows. Let $m$ be the height of a given Dyck path. Draw $m - 1$ circles with origin $O$. Put the initial vertex of the tree (i.e., the original vertex of the root) at $O$. Start walking along the Dyck path. If the path goes up, then put a new vertex on the circle with the larger radius. If we have already placed some points on this circle, then make sure that the new point is to the right (i.e., in clockwise direction) of the last point that we placed on the circle and to the left of the first point that we placed on the circle. Then connect this point to the current point on the tree.

If the path goes down, then we simply travel to the circle with smaller radius along the available edge of the tree. It is possible to show that this construction gives a bijection between the set of all Dyck paths and all non-isomorphic oriented rooted trees. An example of this bijection is shown in Figure 8.

Another important bijection is between oriented rooted trees and non-crossing pairings of the set $\{1, 2, \ldots, 2n\}$. A pairing is a partition of the set into the collection of subsets that consist of two elements each. A pairing has a crossing if there
are two pairs \( \{i, j\} \) and \( \{k, l\} \) such that \( i < k < j < l \). A pairing is non-crossing if it has no crossings.

Now, let us take an oriented rooted tree and let us travel along the corresponding fat tree as we described when we discussed fat trees. Namely, let us start with the initial vertex of the root edge and go on the left-hand side of the root edge. As we go along, let us mark this side of the root by number 1. Then, as we go along the tree, mark the sides we pass by a sequence of consecutive integers. Hence, when we finish our travel along the tree with \( n \) edges, we will use integers from 1 to \( 2n \). We travel exactly twice next to each of the edges, hence we can pair the numbers written on the opposite sides of the edge.

Graphically, we can represent this pairing by drawing a curve that connects two opposite sides of the tree and contains all the edges that are passed between the times when we first left the edge and when we returned back to it. It can be seen after some reflection that the resulting partition is non-crossing. Moreover, this construction furnish a bijection between the non-isomorphic oriented rooted trees and non-crossing partitions.

We can represent this bijection graphically. Imagine that we cut the plane along the tree and straighten the cut. Then the closed curves defined above become arcs that connect pieces of the cut. These \( 2n \) pieces represent edges of the former tree. An example of this process is presented in Figure 9.

It is sometimes convenient to connect the extreme points of the cut so that the non-crossing pairing connects pairs of \( 2n \) pieces of a circle. The arcs that represents the pairing can be drawn either outside or inside the circle. Note that the circle is oriented and the pieces that correspond to the root are marked. See Figure 10 for an illustration. Such pictures are called planar maps and they often arise in
Figure 9. An oriented rooted tree and corresponding non-crossing pairing

Figure 10. Planar maps corresponding to a non-crossing pairing

the theory of large random matrices. See Zvonkin and Lando and Zvonkin for an excellent treatment of planar maps.
3. LECTURE 3. FREENESS

In general, it is difficult to say anything about the joint distribution of non-commutative random variables. In the classical case, the crucial role is played by the concept of independence. We can easily adapt this concept to the non-commutative case as follows.

**Definition 3.1.** Let $A_1$ and $A_2$ be sub-algebras of algebra $A$. They are called independent if they commute and if $E(a_1a_2) = E(a_1)E(a_2)$ for all elements $a_1 \in A_1$ and $a_2 \in A_2$.

Two random variables are called independent if the algebras that they and their adjoints generate are independent.

Clearly these definitions can be easily generalized to any finite or countable number of algebras.

Note especially that by the definition, independent algebras $A_1$ and $A_2$ must commute. Hence, non-commutativity plays role only because the subalgebras $A_1$ and $A_2$ by themselves can be non-commutative.

In free probability theory, the main concept is that of freeness.

**Definition 3.2.** Let $A_1, A_2, \ldots, A_n$ be sub-algebras of algebra $A$. They are called free if

$$E(a_1a_2 \ldots a_s) = 0$$

for all such sequences of $a_1 \in A_{i(1)}$, $a_2 \in A_{i(2)}$, $\ldots$, $a_s \in A_{i(s)}$ that $E(a_k) = 0$ for all $k \leq s$, and $i(k) \neq i(k+1)$ for all $k \leq s - 1$.

Random variables are called free if the algebras that they and their adjoints generate are free.

It turns out that the rule in Definition 3.2 allows us, in fact, compute the expectation of all possible products of free variables. Indeed, suppose that the sequence of $a_1, a_2, \ldots, a_s$ is alternating (i.e., if $a_k \in A_{i(k)}$, then $i(k) \neq i(k+1)$ for all $k \leq s - 1$) but that $E(a_k)$ can be different from zero. Then, we can write

$$E[(a_1 - E a_1)(a_2 - E a_2) \ldots (a_s - E a_s)] = 0,$$

which holds by the definition of freeness.

If we expand this expression and move all terms except $E(a_1a_2 \ldots a_s)$ on the right, then we will obtain the expression for $E(a_1a_2 \ldots a_s)$ in terms of the expectations of shorter products.

For example,

$$E(\text{xyxy}) = E(x^2)E(y^2) - E(x^2)(E(x))^2 = E(x^2)E(y^2) - \text{Var}(x)\text{Var}(y),$$

where we defined variance of a variable $x$ by the formula $\text{Var}(x) := E(x^2) - (E(x))^2$.

In order to see the difference between the concepts of independence and freeness, suppose that $x$ and $y$ are independent. Then we can compute $E(x\text{xy}y) = E(x^2)E(y^2)$. Hence, these variables can be free if and only if $\text{Var}(x)$ or $\text{Var}(y)$. 
If the expectation function is faithful, then this implies that one of these variables must be constant.

Example

Let $\mathcal{A} = \mathbb{C}F(x, y)$ be the group algebra of the free group with two generators $x$ and $y$, and let $(\mathcal{A}, E)$ be the corresponding probability space with expectation $E$ defined as in Example 4 on page 3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subalgebras generated by $\{x, x^{-1}\}$ and $\{y, y^{-1}\}$, respectively. Then subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are free.

Indeed, let us compute, for example,

$$E \left[ \sum_k a_{k,1}x^k \sum_k a_{k,2}y^k \cdots \sum_k a_{k,s}y^k \right],$$

where $a_{0,i} = 0$ for all $i \leq s$. (That is, we assume that the coefficient before the identity group element is zero for all sums.)

It is easy to check by induction that the length of the terms in the expansion of the product is greater or equal than $s$. Hence, the coefficient before the identity element is 0 and therefore, the expectation is 0.

This means that the condition of the definition of freeness is satisfied.

Exercise 3.3. Let $\xi$ and $\eta$ are two orthogonal vectors in a Hilbert space $H$ and let $\mathcal{A} = \mathbb{C} \langle a(\xi), a^*(\xi), a(\eta), a^*(\eta) \rangle$ be the algebra generated by their creation and annihilation operators. Suppose that $(\mathcal{A}, E)$ is the probability space with the expectation $E$ as defined in Example 5 on page 3. Let subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ be defined as $\mathcal{A}_1 = \mathbb{C} \langle a(\xi), a^*(\xi) \rangle$ and $\mathcal{A}_2 = \mathbb{C} \langle a(\eta), a^*(\eta) \rangle$. Check that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free.
4. LECTURE 4. ASYMPTOTIC FREENESS OF RANDOM MATRICES

Let $A^{(N)}$ be a sequence of $N$-by-$N$ independent Hermitian Gaussian random matrices. This means that entries of a matrix $A^{(N)}$ are complex Gaussian random variables with zero expectation and the following covariance structure:

$$\langle a_{ij}a_{kl} \rangle = \delta_{il}\delta_{jk}N^{-1}. \quad (6)$$

Here $a_{ij}$ are entries of the matrix $A^{(N)}$ and we write $\langle a \rangle$ to denote the expectation of the random variable $a$.

Let $D^{(N)}$ be a sequence of $N$-by-$N$ (non-random) matrices. (These matrices are not necessarily Hermitian.) We think about these matrices as non-commutative random variables and we assume that $D^{(N)}$ and $A^{(N)}$ belong to the same non-commutative probability space.

Suppose that $D^{(N)}$ converges in distribution to a non-commutative random variable $d$. Here the convergence in distribution is understood as in Definition 2.1.

**Theorem 4.1.** The sequence of $(A^{(N)}, D^{(N)})$ converges in distribution to $(s, d)$ where $s$ has the semicircle distribution and $s$ and $d$ are free.

This phenomenon of the convergence of random matrices to free random variables is often called asymptotic freeness of random matrices. It is not really necessary that $D^{(N)}$ be non-random. They can be random but in this case they must be independent of the Gaussian matrices $A^{(N)}$.

**Proof:** Let $D^1, D^2, \ldots, D^n$ denote arbitrary polynomials of the matrix $D^{(N)}$, and let us for conciseness omit the superscript in $A^{(N)}$. We need to compute $E(A D^1 A D^2 \ldots A D^n)$. Note that $D^1$ can be equal to the identity operator and therefore the formula for $E(A D^1 A D^2 \ldots A D^n)$ will cover the expectations of the products which include powers of $A$.

We are going to show that

$$\langle \text{tr} (A D^1 A D^2 \ldots A D^n) \rangle - \sum_{\pi \in NP^2(n)} \text{tr}_{\pi\gamma}(D^1, D^2, \ldots, D^n) \to 0 \quad (7)$$

as $N$ approaches infinity.

Here $\pi$ stands for a permutation of the set $\{1, 2, \ldots, n\}$. The notation $\pi \in NP^2(n)$ means that $\pi$ is a “non-crossing pairing”. This means that (i) $\pi$ is a product of disjoint transpositions with no fixed points, and (ii) $\pi$ is non-crossing. The first requirement implies, of course, that $n$ must be even. The meaning of the second requirement is that there are no transpositions $(ij)$ and $(kl)$ in the product such that $i < k < j < l$.

Next,

$$\gamma := (12 \ldots n), \quad (8)$$

and $\pi\gamma$ is the product of permutations $\pi$ and $\gamma$.

Finally, $\text{tr}_{\pi\gamma}(X^1, \ldots, X^n)$ denotes the “cyclic trace” of matrices $X^1, \ldots, X^n$.

Let the substitution $\pi\gamma$ equals the product of $s$ cycles $c_i$, $i = 1, \ldots, s$, and let the
elements of cycle $c_i$ be denoted $c_{i,1}, c_{i,2}, \ldots, c_{i,k_i}$. Then
\[
\text{tr}_{\pi\gamma} (X^1, \ldots, X^n) = \prod_{i=1}^{s} \text{tr} (X^{c_{i,1}} X^{c_{i,2}} \ldots X^{c_{i,k_i}}) = N^{-s} \prod_{i=1}^{s} \text{Tr} (X^{c_{i,1}} X^{c_{i,2}} \ldots X^{c_{i,k_i}}).
\]

For example, if $n = 6$ and $\pi = (14)(23)(56)$, then $\pi\gamma = (13)(2)(46)(5)$, and
\[
\text{tr}_{\pi\gamma} (X^1, \ldots, X^6) = N^{-4} \text{Tr} (X^1 X^3) \text{Tr} (X^2) \text{Tr} (X^4 X^6) \text{Tr} (X^5).
\]

After we prove formula (7), we will refer to a corresponding formula for semicircle variables, which says that if $s$ and $\{d^1, d^2, \ldots, d^N\}$ are free, and if $s$ is a semicircle variable, then
\[
E (sd^1 sd^2 \ldots sd^n) = \sum_{\pi \in NP_2(n)} E_{\pi\gamma} (d^1, d^2, \ldots, d^n),
\]
where $E_{\pi\gamma}$ is defined similarly to $\text{tr}_{\pi\gamma}$. This formula will be proved later in the course.

A comparison of formulas (7) and (9) shows that the joint moments of matrices $A^{(N)}$ and $D^{(N)}$ converge to the corresponding joint moments of random variables $s$ and $d$, and this is exactly what we are required to prove in order to establish validity of Theorem 4.1.

Let us expand the left-hand side of (7):
\[
\langle \text{tr} (AD^1 AD^2 \ldots AD^n) \rangle = \frac{1}{N} \sum_{i_1 j_1 \ldots} \langle a_{i_1 j_1} d_{j_1 i_2} a_{i_2 j_2} \ldots d_{j_n i_1}^n \rangle.
\]

Since $d$ variables are not random, we can take them outside the average signs. In addition, we can use the Wick formula in order to compute the expectations of the products of Gaussian variables. As a result, we obtain the following expression:
\[
= \frac{1}{N} \sum_{i_1 j_1 \ldots} \sum_{\pi \in P_2(n)} \prod_{s=1}^{n} \langle a_{i_s j_s} a_{i_{\pi(s)} j_{\pi(s)}} \rangle^{1/2} d_{j_s i_{s+1}}^n.
\]

Here, $P_2(n)$ is the set of all possible pairings of $n$ elements. (In particular, this set is empty if $n$ is odd.) The power $1/2$ in this formula is needed to avoid double counting. Also, by convention $j_{n+1} := j_1$.

For convenience, we identify pairings with transpositions. So, by definition, $\pi(s)$ is the second element in the same block of pairing $\pi$ that contain $s$.

Recall our assumption about the covariance structure of entries, (6). It implies that for all non-zero terms in this sum, we have $i_s = j_{\pi(s)}$ and $j_s = i_{\pi(s)}$. This allows us to express everything in terms of $j$-indices. Namely, we can write the
previous expression as

\[
\frac{1}{N^{n/2+1}} \sum_{j_1, j_2, \ldots, j_n} \sum_{\pi \in P_2(n)} \prod_{s=1}^n d_{j_s j_{\pi(s)}}^s.
\]

If we use the permutation \( \gamma \) defined in (8), then we find the following exact formula:

\[
\langle \text{tr} \left( A D^1 A D^2 \ldots A D^n \right) \rangle = \frac{1}{N^{n/2+1}} \sum_{j_1, j_2, \ldots, j_n} \sum_{\pi \in P_2(n)} \prod_{s=1}^n d_{j_s j_{\pi(s)}}^s \tag{13}
\]

\[
= \sum_{\pi \in P_2(n)} N^{} \#(\pi \gamma) - n/2 - 1 \text{tr}_{\pi \gamma} (D^1, \ldots, D^n).
\]

Here \#(\pi \gamma) denotes the number of cycles (including trivial) in the permutation \( \pi \gamma \).

By assumption, the normalized traces of the polynomials of variables \( D^i \) converge to certain limits. Hence, in order to find the asymptotic behavior of the trace we need to find those pairings \( \pi \) for which \#(\pi \gamma) takes the maximal value.

**Lemma 4.2.** Let \( \gamma = (1, 2, \ldots, 2m) \) and let \( \pi \) be a permutation corresponding to a pairing of the set \( \{1, 2, \ldots, 2m\} \). Then \#(\pi \gamma) \leq m + 1, and the equality is achieved if and only if \( \pi \) is non-crossing.

Indeed, \( \gamma \) has just one cycle, and a multiplication by a transposition can increase the number of cycles by no more than 1. Hence, \#(\pi \gamma) \leq m + 1.

For non-crossing pairings \( \pi \), we can show that the number of cycles in \( \pi \gamma \) is \( m + 1 \) by induction. Indeed, let \( (ij) \) is one of the outer-most pairs of \( \pi \), that is, there is no pair \( (i'j') \) such that \( i' < i \) and \( j' > j \). If we compute \( (ij) \gamma \), then we will get two cycles: \( (1, \ldots, i - 1, j, \ldots, 2m) \) and \( (i, \ldots, j - 1) \). The pairs that are outside of \( (ij) \) will operate on the first cycle, and the pairs that are inside of \( (ij) \) will operate on the second cycle. By induction hypothesis the multiplication by these transpositions will always increase the number of cycles by one, hence the total number of cycles will be \( m + 1 \).

In contrast, if there is a crossing and the pairing \( \pi \) contains \( (ij) (kl) \) with \( i < k < j < l \), then it is easy to check that \( (ij) (kl) \gamma \) is a cycle. Multiplication by remaining transpositions can increase the number of cycles by \( m - 2 \) at most. Hence, \#(\pi \gamma) \leq m - 1 in this case. This completes the proof of Lemma 4.2.

Lemma 4.2 shows that the right-hand side in (13) converges to the sum over non-crossing pairings only, and therefore,

\[
\langle \text{tr} \left( A D^1 A D^2 \ldots A D^n \right) \rangle = \sum_{\pi \in \mathcal{N} P_2(n)} \text{tr}_{\pi \gamma} (D^1, \ldots, D^n).
\]

This completes the proof of the theorem.

In order to see better what is going on in the proof of the theorem, let us represent the sum in (10) by an oriented polygon with labeled vertices. See Figure 11.
The vertex labels can take values from 1 to $N$. However, not all labellings survive after we take expectations.

Indeed, the Wick formula allows us to compute the expectation as a sum over pairings. Each pairing can be thought of as a gluing of the polygon and this gluing and the assumed covariance structure forces some of the labels to be identified. For example presented in Figure 11, we have,

\[\begin{align*}
i_1 &= j_3, \quad i_2 = j_4, \\
i_3 &= j_1, \quad i_4 = j_2.
\end{align*}\]

Next, it turns out that arcs that correspond to $D$-variables form cycles and the number of these cycles equals the number of cycles in the permutations $\pi\gamma$.

Let us ignore these cycles for a second and think about them as vertices that remain distinct after we glued the polygon. Then we get a closed surface and the Euler characteristic of this surface can be computed as

\[\chi = F - E + V,\]

where $F$, $E$, and $V$ are the number of faces, edges, and vertices in the map that we results from the edges and vertices of the polygon after the gluing. Clearly, $F = 1$ and $E = n/2$. The number of distinct vertices $V = \# (\pi\gamma)$. Hence, we have

\[\chi = 1 - n/2 + \# (\pi\gamma).\]

Recall that the genus of a surface $g$ is related to its Euler characteristic by the formula $2g = 2 - \chi$. Hence, we have

\[2g = 1 + n/2 - \# (\pi\gamma).\]

This means that the exact formula for the trace (13) can be written in the following way:

\[\langle \text{tr} \left( AD^1 AD^2 \ldots AD^n \right) \rangle = \sum_{\pi \in P_2(n)} \frac{1}{N-2g(\pi)} \text{tr}_{\pi\gamma} \left( D^1, \ldots, D^n \right),\]
where \( g(\pi) \) denotes the genus of the closed surface constructed by gluing the \( n \)-polygon according to the pairing \( \pi \).

\[
\begin{align*}
\langle \text{tr} (A^n) \rangle &= \sum_g k_n(g) N^{-2g},
\end{align*}
\]

where \( k_n(g) \) is the number of distinct gluings of a polygone with \( n \) sides that induce the surface of genus \( g \). For example, if \( n = 2 \), then

\[
\langle \text{tr} (A^2) \rangle = 1;
\]

if \( n = 4 \), then

\[
\langle \text{tr} (A^4) \rangle = 2 + N^{-2};
\]

if \( n = 6 \), then

\[
\langle \text{tr} (A^4) \rangle = 5 + 10N^{-2},
\]

and so on.

It is clear from the proof of the theorem that instead of one matrix \( D_N \), we can use several matrices \( D_N^{(1)}, \ldots, D_N^{(m)} \), provided that they converge in distribution to some variables \( d_1, \ldots, d_m \) as \( N \to \infty \).

Moreover, instead of non-random matrices \( D \), we can use random matrices which are independent of Gaussian matrices \( A_N \).

Finally, we can use independent Gaussian matrices \( A_N^{(i)} \) as some of these \( D_N^{(i)} \). This should convince the reader that the following theorem is true.
Theorem 4.3. Let $A_N^{(1)}, \ldots, A_N^{(p)}$ be independent Hermitian Gaussian random matrices and let $D_N^{(1)}, \ldots, D_N^{(r)}$ be random matrices which are independent of $A$ matrices. Assume that the $r$-tuple $D_N^{(1)}, \ldots, D_N^{(r)}$ converge in distribution to an $r$-tuple $(d_1, \ldots, d_r)$. Then $A_N^{(1)}, \ldots, A_N^{(p)}, D_N^{(1)}, \ldots, D_N^{(r)}$ converge in distribution to $(s_1, \ldots, s_p, d_1, \ldots, d_p)$ where $s_i$ are semicircle random variables and $s_i$ is free from $\{s_1, \ldots, s_p, d_1, \ldots, d_p\} \setminus \{s_i\}$.

As a next step, we can consider $C_N := (A_N + iB_N) / \sqrt{2}$, where $A_N$ and $B_N$ are two independent Hermitian Gaussian random matrices. The entries of this matrix are complex Gaussian variables and the different entries are independent.

We say that a random variable $c$ has the circle distribution if it is equivalent to $(s_1 + is_2) / \sqrt{2}$ where $s_1$ and $s_2$ are two free semicircle variables.

By an application of the previous theorem we can get the following statement.

Let $D_N$ be random matrices independent of $C_N$ and let $D_N$ converge in $*$-distribution to $d$. Then $(C_N, D_N)$ converge in $*$-distribution to $(c, d)$ where $c$ has the circle distribution.

As the next step we consider the polar decomposition $C_N = U_N T_N$ where $U_N$ is unitary and $T_N$ is Hermitian and positive.

It is possible to show that $U_N$ has the Haar distribution on the unitary group $U(N)$. Matrix $U_N$ can be approximated by polynomials in $C_N$ and $C_N^*$. It turns out that this can be used to prove the following result.

Theorem 4.4. Let $U_N^{(1)}, \ldots, U_N^{(p)}$ be independent unitary matrices with Haar distribution and let $D_N^{(1)}, \ldots, D_N^{(r)}$ be random matrices which are independent of $U$ matrices. Assume that the $r$-tuple $D_N^{(1)}, \ldots, D_N^{(r)}$ converge in $*$-distribution to an $r$-tuple $(d_1, \ldots, d_r)$. Then $U_N^{(1)}, \ldots, U_N^{(p)}, D_N^{(1)}, \ldots, D_N^{(r)}$ converge in $*$-distribution to $(u_1, \ldots, u_p, d_1, \ldots, d_p)$ where $u_i$ are random variables with Haar unitary distribution and $u_i$ is $*$-free from $\{u_1, \ldots, u_p, d_1, \ldots, d_p\} \setminus \{u_i\}$.

This is somewhat roundabout way to prove the asymptotic freeness of Haar unitary matrices, and in the next lecture we will consider a more direct approach.
5. Asymptotic Freeness of Haar Unitary Matrices

We are going to prove the following theorem.

**Theorem 5.1.** Let $A_N$ and $B_N$ be $N$-by-$N$ Hermitian matrices that converge in distribution to the pair $\{a, b\}$. Let $U_N$ be a sequence of $N$-by-$N$ independent random unitary matrices that have the Haar distribution on the unitary group $U(N)$. Then $A_N$ and $U_NB_NU_N^*$ converge in distribution to random variables $a$ and $\tilde{b}$, where $\tilde{b}$ has the same distribution as $b$, and $a$ and $\tilde{b}$ are free.

Theorem 5.1 essentially says that if we take two large random matrices $A_N$ and $B_N$ and if we conjugate one of them by a uniformly random unitary transformation $U_N$, then the resulting pair of matrices $A_N$ and $U_NB_NU_N^*$ will be approximately free.

As a slogan, this can be put as follows:

"Two large random matrices in the general position are asymptotically free!"

The proof of Theorem 5.1 is similar to the proof of Theorem 4.1. We are going to derive an asymptotic formula for the following expected trace:

$$\left\langle \text{tr} \left( A^{(1)}_N U_N B^{(1)}_N U_N^* A^{(2)}_N \ldots U_N B^{(n)}_N U_N^* \right) \right\rangle. \quad (14)$$

Here $A^{(i)}_N$ and $B^{(i)}_N$ denote polynomials of matrices $A_N$ and $B_N$, respectively.

The main difficulty in the derivation of a formula for (14) is that we need an analogue of the Wick formula for the expectations of products of the elements of a Haar-distributed unitary matrix.

Such formulas are called Weingarten formulas (see [22]).

Namely, note that the distribution of matrix $U_N$ is invariant if we multiply it by $e^{i\alpha I}$. This implies that the expectation of $u_{i_1 j_1} \ldots u_{i_p j_p} \overline{u}_{i_1' j_1'} \ldots \overline{u}_{i_q j_q'}$ is zero if $p \neq q$. If $p = q$, then we have the following formula:

$$\left\langle u_{i_1 j_1} \ldots u_{i_q j_q} \overline{u}_{i_1' j_1'} \ldots \overline{u}_{i_q j_q'} \right\rangle = \sum_{\alpha, \beta} \delta_{i_1 \beta(1)} \ldots \delta_{i_q \beta(1)} \ldots \delta_{j_1 \alpha(1)} \ldots \delta_{j_q \alpha(1)} \cdot Wg\left( N, \beta \alpha^{-1} \right). \quad (15)$$

Here the sum is over permutations $\alpha$ and $\beta$ of the set $\{1, 2, \ldots, q\}$. The coefficient $Wg\left( N, \beta \alpha^{-1} \right)$ is called the Weingarten function. We can define it by the following equality:

$$Wg\left( N, \alpha \right) := E \left( u_{i_1 \ldots u_{i_q} \overline{u}_{i_1(1)} \ldots \overline{u}_{i_q(1)}} \right).$$

There is a beautiful explicit formula for $Wg\left( N, \alpha \right)$ due to B. Collins (see [7]). Namely, let $N \geq q$. Then,

$$Wg\left( N, \alpha \right) = \frac{1}{(q!)^2} \sum_{\lambda \vdash q} \chi^\lambda(\alpha) \chi^\lambda(\alpha) s_{\lambda, N}(id).$$

Here, the sum is over all partitions of $q$, $\chi^\lambda$ is the character of the irreducible representation of the symmetric group $S_q$ that corresponds to the partition $\lambda$, and...
$s_{\lambda, N}$ is the character of the irreducible representation of the unitary group $U(N)$, that corresponds to the partition $\lambda$.

It is clear from this formula that $W_{g}(N, \alpha)$ depends only on the conjugacy class of the permutation $\alpha$.

**Exercise 5.2.** *(from [7])* Let us use the notation for the partition class as the second argument, so, for example, the identity permutation in $S_n$ corresponds to $1^n$. Check the following formulas:

\[
\begin{align*}
W_{g}(N, 1) &= \frac{1}{N}, \\
W_{g}(N, 1^2) &= \frac{1}{N^2 - 1}, \\
W_{g}(N, 1^3) &= \frac{N^2 - 2}{N (N^2 - 1) (N^2 - 4)}, \\
W_{g}(N, 2) &= \frac{-1}{N (N^2 - 1)}, \\
W_{g}(N, 21) &= \frac{-1}{(N^2 - 1) (N^2 - 4)}, \\
W_{g}(N, 3) &= \frac{2}{N (N^2 - 1) (N^2 - 4)}.
\end{align*}
\]

For our purposes, we are more interested in the asymptotic behavior of the Weingarten function. It turns out that it is given by the following formula:

\[
W_{g}(N, \alpha) = \phi(\alpha) N^{\#(\alpha) - 2q} + O\left(N^{\#(\alpha) - 2q - 2}\right), \tag{16}
\]

where $\phi(\alpha)$ is not zero.

The explicit formula for $\phi(\alpha)$ is as follows. Let the conjugacy class for $\alpha$ be described by partition $\lambda = (\lambda_1, \ldots, \lambda_s)$. Then,

\[
\phi(\alpha) = \prod_{i=1}^{s} (-1)^{\lambda_i - 1} C_{\lambda_i},
\]

where $C_n$ denote Catalan numbers.

For example, $\phi(id) = 1$, $\phi((12)) = -1$, $\phi((123)) = 2$, $\phi((12)(34)) = 1$.

Now let us return to the proof of Theorem 5.1. By using formulas (15) and (16), it is possible to derive the following asymptotic formula for the trace:

\[
\lim_{N \to \infty} \left\langle \text{tr} \left( A^{(1)}_{N} U_{N} B^{(1)}_{N} U^{*}_{N} A^{(2)}_{N} \ldots U_{N} B^{(n)}_{N} U^{*}_{N} \right) \right\rangle = \sum_{\alpha, \beta \in S_n} E_{\alpha}(a_1, \ldots, a_n) E_{\beta-1, \gamma}(b_1, \ldots, b_n) \phi(\alpha^{-1} \beta) \tag{17}
\]
In this formula \(a_i\) and \(b_j\) are the polynomials of variables \(a\) and \(b\), respectively, that correspond to the polynomials \(A_N^{(i)}\) and \(B_N^{(j)}\) of matrices \(A_N\) and \(B_N\). The permutation \(\gamma\) equals by definition \((1, 2, \ldots, n)\), and \(|\sigma|\) denotes the length of transposition \(\sigma\), that is, the minimal number of transpositions which is needed to represent \(\sigma\) as a product of transpositions.

Note that the length function defines a distance on the set of all permutations: \(d(\sigma, \rho) = d(\sigma^{-1}\rho)\) and it turns out that this distance is actually a metric. Hence we can define a notion of a geodesic: a set of permutations is a geodesic if (i) for every triple of permutations from this set, we can order them in such a way \((\rho, \sigma, \tau)\) that \(|\rho^{-1}\sigma| + |\sigma^{-1}\tau| = |\rho^{-1}\tau|\), and (ii) it is not possible to add another permutation without violating property (i).

![Figure 13](https://via.placeholder.com/150)

**Figure 13.** A geodesic from \(id\) to \(\gamma\)

We are interested in the geodesics from the identity element \(id\) to the cycle \(\gamma\). It is possible to introduce the total ordering on each of these geodesic. The permutation \(\alpha\) precedes \(\beta\) if \(\alpha\) is between \(id\) and \(\beta\). In this ordering \(id\) and \(\gamma\) are largest and the smallest element respectively.

In this terms, the condition

\[
|\alpha^{-1}\beta| + |\alpha| + |\beta^{-1}\gamma| = n - 1
\]

means that \(\alpha\) and \(\beta\) lie on a geodesic between \(id\) and \(\gamma\) and that \(\alpha\) precedes \(\beta\). This situation is shown in Figure 13.

**Lemma 5.3.** Suppose that

\[
|\alpha^{-1}\beta| + |\alpha| + |\beta^{-1}\gamma| = n - 1.
\]

Then either \(\alpha\) or \(\beta^{-1}\gamma\) has a fixed point.

Proof: The assumption implies that either \(|\alpha| \leq (n - 1)/2\), or \(|\beta^{-1}\gamma| \leq (n - 1)/2\). This means that one of these permutations can be represented as a product of less than or equal to \((n - 1)/2\) transposition. Since a transposition
moves only 2 elements, therefore the product of no more than \((n - 1)/2\) transpositions moves no more than \(n - 1\) elements. Hence, at least one element remains unmoved. QED.

Suppose now that \(\lim_{N \to \infty} \text{tr} \left( A_N^{(i)} \right) = 0\) and \(\lim_{n \to \infty} \text{tr} \left( B_N^{(i)} \right) = 0\) for all \(i\). Then, Lemma 5.3 and formula (17) imply that
\[
\lim_{N \to \infty} \left\langle \text{tr} \left( A_N^{(1)} U_N B_N^{(1)} U_N^* A_N^{(2)} U_N \cdots U_N B_N^{(n)} U_N^* \right) \right\rangle = 0
= E \left( a_1 b_1 a_2 \cdots b_n \right)
\]
where \(\tilde{b}_k\) is the same polynomial as \(b_k\) except that it is applied to \(\tilde{b}\), which is free of \(a\), and not to \(b\). This implies that even if the traces of \(A_N^{(i)}\) and \(B_N^{(i)}\) have non-zero limits, it is still true that
\[
\lim_{N \to \infty} \left\langle \text{tr} \left( A_N^{(1)} U_N B_N^{(1)} U_N^* A_N^{(2)} U_N \cdots U_N B_N^{(n)} U_N^* \right) \right\rangle = E \left( a_1 \tilde{b}_1 a_2 \cdots \tilde{b}_n \right).
\]
This completes the proof of the theorem.

**Corollary 5.4.** If \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) are elements of sub-algebras \(A\) and \(B\) of a probability space with faithful tracial expectation \(E\), and if \(A\) and \(B\) are generated by free variables \(a\) and \(b\), respectively, then
\[
E \left( a_1 b_1 a_2 \cdots b_n \right) = \sum_{\alpha, \beta \in S_n, \text{ } |\alpha^{-1} \beta| + |\alpha| + |\beta^{-1} \gamma| = n-1} E_{\alpha} (a_1, \ldots, a_n) E_{\beta^{-1}} (b_1, \ldots, b_n) \phi (\alpha^{-1} \beta)
\]

Here is an example of computation that uses formula (18). Consider \(E (abab)\), where \(a\) and \(b\) are free. Then we set \(n = 2\) in formula (18), \(a_1 = a_2 = a\), \(b_1 = b_2 = b\), and we have three possibilities: (i) \(\alpha = \beta = id\), (ii) \(\alpha = \beta = \gamma = (1, 2)\), and (iii) \(\alpha = id\), \(\beta = (12)\). Note that the case when \(\alpha = (12)\) and \(\beta = id\) is impossible, because \(\alpha\) must precede \(\beta\) on the geodesic from \(id\) to \(\gamma\). To these cases, we have the correspondent summands: (i) \(E (a^2) E (b^2)\), (ii) \(E (a^2) E (b^2)\), and (iii) \(-E (a^2) E (b^2)\). Hence,
\[
E (aubu^* au bu^*) = E (a^2) E (b^2) + E(a^2)E (b^2) - E (a^2) E (b^2).
\]
It is easy to check that this coincide with the formula for free variables 5 on page 14.
6. Lecture 6. Free Products of Probability Spaces

If $G_1$ and $G_2$ are two finitely generated groups, then their free product $G_1 \ast G_2$ is the group with the combined sets of generators and relations. It is natural to call the group algebra of $G_1 \ast G_2$ as the free product of group algebras of $G_1$ and $G_2$. Clearly, $G_1$ and $G_2$ can be included in $G_1 \ast G_2$. The images of their elements generate two subalgebras, $A_1$ and $A_2$ of the group algebra $\mathbb{C} (G_1 \ast G_2)$. It is easy to see that these sub-algebras are free.

It turns out that this construction can be generalized and we can define a free product of arbitrary probability spaces.

**Theorem 6.1.** Let $(A_i, E_i)$ be non-commutative probability spaces ($i = 1, \ldots, n$). Then there exists a non-commutative probability space $(A, E)$ and unital homomorphisms $\Phi_i : A_i \to A$ such that

(i) $E \circ \Phi_i = E_i$,

(ii) $\Phi_i (A_i)$ are free in $A$.

If $(A_i, E_i)$ are $C^*$-probability spaces with faithful tracial expectation $E_i$, then $(A, E)$ can be chose to be a $C^*$-probability space with faithful tracial expectation $E$. In this case, $\Phi_i$ are norm-preserving.

I am not going to prove this theorem in complete detail. Instead, I will explain the construction and highlight some subtle points.

First of all, we build $A$ as a vector space. Let

$$A_1^0 := \{ x \in A_i | E_i (x) = 0 \}.$$  

Clearly, this is a linear space. Define

$$A := \mathbb{C} + \bigoplus_{i_1 \neq i_2} A_{i_1}^0 + \bigoplus_{i_1 \neq i_2, i_3} A_{i_1}^0 \otimes A_{i_2}^0 + \bigoplus_{i_1 \neq i_2 \neq i_3} A_{i_1}^0 \otimes A_{i_2}^0 \otimes A_{i_3}^0 + \ldots, \quad (19)$$

where it is assumed that every sum consists of a finite number of elements and where the notation $i_1 \neq i_2 \neq \ldots \neq i_n$ means that $i_1 \neq i_2$, $i_2 \neq i_3$, $i_3 \neq i_4$, and $i_{n-1} \neq i_n$. A typical element of $A$ can look like $5 + 2x \otimes y + y \otimes x \otimes w$, where the expectations of $x, y$, and $w$ are zero.

For convenience, let us also define the linear spaces $W_I$ follows: $W_0 := \mathbb{C}$, and if $I = [i_1, i_2, \ldots, i_n]$ is an alternating sequence of indices, then

$$W_I = A_{i_1}^0 \otimes A_{i_2}^0 \otimes \ldots \otimes A_{i_n}^0.$$  

Next step is to convert $A$ into an algebra. In other words, we need to define the product on the elements of $A$. Since the product must be linear, it is enough to define it on the individual summands in (19), that is, on those elements of $A$, which can be represented as tensor products. Let us define the product for two such terms $x$ and $y$. Let the tensor product for $x$ equal $x_1 \otimes a_i$, where $a_i \in A_i$, and let the tensor product for $y$ equal $a_j \otimes y_1$, where $a_j \in A_j$.

If $i \neq j$, then we can simply define the product of $x$ and $y$ as $x \otimes y$. 


In contrast, if \( i = j \), then we write \( a_ia_j = E(a_ia_j) + (a_ia_j)^0 \), where \( (a_ia_j)^0 := a_ia_j - E(a_ia_j) \in \mathcal{A}_i^0 \). Then,
\[
xy := x_1 \otimes (a_ia_j)^0 \otimes y_1 + E(a_ia_j)x_1y_1.
\]
If we use induction over the length of the tensor products, we can safely assume that \( xy \) is already defined.

The following example illustrate this procedure. Let \( x \in \mathcal{A}_1 \) and \( y \in \mathcal{A}_2 \). Then,
\[
(xy)(y \otimes x) = x \otimes (y^2 - E(y^2)) \otimes x + E(y^2)xx \\
= x \otimes (y^2 - E(y^2)) \otimes x + E(y^2)(x^2 - E(x^2)) + E(y^2)E(x^2).
\]

It is possible to verify that the product defined in this way makes \( \mathcal{A} \) into an algebra.

The involution is defined very naturally. For example,
\[
[\lambda(a_1 \otimes a_2)]^* = \overline{\lambda}(a_2^* \otimes a_1^*).
\]

Next, we need to define the expectation. We define \( E(x) = x \) if \( x \in W_I = \mathbb{C} \), and \( E(x) = 0 \), if \( x \in W_I \) with \( I \neq \emptyset \). We can extend this definition to all elements in \( \mathcal{A} \) by linearity.

In order to show that linear functional \( E \) is a state, let us show that \( E(a^*a) \geq 0 \). If \( a \in W_I \) and \( b \in W_J \) and \( I \neq J \), then \( E(a^*b) = 0 \). Hence, it is enough to check that the inequality \( E(a^*a) \geq 0 \) holds for \( a \in W_I \) for all \( I \).

Let us do the proof for a particular example. Namely, let
\[
a = \lambda a_1 \otimes a_2 + \mu b_1 \otimes b_2
\]
Then, we can calculate,
\[
E(a^*a) = |\lambda|^2 E(a_1^*a_1) E(a_2^*a_2) + \lambda \mu E(a_1^*b_1) E(a_2^*b_2) \\
+ \lambda \mu E(b_1^*a_1) E(b_2^*a_2) + |\mu|^2 E(b_1^*b_1) E(b_2^*b_2).
\]

By using the matrix notation, we can write this as follows.
\[
E(a^*a) = v^*Qv,
\]
where \( v = (\lambda, \mu)^t \), and \( Q = Q^{(1)} \ast Q^{(2)} \). Here the operation \( \ast \) means entry-by-entry multiplication of matrices, and
\[
Q^{(i)} = \begin{pmatrix}
E(a_1^*a_1) & E(b_1^*a_1) \\
E(a_2^*b_1) & E(b_2^*b_1)
\end{pmatrix}.
\]

Hence, if we want to show that \( E(a^*a) \geq 0 \), then it is enough to show the following lemma.

**Lemma 6.2.** If matrices \( Q^{(i)} \) are positive semidefinite, then \( Q = Q^{(1)} \ast Q^{(2)} \) is also positive semidefinite.

**Proof:** Write \( Q_1 = A^*A \), where \( A \) is the matrix of the same size as \( Q_1 \). This is possible because \( Q_1 \) is positive semidefinite. Then,
\[
Q_{ij} = \sum_k A_{ik}^* A_{kj} Q_{ij}^{(2)}.
\]
Therefore, we can write:

\[ v^* Q v = \sum_{i,j,k} \overline{v}_i A_{ik} A_{kj} Q^{(2)}_{ij} \]

\[ = w^* Q^{(2)} w, \]

where \( w = Av \). Since \( Q^{(2)} \) is positive semidefinite, we can conclude that \( v^* Q v \geq 0 \) for all \( v \). QED.

**Exercise 6.3.** Check that if \( E_i \) are faithful, then \( E \) is faithful.

**Exercise 6.4.** Check that if \( E_i \) are tracial, then \( E \) is tracial.

**Exercise 6.5.** Check that \( A_i = C + A_i^0 \) are free from each other.

Theorem 6.1 is a convenient tool. It shows that if we have a collection of distributions, we can always find a probability space and random variables such that these random variables are free and have the prescribed distributions. Indeed, we can realize each of the distributions in a separate probability space and then take their free product.
This lecture is about addition of free random variables. However, I would like to start with pointing out a connection between free probability and classical probability theory. The connection comes from the theory of random walks on graphs.

Let $G$ be a rooted graph, that is, a graph with one marked vertex. Let $p_{xy}$ denote a probability of transition from vertex $x$ to $y$. We can think about the collection of $p_{xy}$ as a matrix with rows and columns indexed by vertices. We call this matrix $P$ and we will assume that each row and each column have only a finite number of non-zero entries, so that a particle can move from $x$ to only a finite number of other vertices, and a particle can come to $y$ from only a finite number of other vertices. Then, the $n$-th power $P^n$ is well defined for each $n$, and the meaning of the entry $(P^n)_{xy}$ is the probability to go from $x$ to $y$ in $n$ steps.

The generating series

$$\varphi(x, y | z) := \sum_{n=1}^{\infty} (P^n)_{xy} z^n$$

is called the Green function of the random walk specified by $P$. This function encodes many of the properties of the random walk, such as transience and recurrence. Moreover, for many of the properties it is enough to know only $\varphi(z) := \varphi(e, e | z)$, where $e$ is the marked vertex of the graph (i.e., root). We will call this function Green function as well.

Let us see the connection of the Green functions to some of the concepts that we introduced above. Note that the random walk $P$ defines an operator on the functions on the graph.

$$P f(x) = \sum_{y} P_{xy} f(y).$$

Consider the linear space of square-summable functions on the graph $G$. We can introduce the structure of the Hilbert space on this space by the formula

$$\langle f, g \rangle = \sum_{x} \overline{f(x)} g(y).$$

Then, the expectation on operators can be defined by

$$E(X) = \langle \delta_e, X \delta_e \rangle,$$

where $\delta_e$ is the function that takes value 1 on the root of the graph and value 0 everywhere else. In this terms, the Green function can be written as follows:

$$\varphi(z) = \sum_{n=1}^{\infty} E(P^n) z^n$$

$$= E\left(\frac{zP}{1 - zP}\right).$$

This function is closely related to the expectation of the resolvent of $P$, that is, to the Cauchy transform of the operator $P$. 
All this motivates a question: suppose we are given a random walk on a graph. How can we calculate the associated Green function? In general, this is a difficult question. We consider the special case when the graph is the free product of two other graphs.

What is the free product of graphs? If we have several rooted graphs $G_i$, then it is possible to define their free product $G_1 \ast \ldots \ast G_n$. The set of its vertices consists of the root $e$ and the points indexed by alternating sequences of the vertices of $G_i$. The vertex $(x_1, \ldots, x_n)$ is connected to $(y_1, \ldots, y_n)$ if one of the two situations holds. The first situation is when $x_i = y_i$ for $i = 1, \ldots, n-1$, the vertices $x_n$ and $y_n$ are from the same graph, and $x_n$ is connected to $y_n$ in this graph. The second situation is when the first vertex is $(x_1, \ldots, x_{n-1})$, the second vertex is $(x_1, \ldots, x_n)$, and $x_n$ is connected to the root in its graph. One particular case of the second situation is the root $e$ and any the vertices $(x_i)$ where $x_i$ is connected to the root in its own graph.

Let $Z$ is graph with the vertices indexed by integers and the edges $(k, k+1)$. The graph of $Z \ast Z$ is shown in Figure 14. Note that $Z$ is the Cayley graph of the group of integers $\mathbb{Z}$, and the free product $Z \ast Z$ is the Cayley graph of the free product of the groups, $\mathbb{Z} \ast \mathbb{Z}$, which is simply free group with two generators. More generally, the free product of graphs can be considered as a generalization of the free product of finitely generated groups.

If $P$ is the matrix of a random walk on a rooted graph $G_1$ and $Q$ is a matrix of a random walk on $G_2$, then we can define a random walk on $G_1 \ast G_2$. Here is how. Suppose we are at the vertex $(x_1, \ldots, x_n)$ and $x_n \in G_1$. Toss a coin. If the result is heads, use the transition probabilities $P_{x_1,y}$ to determine the next vertex $(x_1, \ldots, x_{n-1}, y)$. If the result of the toss is tails, then use the transition probabilities $Q_{e,z}$ to determine the next vertex $(x_1, \ldots, x_n, z)$. (If the particle at the root of $G_1 \ast G_2$, then this rule should be slightly modified.)

Suppose that we know the Green functions for $P$ and $Q$. How can we compute the Green function of the combined random walk?

Note that we can construct the operators for random walks $P$ and $Q$, which acts on functions on $G_1 \ast G_2$. It is easy to see that these operators, $\tilde{P}$ and $\tilde{Q}$, are free,
and that the operator of the combined walk is \( \left( \tilde{P} + Q \right) / 2 \). Hence, we need to compute the Cauchy transform of \( \tilde{P} + Q \) in terms of the Cauchy transforms of \( \tilde{P} \) and \( Q \).

Curiously, this question was posed and answered independently from the development of free probability theory and at the approximately the same time as the corresponding question was solved in the context of free probability theory. The history of this discovery is involved and for the details, the reader is referred to the book by Woess.

Recall that the Cauchy transform of a random variable \( X \) is the expectation of its resolvent:
\[
G_X (z) = E \left[ \frac{1}{z - X} \right].
\]

If \( X \) is bounded, then \( G_X (z) \) is analytic for \( z \geq \|X\| \) and we can write it as the following convergent series:
\[
G_X (z) = \frac{1}{z} + \sum_{k=1}^{\infty} E \left( X^k \right) \frac{1}{z^{k+1}}.
\]
If we change variable \( t = z^{-1} \), then we get
\[
G_X (t) = t + \sum_{k=1}^{\infty} E \left( X^k \right) t^{k+1}.
\]
Clearly, this function is holomorphic near \( t = 0 \), and \( G' \) is non-zero at \( t = 0 \). This implies that \( G_X (t) \) is invertible near \( t = 0 \) and the expansion is
\[
t = u + \sum_{k=1}^{\infty} a_k u^{k+1}.
\]
Then, we can make the substitution \( z = 1/t \), and get
\[
z(u) = \frac{1}{u} + \sum_{k=0}^{\infty} c_k u^k.
\]
This function is called the \( K \)-function of \( X \) and we will denote it as \( K_X (u) \).

A slightly modified function is called the \( R \)-transform of \( X \):
\[
R_X (u) = K_X (u) - \frac{1}{u}.
\]

It is possible to generalize the concept of freeness to some unbounded variables. We say that an unbounded variable \( X \) is affiliated with a probability space \( (A, E) \) if all its truncations belong to \( A \). Two unbounded variables affiliated with \( (A, E) \) are said to be free if all their truncations are free. In the following we will assume that the unbounded random variables are affiliated with a probability space.

It is possible to show that for an unbounded self-adjoint variable \( X \) the Cauchy transform is well-defined for \( z \) in the area \( \Gamma_{\alpha,m} = \{ z | \text{Im} z > \alpha \text{Re} z, \text{Im} z > m \} \) for appropriately chosen \( \alpha \) and \( m \). Moreover, for sufficiently large \( m \) the Cauchy transform is invertible in such an area. This allows to define the \( K \)-function and
$R$-transform. We will not pursue this in detail. However, we would like to mention that this is key to the theory of free infinitely-divisible distributions.

**Exercise 7.1.** Let $X$ be a self-adjoint random variable with the semicircle distribution (defined on p. 9). Show that $R_X(z) = z$.

**Exercise 7.2.** Let $X$ be a self-adjoint random variable with the arcsine distribution (defined on p. 7). Show that

$$K_X(z) = \frac{1}{z} \sqrt{1 + 4z^2}.$$

**Exercise 7.3.** Let $X$ be a self-adjoint random variable with the distribution $\frac{1}{2} \{\delta_{-1} + \delta_1\}$. Show that $K_X(z) = \frac{1 + \sqrt{1 + 4z^2}}{2z}$.

**Exercise 7.4.** Let $X$ be an unbounded self-adjoint random variable with the Cauchy distribution, that is, let its spectral probability distribution have the density:

$$p(x) = \frac{1}{\pi} \frac{1}{1 + x^2}.$$ 

Show that

$$G_X(z) = \frac{1}{z + i}$$

for $z \in \mathbb{C}^+$ and that

$$R_X(z) = -i.$$

**Theorem 7.5.** If $X$ and $Y$ are bounded and free random variables, then

$$R_{X+Y}(u) = R_X(u) + R_Y(u).$$

This theorem was originally proved in [21]. We present the proof discovered in [10].

**Proof:** The functions $R_X(z)$ and $R_Y(z)$ determine the Cauchy transforms of $X$ and $Y$. Hence, they determine the moments of these two variables. If $X$ and $Y$ are assumed to be free, the moments of $X$ and $Y$ determine the moments of $X + Y$ and hence they determine the function $R_{X+Y}(z)$. This shows that the function $R_{X+Y}(z)$ is determined by functions $R_X(z)$ and $R_Y(z)$. Our strategy will be to find a convenient probability space and two variables $X$ and $Y$ in this probability space that have the $R$-transforms $R_X(z)$ and $R_Y(z)$. We will be able to compute $R_{X+Y}(z)$ for these particular $X$ and $Y$ and we will find that it equals $R_X(z) + R_Y(z)$. This will show that this formula holds in general.

We will use the full Fock space $F(H)$ as our probability space and we will assume that $\dim H \geq 2$. (See p. 11 for details.) Let $a$ and $a^*$ be operators of creation and annihilation, respectively, corresponding to a vector $e \in H$. Define

$$X = a + \sum_{k=0}^{\infty} c_k (a^*)^k.$$
The symbol of operator $X$ is defined as the function

$$K_X(z) := \frac{1}{z} + \sum_{k=0}^{\infty} c_k z^k.$$ 

**Lemma 7.6.** $K_X(z)$ is the $K$-function of the operator $X$.

**Proof of Lemma:** We need to show that

$$E \left[ \frac{1}{(K_X(z) - X)^{-1}} \right] = z.$$ 

Let us calculate how $K_X(z) - X$ acts on a vector

$$w = \Omega + \sum_{n=1}^{\infty} z^n \, e^{\otimes n},$$

where $\Omega$ is the vacuum vector.

Note that

$$a^* w = z w, \quad a w = \frac{w - \Omega}{z}.$$ 

Hence,

$$X w = K_X(z) \, w - \frac{\Omega}{z},$$

and

$$(K_X(z) - X)^{-1} \, \Omega = zw.$$ 

(The inverse of $K_X(z) - X$ exists for all sufficiently small $z$ because for all sufficiently small $z$, $K_X(z)$ is larger than $\|X\|$ due to a pole at $z = 0$.)

Therefore,

$$E \left[ (K_X(z) - X)^{-1} \right] = \left< \Omega, (K_X(z) - X)^{-1} \, \Omega \right> = \left< \Omega, zw \right> = z.$$ 

QED.

Now, let $(a, a^*)$ and $(b, b^*)$ be creation and annihilation operators corresponding to two orthogonal vectors $e$ and $f$ in $H$. We know that orthogonality of vectors implies that the subalgebras generated by $(a, a^*)$ and $(b, b^*)$ are free. Define

$$X = a + \sum_{k=0}^{\infty} x_k (a^*)^k,$$

and

$$Y = b + \sum_{k=0}^{\infty} y_k (b^*)^k.$$ 

The random variables $X$ and $Y$ are free. We need to show that

$$E \left[ \left( K_X(z) + K_Y(z) - \frac{1}{z} - (X + Y) \right)^{-1} \right] = z.$$
Let us define vector $w$ as follows:

$$w = \Omega + \sum_{n=1}^{\infty} z^n (e + f)^{\otimes n}.$$ 

Then,

$$a^*w = zw, \quad b^*w = zw, \quad aw = e \otimes w, \quad bw = f \otimes w, \quad (a + b)w = (e + f) \otimes w = w - \Omega.$$ 

Hence,

$$(X + Y)w = \left( K_X (z) + K_Y (z) - \frac{1}{z} \right) w - \frac{\Omega}{z},$$

and

$$\left[ K_X (z) + K_Y (z) - \frac{1}{z} - (X + Y) \right]^{-1} \Omega = zw.$$ 

Therefore,

$$E \left[ \left( K_X (z) + K_Y (z) - \frac{1}{z} - (X + Y) \right)^{-1} \right] = z.$$ 

This implies that

$$K_{X+Y} (z) = K_X (z) + K_Y (z) - \frac{1}{z}$$

and

$$R_{X+Y} (z) = R_X (z) + R_Y (z).$$ 

This completes the proof of the theorem.

**Exercise 7.7.** Let $X$ and $Y$ be two free semicircle variables with zero mean and variances $a$ and $b$, respectively. Show that $X + Y$ is a semicircle variable with zero mean and variance $a + b$.

**Exercise 7.8.** Let $X$ and $Y$ be two identically distributed free random variables with the distribution $\frac{1}{2} \{ \delta_{-1} + \delta_1 \}$. Show that $X + Y$ has the arcsine distribution.

Let $\mu$ and $\nu$ be two compactly-supported probability measures. Then we can find two random variables $X$ and $Y$ that have $\mu$ and $\nu$ as their spectral probability measures. Moreover, if we use the free product construction, we can ensure that $X$ and $Y$ are free. By Theorem 7.5, the Cauchy transform of $X + Y$ depends only on measures $\mu$ and $\nu$. This Cauchy transform corresponds to the spectral probability measure of $X + Y$. We will call this measure the free additive convolution of measures $\mu$ and $\nu$ and will denote it as $\mu \boxplus \nu$.

This definition can be generalized to probability measures with the unbounded support by the method of truncations. It is possible to prove that the convolutions
of truncated measures converge and the limit is called the free additive convolution of the original measures. The details are somewhat technical and we omit them. The interested reader can find them in the original paper \[1\].

Recall that the classical convolution of measures $\mu$ and $\nu$ is the distribution measure of $X + Y$, when $X$ and $Y$ are independent. We will denote it as $\mu * \nu$.

**Exercise 7.9.** Let $\mu = \delta_x$. Show that $\mu \boxplus \nu$ is a shift of measure $\nu$ by $x$. That is, $\mu \boxplus \nu (A) = \nu (A - x)$.

For the situation in this exercise, the free additive convolution coincides with the classical convolution:

$$\delta_x \boxplus \nu = \delta_x * \nu.$$  

However, in general the free additive convolution is very different from the classical. One distinctive feature of the free additive convolution is its non-linearity. Recall that the linear combinations of measures $\mu$ and $\nu$ is defined as follows:

$$(t \mu + s \nu) (A) = t \mu (A) + s \nu (A)$$

for every measurable set $A$.

The classical convolution is linear with respect to taking convex combinations of probability measures. That is, let $t + s = 1$, then it is true that

$$(t \mu_1 + s \mu_2) * \nu = t \mu_1 * \nu + s \mu_2 * \nu.$$ 

Since we know how to compute the convolution of a measure with a single atom, this property allows us to compute the convolution of arbitrary discrete measures which is supported on a finite number of points. In particular, such a convolution is again discrete and supported on a finite number of points.

This linearity property does not hold with respect to free additive convolution:

$$(t \mu_1 + s \mu_2) \boxplus \nu \neq t \mu_1 \boxplus \nu \boxplus s \mu_2 \boxplus \nu.$$ 

For example, let $\mu = \nu = \frac{1}{2} \{ \delta_{-1} + \delta_1 \}$. By the result in Exercise \[7.8\], $\mu \boxplus \nu$ has the arcsine distribution which is absolutely continuous on the interval $(-2, 2)$.

On the other hand, the free additive convolution has a continuity property which is similar to the continuity property of the classical convolution. Namely, let $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$ in distribution. Then,

$$\mu_n \boxplus \nu_n \rightarrow \mu \boxplus \nu$$  

in distribution. We will leave this fact without proof and refer the interested reader to \[1\]. Note that if $\mu_n$ and $\nu_n$ have unbounded support, then it is possible to give some condition which ensure that the convergence in \[20\] is tight.
The law of addition formulated in Theorem 7.5 allows us to prove the following limit theorem for sums of free random variables. This theorem is an analogue of the Central Limit Theorem for sums of independent identically-distributed variables.

**Theorem 8.1.** Let $X_1, X_2, \ldots$ be a sequence of identically-distributed bounded self-adjoint random variables. Assume that $E(X_i) = 0$, $E(X_i^2) = 1$, and that $X_i$ are free. Define $S_n = X_1 + \ldots + X_n$. Then the sequence $S_n/\sqrt{n}$ converges in distribution to the standard semicircle random variable.

**Proof:** Let the Taylor series for the $R$-transform of a variable $X_i$ be

$$R(z) = z + c_2 z^2 + c_3 z^3 + \ldots$$

Note that $c_0 = 0$ and $c_1 = 1$ because of the assumption that $E(X_i) = 0$ and $E(X_i^2) = 1$.

Then, the $R$-transform of $S_n$ is

$$R_n(z) = nR(z),$$

and it is easy to check that the $R$-transform of $S_n/\sqrt{n}$ is

$$R_{S_n/\sqrt{n}}(z) = \frac{1}{\sqrt{n}} R_n \left( \frac{z}{\sqrt{n}} \right) = \sqrt{n} R \left( \frac{z}{\sqrt{n}} \right) = z + \frac{c_2}{\sqrt{n}} z^2 + \frac{c_3}{n} z^3 + \ldots$$

Hence, all the coefficients in the Taylor expansion of $R_{S_n/\sqrt{n}}(z)$ converge to zero, except the first one which is equal to zero.

The moments of the variable $S_n/\sqrt{n}$ are the coefficients in the power series expansion of the Cauchy transform, and it is easy to see that these coefficients can be expressed as the polynomials of the power series coefficients of $R_{S_n/\sqrt{n}}(z)$. It follows that the moments of $S_n/\sqrt{n}$ converge to the corresponding moments of the random variable with the $R$-transform $R(z) = z$, that is, to the moments of the standard semicircle variable. Hence, the sequence $S_n/\sqrt{n}$ converges in distribution to the standard semicircle random variable. QED.

This theorem leads to many questions which are parallel to the questions of classical probability theory.

1) Is it possible to prove the Central Limit Theorem for unbounded and not necessarily identically distributed variables?

2) Is it possible to develop a theory of infinitely divisible distributions?

3) What can be said about the large deviations of the sums $S_n$?

These questions were deeply studied in the literature and we will present only very limited coverage of these topics. One excellent example of the research in this area can be found in [3].
In relation with large deviations, the following inequality from [21] is pertinent:

\[ \left\| \sum_{i=1}^{r} X_i \right\| \leq \max_{1 \leq i \leq r} \left\| X_i \right\| + \sqrt{\sum_{i=1}^{r} \text{Var} (X_i)}. \]

If \( X_i \) are free identically-distributed bounded variables with zero mean and unit variance, then this inequality implies that

\[ \left\| S_n / \sqrt{n} \right\| \leq 1 + \frac{\left\| X_i \right\|}{\sqrt{n}}. \]

That is, for all \( \varepsilon > 0 \), the distribution of \( S_n / \sqrt{n} \) is contained inside the interval \( [-1 - \varepsilon, 1 + \varepsilon] \) for all sufficiently large \( n \). This, and unusually high smoothness of \( S_n / \sqrt{n} \), were called the superconvergence of sums of free variables in [2].

For unbounded variables, the question about probability of large deviations of sums of free variables is not yet answered.

Let us now discuss a free analogue of another theorem from classical probability theory, which is sometimes called the law of small numbers. In the classical case, this law says that counts of rare events are distributed by Poisson Law. (The classical example is the distribution of the number of deaths of Prussian cavalleray officers from horse kicks.)

First, let us define the free analogue of the Poisson Law.

Let \( \mu \) be a distribution with the density

\[ p(x) = \frac{\sqrt{4x - (1 - \lambda + x)^2}}{2\pi x} \text{ if } x \in \left( (1 - \sqrt{\lambda})^2, (1 + \sqrt{\lambda})^2 \right), \]

where \( \lambda \) is a positive parameter. If \( x \) is outside of this interval, then the density is zero. In addition, if \( \lambda < 1 \), then the distribution has an atom at \( 0 \) with weight \( (1 - \lambda) \).

This distribution is called the free Poisson distribution with parameter \( \lambda \). It is also known as the Marchenko-Pastur distribution because it was discovered in [14].

Recall that the classical Poisson distribution is supported on the set of non-negative integers and given by the formula

\[ \mu (\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}. \]

In contrast to the classical Poisson distribution, the free Poisson distribution is absolutely continuous except for a possible atom at \( 0 \). The reason why this distribution is called the free Poisson distribution is given in the following theorem. A similar theorem with the assumption of freeness substituted by the assumption of independence would result in the classical Poisson distribution.

**Theorem 8.2.** Let \( X_{1,n}, \ldots, X_{n,n} \) be self-adjoint random variables with the Bernoulli distribution

\[ \mu = \left( 1 - \frac{\lambda}{n} \right) \delta_0 + \frac{\lambda}{n} \delta_1. \]
Assume that $X_{i,n}$ are free and define

$$S_n = X_{1,n} + \ldots + X_{n,n}.$$ 

Then, $S_n$ converges in distribution to free Poisson distribution with parameter $\lambda$.

**Proof:** It is easy to compute the Cauchy transform of $X_{i,n}$ as

$$G(z) = \frac{z - 1 + \lambda/n}{(z - 1) z},$$

and the $R$-transform as

$$R(z) = -1 + z - \sqrt{(1 - z)^2 + 4(\lambda/n) z}.$$ 

Hence, the $R$-transform of the sum $S_n$ is

$$R_n(z) = \frac{1 - z}{2z} - \frac{1}{n} \left( -1 + \sqrt{1 + \frac{4\lambda}{n} \frac{z}{(1 - z)^2}} \right).$$

Then,

$$R_n(z) \to \frac{\lambda}{1 - z},$$

and the convergence is uniform in a disc around $z = 0$.

It is easy to check that $\lambda (1 - z)^{-1}$ is the $R$-transform of the free Poisson distribution. Since the convergence of the $R$-transforms implies the convergence of the Cauchy transforms and therefore convergence of moments, we can conclude the $S_n(z)$ converges in distribution to the free Poisson distribution. QED.
9. Lecture 9. Multivariate CLT

Let $\mathcal{P}_2(n)$ denote the set of pairings of the elements of set $\{1, 2, \ldots, n\}$. (If $n$ is odd, then this set is empty.) For example, $\mathcal{P}_2(4)$ consists of three elements: $(12)(34)$, $(13)(24)$, and $(14)(23)$. They are represented graphically in Figure 15.

Next, let us say that a pairing has a crossing if there are four elements $i, j, k, l$, of the set $\{1, 2, \ldots, n\}$, such that $i < j < k < l$, and that $i$ is paired with $k$, and $j$ is paired with $l$. For example, pairing $(13)(24)$ has a crossing. Let $\mathcal{NCP}_2(n)$ denote the set of non-crossing pairings of the set $\{1, 2, \ldots, n\}$, that is, the set of pairings that do not have a crossing. For example, $\mathcal{NCP}_2(4)$ consists of two elements: $(12)(34)$, and $(14)(23)$.

Let $s_i, i = 1, \ldots, n$ be self-adjoint variables with zero mean and the covariances $E(s_is_j) = c_{ij}$. Suppose that higher moments of the variables $s_i$ are given by the following formula:

$$E(s_{i_1}s_{i_2}\ldots s_{i_n}) = \sum_{\pi \in \mathcal{NCP}(n)} \text{prod}_{(p,q) \in \pi} c_{i_pi_q}.$$  \hfill (21)

Then $\{s_i\}$ is called the semicircular family with covariance $c_{ij}$.

Do the semicircle families exist? In other words, can we find random variables $s_1, \ldots, s_n$ with the joint moments defined by (21)?

It turns out that the semicircle family exists if and only if the matrix $c_{ij}$ is non-negative definite. This is similar to the situation in classical probability theory in which the Gaussian multivariate variable with covariance matrix $c_{ij}$ exists if and only if $c_{ij}$ is non-negative definite. Formula (21) is the free probability analogue of the Wick formula.
We postpone the proof of the existence and prove first the multivariate CLT for free random variables.

**Theorem 9.1.** Let \( \{X^{(1)}_i, \ldots, X^{(r)}_i\} \) be a sequence of \( r \)-tuples of selfadjoint bounded random variables. Assume that the joint distribution of an \( r \)-tuple does not depend on \( i \), that \( E \left(X^{(\alpha)}_i\right) = 0 \) and \( E \left(X^{(\alpha)}_i X^{(\beta)}_i\right) = c_{\alpha \beta} \), and that the \( r \)-tuples are free. Let

\[
S^{(\alpha)}_n = \sum_{i=1}^{n} X^{(\alpha)}_i.
\]

Then, the sequence of \( r \)-tuples \( \left\{ S^{(1)}_n / \sqrt{n}, \ldots, S^{(r)}_n / \sqrt{n} \right\} \) converges in distribution to the semicircle family with covariance \( c_{\alpha \beta} \).

**Proof:** Let for simplicity of notation prove the theorem for 2-components vectors \( \{X_i, Y_i\} \). In the general case the proof is similar.

Consider the moment \( E \left(S^{(X)}_n S^{(Y)}_n S^{(X)}_n S^{(Y)}_n\right) \). If we expand the sums, then we will get the following expression:

\[
\sum_{i_1, i_2, i_3, i_4} E \left(X_{i_1} Y_{i_2} X_{i_3} Y_{i_4}\right).
\]

The value of a term in this sum depends only on the partition of indices \( i_1, i_2, i_3, i_4 \) in groups of identical indices but not on the indices themselves. For example, \( E \left(X_{100} Y_{50} X_{50} Y_{100}\right) \) equals \( E \left(X_{1} Y_{2} X_{2} Y_{1}\right) \) but can be different from \( E \left(X_{1} Y_{1} X_{1} Y_{1}\right) \).

If this partition has a singleton, then the corresponding variable is free from any other variable in the product and its expectation can be taken outside as a factor. Since we assumed that this expectation is zero, therefore we can conclude that all partitions with a singleton bring zero contribution.

Next, note that if the partition of indices has \( k \)-blocks, then there are around \( n^k \) terms with this partition structure. (That is, the number of such terms is asymptotically \( n^k \) for large \( n \).) For example, then number of terms that correspond to the partition \( (12) (3456) \) is \( n (n - 1) \sim n^2 \).

It follows that for large \( n \), the asymptotically dominating contribution to the sum will be given by pairings provided that at least one pairing will give non-zero contribution.

Now let us evaluate the contributions given by different pairings.

If there are no pair in a given pairing that connects neighboring indices, then \( i_k \neq i_{k+1} \) for all \( k \). Hence, the consecutive terms in the product always come from different \( r \)-tuples. Since these \( r \)-tuples are assumed to be free, we can conclude that the expectation of the product is zero.

If there is a pair that connects neighboring indices, then we can take the expectation of the product of these elements outside. For example,

\[
E \left(X_3 X_1 X_2 X_2 X_1 X_3\right) = E \left(X_2 X_2\right) E \left(X_3 X_1 X_1 X_3\right) = c_{XX} E \left(X_3 X_1 X_1 X_3\right).
\]

Then, we repeat the procedure.
Let $\alpha(p) = X$ if the index $i_p$ correspond to the random variable $X_{i_p}$ element, and $\alpha(p) = Y$, if $i_p$ corresponds to the random variable $Y_{i_p}$. For example if we have product $X_{50}Y_{100}Y_{100}X_{50}$, then $\alpha(1) = X$, $\alpha(2) = Y$, $\alpha(3) = Y$, and $\alpha(4) = X$.

From the argument above it follows that for a given pairing $\pi$ of a set of $s$ elements, the expectation of a product of $s$ random variables $X_i$ and $Y_j$, which is compatible with the structure of the pairing $\pi$, equals to

$$\prod_{(p,q) \in \pi} c_{\alpha(p)\alpha(q)},$$

provided that in the reduction process we never run into the situation, when no pair connects the neighboring indices. We claim that this situation arises if and only if the pairing has a crossing.

Indeed, if a pair $(ab)$ connects neighbors, then its removal from the pairing does not affect whether the pairing has a crossing. Suppose that we removed all pairs that connects neighbors and now have a pairing which has no pair that connects neighbors. Let $(ab)$ be a pair in this pairing such that there are no pair inside it. Since $a$ and $b$ are not neighbors, there is an element $c$ between them. This element is connected to an element $d$ outside of the pair $(ab)$. This implies existence of a crossing. It is also easy to see that if a pairing has a crossing then at some point in the reduction process we will have the situation when no pair connects the neighboring indices.

Finally, let us note that there are around $n^{r/2}$ products of $X_i$ and $Y_j$ which are compatible with the structure of a pairing $\pi$.

To summarize, the expectation of a product of $r$ random variables $S_n^{(X)}$ and $S_n^{(Y)}$ can be asymptotically approximated by the formula

$$E \left( S_n^{(X)} S_n^{(Y)} \ldots \right) \sim n^{r/2} \sum_{\pi \in \mathcal{CP}_2(r)} \prod_{(p,q) \in \pi} c_{\alpha(p)\alpha(q)}.$$

After we divide variables $S_n^{(X)}$ and $S_n^{(Y)}$ by $\sqrt{n}$, we find that $\left\{ S_n^{(X)}/\sqrt{n}, S_n^{(Y)}/\sqrt{n} \right\}$ converge to the semicircle family with covariance $c_{XX}, c_{XY, c_{YY}}$. This completes proof of Theorem 9.1.

Let $s = (s_1, \ldots, s_r)$ denote an $r$-tuple of the random variables, each of which have the standard semicircle distribution and which are free from each other. We will call such an $r$-tuple a semicircle system. If $s^{(1)}, \ldots, s^{(n)}$ are $n$ semicircle systems which are free from each other, then it is easy to see that $s = (s^{(1)} + \ldots + s^{(n)})/\sqrt{n}$ is also a semicircle system.

Now let $C$ be a non-negative definite $r$-by-$r$ matrix. We can represent it as $C = A'A$, where $A'$ denotes the transposed $A$. Let $x = As$, where $s$ is a semicircle system. Then it is easy to see that the elements of the $r$-tuple $x$ have the semicircle distribution and the covariance of $x_i$ and $x_j$ is given by the element of matrix $C$, $C_{ij}$. 
If \( s^{(1)}, \ldots, s^{(n)} \) are \( n \) free semicircle systems, then we can define \( n \) free \( r \)-tuples \( x^{(i)} \) by the formula \( x^{(i)} = A s^{(i)} \). All of these \( r \)-tuples have the same joint distribution and this distribution coincide with the distribution of \( X^{(n)} = (x^{(1)} + \ldots + x^{(n)}) / \sqrt{n} \). The argument in the proof of Theorem 9.1 establishes that a joint moment of \( X^{(n)} \) converges to the expression

\[
\sum_{\pi \in NCP(n)} \prod_{(p,q) \in \pi} C_{ij}.
\]  

(22)

However, the distribution of \( X^{(n)} \) does not depend on \( n \) and equals the distribution of each of \( x^{(i)} \). Hence, the joint moments of each of \( x^{(i)} \) are given by the expression (22). This means that each of the \( r \)-tuples \( x^{(i)} \) is the semicircle family with the covariance matrix \( C \).

This argument proves the existence of semicircle families.

It is interesting to compare Theorem 9.1 with other central limit theorems in non-commutative probability theory.

As a rule, other CLT are formulated for the case when the terms of the sum \( S_n = X_1 + \ldots + X_n \) are assumed to be independent. In particular, these terms must commute. It follows that it does not make much sense to talk about the non-commutative univariate CLT for independent variables. This theorem is exactly the same as in the CLT in the traditional (i.e., commutative) probability theory.

Hence, we consider \( r \)-tuples \( \{X^{(1)}_i, \ldots, X^{(r)}_i\} \) of the non-commutative random variables and their sums \( S^{(k)}_n = X^{(k)}_1 + \ldots + X^{(k)}_n \). It is assumed that the random variables are independent for different \( i \), so for example \( X^{(k)}_i \) commutes with \( X^{(l)}_j \) and independent of it provided that \( i \neq j \). However, \( X^{(k)}_i \) does not commute with \( X^{(l)}_i \), in general. The main question is what is the limit of joint distributions of variables \( S^{(k)}_n / \sqrt{n} \) for large \( n \).

Consider, as a simple example, pairs \((X_1, Y_1), (X_2, Y_2)\), and the sums \( S_{X,n} = X_1 + \ldots + X_n \) and \( S_{Y,n} = Y_1 + \ldots + Y_n \). If we are interested in the joint moment

\[
E \left( S^{k_1}_{X,n} S^{k_2}_{Y,n} S^{k_3}_{X,n} \ldots \right),
\]

we need to be able to compute or estimate the expectations of the products

\[
X_{i_1} X_{i_2} \ldots X_{i_{k_1}} Y_{j_1} Y_{j_2} \ldots Y_{j_{k_2}} X_{l_1} \ldots
\]

The fact that \( X_i \) and \( Y_i \) do not commute makes the problem difficult.

A usual way to overcome the difficulties is to impose the restriction on the nature of non-commutativity of variables \( X \) and \( Y \). In ..., variables \( X \) and \( Y \) are assumed to be canonically conjugate, that is,

\[
XY - YX = icI,
\]

where \( c \) is a constant. (It is the Planck constant \( h \), if \( X \) and \( Y \) are operators corresponding to the position and momentum of a particle.)

This simplifies the problem in the following aspects.
First, every polynomial in $X$ and $Y$ can be brought to a canonical form. (There are different canonical forms.)

Second, the expectation of the canonical polynomial can be computed using a certain function, which plays the role of the multivariate distribution density. Namely,

$$E(P(X,Y)) = \int W_{XY}(x,y) P(x,y) \, dx \, dy.$$ 

($W(x,y)$ is not necessarily non-negative everywhere.) For a certain choice of the canonical polynomials, this function is called the Wigner function.

Third, $S_{X,n}/\sqrt{n}$ and $S_{Y,n}/\sqrt{n}$ are also canonically conjugate so we can define the Wigner function for this pair of random variables.

Then, the question is about the form of the limit of this Wigner function. Implicitly, this limit was established in [8], although they used a somewhat different terminology.

Later, a central limit theorem was established in [13] for the case of anticommuting random variables:

$$XY + YX = icI,$$

and certain results were achieved for the intermediate case

$$XY - qYX = icI.$$

(The interested reader can consult [20] for details.)
10. LECTURE 10 INFINITELY-DIVISIBLE DISTRIBUTIONS

Probability measure $\mu$ is called infinitely-divisible (in the sense of free probability) if for every integer $n > 0$, it can be represented as a free additive convolution of $n$ measures $\mu_{1/n}$:

$$\mu = \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n},$$

$n$-times

For every infinitely-divisible distribution $\mu$, we can define a family of measures $\{\mu_t\}$, such that $\mu_1 = \mu$, $\mu_0 = \delta_0$, and $\mu_{t+s} = \mu_t \boxplus \mu_s$.

Indeed, it is easy to define $\mu_t$ for all rational $t$. Then it is possible to show that if $t \to 0$, then $\mu_t \to \delta_0$ in a certain sense. Then, the measures $\mu_t$ can be defined for all real $t$, using continuity properties of the free additive convolution.

Clearly, if $R(z)$ is the $R$-transform of $\mu$, then the $R$-transform of $\mu_t$ is $R_t(z) = tR(z)$. It is customary to work not with $R(z)$, but with its modification $\varphi(z) = R(z^{-1})$. We have $\varphi_t(z) = t\varphi(z)$. Let $G(z,t)$ denote the Cauchy transform of $\mu_t$. By using the definition of the $R$-transform, we can write

$$G(t\varphi(z) + z, t) = \frac{1}{z}. \quad (23)$$

Let us take the derivative of (23) with respect to $t$ at $t = 0$. Then we have:

$$\frac{\partial G(z, 0)}{\partial z} \varphi(z) + \frac{\partial G(t, z)}{\partial t} \bigg|_{t=0} = 0.$$ 

We also know that $G(z, 0)$ is the Cauchy transform of $\delta_0$ and, therefore, it equals $z^{-1}$. It follows that

$$\varphi(z) = z^2 \frac{\partial G(t, z)}{\partial t} \bigg|_{t=0}.$$ 

This formula can be used to obtain the following theorem from [1].

**Theorem 10.1.** (i) A probability measure $\mu$ is infinitely divisible with respect to free additive convolution if and only if its $\varphi$-function has an analytic continuation defined everywhere on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$.

(ii) An analytic function $\varphi: \mathbb{C}^+ \to \mathbb{C}^-$ is a continuation of $\varphi$-function for an infinitely-divisible $\mu$ if and only if

$$\lim_{y \to \infty} \frac{\varphi(iy)}{y} = 0,$$

(iii) The following representation holds for $\varphi$-function of an infinitely-divisible $\mu$:

$$\varphi_{\mu}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + sz}{z - s} d\sigma(s)$$

$$= \alpha + \int_{-\infty}^{\infty} \left( \frac{sz}{z - s} - \frac{s}{1 + s^2} \right) \frac{1 + s^2}{s^2} d\sigma(s),$$

where $\alpha$ is real and $d\sigma$ is a finite measure.
This theorem should be compared with the classical result by Levy, Khintchine, and Kolmogorov. Suppose that \( f(t) = E(e^{itX}) \) and \( X \) is infinitely-divisible with respect to the usual convolution of probability measures. Then,

\[
\log f(t) = i\alpha t + \int_{-\infty}^{\infty} \left( \exp(itu) - 1 - \frac{itu}{1 + u^2} \right) \frac{1 + u^2}{u^2} d\sigma(u),
\]

where \( \alpha \) is real and \( d\sigma \) is a finite measure.

Proof of Theorem 10.1. We can write:

\[
\varphi(z) = z^2 \left. \frac{\partial G(t, z)}{\partial t} \right|_{t=0} = z^2 \lim_{\varepsilon \to 0} \frac{G_\varepsilon(z) - G_0(z)}{\varepsilon} = z^2 \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \left[ \frac{1}{z-t} - \frac{1}{z} \right] \mu_\varepsilon(dt) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{zt}{z-t} \mu_\varepsilon(dt). \tag{24}
\]

where \( \mu_\varepsilon \) is the measure of \( \varphi_\varepsilon \). We can expect that \( \mu_\varepsilon \) converges to \( \delta_0 \) as \( \varepsilon \to 0 \), hence we cannot simply take the limit \( \mu_\varepsilon/\varepsilon \). We know, however, that the limit in (24) exists. Let us see what makes the limit of integrals convergent. Let us compute the imaginary part of \( \varphi(iy) \).

\[-\text{Im} \varphi(iy) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{t^2y}{t^2+y^2} \mu_\varepsilon(dt).\]

This suggests that the limit exists because the singularity of the sequence of measures \( \mu_\varepsilon \) at 0 is removed by multiplication by \( t^2/(t^2+y^2) \). In view of this remark, it is natural to divide \( \mu_\varepsilon \) in two parts

\[
\mu_\varepsilon = \frac{t^2}{1+t^2} \mu_\varepsilon + \frac{1}{1+t^2} \mu_\varepsilon.
\]

The first part equals zero at \( t = 0 \), so the limiting atom at 0 is removed. Hence we can hope that if we multiply this part by \( \varepsilon^{-1} \), then the result will converge to a limit measure as \( \varepsilon \to \infty \). So, let us define

\[
\nu_\varepsilon = \frac{1}{\varepsilon} \frac{t^2}{1+t^2} \mu_\varepsilon.
\]

Lemma 10.2. The sequence \( \nu_\varepsilon \) converges to a limit measure \( \nu \).

Proof: For every \( \delta > 0 \), we can find a sufficiently large \( y \) such that \( |\varphi(iy)| < \delta y \). By taking the imaginary part of \( \varphi(iy) \), we see that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{t^2y}{t^2+y^2} \mu_\varepsilon(dt) < \delta y.
\]

This implies that

\[
\varepsilon^{-1} \mu_\varepsilon \left( \{ t : |t| > y \} \right) \leq 2\delta
\]
for all sufficiently small $\varepsilon$. If we make sure that $y > 1$, then this implies that
\[ \nu_\varepsilon \left( \left\{ t : |t| > y \right\} \right) \leq 4\delta, \]
which shows that the sequence $\nu_\varepsilon$ is tight.

In addition, $\nu_\varepsilon$ is positive and the sequence of integrals $\int \nu_\varepsilon \, (dx)$ is bounded for we know that
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{t^2 y}{t^2 + y^2} \mu_\varepsilon \, (dt) = -\text{Im} \varphi (iy) < \infty, \]
and
\[ \frac{t^2}{t^2 + 1} < c_1(y) \frac{t^2 y}{t^2 + y^2}. \]

Hence, the sequence $\nu_\varepsilon$ has a limit point $\nu$. For this limit point we can write the equation
\[ \int \frac{(t^2 + 1) y}{t^2 + y^2} \nu \, (dt) = -\text{Im} \varphi (iy), \]
and this equation determines the measure $\nu$. Hence the limit point is unique and the sequence $\nu_\varepsilon$ converges to $\nu$. This completes the proof of the lemma.

The rest of the proof of the theorem is based on the identity:
\[
\frac{zt}{z-t} = 1 + \frac{zt}{z-t} \frac{t^2}{t^2 + t^2} + \frac{t}{1 + t^2}.
\]
By using this identity, we can write:
\[
\varphi(z) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{zt}{z-t} \mu_\varepsilon \, (dt)
= \lim_{\varepsilon \to 0} \int \frac{1 + zt}{z-t} \nu_\varepsilon \, (dt) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \frac{t}{1 + t^2} \mu_\varepsilon \, (dt)
= \int \frac{1 + zt}{z-t} \nu \, (dt) + \lim_{\varepsilon \to 0} \alpha_\varepsilon.
\]
Clearly, the limit of $\alpha_\varepsilon$ must exist because the left-hand side is well-defined. Therefore,
\[
\varphi(z) = \alpha + \int \frac{1 + zt}{z-t} \nu \, (dt).
\]  (25)
This proves (iii). Also, representation (25) implies that $\varphi(z)$ has an analytic continuation defined everywhere on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$. It also easy to check that
\[
\lim_{y \to \infty} \frac{\varphi(iy)}{y} = 0,
\]  (26)
Conversely, if $\varphi(z)$ has an analytic continuation defined everywhere on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$ and (26) holds, then there is a representation theorem (Nevanlinna representation formula) that shows that (25) must be valid. This proves claims (i) and (ii) and completes the proof of the theorem.

If we consider only bounder random variables, it is easy to understand why the $\varphi$-function must be analytic everywhere. Indeed, let $F_t(z) = G_t(1/z)$. Then,
\[
F_t(z) = z + t \overline{F}(t, z).
\]
For arbitrarily large disc $D$, we can take $t$ so small that, $F_t(z)$ is close to $z$ on $K$. Hence, its inverse $F_t^{-1}(z)$ is defined on a large disc $K'$. It follows that \( \varphi_t(z) \) is defined on disc $K'$. However, the fact that we consider the infinitely-divisible measure implies that $\varphi(z) = \varphi_t^{-1}(z)$. Hence, it also defined on the disc $K'$. This shows that $\varphi(z)$ must be analytical everywhere in the plane.

Example 1.

The semicircle distribution is infinitely divisible. In fact, if $\mu$ is the standard semicircle distribution, then $\mu_t(A) = \mu_t(A/\sqrt{t})$. That is, $\mu_t$ is simply $\mu$ dilated by $\sqrt{t}$.

Example 2.

The free Poisson distribution is infinitely divisible. If $\mu_\lambda$ denote the free Poisson distribution with parameter $\lambda$, then $\mu_\lambda(A) = \mu_\lambda(A/\sqrt{t})$. That is, $\mu_\lambda$ is simply $\mu$ dilated by $\sqrt{t}$.

Example 3.

The Cauchy distribution is infinitely divisible. If $\mu$ denote the standard Cauchy distribution, then $\mu_t(A) = \mu(A/t)$.

Definition 10.3. A family of operators $W_t$ is called a free Brownian motion if

(i) for all $t > s$, $W_t - W_s$ is free from the subalgebra generated by $W_s$, where $s' \leq s$;

(ii) for all $t > s$, $W_t - W_s$ is distributed according to the semicircle law with parameter $t - s$.

How can we show the existence of a free Brownian motion? One way is to take the Hilbert space $H = L^2([0, \infty), dx)$ and let $e_t = 1_{[0,t]}(x)$. Next, we consider the full Fock space, based on $H$ and the creation and annihilation operators $a_t^*$ and $a_t$ that correspond to the vector $e_t$. Then we can define $W_t = a_t^* + a_t$. It turns out that these operators form a free Brownian motion.

Another way to realize a free Brownian motion is by approximating it by Brownian motions on the space of Hermitian random matrices. Let $X_t^N$ be $N$-by-$N$ random matrices and assume that as $t$ grows, every entry of these matrices follow an independent Brownian motion except that entries are constrained by hermiticity of the matrix $(X_{ij} = X_{ji}^*)$. Then, if we take $N \to \infty$ and normalize the variances of the entries suitably, the joint distributions of the process $X_t^N$ considered as a process in the non-commutative probability space of random matrices converge to the corresponding joint distributions of the free Brownian motion $W_t$.

In other words if we take a polynomial in variables $X_t^N$, $t = t_1, \ldots, t_n$, and take the normalized trace and the expected value of this polynomial, then the result will converge to the expectation of the corresponding polynomial in variables $W_t$, $t = t_1, \ldots, t_n$.

Given the concept of the free Brownian motion, it is possible to develop the theory of stochastic integrals. The development is parallel to the similar theory in the classical probability theory. For example, it is possible to define stochastic
integrals by using limits of the partial sums
\[ \sum_i A_t (W_{t+1} - W_t) B_t, \]
where \( A_t \) and \( B_t \) are free of future values of \( W_t \).

Given the concept of the stochastic integral, it is possible to introduce stochastic differential equations like the following.
\[ dX_t = a (X_t) \, dt + b (X_t) \, dW_t \, c (X_t) . \]

For these equations it is possible to write an analogue of the Ito formula. In addition, if \( b = c = 1 \), then it is possible to write the free Fokker-Planck equation for the density of the spectral probability distribution of \( X_t \). If we denote this density by \( p_t \), then the equation is as follows.
\[ \frac{\partial p_t}{\partial t} = - \frac{\partial}{\partial x} \left[ p_t (H p_t + a (x)) \right], \]
where \( H \) denotes a multiple of the Hilbert transform:
\[ Hu(x) := \text{p.v.} \int \frac{u(y)}{x-y}dy. \]

More information about the free Ito formula and the free Fokker-Planck formulas can be found in [4] and [5].

In the case when \( b \) and \( c \) depend on \( X_t \), it is sometimes possible to write a differential equation for the Cauchy transform of \( X_t \). By solving this equation, we can obtain some information about the spectral probability distribution of \( X_t \).

We can also consider the multivariate free stochastic differential equations like the following:
\[ dX_t = f (X_t, Y_t) \, dt + dW_t^{(1)}, \]
\[ dY_t = g (X_t, Y_t) \, dt + dW_t^{(2)}, \]
where \( W^{(1)} \) and \( W^{(2)} \) are two mutually free Brownian motions. However, in this case, little is known about the behavior of the solution. The only exceptions are when the equations separate, or when \( f \) and \( g \) are linear in \( X \) and \( Y \) and the initial condition is semicircle. In the latter case \( X \) and \( Y \) remain semicircle and can be characterized by the evolution of their covariance matrix.
11. LECTURE 11. MULTIPLICATION AND S-TRANSFORM

The problem of calculating the distribution for products of random variables is specific for non-commutative probability theory. In the case of commutative probability theory we can simply take the logarithm and use the fact that $\log (XY) = \log (X) + \log (Y)$. This equality does not hold in the non-commutative case. Moreover, the product $XY$ is not self-adjoint so it is not even clear if we can a multiplicative analogue of free additive convolution of measures.

However, it seems natural to ask if we can compute the distribution of singular values of operator $XY$, if we know the singular values of operators $X$ and $Y$. Recall that for a finite-dimensional matrix $X$, the singular values are defined as square-roots of eigenvalues of the matrix $X^*X$. In particular the largest singular value is equal to the norm of the matrix $X$. In analogy with this definition, we define the squared singular value distribution of a non-commutative random variable $X$, as the spectral probability distribution of $X^*X$.

Let $\mu$ and $\nu$ be the spectral probability measures of $X^*X$ and $Y^*Y$, respectively, and let us assume that $X$ and $Y$ are $*$-free. Then the spectral probability measure of $(XY)^*XY$ is determined by $\mu$ and $\nu$, and is called the free multiplicative convolution of measures $\mu$ and $\nu$. We will denote it as $\mu \boxdot \nu$.

Note that every probability measure supported on $\mathbb{R}^+ = \{x|x \geq 0\}$ can be realized as the spectral probability measure of an operator $X^*X$. Hence, the free multiplicative convolution is defined for all measures supported on $\mathbb{R}^+$.

There is another variant of the theory of free multiplicative convolutions, defined for the measures supported on the unit circle. Indeed, if $X$ and $Y$ are two free unitary operators, then their product is also unitary. The unitary operators are normal and we can talk about their spectral probability distribution, which are supported on the unit circle. The distribution of $XY$ can be expressed in terms of the distributions of $X$ and $Y$, and this gives rise to another concept of free multiplicative convolution. The technique of $S$-transform, which we present below, is applicable to both cases of free multiplicative convolution. However, for definiteness, we will concentrate on the case of measures supported on $\mathbb{R}^+$.

Recall that we say that we call two variables $A$ and $B$ equivalent if have the same $*$-distributions. In this case we write $A \cong B$. We will say that $A$ and $B$ are weakly equivalent and write $A \sim B$ if these variables have the same sequence of moments, i.e. $E(A^n) = E(B^n)$ for all $n > 0$. These concepts coincide for self-adjoint variables. Also, note that if $A$ and $B$ are bounded and self-adjoint and if $A \sim B$, then they have the same spectral probability distribution.

Proposition 11.1. Let $A_1 \sim A_2$ and let both $A_1$ and $A_2$ be free from a subalgebra $\mathcal{B}$. Let $P(A, B_1, \ldots, B_n)$ be a non-commutative polynomial and let $B_1, \ldots, B_n \in \mathcal{B}$. Then

$$P(A_1, B_1, \ldots, B_n) \sim P(A_2, B_1, \ldots, B_n).$$

Proof: It is enough to show that the expectations of these two polynomials coincide. Indeed if we know that this is true and we want to show that the expectations of the $n$-th powers of these polynomials coincide, we can consider new polynomials $Q = P^n$ and apply the result about expectations to these new polynomials.
For the expectations, we note that it is enough to prove the equality of expectations for monomials. And for monomials, we use the freeness to express the expectation of the polynomials $P(A_1, B_1, \ldots, B_n)$ in terms of polynomials from the moments of $A_i$ and the joint moments of variables $B_i$. The moments of $A_1$ are the same as the moments of $A_2$ by assumption and therefore the expectation of $P(A_1, B_1, \ldots, B_n)$ is equal to the expectation of $P(A_2, B_1, \ldots, B_n)$. QED.

Proposition 11.2. If $E$ is tracial, then $A^* A \sim AA^*$.

Proof: By traciality of $E$,

$$E \left[ (A^* A)^n \right] = E \left[ (AA^*)^n \right].$$

QED.

Proposition 11.3. If $X$ and $Y$ are $*$-free, and expectation $E$ is tracial, then

$$(XY)^* XY \sim (X^* X) (Y^* Y) .$$

Proof: We need to prove that for all integer $n > 0$

$$E \left\{ [(XY)^* XY]^n \right\} = E \left\{ [(X^* X) (Y^* Y)]^n \right\}. $$

Let us expand the expression on the left-hand side by using $(XY)^* = Y^* X^*$ and let us move the first $Y^*$ to the back of the expression by using the traciality. Then we will get

$$E \left\{ (X^* XY Y^*) (X^* XY Y^*) \ldots (X^* XY Y^*) \right\}.$$ (27)

By Proposition 11.2 $YY^* \sim Y^* Y$. Hence, we can use Proposition 11.1 to conclude that the expression in (27) equals

$$E \left\{ (X^* XY Y^*) (X^* XY Y^*) \ldots (X^* XY Y^*) \right\} = E \left\{ [(X^* X) (Y^* Y)]^n \right\}.$$ QED.

In other words, the moments of selfadjoint variable $(XY)^* XY$ coincide with the moments of (non-selfadjoint) variable $(X^* X) (Y^* Y)$. It follows that if we want to compute the squared singular value distribution of $XY$, we need to find out how to compute the moments of $(X^* X) (Y^* Y)$.

This observation brings us to the following, more general problem. Suppose that we know the moments of (possibly non-selfadjoint) variables $X$ and $Y$. How can we compute the moments of their product $XY$?

For every random variable $X$, we can define

$$\psi_X (z) = E \left( \frac{1}{1 - zX} \right) - 1.$$ If $X$ is bounded, we can expand it as a series in $z$ which are convergent in a neighborhood of 0:

$$\psi_X (z) = zE (X) + z^2 E (X^2) + \ldots$$
Assume that \( E(X) \neq 0 \). Then, the function \( \psi_X(z) \) is invertible in a neighborhood of zero. Let us call its inverse \( \psi_X^{-1}(z) \). Finally, let us define the \( S \)-transform of variable \( X \) by the formula

\[
S_X(z) = \frac{1 + z}{z} \psi_X^{-1}(z).
\]

**Exercise 11.4.** Consider the Bernoulli distribution \( \mu = (1-p)\delta_0 + p\delta_1 \). Show that

\[
S_\mu(z) = \frac{1 + z}{p + z}.
\]

**Exercise 11.5.** Consider the free Poisson distribution with parameter \( \lambda \) (see p. 37). Show that for this distribution,

\[
S_\mu(z) = \frac{1}{\lambda + z}.
\]

**Theorem 11.6.** Suppose \( X \) and \( Y \) are bounded, \( E(X) \neq 0 \) and \( E(Y) \neq 0 \). Assume that \( X \) and \( Y \) are free. Then,

\[
S_{XY}(z) = S_X(z) S_Y(z).
\]

**Proof:** (We follow the proof in [10]) We will use the notation that we used in the proof of Theorem 7.5 on p. 32. Let

\[
X = (1 + a) f(a^*)
\]

and

\[
Y = (1 + b) g(b^*),
\]

where \( f \) and \( g \) are analytical in a neighborhood of zero.

We claim that \( S_X(z) = 1/f(z) \) and \( S_Y(z) = 1/g(z) \). If this is known, then for every given \( S_X(z) \) and \( S_Y(z) \) we can construct the variables in the form above which will have \( S_X(z) \) and \( S_Y(z) \) as their \( S \)-transforms. Since \( S_X(z) \) and \( S_Y(z) \) determine \( S_{XY}(z) \) in a unique fashion, hence it is enough to compute the \( S \)-transform of \( XY \) for this particular choice of \( X \) and \( Y \) and to check that the result is equal to \( 1/[f(z)g(z)] \).

In order to make sure that \( S_X(z) = 1/f(z) \), let us define

\[
w = \Omega + \sum_{n=1}^{\infty} z^n e^{\otimes n},
\]

and consider the action of \( X \) on \( w \).

\[
a^* w = zw.
\]

Hence,

\[
f(a^*) w = f(z) w.
\]

Then,

\[
a w = \frac{w - \Omega}{z}.
\]
Hence,
\[
(1 + a) f (a^*) w = f (z) w + \frac{f (z)}{z} (w - \Omega) = f (z) \left(1 + \frac{z}{1 + z}\right) w - \frac{f (z)}{z} \Omega.
\]

We can re-write this as follows:
\[
\left( X - f (z) \frac{1 + z}{z} \right) w = -\frac{f (z)}{z} \Omega,
\]
or
\[
\left( 1 - \frac{z}{1 + z} \frac{1}{f (z)} X \right) w = \frac{1}{1 + z} \Omega. \tag{28}
\]

The operator
\[
1 - \frac{z}{1 + z} \frac{1}{f (z)} X
\]
is invertible for \(z\) in an open neighborhood of 0, and we can write:
\[
(1 + z) w = \left(1 - \frac{z}{1 + z} \frac{1}{f (z)} X\right)^{-1} \Omega.
\]

Let
\[
\psi^{(-1)} (z) := \frac{z}{1 + z} \frac{1}{f (z)}.
\]
Then
\[
E \left[ \left(1 - \psi^{(-1)} (z) X\right)^{-1}\right] = \langle \Omega, \left(1 - \psi^{(-1)} (z) X\right)^{-1} \Omega \rangle
\]
\[
= \langle \Omega, (1 + z) w \rangle
\]
\[
= 1 + z.
\]

It follows that \(\psi^{(-1)} (z)\) is the inverse of the function \(\psi (z) = E \left[ (1 - z X)^{-1}\right] - 1\),
and it shows that \(1 / f (z)\) is indeed the \(S\)-transform of \(X\).

Similarly, we can show that the \(S\)-transform of \(Y\) is \(1 / g (z)\).

It remains to show that the \(S\)-transform of \(XY\) is \(1 / [f (z) g (z)]\). We know that
\[
XY = (1 + a) f (a^*) (1 + b) g (b^*).
\]

Let us consider how \(XY\) acts on \(\sigma\), where
\[
\sigma = \Omega + \sum_{n=1}^{\infty} z^n (e + f + e \otimes f)^{\otimes n}.
\]

First, we have
\[
b^* \sigma = z \sigma,
\]
and therefore
\[
g (b^*) \sigma = g (z) \sigma,
\]
and
\[
(1 + b) g (b^*) \sigma = g (z) (1 + b) \sigma.
\]
Next, we note that
\[ a^* \sigma = (1 + f) \sum_{n=1}^{\infty} z^n (e + f + e \otimes f)^n - 1 = z (1 + b) \sigma. \]
This implies that
\[ a^* (1 + b) \sigma = a^* \sigma = z (1 + b) \sigma, \]
that is, \((1 + b) \sigma\) is an eigenvector for \(a^*\) with eigenvalue \(z\), and therefore
\[ f (a^*) (1 + b) \sigma = f (z) (1 + b) \sigma. \]
Hence, we obtain:
\[ XY \sigma = (1 + a) f (a^*) (1 + b) g (b^*) \sigma \]
\[ = f (z) g (z) (1 + a) (1 + b) \sigma. \]
However,
\[ (1 + a) (1 + b) \sigma = \sigma + \sum_{n=1}^{\infty} z^n (e + f + e \otimes f)^n + 1 \]
\[ = \sigma + \frac{\sigma - \Omega}{z} \]
\[ = \frac{z + 1}{z} \sigma - \frac{1}{z} \Omega. \]
Therefore,
\[ XY \sigma = f (z) g (z) \left( \frac{z + 1}{z} \sigma - \frac{1}{z} \Omega \right), \]
or
\[ \left( 1 - \frac{z}{1 + z} \frac{1}{f (z) g (z)} \right) XY \sigma = (1 + z) \Omega. \]
This equation has exactly the same form as equation (28). Therefore, we can repeat the arguments that we made after equation (28) and conclude that \(S_{XY} = 1/ (f (z) g (z)) \). This implies that \(S_{XY} = S_X S_Y\). QED.

Here is an application of the multiplication formula.

**Theorem 11.7.** Suppose that \(\mu_i\) are probability distributions supported on \(\mathbb{R}^+\) such that \(E (\mu_i) = 1\). Let
\[ \mu^{(n)} = \mu_1 \boxtimes \cdots \boxtimes \mu_n \]
Then \( \text{Var} (\mu^{(n)}) = \sum_{i=1}^{n} \text{Var} (\mu_i) \).

**Proof:** Let \(X_i\) be self-adjoint variables with distributions \(\mu_i\). Then \(EX_i^2 = m_2^{(i)} > 1\). We can write
\[ S_{X_i} (z) = 1 + (1 - m_2^{(i)}) z + \ldots \]
Let \(\Pi_n = X_n \cdots X_1\). Then,
\[ S_{\Pi_n} (z) = \prod_{i=1}^{n} S_{X_i} (z) = 1 + \sum_{i=1}^{n} \left( 1 - m_2^{(i)} \right) z + \ldots \]
From this we can conclude that \( E (\Pi_n^2) = 1 - \sum_{i=1}^{n} (1 - m_2^{(i)}) \), or \( E (\Pi_n^2) - 1 = \sum_{i=1}^{n} (m_2^{(i)} - 1) \). The moments of \( \mu^{(n)} \) coincide with the moments of \( \Pi_n \). Hence, 
\[
\text{Var} (\mu^{(n)}) = \sum_{i=1}^{n} \text{Var} (\mu_i).
\]
QED.

This interesting observation is not valid for multiplication of classical random variables.

It is natural to ask what happens if the expectation of one or both of the variables \( X \) and \( Y \) is equal to 0. In this case, either \( S_X \) or \( S_Y \) is not well defined because the corresponding function \( \psi(z) \) has a multiple zero at \( z = 0 \) and therefore cannot be inverted.

If both \( E(X) \) and \( E(Y) \) are zero, then it is easy to see from the definition of freeness that \( E[(XY)^n] = 0 \) for all \( n \). If only one of these expectations is equal to zero, say, \( E(X) = 0 \) and \( E(Y) \neq 0 \) but both variables \( X \) and \( Y \) are assumed to be bounded, then it turns out that it is still possible to define \( S_X(z) \) and \( S_{XY}(z) \) as multivalued functions and then it turns out that it is still true that \( S_{XY}(z) = S_X(z) S_Y(z) \). For details see Speicher and Rao.

**Exercise 11.8.** Consider the semicircle \( X \). Show that \( S_X(z) = 1/\sqrt{z} \).

**Exercise 11.9.** Consider the Bernoulli distribution \( \mu = (\delta_{-1} + \delta_1)/2 \). Show that \( S_\mu(z) = \sqrt{1 + z} \).

**Exercise 11.10.** Consider the arcsine distribution (see p. 8). Show that \( S_\mu(z) = \sqrt{u(2 + u)} \).

One useful concept related to the free multiplicative convolution is that of free compression. Let \( P_t \) be a projection of dimension \( t \). By this we mean that \( P_t^2 = P_t \), and that \( E(P_t) = t \). Assume also that \( A \) is another self-adjoint operator which is free from \( P_t \). Then operator \( A_t = P_t A P_t \) is called the free compression of the operator \( A \).

The distribution of \( P_t A P_t \) depends only on the distribution of \( A \) and \( t \). In particular, it is easy to see that the moments of \( P_t A P_t \) can be computed as the moments of \( A P_t \). Note that the distribution of \( P_t \) is the Bernoulli distribution \( (1 - t) \delta_0 + t \delta_1 \). By using Theorem [11.6] and the result in Exercise [11.4] we see that the \( S \)-transform of the compressed variable \( P_t A P_t \) equals

\[
S_A(z) = \frac{1 + z}{t + z}.
\]

**Exercise 11.11.** Suppose \( A \) has the Bernoulli distribution \( \mu = (1 - p) \delta_0 + p \delta_1 \). Show that the Cauchy transform of the variable \( P_t A P_t \) equals

\[
G(z) = \frac{z + (p + t - 2) - \sqrt{z^2 + 2z(2pt - (p + t)) + (p - t)^2}}{2(z - 1)z}.
\]
Conclude that the absolutely continuous part of the distribution of $P_tA P_t$ is supported on the interval

$$I = \left[ \left( \sqrt{p(1-t)} - \sqrt{(1-p)t} \right)^2, \left( \sqrt{p(1-t)} + \sqrt{(1-p)t} \right)^2 \right],$$

and its density is given by the formula:

$$f(x) = \frac{\sqrt{4pt(1-t)} - [x - (p + t - 2pt)]^2}{2\pi (1-x)}.$$

The distribution has an atom at zero with the mass $1 - t$ if $t < p$, and $1 - p$ if $p \leq t$. If $p + t > 1$, then the distribution has an atom at 1 with the mass $p + t - 1$.

Exercise 11.12. Suppose that $A$ has the standard semicircle distribution. Show that the variable $P_{1/2}AP_{1/2}$ has the Cauchy transform

$$G(z) = z - \frac{\sqrt{z^2 - 2}}{2} + \frac{1}{2z}.$$

Conclude that the distribution of this variable has the semicircle distribution with the radius $\sqrt{2}$ as its continuous part and an atom at zero with the mass 1/2.

Exercise 11.13. Suppose that $A$ has the symmetric Bernoulli distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. Show that the variable $P_{1/2}AP_{1/2}$ has the Cauchy transform

$$G(z) = \frac{1}{2z} + \frac{1}{2\sqrt{z^2 - 1}}.$$

Conclude that the distribution of this variable has the arcsine distribution as its continuous part and an atom at zero with the mass 1/2.

The last exercise is quite interesting because we have seen before that the arcsine distribution is the result of the sum of two symmetric Bernoulli distributions. It turns out that this is a general phenomenon as explained below.

After we compress a variable by $P_t$, the result will always have an atom at zero with the mass of at least $1 - t$. It is natural to remove this atom from the distribution and rescale the remaining distribution so that it remains a probability distribution. Essentially, this is the same as multiplying each expectation by $1/t$.

Exercise 11.14. Check that if the expectation is multiplied by $1/a$, so that $\tilde{E}(X) = a^{-1}E(X)$ for every $X$, then

$$\tilde{S}_X(u) = \frac{1 + u}{a^{-1} + u}S_X(au).$$
Consequently, we obtained the following expression for the rescaled $S$-transform of the compressed variable:

$$
\tilde{S}_{P_t A P_t} (u) = \frac{1 + u}{t^{-1} + u} S_{P_t A P_t} (tu)
= S_A (tu).
$$

**Exercise 11.15.** Let $R(z) = zK(z) - 1$. Check that

$$R^{(-1)}(z) = zS(z)
$$

Hence,

$$
\tilde{R}_{P_t A P_t}^{(-1)} (u) = uS_A (tu)
= \frac{1}{t} R_A^{(-1)} (tu).
$$

It follows that

$$
\tilde{R}_{P_t A P_t} (z) = \frac{1}{t} R_A (tz)
= \frac{1}{t} R_t A (z).
$$

If $1/t = n$ is an integer then we see that the rescaled distribution of $P_t A P_t$ coincides with the free additive convolution of $n$ distributions of $A$ scaled down by $1/n$. This is exactly what happened in Exercise 11.13.

More generally, this construction can be used to define a free additive semigroup $\mu_s$ for $s \geq 1$ and arbitrary $\mu_1$. We simply take the distribution of $s P_{1/s} A P_{1/s}$ as $\mu_s$, with the variable $A$ having the distribution $\mu_1$. In the case, when $\mu_1$ is infinitely-divisible this semigroup coincides with the semigroup of the infinitely-divisible distributions that we defined before.

From the point of view of large random matrices, the free compression corresponds to taking a submatrix of a given random matrix. More precisely, we take a diagonal $N$-by-$N$ matrix and put it in the general position by a random orthogonal change of basis. Then we take the upper-left submatrix of the result with $\lfloor tN \rfloor$ rows and $\lfloor tN \rfloor$ columns, and we diagonalize this submatrix. If the entries of the initial diagonal matrices converged in distribution to the distribution $\mu$, then the entries of the result converged to the free compression of this distribution by the projection with the dimension $t$. 
12. LECTURE 12. PRODUCTS OF FREE RANDOM VARIABLES

Similarly to the free additive convolution, the free multiplicative convolution can be used to define the infinitely-divisible probability distributions with corresponding limit theorems. This development is similar to the theory of infinitely-divisible distributions with respect to the free additive convolution and we will not discuss it further. However, it interesting that there are two different approaches to the limit laws in the case of free multiplicative convolutions.

In the first approach, we look at the products of a large number of variables which deviates from the identity by only a small amount. In other words, we look at the product $\prod_{i=1}^{n} (I + X_{i,n})$, where $X_{i,n}$ are appropriately small. This approach eventually leads to infinitely-divisible distributions as the limits of the products.

The second approach is to multiply many $X_i$, which are not necessarily close to the identity operator and then to normalize the product. In the case of additive convolutions, this distinction did not matter because the operation of normalization commutes with the addition of variables. However, the first and second approaches give different results in the case of multiplication of random variables because the operations of normalization (such as taking logarithms and dividing by $n$, or by taking power roots) do not commute with multiplication.

We will talk about the second approach in this lecture.

Let $\mu$ be a probability measure which is compactly supported on $\mathbb{R}^+$. We are interested in the support of the $n$-time free multiplicative convolution of the measure $\mu$ with itself, which we denote as $\mu_n$:

$$\mu_n = \mu \boxtimes \ldots \boxtimes \mu.$$  

**Theorem 12.1.** Suppose that $\mu$ is a compactly-supported probability measure on $\mathbb{R}^+$, with the expectation 1 and variance $V$. Let $L_n$ denote the upper boundary of the support of $\mu_n$. Then

$$\lim_{n \to \infty} \frac{L_n}{n} = eV,$$

where $e$ denotes the base of natural logarithms, $e = 2.71\ldots$

This theorem can be interpreted as result that relates two different norms of product operators, the usual uniform and operator $L^2$-norm. Let $X_i$ be operators in a von Neumann algebra $\mathcal{A}$ with trace $E$. Assume that $X_i$ are free and identically distributed, and let $\Pi_n = X_1 \ldots X_n$. Note that if $\mu$ is the spectral probability measure of $X_i^*X_i$, then $\mu_n$ is the spectral probability measure of $\Pi_n^*\Pi_n$. Assume further that

$$E (X_i^*X_i) = 1,$$

which means that $L^2$-norm of $X_i$ is 1, and that

$$E \left( (X_i^*X_i)^2 \right) = 1 + V,$$

which means that $\mu$ has expectation 1 and variance $V$. It is easy to check that $\|\Pi_n\|^2 = E (\Pi_n^*\Pi_n) = 1$, and the theorem says that

$$\|\Pi_n\| = \sqrt{eV} \sqrt{n}.$$
More generally, if \( E(X_i^* X_i) \) is arbitrary, we define
\[
\| \Pi_n \|_2 := \sqrt{E(\Pi_n^* \Pi_n)},
\]
and it is easy to check that Theorem 12.1 implies that
\[
\| \Pi_n \| \sim \sqrt{eV} \| \Pi_n \|_2
\]
for sufficiently large \( n \). Here
\[
V := \frac{E(\sqrt{(X_i^* X_i)^2})}{[E(X_i^* X_i)]^2} - 1.
\]

To say it very roughly, the largest singular value of the product approximately equals a certain average of all singular values multiplied by \( \sqrt{n} \) and a proportionality constant.

In the proof we will use the following notation. Let \( \psi_{\mu}(z) = \sum_{k=1}^{\infty} a_k z^k \), where \( a_k = \int t^k d\mu \), and let functional inverse of \( \psi_{\mu}(z) \) be denoted as \( \psi_{\mu}^{(-1)}(z) \). It is a well-defined analytic function in a neighborhood of \( z = 0 \), provided that \( a_1 \neq 0 \).

Suppose that \( \mu \) and \( \nu \) are two probability measures supported on \( \mathbb{R}^+ = \{ x | x \geq 0 \} \) and let \( \psi_{\mu}^{(-1)}(z) \) and \( \psi_{\nu}^{(-1)}(z) \) be their inverse \( \psi \)-transforms. Then, by Voiculescu’s formula
\[
\psi_{\mu \boxtimes \nu}^{(-1)}(z) := \left(1 + z^{-1}\right) \psi_{\mu}^{(-1)}(z) \psi_{\nu}^{(-1)}(z).
\]

Let \( \psi_n(z) \) and \( \psi_n^{(-1)}(u) \) denote the direct and inverse \( \psi \)-functions for measure \( \mu_n \). The main idea of proof of Theorem 12.1 is based on the fact that the radius of convergence of Taylor series for \( \psi_n(z) \) is \( 1/L_n \). Therefore, the function \( \psi_n(z) \) must have a singularity at the boundary of the disc \( |z| = 1/L_n \). Since all the coefficients in this Taylor series are real and positive, the singularity is \( z_n = 1/L_n \). Therefore, the study of \( L_n \) is equivalent to the study of the singularity of \( \psi_n(z) \) which is located on \( \mathbb{R}^+ \) and which is closest to 0.

By Proposition 5.2 in Belinschi and Bercovici (2005), we know that for all sufficiently large \( n \), the measure \( \mu_n \) is absolutely continuous on \( \mathbb{R}^+ \setminus \{ 0 \} \), and its density is analytic at all points where it is different from zero. For these \( n \), the singularity of \( \psi_n(z) \) is neither an essential singularity nor a pole. Hence, the problem is reduced to finding a branching point of \( \psi_n(z) \) which is on \( \mathbb{R}^+ \) and closest to zero.

The branching point of \( \psi_n(z) \) equals a critical value of \( \psi_n^{(-1)}(u) \). Since by Voiculescu’s theorem,
\[
\psi_n^{(-1)}(u) = \left(\frac{1 + u}{u}\right)^{n-1} \left[\psi^{(-1)}(u)\right]^n,
\]
therefore we can find critical points of \( \psi_n^{(-1)}(u) \) from the equation
\[
\frac{d}{du} \left[ n \log \psi^{(-1)}(u) + (n - 1) \log \left(\frac{1 + u}{u}\right) \right] = 0,
\]
or
\[
\frac{d}{du} \log \psi^{(-1)} (u) = \left( 1 - \frac{1}{n} \right) \frac{1}{u (1 + u)}.
\]

Thus, our task is to estimate the root \( u_n \) of this equation which is real, positive and closest to 0, and then study the asymptotic behavior of \( z_n = \psi_n^{(-1)} (u_n) \) as \( n \to \infty \).

Let us repeat our notation conventions: \( L \) and \( L_n \) are the least upper bounds of the support of measures \( \mu \) and \( \mu_n \), respectively; \( V \) and \( V_n \) are variances of these measures; \( \psi (z) \) and \( \psi_n (z) \) are \( \psi \)-transforms for measures \( \mu \) and \( \mu_n \), and \( \psi^{(-1)} (u) \) and \( \psi_n^{(-1)} (u) \) are functional inverses of these \( \psi \)-transforms.

When we work with \( \psi \)-transforms, we use letters \( t, x, y, z \) to denote variables in the domain of \( \psi \)-transforms, and \( b, u, v, w \) to denote the variables in their range.

In our analysis we need some facts about functions \( \psi (z) \) and \( \psi^{(-1)} (u) \). Let the support of a measure \( \mu \) be inside the interval \([0, L]\), and let \( \mu \) have expectation 1 and variance \( V \). Note that for \( z \in (0, 1/L) \), the function \( \psi (z) \) is positive, increasing, and convex. Correspondingly, for \( u \in (0, \psi (1/L)) \), the function \( \psi^{(-1)} (u) \) is positive, increasing and concave.

**Lemma 12.2.** For all positive \( z \) such that \( z < 1/(2L) \), it is true that
\[
\left| \psi (z) - z - (1 + V) z^2 \right| \leq c_1 z^3,
\]
\[
\left| \psi' (z) - 1 - 2 (1 + V) z \right| \leq c_2 z^2,
\]
where \( c_1 \) and \( c_2 \) depend only on \( L \).

**Proof:** Clearly, \( E (X^k) \leq L^k \). Using the Taylor series for \( \psi (z) \) and \( \psi' (z) \), we find that for all positive \( z \) such that \( z < 1/(2L) \),
\[
\left| \psi (z) - z - (1 + V) z^2 \right| \leq \frac{L^3}{1 - Lz} z^3,
\]
and
\[
\left| \psi' (z) - 1 - 2 (1 + V) z \right| \leq L^3 \frac{3 - 2Lz}{(1 - Lz)^2} z^2,
\]
which implies the statement of this lemma. QED.

**Lemma 12.3.** For all positive \( u \) such that \( u < 1/(12L) \), it is true that
\[
\left| \psi^{(-1)} (u) - u + (1 + V) u^2 \right| \leq c_3 u^3,
\]
where \( c_3 \) depends only on \( L \).

**Proof:** Let the Taylor series for \( \psi^{(-1)} (u) \) be
\[
\psi^{(-1)} (u) = u - (1 + V) u^2 + \sum_{k=3}^{\infty} d_k u^k.
\]
Using the Lagrange inversion formula, it is possible to prove that
\[
|d_k| \leq \frac{3}{2} (6L)^{k-1},
\]
see, e.g., proof of Lemmas 3 and 4 in [3]. This implies that the Taylor series for \( \psi^{(-1)}(u) \) are convergent in the disc \( |u| < (6L)^{-1} \). Hence, in this disc,

\[
\left| \sum_{k=3}^{\infty} d_k u^k \right| \leq \frac{54L^2}{1-6Lu^3} u^3,
\]

which implies the statement of this lemma. QED.

The proof of Theorem [12.1] uses the following proposition. Its purpose is to estimate the critical point of \( \psi^{(-1)}(u) \) from below. Later, we will see that this estimate gives the asymptotically correct order of magnitude of the critical point.

**Proposition 12.4.** Let \( u_n \) be the critical point of \( \psi^{(-1)}(u) \) which belongs to \( \mathbb{R}^+ \) and which is closest to 0. Then for all \( \varepsilon > 0 \), there exists such \( n_0(L,V,\varepsilon) \), that for all \( n > n_0 \),

\[
u_n \geq \frac{1}{n (1 + 2V + \varepsilon)}.
\]

**Proof of Proposition 12.4.**

**Claim:** Let \( \varepsilon \) be an arbitrary positive constant. Let \( x_n = (n (1 + 2V + 2\varepsilon))^{-1} \) and \( b_n = \psi(x_n) \). Then for all \( n \geq n_0(\varepsilon, L, V) \) and all \( u \in [0, b_n] \), the following inequality is valid:

\[
\frac{d}{du} \log \psi^{(-1)}(u) > \frac{n-1}{n} \frac{1}{u (1 + u)}.
\]

(30)

If this claim is valid, then since \( u_n \) is the smallest positive root of equation (29), therefore we can conclude that \( u_n > b_n = \psi(x_n) \). By Lemma [12.2] it follows that for all sufficiently large \( n \)

\[
u_n > \psi \left( \frac{1}{n (1 + 2V + 2\varepsilon)} \right) > \frac{1}{n (1 + 2V + \varepsilon)}.
\]

(Indeed, note that the last inequality has \( 2\varepsilon \) and \( \varepsilon \) on the left-hand and right-hand side, respectively. Since Lemma 2 implies that \( \psi(z) \sim z \) for small \( z \), therefore this inequality is valid for all sufficiently large \( n \).)

Hence, Proposition [12.4] follows from the claim, and it remains to prove the claim.

**Proof of Claim:** Let us re-write inequality (30) as

\[
n - 1 > \frac{1}{\psi^{(-1)}(u) (1 + \psi^{(-1)}(u))}.
\]

(31)

where \( z = \psi^{(-1)}(u) \).

Using Lemma [12.2] we infer that inequality (31) is implied by the following inequality:

\[
\frac{1}{z} > \frac{1}{n \psi(z) (1 + \psi(z))},
\]

where \( z = \psi^{(-1)}(u) \).

Using Lemma [12.2] we infer that inequality (31) is implied by the following inequality:
where \( c_2 \) depends only on \( L \). Note that \( \psi (z) \geq z \) because the first moment of \( \mu \) is 1 and all other moments are positive. Therefore, it is enough to show that

\[
\frac{1}{1 + 2 (1 + V) z + c_2 z^2} > \frac{n - 1}{n - 1} \frac{1}{1 + z}
\]

for \( z \leq (n (1 + 2 V + 2 \varepsilon))^{-1} \) and all sufficiently large \( n \). Let us write this inequality as

\[
\frac{1}{n - 1} + \frac{1}{n - 1} z > (1 + 2V) z + c_2 z^2.
\]

If we fix an arbitrary \( \varepsilon > 0 \), then clearly for all \( z \leq (n (1 + 2 V + 2 \varepsilon))^{-1} \) this inequality holds if \( n \) is sufficiently large. QED.

This completes the proof of Proposition 12.4.

Now let us proceed with the proof of Theorem 12.1.

Let \( u_n \) be the critical point of \( \psi_n^{(-1)} (u) \), which is positive and closest to zero, and let \( y_n = \psi_n^{(-1)} (u_n) \). We know that \( y_n \) is a root of the equation

\[
\frac{1}{z \psi' (z)} \frac{1}{z (1 + \psi (z))} = \left( 1 - \frac{1}{n} \right) \frac{1}{\psi' (z)}.
\]

(This is equation (29) in a slightly different form.) After a re-arrangement, we can re-write this equation as

\[
\frac{\psi (z)}{z} (1 + \psi (z)) = \left( 1 - \frac{1}{n} \right) \psi' (z).
\]

On the other hand, from the proof of Proposition 12.4 we know that \( u_n \geq b_n = \psi (x_n) \), so that monotonicity of \( \psi_n^{(-1)} \) implies

\[
y_n = \psi_n^{(-1)} (u_n) \geq x_n = \frac{1}{n (1 + 2V + \varepsilon)}.
\]

Let us look for a root of equation (33) in the range \([x_n, c/n]\) where \( c \) is a fixed positive number. Let us make a substitution \( z = t/n \) in equation (33) and use Lemma 12.2. We get:

\[
\left( 1 + (1 + V) \frac{t}{n} + O \left( n^{-2} \right) \right) \left( 1 + t/n + O \left( n^{-2} \right) \right) = \left( 1 - \frac{1}{n} \right) \left( 1 + 2 (1 + V) \frac{t}{n} + O \left( n^{-2} \right) \right).
\]

After a simplification, we get

\[
t - \frac{1}{V} + O \left( n^{-1} \right) = 0.
\]

Hence, for a fixed \( c > 1 \) and all sufficiently large \( n \), the root is unique in the interval \([0, c]\) and given by the expression

\[
t = \frac{1}{V} + O \left( n^{-1} \right).
\]

Therefore,

\[
y_n = \frac{1}{V n} + O \left( n^{-2} \right).
\]
By Lemma \[12.2\] this implies that
\[ u_n = \psi(y_n) = \frac{1}{Vn} + O\left(n^{-2}\right). \]

This is the critical point of \( \psi_n^{(-1)}(u) \).

The next step is to estimate the critical value of \( \psi_n^{(-1)}(u) \), which is \( z_n = \psi_n^{(-1)}(u_n) \). We write:
\[ z_n = u_n \left[ \frac{\psi_n^{(-1)}(u_n)}{u_n} \right]^n (1 + u_n)^{n-1}. \]

Using Lemma \[12.3\] we infer that
\[ z_n = u_n \left[ 1 - (1 + V) \frac{1}{Vn} + O\left(n^{-2}\right) \right]^n (1 + u_n)^{n-1} \]
\[ = \left( \frac{1}{Vn} + O\left(n^{-2}\right) \right) \times \left[ 1 - (1 + V) \frac{1}{Vn} + O\left(n^{-2}\right) \right]^n \times \left[ 1 + \frac{1}{Vn} + O\left(n^{-2}\right) \right]^n \]
\[ \sim \frac{1}{eVn}. \]

as \( n \to \infty \). Here \( e \) denotes the base of the natural logarithm: \( e = 2.71 \ldots \)

Hence,
\[ \lim_{n \to \infty} \frac{L_n}{n} = \lim_{n \to \infty} \frac{1}{nz_n} = eV. \]

QED.
13. Lecture 13. Free Cumulants

Cumulants were invented by the Dutch scientist T. N. Thiele in the end of XIX century. They were found very useful for analysis of nonlinear transformations of random variables. In particular, Brillinger ([6]) used them as the main technical tool in the spectral analysis of time series.

Why cumulants are useful? Let us fix \( n \) random variables \( X_1, X_2, \ldots, X_n \). The interaction of these variables, or, in other words, their dependence structure can be captured by their moments: \( E(X_{i_1}X_{i_2}\ldots X_{i_n}) \). However, the interactions are not transparent. For a simple example of this phenomenon, note that even if \( X_1 \) and \( X_2 \) are independent \( E(X_1X_2) \) is not necessarily zero. To do better than moments, we can try to isolate the interaction between the \( n \) variables by subtracting from the moment all contributions implied by lower-order interactions. The theory of cumulants is a sophisticated version of this idea.

An especially convenient form to the theory of cumulants was given by Terry Speed ([19]). In this work, cumulants arise as an application of the abstract theory of Möbius inversion on lattices invented by G.-C. Rota ([16]). This abstract Möbius inversion generalizes the following construction in number theory (see Sections 16.3-5 in [12]).

Let \( f(n) \) be a multiplicative function on positive integers (i.e., \( f(nm) = f(n)f(m) \) for relatively prime \( n \) and \( m \)). Then we can define another function:

\[
g(n) = \sum_{d|n} f(d),
\]

where \( d|n \) means that \( d \) is a divisor of \( n \).

This new function \( g(n) \) is also multiplicative. Indeed, if \( n \) and \( m \) are relatively prime, then

\[
g(nm) = \sum_{d|nm} f(d)
\]

\[
= \sum_{d_1|n, d_2|m} f(d_1d_2)
\]

\[
= \sum_{d_1|n} f(d_1) \sum_{d_2|m} f(d_2)
\]

\[
= g(n)g(m).
\]

An important question is how to recover the function \( f(n) \) from \( g(n) \). Let the Möbius function \( \mu(n) \) be defined as follows:

(i) \( \mu(1) = 1 \);
(ii) \( \mu(n) = 0 \), if \( n \) has a squared factor, and
(iii) \( \mu(p_1p_2\ldots p_k) = (-1)^k \), if all the primes \( p_1, p_2, \ldots, p_k \) are different.

Note that \( \mu(n) \) is evidently multiplicative. It follows that

\[
\varsigma(n) = \sum_{d|n} \mu(d)
\]
must be multiplicative. However, we can easily compute that \( \varsigma(1) = 1 \), \( \varsigma(p^k) = 1 + \mu(p) = 0 \). It follows that \( \varsigma(n) = 0 \) for all \( n > 1 \).

Then the answer to our inversion question is given by the following formula:

\[
f(n) = \sum_{d|n} g(d) \mu \left( \frac{n}{d} \right).
\]

(34)

Indeed,

\[
\sum_{d|n} g(d) \mu \left( \frac{n}{d} \right) = \sum_{d|n} \left( \sum_{c|d} f(c) \right) \mu \left( \frac{n}{d} \right)
\]

\[
= \sum_{c|n} f(c) \sum_{b|\frac{n}{c}} \mu(b)
\]

\[
= \sum_{c|n} f(c) \varsigma \left( \frac{n}{c} \right)
\]

\[
= f(n).
\]

The converse statement is also true. If we take any multiplicative function \( g(n) \) and define \( f(n) \) by formula (34), then \( f(n) \) is multiplicative and

\[
g(n) = \sum_{d|n} f(d).
\]

Exercise 13.1. Let \( g(n) = n \). Show that \( f(n) \) is the Euler function \( \phi(n) \), which gives the number of the positive integers not greater than and prime to \( n \).

Another example is of a similar inversion formula is the finite difference formula. Let

\[
g(n) = \sum_{0 \leq d \leq n} f(d).
\]

Then we can recover \( f(n) \) from \( g(n) \) by the simple formula

\[
f(n) = g(n) - g(n-1),
\]

which we can write as

\[
f(n) = \sum_{0 \leq d \leq n} g(d) \mu(d, n),
\]

provided that we define \( \mu(d, n) \) as follows:

(i) \( \mu(d, n) = 1 \) if \( d = n \);
(ii) \( \mu(d, n) = -1 \) if \( d = n - 1 \);
(iii) \( \mu(d, n) = 0 \) if \( d < n - 1 \).

In the general situation, we consider the functions not of integers but rather of elements of a lattice. A lattice is a partially ordered set, also called a poset, which has some additional properties. A poset is a set with an order relation \( \prec \) which is defined only for some pairs of the set. The order relation satisfies the transitivity and reflexivity properties: (i) if \( a \prec b \) and \( b \prec c \), then \( a \prec c \), (ii) \( a \prec a \) for all \( a \).

In addition, it is convenient to assume that if \( a \prec b \) and \( b \prec a \), then \( a = b \).
Here are some examples of posets:
(i) positive integers with respect to divisibility relation;
(ii) subsets of a finite set with respect to inclusion relation (so that \( \{1\} \prec \{1, 2\} \));
(iii) partitions of a finite set with respect to refinement relation (so that \( (\{1\}, \{2, 3\}) \prec (\{1, 2, 3\}) \));
(iv) linear subspaces of a finite-dimensional space with respect to inclusion relation;
(v) Hermitian matrices with the following relation: \( A \prec B \) if and only if \( B - A \) is positive definite.

A poset is a lattice if the following two conditions are satisfied. First, for every two elements \( a \) and \( b \) the set of those elements that dominate both of them has the least element. In other words, we can find such \( c \) that (i) \( a \prec c \) and \( b \prec c \), and (ii) if \( a \prec c' \) and \( b \prec c' \), then \( c \prec c' \). This element is called the join of \( a \) and \( b \) and denoted \( a \lor b \).

Second, for every \( a \) and \( b \) the set of those elements that are less than both of them has the largest element. Such an element is called the meet of \( a \) and \( b \), and denoted \( a \land b \). (A prototypical example of a lattice is the lattice of all subsets of a finite set and operations \( \lor \) and \( \land \) are analogues of the union and the intersection operations \( \cup \) and \( \cap \) for sets.)

The join and meet operations of a lattice are assumed to satisfy certain properties. These properties hold automatically if the lattice was constructed from the poset as above and the underlying poset is finite.

The examples above are all lattices except for the last one. If we have two matrices \( A \) and \( B \), then, in general, there are several matrices \( C \) such that \( C - B \) and \( C - A \) are positive definite and these matrices are not necessarily comparable. Hence, the join is not defined.

As another example note that algebraic integers are posets with respect to the divisibility property but very often they are not lattices. It is necessary to add algebraic ideals to make them lattices.

Now, let us consider a finite lattice which has a maximum and a minimum element. The maximum element, denoted \( 1 \), is greater than any other element in the lattice, and the minimum element, \( 0 \), is smaller than any other element. Let us take a function \( f(x) \) which is defined on the elements of the lattice and let us write:

\[
g(y) = \sum_{x \prec y} f(x).
\]

How do we invert this relation? Following Rota, we write:

\[
f(y) = \sum_{x \prec y} f(x) \mu(x, y),
\]

where \( \mu(x, y) \) is a certain function, which we call the Möbius function.

The analysis of cumulants by Terry Speed is based on the Möbius inversion for the lattice of partitions. Speicher’s construction of free cumulants uses the lattice of non-crossing partitions which we are going to describe next. This lattice and most of its properties were discovered by Kreweras.
Consider the ordered set of $n$ elements, $\Omega = \{1, 2, \ldots, n\}$. A partition of $\Omega$ has a \textit{crossing} if we can find elements $i, j, k, l$, such that $i < j < k < l$, $i$ and $k$ belong to one piece of the partition and $j$ and $k$ belong to another piece.

For example, the partition $\{(\{1, 3\}, \{2, 4\})\}$ has a crossing, and the partition $\{(\{1, 4\}, \{2, 3\})\}$ does not have a crossing.

The partitions that do not have a crossing are called \textit{non-crossing partitions}. The set of all non-crossing partitions forms a lattice with respect to the refinement order. (That is, $a \prec b$ if $a$ is the refinement of $b$.) We will denote this lattice as $NC(n)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{$NC(4)$, non-crossing partitions of a 4-element ordered set.}
\end{figure}

Figure 16 shows all non-crossing partitions of 4 elements together with the lattice structure. Smaller elements are positioned lower on the diagram. It is understood that there are numbers 1, 2, 3, 4 on top of the solid lines and that these lines connect the numbers in the same piece of the partition. The dashed lines show the partial order.

A very important property of the lattice of non-crossing partitions is the existence of the \textit{Kreweras complement} operation. Let $\pi$ be a non-crossing partition of the ordered set $\{1, 2, \ldots, n\}$. Consider the ordered set $\{1, 2, \ldots, n, \pi\}$. The Kreweras complement of $\pi$, denoted by $K(\pi)$, is the largest non-crossing partition of the set $\{1, 2, \ldots, n, \pi\}$ such that union of the partitions $\pi$ and $K(\pi)$, is a
non-crossing partition of the set \( \{1, 2, 3, \ldots, n, \varpi\} \). An example is shown in Figure 17. In this example, it is shown that the Kreweras complement of partition \( \pi = (145)(23) \) is the partition \( K(\pi) = (13)(2)(4)(5) \).

The Kreweras complement is an anti-isomorphism of the lattice of non-crossing partitions in the sense that if \( \lambda \prec \pi \), then \( K(\lambda) \succ K(\pi) \). In particular, \( K(0) = 1 \) and \( K(1) = 0 \). One other property of the Kreweras complement is that \( K(K(\pi)) \) is a shift of \( \pi \). We could remove this small inconvenience and make sure that \( K(K(\pi)) = \pi \) if we considered the set \( \{1, 2, \ldots, n\} \) as the set of \( n \) ordered points on a circle and modified the definitions accordingly.

Let us fix \( n \) non-commutative variables \( X_1, X_2, \ldots, X_n \) and let \( \lambda = (\lambda^1, \ldots, \lambda^k) \) be a partition of the ordered set \( \Omega = \{1, \ldots, n\} \). That is, each \( \lambda^i \) is an ordered sequence of elements of \( \Omega \). We write the elements of \( \lambda^i \) as \( \lambda_1^i, \lambda_2^i, \ldots, \lambda_{|\lambda^i|}^i \). In our considerations we will assume that

\[
\lambda_1^i < \lambda_2^i < \ldots < \lambda_{|\lambda^i|}^i
\]

for all \( i \).

We define the \textit{generalized moment} of variables \( X_1, X_2, \ldots, X_n \) with respect to partition \( \lambda \) as

\[
E_{\lambda}(X_1, \ldots, X_n) := \prod_{i=1}^{k} E\left( X_{\lambda_1^i} X_{\lambda_2^i} \ldots X_{\lambda_{|\lambda^i|}^i} \right).
\]

In other words, the generalized moment is the product of usual moments with the structure of the product determined by partition \( \lambda \).

For example, if the partition is \( \lambda = (\{1, 4\}, \{2, 3\}) \), then the corresponding generalized moment is

\[
E_{\lambda}(X_1, X_2, X_3, X_4) = E(X_1X_4) E(X_2X_3).
\]

In this way the generalized moments are defined for all partitions of the set \( \{1, \ldots, n\} \). In particular, they are defined for all non-crossing partitions. Since we will treat variables \( X_1, \ldots, X_n \) as fixed we will often suppress then in the notation for the generalized moments and will write simply \( E_{\lambda} \) instead of \( E_{\lambda}(X_1, \ldots, X_n) \).
Note that the generalized moments are multiplicative in the following sense. Let us the variables are divided in two groups in such a way that every part of $\lambda$ contains variables from either one or the other group but not from both simultaneously. Then $\lambda$ induces partitions of these two groups of variables, which we denote $\sigma$ and $\pi$. Then, multiplicativity of the generalized moments means that $E_\lambda = E_\sigma E_\pi$.

The generalized moments capture the interaction of variables $X_1, \ldots, X_n$ to a certain degree. However, they are not easy to interprete. Even if the variables are free, the generalized moments can be different from zero. We have to go to free cumulants to obtain easily interpretable numbers.

The free cumulants of variables $X_1, \ldots, X_n$ are the results of the Rota inversion of the generalized moments with respect to the lattice of non-crossing partitions. In other words, let the generalized moments of variables $X_1, \ldots, X_n$ be $E_\sigma$, where $\sigma$ is any non-crossing partition. Then, we define free cumulants $k_\lambda$ as such numbers that

$$E_\sigma = \sum_{\lambda \prec \sigma} k_\lambda,$$

where the order relation $\prec$ is as it is defined in the lattice of non-crossing partitions. The inversion formula can written as follows:

$$k_\sigma = \sum_{\lambda \prec \sigma} \mu(\lambda, \sigma) E_\lambda,$$  \hfill (35)

where $\mu$ is the Mobius function of the lattice of non-crossing partitions.

More generally, if $f_\sigma$ is an arbitrary multiplicative function on the lattice of non-crossing partitions, then we can define $g_\lambda$ by requiring that

$$f_\sigma = \sum_{\lambda \prec \sigma} g_\lambda.$$

The Rota inversion formula says that

$$g_\sigma = \sum_{\lambda \prec \sigma} \mu(\lambda, \sigma) f_\lambda.$$  \hfill (36)

In the following exercises, we show how to compute the Mobius function. First of all, it is not difficult to see that the fact that the generalized moments are multiplicative implies that free cumulants are multiplicative as well.

Moreover, the Mobius function of the lattice of non-crossing partitions is also multiplicative in the following sense. Suppose that $\lambda \prec \sigma$ and $\sigma = \sigma_1\sigma_2$. Then, we can factorize $\lambda$ as $\lambda_1\lambda_2$ so that $\lambda_1 \prec \sigma_1$ and $\lambda_2 \prec \sigma_2$. Then, $\mu(\lambda, \sigma) = \mu(\lambda_1, \sigma_1) \mu(\lambda_2, \sigma_2)$. This multiplicativity property implies that we can compute $\mu(\lambda, \sigma)$ for arbitrary $\lambda$ and $\sigma$ if we know how to compute $\mu(\lambda, 1_n)$ for all $\lambda$ and $n$.

For the next step, we recall that the Kreweras complement is anti-isomorphism of the lattice of non-crossing partitions. This implies that $\mu(\lambda, \sigma) = \mu(K(\sigma), K(\lambda))$, where $K$ denotes the Kreweras complement operation. In particular, $\mu(\lambda, 1_n) = \mu(0_n, K(\lambda))$. From this fact and the multiplicativity of the Mobius function, we can conclude that we can compute $\mu(\lambda, 1_n)$ if we know how to compute $\mu(0_k, 1_k)$ for arbitrary $k$. 


Exercise 13.2. Show that the Mobius function satisfies the following relations:

(i) \( \sum_{\tau \prec \lambda \prec \sigma} \mu(\lambda, \sigma) = 0 \) if \( \tau \neq \sigma \), and

(ii) \( \mu(\sigma, \sigma) = 1 \) for all \( \sigma \).

Exercise 13.3. By using Exercise 13.2, compute explicitly that

\( \mu(0,1) = 1 \), \( \mu(0,2) = -1 \), and \( \mu(0,3) = 2 \).

In order to proceed further, we need a more refined version of the inversion formula (36).

Exercise 13.4. Show that

\[
\sum_{\tau \prec \lambda \prec \sigma} \mu(\lambda, \sigma) f_\lambda = \sum_{\tau \lor \lambda = \sigma} k_\lambda. \tag{37}
\]

Exercise 13.5. Set \( g_\lambda = \mu(0, \lambda) \). Compute corresponding \( f_\sigma \) by using Exercise 13.2. Use (37) in order to show that for every \( \tau \neq 0 \),

\[
\sum_{\tau \lor \lambda = 1} \mu(0, \lambda) = 0.
\]

In other words, for every \( \tau \neq 0 \),

\[
\mu(0, 1) = -\sum_{\tau \lor \lambda = 1, \lambda \neq 1} \mu(0, \lambda). \tag{38}
\]

![Figure 18. Partition \( \tau \)](image)

Theorem 13.6. The Mobius function of the lattice of non-crossing partitions is determined by the formula

\[
\mu(0_n, 1_n) = (-1)^{n-1} C_{n-1},
\]

where \( C_k \) denote Catalan numbers.

Proof: Let us use formula (38) with the partition \( \tau = (\{1\}, \{2\}, \ldots, \{n-2\}, \{n-1, n\}) \) shown in Figure 18. Suppose that \( \lambda \neq 1_n \) and that it satisfies equation \( \tau \lor \lambda = 1_n \).

It is clear that then \( \lambda \) can have one of the two forms shown in Figure 19. By using multiplicativity, we conclude that the Mobius function satisfy the following recursive relation:

\[
\mu(0_n, 1_n) = -\sum_{k=1}^{n-1} \mu(0_k, 1_k) \mu(0_{n-k}, 1_{n-k}).
\]

This implies that \( (-1)^{n-1} \mu(0_n, 1_n) \) satisfy the recursive equation for Catalan numbers. QED.
For example, let $\lambda = (\{1, 4\}, \{2, 3\})$. Then $\mu(\lambda, 1_4) = \mu(0_4, K(\lambda)) = \mu(0_1, 1_1)^2 \mu(0_2, 1_2) = -1$. For another example, let $\lambda = (\{1\}, \{2\}, \{3, 4\})$. Then $\mu(\lambda, 1_4) = \mu(0_4, K(\lambda)) = \mu(0_1, 1_1) \mu(0_3, 1_3) = 2$.

It is a bit more convenient to use notation $k_n$ for $k_1n$, and we will assume this notation in the following.

13.1. **Addendum: Non-crossing partitions and group of permutations.** There is another interpretation of this lattice which is very nice and perhaps more natural. Let $S_n$ be the group of permutation of $n$ elements $\{1, \ldots, n\}$. The length of permutation $\sigma$ is the minimal number of transpositions in a product of transpositions that represent $\sigma$. We denote this quantity as $|\sigma|$. If the permutation is a product of $c(\sigma)$ cycles with disjoint support then $|\sigma| = n - c(\sigma)$. We define the distance between permutations $\sigma$ and $\tau$ as

$$d(\sigma, \tau) = |\sigma^{-1}\tau|.$$  

It can be checked that this is a metric.

**Exercise 13.7.** Check the triangle inequality for this distance.

We say that $\sigma$ belongs to a geodesic between $\rho$ and $\tau$, if $d(\rho, \tau) = d(\rho, \sigma) + d(\sigma, \tau)$.

Let $\tau = (12\ldots n)$, that is, $\tau$ is a cyclic permutation of elements $1, 2, \ldots, n$. Consider the set of all permutations that belong to a geodesic between the identity permutation $e$ and $\tau$. Hence, a permutation $\sigma$ belongs to this set if and only if

$$|\sigma| + |\tau^{-1}\sigma| = n - 1.$$  

In terms of the number of cycles, this can be written as

$$c(\sigma) + c(\tau^{-1}\sigma) = n + 1.$$  

We can define a partial order on this set in the following way. We say that $\sigma \prec \pi$ if $\sigma$ belongs to a geodesic between $e$ and $\pi$. In other words, $\sigma \prec \pi$ if

$$|\pi| = |\sigma| + |\sigma^{-1}\pi|.$$  

**Figure 19.** Two possible types of partition $\lambda$.  

\[
\begin{array}{cccccccc}
1 & 2 & \cdot & \cdot & \cdot & n-2 & n-1 & n \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 2 & \cdot & i & i+1 & n-2 & n-1 & n \\
\end{array}
\]
It turns out that this defines a partial order and the set of permutations becomes a lattice. Moreover, this lattice is isomorphic to the lattice of non-crossing partitions of the set \( \{1, 2, \ldots, n\} \). In order to see this, let us define a bijection between the set of all permutations that belong to a geodesic between \( e \) and \( \tau \), and the set of all noncrossing partitions.

Recall that every permutation can be represented as a product of cycles with disjoint support. The supports of the cycles define a partition of the set \( \{1, 2, \ldots, n\} \). Moreover, if we agree to write the cycles starting with the smallest element in the cycle, then we have a partition with ordered parts. The key observation is as follows.

**Lemma 13.8.** If permutation \( \sigma \) belong to a geodesic between \( e \) and \( \tau = (12 \ldots n) \), then the corresponding partition is non-crossing. Moreover, each non-crossing partition corresponds to a permutation on a geodesic between \( e \) and \( \tau \).

Proof: Since permutation \( \sigma \) belongs to a geodesic between \( e \) and \( \tau \), hence we can write it as a product \( s_k s_{k-1} \ldots s_1 \tau \) where \( s_i \) are transpositions and application of each consequitive \( s_i \) decreases the length of the product by one:

\[
|s_is_{i-1} \ldots s_1\tau| = |s_{i-1} \ldots s_1\tau| - 1.
\]

This means that application of each consequitive \( s_i \) increases the number of cycles in the product by one.

Now, it is easy to check that if a transposition \( s = (ij) \) is applied to a permutation \( \lambda \), then the number of cycles increases if and only if \( i \) and \( j \) belong to the same cycle of \( \lambda \).

Consider for example, what happens if we apply \( s \) to \( \tau = (12 \ldots n) \). Then

\[
s\tau = (12 \ldots i - 1, j, j + 1, \ldots n) (i, i + 1, \ldots, j - 1).
\]

This means that the only block of the partition corresponding to \( \tau \), was split in two blocks which form the partition corresponding to \( s\tau \). Most importantly, these two blocks are non-crossing.

It is easy to see that as we continue multiplying by the transpositions, a block of the partition will be split into two blocks in such a way that the non-crossing property is preserved. Hence, the partition corresponding to \( \sigma \) is non-crossing.

In the other direction, we can always represent a non-crossing partition by a series of split operations that start from partition \( 1_n = (\{1, 2, \ldots, n\}) \). In order to see that this is true, consider an inner block \( b \) of a non-crossing partition \( \pi \). By inner block we mean such a block \( b \) that there are no element of another block between elements of block \( b \). Since \( \pi \) is non-crossing such a block always exists. Let \( b = \{i, i + 1, \ldots, j - 1\} \). Consider then the block \( b' \) that contains element \( j \) and see that these two blocks can be obtained from the union of these two blocks \( b \cup b' \) by the split operation corresponding to the transposition \( (ij) \). Hence we can replace \( \pi \) by the partition \( \pi' \) that have the same blocks as \( \pi \) except that blocks \( b \) and \( b' \) are joined together. Then \( \pi' \) is non-crossing and contains one block less than \( \pi \). Continuing in this fashion we eventually reach partition \( 1_n \).

We define the permutation as the product \( s_k s_{k-1} \ldots s_1 \tau \), where \( s_i \) are the transposition corresponding to the splits. It is easy to see that the partition corresponding
to this permutation is \( \pi \) as required, and that the permutation belongs to a geodesic between \( e \) and \( \tau \). QED.

In a similar fashion it is easy to show that if \( \lambda \) lies on the geodesic between \( \sigma \) and \( \tau \), then the partition corresponding to \( \lambda \) is less refined than the partition corresponding to \( \sigma \). (It requires less splits of blocks to get the partition of \( \lambda \).) It follows that the notion of order for permutations that we introduced above is in agreement with our order for partitions, and we can write \( \sigma \prec \lambda \) for both permutations \( \sigma \) and \( \lambda \) and the corresponding partitions. This fact establishes isomorphism between the lattice of non-crossing partitions and the lattice of permutations that lie on the geodesic between \( e \) and \( \tau = (12 \ldots n) \).

The isomorphism between non-crossing partitions and geodesic permutations allows us to define the Kreweras complement of a geodesic permutation.

**Exercise 13.9.** Show that if \( \pi \) is a geodesic permutation (i.e., it belongs to a geodesic between \( e \) and \( \tau = (12 \ldots n) \)), then its Kreweras complement can be computed by the formula \( K(\pi) = \pi^{-1} \tau \).
14. Lecture 14. Fundamental Properties of Free Cumulants

The main property of free cumulants is that the cumulants of free variables equals to zero. This means that if we are given a set of variables $X_1, X_2, \ldots, X_n$ and we can divide it in two parts, such that the variables in one part are independent from the variables in the other part, then the free cumulant $k_n (X_1, X_2, \ldots, X_n) = 0$.

We will formulate and prove this result a little bit later. In the proof we will need another important property of free cumulants.

Let variables $x_1, \ldots, x_n$ be given and fix $1 = i_0 < i_1 < i_2 < \ldots < i_m = n$. Then we can form products: $X_1 = x_{i_1}, X_2 = x_{i_1 + 1} \ldots x_{i_2}, \ldots, X_m = x_{i_{m-1} + 1} \ldots x_n$. We want to compute the free cumulants of variables $X_1, \ldots, X_m$ in terms of free cumulants of variables $x_1, \ldots, x_n$.

Let us define partition $\hat{0}_m$ as the partition $(\{1, \ldots, i_1\}, \{i_1 + 1, \ldots, i_2\}, \ldots, \{i_{m-1} + 1, i_m = n\})$. See Figure ... for illustration. This is an example of a lift of a partition of $\{1, \ldots, m\}$ to a partition of $\{1, \ldots, n\}$. This lift can be defined for every partition of $\{1, \ldots, m\}$. We simply substitute every element $k$ with elements $i_{k-1}, i_{k-1} + 1, \ldots, i_k$. It is easy to see that a lift of a non-crossing partition is non-crossing. We will denote a lift of partition $\pi$ by $\hat{\pi}$.

**Lemma 14.1.**

$$\mu (\pi, 1_m) = \mu (\hat{\pi}, 1_n)$$

**Proof:** Suppose that $\hat{\pi} \prec \tau$. Then it is easy to see that $\tau$ is a lift of a certain element in $NC(m)$, which we can call $\lambda$, so that $\tau = \hat{\lambda}$. Moreover, $\pi \prec \lambda$.

In the other direction, if $\pi \prec \lambda$ then $\hat{\pi} \prec \hat{\lambda}$. This implies that the lift operation maps interval $[\pi, 1_m]$ on interval $[\hat{\pi}, 1_n]$ isomorphically. Then, the definition of the Mobius function implies that $\mu (\pi, 1_m) = \mu (\hat{\pi}, 1_n)$. QED.

By using this lemma, we can prove the following result.

**Theorem 14.2.**

$$k_m (X_1, \ldots, X_m) = \sum_{\pi \vee \hat{0}_m = 1_n} k_{\pi} (x_1, \ldots, x_n)$$

The condition $\pi \vee \hat{0}_m = 1_n$ means that partition $\pi$ must connect the blocks of $\hat{0}_m$. It is interesting that exactly the same formula holds for classical cumulants, except that in the classical case the sum must be over all partitions that satisfy the condition $\pi \vee 0_m = 1_n$, and in the free case the sum is over non-crossing partitions only.

**Proof:** We can write

$$k_m (X_1, \ldots, X_m) = \sum_{\pi \in NC(m)} \mu (\pi, 1_n) E_{\pi} (X_1, \ldots, X_m)$$

$$= \sum_{\pi \in NC(m)} \mu (\hat{\pi}, 1_n) E_{\hat{\pi}} (x_1, \ldots, x_n),$$
where we used Lemma [14.1] and the fact that
\[ E \left( x_{i_k-1+1} \ldots x_{i_k} \right) = E_{i_k-i_{k-1}} \left( x_{i_k-1+1} \ldots , x_{i_k} \right), \]
which follows from the definition of the generalized moments.

Hence we can continue the calculation as follows:
\[ k_m \left( X_1, \ldots , X_m \right) = \sum_{\mu \neq \lambda} \mu \left( \lambda , 1_n \right) E_\lambda \left( x_1, \ldots , x_n \right) \]
\[ = \sum_{\pi \neq 0_n} k_\pi \left( x_1, \ldots , x_n \right), \]
where the second line follows from formula (37). QED.

By using this theorem, we can prove the fundamental property of free cumulants.

**Theorem 14.3.** Consider variables \( x_1 , \ldots , x_n \) where \( n > 1 \). Let \( 1 \leq m < n \) and suppose that \( m \) of these variables belongs to subalgebra \( A_1 \) and the remaining variables belong to the subalgebra \( A_2 \). Assume that \( A_1 \) and \( A_2 \) are free. Then,
\[ k_n \left( x_1, \ldots , x_n \right) = 0. \]

**Proof:** Let us prove first the following lemma.

**Lemma 14.4.** Consider variables \( x_1 , \ldots , x_n \) where \( n > 1 \) and let \( x_i \) be a constant. Then
\[ k_n \left( x_1, \ldots , x_n \right) = 0. \]

**Proof of lemma:** Suppose without loss of generality that \( x_i = 1 \). It is easy to check that \( k_2 \left( x, 1 \right) = k_2 \left( 1, x \right) = 0 \). In order to prove the statement for \( n > 2 \), let us write
\[ E \left( x_1 \ldots x_{i-1+1} \ldots x_n \right) = \sum_{\lambda \in NC(n)} k_\lambda \left( x_1, \ldots , x_{i-1}, 1, x_{i+1}, \ldots , x_n \right). \]

By induction, all cumulants are equal to zero except for two cases: when \( \lambda = 1_n \) and when \( i \) is a one-element block of partition \( \lambda \). Hence, we obtain:
\[ E \left( x_1 \ldots x_{i-1+1} \ldots x_n \right) = k_n \left( x_1, \ldots , x_{i-1}, 1, x_{i+1}, \ldots , x_n \right) \]
\[ + \sum_{\lambda \in NC(n-1)} k_\lambda \left( x_1, \ldots , x_{i-1}, x_{i+1}, \ldots , x_n \right) \]
\[ = k_n \left( x_1, \ldots , x_{i-1}, 1, x_{i+1}, \ldots , x_n \right) + E \left( x_1 \ldots x_{i-1} x_{i+1} \ldots x_n \right). \]

This implies that
\[ k_n \left( x_1, \ldots , x_{i-1}, 1, x_{i+1}, \ldots , x_n \right) = 0. \]

QED.

In the next step, consider the situation when \( x_1 , \ldots , x_n \) are alternating and centered. That is, let \( x_1 , x_3 , \ldots \) and \( x_2 , x_4 , \ldots \) belong to algebras \( A_1 \) and \( A_2 \) respectively, and let \( E \left( x_i \right) = k_1 \left( x_i \right) = 0 \) for all \( i \). We will proceed by induction on.
Again, it is easy to check that
\[ k(x_1, x_2) = E(x_1 x_2) - E(x_1)E(x_2) = 0 \]
for \( n = 2 \). If \( n > 2 \), then we write:
\[ E(x_1 \ldots x_n) = k_n(x_1, \ldots, x_n) + \sum_{\lambda \in \text{NC}(n)} k_{\lambda}(x_1, \ldots, x_n). \]

If \( \lambda \neq 1 \) and a block of \( \lambda \) connects two variables from different algebras, then \( k_{\lambda} = 0 \) by induction hypothesis. On the other hand, if each block of \( \lambda \) connect variables from the same algebra, then the alternating structure of the sequence \( x_1, \ldots, x_n \) implies that this partition has single-element blocks. Since we assumed that all variables \( x_i \) are centered, this implies that \( k_{\lambda} = 0 \). In sum,
\[ E(x_1 \ldots x_n) = k_n(x_1, \ldots, x_n). \]

However, \( E(x_1 \ldots x_n) = 0 \) by definition of freeness. Hence, \( k_n(x_1, \ldots, x_n) = 0 \), which completes the proof in this case.

It is easy to see that this case and Lemma 14.4 imply that \( k_n(x_1, \ldots, x_n) = 0 \) for alternating sequences \( x_1, \ldots, x_n \) even if variables \( x_i \) are not centered. It is enough to use the fact that the cumulants are multilinear in its arguments and write equations like
\[ k_n(x_1, \ldots, x_n) = k_n(x_1 - E(x_1), x_2, \ldots, x_n) + k_n(E(x_1), x_2, \ldots, x_n) \]
several times.

In the next step, we consider the sequences \( x_1, \ldots, x_n \) which are not alternating. We will use Theorem 14.2. Let \( 0 = i_0 < i_1 < \ldots < i_m = n \) be a sequence such that variables \( x_1, \ldots, x_{i_1} \) belong to \( A_1 \), variables \( x_{i_1+1}, \ldots, x_{i_2} \) belong to \( A_2 \), variables \( x_{i_2+1}, \ldots, x_{i_3} \) belong to \( A_1 \), and so on. (If the sequence \( x_1, \ldots, x_n \) starts with a variable in \( A_2 \), then we choose \( i_1, i_2, \ldots \) in such a way that variables \( x_1, \ldots, x_{i_1} \) belong to \( A_2 \), variables \( x_{i_1+1}, \ldots, x_{i_2} \) belong to \( A_1 \), variables \( x_{i_2+1}, \ldots, x_{i_3} \) belong to \( A_2 \), and so on. The argument is exactly the same in this case.)

Define variables \( X_1 = x_1 \ldots x_{i_1}, X_2 = x_{i_1+1} \ldots x_{i_2}, \) and so on. Then, variables \( X_1, \ldots, X_m \) are alternating and we know that \( k_m(X_1, \ldots, X_m) = 0 \). Therefore by Theorem 14.2, we have:
\[ k_n(x_1, \ldots, x_n) + \sum_{\pi \in \tilde{\text{NC}}_n, \pi \neq 1_n} k_{\pi}(x_1, \ldots, x_n) = 0. \]

If \( \pi \vee \tilde{\rho}_n = 1_n \) then \( \pi \) must connect two variables from different algebras. Hence, if \( \pi \neq 1_n \), we can use induction and conclude that \( k_{\pi} = 0 \). It follows that
\[ k_n(x_1, \ldots, x_n) = 0. \]

QED.

Here is a consequence of this fundamental property of free cumulants. Let us denote \( k_n(X, \ldots, X) \) by \( k_n(X) \).
Corollary 14.5. Let $X$ and $Y$ be free. Then

$$k_n (X + Y) = k_n (X) + k_n (Y).$$

Proof: By definition,

$$k_n (X + Y) = k_n \left( \frac{X + Y, \ldots, X + Y}{n\text{-timed}} \right).$$

We can expand this expression by multicollinearity and note that the mixed cumulants vanish by Theorem 14.3. Hence,

$$k_n (X + Y) = k_n (X, \ldots, X) + k_n (Y, \ldots, Y) = k_n (X) + k_n (Y).$$

QED.

It is natural to form the generating function:

$$\varphi_X (z) = \sum_{n=1}^{\infty} k_n (X) z^n.$$}

Then, Corollary 14.5 implies that

$$\varphi_{X+Y} (z) = \varphi_X (z) + \varphi_X (z).$$

We know that this property is characteristic for the $R$-transforms. It turns out that function $\varphi_X (z)$ coincides with the $R$-transform of variable $X$, as we defined it on p. 31, up to a multiple.

Theorem 14.6. Let

$$\varphi_X (z) = z R_X (z).$$

Proof: Let

$$B_X (z) = 1 + \sum_{n=1}^{\infty} E (X^n) z^n.$$ (39)

Note that $B_X (z) = G_X (z^{-1}) / z$, where $G_X (z)$ is the Cauchy transform of $X$ as defined on p. 31.

By definition of free cumulants, we can write

$$E (X^n) = \sum_{\pi \in NC(n)} k_\pi (X, \ldots, X).$$

If we fix the first the block of $\pi$, then due to the non-crossing part of $\pi$ the remaining part of $\pi$ will be split in portions between the elements of this first block. Then we can use the multiplicativity of free cumulants to write $k_\pi$ as a product. Finally, when we add up all these products and sum over all possible first blocks, then we
can obtain the following formula:

\[
E(\mathbf{X}^n) = \sum_{s=1}^{n} \sum_{i_1+\ldots+i_s+s = n, i_r \geq 0} k_s \left( \sum_{\pi_1 \in NC(i_1)} k_{\pi_1} \right) \cdots \left( \sum_{\pi_s \in NC(i_s)} k_{\pi_s} \right)
\]

\[
= \sum_{s=1}^{n} \sum_{i_1+\ldots+i_s+s = n, i_r \geq 0} k_s E(\mathbf{X}^{i_1}) \cdots E(\mathbf{X}^{i_s})
\]

**Figure 20.** Decomposition of a partition by its first block.

Symbolically, this is shown in Figure 20.

If we substitute this expression in formula (39), the we get

\[
B_X(z) = 1 + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \sum_{i_1+\ldots+i_s+s = n, i_r \geq 0} k_s z^s E(\mathbf{X}^{i_1}) \cdots E(\mathbf{X}^{i_s}) z^{i_s}
\]

\[
= 1 + \sum_{s=1}^{\infty} k_s z^s \left( \sum_{i=0}^{\infty} E(\mathbf{X}^i) z^i \right)^s.
\]

In other words,

\[
B(z) = 1 + \varphi(zB(z)).
\]

In terms of the Cauchy transform, this can be written as

\[
\frac{1}{z} G \left( \frac{1}{z} \right) = 1 + \varphi \left( G \left( \frac{1}{z} \right) \right),
\]

or

\[
tG(t) - 1 = \varphi \left( G(t) \right).
\]

If \(K(u)\) is the functional inverse of \(G(t)\), then \(uR(u) := uK(u) - 1\). If we apply this function to \(G(t)\) then we get \(tG(t) - 1\). Hence, \(\varphi(u)\) and \(uR(u)\) satisfy the same functional equation. It is also easy to check that they have the same first terms in the polynomial expansion around zero. This implies that they are the same function. QED.

Another useful result about moments and cumulants is as follows.

**Theorem 14.7.** Let \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_n\}\) be free. Then

\[
E(a_1 b_1 a_2 b_2 \ldots a_n b_n) = \sum_{\pi \in NC(n)} k_{\pi} (a_1, \ldots, a_n) E_K(\pi) (b_1, \ldots, b_n)
\]

\[
= \sum_{\lambda \prec \pi} E_\lambda (a_1, \ldots, a_n) \mu (\lambda, \pi) E_{\pi-1, \pi} (b_1, \ldots, b_n),
\]

where \(E_K(\pi)\) is the cumulant function.
where \( K(\pi) \) denotes the Kreweras complement of \( \pi \), \( \mu \) is the Mobius function, and \( \tau = (12\ldots n) \).

This result should be compared with formula (17) on page 23 which describes the large \( N \) limit of products of certain random matrices. The formula is very similar, except that here we have \( \mu(\lambda, \pi) \) instead of the asymptotic Weingarten function \( \phi(\lambda^{-1}\pi) \). In fact, it turns out that these two quantities, the Mobius and the asymptotic Weingarten functions, are the same.

Proof of Theorem 14.7. By definition of free cumulants and by vanishing of mixed cumulants, we have

\[
E(a_1 b_1 a_2 b_2 \ldots a_n b_n) = \sum_{\pi_a, \pi_b} k_{\pi_a}(a_1, \ldots, a_n) k_{\pi_b}(b_1, \ldots, b_n),
\]

where \( \pi_a \) is a non-crossing partition of variables \( a_1, \ldots, a_n \), \( \pi_b \) is a non-crossing partition of variables \( \pi_b \), and the joint partition \( \pi_a \cup \pi_b \) is a non-crossing partition of variables \( a_1, b_1, a_2, b_2, \ldots, a_n, b_n \). By recalling the definition of the Kreweras complement, we can see that this expression can be written as follows:

\[
E(a_1 b_1 a_2 b_2 \ldots a_n b_n) = \sum_{\pi_a, \pi_b \prec K(\pi_a)} k_{\pi_a}(a_1, \ldots, a_n) k_{\pi_b}(b_1, \ldots, b_n) = \sum_{\pi} k_{\pi}(a_1, \ldots, a_n) E_{K(\pi)}(b_1, \ldots, b_n).
\]

This proves the first equality in the statement of the theorem. The second equality follows from the first by the Mobius inversion and by the fact that \( K(\pi) = \pi^{-1} \tau \) (Exercise 13.9 on page 72). QED.
15. Lecture 15. Free Cumulants. Examples. \( R \)-diagonal variables.

**Theorem 15.1.** Let \( s \) be the standard semicircle variable. Then \( k_2(s) = 1 \) and \( k_n(s) = 0 \) for all \( n \neq 2 \).

**Proof:** This fact follows from Theorem 14.6 and the fact that \( R_s(z) = z \) (see Exercise 7.1 on page 32). QED.

Recall that a variable \( c \) is called a circle variable if it has the same distribution as \( (s_1 + is_2)/\sqrt{2} \), where \( s_1 \) and \( s_2 \) are two standard semicircle variables.

**Exercise 15.2.** Derive the following formulas for cumulants of variables \( \{c, c^*\} \).

(i) \( k_n(\ldots) = 0 \) for \( n \neq 2 \),

(ii) \( k_2(c, c) = k_2(c^*, c^*) = 0 \), and

(iii) \( k_2(c, c^*) = k_2(c^*, c) = 1 \).

The next natural question is what are the free cumulants of Haar unitaries, that is, of unitary operators that have the uniform spectral distribution supported on the unit circle. The answer is given by the following theorem.

**Theorem 15.3.** Let \( u \) be a Haar unitary. Then,

\[
k_{2n}(u, u^*, \ldots, u, u^*) = k_{2n}(u^*, u, \ldots, u^*, u) = (-1)^{n-1} C_{n-1},
\]

where \( C_n \) denote the Catalan numbers. All other free cumulants of Haar unitaries are zero.

**Proof:** First, note that if the number of \( u \) is different from the number of \( u^* \) in the argument of a free cumulant, then the cumulant must equal zero. Indeed, we can write it as

\[
k_n = \sum \mu(\pi, 1_n) E_{\pi},
\]

and it is easy to see that \( E_{\pi} \) is always zero if the number of \( u \) is different from the number of \( u^* \) in the argument.

Next, suppose that the argument of the free cumulant has the equal number of \( u \) and \( u^* \) but there are two consequitive \( u \) or two consequitive \( u^* \) in the argument. Let us assume that we have already proved that such cumulants are zero for the cumulants of order less than \( 2n \) and let us proceed by induction.

Consider \( k_{2n}(\ldots, u^*, u, u, \ldots) \). (Other cases can be treated similarly.) In this case, we can use the fact that \( u^* u = 1 \) and apply Theorem 14.2 and Lemma 14.4 in order to obtain the following equality:

\[
0 = k_{2n-1}(\ldots, 1, u, \ldots) = \sum_{\pi \in NC(2n) \atop \pi \cap \sigma = 1} k_\pi(\ldots, u^*, u, u, \ldots),
\]

where \( \sigma \) is the partition that pairs \( u^* \) and the first \( u \) and that has all other elements as singletons. Let \( u^* \) be in position \( m \), and the two \( u \) be in positions \( m + 1 \) and \( m + 2 \), respectively. One partition that connects all blocks in \( \sigma \) is \( 1_{2n} \). All other non-crossing partition with this property consist of two blocks one of which is connected to \( m \), and another to \( m + 1 \). If a partition that connects to \( u \) at \( m + 1 \) is
also connected to \( u \) at \( m + 2 \), then the cumulants is zero by inductive assumption. Therefore, \( u \) at \( m + 2 \) must be connected to \( u^* \) at \( m \). However, in this case \( u \) at \( m + 1 \) is a singleton because the partition is non-crossing, and this implies that the cumulant is zero. It follows that

\[
k_{2n}(\ldots, u^*, u, u, \ldots) = 0.
\]

\[\begin{array}{ccccccc}
u & \hat{u} & u & \hat{u} & \ldots & u & \hat{u} & u & \ldots & \hat{u}
\end{array}\]

**Figure 21.** Partition \( \pi \) in the recursion for free cumulants of Haar unitaries

Next, let us calculate \( k_{2n}(u, u^*, \ldots, u, u^*) \). We use the same device as in the argument above. That is, we apply Theorem 14.2 and Lemma 14.4 in order to conclude that

\[
0 = k_{2n-1}(1, u, u^* \ldots)
= \sum_{\pi \in NC(2n)} k_{\pi}(u, u^*, u, u^*, \ldots),
\]

where \( \sigma \) is the partition that pairs the first \( u \) and \( u^* \) and has all other elements as singletons. As in the argument above, one of the partitions in this sum is \( 1_{2n} \), and other partitions have two blocks, one of which is connected to the first \( u \), and the other to the first \( u^* \). We also know that \( u \) and \( u^* \) in this blocks must be alternating. It follows that this requirement enforce that the partition is determined by the position of the first \( u^* \), to which the \( u \) in position 1 is connected. See Figure 21 for illustration. Let this \( u^* \) be in position \( 2p \). Then we have the following recursion: \( e \) is a

\[
k_{2n} = -\sum_{p=1}^{n} k_{2(n-p+1)}k_{2(p-1)},
\]

*which can be recognized as the recursion formula for \((-1)^{n-1} C_{n-1}\). QED.

A generalization of Haar unitaries based on this property is called \( R \)-diagonal variables. This generalization seems to be the simplest class of non-Hermitian operators which can be handled using the methods of free probability.

**Definition 15.4.** An element of a non-commutative probability space, \( X \), is called \( R \)-diagonal if all free cumulants of the set \( \{X, X^*\} \) are zero with a possible exception of alternating cumulants, that is, with an exception of the cumulants that have the form \( k_{2n}(X, X^*, \ldots, X, X^*) \) or \( k_{2n}(X^*, X, \ldots, X^*, X) \).

It turns out that \( R \)-diagonal variables have a couple of nice representations. We present the first one in the two following theorems and we outline the second in the couple of exercises.
Theorem 15.5. Suppose that $U$ is a Haar unitary, $H$ is an arbitrary operator, and $U$ and $H$ are free. Then the variable $X = UH$ is R-diagonal.

**Proof:** We need to show that the cumulants of $X = UH$ are as in Definition 15.4. That is, we need to prove that cumulants with odd number of arguments and cumulants in which $X$ or $X^*$ is repeated equal zero. For example, consider $k_n (\ldots, X, X, \ldots)$. Other cases can be treated similarly. We can use Theorem 14.2 and write

$$k_n (\ldots, UH, UH, \ldots) = \sum_{\pi \vee \sigma = 1_{2n}} k_\pi (\ldots, U, H, U, H, \ldots),$$

where $\sigma = (12)(34)\ldots(2n - 1, 2n)$.

Consider the block of $\pi$, that contains the second $U$. If this block starts with this second $U$, then we have the situation as in Figure 22. Since $U$ is Haar unitary, the last element in this block must be $U^*$. Otherwise, the cumulant $k_\pi$ is zero. However if the last elements in this block of $\pi$ is $U^*$, then it is easy to see that $\pi$ can connect only those blocks of $\sigma$ which are located between the first $U$ and the last $U^*$ of $\pi$. In particular, it cannot connect these blocks to the first $UH$ block depicted in the Figure. Hence $\pi \vee \sigma \neq 1_{2n}$ and such a $\pi$ does not enter the sum.

Next, consider the possibility that the second $U$ is not the first in the block of $\pi$ that contains this $U$. This situation is illustrated in Figure 23. Consider then the first $U^*$ on the left of this $U$, which belongs to this block. Such a $U^*$ must exist by the properties of the Haar unitary elements, or we would have $k_\pi = 0$. Then again, it is clear that $\pi$ cannot connect the first block $UH$ to any block of $\sigma$ which is connected by $\pi$. Hence, $\pi \vee \sigma \neq 1_{2n}$ and such a $\pi$ does not enter the sum.

This argument proves that $k_n (\ldots, UH, UH, \ldots) = 0$. QED.

In the other direction, we have the following result. Note the assumption of traciality.

Theorem 15.6. Suppose that $X$ is an R-diagonal element in a tracial non-commutative probability space. Then it can be represented in distribution by a product $UH$ where $U$ is a Haar unitary and $H$ is a positive operator that has the same distribution as $\sqrt{X^*X}$. 

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**Figure 22.**

**Figure 23.**
Proof: By the previous theorem we know that \( UH \) is \( R \)-diagonal and we only need to show that its non-vanishing cumulants are the same as the corresponding cumulants of variable \( X \). Let

\[
a_s = k_{2s} (X, X^*, \ldots, X, X^*) = k_{2s} (X^*, X, \ldots, X^*, X),
\]

where the equality in the second line holds by traciality of the probability space.

We claim that

\[
k_n (X^* X, \ldots, X^* X) = \sum_{\pi \in NC(n)} a_{|V_1|} \cdots a_{|V_r|}.
\]

Let \( a_s = k_{2s} (X, X^*, \ldots, X, X^*) \)

where the equality in the second line holds by traciality of the probability space.

We claim that

\[
k_n (X^* X, \ldots, X^* X) = \sum_{\pi \in NC(n)} a_{|V_1|} \cdots a_{|V_r|}.
\]

Indeed,

\[
k_n (X^* X, \ldots, X^* X) = \sum_{\pi \in NC(n)} k_\pi (X^*, X, \ldots, X^*, X),
\]

where as usual the sum is over non-crossing \( \pi \) and \( \sigma \) is the partition which pairs \( X^* \) with the following \( X \) (see Figure 24).

Now consider a block \( b \) of the partition \( \pi \). Suppose that \( X \) belongs to \( b \) and that \( X \) is connected to an \( X^* \) on the right. Then it must be that this \( X^* \) is the immediate neighbor of \( X \) on the right. Otherwise, the blocks of \( \sigma \) between these \( X \) and \( X^* \) would be disconnected from \( b \).

Next suppose that \( X \) is not connected to any \( X^* \) on the right. Then, if the cumulant is not-zero, then it must be that this is the ending \( X \) in \( b \) and that \( b \) starts with \( X^* \). Then, it is clear that in this case all blocks of \( \sigma \) on the right of this last \( X \) and all blocks of \( \sigma \) on the left of \( X^* \) are disconnected from block \( b \). Hence, the only possible case is when this \( X \) is the last one (i.e., it the \( X \) in the position \( 2n \)) and the corresponding \( X^* \) is in the position 1.

Let us move the first \( X^* \) to the right end of the sequence of \( X \) and \( X^* \). Note that the resulting partition will still be non-crossing if the original partition was non-crossing. Then the resulting partition has the property: every \( X \) is connected to \( X^* \) on its right. Moreover, every partition with this property comes from a partition that connects all blocks of \( \sigma \).

Now let us identify \( X \) with \( X^* \) on its right. Then, we have a bijection of the non-crossing partitions with the property described above and all non-crossing partitions of the set \( \{ 1, 2, \ldots, n \} \). The corresponding cumulants can be easily seen to be \( a_{|V_1|} \cdots a_{|V_r|} \), where \( V_i \) are the blocks of the partition from \( NC(n) \). This completes the proof of our claim.
We can re-write formula (40) as follows:

\[ k_n (X^* X, \ldots, X^* X) = a_n + \sum_{\pi \neq 1_n} a_{|V_1|} \cdots a_{|V_\pi|}. \]

This form of the formula makes it clear that we can calculate the cumulants \( a_n \) from the sequence \( k_n (X^* X, \ldots, X^* X) \).

However, \( UH \) is also an \( R \)-diagonal element and we know that \((UH)^* (UH) = H^2 = X^* X\). This implies that the sets \( \{UH, (UH)^*\} \) and \( \{X, X^*\} \) have the same cumulants. That is, \( UH \) and \( X \) have the same \(*\)-distribution. QED.

**Exercise 15.7.** Let \( S \) be a self-adjoint operator with distribution \( \frac{1}{2} \{ \delta_{-1} + \delta_1 \} \) and \( A \) be a self-adjoint operator with symmetric distribution. Assume that \( S \) and \( A \) are free. Show that \( SA \) is an \( R \)-diagonal operator.

**Exercise 15.8.** Show that every \( R \)-diagonal operator can be represented in distribution by an operator \( SA \), where \( S \) and \( A \) are as in the previous exercise.

It is easy to see that Theorems 15.5 and 15.6 imply that the sum and the product of two free \( R \)-diagonal elements is \( R \)-diagonal. What is more surprising is that powers of an \( R \)-diagonal element are \( R \)-diagonal. The proof of this fact requires some preparation. The following property is very useful in the study of \( R \)-diagonal random variables.

**Theorem 15.9.** Let \( a_1, a_2, \ldots, a_n \) be free \( R \)-diagonal random variables in a \( C^* \)-probability space \( A_1 \) and \( A_2, A_2, \ldots, A_n \) be self-adjoint random variables in a probability space \( A_2 \). Assume that \( A_i \) has the same probability distribution as \( \alpha_i := \sqrt{a_i^* a_i} \) for every \( i \). Let \( U \) be a Haar unitary in \( A_2 \), which is free from \( \{A_1, A_2, \ldots, A_n\} \). Let \( \Pi = a_n \cdots a_1 \) and \( X = U A_n \cdots U A_1 \). Then \( \Pi \cong X \).

(Cf. Exercise 15.25 on pp. 254-255 in [15])

In other words, if we multiply variables \( A_i \) by the Haar-distributed rotation \( U \), then we will lose all dependencies and the distribution of the product of these variables will be the same as if they all were \( R \)-diagonal and free. The surprising fact is that \( U \) is the same for all \( A_i \).

**Proof:** By Theorem 15.6 \( \Pi \cong u_n |a_n| \cdots u_1 |a_1| \), where \( u_n \) are Haar distributed and all variables \( u_1, \ldots, u_n, |a_1|, \ldots, |a_n| \) are all \(*\)-free.

Next, we can replace \( |a_i| \) with \( V_i A_i V_i^* \) where \( V_i \) are Haar unitaries free from all other variables. We obtain

\[ \Pi \cong u_n V_n A_n V_n^* u_{n-1} V_{n-1} A_{n-1} \cdots u_1 V_1 A_1 V_1^* \]

\[ \cong V_1^* u_n V_n A_n V_n^* u_{n-1} V_{n-1} A_{n-1} \cdots u_1 V_1 A_1. \]

Clearly, the variables \( U_i := V_i^* u_i V_i \) are Haar-distributed. (Here \( V_{n+1} \equiv V_1 \).) We claim that the are also free. Indeed, we can replace each of \( u_i \) with a product of two free Haar-unitalies, \( u'_i \) and \( u''_i \). This will not change the distribution. Then variables \( u'_i V_i \) are \(*\)-free from all \( V_i \). (This holds by Theorem 1 in [17]) The same is true for \( V_i^* u'_i V_i \). This implies that \( V_i^* u'_i u''_i V_i \) is \(*\)-free from all \( V_j^* u'_j u''_j V_j \) for \( j \neq i \).
It follows that
\[ \Pi \cong U_n A_n U_{n-1} A_{n-1} \ldots U_1 A_1, \]
where \( U_j \) are free Haar unitaries. The next step is to show that this variable has the same \( * \)-distribution as
\[ X = U A_n U A_{n-1} \ldots U A_1, \]
where \( U \) is a Haar unitary, which is free from all \( A_i \).

Let \( U^i \) denote the identical copies of \( U \) with superscript \( i \) showing the position of this copy of \( U \) in the product \( U A_n U A_{n-1} \ldots U A_1 \). Hence \( U^i = U^j = U \) for all \( i \) and \( j \), and we write:
\[ \begin{align*}
X &= U^n A_n U^{n-1} A_{n-1} \ldots U^1 A_1, \\
X^* &= A_1 U^1 A_2 U^{2*} \ldots A_n U^{n*}
\end{align*} \]

We are interested in a formula for the \( * \)-moments of \( X \). More particularly, we want to show that this formula is exactly the same as the formula for the corresponding moments of \( \Pi \).

Let us explain the reasoning by an example. Consider the following \( * \)-moment:
\[ E (X^* X X X^*) \]

By applying formula from Theorem 14.7 we obtain the following expression:
\[ E (X^* X X X^*) = \sum_{\pi \in NC(4n-2)} |\kappa_\pi| \left( U_1^*, \ldots, U^{(n-1)*}, U^{n-1}, \ldots, U, U^n, \ldots, U^1, U^{1*}, \ldots, U^{n*} \right) \times \]
\[ \times E_{K(\pi)} \left( A_2, \ldots, (A_n)^2, \ldots, A_1, A_n, \ldots, (A_1)^2, \ldots, A_n, A_1 \right). \]

All these formulas are the same for \( E (\Pi^* \Pi \Pi \Pi \Pi^*) \) except that we would need to use variables \( U_j \) instead of \( U^j \) throughout these expressions.

We claim that if a block of a partition \( \pi \) connects a variable from the subset \( \{ U^i, U^i* \} \) with a variable from the subset \( \{ U^j, U^j* \} \) (where \( i \neq j \)), then the cumulant \( \kappa_\pi \) is equal to zero. Note that this is true for variables \( U_i \) and \( U_j \), because \( U_i \) is \( * \)-free from \( U_j \) by assumption. However, the validity of this claim for \( U^i \) and \( U^j \) needs a proof because \( U^i = U^j \).

In order to explain this fact, we need an addition layer of notation. Namely, we will use a subscript to describe a position of an \( X \) in the product. For example, we will write \( E (X^*_1 X_2 X^*_3 X_1^*) \) instead of \( E (X^* X X X^*) \). It will be understood, however, that all \( X_i \) are identical. Correspondingly, we use notation \( U_{\alpha}^i \) and \( U_{\alpha}^{i*} \) in order to specify the position of variables \( U \) and \( U^* \). For example, \( U_{\alpha}^{i*} \) means the second \( U^* \) in the third \( X \).

In addition we will use variables \( \epsilon \in \{ \emptyset, * \} \) in order to have a uniform notation for \( U \) and \( U^* \). That is, \( U_{\alpha}^{i\epsilon} \) means \( U_{\alpha}^i \) if \( \epsilon = \emptyset \), and it means \( U_{\alpha}^{i*} \) if \( \epsilon = * \).
Suppose that a partition $\pi$ has a block $b$, such that variables $U^{i,\epsilon_1}_\alpha$ and $U^{j,\epsilon_2}_\beta$ belong to $b$ and that $i \neq j$. In addition, suppose that among all the pairs of variables $U^{i,\epsilon_1}_\alpha$ and $U^{j,\epsilon_2}_\beta$ that satisfy this property we choose a pair of variables with the smallest possible distance between them. (Here, the distance between variables is understood as the difference in their positions in the sequence $U^{1,\epsilon_1}_1, \ldots, U^{n,\epsilon_m}_m$, which is the argument of the cumulant $\kappa_\pi$.) This choice implies, in particular that $U^{i,\epsilon_1}_\alpha$ and $U^{j,\epsilon_2}_\beta$ are neighbors in the block $b$, that is, that there are no variables between them that belong to the same block $b$.

An example is shown in Figure 25, where $\alpha = 1, \beta = 3, i = 2, j = 1$.

If $\epsilon_1 = \epsilon_2$, then $\kappa_\pi = 0$ because $U$ is a Haar unitary and a cumulant of a Haar unitary is non-zero if and only if the sequence of $U$ and $U^*$ in its argument is alternating and has the same number of $U$ and $U^*$.

Hence, we can assume that $\epsilon_1 \neq \epsilon_2$ and $\alpha < \beta$. Consider the case when $\epsilon_1 = *$ and $\epsilon_2 = \emptyset$. (The other case is similar.)

Let $S$ be the sequence of $X^{\epsilon_1(\alpha+1)}_\alpha, \ldots, X^{\epsilon_1(\beta-1)}_\beta$. (If $\alpha + 1 > \beta - 1$, then $S$ is empty.) That is, we look on the sequence of $X$ and $X^*$ between $X^*_\alpha$ and $X^*_\beta$ in the defining sequence of the moment. Let $r$ be the number of cases when an $X^*$ is immediately to the left of an $X$ in this sequence. Let $s^*$ be the number of those $X^*$ that have $X^*$ to the right of them, and $s$ be the number of those $X$ that have an $X$ to the left of them.

Let us now count variables $U^*$ in the argument of the cumulant $k_\pi$ that arise from this sequence. (Note that some variables $U^{n*}$ are cancelled out when the product $X^*X$ occurs.). It is easy to see that the number of such variables is

$$c_1(*) = \begin{cases} (n - 1) r + ns^*, & \text{if } \epsilon_{\beta - 1} = \emptyset, \\ (n - 1) (r + 1) + n (s^* - 1), & \text{if } \epsilon_{\beta - 1} = *. \end{cases}$$

A similar count for variables $U$ gives

$$c_1(\emptyset) = \begin{cases} (n - 1) r + ns, & \text{if } \epsilon_{\alpha + 1} = *, \\ (n - 1) (r + 1) + n (s - 1), & \text{if } \epsilon_{\alpha + 1} = \emptyset. \end{cases}$$

Next, let us count those $U^*$ in the argument of the cumulant $k_\pi$ that are between $U^{i,*}_\alpha$ and $U^{j,*}_\beta$ and that come from $X^*_\alpha$. Their number is

$$c_2(*) = \begin{cases} n - i, & \text{if } \epsilon_{\alpha + 1} = *, \\ n - 1 - i, & \text{if } \epsilon_{\alpha + 1} = \emptyset. \end{cases}$$
Similarly, we can count those $U$ that are between $U^{i,*}_\alpha$ and $U^j_\beta$ and that come from $X_\beta$. This count is
$$c_2(\emptyset) = \begin{cases} n - 1 - j, & \text{if } \varepsilon_{\beta - 1} = *, \\ n - j, & \text{if } \varepsilon_{\beta - 1} = \emptyset. \end{cases}$$

We intend to compute the difference between the number of $U^*$ and $U$ which are between $U^{i,*}_\alpha$ and $U^j_\beta$. This number is $\Delta := c_1(*) + c_2(*) - c_1(\emptyset) - c_2(\emptyset)$.

We need to consider the four possible combinations of $\varepsilon_{\alpha + 1}$ and $\varepsilon_{\beta - 1}$.

For example, if $\varepsilon_{\alpha + 1} = \varepsilon_{\beta - 1} = \emptyset$, then
$$\Delta = (n - 1) r + n s^* + n - 1 - i \\
- [(n - 1)(r + 1) + n(s - 1) + n - j] \\
= n (s^* - s) + j - i.$$
16. LECTURE 16. BROWN MEASURE OF R-DIAGONAL VARIABLES

It is not straightforward how one should define a generalization of the eigenvalue distribution for infinite-dimensional non-normal operators. One interesting definition is that of the Brown measure. It is defined only for operators in von Neumann algebras and uses the fact that in these algebras one can define an analogue of the determinant, which is called Fuglede-Kadison determinant.

**Definition 16.1.** Let $X$ be a bounded random variable in a tracial $W^*$-probability space $(A, E)$. Then the Fuglede-Kadison determinant of $X$ is defined as

$$\det X := \exp \left[ \frac{1}{2} E \log (X^*X) \right].$$

This definition is based on the fact that the usual determinant of a finite-dimensional operator $X$ can be written as a product of singular values of $X$.

**Exercise 16.2.** Suppose $S$ is a random variable with distribution \( \{\delta_{-1} + \delta_1\} / 2 \), and let $t$ be a complex number such that $|t| < 1$. Then,

$$\det (1 - tS) = \left( 1 - 2\Re (t^2) + |t|^4 \right)^{1/4}.$$

For another example, let us consider the algebra of $N$-by-$N$ matrices with $E(X) = N^{-1} \sum_{i=1}^{N} X_{ii}$. Then, we can write the Fuglede-Kadison determinant as $\det X = \left( \prod_{i=1}^{N} s_i \right)^{1/N}$, where $s_i$ are the singular values of the matrix $X$, that is, the square roots of eigenvalues of matrix $X^*X$. By using results from linear algebra,

$$\det X = \left| \text{Det} (X^*X) \right|^{1/N},$$

where $\text{Det} (X)$ is the usual determinant. It follows that

$$\log \det (X - \lambda) = \frac{1}{N} \sum_{i=1}^{N} \log |\lambda_i - \lambda|.$$

Here $\lambda_i$ are eigenvalues of $X$ taken with multiplicities that are equal to the number of times that $\lambda_i$ is repeated on the diagonal of the Jordan form of $X$.

Note that in this example $\log \det (X - \lambda)$ is the logarithm of the absolute value of the characteristic polynomial divided by the degree of the polynomial. In a more general situation, we can think about function $\log \det (X - \lambda)$ as a suitable generalization of the logarithm of the modulus of characteristic polynomial.

**Definition 16.3.** The $L$-function of variable $X$ is defined as $L_X(\lambda) := \log \det (X - \lambda) = E \log |X - \lambda|$, where $|X - \lambda| = [(X - \lambda)^* (X - \lambda)]^{1/2}$.

Since the infinite-dimensional determinant is well-defined for all bounded operators in a tracial $W^*$-algebra (and can be with some effort extended to unbounded operators), we can exploit this fact and define the spectral measure of these operators. Indeed, the standard finite-dimensional determinant is equal to the product
of eigenvalues. Hence if we know \( \det (X - \lambda) \) for all \( \lambda \), then we know the characteristic polynomial and can recover the eigenvalues as its zeros. In the infinite-dimensional case, the situation is more complicated since \( L_X (\lambda) = \log \det (X - \lambda) \) can fail to be harmonic on an open set. If we think about \( L_X (\lambda) \) as a potential then this failure means that there is a continuous distribution of charges on this open set. This suggest the following definition.

**Definition 16.4.** Let \( X \) be a bounded random variable in a tracial W*-probability space \((A, E)\). Then its Brown measure is a measure \( \mu_X \) on the complex plane \( \mathbb{C} \) defined by the following equation. Let \( \lambda = x + iy \). Then,

\[
\mu_X = \frac{1}{2\pi} \Delta L_X (\lambda) \, dx \, dy,
\]

where \( \Delta = \partial_x^2 + \partial_y^2 \) is the Laplace operator, and the equality holds in the sense of (Schwarz) distributions.

We are not going to prove here that this construction gives us a well-defined probability measure and refer instead to the original paper by Brown. We only mention that the intuitive reason for this is that the function \( L_X (\lambda) \) is subharmonic and that the Laplacian of \( L_X (\lambda) \) measures the extent to which the function \( L_X (\lambda) \) fails to be harmonic.

In order to gain an intuition of how this definition works, we consider our previous example of \( N \)-by-\( N \) matrices with \( L_X (\lambda) = \frac{1}{N} \sum_{i=1}^{N} \log |\lambda_i - \lambda| \). Then \( L_X (\lambda) \) is harmonic everywhere outside the points \( \lambda_i \). Let \( f \) be a smooth function which is constant in some neighborhood of \( \lambda_1 \) and zero at other \( \lambda_i \). Let \( C \) be a circle of radius \( \varepsilon \) around \( \lambda_1 \), \( C = \{ z : |z - \lambda_1| \leq \varepsilon \} \). Then we can write by using one of Green’s formulas.

\[
\frac{1}{2\pi} \int_C f \Delta L_X \, dx \, dy = \frac{1}{2\pi} \oint_{\partial C} f (\nabla L_X \cdot n) \, ds,
\]

where \( n \) is the outward normal vector to the contour \( \partial C \). We can compute \( \nabla L_X \cdot n = (N\varepsilon)^{-1} \). Hence, the integral on the right equals \( f (\lambda_1) / N \), and we can conclude that \( \mu_X = N^{-1} \sum_{k=1}^{N} \delta_{\lambda_k} \), where \( \delta_x \) denote the Dirac distribution concentrated at point \( x \).

Let us list (without proof) some of the properties of the Brown measure:

(i) It is a unique measure such that

\[ L_X (\lambda) = \int_C \log |z - \lambda| \, d\mu_X (z) ; \]

(ii) For every integer \( n \geq 0 \), we have \( E (X^n) = \int z^n \, d\mu_X \), and

(iii) the Brown measure of a normal operator \( X \) coincides with its spectral probability distribution.

For an \( R \)-diagonal operator, the Brown measure is invariant with respect to rotations of the complex plane around the origin and we can write it as a product of the radial and polar part. Let \( \rho_X \) denote the radial part of the Brown measure. In other words, if \( A \) is a thin annulus between the circles with radii \( r \) and \( r + dr \), then
the measure of this annulus is $\rho_X ([r, r + dr])$. By abusing notation we will use the same letter to denote the density of this measure.

It is useful to have an expression for $\rho_X$ in terms of the $L$-function $L_X (\lambda)$. Note that if the function $f (r, \varphi)$ depends only on $r$, then its Laplacian can be computed as $\Delta f = f_{rr} + f_r / r$.

Lemma 16.5. Let $\lambda = re^{i\varphi}$ and suppose that $L_X (\lambda)$ depends only on $r$, $L_X (\lambda) = L (r)$. Then the radial part of the Brown measure can be computed as follows:

$$\rho_X = (rL_r)_r dr.$$

Proof: We can write for the circular annulus:

$$\mu_X (A) = \int_0^{2\pi} \int_R^{R + \Delta R} \frac{1}{2\pi} (L_{rr} + r^{-1} L_r) r dr d\varphi = \int_R^{R + \Delta R} (rL_{rr} + L_r) dr.$$ 

Since this is supposed to be equal to $\rho_X \Delta R + o (\Delta R)$, we find that $\rho_X = (rL_{rr} + L_r) dr$. QED.

Corollary 16.6. Let $F (r)$ denote the distribution function of the polar measure $\rho_X$. Then

$$F (R) - F (0) = RL_r (R).$$

Another natural distribution which we can consider for an $R$-diagonal operator $X$ is the distribution of the squares of singular values, that is, the spectral probability distribution of $X^* X$. Let us denote this distribution and its density by $\sigma_X$. Note that typically, $X$ is given as the product $UH$, where $U$ is a Haar unitary, and $H$ is a positive operator. Then $X^* X = H^2$ and $\sigma_X$ is simply the spectral distribution of $H^2$.

![Figure 26. Distribution function of the radial measure and its inverse](image)

We can ask the question of how the distributions $\sigma_X$ and $\rho_X$ are related to each other. The answer is given by the following theorem, which was discovered by Haagerup and Larsen in [11].

Theorem 16.7. Let $X$ be an $R$-diagonal operator and $F (r)$ denote the distribution function for the polar measure $\rho_X$. Let $F^{-1} (t)$ denote functional inverse of
$F(t)$. Let $S(t)$ be the $S$-transform of the measure of squared singular values of $X$, $\sigma_X$. Then $\sigma_X(\{0\}) = \rho_X(\{0\}) = w$ and

$$F^{(-1)}(w + x) = \sqrt{\frac{1}{S(x - 1)}}$$

for all $x \in (0, 1 - w]$. In addition, the upper bound for the support of $\rho_X$ equals $\sqrt{E(X^*X)}$.

**Example 1:** Let $X = UH$, where $U$ is Haar unitary and $H$ has the semicircle distribution. Then $X^*X = H^*H$ and therefore $\sigma_X$ is the Marchenko-Pastur distribution with the parameter $\lambda = 1$. It is not difficult to compute the $S$-transform of this distribution as $(1 + z)^{-1}$. Therefore, $F^{(-1)}(t) = \sqrt{t}$ and $F(r) = r^2$, where $r$ changes from 0 to 1. It follows that density of the polar measure is $\rho_X(t) = 2r$. In other words, the Brown measure has the uniform density

$$\mu_X(z) = \frac{1}{\pi} r dr d\theta = \frac{1}{\pi} dx dy$$

in the disc $\{z : |z| \leq 1\}$ and zero outside.

It is interesting to compare this result with the corresponding results from the theory of random matrices. Let $X$ be an $N$-by-$N$ random matrix with independent complex-valued entries such that $\text{Re}X_{ij}$ and $\text{Im}X_{ij}$ are independent Gaussian variables with zero mean and variance $(2N)^{-1}$. It was shown by [9] that the distribution of eigenvalues weakly converges to the uniform distribution on the unit circle. Recently, this result was extended by Tao and Vu to the case when the entries of the matrix are not necessarily Gaussian.

**Example 2:**

**Proof of Theorem 16.7** Let $X = SA$, where $S$ and $A$ are free, $S$ distributed as $(\delta_{-1} + \delta_1)/2$, and $A$ is a self-adjoint variable with symmetric spectral distribution. The possibility of such a representation for an $R$-diagonal operator was stated in Exercise 15.8. Note that $X^*X = A^2$ and therefore $\sigma_X$ is the spectral distribution of $A^2$.

It will be convenient to use the notation

$$L(X) := L_X(0) = \log \det(X)$$

It is easy to see that

$$L(SA - \lambda) = L(A - \lambda S) = L(A + \lambda S).$$

The first equality follows from the fact that $S^2 = I$, the multiplicativity of the determinant, and the fact that $\det(S) = 1$. The second equality reflects the fact that $S$ is symmetric.

Corollary 16.6 implies that

$$F(\lambda) - F(0) = \lambda \frac{d}{d\lambda} L(A + \lambda S).$$

Hence we need to compute $L(A + \lambda S)$. 


Proposition 16.8. Let \( g_A (t) = \int \left( 1 + it^2 x^2 \right)^{-1} d\mu_A (x) \), where \( \mu_A \) is the spectral probability measure of \( A \) and \( t \in \mathbb{R} \). Suppose that \( \lambda \) and \( t \) are related by the following equation \( 1 + \lambda^2 t^2 = g_A (t)^{-1} \). Then,

\[
L (A + \lambda S) = \log (\lambda) - \frac{1}{2} \log \left[ 1 + (\lambda t)^2 \right] + L (1 - itA).
\]

\[
= \log (\lambda) - \frac{1}{2} \log \left[ 1 + (\lambda t)^2 \right] + \frac{1}{2} \int \log [1 + x^2 t^2] d\mu_A (x).
\]

Proof of Proposition [16.8]. The key is the following lemma, which roughly speaking says that the determinant of the sum of the identity and a kind of nilpotent operator is one, provided that the nilpotent operator is sufficiently small.

Lemma 16.9. If a random variable \( a \) in a \( \mathcal{W}^\ast \)-probability space \( (A, E) \) has the property that \( E (a^k) = 0 \) for all integer \( k > 0 \), and if the spectral radius of \( a \) is less than 1, then \( L (1 - a) = 0 \).

This result holds even if the spectral radius of \( a \) is equal to 1 but we will skip the proof of this result which can be found in the original paper by Haagerup and Larsen.

Proof: Let \( x = \log (1 - a) \). Since spectral radius of \( a \) is less than 1, then we can write \( x \) as convergent series: \( x = \sum_{k=1}^{\infty} \frac{a^k}{k} \). It follows immediately that \( E (x) = 0 \). Next, note that \( L (1 - a) = L (e^x) = \frac{1}{2} \int E (e^{pt} e^q) \). By applying the second part of Theorem [16.12] we obtain that \( L (1 - a) = \text{Re} E (x) = 0 \). QED.

In the next step, we intend to write \( A + \lambda S \) as a product. We start with the following expression:

\[
(1 - it_1 A) \left( 1 - \frac{1 - (1 - it_1 A)^{-1}}{g_A (t_1)} \right) \left( 1 - \frac{1 - (1 - it_2 S)^{-1}}{g_S (t_2)} \right) (1 - it_2 S), \tag{43}
\]

where \( g_A (t_1) = E \left[ (1 - it_1 A)^{-1} \right] \) and \( g_S (t_2) = E \left[ (1 - it_2 S)^{-1} \right] \).

Exercise 16.10.

\[
g_S (t) = \frac{1}{1 + t^2},
\]

\[
g_A (t) = \int \frac{1}{1 + t^2 x^2} d\mu_A (x)
\]

The main idea of the proof of Proposition [16.8] is to make expression (43) close to \( (1 - it_1 A) (1 - it_2 S) + A + \lambda S \) by a suitable choice of \( t_1 \) and \( t_2 \). Since

\[
E \left( 1 - \frac{1 - (1 - it_1 A)^{-1}}{g_A (t_1)} \right) = E \left( 1 - \frac{1 - (1 - it_2 S)^{-1}}{g_S (t_2)} \right) = 0,
\]

and \( A \) is free from \( S \), hence

\[
E \left( \left( 1 - \frac{1 - (1 - it_1 A)^{-1}}{g_A (t_1)} \right) \left( 1 - \frac{1 - (1 - it_2 S)^{-1}}{g_S (t_2)} \right) \right)^k = 0,
\]
and we will be able to use Lemma [16.9] in our calculations provided that the spectral radius of the operator is smaller or equal than $1$.

We can re-write expression (43) as

$$(1 - it_1 A) (1 - it_2 S) - \frac{1 - it_2 S}{g_A (t_1)} - \frac{1 - it_1 A}{g_S (t_2)} + \frac{1}{g_A (t_1) g_S (t_2)}$$

where

$$f (t_1, t_2) = \frac{1}{g_A (t_1) g_S (t_2)} - \frac{1}{g_A (t_1)} - \frac{1}{g_S (t_2)}.$$

We impose on $t_1$ and $t_2$ the requirement that

$$\frac{t_2 g_S (t_2)}{t_1 g_A (t_1)} = \lambda,$$  

and

$$f (t_1, t_2) = 0.$$  

It turns out that it is possible to satisfy the two previous equations by a suitable choice of $t_1$ and $t_2$ if $\lambda$ is not too large. Then

$$(1 - t_1 A) \left[ 1 - \left( 1 - \frac{(1 - t_1 A)^{-1}}{g_A (t_1)} \right) \left( 1 - \frac{(1 - t_2 S)^{-1}}{g_S (t_2)} \right) \right] (1 - t_2 S)$$

$$= - \frac{t_1}{g_S (t_2)} (A + \lambda S)$$

$$= h (\lambda) (A + \lambda S),$$

where

$$h (\lambda) := - \frac{it_1}{g_S (t_2)}.$$

An application of Lemma [16.9] shows that

$$\det \left[ 1 - \left( 1 - \frac{(1 - t_1 A)^{-1}}{g_A (t_1)} \right) \left( 1 - \frac{(1 - t_2 S)^{-1}}{g_S (t_2)} \right) \right] = 1,$$

provided that spectral radius of

$$\left( 1 - \frac{(1 - it_1 A)^{-1}}{g_A (t_1)} \right) \left( 1 - \frac{(1 - it_2 S)^{-1}}{g_S (t_2)} \right)$$

is less than or equal to $1$.

Assume for the moment that this fact is established. Then, we see that

$$\det (A + \lambda S) = \frac{1}{|h (\lambda)|} \det (1 - it_1 A) \det (1 - it_2 S)$$  

$$= \frac{1}{t_1 \sqrt{1 + (t_2)^2}} \det (1 - it_1 A),$$  

$$= 1,$$  

$$\det (A + \lambda S) = 1.$$
where we used definition of $h(\lambda)$ and Exercises 16.2 and 16.10.

The next step is to figure out the dependence of $t_1$ and $t_2$ on $\lambda$. Requirements (44) and (45) can be simplified as
\[ \frac{t_2}{1 + (t_2)^2} = \lambda t_1 g_A(t_1), \]
and
\[ \frac{t_2}{1 + (t_2)^2} = \frac{1}{t_2} g_A(t_1). \]
It follows that
\[ \frac{1}{t_2} = \lambda t_1, \]
and
\[ g_A(t_1) = \frac{1}{1 + (\lambda t_1)^2}. \]
To summarize, we have the following formula:
\[ \det(A + \lambda S) = \lambda \frac{1}{\sqrt{1 + (\lambda t)^2}} \det(1 - itA), \]
where $\lambda$ and $t$ are related by the expression:
\[ g_A(t) := \int \frac{1}{1 + t^2 x^2} d\mu_A(x) = \frac{1}{1 + (\lambda t)^2}. \]
We can also write this as
\[ \det(A + \lambda S) = \lambda g_A(t) \det(1 - itA), \]
If we take logarithm we get
\[ L(A + \lambda S) = \log(\lambda) - \frac{1}{2} \log \left[ 1 + (\lambda t)^2 \right] + L(1 - itA). \]
QED. This completes proof of Proposition 16.8

By using the formula in Proposition 16.8 we can compute:
\[ L_\lambda(A + \lambda S) \frac{d\lambda}{dt} = \frac{1}{\lambda \frac{d\lambda}{dt}} - \frac{1}{2} \frac{2t \lambda^2 + 2\lambda^2 \frac{d\lambda}{dt}}{1 + \lambda^2 t^2} + \frac{1}{2} \int \frac{2t x^2}{1 + t^2 x^2} d\mu_A(x). \] (48)
It follows from the definition of $\lambda$ (in the formulation of Proposition 16.8) that
\[ \int \frac{t x^2}{1 + t^2 x^2} d\mu_A(x) = \frac{t \lambda^2}{1 + \lambda^2 t^2}. \]
Miraculously, the terms without $\frac{d\lambda}{dt}$ in (48) cancel out, and we can simplify this formula to the following form:
\[ \lambda L_\lambda(A + \lambda S) = \frac{1}{1 + \lambda^2 t^2} = g_A(t). \]
Hence, from (42) it follows that
\[ F(\lambda) - F(0) = g_A(t). \] (49)
We need to make several simple manipulation with this formula in order to obtain the result in Theorem 16.7.

Let $\sigma_A$ be the spectral distribution of $A^2$. Recall that in the lecture about $S$-transforms we defined function $\psi$ by the formula:

$$
\psi(z) = \psi_{A^2}(z) := E\left[\left(1 - z A^2\right)\right] - 1,
$$

and the $S$-transform of $A^2$ is determined by the identity

$$
S(u) = \frac{1 + u}{u} \psi^{(-1)}(u),
$$

where $\psi^{(-1)}(u)$ is the functional inverse of $\psi(z)$.

Clearly, $g_A(t) - 1 = \psi\left(-t^2\right)$ and therefore $\psi^{(-1)}(g_A(t) - 1) = -t^2$. It follows that

$$
S(g_A(t) - 1) = \frac{g_A(t) t^2}{1 - g_A(t)} = \frac{1}{\lambda^2} = \frac{1}{\left[F(-1)(F(0) + g_A(t))\right]^2},
$$

where we used the key formula (49). If we introduce notation $w = F(0)$ and $x = g_A(t)$, then we have the formula

$$
S(x - 1) = \frac{1}{\left[F(-1)(w + x)\right]^2},
$$

where $x \in [0, 1 - w]$.

Alternatively,

$$
F^{(-1)}(w + x) = \sqrt{\frac{1}{S(x - 1)}},
$$

which proves the formula in Theorem 16.7.

**Lemma 16.11.** Let $a$ and $b$ be two bounded random variables in $W^*$-probability space, and assume that $E(a) = E(b) = 0$. Then the following formula holds for spectral radius of the product:

$$
\rho(ab) = \sqrt{E(a^*a) E(b^*b)}
$$
16.1. **Addendum: Properties of Fuglede-Kadison determinant.** Let us prove some properties of the Fuglede-Kadison determinant following the original paper by Fuglede and Kadison.

**Theorem 16.12.** Let $X$ and $Y$ be two bounded random variables in a tracial $W^*$-probability space $(\mathcal{A}, E)$. Then

(i) $\det (XY) = \det X \det Y$, and

(ii) $\det (e^X) = |e^{E(X)}| = \exp (\text{Re}E(X))$.

**Proof:** First, let us prove the following basic lemma:

**Lemma 16.13.** Let function $f(\lambda)$ be analytical in a domain $\Lambda$ bounded by a curve $\Gamma$ in the complex $\lambda$-plane, and let $X(t), t \in [0,1]$, be a differentiable family of operators such that the spectrum of each $X(t)$ lies in $\Lambda$. Then, $f(X(t))$ is differentiable with respect to $t$, and

$$E \left\{ \frac{d}{dt} f [X(t)] \right\} = E \left\{ g [X(t)] X'(t) \right\},$$

where $g(\lambda) = df(\lambda)/d\lambda$ and $X'(t) = dX(t)/dt$.

(Note that the statement is not valid without the expectation signs.)

**Proof:** First, we write by using the Cauchy formula:

$$f [X(t)] = \int f(\lambda) \frac{1}{\lambda - X(t)} \, d\lambda.$$

It follows that

$$f [X(t+h)] - f [X(t)] = \int f(\lambda) \left[ \frac{1}{\lambda - X(t+h)} - \frac{1}{\lambda - X(t)} \right] \, d\lambda$$

$$= \int f(\lambda) \left[ (\lambda - X(t+h))^{-1} (X(t+h) - X(t)) (\lambda - X(t))^{-1} \right] \, d\lambda.$$

By taking the limit we obtain that

$$\frac{d}{dt} f [X(t)] = \int f(\lambda) \frac{(\lambda - X(t))^{-1} X'(t) (\lambda - X(t))^{-1}}{\lambda - X(t)} \, d\lambda.$$

On the other hand, using integration by parts, we can obtain:

$$g [X(t)] = \int f'(\lambda) (\lambda - X(t))^{-1} \, d\lambda$$

$$= \int f(\lambda) (\lambda - X(t))^{-2} \, d\lambda.$$

It follows that

$$g [X(t)] X'(t) = \int f(\lambda) (\lambda - X(t))^{-2} X'(t) \, d\lambda.$$

Then, the statement of the lemma follows from the fact that integration and taking the expectation commutes, and from the fact that the expectation is tracial. QED.

By applying this result we can obtain the following lemma.
Lemma 16.14. If $H$ is a self-adjoint random variable, and $A$ is an arbitrary bounded variable, then

$$\det \left[ \exp (A^*) \exp (H) \exp (A) \right] = \exp \left( E (A^* + A) \right) \det \left[ \exp (H) \right].$$

It is convenient to think about this lemma as a rather unusual formulation of the invariance of the determinant with respect to unitary changes of coordinates. Indeed if $A = iX$, where $X$ is selfadjoint, then $A^* + A = 0$, and the lemma claims that

$$\det \left[ \exp (A^*) \exp (H) \exp (A) \right] = \det \left[ \exp (H) \right].$$

Proof: Let $X(t) = \exp (A^*t) \exp (H) \exp (At)$ so that $X'(t) = A^* X(t) + X(t) A$, and let $f(\lambda) = \log (\lambda)$. By applying the previous lemma and integrating, we obtain:

$$E \left[ \log X(1) - \log X(0) \right] = \int_0^1 E \left\{ [X(t)]^{-1} X'(t) \right\} dt = E (A^* + A),$$

which implies

$$E \log \left[ \exp (A^*) \exp (H) \exp (A) \right] = E (A^* + A) + E (H),$$

which is equivalent to the statement of the lemma.

Now we can prove the claims of the theorem.

(i) Let $X = U_1 H_1$ and $X_2 = H_2 U_2$ and let $A_1 = \log H_1$, $A_2 = \log H_2$. By using the unitary invariance of the determinant (which can be easily proved) and the product property for the commuting case, it is easy to see that

$$\det (X Y) = \det (H_1 H_2) = \sqrt{\det \left( H_2 (H_1)^2 H_2 \right)}.$$

Next, we can write

$$\det \left( H_2 (H_1)^2 H_2 \right) = \det (\exp (A_2) \exp (2A_1) \exp (A_2)) = \exp \left[ E (2A_2) \right] \exp \left[ E (2A_1) \right] = \det (H_2)^2 \det (H_1)^2 = \det (X)^2 \det (Y)^2.$$

This implies

$$\det (X Y) = \det (X) \det (Y).$$

(ii) This claim follows directly from Lemma 16.14 by setting $H = 0$. QED.
APPENDIX A. DISTRIBUTIONS OF SELF-ADJOINT VARIABLES AND THEIR TRANSFORMS

Note: The Cauchy transform is defined as follows:

$$G(z) := E\left(\frac{1}{z - X}\right);$$

the K-function is the inverse of the Cauchy transform,

$$K(z) := G^{-1}(z);$$

the S-function is

$$S(z) := \frac{1 + z}{z} \psi(-1)(z),$$

where $$\psi(-1)(z)$$ is the inverse of the moment-generating function

$$\psi(z) := E\left(\frac{zX}{1 - zX}\right).$$

If $$R(z) := zK(z) - 1$$ (this is a variant of the definition of the R-transform), then $$S(z)$$ can also be defined by

$$S(z) = \frac{1}{z}R^{-1}(z).$$

### Semicircle

- **Density & atoms**
  $$\frac{1}{2\sqrt{\pi}}\sqrt{4 - x^2} \chi_{[-2,2]}(x)$$
- **Cauchy transform**
  $$\frac{1}{2} \left( z - \sqrt{z^2 - 4} \right)$$
- **K-function**
  $$\frac{1}{z} + z$$
- **S-function**
  $$\pm \frac{1}{\sqrt{z}}$$
- **Moments**
  $$m_{2k} = \frac{1}{k+1} \binom{2k}{k}, \quad m_{2k+1} = 0.$$  
- **Free Cumulants**
  $$c_1 = 1, \quad c_i = 0 \text{ for } i \neq 1$$

### Marchenko-Pastur

- **Density & atoms**
  $$\frac{\sqrt{4x - (1 - \lambda + z)^2}}{2\pi x} \chi\left\{(1 - \sqrt{x})^2, (1 + \sqrt{x})^2\right\}(x)$$
  and an atom at 0 with mass $$(1 - \lambda)$$ if $$\lambda < 1$$
- **Cauchy transform**
  $$\frac{1 - \lambda + z - \sqrt{(1 - \lambda + z)^2 - 4z}}{2z}$$
- **K-function**
  $$\frac{1}{z} + \frac{\lambda}{1 - z}$$
- **S-function**
  $$\lambda \frac{1}{z}$$
- **Moments**
  $$c_i = \lambda$$ for all $$i.$$
### Bernoulli I

- **Density & atoms**: \( p\delta_1 + q\delta_0 \)
- **Cauchy transform**: \( \frac{z-q}{(z-1)z} \)
- **K-function**: \( \frac{1+z-\sqrt{(1+z)^2-4qz}}{2z} \)
- **S-function**: \( \frac{1+z}{p+z} \)
- **Moments**: \( m_i = p \) for all \( i \)
- **Free Cumulants**

### Bernoulli II

- **Density & atoms**: \( \frac{1}{\pi} (\delta_1 + \delta_{-1}) \)
- **Cauchy transform**: \( \frac{1}{\sqrt{1+x^2}} \)
- **K-function**: \( \frac{1+z-\sqrt{1+4z^2}}{2z} \)
- **S-function**: \( \pm \sqrt{1 + \frac{2}{n}} \)
- **Moments**: \( m_{2k} = 1 \) and \( m_{2k+1} = 0 \)
- **Free Cumulants**: \( c_{2k} = (-1)^{k-1} \frac{2^k (2k-3)!!}{k!} \)

### Arcsine

- **Density & atoms**: \( \frac{\chi_{[-2,2]}(t)}{\pi \sqrt{4-t^2}} \)
- **Cauchy transform**: \( \frac{1}{\sqrt{z^2-4}} \)
- **K-function**: \( \frac{1}{z} \sqrt{1+4z^2} \)
- **S-function**: \( \pm \sqrt{u(2+u)} \)
- **Moments**: \( m_n = \binom{2k}{k} \) if \( n = 2k \); \( 0 \) if \( n = 2k + 1 \)
- **Free Cumulants**

### Cauchy

- **Density & atoms**: \( \frac{1}{\pi} \frac{1}{1+t^2} \)
- **Cauchy transform**: \( \frac{1}{t+i} \)
- **K-function**: \( \frac{1}{z} - i \)
- **S-function**: \( i \)
- **Moments**: \( E |X| = \infty \)
- **Free Cumulants**

## Appendix B. Brown Measures

Let \( T = UX \), where \( U \) is Haar unitary and free of \( X \). The density of the Brown measure \( \mu \) of \( T \) is defined as

\[
\frac{1}{\pi} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \det (X + x + iy) 
\]

This measure is invariant with respect to rotation around the origin. The radial eigenvalue measure \( \rho \) of \( T \) defined by

\[
\rho ([0,t]) = \mu \left( B(0,t) \right)
\]
where \( B(0, t) \) is the closed disc with center at 0 and radius \( t \).

\[ X \]

Density of the radial eigenvalue distribution

Semicircle

Bernoulli I

Bernoulli II

Arcsine

Marchenko-Pastur

Cauchy

**APPENDIX C. SCALING PROPERTIES**

It is useful to know how various functions related to the Cauchy transform of random variable \( X \) behave under scaling and translation. For convenience of reference we collect these results in the following propositions:

**Proposition C.1.**

(i) \( G_{aX}(z) = a^{-1}G_X(z/a) \);

(ii) \( K_{aX}(z) = aK_X(az) \);

(iii) \( S_{aX}(z) = a^{-1}S_X(z) \)

**Proposition C.2.**

(i) \( G_{X+b}(z) = G_X(z - b) \);

(ii) \( K_{X+b}(z) = K_X(z) + b \);

The proofs of these propositions directly follow from definitions.

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