A new generalization of the Genocchi numbers and its consequence on the Bernoulli polynomials

Bakir FARHI
Laboratoire de Mathématiques appliquées
Faculté des Sciences Exactes
Université de Bejaia, 06000 Bejaia, Algeria
bakir.farhi@gmail.com
http://farhi.bakir.free.fr/

Abstract

This paper presents a new generalization of the Genocchi numbers and the Genocchi theorem. As consequences, we obtain some important families of integer-valued polynomials those are closely related to the Bernoulli polynomials. Denoting by \((B_n)\) the sequence of the Bernoulli numbers and by \((B_n(X))\) the sequence of the Bernoulli polynomials, we especially obtain that for any natural number \(n\), the reciprocal polynomial of the polynomial \((B_n(X) - B_n)\) is integer-valued.

MSC 2010: Primary 11B68, 13F20, 13F25, 11C08.
Keywords: Genocchi numbers, Bernoulli numbers, Bernoulli polynomials, formal power series, integer-valued polynomials.

1 Introduction and Notations

Throughout this paper, we let \(\mathbb{N}^*\) denote the set of positive integers. For a given prime number \(p\), we let \(\vartheta_p\) denote the usual \(p\)-adic valuation. The rational numbers \(x\) satisfying \(\vartheta_p(x) \geq 0\) are called \(p\)-integers; they constitute a subring of \(\mathbb{Q}\), usually denoted by \(\mathbb{Z}_{(p)}\). For a given natural number \(r\), we let \(\text{den}(r)\) denote the denominator of \(r\); that is the smallest positive integer \(d\) such that \(dr \in \mathbb{Z}\).

Next, we let \(\mathbb{Q}[X]\) denote the ring of polynomials in \(X\) with coefficients in \(\mathbb{Q}\). If \(P \in \mathbb{Q}[X]\), we let \(\deg P\) denote the degree of \(P\). We call the reciprocal polynomial of a polynomial \(P \in \mathbb{Q}[X]\) the polynomial \(P^* \in \mathbb{Q}[X]\) obtained by reversing the order of the coefficients of \(P\); for example \((2X^3 + 5X^2 + 7X + 3)^* = 3X^3 + 7X^2 + 5X + 2\). It is easy to show that for any \(P \in \mathbb{Q}[X]\), we have \(P^*(X) = X^{\deg P} P\left(\frac{1}{X}\right)\). We let \(\Delta\) denote the forward difference operator on \(\mathbb{Q}[X]\); that
is \((\Delta P)(X) := P(X + 1) - P(X)\) \((\forall P \in \mathbb{Q}[X])\). A polynomial \(P \in \mathbb{Q}[X]\) whose value \(P(n)\) is an integer for every integer \(n\) (i.e., \(P(\mathbb{Z}) \subset \mathbb{Z}\)) is called an integer-valued polynomial. The set of integer-valued polynomials is denoted by \(\text{Int}(\mathbb{Z})\) and forms a \(\mathbb{Z}\)-algebra (under the usual operations on polynomials). It is known (see, e.g., \([2, 8]\)) that \(\text{Int}(\mathbb{Z})\) (seen as a \(\mathbb{Z}\)-module) is free with infinite rank and has as a basis the sequences of polynomials \((\frac{X}{n})\) \((n \in \mathbb{N})\), where \((\frac{X}{n}) := \frac{X(X-1)\cdots(X-n+1)}{n!}\) \((\forall n \in \mathbb{N})\). An exhaustive study of the integer-valued polynomials (including the integer-valued polynomials on a general domain) is given in the book of Cahen and Chabert \([2]\).

Further, the Bernoulli polynomials \(B_n(X)\) \((n \in \mathbb{N})\) can be defined by the exponential generating function:

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(X) \frac{t^n}{n!}
\]

and the Bernoulli numbers \(B_n\) are the values of the Bernoulli polynomials at \(X = 0\); that is \(B_n := B_n(0)\) \((\forall n \in \mathbb{N})\). To mark the difference between the Bernoulli polynomials and the Bernoulli numbers, we always put the indeterminate \(X\) in evidence when it comes to polynomials. The Bernoulli polynomials and numbers have many important and remarkable properties; an elementary presentation (but quite rich) can be found in the book of Nielsen \([7]\).

It is known for example that \(\deg B_n(X) = n\) \((\forall n \in \mathbb{N})\) and that \(B_n = 0\) for any odd integer \(n \geq 3\).

Throughout this paper, we deal with formal power series with rational coefficients. We denote by \(\mathbb{Q}[[t]]\) the ring of formal power series in \(t\) with coefficients in \(\mathbb{Q}\). An element \(S\) of \(\mathbb{Q}[[t]]\) is always represented as

\[
S(t) := \sum_{n=0}^{\infty} a_n \frac{t^n}{n!},
\]

where \(a_n \in \mathbb{Q}\) \((\forall n \in \mathbb{N})\). The \(a_n\)’s are called the differential coefficients of \(S\) (because it is immediate that each \(a_n\) is the \(n\)th derivative of \(S\) at 0). If the \(a_n\)’s are all integers, we say that \(S\) is an IDC-series (IDC abbreviates the expression “with Integral Differential Coefficients”). Many usual functions are IDC-series; we can cite for example the functions \(x \mapsto e^x\), \(x \mapsto \sin x\), \(x \mapsto \cos x\), \(x \mapsto \ln(1 + x)\), and so on. The sum of two IDC-series is obviously an IDC-series. The product of two IDC-series is also an IDC-series. Indeed, if \(S_1\) and \(S_2\) are two IDC-series with

\[
S_1(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad S_2(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}
\]

(so \(a_n, b_n \in \mathbb{Z}\), \(\forall n \in \mathbb{N}\)) then we have

\[
S_1(t)S_2(t) = \left(\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}\right) \left(\sum_{\ell=0}^{\infty} b_{\ell} \frac{t^\ell}{\ell!}\right) = \sum_{k,\ell \in \mathbb{N}} \frac{(k + \ell)!}{k!\ell!} a_k b_{\ell} \frac{t^{k+\ell}}{(k + \ell)!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k}\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},
\]
where \[ c_n := \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \quad (\forall n \in \mathbb{N}). \]

Since \( a_k, b_k \in \mathbb{Z} \) (\( \forall n \in \mathbb{N} \)) then \( c_n \in \mathbb{Z} \) (\( \forall n \in \mathbb{N} \)), showing that \( S_1 S_2 \) is an IDC-series.

Showing that a given function is an IDC-series is not always easy. The more famous example is perhaps the function \( t \mapsto \frac{2t}{e^t + 1} \) whose expansion into a power series is

\[
\frac{2t}{e^t + 1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!},
\]

where the \( G_n \)'s are called the Genocchi numbers and have been studied by several authors (see, e.g., [3, 4, 5, 9]). An important theorem of Genocchi [6] states that the \( G_n \)'s are all integers; equivalently, the function \( t \mapsto \frac{2t}{e^t + 1} \) is an IDC-series. A familiar proof of this curious result uses the expression of the \( G_n \)'s in terms of the Bernoulli numbers:

\[
G_n = -2(2^n - 1) B_n \quad (\forall n \in \mathbb{N}),
\]

together with the von Staudt-Clausen theorem and the Fermat little theorem.

In this paper, we generalize the Genocchi numbers by considering for a given integer \( a \geq 2 \), the function \( t \mapsto \frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} \) and its expansion into a power series:

\[
\frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}.
\]

For \( a = 2 \), we simply obtain the usual Genocchi numbers; that is \( G_{n,2} = G_n \) (\( \forall n \in \mathbb{N} \)). In our main Theorem 3.1, we prove that the \( G_{n,a} \)'s are all integers, which generalizes the Genocchi theorem. Next, by interpolating the numbers \( G_{n,a} \) (\( a \geq 2 \)), for a fixed \( n \in \mathbb{N} \), we derive some important families of integer-valued polynomials which are closely related to the Bernoulli polynomials. We particularly obtain that for any natural number \( n \), the polynomial \((B_n(X) - B_n)^*\) is integer-valued.

## 2 Some preliminaries on the IDC-series

In this section, we present some selected elementary properties of the IDC-series. We begin with the following proposition:

**Proposition 2.1.** Let \( f \) be an IDC-series. Then \( \frac{1}{f} \) is an IDC-series if and only if \( f(0) = \pm 1 \).

**Proof.** Write

\[
f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!},
\]

where \( a_n \in \mathbb{Z}, \forall n \in \mathbb{N} \).

- If \( \frac{1}{f} \) is an IDC-series then we have \( (\frac{1}{f})(0) = \frac{1}{f(0)} \in \mathbb{Z} \), which is possible if and only if \( f(0) = \pm 1 \).
(since \( f(0) = a_0 \in \mathbb{Z} \)).

- Conversely, suppose that \( f(0) = \pm 1 \) (that is \( a_0 = \pm 1 \)) and let us show that \( \frac{1}{f} \) is an IDC-series.

The fact that \( f \in \mathbb{Q}[[t]] \) and \( f(0) \neq 0 \) implies that \( \frac{1}{f} \in \mathbb{Q}[[t]] \); so let

\[
\frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!},
\]

where \( b_n \in \mathbb{Q}, \forall n \in \mathbb{N} \). Thus, we have

\[
\left( \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!} \right) \left( \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!} \right) = 1.
\]

Then, by identifying the differential coefficients in both power series of the last identity, we obtain that:

\[
\begin{aligned}
& a_0 b_0 = 1 \\
& \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = 0 \quad (\forall n \geq 1)
\end{aligned}
\]

Hence

\[
\begin{aligned}
& b_0 = \frac{1}{a_0} \\
& b_n = -\frac{1}{a_0} \left[ \binom{n}{1} b_{n-1} a_1 + \binom{n}{2} b_{n-2} a_2 + \cdots + \binom{n}{n} b_0 a_n \right] \quad (\forall n \geq 1)
\end{aligned}
\]

showing that \( b_0 \in \mathbb{Z} \) (since \( a_0 = \pm 1 \) by hypothesis) and then (by a simple induction on \( n \)) that \( b_n \in \mathbb{Z} \) for all \( n \in \mathbb{N} \). Thus \( \frac{1}{f} \) is an IDC-series, as required. Our proof is complete.

From Proposition 2.1, we derive the following corollary:

**Corollary 2.2.** Let \( f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!} \) be an IDC-series with \( a_0 \neq 0 \). Then the formal power series \( \frac{a_0}{f(a_0 t)} \) is also an IDC-series.

**Proof.** We have

\[
\frac{f(a_0 t)}{a_0} = 1 + \sum_{n=1}^{+\infty} a_n \frac{(a_0 t)^n}{n!} = 1 + \sum_{n=1}^{+\infty} a_n a_{n-1} t^n n!,
\]

showing that \( \frac{f(a_0 t)}{a_0} \) is an IDC-series with the first coefficient equal to 1. According to Proposition 2.1, it follows that \( \frac{1}{f(a_0 t)/a_0} = \frac{a_0}{f(a_0 t)} \) is also an IDC-series. This achieves the proof.

Finally, from Corollary 2.2, we derive the following corollary which is essential for our purpose.

**Corollary 2.3.** Let \( f(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!} \) be an IDC-series with \( a_0 \neq 0 \) and let \( \frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!} \in \mathbb{Q}[[t]] \) be the reciprocal of the formal power series \( f \). Then, for all \( n \in \mathbb{N} \), the denominator of the rational number \( b_n \) divides the integer \( a_0^{n+1} \).

**Proof.** From \( \frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!} \), we derive that:

\[
\frac{a_0}{f(a_0 t)} = a_0 + \sum_{n=0}^{+\infty} b_n \frac{(a_0 t)^n}{n!} = \sum_{n=0}^{+\infty} b_n a_0^{n+1} \frac{t^n}{n!}.
\]
But, according to Corollary 2.2, we know that $\frac{a_0}{f(\alpha t)}$ is an IDC-series; equivalently, we have that $b_n a_0^{n+1} \in \mathbb{Z}$ ($\forall n \in \mathbb{N}$). Consequently, the denominator of each of the rational numbers $b_n$ ($n \in \mathbb{N}$) is a divisor of $a_0^{n+1}$, as required. The corollary is proved.

3 The main result

Our main result is the following:

**Theorem 3.1.** Let $a \geq 2$ be an integer. Then for any positive integer $n$, the number $G_{n,a}$ is an integer.

If we take $a = 2$ in Theorem 3.1, we obtain the Genocchi original theorem.

The method of proving Theorem 3.1 consists to show that for any prime number $p$, we have $\vartheta_p(G_{n,a}) \geq 0$ ($a \geq 2, n \in \mathbb{N}^*$). To do so, we distinguish two cases according as $p$ does or does not divide $a$. We begin with the following proposition:

**Proposition 3.2.** Let $a$ and $n$ be two positive integers with $a \geq 2$. Then the denominator of the rational number $G_{n,a}$ divides $a^{n-1}$.

**Proof.** By applying Corollary 2.3 for the IDC-series

$$f(t) := e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1 = a + \sum_{n=1}^{+\infty} \frac{(1^n + 2^n + \cdots + (a-1)^n) t^n}{n!},$$

we obtain that the expansion of $\frac{1}{f(t)}$ into a power series has the form

$$\frac{1}{f(t)} = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}, \quad (3.1)$$

where, for all $n \in \mathbb{N}$, we have $b_n \in \mathbb{Q}$ and $\text{den}(b_n) \mid a^{n+1}$. Then, by multiplying the two sides of (3.1) by $at$, we get

$$\frac{at}{f(t)} = \sum_{n=0}^{+\infty} ab_n \frac{t^{n+1}}{n!} = \sum_{n=1}^{+\infty} ab_{n-1} \frac{t^n}{(n-1)!} = \sum_{n=1}^{+\infty} anb_{n-1} \frac{t^n}{n!}.$$  

But since we have on the other hand $\frac{at}{f(t)} = \sum_{n=0}^{+\infty} G_{n,a} \frac{t^n}{n!}$, we deduce that

$$G_{n,a} = anb_{n-1}, \quad (\forall n \in \mathbb{N}^*).$$

Now, for a given $n \in \mathbb{N}^*$, we have that $\text{den}(b_{n-1}) \mid a^n$; thus $\text{den}(anb_{n-1}) \mid a^{n-1}$; that is $\text{den}(G_{n,a}) \mid a^{n-1}$. This completes the proof.

From Proposition 3.2, we immediately derive the following corollary:

**Corollary 3.3.** Let $a$ and $n$ be two positive integers with $a \geq 2$. Then for any prime number $p$ not dividing $a$, we have

$$\vartheta_p(G_{n,a}) \geq 0.$$
Now, we are going to establish the analog of Corollary 3.3 for the prime numbers \( p \) dividing the considered number \( a \). For this purpose, we first need the following proposition:

**Proposition 3.4.** Let \( a \geq 2 \) be an integer. Then for all positive integer \( n \), we have
\[
G_{n,a} + \sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} = 1.
\]

**Proof.** From the definition of the numbers \( G_{n,a} \), we have
\[
\left( \sum_{n=0}^{+\infty} \frac{G_{n,a} t^n}{n!} \right) \left( \sum_{n=0}^{+\infty} \frac{a^n t^n}{n+1 n!} \right) = \frac{at}{e^{(a-1)t} + e^{(a-2)t} + \cdots + e^t + 1} \cdot \frac{e^{at} - 1}{at} = e^t - 1 = \sum_{n=1}^{+\infty} \frac{t^n}{n!},
\]
that is
\[
\left( \sum_{n=0}^{+\infty} \frac{G_{n,a} t^n}{n!} \right) \left( \sum_{n=0}^{+\infty} \frac{a^n t^n}{n+1 n!} \right) = \sum_{n=1}^{+\infty} \frac{t^n}{n!}, \tag{3.2}
\]
So, for a given \( n \in \mathbb{N}^* \), the identification of the \( n \)th differential coefficients in the two hand-sides of (3.2) gives
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} = 1,
\]
which is nothing else the required identity (since \( G_{0,a} = 0 \)). \( \square \)

Next, we have the following lemma:

**Lemma 3.5.** Let \( a \geq 2 \) be an integer. Then for all prime number \( p \) dividing \( a \) and all natural number \( k \), we have
\[
\vartheta_p \left( \frac{a^k}{k+1} \right) \geq 0.
\]

**Proof.** Let \( p \) be a prime number dividing \( a \) (so \( \vartheta_p(a) \geq 1 \)) and \( k \) be a natural number. Since \( k + 1 \leq 2^k \leq p^k \) then we have \( \vartheta_p(k+1) \leq k \). Hence
\[
\vartheta_p \left( \frac{a^k}{k+1} \right) = k \vartheta_p(a) - \vartheta_p(k+1) \geq k - \vartheta_p(k+1) \geq 0,
\]
as required. \( \square \)

From Proposition 3.4 and Lemma 3.5, we derive the following corollary:

**Corollary 3.6.** Let \( a \) and \( n \) be two positive integers with \( a \geq 2 \). Then for any prime number \( p \) dividing \( a \), we have
\[
\vartheta_p(G_{n,a}) \geq 0.
\]

**Proof.** Let \( p \) be a prime number dividing \( a \). To prove that \( \vartheta_p(G_{n,a}) \geq 0 \), we argue by induction on \( n \in \mathbb{N}^* \) and use the identity of Proposition 3.4 together with Lemma 3.5.

* For \( n = 1 \), we have \( G_{1,a} = 1 \), so \( \vartheta_p(G_{1,a}) = 0 \geq 0 \).
• Let \( n \geq 2 \) be an integer. Suppose that \( \vartheta_p(G_{m,a}) \geq 0 \) for any positive integer \( m < n \) and show that \( \vartheta_p(G_{n,a}) \geq 0 \). By Proposition 3.4, we have

\[
G_{n,a} = - \sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} + 1.
\]

Since the binomial coefficients are known to be integers, the numbers \( G_{n-k,a} \) (\( 1 \leq k \leq n-1 \)) are \( p \)-integers (by the induction hypothesis) and the numbers \( \frac{a^k}{k+1} \) (\( 1 \leq k \leq n-1 \)) are \( p \)-integers (by Lemma 3.5) then the sum \( - \sum_{1 \leq k \leq n-1} \binom{n}{k} \frac{a^k}{k+1} G_{n-k,a} + 1 \) is a \( p \)-integer; that is \( \vartheta_p(G_{n,a}) \geq 0 \), as required. This achieves the induction, and hence, the proof.

The proof of our main result is now immediate:

**Proof of Theorem 3.1.** Let \( a \) and \( n \) be two positive integers with \( a \geq 2 \). According to Corollaries 3.3 and 3.6, we have for any prime number \( p \): \( \vartheta_p(G_{n,a}) \geq 0 \). Thus the number \( G_{n,a} \) is an integer. Our main result is proved.

4 Some consequences of the main result

For the following, we extend the numbers \( G_{n,a} \) to non-integer values of \( a \). Precisely, we define \( G_n(x) \) \((n \in \mathbb{N}, x \in \mathbb{R})\) as the coefficients occurring on the right-hand side of the identity:

\[
\frac{e^{xt}}{e^t - 1}(e^t - 1) = \sum_{n=0}^{+\infty} G_n(x) \frac{t^n}{n!},
\]

where it is understood that \( G_n(0) = \lim_{x \to 0} G_n(x) \) \( = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{otherwise} \end{cases} \) (because \( \lim_{x \to 0} \frac{e^{xt}}{e^t - 1}(e^t - 1) = e^t - 1 \)). Then it is immediate that \( G_n(a) = G_{n,a} \) if \( n \in \mathbb{N} \) and \( a \) is an integer \( \geq 2 \). The following proposition shows that for any \( n \in \mathbb{N} \), the function \( x \mapsto G_n(x) \) is actually a polynomial which depends on the Bernoulli polynomial \( B_n(X) \).

**Proposition 4.1.** For all natural number \( n \), we have

\[
G_n(X) = (B_n(X) - B_n)^* = \sum_{k=0}^{n-1} \binom{n}{k} B_k X^k.
\]

So \( G_n \) \((n \in \mathbb{N})\) is a polynomial with degree \( \leq n - 1 \).

**Proof.** By definition of the Bernoulli polynomials, we have

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(X) \frac{t^n}{n!}.
\]

Then, By substituting in the latter \( X \) by \( \frac{1}{X} \) and \( t \) by \( Xt \), we get

\[
\frac{Xte^t}{e^{Xt} - 1} = \sum_{n=0}^{+\infty} X^n B_n \left( \frac{1}{X} \right) \frac{t^n}{n!}.
\]
that is (since \( \deg B_n = n, \forall n \in \mathbb{N} \))

\[
\frac{X t}{e^{X t} - 1} e^t = \sum_{n=0}^{+\infty} B_n^*(X) \frac{t^n}{n!}.
\]  

(4.1)

On the other hand, we have by definition of the Bernoulli numbers:

\[
\frac{X t}{e^{X t} - 1} = \sum_{n=0}^{+\infty} B_n \frac{(X t)^n}{n!},
\]

that is

\[
\frac{X t}{e^{X t} - 1} = \sum_{n=0}^{+\infty} B_n X^n \frac{t^n}{n!}.
\]

(4.2)

By subtracting side to side (4.2) from (4.1), we finally obtain

\[
\frac{X t}{e^{X t} - 1} (e^t - 1) = \sum_{n=0}^{+\infty} \left( B_n^*(X) - B_n X^n \right) \frac{t^n}{n!}.
\]

Comparing this with the identity defining the \( G_n(X) \)'s, we derive that for all \( n \in \mathbb{N} \), we have

\[
G_n(X) = B_n^*(X) - B_n X^n = (B_n(X) - B_n)^*,
\]

as required. The second equality of the proposition immediately follows from the well-known expression of the Bernoulli Polynomials in terms of the Bernoulli numbers, which is \( B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k} \) (\( \forall n \in \mathbb{N} \)). This completes the proof.

\[ \square \]

**Remark 4.2.** Since we know that \( B_n = 0 \) if and only if \( n \) is an odd integer \( \geq 3 \), then from the formula of Proposition 4.1, we can precise the degree of the polynomial \( G_n \) (\( n \in \mathbb{N}^* \)). We have that:

\[
\deg G_n = \begin{cases} 
  n - 1 & \text{if } n \text{ is odd or } n = 2 \\
  n - 2 & \text{if } n \text{ is even and } n \geq 4
\end{cases}
\]

Further, from Theorem 3.1, we derive the following corollary:

**Corollary 4.3.** For any natural number \( n \), the polynomial \( G_n(X) \) is integer-valued.

To prove this corollary, we lean on the following well-known lemma (see, e.g., [2]):

**Lemma 4.4.** Let \( d \in \mathbb{N} \) and \( P \) be a polynomial of \( \mathbb{Q}[[X]] \) with degree \( d \). For \( P \) to be an integer-valued polynomial, it suffices that \( P \) takes integer values for \((d + 1)\) consecutive integer values of \( X \).

**Proof.** Suppose that \( P \) takes integer values for some \((d + 1)\) consecutive integer values of \( X \), which are: \( a, a+1, \ldots, a+d \) (\( a \in \mathbb{Z} \)) and let us show that \( P \) is an integer-valued polynomial. Since \( \deg P = d \) then we have that \( \Delta^{d+1} P = 0 \); that is

\[
P(X + d + 1) = \sum_{k=0}^{d} (-1)^{d-k} \binom{d+1}{k} P(X + k).
\]
Using this identity, we immediately deduce by induction that:

$$P(x) \in \mathbb{Z} \quad (\forall x \in \mathbb{Z}, x \geq a + d).$$

(4.3)

Next, if we take instead of $P(X)$ the polynomial $P(-X)$, which has the same degree with $P$ and takes integer values for the $(d + 1)$ consecutive integer values $-a - d, -a - d + 1, \ldots, -a$ of $X$, we similarly obtain that:

$$P(-x) \in \mathbb{Z} \quad (\forall x \in \mathbb{Z}, x \geq -a);$$

that is

$$P(x) \in \mathbb{Z} \quad (\forall x \in \mathbb{Z}, x \leq a).$$

(4.4)

From (4.3) and (4.4), we conclude that $P(x) \in \mathbb{Z}, \forall x \in \mathbb{Z}$. Thus $P$ is an integer-valued polynomial. The lemma is proved.

Let us now prove Corollary 4.3:

**Proof of Corollary 4.3.** Let $n \in \mathbb{N}$ be fixed. Since for any integer $a \geq 2$, we have $G_n(a) = G_{n,a} \in \mathbb{Z}$ (according to Theorem 3.1) then the polynomial $G_n(X)$ takes integer values for an infinite number of consecutive integer values of $X$. It follows (according to Lemma 4.4) that $G_n(X)$ is an integer-valued polynomial. This achieves the proof.

Next, from Proposition 4.1 and Corollary 4.3, we derive the following curious result concerning the Bernoulli polynomials of odd degree.

**Corollary 4.5.** For any odd integer $n \geq 3$, the reciprocal polynomial of the Bernoulli polynomial $B_n(X)$ is integer-valued.

**Proof.** This is an immediate consequence of Proposition 4.1, Corollary 4.3 and the well-known fact that $B_n = 0$ for $n$ odd, $n \geq 3$.

We now turn to present another important property concerning the reciprocal polynomials of some particular type of integer-valued polynomials. For a given $n \in \mathbb{N}$, we define

$$\sigma_n(a) := 0^n + 1^n + \cdots + a^n \quad (\forall a \in \mathbb{N}),$$

where we convention that $0^0 = 1$.

It has been known for a very long time that $\sigma_n(a)$ is polynomial on $a$ with degree $(n + 1)$, but a closed form of the polynomial in question was discovered for the first time by Jacob Bernoulli [1] and it is given by:

$$\sigma_n(a) = \frac{1}{n + 1} \sum_{k=0}^{n} \binom{n + 1}{k} B_k a^{n+1-k} + a^n.$$
For a given \( n \in \mathbb{N} \), let us define \( \sigma_n(X) \) as the polynomial interpolating the sequence \( (\sigma_n(a))_{a \in \mathbb{N}} \); that is
\[
\sigma_n(X) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k X^{n+1-k} + X^n.
\] (4.5)

For \( n \in \mathbb{N} \), since the \( \sigma_n(a) \)'s \( (a \in \mathbb{N}) \) are obviously all integers then (according to Lemma 4.4) the polynomial \( \sigma_n(X) \) is an integer-valued polynomial. But what about its reciprocal polynomial? The previous results permit us to obtain something in this direction. We have the following proposition:

**Proposition 4.6.** For any natural number \( n \), the polynomial \((n+1)\sigma_n^*(X)\) is integer-valued.

**Proof.** Let \( n \in \mathbb{N} \) be fixed. According to (4.5), we have
\[
\sigma_n^*(X) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k X^k + X
\]
\[
= \frac{1}{n+1} G_{n+1}(X) + X \quad \text{ (according to Proposition 4.1)}.
\]
Hence
\[
(n+1)\sigma_n^*(X) = G_{n+1}(X) + (n+1)X.
\]
Since \( G_{n+1}(X) \) is integer-valued (according to Corollary 4.3), the last equality shows that also \((n+1)\sigma_n^*(X)\) is integer-valued (as the sum of two integer-valued polynomials). The proposition is proved. \( \square \)

**Two important open problems**

1.—— Since the polynomials \( G_n(X) \) are integer-valued (according to Corollary 4.3) then they admit representations as linear combinations, with integer coefficients, of the polynomials \( \binom{X}{k} \) \( (k \in \mathbb{N}) \). Precisely, there exist integers \( a_{n,k} \) \( (n \in \mathbb{N}^*, k \in \mathbb{N}, 0 \leq k < n) \) for which we have
\[
G_n(X) = a_{n,0} \binom{X}{0} + a_{n,1} \binom{X}{1} + \cdots + a_{n,n-1} \binom{X}{n-1} \quad (\forall n \in \mathbb{N}^*).
\]
The \( a_{n,k} \)'s can be calculated for example by using the Newton interpolation formula:
\[
P(X) = \sum_{k=0}^{\deg P} \left( \Delta^k P \right)(0) \binom{X}{k} \quad (\forall P \in \mathbb{Q}[X]).
\]
So, we have that:
\[
a_{n,k} = \left( \Delta^k G_n \right)(0) \quad (\forall n \in \mathbb{N}^*, \forall k \in \mathbb{N}, 0 \leq k < n).
\]
If we arrange those integers \( a_{n,k} \) \( (0 \leq k < n) \) in a triangle array in which each \( a_{n,k} \) is the entry in the \( n^{th} \) row and \( k^{th} \) column, we obtain (after calculation) the following configuration:
The interesting problem we pose here consists to find a simple and practical rule to construct the above triangle step by step.

2. — For a polynomial $P \in \text{Int}(Z)$, we don’t have in general $(\deg P) \cdot P^* \in \text{Int}(Z)$ (indeed, the polynomial $(X^3) = \frac{X(X-1)(X-2)}{6}$ provides a counterexample); however, the polynomials $\sigma_n(X)$ $(n \in \mathbb{N})$ satisfy this property (according to Proposition 4.6). So, it is interesting to study for which category of integer-valued polynomials, the above property is satisfied.

References

[1] J. Bernoulli. Ars Conjectandi, Thurneysen Brothers, Bâle, 1713.

[2] P-J. Cahen & J-L. Chabert. Integer-valued polynomials, *Mathematical Surveys and Monographs*, Providence, RI: American Mathematical Society, 48, (1997).

[3] L. Comtet. Advanced Combinatorics. The Art of Finite and Infinite Expansions, revised and enlarged ed., *D. Reidel Publ. Co.*, Dordrecht, 1974.

[4] D. Dumont & D. Foata. Une propriété de symétrie des nombres de Genocchi, *Bull. Soc. Math. France*, 104 (1976), p. 433-451.

[5] J. Gandhi. A conjectured representation of Genocchi numbers, *Amer. Math. Monthly*, 77 (1970), p. 505-506.

[6] A. Genocchi. Intorno all espressioni generali di numeri Bernoulliani, *Annali di scienze mat. e fisiche, compilati da Barnaba Tortolini*, 3 (1852), p. 395-405.

[7] N. Nielsen. Traité élémentaire des nombres de Bernoulli, *Gauthier–Villars*, Paris, 1923.
[8] G. Pólya. Über ganzwertige ganze Funktionen, *Palermo Rend.* (in German), 40 (1915), p. 1-16.

[9] J. Riordan & P. Stein. Proof of a conjecture on Genocchi numbers, *Disc. Math.*, 5 (1973), p. 381-388.