Quantum secure non-malleable extractors

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Abstract

Non-malleable extractors introduced by Dodis and Wichs [DW09] have found several applications in the study of tamper resilient cryptography. For example, seeded non-malleable extractor is a key ingredient in the privacy amplification (PA) protocol with an active classical adversary. Similarly, 2-source non-malleable extractors provide a way to construct non-malleable codes introduced by Dziembowski, Pietrzak and Wichs [DPW18] with further applications to non-malleable secret sharing. Thus, it is vital to understand if such non-malleable extractors are secure against quantum adversaries.

We construct several efficient quantum secure non-malleable extractors. All the quantum secure non-malleable extractors we construct are based on the constructions by Chattopadhyay, Goyal and Li [CGL20] and Cohen [Coh15].

• We construct the first efficient quantum secure non-malleable extractor for (source) min-entropy $k \geq \text{poly} \left( \log \left( \frac{n}{\varepsilon} \right) \right)$ and seed length $d = \text{poly} \left( \log \left( \frac{n}{\varepsilon} \right) \right)$ ($n$ is the length of the source and $\varepsilon$ is the error parameter). Previously Aggarwal, Chung, Lin, and Vidick [ACLV19] have shown that an inner-product based non-malleable extractor proposed by Li [Li12] is quantum secure, however it required linear (in $n$) min-entropy and seed length.

• Using a connection between non-malleable extractors and PA (established first in the quantum setting by Cohen and Vidick [CV17]), we get a 2-round PA protocol that is secure against active quantum adversaries with communication $\text{poly} \left( \log \left( \frac{n}{\varepsilon} \right) \right)$. This allows for a trade-off in communication and error in a PA protocol, improving on the result of [ACLV19], where the communication is required to be linear in $n$.

• We construct an efficient quantum secure 2-source non-malleable extractor for min-entropy $k \geq n - n^{\Omega(1)}$, with an output of size $n/4$ and error $2^{-n^{\Omega(1)}}$.

• We also study their natural extensions when the tampering of the inputs is performed $t$-times. We construct efficient quantum secure $t$-non-malleable extractors for both seeded ($t = d^{\Omega(1)}$) as well as 2-source case ($t = n^{\Omega(1)}$).

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1 Introduction

Extractors are functions that transform weak sources to uniform randomness. They hold a lot of importance since randomized algorithms are designed under the assumption that randomness used within the algorithms is uniformly distributed. Extractors have found numerous applications including privacy amplification (PA), pseudo-randomness, derandomization, expanders, combinatorics and cryptography. Some general models of weak sources are the so-called min-entropy sources and conditional min-entropy sources. Please refer to Section 2 for definitions of information-theoretic quantities, extractors and various adversary models. Let random variables $X \in \{0,1\}^n, Y \in \{0,1\}^n, S \in \{0,1\}^d$ and (below $U_d$ is the uniform distribution on $d$ bits and $X \otimes S$ represents independent random variables $X, S$)

$$C_1 = \{ X : H_{\min}(X) \geq k \} \quad ; \quad C_2 = \{ X \otimes S : H_{\min}(X) \geq k \text{ and } S = U_d \} \quad ;$$

$$C_3 = \{ X \otimes Y : H_{\min}(X) \geq k_1 \text{ and } H_{\min}(Y) \geq k_2 \}.$$

It can be argued that no deterministic function can extract even one uniform bit given an (arbitrary) source $X \in C_1$, for $k \leq n - 1$ [CG85]. This led to designing extractors using sources from $C_2$. They use an additional uniform source (aka seed $S = U_d$) called seeded extractors. Subsequent works also considered extraction from class $C_3$, multiple independent weak sources [CG85, Bou05]. In the classical setting, extractors have been studied extensively both in the seeded and the multi-source settings [ILL89, GUV09, DW08, Bou05, CG85, KLRZ08, Rao06, Raz05, KLR09, CGL20, Li15, CZ19].

Consider a situation where the input of an extractor is tampered with. For example, for a source $(X, S) \in C_2$, an adversary may tamper with the seed $S$ to modify it to some other seed $S'$. In this case, one can ask a natural question ‘Does the output of tampered input $(X, S')$ have any correlation with the output of the untampered input $(X, S)$?’ In order to be resilient to this adversarial tampering of input, one natural requirement to consider is that the original input produces an output that is (almost) independent of the one that is generated by the tampered input. Extractors having this property are called non-malleable extractors [DW09]. A non-malleable extractor nmExt, thus produces an output that is nearly uniform and independent of tampering; $(nmExt(X, S))nmExt(X, S') \approx U_m \otimes nmExt(X, S')$.

Applications to cryptography motivate the study of extractors in presence of an adversary holding some side information $E$ on the source. The sources are considered of the form

$$C_4 = \{ X E \otimes S : H_{\min}(X | E) \geq k \text{ and } S = U_d \}.$$

Here, we require the output of the extractor $Ext(X, S)$ to be uniform given the side information $E$; $(Ext(X, S))E \approx U_m \otimes E)$. Additionally, in the case of a non-malleable extractor, we require that it is (nearly) independent of any potential tampering of the seed,

$$(nmExt(X, S))nmExt(X, S')E \approx U_m \otimes nmExt(X, S')E.$$

Similarly one can consider 2-sources with adversary side information (below $k_1, k_2 > 0$ and $|Y| = n$):

$$C = \{ X - E - Y : H_{\min}(X | E) \geq k_1 \text{ and } H_{\min}(Y | E) \geq k_2 \},$$

where $X - E - Y$ represents a Markov-chain (see Definition 10). 2-source extractors have been extensively studied as well [Bou05, CG85, KLRZ08, Rao06, Raz05, KLR09, CGL20, Li15, CZ19].

\footnote{For simplicity and brevity we call the joint systems, including the adversary side information, as a source.}
Similar to seeded extractors, 2-source extractors also have a stronger variant in terms of non-malleability. For a 2-source non-malleable extractor, we allow tampering on both \( X \) and \( Y \). An adversary can modify \((X,Y)\) to some \((X',Y')\) such that, either \( \Pr[X \neq X'] = 1 \) or \( \Pr[Y \neq Y'] = 1 \). A 2-source non-malleable extractor is a function \( 2\text{nmExt} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^m \) such that:

\[
2\text{nmExt}(X,Y)2\text{nmExt}(X',Y')EYY' \approx_\varepsilon U_m \otimes 2\text{nmExt}(X',Y')EYY'.
\]

2-source non-malleable extractors have been used in the construction of non-malleable codes in the well studied split-state model by Chattopadhyay, Goyal and Li [CGL20]. These non-malleable codes are known to have applications in various cryptographic tasks such as non-malleable secret-sharing and non-malleable commitment [GPR16, GK18a, GK18b, ADN+19, SV19]. Seeded non-malleable extractors were used, by Chattopadhyay and Zuckerman [CZ19], as a key ingredient in their breakthrough construction of 2-source extractors for polylog\((n)\) min-entropy sources. 2-source extractors also find applications in graph-theory and are related to Ramsey graphs, well studied combinatorial objects.

With the advent of quantum computers, it is natural to investigate the security of extractors against a quantum adversary with quantum side information on weak sources. Such sources are of the form

\[
Q_1 = \{ \sigma_{XES} = \sigma_X \otimes \sigma_S : H_{\min}(X|E)_\sigma \geq k \text{ and } \sigma_S = U_\ell \},
\]

where side information \( E \) is quantum and source as well as seed \( (XS) \) are classical. As expected, quantum side information presents many more challenges compared to classical side information. Gavinsky et al. [GKK+07] gave an example of a seeded extractor that is secure against a classical adversary but not secure against a quantum adversary, even with very small side information. Very little is known about the security of non-malleable extractors against quantum side information. The initial trouble comes in defining a non-malleable extractor with quantum side information, since we need to provide security with updated quantum side information as the adversary modifies \((E,S) \rightarrow (E',S')\). Informally, we require (for formal definition see Definition 15)

\[
\text{nmExt}(X,S)\text{nmExt}(X,S')E' \approx U_m \otimes \text{nmExt}(X,S')E'.
\]

In the classical setting it can be argued that, conditioned on \( E = e \), \( X \) and \( S' \) remain independent, since with this conditioning, \( S' \) is a deterministic function of \( S \). However in the quantum setting, since conditioning on quantum side information cannot be done in this manner, this argument does not hold.

In the current paper, we study seeded non-malleable extractors and extend the definition to 2-source non-malleable extractors (see Definition 18). We also study their natural extensions where the tampering is performed \( t \)-times and the adversary is allowed to tamper \((E,S) \rightarrow (E',S^1,\ldots,S^t)\). There we require that the output is nearly independent given the quantum side information and any of the tampered outputs. For example, in the seeded case (see Definition 21),

\[
\text{nmExt}(X,S)\text{nmExt}(X,S^1)\ldots \text{nmExt}(X,S^t)E' \approx U_m \otimes \text{nmExt}(X,S^1)\ldots \text{nmExt}(X,S^t)E'.
\]

Before stating our results, we give a brief overview of some relevant previous works.

**Previous works**

Aggarwal et al. [ACLV19] have shown that an inner-product based non-malleable extractor proposed by Li [Li12] is quantum secure, however it requires linear min-entropy and seed length. Recent work
of Aggarwal et al. [ABJO21] has strengthened this result and showed that Li’s extractor remains quantum secure even when the seed is not uniform (still requiring linear min-entropy in the seed). They do so by introducing a notion of security against a quantum measurement adversary, to which the eventual quantum security of the inner-product is reduced. To the best of our knowledge, the inner-product based non-malleable extractor proposed by Li [Li12] is the only non-malleable extractor for which quantum security is known.

Earlier works of [CV17, BF11] attempted to provide quantum security of non-malleable extractors by introducing a notion of security against a quantum purified adversary. They do so by introducing a notion of security against a quantum measurement adversary [DP07]. Unfortunately, the results were later withdrawn due to subtle issues in the arguments.

**Our results**

Let \( \sigma_{XES} \) be a source from \( \mathcal{Q}_1 \). We have \( \sigma_{XES} = \sigma_X \otimes \sigma_S, \) \( \mathsf{H}_{\min}(X|E)_{\sigma} \geq k \) and \( \sigma_S = U_d \). One may consider register \( S \) as uniform seed and register \( E \) as adversary quantum side information on source \( X \). We consider the pure state extension of \( \sigma_{XES} \) denoted by \( \sigma_{X\hat{E}\hat{E}ESS\hat{S}} = \sigma_{X\hat{E}\hat{E}E} \otimes \sigma_{S\hat{S}\hat{S}} \), as it helps us in our analysis. Here \( \sigma_{X\hat{E}\hat{E}E}, \sigma_{S\hat{S}\hat{S}} \) are canonical purifications of \( \sigma_X \) and \( \sigma_S \) respectively. For simplicity we call the entire pure state as a source, even though the uniform randomness is extracted from classical registers of a pure state (see Definition 3). Note that \( \hat{X}, \hat{S} \) are copies of \( X, S \) respectively (see Definition 4).

Consider,

\[
\mathcal{Q}_2 = \{ \sigma_{X\hat{E}\hat{E}ESS\hat{S}} = \sigma_{X\hat{E}\hat{E}} \otimes \sigma_{S\hat{S}\hat{S}} : \mathsf{H}_{\min}(X|E)_{\sigma} \geq k \text{ and } \sigma_S = U_d \},
\]

where \( \sigma_{X\hat{E}\hat{E}ESS\hat{S}} \) is a pure state. Note the sources in \( \mathcal{Q}_2 \) are purifications of sources in \( \mathcal{Q}_1 \). The conditions in \( \mathcal{Q}_2 \) are equivalent to,

\[
\mathsf{H}_{\min}(X|ESS\hat{S})_{\sigma} \geq k \quad ; \quad \mathsf{H}_{\min}(S|X\hat{E})_{\sigma} = d.
\]

More generally, this leads us to consider the following sources (see Definition 16):

\[
\mathcal{Q} = \{ \sigma_{X\hat{E}\hat{E}Y} : \sigma_{X\hat{E}\hat{E}Y} \text{ is a } (k_1, k_2)\text{-qpa-state} \}.
\]

Here qpa stands for quantum purified adversary. To understand the advantage of considering the sources along with purification registers consider the following example. Consider the Markov-chain \( \sigma_{XEY} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|0\rangle \). Note \( I(X : Y|E)_{\sigma} = 0 \). Let \( \rho_{XEY} \) be the state after applying CNOT gate on qubit \( Y \) conditioned on qubit \( E \) of \( \sigma \). Note \( \rho_{XEY} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \). We have \( I(X : Y|E)_{\rho} \neq 0 \). This points us to one of the key difficulty faced by earlier approach of Markov model. On the other hand, we note that sources in \( \mathcal{Q} \) remains in \( \mathcal{Q} \) after adversarial tampering. This enables us to analyse the constructions of non-malleable extractors step by step and ensuring the parameters \((k_1, k_2)\) for the \((k_1, k_2)\text{-qpa-state} \) at the end are still in control to extract randomness.

Also, note the sources \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{Q}_1, \mathcal{Q}_2 \) can all be seen as special cases of \( \mathcal{Q} \) (in the purification picture). This provides us a general framework to define extractors and non-malleable extractors, both in the seeded and the 2-source settings.

We now state our results. Let \( n, d, t \) be positive integers and \( k, \varepsilon > 0 \). The following result is about seeded non-malleable extractors.

**Theorem 1** (quantum secure non-malleable extractor). Let \( d = \mathcal{O}(\log^7(n/\varepsilon)) \) and \( k = \Omega(d) \). There exists an efficient non-malleable extractor \( \text{nmExt} : \{0, 1\}^n \times \{0, 1\}^d \to \{0, 1\}^{k/4} \) that is \((k, \mathcal{O}(\varepsilon))\text{-quantum secure} \) (see Definition 15).
The adversary in the result above is the qnma (short for quantum non-malleable adversary). As a corollary, we get the following result for the more standard model of sources considered in the literature (sources in \( Q \)).

**Corollary 1.** Let \( d = O(\log^7(n/\varepsilon)) \) and \( k = \Omega(d) \). Let \( \rho_{XEY} \) be a c-q state with registers \((XY)\) classical such that

\[
H_{\min}(X|E)_\rho \geq k \quad ; \quad \rho_{XEY} = \rho_{XE} \otimes U_d \quad ; \quad |X| = n.
\]

Let \( T : \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_Y) \to \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_Y \otimes \mathcal{H}_{Y'}) \) be a (safe) CPTP map such that for \( \sigma_{X'E'Y'} = T(\rho_{XEY}) \), we have registers \( X'E'Y' \) classical and \( \Pr(Y \neq Y')_\sigma = 1 \). Let the function \( \text{nmExt} \) be from Theorem 1, \( L = \text{nmExt}(X,Y) \) and \( L' = \text{nmExt}(X,Y') \). Then,

\[
\|\sigma_{L'Y'Y'} - U_{k/4} \otimes \sigma_{L'Y'Y'}\|_1 \leq O(\varepsilon).
\]

Dodis and Wichs [DW09] gave a two-round protocol for privacy amplification (PA) against active adversaries with classical side information. The main ingredient in their protocol is a non-malleable extractor, which when combined with an information-theoretically secure message authentication code gives security in PA. As shown in [CV17], using quantum secure non-malleable extractor, one can extend the proof of security by Dodis and Wichs to the case of active quantum adversaries. Thus, our quantum secure non-malleable extractor, given by Theorem 1, enables us to obtain a PA protocol against active quantum adversaries (see Definition 2).

**Theorem 2.** Let \( d = O(\log^7(n/\varepsilon)) \), \( k = \Omega(d) \) and \( \delta > 0 \) be a small enough constant. There exists an efficient two-round PA protocol against active quantum adversaries for min-entropy \( k \) sources that can extract \( \left(\frac{n}{2} - \delta\right) k \) bits with communication \( O(d) \) and error \( O(\varepsilon) \).

This result allows for a trade-off in communication and error in a PA protocol, improving on the result of [ACLV19], where the communication is required to be linear in \( n \). We provide a proof of Theorem 2 in Appendix 4. Similar proofs have appeared in [CV17, ABJO21, ACLV19].

We show the following result for 2-source non-malleable extractors.

**Theorem 3** (quantum secure 2-source non-malleable extractor). Let \( k = O(n^{1/4}) \) and \( \varepsilon = 2^{-n^{O(1)}} \). There exists an efficient 2-source non-malleable extractor \( 2\text{nmExt} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^{n/4} \) that is \((n-k,n-k,O(\varepsilon))-\)quantum secure (see Definition 1).

The above result is stated for the adversary qnma. As a corollary, we obtain corresponding results for other models of 2-source adversaries studied in the literature.

Kasher and Kempe [KK10] introduced the quantum independent adversary (qia) model, where the adversary obtains independent side-information from both sources. Informally, qia gets the registers \( \rho_{E_1E_2} \) as quantum side information in \( \rho_{XE_1E_2Y} \) such that

\[
\rho_{XE_1E_2Y} = (\rho_{XE_1} \otimes \rho_{YE_2}) \quad ; \quad H_{\min}(X|E_1)_\rho \geq k_1 \quad ; \quad H_{\min}(Y|E_2)_\rho \geq k_2.
\]

We refer the reader to [KK10] for complete details. We propose to incorporate non-malleable extractor security against qia as follows.

**Definition 1** (2-source non-malleable extractor against qia). Let \( \rho_{XE_1E_2Y} \) be a c-q state with registers \((XY)\) classical such that \( |X| = |Y| = n \),

\[
\rho_{XE_1E_2Y} = (\rho_{XE_1} \otimes \rho_{YE_2}) \quad ; \quad H_{\min}(X|E_1)_\rho \geq k_1 \quad ; \quad H_{\min}(Y|E_2)_\rho \geq k_2.
\]
Let $T_1 : \mathcal{L}(\mathcal{H}_{E_2} \otimes \mathcal{H}_X) \to \mathcal{L}(\mathcal{H}_{E'_2} \otimes \mathcal{H}_X \otimes \mathcal{H}_Y)$, $T_2 : \mathcal{L}(\mathcal{H}_{E_1} \otimes \mathcal{H}_Y) \to \mathcal{L}(\mathcal{H}_{E'_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_Y')$ be (safe) CPTP maps such that for $\sigma_{XX'E_1'E_2YY'} = (T_1 \otimes T_2)(\rho_{XE_1'E_2Y})$, we have registers $(X'Y'Y')$ classical and either $\Pr(X \neq X')_\sigma = 1$ or $\Pr(Y \neq Y')_\sigma = 1$. We say a function $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^m$ is a $(k_1,k_2,\varepsilon)$-quantum secure 2-source non-malleable extractor against $qia$ iff for every $\sigma$ as defined above, we have
\[
\|\sigma_{f(X,Y)f(X',Y')YY'E_1} - U_m \otimes \sigma_{f(X',Y')YY'E_1'}\|_1 \leq \varepsilon.
\]

**Remark 1.** In the Definition[7] one may ask if we can provide both the registers $E'_1$ and $E'_2$ as side-information to the adversary along with $YY'$. However this may allow adversary to gain complete knowledge of $X,Y$. Thus we settle on the model as in Definition[4].

We have the following corollary of Theorem[3]

**Corollary 2** ($2nmExt$ is a 2-source non-malleable extractor against qia). Let the function $2nmExt$ be from Theorem[3]. $2nmExt$ is an $(n-k,n-k,O(\varepsilon))$-quantum secure 2-source non-malleable extractor against $qia$.

Arnon-Friedman, Portmann and Scholz [AFPS16] introduced the quantum Markov adversary (qMara). Informally, qMara gets the register $\rho_{E}$ as quantum side information in a Markov-chain $\rho_{XEY}$. We propose to incorporate 2-source non-malleable extractor security against qMara as follows.

**Definition 2** (2-source non-malleable extractor against qMara). Let $\rho_{XEY}$ be a c-q state with registers $(XY)$ classical such that
\[
\rho_{XEY} = \sum_t \Pr(T = t) |t\rangle \langle t| \otimes (\rho_{XE_1}^t \otimes \rho_{E_2}^t); \quad H_{\min}(X|E)_\rho \geq k_1; \quad H_{\min}(Y|E)_\rho \geq k_2,
\]
where $T$ is classical register over a finite alphabet. Let $T_1 : \mathcal{L}(\mathcal{H}_{E_2} \otimes \mathcal{H}_X \otimes \mathcal{H}_T) \to \mathcal{L}(\mathcal{H}_{E'_2} \otimes \mathcal{H}_X \otimes \mathcal{H}_X' \otimes \mathcal{H}_T)$, $T_2 : \mathcal{L}(\mathcal{H}_{E_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_T) \to \mathcal{L}(\mathcal{H}_{E'_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_Y' \otimes \mathcal{H}_T)$ be (safe) CPTP maps such that for $\sigma_{XX'E_1'E_2YY'} = (T_1 \otimes T_2)(\rho_{XEY})$, we have registers $(XX'Y'Y')$ classical and either $\Pr(X \neq X')_\sigma = 1$ or $\Pr(Y \neq Y')_\sigma = 1$. We say a function $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^m$ is a $(k_1,k_2,\varepsilon)$-quantum secure 2-source non-malleable extractor against qMara iff for every $\sigma$ as defined above, we have
\[
\|\sigma_{f(X,Y)f(X',Y')YY'E_1} - U_m \otimes \sigma_{f(X',Y')YY'E_1'}\|_1 \leq \varepsilon.
\]

**Remark 2.** For reasons similar to that of Remark[4] in Definition[2] we do not allow the registers $E'_1$ and $E'_2$ as side-information to the adversary along with $YY'T$.

We have the following corollary of Theorem[3]

**Corollary 3** ($2nmExt$ is a 2-source non-malleable extractor against qMara). Let the function $2nmExt$ be from Theorem[3]. $2nmExt$ is an $(n-k,n-k,O(\varepsilon))$-quantum secure 2-source non-malleable extractor against qMara.

The following are the $t$-tampering extensions of Theorem[1] and Theorem[3].

\[2\]This holds for a Markov-chain $(X-E-Y)$. 

5
Theorem 4 (quantum secure $t$-non-malleable extractor). Let $d = O(\log^7(n/\varepsilon))$, $t = d^{O(1)}$ and $k = \Omega(d)$. There exists an efficient non-malleable extractor $t$-nmExt $: \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^{k/4t}$ that is $(t; k, O(\varepsilon))$-quantum secure (see Definition 27).

Theorem 5 (quantum secure 2-source $t$-non-malleable extractor). Let $k = O(n^{1/4})$, $\varepsilon = 2^{-n^{\Omega(1)}}$ and $t = n^{\Omega(1)}$. There exists an efficient 2-source non-malleable extractor $t$-2nmExt $: \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^{n^{1/4t}}$ that is $(t; n-k, n-k, O(\varepsilon))$-quantum secure (see Definition 23).

All the above non-malleable extractors are based on the powerful technique of alternating extraction along with advice generators and correlation breakers with advice [CGL20, Coh16] that uses the clever flip flop primitive [Coh15].

Proof overview

As noted before, a key difficulty one faces in analyzing non-malleable extractors in the quantum setting is formulating and manipulating conditional independence relations between random variables since conditioning on quantum side information is tricky. Cohen and Vidick [CV17] attempted to deal with the difficulty by using the formalism of quantum Markov-chains. However, as noted by them, after the adversary tampers with the source, a quantum Markov-chain no longer necessarily remains a quantum Markov-chain. Hence it appeared that a generalization of the quantum Markov-chain model is needed.

We use $qpa$-states instead and consider sources in $\mathcal{Q}$. Additionally, we relate the sources produced by $qma$ model, $l$-$qma$-states [ABJO21] (see Definition 12) and $(k_1, k_2)$-$qpa$-states (see Lemmas 4 and 5). We note that a source in $\mathcal{Q}$ remains in $\mathcal{Q}$ after adversary tampering, thereby getting over the key difficulty faced by earlier models, including quantum Markov-chains. Since $qma$ can simulate other adversary models, we are able to derive as corollaries, the existence of seeded and 2-source non-malleable extractors in the more standard models of adversaries studied previously in the literature. Similarly, our application for privacy amplification against active quantum adversary is in the standard model of adversary considered in previous works e.g. [CV17, ACLV19].

Our proof follows on the lines of [CGL20, Coh16]. The key technical lemmas that we use repeatedly in the analysis are,

- a quantum analogue of alternating extraction in a $(k_1, k_2)$-$qpa$-state with approximately uniform seed (Lemma 1), and
- min-entropy loss under classical interactive communication to ensure enough conditional min-entropy is left for alternating extraction (Lemma 2).

Lemma 1 makes use of the powerful Uhlmann’s theorem. Lemma 2 follows using arguments similar to that of a result of Jain and Kundu [JK21] for quantum communication. In the technique of alternating extraction, we repeatedly extract and generate several approximately uniform random variables. In our analysis, the generation of random variables is viewed as communication protocols (see Protocols 1 to 9 for seeded non-malleable extractor analysis). We consider this approach more intuitive and makes the analysis more fine-grained.

As the analysis progresses, several additional classical random variables need to be generated and considered. We generate them in a manner such that the requirement of conditional-min-entropy is met for alternating extraction. It is a priori not clear what the sequence of generation of
classical random variables should be (for original inputs and tampered inputs) because of the non-commutative nature of quantum side information. Careful analysis of the classical non-malleable extractor constructions leads us to show that such a sequence of generation of random variables exists. The communication protocols (Protocols 1 to 9) specify the exact sequence in which the additional random variables are generated in various cases.

In the analysis of 2-source non-malleable extractors, we additionally need a 2-source extractor for sources in $Q$. [ABJO21] provide security of an inner-product 2-source extractor for an $l$-qma-state (see Definition 12) as long as $l < n$. To use this result we need to relate a $(k_1, k_2)$-qpa-state (see Definition 10) with some $l$-qma-state. [ABJO21] show that a pure state $\sigma_{X\bar{X}NMY\bar{Y}}$ can be generated in the $l$-qma-state framework if

$$
\begin{align*}
H_{\min}(XMY)_{\sigma} &\geq k_1 ; \\
\tilde{H}_{\min}(YNX\bar{X})_{\sigma} &\geq k_2.
\end{align*}
$$

Note that one of the min-entropy bounds is in terms of the modified-conditional-min-entropy $\tilde{H}_{\min}(\cdot|\cdot)$. Next we prove a result that connects $H_{\min}(\cdot|\cdot)$ and $\tilde{H}_{\min}(\cdot|\cdot)$. We show for any quantum state $\rho_{XE}$,

$$
\tilde{H}_{\min}(X|E)_{\rho} \leq H_{\min}(X|E)_{\rho} \leq \tilde{H}_{\min}(X|E)_{\rho'} + 2\log(1/\varepsilon)
$$

for some $\rho' \approx \varepsilon \rho$. This is a result of independent interest (see Lemma 3).

In the proof, while relating $(k_1, k_2)$-qpa-state with some $l$-qma-state, we face an additional technical difficulty of correcting a marginal state. For this we use the state perturbation lemma due to [JK21] and circumvent the issue at the cost of minor loss in parameters (see Lemma 4). Thus, we are able to show that any $(k_1, k_2)$-qpa-state (say $\sigma$) can be approximated by an $l$-qma-state (say $\sigma'$) for $l \approx 2n - k_1 - k_2$. We also show that $l$-qma-state, $\sigma'$ also has the appropriate $\tilde{H}_{\min}(\cdot|\cdot)$ bounds, i.e.

$$
\tilde{H}_{\min}(X|MY\bar{Y})_{\sigma'} \approx k_1 ; \\
\tilde{H}_{\min}(Y|NX\bar{X})_{\sigma'} \approx k_2.
$$

Now the security of inner-product in an $l$-qma-state (with appropriate $\tilde{H}_{\min}(\cdot|\cdot)$ bounds) from [ABJO21] for $l < n$ implies the security of inner-product in a $(k_1, k_2)$-qpa-state for $k_1 + k_2 > n$ which is then used in the 2-source non-malleable extractor construction. The analysis for 2-source non-malleable extractor then proceeds on similar lines of seeded non-malleable extractor.

Analysis of the $t$-tampered counterparts proceeds on similar lines by appropriate adjustment of parameters to account for increased communication in communication protocols.

### Comparison with [ACLV19, ABJO21]

Both [ACLV19] and [ABJO21] have considered the inner-product based non-malleable extractor proposed by Li [Li12]. [ACLV19] extends the first step of classical proof, the reduction provided by the non-uniform XOR lemma, to the quantum case. This helps in reducing the task of showing non-malleable extractor property of inner-product to showing security of inner-product in a certain communication game. They then approach the problem of showing security of inner-product in a communication game by using the “reconstruction paradigm” of [DPVR12] to guess the entire input $X$ from the modified side information.

On the other hand, the work of [ABJO21] reduces the security of inner-product in a communication game to the security of inner-product against the quantum measurement adversary. In the process, both [ACLV19] and [ABJO21] crucially use the combinatorial properties of inner-product. For example, [ABJO21] heavily uses the pair wise independence property of inner-product.
Note that the communication protocols we use in our analysis are not related to the earlier approach of reduction to a communication game, which is more specific to inner-product.

Comparison with classical constructions

To the best of our knowledge, [Li19] provides the construction of seeded non-malleable extractor that works for seed-length \( d = \mathcal{O}\left(\log n + \log^{1+o(1)}\left(\frac{1}{\varepsilon}\right)\right) \), source min-entropy \( k \geq \mathcal{O}\left(\log \log n + \log\left(\frac{1}{\varepsilon}\right)\right) \) and output length \( m = \Omega(k) \). In the case of 2-source non-malleable extractor, [Li19] construction works for sources with min-entropy \( k_1, k_2 \geq (1 - \delta)n \), output length \( m = \Omega(n) \) and error \( \varepsilon = 2^{-\Omega\left(\frac{n \log \log n}{\log n}\right)} \).

In the quantum setting, [ACLV19] provided the first construction of quantum secure seeded non-malleable extractor for seed-length \( d = \frac{n}{2} \), source min-entropy \( k \geq \left(\frac{1}{2} + \delta\right)n \), output length \( m = \Omega(n) \) and error \( \varepsilon = 2^{-\Omega(n)} \). Our work exponentially improves the parameters for source min-entropy \( k \geq \text{poly}\left(\log\left(\frac{n}{\varepsilon}\right)\right) \), seed-length \( d = \text{poly}\left(\log\left(\frac{n}{\varepsilon}\right)\right) \) and output length \( m = \Omega(k) \). In the setting of quantum secure 2-source non-malleable extractors, we provide the first construction for sources with min-entropy \( k_1, k_2 \geq (1 - o(1))n \), output length \( m = n/4 \) and error \( \varepsilon = 2^{-n^{\Omega(1)}} \). We note though we are still far from achieving close to optimal constructions in the quantum-setting, we hope our techniques find new applications in proving quantum security of other classical non-malleable extractors.

Subsequent works

- [ABJ22] have extended the connection of [CG16] between 2-source non-malleable extractors and 2-split-state non-malleable codes (for classical messages) secure against quantum adversaries. They used our quantum secure 2-source non-malleable extractors to construct the first explicit quantum secure 2-split-state non-malleable codes (for classical messages) of length \( m = n^{\Omega(1)} \), error \( \varepsilon = 2^{-n^{\Omega(1)}} \) and codeword length \( 2n \).

- Using the techniques introduced in this work, [BBL23] constructed a rate 1/2 quantum secure non-malleable randomness encoder. They use this in a black-box manner, to construct the following:
  - rate 1/11, 3-split-state non-malleable code for quantum messages
  - rate 1/3, 3-split-state non-malleable code for classical messages against quantum adversaries
  - rate 1/5, 2-split-state non-malleable code for (uniform) classical messages against quantum adversaries.

- Furthermore, [BGJR23] have constructed
  - rate 1/11, 2-split-state non-malleable code for (uniform) quantum messages
  - 2-split-state non-malleable code for quantum messages of length \( m = n^{\Omega(1)} \), error \( \varepsilon = 2^{-n^{\Omega(1)}} \) and codeword length \( \mathcal{O}(n) \).
  - They showed something stronger: the explicit 2-split-state non-malleable code for quantum messages is, in fact, a 2-out-of-2 non-malleable secret sharing scheme for quantum messages with share size \( n \), any message of length at most \( n^{\Omega(1)} \), and error \( \varepsilon = 2^{-n^{\Omega(1)}} \).
Organization

In Section 2 we describe quantum information theoretic and other preliminaries. Section 3 contains useful lemmas and claims. We describe the construction and security analysis of seeded non-malleable extractor in Section 4 of 2-source non-malleable extractor in Section 5 and of t-tampered versions of these in Appendix A and B respectively. The communication protocols used in all the analysis appear in the Appendix C. We provide a proof of PA against active adversaries in Appendix D.

2 Preliminaries

Let $n, m, d, t$ represent positive integers and $l, k, k_1, k_2, \delta, \gamma, \varepsilon \geq 0$ represent reals.

Quantum information theory

All the logarithms are evaluated to the base 2. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be finite sets (we only consider finite sets in this paper). Let $|\mathcal{X}|$ represent the size of $\mathcal{X}$, that is the number of elements in $\mathcal{X}$. For a random variable $X \in \mathcal{X}$, we use $X$ to denote both the random variable and its distribution whenever it is clear form the context. We use $x \leftarrow X$ to denote $x$ drawn according to $X$. We call random variables $X, Y$, copies of each other if $\Pr[X = Y] = 1$. For a random variable $X \in \{0, 1\}^n$ and $d \leq n$, let $\text{Prefix}(X, d)$ represent the first $d$ bits of $X$. Let $U_d$ represent the uniform distribution over $\{0, 1\}^d$. Let $Y^1, Y^2, \ldots, Y^t$ be random variables. We denote the joint random variable $Y^1Y^2\ldots Y^t$ by $Y^t$. Similarly for any subset $S \subseteq [t]$, we use $Y^S$ to denote the joint random variable comprised of all the $Y^s$ such that $s \in S$.

Consider a finite-dimensional Hilbert space $\mathcal{H}$ endowed with an inner-product $\langle \cdot, \cdot \rangle$ (we only consider finite-dimensional Hilbert-spaces). A quantum state (or a density matrix or a state) is a positive semi-definite operator on $\mathcal{H}$ with trace value equal to 1. It is called pure iff its rank is 1. Let $|\psi\rangle$ be a unit vector on $\mathcal{H}$, that is $\langle \psi, \psi \rangle = 1$. With some abuse of notation, we use $\psi$ to represent the state and also the density matrix $|\psi\rangle\langle\psi|$, associated with $|\psi\rangle$. Given a quantum state $\rho$ on $\mathcal{H}$, support of $\rho$, called supp($\rho$) is the subspace of $\mathcal{H}$ spanned by all eigenvectors of $\rho$ with non-zero eigenvalues.

A quantum register $A$ is associated with some Hilbert space $\mathcal{H}_A$. Define $|A| := \log (\dim(\mathcal{H}_A))$. Let $L(\mathcal{H}_A)$ represent the set of all linear operators on the Hilbert space $\mathcal{H}_A$. For operators $O, O' \in L(\mathcal{H}_A)$, the notation $O \leq O'$ represents the Löwner order, that is, $O' - O$ is a positive semi-definite operator. We denote by $D(\mathcal{H}_A)$, the set of all quantum states on the Hilbert space $\mathcal{H}_A$. State $\rho$ with subscript $A$ indicates $\rho_A \in D(\mathcal{H}_A)$. If two registers $A, B$ are associated with the same Hilbert space, we shall represent the relation by $A \equiv B$. For two states $\rho, \sigma$, we let $\rho \equiv \sigma$ represent that they are identical as states (potentially in different registers). Composition of two registers $A$ and $B$, denoted $AB$, is associated with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. For two quantum states $\rho \in D(\mathcal{H}_A)$ and $\sigma \in D(\mathcal{H}_B)$, $\rho \otimes \sigma \in D(\mathcal{H}_{AB})$ represents the tensor product (Kronecker product) of $\rho$ and $\sigma$. The identity operator on $\mathcal{H}_A$ is denoted $\mathbb{I}_A$. Let $U_A$ denote maximally mixed state in $\mathcal{H}_A$. Let $\rho_{AB} \in D(\mathcal{H}_{AB})$. Define

$$\rho_B \overset{\text{def}}{=} \text{Tr}_A \rho_{AB} \overset{\text{def}}{=} \sum_i (|i\rangle \otimes \mathbb{I}_B) \rho_{AB} (|i\rangle \otimes \mathbb{I}_B),$$

$^3$Some works use $P_X$ to denote distribution of $X$, however we use this non-standard notation for brevity.
singleton set, we shorthand is not unique. Suppose \( \rho \) of \( \rho \in \mathcal{D}(\mathcal{H}_B) \) such that \( \text{Tr}_{BPAB} = \rho \). Purification of a quantum state is not unique. Suppose \( A \equiv B \). Given \{\ket{i}_A\} and \{\ket{i}_B\} as orthonormal bases over \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively, the canonical purification of a quantum state \( \rho_A \) is \( |\rho_A\rangle \equiv (\rho_A^\perp \otimes I_B)(\sum_i \ket{i}_A\ket{i}_B) \).

A quantum map \( E : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B) \) is a completely positive and trace preserving (CPTP) linear map. A Hermitian operator \( H : \mathcal{H}_A \to \mathcal{H}_A \) is such that \( H = H^\dagger \). Let \( \Lambda_+(H) \) denote the set of eigenvectors with positive eigenvalues, i.e. \( \Lambda_+(H) = \{v \in \mathcal{H}_A : Hv = \lambda_v v, \lambda_v > 0\} \). Let \( H_+ \) be the vector space generated by \( \Lambda_+(H) \). We say that \( H_+ \) is the positive part of \( H \). A projector \( \Pi \in \mathcal{L}(\mathcal{H}_A) \) is a Hermitian operator such that \( \Pi^2 = \Pi \). A unitary operator \( V_A : \mathcal{H}_A \to \mathcal{H}_A \) is such that \( V_A^\dagger V_A = V_A V_A^\dagger = I_A \). The set of all unitary operators on \( \mathcal{H}_A \) is denoted by \( \mathcal{U}(\mathcal{H}_A) \). An isometry \( V_A : \mathcal{H}_A \to \mathcal{H}_B \) is such that \( V_A^\dagger V_A = I_A \) and \( V_A V_A^\dagger = I_B \). A POVM element is an operator \( 0 \leq M \leq I \). We use the shorthand \( M \) to represent \( M \otimes I \), where \( I \) is clear from the context. We use shorthand to represent \( M \otimes I \), where \( I \) is clear from the context.

**Definition 3** (Classical register in a pure state). Let \( X \) be a set. A classical-quantum (c-q) state \( \rho_{XE} \) is of the form

\[
\rho_{XE} = \sum_{x \in X} p(x) \ket{x}\bra{x} \otimes \rho^x_E,
\]

where \( \rho^x_E \) are states.

Let \( \rho_{XEA} \) be a pure state. We call \( X \) a classical register in \( \rho_{XEA} \), if \( \rho_{XE} \) (or \( \rho_{XA} \)) is a c-q state. We identify random variable \( X \) with the register \( X \), with \( \Pr(X = x) = p(x) \).

**Definition 4** (Copy of a classical register). Let \( \rho_{X\hat{X}E} \) be a pure state with \( X \) being a classical register in \( \rho_{X\hat{X}E} \) (see Definition 3) taking values in \( X \). Similarly, let \( \hat{X} \) be a classical register in \( \rho_{X\hat{X}E} \) taking values in \( \hat{X} \). Let \( \Pi_{\text{Eq}} = \sum_{x \in X} \ket{x}\bra{x} \otimes \rho^x_{\hat{X}} \) be the equality projector acting on the registers \( X \hat{X} \). We call \( X \) and \( \hat{X} \) copies of each other (in the computational basis) if \( \text{Tr}(\Pi_{\text{Eq}} \rho_{X\hat{X}}) = 1 \).

**Definition 5** (Conditioning). Let

\[
\rho_{XE} = \sum_{x \in \{0,1\}^n} p(x) \ket{x}\bra{x} \otimes \rho^x_E,
\]

be a c-q state. For an event \( S \subseteq \{0,1\}^n \), define

\[
\Pr(S)_{\rho} \equiv \sum_{x \in S} p(x);
\]

\[
(\rho|X \in S) \equiv \frac{1}{\Pr(S)_{\rho}} \sum_{x \in S} p(x) \ket{x}\bra{x} \otimes \rho^x_E.
\]

We sometimes shorthand \( (\rho|X \in S) \) as \( (\rho|S) \) when the register \( X \) is clear from the context.

Let \( \rho_{AB} \) be a state with \( |A| = n \). We define \( (\rho|A \in S) \equiv (\sigma|S) \), where \( \sigma_{AB} \) is the c-q state obtained by measuring the register \( A \) in \( \rho_{AB} \) in the computational basis. In case \( S = \{s\} \) is a singleton set, we shorthand \( (\rho|A = s) \equiv \text{Tr}_A(\rho|A = s) \).

10
Definition 6 (Extension). Let
\[ \rho_{XE} = \sum_{x \in \{0,1\}^n} p(x) |x \rangle \langle x| \otimes \rho_E^x, \]
be a c-q state. For a function \( Z : \mathcal{X} \rightarrow \mathcal{Z} \), define the following extension of \( \rho_{XE} \),
\[ \rho_{ZXE} \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} p(x) |Z(x) \rangle \langle Z(x)| \otimes |x \rangle \langle x| \otimes \rho_E^x. \]

Definition 7 (Safe maps). We call an isometry \( V : \mathcal{H}_X \otimes \mathcal{H}_A \rightarrow \mathcal{H}_X \otimes \mathcal{H}_B \), safe on \( X \) iff there is a collection of isometries \( V_x : \mathcal{H}_A \rightarrow \mathcal{H}_B \) such that the following holds. For all states \( |\psi \rangle_{XA} = \sum_x \alpha_x |x \rangle_X |\psi^x \rangle_A \),
\[ V|\psi \rangle_{XA} = \sum_x \alpha_x |x \rangle_X V_x |\psi^x \rangle_A. \]
We call a CPTP map \( \Phi : \mathcal{L}((\mathcal{H}_X \otimes \mathcal{H}_A)) \rightarrow \mathcal{L}((\mathcal{H}_X \otimes \mathcal{H}_B)) \), safe on classical register \( X \) iff there is a collection of CPTP maps \( \Phi_x : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B) \) such that the following holds. For all c-q states \( \rho_{XA} = \sum_x \Pr(X = x) |x \rangle \langle x| \otimes \rho_A^x \),
\[ \Phi(\rho_{XA}) = \sum_x \Pr(X = x) |x \rangle \langle x| \otimes \Phi_x(\rho_A^x). \]

All isometries (or in general CPTP maps) considered in this paper are safe on classical registers that they act on. CPTP maps applied by adversaries can be assumed w.l.o.g as safe on classical registers, by the adversary first making a (safe) copy of classical registers and then proceeding as before. This does not reduce the power of the adversary.

For a pure state \( \rho_{X_{E_A}} \) (with \( X \) classical) and a function \( Z : \mathcal{X} \rightarrow \mathcal{Z} \), define \( \rho_{Z_{Z_{X_{E_A}}}E_A} \) to be a pure state extension of \( \rho_{X_{E_A}} \) generated via a safe isometry \( V : \mathcal{H}_X \rightarrow \mathcal{H}_X \otimes \mathcal{H}_Z \otimes \mathcal{H}_{\hat{Z}} \) (\( Z \) classical with copy \( \hat{Z} \)).

Definition 8. 1. For \( p \geq 1 \) and matrix \( A \), let \( \| A \|_p \) denote the Schatten \( p \)-norm.
2. For \( p \geq 1 \): \( \| A \|_p = (\text{Tr}(A^\dagger A)^{p/2})^{\frac{1}{p}}. \)
3. For states \( \rho, \sigma : \Delta(\rho, \sigma) \overset{\text{def}}{=} \frac{1}{2} \| \rho - \sigma \|_1. \)
4. Fidelity: For states \( \rho, \sigma : \mathbb{F}(\rho, \sigma) \overset{\text{def}}{=} \| \sqrt{\rho} \sqrt{\sigma} \|_1. \)
5. Bures metric: For states \( \rho, \sigma : \Delta_B(\rho, \sigma) \overset{\text{def}}{=} \sqrt{1 - \mathbb{F}(\rho, \sigma)} \). We write \( \rho \approx \varepsilon \sigma \) to denote \( \Delta_B(\rho, \sigma) \leq \varepsilon \). Being a metric, it satisfies the triangle inequality.
6. Define \( d(X)_\rho \overset{\text{def}}{=} \Delta_B(\rho_X, U_X) \) and \( d(X|Y)_\rho \overset{\text{def}}{=} \Delta_B(\rho_{XY}, U_X \otimes \rho_Y) \).
7. Max-divergence ([Dat09], see also [JRS02]): For states \( \rho, \sigma \) such that \( \text{supp}(\rho) \subset \text{supp}(\sigma) \),
\[ D_{\text{max}}(\rho\|\sigma) \overset{\text{def}}{=} \min\{\lambda \in \mathbb{R} : \rho \leq 2^\lambda \sigma\}. \]
8. Min-entropy and conditional-min-entropy: For a state $\rho_{XE}$, the min-entropy of $X$ is defined as,

$$H_{\min}(X)_\rho \overset{\text{def}}{=} -D_{\max}(\rho_X\|I_X).$$

The conditional-min-entropy of $X$, conditioned on $E$, is defined as,

$$H_{\min}(X|E)_\rho \overset{\text{def}}{=} -\inf_{\sigma_E \in \mathcal{D}(\mathcal{H}_E)} D_{\max}(\rho_{XE}\|I_X \otimes \sigma_E).$$

The modified-conditional-min-entropy of $X$, conditioned on $E$, is defined as,

$$\tilde{H}_{\min}(X|E)_\rho \overset{\text{def}}{=} -D_{\max}(\rho_{XE}\|I_X \otimes \rho_E).$$

For the facts stated below without citation, we refer the reader to standard text books [NC00, Wat11].

Fact 1 (Uhlmann’s Theorem [Uhl76]). Let $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$. Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ be a purification of $\rho_A$ and $\sigma_{AC} \in \mathcal{D}(\mathcal{H}_{AC})$ be a purification of $\sigma_A$. There exists an isometry $V$ (from a subspace of $\mathcal{H}_C$ to a subspace of $\mathcal{H}_B$) such that,

$$\Delta_B(\rho, \sigma) = \Delta_B(\rho_A, \sigma_A),$$

where $|\theta\rangle_{AB} = (I_A \otimes V)|\sigma\rangle_{AC}$.

Fact 2 ([JRS02]). Let $\rho_{AB}, \sigma_{AB}$ be pure states such that $D_{\max}(\rho_B\|\sigma_B) \leq k$. Let Alice and Bob share $\sigma_{AB}$. There exists an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \mathcal{H}_C$ such that,

1. $(V \otimes I_B)\rho_{AB}(V \otimes I_B)^\dagger = \phi_{A'BC}$, where $C$ is a single qubit register.
2. Let $C$ be the outcome of measuring $\phi_C$ in the standard basis. Then $\Pr(C = 1) \geq 2^{-k}$.
3. Conditioned on outcome $C = 1$, the state shared between Alice and Bob is $\rho_{A'B}$.

Fact 3 ([CLWL14]). Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_M) \rightarrow \mathcal{L}(\mathcal{H}_M')$ be a CPTP map and let $\sigma_{XM'} = (I \otimes \mathcal{E})(\rho_{XM})$. Then,

$$H_{\min}(X'|M')_\rho \geq H_{\min}(X|M)_\rho.$$

Above is equality if $\mathcal{E}$ is a map corresponding to an isometry.

Fact 4 (Lemma B.3. in [DPVR12]). For a c-q state $\rho_{ABC}$ (with $C$ classical),

$$H_{\min}(A|B|C)_\rho \geq H_{\min}(A|B)_\rho - |C|.$$

Fact 5. Let $\rho_{XE}, \sigma_{XE}$ be two c-q states. Then,

- $\|\rho_{XE} - \sigma_{XE}\|_1 \geq E_{x \sim \rho_X}\|\rho^x_E - \sigma^x_E\|_1$.
- $\Delta_B(\rho_{XE}, \sigma_{XE}) \geq E_{x \sim \rho_X} \Delta_B(\rho^x_E, \sigma^x_E)$.

The above inequalities are equalities iff $\rho_X = \sigma_X$.

Fact 6 ([FvdG06]). Let $\rho, \sigma$ be two states. Then,

$$1 - F(\rho, \sigma) \leq \Delta(\rho, \sigma) \leq \sqrt{1 - F^2(\rho, \sigma)} ; \quad \Delta^2_B(\rho, \sigma) \leq \Delta(\rho, \sigma) \leq \sqrt{2}\Delta_B(\rho, \sigma).$$
Fact 7 (Data-processing). Let $\rho, \sigma$ be two states and $\mathcal{E}$ be a CPTP map. Then

- $\Delta(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \Delta(\rho, \sigma)$.
- $\Delta_B(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \Delta_B(\rho, \sigma)$.
- $D_{\max}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D_{\max}(\rho\|\sigma)$.

Above are equalities if $\mathcal{E}$ is a map corresponding to an isometry.

Fact 8. Let $M, A \in \mathcal{L}(\mathcal{H})$. If $A \geq 0$ then $M^\dagger AM \geq 0$.

Fact 9. Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a state and $M \in \mathcal{L}(\mathcal{H}_B)$ such that $M^\dagger M \leq 1_B$. Let $\hat{\rho}_{AB} = \frac{M\rho_{AB}M^\dagger}{\text{Tr} M\rho_{AB}M^\dagger}$. Then,

$$D_{\max}(\hat{\rho}_A\|\rho_A) \leq \log \left( \frac{1}{\text{Tr} M\rho_{AB}M^\dagger} \right).$$

Fact 10 (Substate Perturbation Lemma (Lemma 9 in [JK21])). Let $\sigma_{XB}$, $\psi_X$ and $\rho_B$ be states such that,

$$\sigma_{XB} \leq 2^c (\psi_X \otimes \sigma_B) \quad ; \quad \Delta_B(\sigma_B, \rho_B) \leq \delta_1.$$

For any $\delta_0 > 0$, there exists state $\rho'_{XB}$ satisfying

$$\Delta_B(\rho'_{XB}, \sigma_{XB}) \leq \delta_0 + \delta_1 \quad ; \quad \rho'_{XB} \leq 2^{c+1} \left( 1 + \frac{4}{\delta_0} \right) \psi_X \otimes \rho_B \quad ; \quad \rho_B' = \rho_B$$

Fact 11. Let $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A)$ be two states and $M \in \mathcal{L}(\mathcal{H}_A)$ such that $M^\dagger M \leq 1_A$. Then,

$$|\text{Tr} M\rho M^\dagger - \text{Tr} M\sigma M^\dagger| \leq \frac{\|\rho - \sigma\|_1}{2}.$$

Fact 12 (Gentle Measurement Lemma [Wil13]). Let $\rho \in \mathcal{D}(\mathcal{H}_A)$ be a state and $M \in \mathcal{L}(\mathcal{H}_A)$ such that $M^\dagger M \leq 1_A$ and $\text{Tr}(M\rho M^\dagger) \geq 1 - \varepsilon$. Let $\hat{\rho} = \frac{M\rho M^\dagger}{\text{Tr} M\rho M^\dagger}$. Then, $\Delta_B(\rho, \hat{\rho}) \leq \sqrt{\varepsilon}$.

Fact 13 (Corollary 5.2 in [CGL20]). For any constant $\delta \in (0,1)$, there exist constants $\alpha, \beta < 1/14$ such that for all positive integers $\nu, r, t$, with $r \geq \nu^\alpha$ and $t = O(\nu^\beta)$ the following holds.

There exists a polynomial time computable function $\text{Samp} : \{0,1\}^r \rightarrow [\nu]^t$, such that for any set $S \subset [\nu]$ of size $\delta\nu$,

$$\Pr(|\text{Samp}(U_r) \cap S| \geq 1) \geq 1 - 2^{-\Omega(\nu^\alpha)}.$$

Definition 9. Let $M = 2^m$. The inner-product function, $\text{IP}_M^n : \mathbb{F}_M^n \times \mathbb{F}_M^n \rightarrow \mathbb{F}_M$ is defined as follows:

$$\text{IP}_M^n(x, y) = \sum_{i=1}^{n} x_i y_i,$$

where the operations are over the field $\mathbb{F}_M$.

[JK21] does not explicitly mention in the statement of the substate perturbation lemma that $\rho_B' = \rho_B$. But it can be easily verified from their proof that this holds. The statement in [JK21] is more general and is stated for purified distance, however it holds for any fidelity based distance including the Bures metric.
Definition 10 (Markov-chain). A state $\rho_{XEY}$ forms a Markov-chain (denoted $(X \rightarrow E \rightarrow Y)_{\rho}$) iff $I(X : Y | E)_{\rho} = 0$.

Fact 14 ([HJPW04]). A Markov-chain $(X \rightarrow E \rightarrow Y)_{\rho}$ can be decomposed as follows:

$$
\rho_{XEY} = \sum_{t} \Pr(T = t) |t\rangle \langle t| \otimes (\rho_{XE_{1}} \otimes \rho_{Y_{2}}),
$$

where $T$ is classical register over a finite alphabet.

Fact 15 ([AHJ+21]). For a Markov-chain $(X \rightarrow E \rightarrow Y)_{\rho}$, there exists a CPTP map $\Phi : \mathcal{L}(\mathcal{H}_{E}) \rightarrow \mathcal{L}(\mathcal{H}_{E} \otimes \mathcal{H}_{Y})$ such that $\rho_{XEY} = (I_{X} \otimes \Phi)\rho_{XE}$.

Fact 16 (Corollary 5.5 in [Wat11]). Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{A} \otimes \mathcal{H}_{B})$ be a state and $V_{B} : \mathcal{L}(\mathcal{H}_{B}) \rightarrow \mathcal{L}(\mathcal{H}_{E} \otimes \mathcal{H}_{C})$ be an isometry such that $|C| = 1$. Let $\sigma_{AB'C} = (I_{A} \otimes V_{B})\rho_{AB}(I_{A} \otimes V_{B})^{\dagger}$ and $\Phi_{AB'} = (\sigma_{AB'C}|C = 1)$. There exists an operator $M_{B}$ such that $0 \leq M_{B}^{\dagger}M_{B} \leq I_{B}$ and

$$
\Phi_{AB'} = \frac{(I_{A} \otimes M_{B})\rho_{AB}(I_{A} \otimes M_{B})^{\dagger}}{\text{Tr}(I_{A} \otimes M_{B})\rho_{AB}(I_{A} \otimes M_{B})^{\dagger}} ; \quad \Pr[C = 1]_{\sigma} = \text{Tr} \left( M_{B}\rho_{B}M_{B}^{\dagger} \right).
$$

Extractors

Throughout the paper we use extractor to mean seeded extractor unless stated otherwise.

Definition 11 (quantum secure extractor). An $(n,d,m)$-extractor $\text{Ext} : \{0,1\}^{n} \times \{0,1\}^{d} \rightarrow \{0,1\}^{m}$ is said to be $(k,\varepsilon)$-quantum secure if for every state $\rho_{XES}$, such that $H_{\text{min}}(X | E)_{\rho} \geq k$ and $\rho_{XES} = \rho_{XE} \otimes U_{d}$, we have

$$
\| \rho_{\text{Ext}(X,S)}E - U_{m} \otimes \rho_{E} \|_{1} \leq \varepsilon.
$$

In addition, the extractor is called strong if

$$
\| \rho_{\text{Ext}(X,S)SE} - U_{m} \otimes U_{d} \otimes \rho_{E} \|_{1} \leq \varepsilon.
$$

$S$ is referred to as the seed for the extractor.

Fact 17 ([DPVR12] [CV17]). There exists an explicit $(2m,\varepsilon)$-quantum secure strong $(n,d,m)$-extractor $\text{Ext} : \{0,1\}^{n} \times \{0,1\}^{d} \rightarrow \{0,1\}^{m}$ for parameters $d = 0(\log^{2}(n/\varepsilon) \log m)$.

Definition 12 (l-qma-state [ABJO21]). Let $\tau_{X\hat{X}}, \tau_{Y\hat{Y}}$ be the canonical purifications of independent and uniform sources $X,Y$ respectively. Let $\tau_{NM}$ be a pure state. Let

$$
\theta_{X\hat{X}NMY\hat{Y}} = \tau_{X\hat{X}} \otimes \tau_{NM} \otimes \tau_{Y\hat{Y}}.
$$

Let $U : \mathcal{H}_{X} \otimes \mathcal{H}_{N} \rightarrow \mathcal{H}_{X} \otimes \mathcal{H}_{N'} \otimes \mathcal{H}_{A}$ and $V : \mathcal{H}_{Y} \otimes \mathcal{H}_{M} \rightarrow \mathcal{H}_{Y} \otimes \mathcal{H}_{M'} \otimes \mathcal{H}_{B}$ be isometries such that registers $A,B$ are single qubit registers. Let $\rho_{X\hat{X}AN'M'BY\hat{Y}} = (U \otimes V)\theta_{X\hat{X}NMY\hat{Y}}(U \otimes V)^{\dagger}$

and

$$
l = \log \left( \frac{1}{\Pr(A = 1, B = 1)_{\rho}} \right) ; \quad \sigma_{X\hat{X}N'M'Y\hat{Y}} = (\rho_{X\hat{X}N'M'BY\hat{Y}}|A = 1, B = 1).
$$

We call $\sigma_{X\hat{X}N'M'Y\hat{Y}}$ an l-qma-state.
Definition 13 ((k)-qpa-state). We call a pure state $\sigma_{X^X Y^Y}$, with $(XY)$ classical and $(X^X Y^Y)$ copy of $(XY)$, a $(k)$-qpa-state if
\[ H_{\text{min}}(X|MY) \geq k ; \quad \sigma_{X^X Y^Y} = \sigma_{X^X N} \otimes U_Y. \]

Definition 14 ((k)-qnm-state). Let $\sigma_{X^X Y^Y}$ be a $(k)$-qpa-state. Let $V : H_Y \otimes H_M \to H_Y \otimes H_Y \otimes H_{X'} \otimes H_{M'}$ be an isometry such that for $\rho = V\sigma V^\dagger$, we have $Y'$ classical (with copy $Y'$) and $\Pr(Y \neq Y') = 1$. We call state $\rho$ a $(k)$-qnm-state.

Remark 3. In Definition 14 (and in similar such definitions) previous works consider the notion of CPTP maps with no fixed points. However we replace it with the condition $\Pr(Y \neq Y') = 1$, which suffices for our purposes.

We require the non-malleable extractor to extract from every $(k)$-qnm-state, chosen by the adversary $qnm$ (short for quantum non-malleable adversary). We follow similar convention for 2-source non-malleable extractors and their extensions to $t$-tampering setting.

Definition 15 (quantum secure non-malleable extractor). An $(n, d, m)$-non-malleable extractor $nmExt : \{0, 1\}^n \times \{0, 1\}^d \to \{0, 1\}^m$ is $(k, \varepsilon)$-secure against $qnm$ if for every $(k)$-qnm-state $\rho$ (chosen by the adversary $qnm$),
\[ \|\rho_{nmExt(X,Y)nmExt(X,Y')YY'M'} - U_m \otimes \rho_{nmExt(X,Y')YY'M'}\|_1 \leq \varepsilon. \]

Definition 16 ((k1, k2)-qpa-state). We call a pure state $\sigma_{X^X Y^Y}$, with $(XY)$ classical and $(X^X Y^Y)$ copy of $(XY)$, a $(k_1, k_2)$-qpa-state if
\[ H_{\text{min}}(X|MY) \geq k_1 ; \quad H_{\text{min}}(Y|NX) \geq k_2. \]

Definition 17 ((k1, k2)-qnm-state). Let $\sigma_{X^X Y^Y}$ be a $(k_1, k_2)$-qpa-state. Let $U : H_X \otimes H_N \to H_X \otimes H_X \otimes H_X \otimes H_N$ and $V : H_Y \otimes H_M \to H_Y \otimes H_D \otimes H_{\hat{X}} \otimes H_{M'}$ be isometries such that for $\rho = (U \otimes V)\sigma(U \otimes V)^\dagger$, we have $(XY')$ classical (with copy $X'Y'$) and,
\[ \Pr(Y \neq Y') = 1 \quad \text{or} \quad \Pr(X \neq X') = 1. \]

We call state $\rho$ a $(k_1, k_2)$-qnm-state.

Definition 18 (quantum secure 2-source non-malleable extractor). An $(n, n, m)$-non-malleable extractor $2nmExt : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}^m$ is $(k_1, k_2, \varepsilon)$-secure against $qnm$ if for every $(k_1, k_2)$-qnm-state $\rho$ (chosen by the adversary $qnm$),
\[ \|\rho_{2nmExt(X,Y)2nmExt(X',Y')YY'M'} - U_m \otimes \rho_{2nmExt(X',Y')YY'M'}\|_1 \leq \varepsilon. \]

Fact 18 (IP security against states with $\tilde{H}_{\text{min}}(|\cdot|)$ bounds [ABJO21]). Let $n = \frac{n_1}{m}$ and $k_1 + k_2 - n_1 \geq 2\log\left(\frac{1}{\varepsilon}\right) + m$. Let $\sigma_{X^X Y^Y M'}$ be a state with $|X| = |Y| = n_1$, registers $XY$ classical (with copies $X Y$) and
\[ \tilde{H}_{\text{min}}(X|YM) \geq k_1 ; \quad \tilde{H}_{\text{min}}(Y|XM) \geq k_2. \]

Then
\[ \|\sigma_{1^{IP}_{nm}(X,Y)X'N'} - U_m \otimes \sigma_{X'N'}\|_1 \leq \varepsilon ; \quad \|\sigma_{2^{IP}_{nm}(X,Y)Y'M'} - U_m \otimes \sigma_{Y'M'}\|_1 \leq \varepsilon. \]
Error correcting codes

**Definition 19.** Let $\Sigma$ be a finite set. A mapping $\text{ECC} : \Sigma^k \to \Sigma^n$ is called an error correcting code with relative distance $\gamma$ if for any $x, y \in \Sigma^k$ such that $x \neq y$, the Hamming distance between $\text{ECC}(x)$ and $\text{ECC}(y)$ is at least $\gamma n$. The rate of the code denoted by $\delta$, is defined as $\delta \overset{\text{def}}{=} \frac{k}{n}$. The alphabet size of the code is the number of elements in $\Sigma$.

**Fact 19 ([CS95]).** Let $p$ be a prime number and $m$ be an even integer. Set $q = p^m$. For every $\delta \in (0, 1)$ and for any large enough integer $n$ there exists an efficiently computable linear error correcting code $\text{ECC} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ with rate $\delta$ and relative distance $1 - \gamma$ such that

$$\delta + \frac{1}{\sqrt{q} - 1} \geq \gamma.$$

3 Useful claims and lemmas

**Claim 1.** Let $\rho_{ZA}, \rho'_{ZA}$ be states such that $\Delta_B(\rho, \rho') \leq \varepsilon'$. If $d(Z|A)_\rho \leq \varepsilon$ then $d(Z|A)_\rho \leq 2\varepsilon' + \varepsilon$.

**Proof.** Consider,

$$d(Z|A)_\rho \leq \Delta_B(\rho_{ZA}, \rho'_{ZA}) + \Delta_B(\rho'_{ZA}, U_Z \otimes \rho_A)$$

(Triangle inequality)

$$\leq \varepsilon' + \Delta_B(\rho'_{ZA}, U_Z \otimes \rho_A)$$

$$\leq \varepsilon' + \Delta_B(\rho'_{ZA}, U_Z \otimes \rho_A) + \Delta_B(U_Z \otimes \rho'_A, U_Z \otimes \rho_A)$$

(Triangle inequality)

$$\leq \varepsilon' + \varepsilon + \varepsilon' = 2\varepsilon' + \varepsilon.$$  

\[\square\]

**Claim 2.** Let $\rho_{XE} \in \mathcal{D}(\mathcal{H}_X \otimes \mathcal{H}_E)$ be a c-q state such that $|X| = n$ and $H_{\min}(X|E)_\rho \geq n - k$. Let $X_d = \text{Prefix}(X, d)$ for some integer $k \leq d \leq n$. Then $H_{\min}(X_d|E)_\rho \geq d - k$.

**Proof.** Since $H_{\min}(X|E)_\rho \geq n - k$, there exists a state $\sigma_E$ such that

$$D_{\max}(\rho_{XE} \| U_X \otimes \sigma_E) \leq k.$$  

Using Fact 7, we have

$$D_{\max}(\rho_{X_dE} \| U_{X_d} \otimes \sigma_E) \leq k \implies D_{\max}(\rho_{X_dE} \| I_{X_d} \otimes \sigma_E) \leq k - d.$$  

Thus,

$$H_{\min}(X_d|E)_\rho = -\inf_{\tau_E} D_{\max}(\rho_{X_dE} \| I_{X_d} \otimes \tau_E) \geq d - k,$$

which completes the proof. \[\square\]

**Claim 3.** Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a state and $V_B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$ be an isometry such that $|C| = 1$. Let $\sigma_{AB'C} = (I_A \otimes V_B)\rho_{AB}(I_A \otimes V_B)^\dagger$ and $\Phi_{AB'} = (\sigma_{AB'C}|C = 1)$. Then,

$$\bar{H}_{\min}(A|B')_\Phi \geq \bar{H}_{\min}(A|B)_\rho.$$  

\[5\text{Claim holds even when } \Delta_B() \text{ is replaced with } \Delta().\]
Proof. Since $\tilde{H}_{\min}(A|B)_\rho = -D_{\max}(\rho_{AB}\|I_A \otimes \rho_B)$, we have
\[
\rho_{AB} \leq 2^{-\tilde{H}_{\min}(A|B)_\rho} (I_A \otimes \rho_B).
\]
By Fact 16 there exists an operators $M_B$ such that $0 \leq M_B^\dagger M_B \leq I_B$ and
\[
\Phi_{AB'} = \frac{(I_A \otimes M_B)\rho_{AB}(I_A \otimes M_B)^\dagger}{\text{Tr}(I_A \otimes M_B)\rho_{AB}(I_A \otimes M_B)^\dagger}.
\]
This further implies,
\[
\Phi_{AB'} \leq 2^{-\tilde{H}_{\min}(A|B)_\rho} \left( I_A \otimes \frac{M_B\rho_B M_B^\dagger}{\text{Tr}(M_B\rho_B M_B^\dagger)} \right) = 2^{-\tilde{H}_{\min}(A|B)_\rho} (I_A \otimes \Phi_{B'}).
\]
Thus,
\[
\tilde{H}_{\min}(A|B')_\Phi = -D_{\max}(\rho_{AB'}\|I_A \otimes \Phi_B') \geq \tilde{H}_{\min}(A|B)_\rho.
\]
\[\square\]

Claim 4. Let $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ be a state and $M \in \mathcal{L}(\mathcal{H}_C)$ such that $M^\dagger M \leq I_C$. Let $\hat{\rho}_{ABC} = \frac{M\rho_{ABC}M^\dagger}{\text{Tr}M\rho_{ABC}M^\dagger}$. Then,
\[
H_{\min}(A|B)_\hat{\rho} \geq H_{\min}(A|B)_\rho - \log \left( \frac{1}{\text{Tr}M\rho_{ABC}M^\dagger} \right).
\]
Proof. Let $H_{\min}(A|B)_\rho = -D_{\max}(\rho_{AB}\|I_A \otimes \sigma_B)$ for some state $\sigma_B$. Thus,
\[
\rho_{AB} \leq 2^{-H_{\min}(A|B)_\rho} (I_A \otimes \sigma_B).
\]
Since from Fact 9 we have
\[
\hat{\rho}_{AB} \leq \frac{1}{\text{Tr}M\rho_{ABC}M^\dagger} \rho_{AB},
\]
we finally get $D_{\max}(\rho_{AB}\|I_A \otimes \sigma_B) \leq -H_{\min}(A|B)_\rho + \log \left( \frac{1}{\text{Tr}M\rho_{ABC}M^\dagger} \right)$. Thus,
\[
H_{\min}(A|B)_\hat{\rho} = -\inf_{\theta_B} D_{\max}(\hat{\rho}_{AB}\|I_A \otimes \theta_B) \geq H_{\min}(A|B)_\rho - \log \left( \frac{1}{\text{Tr}M\rho_{ABC}M^\dagger} \right).
\]
\[\square\]

Claim 5 (IP security against $(k_1, k_2)$-qpa-state). Let $n = \frac{n_1}{m}$ and $k_1 + k_2 \geq n_1 + m + 4 + 8 \log \left( \frac{1}{\varepsilon} \right)$. Let $\sigma_{XNN'}$ be a $(k_1, k_2)$-qpa-state with $|X| = |Y| = n_1$. Let $Z = I_{2^n}^\perp(X, Y)$. Then
\[
\|\sigma_{XNN'} - U_m \otimes \sigma_{XN'}\|_1 \leq 17\varepsilon \quad ; \quad \|\sigma_{ZYM'} - U_m \otimes \sigma_{YM'}\|_1 \leq 17\varepsilon.
\]
Proof. Let state $\rho^{(1)}$ be from Lemma 4 such that
\[
\rho^{(1)} \approx_{6\varepsilon} \sigma \quad ; \quad \tilde{H}_{\min}(X|YYM')^{(1)} \geq k_1 - 2 \log \left( \frac{1}{\varepsilon} \right) \quad ; \quad \tilde{H}_{\min}(Y|XXN)^{(1)} \geq k_2 - 4 - 4 \log \left( \frac{1}{\varepsilon} \right).
\]
Using Fact 18, we have
\[ \| \rho_{IP_{2m}^{(1)}(X,Y)XN'} - U_m \otimes \rho_X^{(1)} |X_N' \|_1 \leq \varepsilon ; \quad \| \rho_{IP_{2m}^{(1)}(X,Y)YM'} - U_m \otimes \rho_Y^{(1)} |YM' \|_1 \leq \varepsilon. \]

Using Claim 1 and Fact 6 after noting \( \rho^{(1)} \approx_{6\varepsilon} \sigma \), we finally get,
\[ \| \sigma_{XN'} - U_m \otimes \sigma_{XN'} \|_1 \leq 17\varepsilon ; \quad \| \sigma_{YM'} - U_m \otimes \sigma_{YM'} \|_1 \leq 17\varepsilon. \]

\[ \square \]

**Claim 6.** Let \( \rho_{XAYB} \) be a pure state. Let \( d = |X| \). There exists a pure state \( \hat{\rho}_{XAYB} \) such that,
\[ \Delta_B(\hat{\rho}_{XAYB}, \rho_{XAYB}) = d |X| \rho_{XAYB} \otimes \rho_{XAYB} \otimes \rho_{XAYB} \;
\]
where
\[ \hat{\rho}_{XAYB} = V_{XAYB} \rho_{XAYB} V_{XAYB}^\dagger. \]

From Fact 1 we get an isometry \( V \) such that
\[ \Delta_B(\rho_{XAYB}, \hat{\rho}_{XAYB}) = \Delta_B(\rho_{XAYB}, U_d \otimes \rho_{YB}), \]

where,
\[ \hat{\rho}_{XAYB} = V_{XAYB} \rho_{XAYB} V_{XAYB}^\dagger. \]

From Fact 6
\[ \rho_{XAYB} = U_d \otimes \rho_{YB} \]

Noting that isometry \( V \) acts trivially on \( \theta_{XYB} \), we have \( \hat{\rho}_{XYB} = \theta_{XYB} = U_d \otimes \rho_{YB} \). Thus, \( \hat{\rho}_{XYB} = U_d \otimes \rho_{YB} \) which completes the proof. \[ \square \]

**Lemma 1** (Alternating extraction). Let \( \theta_{XASB} \) be a pure state with \( (XS) \) classical, \( |X| = n, |S| = d \) and
\[ \rho_{XASB} = H_{min}(X|SB) \rho_{XASB} \leq k ; \quad \Delta_B(\theta_{XASB}, \theta_{XA} \otimes U_d) \leq \varepsilon'. \]

Let \( T \stackrel{def}{=} \text{Ext}(X, S) \) where \( \text{Ext} \) is a \((k, \varepsilon)\)-quantum secure strong \((n, d, m)\)-extractor. Then,
\[ \Delta_B(\theta_{TB}, U_m \otimes \theta_{B}) \leq 2\varepsilon' + \sqrt{\varepsilon}. \]

**Proof.** We use Fact 1 with the following assignment of registers (below the registers on the left are from Fact 1 and the registers on the right are the registers in this proof)
\[ (\sigma_A, \rho_A, \sigma_{AC}, \rho_{AB}) \leftarrow (\theta_{XASB}, \theta_{XA} \otimes U_d, \theta_{XASB}, \beta_{XASB} \otimes \tau_{SS'}), \]

where \( \tau_{SS'} \) is the canonical purification of \( \tau_S \equiv U_d \) and \( \beta_{XASB} \equiv \theta_{XASB} \). From Fact 1 we get an isometry \( V \) (acts trivially on \( \theta_{XASB} \)) such that,
\[ \Delta_B(V \theta_{XASB} V^\dagger, \beta_{XASB} \otimes \tau_{SS'}) = \Delta_B(\theta_{XASB}, \theta_{XA} \otimes U_d) \leq \varepsilon'. \] (1)
Let $\gamma_{TAS_1BSS'}$ be the state after $T = \text{Ext}(X,S)$ is generated using $\beta_{XAS_1B} \otimes \tau_{SS'}$. From Definition 6 and noting that,

$$H_{\min}(X|S_1B)_\beta = H_{\min}(X|SB)_\theta \geq k \quad ; \quad \gamma_{TSS_1B} \equiv \gamma_{TSS_1B},$$

we get,

$$\Delta_B(\gamma_{TSS_1B}, U_m \otimes U_d \otimes \beta_{S_1B}) \leq \sqrt{\varepsilon}. \quad (2)$$

Consider,

$$\Delta_B(\theta_{TB}, U_m \otimes \theta_B)$$

$$= \Delta_B(V\theta_{TB}V^\dagger, U_m \otimes V\theta_BV^\dagger) \quad \text{(Fact 7)}$$

$$\leq \Delta_B(V\theta_{TB}V^\dagger, U_m \otimes U_d \otimes \beta_{S_1B}) + \Delta_B(U_m \otimes U_d \otimes \beta_{S_1B}, U_m \otimes V\theta_BV^\dagger) \quad \text{(Triangle inequality)}$$

$$\leq \Delta_B(V\theta_{TB}V^\dagger, U_m \otimes U_d \otimes \beta_{S_1B}) + \varepsilon'$$

$$\leq \Delta_B(V\theta_{TB}V^\dagger, \gamma_{TSS_1B}) + \Delta_B(\gamma_{TSS_1B}, U_m \otimes U_d \otimes \beta_{S_1B}) + \varepsilon' \quad \text{(Triangle inequality)}$$

$$\leq \Delta_B(\beta_{XASB}V^\dagger, \beta_{XAS_1B} \otimes \tau_{SS'}) + \Delta_B(\gamma_{TSS_1B}, U_m \otimes U_d \otimes \beta_{S_1B}) + \varepsilon' \quad \text{(Fact 7)}$$

$$\leq 2\varepsilon' + \sqrt{\varepsilon}. \quad \text{(Eq. (1) and (2))}$$

\[ \square \]

**Lemma 2** (Min-entropy loss under classical interactive communication). Let $\rho_{XNM}$ be a pure state where Alice holds registers $(XN)$ and Bob holds register $M$, such that register $X$ is classical and

$$H_{\min}(X|M)_\rho \geq k.$$

Let Alice and Bob proceed for $t$-rounds, where in each round Alice generates a classical register $R_i$ and sends it to Bob, followed by Bob generating a classical register $S_i$ and sending it to Alice. Alice applies a (safe on $X$) isometry $V^i : \mathcal{H}_X \otimes \mathcal{H}_{N_{i-1}} \rightarrow \mathcal{H}_X \otimes \mathcal{H}_{N_{i-1}} \otimes \mathcal{H}_{R_i}$ (in round $i$) to generate $R_i$. Let $\rho_{XNM_i}$ be the state at the end of round-$i$, where Alice holds registers $XN_i$ and Bob holds register $M_i$. Then,

$$H_{\min}(X|M_i)_{\rho_{t}} \geq k - \sum_{j=1}^{t} |R_j|. \quad (3)$$

Proof. Proof proceeds by induction on $i$. For $i = 0$, the bound follows from initiation (we take $\theta^0 = \rho$). Let us assume the bound for round $i$

$$H_{\min}(X|M_i)_{\rho_{i}} \geq k - \sum_{j=1}^{i} |R_j|,$$

and show the bound for round $i + 1$. Let $\tau_{XN_iR_{i+1}M_i}$ be the state after Alice generates $R_{i+1}$. From Fact 6 we have

$$H_{\min}(X|M_iR_{i+1}) \geq H_{\min}(X|M_i) - |R_{i+1}|.$$
Note that since Alice’s operations are safe on $X, \tau_{X_M} = \theta_{X_M}^i$. Hence,

$$H_{\min}(X|M_i) = H_{\min}(X|M_i|_{\theta^i})$$

From Fact 3 we have

$$H_{\min}(X|M_{i+1}) = H_{\min}(X|M_{i+1}|_{\theta^i}) \geq k - \sum_{j=1}^{i+1} |R_j|,$$

which shows the desired. \hfill \Box

**Lemma 3.** Let $\rho \in D(\mathcal{H}_{AB})$. There exists $\rho' \in D(\mathcal{H}_{AB})$ such that

$$\Delta_B(\rho, \rho') \leq \varepsilon ; \quad \tilde{H}_{\min}(A|B)_\rho \leq H_{\min}(A|B)_\rho \leq \tilde{H}_{\min}(A|B)_{\rho'} + 2 \log \left( \frac{1}{\varepsilon} \right).$$

**Proof.** The inequality $\tilde{H}_{\min}(A|B)_\rho \leq H_{\min}(A|B)_\rho$ is clear from definitions.

Let $H_{\min}(A|B)_\rho = u$ and $|A| = n$. Let $\sigma_B \in D(\mathcal{H}_B)$ be a state such that $D_{\max}(\rho_{AB}|U_A \otimes \sigma_B) = n-u$. Set $t = 2 \log \left( \frac{1}{\varepsilon} \right)$. Let $\Pi$ denote the projector on $(\sigma_B - 2t^i \rho_B)$. Hence, $\text{Tr}(\Pi \sigma_B) > 0$.

This gives us,

$$2t \text{Tr}(\Pi \rho_B) < \text{Tr}(\Pi \sigma_B) \leq 1,$$

and thus,

$$\text{Tr}(\Pi \rho_B) < 2^{-t} = \varepsilon^2 ; \quad \text{Tr}(\Pi \sigma_B) > 1 - \varepsilon^2.$$

Also, since $\text{Tr}((I_A \otimes \Pi) \rho_{AB}) = \text{Tr}(\Pi \rho_B)$, we have,

$$\text{Tr}((I_A \otimes \Pi) \rho_{AB}) > 1 - \varepsilon^2. \quad (3)$$

Note by construction, $\Pi (\sigma_B - 2t^i \rho_B) \Pi \leq 0$, and hence,

$$\Pi \sigma_B \Pi \leq 2t^i \Pi \rho_B \Pi. \quad (4)$$

Consider

$$\rho'_{AB} = \frac{(I_A \otimes \Pi) \rho_{AB} (I_A \otimes \Pi)}{\text{Tr}((I_A \otimes \Pi) \rho_{AB})} ; \quad \rho'_B = \frac{\Pi \rho_B \Pi}{\text{Tr}(\Pi \rho_B)}.$$

Using Fact 3, Fact 7 and Eq. (3), we have

$$\Delta_B \left( \rho'_B, \rho_B \right) \leq \Delta_B \left( \rho'_{AB}, \rho_{AB} \right) \leq \varepsilon.$$

Since, $D_{\max}(\rho_{AB}|U_A \otimes \sigma_B) = n-u$, we get

$$\Pi \rho_{AB} \Pi \leq 2^{n-u} \cdot U_A \otimes \Pi \sigma_B \Pi \quad \text{(Fact 5)}$$

$$\leq 2^{n-u+t} \cdot U_A \otimes \Pi \rho_B \Pi \quad \text{(Eq. (4))}$$

Normalizing by the trace, we get, $\rho'_{AB} \leq 2^{n-u+t} (U_A \otimes \rho'_B) = 2^{t-u} (I_A \otimes \rho'_B)$, which gives us

$$\tilde{H}_{\min}(A|B)_{\rho'} \geq H_{\min}(A|B)_{\rho} - 2 \log \left( \frac{1}{\varepsilon} \right).$$
Lemma 4. Let \( \rho_{X \tilde{X} N \tilde{Y} M} \) be a \((k_1, k_2)\)-qpa-state such that \( |X| = |\tilde{X}| = |Y| = |\tilde{Y}| = n \). There exists an \( l\)-qpa-state \( \rho_{X \tilde{X} N \tilde{Y} M}^{(l)} \), such that,

\[
\Delta_B(\rho^{(l)}, \rho) \leq 6\varepsilon \quad \text{and} \quad l \leq 2n - k_1 - k_2 + 4 + 6 \log \left(\frac{1}{\varepsilon}\right). 
\]

Furthermore,

\[
\tilde{H}_{\min}(X|Y \tilde{Y} M)_{\rho^{(l)}} \geq k_1 - 2 \log \left(\frac{1}{\varepsilon}\right) \quad ; \quad \tilde{H}_{\min}(Y|X \tilde{X} N)_{\rho^{(l)}} \geq k_2 - 4 - 4 \log \left(\frac{1}{\varepsilon}\right).
\]

Proof. For the ease of notation, let us denote \( \tilde{A} = X \tilde{X} N \) and \( \tilde{B} = Y \tilde{Y} M \). Since, \( \tilde{H}_{\min}(X|\tilde{B})_{\rho} \geq k_1 \), using Lemma 3 (on state \( \rho_{\tilde{B}X} \)) with the assignment of registers \((A, B) \leftarrow (X, \tilde{B})\), we know that there exists a state \( \rho_{\tilde{B}X}^{\prime} \), such that

\[
\Delta_B\left(\rho_{\tilde{B}X}^{\prime}, \rho_{\tilde{B}X}\right) \leq \varepsilon \quad ; \quad D_{\max}\left(\rho_{X \tilde{B}}^{\prime}\|U_{X} \otimes \rho_{\tilde{B}}^{\prime}\right) \leq |X| - k_1 + 2 \log \left(\frac{1}{\varepsilon}\right) \overset{\text{def}}{=} c_1. \quad (5)
\]

Consider a purification of \( \rho_{\tilde{B}X}^{\prime} \) denoted as \( \rho_{\tilde{B}XE}^{\prime} \). Using Fact 1 with the following assignment of registers,

\[
(\sigma_A, \rho_A, \sigma_{AC}, \rho_{AB}, \theta_{AB}) \leftarrow \left(\rho_{\tilde{B}X}^{\prime}, \rho_{\tilde{B}X}, \rho_{\tilde{B}XE}^{\prime}, \rho_{\tilde{A}B}, \rho_{\tilde{A}B}^{\prime}\right),
\]

there exists a pure state \( \rho_{\tilde{A}B}^{\prime} \) such that

\[
\Delta_B\left(\rho_{\tilde{A}B}^{\prime}, \rho_{\tilde{A}B}\right) \leq \varepsilon \quad ; \quad D_{\max}\left(\rho_{X \tilde{B}}^{\prime}\|U_{X} \otimes \rho_{\tilde{B}}^{\prime}\right) \leq c_1,
\]

where the inequalities follow from Eq. (3) and noting that isometry taking \( \rho_{\tilde{B}XE}^{\prime} \) to \( \rho_{\tilde{A}B}^{\prime} \) acts trivially on registers \( \tilde{B}X \). Similarly, there exists a pure state \( \rho_{\tilde{A}B}^{\prime\prime} \) such that

\[
\Delta_B\left(\rho_{\tilde{A}B}^{\prime\prime}, \rho_{\tilde{A}B}\right) \leq \varepsilon \quad ; \quad D_{\max}\left(\rho_{Y \tilde{A}}^{\prime\prime}\|U_{Y} \otimes \rho_{\tilde{A}}^{\prime\prime}\right) \leq |Y| - k_2 + 2 \log \left(\frac{1}{\varepsilon}\right) \overset{\text{def}}{=} c_2. \quad (7)
\]

Consider the following state:

\[
\theta = \tau_{X \tilde{X}} \otimes \rho_{\tilde{A}B}^{\prime\prime} \otimes \tau_{Y \tilde{Y} Y_1}^{Y_1} \,
\]

where \( \tau_{X \tilde{X}}, \tau_{Y \tilde{Y} Y_1} \) are canonical purifications of \( \tau_X \equiv U_X, \tau_{Y_1} \equiv U_Y \) respectively. Let Alice hold registers \( \tilde{A}X \), Bob hold registers \( \tilde{B}Y_1 \) and Referee hold registers \( XY_1 \). Now using Fact 2 with the following assignment of registers (below the registers on the left are from Fact 2 and the registers on the right are the registers in this proof)

\[
(\rho_{B}, \sigma_{B}, \rho_{AB}, \sigma_{AB}) \leftarrow \left(\rho_{X \tilde{B}}, \tau_{X} \otimes \rho_{\tilde{B}}^{\prime}, \rho_{X \tilde{X}N \tilde{B}}, \tau_{X \tilde{X}} \otimes \rho_{\tilde{A}B}^{\prime}\right),
\]

it follows from Fact 2 that there exists an isometry \( V_{\text{Alice}} : \mathcal{H}_{\tilde{A}X} \rightarrow \mathcal{H}_{XN} \otimes \mathcal{H}_{CA} \) such that the following hold:

\[
\phi_{\tilde{B}X \tilde{X} NCA} = V_{\text{Alice}} \otimes I_{X \tilde{B}} \begin{pmatrix} \rho_{\tilde{A}B}^{\prime} \otimes \tau_{X \tilde{X}} \end{pmatrix} \left(V_{\text{Alice}} \otimes I_{X \tilde{B}}\right)^{\dagger}. \quad (8)
\]

\[
\Pr(C_A = 1)_{\theta} = p_1 \geq 2^{-c_1} \quad \text{and} \quad (\phi|C_A = 1) = \rho_{\tilde{A}B}^{\prime\prime}. \quad (9)
\]

\[
21
\]
Thus starting from state $\theta$, there exists an isometry $V_{\text{Alice}}$ (acting solely on Alice’s registers) followed by measuring $C_A$, to get a state which we will denote as $\theta^{(1)}$. Hence, we get the following:

$$
\phi_B^{(1)} = \left( V_{\text{Alice}} \otimes I_X^B Y_1^Y \right) \theta \left( V_{\text{Alice}} \otimes I_X^B Y_1^Y \right)^\dagger
$$

$$
\Pr (C_A = 1)_{\phi^{(1)}} = p_1 \geq 2^{-c_1}
$$

$$
\theta^{(1)} = \left( \phi^{(1)} | C_A = 1 \right) = \rho'_{\overline{AB}} \otimes \tau_{Y_1 Y_1}.
$$

Note that Eq. (11)-(13) additionally contain $\tau_{Y_1 Y_1}$ when compared to Eq. (8)-(10). But as the isometry acts trivially on $\tau_{Y_1 Y_1}$, they follow trivially from Eq. (8)-(10).

Using Eq. (6) and Eq. (7) along with triangle inequality, we have

$$
\Delta_B \left( \rho''_{\overline{AB}}, \rho'_{\overline{AB}} \right) \leq 2\varepsilon.
$$

Using Fact 7, we further have $\Delta_B \left( \rho''_{\overline{AB}}, \rho'_{\overline{AB}} \right) \leq 2\varepsilon$. Now, using Fact 10 with the following assignment,

$$
(\sigma_X B, \psi_X, \rho_B, \rho'_{X B}, c_2, \delta_0, \delta_1) \leftarrow \left( \rho''_{\overline{AB}}, U_Y, \rho'_{\overline{AB}}, \rho^{(0)}_{Y \overline{A}}, c_2, \varepsilon, 2\varepsilon \right)
$$

there exists a state $\rho^{(0)}_{\overline{AY}}$ such that,

$$
\Delta_B \left( \rho^{(0)}_{\overline{AY}}, \rho''_{\overline{AY}} \right) \leq 3\varepsilon ; \quad \rho^{(0)}_{\overline{AY}} \leq 2^{c_2 + 1} \left( 1 + \frac{4}{\varepsilon^2} \right) \cdot \left( U_Y \otimes \rho'_{\overline{AB}} \right) \leq 2^{c'} \cdot \left( U_Y \otimes \rho'_{\overline{AB}} \right) ; \quad \rho^{(0)}_{\overline{AB}} = \rho'_{\overline{AB}},
$$

where $c' \overset{\text{def}}{=} c_2 + 4 + 2 \log \left( \frac{1}{\varepsilon} \right)$. Using Eq. (11), Eq. (17) and above, we get,

$$
\Delta_B \left( \rho^{(0)}_{\overline{AY}}, \rho''_{\overline{AY}} \right) \leq 5\varepsilon ; \quad \rho^{(0)}_{\overline{AY}} \leq 2^{c'} \cdot \left( U_Y \otimes \rho'_{\overline{AB}} \right) ; \quad \rho^{(0)}_{\overline{AB}} = \rho'_{\overline{AB}}.
$$

Consider a purification of $\rho^{(0)}_{\overline{AY}}$ denoted as $\rho^{(0)}_{\overline{AY} E}$. Using Fact 1 with the following assignment of registers,

$$
(\sigma_A, \rho_A, \sigma_{AC}, \rho_{AB}, \theta_{AB}) \leftarrow \left( \rho^{(0)}_{\overline{AY}}, \rho''_{\overline{AY}}, \rho^{(0)}_{\overline{AY} E}, \rho'_{\overline{AB}}, \rho^{(1)}_{\overline{AB}} \right),
$$

there exists a state $\rho^{(1)}_{\overline{AB}}$ such that,

$$
\Delta_B \left( \rho^{(1)}_{\overline{AB}}, \rho'_{\overline{AB}} \right) \leq 5\varepsilon ; \quad \text{D}_{\text{max}} \left( \rho^{(1)}_{\overline{AB}} \parallel U_Y \otimes \rho'_{\overline{AB}} \right) \leq c' ; \quad \rho^{(1)}_{\overline{AB}} = \rho'_{\overline{AB}}.
$$

where the inequalities follow from Eq. (15) and noting that isometry taking $\rho^{(0)}$ to $\rho^{(1)}$ acts trivially on registers $\overline{AY}$. Consider Fact 2 with the following assignment of registers,

$$
(\sigma_B, \rho_B, \rho_{AB}, \sigma_{AC}) \leftarrow \left( \rho^{(1)}_{\overline{AB}}, \rho'_{\overline{AB}} \otimes U_Y, \rho^{(1)}_{\overline{AB}}, \rho'_{\overline{AB}} \otimes \tau_{Y_1 Y_1} \right).
$$

From Fact 2 there exists an isometry $V_{\text{Bob}} : \mathcal{H}_{\overline{BY}_1} \rightarrow \mathcal{H}_{\overline{MY}_1} \otimes \mathcal{H}_B$ such that the following hold:

$$
\phi^{(2)}_{\overline{AMY_1 Y_1}} = \left( V_{\text{Bob}} \otimes I_{\overline{AY}} \right) \theta^{(1)} \left( V_{\text{Bob}} \otimes I_{\overline{AY}} \right)^\dagger
$$

$$
\Pr (C_B = 1)_{\phi^{(2)}} = p_2 \geq 2^{-c'}
$$

$$
\rho^{(1)}_{\overline{AB}} \equiv \left( \phi^{(2)} | C_B = 1 \right)
$$

22
For the ease of notation, let us set \( \zeta = (V_{\text{Alice}} \otimes V_{\text{Bob}}) \theta (V_{\text{Alice}} \otimes V_{\text{Bob}})^\dagger \). From Eq. (11)-(13) and Eq. (17)-(19), it follows that,

\[
\rho_{AB}^{(1)} \equiv (\zeta|C_A = 1, C_B = 1) \quad \text{(From Eq. (11), (13), (17) and (19))}
\]

\[
\Pr(C_A = 1, C_B = 1) \zeta \geq 2^{-c_1} 2^{-c'} \quad \text{(From Eq. (9) and (18)).}
\]

To summarize, the following properties hold in \( \rho_{AB}^{(1)} \), which completes the proof.

- From construction, it follows that \( \rho_{AB}^{(1)} \) is an \( l \)-qma-state with

\[
l \leq c_1 + c' = 2n - k_1 - k_2 + 4 + 6 \log(1/\epsilon).
\]

- \( \rho_{AB}^{(1)} \approx_{\delta} \rho_{AB} \) follows from Eq. (7), Eq. (16) and the triangle inequality.

- \( \bar{H}_{\min} \left| Y \right| \hat{A} \rangle_{\rho^{(1)}} \geq k_2 - 4 - 4 \log(1/\epsilon) \) follows from Eq. (16).

- \( \bar{H}_{\min} \left| X \right| \hat{B} \rangle_{\rho^{(1)}} \geq \bar{H}_{\min} \left| \hat{B} \right| \hat{Y}_1 \rangle_{\varphi^{(1)}} = \bar{H}_{\min} \left| \hat{B} \right| \hat{Y} \rangle_{\rho'} \geq k_1 - 2 \log(1/\epsilon) \). Here, the first inequality follows from Claim 3 and the last inequality follows from Eq. (5).

\[\square\]

**Lemma 5.** Let \( \sigma_{X \hat{X} N' M' M' \hat{Y}} \) be an \( l \)-qma-state such that \( |X| = |\hat{X}| = |Y| = |\hat{Y}| = n \). There exists \( k_1, k_2 \) such that \( k_1 \geq n - l, k_2 \geq n - l \) and \( \sigma \) is a \((k_1, k_2)\)-qpa-state.

**Proof.** Let \( \theta_{X \hat{X} N \hat{M}' Y \hat{Y}} = \tau_{X \hat{X}} \otimes \tau_{NM} \otimes \tau_{Y \hat{Y}} \) be the initial state as in Definition 12 (corresponding to an \( l \)-qma-state \( \sigma \)). Let \( U : H_X \otimes H_N \rightarrow H_X \otimes H_{N'} \otimes H_A \) and \( V : H_Y \otimes H_M \rightarrow H_Y \otimes H_{M'} \otimes H_B \) be isometries as in Definition 12. Let \( \rho^{(1)} = U \theta U^\dagger \). Noting isometry \( U \) is safe on classical register \( X \), we have

\[
H_{\min} \left| X \right| MY \hat{Y} \rangle_{\rho^{(1)}} = H_{\min} \left| X \right| MY \hat{Y} \rangle_{\theta} = n.
\]

Let \( p_1 = \Pr(A = 1)_{\rho^{(1)}} \) and \( \theta^{(1)} = (\rho^{(1)}|A = 1) \). Using Claim 4 with the following assignment (terms on the left are from Claim 4 and on the right are from here),

\[
(A, B, C, \rho, \varphi) \leftrightarrow (X, MY \hat{Y}, A, \rho^{(1)}, \theta^{(1)})
\]

we get,

\[
H_{\min} \left| X \right| MY \hat{Y} \rangle_{\theta^{(1)}} \geq H_{\min} \left| X \right| MY \hat{Y} \rangle_{\rho^{(1)}} - \log \left( 1/p_1 \right) = n + \log(p_1). \quad (20)
\]

Furthermore, let \( \rho^{(2)} = V \theta^{(1)} V^\dagger \). Again using Fact 3 and noting \( V \) is an isometry, we have

\[
H_{\min} \left| X \right| M' Y \hat{Y} \rangle_{\rho^{(2)}} \geq H_{\min} \left| X \right| M' Y \hat{Y} B \rangle_{\rho^{(2)}} = H_{\min} \left| X \right| M Y \hat{Y} \rangle_{\theta^{(1)}}. \quad (21)
\]

Let \( p_2 = \Pr(B = 1)_{\rho^{(2)}} \). Note \( \sigma = (\rho^{(2)}|B = 1) \) and \( l = \log \left( 1/p_1 p_2 \right) \). Now we use Claim 4 with the following assignment (terms on the left are from Claim 4 and on the right are from here),

\[
(A, B, C, \rho, \varphi) \leftrightarrow (X, M' Y \hat{Y}, B, \rho^{(2)}, \sigma)
\]
we get $H_{\min}(X|Y^\prime M^\prime) \geq H_{\min}(X|Y^\prime M^\prime)_{\rho(2)} + \log(p_2)$. Using Eq. (20) and (21), we get

$$H_{\min}(X|Y^\prime M^\prime) \geq n + \log(p_1 \cdot p_2) = n - l.$$ 

Using similar argument, we get

$$H_{\min}(Y|X^\prime N^\prime) \geq n - l.$$ 

Thus, $\sigma$ is a $(k_1, k_2)$-qpa-state such that both $k_1, k_2 \geq (n - l)$.

4 A quantum secure non-malleable extractor

These parameters hold throughout this section.

Parameters

Let $\delta > 0$ be a small enough constant and $q$ be a prime power. Let $n, d, d_1, a, d_2, s, b, h$ be positive integers and $k, \varepsilon', \gamma, \varepsilon > 0$ such that:

$$d = \mathcal{O}\left(\log^2\left(\frac{n}{\varepsilon'}\right)\right) ; \quad v = \frac{d}{\varepsilon} ; \quad d_1 = \mathcal{O}\left(\log^2\left(\frac{n}{\varepsilon}\right) \log(\log(v))\right) ; \quad q = \mathcal{O}\left(\frac{1}{\varepsilon^2}\right) ;$$

$$a = d_1 + \log q ; \quad \gamma = \mathcal{O}(\varepsilon) ; \quad 2^{\mathcal{O}(a)\sqrt{\varepsilon'}} = \varepsilon ; \quad d_2 = \mathcal{O}\left(\log^2\left(\frac{n}{\varepsilon}\right) \log d\right) ;$$

$$s = \mathcal{O}\left(\log^2\left(\frac{d}{\varepsilon'}\right) \log d\right) ; \quad b = \mathcal{O}\left(\log^2\left(\frac{d}{\varepsilon'}\right) \log d\right) ; \quad h = 10s ; \quad k \geq 5d.$$

Let

- $\text{Ext}_0$ be $(2 \log v, \varepsilon^2)$-quantum secure $(n, d_1, \log v)$-extractor,
- $\text{Ext}_1$ be $(2b, \varepsilon')$-quantum secure $(d, s, b)$-extractor,
- $\text{Ext}_2$ be $(2s, \varepsilon')$-quantum secure $(h, b, s)$-extractor,
- $\text{Ext}_3$ be $(2h, \varepsilon')$-quantum secure $(d, b, h)$-extractor,
- $\text{Ext}_4$ be $(d/4, \varepsilon^2)$-quantum secure $(d, h, d/8)$-extractor,
- $\text{Ext}_5$ be $(2d, \varepsilon')$-quantum secure $(n, d_2, d)$-extractor,
- $\text{Ext}_6$ be $(k, \varepsilon^2)$-quantum secure $(n, d/8, k/4)$-extractor,

be the quantum secure extractors from Fact 17.

Definition of non-malleable extractor

Let $\mathbb{F}_q$ be the finite field of size $q$. Let $\text{ECC}: \mathbb{F}_q^d \rightarrow \mathbb{F}_q^v$ be an error correcting code with relative distance $1 - \gamma$ and rate $\varepsilon$ (which exists from Fact 18 for our choice of parameters). We identify $I$ as an element from $\{1, \ldots, v\}$. By $\text{ECC}(Y)_I$, we mean the $I$-th entry of the code-word $\text{ECC}(Y)$, interpreted as a bit string of length $\log q$. 

24
Algorithm 1: $nmExt : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^{k/4}$

Input: $X,Y$
1. Advice generator: $Y_1 = \text{Prefix}(Y,d_1)$ ; $I = \text{Ext}_0(X,Y_1)$ ; $G = Y_1 \circ \text{ECC}(Y)_I$
2. $Y_2 = \text{Prefix}(Y,d_2)$ ; $T = \text{Ext}_5(X,Y_2)$
3. Correlation breaker with advice: $S = \text{AdvCB}(Y,T,G)$
4. $L = \text{Ext}_6(X,S)$
Output: $L$

Algorithm 2: $\text{AdvCB} : \{0,1\}^d \times \{0,1\}^d \times \{0,1\}^a \rightarrow \{0,1\}^\#$

Input: $Y,T,G$
1. $Z_0 = \text{Prefix}(T,h)$
2. For $i = 1,2,\ldots,a$ :
   Flip flop: $Z_i = \text{FF}(Y,T,Z_{i-1},G_i)$
3. $S = \text{Ext}_4(Y,Z_a)$
Output: $S$

Algorithm 3: $\text{FF} : \{0,1\}^d \times \{0,1\}^d \times \{0,1\}^h \times \{0,1\} \rightarrow \{0,1\}^h$

Input: $Y,T,Z,G$
1. $Z_0 = \text{Prefix}(Z,s)$, $A = \text{Ext}_1(Y,Z_s)$, $C = \text{Ext}_2(Z,A)$, $B = \text{Ext}_1(Y,C)$
2. If $G = 0$ then $\overline{Z} = \text{Ext}_3(T,A)$ and if $G = 1$ then $\overline{Z} = \text{Ext}_3(T,B)$
3. $Z_s = \text{Prefix}(Z,s)$, $\overline{A} = \text{Ext}_1(Y,\overline{Z_s})$, $\overline{C} = \text{Ext}_2(\overline{Z},\overline{A})$, $\overline{B} = \text{Ext}_1(Y,\overline{C})$
4. If $G = 0$, then $O = \text{Ext}_3(T,\overline{B})$ and if $G = 1$, then $O = \text{Ext}_3(T,\overline{A})$
Output: $O$
Result

In Protocol 1, Alice and Bob generate new classical registers using safe isometries on old classical registers. At any stage of Protocol 1, we use N to represent all the registers held by Alice other than the specified registers at that point. Similarly M represents all the registers held by Bob other than the specified registers. At any stage of the protocol, we use \( \hat{A}, \hat{B} \) to represent all the registers held by Alice and Bob respectively. We use the same convention for communication protocols in later sections as well.

The following theorem shows that the function \( \text{nmExt} \) as defined in Algorithm 1 is \((k, O(\varepsilon))\)-secure against qnm by noting that \( L = \text{nmExt}(X, Y) \) and \( L' = \text{nmExt}(X, Y') \).

**Theorem 6** (Security of \( \text{nmExt} \)). Let \( \rho_{XX'YY'} \) be a \((k)\)-qnm-state with \(|X| = n \) and \(|Y| = d \).

Let Protocol 1 start with \( \rho \). Let \( \Lambda \) be the state at the end of the protocol. Then,

\[
\|\rho_{LL'YY'} - U_{k/4} \otimes \rho_{L'Y'}\|_1 \leq d(L|B)_\Lambda \leq O(\varepsilon).
\]

**Proof.** The first inequality follows from Fact 7.

Note that total communication from Alice to Bob in Protocol 1 is at most (from our choice of parameters)

\[
2\log \left(\frac{d}{\varepsilon}\right) + 6ah + h + \frac{k}{4} \leq (1/4 + \delta) k.
\]

This implies, using Lemma 2, that throughout Protocol 1, \( H_{min}(X|\hat{B}) \geq (3/4 - \delta)k > k/2 \).

Total communication from Bob to Alice in Protocol 1 is at most

\[
2d_1 + 2d_2 + 2a + 6ab + \frac{d}{4} \leq (1/4 + \delta)d.
\]

Again using Lemma 2, throughout Protocol 1, \( H_{min}(Y|\hat{A}) \geq (3/4 - \delta)d \).

Let \( \Phi \) be the joint state in Protocol 1 after registers \( Z_0, Z'_0 \) are generated by Alice. From Claim 7 we have,

1. \( \Pr(G = G')_\Phi = O(\varepsilon) \).

2. \( d(T|\hat{B})_\Phi \leq \varepsilon \).

Let \( \tilde{\Phi} \) be the state obtained from Claim 8 (by letting \( \rho \) in Claim 8 as \( \Phi \) here) such that,

\[
H_{min}(Y|\hat{A})_{\tilde{\Phi}} \geq (3/4 - \delta)d \quad ; \quad \tilde{\Phi}_{\hat{T}\hat{B}} = U_d \otimes \tilde{\Phi}_{\hat{B}} \quad ; \quad \Delta_B(\tilde{\Phi}, \Phi) \leq \varepsilon.
\]  (22)

Let \( S_1 \) \( \overset{\text{def}}{=} \{ (\alpha, \alpha') : \alpha = \alpha' \} \). Note \( \Pr((\alpha, \alpha') \in S_1)_\Phi \leq \Pr((\alpha, \alpha') \in S_1)_\Phi + \varepsilon \leq O(\varepsilon) \). Let

\[
\tilde{\Phi}^{(\alpha, \alpha')} = \tilde{\Phi}|(G, G') = (\alpha, \alpha') \),
\]

and \( S_2 \) \( \overset{\text{def}}{=} \{ (\alpha, \alpha') : \Pr((G, G') = (\alpha, \alpha'))_\Phi \leq \frac{\varepsilon}{2(\log \varepsilon)} \} \). Note \( \Pr((\alpha, \alpha') \in S_2)_\Phi \leq \varepsilon \). For every \( (\alpha, \alpha') \notin S_2 \), we have (using Fact 9 and noting that, in \( \tilde{\Phi} \), a copy of \( (G, G') \) is part of \( \hat{B} \)),

\[
\Phi_{Y^\hat{A}}^{(\alpha, \alpha')} \leq \frac{\Phi_{Y^\hat{A}}}{\Pr((G, G') = (\alpha, \alpha'))_\Phi} \leq 2^{2|G| + \log(1/\varepsilon)} \cdot \Phi_{Y^\hat{A}}. \]  (23)

26
Eq. (22) and (23) imply that for every \((\alpha, \alpha') \notin S_2\), we have
\[
H_{\min} \left( Y \left| \hat{A} \right. \right) \hat{\Phi}(\alpha, \alpha') \geq (3/4 - \delta)d - 2a - \log \left( \frac{1}{\varepsilon} \right) \geq (3/4 - 2\delta)d > d/4.
\]

Let \( S \stackrel{\text{def}}{=} S_1 \cup S_2 \). From the union bound, \( \Pr((\alpha, \alpha') \in S) \hat{\Phi} \leq O(\varepsilon) \).

Let \( \hat{\Gamma}(\alpha, \alpha'), \hat{\Gamma}, \Gamma \) be the joint states at the end of the Protocol 2 (for \( i = a \)) when starting with the states \( \hat{\Phi}(\alpha, \alpha'), \hat{\Phi}, \Phi \) respectively. From Claim 8, we have for every \((\alpha, \alpha') \notin S\),
\[
d(Z|\hat{B})_{\hat{\Gamma}(\alpha, \alpha')} \leq O(\varepsilon). \quad (24)
\]

Consider (register \( Z \) is held by Alice and \( \hat{B} = GG'M \) in the state \( \hat{\Gamma} \)),
\[
d(Z|\hat{B})_{\hat{\Gamma}} = \Delta_B(\hat{\Gamma}_{ZZGM}, U_h \otimes \hat{\Gamma}_{GG'M}) = \mathbb{E}_{(\alpha, \alpha') \sim (G, G')} \Delta_B(\hat{\Gamma}_{Z\hat{B}}, U_h \otimes \hat{\Gamma}_{\hat{B}}) \\
\leq \Pr((\alpha, \alpha') \notin S)_{\hat{\Gamma}} \cdot O(\varepsilon) + \Pr((\alpha, \alpha') \in S)_{\hat{\Gamma}} \\
\leq O(\varepsilon),
\]

where the first equality follows from Fact 5, the first inequality follows from Eq. (24) and the last inequality follows since \( \Pr((\alpha, \alpha') \in S)_{\hat{\Gamma}} = \Pr((\alpha, \alpha') \in S)_{\hat{\Phi}} \leq O(\varepsilon) \). Since (using Fact 7 and Eq. (22)) \( \Delta_B(\hat{\Gamma}, \Gamma) \leq \Delta_B(\hat{\Phi}, \Phi) \leq \varepsilon \), we have,
\[
d(Z|\hat{B})_{\Gamma} = d(Z|\hat{B})_{\hat{\Gamma}} + \varepsilon = O(\varepsilon).
\]

Using arguments as before (involving Lemma 1), we have \( d(L|\hat{B})_{\Lambda} \leq O(\varepsilon) \).

\[\Box\]

Claim 7 (Advice generator). \( \Pr(G = G')_{\Phi} = O(\varepsilon) \) and \( d(T|\hat{B})_{\Phi} \leq \varepsilon \).

Proof. We first prove \( \Pr(G = G')_{\Phi} = O(\varepsilon) \). Let \( \sigma_{XY'Y'Y'} \) be the state after Alice has generated register \( I \) (before sending to Bob). Let \( \beta_{YY'} = U_{\log v} \otimes \Phi_{YY'} \). We have,
\[
\Delta(\Phi_{YY'Y'}, \beta_{YY'}) = \Delta(\sigma_{YY'Y'}, U_{\log v} \otimes \sigma_{YY'}) \\
\leq \Delta(\sigma_{\hat{B}}, U_{\log v} \otimes \sigma_{\hat{B}}) \quad \text{(Fact 7)} \\
\leq \sqrt{2}\Delta_B(\sigma_{\hat{B}}, U_{\log v} \otimes \sigma_{\hat{B}}) \quad \text{(Fact 6)} \\
\leq \sqrt{2}\varepsilon. \quad \text{(Lemma 1)} \quad (25)
\]

Consider,
\[
\Pr(G = G')_{\Phi} \leq \Pr(G = G')_{\beta} + \sqrt{2}\varepsilon \\
= \Pr(Y_1 = Y_1')_{\beta} \Pr(G = G' \mid Y_1 = Y_1')_{\beta} + \Pr(Y_1 \neq Y_1')_{\beta} \Pr(G = G' \mid Y_1 \neq Y_1')_{\beta} + \sqrt{2}\varepsilon \\
= \Pr(Y_1 = Y_1')_{\beta} \Pr(G = G' \mid Y_1 = Y_1')_{\beta} + \sqrt{2}\varepsilon \\
\leq \gamma + \sqrt{2}\varepsilon = O(\varepsilon).
\]

Note that conditioned on \( Y_1 = Y_1' \), we have \( I = I' \). The first inequality above follows from Eq. (25) and noting that the predicate \( (G = G') \) is determined from \( (I, Y, Y') \). The second equality follows from definition of \( G \). Second inequality follows since ECC has relative distance \( 1 - \gamma \).
We now prove $d(T|\tilde{B})_\Phi \leq \varepsilon$. Consider Protocol 8. Let $(E_A, E_A') \leftrightarrow (E_B, E_B')$ represent $2d_1$ distinct EPR pairs each shared between Alice and Bob where $|E_A| = |E_A'| = |E_B| = |E_B'| = d_1$.

Let $\tau''$ be the joint state just before Alice receives $Y_2$ and $\tau'$ be the joint state just after Alice generates $T$. Note, $\tau''_{A Y_2} = \tau''_A \otimes U_{d_2}$. Hence from Lemma 11,

$$\Delta_B(\tau''_B, U_{d} \otimes \tau'_B) \leq \sqrt{\varepsilon}.$$  

(26)

Let $\tau$ be the joint state just before Bob checks $(E_B, E_B') = (Y_1, Y_1')$. Let $C$ be the predicate $((E_B, E_B') = (Y_1, Y_1'))$. Note

$$\Pr(C = 1)_\tau = \Pr((E_B, E_B') = (Y_1, Y_1'))_\tau \geq 2^{-2d_1}.$$  

(27)

Let $\Phi'$ be the state $\Phi$ with two additional copies of $Y_1 Y'_1$. Note that the state at the end of Protocol 3 conditioned on Bob not aborting is $\Phi'$. Consider,

$$2^{-2d_1} \Delta_B(\Phi_{TB'}, U_d \otimes \Phi'_B) \leq 2^{-2d_1} \Delta_B(\Phi'_{TB'}, U_d \otimes \Phi'_B)$$  

(Fact 7)

$$\leq \Pr(C = 1)_\tau \Delta_B(\Phi'_{TB'}, U_d \otimes \Phi'_B)$$  

(Eq. (27))

$$\leq \Delta_B(\tau'_{TB'}, U_d \otimes \tau'_B)$$  

(Fact 7)

$$\leq \Delta_B(\tau'_{TB'}, U_d \otimes \tau'_B)$$  

(Fact 7)

$$\leq \sqrt{\varepsilon}$$  

(Eq. (26))

which shows the desired.

\[ \Box \]

Claim 8 (Correlation breaker with advice). Let Alice and Bob proceed as in Protocol 2 with the starting state as $\hat{\Phi}^{(\alpha, \alpha')}$, where $(\alpha, \alpha') \notin \mathcal{S}$. Let $\hat{\Gamma}^{(\alpha, \alpha')}$ be the joint state at the end of the Protocol 2 (at $i = a$). Then,

$$d(Z|\tilde{B})_{\hat{\Gamma}^{(\alpha, \alpha')}} = \mathcal{O}(\varepsilon).$$

Proof. We have

$$\hat{\Phi}^{(\alpha, \alpha')}_{Z_a \hat{B}} = U_h \otimes \hat{\Phi}^{(\alpha, \alpha')}_{\hat{B}} ; \quad H_{\min} \left( Y | \tilde{A} \right)_{\hat{\Phi}^{(\alpha, \alpha')}} \geq (3/4 - 2\delta)d ; \quad H_{\min} \left( T \middle| \tilde{B} \right)_{\hat{\Phi}^{(\alpha, \alpha')}} = d.$$

The total communication from Alice to Bob in Protocol 2 is at most $6ah \leq \delta d$. From Lemma 2 throughout Protocol 2 we have $H_{\min} \left( T \middle| \tilde{B} \right) \geq (1 - \delta)d > 2h$. From repeated applications of Claim 7 we have,

$$d(Z|\tilde{B})_{\hat{\Gamma}^{(\alpha, \alpha')}} \leq 2^{\mathcal{O}(\alpha)} \sqrt{\varepsilon} = \mathcal{O}(\varepsilon).$$

\[ \Box \]

Claim 9 (Flip flop). Let $\mathcal{P}$ be any of the Protocols 4, 5, 6, 7, 8 or 9 and $i \in [a]$. Let $\alpha$ be the initial joint state in $\mathcal{P}$ such that $d(Z|\tilde{B})_\alpha \leq \eta$. Let $\theta$ be the final joint state at the end of $\mathcal{P}$. Then,

$$d(O|\tilde{B})_\theta \leq \mathcal{O}(\eta + \sqrt{\varepsilon}).$$

Proof. We prove the claim when $\mathcal{P}$ is Protocol 4 and $i = 1$ and the proof for other cases follows analogously. From Fact 7

$$d(Z_a|\tilde{B})_\alpha \leq d(Z|\tilde{B})_\alpha \leq \eta.$$  

28
Let $\gamma$ be the joint state just after Bob generates register $A$. From Lemma 11 we have
\[
d(\rho_{\tilde{A}\tilde{A}A}|A) \leq 2\epsilon + \sqrt{\epsilon'}.
\]
Let $\zeta$ be the joint state after Alice sends register $Z'$ to Bob and Bob generates registers $(A'C'B')$. From Fact 7 we have
\[
d(\rho_{\tilde{A}\tilde{A}A}|A) \leq d(\rho_{\tilde{A}\tilde{A}A}|A) \leq 2\epsilon + \sqrt{\epsilon'}.
\]
Let $\beta$ be the joint state just after Alice generates register $Z$. From Lemma 1
\[
d(\rho_{\tilde{Z}\tilde{Z}A}|A) \leq 4\epsilon + 3\sqrt{\epsilon'}.
\]
Let $\hat{\beta}$ be the state obtained from Claim 6 (by letting $\rho$ in Claim 6 as $\beta$ here) such that
\[
H_{\min}(Y|\tilde{A})_{\beta} = H_{\min}(Y|\tilde{A})_{\hat{\beta}} \geq d/4 ; \quad \hat{\beta}_{Z\tilde{B}} = U_{h} \otimes \hat{\beta}_{\tilde{B}} ; \quad \Delta_{B}(\hat{\beta},\beta) \leq 4\epsilon + 3\sqrt{\epsilon'}.
\]
Let $\theta', \theta''$ be the joint states just after Alice generates register $\overline{C}$, proceeding from the states $\beta, \hat{\beta}$ respectively. Since communication between Alice and Bob after Alice generates register $\overline{Z}$ and before generating $\overline{C}$ is $2s + 2b$, from arguments as before involving Lemma 1 and Lemma 2
\[
d(\rho_{\overline{Z}|\tilde{B}}) \leq O(\eta + \sqrt{\epsilon'}).
\]
From Eq. (28),
\[
d(\rho_{\overline{Z}|\tilde{B}}) \leq d(\rho_{\overline{Z}|\tilde{B}}) + \Delta_{B}(\hat{\beta},\beta) = O(\eta + \sqrt{\epsilon'}).
\]
Proceeding till the last round and using similar arguments involving Lemma 11 Lemma 2 and Claim 6 we get the desired.

We have the following corollary of Theorem 6.

**Corollary 4.** Let $\rho_{XY}$ be a $c$-$q$ state with registers $(XY)$ classical such that
\[
H_{\min}(X|E)_{\rho} \geq k ; \quad \rho_{XY} = \rho_{XE} \otimes U_{d} ; \quad |X| = n.
\]
Let $T : \mathcal{L}(H_{E} \otimes H_{Y}) \rightarrow \mathcal{L}(H_{E} \otimes H_{Y} \otimes H_{Y'})$ be a (safe) CPTP map such that for $\sigma_{XY} = T(\rho_{XY})$, we have registers $XY'$ classical and $Pr(Y \neq Y')_{\sigma} = 1$. Let the function $nmExt$ be as defined in Algorithm 7. Let $L = nmExt(X,Y)$ and $L' = nmExt(X,Y')$. Then,
\[
\|\sigma_{L|YY'E'} - U_{d/4} \otimes \sigma_{L|YY'E'}\|_{1} \leq O(\epsilon).
\]

**Proof.** Let $\rho_{X\tilde{X}\tilde{E}YY'}$ be a pure state extension of $\rho_{XY}$ such that,
\[
\rho_{X\tilde{X}\tilde{E}YY'} = \rho_{X\tilde{X}\tilde{E}} \otimes \rho_{YY'} ; \quad H_{\min}(X|EY)_{\rho} \geq k ; \quad \rho_{Y} = U_{d},
\]
where registers $(XY)$ classical (with copies $\tilde{X}\tilde{Y}$) and $\rho_{X\tilde{X}\tilde{E}}$ is the canonical purification of $\rho_{XE}$.

For the state $\rho$ with the following assignment (terms on the left are from Definition 16 and on the right are from here),
\[
(X, \tilde{X}, N, M, Y, \tilde{Y}) \leftarrow (X, \tilde{X}, \tilde{E}, E, Y, \tilde{Y}),
\]

29
$$H_{\text{min}} \left( X | E Y \hat{Y} \right) = H_{\text{min}}(X|E) \geq k$$ and $$\rho_{\hat{E}X\hat{X}Y} = \rho_{\hat{E}X\hat{X}} \otimes U_d$$, we have $$\rho$$ is a $$(k)$$-qpa-state. Let $$V : \mathcal{H}_E \otimes \mathcal{H}_Y \to \mathcal{H}_{E'} \otimes \mathcal{H}_Z \otimes \mathcal{H}_Y \otimes \mathcal{H}_{Y'} \otimes \mathcal{H}_{Y'}$$, be the Stinespring isometry extension of CPTP map $$T$$ with additional copy $$\hat{Y}$$ of $$Y$$, i.e. $$T(\theta) = Tr_{Z\hat{Y}}(V \theta V^\dagger)$$ for every c-q state $$\theta_{YE}$$. Let $$\sigma = V \rho V^\dagger$$. Using Theorem 6, we have

$$\|\sigma_{LL'YY'E'Z} - U_{k/4} \otimes \sigma_{L'Y'E'Z}\|_1 \leq O(\varepsilon).$$

Using Fact 7, we further have

$$\|\sigma_{LL'YY'E'} - U_{k/4} \otimes \sigma_{L'Y'E'}\|_1 \leq O(\varepsilon),$$

which completes the proof.

5 A quantum secure 2-source non-malleable extractor

The following parameters hold throughout this section.

Parameters

Let $$q$$ be a prime power and $$\delta, \delta_1 > 0$$ be small enough constants. Let $$n, a, v, s, b, h$$ be positive integers and $$k, \varepsilon, \gamma, \varepsilon > 0$$ such that:

$$v = \frac{n}{\varepsilon}; \quad q = O\left(\frac{1}{\varepsilon^2}\right); \quad \varepsilon = 2^{-O(n^{\delta_1})};$$

$$a = 6k + 2 \log q = O(k); \quad \gamma = O(\varepsilon); \quad 2^{O(a)} \sqrt{\varepsilon^2} = \varepsilon;$$

$$s = O\left(\log^2\left(\frac{n}{\varepsilon^2}\right) \log n\right); \quad b = O\left(\log^2\left(\frac{n}{\varepsilon^2}\right) \log n\right); \quad h = 10s; \quad k = O(n^{1/4})$$

- $$\text{IP}_1$$ be $$\text{IP}^{3k/\log v}_v$$,
- $$\text{Ext}_1$$ be $$(2b, \varepsilon')$$-quantum secure $$(n, s, b)$$-extractor,
- $$\text{Ext}_2$$ be $$(2s, \varepsilon')$$-quantum secure $$(h, b, s)$$-extractor,
- $$\text{Ext}_3$$ be $$(2h, \varepsilon')$$-quantum secure $$(n, b, h)$$-extractor,
- $$\text{Ext}_4$$ be $$(n/4, \varepsilon^2)$$-quantum secure $$(n, h, n/8)$$-extractor,
- $$\text{IP}_2$$ be $$\text{IP}^{3k/2}_h$$,
- $$\text{Ext}_6$$ be $$(\frac{n}{2}, \varepsilon^2)$$-quantum secure $$(n, n/8, n/4)$$-extractor.

Note the Stinespring isometry extension is safe on register $$Y$$.  

30
Definition of 2-source non-malleable extractor

Let $\text{ECC} : \mathbb{F}_q^d \to \mathbb{F}_q^v$ be an error correcting code with relative distance $1 - \gamma$ and rate $\varepsilon$ (which exists from Fact 19 for our choice of parameters). We identify $R$ as an element from $\{1, \ldots, v\}$. By $\text{ECC}(Y)_R$, we mean the $R$-th entry of the code-word $\text{ECC}(Y)$, interpreted as a bit string of length $\log q$.

Algorithm 4: $2\text{nmExt} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^{n/4}$

**Input:** $X, Y$

1. Advice generator:

   $X_1 = \text{Prefix}(X, 3k) ; \ Y_1 = \text{Prefix}(Y, 3k) ; \ R = \text{IP}_1(X_1, Y_1) ;$
   
   $G = X_1 \circ Y_1 \circ \text{ECC}(X)_R \circ \text{ECC}(Y)_R$

2. $X_2 = \text{Prefix}(X, 3k^3) ; \ Y_2 = \text{Prefix}(Y, 3k^3) ; \ Z_0 = \text{IP}_2(X_2, Y_2)$

3. Correlation breaker with advice: $S = 2\text{AdvCB}(Y, X, Z_0, G)$

4. $L = \text{Ext}_6(X, S)$

**Output:** $L$

Algorithm 5: $2\text{AdvCB} : \{0,1\}^n \times \{0,1\}^n \times \{0,1\}^h \times \{0,1\}^a \to \{0,1\}^h$

**Input:** $Y, X, Z_0, G$

1. For $i = 1, 2, \ldots, a$:
   
   Flip flop: $Z_i = 2\text{FF}(Y, X, Z_{i-1}, G_i)$

2. $S = \text{Ext}_4(Y, Z_a)$

**Output:** $S$

Algorithm 6: $2\text{FF} : \{0,1\}^n \times \{0,1\}^n \times \{0,1\}^h \times \{0,1\}^a \to \{0,1\}^h$

**Input:** $Y, X, Z, G$

1. $Z_s = \text{Prefix}(Z, s) , \ A = \text{Ext}_1(Y, Z_s) , \ C = \text{Ext}_2(Z, A) , \ B = \text{Ext}_1(Y, C)$

2. If $G = 0$ then $\overline{Z} = \text{Ext}_3(X, A)$ and if $G = 1$ then $\overline{Z} = \text{Ext}_3(X, B)$

3. $\overline{Z}_s = \text{Prefix}(\overline{Z}, s) , \ \overline{A} = \text{Ext}_1(Y, \overline{Z}_s) , \ \overline{C} = \text{Ext}_2(\overline{Z}, \overline{A}) , \ \overline{B} = \text{Ext}_1(Y, \overline{C})$

4. If $G = 0$, then $O = \text{Ext}_3(X, \overline{B})$ and if $G = 1$, then $O = \text{Ext}_3(X, \overline{A})$

**Output:** $O$

Result

The following theorem shows that the function $2\text{nmExt}$ as defined in Algorithm 4 is $(n - k, n - k, O(\varepsilon))$-secure against $q_{\text{mna}}$ by noting that $L = \text{nmExt}(X, Y)$ and $L' = \text{nmExt}(X', Y')$. 

31
**Theorem 7** (Security of 2nmExt). Let \( \rho_{XX'YY'\tilde{Y}'M} \) be a \((n-k,n-k)\)-qnm-state with \(|X| = |Y| = n\). Let Protocol \( \Pi \) start with \( \rho \). Let \( \Lambda \) be the state at the end of the protocol. Then,

\[
\| \rho_{LLYY'M} - U_{n/4} \otimes \rho_{LYY'M} \|_1 \leq d(L|\tilde{B})_\Lambda \leq O(\varepsilon).
\]

**Proof.** The first inequality follows from Fact \(7\). Most of our arguments here are similar to the case of seeded extractor; so we note the modifications that we need to take care of in case of 2nmExt.

First note that using Lemma \(2\) throughout Protocol \( \Pi \) \(X\) and \(Y\) have enough conditional min-entropy left for necessary extractions since the total communication from Alice to Bob is at most (from our choice of parameters)

\[
6k + 2 \log q + 6ah + h + \frac{n}{4} \leq (1/4 + \delta) n
\]

and the total communication from Bob to Alice is at most

\[
6k + 6k^3 + 2a + 6ab + \frac{n}{4} \leq (1/4 + \delta)n.
\]

Thus, at any state \( \rho \) in Protocol \( \Pi \) \(H_{\min}(X|\tilde{B})_\rho \geq n - k - (1/4 + \delta) n \geq (3/4 - 2\delta) n \geq n/2\).

Similarly, \(H_{\min}(Y|\tilde{A})_\rho \geq (3/4 - 2\delta) n \geq n/2\).

We start with a state \( \rho_{XX'YY'\tilde{Y}'M} \) such that \(H_{\min}(X|\tilde{B})_\rho \geq n - k \) and \(H_{\min}(Y|\tilde{A})_\rho \geq n - k\). From Claim \(2\) we have,

\[
H_{\min}(X_1|\tilde{B})_\rho \geq 3k - k = 2k \quad ; \quad H_{\min}(Y_1|\tilde{A})_\rho \geq 3k - k = 2k.
\]

Now from Claim \(5\) with the below assignment of registers (and noting registers \((XX',YY')\) are included in \((A, B)\) respectively),

\[
(Z, X, Y, \sigma) \leftarrow (R, X_1, Y_1, \rho) \quad ; \quad (k_1, k_2, m, n_1, \varepsilon) \leftarrow (2k, 2k, \log(n/\varepsilon), 3k, \varepsilon^2)
\]

we have,

\[
\Delta(\rho_{RYY'}, U_R \otimes \rho_{YY'}) \leq O(\varepsilon^2) \quad ; \quad \Delta(\rho_{RXX'}, U_R \otimes \rho_{XX'}) \leq O(\varepsilon^2).
\]

Using Fact \(6\) we get

\[
\Delta_B(\rho_{RYY'}, U_R \otimes \rho_{YY'}) \leq O(\varepsilon) \quad ; \quad \Delta_B(\rho_{RXX'}, U_R \otimes \rho_{XX'}) \leq O(\varepsilon).
\]

Let \(\kappa\) be the state just before Bob sends \(Y_2\). Note that till then, communication from Alice to Bob and Bob to Alice is at most \(7k\) each. Hence, by Lemma \(2\) \(H_{\min}(X|\tilde{A})_\kappa \geq n - 8k\) and \(H_{\min}(Y|\tilde{B})_\kappa \geq n - 8k\); which implies (from Claim \(2\)), \(H_{\min}(X_2|\tilde{A})_\kappa \geq 3k^3 - 8k \geq 2k^3\) and \(H_{\min}(Y_2|\tilde{B})_\kappa \geq 2k^3\) respectively using Fact \(2\). Let \(\eta\) be the state just after Alice generates \(Z_0\).

Using similar argument as before involving Claim \(5\) we have \(d(Z_0|\tilde{B})_\eta \leq O(\varepsilon)\).

Note that the state obtained in Protocol \( \Pi \) just before Protocol \( \Pi \) starts as a subroutine, is similar to the state obtained in Protocol \( \Pi \) (just before Protocol \( \Pi \) starts as a subroutine) with the below assignment of registers,

\[
(Z_0, T, Y) \leftarrow (Z_0, X, Y).
\]

32
Here the variables on the left are from Protocol 1 and variables on the right are from Protocol 10. The proof then proceeds using similar arguments as Theorem 6 involving Lemma 1, Lemma 2, Claim 6 after noting Claim 10.

We can verify the following claim regarding the state $\Phi$ (the state obtained in Protocol 10 just before Protocol 2 starts as a subroutine) using similar arguments as proof of Claim 7.

**Claim 10.** 1. $\Pr(G = G')_\Phi = O(\varepsilon)$ and 2. $d(Z_0|\tilde{B})_\Phi \leq O(\varepsilon)$.

Since we have either $\Pr(Y \neq Y') = 1$ or $\Pr(X \neq X') = 1$, we show it for the first case and second case will follow analogously. Now note that the event $G = G'$ is a sub-event of $G_1 = G'_1$ where $G_1 = Y_1 \circ \text{ECC}(Y)_R$ and $G'_1 = Y'_1 \circ \text{ECC}(Y')_R$. Thus we get, $\Pr(G = G')_\Phi \leq \Pr(G_1 = G'_1)_\Phi$. Rest of the argument follows similar lines to Claim 7.

This completes the proof.

We have the following corollary of Theorem 7.

**Corollary 5.** Let the function $2nm\text{Ext}$ be as defined in Algorithm 4. $2nm\text{Ext}$ is an $(n-k,n-k,O(\varepsilon))-\text{quantum secure 2-source non-malleable extractor}$ against qia.

**Proof.** Let $\rho_{X_1E_1E_2Y}$ be a state (for $k_1 = k_2 = n-k$), $T_1$ and $T_2$ be CPTP maps as defined in Definition 14. Let $\rho_{X_1E_1E_2E_2Y\tilde{Y}}$ be a pure state extension of $\rho_{X_1E_1E_2Y}$ such that,

$$\rho_{X_1E_1E_2E_2Y\tilde{Y}} = \rho_{X_1\tilde{X}E_1} \otimes \rho_{Y\tilde{Y}E_2E_2} \quad ; \quad \text{H}_{\text{min}}(X|E_1)_\rho \geq n-k \quad ; \quad \text{H}_{\text{min}}(Y|E_2)_\rho \geq n-k,$$

where registers $(XY)$ are classical (with copies $\tilde{X}\tilde{Y}$), $\rho_{X_1\tilde{X}E_1}$ is canonical purification of $\rho_{X_1E_1}$ and $\rho_{Y\tilde{Y}E_2E_2}$ is canonical purification of $\rho_{Y_2E_2}$.

Let $U : \mathcal{H}_E \otimes \mathcal{H}_X \rightarrow \mathcal{H}_{E_2} \otimes \mathcal{H}_{Z_1} \otimes \mathcal{H}_X \otimes \mathcal{H}_X \otimes \mathcal{H}_{\tilde{X}}$, be the Stinespring isometry extension of CPTP map $T_1$ with additional copy $\tilde{X}$ of $X$, i.e. $T_1(\theta) = \text{Tr}_{Z_2X}(U\theta U^\dagger)$ for every c-q state $\theta_{X_2E_2}$. Similarly let $V : \mathcal{H}_{E_1} \otimes \mathcal{H}_Y \rightarrow \mathcal{H}_{E_1} \otimes \mathcal{H}_{Z_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_Y \otimes \mathcal{H}_{\tilde{Y}}$, be the Stinespring isometry extension of CPTP map $T_2$ with additional copy $\tilde{Y}$ of $Y$. Since $\rho_{X_1E_1E_1\tilde{Y}E_2} = \rho_{X_1E_1} \otimes \rho_{Y\tilde{Y}E_2E_2}$, we have

$$\text{H}_{\text{min}}(X|Y\tilde{Y}\tilde{E}_2E_1)_\rho = \text{H}_{\text{min}}(X|E_1)_\rho \geq n-k.$$

Similarly since $\rho_{Y_2E_2X\tilde{X}E_1} = \rho_{Y_2E_2} \otimes \rho_{X\tilde{X}E_1}$, we have

$$\text{H}_{\text{min}}(Y|X\tilde{X}\tilde{E}_1E_2)_\rho = \text{H}_{\text{min}}(Y|E_2)_\rho \geq n-k.$$

Thus $\rho$ is a $(n-k,n-k)$-qpa-state, with the following assignment (terms on the left are from Definition 14 and on the right are from here),

$$(X,\tilde{X},N,M,Y,\tilde{Y}) \leftarrow (X,\tilde{X},\tilde{E}_1E_2,\tilde{E}_2E_1,Y,\tilde{Y}).$$

Let $\sigma = (U \otimes V)\rho(U \otimes V)^\dagger$. Note $\sigma_{X_1'\tilde{E}_1'E_2'E_2'Y'} = (T_1 \otimes T_2)(\rho_{X_1E_1E_2Y})$ and $\sigma$ is a $(n-k,n-k)$-qnm-state. Using Theorem 7 we have

$$\|\sigma_{2nm\text{Ext}(X,Y)2nm\text{Ext}(X',Y')Y'Y'E_1'E_2'Z_1} - U_{n/4} \otimes \sigma_{2nm\text{Ext}(X,Y)Y'Y'E_1'E_2'Z_1}\|_1 \leq O(\varepsilon).$$

*Note the Stinespring isometry extension is safe on register $X$. 33*
Using Fact 7, we further have

\[ \| \sigma_{2nmExt(X,Y)2nmExt(X',Y')YY'} - U_{n/4} \otimes \sigma_{2nmExt(X',Y')YY'} \|_1 \leq O(\varepsilon). \]

which completes the proof. \( \square \)

We have the following additional corollary of Theorem 7.

**Corollary 6.** Let the function 2nmExt be as defined in Algorithm 4. 2nmExt is an (n - k, n - k, O(\varepsilon))-quantum secure 2-source non-malleable extractor against qMara.

**Proof.** Let \( \rho_{XEY} \) be a state (for \( k_1 = k_2 = n - k \)), \( T_1 \) and \( T_2 \) be CPTP maps as defined in Definition 2. Let \( \rho_{X\hat{X}T\hat{E}_1\hat{E}_1\hat{E}_2\hat{E}_2YY} \) be a pure state extension of \( \rho_{XEY} \equiv \rho_{XE_1TE_2Y} \) such that,

\[ \rho_{X\hat{X}T\hat{E}_1\hat{E}_1\hat{E}_2\hat{E}_2YY} = \sum_t \sqrt{\text{Pr}(T = t)} |tt\rangle_{TT} |\rho_t\rangle_{X\hat{X}E_1\hat{E}_1\hat{E}_2\hat{E}_2YY}; \]

\[ H_{\min}(X|E)_{\rho} \geq n - k ; \quad H_{\min}(Y|E)_{\rho} \geq n - k, \]

registers (XYT) are classical (with copies \( \hat{X}\hat{Y}\hat{T} \)), \( |\rho_t\rangle_{X\hat{X}E_1\hat{E}_1\hat{E}_2\hat{E}_2YY} = |\rho_t\rangle_{X\hat{X}E_1\hat{E}_1} \otimes |\rho_t\rangle_{\hat{E}_2\hat{E}_2YY} \) is the pure state extension of \( \rho_t^{X\hat{X}E_1\hat{E}_1} \otimes \rho_t^{\hat{E}_2\hat{E}_2YY} \) with \( |\rho_t^{X\hat{X}E_1\hat{E}_1} \rangle \) canonical purifications of \( \rho_t^{X\hat{X}E_1} \) respectively. Since \( E \equiv E_1TE_2 \), using Fact 8 we have

\[ H_{\min}(X|E_1T)_{\rho} \geq H_{\min}(X|E)_{\rho} \geq n - k. \]  

(29)

Similarly, \( H_{\min}(Y|E_2T)_{\rho} \geq n - k \). Note,

\[ \rho_{XY\hat{Y}\hat{E}_2E_1\hat{E}_1} \equiv \rho_{XY\hat{Y}E_2E_1\hat{T}} \]  

(30)

The first equivalence follows since for every \( T = \hat{T} = t \), \( |\rho_t^{E_2\hat{E}_2YY} \rangle \) is the canonical purification of \( \rho_{Y\hat{E}_2} \) implying \( \rho_{Y\hat{E}_2} = \rho_{Y\hat{E}_1} \). Consider,

\[ H_{\min}(X|Y\hat{Y}\hat{E}_2E_1\hat{T})_{\rho} = H_{\min}(X|Y\hat{Y}TE_1E_2)_{\rho} \]

\[ = H_{\min}(X|TE_1)_{\rho} \geq n - k. \]

First equality follows from Eq. (30), second equality follows from Fact 8 noting for every \( T = t, \rho_t^{XY\hat{Y}E_1E_2} = \rho_t^{X\hat{X}E_1} \otimes \rho_t^{\hat{Y}E_2YY} \) and first inequality follows from Eq. (29). Similarly, \( H_{\min}(Y|X\hat{X}\hat{E}_1E_2T)_{\rho} \geq n - k \). Thus, \( \rho \) is an (n - k, n - k)-qpa-state, with the following assignment (terms on the left are from Definition 16 and on the right are from here),

\[ (X, \hat{X}, N, M, Y, \hat{Y}) \leftarrow (X, \hat{X}, \hat{E}_1E_2T, \hat{E}_2E_1\hat{T}, Y, \hat{Y}). \]

Let \( U : \mathcal{H}_{E_2} \otimes \mathcal{H}_X \otimes \mathcal{H}_T \rightarrow \mathcal{H}_{E_2} \otimes \mathcal{H}_{Z_2} \otimes \mathcal{H}_X \otimes \mathcal{H}_{X'} \otimes \mathcal{H}_{\hat{X}} \otimes \mathcal{H}_T \) be the Stinespring isometry extension of CPTP map \( T_1 \) with additional copy \( \hat{X}' \) of \( X' \), i.e. \( T_1(\theta) = \text{Tr}_{Z_2\hat{X}'}(U\theta U^\dagger) \) for every state \( \theta \). Similarly let \( V : \mathcal{H}_{E_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_T \rightarrow \mathcal{H}_{E_1} \otimes \mathcal{H}_{Z_1} \otimes \mathcal{H}_Y \otimes \mathcal{H}_{Y'} \otimes \mathcal{H}_T \) be the Stinespring

\[ \text{Note the Stinespring isometry extension is safe on classical registers.} \]
isometry extension of CPTP map $T_2$ with additional copy $Y'$ of $Y$ (and treating register $T$ as $\hat{T}$ since $\hat{T} \equiv T$). Let $\sigma = (U \otimes V)\rho(U \otimes V)$. Note $\sigma_{XX'E_1'Y'1E_2'E_2'Y'2} = (T_1 \otimes T_2)(\rho_{X'Y'})$. Thus, $\sigma$ is an $(n-k, n-k)$-qnm-state. Thus, using Theorem 7, we have
\[
\|\sigma_{\text{2nmExt}(X,Y)\text{2nmExt}(X',Y')YY'E_1'E_2Z_1\hat{T}} - U_{n/4} \otimes \sigma_{\text{2nmExt}(X',Y')YY'E_1'E_2Z_1\hat{T}}\|_1 \leq O(\varepsilon).
\]
Using Fact 7 we further have
\[
\|\sigma_{\text{2nmExt}(X,Y)\text{2nmExt}(X',Y')YY'E_1'E_2\hat{T}} - U_{n/4} \otimes \sigma_{\text{2nmExt}(X',Y')YY'E_1'E_2\hat{T}}\|_1 \leq O(\varepsilon).
\]
The desired follows noting $T \equiv \hat{T}$ in $\sigma$ which completes the proof.

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A  A quantum secure $t$-non-malleable extractor

**Definition 20** (($t; k$)-$qnm$-state). Let $\sigma_{X^N M N Y^T}$ be a ($k$)-$qpa$-state. Let $V : H_Y \otimes H_M \rightarrow H_Y \otimes H_{Y[t]} \otimes H_{M'}$ be an isometry such that for $\rho = V \sigma V^\dagger$, we have $Y[Y[t]]$ classical (with copy $Y[i]$) and $\forall i \in [t] : \Pr(Y \neq Y') = 1$. We call $\rho$ a ($t; k$)-$qnm$-state.

**Definition 21** (quantum secure $t$-non-malleable extractor). An $(n, d, m)$-non-malleable extractor $t$-$nmExt : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m$ is ($t; k, \varepsilon$)-secure against $qnma$ if for every ($t; k$)-$qnm$-state $\rho$ (chosen by the adversary $qnma$),

$$\|\rho_{t-nmExt(X, Y)\cdot t-nmExt(X, Y^T)\cdot t-nmExt(X, Y^T)\cdot Y[Y[t]] \cdot M'} - U_m \otimes \rho_{t-nmExt(X, Y^T)\cdot t-nmExt(X, Y^T)\cdot Y[Y[t]] \cdot M'}\|_1 \leq \varepsilon.$$

We define the parameters we use in the construction of quantum secure $t$-non-malleable extractor as follows. These parameters hold throughout this section.

**Parameters**

Let $\delta, \delta_3 > 0$ be small enough constants and $\delta_1, \delta_2 < \frac{1}{11}$ be constants chosen according to Fact [13]. Let $n, n_1, d, d_1, d_2, a, v, s, b, h, t$ be positive integers and $k, \varepsilon, \varepsilon', \varepsilon'' > 0$ such that:

$$d = O\left(\log^7\left(\frac{n}{\varepsilon}\right)\right) ; \quad v = 5d ; \quad n_1 \geq v^{\delta_1} ; \quad d_1 = O\left(\log^2\left(\frac{n t^2}{\varepsilon^2}\right) \log d\right) ; \quad a = d_1 + O(v^{\delta_2}) ;
$$

$$t = \min\{O(d^{\delta_3}), 2^{O(d^{\delta_4})} \cdot \log\left(\frac{1}{\varepsilon}\right)\} ; \quad 2^{O(a)} \sqrt{\varepsilon'} = \varepsilon ; \quad d_2 = O\left(\log^2\left(\frac{n^{\delta_4}}{\varepsilon''}\right) \log d\right) ; \quad \varepsilon'' = 2^{2(t+1)d_1 \varepsilon^2} ;
$$

$$q = O(1) ; \quad s = O\left(\log^2\left(\frac{d}{\varepsilon'}\right) \log d\right) ; \quad b = O\left(\log^2\left(\frac{d}{\varepsilon'}\right) \log d\right) ; \quad h = 10ts ; \quad k \geq 5d.$$

Let

- $Ext_0$ be $(2n_1, \varepsilon^2/t^2)$-quantum secure $(n, d_1, n_1)$-extractor,
- $Ext_1$ be $(2b, \varepsilon')$-quantum secure $(d, s, b)$-extractor,
- $Ext_2$ be $(2s, \varepsilon')$-quantum secure $(h, b, s)$-extractor,
- $Ext_3$ be $(2h, \varepsilon')$-quantum secure $(d, b, h)$-extractor,
- $Ext_4$ be $(d/4t, \varepsilon^2)$-quantum secure $(d, h, d/8t)$-extractor,
- $Ext_5$ be $(2d, \varepsilon'')$-quantum secure $(n, d_2, d)$-extractor,
- $Ext_6$ be $(k/4t, \varepsilon^2)$-quantum secure $(n, d/8t, k/8t)$-extractor,

be the quantum secure extractors from Fact [17]
Algorithm 7: \textit{t-nmExt} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^{k/8t}

\textbf{Input:} \(X,Y\)
1. \(t\)-advice generator:
   \[Y_1 = \text{Prefix}(Y, d_1) \quad \text{I} = \text{Ext}_0(X, Y_1) \quad G = Y_1 \circ \text{ECC}(Y)_{\text{Samp}(I)}\]
2. \(Y_2 = \text{Prefix}(Y, d_2) \quad T = \text{Ext}_5(X, Y_2)\)
3. Correlation breaker with advice: \(S = \text{AdvCB}(Y, T, G)\) \(\triangleright \text{Algorithm} 2\)
4. \(L = \text{Ext}_0(X, S)\)
\textbf{Output:} \(L\)

\textbf{Definition of \(t\)-non-malleable extractor}

Let \(\mathbb{F}_q\) be the finite field of size \(q\). Let \(\text{ECC} : \mathbb{F}_q^d \rightarrow \mathbb{F}_q^r\) be an error correcting code with relative distance \(\frac{1}{10}\) and rate \(\frac{1}{5}\) (which exists from Fact 19 for our choice of parameters) for this section. Let \(\text{Samp} : \{0,1\}^r \rightarrow [v]^{t_1}\) be the sampler function from Fact 13 where \(t_1 = O(v^\delta_2)\) and \(r \geq v^\delta_1\). We identify the output of \(\text{Samp}\) as \(t_1\) samples from the set \([v]\). By \(\text{ECC}(Y)_{\text{Samp}(I)}\), we mean the \(\text{Samp}(I)\) entries of codeword \(\text{ECC}(Y)\), interpreted as a bit string.

\textbf{Result}

The following theorem shows that the function \(t\text{-nmExt}\) as defined in Algorithm 7 is \((t;k, O(\varepsilon))\)-secure against \(\text{qnma}\) by noting that \(L = t\text{-nmExt}(X, Y)\) and \(L^i = t\text{-nmExt}(X, Y^i)\) for every \(i \in [t]\).

**Theorem 8** (Security of \(t\text{-nmExt}\)). Let \(\rho_{X,XY^n|Y,Y^n|MM} \) be a \((t;k)\)-\(\text{qnma}\) state. Let Alice and Bob proceed with Protocol 11 starting with \(\rho\). Let \(\Lambda\) be the state at the end of the protocol. Then,

\[\|\|\rho_{LL|YY^n|M} - U_k/8t \otimes \rho_{L|YY^n|M}\| \leq d(L|B)\Lambda \leq O(\varepsilon).\]

\textit{Proof.} The first inequality follows from Fact 7. Note that the total communication from Alice to Bob in Protocol 11 is at most (from our choice of parameters)

\[(t+1)n_1 + 6ah(t+1) + h + (t+1)\frac{k}{8t} \leq (1/4 + \delta)k.\]

This implies (using Lemma 2) that throughout Protocol 11 \(H_{\min}(X|B) \geq (3/4 - \delta)k > \frac{k}{2}\).

The total communication from Bob to Alice in Protocol 11 is at most

\[(t+1)d_1 + (t+1)d_2 + (t+1)a + (t+1)6ab + (t+1)\frac{d}{8t} \leq (1/4 + \delta)d.\]

Again using Lemma 2 throughout Protocol 11 \(H_{\min}(Y|A) \geq (3/4 - \delta)d\).

The proof then proceeds using similar arguments as Theorem 6 involving Lemma 1, Lemma 2, Claim 6 after noting Claim 11. \(\square\)

**Claim 11** (\(t\)-advice generator). Let \(\Phi\) be the joint state after registers \(Z_0, Z_0^t\) are generated by Alice. Then,

\[\Pr(\forall i \in [t] : (G \neq G^i))_\Phi \geq 1 - O(\varepsilon) \quad \text{and} \quad d(T|B)_\Phi \leq \varepsilon.\]
Proof. We first prove $\Pr(\forall i \in [t]: (G \neq G_i))_\Phi \geq 1 - O(\varepsilon)$. Let $\sigma_{XNIMYY'[t]}$ be the state after Alice has generated register $I$. Let $\beta_{IY'[t]} = U_{n_1} \otimes \Phi_{YY'[t]}$. We have,

$$\Delta(\Phi_{IY'[t]}, \beta_{IY'[t]}) = \Delta(\sigma_{IY'[t]}, U_{n_1} \otimes \sigma_{YY'[t]}) \leq \sqrt{2}\Delta_B(\sigma_{IY'[t]}, U_{n_1} \otimes \sigma_{YY'[t]}) \leq \sqrt{2}\varepsilon/t.$$  (Fact 6)

Fix an integer $i \in [t]$ and consider,

$$\Pr(G = G_i)_\Phi \leq \Pr(G = G_i)_\beta + \sqrt{2}\varepsilon/t \leq 2^{-\Omega(n_1)} + \sqrt{2}\varepsilon/t \leq O\left(\frac{\varepsilon}{t}\right).$$

Note that conditioned on $Y_1 = Y_i$, we have $I = I_i$. The first inequality above follows from Eq. (31) and noting that the predicate $(G = G_i)$ is determined from $(I, Y, Y_i)$. The second equality follows from definition of $G$. Let $S_i \overset{\text{def}}{=} \{j \in [v]: \text{ECC}(Y)_j \neq \text{ECC}(Y_i)_j\}$. Second inequality follows since ECC has relative distance $\frac{1}{10}$ and considering Samp with $(r, \delta, \nu, S)$ in Fact 13 as $(\frac{1}{10}, \frac{1}{10}, v, S_i)$ here. Third inequality follows by our choice of parameters. Now the desired follows from the union bound.

Using arguments similar to the proof of Point 2 of Claim 7, we get,

$$d(T|\tilde{B})_\Phi \leq 2^{(t+1)d_1}\sqrt{\varepsilon''} \leq \varepsilon.$$
B  A quantum secure 2-source $t$-non-malleable extractor

**Definition 22** $(t; k_1, k_2)$-qm-state. Let $\sigma_{XNMY}$ be a $(k_1, k_2)$-qa-state. Let $U : \mathcal{H}_X \otimes \mathcal{H}_N \rightarrow \mathcal{H}_X \otimes \mathcal{H}_{X[t]} \otimes \mathcal{H}_{X[t]} \otimes \mathcal{H}_N$ and $V : \mathcal{H}_Y \otimes \mathcal{H}_M \rightarrow \mathcal{H}_Y \otimes \mathcal{H}_{Y[t]} \otimes \mathcal{H}_{Y[t]} \otimes \mathcal{H}_M$ be isometries such that

$$\forall i \in [t], \Pr(Y \neq Y^i) = 1 \quad \text{or} \quad \Pr(X \neq X^i) = 1.$$  

We call $\rho$ a $(t; k_1, k_2)$-qm-state.

**Definition 23** (quantum secure 2-source $t$-non-malleable extractor). An $(n, n, m)$-non-malleable extractor $t$-$nmExt : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^m$ is $(t; k_1, k_2, \varepsilon)$-secure against qm if for every $(t; k_1, k_2)$-qm-state $\rho$ (chosen by the adversary qm),

$$\|\rho_t-nmExt(X,Y)t-nmExt(X^1,Y^1)...t-nmExt(X^t,Y^t)Y^{[t]}M'-U_{n}\otimes\rho_t-nmExt(X^1,Y^1)...t-nmExt(X^t,Y^t)YY^{[t]}M'\| \leq \varepsilon.$$  

**Parameters**

Let $\delta, \delta_3 > 0$ be small enough constants. Let $\delta_1, \delta_2 < \frac{1}{101}$ be constants chosen according to Fact [14] $n, v, n_1, t, a, s, b, h > 0$ be positive integers and $k, \varepsilon, \varepsilon' > 0$ be such that,

$$v = 5n \quad ; \quad n_1 = v^{\delta_1} \quad ; \quad \varepsilon = 2^{-O(n^{\delta_2})} \quad ; \quad t \leq n^{\delta_3} ;$$

$$k = O(n^{1/4}) \quad ; \quad a = 6k + 2O(v^{\delta_2}) = O(k) \quad ; \quad 2^{O(a)} \sqrt{\varepsilon'} = \varepsilon ;$$

$$s = O \left( \log^2 \left( \frac{n}{\varepsilon} \right) \log n \right) \quad ; \quad b = O \left( \log^2 \left( \frac{n}{\varepsilon'} \right) \log n \right) \quad ; \quad h = 10ts .$$

- $\text{IP}_1$ be $\text{IP}_{3k/n_1}$,  
- $\text{IP}_2$ be $\text{IP}_{3k^3/h}$,

- $\text{Ext}_1$ be $(2b, \varepsilon')$-quantum secure $(n, s, b)$-extractor,

- $\text{Ext}_2$ be $(2s, \varepsilon')$-quantum secure $(h, b, s)$-extractor,

- $\text{Ext}_3$ be $(2h, \varepsilon')$-quantum secure $(n, b, h)$-extractor,

- $\text{Ext}_4$ be $(\frac{n}{4t}, \varepsilon^2)$-quantum secure $(n, h, \frac{n}{M})$-extractor,

- $\text{Ext}_6$ be $(\frac{n}{2t}, \varepsilon^2)$-quantum secure $(n, \frac{n}{M}, \frac{n}{4t})$-extractor.

**Definition of 2-source $t$-non-malleable extractor**

Let $\mathbb{F}_q$ be the finite field of size $q = O(1)$. Let $\text{ECC} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^v$ be an error correcting code with relative distance $\frac{1}{\sqrt{n}}$ and rate $\frac{1}{2}$ (which exists from Fact [19] for our choice of parameters) for this section. Let $\text{Samp} : \{0, 1\}^r \rightarrow [v]^{t_1}$ be the sampler function from Fact [13] where $t_1 = O(\sqrt{v})$ and $r \geq v^{\delta_1}$. We identify the output of $\text{Samp}$ as $t_1$ samples from the set $[v]$. By $\text{ECC}(Y)_{\text{Samp}(l)}$, we mean the $\text{Samp}(l)$ entries of codeword $\text{ECC}(Y)$, interpreted as a bit string.
Algorithm 8 \( t \cdot 2^n \text{mExt} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{n/4t} \)

**Input:** \( X, Y \)

1. Advice generator:
   \[
   X_1 = \text{Prefix}(X, 3k) \quad ; \quad Y_1 = \text{Prefix}(Y, 3k) \quad ; \quad R = \text{IP}_1(X_1, Y_1) \quad ;
   \]
   \[ G = X_1 \circ Y_1 \circ \text{ECC}(X)_{\text{Samp}(R)} \circ \text{ECC}(Y)_{\text{Samp}(R)} \]

2. \( X_2 = \text{Prefix}(X, 3k^3) \quad ; \quad Y_2 = \text{Prefix}(Y, 3k^3) \quad ; \quad Z_0 = \text{IP}_2(X_2, Y_2) \)

3. Correlation breaker with advice: \( S = 2\text{AdvCB}(Y, X, Z_0, G) \)

4. \( L = \text{Ext}_0(X, S) \)

**Output:** \( L \)

**Result**

The following theorem shows that the function \( t \cdot 2^n \text{mExt} \) as defined in Algorithm 8 is \((t; k_1, k_2, \mathcal{O}(\varepsilon))\)-secure against \( \text{qnm} \) by noting that \( L = t \cdot 2^n \text{mExt}(X, Y) \) and \( L^i = t \cdot 2^n \text{mExt}(X^i, Y^i) \) for every \( i \in [t] \).

**Theorem 9.** Let \( \rho_{XX^i[\varepsilon]XX^i[\varepsilon]YY^i[\varepsilon]YY^i[\varepsilon]M} \) be a \((t; n-k, n-k)\)-\text{qnm-state}. Let Alice and Bob proceed with Protocol 16 starting with \( \rho \). Let \( \Lambda \) be the state at the end of the protocol. Then,

\[
||\rho_{XX^iYY^iM} - U_{n/4t} \otimes \rho_{XX^iYY^iM}||_1 \leq d(L|\tilde{B})_\Lambda \leq \mathcal{O}(\varepsilon).
\]

**Proof.** First inequality follows from Fact 7.

Note that the total communication from Alice to Bob in Protocol 16 is at most

\[
(t + 1)(n_1 + 3k + 6ah) + h + \frac{n}{8t}(t + 1) \leq \left(\frac{1}{4} + \delta\right)n.
\]

Similarly, total communication from Bob to Alice is at most

\[
(3k + a + 6ab)(t + 1) + 3k^3(t + 1) + \frac{n}{8t}(t + 1) \leq \left(\frac{1}{4} + \delta\right)n.
\]

Hence, using Lemma 2 at any stage \( \varrho \) in Protocol 16 we have, \( \text{H}_{\min}(X|\tilde{B})_\varrho \geq n - k - (1/4 + \delta)n \geq n/2 \) and similarly, \( \text{H}_{\min}(Y|\tilde{A})_\varrho \geq n/2 \). Thus, both Alice and Bob have enough entropy throughout the protocol for necessary extractions.

We start with a state \( \rho_{XX^i[\varepsilon]YY^i[\varepsilon]M} \) such that \( \text{H}_{\min}(X|\tilde{B})_\rho \geq n - k \) and \( \text{H}_{\min}(Y|\tilde{A})_\rho \geq n - k \).

From Fact 2 we have,

\[
\text{H}_{\min}(X_1|\tilde{B})_\rho \geq 3k - k = 2k \quad ; \quad \text{H}_{\min}(Y_1|\tilde{A})_\rho \geq 3k - k = 2k.
\]

Now from Claim 5 with the below assignment of registers (and noting registers \( XX^i[\varepsilon], YY^i[\varepsilon] \) are included in \( \langle A, B \rangle \) respectively),

\[
(Z, X, Y, \sigma) \leftarrow (R, X_1, Y_1, \rho) \quad ; \quad (k_1, k_2, m, n_1, \varepsilon) \leftarrow (2k, 2k, n_1, 3k, (\varepsilon/t)^2)
\]

\[
40
\]
we have,

\[
\Delta(\rho_{RY\rho_{YY}[t]}, U_R \otimes \rho_{YY}[t]) \leq O((\varepsilon/t)^2) ; \quad \Delta(\rho_{RX\rho_{XX}[t]}, U_R \otimes \rho_{XX}[t]) \leq O((\varepsilon/t)^2).
\]

Using Fact 6, we get

\[
\Delta_B(\rho_{RY\rho_{YY}[t]}, U_R \otimes \rho_{YY}[t]) \leq O(\varepsilon/t) ; \quad \Delta_B(\rho_{RX\rho_{XX}[t]}, U_R \otimes \rho_{XX}[t]) \leq O(\varepsilon/t).
\]

Let \(\kappa\) be the state just before Bob sends \(Y_2\). Note that till then, communication from Alice to Bob and Bob to Alice is at most \(O(tk) \leq O(k^2)\) each. Hence, by Lemma 2 \(H_{\min}(X|\hat{B})_{\kappa} \geq n - k - O(k^2)\) and thus \(H_{\min}(X_2|\hat{B})_{\kappa} \geq 3k^3 - k - O(k^2) \geq 2k^3\). Similarly, \(H_{\min}(Y_2|\hat{A})_{\kappa} \geq 3k^3 - k - O(k^2) \geq 2k^3\). Let \(\eta\) be the state just after Alice generates \(Z_0\). Using similar argument as before involving Claim 5 we have \(d(Z_0|\hat{B})_{\eta} \leq O(\varepsilon)\).

Rest of the proof follows similar lines to that of Theorem 7 with the following change in the Claim 10 as follows:

Claim 12. Let \(\Phi\) be the joint state after registers \(Z_0, Z_0^{[t]}\) are generated by Alice. Then,

\[
\Pr(\forall i \in [t], G \neq G^i)_{\Phi} \geq 1 - O(\varepsilon) \quad \text{and} \quad d(Z_0|\hat{B})_{\Phi} \leq O(\varepsilon).
\]

The proof of the above claim follows from that of Claim 11 after noting that \(G = G^i\) is a sub-event of both \(G_a = G_a^i\) and \(G_b = G_b^i\) where \(G_a = X_1 \circ \text{ECC}(X)_{\text{Samp}(R)}\), \(G_a^i = X_1^i \circ \text{ECC}(X^i)_{\text{Samp}(R^i)}\), \(G_b = Y_1 \circ \text{ECC}(Y)_{\text{Samp}(R)}\) and \(G_b^i = Y_1^i \circ \text{ECC}(Y^i)_{\text{Samp}(R^i)}\).

This completes our proof.
## C Communication Protocols

### Protocols for nmExt

#### Protocol 1 \((X, \hat{X}, N, Y, Y', \hat{Y}, \hat{Y}', M)\).

| Alice: \((X, \hat{X}, N)\) | Bob: \((Y, Y', \hat{Y}, \hat{Y}', M)\) | Analysis |
|-----------------------------|----------------------------------|----------------|
| \(I = \text{Ext}_0(X, Y_1)\) | \(Y_1 \leftarrow Y_1\) | \(d(Y_1|\hat{A}) = 0\) |
| \(I' = \text{Ext}_0(X, Y'_1)\) | \(Y'_1 \leftarrow Y'_1\) | \(d(I'|\hat{B}) \leq \varepsilon\) |
| \(I \rightarrow I\) | \(G = Y_1 \circ \text{ECC}(Y)_I\) |
| \(I' \rightarrow I'\) | \(G' = Y'_1 \circ \text{ECC}(Y')_{I'}\) |
| \(T = \text{Ext}_5(X, Y_2)\) | \(Y_2 \leftarrow Y_2\) | \(d(T|\hat{B}) \leq \varepsilon\) |
| \(T' = \text{Ext}_5(X, Y'_2)\) | \(Y'_2 \leftarrow Y'_2\) | \(d(T'|\hat{B}) \leq \varepsilon\) |
| \(Z_0 = \text{Prefix}(T, h)\) | \(G \leftarrow G\) |
| \(Z'_0 = \text{Prefix}(T', h)\) | \(G' \leftarrow G'\) | \(d(T'|\hat{B}) \leq \varepsilon\) |

### Protocol 2 \((T, T', Z_0, Z'_0, G, G', N)\).

| Alice: \((T, T', Z_0, Z'_0, G, G', N)\) | Bob: \((Y, Y', G, G', M)\) |
|---------------------------------------------|---------------------|
| \(L' = \text{Ext}_6(X, S')\) | \(S' \leftarrow S'\) | \(d(Z|\hat{B}) \leq O(\varepsilon)\) |
| \(Z \rightarrow Z\) | \(S = \text{Ext}_4(Y', Z)\) | \(d(S|\hat{A}) \leq O(\varepsilon)\) |
| \(L' \rightarrow L'\) | \(L = \text{Ext}_6(X, S)\) | \(d(S|\hat{A}) \leq O(\varepsilon)\) |
| \(L \leftarrow L\) | \(S \leftarrow S\) | \(d(L|\hat{B}) \leq O(\varepsilon)\) |

| Alice: \((L, N)\) | Bob: \((L', Y, Y', M)\) |
|-----------------|---------------------|
Protocol 2 \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\).

For \(i = 1, 2, \ldots, a\):

- Protocol \(4\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (0,1)\).
- Protocol \(5\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (1,0)\).
- Protocol \(6\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (0,0)\) and \(\alpha_j = \alpha'_j\) for \(j < i\).
- Protocol \(7\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (0,0)\) and \(\alpha_j \neq \alpha'_j\) for some \(j < i\).
- Protocol \(8\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (1,1)\) and \(\alpha_j = \alpha'_j\) for \(j < i\).
- Protocol \(9\) \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\) for \((\alpha_i, \alpha'_i) = (1,1)\) and \(\alpha_j \neq \alpha'_j\) for some \(j < i\).

\((Z,Z') = (O,O')\).
**Protocol 3** Modified advice generator.

| Alice: \((X, N)\) | Bob: \((Y, Y', M)\) | Analysis |
|------------------|------------------|---------|
| \((E_A, E'_A) \leftrightarrow (E_B, E'_B)\) | \(Y_1 = \text{Prefix}(Y, d_1)\) | \(Y'_1 = \text{Prefix}(Y', d_1)\) |
| \(I = \text{Ext}_0(X, E_A)\) | \(d(I|\bar{B}) \leq \epsilon\) | |
| \(I' = \text{Ext}_0(X, E'_A)\) | \(I' \rightarrow I'\) | |
| \(T = \text{Ext}_5(X, Y_2)\) | \(Y_2 \leftarrow Y_2\) | \(Y_2 = \text{Prefix}(Y, d_2)\) |
| \(T' = \text{Ext}_5(X, Y'_2)\) | \(Y'_2 \leftarrow Y'_2\) | \(Y'_2 = \text{Prefix}(Y', d_2)\) |
| \(Z_0 = \text{Prefix}(T, h)\) | \(G \leftarrow G\) | \(G = Y_1 \circ \text{ECC}(Y)_t\) |
| \(Z'_0 = \text{Prefix}(T', h)\) | \(G' \leftarrow G'\) | \(G' = Y'_1 \circ \text{ECC}(Y')_t\) |
| \(\text{If:} (E_B, E'_B) \neq (Y_1, Y'_1)\) | \(d(T|\bar{B}) \leq \sqrt{\epsilon}\) | |
| \(\text{abort}\) | \(|T|\bar{B}) \leq 2^{d_1}\sqrt{\epsilon} \leq \epsilon\) | |
| \(\text{else: continue}\) | \(|T|\bar{B}) \leq 2^{d_1}\sqrt{\epsilon} \leq \epsilon\) | |

Alice: \((T, T', Z_0, Z'_0, G, G', N)\) Bob: \((Y, Y', G, G', M)\)
Protocol 4 \((T, T', Z, Z', G, G', N, Y, Y', G, G', M)\).

| Alice: \((T, T', Z, Z', G, G', N)\) | Bob: \((Y, Y', G, G', M)\) | Analysis |
|--------------------------------|----------------------------|-----------|
| \(Z_s = \text{Prefix}(Z, s)\) | \(d(Z_s|B) \leq \eta\) | \(Z_s \rightarrow Z_s\) | \(A = \text{Ext}_1(Y, Z_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z' \rightarrow Z'\) | \(A' = \text{Ext}_1(Y', Z_s')\) | \(C' = \text{Ext}_2(Z', A')\) | \(B' = \text{Ext}_1(Y', O')\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z = \text{Ext}_3(T, A)\) | | \(A \leftarrow A\) | | \(d(Z_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z_s = \text{Prefix}(Z, s)\) | | \(Z' = \text{Ext}_3(T', B')\) | | \(B' \leftarrow B'\) | \(d(Z_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z' \rightarrow Z'\) | \(Z' = \text{Ext}_3(T', A')\) | \(Z_s \rightarrow Z_s\) | \(\overline{A} = \text{Ext}_1(Y, Z_s)\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z' \rightarrow Z'\) | | \(Z_s = \text{Prefix}(Z', s)\) | | \(\overline{A} = \text{Ext}_1(Y', \overline{Z}_s)\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{C} = \text{Ext}_2(Z, \overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | | \(d(\overline{C}|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{O'} = \text{Ext}_3(T', \overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | \(\overline{O'} \rightarrow \overline{O'}\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{C} \rightarrow \overline{C}\) | \(\overline{C} \leftarrow \overline{C}\) | \(\overline{O} = \text{Ext}_3(T, \overline{B})\) | \(d(\overline{O}|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{B} \leftarrow \overline{B}\) | | | | |
Protocol 5 \((T,T',Z,Z',G,G',N,Y,Y',G,G',M)\).

| Alice: \((T,T',Z,Z',G,G',N)\) | Bob: \((Y,Y',G,G',M)\) | Analysis |
|-------------------------------|-------------------|----------|
| \(Z_s = \text{Prefix}(Z,s)\) | | \(d(Z_s | \overline{B}) \leq \eta\) |
| \(Z_s \rightarrow Z_s\) | \(A = \text{Ext}_1(Y,Z_s)\) | \(d(A | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(Z'_s = \text{Prefix}(Z',s)\) | \(Z'_s \rightarrow Z'_s\) | \(A' = \text{Ext}_1(Y',Z'_s)\) | \(d(A' | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(C = \text{Ext}_2(Z,A)\) | \(A \leftarrow A\) | \(d(C | \overline{B}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}' = \text{Ext}_3(T',A')\) | \(A' \leftarrow A'\) | \(d(C | \overline{B}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(C \rightarrow C\) | \(B = \text{Ext}_1(Y,C)\) | \(d(B | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}' \rightarrow \overline{Z}'\) | \(\overline{A}' = \text{Ext}_1(Y',\overline{Z}'_s)\) | \(d(B | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z} = \text{Ext}_3(T,B)\) | \(B \leftarrow B\) | \(d(\overline{Z}_s | \overline{B}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(O' = \text{Ext}_3(T',\overline{B}')\) | \(\overline{B}' \leftarrow \overline{B}'\) | \(d(\overline{Z}_s | \overline{B}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s \rightarrow \overline{Z}_s\) | \(\overline{A} = \text{Ext}_1(Y,\overline{Z}_s)\) | \(d(\overline{A} | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(O' \rightarrow O'\) | \(d(\overline{A} | \overline{A}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
| \(O = \text{Ext}_3(T,\overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(O | \overline{B}) \leq \mathcal{O}(\eta + \sqrt{\varepsilon'})\) |
**Protocol 6** \((T, T', Z, Z', G, G', N, Y, Y', G, G', M)\).

| Alice: \((T, T', Z, Z', G, G', N)\) | Bob: \((Y, Y', G, G', M)\) | Analysis |
|--------------------------------------|--------------------------|----------|
| \(Z_s = \text{Prefix}(Z, s)\) | | \(d(Z_s | \bar{B}) \leq \eta\) |
| \(Z_s \rightarrow Z_s\) | \(A = \text{Ext}_1(Y, Z_s)\) | \(d(A | \bar{A}) \leq \eta\) |
| \(Z'_s = \text{Prefix}(Z', s)\) | \(Z'_s \rightarrow Z'_s\) | \(A' = \text{Ext}_1(Y', Z'_s)\) | \(d(A' | \bar{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{Z} = \text{Ext}_3(T, A)\) | \(\bar{Z}_s = \text{Prefix}(\bar{Z}, s)\) | | \(d(\bar{Z}_s | \bar{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{Z}' = \text{Ext}_3(T', A')\) | \(\bar{Z}_s \rightarrow \bar{Z}_s\) | \(A' \leftarrow A'\) | \(d(\bar{Z}_s | \bar{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{Z}'_s = \text{Prefix}(\bar{Z}', s)\) | \(\bar{Z}'_s \rightarrow \bar{Z}'_s\) | \(\bar{A}' = \text{Ext}_1(Y', \bar{Z}'_s)\) | \(d(\bar{A}' | \bar{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{C} = \text{Ext}_2(\bar{Z}, \bar{A})\) | \(\bar{A} \leftarrow \bar{A}\) | | \(d(\bar{C} | \bar{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{C}' = \text{Ext}_2(\bar{Z}', \bar{A}')\) | \(\bar{A}' \leftarrow \bar{A}'\) | | \(d(\bar{C}' | \bar{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{C} \rightarrow \bar{C}\) | \(\bar{B} = \text{Ext}_1(Y, \bar{C})\) | \(d(\bar{B} | \bar{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\bar{C}' \rightarrow \bar{C}'\) | \(\bar{B}' = \text{Ext}_1(Y', \bar{C}')\) | | \(d(\bar{B}' | \bar{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O = \text{Ext}_3(T, \bar{B})\) | \(\bar{B} \leftarrow \bar{B}\) | | \(d(O | \bar{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O' = \text{Ext}_3(T', \bar{B}')\) | \(\bar{B}' \leftarrow \bar{B}'\) | | \(d(O' | \bar{B}') \leq O(\eta + \sqrt{\varepsilon'})\) |
**Protocol 7** \((T, T', Z, Z', G, G', N, Y, Y', G, G', M)\).

| Alice: \((T, T', Z, G, G', N)\) | Bob: \((Y, Y', Z', G, G', M)\) | Analysis |
|---|---|---|
| \(Z_s = \text{Prefix}(Z, s)\) | | \(d(Z_s | \tilde{B}) \leq \eta\) |
| \(\overline{Z} = \text{Ext}_3(T', A')\) | \(A' \leftarrow A'\) | \(A' = \text{Ext}_1(Y', Z'_s)\) | \(d(Z_s | \tilde{B}) \leq \eta\) |
| \(Z_s \rightarrow Z_s\) | | \(A = \text{Ext}_1(Y, Z_s)\) | \(d(A | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}' \rightarrow \overline{Z}\) | | \(\overline{A} = \text{Ext}_1(Y', \overline{Z}_s)\) | \(d(A | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z} = \text{Ext}_3(T, A)\) | \(A \leftarrow A\) | \(d(Z_s | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s = \text{Prefix}(\overline{Z}, s)\) | | \(d(Z_s | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O' = \text{Ext}_3(T', \overline{B}')\) | \(\overline{B}' \leftarrow \overline{B}'\) | \(d(Z_s | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s \rightarrow \overline{Z}_s\) | \(\overline{A} = \text{Ext}_1(Y, \overline{Z}_s)\) | \(d(A | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{C} = \text{Ext}_2(Z, \overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(\overline{C} | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{C} \rightarrow \overline{C}\) | \(\overline{B} = \text{Ext}_1(Y, \overline{C})\) | \(d(\overline{B} | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O' \rightarrow O'\) | \(O = \text{Ext}_3(T, \overline{B})\) | \(d(\overline{B} | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{B} \leftarrow \overline{B}\) | | \(d(O | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
### Protocol 8 \((T, T', Z, Z', G, G', N, Y, Y', G, G', M)\).

| Alice: \((T, T', Z, Z', G, G', N)\) | Bob: \((Y, Y', G, G', M)\) | Analysis |
|----------------------------------------|---------------------------------|----------|
| \(Z_s = \text{Prefix}(Z, s)\)          | \(d(Z_s | B) \leq \eta\)        |          |
| \(Z_s \rightarrow Z_s\)               | \(A = \text{Ext}_1(Y, Z_s)\)   | \(d(A | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z'_s = \text{Prefix}(Z', s)\)       | \(Z'_s \rightarrow Z'_s\)      | \(A' = \text{Ext}_1(Y', Z'_s)\) | \(d(A' | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(C = \text{Ext}_2(Z, A)\)            | \(A \leftarrow A\)             | \(d(C | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(C' = \text{Ext}_2(Z', A')\)         | \(A' \leftarrow A'\)           | \(d(C' | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
|                                         | \(C \rightarrow C\)            | \(B = \text{Ext}_1(Y, C)\)   | \(d(B | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
|                                         | \(C' \rightarrow C'\)          | \(B' = \text{Ext}_1(Y', C')\) | \(d(B' | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z} = \text{Ext}_3(T, B)\) | \(B \leftarrow B\)             | \(d(\overline{Z}_s | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s = \text{Prefix}(\overline{Z}, s)\) | \(|\overline{Z}_s|\) \leq \eta \) |          |
| \(\overline{Z}' = \text{Ext}_3(T', B')\) | \(B' \leftarrow B'\)           | \(d(\overline{Z}_s | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s \rightarrow \overline{Z}_s\) | \(\overline{A} = \text{Ext}_1(Y, \overline{Z}_s)\) | \(d(\overline{A} | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}'_s = \text{Prefix}(\overline{Z}', s)\) | \(\overline{Z}'_s \rightarrow \overline{Z}'_s\) | \(\overline{A}' = \text{Ext}_1(Y', \overline{Z}'_s)\) | \(d(\overline{A}' | \tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O = \text{Ext}_3(T, \overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(O | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O' = \text{Ext}_3(T', \overline{A}')\) | \(\overline{A}' \leftarrow \overline{A}'\) | \(d(O | \tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
**Protocol 9** \((T, T', Z, Z', G, G', N, Y, Y', G', M)\).

| Alice: \((T, T', Z, G, G', N)\) | Bob: \((Y, Y', Z', G, G', M)\) | Analysis |
|-----------------------------|-----------------------------|---------|
| \(Z_s = \text{Prefix}(Z, s)\) | \(d(Z_s|\tilde{B}) \leq \eta\) | \(Z'_s = \text{Prefix}(Z', s)\) |
| \(A' = \text{Ext}_1(Y', Z'_s)\) | \(A = \text{Ext}_1(Y, Z_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(C' = \text{Ext}_2(Z', A')\) | \(\overline{A} = \text{Ext}_1(Y', Z'_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(B' = \text{Ext}_1(Y', G')\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z' = \text{Ext}_3(T', B')\) | \(B' \leftarrow B'\) | \(d(Z_s|\tilde{B}) \leq \eta\) |
| \(Z_s \rightarrow Z_s\) | \(A \leftarrow A\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O' = \text{Ext}_3(T', \overline{A}\) | \(C \rightarrow C\) | \(B = \text{Ext}_1(Y, C)\) | \(d(B|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(B \leftarrow B\) | \(O' \rightarrow O'\) | \(d(B|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z = \text{Ext}_3(T, B)\) | \(d(\overline{Z}_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s = \text{Pref}(\overline{Z}, s)\) | \(d(\overline{Z}_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{A} = \text{Ext}_1(Y, \overline{Z}_s)\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O = \text{Ext}_3(T, \overline{A}\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(O|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
Protocols for 2nmExt

Protocol 2 is exactly same as Protocol 2 except replacing extractors and parameters as in Section 5

Protocols for 2

Protocol 10 \((X, X', X', N, Y, Y', Y', M)\).

| Alice: \((X, X', X', N)\) | Bob: \((Y, Y', Y', M)\) | Analysis |
|--------------------------|--------------------------|---------|
| \(X_1 = \text{Prefix}(X, 3k)\) | \(Y_1 = \text{Prefix}(Y, 3k)\) | \(H_{\text{min}}(X_1 | \hat{B}) \geq 2k\) |
| \(X_1 \rightarrow X_1\) | \(R = t\text{P}_1(X_1, Y_1)\) | \(H_{\text{min}}(Y_1 | \hat{A}) \geq 2k\) |
| \(V = \text{ECC}(X)_R\) | \(V \rightarrow V\) | \(d(R | X') \leq O(\varepsilon)\) |
| \(X'_1 = \text{Prefix}(X', 3k)\) | \(Y'_1 = \text{Prefix}(Y', 3k)\) | \(d(R | Y') \leq O(\varepsilon)\) |
| \(X'_1 \rightarrow X'_1\) | \(R' = t\text{P}_1(X'_1, Y'_1)\) | \(|G| = X_1 \circ Y_1 \circ V \circ W\) |
| \(V' = \text{ECC}(X')_{R'}\) | \(V' \rightarrow V'\) | \(|G'| = X'_1 \circ Y'_1 \circ V' \circ W'\) |
| \(X_2 = \text{Prefix}(X, 3k^3)\) | \(Y_2 = \text{Prefix}(Y, 3k^3)\) | \(H_{\text{min}}(X_2 | \hat{B}) \geq 2k^3\) |
| \(Z_0 = t\text{P}_2(X_2, Y_2)\) | \(Y_2 \leftarrow Y_2\) | \(d(Z_0 | \hat{B}) \leq O(\varepsilon)\) |
| \(X'_2 = \text{Prefix}(X', 3k^3)\) | \(Y'_2 = \text{Prefix}(Y', 3k^3)\) | \(d(Z_0 | \hat{B}) \leq O(\varepsilon)\) |
| \(Z'_0 = t\text{P}_2(X'_2, Y'_2)\) | \(Y'_2 \leftarrow Y'_2\) | \(|G'\rangle \leftarrow (G, G')\) |
| \((G, G')\) | | \(|G'\rangle \leftarrow (G, G')\) |

Alice: \((X, X', Z_0, G, G', N)\) Bob: \((Y, Y', G, G', M)\)

Protocol 2 \((X, X', Z_0, Z'_0, G, G', N, Y, Y', G, G', M)\)

| Alice: \((X, X', Z, N)\) | Bob: \((Y, Y', Z', M)\) |
|--------------------------|--------------------------|
| \(L' = \text{Ext}_6(X, S')\) | \(S' \leftarrow S'\) | \(S' = \text{Ext}_4(Y', Z')\) | \(d(Z | \hat{B}) \leq O(\varepsilon)\) |
| \(Z \rightarrow Z\) | \(S = \text{Ext}_4(Y, Z)\) | \(d(S | \hat{A}) \leq O(\varepsilon)\) |
| \(L' \rightarrow L'\) | | \(d(S | \hat{A}) \leq O(\varepsilon)\) |
| \(L = \text{Ext}_6(X, S)\) | \(S \leftarrow S\) | \(d(L | \hat{B}) \leq O(\varepsilon)\) |

Alice \((L, N)\) Bob \((L', Y, Y', M)\)
### Protocols for $t$-nmExt

#### Protocol 11 $(X, X, N, M, Y, \tilde{Y}, Y^{[t]}, \tilde{Y}^{[t]}).$

| Alice: $(X, X, N)$ | Bob: $(M, Y, \tilde{Y}, Y^{[t]}, \tilde{Y}^{[t]})$ | Analysis |
|---------------------|---------------------------------|----------|
| $Y_1 = \text{Prefix}(Y, d_1)$ | $d(Y_1 | \tilde{A}) = 0$ | |
| $I = \text{Ext}_0(X, Y_1)$ | $Y_1 \leftarrow Y_1$ | $d(I | B | \tilde{B}) \leq \varepsilon / t$ |
| $I^1 = \text{Ext}_0(X, Y_i^1)$ | $Y_i^1 \leftarrow Y_i^1$ | $d(I^1 | B | \tilde{B}) \leq \varepsilon / t$ |
| | ... | ... | ... |
| $I^t = \text{Ext}_0(X, Y_i^t)$ | $Y_i^t \leftarrow Y_i^t$ | $d(I^t | B | \tilde{B}) \leq \varepsilon / t$ |
| $T = \text{Ext}_5(X, Y_2)$ | $Y_2 \leftarrow Y_2$ | $d(T | B) \leq \varepsilon$ |
| $T^1 = \text{Ext}_5(X, Y_2^1)$ | $Y_2^1 \leftarrow Y_2^1$ | $d(T^1 | B) \leq \varepsilon$ |
| | ... | ... | ... |
| $Z_0 = \text{Prefix}(T, h)$ | $G \leftarrow G$ | $d(Z_0 | B | \tilde{B}) \leq \varepsilon$ |
| $Z_0^1 = \text{Prefix}(T^1, h)$ | $G^1 \leftarrow G^1$ | $d(Z_0^1 | B | \tilde{B}) \leq \varepsilon$ |
| | ... | ... | ... |

#### Protocol 12 $(T, T^{[t]}, Z_0, G, G^{[t]}, N, Y, Y^{[t]}, G, G^{[t]}, M)$

| Alice: $(X, Z, N)$ | Bob: $(Y, Y^{[t]}, Z^{[t]}, M)$ |
|---------------------|---------------------------------|
| $L^1 = \text{Ext}_0(X, S^1)$ | $S^1 \leftarrow S^1$ | $S^1 = \text{Ext}_4(Y^1, Z^1)$ | $d(Z | \tilde{B}) \leq O(\varepsilon)$ |
| | ... | ... | ... |
| $L^t = \text{Ext}_0(X, S^t)$ | $S^t \leftarrow S^t$ | $S^t = \text{Ext}_4(Y^t, Z^t)$ | $d(Z | \tilde{B}) \leq O(\varepsilon)$ |
| $Z \rightarrow Z$ | $S = \text{Ext}_4(Y, Z)$ | $d(S | \tilde{A}) \leq O(\varepsilon)$ |
| $L^1 \rightarrow L^1$ | $d(S | \tilde{A}) \leq O(\varepsilon)$ |
| | ... | ... | ... |
| $L = \text{Ext}_0(X, S)$ | $S \leftarrow S$ | $d(L | \tilde{B}) \leq O(\varepsilon)$ |

Alice($L, N$) Bob($L^{[t]}, Y, Y^{[t]}, M$)
Protocol 12 \((T, T^t, Z, Z^t, G, G^t, N, Y, Y^t, G, G^t, M)\).

For \(i = 1, 2, \ldots, a\):

Let \(S_i = \{j : G_i = G_j^i\}\) and \(\overline{S}_i = [t] \setminus S_i\). Let \(S^0_i = \{j : (G_i = G_j^i) \land (G_k = G_k^j \text{ for every } k < i)\}\) and \(S^1_i = \{j : (G_i = G_j^i) \land (G_k \neq G_k^j \text{ for some } k < i)\}\).

- Protocol 13 \((T, T^t, Z, Z^t, G, G^t, N, Y, Y^t, G, G^t, M)\) for \((G_i = 0)\).
- Protocol 14 \((T, T^t, Z, Z^t, G, G^t, N, Y, Y^t, G, G^t, M)\) for \((G_i = 1)\).

\((Z, Z^1, \ldots, Z^t) = (O, O^1, \ldots O^t)\).
### Protocol 13

$\forall p \in S_1^0, \forall q \in S_1^1, \forall r \in S_0$. 

| Alice: $(T, T^{[t]}, Z, Z^{[t]}, G, G^{[t]}, N, Y, Y^{[t]}, G, G^{[t]}, M)$ | Bob: $(Y, Y^{[t]}, Z^{S_1^1}, G, G^{[t]}, M)$ | Analysis |
|---|---|---|
| $Z_s = \text{Prefix}(Z,s)$ | $d(Z_s|B) \leq \eta$ |
| $\overline{Z}^q = \text{Ext}_3(T^q, A^q)$ | $A^q \leftarrow A^q$ | $A^q = \text{Ext}_1(Y^q, Z_s^q)$ | $d(Z_s|B) \leq \eta$ |
| $Z_s \rightarrow Z_s$ | $A = \text{Ext}_1(Y, Z_s)$ | $d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $Z_p^q \rightarrow Z_p^q$ | $A^p = \text{Ext}_1(Y^p, Z_s^p)$ | $d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{Z}^r \rightarrow \overline{Z}^r$ | $A^r = \text{Ext}_1(Y^r, Z_s^r)$ | $d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{Z}^r \rightarrow \overline{Z}^r$ | $C^r = \text{Ext}_1(Z^r, A^r)$ | $d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $Z_r \rightarrow Z_r$ | $B^r = \text{Ext}_1(Y^r, C^r)$ | $d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{Z} = \text{Ext}_3(T, A)$ | $A \leftarrow A$ | $d(Z_s|B) \leq O(\eta + \varepsilon')$ |
| $\overline{Z}^p = \text{Ext}_3(T^p, A^p)$ | $A^p \leftarrow A^p$ | $d(Z_s|B) \leq O(\eta + \varepsilon')$ |
| $O^q = \text{Ext}_3(T^q, B^q)$ | $B^q \leftarrow B^q$ | $d(Z_s|B) \leq O(\eta + \varepsilon')$ |
| $\overline{Z}^r = \text{Ext}_3(T^r, B^r)$ | $B^r \leftarrow B^r$ | $d(Z_s|B) \leq O(\eta + \varepsilon')$ |
| $\overline{Z}_s \rightarrow \overline{Z}_s$ | $\overline{A} = \text{Ext}_1(Y, \overline{Z}_s)$ | $d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{Z}_s^p \rightarrow \overline{Z}_s^p$ | $\overline{A}^p = \text{Ext}(Y^p, \overline{Z}_s^p)$ | $d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{Z}_s^r \rightarrow \overline{Z}_s^r$ | $\overline{A}^r = \text{Ext}_1(Y^r, \overline{Z}_s^r)$ | $d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{C} = \text{Ext}_2(Z, A)$ | $\overline{A} \leftarrow A$ | $d(\overline{C}|\tilde{B}) \leq O(\eta + \varepsilon')$ |
| $\overline{C}^p = \text{Ext}_2(\overline{Z}^p, \overline{A}^p)$ | $\overline{A}^p \leftarrow A^p$ | $d(\overline{C}|\tilde{B}) \leq O(\eta + \varepsilon')$ |
| $O^r = \text{Ext}_3(T^r, \overline{A}^r)$ | $\overline{A}^r \leftarrow A^r$ | $d(\overline{C}|\tilde{B}) \leq O(\eta + \varepsilon')$ |
| $\overline{C} \rightarrow \overline{C}$ | $\overline{B} = \text{Ext}_1(Y, \overline{C})$ | $d(\overline{B}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $\overline{C}^p \rightarrow \overline{C}^p$ | $\overline{B}^p = \text{Ext}_1(Y^p, \overline{C}^p)$ | $d(\overline{B}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $O^q \rightarrow O^q$ | $d(\overline{B}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $O^r \rightarrow O^r$ | $d(\overline{B}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $O = \text{Ext}_3(T, \overline{B})$ | $\overline{B} \leftarrow \overline{B}$ | $d(O|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})$ |
| $O^p = \text{Ext}_3(T^p, \overline{B}^p)$ | $\overline{B}^p \leftarrow \overline{B}^p$ | $d(O|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})$ |
**Protocol 14** \((T, T^{[i]}, Z, Z^{[i]}, G, G^{[i]}, N, Y, Y^{[i]}, G, G^{[i]}, M)\).  

Execute \(\forall p \in S_1^0, \forall q \in S_1^1, \forall r \in S_i\).

| Alice: \((T, T^{[i]}, Z, Z^{[i]}, G, G^{[i]}, N)\) | Bob: \((Y, Y^{[i]}, Z^{[i]}, G, G^{[i]}, M)\) | Analysis |
|---|---|---|
| \(Z_s = \text{Prefix}(Z, s)\) | \(A^q = \text{Ext}_1(Y^q, Z^q_s)\) | \(d(Z_s|B) \leq \eta\) |
| \(Z^q = \text{Ext}_3(T^q, B^q)\) | \(B^q \leftarrow B^q\) | \(d(Z^q|\tilde{B}) \leq \eta\) |
| \(Z^q \rightarrow Z^q\) | \(A = \text{Ext}_1(Y^q, Z^q_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z^q_s = \text{Prefix}(Z^q, s)\) | \(Z^q_s \rightarrow Z^q_s\) | \(A^p = \text{Ext}_1(Y^p, Z^p_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(Z^q_s \rightarrow Z^q_s\) | \(A^p = \text{Ext}_1(Y^p, Z^p_s)\) | \(d(A|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(C = \text{Ext}_2(Z, A)\) | \(A \leftarrow A\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(C^p = \text{Ext}_2(Z^p, A^p)\) | \(A^p \leftarrow A^p\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O^q = \text{Ext}_3(T^q, A^q)\) | \(\overline{A^q} \leftarrow A^q\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{A} = \text{Ext}_1(Y^q, Z^q_s)\) | \(\overline{A} \leftarrow A^q\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{A} = \text{Ext}_1(Y^q, Z^q_s)\) | \(\overline{A} \leftarrow A^q\) | \(d(C|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z} = \text{Ext}_3(T, B)\) | \(B \leftarrow B\) | \(d(\overline{Z}_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}^{p} = \text{Ext}_3(T^p, B^p)\) | \(B^p \leftarrow B^p\) | \(d(\overline{Z}_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{O}^r = \text{Ext}_3(T^r, \overline{B^r})\) | \(\overline{B^r} \leftarrow \overline{B^r}\) | \(d(\overline{Z}_s|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}_s \rightarrow \overline{Z}_s\) | \(\overline{A} = \text{Ext}_1(Y^q, Z^q_s)\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{Z}^{p} \rightarrow \overline{Z}^{p}\) | \(\overline{A} = \text{Ext}_1(Y^p, Z^p_s)\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(\overline{O}^r \rightarrow \overline{O}^r\) | \(d(\overline{A}|\tilde{A}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O = \text{Ext}_3(T, \overline{A})\) | \(\overline{A} \leftarrow \overline{A}\) | \(d(O|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
| \(O^p = \text{Ext}_3(T^p, \overline{A}^p)\) | \(\overline{A} \leftarrow \overline{A}^p\) | \(d(O|\tilde{B}) \leq O(\eta + \sqrt{\varepsilon'})\) |
**Protocols for $t$-2nmExt**

Protocol 12 is the same as Protocol 12 except replacing extractors and parameters as in Section 13.

**Protocol 15** ($X, X, X^{[i]}, X^{[i]}, N, Y, Y^{[i]}, Y^{[i]}, M$).

| Alice: $(X, X^{[i]}, X^{[i]}, X^{[i]}, N)$ | Bob: $(Y, Y^{[i]}, Y^{[i]}, M)$ | Analysis |
|--------------------------------------------|---------------------------------|----------|
| $X_1 = \text{Prefix}(X, 3k)$              | $Y_1 = \text{Prefix}(Y, 3k)$    | $H_{\min}(X_1 | \hat{B}) \geq 2k$ |
| $X_1 \rightarrow X_1$                     | $R = \text{IP}_1(X_1, Y_1)$     | $H_{\min}(Y_1 | \hat{A}) \geq 2k$ |
| $R = \text{IP}_1(X_1, Y_1)$              | $V = \text{ECC}(X)_{\text{Samp}}(R)$ | $d(R|XX^{[i]}) \leq O(\varepsilon/t)$ |
| $V \leftarrow Y_1$                       | $V \rightarrow V$               | $d(R|YY^{[i]}) \leq O(\varepsilon/t)$ |
| $X_1^i = \text{Prefix}(X^i, 3k)$         |                                 |          |
| $X_1^i = \text{Prefix}(X^i, 3k)$         | $X_1^i = \text{Prefix}(X^i, 3k)$ |          |
| $X_1^i = \text{Prefix}(X^i, 3k)$         |                                 |          |
| $(X_1^1, \ldots, X_1^i) \rightarrow (X_1^1, \ldots, X_1^i)$ | $(Y_1^1, \ldots, Y_1^i) \leftarrow (Y_1^1, \ldots, Y_1^i)$ |          |
| $(Y_1^1, \ldots, Y_1^i) \leftarrow (Y_1^1, \ldots, Y_1^i)$ | |          |
| $R^1 = \text{IP}_1(X_1^1, Y_1^1)$        |                                 |          |
| $R^t = \text{IP}_1(X_1^1, Y_1^1)$        |                                 |          |
| $V^1 = \text{ECC}(X^1)_{\text{Samp}}(R^1)$ |                                 |          |
| $V^t = \text{ECC}(X^t)_{\text{Samp}}(R^t)$ | $W^1 = \text{ECC}(Y^1)_{\text{Samp}}(R^1)$ |          |
| $(V^1, \ldots, V^t) \rightarrow (V^1, \ldots, V^t)$ | $(V^1, \ldots, V^t) \rightarrow (V^1, \ldots, V^t)$ |          |
| $X_2 = \text{Prefix}(X, 3k^3)$            | $Y_2 \leftarrow Y_2$            | $d(Z_0 | \hat{B}) \leq O(\varepsilon)$ |
| $Z_0 = \text{IP}_2(X_2, Y_2)$            |                                 |          |
| $X_2^i = \text{Prefix}(X^i, 3k^3)$       | $Y_2^1 = \text{Prefix}(Y^1, 3k^3)$ |          |
| $X_2^i = \text{Prefix}(X^i, 3k^3)$       | $Y_2^t = \text{Prefix}(Y^t, 3k^3)$ |          |
| $Y_2^i = \text{Prefix}(Y^i, 3k^3)$       |                                 |          |
| $(Y_2^1, \ldots, Y_2^i) \leftarrow (Y_2^1, \ldots, Y_2^i)$ | $Z_0^i = \text{IP}_2(X_2^1, Y_2^1)$ |          |
| $Z_0^t = \text{IP}_2(X_2^1, Y_2^1)$      | $Z_0^t = \text{IP}_2(X_2^1, Y_2^1)$ |          |
| $(G^1, \ldots, G^t) \leftarrow (G^1, \ldots, G^t)$ | |          |
**Protocol 16** \((X, \tilde{X}, X^t, \tilde{X}^t, N, Y, \tilde{Y}, Y^t, \tilde{Y}^t, M)\).

| Alice: \((X, \tilde{X}, X^t, \tilde{X}^t, N)\) | Bob: \((Y, \tilde{Y}, Y^t, \tilde{Y}^t, M)\) | Analysis |
|---|---|---|
| **Protocol 15** \((X, X^t, N, Y, Y^t, M)\) | | \(d(Z_0|\tilde{B}) \leq O(\varepsilon)\) |
| Alice: \((X, X^t, Z_0, \tilde{Z}_0, G, G^t, N)\) | Bob: \((Y, Y^t, G, G^t, M)\) | Protocol 12' \((X, X^t, Z_0, \tilde{Z}_0, G, G^t, N, Y, Y^t, G, G^t, M)\) |
| Alice: \((X, X^t, Z, N)\) | Bob: \((Y, Y^t, Z^t, M)\) | |

\(L^1 = \text{Ext}_6(X, S^1)\)  
\(S^1 \leftarrow S^1\)  
\(S^1 = \text{Ext}_4(Y^1, Z^1)\)  
\(d(Z|\tilde{B}) \leq O(\varepsilon)\)  

\(L' = \text{Ext}_6(X, S^t)\)  
\(S^t \leftarrow S^t\)  
\(S^t = \text{Ext}_4(Y^t, Z^t)\)  
\(d(Z|\tilde{B}) \leq O(\varepsilon)\)  

\(Z \rightarrow Z\)  
\(S = \text{Ext}_4(Y, Z)\)  
\(d(S|\tilde{A}) \leq O(\varepsilon)\)  

\(L^1 \rightarrow L^1\)  
\(\vdots\)  
\(d(S|\tilde{A}) \leq O(\varepsilon)\)  

\(L = \text{Ext}_6(X, S)\)  
\(S \leftarrow S\)  
\(d(L|\tilde{B}) \leq O(\varepsilon)\)  

Alice\((L, N)\)  
Bob\((L^t, Y, Y^t, M)\)
D Privacy amplification against an active adversary

Preliminaries

We begin with some useful definitions, facts and claims. Let \( n, m, d, z \) be positive integers and \( k, \varepsilon > 0 \).

**Definition 24.** A function \( \text{MAC} : \{0,1\}^{2m} \times \{0,1\}^m \rightarrow \{0,1\}^m \) is an \( \varepsilon \)-secure one-time message authentication code if for all \( A : \{0,1\}^m \times \{0,1\}^m \rightarrow \{0,1\}^m \times \{0,1\}^m \) and \( b' \in \{0,1\}^m \),

\[
\Pr_{s \leftarrow U_{2m}} [ \Pr(s, A(b', \text{MAC}(s, b')), b') = 1] \leq \varepsilon,
\]

where predicate \( P : \{0,1\}^{2m} \times \{0,1\}^m \times \{0,1\}^m \rightarrow \{0,1\} \) is defined as

\[
P(s, b, t, b') \overset{\text{def}}{=} (\text{MAC}(s, b) = t) \land (b' \neq b).
\]

Efficient constructions of MAC satisfying the conditions of Definition 24 are known.

**Fact 20** (Proposition 1 in [KR09]). For any integer \( m > 0 \), there exists an efficient family of \( 2^{-m} \)-secure one-time message authentication code \( \text{MAC} : \{0,1\}^{2m} \times \{0,1\}^m \rightarrow \{0,1\}^m \).

**Definition 25.** We say joint random variables \( ABC \), form a Markov-chain, denoted \( A \leftrightarrow B \leftrightarrow C \), if

\[
\forall b \in \text{supp}(B) : (AC|B=b) = (A|B=b) \otimes (C|B=b).
\]

We have the following corollaries of Theorem 1.

**Corollary 7.** Let \( d = O\left(\log^7\left(\frac{n}{\varepsilon}\right)\right) \) and \( k \geq 5d \). Let \( \sigma_{XNYYM}^{\text{a}X} \) be a \( (k) \)-qpa-state with \( |X| = n \) and \( |Y| = d \). Let \( \text{nmExt} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^{k/4} \) be an efficient \( (k, \varepsilon) \)-quantum secure non-malleable extractor from Theorem 7. Let \( S = \text{nmExt}(X, Y) \). Then,

\[
\| \sigma_{SYM} - U_{k/4} \otimes \sigma_{YM} \|_1 \leq \varepsilon.
\]

**Proof.** Let \( V : \mathcal{H}_Y \rightarrow \mathcal{H}_Y \otimes \mathcal{H}_{Y}' \otimes \mathcal{H}_{Y}^{\dagger} \) be a (safe) isometry such that for \( \rho = V \sigma V^\dagger \), we have \( Y' \) classical (with copy \( Y' \) of \( Y' \)) and \( \Pr(Y \neq Y')_\rho = 1 \).\(^{10}\) Notice the state \( \rho \) is a \( (k) \)-qnm-state. Since \( \text{nmExt} \) is a \( (k, \varepsilon) \)-quantum secure non-malleable extractor (see Definition 15), we have

\[
\| \rho_{SSYY'M} - U_{k/4} \otimes \rho_{SYY'M} \|_1 \leq \varepsilon.
\]

Using Fact 7, we get

\[
\| \rho_{SYM} - U_{k/4} \otimes \rho_{YM} \|_1 \leq \varepsilon.
\]

The desired now follows by noting \( \sigma_{XNYM} = \rho_{XNYM} \).

**Corollary 8** (\( \text{nmExt} \) is a quantum secure extractor). Let \( \text{nmExt} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^{k/4} \) be an efficient \( (k, \varepsilon) \)-quantum secure non-malleable extractor from Theorem 7. \( \text{nmExt} \) is a \( (k, \varepsilon) \)-quantum secure \( (n, d, k/4) \)-extractor for parameters \( d = O(\log^7(n/\varepsilon)) \) and \( k \geq 5d \).

\(^{10}\)It is easily seen that such an isometry exists.
Proof. Let \( \rho_{XEY} = \rho_{XE} \otimes U_d \) be a c-q state (XY classical) such that \( H_{\min}(X|E)_{\rho} \geq k \). Consider the following purification \( \rho_{\bar{X}\bar{X}EY} \) of \( \rho_{XEY} \),

\[
\rho_{\bar{X}\bar{X}EY} = \rho_{XXE} \otimes \rho_{Y\bar{Y}},
\]

where \( \rho_{XXE} \) is a purification of \( \rho_{XEY} \) (\( \bar{X} \) a copy of \( X \)) and \( \rho_{Y\bar{Y}} \) is the canonical purification of \( \rho_Y \). Note \( \rho \) is a \((k)-qpa\)-state. Let \( S = \text{nmExt}(X,Y) \). Using Corollary \( \ref{cor:qpa-state} \) (by setting \( \sigma_{XXNY\bar{Y}\bar{M}} = \rho_{XX\bar{X}EY\bar{Y}E} \)), we get

\[
\| \rho_{SYE} - U_{k/4} \otimes \rho_{YE} \|_1 \leq \varepsilon. \tag*{\square}
\]

Claim 13. Let MAC be an \( \varepsilon \)-secure one-time message authentication code from Definition \( \ref{def:mac} \). Let \( SB'T' \) be such that \( SB' = U_{2m} \otimes U_{m} \) and \( T' = \text{MAC}(S,B') \). Let \( BT \) be such that \( S \leftrightarrow B'T' \leftrightarrow BT \) and \( |B| = |T'| = m \). Then,

\[
\Pr \left[ P(S,B,T,B') = 1 \right] \leq \varepsilon. \tag*{\ref{claim-13}}
\]

Proof. For each \((b',t')\), define \( g_{(b',t')} : \{0,1\}^m \times \{0,1\}^m \rightarrow [0,1] \) as

\[
g_{(b',t')} \overset{\text{def}}{=} E_{s \leftarrow S^{b't'}} \left[ P(s,b,t,b') = 1 \right].
\]

Define, \( A : \{0,1\}^m \times \{0,1\}^m \rightarrow \{0,1\}^m \times \{0,1\}^m \) as

\[
A(b',t') \overset{\text{def}}{=} \arg \max \{ g_{(b',t')}(b,t) \}. \tag*{\ref{claim-13}}
\]

Consider,

\[
\Pr \left[ P(S,B,T,B') = 1 \right]
\begin{align*}
&= \mathbb{E}_{(b',t') \leftarrow B'T'} \left[ \mathbb{E}_{(s,t) \leftarrow BT^{b't'}} \left[ P(s,b,t,b') = 1 \right] \right] \\
&\leq \mathbb{E}_{(b',t') \leftarrow B'T'} \left[ \mathbb{E}_{s \leftarrow S^{b't'}} \left[ P(s,A(b',t'),b') = 1 \right] \right] \\
&= \mathbb{E}_{(s,b) \leftarrow SB'} \left[ P(s,A(b',MAC(s,b')),b') = 1 \right] \\
&\leq \varepsilon. \tag*{\text{Definition of } A, T'}
\end{align*}
\]

Claim 14. Let MAC be an \( \varepsilon \)-secure one-time message authentication code from Definition \( \ref{def:mac} \). Let \( \rho_{SB'T'E'} \) be a c-q state (\( SB'T' \) classical) such that \( \rho_{SB'T'E'} = \rho_{SB'T'} \otimes \rho_{E'} \); \( \rho_{SB'} = U_{2m} \otimes U_{m} \); \( T' = \text{MAC}(S,B') \).

Let classical registers \( BT \) be generated by a quantum adversary using (safe on classical registers) isometry \( V : H_{E'} \otimes H_{B'} \otimes H_T \rightarrow H_{E''} \otimes H_{B'} \otimes H_{T'} \otimes H_B \otimes H_T \). Let \( \sigma_{SB'T'E''} = V \rho_{SB'T'E'} V^\dagger \). Then,

\[
\Pr \left[ P(S,B,T,B') = 1 \right]_{\sigma} \leq \varepsilon.
\]

Proof. Note in state \( \rho \), for every \( b't' \in \text{supp}(B'T') \), we have \( \rho_{SE}^{b't'} = \rho_{S}^{b't'} \otimes \rho_{E'} \). Since \( V \) is safe on registers \( B'T' \), we have

\[
\sigma_{SB'T'E''} = \sigma_{SB'T'} \otimes \sigma_{E''},
\]

where \( \sigma_{S}^{b't'} = \rho_{S}^{b't'} \). Using Fact \( \ref{fact:qpa-state} \) we get \( \sigma_{SB'T'}^{b't'} = \sigma_{S}^{b't'} \otimes \sigma_{BT'}^{b't'} \). Using Definition \( \ref{def:mac} \) we have (in state \( \sigma \), \( S \leftrightarrow B'T' \leftrightarrow BT \)). Using Claim \( \ref{claim-13} \) (for state \( \sigma_{SB'T'} \)) while noting \( \sigma_{SB'T'} = \rho_{SB'T'} \), we get the desired.

\[\footnote{If there are more than one achieving maximum, pick one of them arbitrarily.}\]

\[\footnote{We included the copies of classical registers \( B,T \) in register \( E'' \).}\]

56
Claim 15. Let $\text{MAC}$ be an $\varepsilon$-secure one-time message authentication code from Definition 24. Let $\text{STBB}'$ be random variables such that, $\text{STBB'} = U_{2m} \otimes \text{TBB}'$ and $B' = U_m$. Then,

$$\Pr[\mathcal{P}(S,B,T,B') = 1] \leq \varepsilon.$$  

Proof. Note that $S \leftrightarrow B' \leftrightarrow BT$. This implies $S \leftrightarrow B'T' \leftrightarrow BT'$, where $T' = \text{MAC}(S, B')$. Using Claim 13 the desired follows. \hfill $\square$

We start with the definition of a quantum secure privacy amplification (PA) protocol against active adversaries. The following description is from [ACLV19]. A PA protocol $(\mathcal{P}_A, \mathcal{P}_B)$ is defined as follows. The protocol is executed by two parties (Alice and Bob) sharing a secret $X \in \{0,1\}^n$. Their actions are described by $\mathcal{P}_A$ and $\mathcal{P}_B$ respectively. In addition there is an active, computationally unbounded adversary Eve, who might have some quantum side information $E$ correlated with $X$, where $\rho$ denotes the initial state of the protocol. Note that, for the definition, it is not necessary to specify exactly how the protocols are formulated; informally, each player’s action is described by a sequence of efficient algorithms that compute the player’s next message, given the past interaction.

The protocol should have the following property: in case protocol does not terminate with a rejection, output keys $R_A, R_B$ should be random and statistically independent of Eve’s view. Moreover, they must output the same keys $R_A = R_B$ with overwhelming probability. We assume that Eve is in full control of the communication channel between Alice and Bob, and can arbitrarily insert, delete, reorder or modify messages sent by Alice and Bob. At the end of protocol, Alice outputs $R_A \in \{0,1\}^z \cup \{\perp\}$, where $\perp$ is a special symbol indicating rejection. Similarly, Bob outputs $R_B \in \{0,1\}^z \cup \{\perp\}$. For a random variable $R \in \{0,1\}^z \cup \{\perp\}$, let $\text{purify}(R)$ be a random variable on $z$-bit strings that is deterministically equal to $\perp$ if $R = \perp$, and is otherwise uniformly distributed over $\{0,1\}^z$. The following definition generalizes the classical definition in [DLWZ14].

Definition 26 ([ACLV19]). Let $\Theta$ be the joint state of Alice, Bob and Eve at the end of the protocol given by $(\mathcal{P}_A, \mathcal{P}_B)$ including $\text{purify}(R_A)$ and $\text{purify}(R_B)$. We say that a PA protocol $(\mathcal{P}_A, \mathcal{P}_B)$ is $(k,z,\varepsilon)$-secure against quantum adversaries if for any initial state $\rho_{XE}$ such that $H_{\text{min}}(X|E)_{\rho} \geq k$ it satisfies the following three properties.

1. Correctness. If the adversary does not interfere with the protocol, then $\Pr(R_A = R_B \neq \perp)_{\Theta} = 1$.

2. Robustness. In the presence of an active adversary, $\Pr(Q(R_A, R_B) = 1)_{\Theta} \leq \varepsilon$, where $Q(R_A, R_B)$ is the predicate $(R_A \neq R_B \land R_A \neq \perp \land R_B \neq \perp)$.

3. Extraction. Let $\Theta_{\tilde{E}}$ be the final quantum state possessed by Eve (including the transcript of the protocol). The following should hold:

$$\|\Theta_{R_A\tilde{E}} - \Theta_{\text{purify}(R_A)\tilde{E}}\|_1 \leq \varepsilon \quad \text{and} \quad \|\Theta_{R_B\tilde{E}} - \Theta_{\text{purify}(R_B)\tilde{E}}\|_1 \leq \varepsilon.$$  

In other words, whenever a party does not reject, the party’s key is (approximately) indistinguishable from a fresh random string to the adversary.
Let $\delta > 0$ be a small constant.

- Let $\text{nmExt} : \{0,1\}^n \times \{0,1\}^d \rightarrow \{0,1\}^{2m}$ be a $(k,\varepsilon)$-quantum secure $(n,d,2m)$-non-malleable extractor from Theorem 1 with following choice of parameters,
  
  $$
  d = O \left( \log^7 \left( \frac{n}{\varepsilon} \right) \right) ; \quad k \geq 5d ; \quad k \geq 8m.
  $$

- Let $\text{MAC} : \{0,1\}^{2m} \times \{0,1\}^m \rightarrow \{0,1\}^m$ be a 2$^{-m}$-secure one-time message authentication code from Fact 20 for $m = O \left( \log^3 \left( \frac{2}{\varepsilon} \right) \right)$. Note $2^{-m} \leq \varepsilon$.

- Let $\text{Ext} : \{0,1\}^n \times \{0,1\}^m \rightarrow \{0,1\}^z$ be a $(2^z,\varepsilon)$-quantum secure strong extractor from Fact 17. Taking $2^z = (1 - 2\delta)k$ suffices for the PA application.

### Protocol 17: PA protocol.

| Alice: $X$ | Eve: $E$ | Bob: $X$ |
|------------|----------|----------|
| Generate $Y \leftarrow U_d$ | $Y \xrightarrow{T_1} Y'$ | $S = \text{nmExt}(X,Y)$ |
| $S' = \text{nmExt}(X,Y')$ | Generate $B' \leftarrow U_m$ | $R_B = \text{Ext}(X,B')$ |
| $T' = \text{MAC}(S',B')$ | $B,T \xleftarrow{T_2} B',T'$ |

If $T \neq \text{MAC}(S,B)$: reject ($R_A = \bot$)

Otherwise:

$R_A = \text{Ext}(X,B)$

### Remark 4.

In Protocol 17, registers $Y$, $B'$ are generated uniformly and independently of the state of the protocol at that point.

### Definition 27 (Active attack).

An active attack against PA protocol is described by 3 parameters.

- A c-q state $\rho_{XE}$ (of adversary choice) such that $H_{\min}(X|E)_{\rho} \geq k$.

- A CPTP map $T_1 : \mathcal{H}_E \otimes \mathcal{H}_Y \rightarrow \mathcal{H}_{E'} \otimes \mathcal{H}_Y \otimes \mathcal{H}_{Y'}$.

- A CPTP map $T_2 : \mathcal{H}_{E'} \otimes \mathcal{H}_{B'} \otimes \mathcal{H}_{T'} \rightarrow \mathcal{H}_{E''} \otimes \mathcal{H}_{B'} \otimes \mathcal{H}_{T'} \otimes \mathcal{H}_B \otimes \mathcal{H}_T$.

### Result

**Theorem 10.** For any active attack $(\rho_{XE}, T_1, T_2)$, Protocol 17 is $(k, (\frac{1}{2} - \delta)k, O(\sqrt{\varepsilon}))$-secure as defined in Definition 26 with communication $O(d)$. 

58
Proof. For the purpose of this proof, without any loss of generality, we can consider $T_1$ and $T_2$ to be isometries because tracing out registers after applying an isometry (which amounts to applying a CPTP map) will only weaken the adversary holding the side information. We can assume isometries to be safe and registers ($YY'B'T'BT$) to be classical, since both Alice and Bob when executing the protocol, keep a copy of the registers they send and also make a copy of the registers they receive\footnote{We do not mention it in the protocol or in the analysis, instead we assume and proceed for simplifying the analysis.}. We keep the transcript of the protocol in adversary side information at different stages of the protocol.

Correctness of the protocol follows by observation. Let the adversary choose state $\rho_{XE}$. Let $\rho_{X^ey} = \rho_{X^eE} \otimes \rho_{Y^y}$ be a pure state such that,

$$\rho_{X^eEY^y} = \rho_{X^eE} \otimes \rho_{Y^y} ; \quad H_{\text{min}}(X|E)_{\rho} \geq k ; \quad \rho_Y = Ud,$$

where registers $XY$ are classical with copies $\hat{X}Y$. Note in Protocol 17, register $X$ (is held by both Alice and Bob), register $E$ is the quantum side information with Eve, register $Y$ is generated by Alice in the first step and $\hat{E}$ is the purification inaccessible to any of Alice, Bob or Eve throughout the protocol.

Let $\sigma = T_1(\rho)$. Note $\sigma$ is $(k)$-qpa-state since $H_{\text{min}}(X|E')_{\sigma} \geq k$ and $\sigma_Y = U_{|Y|}$. Using Corollary 7 we get

$$\|\sigma_{SY} - U_{2m} \otimes \sigma_{YY'}\|_1 \leq \varepsilon. \quad (32)$$

Let $\tau$ be a pure state after Alice generates $S$ (with copy $\hat{S}$), Bob generates classical registers $S'B'R_BT'$ (with copies $\hat{S}'B'\hat{R}_B\hat{T}'$). Note,

$$\tau_{XXS'S'} YY'E' = \sigma_{XSS'} YY'E' \otimes Ud. \quad (33)$$

Let $\Theta$ be the final pure state at the end of protocol including purify($R_A$)purify($R_B$) along with their copies purify($R_A$)purify($R_B$).

**Robustness property:** We now show the robustness property of Protocol 17. Let $p = \Pr(Y' \neq Y|_\sigma)$. Based on the value of $p$, we divide our analysis into three parts.

**Case 1 ($p = 0$):** In this case we have $\Pr(S = S')_\tau = 1$. From Eq. (32) and (33) we have

$$\|\tau_{SBYY'} - U_{2m} \otimes Ud \otimes \sigma_{YY'}\|_1 \leq \varepsilon. \quad (34)$$

Let $\hat{\tau}_{SBYY'} = U_{2m} \otimes Ud \otimes \sigma_{YY'}$. Using Fact 7 we get

$$\|\tau_{SBTYY'} - \hat{\tau}_{SBTYY'}\|_1 \leq \varepsilon. \quad (35)$$

Let $\hat{\Theta}$ be the final state if protocol is run on $\hat{\tau}$ instead of $\tau$. Using Fact 7 we get

$$\|\Theta_{SBTBYY'} - \hat{\Theta}_{SBTBYY'}\|_1 \leq \varepsilon.$$
From Claim 14 with the following assignment of registers and isometry,
\[(\rho_{SB'T'}, \rho_{E'}) \leftarrow (\hat{\tau}_{SB'T'}, \hat{\tau}_{YY'E'}), \quad V \leftarrow T_2,\]
we get
\[\Pr \left( P(S, B, T, B') = 1 \right)_{\Theta} \leq \varepsilon.\]
Using Eq. (35) and Fact 11 we get,
\[\Pr \left( P(S, B, T, B') = 1 \right)_{\Theta} \leq 2\varepsilon.\]
Noting,
\[\Pr \left( Q(R_A, R_B) = 1 \right) \leq \Pr \left( P(S, B, T, B') = 1 \right)_{\Theta}\]
we get
\[\Pr \left( Q(R_A, R_B) = 1 \right)_{\Theta} \leq 2\varepsilon.\] (36)
This establishes the robustness property in Case 1.

**Case 2** ($p = 1$): In this case, $\sigma$ is a $(k)$-qnm-state. Since $\text{nmExt}$ is $(k, \varepsilon)$-quantum secure non-malleable extractor (Definition 15), we get
\[\| \sigma_{SS'YY'E'} - U_{2m} \otimes \sigma_{SS'YY'E'} \|_1 \leq \varepsilon.\]
Since $\tau_{SS'YY'E'} = \sigma_{SS'YY'E'} \otimes U_m$, we have
\[\| \tau_{SB'SYY'E'} - U_{2m} \otimes U_m \otimes \sigma_{SS'YY'E'} \|_1 \leq \varepsilon.\]
Let $\hat{\tau}_{SB'SYY'E'} = U_{2m} \otimes U_m \otimes \sigma_{SS'YY'E'}$. Using Fact 7, we get
\[\| \tau_{SB'T'SYY'E'} - U_{2m} \otimes \hat{\tau}_{B'T'SYY'E'} \|_1 \leq \varepsilon.\]
Also note $\hat{\tau}_{BB'} = U_m$. Let $\hat{\Theta}$ be the final state if protocol is run on $\hat{\tau}$ instead of $\tau$. Using Fact 7 we get
\[\| \Theta_{SB'T'BTSYY'E'} - U_{2m} \otimes \hat{\Theta}_{B'T'BTSYY'E'} \|_1 \leq \varepsilon.\]
Using Fact 7 again, we get
\[\| \Theta_{SB'BT} - U_{2m} \otimes \hat{\Theta}_{B'BT} \|_1 \leq \varepsilon.\] (37)
Since $T_2$ is safe on register $B'$, we also have $\hat{\Theta}_{B'} = U_m$. From Claim 15 with the following assignment of registers (below the registers on the left are from Claim 15 and the registers on the right are the registers in this proof)
\[(SB'TB) \leftarrow (\hat{\Theta}_{SB'TB}),\]
we get
\[\Pr \left( P(S, B, T, B') = 1 \right)_{\hat{\Theta}} \leq \varepsilon.\]
Using Eq. (37) and Fact 11 we get,
\[\Pr \left( P(S, B, T, B') = 1 \right)_{\Theta} \leq 2\varepsilon.\]
Noting,
\[ \Pr(Q(R_A, R_B) = 1)_\Theta \leq \Pr(P(S, B, T, B') = 1)_\Theta \]
we get
\[ \Pr(Q(R_A, R_B) = 1)_\Theta \leq 2\varepsilon. \]  \hfill (38)

This establishes the robustness property in Case 2.

**Case 3:** \( 0 < p < 1 \). In the analysis, we consider a pure state \( \tilde{\sigma} \) which is generated from \( \rho \), in the following way:

- Generate \( \sigma = T_1(\rho) \).
- Generate one bit classical register \( C \) (with copy \( \hat{C} \)) such that \( C = 1 \) indicates \( Y \neq Y' \) in state \( \sigma \).
- Conditioned on \( C = 0 \), generate classical register \( Y'' \) (with copy \( \hat{Y}'' \)) such that \( Y'' \neq Y \).
- Conditioned on \( C = 1 \), generate classical register \( Y'' \) (with copy \( \hat{Y}'' \)) such that \( Y'' = Y' \).

Note \( \Pr(Y \neq Y'')_{\tilde{\sigma}} = 1 \) and \( \tilde{\sigma} \) is a \((k)\)-qm-state (by Fact 4). Since \( \text{nmExt} \) is \((k, \varepsilon)\)-quantum secure non-malleable extractor (Definition 15), we get
\[ \|\tilde{\sigma}_{SS'YY'Y''CE'} - U_{2m} \otimes \tilde{\sigma}_{SS'YY'Y''CE'}\|_1 \leq \varepsilon, \]  \hfill (39)
where \( S = \text{nmExt}(X, Y) \) and \( S' = \text{nmExt}(X, Y'') \). Note by construction of state \( \tilde{\sigma} \), we have \( \Pr(C = 1)_{\tilde{\sigma}} = p \). Let \( \tilde{\sigma}^1 = \tilde{\sigma}|(C = 1) \) and \( \sigma^1 = \sigma|(Y \neq Y') \). Thus, from Eq. (39) and Fact 5 we get
\[ \Pr(C = 1)_{\tilde{\sigma}} \|\tilde{\sigma}_{SS'YY'Y''E'}^1 - U_{2m} \otimes \tilde{\sigma}_{SS'YY'Y''E'}^1\|_1 \leq \varepsilon, \]  \hfill (40)
where \( S = \text{nmExt}(X, Y) \), \( S' = \text{nmExt}(X, Y'') \) and \( Y'' \) is a copy of \( Y' \) in \( \tilde{\sigma}^1 \). Noting \( \Pr(C = 1)_{\tilde{\sigma}} = p \) and \( \tilde{\sigma}_{XY'Y''E'}^1 \) is the same state as \( \sigma_{XY'Y''E'}^1 \) (with additional copy of \( Y' \) in \( Y'' \)), from Eq. (40) and using Fact 7 we further get
\[ \|\sigma_{SS'YY'CE'}^1 - U_{2m} \otimes \sigma_{SS'YY'CE'}^1\|_1 \leq \frac{\varepsilon}{p}. \]  \hfill (41)

We further consider a pure state \( \hat{\sigma} \) which is generated from \( \rho \), in the following way:

- Generate \( \sigma = T_1(\rho) \).
- Generate one bit classical register \( C \) (with copy \( \hat{C} \)) such that \( C = 1 \) indicates \( Y \neq Y' \) in state \( \sigma \).

Note by construction of state \( \hat{\sigma} \), we have \( \Pr(C = 0)_{\hat{\sigma}} = 1 - p \) and \( \hat{\sigma} \) is a \((k)\)-qpa-state (by Fact 3). Let \( \hat{\sigma}^0 = \hat{\sigma}|(C = 0) \) and \( \sigma^0 = \sigma|(Y = Y') \). Using Corollary 7 we get
\[ \|\hat{\sigma}_{SYCE'} - U_{2m} \otimes \hat{\sigma}_{SYCE'}\|_1 \leq \varepsilon. \]

Using Fact 5 we get
\[ \|\hat{\sigma}_{SYCE'}^0 - U_{2m} \otimes \hat{\sigma}_{SYCE'}^0\|_1 \leq \frac{\varepsilon}{1 - p}. \]
Noting $\sigma^0_{XY'E'}$ is the same state as $\sigma^1_{XY'E'}$, we get
\[ \|\sigma^0_{SY'E'} - U_{2m} \otimes \sigma^0_{SY'E'}\|_1 \leq \frac{\epsilon}{1-p}, \] (42)

Note,
\[ \sigma_{SS'Y'E'} = (1-p) \cdot \sigma^0_{SS'Y'E'} + p \cdot \sigma^1_{SS'Y'E'}. \] (43)

Let $\Theta^0$ and $\Theta^1$ be the final states if we proceed PA protocol with states $\sigma^0$ and $\sigma^1$ after the first round. Using Eq. (42) and arguments similar to case 1, we get
\[ \Pr (Q(R_A, R_B) = 1)_{\Theta^0} \leq \frac{\epsilon}{1-p} + \epsilon. \] (44)

Similarly, using Eq. (41) and similar arguments of case 2, we get
\[ \Pr (Q(R_A, R_B) = 1)_{\Theta^1} \leq \frac{\epsilon}{p} + \epsilon. \] (45)

Thus, from Eq. (43), (44) and (45), we have
\[ \Pr (Q(R_A, R_B) = 1)_{\Theta} \leq (1-p) \left( \frac{\epsilon}{1-p} + \epsilon \right) + p \left( \frac{\epsilon}{p} + \epsilon \right) = 3\epsilon, \] (46)

i.e. the robustness property in Case 3.

**Extraction property:** We now show the extraction property of PA protocol. From Eq. (33), we have $\tau_{XE'E'S'B'} = \sigma_{XE'E'S'} \otimes U_m$. Consider,
\[ H_{\min}(X|E'E'Y'S')_{\tau} = H_{\min}(X|E'E'Y'S')_{\sigma} \]
\[ \geq H_{\min}(X|E'E')_{\sigma} - |SS'| \] (Fact 4)
\[ \geq H_{\min}(X|E)_{\rho} - |SS'| \] (Fact 3 and $\rho_{XEY} = \rho_{XE} \otimes U_d$
\[ \geq k - 2m \]
\[ \geq (1 - 2\delta)k. \]

Thus, noting $2z = (1 - 2\delta)k$ and Ext is a quantum secure extractor (see Definition 11), we get
\[ \|\tau_{RB'E'Y'S'S'B'} - U_z \otimes \tau_{E'Y'S'S'B'}\|_1 \leq \epsilon. \]

Using Fact 7, we get
\[ \|\tau_{RB'E'Y'T'S'S'B'} - U_z \otimes \tau_{E'Y'T'S'S'B'}\|_1 \leq \epsilon, \]

and further,
\[ \|\Theta_{RB'E'S'} - U_z \otimes \Theta_{E'S'}\|_1 \leq \epsilon, \] (47)

where $\tilde{E}$ denotes the registers held by Eve and transcript of the protocol. Using Fact 7 again, we have
\[ \|\Theta_{RB'E} - U_z \otimes \Theta_{E}\|_1 \leq \epsilon, \] (48)

the extraction property for Bob. From the robustness property of the protocol, i.e. Eq. (36), (38) and (46), we have
\[ \Pr (Q(R_A, R_B) = 1)_{\Theta} \leq O(\epsilon). \]
Thus, \( \Pr((R_A = R_B) \lor (R_A = 1)) \Theta \geq 1 - O(\varepsilon) \). Using Fact \ref{fact:17} and Fact \ref{fact:15}

\[
\| \Theta - \tilde{\Theta} \|_1 \leq O(\sqrt{\varepsilon}) \text{, where } \tilde{\Theta} = \Theta((R_A = R_B) \lor (R_A = 1)) \tag{49}
\]

Let \( C \) be the predicate to indicate \((\text{MAC}(S, B) = T)\). Let \( \tilde{\Theta}^\perp = \tilde{\Theta}|(C = 0) \) and \( \tilde{\Theta}^\not\perp = \tilde{\Theta}|(C = 1) \).

Note \( \Pr(R_A = R_B)\tilde{\Theta}^\not\perp = 1 \). Consider,

\[
\begin{align*}
\| \tilde{\Theta}_{R_A E} - \tilde{\Theta}_{\text{purify}(R_A)\tilde{E}} \|_1 & \leq \| \tilde{\Theta}_{R_A ESS'} - \tilde{\Theta}_{\text{purify}(R_A)\tilde{ESS'}} \|_1 \\
& = (\Pr(C = 1)\tilde{\Theta})\| \tilde{\Theta}_{R_A ESS'} - U_z \otimes \tilde{\Theta}_{\tilde{ESS'}} \|_1 \\
& = (\Pr(C = 1)\tilde{\Theta})\| \tilde{\Theta}_{R_A ESS'} - U_z \otimes \tilde{\Theta}_{\tilde{ESS'}} \|_1 \tag{Fact \ref{fact:16}} \\
& \leq \| \tilde{\Theta}_{R_B \tilde{ESS}'} - U_z \otimes \tilde{\Theta}_{\tilde{ESS}'} \|_1 \tag{Fact \ref{fact:14}} \\
& = \| \tilde{\Theta}_{R_B \tilde{ESS}'} - U_z \otimes \tilde{\Theta}_{\tilde{ESS}'} \|_1 \tag{C can be generated from \tilde{ESS}'} \\
& \leq O(\sqrt{\varepsilon}). \tag{Claim \ref{claim:1} and Eq. \ref{eq:17}} (50)
\end{align*}
\]

\[
\begin{align*}
\| \Theta_{R_A E} - \Theta_{\text{purify}(R_A)\tilde{E}} \|_1 & \leq \| \Theta_{R_A E} - \tilde{\Theta}_{R_A \tilde{E}} \|_1 + \| \tilde{\Theta}_{R_A \tilde{E}} - \Theta_{\text{purify}(R_A)\tilde{E}} \|_1 \\
& \leq O(\sqrt{\varepsilon}) + \| \tilde{\Theta}_{R_A \tilde{E}} - \Theta_{\text{purify}(R_A)\tilde{E}} \|_1 \tag{Triangle inequality} \tag{51} \\
& \leq O(\sqrt{\varepsilon}) + \| \tilde{\Theta}_{R_A \tilde{E}} - \Theta_{\text{purify}(R_A)\tilde{E}} \|_1 + \| \Theta_{\text{purify}(R_A)\tilde{E}} - \Theta_{\text{purify}(R_A)\tilde{E}} \|_1 \tag{Triangle inequality} \tag{Eq. \ref{eq:49}} \\
& \leq O(\sqrt{\varepsilon}) \tag{Eq. \ref{eq:50} and Eq. \ref{eq:49}) (52)
\end{align*}
\]

By our choice of parameters, using Eq. \ref{eq:36}, \ref{eq:38}, \ref{eq:46}, \ref{eq:48}, \ref{eq:52}, the theorem follows. \( \square \)

**Corollary 9.** For any active attack \((\rho_{XE}, T_1, T_2)\), Protocol \ref{protocol:17} is \( (k, (\frac{1}{2} - \delta)E, O(2^{-n^s/2})) \)-secure as defined in Definition \ref{def:26} with communication \( O(n^7) \) as long as \( k \geq \Omega(n^7) \).

**Proof.** Choosing \( \varepsilon = 2^{-n^s} \), the corollary follows from Theorem \ref{thm:10}. \( \square \)

**Corollary 10.** For any active attack \((\rho_{XE}, T_1, T_2)\), Protocol \ref{protocol:17} is \( (k, (\frac{1}{2} - \delta)E, \frac{1}{\text{poly}(n)}) \)-secure as defined in Definition \ref{def:26} with communication \( O(\log^7(n)) \) as long as \( k \geq \Omega(\log^7(n)) \).

**Proof.** Choosing \( \varepsilon = \frac{1}{\text{poly}(n)} \), the corollary follows from Theorem \ref{thm:10}. \( \square \)
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