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Lie Symmetry Analysis, Analytical Solution, and Conservation Laws of a Sixth-Order Generalized Time-Fractional Sawada-Kotera Equation

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Abstract: To discuss the invariance properties of a sixth-order generalized time-fractional Sawada-Kotera equation, on the basis of the Riemann-Liouville derivative, the Lie point symmetry and symmetry reductions are derived. Then the power series theory is used to construct the exact power series solution of the equation. Finally, the conservation laws for a sixth-order generalized time-fractional Sawada-Kotera equation are computed.

Keywords: sixth-order generalized time-fractional Sawada-Kotera equation; Lie symmetry analysis; symmetry reductions; power series method; conservation laws

1. Introduction

As is well-known, constructing exact solutions of a partial differential equations (PDEs) is a vital theme in nonlinear science. A considerable number of methods have been developed, such as Lie symmetry analysis, the Painlevé test, inverse scattering transformation (IST), Darboux transformation (DT), and Clarkson-Kruskal (CK) transformation. The Lie symmetry analysis method is generally preferred for obtaining exact solutions of a PDE. This method was introduced by Sophus Lie in the early 19th century.

Recently, the symmetry analysis of fractional partial differential equations (FPDEs) and fractional derivatives were proposed [1]. On this basis, several works have been published on time-fractional differential equations. Baleanua et.al obtained the exact traveling solutions of the time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera equation [2]. Saberi and Hejazi obtained some exact solutions of the time-fractional generalized Hirota–Satsuma-coupled KdV system by means of the invariant subspace method [3]. The coefficient of a fractional-order equation was extended from a constant coefficient to a variable coefficient, and the four generators of the equation were obtained by a classified discussion of the variable coefficient [4]. Two diverse approaches were used to obtain the similarity solutions of the time-fractional Burgers system, and the similarity solutions were approximated by a numerical simulation method [5]. Explicit analytical solutions of several time-fractional equations were obtained by using the singular manifold method [6]. Roul and Prasad Goura designed a high-order numerical method to approximate the solution of time-fractional fourth-order PDEs. The result showed that this new method was more accurate than the previous method [7]. A Crank–Nicolson Legendre spectral method was developed to solve the two-dimensional nonlinear time-fractional diffusion-wave equation [8]. Zou et al. considered the following generalized time-fractional Sawada-Kotera equation [9]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + au^2 u_x + bu_x u_{xx} + bu_{xxx} + cu_{xxxx} = 0. \quad (0 < \alpha \leq 1). \quad (1)$$
The analytical solution and conservation laws (CLs) were then derived. In this work, we consider a sixth-order generalized time-fractional Sawada-Kotera equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^2 u_{xx} + bu_x u_{xxx} + cu_{xxxx} + du_{xxxxx} = 0. \quad (0 < \alpha < 2)$$

(2)

where $u = u(x,t), a, b, c, d$ are arbitrary constants, and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Riemann-Liouville (RL) fractional derivative of order $\alpha$ about $t$. Equation (2) is a vital mathematical model that appears in many differential physical environments. It has been widely used in microscopic particle phoronomics, conformal field theory, and nonlinear optics. The long-wave motion in shallow water under the action of gravity and the long-wave motion in a one-dimensional nonlinear lattice can also be described. Equation (2) is a extension of Equation (1), which is more sophisticated and powerful than Equation (1). It is a new equation that has not been studied by other scholars. Therefore, Equation (2) enriches the study of time fractional PDE and has a certain significance of research value.

With $\alpha = 1$, Equation (2) becomes

$$u_t + au^2 u_{xx} + bu_x u_{xxx} + cu_{xxxx} + du_{xxxxx} = 0,$$

(3)

whose Lie symmetry analysis and exact solutions were studied in [10].

The rest of the article is arranged as follows: in Section 2, some information about the Lie group method is given to analyze the FPDE. In Section 3, symmetry reductions are investigated on the basis of the symmetry groups of the FPDE. In Section 4, by using the power series method, we obtain the explicit analytical power series solution. In Section 5, conservation laws are derived using Ibragimov’s nonlocal conservation theorem and the fractional Noether operators. Lastly, several remarks and conclusions are given in Section 6.

2. Lie Symmetry Analysis

First, let us recall the definition of the RL fractional derivative of order $\alpha$:

**Definition 1 ([11,12]).**

$$\partial^\alpha_t f(t) = \left\{ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{n-\alpha}} d\tau, n-1 < \alpha < n, n \in \mathbb{N} \right\}$$

Let us also consider the following scalar time FPDE:

$$E = \frac{\partial^\alpha u}{\partial t^\alpha} - F(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx})$$

(4)
where $\alpha (0 < \alpha < 2)$ is a parameter. Next, we focus on a one-parameter Lie group of infinitesimal transformations as follows:

\[
\begin{align*}
\tau & = x + \varepsilon \xi(x, t, u) + o(\varepsilon^2) \\
\eta & = t + \varepsilon \eta(x, t, u) + o(\varepsilon^2) \\
\eta & = u + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\end{align*}
\]

\[
\frac{\partial^\alpha \eta}{\partial t^\alpha} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \varepsilon \eta(0)(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial x} + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial^2 \eta}{\partial \xi^2} = \frac{\partial^2 \eta}{\partial \xi^2} + \varepsilon \eta(0)(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial^3 \eta}{\partial \eta^3} = \frac{\partial^3 \eta}{\partial \eta^3} + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial^4 \eta}{\partial \xi^4} = \frac{\partial^4 \eta}{\partial \xi^4} + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial^5 \eta}{\partial \xi^5} = \frac{\partial^5 \eta}{\partial \xi^5} + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\]

\[
\frac{\partial^6 \eta}{\partial \xi^6} = \frac{\partial^6 \eta}{\partial \xi^6} + \varepsilon \eta(x, t, u) + o(\varepsilon^2)
\]

(5)

where

\[
\begin{align*}
\eta_x & = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau) \\
\eta_{xx} & = D_x(\eta_x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau) \\
\eta_{xxx} & = D_x(\eta_{xx}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau) \\
\eta_{xxxx} & = D_x(\eta_{xxx}) - u_{xxxx} D_x(\xi) - u_{xxxt} D_x(\tau) \\
\eta_{xxxxx} & = D_x(\eta_{xxxx}) - u_{xxxxx} D_x(\xi) - u_{xxxxxt} D_x(\tau)
\end{align*}
\]

(6)

\(D_x\) is the total differential operator and is defined as follows:

\[
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \ldots
\]

Then, the associated Lie algebra of symmetries is generated by the vector field below:

\[
X = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}.
\]

(7)

On the basis of the infinitesimal invariance criterion, we can easily obtain

\[
pr(6) X |_{\nabla = 0} = 0.
\]

(8)

Additionally, the variance condition yields [13]

\[
\tau(x, t, u)|_{t = 0} = 0.
\]

(9)

The \(\alpha\)th extended infinitesimal about the RL fractional time derivative with Equation (9) is

\[
\eta_a^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \left[ \eta_x - \alpha D_t(\tau) \right] \frac{\partial^\alpha \eta}{\partial t^\alpha} + \mu \frac{\partial^2 \eta}{\partial \xi^2} + \sum_{n=1}^{\infty} \left( \frac{\alpha}{n} \right) D_t^n(\xi) D_t^{(\alpha-n)}(u_x)
\]

The extended infinitesimal about the RL fractional time derivative with Equation (9) is
\[ + \sum_{n=1}^{\infty} \left[ \left( \frac{\alpha}{n} \right) \frac{\partial^n \eta u}{\partial t^n} - \left( \frac{\alpha}{n+1} \right) D_{i}^{n+1} \right] \]

where

\[ \mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \sum_{r=0}^{k-1} \left( \frac{\alpha}{n} \right) \left( \begin{array}{c} n \\ m \\ k \\ r \end{array} \right) \frac{1}{k!} \Gamma(n+1-\alpha) \left[-u\right]^{r} \frac{\partial^{m}}{\partial t^{m}} \times \left[u^{k-r}\right] \frac{\partial^{n-m+k}\eta}{\partial u^{k}}. \]

Let us assume that Equation (2) is invariant under a one-parameter transformation (6); then, the transformed equations are

\[ \frac{\partial^{n} \Pi}{\partial t^{n}} + a \Pi \frac{\partial^{2} \pi_{xx}}{\partial x^{2}} + b \Pi \frac{\partial \pi_{xxx}}{\partial x} + c \Pi \frac{\partial \pi_{xxxx}}{\partial x} + d \Pi = 0 \]

provided that \( u = u(x, t) \) satisfies Equation (2). From the invariant Equation (8), we have

\[ \eta^{0} + (2auu_{xx} + cu_{xxx})\eta + bu_{xx} \eta^{x} + au^{2} \eta^{xx} + bu_{x} \eta^{xxx} + cu \eta^{xxxx} + d \eta^{xxxxx} = 0. \]

Inserting Equations (6) and (10) into Equation (13), we obtain the following determining equations:

\[ \begin{aligned}
\zeta_{t} = \zeta_{u} = \xi_{xx} = 0, \tau_{x} = \tau_{u} = 0, \eta_{x} = \eta_{uu} = 0 \\
(\alpha t + 4\eta_{u})u - \eta = 0, 6\xi'(x) - \alpha \tau'(t) = 0 \\
\left( \frac{\alpha}{n} \right) \frac{\partial^{n}(\eta u)}{\partial t^{n}} - \left( \frac{\alpha}{n+1} \right) D_{i}^{n+1} (\tau) = 0.
\end{aligned} \]

Solving the above equations, we get

\[ \xi = c_{1} x + c_{2}, \tau = \frac{6c_{1}}{\alpha}, \eta = -2c_{1} u \]

where \( c_{1} \) and \( c_{2} \) are arbitrary constants. Hence, the infinitesimal symmetry group for Equation (2) is generated by the following two vector fields:

\[ X_{1} = \frac{\partial}{\partial x}, X_{2} = x \frac{\partial}{\partial x} + \frac{6t}{\alpha} \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u}. \]

It is easy to verify that the above generator is closed under the Lie bracket:

\[ [X_{1}, X_{2}] = X_{1}, [X_{2}, X_{1}] = -X_{1}. \]

3. Symmetry Reductions

Next, according to the Equation (14), we consider the similarity reductions of Equation (2).

Case 1 \( X_{1} = \frac{\partial}{\partial t} \) For \( X_{1} \), the characteristic equation is

\[ \frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \]

Therefore, we can easily construct the expression \( u = f(t) \) by solving the above characteristic equation. If this expression is put into Equation (2), then the fractional differential equation is

\[ \frac{d^{\alpha} f(t)}{dt^{\alpha}} = 0. \]
Consequently, Equation (2) has a group invariant solution of the form

\[ u = k \cdot t^{\alpha - 1} \]

where \( k \) is an arbitrary constant of integration.

**Case 2** \( X_2 = x \frac{\partial}{\partial x} + \frac{6t}{n} \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u} \)

For \( X_2 \), the characteristic equation is

\[ \frac{dx}{x} = \frac{dt}{t} = \frac{du}{-2u}. \]

Solving the above equations, we get

\[ \xi = xt^{-\frac{\alpha}{n}}, u = t^{-\frac{n}{\alpha}} f(\xi). \]  

**Theorem 1.** Putting Equation (15) into Equation (2), we obtain the following reduced equation:

\[ (P_{\beta}^{\tau, a, \alpha} f)(\xi) + af(\xi)^2 \frac{d^2}{d\xi^2} f''(\xi) + bf'(\xi)f'''(\xi) + cf(\xi)f^{(4)}(\xi) + df^{(6)}(\xi) = 0 \]  

with the Erdélyi–Kober (EK) fractional differential operator

\[ (P_{\beta}^{\tau, a, \alpha} g)(\xi) := \prod_{j=0}^{n-1} (\tau + j - \frac{1}{\beta} \xi, \alpha > N) \frac{d}{d\xi} (K_{\beta}^{\tau, a, \alpha} g)(\xi), n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N} \\ \alpha \in \mathbb{N} & \end{cases} \]  

where the EK fractional integral operator is

\[ (K_{\beta}^{\tau, a, \alpha} g)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{1}^{\infty} (u - 1)^{\alpha - 1} u^{-\alpha} (\xi u^\frac{1}{\alpha}) du, & \alpha > 0 \\ 0 & \end{cases} \]  

**Proof.** Let \( n - 1 < \alpha < n, n = 1, 2, 3 \ldots \)

Then, the similarity transformation of RL fractional derivatives is

\[ \frac{\partial^n u}{\partial t^n} = \frac{\partial^n u}{\partial t^n} \left[ \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f(xs^{-\alpha}) ds \right]. \]

By setting \( v = \frac{1}{s} \), we have \( ds = -(\frac{1}{s^2}) dv \). Then, Equation (19) can be written as

\[ \frac{\partial^n u}{\partial t^n} = \frac{\partial^n u}{\partial x^n} \left[ \frac{1}{\Gamma(n - \alpha)} \int_1^\infty v^{-n-\frac{1}{2}n-1} \frac{1}{\Gamma(n - \alpha)} (v - 1)^{n-\alpha-1} f(\xi v^\frac{1}{\alpha}) dv \right]. \]

If we apply the definition of EK integral operator (18) to formula (19), then we obtain the following expression:

\[ \frac{\partial^n u}{\partial t^n} = \frac{\partial^n u}{\partial \xi^n} \left[ \frac{1}{\Gamma(n - \alpha)} (K_{\beta}^{\tau, a, \alpha} f)(\xi) \right]. \]

Let us simplify Equation (21) further: considering \( \xi = xt^{-\frac{\alpha}{n}}, \phi \epsilon C^1(0, \infty) \), we can obtain

\[ \frac{\partial}{\partial t} \phi(\xi) = t(x(-\frac{\alpha}{6})t^{-\frac{n}{\alpha}-1} \phi'(\xi) = -\frac{\alpha}{6} \frac{\partial}{\partial \xi} \phi(\xi). \]
Thus, we have
\[
\frac{\partial^n u}{\partial t^n} = \frac{\partial^n}{\partial t^n} [\mu - \frac{1}{2} a(K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)] = \frac{\partial^n-1}{\partial t^{n-3}} \frac{\partial}{\partial t} (t^{n-\frac{1}{2}a} (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)) \\
= \frac{\partial^n-1}{\partial t^{n-1}} [t^{n-\frac{1}{2}a-1} (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi}) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)].
\]  

(23)

If we perform the same step \( n - 1 \) times, then the following result can be obtained:

\[
\frac{\partial^n}{\partial t^n} [t^{n-\frac{1}{2}a} (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)] = \frac{\partial^n-1}{\partial t^{n-1}} [t^{n-\frac{1}{2}a-1} (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi}) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)] \\
= \frac{\partial^n-2}{\partial t^{n-2}} [t^{n-\frac{3}{2}a-2} (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi}) (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi} - 1) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)] \\
= \frac{\partial^n-3}{\partial t^{n-3}} [t^{n-\frac{5}{2}a-3} (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi}) (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi} - 1) (n - \frac{4}{3} a - \alpha \frac{\xi}{6 \xi} \frac{\partial}{\partial \xi} - 2) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi)] \\
= \ldots \\
= t^{-\frac{1}{2}a} \prod_{j=0}^{n-1} (1 - \frac{4}{3} a + \frac{1}{6 \xi} \xi \frac{\partial}{\partial \xi}) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi). \tag{24}
\]

Putting operator (17) into Equation (24) leads to the following expression:

\[
t^{\frac{1}{2}a} \prod_{j=0}^{n-1} (1 - \frac{4}{3} a + \frac{1}{6 \xi} \xi \frac{\partial}{\partial \xi}) (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi) = t^{\frac{1}{2}a} (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi). \tag{25}
\]

Then, we can use the form of the EK fractional differential operator to express the time-fractional derivative:

\[
\frac{\partial^a u}{\partial t^a} = t^{\frac{1}{2}a} (K^{1 - \frac{1}{2}a,n-a}_\xi) (\xi). \tag{26}
\]

Given the above, Equation (2) is transformed into the equation below:

\[
(K^{1 - \frac{1}{2}a}_\xi) (\xi) + a f(\xi) + b f'(\xi) + c f''(\xi) + d f'''(\xi) + f(\xi) + df^{(4)}(\xi) + df^{(5)}(\xi) = 0. \tag{27}
\]

4. Exact Power Series Solution

Here, on the basis of the power series theory and symbolic computation, a kind of exact power series solution of Equation (27) is obtained with a detailed derivation. If we set

\[
f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n, \tag{28}
\]
then we can obtain

\[ f'(\xi) = \sum_{n=0}^{\infty} (n+1)c_{n+1}\xi^n \]
\[ f''(\xi) = \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2}\xi^n \]
\[ f'''(\xi) = \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)c_{n+3}\xi^n \]
\[ f^{(4)}(\xi) = \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4)c_{n+4}\xi^n \]
\[ f^{(6)}(\xi) = \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)c_{n+6}\xi^n. \]  

(29)

Substituting Equations (28) and (29) into Equation (27), we obtain

\[ \sum_{n=0}^{\infty} \Gamma(2 - \frac{1}{3}a + \frac{2n}{6})c_n\xi^n + a \sum_{n=0}^{\infty} \left[ (\sum_{i=0}^{n} c_i c_{n-i})\xi^n \right] \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}\xi^n + b \sum_{n=0}^{\infty} c_{n+1}\xi^n \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)c_{n+3}\xi^n + c \sum_{n=0}^{\infty} c_n\xi^n \sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1)c_{n+4}\xi^n + d \sum_{n=0}^{\infty} (n+6)(n+5)(n+3)(n+2)(n+1)c_{n+6}\xi^n = 0. \]

(30)

Observing coefficients in Equation (30) carefully, when \( n = 0 \), we have

\[ c_6 = -\frac{1}{720d^2} \frac{\Gamma(2 - \frac{1}{3}a)c_0 + 2ac_0^2c_2 + 6bc_1c_3 + 24c_0c_2c_4}. \]

(31)

When \( n \geq 1 \), we get

\[ c_{n+6} = -c_n \left[ d \prod_{i=1}^{6} (n+i) \right]^{-1} \frac{\Gamma(2 - \frac{1}{3}a + \frac{2n}{6})}{\Gamma(2 - \frac{1}{3}a + \frac{2n}{6})} \]
\[ -a \left[ d \prod_{i=1}^{6} (n+i) \right]^{-1} \sum_{i=0}^{n} c_i \left[ \sum_{k=0}^{n-i} (n-i+2-k)(n-i+1-k)c_k c_{n-i+2-k} \right] \]
\[ -b \left[ d \prod_{i=1}^{6} (n+i) \right]^{-1} \sum_{i=0}^{n} (i+3)(i+2)(i+1)c_{i+3}c_{n-i+1} \]
\[ -c \left[ d \prod_{i=1}^{6} (n+i) \right]^{-1} \sum_{i=0}^{n} (i+4)(i+3)(i+2)(i+1)c_{i+4}c_{n-i}. \]
Then, the explicit solution for Equation (27) can be expressed in the following form:

\[ f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n = c_0 + c_1 \xi + \ldots + c_6 \xi^6 + \sum_{n=1}^{\infty} c_n \xi^{n+6} \]

\[ = c_0 + c_1 \xi + \ldots + c_6 \xi^5 - \frac{1}{720d} \left[ \frac{\Gamma(2 - \frac{1}{2} \alpha)}{\Gamma(2 - \frac{3}{2} \alpha)} \right] c_0 + 2a c_2 c_2 + 6bc_1 c_3 + 24c_0 c c_4 \xi^6 \]

\[ - \sum_{n=1}^{\infty} c_n [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \frac{\Gamma(2 - \frac{1}{2} \alpha + \frac{an}{6})}{\Gamma(2 - \frac{3}{2} \alpha + \frac{an}{6})} \xi^{n+6} \]

\[ - \sum_{n=1}^{\infty} a [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \sum_{i=0}^{n} \left\{ c_i \sum_{k=0}^{n-i} (n-i+2-k)(n-i+1-k)c_k c_{n-i+2-k} \right\} \xi^{n+6} \]

\[ - \sum_{n=1}^{\infty} b [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \sum_{i=0}^{n} (i+3)(i+2)(i+1)c_{i+3}(n-i+1)c_{n-i+1} \xi^{n+6} \]

Thus, the exact power series solution for Equation (2) is

\[ u(x, t) = c_0 t^{-\frac{3}{2}} + c_1 x t^{-\frac{5}{2}} + c_2 x^2 t^{-\frac{7}{2}} + c_3 x^3 t^{-\frac{9}{2}} + c_4 x^4 t^{-\alpha} + c_5 x^5 t^{-\frac{7\alpha}{2}} \]

\[ - \frac{1}{720d} \left[ \frac{\Gamma(2 - \frac{1}{2} \alpha)}{\Gamma(2 - \frac{3}{2} \alpha)} \right] c_0 + 2a c_2 c_2 + 6bc_1 c_3 + 24c_0 c c_4 \xi^6 \]

\[ - \sum_{n=1}^{\infty} c_n [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \frac{\Gamma(2 - \frac{1}{2} \alpha + \frac{an}{6})}{\Gamma(2 - \frac{3}{2} \alpha + \frac{an}{6})} \xi^{n+6} t^{-\frac{(n+8)\alpha}{8}} \]

\[ - \sum_{n=1}^{\infty} a [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \sum_{i=0}^{n} \left\{ c_i \sum_{k=0}^{n-i} (n-i+2-k)(n-i+1-k)c_k c_{n-i+2-k} \right\} \xi^{n+6} t^{-\frac{(n+8)\alpha}{8}} \]

\[ - \sum_{n=1}^{\infty} b [d \prod_{i=1}^{6} (n+i)]^{-1} \cdot \sum_{i=0}^{n} (i+3)(i+2)(i+1)c_{i+3}(n-i+1)c_{n-i+1} \xi^{n+6} t^{-\frac{(n+8)\alpha}{8}} \].

5. Conservation Laws

In this section, we investigate the CLs of Equation (2). The RL left-sided time-fractional derivative is applied as

\[ \partial D_t^\alpha u = D_t^\alpha (\partial I_t^{n-\alpha} u). \]  

The left-sided time-fractional integral of order \( n - \alpha \), namely, \( \partial I_t^{n-\alpha} u \), is defined by

\[ (\partial I_t^{n-\alpha} u)(x, t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{u(\theta, x)}{(t - \theta)^{1-n+\alpha}} d\theta. \]

In the formula above, \( \Gamma(z) \) is the Gamma function, \( D_t \) represents the operator of differentiation about \( t \), and \( n = [\alpha] + 1 \).

A CL for Equation (2) is given as the following formula:

\[ D_t (C^1) + D_x (C^5) = 0 \]

which holds for all solutions \( u(x, t) \) of Equation (2).
A formal Lagrangian for Equation (2) is written as
\[ L = v(x,t) \frac{\partial^2 u}{\partial t^2} + au^2 u_{xx} + bu_x u_{xxx} + cu u_{xxxx} + du_{xxxxx} \]  
(36)

where \( v(t,x) \) is a new dependent variable. The Euler–Lagrange operator is given by \[ \frac{\delta}{\delta u} \] \[ = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} + D_{xx} \frac{\partial}{\partial u_{xx}} - D_{xxx} \frac{\partial}{\partial u_{xxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxx}} \] \[ + D_{xxxxx} \frac{\partial}{\partial u_{xxxxx}} \] \[ = \frac{\partial}{\partial (\partial_t^\alpha + \partial_x + \partial_{xx} - \partial_{xxx} + \partial_{xxxx} + \partial_{xxxxx})} \] \[ = \frac{\partial}{\partial (\partial_t^\alpha + \partial_x + \partial_{xx} - \partial_{xxx} + \partial_{xxxx} + \partial_{xxxxx})} \] \[ + D_{xxxxx} \frac{\partial}{\partial u_{xxxxx}}. \] \[ (37) \]

The adjoint equation to Equation (2) is defined by \[ \frac{\delta L}{\delta u} = 0 \] \[ (38) \]

We also have
\[ \Xi + D_t(\tau) I + D_x(\xi) I = W \frac{\delta}{\delta u} + D_t N^l + D_x N^x \] \[ (39) \]

where \( I \) represents the identity operator, and \( N^l \) and \( N^x \) represent Noether operators. The prolonged vector field \( \Xi \) is defined by the expression below.
\[ \Xi = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \eta^0 \frac{\partial}{\partial D_t^\alpha u} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}} \] \[ + \eta^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \eta^{xxxxx} \frac{\partial}{\partial u_{xxxxx}}. \] \[ (40) \]

The Lie characteristic function \( W \) is defined by
\[ W = \eta - \xi u_x - \tau u_t. \] \[ (41) \]

If we use the RL time-fractional derivative in Equation (2), then \( N^l \) is given by
\[ N^l = \tau I + \sum_{k=0}^{n-1} (-1)^k \cdot 0 D_t^{1-k} (W) D_t^k \frac{\partial}{\partial (\partial_t^\alpha u)} - (-1)^n J(W, D_t^\alpha \frac{\partial}{\partial (\partial_t^\alpha u)})[14]. \] \[ (42) \]

with \( J \) given by
\[ J(f,g) = \frac{1}{\Gamma(n - \alpha)} \int_0^T \int_0^T f(\tau,x) g(\mu,x) (\mu - \tau)^{\alpha + 1 - n} d\mu d\tau. \] \[ (43) \]

For Equation (2), the operator \( N^x \) is given by
\[ N^x = \xi I + W \left( \frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_{xx} \frac{\partial}{\partial u_{xxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxx}} \right) \] \[ + D_x(W) \left( \frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}} + D_{xx} \frac{\partial}{\partial u_{xxxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxxx}} \right) \] \[ + D_{xx}(W) \left( \frac{\partial}{\partial u_{xxx}} - D_x \frac{\partial}{\partial u_{xxxx}} + D_{xx} \frac{\partial}{\partial u_{xxxxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxxx}} \right) \] \[ + D_{xxx}(W) \left( \frac{\partial}{\partial u_{xxxx}} - D_x \frac{\partial}{\partial u_{xxxxx}} + D_{xx} \frac{\partial}{\partial u_{xxxxxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxxxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxxx}} \right) \] \[ + D_{xxxx}(W) \left( \frac{\partial}{\partial u_{xxxxx}} - D_x \frac{\partial}{\partial u_{xxxxxx}} + D_{xx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxxxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxxx}} \right) \] \[ + D_{xxxxx}(W) \left( \frac{\partial}{\partial u_{xxxxxx}} - D_x \frac{\partial}{\partial u_{xxxxxxx}} + D_{xx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxx} \frac{\partial}{\partial u_{xxxxxxx}} + D_{xxxx} \frac{\partial}{\partial u_{xxxxxxx}} - D_{xxxxx} \frac{\partial}{\partial u_{xxxxxxx}} \right). \] \[ (44) \]
Any given generator $X$ of Equation (2) has a corresponding invariance condition, and its solution is

$$[\mathcal{L} \tau + D_t(\tau)\mathcal{L} + D_x(\xi)\mathcal{L}]|_{2} = 0$$  \hspace{1cm} (45)$$

Consequently, the conservation laws of Equation (2) can be represented as

$$D_t(N^1 L) + D_x(N^2 L) = 0.$$  \hspace{1cm} (46)$$

Now, we present the conservation laws for Equation (2) using the basic definitions presented above. On the basis of different values of the order $\alpha$, let us reflect on two diverse cases.

**Case 1** For the case in which $\alpha \in (0,1)$,

$$C_1 = \xi L + (-1)^0 D_t^{-1}(W_i)\frac{\partial L}{\partial \theta_0 D_{t} u} + f(W, D_t \frac{\partial L}{\partial \partial \theta_{D_{t}} u}) = 0$$

$$C_2 = \xi L + W_i(\frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u} + D_x \frac{\partial L}{\partial u} - D_{xxx} \frac{\partial L}{\partial u} + D_{xxxx} \frac{\partial L}{\partial u} - D_{xxxxx} \frac{\partial L}{\partial u})$$

$$+ D_x(W_i)(\frac{\partial L}{\partial \theta_0 D_{t} u} - D_t \frac{\partial L}{\partial \theta_0 D_{t} u} + D_x \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxx} \frac{\partial L}{\partial \theta_0 D_{t} u} + D_{xxxx} \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxxxx} \frac{\partial L}{\partial \theta_0 D_{t} u})$$

$$+ D_{xxx}(W_i)(\frac{\partial L}{\partial \theta_0 D_{t} u} - D_t \frac{\partial L}{\partial \theta_0 D_{t} u} + D_x \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxx} \frac{\partial L}{\partial \theta_0 D_{t} u} + D_{xxxx} \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxxxx} \frac{\partial L}{\partial \theta_0 D_{t} u})$$

$$+ D_{xxxxx}(W_i)\frac{\partial L}{\partial \theta_0 D_{t} u}$$

$$= W_i(2bvuxx - avux^2 - 2avuxu + 2buxxv + buvxxv - cuxxxv - cvxxxu - 3cuxxxv - 3cuuxv - 3cuxv - 3cu - 3dvxxx)$$

$$+ D_x(W_i)(avu^2 - buuxv + buuxv -audvxxx)$$

$$+ D_{xxx}(W_i)(buu - cuv - cuv - cuv - duvxxx) + D_{xxxx}(W_i)(cvu + dvxx) - duvD_{xxxxx}(W_i) + D_{xxxxxx}(W_i)dv$$

where $i = 1,2$, and the Lie characteristic functions $W_i$ have the form

$$W_1 = -u_x, W_2 = -2u + \frac{6}{\alpha} tu - xu_x.$$

**Case 2** For the case in which $\alpha \in (1,2)$,

$$C_1 = \xi L + W_i(\frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u} + D_x \frac{\partial L}{\partial u} - D_{xxx} \frac{\partial L}{\partial u} + D_{xxxx} \frac{\partial L}{\partial u} - D_{xxxxx} \frac{\partial L}{\partial u})$$

$$+ D_x(W_i)(\frac{\partial L}{\partial \theta_0 D_{t} u} - D_t \frac{\partial L}{\partial \theta_0 D_{t} u} + D_x \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxx} \frac{\partial L}{\partial \theta_0 D_{t} u} + D_{xxxx} \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxxxx} \frac{\partial L}{\partial \theta_0 D_{t} u})$$

$$+ D_{xxx}(W_i)(\frac{\partial L}{\partial \theta_0 D_{t} u} - D_t \frac{\partial L}{\partial \theta_0 D_{t} u} + D_x \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxx} \frac{\partial L}{\partial \theta_0 D_{t} u} + D_{xxxx} \frac{\partial L}{\partial \theta_0 D_{t} u} - D_{xxxxx} \frac{\partial L}{\partial \theta_0 D_{t} u})$$

$$+ D_{xxxxx}(W_i)\frac{\partial L}{\partial \theta_0 D_{t} u}$$

$$= W_i(2bvuxx - avux^2 - 2avuxu + 2buxxv + buvxxv - cuxxxv - cvxxxu - 3cuxxxv - 3cuuxv - 3cuxv - 3cu - 3dvxxx)$$

$$+ D_x(W_i)(avu^2 - buuxv + buuxv - audvxxx)$$

$$+ D_{xxx}(W_i)(buu - cuv - cuv - cuv - duvxxx) + D_{xxxx}(W_i)(cvu + dvxx) - duvD_{xxxxx}(W_i) + D_{xxxxxx}(W_i)dv.$$
where \( i = 1, 2 \), and the Lie characteristic functions \( W_i \) have the form

\[
W_1 = -u_x, \quad W_2 = -2u - \frac{6}{\alpha} t u_t - xu_x.
\]

6. Concluding Remarks

In this paper, we apply the Lie symmetry analysis method to study a sixth-order generalized time-fractional Sawada-Kotera equation. First, we obtain its vector fields and symmetry reductions. Furthermore, we use the power series method to obtain explicit analytical solutions of the equation. Finally, the CLs for the original equation are computed. This paper shows that the Lie symmetry analysis method and the power series method play significant roles in the field of mathematical physics. However, we still have several problems to address: Can we generalize time fractional Sawada-Kotera equations to a higher order? Can we use Lie symmetry analysis method to solve \((n + 1)\)-dimensional time fractional equations? How do we solve time-dependent coefficient time fractional PDEs? How can we study the space fractional equation or time-space fractional equation? These are questions that can be investigated in the future.

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**Abbreviations**

The following abbreviations are used in this manuscript:

- FPDE: Fractional Partial Differential Equation
- IST: Inverse Scattering Transformation
- DT: Darboux Transformation
- EK: Erdélyi-Kober
- PDE: Partial Differential Equation
- CK: Clarkson-Kruskal
- FPDEs: Fractional Partial Differential Equations
- KdV: Korteweg-de Vries
- RL: Riemann-Liouville
- CLs: Conservation Laws
- CL: Conservation Law

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