THE CONSTRUCTION OF MECHANICS. A NEW PATH TO NEWTON’S EQUATIONS

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Abstract. In the present essay we attempt to reconstruct the Newtonian system under the guidance of logical principles and a constructive approach related to the genetic epistemology of J. Piaget and R. García [19]. Instead of addressing Newton’s equations as a set of axioms, ultimately given by the revelation of a prodigious mind, we search for the fundamental knowledge, beliefs and provisional assumptions that can produce classical mechanics. The work establishes a precedence between these contributions which can be better appreciated by considering the consequences of removing them: (a) The consequence of renouncing to logic and to the laws of understanding is not being able to understand the world, (b) renouncing to the early elaborations of primary concepts such as time and space leads to a dissociation between everyday life and physics, the latter becoming entirely pragmatic and justified a-posteriori (because it is convenient), (c) changing our temporary beliefs has no real cost other than effort.

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1. Introduction

On the utilitarian side, reasoning is the most powerful adaptation of the animal we call human being to the conditions of Nature, for reason allows humans to foresee the outcomes of events and to act taking advantage of opportunities, avoiding dangers and unfavourable conditions. As a consequence, inasmuch reasoning enhances adaptive capabilities, it is a part of natural selection. This selection took place in front of the difficulties placed by natural phenomena. Understanding Nature requires a process that much later, already in human history, Galileo (1564-1642) called idealisation (see e.g., Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze [8, (day 4)]), a process in which, in relation to the matter under discussion, observations are stripped off the particularities that dress them in any actual observation, thus disclosing properties common to them and relations that often have been named the essence. We emphasise here that in these terms, essence is not absolute but instead relative to the posed questions. For example, for any discussion of the biomechanics of a horse, the colour of his coat is irrelevant, yet if what we are to discuss is the “market price” of the horse, the coat usually plays a significant role in a predictive theory. It is in this process of abstraction in which through synthetic judgements (Kant) experience is put in terms of relations between idealised (abstract) objects. These relations are tentative and are always to be suspected of being in contradiction with new observations; we call them theories.

Descartes (1596-1650) introduced the expression “laws of Nature” referring to principles of physics concerning the motion of bodies (Discourse on method [(pp. 19)]). The dogma that Nature follows its own laws, which can then be discovered,
soon gave birth to a new child: Newton’s laws (Newton, 1643-1727) presented in his *Philosophiæ Naturalis Principia Mathematica*, \[18\]. These developments represent the beginning of the scientific revolution which includes the mathematisation of the idealised relations (Galileo) and a reconstruction of the idea of 'cause' to be compatible with idealised relations. Yet, Aristotle’s dictum ‘we think we do not have knowledge of a thing until we have grasped its why, that is to say, its cause’ (Physics, \[1\]) remained valid. In this sense, the quest of knowledge maintained a continuity form the antique Greeks to the modern scientists.

The understanding of causative relations is the long-term program of science. However, the relation of cause and effect is not free of debate. In the crusade for the “elimination of metaphysics”, Carnap \[4\] asserts 'we say of a thing or process \(y\) that it "arises out of" \(x\) when we observe that things or processes of kind \(x\) are frequently or invariably followed by things or processes of kind \(y\) (causal connection in the sense of a lawful succession).’ Thus, cause in Carnap takes the form of a relation between observed things or processes. Let us challenge such extreme empiricist notion. Consider a rectangular cloth suspended by strings tied to its four corners, if we let a ball to rest on the cloth we observe a deformation of the cloth and the formation of some wrinkles that evidence tensions. Is the ball the cause of the wrinkles and the deformation? Or perhaps the wrinkles have attracted the ball? Such a simple matter is not easy to decide using Carnap dictum. The deformation and the presence of the ball resting on the cloth are simultaneous, there is no precedence in time. However, it has never been observed that by making wrinkles on a cloth we can attract balls in the surroundings. Clearly, we say that the ball is the cause of the deformation and the wrinkles. (This example is adapted from \((p. 183)\) 14). Thus, causes can very well be simultaneous with effects and the decision is taken based on reason and other experiences. But physics requires more than this type of analysis. Take the ball, lift it and let it drop. Is there a relation between what makes the ball drop and the cloth to wrinkle? We all know that there is a relation between not-falling (staying on the cloth) and wrinkles, and both events point towards an element left outside in the previous analysis: gravitation. Without gravitation (or with lesser gravitation) we would observe no (or smaller) deformation, and no (or slower) fall. Gravitation can be inferred but not observed (the same can be said of tension). In our observations, gravitation is acting as a necessary element in our explanation, a cause in Aristotel’s perception. Thus, if we want to construct knowledge we have to abandon the crusade against metaphysics and give a share in the real to what is produced by the mind: elements such as gravitation that are not sensible but rather inferred (call them meta-sensible if it pleases the reader). We shall call them *metaphysical* (where does physics ends? and what is beyond physics?). We notice that by so doing the time-order between causes and effects is erased to a good extent, for gravitation is there before, during and after the time the ball rests on the cloth, although what it counts for us is that it was before and during the experiment. Yet, there is no time for the causes, the fall of the ball on the cloth (with the subsequent coming to rest) and the wrinkles are the result of at least one common cause. In the empiricist view, metaphysics is not absent but rather it is hidden in habits and common places, we cannot account for experience without some degree of idealisation, i.e., metaphysics.

In the present work we will try to distinguish the diverse elements that contribute to the construction of classical (Newtonian) mechanics. We find three clear sources
for these contributions, namely: (a) requisites of reason, of which we will emphasise the principle of no arbitrariness, (b) intuitive building blocks that are constitutive of our understanding since their production in our early contact with the world and (3) provisional assumptions with respect to the organisation of the sensible world which, as pragmatical believes, are the main target of procedures such as falsation and retroduction (often called abduction).

2. No arbitrariness principle

In any rational construction attempting to logically articulate the Natural world, we are bound to keep elements not based in reason or evidence outside the construction if possible. Since what is determined by chance, whim, or impulse, and not by necessity, reason, or principle is called arbitrary, we name this principle after the rejection of arbitrary practices. If it were not possible to achieve this goal, we must make sure that the arbitrariness introduced plays no role in answering valid questions with respect to Nature. While it is certainly possible to uproot the questions whose answers depend on arbitrary decisions, in constructing a theory we must strive to avoid the arbitrariness as much as possible and to keep the valid questions as broad as possible.

Struggling to avoid arbitrariness in a physical description immediately reflects in the invariance of this description with respect to elections for which we do not have a reasonable foundation.

Actually, measurements imply comparisons and the units of measurement are arbitrary to a good extent, yet velocity will always be written in terms of a ratio between distance and time, and this form is independent of the arbitrary choices, although the figure of the velocity changes according to the chosen units.

Axiom 1. There is a material world we perceive with our senses (including experiments)

We call this world Nature and we strive to find laws that describe its perceived organisation. In this context we state the main idea of this work:

Axiom 2. No knowledge of Nature depends on arbitrary decisions. Any question about Natural facts will have an answer which is either the same or exactly equivalent under a one-to-one transformation, when the answers involve different arbitrary assumptions.

In short, the no arbitrariness principle establishes that: If there is no reason for it, we shall make no difference. In this simple form it can be recognised as Leibniz’ principle of sufficient reason put in a cognitive context.

The arbitrary decisions adopted while constructing a theory can be viewed as forming a group of transformations operating over the laws and/or the magnitudes involved in the theory. Each choice of the elements in the set of arbitrary assumptions is then used to produce a presentation of the theory, and all presentations must be equivalent (conjugated by elements of the group of arbitrary decisions).

It is important to notice that this approach continues the construction of the reality by the child. We quote Piaget on groups: “There is a mutual dependence between group and object; the permanence of objects presupposes elaboration of the group of their displacements and viceversa. On the other hand, everything justifies us in centering our description of the genesis of space around that of the concept of
group. Geometrically, ever since H. Poincaré this concept has appeared as a prime essential to the interpretation of displacements ... But it is necessary to remember that we shall attribute the widest meaning to this concept for if, as recent works have shown, the logical definition of the group is inexhaustible and involves the most essential processes of thought, it is possible, purely from our psychological point of view, to consider as a group every system of operations capable of permitting a return to the point of departure.” [12]

2.1. Objectivity and intersubjectivity. If we observe a cat stalking a bird in our garden, we perceive a distance between cat and bird, and we have indications that both cat and bird perceive something similar, since the cat does not launch the attack until the distance is short enough and the bird does not flies away until the cat is threatening close. We actually do not know whether cat, bird and observer made the same internal representation of distance and direction, but we know they perceived something equivalent.

In the next Section we will formalise these perceptions in the concept of relative position, saying that it is objective (as well as cat and bird are), while our perceptions of it are intersubjective, meaning that the cat’s, bird’s and any observer’s perceptions can be put in a one to one correspondence. They need not be the same representation but just corresponding ones. Let \( x_{CB} \) be a symbol for the idea of relative position between cat and bird. We are used to think that the observer \( a \) represents it as \( x_{CB} = x_C^a - x_B^a \), which is in fact a synthetic judgement – not a mathematical one – since it proposes a relation between three different concepts: the position \( (x_C^a, x_B^a) \) of cat and bird relative to the observer (or to a reference of her/his choice) and the relative position between cat and bird \( x_{CB} \). Since the observer entered the picture by the arbitrary action of referring positions with respect to her/himself, this expression must be subject to the no arbitrariness principle, Axiom [2]. It is important to realise that \( x_{AB} \) is a symbol that relates to the sensory perceived world, not an internal mathematical object.

2.2. Mathematisation of objectivity. Consider a set of concepts/magnitudes \( e, (e \in E) \) and a set of arbitrary decisions \( (a \in A) \), necessary for its representation. Let \( R_a(e) \) be a representation of the concepts depending on a particular set of arbitrary decisions. Further, let \( F \) be a “natural law” expressed as \( F(R_a(e)) = 0 \).

Definition 1. Objectivity: A law is objective if \( F(R_a(e)) = 0 \) holds for any \( a \in A \).

Consider an invertible transformation \( T_{ab} \) that maps a representation \( a \) onto another representation \( b \) \( (T_{ab} \circ T_{ba} = \text{Id}) \). By Axiom [2] Definition [1] can be restated as:

\[
(2.1) \quad F(R_a(e)) = 0 \iff F(T_{ba}R_a(e)) = F(R_b(e)) = 0.
\]

The transformations \( T_{ab} \) form a un group \( T \). We assume that:

a) \( T_{ab}T_{bc} = T_{ac} \) There exists a composition law
b) \( T_{aa} = \text{Id} \) There exists an identity
c) \( T_{ab} = (T_{ba})^{-1} \) or \( (T_{ba} \circ T_{ab}) = \text{Id} \), An inverse exists
d) \( T_{ab} \circ (T_{bc} \circ T_{cd}) = (T_{ab} \circ T_{bc}) \circ T_{cd} \) Associativity.

As a consequence, \( T_{ab} \circ (T_{bc} \circ T_{cd}) = \text{Id} = (T_{ab} \circ T_{bc}) \circ T_{cd} \) and \( T_{ab} \circ (T_{bc} \circ T_{cd}) = T_{ab} \circ T_{bd} = T_{ad} \circ T_{cd} = (T_{ab} \circ T_{bc}) \circ T_{cd} \).
3. Classical mechanics

The system of concepts that allow the organisation of reality is dialectical, with multiple concepts coming into existence at once. Thus, concepts arise that are meaningful only in relation to other concepts introduced simultaneously. In all dialectical constructions, the tension of the opposites is the engine of understanding, but the terms in opposition present no difference between them in as much they are just the terms of the opposition. Hegel’s dialectic of being and not-being, is perhaps the clearest example of this constructive procedure [11]. In our case, the concepts: ego, alter, identity, object, space and time are the fundamental building blocks for the construction of understanding; they cannot be referred to previous concepts because understanding can only be referred to its opposite: not-understanding. Thus, any attempt at defining these terms will never be completely satisfactory since we will have to resort to some complicity in the intuitions of the reader. In practical terms, when we mention a concept (in this section) using a word that is defined later in the text, it should be understood in its intuitive form, later to be formalised in a compatible form. We can identify the dialectical pairs ego-alter and change-permanence (time-identity). As for space, considering that it implies the concept of distance and with it the idea of what is within reach and what belongs to our world but cannot be reached without effort, it is the result of the dialectic near me (reachable) and at distance (not reachable, i.e., not near me). Notice that the notion of oppositions as first principles is a subject already considered in Aristotle’s Physics [1], also referring to previous philosophers.

Notation 1. In this Section we will try as much as possible to use superindices \(a, b\) to indicate subjective (i.e., depending on arbitrary decisions) concepts, quantities, etc., while related objective entities will have no superindex.

3.1. Space and time.

3.1.1. Space. Space emerges spontaneously conceived by children along the construction of reality (J. Piaget, The construction of reality in the child[12]). It is not possible to speak separately of object, space, time and ego, because the construction produces all of them as a single dialectic system. We can try to straighten this matter up philosophically, although we will not be loyal to the child’s structuring of thought. We start by memory and the perception of some degree of permanence of ourself, ego, and then an idea of identity (permanence). The real world acquires continuity, we do not longer watch the movie frame by frame. But with our recognising ourselves comes alter, i.e., not-ego, and with the help of our memory some sort of permanency of alter emerges, the object that later, stripped of all other characteristics will be the “body” referenced in physics. Therefore, objects, people and more acquire some sort of identity, in as much as we remove some attributes of them, mainly space. Then, if an object (or ourselves) is in a place in one frame and in a different place in another frame, we do not longer say they are different objects but rather that there was a change in the positions as well as some permanence: the object. The sequence of changes in places is the primary idea of time. Much later the child will conceive herself as being of the same condition than family, pets and toys, this is, she will place herself in the space. Space, with its implication of distance (a concept easy to root in sensorymotive intelligence) is opposed to the unity of the cosmos of the child.
The perceived space-time, centred in the observer (ego) always has ego distinguishing a reference point. This primary, subjective, notion of space is compatible with an empty space, a holder of objects, an idea that in physics goes back, at least, to Newton. Emptiness is in fact a resource for the subsequent suppression of the observer. By NAP, the required objectivity of the laws of physics manifests in that no law is objective if it depends on ego. The mathematisation of this idea is expressed by eq. (2.1), where the group $T$ corresponds to spatial translations and rotations.

**Axiom 3.** *Objective space is built following Descartes.* A set of three orthogonal directions in the real (sensible) space is selected and represented by the symbols $e_i$. Any objective position is then represented by $x_{AB} = \sum_i (x_i e_i)$ with $x_i$ real numbers. We have that the space is represented as a vector space.

Any other choice of reference vectors can be related to the initial one by a linear transformation, $e'_j = \sum_i R_{ji} e_i$. Since relative position is objective, it follows that $\sum_j x'_j e'_j = \sum_i x_i e_i$ for any objective position, hence $x'_j = \sum_i (R^{-1})_{ij} x_i$. The observer can then select a reference point and three independent orientations to describe the position of objects in the world and yet leaving the distance as objective, the class of equivalence thus generated is described by the group $IO(3)$, the group of isometries (translations, rotations and reflexions).

The subjective space, being empty, has no form of distinguishing one direction from another or one reference point from another. As a consequence of NAP, subjective space is then isotropic and homogeneous. We notice that objective space bears some relation to Leibniz’ relational space, while subjective space is related to Newton’s relative space.

**3.1.2. Time.** Unlike space, time is undoubtedly related to changes, sequences of changes, rapidity and causal-relations (12). This is the genetic episteme of time; to call “time” any other kind of object is simply to ask for confusion. Time is the word we use to express our perception of change; it is change in its most abstract form.

The perception of change we use to construct our intuition of time rests on natural processes. An invariant characteristic of our perception of Nature is that processes have *beginning* and *end*. It is a verbalisation of the dialectical tension of the pair being/not-being. In principle, time is given by the order of the sequence of changes (between sundown and sunrise the rooster sings, and if the rooster wakes me up every morning, it sings before-or rather while- I wake up, because this song is the *cause* of my awakening), time does not reach us as a perception, but it is rather the result of memory, a log of changes and a logical process that discriminates between the relative order of effects. It follows that time is measured by comparing sequences of changes. By Axiom 2 any transformation relating time-perceptions of different references (egos) is constrained to preserve this ordering. Therefore, while arbitrary individual subjective time may differ among observers, they are all related by strictly monotonic (bijective) mappings. The underlying group is the set of strictly increasing functions $f : R \rightarrow R$ with standard composition of functions as the group product.

**Axiom 4.** *The logical order between effects related to the same causes is universal.* The measure of time rests on the concept of real numbers.
In simpler words, first the vase falls, then it breaks. It is a vase as long as there exists a particular cohesion in the material. The alteration produced by the impact reorders the material in smaller pieces. Hence, without fall there is no impact, without impact no material stress. Causes here are gravitation and stress. Observables are fall, impact, breakup (rupture).

When we measure the times of a phenomenon, say the change in position of a body, we use changes not involved in the process as references for time. Indeed, we resort to the idealised and imagined order of all events of the universe, leaving no “room” between them since a time without changes is a contradiction in the terms. In mathematical terms, time is well represented by a one dimensional mathematical space such as the real numbers, a notion that we have been using and will continue to use, along this work. Time-intervals are referred to this “background” of events that are present irrespective of the phenomenon in study. This background of changes constitutes a clock. We try to use as clocks devices or observations that appear to us as regular. As long as we cannot present evidence that a process runs faster in one circumstance or another, we expect the relative order between changes in the clock and changes in the phenomena to remain the same, hence

**Assumption 1.** We assume that we can define an objective time by convening on a process to define a time-unit.

Absolute time, a time encompassing all changes, can be regarded as it appears to us that in the same manner in which ego, alter, object, space and time emerge in the development of the child to produce a useful organisation of the world. The laws of physics and absolute time emerge in the construction of physics at the same step as a consequence of the same class of dialectic opening that creates the terms of an opposition that produces understanding. In this case, system-environment (not system) implies absolute time.

**Definition 2.** *Events.* An event is a change in the sensible world that occurs in a relatively short interval of time and as such is idealised as instantaneous. We remove from the event the determination of time and space that are thought of as its circumstances. Thus, events occur at a given location and a given time.

3.2. **Observers.** The observer (*ego*) describes the world by measuring all distances with respect to a point of her/his election. We use the notation $x^a_A$ for the position of body $A$ as measured by observer $a$. When facing the need of relating positions of different observed bodies, conforming to NAP and the search for an objective description of physical entities, we define:

**Definition 3.** *(relative position)*: We call $x_{AB} = x^a_A - x^a_B$ the relative position between observed bodies $A$ and $B$, as seen from the *ego* $a$ at a given moment.

**Corollary 1.** It is a demand of objectivity *(by Axiom 2)* that $x^a_A - x^a_B = x^h_A - x^h_B$ *(hence, there is no superindex in $x_{AB}$).*

In a sense related to Leibniz, objective space is relational. Indeed, one may suspect that vector spaces have been constructed to this end. The group relating the arbitrary choices of different observers is $IO(3)$, the semidirect product of isometries and translations.

In the sequel, we will call *subjective* the space created by an arbitrary choice of reference point and base vectors orientations.
**Definition 4.** Relative velocity is the rate of change of relative position between two bodies with respect to the change in time.

\[ v_{AB} = \frac{d(x_A - x_B)}{dt} = \lim_{\Delta t \to 0} \frac{(x_A(t + \Delta t) - x_B(t + \Delta t)) - (x_A(t) - x_B(t))}{\Delta t} \]

3.3. **Subjective velocity and Galileo transformations.** The relative velocity between the reference point chosen by the observer \( a \) and a body \( A \), follows the just given Definition (4),

\[ v_{Aa} = \frac{d(x_a^a - x_a^a)}{dt} = \lim_{\Delta t \to 0} \frac{(x_A^a(t + \Delta t) - x_a^a(t + \Delta t)) - (x_A^a(t) - x_a^a(t))}{\Delta t} \]

We now focus on the subjective operation consisting in setting \( x_a^a(t + \Delta t) = 0^a \) and similarly \( v_a^a = \lim_{\Delta t \to 0} \frac{x_a^a(t + \Delta t) - x_a^a(t)}{\Delta t} = 0^a \), this is to say that for the observer, the point designated as reference by her/his arbitrary decision does not move (we have added the superscript \( a \) to the zero to indicate the subjectivity).

Now the relative velocity reads

\[ v_{Aa} = \frac{d(x_a^a - x_a^a)}{dt} = \lim_{\Delta t \to 0} \frac{x_A^a(t + \Delta t) - x_a^a(t)}{\Delta t} \equiv v_A^a - 0^a, \]

where we call \( v_A^a \) the subjective velocity of \( A \) as established by observer \( a \).

**Proposition 1.** The Galilean transformation between observers (reference points) \( a \) and \( b \) is given by,

\[ v_A^a = v_A^b + v_b^a \]

It is an operation that belongs to the group associated by Axiom 2 to the concept of subjective velocity.

**Proof.** We begin by writing the equality

\[ x_A^a(t + dt) - x_a^a(t) - x_a^a(t + dt) - x_A^a(t) = x_b^b(t + dt) - x_b^b(t) - x_b^b(t + dt) - x_a^a(t) \]

which after a rearrangement reads

\[ (x_A^a(t + dt) - x_a^a(t)) - 0^a = (x_b^b(t + dt) - x_a^a(t)) - 0^b + (x_b^b(t + dt) - x_b^b(t)) \]

Next we observe that

\[ 0^b - (x_b^b(t + dt) - x_a^a(t)) = (x_b^b(t + dt) - x_a^a(t)) - (x_b^b(t) - x_a^a(t)) = x_b^b(t + dt) - x_b^b(t) \]

with \( x_b^b \) an objective relative distance. We then have

\[ (x_A^a(t + dt) - x_a^a(t)) - 0^a = (x_A^a(t + dt) - x_A^a(t)) + (x_b^b(t + dt) - x_b^b(t)) - 0^b \]

Dropping the zeroes, dividing by \( dt \) and taking the limit \( dt \to 0 \), we obtain the Galilean transformation between observers. We further notice that the limit is not a necessary step.

**Lemma 1.** Galilean transformations form a group having vector addition as internal operation.

**Proof.** Repeated use of Proposition 1 gives,

\[ v_A^a = v_A^b + v_b^a = (v_A^c + v_{cb}) + v_b^a = v_A^a + (v_{cb} + v_{ba}) = v_A^a + v_{ca} \]
4. The law of inertia

4.1. Laws of nature. The meaning of Laws of Nature deserves some examination. In western culture before the Enlightenment, it remitted us to God’s blueprints for the universe as in the early times of Descartes, Newton and Leibniz. With humanisation brought around by the Enlightenment, the Laws of Nature must rest on reason [13, 15]. The laws of nature correspond to fundamental relations in situations in which a small (minimal) portion of the universe, the system, is considered (through a process of idealisation) as isolated from its environment, i.e., the complement in the Universe of the ideally-isolated system. Such a notion implies that the internal organisation of the system—which the law will make explicit—must be independent of the environment since this is the nature of the concept we are seeking. Hence, the law must hold with independence of the relative location of the system with respect to the environment and shall not be affected by the background of changes occurred in the interval of our ideal consideration, before and after it. Thus, in our perspective, there are no laws of nature but rather Laws for the understanding of Nature, which themselves are subject to the laws of reasoning and include in their ontogeny both experience and usefulness: the object and the subject.

4.2. The law of inertia.

Definition 5. An isolated body is an idealisation consisting in extrapolating the (short-time) movement of bodies that are perceived to be not interacting with other bodies.

This perception could originate in the fact that when the distance between bodies is sufficiently large or the time of the observation is sufficiently short, interactions do not show any appreciable effect (i.e., it is an idealisation).

An isolated body can be regarded as being alone in the universe. As such, it defines by itself a privileged place and reference point. However, when we deal with several ideally isolated bodies, we must consider the problem of their changing (relative) distances (which is a result of the same idealising process). By NAP, the description given from the perspective of an isolated body must be equivalent to the description given by any other isolated body in as much as the particularity of the description is only that the observer is isolated (i.e., not influenced in its movement by other bodies). The condition of a body as isolated is a condition of permanence, hence, when compared with time (the measure of change) it has to be represented by a zero derivative of its state. Therefore, we define,

Definition 6. Inertial set is the collection of isolated bodies. The inertial class of observers are those that use an element of the inertial set as point of reference.

When seeking a fundamental law for isolated bodies we must consider first the possibility of giving them fixed relative positions, but such a law contradicts our current perceptions (we recall that in the Aristotelian physics the motion of bodies required causes, i.e., “forces”).

Assumption 2. Isolated bodies can move relative to each other

The next possibility to be considered is:

Lemma 2. Isolated bodies move with constant relative velocity.
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Proof. Consider the law of motion of an isolated body, \( A \), as described by an observer \( a \) in the inertial class. By Assumption 2, the law of motion cannot be \( \frac{d}{dt} x_{Aa} = 0 \). The most general law of second order is

\[
\frac{d^2}{dt^2} x_{Aa} = \alpha(x_{Aa}, v_{Aa}, t)
\]

However, the law must be the same for all times since there is no privileged time, \( \alpha(x_{Aa}, v_{Aa}, t) = \alpha(x_{Aa}, v_{Aa}, 0) \). Additionally, by Axiom 2, a second observer that has selected its reference position at a fixed distance from \( a \) must produce the same law. It follows that the law cannot depend on \( x_{Aa} \) and by the same reasoning an observer that moves with constant relative speed with respect to \( a \) must observe the same law, hence \( \alpha(x_{Aa}, v_{Aa}, t) = \Upsilon \), being \( \Upsilon \) a constant vector. However, such constant would indicate a particular direction in space unless \( \Upsilon = 0 \). Hence, we arrive to the expression

\[
(4.1) \quad \frac{d^2}{dt^2} x_{Aa} = 0
\]

that satisfies the established requisites of permanency. \( \square \)

Corollary 2. An inertial observer (i.e., an element of the inertial class) is one in which only interactions are associated to changes in the velocities.

Proof. The result is just Lemma 2 in negative form. \( \square \)

Theorem 1. Galilean transformations belong to the group of transformations taking from one inertial observer to another.

Proof. By Definition 2 and Lemma 2, two inertial observers behave as two isolated bodies with constant (objective) relative velocity. The subjective velocities determined by each system are related via Proposition 1. \( \square \)

5. Interacting Bodies

We have shown that the only law of motion compatible with isolated bodies is the law of inertia. It is time to consider bodies with interactions. The same law in its negative form is: accelerations (i.e., changes in the instantaneous velocity) are the result of interactions between bodies. The simplest form of interaction to be considered involves just two material bodies. Consider an observer in the
inertial class (hence, it does not interact with the bodies under consideration). From its perspective, if bodies $A$ and $B$ do not follow the inertial law, they must be interacting, hence the minimal setup for non-inertial movement requires two bodies that are described by the inertial (isolated) observer, which establishes the following relations:

\[
\frac{d}{dt}v_A^a = \alpha_A^a \\
\frac{d}{dt}v_B^a = \alpha_B^a \\
\frac{d}{dt}(v_A^a - v_B^a) = \frac{d}{dt}v_{AB} = \alpha_{AB}
\]

where $\alpha_{AB}$ is the (objective) relative acceleration of body $A$ with respect to body $B$.

**Assumption 3.** The effect of interactions is additive, i.e., the dynamical consequences on $A$ of the interactions with $B$ and $C$ is the sum of the individual consequences of the pairwise interactions.

With dynamical consequences we intend the effects of interactions on the motion, e.g., the resulting acceleration.

### 5.1. Gravitation and mass.

The fundamental interaction between two bodies, the one that is always present and cannot be compensated is called gravitation of matter. A fundamental ingredient at this point is Galileo's proof that all bodies experiment the same acceleration in vertical motion \cite{8, Third day, pp. 173-}]. This achievement rests both on the logic need of investigating accelerated bodies after inquiring about non-accelerated bodies (eq. 4.1) and on experimental observation.

**Assumption 4.** The gravitational acceleration $\alpha_A^a(B)$ produced onto a material body ($A$) by a given one ($B$) is proportional to a characteristic of the body $B$ named the mass (all other circumstances being identical). The mass of body $B$ is an objective property (i.e., the same, up to choice of units, for any inertial observer).

In mathematical terms, the accelerations are, letting $g_A^a(B)$ summarise the rest of the dependencies,

\[
\alpha_A^a(B) = m_B g_A^a(B) \\
\alpha_B^a(A) = m_A g_B^a(A)
\]

**Corollary 4.** The mass of an aggregation of matter is the aggregated mass of the parts.

When considering the gravitational interaction of two bodies, $A$, localised in $x_A$ and, $B$, localised in $x_B$ according to an inertial observer, we notice that exchanging attributes between bodies is the same as exchanging positions, hence $g_A^a(B) = -g_B^a(A)$. As a consequence, to render this symmetry explicit it is convenient to consider the hereby defined gravitational force $f_{AB} = m_A \alpha_A^a(B) = m_A m_B g_A^a(B)$, rather than acceleration.

**Notation 2.** From here on, we will drop the index of ego, since we are dealing with just one observer belonging to the inertial class. For example we write $\alpha_A \equiv \alpha_A^a$. However, quantities with two “body”-subindices, such as $f_{AB}$ are always objective.
Definition 7. The linear momentum of a material body is the product of its mass times its velocity \( p = mv \)

Thus, the gravitational interaction is described by the symmetric form

\[
\frac{d}{dt} p_A = m_A \frac{d}{dt} v_A = m_A \alpha_B(B) \equiv F_{AB} = m_A m_B g_A(B)
\]

\[
\frac{d}{dt} p_B = m_B \frac{d}{dt} v_B = m_B \alpha_A(A) \equiv F_{BA} = m_B m_A g_B(A)
\]

the latter being a definition of the concept of gravitational force and of force, the cause for the changes in velocities.

Corollary 5. The total momentum \( p_A + p_B \) is constant in time when the bodies interact gravitationally, i.e.,

\[
\frac{d}{dt} (p_A + p_B) = \frac{d}{dt} p_A + \frac{d}{dt} p_B = m_A m_B g_A(B) + m_B m_A g_B(A) = 0.
\]

The previous corollary is an immediate consequence of \( g_A(B) = -g_B(A) \). Much of the work of mechanics is then to identify and classify forces. We can reformulate these observations as:

Assumption 5. Force is the cause of acceleration. The latter is proportional to the ratio between force and mass.

We have already realised that a consequence of Lemma 2 is that the presence of accelerations is an indication (or a symptom) of the existence of interactions. This assumption states that the totality of the intervening interactions is exhausted in the acceleration, they are its cause. We should notice that this assumption must be contrasted with experiments. For example, Weber’s formalisation of Faraday’s induction law \([23]\) requires for the induction (electromotive) force to depend on the acceleration (as Faraday’s law also does). This may be interpreted as having a mass that depends on the relative state of motion of the interacting charges.

5.2. Other forces. Let us now consider other forces following the previous scheme. We write,

\[
\frac{m_A}{dt^2} x_A = f(Q_A, Q_B, x_A, x_B, \ldots)
\]

\[
\frac{m_B}{dt^2} x_B = k(Q_A, Q_B, x_A, x_B, \ldots)
\]

where \( Q_i \) denotes the defining property of body \( i \) associated to this force, i.e., the necessary information in order to determine the force. Specifically, we consider the class of forces fulfilling

Assumption 6. Following Newton, we consider in the sequel forces of instantaneous action at a distance.

5.2.1. About masses. In this presentation there is no need to introduce inertial mass as something different from gravitatory mass. Gravitation is taken to be a fundamental (unavoidable) interaction and its associated mass is what enters in eqs \([5.2]\) and \([5.3]\). This is a consequence of the symmetry of the presentation, by which whatever pertains the gravitational interaction is explicitly displayed.
5.2.2. Consequences. The above symmetry considerations may be reproposed for eqs. (5.2 and 5.3). In fact, \( f(Q_A, Q_B, x_A, x_B, \ldots) \) is the force that the body \( Q_B \) with geometrical parameters \( x_B, \ldots \) exerts on the body \( Q_A \) (with \( x_A, \ldots \)). The dots indicate that forces may depend on other geometrical properties than just position (velocity, for example). Similarly, \( k(Q_A, Q_B, x_A, x_B, \ldots) \) is the corresponding force with \( A \) and \( B \) exchanged, i.e.,

\[
(5.4) \quad k(Q_A, Q_B, x_A, x_B, \ldots) = f(Q_B, Q_A, x_A, x_B, \ldots)
\]

Assumption 7. Force, as a vector, depends only on relative position and relative velocity.

Lemma 3. Under the previous assumptions, for a pair of bodies \( A \) and \( B \) in interaction, the forces \( f \) (force on \( A \) caused by \( B \)) and \( k \) (force on \( B \) caused by \( A \)) satisfy the Generalised Principle of Action and Reaction, namely

\[
\begin{align*}
    f &= \phi_\epsilon(x - y) + \phi_\nu(u - v) + \phi_\perp((x - y) \times (u - v)) \\
    k &= -\phi_\epsilon(x - y) - \phi_\nu(u - v) + \phi_\perp((x - y) \times (u - v))
\end{align*}
\]

where \( \phi_\epsilon, \phi_\nu, \phi_\perp \) are scalar functions of relative position, relative velocity and possibly other parameters.

Proof. Let \( T \) denote the operation of exchanging the geometrical properties associated to \( Q_A \) and \( Q_B \) (exchanging the charges of \( A \) and \( B \)). We have,

\[
\begin{align*}
    Tf(Q_B, Q_A, x, y, \ldots, t) &= k(Q_A, Q_B, x, y, \ldots, t) \\
    Tk(Q_B, Q_A, x, y, \ldots, t) &= f(Q_A, Q_B, x, y, \ldots, t)
\end{align*}
\]

and consequently \( T^2 = Id \).

Let us now suppose that a given force depends on both relative position and relative velocity. By Assumption 7 there is no other geometric dependency. To lighten the notation we write

\[
Q_A = q, Q_B = Q, x_A = x, x_B = y, v_A = v, v_B = u
\]

Regarding two interacting bodies, Axiom 2 demands that this dependence is actually only on the relative distance and velocity, namely that

\[
\begin{align*}
    f(Q, q, x, y, u, v, t) &= f(Q, q, x - y, u - v, t) \\
    &= \phi_\epsilon(x - y) + \phi_\nu(u - v) + \phi_\perp((x - y) \times (u - v)) \\
    \phi_\epsilon(x - y) + \phi_\nu(u - v) + \phi_\perp((x - y) \times (u - v))
\end{align*}
\]

where the multiplicative factors \( \phi_\epsilon, \phi_\nu, \phi_\perp, \psi_\epsilon, \psi_\nu, \psi_\perp \) are scalar functions of relative position, relative velocity and possibly other parameters. The second line arises since relative position and relative velocity define only three vectors in space. In other words, \( \phi_\epsilon(x - y) \) may depend on various properties (charge, velocity, etc.) but as a vector it lies in the direction of relative distance \( x - y \). The same is intended, correspondingly, for the other two contributions.

Exchanging the properties \( q \) and \( Q \) of \( A \) and \( B \) is equivalent to keeping the properties and exchanging their masses, positions and velocities. Hence, again by
Axiom. We have,
\[
\phi_e(x - y) + \phi_v(u - v) + \phi_\perp((x - y) \times (u - v)) = \psi_e(y - x) + \psi_v(v - u) + \psi_\perp((x - y) \times (u - v))
\]
(5.6)
\[
= -\psi_e(x - y) - \psi_v(u - v) + \psi_\perp((x - y) \times (u - v))
\]

which produces
\[
\phi_e = -\psi_e, \quad \phi_v = -\psi_v, \quad \phi_\perp = \psi_\perp
\]
(5.7)

Remark 1. Gravitation, as conceived by Newton, has vanishing $\phi_v$ and $\phi_\perp$.

Lemma 4. Under the Generalized Principle of Action and Reaction the total momentum, $p_A + p_B$, is constant in time if and only if $\phi_\perp = 0$.

Proof. We just compute
\[
\frac{d}{dt}(p_A + p_B) = F_{AB} + F_{BA}
\]
\[
= (\phi_e - \phi_v)(x - y) + (\phi_v - \phi_v)(u - v) + (\phi_\perp + \phi_\perp)((x - y) \times (u - v))
\]
\[
= 2\phi_\perp((x - y) \times (u - v))
\]
Moreover, if $(x - y) \parallel (u - v)$ (unidimensional relative motion) the actual value of $\phi_\perp$ is irrelevant and we can choose to set $\phi_\perp = 0$ also in that situation.

We must notice that the total momentum as perceived by the observer is a conserved quantity if and only if $\phi_\perp = 0$ for all forces in classical mechanics. The idea of conservation of total momentum is a consequence of Newtonian tradition and not a demand of reason. It must be established in every new theory of nature as an additional assumption subject to empirical consideration.

Lemma 5. Under the Generalized Principle of Action and Reaction there is no internal torque if and only if $\phi_v = \phi_\perp = 0$.

Proof. Following Corollary we compute,
\[
(x - y) \times \frac{d}{dt}(v - u) = (x - y) \times \left(\frac{f}{m_A} - \frac{k}{m_B}\right)
\]
\[
= \left(\frac{1}{m_A} - \frac{1}{m_B}\right) \phi_\perp(x - y) \times ((x - y) \times (u - v))
\]
\[
+ \left(\frac{1}{m_A} + \frac{1}{m_B}\right) \phi_v(x - y) \times (v - u)
\]
Since all terms in the rhs have contributions in different directions,
\[
\frac{d}{dt}\left[(x - y) \times \frac{m_A m_B}{m_A + m_B}(v - u)\right] = 0 \Leftrightarrow \phi_v = \phi_\perp = 0.
\]
In Classical Mechanics the square brackets is called angular momentum and its time-derivative is called torque.
5.3. Central forces. An important issue in Classical Mechanics following the Newtonian tradition is that of Central forces depending only on the relative distance. From the previous results, the following particular case can be highlighted:

**Lemma 6.** Under the previous assumptions, there is a conserved internal energy if $φ_v = 0$ and $ϕ_e = h(|x - y|)$ (a scalar function of relative distance only).

**Proof.** A standard computation using the relative (objective) quantities defined previously yields,

$$(u - v) \cdot \frac{d}{dt}(u - v) \equiv \frac{1}{2} \frac{d}{dt}(u - v)^2 = (u - v) \cdot \left(\frac{f}{m_A} - \frac{k}{m_B}\right)$$

which by eqs. (5.6 and 5.7) reads,

$$\frac{1}{2} \frac{d}{dt}(u - v)^2 = \left(\frac{1}{m_A} + \frac{1}{m_B}\right)(u - v) \cdot (ϕ_e(x - y) + φ_v(u - v)).$$

In other words, under the conditions of this Lemma, we have,

$$\frac{1}{2} \frac{m_A m_B}{m_A + m_B} \frac{d}{dt}(u - v)^2 = h(|x - y|)(x - y) \cdot (u - v).$$

We identify in the lhs the kinetic energy for the relative moment. Let $-V(|x - y|)$ be such that $-\nabla V = h(|x - y|)(x - y)$. Hence, $-\frac{dV}{dt} = h(|x - y|)(x - y) \cdot (u - v)$.

Naming the quantity $\frac{1}{2} \frac{m_A m_B}{m_A + m_B} |u - v|^2 + V(|x - y|)$ as internal energy, we have

$$\frac{d}{dt}\left(\frac{1}{2} \frac{m_A m_B}{m_A + m_B} |u - v|^2 + V(|x - y|)\right) = 0.$$

5.4. The tension between inertial and non-inertial. One of the pillars of our understanding of Nature is observation, which invariably takes place on Earth or (since recently) on its neighbouring (galactic) surroundings. Both the surface of the Earth and the Solar System are regarded as non-inertial references by Classical Mechanics. Indeed, to the extent of our daily experience, we have no reasons to doubt of this non-inertiality since we experience its effects. However, since the Leibniz-Clarke discussion and lately Mach (see below) it has been an important issue the fact we have no (other) means to decide whether a reference system is inertial (fully free from interactions) or not.

Despite this issue, physics has succeeded in conceiving different fundamental interactions, such as gravitation or the electrostatic interaction as described by Coulomb. One may wonder how this could be done in the first place, without having a clue of “how inertial” we are. The solution presented here finds its support on three concepts: (a) The process of idealisation described by Galileo, by which one identifies and eliminates from observation what is conceived to be foreign to the interaction under study, e.g., the effect of friction forces, or the presence of other interactions influencing both the system and the observer, (b) A fully-objective approach where the description of an interaction is performed using objective (invariant) quantities belonging to the interaction pair (relative positions, charges and the like) and (c) The honest effort to keep the description free from influences coming from the observer to the largest possible extent (this description is always provisional until a new, so-far neglected, influence of the observer is detected). We return to this issue in Subsection 6.1.
It is illustrative to consider the way in which the founders of electromagnetism studied the new interactions. Prior to experimenting with the new interaction, they first let the system to be at rest as a result of the equilibrium of the (other) present forces and accelerations. Next, the new influential condition is established and when a new equilibrium is reached they measured the balancing force judged to be equivalent (but opposite in sign) to the newly introduced force. Thus, forces are introduced as members of an equivalent class rather than by a direct application of the definition, i.e., no acceleration is truly measured in a first instance. Such are the methods of Coulomb and Ampère for example.

6. ON NEWTON’S LAWS OF CLASSICAL MECHANICS

Those, like us, that have tried to reach a deeper understanding of Newton’s ideas by reading the *Principia*, might have come to the conclusion that Newton left behind little or no clues of the fundamentals behind his axioms. In the Scholium to the Definitions [18], Newton writes his famous notion of absolute space and time. Although he recognises them as related to the common intuition of “the vulgar” he proposes that the vulgar conceptions are imperfect images of absolute time and absolute space, a position that reminds us of the precedence of Platonic worlds, being the world of ideas the real world [20]. Needless to say, Newton relies on the vulgar notions to deliver and argument for his notions of space and time, place and motion. He indeed introduces without explicit recognition the idea of Galilean transformations: “Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; ... But real, absolute rest, is the continuance of the body in the same part of that immovable space in which the ship itself, its cavity, and all that it contains is moved. Wherefore if the earth is really at rest, the body which relatively rests in the ship, will really and absolutely move and with the same velocity which the ship has on the earth. But if the earth also moves the truly and absolute motion of the body will arise, partly from the true motion of the earth in immovable space; partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship; its true motion will arise, partly from the true motion of the earth in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body on the ship;...” [(Motte) 18]

For Newton, it is relative space what can be conceived by the vulgar as fixed to the heavens: “All things are placed in time as to order of succession; and in space as to order of situation. It is from their essence or nature that their are places; an that the primary place of things should be movable, is absurd... But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places we estimate all motions... And so, instead of absolute places and motions we use relative ones, and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body at rest to which the places and motions of others may be referred [(Motte) 18]. (Cajou corrected the last phrase to: “And therefore, as it is possible, that in the remote region of fixed stars, and perhaps far beyond them, there may be some body absolutely at rest...”[(Motte-Cajou) 18]).
Newton’s construction cannot come to an end unless the existence of an end, absolute space, is introduced ad-hoc. In terms of the present work, Newton’s construction is moving from the subjectivity of one reference into the subjectivity of another, each one carrying its relative space. The sequence of subjective views can only finish in an objective view. Is God the observer of absolute space? Thus, rather than fixed to the heavens, absolute space is fixed to the Heavens, i.e., objective space would be God’s perspective. The same sort of recursion is present in absolute time.

Concerning time, Newton did not conceive a method of measuring absolute time, but only approximations to the ideal time: “Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the vulgar time. For the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality for their more accurate deducing of the celestial motions. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded; but the true, or equable, progress of absolute time is liable to no change.” [(Motte) 18]

Newton conceived absolute space and time as the limit of a process in which relative movements were added and clocks were perfected in their regularity. The opposite view of space in those years was that of Leibniz who sustained the relational view. We show in this work that both views are intimately related and that absolute space and time consist in a final attempt to maintain the observer in the scene. In our discussion, in contrast, subjective space and motion are the result of the introduction of the idealised observer in the scene. This introduction is an arbitrary act with the only benefit of addressing the perspective of our very early childhood, which is not suppressed in more advanced elaborations but is rather the foundation of them and as such survives in them [19]. It is important to notice that although relative space appears as objective, relative time appears only as intersubjective, not being possible for us to define an objective measure of time without an ad-hoc assumption.

In *Principia Mathematica* Newton founded the concept of mass on the intuition of weight: “The quantity of matter is the measure of the same, arising from its density and bulk conjointly... Is this quantity that I mean here after everywhere of body or mass. That becomes known by the weight of each body: for the proportional to the weight as I have found by experiments with pendulums very accurately made, which shall be shown later.” [(Motte) 18] Such an idea was later challenged in an *a-posteriori* empiricist view by Mach [17]. We have shown in this work that Newton’s perspective is proper of the construction of knowledge, this is, the genetic meaning of mass emerges from interactions. We realise that the gravitational interaction is the most notorious one, and pre-exists the concept of force. In the terms of Popper, Newton’s theory is simpler than Mach’s proposal, since it lends itself to refutation more generously [22].

It is worth noting that *Principia Mathematica* dedicates special attention to the issue of relative vs. absolute rotations. A fundamental difference between rotational and translational motion is that rotations cannot be conceived without acceleration. We find, also in the Scholium of the first chapter, that "The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion
purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion.” [Motte] 18 This view contrasts with Mach’s attempts to wipe out the difference between relative and absolute rotations: “The principles of mechanics can, indeed, be so conceived, that even for relative rotations centrifugal forces arise.” [II.VI.5, pp. 232] 17. In that paragraph Mach remains within the cinematic aspect of relative rotations; what we may call the visual intuition of rotation. However, in the construction of knowledge the child gradually adds other sensations to the visual, noticeably the physical effort required to sustain a rotation. Mach makes an enormous metaphysical effort - thus betraying his own ideas- to disregard the difference in sensory effects arising in actual rotations as opposed to apparent ones (called in this context “relative” both by Newton and Mach). Poincaré [Chapter VII] 21 criticises Mach’s view accurately indicating that in order to confuse apparent rotations with true ones, we need to assign to the apparent rotation a force which is contrary to intuition (it increases with distance), a sort of conspiracy of the rest of the universe to deceive the observer. Mach’s attempt to suppress the subject comes short of Newton’s insight about the difference between subjective and objective descriptions.

6.1. The present view. In this essay we explore the consequences of recognising our humanity, the acceptance of the undeniable fact that our thoughts have their grounds in the constructions we produced as babies and infants, integrated later with our cultural background. This structure is risen always under the supervision of reason, since reason protects the unity of our conception of the world: the construction by dialectical oppositions that we have come to call understanding. Thus, the starting point of our construction is a principle of knowledge, an internal requirement of both what we are ready to accept as knowledge of the world and what has to be rejected as such. We named this principle the No Arbitrariness Principle (NAP). On the positive side, it enables us to pursue the quest of developing an objective knowledge of the world. Since knowledge means to put the input of the sensory system in correspondence with the organisational labour of our brain, truly objective knowledge can only be seeked by a second movement in which the subject (the observer) removes her/himself from the scene. This is the master plan of this work, reaching as much objectivity as possible by going across (transcending) subjectivity.

If space and time have a foundation in the early experiences of life, interactions are a rather different matter. Interactions are not objects in space. Forces are metaphysical entities responsible for the departure from inertial motion, they have no time since causes are only made apparent by their effects: the gravitational interaction is present before, during and after the fall of an object. Forces can only be recognised by their effects and characterised using the established framework in a process of inference. The first law of motion has then a stronger support that the other two: The result is independent of the way we characterise interactions. The second law, and even more the third are in a good degree the result of promissory assumptions. Newton, in search of credibility for the action and reaction principle, relies on contact forces that impress us in a sensory form (“If you press a stone with your finger, the finger is also pressed by the stone” [Motte] 18). In our discussion we explore to what extent the principle of action and reaction is the result of NAP, restricting our discusion to forces of instantaneous action. The result is a generalised principle of action and reaction. Thus, Newton’s third law has a weaker
logical support than the first and second laws. If classical forces are ever found that break reflexion symmetry, we expect total linear momentum not to be conserved by the interaction.

Apart from assumptions of technical character and others suggested by observation, the present approach rests on a principle of knowledge (derived from NAP), namely that the relative properties of a pair of interacting objects depend only on the objects themselves and their interaction. Thus, we develop the concept of objective relative distance. Indeed, the construction rests on the resolution of the tension between objective and subjective. In this sense, we get past the traditional issue present in Leibniz, Newton and later Mach and Poincaré (see below) about “absolute” and “relative” distances. The quest here is to objectivise relative distances. The present work disposes of absolute space, while retaining objectivity.

Another basic underlying idea is that the concepts of space and time are different. Space is related to permanence and time to change. In particular, space and time are not interchangeable. This is based in the way we construct our knowledge.

7. On the laws of Nature

The idea of laws of Nature fits well a society that rests on religious concepts. In such a context we can consider that Nature is a creation of God and assume She/He endowed Nature with laws. But in as much as religious beliefs decline and faith in our own strength grows, we come to accept that the laws of Nature are actually laws produced by humanity to understand Nature, they are "Laws for the understanding of Nature", constructs generated to organise the sensory world, the continuation at the level of civilisation of the work autonomously initiated by every child. As such, the laws of Nature are subject to a higher set of rules, those discovered by every member of our species in our early quest for survival. They include the conviction that “there is something out there” that reaches us through our senses (call it Nature) and then, the possibility of objective knowledge.

Unlike social rules on which we agree in order to preserve energies for other relevant social matters and are established by consensus or power, unlike the “agreement on disagreeing” that makes room for cooperation by setting aside disputes, Truth is the only form of agreement with Nature, as Nature is not a social actor that can change behaviour or negotiate the rules. But Truth requires an agreement with our humanity as well. Thus, the laws of Nature or Truth about Nature come as a result of the dialectic interplay between an universal (humanwise or civilisationwise) subject and the universe of sensory observations, Nature. None of them can be absent in the laws of Nature. In turn, the universal subject requires a correspondence between the individuals, it requires intersubjectivity. Intersubjectivity sets the lower level of requisites on the side of the subject for calling something a law of Nature.

Much of our understanding of Nature comes from what we call “dialectical openings to knowledge”. Thus, the recognition of ourselves requires the simultaneous recognition of not-ourselves, our environment. Likewise, the movement of isolated bodies, the inertial movement, has it necessary opponent in the movement of bodies that interact. At first sight, we could have introduced just unilateral action since the opponent of isolated is not-isolated, i.e., influenced by (or influencing) others. But such an idea contradicts the higher level of reasoning, the rules on rules, one
of them being the principle of no-arbitrariness (NAP). Do we have a reason to support that one body can influence another without being influenced, that there is an asymmetry among bodies? So far, the answer is no.

The present approach is not just discursive, it is constructive as well. Let us see how it works on a much debated matter.

7.1. The speed of light. During a large part of the XIX-th century and well into the following century, a number of attempts to measure the speed of light and to understand its constancy were done. These attempts were mostly independent of the electromagnetic theory of light (at most, the wave-like properties of light were used). Experiments such as those by Fizeau in 1848, later developed and improved by Cornu in 1872-74\[5\] (using a light source, a rotating cogwheel and a mirror) consider that there exists a light path and that the event “detect the light” occurs at a later time than the event “turn on the light”. The measuring issue can hence be regarded as purely cinematical, although the connection between light and electrodynamics was advanced already in the mid 1800’s. The final outcome of these efforts has been that the “speed of light” is constant and also that this constancy is incompatible with Newtonian mechanics. In this Section we will show that under the present approach based on NAP the last assertion is incorrect.

7.1.1. The view of this work. In the first place we must raise the objection that light, being a perception resulting from the electromagnetic interaction, is not an object. Hence, what is meant by “the speed of light” needs to be explained. What is truly measured in Fizeau’s or Cornu’s experiments is a distance and a time-interval. The quotient of both has the dimensions of a velocity so we may agree to consider it a velocity. But the next step in the traditional conception of light is to interpret this quantity so measured in terms of the velocity of a body or a material wave, something that, being matter, has a place in space. Since interactions are not matter, in principle this interpretation introduces a fundamental belief: interactions are mediated by quasi-material entities that as such have a place in space. Neither this assumption or its rational/philosophical basis is (as far as we know) ever stated. Interactions do not have a place, interactions require two places and relative distances, such as the relative distance between source and mirror. Thus, what is measured is an objective distance and an intersubjective time-interval between three events. If we want to associate these events in terms of traditional causal relations it will have to be in the order: turn on the light, reflect the light at the mirror, detect the light. Yet the cause of our detection of light cannot have ceased by the time of the detection, because in such a case we would detect nothing as Kant teaches us [(pp. 183)14]. Since without turning on the light there is no light detected, and the same can be said when the mirror is not in place, both the presence of the mirror and the turning on of the light are within the causes of the light detected. Also, the precedence of the causes indicates the time-order attributed to the three events. But the observed events are not the causes in themselves since the first event might very well have ceased when the third event happens. Similarly, the detected light immediately associated with the turn on event could only be the cause if it travelled (matter-like) in space. An alternative option, on the other hand, which is fully compatible with NAP, is to consider that all three events are actually the result of a common cause, namely, the electromagnetic interaction. Moreover, the time for a common cause is an undetermined time before the first detected event associated
with it. Whatever we make in our interpretation of light, it is undeniable that what we have called the speed of light is the quotient of an objective distance and an intersubjective time interval. The result is objective (or rather intersubjective) and is subject to the laws of transformations of objective quantities, this is, it is the same for all observers. The interpretation in terms of a material analogous allows for questions that do not correspond to what we are truly measuring.

7.1.2. The mediator view. On the other hand, equating a subjective quantity such as $|v_a^L|$ (the velocity of an interpreted material point $L$ with respect to a reference $a$) with an objective quantity such as $C$ (namely a constant, and hence objective and invariant) is, to say the least, confusing. A possible solution confronting experimental evidence (which is not the above-mentioned experiments) could be to consider $\Delta t$ as subjective, since after all, absolute time was an assumption of Newtonian mechanics. If space is still to be conceived as absolute, then time might very well depend on the velocity of the reference system with respect to absolute space.

Let us work out an exercise along this line of thought. We will tentatively assume as a new axiom a deeply rooted belief that we are not going to offer for examination:

**Tacit belief 1.** Interactions are mediated by quasi-material entities that as such have a place in space and a velocity.

Then, if the quasi-material entity moves in absolute space with a characteristic velocity, we would expect that it appears to us with different (and measurable) relative velocities depending on the relative motion of our system of references with respect to absolute space. However, experiments (interpreted in this frame of mind) indicate otherwise: we measure always the same velocity. Then, absolute time, absolute space, these measurements and the present Tacit belief have come into collision. We can then ask what can we save from the wreckage? For the sake of the exercise, we insist in keeping the Tacit belief alive. Again, it has been argued that such state of things is incompatible with Newtonian mechanics, but again, this is not the case. Is it possible that velocities transform between different reference systems satisfying the existence of a universal velocity and the structure of Newtonian mechanics? The following Theorems give an affirmative solution.

**Definition.** Let $U$ be a nonnegative real number and $g(U) = G(|U|)$, where $G$ is a strictly increasing function such that $G(0) = 1$ and $\lim_{|U| \to C^-} g(U) = \infty$. We advance that $U$ will play the role of a velocity in the sequel. Then we define velocity addition as

$$W = U \oplus V \iff g(W)W = g(U)U + g(V)V,$$

where addition in the rhs denotes standard vector addition in 3-space.

**Theorem.** (Nonlinear presentation of Galileo’s group) Velocity addition has the following properties:

a) $|U|, |V| < C \Rightarrow |W| = C$.
b) $U \oplus 0 = 0$, i.e., 0 is the neutral element of velocity addition.
c) $U \oplus V = V \oplus U$.
d) For all $U$, the inverse of $U$ is $-U$, i.e., $U \oplus (-U) = 0$.
e) **Associativity**: $(W \oplus V) \oplus U = W \oplus (V \oplus U)$. 
Definition. We call \( t = g(V)T \) subjective time, where \( T \) is the time measured by an observer at rest and we call velocity \( V \) the vector satisfying \( \Delta x = V \Delta t \).

Theorem. \( T \) is invariant if and only if space is invariant.

Proof. Consider two reference systems \( S, S' \) in relative motion with velocity \( W \). An object \( O \) with velocity \( V \) in \( S \) has a velocity \( U \) in \( S' \), satisfying the following equations:

\[
\begin{align*}
\Delta x &= V \Delta t = g(V) V \Delta T \\
\Delta x' &= U \Delta t' = g(U) U \Delta T \\
g(V)V &= g(U)U + g(W)W
\end{align*}
\]

Seen from \( S' \), while \( O \) has moved a distance \( d'_{O} = g(U) U \Delta T \), the origin of \( S \) has moved a distance \( d'_{S} = -g(W) W \Delta T \). Hence \( S' \) computes the distance between \( O \) and \( S \) as:

\[
d_{OS} = d'_{O} - d'_{S} = (g(U) U + g(W) W) \Delta T.
\]

Formally, this is the same result computed by \( S \) for the distance. Hence, distances are invariant if and only if \( \Delta T \) is the same for all observers. \( \square \)

As far as we know, the physics emerging from this picture has not been studied. Certainly, we may imagine other alternatives to exit the contradictory situation. For instance, we could drop entirely Tacit belief \( 1 \) what leads to a full return to the view of this work, or try other forms of reconciliation such as e.g., keeping the Tacit belief while dropping both absolute space and time, as well as relative distance. This part of the story is well known.

We have then seen that the argued need to abandon the Newtonian frame upon the requirement of a constant speed of light is not mandatory. On the contrary, this course of action is related to a bolder decision: to associate interactions with the exchange of quasi-material entities (i.e., to consider light an object) and subsequently opting for a solution without all three of absolute space, absolute time and relative distance.

7.2. Poincaré’s principle of relativity and related principles. As we have already mentioned, the No Arbitrariness Principle relates to Leibniz’ principle of sufficient reason. Mach also used the principle of sufficient reason in the form of a “principle of symmetry” [(pp. 9) 17]. At the beginning of the XX-th century, a related principle became notorious, the principle of relativity presented by Poincaré.
“The motion of any system must obey the same laws, whether it be referred to fixed axes or to moving axes carrying along a rectilinear uniform motion. This is the principle of relative motion that forces upon us for two reasons: first, the commonest experiences confirm it, and second, the contrary hypothesis is singularly repugnant to the mind”. Poincaré hints us about the principle being a principle of knowledge (“not solely the result of experiment”), but does not elaborate further. As we have shown, Poincaré’s principle belongs to the context of interactions, for in the description of interactions it should not matter which isolated body we take (arbitrarily) as reference for our calculations and descriptions. This issue was advanced before in this work. Poincaré and Mach inherit the discussion between absolute and relative spaces already present in Leibniz-Clarke and Newton. They attempt to resolve this issue in different ways, most remarkably Mach reducing the cosmos to a set of empirical relations without subjects (observers). The present approach goes past this controversy introducing the arbitrariness of the observer, and with her/him the concept of space, subsequently eliminating the arbitrariness by focusing the understanding in objective or intersubjective elements. In short, this work takes sides for an objective and relative description of interactions, with the stress resting on its objective nature.

8. Summary and conclusions

We have reconstructed Newtonian mechanics starting with a minimal realism that accepts as starting point the existence of a real, or objective, world. The analysis of this physical world, we called it Nature, proceeds through several constructive steps. Some of them consist in dialectical openings to understanding. The first opening consists precisely in the opposition between objective and subjective: objective opposes not-objective, i.e., subjective, which is what comes from the knowing subject. Thus, understanding Nature is the result of a dialectic interplay between the knowing subject and the known object and it is this dialog what we call understanding. It is worth mentioning that the subject is not an individual in principle but rather humanity or society. Thus knowledge in this primary form cannot be objective, the subject needs to transcend (negate) his subjectivity if she/he is going to reach any possible objectivity. Hence, the subject establishes principles of reason such as the principle of sufficient reason that we have recast in mathematical terms under the name of No Arbitrariness Principle. Actually, it is not the subject by her/himself who establishes the principle but the subject in relation to Nature. Thus, the principle is likely to exist because it represented an adaptive advantage for the early humanity. There is no form in which the dialectic objective-subjective can be separated giving preeminence to one of the terms.

The next contributions to mechanics come from the early childhood where the notions of time and space are formed as a consequence of active adaptation of the child to her/his environment. In this stage of our construction, concepts point toward perceptions rather than to other concepts. With the opening produced by the concepts of isolated and interacting comes the first mechanical law, the only law acceptable to NAP for the condition of permanence of isolated objects. Repeated use of NAP in different contexts will finally unravel Newton’s laws. But these laws require an increasing number of assumptions regarding Nature.

We address the laws of Nature not as rules proper to the objects, as in metaphysical realism, but rather as Laws (for the understanding) of Nature generated
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by the interplay between subject and object. This position becomes quite evident when Galileo transformations of velocities are not alleged to be experimental laws but rather considered as a requisite for the suppression of contributions arbitrarily introduced by the observer.

The adopted approach allows us to attain a higher level of consciousness on the strength of the different components that contribute to the construction of mechanics. There is a precedence between these components which can be better appreciated by considering the consequences of removing them: (a) The consequence of renouncing to logic and to the laws of understanding is not being able to understand the world, (b) renouncing to the early elaborations of primary concepts such as time and space leads to a dissociation between every day life and physics, the latter becoming entirely pragmatic and justified a-posteriori (because it is convenient), (c) in contrast, modifying our temporary beliefs has no real cost other than the effort of reconstructing our understanding on a more solid basis. Moreover, the present approach allows for a critical view of further developments, since it opens for new alternatives to the current views in different issues, alternatives that we believe should be explored to decide their worth.

An important reflection corresponds to the use of tacit beliefs (also known, in mathematics, as hidden assumptions or hidden lemmas). In as much as they are hidden and as such can escape the scrutiny of reason, they must be considered dangerous. They elude the principle of necessary reason and usually protect something that is not explicitly mentioned. In the case of Tacit belief it hides the fact that we support the argument in a material model, be it wave or particle, on the basis that we cannot imagine it otherwise. Our imagination works with memories of the sensible world and in doing so it limits its value to the material experience. “Because we are not able to imagine otherwise” appears to us as an extremely weak argument: the confession of a limitation does not constitute a reason. We can explore the possibilities of the hidden assumption but we should be ready to abandon it if it comes into conflict with higher principles.

The disciplines are not autonomous, they are ruled by reason, which is the only autonomous entity. The application of the principles of reason to the understanding is known as critic. Critic in science needs not only be declared but it should be truly exercised. This work is indeed an exercise of critic. It seems proper to quote Kant [(The discipline of reason) on this respect “...where reason is not held in a plain track by the influence of empirical or of pure intuition, that is, when it is employed in the transcendental sphere of pure conceptions, it stands in great need of discipline, to restrain its propensity to overstep the limits of possible experience and to keep it from wandering into error. In fact, the utility of the philosophy of pure reason is entirely of this negative character.” (The terms pure intuition and pure conceptions in Kant can be understood from. “Pure intuition consequently contains merely the form under which something is intuited, and pure conception only the form of the thought of an object. Only pure intuitions and pure conceptions are possible a priori; the empirical only a posteriori.” ]

In conclusion, the construction of knowledge is not only a philosophical perspective but it is a constructive and critic method as well. Indeed, the history of knowledge has to acknowledge the contributions from metaphysical (unverifiable/irrefutable) ideas such as absolute space and later the reification of interactions. Changing the underlying metaphysics, great revolutions operate in physics,
which is not automatically equal to great progresses. In these lines, recognising the need for introducing some metaphysics, the idea of a pluralist realism should be considered not only as a description but rather as a programme. We might ask if the efforts of critical revision such as those of Mach or the present work could have been possible without the preceding developments and their metaphysical ingredients. Critic, as the negative tool of philosophy, cannot operate in the void but can operate as self-criticism in the process of construction of knowledge.

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