UNBIASED MONTE CARLO ESTIMATION FOR SOLVING
OF LINEAR INTEGRAL EQUATION,
with error estimate.

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Abstract.

We offer a new Monte-Carlo method for solving linear integral equation which gives the unbiased estimation for solution of Volterra’s and Fredholm’s type, and consider the problem of confidence region building.

We study especially the case of the so-called equations with weak singularity in the kernel of Abelian type.

Key words and phrases: Integral equations, Neuman series, Monte Carlo method, random variables and vectors (r.v.), Poisson, Mittag-Leffler and Geometrical integer distributions, constrained optimization, random number of elapsed r.v., Kroneker’s square of the linear operator, Central Limit Theorem (CLT), conditional probability and expectation, tail estimation.

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary 34A34, 65M20, 42B25.

1 Notations. Statement of problem.

We intent in this article to study the numerical Monte-Carlo method for solving of the linear integral equation, for example, of a form

\[ x(t) = f(t) + \lambda \int_0^t K(t, s) x(s) \, ds, \quad (1.0) \]

Volterra’s equation. (The case of Fredholm’s equation will be considered further.)

Here \( x(t) \) is unknown function, \( K(t, s) \) is kernel, the function \( f(t) \) is ”right-hand” side, \( \lambda \) is positive number.
Briefly:

\[ x = f + \lambda K[x], \quad (1.0a) \]

where \( K[x] = K[x](t) \) is linear integral operator of Volterra’s type:

\[ K[x](t) = \int_0^t K(t, s) x(s) \, ds. \]

The offered here method gives as ordinary the optimal rate of convergence, exponential tail estimation for confidence probability, but in addition our (random!) estimates of solution are unbiased.

We will consider the equation (1.0), as well as all the next equations, in the space of continuous functions \( C(T) \) defined on the set \( t \in [0, T] \), where \( T = \text{const} \in (0, \infty) \), suppose therefore \( f(\cdot) \in C(T) \), \( K(\cdot, \cdot) \in C(T \times T) \), and denote as usually

\[ ||f|| = \max_{t \in [0,T]} |f(t)|, \quad ||K|| = \max_{s,t \in [0,T]} ||K(t, s)||. \quad (1.1) \]

These equations appear in particular in the reliability theory: renewal equation, equating of periodical checking etc., see, e.g. the article [25], where is explained in particular why the Monte-Carlo method is natural for this problem.

It is well known that this problem is well posed: the solution \( x(t) \) there exists, is unique and dependent continuously on the entries: \( f(\cdot), K(\cdot, \cdot) \) and may be computed by means of the standard recursion procedure:

\[ x_n = f + \lambda K[x_{n-1}], \quad x_0 = 0, \quad x_1 = f; \quad n = 2, 3, \ldots \quad (1.2) \]

\section{2 Essence of offered method.}

The solution \( x(\cdot) \) may be represented by means of the so - called Neuman’s series:

\[ x = f + \sum_{n=1}^{\infty} \lambda^n K^n[f] = \sum_{n=0}^{\infty} \lambda^n K^n[f], \quad (2.1) \]

where \( K^n \) denotes the \( n^{th} \) iteration (power) of the operator \( K \), so that \( K^0[f] = f \); which converges uniformly.

Denote also \( y_n(t) = K^n[f](t) \); then \( x = \sum_{n=0}^{\infty} y_n \) and we have for the values \( n = 1, 2, \ldots \): \( y_n(t) = \lambda^n \times \)

\[
\int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \cdot K(t, s_1)K(s_1, s_2) \cdots K(s_{n-1}, s_n) f(s_n) = \lambda^n t^n \times
\]

\[
\int_0^1 ds_1 \int_0^{s_2} \cdots \int_0^{s_{n-1}} ds_n \cdot K(t, ts_1)K(ts_1, ts_2) \cdots K(ts_{n-1}, ts_n) f(ts_n). \quad (2.2)
\]
Let us introduce the following \( n \) - dimensional simplex (polygon) \( S(n) := \{ s_1, s_2, \ldots, s_n : 0 < s_1 < 1, 0 < s_2 < s_1, 0 < s_3 < s_2, \ldots, 0 < s_n < s_{n-1} \} \),

and denote \( L_n(t, \vec{s}) = L_{n,K,f}(t, \vec{s}) = L_n(t, s) = K(t, ts_1)K(ts_1, ts_2) \ldots K(ts_{n-1}, ts_n) f(ts_n), \quad \vec{s} = s = (s_1, s_2, \ldots, s_n) \);

then

\[
y_n(t) = \int_{S(n)} L_n(t, \vec{s}) \, ds. \tag{2.4}\]

Note that the volume of the polygon \( S(n) \) is equal to \( 1/n! \), and if we introduce the probability measure \( \mu_n(\cdot) \) on the Borelian subsets of the simplex \( S(n) \) with a density \( n! \, ds \):

\[
\mu_n(A) = n! \int_A ds,
\]

then the expression for \( x(t) \) may be rewritten as follows:

\[
x(t) = f(t) + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} \int_{S(n)} L_n(t, \vec{s}) \, \mu_n(ds). \tag{2.5}\]

If we denote \( x_\lambda = x_\lambda(t) = e^{-\lambda t} \cdot x(t) \), then

\[
x_\lambda(t) = e^{-\lambda t} f(t) + \sum_{n=1}^{\infty} \frac{e^{-\lambda t} \lambda^n t^n}{n!} \int_{S(n)} L_n(t, \vec{s}) \, \mu_n(ds), \tag{2.6}\]

which may be formally rewritten as follows:

\[
x_\lambda(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t} \lambda^n t^n}{n!} \int_{S(n)} L_n(t, \vec{s}) \, \mu_n(ds), \tag{2.6a}\]

Let us introduce a sufficiently rich probability space \((\Omega, B, P)\) with probability \( P \), expectation \( E \) and variance \( \text{Var} \); and the r.v. \( \tau \) which has a Poisson distribution with parameter \( \lambda \cdot t \) (which is closely related with Poisson flow of intensity \( \lambda \)):

\[
P(\tau = n) = e^{-\lambda t} \frac{\lambda^n t^n}{n!}; \tag{2.7}\]

then the expression (2.6a) takes the form

\[
x_\lambda(t) = E \int_{S(\tau)} L_\tau(t, \vec{s}) \, \mu_\tau(ds), \tag{2.6a}\]

Further, the integral in the right - hand side (2.6a) may be represented as follows. We introduce the random vector \( \xi_\tau \) of a dimension \( \tau \) which has the uniform (conditional) distribution in the simplex \( S(\tau) \):

\[
P(\xi_\tau \in A)/\tau = \mu_\tau(A), \tag{2.7}\]
then
\[ \int_{S(\tau)} L_{\tau}(t, s) \, \mu_{\tau}(ds) = \mathbf{E}_{\tau} L_{\tau}(t, \xi^\tau_{\tau}), \] (2.8)

so that
\[ x_{\lambda}(t) = \mathbf{E} L_{\tau}(t, \xi^\tau_{\tau}). \] (2.9)

By the fixed value $\tau$ the integral in the expression (2.8) may be computed by means of the Monte Carlo method:
\[ \int_{S(\tau)} L_{\tau}(t, s) \, \mu_{\tau}(ds) \approx \frac{1}{k} \sum_{j=1}^{k} L_{\tau}(t, \xi^{(j)}_{\tau}), \] (2.10)

where $\xi^{(j)}_{\tau}$ are independent copies of the r.v. $\xi^\tau_{\tau}$.

Correspondingly, the Monte Carlo approximation for the whole value $x_{\lambda}(t)$ may be offered as follows:
\[ \hat{x}_{\lambda} = \hat{x}_{\lambda,N,Z}(t) \overset{\text{def}}{=} \frac{1}{Z} \sum_{i=1}^{Z} \left[ \frac{1}{N(\tau(i))} \sum_{j=1}^{N(\tau(i))} L_{\tau(i)}(t, \xi^{(j)}_{\tau(i)}) \right], \] (2.11)

where $\tau(i)$ are independent copies of the r.v. $\tau$, i.e. are independent Poisson distributed r.v. with parameter $\Lambda = \lambda t$, and $N = N(n)$ be some non-random positive numerical sequence, her choice will be specified later.

We state by definition
\[ \frac{1}{N(n)} \sum_{j=1}^{N(n)} L_{j} \overset{\text{def}}{=} 0, \] (2.11a)

if $N(n) = 0$.

Evidently, the approximation $\hat{x}_{\lambda} = \hat{x}_{\lambda,N,Z}(t)$ of the solution $x_{\lambda}(t)$ is unbiased: $\mathbf{E} \hat{x}_{\lambda,N,Z}(t) = x_{\lambda}(t)$.

Another approach for Monte Carlo solving of linear integral equation which gives biased estimation see in the article [17].

Let’s count the amount $R$ of elapsed r.v. for the $\hat{x}_{\lambda} = \hat{x}_{\lambda,N,Z}(t)$ computation.
\[ R = Z \cdot \sum_{i=1}^{Z} \tau(i)N(\tau(i)), \] (2.12)

so that $R$ is random variable with the expectation
\[ \Theta \overset{\text{def}}{=} \mathbf{E}R \approx Z \cdot \sum_{n=1}^{\infty} e^{-\Lambda} \frac{\Lambda^{n}}{n!} \cdot (nN(n)), \] (2.13)

where $\Lambda = \lambda t$. 


3 Error estimate.

Let us estimate the variance of the approximate solution \( \hat{x}_\lambda = \hat{x}_{\lambda,N,Z}(t) \). Note first of all that

\[
\text{Var} [\hat{x}_\lambda] = \frac{1}{Z} \cdot \text{Var} \left[ \frac{1}{N(\tau)} \sum_{j=1}^{N(\tau)} L_\tau \left( t, \hat{\zeta}_\tau^{(j)} \right) \right].
\]  

(3.1)

Further, we will use the next formula

\[
\text{Var}(\zeta) = E \left\{ E (\zeta - E \zeta)^2 / \tau \right\} : \]

\[
Z \cdot \text{Var} [\hat{x}_\lambda] \leq ||f||^2 \cdot e^{-\Lambda} \sum_n \frac{\Lambda^n ||K||^{2n}}{N(n) n!^2} = ||f||^2 \cdot e^{-\Lambda} \sum_n \frac{Q^n}{N(n) n!^2}, \quad Q := \Lambda ||K||^2.
\]  

(3.2)

Lemma 3.1. Let us consider the following constrained optimization problem:

\[
D := \min_{\{N(n)\} \geq 1} \sum_n \frac{A(n)}{N(n)} / \left\{ \sum_n B(n) N(n) \leq M \right\}.
\]  

(3.3)

Here \( \{A(n)\}, \{B(n)\} \) are (may be unbounded) positive sequences, \( M = \text{const} >> 1, \ N(n) > 0 \).

We derive using the Lagrange’s factors method that

\[
D = \left[ \sum_n \sqrt{A(n) B(n)} \right]^2 M,
\]  

(3.4)

and is achieved iff

\[
N(n) := N_0(n) \overset{def}{=} \frac{M \sqrt{A(n)}}{\sum_n \sqrt{A(n) B(n)}} \cdot \frac{\sqrt{A(n)}}{B(n)}.
\]  

(3.5)

up to rounding to the nearest greatest integer number:

\[
N(n) := \text{Ent}[N_0(n)] + 1,
\]

where \( \text{Ent}(z) \) denotes the integer part of real positive number \( z \), if it is known that the numbers \( N(n) \) are integer.

In the last case we have only

\[
D \approx \left[ \sum_n \sqrt{A(n) B(n)} \right]^2 M.
\]  

(3.6)

Let \( \Theta = ER \) be a fixed ”great” number, for instance, \( 10^7 - 10^8 \). It seems the following constrained optimization problem of the variance minimization:
\[
\sum_n \frac{Q^n}{N(n) \cdot n!} \rightarrow \min / \sum_{n=1}^\infty \frac{\Lambda^n}{n!} \cdot (nN(n)) = e^\Theta / Z.
\]

Of course, it will be supposed that \(\Theta e^\Lambda >> Z\).

We observe using lemma 3.1

\[
\min \left\{ N(n) \right\} \left[ \text{Var}(Z \cdot \hat{x}_\lambda) \right] \approx \frac{1}{\Theta} \cdot \left( \sum_n \frac{\Lambda^n \cdot ||K||^n \cdot \sqrt{n}}{(n!)^{3/2}} \right)^2,
\]

wherein the (quasi - ) optimal values \(N(n) = N_0(n)\) are following:

\[
N_0(n) = \text{Ent} \left[ \frac{e^\Lambda Q/Z}{\sum_m \Lambda^m \cdot ||K||^m \cdot (m!)^{-3/2} \cdot \sqrt{m}} \cdot \sqrt{n} \right] + 1,
\]

wherein by the practical using in the case when the value \(N_0(n)\) is sufficiently small, for instance, if \(N_0(n) \leq 10\), then we oblige to take \(N_0(n) := 0\); see (2.11a).

This imply that the rate of convergence of offered method is equal to \(1/\sqrt{Z}\), as in the classical Monte Carlo method.

It remains to use the classical Central Limit Theorem (CLT) to construct the confidence interval for the solution \(x_\lambda(t)\).

### 4 Fredholm’s equations.

We consider in this section the Monte - Carlo approach for solution of Fredholm’s [22] linear integral equation of a second kind

\[
x(u) = f(u) + \lambda \int_V K(u, v) \cdot x(v) \cdot \nu(dv),
\]

or equally

\[
x(u) = f(u) + \lambda \cdot K[x](u),
\]

where \(\lambda = \text{const} \in (0, 1)\), \((V, A, \nu)\) with a distance \(\rho = \rho(v_1, v_2)\) is compact metric measurable probability space: \(\nu(V) = 1\); \(u, v \in V\), and both the functions \(f(\cdot), K(\cdot, \cdot)\) are continuous, and we denote as in the first section

\[
||f|| = \max_{u \in V} |f(u)|, \quad ||K|| = \max_{u, v \in V} |K(u, v)|.
\]

Another (deterministic) approach via the so-called Fredholm’s determinants [22] computing implementation see in the article [21].

The norm of linear operator \(K[x](u) = \int_V K(u, v) \cdot x(v) \cdot \nu(dv)\) in the space of continuous functions \(C(V) = C(V, \rho)\) will be denoted by \(|||K||||\); it is known that

\[
|||K||| = \max_{u \in V} \int_V |K(u, v)| \cdot \nu(dv).
\]

Evidently, \(|||K|||| \leq ||K||\).

Further, let us denote
\[ r_n = r_n(K) = |||K^n|||^{1/n}, \quad r = r(K) = \lim_{n \to \infty} r_n(K). \]  

The last limit there exists and is named as spectral radius of the operator \( K \); see [24], chapters 4, 5.

We suppose at first that \( |||K||| \leq 1 \); then the continuous solution \( x = x(u) \) of (4.1) there exists, is unique and may be represented by means of uniform converge Neuman series:

\[
x(u) = f(u) + \sum_{n=1}^{\infty} \lambda^n K^n[f](u) = f(u) + \sum_{n=1}^{\infty} \lambda^n y_n(u), \quad y_n(u) := \int_V \nu(ds_1) \int_V \nu(ds_2) \ldots \int_V \nu(ds_n) K(u, s_1) K(s_1, s_2) \ldots K(s_{n-1}, s_n) f(s_n). \]

Recall that \( 0 < \lambda < 1 \).

Let \( \gamma(i), \ i = 1, 2, \ldots, n \) be independent random variables with distribution \( \nu \):

\[ P(\gamma(i) \in A) = \nu(A). \]

Then the function \( y_n(\cdot) \) has a probabilistic representation: \( y_n(u) = E[L_n(u, \bar{\theta}_n)] \).

If we denote (in this section)

\[
x_{\lambda}(u) = (1 - \lambda)x(u), \quad \bar{s}_n = \{s(1), s(2), \ldots, s(n)\},
\]

\[
L_n(u, \bar{s}_n) = K(u, s(1)) K(s(1), s(2)) \ldots K(s(n-1), s(n)),
\]

then

\[ y_n(u) = E[L_n(u, \bar{\theta}_n)] \]

and

\[
x_{\lambda} = \sum_{n=0}^{\infty} (1 - \lambda) \lambda^n E[L_n(u, \bar{\theta}_n)]. \]

The non-negative integer valued random variable \( \Delta \) with distribution

\[ P(\Delta = n) = (1 - \lambda) \lambda^n, \quad n = 0, 1, 2, \ldots \]

is named (integer) geometrical distributed, write: \( \text{Law}(\Delta) = G_{\lambda} \).

We can rewrite the expression (4.7) using this notation as follows

\[ x_{\lambda} = E[L_{\Delta}(u, \bar{\theta}_{\Delta})]. \]
The Monte-Carlo approximation \( \hat{x}_\lambda = \hat{x}_\lambda(u) = \hat{x}_{\lambda,N,Z}(u) \) for \( x_\lambda \) may be written as before

\[
\hat{x}_\lambda(u) = \frac{1}{Z} \sum_{i=1}^{Z} \frac{1}{N(\Delta(i))} \sum_{j=1}^{N(\Delta(i))} L_{\Delta(i)}(u, \theta_{\Delta(i)}^{(j)}),
\]

(4.9)

where \( \{\Delta(i)\} \) are independent copies of the r.v. \( \Delta \) and \( \theta_{\Delta(i)}^{(j)} \) are independent copies of the random vector \( \theta_{\Delta(i)} \).

In order to calculate (estimate) the variance of \( \hat{x}_\lambda(u) \), we need to use the following definition. Let \( K[x](u) \) be any linear integral operator with kernel \( K \):

\[
K[x](u) = \int_V K(u, v) x(v) \nu(dv).
\]

The Kroneker’s square \( K^{[2]} \) of the operator \( K \) is an operator acting as follows:

\[
K^{[2]}[x](u) \overset{\text{def}}{=} \int_V K^2(u, v) x(v) \nu(dv).
\]

(4.10)

We impose another condition on the coefficient \( \lambda \) and on the kernel \( K \):

\[
\lambda \cdot |||K^{[2]}||| < 1.
\]

(4.11)

Then

\[
\frac{Z}{(1 - \lambda)||f||^2} \text{Var} \hat{x}_\lambda \leq \sum_n \frac{\lambda^n|||K^{[2]}|||^n}{N(n)}.
\]

(4.12)

Let’s count now the amount \( R_F \) of elapsed standard, i.e. uniform \([0,1]\) distributed, r.v. for the \( \hat{x}_\lambda = \hat{x}_{\lambda,N,Z}(t) \) computation.

We suppose that for one \( \theta_{\lambda}^{(j)} \) random vector number generation are need exactly \( k \cdot d, \quad d = \text{const} = 1, 2, \ldots \) standard r.v.

Then

\[
R_F = Z \cdot \sum_{i=1}^{Z} \tau(i) N(\tau(i)),
\]

(4.13)

so that \( R_F \) is random variable with the expectation

\[
\Theta_F \overset{\text{def}}{=} \mathbb{E}R_F \asymp Z \cdot d \cdot \sum_{n=0}^{\infty} [(1 - \lambda) \lambda^n \cdot (nN(n))].
\]

(4.14)

We get denoting \( M_F \)

\[
M_F = \frac{\Theta_F}{(1 - \lambda) Z d}
\]

to the following constrained optimization problem assuming \( M_F \) a fixed number of large:
\[
\left(1 - \lambda \|f\|^2\right) \times \sum_n \frac{\lambda^n \|K^{[2]}\|^n}{N(n)} \rightarrow \min \left[ \sum_n \lambda^n \cdot (nN(n)) = M_F \right]. \quad (4.14)
\]

Let us introduce the following function:

\[ G_\alpha(z) = \sum_{n=0}^{\infty} n^\alpha z^n; \quad \alpha = \text{const} \geq 0, \ 0 \leq x < 1. \]

We find tacking into account the proposition of lemma 3.1:

\[ D_F := \min_{\{N(n)\}} \text{Var} \hat{x}_\lambda \asymp \frac{(1 - \lambda)^2 d \|f\|^2}{\Theta_F} \cdot G_{1/2} \left( \lambda \sqrt{\|K^{[2]}\|} \right) \quad (4.15) \]

and this minimum is attained iff

\[ N(n) := 1 + \text{Ent}(N_0(n)), \]

where

\[ N_0(n) \overset{\text{def}}{=} \left[ \frac{M_F}{G_{1/2} \left( \lambda \sqrt{\|K^{[2]}\|} \right)} \right] \times \frac{\|K^{[2]}\|^n}{\sqrt{n}}. \quad (4.16) \]

Of course, if \( N_0(n) \leq 10 \), we must take \( N(n) = 0 \).

We conclude again that the rate of convergence offered estimate is equal to \( 1/\sqrt{\Theta_F} \), as in the one dimensional case.

\textbf{Remark 4.1.} Note that as \( z \to 1 - 0 \)

\[ G_{1/2}(z) \sim 0.5 \sqrt{\pi} |\ln z|^{-3/2}. \]

5 \hspace{1em} \textbf{Equations with weak singularity.}

We consider in this section the Volterra’s type integral equation with weak (Abel’s) singularity of a form

\[ x(t) = f(t) + \lambda \int_0^t K(t, s) \ x(s) \ ds, \quad (5.1) \]

where as before \( \lambda = \text{const} > 0, \ t \in [0, T], \ T = \text{const} > 0, \ f(\cdot), \ K(\cdot, \cdot) \) are continuous functions, \( \alpha := 1 - \beta = \text{const} \in (0, 1); \) the case \( \alpha = 0 \) was considered in the second section.

This case \( \alpha > 0 \) has a number of interesting features, and we will briefly enumerate.

First of all \( x(t) = f(t) + \sum_{n=1}^{\infty} y_n(t), \) where \( y_n(t) = \lambda^n \times \)

\[
\int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \frac{K(t, s_1) K(s_1, s_2) \ldots K(s_{n-1}, s_n) f(s_n)}{[(t - s_1)(s_1 - s_2) \ldots (s_{n-1} - s_n)]^\alpha} = [\lambda t^\beta]^n \times \]

\[ \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \frac{K(t, s_1) K(s_1, s_2) \ldots K(s_{n-1}, s_n) f(s_n)}{[(t - s_1)(s_1 - s_2) \ldots (s_{n-1} - s_n)]^\alpha} = [\lambda t^\beta]^n \times \]
\[\int_{0}^{1} \int_{0}^{s_{2}} \ldots \int_{0}^{s_{n-1}} ds_{\eta} K(t, ts_{1}) K(ts_{1}, ts_{2}) \ldots K(ts_{n-1}, ts_{n}) f(ts_{n}) \left(\frac{1}{(1 - s_{1})(s_{1} - s_{2}) \ldots (s_{n-1} - s_{n})}\right)^{\alpha}.\] 

(5.2)

Denote

\[R_{\alpha,n}(\bar{s}) = (1 - s_{1})^{-\alpha}(s_{2} - s_{1})^{-\alpha}(s_{3} - s_{2})^{-\alpha} \ldots (s_{n} - s_{n-1})^{-\alpha},\]

\[W_{n}(\beta) = \frac{\Gamma^{n}(\beta)}{\Gamma(1 + n\beta)}\]

where \(\Gamma(\cdot)\) is ordinary Gamma function. Evidently, \(\lim_{n \to \infty} W_{n}(\beta) = 0\).

Note that

\[\int \int \ldots \int_{S(n)} \frac{ds_{1}ds_{2}\ldots ds_{n}}{(1 - s_{1})^{\alpha}(s_{2} - s_{1})^{\alpha}(s_{3} - s_{2})^{\alpha} \ldots (s_{n} - s_{n-1})^{\alpha}} = W_{n}(\beta).\]

The following function \(h_{\alpha}(s) = h_{\bar{s}}(s), \ s \in S(n)\), could be chosen as a density of distribution with support on the simplex \(S(n)\):

\[h_{\alpha}(s) = \frac{R_{\alpha,n}(s)}{W_{n}(\beta)}.\]

**Definition 5.1.** (See [17].) The random vector \(\kappa = \kappa_{\alpha,n} = \bar{\kappa} = \bar{\kappa}_{\alpha,n}\) with values in the polygon \(S(n)\) has by definition a polygonal Beta distribution, write: \(\text{Law}(\kappa) = PB(\alpha, n)\), iff it has a density \(h_{\alpha}(s), \ s \in S(n)\).

On the other word,

\[P(\kappa \in G) = \int_{G} h_{\alpha}(s) \ ds \overset{def}{=} \mu_{\alpha,n}(G), \ G \subset S(n).\]

Evidently, \(\mu_{\alpha,n}(\cdot)\) is the probabilistic Borelian measure on the set \(S(n)\).

The expression for the function \(y_{\alpha}(\cdot)\) may be rewritten as follows:

\[y_{\alpha}(t) = \lambda^{n} t^{n\beta} \cdot W_{n}(\beta) \cdot E_{L_{n}}(t \eta_{n}),\]

where the random vector \(\eta_{n}\) has the polygonal beta distribution of dimension \(n\) with parameters \((\alpha, n)\).

Recall that the function of Mittag - Leffler \(E_{\beta}(z)\), more exactly, the family of the functions which dependent on the positive real parameter \(\beta, \ \beta > 0\), is defined for all (may be complex) values \(z\) by the formula

\[E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(1 + n\beta)}, \ \beta = \text{const} > 0.\]

We define also some slight generalization of the Mittag - Leffler function:

\[E_{\beta,\alpha,\delta}(z) = \sum_{n=1}^{\infty} \frac{z^{n}}{n^{\delta} \Gamma^{\alpha}(1 + n\beta)}, \ \alpha, \beta = \text{const} > 0, \ \delta = \text{const}.\]
Evidently, \( E_{\beta,1,0}(z) = E_{\beta}(z) - 1 \).

This definition with investigation of properties of this function belongs to G. Mittag-Lefler [15]. See also a recent article [11], (with reference therein,) where are described in particular some interest applications of these functions.

**Definition 5.2.** The integer valued non-negative random variable \( \zeta \) has by definition Mittag-Lefler distribution with parameters \((\beta, \mu)\), \( \beta, \mu > 0 \), write:

\[
\text{Law}(\zeta) = R_{\beta}(\mu),
\]

iff

\[
P(\zeta = n) = \frac{\mu^n/\Gamma(1 + \beta n)}{E_{\beta}(\mu)}, \quad n = 0, 1, 2, \ldots
\]

(5.4)

**Remark 5.1.** Our definition (5.2) is unlike from ones in the article [11], where was introduced the so-called continuous Mittag-Lefler distribution.

Denote

\[
x_{\lambda, \beta}(t) = \frac{x(t)}{E_{\beta}(\Lambda_{\beta}(t))}.
\]

The function \( x_{\lambda, \beta}(t) \), which is proportional to the true solution \( x(t) \) of the equation (5.1), may be probability represented as follows

\[
x_{\lambda, \beta}(t) = E L_{\zeta}(t, \tilde{\eta}_\zeta).
\]

(5.5)

By the fixed value \( \zeta \) the integral in the expression (5.5) may be computed by means of the Monte Carlo method:

\[
\int_{S(\zeta)} L_{\zeta}(t, \bar{s}) \mu_{\zeta}(ds) \approx \frac{1}{k} \sum_{j=1}^{k} L_{\zeta}(t, \tilde{\eta}_{\zeta}^{(j)});
\]

where \( \tilde{\eta}_{\zeta}^{(j)} \) are independent copies of the r.v. \( \tilde{\eta}_\zeta \).

Correspondingly, the Monte Carlo approximation for the whole value \( x(t) \) may be offered as in the second section as follows:

\[
\hat{x}_{\lambda, \beta} = \hat{x}_{\lambda, \beta;N,Z}(t) \overset{def}{=} \frac{1}{Z} \sum_{i=1}^{Z} \left[ \frac{1}{N(\zeta(i))} \sum_{j=1}^{N(\zeta(i))} L_{\zeta(i)}(t, \tilde{\eta}_{\zeta(i)}^{(j)}) \right],
\]

(5.6)

where \( \zeta(i) \) are independent copies of the r.v. \( \zeta \), i.e. are independent Mittag-Lefler’s distributed r.v. with parameter \( \beta, \Lambda_{\beta} = \Lambda_{\beta}(t) \overset{def}{=} \lambda t^{\beta} \), and \( N = N(n) \) be some non-random integer positive numerical sequence, her choice will be specified later.

Further, let us estimate the variance of our estimation \( \hat{x} \):

\[
\text{Var} \hat{x}_{\lambda, \beta} \leq \sum_{n} W_n^2(\beta) \frac{\Lambda_{\beta}^2(t)}{N(n)} \int_{S(n)} L^2(t, s) \mu_n(ds) \leq ||f||^2 \sum_{n} W_n^2(\beta) \frac{\Lambda_{\beta}^2(t) ||K||^{2n}}{N(n)}.
\]

(5.7)
Let us discuss now the question of amount \( R_\beta \) elapsed random variables. We find analogously to the second section

\[
R_\beta = Z \cdot \sum_{i=1}^{Z} \zeta(i) N(\zeta(i)), \tag{5.8}
\]

so that \( R_\beta \) is random variable with the expectation

\[
\Theta_\beta \overset{\text{def}}{=} \mathbb{E} R_\beta \simeq Z \cdot \sum_{n=1}^{\infty} \frac{\Gamma^n(\beta) \Lambda^n_\beta / \Gamma(1 + \beta n)}{E_\beta(\Lambda_\beta)} \cdot (n N(n)). \tag{5.9}
\]

Let \( \Theta_\beta = \mathbb{E} R \) be a fixed "great" number. Denote

\[
M_\beta = E_\beta(\Lambda_\beta(t)) \cdot \frac{\Theta_\beta}{Z};
\]

it will be presumed that \( M_\beta \) is also a great number, for instance, \( 10^7 - 10^8 \).

It seems the following constrained optimization problem of the variance minimization:

\[
||f||^2 \sum_n W_n^2(\beta) \frac{||K||^{2n} \Lambda^n_\beta}{N(n)} \rightarrow \min \sum_n \frac{\Lambda^n_\beta \Gamma^n(\beta)}{\Gamma(1 + \beta n)} \cdot (n N(n)) \leq M_\beta. \tag{5.10}
\]

We deduce using again lemma 3.1

\[
\min_{\{N(n)\}} \text{Var} x_{\lambda, \beta} \approx \frac{||f||^2}{M_\beta} \cdot E_{\beta, \frac{3}{2}, \frac{1}{2}} \left( \Lambda_\beta \frac{||K||}{\Gamma^{3/2}(\beta)} \right), \tag{5.11}
\]

wherein the (quasi-) optimal values \( N(n) = N_0(n) \) are following: \( N_0(n) = \)

\[
\text{Ent} \left[ \frac{M_\beta}{E_{\beta, \frac{3}{2}, \frac{1}{2}} \left( \Lambda_\beta \frac{||K||}{\Gamma^{3/2}(\beta)} \right)} \cdot \frac{W_n(\beta) \frac{||K||^n}{\sqrt{n} \Gamma^{n/2}(\beta)}}{\sqrt{n} \Gamma^{n/2}(\beta)} \right] + 1. \tag{5.12}
\]

Evidently, by the practical using in the case when the value \( N_0(n) \) is sufficiently small, for instance, if \( N_0(n) \leq 10 \), then we must take \( N_0(n) := 0 \); see (2.11a).

This imply that the rate of convergence of offered method is equal to \( 1/\sqrt{Z} \), as in the classical Monte Carlo method.

It remains as ordinary to use the classical Central Limit Theorem (CLT) to construct the confidence interval for the solution \( x_{\lambda, \beta}(t) \).

\section{Concluding remarks.}

\textbf{A. Confidence region for solution in the uniform norm.}

All the offered (Monte Carlo approximated) solutions, see for example (4.9), \( \hat{x} = \hat{x}(u), \ u \in V \) have a form
\[ \hat{x}(u) = \hat{x}_Z = \frac{1}{Z} \sum_{i=1}^{Z} \frac{1}{N(\Delta(i))} \sum_{j=1}^{N(\Delta(i))} L_{\Delta(i)} \left( u, \theta_{\Delta(i)}^{(j)} \right), \] (6.1)

and are unbiased.

We restrict ourselves for definiteness in this section only the case of the Fredholm’s equation; another cases may be considered analogously.

Denote for simplicity

\[ \xi_i(u) = \frac{1}{N(\Delta(i))} \sum_{j=1}^{N(\Delta(i))} L_{\Delta(i)} \left( u, \theta_{\Delta(i)}^{(j)} \right) - x(u), \] (6.2)

then the random fields \( \{\xi_i(u)\} \) are continuous, mean zero, identical distributed and

\[ \sqrt{Z}(\hat{x}_Z(u) - x(u)) = \frac{1}{\sqrt{Z}} \sum_{i=1}^{Z} \xi_i(u). \] (6.3)

In order to build the confidence region in the uniform norm \( ||x|| = \max_{u \in V} |x(u)| \) for the solution \( x = x(u) \), we need to use the so-called Central Limit Theorem (CLT) in the space of continuous functions \( C(V) \). see [1], [2], [13], [14], [17], [23], [25].

Namely, if the sequence of random fields \( \xi_i = \xi_i(u), u \in V \) satisfies this CLT, then

\[ \lim_{Z \to \infty} \mathbb{P} \left( \sup_{u \in V} |(\hat{x}_Z(u) - x(u))| > Q \right) = \mathbb{P} \left( \sup_{u \in V} |\zeta(u)| > Q \right), \] (6.4)

where \( \zeta(u) \) is continuous centered Gaussian random field with at the same covariation function \( R(u_1, u_2) \) as \( \xi_1(u) \):

\[ R(u_1, u_2) = \mathbb{E} \zeta(u_1)\zeta(u_2) = \text{Cov}(\xi_1(u_1), \xi_1(u_2)). \]

Many sufficient conditions for CLT in the Banach space \( C(V) \) may be found in [3], [5], [6], [7], [9], [10], [12] etc. CLT in another separable Banach spaces is investigated, e.g. in [4], [8], [10], [6], [18], [27], [28], [19], [20].

The technology of application of the Banach space valued Central Limit Theorem in the parametric Monte Carlo method is described in [23], [25], [1], [26].

B. Analogously may be considered the integral equations of a form

\[ x(t_1, t_2) = f(t_1, t_2) + \int_{0}^{t_1} ds_1 \int_{0}^{t_2} ds_2 \cdot K(t_1, t_2, s_1, s_2) x(s_1, s_2) \, ds_1 \, ds_2, \]

with or without weak singularities.

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