The gravitational wave spectrum of non–axisymmetric, freely
precessing neutron stars

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Abstract

Evidence for free precession has been observed in the radio signature of several pulsars. Freely
precessing pulsars radiate gravitationally at frequencies near the rotation rate and twice the rotation
rate, which for rotation frequencies greater than \( \sim 10 \) Hz is in the LIGO band. In older work, the
gravitational wave spectrum of a precessing neutron star has been evaluated to first order in a small
precession angle. Here we calculate the contributions to second order in the wobble angle, and we
find that a new spectral line emerges. We show that for reasonable wobble angles, the second-order
line may well be observable with the proposed advanced LIGO detector for precessing neutron stars
as far away as the galactic center. Observation of the full second-order spectrum permits a direct
measurement of the star’s wobble angle, oblateness, and deviation from axisymmetry, with the
potential to significantly increase our understanding of neutron star structure.

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I. INTRODUCTION

For some decades now, freely precessing neutron stars have been considered as possible sources of detectable gravitational radiation. Early treatments include the ones by Zimmermann et al. [1, 2, 3] and by Alpar and Pines [4]. More recently interest in the subject was revived by compelling observational evidence for free precession in the radio signature of PSR B1828-11 [5] as well as PSR B1642-03 [6]. Evidence for free precession had also been put forward for PSR B0531+21 (the Crab pulsar) [7, 8] and PSR B0833-45 (the Vela pulsar) [9]. These examples suggest the existence of more such objects, some of which might be viable sources for current or future ground-based gravitational wave detectors such as LIGO.

Prompted by the evidence for free precession in PSR B1828-11, Cutler and Jones [10] revisited earlier work which seemed to suggest that when a neutron star is treated as an elastic body, gravitational wave back-reaction will quickly damp the precessing motion. Careful consideration led them to conclude instead that in realistic precessing neutron stars, gravitational wave damping will play but a minor role in the star’s evolution, although there will be other sources of damping. Jones and Andersson [11] then constructed a detailed model of freely precessing neutron stars and compared it with the observational data. They concluded that the precessional interpretation of these observations required superfluid vortex pinning of the star’s crust to its core to be inhibited, thereby challenging some of the then-prevailing ideas concerning a neutron star’s liquid interior. Nevertheless, an analysis by Link and Epstein [12] supported the precessional hypothesis for PSR B1828-11.

In [13], Jones and Andersson evaluated the viability of precessing neutron stars as sources for gravitational wave detection in the near future. To estimate the strength of the signal, they used the waveforms computed by Zimmermann et al. [2, 3]. Their conclusions appear to imply that gravitational waves from precessing neutron stars are unlikely to be detectable even with an advanced LIGO detector. However, their paper was written before the more recent analysis of the expected capabilities of LIGO II by Fritschel [14, 15], which is the one we will use.

Here we re-examine the gravitational waves generated by rapidly rotating, precessing neutron stars, evaluating whether the radiation from such sources might indeed be detectable, and what we may learn from their detection.
The motion of a freely precessing neutron star has two characteristic frequencies associated with it: the precession frequency and the rotation frequency. The corresponding gravitational wave spectrum consists of a series of discrete spectral lines, with a principal line at or near a harmonic of the rotation frequency, surrounded by companion lines separated from the main line by multiples of the precessional frequency.

In [3], Zimmermann evaluated to first order in small wobble angle the quadrupole gravitational radiation emitted by freely precessing, non-axisymmetric neutron stars, approximated as rigid bodies. At first order the spectrum includes two spectral lines: one at approximately the pulsar frequency and another at its first harmonic. Unfortunately, knowledge of the first-order spectrum yields little or no information about the physical characteristics of the neutron star emitting the radiation. From an observational point of view, this makes it imperative to study the spectrum in more detail.

Our goal here is threefold. First, we want to show that it is reasonable to take the precession angle as the primary expansion parameter among the various ‘small’ parameters that characterize the problem. Secondly, we want to determine the conditions under which the second-order contributions to the gravitational wave spectrum of a precessing neutron star are observable. Finally, we would like to explore what observation of these second-order corrections can reveal about the physical properties of the source.

Throughout this paper, we will use units where $G = c = 1$ unless stated otherwise. Wherever estimates are needed for the mass and radius of a neutron star, we will use $M = 1.4M_\odot$ and $R = 10^6$ cm. The neutron stars of interest to us are fast-spinning ones; in estimates involving rotation frequencies we use 500 Hz as a reference. We shall treat neutron stars as rigid bodies and limit ourselves to the quadrupole approximation to the gravitational radiation. When describing rigid body motion we will follow [3, 16, 17], i.e., we use the physical conventions of [16] and the mathematical notation of [3, 17]. In section III we review the closed expressions for the gravitational waveforms found in [3] and in III the first-order approximation of [3]. In section IV the closed expressions are used to construct a second-order expansion, and a new contribution to the gravitational radiation spectrum of precessing neutron stars is found. Section V first discusses the detectability of all three spectral lines. We then explain how physical information can be extracted from the second-order spectrum. Finally, section VI provides a summary of the results and conclusions.
II. GRAVITATIONAL RADIATION AND PRECESSION

The motion of an isolated, freely precessing rigid body is most conveniently described in terms of the Euler angles $\theta$, $\phi$ and $\psi$ that orient the body with respect to an inertial coordinate system. Following the conventions of [16], denote the coordinates of the inertial coordinate system centered on the body by $X$, $Y$, and $Z$, with the coordinate vector $\hat{e}_Z$ in the direction of the body’s angular momentum. Similarly, denote the coordinates of the body-fixed frame as $x$, $y$, and $z$, with coordinate vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ parallel to the eigenvectors of the body’s moment of inertia tensor, and with the corresponding principal moments of inertia satisfying $I_z > I_y \geq I_x$. The Euler angle $\theta$ is then the angle between $\hat{e}_Z$ and $\hat{e}_z$, $\phi$ is the angle between $\hat{e}_X$ and the line of nodes that is the intersection of the $\hat{e}_X \times \hat{e}_Y$ plane and the $\hat{e}_x \times \hat{e}_y$ plane, while $\psi$ is the angle in the $\hat{e}_x \times \hat{e}_y$ plane between the line of nodes and $\hat{e}_x$ (cf. [16, figure 47]).

Denote the time-dependent rotation matrix that transforms from the body-frame to the inertial frame by $R_{j\mu}$, where Latin indices refer to the inertial frame and Greek indices refer to the body-fixed frame. The quadrupole gravitational radiation from the precessing body is proportional to the second time derivative of the moment of inertia tensor in the inertial frame, which depends on the principal moments of inertia and the time-dependent rotation matrix.

The standard quadrupole formula for the transverse-traceless (TT) gauge metric perturbation is

$$h_{TT}^{jk} = \frac{2}{r} \frac{d^2 t_{jk}}{dt^2}, \quad (2.1)$$

where

$$t_{jk} = \int d^3 X \left( X_j X_k - \frac{1}{3} X^2 \delta_{jk} \right) \rho(X), \quad (2.2)$$

with $\rho(X)$ the density of the source. Eq. (2.1) can be written in terms of the rotation matrix $R_{j\mu}$, the components $\Omega_{\mu}$ of the angular velocity in the body-fixed frame, and the eigenvalues of the rigid body’s moment of inertia tensor $I_1$, $I_2$, and $I_3$ [3]:

$$h_{TT}^{jk} = \frac{2}{r} R_{j\mu} A_{\mu\nu} R^T_{\nu k}, \quad (2.3)$$

where summation over $\mu$ and $\nu$ is understood. $A_{\mu\nu}$ is given by

$$A_{11} = 2(\Delta_2 \Omega_2^2 - \Delta_3 \Omega_3^2), \quad (2.4a)$$

$$A_{12} = \Gamma_3 \Omega_1 \Omega_2, \quad (2.4b)$$
with the other components being defined by symmetry and cyclic permutation of the indices. \( \Delta_{\mu} \) and \( \Gamma_{\mu} \) are shorthand for

\[
\Delta_1 = I_2 - I_3, \quad (2.5a)
\]

\[
\Gamma_1 = \Delta_2 - \Delta_3 + \frac{\Delta_1^2}{I_1}, \quad (2.5b)
\]

where the other components are again given by cyclic permutation of the indices.

To facilitate the description of the radiation, introduce two orthogonal unit vectors \( \hat{v} \) and \( \hat{w} \) with \( \hat{v} \times \hat{w} \) directed toward a distant observer of the emitted gravitational waves. Without loss of generality we may assume that the observer lies in the \( \hat{e}_Y \times \hat{e}_Z \) plane. Let \( i \) denote the angle between \( \hat{e}_Z \) and the line of sight to the observer, and define

\[
\hat{v} = \hat{e}_Y \cos(i) - \hat{e}_Z \sin(i), \quad (2.6a)
\]

\[
\hat{w} = -\hat{e}_X. \quad (2.6b)
\]

In this frame the TT gauge metric perturbation corresponding to the observed radiation can be described in terms of two scalar functions, \( h_+(t) \) and \( h_\times(t) \), representing the radiation in the two gravitational wave polarizations:

\[
h_{ij}^{TT}(t) = h_+(t) (\hat{e}_+)_{ij} + h_\times(t) (\hat{e}_\times)_{ij}, \quad (2.7a)
\]

with

\[
\hat{e}_+ = \hat{v} \otimes \hat{v} - \hat{w} \otimes \hat{w}, \quad (2.7b)
\]

\[
\hat{e}_\times = \hat{v} \otimes \hat{w} + \hat{w} \otimes \hat{v}, \quad (2.7c)
\]

\[
h_+ = \frac{1}{2} h_{jk}^{TT} e_+^{jk}, \quad (2.7d)
\]

\[
h_\times = \frac{1}{2} h_{jk}^{TT} e_\times^{jk}. \quad (2.7e)
\]

Using equation (2.3) for the TT gauge metric perturbation we have

\[
h_+ = -\frac{1}{r} [(R_{\mu\nu} \cos(i) - R_{z\mu} \sin(i)) (R_{\nu\mu} \cos(i) - R_{z\nu} \sin(i)) - R_{x\mu} R_{x\nu}] A_{\mu\nu}, \quad (2.8a)
\]

\[
h_\times = \frac{2}{r} (R_{\mu\nu} \cos(i) - R_{z\mu} \sin(i)) R_{x\nu} A_{\mu\nu}. \quad (2.8b)
\]

In these expressions time evolution arises from the time dependence of the angular velocity in the body frame and the rotation matrix.
When \( J^2 \geq 2EI_2 \), with \( J \) the magnitude of the angular momentum and \( E \) the energy in the rigid body motion, the components of the angular velocity in the body frame are given by

\[
\begin{align*}
\Omega_1 &= a \cn(\tau, m), \\
\Omega_2 &= a \left[ \frac{I_1 (I_3 - I_1)}{I_2 (I_3 - I_2)} \right]^{1/2} \sn(\tau, m), \\
\Omega_3 &= b \dn(\tau, m),
\end{align*}
\]

where \( \tau \) is a rescaled time variable,

\[
\tau = bt \left[ \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2} \right]^{1/2},
\]

\( \cn, \sn, \) and \( \dn \) are elliptic functions, and \( m \) is given by

\[
m = \frac{(J_2 - I_1)I_1a^2}{(I_3 - I_2)I_3b^2}.
\]

The constants \( a \) and \( b \) are such that at \( t = 0 \), \( \Omega_1 = a \) and \( \Omega_3 = b \). When the precession angle is small, it will approximately equal \( a/b \).

The rotation matrix \( R_{j\mu} \) is given in terms of the Euler angles \( \theta, \phi, \psi \). The time dependence of these can in turn be expressed in closed form in terms of elliptic and theta functions:

\[
\begin{align*}
\cos \theta &= \frac{I_3b}{J} \dn(\tau), \\
\tan \psi &= \left[ \frac{I_1 (I_3 - I_2)}{I_2 (I_3 - I_1)} \right]^{1/2} \frac{\cn(\tau, m)}{\sn(\tau, m)}, \\
\phi &= \phi_1 + \phi_2, \quad \phi_1 + \phi_2 = 2\pi t T' \Theta,
\end{align*}
\]

with

\[
\exp[2i\phi_1(t)] = \frac{\vartheta_4 \left( \frac{2\pi t}{T'} - i\pi \alpha \right)}{\vartheta_4 \left( \frac{2\pi t}{T'} + i\pi \alpha \right)},
\]

\[
2\pi T' = \frac{J}{I_1} - \frac{2i}{T} \vartheta_4'(i\pi \alpha),
\]

The constants \( \alpha \) and \( T \) that appear above are given by

\[
\begin{align*}
\sn[2i\alpha K(m)] &= \frac{iI_3b}{I_1a}, \\
T &= 4K(m) \left[ \frac{I_1 I_2}{(I_3 - I_2)(I_3 - I_1)} \right]^{1/2}.
\end{align*}
\]
where \( \vartheta_4 \) is the \( 4^{th} \) Jacobi theta function with nome

\[
q = \exp \left( -\pi \frac{K(1 - m)}{K(m)} \right),
\]

(2.10i)

and \( K(m) \) is the complete elliptic integral of the first kind [17].

Within the weak-field, quadrupole approximation these closed expressions fully describe the gravitational wave signal from a freely precessing, rigid body. However, after all is said and done, the problem involves just two periodicities determined by \( T \) and \( T' \). Accordingly, the radiation should be resolvable into a discrete spectrum. Knowledge of the amplitude and frequencies of the spectral lines greatly simplifies the analysis of gravitational wave data. The discrete nature of the spectrum becomes apparent when \( h_+ \) and \( h_\times \), as given in equations (2.8), are expanded in terms of ‘small’ parameters: the ‘wobble angle parameter’ \( a/b \), the rigid body’s oblateness, and its deviation from axisymmetry. Finding the second-order contributions to the spectrum in these parameters, evaluating the observability of the associated spectral lines, and exploring what can be learned from such observations will be our main considerations.

### III. FIRST ORDER CONTRIBUTIONS TO THE GRAVITATIONAL WAVE SPECTRUM

In his first-order expansion of \( h_+ \) and \( h_\times \), Zimmermann identified two expansion parameters, \( m \) and \( \delta \):

\[
m = \frac{I_2 - I_3 I_1}{I_3 - I_2 I_1} \left( \frac{a}{b} \right)^2,
\]

(3.1a)

\[
\delta = 1 - \left( \frac{I_1 I_3 - I_2}{I_2 I_3 - I_1} \right)^{1/2}.
\]

(3.1b)

To find the ‘first-order’ contributions to the radiation spectrum, Zimmermann

- expanded the rotation matrix \( R_{ij} \) to first order in \( m \) and \( \delta \);

- substituted the expanded rotation matrix into the relation between the inertia tensor in the inertial frame and that in the body frame;

- substituted the inertia tensor in the inertial frame into the standard quadrupole formulae for the strains (2.1);
• truncated the resulting \( h_+ \) and \( h_\times \) to first order in small \( a/b \), small oblateness \((I_3 - I_1)/I_3\), and small non-axisymmetry \((I_2 - I_1)/(I_3 - I_2)\).

This leads to

\[
\begin{align*}
    h_+^{Z} &= -\frac{2I_3\Omega_{\text{rot}}^2}{r} \frac{(I_2 - I_1)}{I_3} \left(1 + \cos^2(i)\right) \cos(2\Omega_{\text{rot}}t) \\
    &+ \frac{I_3(\Omega_{\text{rot}} + \Omega_{\text{prec}})^2}{2I_3} \frac{2I_3 - (I_1 + I_2)}{r} \left(\frac{I_1a}{I_3b}\right) \sin(2i) \cos[(\Omega_{\text{rot}} + \Omega_{\text{prec}})t], \quad (3.2a) \\
    h_\times^{Z} &= -\frac{4I_3\Omega_{\text{rot}}^2}{r} \frac{(I_2 - I_1)}{I_3} \cos(i) \sin(2\Omega_{\text{rot}}t) \\
    &+ \frac{2I_3(\Omega_{\text{rot}} + \Omega_{\text{prec}})^2}{r} \frac{2I_3 - (I_1 + I_2)}{2I_3} \left(\frac{I_1a}{I_3b}\right) \sin(i) \sin[(\Omega_{\text{rot}} + \Omega_{\text{prec}})t]. \quad (3.2b)
\end{align*}
\]

The fundamental angular frequencies in the problem are

\[
\begin{align*}
    \Omega_{\text{prec}} &= \frac{2\pi}{T} \\
    &= \frac{\pi b}{2K(m)} \left(\frac{(I_3 - I_2)(I_3 - I_1)}{I_1I_2}\right)^{1/2}, \quad (3.3a) \\
    \Omega_{\text{rot}} &= \frac{2\pi}{T'} - \frac{2\pi}{T} \\
    &= \frac{J}{I_1} - \left(1 + \frac{i}{\pi} \frac{\vartheta_4(i\pi\alpha)}{\vartheta_4(i\pi\alpha)}\right) \Omega_{\text{prec}}. \quad (3.3b)
\end{align*}
\]

The expressions (3.2) represent Fourier expansions of the waveforms, with gravitational spectral lines at the angular frequencies

\[
\begin{align*}
    \Omega^I &= 2\Omega_{\text{rot}} \quad \text{(line I)}, \\
    \Omega^{II} &= \Omega_{\text{rot}} + \Omega_{\text{prec}} \quad \text{(line II)}. \quad (3.4)
\end{align*}
\]

Line I is a consequence of the body’s deviation from axisymmetry. Non-axisymmetry is often discussed in terms of an ‘equatorial eccentricity’: if the body is approximated as an ellipsoid rotating about an axis associated with a principal moment of inertia, then it will look identical after half a rotation period, hence the frequency \( 2\Omega_{\text{rot}} \). Line II encodes the precessing motion; recall that \( a/b \) is approximately the wobble angle. The coupling of precession and rotation causes the frequency of this line to be shifted upwards by \( \Omega_{\text{prec}} \); note that both motions are in the same sense. As we shall see later on, this coupling also has an effect near line I, although it only becomes visible at higher orders in the precession angle.
IV. SECOND ORDER CONTRIBUTIONS TO THE SPECTRUM

Zimmermann’s calculation of the first order contribution to the gravitational radiation from rigid body free precession involved the expansion parameters $m$ and $\delta$ defined by the somewhat awkward expressions (3.1a) and (3.1b), and at the end further truncations were made in terms of different small quantities. Here we will expand in terms of small parameters that have a more direct physical meaning.

The wobble angle is only one of several small parameters in the problem. To set up a second-order expansion, we will first specify a suitable set of expansion parameters and then estimate the size of these parameters for realistic neutron stars. These estimates will tell us to what order we need to expand in parameters other than the wobble angle parameter $a/b$ to arrive at a sufficiently good approximation for the waveforms. They will also yield information about the amplitudes of the different spectral lines at first and second order.

Unlike Zimmermann, who used the general expressions for the quadrupole radiation (2.1), our starting point will be the closed forms (2.8) for $h_+\, h_\times$. There is no significant difference between the two approaches, but expansions of the closed formulae will lead to somewhat more elegant expressions for the spectral line amplitudes. Overall, our aim is to make the derivation and physical meaning of both the first-order and second-order results as transparent as possible.

A. Expansion parameters

The parameter $m$ defined by eq. (3.1a) encodes two pieces of information at once: the deviation from axisymmetry and the precession angle. To understand the role played by each, factor $m$ as

$$m = 16 \kappa \gamma^2,$$

(4.1)

where the new small parameters $\gamma$ and $\kappa$ are defined by

$$\gamma = \frac{I_1}{I_3} \frac{a}{b},$$

(4.2a)

$$\kappa = \frac{1}{16} \frac{I_3 I_2 - I_1}{I_1 I_3 - I_2}.$$

(4.2b)

The parameter $\gamma$ describes the precession angle while $\kappa$ describes the non-axisymmetry $(I_2 - I_1)$ relative to the axisymmetric non-sphericity $(I_3 - I_2)$. Note that in all cases that
are of astrophysical interest, \( I_1/I_3 \simeq 1 \); this and the factor of 1/16 in \( \kappa \) have been included for computational convenience. The parameter \( \delta \) (eq. 3.1b) is related to the new parameter \( \kappa \); it will be discussed in subsection C below.

We would like to stress that specifying these different expansion parameters from the outset does not imply a departure from the approach of [3]. Close inspection of that analysis shows that \( \gamma \) and \( \kappa \) are in fact used implicitly to truncate to first order in \( a/b \) at the end of the computation. Introducing them from the beginning merely facilitates the calculation.

In order to develop appropriate parameters for the second order expansion, we need to find out what astrophysics is involved in the new parameters \( \gamma \) and \( \kappa \), and any other relevant ‘small’ quantities. Note that in realistic cases, \( \gamma \) and \( \kappa \) will not be independent, since the precession is determined for the most part by the loss of axisymmetry. Ideally, it would be a simple exercise in classical mechanics to find the relationship: one could start from a non-precessing, axisymmetric body and then study what kind of precessing motions are induced by various continuous deformations. However, we may assume that during a starquake other factors will be at play, such as torques induced by partial crustal pinning of vortices in the interior. To estimate their effect we would have to take recourse to the existing models of pinning, which seem to be contradicted by the observations of apparently free precession in pulsars [5, 6, 12]. For this reason, we will mostly disregard any a priori relation between \( \gamma \) and \( \kappa \) and keep the discussion as general as possible.

### B. Astrophysics of precessing neutron stars

Let us first look into some estimates for oblateness, non-axisymmetry, and the precession angle.

Oblateness was described by Baym and Pines [18] as

\[
\frac{I_3 - I_1}{I_3} = \frac{3}{2} \beta \epsilon_0.
\]

Here \( \beta \) is the rigidity parameter, which depends on the equation of state; in [19] it was derived to be

\[
\beta \simeq 1.6 \times 10^{-5}.
\]

For very young neutron stars, the zero strain oblateness \( \epsilon_0 \) will depend on the shape of the star when the crust first solidified; subsequent starquakes as well as less violent plastic
deformations will change it. We will take it to be of the order of the ratio of rotational kinetic energy to gravitational binding energy:

$$\epsilon_0 \simeq \frac{\Omega_{\text{rot}}^2 R_t^3}{GM}$$

$$= 5.2 \times 10^{-2} \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^2,$$

where \( f_{\text{rot}} = \Omega_{\text{rot}}/2\pi \). We have set \( M = 1.4M_\odot \) and \( R = 10^6 \) cm, which are the values for mass and radius we will use throughout this paper.

Eqns. (4.3), (4.4), and (4.5) lead to an oblateness of

$$\frac{I_3 - I_1}{I_3} \simeq 7.7 \times 10^{-7} \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^2 \left( \frac{\beta}{10^{-5}} \right).$$

The implied range agrees with estimates by Alpar and Pines [4]; see also the discussion in Cutler and Jones [10].

To arrive at an estimate of \( \kappa \), we need information as to how a neutron star loses its axisymmetry; here we will follow the analysis by Link, Franco, and Epstein [20]. According to their model, a starquake is expected to induce fault lines at angles of 30 to 45 degrees relative to the equator. Crust material shifts along the faults and accumulates in mountains with a typical height of \( 10^{-2} \) cm. We may assume

$$I_2 - I_1 \sim M_m R^2,$$

with \( M_m \) the mass contained in a mountain and \( R \) the radius of the neutron star. One has

$$M_m \sim \frac{\Delta R L}{R^2} M_c,$$

where \( \Delta R \) is the change in equatorial radius due to the decrease in oblateness, \( L \) is the fault length, and \( M_c \) is the mass of the crust. The factor in front of \( M_c \) in (4.8) is the fraction of the star’s area that moves with the fault. Assuming an outer crust density [23] of \( \rho \sim 10^{10} \) g/cm\(^3\) and a crust thickness in the order of \( R/10 \), we can use the expression for \( \Delta R/R \) in [20] to arrive at

$$\frac{I_2 - I_1}{I_3} \simeq 2.7 \times 10^{-9} \frac{L}{R} \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^2 \left( \frac{t_q}{10 \text{ yr}} \right) \left( \frac{t_{\text{age}}}{10^3 \text{ yr}} \right)^{-1},$$

where \( t_q \) is the average time between large starquakes and \( t_{\text{age}} \) is the spin-down age.
We now have sufficient information to estimate $\kappa$. Using (4.2b), (4.6), and (4.9), we find
\[ \kappa \simeq 2.2 \times 10^{-4} \frac{L}{R} \left( \frac{\beta}{10^{-3}} \right)^{-1} \left( \frac{t_q}{10 \text{ yr}} \right) \left( \frac{t_{\text{age}}}{10^3 \text{ yr}} \right)^{-1}. \] (4.10)

The most important ‘small’ parameter in our analysis will be the wobble angle $\gamma$. For sufficiently small rotation frequencies, the wobble angle is essentially unconstrained. For rapidly rotating neutron stars (which are the ones of interest for gravitational wave detection), Jones and Andersson \[11\] propose the following expression for the maximum wobble angle that can be supported by the crust:
\[ \gamma_{\text{max}} \simeq 1.8 \times 10^{-2} \left( \frac{500 \text{ Hz}}{f_{\text{rot}}} \right)^2 \left( \frac{u_{\text{break}}}{10^{-3}} \right), \] (4.11)
where $u_{\text{break}}$ is the strain at which the crust fractures. It is easy to understand this expression qualitatively; if $f_{\text{rot}}$ is large, so will be the oblateness and hence the equatorial bulge. The star will then have to displace more matter while precessing, leading to higher crustal strains. In [21], the breaking strains of terrestrial materials were extrapolated to yield estimates in the range $10^{-4} < u_{\text{break}} < 10^{-2}$, although the reliability of such extrapolations is unclear. Note that if the breaking strain is near the upper limit of this range, $u_{\text{break}} \simeq 10^{-2}$, the expression (4.11) implies an unconstrained precession angle even at a rotation frequency of 100 Hz.

C. Setting up the second order expansion

From the estimates (4.6), (4.9), (4.10) and (4.11) of the previous subsection, we infer that in the frequency band of interest,
\[ \left| \frac{I_\mu - I_\nu}{I_\rho} \right| \ll \kappa \ll \gamma_{\text{max}} \] (4.12)
for $\mu, \nu, \rho = 1, 2, 3$.

We will choose $\gamma$ as our primary expansion parameter. The approximation (3.2) for the waveforms is of first order in $\gamma$; here we will go to order $\gamma^2$. However, we will only retain first-order terms in $\kappa$. The latter will be treated on the same footing as $\gamma^2$, so that terms in, e.g., $\gamma \kappa$ will be discarded.

Considering that quantities of the form $|I_\mu - I_\nu|/I_\rho|$ are our smallest parameters, we will also take those to be of order $\gamma^2$. Accordingly, the quantities $\Gamma_\mu$ defined in (2.5) will be
approximated as $\Gamma_1 \simeq \Delta_2 - \Delta_3$ whenever multiplied with $\gamma$ or $\kappa$, and similarly for $\Gamma_2$, $\Gamma_3$ with cyclic permutation of the indices.

In the expansions of the waveforms, linear combinations of $\Delta_\mu$ and $\Gamma_\mu$ defined in (2.5) will appear, multiplied with powers of $\gamma$ and $\kappa$. With the approximations above, these can be written as linear combinations of $I_3 - (I_1 + I_2)/2$ and $I_2 - I_1$, the first of which measures oblateness. Note that we have

$$I_2 - I_1 \simeq 16 \kappa [I_3 - (I_1 + I_2)/2],$$

(4.13)

the correction being of order $\kappa^2$. Also, the non-axisymmetry parameter $\delta$ defined in (3.1b) can be expressed in terms of $\kappa$ as

$$\delta \simeq 1 - \left(1 - \frac{I_2 - I_1}{I_3 - I_2}\right)^{1/2}$$

$$\simeq 1 - (1 - 16\kappa)^{1/2}$$

$$\simeq 8\kappa,$$

(4.14)

again up to terms in $\kappa^2$.

We are now ready to calculate the spectrum at second order in the wobble angle.

**D. The spectrum at second order**

The expressions for $R_{ij\mu}$ and $\Omega_{\mu}$ given in [3] can be expanded to second order in $\gamma$ and to first order in $\kappa$. By substituting the results into (2.8) and truncating appropriately we find the desired second-order expansion of the waveforms.

In the expansions, the time dependence of the waveforms is given entirely in terms of linear combinations of products of trigonometric functions, with frequencies that are integer linear combinations of the basic frequencies $\Omega_{\text{rot}}$ and $\Omega_{\text{prec}}$. These trivially allow for a Fourier expansion, and after collecting the prefactors of the simple sines and cosines, we find that the second-order waveforms $h^{(2)}_+$ and $h^{(2)}_\times$ are of the form

$$h^{(2)}_+ = \sum_{k=0}^{1} [A_{+,k}^I \cos (\Omega_{2k}^I t) + A_{+,k}^{II} \cos (\Omega_{2k}^{II} t)],$$

$$h^{(2)}_\times = \sum_{k=0}^{1} [A_{-,k}^I \sin (\Omega_{2k}^I t) + A_{-,k}^{II} \sin (\Omega_{2k}^{II} t)],$$

(4.15)
where we defined
\[ \Omega_{2k} = \Omega^I + 2k \Omega_{\text{prec}}, \]
\[ \Omega_{2k}^I = \Omega^{II} + 2k \Omega_{\text{prec}}. \] (4.16)

The contributions with frequencies \( \Omega^I \) and \( \Omega^{II} \) are the first-order ones in small wobble angle found by Zimmermann [3]. The second-order contributions appear as sidelobes to the first order lines.

We find the following non-zero amplitudes. At first order, we retrieve Zimmermann’s lines in [3]:
\[ A^I_{+,0} = -\frac{2}{r} b^2 (1 + \cos^2(i)) (I_2 - I_1), \]
\[ A^I_{+,0} = \frac{1}{r} b^2 \sin (2i) [I_3 - (I_1 + I_2)/2] \gamma. \] (4.17)

To compare these with the amplitudes from (3.2), note that with the approximations we are making, \( \Omega_{\text{prec}} \ll \Omega_{\text{rot}}, \) so that \( b^2 \simeq \Omega_{\text{rot}}^2 \simeq (\Omega_{\text{rot}} + \Omega_{\text{prec}})^2 \) (indeed, from (3.3a) and (3.3b) it can be seen that the difference between the three is of second order in \( \gamma \)). The reason why we find a simpler frequency prefactor stems from the fact that we have used the closed expressions (2.8) instead of the more general formulae (2.1) as the basis of our computation, and that the former do not contain any explicit time derivatives.

At second order in the wobble angle, we find a single extra line:
\[ A^I_{+,+1} = \frac{2}{r} b^2 (1 + \cos^2(i)) [I_3 - (I_1 + I_2)/2] \gamma^2. \] (4.18)

The results for the ‘cross’ polarization are quite similar:
\[ A^I_{x,0} = -\frac{4}{r} b^2 \cos (i) (I_2 - I_1), \]
\[ A^I_{x,0} = \frac{2}{r} b^2 \sin (i) [I_3 - (I_1 + I_2)/2] \gamma \] (4.19)
at first order in \( \gamma \), and
\[ A^I_{x,+1} = \frac{4}{r} b^2 \cos (i) [I_3 - (I_1 + I_2)/2] \gamma^2 \] (4.20)
at second order.

Henceforth, the second-order line will be referred to as line III.

Figure 1 gives a schematic representation of the spectrum at second order in the precession angle.
FIG. 1: A schematic representation of the spectrum at second order in the wobble angle. The lines I and II found by Zimmermann et al. [2, 3] arise at first order in the small wobble angle. Line I is due to the combination of rotation and non-axisymmetry. Line II has an amplitude proportional to the wobble angle parameter $\gamma$. The amplitude of line III is proportional to $\gamma^2$; it appears as a sidelobe to line I.

V. DETECTABILITY AND PHYSICAL CONTENT OF THE SPECTRUM

A. Detectability

In subsection IV B, astrophysical estimates were obtained for the small parameters governing the problem: the wobble angle $\gamma$, the deviation from axisymmetry $\kappa$, and the oblateness. With the expressions for the amplitudes of the spectral lines found in the previous subsection, these can now be used to determine the detectability of the lines. For concreteness, we will be interested in distances of $r \sim 10$ kpc, rotation frequencies $f_{\text{rot}} \sim 500$ Hz, and a rigidity parameter $\beta \sim 10^{-5}$. Factors of $G$ and $c$ in the amplitudes are reinstated.
The (‘cross’ polarization) amplitude for line I is of the form

\[ A^I = -\frac{G^4}{c^4 r} b^2 f(i) (I_2 - I_1) \]
\[ = -I_3 \frac{G 64}{c^4 r^2} b^2 f(i) \frac{I_3 - (I_1 + I_2)/2}{I_3} \kappa \]  

(5.1)

with \( f(i) \) some function of order 1 depending on the polarization; in the second line we have used (4.13). Using the estimates of subsection IV B and setting \( I_3 = (2/5)(1.4 M_\odot)(10^6 \text{ cm})^2 \), we find

\[ A^I \simeq 3.3 \times 10^{-28} \left( \frac{10 \text{ kpc}}{r} \right) \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^4 \frac{L}{R} \left( \frac{t_q}{10 \text{ yr}} \right) \left( \frac{t_{\text{age}}}{10^3 \text{ yr}} \right)^{-1}. \]  

(5.2)

For line II we have (under the same assumptions):

\[ A^{II} = I_3 \frac{G^2}{c^4 r^2} b^2 g(i) \frac{I_3 - (I_1 + I_2)/2}{\gamma} \]
\[ \simeq 4.6 \times 10^{-26} \left( \frac{10 \text{ kpc}}{r} \right) \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^4 \left( \frac{\beta}{10^{-5}} \right) \gamma, \]  

(5.3)

where \( g(i) \) is again a function of order 1. Finally, for line III:

\[ A^{III} \simeq 9.4 \times 10^{-26} \left( \frac{10 \text{ kpc}}{r} \right) \left( \frac{f_{\text{rot}}}{500 \text{ Hz}} \right)^4 \left( \frac{\beta}{10^{-5}} \right) \gamma^2. \]  

(5.4)

In Jones and Andersson [13], detectability of spectral lines from precessing neutron stars was discussed using estimates of the sensitivity of interferometric detectors that differ from the ones we will use. Their paper was written before the more recent analysis of the expected capabilities of LIGO II by Fritschel [14, 15]. Here we will assume a LIGO II detector, optimized for isolated neutron star observations by adding a signal recycling mirror at the output, the position of which can be changed macroscopically. For frequencies between 500 and 1000 Hz, this leads to a strain sensitivity of \( 10^{-24} \text{ Hz}^{-1/2} \) [14, 15]. A one-year integration time would then allow for the detection of signals with amplitudes as low as

\[ h_{\text{min}} \simeq 10^{-24} \left( \frac{1}{3.15 \times 10^7} \right)^{1/2} \]
\[ \simeq 2 \times 10^{-28}. \]  

(5.5)

From (5.2), we see that line I should be detectable for a spin-down age much less than \( 10^{3} \) years. (Also recall that our estimate for the magnitude of \( \kappa \) should be considered pessimistic.) Eq. (5.3) indicates that to observe line II, one would need \( \gamma > 4.3 \times 10^{-3} \). This is to be
compared with the physical upper limit implied by eq. (4.11), which is $\gamma_{\text{max}} \simeq 1.8 \times 10^{-2}$ for a breaking strain of $u_{\text{break}} \sim 10^{-3}$.

Finally, we come to line III. Eq. (5.4) shows that one needs

$$\gamma > 4.6 \times 10^{-2}$$

(5.6)

for the second-order line to be detectable. Now, recall that there is much uncertainty about the value of the breaking strain of the crust [21]. If, e.g., $u_{\text{break}} \sim 10^{-2}$, then for the frequency of $f_{\text{rot}} \sim 500$ Hz we are considering, the maximum possible wobble angle would be 0.18 rad. Since we have been assuming a distance of $r \sim 10$ kpc, this means that line III may well be observable even for sources as distant as the galactic center.

**B. Extracting physical information**

We now investigate what information about a neutron star’s shape and motion can be deduced from the spectrum computed in the previous section.

If the neutron star is not precessing, then at most one line will be present in the spectrum, namely the first-order line I associated with non-axisymmetry. We will not consider this case any further.

Even if the non-axisymmetry of the neutron star is insignificant, two lines are present; both lines II and III are generated purely by the precessing motion. The inclination angle $i$ can always be determined by comparing the two polarizations for two different lines. Introducing non-axisymmetry, line I appears, which is separated from line III by two times the precession frequency. Let us assume that line III has enough power to be detectable. We can then distinguish between two cases.

(a) The precession period is large compared to the integration time. In that case we see the precession line II at a frequency $\Omega_{\text{rot}} + \Omega_{\text{prec}} \simeq \Omega_{\text{rot}}$. Even though lines I and III may not be resolvable, the first-order non-axisymmetry line I gets modulated by the second-order precession line III. When studying data, one would know what function to fit the time dependence of this modulation by. The unknowns are the precession frequency, and the amplitudes of lines I and III. Assuming the fit would allow one to estimate these quantities with some
degree of accuracy, the rest of the discussion for this case is identical to that of case (b) below.

(b) If lines I and III can be resolved, three distinct lines are seen. The parameters encoding the physical information about the neutron star can then be found as follows. The ratio of the amplitudes of lines III and II is proportional to $\gamma$, the proportionality factor being a function of the inclination angle $i$. Next, one can look at the ratio of the amplitudes of lines I and II. This is proportional to $\kappa/\gamma$, where the proportionality factor is again a function of $i$; knowing $\gamma$, we obtain a value for $\kappa$. The frequencies at which the lines occur allow us to determine $\Omega_{\text{prec}}$ and $\Omega_{\text{rot}}$. Now, the expression (3.3a) for the precession frequency can be written as

$$\Omega_{\text{prec}} \simeq \frac{\pi}{2K(m)} \frac{I_3 - (I_1 + I_2)/2}{I_3} \Omega_{\text{rot}}.$$  \hfill (5.7)

Using $m = 16\kappa\gamma^2$, from $\Omega_{\text{rot}}$ and $\Omega_{\text{prec}}$ we can then compute the oblateness parameter

$$\frac{I_3 - (I_1 + I_2)/2}{I_3}.$$  \hfill (5.8)

It is important to note that, to extract the physical information one is interested in (precession angle, deviation from axisymmetry, and oblateness), the first-order spectrum would not suffice.

We end this section with a brief comment concerning the distance to the neutron star emitting the radiation. Consider the amplitude of, e.g., line I:

$$A^I = -I_3 \frac{G}{c^4} \frac{64}{r} b^2 f(i) \frac{I_3 - (I_1 + I_2)/2}{I_3} \kappa,$$  \hfill (5.9)

where $f(i)$ is again some function of the inclination angle of order one. As we have just seen, knowledge of the amplitudes and frequencies of the three spectral lines at second order suffices to infer values for $i$, $\kappa$, $[I_3 - (I_1 + I_2)/2]/I_3$, and of course $b^2 \simeq \Omega_{\text{rot}}^2$. From the expression above, it is then clear that by inserting an educated guess for the principal moment of inertia $I_3$ (as we have been doing to assess the detectability of the lines), we can obtain a rough estimate of the distance to the source.
VI. SUMMARY AND CONCLUSIONS

We derived the quadrupole gravitational wave spectrum of a rigidly rotating and freely precessing neutron star up to second order in the precession angle.

Before doing so, we reviewed Zimmermann’s first-order expansion of the waveforms \[3\]. We then introduced new ‘small’ parameters which are more directly related to the physical quantities of interest, with a view on facilitating the second order expansion and its interpretation, and also clarifying the earlier work. These parameters correspond to the oblateness, the deviation from axisymmetry, and the precession angle of the neutron star.

Next, we made careful astrophysical estimates of physical quantities characterizing neutron stars. This led to estimates of the ‘small’ parameters, which justified the use of the wobble angle as the primary expansion parameter for a higher-order expansion of the waveforms, putting the other small parameters on the same footing as the square of the wobble angle. In addition, the estimates explained why the truncations implicitly made by Zimmermann were physically reasonable.

We then used the new small parameters to set up an expansion of the waveforms to second order in precession angle. The approach taken differed somewhat from that of Zimmermann’s in that we used as a starting point the closed expressions for the specific case of quadrupole radiation from rigidly rotating and precessing bodies instead of the general formulae, truncating directly in terms of parameters with a clear physical interpretation. Our method does not imply a significant departure from Zimmermann’s approach, but it does make the calculations and the end result more transparent.

At first order, the two spectral lines predicted by Zimmermann were retrieved: line I, which is associated with non-axisymmetry, and line II which results from the precessing motion. At second order, a single additional line was found (line III). It appears as a ‘sidelobe’ to line I and is separated from it by twice the precession frequency.

On the basis of our estimates for the small parameters, we evaluated the detectability of the three spectral lines with a LIGO II detector optimized for isolated neutron star observations \[14, 15\], assuming a one-year integration time. For a rotation frequency of 500 Hz and a distance of 10 kpc, together with some conservative assumptions concerning neutron star structure, we found that:

- Line I should be detectable if the neutron star’s spin-down age is much less than \(10^3\)
years;

- Observability of line II depends on the crustal breaking strain, which is subject to much uncertainty. However, the line can be observable even if the breaking strain is near the lower limit of the range proposed by Ruderman [21];

- If the breaking strain is near the upper end of this range, line III can also be detectable.

Hence, it may well be possible to observe the entire second-order spectrum of neutron stars as far away as the galactic center.

Finally, we investigated what gravitational wave observations might tell us about the neutron star emitting the radiation. The two first-order spectral lines found by Zimmermann would not allow one to infer much about the intrinsic physical properties of the source beyond the rotation and precession frequencies. To separate the precession angle, the deviation from axisymmetry and the oblateness from given data, at least one additional line is needed. Our results provide the missing information: if the second order line can be seen, then it does become possible to infer these quantities.

The direct measurement of these characteristics would permit an evaluation of the models for neutron star structure and evolution that led to our estimates of the small parameters [4, 10, 11, 18, 19, 20, 21], in a way that would be impossible by conventional astronomical means. The detailed implications for neutron star modeling will be investigated elsewhere.

Given our assessment of the observability of the spectral lines we found, it may be of interest to extend our results by constructing still higher-order expansions. The scheme presented here can easily be used to compute any number of discrete contributions to the quadrupole gravitational wave spectrum of freely precessing neutron stars, modeled as rigid bodies.

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3, describe the motion.

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