Poisson integrators for Volterra lattice equations

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Abstract

The Volterra lattice equations are completely integrable and possess bi-Hamiltonian structure. They are integrated using partitioned Lobatto IIIA-B methods which preserve the Poisson structure. Modified equations are derived for the symplectic Euler and second order Lobatto IIIA-B method. Numerical results confirm preservation of the corresponding Hamiltonians, Casimirs, quadratic and cubic integrals in the long-term with different orders of accuracy.

Keywords: Volterra lattice equations, Korteweg-de Vries equation, bi-Hamiltonian systems, Poisson structure, Lobatto methods, symplectic Euler method

1 Introduction

The preservation of qualitative properties of discretized models of continuous systems became more and more important in recent years. The development of symplectic integrators for Hamiltonian systems, construction of geometric integrators which preserve symmetries, reversing symmetries and phase space volume of the underlying differential equations are some examples. For Hamiltonian systems in non-canonical form with a non-linear structure matrix, i.e Poisson systems, there does not exist any general structure preserving integrator similar to the symplectic methods for canonical Hamiltonian systems (Sec. VII.2 [5, 13]). There are some well developed methods based on generating functions and Hamiltonian splitting for the Lie-Poisson systems, i.e. non-canonical Hamiltonian systems with a linear structure matrix (Sec. VII.2.6 [5]). Poisson systems arise especially as Hamiltonian pde’s like Korteweg de Vries (KdV) and nonlinear Schrödinger equation with infinitely many integrals. Semi-discretization of these pde’s in space by preserving the integrals and Hamiltonian structure results in integrable lattice equations; ode’s with a Poisson structure and a finite number of integrals.

After introducing the Volterra lattice equations in the next section, we discuss in Section 3 symplectic integrators like the implicit mid-point and the symplectic Euler method applied to the well-known two dimensional Lotka-Volterra equation. Preservation of the Poisson structure of the Volterra lattice equation by symplectic Euler and Lobatto IIIA-B methods is proved and the corresponding modified equations are derived. The numerical results show the preservation of the conserved quantities by these methods.

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2 Volterra lattice equation

The $m$-dimensional Volterra lattice

$$\dot{y}_i = y_i(y_{i+1} - y_{i-1}), \quad i = 1, \ldots, m \quad (1)$$

for even $m$ and with periodic boundary conditions $y_{m+i} = y_i$, $i = 0, 1, \ldots$ and with $y_i > 0$ was studied first in [10] as an integrable system. It was shown that the Volterra lattice equation represents an integrable discretization of the KdV equation [10, 19] and of inviscid Burger's equation [12]. Besides these, the Volterra lattice equation describes many phenomena such as the vibrations of particles on lattices (Liouville model on the lattice), waves in plasmas and the evolution of populations in a hierarchical system of competing species [1, 3, 19].

The system (1) possesses a bi-Hamiltonian structure [1, 19]

$$\dot{y} = J_0(y)\nabla H_1 = J_1(y)\nabla H_0 \quad (2)$$

with respect to the quadratic and the cubic Poisson brackets

$$\{y_i, y_{i+1}\}_0 = y_i y_{i+1}, \quad (3)$$

$$\{y_i, y_{i+1}\}_1 = y_i y_{i+1}(y_i + y_{i+1}), \quad \{y_i, y_{i+2}\}_1 = y_i y_{i+1} y_{i+2}. \quad (4)$$

The corresponding Hamiltonians are

$$H_1 = \sum_{i=1}^m y_i, \quad H_0 = \frac{1}{2} \sum_{i=1}^m \log(y_i). \quad (5)$$

and the structure matrices $J_0(y)$ corresponding to the quadratic Poisson bracket has the form

$$J_0(y) = \begin{pmatrix}
0 & y_1 y_2 & \cdots & \cdots & \cdots & y_1 y_m \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \cdots & -y_{i-1} y_i & 0 & y_i y_{i+1} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
y_1 y_m & \cdots & \cdots & \cdots & -y_{m-1} y_m & 0
\end{pmatrix}.$$  

If the Poisson brackets are compatible, i.e. the sum of $\{\cdot, \cdot\}_0 + \{\cdot, \cdot\}_1$ is again a Poisson bracket, then the bi-Hamiltonian system (1) has a finite number of functionally independent first integrals $I_i, i = 1, \ldots, m$ in involution i.e. $\{I_i, I_j\}_k = 0$ for $k = 0, 1$ and $i \neq j$, i.e. the Volterra lattice is completely integrable with respect to both brackets.

The Hamiltonian $H_0$ is a Casimir with respect to the Poisson bracket $\{\cdot, \cdot\}_0$, i.e. $\{H_0, F\}_0 = 0$ for any function $F(y)$.

The Volterra lattice (1) represents an integrable discretization of KdV equation [19]

$$\frac{\partial u}{\partial \tau} + 6u \frac{\partial u}{\partial \xi} \frac{\partial^2 u}{\partial \xi^2} = 0, \quad (6)$$

which also possesses a bi-Hamiltonian structure [10] and has infinitely many integrals. The first three integrals of (6) are [6, 7]

$$\mathcal{I}_1(u) = \int_{-a}^{a} u d\xi, \quad \mathcal{I}_2(u) = \frac{1}{2} \int_{-a}^{a} u^2 d\xi, \quad \mathcal{I}_3(u) = \int_{-a}^{a} \left( \frac{1}{2} \left( \frac{\partial u}{\partial \xi} \right)^2 - \frac{1}{6} u^3 \right) d\xi.$$
The corresponding conserved quantities of the Volterra lattice (1) are the quadratic and cubic integrals of
\[
I_q = \sum_{i=1}^{m} \frac{1}{2} y_i^2 + y_i y_{i+1}, \quad I_c = \sum_{i=1}^{m} \frac{1}{3} y_i^3 + y_i y_{i+1} + (y_i + y_{i+1} + y_{i+2}).
\]

In discretized form, the mass conservation \(I_1\) corresponds to the Hamiltonian \(H_1\), the momentum and energy integrals \(I_2, I_3\) of the KdV equation (5) correspond to the first integrals \(I_q\) and \(I_c\) of the Volterra lattice (1) respectively.

The Volterra lattice is time-reversible and is closely connected to the Toda lattice, which is also completely integrable and has a tri-Hamiltonian structure \([2], [19]\). The transformation of variables \(a_i = y_{2i} y_{2i-1}, \ b_i = y_{2i-1} - y_{2i-2}\) in (1) gives the Toda lattice

\[
\dot{a}_i = a_i(b_{i+1} - b_i), \quad \dot{b}_i = a_i - a_{i-1}, \quad i = 1, \ldots, m/2
\]

with periodic boundary conditions

\[
a_0 = a_{m/2}, \quad b_{m/2+1} = b_1.
\]

3 Poisson integrators

Geometric integrators for the Poisson systems

\[
\dot{y} = J(y) \nabla H
\]

with a skew-symmetric non-constant structure matrix \(J(y)\) were studied recently in several papers. For recent surveys on Poisson integrators see (Sec. VII.2, \([8]\) and \([11]\)). For related material on Poisson systems see (Sec. VII.2, \([8]\), (Ch. 10, \([13]\)) and (Ch. 6 & 7, \([16]\)).

The Poisson bracket \(\{F, G\}\) for two smooth functions \(F(y)\) and \(G(y)\) is defined by

\[
\{F, G\}(y) = \nabla F(y)^T J(y) \nabla G(y)
\]

which is bilinear, skew-symmetric \(\{G, F\} = -\{F, G\}\) and satisfies the Leibniz' rule \(\{E \cdot F, G\} = E \cdot \{F, G\} + F \cdot \{E, G\}\) as well as the Jacobi identity \(\{E, \{F, G\}\} + \{F, \{G, E\}\} + \{G, \{E, F\}\}\).

The structure matrix \(J(y)\) of Poisson systems does not need to be invertible as for canonical Hamiltonian systems with \(J^{-1}\). All odd-dimensional skew-symmetric structure matrices \(J(y)\) are singular. The structure matrices \(J_0(y)\) and \(J_1(y)\) of the periodic Volterra lattice (1) are also singular. Such systems are called degenerate Poisson systems.

The functions \(C_i(y)\) satisfying \(\{C_i, H\} = 0\) are called Casimirs or distinguished functions, which are first integrals whatever \(H(y)\) is.

There are two characteristics for the flow \(\varphi_t(y)\) of the Poisson system (8):

- the flow \(\varphi_t(y)\) of the differential equation (8) is a Poisson map, i.e.

\[
\dot{\varphi}_t(y) J(y) \dot{\varphi}_t(y)^T = J(\varphi_t(y))
\]

where \(\dot{\varphi}_t(y)\) denotes the Jacobian of \(\varphi_t\),

- and it respects the Casimirs of \(J(y)\), i.e. \(C_i(\varphi_t(y)) = Const.\)
A numerical method \( y_{n+1} = \phi_h(y_n) \) is called a Poisson integrator for the structure matrix \( J(y) \) if the transformation \( y_n \rightarrow y_{n+1} \) is a Poisson map that respects the Casimirs. The Casimirs should be preserved by the Poisson integrator. But the Casimir functions can be arbitrary, in case of the Volterra lattice they are in logarithmic form, therefore their conservation depends on the special structure of the problem.

Because each Poisson system is distinguished by the structure matrix \( J(y) \), a method will be Poisson integrator only for a specific class of structure matrices. Therefore symplectic methods used for Hamiltonian systems can not be directly applied to Poisson systems. But some of the symplectic integrators can preserve certain Poisson structures. An example of this kind is the Poisson system resulting from the Ablowitz-Ladik integrable discretization of the nonlinear Schrödinger equation [18] which is preserved by the symplectic Euler method.

The two-dimensional Lotka-Volterra equation was studied by several authors. The symplectic Euler and Störmer-Verlet methods preserve the Poisson structure of (10) whereas the implicit mid-point rule does not (see pp. 238 [8]). There are also some non-standard methods ([15], [17]) which preserve the Poisson structure of the two-dimensional Lotka-Volterra equation.

In the following we will apply the symplectic Euler method and Lobatto IIIA-B methods to the splitted form of the Volterra lattice equation (1). Both belong to the class of splitting methods which have been successfully used as geometric integrators in recent years. For a survey of the splitting methods see [9] and [11].

For the application of partitioned Runge-Kutta methods of Lobatto type, we split the equation (1) into two parts

\[
\dot{u}_i = u_i(v_{i+1} - v_i), \quad \dot{v}_i = v_i(u_{i+1} - u_i), \quad i = 1, \ldots, m/2
\]

by grouping the variables into odd \( u_i = y_{2i-1} \) and even \( v_i = y_{2i} \) parts. Equation (10) is bi-Hamiltonian with the quadratic Poisson bracket and Hamiltonian like the Volterra lattice [19]:

\[
\{u_i, v_i\}_0 = u_i v_i \quad \{v_i, u_{i+1}\}_0 = v_i u_{i+1}.
\]

The Hamiltonians and the first integrals can be written in the new variables:

\[
H_1(u) = \frac{1}{2} \sum_{i=1}^{m/2} (u_i + v_i), \quad H_0(u) = \frac{1}{2} \sum_{i=1}^{m/2} \log(u_i) + \log(v_i),
\]

\[
I_q = \sum_{i=1}^{m/2} \frac{1}{2} u_i^2 + u_i v_i, \quad I_c = \sum_{i=1}^{m/2} \frac{1}{3} u_i^3 + u_i v_i(u_i + v_i + u_{i+1}).
\]

Symplectic and time-reversible partitioned Runge-Kutta methods like the Lobatto IIIA-B methods [8] can be easily applied to Volterra lattice equations in the partitioned form. The symplectic Euler method which consists of a combination of explicit and implicit Euler is a first order Lobatto IIIA-B method. For the partitioned Volterra equations (10) the symplectic Euler method becomes

\[
u_{i+1}^n = u_i^n + h v_{i+1}^n (v_{i+1}^n - v_{i-1}^n), \quad v_{i+1}^n = v_i^n + h u_{i+1}^n (u_{i+1}^n - u_i^n).
\]

The second order Lobatto IIIA-B method for the partitioned Volterra lattice (10) gives

\[
k_i^1 = \left( v_i^n + \frac{h}{2} k_i^1 \right), \quad l_i^1 = \left( v_i^n + \frac{h}{2} l_i^1 \right)
\]

\[
k_i^2 = \left( u_i^n + \frac{h}{2} (k_i^1 + k_i^2) \right), \quad l_i^2 = \left( u_i^n + \frac{h}{2} (l_i^1 + l_i^2) \right)
\]

\[
u_{i+1}^n = u_i^n + \frac{h}{2} (k_i^1 + k_i^2), \quad v_{i+1}^n = v_i^n + \frac{h}{2} (l_i^1 + l_i^2).
\]
The internal stage vectors $k^2$ and $l^1$ are computed by solving a system of linear equations, whereas the vectors $k^1$ and $l^2$ are obtained explicitly. The second order Lobatto IIIA-B method is known as Störmer-Verlet method for separable Hamiltonian systems.

In order to show the preservation of the Poisson structure with the quadratic brackets (11) we consider the corresponding two-form formulation \[16\]

$$
\sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i \wedge dv_i + \sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i \wedge dv_{i+1}. \tag{15}
$$

**Theorem 1** The Poisson structure of the Volterra lattice with the quadratic brackets (11) is preserved by the symplectic Euler method.

**Proof:** In order to show the equality

$$
\sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i^{n+1} \wedge dv_i^{n+1} + \sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i^{n+1} \wedge dv_{i+1}^{n+1} = \sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i^n \wedge dv_i^n + \sum_{k=0}^{m/2} \frac{m/2}{u_i v_i} du_i^n \wedge dv_{i+1}^n,
$$

we differentiate (11) to get

$$
\begin{align*}
du_i^{n+1} &= du_i^n + h(v_i^{n+1} - v_i^{n+1})du_i^n + hu_i^n dv_i^{n+1} - hu_i^n dv_{i+1}^{n+1}, \\
dv_i^{n+1} &= dv_i^n + h(u_i^{n+1} - u_i^{n+1})dv_i^n + hv_i^{n+1} du_i^{n+1} - hv_i^{n+1} du_i^{n+1}.
\end{align*} \tag{16, 17}
$$

Using the equations (16) and (17) successively we obtain

$$
\begin{align*}
&du_i^{n+1} \wedge dv_i^{n+1} = (1 + h(v_i^{n+1} - v_i^{n+1}))du_i^n \wedge dv_i^n + hu_i^n dv_i^{n+1} \wedge dv_i^{n+1}, \\
&dv_i^{n+1} \wedge du_i^{n+1} = (1 + h(u_i^{n+1} - u_i^{n+1}))dv_i^n \wedge du_i^n + hv_i^{n+1} du_i^{n+1} \wedge dv_i^{n+1}.
\end{align*}
$$

The equations (14) can be written in the equivalent form:

$$
1 + h(v_i^{n+1} - v_i^{n+1}) = \frac{u_i^{n+1}}{u_i^n}, \quad 1 - h(u_i^{n+1} - u_i^{n+1}) = \frac{v_i^{n+1}}{v_i^n}.
$$

Combining all these into the two-forms in (15) we obtain

$$
\begin{align*}
&du_i^{n+1} \wedge dv_i^{n+1} = \frac{u_i^{n+1} v_i^{n+1}}{u_i^n v_i^n} (du_i^n \wedge dv_i^n + hv_i^{n+1} du_i^n \wedge dv_i^{n+1}) - hu_i^n dv_i^{n+1} \wedge dv_i^{n+1}, \\
&dv_i^{n+1} \wedge du_i^{n+1} = \frac{u_i^{n} v_i^{n+1}}{u_i^{n+1} v_i^n} (dv_i^n \wedge du_{i+1}^n - hv_i^{n+1} du_i^n \wedge du_{i+1}^n) + hv_i^{n+1} du_i^{n+1} \wedge dv_i^{n+1}.
\end{align*}
$$

Because $du_i^n \wedge dv_{i+1}^n = 0$, the $h$-order terms in the parenthesis vanish and taking in the summation over $i$ and considering periodicity of the Volterra lattice one can easily see that the second terms of order $h$ cancel and the quadratic Poisson brackets (11) are preserved by the symplectic Euler method.

Because the second order Lobatto IIIA-B method is a composition of symplectic Euler methods with step sizes $h/2$, it preserves the quadratic Poisson bracket of the Volterra lattice. But higher order Lobatto IIIA-B methods can not preserve the Poisson structure, because they can not be written as combination of symplectic Euler method. Only diagonally implicit partitioned
Runge-Kutta methods can be written as combination of symplectic Euler method (see pp. 180, 8).

Recently finite dimensional systems with a Poisson structure arising after semi-discretization of certain partial differential equations have been integrated by splitting methods. An example of this is the preservation of the Lie-Poisson structure of Landau-Lifschitz lattice in partitioned form using a staggered scheme which corresponds to the second order Lobatto IIIA-B [5]. It was shown in [4] that the finite dimensional system which arises using a variational approximation of the time-dependent Schrödinger equation by Gaussian wave packets inherits a Poisson structure and various splitting methods were considered for its integration.

The linear integrals are preserved exactly by all Runge-Kutta methods. The implicit midpoint rule preserves the quadratic integrals exactly assuming that the underlying system of nonlinear equations is solved within the machine accuracy. The Lobatto IIIA-B methods preserve only the quadratic integrals of the form \( Q = p^T D q \), where \( D \) is an arbitrary matrix of appropriate dimension (Ch. 4, 8). Unfortunately the quadratic integral \( I_q \) (13) of the splitted Volterra lattice is not in this form. Higher order polynomial integrals like the cubic integral \( I_c \) and non-polynomial integrals like the Casimir function \( H_0 \) are not preserved exactly by the symplectic Euler and Lobatto IIIA-B methods.

For the periodic Volterra lattice we have used the following initial condition

\[
y(x_i) = 1 + \frac{1}{2m^2} \text{sech}^2(x_i), \quad x_i = -1 + (i - 1) \frac{1}{2m}, \quad i = 1, \ldots, m.
\]

All computations are done with a constant time step \( \Delta t = 0.1 \) over the time interval \( t \in [0, 2000] \) for a Volterra lattice of dimension \( m = 20, 40, 80 \). The errors in the Hamiltonians and conserved quantities are given in Table 1 in the mean square root norm \( \sqrt{\sum_{i=1}^{N} (I^i - I_0)^2 / N} \), where \( I^i \) denote the computed Hamiltonians or first integrals at time step \( t_i \) and \( N \) is the number of time steps.

For all values of \( m \) and \( \Delta t \), the Hamiltonian \( H_1 \) is preserved with almost the same high accuracy for both methods. The Casimir \( H_0 \), the quadratic and cubic first integrals can not be preserved exactly, but the errors do not grow with time as in non-symplectic methods. Similar numerical results are obtained for the Toda lattice (see pp. 385-386 8 and pp. 430-431 9). It was shown in 8, Theorem 3.1, pp. 353, that for completely integrable systems, the symplectic integrators preserve the first integrals over long-time with an error \( O(\Delta t^p) \), where \( p \) denotes the order of the method. The Lobatto IIIA-B method results in smaller errors than the symplectic Euler method because it is a second order accurate method. One can also observe that the Casimir \( H_0 \) is preserved slightly better than the quadratic and cubic first integrals.
Figure 1: Errors in the Hamiltonians and first integrals: symplectic Euler method, $m = 40$

Figure 2: Errors in the Hamiltonians and first integrals: Lobatto IIIA-B method, $m = 40$
Table 1: Average errors of the Hamiltonians and first integrals

| m | $\Delta t$ | symplectic Euler method | Lobatto IIIA-B method |
|---|---|---|---|
|   |   | $H_1$ | $H_0$ | $I_q$ | $I_c$ | $H_1$ | $H_0$ | $I_q$ | $I_c$ |
| 20 | 0.2 | 2.589 -16 | 7.021 -08 | 6.200 -08 | 2.380 -07 | 3.042 -16 | 2.698 -11 | 1.879 -10 | 1.597 -09 |
| 20 | 0.1 | 2.740 -16 | 1.240 -08 | 1.072 -08 | 4.100 -08 | 2.526 -16 | 5.157 -12 | 3.268 -11 | 2.765 -10 |
| 20 | 0.05 | 1.652 -16 | 2.229 -09 | 1.853 -09 | 7.036 -09 | 1.609 -16 | 9.373 -13 | 5.793 -12 | 4.896 -11 |
| 40 | 0.2 | 3.016 -16 | 2.608 -10 | 2.158 -10 | 1.746 -09 | 1.752 -16 | 7.557 -13 | 6.870 -12 | 6.036 -11 |
| 40 | 0.1 | 1.283 -16 | 4.820 -11 | 4.021 -11 | 3.218 -10 | 4.824 -16 | 1.401 -13 | 1.202 -12 | 1.053 -11 |
| 40 | 0.05 | 2.309 -16 | 9.300 -12 | 7.863 -12 | 6.163 -11 | 2.562 -16 | 2.540 -14 | 2.115 -13 | 1.857 -12 |
| 80 | 0.2 | 2.903 -16 | 7.256 -12 | 7.162 -12 | 5.650 -11 | 9.837 -16 | 1.409 -14 | 1.396 -13 | 1.229 -12 |
| 80 | 0.1 | 5.237 -16 | 1.314 -12 | 1.296 -12 | 1.017 -11 | 2.244 -16 | 2.968 -15 | 2.342 -14 | 2.137 -13 |
| 80 | 0.05 | 6.117 -16 | 2.439 -13 | 2.396 -13 | 1.864 -12 | 1.547 -16 | 6.175 -16 | 3.830 -15 | 3.745 -14 |

4 Conclusion

We have shown that the symplectic Euler method preserves the quadratic Poisson structure of the periodic Volterra lattice. The numerical results show excellent long time preservation of the Hamiltonian, Casimirs and the first integrals. Because of the singularity of the structure matrices we can obtain only local results for the backward error analysis in contrast to the global results obtained for Poisson systems with invertible structure matrices like in (Sec. IX.3.3, pp.297 [8]).

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