Weighted Bergman-Dirichlet and Bargmann-Dirichlet spaces of order \( m \) in high dimensions

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Abstract

We introduce and study a generalization of the classical weighted Bergman and Dirichlet spaces on the unit ball in high dimension, the Bergman-Dirichlet spaces. Their counterparts on the whole \( n \)-complex space \( \mathbb{C}^n \), the Bargmann-Dirichlet spaces, are also introduced and studied. Mainly, we give a complete description of the considered spaces, including orthonormal basis and the explicit formulas for their reproducing kernel functions. Moreover, we investigate their asymptotic behavior when the curvature goes to 0.

Keywords: Weighted Bergman-Dirichlet spaces, Weighted Bargmann-Dirichlet spaces, Reproducing kernel function, Hypergeometric function

1. Introduction and statement of main results

The Segal-Bargmann space, on the \( n \)-complex space \( \mathbb{C}^n \) endowed with its standard inner product \( \langle z, w \rangle \), and the so-called weighted Bergman and Dirichlet spaces, on the open unit ball \( \mathbb{B}^n = \{ z \in \mathbb{C}^n; |z| := \sqrt{\langle z, z \rangle} < 1 \} \), are basic examples of functional spaces in the theory of analytic functions. Such spaces play important roles in function theory and operator theory, as well as in modern analysis, probability and statistical analysis. For a nice introduction and surveys of these spaces in the context of function and operator theories, see for example [8, 9, 12, 13, 1, 14, 3] and the references therein.

Recently, two new classes of analytic function spaces of Sobolev type, labeled by a nonnegative integer \( m \), have been introduced and studied in [4]. The first one is the Bergman-Dirichlet space generalizing the weighted Bergman and Dirichlet spaces on the disk \( D(0, R) \) in the complex plane \( \mathbb{C} \). The second is the Bargmann-Dirichlet space generalizing the Segal-Bargmann space on the complex plane \( \mathbb{C} = D(0, +\infty) \). They are reproducing kernel Hilbert spaces. Their reproducing kernel functions have been calculated explicitly and expressed in terms of the hypergeometric functions.

Our purpose in the present paper is to introduce the spaces \( \mathcal{A}_m^{2\alpha}(\mathbb{B}^n) \), the analogue of the considered Bergman-Dirichlet spaces in high dimension. What we do in the construction of \( \mathcal{A}_m^{2\alpha}(\mathbb{B}^n) \) works mutatis mutandis to introduce their counterparts on the whole \( n \)-complex space.
\( \mathbb{C}^n \), the Bargmann-Dirichlet spaces \( \mathcal{F}_m^{2,2}(\mathbb{C}^n) \) of order \( m \). We investigate their spectral properties and generalize the results obtained in [4] to high dimensions \( n \geq 1 \). A part of some special techniques introduced in the calculation, the approach used here to prove our main results is quite similar to the one-dimensional setting. The motivations for studying such a generalization are various and meaningful.

The paper is organized as follows. In Section 2, we introduce the weighted Bergman-Dirichlet and Bargmann-Dirichlet spaces and state our main results (Theorems 2.3 and 2.5) giving the explicit expression of the corresponding reproducing kernel functions. Section 3 is devoted to the concrete description of the Bergman-Dirichlet spaces and to the proof of Theorem 2.3. In Section 4 we are concerned with the Bargmann-Dirichlet spaces and the proof of Theorem 2.5. We conclude the paper by studying the asymptotic behavior of the \( L^2 \)-eigenprojector kernel of \( A_m^{2,\alpha}(\mathbb{B}_R^n) \) when \( R \) tends to infinity, and show that it gives rise to the \( L^2 \)-eigenprojector kernel of \( \mathcal{F}_m^{2,2}(\mathbb{C}^n) \).

2. Statement of main results

For given \( \alpha > -1 \), we consider the measure \( d\mu_\alpha(z) := (1 - |z|^2)\alpha^\alpha \cdot d\lambda(z) \), where \( d\lambda \) stands for the Lebesgue measure. Then, the weighted Bergman space \( \mathcal{A}^{2,\alpha}(\mathbb{B}^n) \) ([8, 11, 7, 13]) can be defined as the functional space of all holomorphic functions \( f \) on \( \mathbb{B}^n \), \( f \in \mathcal{H}ol(\mathbb{B}^n) \), that are \( d\mu_\alpha \)-square integrable, \( f \in L^{2,\alpha}(\mathbb{B}^n) := L^2(\mathbb{B}^n; d\mu_\alpha) \). That is

\[
\mathcal{A}^{2,\alpha}(\mathbb{B}^n) := L^{2,\alpha}(\mathbb{B}^n) \cap \mathcal{H}ol(\mathbb{B}^n).
\]

It is supplied with the norm \( \|f\|_\alpha := \|f\|_{L^{2,\alpha}(\mathbb{B}^n)} \) associated to the inner product defined as

\[
\langle f, g \rangle_\alpha := \int_{\mathbb{B}^n} f(z)|g(z)|^2(1 - |z|^2)^\alpha \cdot d\lambda(z).
\]

The weighted Dirichlet space \( \mathcal{D}^{2,\alpha}(\mathbb{B}^n) \) is the analytic function space on \( \mathbb{B}^n \) defined by [10, 13]

\[
\mathcal{D}^{2,\alpha}(\mathbb{B}^n) := \left\{ f(z) = \sum_{p \in (\mathbb{Z}^+)^n} a_pz^p; \sum_{p \in (\mathbb{Z}^+)^n} |p|! |a_p|^2 < +\infty \right\}
\]

Here \(|p| = p_1 + \cdots + p_n\) and \(|p|! = p_1! \cdots p_n!\) for given multi-index \( p = (p_1, \cdots, p_n) \in (\mathbb{Z}^+)^n \) and \( z^p = z_1^{p_1} \cdots z_n^{p_n} \) for given \( z = (z_1, \cdots, z_n) \in \mathbb{C}^n \).

Now, for every fixed nonnegative integer \( m \), every complex valued holomorphic function \( f \) with the Taylor expansion \( f(z) = \sum_{p \in (\mathbb{Z}^+)^n} a_pz^p \) can be written as

\[
f(z) = f_{1,m}(z) + f_{2,m}(z),
\]

where \( f_{1,m} \) and \( f_{2,m} \) stand for \( f_{1,m}(z) = \sum_{|p| < m} a_pz^p \) and \( f_{2,m}(z) = \sum_{|p| \geq m} a_pz^p \). We define

\[
\|f\|^2_{\alpha,m} = \left\| f_{1,m} \right\|_\alpha^2 + m! \sum_{|p| = m} \frac{1}{|p|!} \left\| D^p f_{2,m} \right\|_\alpha^2
\]

to be the norm on \( L^{2,\alpha}(\mathbb{B}^n) \) associated to the inner product

\[
\langle f, g \rangle_{\alpha,m} = \langle f_{1,m}, g_{1,m} \rangle_\alpha + m! \sum_{|p| = m} \frac{1}{|p|!} \langle D^p f_{2,m}, D^p g_{2,m} \rangle_\alpha.
\]
Here $D^p$ is the partial differential operator of total degree $|p|$, $p = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$, defined by
\[
D^p = \frac{\partial^{|p|}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}.
\] (7)

**Definition 2.1.** The functional space $\mathcal{F}^2_m(\mathbb{B}^n) = \{ f \in \mathcal{H}ol(\mathbb{B}^n) \mid \|f\|_{m, \infty} < +\infty \}$ endowed with the norm (5) will be called the generalized Bergman-Dirichlet space of order $m$ on $\mathbb{B}^n$.

**Remark 2.2.** For the special cases $m = 0$ and $m = 1$, we recover the weighted Bargmann space (1) and the classical Dirichlet space (3), respectively.

Our central result for these spaces can be stated as follows

**Theorem 2.3.** Keep notations as above. Then, the space $\mathcal{F}^2_m(\mathbb{B}^n)$ is a reproducing kernel Hilbert space. Its reproducing kernel function is given in terms of the $3F_2$-hypergeometric function by the following closed form
\[
K_{\mathcal{F}^2_m(\mathbb{B}^n)}(z, w) = \frac{\Gamma(\alpha + n + 1)}{\pi^2 \Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \frac{(\alpha + n + 1)_k}{k!} (z, w)^k
\]
\[
+ \frac{\Gamma(\alpha + n + 1)}{\pi^2 \Gamma(\alpha + 1)} \frac{(z, w)^m}{(m!)^2} 3F_2\left( \begin{array}{c} 1, 1, \alpha + n + 1 \\ m + 1, m + 1 \end{array} \mid (z, w) \right).
\]
(8)

On the whole $n$-dimensional euclidean complex space $\mathbb{C}^n$ and for given $\nu > 0$, we denote by $L^2(\mathbb{C}^n) := L^2(\mathbb{C}^n; e^{-\nu |z|^2} d\lambda)$ the space of $e^{-\nu |z|^2} d\lambda$-square integrable functions on $\mathbb{C}^n$. The Segal-Bargmann Hilbert space $\mathcal{F}^2(\mathbb{C}^n)$ is then defined to be the space of all holomorphic functions belonging to $L^2(\mathbb{C}^n)$. That is $\mathcal{F}^2(\mathbb{C}^n) = L^2(\mathbb{C}^n) \cap \mathcal{H}ol(\mathbb{C}^n)$, supplied with the norm
\[
\|f\|^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\nu |z|^2} d\lambda(z).
\] (9)

As a generalization of this space, we perform the space of all holomorphic functions on $\mathbb{C}^n$ such that
\[
\|f\|_{\nu, m} := \|f_{1,m}\|^2 + \sum_{l=0}^{m} \frac{m^l}{l!} \|D^l f(z)\|_{\nu}^2 < +\infty,
\] (10)

where we have split $f$ as in (4), to wit $f = f_{1,m} + f_{2,m}$. That is

**Definition 2.4.** The functional space $\mathcal{F}^2_m(\mathbb{C}^n) = \{ f \in \mathcal{H}ol(\mathbb{C}^n) \mid \|f\|_{\nu, m} < +\infty \}$ endowed with the norm (10) will be called the generalized Bargmann-Dirichlet space of order $m$ on $\mathbb{C}^n$.

The analogue of Theorem 2.3 for $\mathcal{F}^2_m(\mathbb{C}^n)$ is the following

**Theorem 2.5.** The space $\mathcal{F}^2_m(\mathbb{C}^n)$ is a reproducing kernel Hilbert space. Its reproducing kernel function is given in terms of the $3F_2$-hypergeometric function by
\[
K_{\mathcal{F}^2_m(\mathbb{C}^n)}(z, w) = \left( \frac{\nu}{\pi} \right)^m \sum_{k=0}^{\infty} \frac{(\nu (z, w))^k}{k!} + \frac{(z, w)^m}{(m!)^2} 3F_2\left( \begin{array}{c} 1, 1 \\ m + 1, m + 1 \end{array} \mid \nu (z, w) \right).
\] (11)
What we have done for the unit ball can be extended in an appropriate way to any 0-centered ball $B_R^n$ of radius $R$. Doing so, one shows that the Bargmann-Dirichlet spaces $\mathcal{F}_m^\nu(\mathbb{C}^n)$ can be seen as the limit of the Bergman-Dirichlet spaces $\mathcal{A}_m^{\alpha}(B_R^n)$, with $\alpha = R^2$, as $R$ goes to infinity, in the sense that we have

**Theorem 2.6.** For every fixed nonnegative integer $m$ and real number $\nu > 0$, the reproducing kernel $K_{\mathcal{A}_m^{\alpha}(B_R^n)}$ of the weighted Bergman-Dirichlet space $\mathcal{A}_m^{\alpha}(B_R^n)$, with $\alpha = R^2$, converges pointwise and uniformly on compact sets of $\mathbb{C}^n \times \mathbb{C}^n$ to the reproducing kernel function $K_{\mathcal{F}_m^\nu(\mathbb{C}^n)}$ of weighted Bargmann-Dirichlet space $\mathcal{F}_m^\nu(\mathbb{C}^n)$.

This is motivated by the fact that the flat Hermitian geometry on $\mathbb{C}^n$ can be approximated by the complex hyperbolic geometry of the balls $B_R^n$ of radius $R > 0$ associated to an appropriate scaled Bergman Kähler metric $[6]$. Indeed, we have to deal with the scaled measure

$$d\mu_{\alpha,R}(z) := \left(1 - \frac{|z|^2}{R^2}\right)^\nu d\lambda(z).$$

**Remark 2.7.** To not cumbersome with additional notations in proving our main results, we restrict ourself to the case $n = 2$, the general case can be investigated in a similar way.

### 3. The generalized Bergman-Dirichlet space on $\mathbb{B}^2$

The 2-dimensional complex space $\mathbb{C}^2$ is endowed with the inner product $(z, w) = z_1\bar{w}_1 + z_2\bar{w}_2$ for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{C}^2$. Its associated norm is given by $|z| = \sqrt{|z_1|^2 + |z_2|^2}$. Let us denote by $S^3 = \partial \mathbb{B}^2$ be the unit sphere of $\mathbb{C}^2$ viewed as the boundary of the unit ball $\mathbb{B}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z|^2 < 1\}$. On $\mathbb{B}^2$, we consider the weighted measure

$$d\mu_\alpha(z) = \left(1 - |z|^2\right)\nu d\lambda(z),$$

where $\alpha \in \mathbb{R}$ and $d\lambda(z) = \left(\frac{2}{\pi}\right)^2 dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2$ is the usual Lebesgue measure. Notice that the measure $\mu_\alpha$ is finite on $\mathbb{B}^2$ if and only if $\alpha > -1$, indeed we have

$$\int_{\mathbb{B}^2} d\mu_\alpha = \pi^\frac{1}{2} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 3)}.$$

For given multi-index $p = (p_1, p_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, we use as usual $|p| = p_1 + p_2$, $p! = p_1!p_2!$ and set

$$z^p = z_1^{p_1}z_2^{p_2} = : \varphi_p(z).$$

In order to prove Theorem 2.6, we begin with the following

**Proposition 3.1.** The monomials $\varphi_p(z) = z^p; p \in \mathbb{Z}^+ \times \mathbb{Z}^+$, belong to $\mathcal{A}_m^{\alpha}(\mathbb{B}^2)$ if and only if $\alpha > -1$. Moreover, for $\alpha > -1$, they form an orthogonal system in $\mathcal{A}_m^{\alpha}(\mathbb{B}^2)$ with

$$||\varphi_p||^2_{m,\alpha} = \pi^2 \Gamma(\alpha + 1) \begin{cases} m! & \text{if } |p| < m \\ \Gamma(|p| + \alpha + 3) p! & \text{if } |p| \geq m \end{cases}.$$  \hspace{1.5cm} (12)
Proof. Orthogonality of the monomials \( \varphi_p(z) = z^p \); \( p \in \mathbb{Z}^+ \times \mathbb{Z}^+ \), with respect to the inner product \( (\cdot, \cdot)_{\alpha, m} \) follows from the fact that \( D^q \varphi_p(z) = \frac{p!}{(p-q)!} z^{p-q} \) and the orthogonality of the monomials in classical weighted Bergman space. Furthermore, for \( |p| < m \), we have
\[
\|\varphi_p\|_{\alpha, m}^2 = \int_{\mathbb{D}} |\varphi_p|^2 \, d\mu_z.
\]
By integrating in polar coordinates \( z = re^{i\theta} \) for \( r \in [0, 1] \) and \( \theta \in S^1 \), we get
\[
\|\varphi_p\|_{\alpha, m}^2 = \int_0^1 r^{2|p|+1} (1 - r^2)^\alpha \, dr \int_{S^1} |\varphi_p|^2 \, d\sigma(\xi),
\]
where \( d\sigma \) is the area measure on \( S^1 \). Now, to compute the integral \( \int_{S^1} |\varphi_p|^2 \, d\sigma(\xi) \), we use the coordinates \( \xi_1 = e^{i\theta_1} \sin(\phi) \), \( \xi_2 = e^{i\theta_2} \cos(\phi) \), where \( \phi \in [0, \pi/2] \) and where \( \theta_1 \) and \( \theta_2 \) can take any value between 0 and \( 2\pi \), we find
\[
\int_{S^1} |\varphi_p|^2 \, d\sigma(\xi) = (2\pi)^2 \int_0^{\pi/2} (\sin(\phi))^2 |\varphi_p|^2 \, d\phi = 2\pi^2 \frac{p!}{(|p|+1)!}.
\]
Making the change of variable \( t = r^2 \) yields
\[
\|\varphi_p\|_{\alpha, m}^2 = \pi^2 \frac{p!}{\Gamma(|p|+2)} \int_0^1 t^{|p|+1} (1-t)^\alpha \, dt.
\]
The involved integral is the Euler function, which converges if and only if \( \alpha > -1 \). Now, we consider the case of \( |p| \geq m \). Indeed, in this case, for \( q = (q_1, q_2) \), we have \( D^q \varphi_p(z) = \frac{p!}{(p-q)!} z^{p-q} \) with the convention \( D^q \varphi_p(z) = 0 \) if \( p_1 < q_1 \) or \( p_2 < q_2 \). Therefore,
\[
\|\varphi_p\|_{\alpha, m}^2 = \sum_{|q| = m} \frac{m!}{q!} \left( \frac{p!}{(p-q)!} \right)^2 \|\varphi_{q-p}\|_{\alpha}
\]
\[
= \pi^2 \frac{p!}{\Gamma(|p|-m+2)} \int_0^1 t^{|p|-m+1} (1-t)^\alpha \, dt \sum_{|q| = m} \frac{m!}{q!} \left( \frac{p!}{(p-q)!} \right).
\]
Thus, the norm \( \|\varphi_p\|_{\alpha, m} \) is finite if and only if \( \alpha > -1 \). In this case, (13) reduces further to
\[
\|\varphi_p\|_{\alpha, m}^2 = \pi^2 \Gamma(\alpha+1) \begin{cases} \Gamma(|p|+\alpha+3) & \text{if } |p| < m \\ \frac{1}{\Gamma(|p|-m+\alpha+3)} & \text{if } |p| \geq m \end{cases}
\]
thanks to the multi-monomial formula
\[
\prod_{j=0}^{k-1} (z_1 + z_2 - j) = k! \sum_{p=0}^{k-1} \frac{\prod_{j=0}^{p-1} (z_1 - j) \prod_{j=0}^{p-1} (z_2 - j)}{p!},
\]
which will be used systematically in the sequel. \( \square \)
The previous proposition shows that the space \( \mathcal{A}_m^\alpha(\mathbb{B}^2) \) is non trivial if and only if \( \alpha > -1 \). From now on, we assume that \( \alpha > -1 \).

**Proposition 3.2.** A holomorphic function \( f \) belongs to \( \mathcal{A}_m^\alpha(\mathbb{B}^2) \) if and only if its Taylor coefficients satisfy the condition

\[
\sum_{|p|\geq m} \frac{|p|!(|p| - 1)\cdots(|p| - m + 1)p!}{\Gamma(|p| - m + \alpha + 3)}|a_p|^2 < +\infty.
\]

Furthermore, we have

\[
\|f\|_{\mathcal{A}_m}^2 = \pi^2\Gamma(\alpha + 1) \sum_{|p|\geq m} \gamma_{\alpha,p}|a_p|^2,
\]

where \( \gamma_{\alpha,p} \) stands for

\[
\gamma_{\alpha,p} = \begin{cases} \frac{p!}{\Gamma(|p| + \alpha + 3)} & \text{if } |p| < m \\ \frac{p!}{p!|p|!(|p| - 1)\cdots(|p| - m + 1)} & \text{if } |p| \geq m \end{cases}.
\]

**Proof.** Notice first that according to Proposition 3.1, we have

\[
\|f\|_{\mathcal{A}_m}^2 = \pi^2\Gamma(\alpha + 1) \sum_{|p|\geq m} \frac{p!}{\Gamma(\alpha + |p| + 3)}|a_p|^2.
\]

For \( l = (l_1, l_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) such that \( |l| = m \), we have \( D^l \varphi_p(z) = \frac{d^l}{(p - l)!} z^{p - l} \), thus

\[
\int_{\mathbb{B}^2} |D^l f(z)|^2 d\mu_\alpha(z) = \int_{\mathbb{B}^2} \left( \sum_{|p|\geq m} \frac{p!}{(p - q)!} a_p \zeta^{p - q} \right)^2 \mu_\alpha(z) = \lim_{r \to 1} \int_{\mathbb{B}^2(r)} \left( \sum_{|p|\geq m} \frac{p!}{(p - q)!} a_p \zeta^{p - q} \right)^2 d\mu_\alpha(z).
\]

From the compacticity of \( \mathbb{B}^2(\rho) \) and the orthogonality of \( (\varphi_p)_p \) in the space \( L^2(\mathbb{B}^2(\rho), d\mu_\alpha) \), we get

\[
\int_{\mathbb{B}^2} |D^l f(z)|^2 d\mu_\alpha(z) = \lim_{r \to 1} \sum_p \frac{p!}{(p - q)!} |a_p|^2 \int_{\mathbb{B}^2(r)} |\zeta^{p - q}|^2 d\mu_\alpha(z).
\]

By applying discrete monotone convergence theorem, we obtain

\[
\int_{\mathbb{B}^2} |D^l f(z)|^2 d\mu_\alpha(z) = \pi^2\Gamma(\alpha + 1) \sum_{|p|\geq m} \frac{p!}{(p - q)!} \frac{(p - q)!}{\Gamma(\alpha + |p| - q + 3)} |a_p|^2.
\]

Finally, in view of (14), we get

\[
\sum_{|p|\geq m} \int_{\mathbb{B}^2} |D^l f(z)|^2 d\mu_\alpha(z) = \pi^2\Gamma(\alpha + 1) \sum_{|p|\geq m} \frac{|p|!(|p| - 1)\cdots(|p| - q + 1)p!}{\Gamma(\alpha + |p| - m + 3)} |a_p|^2.
\]

This completes the proof. \( \Box \)
In order to establish the second result in this section we need to the following

**Lemma 3.3.** Let $f$ be a given holomorphic function $f$ on $\mathbb{B}^2$. Then, for every fixed $z \in \mathbb{B}^2$ we have

$$|f(z)| \leq \frac{1}{\pi^2} \frac{\Gamma(\alpha + 3)}{\Gamma(\alpha + 1)} \left( \sum_{k=m}^{\infty} \frac{(\alpha + 3)_k |z|^k}{k!} + \frac{1}{(1 - |z|^2)^{\alpha+3}} \right) \|f\|_{\alpha,m}. \quad (15)$$

Moreover, for every compact set $K$ of $\mathbb{B}^2$ there is a constant $c_K$ such that

$$|f(z)| \leq c_K \|f\|_{\alpha,m}; \quad z \in K. \quad (16)$$

**Proof.** By Cauchy-Schwarz inequality, we get

$$|f(z)| \leq \frac{\|f\|_{\alpha,m}}{\pi^2 \Gamma(\alpha + 1)} \left( \sum_{|p| \leq m} \frac{\Gamma(|p| + \alpha + 3)}{|p|!} |z|^{|p|}^2 + \sum_{|p| \geq m} \frac{\Gamma(|p| - m + \alpha + 3)}{|p|! |p|(|p| - 1) \cdots (|p| - m + 1)} |z|^{|p|}^2 \right).$$

For $|p| \geq m$, we get $|p|(|p| - 1) \cdots (|p| - m + 1) \geq 1$ and therefore we get

$$\sum_{|p| \geq m} \frac{\Gamma(|p| - m + \alpha + 3)}{|p|! |p|(|p| - 1) \cdots (|p| - m + 1)} |z|^{|p|}^2 \leq \sum_{|p| \geq m} \frac{\Gamma(|p| - m + \alpha + 3)}{|p|!} |z|^{|p|}^2$$

$$\leq \sum_{|p| \geq m} \frac{\Gamma(|p| + \alpha + 3)}{|p|!} \sum_{k=0}^{\infty} \frac{|p|!}{|p|^{|p|}} |z|^{|p|}^2$$

$$\leq \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha + 3)}{k!} (|z|^2)^k$$

$$\leq \Gamma(\alpha + 3) \frac{1}{(1 - |z|^2)^{\alpha+3}}.$$

Whence

$$|f(z)| \leq \frac{1}{\pi^2} \frac{\Gamma(\alpha + 3)}{\Gamma(\alpha + 1)} \left( \sum_{k<m} (\alpha + 3)_k |z|^k + \frac{1}{(1 - |z|^2)^{\alpha+3}} \right) \|f\|_{\alpha,m}.$$ 

Since, the function $z \rightarrow \sum_{k \leq m} (\alpha + 3)_k |z|^k + \frac{1}{(1 - |z|^2)^{\alpha+3}}$ is bounded on $\mathbb{B}^2$ for being continuous, it follows that for every compact set $K$ of $\mathbb{B}^2$ there exists a constant $c_K$ such that $|f(z)| \leq c_K \|f\|_{\alpha,m}$ for every $z \in K$. □

We assert

**Proposition 3.4.** The space $\mathcal{A}^{\alpha}_{\mathbb{B}^2}$ is a Hilbert space and the monomials $\varphi_p; \ p \in \mathbb{Z^+} \times \mathbb{Z^+}$, form an orthogonal basis of it.

**Proof.** In view of [16], it follows that the space $\mathcal{A}^{\alpha}_{\mathbb{B}^2}$ is a Hilbert space. Indeed, any Cauchy sequence $(f_p)_p$ in $\mathcal{A}^{\alpha}_{\mathbb{B}^2}$ is uniformly Cauchy sequence on every compact subset of $\mathbb{B}^2$. Thence, by Weierstrass’ theorem, $(f_p)_p$ converges uniformly to a holomorphic function $f$ on $\mathbb{B}^2$ as well as $(D^\alpha f_p)$ to $D^\alpha f$. On the other hand, $(D^\alpha f_p)$ is also a Cauchy sequence in the Hilbert space $L^2(\mathbb{B}^2, d\mu_\alpha)$. Thus, there exists a subsequence $(D^\alpha f_{p_q})_q$ of $(D^\alpha f_p)_p$ converging to $g \in L^2(\mathbb{B}^2, d\mu_\alpha(z))$ pointwise almost everywhere. It follows that $D^\alpha f = g \in L^2(\mathbb{B}^2, d\mu_\alpha)$
and therefore \( f \in \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \) and \((f_{p})_{p}\) converges to \( f \) in \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \). This proves that \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \) is a Hilbert space for the norm \( \| \|_{\alpha,m} \). To conclude, we need only to prove that the monomials \( z^{\alpha} \) form a basis of \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \). For this end, let \( f(z) = \sum_{\alpha} a_{\alpha}z^{\alpha} \) be a function belonging to \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \) and observe that for every integer \( k \), the function \( f_{k} = \sum_{|\alpha| \leq k} a_{\alpha}z^{\alpha} \) belongs to \( \text{span} \{ \varphi_{\alpha} ; \ p \in \mathbb{Z}^{+} \times \mathbb{Z}^{+} \} \), the linear span of \( (\varphi_{\alpha})_{\alpha} \). Thus by Proposition 3.2, we get

\[
\| f - f_{k} \|_{\alpha,m} = \pi^{2} \Gamma(\alpha + 1) \sum_{|\alpha| \geq k} \gamma_{\alpha,m}|a_{\alpha}|^{2},
\]

for \( k \) large enough. Since the involved sum is the rest of a convergent series, the sequence \((f_{k})_{k}\) converges to \( f \) with respect the norm \( \| \|_{\alpha,m} \). This proves that

\[
\mathcal{A}^{2}_{m}(\mathbb{B}^{2}) = \overline{\text{span} \{ \varphi_{\alpha} ; \ p \in \mathbb{Z}^{+} \times \mathbb{Z}^{+} \}}_{\| \|_{\alpha,m}}.
\]

Lemma 3.3 shows that the evaluation map \( f \rightarrow f(z) \) is continuous, and therefore \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \) is a reproducing kernel function, according to Riesz representation theorem. More explicitly, we have

**Proposition 3.5.** The reproducing kernel function of \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \) is given explicitly in terms of \( \text{3F}_{2} \)-sum by following closed form

\[
K_{\mathcal{A}^{2}_{m}(\mathbb{B}^{2})}(z, w) = \frac{\Gamma(\alpha + 3)}{\pi^{2} \Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \left( \frac{\langle z, w \rangle^{k}}{k!} \right)
\]

\[
+ \frac{\Gamma(\alpha + 3)}{\pi^{2} \Gamma(\alpha + 1)} \sum_{m=1}^{\infty} \frac{\langle z, w \rangle^{m}}{m!} \text{3F}_{2} \left( \begin{array}{c}
1, 1, \alpha + 3 \\
0, m, m + 1
\end{array} \left| \frac{\langle z, w \rangle}{m + 1, m + 1} \right. \right).
\]

**Proof.** Since the monomials \( \varphi_{\alpha} \) constitute an orthogonal basis of \( \mathcal{A}^{2}_{m}(\mathbb{B}^{2}) \), its reproducing kernel function can be computed by the formula

\[
K_{\mathcal{A}^{2}_{m}(\mathbb{B}^{2})}(z, w) = \sum_{p \in \mathbb{C}^{2}} \frac{\varphi_{p}(z)\overline{\varphi_{p}(w)}}{\| \varphi_{p} \|_{\alpha,m}^{2}}.
\]

More explicitly, by writing \( \sum_{|\alpha| \leq k} = \sum_{k=0}^{\infty} \sum_{|\alpha| = k} \) and \( \sum_{|\alpha| = k} = \sum_{k=0}^{\infty} \sum_{|\alpha| = k} \) and using the multimonomial formula

\[
\sum_{|\alpha| = k} \frac{z^{\alpha}w^{\alpha}}{p!k!} = \frac{(z_{1}w_{1} + z_{2}w_{2})^{k}}{k!} = \frac{\langle z, w \rangle^{k}}{k!},
\]

we get

\[
K_{\mathcal{A}^{2}_{m}(\mathbb{B}^{2})}(z, w) = \frac{1}{\pi^{2} \Gamma(\alpha + 1)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(|p| + \alpha + 3)}{p!} \sum_{|\alpha| = k} \frac{\Gamma(|p| - m + \alpha + 3)}{k!} \frac{z^{\alpha}w^{\alpha}}{p!} \right\}
\]

\[
= \frac{1}{\pi^{2} \Gamma(\alpha + 1)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha + 3)}{p!} \sum_{|\alpha| = k} \frac{\Gamma(k - m + \alpha + 3)}{k!} \frac{z^{\alpha}w^{\alpha}}{p!} \right\}
\]

\[
= \frac{1}{\pi^{2} \Gamma(\alpha + 1)} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha + 3)}{k!(k - 1)\cdots(k - m + 1)} \frac{z^{\alpha}w^{\alpha}}{p!} \right\}.
\]
Therefore, making use of the Pochhammer symbol, it follows

\[
K_{\mathcal{H}^m_2(B_2)}(z, w) = \frac{\Gamma(\alpha + 3)}{\pi^2 \Gamma(\alpha + 1)} \left\{ \sum_{k=0}^{\infty} \frac{(\alpha + 3)_k (z, w)^k}{k!} + \sum_{k=m}^{\infty} \frac{(\alpha + 3)_{k-m}}{(k-m)!} (z, w)^k \right\}
\]

This completes the proof. \( \square \)

We conclude this section by noting that the proof of Theorem 2.3 is contained in the above propositions.

4. The generalized Bargmann-Fock spaces on \( \mathbb{C}^2 \)

In this section we prove Theorem 2.3. Namely, we show that the generalized Bargmann-Fock spaces are reproducing kernel Hilbert spaces, and we explicit their reproducing kernel function. To this end, we proceed in a similar way as in the previous section. Thus, the proof of Theorem 2.3 is contained in the following propositions, that we claim without proofs.

Proposition 4.1. The monomials \( \varphi_p(z) := z^p \) belongs to \( \mathbb{F}_m^{2,2}(\mathbb{C}^2) \) and their square norms are given by

\[
\|\varphi_p\|_{\mathbb{F}_m}^2 = \frac{\pi^2}{\nu} \left\{ \frac{p!}{\nu^p} \right\} \begin{cases} 1 & \text{for } |p| < m \\ \frac{1}{\nu^{p-m+1}} & \text{for } |p| \geq m \end{cases}
\]

Proposition 4.2. The space \( \mathbb{F}_m^{2,2}(\mathbb{C}^2) \) is a Hilbert space and the monomials \( z^p \); \( p \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) constitute an orthogonal basis of \( \mathbb{F}_m^{2,2}(\mathbb{C}^2) \).

Proposition 4.3. The space \( \mathbb{F}_m^{2,2}(\mathbb{C}^2) \) is a reproducing kernel Hilbert space. Its reproducing kernel function is given by

\[
K_{\mathbb{F}_m^{2,2}(\mathbb{C}^2)}(z, w) = \frac{\nu^2}{\pi^2} \left\{ \sum_{l=0}^{\infty} \frac{(z, w)^l}{l!} + \frac{(z, w)^m}{(m!)^2} {}_3F_2\left( \begin{array}{c} 1, 1, \alpha + 3 \\ m+1, m+1 \end{array} \left| v(z, w) \right. \right) \right\}
\]

5. Asymptotic: Proof of Theorem 2.6

The proof of Theorem 2.6 lies essentially on following lemma giving the asymptotic of the \( {}_3F_2 \)-hypergeometric function. Namely, we claim

Lemma 5.1. Let \( z \in \mathbb{C} \) and \( a, b, c \) are complex number. Then we have

\[
\lim_{z \to +\infty} {}_3F_2\left( \begin{array}{c} b, c, x+a \\ d, e \end{array} \left| \frac{z}{x} \right. \right) = {}_2F_2\left( \begin{array}{c} b, c \\ d, e \end{array} \left| z \right. \right).
\]
The results in the third section on the unit ball can be generalized easily to the ball $B_R^2$ centered at 0 and of radius $R$. Thus, the generalized Bergman-Dirichlet space is given by

$$\mathcal{A}_{R,m}^{2,p}(B_R^2) = \left\{ f \in \mathcal{H}(B_R^2) \mid \|f\|_{\alpha,m,R}^2 = \|f\|_{\alpha,R}^2 + \sum_{|\lambda|=m} \|D^\lambda f\|_{2,R}^2 < \infty \right\},$$

supplied with the norm

$$\|f\|_{\alpha,R}^2 = \int_{B_R^2} |f(z)|^2 d\mu_{\alpha,R}(z),$$

where

$$d\mu_{\alpha,R}(z) = \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha} dA(z)$$

is the density measure. We make the change of variable $w = \frac{z}{R}$, $z \in B_R^2$, we get

$$\int_{B_R^2} |z|^p d\mu_{\alpha,R}(z) = R^{2(p+2)} \int_{B_R^2} w^p d\mu_{\alpha,R}(w) = \pi^2 R^{2p} R^{2(p+1)} \Gamma(\alpha + 1) \Gamma(\alpha + 3).$$

It follows that the norm of $\varphi_p$ in $\mathcal{A}_{R,m}^{2,p}(B_R^2)$ is given by

$$\|\varphi\|_{\alpha,R,m}^2 = \pi^2 R^4 \Gamma(\alpha + 1) \frac{R^{2(p+1)}}{\Gamma(|p| + \alpha + 3)}$$

for $|p| < m$ and

$$\|\varphi\|_{\alpha,R,m}^2 = \pi^2 R^4 \Gamma(\alpha + 1) \frac{R^{2(|p|-m)|p|(|p|-m) \cdots (|p|-m+1)p!}}{\Gamma(|p| - m + \alpha + 3)}$$

for $|p| \geq m$. Moreover, the reproducing kernel function of $\mathcal{A}_{R,m}^{2,p}(B_R^2)$ reads

$$K_{\mathcal{A}_{R,m}^{2,p}(B_R^2)}(z,w) = \frac{\Gamma(\alpha + 3)}{\pi^2 R^4 \Gamma(\alpha + 1)} \sum_{k=0}^{m} \frac{(\alpha + 3)_k (z,w)^k}{k! R^{2m}} \left(\frac{z,w}{R}\right)^{m+1} F_2\left(\binom{\alpha + 3}{m+1}, \binom{1}{m+1} \left| \frac{z,w}{R}\right| \right).$$

Now, for the specific choice of $\alpha = \nu R^2$, we see that the density $(1 - \frac{|z|^2}{R^2})^\alpha$ leads to the Gaussian density $e^{-|z|^2}$. Furthermore, using the Binet formula [5, page 47],

$$\frac{\Gamma(x + a)}{\Gamma(x - b)} = x^{a-b}(1 + O(\frac{1}{x}))$$

for $x = \nu R^2$, $a = n + 1$ and $b = 1$, we get

$$\lim_{R \to +\infty} \frac{1}{\pi^2 R^4 \Gamma(\nu R^2 + 1)} \left(\frac{\nu}{\pi}\right)^3.$$

Finally, by means of Lemma [5.1] we deduce that $K_{\mathcal{A}_{R,m}^{2,p}(B_R^2)}(z,w)$ converges pointwise to $K_{\mathcal{A}_{R,m}^{2,p}(B_R^2)}(z,w)$ as $R \to +\infty$.

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