THE CONVEX AND WEAK CONVEX DOMINATION NUMBER OF CONVEX POLYTOPES

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Abstract. This paper is devoted to solving the weakly convex dominating set problem and the convex dominating set problem for some classes of planar graphs—convex polytopes. We consider all classes of convex polytopes known from the literature and present exact values of weakly convex and convex domination number for all classes, namely $A_n$, $B_n$, $C_n$, $D_n$, $E_n$, $R_n$, $R'_n$, $Q_n$, $S_n$, $S'_n$, $T_n$, $T'_n$ and $U_n$. When $n$ is up to 26, the values are confirmed by using the exact method, while for greater values of $n$ theoretical proofs are given.

1. Introduction

This paper is dedicated to solving the weak convex and convex dominating set problems (WCDDSP/CDSP) for some special classes of graphs. A dominating set of a graph $G$ can be defined as a set of vertices in $G$ such that any vertex $u \in V(G)$ either belongs to the dominating set or is adjacent to some vertex that belongs to the dominating set. A closely related problem is that of finding the domination number, i.e. finding the dominating set with the minimum cardinality. The minimal cardinality of a dominating set will be denoted as $\gamma(G)$.

In this paper, we consider only simple graphs, i.e. graphs without loops or parallel edges. In order to introduce the convexity property in graphs let us define the distance $d(u, v)$ between any two vertices $u$ and $v$ of $G$ as the length of the shortest path between them. Since the graph is not weighted, the distance $d(u, v)$ is equal to the number of edges in the shortest path.

We say that a set of vertices $S$, $S \subset V(G)$, is a weakly convex (or isometric [17]) set in $G$ if for any two vertices $u, v \in S$, there exists at least one shortest $u$–$v$ path that contains only vertices belonging to $S$. Now, a weakly convex dominating set is a set $S$ that is both weakly convex and dominating. Similarly to the domination number, the weakly convex domination number $\gamma_{wconv}(G)$ is the minimal cardinality among all weakly convex dominating sets. The weakly convex dominating set problem (WCDDSP) is the problem of finding such minimal cardinality.

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As expected, convex domination requires stricter conditions. We will say that a set of vertices $S$, $S \subset V(G)$, is a convex set in $G$ if for any two vertices $u, v \in S$ all vertices in all shortest $u-v$ paths belong to $S$. A set $S$ is convex dominating if it is convex and dominating at the same time. The convex domination number $\gamma_{\text{conv}}(G)$ is the minimal cardinality among all convex dominating sets. The convex dominating set problem (CDSP) can be now defined as the problem of determining such minimal cardinality. There are other domination related problems depending on additional conditions such as Roman domination, signed Roman domination etc.

The convex domination number was defined during verbal communication between Jerzy Topp and Magdalena Lemanska in 2002 (stated in [15]). In [18] it was proven that decision problems of WCDSP and CDSP are NP-complete even for bipartite and split graphs. So, determining the weakly convex domination number and the convex domination number is NP-hard in a general case.

In [14], relations between $\gamma_{\text{conv}}$ and $\gamma_{\text{wconv}}$ were discussed for certain classes of graphs, and the following lemma was proposed for connected graphs:

Lemma 1.1 ([14]). For any connected graph $G$, we have

$$\gamma(G) \leq \gamma_{\text{wconv}}(G) \leq \gamma_{\text{conv}}(G).$$

In the paper [10] different bounds of the weakly convex domination and the convex domination number were obtained. It was shown in [4] that the convex domination number can be arbitrarily increased or decreased by an edge subdivision.

In this paper we will consider finding the weakly convex and the convex domination number for some classes of planar graphs. Some of these classes, called convex polytopes, were for the first time considered in [2], where they were denoted as $R_n$ and $Q_n$. Other classes, such as $A_n, B_n, C_n, D_n, E_n, R''_n, S_n, S'_n, T_n$ and $U_n$, were introduced in [5, 6, 9, 8, 7]. Certain graph invariants of convex polytopes were considered in [11]. For all these we assume that $n \geq 5$, and all indices are taken modulo $n$.

For $n \geq 27$, the values of both the weak convex and the convex domination number are stated by theorems. For $n \leq 26$, the exact method from [12] is used. The largest graph for which the exact method was used is $R''_{26}$, with 156 vertices and 234 edges. The exact method found that the weak convex and the convex domination numbers are equal to 156.

2. Convex polytopes $D_n$

The graph of convex polytope $D_n$, $n \geq 5$ (Figure 1), introduced in [1], consists of $2n$ 5-sided faces and a pair of $n$ sided faces. We will use the standard notation $[n] = \{1, 2, \ldots, n\}$ and $[n]_0 = \{0, 1, 2, \ldots, n\}$. 

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1A. Kartelj, D. Matić and V. Filipović, Solving the signed Roman domination and signed total Roman domination problems by exact and heuristic methods (unpublished manuscript).

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For every convex polytope $D_n$.

\[ V(D_n) = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\}, \]
\[ E(D_n) = \{a_i a_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_{i+1} c_i \mid i \in [n-1]_0\}. \]

**Theorem 2.1.** For every convex polytope $D_n$ with $n \geq 27$ it holds that $\gamma_{\text{wconv}}(D_n) = |V(D_n)| = 4n$.

**Proof.** Let $S$ be a weak convex dominating set of $D_n$. We will prove that $S = V(D_n)$. First we will show that $a_i \in S$ for every $i \in [n-1]_0$. From the fact that $S$ is a dominating set, for arbitrary $i$ it follows that there exist $u$ from $\mathcal{N}[a_i] \cap S$ and $w$ from $\mathcal{N}[a_{i+3}] \cap S$. Since $n \geq 27$, $\mathcal{N}[a_i] = \{a_i, a_{i-1}, a_{i+1}, b_i\}$ and $\mathcal{N}[a_{i+3}] = \{a_{i+3}, a_{i+2}, a_{i+4}, b_{i+3}\}$, we have the 16 subcases displayed in Table [1].

As it can be seen from Table [1] in all 16 subcases the shortest path is unique and all 16 shortest paths contain vertices $a_{i+1}$ and $a_{i+2}$. Since for any $u \in \mathcal{N}[a_i]$ and any $w \in \mathcal{N}[a_{i+3}]$, vertex $a_{i+1}$ belongs to a unique shortest $u\nobreakdash-w$ path and $S$ is a weak convex dominating set, vertex $a_{i+1}$ must belong to $S$. As $i$ is arbitrary we have $a_i \in S$ for every $i \in [n-1]_0$.

Note that the symmetry property holds for convex polytopes $D_n$, so vertices $a_i$, $b_i$, $c_i$, $d_i$ can be relabeled as $d_i$, $c_i$, $b_i$, $a_i$, which gives the same graph $D_n$. Therefore, using the same argumentation as above, we have $d_i \in S$ for every $i \in [n-1]_0$.

As it can be shown, for arbitrary $i \in [n-1]_0$ the shortest path between $a_i$ and $d_i$ is unique $(a_i-b_i-c_i-d_i)$ and has both endpoints in $S (a_i, d_i \in S)$. From the fact that $S$ is a weak convex dominating set it must hold that vertices $b_i$ and $c_i$ belong to $S$ for all $i \in [n-1]_0$. Due to the inclusion $\{b_i, c_i \mid i \in [n-1]_0\} \subseteq S$ and previous facts, it stands that $\{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\} \subseteq S$, implying that $S = V(D_n)$.

**Corollary 2.2.** For every convex polytope $D_n$ with $n \geq 27$ it holds that $\gamma_{\text{conv}}(D_n) = |V(D_n)| = 4n$.

**Proof.** Since for each graph $G$ it holds that $\gamma_{\text{wconv}}(G) \leq \gamma_{\text{conv}}(G) \leq |V(G)|$ and $\gamma_{\text{wconv}}(D_n) = |V(D_n)| = 4n$, we have $\gamma_{\text{conv}}(D_n) = 4n$. □

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**Figure 1.** The graph of convex polytope $D_n$
Table 1. Shortest paths used in Theorems 2.1 and 3.1

| u   | w   | Shortest path                                      |
|-----|-----|----------------------------------------------------|
| $a_i$ | $a_{i+3}$ | $a_i-a_{i+1}-a_{i+2}-a_{i+3}$                     |
|     | $a_{i+2}$ | $a_i-a_{i+1}-a_{i+2}$                              |
|     | $a_{i+4}$ | $a_i-a_{i+1}-a_{i+2}-a_{i+3}-a_{i+4}$              |
|     | $b_{i+3}$ | $a_i-a_{i+1}-a_{i+2}-a_{i+3}-b_{i+3}$              |
| $a_{i-1}$ | $a_{i+3}$ | $a_{i-1}-a_i-a_{i+1}-a_{i+2}-a_{i+3}$             |
|     | $a_{i+2}$ | $a_{i-1}-a_i-a_{i+1}-a_{i+2}$                      |
|     | $a_{i+4}$ | $a_{i-1}-a_i-a_{i+1}-a_{i+2}-a_{i+3}-a_{i+4}$     |
|     | $b_{i+3}$ | $a_{i-1}-a_i-a_{i+1}-a_{i+2}-a_{i+3}-b_{i+3}$     |
| $a_{i+1}$ | $a_{i+3}$ | $a_{i+1}-a_{i+2}-a_{i+3}$                         |
|     | $a_{i+2}$ | $a_{i+1}-a_{i+2}$                                  |
|     | $a_{i+4}$ | $a_{i+1}-a_{i+2}-a_{i+3}-a_{i+4}$                  |
|     | $b_{i+3}$ | $a_{i+1}-a_{i+2}-a_{i+3}-b_{i+3}$                  |
| $b_i$ | $a_{i+3}$ | $b_i-a_i-a_{i+1}-a_{i+2}-a_{i+3}$                  |
|     | $a_{i+2}$ | $b_i-a_i-a_{i+1}-a_{i+2}$                          |
|     | $a_{i+4}$ | $b_i-a_i-a_{i+1}-a_{i+2}-a_{i+3}-a_{i+4}$         |
|     | $b_{i+3}$ | $b_i-a_i-a_{i+1}-a_{i+2}-a_{i+3}-b_{i+3}$         |

3. Convex polytopes $R''_n$

The graph of convex polytope $R''_n$, $n \geq 5$ (Figure 2), introduced in [16], has the following vertex and edge sets:

$V(R''_n) = \{a_i, b_i, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\}$

$E(R''_n) = \{(a_i, a_{i+1}), (f_i, f_{i+1}), (a_i, b_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (e_i, f_i), (b_{i+1}, c_i), (d_{i+1}, e_i) \mid i \in [n-1]_0\}$.

![Figure 2. Polytope $R''_n$](image)
Theorem 3.1. For every convex polytope $R''_n$ with $n \geq 27$ it holds that $\gamma_{w\text{conv}}(R''_n) = |V(R''_n)| = 6n$.

Proof. Let $S$ be a weak convex dominating set of $R''_n$. We will prove that $S = V(R''_n)$. First we will show that $a_i \in S$ for every $i \in [n-1]_0$. Similarly to Theorem 2.1, from the fact that $S$ is a dominating set for an arbitrary $i$, it stands that there exist $u \in N[a_i] \cap S$ and $w \in N[a_{i+3}] \cap S$. Because $n \geq 27$, $N[a_i] = \{a_i, a_{i-1}, a_{i+1}, b_i\}$ and $N[a_{i+3}] = \{a_{i+3}, a_{i+2}, a_{i+4}, b_{i+3}\}$, we have 16 subcases, same as for $D_n$, displayed in Table 1.

Again, as it can be seen from Table 1, in all 16 subcases the shortest path is unique and all 16 shortest paths contain vertices $a_{i+1}$ and $a_{i+2}$. Same as for $D_n$, $a_{i+1}$ must belong to $S$. This is because $S$ is a weak dominating set and some $u, w \in S$ contain $a_{i+1}$ in its unique shortest $u-w$ path. As $i$ is arbitrary we have $a_i \in S$ for every $i \in [n-1]_0$.

Note that the symmetry property also holds for convex polytopes $R''_n$, so vertices $a_i, b_i, c_i, d_i, e_i, f_i$ can be relabeled as $f_1, e_1, d_i, c_i, b_i, a_i$, giving the same graph $R''_n$. Therefore, because of the same arguments used for $a$-vertices, we have that all $f$-vertices belong to $S$, i.e. $\{f_i \mid i \in [n-1]_0\} \subset S$.

As it can be shown, for arbitrary $i \in [n-1]_0$ the shortest path between $a_{i+1}$ and $f_i$ is unique $(a_{i+1} - b_{i+1} - c_i - d_i - e_i - f_i)$, and has both endpoints in $S$ ($a_{i+1}, f_i \in S$), so from the fact that $S$ is a weak convex dominating set it must hold that for all $i \in [n-1]_0$ we have $b_{i+1}, c_i, d_i, e_i \in S$. Due to the inclusion $\{b_{i+1}, c_i, d_i, e_i \mid i \in [n-1]_0\} \subset S$ and previous facts, we have $\{a_{i+1}, b_{i+1}, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\} \subset S$, implying that $V(R''_n) \subseteq S$. Therefore, it holds that $S = V(R''_n)$.

\[\square\]

Corollary 3.2. For every convex polytope $R''_n$ with $n \geq 27$ it holds that $\gamma_{\text{conv}}(R''_n) = |V(R''_n)| = 6n$.

Proof. Since, for each graph $G$, it holds that $\gamma_{\text{wconv}}(G) \leq \gamma_{\text{conv}}(G) \leq |V(G)|$ and $\gamma_{\text{wconv}}(R''_n) = |V(R''_n)| = 6n$, we have $\gamma_{\text{conv}}(R''_n) = 6n$. \[\square\]

4. Convex Polytopes $A_n$ and $R_n$

The classes of convex polytopes $A_n$ and $R_n$ (Figure 3) were introduced in [6] and [2] respectively. Their sets of vertices are

\[V(A_n) = V(R_n) = \{a_i, b_i, c_i \mid i \in [n-1]_0\},\]

while their sets of edges are

\[E(R_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i \mid i \in [n-1]_0\},\]

\[E(A_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i b_i, b_i c_i, a_{i+1} b_i, b_{i+1} c_i \mid i \in [n-1]_0\}.\]

Theorem 4.1. For every convex polytope $A_n$ with $n \geq 27$ it holds that $\gamma_{\text{wconv}}(A_n) = \gamma_{\text{conv}}(A_n) = n$.

Proof. First, we will prove that $\gamma_{\text{conv}}(A_n) \leq n$. Let $S = \{b_i \mid i \in [n-1]_0\}$. Now, we will prove that $S$ is a convex dominating set of $A_n$. The set $S$ is a dominating set of $A_n$, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i \in N[b_i]$, so
Theorem 4.2. For every convex polytope $R_n$ with $n \geq 27$ it holds that $\gamma_{\text{conv}}(R_n) = n$.

Proof. The proof goes along similar lines as the proof of Theorem 4.1 and will be omitted. \hfill \square
these convex polytopes are introduced in [6]. The sets of vertices and edges defining these convex polytopes are

\[ V(S_n) = \{a_i, b_i, c_i, d_i \mid i \in [n - 1]\}, \]
\[ E(S_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, a_ib_i, b_ic_i, c_id_i, a_{i+1}b_i \mid i \in [n - 1]\}. \]

The class of convex polytopes \( S''_n \) (Figure 5) was introduced in [8], where the authors also determined its metric dimension. The sets of vertices and edges defining these convex polytopes are

\[ V(S''_n) = \{a_i, b_i, c_i, d_i \mid i \in [n - 1]\}, \]
\[ E(S''_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, a_ib_i, b_ic_i, c_id_i, a_{i+1}b_i, c_i+1c_i \mid i \in [n - 1]\}. \]

The graph of convex polytope \( T_n \) (Figure 6), introduced in [8], consists of \( 4n \) 3-sided faces, \( n \) 4-sided faces and a pair of \( n \) sided faces. In this case, the vertices and edges are

\[ V(T_n) = \{a_i, b_i, c_i, d_i \mid i \in [n - 1]\}, \]
\[ E(T_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, a_ib_i, b_ic_i, c_id_i, a_{i+1}b_i, c_i+1d_i \mid i \in [n - 1]\}. \]

**Theorem 5.1.** For every \( n \geq 27 \) it holds that \( \gamma_{wconv}(S_n) = \gamma_{conv}(S_n) = 2n. \)

**Proof.** First let us prove that \( \gamma_{conv}(S_n) \leq 2n. \) In order to accomplish this let us prove that the set \( S = \{b_i, c_i \mid i \in [n - 1]\} \) is a convex dominating set for \( S_n. \) The set \( S \) is obviously a dominating set of \( S_n, \) since for all \( i \in [n - 1] \) it holds...
that $a_i, b_i, c_i, d_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] = \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\} = V(S_n)$. Also, $S$ is a convex set, since:

- All shortest paths between $b$-vertices contain only $b$-vertices;
- All shortest paths between $c$-vertices contain only $c$-vertices;
- All shortest paths between any $b$-vertex and any $c$-vertex contain only some $b$-vertices and some $c$-vertices.

Therefore, $S$ is a convex dominating set for $S_n$ and $\gamma_{\text{conv}}(S_n) \leq |S| = 2n$. Note that the shortest paths mentioned are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{\text{wconv}}(S_n) \geq 2n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every $b$-vertex is in $S$, i.e. for all $i \in [n-1]_0$ it holds that $b_i \in S$, which implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Similarly, it can be concluded that $c$-vertices belong to the set $S$, i.e. $\{c_i \mid i \in [n-1]_0\} \subseteq S$. Therefore, $\{b_i, c_i \mid i \in [n-1]_0\} \subseteq S$ so $|S| \geq 2n$.

Finally, since $\gamma_{\text{wconv}}(S_n) \leq \gamma_{\text{conv}}(S_n)$, $\gamma_{\text{wconv}}(S_n) \geq 2n$ and $\gamma_{\text{conv}}(S_n) \leq 2n$, we have $\gamma_{\text{wconv}}(S_n) = \gamma_{\text{conv}}(S_n) = 2n$.

\begin{theorem}
For every $n \geq 27$ it holds that
$$
\gamma_{\text{wconv}}(S_n') = \gamma_{\text{conv}}(S_n') = \gamma_{\text{wconv}}(T_n') = \gamma_{\text{conv}}(T_n') = 2n.
$$
\end{theorem}

\begin{proof}
The proof goes along similar lines as the proof of Theorem 5.1 so it will be omitted.
\end{proof}

\section{Convex polytopes $Q_n$ and $T_n''$}

Convex polytopes of the class $Q_n$ (Figure 7) were introduced in [2]. They are specified by these sets of vertices and edges:

\begin{align*}
V(Q_n) &= \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\}, \\
E(Q_n) &= \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_i+1 c_i \mid i \in [n-1]_0\}.
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{polytope_Q_n}
\caption{Polytope $Q_n$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{polytope_T_n''}
\caption{Polytope $T_n''$}
\end{figure}

Convex polytopes of the class $T_n''$ (Figure 8) were introduced in [6]. They are specified by these sets of vertices and edges:

\begin{align*}
V(T_n'') &= \{a_i, b_i, c_i, d_i \mid i \in [n-1]_0\}, \\
E(T_n'') &= \{a_i a_{i+1}, b_i b_{i+1}, d_i d_{i+1}, a_i b_i, b_i c_i, c_i d_i, b_i+1 c_i, a_i+1 b_i \mid i \in [n-1]_0\}.
\end{align*}
Theorem 6.1. For every \( n \geq 27 \) it holds that \( \gamma_{wconv}(Q_n) = \gamma_{conv}(Q_n) = 2n \).

Proof. We start by proving that \( \gamma_{conv}(Q_n) \leq 2n \). Let \( S = \{ b_1, c_1 \mid i \in [n - 1]_0 \} \).

Now, we will prove that \( S \) is a convex dominating set of \( Q_n \). The set \( S \) is obviously a dominating set of \( Q_n \), since for all \( i \in [n - 1]_0 \) it holds that \( a_i, b_i, c_i, d_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i] \), so \( \bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] = \{ a_i, b_i, c_i, d_i \mid i \in [n - 1]_0 \} = V(Q_n) \). Also, \( S \) is a convex set, since:

- All shortest paths between \( b \)-vertices contain only \( b \)-vertices;
- All shortest paths between \( c \)-vertices, except endpoints, contain only \( b \)-vertices;
- All shortest paths between any \( b \)-vertex and any \( c \)-vertex, except \( c \)-endpoint, contain only \( b \)-vertices.

Therefore, \( S \) is a convex dominating set for \( Q_n \) and \( \gamma_{conv}(Q_n) \leq |S| = 2n \). Note that the shortest paths mentioned are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that \( \gamma_{wconv}(Q_n) \geq 2n \). Using the same argument as in the proof of Theorem 4.1, it can be concluded that every \( b \)-vertex is in \( S \), i.e. for all \( i \in [n - 1]_0 \) it holds that \( b_i \in S \), which implies that \( \{ b_i \mid i \in [n - 1]_0 \} \subseteq S \).

Since vertices \( b_i \) such that \( i \in [n - 1]_0 \) are in \( S \), and \( S \) is a weak convex dominating set, from the uniqueness of the shortest path \( b_j - c_j - d_j \) the following implication holds:

\[
Q_n: \quad d_j \in S \Rightarrow c_j \in S. \tag{6.1}
\]

Let \( k = \lfloor \frac{n-1}{2} \rfloor \) and \( i \in [n - 1]_0 \). Therefore, it is easy to see that \( 2k + 1 \leq n \leq 2k + 2 \). To obtain domination for all \( d \)-vertices, \( S \) must contain some of \( c \)-vertices or/and \( d \)-vertices. Precisely, for \( \mathcal{N}[d_i] = \{ d_i, d_{i-1}, d_{i+1}, c_i \} \) and \( \mathcal{N}[d_{i+k}] = \{ d_{i+k}, d_{i+k-1}, d_{i+k+1}, c_{i+k} \} \), the following conditions must hold: \( \mathcal{N}[d_i] \cap S \neq \emptyset \) and \( \mathcal{N}[d_{i+k}] \cap S \neq \emptyset \).

There are 9 nontrivial possible cases. As it can be seen from Table 2 each of the 7 other possibilities can be directly reduced to one of the following cases.

**Case 1:** There is \( i \) such that \( d_i \in S \) and \( c_{k+i} \in S \). Let us consider the unique shortest path \( d_i - d_{i+1} - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i} \) whose endpoints are in \( S \). Since \( S \) is a weak convex dominating set, we have that \( d_i, d_{i+1}, \ldots, d_{k+i-1}, d_{k+i} \in S \). From (6.1) it follows that \( c_i, c_{i+1}, \ldots, c_{k+i-1}, c_{k+i} \in S \). As \( S \) contains at least \( k + 1 \) \( c \)-vertices and \( k + 1 \) \( d \)-vertices and all \( n \) \( b \)-vertices, and \( 2k + 2 \geq n \), we have \( |S| \geq n + 2k + 2 \geq 2n \).

**Case 2:** There is \( i \) such that \( c_i \in S \) and \( d_{k+i} \in S \). Similarly to Case 1, we can consider the unique shortest path \( c_i - d_i - d_{i+1} - \cdots - d_{k+i-1} - d_{k+i} \) whose endpoints are in \( S \). Again, since \( S \) is a weak convex dominating set, \( d_i, d_{i+1}, \ldots, d_{k+i-1}, d_{k+i} \in S \). In the same way, from (6.1) it follows that \( c_i, c_{i+1}, \ldots, c_{k+i-1}, c_{k+i} \in S \), again giving \( |S| \geq n + 2k + 2 \geq 2n \).

**Case 3:** There is \( i \) such that \( c_i \in S \) and \( d_{k+i-1} \in S \). Similarly to Case 2, we can consider the unique shortest path \( c_i - d_i - d_{i+1} - \cdots - d_{k+i-2} - d_{k+i-1} \) whose endpoints are in \( S \). Again, since \( S \) is a weak convex dominating set,
are mutually disjoint and, since $N$ is a dominating set, at least one vertex from neighbourhoods $d_i, d_{i+1}, \ldots, d_{k+i-2}, d_{k+i-1}$ in $S$. In the same way, from (6.1) it follows that $c_i, c_{i+1}, \ldots, c_{k+i-2}, c_{k+i-1}$ in $S$. Let $S' = \{b_l \mid l \in [n-1]_0 \} \cup \{c_j, d_j \mid j \in [k + i - 1]_0, j \geq i\}$; then $S' \subseteq S$ and $|S'| \geq 2n - 2$. Since $S$ is a dominating set, at least one vertex from $N[d_{i-2}]$ and at least one vertex from $N[d_{i-5}]$ must be in $S$. We have $N[d_{i-2}] \cap N[d_{i-5}] = \emptyset$, and $n \geq 27$, which implies that $N[d_{i-2}] \cap S' = \emptyset$ and $N[d_{i-5}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 2 = 2n$.

Case 4: There is $i$ such that $c_i \in S$ and $d_{k+i+1} \in S$. If $n$ is odd ($n = 2k + 1$), we can consider the unique shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-1} - d_i - c_i$ whose endpoints are in $S$. Again, since $S$ is a weak convex dominating set, $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-1}, d_i \in S$. In the same way, from (6.1) it follows that $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-1}, c_i \in S$, giving $|S| \geq n + 2(k+1) = n+2k+2 = 2n+1 > 2n$. Otherwise, if $n$ is even ($n = 2k+2$), we can consider the two shortest paths between vertices $c_i$ and $d_{k+i+1}$:

- shortest path $c_i - d_i - d_{i+1} - \cdots - d_{k+i} - d_{k+i+1}$;
- shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-1} - d_i - c_i$.

Since both endpoints $c_i$ and $d_{k+i+1}$ are in $S$, and $S$ is a weak convex dominating set, at least one shortest path must be in $S$. Therefore, we have that $d_i, d_{i+1}, \ldots, d_{k+i}, d_{k+i+1} \in S$ or $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-1}, d_i \in S$, which leads to $c_i, c_{i+1}, \ldots, c_{k+i}, c_{k+i+1} \in S$ or $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-1}, c_i \in S$. In both subcases, $S$ must have at least $n$ $b$-vertices, $k + 2$ $c$-vertices and $k + 2$ $d$-vertices, giving $|S| \geq n + 2(k+2) = n+2k+4 = 2n+2 > 2n$.

Case 5: There is $i$ such that $d_{i+1} \in S$ and $c_{k+i} \in S$. Similarly to Case 3, we can consider the unique shortest path $d_{i+1} - d_{i+2} - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i}$ whose endpoints are in $S$. Again, since $S$ is a weak convex dominating set, $d_{i+1}, d_{i+2}, \ldots, d_{k+i-1}, d_{k+i} \in S$. In the same way, (6.1) implies that $c_{i+1}, c_{i+2}, \ldots, c_{k+i-1}, c_{k+i} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0 \} \cup \{c_j, d_j \mid j \in [k + i], j \geq i+1\}$; it follows that $S' \subseteq S$ and $|S'| \geq 2n - 2$. Since $S$ is a dominating set, at least one vertex from $N[d_{i-1}]$ and at least one vertex from $N[d_{i-4}]$ must be in $S$. We have $N[d_{i-1}] \cap N[d_{i-4}] = \emptyset$ and $n \geq 27$, which imply that $N[d_{i-1}] \cap S' = \emptyset$ and $N[d_{i-4}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 2 = 2n$.

Case 6: There is $i$ such that $d_{i+1} \in S$ and $d_{k+i+1} \in S$. Similarly to Cases 3 and 5, we can consider the unique shortest path $d_{i+1} - d_{i+2} - \cdots - d_{k+i-2} - d_{k+i-1}$ whose endpoints are in $S$. Again, since $S$ is a weak convex dominating set, $d_{i+1}, d_{i+2}, \ldots, d_{k+i-2}, d_{k+i-1} \in S$. In the same way, (6.1) implies that $c_{i+1}, c_{i+2}, \ldots, c_{k+i-2}, c_{k+i-1} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0 \} \cup \{c_j, d_j \mid j \in [k + i - 1], j \geq i+1\}$, which leads to $S' \subseteq S$ and $|S'| \geq 2n - 4$. Since $S$ is a dominating set, at least one vertex from neighbourhoods $N[d_{i-1}], N[d_{i-4}], N[d_{i-7}]$ and $N[d_{i-10}]$ must be in $S$. It is easy to see that all four mentioned neighbourhoods are mutually disjoint and, since $n \geq 27$, we conclude that each neighbourhood has an empty intersection with $S'$, so $|S| \geq |S'| + 4 = 2n$.  

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For all Case 9: Since $\forall n \in \mathbb{N}$, $d_{i-1} \in S$ and $d_{k+i} \in S$. Similarly to Case 4, for odd $n$ ($n = 2k + 1$) we have the unique shortest path $c_{k+i} - d_{k+i} - d_{k+i+1} - \cdots - d_{i-2} - d_{i-1}$ whose endpoints are in $S$. Again, since $S$ is a weak convex dominating set, $d_{k+i}, d_{k+i+1}, \ldots, d_{i-2}, d_{i-1} \in S$. In the same way, (6.1) implies that $c_{k+i}, d_{k+i+1}, \ldots, d_{i-2}, d_{i-1} \in S$, giving $|S| \geq n + (k+1) = n + 2k + 2 = 2n + 1 > 2n$. Otherwise, for even $n$ ($n = 2k + 2$), we can consider the two shortest paths between vertices $d_{i-1}$ and $c_{k+i}$:

- shortest path $d_{i-1} - d_{i} - \cdots - d_{k+i-1} - d_{k+i} - c_{k+i};$
- shortest path $c_{k+i} - d_{k+i} - d_{k+i+1} - \cdots - d_{i-2} - d_{i-1}.$

Since both endpoints $d_{i-1}$ and $c_{k+i}$ are in $S$, and $S$ is a weak convex dominating set, at least one shortest path must be in $S$. Therefore, $d_{i-1}, d_{i}, \ldots, d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-2}, d_{i-1} \in S$, which leads to $c_{k+i}, d_{k+i}, c_{k+i+1}, \ldots, c_{i-2}, c_{i-1} \in S$. In both subcases, $S$ must have at least $n$ b-vertices, $k + 2$ c-vertices and $k + 2$ d-vertices, giving $|S| \geq n + 2(k + 2) = n + 2k + 4 = 2n + 2 > 2n$.

Case 8: There is $i$ such that $d_{i-1} \in S$ and $d_{k+i+1} \in S$. Let us consider the unique shortest path $d_{k+i+1} - d_{k+i+2} - \cdots - d_{i-2} - d_{i-1}$ whose endpoints are in $S$. Since $S$ is a weak convex dominating set, $d_{k+i+1}, d_{k+i+2}, \ldots, d_{i-2}, d_{i-1} \in S$. From (6.1) it follows that $c_{k+i+1}, c_{k+i+2}, \ldots, c_{i-2}, c_{i-1} \in S$. Let $S' = \{b_l \mid l \in [n-1]_0\} \cup \{c_j, d_j \mid j \in [n+i-1], j \geq k+i+1\}$. First, $S'$ is a subset of $S$. Second, since $n + i - 1 - (k + i + 1) + 1 = n - k - 1$, $S'$ must have $n$ b-vertices, $n - k - 1$ c-vertices and $n - k - 1$ d-vertices, giving $|S'| = n + 2(n - k - 1) = 3n - 2k - 2 \geq 2n - 1$. Since $S$ is a dominating set, at least one vertex from $N[d_{i+1}]$ must be in $S$. From $n \geq 27$ we have $N[d_{i+1}] \cap S' = \emptyset$. Therefore, we have $|S| \geq |S'| + 1 \geq 2n$.

Case 9: For all $i$ it holds that $c_i \in S$ and $c_{k+i} \in S$. Since $i \in [n-1]_0$, it holds that all c-vertices are in $S$. As all b-vertices are already in $S$, we have $|S| \geq 2n$.

| Possibility | Reduced to case | Reason |
|-------------|----------------|--------|
| (3) $(d_i \in S \land d_{k+i} \in S)$ | 1 | $d_k+i \in S \Rightarrow c_k+i \in S$ |
| (3) $(d_i \in S \land d_{k+i-1} \in S)$ | 3 | $d_i \in S \Rightarrow c_i \in S$ |
| (3) $(d_i \in S \land d_{k+i+1} \in S)$ | 4 | $d_i \in S \Rightarrow c_i \in S$ |
| (3) $(d_{i-1} \in S \land d_{k+i} \in S)$ | 7 | $d_{k+i} \in S \Rightarrow c_{k+i} \in S$ |
| (3) $(d_{i-1} \in S \land d_{k+i-1} \in S)$ | 1 | $i' = i - 1, d_{k+i} \in S \Rightarrow c_{k+i} \in S$ |
| (3) $(d_{i+1} \in S \land d_{k+i} \in S)$ | 5 | $d_{k+i} \in S \Rightarrow c_{k+i} \in S$ |
| (3) $(d_{i+1} \in S \land d_{k+i+1} \in S)$ | 1 | $i' = i + 1, d_{k+i} \in S \Rightarrow c_{k+i} \in S$ |
Finally, since \(|S| \geq 2n\) for all nine cases, \(\gamma_{wconv}(Q) \geq 2n\). Since \(\gamma_{wconv}(Q) \leq \gamma_{conv}(Q)\), \(\gamma_{wconv}(Q) \geq 2n\) and \(\gamma_{conv}(Q) \leq 2n\), we have that \(\gamma_{wconv}(Q) = \gamma_{conv}(Q) = 2n\).

**Theorem 6.2.** For every \(n \geq 27\) it holds that \(\gamma_{wconv}(T''_n) = \gamma_{conv}(T''_n) = 2n\).

**Proof.** The proof goes along similar lines as the proof of Theorem 6.1 so it will be omitted. \(\square\)

### 7. Convex Polytopes \(B_n, C_n\) and \(E_n\)

The graph of convex polytope \(B_n\) (Figure 9) was introduced in \([9]\) and consists of \(2n\) 4-sided faces, \(n\) 3-sided faces, \(n\) 5-sided faces and a pair of \(n\)-sided faces. The set of vertices is

\[
V(B_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}
\]

and the set of edges is

\[
E(B_n) = \{a_ia_{i+1}, b_ib_{i+1}, d_id_{i+1}, e_ie_{i+1}, a_ib_i, b_ic_i, b_{i+1}c_i, c_id_i, d_id_i \mid i \in [n-1]_0\}.
\]

![Figure 9. Polytope \(B_n\)](image)

Convex polytopes \(C_n\) (Figure 10) were introduced in \([9]\) and consist of \(3n\) 3-sided faces, \(n\) 4-sided faces, \(n\) 5-sided faces and a pair of \(n\)-sided faces. Their sets of vertices and edges are

\[
V(C_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}
\]

and

\[
E(C_n) = \{a_ia_{i+1}, b_ib_{i+1}, d_id_{i+1}, e_ie_{i+1}, a_ib_i, b_ic_i, c_id_i,
\]

\[
d_id_i, b_{i+1}c_i, d_{i-1}e_i \mid i \in [n-1]_0\}.
\]

The graph of convex polytope \(E_n\) (Figure 11), similar to the \(C_n\), introduced in \([9]\), consists of \(5n\) 3-sided faces, \(n\) 5-sided faces and a pair of \(n\)-sided faces, where

\[
V(E_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}
\]

\[
E(E_n) = \{a_ia_{i+1}, b_ib_{i+1}, d_id_{i+1}, e_ie_{i+1}, a_ib_i, b_ic_i, c_id_i,
\]

\[
d_id_i, b_{i+1}c_i, d_{i-1}e_i, a_{i-1}b_i \mid i \in [n-1]_0\}.
\]

**Theorem 7.1.** For every \(n \geq 27\) it holds that \(\gamma_{wconv}(B_n) = \gamma_{conv}(B_n) = 3n\).
Proof. First we will prove that $\gamma_{\text{conv}}(B_n) \leq 3n$. Let $S = \{b_i, c_i, d_i \mid i \in [n - 1]_0\}$. Now, we will prove that $S$ is a convex dominating set for $B_n$. The set $S$ is obviously a dominating set of $B_n$, since for all $i \in [n - 1]_0$ it holds that $a_i, b_i, c_i, d_i, e_i \in N[b_i] \cup N[c_i] \cup N[d_i], \text{ so } \bigcup_{i=0}^{n-1} N[b_i] \cup \bigcup_{i=0}^{n-1} N[c_i] \cup \bigcup_{i=0}^{n-1} N[d_i] = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n - 1]_0\} = V(B_n)$. Also, $S$ is a convex set, since each shortest path between two vertices from $\{b_i, c_i, d_i \mid i \in [n - 1]_0\}$ contains only $b$-vertices, $c$-vertices or $d$-vertices.

Therefore, $S$ is a convex dominating set for $B_n$ and $\gamma_{\text{conv}}(B_n) \leq |S| = 3n$. Note that, as before, shortest paths are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{\text{wconv}}(B_n) \geq 3n$. Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that every $b$-vertex is in $S$, i.e. $\{b_i \mid i \in [n - 1]_0\} \subseteq S$.

Using the same argumentation as in the proof of Theorem 4.1, it can be concluded that $\{d_i \mid i \in [n - 1]_0\} \subseteq S$.

For an arbitrary $i \in [n - 1]_0$ the shortest path between $b_i$ and $d_i$ is unique $(b_i - c_i - d_i)$, and has both endpoints in $S (b_i, d_i \in S)$, so from the fact that $S$ is a weak convex dominating set it must hold that for all $i \in [n - 1]_0$, $c_i \in S$. Due to the inclusion $\{c_i \mid i \in [n - 1]_0\} \subseteq S$ and previous facts, we have $\{b_i, c_i, d_i \mid i \in [n - 1]_0\} \subseteq S \Rightarrow |S| \geq 3n$.

Finally, since $\gamma_{\text{wconv}}(B_n) \leq \gamma_{\text{conv}}(B_n), \quad \gamma_{\text{wconv}}(B_n) \geq 3n \quad \text{and} \quad \gamma_{\text{conv}}(B_n) \leq 3n,$

we have that $\gamma_{\text{wconv}}(B_n) = \gamma_{\text{conv}}(B_n) = 3n$. \qed
Theorem 7.2. For every $n \geq 27$ it holds that
\[ \gamma_{wconv}(C_n) = \gamma_{conv}(C_n) = \gamma_{wconv}(E_n) = \gamma_{conv}(E_n) = 3n. \]

Proof. The proof goes along similar lines as the proof of Theorem 7.1 so it will be omitted. \hfill \square

8. Convex polytopes $U_n$

The convex polytopes $U_n$, $n \geq 5$ (Figure 12), were introduced in [8]. They consist of $n$ 4-sided faces, $2n$ 5-sided faces and a pair of $n$-sided faces, having these vertex and edge sets:

\[
V(U_n) = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\},
\]

\[
E(U_n) = \{a_ia_{i+1}, b_ib_{i+1}, e_ie_{i+1}, a_ib_i, b_ic_i, d_id_i, d_ie_i, c_{i+1}d_i \mid i \in [n]\}.
\]

![Figure 12. Polytope $U_n$](image)

Theorem 8.1. For every $n \geq 27$ it holds that $\gamma_{wconv}(U_n) = \gamma_{conv}(U_n) = 4n$.

Proof. First, we will prove that $\gamma_{conv}(U_n) \leq 4n$. Let $S = \{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$. Now, we will prove that $S$ is a convex dominating set for $U_n$. The set $S$ is obviously a dominating set of $U_n$, since for all $i \in [n-1]_0$ it holds that $a_i, b_i, c_i, d_i, e_i \in \mathcal{N}[b_i] \cup \mathcal{N}[c_i] \cup \mathcal{N}[d_i] \cup \mathcal{N}[e_i]$, so $\bigcup_{i=0}^{n-1} \mathcal{N}[b_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[c_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[d_i] \cup \bigcup_{i=0}^{n-1} \mathcal{N}[e_i] = \{a_i, b_i, c_i, d_i, e_i \mid i \in [n-1]_0\} = V(U_n)$. Also, $S$ is a convex set, since each shortest path between two vertices from $\{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$ contains only $b$-vertices, $c$-vertices, $d$-vertices or $e$-vertices.

Therefore, $S$ is a convex dominating set for $U_n$ and $\gamma_{conv}(U_n) \leq |S| = 4n$. Note that, as previously mentioned shortest paths are not always unique, so this argument can be used only for the convex domination, and not for the weak convex domination.

Next, we will prove that $\gamma_{wconv}(U_n) \geq 4n$. Using the same argumentation as in the proof of Theorem 7.1 it can be concluded that every $b$-vertex is in $S$, i.e. for all $i \in [n-1]_0$ it holds that $b_i \in S$ implies that $\{b_i \mid i \in [n-1]_0\} \subseteq S$.

Also, we will prove that all $e$-vertices must be in the weak convex dominating set of $U_n$, i.e. for all $i \in [n-1]_0$ it holds that $e_i \in S$. We can notice that by removing vertices $\{a_i \mid i \in [n-1]_0\}$ from $U_n$ we obtain $D_n$, and $\{b_i \mid i \in [n-1]_0\} \subseteq S$. Using
the same argumentation for vertices \( \{d_i \mid i \in [n-1]\} \) as in Theorem 2.1, we can conclude that \( \{e_i \mid i \in [n-1]\} \subseteq S \).

Similarly to the proof of Theorem 2.1, for an arbitrary \( i \in [n-1] \) the shortest path between \( b_i \) and \( e_i \) is unique \( (b_i-c_i-d_i-e_i) \), and has both endpoints in \( S \) \( (b_i, e_i \in S) \). From the fact that \( S \) is a weak convex dominating set it must hold that for all \( i \in [n-1] \) the vertices \( b_i \) and \( d_i \) belong to \( S \). Since \( \{c_i, d_i \mid i \in [n-1]\} \subseteq S \), and from previous facts, it stands that \( \{b_i, c_i, d_i, e_i \mid i \in [n-1]\} \subseteq S \), implying that \( |S| \geq 4n \).

Finally, since \( \gamma_{\text{wconv}}(U_n) \leq \gamma_{\text{conv}}(U_n) \), \( \gamma_{\text{wconv}}(U_n) \geq 4n \) and \( \gamma_{\text{conv}}(U_n) \leq 4n \), we have that \( \gamma_{\text{wconv}}(U_n) = \gamma_{\text{conv}}(U_n) = 4n \). \( \square \)

9. Conclusion

This paper presents values of the weak convex and the convex domination numbers for all known convex polytopes. Validity of these values is proven for \( n \geq 27 \), while for \( n \leq 26 \) the values are computed using the exact method. Values obtained by the exact method satisfy formulas obtained theoretically, except in some particular cases which will be mentioned below.

The results obtained are summarized in Table 3. Each row describes the weak convex and the convex dominating sets with minimal cardinality, and the corresponding weak convex and convex domination numbers for a certain polytope. The first column, labeled with \( G \), contains the type of polytope. The second and third columns, labeled with \( |V(G)| \) and \( |E(G)| \), contain the number of vertices and edges, respectively. The fourth column, labeled Cond., denotes the polytopes for which a certain result applies. In this column, ‘gc’ denotes that the result stands in the general case, while \( n = 5 \) means that the result presented with given row stands for the polytope where \( n = 5 \). The fifth column, labeled Type, describes the type of domination: ‘wconv’ for the weak convex domination and ‘conv’ for the convex domination, while ‘wconv, conv’ in a certain row means that the result stands for both weak convex and convex domination. The sixth column, labeled Card., contains the weak convex and convex domination numbers. The seventh column, labeled Basis, contains the weak convex and the convex dominating set with minimal cardinality. For example, polytope \( B_n \), in general, has the weak convex and convex dominating sets \( \{b_i, c_i, d_i \mid i \in [n-1]\} \) with \( 3n \) elements, with one exception for \( n = 5 \) where the weak convex dominating set with minimal cardinality 14 is \( \{b_i, c_i, d_i, e_i \mid i \in \{0, 1, 2\}\} \cup \{b_3, b_4\} \).

Future work could be directed to obtaining the weakly convex and convex domination numbers of some other classes of graphs. Another promising direction for future research would be to calculate other graph invariants of convex polytopes.
Table 3. Summary of results

| $G$  | $|V(G)|$ | $|E(G)|$ | Cond. | Type            | Card. | Basis                                                                 |
|------|---------|---------|-------|-----------------|-------|----------------------------------------------------------------------|
| $A_n$ | $3n$    | $7n$    | gc    | wconv, conv     | $n$   | $\{b_i \mid i \in [n-1]_0\}$                                        |
| $B_n$ | $5n$    | $9n$    | gc    | wconv, conv     | $3n$  | $\{b_i, c_i, d_i \mid i \in [n-1]_0\}$                               |
|       |         |         |       |                 | $\text{n = 5}$ | $\text{wconv}$ 14 $\{b_i, c_i, d_i, e_i \mid i = 0, 1, 2 \} \cup \{b_3, b_4\}$ |
| $C_n$ | $5n$    | $10n$   | gc    | wconv, conv     | $3n$  | $\{b_i, c_i, d_i \mid i \in [n-1]_0\}$                               |
| $D_n$ | $4n$    | $6n$    | gc    | wconv, conv     | $4n$  | $\{a_i b_i, c_i, d_i \mid i \in [n-1]_0\}$                           |
|       |         |         |       |                 | $\text{n = 5}$ | $\text{wconv}$ 10 $\{a_2, a_3, a_4, b_2, b_4, c_1, c_4, d_0, d_4, d_1\}$ |
| $E_n$ | $5n$    | $11n$   | gc    | wconv, conv     | $3n$  | $\{b_i, c_i, d_i \mid i \in [n-1]_0\}$                               |
| $Q_n$ | $4n$    | $7n$    | gc    | wconv, conv     | $2n$  | $\{b_i, c_i \mid i \in [n-1]_0\}$                                    |
| $R_n$ | $3n$    | $6n$    | gc    | wconv, conv     | $n$   | $\{b_i \mid i \in [n-1]_0\}$                                        |
| $R''_n$ | $6n$ | $9n$    | gc    | wconv, conv     | $6n$  | $\{a_i b_i, c_i, d_i, e_i, f_i \mid i \in [n-1]_0\}$                 |
|       |         |         |       |                 | $\text{n = 5}$ | $\text{wconv}$ 20 $\{b_i, c_i, d_i, e_i \mid i \in \{0, 1, 2, 3, 4\}\}$ |
| $S_n$ | $4n$    | $8n$    | gc    | wconv, conv     | $2n$  | $\{b_i, c_i \mid i \in [n-1]_0\}$                                    |
| $S''_n$ | $4n$ | $8n$    | gc    | wconv, conv     | $2n$  | $\{b_i, c_i \mid i \in [n-1]_0\}$                                   |
| $T_n$ | $4n$    | $9n$    | gc    | wconv, conv     | $2n$  | $\{b_i, c_i \mid i \in [n-1]_0\}$                                    |
|       |         |         |       |                 | $\text{n = 5}$ | $\text{wconv}$ 8 $\{a_1, b_0, b_3, b_4, c_0, c_3, c_4, d_0\}$          |
| $T''_n$ | $4n$ | $8n$    | gc    | wconv, conv     | $2n$  | $\{b_i, c_i \mid i \in [n-1]_0\}$                                  |
| $U_n$ | $5n$    | $8n$    | gc    | wconv, conv     | $4n$  | $\{b_i, c_i, d_i, e_i \mid i \in [n-1]_0\}$                          |
|       |         |         |       |                 | $\text{n = 5}$ | $\text{wconv}$ 13 $\{a_0, a_3, a_4, b_0, b_3, b_4, c_0, c_3, d_0, d_2, e_0, e_1, e_2\}$ |
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REFERENCES

[1] M. Bača, Labelings of two classes of convex polytopes, *Utilitas Math.* 34 (1988), 24–31. MR 0982242
[2] M. Bača, On magic labellings of convex polytopes, in *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity* (Prachatice, 1990), 13–16, *Ann. Discrete Math.*, 51, North-Holland, Amsterdam, 1992. MR 1206237
[3] J. Cyman, M. Lemańska and J. Raczek, Graphs with convex domination number close to their order, *Discuss. Math. Graph Theory* 26 (2006), no. 2, 307–316. MR 2330331
[4] M. Dettlaff and M. Lemańska, Influence of edge subdivision on the convex domination number, *Australas. J. Combin.* 53 (2012), 19–30. MR 2962128
[5] M. Imran, A. Q. Baig and A. Ahmad, Families of plane graphs with constant metric dimension, *Util. Math.* 88 (2012), 43–57. MR 2975822
[6] M. Imran, A. Q. Baig, M. K. Shafiq and A. Semaničová-Fečová, Classes of convex polytopes with constant metric dimension, *Util. Math.* 90 (2013), 85–99. MR 3059386
[7] M. Imran, S. A. U. H. Bokhary, A. Ahmad and A. Semaničová-Fečová, On classes of regular graphs with constant metric dimension, *Acta Math. Sci. Ser. B (Engl. Ed.)* 33 (2013), no. 1, 187–206. MR 3003753
[8] M. Imran, S. A. U. H. Bokhary and A. Q. Baig, On families of convex polytopes with constant metric dimension, *Comput. Math. Appl.* 60 (2010), no. 9, 2629–2638. MR 2728092
[9] M. Imran, S. A. U. H. Bokhary and A. Q. Baig, On the metric dimension of rotationally-symmetric convex polytopes, *J. Algebra Comb. Discrete Struct. Appl.* 3 (2016), no. 2, 45–59. MR 3493642
[10] T. N. Janakiraman and P. J. A. Alphonse, Weak convex domination in graphs. *Int. J. Eng. Sci. Adv. Comput. Biotech.* 1 (2010), no. 1, 1–13.
[11] J. Kratica, V. Kovačević-Vujčić, M. Ćangalović and M. Stojanović, Minimal doubly resolving sets and the strong metric dimension of some convex polytopes, *Appl. Math. Comput.* 218 (2012), no. 19, 9790–9801. MR 2916158
[12] J. Kratica, V. Filipović, D. Matić and A. Kartelj, An integer linear programming formulation for the convex dominating set problems, *https://arxiv.org/abs/1901.02541 [math.OC]*, 2019.
[13] M. A. Labendia and S. R. Canoy, Jr., Convex domination in the composition and Cartesian product of graphs, *Czechoslovak Math. J.* 62(137) (2012), no. 4, 1003–1009. MR 3010253
[14] M. Lemańska, Weakly convex and convex domination numbers, *Opuscula Math.* 24 (2004), no. 2, 181–188. MR 2100881
[15] M. Lemańska and R. Zuazua, Convex universal fixers, *Discuss. Math. Graph Theory* 32 (2012), no. 4, 807–812. MR 2993524
[16] M. Miller, M. Bača and J. A. MacDougall, Vertex-magic total labeling of generalized Petersen graphs and convex polytopes, *J. Combin. Math. Combin. Comput.* 59 (2006), 89–99. MR 2277342
[17] N. Polat, On isometric subgraphs of infinite bridged graphs and geodesic convexity, *Discrete Math.* 244 (2002), no. 1-3, 399–416. MR 1844048
[18] J. Raczek, NP-completeness of weakly convex and convex dominating set decision problems, *Opuscula Math.* 24 (2004), no. 2, 189–196. MR 2100882
