Klein-Gordon-Langevin Quantum Geometry.

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Abstract

We present a quantum geometric framework for stochastic quantisation in the case of a free Klein-Gordon field on Euclidean space. In this approach the noise is part of the background space, spacetime is fuzzy. We extend the notion of sharp point limit and show how fuzzy spacetime and the Klein-Gordon field gives the Euclidean space and the stochastically quantised Klein-Gordon field respectively.

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1 Prologue

‘Antonio Stradivarius voyait bien le problème soumis
à tous les luthiers de son temps.
Il savait aussi qu’il n’arriverait pas à le résoudre
en modifiant frileusement telle ou telle partie du violon de l’époque
mais en redessinant celui-ci, bref en en modifiant la structure.
C’est ce qui arriva, un peu par hasard, en 1679 ..’

Jean Diwo Les Violons du Roi, 1992.

Stochastic quantisation is one of the many quantisation methods that is available for quantising a field. It has the advantage that it is simple; for a given action \( S \) one first solves the Langevin equation

\[
\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta S}{\delta \phi(x, \tau)} + \eta(x, \tau)
\]

in terms of the Gaussian white-noise field \( \eta \) and the Green’s functions are then the \( \tau \to 0 \) limit of the \( \eta \)-expectation values

\[
\langle \phi(x_1) \ldots \phi(x_n) \rangle = \lim_{\tau \to 0} \langle \phi(x_1, \tau) \ldots \phi(x_n, \tau) \rangle_{\eta}.
\]

One also usually assumes that the background space is Euclidean [1].

It is well known that the white-noise \( \eta \) in the description of the Brownian motion of a ‘particle’ in a fluid represents the average force of the fluid on this particle [3]. What is the analog in the case of eq. [1] ? If the field is the particle analog, it seems natural to infer that the background space is fluid analog. Stated in another way, can one make a connection between \( \eta \) and the geometry of the background space, has \( \eta \) molecule-like properties ? There is also the ‘time’ parameter \( \tau \), what is its origin ? In the same way that the Brownian motion and the fluid fluctuations are not present on a large scale, the geometry producing the random fluctuation \( \eta \) should behave on a large scale as a solid space. Such a geometry would carry the Langevin noise \( \eta \), and as such would quantise any field living on this geometry through the Langevin equation. In this paper we argue that, in the case of a Klein-Gordon field, it is possible to define a geometrical structure with the properties mentioned above. Many of the ideas presented below have their origin in the beautiful and well-elaborated work by Prugovečki (see [3] and references therein) on Quantum Geometry. In the next section we introduce the necessary concepts, of which the quantum noise field is the most important, and define the sharp point limit of a process on stochastic phase space. This limit is used in section three to show, first, how it damps out the vielbein fluctuations of the space to produce a Ricci flat space and, second, how the stochastic quantised Klein-Gordon field emerges from the diffusing quantumnoise field. We conclude with some (yet) unsolved problems.
2 About Spaces and Stochasticity.

In this section we introduce two basic objects, the stochastic phase space (abbreviated as SPS) and the quantum noise field (abbreviated as QNF). The SPS is, loosely speaking, a set of points with fuzzy character. This fuzzyness will enable us to define the QNF, which will produce the quantisation of the Klein-Gordon field (abbreviated as KG field) and a stochastic metric in the next section.

In the first part of this section we describe the spaces, in the second stochasticity.

2.1 Spaces

The spacemanifold. Assume that $\mathcal{M}$ is an Euclidean manifold of dimension $D$ and $q \in \mathcal{M}$, in a certain coordinate system $q = (q^\alpha)_{\alpha=1,\ldots,D}$. This is a classical or deterministic manifold in the sense that to each point we can associate a delta distribution describing the rigidness of this point. So, let us define a deterministic point as a two-component object

$$Q = (q, \chi_q^d(q')),$$

where $\chi_q^d(q') = \delta(q - q')$ is called the confidence function and represents the ‘probability’ of the point $q \in \mathcal{M}$. By ‘probability’ I mean that the delta function reflects the fact that the point has no extension, it is fixed or sharp defined.

The stochastic spacemanifold. Suppose now that $\chi_q$ is a complex (random) function which in some limit, in a sense to be made precise later, collapses to $\chi_q^d$. The collapsing procedure will be called the classical limit or sharp point limit. In this way we extend the spacemanifold $\mathcal{M}$ to a stochastic spacemanifold $\mathcal{M}_s$ with points $Q = (q, \chi_q)$.

Phasespace. By $\Gamma_d(\mathcal{M})$ we denote the classical phasespace associated to $\mathcal{M}$. In a certain coordinate system $(q, p) = (q^\alpha, p^\beta) \in \Gamma_d(\mathcal{M})$, $\alpha, \beta = 1, \ldots, D$.

Stochastic Phasespace. In a similar way that we attached to each $q \in \mathcal{M}$ a confidence function, we attach to each $p$ of phasespace a confidence function $\chi_p$. The elements $\Pi$ of the stochastic phasespace are then constructed by a Cartesian product

$$\Pi = Q \times P$$

$$= (q, \chi_q) \times (p, \chi_p)$$

$$= ((q, p), \chi_q \chi_p)$$

$$= (\pi, \chi_\pi).$$

In the next section we will give a concrete expression for $\chi_\pi$. The confidence functions should be seen as an extra structure on a space in the same way that a connection or a metric can be added on a manifold. It should be noted that eq.(4) strongly suggests a description in terms of fiber bundles[4].
2.2 Stochasticity.

The $\eta$-proces. At each point $\pi \in \Gamma_d(M)$ we attach a multiplet of random functions $\eta_{\pi}^{a}, \ a = 1, \ldots, 2D$ parametrized by $t \in R$ and $k \in R^D$. This random field on SPS is called the $\eta$-proces. We make the following assumptions about the $\eta$-proces:

- the proces is independent of $q$ :
  \[ \eta_{\pi}^{a} = \eta_{p}^{a}, \tag{5} \]

- the proces is Gaussian, such that all the odd correlators are zero while the even ones can be computed using the Wick-property and the fact that
  \[ \langle \eta_{p}^{a}(k, t) \eta_{p^{'}}^{b}(k', t') \rangle = \delta^{ab} \delta(p - p') \delta(k + k') \delta(t - t'). \tag{6} \]

The confidence functions. The confidence functions we will use in the rest of this work are

\[ \chi_{q}(k) = \left( \frac{1}{2\pi} \right)^{D/2} \exp -ikq \tag{7} \]
\[ \chi_{\lambda,a}^{\lambda,a}(k, t) = \left( \frac{\lambda^2}{\pi} \right)^{D/4} \eta_{p}^{a}(k, t) \exp -\frac{1}{2} \lambda^2 k^2, \tag{8} \]

where $\lambda \in R$. The sharp point limit, denoted by $[ ]^{\#}$, is now defined as

\[ [ \chi_{\lambda,a}^{\lambda,a}]^{\#} = \lim_{\lambda \to 0} \langle \chi_{\lambda,a}^{\lambda,a} \rangle_{\eta} = 0 \tag{9} \]
\[ [ \chi_{q}]^{\#} = \lim_{\lambda \to 0} \langle \chi_{q} \rangle_{\eta} = \chi_{q} \tag{10} \]

and we see that the confidence function of the $p$-submanifold collapses to zero while

\[ [ \chi_{q}]^{\#} = \delta(q - q'), \tag{11} \]

which means that we recover the deterministic manifold $\mathcal{M}$.

The quantum noise field. The confidence functions as used in the context of Quantumgeometry (à la Prugovečki) are nonstochastic and play a fundamental role throughout the whole theory. I would like to show that the use of random $\chi$-functions introduce new possibilities in Quantumgeometry. Define the quantum noise vector as

\[ n^a(q, p, t) = \int dk \ \chi_{q}(k) \chi_{\lambda,a}^{\lambda,a}(k, t), \tag{12} \]

and note that it disappears in the sharp point limit:

\[ [ n^a]^{\#} = 0. \tag{13} \]

This last equation is of course a direct consequence of

\[ \langle n^a(q, p, t) \rangle_{\eta} = 0, \tag{14} \]
which seems to be a first indication that we could use $n^a$ in the sharp point limit as a candidate for a Gaussian process. A short computation shows that is indeed possible:

$$\langle n^a(q, p, t) n^b(q', p', t') \rangle_\eta = \left( \frac{1}{2\pi} \right)^D \delta^{ab} \delta(t - t') \delta(p - p') \exp \left[ -\frac{(q - q')^2}{4\lambda^2} \right], \quad (15)$$

such that in the sharp point limit

$$\left[ \lambda^{-D/2} n^a(q, p, t) n^b(q', p', t') \right]^2 = \left( \frac{1}{2\pi} \right)^D \delta^{ab} \delta(t - t') \delta(p - p') \delta(q - q'). \quad (16)$$

Which proves that the renormalized QNF behaves like a Gaussian source in the limit $\lambda \to 0$.

3 The Klein-Gordon System.

In the first two parts of this section we describe how the QNF on SPS acts as a quantising stochastic source for the KG field on a flat Euclidian manifold. The quantum field, however, is only the sharp point limit of a diffusing field produced by the diffusion of the QNF. In the last part we define several stochastic objects and show that in the sharp point limit the Feynman propagator, the Euclidean metric and the Ricci flatness is reproduced. The idea that quantum field theory dictates geometry is not new, see [2] however for a nice approach.

Starting with diffusion on SPS ...

The classical diffusion equation on SPS is

$$\left[ \frac{\partial}{\partial \tau} - \Delta \right] \psi(q, p, \tau) = 0, \quad (17)$$

with $\Delta = \delta^{\alpha\beta} \frac{\partial}{\partial q^\alpha} \frac{\partial}{\partial q^\beta}$ the Laplace operator. Inserting the QNF as a source on the right hand side gives

$$\left[ \frac{\partial}{\partial \tau} - \Delta \right] \psi^a(q, p, \tau) = n^a(q, p, \tau) \quad (18)$$

and solving for $\psi^a$ results in

$$\psi^a(q, p, t) = \int dk \chi_q(k) \Xi^\lambda_a(k, t) \quad (19)$$

$$\Xi^\lambda_a(k, t) = \int_0^t d\tau e^{-k^2(t-\tau)} \chi^\lambda_a(k, \tau). \quad (20)$$

A short inspection shows that $\psi^a$ is nothing but the inverse Fourier transform of the convolution of the Green function with $\hat{\chi}_p^\lambda a$. Let us put a mass term in eq.(18):

$$\left[ \frac{\partial}{\partial \tau} - (\Delta - m^2) \right] \psi^a(q, p, \tau) = n^a(q, p, \tau). \quad (21)$$
This gives a modified $\Xi$:

$$\Xi_{m,p}^{\lambda,a}(k, t) = \int_0^t d\tau \, e^{-(k^2+m^2)(t-\tau)} \chi_{p}^{\lambda,a}(k, \tau), \quad (22)$$

as is easily checked.

...passing along the quantum KG field...

From eq.(21) it is now a small step to the Langevin equation of the free KG field

$$\frac{\partial}{\partial \tau} \psi^{a}(q, p, t) = (\Delta - m^2) \psi^{a}(q, p, t) + n^{a}(q, p, t) \quad (23)$$

with QNF as noise source. This stochastic differential equation produces a certain ‘diffusion’ field which, as proved above, will converge to the quantised KG field in the sharp point limit since the stochastic noise has Gaussian in this limit. The solution for the stochastic differential equation is

$$\psi^{a}_{KG}(q, p, t) = \int dk \, \chi_{q}(k) \, \Xi_{p}^{\lambda,a}(k, t) \quad (24)$$

$$\Xi_{p}^{\lambda,a}(k, t) = \int_0^t d\tau \, e^{(k^2+m^2)(t-\tau)} \chi_{p}^{\lambda,a}(k, \tau). \quad (25)$$

The linear combinations of $\psi^{a}_{KG}$ with arbitrary normalised functions $c_{a}(p)$

$$\phi_{KG}^{\lambda}(q, t) = \int dp \, c_{a}(p) \psi^{a}_{KG}(q, p, t) \quad (26)$$

is also a solution of eq.(23). Because the QNF behaves as a proper Langevin/Gauss noise only in the limit $\lambda \to 0$ (and after renormalisation), we see that the quantised KG field is the sharp point limit of the renormalised $\phi_{KG}^{\lambda}$:

$$\phi_{quant.}(q) = \lim_{t \to \infty} \left[ \lambda^{-D/2} \phi_{KG}^{\lambda}(q, t) \right]^{\sharp}. \quad (27)$$

Note that all the renormalisation factors are infinite.
To make the connection between Langevin quantisation and geometry we define a vielbein field on SPS as follows

$$e^A_{\alpha}(q,p,t) = \frac{\partial}{\partial q^\alpha} n^A(q,p,t),$$

(28)

for $\alpha, A = 1, \ldots, D$. To define the inverse vielbein we should know the precise expression of the vielbein, but we can proceed in another way. Define the ‘stochastically inverse’ vielbein as

$$e_A^\alpha = \delta_{AB} \delta^{\alpha\beta} e^B_\beta,$$

(29)

and the stochastic metric in the usual way

$$g_{\alpha\beta}(q | p, p', t, t') = \left[ e^A_\alpha \delta_{AB} e^B_\beta \right]_{q=q'},$$

(30)

Now, of course the objects defined above are ill defined since we don’t know the form of the $\eta$-process, but they make sense if we average out the randomness, i.e. taking the sharp point limit:

$$\left[ \lambda^{-D} g_{\alpha\beta} \right]^\sharp \sim \delta_{\alpha\beta},$$

(31)

$$\left[ \lambda^{-2D} g_{\alpha\beta} g^{\beta\rho} \right]^\sharp \sim \delta_{\alpha\rho}.$$

(32)

Here, just as before, we have to renormalise to get a sensible answer. All this seems to indicate that the stochastic objects we defined before only make sense in the deterministic limit, that quantisation is present because of the fuzzyness but we have to average it out if we want a precise expression. At this point the QNF is the origin of both the quantisation and the metric fluctuation as defined above. A last argument showing that the QNF behaves in a nice way as a geometric object is the following

**Statement:**

The expectations of the stochastic connection and the stochastic Ricci scalar are zero, and a fortiori their sharp point limit:

$$\left[ \Gamma^\alpha_{\beta\gamma} \right]^\sharp = 0,$$

(33)

$$\left[ R \right]^\sharp = 0.$$

(34)

We note that although one expects these properties from eq.(31) and consistency requirements (remember that we started with a flat Euclidean space), the sharp point limit does not act transitively on a product such that eq.(33) and eq.(34) are nontrivial results.

The stochastic connection and the stochastic Riemann tensor are, of course, defined in the usual way [4]. The vanishing of the connection is then a direct consequence of the Wick property and the fact that

$$\forall \alpha, \beta, \rho, A, B : \left\langle e^A_{\alpha\beta} e^B_\rho \right\rangle_\eta = 0,$$

(35)

which is easily verified, while eq.(34) on the other hand is obtained after a long but straightforward computation using the same tricks as for the connection.
4 Conclusion.

We have shown how the Gaussian noise in stochastic quantisation is related to the geometry of the background space. The link between both is the quantum noise field, playing a double role: it is used as random source in the Langevin equation and it produces a vielbein field on stochastic phase space. In this way the ‘time’ parameter, which seems to have only a minor place throughout stochastic quantisation, has its origin in the fuzzy points of the stochastic phase space. There are, however, many open questions:

• can one proceed, even for this simple KG system, in a similar way if the background space is curved?

• adding a potential to the KG system does not seem to give many problems, but what about generalising to other theories?

• a derivation of the Langevin equation, based on ‘coarse graining’ and renormalisation, is still missing. Could a geometric principle really produce the Langevin equation or the other way round?

• can one incorporate in some way the $t \to \infty$ limit into the sharp point limit?

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References

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