Heat Kernel Measure on Central Extension of Current Groups in any Dimension

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Received October 30, 2005, in final form January 13, 2006; Published online January 13, 2006
Original article is available at http://www.emis.de/journals/SIGMA/2006/Paper003/

Abstract. We define measures on central extension of current groups in any dimension by using infinite dimensional Brownian motion.

Key words: Brownian motion; central extension; current groups

2000 Mathematics Subject Classification: 22E65; 60G60

1 Introduction

If we consider a smooth loop group, the basic central extension associated to a suitable Kac–Moody cocycle plays a big role in mathematical physics [3, 11, 21, 24]. Léandre has defined the space of $L^2$ functionals on a continuous Kac–Moody group, by using the Brownian bridge measure on the basis [16] and deduced the so-called energy representation of the smooth Kac–Moody group on it. This extends the very well known representation of a loop group of Albeverio–Hoegh–Krohn [2].

Etingof–Frenkel [13] and Frenkel–Khesin [14] extend these considerations to the case where the parameter space is two dimensional. They consider a compact Riemannian surface $\Sigma$ and consider the set of smooth maps from $\Sigma$ into a compact simply connected Lie group $G$. We call $C_r(\Sigma; G)$ the space of $C^r$ maps from $\Sigma$ into $G$ and $C_\infty(\Sigma; G)$ the space of smooth maps from $\Sigma$ into $G$. They consider the universal cover $\hat{C}_\infty(\Sigma; G)$ of it and construct a central extension by the Jacobian $J$ of $\Sigma$ of it $\hat{C}_{\infty}(\Sigma; G)$ (see [7, 8, 25] for related works).

We can repeat this construction if $r > s$ big enough for $C_r(\Sigma; G)$. We get the universal cover $\hat{C}_r(\Sigma; G)$ and the central extension by the Jacobian $J$ of $\Sigma$ of it $\hat{C}_r(\Sigma; G)$ denoted by $\hat{C}_r(\Sigma; G)$.

By using Airault–Malliavin construction of the Brownian motion on a loop group [11], we have defined in [19] a probability measure on $\hat{C}_r(\Sigma; J)$, and since the Jacobian is compact, we can define in [19] a probability measure on $\hat{C}_r(\Sigma; G)$.

Maier–Neeb [20] have defined the universal central extension of a current group $C_\infty(M; G)$ where $M$ is any compact manifold. The extension is done by a quotient of a certain space of differential form on $M$ by a lattice. We remark that the Maier–Neeb procedure can be used if we replace this infinite dimensional space of forms by the de Rham cohomology groups $H(M : \text{Lie} G)$ of $M$ with values in Lie $G$. Doing this, we get a central extension by a finite dimensional Abelian groups instead of an infinite dimensional Abelian group. On the current group $C_r(M; G)$ of $C^r$ maps from $M$ into the considered compact connected Lie group $G$, we use heat-kernel measure deduced from the Airault–Malliavin equation, and since we get a central extension $\hat{C}_r(M; G)$ by a finite dimensional group $Z$, we get a measure on the central extension of the current group. Let us recall that studies of the Brownian motion on infinite dimensional manifold have a long history (see works of Kuo [15], Belopolskaya–Daletskii [6, 12], Baxendale [4, 5], etc.).
Let us remark that this procedure of getting a random field by adding extra-time is very classical in theoretical physics, in the so called programme of stochastic-quantization of Parisi–Wu [23], which uses an infinite-dimensional Langevin equation. Instead to use here the Langevin equation, we use the more tractable Airault–Malliavin equation, that represents infinite-dimensional Brownian motion on a current group.

2 A measure on the current group in any dimension

We consider \( C_r(M; G) \) endowed with its \( C^r \) topology. The parameter space \( M \) is supposed compact and the Lie group \( G \) is supposed compact, simple and simply connected. We consider the set of continuous paths from \([0, 1]\) into \( C_r(M; G) \) \( t \mapsto g_t(\cdot) \), where \( S \in M \mapsto g_t(S) \) belongs to \( C_r(M; G) \) and \( g_0(S) = e \). We denote \( P(C_r(M; G)) \) this path space.

Let us consider the Hilbert space \( H \) of maps \( h \) from \( M \) into \( \text{Lie} \, G \) defined as follows:

\[
\int_S \langle (\Delta^k + 1)h, h \rangle dS = \|h\|_{H}^2,
\]

where \( \Delta \) is the Laplace Beltrami operator on \( M \) and \( dS \) the Riemannian element on \( M \) endowed with a Riemannian structure.

We consider the Brownian motion \( B_t(\cdot) \) with values in \( H \).

We consider the Airault–Malliavin equation (in Stratonovitch sense):

\[
dg_t(S) = g_t(S)dB_t(S), \quad g_0(S) = e.
\]

Let us recall (see [17]):

**Theorem 1.** If \( k \) is enough big, \( t \mapsto \{S \mapsto g_t(S)\} \) defines a random element of \( P(C_r(M; G)) \).

We denote by \( \mu \) the heat-kernel measure \( C_r(M; G) \): it is the law of the \( C^r \) random field \( S \mapsto g_t(S) \). It is in fact a probability law on the connected component of the identity \( C_r(M; G)_e \) in the current group.

3 A brief review of Maier–Neeb theory

Let us consider \( \Pi_2(C_r(M; G)_e) \) the second fundamental group of the identity in the current group for \( r > 1 \). The Lie algebra of this current group is \( C_r(M; \text{Lie} \, G) \) the space of \( C^r \) maps from \( M \) into the Lie algebra \( \text{Lie} \, G \) of \( G \) [22]. We introduce the canonical Killing form \( k \) on \( \text{Lie} \, G \).

\( \Omega^1(M; \text{Lie} \, G) \) denotes the space of \( C^{r-1} \) forms of degree \( i \) on \( M \) with values in \( \text{Lie} \, G \). Following [20], we introduce the left-invariant 2-form \( \Omega \) on \( C_r(M; G) \) with values in the space of forms \( Y = \Omega^1(M; \text{Lie} \, G) / d\Omega^0(M; \text{Lie} \, G) \) which associates

\[
k(\eta, d\eta).
\]

to \((\eta, \eta_1)\), elements of the Lie algebra of the current group.

For that, let us recall that the Lie algebra of the current group is the set of \( C^r \) maps \( \eta \) from the manifold into the Lie algebra of \( G \). \( d\eta \) is a \( C^{r-1} \) 1-form into the Lie algebra of \( G \). Therefore \( k(\eta, d\eta_1) \) appears as a \( C^{r-1} \) 1-form with values in the Lie algebra of \( G \). Moreover

\[
dk(\eta, \eta_1) = k(d\eta, \eta_1) + k(\eta, d\eta_1).
\]

This explains the introduction of the quotient in \( Y \). Following the terminology of [20], we consider the period map \( P_1 \) which to \( \sigma \) belonging to \( \Pi_2(C_r(M; G)_e) \) associates \( \int_0^1 \Omega \). Apparently \( P_1 \) takes its values in \( Y \), but in fact, the period map takes its values in a lattice \( L \) of \( H^1(M; \text{Lie} \, G) \).
Measures on Current Groups

It is defined on $\Pi_2(C_r(M;G))$ since $\Omega$ is closed for the de Rham differential on the current group, as it is left-invariant and closed and it is a 2-cocycle in the Lie algebra of the current group [20]. We consider the Abelian group $Z = H^1(M;\text{Lie } G)/L$. $Z$ is of finite dimension.

We would like to apply Theorem III.5 of [20]. We remark that the map $P_2$ considered as taking its values in $Y/L$ is still equal to 0 when it is considered by taking its values in $H^1(M;\text{Lie } G)/L$.

We deduce the following theorem:

**Theorem 2.** *We get a central extension $\hat{C}_r(M;G)$ by $Z$ of the current group $C_r(M;G)_e$ if $r > 1$.*

Since $Z$ is of finite dimension, we can consider the Haar measure on $Z$. We deduce from $\mu$ a measure $\hat{\mu}$ on $\hat{C}_r(M;G)$.

**Remark 1.** Instead of considering $C_r(M;G)$, we can consider $W_{\theta,p}(M;\text{Lie } G)$, some convenient Sobolev–Slobodetsky spaces of maps from $M$ into $\text{Lie } G$. We can deduce a central extension $\hat{C}_{\theta,p}(M;G)$ of the Sobolev–Slobodetsky current group $C_{\theta,p}(M;G)_e$. This will give us an example of Brzezniak–Elworthy theory, which works for the construction of diffusion processes on infinite-dimensional manifolds modelled on M-2 Banach spaces, since Sobolev–Slobodetsky spaces are M-2 Banach spaces [2, 10, 13]. We consider a Brownian motion $B^1_t$ with values in the finite dimensional Lie algebra of $Z$ and $B_t = (B_t(\cdot), B^1_t)$ where $B_t(\cdot)$ is the Brownian motion in $H$ considered in the Section 2. Then, following the ideas of Brzezniak–Elworthy, we can consider the stochastic differential equation on $\hat{C}_{\theta,p}(M;G)$ (in Stratonovitch sense):

$$d\hat{g}_t(\cdot) = \hat{g}_t(\cdot)dB_t.$$ 

**Acknowledgements**

The Author thanks Professor K.H. Neeb for helpful discussions.

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