EXCEPTIONAL DISCRETE MAPPING CLASS GROUP ORBITS IN MODULI SPACES

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ABSTRACT. Let $M$ be a four-holed sphere and $\Gamma$ the mapping class group of $M$ fixing $\partial M$. The group $\Gamma$ acts on the space $M_B(SU(2))$ of $SU(2)$-gauge equivalence classes of flat $SU(2)$-connections on $M$ with fixed holonomy on $\partial M$. We give examples of flat $SU(2)$-connections whose holonomy groups are dense in $SU(2)$, but whose $\Gamma$-orbits are discrete in $M_B(SU(2))$. This phenomenon does not occur for surfaces with genus greater than zero.

1. Introduction

Let $M$ be a Riemann surface of genus $g$ with $n$ boundary components (circles). Let

$$\{\gamma_1, \gamma_2, ..., \gamma_n\} \subset \pi_1(M)$$

be the elements in the fundamental group corresponding to these $n$ boundary components. Assign each $\gamma_i$ a conjugacy class $B_i \subset SU(2)$ and let

$$B = \{B_1, B_2, ..., B_n\},$$

$$H_B = \{\rho \in \text{Hom}(\pi_1(M), SU(2)) : \rho(\gamma_i) \in B_i, 1 \leq i \leq n\}.$$ 

A conjugacy class in $SU(2)$ is determined by its trace which is in $[-2, 2]$. Hence we might consider $B$ as an element in $[-2, 2]^n$. The group $SU(2)$ acts on $H_B$ by conjugation.

Definition 1.1. The moduli space with fixed holonomy $B$ is

$$M_B = H_B / SU(2).$$

Denote by $[\rho]$ the image of $\rho \in H_B$ in $M_B$. The set of smooth points of $M_B$ possesses a natural symplectic structure which gives rise to a finite measure $\mu$ on $M_B$ (see [4, 5]).

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Let $\text{Diff}(M, \partial M)$ be the group of diffeomorphisms of $M$ fixing $\partial M$. The mapping class group $\Gamma$ is $\pi_0(\text{Diff}(M, \partial M))$. The group $\Gamma$ acts on $\pi_1(M)$ fixing the $B_i$’s. This action induces a $\Gamma$-action on $\mathcal{M}_B$.

**Theorem 1.2** (Goldman). *The mapping class group $\Gamma$ acts ergodically on $\mathcal{M}_B$ with respect to the measure $\mu$.***

Since $\mathcal{M}_B$ has a natural topology, one may also study the topological dynamics of the mapping class group action and we have [4, 5]:

**Theorem 1.3.** *Suppose $M$ is an orientable surface with boundary and $g > 0$. Let $\rho \in \mathcal{H}_B$ such that $\rho(\pi_1(M))$ is dense in $\text{SU}(2)$. Then the $\Gamma$-orbit of the conjugacy class $[\rho] \in \mathcal{M}_B$ is dense in $\mathcal{M}_B$.***

In this paper we show:

**Theorem 1.4.** *Let $M$ be a four-holed sphere. Then there exists a subset $F \subset [-2, 2]^4$ of two real dimensions with the following property: Suppose $B \in F$. Then there exists $\rho \in \mathcal{H}_B$ with $\rho(\pi_1(M))$ dense in $\text{SU}(2)$, but the $\Gamma$-orbit of the conjugacy class $[\rho]$ is discrete in $\mathcal{M}_B$.***

Let $G$ be a subgroup of $\text{SU}(2)$. We say that a representation $\rho$ is a $G$-representation if $\rho(\pi_1(M)) \subset G$ up to conjugation by $\text{SU}(2)$. The group $\text{SU}(2)$ is a double cover of $\text{SO}(3)$:

$$p : \text{SU}(2) \longrightarrow \text{SO}(3).$$

The group $\text{SO}(3)$ contains $\text{O}(2)$, and the symmetry groups of the regular polyhedra: $T'$ (the tetrahedron), $C'$ (the cube), and $D'$ (the dodecahedron). Let $\text{Pin}(2)$, $\mathcal{T}$, $\mathcal{C}$, and $\mathcal{D}$ denote the groups $p^{-1}(\text{O}(2))$, $p^{-1}(T')$, $p^{-1}(C')$, and $p^{-1}(D')$, respectively. The proper closed subgroups of $\text{SU}(2)$ consist of $\mathcal{T}$, $\mathcal{C}$, $\mathcal{D}$, and the closed subgroups of $\text{Pin}(2)$. The group $\text{Pin}(2)$ has two components, and we write

$$\text{Pin}(2) = \text{Spin}(2) \cup \text{Spin}_-(2),$$

where $\text{Spin}(2)$ is the identity component of $\text{Pin}(2)$.

**Remark 1.5.** *Suppose $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$. If $\rho(\pi_1(M))$ is not contained in any of the aforementioned closed subgroups, then it is dense in $\text{SU}(2)$.***

We adopt the following notational conventions: For a fixed representation $\rho$, $X \in \pi_1(M)$, we write $X$ for $\rho(X)$ when there is no ambiguity. A small letter denotes the trace of the matrix represented by the corresponding capital letter.
2. The moduli space of the four-holed sphere

We first review some results that appear in [1, 2, 5]. Suppose $M$ is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I \rangle,$$

where $A, B, \text{ and } C$ represent the homotopy classes of the three boundaries of $M$.

**Proposition 2.1.**

1. A representation $\rho$ on a three-holed sphere is a $\text{Spin}(2)$-representation if and only if

   $$a^2 + b^2 + c^2 - abc - 4 = 0.$$

2. A representation $\rho$ on a three-holed sphere is a $\text{Pin}(2)$-representation and not a $\text{Spin}(2)$-representation if and only if

   $$a^2 + b^2 + c^2 - abc - 4 \neq 0 \quad \text{and at least two of the three: } A, B, AB, \text{ have zero trace.}$$

**Proof.** See [2, 5].

Suppose $M$ is a four-holed sphere. Then the fundamental group $\pi_1(M)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$

Set $X = AB, Y = BC,$ and $Z = CA$. Let $\kappa = (a, b, c, d) \in [-2, 2]^4$ be the holonomies on the boundary. Then the moduli space $\mathcal{M}_\kappa$ is the subspace of $[-2, 2]^3$ given by the equation

$$x^2 + y^2 + z^2 + xyz = (ab + cd)x + (ad + bc)y + (ac + bd)z -(a^2 + b^2 + c^2 + d^2 + abcd - 4).$$

**Remark 2.2.** If two representations in $\mathcal{M}_\kappa$ share $(x, y, z)$, then they are conjugate.

Let

$$I_{a,b} = \left[\frac{ab - \sqrt{(a^2 - 4)(b^2 - 4)}}{2}, \frac{ab + \sqrt{(a^2 - 4)(b^2 - 4)}}{2}\right].$$

If $I_{a,b} \cap I_{c,d} \neq \emptyset$, then $\mathcal{M}_\kappa$ is a (possibly degenerate) topological sphere (see Figure 1).

The mapping class group $\Gamma$ of the 4-holed sphere is generated by three Dehn twists $\tau_X, \tau_Y, \tau_Z$. In local coordinates, the actions are

$$\tau_X \begin{bmatrix} y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{bmatrix},$$

$$\tau_Y \begin{bmatrix} z \\ x \end{bmatrix} \rightarrow \begin{bmatrix} bd + ca - y(ba + cd - yz - x) - z \\ ba + cd - yz - x \end{bmatrix},$$

$$\tau_Z \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} cd + ab - z(cb + ad - zx - y) - x \\ cb + ad - zx - y \end{bmatrix}.$$
Consider
\[ e^{i\theta} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \lambda = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \]
in Pin(2).

**Proposition 3.1.** Suppose \( \rho \in \mathcal{H}(a,b,c,d) \) with \( a, b, c, d \not\in \{\pm 2\} \) and \( [\rho] = (x, y, z) \in \mathcal{M}_\kappa \). Then the representation \( \rho \) is a Spin(2)-representation if and only if \( x \) is an endpoint of both \( I_{a,b} \) and \( I_{c,d} \), \( y \) is an endpoint of both \( I_{b,c} \) and \( I_{a,d} \), and \( z \) is an endpoint of both \( I_{a,c} \) and \( I_{b,d} \).

**Proof.** First, suppose that \( \rho \) is a Spin(2)-representation. Then, up to conjugation,
\[ \rho(A) = e^{i\theta_a}, \rho(B) = e^{i\theta_b}, \rho(C) = e^{i\theta_c}, \rho(D) = e^{i\theta_d}, \]
where \( \theta_a + \theta_b + \theta_c + \theta_d = 0 \). The endpoints of \( I_{a,b} \) are given by
\[ \frac{1}{2} (ab \pm \sqrt{(4 - a^2)(4 - b^2)}) \]
\[ = \cos(\theta_a + \theta_b) + \cos(\theta_a - \theta_b) \pm \frac{1}{2} \sqrt{(4 - 4 \cos^2(\theta_a))(4 - 4 \cos^2(\theta_b))} \]
\[ = \cos(\theta_a + \theta_b) + \cos(\theta_a - \theta_b) \pm 2|\sin(\theta_a)\sin(\theta_b)| \]
\[ = \cos(\theta_a + \theta_b) + \cos(\theta_a - \theta_b) \pm |\cos(\theta_a - \theta_b) - \cos(\theta_a + \theta_b)|. \]

This implies that an endpoint of \( I_{a,b} \) is equal to \( 2\cos(\theta_a + \theta_b) \).

Similarly, an endpoint of \( I_{c,d} \) is equal to \( 2\cos(\theta_c + \theta_d) \) which is equal to \( 2\cos(\theta_a + \theta_b) \).

Thus \( x \) is equal to an endpoint of both \( I_{a,b} \) and \( I_{c,d} \). A similar argument shows that \( y \) must be an endpoint of \( I_{b,c} \) and \( I_{a,d} \), and also \( z \) must be an endpoint of \( I_{a,c} \) and \( I_{b,d} \).

To prove the converse, suppose that \( \rho \) is such that \( x \) is an endpoint
of both \( I_{a,b} \) and \( I_{c,d} \), \( y \) is an endpoint of both \( I_{b,c} \) and \( I_{a,d} \), and \( z \) is an endpoint of both \( I_{a,c} \) and \( I_{b,d} \).

Then \( 2x = ab \pm \sqrt{(4 - a^2)(4 - b^2)} \) which implies that
\[ 4x^2 = a^2b^2 + 16 - 4a^2 - 4b^2 + a^2b^2 \pm 2ab\sqrt{(4 - a^2)(4 - b^2)} \]
\[ = a^2b^2 + 16 - 4a^2 - 4b^2 + a^2b^2 \pm 2ab(\pm(2x - ab)). \]

Hence
\[ x^2 + a^2 + b^2 - xab = 4 \]
which implies that \( \rho \) is a Spin(2)-representation on the three-holed sphere \((A, B, X)\) by Proposition 2.1. Similarly, \((C, D, X)\), \((A, C, Z)\), \((B, D, Z)\), \((A, D, Y)\), and \((B, C, Y)\) are all Spin(2)-representations. As \( A, B, C, \) and \( D \) all pairwise commute, we have that \( \rho \) is a Spin(2)-representation on the entire four-holed sphere.

**Proposition 3.2.** Let \( \rho \in \mathcal{H}_\kappa \) and \([\rho] = (x, y, z) \in \mathcal{M}_\kappa \). Suppose \( \rho \)
is a Pin(2)-representation but not a Spin(2)-representation then one of
the following two conditions holds:

1. \( \kappa = (0, 0, 0, 0) \),
2. \( \kappa = (0, 0, c, d) \), where \( y = 0 \) and \( z = 0 \), along with the five other
symmetric cases.

If \( \rho \) satisfies one of the two conditions above, then \( \rho \) is a Pin(2)-representation.

**Proof.** Let \( \rho \) be a Pin(2)-representation but not a Spin(2)-representation.
Then at least one of \( A, B, C, \) or \( D \) must be in Spin\(_{-}(2)\). However, since \( ABCD = I \), at least two of \( A, B, C, \) or \( D \) must be in Spin\(_{-}(2)\).
Suppose \( A, B \in \text{Spin}_{-}(2) \). If \( C \in \text{Spin}_{-}(2) \), then \( D \in \text{Spin}_{-}(2) \), then we obtain \( \kappa = (0, 0, 0, 0) \). If \( C \in \text{Spin}(2) \), then \( D \in \text{Spin}(2) \), which implies that \( AC, BC \in \text{Spin}_{-}(2) \), i.e., \( y = z = 0 \).

Now consider
\[ A = \iota, B = -\iota e^{i\theta} \]
which are contained in a Pin(2) subgroup.
Case 1: Let $\rho \in H_\kappa$ with $\kappa = (0, 0, 0, 0)$ with $x, y, z$ satisfying the equation $x^2 + y^2 + z^2 + xyz = 4$. We construct a Pin(2)-representation conjugate to $\rho$ by setting $x = 2\cos\theta$ (in $A$ and $B$ above) and setting $C$ equal to one of $e^{\pm i\psi}$, where $z = -2\cos\psi$. As $CA = -e^{\pm i\psi}$ and $Y = BC$ is either $e^{i(\theta + \psi)}$ or $e^{i(\theta - \psi)}$ whose traces are the two solutions of $x^2 + y^2 + z^2 + xyz = 4$ for fixed $x$ and $z$. Therefore, this Pin(2)-representation is conjugate to $\rho$.

Case 2: Let $\rho \in H_\kappa$ with $\kappa = (0, 0, c, d)$ with $y = z = 0$. Thus $x, c, d$ satisfy: $x^2 = cdx - c^2 - d^2 + 4$ implying that $\rho$ restricted to $(X, C, D)$ is a Spin(2)-representation by Proposition 2.1. We construct a Pin(2)-representation conjugate to $\rho$ by setting $x = 2\cos\theta$ (in $A$ and $B$ above) and setting $C$ to be $e^{i\psi}$ and $D = e^{-i(\psi + \theta)}$. As the traces of $Y = BC$ and $Z = AC$ are zero, this Pin(2)-representation is conjugate to $\rho$.

Propositions 3.1 and 3.2 provide a complete characterization of the Pin(2)-representation classes.

4. Examples

A direct computation shows that the traces of elements in the groups $C, D$ are in the set $S = \{0, \pm 1, \pm \sqrt{2}, \pm \frac{\sqrt{5} + 1}{2}, \pm \frac{\sqrt{5} - 1}{2}, \pm 2\}$.

Let $F$ be the set of $\kappa = (a, a, c, -c) \in [-2, 2]^4$ satisfying the following conditions:

1. $a^2 + c^2 < 4$,
2. $a \neq 0$ and $c \neq 0$,
3. $a \notin S$ or $c \notin S$.

Consider the space $\mathcal{M}_\kappa$ with $\kappa \in F$. A direct computation shows

$$\mathcal{O} = \{(a^2 - 2, 0, 0), (2 - c^2, 0, 0)\} \subset \mathcal{M}_\kappa$$

is $\Gamma$-invariant. By condition 1,

$$I_{a,a} \cap I_{c,-c} = [a^2 - 2, 2] \cap [-2, 2 - c^2] = [a^2 - 2, 2 - c^2]$$

is a closed interval. Again by condition 1, $a, c \neq \pm 2$. Hence Proposition 3.1 implies that elements in $\mathcal{O}$ do not correspond to Spin(2)-representations. By condition 2, $a, c \neq 0$, so the elements in $\mathcal{O}$ do not correspond to Pin(2)-representations by Proposition 3.2. Finally, by condition 3, they do not correspond to $C, D$-representations. Thus, by Remark 1.3, the elements in the discrete orbit $\mathcal{O}$ correspond to representations with dense images in SU(2). This proves Theorem 1.4.

Figure 1 shows one such case with $\kappa = (\sqrt{2}, \sqrt{2}, \frac{1}{2}, -\frac{1}{2})$. The special orbit $\mathcal{O}$ consists of the two points that are intersections of the $x$-axis
with $\mathcal{M}_\kappa$, i.e. $\mathcal{O} = \{(0, 0, 0), (\frac{7}{4}, 0, 0)\}$. Below is a representation in the conjugacy class $(0, 0, 0) \in \mathcal{O} \subset \mathcal{M}_\kappa$:

$$A = B = \begin{bmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\ 0 & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{bmatrix} \quad \text{and} \quad C = -D = \begin{bmatrix} \frac{1}{4} + \frac{1}{4}i & \frac{\sqrt{13}}{4} \\ -\frac{\sqrt{13}}{4} & \frac{1}{4} - \frac{1}{4}i \end{bmatrix}.$$