EULER OBSTRUCTION OF ESSENTIALLY ISOLATED DETERMINANTAL SINGULARITIES

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Abstract. We study the Euler obstruction of essentially isolated determinantal singularities (EIDS). The EIDS were defined by W. Ebeling and S. Gusein-Zade [14], as a generalization of isolated singularity. We obtain some formulas to calculate the Euler obstruction for the determinantal varieties with singular set an ICIS.

1. Introduction

In this work, we study the Euler obstruction of essentially isolated determinantal singularities (EIDS). The EIDS have been defined by W. Ebeling and S. M. Gusein-Zade in [14]. A generic determinantal variety $M_{m,n}^t$ is the subset of the space of $m \times n$ matrices, given by the matrices of rank less than $t$, where $0 \leq t \leq \min\{m, n\}$. A variety $X \subset \mathbb{C}^N$ is determinantal if $X$ is the pre-image of $M_{m,n}^t$ by a holomorphic function $F : \mathbb{C}^N \to M_{m,n}$ with the condition that the codimension of $X$ in $\mathbb{C}^N$ is the same as the codimension of $M_{m,n}^t$ in $M_{m,n}$.

Determinantal varieties have isolated singularities if $N \leq (m-t+2)(n-t+2)$ and they admit smoothing if $N < (m-t+2)(n-t+2)$, see [19]. Several recent works investigate determinantal varieties with isolated singularities. The Milnor number of a determinantal surface was defined in [7, 16, 18] while the vanishing Euler characteristic of a determinantal variety was defined in [7, 16]. Other recent results on isolated determinantal varieties related to this paper appear in particular in [9, 11, 12]. Section of EIDS are studied in [3].

In this work we study the Euler obstruction of EIDS, specially for EIDS admitting a stratification with at most 3 strata. The main result is a formula to compute the Euler obstruction in terms of the singular vanishing Euler characteristic.

2. Essentially isolated determinantal singularity

We denote by $M_{m,n}$ the set of matrices $m \times n$ with complex entries.
Definition 2.1. For all \( t, 1 \leq t \leq \min\{m, n\} \), let \( M^t_{m,n} \) be the subset of \( M_{m,n} \) whose elements are matrices of rank less than \( t \):

\[
M^t_{m,n} = \{ A \in M_{m,n} | \text{rank}(A) < t \}.
\]

This set is a singular variety of codimension \((m - t + 1)(n - t + 1)\) in \( M_{m,n} \), called generic determinantal variety.

The singular set of \( M^t_{m,n} \) is \( M^{t-1}_{m,n} \). The partition of \( M^t_{m,n} \) defined by

\[
M^t_{m,n} = \bigcup_{i=1,\ldots,t} (M^i_{m,n} \setminus M^{i-1}_{m,n})
\]

is a Whitney stratification [1].

Let \( F : \mathbb{C}^N \to M_{m,n} \) be a map defined by \( F(x) = (f_{ij}(x)) \), whose entries are complex analytic functions defined on an open domain \( U \subset \mathbb{C}^N \).

Definition 2.2. The map \( F \) determines a determinantal variety of type \((m, n, t)\) in \( U \subset \mathbb{C}^N \) as the analytic variety \( X = F^{-1}(M^t_{m,n}) \), such that \( \text{codim} X = \text{codim} M^t_{m,n} = (m - t + 1)(n - t + 1) \).

A generic map \( F \) intersects transversally the strata \( M^i_{m,n} \setminus M^{i-1}_{m,n} \) of the variety \( M^t_{m,n} \). The following definition was introduced in [14].

Definition 2.3. A point \( x \in X = F^{-1}(M^t_{m,n}) \) is called essentially nonsingular if at this point the map \( F \) is transversal to the corresponding stratum of the variety \( M^t_{m,n} \) (that is, to \( M^i_{m,n} \setminus M^{i-1}_{m,n} \), where \( i = \text{rank } F(x) + 1 \)).

Definition 2.4. A germ \((X, 0) \subset (\mathbb{C}^N, 0)\) of a determinantal variety of type \((m, n, t)\) has an essentially isolated singular point at the origin (or is an essentially isolated determinantal singularity; EIDS) if it has only essentially nonsingular points in a punctured neighborhood of the origin in \( X \).

An EIDS \( X \subset \mathbb{C}^N \) has isolated singularity if and only if \( N \leq (m-t+2)(n-t+2) \). An EIDS with isolated singularity will be called isolated determinantal singularity, denoted by IDS [16, 18].

We want to consider deformations of an EIDS that are themselves determinantal varieties of the same type.

Definition 2.5. An essential smoothing \( \tilde{X}_s \) of the EIDS \((X, 0)\) is a subvariety lying in a neighborhood of the origin in \( \mathbb{C}^N \) and defined by \( \tilde{X}_s = \tilde{F}^{-1}(M^t_{m,n}) \) where \( \tilde{F} : U \times \mathbb{C} \to M_{m,n} \) is a perturbation of the germ \( F \), with \( \tilde{F}_s(x) = \tilde{F}(x, s), \tilde{F}_0(x) = F(x) \) such that \( \tilde{F}_s : U \to M_{m,n} \) is transversal to all strata \( M^i_{m,n} \setminus M^{i-1}_{m,n} \).
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An essential smoothing is in general not smooth (when \( N \geq (m-t+2)(n-t+2) \)) as we see in the following theorem.

**Theorem 2.1.** [19] Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be the germ of a determinantal variety with isolated singularity at the origin. Then, \(X\) has a smoothing if and only if \( N < (m-t+2)(n-t+2) \).

When the essential smoothing \( \tilde{X}_s \) is singular, its singular set is \( \tilde{F}^{-1}(M^{i-1}_{m,n}) \). Since \( \tilde{F} \) is transversal to the strata of the Whitney stratification \( M^{i}_{m,n} \), the partition \( \tilde{X}_s = \bigcup_{1 \leq i \leq t} \tilde{F}^{-1}_{s}(M^{i}_{m,n} \backslash M^{i-1}_{m,n}) \) is a Whitney stratification of \( \tilde{X}_s \).

**Example 1.** Let \( X = F^{-1}(M^{2}_{2,3}) \) be the 4-dimensional variety in \( \mathbb{C}^6 \) where

\[
F : \mathbb{C}^6 \rightarrow M^{2}_{2,3} \quad (x, y, z, w, v, u) \mapsto \begin{pmatrix} x & y & v \\ z & w & x+u^2 \end{pmatrix}.
\]

The following matrix defines an essential smoothing \( \tilde{X}_s = \tilde{F}^{-1}_{s}(M^{2}_{2,3}) \) of \( X \)

\[
\tilde{F} : \mathbb{C}^6 \times \mathbb{C} \rightarrow M^{2}_{2,3} \quad (x, y, z, w, v, u, s) \mapsto \begin{pmatrix} x+s & y & v \\ z & w & x+u^2 \end{pmatrix}.
\]

In this case, \( \tilde{X}_s \) is singular.

3. Lé-Greuel formula type for IDS with smoothing

In this section we review some results about isolated singularities following [18].

Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be the germ of a \( d \)-dimensional variety with isolated singularity at the origin. Suppose that \(X\) has a smoothing. Then, there exists a flat family \( \pi : \tilde{X} \subset U \times \mathbb{C} \rightarrow \mathbb{C} \) such that the fiber \( X_s = \pi^{-1}(s) \) is smooth for all \( s \neq 0 \) and \( X_0 = X \).

Let \( p : (X, 0) \rightarrow \mathbb{C} \) be a complex analytic function defined in \( X \) with isolated singularity at the origin. Let us consider a function

\[
\tilde{p} : \mathbb{C}^N \times \mathbb{C} \rightarrow \mathbb{C} \quad (x, s) \mapsto \tilde{p}(x, s),
\]

such that \( \tilde{p}(x, 0) = p(x) \) and for all \( s \neq 0 \), \( \tilde{p}(\cdot, s) = p_s \) is a Morse function on \( X_s \).

Thus we have the following diagram.
Proposition 3.1. [18] Proposition 4.1] Let $X$ be a $d$-dimensional variety with isolated singularity at the origin admitting smoothing and $p_s : X_s \rightarrow \mathbb{C}$, $p_s = \tilde{p}(\cdot, s)$ as above. Then,

(a) If $s \neq 0$, $X_s \simeq p_s^{-1}(0) \cup \{\text{cells of dimension } d \}$,

(b) $\chi(X_s) = \chi(p_s^{-1}(0)) + (-1)^d n_0$,

where $n_0$ is the number of critical points of $p_s$ and $\chi(X_s)$ denotes the Euler characteristic of $X_s$.

Let us recall how the invariant $n_0$ is related to the polar multiplicity of $X$, $m_d(X)$ ([18], see also [10]).

Definition 3.1. (The $d$-Polar multiplicity) Let $X$, $\tilde{X}$, $p$ and $\tilde{p}$ as above. Let $P_d(X, \pi, p) = \frac{\Sigma(\pi, \tilde{p})}{|X_{reg}|}$ be the relative polar variety of $X$ related to $\pi$ and $p$. We define $m_d(X, \pi, p) = m_0(P_d(X, \pi, p))$.

In general $m_d(X, \pi, p)$ depends on the choices of $\tilde{X}$ and $\tilde{p}$. When the variety $X$ has a unique smoothing $\tilde{X}$, then $m_d(X, \pi, p)$ depends only on $X$ and $p$. If $p$ is a generic linear embedding, $m_d(X, p)$ is an invariant of the EIDS $X$, denoted by $m_d(X)$.

Proposition 3.2. [18] Under the conditions of Proposition 3.1, $n_0 = m_d(X)$.

Theorem 3.3. [13] Let $X_s$ be a smoothing of a normal isolated singularity, then $b_1(X_s) = 0$.

Let $X$ be a determinantal variety of type $(m, n, t)$ in $\mathbb{C}^N$ with $N < (m - t + 2)(n - t + 2)$. Then $X$ has isolated singularity and admits smoothing.

Definition 3.2. [18] Let $X$ be a determinantal surface in $\mathbb{C}^N$, with isolated singularity at the origin. The Milnor number of $X$, denoted by $\mu(X)$, is defined as the second Betti number of the generic fiber $X_s$,

$$\mu(X) = b_2(X_s).$$

The following result appears in [7] [16] [18], for determinantal surfaces $X \subset \mathbb{C}^4$, but it also holds for any surface with isolated singularity in $\mathbb{C}^N$ admitting smoothing.
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**Proposition 3.4.** Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be the germ of a determinantal surface in \(\mathbb{C}^N\) with isolated singularity at the origin admitting smoothing. Let \(p : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)\) be a linear function whose restriction to \(X\) has an isolated singularity at the origin. Then one has the Lê-Greuel formula

\[
\mu(X) + \mu(X \cap p^{-1}(0)) = m_2(X, p).
\]

When \(d = \dim X > 2\), the Betti numbers \(b_i(X)\), \(2 \leq i < d\) are not necessarily zero (see [9]). In [7, 16] the authors define the vanishing Euler characteristic of varieties admitting smoothing.

**Definition 3.3.** [16] Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be an IDS such that \(N < (m-t+2)(n-t+2)\). The vanishing Euler characteristic is defined by

\[
\nu(X) = (-1)^d(\chi(X_s) - 1),
\]

where \(X_s\) is a smoothing of \(X\) and \(\chi(X_s)\) is the Euler characteristic of \(X_s\).

**Theorem 3.5.** [16] Let \((X, 0) \subset (\mathbb{C}^N, 0)\) be an IDS such that \(N < (m-t+2)(n-t+2)\) and let \(p : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)\) be a linear projection whose restriction to \(X\) has isolated singularity at the origin. Then,

\[
\nu(X) + \nu(X \cap p^{-1}(0)) = m_d(X, p).
\]

When \(p\) is a generic linear projection, then \(m_d(X, p) = m_d(X)\).

**Remark.** When \(d = 2\), then \(\nu(X) = \mu(X)\).

**Example 2.** [17] Let \(X = F^{-1}(M_{2,3}^2) \subset \mathbb{C}^4\) be the variety defined by:

\[
F : \mathbb{C}^4 \to M_{2,3}
\]

\[
(x, y, z, w) \mapsto \begin{pmatrix} z & y + w & x \\ w & x & y \end{pmatrix}.
\]

The following matrix defines an essential smoothing \(\tilde{X}_s = \tilde{F}_s^{-1}(M_{2,3}^2)\) of \(X\)

\[
\tilde{F} : \mathbb{C}^4 \times \mathbb{C} \to M_{2,3}
\]

\[
(x, y, z, w, s) \mapsto \begin{pmatrix} z & y + w & x + s \\ w & x & y \end{pmatrix}.
\]

In this case \(\tilde{X}\) is an smoothing of \(X\). Consider \(p : \mathbb{C}^4 \to \mathbb{C}\) given by \(p(x, y, z, w) = w\), then it follows that \(m_2(X) = 3\) and \(\mu(X \cap p^{-1}(0)) = 2\), then \(\mu(X) = 1\).
4. EULER OBSTRUCTION

Let \((X,0) \subset (\mathbb{C}^N,0)\) be the germ of a complex analytic variety. Let \(Gr(d,N)\) be the Grassmanian of complex \(d\)-planes in \(\mathbb{C}^N\). On the regular part \(X_{\text{reg}}\) of \(X\) the Gauss map \(\phi : X_{\text{reg}} \to \mathbb{C}^N \times Gr(d,N)\) is well defined by \(\phi(x) = (x, T_x(X_{\text{reg}}))\), where \(T_x(X_{\text{reg}})\) denotes the tangent space of \(X_{\text{reg}}\) at the point \(x\).

**Definition 4.1.** The Nash transformation \(\tilde{X}\) of \(X\) is the closure of the image \(\text{im} \phi\) in \(\mathbb{C}^N \times Gr(d,N)\). That is,

\[
\tilde{X} = \{(x,W)/ x \in X_{\text{reg}}, W = T_xX_{\text{reg}}\} \subset X \times Gr(d,m).
\]

It is a complex analytic space endowed with an analytic projection map \(\nu : \tilde{X} \to X\) which is a biholomorphism away from \(\nu^{-1}(\text{Sing}(X))\).

Denote by \(E\) the tautological bundle over \(Gr(d,N)\). We denote still by \(E\) the corresponding trivial extension bundle over \(\mathbb{C}^N \times Gr(d,N)\) with projection map \(\pi\). We define the Nash bundle \(\tilde{T}\) on \(\tilde{X}\), as the restriction of \(E\) to \(\tilde{X}\).

\[
\begin{array}{ccc}
\tilde{T} & \rightarrow & E \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & \mathbb{C}^N \times Gr(d,N) \\
\downarrow & & \downarrow \\
X & \rightarrow & M.
\end{array}
\]

Let \(X\) be a small representative of the germ \((X,0)\) and \(\{V_i\}\) a Whitney stratification of \(X\). Suppose that the point \(\{0\}\) is a stratum. Let \(\cup TV_i\) be the union of the tangent bundles of all the strata. This union can be seen as a subset of the tangent bundle \(T\mathbb{C}^N|_X\). We follow the references [2, 4, 5].

**Definition 4.2.** A stratified vector field \(v\) on \(X\) is a continuous section of \(T\mathbb{C}^N|_X\) such that, if \(x \in V_i \cap X\), then \(v(x) \in T_x(V_i)\).

**Definition 4.3.** A radial vector field \(v\) in a neighbourhood of \(\{0\}\) in \(X\) is a stratified vector field so that there is \(\epsilon_0\) such that for every \(0 < \epsilon \leq \epsilon_0\), \(v(x)\) is pointing outwards the ball \(B_\epsilon\) over the boundary \(S_\epsilon = \partial B_\epsilon\).

**Lemma 4.1.** [5] Let \(X\) be a stratified non-zero vector field on \(Z \subset X\). Then \(v\) can be lifted as a section \(\tilde{v}\) of \(\tilde{T}\) over \(\nu^{-1}(Z)\).
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The definition of Euler obstruction was given by R. MacPherson \[15\]. Here we give an equivalent definition given by J.-P. Brasselet and M.-H. Schwartz.

**Definition 4.4.** Let $v$ be a radial vector field on $X \cap \partial B_\epsilon$ and $\tilde{v}$ the lifting of $v$ on $\nu^{-1}(X \cap \partial B_\epsilon)$. The vector $\tilde{v}$ defines a cocycle of obstruction $\text{Obs}(\tilde{v})$, that measures the obstruction to extending $\tilde{v}$ as a non zero section of $\tilde{T}$ over $\nu^{-1}(X \cap B_\epsilon)$:

$$\text{Obs}(\tilde{v}) \in Z^2_N(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(V \cap \partial B_\epsilon))$$

The local Euler obstruction (or Euler obstruction) denoted by $\text{Eu}_X(0)$ is the evaluation of the cocycle $\text{Obs}(\tilde{v})$ over the fundamental class of $X$, $[\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(V \cap \partial B_\epsilon)]$, that is,

$$\text{Eu}_X(0) = \langle \text{Obs}(\tilde{v}), [\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(V \cap \partial B_\epsilon)] \rangle$$

**Remark.** The following properties of the Euler obstruction hold

1. $\text{Eu}_x(X)$ is constant along the strata of a Whitney stratification of $X$.
2. $\text{Eu}_x(X) = 1$, if $x \in X_{\text{reg}}$.
3. Assume that $X$ is a local embedding in $\mathbb{C}^N$ and $W$ a smooth variety $\mathbb{C}^N$ that intersects the Whitney stratification of $X$ transversely. Then $\text{Eu}_x(X \cap W) = \text{Eu}_x(X)$ for all $x \in X \cap W$.
4. $\text{Eu}_{(x,y)}(X \times Y) = \text{Eu}_x(X) \times \text{Eu}_y(Y)$ for $x \in X$ and $y \in Y$.

The following theorem gives a Lefschetz type formula for the local Euler obstruction \[4\] that is important tool to compute the Euler obstruction.

**Theorem 4.2.** Let $(X,0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^N$. Let $V_i$, $i = 1, \ldots, l$ be the strata of the Whitney stratification of a small representative $X$ of $(X,0)$. Then there is an open dense Zariski subset $\Omega$ in the space of complex linear forms on $\mathbb{C}^N$, such that for every $\ell \in \Omega$, there is $\epsilon_0$, such that for any $\epsilon, \epsilon_0 > \epsilon > 0$ and $t_0 \neq 0$ sufficiently small, such that the following formula for the Euler obstruction holds

$$\text{Eu}_0(X) = \sum_{i=1}^{l} \chi(V_i \cap B_\epsilon \cap l^{-1}(t_0)) \text{Eu}_{V_i}(X),$$

where $\text{Eu}_{V_i}(X)$ is the value of the Euler obstruction of $X$ at any point of $V_i$, $i = 1, \ldots, l$.

To compute the Euler obstruction of determinantal varieties, the following formula given in \[14\] is an useful tool. For determinantal varieties $X = F^{-1}(M_{m,n}^t)$ outside the origin, a normal section to
the stratum \( V_i = X_i \backslash X_{i-1} \) (where \( X_i = F^{-1}(M^i_{m,n}) \)) is isomorphic to 
\((M^{i+1}_{m-i+1,n-i+1,0})\).

**Theorem 4.3.** [14] Let \( l : M_{m,n} \to \mathbb{C} \) be a generic linear form and let 
\( L^t_{m,n} = M^t_{m,n} \cap l^{-1}(1) \). Then, for \( t \leq m \leq n \).

\[
\chi(L^t_{m,n}) = (-1)^t \binom{m-1}{t-1},
\]
where \( \chi(L^t_{m,n}) = \chi(L^t_{m,n}) - 1 \).

5. **Singular Vanishing Euler Characteristic of Determinantal Varieties**

In the case that \( V \) is not a complete intersection, the authors in [7] define the singular vanishing Euler characteristic as follows:

**Definition 5.1.** [7] Let \( V \subset \mathbb{C}^p \) be a complex analytic variety (not necessarily a complete intersection). Suppose that \( F : \mathbb{C}^N \to \mathbb{C}^p \) is transverse to the strata of \( V \) outside the point \( \{0\} \) and consider \( F_s : \mathbb{C}^N \to \mathbb{C}^p \) a 1-parameter deformation of \( F \) such that \( F_s \) is transversal to \( V \) for all \( s \neq 0 \) small. We say that \( F_s \) is a "stabilization" of \( F \). Write \( X = F^{-1}(V) \). Then the singular vanishing Euler characteristic of \( X \) is given by:

\[
\tilde{\chi}(X) = \tilde{\chi}(F^{-1}_s(V)) = \chi(F^{-1}_s(V)) - 1.
\]

**Remark.** [7] If \( V \) is a \( k \)-dimensional complete intersection and \( X = F^{-1}(V) \) then

\[
\tilde{\chi}(X) = (-1)^{N-k} \mu(X).
\]

In the work [7], the authors obtain several formulas to calculate the singular vanishing Euler characteristic of determinantal varieties defined by \( 2 \times 3 \) matrices. Proposition 5.1 is a Corollary of Theorem 8.1 in [7].

If \( F : \mathbb{C}^N \to M_{2,3} \) is a generic germ of corank 1, then

\[
F(x, y) = \sum_{i=1}^{5} x_i w_i + g(y)w_0,
\]
where \( \{w_1, w_2, w_3, w_4, w_5\} \) is a basis for \( W = dF(0)(\mathbb{C}^N), w_0 \notin W \) and \( g(y) \) define a isolated singularity on \( \mathbb{C}^{N-5} \).

**Proposition 5.1.** [7] Corollary 11.5] If \( F \) is a generic germ of corank 1, \( N \geq 6 \),

\[
\tilde{\chi}(X) = (-1)^{N-1} \mu(g).
\]
If \( g \) is quasi-homogeneous, \( \mu(g) \) is \( \tau(X) \).
6. Euler obstruction of determinantal singularities

Let \( X = F^{-1}(M_{m,n}^{t}) \) be an EIDS, defined by \( F : \mathbb{C}^{N} \rightarrow M_{m,n} \). If \( N \leq (m-t+3)(n-t+3) \) then the singular set \( \Sigma X = F^{-1}(M_{m,n}^{t}) \) is a determinantal variety with isolated singularity. Hence, the variety \( X \) admits at most 3 strata \( \{V_0,V_1,V_2\} \), where \( V_0 = \{0\} \), \( V_1 = \Sigma X \setminus \{0\} \), \( V_2 = \{\text{reg} \} \). These are the determinantal varieties we consider in this section.

For the calculus of the Euler obstruction we use the formula of the Theorem [12] then we have

\[
Eu_{0}(X) = \chi(V_0 \cap l^{-1}(r) \cap B_{c})Eu_{0}(X) + \chi(V_1 \cap l^{-1}(r) \cap B_{c})Eu_{1}(X) + \chi(V_2 \cap l^{-1}(r) \cap B_{c})Eu_{2}(X).
\]

As \( V_0 \cap l^{-1}(r) \cap B_{c} = \emptyset \) then \( \chi(V_0 \cap l^{-1}(r) \cap B_{c}) = 0 \). We also have \( Eu_{V_2}(X) = 1 \), then

\[
(3) \quad Eu_{0}(X) = \chi(V_1 \cap l^{-1}(r) \cap B_{c})(Eu_{1}(X) - 1) + \chi(X \cap l^{-1}(r) \cap B_{c})
\]

Note that \( \chi(V_1 \cap l^{-1}(r) \cap B_{c}) = \chi(\Sigma X \cap l^{-1}(r) \cap B_{c}) \). Then, given \( a \in V_{1} \) there is an open set \( U_{a} \) containing \( a \) such that \( U_{a} \cong B_{3} \times c(L_{V_{1}}) \), where the dimension of \( B_{3} \) is \( \dim V_{1} = N - (m-t+2)(n-t+2) \) and \( c(L_{V_{1}}) \) is the cone over the complex link of \( V_{1} \) in \( X \), \( L_{V_{1}} = X \cap N \cap p^{-1}(s) \), with \( N \) transversal to \( V_{1} \) at \( a \), \( \text{codim} N = \dim V_{1} \), then

\[
Eu_{V_{1}}(X) = Eu_{a}(B_{3}) \times Eu_{a}(c(L_{V_{1}})) = Eu_{a}(c(L_{V_{1}})) = \chi(L_{V_{1}}) = \chi(X \cap N \cap p^{-1}(s)).
\]

The equality \( Eu_{a}(c(L_{V_{1}})) = \chi(L_{V_{1}}) \) is given by applying again Theorem [14] to \( c(L_{V_{1}}) \), observing that \( c(L_{V_{1}}) \) has a isolated singularity and \( c(L_{V_{1}}) \cap p^{-1}(s) \) is isomorphic to \( L_{V_{1}} \). Here \( X \cap N \) is an essential smoothing of a determinantal variety of type \( (m,n,t) \) in \( \mathbb{C}^{r} \), where \( r = \text{codim} V_{1} = (m-t+2)(n-t+2) \), therefore with isolated singularity.

Substituting \( Eu_{V_{1}}(X) \) in (3), we have

\[
(4) \quad Eu_{0}(X) = \chi(\Sigma X \cap l^{-1}(r) \cap B_{c})(\chi(L_{V_{1}}) - 1) + \chi(X \cap l^{-1}(r) \cap B_{c}).
\]

This formula can be expressed in terms of the singular vanishing Euler characteristic of the Definition [5, 1]

\[
Eu_{0}(X) = (\tilde{\chi}(\Sigma X \cap l^{-1}(r) \cap B_{c}) + 1)(\chi(L_{V_{1}}) - 1) + \tilde{\chi}(X \cap l^{-1}(0) \cap B_{c}) + 1.
\]

**Proposition 6.1.** Let \( X = F^{-1}(M_{m,n}^{t}) \) be an EIDS, defined by \( F : \mathbb{C}^{N} \rightarrow M_{m,n} \). If \( N \leq (m-t+3)(n-t+3) \) and \( \Sigma X \) is an ICIS, then

\[
(5) Eu_{0}(X) = ((-1)^{\dim(\Sigma X \cap l^{-1}(0))}) \mu(\Sigma X \cap l^{-1}(0)) + 1)(\chi(L_{V_{1}}) - 1) + \tilde{\chi}(X \cap l^{-1}(0) \cap B_{c}) + 1
\]
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where \( l : \mathbb{C}^N \to \mathbb{C} \) is a generic linear projection, \( L_{V_1} \) is the complex link of the stratum \( V_1 \) em \( X \) and \( B_\epsilon \) is the ball of radius \( \epsilon \) in \( \mathbb{C}^N \).

**Proof:**

If \( \Sigma X \) is an ICIS, then

\[
\chi(\Sigma X \cap l^{-1}(r)) = 1 + (-1)^{\dim(\Sigma X \cap l^{-1}(0))} \mu(\Sigma X \cap l^{-1}(0)).
\]

Substituting this formula in (5) we have the result. \( \square \)

The following result shows the existence of determinantal varieties whose singular set is an ICIS.

**Proposition 6.2.** Let \( X = F^{-1}(M_{2,n}^{t}) \subset \mathbb{C}^N \) be an EIDS defined by the function \( F : \mathbb{C}^N \to M_{2,n} \), \( n \geq 2 \) then the singular set of \( X \), \( \Sigma X \subset \mathbb{C}^N \) is an ICIS.

**Proof:** As \( X = F^{-1}(M_{2,n}^{t}) \) is an EIDS then \( \Sigma X = F^{-1}(M_{2,n}^{1}) \). As \( M_{2,n}^{1} = \{0\} \), then \( \Sigma X = F^{-1}(0) \) and \( \text{codim} \ \Sigma X = (2-2+2)(n-2+2) = 2 \cdot n \), therefore \( \Sigma X \) is an ICIS.

Let \( x \in \Sigma X \), \( x \neq 0 \) then \( \text{rank} \ F(x) = 0 \), by the definition of EIDS, we have that \( F \) is transversal to the corresponding stratum, in this case \( F \) is transversal to \( \{0\} \) at the origin. Then \( dF \) is an embedding in \( x \neq 0 \), so \( \Sigma X \) is an ICIS. \( \square \)

6.1. **Euler obstruction of IDS, case** \( N < (n - t + 2)(m - t + 2) \).

In this section we will treat the varieties IDS, defined by the function \( F : \mathbb{C}^N \to M_{m,n} \), such that \( N < (n - t + 2)(m - t + 2) \). These varieties admit smoothing and the Euler obstruction was studied in the work \[16\], using the vanishing Euler characteristic and the multiplicity of the variety.

**Theorem 6.3.** \[16\] Let \( X = F^{-1}(M_{m,n}^{t}) \) be the determinantal variety defined by \( F : \mathbb{C}^N \to M_{m,n} \) with \( N < (n - t + 2)(m - t + 2) \), then

\[
Eu_0(X) = 1 + (-1)^d \nu(X, 0) + (-1)^{d+1} m_d(X, 0)
\]

**Example 3.** Let \( X = F^{-1}(M_{2,3}^{2}) \subset \mathbb{C}^4 \) be a determinantal variety with isolated singularity, defined by \( F \).

\[
F : \mathbb{C}^4 \to M_{2,3}
\]

\[
(x, y, z, w) \mapsto \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}
\]

By \[16\] we have that \( m_2(X) = 3 \), \( \nu(X) = 1 \), then \( Eu_0(X) = -1 \).

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6.2. Euler obstruction of an EIDS, case \( N = 6 \). The next result is for the case \( N = 6 \). In this case the determinant variety does not admit smoothing. The formula of the Theorem 6.3 given above does not hold.

**Theorem 6.4.** Let \( X = F^{-1}(M_{2,3}^2) \) defined by the \( F : \mathbb{C}^N \to M_{2,3} \) with \( N = 6 \), then

\[
Eu_0(X) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)) + 1.
\]

**Proof:** The formula comes from the fact that

\[
\chi(X \cap l^{-1}(r)) - 1 = \tilde{\chi}(X \cap l^{-1}(0)) = b_2(X \cap l^{-1}(r)) - b_3(X \cap l^{-1}(r)).
\]

In the Tables 1 and 2 we calculate the Euler obstruction of 4-dimensional determinantal varieties in \( \mathbb{C}^6 \), classified in [8]. We use the Table 5 of article [7] to obtain the results. The invariant \( \tau \) in Tables 1 and 2 is the Tjurina number.

6.3. Euler obstruction of EIDS, case \( N \geq 7 \).

**Theorem 6.5.** Let \( X \subset \mathbb{C}^N \) be an EIDS, defined by the function \( F : \mathbb{C}^N \to M_{2,3} \), with \( N \geq 7 \). Then

(6) \[
Eu_0(X) = (-1)^{N-7} \mu(\Sigma X \cap l^{-1}(0)) + \tilde{\chi}(X \cap l^{-1}(0)) + 2.
\]

**Proof:** In this case, \( X \) has three strata, \( \{ V_0, V_1, V_2 \} \), \( V_0 = \{ 0 \}, V_1 = \Sigma \setminus \{ 0 \}, V_2 = X_{reg} \) and \( V_1 = F^{-1}(M_{2,3}^1) \setminus F^{-1}(M_{2,3}^0) \). We have that \( L_{V_1} = X \cap \mathcal{N} \cap p^{-1}(s) \), where \( X \cap \mathcal{N} \cong M_{2,3}^2 \). So \( \chi(L_{V_1}) - 1 = 1 \) by Theorem 4.3. Substituting this value in the Proposition 6.1 we have the result.

**Corollary 6.6.** With the hypotheses of the Theorem 6.5, if \( F \) has corank 1, then

\[
Eu_0(X) = 2.
\]

**Proof:** As \( F \) have corank 1, then

\[
F(x, y) = \sum_{i=1}^{5} x_i w_i + g(y)w_0,
\]

where \( \{ w_1, w_2, w_3, w_4, w_5 \} \) is a base for \( W = d_0 F(\mathbb{C}^N), w_0 \notin W \) and \( g(y_1, y_2, \ldots, y_{N-5}) \) define a isolated singularity on \( \mathbb{C}^{N-5} \). Suppose without loss of generality that \( l(x, y) = y_1 \) and consider \( \tilde{g}(y_2, \ldots, y_{N-5}) = g(0, y_2, \ldots, y_{N-5}) \)

Using the formula of the Proposition 5.1 for \( X \cap l^{-1}(0) \subset \mathbb{C}^{N-1} \), we have that \( \tilde{\chi}(X \cap l^{-1}(0)) = (-1)^{N-2} \mu(\tilde{g}) \).
| Type      | Form of the matrix                                      | Conditions | $\tau$ | $Eu_0(X)$ |
|----------|--------------------------------------------------------|------------|--------|----------|
| $\Omega_1$ | $(x \ y \ v \ z \ w \ u)$                           |            | 0      | 2        |
| $\Omega_k$ | $(x \ y \ v \ z \ w \ x + u^k)$                       | $k \geq 2$ | $k - 1$ | 2        |
| $A^\dagger_k$ | $(x \ y \ u^2 + x^{k+1} + y^2)$                     | $k \geq 1$ | $k - 2$ | 1        |
| $D^\dagger_k$ | $(x \ y \ u^2 + xy^2 + x^{k-1})$                   | $k \geq 4$ | $k + 2$ | -1       |
| $E^\dagger_6$ | $(x \ y \ z \ w \ u^2 + y^4)$                        |            | 8      | 0        |
| $E^\dagger_7$ | $(x \ y \ z \ w \ u^2 + x^3 + xy^4)$               |            | 9      | 0        |
| $E^\dagger_8$ | $(x \ y \ z \ w \ u^2 + x^3 + y^5)$               |            | 10     | 0        |
|            | $(x \ y \ z \ w \ u \ y + y^k + u^l)$                | $k \geq 2$, $l \geq 3$ | $k + l - 1$ | 3 - $k$ |
|            | $(x \ y \ z \ w \ u \ x^2 + y^2 + u^3)$                |            | 6      | 1        |
| $F^\dagger_{q,r}$ | $(w \ y \ x \ z \ w + vu \ y + v^q + u^r)$ | $q, r \geq 2$ | $q + r$ | 2        |
| $G^\dagger_5$ | $(w \ y \ x \ z \ w + v^2 \ y + u^3)$                   |            | 7      | 2        |
| $G^\dagger_7$ | $(w \ y \ x \ z \ w + v^2 \ y + u^4)$                   |            | 10     | 2        |

**Table 1.** Euler obstruction, $X \subset \mathbb{C}^w$
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| Type          | Form of the matrix | Conditions | $\tau$ | $E_{u_0}(X)$ |
|---------------|--------------------|-----------|--------|-------------|
| $H_{q+3}^+$   | \( \begin{pmatrix} w & y & x \\ z & w + v^2 + u^q & y + vu^2 \end{pmatrix} \) | $q \geq 3$ | $q + 5$ | 2           |
| $I_{2q-1}^+$ | \( \begin{pmatrix} w & y & v \\ z & w + v^2 + u^3 & y + u^q \end{pmatrix} \) | $q \geq 4$ | $2q + 1$ | 2           |
| $I_{2r+2}^+$ | \( \begin{pmatrix} w & y & x \\ z & w + v^2 + u^3 & y + vu^r \end{pmatrix} \) | $r \geq 3$ | $2r + 4$ | 2           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + v^2 & u^2 + yv \end{pmatrix} \) |       | 6      | 1           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + uv & u^2 + yv + v^k \end{pmatrix} \) | $k \geq 3$ | $k + 4$ | 1           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + v^k & u^2 + yv + v^3 \end{pmatrix} \) | $k \geq 3$ | $2k + 2$ | 1           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + uv^k & u^2 + yv + v^3 \end{pmatrix} \) | $k \geq 2$ | $2k + 5$ | 1           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + v^3 & u^2 + yv \end{pmatrix} \) |       | 9      | 1           |
|               | \( \begin{pmatrix} w & y & x \\ z & w + v^k & u^2 + y^2 = v^3 \end{pmatrix} \) | $k \geq 3$ | $2k + 3$ | 1           |
| $H_{q+3}^+$   | \( \begin{pmatrix} w & y & x \\ z & w + uv^k & u^2 + y^2 + v^3 \end{pmatrix} \) | $k \geq 2$ | $2k + 6$ | 1           |

Table 2. Euler obstruction, $X \subset \mathbb{C}^6$.

Also, we have that

$$\Sigma X \cap l^{-1}(0) = \{(0, \ldots, 0, y_2, \ldots, y_{N-5})|\tilde{g}(y_2, \ldots, y_{N-5}) = 0\}$$

and hence $\mu(\Sigma X \cap l^{-1}(0)) = \mu(\tilde{g})$. Substituting in the formula (3), we have that $E_{u_0}(X) = 2$. \(\square\)
Example 4. Let $X$ be the variety defined by the function $F$.

$$F : \mathbb{C}^8 \rightarrow M_{2,3}$$

$$(x, y) \mapsto \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_1 + y_1^2 - y_2^2 + y_3^2 \end{pmatrix}$$

where $x = (x_1, x_2, x_3, x_4, x_5)$ and $y = (y_1, y_2, y_3)$. The variety $X$ is an EIDS of dimension 6 in $\mathbb{C}^8$. The singular set has dimension 2 and is the set

$$\Sigma X = \{(0, 0, 0, 0, y_1, y_2, y_3) | y_2^2 - y_2^2 + y_3^2 = 0\}.$$ 

This variety has three strata $V_0 = 0$, $V_1 = \Sigma X - \{0\}$ and $V_2 = X_{\text{reg}}$. As the corank of $F$ is 1, then by the Corollary 6.6 we have

$$Eu_0(X) = 2.$$ 

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