A simple method for sampling random Clifford operators

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Abstract

We describe a simple algorithm for sampling $n$-qubit Clifford operators uniformly at random. The algorithm outputs the Clifford operators in the form of quantum circuits with at most $5n + 2n^2$ elementary gates and a maximum depth of $O(n \log n)$ on fully connected topologies. The circuit can be output in a streaming fashion as the algorithm proceeds, and different parts of the circuit can be generated in parallel. The algorithm has an $O(n^2)$ time complexity, which matches the current state of the art. The main advantage of the proposed algorithm, however, lies in its simplicity and elementary derivation.

1 Introduction

The $n$-qubit Clifford group, $C_n$, consists of all unitary operators for which the signed $n$-qubit Pauli group, $P_n$, is closed under conjugation. Operators from the Clifford group can be implemented as quantum circuits consisting of only Hadamard, phase, and controlled-not (CNOT) gates. Conversely, any circuit made up of only these and derived gates, such as single-qubit Pauli gates and $cz$ gates, so-called Clifford circuits, implements an element of the Clifford group. An important property of Clifford circuits is that they can be efficiently simulated when the initial state is a computational basis state [1]. In addition to simulation and various other applications [2], Clifford operators play an important role in the characterization of noise channels using techniques such as randomized benchmarking [3, 4, 6]. Some of these applications, randomized benchmarking included, require that we can efficiently sample elements from $C_n$ uniformly at random.

The first efficient algorithm for sampling from the Clifford group was given by Koenig and Smolin [5]. They start with the observation that the cardinality of $C_n$ is finite:

$$|C_n| = 2^{n^2 + 2n} \prod_{j=1}^{n} (4^j - 1).$$

It then follows that randomly sampling of the group is equivalent to sampling an integer index between 0 and $|C_n| - 1$ and providing a one-to-one mapping between indices and the elements of $C_n$. As a next step, they observe that $C_n/P_n \cong Sp(2n, \mathbb{F}_2)$, the symplectic group on $\mathbb{F}_2^{2n}$. Given that the mapping from integers to the Pauli group is trivial, it remains to find a mapping to elements of the symplectic group. Efficient algorithms for this and the inverse mapping based on transvections are given in [5], resulting in a sampling algorithm with time complexity $O(n^3)$. In recent work, Bravyi and Maslov [2] study the structure of the Clifford group and provide a canonical representation for elements in the group based on the Bruhat decomposition [7]. Leveraging this canonical form, they obtain an $O(n^2)$ algorithm for sampling elements from the Clifford group uniformly at random. They also describe how variants of the canonical form can be used to implement arbitrary Clifford unitaries with a circuit depth of at most $9n$ on a linear nearest-neighbor architecture.

In this work we provide a simple algorithm for uniform sampling from the Clifford group. The algorithm is based on the tableau representation of Paulis and, similar to [5], takes advantage of the hierarchical structure of the Clifford group. The algorithm has an $O(n^2)$ runtime and, unlike other algorithms, directly generates circuits...
Pauli operators are formed as tensor products of the $2 \times 2$ Paulis and tableau representation concludes with a brief discussion in Section 5. The motivation behind the proposed algorithm is given in Section 3, followed by a derivation of the algorithm in Section 4. We describe this convenient representation in detail in Section 2. The algorithm, however, lies in its simplicity and elementary derivation. The algorithm leverages the tableau with $O$ latency and segments of the circuit can be generated fully in parallel. Perhaps the main advantage of the algorithm, however, lies in its simplicity and elementary derivation. The algorithm leverages the tableau representation introduced in [1]. We describe this convenient representation in detail in Section 2. The motivation behind the proposed algorithm is given in Section 3, followed by a derivation of the algorithm in Section 4. We conclude with a brief discussion in Section 5.

2 Paulis and tableau representation

Pauli operators are formed as tensor products of the $2 \times 2$ identity matrix $I$, and the three Pauli matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Elements from the $n$-qubit Pauli group are generated using $n$ components and can be written as $2^n \times 2^n$ operators $P = \otimes_i P_i$. Multiplying two Pauli operators reduces to multiplying the corresponding components: given a second Pauli operator $Q = \otimes_i Q_i$, we have $PQ = \otimes_i (P_i Q_i)$. An important property of the Pauli group is that any two elements either commute $(PQ = QP)$ or anticommute $(PQ = -QP)$. Each Pauli matrix commutes with itself and the identity matrix, and anticommutes with the other two Pauli matrices. It follows from the multiplication rule given above that elements $P$ and $Q$ anticommute if and only if an odd number of components $P_i$ and $Q_j$ anticommute. Denote by $X_j$ the tensor product $\otimes_i P_i$ with $P_j = X$ and $P_j = I$ for all $i \neq j$, and define $Y_j$ and $Z_j$ is a similar manner. We can then conveniently represent Paulis in terms of binary vectors $x, z \in \mathbb{F}_2^n$:

$$P(x, z) = \prod_j i^{x_j z_j} X_j^{x_j} Z_j^{z_j}. \quad (2)$$

The factors $i^{x_j z_j}$ are added to correct for the phase resulting from the multiplication $X_j Z_j = -iY_j$. We can represent a set of $k$ Paulis as a $k \times 2^n$ binary matrix, or tableau, where each row contains the $x$ and $z$ coefficients of a single Pauli. The leftmost plot in Figure 1 illustrates a tableau representing the Pauli operators $XZI = X \otimes Z \otimes I$, IIY, and YXI with the $x$ and $z$ coefficients grouped together into $X$ and $Z$ blocks. The order of the columns in the tableau can be changed to suit our needs and we sometimes use tableaus in which the $x$ and $z$ coefficients alternate. Tableaus can additionally be augmented with a column of sign bits $s$ that indicate a phase $(-1)^s$. Although all tableaus in the paper will have such a sign column, we sometimes omit them from illustrations to keep the exposition clean.

As mentioned in the introduction, the Clifford group is generated by the single-qubit Hadamard (H) and phase (S) gates, and the two-qubit controlled-not gate (CX). Any quantum circuit consisting of only these and derived gates (including all Pauli operators and the controlled-Z gate) implements a Clifford operator $C(\rho) = C_p C_\rho$. (Note that such a circuit implementation is by far unique: any given Clifford operator has infinitely many circuit representations.) The power of the tableau notation lies in the ease with which it represents the mapping of Pauli operators under Clifford operators. For instance, as illustrated in Figure 1 applying the Hadamard gate on qubit $a$ results in the exchange of the corresponding columns in the tableau. Conjugation with the phase gate results in
addition of $x_a$ to $z_a$, modulo two, for each of the Paulis in the tableau, and finally, application of CX on qubits $a$ and $b$ adds $x_a$ to $x_b$ and $z_b$ to $z_a$. Each of these operations has an associated update to the sign vector, as detailed in \[1\]. Since Clifford operators can be written as products of unitary operators, they themselves must be unitary. Given Clifford operator $C$ and any two Pauli operators $P$ and $Q$ we therefore have

$$C(PQ) = C(P)C(Q) = CP(C^\dagger C)QC = C(P)C(Q).$$

This means that if we know the Pauli operators resulting from conjugation of $P$ and $Q$, we know the resulting operator for Pauli $PQ$. We know from \[3\] that any Pauli can be written as a product of basis terms $X_j$ and $Z_j$, so once we know $C(X_j)$ and $C(Z_j)$ for all $j$, we can determine the action of $C$ on all other Paulis. In order to fully describe a Clifford operator it thus suffices to prescribe the mapping of the $2n$ basis terms. These mappings have a number of restrictions. First, the identity always maps to itself: $CIC^\dagger = ICC^\dagger = I$. Second, the mapping is a bijection; no two Paulis map to the same Pauli. Third, commutation relations between elements remain invariant under conjugation. In other words, if Paulis $P$ and $Q$ commute then so do $P' = C(P)$ and $Q' = C(Q)$:

$$0 = PQ - QP = C(PQ - QP)C^\dagger = CP(C^\dagger C)QC^\dagger - CQ(C^\dagger C)PC^\dagger = P'Q' - Q'P'.$$

Note that the conjugation of a Pauli with a Clifford operator can results in a signed Pauli. For instance, $HYH^\dagger = -Y$.

### 3 Motivation

Consider an initial tableau that contains the interleaved set of basis Paulis $X_j$ and $Z_j$, as shown in Figure 2(a). By applying Clifford operator $C$ we obtain a new tableau that contains Paulis $C(X_j)$ and $C(Z_j)$, along with their sign. Since the new tableau specifies the output of the map from all basis Paulis, it completely represents the Clifford operator. Suppose now that we are given a tableau corresponding to a random Clifford operator $C$, as illustrated in Figure 2(b), where gray entries represents possibly non-zero elements, and where an asterisk (*) denotes any one of the $I$, $X$, $Y$, or $Z$ components. We can convert the tableau back to a Clifford operator as follows. First, using an appropriate combination of operations from Figure 4 on qubits 1, 2, and 3, we can normalize the first two rows and obtain the tableau as shown in Figure 2(c). The successive operators applied to the tableau correspond to Clifford operators, and we denote their product by $C_1$. Repeating a similar sweeping procedure on qubits 2 and 3, we generate Clifford operator $C_2$ and obtain the tableau given in Figure 2(c). Finally, operations on qubit 3 with operator $C_3$ results in the basis tableau shown in Figure 2(a). Given that the tableau in Figure 2(b) was determined by applying the random operator $C$ on the basis tableau, it follows that the reverse operator $C^\dagger$ is given by $C_3C_2C_1$. Taking the adjoint of this operator therefore allows us to obtain the original operator $C$ (the circuit representation of the operator could differ).
3 Proposed algorithm

The proposed algorithm is similar to the approach described in the previous section but operates on two rows at a time rather than on the full tableau. We initialize an empty tableau with \( n \) rows and populate the first two rows with randomly selected anticommuting Pauli operators and random signs, as shown in Figure 3(a). As before, we then manipulate these rows such that the first Pauli becomes \( X_1 \) and the second becomes \( Z_1 \), as illustrated in Figure 3(b). If we had applied to same manipulations on a fully populated tableau shown in Figure 2(b), we would now have the tableau shown in Figure 2(c). However, instead of sampling a full tableau and updating it with every sweep, we simply randomly sample what the next two rows would have been after sweeping. The crucial point here is that it does not matter whether we first sample the rows randomly at the beginning and the sweep, or first sweep and then randomly sample what they would have been after the transformation. To be consistent with the result of the sweeping operation, however, we do have to sample the Paulis such that their first component is the identity. Doing so, we arrive at the tableau shown in Figure 3(c). We then sweep the second random pair of anticommuting Paulis to \( X_2 \) and \( Z_2 \), as shown in Figure 3(d), and continue in a similar fashion until we arrive at the final tableaus given by Figure 2(d) and Figure 2(a).

In this procedure it can be seen that we operate on increasingly smaller subtableaus consisting of two rows. The sweeping operations have no effect on the entries outside the subtableau, since all relevant entries are zero and therefore represent identity Paulis. That means that we can start with an empty tableau with exactly two rows, and perform all operations in-place on the same tableau. Since every operation in the sweep routine corresponds to a gate in the quantum circuit, we can output the circuit in a streaming fashion while sweeping. In summary, the proposed algorithm performs the following operations in successive iterations \( \ell = 1, \ldots, n \):

Step 1 – Consider the signed subtableau consisting of rows \( 2i-1 \) and \( 2i \), and columns \( x_j \) and \( z_j \) for \( j \geq \ell \). For the in-place version, simply restrict the columns of the two-row tableau.

Step 2 – Randomly sample two anticommuting Pauli operators on \( n+1-\ell \) qubits with random signs and assign them to the rows of the subtableau. This can be done using the algorithm described in Section 4.1.

Step 3 – Sweep the subtableau to basis Paulis \( X_1 \) and \( Z_1 \), which in the larger tableau corresponds to Paulis \( X_\ell \) and \( Z_\ell \). One possible algorithm for doing so is given in Section 4.2.

According to the discussion in Section 3, the above procedure outputs a quantum circuit corresponding to the adjoint of the randomly sampled Clifford operator \( C \). However, the adjoint of a Clifford operator is itself a Clifford operator, and given the one-to-one mapping between \( C \) and \( C^\dagger \) it follows that the generated circuits are indeed sampled uniformly at random from the Clifford group. We prove correctness of the algorithm in Section 4.3.

4.1 Sampling rows

As part of the proposed algorithm we need to randomly sample pairs of anticommuting Pauli operators. For \( k \) qubits we can sample individual Pauli operators by randomly sampling \( 2k \) bits (a single row of the tableau). There are a total of \( 4^k \) times \( 4^k \) possible Pauli pairs, but not all pairs anti-commute. For the pair to be valid we require
If the pair is found to commute, we can simply discard them and repeat the process. Depending on that the first Pauli is not the identity, which gives \(4^k - 1\) options. For each such Pauli there are \(4^k / 2 = 2^{2k-1}\) Paulis that anticommute. The probability of sampling a valid pair is therefore

\[
\frac{(4^k - 1)2^{2k-1}}{4^{2k}} = (1 - 1/4^k)/2 \geq 3/8.
\]

If the pair is found to commute, we can simply discard them and repeat the process. Depending on \(k\), it takes between 2 and \(8/3\) trials on average before we find an anticommuting pair. We have \(k \leq n\), and sampling the bits and determining commutativity therefore has an expected time complexity of \(O(n)\). The algorithm has \(n\) iterations, which thus gives an overall sample complexity of \(O(n^2)\). This complexity matches the minimum, since we have to sample at least \(\log_2(|C_n|)\) random bits, where \(|C_n|\) is the cardinality of the Clifford group given by equation [1].

### 4.2 Sweeping

For the sweeping step we are given a subtableau with anticommuting \(k\)-Paulis \(P(x^a, z^a)\) and \(P(x^b, z^b)\), illustrated in Figure 4(a) with separate \(X\) and \(Z\) blocks. The goal of this step is to manipulate the tableau using the operations from Figure 1 to obtain a tableau that represents Pauli operators \(X_1\) and \(Z_1\). The algorithm proceeds as follows:

**Step 1** – Clear the elements in the \(Z\)-block of the first row. This is done by finding all indices \(j\) for which \(z^a_j = 1\) and applying \(H(j)\) when \(x^a_j = 0\), and \(S(j)\) otherwise. This step generates a maximum of \(k\) gates with circuit depth one and has a time complexity of \(O(k)\). An example of the result of this step is illustrated in Figure 4(b).

**Step 2** – Determine the (sorted) list of indices \(J = \{j \mid x^a_j = 1\}\) where the \(x^a\) coefficients are nonzero. The set is guaranteed to be nonempty, otherwise the row would correspond to the identity. Assuming one-based indexing, we can apply \(\text{CX}(J_i, J_{i+1})\) in parallel for all odd indices \(i < |J|\) to clear up to half of the nonzero coefficients in \(x^a\).

We then update \(J\) by retaining only the values at the odd locations, namely \(\{J_i \mid \text{odd } i\}\), and repeat the procedure until \(J\) is a singleton. In the example in Figure 4(b) we would start with \(J = \{2, 4, 5\}\) and apply \(\text{CX}(2, 4)\) in the first stage. We then update the index set to \(J = \{2, 5\}\) and apply \(\text{CX}(2, 5)\). One final update to the index set then gives \(J = \{2\}\). This step has an \(O(k)\) time complexity and generates a circuit with at most \(k - 1\) \(\text{CX}\) gates. The circuit depth is bounded by \(\lceil \log_2(k) \rceil\), which is the maximum number of updates to \(J\).

**Step 3** – When \(J \neq \{1\}\) we need to move the remaining nonzero coefficient in \(x^a\) to the first location. This is done by swapping qubits 1 and \(J_1\), which can be implemented using three \(\text{CX}\) gates. The tableau at the end of this step is as shown in Figure 4(c). For the second Pauli to anticommute with \(X_1\) it must have either \(Y\) or \(Z\) as the first component. In either case that means that \(z^b_1 = 1\), as indicated in the figure.

**Step 4** – If the second Pauli is equal to \(\pm Z_1\) we skip this step. Otherwise we first apply the single-qubit gate \(H(1)\) to arrive at the tableau given by Figure 4(d). We then repeat steps 1 and 2 with \(x^a\) and \(z^a\) replaced by \(x^b\) and \(z^b\): we first use single-qubit operations to clear the elements in \(z^b\), as illustrated in Figure 4(e), and then zero out all but one of the elements in \(x^b\). The application of the Hadamard gate ensures that the remaining element in the set \(J\) is 1, and we are thus left with the tableau shown in Figure 4(f). We again apply \(H(1)\) to obtain Paulis \(X_1\) and \(Z_1\).

**Step 5** – As a final step we clear any sign bits by applying the appropriate Pauli operator on the first qubit. If this operator anticommutes with the corresponding element in the Pauli represented by the row it will cause the sign
bit to flip. When \( s^a = 0 \) and \( s^b = 1 \) we apply \( X_1 \) since it commutes with \( X_1 \) and anticommutes with \( Z_1 \). When \( s^a = 1 \) we apply \( Y_1 \) if \( s^b = 1 \) and \( Z_1 \) otherwise. Note that, if we are interested only in the operator, we could avoid maintaining signs throughout the sweeping and simply sample the sign here to decide whether or not to apply the above gates.

The maximum number of single-qubit gates generate during the sweeping process is \( 2k + 2 \), while the maximum number of CX gates is \( 2(k - 1) + 3 \). The circuit depth for fully connected qubits is at most \( 8 + 2\lceil \log_2(k) \rceil \). Applying the sweeping step for \( k \) ranging from 1 to \( n \) gives a maximum of \( 5n + 2n^2 \) gates and a maximum circuit depth of

\[
\sum_{k=1}^{n} 8 + 2\lceil \log_2(k) \rceil \leq 10n + 2\log_2(n!) = \mathcal{O}(n \log n).
\]

Figure 5 illustrates a random four-qubit Clifford circuit generated by the proposed algorithm. The corresponding tableaus, at various stages of the algorithm, are shown in Figure 6 along with a detailed explanation in the caption.

### 4.3 Correctness of the algorithm

At iteration \( k \) of the algorithm we randomly sample one of the \( 2^{2k-1}(4^k - 1) \) pairs of anticommuting Paulis. Adding signs gives a total of \( 2^{2k+1}(4^k - 1) \) different settings. Multiplying the number of settings for iterations \( k \in [1, n] \), we exactly obtain the cardinality of the Clifford group given in Equation 1. Each of these settings has the same probability of being sampled, and in order to prove that sampling is uniform it therefore remains to show that each settings generates a different Clifford operator. Consider two settings whose sampled row pairs first differ at iteration \( k \), and denote the different pairs by \( (R_x, R_z) \) and \( (\bar{R}_x, \bar{R}_z) \). Assume, without loss of generality, that \( R_x \neq \bar{R}_x \). Now, denote the Clifford operator generated by the sweeping operations up to that point by \( C \), which we define to be the identity operator if \( k \) is one. At iteration \( k \) we sweep the tableau with operations that correspond to operators \( C_k \) and \( \bar{C}_k \). These operations are clearly different since

\[
C^\dagger_k X_k C_k = R_x \neq \bar{R}_x = C^\dagger_k X_k \bar{C}_k.
\]

The operations in subsequent iterations have no effect on \( X_k \), and the final Clifford operators therefore map \( X_k \) to \( C^\dagger R_x C \), and \( C^\dagger \bar{R}_x C \), respectively. Conjugation with Clifford operators never maps two Paulis to the same value, and we therefore conclude that the sampled Clifford operators must differ. This completes the proof.

### 5 Discussion

In this work we have proposed a simple algorithm for sampling operators from the Clifford group uniformly at random. The algorithm uses the tableau representation of Paulis and, similar to work by Koenig and Smolin [5], takes advantage of the hierarchical structure of the Clifford group. Unlike existing algorithms, the algorithm can directly output the quantum circuit corresponding to the random Clifford operator in a streaming fashion with minimal overhead. The circuits generated for \( n \)-qubit Cliffrords contain at most \( 5n + 2n^2 \) single- or two-qubit gates and have a maximum depth of \( \mathcal{O}(n \log n) \). The runtime of the algorithm matches the \( \mathcal{O}(n^2) \) complexity obtained by Bravyi and Maslov [2]. Each iteration of the algorithm consists of a sampling step and a sweeping step. The sampling step randomly samples a pair of anticommuting Paulis in tableau form. The sweeping step then normalizes the tableau using operations that directly correspond to elementary quantum gates. Each iteration of the algorithm therefore results in a small quantum circuit. By stitching together the circuits generated by successive iterations we obtain a quantum circuit that implements the randomly sampled Clifford operator. Instead of stitching the circuits together at the end we can also emit the gates as they are generated during the sweeping step and thus output the circuit in a streaming manner. Interestingly, the iterations in the algorithm are completely decoupled, which means that the \( n \) circuit segments could in principle be generated in parallel in \( \mathcal{O}(n) \) time.

Clifford operators are completely characterized by their mapping of the basis Paulis \( X_i \) and \( Z_j \). If needed, we can obtain this mapping by initializing the basis tableau shown in 2(a) and applying the adjoint operations of each of the iterations, with the iteration order reversed. The Paulis in the resulting tableau then represent the exact mapping. This process requires operations on increasingly large \( k \times k \) subblocks of the tableau and results in an \( \mathcal{O}(n^3) \) algorithm, which matches the complexity obtained by [5]. The algorithm presented in [2] can find the same mapping with a complexity matching that of matrix multiplication and is therefore preferable, at least in theory.
Figure 5: Random four-qubit Clifford circuit generated by the proposed algorithm.

Figure 6: Tableaus involved in generating the circuit in Figure 5. (a) the randomly sampled tableau at the first iteration of the algorithm. Step 1 of sweeping procedure applies phase gates to qubits 2 and 3 to clear entries $Z_2$ and $Z_3$ in the first row, as indicated by the white dots. The resulting tableau, shown in (b), has nonzero entries in $X_2-X_4$ of the first row, which we wish to clear in step 2. This is done pairwise, by applying CX gates on qubits (1,2) and (3,4), and then on (1,3), leading respectively to the tableaus shown in (c) and (d). Step 3 does not apply to (d), whereas application of the Hadamard gate in step 4 results in tableau (e). Sweeping steps 1 and 2 are then repeated for the second row. Step 1 clears $Z_1$ using a phase gate while, as seen in (f), no additional entries need to be cleared in step 2. As before, step 3 does not apply and step 4 applies another Hadamard to obtain the normalized tableau in (g). Although omitted in this example, Step 5 would then clear any sign bits by applying an appropriate Pauli gate. The gates corresponding to this first iteration are seen in the leftmost shaded region on qubits 1 through 4 in Figure 5. The second iteration of the sampling algorithm applies to qubits 2 through 4, and we start with the randomly sampled tableau shown in (h). Step 1, applied to the first row, applies a Hadamard gate to ‘exchange’ the $Z_3$ and $X_3$ entries, resulting in tableau (i). Step 2 only finds a single nonzero entry in the X block and therefore does not apply any gates. Since the first entry in the tableau (corresponding to qubit 2) is zero, step 3 applies a swap operation to normalize the first row, as shown in (j). Step 4 again applies a Hadamard gate to give tableau (k). We then repeat steps 1 and 2 on the second row. Step 1 clears the $Z_2$ and $Z_3$ entries, as indicated by the white dots, resulting in tableau (l). Step 2 then clears $X_3$ using a CX gate. Applying a Hadamard gate on the resulting tableau (m) again results in the normalized tableau, which we omit here. The randomly sampled tableau for iteration three is shown in (n). It can be seen that only the swap operation in step 3 applies. Finally, the random tableau for iteration four, shown in (o), is normalized by applying a Hadamard gate in sweeping step 1.
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