Optimization of Bit Mapping and Quantized Decoding for Off-the-Shelf Protograph LDPC Codes with Application to IEEE 802.3ca

Fabian Steiner, Gerhard Kramer
Institute for Communications Engineering
Technical University of Munich
Email: {fabian.steiner, gerhard.kramer}@tum.de

Abstract—Protograph-based, off-the-shelf low-density parity-check (LDPC) codes are optimized for higher-order modulation and quantized sum-product decoders. As an example, for the recently proposed LDPC code from the upcoming IEEE 802.3ca standard for passive optical networks (PONs), an optimized mapping of the bit channels originating from bit-metric decoding to the protograph variable nodes gains 0.4 dB and 0.3 dB at a bit-error rate of $10^{-6}$ for shaped and uniform signaling, respectively. Furthermore, the clipping value for a quantized sum-product LDPC decoder is optimized via discretized density evolution.

I. INTRODUCTION

Passive optical networks (PONs) use unpowered fiber optic splitters to serve multiple terminals with one common optical fiber. Their predominant use is in fiber-to-the-home (FTTH) networks. Most PONs operate in a time division multiple access (TDMA) manner and provide up to 10 Gbit/s with simple noncoherent on-off keying (OOK) modulation. New standards for 25 Gbit/s, 50 Gbit/s and 100 Gbit/s PONs are underway, which will result in the IEEE 802.3ca standard in 2019. The standardization consortium agreed on low-density parity-check (LDPC) codes [1] as the preferred forward error correction (FEC) solution.

Coherent detection and higher-order modulation formats such as $M$-quadrature amplitude modulation (QAM) are becoming increasingly important also for PON networks to enable a greater flexibility in data rates [2]. To operate LDPC codes with higher order modulation, usually bit-metric decoding (BMD) is employed, where the demapper calculates a bit-wise soft information for each of the $m = \log_2(M)$ bits indexing a constellation symbol. BMD is also the key component of bit-interleaved coded modulation (BICM) [3]. Various optimization techniques have been proposed to map the BMD bit levels to the different variable node (VN) degrees of an irregular LDPC code or to the VN types of a protograph [4]. Two approaches can be distinguished in literature. The first approach considers the bit mapping optimization for an off-the-shelf LDPC code [5]–[10]. The second designs the code and the bit mapping jointly [11], [12]. While the second approach can exploit all degrees of freedom, practical constraints often require to reuse a given code for a new scenario and the best bit mapping needs to be found. This is the case when the proposed PON LDPC code is used for higher order modulation.

None of the previously suggested optimization approaches is tailored to the code defined in IEEE 802.3ca and also other standards (e.g., IEEE 802.11, G.hn): Some assume unstructured LDPC codes (e.g., [6], [8]) and others are prohibitively complex to work with the protograph sizes in standards. For example, the authors of [9] use differential evolution to optimize the bit mapping for protograph based spatially coupled LDPC codes with window decoding and exploit the periodicity imposed by window decoding to limit the optimization space. Furthermore, sampling from the high-dimensional polytope to generate populations for differential evolution becomes rather time consuming.

In this work, we propose an algorithm that optimizes the bit mapping of protograph based LDPC codes one level after the other. We use the surrogate approach of [12] and P-EXIT [13] analysis to determine the decoding threshold for a given mapping and use the patternsearch algorithm [14].
to find the best bit mapping. We validate this approach by comparing the predicted P-EXIT thresholds with discretized density evolution (DDE).

Additionally, practical LDPC decoders often quantize the exchanged messages with a finite number of bits. This is particularly important for optical communication with its high throughput requirements [15 Sec. III]. Previous works [16, 17] noted that the performance of quantized decoders depends on the clipping of messages and found the optimal clipping by time consuming finite length simulations.

As a second contribution, we therefore optimize the clipping of a quantized sum-product LDPC decoder exemplarily for three and four bits resolution. We use the decoding threshold of an ensemble as the objective and show that DDE accurately predicts the finite length performance, making it an important tool to facilitate the design process.

The paper is structured as follows. In Sec. II we present the system model, introduce the basic information theoretic quantities and provide a brief introduction to LDPC codes and their asymptotic analysis with DDE. Sec. III formalizes the bit mapping optimization problem and presents our successive bit allocation mapping approach. In Sec. IV we illustrate the effect of clipping for quantized LDPC decoders and optimize it for different number of bits. We conclude in Sec. V.

II. PRELIMINARIES

A. System Model

Consider transmission over the discrete time additive white Gaussian noise (AWGN) channel

\[ Y_i = X_i + Z_i \] (1)

for \( i = 1, \ldots, n \). The channel input \( X_i \) is taken from the \( M \)-ary amplitude shift keying (ASK) constellation \( \mathcal{X} \). The noise \( Z_i \) is a Gaussian random variable (RV) with zero mean and variance \( \sigma^2 \), i.e., \( Z_i \sim \mathcal{N}(0, \sigma^2) \). The signal-to-noise ratio (SNR) is \( 1/\sigma^2 \). All further results are readily applicable to QAM which has an ASK constellation on the inphase and quadrature component.

Mutual information is maximized under an average power constraint by a zero mean Gaussian input \( X \) with unit variance, and the capacity expression is

\[ C_{\text{AWGN}}(\text{SNR}) = \frac{1}{2} \log_2(1 + \text{SNR}) \] (2)

At the receiver, the decoder uses a decoding metric \( q^n(x^n, y^n) : \mathcal{X}^n \times \mathbb{R}^n \to \mathbb{R}^+ \) to detect the transmitted sequence \( x^n \) in the codebook \( \mathcal{C} \) from the noisy observations \( y^n \). The decoding decision is \( \hat{x}^n \) if

\[ \hat{x}^n = \arg\max_{x^n \in \mathcal{C}} q^n(x^n, y^n) = \prod_{i=1}^{n} q(x_i, y_i) \] (3)

In [13], it is shown that an achievable rate is

\[ R_a = [H(X) - U(q)]^+ \] (4)

where \([\cdot]^+ = \max(0, \cdot)\) and \( H(X) \) is the entropy of the discrete RV \( X \). The cross entropy \( U(\cdot) \) is the uncertainty a FEC decoder needs to resolve [18]:

\[ U(q) = \mathbb{E} \left[ -\log_2 \left( \frac{q(X, Y)}{\sum_{x \in \mathcal{X}} q(x, y)} \right) \right] \] (5)

For BMD, we label each constellation point \( x \in \mathcal{X} \) with an \( m \)-bit label, i.e., \( \chi : \mathcal{X} \to \{0, 1\}^m \) and \( \chi(x) = b_1 b_2 \ldots b_m = b \). Its inverse is \( \chi^{-1} : \{0, 1\}^m \to \mathcal{X} \). A binary reflected Gray code (BRGC) [19] usually performs well for BMD and the BMD decoder uses the metric

\[ q(x, y) = q_{BMD}(\chi(x), y) = q_{BMD}(b, y) = \prod_{j=1}^{m} P_{B_j|Y}(b_j|y) \] (6)

Using (6) in (4), the achievable rate (4) becomes

\[ R_{\text{BMD}}(\text{SNR}; P_X) = \left[ H(B) - \sum_{j=1}^{m} H(B_j|Y) \right]^+ \] (7)

The FEC decoder inputs are the soft information values

\[ l_j = \log \left( \frac{\sum_{x \in \mathcal{X}^b_j} P_X(x) Y(x|y) P_X(x)}{\sum_{x \in \mathcal{X}^b_j} P_Y(x|y) P_X(x)} \right), \quad j = 1, \ldots, m \] (8)

and \( \mathcal{X}^b_j = \{ x \in \mathcal{X} : \chi(x)_j = b \} \). For this choice, (5) can be written as

\[ U(q_{\text{BMD}}) = \sum_{j=1}^{m} \mathbb{E} \left[ -\log_2 \left( \frac{e^{(1-2B_j)(L_j/2)}}{e^{-L_j/2} + e^{L_j/2}} \right) \right] \] (9)

B. Probabilistic Amplitude Shaping (PAS)

Probabilistic amplitude shaping (PAS) is a coded modulation (CM) scheme that combines probabilistic shaping (PS) with FEC [20]. It builds upon two important properties. First, the capacity achieving distribution \( P_A \) on the AWGN channel is symmetric. We therefore factor the input distribution into an amplitude and sign part as \( P_X(x) = P_A(|x|) \cdot P_S(\text{sign}(x)) \), where \( P_A \) is non-uniform on \( \{|x|, x \in \mathcal{X}\} \) and \( S \) is uniform on \( \{-1, +1\} \). Second, the scheme exploits systematic encoding to preserve the non-uniform \( P_A \). It copies the binary representation of the amplitudes into the information part of the codeword and uses the approximately uniform distributed parity bits as signs.

The distribution matcher (DM) [21] realizes the non-uniform distribution \( P_A \) on the amplitudes. It takes \( k \) uniformly distributed input bits and maps them to a length \( n \) sequence of symbols with the empirical distribution \( P_A \). For PAS, the DM output set consists of amplitude values \( \{|x|, x \in \mathcal{X}\} \). The DM rate is \( R_{\text{dm}} = k/n \). The spectral efficiency (SE) of PAS is [20 Sec. IV-D]

\[ \eta = R_{\text{dm}} + 1 - (1 - R_c) \cdot m, \] (10)

for uniform signaling we have \( R_{\text{dm}} = m-1 \) and (10) becomes

\[ \eta = R_c \cdot m \] (11)

where \( R_c \) is the code rate of the FEC code.
C. Low-Density Parity-Check (LDPC) Codes

Binary LDPC codes are linear block codes with a sparse parity-check matrix \( H \in \{0,1\}^{m_c \times n_c} \). LDPC codes can be represented via their Tanner graph that are directly related to their parity-check matrices. A Tanner graph is a bipartite graph \( G = (\mathcal{V} \cup \mathcal{C}, \mathcal{E}) \) consisting of \( n_v \) VNs and \( m_c \) check nodes (CNs). The set \( \mathcal{E} \) of edges contains the element \( e_{ij} \), denoting an edge between VN \( V_i \in \mathcal{V} \) and CN \( C_j \in \mathcal{C} \), if \( h_{ji} = [H]_{ji} = 1 \). The sets \( \mathcal{N}(V_i) \) and \( \mathcal{N}(C_j) \) denote the neighbors of VN \( V_i \) and CN \( C_j \), respectively. The degree \( d_{v,i} \) \( (d_{c,j}) \) of VN \( V_i \) (CN \( C_j \)) is the cardinality of the sets \( \mathcal{N}(V_i) \) and \( \mathcal{N}(C_j) \).

Practical LDPC code designs are often based on protographs [4]. The latter are defined via a basematrix \( \mathcal{B} \) represented as a Tanner graph (also referred to as a protograph), using the probability density function (PDF) of the involved distributions. Discretized Density Evolution (DDE) [22] approximates real density evolution [22] by discretizing the probability density function of the involved distributions. For writing this manuscript, the exact shortening pattern for the different BMD bit levels to the different VN degrees of an irregular LDPC code, or to VN types of a protograph to improve the performance [5][10].

In the following, we use the ideas of [8], [9] to formulate an optimization procedure that optimizes the assignment of the \( m \) bit channels to the \( n_{p,t} \) transmitted VN types of a given protograph basematrix. This bit mapping can be expressed as a non-negative matrix \( A = [a_{1}, \ldots, a_{n_{p,t}}] \) of dimension \( m \times n_{p,t} \), where the entry \( a_{ji} = [A]_{ji} \) denotes the fraction of bit level \( j \) that is assigned to the \( i \)-th transmitted VN type. The matrix \( A \) needs to fulfill the constraints

\[
\sum_{j'=1}^{n_{p,t}} a_{ji} = \frac{1}{n_{p,t}}, \quad \sum_{j'=1}^{n_{p,t}} a_{j'i} = 1, \quad \text{for all } j \in \{1,2,\ldots,m\} \text{ and } i \in \{1,2,\ldots,n_{p,t}\}. \tag{13}
\]

D. Discretized Density Evolution (DDE)

DDE approximates real density evolution [22] by discretizing the probability density function (PDF) of the involved belief propagation (BP) messages. It was first used to design capacity approaching LDPC codes in [24] and quantizes the decoder soft-information [8] with a bit \( b \in \mathbb{N} \) quantization function, which first clips its input to \( B \in \mathbb{R}^{+} \) or \(-B \) via \( \text{clip}(\cdot) \) and maps the result to \( q = 2^b - 1 \) quantization levels. We define this quantization function as \( Q(\cdot) : \mathbb{R} \rightarrow \mathbb{Q} \), where \( \mathbb{Q} = \{-q,0,q,2q,\ldots\} \), \( \Delta = (2B)/(q-1) \), and have

\[
Q(l) = \begin{cases} 
\text{clip}(l)/\Delta + 1, & l > \Delta/2 \\
\text{clip}(l)/\Delta - 1, & l < -\Delta/2 \\
0, & \text{otherwise.}
\end{cases} \tag{12}
\]

We use this type of quantization to represent \( l = 0 \) without quantization error. This is important for punctured VN.

III. Optimizing the Bit Mapping for Off-the-Shelf Protograph LDPC Codes

A. Problem Formulation

The bit channels \( P_{B_{\text{BMD}}} \) resulting from BMD have different qualities. Previous works noted the importance of mapping the different BMD bit levels to the different VN degrees of an irregular LDPC code, or to VN types of a protograph to improve the performance [5][10].

In the following, we use the ideas of [8], [9] to formulate an optimization procedure that optimizes the assignment of the \( m \) bit channels to the \( n_{p,t} \) transmitted VN types of a given protograph basematrix. This bit mapping can be expressed as a non-negative matrix \( A = [a_{1}, \ldots, a_{n_{p,t}}] \) of dimension \( m \times n_{p,t} \), where the entry \( a_{ji} = [A]_{ji} \) denotes the fraction of bit level \( j \) that is assigned to the \( i \)-th transmitted VN type. The matrix \( A \) needs to fulfill the constraints

\[
\sum_{j'=1}^{n_{p,t}} a_{ji} = \frac{1}{n_{p,t}}, \quad \sum_{j'=1}^{n_{p,t}} a_{j'i} = 1, \quad \text{for all } j \in \{1,2,\ldots,m\} \text{ and } i \in \{1,2,\ldots,n_{p,t}\}. \tag{13}
\]

The optimization requires an efficient way to evaluate the objective many times is therefore limited. Instead, we combine the approaches of [9], [12] and use P-EXIT [13]. In [12], the \( m \) BMD bit channels are matched to \( m \) parallel binary-input AWGN (biAWGN) surrogate channels for which a P-EXIT analysis is feasible. To perform this matching, we use the conditional entropy \( H(B_j|Y) \), and the corresponding biAWGN surrogate channel \( Y_j = \tilde{X}_j + Z_j \), with input \( \tilde{X}_j \in \{-1,1\} \) and noise \( \tilde{Z}_j \sim \mathcal{N}(0,\tilde{\sigma}_j^2) \) that is determined by solving

\[
\tilde{\sigma}_j^2 : H(\tilde{X}_j|Y_j) = H(B_j|Y), \quad j = 1,\ldots,m. \tag{16}
\]

For a given mixing vector \( \alpha = (a_1,\ldots,a_m) \), i.e., a column of \( A \), we find the biAWGN surrogate with parameter \( \tilde{\sigma}_j^2 \) as

\[
\tilde{\sigma}_j^2 : H(\tilde{X}_j|Y_j) = \sum_{j'=1}^{m} a_{j'} H(B_{j'}|Y_j), \quad j = 1,\ldots,m. \tag{17}
\]

B. Successive Bit Mapping Optimization

Performing the optimization (15) jointly over all bit levels is a complicated task, as it involves a large number of optimization variables for large constellation sizes and protograph VN. Instead, we propose a successive method that optimizes each bit level one at a time while leaving the mappings of the other bit levels fixed. As a consequence, we do not optimize
over the whole bit mapping matrix $A$, but only over one row of $A$, where the ordering is chosen as a parameter. All other bit levels are assigned uniformly. The algorithm for uniform signaling is summarized in Algorithm 1. For PAS, the function $\text{MAKE}_A$ is modified accordingly to account for the additional constraints (14). For the optimization in line 3, we use $\text{patternsearch}$ [13], a derivative free optimization approach that starts from a feasible initial point $x$ (i.e., one that fulfills the constraints) and then performs a search with a set of vectors to find a direction in which the objective value improves. For our setting, we use a so-called $2N$ basis which consists of the $2N$ canonical unit vectors $e_i$, $i = 1, \ldots, N$ of $\mathbb{R}^N$ and their negative counterparts, where $N$ is the number of independent optimization variables. The algorithm then polls all possible new points $x \pm s \cdot e_i$ after an appropriate scaling ($s \in \mathbb{R}^+$) of the basis vectors and selects the one with the best objective value as the starting point for the next iteration.

C. Numerical Results

We focus on a scenario with an SE of $\eta = 2.545$ bits per channel use (bpcu). Uniform signaling uses 8-ASK, whereas PAS uses 16-ASK with an appropriately chosen Maxwell-Boltzmann (MB) input distribution [23]. The different constellation sizes are chosen such that the best performance for both signaling modes is ensured. The DM rate for PAS is $R_{\text{dm}} = 2.152$ bits.

We validate the P-EXIT thresholds by DDE and use a 8-bit quantization ($q = 255$) with $B = 15$. These values are motivated by the observations in Sec. IV which show that a decoder with these parameters operates with almost no loss as compared to a full resolution, floating point implementation. The obtained decoding thresholds are summarized in Table I.

As a reference we choose a bit mapping $A_{\text{ref}}$ which assigns each bit level uniformly to each VN type, i.e., $A_{\text{ref}} = 1/m \cdot 1$, where $1$ is the all-ones matrix of size $m \times n_P t$. For PAS, $A_{\text{ref}}$ is additionally adjusted to meet the constraints of (14).

We observe that the P-EXIT and DDE values are in good agreement with a maximum difference of 0.12 dB, which is caused by the surrogate analysis and the mixing of the bit channels. For uniform signaling the gain is 0.23 dB, and for PAS the gain is 0.37 dB based on the DDE thresholds. In both cases, the optimization yields a bit mapping $A_{\text{opt}}$, which favors the assignment of the most reliable bit-channel (i.e., the one with the smallest uncertainty $H(B_j|Y)$) to the degree six VNs in the protograph. For uniform signaling, this means bit-level one is mapped to the degree 6 VN types, whereas for PAS bit-level two (which has the largest prior $\log(P_{B_j}(0)/P_{B_j}(1))$) is the most reliable one and is assigned to them. Empirical studies show that the ordering $O$ (cf. the input of Algorithm 1) plays an important role and that the best decoding threshold is achieved by starting with the bit channel having the smallest uncertainty. This result is intuitive as the first bit channel has the largest degree of freedom for the bit mapping optimization. We validate the asymptotic results by finite length simulations in Fig. 1.

IV. Clipping Optimization for Quantized Decoders

We now investigate the influence of the number of quantization levels $q$ and the clipping $B$. As noted in [17], clipping the soft-information can greatly influence the performance of the decoder and depends on the considered code ensemble.

We examine two scenarios with $b = 3$ and $b = 4$ bits resolution and investigate different approaches to find the best $B$. The first approach considers the uncertainty expression in (9). We evaluate the metric by generating soft-information values according to (8), quantizing them (12) and approximating the expectation by its empirical mean in a Monte-Carlo manner. The second approach uses DDE, as defined in Sec. II-D and determines the decoding threshold of the LDPC code ensemble given the selected quantization and clipping parameters.

We depict the results of this analysis in Fig. 2 for the setting of Sec. III-C. The optimized clipping is given by $B \approx 6 \pm 1$ for three bits and by $B \approx 8 \pm 2$ for four bits. The lines without markers represent the DDE thresholds, whereas the lines with markers are finite length simulation results and denote the required SNR to obtain a target FER of $10^{-3}$. Observe that the simulation results closely follow the DDE thresholds. Observe also that the uncertainty provides a good indication for the optimal clipping value, but does not reflect the overall qualitative behavior.

In Fig. 1 we show the bit error rate (BER) curves for the quantized decoders discussed in this section. We see that the loss due to quantization is about 0.25 dB for 4 bits and 1.50 dB for 3 bits. A quantized decoder with $B = 15$ and $q = 8$ operates with almost no loss as compared to a floating point implementation with full double resolution.

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**Algorithm 1** Algorithmic description of the successive bit mapping optimization.

**INPUT:** Protograph $B$, Distribution $P_X$, Ordering $O$, Set of fixed row indices $I$

1. $A_0 \leftarrow [I, T] \leftarrow []$
2. for $j \in O$ do
3. $A_{\text{opt}} = \arg \min_{A} \text{SNR}(\text{MAKE}_A(a_{\text{ref}}; j, N, T); B, P_X)$ subject to $0 \leq a_{i,j} \leq 1 - \frac{\text{sum}(A_{\text{ref}}(i,:), 1))}{|I|} \forall i \in \{1, \ldots, m_P t\}$
4. $A_{\text{opt}} \leftarrow A_{\text{opt}}$, $I \leftarrow I \cup \{j\}$
5. end for
6. $A_{\text{opt}} = \text{MAKE}_A(I, \{\}, N, T)$

**function** $\text{MAKE}_A(a_{\text{ref}}; j, N, T)$

7. $A(j,:) \leftarrow a_{\text{ref}}$
8. $A(I, :) \leftarrow A_{\text{opt}}$
9. $A([1:m] \setminus I, :) = (1/(m - |I|)) \cdot (1 - \text{sum}(A_{\text{ref}}, 1))$
10. **return** $A$
11. **end function**

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**TABLE I** Comparison of decoding thresholds with P-EXIT and DDE for PAS and uniform signaling.

| Signaling | Bit mapping | P-EXIT [dB] | DDE [dB] |
|-----------|-------------|-------------|----------|
| PAS       | reference   | 16.00       | 16.12    |
|           | optimized   | 15.67       | 15.75    |
| uniform   | reference   | 17.12       | 17.21    |
|           | optimized   | 16.90       | 16.98    |
V. Conclusion
Using the example of the proposed IEEE 802.3ca LDPC code, we have shown how off-the-shelf protograph based LDPC codes can be optimized for higher-order modulation and a quantized decoder. The optimized bit mapping improves the finite-length performance by 0.4 dB for PS and by 0.3 dB for uniform signaling without increasing the complexity. We also found the best clipping values for a quantized decoder with DDE and showed its accuracy with finite length simulations. Future work can further optimize the quantization levels and look-up table (LUT) entries for the CN operation.

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![Diagram](https://example.com/diagram.png)