New Constructions of Complementary Sequence Pairs over 4\(^q\)-QAM

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Abstract

The previous constructions of quadrature amplitude modulation (QAM) Golay complementary sequences (GCSs) were generalized as 4\(^q\)-QAM GCSs of length 2\(^m\) by Li (the generalized cases I-III for \(q \geq 2\)) in 2010 and Liu (the generalized cases IV-V for \(q \geq 3\)) in 2013 respectively. Those sequences are represented as the weighted sum of \(q\) quaternary standard GCSs. In this paper, we present two new constructions for 4\(^q\)-QAM GCSs of length 2\(^m\), where the proposed sequences are also represented as the weighted sum of \(q\) quaternary standard GCSs. It is shown that the generalized cases I-V are special cases of these two constructions. In particular, if \(q\) is a composite number, a great number of new GCSs other than the sequences in the generalized cases I-V will arise. For example, in the case \(q = 4\), the number of new GCSs is seven times more than those in the generalized cases IV-V. In the case \(q = 6\), the ratio of the number of new GCSs and the generalized cases IV-V is greater than six and will increase in proportion with \(m\). Moreover, our proof implies all the mentioned GCSs over QAM in this paper can be regarded as projections of Golay complementary arrays of size 2 \(\times\) 2 \(\times\) \(\cdots\) \(\times\) 2.

1 Introduction

A pair of sequences is called a Golay complementary pair (GCP) \([11]\) if their aperiodic autocorrelation sums for any nonzero shifts are all equal to zero. Each sequence in the GCP is called a Golay complementary sequence (GCS). The concept of binary GCP was extended later to the polyphase case \([22]\) and complementary sequence sets \([23]\). These sequences have found numerous applications in various fields of science and engineering, especially in orthogonal frequency-division multiplexing (OFDM) systems. One of the major impediments to deploying OFDM is the high peak-to-mean envelope power ratio (PMEPR) of uncoded OFDM signals. PMEPR reduction in OFDM transmission can be implemented \([1]\) \([19]\) by using codes constructed from the sequences in the complementary sequence sets, especially GCSs.
GCPs were initially constructed by the recursive methods [11][3]. An extensive study on this topic was made by Davis and Jedwab in [7] by a direct construction of polyphase complementary sequences based on generalized Boolean functions (GBFs), which have been referred to as the standard GCSs subsequently. Non-standard GCPs were studied in [16][8][9] and complementary sequence sets were constructed in [18][21] based on GBFs later on.

All the aforementioned sequences are constructed over the phase-shift keying (PSK) constellations. Since quadrature amplitude modulation (QAM) are widely employed in high rate OFDM transmissions, 16-QAM sequences based on weighted quaternary PSK (QPSK) GCSs were studied by Rößing and Tarokh [20] in 2001. Chong et al. [5] then proposed a construction of 16-QAM GCSs based on standard GCSs over QPSK and first-order offsets in three cases. It was pointed out in [5] that an OFDM system with 16-QAM GCSs has a higher code rate than that with binary or quaternary standard GCSs, given the same PMEPR constraint. In 2006, Lee and Golomb [13] proposed a construction of 64-QAM GCSs with the weighted-sum of three standard GCSs over QPSK and first-order offsets in five cases. Further improvements of constructions of GCSs over 16-QAM and 64-QAM were given by Li et al. [14] and Chang et al. [6] later on. These results were extended to the general construction GCSs over 4^q-QAM by Li et al. [15] in 2010 and Liu et al. [17] in 2013, respectively. All these GCSs over QAM are constructed based on standard QPSK GCSs and compatible offsets. Depending on the algebraic structure of the compatible offsets, the GCSs proposed in [15] and [17] are referred to as the generalized cases I-III and the generalized cases IV-V, respectively.

In 2018, Budišin and Spasojević [2] introduced a new recursive algorithm in multiplicative form to generate GCPs over QAM by para-unitary (PU) matrices, where any element of a sequence can be generated by indexing the entries of unitary matrices with the binary representation of the discrete time index. Sequences derived from M unitary matrix over QAM constellation are referred to as the M-Qum case. It is shown that the 1-Qum case and 2-Qum case can generate the sequences in the generalized cases I-III [15] and cases IV-V [17], respectively. Moreover, a large number of new GCSs over QAM are produced from the M-Qum case when M ≥ 2. For instance, the numerical results show that the overall increase in the total number of GCSs including 1-Qum and 2-Qum cases of length 1024 is up to 59%, 242%, and 340%, for 64-, 256-, and 1024-QAM, respectively. However, for given q and sequence length 2^m, the acceptable unitary matrices over QAM can be obtained only by exhaustive search, and the lack of explicit algebraic expression of GCSs leads to unexpected duplication for M-Qum case when M ≥ 2.

In this paper, we propose two new general constructions of GCSs over 4^q-QAM of length 2^m by explicit GBFs based on standard GCSs over QPSK and compatible offsets. The known generalized cases I-V in [15][17] are special cases of our new constructions. Moreover, if q is a composite number, the proposed constructions significantly increase the number of the GCSs given in [15][17]. An example
for \( q = 4 \) shows that the number of new GCSs from our constructions is seven times more than that in generalized cases IV-V [17]. Another example for \( q = 6 \) shows that the ratio of the number of new GCSs and the generalized cases IV-V is greater than six and will increase in proportion with \( m \). Note that the number of GCSs in generalized cases IV-V is much larger than that in cases I-III when \( m \) is large.

Although the GCSs over QAM proposed in our construction are represented by explicit algebraic expression of weighted-sum of standard GCSs over QPSK, the ideas here are inspired by the para-unitary algorithm [2] and the Golay array pairs (GAPs) [10]. GAPs and their relationship with GCPs over PSK by a three-stage construction process were introduced by F. Fiedler et al. [10] in 2008. We extend this idea from PSK modulation to QAM modulation, and propose a mapping from a GAP of size \( 2 \times 2 \times \cdots \times 2 \) to a large number of GCPs over QAM of length \( 2^m \) instead of previous three-stage construction process. We also make a connection of the construction of the GAPs and the specified PU matrices with multi-variables over \( 4^q \)-QAM. Finally, by the technique to derive GBFs from PU matrices introduced in our recent work [24], two new general constructions of GCSs over \( 4^q \)-QAM are obtained.

For the general cases I-III [15], general cases IV-V [17], and the constructions in this paper, the GCSs over \( 4^q \)-QAM of length \( 2^m \) are all expressed by the standard GCSs over QPSK and compatible offsets. If \( q \) is a prime, the proposed constructions are identical to the general case I-V. However, if \( q \) is a composite number, the proposed constructions comprise of not only the general case I-V, but also a great number of new GCSs. The numbers of the GCSs proposed in the general cases I-V and this paper are all equal to the product of the number of the standard GCSs over QPSK (which is determined) and the number of the compatible offsets. It has been shown that the numbers of the compatible offsets in general cases I-III and cases IV-V are linear polynomials of \( m \) and quadratic polynomial of \( m \), respectively. We show that the numbers of the new compatible offsets in this paper is seven times more than that in general cases IV-V for \( q = 4 \), and is at least a cubic polynomial of \( m \) for \( q = 6 \).

On the other hand, from the proof of the proposed constructions, the GCSs proposed in this paper give the explicit algebraic expressions of GCSs from the PU construction in [2] for special \( M \)-Qum cases when \( M \geq 1 \). Moreover, our proof implies all the mentioned GCSs over QAM in this paper can be regarded as projections of Golay complementary arrays of size \( 2 \times 2 \times \cdots \times 2 \), so the results in this paper provide a partial solution to an open problem from [10] for GAPs of size \( 2 \times 2 \times \cdots \times 2 \):

**How can the three-stage construction process be used to simplify or extend known results on the construction of Golay sequences in QAM modulation?**

The rest of this paper is organized as follows. In the next section, we introduce the definitions of GCP and GCS, and the known constructions of the GCPs over QAM. In Section 3, we present two new general constructions of \( 4^q \)-QAM GCSs including the generalized cases I-V as special cases in Theorem
1 and 2. Enumerations of the new GCSs other than the generalized cases I-V for \( q = 4 \) and \( q = 6 \) are given in Section 4. The proofs of our main results are presented in the following two sections. In Section 5, we explain our main theory on GAPs, PU matrices and corresponding GBFs. What’s more, we propose the array form of our result in Theorem 5 and 6, namely \( 4^q \)-QAM GAPs based on PU matrices, from which the main result on GCSs can be derived directly. The proofs of Theorem 5 and 6 are shown in Section 6. We conclude the paper in Section 7.

2 Preliminary

The following notations will be used throughout the paper.

- \( q, p, m, L \) are all positive integers, where \( 0 \leq p < q \).
- For any positive integer \( N \), \( \mathbb{Z}_N = \{0, 1, \ldots, N-1\} \) is the residue class ring modulo \( N \).
- For positive integers \( n \) and \( N \), define \( n \cdot \mathbb{Z}_N \) as a set \( \{0, n, \ldots, n(N-1)\} \) and \( n \cdot \mathbb{Z}_1 = \{0\} \).
- \( \mathbb{F}_2 \) is the finite field with two elements, and \( \mathbb{F}_2^n \) is m-dimension vector space over \( \mathbb{F}_2 \).
- \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{F}_2^m \), where each \( x_i \) is a indeterminate for \( 1 \leq i \leq m \).
- \( \mathbb{C} \) is the complex field, and \( \mathbb{R} \) is the real number field.
- For any \( \alpha \in \mathbb{C} \), \( \overline{\alpha} \) is the conjugation of \( \alpha \).
- \( \pi \) is a permutation of symbols \( \{1, 2, \ldots, m\} \), and \( \sigma \) is a permutation of symbols \( \{0, 1, \ldots, m\} \).

2.1 Golay Complementary Pair

A complex-valued sequence of length \( L \) can be expressed by a function \( F(y) : \mathbb{Z}_L \to \mathbb{C} \), i.e.,

\[
F(y) = (F(0), F(1), \ldots, F(L-1)).
\]

The aperiodic auto-correlation of \( F(y) \) at shift \( \tau \) \( (1 - L \leq \tau \leq L - 1) \) is defined by

\[
C_F(\tau) = \sum_y F(y+\tau) \cdot \overline{F(y)},
\]

where \( F(y+\tau) \cdot \overline{F(y)} = 0 \) if \( F(y+\tau) \) or \( F(y) \) is not defined.

A pair of sequences \( \{F(y), G(y)\} \) of length \( L \) is said to be Golay complementary pair (GCP) if

\[
C_F(\tau) + C_G(\tau) = 0, \quad (\forall \tau \neq 0). \tag{1}
\]

And either sequence in a GCP is called a Golay complementary sequence (GCS) [11].
2.2 GCPs over QPSK

A generalized Boolean function (GBF) $f(x)$ (or $f(x_1, x_2, \cdots, x_m)$) over $\mathbb{Z}_4$ is a function from $\mathbb{F}_2^m$ to $\mathbb{Z}_4$. Such a function can be uniquely expressed as a linear combination over $\mathbb{Z}_4$ of the monomials

$$1, x_1, x_2, \cdots, x_m, x_1x_2, x_1x_3, \cdots, x_{m-1}x_m, \cdots, x_1x_2x_3 \cdots x_m,$$

where the coefficient of each monomial belongs to $\mathbb{Z}_4$.

For $0 \leq y < 2^m$, $y$ can be written uniquely in a binary expansion as $y = \sum_{k=1}^{m} x_k \cdot 2^{k-1}$ where $x_k \in \mathbb{Z}_2$. Then a sequence $F(y)$ of length $L = 2^m$ over QPSK can be associated with a GBF $f(x)$ over $\mathbb{Z}_4$ by

$$F(y) = \xi^{f(x)},$$

where $\xi = \sqrt{-1}$ is a fourth primitive root of unity.

There are several constructions of GCPs over QPSK based on GBFs, such as [7], [16], [8] and [9]. The most typical GCPs are so called standard GCPs given in [7], which are associated with GBFs over $\mathbb{Z}_4$ given below.

**Fact 1 [7]** For GBF

$$f(x) = 2 \cdot \sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k \cdot x_k + c_0,$$

where $c_k \in \mathbb{Z}_4(0 \leq k \leq m)$, and $c' \in \mathbb{Z}_4$, the sequence pair associated with the GBFs over $\mathbb{Z}_4$

$$\begin{cases}
  f(x), \\
  f(x) + 2x_{\pi(1)} + c',
\end{cases} \quad \text{or} \quad \begin{cases}
  f(x), \\
  f(x) + 2x_{\pi(m)} + c',
\end{cases}$$

form a GCP over QPSK.

2.3 GCPs over QAM

In this subsection, we show some of the known results about sequence and GCPs over QAM based on the weighted sums of sequences over QPSK.

A vectorial GBF (V-GBF) is a function from $\mathbb{F}_2^m$ to $\mathbb{Z}_4^q$, denoted by

$$f\left(\mathbf{\hat{x}}\right) = \left(f^{(0)}(x), f^{(1)}(x), \cdots, f^{(q-1)}(x)\right),$$

where each component function $f^{(p)}(x)(0 \leq p < q)$ is a GBF over $\mathbb{Z}_4$.

A sequence over $4^q$-QAM can be viewed as the weighted sums of $q$ sequences over QPSK, with weights in the ratio of $2^{q-1} : 2^{q-2} : \cdots : 1$. Then a sequence over $4^q$-QAM of length $2^m$ can be
associated with a V-GBF \( \tilde{f}(x) = (f^{(0)}(x), f^{(1)}(x), \cdots, f^{(q-1)}(x)) \) over \( \mathbb{Z}_4 \) by

\[
F(y) = \sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(y)},
\]

where \( y = \sum_{k=1}^{m} x_k \cdot 2^{k-1} \) and \( f^{(p)}(y) = f^{(p)}(x)(0 \leq p < q) \). Obviously, the sequence over QPSK can be seen as a special case of QAM sequence when \( q = 1 \).

The GCPs \( \{F(y), G(y)\} \) of length \( 2^m \) over \( 4^q \)-QAM were well studied by their associated V-GBFs \( \{\tilde{f}(x), \tilde{g}(x)\} \) in the literature. Such V-GBFs \( \{\tilde{f}(x), \tilde{g}(x)\} \) are usually given by

- standard GCSs \( f(x) \) in form \( \mathbb{3} \),
- offset V-GBF \( \tilde{s}(x) = (s^{(0)}(x) = 0, s^{(1)}(x), \cdots, s^{(q-1)}(x)) \),
- paring difference V-GBF \( \tilde{\mu}(x) = (\mu^{(0)}(x), \mu^{(1)}(x), \cdots, \mu^{(q-1)}(x)) \),

or more clearly,

\[
\begin{cases}
\tilde{f}(x) = \tilde{1} \cdot f(x) + \tilde{s}(x) \\
\tilde{g}(x) = \tilde{f}(x) + \tilde{\mu}(x)
\end{cases}
\]

where \( \tilde{n} \) denotes the \( q \)-dimension vector \((n, n, \cdots, n)\). The offset V-GBFs \( \tilde{s}(x) \) and paring difference V-GBFs \( \tilde{\mu}(x) \) proposed in the generalised cases I-III given by Li et al. \[15\] and the generalised cases IV-V given by Liu et al. \[17\] are shown below respectively.

**Fact 2 (The generalised cases I-III [12])** \( \{F(y), G(y)\} \) form a \( 4^q \)-QAM GCP of length \( 2^m \) if the offset V-GBF \( \tilde{s}(x) \) and paring difference V-GBF \( \tilde{\mu}(x) \) in their associated V-GBFs \( \{\tilde{f}(x), \tilde{g}(x)\} \) satisfy one of the following cases:

1. **The generalised case I:**

\[
s^{(p)}(x) = d^{(p)}_0 + d^{(p)}_1 x_{\pi(1)}, 1 \leq p \leq q - 1, \tilde{\mu}(x) = \tilde{2} \cdot x_{\pi(m)},
\]

2. **The generalised case II:**

\[
s^{(p)}(x) = d^{(p)}_0 + d^{(p)}_1 x_{\pi(m)}, 1 \leq p \leq q - 1, \tilde{\mu}(x) = \tilde{2} \cdot x_{\pi(1)},
\]

3. **The generalised case III:**

\[
s^{(p)}(x) = d^{(p)}_0 + d^{(p)}_1 x_{\pi(\omega)} + d^{(p)}_2 x_{\pi(\omega+1)}, 1 \leq p \leq q - 1,
\]

with \( \tilde{\mu}(x) = \tilde{2} \cdot x_{\pi(1)} \) or \( \tilde{2} \cdot x_{\pi(m)} \), \( 1 \leq \omega \leq m - 1 \), and \( 2d^{(p)}_0 + d^{(p)}_1 + d^{(p)}_2 = 0 \).
**Definition 1** ([17]) A complex number is called a Gaussian integer if its real and imaginary part are both integers. Define

\[ Q(b_1, b_2, \ldots, b_{q-1}) = 2^{q-1} + \sum_{p=1}^{q-1} 2^{q-1-p} \xi^{b_p}, \quad b_p \in \mathbb{Z}_4. \]  

(6)

Then \( Q \) a one-to-one mapping from \( \mathbb{Z}_4^{q-1} \) to \( \mathbb{Q}_q \), which is a set consisting of \( 4^{q-1} \) Gaussian integers.

For \( Q_0 = Q(b_1, b_2, \ldots, b_{q-1}) \in \mathbb{Q}_q \) and \( Q_1 = Q(b'_1, b'_2, \ldots, b'_{q-1}) \in \mathbb{Q}_q \), a pair of distinct Gaussian integers with identical magnitude, and which are not conjugate with each other, namely:

\[ |Q_0| = |Q_1|, \quad Q_0 \neq Q_1, \quad \text{and} \quad Q_0 \neq \overline{Q_1}, \]  

(7)

\( (Q_0, Q_1) \) is called a non-symmetrical Gaussian integer pair (NSGIP).

**Fact 3** (The generalized cases IV-V [17]) Given a non-symmetrical Gaussian integer pair \( (Q_0, Q_1) \), \( \{F(y), G(y)\} \) forms a \( 4^q \)-QAM GCP of length \( 2^m \) if the offset V-GBFs \( \vec{s}(x) \) and paring difference V-GBFs \( \vec{\mu}(x) \) in their associated V-GBFs \( \{\vec{f}(x), \vec{g}(x)\} \) satisfy one of the following cases:

(4) The generalized case IV:

\[ s^{(p)}(x) = b_p + (b'_p - b_p)x_{\pi(\omega)}, \quad 1 \leq p \leq q - 1, \]

with \( \vec{\mu}(x) = 2 \cdot x_{\pi(1)} \) or \( 2 \cdot x_{\pi(m)} \), \( 2 \leq \omega \leq m - 1. \)

(5) The generalized case V:

\[ s^{(p)}(x) = b_p + (b'_p - b_p)x_{\pi(\omega)} + (-b'_p - b_p)x_{\pi(\nu)}, \quad 1 \leq p \leq q - 1, \]

with \( \vec{\mu}(x) = 2 \cdot x_{\pi(1)} \) or \( 2 \cdot x_{\pi(m)} \), \( 1 \leq \omega \leq m - 2, \quad \omega + 2 \leq \nu \leq m. \)

### 3 Main Results

In this section, we will propose two new constructions of GCPs over \( 4^q \)-QAM. The generalized cases I-III [15] and generalized cases IV-V [17] are special cases of our first and second constructions, respectively. Different from the offset V-GBFs \( \vec{s}(x) \) and paring difference V-GBFs \( \vec{\mu}(x) \) shown in Fact 2 and 3, the new proposed \( \vec{s}(x) \) and \( \vec{\mu}(x) \) relate to the factorization of the integer \( q \).

**Definition 2** Let \( q = q_0 \cdot q_1 \cdots q_m \) be a factorization of \( q \), where \( q_k (0 \leq k \leq m) \) are positive integers. For \( 1 \leq k \leq m \), define \( T_k \) as sets of integers by

\[ T_k = (q_0 \cdot q_1 \cdots q_{k-1}) \cdot \mathbb{Z}_{q_k} = \left( \prod_{i=0}^{k-1} q_i \right) \cdot \mathbb{Z}_{q_k}. \]  

(8)
For the case $k = 0$, the above definition is extended to $T_0 = \mathbb{Z}_{q_0}$. For any given permutation $\sigma$ of symbols $\{0, 1, \cdots, m\}$, we can define mappings $\rho_k : \mathbb{Z}_q \to T_{\sigma(k)}$ for $0 \leq k \leq m$, such that $p = \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p)$.

The rationale of the definition of $\rho_k : \mathbb{Z}_q \to T_{\sigma(k)}$ is guaranteed by the following remark.

**Remark 1** Any $p \in \mathbb{Z}_q$ can be uniquely decomposed as $p = p_0 + p_1 + \cdots + p_m$, where $p_k \in T_k$. In fact, $p_k$ ($0 \leq k \leq m$) can be uniquely determined by the following recursive formula:

\[
\begin{align*}
\rho_0 & \equiv p \pmod{q_0}, \\
\rho_k & \equiv p - \sum_{i=0}^{k-1} p_i \pmod{\prod_{i=0}^{k} q_i}, 1 \leq k \leq m.
\end{align*}
\]

It is easy to verify $\rho_k(p) = p_{\sigma(k)}$.

**Example 1** For $q = 6$ and the factorization $q = q_0 \cdot q_1 \cdots q_m = 3 \times 2 \times \cdots \times 1$, we have

| $k$ | $q_k$ | $\prod_{i=0}^{k-1} q_i \cdot \mathbb{Z}_{q_k}$ | $T_k$ |
|-----|-------|------------------------------------------|-------|
| 0   | 3     | $\mathbb{Z}_3$                          | $\{0, 1, 2\}$ |
| 1   | 2     | $3 \cdot \mathbb{Z}_2$                  | $\{0, 3\}$ |
| 2   | 1     | $3 \cdot \mathbb{Z}_1$                  | $\{0\}$ |
| :   | :     | :                                        | :     |
| $m$ | 1     | $3 \cdot \mathbb{Z}_1$                  | $\{0\}$ |

Suppose that the permutation $\sigma$ satisfies $\sigma(m) = 0$ and $\sigma(\omega) = 1$. For $p \in \mathbb{Z}_6$, the decomposition of $p = \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p)$ ($\rho_k(p) \in T_{\sigma(k)}$) is given below:

| $k$ | $T_{\sigma(k)}$ | $\rho_k(p)$ | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|-----|-----------------|-------------|--------|--------|--------|--------|--------|--------|
| $k \neq \omega, m$ | $\{0\}$ | $\rho_k(p)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\omega$ | $\{0, 3\}$ | $\rho_\omega(p)$ | 0 | 0 | 0 | 3 | 3 | 3 |
| $m$ | $\{0, 1, 2\}$ | $\rho_m(p)$ | 0 | 1 | 2 | 0 | 1 | 2 |

**Definition 3** For $0 \leq p \leq q - 1$, define $d_0^{(p)}, d_1^{(p)}, d_2^{(p)} \in \mathbb{Z}_4$ such that $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$. In particular, we always define $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$.

For any $1 \leq p \leq q - 1$, there are 16 possible values of $(d_0^{(p)}, d_1^{(p)}, d_2^{(p)})$, which are given as follows

$(0, 0, 0), (0, 1, 3), (0, 2, 2), (0, 3, 1), (1, 0, 2), (1, 1, 1), (1, 2, 0), (1, 3, 3), (2, 0, 0), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 2), (3, 1, 1), (3, 2, 0), (3, 3, 3)$.
3.1 The First Construction

Our first construction includes the results in the generalized cases I-III in [15] as a special case.

**Theorem 1** For any factorization \( q = q_0 \cdot q_1 \cdots q_m \) and permutation \( \sigma \), let \( \rho_k : \mathbb{Z}_q \rightarrow \mathcal{T}_{\sigma(k)} \) \((1 \leq k \leq m)\) be mappings given in Definition 2, and \( d_0^{(p)}, d_1^{(p)}, d_2^{(p)} \) \((0 \leq p \leq q-1)\) elements over \( \mathbb{Z}_4 \) given in Definition 3. Denote the vectors \( \vec{d}_1 \) and \( \vec{d}_2 \) respectively by

\[
\vec{d}_1 = (d_1^{(\rho_0(0))}, d_1^{(\rho_0(1))}, \ldots, d_1^{(\rho_0(q-1))}) \quad \text{and} \quad \vec{d}_2 = (d_2^{(\rho_m(0))}, d_2^{(\rho_m(1))}, \ldots, d_2^{(\rho_m(q-1))}).
\]

Sequences over \( 4^n \)-QAM of length \( 2^m \) associated with V-GBFs in (3) form a GCP if the offset V-GBFs \( \vec{s}(x) \) and paring difference V-GBFs \( \vec{\mu}(x) \) satisfy the following conditions:

\[
\begin{align*}
\{s^{(p)}(x) & = \sum_{k=1}^{m} \left(d_1^{(\rho_k(0))} + d_2^{(\rho_k-1)(p))}\right)x_{\pi(k)} + \sum_{k=0}^{m} d_0^{(\rho_k(p))}, \\
\vec{\mu}(x) & = 2 \cdot x_{\pi(1)} + \vec{d}_1 \quad \text{or} \quad 2 \cdot x_{\pi(m)} + \vec{d}_2.
\end{align*}
\]

The detailed proof of Theorem 1 will be given in Sections 5 and 6. We first explain the sequences in Theorem 1 by the following examples.

Let the factorization of \( q \) be trivial, i.e., \( q = q_0 \cdot q_1 \cdots q_m \) such that \( \omega_k = q \) for \( k = \omega \) and \( \pi_k = 1 \) for \( k \neq \omega \). Let \( \sigma \) be the identity permutation. We have

\[
\mathcal{T}_k = \begin{cases} 
\mathbb{Z}_q, & \text{if } k = \omega; \\
\{0\}, & \text{otherwise,}
\end{cases} \quad \text{and} \quad \rho_k(p) = \begin{cases} 
p, & \text{if } k = \omega; \\
0, & \text{otherwise.}
\end{cases}
\]

From Definition 2 we obtain \( (d_0^{(\rho_k(p))}, d_1^{(\rho_k(p))}, d_2^{(\rho_k(p))}) \) equals \( (d_0^{(p)}, d_1^{(p)}, d_2^{(p)}) \) if \( k = \omega \), and equals \((0, 0, 0)\) otherwise. Then the offset and paring difference set V-GBFs for different \( \omega \) can be obtained immediately by Theorem 1.

1. If \( \omega = 0 \), i.e., \( q = q_0 \cdot q_1 \cdots q_m = q \times 1 \times \cdots \times 1 \), then \( \vec{d}_1 = (d_1^{(0)}, d_1^{(1)}, \ldots, d_1^{(q-1)}) \) and \( \vec{d}_2 = \vec{0} \). We have

\[
\begin{align*}
\{s^{(p)}(x) & = d_1^{(p)} + d_2^{(p)}x_{\pi(1)}, \\
\vec{\mu}(x) & = 2 \cdot x_{\pi(m)}
\end{align*}
\]

or

\[
\begin{align*}
\{s^{(p)}(x) & = d_1^{(p)} + d_2^{(p)}x_{\pi(1)}, \\
\vec{\mu}(x) & = 2 \cdot x_{\pi(1)} + \vec{d}_1.
\end{align*}
\]
(2) If $\omega = m$, i.e., $q = q_0 \cdot q_1 \cdots q_m = 1 \times 1 \times \cdots \times q$, then $d_2 = (d_2^{(0)}, d_2^{(1)}, \cdots, d_2^{(q-1)})$ and $d_1 = 0$. We have

$$
\begin{align*}
\{ s^{(p)}(x) &= d_0^{(p)} + d_1^{(p)} x_{\pi(m)}, \\
\tilde{\mu}(x) &= \tilde{2} \cdot x_{\pi(1)}; \\
\}
\end{align*}
$$

or

$$
\begin{align*}
\{ s^{(p)}(x) &= d_0^{(p)} + d_1^{(p)} x_{\pi(m)}, \\
\tilde{\mu}(x) &= \tilde{2} \cdot x_{\pi(m)} + d_2. \\
\}
\end{align*}
$$

(3) If $1 \leq \omega \leq m - 1$, i.e., $q = q_0 \cdot q_1 \cdots q_m = 1 \times \cdots q \cdots \times 1$, then $d_1 = d_2 = 0$. We have

$$
\begin{align*}
\{ s^{(p)}(x) &= d_0^{(p)} + d_1^{(p)} x_{\pi(\omega)} + d_2^{(p)} x_{\pi(\omega + 1)}, \\
\tilde{\mu}(x) &= \tilde{2} \cdot x_{\pi(1)} \quad \text{or} \quad \tilde{\mu}(x) = \tilde{2} \cdot x_{\pi(m)}. \\
\end{align*}
$$

It is obvious that the offset V-GBFs $\tilde{s}(x)$ and paring difference V-GBFs $\tilde{\mu}(x)$ shown in (11), (13), (15) agree with the generalized cases I-III in [15]. New paring difference V-GBFs shown in (12), (14) lead to new GCPs over 4$q$-QAM, but they do not produce new GCSs.

Moreover, new GCPs and GCSs over QAM can be constructed from Theorem II if $q$ is a composite number and $q = q_0 \cdot q_1 \cdots q_m$ is a non-trivial factorization. Two examples are given below to illustrate it.

**Example 2** For $q = 4$, suppose that the factorization $q = q_0 \cdot q_1 \cdots q_m = 2 \times 1 \times \cdots \times 2$ and $\sigma$ is the identity permutation. Then we have

| $k$ | $q_k$ | $\prod_{i=0}^{k-1} q_i = \mathbb{Z}_{q_k}$ | $T_{\sigma(k)} = T_k$ |
|-----|------|---------------------------------|----------------|
| 0   | 2    | $\mathbb{Z}_2$                  | $\{0, 1\}$    |
| 1   | 1    | $2 \cdot \mathbb{Z}_1$         | $\{0\}$       |
|     |      |                                 |                |
| $m$ | 2    | $2 \cdot \mathbb{Z}_2$         | $\{0, 2\}$    |

From Theorem [1] the 256-QAM sequence pair $\{F(y), G(y)\}$ associated with the V-GBFs

$$
\begin{align*}
\hat{f}(x) &= \hat{1} \cdot f(x) + \left( s^{(0)}(x), s^{(1)}(x), s^{(2)}(x), s^{(3)}(x) \right), \\
\hat{g}(x) &= \hat{f}(x) + \hat{2} \cdot x_{\pi(1)} + \hat{d}_1 \quad \text{or} \quad \hat{f}(x) + \hat{2} \cdot x_{\pi(m)} + \hat{d}_2
\end{align*}
$$
form a GCP of length \( L = 2^m \), where \( f(x) \) is given in (5). Considering \( \rho_k(p) \equiv 0 \) for \( k \neq 0, m \), the offset \( s^{(p)}(x) \) and vectors \( \vec{d}_1, \vec{d}_2 \) are given in the table respectively:

| \( p \) | \( \rho_0(p) \) | \( \rho_3(p) \) | \( d^{(\rho_0(p))}_1 \) | \( d^{(\rho_0(p))}_2 \) | \( \text{offset} : s^{(p)}(x) \) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | \( d^{(0)}_1 \) | \( d^{(0)}_2 \) | 0 |
| 1 | 1 | 0 | \( d^{(1)}_1 \) | \( d^{(0)}_2 \) | \( d^{(1)}_2 x_{\pi(1)} + d^{(0)}_0 \) |
| 2 | 0 | 2 | \( d^{(0)}_1 \) | \( d^{(2)}_2 \) | \( d^{(2)}_1 x_{\pi(m)} + d^{(2)}_0 \) |
| 3 | 1 | 2 | \( d^{(1)}_2 \) | \( d^{(0)}_2 \) | \( d^{(1)}_2 x_{\pi(1)} + d^{(1)}_1 x_{\pi(m)} + d^{(0)}_1 + d^{(0)}_0 \) |

where \( 2d^{(p)}_0 + d^{(p)}_1 + d^{(p)}_2 = 0 \) for \( 1 \leq p \leq q - 1 \) and \( d^{(0)}_1 = d^{(0)}_2 = d^{(0)}_0 = 0 \).

**Example 3** For \( q = 6 \) and \( 1 \leq \omega \leq m - 1 \), suppose that the factorization is given by \( q = q_0 \cdot q_1 \cdots q_m = 3 \times 2 \times \cdots \times 1 \) and permutation \( \sigma \) satisfies \( \sigma(0) = 0 \) and \( \sigma(\omega) = 1 \). Then \( T_k (k \in \{0, 1, \ldots, m\}) \) and the decomposition of \( p = \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p) \) (\( \rho_k(p) \in T_{\pi(k)} \)) are shown in Example 4. Notice that \( \rho_k(p) \equiv 0 \) for \( k \neq \omega, m \). For \( 0 \leq p \leq 5 \), the offset \( s^{(p)}(x) \) in (11) is simplified to

\[
s^{(p)}(x) = d^{(\rho_0(p))}_1 x_{\pi(\omega)} + d^{(\rho_0(p))}_2 x_{\pi(\omega+1)} + d^{(\rho_0(p))}_3 x_{\pi(m)} + d^{(\rho_0(p))}_4 x_{\pi(m)} = (16),
\]

where \( 2d^{(p)}_0 + d^{(p)}_1 + d^{(p)}_2 = 0 \) for \( 1 \leq p \leq q - 1 \) and \( d^{(0)}_1 = d^{(0)}_2 = d^{(0)}_0 = 0 \), i.e.,

| \( p \) | \( \rho_\omega(p) \) | \( \rho_m(p) \) | \( d^{(\rho_m(p))}_2 \) | \( \text{offset} : s^{(p)}(x) \) |
|---|---|---|---|---|
| 0 | 0 | 0 | \( d^{(0)}_2 \) | 0 |
| 1 | 0 | 1 | \( d^{(1)}_2 \) | \( d^{(1)}_1 x_{\pi(m)} + d^{(1)}_0 \) |
| 2 | 0 | 2 | \( d^{(2)}_2 \) | \( d^{(2)}_1 x_{\pi(m)} + d^{(2)}_0 \) |
| 3 | 3 | 0 | \( d^{(0)}_2 \) | \( d^{(3)}_1 x_{\pi(\omega)} + d^{(3)}_2 x_{\pi(\omega+1)} + d^{(3)}_0 \) |
| 4 | 3 | 1 | \( d^{(1)}_2 \) | \( d^{(3)}_1 x_{\pi(\omega)} + d^{(3)}_2 x_{\pi(\omega+1)} + d^{(3)}_0 + d^{(3)}_1 x_{\pi(m)} + d^{(3)}_0 \) |
| 5 | 3 | 2 | \( d^{(2)}_2 \) | \( d^{(3)}_1 x_{\pi(\omega)} + d^{(3)}_2 x_{\pi(\omega+1)} + d^{(3)}_0 + d^{(3)}_1 x_{\pi(m)} + d^{(2)}_0 \) |

Thus the sequence pair \( \{F(y), G(y)\} \) over \( 4^6\)-QAM associated with the V-GBFs

\[
\begin{align*}
\vec{f}(x) &= \vec{1} \cdot f(x) + (s^{(0)}(x), s^{(1)}(x), \ldots, s^{(5)}(x)), \\
\vec{g}(x) &= \vec{f}(x) + \vec{2} \cdot x_{\pi(1)} \quad \text{or} \quad \vec{f}(x) + \vec{2} x_{\pi(m)} + \vec{d}_2
\end{align*}
\]

form a GCP of length \( L = 2^m \), where \( f(x) \) is given in (5) and vector \( \vec{d}_2 = \left( d^{(\rho_m(0))}_2, \ldots, d^{(\rho_m(5))}_2 \right) \) is given in the table above.

To the best of our knowledge, the expressions of offset V-GBFs shown in the above examples have never been reported before.
3.2 The Second Construction

In this subsection, we slightly modify the conditions in the first construction, and obtain our second construction which include the generalized cases IV-V in [17].

**Definition 4** Let \( q = q' \cdot q_0 \cdot q_1 \cdots q_m \) be a factorization of \( q \), where \( q' \geq 3 \) and \( q_k (0 \leq k \leq m) \) are positive integers. For \( 1 \leq k \leq m \), define \( \mathcal{T}_k' \) as sets of integers by

\[
\mathcal{T}_k' = (q' \cdot q_0 \cdot q_1 \cdots q_{k-1}) \cdot \mathbb{Z}_{q_k} = \left( q' \cdot \prod_{i=0}^{k-1} q_i \right) \cdot \mathbb{Z}_{q_k}.
\]

(17)

For the case \( k = 0 \), we extend the definition to \( \mathcal{T}_0' = q' \cdot \mathbb{Z}_{q_0} \). For any given permutation \( \sigma \) of symbols \( \{0, 1, \cdots, m\} \), define mappings \( \rho_k' : Q_q \rightarrow \mathcal{T}_{\sigma(k)}(0 \leq k \leq m) \) and \( \rho' : Q_q \rightarrow Q_{q'} \), such that \( p = \rho'(p) = \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p) \).

Similar to the mappings \( \rho_k \) in the first construction, \( p \in Q_q \) can be uniquely decomposed as \( p = p' + \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p) \) such that \( p' \in Q_{q'} \) and \( \rho_k' \in \mathcal{T}_{\sigma(k)}' \) here, so the above definition is reasonable.

**Theorem 2** For any factorization \( q = q' \cdot q_0 \cdot q_1 \cdots q_m \) and permutation \( \sigma \), let \( \rho_k' \) and \( \rho' \) be mappings given in Definition 1 and \( d^{(0)}_0, d^{(1)}_1, d^{(2)}_2 (0 \leq p \leq q-1) \) elements over \( \mathbb{Z}_4 \) given in Definition 3. Suppose that \( G_0 = G(b_1, b_2, \ldots, b_{q-1}) \) and \( G_1 = G(b'_1, b'_2, \ldots, b'_{q'-1}) \) are NSGIP introduced in Definition 4.

Denote the vectors \( \vec{d}_1 \) and \( \vec{d}_2 \) here respectively by

\[
\vec{d}_1 = \left( d^{(0)}_1, d^{(1)}_1, \ldots, d^{(q-1)}_1 \right) \quad \text{and} \quad \vec{d}_2 = \left( d^{(0)}_2, d^{(1)}_2, \ldots, d^{(q-1)}_2 \right). \quad (18)
\]

Sequences over \( 4^q \)-QAM of length \( 2^m \) associated with V-GBFs in \( \vec{s}(x) \) form a GCP if the offset V-GBFs \( \vec{s}(x) \) satisfy

Case (a): for \( 2 \leq \omega \leq m - 1 \),

\[
s^{(p)}(x) = \sum_{k=1}^{m} \left( d^{(0)}_1(b'_k(p)) + d^{(1)}_1(b'_{k-1}(p)) \right) x^{(k)} + \sum_{k=0}^{m} d^{(0)}_0(b'_k(p)) + \left( (b'_p(p) - b'_k(p)) x^{(\omega)} + b'_p(p) \right),
\]

(19)

Case (b): for \( 1 \leq \omega \leq m - 2, \omega + 2 \leq v \leq m \),

\[
s^{(p)}(x) = \sum_{k=1}^{m} \left( d^{(0)}_1(b'_k(p)) + d^{(1)}_1(b'_{k-1}(p)) \right) x^{(k)} + \sum_{k=0}^{m} d^{(0)}_0(b'_k(p))
\]

\[
+ \left( (b'_p(p) - b'_k(p)) x^{(\omega)} + (b'_p(p) - b'_k(p)) x^{(v)} + b'_p(p) \right),
\]

(20)

and paring difference V-GBFs are given by

\[
\vec{\mu}(x) = \vec{2} \cdot x^{(1)} + \vec{d}_1 \quad \text{or} \quad \vec{\mu}(x) = \vec{2} \cdot x^{(m)} + \vec{d}_2.
\]

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We explain Theorem 2 by the following examples and a proof in details will be offered in Sections 5 and 6.

Let the factorization $q = q' \cdot q_0 \cdot q_1 \cdots q_m$ be restricted as $q' = q$ and $q_k = 1$ for $0 \leq k \leq m$, and $\sigma$ be the identity permutation. We have $\rho'(p) = p$ and $\rho'_k(p) = 0$. Notice $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$, we have $d_0^{(\rho_k(p))} = d_1^{(\rho_k(p))} = d_2^{(\rho_k(p))} = 0$ and $\vec{d}_1 = \vec{d}_2 = \vec{0}$. Then the offset V-GBFs in formula (19) and (20), can be simplified to

$$s^{(p)}(\vec{x}) = b_p + (b'_p - b_p)x_{\pi(\omega)}$$

and

$$s^{(p)}(\vec{x}) = b_p + (b'_p - b_p)x_{\pi(\omega)} + (-b'_p - b_p)x_{\pi(\nu)},$$

respectively, which coincide with the generalized cases IV-V in [17].

Moreover, New GCPs and GCSs over QAM can be derived from Theorem 2 if $q \neq q'$. We give an example of Case (b) to illustrate it. To the best of our knowledge, the expressions of offset V-GBFs shown below have never been reported before.

**Example 4** For $0 < \kappa < m$, let $q = q' \cdot q_0 \cdot q_1 \cdots q_m$ be $q' = 3$, $q_2 = 2$ and $q_k = 1$ for $0 \leq k \leq m$, and $\sigma$ be the identity permutation. Then we have $Z_{q'} = \{0, 1, 2\},$ and

| $k$ | $q_k$ | $q' \prod_{i=0}^{k-1} q_i \cdot Z_{q_k}$ | $T'_{\sigma(k)} = T_k$ |
|----|------|----------------------------------|---------------------|
| $1 \leq k < \kappa$ | 1 | $3 \cdot Z_1$ | $\{0\}$ |
| $k = \kappa$ | 2 | $3 \cdot Z_2$ | $\{0, 3\}$ |
| $\kappa < k \leq m$ | 1 | $3 \cdot 2 \cdot Z_1$ | $\{0\}$ |

Considering $\rho'_k(p) = 0$ for $k \neq \kappa$, we have $\vec{d}_1 = \vec{d}_2 = \vec{0}$. In Case (b) of Theorem 2 for $1 \leq \omega \leq m - 2$, $\omega + 2 \leq \nu \leq m$, offset $s^{(p)}(\vec{x}) = f^{(p)}(\vec{x}) - f(\vec{x})$ are given as follows,

| $p$ | $\rho'_1(p)$ | $\rho'_2(p)$ | offset : $s^{(p)}(\vec{x})$ |
|----|-------------|-------------|-----------------|
| 0  | 0           | 0           | 0               |
| 1  | 0           | 1           | $(b'_1 - b_1)x_{\pi(\omega)} + (-b'_1 - b_1)x_{\pi(\nu)} + b_1$ |
| 2  | 0           | 2           | $(b'_2 - b_2)x_{\pi(\omega)} + (-b'_2 - b_2)x_{\pi(\nu)} + b_2$ |
| 3  | 3           | 0           | $d_1^{(3)} x_{\pi(\kappa)} + d_2^{(3)} x_{\pi(\kappa + 1)} + d_0^{(3)}$ |
| 4  | 3           | 1           | $d_1^{(3)} x_{\pi(\kappa)} + d_2^{(3)} x_{\pi(\kappa + 1)} + (b'_1 - b_1)x_{\pi(\omega)} + (-b'_1 - b_1)x_{\pi(\nu)} + d_0^{(3)} + b_1$ |
| 5  | 3           | 2           | $d_1^{(3)} x_{\pi(\kappa)} + d_2^{(3)} x_{\pi(\kappa + 1)} + (b'_2 - b_2)x_{\pi(\omega)} + (-b'_2 - b_2)x_{\pi(\nu)} + d_0^{(3)} + b_2$ |

where $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$ for $1 \leq p \leq q - 1$, and $G(b_1, b_2)$, $G(b'_1, b'_2)$ forms an NSGIP. Thus the
\[\{F(y), G(y)\}\] associated with the V-GBFs

\[
\begin{align*}
\vec{f}(x) &= \vec{1} \cdot f(x) + \left(s^{(0)}(x), s^{(1)}(x), \ldots, s^{(5)}(x)\right), \\
\vec{g}(x) &= \vec{f}(x) + \vec{2} \cdot x_{\pi(1)} \quad \text{or} \quad \vec{f}(x) + \vec{2} x_{\pi(m)},
\end{align*}
\]

form a GCP of length \(L = 2^m\), where \(f(x)\) is given in (4).

### 4 Enumeration

The number of the GCSs over \(4^q\)-QAM of length \(2^m\) constructed in Theorem 1 and 2 is equal to the product of the number of the standard GCSs \(f(x)\) over QPSK and the number of the compatible offset \(\vec{s}(x)\), i.e.,

\[\#\{\vec{s}(x)\} \times \#\{f(x)\}.\]

It is well known that the number of the standard GCSs over QPSK is given by \(\#\{f(x)\} = (m!)/2\)\(4^{(m+1)}\).

So the enumeration of the GCSs is determined by the number of the compatible offsets.

It was shown in [15] that the number of the compatible offsets in the generalized cases I-III, denoted as \(N_{m,q}^{123}\), is

\[N_{m,q}^{123} = (m + 1)4^{2(q-1)} - (m + 1)4^{(q-1)} + 2^{(q-1)}, \quad m \geq 2.\]

The enumeration of the compatible offsets in the generalized cases IV-V, denoted as \(N_{m,q}^{45}\), was given in [17] for \(q \geq 3, m \geq 3\), especially

\[N_{m,4}^{45} = 14 \cdot (m - 2)(m + 1) \quad \text{and} \quad N_{m,6}^{45} = 584 \cdot (m - 2)(m + 1).\]

If \(q\) is a prime, the constructions in this paper are identical to the generalized cases I-V. If \(q\) is a composite number, new GCSs over QAM are produced. Notice that the offset \(\vec{s}(x)\) in the generalized cases I-V has no more than two Boolean variables \(x_k\) with non-zero coefficients. By studying the offset \(\vec{s}(x)\) with three and four Boolean variables with non-zero coefficients, lower bounds of the numbers of new GCSs other than the generalized cases I-V for \(q = 4\) and \(q = 6\) are obtained in this section, as shown in Table 1. The comparisons between the numbers of the compatible offsets in generalized case I-III, case IV-V and new in this paper for \(q = 4\) and \(q = 6\) are also given in Table 1.

Before listing the new compatible offsets, recall the values of \(\vec{d}(p) = (d_0^{(p)}, d_1^{(p)}, d_2^{(p)})\) in Definition 3. According to the values of \(d_1^{(p)}\) and \(d_2^{(p)}\), all the 16 possible values of \(\vec{d}(p)\) can be classified into four
Table 1: Comparisons of the number of the compatible offsets

|                              | q = 4                                      | q = 6                                      |
|------------------------------|--------------------------------------------|--------------------------------------------|
| The generalized cases I-III  | $4032m + 4040$                            | $1047552m + 1047584$                      |
| The generalized case IV-V    | $14(m^2 - m - 2)$                          | $584(m^2 - m - 2)$                        |
| New in this paper            | $\geq 100(m^2 - m - 2)$                    | $\geq (3700 + 20m)(m^2 - m - 2)$          |

classes:

$C_1 = \{(1, 1, 1), (3, 1, 1), (0, 1, 3), (2, 1, 3), (0, 2, 2), (2, 2, 2), (0, 3, 1), (2, 3, 1), (1, 3, 3), (3, 3, 3)\}$,

$C_2 = \{(1, 0, 2), (3, 0, 2)\}$,

$C_3 = \{(1, 2, 0), (3, 2, 0)\}$,

$C_4 = \{(0, 0, 0), (2, 0, 0)\}$.

4.1 Enumeration for $q = 4$

Since the number of the compatible offsets is independent of the per mutation $\pi$, without loss of generality, we restrict $\pi$ to be the identity permutation in this and the next subsections. In addition, we define $x_0$ and $x_{m+1}$ as ‘fake’ variables with fixed value 0 for convenience.

For the factorization $q = 4 = 2 \times 2$, we have $\mathbb{Z}_4 = \mathbb{Z}_2 + 2\mathbb{Z}_2$ and the offset $s(x)$ in Theorem 1 can be expressed by

$$s^{(p)}(x) = \left(d^{\rho_w(p)}_1 x_\omega + d^{\rho_w(p)}_2 x_{\omega+1} + d^{\rho_w(p)}_0 \right) + \left(d^{\rho_v(p)}_1 x_v + d^{\rho_v(p)}_2 x_{v+1} + d^{\rho_v(p)}_0 \right), \ (0 \leq p < q),$$

where the values of $p = \rho_w(p) + \rho_v(p) \ (0 \leq \omega \neq v \leq m)$ are given below.

| $p$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $\rho_w(p)$ | 0 | 0 | 0 | 1 |
| $\rho_v(p)$ | 0 | 0 | 2 | 2 |

**Proposition 1** For $m \geq 3$, $\bar{d}^{(1)}$, $\bar{d}^{(2)} \in C_1$, and the ordered pairs $(\omega, v)$ satisfying $(\omega, v) \neq (0, m), (m, 0)$ and $|\omega - v| \geq 2$, different choices of $\left(\bar{d}^{(1)}, \bar{d}^{(2)}, (\omega, v)\right)$ determines different offsets in the form of (21) with at least three Boolean variables with non-zero coefficients.

**Proof** The conditions $(\omega, v) \neq (0, m), (m, 0)$ and $|\omega - v| \geq 2$ guarantee that there are at least three ‘real’ Boolean variables in $x_\omega, x_{\omega+1}, x_v, x_{v+1}$. Then above proposition can be verified immediately from the observation of the Table 2 since none of $d^{(1)}_1, d^{(1)}_2, d^{(2)}_1, d^{(2)}_2$ equals 0. □
Table 2: Coefficients of offsets for $q = 4$

| $p$ | $s_1^0(x)$ | $s_2^0(x)$ | $s_3^0(x)$ | $s_4^0(x)$ |
|-----|-------------|-------------|-------------|-------------|
| 0   | 0           | 0           | 0           | 0           |
| 1   | $d_1^{(1)}$ | $d_2^{(1)}$ | 0           | 0           |
| 2   | 0           | 0           | $d_1^{(2)}$ | $d_2^{(2)}$ |
| 3   | $d_1^{(1)}$ | $d_2^{(1)}$ | $d_1^{(2)}$ | $d_2^{(2)}$ |

**Proposition 2** For $q = 4$ and $m \geq 3$, Proposition 1 identifies $100(m + 1)(m - 2)$ distinct compatible offsets other than the generalized cases I-V.

**Proof** If $\omega = 0$, we can select $\upsilon$ such that $2 \leq \upsilon \leq m - 1$. If $\omega = m$, we can choose $\upsilon$ such that $1 \leq \upsilon \leq m - 1, m, m + 1$. So there are totally $(m + 1)(m - 2)$ ordered pairs $(\omega, \upsilon)$. For each ordered pair, there are $10 \times 10 = 100$ choices of $\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)}$ such that $\overrightarrow{d}^{(1)} = \overrightarrow{d}^{(2)} \in \mathcal{C}_1$. Thus, Proposition 1 identifies $100(m + 1)(m - 2)$ distinct compatible offsets with at least three Boolean variables with non-zero coefficients. □

**4.2 Enumeration for $q = 6$**

For the factorization $q = 6 = 3 \cdot 2$, we have $\mathcal{Z}_6 = \mathcal{Z}_3 + 2 \mathcal{Z}_2$. Then the offset $\overrightarrow{s}(x)$ in Theorem 1 can be expressed by

$$s^p(x) = \left(d_1^{(p)}x_\kappa + d_2^{(p)}x_{\kappa+1} + d_0^{(p)}\right) + \left(b^{(p)}x_\omega + b^{(p)}\right), \quad (0 \leq p < q),$$

where $0 \leq \kappa \leq m, 2 \leq \omega \leq m - 1$ for Case (a) and

$$s^p(x) = \left(d_1^{(p)}x_\kappa + d_2^{(p)}x_{\kappa+1} + d_0^{(p)}\right) + \left(b^{(p)} - b^{(p)}\right)x_\omega + b^{(p)}\right), \quad (0 \leq p < q),$$

where $0 \leq \kappa \leq m, 1 \leq \omega \leq m - 2, \omega + 2 \leq \upsilon \leq m$ for Case (b). And the decomposition $p = \rho'(p) + \rho''(p)$ is given as follows.

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| $\rho''(p)$ | 0 | 0 | 0 | 3 | 3 | 3 |
| $\rho'(p)$ | 0 | 1 | 2 | 0 | 1 | 2 |

We first show four cases of offsets for $q = 6$ from Theorem 1 and Theorem 2.

Case (1): For $q = 2 \cdot 3$ and $\mathcal{Z}_6 = \mathcal{Z}_2 + 3 \mathcal{Z}_2$, the offset $\overrightarrow{s}(x)$ in Theorem 1 can be expressed in the
form of (21), where $p = \rho_\omega(p) + \rho_\upsilon(p)$ ($0 \leq \omega \neq \upsilon \leq m$) are given below.

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p$ |   |   |   |   |   |   |
| $\rho_\omega(p)$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $\rho_\upsilon(p)$ | 0 | 0 | 2 | 2 | 4 | 4 |

Let the coefficients $\vec{d}^{(p)}$ ($p = 1, 2, 4$) and the ordered pairs $(\omega, \upsilon)$ satisfy the following conditions:

1. $\vec{d}^{(1)} \in C_1, \vec{d}^{(2)} \notin C_4, (\vec{d}^{(1)}, \vec{d}^{(2)}) \notin (C_2, C_2) \cup (C_3, C_3)$;
2. $(\omega, \upsilon) \neq (0, m), (m, 0)$ and $|\omega - \upsilon| \geq 2$.

Case (2): For $q = 3 \cdot 2$ and $\mathbb{Z}_6 = \mathbb{Z}_3 + 2\mathbb{Z}_2$, the offset $\vec{s}(x)$ in Theorem $\text{I}$ can be expressed in the form of (21), where $p = \rho_\omega(p) + \rho_\upsilon(p)$ ($0 \leq \omega \neq \upsilon \leq m$) are given below.

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p$ |   |   |   |   |   |   |
| $\rho_\omega(p)$ | 0 | 0 | 3 | 3 | 3 |   |
| $\rho_\upsilon(p)$ | 0 | 1 | 2 | 0 | 1 | 2 |

Let the coefficients $\vec{d}^{(p)}$ ($p = 1, 2, 3$) and the ordered pairs $(\omega, \upsilon)$ satisfy the following conditions:

1. $\vec{d}^{(3)} \in C_1, \vec{d}^{(2)} \notin C_4, (\vec{d}^{(1)}, \vec{d}^{(2)}) \notin (C_2, C_2) \cup (C_3, C_3)$;
2. $(\omega, \upsilon) \neq (0, m), (m, 0)$ and $|\omega - \upsilon| \geq 2$.

Case (3): For $q = 3 \cdot 2$ and $\mathbb{Z}_6 = \mathbb{Z}_3 + 2\mathbb{Z}_2$, the offset $\vec{s}(x)$ in Case (a) of Theorem $\text{II}$ can be expressed in the form of (21). Let the coefficients $\vec{d}^{(p)}$ ($p = 1, 2, 3$) and the ordered pairs $(\kappa, \omega)$ satisfy the following conditions:

1. $\vec{d}^{(3)} \in C_1$;
2. $2 \leq \omega \leq m - 1, 1 \leq \kappa \leq m, \text{ and } \kappa \neq \omega, \omega - 1, m$.

Case (4): For $q = 3 \cdot 2$ and $\mathbb{Z}_6 = \mathbb{Z}_3 + 2\mathbb{Z}_2$, the offset $\vec{s}(x)$ in Case (b) of Theorem $\text{II}$ can be expressed in the form of (21). Let the coefficients $\vec{d}^{(p)}$ ($p = 1, 2, 3$) and the ordered triples $(\kappa, \omega, \upsilon)$ satisfy the following conditions:

1. $\vec{d}^{(3)} \in C_1$;
2. $1 \leq \omega \leq m - 2, \omega + 2 \leq \upsilon \leq m, \text{ and } 0 \leq \kappa \leq m, \kappa \neq \omega, \omega - 1, \upsilon, \upsilon - 1$.
Table 3: Coefficients of offsets for $q = 6$

| $p$ | $s^{(p)}(\mathbf{x})$ in Case (1) | $s^{(p)}(\mathbf{x})$ in Case (2) | $s^{(p)}(\mathbf{x})$ in Case (3) and (4) |
|-----|----------------------------------|----------------------------------|----------------------------------|
|     | $x_0$  $x_{\omega}$  $x_{\omega+1}$  $x_0$  $x_0$  $x_{\omega}$  $x_{\omega+1}$  $x_0$  $x_{\omega}$  $x_{\omega+1}$  $x_{\omega}$  $x_{\omega+1}$ | $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$ | $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$  $x_{\omega}$ |
| 0   | 0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0 |
| 1   | $d_1^{(1)}$  $d_2^{(1)}$  0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0 |
| 2   | 0      0      $d_1^{(2)}$  $d_2^{(2)}$  0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0 |
| 3   | $d_1^{(1)}$  $d_2^{(1)}$  $d_1^{(2)}$  $d_2^{(2)}$  $d_1^{(3)}$  $d_2^{(3)}$  0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0      0 |
| 4   | 0      0      $d_1^{(4)}$  $d_2^{(4)}$  $d_1^{(3)}$  $d_2^{(3)}$  $d_1^{(1)}$  $d_2^{(1)}$  0      0      0      0      0      0      0      0      0      0      0      0      0      0 |
| 5   | $d_1^{(1)}$  $d_2^{(1)}$  $d_1^{(4)}$  $d_2^{(4)}$  $d_1^{(3)}$  $d_2^{(3)}$  $d_1^{(2)}$  $d_2^{(2)}$  $d_1^{(1)}$  $d_2^{(1)}$  0      0      0      0      0      0      0      0      0      0      0      0 |

**Proposition 3** For $q = 6$ and $m \geq 3$, the above Case (1)-(4) identifies $(3700 + 20m)(m + 1)(m - 2)$ distinct compatible offsets other than the generalized cases I-V.

**Proof** We verify this enumeration from the observation of the Table 3. Denote the collection of the offsets in Case (i) by $S_i$ for $i = 1, 2, 3, 4$.

First of all, every offset in $S_i$ ($i = 1, 2, 3, 4$) must have at least three Boolean variables with non-zero coefficients, so these offsets must be different from those in the generalized cases I-V.

Secondly, we prove $S_i(i = 1, 2, 3, 4)$ has empty intersection with each other. From the positions of the non-zero coefficients in $s^{(p)}(\mathbf{x})$ in Table 3, it is obviously $S_1 \cap S_2 = \emptyset$. From the positions of the non-zero coefficients of $x_0$ and $x_{\omega+1}$, we have $S_1 \cap (S_3 \cup S_4) = \emptyset$. From the definition of the NSGIP, we have both $(b_1' - b_1, b_2' - b_2) \neq (0, 0)$ and $(-b_1' - b_1, -b_2' - b_2) \neq (0, 0)$. Together with the positions of the non-zero coefficients of $x_0$ and $x_{\omega+1}$, we obtain $S_1 \cap S_3 = \emptyset$. If the offset $s^{(p)}(\mathbf{x})$ belongs to both $S_2$ and $S_3$, we have $v' = \omega$ and $v' + 1 = m + 1$ in Case (2) and (3) in Table 3 which contradicts to $\omega \neq m$ in Case (3). If the offset $s^{(p)}(\mathbf{x})$ belongs to both $S_2$ and $S_4$, we have $v' = \omega$ and $v' + 1 = v$ in Cases (2) and (4) in Table 3 which contradicts to $\omega + 2 \leq v$ in Case (4). Thus we obtain $S_2 \cap (S_3 \cup S_4) = \emptyset$.

Thirdly, it is straightforward that different parameters in each case lead to different offset.

With the same arguments in Proposition 2, we can prove that there are totally $(m + 1)(m - 2)$ ordered pairs $(\omega, v)$ in case (1). For each ordered pair, there are 10 choices of $\overrightarrow{d}^{(1)}$ such that $\overrightarrow{d}^{(1)} \in C_1$, and there are $(14^2 - 2 \times 2^2)$ choices of $\overrightarrow{d}^{(2)}$ and $\overrightarrow{d}^{(4)}$ such that $\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)} \notin C_4, (\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)}) \notin (C_2, C_2) \cup (C_3, C_3)$. Thus we have $\#(S_1) = 1880 \times (m + 1)(m - 2)$. Similar to Case (1), we also have $\#(S_2) = 1880 \times (m + 1)(m - 2)$. We consider the Case (3) and (4) together. It was proved in [17] that different choice of the subscript $\omega, v$ and NSGIPs are $2 \cdot (m - 2)(m + 1)$. Moreover, we have 10 choices of $\overrightarrow{d}^{(3)}$ such that $\overrightarrow{d}^{(3)} \in C_1$, and $(m - 3)$ choices of $\kappa$ satisfying the conditions in Cases (3) and (4). Thus we have $\#(S_3 \cup S_4) = 20 \times (m - 3)(m + 1)(m - 2)$. 

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From the discussion above, we obtain

\[ \# \{ S_1 \cup S_2 \cup S_3 \cup S_4 \} = \# \{ S_1 \} + \# \{ S_2 \} + \# \{ S_3 \} + \# \{ S_4 \} = (3700 + 20m)(m^2 - m - 2), \]

which completes the proof. \( \square \)

5 A Theory of Golay array pair over QAM

In this section, a simplified and completely elementary process for constructing GCPs over 4\(^q\)-QAM from the viewpoint of array is given.

Firstly, we generalize the concept of Golay array pair (GAP) from PSK [12, 10] to 4\(^q\)-QAM, and demonstrate that a large number of GCPs can be constructed from a single GAP over 4\(^q\)-QAM. Secondly, inspired by the idea in [2, 24], we introduce the generating function of array, and make a connection between GAP and specified para-unitary (PU) matrix over 4\(^q\)-QAM. Finally, we introduce our method to construct the desired PU matrix over 4\(^q\)-QAM, and conclude that Theorem 1 and 2 can be proved immediately if Theorem 5 and 6 shown in this section are valid.

5.1 Golay Array and Sequence Pair over QAM

A complex-valued array of size 2 \times 2 \times \cdots \times 2 can be expressed by a function \( F(x_1, x_2, \cdots, x_m) \) (or \( F(x) \) for short) from \( \mathbb{Z}_2^m \) to \( \mathbb{C} \). An array is in a sense a multidimensional sequence. For example, the array for \( m = 3 \) can be viewed as the following cube:

\[
F(x_1, x_2, x_3) =
\begin{array}{cccc}
F(001) & F(011) \\
F(000) & F(010) \\
F(101) & F(111) \\
F(100) & F(110) \\
\end{array}
\]

**Definition 5** Let \( F(x) \) be an array of size 2 \times 2 \times \cdots \times 2. The aperiodic auto-correlation of \( F(x) \) at shift \( \tau = (\tau_1, \tau_2, \cdots, \tau_m) \) (\( \tau_k = -1, 0 \ or \ 1 \)) is defined by

\[
C_F(\tau) = \sum_x F(x + \tau) \cdot \overline{F(x)}, \quad (24)
\]

where “\( y + \tau \)” is the element-wise addition of vectors over \( \mathbb{Z} \), and \( F(x + \tau) \cdot \overline{F(x)} = 0 \) if \( F(x + \tau) \) or \( F(x) \) is not defined.
Definition 6 A pair of arrays \( \{F(x), G(x)\} \) of size \( 2 \times 2 \times \cdots \times 2 \) is said to be a Golay array pair (GAP) if

\[
C_F(\tau) + C_G(\tau) = 0, \forall \tau \neq 0.
\] (25)

For further results on GAPs, see [10, 24]. An array over QPSK of size \( 2 \times 2 \times \cdots \times 2 \) can be described by a GBF over \( \mathbb{Z}_4 \) [24]. Similarly, an array of size \( 2 \times 2 \times \cdots \times 2 \) over \( 4^q \)-QAM can be described by the weighted sum:

\[
F(x) = \sum_{p=0}^{q-1} 2^p \cdot \xi^{f(p)}(x).
\] (26)

Arrays over QPSK can be obviously regarded as arrays over \( 4^q \)-QAM for \( q = 1 \).

Note that a sequence \( F(y) \) of length \( 2^m \) can be connected with an array \( F(x) \) by setting \( y = \sum_{k=1}^{m} x_k \cdot 2^{k-1} \). The aperiodic auto-correlation \( C_F(\tau) \) of sequence \( F(y) \) can be derived from the sum of aperiodic auto-correlation \( C_F(\tau) \) of array \( F(x) \) by restricting \( \tau = \sum_{k=1}^{m} 2^{k-1} \tau_k \), i.e.,

\[
C_F(\tau) = \sum_{\tau \in \mathcal{D}(\tau)} C_F(\tau),
\]

where \( \mathcal{D}(\tau) = \{\tau|\tau = \sum_{k=1}^{m} 2^{k-1} \tau_k\} \). Thus, if the arrays \( F(x) \) and \( G(x) \) over QAM described by V-GBFs \( \vec{f}(x) \) and \( \vec{g}(x) \) form a GAP, the sequences associated with V-GBFs \( \vec{f}(x) \) and \( \vec{g}(x) \) must form a GCP. Denote the permutation acting on GBF and V-GBF by \( \pi \cdot \vec{f}(x) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}) \) and \( \pi \cdot \vec{f}(x) = \vec{f}(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}) \), respectively. Furthermore, we have the following results.

Theorem 3 If a pair of arrays over QAM described by V-GBFs \( \{\vec{f}(x), \vec{g}(x)\} \) form a GAP, the arrays described by V-GBFs

\[
\left\{ \pi \cdot \vec{f}(x) + f'(x) \cdot \vec{1}, \pi \cdot \vec{g}(x) + f'(x) \cdot \vec{1} \right\}
\]

form a GAP, where \( f'(x) = \sum_{k=1}^{m} c_k x_k + c_0 \) is an arbitrary affine GBF from \( \mathbb{Z}_4^m \) to \( \mathbb{Z}_4 \) for \( c_k \in \mathbb{Z}_4 \) and \( \pi \) is an arbitrary permutation. Consequently, the sequences associated with the above V-GBFs form a GCP.

Proof We first prove the arrays described by V-GBFs \( \{\pi \cdot \vec{f}(x), \pi \cdot \vec{g}(x)\} \) form a GAP. The aperiodic auto-correlation of array described by \( \vec{f}(x) \) is given by

\[
C_{\vec{f}}(\tau) = \sum_{x} F(x + \tau) \cdot \overline{F(x)}.
\]
Then the aperiodic autocorrelation of array described by $\pi \cdot \mathbf{f}(x)$ is given by

$$C_{\pi \cdot \mathbf{f}}(\tau) = \sum_x F(\pi \cdot (x + \tau)) \cdot \overline{F(\pi \cdot x)} = \sum_x F(\pi \cdot x + \pi \cdot \tau) \cdot \overline{F(\pi \cdot x)}.$$  

As $x$ goes over $\mathbb{Z}_m^2$, $\pi \cdot x$ also runs over $\mathbb{Z}_m^2$. So $C_{\pi \cdot \mathbf{f}}(\tau) = C_{\mathbf{f}}(\pi \cdot \tau)$ follows. Thus for $\tau \neq 0$, we have

$$C_{\pi \cdot \mathbf{f}}(\tau) + C_{\pi \cdot \mathbf{g}}(\tau) = C_{\mathbf{f}}(\pi \cdot \tau) + C_{\mathbf{g}}(\pi \cdot \tau) = 0.$$

Next, we prove the arrays described by V-GBFs $\left\{\mathbf{f}(x) + f'(x) \cdot \mathbf{1}, \mathbf{g}(x) + f'(x) \cdot \mathbf{1}\right\}$ form a GAP.

The aperiodic autocorrelation of array described by $\mathbf{f}(x)$ is given by

$$C_{\mathbf{f}}(\tau) = \sum_x \left(\sum_{p=0}^{q-1} 2^p \cdot \xi f^{(p)}(x+\tau)\right) \cdot \left(\sum_{p=0}^{q-1} 2^p \cdot \overline{\xi f^{(p)}(x)}\right).$$

Then the aperiodic autocorrelation of array described by $\mathbf{f}(x) + f'(x) \cdot \mathbf{1}$ is given by

$$C_{\mathbf{f} + f'}(\tau) = \sum_x \left(\sum_{p=0}^{q-1} 2^p \cdot \xi f^{(p)}(x+\tau) + f'(x+\tau)\right) \cdot \left(\sum_{p=0}^{q-1} 2^p \cdot \overline{\xi f^{(p)}(x) - f'(x)}\right)$$

$$= \sum_x \xi f^{(p)}(x+\tau) - f'(x) \left(\sum_{p=0}^{q-1} 2^p \cdot \xi f^{(p)}(x+\tau)\right) \cdot \left(\sum_{p=0}^{q-1} 2^p \cdot \overline{\xi f^{(p)}(x)}\right)$$

$$= \xi \sum_{k=1}^{n} c_k \tau_k \sum_x \left(\sum_{p=0}^{q-1} 2^p \cdot \xi f^{(p)}(x+\tau)\right) \cdot \left(\sum_{p=0}^{q-1} 2^p \cdot \overline{\xi f^{(p)}(x)}\right)$$

$$= \xi \sum_{k=1}^{n} c_k \tau_k \cdot C_{\mathbf{f}}(\tau).$$

Similarly, the aperiodic autocorrelation of array described by $\mathbf{g}(x) + f'(x) \cdot \mathbf{1}$ is given by

$$C_{\mathbf{g} + f'}(\tau) = \xi \sum_{k=1}^{n} c_k \tau_k \cdot C_{\mathbf{g}}(\tau).$$

Then we have

$$C_{\mathbf{f} + f'}(\tau) + C_{\mathbf{g} + f'}(\tau) = \xi \sum_{k=1}^{n} c_k \tau_k \cdot (C_{\mathbf{g}}(\tau) + C_{\mathbf{g}}(\tau)) = 0(\tau \neq 0),$$

which completes the proof.  

From Theorem 3, we are able to construct a large number of GCPs over QAM from GAPs over QAM.
5.2 GAP and PU Matrix over QAM

For a complex-valued array $F(x)$ of size $2 \times 2 \times \cdots \times 2$, we can define its generating function by

$$F(z_1, z_2, \cdots, z_m) = \sum_{x_1, x_2, \cdots, x_m} F(x_1, x_2, \cdots, x_m) z_1^{x_1} z_2^{x_2} \cdots z_m^{x_m}, \quad (27)$$

or denoted by $F(z) = \sum_x F(x) \cdot z^x$ for short. It is easy to verify

$$F(z) \cdot \overline{F}(z^{-1}) = \sum_{\tau} C_F(\tau) z_1^{\tau_1} z_2^{\tau_2} \cdots z_m^{\tau_m}, \quad (28)$$

where $z^{-1} = (z_1^{-1}, z_2^{-1}, \cdots, z_m^{-1})$. So arrays $\{F(x), G(x)\}$ of size $2 \times 2 \times \cdots \times 2$ form a GAP if and only if their generating functions $\{F(z), G(z)\}$ satisfy

$$F(z) \cdot \overline{F}(z^{-1}) + G(z) \cdot \overline{G}(z^{-1}) = c, \quad (29)$$

where $c$ is a real constant.

Note that the array described by $\vec{f}(x)$ over QAM (or $f(x)$ over QPSK) is uniquely determined by the generating function $F(z)$, and vice versa.

Let $F_{i,j}(x)$ ($0 \leq i, j \leq 1$) be $m$-dimensional arrays over QPSK of size $2 \times 2 \times \cdots \times 2$ corresponding to GBF $f_{i,j}(x)$ and generating function $F_{i,j}(z)$. These arrays can be expressed by a formalized array matrix $M(x)$, where each entry is $F_{i,j}(x)$, i.e.,

$$M(x) = \begin{bmatrix} F_{0,0}(x) & F_{0,1}(x) \\ F_{1,0}(x) & F_{1,1}(x) \end{bmatrix}. \quad (30)$$

Also, these arrays can be described by a formalized matrix with the GBF entry, i.e.,

$$\tilde{M}(x) = \begin{bmatrix} f_{0,0}(x) & f_{0,1}(x) \\ f_{1,0}(x) & f_{1,1}(x) \end{bmatrix}, \quad (31)$$

and described by a formalized matrix with the generating-function entry, i.e.,

$$M(z) = \begin{bmatrix} F_{0,0}(z) & F_{0,1}(z) \\ F_{1,0}(z) & F_{1,1}(z) \end{bmatrix}. \quad (32)$$

$M(z)$ is called the generating-function matrix of $M(x)$ and $\tilde{M}(x)$.

Furthermore, suppose that $F_{i,j}(x)$ ($0 \leq i, j \leq 1$) are $m$-dimensional arrays over $4^q$-QAM of size $2 \times 2 \times \cdots \times 2$ corresponding to V-GBF $\vec{f}_{i,j}(x) = (f_{i,j}^{(0)}(x), f_{i,j}^{(1)}(x), \cdots, f_{i,j}^{(q-1)}(x))$ and generating
function $F_{i,j}(z)$. We can correspondingly define the formalized array matrix

$$M(x) = \begin{bmatrix} F_{0,0}(x) & F_{0,1}(x) \\ F_{1,0}(x) & F_{1,1}(x) \end{bmatrix},$$

(33)

the formalized V-GBF matrix

$$\tilde{M}(x) = \begin{bmatrix} \tilde{f}_{0,0}(x) & \tilde{f}_{0,1}(x) \\ \tilde{f}_{1,0}(x) & \tilde{f}_{1,1}(x) \end{bmatrix},$$

(34)

and the generating-function matrix

$$\mathbb{M}(z) = \begin{bmatrix} F_{0,0}(z) & F_{0,1}(z) \\ F_{1,0}(z) & F_{1,1}(z) \end{bmatrix}.\quad (35)$$

Denote the component GFB matrix of the V-GBF matrix $\tilde{M}(x)$ by

$$\tilde{M}^{(p)}(x) = \begin{bmatrix} f_{0,0}^{(p)}(x) & f_{0,1}^{(p)}(x) \\ f_{1,0}^{(p)}(x) & f_{1,1}^{(p)}(x) \end{bmatrix},$$

in the form of (31) for $0 \leq p < q$. Let $M^{(p)}(z)$ be the corresponding generating-function matrix of $\tilde{M}^{(p)}(x)$. Since an array over $4^q$-QAM is the weighted sums of $q$ arrays over QPSK, the generating-function matrix $\mathbb{M}(z)$ is the weighted sums of $M^{(p)}(z)$, i.e.,

$$\mathbb{M}(z) = \sum_{p=0}^{q-1} 2^p \cdot M^{(p)}(z).$$

(36)

**Theorem 4** Let $\mathbb{M}(z)$ in the form (35) be the generating-function matrix of a V-GBF matrix $\tilde{M}(x)$ in the form (34). If $\mathbb{M}(z)$ is a para-unitary (PU) matrix, i.e.,

$$\mathbb{M}(z)\mathbb{M}^\dagger(z^{-1}) = c \cdot I,$$

(37)

where $c$ is a real number, $(\cdot)^\dagger$ denotes the Hermitian transpose and $I$ is an identity matrix of order 2, the arrays over QAM described by every row (or column) of $\mathbb{M}(x)$ form a GAP.

**Proof** It is straightforward from an alternative definition of the GAP in formula (29). \qed

From Theorem 4, GAPs can be constructed by studying the array matrix $\mathbb{M}(z)$ over QAM satisfying PU condition.
5.3 Construction of the desired PU matrices

The following notations of matrices of order 2 will be used in the rest of the paper.

- \( D(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, D(x) = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix} \).

- \( J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \).

The most simple PU matrix in the form of (32) over QPSK is the Butson-type Hadamard matrix of order 2 with entries being fourth roots of unity, which corresponds to the GAP of dimension 1 and length 1 [24]. Such a Butson-type Hadamard matrix \( H \) can be uniquely expressed by

\[
H(d_0, d_1, d_2) = \xi^{d_0} \cdot \begin{bmatrix} 1 \\ \xi^{d_1} \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \xi^{d_2} \end{bmatrix} = \begin{bmatrix} \xi^{d_0} \\ \xi^{d_0 + d_1} \\ -\xi^{d_0 + d_1 + d_2} \end{bmatrix},
\]

where \( d_0, d_1, d_2 \in \mathbb{Z}_4 \). Then this Hadamard matrix \( H(d_0, d_1, d_2) \) is uniquely determined by the values of \( (d_0, d_1, d_2) \).

Furthermore, denote GBF matrix of \( H \) by \( \tilde{H} \), we have

\[
\tilde{H}(d_0, d_1, d_2) = \begin{bmatrix} d_0 & d_0 + d_2 \\ d_0 + d_1 & d_0 + d_1 + d_2 + 2 \end{bmatrix} = d_0 \cdot J + d_1 \cdot A + d_2 \cdot B + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.
\]

In this paper, we are only interested in the following Hadamard matrices.

**Definition 7** For \( 0 \leq p < q \) and \( d_1^{(p)}, d_2^{(p)}, d_0^{(p)} \in \mathbb{Z}_4 \) given in definition 3, define Hadamard matrices

\[
H_p = H(d_0^{(p)}, d_1^{(p)}, d_2^{(p)}).
\]

For \( p = 0 \), we have \( d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0 \) from Definition 3 and

\[
H_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \tilde{H}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.
\]

**Lemma 1** For \( 0 \leq p < q \), let \( H_p \) be Hadamard matrices given in Definition 7 and \( T \) an arbitrary nonempty subset of \( \mathbb{Z}_q \). Then

\[
\mathbb{H} = \sum_{p \in T} 2^p \cdot H_p.
\]
is a unitary matrix, i.e.,

$$\mathbb{H}\mathbb{H}^\dagger = c \cdot \mathbf{I},$$

where $c$ is a real number.

**Proof** Since $2d_0 + d_1 + d_2 = 0$ over $\mathbb{Z}_4$, we have

$$H(d_0, d_1, d_2) = \begin{bmatrix} \xi^{d_0} & \xi^{d_0+d_2} \\ \xi^{d_0+d_1} & -\xi^{d_0+d_1} \end{bmatrix} = \begin{bmatrix} \xi^{d_0} & \overline{\xi^{d_0+d_1}} \\ \xi^{d_0+d_1} & -\xi^{d_0} \end{bmatrix}. $$

Then $\mathbb{H}$ can be re-expressed by

$$\mathbb{H} = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

where $\alpha = \sum_{p \in T} 2^p \cdot \xi^{d_0^{(p)}}$ and $\beta = \sum_{p \in T} 2^p \cdot \xi^{d_1^{(p)}+d_2^{(p)}}$. It's easy to verify that

$$\mathbb{H} \cdot \mathbb{H}^\dagger = (|\alpha|^2 + |\beta|^2) \cdot \mathbf{I},$$

which completes the proof. \qed

**Theorem 5** Suppose that the sets $T_k$, the permutation $\sigma$, the mappings $\rho_k$, the elements $d_0^{(p)}, d_1^{(p)}, d_2^{(p)}$, and the vectors $\vec{d}_1, \vec{d}_2$ are the same as those in Theorem 1. Let

$$\mathbb{H}^{(k)} = \sum_{p \in T_{\sigma(k)}} 2^{\rho_k(q-1)-p} \cdot H_p,$$

where $H_p$ is given in Definition 7. Then

$$M_{Thm-1}(z) = \mathbb{H}^{(0)} \cdot \left( \prod_{k=1}^{m} D(z_k) \cdot \mathbb{H}^{(k)} \right)$$

is a PU matrix over $4^q$-QAM. Moreover, $M_{Thm-1}(z)$ is the generating matrix of the V-GBF matrix

$$\widetilde{M}_{Thm-1}(\mathbf{x}) = \left( \mathbf{I} \cdot f(\mathbf{x}) + \mathbf{s}(\mathbf{x}) \right) \cdot \mathbf{J} + \left( 2 \cdot x_1 + \vec{d}_1 \right) \cdot \mathbf{A} + \left( 2 \cdot x_m + \vec{d}_2 \right) \cdot \mathbf{B},$$

where $f(\mathbf{x}) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}$, $\mathbf{s}(\mathbf{x}) = (0, s^{(1)}(\mathbf{x}), \ldots, s^{(q-1)}(\mathbf{x}))$ with

$$s^{(p)}(\mathbf{x}) = \sum_{k=1}^{m} \left( d_1^{\rho_k(p)} + d_2^{\rho_{k-1}(p)} \right) x_k + \sum_{k=0}^{m} d_0^{\rho_k(p)}, \quad (0 \leq p \leq q-1).$$


Notice that the V-GBFs in first row and first column of \( \tilde{M}_{\text{Thm-1}}(x) \) are
\[
\begin{align*}
\tilde{f}(x) &= \tilde{1} \cdot f(x) + \tilde{s}(x), \\
\tilde{g}(x) &= \tilde{f}(x) + \tilde{2} \cdot x_1 + \tilde{d}_1,
\end{align*}
\]
and
\[
\begin{align*}
\tilde{f}(x) &= \tilde{1} \cdot f(x) + \tilde{s}(x), \\
\tilde{g}(x) &= \tilde{f}(x) + \tilde{2} \cdot x_m + \tilde{d}_2,
\end{align*}
\]
which are both GAPs. By applying Theorem 3, Theorem 1 can be proved immediately if Theorem 5 is valid.

We introduce another construction of PU matrices over 4\(^a\)-QAM involving NSGIP as follows.

For NSGIP \( Q_0 = Q(b_1, b_2, \ldots, b_{q'-1}) \) and \( Q_1 = Q(b_1', b_2', \ldots, b_{q'-1}') \), define two matrices
\[
\text{diag}\{Q_0, Q_1\} = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix}
\]
and \( Q = \begin{bmatrix} Q_0 & Q_1 \\ Q_1 & Q_0 \end{bmatrix} \).

Then \( \text{diag}\{Q_0, Q_1\} \) is a PU matrix. Moreover, if \( M(z) = \begin{bmatrix} F_{0,0}(z) & F_{0,1}(z) \\ F_{1,0}(z) & F_{1,1}(z) \end{bmatrix} \) is PU matrix, then
\[
Q \odot M(z) = \begin{bmatrix} Q_0 \cdot F_{0,0}(z) & Q_1 \cdot F_{0,1}(z) \\ Q_1 \cdot F_{1,0}(z) & Q_0 \cdot F_{1,1}(z) \end{bmatrix}
\]
is also a PU matrix, where the symbol \( \odot \) means the element-wise product of matrices.

**Theorem 6** Suppose that the sets \( T'_k \), the permutation \( \sigma \), the mappings \( \rho'_k \) and \( \rho' \), the elements \( d_0^{(p)}, d_1^{(p)}, d_2^{(p)} \), and the vectors \( \tilde{d}_1, \tilde{d}_2 \) are the same as those in Theorem 2. Let
\[
\mathbb{H}^{(k)} = \sum_{p \in T'_k} 2^{\rho'_k(q-1)-p} \cdot H_p,
\]
where \( H_p \) is given in Definition 7. Then both

**Case (a):**
\[
\mathbb{M}_{\text{Thm-2}}(z) = \mathbb{H}^{(0)} \cdot \left( \prod_{k=1}^{\omega-1} D(z_k) \cdot \mathbb{H}^{(k)} \right) \cdot \text{diag}\{Q_0, Q_1\} \cdot \left( \prod_{k=\omega}^{m} D(z_k) \cdot \mathbb{H}^{(k)} \right),
\]
where \( 2 \leq \omega \leq m - 1 \), and

**Case (b):**
\[
\mathbb{M}_{\text{Thm-2}}(z) = \left( \prod_{k=0}^{\omega-1} \mathbb{H}^{(k)} \cdot D(z_{k+1}) \right) \cdot \left( Q \odot \left( \mathbb{H}^{(\omega)} \cdot \left( \prod_{k=\omega+1}^{\omega-1} D(z_k) \cdot \mathbb{H}^{(k)} \right) \right) \right) \cdot \left( \prod_{k=\omega}^{m} D(z_k) \cdot \mathbb{H}^{(k)} \right),
\]
(45)
where \(1 \leq \omega \leq m - 2\), \(\omega + 2 \leq \nu \leq m\),
are PU matrices over \(4^q\)-QAM. Moreover, it is the generating matrix of the V-GBF matrix
\[
\tilde{M}_{\text{Thm-2}}(x) = \left( \mathbf{1} \cdot f(x) + \tilde{s}(x) \right) \cdot \mathbf{J} + \left( \tilde{\mathbf{2}} \cdot x_1 + \tilde{\mathbf{d}}_1 \right) \cdot \mathbf{A} + \left( \tilde{\mathbf{2}} \cdot x_m + \tilde{\mathbf{d}}_2 \right) \cdot \mathbf{B}
\]  
(46)
where \(f(x) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}\) and \(s^{(p)}(x) = s^{(p)}_0(x) + s^{(p)}'(x)\) with
\[
s^{(p)}_0(x) = \sum_{k=1}^{m} \left( d_1^{(p)(\nu)} + d_2^{(p)(\nu-1)} \right) x_k + \sum_{k=0}^{m} d_0^{(p)(\omega)}
\]
and
\[
s^{(p)}'(x) = \begin{cases} 
(b^{(p)(\nu)} - b^{(p)(\nu)})x_\omega + b^{(p)(\nu)}; & \text{Case (a);} \\
(b^{(p)(\nu)} - b^{(p)(\nu)})x_\omega + (b^{(p)(\nu)} - b^{(p)(\nu)})x_v + b^{(p)(\nu)}; & \text{Case (b)}
\end{cases}
\]
for \(0 \leq p \leq q - 1\).

Similar to Theorem 5, by applying Theorem 3 and 6, Theorem 2 is proved immediately.

Since \(D(z_k), \mathbb{H}^{(k)}, \text{diag}\{Q_0, Q_1\}\) and \(Q \odot M(z)\) are all PU matrices if \(M(z)\) is a PU matrix, it is easy to verify both \(\tilde{M}_{\text{Thm-1}}(z)\) and \(\tilde{M}_{\text{Thm-2}}(z)\) in Theorem 5 and 6 are PU matrices. We will prove \(\tilde{M}_{\text{Thm-1}}(x)\) and \(\tilde{M}_{\text{Thm-2}}(x)\) are their respectively corresponding V-GBF matrices in the next section.

6 Proof of the Main Theorems

From the discussion in Subsection 5.3, it is sufficient to prove Theorem 1 and 2 if Theorem 5 and 6 are valid. To prove Theorem 1 and 2, we should develop a method to extract the corresponding V-GBFs from the desired PU matrices over QAM. Such a method has been deeply studied in [24] for PU matrices of arbitrary order over PSK. We introduce this method for PU matrices of order 2 over QPSK and prove Theorem 5 and 6 in this section.

6.1 GBF Matrix and Its Generating Matrix

In this subsection, we introduce some basic results on how to extract GBF matrix in the form of \(30\) from its generating matrix over QPSK in the form of \(32\).

**Lemma 2** Let the matrices \(D(x), J, A\) and \(B\) be the same at those given in Subsection 5.3, we have

(1) \(J \cdot D(x) \cdot J = J\);

(2) \(A \cdot D(x) \cdot J = A\);
(3) \( B \cdot D(x) \cdot J = x \cdot J \);

(4) \( J \cdot D(x) \cdot A = x \cdot J \);

(5) \( J \cdot D(x) \cdot B = B \).

Lemma 3 Let \( M^{(0)}(z_0) \) and \( M^{(1)}(z_1) \) be generating matrices of GBF matrices \( \tilde{M}^{(0)}(x_0) \) and \( \tilde{M}^{(1)}(x_1) \) in the form of (30) over QPSK respectively, where \( x_0 \) and \( x_1 \) are two non-intersecting multivariate Boolean variables. Denote \( z = \{z_0, z_1, z\} \) and \( x = \{x_0, x_1, x\} \). Then

\[
M(z) = M^{(0)}(z_0) \cdot D(z) \cdot M^{(1)}(z_1)
\] (47)

is a PU matrix and its corresponding GBF matrix is given by

\[
\tilde{M}(x) = \tilde{M}^{(0)}(x_0) \cdot D(x) \cdot J + J \cdot D(x) \cdot \tilde{M}^{(1)}(x_1).
\] (48)

Proof It is obvious that \( M(z) \) is PU matrix. Let \( M(x) \), \( M^{(0)}(x_0) \), and \( M^{(1)}(x_1) \) be the array matrices of the generating matrices \( M(z) \), \( M^{(0)}(z_0) \) and \( M^{(1)}(z_1) \) respectively. Recall that \( D(z) = \sum_{x=0}^{1} D(x) \cdot z^x \), we have

\[
M(z) = \sum_{x_0} M^{(0)}(x_0) \cdot z^{x_0} \cdot \sum_{x} D(x) \cdot z^x \cdot \sum_{x_1} M^{(1)}(x_1) \cdot z^{x_1}.
\]

From another expansion:

\[
M(z) = \sum_{x} M(x) \cdot z^x,
\]

we obtain

\[
M(x) = M^{(0)}(x_0) \cdot D(x) \cdot M^{(1)}(x_1)
\]

for \( \forall x \). Then we can verify that each entry of \( M(x) \) can be expressed by

\[
M_{i,j}(x) = M^{(0)}_{i,x}(x_0) \cdot M^{(1)}_{x,j}(x_1).
\]

Alternatively, we have

\[
\tilde{M}_{i,j}(x) = \tilde{M}^{(0)}_{i,x}(x_0) + \tilde{M}^{(1)}_{x,j}(x_1)
\] (49)
for \(i, j, x = 0, 1\).

On the other hand, one can check that the formula (48) and (49) are equivalent by specifying \(x = 0\) and 1. This complete the proof. 

We are able to extend Lemma 3 to a general case. Note that here both \(z'_k\) and \(x'_k\) are multivariate variables, and both \(z_k\) and \(x_k\) are single variables. Suppose that \(\{z'_0, z'_1, \ldots, z'_m, z_1, \ldots, z_m\}\) are multivariate variables which do not intersect with each other, and \(\{x'_0, x'_1, \ldots, x'_m, x_1, \ldots, x_m\}\) are their corresponding Boolean variables respectively. Denote

\[
\begin{align*}
  z &= (z'_0, z'_1, \ldots, z'_m, z_1, \ldots, z_m), \\
  x &= (x'_0, x'_1, \ldots, x'_m, x_1, \ldots, x_m).
\end{align*}
\]

By applying item (1) in Lemma 2 and Lemma 3 iteratively, we obtain a general result.

**Theorem 7** For \(0 \leq k \leq m\), let \(M^{(k)}(z'_k)\) be generating matrices of GBF matrices \(\tilde{M}^{(k)}(x'_k)\) over QPSK. Then

\[
M(z) = M^{(0)}(z'_0) \cdot \left( \prod_{k=1}^{m} (D(z_k) \cdot M^{(k)}(z'_k)) \right)
\]

is a PU matrix and its corresponding GBF matrix is given by

\[
\tilde{M}(x) = \tilde{M}^{(0)}(x'_0) \cdot D(x_1) \cdot J + \sum_{k=1}^{m-1} J \cdot D(x_k) \cdot \tilde{M}^{(k)}(x'_k) \cdot D(x_{k+1}) \cdot J + J \cdot D(x_m) \cdot \tilde{M}^{(m)}(x'_m).
\]

The following corollary is an immediate consequence of Theorem 7. Note that it has been provided in both [2] and [24] by other expressions, but the matrix expression here will simplify the process of the proof of the Theorem 5 and 6 in the next two subsections.

**Corollary 1** Let \(H_0\) be shown in Definition 7. The corresponding GBF matrix of the following PU matrix

\[
U(z) = H_0 \cdot \left( \prod_{k=1}^{m} (D(z_k) \cdot H_0) \right)
\]

is given by

\[
\tilde{U}(x) = f(x) \cdot J + 2x_1 \cdot A + 2x_m \cdot B,
\]

where \(f(x) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}\). 

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Proof  For $0 \leq k \leq m$, let $M^{(k)}(z'_k) = H_0$ in Theorem 7 we have

\[
\tilde{H}_0 \cdot D(x_1) \cdot J = 2x_1 \cdot A,
\]
\[
J \cdot D(x_k) \cdot \tilde{H}_0 \cdot D(x_{k+1}) \cdot J = 2x_kx_{k+1} \cdot J,
\]
\[
J \cdot D(x_m) \cdot \tilde{H}_0 = 2x_m \cdot B.
\]

Then the proof is completed by formula (52). □

6.2 Proof of Theorem 5

According to the definition of $\rho_k$ in Definition 2 and $H^{(k)}$ in Theorem 5, we have

\[
M_{Thm-1}(z) = \sum_{p=0}^{q-1} 2^{q-1-p} \cdot M^{(p)}_{Thm-1}(z)
\]

where

\[
M^{(p)}_{Thm-1}(z) = H_{\rho_0(p)} \cdot \left( \prod_{k=1}^{m} D(z_k) \cdot H_{\rho_k(p)} \right).
\]

According to Theorem 7 their corresponding GBF matrices are given by

\[
\tilde{M}^{(p)}_{Thm-1}(x) = \tilde{H}_{\rho_0(p)} \cdot D(x_1) \cdot J + \sum_{k=1}^{m-1} J \cdot D(x_k) \cdot \tilde{H}_{\rho_k(p)} \cdot D(x_{k+1}) \cdot J + J \cdot D(x_m) \cdot \tilde{H}_{\rho_m(p)}.
\]

In particular, for $p = 0$, we have $H_{\rho_k(p)} = H_0$ shown in Definition 7. Moreover, $M^{(0)}_{Thm-1}(z) = U(z)$ and

\[
\tilde{M}^{(0)}_{Thm-1}(x) = \tilde{U}(x)
\]

has been shown in Corollary 1.

The difference between $\tilde{M}^{(p)}_{Thm-1}(x)$ and $\tilde{M}^{(0)}_{Thm-1}(x)$ can be calculated by

\[
\tilde{M}^{(p)}_{Thm-1}(x) - \tilde{M}^{(0)}_{Thm-1}(x) = (\tilde{H}_{\rho_0(p)} - \tilde{H}_0) \cdot D(x_1) \cdot J
\]

\[
+ \sum_{k=1}^{m-1} J \cdot D(x_k) \cdot (\tilde{H}_{\rho_k(p)} - \tilde{H}_0) \cdot D(x_{k+1}) \cdot J
\]

\[
+ J \cdot D(x_m) \cdot (\tilde{H}_{\rho_m(p)} - \tilde{H}_0)
\]

According to (39), the difference of $\tilde{H}_p$ and $\tilde{H}_0$ can be expressed by

\[
(\tilde{H}_p - \tilde{H}_0) = d^{(p)}_0 \cdot J + d^{(p)}_1 \cdot A + d^{(p)}_2 \cdot B.
\]
According to Lemma 4, the items in the above difference between $\tilde{M}_{\text{Thm-1}}^{(p)}(x)$ and $\tilde{M}_{\text{Thm-1}}^{(0)}(x)$ can be simplified as

$$(\tilde{H}_{p_0(p)} - \tilde{H}_0) \cdot D(x_1) \cdot J = (d_2^{(p_0(p))}x_1 + d_0^{(p_0(p))}) \cdot J + d_1^{(p_0(p))} \cdot A,$$

$$J \cdot D(x_k) \cdot (\tilde{H}_{p_k(p)} - \tilde{H}_0) \cdot D(x_{k+1}) \cdot J = (d_2^{(p_k(p))}x_k + d_0^{(p_k(p))}x_{k+1} + d_0^{(p_k(p))})J,$$

$$J \cdot D(x_m) \cdot (\tilde{H}_{p_m(p)} - \tilde{H}_0) = (d_2^{(p_m(p))}x_m + d_0^{(p_m(p))}) \cdot J + d_2^{(p_m(p))} \cdot B.$$

Consequently, we have

$$\tilde{M}_{\text{Thm-1}}^{(p)}(x) - \tilde{M}_{\text{Thm-1}}^{(0)}(x) = s^{(p)}(x) \cdot J + d_1^{(p_0(p))} \cdot A + d_2^{(p_m(p))} \cdot B,$$

where

$$s^{(p)}(x) = \sum_{k=1}^{m} \left( d_1^{(p_k(p))} + d_2^{(p_k-1)(p))} \right) x_k + \sum_{k=0}^{m} d_0^{(p_k(p))}.$$

By applying Corollary 1, we obtain

$$\tilde{M}_{\text{Thm-1}}^{(p)}(x) = \left( f(x) + s^{(p)}(x) \right) \cdot J + \left( 2x_1 + d_1^{(p_0(p))} \right) \cdot A + \left( 2x_m + d_2^{(p_m(p))} \right) \cdot B,$$

where $f(x) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}$. This completes the proof.

### 6.3 Proof of Theorem 6

The give the following lemma before we prove Theorem 6.

**Lemma 4** Let $M(z)$ be the generating matrix of GBF matrix $\tilde{M}(x)$ over QPSK. Suppose $\alpha, \beta, c_{i,j} (0 \leq i, j \leq 1) \in \mathbb{Z}_4$. We have

1. $M(z) \cdot \text{diag}\{\xi^a, \xi^b\}$ is the generating matrix of GBF matrix $\tilde{M}(x) + J \cdot \text{diag}\{\alpha, \beta\}$;

2. $M(z) \odot C$ (or $C \odot M(z)$) is the generating matrix of GBF matrix $\tilde{M}(x) + \tilde{C}$, where $C = \begin{bmatrix} \xi^{c_{0,0}} & \xi^{c_{0,1}} \\ \xi^{c_{1,0}} & \xi^{c_{1,1}} \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} \end{bmatrix}$.

According to the definition of $p_k'$ and $p'$ in Definition 4 and $H^{(k)}$ in Theorem 5, we have

$$\tilde{M}_{\text{Thm-2}}(z) = \sum_{p=0}^{q-1} 2^{q-p-1} \cdot \tilde{M}_{\text{Thm-2}}^{(p)}(z)$$

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where

\[ M_{\text{Thm-2}}^{(p)}(z) = H_{\rho_k(p)} \prod_{k=1}^{\omega-1} \left( D(z_k) \cdot H_{\rho_k(p)} \right) \cdot \text{diag}\{\xi_{b_\rho(p)}, \xi_{b_\rho'(p)}\} \cdot \prod_{k=\omega}^{m} \left( D(z_k) \cdot H_{\rho_k(p)} \right) \]

for Case (a), and

\[ M_{\text{Thm-2}}^{(p)}(z) = \prod_{k=0}^{\omega-1} \left( H_{\rho_k(p)} \cdot D(z_{k+1}) \right) \cdot \left( \left[ \begin{array}{cc} \xi_{b_\rho'(p)} & \xi_{b_\rho'(p)} \\ \xi_{-b_\rho(p)} & \xi_{-b_\rho'(p)} \end{array} \right] \otimes \left( H_{\rho_k(p)} \cdot \prod_{k=\omega+1}^{\omega-1} \left( D(z_k) \cdot H_{\rho_k(p)} \right) \right) \right) \cdot \prod_{k=\omega}^{m} \left( D(z_k) \cdot H_{\rho_k(p)} \right) \]

for Case (b).

According to Lemma\[3\] by iteratively using Theorem\[4\] their corresponding GBF matrices are given by

\[ \tilde{M}_{\text{Thm-2}}^{(p)}(x) = \tilde{M}^{(p)}(x) + J \cdot D(x_{\omega-1}) \cdot J \cdot \text{diag}\{b_\rho(p), b_\rho'(p)\} \cdot D(x_\omega) \cdot J \]

for Case (a), and

\[ \tilde{M}_{\text{Thm-2}}^{(p)}(x) = \tilde{M}^{(p)}(x) + J \cdot D(x_{\omega}) \begin{bmatrix} b_\rho'(p) & b_\rho'(p) \\ -b_\rho'(p) & -b_\rho'(p) \end{bmatrix} \cdot D(x_\omega) \cdot J \]

for Case (b), where

\[ \tilde{M}^{(p)}(x) = \tilde{H}_{\rho_0(p)} \cdot D(x_1) \cdot J + \sum_{k=1}^{m-1} J \cdot D(x_k) \cdot \tilde{H}_{\rho_k(p)} \cdot D(x_{k+1}) \cdot J + J \cdot D(x_m) \cdot \tilde{H}_{\rho_m(p)} \cdot J \]

The last term in \( \tilde{M}_{\text{Thm-2}}^{(p)}(x) \) can be calculated by

\[ J \cdot D(x_{\omega-1}) \cdot J \cdot \text{diag}\{b_\rho(p), b_\rho'(p)\} \cdot D(x_\omega) \cdot J = \left( (b_\rho'(p) - b_\rho(p))x_{\omega} + b_\rho(p) \right) \cdot J \quad (56) \]

for Case (a), and

\[ J \cdot D(x_\omega) \begin{bmatrix} b_\rho(p) & b_\rho'(p) \\ -b_\rho'(p) & -b_\rho(p) \end{bmatrix} \cdot D(x_\omega) \cdot J = \left( (b_\rho'(p) - b_\rho(p))x_{\omega} + (-b_\rho'(p) - b_\rho'(p))x_\omega + b_\rho(p) \right) \cdot J \quad (57) \]

for Case (b). The term \( \tilde{M}^{(p)}(x) \) is well studied in the proof of Theorem\[5\]. By replacing the subscript \( \rho(p) \) by \( \rho'(p) \) in formula \(53\), we have

\[ \tilde{M}^{(p)}(x) = \left( f(x) + d_0^{(p)}(x) \right) \cdot J + \left( 2x_1 + d_1^{(p_0(p))} \right) \cdot A + \left( 2x_m + d_2^{(p_m(p))} \right) \cdot B \quad (58) \]
where \( f(x) = 2 \sum_{k=1}^{m-1} x_k x_{k+1} \), and \( s_0^{(p)}(x) = \sum_{k=1}^{m} \left( d_1^{(p)}(\rho_k) + d_2^{(p)}(\rho_{k-1}) \right) x_k + \sum_{k=0}^{m} d_0^{(p)}(\rho_k) \).

Combining the formulae (56), (57) and (58), we complete the proof.

7 Conclusion

This paper is devoted to the constructions of GCPs and GCSs over 4\(^q\)-QAM.

The first contribution of this paper is that we demonstrate a large number of GCPs can be constructed from a GAP over 4\(^q\)-QAM in Theorem 3. This argument greatly simplifies the process for constructing GCPs and GCSs over 4\(^q\)-QAM. And it answers an open problem posed in [10] for GAPs of size 2 \( \times \) 2 \( \times \) \( \cdots \) \( \times \) 2 over QAM. Although the process in [10] are not involved here, Theorem 3 in this paper has the same power as the three-stage process in [10] for GAPs of size 2 \( \times \) 2 \( \times \) \( \cdots \) \( \times \) 2. The results in this paper shows that the proposed GAPs over 4\(^q\)-QAM can not only explain GCPs in the generalized cases I-V [15] [17], but also produce a large number of new GCSs over 4\(^q\)-QAM.

The second contribution of this paper is that we make a connection between GAP and specified PU matrix with multi-variables over 4\(^q\)-QAM in Theorem 4, which generalized the idea in [2] which make a connection between GCP and specified PU matrix with a single variable over 4\(^q\)-QAM. This observation greatly simplifies the process for constructing the desired PU matrices. It should be pointed out that the GCSs proposed here belong to the so-called \( M \)-Qum cases, which was mentioned but could not be explicit given in [2].

The most important contribution of this paper is that we develop a method to extract the corresponding V-GBFs from the desired PU matrices in Section 5. Note that many new GCSs over QAM different from the generalized cases I-V [15] [17] were also found in [2] by PU method and exhaustive search, but these GCSs were realized by the algorithm in [2] instead of explicit V-GBFs. The new proposed method overcomes the disadvantage in [2]. A large number of the GCSs over 4\(^q\)-QAM with explicit V-GBFs, including the generalized cases I-V, are given in Theorem 1 and 2 in this paper. If \( q \) is a composite number, new GCSs arise in our construction. For example, if \( q = 4 \), the number of new GCSs is seven times more than those in the generalized cases IV-V [17], and if \( q = 6 \), the ratio of the number of new and generalized cases IV-V is greater than six and will increase in proportion with \( m \).

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