PROOF OF A CONJECTURE OF KULAKOVA ET AL. RELATED TO
THE $\mathfrak{sl}_2$ WEIGHT SYSTEM

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Abstract. In this article, we show that a conjecture raised in [KLMR], which regards the
coefficient of the highest term when we evaluate the $\mathfrak{sl}_2$ weight system on the projection of
a diagram to primitive elements, is equivalent to the Melvin-Morton-Rozansky conjecture,
proved in [BNG].

1. Introduction

In this section, we briefly recall a conjecture of [KLMR] together with the relevant terminologies. A more complete treatment can be found in [KLMR]. Given a chord diagram $D$ with $m$ chords, its labelled intersection graph $\Gamma(D)$ is the simple labelled graph whose vertices are the chords of $D$, labelled from 1 to $m$, and two vertices are connected by an edge if the two corresponding chords intersect.

Following [KLMR], by orienting the chords of $D$ arbitrarily, we can turn $\Gamma(D)$ into an oriented graph as follows. Given two intersecting oriented chords $a$ and $b$, the edge connecting $a$ and $b$ goes from $a$ to $b$ if the beginning of the chord $b$ belongs to the arc of the outer circle of $D$ which starts at the tail of $a$ and goes in the positive (counter-clockwise) direction to the head of $a$ (see Figure 1). We also have another description of the orientation. Given two intersecting oriented chords $a$ and $b$, we look at the smaller arc of the outer circle of $D$ that contains the tails of $a$ and $b$. Then we orient the edge connecting $a$ and $b$ from $a$ to $b$ if we go from the tail of $a$ to the tail of $b$ along the smaller arc in the counter-clockwise direction. The reader should check that the two definitions of orientation are equivalent.

Consider a circuit of even length $l = 2k$ in the oriented graph $\Gamma(D)$. By a circuit we mean a closed path in $\Gamma(D)$ with no repeated vertices. Choose an arbitrary orientation of the circuit. For each edge, we assign a weight +1 if the orientation of the edge coincides with the orientation of the circuit and −1 otherwise. The sign of a circuit is the product of the weights over all the edges in the circuit. We say that a circuit is positively oriented if its sign is positive and negatively oriented if its sign is negative. We define

$$R_k(D) := \sum_{s} \text{sign}(s),$$

Figure 1. Orienting $\Gamma(D)$
where the sum is over all (un-oriented) circuits $s$ in $\Gamma(D)$ of length $2k$.

It is well-known that given a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant non-degenerate bilinear form, we can construct a weight system $w_{\mathfrak{g}}$ with values in the center $ZU(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ (see, for instance [CDM, Section 6]). In the case of the Lie algebra $\mathfrak{sl}_2$, we obtain a weight system with values in the ring $\mathbb{C}[c]$ of polynomials in a single variable $c$, where $c$ is the Casimir element of the Lie algebra $\mathfrak{sl}_2$. Note that the Casimir element $c$ also depends on the choice of the bilinear form. For the case of $\mathfrak{sl}_2$, an ad-invariant non-degenerate bilinear form is given by

$$\langle x, y \rangle = \text{Tr}(\rho(x)\rho(y)), \quad x, y \in \mathfrak{sl}_2,$$

where $\rho: \mathfrak{sl}_2 \to \mathfrak{gl}_2$ is the standard representation of $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is simple, any invariant form is of the form $\lambda\langle \cdot, \cdot \rangle$ for some constant $\lambda$. If we let $c_\lambda$ be the corresponding Casimir element and $c = c_1$, then $c_\lambda = c/\lambda$. If $D$ is a chord diagram with $n$ chords, it is known that

$$w_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + \cdots + a_1c$$

and the weight system corresponding to $\lambda\langle \cdot, \cdot \rangle$ is

$$w_{\mathfrak{sl}_2,\lambda}(D) = c^n_\lambda + a_{n-1,\lambda}c^{n-1}_\lambda + \cdots + a_{1,\lambda}c_\lambda.$$

Therefore the relationship between these two weight systems is given by

$$w_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n} w_{\mathfrak{sl}_2}(D)_{|c = \lambda c_\lambda}.$$

Following [L] we now define a map which sends a chord diagram into the set of primitive elements in the space of chord diagrams. Let $D$ be a chord diagram with $n$ chords, $V = V(D)$ its set of chords. Then the map $\pi_n$ from the space of chord diagrams to its primitive elements is given by

$$\pi_n(D) = D - 1! \sum_{\{V_1, V_2\}} D_{|V_1} \cdot D_{|V_2} + 2! \sum_{\{V_1, V_2, V_3\}} D_{|V_1} \cdot D_{|V_2} \cdot D_{|V_3} - \cdots,$$

where sums are taken over all unordered disjoint partitions of $V$ into non-empty subsets and $D_{|V_i}$ denotes $D$ with only chords from $V_i$ and multiplication is the usual multiplication in the space of chord diagrams. If we change unordered partitions to ordered ones, we obtain

$$(1) \quad \pi_n(D) = D - \frac{1}{2} \sum_{V = V_1 \sqcup V_2} D_{|V_1} \cdot D_{|V_2} + \frac{1}{3} \sum_{V = V_1 \sqcup V_2 \sqcup V_3} D_{|V_1} \cdot D_{|V_2} \cdot D_{|V_3} - \cdots.$$

It is shown (see [L]) that $\pi_n(D)$ is indeed a primitive element. We are now ready to state the conjecture raised in [KLMR].

**Conjecture 1.** Let $D$ be a chord diagrams with $2m$ chords, and $w_{\mathfrak{sl}_2,2}$ be the weight system associated with $\mathfrak{sl}_2$ and $2\langle \cdot, \cdot \rangle$. Then

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = 2R_m(D)c_2^m + \text{terms of degree less than } m \text{ in } c_2.$$
2. Proof of the conjecture

The conjecture is a consequence of the Melvin-Morton-Rozansky (MMR) conjecture, which was proved in [BNG]. We recall the statement of the MMR conjecture below. Let $J^k(q)$ be the “framing independent” colored Jones polynomial associated with the $k$-dimensional irreducible representation of $\mathfrak{sl}_2$. Set $q = e^h$, write $J^k(q)$ as power series in $h$:

$$J^k = \sum_{n=0}^{\infty} J^k_n h^n.$$  

It is known that $J^k_n$ is given by (see [O, Theorem 6.14] and [CDM, Section 11.2.3])

$$J^k_n = \text{Tr} \left( w'_{\mathfrak{sl}_2} |_{c = \frac{k^2-1}{2}} I_k \right).$$  

Here $I_k$ is the $k \times k$ identity matrix and $w'_{\mathfrak{sl}_2}$ is the “deframing” of the weight system $w_{\mathfrak{sl}_2}$ (see [CDM, Section 4.5.4]). For any chord diagram $D$ of degree $n$ (modulo the framing independent relation), the value $w'_{\mathfrak{sl}_2}(D)$ is a polynomial in $c$ of degree at most $\lfloor n/2 \rfloor$ (see [CDM, Exercise 6.25]). It follows that $J^k_n$ is a polynomial in $k$ of degree at most $n+1$. Dividing $J^k_n$ by $k$ we then obtain

$$\frac{J^k}{k} = \sum_{n=0}^{\infty} \left( \sum_{0 \leq j \leq n} b_{n,j} k^j \right) h^n,$$

where $b_{n,j}$ are Vassiliev invariants of order $\leq n$. We denote the highest order part of the colored Jones polynomial by

$$JJ : = \sum_{n=0}^{\infty} b_{n,n} h^n.$$  

Next we recall the definition of the Alexander-Conway polynomial of link diagrams. The Conway polynomial $C(t)$ can be defined by the skein relation:

(i) $C(\text{unknot}) = 1,$
(ii) $C(L_+) - C(L_-) = tc(L_0),$ where $L_+$, $L_-$ and $L_0$ are identical outside the regions consisting of a positive crossing, a negative crossing and no crossing, respectively.

The Alexander-Conway polynomial is a Vassiliev power series:

$$\tilde{C}(h) : = \frac{h}{e^{h/2} - e^{-h/2}} C|_{t = e^{h/2} - e^{-h/2}} = \sum_{n=0}^{\infty} c_n h^n.$$  

Now we are ready to state the MMR conjecture, which was proved in [BNG].

Theorem. With the notations as above, we have

$$JJ(h)(K) \cdot \tilde{C}(h)(K) = 1$$

for any knot $K$.  

The proof of the MMR conjecture found in [BNG] consists of reducing the equality of Vassiliev power series to an equality of weight systems. Recall that a Vassiliev invariant $\nu$
of order $n$ gives us a weight system $W_n(\nu)$ of order $n$ by $W_n(\nu)(D) = \nu(K_D)$, where $D$ is a chord diagram of degree $n$ and $K_D$ is a singular knot whose chord diagram is $D$. Let

$$W_{JJ} = \sum_{n=0}^{\infty} W_n(b_{n,n})$$

and

$$W_C = \sum_{n=0}^{\infty} W_n(c_n).$$

Then it is shown in [BNG] that the equality (2) is equivalent to

$$W_{JJ} \cdot W_C = 1.$$ 

Here 1 denotes the weight system that takes value 1 on the empty chord diagram and 0 otherwise. Recall also that the product of two weight systems is given by

$$W_1 \cdot W_2(D) = m(W_1 \otimes W_2)(\Delta(D)),$$

where $m$ denotes the usual multiplication in $Q$ and $\Delta$ denotes co-multiplication in the space of chord diagrams. Specifically,

$$\Delta(D) = \sum_{V(D) = V_1 \sqcup V_2} D|_{V_1} \otimes D|_{V_2},$$

where the sum is taken over all ordered disjoint partitions of $V(D)$, the set of chords of $D$. (Note that $V_1$ or $V_2$ can be empty.) When $D$ is primitive, we have

$$0 = W_{JJ} \cdot W_C(D) = m(W_{JJ} \otimes W_C)(D \otimes 1 + 1 \otimes D) = W_{JJ}(D) + W_C(D).$$

Thus we obtain

**Lemma 1.** If $D$ is a chord diagram of degree $2m$, then

$$W_{JJ}(\pi_{2m}(D)) = -W_C(\pi_{2m}(D)).$$

To prove conjecture [1], we need the notion of **logarithm** of a weight system (see [LZ, Chapter 6]). Let $w$ be a weight system and suppose $w$ can be written as $w = 1 + w_0$, where $w_0$ vanishes on chord diagrams of degree 0. Then

$$\log w = \log(1 + w_0) = w_0 - \frac{1}{2} w_0^2 + \frac{1}{3} w_0^3 - \cdots$$

is well-defined since for each chord diagram we only have finitely many non-zero summands.

**Lemma 2.** Let $w$ be a multiplicative weight system, i.e. $w(D_1 \cdot D_2) = w(D_1)w(D_2)$, and $w(\text{empty chord diagram}) = 1$. If $D$ is a chord diagram of degree $2m$, then

$$\log w(D) = w(\pi_{2m}(D)).$$

**Proof.** From the definition of the logarithm of a weight system we have

$$\log w = \log(1 + (w - 1))$$

$$= (w - 1) - \frac{1}{2}(w - 1)^2 + \frac{1}{3}(w - 1)^3 - \cdots$$
Now if $D$ is a chord diagram, then $(w - 1)(\text{empty chord diagram}) = 0$ and $(w - 1)(D) = w(D)$ if $D$ has degree $> 0$. Therefore,

$$(w - 1)^k(D) = \sum_{V_1 \cup V_2 \cup \ldots \cup V_k = V(D)} w(D|_{V_1})w(D|_{V_2})\cdots w(D|_{V_k})$$

$$= \sum_{V_1 \cup V_2 \cup \ldots \cup V_k = V(D)} w(D|_{V_1} \cdot D|_{V_2} \cdots D|_{V_k}),$$

where the sum is over ordered disjoint partition of $V(D)$ into non-empty subsets and the last equality follows from the multiplicativity of $w$. Comparing with equation (1) we obtain our desired equality. □

It is known that the weight system $W_C$ is multiplicative. Therefore for a chord diagram $D$ of degree $2m$,

$$(\log W_C)(D) = W_C(\pi_{2m}(D)).$$

Given an oriented circuit $H$ in a labelled intersection graph, we define the descent $d(H)$ of the circuit to be the number of label-decreases of the vertices when we go along the circuit in the specified orientation. We have the following lemma.

**Lemma 3.** Given a chord diagram $D$ of degree $2m$, we have

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D),$$

where the sum is over all oriented circuits $H$ of length $2m$ in $\Gamma(D)$.

**Proof.** To prove the first equality, we show that by labelling the chords of $D$ appropriately, the labelled intersection graph $\Gamma(D)$ of $D$ has the property that the edges always go in the direction of increasing indices. To get a required labelling, we cut the outer circle of $D$ to obtain a long chord diagram and then we label the chords by integers $1, 2, \ldots, 2m$ as we encounter them when we go from left to right in an increasing fashion. Then it’s clear that a descent will correspond to an edge with weight $-1$. Every circuit $H$ will have two possible orientations $H_+$ and $H_-$. However, since the circuit has even length, $d(H_+)$ and $d(H_-)$ have the same parity and the first equality follows.

The second equality is proved in [BNG, Proposition 3.13]. Here we briefly describe the main idea for completeness. The key identity is $W_C(D) = \det \text{IM}(D)$, where $\text{IM}(D)$ is the intersection matrix of the chord diagram $D$, which is defined as follows: label the chords of $D$ as above, then $\text{IM}(D)$ is the $2m \times 2m$ matrix given by

$$\text{IM}(D)_{ij} = \begin{cases} 
\text{sign}(i - j) & \text{if chords } i \text{ and } j \text{ of } D \text{ intersect,} \\
0 & \text{otherwise.}
\end{cases}$$

It turns out that $\text{IM}(D)$ only depends on $\Gamma(D)$. The identity is proved by showing that $\det \text{IM}$ satisfies the defining relations of $W_C$ (see [BNG, Theorem 3]). Now expanding $\det \text{IM}(D)$ we obtain

$$W_C(D) = \sum_{H=\bigcup_{\alpha} H_\alpha} (-1)^{\text{sign}(\sigma_H)}(-1)^{d(H)}.$$
Here \( H \) is an oriented circuit of length \( 2m \) in \( \Gamma(D) \), \( \sigma_H \) is the permutation of the vertices of \( \Gamma(D) \) underlying \( H \) and \( \bigcup_a H_a \) is the (unordered) cycle decomposition of \( \sigma_H \). From there it follows that

\[
(\log W_C)(D) = - \sum_H (-1)^{d(H)}.
\]

The readers can consult \cite{BNG} for more details.

\[
\square
\]

**Proof of Conjecture** \cite{BNG} Let \( D \) be a chord diagram of degree \( 2m \), we have a chain of equalities from the above lemmas

\[
2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D) = - W_C(\pi_{2m}(D)) = W_{JJ}(\pi_{2m}(D)).
\]

Therefore,

\[
\frac{J_{2m}^k(\pi_{2m}(D))}{k} = 2R_m(D)k^{2m} + \cdots.
\]

Plug in \( c = (k^2 - 1)/2 \) or \( k^2 = 2c + 1 \) we obtain

\[
w_{\mathfrak{sl}_2}(\pi_{2m}(D)) = 2^{m+1}R_m(D)c^m + \cdots.
\]

Now we just need to do a change of variable

\[
w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = \frac{1}{2^{2m}} w_{\mathfrak{sl}_2}(\pi_{2m}(D))|_{c=2c_2} = 2R_m(D)c_2^m + \cdots
\]

and the proof is complete.

\[
\square
\]

**Remark.** Technically we need to consider \( w'_{\mathfrak{sl}_2} \) instead of \( w_{\mathfrak{sl}_2} \). However for primitive elements, deframing does not affect the value of the highest terms (see \cite[Section 4.5.4]{CDM}).

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