FORMAL GENUS 0 GROMOV-WITTEN THEORIES AND GIVENTAL'S FORMALISM

Abstract. In [Gi3] Givental introduced and studied a space of formal genus zero Gromov-Witten theories $GW_0$, i.e. functions satisfying string and dilaton equations and topological recursion relations. A central role in the theory plays the geometry of certain Lagrangian cones and a twisted symplectic group of hidden symmetries. In this note we show that the Lagrangian cones description of the action of this group coincides with the genus zero part of Givental's quantum Hamiltonian formalism. As an application we identify explicitly the space of $N = 1$ formal genus zero GW theories with lower-triangular twisted symplectic group modulo the string flow.

Introduction

Let $X$ be a compact Kahler manifold and $H = H^*(X)$ be the space of cohomology of $X$ equipped with a non-degenerate Poincare pairing $(\cdot, \cdot)$. Let $\mathcal{H} = H \otimes \mathbb{C}[z]$ be the big phase space. The genus $g$ Gromow-Witten descendant potential of $X$ is a generating function $F^X_g$ for Gromov-Witten invariants ([LT]). The series $F^X_g$ depends on variables $\{t_i\}_{i=0}^{\infty}$, $t_i \in H$ and thus can be considered as a function on the big phase space.

There exists a large family of relations which are satisfied by all descendant potentials ([BP], [EX], [FSZ], [Ge1], [Ge2], [L1], [LSL], [Ma]). These relations come from the geometry of the moduli spaces $M_{g,n}$. They include the string and the dilaton equations (see [W1]), and the set of topological recursion relations (see [Ge1]). In [Gi2], [Gi3] axiomatic approach to the study of the so called formal Gromow-Witten theories is developed. Namely the idea is to capture the main properties of geometric Gromow-Witten potentials and to consider the space of functions satisfying those properties. Recent applications of Givental’s formalism include Frobenius structures (see [L1], [L2], [L3], [LP]), r-spin Gromow-Witten theory (see [FSZ], [CZ]), integrable systems ([Gi4], [FSZ], [Mi]), topological and cohomological field theories ([I], [K], [S]).

The simplest and the most concrete is the genus zero case. The space of formal genus zero Gromow-Witten theories $GW_0$ is defined as follows. Let $H$ be an $N$-dimensional vector space equipped with a non-degenerate bilinear form $(\cdot, \cdot)$ and a distinguished vector $1 \in H$. Consider a space of functions $F(t_0, t_1, \ldots)$ satisfying the string equation, the dilaton equation and the set of genus zero topological recursion relations. In particular, genus zero parts...
of geometric Gromov-Witten potentials and genus zero parts of Witten’s $r$-spin potentials (see [W1], [JKV]) are elements of $GW_0$. It is proved in [CG], [Gi1] that the space of formal genus zero theories can be equipped with an action of infinite-dimensional symplectic Lie groups $G_+$ and $G_-$ (the so-called upper- and lower-triangular twisted loop groups). The existence of these groups of hidden symmetries is a very powerful tool for the study of formal theories (see [LP], [L3] and references within).

There exist two definitions of the action of twisted loop groups. The first one is based on the quantization of certain Hamiltonians (see [Gi2]) and goes through the infinitesimal action of the corresponding Lie algebras on the space of total descendant potentials. We recall some details in Section 1. The second definition is based on the Lagrangian cones formalism (see [CG], [Gi3]). Namely let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_\pm$ be the space of Laurent series with values in $\mathcal{O}$ equipped with a symplectic form

$$\Omega(f(z), g(z)) = \frac{1}{2\pi i} \int (f(-z), g(z)) dz.$$ 

The Lagrangian decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_\pm$ ( $\mathcal{H}_+$ is a big phase space) leads to the identification of $\mathcal{H}$ with the cotangent bundle $T^*\mathcal{H}_+$. For $F \in GW_0$ let $\mathcal{L}_F \subset \mathcal{H}$ be a graph of $dF$ (recall that $F$ can be viewed as a function on $\mathcal{H}_+$). In [Gi3] the system $DE + SE + TRR$ is rewritten in terms of geometric properties of $\mathcal{L}_F$. This approach allows to define the action of $G_-^\pm$ on $GW_0$ as well as to study the properties of this action on the subspace $GW_0^{ss}$ of semi-simple theories.

In this paper we suggest another – more algebraic – approach to the study of the space of formal genus 0 Gromov-Witten theories. The main points are the exact formula for the action of the lower-triangular twisted loop group and the form of the expansion of elements of $GW_0$ as Laurent series in a distinguished variable. We will deal only with the action of the lower-triangular group $G_-$ since the "opposite" group $G_+$ doesn’t preserve the space $\mathcal{H}$ (see [Gi3] for the discussion of possible completions). We briefly describe our approach below.

First, starting from the Lagrangian cones formalism we derive exact formula for the action of $G_-$. The resulting formula coincide with a genus zero restriction of the Givental’s quantum action of the twisted loop group on the space of total descendent potentials (see [Gi2]). Thus we prove that the Lagrangian cones approach and quantization procedure produce the same actions of $G_-$ on genus 0 formal theories (see also [Gi3], [L1], [L3]).

Next, using the dilaton equation we write elements of $GW_0$ as Laurent series in a distinguished variable. Namely we fix a basis $\{\phi_\alpha\}_{\alpha=1}^N$ of $\mathcal{H}_+$ such that $\phi_1 = 1$. This gives a basis $\phi_\alpha \otimes z^n$ of $\mathcal{H}_+$ and the corresponding coordinates $t_\alpha^n$. We show that any element $F \in GW_0$ can be written in a form

$$F = \sum_{n \geq 0} (t_1^n - 1)^{2-n} c_n,$$ (1)
where \( c_n \) are homogeneous degree \( n \) polynomials in variables \( t_{\alpha}^n \), \((\alpha, n) \neq (1, 1)\). As an application we study \( N = 1 \) case. We show explicitly that the group \( G_- \) acts transitively and, modulo the string flow, freely on the space \( GW_0 \). We note that for \( N = 1 \) the group \( G_\pm \) acts trivially on \( GW_0 \) (see [Gi2], [FP]). Thus our results agree with Givental’s theorem which states that for general \( N \) one needs the action of both groups \( G_\pm \) in order to generate the whole space \( GW_0 \) starting from the Gromov-Witten potential of \( N \) points.

We finish the introduction part with a following remark. The results of our paper seems to be known to experts. The novelty is an algebraic approach to the Givental’s theory. The advantage of this approach is twofold. Firstly, it makes some constructions more clear and allows to prove certain statements or simplify the existed proofs. Secondly, we hope that our approach can be applied to the study of non-semisimple GW theories, which are not covered by the geometric theory (see a discussion in [L3]).

Our paper is organized as follows. In Section 1 we recall main points of the Givental’s formalism. In Section 2 we derive exact formulas for the action of the lower-triangular subgroup on \( GW_0 \). In Section 3 we recall the connection with Frobenius structures. In Section 4 we describe explicitly the space \( GW_0 \) for \( N = 1 \).

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1. Givental’s formalism

In this section we recall main points of the formalism developed in [CG], [Gi1], [Gi2].

Let \( H \) be an \( N \)-dimensional vector space equipped with a symmetric non-degenerate bilinear form \((\cdot, \cdot)\) and with a distinguished nonzero element \( 1 \). Let \( \mathcal{H} \) be the space of Laurent series \( H((z^{-1})) \) in \( z^{-1} \). This space carries a symplectic form \( \Omega \):

\[
\Omega(f(z), g(z)) = \frac{1}{2\pi i} \oint (f(-z), g(z))dz.
\]

The space \( \mathcal{H} \) is naturally a \( \mathbb{C}[z, z^{-1}] \)-module. Any operator \( M(z) \in \text{End}(\mathcal{H}) \), which commutes with the action of \( z \) can be written in a form \( M(z) = \sum_{i \in \mathbb{Z}} M_i z^i \). If \( M(z) \) preserves \( \Omega \) then one has

\[
M^*(-z)M(z) = \text{Id},
\]

where * is adjoint with respect to \((\cdot, \cdot)\). An algebra of \( \text{End}(H) \)-valued Laurent series \( \text{End}(H)((z^{-1})) \) acts on the space \( \mathcal{H} \).

Lemma 1.1. Let \( M(z) \in \text{End}H((z^{-1})) \) be a series such that \( M^*(-z)M(z) = \text{Id} \). Then \( M_i = 0 \) for \( i > 0 \).
**Proof.** Suppose $M_i = 0$ for $i > N > 1$. Then because of $M^*(-z)M(z) = \text{Id}$ one gets $M^*_N M_N = 0$. Therefore $M_N = 0$ since $(\cdot, \cdot)$ is non-degenerate. □

The group of operators $S(z) = \text{Id} + \sum_{i>0} S_i z^i$ satisfying $S^*(-z)S(z) = \text{Id}$ is called the lower-triangular group. We denote it by $G_-$. The elements of this group are sometimes called the calibrations of Frobenius manifolds (see Section 3). They are also involved in the ancestor-descendant potentials correspondence (see [Gi2],[KM]).

**Remark 1.1.** The upper-triangular group $G_+$ consists of the operators

$$R(z) = \text{Id} + \sum_{i>0} R_i z^i.$$

These operators play the central role in the Givental’s theory of Frobenius manifolds and formal Gromov-Witten potentials (see [Gi1]). We are not considering the group $G_+$ in this paper because of two reasons. First because of Lemma 1.1 the upper-triangular group does not exactly fit the Lagrangian cone formalism (see [CG],[Gi3] for discussion of possible modifications). Another reason is that in our main application ($N = 1$) we only need the action of the group $G_-$ (because of Faber-Pandharipande relations, see [Gi2],[FP]).

We now describe the action of the Givental’s group on the axiomatic genus 0 theories and the quantization formalism. Let $\{\phi_\alpha\}$ be a basis of $H$, such that $\phi_1 = 1$, $(\phi_\alpha,\phi_\beta) = g_{\alpha,\beta}$. Let $t_n$ be an element of $H \otimes z^n$. Then we set $t_n = \sum_\alpha t_n^\alpha \phi_\alpha$. In what follows we deal with functions $F(t_0,t_1,\ldots)$, $t_i \in H$. We write $\partial_{\alpha,n}$ for the partial derivative with respect to $t_n^\alpha$. As usual, $(g^{\alpha,\beta}) = (g_{\alpha,\beta})^{-1}$.

By definition, a function $F(t_0,t_1,\ldots)$, $t_i \in H$ is called axiomatic genus zero Gromov-Witten theory if it satisfies the string equation (SE)

$$\sum_{i \geq 0} t_i^\alpha \partial_{\alpha,i} F(t) - \partial_{1,1} F(t) = 2 F(t),$$

the dilaton equation (DE)

$$\sum_{i \geq 0} t_i^\alpha \partial_{\alpha,i} F(t) - \partial_{1,1} F(t) = 2 F(t),$$

and the system of topological recursion relations (TRR), labeled by triples of integers $k,l,m \geq 0$ and $\alpha, \beta, \gamma = 1, \ldots, \dim H$:

$$\partial_{\alpha,k+1} \partial_{\beta,l} \partial_{\gamma,m} F(t) = \partial_{\alpha,k} \partial_{\mu,0} F(t) g^{\mu,\nu} \partial_{\nu,0} \partial_{\beta,l} \partial_{\gamma,m} F(t).$$

We denote the set of functions satisfying string, dilaton and topological recursion relations by $GW_0$. It is convenient to introduce the dilaton shifted variables

$$q_n^\alpha = t_n^\alpha - \delta_{\alpha,1} \delta_{n,1}.$$
In new variables the system (DE), (SE) and (TRR) for the function $F(q) = F(q_0, q_1, \ldots)$ reads as

\[ \sum_{i \geq 0} q_i^\alpha \partial_{\alpha,i} F(q) = 2F(q), \quad (DE) \]
\[ \sum_{i \geq 0} q_i^{\alpha+1} \partial_{\alpha,i} F(q) = -\frac{1}{2} (q_0, q_0), \quad (SE) \]
\[ \partial_{\alpha,k+1} \partial_{\beta,l} \partial_{\gamma,m} F(q) = \partial_{\alpha,k} \partial_{\mu,0} F(q) g^{\mu,\nu} \partial_{\nu,0} \partial_{\beta,l} \partial_{\gamma,m} F(q). \quad (TRR) \]

**Lemma 1.2.** Any axiomatic genus zero theory $F$ can be written in a form

\[ F(q) = \sum_{n \geq 0} (q_1^1)^{2-n} c_n, \]

where $c_n$ are degree $n$ series in variables $q_0^n$, $(\alpha, n) \neq (1, 1)$.

**Proof.** Let

\[ t^{\alpha}_k = t^{\alpha_1}_{k_1} \cdots t^{\alpha_n}_{k_n} \]

be a monomial satisfying $(\alpha_i, k_i) \neq (1, 1)$. Let $a^{\alpha}_k(l)$ be a coefficient of $t^{\alpha}_k(t_1^1)^l$ in $F(t_0, t_1, \ldots)$. Comparing the coefficients of $t^{\alpha}_k(t_1^1)^l$ in the right and left hand sides of the dilaton equation one gets

\[ a^{\alpha}_k(l + 1) = a^{\alpha}_k(l) \frac{l + n - 2}{l + 1}. \]

But the same relation holds for the coefficients $b(l)$ of $t^l$ in the expansion of $(1 - t)^{2-k}$. Now Lemma follows from the relation $q_1^1 = t_1^1 - 1$. \qed

**Remark 1.2.** For the genus 0 Gromov-Witten potential of the point $F^pt_0$ we have

\[ c_3 = -\frac{1}{6} (q_0^1)^3, \quad c_0 = c_1 = c_2 = 0, \]

because the Deligne-Mumford spaces $\overline{M}_{0,0}$, $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ are missing. In general, the coefficients $c_0$, $c_1$ and $c_2$ do not vanish.

In [G3] Givental constructed an action of the group of $\Omega$-preserving operators on the space $GW_0$. We briefly outline his construction. Let $n$ be some nonnegative integer. Recall the coordinates $q_0^n$ in the space $H \otimes z^n$, $n \geq 0$. We denote by $p^n_\alpha$ the Darboux coordinates with respect to $\Omega$ in the space $H \otimes z^{-n-1}$ and set

\[ q_n = \sum_{\alpha=1}^N q_0^{\alpha,n} \phi_\alpha, \quad p_n = \sum_{\alpha=1}^N p_0^{\alpha,n} \phi_\alpha, \]

where $\phi_\alpha = \sum_{\beta=1}^N g^{\alpha,\beta} \phi_\beta$. Thus any element of $\mathcal{H}$ can be written in a form

\[ \sum_{n \geq 0} q_n z^n + \sum_{n \geq 0} p_n (-z)^{-n-1}. \]
For any \( F \in GW_0 \) let \( \mathcal{L}_F \subset \mathcal{H} \) be a graph of \( dF \). Namely,
\[
\mathcal{L}_F = \left\{ \sum_{i \geq 0} p_i (-z)^{-i+1} + \sum_{i \geq 0} q_i z^i : p_i^\alpha = \frac{\partial F}{\partial q_i^\alpha} \right\}.
\]
Because of the dilaton equation (DE) the subset \( \mathcal{L}_F \) is a cone.

**Theorem 1.1.** \([Gi3]\) A cone \( \mathcal{L} \subset \mathcal{H} \) is equal to some \( \mathcal{L}_F \) for \( F \in GW_0 \) if and only if \( \mathcal{L} \) is Lagrangian and for any \( f \in \mathcal{L} \) the tangent space \( L_f = T_f \mathcal{L} \) is tangent to \( \mathcal{L} \) exactly along \( zL \).

**Corollary 1.1.** The group \( G_- \) acts on \( GW_0 \).

The following simple Lemma will be important for our computations:

**Lemma 1.3.** Let \( F \) be an element of \( GW_0 \) and let \( \mathcal{L}_F ^\rightarrow H \) be the corresponding cone with the projection \( \pi : \mathcal{L}_F \to \mathcal{H}_\perp \). Then
\[
2F(q) = \sum_{i \geq 0} (p_i, q_i), \quad p = \pi^{-1} q.
\]

**Proof.** Follows from the definition of \( \mathcal{L}_F \) and the dilaton equation. \( \square \)

Another description of the action of \( G_- \) can be given in terms of the genus zero restriction of the Givental’s quantization formalism. This theory defines an action of the groups \( G_\pm \) on the space of total descendent potentials, involving all genera. Namely, let \( g_+, g_- \) be Lie algebras of the groups \( G_+, G_- \). Explicitly,
\[
g_\pm = \left\{ \sum_{i > 0} a_i z^i : a_i^* = (-1)^{i+1} a_i \right\}.
\]
For any \( a \in g_\pm \) consider a Hamiltonian
\[
H_a(f) = \frac{1}{2} \Omega(af, f).
\]
The element \( f \in \mathcal{H} \) can be written in a form
\[
f = \sum_{i \geq 0} p_i (-z)^{-i+1} + \sum_{i \geq 0} q_i z^i, \quad p_i, q_i \in H.
\]
Thus \( H_a \) is a quadratic function in variables \( p_i^\alpha, q_j^\beta \). The quantization rule
\[
p_i^\alpha p_j^\beta \mapsto \hbar \partial_{i, \alpha} \partial_{j, \beta}, \quad p_i^\alpha q_j^\beta \mapsto q_j^\alpha \partial_{i, \alpha}, \quad q_i^\alpha q_j^\beta \mapsto \hbar^{-1} q_i^\alpha q_{j-1, \beta},
\]
defines operators \( \widehat{H}_a \) on \( \hbar \)-series with values in the space of functions on \( \mathcal{H}_+ = \mathcal{H} \otimes \mathbb{C}[[z]] \). The exact formulas are given in \([L1]\) (see also \([FSZ]\)). Namely
\begin{align}
\widehat{H}_{s_i z^{-i}} &= \sum_{n \geq 0} \sum_{\alpha, \beta} (s_i)_{\alpha, \beta} q_{i+n}^\alpha \partial_{n, \beta} + \frac{1}{\hbar} \sum_{n=0}^{l-1} \sum_{\alpha, \beta} (s_i)_{\alpha, \beta} (-1)^n q_n^\alpha q_{l-1-n}^\beta, \\
\widehat{H}_{r_i z^i} &= \sum_{n \geq 0} \sum_{\alpha, \beta} (r_i)_{\alpha, \beta} q_{n}^\alpha \partial_{i+n, \beta} + \frac{1}{2} \sum_{n=0}^{l-1} \sum_{\alpha, \beta} (r_i)_{\alpha, \beta} (-1)^n \partial_{n, \alpha} \partial_{l-1-n, \beta},
\end{align}
where \( l > 0 \) and \( (a)_{\alpha,\beta} \) are entries of a matrix \( a \) in the basis \( \phi_\alpha \) of \( H \). These formulas give rise to the action of the groups \( G_+ \) and \( G_- \) on the space of formal total descendant potentials of the form

\[
\mathcal{F} = \exp(h^{-1}F_0 + F_1 + hF_2 + \ldots).
\]

In the following Corollary we extract the genus zero part of the action of \( G_- \). We use the notation \( q(z) = \sum_{i \geq 0} q_i z^i \) and write \( F(q(z)) \) instead of \( F(q) \).

**Corollary 1.2.** Let \( F \) be a genus 0 part of a total descendent potential \( Z \). Then the genus zero part of the image \( \hat{S}(z)F \) is given by the formula

\[
(\hat{S}(z)F)(q(z)) = \frac{1}{2} \sum_{k,l \geq 0} (W_{k,l}q_k, q_l) + F([S^{-1}(z)q(z)]_+),
\]

where the operators \( W_{k,l} \) are defined by

\[
\sum_{k,l \geq 0} W_{k,l} z^k w^l = (S^*)^{-1}(w)S^{-1}(z) - \text{Id} \]

and \([S^{-1}(z)q(z)]_+\) is a power series truncation of the series \( S^{-1}(z)q(z) \).

**Proof.** Follows either from the formula (6) or from the genus zero restriction of the formula from [Gi2, Proposition 5.3].

**Remark 1.3.** The change of variables \( q(z) \rightarrow [S^{-1}(z)q(z)]_+ \) is not always well-defined operation. One should be careful to avoid infinite coefficients in the decomposition \([2]\) after the substitution. In the following Lemma we formulate a sufficient condition.

**Lemma 1.4.** Let \( S_1 = 0 \). Then the expression \( F_0([S^{-1}(z)q(z)]_+) \) is well-defined.

**Proof.** We consider a summand \((q_1^1)^{2-n}c_n\) from the decomposition \([2]\), where \( c_n \) is a degree \( n \) polynomial in variables \( q_\alpha^n \), \( (\alpha, n) \neq (1, 1) \). After the change of variables \( q(z) \rightarrow [S^{-1}(z)q(z)]_+ \) the series \( c_n \) remains independent of \( q_1^1 \) because of the condition \( S_1 = 0 \). Therefore the sum

\[
(\sum_{n \geq 0}(q_1^1)^{2-n}c_n)([S^{-1}(z)q(z)]_+)
\]

is well-defined. \( \square \)

2. "Negative" (lower triangular) subgroup

Let \( S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \ldots \) be an element of the group \( G_- \),

\[
S^*(-z)S(z) = 1.
\]

For instance

\[
S_1^* + S_1 = 0, \ S_2^* + S_1^* S_1 + S_2 = 0.
\]
Let $F(q)$ be a formal theory, $\mathcal{L}_F$ be the corresponding Lagrangian cone. We use Lemma 1.3 to compute the action of $G_-$ on $GW_0$. By definition $S(z)(p, q) = (p', q')$, where

$$q'_i = \sum_{n \geq 0} S_n q_{i+n},$$

$$p'_i = \sum_{n=0}^i (-1)^n S_n p_{i-n} - (-1)^i \sum_{n \geq 0} S_{i+n+1} q_n. \tag{9}$$

**Lemma 2.1.**

$$\sum_{i \geq 0} (p'_i, q'_i) = \sum_{i \geq 0} (p_i, q_i) - \sum_{i \geq 0} (-1)^i (S_{i+1} q_0 + S_{i+2} q_1 + \ldots, q'_i). \tag{11}$$

**Proof.** Using the formulas (9) and (10) we obtain

$$\sum_{i \geq 0} (p'_i, q'_i) = \sum_{i \geq 0} \left( \sum_{n=0}^i (-1)^n S_n p_{i-n} + (-1)^{n+1} \sum_{n \geq 0} S_{i+n+1} q_n, q'_i \right).$$

The direct computation shows that because of the relation $S^*(-z)S(z) = 1$ all the terms of the form $(p'_i, q'_j)$ in the right hand side vanish. The rest terms are exactly the right hand side of (11). \qed

**Lemma 2.2.** Let $W_{k,l}, k, l \geq 0$ be operators defined by

$$S^{-1*}(w)S^{-1}(z) - 1 = \sum_{k, l \geq 0} \frac{(W_{k,i} q_k, q_l)}{z^k w^l}. \tag{12}$$

Then for any $i \geq 0$

$$(-1)^{i+1} \sum_{n \geq 0} S_{i+n+1} q_n = \sum_{k \geq 0} W_{k,i} q_k. \tag{13}$$

**Proof.** Using the definition of $q'_k$ we obtain

$$\sum_{k \geq 0} W_{k,i} q'_k = \sum_{k \geq 0} \left( \sum_{m=0}^k W_{m,i} S_{k-m} \right) q_k.$$

To compute $\sum_{m=0}^k W_{m,i} S_{k-m}$ we multiply (12) by $S(z)$ from the right:

$$\sum_{k, l \geq 0} \frac{W_{k,l}}{z^k w^l} S(z) = \frac{S^{-1*}(w)S^{-1}(z) - 1}{w + z} = \frac{S(-w) - S(z)}{w + z}. \tag{14}$$

We now decompose the rightmost and leftmost expressions in (14).

$$\sum_{k, l \geq 0} \frac{W_{k,l}}{z^k w^l} S(z) = \sum_{k, l \geq 0} \frac{1}{w^l} \sum_{m=0}^k W_{m,i} S_{k-m},$$

$$\frac{S(-w) - S(z)}{w + z} = \sum_{k, l \geq 0} \frac{1}{z^k w^l} (-1)^{l+1} S_{k+l+1}.$$
We conclude that \( \sum_{m=0}^{k} W_{m,l} S_{k-m} = (-1)^{l+1} S_{k+l+1} \). Lemma is proved. \( \square \)

**Theorem 2.1.**

(a) The action of the group \( G_- \) is given by
\[
(\hat{S}(z)F)(q) = F([S^{-1}(z)q(z)]_+) + \frac{1}{2} \sum_{k,l \geq 0} (W_{k,l} q_k, q_l),
\]
where \([S^{-1}(z)q(z)]_+\) is a power series truncation.

(b) The action of the Lie algebra \( g_- \) is given by the genus zero restriction of the formula (4).

**Proof.** Formula (11) and Lemma 2.2 show that
\[
(S(z)F)(q') = F(q(z)) + \frac{1}{2} W(q', q').
\]
Now part a) of our theorem follows from the equation
\[
\sum_{i \geq 0} q'_i z^i = [S(z)q(z)]_+.
\]
The statement b) of the theorem follows from the part a). \( \square \)

**Corollary 2.1.** The definition of the action of \( G_- \) via the Lagrangian cones formalism is equivalent to those given via the genus zero restriction of the Givental’s quantization procedure.

**Proof.** Follows from Corollary 1.2 and Theorem 2.1. \( \square \)

3. Frobenius structures.

In [G3] Givental used the Lagrangian cones technique to study Frobenius structures on \( H \) ([D]). In particular he proved that each element of \( GW_0 \) determines a Frobenius manifold structure on \( H \). In this section we reprove this statement algebraically.

**Lemma 3.1.** Let \( F \in GW_0 \). Then the function \( \Phi(q), q \in H \) defined by
\[
\Phi(q) = F(q, 0, 0, \ldots)
\]
satisfies the WDVV equation.

**Proof.** Let \( 1 \leq a, b, c, d \leq N = \dim H \). We introduce the notation
\[
(15) \quad D = \partial_{a,0} \partial_{b,1} \partial_{c,0} \partial_{d,0} F.
\]
We denote by \( TRR_{a,k;\beta,l;\gamma,m} \) the topological recursion relation which corresponds to the triples \((k, l, m), (\alpha, \beta, \gamma)\). Differentiating both sides of the relation \( TRR_{a,k;\beta,l;\gamma,m} \) with respect to \( q_0^0 \) we obtain
\[
(16) \quad D = \partial_{a,0} \partial_{b,0} \partial_{c,0} F g_{ef} \partial_{f,0} \partial_{c,0} \partial_{d,0} F + \partial_{b,0} \partial_{c,0} F g_{ef} \partial_{f,0} \partial_{a,0} \partial_{c,0} \partial_{d,0} F.
\]
Similarly, differentiating both sides of \( TRR_{b,1;\alpha,0;\gamma,0} \) with respect to \( q_0^0 \) we obtain
\[
(17) \quad D = \partial_{c,0} \partial_{b,0} \partial_{e,0} F g_{ef} \partial_{f,0} \partial_{a,0} \partial_{d,0} F + \partial_{b,0} \partial_{e,0} F g_{ef} \partial_{f,0} \partial_{a,0} \partial_{c,0} \partial_{d,0} F.
\]
The equality of the right hand sides of (16) and (17) imply the WDVV equation for $\Phi(q)$:

$$
\partial_a \partial_b \partial_c \Phi g^{ef} \partial_f \partial_g \Phi = \partial_c \partial_b \partial_e \Phi g^{ef} \partial_f \partial_a \Phi.
$$

□

Lemma above is equivalent to the following Corollary (see [Gi3]).

**Corollary 3.1.** Each $F \in GW_0$ determines a Frobenius manifold structure on $H$ via the multiplication

$$
\phi_\alpha \cdot_0 \phi_\beta = \sum A^\gamma_{0,\alpha}(F) \phi_\gamma, \quad A^\gamma_{0,\alpha}(F) = \sum_{\lambda=1}^{N} g^{\gamma\lambda} (\partial_{t_0,0} \partial_{t_\lambda,0} F)_{t_1=t_2=\cdots=0}.
$$

The exact formulas for the action of $G_-$ provide an immediate corollary (see [Gi3]).

**Corollary 3.2.** Let $S(z)$ be an element of $G_-$ such that $S_1 = 0$ (see Lemma [L.4]). Then the action of $S(z)$ does not change the Frobenius structure.

**Proof.** By definition

$$
A^\gamma_{0,\alpha}(S(z)F) = \sum_{\lambda=1}^{N} g^{\gamma\lambda} (\partial_{t_0,0} \partial_{t_\lambda,0} F([S^{-1}(z)q(z)]_+)),
$$

where derivatives are taken after the substitution $t_1 = t_2 = \cdots = 0$. But

$$
(F([S^{-1}(z)q(z)]_+ - F(q(z)))|_{t_1=t_2=\cdots=0} = 0.
$$

Corollary is proved. □

4. THE RANK 1 CASE

In this section we consider the case $N = \dim H = 1$. We show that the space $GW_0$ form a single orbit of the lower-triangular group.

We first recall that the genus 0 Gromov-Witten potential of a point is defined by

$$
F^{pt}_g(t_0, t_1, \ldots) = \sum_{n \geq 3} \frac{1}{n!} \sum_{i_1, i_2, \ldots, i_n \geq 0} t_1^{i_1} \cdots t_n^{i_n} \int_{\overline{M}_{0,n}} \psi_1^{i_1} \cdots \psi_n^{i_n},
$$

where $\psi_i$ denotes the first Chern class of the line bundle on $\overline{M}_{0,n}$, whose fiber at the point $(C, x_1, \ldots, x_n)$ is equal to $T^*_x C$. The Laurent series expansion of $F^{pt}_0(q_0, q_1, \ldots)$ in variable $q_1$ starts with

$$
F^{pt}_0 = -\frac{q_0^3}{6q_1} - \frac{1}{24} \frac{q_0^4 q_2}{q_1^4} + \ldots.
$$

It is easy to see that if $N = 1$ then

$$
g_- = \{ \sum a_i z^{-2i+1}, \ a_i \in \mathbb{C} \}, \quad G_- = \{ \exp(\sum a_i z^{-2i+1}), \ a_i \in \mathbb{C} \} 
$$
are abelian Lie algebra and Lie group. The topological recursion relations are labeled by triples of nonnegative integers. We set \( g_{1,1} = 1 \) and denote the relation
\[
\partial_{k+1} \partial_0 \partial_m F(q) = \partial_k \partial_0 F(q) \partial_0 \partial_{l} \partial_n F(q)
\]
by \( TRR_{k,l,m} \). We also use a following notation: let \((A)\) be an equality of two series in the variable \( q_1 \):
\[
\sum_{n \in \mathbb{Z}} G_n q_1^n = \sum_{n \in \mathbb{Z}} H_n q_1^n. \tag{A}
\]
We denote an equality \( G_n = H_n \) by \((A(n))\).

Recall (see Lemma 1.2) that any element \( F \in GW_0 \) can be written in a form
\[
F(q) = \sum_{n \geq 0} q_1^{-n} c_n(q_0, q_2, q_3, \ldots), \tag{18}
\]
where \( c_n \) are degree \( n \) series in variables \( q_0, q_2, \ldots \). Consider the string equation \((SE)\). We note that equations \((SE(n))\) are trivial for \( n > 2 \) and the first 3 nontrivial equations are of the form:
\[
\begin{align*}
\partial_0 c_1 &= 0, \quad (SE(2)), \\
\partial_0 c_2 + \sum_{i \geq 2} q_{i+1} \partial_i c_1 &= 0, \quad (SE(1)), \\
\partial_0 c_3 + q_2 c_1 + \sum_{i \geq 2} q_{i+1} \partial_i c_2 &= -\frac{1}{2} q_0^2, \quad (SE(0)).
\end{align*}
\]
In general the following Lemma holds.

**Lemma 4.1.** An explicit form of \((SE(n))\), \( n < 0 \) is given by
\[
\partial_0 c_{-n+3} + (-n+1) q_2 c_{-n+1} + \sum_{i \geq 2} q_{i+1} \partial_i c_{-n+2} = 0.
\]

It turns out that among the TRR equations we only need the \( TRR_{0,1,1} \).

In the following lemma we write this equation explicitly.

**Lemma 4.2.** Equations \( TRR_{0,1,1}(n) \) are trivial for \( n > -4 \). Equations \( TRR_{0,1,1}(n) \), \( n \leq -4 \) are given by
\[
(n + 3)(n + 2)(n + 1)c_{-n-1} = \sum_{n_1 + n_2 = 2-n} (2 - n_2)(1 - n_2) \partial_0^2 c_{n_1} \partial_0 c_{n_2}.
\]

**Lemma 4.3.** For any \( F \in GW_0 \) the coefficient of \( q_1^{-1} \) is a cube of some linear form. That is
\[
\begin{align*}
c_3 &= -\frac{1}{6} (q_0 + \alpha_2 q_2 + \alpha_3 q_3 + \ldots)^3 \tag{19}
\end{align*}
\]
for some constants \( \alpha_i, i \geq 2 \).
Proof. Using relations \((SE(0))\) and \((SE(1))\) we obtain \(\partial_0^2 c_2 = 0\). Therefore the relation \((TRR_{0,1,1}(-4))\) reads as
\[-6c_3 = 2\partial_0^2 c_3\partial_0 c_3.\]
Using the relation \(\partial_0^3 c_3 = 1\) (which comes from \((SE(0))\)) we derive that
\[6c_3 = (\partial_0^2 c_3)^3.\]
Lemma follows. \(\square\)

**Corollary 4.1.** The group \(G_- = G_-/\{\exp(az^{-1}), a \in \mathbb{C}\}\) acts freely on the set \(GW_0\).

**Proof.** We first recall that because of the string equation the action of \(z^{-1} \in g_-\) on \(GW_0\) is trivial. Next, Theorem 2.1 shows that the coefficient of \(q_1^{-1}\) in \(\exp(-az^{2k+1})F, k \geq 1\) is equal to
\[c_3(q_0 + aq_{2k+1} + \frac{a^2}{2}q_{2(2k+1)} + \ldots, q_2 + aq_{2+2k+1} + \frac{a^2}{2}q_{2+2(2k+1)} + \ldots, \ldots).\]
Corollary is proved. \(\square\)

In the rest of this section we prove that \(G_-\) acts transitively on \(GW_0\).

**Lemma 4.4.** Let \(F \in GW_0\) and let \(c_3 = -\frac{1}{8}(q_0 + \alpha_2 a_2 + \ldots)\) be the coefficient of \(q_1^{-1}\) in \(F\). Then
- \(\alpha_2 = 0;\)
- if \(\alpha_i = 0\) for \(i \leq 2N - 1, N = 1, 2, \ldots, \) then \(\alpha_{2N} = 0.\)

**Proof.** We apply derivatives \(\partial_0 \partial_2\) to both sides of \((SE(0))\) and obtain
\[\partial_2 \partial_0^2 c_3 + \partial_2 \partial_0 (q_2 c_1) + \sum_{i \geq 2} q_{i+1} \partial_2 \partial_0 \partial_i c_2 = 0.\]
This gives \(\alpha_2 = 0\), because \(\partial_0 c_1 = 0\) and \(c_2\) is a degree 2 series.
Now suppose \(\alpha_2 = \alpha_3 = \ldots = \alpha_{2N-1} = 0\). Let
\[c_2 = \frac{1}{2} \sum_{i,j \neq 1} \beta_{ij} q_i q_j.\]
The equation \((SE(1))\) gives
\[c_0 = -\beta_{2,0}, \ c_1 = -\beta_{3,0} q_2 - \beta_{4,0} q_3 - \ldots.\]
Thus, the equation \((SE(0))\) reads as
\[-\frac{1}{2}(q_0 + \alpha_2 q_2 + \ldots)^2 - q_2(\beta_{3,0} q_2 + \beta_{4,0} q_3 + \ldots) + \sum_{i \geq 2} q_{i+1} \sum_{j \geq 0} q_j \beta_{i,j} = -\frac{1}{2} q_0^2.\]
Extracting the coefficient of \(q_nq_0\) we obtain \(\alpha_2 = 0\) and
\[\alpha_n = \beta_{n-1,0}, n \geq 3.\]
The coefficient of \(q_2 q_n, n \geq 2\) gives
\[\beta_{n+1,0} = \beta_{n-1,2}.\]
Finally the coefficient of $q_i q_j$, $i, j \geq 3$ yields
(23) \[ \beta_{i-1,j} + \beta_{i,j-1} = \alpha_i \alpha_j. \]
Thus we obtain the following equations:
\[ \alpha_{2n} = \beta_{2n-1,0} = \beta_{2n-3,2} = \alpha_{2n-3} \alpha_3 - \beta_{2n-4,3} = \ldots \]
\[ = \alpha_{2n-3} \alpha_3 - \alpha_{2n-4} \alpha_4 + \ldots \pm \frac{1}{2} \alpha_{n+1}^2. \]
This proves our Lemma. \qed

Lemma 4.5. A point $F \in GW_0$ is uniquely determined by the $q_1^{-1}$-coefficient $c_3$.

Proof. Formulas (21), (22) and (23) show that $\beta_{i,j}$ can be expressed via $\alpha_i$. Therefore $c_0, c_1, c_2$ are determined by $c_3$.

We now show that the same is true for $c_n$ with $n \geq 4$. In fact, the equation $TRR_{0,1,1}(-n-1)$ allows to express $c_n$ in terms of $\partial_0 c_n$ and $c_m$, $m < n$. But using $SE(-n+3)$ we can write $\partial_0 c_n$ via $c_{n-1}$ and $c_{n-2}$. Lemma is proved. \qed

Corollary 4.2. The group $G_-$ acts transitively on $GW_0$.

Proof. Note that the coefficient of $q_1^{-1}$ in $F_0^{pt}$ is equal to $-\frac{a_3^3}{6}$. Therefore it suffices to show that for any $F \in GW_0$ there exist complex numbers $a_3, a_5, \ldots$ such that the coefficient of $q_1^{-1}$ in
\[ \exp(a_3 z^{-3} + a_5 z^{-5} + \ldots) F \]
is equal to $-\frac{a_3^3}{6}$. We find $a_{2k+1}$ by induction on $k$. Let
\[ \frac{1}{6} (q_0 + \alpha_3 q_3 + \ldots)^3 \]
be a coefficient of $q_1^{-1}$ in $F$. Then the coefficient of $q_1^{-1}$ in $\exp(\alpha_3 z^{-3}) F$ is a cube of a linear form that contains no $q_3$ term and therefore (because of the Lemma 4.4) no $q_4$ term. Now suppose that the coefficient of $q_1^{-1}$ in $F$ is given by
\[ -\frac{1}{6} (q_0 + \alpha_{2k+1} q_{2k+1} + \alpha_{2k+2} q_{2k+2} + \ldots)^3. \]
Then the coefficient of $q_1^{-1}$ in $\exp(\alpha_{2k+1} z^{-2k-1}) F$ is a cube of a linear form that contains no $q_i$ terms, $i \leq 2k + 1$ and therefore (because of the Lemma 4.4) no $q_{2k+2}$ term. This yields our Corollary. \qed

Theorem 4.1. The set of solutions of $(DE)$, $(SE)$ and $(TRR)$ is isomorphic to $\mathbb{C}^\infty$ via the map
\[ (a_3, a_5, \ldots) \mapsto \exp \left( \sum_{i \geq 1} a_{2i+1} z^{-2i-1} \right) F_0^{pt}. \]

Corollary 4.3. The system $(DE) + (SE) + (TRR)$ is equivalent to $(DE) + (SE) + (TRR_{0,1,1})$. 
Proof. Follows from the proof. □

Remark 4.1. For general $N$ one needs both groups $G_\pm$ to generate the space $GW_0$ starting from the potential of $N$ points. But for $N = 1$ the Lie algebra $g_+$ (see (4)) acts trivially on $F_0^\text{pt}$ due to the Faber-Pandharipande relations (see [Gi2], [FP]).

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