THE PLANAR ALGEBRA OF DIAGONAL SUBFACTORS

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Abstract. There is a natural construction which associates to a finitely generated, countable, discrete group $G$ and a 3-cocycle $\omega$ of $G$ an inclusion of II$_1$ factors, the so-called diagonal subfactors (with cocycle). In the case when the cocycle is trivial these subfactors are well studied and their standard invariant (or planar algebra) is known. We give a description of the planar algebra of these subfactors when a cocycle is present. The action of Jones’ planar operad involves the 3-cocycle $\omega$ explicitly and some interesting identities for 3-cocycles appear when naturality of the action is verified.

1. Introduction

The theory of subfactors ([10]) has experienced several new developments in the last few years through the introduction of planar algebra technology ([11]). Every subfactor comes with a very rich mathematical object, the standard invariant or planar algebra of the subfactor, which in nice situations is a complete invariant of the subfactor ([20], [21]). It can be described in many interesting ways, as for instance a certain category of bimodules ([18], see also [2]), as lattices of multi-matrix algebras ([19]), or as a planar algebra ([11]). The planar algebra approach is particularly powerful since it allows one to use algebraic-combinatorial methods in conjunction with topological ones to investigate the structure of subfactors. A number of examples of explicit planar algebras associated to subfactors have been computed (see for instance [3], [4], [8], [11], [14], [15], [16]) but there is a need for more concrete examples. This is what we accomplish in this paper. We give a description of the planar algebra of the diagonal subfactors associated to a $G$-kernel.

Let $P$ be a II$_1$ factor and let $\theta_1, \ldots, \theta_n$ be automorphisms of $P$ (we may assume without loss of generality that $\theta_1 = id$). Consider the subfactor $N = \{\sum_{i=1}^{n} \theta_i(x)e_{ii} \mid x \in P\} \subset M = P \otimes M_n(\mathbb{C})$, where $(e_{ij})_{1 \leq i, j \leq n}$ denote matrix units in $M_n(\mathbb{C})$. $N \subset M$ is then called the diagonal subfactor associated to $\{\theta_i\}_{1 \leq i \leq n}$. These subfactors were proposed by Jones in 1985 to provide examples of potentially non-classifiable subfactors, since this construction allows one to translate problems on classification of group actions into problems on subfactors. Popa used them to prove vanishing of 2-cohomology results for cocycle actions of finitely generated, strongly amenable groups on an arbitrary II$_1$ factor ([23]). Ocneanu had proved such a result for cocycle actions of amenable groups on the hyperfinite II$_1$ factor using different techniques ([17]). The diagonal subfactors are of course reducible and have Jones index $n^2$. They provide a wealth of simple examples of infinite depth subfactors whose structure theory is well understood. In particular, the standard

Key words and phrases. G-kernel, group cocycle, planar algebra, planar operad, subfactor.

The authors were supported by NSF under Grant No. DMS-0301173 and DMS-0653717.
invariant or planar algebra of these subfactors has been determined in ([20], [11], [1]).

Let $G$ be the group generated by the images $g_i$ of $\theta_i$, $1 \leq i \leq n$, in $\text{Out } P = \text{Aut } P/\text{Int } P$. Popa showed that analytical properties of these subfactors are reflected in the corresponding properties of the group $G$. For instance, if $P$ is hyperfinite, then the diagonal subfactor is amenable (in the sense of Popa) if and only if $G$ is an amenable group ([20]). The subfactor has property (T) in the sense of Popa if and only if $G$ has property (T) of Kazhdan ([22]). The principal graphs of these subfactors are then Cayley-like graphs of $G$ with respect to the generators $g_1, \ldots, g_n$ and their inverses (see [20] or [1] for the precise statement). The higher relative commutants, Jones projections and conditional expectations have all been worked out in ([20], [11], [1]).

A well-known variation of the diagonal subfactors is obtained as follows (see e.g. [20]). Consider a $G$-kernel, that is an injective homomorphism $\chi$ from a (countable, discrete, finitely generated) group $G$ into $\text{Out } P$. Denote by $\epsilon : \text{Aut } P \to \text{Out } P$ the canonical homomorphism, and let $\alpha : G \to \text{Aut } P$ be a lift of $\chi$ such that $\epsilon \circ \alpha_s = (s)$, for all $s \in G$. It follows that $\alpha_s \alpha_t = \text{Ad}(u(s,t))\alpha_{st}$, for all $s, t \in G$, and some unitaries $u(s, t) \in P$. Associativity of composition of automorphisms implies that $\text{Ad}(u(r,s)u(rs,t)) = \text{Ad}(\alpha_r(\text{Ad}(\alpha_t(\text{Ad}(\alpha_s)))))$. Hence there is a function $\omega : G \times G \times G \to \mathbb{T}$ with $u(r,s)u(rs,t) = \omega(r,s,t)\alpha_r(\text{Ad}(\alpha_t(\text{Ad}(\alpha_s))))u(rs,t)$. It is easy to check that $\omega$ is a 3-cocycle and that its class in $H^3(G, \mathbb{T})$ does not depend on the choices made. One usually denotes the class by $\text{Ob}(G)$ or $\text{Ob}(\chi)$, the obstruction of $\chi$. It is an obstruction to lifting $G$ to an action on the $\text{II}_1$ factor $P$. Clearly, if two $G$-kernels $\chi$ and $\eta$ are conjugate (in $\text{Out } P$), then $\text{Ob}(\chi) = \text{Ob}(\eta)$. It was shown in [17] that $\text{Ob}$ is a complete conjugacy invariant for $G$-kernels if $P$ is the hyperfinite $\text{II}_1$ factor and $G$ is a countable, discrete, amenable group. Note that in general, even if $\text{Ob}(G) = 0$, there may be no lifting of the $G$-kernel to an action on $P$. Connes and Jones found in [7] the first such example of a non-liftable $G$-kernel with vanishing obstruction, where $G$ is a group with property (T). Vanishing of the obstruction is a necessary and sufficient condition for $G$ to lift to a cocycle action on the $\text{II}_1$ factor $P$. See [9], [12], [17], [20], [24] for more on this.

Connes showed that if $G$ is cyclic, one can construct $G$-kernels in $\text{Out } R$ with arbitrary obstructions, where $R$ denotes the hyperfinite $\text{II}_1$ factor ([9]). It was an open problem whether this result is true for more general groups, and Jones showed in [9] that this is indeed the case for $G$ an arbitrary discrete group. Thus, given a discrete group $G$ and a class $\pi \in H^3(G, \mathbb{T})$, there is $G$-kernel $\chi : G \to \text{Out } R$ with $\text{Ob}(\chi) = \pi$. Sutherland constructed $G$-kernels with arbitrary obstructions in non-hyperfinite $\text{II}_1$ factors ([24]).

Given a finitely generated, countable group $G$ and a 3-cocycle $\omega$, we can associate a (hyperfinite) subfactor to $(G, \omega)$ as follows. Let $\chi : G \to \text{Out } P$ be a $G$-kernel with $\text{Ob}(G) = [\omega] \in H^3(G, \mathbb{T})$ (choose for instance $P = R$, and use [9]). Fix generators $\{g_1, \ldots, g_n\}$ of $G$, let $\alpha$ be a lift of $\chi$, and consider the diagonal subfactor associated to the automorphisms $\alpha_{g_1}, \ldots, \alpha_{g_n}$ (let us choose $g_1 = e$, the identity of $G$, and $\alpha_e = \text{id}$). If $\eta$ is another $G$-kernel with lift $\beta$, and automorphisms $\beta_{g_1}, \ldots, \beta_{g_n}$, then the diagonal subfactors associated to $(\alpha_{g_j})_j$ and $(\beta_{g_j})_j$ are isomorphic if and only if there is an automorphism $\theta$ of the underlying $\text{II}_1$ factor $P$ such that $\alpha_{g_{s(i)}} = \beta_{g_j} \theta^{-1}$ mod $\text{Int } P$, where $\pi$ is a permutation of the indices (fixing 1) (see e.g. [23]). Thus, in particular, isomorphism of these diagonal subfactors implies
that \( \text{Ob}(\chi) = \text{Ob}(\eta) \) (up to a possible modification of \( \chi \) by the permutation \( \pi \)). Isomorphism of standard invariants is weaker than isomorphism of subfactors, but we still have the following. If \( (G, \chi) \) and \( (G, \eta) \) are two \( G \)-kernels as above with \( \text{Ob}(\eta) = \text{Ob}(\chi) \), then the standard invariants of the associated diagonal subfactors are isomorphic (see (20), page 228 ff.). The converse is not true due to the fact that group isomorphisms can change the class of a 3-cocycle. If one constructs diagonal subfactors where the automorphisms are repeated with distinct multiplicities, one can show a converse, e.g. if \( G \) is strongly amenable and \( P \) is hyperfinite, using Popa's classification results of amenable subfactors (20, page 229 ff.). Clearly, the 3-cocycle \( \omega \) giving rise to the obstruction of the \( G \)-kernel will appear in the standard invariant of these diagonal subfactors (with cocycle) and the purpose of this paper is to give a precise description of this occurrence.

Here is a more detailed outline of the sections of this paper. We review group cocycles in section 2. In section 3, we define an abstract planar algebra \( P_{(g_i;i \in I)}^\omega \) associated to a finitely generated group \( G \) with a fixed finite generating set \( \{g_i\}_{i \in I} \), and a 3-cocycle \( \omega \in Z^3(G,\mathbb{T}) \). The vector spaces underlying this planar algebra are spanned by multi-indices in \( I^{2n} \) such that the corresponding alternating word on generators and their inverses is the identity in \( G \). The action of Jones’ planar operad is defined explicitly, and of course \( \omega \) appears prominently in this definition. It should be noted that the definition of the action of a tangle involves a labelling of the strings in the tangle whereas the planar algebra description of the group-type subfactors in [3] involved a labelling of boundary segments (called “openings” in [3]) of the internal and external discs of the tangle. We would like to point out that the 3-cocycle \( \omega \) does not appear in our definition of the action of the multiplication, inclusion, Jones projection and right conditional expectation tangles. It does appear in the definition of the action of the left conditional expectation tangles (and hence the rotation tangles). We verify in this section that our definition of the action of tangles is indeed natural with respect to composition of tangles. This takes a little work, but some interesting identities for 3-cocycles appear along the way.

In section 4 we give a model for the higher relative commutants of the diagonal subfactor with cocycle and describe the associated concrete planar algebra. We choose an appropriate basis of the higher relative commutants which allows us to identify this concrete planar algebra with the abstractly defined one in section 3. This isomorphism is obtained in the usual way by constructing a filtered \( \ast \)-algebra isomorphism between the abstract planar algebra of section 3 and the concrete one of section 4, that preserves Jones projection and conditional expectation tangles. The main feature of this planar algebra is the fact that the distinguished basis of the higher relative commutants that we found here, matches with the one coming from the description of the planar algebra as a path algebra associated to the principal graphs of the subfactor (see e.g. [13], [11]). Conversely, we prove that any finite index extremal subfactor whose standard invariant is given by the abstract planar algebra (in Section 3), must necessarily be (isomorphic to) a diagonal subfactor.

2. A brief review of group cocycles

For the convenience of the reader, we recall in this brief section the definition of a cocycle of a group \( G \). \( G \) will denote throughout this article a countable, discrete group, and we will denote the identity of \( G \) by \( e \). Define \( C^n = \text{Fun}(G^n,\mathbb{T}) \) the space of functions from \( G^n \) to \( \mathbb{T} \) and \( \partial^n : C^n \rightarrow C^{n+1} \) by
\[\partial^n(\phi)(g_1, \cdots, g_{n+1}) = \phi(g_2, \cdots, g_{n+1}) \phi(g_1 g_2 g_3, g_4, \cdots, g_{n+1})^{-1} \phi(g_1, g_2 g_3, g_4, \cdots, g_{n+1}) \cdots \]

It follows that \((\partial^{n+1} \circ \partial^n)(\cdot) = 1_{C_{n+2}}\) where 1 denotes the constant function 1. Denote \(\ker(\partial)\) by \(Z^n(G, \mathbb{T})\) (whose elements are called \(n\)-cocycles) and \(\text{Im}(\partial^{n-1})\) by \(B^n(G, \mathbb{T})\) (whose elements are called coboundaries). Note that \(B^n(G, \mathbb{T}) \subset Z^n(G, \mathbb{T})\).

In this paper we will be dealing mostly with a 3-cocycle \(\omega\). Thus \(\omega\) satisfies the identity

\[
(2.1) \quad \omega(g_1, g_2, g_3) \omega(g_1, g_2 g_3, g_4) \omega(g_2, g_3, g_4) = \omega(g_1 g_2, g_3, g_4) \omega(g_1, g_2, g_3 g_4)
\]

We call \(\omega\) normalized if \(\omega(g_1, g_2, g_3) = 1\) whenever either of \(g_1, g_2, g_3\) is \(e\).

Any cocycle \(\omega\) is coboundary equivalent to a normalized cocycle. In particular, \((\omega \cdot \partial^2(\phi))\) is a normalized 3-cocycle where \(\phi \in C^2\) is defined as \(\phi(g_1, g_2) = \omega(g_1, e, e) \varphi(e, e, g_2)\) for all \(g_1, g_2 \in G\).

3. An abstract planar algebra

In this section we will construct an abstract planar algebra which, in section 4, will be shown to be isomorphic to the planar algebra of a diagonal subfactor with cocycle.

Let \(G\) be a countable, discrete group generated by a finite subset \(\{g_i\}_{i \in I}\), and let \(\omega \in Z^3(G, \mathbb{T})\) be a normalized 3-cocycle. We will construct a planar algebra \(P\{g_i; i \in I\}; \omega\) (= \(P\)) as follows. Let \(e\) denote the identity of \(G\). We define first a map \(alt\) from multi-indices \(\prod_{n \geq 0} I^n\) to \(G\) by

\[
\left(\prod_{n \geq 0} I^n\right) \ni \underline{i} = (i_1, \cdots, i_n) \overset{alt}{\longrightarrow} g_{i_1}^{-1} g_{i_2} \cdots g_{i_n}^{-1} = alt(\underline{i}) \in G
\]

where \(alt\) of the empty multi-index is defined to be the identity element of the group. To define the planar algebra we need to define vector spaces \(P_n\) and an action of Jones’ planar operad on these vector spaces. We refer to \(\text{III}\) for the planar algebra terminology used in this paper.

The vector spaces. For \(n \geq 0\), define \(P_n = \{ \underline{i} \in I^{2n} : alt(\underline{i}) = e \} \) if \(n > 0\), \(\subset \) if \(n = 0\).

Action of tangles. Let \(T\) be an \(n_0\)-tangle having internal discs \(D_1, \cdots, D_b\) with colors \(n_1, \cdots, n_b\) respectively (or no internal discs of course). A state \(\sigma\) on \(T\) is a map from \(\{\text{strings in } T\}\) to \(I\) such that \(alt(\sigma|_{\partial D_c}) = e\) for all \(1 \leq c \leq b\) where \(\sigma|_{\partial D_c}\) denotes the element of \(I^{2n_c}\) obtained by reading the elements of \(I\) induced at the marked points on the boundary of \(D_c\) by the strings via the map \(\sigma\), starting from the first marked point and moving clockwise. This has been illustrated in Figure \(\text{IV}\) where \(alt(\sigma|_{\partial D_1})\) and \(alt(\sigma|_{\partial D_2})\) are just the products \(g_{i_1}^{-1} g_{i_2} g_{i_3}^{-1} g_{i_4} g_{i_5}^{-1} g_{i_6}\) and \(g_{i_2}^{-1} g_{i_4} g_{i_5}^{-1} g_{i_3}\) respectively, and are thus required to be the identity element. It is a
consequence that $\text{alt}(\sigma|_{\partial D_0}) (= g_{i_8}^{-1} g_{i_6}^{-1} g_{i_5}^{-1} g_{i_1}^{-1} g_{i_8} = e$ in the figure) holds for the external disc. Let $S(T)$ denote the states on $T$.

In order to define the action $Z_T$ of $T$, it is enough to define the coefficient $\langle Z_T(k^1, \ldots, k^b) | k^0 \rangle$ of $k^0$ in the linear expansion of $Z_T(k^1, \ldots, k^b)$ where $k^c \in I^{2n_c}$ such that $\text{alt}(k^c) = e$ for $1 \leq c \leq b$. For this, we choose a picture $T_1$ in the isotopy class of $T$ and then choose a simple path $p_c$ in $D_0 \setminus [\bigcup_{c=1}^b \text{Int}(D_c)]$ starting from the $\ast$ of $D_0$ to that of $D_c$ for $1 \leq c \leq b$ such that:

(i) $p_c$ intersects the strings of $T_1$ transversally for $1 \leq c \leq b$,

(ii) $p_{c_1}$ and $p_{c_2}$ intersects exactly at the $\ast$ of $D_0$ for $1 \leq c_1 < c_2 \leq b$.

Note that any state $\sigma$ on $T$ gives an element $\sigma|_{p_c} \in I^{m_c}$ obtained by reading the elements of $I$ induced by $\sigma$ at the crossings of the path $p_c$ and the strings along the direction of the path where $m_c$ (necessarily even) is the number of strings cut by $p_c$.

Define

$$\langle Z_T(k^1, \ldots, k^b) | k^0 \rangle = \sum_{\sigma \in S(T) \text{ s.t. } \sigma|_{\partial D_0} = k^d, \sigma|_{\partial D_0} = k^d} \prod_{c=1}^b \lambda_{\sigma|_{p_c}}(k^c)$$

where $\lambda_\omega(\hat{i}) = \prod_{s=1}^n \lambda_{\omega}(\hat{i}, s)$ and

$$\lambda_{\omega}(\hat{i}, s) = \begin{cases} \omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_s), g_{i_s}) & \text{if } s \text{ is odd} \\ \omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_{s-1}), g_{i_s}) & \text{if } s \text{ is even} \end{cases}$$

for $\hat{i} \in I^n$, $\hat{j} \in I^m$. If there is no compatible state on $T$, then we take the coefficient to be 0 and if there is no internal disc in $T$, then the scalar inside the sum is considered to be 1. (Note that $\lambda_\omega(\hat{i})$ depends only on $\text{alt}(\hat{j})$ and $\hat{j}$)

We need to show first that the multi-linear map $Z_T$ is well-defined. Two configurations of paths $\{p_c\}_{c=1}^b$ and $\{p'_c\}_{c=1}^b$ in $T_1$ can be obtained from each other using

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1 Recall $\ast$ of a disc $D$ is a point chosen on the boundary of $D$ strictly between the last and the first marked points, moving clockwise.
a finite sequence of the following moves:

I. isotopy

\[ A \xrightarrow{p_c} B \sim A \xrightarrow{p_c} B \]

II. cap-sliding moves

\[ A \xrightarrow{p_c} B \sim A \xrightarrow{p_c} B \]

III. disc-sliding moves

\[ D_0 \xrightarrow{p_c} D_d \sim D_0 \xrightarrow{p_d} D_d \]

IV. rotation moves

\[ D_0 \xrightarrow{p_c} D_c \sim D_0 \xrightarrow{p_c} D_c \]

It is enough to check that the definition of the action is independent under each of the above moves. Invariance under isotopy moves are the easiest to check since \( \sigma|_{p_c} = \sigma|_{p'_c} \) for all \( c \in \{1, \ldots, b\} \). To see invariance under the remaining three moves, we show that \( \text{alt}(\sigma|_{p_c}) = \text{alt}(\sigma|_{p'_c}) \) for \( 1 \leq c \leq b \). For a cap-sliding move, note that the cap induces the same index at the two consecutive crossing with the path but after applying the \( \text{alt} \) map, the corresponding group elements cancel each other since they inverses of each other. For the disc-sliding (resp., rotation moves), the invariance follows from the fact that \( \text{alt}(\sigma|_{\partial D_d}) = e \) (resp., \( \text{alt}(\sigma|_{\partial D_c}) = e \)).
We will show next that the action is compatible with composition of tangles.

**Action is natural with respect to composition of tangles.** Let $S$ be an $m_0$-tangle containing the internal discs $D_1', \ldots, D_a'$ with colors $m_1, \ldots, m_a$ and $T$ be an $m_1$-tangle containing internal discs $D_1, \ldots, D_b$ with colors $n_1, \ldots, n_b$. Let $D_0'$ and $D_0$ denote the external discs of $S$ and $T$ respectively. We need to show that for all $j^c \in I^{2m_1}$ and $j^c' \in I^{2n_1}$ where $c' \in \{0, 2, \ldots, a\}$ and $c \in \{1, \ldots, b\}$

\[
(3.1) \quad \langle Z_S(Z_T(j^1, \ldots, j^b), \ldots, j^a) \rangle^0 = \langle Z_{S \circ_D T}(j^1', \ldots, j^b', \ldots, j^a) \rangle^0
\]

The left-hand side of $(3.1)$ can be expanded as

\[
\sum_{\substack{j^c \in I^{2m_1} \text{ s.t. } \alpha(j^c) = c \in \{1, \ldots, b\} \text{ for } c' \in \{0, 2, \ldots, a\} }} \prod_{c' = 1}^{a} \lambda_{\sigma|_{D_{c'}^{j_{c'}}}}(j^c') \left( \prod_{c_1 = 1}^{b} \lambda_{\tau|_{D_{c}^{j_{c}}}}(j^c) \right)
\]

where $p_{c'}$'s and $p_c$'s are paths in the tangles $S$ and $T$ respectively, required to define their actions. For the action of $S \circ_D T$, we consider the paths $p_{c'}$ for $2 \leq c' \leq a$ and $p_c$ for $1 \leq c \leq b$. (Strictly speaking, one has to disjointify the $p_{c'}$-portion of the paths $(p_{c'} \circ p_c)$ for different values of $c$ in order to define the action of $S \circ_D T$.) So, the right-hand side of $(3.1)$ becomes

\[
\sum_{\gamma \in S(S \circ_D T) \text{ s.t. } \gamma|_{D_{c'}^{j_{c'}}} = j^c' \text{ for } c' \in \{0, 2, \ldots, a\}, \gamma|_{D_{c}^{j_{c}}} = j^c \text{ for } c \in \{1, \ldots, b\} } \prod_{c' = 2}^{a} \lambda_{\gamma|_{D_{c'}^{j_{c'}}}}(j^c') \left( \prod_{c_1 = 1}^{b} \lambda_{\gamma|_{D_{c}^{j_{c}}}}(j^c) \right)
\]

Observe that \( \{ (\sigma, \tau) \in S(S) \times S(T) : \sigma|_{D_{c'}^{j_{c'}}} = j^c' \text{ for } c' \in \{0, 2, \ldots, a\}, \tau|_{D_{c}^{j_{c}}} = j^c \text{ for } 1 \leq c \leq b, \sigma|_{D_0'} \tau|_{D_0} = \tau|_{D_0} \sigma \} \)

is clearly in bijection with \( \{ \gamma \in S(S \circ_D T) : \gamma|_{D_{c'}^{j_{c'}}} = j^c' \text{ for } c' \in \{0, 2, \ldots, a\}, \gamma|_{D_{c}^{j_{c}}} = j^c \text{ for } 1 \leq c \leq b \} \). A bijection is obtained by sending $(\sigma, \tau)$ to the state defined by $\sigma$ (resp. $\tau$) on the $S$-part (resp. $T$-part) of $S \circ_D T$ and the well-definedness of such a state is a consequence of the condition $\sigma|_{D_0'} = \tau|_{D_0}$; we denote this state by $\sigma \circ \tau$. If these sets are empty, equation $(3.1)$ holds trivially since both sides have value 0. Let us assume that the sets are nonempty. It is enough to prove for $\sigma \in S(S)$ and $\tau \in S(T)$ such that $\sigma|_{D_{c'}^{j_{c'}}} = j^c'$ for $0 \leq c' \leq a$ and $\tau|_{D_{c}^{j_{c}}} = j^c$ for $1 \leq c \leq b$ and $\tau|_{D_0} = j^1$ we have

\[
(3.2) \quad \lambda_{\sigma|_{D_1'}^{j^1}} \prod_{c_1 = 1}^{b} \lambda_{\tau|_{D_1'}^{j^c}}(j^c) = \prod_{c_1 = 1}^{b} \lambda_{(\sigma \circ \tau)|_{D_1'}^{j^c}}(j^c)
\]

We prove this in two cases.

**Case 1**: $T$ has no internal disc or closed loop, that is, $T$ is a Temperley-Lieb diagram. Then the right-hand side of equation $(3.2)$ is 1. It remains to show
\( \lambda_{\sigma | \mu_1}(1^1) = 1 \). This follows from the next lemma and the fact \( i^1 = Z_T(1) \) is a sequence of non-crossing matched pairings of indices from \( I \).

**Lemma 3.1.** If \( i \in I, j \in I^m, \hat{i} = (i_1, \ldots, i_n) \in I^n \) and \( 0 \leq s \leq n \), then we have
\[
\lambda_{\hat{i}}(i) = \lambda_{\hat{i}}(i_1, \ldots, i_s, i, i, i, i, i, i_{s+1}, \ldots, i_n).
\]

**Proof:** Note that
\[
\lambda_{\hat{i}}(i) = \lambda_{\hat{i}}((i_1, \ldots, i_s, i, i, i_{s+1}, \ldots, i_n), r) \text{ for } 1 \leq r \leq s, \text{ and }
\lambda_{\hat{i}}(i) = \lambda_{\hat{i}}((i_1, \ldots, i_s, i, i, i, i, i, i_{s+1}, \ldots, i_n), r + 2) \text{ for } s + 1 \leq r \leq n.
\]

We compute then,
\[
\lambda_{\hat{i}}(i_1, \ldots, i_s, i, i, i, i, i, i_{s+1}, \ldots, i_n), s + 1) = \lambda_{\hat{i}}((i_1, \ldots, i_s, i, i, i, i, i, i, i_{s+1}, \ldots, i_n), s + 2) \]
\[
= \{ \omega(alt(\lambda_{\hat{i}}(i_1, \ldots, i_s, i, i, i, i, i, i, i_{s+1}, \ldots, i_n), r + 1)) \} \text{ if } s \text{ is even}
\]
\[
= \{ \omega(alt(\lambda_{\hat{i}}(i_1, \ldots, i_s, i, i, i, i, i, i, i_{s+1}, \ldots, i_n), r + 1)) \} \text{ if } s \text{ is odd}
\]
\[
= 1.
\]

**Remark 3.2.** In Lemma 3.1, if \( \hat{i} \) is a sequence of indices with non-crossing matched pairings, then we can apply the lemma several times to reduce all the consecutive matched pairings to get \( \lambda_{\hat{i}}(i) = 1 \). □

**Case 2:** \( T \) has at least one internal disc. Any unlabelled tangle \( T \) can be expressed as composition of elementary annular tangles of four types as described in \( S \), namely, capping, cap-inclusion, left-inclusion and disc-inclusion tangles. It is enough to prove equation (3.2) for any tangle \( S \) and compatible tangle \( T \) in \( E \) (= the set of all elementary tangles). If \( T \) is of capping or cap-inclusion type annular tangle, the proof directly follows from Lemma 3.1 and is left to the reader.

If \( T \) is a left-inclusion annular tangle, equation (3.2) is implied from the following lemma.

**Lemma 3.3.** For all \( \hat{i} \in I^{2n}, \hat{j} \in I^m \) and \( \hat{k} \in I^{2m} \), such that \( alt(\hat{i}) = e \), we have
\[
\lambda_{(\hat{k}, \hat{j})}(\hat{i}) = \lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}) \lambda_{\hat{i}}(i)
\]
where \( \hat{j} \) is the sequence of indices from \( \hat{i} \) in the reverse order.

**Proof:** We rearrange the terms of the right-hand side of the identity in the lemma in the following way:
\[
\lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}) \lambda_{\hat{i}}(i) = \left( \prod_{r=1}^{2n} \lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}), r \right) \lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}, 2(m + 2n) - r + 1) \left( \prod_{s=1}^{2m} \lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}, 2n + s) \lambda_{\hat{i}}(i, s) \right)
\]
Let \( \hat{i} = (i_1, \ldots, i_{2m}) \) and \( \hat{j} = (j_1, \ldots, j_{2m}) \). Note that for \( 1 \leq r \leq 2n \),
\[
\lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}, 2(m + 2n) - r + 1) = \{ \omega(alt(\hat{k}), alt(j_1, \ldots, j_r), g_{j_r}) \} \text{ if } r \text{ is odd}
\]
\[
= \{ \omega(alt(\hat{k}), alt(j_1, \ldots, j_{r-1}), g_{j_r}) \} \text{ if } r \text{ is even}
\]
\[
= \lambda_{\hat{k}}(\hat{i}, \hat{j}, \hat{k}, r)
\]
since \( \text{alt}(\cdot) = e \) and \( \text{alt}(\hat{j}, \hat{i}, \hat{j}) = \text{alt}(\hat{j}) \text{alt}(\hat{j})(\text{alt}(\hat{j}))^{-1} = e \). Thus the first product of the terms in the rearrangement vanishes. For the second product, if \( s \in \{1, \cdots, 2m\} \) is odd, then

\[
\lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2n + s) \lambda_{s}(\hat{i}, s)
\]

\[
= \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}) \text{alt}(i_{1}, \cdots, i_{s}), g_{i_{s}}) \mathcal{V}(\text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s}), g_{i_{s}})
\]

\[
= \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s-1})) \mathcal{V}(\text{alt}(k), \text{alt}(i_{1}, \cdots, i_{s}), g_{i_{s}}).
\]

\[
\omega(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s}))
\]

\[
= \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s-1})) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}) \omega(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s}))
\]

where we used the defining equation (2.1) of the 3-cocycle \( \omega \) for the second equality.

Similarly, if \( s \in \{1, \cdots, 2m\} \) is even, then

\[
\lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2n + s) \lambda_{s}(\hat{i}, s)
\]

\[
= \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s-1})) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}) \omega(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{s}))
\]

Thus,

\[
\prod_{s=1}^{2m} \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2n + s) \lambda_{s}(\hat{i}, s)
\]

\[
= \prod_{t=1}^{m} \left( \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2n + 2t - 1) \lambda_{s}(\hat{i}, 2t - 1) \right) \left( \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2n + 2t) \lambda_{s}(\hat{i}, 2t) \right)
\]

\[
= \prod_{t=1}^{m} \left( \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{2t-2})) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2t - 1) \right) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2t) \omega(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{2t}))
\]

\[
= \mathcal{V}(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{2t})) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2t - 1) \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), 2t) \omega(\text{alt}(k), \text{alt}(\hat{j}), \text{alt}(i_{1}, \cdots, i_{2t}))
\]

\[
= \lambda_{k}(\hat{j}, \hat{i}, \hat{j})
\]

since \( \text{alt}(\hat{i}) = e \) and \( \omega \) is normalized. \( \square \)

**Remark 3.4.** The proof of Lemma 3.3 also implies the following identity:

\[
\lambda_{k}(\hat{j}, \hat{i}, \hat{j}) = \prod_{s=2n+1}^{2m+2n} \lambda_{k}(\hat{j}, \hat{i}, \hat{j}), s)
\]

\( \square \)

Now, suppose \( T \) is a disc-inclusion tangle as shown in Figure 2. Note that \( n_{1} = m_{1} \). Without loss of generality, we can assume that \( T \) be given by the following picture in which we also indicate the paths \( p_{1} \) and \( p_{2} \).

Observe that since \( p_{1} \) does not intersect any string, we have that \( \tau|_{p_{1}} \) is the empty multi-index. So it is enough to prove

\[
(3.3) \quad \lambda_{\sigma|p_{1}}(\hat{j}_{1}) \lambda_{\tau|p_{2}}(\hat{j}_{2}) = \lambda_{\sigma|p_{1}}(\hat{j}_{1}) \lambda_{\sigma|p_{1}, \tau|p_{2}}(\hat{j}_{2})
\]

Let us denote \( \tau|_{p_{2}} \) by \( k \in I^{2r} \). Since \( \tau \) is a state, the following relations clearly follow from Figure 2
(i) $i_s^1 = j_s^1$ for $1 \leq s \leq 2r$ and $2r + n_2 + 1 \leq s \leq 2n_1$,  
(ii) $i_{2r+s}^1 = j_s^2$ for $1 \leq s \leq n_2$,  
(iii) $i_s^2 = k_s = j_s^1$ for $1 \leq s \leq 2r$,  
(iv) $j_{2r+s}^2 = j_{2n_2-s+1}^2$ for $1 \leq s \leq n_2$.

We now express $\lambda_{\sigma|_{\nu_1}^t}(\underline{1})$ as a product of three terms with which we work separately.

$$\lambda_{\sigma|_{\nu_1}^t}(\underline{1}) = \left(\prod_{s=1}^{2r} \lambda_{\sigma|_{\nu_1}^t}(\underline{1}\cdot s) \right) \left(\prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|_{\nu_1}^t}(\underline{1}\cdot s) \right) \left(\prod_{s=2r+n_2+1}^{2n_1} \lambda_{\sigma|_{\nu_1}^t}(\underline{1}\cdot s) \right)$$

First term: For $1 \leq s \leq 2r$,

$$\lambda_{\sigma|_{\nu_1}^t}(\underline{1}\cdot s) = \left\{ \begin{array}{ll} \varnothing(alt(\sigma|_{\nu_1}^t), alt(i_1^1, \ldots, i_s^1), g_i^1) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu_1}^t), alt(i_1^1, \ldots, i_{s-1}^1), g_i^1) & \text{if } s \text{ is even} \end{array} \right.$$  

Second term: For $2r + 1 \leq s \leq 2r + n_2$,

$$\lambda_{\sigma|_{\nu_1}^t}(\underline{1}\cdot s) = \left\{ \begin{array}{ll} \varnothing(alt(\sigma|_{\nu_1}^t), alt(i_1^1, \ldots, i_s^1), g_i^1) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu_1}^t), alt(i_1^1, \ldots, i_{s-1}^1), g_i^1) & \text{if } s \text{ is even} \end{array} \right.$$  

Figure 2. Disc inclusion tangle
Third term: Note that

\[(v) \quad alt(i^1_{2r+1}, \ldots, i^1_{2r+n_2}) = alt(j^2_1, \ldots, j^2_{n_2}) \text{ (using (iii))} = alt(j^2_{2n_2}, \ldots, j^2_{n_2+1}) \quad \text{(since } alt(\underline{2}) = e) = alt(j^1_{2r+1}, \ldots, j^1_{2r+n_2}) \text{ (using (iv))}.\]

Thus, for \(2r + n_2 + 1 \leq s \leq 2n_1\),

\[\lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) = \begin{cases} \omega(alt(\sigma|_{\nu}^1), alt(i^1_1, \ldots, i^1_{2r+1}, g_{i^1_1}) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu}^1), alt(i^1_1, \ldots, i^1_{2r+1}, g_{i^1_1}) & \text{if } s \text{ is even} \\ \omega(alt(\sigma|_{\nu}^1), alt(j^1_1, \ldots, j^1_{2r+1}, g_{j^1_1}) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu}^1), alt(j^1_1, \ldots, j^1_{2r+1}, g_{j^1_1}) & \text{if } s \text{ is even} (using (i) and (v)) \end{cases} \]

Combining the three terms, we get

\[\lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) = \left( \prod_{s=1}^{2r} \lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|_{\nu}^1} ((\underline{k}, \underline{j}^2, \underline{K}), s) \right) \left( \prod_{s=2r+n_2+1}^{2n_1} \lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) \right) \]

Now, for \(2r + 1 \leq s \leq 2r + n_2\), compute

\[\lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) = \begin{cases} \omega(alt(\sigma|_{\nu}^1), alt(j^1_1, \ldots, j^1_{2r+1}, g_{j^1_1}) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu}^1), alt(j^1_1, \ldots, j^1_{2r+1}, g_{j^1_1}) & \text{if } s \text{ is even} \\ \omega(alt(\sigma|_{\nu}^1), alt(j^2_{2n_2+2r-s+1}, g_{j^2_{2n_2+2r-s+1}}) & \text{if } s \text{ is odd} \\ \omega(alt(\sigma|_{\nu}^1), alt(j^2_{2n_2+2r-s+1}, g_{j^2_{2n_2+2r-s+1}}) & \text{if } s \text{ is even} \end{cases} \]

(since \(alt(\underline{2}) = e\))

\[= \lambda_{\sigma|_{\nu}^1} ((\underline{k}, \underline{j}^2, \underline{K}), 2n_2 + 4r - s + 1).\]

Hence, we obtain

\[\lambda_{\sigma|_{\nu}^1} (\underline{1}^1, s) = \lambda_{\sigma|_{\nu}^1} (\underline{1}^1) \left( \prod_{s=2r+n_2+1}^{2n_1} \lambda_{\sigma|_{\nu}^1} ((\underline{k}, \underline{j}^2, \underline{K}), s) \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|_{\nu}^1} ((\underline{k}, \underline{j}^2, \underline{K}), s) \right) \]

(by Remark 3.4).
We can now proceed with the proof of equation (3.3):
\[
\lambda_{\sigma|\iota_1}((1)^1) \lambda_{\tau|\iota_2}((1)^2) = \lambda_{\sigma|\iota_1}((1)^1) \lambda_{\sigma|\iota_1}((2)^2, \overline{\lambda}) \lambda_{\sigma|\iota_2}(1^2)
\]

where we applied Lemma 3.3 for the last identity. This completes the proof that the action of tangles defined above is compatible with composition of tangles (called naturality of composition in [11]). Hence, \(P(g; i \in I) \omega\) is a planar algebra.

We will define next a \(*\)-structure on \(P(g; i \in I) \omega\). Note that if \(\overline{1} \in I^{2n}\), then \(alt(\overline{1}) = e\) if and only if \(alt(\overline{1}) = e\). Extend the operation \(\sim\) conjugate linearly to define a \(*\)-structure on \(P_n(g; i \in I) \omega\). Clearly, this is an involution. We need to check whether the action of tangles preserves \(\ast\), that is, \(Z_T \circ (\ast \times \cdots \times \ast) = \ast \circ Z_T\). It is enough to check this identity when \(T\) has no internal discs or closed loops, and when \(T\) is an elementary annular tangle.

If \(T\) has no internal discs or closed loops, then it is a Temperley-Lieb diagram and hence \(Z_T\) is the sum of all sequences of indices from \(I\) which have non-crossing matched pairings where the position of the pairings are given by the numberings of the marked points on the boundary of \(T\) which are connected by a string. Now, in the tangle \(T^\ast\), the \(m\)-th and the \(n\)-th marked points are connected by a string if and only if the \(m\)-th and the \(n\)-th marked points starting from the last point in \(T\) reading anticlockwise, are connected. So, \(Z_T^\ast\) is indeed the sum of all sequences featuring in the linear expansion of \(Z_T\) in the reverse order (that is, applying \(\sim\)).

If \(T\) is an elementary annular tangle of capping (resp. cap-inclusion) type with \(m\)-th and \((m+1)\)-th marked points of the internal (resp. external) disc being connected by a string, then \(T^\ast\) is also the same kind of elementary annular tangle but the ‘capping’ occurs at the \(m\)-th and \((m+1)\)-th marked points of the internal (resp. external) disc starting from the last point reading anticlockwise. The identity easily follows from this observation.

If \(T\) is an elementary tangle of left-inclusion type, then \(T = T^\ast\). The identity will then follow from the next lemma.

Lemma 3.5. If \(\overline{j} \in I^m\) and \(\overline{i} = (i_1, \cdots, i_{2n}) \in I^{2n}\) such that \(alt(\overline{i}) = e\), then \(\lambda_{\overline{j}}(\overline{i}) = X_{\overline{j}}(\overline{i})\).

Proof: The proof is an immediate consequence of \(alt(i_1, \cdots, i_s) = alt(i_{2n}, \cdots, i_{s+1})\), which holds since \(alt(\overline{i}) = e\).

Lastly, if \(T\) is an elementary tangle of disc-inclusion type given by Figure 2, then we need to show \(\langle Z_T, (\overline{j}), \overline{k} \rangle = \langle Z_T, (\overline{i}), \overline{\iota} \rangle\) for \(\overline{i}, \overline{j}, \overline{k} \in I^{2n}\). \(\overline{j}, \overline{k}\) define a state on \(T\) if and only if \(\overline{j}, \overline{k}\) define the same on \(T^\ast\). If they fail to define a state, then both sides are zero. If they define a state, then the scalars appearing on both sides can be made equal by applying Lemma 3.5.

Remark 3.6. If \(\omega\) is trivial, that is, a coboundary, then \(P(g; i \in I) \omega\) is isomorphic to example 2.7 in [11]. Jones constructed this example of a planar algebra by considering a certain subspace of the tensor planar algebra (TPA) over the vector space with the indexing set \(I\) as a basis. He then showed that this subspace is closed under the TPA-action of tangles, thus showing that the subspace is indeed a planar algebra. One can view our planar algebra \(P(g; i \in I) \omega\) as a subspace of the
TPA in an obvious way but the action of planar tangles induced by TPA will not be the same as our action which involves the extra data of a 3-cocycle. It does not seem clear if our planar algebra \( P^{(g_i : i \in I), \omega} \) can be viewed as a planar subalgebra of the tensor planar algebra over the vector space with basis \( I \) if \( \omega \) is nontrivial.

4. THE PLANAR ALGEBRA OF THE DIAGONAL SUBFACTOR WITH COCYCLE

In this section we will compute the relative commutants of the diagonal subfactor associated to a \( G \)-kernel. We will determine the filtered *-algebra structure, Jones projections and the conditional expectations. Note that some of this can already be found in [20, [11]. The main point here is that we are able to choose an appropriate basis of the higher relative commutants such that the action of planar tangles on these allows us to identify this concrete planar algebra with the abstract one defined in the previous section.

Let \( N \) be a \( \text{II}_1 \) factor, \( I \) be a finite set and for \( i \in I \), choose \( \theta_i \in \text{Aut} N \). Set \( M = M_I \otimes N \) where \( M_I \) denotes the matrix algebra whose rows and columns are indexed by \( I \). As in the introduction, consider the diagonal subfactor \( N \mapsto M \) given by

\[
N \ni x \mapsto \sum_{i \in I} E_{i,i} \otimes \theta_i(x) \in M
\]

that is, an element \( x \) of \( N \) sits in \( M \) as a diagonal matrix whose \( i \)th diagonal is \( \theta_i(x) \). If we have another collection of automorphisms \( \theta'_i \in \text{Aut} N \) for \( i \in I \) such that \( \theta_i = \theta'_i \mod \text{Int} N \), for all \( i \in I \) (up to a permutation of \( I \)), then the associated diagonal subfactors are isomorphic. It is therefore natural to associate a diagonal subfactor to a collection of elements \( g_i \in \text{Out} N, i \in I \). We consider the subgroup \( G = \langle g_i : i \in I \rangle \) of \( \text{Out} N \), which can be viewed as a \( G \)-kernel in the obvious way, and choose a lift

\[
G \ni g \mapsto \alpha_g \in \text{Aut} N
\]

such that \( \alpha_e = \text{id}_N \). Set \( \alpha_i = \alpha_{g_i} \) for all \( i \in I \). Consider the diagonal subfactor \( N \subset M = M_I \otimes N \) where the \( i \)th diagonal entry of an element of \( N \) viewed in \( M \) is twisted by the action of \( \alpha_i \). The index of this subfactor is \( |I|^2 \). Set \( M_n = M_{I^{n+1}} \otimes N \) for \( n \geq 0 \). We will often identify \( E_{i,j} \otimes E_{k,l} \in M_{I^m} \otimes M_{I^n} \) with \( E_{i,j} \otimes E_{k,l} \in M_{I^{m+n}} \) for \( i,j \in I^m \) and \( k,l \in I^n \). Now, \( M_{n-1} \) is included in \( M_n \) in the following way:

\[
M_{n-1} = M_{I^n} \otimes N \ni \sum_{i,j} E_{i,j} \otimes \psi_n(x) \mapsto \sum_{i,j} E_{i,j} \otimes \psi_n(x) \in M_{I^n} \otimes M_I \otimes N = M_{I^{n+1}} \otimes N = M_n
\]

for all \( i,j \in I^n \) and \( x \in N \) where \( \psi_n : N \to M_I \otimes N \) is defined as:

\[
\psi_n(x) = \begin{cases} 
\sum_{k \in I} E_{k,k} \otimes \alpha_k(x) & \text{if } n \text{ is even} \\
\sum_{k \in I} E_{k,k} \otimes \alpha_k^{-1}(x) & \text{if } n \text{ is odd}
\end{cases}
\]

It is easy to check (see [20, [11]) that \( N \subset M \subset M_1 \subset M_2 \subset \cdots \) is isomorphic to the Jones tower of \( \text{II}_1 \) factors associated to \( N \subset M \) where the Jones projections and conditional expectations are given by:

\[
e_n = |I|^{-1} \sum_{k \in I^{n+1}, i,j \in I} E_{i,j} \otimes \alpha_k^{-1}(x) \in M_n
\]

\[
\mathbb{E}_{M_{n-1}}^{M_n}(E_{i,k} \otimes x) = \delta_{k,l} |I|^{-1} E_{i,j} \otimes \alpha_k^{-1}(x)
\]
for all $i,j \in I^n$, $k,l \in I$, $x \in N$. Moreover, \( \{ \sqrt{|I'|}(E_{i,j} \otimes 1) : i,j \in I \} \) forms a Pimsner–Popa basis of $M$ over $N$. This basis will be used to write the conditional expectation of commutant of $N$ onto the commutant of $M$ (see [2]).

To find the relative commutant $N' \cap M_{n-1}$, first note that $N$ is included in $M_{n-1}$ by the following map:

\[
N \ni x \mapsto \sum_{i \in I^n} E_{i,i} \otimes \alpha^{-1}_{n}(i)(x) \in M_{n-1}
\]

where $\alpha^{-1}_{n}(i) = \alpha^{-1}_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n-1}}$ for all $i = (i_1, \ldots, i_n) \in I^n$. Now, if $\sum_{i,j \in I^n} x_{i,j} (E_{i,j} \otimes 1) \in N' \cap M_{n-1}$, then

\[
\sum_{i,j \in I^n} x_{i,j} (E_{i,j} \otimes 1) y = y \sum_{i,j \in I^n} x_{i,j} (E_{i,j} \otimes 1) \text{ for all } y \in N
\]

\[
\iff x_{i,j} (\alpha^{-1}_{n}(i)\alpha^{-1}_{n}(j)) (y) = y x_{i,j} \text{ for all } y \in N, \ i,j \in I^n
\]

\[
\iff x_{i,j} \alpha^{-1}_{n}(i,j) = y x_{i,j} \text{ for all } y \in N, \ i,j \in I^n
\]

Thus, for $i,j \in I^n$, if $x_{i,j} \neq 0$, then $x_0 = \frac{x_{i,j}}{\|x_{i,j}\|} \in U(N)$ and $Ad x_0 \circ \alpha^{-1}_{n}(i,j) = \text{id}_N$ which implies $\alpha^{-1}(i,j) = e$. Similarly, if there exist $i,j \in I^n$ such that $\alpha^{-1}(i,j) = e$, then $u(E_{i,j} \otimes 1) \in N' \cap M_{n-1}$ where $u \in U(N)$ satisfies $Ad u \circ \alpha^{-1}(i,j) = \text{id}_N$.

Thus,

\[
N' \cap M_{n-1} = \text{span} \left\{ u(E_{i,j} \otimes 1) \in M_{n-1} \mid i,j \in I^n \text{ and } u \in U(N) \text{ s.t. } Ad u \circ \alpha^{-1}(i,j) = \text{id}_N \right\}
\]

The elements in this set do not yet form a basis since the unitary $u$ can be modified by a scalar of absolute value 1. To get a good basis of $N' \cap M_{n-1}$ we need to choose $u$ in such a way that the planar algebra associated to $N \subset M$ can easily be identified with the abstract one defined in Section [3].

We now digress a little bit to set up some notations. Let $u : G \times G \rightarrow U(N)$ be a unitary defined by

\[
\alpha_{g_1, g_2} = Ad u(g_1, g_2) \circ \alpha_{g_1, g_2} \text{ for all } g_1, g_2 \in G
\]

such that $u(g_1, g_2) = 1$ whenever either of $g_1$ or $g_2$ is $e$. For $i = (i_1, \ldots, i_n) \in I^n$, define

\[
v_m(i) = \begin{cases} u^* \alpha^{-1}(i_1, \ldots, i_m, g_m) & \text{if } m \text{ is odd} \\ u(\alpha^{-1}(i_1, \ldots, i_{m-1}, g_{m-1})) & \text{if } m \text{ is even} \end{cases}
\]

and set $v(i) = v_1(i) \cdots v_m(i)$. Next, we prove a several useful lemmas involving $v$. The first lemma motivates our choice of the basis.

**Lemma 4.1.** $\alpha^{-1}_{n}(i) = Ad v(i) \circ \alpha^{-1}_{n}(i)$ for all $i \in I^n$.

**Proof:** Using the definition of $u$, note that for all $m \geq 1$,

\[
Ad v_m(i) = \begin{cases} \alpha^{-1}(i_1, \ldots, i_{m-1}, g_m) & \text{if } m \text{ is odd} \\ \alpha(i_1, \ldots, i_{m-1}, g_{m-1}) \alpha^{-1}(i_1, \ldots, i_{m-1}) & \text{if } m \text{ is even} \end{cases}
\]

Hence

\[
Ad v(i) = Ad v_1(i) \cdots Ad v_n(i) = \alpha_{i_1}^{-1} \alpha_{i_2}^{-1} \cdots \alpha_{i_n}^{-1} \alpha_{\alpha^{-1}(i_1, \ldots, i_n)}^{-1} = \alpha^{-1}_{n}(i)\alpha^{-1}_{n}(i).
\]

\[\square\]
Lemma 4.2. For any \( \underline{k} = (k_1, \ldots, k_{2n}) \in I^{2n} \) such that alt(\( \underline{k} \)) = e, we have \( v(\underline{k})v(\underline{k}) = 1 \).

Proof: First we expand \( v(\underline{k}) \) and \( v(\underline{k}) \) into products of unitaries arising from the definition of \( v \), and then we consider that the product of the \( p \)-th unitary of \( v(\underline{k}) \) from the right and \( p \)-th unitary of \( v(\underline{k}) \) from the left, that is,

\[
v_{2n-p+1}(\underline{k})v_p(\underline{k}) = \begin{cases} u(alt(k_{2n}, \ldots, k_{p+1}), k_p) u^*(alt(k_1, \ldots, k_p), k_p) & \text{if } p \text{ is odd } \\ u^*(alt(k_{2n}, \ldots, k_p), k_p) u(alt(k_1, \ldots, k_{p-1}), k_p) & \text{if } p \text{ is even } \\ \end{cases}
\]

for \( 1 \leq p \leq 2n \).

\( \square \)

Lemma 4.3. For \( \underline{i} = (i_1, \ldots, i_n) \), \( \underline{j} = (j_1, \ldots, j_n) \), \( \underline{k} = (k_1, \ldots, k_n) \in I^n \) such that alt(\( \underline{j} \underline{k} \)) = e = alt(\( \underline{i} \underline{k} \)), we have \( v(\underline{i} \underline{j}) v(\underline{i} \underline{k}) = v(\underline{i} \underline{k}) \).

Proof: Using an argument similar to the proof of Lemma 4.2, one can prove that the product of the \( p \)-th unitary of \( v(\underline{i} \underline{j}) \) from the right and the \( p \)-th unitary of \( v(\underline{i} \underline{k}) \) from the left, is \( 1 \) for \( 1 \leq p \leq n \). Again, for \( n+1 \leq p \leq 2n \),

\[
v_p(\underline{i} \underline{k}) = \begin{cases} u^*(alt(j_p) k_{2n-p+1}^{(-1)^{n+1}} \cdots k_{2n-p+1}^{(-1)^1}, k_{2n-p+1}) & \text{if } p \text{ is odd } \\ u(alt(j_p) k_{2n-p+2}^{(-1)^{n+1}} \cdots k_{2n-p+2}^{(-1)^1}, k_{2n-p+1}) & \text{if } p \text{ is even } \\ \end{cases}
\]

Thus,

\[
v(\underline{i} \underline{j}) v(\underline{i} \underline{k}) = v(\underline{i} \underline{j}) v_n(\underline{i} \underline{k}) v_{n+1}(\underline{i} \underline{k}) \cdots v_{2n}(\underline{i} \underline{k}) = v(\underline{i} \underline{k})
\]

By applying Lemma 4.3 we see that the set \( \left\{ u^*(\underline{i} \underline{j})(E_{\underline{i} \underline{j}} \otimes 1) \mid \underline{i} \underline{j} \in I^n \text{ s.t. } alt(\underline{i} \underline{j}) = e \right\} \) is a basis for \( N' \cap M_{n-1} \). We will use this basis to establish an isomorphism between the planar algebra associated to \( N \subseteq M \) and the abstract planar algebra defined in Section 3. Let \( \omega : G \times G \times G \to T \) be the 3-cocycle associated to \( G \), that is, for all \( g_1, g_2, g_3 \in G \) we have

\[
u(g_1, g_2)u(g_1 g_2, g_3) = \omega(g_1, g_2, g_3)u(g_1, g_2 g_3).
\]

As before, this is a consequence of associativity of the multiplication in \( G \).

We prove next a useful lemma relating \( v \) and \( \omega \).

Lemma 4.4. If \( i \in I \) and \( \underline{k} = (k_1, \ldots, k_{2n}) \in I^{2n} \) such that alt(\( \underline{k} \)) = e and \( k_1 = k_{2n} \), then alt(\( i, k_1 \))(v(\( \underline{k} \))) = \sum_{(i, k_1)}(\underline{k}) v(i, k_2, \ldots, k_{2n-1}, i).
Proof: We expand \( \text{alt}_n(i, k_1)(v(k)) \) as a product of unitaries and work with them separately. For \( 1 \leq p \leq n \), we have

(i) \[
(\omega)_{\alpha^{-1}}(u^{-1}_{g_i}, g_i) (u^{-1}_{g_i} \circ \alpha^{-1}_{g_i} g_i) \left( u^*(alt(k_1, \ldots, k_{2p-1}, g_{k_{2p-1}})) \right)
\]

(\text{using definition of } \omega)

(ii) \[
(\omega)_{\alpha^{-1}}(u^{-1}_{g_i}, g_i) (u^{-1}_{g_i} \circ \alpha^{-1}_{g_i} g_i) \left( u(alt(k_1, \ldots, k_{2p-1}, g_{k_{2p-1}})) \right)
\]

(\text{using definition of } \omega)

Multiplying (i) and (ii), we get,

\[
\text{alt}_n(i, k_1) (v_{2p-1}(k)) \left( v_{2p}(k) \right)
\]

\[
= \bar{\lambda}_{(i,k_1)}(k, 2p-1) \bar{\lambda}_{(i,k_1)}(k, 2p)
\]

\[
\text{Ad}_{u^*}(g_{i}^{-1}, g_{i}) u(g_{i}^{-1}, g_{j}) \left( u(alt(i, k_1), alt(k_1, \ldots, k_{2p-2}, k_{2p-1}, g_{k_{2p-1}})) \right)
\]

\[
= \bar{\lambda}_{(i,k_1)}(k, 2p-1) \bar{\lambda}_{(i,k_1)}(k, 2p)
\]

\[
\left\{ \begin{array}{ll}
\text{Ad}_{u^*}(g_{i}^{-1}, g_{i}) u(g_{i}^{-1}, g_{j}) \left( u(alt(i, k_1), alt(k_1, \ldots, k_{2p-2}, k_{2p-1}, g_{k_{2p-1}})) \right) & \text{if } p = 1 \\
\text{Ad}_{u^*}(g_{i}^{-1}, g_{i}) u(g_{i}^{-1}, g_{j}) \left( u(alt(i, k_1), alt(k_1, \ldots, k_{2p-2}, k_{2p-1}, g_{k_{2p-1}})) \right) & \text{if } 2 \leq p \leq n - 1 \\
\text{Ad}_{u^*}(g_{i}^{-1}, g_{i}) u(g_{i}^{-1}, g_{j}) \left( u(alt(i, k_1), alt(k_1, \ldots, k_{2p-2}, k_{2p-1}, g_{k_{2p-1}})) \right) & \text{if } p = n
\end{array} \right.
\]

\[
= W_p
\]
Thus,
\[
alt_n(i, k_1)(v(k_i)) = W_1 \cdots W_n = \overline{\lambda}_{(i, k_1)}(k_i).
\]

\[
Adu^*(g_i^{-1}, g_i)u(g_i^{-1}, g_i) = (u^*(g_i^{-1}, g_i))v_2(i, k_2, \ldots, k_{2n-1}, i) \cdots v_{2n-1}(i, k_2, \ldots, k_{2n-1}, i)u(g_i^{-1}, g_i)
\]

\[
\overline{\lambda}_{(i, k_1)}(k_i)u^*(g_i^{-1}, g_i)v_2(i, k_2, \ldots, k_{2n-1}, i) \cdots v_{2n-1}(i, k_2, \ldots, k_{2n-1}, i)u(g_i^{-1}, g_i).
\]

Let us recall the following well-known fact about isomorphisms of two planar algebras which will be used in the next theorem (\[\[\]). Let \(P^1\) and \(P^2\) be two planar algebras. Then \(P^1 \cong P^2\) (as planar algebras) if and only if there exist a vector space isomorphism \(\psi : P^1_n \to P^2_n\) such that:

(i) \(\psi\) preserves the filtered algebra structure,
(ii) \(\psi\) preserves the actions of all Jones projection tangles and the (two types of) conditional expectation tangles.

If \(P^1\) and \(P^2\) are \(*\)-planar algebras, then we require \(\psi\) to be \(*\)-preserving.

**Theorem 4.5.** The planar algebra \(P^{sf}\) associated to the diagonal subfactor obtained from a II\(_1\) factor \(N\) and a finite collection of automorphisms \(\alpha_i \in Aut N\) for \(i \in I\), is isomorphic to \(P^{(g_i ; i \in I), \omega}\) where \(g_i = [\alpha_i] \in Out N\) for all \(i \in I\), and \(\omega\) is the normalized 3-cocycle associated to \(G = \langle g_i : i \in I \rangle \subseteq Out N\) as above.

**Proof:** Let \(G = \langle g_i : i \in I \rangle\) and without loss of generality, let us assume that \(\alpha\) is a lift of \(G\) such that \(\alpha_i = \alpha g_i\). By \(\[\[\) we have \(P^{sf}_n = N' \cap M_{n-1}\) for all \(n \geq 0\).

Define the map \(\phi : P^{sf} \to P^\defeq P^{(g_i ; i \in I), \omega}\) by first defining it on basis elements as \(\phi(u^*(\underline{i}, \underline{j}))(E_{\underline{i}, \underline{j}} \otimes 1) = (\underline{i}, \underline{j})\) for all \(\underline{i}, \underline{j} \in I^n\) such that \(alt(\underline{i}, \underline{j}) = e\), and then extending it linearly. Clearly, \(\phi\) is a vector space isomorphism. We will show that \(\phi\) is \(*\)-planar algebra isomorphism. We make first the following observation:

For \(i \in I\), \(\underline{i} = (i_1, \ldots, i_n) \in I^n\) and \(0 \leq s \leq n\), we have the identity \(v(\underline{i}) = v(i_1, \ldots, i_s, i, i, i_{s+1}, \ldots, i_n)\). The proof is similar to that of Lemma 3.1. Thus, if \(\underline{i}\) is a sequence of indices with non-crossing matched pairings, then using this identity several times to reduce all consecutive matched pairings, we get \(v(\underline{i}) = 1\).

We show now that \(\phi\) is indeed a planar algebra isomorphism following the remark just before the theorem.

(a) \(\phi\) is unital: Since \(1_{P^{sf}_n} = \sum_{\underline{i} \in I^n}(E_{\underline{i}, \underline{i}} = 1)\) and \(v(\underline{i}, \underline{i}) = \sum_{\underline{i} \in I^n}(\underline{i}, \underline{i}) = 1\), by the above observation, we get \(\phi(1_{P^{sf}_n}) = \sum_{\underline{i} \in I^n}(\underline{i}, \underline{i}) = 1\).

(b) \(\phi\) preserves Jones projection tangles: By Theorem 4.2.1 in \(\[\[\), the \(n\)-th Jones projection tangle \(E_n\) acts as \(Z_{E_n}^{P^{sf}} = |I|e_n = \sum_{\underline{k} \in I^{n-1}, i, j \in I}(E_{\underline{k}, i, j}(\underline{k}, j, j) \otimes 1)\). Since \((\underline{k}, i, i, j, j, \underline{k})\) is a sequence of indices with non-crossing matched pairings, by the above note \(v(\underline{k}, i, i, j, j, \underline{k}) = 1\). So, \(\phi(Z_{E_n}^{P^{sf}}) = \sum_{\underline{k} \in I^{n-1}, i, j \in I}(\underline{k}, i, i, j, j, \underline{k}) = Z_{E_n}^P\).

(c) \(\phi\) preserves the action of conditional expectation tangle: By Theorem 4.2.1 in \(\[\[\), the action of conditional expectation tangle \(E_{n+1}^{P^{sf}}\) is given by \(Z_{E_{n+1}}^{P^{sf}} =\)
defined in Section 3, it is easy to check $Z_{E_n^p}(f) = \mathbb{M}_n$. We compute

$$Z_{E_n^p}(v^*(\vec{k},l,\vec{j})(E_{i,j}\otimes 1)) = v^*(\vec{k},l,\vec{j})\mathbb{M}_n(E_{i,j}\otimes 1)$$

$$= \delta_{k,l} v^*(\vec{k},\vec{k},\vec{j})(E_{\vec{j}}\otimes 1)$$

$$= \delta_{k,l} v^*(\vec{j},\vec{j})(E_{\vec{j}}\otimes 1) \xrightarrow{\phi} \delta_{k,l} (\vec{j},\vec{j}) \in P_n$$

for all $\vec{i},\vec{j} \in I^n, k,l \in I$ such that $alt(\vec{i},k,l,\vec{j}) = e$. From the action of tangles defined in Section 3 it is easy to check $Z_{E_n^p}(\vec{k},l,\vec{j}) = \delta_{k,l} (\vec{j},\vec{j})$.

(d) $\phi$ preserves $*$: Applying $*$ on a basis element of $P_n$ we get

$$\left(v^*(\vec{j})(E_{\vec{j}}\otimes 1)\right)^* = (E_{\vec{j}}\otimes 1)v(\vec{j}) = (alt(\vec{j})alt^{-1}(\vec{j})) (v(\vec{j}))(E_{\vec{j}}\otimes 1)$$

$$= v(\vec{j}) v(\vec{j}) (E_{\vec{j}}\otimes 1)$$

$$= v^*(\vec{j})(E_{\vec{j}}\otimes 1) \xrightarrow{\phi} (\vec{j},\vec{j})^*$$

for $\vec{i},\vec{j} \in I^n$ such that $alt(\vec{j}) = e$. Note that we used Lemma 4.1 for the third equality and Lemma 4.2 for the fourth one.

(e) $\phi$ preserves multiplication: Suppose $\vec{i},\vec{j},\vec{k},\vec{l} \in I^n$ such that $alt(\vec{j}) = e = alt(\vec{l})$. Then,

$$\left(v^*(\vec{j})(E_{\vec{j}}\otimes 1)\right) \cdot \left(v^*(\vec{l})(E_{\vec{l}}\otimes 1)\right)$$

$$= v^*(\vec{j}) (alt(\vec{j}))(E_{\vec{j}}\otimes 1) (v^*(\vec{l}))(E_{\vec{l}}\otimes 1)$$

$$= v^*(\vec{j}) v(\vec{j}) v^*(\vec{l}) v(\vec{l}) \delta_{\vec{j},\vec{k}} (E_{\vec{l}}\otimes 1) \text{ (using Lemma 4.1)}$$

$$= \delta_{\vec{j},\vec{k}} (v(\vec{j})(E_{\vec{j}}\otimes 1) = \delta_{\vec{j},\vec{k}} v^*(\vec{j})(E_{\vec{j}}\otimes 1) \text{ (using Lemma 4.3)}$$

On the other hand, one can easily deduce from the action of the multiplication tangle in $P$ that $(\vec{j},\vec{l}) = \delta_{\vec{j},\vec{k}} (\vec{j},\vec{l})$.

(f) $\phi$ preserves the action of the left conditional expectation tangle: By Theorem 4.2.1 in [11], the action of the left conditional expectation tangle $E_n^l$ is given by $Z_{E_n^p} = [I]\mathbb{E}_{M'\cap N_{n-1}}$. Using the basis of $M$ over $N$ mentioned before, the conditional expectation onto $M' \cap N_{n-1}$ can be expressed as (see [2])

$$\mathbb{E}_{M'\cap N_{n-1}}(x) = \left|I\right|^{-2} \sum_{i,j \in I} \left(\sqrt{|I|} (E_{i,j}\otimes 1) x \left(\sqrt{|I|} (E_{j,i}\otimes 1)\right)\right)$$

$$\left|I\right|^{-1} \sum_{i,j \in I} (E_{i,j}\otimes 1)x(E_{j,i}\otimes 1)$$
for $x \in N^i \cap M_{n-1}$. Hence, for $i = (i_1, \cdots, i_{n-1})$, $j = (j_1, \cdots, j_{n-1}) \in I^{n-1}$ and $k, l \in I$ such that $\text{alt}(k, \tilde{i}, \tilde{j}, l) = e$, we have

$$Z^{'n}_{E_n} \left( v^*(k, \tilde{i}, \tilde{j}, l)(E(k, \tilde{i}, \tilde{j}, l) \otimes 1) \right)$$

$$= \sum_{i,j \in I} (E_{i,j} \otimes 1)v^*(k, \tilde{i}, \tilde{j}, l)(E_{k,\tilde{i},\tilde{j},l} \otimes 1)(E_{j,i} \otimes 1)$$

$$= \sum_{i,j \in I} \text{alt}_G(i, j)(v^*(k, \tilde{i}, \tilde{j}, l))(E_{i,j} \otimes 1)(E_{k,\tilde{i},\tilde{j},l} \otimes 1)(E_{j,i} \otimes 1)$$

$$= \delta_{k,l} \sum_{i \in I} \text{alt}_G(i, k)(v^*(k, \tilde{i}, \tilde{j}, k))(E_{i,\tilde{i},\tilde{j},k} \otimes 1)$$

$$= \delta_{k,l} \sum_{i \in I} \lambda_{i,k}(k, \tilde{i}, \tilde{j}, k)v^*(i, \tilde{i}, \tilde{j}, i)(E_{i,\tilde{i},\tilde{j},i} \otimes 1) \quad (\text{using Lemma 4.4})$$

On the other hand, one can easily check that the action of $E_n'$ is given by

$$Z^{'n}_{E_n'}(k, \tilde{i}, \tilde{j}, l) = \delta_{k,l} \sum_{i \in I} \lambda_{i,k}(k, \tilde{i}, \tilde{j}, k) (i, \tilde{i}, \tilde{j}, i).$$

\[\square\]

Corollary 4.6. Given a group $G$ generated by a finite collection $g_i$ for $i \in I$ and given a normalized 3-cocycle $\omega \in Z^3(G, T)$, there exists a hyperfinite subfactor whose associated planar algebra is isomorphic to $P^{(g_i; i \in I), \omega}$.

Proof: The proof follows from [9] and Theorem 4.5. \[\square\]

Remark 4.7. Note that the isomorphism $\phi$ in the proof of Theorem 4.6 uses the 3-cocycle $\omega$ only in the step involving the conditional expectation onto the commutant of $M$. In particular, the filtered $*$-algebra structure does not involve $\omega$.

Analyzing the filtered $*$-algebra structure of our planar algebra, one can easily find that the principal graph $\Gamma$ of $N \subset M$ is a Cayley-like graph. More precisely, if $G_n = \{\text{alt}(\tilde{i}) : \tilde{i} \in I^n\}$ for $n \geq 1$, and $G_0 = \{e\}$, then $V_n(\Gamma) = G_n \setminus G_{n-2}$ denotes the set of vertices of $\Gamma$ which are at a distance $n$ from the distinguished vertex for $n \geq 1$, and $V_0(\Gamma) = \{e\}$. The number of edges between $g \in V_n(\Gamma)$ and $h \in V_{n+1}(\Gamma)$ is $\sum_{i \in I} \delta_{g.h_{i}}$ (resp. $\sum_{i \in I} \delta_{g_{i},h}$) if $n$ is odd (resp. even). Note that this is well-known.

The most elegant feature of the planar algebra $P^{(g_i; i \in I), \omega}$ is that the distinguished basis forms the ‘loop-basis’ of the filtered $*$-algebra arising from paths on the principal graph. Note that the 3-cocycle $\omega$ does not enter in the definition of the actions of multiplication, inclusion and unit tangles (defined in Section 3) or in the $*$-operation. Of course, we found the abstract planar algebra by first computing the action of tangles on the relative commutants. We then deduced from it an abstract prescription of the planar algebra associated to a $G$-kernel and a 3-cocycle, which is the one presented in Section 3.

The path algebra associated to the principal graph can always be used to obtain a description of the filtered $*$-algebra structure of a subfactor planar algebra (see for instance [13]). The extra information encoded in the planar algebra which the principal graph cannot provide, is the action of the left conditional expectation tangle (or equivalently, the rotation tangle or the left-inclusion tangle with an even number of strings). The main issue in this paper was the choice of the unitaries
associated group and its generators of the diagonal planar algebra is given by a diagonal planar algebra is a diagonal subfactor. Moreover, if the graph $\Gamma$.

Any finite index extremal subfactor $N \subset M$ whose standard invariant is given by a diagonal planar algebra is a diagonal subfactor. Moreover, if the associated group and its generators of the diagonal planar algebra is given by $G$ and $\{g_i : i \in I\}$ repectively, then for every $i_0 \in I$, there exists $\alpha_i \in \text{Aut} N$, $i \in I$, such that:

(i) $(N \subset M) \cong (N \hookrightarrow M_f \otimes N)$ where $N \hookrightarrow M_f \otimes N$ is the diagonal subfactor with respect to the automorphisms $\alpha_i$ for $i \in I$.

(ii) There exists a group isomorphism $\psi : (\alpha_i : i \in I) \text{Out} N \rightarrow \{g_i^{-1} g_i : i \in I\} \leq G$ sending $\alpha_i$ to $g_{i_0}^{-1} g_i$ for all $i \in I$.

Proof: Let $P$ be the planar algebra associated to $N \subset M$, $P^\Delta$ be a diagonal planar algebra associated to $G$ and $\{g_i : i \in I\}$, and $\phi : P^\Delta \rightarrow P$ a $*$-planar algebra isomorphism.

Setting up matrix units:
For all $i, j \in I$ such that $g_i = g_j$, set $q_j^i := \phi((i, j)) \in P_1 = N' \cap M$. Note that $\sum_{i \in I} q_j^i = 1$. So, to create other off-diagonal matrix units $q_j^i$, we partition $I = \bigsqcup_{n=0}^m I_n$ such that:

(i) $i_0 \in I_0$,
(ii) $g_i = g_j \Leftrightarrow i, j \in I_n$ for some $0 \leq n \leq m$.

For each $n \in \{1, 2, \cdots, m\}$, choose $i_n \in I^n$ and partial isometry $q_{i_n}^{i_0} \in M$ such that $q_{i_n}^{i_0} (q_{i_n}^{i_0}^*) = q_{i_0}^{i_0}$ and $(q_{i_n}^{i_0})^* q_{i_n}^{i_0} = q_{i_n}^{i_0}$. (Note that $q_{i_0}^{i_0} = \phi((i_0, i_0))$ and $q_{i_n}^{i_0} = \phi((i_n, i_n))$ have the same trace $|I|^{-1}$). Extend $q$ by defining $q : I \times I \rightarrow M$ by $I \times I \supset I^* \times I^* \ni (i, j) \mapsto q_j^i := q_j^i (q_{i_n}^{i_0})^* q_{i_n}^{i_0} q_j^i = \phi((i, i_n)) (q_{i_n}^{i_0})^* q_{i_n}^{i_0} \phi((i_n, j)) \in M$.

It is completely routine to check (using properties of partial isometry and action of multiplication tangle in $P^\Delta$) that (i) $q$ is well-defined, (ii) $q_j^i q_k^l = \delta_{j,k} q_j^i$, and (iii) $(q_j^i)^* = q_j^i$.

Finding automorphisms:
Using extremality of $N \subset M$, for each $i \in I$, we get

\[ q_i^i M q_i^i : N q_i^i ] = 1 \Rightarrow N q_i^i = q_i^i M q_i^i \]

\[ \Rightarrow N q_j^i \subset q_i^i M q_j^i \subset q_j^i N \text{ and } N q_j^i \supset q_j^i M q_j^i \supset q_j^i N \]

\[ \Rightarrow N q_j^i = q_i^i M q_j^i = q_j^i N \Rightarrow M \cong \bigsqcup_{i, j \in I} q_j^i N \]

where we use $[q_i^i, N] = 0$ in the second implication.
For each $i \in I$, define $\alpha_i : N \to N$ by $q_i^{10}x = \alpha_i(x)q_i^{10}$ for all $x \in N$. Since $N$ is a factor and $[q_j^{10}, N] = 0$ for $j \in I$, $\alpha_i$ is well-defined and injective; surjectivity follows from $Nq_i^{10} = q_i^{10}N$. Linearity and homomorphism property of $\alpha_i$ follow immediately, and we also have the identity $xq_i^{10} = q_i^{-1}\alpha_i^{-1}(x)$. To show $q_j^{10}x = (\alpha_i^{-1}\alpha_j)(x)q_j^{10}$, note that

$$q_i^{10}x = q_i^{10}q_j^{10}q_i^{10} = q_i^{10}\alpha_i^{-1}(x)q_i^{10} = \alpha_i^{-1}(x)q_i^{10}$$

$$\Rightarrow q_j^{10}x = q_i^{10}\alpha_j(x)q_j^{10} = (\alpha_i^{-1}\alpha_j)(x)q_j^{10}, \text{ for all } x \in N, \text{ for all } i, j \in I.$$

Using this relation, it is easy to show that $\alpha_i$ is $*$-preserving. Two other easy consequences are $\alpha_i = \alpha_j$ if and only if $g_i = g_j$ and $\alpha_{10} = id_N$.

**Structure of diagonal subfactor:**

Define:

(i) $\kappa : \hat{M} := M_I \otimes N \to M$ by $\kappa(E_{i,j} \otimes x) = q_i^{10}\alpha_j^{-1}(x) = \alpha_i^{-1}(x)q_j^{10}$,

(ii) $\lambda : M \to \hat{M}$ by $\lambda(x) = \sum_{i,j \in I} E_{i,j} \otimes \lambda_{i,j}(x)$,

where $\lambda_{i,j} : M \to N$ is the map given by the relation $q_i^{10}q_j^{10} = q_i^{10}\alpha_j^{-1}(\lambda_{i,j}(x))$ for all $x \in M$.

Clearly, $\kappa \circ \lambda = id_M$, $\lambda \circ \kappa = id_{\hat{M}}$ and $\kappa$ is a $*$-isomorphism. Set $\hat{N} := \lambda(N) = \left\{ \sum_{i \in I} E_{i,i} \otimes \alpha_i(x) \mid x \in N \right\} \subset \hat{M}$.

This proves that $N \subset M$ is a diagonal subfactor as claimed. The rest of the proof pertains to (ii).

**Matrix units for the tower of basic construction:**

Let $M_n$ denotes the $\Pi_1$ factor obtained from $N \subset M$ by iterating the basic construction $n$ times. We will first define $q_{ij}^{10} \in M_{n-1}$ for $i, j \in I^n$ and $n \geq 2$ satisfying:

(i) $\left(q_{ij}^{10}\right)^* = q_{ij}^{10}$ for all $i, j \in I^n$,

(ii) $q_{ij}^{10}x = alt_\alpha(i, j)(x)q_{ij}^{10}$ for all $i, j \in I^n$ and $x \in N$,

(iii) $q_{ij}^{10} = \phi((\tilde{i}, \tilde{j}))$ for all $i, j \in I^n$,

(iv) $q_{ij}^{10}q_{kl}^{10} = \delta_{ik}q_{jl}^{10}$ for all $i, j, k, l \in I^n$,

(v) $q_{ij}^{10}\left(q_{ij}^{10}\right)^* = \delta_{ss}q_{(s,t)}^{10}$ for all $s, t \in I^{n-1}$ and $s, t \in I$

by induction on $n$ where $alt_\alpha$ is defined by $alt_\alpha(i_1, \ldots, i_m) = \alpha_{i_1}^{-1}\alpha_{i_2}\alpha_{i_3}^{-1}\alpha_{i_4} \cdots \alpha_{i_m}^{-1}$.

Suppose we have defined such $q_{ij}^{10}$’s for all $i, j \in I^n$ and $m \leq n$. Now, for $i, j \in I$ and $i, j \in I^n$, set $q_{(i, j)}^{(1, s)} := q_{(i, j)}^{10}E_nq_{(s, t)}^{(2, j)} \in M_n$ for some $s \in I^{n-1}$, where $E_n = |I|e_n$ is the element in $M_n$ given by the $n$-th Jones projection tangle. To show that the definition of $q_{(i, j)}^{(1, s)}$ is independent of $s \in I^{n-1}$, observe that

$q_{(i, j)}^{(1, s)} = q_{(i, j)}^{10}\phi((s, \tilde{s}))E_nq_{(s, t)}^{(2, j)} = q_{(i, j)}^{10}E_nq_{(s, t)}^{(2, j)} = q_{(i, j)}^{10}E_nq_{(i, j)}^{(1, s)}$ for all $i, j \in I^{n-1}$. Properties (i) and (v) hold trivially. For (ii), note that

$q_{(i, j)}^{(1, s)}x = alt_\alpha(i, j, \tilde{s}, t)(x)q_{(i, j)}^{(1, s)} = alt_\alpha((i, \tilde{i}), (j, \tilde{j}))(x)q_{(i, j)}^{(1, s)}$ for all $x \in N$.

Next, we prove property (iv). For $i \in I$ and $m \geq 1$, let $\eta_m(i)$ denote the element
in $M'_{m-2} \cap M_{m-1}$ (where $M_{-1} := N$, $M_0 := M$). Two important relations which we will often use, are $\eta_m(i)E_m = \eta_{m+1}(i)E_m$ and $E_m\eta_m(i) = E_m\eta_{m+1}(i)$. Getting back to property (iv), we have

$$
\eta_m(i) = \begin{cases} 
1 & \text{if } m \text{ is odd} \\
2 & \text{if } m \text{ is even}
\end{cases}
$$

It remains to establish property (iii). Now, for $k \in \mathbb{I}^n$, $i \in I$ and $s \in \mathbb{I}^{n-1}$,

$$
\phi((\underline{i}, j, s)) q_{(\underline{s}, \underline{i})}^{(i)} = q_{(\underline{s}, \underline{i})}^{(i)} E_n q_{(\underline{s}, \underline{i})}^{(i)} = q_{(\underline{s}, \underline{i})}^{(i)} E_n = q_{(\underline{s}, \underline{i})}^{(i)} E_n = q_{(\underline{s}, \underline{i})}^{(i)} = q_{(\underline{s}, \underline{i})}^{(i)} = q_{(\underline{s}, \underline{i})}^{(i)}.
$$

Since $q_{(\underline{s}, \underline{i})}^{(i)} \in \mathcal{P}(N' \cap M_i)$ (by property (ii)) and $\phi((\underline{i}, j, s))$ is a minimal projection of $P_{n+1} = N' \cap M_n$, therefore $q_{(\underline{s}, \underline{i})}^{(i)} = \phi((\underline{i}, j, s))$.

The proof for the initial case of $n = 2$ is similar and is left to reader.

**Correspondence between relations satisfied by $\alpha_i$'s and $g_i$'s:**

In this part, we will prove that for $k, j \in \mathbb{I}^n$, $alt(k, j) = e$ if and only if $alt_{\alpha}(k, j) \in Int N$.

Following the construction of the isomorphism $\lambda$ between $M$ and $\widetilde{M}$, one can define an isomorphism $\lambda^{(n)}$ as follows:

$$
M_{n-1} \ni x \xrightarrow{\lambda^{(n)}} \lambda^{(n)}(x) = \sum_{k, l \in \mathbb{I}^n} E_{k, l} \otimes \lambda^{(n)}_{k, l}(x) \in \widetilde{M}_{n-1} := M_{I^n} \otimes N
$$

where $\lambda^{(n)}_{k, l} : M_{n-1} \to N$ is the map given by the relation

$$
\phi_{k, l}^i x_{k, l}^j = \phi_{k, l}^i alt_{\alpha}(k, j) \left( \lambda^{(n)}_{k, l}(x) \right).
$$
Thus, $\lambda^{(n)}(N) = \left\{ \sum_{k \in I^n} E_{k/u} \otimes \alpha^{-1}(k) : x \in N \right\}$, for $n \geq 1$. Note that $\lambda^{(1)} = \lambda$.

Let $\alpha^{-1}(\langle k \rangle) = e$ for $\langle k \rangle \in I^n$. Note that $\lambda^{(n)}(\langle k \rangle) = E_{k/u} \otimes 1$ and $\lambda^{(n)}(\phi(\langle k \rangle))$ are partial isometries between $\lambda^{(n)}(\langle k \rangle) = E_{k/u} \otimes 1$ and $\lambda^{(n)}(\alpha^{-1}(k)) = \lambda^{(n)}(\phi(\langle k \rangle)) = E_{k/u} \otimes 1$. So, there exists $u \in \mathcal{U}(N)$ such that $\lambda^{(n)}(\phi(\langle k \rangle)) = E_{k/u} \otimes u = \lambda^{(n)}(\alpha^{-1}(k))$ where $v = \alpha^{-1}(k)(u) \in \mathcal{U}(N)$. Hence,

$$
\phi(\langle k \rangle) \in \mathcal{N} \cap M_{n-1}
$$

Thus, $v = y q_{\frac{1}{2}}(y \cdot v) = (\alpha^{-1}(k)) \circ \mathcal{A}(y) q_{\frac{1}{2}}(v)$ for all $y \in N$

$$
(\alpha^{-1}(k)) \circ \mathcal{A}(y) = y \text{ for all } y \in N
$$

$$
(\alpha^{-1}(k)) \in \text{Int } N.
$$

Conversely, if $\alpha^{-1}(k) \in \text{Int } N$, then $\phi(\langle k \rangle), \mathcal{A}(y) q_{\frac{1}{2}}(v) \in (\lambda^{(n)}(N))' \cap M_{n-1}$. Now, $\alpha^{-1}(k) \neq e$ implies $\text{dim} (\lambda^{(n)}(N))' \cap M_{n-1}$ is a contradiction. Hence, $\alpha^{-1}(k) = e$.

The group generated by $\alpha_i$'s:

Let $H := \langle \alpha_i \rangle \leq \text{Out } N$, $\tilde{H} := \langle g_i^{-1} : i \in I \rangle \leq G$, $J := I \times \{1,-1\}$. Define maps $w_H : \prod_{n \geq 1} J^n \to H$ and $w_{\tilde{H}} : \prod_{n \geq 1} J^n \to \tilde{H}$ by

$$
J^n \ni (i_1, \epsilon_1, i_2, \epsilon_2, \cdots , i_n, \epsilon_n) \mapsto \phi_{i_1}^\epsilon_1 \phi_{i_2}^\epsilon_2 \cdots \phi_{i_n}^\epsilon_n \in H
$$

$$
J^n \ni (i_1, \epsilon_1, i_2, \epsilon_2, \cdots , i_n, \epsilon_n) \mapsto g_{i_1}^{-1} g_{i_2}^{-1} \epsilon_2 g_{i_2}^{-1} \cdots (g_{i_n}^{-1} g_{i_n} g_{i_n})^\epsilon_n \in \tilde{H}
$$

where $\epsilon_i \in \{1,-1\}$ for $1 \leq i \leq n$. Define $\gamma : H \to \tilde{H}$ by $\gamma (w_H(i)) = w_{\tilde{H}}(i)$. For $\gamma$ to be an isomorphism, it is enough to show $\gamma$ is well-defined and injective. Suppose the map $\rho : J \to I^2$ sends $(i,1)$ (resp. $(i,-1)$) to $(i_0,i)$ (resp. $(i_0,i_0)$). Extend $\rho$ to $J^n \to I^{2n}$ entrywise. Note that $w_H(j) = \alpha H (\rho(j))$ and $w_{\tilde{H}}(j) = \alpha (\rho(j))$. This implies

$$
w_H(j) = 1_H \iff \alpha H (\rho(j)) \in \text{Int } N \iff \alpha (\rho(j)) = e \iff w_{\tilde{H}}(j) = e.
$$

Hence, $H$ and $\tilde{H}$ are isomorphic.

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