Dyonic Black Hole in Heterotic String Theory

Dileep P. Jatkar\textsuperscript{1}, Sudipta Mukherji\textsuperscript{2} and Sudhakar Panda\textsuperscript{1}

\textsuperscript{1}Mehta Research Institute of Mathematics \\& Mathematical Physics
10, Kasturba Gandhi Marg, Allahabad 211 002, India

\textsuperscript{2}Center for Theoretical Physics, Department of Physics
Texas A\& M University, College Station, Texas 77843-4242, USA

We study some features of dyonic Black hole solution in heterotic string theory on a six torus. This solution has 58 parameters. Of these, 28 parameters denote the electric charge of the black hole, another 28 correspond to the magnetic charge, and the other two parameters being the mass and the angular momentum of the black hole. We discuss the extremal limit and show that in various limits, it reduces to the known black hole solutions. The solutions saturating the Bogomolnyi bound are identified. Explicit solution is presented for the non-rotating dyonic black hole.
1. Introduction

In recent years a variety of approximate and exact solitonic solutions to the string theory were obtained\[1,2,3\]. For a comprehensive review of the subject see ref.[4]. The study of soliton solutions is useful for understanding the non-perturbative aspects of the string theory, as it has become abundantly clear from the recent progress in the S and U duality symmetries of the string theory in various dimensions. Black hole solutions are solitonic solutions of the string theory and they play an important role in the duality symmetries of the string theory. From the point of view of the S-duality, which in general relates the perturbative spectrum of one string theory to the non-perturbative spectrum of another string theory, it is therefore important to understand the qualitative features of solitonic solutions in string theory. For example, in case of black holes it is known that the entropy of the black hole is proportional to the area of the horizon. Using the relation of the area of the horizon and the mass of the black hole one finds that the entropy of the black hole depends quadratically on its mass. The density of string states at a given mass level is proportional to the exponential of the mass and therefore the entropy is a linear function of the mass. Since black holes are also part of the string spectrum it is desirable to understand the relation between the massive string states and the black hole\[3\]. Explicit form of the black hole solution may be useful in addressing some of these questions. Recently there has been a lot of progress in understanding the relation between extremal black holes and the elementary string states\[6,7,8\]. Black hole solution to the string equations of motion has been discussed by several groups in the past\[10\]. Most general electrically charged black hole solution in the heterotic string theory was discussed by Sen\[11\].

In this paper we present the general dyonic black hole solution to the heterotic string theory. We obtain the solution using the technique outlined in ref.[11]. The plan of the paper is as follows. Section 2 contains a short exposition to the duality transformations\[11\] within the context of the Kerr solution. Section 3 contains the derivation of the dyonic black hole solution starting from the Kerr metric and discussion of the characteristics of the new solution. In section 4 we discuss various limits of the solution. We show that in certain limits we get the extremal black hole solution which saturates the Bogomolnyi bound. We also discuss various limits in which this solution reduces to pure electrically charged, pure magnetically charged as well as neutral black hole solution. In section 5 we discuss the non-rotating black hole in detail.
2. Duality Transformations

We wish to study a general black hole solution to the heterotic string compactified on a six dimensional torus. The spectrum of massless fields for a generic six torus compactification of the heterotic string contains the metric $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, the dilaton $\Phi$, twenty eight abelian gauge fields $A_{\mu}^{(a)}$ and a $28 \times 28$ matrix scalar field $M$ which contains the scalar fields coming from the internal components of the ten dimensional metric, antisymmetric tensor and gauge fields. The matrix valued scalar field $M$ satisfies following relations

$$MLM^T = L \quad M^T = M. \tag{2.1}$$

Here $L$ is a $28 \times 28$ symmetric matrix and we choose it to be

$$L = \begin{pmatrix} -I_{22} & \\ I_6 & \end{pmatrix} \tag{2.2}$$

where and in the rest of the paper $I_n$ denotes $n \times n$ identity matrix. The low energy effective field theory of these fields is given by

$$S = \int d^4x \sqrt{-G} e^{-\Phi} [R_G + G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu M L \partial_\nu M L) - \frac{1}{12} G^{\mu\nu'} G^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} - G^{\mu\nu'} G^{\rho\rho'} F_{\mu\nu}^{(a)} (LML)_{ab} F_{\mu'\nu'}^{(b)}], \tag{2.3}$$

where,

$$F_{\mu\nu}^{(a)} = \partial_\mu A_{\nu}^{(a)} - \partial_\nu A_{\mu}^{(a)}, \tag{2.4}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2A_{\mu}^{(a)} L_{ab} F_{\nu\rho}^{(b)} + \text{c.p.in} \mu, \nu \text{and} \rho, \tag{2.5}$$

and $R_G$ is the Ricci scalar corresponding to the metric $G_{\mu\nu}$. This action is invariant under an $O(6, 22)$ transformations $\Omega$ which acts on the fields as follows

$$M \rightarrow \Omega M \Omega^T, \quad A_{\mu}^{(a)} \rightarrow \Omega_{ab} A_{\mu}^{(b)}, \quad \Omega L \Omega^T = L, \tag{2.6}$$

and leaves the four dimensional metric $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton $\Phi$ invariant.

We intend to obtain the general black hole solution using the solution generating technique[12,13] which uses the global symmetries of the effective field theory to generate new solutions of the equation of motion from the known solution. This technique has
been used extensively to get various new solutions to the string equations of motion. For definiteness let us start with the Kerr solution

\[
ds^2 \equiv G_{\mu\nu}dx^\mu dx^\nu
\]
\[
= -\frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta} dt^2 + \frac{\rho^2 + a^2 \cos^2 \theta}{\rho^2 + a^2 - 2m\rho} d\rho^2 + (\rho^2 + a^2 \cos^2 \theta) d\theta^2
\]
\[
+ \frac{\sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} [(\rho^2 + a^2)(\rho^2 + a^2 \cos^2 \theta) + 2m\rho a^2 \sin^2 \theta] d\phi^2
\]
\[\Phi = 0, \quad B_{\mu\nu} = 0, \quad A^{(a)}_{\mu} = 0, \quad M = I_{28}.\]

Since the dilaton, the antisymmetric tensor, all 28 gauge fields and the matrix valued scalar field is trivial, it reduces the equations of motion of the effective field theory to Einstein equations. Thus the Kerr metric is a solution to the equations of motion of the effective field theory. Since we are considering the static solutions, the action is three dimensional. It is known that the action has T and S duality symmetries that are given by $O(6, 22)$ and $SL(2, Z)$ symmetry groups respectively. In fact, these two groups combine together to a $O(8, 24)$ symmetry group for the effective three dimensional action [14]. The matrix valued scalar field in this case is given by a $32 \times 32$ matrix. The off-diagonal component of the metric can be treated as gauge fields in lower dimensions. By dualising gauge fields to scalar fields in three dimensions we can write a $32 \times 32$ matrix

\[
\hat{M} = \begin{pmatrix}
M - e^{2\Phi}\psi^T \psi & e^{2\Phi}\psi & e^{2\Phi}\psi^T L \psi - \frac{1}{2} e^{2\Phi}\psi (\psi^T \bar{L} \psi) \\
e^{2\Phi}\psi^T & -e^{2\Phi} & \frac{1}{2} e^{2\Phi}\psi^T \bar{L} \psi \\
\psi^T \bar{L} M - \frac{1}{2} e^{2\Phi}\psi^T (\psi^T \bar{L} \psi) & \frac{1}{2} e^{2\Phi}\psi^T \bar{L} \psi & -e^{-2\Phi} + \psi^T \bar{L} M \bar{L} \psi - \frac{1}{4} e^{2\Phi}(\psi^T \bar{L} \psi)^2
\end{pmatrix}
\]

\[
\hat{L} = \begin{pmatrix}
\bar{L} & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \bar{L} = \begin{pmatrix}
-I_{22} & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \bar{g}_{ij} = e^{-2\Phi}\bar{G}_{ij}.
\]
In writing the matrix $\mathcal{M}$ we used the following duality relations

$$\bar{A}_i^{(a)} = A_i^{(a)} - (G_{tt})^{-1}G_{ti}A_t^{(a)}, \quad 1 \leq a \leq 28, \quad 1 \leq i \leq 3,$$

$$\bar{A}_i^{(29)} = \frac{1}{2}(G_{tt})^{-1}G_{ti},$$

$$\bar{A}_i^{(30)} = \frac{1}{2}B_{ti} + A_t^{(a)}L_{ab}\bar{A}_i^{(b)},$$

$$\bar{G}_{ij} = G_{ij} - (G_{tt})^{-1}G_{ti}G_{tj}, \quad \bar{B}_{ij} = B_{ij} + (G_{tt})^{-1}(G_{ti}A_j^{(a)} - G_{tj}A_i^{(a)})L_{ab}A_t^{(b)}$$

$$+ \frac{1}{2}(G_{tt})^{-1}(B_{ti}G_{tj} - B_{tj}G_{ti})$$

$$\bar{\Phi} = \Phi - \frac{1}{2} \ln(-G_{tt})$$

$$\bar{\mathcal{M}} = \begin{pmatrix}
M + 4(G_{tt})^{-1}A_tA_t^T & -2(G_{tt})^{-1}A_t & 2MLA_t \\
-2(G_{tt})^{-1}A_t^T & (G_{tt})^{-1} & -2(G_{tt})^{-1}A_t^TLA_t \\
2A_t^TLM + 4(G_{tt})^{-1}A_t^T(A_t^TMLA_t) & -2(G_{tt})^{-1}A_t^TLA_t & G_{tt} + 4A_t^TMLMLA_t + 4(G_{tt})^{-1}(A_t^TMLA_t)^2
\end{pmatrix}$$

(2.10)

and

$$\bar{L} = \begin{pmatrix}
L & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$$

(2.11)

where $\psi$ is obtained by dualising the gauge fields. The duality relation between the gauge fields and $\psi$ is

$$\sqrt{\text{det} \bar{G} e^{-\bar{\Phi}(\bar{M}\bar{L})_{\bar{a}\bar{b}}}G^{\bar{i}\bar{j}}G_j^{(b)}\bar{F}_{\bar{i}\bar{j}}^{(b)} = \frac{1}{2} \epsilon_{ijk} \partial_k \psi^\bar{a}},$$

(2.12)

and the Bianchi identity for the field strength can now be written in terms of $\psi$ as

$$\bar{D}^i(e^{\bar{\Phi}(\bar{M}\bar{L})_{\bar{a}\bar{b}}}\partial_i \psi^\bar{b})) = 0.$$  

(2.13)

In the above $\bar{a}$ takes values from 1 to 30. The corresponding three dimensional action from which these equations can be derived is given by

$$S_3 = \int d^3x \sqrt{\text{det} \bar{g}} [R_{\bar{a}} + \frac{1}{8} \bar{g}^{ij}Tr(\partial_i \bar{M}L \partial_j \bar{M}L)].$$

(2.14)

This action is invariant under the $O(8,24)$ transformation

$$\mathcal{M} \rightarrow \bar{\Omega}\mathcal{M}\bar{\Omega}^T, \quad \bar{g}_{ij} \rightarrow \bar{g}_{ij},$$

(2.15)
where $\tilde{\Omega}$ is a $32 \times 32$ matrix which leaves $\mathcal{L}$ invariant.

Transformation that diagonalises $\mathcal{L}$ is,

$$U = \begin{pmatrix}
I_{28} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}.$$  \hspace{1cm} (2.16)

In the next section we will work in the frame where $\mathcal{L}$ is diagonal. This choice, though, is just a matter of convenience.

3. Dyonic Black Hole

In this section we will give the explicit ansatz of $\Omega$ the matrices used for duality rotation of $\mathcal{M}$. We will use the strategy outlined in the previous section to get the matrix valued scalar field $\mathcal{M}$ starting from the Kerr solution. After implementing the transformations we will extract the expressions for various charges characterising the transformed solution.

Using the duality transformations given in the last section we find that the Kerr solution can be written as,

$$\bar{g}_{ij} dx^i dx^j = \left( \rho^2 + a^2 \cos^2 \theta - 2m\rho \right) \left[ \frac{1}{\rho^2 + a^2 - 2m\rho} d\rho^2 + d\theta^2 \\
+ \frac{\rho^2 + a^2 - 2m\rho}{\rho^2 + a^2 \cos^2 \theta - 2m\rho} \sin^2 \theta d\phi^2 \right]$$

$$\psi^a = \delta_{a,30} \frac{2ma \cos \theta}{\rho^2 + a^2 \cos^2 \theta};$$

and

$$\mathcal{M} = \begin{pmatrix}
I_{28} & 0 & 0 & 0 & 0 \\
0 & -f^{-1} & 0 & 0 & -g \\
0 & 0 & -f - fg^2 & g & 0 \\
0 & 0 & g & -f^{-1} & 0 \\
0 & -g & 0 & 0 & -f - fg^2 \\
\end{pmatrix}.$$  \hspace{1cm} (3.2)

$$\tilde{\mathcal{M}} = U \mathcal{M} U^T = \begin{pmatrix}
I_{28} & 0 & 0 & 0 & 0 \\
0 & A - 1 & B & 0 & g \\
0 & B & A - 1 & -g & 0 \\
0 & 0 & -g & A - 1 & B \\
0 & g & 0 & B & A - 1 \\
\end{pmatrix},$$

5
where \( A = 1 - f^{-1}/2 - f(1 + g^2)/2, \ B = f(1 + g^2)/2 - f^{-1}/2 \) and

\[
f = \frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta - 2m\rho}, \quad g = \frac{2ma \cos \theta}{\rho^2 + a^2 \cos^2 \theta - 2m\rho}
\]

\[
A = \frac{-2m^2}{\rho^2 + a^2 \cos^2 \theta - 2m\rho}, \quad B = \frac{2m(m - \rho)}{\rho^2 + a^2 \cos^2 \theta - 2m\rho}.
\]

At this point let us also note down the asymptotic behaviour of \( A, B, \) and \( g \) as \( \rho \to \infty \)

\[
A \sim -\frac{2m^2}{\rho^2} + \ldots \quad B \sim -\frac{2m}{\rho} + \ldots \quad g \sim \frac{2ma \cos \theta}{\rho^2} + \ldots
\]

These expressions will be useful for determining the mass, angular momentum and the charges of the black hole. Matrices used for the duality rotations are

\[
\Omega_1 = \begin{pmatrix}
I_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & O_1 & O_2 & 0 & 0 & 0 & O_3 & 0 & O_4 & 0 \\
0 & O_5 & O_6 & 0 & 0 & 0 & O_7 & 0 & 0 & 0 \\
0 & 0 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_1 & P_2 & 0 & P_3 & 0 & P_4 \\
0 & 0 & 0 & 0 & P_5 & P_6 & 0 & P_7 & 0 & 0 \\
0 & O_8 & P_8 & 0 & 0 & 0 & Q_1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & Q_3 & Q_4 & 0 & Q_5 & 0 & Q_6 \\
0 & 0 & 0 & 0 & Q_7 & Q_8 & 0 & Q_9 & 0 & Q_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Omega_2 = \begin{pmatrix}
R_{22} & 0 & 0 \\
0 & R_6 & 0 \\
0 & 0 & I_4
\end{pmatrix},
\]

where,

\[
O_1 = \cosh \beta \cos T, \quad O_2 = \cosh \alpha \cosh \beta \sin T, \quad O_3 = \sinh \alpha \cosh \beta \sin T \\
O_4 = \sinh \beta, \quad O_5 = -\sin T, \quad O_6 = \cosh \alpha \cos T, \quad O_7 = \sinh \alpha \cos T \\
O_8 = \sinh \beta \sin u \cos T, \quad O_9 = \sinh \beta \cos u \cos T
\]

\[
P_1 = \cosh \delta \cos R, \quad P_2 = \cosh \gamma \cosh \delta \sin R, \quad P_3 = \sinh \gamma \cosh \delta \sin R \\
P_4 = \sinh \delta, \quad P_5 = -\sin R, \quad P_6 = \cosh \gamma \cos R, \quad P_7 = \sinh \gamma \cos R \\
P_8 = \cosh \alpha \sinh \beta \sin u \sin T + \sinh \alpha \cos u \\
P_9 = \cosh \alpha \sinh \beta \cos u \sin T - \sinh \alpha \sin u
\]

(3.8)
\[ Q_1 = \sinh \alpha \sinh \beta \sin u \sin T + \cosh \alpha \cos u, \quad Q_2 = \cosh \beta \sin u \]
\[ Q_3 = \sinh \gamma, \quad Q_4 = \cosh \gamma, \quad Q_5 = \sinh \alpha \sinh \beta \cos u \sin T - \cosh \alpha \sin u \]
\[ Q_6 = \cosh \beta \cos u, \quad Q_7 = \sinh \delta \cos R, \quad Q_8 = \cosh \gamma \sinh \delta \sin R \]
\[ Q_9 = \sinh \gamma \sinh \delta \sin R, \quad Q_{10} = \cosh \delta \]

and \( R_N \) denotes any \( N \)-dimensional rotation matrix. Some comments regarding the choice of these matrices are in order. The choice of the matrix \( \Omega_1 \) is made in a following way. It was noticed earlier [11] that the most general transformation that preserves asymptotic form of all the field configurations and gives inequivalent solutions belongs to the coset

\[(O(22, 2) \times O(6, 2))/(O(22) \times O(6) \times SO(2)).\]  

(3.10)

We choose to perform this transformation in five steps. The matrix \( \Omega_1 \) contains four of them, two hyperbolic rotations corresponding to \( O(22, 2)/O(22) \) and similarly for \( O(6, 2)/O(6) \). These transformations are chosen in such a way that the electric and the magnetic charge vectors lie in a plane with non-zero dot and cross product. We then embed this plane in 22 and 6 directions by a general rotation performed by \( \Omega_2 \).

After doing the duality rotation of \( \mathcal{M} \) with respect to both \( \Omega \)s we can extract the expressions for various fields in the new solution. Most of these expressions are very lengthy and cumbersome \(^1\), but to determine exact expressions for the mass, the angular momentum and the charges it suffices to look at the asymptotic form of these fields. Substituting the asymptotic form of \( A, B \) and \( g \) in the expressions of \( G_{tt}, \Phi, \psi \) and \( A_t \) we find that the black hole solution obtained using (3.6) has

Mass:
\[
\text{Mass} = \frac{m}{2} \Delta^{1/2} 
\]

Angular Momentum:
\[
J = \frac{ma}{2\Delta^{1/2}} (\cosh \alpha \cosh \beta (s_r^2) + \cosh \gamma \cosh \delta (s_t^2) \\
+ (\cosh \alpha \cosh \gamma + \cosh \beta \cosh \delta)(\cosh \alpha \cosh \delta + \cosh \beta \cosh \gamma)) \]

(3.12)

\(^1\) Details will be published elsewhere\(^{13}\), however, the spherically symmetric case is discussed in detail in section (5).

\(^2\) notations are given in the appendix
Electric Charge: \((1 \leq n \leq 22 \text{ and } 1 \leq p \leq 6)\)

\[ Q_{el}^{(n)} = \frac{m}{\sqrt{2}} R_{22} \begin{pmatrix} 0_{20} \\ O_3 Q_4 \\ O_7 Q_4 \end{pmatrix} \]

\[ Q_{el}^{(p)} = \frac{m}{\sqrt{2}} R_{6} \begin{pmatrix} 0_4 \\ P_3 Q_1 + P_4 Q_2 \\ P_7 Q_1 \end{pmatrix} \]  \hspace{1cm} (3.13)

Magnetic charge: \((1 \leq n \leq 22 \text{ and } 1 \leq p \leq 6)\)

\[ Q_{mag}^{(n)} = \frac{m}{\sqrt{2}} R_{22} \begin{pmatrix} 0_{20} \\ O_3 Q_9 + O_4 Q_{10} \\ O_7 Q_9 \end{pmatrix} \]

\[ Q_{mag}^{(p)} = -\frac{m}{\sqrt{2}} R_{6} \begin{pmatrix} 0_4 \\ P_3 Q_5 + P_4 Q_6 \\ P_7 Q_5 \end{pmatrix} \]  \hspace{1cm} (3.14)

Electric dipole moment: \((1 \leq n \leq 22 \text{ and } 1 \leq p \leq 6)\)

\[ \mu_{el}^{(n)} = \frac{ma}{\sqrt{2}} R_{22} \begin{pmatrix} 0_{20} \\ O_4 Q_4 \\ 0 \end{pmatrix} \]

\[ \mu_{el}^{(p)} = \frac{ma}{\sqrt{2}} R_{6} \begin{pmatrix} 0_4 \\ P_3 Q_2 - P_4 Q_1 \\ P_7 Q_1 \end{pmatrix} \]  \hspace{1cm} (3.15)
Magnetic dipole moment: \(1 \leq n \leq 22\) and \(1 \leq p \leq 6\)

\[
\mu^{(n)}_{\text{mag}} = \frac{ma}{\sqrt{2}} R_{22} \begin{pmatrix}
0_{20} \\
O_3 Q_{10} - O_4 Q_9 \\
O_7 Q_{10} \\
0_4
\end{pmatrix}
\]

\[
\mu^{(p)}_{\text{mag}} = \frac{ma}{\sqrt{2}} R_6 \begin{pmatrix}
P_3 Q_6 - P_4 Q_5 \\
P_7 Q_6
\end{pmatrix}
\]

where \(R_{22}\) and \(R_6\) are 22 and 6 dimensional rotation matrices. It is known that when we use the equation (3.6) to get new solutions with the magnetic charge it also generates the NUT charge, which corresponds to the singularity in the \(G_{t\phi}\) component of the metric. Since we are interested in the non-singular metrics, the NUT charge of the new black hole should be zero. The condition for vanishing of the NUT charge is

\[
\tan u = \frac{\sinh \alpha \sinh \beta \cosh \gamma \sin T - \cosh \alpha \sinh \gamma \sinh \delta \sin R}{\sinh \alpha \sinh \beta \sinh \gamma \sinh \delta \sin T \sin R + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma}.
\]

(3.17)

All the charges as well as the moments carried by the black hole are subject to this constraint. We will implement this constraint in the rest of the paper. Since the \(G_{\rho\rho}\) part of the metric in the heterotic string frame is unaltered the space-time structure remains same as the original Kerr solution, i.e. this solution in general has two horizons which are located at

\[
\rho = m \pm \sqrt{m^2 - a^2}.
\]

(3.18)

The case \(a > m\) corresponds to the naked singularity whereas \(a \to m\) is the extremal limit. The curvature singularity is time-like and is concentrated on a ring at \(\phi = 0\) and \(\theta = \pi/2\).

4. Various Limits

We can take various limits of the black hole solution described in the previous section by changing parameters. In particular, we will see that in certain limit this solution can be reduced either to pure electrically charged black hole or pure magnetically charged black hole solution. We will examine the extremal limits in this section. We will show that in certain limits this solution saturates the Bogomolnyi bound. We should point out at this stage that saturation of Bogomolnyi bound is a necessary condition for the solution to be
supersymmetric, but it is yet not so clear if it is sufficient \[16\]. One would thus need to figure out the killing spinor in order confirm the solution to be supersymmetric. However, in most of the cases in the past, solutions saturating Bonomolnyi bound turned out to preserve partial supersymmetry.

Case I: $\alpha \to \infty$, $m \to 0$, $a \to 0$ such that $m \cosh \alpha \equiv m_0$ finite

$$\text{Mass} = \frac{m_0}{2}[N_1 (\cosh \gamma + \sinh \beta \sinh \gamma \sinh \delta \sin T \sin R)$$

$$\quad + N_2 (\sinh \beta \cosh \gamma \sin T - \sinh \gamma \sinh \delta \sin R)]$$

$$Q_{el}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \begin{pmatrix} 0_{20} \\ \cosh \beta \cosh \gamma \sin T \\ \cosh \gamma \cos T \end{pmatrix}$$

$$Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \begin{pmatrix} 0_{20} \\ \cosh \beta \sinh \gamma \sinh \delta \sin T \sin R \\ \sinh \gamma \sinh \delta \cos T \sin R \end{pmatrix}$$

$$Q_{el}^{(p)} = \frac{-m_0}{\sqrt{2}} R_6 \begin{pmatrix} 0_{4} \\ N_1 \sinh \gamma \cosh \delta \sin R + N_2 \sinh \beta \sinh \gamma \cosh \delta \sin T \sin R \\ N_1 \sinh \gamma \cosh \delta \sin R + N_2 \sinh \beta \sinh \gamma \sin T \cos R \end{pmatrix}$$

$$Q_{mag}^{(p)} = \frac{-m_0}{\sqrt{2}} R_6 \begin{pmatrix} 0_{4} \\ N_1 \sinh \beta \sinh \gamma \cosh \delta \sin T \sin R - N_2 \sinh \gamma \cosh \delta \sin R \\ N_1 \sinh \beta \sinh \gamma \sin T \cos R - N_2 \sinh \gamma \cos R \end{pmatrix}$$

where in the above and in the rest of the paper we have defined $N_1 = ((s_t)(s_r) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma)\Delta^{-1/2}$ and $N_2 = ((s_t) \cosh \gamma - (s_r) \cosh \alpha)\Delta^{-1/2}$. All other quantities like electric and magnetic moments, angular momentum vanish since $a$ is taken to zero. In this limit, the relation between the mass and the charges is given by

$$\text{(Mass)}^2 = \frac{1}{2}[(Q_{el}^{(n)})^2 + (Q_{mag}^{(n)})^2]$$

(4.3)
and \[ (Q_{el}^{(p)})^2 + (Q_{mag}^{(p)})^2 \] vs \[ (Q_{el}^{(n)})^2 + (Q_{mag}^{(n)})^2 \], therefore case I does not lead to a black hole preserving half supersymmetry. Nevertheless, it can, in principle, preserve 1/4 supersymmetry. The mass formula for the latter case\[17\] in our context is given by

\[
M_{BPS}^2 = Q_{el}^{(p)T} Q_{el}^{(p)} + Q_{mag}^{(p)T} Q_{mag}^{(p)} + 2[Q_{el}^{(p)T} Q_{el}^{(p)} Q_{mag}^{(p)T} Q_{mag}^{(p)} - (Q_{el}^{(p)T} Q_{mag}^{(p)})^2]^{\frac{1}{2}}. \tag{4.4}
\]

We find that this configuration does not satisfy the 1/4 supersymmetry constraint as well. Thus, we conclude that this black hole breaks all the supersymmetry.

**Case II:** \( \beta \to \infty \), \( m \to 0 \), \( a \to 0 \) such that \( m \cosh \beta \equiv m_0 \) finite

\[
\text{Mass} = \frac{m_0}{2} \left[ N_1 (\cosh \delta + \sinh \alpha \sinh \gamma \sin \delta \sin T \sin R) \right. \\
\left. + N_2 \sinh \alpha \cosh \gamma \sin T \right]
\]

\[
Q_{el}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \begin{pmatrix}
0_{20} \\
\sinh \alpha \cosh \gamma \sin T \\
0_{20}
\end{pmatrix}, \tag{4.5}
\]

\[
Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \begin{pmatrix}
\cosh \delta + \sinh \alpha \sinh \gamma \sin \delta \sin T \sin R \\
0 \\
0_{4}
\end{pmatrix},
\]

\[
Q_{el}^{(p)} = \frac{m_0 N_2}{\sqrt{2}} R_{6} \begin{pmatrix}
(\sinh \delta + \sinh \alpha \sinh \gamma \cosh \delta \sin T \sin R) \\
\sinh \alpha \sinh \gamma \sin T \cos R \\
0_{4}
\end{pmatrix}, \tag{4.6}
\]

\[
Q_{mag}^{(p)} = \frac{-m_0 N_1}{\sqrt{2}} R_{6} \begin{pmatrix}
0_{4} \\
(\sinh \alpha \sinh \gamma \cosh \delta \sin T \sin R + \sinh \delta) \\
\sinh \alpha \sinh \gamma \sin T \cos R
\end{pmatrix}.
\]

All other physical quantities vanish. In this limit, the relation between the mass and the charges is again given by

\[
(Mass)^2 = \frac{1}{2} [(Q_{el}^{(n)})^2 + (Q_{mag}^{(n)})^2], \tag{4.7}
\]

11
and \[\left( Q_{el}^{(p)} \right)^2 + \left( Q_{mag}^{(p)} \right)^2 \right] < \left( Q_{el}^{(n)} \right)^2 + \left( Q_{mag}^{(n)} \right)^2 \], therefore this limit too does not correspond to a black hole preserving half supersymmetry. It is easy to check that it also fails to satisfy the general BPS mass formula (4.4) and hence breaks all supersymmetries.

**Case III:** \( \gamma \to \infty, \; m \to 0, \; a \to 0 \) such that \( m \cosh \gamma \equiv m_0 \) finite

The nonzero physical quantities are:

\[
\text{Mass} = \frac{m_0}{2} \left[ N_1 (\cosh \alpha + \sinh \alpha \sinh \beta \sin \delta \sin T \sin R) \\
+ N_2 (\sinh \alpha \sinh \beta \sin T - \cosh \alpha \sinh \delta \sin R) \right]
\]

\[
Q_{el}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \left( \begin{array}{c} 0_{20} \\
\sinh \alpha \cosh \beta \sin T \\
\sinh \alpha \cos T \end{array} \right),
\]

\[
Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \left( \begin{array}{c} 0_{20} \\
\sinh \alpha \cosh \beta \sinh \delta \sin T \sin R \\
\sinh \alpha \sinh \delta \sin R \cos T \end{array} \right)
\]

\[
Q_{el}^{(p)} = \frac{m_0}{\sqrt{2}} R_{6} \left( \begin{array}{c} 0_{4} \\
N_1 \cosh \alpha \cosh \delta \sin R \\
+ N_2 \sinh \alpha \sinh \beta \cosh \delta \sin T \sin R \\
N_1 \cosh \alpha \cos R \\
+ N_2 \sinh \alpha \sinh \beta \sin T \cos R \end{array} \right)
\]

\[
Q_{mag}^{(p)} = \frac{-m_0}{\sqrt{2}} R_{6} \left( \begin{array}{c} 0_{4} \\
N_1 \sinh \alpha \sinh \beta \cosh \delta \sin T \sin R \\
- N_2 \cosh \alpha \cosh \delta \sin R \\
N_1 \sinh \alpha \sinh \beta \sin T \cos R \\
- N_2 \cosh \alpha \cos R \end{array} \right)
\]

In this limit, the relation between the mass and the charges is

\[
(Mass)^2 = \frac{1}{2} \left( Q_{el}^{(p)} \right)^2 + \left( Q_{mag}^{(p)} \right)^2,
\]

therefore in the extremal limit this solution corresponds to a BPS saturated black hole carrying both electric and magnetic charge. This is an extremal dyonic black hole which is specified by 57 parameters, 28 dimensional electric charge vector, 28 dimensional magnetic charge vectors and the mass of the black hole.
Let us study this limit in more details. If we take the limit \( \sin R \to 0 \), this solution reduces to a pure electrically charged solution, i.e., \((\text{Mass})^2 = (Q_{el}^{(p)})^2 / 2\). If we also take \( \sin T \to 0 \) we recover the supersymmetric solution (case II) of [11].

Case IV: \( \delta \to \infty, \ m \to 0, \ a \to 0 \) such that \( m \cosh \delta \equiv m_0 \) finite.

The non-zero physical quantities are:

\[
\begin{align*}
\text{Mass} &= \frac{m_0}{2} [N_1 (\cosh \beta + \sinh \alpha \sinh \beta \sinh \gamma \sin T \sin R) \\
&\quad - N_2 \cosh \alpha \sinh \gamma \sin R]
\end{align*}
\]

\[
Q_{el}^{(n)} = 0, \quad Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \left( \begin{array}{c} 0_2 \cr \sinh \alpha \cosh \beta \sinh \gamma \sin T \sin R + \sinh \beta \cr \sinh \alpha \sinh \gamma \sin R \cos T \end{array} \right)
\]

\[
Q_{el}^{(p)} = 0, \quad Q_{mag}^{(p)} = -\frac{m_0}{\sqrt{2}} R_6 \left( \begin{array}{c} N_1 (\sinh \alpha \sinh \beta \sinh \gamma \sin T \sin R + \cosh \beta) \\
- N_2 \cosh \alpha \sinh \gamma \sin R \cr 0 \end{array} \right)
\]

In this limit, the relation between the mass and the charges is

\[
(Mass)^2 = \frac{1}{2} (Q_{mag}^{(p)})^2,
\]

which satisfies the Bogomolnyi bound for mass and therefore in the extremal limit this solution may lead to a supersymmetric black hole carrying only magnetic charge. This is an extremal magnetically charged black hole which is specified by 28 dimensional magnetic charge vectors and the mass of the black hole.

Case V: \( \alpha = \gamma \to \infty, \ m \to 0, \ a \to 0 \) such that \( m \cosh^2 \alpha \equiv m_0 \) finite.

The non-zero physical quantities in this limit are:

\[
\begin{align*}
\text{Mass} &= \frac{m_0}{2} [N_1 (1 + \sinh \beta \sinh \delta \sin T \sin R) \\
&\quad + N_2 (\sinh \beta \sin T - \sinh \delta \sin R)]
\end{align*}
\]

\[
Q_{el}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \left( \begin{array}{c} 0_2 \cr \cosh \beta \sin T \cr \cos T \end{array} \right), \quad Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2}} R_{22} \left( \begin{array}{c} 0_2 \cr \cosh \beta \sinh \delta \sin T \sin R \cr \sinh \delta \sin R \cos T \end{array} \right)
\]

13
\[
Q_{el}^{(p)} = \frac{m_0}{\sqrt{2} R_6} \begin{pmatrix}
N_1 \cosh \delta \sin R + N_2 \sinh \beta \cosh \delta \sin T \sin R \\
n_1 \cos R + N_2 \sinh \beta \sin T \cos R
\end{pmatrix}^{04}
\]

\[
Q_{mag}^{(p)} = -\frac{m_0}{\sqrt{2} R_6} \begin{pmatrix}
N_1 \sinh \beta \cosh \delta \sin T \sin R - N_2 \cosh \delta \sin R \\
n_1 \sinh \beta \sin T \cos R - N_2 \cos R
\end{pmatrix}^{04}.
\]

In this limit, the relation between the mass and the charges is

\[
(Mass)^2 = \frac{1}{2} [(Q_{el}^{(p)})^2 + (Q_{mag}^{(p)})^2] = \frac{1}{2} [(Q_{el}^{(n)})^2 + (Q_{mag}^{(n)})^2],
\]

therefore again this is BPS saturated and may lead to a supersymmetric dyonic black hole carrying both electric and magnetic charge. This is an extremal dyonic black hole which is specified by 57 parameters, 28 dimensional electric charge vector, 28 dimensional magnetic charge vectors and the mass of the black hole.

*case VI: \(\beta = \delta \rightarrow \infty\), \(m \rightarrow 0\), \(a \rightarrow 0\) such that \(m \cosh^2 \beta \equiv m_0\) finite

\[
\text{Mass} = \frac{m_0 N_1}{2} (1 + \sinh \alpha \sinh \gamma \sin T \sin R)
\]

\[
Q_{el}^{(n)} = 0, \quad Q_{mag}^{(n)} = \frac{m_0}{\sqrt{2} R_{22}} \begin{pmatrix}
1 + \sinh \alpha \sinh \gamma \sin T \sin R \\
0
\end{pmatrix}^{020}
\]

\[
Q_{el}^{(p)} = 0 \quad Q_{mag}^{(p)} = -\frac{m_0}{\sqrt{2} R_6} \begin{pmatrix}
1 + \sinh \alpha \sinh \gamma \sin T \sin R \\
0
\end{pmatrix}^{04}.
\]

All the other physical quantities such as electric and magnetic moments and angular momentum vanish. In this limit, the relation between the mass and the charges is

\[
(Mass)^2 = \frac{1}{2} (Q_{mag}^{(p)})^2 = \frac{1}{2} (Q_{mag}^{(n)})^2.
\]

Here also mass saturates the Bogomolnyi bound and hence this solution is a potential candidate for a supersymmetric black hole carrying only magnetic charge. This is an extremal magnetically charged black hole which is specified by 28 dimensional magnetic charge vectors and the mass of the black hole.
5. Non-rotating Dyonic Black Hole

In this section we give the explicit solutions for the non-rotating dyonic black holes by restricting the Kerr metric to the case $a = 0$. After carrying out the duality transformations, as discussed earlier, we find the metric to be

$$G_{\mu\nu}dx^\mu dx^\nu = G_{tt}dt^2 + \left(\frac{\rho^2 - 2m\rho(1 - b\Delta) + m^2(a\Delta - 2b\Delta)}{\rho^2 - 2m\rho}\right)(d\rho^2 + (\rho^2 - 2m\rho)(d\theta^2 + \sin^2\theta d\phi^2))$$

where (notations used in this section are explained in the appendix)

$$G_{tt} = -\frac{(\rho^2 - 2m\rho)(\rho^2 - 2m\rho(1 - b\Delta) + m^2(a\Delta - 2b\Delta))}{\rho^4 + 2m\rho^3d_3 + m^2\rho^2d_2 + 2m^3\rho d_1 + m^4d_0}$$

and the dilaton is given by

$$\Phi = \frac{1}{2} \ln \left(\frac{\rho^4 - 4m\rho^3n + 2m^2\rho^2n_2 - 4m^3\rho n_1 + m^4n_0}{\rho^4 + 2m\rho^3d_3 + m^2\rho^2d_2 + 2m^3\rho d_1 + m^4d_0}\right) = \frac{1}{2} \ln \frac{\Phi_n}{\Phi_d}.$$  \hspace{1cm} (5.1)

The time components of the gauge fields are

$$A_t^{(n)} = \frac{R_{22}}{\sqrt{2}\Phi_d} \left(\begin{array}{c} 0_{20} \\ G_nM_1 - G_p\psi_1 \\ G_nM_2 - G_p\psi_2 \end{array}\right)$$

$$A_t^{(p)} = \frac{R_{6}}{\sqrt{2}\Phi_d} \left(\begin{array}{c} 0_4 \\ G_nM_3 - G_p\psi_3 \\ G_nM_4 - G_p\psi_4 \end{array}\right)$$

where,

$$G_n = \rho^2 - 2m\rho(1 - b\Delta) + m^2(a\Delta - 2b\Delta)$$

$$G_p = m^2(a - b_1 + \cosh \gamma(s_r)) + m\rho b_1$$

with

$$M_1 = -\frac{m^2}{\Delta^{1/2}} \{(s_i) \coth \beta(\cosh \alpha \cosh \beta \cosh \delta + \cosh^2 \alpha \cosh \gamma + (s_i^2) \cosh \gamma)$$

$$+ \cosh \beta \sinh \beta(\cosh \gamma(s_i) - \cosh \alpha(s_r))\} + m(m - \rho)(s_i) \cosh \gamma \coth \beta$$

$$M_2 = \cos T \sinh \alpha \{-\frac{m^2}{\Delta^{1/2}} (\cosh \alpha \cosh \beta \cosh \delta + \cosh^2 \alpha \cosh \gamma)$$

$$+ (s_i^2) \cosh \gamma) + m(m - \rho) \cosh \gamma\}$$

$$M_3 = -m^2(s_r) \cosh \gamma \coth \delta + \frac{m(m - \rho)}{\Delta^{1/2}} \{\coth \delta(s_r)(\cosh \alpha \cosh \beta \cosh \delta$$

$$+ \cosh^2 \alpha \cosh \gamma + \cosh \gamma(s_i^2) + \cosh \beta \sinh \beta(\cosh \gamma(s_i) - \cosh \alpha(s_r))\})$$

$$M_4 = \cos R \sinh \gamma \{-m^2 \cosh \gamma + \frac{m(m - \rho)}{\Delta^{1/2}} (\cosh \alpha \cosh \beta \cosh \delta$$

$$+ \cosh^2 \alpha \cosh \gamma + \cosh \gamma(s_i^2))\}.$$
\[ \psi_1 = -m^2(b_4 \cosh \beta \sinh \beta + b_5 \coth \beta) + m(m - \rho) \sinh \beta \cosh \delta \\
+ (s_t s_r) \coth \beta \]

\[ \psi_2 = \cos T \sinh \alpha \left[ -\frac{m^2 b_1}{2 \cosh \gamma} + m(m - \rho)(s_r) \right] \]

\[ \psi_3 = -m^2(\cosh \delta \sinh \delta (1 + \sinh^2 \gamma \sin^2 R)) + m(m - \rho)(b_2 \tanh \delta \\
+ b_3 \coth \delta) \]

\[ \psi_4 = \cos R \left[ -m^2 \sinh \gamma (s_r) + \frac{m(m - \rho)b_1}{2} \tanh \gamma \right]. \]
\[ H^{\mu\nu\lambda} = -\exp(2\Phi)(-G)^{-1/2} \epsilon^{\mu\nu\lambda\sigma} \partial_{\sigma} \chi \] and is given by

\[
\chi = \frac{\sin^2 \theta}{4\sqrt{2}} \int \frac{d\rho}{(\rho^2 - 2m\rho)^2} \sqrt{\Phi} \left[ (G_n M_1 - G_p \psi_1)(-\bar{M}_1 \Psi_1 - \bar{M}_2 \Psi_2 + \bar{M}_3 \Psi_3 + \bar{M}_4 \Psi_4 + \bar{M}_5 \Psi_5 + \bar{M}_6 \Psi_6) \\
+ (G_n M_2 - G_p \psi_2)(-\bar{M}_2 \Psi_1 - \bar{M}_5 \Psi_2 + \bar{M}_8 \Psi_3 + \bar{M}_9 \Psi_4 + \bar{M}_{10} \Psi_5 + \bar{M}_{11} \Psi_6) \\
- (G_n M_3 - G_p \psi_3)(-\bar{M}_3 \Psi_1 - \bar{M}_8 \Psi_2 + \bar{M}_{12} \Psi_3 + \bar{M}_{13} \Psi_4 + \bar{M}_{14} \Psi_5 + \bar{M}_{15} \Psi_6) \\
- (G_n M_4 - G_p \psi_4)(-\bar{M}_4 \Psi_1 - \bar{M}_9 \Psi_2 + \bar{M}_{13} \Psi_3 + \bar{M}_{16} \Psi_4 + \bar{M}_{17} \Psi_5 + \bar{M}_{18} \Psi_6) \right] \] (5.11)

The electric and magnetic charges are identical to those mentioned in the section (3) however, both electric and magnetic dipole moments as well as the angular momentum vanish. The Einstein metric is given by

\[ g_{\mu\nu} dx^\mu dx^\nu = e^{-\Phi} G_{\mu\nu} dx^\mu dx^\nu = \left( \frac{\Phi_d}{\Phi_n} \right)^{1/2} (G_{tt} \, dt^2 + \frac{G_n}{\rho^2 - 2m\rho} (d\rho^2 + (\rho^2 - 2m\rho)(d\theta^2 + \sin^2 \theta d\phi^2))) \] (5.12)

Using this metric it is easy to see that the event horizon is located at \( \rho = 2m \) and the area of the event horizon is given by

\[
A = \int d\theta d\phi \left( \frac{\Phi_d}{\Phi_n} \right)^{1/2} G_n \sin \theta \big|_{\rho=2m} \\
= 4\pi m^2 (4 \cosh \alpha \cosh \beta \cosh \gamma \cosh \delta - 2c_4 - c_3)^{1/2}. \] (5.13)

The surface gravity is of the black hole is given by,

\[
\kappa = \lim_{\rho \to 2m} \frac{\sqrt{g_{\rho\rho} \partial_{\rho} \sqrt{-g_{tt}}}}{\sqrt{-g_{tt}}} = \frac{1}{m(4 \cosh \alpha \cosh \beta \cosh \gamma \cosh \delta - 2c_4 - c_3)^{1/2}}. \] (5.14)

In the limit \( m \to 0 \), event horizon touches the singularity. Various limits of the boost parameters can be considered for this solution, which is similar in spirit to previous section. All the limiting cases of section (3) are trivially applicable to this solution. This solution therefore represents a non-rotating black hole solution carrying 28 electric and 28 magnetic charges. In the limit \( \beta, \delta, R \) and \( T \to 0 \) our results reduces to the results those of \[11\] in the \( a \to 0 \) limit.
6. Conclusion

In this paper, we have presented a general black hole solution to the heterotic string theory compactified on a six dimensional torus. We also studied the solution in various limits and we find that in certain cases the black hole mass saturates the Bogomolnyi bound. One of the motivations behind constructing this solution is to study if black hole can have elementary particle like behaviour. We studied the non-rotating black hole in detail. All the limits studied in the earlier sections also apply to this solution as well. We thus find the most general non-rotating black hole solution in the heterotic string theory on a torus which carries 28 electric and 28 magnetic charges.

While preparing this manuscript, a recent preprint [18] appeared which addresses similar issues.

Acknowledgements: We would like to thank A. Sen for several discussions and R. Kallosh for a fruitful communication. SP acknowledges the hospitality of ICTP, Trieste, during which this work was initiated. Work of SM was partially supported by NSF Grant No. PHY-9411543
7. Appendix

This appendix contains the shorthand notation used in the paper:

\[(s_t) = \sinh \alpha \sinh \beta \sin T \quad (s_r) = \sinh \gamma \sinh \delta \sin R\]

\[
\Delta = (s_t^2 s_r^2) + 2 \cosh \beta \cosh \delta (s_t s_r) + \cosh^2 \alpha (s_r^2)
\]

\[
+ \cosh^2 \gamma (s_t^2) + (\cosh \alpha \cosh \gamma + \cosh \beta \cosh \delta)^2
\]

\[
a_\Delta = \Delta^{-1}[(s_t^4 s_r^2) + (s_t^2 s_r^4) + \cosh^2 \alpha (s_r^4) + (s_t^2 s_r^2)(2 \cosh^2 \alpha + \cosh^2 \beta
\]

\[
+ \cosh^2 \gamma + \cosh^2 \delta) + 2 \cosh \beta \cosh \delta (s_t^3 s_r + s_t s_r^3)
\]

\[
+ (\cosh^2 \beta \cosh^2 \delta + \cosh^2 \delta \cosh^2 \gamma) (s_t^2) + (\cosh^4 \alpha + \cosh^2 \alpha \cosh^2 \delta
\]

\[
+ \cosh^2 \alpha \cosh^2 \gamma + \cosh^2 \beta \cosh^2 \delta + 2 \cosh \alpha \cosh \beta \cosh \gamma \cosh \delta (s_r^2)
\]

\[
+ 2 \cosh \beta (\cosh^2 \alpha \cosh \delta + \cosh \alpha \cosh \beta \cosh \gamma + \cosh^2 \beta \cosh \delta
\]

\[
+ \cosh^3 \delta) (s_t s_r) + (\cosh^2 \alpha \cosh^2 \gamma + \cosh^2 \beta \cosh^2 \delta) (\cosh^2 \beta + \cosh^2 \delta)
\]

\[
+ 2 \cosh \alpha \cosh^3 \beta \cosh \gamma \cosh \delta + 2 \cosh \alpha \cosh \beta \cosh \gamma \cosh^3 \delta]
\]

\[
b_\Delta = \Delta^{-1/2}[(s_t^2 s_r^2) + 2 \cosh \beta \cosh \delta (s_t s_r)
\]

\[
+ \cosh^2 \alpha (s_r^2) + (\cosh \alpha \cosh \gamma + \cosh \beta \cosh \delta \cosh \beta \cosh \delta]
\]

\[
c_1 = \cosh^2 \alpha + \cosh^2 \beta + \cosh^2 \gamma + \cosh^2 \delta + (s_t^2) + (s_r^2)
\]

\[
+ 4 \cosh \alpha \cosh \beta \cosh \gamma \cosh \delta \quad c_2 = \Delta^{1/2}
\]

\[
c_3 = \Delta^{-1}[- \cosh^2 \gamma \cosh^2 \delta (s_t^4) - (2 \cosh^2 \beta \cosh^2 \gamma \cosh^2 \delta
\]

\[
+ \cosh^2 \alpha \cosh^2 \beta \cosh^2 \gamma + \cosh^4 \gamma \cosh^2 \delta
\]

\[
+ 2 \cosh \alpha \cosh \beta \cosh \gamma \cosh^3 \delta + 2 \cosh^2 \alpha \cosh^2 \gamma \cosh^2 \delta) (s_t^2)
\]

\[
- (2 \cosh^2 \alpha \cosh^2 \beta \cosh^2 \delta + \cosh^4 \alpha \cosh^2 \beta + \cosh^2 \alpha \cosh^2 \gamma \cosh^2 \delta
\]

\[
+ 2 \cosh \alpha \cosh^3 \beta \cosh \gamma \cosh \delta + 2 \cosh^2 \alpha \cosh^2 \beta \cosh^2 \gamma) (s_r^2)
\]

\[
+ (4 \cosh \alpha \cosh^2 \beta \cosh \gamma \cosh^2 \delta - 2 \cosh^2 \alpha \cosh^3 \beta \cosh \delta
\]

\[
- 2 \cosh \beta \cosh^2 \gamma \cosh^3 \delta) (s_t s_r) - (\cosh^2 \alpha \cosh^2 \beta + \cosh^2 \beta \cosh^2 \gamma
\]

\[
+ \cosh^2 \gamma \cosh^2 \delta + \cosh \delta \cosh^2 \alpha \cosh \gamma \cosh \beta \cosh \delta)^2
\]

\[
- \cosh^2 \alpha \cosh^2 \beta (s_t^4) - (\cosh^2 \alpha \cosh^2 \beta + \cosh^2 \gamma \cosh^2 \delta) (s_t^2 s_r^2)\]

19
\[ c_4 = -\Delta^{-1/2}(\cosh \alpha \cosh \beta + \cosh \gamma \cosh \delta)[\cosh \alpha \cosh \beta(\cosh^2 \gamma \\
+ \cosh^2 \delta + (s_t^2)) + \cosh \gamma \cosh \delta(\cosh^2 \alpha + \cosh^2 \beta + (s_t^2))] \]

\[ d_0 = 2c_4 - c_3 + 4 \cosh \alpha \cosh \beta \cosh \gamma \cosh \delta \quad d_1 = 2c_2 - c_1 - c_4 \quad (7.2) \]

\[ d_2 = 4 + c_1 - 6c_2 \quad d_3 = c_2 - 2 \quad n_0 = (a_\Delta - 2b_\Delta)^2 \]

\[ n_1 = a_\Delta + b_\Delta(2b_\Delta - a_\Delta - 2) \quad n_2 = 2 - 6b_\Delta + 2b_\Delta^2 + a_\Delta \quad n_3 = 1 - b_\Delta \]

\[ a_1 = \frac{1}{\Delta}[\cosh \gamma(s_t^4 s_r) + \cosh \beta \cosh \gamma \cosh \delta(s_t^3) \\
+ (2 \cosh^2 \alpha \cosh \gamma + \cosh \alpha \cosh \beta \cosh \delta + \cosh^2 \beta \cosh \gamma)(s_t^2 s_r) \\
- \cosh \alpha \cosh^2 \beta(s_t s_r^2) + (\cosh \alpha \cosh^2 \beta(\cosh^2 \gamma + \cosh^2 \delta) \\
+ \cosh \beta \cosh \gamma \cosh \delta(\cosh^2 \alpha + \cosh^2 \beta))(s_t) \\
+ \cosh \alpha(\cosh^2 \alpha - \cosh^2 \beta)(\cosh \alpha \cosh \gamma + \cosh \beta \cosh \delta)(s_r)] \]

\[ b_1 = \frac{2 \cosh \gamma}{\Delta^{1/2}}[(s_t^4 s_r) + \cosh^2 \alpha(s_r) + \cosh \beta \cosh \delta(s_t)] \quad (7.3) \]

\[ b_2 = \frac{1}{\Delta^{1/2}}[\cosh \beta \cosh \delta(\cosh \alpha \cosh \gamma + \cosh \beta \cosh \delta + (s_t s_r))] \]

\[ b_3 = \frac{1}{\Delta^{1/2}}[(s_t^2 s_r^2) + \cosh \beta \cosh \delta(s_t s_r) + \cosh \alpha(s_r^2)] \]

\[ b_4 = \frac{1}{\Delta^{1/2}}[(s_t s_r) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma] \]

\[ b_5 = \frac{1}{\Delta^{1/2}}[(s_t^3 s_r) + \cosh \beta \cosh \delta(s_t^2) + \cosh^2 \alpha(s_t s_r)] \]

\[ \tilde{M}_1 = 1 + \frac{2}{\rho^2 - 2m \rho}[ - m^2((s_t)^2 \coth^2 \beta + \sinh^2 \beta) + \frac{1}{G_n}[m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - m(m - \rho)(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta)^2] \]

\[ M_2 = \frac{1}{\rho^2 - 2m \rho}[ - 2m^2 \cosh \beta \sinh^2 \alpha \cos T \sin T \\
+ \frac{\cos T \sinh \alpha}{G_n}[m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - m(m - \rho) \\
(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta)][(m^2 \cosh \gamma b_1 + 2m(m - \rho)(s_r)]] \quad (7.4) \]

20
\[ M_3 = \frac{1}{\rho^2 - 2m\rho} [2m(m - \rho)(\sinh \beta \sinh \delta - \sin T \sin R \sinh \alpha \cosh \beta \sinh \gamma \cosh \delta) + \frac{1}{G_n} \{ m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - m(m - \rho) \} \] 
\[(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta)][-2m^2 \sinh \delta \cosh \delta (1 + \sinh^2 \gamma \sin^2 R) + 2m(m - \rho)(b_2 \tanh \delta + b_3 \coth \delta)]\] 
\[ M_4 = \frac{1}{\rho^2 - 2m\rho} [-2m(m - \rho) \sinh \alpha \cosh \beta \sinh \gamma \cos R \sin T + \cos R \frac{G_n}{G_n} \{ m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - m(m - \rho)(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta) \}[-2m^2 \sinh \gamma(s_r) + m(m - \rho) \tanh \gamma b_1] \] 
\[ (7.5) \] 
\[ M_5 = \frac{1}{\sqrt{2}(\rho^2 - 2m\rho)} [-2m(m - \rho) \sinh \alpha \cosh \beta \cosh \gamma \sin T - \frac{2m^2}{\sqrt{\Delta}} \{ \sinh \alpha \cosh \alpha \cosh \beta \sin T((s_t s_r) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma) + \cosh \beta \sinh \beta(1 + \sinh^2 \alpha \sin^2 T)(\cosh \gamma(s_t) - \cosh \alpha(s_r)) \} + m^2(a_1 - b_1 + \cosh \gamma(s_r)) + m \rho b_1 \} [2m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - 2m(m - \rho)(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta)] \] 
\[ (7.6) \] 
\[ M_6 = \frac{1}{\sqrt{2}(\rho^2 - 2m\rho)} [-2m(m - \rho) \sinh \alpha \cosh \beta \cosh \gamma \sin T - \frac{2m^2}{\sqrt{\Delta}} \{ \sinh \alpha \cosh \alpha \cosh \beta \sin T((s_t s_r) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma) + (\cosh \gamma(s_t) - \cosh \alpha(s_r))(\cosh \beta \sinh \beta + (s_t)^2 \coth \beta) \} + \frac{m^2(a_1 - \cosh \gamma(s_r))}{G_n} [2m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta) - 2m(m - \rho)(\sinh \beta \cosh \delta + (s_t s_r) \coth \beta)] \] 
\[ \] 
\[ M_7 = 1 + \frac{\cos^2 T \sinh^2 \alpha}{\rho^2 - 2m\rho} (-2m^2 + \frac{1}{2G_n} (-\frac{m^2}{\cosh \gamma}b_1 + 2m(m - \rho)(s_r))^2) \] 
\[ M_8 = \frac{2 \cos T \sinh \alpha}{\rho^2 - 2m\rho} [m(m - \rho) \sin R \sinh \gamma \cosh \delta + \frac{1}{2G_n} (\frac{m^2 b_2}{\cosh \gamma}) - \frac{2m(m - \rho)(s_r)}{G_n} (2m^2 \sinh \delta \cosh \delta(1 + \sinh^2 \gamma \sin^2 R) - m(m - \rho)(b_2 \tanh \delta + b_3 \coth \delta)] \]
\[
\begin{align*}
\bar{M}_9 &= \frac{\cos R \cos T \sinh \alpha \sinh \gamma}{\rho^2 - 2m\rho} \left[2m(m - \rho) + \frac{1}{2G_n} \left(\frac{m^2b_1}{\cosh \gamma} - 2m(m - \rho)(s_r)\right)\right] \left(2m^2(s_r) - \frac{m(m - \rho)b_1}{\cosh \gamma}\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{10} &= \frac{\sqrt{2} \cos T \sinh \alpha}{\rho^2 - 2m\rho} \left[m(m - \rho) \cosh \gamma - \frac{m^2}{\sqrt{\Delta}} \left(\cosh \alpha \cosh \beta \cosh \delta + \cosh^2 \alpha \cosh \gamma + \cosh \gamma (s_t)^2\right) + \frac{G_p}{2G_n} \left(\frac{m^2b_1}{\cosh \gamma} - 2m(m - \rho)(s_r)\right)\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{11} &= -\frac{\sqrt{2} \cos T \sinh \alpha}{\rho^2 - 2m\rho} \left[m(m - \rho) \cosh \gamma + \frac{m^2}{\sqrt{\Delta}} \left(\cosh \alpha \cosh \beta \cosh \delta + \cosh^2 \alpha \cosh \gamma + \cosh \gamma (s_t)^2\right) - \frac{1}{2G_n} \left[m^2(a_1 - \cosh \gamma (s_r))(\frac{m^2b_1}{\cosh \gamma} - 2m(m - \rho)(s_r))\right]\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{12} &= 1 - \frac{2m^2}{\rho^2 - 2m\rho} \left[(\sin^2 \delta + (s_r)^2 \coth^2 \delta) + \frac{1}{G_n} \left[m \cosh \delta \sinh \delta (1 + \sinh^2 \gamma \sin^2 R) - (m - \rho) (b_2 \tanh \delta + b_3 \coth \delta)\right]^2\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{13} &= \frac{\cos R}{\rho^2 - 2m\rho} \left[-2m^2 \sin R \sinh^2 \gamma \cosh \delta + \frac{1}{G_n} \left[m^2 \cosh \delta \sinh \delta (1 + \sinh^2 \gamma \sin^2 R) - m(m - \rho) (b_2 \tanh \delta + b_3 \coth \delta)\right]\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{14} &= -\frac{\sqrt{2}}{\rho^2 - 2m\rho} \left[2m^2 \sin R \sinh \gamma \cosh \gamma \cosh \delta - \frac{m(m - \rho)}{\sqrt{\Delta}} \left(\sin R \cosh \alpha \sinh \gamma \cosh \delta (s_r s_t) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma\right) + (\cosh \gamma (s_t) - \cosh \alpha (s_r))(\cosh \beta \sinh \delta + (s_t) \sin R \sinh \gamma \cosh \delta) + \frac{G_p}{G_n} \left(m^2 \cosh \delta \sinh \delta (1 + \sinh^2 \gamma \sin^2 R) + 2m(m - \rho)(b_2 \tanh \delta + b_3 \coth \delta)\right)\right]
\end{align*}
\]
\[
\begin{align*}
\bar{M}_{15} &= \frac{\sqrt{2}}{\rho^2 - 2m\rho} \left[m^2 \sin R \sinh \gamma \cosh \gamma \cosh \delta + \frac{m(m - \rho)}{\sqrt{\Delta}} \left(\sin R \cosh \alpha \sinh \gamma \cosh \delta (s_r s_t) + \cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma\right) + (\cosh \gamma (s_t) - \cosh \alpha (s_r))(\cosh \beta \sinh \delta + (s_t) \sin R \sinh \gamma \cosh \delta) + \frac{1}{G_n} \left(m^2(a_1 - \cosh \gamma (s_r))(m^2 \cosh \delta \sinh \delta (1 + \sinh^2 \gamma \sin^2 R) + 2m(m - \rho)(b_2 \tanh \delta + b_3 \coth \delta)\right)\right]
\end{align*}
\]
\[ M_{16} = 1 - \frac{2m^2 \cos^2 R \sinh^2 \gamma}{4G_n(\rho^2 - 2m\rho)} \left[ 4G_n - (2m \sin R \sinh \gamma - (m^2 - m\rho) \frac{b_1}{\cosh \gamma} \right]^2 \]

\[ M_{17} = \frac{\sqrt{2} \cos R \sinh \gamma}{\rho^2 - 2m\rho} \left[ -m^2 \cosh \gamma + \frac{m(m - \rho)}{\sqrt{\Delta}} (\cosh \gamma(s_i))^2 \right. \\
\left. + \cosh \alpha(\cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma) + \frac{G_p}{2G_n}(2m^2(s_r) - m(m - \rho) \frac{b_1}{\cosh \gamma} \right] \]

\[ M_{18} = \frac{\sqrt{2} \cos R \sinh \gamma}{\rho^2 - 2m\rho} \left[ m^2 \cosh \gamma + \frac{m(m - \rho)}{\sqrt{\Delta}} (\cosh \gamma(s_i))^2 \right. \\
\left. + \cosh \alpha(\cosh \beta \cosh \delta + \cosh \alpha \cosh \gamma) + \frac{1}{2G_n} m^2(a_1 - \cosh \gamma(s_r)) \right. \\
\left. (2m^2(s_r) - m(m - \rho) \frac{b_1}{\cosh \gamma} \right] \]

\[ \Psi_1 = \sqrt{2}[2m^2(b_4 \sinh \beta \cosh \beta + b_5 \coth \beta - \sinh \beta \cosh \delta - (s_ts_r) \coth \beta) \]

\[ (\rho^3 - m\rho^2(3 - b_\Delta) + 2m\rho(1 - b_\Delta)) + m\rho(\sinh \beta \cosh \delta + (s_ts_r) \coth \beta) \]

\[ (\rho^3 - 2m\rho^2 - m^2(\rho - 2m)(a_\Delta - 2b_\Delta)) \]

\[ \Psi_2 = \sqrt{2} \cos T \sinh \alpha[m^2 \rho^3(\frac{b_1}{\cosh \gamma} - 4(s_r)) + \rho^2 m^3(\frac{b_1}{\cosh \gamma}(b_\Delta - 3) \]

\[ + (s_r)(6 - a_\Delta) + m\rho^4(2 \frac{b_1}{\cosh \gamma}(1 - b_\Delta) + 2(s_r)(a_\Delta - 2) + m\rho^4(s_r)] \]

\[ \Psi_3 = \sqrt{2}[(b_2 \tanh \delta + b_3 \coth \delta)(\rho^4 m + 2\rho m^3(a_\Delta - 2) + m^3 \rho^2(6 - a_\Delta) \]

\[ - 4m^2 \rho^3) + 2 \cosh \delta \sinh \delta(1 + \sinh^2 \gamma \sin^2 R)(2m^4 \rho(1 - b_\Delta) \]

\[ + m^3 \rho^2(b_\Delta - 3) + m^2 \rho^3) \]

\[ \Psi_4 = \sqrt{2} \cos R[m^2 \rho^3(2 \sinh \gamma(s_r) - 2b_1 \tanh \gamma) + \rho^2 m^3(2 \sinh \gamma(s_r)(b_\Delta - 3) \]

\[ + \frac{b_1}{2} \tanh \gamma(6 - a_\Delta) + \rho m^4(4 \sinh \gamma(s_r)(1 - b_\Delta) + b_1 \tanh \gamma(a_\Delta - 2) \]

\[ + m\rho^4(b_1 \tanh \gamma) \]

\[ \Psi_5 = m\rho[b_1 \rho^3 + 2(a_1 - b_1 + \cosh \gamma(s_r)) \rho m^2 + m^2 \rho(2b_\Delta(a_1 + \cosh \gamma(s_r)) \]

\[ - b_1 a_\Delta - 6(a_1 - b_1 + \cosh \gamma(s_r)) + 2m^3(b_1 a_\Delta + 2(a_1 - b_1 + \cosh \gamma(s_r)) \]

\[ - 2b_\Delta(a_1 + \cosh \gamma(s_r))] \]

\[ \Psi_6 = 2m^2 \rho(a_1 - \cosh \gamma(s_r))[\rho^2 - m\rho(3 - b_\Delta) + 2m^2(1 - b_\Delta)] \]
\[ p_1 = (\bar{M}_1 - 1) + \frac{2(G_n M_1 - G_p \psi_1)^2}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ p_2 = \bar{M}_2 + \frac{2(G_n M_1 - G_p \psi_1)(G_n M_2 - G_p \psi_2)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ p_3 = (\bar{M}_7 - 1) + \frac{2(G_n M_2 - G_p \psi_2)^2}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ q_1 = \bar{M}_3 + \frac{2(G_n M_1 - G_p \psi_1)(G_n M_3 - G_p \psi_3)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ q_2 = \bar{M}_4 + \frac{2(G_n M_1 - G_p \psi_1)(G_n M_4 - G_p \psi_4)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ q_3 = \bar{M}_8 + \frac{2(G_n M_3 - G_p \psi_3)(G_n M_2 - G_p \psi_2)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ q_4 = \bar{M}_9 + \frac{2(G_n M_2 - G_p \psi_2)(G_n M_4 - G_p \psi_4)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ r_1 = (\bar{M}_{12} - 1) + \frac{2(G_n M_3 - G_p \psi_3)^2}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ r_2 = \bar{M}_{13} + \frac{2(G_n M_3 - G_p \psi_3)(G_n M_4 - G_p \psi_4)}{(\rho^2 - 2m\rho) \Phi_d G_n} \]

\[ r_3 = (\bar{M}_{16} - 1) + \frac{2(G_n M_4 - G_p \psi_4)^2}{(\rho^2 - 2m\rho) \Phi_d G_n} \]
References

[1] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, Nucl. Phys. B340 (1990) 33.
[2] A. Dabholkar and J. A. Harvey, Phys. Rev. Lett. 63 (1989) 478.
[3] C. G. Callan, J. A. Harvey and A. Strominger, Nucl. Phys. B359 (1991) 611.
[4] M. J. Duff, R. R. Khuri and J. X. Lu, String Solitons, hep-th/9412187 (submitted to Phys. Rep.).
[5] M. J. Duff and J. Rahmfeld, Phys. Lett. B345 (1995) 441.
[6] A. Sen Extremal Black Holes and Elementary String States, hep-th/9504147.
[7] C. G. Callan, J. M. Maldacena and A. W. Peet, Extremal Black Holes as Fundamental Strings, hep-th/9510134.
[8] A. Dabholkar, J. Gauntlett, J. Harvey and D. Waldram, hep-th/9511053.
[9] G. Mandal and S. R. Wadia, Black Hole Geometry around an elementary BPS String State, hep-th/9511218.
[10] A. Sen, Black Holes and Solitons in String Theory, hep-th/9210050; G. Horowitz, The Dark Side of String Theory: Black Holes and Black Strings, hep-th/9210119; R. R. Khuri, Black Holes and Solitons in String Theory, hep-th/9506065.
[11] A. Sen Black Hole Solutions in Heterotic String Theory on a Torus, TIFR-TH-94-47, hep-th/9411187.
[12] A. Sen, Phys. Lett. B271 (1991) 295.
[13] S. F. Hassan and A. Sen, Twisting Classical Solutions in Heterotic String Theory, TIFR-TH-94-47, hep-th/9109038.
[14] A. Sen. Strong-Weak Coupling Duality in Three Dimensional String Theory, hep-th/9408083.
[15] D. P. Jatkar, S. Mukherji and S. Panda, in preparation.
[16] R. Kallosh, private communication.
[17] M Cvetic and D. Youm, Dyonic BPS Saturated Black Holes of Heterotic string on a Six-torus, hep-th/9507090.
[18] M. Cvetic and D. Youm, All the static spherically symmetric Black Holes of Heterotic String on a Six Torus, IASSNS-HEP-95/107, hep-th/9512127.