Plancherel Inversion as Unified Approach to Wavelet Transforms and Wigner Functions

S. Twareque Ali\textsuperscript{1}, Hartmut Führ\textsuperscript{2} and Anna E. Krasowska\textsuperscript{3}

\textit{Department of Mathematics and Statistics, Concordia University, Montréal, Quebec, CANADA H3B 1R6}
\textit{Institut für Biomathematik und Biometrie, GSF, 85764 Neuherberg, Deutschland}

Abstract

We demonstrate that the Plancherel transform for Type-I groups provides one with a natural, unified perspective for the generalized continuous wavelet transform, on the one hand, and for a class of Wigner functions, on the other. The wavelet transform of a signal is an $L^2$-function on an appropriately chosen group while the Wigner function is defined on a coadjoint orbit of the group and serves as an alternative characterization of the signal, which is often used in practical applications. The Plancherel transform maps $L^2$-functions on a group unitarily to fields of Hilbert-Schmidt operators, indexed by unitary irreducible representations of the group. The wavelet transform can essentially be looked upon as a restricted inverse Plancherel transform, while Wigner functions are modified Fourier transforms of inverse Plancherel transforms, usually restricted to a subset of the unitary dual of the group. Some known results on both Wigner functions and wavelet transforms, appearing in the literature from very different perspectives, are naturally unified within our approach. Explicit computations on a number of groups illustrate the theory.

\textsuperscript{1}e-mail: stali@mathstat.concordia.ca
\textsuperscript{2}e-mail: fuehr@gsf.de
\textsuperscript{3}e-mail: ankras@alcor.concordia.ca
1 Introduction

The continuous wavelet transform is used extensively in image processing and signal analysis and its group theoretical origin is well known 1, 2. The Wigner function has also been employed in the analysis of signals as well as in numerous quantum optical and quantum statistical computations 3, 4, 20, 34, 35. It has been argued before that both the wavelet transform and the Wigner function owe their origin to the square integrability of certain group representations. This point was discussed extensively in 3, where the square integrability of a single unitary irreducible representation of a group was exploited to build both a generalized wavelet transform and a class of Wigner functions. It is the purpose of this paper to show, quite generally, how both these concepts can be unified using the Plancherel transform for Type-I groups. The Plancherel transform sets up a unitary isomorphism between the Hilbert space of square integrable (with respect to the Haar measure) functions on the group and the direct integral Hilbert space (with respect to the Plancherel measure) built out of the spaces of Hilbert-Schmidt operators on the Hilbert spaces of unitary irreducible representations of the group. It is the inverse of this unitary map which, when appropriately restricted, leads to a generalized wavelet transform. On the other hand, taking the inverse Plancherel transform and following it up with a Fourier type of transform leads to functions on the dual of the Lie algebra of the group. The final function, when restricted to appropriate coadjoint orbits, then yields a wide class of generalized Wigner functions, which share many of the interesting properties of the original Wigner function 34, but now is definable for a vast array of groups and representations. Generalized wavelet transforms can also be seen as coherent state transforms of vectors in the Hilbert spaces of group representations 1. However, in most cases one defines the coherent state transform on Hilbert spaces carrying a single unitary irreducible representation of the group. This requires that the representation in question be square integrable, or in other words, that it belong to the discrete series of representations of the group. If, on the other hand, the group in question does not admit square integrable representations, the above construction of coherent states and related transforms clearly fails. In such cases, in specific examples, it has been demonstrated 23 how the use of direct integral representations, over some convenient subset of the unitary dual of the group, leads once more to the existence of a coherent state transform. We show here that this situation is generic and is again a simple consequence of the
The rest of this paper is organized as follows: In Section 2, we briefly recall the Plancherel transform and its inverse for Type-I groups. In Section 3, we derive explicit expressions for the inverse Plancherel transform and demonstrate how it can be used to define coherent states and a generalized wavelet transform. We carry out the construction explicitly in Section 4 for the case of the Poincaré group in a two-dimensional space-time. Section 5 is devoted to a definition and construction of the generalized Wigner function. This function is defined on the coadjoint orbits, foliating the dual of the Lie algebra of the group, and we introduce a modified Fourier transform on the range of the Plancherel transform to arrive at it. We also discuss general properties of the Wigner function, which follow immediately from the definition. As examples, we compute in Section 6 Wigner functions for the cases of three commonly used groups: the two-dimensional Poincaré group, the affine Poincaré group (the Poincaré group including dilations) and the Weyl-Heisenberg group, which leads us back to the original quasi-probability distribution function introduced by Wigner. Finally, in the Appendix we collect together a few results, of a computational nature, used in working out the examples.

2 Plancherel Measure

Let us first fix some notation: $G$ denotes a second countable, locally compact group. All representations will be understood to be unitary and strongly continuous. By $\hat{G}$ we denote the set of equivalence classes of irreducible representations of $G$, equipped with the Mackey Borel structure (see, e.g., [16]). It will often be necessary to distinguish between an equivalence class $\sigma \in \hat{G}$ of representations and a specific realization of a representation $U_\sigma$, in this equivalence class and acting on a particular Hilbert space $H_\sigma$. In the direct integrals below, a measurable realization of the representations $U_\sigma$ used is provided by the theory [31, Theorem 10.2]. $\mu_G$ denotes the left Haar measure, and $L^p(G)$ is the corresponding $L^p$-space. $C_c(G)$ denotes the space of compactly supported continuous functions on $G$. The modular function of $G$ is denoted by $\Delta_G$, the convention being,

$$d\mu_G(x) = \Delta_G(x) \, d\mu_r(x) \, ,$$  \hspace{1cm} (1)
where $\mu_r$ is the right invariant Haar measure. For a function $f$ on $G$, we write

$$\tilde{f}(x) := \overline{f(x^{-1})}.$$ 

For a given Hilbert space $\mathfrak{H}$, $B_2(\mathfrak{H})$ denotes the space of Hilbert-Schmidt operators. It is a Hilbert space, endowed with the scalar product $\langle A|B \rangle_2 = \text{tr}(A^*B)$; the corresponding norm shall be denoted by $\| \cdot \|_2$. Furthermore, $B_1(\mathfrak{H})$ denotes the subspace of trace class operators, endowed with the norm $\|A\|_1 = \text{tr}(|A|)$, where $|A| := (AA^*)^{1/2}$. Elements of special interest in both spaces are the rank-one operators, denoted by $|\eta\rangle\langle\phi|$, (for $\eta, \phi \in \mathfrak{H}$), which are defined by $|\eta\rangle\langle\phi|(\psi) = \langle\phi|\psi\rangle\eta$, for any $\psi \in \mathfrak{H}$. We have $\|\eta\langle\phi\rangle\|_1 = \|\eta\|\|\phi\|_2 = \|\eta\|\|\phi\|$. The usual operator norm is denoted by $\|\cdot\|_{\infty}$. If a densely defined operator $A$ has a bounded extension, we denote the extension by $[A]$. A simple and often used fact is that for linear operators $A, B, T$ with $[A], [B]$ bounded, such that $[AT]$ and $[TB]$ exist, $[ATB] = [AT][B] = A[TB]$.

The central object of interest in this paper is the left regular representation $\lambda_G$ of $G$, acting on $L^2(G)$ via $(\lambda_G(x)f)(y) := f(x^{-1}y)$. Another representation acting on $L^2(G)$ is the right regular representation $\rho_G$, defined by $(\rho_G(x)f)(y) := \Delta_G(x)^{1/2}f(yx)$. The left and the right regular representations commute and are unitarily equivalent. Finally, the two-sided representation is denoted by $\lambda_G \times \rho_G$. This is a representation of the product group $G \times G$, defined by $(\lambda_G \times \rho_G)(x, y) := \lambda_G(x)\rho_G(y)$. The Plancherel theory can be seen as the theory of a direct integral decomposition of the two-sided representation into irreducibles, where the intertwining operator is given by the operator-valued Fourier transform. In this paper, we shall only be concerned with groups $G$ such that $\lambda_G$ is a Type-I factor, i.e., the von Neumann algebra generated by the left (right) regular representation of $G$ is a Type-I factor. (Such von Neumann algebras are algebraically isomorphic, as $C^*$-algebras, to the full algebra of bounded operators on some Hilbert space.)

Recall that the operator-valued Fourier transform on $G$ maps each $f \in L^1(G)$ to the family $\{U_\sigma(f)\}_{\sigma \in \hat{G}}$ of operators, where each $U_\sigma(f)$ is defined by the weak operator integral

$$U_\sigma(f) := \int_G f(x)U_\sigma(x)d\mu_G(x) \quad .$$

This defines a field of bounded operators, in fact, we have

$$\|U_\sigma(f)\|_{\infty} \leq \|f\|_1$$

(3)
Another feature of the operator-valued Fourier transform, reminiscent of the well-known Fourier transform over the reals, is that convolution becomes operator multiplication on the Fourier side, more precisely, \( U_\sigma(f * g) = U_\sigma(f) \circ U_\sigma(g) \). In order to invert this transform, we have to find a Hilbert space \( \mathfrak{H} \), such that \( f \mapsto \{ U_\sigma(f) \}_{\sigma \in \hat{G}} \) extends from \( L^1(G) \cap L^2(G) \) to a unitary equivalence \( L^2(G) \to \mathfrak{H} \). To see the relationship of this definition to the usual Fourier transform (over the reals, say), let us suppose for a moment that \( G \) is abelian. Then each \( U_\sigma(f) \) is a scalar, since each \( U_\sigma \) is a character, and the above mapping yields the usual Fourier transform \( \hat{f} \) (except that in the generally used definition one integrates over \( U_\sigma^* \) rather than \( U_\sigma \)). Also, in this case, \( \hat{G} \) is a locally compact abelian group, and the abelian Plancherel theorem states that we may take the Haar measure on \( \hat{G} \) as the Plancherel measure, i.e. \( \mathfrak{H} = L^2(\hat{G}) \), in the stated unitary equivalence.

Returning to the general case, let us try to motivate the construction of the Hilbert space \( \mathfrak{H} \). The Fourier transform \( \{ U_\sigma(f) \}_{\sigma \in \hat{G}} \) forms a field of bounded operators on \( \hat{G} \). Furthermore, this field is measurable, as follows from the definition of the Mackey Borel structure on \( \hat{G} \). It is thus reasonable to expect \( \mathfrak{H} \) to be the direct integral over a measure space \( (\hat{G}, \nu_G) \), where each fibre is some Hilbert space of operators, and the measure \( \nu_G \) is to be determined. The natural choice for the fibres is given by the Hilbert-Schmidt operators on the representation spaces \( \mathfrak{H}_\sigma \). At this point, the Plancherel theory splits into the unimodular and the nonunimodular cases: In the unimodular case, \( U_\sigma(f) \) is automatically Hilbert-Schmidt, for every \( f \in L^1(G) \cap L^2(G) \) and almost every \( \sigma \in \hat{G} \). In the nonunimodular case we have to employ a family \( \{ C_\sigma \}_{\sigma \in \hat{G}} \) of densely defined unbounded operators \( C_\sigma \) on \( \mathfrak{H}_\sigma \), with densely defined inverses, such that \( U_\sigma(f)C_\sigma^{-1} \) is Hilbert-Schmidt (more precisely: for almost all \( \sigma \) (with respect to the measure \( \nu_G \)), the closure \( [U_\sigma(f)C_\sigma^{-1}] \) is Hilbert-Schmidt). These operators can indeed be constructed in such way that the operator Fourier transform extends to a unitary map.

Let us now give the exact statement of the Plancherel theorem in the form we are going to use.

**Theorem 2.1** Let \( G \) be a second countable locally compact group having a type-I regular representation. Then there exists a measure \( \nu_G \) on \( \hat{G} \), called the **Plancherel measure**, and a measurable field \( \{ C_\sigma \}_{\sigma \in \hat{G}} \) of self adjoint positive operators with densely defined inverses, such that the following hold:

(i) For \( f \in L^1(G) \cap L^2(G) \) and \( \nu_G \)-almost all \( \sigma \in \hat{G} \), the closure of the
operator $U_\sigma(f)C_\sigma^{-1}$ is a Hilbert-Schmidt operator on $\mathcal{S}_\sigma$.

(ii) The map $L^1(G) \cap L^2(G) \ni f \mapsto \{[U_\sigma(f)C_\sigma^{-1}]\}_{\sigma \in \hat{G}}$ extends to a unitary equivalence

$$\mathcal{P} : L^2(G) \to \int_{\hat{G}} \mathcal{B}_2(\mathcal{H}_\sigma) d\nu_G(\sigma).$$

This unitary operator is called the **Plancherel transform** of $G$. It has the intertwining property

$$\mathcal{P}(\lambda_G(x)\rho_G(y)f)(\sigma) = U_\sigma(x)(\mathcal{P}(f)(\sigma))U_\sigma(y)^*.$$

(iii) There exists a subspace $\mathcal{D}(G) \subset L^1(G) \cap L^2(G)$, dense in $L^2(G)$, such that for all $f \in \mathcal{D}(G)$ and $\nu_G$-almost all $\sigma \in \hat{G}$, the operator

$$[U_\sigma(f)C_\sigma^{-2}] = [[U_\sigma(f)C_\sigma^{-1}]C_\sigma^{-1}]$$

is densely defined and has a trace class extension, and we have the Fourier inversion formula,

$$f(x) = \int_{\hat{G}} \text{tr}([U_\sigma(x)^*U_\sigma(f)C_\sigma^{-2}]) d\nu_G(\sigma).$$

(iv) The Plancherel measure is essentially unique: The covariance relation

$$U_\sigma(x)C_\sigma U_\sigma(x)^* = \Delta_G(x)^{1/2}C_\sigma$$

fixes each $C_\sigma$ uniquely up to multiplication by a scalar, and once these are fixed, so is $\nu_G$. Conversely, one can fix $\nu_G$ (which is a priori only unique up to equivalence) and thereby determine the $C_\sigma$ uniquely.

(v) $G$ is unimodular if and only if for $\nu_G$-almost all $\sigma$, $C_\sigma$ is a multiple of the identity $I_\sigma$ on $\mathcal{S}_\sigma$. In this case we require that $C_\sigma = I_\sigma$, which then determines $\nu_G$ completely. If $G$ is nonunimodular, $C_\sigma$ is an unbounded operator for ($\nu_G$-almost all) $\sigma \in \hat{G}$.

**Remark 2.2** The inversion formula (5) was shown in [15] to hold for the space of Bruhat functions introduced in [11]. It can be written as

$$\mathcal{D}(G) = \bigcup \{C_c^\infty(G/K) : K \subset G \text{ compact such that } G/K \text{ is a Lie group} \}$$

where $C_c^\infty(G/K)$ is the space of arbitrarily smooth functions on $G/K$ with compact support, canonically embedded into $C_c(G)$.

$\diamond$
In the following, we suppose that $G$ is a second countable group with type-I regular representation. We use $\hat{\cdot}$ to denote the Plancherel transform. So, for $f \in L^1(G) \cap L^2(G)$ we have $(Pf)(\sigma) = \hat{f}(\sigma) := [U_\sigma(f)C^{-1}_\sigma]$. The direct integral space of Hilbert-Schmidt spaces in $\hat{\Pi}$ will be denoted by $B^2_2$.

The scalar product of two elements, $A^i \in B_2^2$, $i = 1, 2$, consisting of the measurable fields $\{A^i(\sigma) \in B_2(\mathcal{H}_\sigma)\}_{\sigma \in \hat{G}}$, is given by

$$\langle A^1 | A^2 \rangle_{B^2_2} = \int_{\hat{G}} \text{tr} [A^1(\sigma)^* A^2(\sigma)] \, d\nu_G(\sigma).$$

3 The wavelet transform as inverse Plancherel transform

Let us quickly recall the group-theoretical formalism for the construction of wavelet transforms: Suppose we are given a unitary (not necessarily irreducible) representation $U$ of $G$, on a Hilbert space $\mathcal{H}$ and a vector $\eta \in \mathcal{H}$. We can then define the (generalized) wavelet transform of $\phi \in \mathcal{H}$ as the function $V_\eta \phi$ on $G$, defined by

$$(V_\eta \phi)(x) := \langle U(x)\eta | \phi \rangle.$$  \hfill (7)

Generally, this construction gives an injective operator $V_\eta : \phi \mapsto V_\eta \phi$, whenever $\eta$ is a cyclic vector (which means that the orbit $U(G)\eta$ is total in $\mathcal{H}$). However, in order to have an efficient way of inverting $V_\eta$, we require more, viz, that $V_\eta : \mathcal{H} \rightarrow L^2(G)$ be an isometry (possibly up to a scalar factor $c_\eta$). Note that generally, even the well-definedness, that is $V_\eta(\mathcal{H}) \subset L^2(G)$, is not guaranteed. However, if $V_\eta$ is an isometry (in which case we say that $\eta$ is an admissible vector), we can rewrite the isometry property in the form of an inversion formula,

$$\phi = \frac{1}{c_\eta} \int_G (V_\eta \phi)(x) \, U(x)\eta \, d\mu_G(x),$$ \hfill (8)

where the integral is understood in the weak operator sense or, equivalently, as a resolution of the identity:

$$\text{Id}_\mathcal{H} = \frac{1}{c_\eta} \int_G U(x)|\eta\rangle\langle \eta|U(x)^* \, d\mu_G(x).$$ \hfill (9)
The relationship to the left regular representation is quite obvious: Besides being an isometry, the wavelet transform $V_\eta$ is easily seen to intertwine the representation $U$ of $G$ with its left regular representation $\lambda_G$. Hence wavelet transforms fall quite naturally in the domain of the Plancherel theory. In fact, as will become clear below, if $U$ is given as a direct integral of irreducible representations, a wavelet transform is just the inverse Plancherel transform, applied to certain operator fields.

In order to motivate the last statement, let us take a closer look at the case where $U = U_\sigma$ is an irreducible, square-integrable representation of $G$. This means that there exist explicit admissibility conditions involving $C_\sigma$, which are usually cited in the following form [15, Theorem 3]:

**Theorem 3.1** Let $U_\sigma$ be an irreducible subrepresentation of $\lambda_G$.

(i) For $\eta, \phi \in \mathcal{H}_\sigma$, the wavelet transform $V_\eta \phi$ is square integrable (i.e., is an element of $L^2(G)$) iff $\eta \in \text{dom}(C_\sigma)$.

(ii) For $\eta_1, \eta_2 \in \text{dom}(C_\sigma)$ and $\phi_1, \phi_2 \in \mathcal{H}_\sigma$, we have the orthogonality relation,

$$\langle V_{\eta_1}\phi_1 | V_{\eta_2}\phi_2 \rangle_{L^2(G)} = \langle C_\sigma \eta_2 | C_\sigma \eta_1 \rangle_{\mathcal{H}} \langle \phi_1 | \phi_2 \rangle_{\mathcal{H}} .$$

(10)

To see the relationship with the Plancherel transform, let us consider the rank-one operators $A_i = |\phi_i \rangle \langle C_\sigma \eta_i | (i = 1, 2)$. Then $V_{\eta_i}\phi_i(x) = \langle U_\sigma(x) | \eta_i \rangle \langle \phi_i | = \text{tr}(\langle \phi_i | \langle C_\sigma \eta_i | C_\sigma^{-1} U_\sigma(x)^* \rangle) = \text{tr}(A_i C_\sigma^{-1} U_\sigma(x)^* \rangle)$, which is essentially the Plancherel inversion formula (3) (up to an ordering of operators), with the operator fields supported only at the point $\sigma$. Here we have taken account of the fact that $U_\sigma$ is a subrepresentation of $\lambda_G$ iff $\nu_G(\{\sigma\}) > 0$, and hence, without loss of generality, we may take $\nu_G(\{\sigma\}) = 1$. Assuming that the inversion formula holds for both $V_{\eta_i}\phi_i, \ i = 1, 2$, we obtain $(\mathcal{P}(V_{\eta_i}\phi_i))(\pi) = (\overline{V_{\eta_i}\phi_i})(\pi) = A_i, \ \text{for} \ \pi = \sigma \ \text{and} \ 0 \ \text{elsewhere}$. Thus, the orthogonality relations, and in particular the isometry property of the generalized wavelet transform $V_\eta$, are immediate consequences of the unitarity of the Plancherel transform.

While this way of showing the isometry property of $V_\eta$, using the Plancherel transform, is much too complicated in the irreducible case (which is easily dealt with using Schur's lemma), it has the advantage of being readily generalizable to direct integral representations, once we have extended the inversion formula (3) to a wider class of functions.
Let us first establish a few preliminary facts. The first lemma deals with the operators \( C_\sigma \) and their relation to convolution.

**Lemma 3.2** Let \( f \in C_c(G) \).

(i) For \( \nu_G \)-almost every \( \sigma \), we have \( \hat{f}(\sigma)^* = C^{-1}_\sigma U_\sigma(\Delta^{-1}_G \tilde{f}) \). In particular the right hand side is everywhere defined and bounded.

(ii) For \( \nu_G \)-almost every \( \sigma \), we have

\[
[U_\sigma(f)C^{-1}_\sigma] = C^{-1}_\sigma U_\sigma(\Delta^{-1/2}_G f) ,
\]

in particular the right hand side is everywhere defined and bounded.

(iii) For all \( g \in L^2(G) \), we have

\[
\langle \hat{g} \ast f \rangle(\sigma) = \hat{g}(\sigma) U_\sigma(\Delta^{-1/2}_G f) ,
\]
\[
\langle f \ast g \rangle(\sigma) = U_\sigma(f) \hat{g}(\sigma) .
\]

**Proof.** For part (i) we invoke [32, Theorem 13.2], to find that, since \( C^{-1}_\sigma \) is self-adjoint and \( U_\sigma(f) \) is bounded, \( (U_\sigma(f)C^{-1}_\sigma)^* = C^{-1}_\sigma U_\sigma(f)^* \). Moreover, since \( \tilde{f}(\sigma) \) is bounded, the right hand side of the last equation is everywhere defined. Calculating \( U_\sigma(f)^* \) is routine.

For (ii) we first note that by (i), applied to \( \Delta^{-1/2}_G \tilde{f} \in C_c(G) \), the right hand side is bounded and everywhere defined. Moreover, the left-hand side is bounded since \( f \in L^1(G) \cap L^2(G) \). It thus remains to show that the equality holds on the dense subspace dom\((C^{-1}_\sigma)\): For \( \phi, \eta \in \text{dom}(C^{-1}_\sigma) \) the definition of the weak operator integral yields

\[
\langle \phi | U_\sigma(f)C^{-1}_\sigma \eta \rangle = \int_G \langle \phi | U_\sigma(x)C^{-1}_\sigma \eta \rangle f(x) d\mu_G(x)
\]
\[
= \int_G \langle \phi | \Delta_G(x)^{-1/2} C^{-1}_\sigma U_\sigma(x) \eta \rangle f(x) d\mu_G(x)
\]
\[
= \int_G \langle C^{-1}_\sigma \phi | U_\sigma(x) \eta \rangle (\Delta_G(x)^{-1/2} f(x)) d\mu_G(x)
\]
\[
= \langle C^{-1}_\sigma \phi | U_\sigma(\Delta^{-1/2}_G f) \eta \rangle
\]
\[
= \langle \phi | C^{-1}_\sigma U_\sigma(\Delta^{-1/2}_G f) \eta \rangle ,
\]
where the second equality uses the covariance relation (3), and the self-
adjointness of $C^{-1}_\sigma$ was used on various occasions. This shows (ii).

Part (iii) is then immediate from (i) and (ii), at least for $g \in L^1(G) \cap
L^2(G)$. It extends by continuity to all of $L^2(G)$: The left-hand sides are
continuous operators, being convolution operators with $f \in C_c(G)$, and the
right hand sides are continuous because of inequality (3).

The next lemma defines the space $\mathcal{B}^{\oplus}_1$, which arises very naturally when
dealing with inversion formulae. In fact, there is a natural representation-
theoretic interpretation of $\mathcal{B}^{\oplus}_1$ as the space of Fourier transforms of the Fourier
algebra $A(G)$. This was noted for the unimodular case by Lipsman [27], but
the arguments go through for the non-unimodular case as well.

**Lemma 3.3** Let $\mathcal{B}^{\oplus}_1$ be the space of measurable fields \( \{ B(\sigma) \}_{\sigma \in \hat{G}} \) of trace
class operators, for which the norm

\[
\| B \|_{\mathcal{B}^{\oplus}_1} := \int_{\hat{G}} \| B(\sigma) \|_1 \, d\nu_G(\sigma)
\]

is finite. Here we identify operator fields which agree $\nu_G$-almost everywhere. Then $\left( \mathcal{B}^{\oplus}_1, \| \cdot \|_{\mathcal{B}^{\oplus}_1} \right)$ is a Banach space and the set of measurable fields of rank
one operators in $\mathcal{B}^{\oplus}_1$ spans a dense subspace.

The proof consists of standard arguments and is omitted here.

Now we can show that the inversion formula holds almost everywhere,
whenever it makes sense (i.e., whenever all quantities involved can be expected to converge). This is the nonabelian analogue of the well known Fourier inversion formula for an $L^2$-function whose Plancherel transform is
in $L^1$. The statement for the unimodular case was in fact given in [27].

**Theorem 3.4** Let $A \in \mathcal{B}^{\oplus}_2$ be such that for almost all $\sigma \in \hat{G}$, $A(\sigma)C^{-1}_\sigma$
extends to a trace class operator and such that $\{ [A(\sigma)C^{-1}_\sigma] \}_{\sigma \in \hat{G}} \in \mathcal{B}^{\oplus}_1$. Let $a \in L^2(G)$ be the inverse Plancherel transform of $A$. Then we have (for
almost every $x \in G$)

\[
a(x) = \int_{\hat{G}} \text{tr}(U_\sigma(x)^* [A(\sigma)C^{-1}_\sigma]) \, d\nu_G(\sigma) . \tag{11}
\]

If we assume that $C^{-1}_\sigma A(\sigma)$ is trace-class, and that $\{ [C^{-1}_\sigma A(\sigma)] \}_{\sigma \in \hat{G}} \in \mathcal{B}^{\oplus}_1$, then $\sigma \mapsto \text{tr}(\| [C^{-1}_\sigma A(\sigma)] \|$ is integrable and we obtain (almost everywhere)

\[
a(x) = \int_{\hat{G}} \text{tr}( [C^{-1}_\sigma U_\sigma(x)^* A(\sigma)] ) \, d\nu_G(\sigma) . \tag{12}
\]
Proof. Let \((f_n)_{n \in \mathbb{N}} \subset C_c(G)\) be a sequence with decreasing supports, satisfying the following requirements: \(f_n \geq 0\), \(\|\tilde{f}_n\|_{L^1} = 1\), and \(\text{supp}(f_n)\) runs through a neighborhood base at unity. Then \((f_n)_{n \in \mathbb{N}}\) is a bounded approximate identity with respect to right convolution, i.e., for all \(g \in L^2(G)\) we have \(g * f_n \to g\) in \(L^2(G)\), and the operator norms of \(g \mapsto g * f_n\) are bounded by a constant. In addition, \((\Delta_G^{−1/2}f_n)_{n \in \mathbb{N}}\) has the same properties, since \(\|\Delta_G^{−1/2}\tilde{f}_n\|_{L^1} \to 1\).

By passing to a subsequence, if necessary, we may assume that \(a * f_n \to a\) pointwise almost everywhere. We first evaluate the convolution using the unitarity of the Plancherel transform, obtaining for almost every \(x \in G\):

\[
(a * f_n)(x) = \langle \lambda(x)\tilde{f}_n|a \rangle = \int_{\hat{G}} \text{tr} \left( A(\sigma) \left[ U_\sigma(x)U_\sigma(\tilde{f}_n)C^{-1}_\sigma \right]^* \right) d\nu_G(\sigma) = \int_{\hat{G}} \text{tr}(A(\sigma)[C^{-1}_\sigma U_\sigma(\tilde{f}_n)^*U_\sigma(x)^*])d\nu_G(\sigma)
\]

\[
= \int_{\hat{G}} \text{tr}([A(\sigma)C^{-1}_\sigma U_\sigma(\Delta_G^{-1}f_n)U_\sigma(x)^*])d\nu_G(\sigma)
\]

\[
= \int_{\hat{G}} \text{tr}([A(\sigma)C^{-1}_\sigma]U_\sigma(U_\sigma(\Delta_G^{-1}f_n)U_\sigma(x)^*))d\nu_G(\sigma) .
\]

Here we have used the fact that \(C^{-1}_\sigma U_\sigma(\Delta_G^{-1}f_n)U_\sigma(x)^*\) is bounded, by Lemma 3.2 (i), as well as the existence of \([A(\sigma)C^{-1}_\sigma]\), as assumed.

From the definition of \(B^\oplus_1\), it is clear that

\[
(B(\sigma))_{\sigma \in \hat{G}} \to \int_{\hat{G}} \text{tr}(B(\sigma))d\nu_G(\sigma)
\]

defines a bounded linear functional; this was our motivation for introducing the space. Comparing the right-hand side of (11) with (14), we find that it suffices to show that the sequence of operators

\[
T^{(n)}_1 : B^\oplus_1 \to B^\oplus_1 \]

\[
(B(\sigma))_{\sigma \in \hat{G}} \to (B(\sigma)U_\sigma(\Delta_G^{-1}f_n))_{\sigma \in \hat{G}}
\]

converges strongly to the identity operator. For this purpose, let us write \(T^{(n)}_2 : B^\oplus_2 \to B^\oplus_2\) for the identically defined operators on \(B^\oplus_2\). Let us first note that \(\|U_\sigma(\Delta_G^{-1}f_n)\|_\infty \leq \|\Delta_G^{-1}f_n\|_{L^1} \leq K\), with \(K\) independent of \(n\), thus both sequences of operators are norm-bounded.
Applying Lemma 3.2 (ii), we find that the Plancherel transform conjugates \( (T_2^{(n)})_{n \in \mathbb{N}} \) with the family of convolution operators \( S^{(n)} : g \mapsto g * (\Delta_G^{-1/2} f_n) \), which strongly converges to the identity operator. Moreover, it strongly converges with respect to the \( \mathcal{B}_1^\oplus \)-norm on the subspace generated by the fields of rank one operators: Let \( B = \{ \| \phi(\sigma) \rangle \langle \eta(\sigma) \| : \sigma \in \widehat{G} \} \) be such a field. We may assume \( \| \phi(\sigma) \rangle \langle \eta(\sigma) \| = \| \eta(\sigma) \| \). Then \( (T_1^{(n)} B)(\sigma) = \| \phi(\sigma) \rangle \langle \hat{T}_n(\sigma)^* \eta(\sigma) \|, \ \sigma \in \widehat{G} \), with \( T_1^{(n)} = U_\sigma(\Delta_G^{-1} f_n) \). Hence

\[
\left\| T_1^{(n)} B - B \right\|_{\mathcal{B}_1^\oplus} = \int_{\widehat{G}} \| \phi(\sigma) \| \left\| T_1^{(n)} \eta(\sigma) - \eta(\sigma) \right\| d\nu_G(\sigma)
\leq \left( \int_{\widehat{G}} \| \phi(\sigma) \|^2 d\nu_G(\sigma) \right)^{1/2}
\times \left( \int_{\widehat{G}} \| T_1^{(n)} \eta(\sigma) - \eta(\sigma) \|^2 d\nu_G(\sigma) \right)^{1/2},
\]

by the Cauchy-Schwarz inequality. Here we have used that \( \| \phi(\sigma) \rangle \langle \eta(\sigma) \|_1 = \| \phi(\sigma) \| \| \eta(\sigma) \| = \| \phi(\sigma) \|^2 = \| \eta(\sigma) \|^2 \), hence all integrals converge. Picking any measurable family \( \{ \xi(\sigma) \}_{\sigma \in \widehat{G}} \) of unit vectors, we can define the operator field \( B' = \{ \langle \xi(\sigma) \rangle \langle \eta(\sigma) \| \}_{\sigma \in \widehat{G}} \in \mathcal{B}_2^\oplus \), and find that

\[
\int_{\widehat{G}} \| T_1^{(n)} \eta(\sigma) - \eta(\sigma) \|^2 d\nu_G(\sigma) = \left\| T_1^{(n)} B' - B' \right\|_{\mathcal{B}_2^\oplus}^2
\]

converges to zero.

Thus \( (T_1^{(n)})_{n \in \mathbb{N}} \) is a bounded sequence of operators converging strongly on a dense subspace, which entails strong convergence on \( \mathcal{B}_1^\oplus \). Hence,

\[
\lim_{n \to \infty} \int_{\widehat{G}} \text{tr}(\left[ A(\sigma) C_{\sigma}^{-1} \right] U_\sigma(\Delta_G^{-1} f_n) U_\sigma(x)^*) d\nu_G(\sigma) = \int_{\widehat{G}} \text{tr}(\left[ A(\sigma) C_{\sigma}^{-1} \right] U_\sigma(x)^*) d\nu_G(\sigma),
\]

and the first equation is proved.

The second formula is proved by modifying the argument for the first: We employ

\[
\text{tr} \left( A(\sigma) \left[ U_\sigma(x) U_\sigma(\hat{f}_n) C_{\sigma}^{-1} \right]^* \right) = \text{tr} \left( \left[ U_\sigma(x) U_\sigma(\hat{f}_n) C_{\sigma}^{-1} \right]^* A(\sigma) \right)
\]

in equation (13).
After using

\[ [C^{-1}_\sigma U_\sigma(\Delta_G^{-1/2} f_n)U_\sigma(x)^* A(\sigma)] = U_\sigma(\Delta_G^{-1/2} f_n)[C^{-1}_\sigma U_\sigma(x)^* A(\sigma)] \]

the fact that \((\Delta_G^{-1/2} f_n)_{n \in \mathbb{N}}\) is a bounded approximate identity with respect to left convolution now gives the desired convergence of the traces, and we are done.

\[ \diamond \]

**Remark 3.5** Before we apply the theorem to general direct integral representations, let us first consider the relevance of the two inversion formulae, (11) and (12), for the irreducible case. So let \(\pi \in \hat{G}\), \(U_\pi\) a representation in this class and assume it to be an irreducible subrepresentation of \(\lambda_G\). Let \(\phi, \eta \in \mathcal{H}_\pi\) with \(\eta \in \text{dom}(C_\pi)\). The rank one operator \(|\phi\rangle\langle C_\pi \eta|\) fulfills the requirement for the first inversion formula, hence \(V_\eta \phi\) is the inverse Plancherel transform of this operator. But we can also consider the operator \(|C_\pi \eta\rangle\langle \phi|\), suitable for the second inversion formula, which gives

\[ \text{tr}(C^{-1}_\pi |U_\pi(x)^* C_\pi \eta\rangle\langle \phi|) = \langle \phi|C^{-1}_\pi U_\pi(x)^* C_\pi \eta\rangle = \langle \phi|\Delta_G^{-1/2}(x)U_\pi(x)^* \eta\rangle = (\Delta_G^{-1/2} \tilde{V}_\eta \phi)(x) . \]

This reveals the general relationship between the two inversion formulae: The operators \(f \mapsto \Delta_G^{-1/2} \tilde{f}\) and \(\{A(\sigma)\}_{\sigma \in \hat{G}} \mapsto \{A(\sigma)^*\}_{\sigma \in \hat{G}}\) are conjugate under the Plancherel transform, hence an inversion formula for \(f\) gives rise to an inversion formula for \(\Delta_G^{-1/2} \tilde{f}\), and vice versa.

\[ \diamond \]

Now let \(U_\pi\) be a multiplicity-free subrepresentation of \(\lambda_G\). Since \(G\) has a type-I regular representation, we may assume

\[ U_\pi = \int_\Sigma^{\oplus} U_\sigma d\nu_G(\sigma) , \]

for some measurable subset \(\Sigma \subset \hat{G}\). A simple method for the construction of admissible vectors is then given in the following corollary:

**Corollary 3.6** Let \(\phi = \{\phi(\sigma)\}_{\sigma \in \Sigma}\), \(\{\eta(\sigma)\}_{\sigma \in \Sigma} \in \mathcal{H}_\pi\) be given. Assume, moreover, that \(\eta(\sigma) \in \text{dom}(C_\sigma)\), and that the field \(A(\sigma) := |\phi(\sigma)\rangle\langle C_\sigma \eta(\sigma)|\), extended trivially outside \(\Sigma\), is in \(\mathcal{B}_2^{\oplus}\). Then \(V_\eta \phi \in L^2(G)\), with \((V_\eta \phi) = A\), and hence

\[ \|V_\eta \phi\|^2 = \int_\Sigma \|\phi(\sigma)\|^2 \|C_\sigma \eta(\sigma)\|^2 d\nu_G(\sigma) . \]
Thus, \( \eta \) is admissible iff \( \{ \eta(\sigma) \}_{\sigma \in \Sigma} \) can be chosen such that \( \| C_\sigma \eta(\sigma) \| = 1 \), for \( \nu_G \)-almost every \( \sigma \in \Sigma \).

**Proof.** Let \( a \in L^2(G) \) be the inverse Plancherel transform of \( A \). Then, observing that \( A(\sigma)C_\sigma^{-1} = |\phi(\sigma)\rangle \langle \eta(\sigma)| \), we see that as a function of \( \sigma \),

\[
\text{tr}(A(\sigma)C_\sigma^{-1}) = \| \phi(\sigma) \| \| \eta(\sigma) \| \text{ is integrable, since } \phi \text{ and } \eta \text{ are square-integrable vector fields. Hence all requirements of Theorem 3.4 are met, and we obtain almost everywhere }
\]

\[
a(x) = \int_G \text{tr}(A(\sigma)C_\sigma^{-1}U_\sigma(x)^*)d\nu_G(\sigma) = \int_G (U_\sigma(x)\eta(\sigma)|\phi(\sigma)\rangle d\nu_G(\sigma) = (V_\eta\phi)(x)
\]

The equality of norms is then immediate, since the right hand side is the norm squared of \( A \) in \( B^2 \); and the admissibility condition is an immediate corollary.

The construction of admissible vectors for representations with multiplicities can be a subtle task. For instance, it is known that \( \lambda_G \) has admissible vectors, for \( G \) non-unimodular with type-I regular representation [18], but a direct construction of such vectors, without the use of Plancherel transform, could not be given. By contrast, the admissibility condition of the corollary is fairly easy to handle, once the direct integral decomposition of the representation is obtained. One important class of representations which fall under this category are the quasi-regular representations of certain semidirect product groups, see [17].

**Remark 3.7** At the moment, we do not know whether the requirement \( \eta_\sigma \in \text{dom}(C_\sigma) \) is necessary for admissibility, i.e., for the finiteness of \( \| V_\eta\phi \|^2 \), though we expect it to be true.

A criterion for the existence of admissible vectors is given in the following theorem. The proof for the unimodular part is a straightforward consequence of Corollary 3.4, for the non-unimodular case see [18], where in fact all representations having admissible vectors are classified.

**Theorem 3.8** The unitary representation \( U_\pi \) has admissible vectors iff \( G \) is nonunimodular or \( G \) is unimodular and \( 0 < \nu_G(\Sigma) < \infty \).
4 Example of the (1+1)-Poincaré group

In this section we want to calculate the Plancherel measure of the Poincaré group in (1 + 1)-dimensional space-time. This is the group $\mathcal{P}^+_1(1,1) = \mathbb{R}^2 \rtimes \mathrm{SO}_0(1,1)$ (connected part of $\mathrm{SO}(1,1)$) and we shall explicitly construct admissible vectors for some of its representations. Note that $\mathrm{SO}_0(1,1)$ is the proper Lorentz group in a space-time of (1 + 1)-dimensions. In computing the Plancherel measure, we follow the procedure given by Kleppner and Lipsman \[24\], which employs the Mackey machinery for this purpose. Recall that it follows from Mackey’s theory of induced representations \[29, 30\], that (almost all of) the unitary irreducible representations of $\mathbb{R}^2 \rtimes \mathrm{SO}_0(1,1)$ are in one-to-one correspondence with the orbits of $\mathrm{SO}_0(1,1)$ in the dual space $\hat{\mathbb{R}}^2$ (which we identify here with $\mathbb{R}^2$ itself). Here we have used the fact that $\mathrm{SO}_0(1,1)$ operates freely on $\hat{\mathbb{R}}^2 \setminus \{0\}$, such that each dual orbit contributes precisely one irreducible representation, and we have dropped the one-dimensional representations arising from the dual orbit $\{0\}$.

We parametrize the Lorentz group by

$$\mathbb{R} \ni \theta \mapsto \Lambda_\theta = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}. \quad (15)$$

In this parametrization, $d\theta$ is invariant, under both left and right actions, and we choose this for the Haar measure on $\mathrm{SO}_0(1,1)$. We write a generic element of $G$ as $(x,h)$, with $x = \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^2$ and $h$ a matrix of the form $(15)$. As Haar measure on $\mathcal{P}^+_1(1,1)$ we may take $d\mu(x,h) = dx d\theta$, $dx$ being the Lebesgue measure on $\mathbb{R}^2$, and note that this group is unimodular.

The first step for the calculation of the Plancherel measure is the computation of the dual orbits. They are conveniently represented by the set

$$\left\{ yv \mid v \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, y \in \mathbb{R}^* \right\} \cup \left\{ \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad (16)$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The last five points represent Lebesgue-null sets; and hence the set of representations arising from these orbits will have Plancherel measure zero (see below). We therefore drop them from further discussion. (It ought to be pointed out, however, that the first four of these orbits correspond, physically, to zero-mass systems. Thus, while they do not play any role in the Plancherel theory, they are by no means physically negligible.) On
any of the remaining orbits, $O_{v,y} = ySO_0(1,1)v$, the Lorentz group operates freely. Hence we obtain the parametrization

$$\mathbb{R} \ni \theta \mapsto k = \begin{pmatrix} k_0 \\ k \end{pmatrix} := y \Lambda_{\theta} v \in O_{v,y} ,$$

of $O_{v,y}$, and the measure $d\theta$ is the image of the Haar measure of $SO_0(1,1)$, under this parametrization. Hence, up to a null set, $\hat{\mathbb{R}}^2$ is parametrized by

$$\mathbb{R} \times \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \times \mathbb{R}^* \ni (\theta, v, y) \mapsto y \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} v ,$$

where $\theta$ parametrizes $O_{v,y}$, and $(v, y)$ parametrizes the orbit space.

By Mackey’s theory of induced representations [29, 30], each $O_{v,y}$ contributes exactly one representation class $\sigma_{v,y} \in \mathcal{P}_+^+(1,1)$. Denoting the corresponding induced representation in this class by $U_{v,y}$, its action on $L^2(O_{v,y}, d\theta)$ is given by

$$(U_{v,y}(x, h)f)(k) = e^{i\langle k, x \rangle} f(h^{-1}k) ,$$

$\langle , \rangle$ denoting the dual pairing between $\mathbb{R}^2$ and $\hat{\mathbb{R}}^2$, which we take (following the physicists’ convention) as $\langle k, x \rangle = k_0 x_0 - k x$. Note that this choice of dual pairing, as opposed to the more conventional mathematician’s choice, $\langle k, x \rangle = k_0 x_0 + k x$, does not change the dual action of $SO_0(1,1)$.

Hence the Plancherel measure $\nu_P$, of the semidirect product group $\mathcal{P}_+^+(1,1) = \mathbb{R}^2 \rtimes SO_0(1,1)$, can be viewed as a measure on the orbit space $\hat{\mathbb{R}}^2/SO_0(1,1)$, or, equivalently, on $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \times \mathbb{R}^*$. It is obtained by decomposing the Lebesgue measure of $\hat{\mathbb{R}}^2$ along the orbits; in other words, we are looking for a measure $\overline{\lambda}$ on $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \times \mathbb{R}^*$ such that in the parametrization (17) of $\hat{\mathbb{R}}^2$, the Lebesgue measure is given by $d\theta d\overline{\lambda}(v, y)$. By computing the Jacobian of (17), we obtain

$$d\nu_P(\sigma_{v,y}) = d\overline{\lambda}(v, y) = dv \, dy ,$$

where $dv$ is just the counting measure on the two-element set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

That we have indeed computed the Plancherel measure is due to [17], Theorem 3.3. (An alternative argument could be derived from [24], II,
Theorem 2.3], or rather, the proof of that result.) Generally, the procedure for the computation of the Plancherel measure of semidirect products $\mathbb{R}^k \rtimes H$ following Kleppner and Lipsman [24], involves three steps, which in this (unimodular) setting may be roughly sketched as follows: First compute invariant measures on the orbits (in our case, this was the measure $d\theta$). Then compute a unique measure on the orbit space (our $\nu_P$) such that first integrating along the orbits and then integrating over the orbit space gives Lebesgue measure on the dual. Finally the Plancherel measures of the little fixed group and the measure on the orbit space can be combined to give the Plancherel measure of the semidirect product. In our case, the little fixed groups are trivial, and in this case the last step reduces to a – still somewhat subtle – normalization issue. (This is discussed at length in [17].)

The role played by the decomposition of Lebesgue-measure for the construction of the Plancherel measure also justifies dropping the five orbits from our discussion: They constitute a Lebesgue-null set, hence the representations arising from the orbits are a null set with respect to the Plancherel measure.

The Poincaré group above also provides us with an easy example of the use of Theorem 3.8. Since the individual points $\sigma_{v,y} \in \mathcal{P}_+(1,1)$ have Plancherel measure zero, none of the (irreducible) representations $U_{v,y}$ is by itself square-integrable and hence does not have admissible vectors. However, it is known [11, 2] that if one works on the homogeneous space $\mathcal{P}_+(1,1)/T$, where $T = \{ (x_0,0) \mid x_0 \in \mathbb{R} \}$ is the subgroup of time translations, it is possible to again obtain admissible vectors for these irreducible representations. On the other hand, it should also be possible, according to Theorem 3.8, to take sets of these representations, of finite Plancherel measure such that the corresponding (reducible) direct integral representations possess admissible vectors. Such a construction was done in [23] (without, however, relating it to the Plancherel theorem). Let us briefly work out the construction again, in the light of Theorem 3.8.

Let $v = (1,0)$ and $\Sigma$ be any Borel subset of $\mathbb{R}_+$ for which

$$0 < \nu_P(\Sigma) = \int_{\Sigma} y \, dy < \infty.$$ 

Consider the direct integral Hilbert space and the direct integral representation on it,

$$\mathcal{H}_\Sigma = \int_{\Sigma}^\oplus L^2(O_{v,y}, \, d\theta) \, y \, dy, \quad U_{\Sigma}(x, h) = \int_{\Sigma}^\oplus U_{v,y}(x, h) \, y \, dy.$$
Elements $\phi \in H_\Sigma$ are fields of vectors $\phi_{v,y} \in L^2(O_{v,y}, d\theta)$, $y \in \Sigma$, representable by functions on $\mathbb{R}^2$ of the type,

$$\phi_{v,y}(k) = \phi_{v,y}(k_0, k) = \phi_{v,y}(y \cosh \theta, y \sinh \theta), \quad y = \frac{k_0}{|k_0|} \sqrt{k_0^2 - k^2} \in \Sigma.$$ 

Explicitly, the representations $U_{v,y}(x, h)$ act on the Hilbert spaces $L^2(O_{v,y}, d\theta)$ in the manner,

$$(U_{v,y}(x, h)\phi_{v,y})(y \cosh \theta, y \sinh \theta) = \exp \left[ i y (x_0 \cosh \theta - x \sinh \theta) \right] \times \phi_{v,y}(y \cosh(\theta - \xi), y \sinh(\theta - \xi)),$$

where we have written

$$h = \Lambda_\xi = \begin{pmatrix} \cosh \xi & \sinh \xi \\ \sinh \xi & \cosh \xi \end{pmatrix}, \quad x = \begin{pmatrix} x_0 \\ x \end{pmatrix},$$

$$<k, x> = k_0 x_0 - k x = x_0 \cosh \theta - x \sinh \theta.$$

If we use the variables $(k_0, k)$ rather than $(y, \theta)$ to designate points in the orbits, then

$$y \ dy \ d\theta = dk_0 \ dk,$$

and

$$(U_{v,y}(x, h)\phi_{v,y})(k) = \exp[i(k_0 x_0 - k x)] \phi_{v,y}(h^{-1}k), \quad y = \frac{k_0}{|k_0|} \sqrt{k_0^2 - k^2}. \quad (19)$$

Let $\eta = \{\eta_{v,y} | y \in \Sigma\} \in H_\Sigma$ be a vector such that $\|\eta_{v,y}\| = \frac{1}{2\pi}$, for almost all $y$. Then

$$\langle U_\Sigma(x, h)\eta | \phi \rangle = \int_{\Sigma \times \mathbb{R}} e^{-i(k_0 x_0 - k x)} \overline{\eta_{v,y}(h^{-1}k)} \phi_{v,y}(k) \ dk_0 \ dk, \quad (20)$$

and a straightforward computation shows that

$$\int_G |\langle U_\Sigma(x, h)\eta | \phi \rangle|^2 \ d\mu(x, h) = \|\phi\|^2, \quad \phi \in H_\Sigma.$$ 

Thus, the vector $\eta$ is admissible for the representation $U_\Sigma(x, h)$, and defining coherent states, $\eta(x, h) = U_\Sigma(x, h)\eta$, we get the resolution of the identity on $H_\Sigma$,

$$\int_G |\eta(x, h)\rangle \langle \eta(x, h) | \ d\mu(x, h) = \text{Id}_\Sigma. \quad (21)$$
Before leaving this section, it is worthwhile looking also at the affine Poincaré group, which is the Poincaré group \( \mathcal{P}^\dagger(1,1) \) just considered, together with dilations. Writing this group as \( \mathcal{P}_{\text{Aff}}(1,1) = \mathbb{R}^2 \rtimes H \), where \( H \) now consists of matrices of the type
\[
a \Lambda_\theta = \begin{pmatrix} a \cosh \theta & a \sinh \theta \\ a \sinh \theta & a \cosh \theta \end{pmatrix}, \quad a > 0,
\]
we see that the orbits of \( H \) in \( \mathbb{R}^2 \) consist of the four open cones,
\[
C_{\pm}^{\tau, \dagger} = Hv_{\pm}^{\tau, \dagger}, \quad v_{\pm}^{\tau, \dagger} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \quad v_{\pm}^\dagger = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix},
\]
the four semi-infinite lines,
\[
\ell_{\pm}^\tau = Hv_{\pm}^\tau, \quad v_{\pm}^\tau = \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix},
\]
and the singleton consisting of the origin. The first four are open free orbits, which are unions of orbits of the Poincaré group \( \mathcal{P}^\dagger(1,1) \) (see \((16)-(17)\). For example,
\[
C_+^\dagger = \bigcup_{y>0} O_{v,y}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_-^\tau = \bigcup_{y<0} O_{v,y}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
\((24)\)

etc. The remaining five orbits of \( \mathcal{P}_{\text{Aff}}(1,1) \) coincide with the five orbits of \( \mathcal{P}^\dagger(1,1) \) which have Plancherel measure zero. The Plancherel measure of \( \mathcal{P}_{\text{Aff}}(1,1) \) is just the counting measure on the first four orbits, the last five orbits again having Plancherel measure zero. The unitary irreducible representations corresponding to the orbits \( C_{\pm}^{\tau, \dagger} \) are again induced representations (from the subgroup of \( H \) consisting of the identity element alone) and are square-integrable. However, the group \( \mathcal{P}_{\text{Aff}}(1,1) \) is nonunimodular and hence not every vector in these representations is admissible (see, e.g., \([1, 10]\)).

5 Wigner functions

Wigner functions are a class of transforms associated to elements of the direct integral Hilbert space appearing in \((4)\). We denoted this space by \( \mathcal{B}_2^\oplus \) in Section \([3]\). The Wigner map associates its elements isometrically to
square-integrable functions on the dual of the Lie algebra of $G$. This dual space foliates into orbits under the coadjoint action of the group, the invariant components being often identifiable with phase spaces of physical systems. Motivated by the properties of such a function, originally introduced in the context of quantum statistical mechanics by Wigner [34], a general procedure for constructing analogous maps (applicable to a class of groups admitting square integrable representations) was introduced in [4] and further discussed in a specific context in [3]. Here we extend the definition of a Wigner function given in [4] to representations which are not necessarily square integrable, using the Plancherel transform. This will also bring into focus the fact that the Wigner function, like the wavelet transform, owes its existence to the Plancherel transform.

It will first be necessary to set out a few details about Lie groups and their duals. Again, let $G$ be a Lie group with a Type-I regular representation, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. We make the assumption that the range of the exponential map, $\mathfrak{g} \ni X \mapsto e^X \in G$, is a dense set in $G$, and such that its complement has Haar measure zero. By an exponential group we mean a simply connected, connected solvable Lie group for which the exponential map is a homeomorphism. A nilpotent group is understood to be a simply connected, connected nilpotent Lie group. In particular, nilpotent groups are exponential. A Lie group has a natural action on its Lie algebra, the adjoint action, $X \mapsto \text{Ad}_{x_0} X$, $x_0 \in G$, defined by, $x_0^{-1} e^{X} x_0 = e^{[\text{Ad}_{x_0} X]}$. The dual of this map, acting on $\mathfrak{g}^*$, defines the coadjoint action, $\mathfrak{g}^* \ni X^* \mapsto \text{Ad}^*_{x_0} X^*$, $x_0 \in G$, via $\langle \text{Ad}^*_{x_0} X^* ; X \rangle = \langle X^* ; \text{Ad}_{x_0^{-1}} X \rangle$, where $\langle ; \rangle$ denotes the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Orbits of vectors in $\mathfrak{g}^*$ under the coadjoint action are the coadjoint orbits of the group $G$. The corresponding orbit space, denoted $O(G)$, has a natural quotient topology, and according to the Kirillov theory [21, 22] for nilpotent groups, later extended to exponential groups [20], this space is homeomorphic to the unitary dual, $\hat{G}$, of the group, via the so-called Kirillov map. One example is the Poincaré group $\mathcal{P}^*_+(1,1)$. For this group, each coadjoint orbit can be naturally identified with the cotangent bundle of a corresponding dual orbit. More generally speaking, it is known that coadjoint orbits have the structure of symplectic manifolds and carry natural invariant measures under the coadjoint action, making them resemble physical phase spaces. The collection of coadjoint
orbits exhausts \( \mathfrak{g}^* \), allowing for a foliation of the type

\[
\mathfrak{g}^* = \bigcup_{\lambda \in J} \mathcal{O}_\lambda,
\]

where \( \mathcal{O}_\lambda \in \mathcal{O}(G) \) denotes an orbit, parametrized by an index (or collection of indices) \( \lambda \), and \( J \) is the corresponding index set. We make the assumption that the orbit space is a countably separated Borel space, in which case the Lebesgue measure on \( \mathfrak{g}^* \) can be decomposed along these orbits, i.e., if \( dX^* \) denotes this Lebesgue measure, then it is possible to write,

\[
dX^* = \sigma_\lambda(X^*_\lambda) \, d\Omega_\lambda(X^*_\lambda), \quad X^*_\lambda \in \mathcal{O}_\lambda,
\]

where \( \sigma_\lambda \) is a positive density defined on the orbit \( \mathcal{O}_\lambda \) and \( d\Omega_\lambda \) the (coad)-invariant measure on \( \mathcal{O}_\lambda \). Note that the assumption on the coadjoint orbits entails that also the dual space \( \hat{G} \) is a countably separated Borel space, which is equivalent to the Type-I property of \( G \) (and thus of \( \lambda G \)).

The measure \( \kappa \) on the orbit space could be continuous or discrete; whenever it has an atom, it is in fact supported on finitely many of them. It is only necessary to assume that the above disintegration holds on an open dense set of \( \mathfrak{g}^* \), such that its complement has Lebesgue measure zero. (Such a decomposition, which is sort of a regularity condition, certainly holds for nilpotent groups [5] and semi-direct product groups admitting open free orbits [25], and in these cases, the measure \( \kappa \) is essentially the Plancherel measure.)

For each orbit \( \mathcal{O}_\lambda \), consider the Hilbert space \( L^2(\mathcal{O}_\lambda, d\Omega_\lambda) \) and denote by \( \mathcal{H}^\sharp \) the direct integral Hilbert space,

\[
\mathcal{H}^\sharp = \int J L^2(\mathcal{O}_\lambda, d\Omega_\lambda) \, d\kappa(\lambda) \simeq L^2(\mathfrak{g}^*),
\]

where we used the measure disintegration [23] to canonically identify \( L^2(\mathfrak{g}^*) \) with the direct integral. Elements in \( \mathcal{H}^\sharp \) are fields of vectors, \( \Phi = \{ \Phi_\lambda \in L^2(\mathcal{O}_\lambda, d\Omega_\lambda) \}_{\lambda \in J} \) with the norm,

\[
\| \Phi \|^2 = \int J \| \Phi_\lambda \|^2 \, d\kappa(\lambda),
\]

the norm inside the integral being taken in \( L^2(\mathcal{O}_\lambda, d\Omega_\lambda) \). If the measure \( \kappa \) is discrete, then clearly the integral would just be a sum. The Wigner map will be defined as a linear isometry, \( W : \mathcal{B}^\oplus_2 \rightarrow \mathcal{H}^\sharp \). Let \( N_0 \in \mathfrak{g} \) be the
maximal symmetric set (i.e., $N_0$ includes the origin and $X \in g \Rightarrow -X \in g$) such that its image under the exponential map is dense in $G$ and such that the complement of this image set has Haar measure zero. For any $f \in L^2(G)$, $f(e^X)$ defines a function on $N_0$. We transfer the left Haar measure $\mu_G$ to $N_0$, using the exponential map and write,

$$d\mu_G(g) = d\mu_G(e^X) = m(X) \, dX, \quad X \in N_0,$$

where $dX$ is the Lebesgue measure of $g$ and $m$ an appropriate, positive density function. It is not hard to see that

$$m(X) = |\det [-F(\text{ad}X)]|,$$

where $F$ is the function (25) defined in the Appendix and $\text{ad}X$ the linear transformation on $g$ defined by $\text{ad}X(Y) = [X,Y]$, $Y \in g$.

Let us next define a modified Fourier transform, $\mathcal{F} : L^2(G) \rightarrow \mathfrak{h}^\flat$ as

$$(\mathcal{F}f)_{\lambda}(X_\lambda^*) = \frac{[\sigma_{\lambda}(X_\lambda^*)]^\frac{1}{2}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i(X_\lambda^*:X)} f(e^X)[m(X)]^\frac{1}{2} \, dX,$$

at least on $L^1(G) \cap L^2(G)$, and extend by continuity. (Note, we are assuming the dimension of the group $G$, and hence of its Lie algebra, to be $n$). Since the complement of the set $N_0$ is of (Haar) measure zero, this map is easily seen to be an isometry. For nilpotent groups (28) simplifies considerably since all involved density functions (i.e., $\sigma_{\lambda}$, $m$) are identical one.

**Definition 5.1** The composite transformation

$$\mathfrak{M} := \mathcal{F} \circ P^{-1} : B_2^\oplus \rightarrow \mathfrak{h}^\flat,$$

where $P$ is the Plancherel transform in (4), is called the Wigner map and for any $A \in B_2^\oplus$, the function

$$W(A|X_\lambda^*) := (\mathfrak{M}A)_{\lambda}(X_\lambda^*), \quad X_\lambda^* \in \mathcal{O}_\lambda$$

is called the Wigner function of $A$, restricted to the orbit $\mathcal{O}_\lambda$.

For any $A \in \mathfrak{h}^\sharp$ which satisfies the conditions of Theorem 3.4, using (11) we obtain the following explicit expression for its Wigner function:

$$W(A|X_\lambda^*) = \frac{[\sigma_{\lambda}(X_\lambda^*)]^\frac{1}{2}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i(X_\lambda^*:X)}$$

$$\times \left[ \int_{\hat{G}} \text{tr} \left( U_\sigma(\text{e}^{-X})[A(\sigma)C_{\sigma}^{-1}] \right) [m(X)]^\frac{1}{2} \, d\nu_G(\sigma) \right] \, dX,$$

where $\sigma$ is the unitary representation of $G$ on $L^2(G)$. The above expression is identical to (31) and simplifies considerably for nilpotent groups.
provided the inverse Plancherel transform of $A$ is integrable. The inverse of this transform can be computed using (11) and standard Fourier transform methods. We get,

$$A(\sigma) = \frac{1}{(2\pi)^{n+1}} \left\{ \int_{N_0} \left[ \int_{O_\lambda} \int_{\mathcal{O}_\lambda} e^{i\langle X_\lambda^* : X \rangle} W(A|X_\lambda^*) U_\sigma(e^X) \right. \right. $$

$$\times \left[ \sigma_\lambda(X_\lambda^*) m(X) \right]^{\frac{1}{2}} \, d\Omega_\lambda \, d\kappa(\lambda) \right\} C^{-1}_{\sigma},$$

(32)

the extreme pair of square brackets implies taking the closure of the operator involved.

A few properties of the Wigner map can easily be established from its definition. We collect these into the theorem below. The proof involves straightforward computations, similar to those done to obtain analogous results in [4], and we omit it.

On $\mathcal{B}_2^\oplus$ and $\mathcal{H}^\sharp$ we define the two unitary representations, $U^\oplus$ and $U^\sharp$, of the group $G$:

$$(U^\oplus(x)A)(\sigma) = U_\sigma(x)A(\sigma)U^*_\sigma(x), \quad x \in G,$$

(33)

the above relations holding for almost all $\sigma \in \hat{G}$ (w.r.t. $\nu_G$), and

$$(U^\sharp(x)\Phi)_\lambda(X_\lambda^*) = \Phi_\lambda(\text{Ad}^\sharp_{x^{-1}} X_\lambda^*), \quad x \in G,$$

(34)

holding for almost all $\lambda \in J$ (w.r.t. $\kappa$).

**Theorem 5.2** The Wigner map is a linear isometry, which intertwines the representation $U^\oplus$ of $G$ with the representation $U^\sharp$. The corresponding Wigner function satisfies the **overlap condition**,

$$\int_J \left[ \int_{\mathcal{O}_\lambda} \frac{W(A^1|X_\lambda^*) W(A^2|X_\lambda^*) \, d\Omega_\lambda}{d\kappa(\lambda)} \right] \, d\kappa(\lambda) = \langle A^1|A^2 \rangle_{\mathcal{B}_2^\oplus}, \quad A^1, A^2 \in \mathcal{B}_2^\oplus,$$

(35)

and the **covariance condition**,\n
$$W(U^\oplus(x)A|X_\lambda^*) = W(A|\text{Ad}^\sharp_{x^{-1}} X_\lambda^*),$$

(36)

for all $x \in G$, and almost all $X_\lambda^* \in \mathcal{O}_\lambda$ (w.r.t. $\Omega_\lambda$). If $A = A^*$ is self adjoint, its Wigner function is real, i.e.,

$$W(A|X_\lambda^*) = W(A|X_\lambda^*),$$

(37)

almost everywhere.
Note that if \( A^1, A^2 \in B_2^\oplus \) satisfy the conditions of Theorem 3.4, then Theorem 2.1 together with (11) implies the orthogonality relation

\[
\int_{\hat{G}} \left\{ \int_{G} \text{tr}(U_{\sigma}(x)^* A^1(\sigma) C^{-1}_{\sigma}) d\nu_G(\sigma) \times \int_{G} \text{tr}(U_{\sigma'}(x)^* A^2(\sigma') C^{-1}_{\sigma'}) d\nu_G(\sigma') \right\} d\mu(x) = \langle A^1 \mid A^2 \rangle_{B_2^\oplus}, \quad (38)
\]

which is equivalent to the overlap condition (35). If now \( \sigma \in \hat{G} \) has positive Plancherel measure, then the above relation implies the restricted orthogonality relation,

\[
\int_{G} \text{tr}(U_{\sigma}(x)^* A^1(\sigma) C^{-1}_{\sigma}) \text{tr}(U_{\sigma'}(x)^* A^2(\sigma') C^{-1}_{\sigma'}) d\mu(x) = \text{tr} [A^1(\sigma)^* A^2(\sigma)],
\]

familiar from the theory of square integrable group representations [1, 19]. This equation was the basis for the construction of Wigner functions, for square integrable group representations, in [4].

**Remarks 5.3** A few comments are in order here:

(a) Generally, the range of the Wigner map \( \mathcal{W} \) is a closed, proper subspace of \( \mathfrak{h}^\sharp \), which we denote by \( \mathfrak{h}_W^\sharp \). If we restrict the Wigner map to a subspace of \( B_2^\oplus \) of the type

\[
B_2^\Sigma = \int_{\Sigma} B_2(\mathfrak{h}_\sigma) \ d\nu_G(\sigma),
\]

where \( \Sigma \) is subset \( \sigma \) of \( \hat{G} \), such that \( \nu_G(\Sigma) \neq 0 \), then clearly its range \( \mathcal{W}(B_2^\Sigma) \) is a closed subspace of \( \mathfrak{h}_W^\sharp \). In this case, the integral over \( \hat{G} \) in (31) has to be replaced by an integral over \( \Sigma \), however the expressions (32) and (35) remain unchanged. In particular, if \( \Sigma \) is a discrete subset, the representations \( \sigma \in \Sigma \) are square-integrable and we recover the results of [4].

(b) Suppose that the group \( G \) is exponential and assume, moreover, that it is a Type-I group. Note that the homeomorphism property of the Kirillov map entails that the coadjoint orbit space is a countably separated Borel space; in particular the measure disintegration (23) exists.
Then we have on the one hand a mapping

\[ W : \int_{\hat{G}} \mathcal{B}_2(\mathcal{F}_\sigma) d\nu_G(\sigma) \to \int_J \mathcal{L}^2(\mathcal{O}_\lambda, d\Omega_\lambda) d\kappa(\lambda) \simeq \mathcal{L}^2(\mathfrak{g}^*), \]

between the direct integral spaces, and on the other hand the inverse of the Kirillov map, which gives rise to a bijection \( \hat{G} \to J, \sigma \mapsto \lambda_\sigma \). It is thus a natural question to ask whether the Wigner map is decomposable, i.e., if there exists a field of operators

\[ W_{\sigma} : \mathcal{B}_2(\mathcal{F}_\sigma) \to \mathcal{L}^2(\mathcal{O}_{\lambda_\sigma}, d\Omega_{\lambda_\sigma}) \]

such that for almost all \( \sigma \in \hat{G} \) and almost all \( X^*_{\lambda_\sigma} \in \mathcal{O}_{\lambda_\sigma} \),

\[ W(A|X^*_{\lambda_\sigma}) = [W_{\sigma}A(\sigma)](X^*_{\lambda_\sigma}) \, . \]

The existence of such a field of operators is not just of mathematical interest, but also desirable from a physical point of view: The coadjoint orbits have a natural interpretation as phase spaces of physical systems, but the dual \( \mathfrak{g}^* \), as a disjoint union of such phase spaces, does not usually have a natural interpretation, except in some cases, where one might look upon a set of orbits as constituting the phase space of a composite physical system. Correspondingly, the space \( \mathcal{L}^2(\mathcal{O}_{\lambda_\sigma}, d\Omega_{\lambda_\sigma}) \) has a simpler interpretation than \( \mathcal{L}^2(\mathfrak{g}^*) \). A similar reasoning applies to the representations.

A related question concerns the supports of the Wigner functions. Even when the Wigner map is restricted to a subspace such as \( \mathcal{B}_2^\Sigma \) as in part (a), the corresponding Wigner functions could in general have supports on orbits which are not associated to the representations in \( \Sigma \) (see example of the Poincaré group below). This is possible even for representations which arise from semidirect product groups admitting open free orbits [25]. It is obvious that whenever the Wigner map is decomposable, the supports of the Wigner functions of elements in \( \mathcal{B}_2^\Sigma \) are contained (up to a null set) in the coadjoint orbits corresponding to \( \Sigma \); we expect the converse of this statement to hold as well.

It turns out that these questions have been addressed, and to a large extent solved, in the context of star products: First of all, the nilpotent Lie groups for which the Wigner transform is decomposable are precisely those for which almost all coadjoint orbits are affine subspaces
If a nilpotent Lie group does not fulfill this condition, the modified Fourier transform (28) can be replaced by an adapted Fourier transform. Following [5, 7], the adaptation consists in constructing a suitable mapping \( \alpha : g \times V \to \mathbb{R} \), polynomial in the elements of \( g \) and rational in the elements of a suitably chosen open conull subset \( V \subset g^* \). The specific construction of \( \alpha \) first ensures that defining

\[
\mathcal{F}_{ad}(f)(X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_g e^{-i\alpha(X,X^*)} f(e^X) \, dX ,
\]

for \( f \) in the Schwartz space of the group, induces a a unitary map \( L^2(G) \to L^2(g^*) \). Secondly, the adapted Wigner map \( \mathcal{W}_{ad} = \mathcal{F}_{ad} \circ \mathcal{P}^{-1} \) has all the properties of the Wigner map collected in Theorem 5.2, and is in addition decomposable. Extensions of this construction to certain solvable groups exist [6]. It seems worthwhile to explicitly work out adapted Wigner transforms for concrete examples. This might also provide additional criteria for the choice of \( \alpha \), which is apparently not unique.

6 Some examples

Let us go back to the Poincaré groups \( \mathcal{P}_+^\uparrow(1,1) \) and \( \mathcal{P}_{\text{Aff}}(1,1) \), studied in Section 4, and explicitly compute the Wigner functions for them.

6.1 The Poincaré group \( \mathcal{P}_+^\uparrow(1,1) \)

We start by writing a general element of \( \mathcal{P}_+^\uparrow(1,1) \) as a \( 3 \times 3 \) matrix,

\[
(x, h) = \begin{pmatrix} h & x & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad 0^T = (0, 0),
\]

where \( x \) and \( h \) are as defined earlier (in Section 4). The Lie algebra \( \mathfrak{p} \) is generated by the three elements,

\[
Y^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\( (40) \)
which satisfy the commutation relations,

\[ [Y^1, Y^2] = Y^3, \quad [Y^1, Y^3] = Y^2, \quad [Y^2, Y^3] = 0. \]

A general element \( X \in \mathfrak{p} \) can be written as (see (57) in the Appendix),

\[
X = \begin{pmatrix}
X_q & x_p \\
0 & 0
\end{pmatrix}, \quad X_q := X_q(0, \theta) = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}, \quad x_p = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2,
\]

so that

\[
e^X = \begin{pmatrix} e^{X_q} & F(X_q)x_p \\ 0 & 1 \end{pmatrix},
\]

\( F(X_q) \) being the matrix function defined in (59) in the Appendix. Following (26), the Haar measure \( d\mu(x, h) = dx \, d\theta \), expressed in terms of the Lie algebra variables \( x_p, \theta \), becomes

\[
d\mu(x, h) = m(x_p, \theta) \, dx_p \, d\theta, \quad dx_p = d\xi_1 \, d\xi_2,
\]

and the density \( m(x_p, \theta) \) is easily calculated to be (see (59) in the Appendix),

\[
m(x_p, \theta) = \det [F(X_q)] = \sinh^2(\frac{\theta}{2}). \tag{41}
\]

The adjoint action of \( \mathcal{D}_+^+(1, 1) \) on \( \mathfrak{p} \), given by \( X \rightarrow X' = (x, h)X(x, h)^{-1} \), leads to the transformation

\[
\begin{pmatrix} x'_p \\ \theta' \end{pmatrix} = M(x, h) \begin{pmatrix} x_p \\ \theta \end{pmatrix}, \quad M(x, h) = \begin{pmatrix} h & \sigma_1 x \\ 0 & 1 \end{pmatrix},
\]

of the variables \( x_p, \theta \), where \( \sigma_1 \) is the \( 2 \times 2 \) matrix defined in the Appendix. Let \( \mathfrak{p}^* \) denote the dual space of \( \mathfrak{p} \). We write elements \( X^* \in \mathfrak{p}^* \) in terms of the dual basis \( \{Y_1^*, Y_2^*, Y_3^*\} \) as \( X^* = \gamma Y_1^* + k_0 Y_2^* + k Y_3^* \) and compute the coadjoint action, in terms of a matrix \( M^T(x, h) \) acting on the variables \( k, \gamma \),

\[
\begin{pmatrix} k' \\ \gamma' \end{pmatrix} = M^T(x, h) \begin{pmatrix} k \\ \gamma \end{pmatrix} = M(-h^{-1}x, h^{-1})^T \begin{pmatrix} k \\ \gamma \end{pmatrix}, \quad k = \begin{pmatrix} k_0 \\ k \end{pmatrix} \in \mathbb{R}^2,
\]

to obtain,

\[
k' = h^{-1}k \\
\gamma' = \gamma - x^T \sigma_1 k', \quad x^T = (x_0, x).
\]
Using these relations, all the coadjoint orbits of $P^+_+(1,1)$ in $\mathbb{R}^3 \simeq p^*$ can now be calculated. Indeed, introducing the vectors $yv$, defined in (16), we get the coadjoint orbits

$$O^*_{v,y} = \left\{ yM^2(x,h) \begin{pmatrix} v \\ 0 \end{pmatrix} \, \big| \, (x,h) \in P^+_+(1,1) \right\},$$

(43)

which, taken together for all $y,v$, exhaust $\mathbb{R}^3$. It is also clear from (42) that these orbits are precisely the cotangent bundles, $O^*_{v,y} = T^*O_{v,y}$, of the orbits $O_{v,y} = ySO_0(1,1)v$ computed in Section (3).

Explicitly, let us take the set

$$\Sigma = \{ yv \mid y > 0 \} \subset \hat{P}^+_+(1,1), \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (44)$$

Points in the corresponding coadjoint orbits can then be parametrized as

$$O^*_{v,y} = \{(k,\gamma) \in \mathbb{R}^3 \mid k = (k_0,k), \ k_0 > 0, \ y = \sqrt{k_0^2 - k^2}\},$$

and the invariant measure under the coadjoint action calculated to be

$$d\Omega_{v,y}(k,\gamma) = \frac{dk}{k_0} d\gamma.$$

Of course, we could also use the alternative coordinatization $(\theta, -\gamma)$ for $O^*_{v,y}$, where $\theta = -\tanh^{-1}(k/k_0)$, which are actually the Darboux coordinates for this orbit, and then the invariant measure would simply be $d\theta d\gamma$. However it will be more useful, for the purposes of computing the Wigner function, to use the $(k,\gamma)$ coordinates. The Lebesgue measure on $p^*$ in the $(k,\gamma)$ coordinates is $dk_0 \, dk \, d\gamma$, and making the change of variables, $(k_0,k,\gamma) \rightarrow (y,k,\gamma)$ we get the measure disintegration along the coadjoint orbits in $\Sigma$ (see (25)),

$$dk_0 \, dk \, d\gamma = y \, dy \, \frac{dk}{\sqrt{k^2 + y^2}} \, d\gamma = \sigma_{v,y}(k,\gamma) \, d\kappa(v,y) \, d\Omega_{v,y}(k,\gamma),$$

with $\sigma_{v,y}(k,\gamma) \equiv 1$, $d\kappa(v,y) = y \, dy$. \quad (45)

Thus, in this case, the measure $\kappa$ is precisely equal to the Plancherel measure $\nu_P$, restricted to $\Sigma$ (see (18)).

We are now in a position to explicitly compute the Wigner map

$$\mathfrak{W} : B^\Sigma_2 \longrightarrow \hat{\mathfrak{h}}^\#,$$
for $P^+_1(1,1)$, restricted to the subset $\Sigma$ defined in (44). Since $P^+_1(1,1)$ is unimodular, the Duflo-Moore operators $C^\sigma$ are trivial and in fact, using relations such as (20), it can be seen that for almost all $yv \in \Sigma$, the corresponding Duflo-Moore operator is $C_{v,y} = 2\pi \text{Id}_H$ on the representation space $H = L^2(O_{v,y}, d\theta)$. Let us consider elements in $B^\Sigma_2$ which are of the type

$$A = \{ A(v,y) \}_{yv \in \Sigma}, \quad A(v,y) = |\phi_{v,y}\rangle\langle\psi_{v,y}|, \quad \phi_{v,y}, \psi_{v,y} \in L^2(O_{v,y}, d\theta).$$

A tedious but straightforward manipulation, after inserting the various quantities into the expression (31) for the Wigner function and using relations such as (59), yields the final expression,

$$W(A|k_{v,y}, \gamma) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i \theta_1 \gamma} \psi_{v,y}(\theta) \left( e^{-\frac{i \theta_1}{2} \sigma_3} \frac{k_{v,y}}{\text{sinc} \left( \frac{\theta}{2} \right)} \right) \frac{1}{\text{sinc} \left( \frac{\theta}{2} \right)} \times \phi_{v,y}(\theta) \left( e^{\frac{i \theta_1}{2} \sigma_3} \frac{k_{v,y}}{\text{sinc} \left( \frac{\theta}{2} \right)} \right) d\theta, \quad y(\theta) = \frac{y}{\text{sinc} \left( \frac{\theta}{2} \right)}.$$ (46)

Here, $k_{v,y} \in O_{v,y}$, and the point $(k_{v,y}, \gamma) \in T^*O_{v,y}$ and hence the above expression is for the Wigner function restricted to the orbit $T^*O_{v,y}$. However, it ought to be noted that its value on any orbit receives contributions from vectors $\phi_{v,y}(\theta), \psi_{v,y}(\theta)$ coming from representations associated to all the orbits in $\Sigma$. Hence we see that the Wigner map is not decomposable. (Completely the opposite situation is true for the affine Poincaré group, as will be shown in Theorem 6.1 below.)

Using (42) and (60) we directly verify the covariance condition (36),

$$W(U_{\Sigma}^\oplus(x,h)A|k_{v,y}, \gamma) = W(A|k'_{v,y}, \gamma'), \quad \left( \begin{array}{c} k'_{v,y} \\ \gamma' \end{array} \right) = M^2(x,h)^{-1} \left( \begin{array}{c} k_{v,y} \\ \gamma \end{array} \right),$$

$(x,h) \in P^+_1(1,1)$, with

$$(U_{\Sigma}^\oplus(x,h)A)(v,y) = U_{v,y}(x,h)|\phi_{v,y}\rangle\langle\psi_{v,y}|U_{v,y}(x,h)^*, \quad yv \in \Sigma,$$

and the overlap condition,

$$\int_{O_{v,y} \times \mathbb{R}^+} W(A^1|k_{v,y}, \gamma)W(A^2|k_{v,y}, \gamma) d\Omega_{v,y}(k_{v,y}, \gamma) y dy =$$

$$\int_{\mathbb{R}^+} \langle\psi_{v,y}^1|\psi_{v,y}^2\rangle \langle\phi_{v,y}^1|\phi_{v,y}^2\rangle y dy,$$
where, $A^i = \{|\phi_{v,y}^i\rangle\langle\psi_{v,y}^i|\}_{y \in \Sigma}$, $i = 1, 2$.

Consider now the open forward light cone $C^\uparrow_+$ (see (24)),

$$C^\uparrow_+ = \left\{ \begin{pmatrix} k_0 \\ k \end{pmatrix} \in \mathbb{R}^2 \mid k_0 > 0, \ k_0^2 > k^2 \right\}.$$  

Using the coordinates $(y, \theta) = (\sqrt{k_0^2 - k^2}, -\tanh^{-1}(k/k_0))$, the invariant measure under the action of $SO_0(1, 1)$ is clearly $y \, dy \, d\theta$ and the Hilbert space $L^2(C^\uparrow_+, y \, dy \, d\theta)$ is naturally isomorphic to the direct integral Hilbert space

$$H_\Sigma = \int_{\Sigma}^{\oplus} L^2(O_{v,y}, d\theta) \, y \, dy.$$  

The corresponding direct integral representation $U_\Sigma$ can thus be expressed by its action on $L^2(C^\uparrow_+, y \, dy \, d\theta)$ in the manner

$$(U_\Sigma(x, h)\phi)(y, \theta) = \exp\left[iy(x_0 \cosh \xi - x \sinh \xi)\right] \phi(y, \theta - \xi),$$

$\phi \in L^2(C^\uparrow_+, y \, dy \, d\theta)$ and $\xi$ being the hyperbolic angle of the transformation $h$. Thus, the representation $U_\Sigma$ is precisely the Fourier transform of the quasi-regular representation of $\mathcal{P}^\uparrow_+(1, 1)$, restricted to the Hilbert space $L^2(C^\uparrow_+, y \, dy \, d\theta)$, of functions with support in the forward light cone. The Wigner function (46) can now be thought of as a function on $C^\uparrow_+ \times \mathbb{R} \simeq \bigcup_{y > 0} T^* O_{v,y}$ and, written in terms of the variables $(y, \theta, -\gamma)$ (the invariant measure under the coadjoint action being $y \, dy \, d\theta \, d\gamma$), it becomes

$$W'(\psi, \phi \mid \theta, \gamma; y) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{-i\xi \gamma} \psi\left(\frac{y}{\sinh(\xi/2)}, \theta - \frac{\xi}{2}\right) \frac{1}{\sinh(\xi/2)} \times \phi\left(\frac{y}{\sinh(\xi/2)}, \theta + \frac{\xi}{2}\right) d\xi, \ \theta, \gamma \in \mathbb{R}, \ y > 0,$$  

$\phi, \psi \in L^2(C^\uparrow_+, y \, dy \, d\theta)$. It ought to be noted, however, that although in this way of writing the Wigner function, $W'(\psi, \phi \mid k, \gamma)$ is sesquilinear in $\psi, \phi \in L^2(C^\uparrow_+, y \, dy \, d\theta)$, and can be extended to the linear span of rank-one operators $|\phi\rangle\langle\psi|$, it cannot be used to define an isometric map between $B_2(L^2(C^\uparrow_+, y \, dy \, d\theta))$ and $L^2(C^\uparrow_+ \times \mathbb{R}, y \, dy \, d\theta \, d\gamma)$, since

$$\int_{C^\uparrow_+ \times \mathbb{R}} |W'(\psi, \phi \mid \theta, \gamma; y)|^2 \, y \, dy \, d\theta \, d\gamma = \int_{\mathbb{R}^+ \times \mathbb{R}^2} |\psi(y, \theta)|^2 |\phi(y, \xi)|^2 \times y \, dy \, d\theta \, d\xi \neq \|\phi\|^2 \|\psi\|^2.$$  

30
Thus $W'(\phi, \psi | \theta, \gamma; y)$ is not a Wigner function for the operator $|\phi\rangle\langle\psi| \in B_2(L^2(C_+^\uparrow, y dy d\theta))$, in the sense of our definition (hence the use of the altered notation $W'$). On the other hand, physically the representation $U_\Sigma$ refers to systems of relativistic particles of all possible (positive) masses and $W'(\psi, \psi | y, \theta, \gamma)$ can serve as the Wigner function for the state of a system consisting of a cluster of masses. Furthermore, this form of the Wigner function is particularly simple looking and bears a striking resemblance to the original Wigner function [34] (see (52) below).

### 6.2 The affine Poincaré group $\mathcal{P}_{\text{Aff}}(1, 1)$

Elements, $(x, ah) \in \mathcal{P}_{\text{Aff}}(1, 1)$, $a > 0$, $h \in SO_0(1, 1)$ can be represented by matrices of the form

\[
(x, ah) = \begin{pmatrix} ah & x \\ 0 & 1 \end{pmatrix}.
\]

The group is nonunimodular, with left and right Haar measures,

\[
d\mu_\ell(a, h, x) = \frac{1}{a^3} \, dx_0 \, dx \, da \, d\theta, \quad d\mu_\ell(a, h, x) = \frac{1}{a} \, dx_0 \, dx \, da \, d\theta.
\]

As discussed at the end of Section 4, there are four irreducible representations of $\mathcal{P}_{\text{Aff}}(1, 1)$, corresponding to the four open free orbits $C_+^\uparrow, \downarrow$ (see (22)) which are square integrable, and these are the only ones which contribute to the Plancherel measure. It will be enough to work out the Wigner function for the one orbit $C_+^\uparrow$, for the other three are entirely similar. The Hilbert space of the irreducible representation $U_\uparrow^\rightarrow$, associated to this orbit, is $L^2(C_+^\uparrow, dk_0 \, dk)$ and

\[
(U_\uparrow^\rightarrow(x, ah)\phi)(k) = ae^{i(k \cdot x)} \phi(ah^{-1}k).
\]

The Duflo-Moore operator $C$ for this representation is unbounded, acting on $L^2(C_+^\uparrow, dk_0 \, dk)$ in the manner (see [1, 10]),

\[
(C\phi)(k) = \frac{2\pi}{|k_0^2 - k^2|^2} \phi(k).
\]

Recall that the orbit $C_+^\uparrow$ is characterized by $k_0 > 0$, $k_0^2 > k^2$ and the invariant measure on it under the action of the group elements $ah$ is $\frac{dk_0 \, dk}{k_0^2 - k^2}$.

The Lie algebra $\mathfrak{p}_{\text{Aff}}$ is four dimensional, being generated by the three elements (13) of the Lie algebra of $\mathcal{P}_\uparrow^\rightarrow(1, 1)$ together with $I_3$, the $3 \times 3$ identity.
matrix. Computation of the coadjoint orbits is routine. The one which concerns us here is the cotangent bundle, $T^*C_+^\uparrow$, of the orbit $C_+^\uparrow$. Denoting its elements by $(\gamma, k)$, $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathbb{R}^2$, $k \in C_+^\uparrow$, using relations such as $(60)-(61)$ the coadjoint action is computed to be,

$$k \rightarrow k' = \frac{1}{a} h^{-1} k,$$

$$\gamma \rightarrow \gamma' = \gamma + X_q(x_0, x) \frac{1}{a} h^{-1} k. \tag{48}$$

The invariant measure on $T^*C_+^\uparrow$ under this action is

$$d\Omega_+^\uparrow(k, \gamma) = \frac{dk_0}{k_0^2 - k^2} \frac{dk}{k} d\gamma_1 d\gamma_2,$$

and thus the densities $\sigma_\lambda$ and $m$ appearing in the Wigner function (see $(25)$, $(26)$ and $(31)$ become in this case,

$$\sigma_+^\uparrow(k, \gamma) = k_0^2 - k^2, \quad m(\lambda, \theta) = \frac{2(\cosh \lambda - \cosh \theta)}{e^{\lambda^2 - \theta^2}}.$$

The final expression for the Wigner function is obtained after a routine computation, starting with $(31)$ and using expressions such as $(59)-(61)$ in the appendix. We get,

$$W(\psi, \phi \mid k, \gamma) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\theta \left. e^{-i(\gamma_1 \lambda + \gamma_2 \theta)} \phi \left( \frac{\sigma_3 e^{X_q(\lambda, \theta)/2}}{\sinh (X_q(\lambda, \theta)/2) k} \right) \right|_{k_0}^{k} \times \frac{(k_0^2 - k^2)(\lambda^2 - \theta^2)}{\cosh \lambda - \cosh \theta} \phi \left( \frac{\sigma_3 e^{-X_q(\lambda, \theta)/2}}{\sinh (X_q(\lambda, \theta)/2) k} \right), \tag{49}$$

(where we have written $\frac{1}{\sinh A}$ for $[\sinh A]^{-1}$). The above expression should be compared with $(43)$ and $(47)$. Also, by virtue of Lemma 7.1 and (58) in the Appendix, if $k \in C_+^\uparrow$ then so also are the arguments of the functions $\psi$ and $\phi$ in the above expression for the Wigner function. Thus we have the important result:

**Theorem 6.1** The Wigner function $W(\psi, \phi \mid k, \gamma)$ in $(49)$ has support inside the coadjoint orbit, $T^*C_+^\uparrow$, associated to the representation $U_+^\uparrow$. 

32
6.3 The Weyl-Heisenberg group $G_{WH}$

The Wigner function arising from the Weyl-Heisenberg group is the original phase space distribution introduced by Wigner \[34\] in 1932. Although this function is well known, to our knowledge, it has not been obtained by the methods introduced in this paper, linking it to the Plancherel transform. We, therefore, give a somewhat detailed derivation of it. Since this function was the original motivation for developing our general analysis, it is also worthwhile to put it in this context. (A somewhat different derivation, based on the theory of square integrability of group representations, modulo subgroups, was given in \[3\]).

The group $G_{WH}$ consists of $4 \times 4$ matrices

$$g(\theta, \xi, \eta) = \begin{pmatrix} 1 & \frac{1}{2} \xi^T \omega & \theta \\ 0 & \frac{1}{2} \xi \omega & \zeta \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with

$$\theta \in \mathbb{R}, \quad \zeta = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \in \mathbb{R}^2, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

This group is unimodular and nilpotent. The Lie algebra $\mathfrak{g}_{WH}$ is generated by the three elements,

$$X^0 = \begin{pmatrix} 0^T & 1 \\ \emptyset & 0 \end{pmatrix}, \quad X^1 = \begin{pmatrix} \frac{1}{2} e_3^T \\ \emptyset \end{pmatrix} e_2, \quad X^2 = \begin{pmatrix} -\frac{1}{2} e_2^T \\ \emptyset \end{pmatrix} e_1, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$\emptyset$ being the $3 \times 3$ zero matrix and $e_1, e_2, e_3$ the canonical basis vectors in $\mathbb{R}^3$. They satisfy the commutation relations,

$$[X^0, X^1] = [X^0, X^2] = 0, \quad [X^1, X^2] = X^0.$$

Writing a general element of $\mathfrak{g}_{WH}$ as $Y = -\theta X^0 - \eta X^1 + \xi X^2$, and noting that $(Y)^2$ is the null matrix, we see that

$$e^Y = g(-\theta, \xi, -\eta) = \mathbb{I}_4 + Y.$$

Thus, the group and the Lie algebra have essentially the same parametrization; the Haar measure of $G_{WH}$ is $d\theta \ d\xi d\eta = d\theta \ d\xi d\eta$ and the density $m(X) = 1$ (see (26)), almost everywhere.
Let \( \{X^*_0, X^*_1, X^*_2\} \) be the dual basis in \( g^*_{WH} \) and denote a general element in it by \( X^* = \gamma^0 X^*_0 + \gamma^1 X^*_1 + \gamma^2 X^*_2 \). The computation of the coadjoint action of \( g(\theta, \xi, \eta) \in G_{WH} \) on \( g^*_{WH} \) is now a routine matter. The coordinates \( \gamma^i, \ i = 1, 2, 3 \), transform under this action in the manner

\[
\begin{align*}
\gamma^0 & \rightarrow \gamma^0' = \gamma^0, \\
\gamma^1 & \rightarrow \gamma^1' = \gamma_1 - \xi \gamma^0, \\
\gamma^2 & \rightarrow \gamma^2' = \gamma_2 - \eta \gamma^0.
\end{align*}
\]

The (physically) non-trivial coadjoint orbits are the planes,

\[
O_{\gamma_0} = \left\{ \left( \begin{array}{c} \gamma_0 \\ \gamma \end{array} \right) \bigg| \gamma = \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right) \in \mathbb{R}^2, \ \gamma_0 \neq 0 \right\},
\]

which carry the (coad-)invariant measures \( d\Omega_{\gamma_0}(\gamma) = d\gamma = d\gamma_1 \ d\gamma_2 \), and comparing with \( (23) \) we get, \( d\kappa(\gamma_0) = d\gamma_0 \) and \( \sigma_{\gamma_0}(\gamma) = 1 \), for all \( \gamma_0 \neq 0 \) and almost all \( \gamma = \mathbb{R}^2 \).

The (physically) non-trivial unitary irreducible representations of \( G_{WH} \) are in one-to-one correspondence with the orbits \( O_{\gamma_0} \) and thus the unitary dual \( \hat{G}_{WH} \) is identifiable with \( \mathbb{R}\{0\} \). We choose the realization, \( U_\lambda \), of the UIR (corresponding to \( \lambda \in \hat{G}_{WH} \)), which is carried by the Hilbert space \( \mathcal{H}_\lambda = L^2(\mathbb{R}, dx) \) and acts in the manner,

\[
(U_\lambda(\theta, \xi, \eta) \phi_\lambda)(x) = e^{i\lambda \theta} e^{i\lambda (x-\xi)} \phi_\lambda(x - \xi).
\]

Since \( G_{WH} \) is unimodular, the Duflo-Moore operator \( C_\lambda \) is a multiple of the identity, which we denote by \( N_\lambda \) \((> 0)\). Writing the Plancherel measure as \( d\nu_{G_{WH}}(\lambda) = \rho(\lambda) \ d\lambda \), where \( \rho \) is some density function, we may compute both \( N_\lambda \) and \( \rho \) by noting that the orthogonality condition \( (38) \) leads in this case to the explicit relation,

\[
\frac{1}{N^2_\lambda} \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}\{0\}} \frac{|\psi_\lambda| U_\lambda(\theta, \xi, \eta) \phi_\lambda\rangle \rho(\lambda) \ d\lambda}{\int_{\mathbb{R}\{0\}}} \langle \psi_\lambda' | U_{\lambda'}(\theta, \xi, \eta) \phi_{\lambda'} \rangle \times \rho(\lambda') \ d\lambda' \right] d\theta \ d\xi \ d\eta = (2\pi)^2 \int_{\mathbb{R}\{0\}} \|\phi_\lambda\|^2 \|\psi_\lambda\|^2 \left[ \frac{(\rho(\lambda))^2}{\lambda} \right] \ d\lambda,
\]

for all \( \phi_\lambda, \psi_\lambda \in \mathcal{H}_\lambda \). We easily obtain, \( N_\lambda = \frac{1}{2\pi} \) and \( \rho(\lambda) = |\lambda| \), almost everywhere, so that the Plancherel measure of \( G_{WH} \) is \( |\lambda| \ d\lambda \).

34
The Wigner function is now obtained after a routine computation, using (31):

\[
W(A | \gamma_1, \gamma_2; \lambda) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{i x \gamma_2} \psi_{\lambda}( - \frac{\gamma_1}{\lambda} - \frac{x}{2}) \phi_{\lambda}( - \frac{\gamma_1}{\lambda} + \frac{x}{2}) \, dx,
\]

for \( A \in B_2^\oplus \) such that \( A = \{|\phi_{\lambda}\rangle\langle\psi_{\lambda}| \in B_2(\mathfrak{h}_\lambda)\}_{\lambda \in \mathbb{R}\{0\}} \) and \( \gamma_1, \gamma_2 \in \mathcal{O}_\lambda \). Note that again, in this case, the support of the Wigner function is concentrated on the orbit \( \mathcal{O}_\lambda \) which corresponds to the UIR \( U_\lambda \).

The above formula for the Wigner function is particularly interesting, since for fixed \( \lambda \),

\[
\int_{\mathbb{R}^2} |W(A | \gamma_1, \gamma_2; \lambda)|^2 \, d\gamma_1 \, d\gamma_2 = |\lambda| \|\phi_{\lambda}\|^2 \|\psi_{\lambda}\|^2.
\]

This means that, for fixed \( \lambda \), the expression (51) can be used to define a function \( W(A_\lambda | \gamma_1, \gamma_2; \lambda) \) on \( \mathcal{O}_\lambda \simeq \mathbb{R}^2 \) for any \( A_\lambda \in B_2(\mathfrak{h}_\lambda) \), such that the map

\[
A_\lambda \mapsto \frac{1}{|\lambda|^{\frac{1}{2}}} W(A_\lambda | \cdot; \lambda) \in L^2(\mathcal{O}_\lambda, d\gamma),
\]

is an isometry. Indeed, writing

\[
W_{QM}(\psi_{\lambda}, \phi_{\lambda} | q, p; \hbar) = \frac{1}{(2\pi)^{\frac{1}{2}}\hbar} W\left(A_\lambda \big| - \frac{q}{\hbar}, - \frac{p}{\hbar}; \frac{1}{\hbar}\right)
= \frac{1}{2\pi\hbar} \int_{\mathbb{R}} e^{-i \frac{\pi}{\hbar} \psi_{\lambda} \left( q - \frac{x}{2} \right)} \phi_{\lambda} \left( q + \frac{x}{2} \right) \, dx,
\]

we recover the well known function originally introduced by Wigner [34]. Thus, effectively, in this case Wigner functions can be defined for each UIR, \( U_\lambda \), that appears in the direct integral decomposition of the regular representation and the support of the Wigner function is concentrated on the corresponding coadjoint orbit.

7 Conclusion

The procedure outlined and illustrated in this paper is general enough to cover most groups of practical importance, for constructing wavelet transforms and Wigner functions. There are still, however, group representations
which are used in practical applications, but which are not amenable to the present technique. Representations which are not in the support of the Plancherel measure fall into this category. For example, in the case of the two Poincaré groups discussed here, the representations corresponding to the boundaries of the cones (the “mass zero representations”) fall outside of our scheme. One interesting direction for further research concerns the use of adapted Fourier transforms for the construction of decomposable Wigner maps. Also, the precise relationship between Wigner maps and deformation quantization (which is where the adapted Fourier transforms originate) should be worked out explicitly.

Acknowledgements

We would like to thank M. Cahen and S. Gutt for stimulating discussions and pointing out various references to us. The authors would like to acknowledge financial support from the Natural Sciences and Engineering Research Council (NSERC), Canada and the Fonds pour la Formation de Chercheurs et l’Aide à la Recherche (FCAR), Québec. HF would like to thank the Department of Mathematics and Statistics of Concordia University, Montréal, for their hospitality. STA would also like to thank G. Schlichting for hospitality at the Zentrum Mathematik der Technischen Universität München, where part of this work was completed.

Appendix

We collect in this appendix a few formulae and results for the for the various special matrix functions which appear in this paper. We begin by defining three real valued functions on \( \mathbb{R} \):

\[
\sinh x \quad := \quad 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \ldots
\]

\[
= \frac{\sinh x}{x}, \quad \text{if } x \neq 0,
\]

and

\[
\cosh x \quad := \quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots
\]

\[
= \frac{\cosh x + 1}{x}, \quad \text{if } x \neq 0,
\]
\[
F(x) := \sinh x + \cosh x = e^x - 1 \quad \text{if } x \neq 0.
\]

Note that \( \sinh x \) is an even function of \( x \), while \( \cosh x \) is an odd function.

The inverse of \( F(x) \) has an interesting series expansion:
\[
F^{-1}(x) = e^{-x} F(-x)^{-1} = \frac{e^{-x}}{\sinh \left( \frac{x}{2} \right)} = 1 - \frac{x}{2} \sum_{k \geq 1} \frac{B_{2k}}{2k!(2k)!} x^{2k}
\]

where the \( B_k \) are the Bernoulli numbers, \( B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, \) etc., and generally,
\[
B_k = \frac{(2k)!}{\pi^{2k} 2^{2k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}.
\]

If \( A \) is an \( n \times n \) matrix, then using the series expansions, we can define the matrix versions \( \sinh A, \cosh A, F(A) \) of these functions. In addition, if \( \det A \neq 0 \), then we can also write, \( \sinh A = A^{-1} \sinh A, \cosh A = A^{-1} (\cosh A - 1) \), etc. In particular, for the matrix
\[
X_q(\lambda, \theta) = \lambda I_2 + \theta \sigma_1, \quad \lambda, \theta \in \mathbb{R}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (57)
\]
we easily compute,
\[
e^{X_q(\lambda, \theta)} = e^\lambda \cosh \theta \ I_2 + e^\lambda \sinh \theta \ \sigma_1, \quad \det [e^{X_q(\lambda, \theta)}] = e^{2\lambda},
\]
\[
F(X_q(\lambda, \theta)) = e^{\frac{X_q(\lambda, \theta)}{2}} \sinh \left( \frac{X_q(\lambda, \theta)}{2} \right), \quad (58)
\]
\[
\det [F(X_q(\lambda, \theta))] = \frac{2e^\lambda (\cosh \lambda - \sinh \theta)}{\lambda^2 - \theta^2}, \quad \det [F(X_q(0, \theta))] = \sinh^2 \left( \frac{\theta}{2} \right),
\]
\[
\sinh X_q(0, \theta) = \sinh \theta \ I_2, \quad \cosh X_q(0, \theta) = \cosh \theta \ \sigma_1,
\]
\[
F(X_q(0, \theta)) = \sinh \theta \ I_2 + \cosh \theta \ \sigma_1 = \sinh \left( \frac{\theta}{2} \right) e^{\frac{X_q(0, \theta)}{2}}, \quad (59)
\]

To these we add the useful relationships,
\[
\sigma_3 e^{X_q(\lambda, \theta)} \sigma_3 = e^{X_q(\lambda, -\theta)}, \quad \sigma_3 F(X_q(\lambda, \theta)) \sigma_3 = F(X_q(\lambda, -\theta)),
\]
\[
\sigma_1 e^{X_q(\lambda, \theta)} \sigma_1 = e^{X_q(\lambda, \theta)}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1^2 = \sigma_3^2 = I_2. \quad (60)
\]
Note that the matrices $X_q(\lambda, \theta)$ span the Lie algebra of the $SO_{\text{Aff}}(1,1)$, which is the group $SO_0(1,1)$ together with dilations. Let us denote this Lie algebra by $\mathfrak{so}_{\text{Aff}}(1,1)$ and note that the group $SO_{\text{Aff}}(1,1)$ itself consists of all invertible elements of this algebra. The set $\mathfrak{so}_{\text{Aff}}(1,1)$ is closed under ordinary matrix multiplication and under this multiplication it is a commutative set. Furthermore, the matrices $F(X_q(\lambda, \theta))$ are elements of $\mathfrak{so}_{\text{Aff}}(1,1)$, for all $\lambda, \theta \in \mathbb{R}$. For any two vectors, $k = \begin{pmatrix} k_0 \\ k \end{pmatrix}$, $u = \begin{pmatrix} u_0 \\ u \end{pmatrix} \in \mathbb{R}^2$, and any $2 \times 2$ matrix $A$,

$$X_q(u_0, u)k = X_q(k_0, k)u,$$

$$\langle Ak, \sigma_3 Au \rangle = \det A \langle k, u \rangle = \det A (k_0 u_0 - ku).$$

(61)

Diagonalizing the matrices $X_q(\lambda, \theta)$,

$$VX_q(\lambda, \theta)V^T = \begin{pmatrix} \lambda + \theta & 0 \\ 0 & \lambda - \theta \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we get,

$$VF(X_q(\lambda, \theta))V^T = \begin{pmatrix} F(\lambda + \theta) & 0 \\ 0 & F(\lambda - \theta) \end{pmatrix},$$

$$\det [F(X_q(\lambda, \theta))] = F(\lambda + \theta) F(\lambda - \theta) > 0.$$  

(62)

Writing

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = V \begin{pmatrix} k_0 \\ k \end{pmatrix} = V k,$$

we see that the condition that $k \in \mathcal{C}^+_+$ (i.e., $k_0^2 > k^2$, $k_0 > 0$), is equivalent to having $\zeta_1, \zeta_2 > 0$. Thus we have the result,

**Lemma 7.1** If $k \in \mathcal{C}^+_+$, then $F(X_q(\lambda, \theta))k \in \mathcal{C}^+_+$, for all $\lambda, \theta \in \mathbb{R}$.

**Proof.** By (62),

$$VF(X_q(\lambda, \theta))V^T \zeta = \begin{pmatrix} F(\lambda + \theta) \zeta_1 \\ F(\lambda - \theta) \zeta_2 \end{pmatrix},$$

and since both $F(\lambda + \theta), F(\lambda - \theta) > 0$, the condition $\zeta_1, \zeta_2 > 0$ is preserved under the action of $VF(X_q(\lambda, \theta))V^T$. Hence the condition $k \in \mathcal{C}^+_+$ is preserved under the action of $F(X_q(\lambda, \theta))$. \hfill \Box
References

[1] S.T. Ali, J.-P. Antoine and J.-P. Gazeau: Coherent States, Wavelets and their Generalizations. Springer, New York, 2000.

[2] S.T. Ali, J.-P. Antoine and J.-P. Gazeau: Square integrability of group representations on homogeneous spaces. II. Coherent and quasi-coherent states. The case of the Poincaré group. Ann. Inst. Henri Poincaré 55 (1991), 857-890.

[3] S.T. Ali, A.E. Krasowska and R. Murenzi: Wigner functions from the two-dimensional wavelets group. J. Opt. Soc. Am. A17 (2000), 1-11.

[4] S.T. Ali, N.M. Atakishiyev, S.M. Chumakov and K.B. Wolf: The Wigner function for general Lie groups and the wavelet transform. Ann. H. Poincaré 1 (2000), 685-714.

[5] D. Arnal and J.C. Cortet: Nilpotent Fourier transform and applications. Letters in Math. Phys. 9 (1985), 25-34.

[6] D. Arnal, J.C. Cortet and J. Ludwig: Moyal product and representations of solvable Lie groups. J. Funct. Anal. 133 (1995), 402–424.

[7] D. Arnal and S. Gutt: Décomposition de $L^2(G)$ et transformation de Fourier adaptée pour un groupe $G$ nilpotent. C.R. Acad. Sci. Paris, t. 306, Série I (1988), 25-28.

[8] M.J. Bastiaans: The Wigner distribution function applied to optical signals and systems. Opt. Comm. 25 (1978), 26–30, and Wigner distribution functions and its application to first-order optics. J. Opt. Soc. Am. 69 (1979), 1710–1716.

[9] J. Bertrand and P. Bertrand: Représentations temps-fréquence des signaux. C.R. Acad. Sc. Paris 299, Série I (1984) 635-638, and A class of Wigner functions with extended covariance properties. J. Math. Phys. 33 (1992), 2515-2527.

[10] D. Bernier and K. Taylor: Wavelets from square-integrable representations. SIAM J. Math. Anal. 27 (1996), 594-608.
[11] F. Bruhat: *Distributions sur un groupe localement compact et applications à l’étude des représentations des groupes p-adiques*. Bull. Soc. math. France 89 (1981), 43-75.

[12] I. Daubechies: *Ten lectures on wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics, 61, 1992.

[13] J. Dixmier: *Von Neumann-Algebras*. North Holland, Amsterdam, 1981.

[14] J. Dixmier: *$C^*$-Algebras*. North Holland, Amsterdam, 1977.

[15] M. Duflo and C.C. Moore: *On the regular representation of a nonunimodular locally compact group*. J. Funct. Anal. 21 (1976), 209-243.

[16] G.B. Folland: *A Course in abstract harmonic analysis*. CRC Press, Boca Raton, 1995.

[17] H. Führ and M. Mayer: *Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure*. to appear in J. Fourier Anal. Appl.

[18] H. Führ: *Admissible vectors for the regular representation*, to appear in Proc AMS.

[19] A. Grossmann, J. Morlet and T. Paul: *Transforms associated to square integrable group representations I: General Results*. J. Math. Phys. 26 (1985), 2473-2479.

[20] M. Hillery, R.F. O’Connel, M.O. Scully, and E.P. Wigner: *Distribution functions in physics: fundamentals*. Phys. Rep. 106 (1984), 121-167.

[21] A.A. Kirillov in *Representation Theory and Noncommutative Harmonic Analysis I*. Springer Verlag, New York, 1994.

[22] A. A. Kirillov: *Elements of the Theory of Representations*. Springer Verlag, Berlin, 1976.

[23] J.R. Klauder and R.F. Streater: *A wavelet transform for the Poincaré group*. J. Math. Phys. 32 (1991), 1609-1611.

[24] A. Kleppner and R.L. Lipsman: *The Plancherel formula for group extensions, I and II*, Ann.Sci.Ecole Norm.Sup. 5 (1972), 459-516; ibid. 6 (1973), 103-132.
[25] A. E. Krasowska and S.T. Ali: Wigner functions for a class of semi-direct product groups, to appear.

[26] H. Leptin and J. Ludwig: Unitary representation theory of exponential Lie groups., de Gruyter, Berlin, 1994.

[27] R.L. Lipsman: Non-abelian Fourier analysis. Bull. Sci. Math. 98 (1974), 209-233.

[28] J. Ludwig: On the Hilbert-Schmidt Semi-Norms of $L^1$ of a nilpotent Lie group, Math. Ann. 273 (1986) 383-395.

[29] G. W. Mackey: Induced representations of groups and quantum mechanics. W. A. Benjamin Inc., New York, 1968.

[30] G. W. Mackey: Induced representations of locally compact groups, I. Ann. of Math. 55 (1952), 101-139.

[31] G. W. Mackey: Borel structure in groups and their duals. Trans. Amer. Math. Soc. 85 (1957), 134-165.

[32] W. Rudin: Functional Analysis. McGraw-Hill, New York, 1973.

[33] W. Rudin: Real and Complex Analysis. McGraw-Hill, New York, 1966.

[34] E.P. Wigner: On the quantum correction for thermodynamic equilibrium. Phys. Rev. 40 (1932), 749-759.

[35] K.B. Wolf: Wigner distribution function for paraxial polychromatic optics. Opt. Comm. 132 (1996), 343-352.