On syzygies of Segre embeddings

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Abstract

We study the syzygies of the ideals of the Segre embeddings. Let $d \in \mathbb{N}$, $d \geq 3$; we prove that the line bundle $\mathcal{O}(1, \ldots, 1)$ on the $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($d$ copies) satisfies Property $N_p$ of Green-Lazarsfeld if and only if $p \leq 3$. Besides we prove that if we have a projective variety not satisfying Property $N_p$ for some $p$, then the product of it with any other projective variety does not satisfy Property $N_p$. From this we deduce also other corollaries about syzygies of Segre embeddings.

1 Introduction

Let $M$ be a very ample line bundle on a smooth complex projective variety $Y$ and let $\varphi_M : Y \to \mathbb{P}(H^0(Y, M)^*)$ be the map associated to $M$. We recall the definition of Property $N_p$ of Green-Lazarsfeld, studied for the first time by Green in [Gr1-2] (see also [G-L], [Gr3]): let $Y$ be a smooth complex projective variety and let $L$ be a very ample line bundle on $Y$ defining an embedding $\varphi_L : Y \to \mathbb{P} = \mathbb{P}(H^0(Y, L)^*)$; set $S = S(L) = \text{Sym}^* H^0(L)$, the homogeneous coordinate ring of the projective space $\mathbb{P}$, and consider the graded $S$-module $G = G(L) = \bigoplus_n H^0(Y, L^n)$; let $E_* : 0 \to E_n \to E_{n-1} \to \ldots \to E_0 \to G \to 0$ be a minimal graded free resolution of $G$; the line bundle $L$ satisfies Property $N_p$ ($p \in \mathbb{N}$) if and only if

$E_0 = S$

$E_i = \oplus S(-i - 1)$ for $1 \leq i \leq p$.

(Thus $L$ satisfies Property $N_0$ if and only if $Y \subset \mathbb{P}(H^0(L)^*)$ is projectively normal, i.e. $L$ is normally generated; $L$ satisfies Property $N_1$ if and only if $L$ satisfies Property $N_0$ and the homogeneous ideal $I$ of $Y \subset \mathbb{P}(H^0(L)^*)$ is generated by quadrics; $L$ satisfies Property $N_2$ if and only if $L$ satisfies Property $N_1$ and the module of syzygies among quadratic generators $Q_i \in I$ is spanned by relations of the form $\sum L_i Q_i = 0$, where $L_i$ are linear polynomials; and so on.)

Now let $L = \mathcal{O}_{\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_d}}(a_1, \ldots, a_d)$, where $d, a_1, \ldots, a_d, n_1, \ldots, n_d$ are positive integers. The known results on the syzygies in this case are the following:

Case $d = 1$, i.e. the case of the Veronese embedding:

**Theorem 1 (Green)** [Gr1-2]. Let $a$ be a positive integer. The line bundle $\mathcal{O}_{\mathbb{P}^n}(a)$ satisfies Property $N_a$.  

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Theorem 2 (Ottaviani-Paoletti) \([O-P]\): If \(n \geq 2, a \geq 3\) and the bundle \(\mathcal{O}_{\mathbb{P}^n}(a)\) satisfies Property \(N_p\), then \(p \leq 3a - 3\).

Theorem 3 (Josefiak-Pragacz-Weyman) \([J-P-W]\): The bundle \(\mathcal{O}_{\mathbb{P}^n}(2)\) satisfies Property \(N_p\) if and only if \(p \leq 5\) when \(n \geq 3\) and for all \(p\) when \(n = 2\).

(See \([O-P]\) for a more complete bibliography.)

Case \(d = 2:\)

Theorem 4 (Gallego-Purnapranja) \([G-P]\): Let \(a, b \geq 2\). The line bundle \(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)\) satisfies Property \(N_p\) if and only if \(p \leq 2a + 2b - 3\).

Theorem 5 (Lascoux-Pragacz-Weymann) \([Ls], [P-W]\): Let \(n_1, n_2 \geq 2\). The line bundle \(\mathcal{O}_{\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}}(1,1)\) satisfies Property \(N_p\) if and only if \(p \leq 3\).

Here we consider \(\mathcal{O}(1, ..., 1)\) on \(\mathbb{P}^1 \times ... \times \mathbb{P}^1\) \((d\ times, \ for \ any \ d)\). We prove (Section 2):

Theorem 6: The line bundle \(\mathcal{O}(1, ..., 1)\) on \(\mathbb{P}^1 \times ... \times \mathbb{P}^1\) \((d\ times)\) satisfies Property \(N_3\) for any \(d\).

Besides we prove (Section 3):

Proposition 7: Let \(X\) and \(Y\) be two projective varieties and let \(L\) be a line bundle on \(X\) and \(M\) a line bundle on \(Y\). Let \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) be the canonical projections. Suppose \(L\) and \(M\) satisfy Property \(N_1\). Let \(p \geq 2\). If \(L\) does not satisfy Property \(N_p\), then \(\pi_X^* L \otimes \pi_Y^* M\) does not satisfy Property \(N_p\), either.

Corollary 8: Let \(a_1, ..., a_d\) be positive integers with \(a_1 \leq a_2 \leq ... \leq a_d\). Suppose \(k = \max\{i|a_i = 1\}\). If \(k \geq 3\) the line bundle \(\mathcal{O}_{\mathbb{P}^{a_1} \times ... \times \mathbb{P}^{a_d}}(1, ..., a_d)\) does not satisfy Property \(N_4\) and if \(d - k \geq 2\) it does not satisfy Property \(N_{2a_{k+1} + 2a_{k+2} - 2}\).

In particular, from Corollary 8 and Theorem 5, we have:

Corollary 9: Let \(d \geq 3\). The line bundle \(\mathcal{O}_{\mathbb{P}^1 \times ... \times \mathbb{P}^1}(1, ..., 1)\) \((d\ times)\) satisfies Property \(N_p\) if and only if \(p \leq 3\).

2 Proof of Theorem 5

First we have to recall some facts on toric ideals from \([St]\).

Let \(k \in \mathbb{N}\). Let \(A = \{a_1, ..., a_n\}\) be a subset of \(\mathbb{Z}^k\). The toric ideal \(\mathcal{I}_A\) is defined as the ideal in \(\mathbb{C}[x_1, ..., x_n]\) generated as vector space by the binomials

\[
x_1^{u_1} ... x_n^{u_n} - x_1^{v_1} ... x_n^{v_n}
\]

for \((u_1, ..., u_n), (v_1, ..., v_n) \in \mathbb{N}^n\), with \(\sum_{i=1}^{n} u_i a_i = \sum_{i=1}^{n} v_i a_i\).

We have that \(\mathcal{I}_A\) is homogeneous if and only if \(\exists \omega \in \mathbb{Q}^k\) s.t. \(\omega \cdot a_i = 1 \ \forall i = 1, ..., n\); the rings \(\mathbb{C}[x_1, ..., x_n]\) and \(\mathbb{C}[x_1, ..., x_n]/\mathcal{I}_A\) are multigraded by \(\mathbb{N}A\) via \(\deg x_i = a_i\); the element \(x_1^{u_1} ... x_n^{u_n}\) has multidegree \(b = \sum_i u_i a_i \in \mathbb{N}A\) and degree \(\sum_i u_i = b \cdot \omega\); we define \(\deg b = b \cdot \omega\).
Theorem 12.12 p.120 in [St] studies the syzygies of the ideal $\mathcal{I}_A$; for each $b \in \mathbb{N}A$, let $\Delta_b$ be the simplicial complex on the set $\{1, \ldots, n\}$ defined as follows:

$$\Delta_b = \{ F \subseteq \{1, \ldots, n\} : b - \sum_{i \in F} a_i \in \mathbb{N}A \}$$

(thus, by identifying $\{1, \ldots, n\}$ with $A$, we have:

$$\Delta_b = \bigcup_{k \in \mathbb{N}, a_{i_1}, \ldots, a_{i_k} \in A, a_{i_1} + \ldots + a_{i_k} = b} < a_{i_1}, \ldots, a_{i_k} >,$$

where $< a_{i_1}, \ldots, a_{i_k} >$ is the simplex generated by $a_{i_1}, \ldots, a_{i_k}$).

Theorem 10 (Campillo-Pison-Sturmfels) [C-F], [S] (Thm. 12.12). Let $A = \{a_1, \ldots, a_n\}$ be a subset of $\mathbb{Z}^k$ and $\mathcal{I}_A$ be the associated toric ideal. Let $0 \to E_n \to \ldots \to E_1 \to E_0 \to G \to 0$ be a minimal free resolution of $G = C[x_1, \ldots, x_n]/\mathcal{I}_A$ on $C[x_1, \ldots, x_n]$. Each of the generators of $E_j$ has a unique multidegree. The number of the generators of multidegree $b \in \mathbb{N}A$ of $E_{j+1}$ equals the rank of the $j$-th reduced homology group $\tilde{H}_j(\Delta_b, C)$ of the simplicial complex $\Delta_b$.

Notation 11. If $\alpha$ is a chain in a topological space, $sp(\alpha)$ will denote the support of $\alpha$, i.e. the union of the supports of the simplexes $\sigma_i$ s.t. $\alpha = \sum_i c_i \sigma_i$, $c_i \in \mathbb{Z}$. If $X$ is a simplicial complex, $sk^i(X)$ will denote the $i$-skeleton of $X$.

Proof of Theorem 11. If we take $A = A_d = \{(1, \epsilon_1, \ldots, \epsilon_d) | \epsilon_i \in \{0, 1\}\}$, we have that $\mathcal{I}_{A_d}$ is the ideal of the Segre embedding of $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($d$ times), i.e. the ideal of the embedding of $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($d$ times) by the line bundle $O(1, \ldots, 1)$. In this case $\omega = \omega_d = (1, 0, \ldots, 0)$ ($0$ repeated $d$ times) and $n = 2^d$.

Let $b \in \mathbb{N}A_d$: we have that $\deg b = (b \cdot \omega) = k$ if and only if $b$ is the sum of $k$ (not necessarily distinct) elements of $A_d$. By identifying the set $\{1, \ldots, 2^d\}$ with $A_d$, we have that, if $k = \deg b$, $\Delta_b = \bigcup_{a_{i_1}, \ldots, a_{i_k} \in A_d, a_{i_1} + \ldots + a_{i_k} = b} < a_{i_1}, \ldots, a_{i_k} >$; we say that $< a_{i_1}, \ldots, a_{i_k} >$ is a degenerate $k$-simplex if $\exists l, m \in \{1, \ldots, k\}$ with $l \neq m$ s.t. $a_{i_l} = a_{i_m}$; thus $\Delta_b$ is equal to the union of the (possibly degenerate) $k$-simplexes $S$ with vertices in $A_d$ such that the sum of the vertices (with multiplicities) of $S$ is $b$.

By Theorem 1, in order to prove that $\mathcal{O}_{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}(1, \ldots, 1)$ ($d$ times) satisfies $N_2$, we have to prove that $H_1(\Delta_b) = 0$ for each $b \in \mathbb{N}A_d$ with $\deg b \geq 4$. Analogously in order to prove that $\mathcal{O}_{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}(1, \ldots, 1)$ ($d$ times) satisfies $N_3$, we have to prove that $H_2(\Delta_b) = 0$ for each $b \in \mathbb{N}A_d$ with $\deg b \geq 5$.

The proof is by induction on $d$. Observe that any $b' \in \mathbb{N}A_{d+1}$ with $\deg b' = k$ is equal to $\binom{b}{\epsilon}$ for some $b \in \mathbb{N}A_d$ with $\deg b = k$ and for some $\epsilon \in \{0, 1, \ldots, k\}$. Then, in order to prove $N_2$ we suppose (by induction) that $H_1(\Delta_b) = 0 \ \forall b \in \mathbb{N}A_d$ with $\deg b = k, k \geq 4$ and we show that $H_1(\Delta_b) = 0$ for $\epsilon \in \{0, \ldots, k\}$ and in order to prove $N_3$ we suppose (by induction) that $H_2(\Delta_b) = 0 \ \forall b \in \mathbb{N}A_d$ with $\deg b = k, k \geq 5$, and we show that $H_2(\Delta_b) = 0$ for $\epsilon \in \{0, \ldots, k\}$.

Observe that, if $\epsilon \in \{0, k\}$ ($k := \deg b$), then obviously $\Delta_b \binom{\epsilon}{\epsilon}$ and $\Delta_b$ are isomorphic; besides $\Delta_b \binom{\epsilon}{\epsilon}$ is isomorphic to $\Delta_b \binom{k-\epsilon}{\epsilon}$ (the isomorphism is given by substituting $0$ with $1$ and $1$ with $0$ in the last coordinate). Thus we may consider only the cases $\epsilon \in \{1, \ldots, \binom{k}{2}\}$.

First we need some preliminary notation and lemmas.
Notation 12 Let $S = < a_1, ..., a_k >$ be a (possibly degenerate) $k$-simplex, $a_i \in A_d$. Let $\varepsilon \in \{0, ..., k\}$. We denote

$$S'_{\varepsilon} = \bigcup_{\chi_j \in \{0,1\} \text{ for } j=1,...,k} \text{ and exactly } \varepsilon \text{ of } \chi_1,...,\chi_k \text{ are equal to } 1 < \left( \frac{a_1}{\chi_1}, ..., \frac{a_k}{\chi_k} \right).$$

Example 13 Let $S = < a_1, a_2, a_3, a_4 >$ is a (possibly degenerate) tetrahedron, $a_i \in A_d$. The set $S'_1$ is the union of the four (possibly degenerate) tetrahedrons

$$< \left( \frac{a_1}{0}, \frac{a_2}{1}, \frac{a_3}{0}, \frac{a_4}{1} \right), < \left( \frac{a_1}{0}, \frac{a_2}{0}, \frac{a_3}{1}, \frac{a_4}{1} \right), < \left( \frac{a_1}{0}, \frac{a_2}{0}, \frac{a_3}{0}, \frac{a_4}{1} \right).$$

Thus $S'_1$ can be obtained from $S$ by “constructing a tetrahedron on everyone of the four faces of $S$” and considering the union of these four tetrahedrons. The set $S'_2$ is the union of the following six (possibly degenerate) tetrahedrons:

$$< \left( \frac{a_1}{0}, \frac{a_2}{0}, \frac{a_3}{1}, \frac{a_4}{1} \right), < \left( \frac{a_1}{0}, \frac{a_2}{1}, \frac{a_3}{1}, \frac{a_4}{1} \right), < \left( \frac{a_1}{0}, \frac{a_2}{1}, \frac{a_3}{0}, \frac{a_4}{0} \right),$$

Then $S'_2$ can be obtained from $S$ by “constructing a tetrahedron on everyone of the six edges of $S$” and considering the union of these six tetrahedrons (see Fig. 1, representing $S'_\varepsilon$ in the case $S$ is not degenerate).

![Figure 1](image_url)

**NOTE TO FIG. 1.** In the representation of $S'_2$, for the sake of simplicity, we do not represent the tetrahedrons

$$< \left( \frac{a_1}{0}, \frac{a_2}{1}, \frac{a_3}{0}, \frac{a_4}{1} \right) \text{ and } < \left( \frac{a_1}{1}, \frac{a_2}{0}, \frac{a_3}{1}, \frac{a_4}{0} \right).$$

Let $b \in NA_d$ with $\deg b = k$ and $\varepsilon \in \{0, ..., k\}$. Obviously

$$\Delta_{\varepsilon} = \bigcup_{S = < a_1, ..., a_k > \text{ with } a_1 + ... + a_k = b} S'_\varepsilon.$$

Notation 14 Let $b \in NA_d$ with $\deg b = k$. For $l \in \mathbb{N}$, $0 \leq l \leq k - 1$, let

$$F^l(\Delta_b) = \bigcup_{a_1, ..., a_k \in A_d \text{ s.t. } a_1 + ... + a_k = b} \bigcup_{i_0, ..., i_l \in \{1, ..., k\}} < \left( \frac{a_0}{i_0}, ..., \frac{a_l}{i_l} \right).$$

Observe that $F^l(\Delta_b) \subseteq \Delta_{\frac{l}{k}}$ iff $k - \varepsilon \geq l + 1$.

The idea of the proof is to consider a $l$-cycle (for $l = 1, 2$) in $\Delta_{\frac{l}{k}}$ and to show that it is homologous to a $l$-cycle in $F^l(\Delta_b)$ and then to show that it is homologous to 0 by using that $H_l(\Delta_b) = 0$. 


Remark 15 Let $S = \{a_1, \ldots, a_k\}$, where $a_i \in A_d$. If $k \geq 4$ and $1 \leq \varepsilon \leq k - 2$, the set $S^\varepsilon$ contains the cone with vertex $\left(\frac{a_{1j}}{1}\right)$ on the border of $\left(\frac{a_{1i}}{0}, \frac{a_{ij}}{0}, \frac{a_{ij}}{0}\right)$ for any $i_1, i_2, i_3, l \in \{1, \ldots, k\}$ with $l \neq i_j$ for $j = 1, 2, 3$. This is true in particular if $k \geq 4$ and $\varepsilon \in \{1, \ldots, \frac{k}{2}\}$.

If $k \geq 5$ and $1 \leq \varepsilon \leq k - 3$, the set $S^\varepsilon_k$ contains the cone with vertex $\left(\frac{a_{1j}}{1}\right)$ on the border of $\left(\frac{a_{1i}}{0}, \ldots, \frac{a_{1i}}{0}\right)$ for any $i_1, \ldots, i_4, l \in \{1, \ldots, k\}$ with $l \neq i_j$ for $j = 1, 2, 3, 4$. This is true in particular if $k \geq 5$ and $\varepsilon \in \{1, \ldots, \frac{k}{2}\}$.

Definition 16 For any $c \in \mathbb{N}A_d$ with $\deg c = s$ and $\varepsilon \in \{1, \ldots, s\}$, we define $R_{c, \varepsilon}$ the following set:

$$\bigcup_{a_1, \ldots, a_s \in A_d \text{ s.t. } a_1 + \ldots + a_s = c} \bigcup_{i_1, \ldots, i_{s-1} \in \{1, \ldots, s\}, i_{s-1} \neq i_m} \left(\frac{a_{i_1}}{1}, \ldots, \left(\frac{a_{i_{s-1}}}{1}\right), \left(\frac{a_i}{0}\right), \ldots, \left(\frac{a_{i_{s-1}}}{0}\right)\right).$$

Lemma 17 Let $c \in \mathbb{N}A_d$ with $\deg c = s$. We have that $\tilde{H}_i(\Delta_\varepsilon(\varepsilon - 1)) = 0$ implies $\tilde{H}_i(R_{c, \varepsilon}) = 0$ if we are in one of the following cases: a) $i = 0$, $s \geq 3$, $\varepsilon \in \{1, \ldots, \frac{s+1}{2}\}$ b) $i = 1$, $s \geq 4$, $\varepsilon \in \{1, \ldots, \frac{s+1}{2}\}$

Proof. Observe that $R_{c, \varepsilon} \subseteq \Delta_\varepsilon(\varepsilon - 1)$. Since $\tilde{H}_i(\Delta_\varepsilon(\varepsilon - 1)) = 0$, we have $\tilde{H}_i(sk^{i+1}(\Delta_\varepsilon(\varepsilon - 1))) = 0$. Obviously $sk^{i+1}(R_{c, \varepsilon}) \subseteq sk^{i+1}(\Delta_\varepsilon(\varepsilon - 1))$. We want to show $\tilde{H}_i(sk^{i+1}(R_{c, \varepsilon})) = 0$. Let $\beta$ be a $i$-cycle in $sk^{i+1}(R_{c, \varepsilon})$. Since $\tilde{H}_i(sk^{i+1}(\Delta_\varepsilon(\varepsilon - 1))) = 0$, there exists a $(i + 1)$-chain $\eta$ in $sk^{i+1}(\Delta_\varepsilon(\varepsilon - 1))$ s.t. $\partial \eta = \beta$. Suppose $sp(\eta) = \bigcup_j F_j$, where $F_j$ are $(i + 1)$-simplexes in $sk^{i+1}(\Delta_\varepsilon(\varepsilon - 1))$; consider now a $(i + 1)$-chain $\psi$ in $sk^{i+1}(R_{c, \varepsilon})$ whose support is $\bigcup_j \tilde{F}_j$, where $\tilde{F}_j = F_j$ if $F_j \subseteq sk^{i+1}(R_{c, \varepsilon})$ and $\tilde{F}_j$ is a cone on the border of $F_j$ if $F_j \nsubseteq sk^{i+1}(R_{c, \varepsilon})$, in such way that $\partial \psi = \beta$ (observe that in our cases such cones exist, in fact: $R_{c, \varepsilon}$ is the union of the (possibly degenerate) $(s - 1)$-simplexes “obtained from the (possibly degenerate) $s$-simplexes of $\Delta_\varepsilon(\varepsilon - 1)$ by taking off a vertex whose last coordinate is $0'$; in the case $i = 0$ one can check that the 1-simplexes whose vertices have the last coordinates equal to 1, 1 or to 1, 0 are contained in $R_{c, \varepsilon}$, while for a 1-simplex $F$ whose vertices have the last coordinates equal to 0, there exists a cone, $\tilde{F}$, on the border of $F$ with $\tilde{F} \subseteq R_{c, \varepsilon}$, since $s \geq 3$; analogously the case b)). Thus we proved $\tilde{H}_i(sk^{i+1}(R_{c, \varepsilon})) = 0$. Thus $\tilde{H}_i(R_{c, \varepsilon}) = 0$.

Q.e.d. in Lemma 17

PROOF THAT $O_{P^1 \times \ldots \times P^1}(1, \ldots, 1)$ SATISFIES PROPERTY $N_2$.

Lemma 18 Let $b \in \mathbb{N}A_d$, $\deg b = k$, $k \geq 4$ and $\varepsilon \in \{1, \ldots, \frac{k}{2}\}$. Every 1-cycle $\gamma$ in $\Delta_\varepsilon(\varepsilon)$ is homologous to a 1-cycle in $F_1(\Delta_b)$ (which is $\subseteq \Delta_\varepsilon(\varepsilon)$ since $k - \varepsilon \geq 2$).

Proof. Obviously we can suppose $sp(\gamma) \subseteq sk(\Delta_\varepsilon(\varepsilon))$. The proof is by induction on the cardinality of $(sp(\gamma) \cap sk(\Delta_\varepsilon(\varepsilon))) - F_1(\Delta_b)$, i.e. we will prove that $\gamma$ is homotopically equivalent a 1-cycle $\gamma$ s.t. $\sharp((sp(\gamma) \cap sk(\Delta_\varepsilon(\varepsilon))) - F_1(\Delta_b)) < \sharp((sp(\gamma) \cap sk(\Delta_\varepsilon(\varepsilon))) - F_1(\Delta_b))$. 


Let \( \binom{a}{1} \in (sp(\gamma) \cap sk^0(\Delta_{\binom{b}{a}^\gamma})) - F^1(\Delta_b), (a \in A_d) \). Let \( \binom{a}{1}, P_1 > \cup < \binom{a}{1}, P_2 > \subseteq sp(\gamma), \) with \( P_j \in sk^0(\Delta_{\binom{b}{a}^\gamma}) \) for \( j = 1, 2 \). Precisely let \( \gamma = \sigma_1 + \sigma_2 + \ldots, \) where \( \sigma_1 \) and \( \sigma_2 \) are two simplexes \( \sigma_1 : [0,1] \rightarrow < \binom{a}{1}, P_1 > \) and \( \sigma_2 : [0,1] \rightarrow < \binom{a}{1}, P_2 > \) s.t. \( \sigma_1(0) = P_1, \sigma_1(1) = \binom{a}{1} = \sigma_2(0), \sigma_2(1) = P_2. \)

Let \( \alpha \) be the 1-cycle \( -\sigma_1 - \sigma_2 + \sigma'_1 + \sigma'_2, \) where \( \sigma'_1 \) and \( \sigma'_2 \) are two simplexes \( \sigma'_1 : [0,1] \rightarrow < \binom{a}{0}, P_1 > \) and \( \sigma'_2 : [0,1] \rightarrow < \binom{a}{0}, P_2 > \) s.t. \( \sigma'_1(0) = P_1, \sigma'_1(1) = \binom{a}{0} = \sigma'_2(0), \sigma'_2(1) = P_2. \)

The support of \( \alpha \) is the union of the two cones with vertices \( \binom{a}{1} \) and \( \binom{a}{0} \) on \( \{P_1, P_2\} \).

We state that \( P_i \in R_{b-a,\varepsilon} \) for \( i = 1, 2 \). In fact, \( < P_i, \binom{a}{1} > \subseteq \Delta_{\binom{b}{a}^\gamma}, \) then \( P_i \in \Delta_{\binom{b}{a}^\gamma}; \) we recall that \( R_{b-a,\varepsilon} \) is

\[
\bigcup_{\alpha_1, \ldots, \alpha_{k-1} \in A_d \text{ s.t. } \alpha_1 + \ldots + \alpha_{k-1} = b-a} \bigcup_{i_1, \ldots, i_{k-2} \in \{1, \ldots, k-1\}, i_l \neq i_m} < \binom{\alpha_{i_1}}{1}, \ldots, \binom{\alpha_{i_{k-1}}}{1} > \cup < \binom{\alpha_i}{0}, \binom{\alpha_{i+1}}{0}, \ldots, \binom{\alpha_{i_{k-2}}}{0} >, \]

i.e. \( R_{b-a,\varepsilon} \) is the union of the (possibly degenerate) \((k-2)\)-simplexes “obtained from the (possibly degenerate) \((k-1)\)-simplexes of \( \Delta_{\binom{b}{a}^\gamma} \) by taking off a vertex whose last coordinate is 0”; then, if the last coordinate of \( P_i \) is 1, we may conclude at once that \( P_i \in R_{b-a,\varepsilon} \); also if the last coordinate of \( P_i \) is 0, we may conclude that \( P_i \in R_{b-a,\varepsilon} \) because the number of the vertices whose last coordinate is 0 in a (possibly degenerate) \((k-1)\)-simplex of \( \Delta_{\binom{b}{a}^\gamma} \) is \( k-1 - (\varepsilon - 1) \geq 2 \).

Thus \( sp(\alpha) \subseteq C, \) where \( C \) is the union of the two cones \( < \binom{a}{1}, R_{b-a,\varepsilon} > \) and \( < \binom{a}{0}, R_{b-a,\varepsilon} >. \)

Observe that \( C \subseteq \Delta_{\binom{b}{a}^\gamma}. \)

We want to show that \( H_1(C) = 0 \); by Theorem 11, since \( \mathcal{O}_{p_1 \times \ldots \times p_1}(1, \ldots, 1) \) (d times) satisfies Property \( N_1 \) \( \forall d, \) we have \( \tilde{H}_0(\Delta_g) = 0 \) \( \forall g \in \mathbb{N}A_d \) with deg \( g \geq 3, \) \( \forall d \) (this can be easily proved directly without using that \( \mathcal{O}_{p_1 \times \ldots \times p_1}(1, \ldots, 1) \) satisfies Property \( N_1 \)); then \( \tilde{H}_0(\Delta_{\binom{b}{a}^\gamma}) = 0; \) thus \( \tilde{H}_0(R_{b-a,\varepsilon}) = 0 \) by Lemma 17 we have \( \tilde{H}_1(C) = \tilde{H}_{i-1}(R_{b-a,\varepsilon}); \) thus \( H_1(C) = \tilde{H}_0(R_{b-a,\varepsilon}) = 0. \)

Thus we have that \( \alpha \) is homologous to 0.

Thus \( \gamma \) is homologous to \( \gamma + \alpha; \) obviously \( \gamma + \alpha \) (and then \( \gamma \)) is homologous to a 1-cycle \( \tilde{\gamma} \) whose support can be obtained from \( sp(\gamma) \) by substituting the cone with vertex \( \binom{a}{1} \) on \( \{P_1, P_2\} \) with the cone with vertex \( \binom{a}{0} \) on \( \{P_1, P_2\}. \) Then \( \sharp((sp(\tilde{\gamma}) \cap sk^0(\Delta_{\binom{b}{a}^\gamma}))) - F^1(\Delta_b)) < \sharp((sp(\gamma) \cap sk^0(\Delta_{\binom{b}{a}^\gamma}))) - F^1(\Delta_b)); \) thus we conclude.

\[Q.e.d. \text{ in Lemma 13}\]

In order to prove that \( \mathcal{O}_{p_1 \times \ldots \times p_1}(1, \ldots, 1) \) (d times) satisfies \( N_2 \) for any \( d, \) we suppose (by induction) that \( H_1(\Delta_b) = 0 \) \( \forall b \in \mathbb{N}A_d \) with deg \( b = k, k \geq 4 \) and we show that \( H_1(\Delta_{\binom{a}{1}}^\gamma) = 0 \) for \( \varepsilon \in \{1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}. \)

Cases \( \varepsilon \leq k - 3. \) We know that every 1-cycle \( \gamma \) in \( \Delta_{\binom{a}{1}} \) is homologous to a 1–cycle in \( F^1(\Delta_b) \) by Lemma 18. Thus, since \( F^2(\Delta_b) \subseteq \Delta_{\binom{a}{1}} \) and \( H_1(F^2(\Delta_b)) = 0 \) (because, by induction hypothesis, \( H_1(\Delta_b) = 0), \) we have that \( H_1(\Delta_{\binom{a}{1}}) = 0. \)
Cases \( \varepsilon > k - 3 \). These cases are slightly more difficult. By Lemma 18 every 1-cycle \( \gamma \) in \( \Delta_{b}(\delta) \) is homologous to a 1-cycle \( \gamma' \) in \( F^{1}(\Delta_{b}) \). But in these cases we have not the inclusion \( F^{2}(\Delta_{b}) \subseteq \Delta_{b}(\delta) \), thus we have to conclude the proof in another way.

Since \( H_{1}(F^{2}(\Delta_{b})) = 0 \), there exists a 2-chain \( \mu \in F^{2}(\Delta_{b}) \) s.t. \( \partial \mu = \gamma' \). Let \( sp(\mu) = \cup_{i} F_{i} \), \( F_{i} \) triangles in \( F^{2}(\Delta_{b}) \). Consider a 2-chain \( \psi \) in \( \Delta_{b}(\delta) \) whose support is \( \cup_{i} F_{i} \), where \( F_{i} \) is a cone \( \subseteq \Delta_{b}(\delta) \) on the border of \( F_{i} \) (there exists by Remark 13), in such way that \( \partial \psi = \gamma' \); thus \( [\gamma'] = 0 \) in \( H_{1}(\Delta_{b}(\delta)) \), thus \( [\gamma] = 0 \) in \( H_{1}(\Delta_{b}(\delta)) \). Thus \( H_{1}(\Delta_{b}(\delta)) = 0 \).

**Proof that** \( \mathcal{O}_{P^{1} \times \ldots \times P^{1}}(1, \ldots, 1) \) **satisfies property** \( N_{3} \)

**Lemma 19** Let \( b \in N_{A_{d}} \) with \( \deg b = k \) and \( \varepsilon \in \{1, \ldots, \lfloor \frac{k}{2} \rfloor \} \). If \( k \geq 5 \), every 2-cycle \( \mu \) in \( \Delta_{b}(\delta) \) is homologous to a 2-cycle in \( F^{2}(\Delta_{b}) \) (which is \( \subseteq \Delta_{b}(\delta) \) since \( k - \varepsilon \geq 3 \)).

**Proof.** Obviously we can suppose that \( sp(\mu) \subseteq sk^{2}(\Delta_{b}(\delta)) \). The proof is by induction on the cardinality of \( (sp(\mu) \cap sk^{0}(\Delta_{b}(\delta))) - F^{2}(\Delta_{b}) \), i.e. we will prove that \( \mu \) is homotopically equivalent to a 2-cycle \( \tilde{\mu} \) s.t. \( \#((sp(\tilde{\mu}) \cap sk^{0}(\Delta_{b}(\delta))) - F^{2}(\Delta_{b})) < \#((sp(\mu) \cap sk^{0}(\Delta_{b}(\delta))) - F^{2}(\Delta_{b})) \).

Let \( \begin{pmatrix} a \\ 1 \end{pmatrix} \in (sp(\mu) \cap sk^{0}(\Delta_{b}(\delta))) - F^{2}(\Delta_{b}), \ (a \in A_{d}) \). Let \( < \begin{pmatrix} a \\ 1 \end{pmatrix}, P_{1}, P_{2} > \cup < \begin{pmatrix} a \\ 1 \end{pmatrix}, P_{2}, P_{3} > \cup \ldots \cup < \begin{pmatrix} a \\ 1 \end{pmatrix}, P_{r-1}, P_{r} > \cup < \begin{pmatrix} a \\ 1 \end{pmatrix}, P_{r}, P_{1} > \subseteq sp(\mu), \) with \( P_{3} \in sk^{0}(\Delta_{b}(\delta)) \).

Let \( \alpha \) be a 2-cycle whose support is the union of the two cones with vertices \( \begin{pmatrix} a \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} a \\ 0 \end{pmatrix} \) on the polygon with vertices \( P_{1}, \ldots, P_{r} \); choose \( \alpha \) in such way that \( \mu + \alpha \) is homologous to a 2-cycle \( \tilde{\mu} \) whose support can be obtained from \( sp(\mu) \) by substituting the cone with vertex \( \begin{pmatrix} a \\ 1 \end{pmatrix} \) on the polygon with vertices \( P_{1}, \ldots, P_{r} \) with the cone with vertex \( \begin{pmatrix} a \\ 0 \end{pmatrix} \) on the polygon with vertices \( P_{1}, \ldots, P_{r} \).

We state that \( < P_{i}, P_{i+1} > \subseteq R_{b-a,\varepsilon} \) for \( i = 1, \ldots, r-1 \) and \( < P_{r}, P_{1} > \subseteq R_{b-a,\varepsilon} \). In fact:

\( < P_{i}, P_{i+1} > \subseteq \Delta_{b}(\delta) \), then \( < P_{i}, P_{i+1} > \subseteq \Delta_{b}(\delta_{b-a,\varepsilon}) \); since \( R_{b-a,\varepsilon} \) is the union of the (possibly degenerate) \( (k-2) \)-simplexes “obtained from the (possibly degenerate) \( (k-1) \)-simplexes of \( \Delta_{b}(\delta_{b-a,\varepsilon}) \) by taking off a vertex whose last coordinate is 0” and since the number of the vertices whose last coordinate is 0 in a (possibly degenerate) \( (k-1) \)-simplex of \( \Delta_{b}(\delta_{b-a,\varepsilon}) \) is \( k - 1 - (\varepsilon - 1) \geq 3 \), we have \( < P_{i}, P_{i+1} > \subseteq R_{b-a,\varepsilon} \) (obviously in a completely analogous way we have \( < P_{r}, P_{1} > \subseteq R_{b-a,\varepsilon} \)).

Thus \( sp(\alpha) \subseteq C \), where \( C \) is the union of the two cones \( < \begin{pmatrix} a \\ 1 \end{pmatrix}, R_{b-a,\varepsilon} > \) and \( < \begin{pmatrix} a \\ 0 \end{pmatrix}, R_{b-a,\varepsilon} > \).

Observe that \( C \subseteq \Delta_{b}(\delta) \).
We want to show that \( H_2(C) = 0 \): we have already proved that \( \mathcal{O}_{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}(1, \ldots, 1) \) satisfies Property \( N_2 \) i.e. \( H_1(\Delta_a) = 0 \) for \( g \) with \( \deg g \geq 4 \); thus \( H_1(\Delta_{\binom{b}{\varepsilon}}) = 0 \); then \( H_1(R_{b-a,\varepsilon}) = 0 \) by Lemma 17.

Thus we have that \( \alpha \) is homologous to \( 0 \).

3 Proof of Proposition 7

Let \( X \) and \( Y \) be two projective varieties and \( L \) a line bundle on \( X \) and \( M \) a line bundle on \( Y \). Let \( \{ \sigma_0, \ldots, \sigma_k \} \) be a basis of \( H^0(X, L) \) and let \( \{ s_0, \ldots, s_l \} \) be a basis of \( H^0(Y, M) \); we can suppose \( \exists \, \mathbf{f} \in Y \) s.t. \( s_0(\mathbf{f}) \neq 0 \), \( s_j(\mathbf{f}) = 0 \) for \( j \neq 0 \); let \( t_{i,j} \) be the coordinates corresponding to \( \{ \sigma_i \otimes s_j \}_{i,j} \) of the embedding of \( X \times Y \) by \( \pi_X^*L \otimes \pi_Y^*M \) (where \( \pi \) is the projection on \( \cdot \)) and let \( t_i \) be the coordinates corresponding to \( \{ \sigma_0, \ldots, \sigma_k \} \) of the embedding of \( X \) by \( L \).

Remark 20 By setting \( t_{i,j} = 0 \) for \( j \neq 0 \) in an equation of \( X \times Y \) and then taking off the last index (a 0) of each variable, we get an equation of \( X \) (to prove this, use \( \mathbf{f} \)).

Remark 21 Let \( M \) be a graded module on \( \mathbb{C}[x_1, \ldots, x_n] \) with a minimal set of generators of degree \( s \); then a subset of elements of degree \( s \) of \( M \) can be extended to a minimal set of generators if and only if these elements are linearly independent on \( \mathbb{C} \).
Proof of Proposition 3. Suppose \( L \) satisfies Property \( N_{p-1} \) but not \( N_p \). We want to show \( \pi_X L \otimes \pi_Y^* M \) does not satisfy Property \( N_p \); we can suppose \( \pi_X^* L \otimes \pi_Y^* M \) satisfies Property \( N_{p-1} \). Let \( l_m \) and \( q_m \) be the ranks of the \( m \)-module of a minimal free graded resolution respectively of \( G(L) \) and of \( G(\pi_X^* L \otimes \pi_Y^* M) \). Let \( \{ g_j^{2m} \}_{j=1,...,l_m} \) be a minimal set of generators of the \( m \)-module \( E_m \) of a minimal resolution, \( \ldots \to E_m \to E_{m-1} \to \ldots \to E_0 \to G(L) \to 0 \), of \( G(L) \).

Since \( L \) satisfies Property \( N_{p-1} \) but not \( N_p \), there exists a syzygy \( S \) of \( \{ g_j^{2m-1},...,g_{l_{p-1}}^{2m-1} \} \), s.t. \( S \) is not generated by linear syzygies of \( \{ g_j^{2m-1},...,g_{l_{p-1}}^{2m-1} \} \). Add a 0 to the indices of the variables appearing in \( S \) and call \( \tilde{S} \) the so obtained vector of polynomials; let \( \tilde{S}' = (\tilde{S}, 0, ..., 0) \) with 0 repeated \( q_{p-1} - l_{p-1} \) times.

Obviously by adding a 0 to the indices of each variable appearing in the equations of \( X \), we get equations of \( X \times Y \) and by adding a 0 to the indices of every variable appearing in the syzygies of \( X \) we get syzygies of \( X \times Y \).

Add a 0 to the indices of the variables appearing in \( g_j^{2m} \) and call \( \tilde{g}_j^{2m} \) the so obtained vector of polynomials for \( j = 1, ..., l_m \); set \( f_j^1 = \tilde{g}_j^{1m} \) for \( j = 1, ..., l_1 \) and \( f_j^m = (\tilde{g}_j^{2m}, 0, 0, ..., 0) \) (0 repeated \( q_{m-1} - l_{m-1} \) times) for \( j = 1, ..., l_m \) and \( 2 \leq m \leq p - 1 \); \( f_j^m \) for \( j = 1, ..., l_m \) are vectors of linear polynomials for \( 2 \leq m \leq p - 1 \) and they are quadratic if \( m = 1 \), thus, by induction on \( m \) and by Remark 21, one can extend this set to a minimal set of generators \( \{ f_j^m \}_{j=1,...,q_m} \), of the \( m \)-module of a minimal resolution of \( G(\pi_X^* L \otimes \pi_Y^* M) \) for \( m \leq p - 1 \) (we recall that we supposed \( \pi_X^* L \otimes \pi_Y^* M \) satisfies Property \( N_{p-1} \)); we can do it in such way that, when we set \( t_{i,j} = 0 \) for \( j \neq 0 \), we have that \( f_j^1 \) is zero for \( j = l_1 + 1, ..., q_1 \) and the \( r \)-th coordinate of \( f_{j}^{2m} \) is zero for \( r \leq l_{m-1} \) and \( j = l_{m-1} + 1, ..., q_{m} \) (we can prove this by induction on \( m \), by using Remark 20 for the case \( m = 1 \); it is sufficient to subtract linear combination of \( f_j^{2m} \) for \( j = 1, ..., l_m \) to \( f_j^{2m} \) for \( j = l_{m} + 1, ..., q_{m} \)).

Obviously \( \tilde{S}' \) is a syzygy of \( (f_{1}^{p-1}, ..., f_{q_{p-1}}^{p-1}) \).

If \( \pi_X^* L \otimes \pi_Y^* M \) satisfies Property \( N_p \), then \( \tilde{S}' \) would be generated by linear syzygies of \( (f_{1}^{p-1}, ..., f_{q_{p-1}}^{p-1}) \).

We state that \( \tilde{S}' \) cannot be generated by linear syzygies of \( (f_{1}^{p-1}, ..., f_{q_{p-1}}^{p-1}) \). In fact, if it were, say \( \tilde{S}' = \sum S_\alpha \) (\( S_\alpha \) linear syzygies of \( (f_{1}^{p-1}, ..., f_{q_{p-1}}^{p-1}) \)), we set \( t_{i,j} = 0 \) for \( j \neq 0 \) in each member of the equality \( \tilde{S}' = \sum S_\alpha \) and, by taking off the last index (a 0) of every variable and considering only the first \( l_{p-1} \) coordinates of \( S_\alpha \) and \( S_\alpha \), one would obtain that \( S \) would be generated by linear syzygies of \( (g_{1}^{p-1}, ..., g_{q_{p-1}}^{p-1}) \) (observe that by setting \( t_{i,j} = 0 \) for \( j \neq 0 \) in \( S_\alpha \) and taking the first \( l_{p-1} \) coordinates, we get a syzygy of \( (f_1^{p-1}, ..., f_{l_{p-1}}^{p-1}) \)).

But \( S \) cannot be generated by linear syzygies by assumption.  

Q.e.d.

By using the program Macaulay 3 one can check that \( \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(1,1,1) \) does not satisfy Property \( N_4 \), precisely the resolution, with the notation of Introduction, is:

\[
0 \to S(-6) \to S(-4)^9 \to S(-3)^{16} \to S(-2)^9 \to S \to G \to 0.
\]

From this and from Prop. 3 we deduce that \( \mathcal{O}_{\mathbb{P}_1 \times \ldots \times \mathbb{P}_1}(1, ..., 1) \) (\( d \) times) does not satisfy Property \( N_4 \) for \( d \geq 3 \). By using also Gallego-Purnapranja’s Theorem 4, we deduce that, if \( a_1, ..., a_d \) are integer numbers with \( a_1 \leq a_2 \leq \ldots \leq a_d \) and \( a_1 = \ldots = a_k = 1 \), the line bundle \( \mathcal{O}_{\mathbb{P}_1 \times \ldots \times \mathbb{P}_1}(a_1, ..., a_d) \) does not satisfy Property \( N_4 \) if \( k \geq 3 \) and it does not satisfy Property \( N_{d-k+1+2a_{k+2}-2} \) if \( d-k \geq 2 \).

With the same argument as in Remark in Section 2 of part II of [15] we deduce Corollary 8.

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References

[B-S] D. Bayer, M. Stillman Macaulay: A system for computation in algebraic geometry and commutative algebra. It can be downloaded from math.columbia.edu/bayer/macaulay via anonymous ftp.

[C-P] A. Campillo, P. Pison L’ideal d’un semigroup de type fini Comptes Rendus Acad. Sci. Paris Serie I, 316 1303-1306

[G-P] F.J. Gallego, B.P. Purnaprajna Some results on rational surfaces and Fano varieties preprint math.AG/0001107

[Gr1-2] M. Green Koszul cohomology and the geometry of projective varieties I,II J. Differ. Geom. 20, 125-171, 279-289 (1984)

[Gr3] M. Green Koszul cohomology and geometry, in: M. Cornalba et al. (eds), Lectures on Riemann Surfaces, World Scientific Press (1989)

[G-L] M. Green, R. Lazarsfeld On the projective normality of complete linear series on an algebraic curve Invent. math. 83, 73-90 (1986)

[J-P-W] T. Josefiak, P. Pragacz, J. Weyman Resolutions of determinantal varieties and tensor complexes associated with symmetric and anti-symmetric matrices Asterisque 87-88, 109-189 (1981)

[Las] A. Lascoux Syzygies des variétés determinantes Adv. in Math. 30, 202-237 (1978)

[O-P] G. Ottaviani, R. Paoletti Syzygies of Veronese embeddings preprint math.AG/9811131 to appear in Compositio Mathematica

[P-W] P. Pragacz, J. Weyman Complexes associated with trace and evaluation. Another approach to Lascoux’s resolution Adv. Math. 57, 163-207 (1985)

[St] B. Sturmfels, Gröbner bases and convex polytopes, University Lecture Series American Mathematical Society 8 (1996).