An Observable Canonical Form for a Rational System on a Variety

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Abstract—An observable canonical form is formulated for the set of rational systems on a variety each of which is a single-input-single-output, affine in the input, and a minimal realization of its response map. The equivalence relation for the canonical form is defined by the condition that two equivalent systems have the same response map. A proof is provided that the defined form is well-defined canonical form. Special cases are discussed.

I. INTRODUCTION

The purpose of the paper is to define an observable canonical form for a rational system on a variety which is single-input-single-output and affine in the input.

The motivation of a canonical form for this set of systems is their use in system identification. In case of blackbox modeling with a rational system, a canonical form is needed. If no canonical form is used then there arises a problem of nonidentifiability of the system parametrization with serious consequences. The canonical form in this case is based on the equivalence relation which relates two rational systems if they have the same response map.

There is another motivation which is control synthesis of rational systems. For this a canonical form is needed for both response-map equivalence and for feedback equivalence. The research issue to define a canonical form of rational systems for these combined equivalence relations is briefly discussed in the paper but it is not the main focus.

The main results of the paper are the formulation of the concept of the observable canonical form of a rational system and the theorem that the observable canonical form is well defined.

A summary of the remainder of the paper follows. The next section provides a more detailed problem formulation. Rational systems are defined in Section III while the canonical form is defined in Section IV. That the defined form is well-defined canonical form. Special cases are discussed.

II. PROBLEM FORMULATION

To be able to motivate the use of canonical forms, it has to be explained what they are.

Consider a set of control systems, each system of which has an input, a state, and an output. Associate with each system its response map which maps, for every time, the past input trajectory to the output at that time. The realization problem is the converse issue. Consider an arbitrary response map, which for every time maps a past input trajectory to the output at that time. The realization problem is to construct, for a considered response map, a system in a considered set of which the associated response map equals the considered response map. A condition must hold for an arbitrary response map to have a realization as a system in the considered set. But if the condition holds, then there may exist not one but many systems having the considered response map. Hence one restricts attention to those systems representing the response map which are minimal in a sense to be defined, usually related to the dimension of the state set. Again, there is not a unique minimal system representing the considered response map but a set of systems. Consider thus the subset of systems each of which is a minimal realization of its own response map. Define then an equivalence relation on any tuple of that subset of systems if both systems have the same response map.

A canonical form or normal form is now a subset of the subset of the considered systems for the above defined equivalence relation. A canonical form requires that each system in the considered subset is equivalent to a unique element of the canonical form.

A major motivation for canonical forms is system identification. In system identification, a canonical form of a set of systems can be used to restrict the problem of how to estimate the parameters of the system. On the set of minimal systems one defines the equivalence relation of two systems having the same response map. Without a canonical form there is an identifiability problem meaning that there exist two or more different minimal systems which represent the same response map.

A second application of a canonical form is control synthesis. In this case the equivalence relation is based on (1) the equality of the response map, and (2) feedback equivalence. Feedback equivalence is defined for two systems if the second system can be obtained from the first system by a state-feedback control law and a new input. A canonical form for these properties simplifies the control synthesis.

Canonical forms for time-invariant finite-dimensional linear systems have been formulated and proven. P. Brunovsky formulated a canonical form, [1]. W.M. Wonham and A. Morse, [2], have formulated a canonical form for the equiv-
alence of the response map and feedback equivalence. The well known text book, [3], also describes this well. Another early paper is [4]. Books which contain a discussion on canonical forms for linear systems include: [5, Section 9.2]; [6, pp. 187-192, pp. 198-199, Section 5.5, Section 6.3, Section 6.4]; [7, Sections 6.7.3, 7.1]; [8, pp. 494-508]; [9, p. 292]; and [10, p. 39, Section 5.5].

In the book [11, pp. 137–142] there is defined a normal form in local coordinates for a single-input-single-output and affine-in-the-input smooth nonlinear system on the state set \( \mathbb{R}^n \). But that form is not a canonical form as defined in this paper. Note that the normal form of that reference is obviously not controllable. There is neither a proof nor a claim in that reference that the normal form is a well-defined canonical form.

I.A. Tall and W. Respondek, [12], have formulated a canonical form for smooth nonlinear systems on a differential manifold for the equivalence relation of feedback equivalence. The approach of this paper differs from that of the paper of Tall and Respondek in that they consider feedback equivalence while in this paper only the equivalence of the response map is addressed and that in this paper the variety plays a role in the formulation of the canonical form. A rational system on a variety is characterized by both the variety and by the system. This makes the problem more complicated than the contribution of [12].

Problem II.1 Consider the set of rational systems on a variety, each of which is single-input-single-output, affine in the input, and each of which is a minimal realization of its response map. Formulate a candidate canonical form for this set of systems and prove that it is a well-defined canonical form.

The extensions to multi-input-multi-output and to other subsets of rational systems require much more space then is available in this short paper.

III. RATIONAL SYSTEMS

Terminology and notation of commutative algebra is used from the books [13], [14], [15], [16], [17]. References on algebraic geometry include, [18], [19], [20], [21], [22].

The notation of the paper is simple. The set of the integers is denoted by \( \mathbb{Z} \) and the set of the strictly positive integers by \( \mathbb{Z}_+ = \{1, 2, \ldots\} \). For \( n \in \mathbb{Z}_+ \), define \( \mathbb{Z}_n = \{1, 2, \ldots, n\} \). The set of the natural numbers is denoted by \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and, for \( n \in \mathbb{Z}_+ \), \( \mathbb{N}_n = \{0, 1, 2, \ldots, n\} \). The set of the real numbers is denoted by \( \mathbb{R} \) and that of the positive and the strictly-positive real numbers respectively by \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{R}_+ = (0, \infty) \). The vector space of \( n \)-tuples of the real numbers is denoted by \( \mathbb{R}^n \).

A subset \( X \subseteq \mathbb{R}^n \) for \( n \in \mathbb{Z}_+ \) is called a variety if it is determined by a finite set of polynomial equalities. Such a set is also called an algebraic set.

A variety is called irreducible if it cannot be written as a union of two disjoint nonempty varieties. Any variety determined by a set of polynomials \( p_1, \ldots, p_k \) of the form,

\[
X = \{ x \in \mathbb{R}^n \mid 0 = p_i(x), \forall i \in \mathbb{Z}_+ \}, \quad n, k \in \mathbb{Z}_+, 
\]

is an irreducible variety according to [18, Section 5.5, Proposition 5]. A canonical form for an irreducible variety can be formulated based on the concept of a decomposition of such a variety using prime ideals. This will not be detailed in this short paper.

Denote the algebra of polynomials in \( n \in \mathbb{Z}_+ \) variables with real coefficients by \( \mathbb{R}[X_1, \ldots, X_n] \). For a variety \( X \), denote by \( I(X) \) the ideal of polynomials of \( \mathbb{R}[X_1, \ldots, X_n] \) which vanish on the variety \( X \). The elements of \( \mathbb{R}[X_1, \ldots, X_n]/I(X) \) are referred to as polynomials on the variety \( X \). The ring of all such polynomials is denoted by \( A_X \) which is also an algebra. If the variety \( X \) is irreducible then the ring \( A_X \) is an integral domain hence one can define the field of rational functions on the variety \( X \) as a field of fractions \( Q_X \) of the algebra \( A_X \).

A polynomial and a rational function for \( n \) variables are denoted respectively by the representations (assumed to be defined over a finite sum),

\[
p(x) = \sum_{k \in \mathbb{N}_n} c_p(k) \prod_{i=1}^n X_i^k, \quad \forall k \in \mathbb{N}_n, \; c_p(k) \in \mathbb{R},
\]

\[
r(x) = \frac{p(x)}{q(x)}.
\]

For a rational function the following special form or canonical form is defined: (1) there are no common factors in the numerator and the denominator; such factors when present can be eliminated by cancellation; and (2) the constant factor in the denominator polynomial, assumed to be present, is set to one by multiplication of the numerator and the denominator with a real valued number. In case there is no constant term in the denominator then the coefficient of the highest degree in a defined ordering of the denominator polynomial, has to be set to the real number one.

\[
r(x) = \frac{p(x)}{q(x)},
\]

\[
q(x) = 1 + \sum_{k \in \mathbb{N}_n \setminus \{0\}} c_q(k) \prod_{i=1}^n X_i^k,
\]

\[
Q_X,can = \{ r(x) \in Q_X, \text{ as defined above} \}.
\]

The transcendence degree of a field \( F \), denoted by \( \text{trdeg}(F) \), is defined to be the greatest number of algebraically-independent elements of \( F \) over \( \mathbb{R} \), [14, Section 7.1, p. 293, p. 304] and [16, Ch. 2, Sections 3 and 12].

For the detailed definitions of the concepts introduced below, a rational system on a variety etc., the reader is referred to the papers, [23], [24], [25]. This includes the concept of a differential equation on a variety, see also [26], and the fact that the variety is forward invariant with respect to the differential equation. Controlled invariant hypersurfaces of polynomial control systems on varieties are considered in [27], [28].
Definition III.1 A rational system on a variety, in particular, a single-input-single-output-system which is affine in the input, is defined as a control system as understood in control theory with the representation,

\[
\begin{align*}
\frac{dx(t)}{dt} &= f_0(x(t)) + f_1(x(t))u(t), \quad x(0) = x_0, \quad (1) \\
y(t) &= h(x(t)), \quad (2) \\
f_{\alpha} &= \sum_{i=1}^{n} [f_{0,i}(x) + f_{1,i}(x)] \alpha_i \frac{\partial}{\partial x_i}, \quad \forall \alpha \in U, \quad (3) \\
f &= \{f_{\alpha}, \quad \alpha \in U\}, \quad (4) \\
s &= (X, U, Y, f_0, f_1, h, x_0) \in S_r; \quad (5)
\end{align*}
\]

where \( n \in \mathbb{Z}_+ \), \( X \subseteq \mathbb{R}^n \) is an irreducible nonempty variety called the state set, \( U \subseteq \mathbb{R} \) is called the input set, it is assumed that \( \{0\} \subseteq U \) and \( U \) contains at least two distinct elements. \( Y = \mathbb{R} \) is called the output set, \( x_0 \in X \) is called the initial state, \( f_{0,1}, \ldots, f_{0,n}, f_{1,1}, \ldots, f_{1,n} \in Q_X \) and \( h \in Q_{\mathbb{R}} \) are rational functions on the variety, \( u : [0, \infty) \rightarrow U \) is a piecewise-constant input function, and \( S_r \) denotes the set of rational systems as defined here.

One defines the set of piecewise-constant input functions which are further restricted by the existence of a solution of the rational differential equation of the system. Denote for a rational system \( s \in S_r \) as defined above, the admissible set of piecewise-constant input functions by \( U_{pc}(s) \). Further, for \( u \in U_{pc}(s), \ t_0 \in \mathbb{R}, \) denotes the life time of the solution of the differential equation for \( x \) with input \( u \). For any \( u \in U_{pc}(s) \) and any \( t \in [0, t_0) \) denote by \( u(0, t) \) the restriction of the function \( u \) to the interval \( [0, t) \). Denote the solution of the differential equation \( (7) \) by \( x(t; x_0, u(0, t)) \), \( \forall t \in [0, t_0) \).

Define the dimension of this rational system as the transcendence degree of the set of rational functions on \( X \), \( \dim(X) = \text{trdeg}(Q_X) \). In the remainder of the paper, a rational system will refer to a rational system on a variety.

Definition III.2 Associate with any rational system in the considered set, its response map as,

\[
r_s : U_{pc}(s) \rightarrow \mathbb{R}, \quad r_s \in A(U_{pc} \rightarrow \mathbb{R}),
\]

such that for \( e = \text{empty input}, \ r_s(e) = 0, \)

and such that for all \( u \in U_{pc}(s), \) if \( x : [0, t_0) \rightarrow X \) is a solution of the rational system \( s \) for \( u \), i.e. \( x \) satisfies the differential equation \( (7) \) and the output \( (2) \) is well defined, then

\[
r_s(u(0, t)) = y(t) = h(x(t; x_0, u(0, t))), \quad \forall t \in [0, t_0).
\]

The realization problem for the considered set of rational systems is, when considering an arbitrary response map, to determine whether there exists a rational systems whose response map equals the considered response map, [23], [24].

Definition III.3 Consider a response map, \( p : U_{pc} \rightarrow \mathbb{R}. \) Call the system \( s \in S_r \) a realization of the considered map \( p \) if

\[
\forall u \in U_{pc}, \ \forall t \in [0, t_0), \ p(u(0, t)) = r_s(u(0, t)).
\]

Call the system a minimal realization of the response map if \( \dim(X) = \text{trdeg}(Q_{\text{obs}}(p)) \). Define a minimal rational system to be a rational system which is a minimal realization of its own response map.

In general, a realization is not unique. Attention will be restricted to minimal realizations characterized by a condition of controllability and of observability defined next. Observability of discrete-time polynomial systems was defined in [29], observability of continuous-time polynomial systems was defined in [30], and rational observability and algebraic controllability of rational systems were defined in [24]. The formal definitions are recalled for ease of reference.

Definition III.4 Consider a rational system as defined in Def. III.1. The observation algebra \( Q_{\text{obs}}(s) \subseteq Q_X \) of a rational system \( s \in S_r \) is defined as the smallest subalgebra of \( Q_X \) which contains the \( \mathbb{R} \)-valued output map \( h \) and is closed with respect to taking Lie derivatives of the vector field of the system.

Denote by \( Q_{\text{obs}}(s) \subseteq Q_X \) the field of fractions of \( Q_{\text{obs}}(s) \) and call this set the observation field of the system. Call the system \( s \in S_r \) rationally observable if

\[
Q_X = Q_{\text{obs}}(s). \quad (6)
\]

Definition III.5 Consider a rational system as defined in Def. III.1. Call the system algebraically controllable or algebraically reachable if

\[
X = \mathbb{Z} - \text{cl}(\{x(t_0) \in X | u \in U_{pc}\}).
\]

where \( \mathbb{Z} - \text{cl}(S) \) of a set \( S \subseteq \mathbb{R}^n \) denotes the smallest variety containing the set \( S \), also called the Zariski-closure of \( S \).

See [31] for procedures how to check that algebraic controllability holds.

It follows from the existing realization theory that if a rational system is algebraically controllable and rationally observable then it is a minimal realization of its response map, [24, Proposition 6].

A minimal realization of a response map is not unique. It has been proven that any two minimal rational realizations are birationally equivalent if the condition holds that the elements of \( Q_X \backslash Q_{\text{obs}}(s) \) are not algebraic over \( Q_{\text{obs}}(s) \) for both systems \( s \). Let \( X \subseteq \mathbb{R}^n \) and \( X' \subseteq \mathbb{R}^{n'} \) be two irreducible varieties. A birational map from \( X \) to \( X' \) is a map which has \( n' \) components which are all rational functions of \( Q_X \) and for which an inverse exists such that it is a map which has \( n \) components which are all rational functions of \( Q_{X'} \). A birational map transforms a rational system on a variety to another rational system on another variety, see [24]. A reference on birational geometry is [32].

IV. CANONICAL FORMS

A canonical form is defined for the case one has a set with an equivalence relation defined on it. The terms of normal form, canonical form, or canonical normal form, all refer to the same concept. The authors prefer the expression canonical form. The following books define the concept
of canonical form, [33, p. 277], [34, Section 0.3], [35, Subsection 2.2.1], and [14, Section 4.5 Reduction Relations].

**Definition IV.1** Consider a set $X$ and an equivalence relation $E \subseteq X \times X$ defined on it. A canonical form or a normal form for this set and this equivalence relation, consists of a subset $X_{c} \subseteq X$ such that, for any $x \in X$, there exists a unique element of the canonical form $x_{c} \in X_{c}$, such that $(x, x_{c}) \in E$.

A canonical form is nonunique in general. One may impose conditions on the canonical form if there is an algebraic structure on the underlying set.

In system theory, a canonical form is needed for the realization of a response map. Realization theory provides a condition for the existence of a realization of a response map. One then restricts attention to the subset of minimal systems; equivalently, those systems which are minimal realizations of their own response map. One then defines an equivalence relation on the set of minimal systems if the response maps of the two considered minimal systems are equal.

In system theory there have been defined canonical forms for two different equivalence relations defined next:

1) response-map equivalence; or
2) feedback-and-response-map equivalence.

**Definition IV.2** Consider the set $S_{r, \text{min}}$ of minimal rational systems on a variety $X$. Define the response-map equivalence relation $E_{rm}$ on this set of systems by the condition that two systems are equivalent if their response maps are equal. One then says that the considered two systems are response-map equivalent.

**Definition IV.3** Consider the set $S_{r, \text{min}}$ of minimal rational systems on a variety $X$. Define the feedback-and-response-map equivalence relation $E_{frm}$ on this set of systems by the condition that two systems are response-map equivalent and state-feedback equivalent. System 1 and System 2 are called state-feedback equivalent if System 1 is response-map equivalent with System 2 after closing the loop with a state-feedback and a new input variable $v$ according to,

$$
\begin{align*}
    u &= g_{0}(x) + g_{1}(x)v, \quad g_{0}, g_{1} \in Q_{X}, \\
    dx(t)/dt &= \left[f_{0}(x(t)) + f_{1}(x(t))g_{0}(x(t))\right] + f_{1}(x(t))g_{1}(x(t))v(t), \\
    y(t) &= h(x(t)),
\end{align*}
$$

One then says that the considered two systems are feedback-and-response-map equivalent.

Feedback equivalence is best considered in combination with the control canonical form. For an observable canonical form one may define an observer-feedback equivalence relation not further discussed in this paper.

**Problem IV.4** Consider the set $S_{r, \text{min}}$ of rational systems each of which is single-input-single-output and affine in the input, and each of which is a minimal realization of its response map. The problem is to define a canonical form for (1) the response-map equivalence and for (2) the feedback-and-response-map equivalence. Prove that each of the defined forms is a well-defined canonical form.

In this paper only a canonical form for the first equivalence relation is provided. The solution for the second equivalence relation is postponed.

For a set of systems and for the response-map equivalence relation there is no unique canonical form. For time-invariant finite-dimensional linear systems there have been defined both a control canonical form and an observable canonical form. Which of the many canonical form is most appropriate depends on other objectives of the user.

The formulation of a canonical form for the set of minimal rational systems as specified above has to involve: (1) the variety of the state set $X$ and (2) the functions specifying the minimal rational system, both the dynamics and the output map. This makes the problem different from that of a canonical form for a time-invariant linear system which are defined on the state-space $\mathbb{R}^{n}$ and by the functions of the dynamics and the output equation. In the linear case there is no restriction on the state space except its dimension. In the book [29, Sections 27 and 28] there is a discussion of these two cases for discrete-time polynomial systems.

Due to the above remark, there are two types of canonical forms for rational systems:

1) with a structured rational systems and an unrestricted arbitrary variety; and
2) with a variety of a given structure and with unstructured or partly-structured rational system.

One may also define a canonical form for the description of the variety of the state set. The classification of algebraic varieties up to a birational equivalence is the main problem studied within birational geometry, see [32]. It is proven that every $n$-dimensional irreducible variety over an algebraically closed field is birationally equivalent to a hypersurface in $\mathbb{R}^{n+1}$. Hence, good candidates for a canonical form for the description of the variety of the state set are hypersurfaces in $\mathbb{R}^{n+1}$. Note that a hypersurface is given by a homogeneous polynomial, a polynomial whose nonzero terms all have the same degree.

**Definition IV.5** Consider the set of rational systems as defined in Def. [11.7]. Assume that the class of systems is restricted from $S_{r}$ to $S_{rr}$ so that for any system $s$ in the considered class $S_{rr}$, $Q_{X}/Q_{\text{obs}}(s)$ is not algebraic over $Q_{\text{obs}}(s)$.

Define the observable canonical form on the set of minimal rational systems $S_{rr}$, for the response-map equivalence relation as the algebraic structure described by the equations,

$$
\begin{align*}
    X &= \{x \in \mathbb{R}^{n} | 0 = p_{i}(x), \forall i \in \mathbb{Z}_{k}\}, \\
    d &= \text{trdeg}(Q_{\text{obs}}(s)) \in \mathbb{Z}_{+}, \quad n, k \in \mathbb{Z}_{+}, \\
    dx_{1}/dt &= x_{2}(t) + f_{1}(x(t))u(t),
\end{align*}
$$

(10)
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= x_{i+1}(t) + f_i(x(t)) \ u(t), \\
& \quad i = 2, 3, \ldots, n-1, \\
\frac{dx_n(t)}{dt} &= f_{n,0}(x(t)) + f_{n,1}(x(t)) \ u(t), \\
y(t) &= x_1(t), \\
\end{align*}
\] (11)

Every system of the observable canonical form is assumed to be algebraically controllable.

The assumptions algebraic controllability in the above definition of an observable canonical form is necessary due to the focus on observability. Even in the case of the observable canonical form of a time-invariant minimal linear system, the condition of controllability has to be imposed. It is conjectured that the condition of controllability has to be imposed. It is conjectured that the condition of controllability has to be imposed. It is conjectured that the condition of controllability has to be imposed.

The proofs of the above items are provided in Section V.

There follow the special cases of rational systems in the observable canonical form for systems with the state-space dimensions one and two.

Example IV.6 Consider the rational system with state-space dimension one, \( n = 1 \).

\[
\begin{align*}
X &= \{ x \in \mathbb{R}^1 \mid 0 = p_1(x) = \ldots = p_k(x) \}, \ k \in \mathbb{Z}_+, \\
\frac{dx(t)}{dt} &= f_0(x(t)) + f_1(x(t)) \ u(t), \ x(0) = x_0, \\
y(t) &= x(t), \\
f_0, f_1 \in Q_{X,can}, \ (\forall x \in X \setminus A_e, \ f_1(x) \neq 0), \\
A_e \subset A_X \text{ an algebraic set,}
\end{align*}
\] (14)

Assume in addition that the system is algebraically controllable. Then this particular system is in the observable canonical form. The state set of this system is an irreducible variety in \( \mathbb{R}^1 \). An irreducible variety in \( \mathbb{R}^1 \) is either a singleton or all of \( \mathbb{R}^1 \).

Example IV.7 Consider the rational system with state-space dimension two, \( n = 2 \).

\[
\begin{align*}
X &= \{ x \in \mathbb{R}^2 \mid 0 = p_1(x) = \ldots = p_k(x) \}, \ k \in \mathbb{Z}_+, \\
\frac{dx_1(t)}{dt} &= x_2(t), \\
\frac{dx_2(t)}{dt} &= -\frac{x_2(t)}{1 + x_1(t)^2} \ u(t), \\
y(t) &= x_1(t),
\end{align*}
\] (19)

Assume in addition that the system is algebraically controllable. Then this particular system is in the observable canonical form.

Definition IV.9 Consider a minimal rational system. Define the observability index with respect to response-map equivalence of this system, as the integer:

\[
\begin{align*}
n_o &= \min_{k \in \mathbb{Z}_+, \ Q_{obs}(x) = Q_{G_k}} |G_k|, \\
G_k &= \left\{ h, L_{f_{a_1}} h, L_{f_{a_2}} L_{f_{a_1}} h, \ldots \right\}, \\
L_{f_a} &= \sum_{i=1}^n \left[ f_0(x) + f_1(x) \alpha_i \right] \frac{\partial}{\partial x_i}. \quad (23)
\end{align*}
\] (23)

In words, the observability index is the minimal number of elements of the set \( G_k \) consisting of the output map \( h \) and its Lie derivatives up to order \( k-1 \) such that this set is a generator set of the observability field \( Q_{obs}(x) \) of the system. Here \( |G_k| \) denotes the number of elements in the set \( G_k \).

The dimension of an irreducible variety \( X \) is defined as its Krull dimension and coincides with the transcendence degree of its function field \( Q_X \). Hence, the dimension of the state set \( X \subset \mathbb{R}^n \) of a rational system \( s \) as well as the transcendence basis of \( Q_X \) is finite \( (\leq n) \). Recall that the transcendence degree is defined as the smallest number of generators of a field which are algebraically independent. If the system \( s \) is rationally observable, such that \( Q_X = \ldots \)
Example IV.10 Consider the polynomial system $s$,

$$
\begin{align*}
X &= \mathbb{R}^2, \\
\dot{x}(t) &= \begin{pmatrix} x_1(t) - x_2^2(t) + x_2(t) \\ x_2(t) \end{pmatrix}, \quad x(0) = x_0, \\
y(t) &= x_1(t) = h(x(t)).
\end{align*}
$$

Calculations then yield that,

$$
\begin{align*}
b_1(x) &= h(x) = x_1, \\
Q_x \neq Q(\{b_1\}), \\
b_2(x) &= L_f h(x) = x_1 - x_2^2 + x_2, \\
Q_x \neq Q(\{b_1, b_2\}), \\
b_3(x) &= (L_f)^2 h = x_1 - 3x_2^2 + 2x_2, \\
Q_x = Q(\{b_1, b_2, b_3\}) = Q_{\text{obs}}(s), \\
n_0 &= 3 > 2 = n.
\end{align*}
$$

The conclusion is that the observability index can be strictly higher than the state-space dimension of a rational system.

Definition IV.11 Consider the set of rational systems each of which is single-input-single-output and affine in the input, and each of which is a minimal realization of its response map. Define the variety-structured canonical form of such a rational system for the response-map equivalence relation as the algebraic structure described by the equations,

$$
\begin{align*}
X &= \mathbb{R}^n, \text{ or a hypersurface of } \mathbb{R}^{n+1}, \\
\dot{x}(t) &= f_0(x(t)) + f_1(x(t)) u(t), \quad x(0) = x_0, \\
y(t) &= h(x(t)), \\
f_0, f_1 \in Q_{X, \text{can}}, \quad h \in Q_{X, \text{can}}.
\end{align*}
$$

It could be that the functions $f$ and $h$ are partly structured for which in each case a particular birational equivalence form has to be formulated.

A canonical form of the set of minimal rational system could also be formulated in an algebraic way by a subfield of the observation field. A rational system is basically equivalent with its observation field. It follows from [23, Th. 6.1] that for a response map there exists a rationally-observable realization if and only if the observation field of the response map has a finite transcendence degree. Equivalently, if for the observation field of the response map there exists a finite set of generators. A rationally-observable rational system realizing the response map is therefore completely described by the observation field of its response map. Because the observation field of the response map and the observation field $Q_{\text{obs}}(s)$ of the rational system $s$ are isomorphic, the above condition on the observation field of the response map can be rewritten in terms of the observation field of the systems, $Q_{\text{obs}}(s)$. This relation allows one then to formulate a canonical form in terms of the observation field of a system rather than in terms of the rational system representation.

The algebraic formulation of an observable canonical form follows.

Definition IV.12 Consider the set of rational systems each of which is single-input-single-output and affine in the input, and each of which is a minimal realizations of its response map. Define the sequence of canonical observation subfields of this system on the basis of the observable canonical form by the equations,

$$
\begin{align*}
Q_{X,i} &= \{Q_{X,i} \subset Q_{\text{obs}}(s), \quad \forall i \in \mathbb{Z}_n\}, \\
Q_{X,i} &= \{r \in Q_{\text{obs}}(s) \subseteq Q_X | x_{i+1} = 0, \ldots, x_n = 0\}, \\
Q_{X,d} &= Q_{\text{obs}}(s) \quad \forall i \in \mathbb{Z}_d.
\end{align*}
$$

This algebraic formulation of the observable canonical form allows also an extension to controlled-invariant observation subfields by state feedback. This is then the analogon for rational systems of the concept of a dynamic cover defined in [2]. This topic will be addressed in a future publication.

V. THEOREM

The reader finds in this section a proof that the observable canonical form is a well-defined canonical form.

Proposition V.1 Consider a rational system in the observable canonical form of Def. IV.5. This system is:

(a) rationally observable; and
(b) a minimal realization of its response map.

Proof: (a) The proof is provided for the case $n = 2$ from which the general case is easily deduced. The rational system in the observable canonical form is represented by the equations,

$$
\begin{align*}
\dot{x}_1(t) &= x_2(t) + f_1(x(t)) u(t), \\
\dot{x}_2(t) &= f_2(x(t)) + f_2(x(t)) u(t), \\
y(t) &= x_1(t) = h(x(t)).
\end{align*}
$$
The observation field is calculated according to,

\[ L_{fa} = \left[ x_2 + f_1(x)\alpha_1 \frac{\partial}{\partial x_1} + f_2(x) + f_2(1)\alpha_2 \frac{\partial}{\partial x_2} \right], \]

\[ Q_{obs} = Q \left( \{ h, L_{fa_1} \ldots L_{fa_k} h, \right. \]

with only a constant input, the system is not rationally observable. With this, the input has to be varied to make the system rationally observable, see [24, Proposition 8]. Note that the inclusion \( r_{Q_{obs}} \subseteq r_{Q_X} \) follows from the observation field of \( s \) and from the assumption \( \{0\} \subseteq U \). The equality \( r_{Q_{obs}}(s) = r_{Q_X} \) follows from the rational observability of the system \( s \). By induction it follows that,

\[ \exists_i = b_i(x) \in Q_X, \]

\[ d\exists_i(t)/dt = \exists_{i+1}(t) + g_{i+1}(x(t))u(t), i = 2, \ldots \]

It follows from Definition [IV.9] and the discussion below it that there exists a finite observability index \( n_0 \) such that \( b_1(x), \ldots, b_{n_0}(x) \) generate \( Q_X \). Because \( Q_X \) is the smallest set of rational functions on \( X \) which distinguishes the points of \( X \), its generators \( b_1(x), \ldots, b_{n_0}(x) \) distinguish the points of \( X \) as well. Therefore, the map

\[ b : X \to \mathbb{X} \]

is injective and maps \( X \) onto \( b(X) \). Let us define

\[ \mathbb{X} = \mathbb{Z} - \text{cl}(b(X)). \]

Then the rational map \( b = (b_1, \ldots, b_{n_0}) : X \to \mathbb{X} \) is invertible with the rational inverse \( b^{-1} : \mathbb{X} \to X \). The fact that the components of \( b^{-1} \) are elements of \( Q_{\mathbb{X}} \) follows from the construction of \( b \) (its components contain the components of \( h, L_{fa} \) and transcendence basis elements of \( Q_{obs} \)), see

**Proposition V.3** Any minimal rational system \( s \) such that \( Q_X \setminus Q_{obs}(s) \) is not algebraic over \( Q_{obs}(s) \), can be transformed by a birational map to a rational system in the observable canonical form of Def. [IV.5] such that both systems are response-map equivalent.

**Proof:** Consider a minimal rational system \( s \) in the representation,

\[ dx(t)/dt = f_0(x(t)) + f_1(x(t))u(t), \]

\[ y(t) = h(x(t)), x(t) \in \mathbb{R}^n. \]

Define the new state variable,

\[ x_1 = h(x) = b_1(x); \text{ then,} \]

\[ dx_1(t)/dt = \sum_{i=1}^n \frac{\partial h(x(t))}{\partial x_i} f_i(x(t)) + \sum_{i=1}^n \frac{\partial h(x(t))}{\partial x_i} f_i(x(t))u(t) \]

Define the rational function and the variable,

\[ g_{1,1}(x) = \sum_{i=1}^n \frac{\partial h(x(t))}{\partial x_i} f_i(x) \in Q_{obs}(s) = Q_X, \]

\[ \exists_2 = \frac{dx_1(t)}{dt} - g_{1,1}(x(t))u(t) \]

\[ = \sum_{i=1}^n f_i(x) \frac{\partial h(x(t))}{\partial x_i} = b_2(x) \in Q_{obs}(s) = Q_X, \]

\[ dx_1(t)/dt = \exists_2(t) + g_{1,1}(x(t))u(t). \]

The proof will be provided for the case of dimension \( n = 2 \) from which the general case follows by induction. Consider System 1 with state \( x \) and System 2 with state \( \exists \) both in the observable canonical form with the same state-set dimension. Assume that the systems are birationally related by the map \( b : X \to \mathbb{X} \) hence \( \exists_1 = b_1(x_1, x_2) \) and \( \exists_2 = b_2(x_1, x_2) \). Then it follows from the observable canonical form for both systems that,

\[ x_1 = y = \exists_1 = b_1(x_1, x_2); \]

\[ d\exists_1(t)/dt = \exists_2 + f_1(x)\alpha = dx_1(t)/dt = x_2 + f_1(x)\alpha; \]

\[ x_2 - x_2 = [f_1(x) - \frac{\partial}{\partial \alpha}]\alpha; \text{ by assumption on } U \Rightarrow \alpha_1, \alpha_2 \in U, \alpha_1 \neq \alpha_2, \]

\[ x_2 - x_2 = [f_1(x) - \frac{\partial}{\partial \alpha}]\alpha_1 = [f_1(x) - \frac{\partial}{\partial \alpha}]\alpha_2, \]

0 \quad (\alpha_1 - \alpha_2)[f_1(x) - \frac{\partial}{\partial \alpha}]\alpha_2, \]

and, by \( (\alpha_1 - \alpha_2) \neq 0 \),

0 \quad [f_1(x) - \frac{\partial}{\partial \alpha}] \Rightarrow \exists_2 - x_2 = 0, \]

\[ x_2 = \exists_2 = b_2(x_1, x_2) \Rightarrow b(x) = x. \]

**Proposition V.2** If two rational systems on a variety are both in the observable canonical form, of the same state-space dimension, and response-map equivalent and hence birationally equivalent, then they are identical.

**Proof:** It follows from [24, Th. 8] that if two rational systems have the same response map and if they are both rationally observable and algebraically controllable then they are birationally related.

The proof will be provided for the case of dimension \( n = 2 \) from which the general case follows by induction. Consider System 1 with state \( x \) and System 2 with state \( \exists \) both in the observable canonical form with the same state-set dimension. Assume that the systems are birationally related by the map \( b : X \to \mathbb{X} \) hence \( \exists_1 = b_1(x_1, x_2) \) and \( \exists_2 = b_2(x_1, x_2) \). Then it follows from the observable canonical form for both systems that,
It follows by induction that,

\[ b^{-1}(b(x)) = x, \quad b(b^{-1}(x)) = x. \]

Next, define the function,

\[ f_{i,1}(x) = g_{i,1}(b^{-1}(x)) \in Q_X, \text{ then,} \]
\[ dx_i(t)/dt = x_i(t) + f_{i,1}(x)(t)u(t). \]

It follows by induction that,

\[ f_{i,1}(x) = g_{i,1}(b^{-1}(x)) \in Q_X, \]
\[ dx_i(t)/dt = x_i(t) + f_{i,1}(x)(t)u(t), \]
\[ x_i \in b_i(x) \in Q_X, \quad i = 1, \ldots, n_0; \]
\[ dx_0(t)/dt = \sum_{i=1}^{n} [f_{i,0}(x) + f_{i,1}(x)u] \frac{\partial b_{n_0}(x)}{\partial x_i} \]
\[ + \sum_{i=1}^{n} f_{i,1}(x) \frac{\partial b_{n_0}(x)}{\partial x_i}u(t) \]
\[ = b_{n_0}(x_0(t)) + g_{n_0,1}(x(t))u(t), \]
\[ = f_{n_0,0}(x(t)) + f_{n_0,1}(x(t))u(t), \text{ where,} \]
\[ f_{n_0,0}(x(t)) = g_{n_0,0}(b^{-1}(x)), \]
\[ f_{n_0,1}(x(t)) = g_{n_0,1}(b^{-1}(x)) \in Q_X. \]

Thus a rational system in the observable canonical form is obtained. Then it follows with [24, Def. 18, 19] that the original system and the constructed system are birationally equivalent and hence response-map equivalent.

**Theorem V.4** Consider the set of rational systems which is such that for any system \( s \) in this set it holds that \( Q_X \setminus Q_{obs}(s) \) is not algebraic over \( Q_{obs}(s) \). Restrict further attention to minimal rational systems and denote the resulting set of systems by \( S_{r,min} \).

The observable canonical form of Def. [V.5] for the specific set of minimal rational systems \( S_{r,min} \) and for response-map equivalence, is a well-defined canonical form.

**Proof:** The following three items prove the theorem. (1) Each rational system in the observable canonical form is a minimal realization of its response map. This follows from Proposition [V.1]. (2) Every rational system of a variety which is a minimal realization of its response map can be transformed to a rational system in the observable canonical form. This follows from Proposition [V.3]. (3) That a rational system can be transformed to a unique rational system in the observable canonical form. This condition, due to minimal realizations being birationally equivalent, is equivalent to proving that two rational systems in the observable canonical form which are birationally related, are identical. This follows from Proposition [V.2].

VI. CONCLUDING REMARKS

Further research is needed on the algebraic formulation of canonical forms, on the control canonical form for the set of rational systems, and on the use of the observable canonical form for system identification.

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