UNIQUENESS OF SOLUTIONS TO $L^p$-CHRISTOFFEL-MINKOWSKI PROBLEM FOR $p < 1$

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Abstract. Since the lack of Brunn-Minkowski inequality and constant rank theorem, the uniqueness of solutions to $L_p$-Christoffel-Minkowski problem for $p < 1$ is a very difficult and challenging problem. In this paper, we make some progresses on this problem and prove a uniqueness theorem for $p < 1$.

Keywords: Christoffel-Minkowski problem, uniqueness.
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1. Introduction

The $L_p$-Christoffel-Minkowski problem which is equivalent to solve the following PDE

\begin{equation}
\sigma_k(u_{ij} + u\delta_{ij}) = \psi(x)u^{p-1} \quad \text{on} \quad S^n,
\end{equation}

arises naturally in the $L_p$-Brunn-Minkowski theory, see [14] [17]. The $L_p$-Minkowski problem ($k = n$) has been extensively studied during the last twenty years after the seminal work of Lutwak [14], see [1] [5] [16] [15] for motivation and see also [17] for the most comprehensive list of results. When $p > 1$, the existence and uniqueness of solutions are well understood. However, when $p < 1$ the uniqueness of solutions to the $L_p$-Minkowski problem is very subtle, and indeed it was shown in [11] that the uniqueness fails when $p < 0$ even restricted to smooth origin-symmetric convex bodies. Recently, Brendle-Choi-Daskaspoulos [2] shows the uniqueness holds true for $1 > p > -1 - n$ and $\psi \equiv 1$, and Chen-Huang-Li-Liu [3] prove the uniqueness for $p$ close to 1 and even positive function $\psi$.

For $k < n$, if $p \geq 1$, under a sufficient condition on the prescribed function $\psi$, the existence and uniqueness of solutions to $L_p$-Christoffel-Minkowski problem are also well understood through Guan-Ma’s work [8] for $p = 1$, Hu-Ma-Shen’s work [12] for $p \geq k + 1$ and Guan-Xia’ work [9] for $1 < p < k + 1$ and even prescribed data, by using the constant rank theorem. See also [10] for the proof of uniqueness and [13] for a simple proof. But for $p < 1$, since the lack of Brunn-Minkowski inequality and constant rank theorem, the uniqueness is a very difficult and challenging problem. As far as I know, the uniqueness for $p < 1$ is unknown until now. In this paper, we make some progresses in this direction for $\psi \equiv 1$.

We consider the uniqueness of solutions to the following $L_p$-Christoffel-Minkowski problem:

\begin{equation}
\sigma_k(u_{ij} + u\delta_{ij}) = u^{p-1} \quad \text{on} \quad S^n.
\end{equation}

where $u_{ij}$ are the second order covariant derivatives with respect to any orthonormal frame $\{e_1, e_2, ..., e_n\}$ on $S^n$, $\delta_{ij}$ is the standard Kronecker symbol and $\sigma_k$ is the $k$-th elementary symmetric function. To ensure the ellipticity of (1.2), we have to restrict the class of functions.

Definition 1.1. A function $u \in C^2(S^n)$ is called $k$-convex if

$$
\lambda[u_{ij} + u\delta_{ij}] = (\lambda_1[u_{ij} + u\delta_{ij}], ..., \lambda_n[u_{ij} + u\delta_{ij}])
$$

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belongs to $\Gamma_k$ for all $x \in \mathbb{S}^n$, where $\Gamma_k$ is the Garding’s cone
$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall \, 1 \leq j \leq k \}.$$  

We mainly get the following result.

**Theorem 1.2.** Assume $u \in C^4(\mathbb{S}^n)$ is a $k$-convex solution to (1.2) satisfying that $u_{ij} + u_\delta_{ij} \geq 0$ on $\mathbb{S}^n$, then $u \equiv \text{constant}$ for $1 > p > 1 - k$.

Our proof is motivated by the idea of Choi-Daskaspoulos [1] and Brendle-Choi-Daskaspoulos [2] in which they show the self-similar solution $\Sigma$ of $\alpha$-Gauss curvature flow satisfying the equation
$$K^\alpha = \langle X, \nu \rangle$$
is a sphere when $\alpha > \frac{1}{n+2}$, where $\Sigma$ is an embedded, strictly convex hypersurface in $\mathbb{R}^{n+1}$ given by $X : \mathbb{S}^n \to \mathbb{R}^{n+1}$, $K$ and $\nu$ are the Gauss curvature and out unit normal of $\Sigma$ respectively. Their result is also equivalent to say that the $L_p$-Minkowski problem (1.2) $(k = n)$ has the unique solution $u \equiv 1$ for $1 > p > -n - 1$. In [1, 2], the authors introduce two important functions:
$$W(x) = K^\alpha \lambda_1^{-1}(h_{ij}) - \frac{n\alpha - 1}{2n\alpha}|X|^2 = u \cdot \lambda_1(b_{ij}) - \frac{n\alpha - 1}{2n\alpha}(u^2 + |Du|^2)$$
and
(1.3)  
$$Z(x) = K^\alpha \text{tr}(b_{ij}) - \frac{n\alpha - 1}{2\alpha}|X|^2 = u \cdot \text{tr}(b_{ij}) - \frac{n\alpha - 1}{2\alpha}(u^2 + |Du|^2),$$
which are the key to their proof, where $\lambda_1(h_{ij})$ and $\lambda_1(b_{ij})$ are the smallest and biggest eigenvalues of the second fundamental form $h_{ij}$ of $\Sigma$ and its inverse matrix $b_{ij}$ respectively, $u : \mathbb{S}^n \to \Sigma \subset \mathbb{R}^{n+1}$ is the support function of $\Sigma$. Later, Gao-Li-Ma [7] and Gao-Ma [6] use these two functions above to study the uniqueness of closed self-similar solutions to $\sigma_k^\alpha$-curvature flow following the idea of [1, 2]. In fact, in [7] the authors consider the following general equation
(1.4)  
$$S^\alpha(\kappa_1, \ldots, \kappa_n) = \langle X, \nu \rangle,$$
where $S$ is a 1-homogeneous smooth symmetric function of the principle curvatures $\kappa_i$. Under some assumptions on $S$, they show $\Sigma = X(\mathbb{S}^n)$ is a round sphere for $\alpha \geq 1$. Examples of $S$ include $S = \sigma_k(\kappa_1, \ldots, \kappa_n)$, but not include $S = \frac{\sigma_n}{\sigma_{n-k}}(\kappa_1, \ldots, \kappa_n)$, for which the equation (1.4) is equivalent to $L_p$-Christoffel-Minkowski problem with $p = 1 - \frac{1}{\alpha}$. The main difficulty lies in the non-positivity of the term (2.3)
$$2\beta \left( k\sigma_k f - n \sum_{i=1}^n \sigma_{k-1}(\lambda|i)\lambda_i^2 \right),$$
if $f = \sigma_1$. (In this case, the $Z$ function (1.3) is just the original $Z$ function (1.2).) To overcome this difficulty, the easiest way is to choose $f$ such that $k\sigma_k f - n \sum_{i=1}^n \sigma_{k-1}(\lambda|i)\lambda_i^2 = 0$. So, we need to modify the $Z$ function. We introduce the following two functions:
$$W(x) = u \cdot \lambda_1(b_{ij}) - \beta(u^2 + |Du|^2)$$
and
(1.5)  
$$Z(x) = uF(b_{ij}) - n\beta(u^2 + |Du|^2),$$
where $\lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1$ are the eigenvalues of the matrix $b_{ij} = u_{ij} + u_\delta_{ij}$, $\beta = \frac{p-1+k}{2k}$ and
$$F(b_{ij}) = f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^n \frac{n\sigma_{k-1}(\lambda|i)\lambda_i^2}{k\sigma_k},$$
Moreover, then we have

Let

Suppose that

here \( \sigma \) is also the \((k-1)\)-th elementary symmetric function of \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \) with \( \lambda_i = 0 \).

**Remark 1.1.** We can propose the following questions:

(i) When \( k = n \) our result does not cover the previous result in [4, 2], then it is natural to ask if one can improve it.

(ii) Can we Theorem 1.2 hold true without the assumption on the positive semi-definite of \( u_{ij} + u \delta_{ij} \)?

(iii) Can we construct some non-uniqueness examples of solutions to (1.2) for \( p < 1 - k \)?

## 2. The proof of Theorem 1.2

We denote by \( \sigma_{k-1}(\lambda|i) = \frac{\partial \sigma_{k-2}(\lambda)}{\partial \lambda_i} \) and \( \sigma_{k-2}(\lambda ij) = \frac{\partial^2 \sigma_{k-1}}{\partial \lambda_i \partial \lambda_j} \). We use the notations \( u_i = D_i u \), \( u_{ij} = D_j D_i u \), \( D_i b_{ij} = b_{ij};p \), and so on, where \( D \) is the standard Levi-Civita connection on \( \mathbb{S}^n \). Set \( b_{ij} = u_{ij} + u \delta_{ij} \), we denote by \( \lambda_n \leq ... \leq \lambda_2 \leq \lambda_1 \) are the eigenvalues of \( \{b_{ij}\} \), arranged in decreasing order. Each eigenvalue defines a Lipschitz continuous function on \( \mathbb{S}^n \).

We recall the following Lemma which is similar to Lemma 5 in [2].

**Lemma 2.1.** Suppose that \( \varphi \) is a smooth function on \( \mathbb{S}^n \) such that

\[
\lambda_1 \leq \varphi \quad \text{everywhere and} \quad \lambda_1(p) = \varphi(p).
\]

Let \( m \) denote the multiplicity of the biggest eigenvalue at \( p \), so that

\[
\lambda_n(p) \leq ... \leq \lambda_{m+1}(p) < \lambda_m(p) = ... = \lambda_1(p).
\]

Then, we have

\[
b_{ij};l = \varphi \delta_{ij} \quad \text{at} \quad p \quad \text{for} \quad 1 \leq i,j \leq m.
\]

Moreover,

\[
\varphi_{ii} \geq b_{11;ii} + 2 \sum_{l>m} \frac{(b_{1l;1})^2}{\lambda_1 - \lambda_l}, \quad \text{at} \quad p.
\]

We also need Lemma 4.4 in [7] which statement as follows.

**Lemma 2.2.** Under the assumptions of Lemma 2.1, we have

\[
\sigma_{ij;pq} b_{ij;1} b_{pq;1} - 2 \sigma_{ij} \sum_{l>m} \frac{(b_{1l;1})^2}{\lambda_1 - \lambda_l} \]

\[
= \sum_{i \neq j} \sigma_{k-2}(\lambda ij)b_{ii;1} b_{jj;1} - 2 \sum_{i>m} \sigma_{k-1}(\lambda |i) \frac{(b_{1l;1})^2}{\lambda_1 - \lambda_i} - 2 \sum_{i>m} \sigma_{k-1}(\lambda |i) \frac{(b_{ii;1})^2}{\lambda_1 - \lambda_i} \]

\[
+ 2 \sum_{i>j>m} \frac{\sigma_{k-1}(\lambda |i)(\lambda_1 - \lambda_i)^2 - \sigma_{k-1}(\lambda |j)(\lambda_1 - \lambda_j)^2}{(\lambda_1 - \lambda_i)(\lambda_1 - \lambda_j)(\lambda_i - \lambda_j)} b_{ij;1}^2.
\]

Now, we begin to prove Theorem 1.2: Set

\[
W(x) = u \cdot \lambda_1(b_{ij}) - \beta (u^2 + |Du|^2),
\]

where \( \beta = \frac{p-1+k}{2k} > 0 \), \( \lambda_1, ..., \lambda_n \) are the eigenvalues of \( \{b_{ij}\} \), \( \lambda_n \leq ... \leq \lambda_1 \). Our proof is divided into two steps.

**Step 1:** we will prove

\[
\lambda_1(x_0) = \lambda_2(x_0) = ... = \lambda_n(x_0) \quad \text{and} \quad |Du|(x_0) = 0
\]
for any $x_0 \in \{ x \in S^n : W(x) = \max_{S^n} W \}$.

Assume $W(x)$ attains its maximum at $x_0$. As above, we denote by $m$ the multiplicity of the biggest eigenvalue at $x_0$. Let us define a smooth function $\varphi$ such that

$$W(x_0) = u \cdot \varphi - \beta(u^2 + |Du|^2).$$

Since $W$ attains its maximum at $x_0$, we have $\varphi(x) \geq \lambda_1$ everywhere and $\lambda_1 = \varphi$ at $x_0$. Choose a coordinate at $x_0$ such that

$$b_{ij}(x_0) = \text{diag}\{\lambda_1(x_0), ..., \lambda_n(x_0)\}$$

with

$$\lambda_n(x_0) \leq ... \leq \lambda_{m+1}(x_0) < \lambda_m(x_0) = ... = \lambda_1(x_0).$$

Since $u \cdot \varphi - \beta(u^2 + |Du|^2) = \text{constant}$, then

$$[\varphi - \beta(u^2 + |Du|^2)]_i = 0$$

and

$$[\varphi - \beta(u^2 + |Du|^2)]_{ii} = 0.$$

Taking (2.1)’s value at $x_0$ results in

$$0 = u_i \lambda_1 + ub_{11;i} - 2\beta u_i \lambda_i.$$

Thus,

$$b_{11;i} = \frac{u_i}{u}(2\beta \lambda_i - \lambda_1),$$

which implies together with Lemma 2.1 $u_i(x_0) = 0$ for $2 \leq i \leq m$. Taking (2.2)’s value at $x_0$ results in

$$0 = u_{ii} \lambda_1 + 2u_i b_{11;i} + u \varphi_{ii} - 2\beta[u_i^2 + uu_{ii} + u_{ii}^2 + u_{i}u_{ii}].$$

Thus,

$$0 = u_{ii} \lambda_1 + 2u_i b_{11;i} + u \varphi_{ii} - 2\beta[u_i^2 + uu_{ii} + u_{ii}^2 + u_{i}u_{ii}]$$

$$\geq (\lambda_i - u) \lambda_1 + 2u_i \frac{u_i}{u}(2\beta \lambda_i - \lambda_1) + u \left( b_{11;i} + 2 \sum_{l>m} \frac{(b_{1l;i})^2}{\lambda_1 - \lambda_l} \right) - 2\beta[\lambda_i \lambda_1 - u \lambda_i + ub_{ii;i}]$$

$$\geq \lambda_i \lambda_1 - u \lambda_i + \frac{2u_i^2}{u}(2\beta \lambda_i - \lambda_1) + u \left( b_{ii;11} + 2 \sum_{l>m} \frac{(b_{1l;i})^2}{\lambda_1 - \lambda_l} \delta_{ii} \right) - 2\beta[\lambda_i^2 - u \lambda_i + ub_{ii;i}],$$

here we use the following Ricci identity

$$b_{ii;11} = b_{11;ii} - b_{11} + b_{ii}$$

to get the last inequality. Differentiating the equation (1.2) shows

$$\sigma_k^{ii} b_{ii;1} = (p - 1) u^{p-2} u_1.$$

Differentiating it again

$$\sigma_k^{ii} b_{ii;11} = (p - 1) u^{p-3} [uu_{11} + (p - 2) u_{11}^2] - \sigma_k^{ij,pq} b_{ij;1} b_{pq;1}$$

$$= (p - 1) u^{p-3} [u \lambda_1 - u^2 + (p - 2) u_{11}^2] - \sigma_k^{ij,pq} b_{ij;1} b_{pq;1}.$$
Due to the concavity of $\sigma_k^p(\lambda)$,

$$- \sum_{i \neq j} \sigma_{k-2}(\lambda|i) b_{ii;1} b_{jj;1} \geq - \frac{(k-1)(p-1)^2}{k} u^{p-3} u_1^2,$$

which results in together with Lemma 2.2 by noticing that the forth and fifth terms in the right hand of the equation in Lemma 2.2 are negative

$$-\sigma^{ij,pq}_{k} b_{ij;1} b_{pq;1} + 2\sigma^i_k \sum_{l>m} \frac{(b_{1;i})^2}{\lambda_l - \lambda_i} \geq - \frac{(k-1)(p-1)^2}{k} u^{p-3} u_1^2 + 2 \sum_{l>m} \sigma_{k-1}(\lambda|i) \frac{(b_{1;i})^2}{\lambda_l - \lambda_i}.$$

Then,

$$0 = \sigma_k^{ij}[u \cdot \varphi - \beta(u^2 + |Du|^2)]_{ij} \geq k u^{p-1} \lambda_1 - k u^p + \sum_{i=1}^n \sigma_{k-1}(\lambda|i) \frac{2 u^2_i}{u} (2 \beta \lambda_i - \lambda_1)$$

$$+ u \left( (p-1) u^{p-3} [u \lambda_1 - u^2 + (p-2) u_1^2] - \frac{(k-1)(p-1)^2}{k} u^{p-3} u_1^2 \right)$$

$$+ \frac{2}{u^2} \sum_{i>m} \sigma_{k-1}(\lambda|i) u^2_i (2 \beta \lambda_i - \lambda_1)^2$$

$$- 2 \beta \left( \sum_{i=1}^n \sigma(\lambda|i) \lambda_i (\lambda_i - \lambda_1) + k u^{p-1} \lambda_1 - k u^p + (p-1) u^{p-2} \sum_{i=1}^n u_i^2 \right)$$

$$\geq u^{p-1} (k \lambda_1 + (p-1) \lambda_1 - 2 \beta k \lambda_1)$$

$$+ u^p (-k - (p-1) + 2 k \beta) + 2 \beta \sigma_{k-1}(\lambda|i) \lambda_i (\lambda_i - \lambda_1)$$

$$+ \frac{u^2}{u} \left( 2 (2 \beta - 1) \sigma_{k-1}(\lambda|1) \lambda_1 + (p-1)(p-2) u^{p-1} \right)$$

$$- \frac{(k-1)(p-1)^2}{k} u^{p-1} - 2 \beta (p-1) u^{p-1}$$

$$+ \frac{2}{u} \sum_{i>m} \sigma_{k-1}(\lambda|i) u^2_i (2 \beta \lambda_i - \lambda_1) \frac{(2 \beta - 1) \lambda_i}{\lambda_1 - \lambda_i}$$

$$+ 2 \beta (1 - p) u^{p-2} \sum_{i>m} u_i^2$$

$$\geq u_1^2 u^{p-3} \frac{2(k-1)(p-1)}{k} + \frac{2}{u} \sum_{i>m} \sigma_{k-1}(\lambda|i) u^2_i (2 \beta \lambda_i - \lambda_1) \frac{(2 \beta - 1) \lambda_i}{\lambda_1 - \lambda_i}$$

$$+ 2 \beta (1 - p) u^{p-2} \sum_{i>m} u_i^2 \geq 0,$$

where we use the following inequality to get the last inequality

$$\sigma_k(\lambda) \geq \lambda_1 \sigma_{k-1}(\lambda|1)$$

in view of the assumption on the positive semi-definite of $u_{ij} + u \delta_{ij}$. Thus, $\lambda_1(x_0) = \lambda_2(x_0) = \ldots = \lambda_n(x_0)$ and $|Du|(x_0) = 0$. 

$L_p$-CHRISTOFFEL-MINKOWSKI PROBLEM 5
Step 2: we want to show that \( \{ x \in S^n : W(x) = \max_{S^n} W \} \) is an open set. We define

\[
Z(x) = uF(b_{ij}) - n_β(u_2^2 + |Du|^2),
\]

where

\[
F(b_{ij}) = f(λ_1, λ_2, ..., λ_n) = \sum_{i=1}^{n} \frac{n_σ_{k-1}(λ_i)λ_i^2}{k_σ_k} = \frac{n}{k}[σ_1 - (k + 1)\frac{σ_{k+1}}{σ_k}],
\]

which is a 1-homogeneous convex function satisfying \( f(1, 1, ..., 1) = n_β \) since \( u \) is \( k_β \)-convex. We will prove for any \( x_0 \in \{ x \in S^n : W(x) = \max_{S^n} W \} \), there exists a small neighborhood \( U(x_0) \) of \( x_0 \) such that

\[
s_{k_β}Z_{ij} + \frac{2}{u}g_{k_β}u_iZ_j \geq 0
\]

and

\[
Z(x_0) = \max_{S^n} Z(x).
\]

Denoting by \( f_i = \frac{∂f}{∂λ_i} \) and \( f_{ij} = \frac{∂^2f}{∂λ_i∂λ_j} \). For any \( x \in U(x_0) \), we choose a coordinate at \( x \) such that

\[
b_{ij}(x) = \text{diag}\{λ_1(x), ..., λ_n(x)\}.
\]

Then, we have at \( x \)

\[
Z_i = ud_i + u \sum_{i=1}^{n} F_{pq}b_{pq;i} - 2n_βu_iλ_i
\]

and

\[
Z_{ii} = ud_{ii}f + 2ud_i \sum_{i=1}^{n} f_ib_{ii;i} + uF_{pq, st}b_{pq;st;i} + u \sum_{i=1}^{n} f_ib_{ii;ii} - 2n_β[u_i^2 + uu_{ii} + u_i^2 + uu_{ii}]
\]

\[
= λ_i f - u \sum_{i=1}^{n} f_iλ_i + u \sum_{i=1}^{n} f_i(2n_βλ_i - f) + uF_{pq, st}b_{pq;st;i} + u \sum_{i=1}^{n} f_ib_{ii;ii} - 2n_β[λ_i^2 - uλ_i + uu_{ii}]
\]

\[
≥ λ_i f - u \sum_{i=1}^{n} f_i(2n_βλ_i - f) + uF_{pq, st}b_{pq;st;i} + u \sum_{i=1}^{n} f_ib_{ii;ii} - 2n_β[λ_i^2 - uλ_i + uu_{ii}]
\]

\[
+ u \sum_{i=1}^{n} fib_{ii;ii} - 2n_β[λ_i^2 - uλ_i + uu_{ii}]
\]

\[
+ u \sum_{i=1}^{n} fib_{ii;ii} - 2n_β[λ_i^2 - uλ_i + uu_{ii}]
\]
in view of the Ricci identity
\[ b_{ii;lt} = b_{lt;ii} - b_{lt} + b_{ii} \]
and we use the convexity of \( f \) to get the last inequality. Differentiating the equation \((1.2)\) shows
\[ \sigma_k^{ij} b_{ii;lt} = (p - 1) u^{p-2} u_t. \]

Differentiating it again
\[ \sigma_k^{ij} b_{ii;lt} = (p - 1) u^{p-3} \left[ uu_{tt} + (p - 2) u^2_t \right] - \sigma_k^{ij,pq} b_{ij;lt} b_{pq;lt} \]
\[ = (p - 1) u^{p-3} \left[ u\lambda - u^2 + (p - 2) u^2_t \right] - \sigma_k^{ij,pq} b_{ij;lt} b_{pq;lt}. \]

Due to the concavity of \( \sigma_k^i \),
\[ \sigma_k^{ij,pq} b_{ij;lt} b_{pq;lt} \leq \frac{k - 1}{k} \left( \sigma_k^{ij} b_{ij;lt} \right)^2. \]

Then,
\[ \sigma_k^{ij} Z_{ij} \]
\[ \geq ku^{p-1}f - ku^{p} \sum_{i=1}^{n} f_i + \frac{2}{u} \sigma_k^{ij} u_i Z_j + \sum_{i=1}^{n} \sigma_{k-1}(\lambda |i) \frac{2u^2}{u} (2n\beta\lambda_i - f) \]
\[ + u \left( (p - 1) u^{p-3} \left[ uf - u^2 \sum_{i=1}^{n} f_i + (p - 2) \sum_{i=1}^{n} u^2 f_i \right] - \frac{k - 1}{k} (p - 1)^2 u^{p-3} \sum_{i=1}^{n} u^2 f_i \right) \]
\[ - 2n\beta \left( \sum_{i=1}^{n} \sigma_{k-1}(\lambda |i) \lambda_i^2 - ku^p + (p - 1) u^{p-2} \sum_{i=1}^{n} u^2 \right) \]
\[ \geq u^{p-1} (kf + (p - 1)f - 2\beta kf) + u^{p} \left( - k \sum_{i=1}^{n} f_i - (p - 1) \sum_{i=1}^{n} f_i + 2nk\beta \right) + \frac{2}{u} \sigma_k^{ij} u_i Z_j \]
\[ + \sum_{i=1}^{n} u^2 f^{p-2} \left( 2\sigma_{k-1}(\lambda |i)(2n\beta\lambda_i - f)u^{1-p} + (p - 1)(p - 2)f_i \right) \]
\[ - \frac{k - 1}{k} (p - 1)^2 f_i + 2n\beta(1 - p) \]
\[ + 2\beta \left( k\sigma_k f - n \sum_{i=1}^{n} \sigma_{k-1}(\lambda |i) \lambda_i^2 \right) \]
\[ \geq u^{p} \left( - k \sum_{i=1}^{n} f_i - (p - 1) \sum_{i=1}^{n} f_i + 2nk\beta \right) + \frac{2}{u} \sigma_k^{ij} u_i Z_j \]
\[ + \sum_{i=1}^{n} u^2 f^{p-2} \left( 2\sigma_{k-1}(\lambda |i)(2n\beta\lambda_i - f)u^{1-p} + (p - 1)(p - 2)f_i \right) \]
\[ - \frac{k - 1}{k} (p - 1)^2 f_i + 2n\beta(1 - p) \right). \]

At \( x_0 \), we have \( \lambda_1 = \lambda_2 = \ldots = \lambda_n \). Thus, at \( x_0 \)
\[ \sum_{i=1}^{n} n\lambda_i = f \quad \text{and} \quad f_i(x_0) = 1 \quad \forall 1 \leq i \leq n. \]
Thus,

\[2\sigma_{k-1}(\lambda_i)(2n\beta \lambda_i - f) = 2k(2\beta - 1)u^{p-1}.\]

So,

\[2\sigma_{k-1}(\lambda_i)(2n\beta \lambda_i - f)u^{1-p} + (p-1)(p-2)f_i - \frac{k-1}{k}(p-1)^2 f_i + 2n\beta(1-p)
\]

for \(1 > p > 1 - k\). Thus, there exists a small neighborhood \(U(x_0)\) such that

\[u^2 u^{p-2} \left(2\sigma_{k-1}(\lambda_i)(2n\beta \lambda_i - f)u^{1-p} + (p-1)(p-2)f_i - \frac{k-1}{k}(p-1)^2 f_i + 2n\beta(1-p)\right) \geq 0.
\]

(2.4)

Moreover, we obtain by Newton-MacLaurin’s inequality

\[\sum_{i=1}^{n} \frac{\partial (2\sigma_{k+1})}{\partial \lambda_i} \geq \frac{n-k}{k+1},\]

which implies

\[\sum_{i=1}^{n} f_i \leq \frac{n}{k} \left[n - (k+1) \frac{n-k}{k+1}\right] = n.
\]

Thus,

(2.5)

\[u^p(-k \sum f_i - (p-1) \sum f_i + 2nk\beta) = u^p(k+p-1)(n-\sum f_i) \geq 0.
\]

Thus, combining (2.4) and (2.5), we can find \(U(x_0)\) such that

\[\sigma_{k}^{ij} Z_{ij} + 2u \sigma_{k}^{ij} u_i Z_j \geq 0.
\]

Since \(f_i(\lambda_0) = 1\), we can choose \(U(\lambda_0)\) such that \(f\) is increase with each \(\lambda_i\) in \(U(\lambda_0)\), where \(\lambda_0 = (\lambda_1(x_0), ..., \lambda_n(x_0))\). Then, we can choose \(U(x_0)\) such that \(\{\lambda(x) : x \in U(x_0)\} \subset U(\lambda_0)\).

So, we have

\[Z(x_0) = nW(x_0) \geq nW(x) \geq Z(x),\]

which implies

\[Z(x_0) = \max_{U(x_0)} Z(x).
\]

Thus, we have by the strong maximum principle

\[W(x_0) = W(x).
\]

for any \(x\) in \(U(x_0)\). Thus, \(W(x) \equiv \max_{\mathbb{S}^n} W\) for any \(x \in \mathbb{S}^n\). So, \(Du = 0\) which implies \(u = \text{constant}\). Thus, we complete our proof.

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References

[1] G. Bianchi, K. J. Böröczky, A. Colesanti, Smoothness in the $L_p$-Minkowski problem for $p < 1$, arXiv:1706.06310 (2017).
[2] S. Brendle, K. Choi, P. Daskalopoulos, Asymptotic behavior of flows by powers of the Gaussian curvature, Acta Mathematica 219, 1-16 (2017)
[3] S. B. Chen, Y. Huang, Q. R. Li, J. K. Liu, $L_p$-Brunn-Minkowski inequality for $p < 1$, arXiv:1811.10181.
[4] K. Choi, P. Daskalopoulos, Uniqueness of closed self-similar solutions to the Gauss curvature flow, arXiv:1609.05487 (2016)
[5] K. S. Chou, X. J. Wang, The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. in Math. 205, 33-83 (2006)
[6] S. Z. Gao, H. Ma, Self-similar solutions of curvature flows in warped products, Differential Geom. Appl. 62, 234-252 (2019)
[7] S. Z. Gao, H. Z. Li, H. Ma, Uniqueness of closed self-similar solutions to $\sigma^p_k$-curvature flow, NoDEA Nonlinear Differential Equations Appl. 25 (2018), no. 5, Art. 45, 26 pp.
[8] P. F. Guan, X. N. Ma, Christoffel-Minkowski problem I: convexity of solutions of a hessian equation, Invent. Math. 151, 553-577 (2003)
[9] P. F. Guan, C. Xia, $L_p$ Christoffel-Minkowski problem: the case $1 < p < k + 1$, Cal. Var. Partial Differential Equations 57-69 (2018)
[10] P. F. Guan, X. N. Ma, N. Trudinger, X. H. Zhu, A Form of Alexandrov-Fenchel Inequality, Pure and Applied Mathematics Quarterly Volume 6, Number 4 (Special Issue: In honor of Joseph J. Kohn, Part 2 of 2 ) 999-1012, 2010
[11] H. Jian, J. Lu, X.J. Wang, Nonuniqueness of solutions to the $L_p$-Minkowski problem. Adv. Math. 281 (2015), 845-856.
[12] C. Q. Hu, X. N. Ma, C. L. Shen, On the Christoffel-Minkowski problem of Firey’s p-sum, Cal. Var. Partial Differential Equations 21, 137-155 (2004)
[13] S. Y. Li, Christoffel-Minkowski problem of Firey’s p-sum, Master thesis, Chinese Academy of Sciences (2015)
[14] E. Lutwak, The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem, J. Differential Geom. 38, 131-150 (1993)
[15] E. Lutwak, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas, Adv. in Math. 118, 244-294 (1996)
[16] E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem, J. Differential Geom. 41, 227-246 (1995)
[17] R. Schneider, Convex bodies: the Brunn-Minkowski theory, Second edition, No. 151. Cambridge University Press, 2013.

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