Infinite Hierarchy of Exact Equations in the Bak-Sneppen Model

Sergei Maslov
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973
and Department of Physics, SUNY at Stony Brook, Stony Brook, New York 11794
(November 17, 2021)

We derive an infinite hierarchy of exact equations for the Bak-Sneppen model in arbitrary dimensions. These equations relate different moments of temporal duration and spatial size of avalanches. We prove that the exponents of the BS model are the same above and below the critical point and express the universal amplitude ratio of the avalanche spatial size in terms of the critical exponents. The equations uniquely determine the shape of the scaling function of the avalanche distribution. It is suggested that in the BS model there is only one independent critical exponent.

05.40+j, 64.60Ak, 64.60Fr, 87.10+e

Recently Bak and Sneppen \[1\] introduced a particularly simple toy model of biological evolution (BS model). The model correctly reproduces such features of real evolution process as punctuated equilibria, power law probability distributions of lifetimes of species and of the sizes of extinction events. In spite of the simplicity of the rules, the model exhibits extremely rich and interesting behavior.

In the one-dimensional version of the model \(L\) numbers \(f_i\) are arranged on a line. At every time step the smallest number in the system and its nearest neighbors are replaced with new uncorrelated random numbers drawn from the uniform distribution between 0 and 1. The generalization to higher dimensions is straightforward. In fact there exists a whole class of models where one selects the site with the extremal (global maximal or minimal) value of some variable and then changes this variable at this site and its nearest neighbors according to some stochastic rule. One of the best known representatives of this class is Invasion Percolation \[2\]. Such models, referred to as extremal models were extensively studied (for a recent review see \[3\]).

The interesting feature of the BS model (as well as other extremal models) is its ability to organize itself into a scale-free stationary state. The dynamics in this critical state is given in terms of bursts of activity or avalanches, which form a hierarchical structure \[4\] of subavalanches within bigger avalanches. Here we introduce a “master” equation for this avalanche hierarchy. It describes the cascade process of smaller avalanches merging into bigger ones as the critical parameter is changed. From this equation we derive an infinite series of exact equations, relating different moments of temporal duration \(S\) and spatial size \(R^d\) of individual avalanches.

The “master” equation connects undercritical and overcritical regions of parameters. Given the existence of the scaling, we rigorously prove that the exponents of the BS model are the same above and below the transition. From our results it follows that all terms of Taylor series of the scaling function \(f(x)\) for the avalanche distribution are uniquely and explicitly determined by two critical exponents of the model. It was suggested that usual restrictions on the shape of \(f(x)\) indirectly relate these two exponents and, therefore, reduce the number of independent critical exponents in the BS model to just one.

To define these avalanches one records the signal of the model, i.e. the value of the global minimal number \(f_{\text{min}}(s)\) as a function of time \(s\). Then for every value of the auxiliary parameter \(f_o\), an \(f_o\)-avalanche of size (temporal duration) \(S\) is defined as a sequence of \(S - 1\) successive events when \(f_{\text{min}}(s) < f_o\) confined between two events when \(f_{\text{min}}(s) > f_o\). In other words, the events when \(f_{\text{min}}(s) > f_o\) divide the time axis into a series of avalanches, following one another. It is easy to see that an avalanche defined by this rule is nothing else but a creation-annihilation branching process where sites with \(f_i < f_o\) play the role of particles. The avalanche is terminated (and the next one is immediately started) when there are no such “particles” left in the system. As in any other creation-annihilation branching processes (such as directed percolation) in BS model there exists a critical value \(f_c\) of the creation probability \(f_o\), for which the creation of particles is just marginally balanced by their annihilation, and avalanches of all sizes can be realized.

In the stationary state of the BS model on the infinite lattice, \(f_{\text{min}}(s) \leq f_c\) for every \(s\). Therefore, the overcritical \((f_o > f_c)\) region of the branching process parameters is not accessible, since there are no events starting or terminating such avalanches. However, if the system is artificially prepared in the overcritical state with \(f_i > f_o\) everywhere, one can observe overcritical avalanches. In this case there is a non-zero probability \(P_{\infty}(f_o)\) to start an infinite avalanche and at the same time the size of finite avalanches has a finite cutoff.

The events within the same avalanche are spatially and causally connected. It is easy to understand that the position of active site at any time step within the avalanche
is connected to the set of sites covered (updated at least once) by the avalanche up to this time step. We characterize an avalanche by two principal numbers: 1) $S$ – the avalanche size, equal to its temporal duration; 2) $n_{cov}$ – the number of covered sites, i.e., sites that had their random number updated at least once during the course of this avalanche. In one-dimensional models the connected nature of the set of covered sites ensure its compactness and, therefore, $n_{cov}$ is equal to the avalanche spatial extent $R$. In higher dimensions (below the upper critical dimension) it was conjectured in [3] that the set of covered sites is a non-fractal object of the same dimensionality as the underlying lattice. In this case the spatial size $R$ of the avalanche can be defined by the relation $n_{cov} = R^d$.

$f_o$-avalanches in the Bak-Sneppen model were shown to be exactly equivalent to the realizations of the BS branching process [4]. In this process one only keeps track of the numbers $f_i < f_o$, and at each time step activates the smallest one of them. The BS branching process is terminated when there are no numbers $f_i < f_o$ left in the system. Besides the fact that the BS branching process is a very effective way to simulate the BS model numerically, it has the additional important advantage that the overcritical region $f_o > f_i$ becomes accessible.

The quantity of primary interest in the BS model is the probability distribution $P(S, f_o)$ of the avalanche sizes $S$ at any given value of the auxiliary parameter $f_o$. The moments in time, when $f_o < f_{\min}(s) < f_o + df_o$ serve as breaking points for $f_o$-avalanches but not for $f_o + df_o$-avalanches. Therefore, when $f_o$ is raised by an infinitesimal amount $df_o$ some of $f_o$-avalanches merge together to form bigger $(f_o + df_o)$-avalanches. In the rest of this paper we study in more detail the properties of this merging process and the avalanche hierarchy that it induces.

The most important observation about $f_o$-avalanches in the BS model (as well as in several other extremal models, such as the Sneppen model [6] or Invasion Percolation [3]) is that when an $f_o$-avalanche is terminated, the numbers $f_i$ on the set of $n_{cov} = R^d$ updated sites are uncorrelated and uniformly distributed between 0 and 1. If such a number $f_i$ is raised by $df_o$ is given by $R^d df_o/1-f_o$. (the merging occurs if at least one of the changed numbers falls in $[f_o, f_o + df_o]$.) For the following arguments to be true it is important that any two subsequent avalanches are mutually uncorrelated. That is: the probability distribution of $f_o$-avalanches, starting immediately after the termination of an $f_o$-avalanche of a given size $S$ is independent of $S$. That is true for the BS model since the dynamics within an $f_o$-avalanche in BS model is completely independent of the particular value of the numbers $f_i > f_o$ in the background that were left by the previous avalanches. This mutual independence may not be the case for other extremal models such as the Sneppen model or Invasion Percolation. To understand to what extent the results of this work apply to these other models is the direction of our current research [7].

Now we are in a position to write down the exact “master” equation describing how avalanche merging changes $P(S, f_o)$ as $f_o$ is raised. Let $R^d(S, f_o)$ be the average number of updated (covered) sites in an $f_o$-avalanche of temporal size $S$. From our simulations of the BS model [3] we know that for $f_o$ close to $f_c$, $R^d(S, f_o)$ scales with $S$ as $S^{d/D}$, where $D$ is the fractal mass dimension of the avalanche. However, for the following arguments any form of $R^d(S, f_o)$ will suffice. The “master” equation for $P(S, f_o)$ can be written as

$$
(1 - f_o) \frac{\partial P(S, f_o)}{\partial f_o} = -P(S, f_o)R^d(S, f_o)
$$

$$
+ \sum_{S_1=1}^{S-1} P(S_1, f_o)R^d(S_1, f_o)P(S - S_1, f_o)
$$

(1)

Here the first term describes the loss of avalanches of size $S$ due to the merging with the subsequent one, while the second term describes the gain in $P(S, f_o)$ due to merging of avalanches of size $S_1$ with avalanches of size $S - S_1$. It is convenient to change variables from $f_o$ to $g = -\ln(1-f_o)$, so that $f_o = 0$ corresponds to $g = 0$, and $f_o = 1$ corresponds to $g = +\infty$, and $df_o = dg = \frac{1}{1-f_o}$. This change is due to the fact that, although traditionally new random numbers are drawn from the flat distribution $P(f_o) = 1$, the “natural” distribution for the BS model has the probability density $P(g) = e^{-g}$. As usual, the critical properties of the model are independent of the particular shape of $P$. In the rest of the paper we will use the “natural” variable $g$ instead of $f_o$.

To proceed further we make the Laplace transformation of $P(S, g)$: $p(\alpha, g) = \sum_{S=1}^{\infty} P(S, g)e^{-\alpha S}$ and of $P(S, g)R^d(S, g)$: $r(\alpha, g) = \sum_{S=1}^{\infty} P(S, g)R^d(S, g)e^{-\alpha S}$. The equation (1) can be conveniently written in terms of Laplace transforms $p(\alpha, g) = \sum_{S=1}^{\infty} P(S, g)e^{-\alpha S}$ and $r(\alpha, g) = \sum_{S=1}^{\infty} P(S, g)R^d(S, g)e^{-\alpha S}$ as $\partial p(\alpha, g)/\partial g = -r(\alpha, g) + p(\alpha, g)r(\alpha, g)$, or simply,

$$
\frac{\partial \ln(1 - p(\alpha, g))}{\partial g} = r(\alpha, g)
$$

(2)

This exact equation is the central result of this paper. It has many interesting physical consequences. When $g < g_c$ all avalanches are finite ($P_{\infty} = 0$) and normalization requires $p(0, g) = \sum_{S=1}^{\infty} P(S, g) = 1$. From the general properties of the Laplace transform one can
write the Taylor series for \( p(\alpha, g) \) and \( r(\alpha, g) \) at \( \alpha = 0 \) as
\[
p(\alpha, g) = 1 - (S_g)_0\alpha + (S_g^2)_0\alpha^2/2 - (S_g^3)_0\alpha^3/6 + \ldots
\]
and
\[
r(\alpha, g) = (R^d)_{g} - (R^d)_{g}S_g\alpha + (R^d)_{g}S_g^2\alpha^2/2 + \ldots.
\]
Substitution of these expressions into Eq. (2) results in
\[
\frac{\partial}{\partial g} \ln((S)_g) - (S_g^2)\alpha^2/2 + \ldots = (R^d)_{g} - (R^d)_{g}S_g\alpha + (R^d)_{g}S_g^2\alpha^2/2 + \ldots.
\]
Since the equation (2) holds for arbitrary \( \alpha \), comparing the coefficients of different powers of \( \alpha \) in the above Taylor series results in an in\-\linebreak finite series of exact equations. Comparison of the coefficients of \( \alpha^0 \) gives
\[
\frac{d \ln(S)_g}{dg} = (R^d)_{g}.
\] (3)
This is exactly the “gamma”-equation derived in [3].
This equation is valid not only for the BS model but also for the Sneppen model and Invasion Percolation as well since it does not rely on the assumption that the sizes of subsequent avalanches are uncorrelated. To show this, one has to take a large number \( N \) of \( \text{f}_\text{c}\)-avalanches and write the balance equation of how this number decreases as \( \text{f}_\text{c} \) is increased [3]. Changing the variables from \( g \) back to \( \text{f}_\text{c} \) gives the more familiar form of the “gamma”-
\equation: \( d\ln(S)_{\text{f}_\text{c}}/df_\text{c} = (R^d)_{\text{f}_\text{c}}/(1 - \text{f}_\text{c}) \).

Higher powers of \( \alpha \) in Eq. (2) give new exact equations. Here we show just the first two:
\[
\frac{d}{dg} \left( \langle S^2 \rangle_g \right) = 2 \langle R^d S \rangle_g;
\] (4)
\[
\frac{d}{dg} \left( \frac{(S^3)_g}{3} - \frac{(S^2)_g^2}{2} \right) = \langle R^d S^2 \rangle_g.
\] (5)

The Taylor expansion changes slightly in the overcritical region, where there is a finite probability \( P_\infty(g) \) to start an infinite avalanche. Since the avalanche distribution \( P(S, g) \) is limited to finite avalanches, it is naturally normalized to \( 1 - P_\infty(g) \). Therefore, when \( g > g_c \), the Fourier series for \( p(\alpha, g) \) can be written as
\[
p(\alpha, g) = 1 - P_\infty(g) - (S)_0\alpha + (S^2)_0\alpha^2/2 + \ldots.
\]
Now the comparison of the coefficients of \( \alpha^0 \) at \( g = \infty \) gives
\[
\frac{d \ln P_\infty(g)}{dg} = (R^d)_{g}.
\] (6)
This new equation is the \( g > g_c \) analog of the “gamma”-equation (3). We will refer to it as “beta”-equation (the exponent \( \beta \) is traditionally used for the scaling of \( P_\infty(g) \)), while \( -\gamma \) is used for \( \langle S \rangle_g \).

There is a more straightforward way to derive equation (6) from the average properties of the merging process. The merging of infinite and infinite avalanches gives an infinite avalanche and, therefore, leads to an increase in \( P_\infty(g) \). A simple analysis gives an equation governing this process as
\[
dP_\infty/dg = (R^d)_{g}P_\infty, \text{ which is just Eq. (6).}
\]
As in the undercritical case, the Taylor expansion of Eq. (6) for \( g > g_c \) determines an infinite series of exact equations. Here are the first two:
\[
\frac{d}{dg} \left( \frac{\langle S \rangle_g}{P_\infty(g)} \right) = -(R^d)_{g}S_g;
\] (7)
\[
\frac{d}{dg} \left( \frac{(S^2)_g}{P_\infty(g)} + \frac{2(S)_g^2}{P_\infty(g)^2} \right) = -(R^d S^2)_{g}.
\] (8)

As in other creation-annihilation branching processes, the avalanche distribution \( P(S, g) \) in BS model for \( g < g_c \) is known to have a scaling form
\[
P(S, g) = S^{-\tau} f(S^\sigma(g - g_c)),
\] (9)
where \( \tau \) and \( \sigma \) are some critical exponents and \( f(x) \) is a scaling function that rapidly decays to zero as \( x \to -\infty \). From (6) it follows that the average avalanche size diverges when \( g \) approaches \( g_c \): from below as \( \langle S \rangle_g \sim (g_c - g)^{-\gamma} \), where \( \gamma = 2 - \sigma \). Substitution of this expression into the “gamma”-equation (3) results in
\[
(R^d)_{g} = \frac{\gamma}{g_c - g}, \text{ for } g < g_c.
\] (10)

In the BS model the critical exponent \( \gamma \) determines the amplitude of \( (R^d)_{g} \) below \( g_c \). The exponent relation derived from (3) connects \( \tau \) to \( D \) and \( \sigma \): \( \tau = 1 + d/D - \sigma \) [3]. It is easy to see that Eqs. (3) do not yield additional exponent relations but further restrict the exact form of the avalanche distribution. In fact it can be shown that Eq. (3) and the exponents \( D \) and \( \tau \) uniquely determine the shape of the scaling function \( f(x) \) [3].

The scaling should work in the overcritical regime as well. However, unlike in the equilibrium statistical mechanics, the critical exponent \( \sigma \) can a priori be different above and below the transition. In what follows we show that at least for the BS model this is not true. Substitution of the scaling form form \( P_\infty(g) \) into the “gamma”-equation (3) results in
\[
(R^d)_{g} = \frac{\beta}{g - g_c}, \text{ for } g > g_c.
\] (11)
Again, similar to the “gamma”-equation (10), the critical exponent \( \beta \) gives the amplitude of \( (R^d)_{g} \) above the transition. From (11) it follows that the same exponent relation \( \tau = 1 + d/D - \sigma \) holds in the overcritical region, and, therefore, the exponent \( \sigma \) is the same above and below the transition. The scaling form (3) can now be extended to include the overcritical region. For this one just lets the argument \( x \) of the scaling function \( f(x) \) be positive. As in various percolation problems [3] the scaling form (3) for \( P(S, g) \) at \( g > g_c \) results in the exponent relation \( \beta = -\tau/\gamma \). An interesting consequence of exact Eqs. (3), (11) is that the universal amplitude ratio for \( (R^d)_{g} \) is given by the ratio of two critical exponents
\[
\frac{(R^d)_{g + \Delta g}}{(R^d)_{g - \Delta g}} = \frac{\beta}{\gamma} = \frac{\tau - 1}{2 - \tau}.
\] (12)
This unusual relation between the universal amplitude ratio and critical exponents is to our knowledge unique for the BS model.

There is a case when the master equation (2) can be written in a closed form. This is the extensively studied [11,3] mean field random neighbor version of the BS model, where at each time step \( K - 1 \) “neighbors” of the active site are selected in an annealed random fashion throughout the whole system. It is easy to see that in the thermodynamic limit of this model the number of updated sites in the avalanche of temporal duration \( S \) is given by \( n_{\text{act}} = (K - 1)S + 1 \). This is the quantity that should be used instead of \( R^d(S,g) \) in our equations. The missing equation connecting \( r(\alpha, g) \) and \( p(\alpha, g) \) is

\[
r(\alpha, g) = -(K - 1)\frac{\partial p(\alpha, g)}{\partial \alpha} + p(\alpha, g)
\]

This equation should be solved with the initial condition \( p(\alpha, 0) = e^{\beta \alpha} \), since \( P(S,0) = \delta_{S,1} \). We checked that for \( K = 2 \) the generating function \( \sum_{S=1}^{\infty} P(S,f_0)x^S = 1-2x(f_1(1-f_0)-1-4x f_0(1-f_0))^{1/2} \) derived in [10] using different methods, after the substitution of \( x = e^{-\alpha} \) and \( f_0 = 1-e^{-g} \) satisfies (13) and has the correct initial condition. That confirms the overall consistency of our approach.

It can be shown [9,10] that the substitution of the scaling form (3) into the Eq. (2) defines recursively all terms in the Taylor series of the scaling function \( f(x) \) at \( x = 0 \):

\[
f^{(n+1)}(0) = \sum_{n_1+n_2=n} \frac{\Gamma(1+\sigma+n_1)\Gamma(\sigma+n_2)}{\Gamma(1-\sigma+\sigma+n_1)} \frac{n_1!}{n_2!} f^{(n_1)}(0) f^{(n_2)}(0)
\]

where \( \Gamma(x) \) is the Euler’s gamma function. Actually Eq. (2) does more than that: for a given \( d/D \) it uniquely selects \( \tau \). We suspect that only for this \( \tau \) the scaling function satisfies all usual requirements, such as \( f(x) \to 0 \), when \( x \to \pm \infty \), and \( \int_0^\infty x^{-\tau} f(0) - f(-x^{1/\nu}) dx = 0 \) (the absence of the infinite avalanche below \( g_c \)). Which of these constraints actually defines \( \tau \) as a function of \( d/D \) remains to be determined. The numerical solution of the Eq. (1) with \( R^d(S,f_0) = AS^{d/D} \) indeed seems to give the correct value for \( \tau \).

In [3] it was shown that all the exponents of a general extremal model are determined in terms of just two independent ones, say \( D \) and \( \tau \). As a result of the approach described here the set of independent exponents for the BS model was narrowed down to just one. It is tempting to extend these arguments to other extremal models. The weak point for this lies in the assumption that sizes of two subsequent avalanches are mutually uncorrelated. At present it is unclear how strong is this correlation and how it influences the scaling.

This work was supported by the U.S. Department of Energy Division of Material Science, under contract DE-AC02-76CH00016. The author thanks Prof. Y.-C. Zhang and the Institut de Physique Théorique, Université de Fribourg for hospitality during the visit when part of this work was accomplished. The author is grateful to P. Bak and M. Paczuski for helpful comments on the manuscript.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The results of the numerical solution of Eq.(1) on the interval \( 1 \leq S \leq 100 \) with \( R^d(S,g) = S^{0.412} \). Values of \( g \) increase from top to bottom. The exponent of power law part was measured to be \( 1.1 \pm 0.1 \).}
\end{figure}