MULTI-BUMP SOLUTIONS FOR FRACTIONAL NIRENBERG PROBLEM

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ABSTRACT. We consider the multi-bump solutions of the following fractional Nirenberg problem

\[(−Δ)^s u = K(x)u^{\frac{n+2s}{n−2s}}, \quad u > 0 \text{ in } \mathbb{R}^n,\]

where \(s \in (0, 1)\) and \(n > 2+2s\). If \(K\) is a periodic function in some \(k\) variables with \(1 ≤ k < \frac{n−2s}{2}\), we proved that (1.1) has multi-bump solutions with bumps clustered on some lattice points in \(\mathbb{R}^k\) via Lyapunov-Schmidt reduction. It is also established that the equation (0.1) has an infinite-many-bump solutions with bumps clustered on some lattice points in \(\mathbb{R}^n\) which is isomorphic to \(\mathbb{Z}^k_+\).

1. Introduction and main results

The classic Nirenberg problem asks that on the standard sphere \((\mathbb{S}^n, g_{\mathbb{S}^n})\) with \(n ≥ 2\), whether there exists a function \(w\) such that the scalar curvature (Gauss curvature in the dimension 2) of the conformal metric \(g = e^w g_{\mathbb{S}^n}\) equals to a prescribed function \(\tilde{K}\). This problem is equivalent to solving the following equations

\[- Δ_{g_{\mathbb{S}^n}} w + 1 = \tilde{K}e^{2w} \text{ on } \mathbb{S}^2 \quad (1.1)\]

and

\[- Δ_{g_{\mathbb{S}^n}} v + \frac{n − 2}{4(n − 1)} R_{g_{\mathbb{S}^n}} v = \frac{n − 2}{4(n − 1)} \tilde{K}v^{\frac{n+2s}{n−2s}} \text{ on } \mathbb{S}^n \text{ for } n ≥ 3, \quad (1.2)\]

where \(R_{g_{\mathbb{S}^n}} = n(n − 1)\) is the scalar curvature of \((\mathbb{S}^n, g_{\mathbb{S}^n})\) and \(v = e^{\frac{n−2w}{4}}\). The linear operators defined on left-hand side of the equation (1.1) and (1.2) are called the conformal Laplacian on \(\mathbb{S}^n\).

For any Riemannian manifold \((M, g)\), the conformal Laplacian is defined by

\[P_g^1 = −Δ_g + \frac{n−2}{4(n−1)} R_g,\]

where \(R_g\) is the scalar curvature of \((M, g)\). Let \(u > 0\) and \(h = u^{\frac{4}{n−2}}g\), the conformal Laplacian has the following conformally invariant property

\[P_g^q(uφ) = u^{\frac{n+2q}{n−2}}P_{h}^q(φ) \quad \text{for } φ \in C^\infty(M).\]

The Paneitz operator \(P_g^2\) is another interesting conformal invariant operator. It was defined in [27] by

\[P_g^2 = Δ_g^2 + div_g(\alpha_n R_g Id − b_n Ric_g)\nabla g + \frac{n−4}{2} Q_g,\]

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where \( a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)} \), \( b_n = -\frac{4}{n-2} \), \( \mathcal{Ric} : TM \to TM \) is a \((1,1)\)-tensor operator defined by \( \mathcal{Ric}_i^j = g^{jk} \mathcal{Ric}_{kj} \); and \( Q_g = -\frac{\frac{2}{(n-2)^2} |\mathcal{Ric}|^2 + \frac{\rho^3}{8(n-1)^2(n-2)} R^2 - \frac{1}{2(n-1)} \Delta_g R_g}{2(n-1)} \) which is called the \( Q \)-curvature of \((M, g)\).

Later on, more conformally covariant elliptic operators were found. The operator \( P^g_1 \) and \( P^g_2 \) were generalized by Graham, Jenne, Mason and Sparling in \([16]\) to a sequence of integer order conformally covariant elliptic operators \( P^g_k \) for \( k \in \mathbb{N}_+ \) if \( n \) is odd; and \( k \in \{1, \ldots, \frac{n}{2}\} \) if \( n \) is even. Furthermore, any real number order conformally covariant pseudo-differential operator was intrinsically defined by Peterson in \([28]\). Graham and Zworski in \([17]\) proved that the operators \( P^g_k \) can be considered as the residue of a meromorphic family of scattering operators \( S(s) \) at \( s = \frac{3}{2} + k \). Then a family of non-integer order conformally covariant pseudo-differential operators \( P^g_s \) \((0 < s < \frac{n}{2})\) were naturally defined. Using the localization method in \([5]\), Chang and González \([6]\) showed that for any \( s \in (0, \frac{n}{2}) \), the operator \( P^g_s \) can also be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold.

The conformally covariant law for \( P^g_s \) means that for any Riemannian manifold \((M, g)\) and a conformal transformation \( h = v^{\frac{4}{n-2s}} g \), \( v > 0 \), there holds

\[
P^g_s(v \phi) = v^{\frac{n+2s}{n-2s}} P^g_s(\phi) \quad \text{for} \quad \phi \in C^\infty(M).
\]

Especially, \( P^g_s(1) \) is called the \( Q_s \) curvature or \( s \)-curvature of \((M, g)\) \( (\text{see } [6] \text{ and } [18] \text{ for example}) \).

The fractional Nirenberg problem was naturally raised on \( Q_s \) curvature, it asks that on the standard sphere \( \mathbb{S}^n \), there exists a function \( v > 0 \) such that the \( Q_s \) curvature of the conformal metric \( g = v^{\frac{4}{n-2s}} g_{\mathbb{S}^n} \) equals to a prescribed function \( \tilde{K} \). It can be reduced to the existence of the solution of the following equation

\[
P^g_{2s}(v) = \tilde{K} v^{\frac{n+2s}{n-2s}}, \quad v > 0 \quad \text{on} \quad \mathbb{S}^n,
\]

where \( s \in (0,1) \), \( n > 2s \) and \( \tilde{K} \) is a given positive function.

It was shown in \([4]\) that the operator \( P^g_{2s} \) is an intertwining operator and can be expressed as

\[
P^g_{2s} = \frac{\Gamma(B + \frac{n}{2} + s)}{\Gamma(B + \frac{n}{2} - s)} \quad \text{where} \quad B = \sqrt{-\Delta_{\mathbb{S}^n} + \left(\frac{n-1}{2}\right)^2},
\]

where \( \Delta_{\mathbb{S}^n} \) is the Beltrami-Laplacian operator. What is more, \( P^g_{2s} \) is more concrete under the stereographic projection. Let

\[
F : \mathbb{R}^n \to \mathbb{S}^n \setminus \{\mathcal{N}\}, \quad x \mapsto \left(\frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1}\right)
\]

be the inverse of stereographic projection, where \( \mathcal{N} \) is the north pole of \( \mathbb{S}^n \). Then it holds that

\[
P^g_{2s}(\phi) \circ F = |J_F|^{-\frac{n+2s}{2n}} (-\Delta)^s (|J_F|^{\frac{n-2s}{2n}} (\phi \circ F)),
\]

where \((-\Delta)^s\) is the fractional Laplacian defined by

\[
(-\Delta)^s \phi(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x-y|^{n+2s}} dy, \quad \text{where} \quad \phi \in C^\infty(\mathbb{R}^n).
\]

If we write \( u = |J_F|^{-\frac{n+2s}{2n}} (v \circ F) \) and \( K = \tilde{K} \circ F \), the equation \((1.3)\) is transformed into

\[
(-\Delta)^s u = K(x) u^{\frac{n+2s}{n-2s}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n,
\]
where $s \in (0, 1)$ and $n > 2s$. The existence of the solutions to the problem (1.4) has been proved under various conditions (see for example [11][12][7][9][11][18][19]). The compactness of the solutions to (1.4) was studied in [18]. Chen and Zheng [10] found a 2-peak solution when $K(x) = 1 + \varepsilon \tilde{K}(x)$ has at least two critical points and satisfies some local conditions. What is more, Liu in [25] constructed infinitely many 2-peak solutions when $K$ has a sequence of strictly local maximum points moving to infinity. When $K$ is a radial symmetric function, in [24] and [26] it was showed that (1.4) has infinitely many non-radial solutions.

In this paper, we continue to study the bump solutions or peak solutions of (1.4). Assume that $K$ satisfies the following conditions

(H1) $0 < \inf_{\mathbb{R}^n} K \leq \sup_{\mathbb{R}^n} K < \infty$;
(H2) $K(x)$ is a $C^{1,1}$ function, and 1-periodic in the first $k$ variables $x_1, \cdots, x_k$;
(H3) $0$ is a critical point of $K$, and in a neighborhood of $0$, there is a number $\beta \in (n - 2s, n)$ such that

$$K(x) = K(0) + \sum_{i=1}^{n} a_i |x_i|^\beta + R(x),$$

where $a_i \neq 0$ for $i = 1, \ldots, n$, $\sum_{i=1}^{n} a_i < 0$, $R(y) \in C^{[\beta]-1,1}$ and $\sum_{j=0}^{[\beta]} |\nabla^j R(y)||y|^{-\beta+j} = o(1)$ as $y \to 0$. Here $\nabla^j R(y)$ denote all of the possible derivatives of $R(y)$ of the order $j$.

We note that the conditions (H1)-(H3) and the condition (K) in [25] have some intersection. When $K$ satisfies both the condition (K) in [25] and our conditions (H1)-(H3), the equation (1.4) has infinitely many 2-peak solutions according to [25].

In this paper, we will show that equation (1.4) has solutions with large number bumps and its bumps located near some lattice points in $\mathbb{R}^k$ with $1 < k < \frac{n-2s}{2}$.

Let $Q_m := ([0, m + 1)^k \times \mathbb{O}) \cap \mathbb{Z}^n$, where $m \in \mathbb{N}_+ \cup \{\infty\}$ and $\mathbb{O}$ is a zero vector in $\mathbb{R}^{n-k}$.

**Theorem 1.1.** Suppose $n > 2s + 2$, $m \in \mathbb{N}_+ \cup \{\infty\}$ and $1 < k < \frac{n-2s}{2}$. If $K$ satisfies the conditions (H1)-(H3), there exists an integer $l_0 \in \mathbb{N}$, such that for any integer $l > l_0$, the equation (1.4) has a solution with its bumps clustered on $IQ_m$.

Notice that $Q_\infty$ is an infinite lattice which isomorphic to $\mathbb{Z}^k_\ast$. So we get an infinite-many-bump solution of the equation (1.5) via Theorem 1.1.

In order to prove Theorem 1.1 we assume $K(0) = 1$ with no loss of generality. For any positive integer $l$, define $\lambda = l^{\frac{n-2s}{n-2s}}$. Then we have $\lambda^\beta = (\lambda l)^{n-2s}$. Using the transformation $u(x) \mapsto \lambda^{-\frac{n-2s}{2}} u(\frac{x}{\lambda})$, we can change the equation (1.4) into

$$(-\Delta)^s u = K\left(\frac{x}{\lambda}\right) u^{\frac{n+2s}{n-2s}}, \quad u > 0, \quad \text{in} \quad \mathbb{R}^n. \quad (1.5)$$

The functional corresponding to equation (1.5) is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{n - 2s}{2n} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) (u_+)^{\frac{n+2s}{2}}, \quad u \in \dot{H}^s(\mathbb{R}^n),$$

where $u_+ = \max\{u, 0\}$. The Hilbert space $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C^\infty_0(\mathbb{R}^n)$ under the Gagliardao semi-norm (cf. [15] for detail)

$$[u]_{\dot{H}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 \frac{1}{|x - y|^{n+2s}}\right)^{\frac{1}{2}} = \left(2C(n, s) \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}|^2\right)^{\frac{1}{2}},$$
where \( C(n, s) \) is a constant depending on \( n \) and \( s \). It is well known that \( \dot{H}^s(\mathbb{R}^n) \) can be imbedded into \( L^{2^*(s)}(\mathbb{R}^n) \) and the following Hardy-Littlewood-Sobolev inequality holds
\[
S \left( \int_{\mathbb{R}^n} |u|^{2^*(s)} \right)^{2/2^*(s)} \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2, \quad u \in C_0^\infty(\mathbb{R}^n),
\]
where \( s \in (0, 1), n > 2s \) and \( 2^*(s) = \frac{2n}{n-2s} \). Lieb \cite{23} proved that the extremals corresponding to the best constant \( S \) of (1.6) are of the form
\[
U_{\xi, \Lambda, C_0} = C_0 \left( \frac{\Lambda}{1 + \Lambda^2|x - \xi|^2} \right)^{\frac{n-2s}{2}},
\]
where \( C_0, \Lambda \in \mathbb{R}_+ \) and \( \xi \in \mathbb{R}^n \).

Choosing a suitable constant \( C_0 = C_0(n, s) \), we see that the function \( U_{\xi, \Lambda} := U_{\xi, \Lambda, C_0} \) solves the equation
\[
(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, \quad u > 0 \text{ in } \mathbb{R}^n.
\]
Under some decay assumptions, \cite{8, 20, 22} proved that all the solutions of (1.7) are only of the form \( U_{\xi, \Lambda} \). Furthermore, it was proved in \cite{13} that the solution \( U_{\xi, \Lambda} \) of the equation (1.7) is nondegenerate, i.e. any bounded solution of the equation \((-\Delta)^s \phi = \frac{n+2s}{n-2s} u^{\frac{4s}{n-2s}} \phi \) is a linear combination of \( \frac{\partial U_{\xi, \Lambda}}{\partial \Lambda} \) and \( \frac{\partial U_{\xi, \Lambda}}{\partial \xi} \), \( i = 1, 2, \ldots, n \).

We will use the functions \( U_{\xi, \Lambda} \) to construct the approximate solutions of the equation (1.5). We define \( X_{l,m} = \{ \lambda l x | x \in Q_m \} \) and arrange it in any way as a sequence \( X_{l,m} = \{ X_i \}_{i=1}^{(m+1)^k} \).

Let \( P^i \in B_{\frac{1}{2}}(X_i) = \{ X \in \mathbb{R}^n | |X - X_i| < \frac{1}{2} \} \), \( \Lambda_i \in [C_1, C_2] \), for \( i = 1, 2, \ldots, (m+1)^k \), where \( C_1 \) and \( C_2 \) are some positive numbers to be defined later (see \cite{3, 0}). Let
\[
W_m(x) := \sum_{i=1}^{(m+1)^k} U_{P_i^i, \Lambda_i}(x)
\]
to be an approximate solution of the problem (1.5).

**Theorem 1.2.** Under the same conditions of Theorem 1.1, there exists an interger \( l > l_0 \) such that for any integer \( l > l_0 \), equation (1.5) has a \( C_2^\text{loc} \) solution \( u_m \) of the form
\[
 u_m = W_m + \phi_m,
\]
where \( m \in \mathbb{N}_+ \cup \{ \infty \} \), \( |\phi|_{L^\infty(\mathbb{R}^n)} \to 0 \) and \( \max_{i=1, \ldots, (m+1)^k} \{ |P^i - X_i^i| \} \to 0 \) as \( l \to \infty \).

As a consequence of Theorem 1.2 we have

**Corollary 1.3.** Under the same conditions of Theorem 1.1, the equation (1.4) has infinitely many multi-bump solutions.

Theorem 1.1 follows from Theorem 1.2. So we only need to prove Theorem 1.2. In this article we use \( l \) as the pertubation parameter and follow the methods developed in \cite{21, 29}. In the section 2 we carry out the Liapunov-Schmidt reduction. Theorem 1.2 is proved in the section 3. Some useful estimations are presented in Appendix A. The expansions of the functional \( \frac{\partial I}{\partial \Lambda_i}(W_m) \) and \( \frac{\partial I}{\partial P^i}(W_m) \) are shown in Appendix B.

In this article, \( C \) denotes a varying constant independent of \( m \).
2. Finite dimensional reduction

In this section, we will carry out the Lyapunov-Schmidt reduction in the case of $m < \infty$. We define two weighted norms

$$
\|u\|_* = \sup_{y \in \mathbb{R}^n} \left( \gamma(y) \sum_{i=1}^{(m+1)k} \frac{1}{(1 + |y - X_i|^\frac{n-2s}{2} + \tau)} \right)^{-1} |u(y)|,
$$

and

$$
\|u\|_{**} = \sup_{y \in \mathbb{R}^n} \left( \gamma(y) \sum_{i=1}^{(m+1)k} \frac{1}{(1 + |y - X_i|^\frac{n+2s}{2} + \tau)} \right)^{-1} |u(y)|,
$$

where

$$
\gamma(y) = \min \left\{ \min_{i=1, \ldots, (m+1)k} \left( \frac{1 + |y - X_i|}{\lambda} \right)^{\tau-s}, 1 \right\},
$$

and $\tau \in (k, \frac{n-2s}{2})$ is a constant.

Consider the following equation

$$
(-\Delta)^s \phi - \frac{n + 2s}{n - 2s} K \left( \frac{x}{\lambda} \right) W_{\frac{4s}{n - 2s}} \phi = g \quad \text{in} \quad \mathbb{R}^n.
$$

(2.1)

**Lemma 2.1.** Let $\phi$ be a solution of the equation (2.1), then we have the following estimate

$$
\left( \gamma(y) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|^\frac{n-2s}{2} + \tau)} \right)^{-1} |\phi(y)|
\leq C\|g\|_{**} + C\|\phi\|_* \left( \frac{1}{(\lambda l)^{\frac{4s}{n-2s}}} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|^\frac{n+2s}{2} + \theta)} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|^\frac{n-2s}{2} + \tau)} \right),
$$

(2.2)

where $\theta > 0$ is a constant and $C$ is independent of $m$.

**Proof.** We rewrite the equation (2.1) into an integral equation

$$
\phi(y) = C_1(n, s) \int_{\mathbb{R}^n} \frac{1}{y - z}^{n-2s} \left( \frac{n + 2s}{n - 2s} K \left( \frac{z}{\lambda} \right) W_{\frac{4s}{n - 2s}}(z) \phi(z) + g(z) \right) dz,
$$

(2.3)

where the constant $C_1(n, s)$ is defined in the Green function of $(-\Delta)^s$ on $\mathbb{R}^n$ (cf. [5]).
From Lemma A.3, we get
\[
\left| \int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} K\left( \frac{z}{\lambda} \right) W_m^{4s} (z) \phi(z) dz \right| \\
\leq C\|\phi\|_s \int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} W_m^{4s} (z) (\gamma (z)) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |z - X^h|)^{\frac{n-2s}{2} + \tau}} dz \\
\leq C\|\phi\|_s \left( \gamma (y) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}} + \frac{1}{(\lambda \tau)^{\frac{n-2s}{2}} \gamma (y) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}} \right). \\
\tag{2.4}
\]

For the second term on the right-hand side of (2.3), we have
\[
\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} |g(z)| dz \leq \|g\|_s \int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \gamma (z) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |z - X^h|)^{\frac{n-2s}{2} + \tau}} dz. \\
\tag{2.5}
\]

Using Lemma A.2, we obtain
\[
\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |z - X^h|)^{\frac{n-2s}{2} + \tau}} dz \leq C \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |z - X^h|)^{\frac{n-2s}{2} + \tau}}. \\
\tag{2.6}
\]

Define \( B_i := B_{r_0}(X^i) \), \( B_{i,m} := B_{r_0}(X^i) \) with \( r_0 = \max\{\frac{4s}{\tau}, 1\} \) and
\[
\Omega_i := \{z \in \mathbb{R}^n : |z - X^i| = \min_{j=1,...,(m+1)k} |z - X^j| \}.
\]

Without loss of generality, we assume \( y \in \Omega_1 \). Make use of Lemma A.3 under different cases, we have
\[
\frac{1}{\lambda^{\tau-s}} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2}}} \leq C \left( \frac{1 + |y - X^1|}{\lambda} \right)^{\tau-s} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}. \\
\tag{2.7}
\]

Using Lemma A.2 and (2.7), we have
\[
\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \min_{i=1,...,(m+1)k} \left\{ \left( \frac{1 + |z - X^i|}{\lambda} \right)^{\tau-s} \right\} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |z - X^h|)^{\frac{n-2s}{2} + \tau}} dz \\
\leq C \frac{\lambda^{\tau-s}}{\lambda^{\tau-s}} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2}}} \\
\leq C \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2}}} \\
\leq C \left( \frac{1 + |y - X^1|}{\lambda} \right)^{\tau-s} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}. \\
\tag{2.8}
\]
From the definition of $\gamma(y)$, (2.5), (2.6) and (2.8), we know
\[
\int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} |g(z)| dz \leq C\|g\|_{**} \gamma(y) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |y - X_h|)^{\frac{n-2s}{2} + \tau}}. \tag{2.9}
\]

Now (2.2) follows from (2.3), (2.4) and (2.9).

Consider the following problem
\[
\begin{cases}
(-\Delta)^{\frac{s}{2}} \phi - \frac{n+2s}{n-2s} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi = g + \sum_{i=1}^{(m+1)k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P_i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j}, \\
\int_{\mathbb{R}^n} U_{P_i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \phi dx = 0, \quad \phi \in \dot{H}^s(\mathbb{R}^n), \quad i = 1, \ldots, (m+1)^k, \quad j = 1, \ldots, n+1,
\end{cases} \tag{2.10}
\]
where $Z_{i,j} = \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_j}$ for $j = 1, \ldots, n$ and $Z_{i,n+1} = \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_{n+1}}$.

**Lemma 2.2.** Assume $\phi$ solves the problem (2.10), there exists $l_0 > 0$, such that for all $l > l_0$, we have $\|\phi\|_* \leq C\|g\|_{**}$, where $C$ is independent of $m$.

**Proof.** If this lemma is not right, then there would be sequences $\{g_l\}_{l=1}^{\infty}$ and $\{\phi_l\}_{l=1}^{\infty}$ satisfying (2.10) with $\|\phi_l\|_* = 1$ and $\|g_l\|_{**} \to 0$ as $l \to +\infty$. For notation simplicity, we suppress $l$ in the argument below.

First, we give an estimate of the parameters $c_{ij}^{(m)}$. Multiplying (2.10) with $Z_{r,t}$ and integrating on both sides, we get
\[
-\frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi Z_{r,t} dx = \int_{\mathbb{R}^n} g Z_{r,t} dx + \sum_{i=1}^{(m+1)k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \int_{\mathbb{R}^n} U_{P_i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} Z_{r,t} dx. \tag{2.11}
\]

For the first term on the right hand side of (2.11), using Lemma A.2 we have
\[
\left| \int_{\mathbb{R}^n} g Z_{r,t} dx \right| \leq C\|g\|_{**} \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X_r|)^{n-2s}} \gamma(x) \sum_{j=1}^{(m+1)k} \frac{1}{(1 + |x - X_j|)^{\frac{n-2s}{2} + \tau}} dx
\]
\[
\leq C \frac{\|g\|_{**}}{\lambda^{r-s}} \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X_r|)^{n-2s}} \sum_{j=1}^{(m+1)k} \frac{1}{(1 + |x - X_j|)^{\frac{n-2s}{2} + \tau}} dx
\]
\[
\leq C \frac{\|g\|_{**}}{\lambda^{r-s}} \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X_r|)^{n+\frac{\tau}{2}}} dx + \sum_{j \neq r} \frac{1}{|X_j - X_r|^\frac{\tau}{2}} \right)
\]
\[
\leq C \frac{\|g\|_{**}}{\lambda^{r-s}},
\]
where we have used the fact that
\[
\sum_{j \neq r} \frac{1}{|X_j - X_r|^\frac{\tau}{2}} \text{ converges for } \frac{n}{2} > k. \tag{2.12}
\]
Since the left hand side of the equation (2.11) is estimated in Lemma A.6, we have
\[
\sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \int_{\mathbb{R}^n} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} Z_{r,t} dx = \frac{1}{\lambda^{s-r}} O \left( \|g\|_{**} + \frac{\|\phi\|_{*}}{\lambda^{s-r}} \right).
\]
As we know \( \int_{\mathbb{R}^n} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} Z_{r,t} dx = C \delta_{j,t} \) and \( \int_{\mathbb{R}^n} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} Z_{r,t} dx \leq \frac{C}{|X^r - X^r|^n s} \) for \( i \neq r \), we obtain
\[
\max_{i,j} \{ |c_{ij}^{(m)}| \} = \frac{1}{\lambda^{s-r}} O \left( \|g\|_{**} + \frac{\|\phi\|_{*}}{\lambda^{s-r}} \right).
\]
An argument similar to the one used in (2.7) yields
\[
\left| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} \right| \leq \frac{C}{\lambda^{s-r}} \left( \|g\|_{**} + \frac{\|\phi\|_{*}}{\lambda^{s-r}} \right) \gamma(y) \sum_{i=1}^{(m+1)^k} \frac{1}{(1 + |y - X^r|^n s + r)}.
\]
From the definition of the norm \( \| \cdot \|_{**} \), we have
\[
\| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} \|_{**} \leq C \left( \|g\|_{**} + \frac{\|\phi\|_{*}}{\lambda^{s-r}} \right).
\]
Applying Lemma 2.1 to the first equation of the system (2.10), one get
\[
\left( \gamma(y) \sum_{h} \frac{1}{(1 + |y - X^h|^n s + r)} \right)^{-1} |\phi(y)| \leq C \left( \|g\|_{**} + \| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \frac{4s}{n-2s} U_{p^r,\Lambda_r} Z_{i,j} \|_{**} + \left( \frac{1}{\lambda} \right)^{\frac{n+2s}{n-2s}} + \frac{1}{\lambda} \right),
\]
As a result, there exist a number \( i_0 \in \mathbb{N} \) and a large constant \( R > 0 \), such that
\[
1 = \|\phi\|_{*} = \sup_{B_R(X^0)} \left( \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|^n s + r)} \right)^{-1} |\phi(y)|.
\]  
Hence there is a constant \( c_0 > 0 \) such that \( |\lambda^{s-r} \phi|_{L^\infty(B_R(X^0))} \geq c_0 \).

Applying Lemma A.3 to the equation (2.10), we know \( \lambda^{s-r} \phi \) is equi-continuous. Also \( \lambda^{s-r} |\phi(\cdot)| \) is uniformly bounded. In fact, we assume that \( y \in \Omega_1 \) with no loss of generality. From the fact (2.12), we have
\[
\lambda^{s-r} |\phi(y)| \leq \|\phi\|_{*} \sum_{h} \frac{1}{(1 + |y - X^h|^n s + r)} \leq C + \sum_{h \neq 1} \frac{1}{|X^h - X^1|^r} \leq C.
\]
Then the Arzelà-Ascoli Theorem yields that there is a function \( \tilde{\phi} \), such that \( \lambda^{r-s}\phi(\cdot + P^{\delta_0}) \) convergent to \( \tilde{\phi} \) uniformly on compact sets. Then

\[
|\tilde{\phi}|_{L^\infty(B_{R+1}(0))} \geq c_0. \tag{2.14}
\]

Using a similar argument as in [12, Lemma 7.3], we know \( \tilde{\phi} \) satisfies

\[
\begin{cases}
(-\Delta)^s\tilde{\phi} - \frac{n+2s}{n-2s}U^\frac{4s}{n-2s} \tilde{\phi} = 0, \\
\int_{\mathbb{R}^n} U_0^{\frac{4s}{n-2s}} \frac{\partial U_0}{\partial \Lambda_0} \tilde{\phi} = 0, \\
\int_{\mathbb{R}^n} U_0^{\frac{4s}{n-2s}} \frac{\partial U_0}{\partial P^j_0} \tilde{\phi} = 0, & j = 1, \ldots, n.
\end{cases}
\]

Then \( \tilde{\phi} = 0 \) by nondegeneracy, which is contradict to (2.14). Hence the solution \( \phi \) of the equation (2.10) satisfies \( \|\phi\|_* \leq C\|g\|_{**} \).

Combining Lemma 2.2, Lemma A.8 and the argument of [14, Proposition 4.1]( cf. [24, Proposition 2.2]), we have

**Proposition 2.3.** For any \( g \) satisfying \( \|g\|_{**} < +\infty \), (2.10) has a unique solution \( \phi = L_m(g) \in \dot{H}^s(\mathbb{R}^n) \cap C^{\alpha,\alpha}(\mathbb{R}^n) \) with \( \alpha = \min\{2s,1\} \), such that \( \|L_m(g)\|_* \leq C\|g\|_{**} \). The constant \( c_{ij}^{(m)} \) satisfies \( |c_{ij}^{(m)}| \leq \frac{C_{ij}}{\lambda^{\frac{m+2s}{n-2s}}}\|g\|_{**} \).

Since we are interested in the solution of the form \( W_m + \phi_m \) of the equation (1.5), we now consider the following problem

\[
\begin{cases}
(-\Delta)^s\phi - \frac{n+2s}{n-2s}K\left(\frac{x}{\lambda}\right)W_m^{\frac{n+2s}{n-2s}} \phi = N(\phi) + l_m + \sum_{i=1}^{(m+1)} \sum_{j=1}^{n+1} c_{ij}^{(m)} U^{\frac{4s}{n-2s}}_{P^i,\Lambda_i} Z_{i,j}, \\
\int_{\mathbb{R}^n} U^{\frac{4s}{n-2s}}_{P^i,\Lambda_i} Z_{i,j} \phi dx = 0, & \phi \in \dot{H}^s(\mathbb{R}^n),
\end{cases}
\tag{2.15}
\]

where

\[
N(\phi) = K\left(\frac{x}{\lambda}\right) \left( (W_m + \phi)^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}} - \frac{n+2s}{n-2s}\left( W_m^{\frac{n+2s}{n-2s}} - \phi\right) \right)
\]

and

\[
l_m = K\left(\frac{x}{\lambda}\right)W_m^{\frac{n+2s}{n-2s}} - \sum_{i=1}^{(m+1)} U^{\frac{n+2s}{n-2s}}_{P^i,\Lambda_i}.
\]

**Lemma 2.4.** For the terms \( N(\phi) \) and \( l_m \) defined above, we have the following estimates

\[
\|N(\phi)\|_{**} \leq C\|\phi\|_{\min\{2,2^*(s)\},}
\]

\[
\|l_m\|_{**} \leq \frac{C}{\lambda^{\frac{m+2s}{n-2s}}}. \leq \frac{C}{\lambda^{\frac{m+2s}{n-2s}}}
\]

**Proof.** The proof of the first estimation is rather standard(cf. [29] Lemma 2.4) for ideas). We only prove the second estimate.
Without loss of generality, we assume \( x \in \Omega \). Then

\[
l_m = K\left(\frac{x}{\lambda}\right) W_{m}^{\frac{n+2s}{n-2s}} U_{P^1,\Lambda_1}^{\frac{n+2s}{n-2s}} - \sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}}
\]

\[
= \left(K\left(\frac{x}{\lambda}\right) - 1\right) U_{P^1,\Lambda_1}^{\frac{n+2s}{n-2s}} + O\left(\sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}} \sum_{h \neq 1} U_{P^h,\Lambda_h} \right). \tag{2.16}
\]

The two error terms in (2.16) can be estimated by using Lemma A.3 under different cases.

**Case 1:** \( x \in \Omega_1 \cap B^c_1 \cap B_{1,m}, \) we have \( \gamma(x) = 1 \). Using Lemma A.3, we have

\[
\left(\sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}}\right) \leq \frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}}} \frac{1}{(1 + |x - X^1|)^{n+2s-\frac{n+2s}{n-2s}}}
\]

\[
\leq \frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}}} \frac{1}{(1 + |x - X^1|)^{\frac{n+2s}{n-2s}+\tau-k}}
\]

\[
\leq \frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}}} \frac{1}{\sum_{h} (1 + |x - X^h|)^{\frac{n+2s}{n-2s}+\tau}}.
\]

**Case 2:** \( x \in \Omega_1 \cap B_1, \) it holds that \( |x - X^i| \geq \frac{1}{2} |X^i - X^1| \geq \frac{1}{2} \lambda l \) for \( i \neq 1 \). From Lemma A.3,

\[
\left(\sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}}\right) \leq \left\{ \begin{array}{ll}
\frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}}} \sum_{h} (1 + |x - X^h|)^{\frac{n+2s}{n-2s}+\tau}, & \text{if } x \in \Omega_1 \cap B_1 \cap B^c_1(X_1),
\frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}}} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\tau-s} \lambda^{\tau-s}} \sum_{h} (1 + |x - X^h|)^{\frac{n+2s}{n-2s}+\tau}, & \text{if } x \in \Omega_1 \cap B_1 \cap B_1(X^1).
\end{array} \right.
\]

**Case 3:** \( x \in \Omega_1 \cap B^c_{1,m}, \) we can get

\[
\left(\sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}}\right) \leq \frac{C m^{\frac{n+2s}{n-2s}}}{(1 + |x - X^1|)^{n+2s}} \leq \frac{C m^{\frac{4s}{n-2s}}}{(1 + |x - X^1|)^{2s}} \leq (1 + C\left[\frac{m}{T}\right])^{k}.
\]

Following the proof of Lemma A.3, we have

\[
\sum_{h} \frac{1}{(1 + |x - X^h|)^n} \geq \frac{1}{(1 + |x - X^1|)^n} \left(1 + 2^{-k} \int_{[0,\left]\frac{m}{T}\right]^{k} \setminus [0,1]^{k}} \frac{1}{1 + \frac{\lambda l}{1 + |x - X^1|}|z|^n} dz \right)^n
\]

\[
\geq \frac{(1 + C\left[\frac{m}{T}\right])^{k}}{(1 + |x - X^1|)^n}.
\]

Since in the domain \( \Omega_1 \cap B^c_{1,m}, \) we can get \( |x - X^h| \geq \frac{1}{2} \lambda l \) for \( h = 1, \ldots, (m + 1)^{k} \), then

\[
\left(\sum_{h \neq 1} U_{P^h,\Lambda_h}^{\frac{n+2s}{n-2s}}\right) \leq \frac{C}{(\lambda l)^{2s}} \sum_{h} \frac{1}{(1 + |x - X^h|)^n} \leq \frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}+\tau}} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{n-2s}+\tau}}.
\]
Combining these three cases above, we have \( \|(\sum_{h\neq 1} U_{p,\lambda_h})^{\frac{n+2s}{2}}\|_{**} \leq \frac{C}{(\lambda_1^{\frac{n+2s}{2}-\tau}}. \) By the same procedure, we can also get the estimation \( \|U_{p_1,\lambda_1}^{\frac{n+2s}{2}} \sum_{h\neq 1} U_{p_h,\lambda_h}\|_{**} \leq \frac{C}{(\lambda_1^{\frac{n+2s}{2}-\tau}}. \)

At last, we estimate the first term in (2.16).

In the case of \( |x - X^{1}| \geq \lambda \), we have \( \gamma(x) = 1. \) Then

\[
|K(\frac{x}{\lambda}) - 1| U_{p_1,\lambda_1}^{\frac{n+2s}{2}} \leq \frac{C}{(1 + |x - X^{1}|)^{n+2s}} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \gamma(x) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |x - X^{h}|)^{\frac{n+2s}{2} + \tau}}. \tag{2.17}
\]

In the case of \( |x - X^{1}| < \lambda \), it holds \( \frac{1+|x-X^{1}|}{\lambda} \leq C. \) The condition \((H_3) \) yields

\[
|K(\frac{x}{\lambda}) - 1| U_{p_1,\lambda_1}^{\frac{n+2s}{2}} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \left(\frac{1+|x-X^{1}|}{\lambda}\right)^{\tau - s} \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |x - X^{h}|)^{\frac{n+2s}{2} + \tau}}.
\]

Summarizing (2.17) and (2.18), we have

\[
\| K(\frac{x}{\lambda}) - 1 \|_{**} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}.
\]

Hence this lemma follows.

**Proposition 2.5.** For \( \lambda \) large enough, the problem (2.15) has a unique solution \( \phi_m \in \tilde{H}^{s}(R^n) \cap C^{0,\alpha}(R^n) \) with \( \alpha = \min\{2s,1\} \), such that \( \|\phi_m\|_{*} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}. \) The constants \( c_{ij}^{(m)} \) satisfy \( |c_{ij}^{(m)}| \leq C\lambda^{-\frac{2}{2}}. \)

**Proof.** We define

\[
E = \left\{ \varphi \in \tilde{H}^{s}(R^n) \cap C(R^n) : \|\varphi\|_{*} \leq \frac{1}{\lambda^{\frac{n+2s}{2}-\tau-\epsilon_1}}, \int_{R^n} U_{p_1,\lambda_1}^{\frac{n+2s}{2}} Z_{i} j \varphi = 0, \quad i = 1, \ldots, (m+1)k, \right. \quad j = 1, \ldots, n+1 \left. \right\},
\]

where \( \epsilon_1 = \min\{\frac{1}{1}, \frac{2s}{n+2s}\} (\frac{n+2s}{2} - \tau) \). Notice that \( (E, \|\cdot\|_{*}) \) is a metric space.

In order to use the contraction map theorem, we define \( A \varphi := L_m(N(\varphi) + l_m) \), where \( L_m \) is an operator defined in Proposition 2.3.

Firstly, we show that \( A \) maps \( E \) into itself for \( \lambda \) large. Combining Proposition 2.3 and Lemma 2.4 we have \( \forall \varphi \in E \),

\[
\|A\varphi\|_{*} \leq C(\|N(\varphi)\|_{**} + \|l_m\|_{**}) \leq C(\|\varphi\|_{*}^{\min\{2,2^*(s)\}-1} + \|l_m\|_{**}) \leq \frac{1}{\lambda^{\frac{n+2s}{2}-\tau-\epsilon_1}}.
\]

Secondly, we prove \( A \) is an contraction map for \( \lambda \) large.
Choose $\varphi_1, \varphi_2 \in E$ with $\varphi_1 \neq \varphi_2$. If $N \geq 6s$, we have
\[
|N(\varphi_1) - N(\varphi_2)| = |N'(t\varphi_1 + (1-t)\varphi_2)(\varphi_1 - \varphi_2)|
\leq C(|\varphi_1^{4s/n-2s}| + |\varphi_2^{4s/n-2s}|)(|\varphi_1 - \varphi_2|)
\leq C(||\varphi_1||_{4s/n-2s}^2 + ||\varphi_2||_{4s/n-2s}^2)||\varphi_1 - \varphi_2||_2 \left( \gamma(x) \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |x - X_h|^{n+2s + \tau})} \right)^{n+2s\over n+2s + \tau}.
\]
We remind that in the last inequality, we have used the Hölder inequality. Hence $||N(\varphi_1) - N(\varphi_2)||_2 \leq C(||\varphi_1||_{4s/n-2s}^2 + ||\varphi_2||_{4s/n-2s}^2)||\varphi_1 - \varphi_2||_2$.

In the case of $N \leq 6s$, we also have $||N(\varphi_1) - N(\varphi_2)||_2 \leq C(||\varphi_1||_{4s/n-2s}^2 + ||\varphi_2||_{4s/n-2s}^2)||\varphi_1 - \varphi_2||_2$ by a similar argument.

Then there hold
\[
||A\varphi_1 - A\varphi_2||_2 \leq C||N(\varphi_1) - N(\varphi_2)||_2 \leq C(||\varphi_1||_{4s/n-2s}^2 + ||\varphi_2||_{4s/n-2s}^2)||\varphi_1 - \varphi_2||_2.
\]

For $\lambda$ large enough, we get $||A\varphi_1 - A\varphi_2||_2 \leq \frac{\lambda}{2}||\varphi_1 - \varphi_2||_2$.

3. Proof of the main theorem

Let $\Lambda := (\Lambda_1, \ldots, \Lambda_{(m+1)k}) \in \mathbb{R}^{(m+1)k}$ and $P := (P^1, \ldots, P^{(m+1)k}) \in \mathbb{R}^{n \times (m+1)k}$, in which $P^i = (P^i_1, \ldots, P^i_n) \in \mathbb{R}^n$ for $i = 1, 2, \ldots, (m+1)k$. We define $J(P, \Lambda) = I(W_m + \phi_m)$, where $\phi_m$ is a unique small solution obtained by Proposition 2.5. A standard argument shows that from a critical point of $J$, we can get a critical point of $I$ of the form $W_m + \phi_m$ (for example, cf. [14] Lemma 6.1 for ideas).

Proposition 3.1. For $\lambda$ large, we have the following expansions
\[
\frac{\partial J}{\partial P^i_j}(P, \Lambda) = -\frac{c_3 d_{ij}}{\Lambda_i^{\beta-2} \lambda^{\beta}} (P^i_j - X^j_i) + O \left( \frac{|P^i_j - X^i_j|^2}{\lambda^{\beta}} \right) + o(\lambda^{-\beta}),
\]
and
\[
\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = -\frac{c_1}{\Lambda_i^{\beta+1} \lambda^{\beta}} + \sum_{h \neq i} \frac{c_2}{\Lambda_i (\Lambda_i \Lambda_h)^{n-2s}} |X^i - X^h|^{n-2s} + O \left( \frac{|P^i - X^i|^\min\{2, \beta-1\}}{\lambda^{\beta}} \right) + o(\lambda^{-\beta}),
\]
where \( i = 1, \ldots, (m+1)^k \) and \( j = 1, \ldots, n \) and the constant \( c_1, c_2, c_3 \) are positive.

**Proof.** A simple calculation yields

\[
\frac{\partial J}{\partial P_j}(P, \Lambda) = \langle I'(W_m + \phi_m), \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} \rangle + \frac{\partial \phi_m}{\partial P_j}
\]

\[
= \frac{\partial I}{\partial P_j}(W_m) + \int_{\mathbb{R}^n} K(\frac{x}{\lambda})(W_m^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx
\]

\[
\quad + \sum_{t=1}^{(m+1)^k} \sum_{h=1}^{n+1} c_{th} \int_{\mathbb{R}^n} U_{P_j}^{P, \Lambda_j} Z_{t,h} \frac{\partial \phi_m}{\partial P_j} dx
\]

The functional \( \frac{\partial I}{\partial P_j}(W_m) \) is expanded in the Proposition \[3.6\]. So we only need to estimate the last two terms in the equality above.

From Lemma \[A.4\] we see that for \( \lambda \) large enough, \( \{ x : W_m \leq -\phi_m \} \subset \{ x : \frac{1}{2} W_m \leq |\phi_m| \} \subset \cup_h (\Omega_h \cap B_\delta_0^c) \). Then we have

\[
\int_{\mathbb{R}^n} K(\frac{x}{\lambda})(W_m + \phi_m)^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx
\]

\[
= \int_{\mathbb{R}^n} K(\frac{x}{\lambda})(W_m + \phi_m)^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx - \int_{|\phi_m| \geq \frac{1}{2} W_m} K(\frac{x}{\lambda})(W_m + \phi_m)^{\frac{n+2s}{n-2s}} \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx
\]

\[
= \frac{n + 2s}{n - 2s} \int_{\mathbb{R}^n} K(\frac{x}{\lambda}) W_m^{\frac{n+2s}{n-2s}} \phi_m \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx + O \left( \int_{|\phi_m| \geq \frac{1}{2} W_m} |\phi_m|^{\frac{n+2s}{n-2s}} U_{P_j}^{P, \Lambda_i} dx \right)
\]

\[
+ \int_{|\phi_m| < \frac{1}{2} W_m} W_m^{\frac{n+2s}{n-2s}} \phi_m^2 \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx + O \left( \int_{\cup_h (\Omega_h \cap B_\delta^c)} |\phi_m|^{\frac{n+2s}{n-2s}} U_{P_j}^{P, \Lambda_i} dx \right)
\]

Using Proposition \[2.5\], Lemma \[A.6\] and Lemma \[A.7\], we have

\[
\int_{\mathbb{R}^n} K(\frac{x}{\lambda})((W_m + \phi_m)^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P_j}^{P, \Lambda_i}}{\partial P_j} dx = O (\lambda^{-n}) = o (\lambda^{-\beta}).
\]

By using the orthogonal condition of \[2.15\] and Lemma \[A.1\], we have

\[
\sum_{t=1}^{(m+1)^k} \sum_{h=1}^{n+1} c_{th} \int_{\mathbb{R}^n} U_{P_j}^{P, \Lambda_j} Z_{t,h} \frac{\partial \phi_m}{\partial P_j} dx \leq \frac{C}{\lambda^{\frac{n}{2}}} \| \phi_m \|_2 \int_{\mathbb{R}^n} (1 + |x - X|^2)^{\frac{n+2s}{2}} \sum_{r=1}^{(m+1)^k} \frac{1}{(1 + |x - X^r|)^{\frac{n+2s}{2}}} dx \leq C \lambda^{-n}.
\]

Hence we can get \[3.1\]. The estimation \[3.2\] can be derived by the same procedure along with Proposition \[B.5\].
Proof of Theorem 1.2. From Proposition 3.1, we know that there exist bounded functions $\Xi_j = \Xi_j(P, \Lambda, \lambda)$ and $\Theta_j = \Theta_j(P, \Lambda, \lambda)$, $i = 1, \ldots, (m + 1)^k$, $j = 1, \ldots, n + 1$ satisfying $|\Xi_j| \leq C$, where $C$ is constant independent of $m$ and $|\Theta_j| \leq C\lambda$, where $C\lambda$ is a constant only depend on $\lambda$, and $C\lambda \to 0$ as $\lambda \to \infty$ such that
\[
\frac{\partial J}{\partial R^i_{p_j}}(P, \Lambda) = -\frac{c_3a_j}{\Lambda^{\beta-2}\lambda^\beta}(P^i_j - X^i_j) + \frac{|P^i_j - X^i_j|^2}{\lambda^\beta} \Xi_j + \lambda^{-\beta}\Theta_j,
\]
and
\[
\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = -\frac{c_1}{\Lambda_i^{\beta+1}\lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_iA_h)^{-2\beta}} |X^i - X^h|^{n-2s} = -\frac{|P^i_j - X^i_j|^{\min\{2, \beta-1\}}}{\lambda^\beta} \Xi_j^{n+1} + \lambda^{-\beta}\Theta_j^{n+1}.
\]

Remark 3.2. From Proposition 3.1, we know that there exist bounded functions $\Xi_j = \Xi_j(P, \Lambda, \lambda)$ and $\Theta_j = \Theta_j(P, \Lambda, \lambda)$, $i = 1, \ldots, (m + 1)^k$, $j = 1, \ldots, n + 1$ satisfying $|\Xi_j| \leq C$, where $C$ is constant independent of $m$ and $|\Theta_j| \leq C\lambda$, where $C\lambda$ is a constant only depend on $\lambda$, and $C\lambda \to 0$ as $\lambda \to \infty$ such that
\[
\frac{\partial J}{\partial R^i_{p_j}}(P, \Lambda) = -\frac{c_3a_j}{\Lambda^{\beta-2}\lambda^\beta}(P^i_j - X^i_j) + \frac{|P^i_j - X^i_j|^2}{\lambda^\beta} \Xi_j + \lambda^{-\beta}\Theta_j,
\]
and
\[
\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = -\frac{c_1}{\Lambda_i^{\beta+1}\lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_iA_h)^{-2\beta}} |X^i - X^h|^{n-2s} = -\frac{|P^i_j - X^i_j|^{\min\{2, \beta-1\}}}{\lambda^\beta} \Xi_j^{n+1} + \lambda^{-\beta}\Theta_j^{n+1}.
\]

Proof of Theorem 1.2. Firstly we look for the solution of (1.5) of the form $W_m + \phi_m$, $m < \infty$.

It is equivalent to solving the system
\[
\begin{cases}
\frac{\partial J}{\partial P^i_{p_j}}(P, \Lambda) = 0, \\
\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = 0,
\end{cases}
\]
that is
\[
\begin{cases}
\frac{c_3a_j}{\Lambda_i^{\beta-2}\lambda^\beta}(P^i_j - X^i_j) + \frac{|P^i_j - X^i_j|^2}{\lambda^\beta} \Xi_j + \lambda^{-\beta}\Theta_j, \\
-\frac{c_1}{\Lambda_i^{\beta+1}\lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_iA_h)^{-2\beta}} |X^i - X^h|^{n-2s} = -\frac{|P^i_j - X^i_j|^{\min\{2, \beta-1\}}}{\lambda^\beta} \Xi_j^{n+1} + \lambda^{-\beta}\Theta_j^{n+1}.
\end{cases}
\]

To simplify the equations (3.3), we denote $d_j = \Lambda_j^{-\frac{n-2s}{2}}$ and $A_{ih} = \left\{ \begin{array}{ll} 0, & \text{if } i = h, \\
\frac{\min\{2, \beta-1\}n-2s}{|X^i - X^h|^{n-2s}}, & \text{if } i \neq h. \end{array} \right.$

The equations (3.3) can be written as
\[
\begin{cases}
P^i_j - X^i_j = \frac{\Lambda_i^{\beta-2}\lambda}{c_3a_j} |P^i_j - X^i_j|^2 + \Lambda_i^{\beta-2}\Theta_j, \\
c_2 \sum_{h \neq i} A_{ih}d_h - c_1d_i^{\frac{2\beta}{n-2s}-1} = \Lambda_i^{\frac{n-2s}{2\beta}+1} \Xi_j^{n+1} + \Lambda_i^{\frac{n-2s}{2\beta}+1} \Theta_j^{n+1},
\end{cases}
\]
where $i = 1, 2, \ldots, (m + 1)^k$ and $j = 1, 2, \ldots, n$.

Define a function $F(z) := \frac{c_2}{2\beta} \sum_{h \neq i} A_{ih}z_h - \frac{(n-2s)c_1}{2\beta} \sum_{h} z_h^{\frac{2\beta}{n-2s}}$, where $z = (z_1, z_2, \ldots, z_{(m+1)^k}) \in \mathbb{R}^{(m+1)^k}$. Obviously, $F(z)$ has a maximum point $b = (b_1, b_2, \ldots, b_{(m+1)^k}) \in \mathbb{R}^{(m+1)^k}$. It holds that
\[
c_2 \sum_{h \neq i} A_{ih}b_h - c_1b_i^{\frac{2\beta}{n-2s}-1} = 0, \quad i = 1, \ldots, (m + 1)^k.
\]

Claim: Each component $b_i$ of $b$ satisfies $0 < C_1' \leq b_i \leq C_2'$ for some constant $C_1'$ and $C_2'$.

Suppose that $b_1 \leq b_i \leq b_2$. Using the definition of $A_{ih}$, we know $\sum_{h \neq i} A_{ih}$ is bounded. From (3.5), we can get
\[
c_1b_2^{\frac{2\beta}{n-2s}-1} = c_2 \sum_{h \neq 2} A_{2ih}b_h \leq C_3b_2,
\]
which tell us \( b_2 \) is bounded from above.

Using (3.5) again, we have

\[
c_1 b_1^{-\frac{2\beta}{n-2s}} = c_2 \sum_{h \neq 1} A_{1h} b_h \geq c_2 \sum_{h \neq 1} A_{1h} b_1 \geq c_2 A_{12} b_1,
\]

which implies \( b_1 \) is bounded from below, away from zero. Hence the Claim follows.

We can choose a small \( \delta_0 > 0 \) such that \( b_2^{-\frac{2}{n-2s}} - \delta_0 > 0 \). The constant \( C_1 \) and \( C_2 \) in the introduction can be defined by

\[
C_1 = b_2^{-\frac{2}{n-2s}} - \delta_0 \quad \text{and} \quad C_2 = b_1^{-\frac{2}{n-2s}} + \delta_0.
\]

(3.6)

For any \( x = (x_1, \ldots, x_{(m+1)k}) \in \mathbb{R}^{(m+1)k} \), we denote \( \|x\|_0 = \max\{|x_j|\} \). Let \( \frac{x_{j0}}{b_{j0}} = \|x\|_0 \). From the claim above, we know \( |x_{j0}| \geq C \|x\|_0 \). Using (3.5), we have

\[
\| (D^2 F(b)x)_{j0} \| = \bigg| c_2 \sum_{h \neq 0} A_{ih} h x_h - c_1 \bigg( \frac{2\beta}{n-2s} - 1 \bigg) b_i^{-\frac{2\beta}{n-2s}} - x_{i0} \bigg| \geq c_1 \bigg( \frac{2\beta}{n-2s} - 1 \bigg) b_i^{-\frac{2\beta}{n-2s}} - 2 |x_{j0}| \geq 0.
\]

From the definition of \( \| \cdot \|_0 \), we get \( \| D^2 F(b)x \|_0 \geq C \|x\|_0 \).

Let \( \theta = (\theta_1, \ldots, \theta_{(m+1)k}) \in \mathbb{R}^{(m+1)k} \) whose component \( \theta_i := d_i - b_i, \ i = 1, \ldots, (m+1)k \). We define \( X := (X_1, \ldots, X_{(m+1)k}) \in \mathbb{R}^{n \times (m+1)k} \), in which \( X_i := (X_{i1}, \ldots, X_{ik}) \in \mathbb{R}^n \) for \( i = 1, \ldots, (m+1)k \). For any \( Y = (Y_1, \ldots, Y_{(m+1)k}) \in \mathbb{R}^{n \times (m+1)k} \), we use the notation \( \|Y\| := \max_{i=1,\ldots,(m+1)k}\{|Y_i|\} \) to denote the maximum norm.

To simplify the equations (3.4), we need to define some vector value functions below. Let \( \Xi^{(1)} := \Xi^{(1)}(P, \lambda, \lambda) \in \mathbb{R}^{n \times (m+1)k} \) and \( \Theta^{(1)} := \Theta^{(1)}(P, \lambda, \lambda) \in \mathbb{R}^{n \times (m+1)k} \) with their exponents defined by

\[
(\Xi^{(1)})^i_j = \frac{\Lambda_i^{\beta-2} \Xi_j |P^i - X_j|^2}{c_3 a_j \|P - X\|^{2}} \quad \text{and} \quad (\Theta^{(1)})^i_j = \frac{\Lambda_i^{\beta-2} \Theta_j^i}{c_3 a_j} \quad i = 1, \ldots, (m+1)k; j = 1, \ldots, n.
\]

Let \( \Xi^{(2)} := \Xi^{(2)}(P, \lambda, \lambda) \in \mathbb{R}^{(m+1)k} \), \( \Theta^{(2)} := \Theta^{(2)}(P, \lambda, \lambda) \in \mathbb{R}^{(m+1)k} \) with exponents defined by

\[
(\Xi^{(2)})^i = \Lambda_i^{\frac{n-2s}{2} + 1} \Xi_i^{|P^i - X^i|^{\min\{2, \beta-1\}}} \quad \text{and} \quad (\Theta^{(2)})^i = \Lambda_i^{\frac{n-2s}{2} + 1} \Theta_i^{|P^i - X^i|^{\min\{2, \beta-1\}}} \quad i = 1, \ldots, (m+1)k.
\]

Define \( \Pi(\theta) := (\Pi(\theta), \ldots, \Pi(\theta)^{(m+1)k}) \), where \( \Pi(\theta)^{(i)}(i = 1, \ldots, (m+1)k) \) is defined by

\[
(\Pi(\theta)^{(i)} = \int_0^1 (D^3 F(b + s \theta) \theta; (1 - s) ds = \int_0^1 c_1 \bigg( \frac{2\beta}{n-2s} - 1 \bigg) \bigg( \frac{2\beta}{n-2s} - 2 \bigg) (b_i + s \theta_i)^{\frac{2\beta}{2-s} - 3} \theta_i^2.
\]

From their definition, we know there is a constant \( C \) and a constant \( C_\lambda \) satisfying \( C_\lambda \to 0 \) as \( \lambda \to \infty \) such that \( \| \Xi^{(1)} \| \leq C, \| \Xi^{(2)} \| \leq C; \| \Theta^{(1)} \| \leq C_\lambda \) and \( \| \Theta^{(2)} \| \leq C_\lambda \).
Using these notations and Taylor expansion, we can write the equations (3.4) into another form:

\[
\begin{align*}
\{ & P - X = \|P - X\|^2 \Xi^{(1)} + \Theta^{(1)}; \\
& D^2 F(b)\theta = \|P - X\|^{\min(2, \beta - 1)} \Xi^{(2)} + \Theta^{(2)} + \Pi(\theta),
\end{align*}
\]

(3.7)

Let
\[
B = \left( \prod_{i=1}^{(m+1)k} B_{2C_\lambda}(X^i) \right) \times B_{3C_\lambda^{-1}}(0) \in \mathbb{R}^{n \times (m+1)^k} \times \mathbb{R}^{(m+1)^k}.
\]

Define a function
\[
G : B \to B
\]

\[
(P, \theta) \mapsto (X + \Xi^{(1)} \|P - X\|^2 + \Theta^{(1)}, D^2 F(b)^{-1}(\|P - X\|^{\beta - 1} \Xi^{(2)} + \Theta^{(2)} + \Pi(\theta))
\]

For each \((P, \theta) \in B\), Choose \(C_\lambda\) small enough, we have
\[
\|\Xi^{(1)} \|P - X\|^2 + \Theta^{(1)}\| \leq C(2C_\lambda)^2 + C_\lambda \leq 2C_\lambda^2,
\]

and
\[
\|D^2 F(b)^{-1}(\|P - X\|^{\min(2, \beta - 1)} \Xi^{(2)} + \Pi(\theta))\|_0 \leq C_\lambda^{-1}(CC_\lambda^{\min(2, \beta - 1)} + C_\lambda + C\theta^2) \leq 3C_\lambda^{-1} C_\lambda.
\]

Since \(C_\lambda \to 0\) as \(\lambda \to \infty\), so for \(\lambda\) large enough, we use the Brouwer fixed-point theorem to get a solution \((P^1, \ldots, P^{(m+1)^k}, \theta)\) of (3.7) in \(B\). It holds that
\[
|P^i - X^i| \leq 2C_\lambda \quad \text{and} \quad |\theta_i| = |b_i - \Lambda_i^{n^2s \alpha}| \leq 3C_\lambda^{-1} C_\lambda.
\]

Hence we find a critical point of \(I\) of the form \(u_m := W_m + \phi_m\) with \(m < \infty\).

Next, we prove \(u_m\) is a positive function. Denote \(u_m = \min\{0, u_m\}\) and \(u_m^+ = u_m - u_m^-\). Then we have
\[
\int_{\mathbb{R}^n} (-\Delta)^s u_m(x)u_m^-(x)dx = \int_{\mathbb{R}^n} K_{\lambda^2}^{x \lambda}(x)u_m^{-\frac{n+2s}{n}}(x)u_m^-(x)dx.
\]

From the definition of \((-\Delta)^s\),
\[
\int_{\mathbb{R}^n} (-\Delta)^s u_m u_m^- dx = \int_{\mathbb{R}^n} (-\Delta)^s u_m^- u_m^- dx + \int_{\mathbb{R}^n} (-\Delta)^s u_m^+ u_m^- dx
\]
\[
= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_m^+(x) - u_m^+(y))u_m^-(x)}{|x - y|^{n+2s}} dxdy
\]
\[
\geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx.
\]

The Hardy-Littlewood-Sobolev inequality yields
\[
\left( \int_{\mathbb{R}^n} |u_m^ {-\frac{n}{n+2s}} dx \right)^{\frac{n+2s}{n}} \leq C \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx
\]
\[
\leq C \int_{\mathbb{R}^n} K_{\lambda^2}^{x \lambda}(x)u_m^{-\frac{n+2s}{n}}(x)u_m^-(x)dx \leq C \int_{\mathbb{R}^n} u_m^{-\frac{n}{n}} dx.
\]
Suppose $u_m^- \neq 0$, we have $\int_{\mathbb{R}^n} |u_m^-|^\frac{2n}{n-2s} dx \geq C$. It is easy to get $u_m^- \leq |\phi_m|$. From this fact,

\[
\int_{\mathbb{R}^n} |u_m^-|^\frac{2n}{n-2s} dx \leq \int_{\mathbb{R}^n} |\phi_m|^\frac{2n}{n-2s} dx
\]

\[
\leq \|\phi_m\|^\frac{2n}{n-2s} \int_{\mathbb{R}^n} \left( \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |x - X^h|)^\frac{n-2s}{n}} \right) dx
\]

\[
\leq \|\phi_m\|^\frac{2n}{n-2s} C(m, k) \sum_{h} \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X^h|)^{n+\frac{2n}{n-2s}}} dx.
\]

So we get $C \leq \int_{\mathbb{R}^n} |u_m^-|^\frac{2n}{n-2s} dx \leq C(m, k)\|\phi_m\| \to 0$ as $\lambda \to \infty$, which is impossible. Hence $u_m \geq 0$. Suppose there is a point $x_0$ such that $u_m(x_0) = 0$, then

\[
0 = K^{\frac{1}{1}}(\frac{x_0}{\lambda}) u_{m, n}^\frac{n}{n-2s} (x_0) = (-\Delta)^s u_m(x_0) = P.V. \int_{\mathbb{R}^n} \frac{u_m(x_0) - u_m(y)}{|x_0 - y|^{n+2s}} dy = P.V. \int_{\mathbb{R}^n} \frac{-u_m(y)}{|x_0 - y|^{n+2s}} dy.
\]

Then $u_m \equiv 0$ which is impossible. Hence $u_m > 0$.

According to Proposition 2.5, $u_m = W_m + \phi_m \in C^{\alpha, \alpha}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$. Using local Schauder estimate [13 Proposition 2.11] and a bootstrap argument, we know $u_m \in C^{2, \alpha'}(\mathbb{R}^n)$, for some $\alpha' \in (0, 1)$.

What is more, $|u_m|_{L^\infty(\mathbb{R}^n)} \leq C$ with $C$ independent of $m$. In fact, Choosing $x \in \Omega_1$ with no loss of generality, we have

\[
W_m(x) \leq \sum_{i=1}^{(m+1)k} \frac{C}{(1 + |x - X^i|)^{n-2s}} \leq C + \sum_{i \neq 1}^{(m+1)k} \frac{C}{|X^i - X^1|^{n-2s}} \leq C + \frac{C}{(\lambda l)^{n-2s}} \leq C,
\]

and

\[
|\phi_m| \leq \|\phi_m\| \sum_{h=1}^{(m+1)k} \frac{1}{(1 + |x - X^h|)^\frac{2n}{n-2s}} \leq \|\phi_m\| \sum_{h=1}^{\infty} \frac{1}{(1 + |x - X^h|)^\frac{2n}{n-2s}} \leq C \|\phi_m\| \leq C.
\]

Since $\phi_m$ satisfies the equation $(-\Delta)^s \phi_m - \frac{n+2s}{n-2s}K(x) W_m^{\frac{4s}{n-2s}} \phi_m = N(\phi_m) + l_m$, then from Lemma [18 Lemma 2.4] and Proposition 2.5, we know that for any $x, y \in \mathbb{R}^n$ with $x \neq y$, there holds

\[
\frac{|\phi_m(x) - \phi_m(y)|}{|x - y|^{\alpha}} \leq \frac{C}{\lambda^\alpha} \max\{\|\phi_m\|, \|N(\phi_m)\|, \|l_m\|\} \leq C, \text{ where } \alpha = \min\{1, 2s\}.
\]

Also from simple calculation, we get for any $x \in \mathbb{R}^n$ and $R > 0$, $\|W_m\|_{C^{0, \alpha}(B_{2R}(x))} \leq C(n, R)$, where $C(n, R)$ is a constant independent of $m$. Hence $\|u_m\|_{C^{0, \alpha}(B_R(x))} \leq C(n, R)$. Local Schauder estimate and a bootstrap argument yields that $\|u_m\|_{C^{2, \alpha'}(B_R(x))} \leq C(n, R)$. Thanks to Azellà-Ascoli theorem, we have $u_m$ convergent uniformly to a $C^{2, \alpha'}_{loc}$ function $u_\infty = W_\infty + \phi_\infty$ on compact sets as $m \to \infty$. We know $u_\infty$ satisfies $|u_\infty|_{L^\infty(\mathbb{R}^n)} \leq C$ and $|u_\infty|_{C^{2, \alpha'}(B_R(x))} \leq C(n, R)$.

We will show that $u_\infty$ satisfies the equation (1.5). Let $v_m = u_m - u_\infty$. From above, we know $v_m$ has the property $|v_m|_{L^\infty(\mathbb{R}^n)} \leq C; |v_m|_{C^{2}(B_1(x))} \leq C$ and $v_m \to 0$ uniformly on compact sets.
From the definition of \((-\Delta)^s\), we have for any \(x \in \mathbb{R}^n\)
\[
C(n, s)^{-1}|(-\Delta)^s v_m(x)| \leq PV \int_{\mathbb{R}^n} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy
\]
\[
= PV \int_{B_{\varepsilon_0}(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy + \int_{B_R(x) \setminus B_{\varepsilon_0}(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy
\]
\[
+ \int_{\mathbb{R}^n \setminus B_R(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy =: T_1 + T_2 + T_3.
\]
For the term \(T_1\), we have
\[
T_1 = \frac{1}{2} PV \int_{B_{\varepsilon_0}(0)} \frac{|v_m(x + y) + v_m(x - y) - 2v_m(x)|}{|y|^{n+2s}} dy
\]
\[
\leq C|v_m|_{C^2 B_1(x)} \int_{B_{\varepsilon_0}(0)} |y|^{2-2s-n} dy \leq C|v_m|_{C^2 B_1(x)} \varepsilon_0^{2-2s} \to 0 \text{ as } \varepsilon_0 \to 0.
\]
For the third term,
\[
T_3 \leq C \int_{\mathbb{R}^n \setminus B_R(x)} \frac{1}{|x - y|^{n+2s}} = CR^{-2s} \to 0 \text{ as } R \to \infty.
\]
Then we estimate the term \(T_2\). For fixed \(R\) large enough and \(\varepsilon_0\) small enough, \(B_R(x) \setminus B_{\varepsilon_0}(x)\) is a compact set. So we have \(T_2 \to 0\) as \(m \to \infty\). Hence \((-\Delta)^s u_m(x) \to (-\Delta)^s u_\infty(x)\) as \(m \to \infty\). Therefore \(u_\infty\) satisfies equation \((1.5)\).

**Proof of Corollary 1.3** Fix the constant \(m < \infty\). Using a similar argument as in [29], we can expand \(I(W_m + \phi_m)\) as
\[
I(W_m + \phi_m) = (m + 1)^k \left( \frac{s}{n} \int U_{0.1}^{0.2s} + o(1) \right), \text{ as } l \to \infty.
\]
For each \(m < \infty\), \(I(W_m + \phi_m) \to (m + 1)^k \frac{s}{n} \int U_{0.1}^{0.2s} \) as \(l \to \infty\). For any \(m_1, m_2 \in \mathbb{N}_+\) such that \(m_1 \neq m_2\), we can find two solutions \(W_{m_1} + \phi_{m_1}\) and \(W_{m_2} + \phi_{m_2}\) of \((1.5)\), such that \(I(W_{m_1} + \phi_{m_1}) \neq I(W_{m_2} + \phi_{m_2})\). Hence we can find infinitely many solutions of \((1.5)\).

**Appendix A. Basic Estimates**

**Lemma A.1.** (cf. [27,29]) For any \(x_i, x_j, y \in \mathbb{R}^n\) and constant \(\sigma \in [0, \min\{\alpha, \beta\}]\), we have
\[
\frac{1}{(1 + |y - x_i|)^\alpha (1 + |y - x_j|)^\beta} \leq \frac{2^\sigma}{(1 + |x_i - x_j|)^\sigma} \left( \frac{1}{(1 + |y - x_i|)^{\alpha + \beta - \sigma}} + \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} \right).
\]

**Lemma A.2.** For any \(\sigma > 0\) with \(\sigma \neq n - 2s\), there is a constant \(C > 0\) such that
\[
\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \frac{1}{(1 + |z|)^{2s+\sigma}} dz \leq \frac{C}{(1 + |y|)^{\min\{\sigma, n-2s\}}}.
\]
For \( \sigma = n - 2s \), there is also a constant \( C > 0 \), such that
\[
\int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} \frac{1}{(1+|z|)^n} \leq C \max(1, \log |y|) / (1+|y|)^{n-2s}
\]

**Proof.** The proof follows from the same argument as [21] Lemma A.2. See also [29], Lemma B.2. \( \square \)

Recall that \( X^i \in X_{i,m} = \{ X^i \}_{i=1}^{(m+1)k}, B_i = B_M(X^i) \) and \( B_{i,m} = B_{\max\{m+1\}M}(X^i) \).

**Lemma A.3.** (cf. [21]) For any \( \theta > k \), there exists a constant \( C(\theta, k, n) > 1 \) independent of \( m \), such that if \( y \in B_i \cap \Omega_i \), there holds
\[
\frac{1}{(1+|y-X^i|)^\theta} \leq \sum_j \frac{1}{(1+|y-X^j|)^\theta} \leq \frac{C}{(1+|y-X^i|)^\theta}.
\]

If \( y \in B_i^c \cap B_{i,m} \cap \Omega_i \), there holds
\[
\frac{1}{C(1+|y-X^i|)^{\theta-k}(\lambda)^k} \leq \sum_j \frac{1}{(1+|y-X^j|)^\theta} \leq \frac{C}{(1+|y-X^i|)^{\theta-k}(\lambda)^k} \tag{A.1}
\]
and if \( y \in B_{i,m}^c \cap \Omega_i \), there holds
\[
\frac{m^k}{C(1+|y-X^i|)^\theta} \leq \sum_j \frac{1}{(1+|y-X^j|)^\theta} \leq \frac{Cm^k}{(1+|y-X^i|)^\theta} \leq \frac{C}{(1+|y-X^i|)^{\theta-k}(\lambda)^k}.
\]

**Lemma A.4.** Let \( n > 2s + 2 \) and \( 0 < \tau < \frac{n+2s}{2} \). If \( \phi \) satisfies \( \| \phi \|_* \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \), then for any \( c > 0 \), there exists \( \lambda_0 > 0 \) such that for any \( \lambda > \lambda_0 \), there holds \( |\phi| \leq cW_m \) in \( \cup_{h}(\Omega_h \cap B_h) \).

**Proof.** We prove this lemma indirectly. Suppose that there exists \( c_0 > 0 \), such that for any \( \lambda_0 > 0 \), there is a \( \lambda > \lambda_0 \) and \( y \in \cup_{h}(\Omega_h \cap B_h) \) such that \( |\phi(y)| \geq c_0W_m(y) \). Then
\[
|\phi(y)| \geq C \sum_{h=1}^{(m+1)k} \frac{1}{(1+|y-X^h|)^{n-2s}} \geq C \gamma(y) \sum_{h=1}^{(m+1)k} \frac{1}{(1+|y-X^h|)^{\theta\frac{n+2s}{2}+\tau}} \frac{1}{(\lambda)^{\frac{n-2s}{2}-\tau}}
\]
If \( 0 < \tau < \frac{n-2s}{2} \), we have
\[
\frac{1}{\lambda^{\frac{n-2s}{2}-\tau}} \geq \| \phi \|_* \geq \frac{C}{(\lambda)^{\frac{n-2s}{2}-\tau}},
\]
which does not hold for \( \lambda \) large enough.

If \( \frac{n-2s}{2} \leq \tau < \frac{n+2s}{2} \), we can also get
\[
\frac{1}{\lambda^{\frac{n-2s}{2}-\tau}} \geq \| \phi \|_* \geq c,
\]
which also is a contradiction for \( \lambda \) large. \( \square \)
Lemma A.5. For $n > 2s + 2$ and $1 \leq k < \frac{n - 2s}{2}$, we have
\[
\int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2s}} W_{m, \Lambda}^{4s} (y) \gamma(y) \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + r}} dy \\
\leq C \left( \gamma(x) \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + r + \theta}} + \frac{1}{(\lambda)^{\frac{n-2s}{2}k}} \gamma(x) \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + r}} \right),
\]
where $\theta > 0$ is a small constant and $C > 0$ does not depend on $m$.

Proof. Without loss of generality, we assume $x \in \Omega_1$. We write
\[
\int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2s}} W_m^{4s} (y) \gamma(y) \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + r}} dy \\
= \left( \int_{\cup_h (\Omega_h \cap B_h)} + \int_{\cup_h (\Omega_h \cap B_h^c \cap B_h, m)} + \int_{\cup_h (\Omega_h \cap B_h^c, m)} \right) \frac{1}{|x - y|^{n-2s}} W_m^{4s} (y) \\
\times \gamma(y) \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + r}} dy \\
=: T_1 + T_2 + T_3.
\]
We now estimate each term $T_i$ ($i = 1, 2, 3$).

Using Lemma [A.2] and Lemma [A.3], we have
\[
T_1 \leq C \int_{\cup_h (\Omega_h \cap B_h)} \frac{1}{|x - y|^{n-2s}} \sum_h \frac{1}{(1 + |y - X^h|)^{4s + \frac{n-2s}{2} + r}} dy \\
\leq C \sum_h \frac{1}{(1 + |x - X^h|)^{\min(4s + \frac{n-2s}{2} + r, n-2s)}} \\
= C \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + r + \theta_1}}, \quad \text{where } \theta_1 = \min\{2s, \frac{n-2s}{2} - \tau \}. \quad (A.2)
\]
Similarly, we also obtain
\[
T_1 \leq C \int_{\cup_h (\Omega_h \cap B_h)} \frac{1}{|x - y|^{n-2s}} \frac{1}{\lambda^{\tau-s}} \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + 4s}} dy \\
\leq C \frac{1}{\lambda^{\tau-s}} \sum_h \frac{1}{(1 + |x - X^h|)^{\min(\frac{n-2s}{2} + 4s, n-2s)}} \\
\leq C \frac{1}{\lambda^{\tau-s}} \sum_h \frac{1}{(1 + |x - X^h|)^{\min(n-3s+2s, \frac{n}{2} + \tau + s)}} \\
= C \frac{1}{\lambda^{\tau-s}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + \tau + \theta_2}}, \quad (A.3)
\]
where $\theta_2 = \min\{\frac{n-4s}{2}, 2s\}$.

Combining the estimate [A.2] and [A.3], we have
\[
T_1 \leq C \gamma(x) \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + r + \theta}},
\]
Thus, this lemma follows. By the same procedure, we have

\[
T_2 \leq \frac{C}{(\lambda) \frac{4s}{n-2s} k} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} \sum_h \frac{1}{1 + |y - X^h|} \frac{1}{\lambda^{-2s} + 4s + \sigma + \frac{4s}{n-2s} x y dy} \\
\leq \frac{C}{(\lambda) \frac{4s}{n-2s} k} \sum_h \frac{1}{(1 + |x - X^h|) \min\{n-2s, \lambda^{-2s} + 4s + \frac{4s}{n-2s} x y dy}.
\]

and

\[
T_2 \leq \frac{C}{\lambda^{\tau-s}(\lambda) \frac{4s}{n-2s} k} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} \sum_h \frac{1}{1 + |y - X^h|} \frac{1}{\lambda^{-2s} + 4s + \sigma + \frac{4s}{n-2s} x y dy} \\
\leq \frac{C}{\lambda^{\tau-s}(\lambda) \frac{4s}{n-2s} k} \sum_h \frac{1}{(1 + |x - X^h|) \lambda^{\tau-s} \min\{n-2s, \lambda^{-2s} + 4s + \frac{4s}{n-2s} x y dy}.
\]

Thus

\[
T_2 \leq \frac{C}{(\lambda) \frac{4s}{n-2s} k} \gamma(x) \sum_h \frac{1}{(1 + |x - X^h|) \frac{\lambda^{-2s} + 4s + \frac{4s}{n-2s} x y dy}}.
\]

By the same procedure, we have

\[
T_3 \leq \frac{C}{(\lambda) \frac{4s}{n-2s} k} \gamma(x) \sum_h \frac{1}{(1 + |x - X^h|) \frac{\lambda^{-2s} + 4s + \frac{4s}{n-2s} x y dy}}.
\]

Hence this lemma follows.

Remember \( Z_{i,j} = \frac{\partial U^{p_i, A_i}}{\partial P_j} \) for \( j = 1, 2, \ldots, n \) and \( Z_{i,n+1} = \frac{\partial U^{p_i, A_i}}{\partial A_i} \).

**Lemma A.6.** For \( t = 1, 2, \ldots, n + 1 \), we have

\[
\left| \int_{\mathbb{R}^n} K(x) W_{m}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq \frac{C \lambda^{\tau-s}}{\lambda^{\tau-s}(\lambda)^{\frac{4s}{2n-2s}}} \left| \phi \right| \frac{\sigma + \frac{4s}{n-2s} x y dy}.
\]

**Proof.** In the proof of this lemma, we denote \( \hat{W}_{m,r} = \sum_{h \neq r} U^{p_h, A_h} \). It is easy to get

\[
\int_{\mathbb{R}^n} K(x) W_{m}^{\frac{4s}{n-2s}} Z_{r,t} \phi = \int_{\mathbb{R}^n} K(x) U^{\frac{4s}{n-2s}} Z_{r,t} \phi + O \left( \int_{W_{m,r} > U^{p_r, A_r}} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right) (A.4)
\]

\[ + O \left( \int_{W_{m,r} \leq U^{p_r, A_r}} U^{\frac{4s}{n-2s}} \hat{W}_{m,r} \phi \right).
\]

We need to estimate each term in the equality above.
For $i \neq r$, from Lemma A.3 we have
\[
\left| \int_{\Omega \cap B_i} \hat{W}_{m,r}^{4s} Z_{r,t} \phi \right| \leq C \| \phi \|_{\lambda^{r-s}} \int_{\Omega \cap B_i} \left( \sum_{h \neq r} \frac{1}{1 + |x - X^h|^{n-2s}} \right)^{\frac{4s}{n-2s}} \frac{1}{(1 + |x - X^r|^{n-2s})} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} \, dx
\]
\[
\leq C \| \phi \|_{\lambda^{r-s}} \int_{\Omega \cap B_i} \frac{1}{(1 + |x - X^r|^{n-2s})} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2} + 4s + r - \frac{4s}{n-2s}}} \, dx \leq C \frac{\| \phi \|_{\lambda^{r-s}}}{\lambda^{r-s}(X^r - X^l)^2}
\]

With the help of Lemma A.1 and Lemma A.3 we get
\[
\left| \int_{\bigcup (\Omega \cap B_k)} \hat{W}_{m,r}^{4s} Z_{r,t} \phi \right| \leq C \| \phi \|_{\lambda^{r-s}} \int_{\bigcup (\Omega \cap B_k)} \frac{1}{(1 + |x - X^r|^{n-2s})} \left( \sum_{h \neq r} \frac{1}{(1 + |x - X^h|)^{n-2s}} \right)^{\frac{4s}{n-2s}} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2} + 3s + r - \frac{4s}{n-2s}}} \, dx
\]
\[
\leq C \| \phi \|_{\lambda^{r-s}} \frac{(\lambda)^{\frac{n-2s}{2} + s + r}}{\lambda^{r-s}(X^r - X^l)^2}
\]

Since in $\Omega \cap B_r$, there holds $\hat{W}_{m,r} \leq \sum_{j \neq r} \frac{C}{|X^j - X^r|^{n-2s}} \leq \frac{C}{(\lambda)^{n-2s}}$. Then if $n \geq 6s$, we have
\[
\left| \int_{\Omega \cap B_r} \hat{W}_{m,r}^{4s} Z_{r,t} \phi \right| \leq C \int_{\Omega \cap B_r} \hat{W}_{m,r}^{4s} Z_{r,t} \phi \leq C \| \phi \|_{\lambda^{r-s}} \frac{1}{\lambda^{r-s}(\lambda)^{\frac{n-2s}{2}}} \int_{\Omega \cap B_r} \frac{1}{(1 + |x - X^r|^{n-2s})} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} \, dx
\]
\[
\leq C \| \phi \|_{\lambda^{r-s}} \frac{\lambda^{r-s}(\lambda)^{\frac{n-2s}{2}}}{\lambda^{r-s}(\lambda)^{n-2s}}
\]

And if $n < 6s$, we get $\frac{n+2s}{2} < 4s$. In this case
\[
\left| \int_{\Omega \cap B_r} \hat{W}_{m,r}^{4s} Z_{r,t} \phi \right| \leq C \| \phi \|_{\lambda^{r-s}(\lambda)^{4s}} \int_{\Omega \cap B_r} \frac{1}{(1 + |x - X^r|^{n-2s})} \sum_{h} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} \, dx
\]
\[
\leq C \frac{\| \phi \|_{\lambda^{r-s}(\lambda)^{\frac{n-2s}{2}}}}{\lambda^{r-s}(\lambda)^{\frac{n-2s}{2}}}
\]
From these arguments above, we arrive

\[
\left| \int_{W_{m,r} > U_{pr,Ar}} W_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq C \frac{\|\phi\|*}{\lambda^{T-s}(\lambda l)^{T}}.
\] (A.6)

By a similar procedure, we get

\[
\left| \int_{W_{m,r} \leq U_{pr,Ar}} U_{pr,Ar}^{\frac{4s}{n-2s}} W_{m,r} \phi \right| \leq C \frac{\|\phi\|*}{\lambda^{T-s}(\lambda l)^{T}}.
\] (A.7)

Now we estimate the first term on the right hand side of the equality (A.4). Since \( \phi \) satisfies the second equality in (2.10), we have

\[
\left| \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{pr,Ar}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| = \left| \int_{\mathbb{R}^n} \left( K\left(\frac{x}{\lambda}\right) - 1 \right) U_{pr,Ar}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right|
\]

\[
\leq \frac{\|\phi\|*}{\lambda^{T-s}} \int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{pr,Ar}^{\frac{4s}{n-2s}} \sum_{h \neq r} \frac{1}{1 + |x - X_h|^{\frac{n}{2}}} \, dx.
\]

On one hand, Lemma A.1 implies that

\[
\int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{pr,Ar}^{\frac{4s}{n-2s}} \sum_{h \neq r} \frac{1}{1 + |x - X_h|^{\frac{n}{2}}} \, dx
\]

\[
\leq C \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X_r|)^{n+2s}} \sum_{h \neq r} \frac{1}{1 + |x - X_h|^{\frac{n}{2}}} \, dx
\]

\[
\leq \frac{C}{(\lambda l)^{T}}.
\]

On the another hand, choose \( \delta \) to be a fixed constant small enough,

\[
\int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{pr,Ar}^{\frac{4s}{n-2s}} \frac{1}{(1 + |x - X_r|)^{n}}\]

\[
= \int_{|x - X_r| \leq \delta \lambda} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{pr,Ar}^{\frac{4s}{n-2s}} \frac{1}{(1 + |x - X_r|)^{n}}
\]

\[
+ \int_{|x - X_r| > \delta \lambda} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{pr,Ar}^{\frac{4s}{n-2s}} \frac{1}{(1 + |x - X_r|)^{n}} =: J_1 + J_2.
\]

From the condition \((H_3)\), we have

\[
|J_1| \leq \frac{C}{\lambda^\beta} \int_{|x - X_r| \leq \delta \lambda} \frac{|x - X_r|^\beta}{(1 + |x - X_r|)^{n+2s+\frac{\beta}{2}}} \leq \begin{cases} \frac{C \log \lambda}{\lambda^{\frac{n+4s}{2}}}, & \text{if } \beta \geq \frac{n+4s}{2}, \\ \frac{C}{\lambda^\beta}, & \text{if } \beta < \frac{n+4s}{2}. \end{cases}
\]
For the term $J_2$, a direct calculation yields

$$J_2 = \int_{|x-X^*| > \delta \lambda} |K\left(\frac{x}{\lambda}\right) - 1| \frac{U_{Pr,L_\lambda}^{n+2s}}{(1+|x-X^*|)^{\frac{n+2s}{2}}}$$

$$\leq \int_{|x-X^*| > \delta \lambda} \frac{1}{(1+|x-X^*|)^{n+2s+\frac{n}{2}}}$$

$$\leq \frac{C}{\lambda^{n+2s}}.$$  

Since $\min\{\beta, \frac{n+4s}{2}\} > \frac{n}{2} \frac{n}{n-2s}$, the definition of $\lambda$ implies

$$\int_{\mathbb{R}^n} |K\left(\frac{x}{\lambda}\right) - 1| \frac{U_{Pr,L_\lambda}^{n+2s}}{(1+|x-X^*|)^{\frac{n+2s}{2}}} \leq \frac{C}{(\lambda \lambda)^{\frac{n+2s}{2}}}.$$  

Hence we obtain

$$\left| \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{Pr,L_\lambda}^{n+2s} Z_r,t \phi \right| \leq C \frac{\|\phi\|_2}{\lambda^{n-s}(\lambda \lambda)^{\frac{n+2s}{2}}}.$$  

(A.8)

Putting (A.6), (A.7) and (A.8) into (A.4), we get this lemma.

\[\square\]

**Lemma A.7.** It holds that

$$\int_{\bigcup_k (\Omega_k \cap B_k^i)} |\phi| \frac{n+2s}{n-2s} U_{P^i,L_i} \leq C \frac{\|\phi\|_2^{n+2s}}{(\lambda \lambda)^{n+2s}},$$

(A.9)

and

$$\left| \int_{\mathbb{R}^n} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i,L_i} \right| \leq C \frac{\|\phi\|_2^2}{\lambda^{2(\tau+s)}}.$$  

(A.10)

**Proof.** The estimate (A.9) follows by the same method as in (A.5). So we only prove the estimation (A.10).

Using the same trick as in (A.5), we have

$$\left| \int_{\bigcup_k (\Omega_k \cap B_k^i)} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i,L_i} \right| \leq C \frac{\|\phi\|_2^2}{\lambda^{2(\tau+s)}}.$$  

(A.11)

According to Lemma [A.3] we have for $t \neq i$,

$$\left| \int_{\Omega_t \cap B_t} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i,L_i} \right| \leq C \frac{\|\phi\|_2^2}{(\lambda \lambda)^{2(\tau-s)}} \int_{\Omega_t \cap B_t} \frac{1}{(1+|x-X^i|)^{\frac{n}{n-2s}}} d\lambda.$$  

If $n \geq 6s$

$$\int_{\Omega_t \cap B_t} \frac{1}{(1+|x-X^i|)^{\frac{n}{n-2s}}} d\lambda \leq \frac{(\lambda \lambda)^{n-6s} \log(\lambda)}{|X^i - X^t|^{n-2s}}.$$  

If otherwise, $n < 6s$, there holds

$$\int_{\Omega_t \cap B_t} \frac{1}{(1+|x-X^i|)^{\frac{n}{n-2s}}} d\lambda \leq \frac{1}{|X^i - X^t|^{n-2s}}.$$  

Hence

$$\sum_{t \neq i} \left| \int_{\Omega_t \cap B_t} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i,L_i} \right| \leq C \frac{\|\phi\|_2^2}{(\lambda \lambda)^{2(\tau-s)}} C \log(\lambda) \frac{C \log(\lambda)}{\lambda^{\min\{n-2s,4s\}}}.$$  

(A.12)
Using Lemma A.3, we also have

\[
\left| \int_{\Omega_i \cap B_i} W_m^{\frac{6s-n}{2s}} \phi^2 U_{P_i, \Lambda_i} \right| \leq C \left\| \phi \right\|_{\infty}^2 \int_{\Omega_i \cap B_i} \frac{1}{(1 + |y - X_i|)^{n+4s}}. \tag{A.13}
\]

So we obtain the estimate (A.10) from (A.11), (A.12) and (A.13).

Lemma A.8. If \( \phi \) is the solution of the equation

\[
(-\Delta)^s \phi(x) - \frac{n + 2s}{n - 2s} K(x) W_m^{\frac{4s}{n-2s}}(x) \phi(x) = g(x), \tag{A.14}
\]

satisfying \( \|\phi\|_\infty < +\infty \), then we have

\[
\sup_{x_1 \neq x_2} \frac{|(\lambda^\tau - s) \phi(x_1) - (\lambda^\tau - s) \phi(x_2)|}{|x_1 - x_2|^\alpha} \leq C \max\{\|\phi\|_\infty, \|g\|_\infty\},
\]

where \( \alpha = \min\{2s, 1\} \) and the constant \( C \) does not depend on \( \lambda \) and \( m \).

Proof. Since \( |(\lambda^\tau - s) \phi(x)| \leq \|\phi\|_\infty \sum_h \frac{1}{(1 + |x - X^h|)^\frac{n}{2}} \leq C \|\phi\|_\infty \), we can assume \( |x_1 - x_2| \leq \frac{1}{3} \) with no loss of generality. Using the Green function of \(-\Delta)^s\) (see [5]), we can write (A.14) into the following form

\[
\phi(x) = C \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2s}} \left( \frac{n + 2s}{n - 2s} K\left(\frac{y}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y) \phi(y) + g(y) \right) dy,
\]

Then we get

\[
|\phi(x_1) - \phi(x_2)| \leq C \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right) K\left(\frac{y}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y) \phi(y) dy \right| \\
+ C \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right) g(y) dy \right| \\
=: C(H_1 + H_2).
\]
Using the definition of the norm $\| \cdot \|_*$, there hold

$$
|H_1| \leq \| \phi \|_* \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s} - \frac{1}{|y|^{n-2s}}} \right) K\left(\frac{y + x_2}{\lambda}\right) W_m^{4s}(y + x_2) \times 

\times \gamma(y + x_2) \sum_h \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-2s}{2} + \tau}} dy \right| 

= \| \phi \|_* \left\{ \int_{|y| \leq 3|x_1 - x_2|} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s} - \frac{1}{|y|^{n-2s}}} \right) K\left(\frac{y + x_2}{\lambda}\right) W_m^{4s}(y + x_2) \times 

\times \gamma(y + x_2) \sum_h \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-2s}{2} + \tau}} dy 

+ \int_{|y| \geq 3|x_1 - x_2|} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s} - \frac{1}{|y|^{n-2s}}} \right) K\left(\frac{y + x_2}{\lambda}\right) W_m^{4s}(y + x_2) \times 

\times \gamma(y + x_2) \sum_h \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-2s}{2} + \tau}} dy \right\} 

=: \| \phi \|_*(K_1 + K_2).
$$

For the term $K_1$, we have

$$
|K_1| \leq \frac{C}{\lambda^{1-s}} \int_{|y| \leq 3|x_1 - x_2|} \frac{1}{|y|^{n-2s}} \leq \frac{C}{\lambda^{1-s}} |x_1 - x_2|^{2s}.
$$

For the term $K_2$, we have

$$
|K_2| \leq C|x_1 - x_2| \int_0^1 dt \int_{|y| \geq 3|x_1 - x_2|} \frac{1}{|t(x_1 - x_2) - y|^{n-2s+1}} W_m^{4s}(y + x_2) \times 

\times \gamma(y + x_2) \sum_h \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-2s}{2} + \tau}} dy 

= \frac{C|x_1 - x_2|}{\lambda^{1-s}} \left( \int_{1 > |y| \geq 3|x_1 - x_2|} + \int_{|y| \geq 1} \right) \frac{1}{|t(x_1 - x_2) - y|^{n-2s+1}} W_m^{4s}(y + x_2) \times 

\times \gamma(y + x_2) \sum_h \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-2s}{2} + \tau}} dy 

=: \frac{C|x_1 - x_2|}{\lambda^{1-s}} (M_1 + M_2).
$$

Since it holds that $|t(x_1 - x_2) - y| \in [2|x_1 - x_2|, \frac{4}{3})$ for $1 > |y| \geq 3|x_1 - x_2|$, we get

$$
|M_1| \leq \frac{1}{\lambda^{1-s}} \int_{2|x_1 - x_2| \leq |y| \leq \frac{4}{3}} \frac{1}{|y|^{n-2s+1}} \leq \frac{1}{\lambda^{1-s}} (C + C|x_1 - x_2|^{2s-1}).
$$
For $|y| \geq 1$, we have $\frac{2}{3} |y| \leq |t(x_1 - x_2) - y| \leq 4\frac{1}{3} |y|$. Lemma A.5 yields

\[
M_2 \leq C \int_{|y| \geq 1} \frac{1}{|y|^{n-2s+1}} W_m^{\frac{4s}{n-2s}} (y + x_2) \gamma(y + x_2) \sum_\lambda \left( \frac{1}{(1 + |y + x_2 - X^h|)^{\frac{n-s}{2}}} \right) dy \\
\leq C \int_{\mathbb{R}^n} \frac{1}{|y - x_2|^{n-2s}} W_m^{\frac{4s}{n-2s}} (y) \gamma(y) \sum_\lambda \left( \frac{1}{(1 + |y - X^h|)^{\frac{n-s}{2}}} \right) dy \\
\leq \frac{C}{\lambda^{n-s}}.
\]

Hence $|H_1| \leq \frac{C}{\lambda^{n-s}} \| \phi \| \cdot |x_1 - x_2|^\alpha$. The same procedure with the help of Lemma A.2 yields that $|H_2| \leq \frac{C}{\lambda^{n-s}} \| g \| \cdot |x_1 - x_2|^\alpha$. Then Lemma A.8 follows.

\[\square\]

**Appendix B. Expansions of the functionals $\frac{\partial}{\partial \Lambda_i} I(W_m)$ and $\frac{\partial}{\partial P^i_j} I(W_m)$**

In this section, we will expand the functionals $\frac{\partial}{\partial \Lambda_i} I(W_m)$ and $\frac{\partial}{\partial P^i_j} I(W_m)$. A direct computation yields

\[
\frac{\partial}{\partial \Lambda_i} I(W_m) = \int_{\mathbb{R}^n} W_m(-\Delta)^s \frac{\partial W_m}{\partial \Lambda_i} - \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial W_m}{\partial \Lambda_i} \\
= \int_{\mathbb{R}^n} \sum_{h=1}^{(m+1)\lambda} U_{P^k, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_h}}{\partial \Lambda_i} - \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_i}}{\partial \Lambda_i}, \quad (B.1)
\]

and

\[
\frac{\partial}{\partial P^i_j} I(W_m) = \int_{\mathbb{R}^n} W_m(-\Delta)^s \frac{\partial W_m}{\partial P^i_j} - \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial W_m}{\partial P^i_j} \\
= \int_{\mathbb{R}^n} \sum_{h \neq i} U_{P^k, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_h}}{\partial P^i_j} - \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_i}}{\partial P^i_j}. \quad (B.2)
\]

In order to get the useful expansions, we need to estimate each term on the right hand side of (B.1) and (B.2) above.

**Lemma B.1.** There holds

\[
\int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) \sum_{h} U_{P^k, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_i}}{\partial \Lambda_i} \\
+ \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P^k, \Lambda_i}^{\frac{n+2s}{n-2s}} \sum_{h \neq i} U_{P^k, \Lambda_h} \frac{\partial U_{P^k, \Lambda_i}}{\partial \Lambda_i} + O \left( (\lambda l)^{-n} \right).
\]

**Proof.** We estimate the integration on different region. By the same method used in (A.5), we have

\[
\int_{B_{\lambda}(h_k)} K \left( \frac{x}{\lambda} \right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^k, \Lambda_i}}{\partial \Lambda_i} = O \left( (\lambda l)^{-n} \right).
\]

(B.3)
In the domain $\Omega_j \cap B_i$, where $j \neq i$, there holds $\hat{W}_{m,j}(y) \leq \sum_{h \neq j} \frac{C}{|x_j - x_h|} \leq \frac{C}{(|\lambda|)^{n-2s}} \leq CU_{p_j,\Lambda_j}$. Taylor expansion yields

$$\int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) W_{m}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = \int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} + O\left(\int_{\Omega_j \cap B_i} U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} \right).$$

For the error term, a direct computation yields

$$\int_{\Omega_j \cap B_i} U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = O\left(\frac{1}{(|\lambda|)^{2s}|X_i - X_j|^{n-2s}}\right).$$

Claim: For $j \neq i$, there holds

$$\int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = \int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{(|\lambda|)^{2s}|X_i - X_j|^{n-2s}}\right). \quad (B.4)$$

From direct computation

$$\int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = O\left(\frac{(|\lambda|)^{2s}}{|X_i - X_j|^{n+2s}}\right). \quad (B.5)$$

Using Lemma [A.1] we can obtain

$$\sum_{h \neq i,j} \int_{\Omega_h \cap B_h} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} \leq \sum_{h \neq i,j} C \frac{1}{|X_i - X_j|^{n-2s}} \int_{\Omega_h \cap B_h} \frac{1}{(1 + |x - X^i|)^{n+2s}} \leq C \frac{(|\lambda|)^n}{|X_i - X_j|^{n-2s}} \sum_{h \neq i} \frac{1}{|X^h - X^i|^{n+2s}} \quad (B.6)$$

and

$$\int_{\cup_{h}(\Omega_h \cap B_h^i)} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = O\left(\frac{1}{(|\lambda|)^{2s}|X_i - X_j|^{n-2s}}\right). \quad (B.7)$$

From (B.5), (B.6) and (B.7), we know the Claim is true.

Hence for $j \neq i$,

$$\int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) W_{m}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} = \int_{\Omega_j \cap B_i} K\left(\frac{x}{\lambda}\right) U_{n-2s}^{P_i,\Lambda_j} \frac{\partial U_{P_i,\Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{(|\lambda|)^{2s}|X_i - X_j|^{n-2s}}\right). \quad (B.8)$$
Now we estimate the integration on $\Omega_i \cap B_i$. By Taylor expansion,

$$
\int_{\Omega_i \cap B_i} K \left( \frac{x}{\lambda} \right) W_m^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\Omega_i \cap B_i} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + \sum_{h \neq i} U_{P_h, \Lambda_h} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \left( \frac{1}{\lambda \lambda_i} \right)^{-n} \right). \tag{B.9}
$$

Since in the domain $\Omega_i^c \cup B_i^c$, we have $|y - X^i| \geq \min \{\lambda_l, \min_{j \neq i} \frac{1}{2} |X^i - X^j| \} \geq \frac{1}{2} \lambda l$. Then

$$
\int_{\Omega_i \cap B_i} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \left( \frac{1}{\lambda \lambda_i} \right)^{-n} \right). \tag{B.10}
$$

By a similar method used in the proof of (B.4), we get

$$
\int_{\Omega_i \cap B_i} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \left( \frac{1}{\lambda \lambda_i} \right)^{-n} \right). \tag{B.11}
$$

Substituting (B.10) and (B.11) into (B.9), we have

$$
\int_{\Omega_i \cap B_i} K \left( \frac{x}{\lambda} \right) W_m^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \left( \frac{1}{\lambda \lambda_i} \right)^{-n} \right). \tag{B.12}
$$

Now Lemma B.1 follows from the estimate (B.3), (B.8) and (B.12).

\[ \square \]

**Lemma B.2.** For $h \neq i$, there holds

$$
\int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\mathbb{R}^n} U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \frac{1}{\lambda^2 \lambda_i} \right), \tag{B.13}
$$

and

$$
\frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} = \int_{\mathbb{R}^n} U_{P_i, \Lambda_i}^{n+2s \over n-2s} \frac{\partial U_{P_i, \Lambda_i}}{\partial \lambda_i} + O \left( \frac{1}{\lambda^2 \lambda_i} \right). \tag{B.14}
$$
Proof. Notice the fact
\[
\int_{\mathbb{R}^n} U_{\frac{n+2s}{2}}^{\frac{n+2s}{2s}} \frac{\partial U_{p^i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} (-\Delta)^s U_{p^i, \Lambda} \frac{\partial U_{p^i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} U_{p^i, \Lambda} (-\Delta)^s U_{p^i} \frac{\partial U_{p^i}}{\partial \Lambda_i} = \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} U_{p^i, \Lambda} \frac{4s}{n} \frac{\partial U_{p^i}}{\partial \Lambda_i}.
\]
So the proof of (B.14) and (B.13) are identical. We only give a proof of (B.13).

Choose \(\delta\) to be a fixed constant some enough. Since \(n > 4s > n + 2s - \beta\), the condition \((H_3)\) implies
\[
\left| \int_{B_\delta(X^h)} |K \left( \frac{x}{\lambda} \right) - 1| U_{\frac{n+2s}{2}}^{\frac{n+2s}{2s}} \frac{\partial U_{p^i}}{\partial \Lambda_i} \right| \leq C \int_{B_\delta(X^h)} \frac{|x - X^h|^{\beta}}{\lambda^\beta} \left( \frac{1}{1 + |x - P^h|^{n+2s}} \right) \left( \frac{1}{1 + |x - P^i|^{n-2s}} \right) \leq C \lambda^{2s} |X^h - X^i|^{n-2s}.
\]
A direct calculation yields
\[
\left| \int_{B_\delta(X^i)} |K \left( \frac{x}{\lambda} \right) - 1| U_{\frac{n+2s}{2}}^{\frac{n+2s}{2s}} \frac{\partial U_{p^i}}{\partial \Lambda_i} \right| \leq \frac{C \lambda^{2s}}{|X^i - X^h|^{n+2s}} \leq \frac{C}{\lambda^{2s} |X^i - X^h|^{n-2s}}.
\]
Using Lemma \(\text{A.1}\) we have
\[
\left| \int_{B_\delta(X^h) \cap B_{\delta}(X^i)} \left| K \left( \frac{x}{\lambda} \right) - 1 \right| U_{\frac{n+2s}{2}}^{\frac{n+2s}{2s}} \frac{\partial U_{p^i}}{\partial \Lambda_i} \right| \leq C \int_{B_{\delta}(X^h) \cap B_{\delta}(X^i)} \frac{1}{|X^h - X^i|^{n-2s}} \int_{B_{\delta}(X^i)} \frac{1}{|x - X^i|^{n+2s}} \leq \frac{C}{\lambda^{2s} |X^i - X^h|^{n-2s}}.
\]
Hence (B.13) follows from (B.15), (B.16) and (B.17).

\[\Box\]

Lemma B.3. We have
\[
\int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{\frac{n+2s}{2}}^{\frac{n+2s}{2s}} \frac{\partial U_{p^i}}{\partial \Lambda_i} = \frac{-n-2s \beta C_0(n,s)}{\lambda^{\beta+1} \lambda^\beta} \left( \frac{\sum a_h}{\sum a_h} \right) \int_{\mathbb{R}^n} \frac{|x_1|^\beta}{(1 + |x|^2)^n} + o \left( \frac{|P^i - X^i|^{\min(2,\beta-1)}}{\lambda^\beta} \right),
\]
and
\[
\int_{\mathbb{R}^n} K\left(\frac{x}{\chi}\right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P^i_j} = \left( n - 2s \right) C_0(n, s)^{\frac{2a}{n-2s}} \beta \int_{\mathbb{R}^n} \frac{|x|^{\beta}}{(1 + |x|^2)^{n+1}} (P^i_j - X^i_j)
\]
\[\quad + O(\frac{|P^i_j - X^i_j|^2}{\lambda^\beta}) + o(\lambda^{-\beta}), \quad (B.19)\]

where \( i = 1, \ldots, (m + 1)^k \) and \( j = 1, \ldots, n \).

**Proof.** The two formulas follows from some standard calculations, see [21, Lemma A.9, Lemma A.10] for details.

**Lemma B.4.** For \( h \neq i \), we have
\[
\int_{\mathbb{R}^n} U_{P^h, \Lambda_h}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = c_0 \frac{\partial \varepsilon_{ih}}{\partial \Lambda_i} + \frac{1}{\lambda^\beta} O(\frac{1}{\varepsilon_{hi}^{n-2s}} \log \varepsilon_{hi}),
\]
where \( c_0 = C_0(n, s)^{\frac{n}{n-2s}} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{n+2s}} \) and \( \varepsilon_{ih} = \left( \frac{1}{\varepsilon_{hi}^{n-2s} + \varepsilon_{i}^{n+2s}} \right) \). 

**Proof.** The proof of this lemma is rather standard. We refer to [3] and [9] for ideas.

**Proposition B.5.** It holds that
\[
\frac{\partial I}{\partial \Lambda_i}(W_m) = -\frac{c_1}{\lambda^\beta + \lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i^\beta} |X^i - X^h|^{n-2s} + O\left( \frac{|P^i_j - X^i_j|^{\min\{2, \beta - 1\}}}{\lambda^\beta} \right) + o(\lambda^{-\beta}),
\]
where \( c_1 = \frac{(n-2s)\beta C_0(n, s)^{\frac{n}{n-2s}}}{2n} > 0 \) and \( c_2 = \frac{n-2s}{2(n-2s)} C_0(n, s)^{\frac{n}{n-2s}} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{n+2s}} \).

**Proof.** This proposition is a consequence of Lemma B.1, B.13, B.14, B.18, Lemma B.4 and the definition of \( \lambda \). We need to remind that
\[
\frac{\partial \varepsilon_{ih}}{\partial \Lambda_i} = -\frac{n - 2s}{2\lambda^\beta |X^i - X^h|^{n-2s}} + O\left( \frac{1}{|X^i - X^h|^{n-2s+1}} \right),
\]
which is directly from \( P_h \in B_4(X^h) \) and the definition of \( \{X^h\}_{h=1}^{(m+1)^k} \).

**Proposition B.6.** We have
\[
\frac{\partial I}{\partial P^i_j}(W_m) = -\frac{c_3 a_j}{\Lambda_i^\beta} (P^i_j - X^i_j) + O\left( \frac{|P^i_j - X^i_j|^2}{\lambda^\beta} \right) + o(\lambda^{-\beta}),
\]
where \( c_3 = (n-2s)C_0(n, s)^{\frac{n}{n-2s}} \int_{\mathbb{R}^n} \frac{|x|^{\beta}}{(1 + |x|^2)^{n+2s}} \).

**Proof.** We need to estimate each term on the right hand side of the equality [B.2]. By simple calculation, we have
\[
\left| W_m^{n+2s} - U_{P^h, \Lambda_h}^{n+2s} \right| \leq \left\{ \begin{array}{ll}
\left( \sum_{h \neq i} U_{P^h, \Lambda_h}^{n+2s} \right)^{\frac{n+2s}{n-2s}}, & \text{if } U_{P^i, \Lambda_i} \leq \sum_{h \neq i} U_{P^h, \Lambda_h}, \\
U_{P^i, \Lambda_i} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^{2}, & \text{otherwise.}
\end{array} \right.
\]
Then we get
\[
\int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) W_{n-2s}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}
= \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} + \frac{n + 2s}{n - 2s} \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) \frac{U_{P^i, \Lambda_i}^{n+2s}}{P_j^i} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^h, \Lambda_h}}{\partial P_j^h}
+ O \left( \int_{\mathbb{R}^n} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \right) + O \left( \int_{U_{P^i, \Lambda_i} > \sum_{h \neq i} U_{P^h, \Lambda_h}} \frac{U_{P^i, \Lambda_i}^{n+2s}}{P_j^i} \sum_{h \neq i} U_{P^h, \Lambda_h} \right) \frac{\partial U_{P^i, \Lambda_i}}{P_j^i} \right). \tag{B.20}
\]

We first estimate the error terms above. Since \(n > 2s + 2 > 4s\), we get \((n - s)\frac{n - 2s}{n + 2s} > \frac{n - 2s}{2} > k\). From Lemma A.1, we have
\[
\int_{\mathbb{R}^n} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \leq C \int_{\mathbb{R}^n} \left( \sum_{h \neq i} \frac{1}{(1 + |y - X^h|)^{n-2s}} \right)^{n+2s} \frac{1}{(1 + |y - X^i|)^{n-2s+1}} \leq C \int_{\mathbb{R}^n} \left( \sum_{h \neq i} \frac{1}{(1 + |y - X^h|)^{n-2s}} \right)^{n+2s} \frac{1}{(1 + |y - X^i|)^{(n-s)^{\frac{n+2s}{n-2s}}}} \leq \left( \sum_{h \neq i} \frac{1}{X^h - X^i} \right)^{n+2s} \frac{1}{(n-s)^{\frac{n+2s}{n-2s}}} \int_{\mathbb{R}^n} \left( 1 + |y - X^i| \right)^{n+2s} \leq C(\lambda)^{-(n-s)}. \tag{B.21}
\]
The similar argument yields
\[
\int_{U_{P^i, \Lambda_i} > \sum_{h \neq i} U_{P^h, \Lambda_h}} \frac{1}{n-2s} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^n \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \leq \int_{\mathbb{R}^n} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^{n-2s} U_{P^i, \Lambda_i} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \leq C(\lambda)^{-(n-s)}. \tag{B.22}
\]
For \(h \neq i\), we see that
\[
\frac{1}{n-2s} \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} = \partial \frac{\partial P_j^i}{\partial P_j^i} \int_{\mathbb{R}^n} K \left( \frac{x}{\lambda} \right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}
= \frac{1}{\lambda} \int_{\mathbb{R}^n} \frac{\partial K}{\partial \lambda^i} \left( \frac{x + P^i}{\lambda} \right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} - \int_{\mathbb{R}^n} K \left( \frac{x + P^i}{\lambda} \right) U_{P^i, \Lambda_i}^{n+2s} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}
= O(\lambda^{-1} \frac{1}{|X^i - X^h|^{n-2s}}) + O(\frac{1}{|X^i - X^h|^{n-s}}). \tag{B.23}
\]
The first part of (B.2) can be estimated as
\[
\int_{\mathbb{R}^n} \sum_{h \neq i} U_{i}^{n+2s} \left| \frac{\partial U_{i}}{\partial P_{j}} \right| \leq \sum_{h \neq i} \int_{\mathbb{R}^n} \frac{C}{(1 + |x - X^{h}|)^{n+2s}} \frac{1}{(1 + |x - X^{i}|)^{n-2s+1}}
\leq \sum_{h \neq i} \frac{C}{|X^{h} - X^{i}|^{n-s}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y - X^{h}|)^{n+s+1}}
\leq C(\lambda)^{-(n-s)}.
\]  
(B.24)

Then the expansion of \( \frac{\partial I}{\partial P_{j}}(W_m) \) follows from (B.19), (B.20), (B.21), (B.22), (B.23) and (B.24). □

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