NULL CONTROLLABILITY FOR A CLASS OF STOCHASTIC SINGULAR PARABOLIC EQUATIONS WITH THE CONVECTION TERM

LIN YAN AND BIN WU*

School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China

(Communicated by Michael Malisoff)

Abstract. This paper concerns the null controllability for a class of stochastic singular parabolic equations with the convection term in one dimensional space. Due to the singularity, we first transfer to study an approximate nonsingular system. Next we establish a new Carleman estimate for the backward stochastic singular parabolic equation with convection term and then an observability inequality for the adjoint system of the approximate system. Based on this observability inequality and an approximate argument, we obtain the null controllability result.

1. Introduction. Null controllability is an very important field of controllability theory, which has been widely studied for various mathematical models, such as parabolic equation [22, 29, 38], wave equation [40], KdV equation [17], Kuramoto-Sivashinsky equation [8, 16], Schrödinger equation [26] and so on. It is well known that the key ingredient for proving the null controllability is to obtain the observability inequality for the corresponding adjoint system. An important tool to study observability inequality is Carleman estimate, which is a class of weighted energy estimates related to some differential operator. Now, Carleman estimate becomes a useful tool to solve the unique continuation property [35], inverse problems [6, 36, 37] and the control theory [12, 15, 19] for deterministic partial differential equations. As for Carleman estimates for stochastic control theory, we refer to the seminal paper [2] and [24, 29]. Recently, Carleman estimates were also applied to inverse problems [25, 39].

There are rich references on the controllability for degenerate/singular parabolic equations. We refer to [1, 4, 5, 33] for deterministic degenerate parabolic equation, [7] for deterministic singular equation. As for the controllability results for the stochastic degenerate parabolic equation, we refer to [23] and [34]. To the best of our knowledge, there is no paper considering the controllability for stochastic singular equations. The goal of this paper is to prove the null controllability of a class of stochastic boundary singular parabolic equations with the convection term.
Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\) is defined such that \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration generated by \(B(\cdot)\), augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Let \(I = (0, 1)\), \(Q_T := I \times (0, T)\). Then for given subdomain \(\omega = (x_1, x_2)\) such that \(0 < x_1 < x_2 < 1\), we consider the following forward stochastic singular parabolic equation with convection term

\[
\begin{aligned}
&x^\alpha du - u_{xx}dt + au_xdt + budy = g|\omega|dt + GdB(t), \quad (x, t) \in Q_T, \\
u(0, t) = u(1, t) = 0, \quad t \in (0, T), \\
u(x, 0) = u_0(x), \quad x \in I,
\end{aligned}
\]

where \(\alpha > 0\), \(|\omega|\) is the characteristic function of \(\omega\). Functions \(a, b\) are suitable coefficients. In system (1), \(u\) is the state variable, \((g, G)\) is a pair of control variables. Obviously, the equation is singular at the boundary \(x = 0\). The singular parabolic equation can describe the propagation of a thermal wave in an inhomogeneous medium \([20, 28]\), where \(u\) is the temperature and \(x^\alpha\) represent the particle density. The thermal phenomena in an inhomogeneous medium can be effected by the externally added heat sources, called control. Therefore, a natural controllability question would rather be: Can we steer the temperature of the thermal wave at time \(T\) to 0 by selecting a suitable source.

Through this paper, for a Banach space \(H\) we use \(C([0, T]; H)\) to denote the Banach space of all \(H\)-valued strongly continuous functionals defined on \([0, T]\). Also, we denote by \(L_2^\beta(0, T; H)\) the Banach space consisting of all \(H\)-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted processes \(X(\cdot)\) such that \(E([X(\cdot)]^2_{L_2(0, T; H)}) < \infty\), with the canonical norm; by \(L_2^\beta(0, T; H)\) the Banach space consisting of all \(H\)-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted bounded processes; and by \(L_2^\beta(\Omega; C([0, T]; H))\) the Banach space consisting of all \(H\)-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted continuous processes \(X\) such that \(E([X]^2_{C([0, T]; H)}) < \infty\), with the canonical norm. Moreover, for some given constant \(\beta\), we define

\[
L_2^\beta(I) := \left\{ u \in L^2(I) \mid x^\beta u \in L^2(I) \right\}
\]

endowed with the norm

\[
||u||_{L_2^\beta(I)} := E \int_I x^\beta u^2 dx.
\]

By applying a coordinate transformation

\[
y = x^{1 + \alpha}, \quad x \in I,
\]

we can rewrite (1) as

\[
\begin{aligned}
d\hat{u} - (\alpha + 1)^2 (y^{\alpha/(\alpha+1)} \hat{u}_y)_y dt + (\alpha + 1)\hat{u}_y dt + y^{-\alpha/(\alpha+1)} \hat{b} \hat{u} dt \\
\hat{u}(0, t) = \hat{u}(1, t) = 0, \\
\hat{u}(y, 0) = u_0(y), \quad y \in I,
\end{aligned}
\]

where \(\hat{\omega} = (x_1^{\alpha+1}, x_2^{\alpha+1})\), \(\hat{u}(y, t) = u(y^{1/(\alpha+1)}, t)\) and other functions \(\hat{a}, \hat{b}, \hat{g}, \hat{G}\) are defined analogously. Here, (2) is degenerate at the boundary \(y = 0\), meanwhile the reaction term and convection term are singular. Therefore the controllability is more complicated than the ones related to the stochastic degenerate equations \([23]\) or \([34]\).

As for deterministic boundary singular parabolic equations, we refer to \([11]\). In this paper the authors considered the following singular parabolic equation without
null controllability was obtained for any $\beta > 0$. We also refer to \[3, 31\] for the analysis of control properties of the internal singular parabolic equation. In \[10\], the author studied the following degenerate equation with convection term:
\[
\begin{align*}
&u_t - (x^\alpha u)_x + a u_x + bu = h, \\
&u(0, t) = u(1, t) = 0, \\
&u(x, 0) = u_0(x),
\end{align*}
\]
where $b, c \in L^\infty(Q_T)$. They proved (4) is null controllable for $\beta \in (0, \frac{1}{2})$. There are also other studies on the controllability for deterministic degenerate parabolic equations \[13, 14\]. Recently, the study of controllability for stochastic non degenerate parabolic equations has attracted the interest of several authors, we refer to the seminal work \[2\] and \[29\]. All these results were obtained using Carleman estimates.

In this paper, we investigate the null controllability of (1). Due to the singularity of system (1), we first transfer to study an approximate nonsingular system. By means of a weighted identity method, we prove a uniform Carleman estimate for this approximate system. In order to deal with convection term, we use a duality technique proposed by Imanuvilov and Yamamoto in \[19\] or Liu \[22\]. Owing to we only assume $a \in L^\infty_F(0, T; L^\infty(I))$, we obtain the Carleman estimate only for the case $\alpha \in (0, 1)$. Combining this Carleman estimate and an approximate argument, we deduce the null controllability result.

Now we state the main result in this paper.

**Theorem 1.1.** Let $\alpha \in (0, 1)$ and $a, b \in L^\infty_F(0, T; L^\infty(I))$. Then for any given $u_0 \in L^2(\Omega, F_0; L^2(I))$, there exists a pair of $(g, G)$ such that $g \in L^2_F(0, T; L^2(\omega))$, $x^{-2}G \in L^2_F(0, T; L^2(I))$ and the solution of (1) satisfies $u(x, T) = 0$ in $I$, $P - a.s.$

**Remark 1.** For deterministic case, in \[10, 32\] the authors pointed out that the restriction $\beta \in (0, \frac{1}{2})$ was optimal for establishing the Carleman estimate for (4) under the convection term coefficient $a \in L^\infty(Q_T)_F$. Therefore the equivalent version (2) of the original problem (1) seems controllable only for $\frac{\alpha}{1+\alpha} \in (0, \frac{1}{2})$, i.e. $\alpha \in (0, 1)$, even if there is no singularity in (2). In other words, $\alpha \in (0, 1)$ could not be improved in Theorem 1.1 under $a \in L^\infty_F(0, T; L^\infty(I))$ based on the method used in this paper. On the other hand, from Lemma 4.3 one also easily see that Carleman estimate holds only for $\alpha \in (0, 1)$.

**Remark 2.** Theorem 1.1 requires the introduction of an extra control $G$ owing to the coefficients $a, b$ in (1) depend on both time and space variables, and on the random parameter $\omega$ as well. While establishing the observability estimate for the corresponding adjoint system of (1), it is unknown how to prove the observability estimate except for some special restrictions on coefficients. Indeed, inspired by \[2\],
if these coefficients are assumed to be independent of the space variable \( x \), we can reduce the problem to a deterministic problem, then the control holds only for a control \( g \). It is still open whether system (1) is controllable with a control \( G \) or when \( G \) acts only on a subdomain of \( I \).

The rest of this paper is organized as follows. In next section, we give the well-posedness of (1). In next two sections, we obtain two Carleman estimates for backward stochastic parabolic equation without lower order terms and the backward stochastic parabolic equation with the convection term, respectively. In section 5, we prove the null controllability for system (1), i.e. Theorem 1.1.

2. Well-posedness. In this section, we prove the well-posedness of the stochastic singular parabolic equation:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
x^\alpha du - u_{xx}dt + au_xdt + bu dt = f dt + FdB(t), & (x, t) \in Q_T, \\
u(0, t) = u(1, t) = 0, & t \in (0, T), \\
u(x, 0) = u_0(x), & x \in I.
\end{array} \right.
\tag{5}
\end{aligned}
\]

We introduce several weighted spaces

\[
\mathcal{P}_T := L^2_T(\Omega; C([0, T]; L^2(I))) \cap L^2_T(0, T; H^1_0(I)),
\]

\[
\mathcal{H}^\alpha_T := L^2_T(\Omega; C([0, T]; L^2(I))) \cap L^2_T(0, T; H^1_0(I)),
\]

\[
S_T^\alpha := L^2_T(0, T; L^2_{-\alpha}(I)).
\]

We endow \( \mathcal{P}_T \) with the following norms making it to be a Banach space:

\[
\|u\|_{\mathcal{H}^\alpha_T} = \left( E \max_{t \in [0, T]} \int_I x^\alpha |u|^2 dx + E \int_{Q_T} |u_x|^2 dx \right)^{\frac{1}{2}},
\]

\[
\|u\|_{S_T^\alpha} = \left( E \int_{Q_T} x^{-\alpha} |u|^2 dx \right)^{\frac{1}{2}}.
\]

**Definition.** A stochastic process \( u \in \mathcal{H}^\alpha_T \) is said to be a weak solution of (5) if \( u(x, 0) = u_0(x) \) in \( I \), \( \mathbb{P} \)-a.s., and it holds that

\[
\int_I x^\alpha u(x, t) \varrho(x) dx - \int_I x^\alpha u_0(x) \varrho(x) dx + \int_0^t \int_I u_x(x, \tau) \varrho_x(x) dx d\tau + \int_0^t \int_I [a(x, \tau)u_x(x, \tau) + b(x, \tau)u(x, \tau)] \varrho(x) dx d\tau
\]

\[
= \int_0^t \int_I f(x, \tau) \varrho(x) dx d\tau + \int_0^T \int_I F(x, \tau) \varrho(x) dB_\tau(\varrho)(x)
\]

for every \( \varrho \in C_0^\infty(T) \) and all \( (\omega, t) \in \Omega \times [0, T] \).

We have the following well-posedness result for system (5).

**Theorem 2.1.** Let \( \alpha \in (0, +\infty) \) and \( a, b \in L^\infty_T(0, T; L^2(I)), \ u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2_0(I)), \ f \in L^2_T(0, T; L^2(I)), \ F \in L^2_T(0, T; L^2_{-\alpha}(I)) \). Then (5) admits a unique weak solution \( u \in \mathcal{H}^\alpha_T \) such that

\[
\|u\|_{\mathcal{H}^\alpha_T} \leq C\|u_0\|_{L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2_0(I))} + C\|f\|_{L^2_T(0, T; L^2(I))} + C\|F\|_{L^2_T(0, T; L^2_{-\alpha}(I))},
\]

where \( C \) depends on \( T, a, b \) and \( \alpha \).
Proof. Due to the system (5) is singular at the boundary \( x = 0 \), we first consider the nonsingular approximate system, then the existence and uniqueness of solution to (5) is based on the uniform energy estimates for solution to the approximate problem and an approximate argument. We first consider the following nonsingular approximate system

\[
\begin{aligned}
(x + \varepsilon)^\alpha d\xi^\varepsilon - u_{x,x}^\varepsilon dt + au_x^\varepsilon dt + bu^\varepsilon dt = f dt + FdB(t), & \quad (x, t) \in Q_T, \\
u^\varepsilon(0, t) = u^\varepsilon(1, t) = 0, & \quad t \in (0, T), \\
u^\varepsilon(x, 0) = u_0^\varepsilon(x), & \quad x \in I,
\end{aligned}
\]

where \( \varepsilon \in (0, 1) \), \( u_0^\varepsilon \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)) \) satisfying

\[
u_0^\varepsilon \to u_0 \text{ in } L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)), \quad \text{as } \varepsilon \to 0.
\]

According to [9], we know (8) admits a unique solution \( u^\varepsilon \in \mathcal{P}_T \).

By using \( d \left[(x + \varepsilon)^\alpha |u^\varepsilon|^2\right] = 2(x + \varepsilon)^\alpha u^\varepsilon du^\varepsilon + (x + \varepsilon)^\alpha (du^\varepsilon)^2 \) and the equation of \( u^\varepsilon \) (8), we have

\[
\int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + 2 \int Q \varepsilon |u_x^\varepsilon|^2 dx d\tau \\
\geq \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + \int Q \varepsilon 2u^\varepsilon(-au_x^\varepsilon - bu^\varepsilon + f) dx d\tau \\
+ 2 \int Q \varepsilon u^\varepsilon Fdx dB(\tau) + \int Q \varepsilon (x + \varepsilon)^{-\alpha} |F|^2 dx d\tau \\
\leq \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + \frac{1}{2} \int Q \varepsilon |u_x^\varepsilon|^2 dx d\tau \quad C \int Q \varepsilon |u^\varepsilon|^2 dx d\tau \\
+ \int Q \varepsilon |f|^2 dx d\tau + 2 \int Q \varepsilon u^\varepsilon Fdx dB(\tau) + \int Q \varepsilon (x + \varepsilon)^{-\alpha} |F|^2 dx d\tau, \tag{9}
\]

where \( C \) depends on \( a, b \).

On the other hand, for any \( \delta \in (0, 1) \), we have

\[
\int Q \varepsilon |u^\varepsilon|^2 dx d\tau = \int_0^\delta |u^\varepsilon|^2 dx + \int_\delta^1 |u^\varepsilon|^2 dx \\
\leq \int_0^\delta \left( \int_0^x u_\varepsilon^\varepsilon(y, t) dy \right)^2 dx + \delta^{-\alpha} \int_\delta^1 (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx \\
\leq \int_0^\delta x \left( \int_0^x |u_\varepsilon^\varepsilon(y, t)|^2 dy \right)^2 dx + \delta^{-\alpha} \int_\delta^1 (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx \\
\leq \frac{\delta^2}{2} \int_0^\delta |u_x^\varepsilon|^2 dx + \delta^{-\alpha} \int_\delta^1 (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx. \tag{10}
\]

Taking \( \delta^2 C < 1 \) and combining (9) with (10), we conclude that

\[
\int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + \int Q \varepsilon |u_x^\varepsilon|^2 dx d\tau \\
\geq \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + C \int Q \varepsilon (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx d\tau + \int Q \varepsilon |f|^2 dx d\tau \\
+ 2 \int Q \varepsilon u^\varepsilon Fdx dB(\tau) + \int Q \varepsilon (x + \varepsilon)^{-\alpha} |F|^2 dx d\tau. \tag{11}
\]
Taking expectation on both sides of (11) yields that
\[ E \int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + E \int_0^t \int_I |u_\varepsilon^\tau|^2 dx d\tau \]
\[ \leq E \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + C_1 \int_0^t \int_I (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx d\tau + E \int_0^t \int_I |f|^2 dx d\tau \]
\[ + E \int_0^t \int_I (x + \varepsilon)^{-\alpha} |F|^2 dx d\tau. \] (12)

By Gronwall’s inequality, we have
\[ \sup_{0 \leq t \leq T} E \int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx \]
\[ \leq C \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + C_1 \int_0^T \int_I |f|^2 dx d\tau + E \int_0^T (x + \varepsilon)^{-\alpha} |F|^2 dx d\tau. \] (13)

Choosing \( t = T \) in (12) and together with (13) yields
\[ E \int_{Q_T} |u_\varepsilon^T|^2 dx dt \]
\[ \leq C \left[ E \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + E \int_{Q_T} |f|^2 dx dt + E \int_{Q_T} (x + \varepsilon)^{-\alpha} |F|^2 dx dt \right]. \] (14)

Furthermore,
\[ \sup_{0 \leq t \leq T} E \int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + E \int_{Q_T} |u_\varepsilon^T|^2 dx dt \]
\[ \leq C \left[ E \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + E \int_{Q_T} |f|^2 dx dt + E \int_{Q_T} (x + \varepsilon)^{-\alpha} |F|^2 dx dt \right]. \] (15)

It follows from Burkholder-Davis-Gundy inequality that
\[ E \sup_{0 \leq t \leq T} \left| \int_{Q_t} u^\varepsilon F dx dB(\tau) \right| \leq E \left[ \int_0^T \left( \int_I |u^\varepsilon|^2 dx \right)^2 dt \right]^\frac{1}{2} \]
\[ \leq C \left[ \int_0^T \left( \int_I (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx \right) \left( \int_I (x + \varepsilon)^{-\alpha} |F|^2 dx \right) dt \right]^\frac{1}{2} \]
\[ \leq \frac{1}{4} E \sup_{0 \leq t \leq T} \int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + C_1 \int_{Q_T} (x + \varepsilon)^{-\alpha} |F|^2 dx dt \]

which together with (11) implies
\[ E \sup_{0 \leq t \leq T} \int_I (x + \varepsilon)^\alpha |u^\varepsilon(x, t)|^2 dx + E \int_{Q_T} |u_\varepsilon^T|^2 dx dt \]
\[ \leq C \left[ E \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + E \int_{Q_T} (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx dt + E \int_{Q_T} |f|^2 dx dt \right] \]
\[ + E \sup_{0 \leq t \leq T} \left| \int_{Q_t} u^\varepsilon F dx dB(\tau) \right| + C_1 \int_{Q_T} (x + \varepsilon)^{-\alpha} |F|^2 dx dt \]
\[ \leq C \left[ E \int_I (x + \varepsilon)^\alpha |u_0^\varepsilon(x)|^2 dx + E \int_{Q_T} (x + \varepsilon)^\alpha |u^\varepsilon|^2 dx dt + E \int_{Q_T} |f|^2 dx dt \right] \]
Null controllability for stochastic singular parabolic equations

By (15) and (16), we find that

\[ E \sup_{0 \leq t \leq T} \int_I (x + \varepsilon)^\alpha |u_\varepsilon(x,t)|^2 \, dx + E \int_{Q_T} |u_\varepsilon|^2 \, dx \, dt \leq C \left[ E \int_I (x + \varepsilon)^\alpha |u_\varepsilon^0(x)|^2 \, dx + E \int_{Q_T} |f|^2 \, dx \, dt + E \int_{Q_T} (x + \varepsilon)^{-\alpha} |F|^2 \, dx \, dt \right]. \] (17)

By a similar argument, for any \( \varepsilon_1, \varepsilon_2 \in (0,1) \), we also have

\[ E \sup_{0 \leq t \leq T} \int_I (x + \varepsilon)^\alpha |u_{\varepsilon_1}(x,t) - u_{\varepsilon_2}(x,t)|^2 \, dx + E \int_{Q_T} |u_{\varepsilon_1} - u_{\varepsilon_2}|^2 \, dx \, dt \leq C E \int_I (x + \varepsilon)^\alpha |u_{\varepsilon_1}^0(x) - u_{\varepsilon_2}^0(x)|^2 \, dx, \] (18)

where \( u_{\varepsilon_1} \) and \( u_{\varepsilon_2} \) are two solutions of (8) corresponding to \( u_{\varepsilon_1}^0 \) and \( u_{\varepsilon_2}^0 \), respectively. Hence, we obtain that \( \{ u_\varepsilon \} \) is a Cauchy sequence in \( H^2_{\alpha} \). Letting \( \varepsilon \to 0 \), we find that (5) admits a weak solution \( u \in H^2_{\alpha} \) (the limit of \( u_\varepsilon \)) such that (7). The uniqueness of the solution of (5) can be deduced from (7) directly. This completes the proof of Theorem 2.1.

By a similar argument to Theorem 2.1, we have the following well-posedness for the backward stochastic singular equation

\[
\begin{align*}
&\left\{ \begin{array}{l}
x^\alpha dv + v_{xx} \, dt + (av)_x \, dt + bvdv = fdt + VdB(t), \\
v(0,t) = v(1,t) = 0, \\
v(x,T) = v_T(x),
\end{array} \right. \\
&v(x,T) = v_T(x), \quad x \in I.
\end{align*}
\] (19)

Theorem 2.2. Let \( a, b \in L^\infty_T(0,T;L^\infty(I)) \), \( v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2_{\alpha}(I)) \) and \( f \in L^2_{\alpha}(0,T;L^2(I)) \). Then (19) admits a unique weak solution \((v,V) \in H^2_{\alpha} \times S^2_{\alpha}\) such that

\[
||v||_{H^2_{\alpha}} + ||V||_{S^2_{\alpha}} \leq C ||v_T||_{L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2_{\alpha}(I))} + C ||f||_{L^2_{\alpha}(0,T;L^2(I))},
\] (20)

where \( C \) depends on \( T, a, b \) and \( \alpha \).

3. Carleman estimate for the backward stochastic parabolic equation without lower order terms. In this section, we will show a uniform Carleman estimate in \( \varepsilon \) for the non-singular approximate problem without lower order terms

\[
\begin{align*}
&\left\{ \begin{array}{l}
(x + \varepsilon)^\alpha dp + p_{xx} \, dt = f_1dt + F_1dB(t), \\
p^\varepsilon(0,t) = p^\varepsilon(1,t) = 0, \\
p^\varepsilon(x,T) = p^\varepsilon(x),
\end{array} \right. \\
&p^\varepsilon(x,T) = p^\varepsilon(x), \quad x \in I,
\end{align*}
\] (21)

where \( 0 < \varepsilon < 1 \) and \( p^\varepsilon \to p_T \) in \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I)) \).

To formulate our Carleman estimate, we first introduce some weight functions. For \( \omega = (x_1, x_2) \), we choose \( \omega_\beta := (x^{(1)}, x^{(2)}) \) such that \( \omega_1 \subseteq \omega \). Let \( \chi(x) \in C^2(T) \) be a cut-off function such that \( 0 \leq \chi(x) \leq 1 \) for \( x \in I, \chi(x) = 1 \) for \( x \in (0, x^{(1)}(\omega)) \) and \( \chi(x) = 0 \) for \( x \in (x^{(2)}, 1) \). For a suitable positive constant \( \beta \), we introduce

\[
\eta_1(x) = \frac{1}{\beta}(x + \varepsilon)^\beta, \quad x \in I,
\]
and $\eta_2 \in C^2(I)$ such that
\[ \eta_2(x) > 0, \quad x \in I, \quad \eta_2(0) = \eta_2(1) = 0 \quad \text{and} \quad |\eta_{2, x}(x)| > 0, \quad x \in \overline{I \setminus \omega_1} \]
and
\[ \eta_1(x) = \eta_2(x), \quad x \in (x^{(1)}, x^{(2)}). \]
Let us define two weight functions respectively for singular part and non-singular part
\[ \Phi_i(x, t) = \varphi_i(x)\xi(t), \quad (x, t) \in Q_T, \quad i = 1, 2, \]
with
\[ \varphi_i(x) = e^{\lambda \eta_i(x)} - e^{2\lambda M}, \quad \xi(t) = \frac{1}{t^4(T - t)^4}. \]
Here $\lambda$ is a positive parameter and $M$ is a sufficiently large constant such that
\[ \varphi_i(x, t) < 0, \quad (x, t) \in Q_T, \quad i = 1, 2. \]
Now we introduce weight function in the global Carleman estimate
\[ \Phi(x, t) = \chi(x)\Phi_1(x, t) + (1 - \chi(x))\Phi_2(x, t), \quad (x, t) \in Q_T. \]
We easily see that
\[ \Phi(x, t) = \begin{cases} 
\Phi_1(x, t), & (x, t) \in (0, x^{(1)}) \times (0, T), \\
\Phi_2(x, t), & (x, t) \in (x^{(1)}, x^{(2)}) \times (0, T), \\
\Phi_2(x, t), & (x, t) \in (x^{(2)}, 1) \times (0, T). 
\end{cases} \tag{22} \]
and
\[ \xi(t) > 0, \quad |\xi(t)| \leq C\xi^2(t), \quad |\xi_u(t)| \leq C\xi^2(t), \tag{23} \]
where $C$ depends only on $T$.

Our main result in this section is stated as follows.

**Theorem 3.1.** Let $\alpha \in (0, +\infty)$, $\beta \in (1 + \alpha, 2 + \alpha]$, $f_1 \in L^2_{\mathbb{P}}(0, T; L^2(I))$ and $F_1 \in L^2_{\mathbb{P}}(0, T; L^2(I))$. Then for any $\varepsilon \in (0, 1)$, there exist positive constants $s_0 = s_0(T, \omega, \alpha, M, \lambda)$ and $C = C(T, \omega, \alpha, M, \lambda)$ independent of $\varepsilon$ such that
\[ \mathbb{E} \int_{Q_T} s^3(x + \varepsilon)^{-\alpha + 3\beta - 4}|p^2_\varepsilon|e^{2s\Phi}dxdt + \mathbb{E} \int_{Q_T} s^3\xi^3(x + \varepsilon)^{-\alpha + 3\beta - 4}|p^2|e^{2s\Phi}dxdt \]
\[ \leq C\mathbb{E} \int_{Q_T} (x + \varepsilon)^{-\alpha}|f_1|^2e^{2s\Phi}dxdt + C\mathbb{E} \int_{Q_T} s^2\xi^2(x + \varepsilon)^{-\alpha}|F_1|^2e^{2s\Phi}dxdt \]
\[ + C\mathbb{E} \int_{Q_T} s^3\xi^3|p^2|e^{2s\Phi}dxdt \tag{24} \]
for all $s > s_0$ and all $p^2 \in \mathcal{P}_T$ satisfying (21).

**Remark 3.** For the deterministic version of (21), the authors [11] established the Carleman estimate by choosing a special weight function related to the singularity rate $\alpha$, and then obtained the Carleman estimate for singular parabolic equation without convection term. In this paper, we apply the dual technique to establish the Carleman estimate for the stochastic singular parabolic equation with the convection term for the case $\alpha \in (0, 1)$. To show and explain the restriction, we first need to choose a new more general weight function to obtain a new Carleman estimate for, i.e. (24).
To prove Theorem 3.1, we first establish two Carleman estimates for singular part and non-singular part, respectively. The proofs of these two results are detailed in the Appendix and omitted here.

**Lemma 3.2.** Let $\alpha \in (0, +\infty)$, $\beta \in (1 + \alpha, 2 + \alpha]$, $f_1 \in L^2_T(0, T; L^2(I))$ and $F_1 \in L^2_T(0, T; L^2(I))$. Then for any $\varepsilon \in (0, 1)$, there exist positive constants $s_1 = s_1(T, \omega, \alpha, M, \lambda)$ and $C = C(T, \omega, \alpha, M, \lambda)$ such that

$$
\begin{align*}
\mathbb{E} \int_0^T \int_0^{s_1} [s \xi(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p_x|^2 e^{2s\Phi_1} dxdt + s^3 \xi^3(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p|^2] e^{2s\Phi_1} dxdt \\
\leq CE \int_0^T \int_0^{s_1} (x + \varepsilon)^{-\alpha}|f_1|^2 e^{2s\Phi_1} dxdt + CE \int_0^T \int_0^{s_1} s^2 \xi^2(x + \varepsilon)^{-\alpha}|F_1|^2 e^{2s\Phi_1} dxdt \\
+ CE \int_0^T \int_{\omega_1} (|p_x|^2 + |p|^2) e^{2s\Phi_1} dxdt
\end{align*}
$$

for all $s > s_1$ and all $p^\varepsilon \in \mathcal{P}_T$ satisfying (21).

**Lemma 3.3.** Let $\alpha \in (0, +\infty)$, $\beta \in (1 + \alpha, 2 + \alpha]$, $f_1 \in L^2_T(0, T; L^2(I))$ and $F_1 \in L^2_T(0, T; L^2(I))$. Then for any $\varepsilon \in (0, 1)$, there exist positive constants $s_2 = s_2(T, \omega, \alpha, M, \lambda)$ and $C = C(T, \omega, \alpha, M, \lambda)$ such that

$$
\begin{align*}
\mathbb{E} \int_0^T \int_0^{s_2} [s \xi(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p_x|^2 + s^3 \xi^3(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p|^2] e^{2s\Phi_2} dxdt \\
\leq CE \int_0^T \int_0^{s_2} (x + \varepsilon)^{-\alpha}|f_1|^2 e^{2s\Phi_2} dxdt + CE \int_0^T \int_0^{s_2} s^2 \xi^2(x + \varepsilon)^{-\alpha}|F_1|^2 e^{2s\Phi_2} dxdt \\
+ CE \int_0^T \int_{\omega_2} (|p_x|^2 + |p|^2) e^{2s\Phi_2} dxdt
\end{align*}
$$

for all $s > s_2$ and all $p^\varepsilon \in \mathcal{P}_T$ satisfying (21).

Now we prove Theorem 3.1.

**Proof of Theorem 3.1.** Combining (25) and (26) and using (22), we find that

$$
\begin{align*}
\mathbb{E} \int_{Q_T} s \xi(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p_x|^2 e^{2s\Phi} dxdt + \mathbb{E} \int_{Q_T} s^3 \xi^3(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p|^2 e^{2s\Phi} dxdt \\
= \mathbb{E} \int_0^T \int_0^{s_1} [s \xi(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p_x|^2 + s^3 \xi^3(x + \varepsilon)^{-\alpha,\beta,\gamma,\delta,2}|p|^2] e^{2s\Phi} dxdt \\
+ \mathbb{E} \int_0^T \int_{\omega_1} (|p_x|^2 + |p|^2) e^{2s\Phi} dxdt \\
\leq CE \int_{Q_T} (x + \varepsilon)^{-\alpha}|f_1|^2 e^{2s\Phi} dxdt + CE \int_{Q_T} s^2 \xi^2(x + \varepsilon)^{-\alpha}|F_1|^2 e^{2s\Phi} dxdt \\
+ CE \int_{\omega_2} (s^2 \xi^3|p|^2 + s^3 \xi^3|p|^2) e^{2s\Phi} dxdt.
\end{align*}
$$

(27)
Next we eliminate the term \( p_x^2 \) on \( \omega_1 \) by using a Caccioppoli inequality. To do this, we introduce a cut-function \( \zeta \in C^2(\mathcal{T}) \) such that

\[
\zeta(x) \begin{cases} 
  = 1, & x \in \omega_1, \\
  \in [0, 1], & x \in \omega \setminus \omega_1, \\
  = 0, & x \in \Omega \setminus \omega.
\end{cases}
\]

By Itô’s formula and the equation of \( p^x \), we have

\[
d [\zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}] \\
= \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| + \varepsilon \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| \left[ -p_x^2 dt + f_1 dt + F_1 dB(t) \right] \\
+ \zeta(x + \varepsilon)^{-\alpha}|f_1|^2|e^{2x\Phi}| dt.
\]

Integrating both sides of (28) in \( Q_T \) and taking mathematical expectation in \( \Omega \) yields

\[
\mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dx dt \\
= -\frac{1}{2} \mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dx dt + \frac{1}{2} \mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dx dt \\
- \mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^{-\alpha}|f_1|^2|e^{2x\Phi}| dx dt
\]

\[
\leq \mathbb{E} s^3(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dx dt + \mathbb{E} \int_{Q_T} (|\zeta| + |\zeta_x| + |\zeta_{xx}|) s^2 \varepsilon^3|p^x|^2|e^{2x\Phi}| dx dt \\
+ \mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^{-\alpha}|f_1|^2|e^{2x\Phi}| dx dt
\]

which implies

\[
\mathbb{E} \int_{Q_T} \zeta(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dx dt \\
\leq \mathbb{E} s^3(x + \varepsilon)^2|p^x|^2|e^{2x\Phi}| dxdt + \mathbb{E} \int_{Q_T} (x + \varepsilon)^{-\alpha}|f_1|^2|e^{2x\Phi}| dx dt
\]

Finally, substituting (30) into (27), we immediately obtain (24) and then complete the proof of Theorem 3.1.

4. Carleman estimate for the backward stochastic singular parabolic equation with convection term. In this section, we give a Carleman estimate for the backward stochastic singular parabolic equation with convection term

\[
\begin{align*}
  \frac{x^a dx}{dt} + v_x dt + (av_x) dt - b v dt &= f_2 dt + F_2 dB(t), & (x, t) \in Q_T, \\
  v(0, t) &= v(1, t) = 0, & t \in (0, T), \\
  v(x, T) &= v_T(x), & x \in I.
\end{align*}
\]

To overcome the singularity, we consider the following approximate non-singular version:
Then for any (31), which is our main result in this section.

Let Theorem 4.2. The proof of Theorem 4.2 is very long, for the reader’s convenience, the first and second steps are carried out in Lemma 4.3, then (34) follows from Lemma 4.3 and obtained. Finally, by a duality argument, we derive the desired Carleman estimate.

Since we only assume \( a \in L^2(0,T;L^\infty(I)) \), the convection term cannot be directly controlled by the terms on the left-hand side of (24). In order to overcome the difficulty, we prove the Carleman estimate by a duality technique proposed by Imanuvilov and Yamamoto in [19] for deterministic parabolic equation, or introduced by Liu [22] for stochastic parabolic equation. For this reason, in this paper we only establish the Carleman estimate for \( \alpha \in (0,1) \) for the backward stochastic singular parabolic equation with convection term.

**Theorem 4.1.** Let \( \alpha \in (0,1), f_2 \in L^2_T(0,T;L^2(I)) \) and \( F_2 \in L^2_T(0,T;L^2(I)) \). Then for any \( \varepsilon \in (0,1) \) there exist two constants \( s_3 = s_3(T,T,\omega,\alpha,M,\lambda) \) and \( C = C(T,T,\omega,\alpha,M,\lambda) \) independent of \( \varepsilon \) such that

\[
\begin{align*}
E \int_{Q_T} s^3 |v_\varepsilon|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t + E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v_\varepsilon|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t \\
\leq CE \int_{Q_T} (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t + CE \int_{Q_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\varepsilon \Phi} \\
+ CE \int_{\omega_T} s^3 \xi^3 |v_\varepsilon|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t
\end{align*}
\]

for all \( s > s_3 \) and all \( v \in \mathcal{P}_T \) satisfying (32).

Since \( v_{\varepsilon} \to v_T \) in \( L^2(\Omega,F_T,\mathbb{P};L^2(I)) \), by a similar process as the end of the proof of Theorem 2.1, together with the standard energy estimate for the backward stochastic parabolic equation [30], we can obtain that \( v_{\varepsilon} \to v \) in \( \mathcal{H}^2_T \) as \( \varepsilon \to 0 \). Then based on Theorem 4.1, we immediately obtain the following Carleman estimate for (31), which is our main result in this section.

**Theorem 4.2.** Let \( \alpha \in (0,1), f_2 \in L^2_T(0,T;L^2(I)) \) and \( F_2 \in L^2_T(0,T;L^2(I)) \). Then there exist two constants \( s_3 = s_3(T,T,\omega,\alpha,M,\lambda) \) and \( C = C(T,T,\omega,\alpha,M,\lambda) \) such that

\[
\begin{align*}
E \int_{Q_T} s^3 \xi^3 |v|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t + E \int_{Q_T} s^3 |v_x|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t \\
\leq CE \int_{Q_T} x^{-\alpha} |f_2|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t + CE \int_{Q_T} s^2 \xi^2 x^{-\alpha} |F_2|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t \\
+ CE \int_{\omega_T} s^3 \xi^3 |v|^2 e^{2\varepsilon \Phi} \text{d}x \text{d}t
\end{align*}
\]

for all \( s > s_3 \) and all \( v \in \mathcal{H}^2_T \) satisfying (31).

For the proof of Theorem 4.2, we borrow some ideas from [22, 23, 38]. The main steps are as follows. First, we construct a family of constrained extremal problems for a controlled forward stochastic parabolic equation, and establish a uniform estimate or these optimal solutions. Second, by taking the limit, a null controllability result is obtained. Finally, by a duality argument, we derive the desired Carleman estimate. The proof of Theorem 4.2 is very long, for the reader’s convenience, the first and second steps are carried out in Lemma 4.3, then (34) follows from Lemma 4.3 and
admits a solution there exists a pair of controls \((\alpha, \beta)\). Then we can trace back to the seminal work J. L. Lions [21]. We first connect the optimal solution \((\alpha, \beta)\) to guarantee the non-emptiness of \(\beta\).

**Proof.** The proof of Lemma 4.3 is based on a classical duality method, which ideas can be traced back to the seminal work J. L. Lions [21]. We first connect the solutions to the optimal problem to the solutions of the controlled problem. Then, we have the following null controllability result for (35).

**Lemma 4.3.** Let \(\alpha \in (0, 1)\) and \(\beta \in (1 + \alpha, \frac{4}{3} + \frac{2}{3}\alpha]\). Then for any \(\epsilon \in (0, 1)\), there exists a pair of controls \((\alpha, \beta)\) such that (35) admits a solution \(h \in P_T\) corresponding to \((\alpha, \beta)\).

Moreover, there exists a positive constant \(C = C(T, \omega, M, \lambda)\) such that

\[
\mathbb{E} \int_{Q_T} (x + \epsilon)^\alpha |h|^2 e^{-2\epsilon\Phi} dxd\tau + \mathbb{E} \int_{Q_T} s^{-2} |\partial_x|^2 e^{-2\epsilon\Phi} dxd\tau \\
+ \mathbb{E} \int_{Q_T} s^{-3} |\partial_x|^3 e^{-2\epsilon\Phi} dxd\tau + \mathbb{E} \int_{Q_T} s^{-2} (x + \epsilon)^{-\alpha} |\partial_x|^2 e^{-2\epsilon\Phi} dxd\tau \\
\leq C \mathbb{E} \int_{Q_T} s^3 (x + \epsilon)^\alpha |\partial_x|^2 e^{2\epsilon\Phi} dxd\tau.
\]

**Remark 4.** In the proof of Lemma 4.3, we need \(\beta \in (1 + \alpha, \frac{2}{3} + \frac{2}{3}\alpha]\), which leads to \(\alpha \in (0, 1)\) to guarantee the non-emptiness of \(\beta\).

**Proof.** The proof of Lemma 4.3 is based on a classical duality method, which ideas can be traced back to the seminal work J. L. Lions [21]. We first connect the solutions to the optimal problem to the solutions of the controlled problem. Then we establish a uniform estimate for these optimal solutions. Finally, by taking the limit, we establish an estimate of the convergence rate of the optimal control solutions to the null control pair.

For any \(\tau > 0\), we set

\[
\Phi_\tau(x, t) = \frac{\chi(x)\varphi_1(x) + (1 - \chi(x))\varphi_2(x)}{(1 + \tau)^4}.
\]

Let us consider the following constrained extremal problem \((P_\tau)\):

\[
\mathcal{J} = \frac{1}{2} \min_{(g_1, g_2) \in \mathcal{U}} \left( \mathbb{E} \int_{Q_T} (x + \epsilon)^\alpha |h|^2 e^{-2\epsilon\Phi} dxd\tau + \mathbb{E} \int_{Q_T} s^{-3} |\partial_x|^3 e^{-2\epsilon\Phi} dxd\tau \\
+ \mathbb{E} \int_{Q_T} s^{-2} (x + \epsilon)^{-\alpha} |\partial_x|^2 e^{-2\epsilon\Phi} dxd\tau + \frac{1}{\tau} \mathbb{E} \int_I (x + \epsilon)^\alpha |h|^2(x, T) dx \right)
\]

subject to (35), where

\[
\mathcal{U} := \left\{ (g_1, g_2) \in L^2_T(0, T; L^2(\omega) \times L^2_T(0, T; L^2(I)) \mid \\
\mathbb{E} \int_{Q_T} s^{-3} |\partial_x|^3 e^{-2\epsilon\Phi} dxd\tau + \mathbb{E} \int_{Q_T} s^{-2} (x + \epsilon)^{-\alpha} |\partial_x|^2 e^{-2\epsilon\Phi} dxd\tau < \infty \right\}.
\]

By the standard variational method [21], for any \(\tau > 0\), \((P_\tau)\) admits a unique optimal solution \((h_\tau, g_1, g_2) \in P_T \times \mathcal{U}\). Moreover,

\[
g_1, g_2 = s^3 \xi^3 e^{2\epsilon W_\tau} \omega, \quad g_1, g_2 = s^2 \xi^2 e^{2\epsilon W_\tau}, \quad \text{in } Q_T, \quad \mathbb{P} - \text{a.s.}
\]
where \((w_\tau, W_\tau)\) is the solution of the following backward stochastic parabolic equation:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
(x + \varepsilon)\alpha dw_\tau + w_\tau,x dt &= (x + \varepsilon)\alpha h_\tau e^{-2s\Phi_\tau}dt + W_\tau dB(t), & (x, t) \in Q_T, \\
\varepsilon w_\tau(0, t) &= w_\tau(1, t) = 0, & t \in (0, T), \\
\varepsilon w_\tau(x, T) &= -\frac{1}{T}h_\tau(x, T), & x \in I. 
\end{array} \right.
\] (39)

We now establish a uniform estimate for \((h_\tau, g_{1, \tau}, g_{2, \tau})\) in \(\tau\) and \(\varepsilon\). Applying Theorem 3.1 to (39), we obtain

\[
E\int_{Q_T} s\xi(x + \varepsilon)^{-\alpha+\beta-2}|w_{\tau,x}|^2e^{2s\Phi}dxdt + \int_{\omega_T} s^3\xi^3(x + \varepsilon)^{-\alpha+3+3\beta-4}|w_\tau|^2e^{2s\Phi}dxdt
\]

\[
\leq CE\int_{Q_T} (x + \varepsilon)^\alpha|h_\tau|^2e^{-4s\Phi_\tau}e^{2s\Phi}dxdt + CE\int_{Q_T} s^2\xi^2(x + \varepsilon)^{-\alpha}|W_\tau|^2e^{2s\Phi}dxdt
\]

\[\quad + CE\int_{\omega_T} s^3\xi^3|w_\tau|^2e^{2s\Phi}dxdt. \] (40)

Itô’s formula gives

\[
d[(x + \varepsilon)\alpha h_\tau w_\tau] = h_\tau(x + \varepsilon)\alpha dw_\tau + w_\tau(x + \varepsilon)\alpha dh_\tau + (x + \varepsilon)\alpha dh_\tau dw_\tau. \] (41)

Integrating both sides of (41) in \(Q_T\), taking mathematical expectation in \(\Omega\) and using the equations of \(w_\tau\) and \(h_\tau\), we find that

\[
E\int_{Q_T} (x + \varepsilon)\alpha|h_\tau|^2e^{-2s\Phi_\tau}dxdt + \int_{\omega_T} w_\tau g_{1, \tau} dxdt
\]

\[\quad + E\int_{Q_T} (x + \varepsilon)^{-\alpha}W_\tau g_{2, \tau} dxdt + \frac{1}{\tau} E\int_I (x + \varepsilon)\alpha|h_\tau|^2(x, T)dx
\]

\[= -E\int_{Q_T} s^3\xi^3(x + \varepsilon)^\alpha w_\tau u^\tau e^{2s\Phi}dxdt. \] (42)

From Young’s inequality with \(\mu\), (38) and (40), we deduce that

\[
E\int_{Q_T} (x + \varepsilon)\alpha|h_\tau|^2e^{-2s\Phi_\tau}dxdt + \int_{\omega_T} s^{-3}\xi^{-3}|g_{1, \tau}|^2e^{-2s\Phi}dxdt
\]

\[\quad + E\int_{Q_T} s^{-2}\xi^{-2}(x + \varepsilon)^{-\alpha}|g_{2, \tau}|^2e^{-2s\Phi}dxdt + \frac{1}{\tau} E\int_I (x + \varepsilon)\alpha|h_\tau|^2(x, T)dx
\]

\[\leq E\int_{Q_T} s^3\xi^3(x + \varepsilon)^{-\alpha+3+3\beta-4}|w_\tau|^2e^{2s\Phi}dxdt
\]

\[\quad + C(\mu) E\int_{Q_T} s^3\xi^3(x + \varepsilon)^{3\alpha-3\beta+4}|\varepsilon|^2e^{2s\Phi}dxdt
\]

\[\leq C(\mu) E\int_{Q_T} (x + \varepsilon)^\alpha|h_\tau|^2e^{-4s\Phi_\tau}e^{-2s\Phi}dxdt + E\int_{Q_T} s^2\xi^2(x + \varepsilon)^{-\alpha}|W_\tau|^2e^{2s\Phi}dxdt
\]

\[\quad + C(\mu) E\int_{Q_T} s^3\xi^3(x + \varepsilon)^{3\alpha-3\beta+4}|\varepsilon|^2e^{2s\Phi}dxdt
\]

\[\quad + C(\mu) E\int_{Q_T} s^3\xi^3|w_\tau|^2e^{2s\Phi}dxdt. \] (43)
we further obtain that

\[ E \int_{Q_T} (x + \varepsilon)^\alpha |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt + E \int_{\omega_T} s^{-3} \xi^{-3} |g_{1, \tau}|^2 e^{-2s\Phi} \, dx \, dt \\
+ E \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^{-\alpha} |g_{2, \tau}|^2 e^{-2s\Phi} \, dx \, dt + \frac{1}{\tau} E \int_I (x + \varepsilon)^\alpha |h_{\tau}|^2 (x, T) \, dx \\
\leq C E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^{3\alpha - 3\beta + 4} |\varphi|^2 e^{2s\Phi} \, dx \, dt. \tag{44} \]

Next we estimate the term of \( h_{\tau,x} \). We easily see that

\[ d \left[ s^{-2} \xi^{-2} (x + \varepsilon)^\alpha |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \right] \\
= s^{-2} (x + \varepsilon)^\alpha \left( \xi^{-2} e^{-2s\Phi_{\tau}} \right) |h_{\tau}|^2 \, dt + 2s^{-2} \xi^{-2} (x + \varepsilon)^\alpha e^{-2s\Phi_{\tau}} h_{\tau} dh_{\tau} \\
+ s^{-2} \xi^{-2} (x + \varepsilon)^\alpha e^{-2s\Phi_{\tau}} (dh_{\tau})^2. \]

Integrating the above equality in \( Q_T \), taking mathematical expectation and using the equation of \( h_{\tau} \), we have

\[ E \int_{Q_T} s^{-2} \xi^{-2} |h_{\tau,x}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
= -\frac{1}{2} E \int_I \left[ s^{-2} \xi^{-2} (x + \varepsilon)^\alpha |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \right]_{t=0}^{t=T} \, dx - E \int_{Q_T} s^{-2} \xi^{-2} (e^{-2s\Phi_{\tau}}) \, dx \, dt \\
+ E \int_{Q_T} s\xi (x + \varepsilon)^\alpha h_{\tau} \varphi e^{-2s\Phi_{\tau}} e^{2s\Phi} \, dx \, dt + E \int_{\omega_T} s^{-2} \xi^{-2} h_{\tau} \varphi e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ \frac{1}{2} E \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^{-\alpha} |g_{1, \tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ \frac{1}{2} E \int_{Q_T} s^{-2} (\xi^{-2} e^{-2s\Phi_{\tau}}) \varphi (x + \varepsilon)^\alpha |h_{\tau}|^2 \, dx \, dt \\
\leq \frac{1}{4} E \int_{Q_T} s^{-2} \xi^{-2} |h_{\tau,x}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt + C E \int_{Q_T} (x + \varepsilon)^{2\beta - 2} |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ C E \int_{Q_T} (x + \varepsilon)^\alpha |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt + C E \int_{Q_T} s^{-3} \xi^{-3} |g_{1, \tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ C E \int_{Q_T} s^2 \xi^2 (x + \varepsilon)^\alpha |\varphi|^2 e^{-2s\Phi_{\tau}} e^{4s\Phi} \, dx \, dt \\
+ C E \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^{-\alpha} |g_{2, \tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt. \tag{45} \]

Noticing \( e^{-2s\Phi_{\tau}} e^{2s\Phi} \leq 1 \), and using (44), (45) and \( 2\beta - 2 \geq \alpha \), we have

\[ E \int_{Q_T} (x + \varepsilon)^\alpha |h_{\tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt + E \int_{Q_T} s^{-2} \xi^{-2} |h_{\tau,x}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ E \int_{\omega_T} s^{-3} \xi^{-3} |g_{1, \tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt + E \int_{Q_T} s^{-2} \xi^{-2} (x + \varepsilon)^{-\alpha} |g_{2, \tau}|^2 e^{-2s\Phi_{\tau}} \, dx \, dt \\
+ \frac{1}{\tau} E \int_I (x + \varepsilon)^\alpha |h_{\tau}|^2 (x, T) \, dx \\
\leq C E \int_{Q_T} \left[ s^3 \xi^3 (x + \varepsilon)^{3\alpha - 3\beta + 4} + s^2 \xi^2 (x + \varepsilon)^\alpha \right] |\varphi|^2 e^{2s\Phi} \, dx \, dt. \tag{46} \]
where $C$ is independent of $\varepsilon$ and $\tau$.

Letting $\tau \to 0$ in (46), we can obtain that there exists $(h, g_1, g_2) \in \mathcal{P}_T \times \mathcal{U}$ such that $(h, g_1, g_2, \tau) \to (h, g_1, g_2)$ weakly in $\mathcal{P}_T \times \mathcal{U}$. By an argument as [22], we can obtain that $h$ is the weak solution of (35) corresponding to $(g_1, g_2)$. Also, by (46) together with $3\alpha - 3\beta + 4 \geq \alpha$ due to $1 + \alpha < \beta \leq \frac{4}{3} + \frac{2\mu}{\omega}$, we get $h(x, T) = 0$ in $I$, $\mathbb{P} - a.s.$ and (36) holds. This completes the proof. \hfill $\Box$

Now, we are in a position to prove Theorem 4.1.

**Proof of Theorem 4.1.** By Lemma 4.3, we know that there exists a pair of controls $(g_1, g_2) \in L^2_T(0, T; L^2(\omega)) \times L^2_T(0, T; L^2(I))$ such that (35) admits a solution $h \in \mathcal{P}_T$ corresponding to $(g_1, g_2)$ satisfying $h(x, T) = 0$ in $I$, $\mathbb{P} - a.s.$ Then using

\[
\begin{align*}
\mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\varepsilon^2} \, dx \, dt \\
= -\mathbb{E} \int_{\omega_T} (av^\varepsilon h + (x + \varepsilon)^\alpha v^\varepsilon dh + (x + \varepsilon)^\alpha dv^\varepsilon dh \\
\quad + h \left[ -v^\varepsilon_{xx} - (av^\varepsilon)_x + bv^\varepsilon + f_2 \right] dt + F_2 dB(t)] \\
\quad + v^\varepsilon \left[ (h_{xx} + s^2 \xi^3 (x + \varepsilon)^\alpha v^\varepsilon e^{2\varepsilon^2} + g_1 |\omega|) dt + g_2 dB(t) \right] + (x + \varepsilon)^{-\alpha} F_2 g_2 dt, \\
\end{align*}
\]

and (36), we obtain

\[
\begin{align*}
\mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\varepsilon^2} \, dx \, dt \\
\leq \mu \mathbb{E} \int_{\mathcal{Q}_T} s^2 \xi^2 |\omega| |h|^2 e^{2\varepsilon^2} \, dx \, dt \\
\quad + \mu \mathbb{E} \int_{\omega_T} (x + \varepsilon)^\alpha |g_1|^2 e^{2\varepsilon^2} \, dx \, dt \\
\quad + C(\mu) \mathbb{E} \int_{\mathcal{Q}_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\varepsilon^2} \, dx \, dt \\
\quad + C(\mu) \mathbb{E} \int_{\omega_T} s^3 \xi^3 (x + \varepsilon)^{-\alpha} |v^\varepsilon|^2 e^{2\varepsilon^2} \, dx \, dt. \\
\end{align*}
\]

We can choose $\mu$ sufficiently small to absorb the first term on the right-hand side of (47) by the term on the left-hand side of (47) and obtain

\[
\begin{align*}
\mathbb{E} \int_{\mathcal{Q}_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\varepsilon^2} \, dx \, dt \\
\leq C \mathbb{E} \int_{\mathcal{Q}_T} (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\varepsilon^2} \, dx \, dt + C \mathbb{E} \int_{\mathcal{Q}_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\varepsilon^2} \, dx \, dt \\
\quad + C \mathbb{E} \int_{\omega_T} s^3 \xi^3 (x + \varepsilon)^{-\alpha} |v^\varepsilon|^2 e^{2\varepsilon^2} \, dx \, dt. \\
\end{align*}
\]
Next we estimate the term of $v^\varepsilon$. Itô’s formula and (32) yields that

\[
d\left[ (x + \varepsilon)^\alpha \xi |v^\varepsilon|^2 e^{2\Phi} \right] = (x + \varepsilon)^\alpha (\xi e^{2\Phi})_{|v^\varepsilon|^2} dt + 2s\xi e^{2\Phi} v^\varepsilon \left[ (\varepsilon e^{2\Phi})_{x} + (av^\varepsilon)_x + b\nu^\varepsilon + f_2^2 \right] dt + F_2 dB(t)
\]

\[
+ s\xi (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dt.
\]

(49)

From (54), we further deduce that

\[
E \int_{Q_T} s\xi |v^\varepsilon_x|^2 e^{2\Phi} dx dt
\]

\[
= -\frac{1}{2} E \int_{Q_T} s(x + \varepsilon)^\alpha (\xi e^{2\Phi})_{|v^\varepsilon|^2} dx dt - E \int_{Q_T} s\xi (e^{2\Phi})_{x} v^\varepsilon_x dx dt
\]

\[
- E \int_{Q_T} as\xi v^\varepsilon v^\varepsilon_x e^{2\Phi} dx dt - E \int_{Q_T} bs\xi |v^\varepsilon|^2 e^{2\Phi} dx dt
\]

\[
- E \int_{Q_T} s\xi f_2 e^{2\Phi} dx dt - \frac{1}{2} E \int_{Q_T} s\xi (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dx dt
\]

\[
\leq \frac{1}{2} E \int_{Q_T} s\xi |v^\varepsilon_x|^2 e^{2\Phi} dx dt + CE \int_{Q_T} (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dx dt
\]

\[
+ CE \int_{Q_T} [s^2 \xi^3 (x + \varepsilon)^\alpha + s^3 \xi^3 (x + \varepsilon)^2\beta - 2 + s\xi + s^2 \xi^2 (x + \varepsilon)^{\beta - 1}] |v^\varepsilon|^2 e^{2\Phi} dx dt,
\]

which implies

\[
E \int_{Q_T} s\xi |v^\varepsilon_x|^2 e^{2\Phi} dx dt
\]

\[
\leq CE \int_{Q_T} (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dx dt + CE \int_{Q_T} s\xi (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\Phi} dx dt
\]

\[
+ CE \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\Phi} dx dt + CE \int_{Q_T} s\xi |v^\varepsilon|^2 e^{2\Phi} dx dt.
\]

(50)

Combining (48) with (50), together with

\[
CE \int_{Q_T} s^2 \xi^2 |v^\varepsilon|^2 e^{2\Phi} dx dt
\]

\[
\leq \mu E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\Phi} dx dt + C(\mu) E \int_{Q_T} s\xi (x + \varepsilon)^{-\alpha} |v^\varepsilon|^2 e^{2\Phi} dx dt,
\]

(51)

we then obtain

\[
E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^\alpha |v^\varepsilon|^2 e^{2\Phi} dx dt + E \int_{Q_T} s\xi |v^\varepsilon_x|^2 e^{2\Phi} dx dt
\]

\[
\leq CE \int_{Q_T} (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dx dt + CE \int_{Q_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |F_2|^2 e^{2\Phi} dx dt
\]

\[
+ CE \int_{Q_T} s^3 \xi^3 |v^\varepsilon|^2 e^{2\Phi} dx dt + CE \int_{Q_T} s\xi (x + \varepsilon)^{-\alpha} |v^\varepsilon|^2 e^{2\Phi} dx dt,
\]

(52)

if we choose $\mu$ sufficiently small.
Finally, we eliminate the last term on the right-hand side of (52). By Hardy inequality, we have
\[
\mathbb{E} \int_{Q_T} s\xi(x + \varepsilon)^{-\alpha} \nu^2 e^{2s\Phi} dx dt \\
\leq C\mathbb{E} \int_{Q_T} (x + \varepsilon)^{2-\alpha} (s\xi|x_x|^2 + s^3\xi^3|\nu|^2) e^{2s\Phi} dx dt.
\]  
(53)

By a similar argument to (46) in [34], we obtain that for any \(\nu \in (0, \frac{1}{\sqrt{T}}\) there exists a constant \(C\) depending on \(T, \omega, \alpha, M\) and \(\nu\), but independent of \(\varepsilon\) such that
\[
\mathbb{E} \int_0^T \int_{3\nu} s\xi|x_x|^2 e^{2s\Phi} dx dt + \mathbb{E} \int_0^T \int_{3\nu} s^3\xi^3|\nu|^2 e^{2s\Phi} dx dt \\
\leq C(\nu)\mathbb{E} \int_0^T \int_{2\nu} (|\nu|^2 + |v_x|^2) e^{2s\Phi} dx dt + C(\nu)\mathbb{E} \int_0^T \int_{2\nu} |f_2|^2 e^{2s\Phi} dx dt \\
+ C(\nu)\mathbb{E} \int_0^T \int_{2\nu} s^2\xi^2|F_2|^2 e^{2s\Phi} dx dt + C(\nu)\mathbb{E} \int_{\omega_T} s^3\xi^3|\nu|^2 e^{2s\Phi} dx dt.
\]  
(54)

On the other hand, we easily obtain that
\[
\mathbb{E} \int_{Q_T} (x + \varepsilon)^{2-\alpha} (s\xi|x_x|^2 + s^3\xi^3|\nu|^2) e^{2s\Phi} dx dt \\
\leq (4\nu)^{2-2\alpha}\mathbb{E} \int_0^T \int_{3\nu} (s\xi|x_x|^2 + s^3\xi^3(x + \varepsilon)|\nu|^2) e^{2s\Phi} dx dt \\
+ 2^{2-\alpha}\mathbb{E} \int_0^T \int_{3\nu} (s\xi|x_x|^2 + s^3\xi^3|\nu|^2) e^{2s\Phi} dx dt
\]  
(55)

for \(\nu \in (0, \frac{1}{T})\) and \(\varepsilon \in (0, \nu)\).

Then substituting (54) into (55) yields that
\[
\mathbb{E} \int_{Q_T} (x + \varepsilon)^{2-\alpha} (s\xi|x_x|^2 + s^3\xi^3|\nu|^2) e^{2s\Phi} dx dt \\
\leq (4\nu)^{2-2\alpha}\mathbb{E} \int_0^T \int_{3\nu} (s\xi|x_x|^2 + s^3\xi^3(x + \varepsilon)|\nu|^2) e^{2s\Phi} dx dt \\
+ C(\nu)\mathbb{E} \int_0^T \int_{2\nu} (|\nu|^2 + |v_x|^2) e^{2s\Phi} dx dt + C(\nu)\mathbb{E} \int_0^T \int_{2\nu} |f_2|^2 e^{2s\Phi} dx dt \\
+ C(\nu)\mathbb{E} \int_0^T \int_{2\nu} s^2\xi^2|F_2|^2 e^{2s\Phi} dx dt + C(\nu)\mathbb{E} \int_{\omega_T} s^3\xi^3|\nu|^2 e^{2s\Phi} dx dt.
\]  
(56)

From (53) and (56), we deduce that
\[
\mathbb{E} \int_{Q_T} s\xi(x + \varepsilon)^{-\alpha} |\nu|^2 e^{2s\Phi} dx dt \\
\leq C(\nu)\mathbb{E} \int_{Q_T} |f_2|^2 e^{2s\Phi} dx dt + C(\nu)\mathbb{E} \int_{Q_T} s^2\xi^2|F_2|^2 e^{2s\Phi} dx dt \\
+ C(\nu)\mathbb{E} \int_{\omega_T} s^3\xi^3|\nu|^2 e^{2s\Phi} dx dt \\
+ (4\nu)^{2-2\alpha} C\mathbb{E} \int_{Q_T} (s\xi|x_x|^2 + s^3\xi^3(x + \varepsilon)|\nu|^2) e^{2s\Phi} dx dt
\]
This completes the proof of Theorem 4.1.

Substituting (57) into (52), we further find that
\[
E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^{\alpha} |v|^2 e^{2\varepsilon \Phi} \, dx \, dt + E \int_{Q_T} s |v_x|^2 e^{2\varepsilon \Phi} \, dx \, dt \\
\leq C(\nu) \int_{Q_T} (x + \varepsilon)^{-\alpha} |f_2|^2 e^{2\varepsilon \Phi} \, dx \, dt + C(\nu) \int_{Q_T} s^3 \xi^3 |v|^2 e^{2\varepsilon \Phi} \, dx \, dt \\
+ C(\nu) \int_{Q_T} s^2 \xi^2 (x + \varepsilon)^{-\alpha} |f|^2 e^{2\varepsilon \Phi} \, dx \, dt \\
+ (4\nu)^2 CE \int_{Q_T} (s \xi |v_x|^2 + s^3 \xi^3 (x + \varepsilon)^{\alpha} |v|^2) e^{2\varepsilon \Phi} \, dx \, dt \\
+ \frac{C(\nu)}{(2\nu)^\alpha} \int_{Q_T} (x + \varepsilon)^{\alpha} |v|^2 e^{2\varepsilon \Phi} \, dx \, dt + C(\nu) \int_{Q_T} |v_x|^2 e^{2\varepsilon \Phi} \, dx \, dt.
\]  
(58)

Finally, for \( \alpha \in (0, 1) \) we can choose \( \nu \) sufficiently small and \( s \) sufficiently large to absorb the last three terms on the right-hand side of (58) and then obtain (34). This completes the proof of Theorem 4.1. \( \square \)

5. Null controllability. This section is devoted to proving the null controllability result for (1), i.e. Theorem 1.1. It is well-known that the null controllability of (1) can be proved by the observability estimate for the corresponding adjoint system of (1)
\[
\begin{cases}
  x^\alpha dv + v_x dt + (av)_x dt - bv dt = V dB(t), & (x, t) \in Q_T, \\
  v(0, t) = v(1, t) = 0, & t \in (0, T), \\
  v(x, T) = v_T(x), & x \in I.
\end{cases}
\]  
(59)

Lemma 5.1. Let \( \alpha \in (0, 1) \). Then for any \( v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2_\alpha(I)) \) there exists positive constant \( C = C(T, a, b) \) such that the solution \((v, V) \in H_T^\alpha \times S^{-\alpha}_T\) of (59) satisfies
\[
E \int_I x^\alpha |v(x, 0)|^2 dx \leq CE \int_{Q_T} x^{-\alpha} |V|^2 dx \, dt + CE \int_{Q_T} |v|^2 dx \, dt.
\]  
(60)

Proof. Applying Theorem 4.2 to (59) yields that
\[
E \int_{Q_T} s^3 \xi^3 x^\alpha |v|^2 e^{2\varepsilon \Phi} \, dx \, dt + E \int_{Q_T} s |v_x|^2 e^{2\varepsilon \Phi} \, dx \, dt \\
\leq CE \int_{Q_T} s^2 \xi^2 x^{-\alpha} |V|^2 e^{2\varepsilon \Phi} \, dx \, dt + CE \int_{Q_T} s^3 \xi^3 |v|^2 e^{2\varepsilon \Phi} \, dx \, dt
\]  
(61)

for all large \( s \). We fix \( s \) and then easily see that
\[
E \int_{Q_T} (\xi^3 x^\alpha |v|^2 e^{2\varepsilon \Phi} + \xi |v_x|^2 e^{2\varepsilon \Phi}) \, dx \, dt \\
\geq \min_{x \in I} \left[ \xi^3 \left( \frac{T}{2} \right) e^{2\varepsilon \Phi(x, \frac{T}{2})}, \xi \left( \frac{T}{2} \right) e^{2\varepsilon \Phi(x, \frac{T}{2})} \right] E \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_I (x^\alpha |v|^2 + |v_x|^2) dx \, dt
\]  
(62)
and
\[ \mathbb{E} \int_{Q_T} \xi^2 x^{-\alpha}|v|^2e^{2\sigma t}dxdt + \mathbb{E} \int_{\omega_T} \xi^3 |v|^2e^{2\sigma t}dxdt \]
\[ \leq \max_{(x,t) \in Q_T} (\xi^2 e^{2\sigma t} + \xi^3 e^{2\sigma t}) \left[ \mathbb{E} \int_{Q_T} x^{-\alpha}|v|^2dxdt + \mathbb{E} \int_{\omega_T} |v|^2dxdt \right]. \quad (63) \]

From (61), (62) and (63), we deduce that
\[ \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_I (x^\alpha |v|^2 + |v_x|^2)dxdt \]
\[ \leq C(s) \left[ \mathbb{E} \int_{Q_T} x^{-\alpha}|v|^2dxdt + \mathbb{E} \int_{\omega_T} |v|^2dxdt \right]. \quad (64) \]

Itô’s formula gives
\[ d(x^\alpha v^2e^{\sigma t}) = \sigma x^\alpha v^2e^{\sigma t}dt + 2x^\alpha ve^{\sigma t}dv + x^\alpha e^{\sigma t}(dv)^2. \quad (65) \]

Here \( \sigma \) is a positive constant, which will be specified below. Therefore, for any \( 0 \leq t_1 < t_2 \leq T \), we obtain
\[ \mathbb{E} \int_{t_1}^{t_2} x^\alpha|v(x,t_2)|^2e^{\sigma t_2}dx - \mathbb{E} \int_{t_1}^{t_2} x^\alpha|v(x,t_1)|^2e^{\sigma t_1}dx \]
\[ = \mathbb{E} \int_{t_1}^{t_2} \int \sigma x^\alpha|v|^2e^{\sigma t}dxdt + 2\mathbb{E} \int_{t_1}^{t_2} \int v[-v_x -(av)_x + bv]e^{\sigma t}dxdt \]
\[ + \mathbb{E} \int_{t_1}^{t_2} \int x^{-\alpha}|v|^2e^{\sigma t}dxdt \geq \mathbb{E} \int_{t_1}^{t_2} \int \sigma x^\alpha|v|^2e^{\sigma t}dxdt + \mathbb{E} \int_{t_1}^{t_2} \int |v|^2e^{\sigma t}dxdt - C^* \mathbb{E} \int_{t_1}^{t_2} \int |v|^2e^{\sigma t}dxdt, \quad (66) \]

where \( C^* = \left( ||a||_{L^\infty(0,T;L^\infty(I))} + 2||b||_{L^\infty(0,T;L^\infty(I))} \right) \). On the other hand, similar to (10) we obtain
\[ \mathbb{E} \int_{t_1}^{t_2} \int |v|^2e^{\sigma t}dxdt \]
\[ \leq \frac{\delta^2}{2} \mathbb{E} \int_{t_1}^{t_2} \int_0^\delta |v_x|^2e^{\sigma t}dxdt + \delta^{-\alpha} \mathbb{E} \int_{t_1}^{t_2} \int_0^1 x^\alpha |v|^2e^{\sigma t}dxdt \quad (67) \]

for any \( \delta \in (0,1) \). Therefore, substituting (67) into (66) and choosing \( \delta \) sufficiently small such that \( C^*\delta^2/2 < 1 \), we have
\[ \mathbb{E} \int_{t_1}^{t_2} x^\alpha|v(x,t_2)|^2e^{\sigma t_2}dx - \mathbb{E} \int_{t_1}^{t_2} x^\alpha|v(x,t_1)|^2e^{\sigma t_1}dx \]
\[ \geq \mathbb{E} \int_{t_1}^{t_2} \int \sigma x^\alpha|v|^2e^{\sigma t}dxdt - C^* \delta^{-\alpha} \mathbb{E} \int_{t_1}^{t_2} \int x^\alpha |v|^2e^{\sigma t}dxdt. \quad (68) \]

Letting \( t_1 = 0 \) and \( t_2 = t \) and choosing \( \sigma = C^*\delta^{-\alpha} \) in (68), we obtain
\[ \mathbb{E} \int_t x^\alpha|v(x,0)|^2dx \leq e^{CT} \mathbb{E} \int_t x^\alpha|v(x,t)|^2dx, \quad 0 < t < T. \quad (69) \]

which implies that
\[ \mathbb{E} \int_t x^\alpha|v(x,0)|^2dx \leq C \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int x^3 |v|^2dxdt. \quad (70) \]
Finally, combining (64) and (70), we arrive at (60). This completes the proof of Lemma 5.1.

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let us set

\[ \mathcal{V} = \{ (v|\omega, V) \in \mathcal{H}_T^2 \times \mathcal{S}_{T}^{-\alpha} \mid (v, V) \text{ solves system } (59) \text{ with some } v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I)) \} \]

endowed with the norm

\[ ||(v|\omega, V)||_V = \left( \mathbb{E} \int_{\omega_T} |v|^2 \mathrm{d}x \mathrm{d}t + \mathbb{E} \int_{Q_T} x^{-\alpha}|V|^2 \mathrm{d}x \mathrm{d}t \right)^{\frac{1}{2}}. \]

Clearly, \( \mathcal{V} \) is a linear subspace of \( L^2_T(0, T; L^2(\omega)) \times L^2_T(0, T; L^2_{-\alpha}(I)) \). Let us define a linear functional \( \mathcal{L} \) on \( \mathcal{V} \) as follows:

\[ \mathcal{L}(v|\omega, V) = -\mathbb{E} \int_I x^\alpha u_0(x)v(x, 0) \mathrm{d}x. \]

It follows from the Hölder inequality and observation inequality (60) that

\[
\begin{align*}
\left| \left(-\mathbb{E} \int_I x^\alpha u_0(x)v(x, 0) \mathrm{d}x \right) \right| & \leq \left( \mathbb{E} \int_I x^\alpha |u_0(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left( \mathbb{E} \int_I |v(x, 0)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
& \leq C \left( \mathbb{E} \int_I x^\alpha |u_0(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left( \mathbb{E} \int_{\omega_T} |v|^2 \mathrm{d}x \mathrm{d}t + \mathbb{E} \int_{Q_T} x^{-\alpha}|V|^2 \mathrm{d}x \mathrm{d}t \right)^{\frac{1}{2}} \\
& \leq C \|u_0\|_{L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I))} \|(v|\omega, V)||_V.
\end{align*}
\]

From (71), we see that \( \mathcal{L} \) is a bounded linear functional on \( \mathcal{V} \). We can extend \( \mathcal{L} \) to be a bounded linear functional on \( L^2_T(0, T; L^2(\omega)) \times L^2_T(0, T; L^2_{-\alpha}(I)) \) and use the same notation for this extension. Now, by the Riesz representation theorem, we know that there is a pair of functions \((g, G) \in L^2_T(0, T; L^2(\omega)) \times L^2_T(0, T; L^2_{-\alpha}(I))\) such that

\[ \mathbb{E} \int_{\omega_T} g v \mathrm{d}x \mathrm{d}t + \mathbb{E} \int_{Q_T} x^{-\alpha}Gv \mathrm{d}x \mathrm{d}t = -\mathbb{E} \int_I x^\alpha u_0(x)v(x, 0) \mathrm{d}x. \]  

(72)

Next we prove that this pair of functions \((g, G)\) is the control functions. Noting that the identity

\[ d(x^\alpha uv) = x^\alpha vdu + x^\alpha udv + x^\alpha duv, \]

and combining the equations (1) and (59) of \( u \) and \( v \), we have

\[
\begin{align*}
\mathbb{E} \int_I x^\alpha u(x, T)v_T(x) \mathrm{d}x & - \mathbb{E} \int_I x^\alpha u_0(x)v(x, 0) \mathrm{d}x \\
= & \mathbb{E} \int_{Q_T} v[(u_{xx} - au_x - bu + g|\omega)] \mathrm{d}t + GdB(t) \mathrm{d}x \\
& + \mathbb{E} \int_{Q_T} u[(-v_{xx} - (av)_x + bv)dt + VdB(t)] \mathrm{d}x + \mathbb{E} \int_{Q_T} x^{-\alpha}Gv \mathrm{d}x \mathrm{d}t \\
= & \mathbb{E} \int_{\omega_T} g v \mathrm{d}x \mathrm{d}t + \mathbb{E} \int_{Q_T} x^{-\alpha}Gv \mathrm{d}x \mathrm{d}t.
\end{align*}
\]

(73)
This completes the proof of Theorem 1.1.

Comments and conclusions. This paper concerns the null controllability for a class of stochastic singular parabolic equations with the convection term in one dimensional space. There are many problems related to the topic of our paper. Some of them are listed as follows.

- As we all know, the lack of compact embedding for the solution spaces related to stochastic PDEs is the main obstacle to study the controllability problem for stochastic semilinear PDEs. Recently, [18] proposed a new twist on a classical strategy for controlling linear stochastic parabolic equation and used the Banach fixed point method to prove the null controllability of forward semilinear stochastic parabolic equations. It seems interesting to study a controllability result for stochastic semilinear singular parabolic equation by the idea of [18], the key step is that we need to modify the weight function in [18] to establish an refined Carleman estimate for the stochastic singular parabolic equation.

- We can deduce the exact controllability to the trajectory for (1) from the null controllability result Theorem 1.1. We fix a trajectory $\bar{u}$, a solution of the problem without control

$$\begin{cases}
x^n d\bar{u} - \bar{u}_{xx} dt + a\bar{u}_x dt + b\bar{u} dt = \bar{g} dt + \bar{G} dB(t), & (x, t) \in Q_T, \\
\bar{u}(0, t) = \bar{u}(1, t) = 0, & t \in (0, T), \\
\bar{u}(x, 0) = \bar{u}_0(x), & x \in I.
\end{cases}$$

(1) is exactly controllable to the trajectory, if for any initial $u_0$, there exists a pair control $(g, G)$ stasfies $g \in L^2_{\mathcal{F}}(0, T; L^2(\omega))$, $x^{-\frac{\alpha}{2}} G \in L^2_{\mathcal{F}}(0, T; L^2(I))$ such that the corresponding solution $u$ of

$$\begin{cases}
x^n du - u_{xx} dt + au_x dt + bu dt = (g_{\omega} + \bar{g}) dt + (G + \bar{G}) dB(t), & (x, t) \in Q_T, \\
u(0, t) = u(1, t) = 0, & t \in (0, T), \\
u(x, 0) = u_0(x), & x \in I,
\end{cases}$$

satisfies $u(\cdot, T) = \bar{u}(\cdot, T)$. We introduce the change of variable $\tilde{u} = u - \bar{u}$, then we can see that the exact controllability to the trajectory of (1) is equivalent to the null controllability result of the system $\tilde{u}$, which can be obtained from Theorem 1.1.

- An immediate and interesting extension of the result we obtained would be the analysis of the multidimensional case. For simplicity, we consider the stochastic singular parabolic equation in two dimensional:

$$\begin{cases}
A(x) du - \Delta u dt = g_{\omega} dt + G dB(t), & (x, t) \in \Omega \times (0, T), \\
u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) = \nu_0(x), & x \in \Omega,
\end{cases}$$

where $\Omega := (0, 1) \times (0, 1)$, the function $A(x)$ is given by $A(x) = x_1^{\alpha_1} + x_2^{\alpha_2}$ with $0 < \alpha_1, \alpha_2 < 1$. Due to the system is singular on the boundary, when establish the Carleman estimate for the corresponding adjoint system, we need to construct a weight function with a specific behavior near the part of the boundary where the singularity occurs. What’s more, the major step of
proving Carleman estimate is to obtain lower bounds of scalar product. It seems challenging and requires new ideas.

- It is very difficult to apply the method in this paper to study the controllability of a coupled system of forward stochastic singular parabolic equations with the same number of controls as equations. Taking a weak coupled system of two stochastic equations as an example, the weak coupled terms could be absorbed by the terms of Carleman estimate (34) by combining Hardy inequality. Therefore, we can obtain an observability inequality with four observations for the corresponding adjoint system by the standard argument. This leads us to have four controls for controlling this kind of coupled system. We don’t know how to prove the null controllability of two stochastic singular parabolic equations with only two controls.

- The Carleman estimate we proved in this paper might not be directly used to the stochastic singular parabolic equation with multiple singular points. On the one hand, if the singular point occurs at the interior of the domain, we do not know any information on the solution at the singular point. So we can’t perform integration by parts near the singularity. On the other hand, it is difficult to construct a weight function such that the Carleman estimates of singular part and nonsingular part are uniform with the whole interval. A feasible way is to follow the idea of Biccari and Zuazua in [3], where the authors studied the controllability of the heat equation with singular potential. However, as shown in (2), our equation is degenerate and singular at the boundary, how to balance the degeneracy and singularity of the equation is still a difficult problem.

- As far as we know, there are no articles on the theory and numerical calculation of stochastic singular parabolic equations. It remains to be further understood.

7. Appendix. This appendix is devoted to proving Lemma 3.2 and Lemma 3.3. We begin with a weighted identity for the backward stochastic singular parabolic operator.

**Lemma 7.1.** Assume that $p^x$ is an $H^2(\mathbb{R})$-valued continuous semi-martingale. Set $z = \theta p^x, \theta = e^{l} t, l = s \Phi_1$ and

$$L_1 = z_{xx} - (x + \varepsilon)^{\alpha} l_t z + l_t^2 z,$$

$$L_2 = (x + \varepsilon)^{\alpha} d z - 2 l_t z_x dt - l_{xx} z dt.$$

Then for a.e. $x \in \mathbb{R}$ and $\mathbb{P} - a.s. \omega \in \Omega$, we have the following weighted identity:

$$(x + \varepsilon)^{- \alpha} L_1 \theta [(x + \varepsilon)^{\alpha} p^x + p^x_{xx} dt]$$

$$= (x + \varepsilon)^{- \alpha} |L_1|^2 dt + \sum_{i=1}^{3} X_i dt + dK + \{\cdot\}_{x} + J,$$  \hspace{1cm} (A.1)

where

$$X_1 = \left[ 2(x + \varepsilon)^{- \alpha} l_{tx} - \alpha (x + \varepsilon)^{- \alpha - 1} l_t \right] z_x^2,$$

$$X_2 = \left[ 2(x + \varepsilon)^{- \alpha} l_{tx} - \alpha (x + \varepsilon)^{- \alpha - 1} l_t \right] l_t^2 z^2,$$

$$X_3 = -2 l_t l_x z^2 - \frac{1}{2} [ (x + \varepsilon)^{- \alpha} l_{xx} ]_{xx} z^2 + \frac{1}{2} (x + \varepsilon)^{\alpha} l_{tt} z^2.$$
\[ K = -\frac{1}{2}z_{x}^{2} - \frac{1}{2}(x + \varepsilon)^{\alpha}l_{x}z^{2} + \frac{1}{2}l_{xx}^{2}, \]
\[ \{ \} = z_{x} \, dz + \left[ -(x + \varepsilon)^{\alpha}l_{x}z_{x}^{2} + l_{x}l_{x}z^{2} - (x + \varepsilon)^{\alpha}l_{xx}z_{x}^{2} \right. \]
\[ \left. + \frac{1}{2}((x + \varepsilon)^{\alpha}l_{xx})z_{x}^{2} \right] \, dt, \]
\[ J = \frac{1}{2}(dz_{x})^{2} + \frac{1}{2}(x + \varepsilon)^{\alpha}l_{x}(dz)^{2} - \frac{1}{2}l_{xx}^{2}(dz)^{2}. \]

**Proof.** Notice that \( \theta = c^{1}, l = s\varphi, \) and \( z = \theta p. \) It is easy to verify that
\[ \theta [(x + \varepsilon)^{\alpha}d\varphi + p_{x}^{\alpha}dt] = L_{1}dt + L_{2}. \] (A.2)

Multiplying both sides of (A.2) by \((x + \varepsilon)^{-\alpha}L_{1},\) we have
\[ (x + \varepsilon)^{-\alpha}L_{1}\theta [(x + \varepsilon)^{\alpha}d\varphi + p_{x}^{\alpha}dt] = (x + \varepsilon)^{-\alpha}L_{1}|L_{1}|^{2}dt + (x + \varepsilon)^{-\alpha}L_{1}L_{2}, \] (A.3)

where
\[ (x + \varepsilon)^{-\alpha}L_{1}L_{2} = L_{1}dz - 2(x + \varepsilon)^{-\alpha}l_{x}z_{x}L_{1}dt - (x + \varepsilon)^{-\alpha}l_{xx}z_{x}L_{1}dt. \] (A.4)

Now we deal with the terms on the right-hand side of (A.4) one by one. For the first term, applying Itô’s formula, we have
\[ L_{1}dz = z_{xx}dz - (x + \varepsilon)^{\alpha}l_{x}z_{x}dz + l_{xx}^{2}dz \]
\[ = (z_{x}dz)_{x} - \frac{1}{2}d(z_{x}^{2}) + \frac{1}{2}(dz_{x})^{2} - \frac{1}{2}d[(x + \varepsilon)^{\alpha}l_{x}z^{2}] + \frac{1}{2}(x + \varepsilon)^{\alpha}l_{xx}z^{2}dt \]
\[ + \frac{1}{2}(x + \varepsilon)^{\alpha}l_{x}(dz)^{2} + \frac{1}{2}d(l_{xx}^{2}z^{2}) - l_{xx}l_{xt}z^{2}dt - \frac{1}{2}l_{xx}^{2}(dz)^{2}. \] (A.5)

The second one can be rewritten as
\[ -2(x + \varepsilon)^{-\alpha}l_{x}z_{x}L_{1}dt \]
\[ = -[(x + \varepsilon)^{-\alpha}l_{x}^{2}z_{x}^{2}]_{x}dx + [(x + \varepsilon)^{-\alpha}l_{x}]_{x}z_{x}^{2}dt + (l_{x}l_{x}z^{2})_{x}dt - (l_{x}l_{x}z^{2})_{x}dt \]
\[ - [(x + \varepsilon)^{-\alpha}l_{xx}^{3}z^{2}]_{x}dt + 3(x + \varepsilon)^{-\alpha}l_{xx}^{3}z^{2}dt - \alpha(x + \varepsilon)^{-\alpha}l_{xx}^{3}z^{2}dt. \] (A.6)

For the last one, we have
\[ -(x + \varepsilon)^{-\alpha}l_{xx}z_{x}L_{1}dt \]
\[ = -[(x + \varepsilon)^{-\alpha}l_{xx}z_{x}^{2}]_{x}dt + \frac{1}{2}[(x + \varepsilon)^{-\alpha}l_{xx}]_{x}z_{x}^{2}dt + \frac{1}{2}l_{xx}l_{xx}z_{x}^{2}dt \]
\[ + (x + \varepsilon)^{-\alpha}l_{xx}z_{x}^{2}dt + l_{xx}l_{xx}z_{x}^{2}dt - (x + \varepsilon)^{-\alpha}l_{xx}^{2}z_{x}^{2}dt. \] (A.7)

By substituting (A.4)-(A.7) into (A.3), we can immediately obtain the desired identity (A.1) and then complete the proof. \( \Box \)

Now we prove Lemma 3.2. We first prove a Carleman estimate with boundary term at \( x = 1 \) based on the weighted identity (A.1). Then we introduce a cut-function to eliminate this boundary term in Carleman estimate.

**Proof of Lemma 3.2.** A direct calculation gives
\[ \begin{aligned}
    l_{x} &= s\lambda \xi(x + \varepsilon)^{\beta-1}\phi_{1}, \\
    l_{t} &= s\xi_{t}\phi_{1}, \\
    l_{tt} &= s\xi_{tt}\phi_{1}, \\
    l_{xx} &= s\lambda^{2}\xi(x + \varepsilon)^{2\beta-2}\phi_{1} + s\lambda\xi(\beta - 1)(x + \varepsilon)^{\beta-2}\phi_{1},
\end{aligned} \] (A.8)
where \( \phi_1(x) = e^{\lambda_1(x)} \). Then integrating both sides of (A.1) in \( Q_T \) and taking mathematical expectation in \( \Omega \), together with (A.8), we find that

\[
E \int_{Q_T} (x + \varepsilon)^{-\alpha} L_1 \theta \left( (x + \varepsilon)^{\alpha} d\rho + p_{xx} \, dt \right) \, dx = E \int_{Q_T} (x + \varepsilon)^{-\alpha} |L_1|^2 \, dx \, dt + \sum_{i=1}^{5} I_i + E \int_{Q_T} dK \, dx + E \int_{Q_T} \{ \cdot \} \, dx + E \int_{Q_T} J \, dx,
\]

where

\[
I_1 = E \int_{Q_T} s \xi \left[ 2\lambda^2 (x + \varepsilon)^{-\alpha + 2\beta - 2} + C_{\alpha,\beta}^{(1)} \lambda (x + \varepsilon)^{-\alpha + \beta - 2} \right] \phi_1 |z|^2 \, dx \, dt,
\]

\[
I_2 = E \int_{Q_T} s^3 \xi^3 \left[ 2\lambda^4 (x + \varepsilon)^{-\alpha + 4\beta - 4} + C_{\alpha,\beta}^{(1)} \lambda^3 (x + \varepsilon)^{-\alpha + 3\beta - 4} \right] \phi_1^3 |z|^2 \, dx \, dt,
\]

\[
I_3 = -C_{\alpha,\beta}^{(2)} E \int_{Q_T} s \lambda \xi (x + \varepsilon)^{-\alpha + \beta - 4} \phi_1 |z|^2 \, dx \, dt,
\]

\[
I_4 = E \int_{Q_T} s \xi O(\lambda^2)(x + \varepsilon)^{-\alpha + 2\beta - 4} \phi_1 |z|^2 \, dx \, dt + \frac{1}{2} E \int_{Q_T} s \xi_{tt}(x + \varepsilon)^{\alpha} \phi_1 |z|^2 \, dx \, dt,
\]

\[
I_5 = E \int_{Q_T} s \xi O(\lambda^4) \left[ (x + \varepsilon)^{-\alpha + 3\beta - 4} + (x + \varepsilon)^{-\alpha + 4\beta - 4} \right] \phi_1 |z|^2 \, dx \, dt + E \int_{Q_T} s^2 O(\lambda^2)(x + \varepsilon)^{2\beta - 2} \xi \phi_1 |z|^2 \, dx \, dt.
\]

Here \( C_{\alpha,\beta}^{(1)} = -\alpha + 2\beta - 2, \) \( C_{\alpha,\beta}^{(2)} = \frac{1}{2}(\beta - 1)(2 + \alpha - \beta)(3 + \alpha - \beta) \) and \( O(\lambda) \) denotes different functions such that \( |O(\lambda)| \leq C\lambda \).

Next we estimate the last four terms on the right-hand side of (A.9). Obviously, we have

\[
2 \sum_{i=1}^{2} I_i \geq C_{\alpha,\beta}^{(1)} E \int_{Q_T} s \lambda \xi (x + \varepsilon)^{-\alpha + \beta - 2} \phi_1 |z|^2 \, dx \, dt + C_{\alpha,\beta}^{(1)} E \int_{Q_T} s^3 \lambda^3 \xi^3 (x + \varepsilon)^{-\alpha + 3\beta - 4} \phi_1^3 |z|^2 \, dx \, dt.
\]

By Hardy inequality, we obtain

\[
|I_3| \leq C_{\alpha,\beta}^{(2)} \frac{4}{(\alpha - \beta + 3)^2} E \int_{Q_T} s \lambda \xi (x + \varepsilon)^{-\alpha + \beta - 4} \left( \phi_1^2 \frac{1}{z} \right)^2 \, dx \, dt
\]

\[
\leq C_{\alpha,\beta}^{(2)} \frac{4}{(\alpha - \beta + 3)^2} E \int_{Q_T} s \lambda \xi (x + \varepsilon)^{-\alpha + \beta - 2} \phi_1 |z|^2 \, dx \, dt
\]

\[
+ C(\lambda) E \int_{Q_T} s \xi (x + \varepsilon)^{-\alpha + 3\beta - 4} \phi_1 |z|^2 \, dx \, dt.
\]

By Hölder inequality and \( \beta \leq 2 + \alpha \), we have

\[
|I_4| \leq C(\lambda) E \int_{Q_T} \xi (x + \varepsilon)^{-\alpha + \beta - 4} \phi_1 |z|^2 \, dx \, dt
\]

\[
+ E \int_{Q_T} s^2 \xi^3 (x + \varepsilon)^{-\alpha + 3\beta - 4} \phi_1 |z|^2 \, dx \, dt.
\]
Moreover,

\[ |I_5| \leq C(\lambda)E \int_{Q_T} s^2\xi^3(x + \varepsilon)^{-\alpha+3\beta-4}\phi_1^2|z|^2dxdt. \tag{A.13} \]

Therefore, from (A.10)-(A.13) it follows that

\[
\sum_{i=1}^{5} I_i \geq E \int_{Q_T} \left( C^{(3)}_{\alpha,\beta}s\lambda - C(\lambda) \right) \xi(x + \varepsilon)^{-\alpha+\beta-2}\phi_1|z_x|^2dxdt
\]

\[ + E \int_{Q_T} \left( C^{(1)}_{\alpha,\beta}s^3\lambda^3 - C(\lambda)s - s^2 \right) \xi^3(x + \varepsilon)^{-\alpha+3\beta-4}\phi_1^2|z|^2dxdt \tag{A.14} \]

with

\[ C^{(3)}_{\alpha,\beta} = C^{(1)}_{\alpha,\beta} - \frac{4}{(\alpha - \beta + 3)^2} > 0 \]

due to \( \beta > 1 + \alpha \). Further for all sufficiently large \( s \) we obtain that

\[
\sum_{i=1}^{5} I_i \geq E \int_{Q_T} s\xi(x + \varepsilon)^{-\alpha+\beta-2}|z_x|^2dxdt
\]

\[ + E \int_{Q_T} s^3\xi^3(x + \varepsilon)^{-\alpha+3\beta-4}|z|^2dxdt. \tag{A.15} \]

It is easy to see that \( z(x,0) = z(x,T) = 0 \) for \( x \in I \) and then

\[ E \int_{Q_T} dKdx = 0. \tag{A.16} \]

Now we deal with \( E \int_{Q_T} \{ \cdot \}\_x dx \). Taking into account \( z(0,t) = z(1,t) = 0 \) for \( t \in (0,T) \) and (A.8), we can obtain

\[ E \int_{Q_T} \{ \cdot \}\_x dx = - E \int_0^T [(x + \varepsilon)^{-\alpha}t_xz_x^2]_{x=0}^{x=1}dt \geq -C(\lambda)E \int_0^T s\xi z_x^2(1,t)dt. \tag{A.17} \]

Finally, noticing that \( (dz)^2 = (x + \varepsilon)^{-2\alpha}\theta^2|F_1|^2dt \) and \(-2\alpha + 2\beta - 2 \geq -\alpha\), we obtain for the term of \( J \) that

\[ E \int_{Q_T} Jdx \geq \frac{1}{2} E \int_{Q_T} (x + \varepsilon)^{\alpha}I_1(dz)^2dx - E \int_{Q_T} I_2^2(dz)^2dx
\]

\[ \geq - C(\lambda)E \int_{Q_T} [s\xi^2(x + \varepsilon)^{-\alpha} + s^2\xi^2(x + \varepsilon)^{-2\alpha+2\beta-2}] \theta^2|F_1|^2dxdt \]

\[ \geq - C(\lambda)E \int_{Q_T} s^2\xi^2(x + \varepsilon)^{-\alpha}\theta^2|F_1|^2dxdt. \tag{A.18} \]

Thus, by substituting (A.15)-(A.18) into (A.9), we find that

\[ E \int_{Q_T} (x + \varepsilon)^{-\alpha}L_1\theta [(x + \varepsilon)^{\alpha}d\rho^2 + p_{xx}^2]dx \]

\[ \geq E \int_{Q_T} (x + \varepsilon)^{-\alpha}|L_1|^2dxdt + E \int_{Q_T} s\xi(x + \varepsilon)^{-\alpha+\beta-2}|z_x|^2dxdt
\]

\[ + E \int_{Q_T} s^3\xi^3(x + \varepsilon)^{-\alpha+3\beta-4}|z|^2dxdt - C(\lambda)E \int_{Q_T} s^2\xi^2(x + \varepsilon)^{-\alpha}\theta^2|F_1|^2dxdt
\]

\[ - C(\lambda)E \int_0^T s\xi z_x^2(1,t)dt. \tag{A.19} \]
On the other hand, using the equation of \( p^\varepsilon \) and
\[
E \int_{Q_T} (x + \varepsilon)^\alpha L_1 \theta F_1 dx dB(t) = 0,
\]
we obtain
\[
E \int_{Q_T} (x + \varepsilon)^{-\alpha} L_1 \theta [(x + \varepsilon)^\alpha dp^\varepsilon + p^\varepsilon_{xx} dt] dx = E \int_{Q_T} (x + \varepsilon)^{-\alpha} L_1 \theta f_1 dx dt
\]
\[
\leq E \int_{Q_T} (x + \varepsilon)^{-\alpha} |L_1|^2 dx dt + \frac{1}{4} E \int_{Q_T} (x + \varepsilon)^{-\alpha} \theta^2 |f_1|^2 dx dt. \tag{A.20}
\]
Combining (A.19) with (A.20), recalling \( \theta = e^{\Phi_1} \) and going back to the original variable \( p^\varepsilon \), we obtain
\[
E \int_{Q_T} s \xi (x + \varepsilon)^{-\alpha+\beta-2} |p^\varepsilon_x|^2 e^{2s\Phi_1} dx dt + E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^{-\alpha+3\beta-4} |p^\varepsilon|^2 e^{2s\Phi_1} dx dt
\]
\[
\leq C E \int_{Q_T} (x + \varepsilon)^{-\alpha} |f_1|^2 e^{2s\Phi_1} dx dt + C(\lambda) E \int_{Q_T} s^2 (x + \varepsilon)^{-\alpha} |F_1|^2 e^{2s\Phi_1} dx dt + C(\lambda) E \int_0^T s \xi |p^\varepsilon_x|^2 (1,t) e^{2s\Phi_1(1,t)} dt. \tag{A.21}
\]
To eliminate the boundary term, we consider the following system of \( \tilde{p}^\varepsilon = \chi p^\varepsilon \)
\[
\begin{cases}
(x + \varepsilon)^\alpha d\tilde{p}^\varepsilon + \tilde{p}^\varepsilon_{xx} dt = \tilde{f}_1 dt + \tilde{F}_1 dB(t), & (x,t) \in Q_T,
\tilde{p}^\varepsilon(0,t) = \tilde{p}^\varepsilon(1,t) = 0, & t \in (0,T),
\tilde{p}^\varepsilon(x,T) = \chi(x)p^\varepsilon_T(x), & x \in I,
\end{cases} \tag{A.22}
\]
where
\[
\tilde{f}_1 = \chi f_1 + 2\chi_{xx} p^\varepsilon_x + \chi_{xx} p^\varepsilon, \quad \tilde{F}_1 = \chi F_1.
\]
Applying (A.21) to \( \tilde{p}^\varepsilon \) and recalling the definition of \( \chi \), we have
\[
E \int_{Q_T} s \xi (x + \varepsilon)^{-\alpha+\beta-2} |\tilde{p}^\varepsilon_x|^2 e^{2s\Phi_1} dx dt + E \int_{Q_T} s^3 \xi^3 (x + \varepsilon)^{-\alpha+3\beta-4} |\tilde{p}^\varepsilon|^2 e^{2s\Phi_1} dx dt
\]
\[
\leq C E \int_{Q_T} (x + \varepsilon)^{-\alpha} (\chi f_1 + 2\chi_{xx} p^\varepsilon_x + \chi_{xx} p^\varepsilon)^2 e^{2s\Phi_1} dx dt
\]
\[
+ C(\lambda) E \int_{Q_T} s^2 (x + \varepsilon)^{-\alpha} \xi^2 |F_1|^2 e^{2s\Phi_1} dx dt. \tag{A.23}
\]
Noticing that \( \tilde{p}^\varepsilon = p^\varepsilon \) for \( x \in (0,(1)) \) and \( \text{supp}(\chi_x) \subset \omega_1 \), we deduce (25) from (A.23) and then complete the proof of Lemma 3.2.

The proof of Lemma 3.3 is based on the classic Carleman estimate for stochastic nondegenerate parabolic equation, e.g. [29].

**Proof of Lemma 3.3.** Letting \( \bar{p} = (1-\chi)p \), then we have
\[
\begin{cases}
d\bar{p} + (x + \varepsilon)^{-\alpha} \bar{p}_{xx} dt = (x + \varepsilon)^{-\alpha} \bar{f}_1 dt + (x + \varepsilon)^{-\alpha} \bar{F}_1 dB(t), & (x,t) \in (1,1),
\bar{p}(x,1) = \bar{p}(1,1) = 0, & t \in (0,T),
\bar{p}(x,T) = (1-\chi(x))p^\varepsilon_T(x), & x \in (1,1),
\end{cases} \tag{A.24}
\]
where
\[
\bar{f}_1 = (1-\chi)f_1 - 2\chi_{xx} p^\varepsilon_x - \chi_{xx} p^\varepsilon, \quad \bar{F}_1 = (1-\chi)F_1.
\]
We apply Theorem 6.1 in [29] to $\bar{\rho}^\epsilon$ to yield that

$$
\mathbb{E} \int_0^T \int_{x(1)} |(1-\chi)f_1 - 2\chi_\delta p^\epsilon - \chi_{xx} p^\epsilon|^2 e^{2\Phi_2} \, dx \, dt + \mathbb{E} \int_0^T \int_{x(1)} |\rho^\epsilon|^2 e^{2\Phi_2} \, dx \, dt
$$

$$
\leq C \mathbb{E} \int_0^T \int_{x(1)} |(1-\chi)f_1 - 2\chi_\delta p^\epsilon - \chi_{xx} p^\epsilon|^2 e^{2\Phi_2} \, dx \, dt
$$

$$
+ C \mathbb{E} \int_0^T \int_{x(1)} \rho^\epsilon|^2 e^{2\Phi_2} \, dx \, dt
$$

$$
+ C \mathbb{E} \int_0^T \int_{x(1)} \rho^\epsilon|^2 e^{2\Phi_2} \, dx \, dt
$$

$$
\leq C \mathbb{E} \int_0^T \int_{x(1)} (x+\epsilon)^{-\alpha}|f_1|^2 e^{2\Phi_2} \, dx \, dt + C \mathbb{E} \int_0^T \int_{x(1)} (|p^\epsilon|^2 + |p^\epsilon|^2) e^{2\Phi_2} \, dx \, dt
$$

$$
+ C \mathbb{E} \int_0^T \int_{x(1)} \rho^\epsilon|^2 e^{2\Phi_2} \, dx \, dt
$$

$$
+ C \mathbb{E} \int_0^T \int_{x(1)} \rho^\epsilon|^2 e^{2\Phi_2} \, dx \, dt,
$$

(A.25)

where $C$ is depending on $x(1), I, T, \omega, \alpha, M, \lambda$, but independent of $\epsilon$. Together with $\bar{\rho} = p$ and $\min\{ |x+\epsilon|^{-\alpha+\beta-2}, (x+\epsilon)^{-\alpha+3\beta-4} \} > 0$ for $x \in (x(2), 1)$, we then obtain (26) from (A.25).

**Acknowledgments.** The authors would like to thank the anonymous referees for their valuable comments and suggestions, which make this paper much improved.

**REFERENCES**

[1] F. Alabau-Boussouira, P. Cannarsa and G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, *J. Evol. Equ.* 6 (2006), 161–204.

[2] V. Barbu, A. Răşcanu and G. Tessitore, Carleman estimate and controllability of linear stochastic heat equations, *Appl. Math. Optim.*, 47 (2003), 97–120.

[3] U. Biccari and E. Zuazua, Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function, *J. Differential Equations*, 261 (2016), 2809–2853.

[4] P. Cannarsa, P. Martinez and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, *SIAM J. Control Optim.*, 47 (2008), 1–19.

[5] P. Cannarsa, P. Martinez and J. Vancostenoble, Carleman estimates and null controllability for boundary-degenerate parabolic operators, *C. R. Math. Acad. Sci. Paris.*, 347 (2009), 147–152.

[6] P. Cannarsa, J. Tort and M. Yamamoto, Determination of source terms in a degenerate parabolic equation, *Inverse Probl.*, 26 (2010), 105003, 20pp.

[7] C. Cazacu, Controllability of the heat equation with an inverse-square potential localized on the boundary, *SIAM J. Control Optim.*, 52 (2014), 2055–2089.

[8] E. Corpa, P. Guzmán and A. Mercado, On the control of the linear Kuramoto-Sivashinsky equation, *ESAIM Control Optim. Calc. Var.*, 23 (2017), 165–194.

[9] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 2014.

[10] R. Du, Null controllability for a class of degenerate parabolic equations with gradient terms, *J. Evol. Equ.*, 19 (2019), 585–613.

[11] R. Du, J. Eichhorn, Q. Liu and C. Wang, Carleman estimates and null controllability of a class of singular parabolic equations, *Adv. Nonlinear Anal.*, 8 (2019), 1057–1082.

[12] S. Ervedoza, Control and stabilization properties for a singular heat equation with an inverse-square potential, *Comm. Partial Differential Equations*, 33 (2008), 1996–2019.

[13] J. C. Flores and L. de Teresa, Null controllability of one dimensional degenerate parabolic equations with first order terms, *Discrete Contin. Dyn. Syst. Ser. B*, 25 (2020), 3963–3981.
[14] G. Fragnelli, Null controllability of degenerate parabolic equations in non divergence form via Carleman estimates, *Discrete Contin. Dyn. Syst. Ser. S*, 6 (2013), 687–701.

[15] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series, 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[16] P. Gao, M. Chen and Y. Li, Observability estimates and null controllability for forward and backward linear stochastic Kuramoto-Sivashinsky equations, *SIAM J. Control Optim.*, 53 (2015), 475–500.

[17] O. Glass and S. Guerrero, Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition, *Systems Control Lett.*, 59 (2010), 390–395.

[18] V. Hernández-Santamaría, K. L. Balch and L. Peralta, Global null-controllability for stochastic semilinear parabolic equations, preprint, *arXiv:2010.08854v1* (2020).

[19] O. Y. Imanuvilov and M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, *Publ. Res. Inst. Math. Sci.*, 39 (2003), 227–274.

[20] S. Kamin and P. Rosenau, Propagation of thermal waves in an inhomogeneous medium, *Comm. Pure Appl. Math.*, 34 (1981), 831–852.

[21] J.-L. Lions, *Optimal Control of System Governed by Partial Differential Equation*, vol. 170, Springer-Verlag, New York, 1971.

[22] X. Liu, Global Carleman estimates for stochastic parabolic equations and its application, *ESAIM Control Optim. Calc. Var.*, 20 (2014), 823–839.

[23] X. Liu and Y. Yu, Carleman estimates of some stochastic degenerate parabolic equations and application, *SIAM J. Control Optim.*, 57 (2019), 3527–3552.

[24] Q. Lü, Exact controllability for stochastic transport equations, *SIAM J. Control Optim.*, 52 (2014), 397–419.

[25] Q. Lü, Carleman estimate for stochastic parabolic equations and inverse stochastic parabolic problems, *Inverse Probl.*, 28 (2012), 045008, 18pp.

[26] Q. Lü, Observability estimate for stochastic Schrödinger equations and its applications, *SIAM J. Control Optim.*, 51 (2013), 121–144.

[27] H. Ockendon, Channel flow with temperature-dependent viscosity and internal viscous dissipation, *J. Fluid Mech.*, 93 (1979), 737–746.

[28] P. Rosenau and S. Kamin, Nonlinear diffusion in a finite mass medium, *Comm. Pure Appl. Math.*, 35 (1982), 113–127.

[29] S. Tang and X. Zhang, Null controllability for forward and backward stochastic parabolic equations, *SIAM J. Control Optim.*, 48 (2009), 2191–2216.

[30] G. Tessitore, Existence, uniqueness and space regularity of the adapted solutions of a backward SPDE, *Stochastic Analysis and Applications*, 14 (1996), 461–486.

[31] J. Vancostenoble and E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials, *J. Funct. Anal.*, 254 (2008), 1864–1902.

[32] C. Wang and R. Du, Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms, *SIAM J. Control Optim.*, 52 (2014), 1457–1480.

[33] C. Wang, Y. Zhou, R. Du and Q. Liu, Carleman estimate for solutions to a degenerate convection-diffusion equation, *Discrete Contin. Dyn. Syst. Ser. B*, 23 (2018), 4207–4222.

[34] B. Wu, Q. Chen and Z. Wang, Carleman estimates for a stochastic degenerate parabolic equation and applications to null controllability and an inverse random source problem, *Inverse Probl.*, 36 (2020), 075014, 38pp.

[35] B. Wu, Y. Gao, Z. Wang and Q. Chen, Unique continuation for a reaction-diffusion system with cross diffusion, *J. Inverse Ill-Posed Probl.*, 27 (2019), 511–525.

[36] B. Wu and J. Yu, Hölder stability of an inverse problem for a strongly coupled reaction-diffusion system, *IMA J. Appl. Math.*, 82 (2017), 424–444.

[37] M. Yamamoto, Carleman estimates for parabolic equations and applications, *Inverse Probl.*, 25 (2009), 123013, 75pp.

[38] Y. Yan, Carleman estimates for stochastic parabolic equations with Neumann boundary conditions and applications, *J. Math. Anal. Appl.*, 457 (2018), 248–272.

[39] G. Yuan, Determination of two kinds of sources simultaneously for a stochastic wave equation, *Inverse Probl.*, 31 (2015), 085003, 13pp.
[40] X. Zhang, Carleman and observability estimates for stochastic wave equations, *SIAM J. Math. Anal.*, 40 (2008), 851–868.

Received September 2020; 1st revision April 2021; 2nd revision May 2021; early access July 2021.

E-mail address: binwu@nuist.edu.cn
E-mail address: yanlin@nuist.edu.cn