Perturbative expansion in gauge theories on compact manifolds

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Abstract

A geometric formal method for perturbatively expanding functional integrals arising in quantum gauge theories is described when the spacetime is a compact riemannian manifold without boundary. This involves a refined version of the Faddeev-Popov procedure using the covariant background field gauge-fixing condition with background gauge field chosen to be a general critical point for the action functional (i.e. a classical solution). The refinement takes into account the gauge-fixing ambiguities coming from gauge transformations which leave the critical point unchanged, resulting in the absence of infrared divergences when the critical point is isolated modulo gauge transformations. The procedure can be carried out using only the subgroup of gauge transformations which are topologically trivial, possibly avoiding the usual problems which arise due to gauge-fixing ambiguities. For Chern-Simons gauge theory the method enables the partition function to be perturbatively expanded for a number of simple spacetime manifolds such as $S^3$ and lens spaces, and the expansions are shown to be formally independent of the metric used in the gauge-fixing.

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1 Introduction

The functional integrals associated with vacuum expectation values in gauge theories have a geometric nature which allows them to be formulated in the setting where the spacetime has curved geometry and non-trivial topology, and where the gauge fields are associated with a topologically non-trivial principal fibre bundle. There are a number of reasons why it is interesting to consider these functional integrals in this general setting: (i) The curved spacetime is equivalent to a background gravitational field (when the geometry of the spacetime is pseudo-riemannian). (ii) Yang-Mills gauge theory on Euclidean $\mathbb{R}^4$ is equivalent (with regards to finite action gauge fields) to Yang-Mills gauge theory on the compact riemannian manifold $S^4$ because of the conformal invariance of the theory, and the gauge fields on $\mathbb{R}^4$ satisfying topologically twisted boundary conditions at $\infty$ are associated with non-trivial principal fibre bundles over $S^4$ (see e.g. [1] for a discussion of this). (iii) In a topological gauge theory, the Chern-Simons theory, these integrals lead to a new quantum field theoretic way of obtaining topological invariants of compact 3-dimensional manifolds, and of linked knots embedded in these manifolds, as discussed in [2] and later (independently) explicitly demonstrated in [3]. In the usual setting where the spacetime is flat the functional integrals are usually evaluated as a perturbation series in a coupling parameter. There is a well-developed formalism for carrying out this perturbative expansion via Feynman diagrams, described e.g. in [4]. This formalism extends to the curved spacetime setting as discussed in [5]: however this presumes a trivialisation of the principal fibre bundle with which the gauge fields are associated, since when the bundle is non-trivial there is no canonical decomposition of the action functional into a quadratic “kinetic” term and higher order “interaction” term.

Our aim in this paper is to provide an extension of the formalism for perturbatively expanding the functional integrals arising in quantum gauge theories to the setting where the spacetime is a general riemannian manifold $M$ and the gauge fields are associated with an arbitrary principal fibre bundle $P$ over $M$ (i.e. the gauge fields are the connection 1-forms on $P$). For technical reasons (discussed in the conclu-
sion) we take the spacetime manifold to be compact and without boundary. This problem has been studied previously by S. Axelrod and I. Singer in the context of Chern-Simons gauge theory with spacetime a general compact 3-dimensional manifold \( \mathbb{S}^3 \). They decomposed the action functional into kinetic- and interaction terms by expanding it about a classical solution, i.e. a flat gauge field, and extended the standard formalism (with BRS gauge fixing) to perturbatively expand the Chern-Simons partition function, showing that the expansion was ultraviolet finite (with a natural point-splitting regularisation) and (essentially) independent of the metric used in the gauge fixing. (A result on the topological nature of the expansion was later extended in \([9]\)). However, the perturbative expansion obtained from their method is infrared divergent unless a very restrictive condition is satisfied by the flat gauge field about which the action is expanded: it must be isolated modulo gauge transformations and irreducible. Because of this their method is not applicable for a number of simple spacetime manifolds such as \( S^3 \) and the lens spaces since all the flat gauge fields on these manifolds are reducible. One of the motivations for this paper is to provide a method for perturbative expansion which in the context of Chern-Simons gauge theory extends the one given in \([6]\) to obtain an expansion of the partition function which is infrared finite as well as ultraviolet finite for simple manifolds such as \( S^3 \) and the lens spaces. This would open up the possibility of explicitly evaluating the terms in the expansions for these manifolds and comparing with the expressions obtained from the non-perturbative prescription of \([2]\). (This would provide a very interesting test of perturbation theory; such tests have already been successfully carried out in the semiclassical approximation as we discuss in the conclusion).

The formalism for perturbative expansion in this paper is presented in a general context which encompasses both Yang-Mills- and Chern-Simons gauge theories. We consider functional integrals of the form

\[
I(\alpha; f, S) = \int_A \mathcal{D}A f(A) e^{-\frac{1}{\alpha^2}S(A)}
\]  

\(2\)Perturbative expansion in Chern-Simons gauge theory on \( \mathbb{R}^3 \) has been extensively studied in the physics literature, see \([6]\) and the references therein. A rigorous treatment of the perturbative definition of knot invariants in \( \mathbb{R}^3 \) up to two loops was given by D. Bar-Natan in \([8]\).
where the formal integration is over a space $\mathcal{A}$ of gauge fields $A$ on a compact riemannian manifold $M$ without boundary, $f(A)$ and $S(A)$ are gauge-invariant functionals on $\mathcal{A}$ and $\alpha \in \mathbb{R}$ is a coupling parameter. (The functional integral (1.1) arises in connection with the vacuum expectation values of a functional $f$ in a gauge theory with action functional $S$). We will describe a method for carrying out formal perturbative expansions of (1.1) in $\alpha$ via a new geometric version of Feynman diagrams analogous to the momentum space version of Feynman diagrams used in the usual flat spacetime setting. In order to obtain a decomposition of the action functional $S(A)$ into a “kinetic term” and “interaction term” we expand about a general critical point $A^c$ (i.e. a classical solution); this gives

$$S(A^c + B) = S(A^c) + \langle B, D_{A^c} B \rangle + S_{A^c}^I(B)$$

(1.2)

where $D_{A^c}$ is an operator which is self-adjoint w.r.t. the inner product $\langle \cdot, \cdot \rangle$ in the space of fields $B$, and $S_{A^c}^I(B)$ is a “polynomial” in $B$ with each term of order $\geq 3$ in $B$. The quadratic- and higher order terms in (1.2) will play the roles of “kinetic”- and “interaction terms” respectively. When the spacetime manifold is compact without boundary the spectrum of the operator $D_{A^c}$ is discrete (for the cases that we are interested in), and the discrete variable labelling the spectrum will play an analogous role to the momentum vector in the flat spacetime setting for constructing the Feynman diagrams.

To rewrite (1.1) in a form which can be perturbatively expanded we develop a refined version of the Faddeev-Popov gauge-fixing procedure [10]. This uses the covariant background field gauge-fixing condition with background gauge field chosen to be the critical point $A^c$ for $S$ in (1.2). Our method for perturbative expansion is formal in the sense that the problem of ultraviolet divergences is not addressed (although these divergences do not arise in Chern-Simons gauge theory with point-splitting regularisation, due to a result in [3]). However, the problem of infrared divergences is considered in detail. The main feature of our method (besides its geometric nature) is that infrared divergences do not arise when the critical point $A^c$ is isolated modulo gauge transformations. (We also briefly sketch how it may be
possible to extend the method to the case where $A^c$ is a completely general critical point, but our arguments for this are incomplete). This is a consequence of our refinement of the Faddeev-Popov procedure, which takes into account the gauge-fixing ambiguities coming from the isotropy subgroup of $A^c$, i.e. the gauge transformations which leave $A^c$ unchanged. It is only when $A^c$ is reducible, i.e. when the isotropy subgroup of $A^c$ is non-trivial, that our refinement leads to a different result than the usual procedure described with the covariant background field condition in [11]. Also, we point out, as was first noted in [11], that the gauge-fixing procedure can be carried out using only the subgroup of gauge transformations which are topologically trivial. This avoids the usual problems which arise due to gauge-fixing ambiguities, provided that all ambiguities which do not come from the isotropy subgroup of $A^c$ come from topologically non-trivial gauge transformations (which we will assume to be the case).

In the context of Chern-Simons gauge theory our method extends the one of Axelrod and Singer in [6] to allow for reducible flat gauge fields $A^c$, providing an ultraviolet- and infrared-finite method for perturbatively expanding the partition function for a number of simple spacetime 3-manifolds such as $S^3$ and the lens spaces. (We discuss this in more detail in §5). We show that the perturbative expansion of the partition function obtained from our method (with $A^c$ isolated modulo gauge transformations) is formally metric-independent. This extends a result in [6].

The contribution to expectation values from field fluctuations about instantons in Yang-Mills gauge theories with compact riemannian spacetime was studied in [12]. In connection with this a formula was derived for the weak coupling ($\alpha \to 0$) limit of (1.1) in [12, App. II]. (This formula was also used in [13] to obtain an expression for the semiclassical approximation for the partition function of a gauge theory). We find that the lowest order term in our perturbative expansion of (1.1) reproduces this formula. This is reassuring, since the formula in [12, App. II] was derived without gauge-fixing, whereas our method does use gauge-fixing.

This paper is organised as follows. In §2 we explain the basic ideas behind the perturbative expansion of the functional integral (1.1), including the geometric ver-
sion of the Feynman diagrams. The precise relationship between infrared divergences, non-existence of the propagator and gauge invariance is determined. This shows precisely what it is that a gauge-fixing procedure needs to do to ensure a well-defined propagator and avoid infrared divergences. In §3 we describe the gauge-theoretic setup to be used in the rest of the paper, fixing notations and stating a few basic formulae that we will be using. In §4 we rewrite the functional integral (1.1) using a refined version of the Faddeev-Popov gauge-fixing procedure to obtain an expression which can be perturbatively expanded without infrared divergences by the method described in §2 (at least when $A^c$ is isolated modulo gauge transformations). In §5 we specialise to Chern-Simons gauge theory. We show that our approach to perturbative expansion extends the approach of [6] to the case where $A^c$ is isolated modulo gauge transformations and show the formal metric-independence of the perturbative expansion of the partition function in this case. In §6 we make some concluding remarks. Most of what we do in §2–§4 is formal. It seems possible that parts of §4 can be made rigorous; the results in [14] may be of use for this.

The method described in this paper was discussed previously by the author in the context of Chern-Simons gauge theory on $S^3$ in [13]. Features of the method in the general case, and their connection with [6] were later pointed out in [16]. In a recent overview paper [17] S. Axelrod has announced that he has extended his previous work with I. Singer [6], [9] on Chern-Simons gauge theory to the very general case where $A^c$ (in (1.2)) is only required to belong to a smooth component of the moduli space of flat gauge fields. The details of the method and arguments used for this have yet to appear (as far as we are aware), and we do not know to what extent they coincide with ours.

**Note added.** When the background gauge field is irreducible it was shown in [18] that there is a very interesting and deep relationship between the Faddeev-Popov determinant and the natural metric on the orbit space of the gauge fields (see also [13]). We expect that this relationship will continue to hold for reducible background gauge fields, with the Faddeev-Popov determinant replaced by our modified expression.
(the inverse of \([1.21]\) below), although we have yet to verify this.

2 A general method for perturbative expansion

In this section we describe a formal method for perturbatively expanding the functional integral \([1.1]\) in a general setting where \(A\) is an arbitrary (infinite-dimensional) affine space modelled on a vectorspace \(\Gamma\) with inner product \(\langle \cdot, \cdot \rangle\). The functional \(f\) may be complex-valued while \(S\) may be real-valued or purely imaginary-valued. (Unless stated otherwise we take \(S\) to be real-valued in the following; the modifications required when \(S\) is replaced by \(iS\) will be clear). The functionals are required to satisfy the following basic condition. (Examples of functionals \(S\) satisfying the condition are the action functionals for Yang-Mills- and Chern-Simons gauge theories, given by \([4.36]\) and \([4.44]\) below; an example of functional \(f\) satisfying the condition is the Wilson loop functional given by \([4.50]\) below). For each critical point \(A^c\) for \(S_{f}(A^c + B)\) and \(S(A^c + B)\) are “polynomials” in \(B \in \Gamma\). More precisely, the functionals can be expanded as

\[
S(A^c + B) = \sum_{k=0}^{s} S^{(k)}_{A^c}(B), \quad f(A^c + B) = \sum_{k=0}^{\infty} f^{(k)}_{A^c}(B) \tag{2.1}
\]

with \(2 \leq s < \infty\), \(S^{(k)}_{A^c}(B) = S^{(k)}_{A^c}(B, \ldots, B)\) and \(f^{(k)}_{A^c}(B) = f^{(k)}_{A^c}(B, \ldots, B)\) where \(S^{(k)}_{A^c}(B_1, \ldots, B_k)\) and \(f^{(k)}_{A^c}(B_1, \ldots, B_k)\) are multilinear functionals of \(B_1, \ldots, B_k \in \Gamma\). Note that \(S^{(0)}_{A^c}(B) = S(A^c)\), \(f^{(0)}_{A^c}(B) = f(A^c)\) and \(S^{(1)}_{A^c} = 0\) since \(A^c\) is a critical point for \(S\). Since \(S^{(2)}_{A^c}(B)\) is a quadratic functional we can write

\[
S^{(2)}_{A^c}(B) = \langle B, D_{A^c}B \rangle \tag{2.2}
\]

where \(D_{A^c}\) is a uniquely determined selfadjoint operator on \(\Gamma\). (If real-valued \(S\) is replaced by \(iS\) then we replace \(D_{A^c}\) by \(iD_{A^c}\) in \([2.2]\)). This leads to the expression \([1.2]\):

\[
S(A^c + B) = S(A^c) + \langle B, D_{A^c}B \rangle + S^{(1)}_{A^c}(B) + S^{(2)}_{A^c}(B) \tag{2.3}
\]

where \(S^{(1)}_{A^c}(B) = \sum_{k \geq 3} S^{(k)}_{A^c}(B)\). We now choose a specific critical point \(A^c\) for \(S\) and change variables in the integration in \([1.1]\) from \(A \in \mathcal{A}\) to \(B = A - A^c \in \Gamma\) to obtain
the following expression for the functional integral:

\[
I(\alpha; f, S) = \int_{\Gamma} DBf(A^c + B)e^{-\frac{1}{\alpha^2}S(A^c + B)}
= e^{-\frac{1}{\alpha^2}S(A^c)} \int_{\Gamma} DBf(A^c + B)e^{-\frac{1}{\alpha^2}(<B, D_{A^c}B> + S_{A^c}'(B))}
\]

(2.4)

In the perturbative expansion of (2.4) the quadratic term \( <B, D_{A^c}B> \) and higher order term \( S_{A^c}'(B) \) in the exponential will play the roles of “kinetic term” and “interaction term” respectively. We change variables from \( B \) to \( B' = \frac{1}{\alpha}B \) in (2.4) to obtain

\[
I(\alpha; f, S) = e^{-\frac{1}{\alpha^2}S(A^c)} \int_{\Gamma} D(\alpha B')f(A^c + \alpha B')e^{-<B', D_{A^c}B'> - \frac{1}{\alpha^2}S_{A^c}'(\alpha B')}
\]

(2.5)

We choose an orthonormal basis \( \{B_j\}_{j=0,1,2,...} \) for \( \Gamma \) and set \( b_j = <B', B_j> \), then \( B' = \sum_j b_jB_j \) and from (2.1) we get the expansions

\[
f(A^c + \alpha B') = \sum_{k=0}^{\infty} \alpha^k \sum_{j_1,...,j_k} f_{A^c}^{j_1,...,j_k} b_{j_1} \cdots b_{j_k}
\]

(2.6)

\[
\frac{1}{\alpha^2} S_{A^c}'(\alpha B') = \sum_{k=1}^{\kappa-2} \alpha^k \sum_{j_1,...,j_{k+2}} S_{A^c}^{j_1,...,j_k+2} b_{j_1} \cdots b_{j_{k+2}}
\]

(2.7)

where \( f_{A^c}^{j_1,...,j_k} := f_{A^c}^{(k)}(B_{j_1},...,B_{j_k}) \) and \( S_{A^c}^{j_1,...,j_k} := S_{A^c}^{(k)}(B_{j_1},...,B_{j_k}) \). Substituting (2.6) in (2.5) leads to

\[
I(\alpha; f, S) = e^{-\frac{1}{\alpha^2}S(A^c)} \sum_{N=0}^{\infty} \alpha^N \sum_{j_1,...,j_N} f_{A^c}^{j_1,...,j_N} G_{A^c}^{(N)}(\alpha; j_1, \ldots, j_N)
\]

(2.8)

where

\[
G_{A^c}^{(N)}(\alpha; j_1, \ldots, j_N) = \int_{\Gamma} D(\alpha B') <B', B_{j_1} > \cdots <B', B_{j_k} > e^{-<B', D_{A^c}B'> - \frac{1}{\alpha^2}S_{A^c}'(\alpha B')}
\]

(2.9)

The functions \( G_{A^c}^{(N)}(\alpha; j_1, \ldots, j_N) \) will play an analogous role to the Greens functions for field theories on flat spacetime. To perturbatively expand (2.8) we must perturbatively expand the Greens functions (2.9). To do this we introduce a variable \( J \in \Gamma \) (the “source” variable for \( B' \), set \( J_j = <J, B_j> \) and rewrite (2.9) via functional derivatives:

\[
G_{A^c}^{(N)}(\alpha; j_1, \ldots, j_N) = \left. \frac{\partial^N}{\partial J_{j_1} \cdots \partial J_{j_k}} \exp\left(-\frac{1}{\alpha^2} S_{A^c}'(\alpha \frac{\partial}{\partial J}) \right) \int_{\Gamma} D(\alpha B') e^{-<B', D_{A^c}B'> + <B', J>} \right|_{J=0}
\]

(2.10)
The r.h.s. of this expression is to be understood as follows. Writing \( <B', J> = \sum_j <B'_j, J_j> \) we consider the integral as an infinite polynomial in \( \{J_j\}_{j=0,1,2,...} \). The functional derivative \( \frac{1}{\alpha} S'_{A^c}(\alpha \frac{\partial}{\partial J}) \) is then the partial derivative operator obtained by replacing the \( b_j \)'s in (2.7) by \( \frac{\partial}{\partial J_j} \)'s. We change variables in the integral in (2.10) from \( B' \) back to the old variable \( B = \alpha B' \) and evaluate the integral using the generalisation of the formula

\[
\int_{-\infty}^{\infty} e^{-\lambda x^2 + ax} dx = \left( \frac{\lambda}{\pi} \right)^{-1/2} e^{\frac{a^2}{4\lambda}}
\]

(2.11)

to obtain

\[
\int_{\Gamma} D(\alpha B') e^{-<B', D_{A^c} B'> + <B', J>} = \det \left( \frac{1}{\pi \alpha^2} D_{A^c} \right)^{-1/2} e^{\frac{1}{4} <J, (D_{A^c})^{-1} J>}
\]

(2.12)

(Of course, \( D_{A^c} \) will have zero-modes in general so the r.h.s. of (2.12) is ill-defined. In the case of gauge theories this problem is circumvented using a gauge-fixing procedure as we will see in §4). Substituting (2.12) in (2.10) enables \( G^{(N)}_{A^c}(\alpha; j_1, \ldots, j_N) \) to be perturbatively expanded via Feynman diagrams, as we now discuss. In order to simplify the expressions we choose the o.n.b. \( \{B_j\}_{j=0,1,2,...} \) to consist of eigenvectors for \( D_{A^c} \) such that

\[
D_{A^c} B_j = \lambda(j) B_j
\]

\[
0 \leq |\lambda(0)| \leq \cdots \leq |\lambda(j)| \leq |\lambda(j+1)| \leq \cdots \to \infty \quad \text{for} \quad j \to \infty \quad (2.13)
\]

Using \( <J, (D_{A^c})^{-1} J> = \sum_j \frac{1}{\lambda(j)} J_j^2 \) we write (2.10) as

\[
G^{(N)}_{A^c}(\alpha; j_1, \ldots, j_N) = \det \left( \frac{1}{\pi \alpha^2} D_{A^c} \right)^{-1/2} \frac{\partial^N}{\partial J_{j_1} \cdots \partial J_{j_N}}
\]

\[
\times \exp \left( -\sum_{k \geq 1} \alpha^k \left( \sum_{i_1, \ldots, i_{k+2}} S^{i_1 \cdots i_{k+2}}_{A^c} \frac{\partial^{k+2}}{\partial J_{i_1} \cdots \partial J_{i_{k+2}}} \right) \right) \exp \left( \sum_{j=0}^{\infty} \frac{1}{4 \lambda(j)} J_j^2 \right) \bigg|_{J=0}
\]

(2.14)

From this we see that the Greens functions can be perturbatively expanded as

\[
G^{(N)}_{A^c}(\alpha; j_1, \ldots, j_N) = \det \left( \frac{1}{\pi \alpha^2} D_{A^c} \right)^{-1/2} \sum_{k \geq 0} \alpha^k G^{(k, N)}_{A^c}(j_1, \ldots, j_N)
\]

(2.15)

\footnote{We are assuming that \( D_{A^c} \) has discrete spectrum; this is the case for Yang-Mills- and Chern-Simons gauge theories on compact riemannian manifolds as we will see in §4.}
where each term $\alpha^k G_{Ae}^{(k,N)}(j_1, \ldots, j_N)$ is obtained by a Feynman diagram technique. The building blocks of the diagrams, and the factors which each of these contribute, are as follows:

Each diagram for $\alpha^k G_{Ae}^{(N,k)}(j_1, \ldots, j_N)$ has $N$ external lines labelled by $j_1, \ldots, j_N$. The term associated with each diagram (a function of $j_1, \ldots, j_N$) is obtained by taking the product of all the factors associated with the lines and vertices of the diagram and summing over all the values of the indices of the internal lines, and then dividing by the symmetry factor of the diagrams (as described e.g. in [4, §6-1-1]). Then $\alpha^k G_{Ae}^{(N,k)}(j_1, \ldots, j_N)$ is the sum of all topologically distinct diagrams which are proportional to $\alpha^k$.

These Feynman diagrams are analogous to the momentum space diagrams for Greens’ functions for field theories on flat spacetime $\mathbb{R}^4$: The discrete index $j \in \{0, 1, 2, \ldots\}$ is analogous to the momentum vector $p \in \mathbb{R}^4$, and the factor $\frac{1}{4\lambda(j)}$ associated with a line labelled by $j$ in the diagrams is analogous to the momentum space propagator. The term in $\alpha^k G_{Ae}^{(N,k)}(j_1, \ldots, j_N)$ corresponding to a given diagram with $q$ internal lines has the form

$$
\frac{1}{4^N \lambda(j_1) \cdots \lambda(j_N)} \sum c_{i_1 \cdots i_q} \frac{1}{4^q \lambda(i_1) \cdots \lambda(i_q)}
$$

(2.16)

(with summation over repeated indices) where $c_{i_1 \cdots i_q}$ is the product of the vertex factors of the diagram together with the inverse of the symmetry factor of the diagram. For the perturbative expansion to be meaningful the terms (2.16) must be finite. There are two reasons why (2.16) may diverge. First, if $D_{Ae}$ has zero-modes then $\lambda(j)$ is zero for sufficiently small $j$ (cf. (2.13)), leading to divergence of (2.16). We call divergences of this type infrared divergences. Secondly, (2.16) diverges if the summand in (2.16) does not converge quickly enough to zero when $i_1, \ldots, i_q$ become
large, or equivalently, if the divergence $\lambda(j) \to \infty$ for $j \to \infty$ is not sufficiently rapid. We call divergences of this type ultraviolet divergences. We will study the problem of infrared divergences below, and show in §4 how they can be avoided in the case of gauge theories using a gauge-fixing procedure. To deal with the problem of ultraviolet divergences methods of regularisation and renormalisation need to be developed. We will not address this problem in this paper, but make two remarks which may be relevant in this context:

(i) There are general theorems which set lower bounds on the rate of divergence $\lambda(j) \to \infty$, for $j \to \infty$, in many cases of interest, see e.g. [20, §1.5]. These may be useful for establishing general convergence criteria for the diagrams.

(ii) The Feynman diagrams in our approach do not have one of the significant features of the diagrams for field theories on flat spacetime, namely there is no general analogue of momentum conservation at the vertices of the diagrams. Conservation of momentum at the vertices of Feynman diagrams for field theories on flat spacetime is intimately related to the translation invariance of the kinetic term in the action functional of the theory. This suggests that for field theories on compact curved spacetime for which the kinetic term in the action is symmetrical (i.e. invariant under a group of isometries of the spacetime manifold) there may be simplifying conditions analogous to momentum conservation at the vertices of the Feynman diagrams. An example of this is when the spacetime is a compact group manifold and the kinetic term in the action of a field theory is invariant under the action of the group on itself: In this case there are simplifying conditions analogous to (but weaker than) momentum conservation at the vertices of the diagrams; these arise due to the orthogonality relations between the characters of the irreducible representations of the group [21].

Substituting (2.15) in (2.8) we finally obtain the perturbative expansion of the functional integral (1.1):

$$I(\alpha; f, S) = \det \left( \frac{1}{\pi \alpha^2} D_{A^c} \right)^{-1/2} e^{-\frac{\alpha}{\lambda} S(A^c)} \left[ f(A^c) + \sum_{k=1}^{\infty} \alpha^k \left( \sum_{N=1}^{k} \sum_{j_1, \ldots, j_N} f_{j_1, \ldots, j_N} G_{A^c}^{(N, k - N)}(j_1, \ldots, j_N) \right) \right]$$

(2.17)
We have seen that the presence of infrared divergences in the preceding perturbative expansion correspond to the zero-modes of $D_{A^c}$, i.e. the nullspace $\ker(D_{A^c})$ of $D_{A^c}$. We show below that $\ker(D_{A^c})$ is related to the critical point $A^c$ of $S$ as follows: Let $\mathcal{C}$ denote the set of critical points for $S$, then

$$T_{A^c}\mathcal{C} \subseteq \ker(D_{A^c}) \quad \text{and} \quad T_{A^c}\mathcal{C} = \ker(D_{A^c}) \quad \text{in the generic case.} \quad (2.18)$$

Here $T_{A^c}\mathcal{C}$ is the set of tangents to $\mathcal{C}$ at $A^c$, i.e.

$$T_{A^c}\mathcal{C} = \left\{ \frac{d}{dt} \bigg|_{t=0} A^c(t) \bigg| \ A^c(t) \text{ smooth curve in } \mathcal{C} \subseteq \mathcal{A} \text{ with } A^c(0) = A^c \right\}. \quad (2.18)$$

From (2.18) we find the precise reason why infrared divergences are unavoidable in gauge theories without gauge-fixing: If $\mathcal{A}$ is a space of gauge fields and $S(A)$ is gauge invariant then $\mathcal{C}$ is gauge invariant, and in particular the orbit $\mathcal{G} \cdot A^c$ of the group $\mathcal{G}$ of gauge transformations through $A^c$ is contained in $\mathcal{C}$, so the tangentspace $T_{A^c}(\mathcal{G} \cdot A^c)$ to the orbit at $A^c$ is contained in $T_{A^c}\mathcal{C}$ and it follows from (2.18) that

$$T_{A^c}(\mathcal{G} \cdot A^c) \subseteq \ker(D_{A^c}). \quad (2.19)$$

This shows that $\ker(D_{A^c})$ is necessarily non-vanishing for gauge theories. (In gauge theories the action of $\mathcal{G}$ on $\mathcal{A}$ does not have any fixed points, so $T_A(\mathcal{G} \cdot A) \neq 0$ for all $A \in \mathcal{A}$.) When $D_{A^c}$ is a positive operator, e.g. for Yang-Mills gauge theories, the result (2.19) can be obtained in a simple, direct way by a standard argument, see e.g. [11]. This argument does not hold in general though, since it assumes that $<B, D_{A^c}B> = 0 \Rightarrow D_{A^c}B = 0$, which is only true if $D_{A^c}$ is positive. Our argument for (2.18) and (2.19) does not require this assumption.

We show (2.18) as follows.\footnote{Our argument goes along similar lines to an argument used in determining the dimensions of instanton modulispaces in Yang-Mills gauge theories, see e.g. [22, Part IV].} Given $A^c \in \mathcal{C}$ any critical point for $S$ can be written as $A^c + B$ and is characterised by

$$\frac{d}{dt} \bigg|_{t=0} S(A^c + B + tC) = 0 \quad \text{for all } C \in \Gamma \quad (2.20)$$
From (2.1) we see that the functional $C \to \frac{d}{dt}
abla t=0 | S(A^c + B + tC)$ is linear, and can therefore be written as

$$\frac{d}{dt}
abla t=0 | S(A^c + B + tC) = \langle C, R_{A^c}(B) \rangle_{1}$$

(2.21)

It follows from (2.1) and (2.2) that

$$R_{A^c}(B) = 2DA^c(B) + \sum_{k=2}^{s-1} R_{A^c}^{(k)}(B)$$

(2.22)

with $R_{A^c}^{(k)}(B) = R_{A^c}^{(k)}(B, \ldots, B)$, where $R_{A^c}^{(k)}(B_1, \ldots, B_k)$ is a multilinear functional of $B_1, \ldots, B_k \in \Gamma$ with values in $\Gamma$. It follows from (2.20) and (2.21) that $A^c + B$ is a critical point for $S$ precisely when $R_{A^c}(B) = 0$. Each element in $T_{A^c}C$ has the form

$$\frac{d}{dt}
abla t=0 | A^c(t) = B'(0)$$

where $A^c(t) = A^c + B(t)$ is a smooth curve in $C$ with $B(0) = 0$. Then $R_{A^c}(B(t)) = 0$ for all $t$, so

$$0 = \frac{d}{dt}
abla t=0 | R_{A^c}(B(t)) = 2DA^c(B'(0))$$

which shows that $T_{A^c}C \subseteq \ker(D_A^c)$. To show the remaining part of (2.18) we note from (2.22) that the differential (i.e. the “Jacobi matrix”) of $R_{A^c}$ at $B = 0$ is $2DA^c$. If we were dealing with a smooth finite-dimensional situation (i.e. if $R_{A^c}$ was a smooth map between finite-dimensional manifolds) then the implicit function theorem would imply that the tangentspace to the solution space of $R_{A^c}(B) = 0$ at $B = 0$ is $\ker(D_{A^c})$, i.e. $T_{A^c}C = \ker(D_{A^c})$. This argument cannot always be extended to infinite-dimensional situations since the implicit function theorem cannot always be extended to these situations. It is reasonable to say that the argument can be extended in the “generic” situation though; for example in Yang-Mills gauge theory it can be extended when $A^c$ is irreducible, and the set of irreducible gauge field is dense in $\mathcal{A}$ (see e.g. [22, Part IV]). The argument also extends to Chern-Simons gauge theory on $S^3$ and lens spaces. However, there are special cases where the argument cannot be extended and where $T_{A^c}C \neq \ker(D_{A^c})$; examples of this in Chern-Simons gauge theory have been discussed for example in [23].

To obtain an infrared-finite perturbative expansion of $I(\alpha; f, S)$ we must rewrite the expression (2.4) for $I(\alpha; f, S)$ in such a way that the integration in the functional
integral is restricted to a subspace of $\Gamma$ which does not contain zero-modes for $D_{A^c}$.

In §4 we will show how this can be done for gauge theories (i.e. when $\mathcal{A}$ is a space of gauge fields and $f$ and $S$ are gauge invariant) using a version of the Faddeev-Popov procedure. The procedure rewrites $I(\alpha; f, S)$ in such a way that the integration over $B$ is restricted to $T_{A^c}(G \cdot A^c)^\perp$, the orthogonal complement to $T_{A^c}(G \cdot A^c)$ in $\Gamma$. (The price to be paid for this is that a divergent factor $V(G_0)$, the volume of the subgroup of topologically trivial gauge transformations, appears in the overall factor multiplying the functional integral. However, this factor can be avoided by normalising $I(\alpha; f, S)$ by $V(G_0)$ to begin with). When $A^c$ is isolated in $\mathcal{C}$ modulo gauge transformations it follows from (2.18) that $T_{A^c}(G \cdot A^c) = T_{A^c} \mathcal{C} = \ker(D_{A^c})$ (in the generic case), so the integration is over $T_{A^c}(G \cdot A^c)^\perp = \ker(D_{A^c})^\perp$ which by definition contains no zero-modes for $D_{A^c}$. In this case the preceding approach leads to a perturbative expansion of $I(\alpha; f, S)$ in which infrared divergences do not arise. (Further details will be given in §4).

We conclude this section by pointing out that the approach to perturbative expansion described here extends in a straightforward way to situations where the functional integration is over more that one field, and to the situation where Grassmannian (anticommuting) fields are involved. We will exploit this in §4, where the gauge-fixed expression obtained for $I(\alpha; f, S)$ involves additional integrations over anticommuting “ghost” fields.

3 The gauge-theoretic setup

In this section we describe the gauge-theoretic setup which we will be using in the rest of this paper. (The definitions and further details can be found in [24, Part IV]). The space $\mathcal{A}$ of gauge fields $A$ is the space of connection 1-forms on a principal fibre bundle $P$ over a compact oriented riemannian manifold $M$ (spacetime) without boundary. We set $n = \dim M$. The structure group (gauge group) of $P$ is a compact semisimple Lie group $G$; we denote its Lie algebra by $\mathfrak{g}$. The bundle $P \times_G \mathfrak{g}$ (where $G$ acts on $\mathfrak{g}$ by the adjoint representation) is denoted by $\mathfrak{g}$, and $\Omega^q(M, \mathfrak{g})$ denotes the
differential forms of degree $q$ on $M$ with values in $\mathfrak{g}$. A riemannian metric on $M$ and invariant inner product in $\mathfrak{g}$ determine an inner product $\langle \cdot, \cdot \rangle_q$ in each $\Omega^q(M, \mathfrak{g})$. Note that $\mathcal{A}$ is an affine vectorspace modelled on $\Omega^1(M, \mathfrak{g})$ (so $\Omega^1(M, \mathfrak{g})$ is the space $\Gamma$ of §2). We will think of $\mathcal{A}$ as an infinite-dimensional manifold; the tangentspace at each $A \in \mathcal{A}$ is $T_A \mathcal{A} = \Omega^1(M, \mathfrak{g})$. The inner product $\langle \cdot, \cdot \rangle_1$ in $\Omega^1(M, \mathfrak{g})$ therefore determines a metric in $\mathcal{A}$, which formally determines a volume form $DA$ on $\mathcal{A}$ (up to a sign). The curvature (force tensor) $F^A = dA + \frac{1}{2} [A, A]$ of each $A \in \mathcal{A}$ can be considered as an element in $\Omega^2(M, \mathfrak{g})$.

The group $\mathcal{G}$ of gauge transformations (an infinite-dimensional Lie group) can be identified with $C^\infty(M, \mathcal{G})$, the smooth maps from $M$ to the bundle $\mathcal{G} = P \times_G G$ (where $G$ acts on itself by the adjoint action) which map each $x \in M$ to the fibre $\mathcal{G}_x$ above $x$. It acts on $\mathcal{A}$ and $\Omega^q(M, \mathfrak{g})$ and we denote the action of $\phi \in \mathcal{G}$ on $A \in \mathcal{A}$ and $B \in \Omega^q(M, \mathfrak{g})$ by $\phi \cdot A$ and $\phi \cdot B$ respectively. Given a trivialisation of $P$ over a coordinate patch $U \subseteq M$ with coordinates $(x^\mu)$ and given a basis $\{\lambda^i\}$ for $\mathfrak{g}$ we can express $A \in \mathcal{A}$ and $B \in \Omega(M, \mathfrak{g})$ in the familiar way:

$$A(x) = A^i(x) \lambda_i dx^i$$

$$B(x) = B_{\mu_1 \ldots \mu_q}(x) \lambda_i dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}$$

The trivialisation allows the restriction of each $\phi \in \mathcal{G}$ to $U$ to be considered as a function from $U$ to $G$, and for all $x \in U$ we have the familiar expressions

$$(\phi \cdot A)(x) = \phi(x) A(x) \phi^{-1}(x) + \phi(x) d\phi^{-1}(x)$$

$$= \left( A^i(x) \phi(x) \lambda_i \phi^{-1}(x) + \phi(x) \partial_\mu \phi^{-1}(x) \right) dx^\mu$$

$$(\phi \cdot B)(x) = \phi(x) B(x) \phi^{-1}(x)$$

$$= \frac{1}{q!} B_{\mu_1 \ldots \mu_q}(x) \phi(x) \lambda_i \phi^{-1}(x) \lambda_i dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}.$$  

Note for $A \in \mathcal{A}$, $B \in \Omega^1(M, \mathfrak{g})$ and $\phi \in \mathcal{G}$ that $A + B \in \mathcal{A}$ and

$$\phi \cdot (A + B) = \phi \cdot A + \phi \cdot B \quad (3.1)$$

The inner product in each $\Omega^q(M, \mathfrak{g})$ is invariant under $\mathcal{G}$ so the metric and volume form $\mathcal{D}A$ on $\mathcal{A}$ are formally invariant under $\mathcal{G}$. The Lie algebra of $\mathcal{G}$ is $\text{Lie}(\mathcal{G}) = T_1 \mathcal{G} = \ldots$
\[ \Omega^0(M, g). \] The inner product in \( T_1G = \Omega^0(M, g) \) determines a metric on \( G \) via the action of \( G \) on itself; this formally determines a \( G \)-biinvariant volume form \( D\phi \) on \( G \) (up to a sign). Note that the subgroup \( G_0 \) of \( G \) consisting of the topologically trivial gauge transformations (i.e. the gauge transformations which can be continuously deformed to the identity) has the same Lie algebra as \( G \), i.e. \( \text{Lie}(G_0) = \text{Lie}(G) = \Omega^0(M, g) \).

The Lie bracket \([\cdot, \cdot]\) in \( g \) determines a graded product in the space \( \Lambda(T_xM)^* \otimes g = \bigoplus_{q=0}^n \Lambda^q(T_xM)^* \otimes g \) for each \( x \in M \), defined by \( [\omega_x \otimes a, \tau_x \otimes b] = \omega_x \wedge \tau_x \otimes [a, b] \); this determines a product \( [\cdot, \cdot] \) in the space \( \Omega(M, g) = \bigoplus_{q=0}^n \Omega^q(M, g) \) making \( \Omega(M, g) \) a graded Lie algebra. Each \( A \in \mathcal{A} \) determines covariant derivatives \( d_{q}^A : \Omega^q(M, g) \to \Omega^{q+1}(M, g), q = 0, 1, \ldots, n \), with the covariance property

\[
(\text{d}^{A}_{q})^{\star} B = \phi \cdot (\text{d}^{A}_{q} B) \quad \forall \phi \in \mathcal{G}, B \in \Omega^{q}(M, g)
\]

(3.2)

\[
(\text{d}^{A}_{q})^{\star} B = \phi \cdot ((\text{d}^{A}_{q})^{\star} B) \quad \forall \phi \in \mathcal{G}, B \in \Omega^{q+1}(M, g)
\]

(3.3)

(where \((\text{d}^{A}_{q})^{\star}\) is the adjoint of \(\text{d}^{A}_{q}\)) and the property

\[
d_{q}^{A+B} C = d_{q}^{A} C + [B, C] \quad \forall B \in \Omega^{1}(M, g), C \in \Omega^{q}(M, g).
\]

(3.4)

The covariant derivative \(d_{0}^{A}\) is minus the generator of infinitesimal gauge transformations of \( A \): For all \( v \in \text{Lie}(\mathcal{G}) = \Omega^0(M, g) \) we have

\[
v \cdot A := \left. \frac{d}{dt} \right|_{t=0} \exp(tv) \cdot A = -d_{0}^{A} v
\]

(3.5)

Some notations: If \( L \) is a linear map we denote the image and nullspace of \( L \) by \( \text{Im}(L) \) and \( \text{ker}(L) \) respectively. If \( L : V \to V \) is selfadjoint w.r.t. an inner product in the vectorspace \( V \) then \( L \) restricts to an invertible map on \( \ker(L)^{\perp} \) (the orthogonal complement to \( \ker(L) \) in \( V \)) which we denote by \( \tilde{L} \), i.e.

\[
\tilde{L} := \left. L \right|_{\ker(L)^{\perp}} : \ker(L)^{\perp} \xrightarrow{\cong} \ker(L)^{\perp}.
\]

(3.6)

We will be using the following general formulae: Let \( M_1 \) and \( M_2 \) be riemannian manifolds with volume forms \( Dx \) and \( Dy \) respectively, and let \( \Phi : M_1 \to M_2 \) be a
smooth invertible map. The differential of \( \Phi \) gives invertible linear maps (the “Jacobi matrix”)
\[
D_x \Phi : T_x M_1 \rightarrow T_y M_2 , \ y = \Phi(x)
\]
for each \( x \in M_1 \). The inner products in \( T_x M_1 \) and \( T_y M_2 \) (given by the riemannian metrics) determine \(|\det(D_x \Phi)| = \det((D_x \Phi)^* D_x \Phi)^{1/2}\) (the Jacobi determinant), and for arbitrary function \( h(y) \) on \( M_2 \) we have the change of variables formula
\[
\int_{M_2} D_y h(y) = \int_{M_1} D_x \det((D_x \Phi)^* D_x \Phi)^{1/2} h(\Phi(x)) .
\] (3.7)
When \( M_2 \) is a vectorspace we define the delta-function \( \delta(y) \) on \( M_2 \) by \( \int_{M_2} D_y h(y) \delta(y) = h(0) \). Then for arbitrary function \( g(x) \) on \( M_1 \) we apply (3.7) to get the formula
\[
\int_{M_1} D_x g(x) \delta(\Phi(x)) = \det((D_{\Phi^{-1}(0)} \Phi)^* D_{\Phi^{-1}(0)} \Phi)^{-1/2} g(\Phi^{-1}(0)) .
\] (3.8)

4 Gauge fixing

In this section we carry out a gauge-fixing of the normalised functional integral
\[
I(\alpha; f, S) = \frac{1}{V(\mathcal{G}_0)} \int_{\mathcal{A}} DA \ f(A) e^{-\frac{1}{\alpha} S(A)}
\] (4.1)
to formally rewrite it in such a way that an infrared-finite perturbative expansion can be obtained via the approach described in §2 when the gauge invariant functionals \( f \) and \( S \) satisfy the condition (2.1). The normalisation factor \( V(\mathcal{G}_0) \) is the volume of \( \mathcal{G}_0 \), a formal, divergent quantity. The gauge-fixing is a version of the Faddeev-Popov procedure with the covariant background field gauge-fixing condition
\[
(d_0^{A^c})^*(A - A^c) = 0
\] (4.2)
where the background gauge field \( A^c \) is a critical point for \( S \) as in (1.2). The perturbative expansion of (1.1) that we obtain will be infrared-finite when \( A^c \) is isolated modulo gauge transformations, and we will briefly discuss the possibility of extending our approach to obtain an infrared-finite expansion in the general case. The Faddeev-Popov functional associated with this gauge-fixing condition is
\[
P_{A^c}(A) = \int_{\mathcal{G}_0} D\phi \delta\left((d_0^{A^c})^*(\phi \cdot A - A^c)\right) .
\] (4.3)
Following [4] we have taken the domain of the formal integration to be the subgroup \( \mathcal{G}_0 \) of topologically trivial gauge transformations rather than the complete group \( \mathcal{G} \). This has the following consequences: (i) To carry out the gauge-fixing procedure the functionals \( f \) and \( S \) need only be invariant under \( \mathcal{G}_0 \) (rather than \( \mathcal{G} \) ). (ii) To formally evaluate (1.3) we only need the solutions \( \phi \) to \((d_0^{A^c})^*(\phi \cdot A - A^c) = 0\) which belong to \( \mathcal{G}_0 \) (rather than the complete set of solutions in \( \mathcal{G} \) ). As we will see, a consequence of (ii) is that under a certain assumption (stated below) problems due to gauge-fixing ambiguities which arise in the usual approach are avoided. Note that the formal functional (1.3) is invariant under \( \mathcal{G}_0 \) since the formal measure \( D\phi \) is invariant under \( \mathcal{G}_0 \). Inserting \( 1 = P_{A^c}(A) / P_{A^c}(A) \) into the integrand in the functional integral (4.1) leads to

\[
I(\alpha; f, S) = \frac{1}{V(\mathcal{G}_0)} \int_{\mathcal{G}_0} D\phi \int_A DA f(A) e^{-\frac{1}{\alpha} S(A)} P_{A^c}(A)^{-1} \delta((d_0^{A^c})^*(\phi \cdot A - A^c))
\]

\[
= \frac{1}{V(\mathcal{G}_0)} \int_A DA f(A) e^{-\frac{1}{\alpha} S(A)} P_{A^c}(A)^{-1} \delta((d_0^{A^c})^*(A - A^c))
\]

\[
= \int_{\Omega^1(M, \mathbf{g})} DB f(A^c + B) e^{-\frac{1}{\alpha} S(A^c + B)} P_{A^c}(A^c + B)^{-1} \delta((d_0^{A^c})^* B)
\]

(4.4)

To obtain the second line we have used the \( \mathcal{G}_0 \)-invariance of \( f, S \) and \( P_{A^c} \). In the last line we have changed variables from \( A \in \mathcal{A} \) to \( B = A - A^c \in \Omega^1(M, \mathbf{g}) \); \( DB \) denotes the formal volume form on \( \Omega^1(M, \mathbf{g}) \) formally determined (up to a sign) by the inner product \(< \cdot, \cdot>_1 \). Decomposing \( \Omega^1(M, \mathbf{g}) = \ker((d_0^{A^c})^*) \oplus \ker((d_0^{A^c})^*)^\perp \), \( B = (\tilde{B}, C) \), \( DB = D\tilde{B} DC \) and noting that \( \ker((d_0^{A^c})^*) = \text{Im}(d_0^{A^c})^\perp \) we use the formula (1.8) to integrate over \( \ker((d_0^{A^c})^*)^\perp \) in (4.4) and get

\[
I(\alpha; f, S) = \det(\tilde{\Delta}_0^{A^c})^{-1/2} \int_{\text{Im}(d_0^{A^c})^\perp} D\tilde{B} f(A^c + \tilde{B}) e^{\frac{1}{\alpha} S(A^c + \tilde{B})} P_{A^c}(A^c + \tilde{B})^{-1}
\]

(4.5)

where \( \Delta_0^{A^c} = (d_0^{A^c})^* d_0^{A^c} \) and \( \tilde{\Delta}_0^{A^c} \) is the restriction to \( \ker(\Delta_0^{A^c})^\perp = \ker(d_0^{A^c})^\perp \) as in (1.6); the determinant will be regularised by zeta-regularisation as discussed below.

\footnote{This is really the decomposition of the closure of \( \Omega^1(M, \mathbf{g}) \) w.r.t. \(< \cdot, \cdot>_1 \), but we ignore technicalities of this kind here and in the following.}
The next step is to formally evaluate the Faddeev-Popov functional

\[ P_{A'}(A^c + \tilde{B}) = \int_{G_0} \mathcal{D}\phi \delta\left((d_0^A)^*(\phi \cdot (A^c + \tilde{B}) - A^c)\right) \]  

appearing in (4.5) with \( \tilde{B} \in \ker((d_0^A)^*) \). To do this we must determine the solutions \( \phi \in G_0 \) to

\[ (d_0^{A^c})^*(\phi \cdot (A^c + \tilde{B}) - A^c) = 0 \]  

since it is only for these that the integrand in (4.6) is non-vanishing. We see immediately that \( \phi = 1 \) (the identity) is a solution because \( \tilde{B} \in \ker((d_0^A)^*) \). We now show that each \( \phi \in H_{A^c} \) is a solution to (4.7) where

\[ H_{A^c} = \{ \phi \in G_0 \mid \phi \cdot A^c = A^c \} . \]  

It suffices to show

\[ (d_0^{A^c})^*(A - A^c) = 0 \Rightarrow (d_0^{A^c})^*(\phi \cdot A - A^c) = 0 \quad \forall \phi \in H_{A^c} . \]  

i.e. the gauge-fixing condition (4.2) has ambiguities coming from \( H_{A^c} \). (The group \( H_{A^c} \) is finite-dimensional and can be identified with a subgroup of \( G \), see e.g. [25, p.111-112] and the references given there). Using (3.1) and (3.3) we see that for \( \phi \in H_{A^c} \)

\[ (d_0^{A^c})^*(\phi \cdot A - A^c) = (d_0^{A^c})^*(\phi \cdot (A - A^c)) = \phi^{-1} \cdot (d_0^{\phi^{-1} \cdot A^c})^*(A - A^c) = \phi^{-1} \cdot (d_0^{A^c})^*(A - A^c) \]  

from which (4.9) follows. This shows that \( H_{A^c} \) is contained in the solution set to (4.7); we now make the following assumption which implies that \( H_{A^c} \) is the complete set of solutions to (4.7).

**Assumption:** If \( \phi \in G_0 \) and \( A \in \mathcal{A} \) satisfy \((d_0^{A^c})^*(A - A^c) = 0 \) and \((d_0^{A^c})^*(\phi \cdot A - A^c) = 0 \) then \( \phi \in H_{A^c} \).

In other words we are assuming that all gauge-fixing ambiguities in the gauge-fixing condition (4.2) come either from gauge transformations in \( H_{A^c} \) or from topologically
non-trivial gauge transformations. We are not able to prove the assumption but it seems to be compatible with what is already known about gauge-fixing ambiguities:

The existence of gauge-fixing ambiguities was first pointed out by Gribov [26] who considered the Coulomb gauge-fixing condition with spacetime $M = S^3 \times \mathbb{R}$ (and gauge fields on a trivial principal fibre bundle). He showed that there is a collection of gauge transformations $\{\phi_n\}, \ n \in \mathbb{Z}$, such that each $\phi_n \cdot A$ satisfies the Coulomb condition when $A = 0$. However, it was subsequently shown in [27] that all of these gauge transformations are topologically non-trivial (except the identity transformation $\phi_0 = 1$). Therefore, if the general features of gauge ambiguities are the same for different gauge-fixing conditions, then Gribov’s example of gauge ambiguities is compatible with our assumption. As far as we are aware there are no examples of gauge ambiguities which contradict the assumption. Also, there does not appear to be any immediate contradiction between the assumption and the work of I. Singer [28] and M. Narasimhan and T. Ramadas [29] on the unavoidability of gauge-fixing ambiguities, since this work did not determine whether the ambiguities came from topologically trivial- or non-trivial gauge transformations.

In any case our evaluation of the Faddeev-Popov functional (4.6), which takes into account the gauge-fixing ambiguities coming from $H_{A^c}$, is a refinement of the usual evaluation which assumes that there are no gauge ambiguities. To evaluate (4.6) we parameterise a neighbourhood of $H_{A^c}$ (the solutions to (4.7)) in $\mathcal{G}_0$ by two coordinates; one of these parameterises directions along $H_{A^c}$, the other parameterises directions transverse to $H_{A^c}$. The idea is to integrate out the delta-function along the transverse coordinate and then integrate over $H_{A^c}$. We decompose $\text{Lie}(\mathcal{G}_0) = \Omega^0(M, \mathfrak{g})$ as $\text{Lie}(\mathcal{G}_0) = \text{Lie}(H_{A^c}) \oplus \text{Lie}(H_{A^c})^\perp$ and define the map

$$ Q : \text{Lie}(H_{A^c})^\perp \times H_{A^c} \to \mathcal{G}_0 \quad Q(v, h) := \exp(v)h $$

illustrated in the figure below:

[The figure is not included; it is available on request from the author.]
showing that it is non-degenerate at \( \{0\} \times H_{A^c} \), i.e. that the “Jacobi matrix” of \( Q \) at 
\[
D_{(0,h)}Q : \operatorname{Lie}(H_{A^c})^\perp \oplus T_h H_{A^c} \to T_h G_0
\]
(4.11)
has non-zero determinant for all \( h \in H_{A^c} \). In fact we will show that (4.11) is an
isometry, from which it follows that
\[
|\det(D_{(0,h)}Q)| = 1 \quad \forall \ h \in H_{A^c}.
\]
(4.12)
For fixed \( h \in H_{A^c} \) consider the composition of maps
\[
\operatorname{Lie}(G_0) = \operatorname{Lie}(H_{A^c})^\perp \oplus \operatorname{Lie}(H_{A^c}) \xrightarrow{\cong} \operatorname{Lie}(H_{A^c})^\perp \oplus T_h H_{A^c} \xrightarrow{D_{(0,h)}Q} T_h G_0 \xrightarrow{\cong} \operatorname{Lie}(G_0)
\]
(4.13)
where the first map is the isometry given by \((w,a) \mapsto (w, \left. \frac{d}{dt} \right|_{t=0} e^{t a} h)\) and the last map is the inverse of the isometry \( \operatorname{Lie}(G_0) \xrightarrow{\cong} T_h G_0 \) given by \( v \mapsto \left. \frac{d}{dt} \right|_{t=0} e^{t v} h \). We will show that the composition of maps (4.13) is the identity map on \( \operatorname{Lie}(G_0) \); it then follows that \( D_{(0,h)}Q \) must be an isometry since all the other maps in (4.13) are isometries. The image of \( v \in \operatorname{Lie}(G_0) \) under the maps in (4.13) is
\[
v = (w,a) \mapsto \left. \frac{d}{dt} \right|_{t=0} e^{t a} h \mapsto Q(tw,e^{t a} h) = \left. \frac{d}{dt} \right|_{t=0} e^{t w} e^{t a} h \mapsto \left. \frac{d}{dt} \right|_{t=0} e^{t w} e^{t a} = w + a = v
\]
so (4.13) is the identity as claimed.

We now choose a sufficiently small neighbourhood \( N \) of \( \{0\} \) in \( \operatorname{Lie}(H_{A^c})^\perp \), as illustrated in the figure above, so that the parameterisation map \( Q \) restricts to an invertible map from \( N \times H_{A^c} \) to a neighbourhood of \( H_{A^c} \) in \( G_0 \). (The non-degeneracy of \( D_{(0,h)}Q \) for all \( h \in H_{A^c} \) indicates that such a neighbourhood exists; this would certainly be the case in a smooth finite-dimensional situation but we have not proved its existence rigorously in the present infinite-dimensional situation). Then, since the integrand in the Faddeev-Popov functional (4.6) vanishes outside of \( H_{A^c} \) (by our assumption that \( H_{A^c} \) is the complete solution set to (4.7)) we can use (3.7) to write
From (3.5) we see that \( \text{Lie}(H_{A^c}) = \ker(d_0^{A^c}) \), and therefore \( \text{Lie}(H_{A^c})^\perp = \ker(d_0^{A^c})^\perp = \text{Im}(d_0^{A^c})^* \). For fixed \( h \in H_{A^c} \) using (3.3) again we see that the Jacobi matrix of the map

\[
v \mapsto (d_0^{A^c})^*(e^v h \cdot (A^c + \tilde{B}) - A^c) \quad v \in \text{Lie}(H_{A^c})^\perp = \ker(d_0^{A^c})^\perp
\]

at \( v = 0 \) is

\[
- (d_0^{A^c})^* d_0^{h \cdot (A^c + \tilde{B})} \bigg|_{\ker(d_0^{A^c})^\perp} : \ker(d_0^{A^c})^\perp \to \ker(d_0^{A^c})^\perp.
\]

Using this together with (3.8) and (4.12) we integrate out the variable \( v \in \mathcal{N} \subseteq \text{Lie}(H_{A^c})^\perp \) in (4.15) to get

\[
P_{A^c}(A^c + \tilde{B}) = \int_{H_{A^c}} D\!h \det \left( (d_0^{A^c})^* d_0^{h \cdot (A^c + \tilde{B})} \bigg|_{\ker(d_0^{A^c})^\perp} \right)^{-1}.
\]

From (3.2) and (3.3) we see that for \( h \in H_{A^c} \),

\[
(d_0^{A^c})^* d_0^{h \cdot (A^c + \tilde{B})} = (d_0^{A^c})^* (h \cdot) d_0^{A^c + \tilde{B}} (h^{-1} \cdot) = (h \cdot) (d_0^{h^{-1} \cdot A^c})^* d_0^{A^c + \tilde{B}} (h^{-1} \cdot)^{-1}
\]

\[
= (h \cdot) (d_0^{A^c})^* d_0^{A^c + \tilde{B}} (h^{-1} \cdot)^{-1}.
\]

The action of \( h \in H_{A^c} \) on \( \Omega^0(M, g) \) leaves \( \ker(d_0^{A^c})^\perp \) invariant (since the action is by isometries and leaves \( \ker(d_0^{A^c}) = \text{Lie}(H_{A^c}) \) invariant) and it follows from (1.18) that

\[
\det \left( (d_0^{A^c})^* d_0^{h \cdot (A^c + \tilde{B})} \bigg|_{\ker(d_0^{A^c})^\perp} \right) = \det \left( (d_0^{A^c})^* d_0^{A^c} \bigg|_{\ker(d_0^{A^c})^\perp} \right)
\]

independent of \( h \). Substituting this into (1.14) leads to

\[
P_{A^c}(A^c + \tilde{B}) = V(H_{A^c}) \left| \det \left( (d_0^{A^c})^* d_0^{A^c + \tilde{B}} \bigg|_{\ker(d_0^{A^c})^\perp} \right) \right|^{-1}
\]

where \( V(H_{A^c}) \) is the volume\(^6\) of \( H_{A^c} \subseteq \mathcal{G}_0 \). Our calculation above contains an implicit assumption that the map (1.16) is non-degenerate (this is a requirement for

\(^6\)If \( H_{A^c} \) is discrete (i.e. if \( A^c \) is weakly irreducible) then \( V(H_{A^c}) \) is replaced by the number \( |H_{A^c}| \) of elements in \( H_{A^c} \).
using the formula (3.8). This assumption is less crucial than our previous one for the following reason. For small $\alpha$ the functional integral (4.13) is (formally) dominated by the contribution from a neighbourhood of 0 in $\text{Im}(d_0 A^c)^\perp$, and the map (4.16) is non-degenerate for such a neighbourhood (provided that it is sufficiently small) since $(d_0 A^c)^* d_0 A^c$ is non-degenerate on $\text{ker}(d_0 A^c)^\perp$. With this assumption we can state at the formal level that the determinant in (4.20) is non-zero and has the same sign for all $\tilde{B}$. Since $(d_0 A^c)^* d_0 A^c$ is a strictly positive map on $\text{ker}(d_0 A^c)^\perp$ the sign of the determinant is positive and we can discard the numerical signs in (4.20). This leads to the final result:

$$P_{A^c}(A^c + \tilde{B}) = V(H_{A^c}) \det\left( (d_0 A^c)^* d_0 A^c + \tilde{B} \mid_{\text{ker}(d_0 A^c)^\perp} \right)^{-1}. \quad (4.21)$$

This expression differs from the one obtained from the usual evaluation of the Faddeev-Popov functional (which does not take into account the gauge ambiguities coming from $H_{A^c}$): The volume factor $V(H_{A^c})$ appears, and the map $(d_0 A^c)^* d_0 A^c + \tilde{B}$ in the determinant is restricted to $\text{ker}(d_0 A^c)^\perp$. This latter feature is crucial for avoiding infrared divergences in the ghost propagator, as we will see below. The volume factor $V(H_{A^c})$ is crucial for the metric-independence of the overall term multiplying the perturbation series for the partition function in Chern-Simons gauge theory, as we will see in §5 ((5.5) and the subsequent discussion), and for reproducing the large $k$ limits of non-perturbative expressions for the Chern-Simons partition function obtained from the prescription of [3] (cf. the discussion in the conclusion).

Substituting (4.21) into the expression (4.5) for the functional integral leads to a factor $\det\left( (d_0 A^c)^* d_0 A^c \mid_{\text{ker}(d_0 A^c)^\perp} \right)$ in the integrand. Following the Faddeev-Popov procedure we write this determinant as a formal Grassmann integral over independent anticommuting variables (“ghost fields”) $\bar{C}, C \in \text{ker}(d_0 A^c)^\perp$:

$$\det\left( (d_0 A^c)^* d_0 A^c + \tilde{B} \mid_{\text{ker}(d_0 A^c)^\perp} \right) = \int_{\text{ker}(d_0 A^c)^\perp \oplus \text{ker}(d_0 A^c)^\perp} D\bar{C} D C \ e^{-<\bar{C}, (d_0 A^c)^* d_0 A^c + \tilde{B} C >_0}. \quad (4.22)$$

Using (3.4) the term in the exponential in the integrand can be written as

$$< C , (d_0 A^c)^* d_0 A^c + \tilde{B} C >_0 = < C , \Delta_0 A^c C >_0 + < C , (d_0 A^c)^* [\tilde{B} , C ] >_0 \quad (4.23)$$
Substituting (4.21) into the expression (4.5) for the functional integral and writing the determinant as in (4.22)–(4.23) we finally arrive at the gauge-fixed expression for \( I(\alpha; f, S) \):

\[
I(\alpha; f, S)\big|_{\mathcal{A}^c} = V(H_{A^c})^{-1} \text{det} \left( \tilde{\Delta}^{A^c}_0 \right)^{-1/2} \int \text{Im}(d^{A^c}_0)^{\perp} \oplus \ker(d^{A^c}_0)^{\perp} \oplus \ker(d^{A^c}_0)^{\perp} \mathcal{D} \tilde{B} \mathcal{D} C \mathcal{D} \{ f(A^c + \tilde{B}) \times \exp \left( -\frac{1}{\alpha^2} S(A^c + \tilde{B}) - <\tilde{C}, \Delta^{A^c}_0 C >_0 - <\tilde{C}, (d^{A^c}_0)^*[\tilde{B}, C] >_0 \right) \}
\]

(4.24)

It is easy to show (using the \( G_0 \)-invariance of \( f \) and \( S \)) that \( I(\alpha; f, S)\big|_{\mathcal{A}^c} \) is (formally) unchanged when \( A^c \) is replaced by \( \phi \cdot A^c \) for any \( \phi \in G_0 \), i.e. depends only on the orbit \([A^c] = G_0 \cdot A^c\) of \( G_0 \) through \( A^c \) (we leave the verification of this to the reader).

The expression (4.24) can be perturbatively expanded by the approach described in §2, with a straightforward modification to take account of the fact that the integration over the variable \( \tilde{B} \) is restricted to \( \text{Im}(d^{A^c}_0)^{\perp} \) and the fact that there are additional integrations of Grassmannian variables \( \tilde{C} \) and \( C \) over \( \ker(d^{A^c}_0)^{\perp} \). The operator \( \Delta^{A^c}_0 \) in the quadratic term for the ghost variables in the exponential in (4.24) plays an analogous role to \( D_{A^c} \) in the perturbative expansion. From (3.5) we see that \( \text{Im}(d^{A^c}_0) = T_{A^c}(\mathcal{G} A^c) \). It follows from (2.19) that \( \text{Im}(d^{A^c}_0)^{\perp} = T_{A^c}(\mathcal{G} A^c)^{\perp} \supset \ker(D_{A^c})^{\perp} \) is invariant under \( D_{A^c} \), since \( \ker(D_{A^c})^{\perp} \) is invariant under \( D_{A^c} \). We can therefore choose orthonormal bases \( \{B_j\}_{j=0,1,2,...} \) and \( \{C_l\}_{l=0,1,2,...} \) for \( \text{Im}(d^{A^c}_0)^{\perp} \) and \( \ker(d^{A^c}_0)^{\perp} = \ker(\Delta^{A^c}_0)^{\perp} \) respectively, consisting of eigenvectors for \( D_{A^c} \) and \( \Delta^{A^c}_0 \) as follows:

\[
D_{A^c} B_j = \lambda(j) B_j \quad j = 0, 1, 2, \ldots \tag{4.25}
\]

\[
0 \leq |\lambda(0)| \leq \ldots \leq |\lambda(j)| \leq |\lambda(j + 1)| \leq \ldots \to \infty \quad \text{for } j \to \infty \tag{4.26}
\]

\[
\Delta^{A^c}_0 C_l = \mu(l) C_l \quad l = 0, 1, 2, \ldots \tag{4.27}
\]

\[
0 < \mu(0) \leq \ldots \leq \mu(l) \leq \mu(l + 1) \leq \ldots \to \infty \quad \text{for } l \to \infty . \tag{4.28}
\]

(The eigenvectors and eigenvalues above depend of course on \( A^c \) but for the sake of notational simplicity we suppress this in the notation). In the case of Yang-Mills- and
Chern-Simons gauge theories the fact that $D_{A^c}$ and $\Delta_{A^c}$ have discrete spectra, and the properties (4.26) and (4.28) of the eigenvalues, follow via standard mathematical results from the relationship that these operators have to the elliptic complexes (4.42) and (4.48) below. (This result relies on $M$ being compact, riemannian and without boundary). Carrying out the perturbative expansion of (4.24) by the method of §2 leads to

$$I(\alpha; f, S)_{[A^c]} = V(H_{A^c})^{-1} \det \left( \frac{1}{\pi \alpha^2} \tilde{D}_{A^c} \right)^{-1/2} \det (\tilde{\Delta}_0^{A^c})^{1/2} e^{-\frac{1}{\alpha^2} S(A^c)} \times \left[ f(A^c) + \sum_{k=1}^{\infty} \alpha^k \left( \sum_{N=1}^{k} \sum_{j_1, \ldots, j_N} f_{A^c}^{j_1 \cdots j_N} G_{A^c}^{(N,k-N)}(j_1, \ldots, j_N) \right) \right]$$

(4.29)

where $\alpha^k G_{A^c}^{(N,k)}(j_1, \ldots, j_N)$ is obtained from Feynman diagrams as in §2. There are now additional building blocks for the diagrams due to the additional integrations over the Grassmannian variables $\bar{C}$ and $C$ in (4.24). The building blocks for the diagrams in this case, and the factors which each of these contribute, are as follows:

where $\lambda(j)$ and $\mu(l)$ are as in (4.25) and (4.27) respectively, $S_A^{i_1 \cdots i_p} = S_A^{(p)}(B_{i_1}, \ldots, B_{i_p})$ as in §2, and the factor for the new vertex is $\alpha^2 S_A^{i_1 \cdots i_p} = \alpha < C_{l_1}, (d^A_0)^* [B_{i}, C_{l_2}] > 0$.

Each diagram for $\alpha^k G_{A^c}^{(N,k)}(j_1, \ldots, j_N)$ has $N$ external unoriented lines (“gauge lines”) as before, however the diagrams may now contain internal oriented lines (“ghost lines”) and the “gauge-ghost” vertex in addition to the unoriented gauge lines and gauge vertices. All closed loops formed by the ghost lines must be oriented. The
only change in the rules stated in §2 for obtaining the expression corresponding to
a given diagram is that a factor \(-1\) must be included for each closed loop formed
by the ghost lines. To explicitly determine the propagators and vertex factors for
the Feynman diagrams the eigenvectors and eigenvalues of \(D_{A^c}\) and \(\Delta_0^{A^c}\) must be
determined. This is a non-trivial problem in general; however in Chern-Simons gauge
theory with spacetime \(S^3\) or a lens space techniques already exist for determining
these as we discuss in the conclusion.

The situation with regard to infrared divergences in the expressions for the Feyn-
man diagrams is as follows. Since \(\Delta_0^{A^c}\) has no zero-modes in \(\ker(\Delta_0^{A^c})^\perp\) the propagator
\(\frac{1}{\mu(l)}\) for the ghost lines is finite for all \(l = 0, 1, 2, \ldots\) (cf. (4.26)) and does not give
rise to infrared divergences. For the gauge line propagator \(\frac{1}{\lambda(j)}\) to be finite for all
\(j = 0, 1, 2, \ldots\) the operator \(D_{A^c}\) must have no zero-modes in \(\text{Im}(d_0^{A^c})^\perp\) (cf. (4.25)).
Since \(\text{Im}(d_0^{A^c}) = T_{A^c}(G \cdot A^c)\) we see from (2.18) that this is the case precisely when
\(A^c\) is isolated modulo gauge transformations. Therefore, when \(A^c\) is isolated mod-
ulo gauge transformations the expressions for the Feynman diagrams are completely
free of infrared divergences. In this case the operators \(\tilde{D}_{A^c}\) and \(\tilde{\Delta}_0^{A^c}\) in (4.29) are
the restrictions of \(D_{A^c}\) and \(\Delta_0^{A^c}\) to invertible maps on \(\ker(D_{A^c})^\perp\) and \(\ker(\Delta_0^{A^c})^\perp\) re-
spectively. We show below that for Yang-Mills- and Chern-Simons gauge theories
the determinants in (4.29) can be given well-defined meaning via zeta-regularisation
-this enables the \(\alpha\)-dependence of \(\det\left(\frac{1}{\pi \alpha^2} \tilde{D}_{A^c}\right)\) to be extracted. A straightforward
consequence of the \(G_0\)-invariance of \(S\) and \(f\) is that the expressions for the Feynman
diagrams depend only on the orbit \([A^c]\) of \(G_0\) through \(A^c\). We omit the details, except
to note that as a consequence of the \(G_0\)-invariance of \(S\) the operator \(D_{A^c}\) has the
covariance property

\[
D_{\phi, A^c}(\phi \cdot B) = \phi \cdot (D_{A^c} B) \quad \forall B \in \Omega^1(M, g), \quad \phi \in G_0
\]

and the operator \(\Delta_0^{A^c}\) has the same covariance property due to (3.3)-(3.4).

When \(A^c\) is not isolated modulo gauge transformations the gauge propagator \(\frac{1}{\lambda(j)}\)
diverges for sufficiently small \(j\) and infrared divergences are present. It would be
very desirable to have a method for perturbative expansion which is also infrared-
finite when $A^c$ is not isolated modulo gauge transformations, particularly for Yang-Mills gauge theory where the instanton modulispaces have non-zero dimension in general. We now give a brief, rough sketch of how it may be possible to achieve this using a version of the preceding approach to perturbative expansion. For small $\alpha$ the functional integral $I(\alpha; f, S)$ is (formally) dominated by the contributions from neighbourhoods of the critical points for $S$. We can therefore approximate $I(\alpha; f, S)$ for small $\alpha$ by

$$I(\alpha; f, S)_{N(C)} = \frac{1}{V(G_0)} \int_{N(C)} DA f(A)e^{-\frac{1}{\alpha^2}S(A)}$$  \hspace{1cm} (4.31)$$

where $N(C) \subseteq A$ is a thin $G_0$-invariant neighbourhood of the space $C$ of critical points for $S$. In fact it is conceivable that the perturbative expansion of

$$\frac{1}{V(G_0)} \int_{A-N(C)} DA f(A)e^{-\frac{1}{\alpha^2}S(A)}$$  \hspace{1cm} (4.32)$$

vanishes (at the formal level); this is claimed (without argument) in [17, p.2] in the context of Chern-Simons gauge theory. If this is the case then a perturbative expansion of $I(\alpha; f, S)$ is obtained by perturbatively expanding (4.31). A perturbative expansion of (4.31) can be obtained using the techniques developed in the preceding. We find

$$I(\alpha; f, S)_{N(C)} = \int_{C/G_0} \mathcal{D}[A^c] \tilde{I}(\alpha; r_C f, S)_{[A^c]}$$  \hspace{1cm} (4.33)$$

where $\tilde{I}(\alpha; r_C f, S)_{[A^c]}$ is given by (4.24) with $f$ replaced by $r_C f$ and $\text{Im}(d_{A^c}^\perp)$ replaced by $\ker(D_{A^c})^\perp$ in the integration (so the gauge propagator no longer gives rise to infrared divergences in the perturbative expansion). Here $r_C$ is a “measure function” which we are not able to determine in general. If the geometry of $C$ (induced by the metric in $A$) happens to be flat in the directions orthogonal to the orbits of $G_0$ in $C$ (or equivalently, if the geometry of $C/G_0$ is flat) then $r_C = 1$. In general we can only say that $r_C(A^c) = 1$ for all $A^c \in C$. In arriving at (4.33) we have used a formal generalisation of the following observation: For $a, b, c > 0$ the asymptotics (i.e. Taylor expansion) of $\int_a^\infty \frac{x^b}{\alpha^c}e^{-\frac{1}{\alpha^2}x^2}dx$ for $\alpha \to 0$ vanishes since

$$\frac{d^p}{d\alpha^p} \bigg|_{\alpha=0} \int_a^\infty \frac{x^b}{\alpha^e}e^{-\frac{1}{\alpha^2}x^2}dx = 0 \quad \text{for all} \quad p = 0, 1, 2, \ldots$$  \hspace{1cm} (4.34)$$
This is easily shown using the fact that $\frac{1}{y}e^{-\lambda y^2} \to 0$ for $y \to 0$ for all $a, \lambda > 0$. (It seems plausible that (4.34) might also be used to show that the asymptotics of (4.32) vanish).

The perturbative expansion of (4.31) is obtained by substituting (4.29) in (4.33) with $f$ replaced by $r_C \cdot f$. The higher order terms in the expansion are undetermined since we have not been able to determine $r_C$ in general. However, since $r_C(A^c) = 1$ for $A^c \in C$ we obtain an expression for the lowest order term in the expansion:

$$\int_{C/\mathcal{G}_0} \mathcal{D}[A^c] V(H_{A^c})^{-1} \det \left( \frac{1}{\pi \alpha^2} \tilde{D}_{A^c} \right)^{-1/2} \det (\tilde{\Delta}_0^{A^c})^{1/2} f(A^c) e^{-\frac{1}{\pi \alpha} S(A^c)}$$

(4.35)

This reproduces the formula [12, App. II (9)] for the weak coupling limit of $I(\alpha; f, S)$.

We conclude this section by considering two specific gauge theories, the Yang-Mills- and Chern-Simons theories, giving expressions for the vertex factors in the Feynman diagrams for these theories and showing that the determinants in (4.33) can be zeta-regularised. The action functional for Yang-Mills gauge theory on 4-dimensional $M$ is

$$S_{YM}(A) = \frac{1}{2} < F^A, F^A >_2$$

(4.36)

In this case we take $C$ to be the set of absolute minima for $S_{YM}$ (rather than the complete set of critical points for $S_{YM}$). The topological number (2nd Chern character) of the principal fibre bundle $P$ is $Q_P = \frac{1}{16 \pi^2} < F^A, \ast F^A >$ (independent of $A \in \mathcal{A}$); we can assume without loss of generality that $Q_P \geq 0$ since $Q_P$ changes sign when the orientation of $M$ is reversed. A standard calculation gives

$$S_{YM}(A) = 8\pi^2 Q_P + < \pi_- F^A, \pi_- F^A >_2$$

(4.37)

where $\pi_-=\frac{1}{2}(1-\ast)$ on $\Omega^2(M,\mathfrak{g})$, showing that $C$ is the space of instantons on $P$, i.e. the solutions to $\pi_- F^{A^c} = 0$. A straightforward calculation using (4.37) and (3.4) shows that for $A^c \in C$,

$$S_{YM}(A^c + B) = 8\pi^2 Q_P + < B, (\pi_- d_1^{A^c})^\ast \pi_- d_1^{A^c} B >_1$$

$$+ < \pi_- d_1^{A^c} B, \pi_- [B, B] >_2 + \frac{1}{4} < \pi_- [B, B], \pi_- [B, B] >_2$$

(4.38)
This shows that $S_{YM}$ satisfies the condition (2.1) with $s = 4$ and

$$D_{Ac} = (\pi_- d_{1}^{Ac})^* \pi_- d_{1}^{Ac}$$

(4.39)

$$S_{Ac}^{(3)}(B_1, B_2, B_3) = <\pi_- d_{1}^{Ac} B_1, \pi_- [B_2, B_3] >_2$$

(4.40)

$$S_{Ac}^{(4)}(B_1, B_2, B_3, B_4) = \frac{1}{4} <\pi_- [B_1, B_2], \pi_- [B_3, B_4] >_2$$

(4.41)

The gauge vertex factors $\alpha S_{Ac}^{(ijj\bar{i}i)}$ and $\alpha^2 S_{Ac}^{(iijj\bar{i}i)}$ for the Feynman diagrams are obtained from (4.40)–(4.41) as described in §2 (below (2.7)). The zeta-regularisability of the determinants in (4.35) follows in this case from the relationships of the operators $D_{Ac}$ and $\Delta_{0}^{Ac}$ to the operators appearing in the elliptic self-dual complex

$$0 \rightarrow \Omega^0(M, g) \overset{d_{0}^{Ac}}{\rightarrow} \Omega^1(M, g) \overset{\pi_{-} d_{1}^{Ac}}{\rightarrow} \Omega^2(M, g) \rightarrow 0$$

(4.42)

where $\Omega^2(M, g) = \pi_{-}(\Omega^2(M, g))$. An argument analogous to the one given in [30] shows that the zeta-regularisations of the determinants in (4.35) are well-defined and lead to

$$\text{det} \left( \frac{1}{\pi \alpha^2} D_{Ac} \right)^{-1/2} \sim |\alpha|^\zeta(D_{Ac})$$

(4.43)

where $\zeta(D_{Ac})$ is the analytic continuation to 0 of the zeta-function for $D_{Ac}$.

The action functional for Chern-Simons gauge theory on 3-dimensional $M$ is

$$-iS_{CS}(A) = -i \frac{1}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

(4.44)

with the trace taken in the fundamental representation. The natural parameter for this theory is $k = \frac{1}{\alpha^2}$. The parameter $k$ is usually required to be integer-valued, since it is only then that $\exp(ikS_{CS}(A))$ is gauge-invariant. However, $S_{CS}(A)$ is invariant under the subgroup $G_0$, and since this is all that is required in our method for perturbative expansion $k$ may take arbitrary real values. The set $C$ of critical points for $iS_{CS}$ consists of the flat gauge fields on $P$, i.e. the solutions to $F_{Ac} = 0$.

\footnote{We assume for simplicity here that the principal fibre bundle $P$ is trivial so that the gauge fields can be identified with the $g$-valued 1-forms on $M$. This is always the case when $G = SU(2)$; see [31] for the general case}
For $A^c \in C$ a simple calculation gives

$$-iS_{CS}(A^c + B) = -iS_{CS}(A^c) + <B, i\left(\frac{1}{4\pi \lambda_g} * d^A_1\right)B >_1 - i\frac{1}{4\pi} \int_M \text{Tr}(B \wedge B \wedge B)$$

(4.45)

where $*$ is the Hodge star operator and the invariant inner product in $g$ used in constructing the inner product $< \cdot, \cdot >_1$ in $\Omega^1(M, g)$ is taken to be $< a, b >_g = -\lambda_g \text{Tr}(ab)$ with $\lambda_g > 0$ an arbitrary scaling parameter. This shows that $-iS_{CS}$ satisfies the condition (2.1) with $s = 3$ and

$$D_{A^c} = \frac{1}{4\pi \lambda_g} * d^A_1$$

(4.46)

$$-iS_{A^c}^{(3)}(B_1, B_2, B_3) = -i\frac{1}{4\pi} \int_M \text{Tr}(B_1 \wedge B_2 \wedge B_3).$$

(4.47)

In this case there is one gauge vertex with factor $-i\frac{1}{\sqrt{k}} S_{A^c}^{i_1 i_2 i_3}$, obtained from (4.47) as described below (2.7) in §2. The zeta-regularisability of the determinants in (4.33) also follows in this case from the relationships of the operators $D_{A^c}$ and $\Delta_{0A^c}$ to an elliptic complex, namely the twisted de Rham complex

$$0 \longrightarrow \Omega^0(M, g) \overset{d^A_0}{\longrightarrow} \Omega^1(M, g) \overset{d^A_1}{\longrightarrow} \Omega^2(M, g) \overset{d^A_2}{\longrightarrow} \Omega^3(M, g) \longrightarrow 0$$

(4.48)

In [30] it was shown that the zeta-regularisations of the determinants in (4.33) are well-defined and, setting $k = \frac{1}{\alpha^2}$,

$$\det\left(\frac{k}{\pi} iD_{A^c}\right)^{-1/2} \sim k^{(- \dim H^0(d^{A^c}) + \dim H^1(d^{A^c}))/2}$$

(4.49)

where $H^q(d^{A^c})$ is the $q$'th cohomology space of (4.48). (A more explicit expression is given in (5.5) in the following section).

Note that in these examples the requirement $\text{Im}(d^A_0) = \ker(D_{A^c})$ for the absence of infrared divergences in the gauge propagator is equivalent to the vanishing of the 1st cohomology space $H^1(A^c)$ for the complex (4.42) or (4.48).

Finally, an example of functional $f$ satisfying the condition (2.1) is the Wilson loop functional,

$$f_{(\gamma, \rho)}(A) = \text{Tr} \left( \mathcal{P} \exp \left( \oint_{\gamma} \rho(A) \right) \right)$$

(4.50)
where $\gamma$ is a closed curve in $M$, $\rho$ is a representation of the gauge group $G$ and $P$ denotes path-ordering. (I.e. $f(\gamma,\rho)(A)$ is the trace of the holonomy of $A$ around $\gamma$ in the representation $\rho$). Setting $A = A^c + B$ in (4.50) and expanding the exponential as a power series it is easy to see that (2.1) is satisfied for arbitrary $A^c$ and $B$.

5 Perturbative expansion in Chern-Simons gauge theory

In this section we specialise to Chern-Simons gauge theory on 3-dimensional $M$. We begin by pointing out that the approach to perturbative expansion given in the preceding coincides with the superfield approach of Axelrod and Singer in [6] when their condition ($A^c$ acyclic) is satisfied. We then go on to show that the perturbative expansion of the partition function is formally metric-independent when $A^c$ is isolated modulo gauge transformations. This was shown by Axelrod and Singer in [6, §5] in the case where $A^c$ is acyclic; however in our more general case new features arise and to deal with these we derive new properties of the superfield propagator. Throughout this section $A^c$ is an arbitrary flat gauge field which is isolated modulo gauge transformations (unless otherwise stated).

Axelrod and Singer used the BRS version of the usual Faddeev-Popov gauge-fixing procedure to derive a gauge-fixed expression for the Chern-Simons partition function as a functional integral over a superfield. This expression can be reproduced in our approach: We change variables in (4.22) from $\bar{C} \in \ker(d_{A^c}^0)^\perp$ to $\bar{C}' = 8\pi k^{-1}\lambda_g*d_{A^c}^0C \in \ker(d_{A^c}^2)^\perp$ and define the superfield variable

$$\hat{A} = C + \tilde{B} + \bar{C}' \in \ker(d_{A^c}^0)^\perp \oplus \ker(d_{A^c}^1)^\perp \oplus \ker(d_{A^c}^2)^\perp = \ker(d_{A^c}^c)^\perp$$

where $d_{A^c}^c$ denotes the covariant derivative on $\Omega(M, g) = \bigoplus_{q=0}^3 \Omega^q(M, g)$ . Substituting the resulting expression for (4.22) in (4.24) a straightforward calculation gives the following expression for the gauge-fixed partition function:

$$I(\frac{1}{\sqrt{k}}; 1, -iS_{CS})_{[A^c]} = V(H_{A^c})^{-1} \det(\bar{\Delta}_0^{A^c})^{-1/2} \det(8\pi k^{-1}\lambda_g(\bar{\Delta}_0^{A^c})^{-1/2})e^{\frac{i}{4}S_{CS}(A^c)}$$
The functional integral in (5.1), from which the higher order terms in the perturbative expansion of the partition function are obtained, is the gauge-fixed expression obtained in [6, (2.17)–(2.19)] with the requirement that $A^c$ is acyclic. This requirement is the same as requiring $A^c$ to be isolated modulo gauge transformations and weakly irreducible. The weak irreducibility means that the isotropy subgroup $H_{A^c}$ is discrete, i.e. its Lie algebra $\text{Lie}(H_{A^c}) = \ker(d_{A^c}^0)$ vanishes. In our approach, using the refined version of the Faddeev-Popov procedure given in §4, the only requirement is that $A^c$ isolated modulo gauge transformations -in Chern-Simons gauge theory this is equivalent to requiring $\text{Im}(d_{A^c}^0) = \ker(d_{A^c}^1)$. Thus we see that our approach to gauge-fixing is equivalent to that of [6] when their condition is satisfied, and extends their approach to the case where $\ker(d_{A^c}^0)$ is non-zero. For a number of simple 3-manifolds, e.g. $S^3$ and the lens spaces, all the flat gauge fields $A^c$ have non-zero $\ker(d_{A^c}^0)$ but satisfy our requirement $\text{Im}(d_{A^c}^0) = \ker(d_{A^c}^1)$.

The remainder of this section is devoted to showing that the terms in the perturbative expansion of the Chern-Simons partition function are formally metric-independent in our setting (i.e. with $A^c$ isolated modulo gauge transformations). We will work with the expressions for these terms derived in [6]. (The derivation of these expressions from (5.1) goes through for arbitrary flat $A^c$). We begin by introducing the ingredients in these expressions and the notations which are required to formulate them (see [6, §3] for more details). In the formulation given in [6] the propagator for the superfield $\hat{A}$ is taken to be a differential form $L^{A^c}(x,y)$ on $M \times M$ with values in $g \otimes g^*$ defined as follows. The operator $d^{A^c}$ appearing in the quadratic term in the exponential in (5.1) restrict to an invertible map $d^{A^c} : \ker(d^{A^c}) \perp \rightarrow \text{Im}(d^{A^c})$. The propagator $L^{A^c}(x,y)$ is taken to be the differential form version of the kernel-function for the operator $\hat{L}^{A^c} : \Omega(M, g) \rightarrow \Omega(M, g)$ defined by $\hat{L}^{A^c} = (d^{A^c})^{-1}$ on $\text{Im}(d^{A^c})$ and $\hat{L}^{A^c} = 0$ on $\text{Im}(d^{A^c}) \perp$. More precisely, let $\{e^a\}_{a=1,\ldots,\dim g}$ be an orthonor-

\[^8\text{No expression was given in [6] for the overall factor multiplying the functional integral in (5.1) or in the perturbative expansion (5.4) below.}\]
mal basis for \( g \), then using the inner product in \( g \) to identify \( g^* \) with \( g \) we have

\[ L^A(x, y) = L^A_{ab}(x, y) \rho^a \otimes \rho^b \]

given by

\[ (L^A \psi)(x) = \int_{M_y} L^A_{ab}(x, y) \wedge \psi^b(y) \quad , \quad \psi = \psi^b \rho^b \in \Omega(M, g) \quad (5.2) \]

where repeated indices are summed over. Here and in the following \( M_x \) denotes a copy of \( M \) parameterised by a variable \( x \), so for example we have \( L^A(x, y) \in \Omega^2(M_x \times M_y; g \otimes g) \). The propagator \( L^A(x, y) \) diverges at the diagonal \( x = y \) in \( M_x \times M_y \) but is smooth away from the diagonal (see \([3] \) p.17–18).

We will be using the following general notations introduced in \([3] \), §3: Each element \( Q(x, y) = Q_{ab}(x, y) \rho^a \otimes \rho^b \) in \( \Omega(M_x \times M_y; g \otimes g) \) corresponds to an element \( Q_{ab}(x, y) \wedge \rho^a(d) \wedge \rho^b(d) \) in \( \Gamma(M_x \times M_y; \Lambda((T^*M_x \oplus g_x) \oplus (T^*M_y \oplus g_y))) \) where \( g_x \) and \( g_y \) are distinct copies of \( g \); this in turn determines an element

\[ Q_{tot}(x_1, \ldots, x_V) \in \Gamma(M_{x_1} \times \cdots \times M_{x_V}; \Lambda(\oplus_{i=1}^V (T^*M_{x_i} \oplus g_i))) \]

defined by

\[ Q_{tot}(x_1, \ldots, x_V) = \sum_{i,j=1}^V Q_{ab}(x_i, x_j) \rho^a(d) \rho^b(d) \quad (5.3) \]

(Here and in the following we will often omit the wedge symbol in wedge multiplication for notational convenience). The perturbative expansion of \((5.1)\) derived in \([3] \) has the form

\[ I(\frac{1}{\sqrt{k}}, 1, -iS_{CS})[A^c] = Z_{sc}(k, A^c) \sum_{V=0,2,4, \ldots} \left( \frac{1}{\sqrt{k}} \right)^V I_V(A^c) \quad (5.4) \]

(see \([3], (3.54)\)). The overall factor \( Z_{sc}(k, A^c) \) multiplying the series is the overall factor in \((4.22)\) with \( D_{A^c} \) given by \((4.40)\) (this expression can also easily be obtained from \((5.1)\)); the notation reflects the fact that this factor is the contribution from \( A^c \) to the semiclassical approximation. Using the techniques of \([30], [6], \S 4.1\) we find

\[ Z_{sc}(k, A^c) = e^{-\frac{i\pi}{16} \eta(d_{A^c})} \left( \frac{4\pi \lambda g}{k} \right)^{\text{dim}H^0(d_{A^c})/2} V_{\lambda g}(\hat{H}_{A^c})^{-1} V(M)^{-\text{dim}H^0(A^c)/2} \]

\[ \times \tau(A^c)^{1/2} e^{\frac{i\pi}{16} S_{CS}(A^c)} \quad (5.5) \]
where $\eta(\ast d_0 A^c)$ is the analytic continuation to 0 of the eta-function of $\ast d_0 A^c$, $\tau(A^c)$ is the Ray-Singer torsion of $A^c$ and we have used the fact that $H_{A^c}$ can be identified with an invariant subgroup $\tilde{H}_{A^c}$ of the gauge group $G$, from which it follows that $V(H_{A^c}) = V(M)^{\dim H^K(A^c)/2} V_{\lambda_0}(\tilde{H}_{A^c})$ where $V_{\lambda_0}(\tilde{H}_{A^c})$ is the volume of $\tilde{H}_{A^c}$ determined by the inner product in $g$. The product $V(M)^{-\dim H^K(A^c)} \tau(A^c)$ is metric-independent (a proof of this is given in [16, §4.1]) so the only metric dependence of $Z_{sc}$ enters through the phase factor in (5.5). ($Z_{sc}$ can be made completely metric-independent by putting in by hand a phase factor with phase given by Witten’s geometric counterterm [3, §2]).

The coefficients $I_V(A^c)$ in the expression (5.4) are given by [4, (3.54)] to be

$$I_V(A^c) = c_V \prod_{i=1}^V \left[ \int_{M_{x_1}} f_a^{ibc} \frac{\partial}{\partial \rho_{(i)}^a} \frac{\partial}{\partial \rho_{(i)}^{b'}} \frac{\partial}{\partial \rho_{(i)}^{c'}} \right] L^A_{tot}(x_1, \ldots, x_V)^{\frac{1}{2V}}$$  (5.6)

where $c_V = (2\pi i)^{\frac{1}{2V}} ((3!)^V (2!)^V (\frac{1}{2!V})^{-1}$, $\{f_{abc}\}$ are the structure constants of $g$ given by $[\rho^a, \rho^b] = f_{abc} \rho^c$, $\frac{\partial}{\partial \rho_{(i)}^a}$ is interior multiplication by $\rho_{(i)}^a$ and $L^A_{tot}(x_1, \ldots, x_V)$ is defined as in (5.3). We choose the $\{\rho^a\}$ such that $f_{abc}$ is totally antisymmetric. (The coefficient $I_V(A^c)$ can be interpreted as the contribution to the perturbative expansion coming from all Feynman diagrams with $V$ vertices; see [3, p.22–24] for the details).

The propagator $L^A_{ab}(x, y)$ can be expressed in terms of the eigenvectors and (non-zero) eigenvalues for $D_{A^c} = \frac{1}{4\pi \lambda_0} \ast d_0^{A^c}$ and $\Delta_0^{A^c}$ in (1.25) and (1.27):

$$L^A_{ab}(x, y) = -\frac{1}{4\pi \lambda_0} \sum_j \frac{1}{\lambda(j)} B_j^a(x) \wedge B_j^b(y)$$
$$+ \sum_l \frac{1}{\mu(l)} \left( C_l^a(x) \wedge (\ast d_0^{A^c} C_l^b)(y) - (\ast d_0^{A^c} C_l^a)(x) \wedge C_l^b(y) \right)$$  (5.7)

where we have followed the convention of [3, (3.53)]. Substituting the expression (5.7) for the propagator into (5.6) it is straightforward to verify that the perturbative expansion (5.4) for the partition function coincides with the one obtained from our “generalised momentum space” formulation in §5; this is a bit tedious though so we omit the details.
As it stands the expression (5.6) for $I_V$ is a formal expression. Axelrod and Singer showed in [3, §3–§4] how it can be given well-defined finite meaning, as we now discuss. The integrand in (5.6) is not well-defined apriori: $L_{ab}^{Ax}(x, y)$ diverges on the diagonal $x = y$ so the terms $L_{ab}^{Ax}(x_i, x_i)\rho_a^{(i)}\rho_b^{(i)}$ in $L_{tot}^{Ax}(x_1, \ldots, x_V)$ are not well-defined. However, as pointed out in [3, p.17–18 and p.20] the propagator can be written as a sum of the form

$$L_{ab}^{Ax}(x, y) = L_{ab}^{Ax}(x, y)_{\text{div}}\delta_{ab} + L_{ab}^{Ax}(x, y)_{\text{cont}}$$

(5.8)

where $L_{ab}^{Ax}(x, y)_{\text{div}}$ diverges on the diagonal $x = y$ and $L_{ab}^{Ax}(x, y)_{\text{cont}}$ is continuous across the diagonal. Since $\delta_{ab}\rho^a \wedge \rho^b = 0$ we have $L_{ab}^{Ax}(x, y)\rho^a\rho^b = L_{ab}^{Ax}(x, y)_{\text{cont}}\rho^a\rho^b$ for $x \neq y$, which extends continuously across the diagonal $x = y$. Thus a well-defined expression for $L_{tot}^{Ax}(x_1, \ldots, x_V)$ is obtained in a natural way by replacing $L_{ab}^{Ax}(x_i, x_i)\rho_a^{(i)}\rho_b^{(i)}$ by $L_{ab}^{Ax}(x, y)_{\text{cont}}\rho_a^{(i)}\rho_b^{(i)}$ for all $i = 1, \ldots, V$. In physics terminology this can be interpreted as a point-splitting regularisation. With this regularisation Axelrod and Singer showed that each $I_V$ in the perturbative expansion (5.4) is finite [3, theorem 4.2]. (As pointed out in [3, §6, remark II(i)] the argument for this does not require any particular conditions on the flat gauge field $A^c$). This remarkable result shows that with regard to perturbative expansion Chern-Simons gauge theory on compact 3-manifold is very different from the usual quantum field theories in that no renormalisation procedure is required to obtain finite expressions for the terms in the expansions. However, whether or not the perturbation series in (5.4) converges is a completely different question which as far as we know has yet to be answered.

The definition of $L_{ab}^{Ax}(x, y)$ requires a choice of metric $g$ on $M$ so the perturbative expansion (5.4) is apriori metric-dependent. We noted in §5 that the overall factor $Z_{sc}(k, A^c)$ given by (5.5) is metric-independent (provided that a phase factor is put in by hand with phase given by Witten’s geometric counterterm). Thus any metric-dependence of (5.4) is contained in the coefficients $I_V(A^c)$ of the expansion.

We now establish the properties of the propagator $L_{ab}^{Ax}(x, y)$ which we will need to show the formal metric-independence of $I_V(A^c)$. The first of these is

$$L_{ba}^{Ax}(y, x) = -L_{ab}^{Ax}(x, y)$$

(5.9)
This is the property (PL3) in [3, §3]; it can be derived for example from the expression (3.7) above. We define the spaces $\mathcal{H}^A_c \subset \Omega^q(M, g)$ by the orthogonal decompositions $\ker(d_q^A) = \text{Im}(d_q^A) \oplus \mathcal{H}_q^A$ then the Hodge decomposition states

$$\Omega(M, g) = \text{Im}((d^A)^*) \oplus \text{Im}(d^A) \oplus \mathcal{H}^A$$

(5.10)

where $\mathcal{H}^A = \oplus_{q=0}^3 \mathcal{H}_q^A$. Let $\pi_{d^A}$, $\pi_{(d^A)^*}$, and $\pi_{\mathcal{H}^A}$ denote the orthogonal projections of $\Omega(M, g)$ onto $\text{Im}(d^A)$, $\text{Im}((d^A)^*)$ and $\mathcal{H}^A$ respectively. Noting that $\text{Im}((d^A)^*) = \ker(d^A)^\perp$ it follows from the definitions that

$$d^A \hat{L}^A = \pi_{d^A} \quad , \quad \hat{L}^A d^A = \pi_{(d^A)^*}$$

(5.11)

(as in [3, (2.22)]). Let $d^A_{M_x \times M_y}$ denote the covariant derivative on $\Omega(M_x \times M_y; g \otimes g)$ determined by the flat gauge field $(A^c, A^c)$ on $M_x \times M_y$, then a straightforward calculation using (5.10) and (5.11) gives

$$d^A_{M_x \times M_y} L^A_{ab}(x, y) = (d^A_{M_x} + d^A_{M_y}) L^A_{ab}(x, y) = -(\delta_{ab} \delta(x, y) - \pi_{ab} A^c(x, y))$$

(5.12)

where $\delta(x, y) \in \Omega^3(M_x, M_y)$ is the differential form version of the kernel-function for the identity map on $\Omega(M)$ (as defined in [3, (3.44)]) and $\pi_{ab} A^c(x, y) \in \Omega^3(M_x \times M_y)$ is the differential form version of the kernel-function for $\pi_{\mathcal{H}^A}$. (Here and in the following we are using the convention defined in [3, (3.53)]). When $A^c$ is acyclic $\mathcal{H}^A = 0$ and (5.12) reduces to the property (PL1) stated in [3, §3]. We denote the variation of $L^A_{ab}(x, y)$ and $\pi_{ab} A^c(x, y)$ under a variation $\delta g$ of the metric $g$ by $\delta_{\delta g} L^A_{ab}(x, y)$ and $\delta_{\delta g} \pi_{ab} A^c(x, y)$. From (5.12), using the fact that $\delta_{ab} \delta(x, y)$ is metric-independent we obtain

$$d^A_{M_x \times M_y} (\delta_{\delta g} L^A_{ab}(x, y)) = \delta_{\delta g} \pi_{ab} A^c(x, y)$$

(5.13)

In the case which we are considering, i.e. where $A^c$ is isolated modulo gauge transformations, we have $\mathcal{H}_0^A = \ker(d_0^A)$, $\mathcal{H}_1^A = \mathcal{H}_2^A = 0$ and $\mathcal{H}_3^A = *\mathcal{H}_0^A = *\ker(d_0^A)$ so $\mathcal{H}^A = \ker(d_0^A) \oplus *\ker(d_0^A)$. From now on we omit $A^c$ from the notation for the sake of notational simplicity, setting $d_q = d_q^A$, $L_{ab}(x, y) = L^A_{ab}(x, y)$ and $\pi_{ab}(x, y) = \pi_{ab}^A(x, y)$. The space $\ker(d_0)$ is independent of metric on $M$ and we can choose
a basis \( \{ h_i = h_i^a \rho^a \}_{i=1, \ldots, \dim \ker(d_0)} \) for \( \ker(d_0) \), independent of metric, such that

\[
<h_i, h_j> \equiv \delta_{ij} \text{ for all } x \in M.
\]

Using this basis we can write \( \pi_{ab}(x, y) \) as

\[
\pi_{ab}(x, y) = h^a_i(x) h^b_i(y) V_g(M)^{-1}(vol_g(y) - vol_g(x))
\]

where \( V_g(M) \) and \( vol_g \) are the volume and volume form of \( M \), determined by the metric \( g \) and orientation of \( M \). In (5.14) \( vol_g(x) \) and \( vol_g(y) \) are the volume forms on \( M_x \) and \( M_y \) respectively, considered as elements in \( \Omega^3(M_x \times M_y) \).

Using this we can write

\[
L_{ab}(x, y) = L_{ab}^{(0,2)}(x, y) + L_{ab}^{(1,1)}(x, y) + L_{ab}^{(2,0)}(x, y) \tag{5.15}
\]

\[
\pi_{ab}(x, y) = \pi_{ab}^{(0,3)}(x, y) + \pi_{ab}^{(3,0)}(x, y) \tag{5.16}
\]

where \( L_{ab}^{(p,q)}(x, y) \in \Omega^{(p,q)}(M_x \times M_y) \) etc. By substituting (5.15) and (5.16) into (5.13) we see that

\[
d_{M_x \times M_y} (\delta g L_{ab}^{(1,1)}(x, y)) = 0 \tag{5.18}
\]

\[
d_{M_x} (\delta g L_{ab}^{(0,2)}(x, y)) = 0 \quad , \quad d_{M_y} (\delta g L_{ab}^{(2,0)}(x, y)) = 0 \tag{5.19}
\]

Our condition on \( A^c \) implies that the cohomology spaces \( H^1(d) \) and \( H^2(d) \) vanish and it follows from the Künneth formula that \( H^2(d_{M_x \times M_y}) = 0 \). It then follows from (5.18) and (5.9) that

\[
\delta g L_{ab}^{(1,1)}(x, y) = d_{M_x \times M_y} B_{ab}(x, y) \tag{5.20}
\]

for some \( B(x, y) \in \Omega^1(M_x \times M_y; g \otimes g) \) of the form

\[
B_{ab}(x, y) = B_{ab}^{(0,1)}(x, y) - B_{ba}^{(0,1)}(y, x) \tag{5.21}
\]

where \( B_{ab}^{(0,1)}(x, y) \in \Omega^{(0,1)}(M_x \times M_y) \) satisfies \( d_{M_y} B_{ab}^{(0,1)}(x, y) = 0 \). In our metric-independence argument below the properties (5.19)–(5.21) of the propagator replace the key property (PL4) in [3, §3].
The expression (5.6) for $I_V(A^c)$ can be written compactly as in [6, (3.55)]:

$$I_V(A^c) = c_V \int_{M^V} \text{TR}(L_{tot}(x_1, \ldots, x_V)^{2V})$$  
(5.22)

(recall from (5.4) that $V$ is even) where $M^V = M_{x_1} \times \cdots \times M_{x_V}$ and TR is a linear operator mapping $L_{tot}(x_1, \ldots, x_V)^{2V}$ to a differential form of top degree in $\Omega(M^V)$. (TR is defined in [6, p.21]; it can be interpreted as a generalised trace). Generalising the calculation in [6, (5.83)] we obtain the following expression for the variation of $I_V(A^c)$ under a variation $\delta g$ of the metric:

$$\delta g I_V(A^c)$$
$$= \frac{3}{2} V c_V \int_{M^V} \text{TR}((\delta g L_{tot})(L_{tot})^{2V-1})$$
$$= \frac{3}{2} V c_V \left[ \int_{M^V} \text{TR}((\delta g L_{tot}^{(1,1)})(L_{tot})^{2V-1}) + 2 \int_{M^V} \text{TR}((\delta g L_{tot}^{(0,2)})(L_{tot})^{2V-1}) \right]$$
$$= 3 V c_V \left[ \int_{M^V} \text{TR}((d_M B_{tot}^{(0,1)})(L_{tot})^{2V-1}) + \int_{M^V} \text{TR}((\delta g L_{tot}^{(0,2)})(L_{tot})^{2V-1}) \right]$$
$$= 3 V c_V \left[ \int_{M^V} \text{TR}((B_{tot}^{(0,1)} d_M L_{tot})^{2V-1}) + \int_{M^V} \text{TR}((\delta g L_{tot}^{(0,2)})(L_{tot})^{2V-1}) \right]$$
$$= -\frac{3}{2} V (\frac{3}{2} V - 1) c_V \left[ \int_{M^V} \text{TR}(B_{tot}^{(0,1)} \delta g^{(L_{tot})^{2V-2}}) - \int_{M^V} \text{TR}(B_{tot}^{(0,1)} \pi_{tot}(L_{tot})^{2V-2}) \right]$$
$$+ 3 V c_V \int_{M^V} \text{TR}((\delta g L_{tot}^{(0,2)})(L_{tot})^{2V-1})$$  
(5.23)

where we have used

$$L_{tot}^{(0,2)}(x_1, \ldots, x_V) = L_{tot}^{(2,0)}(x_1, \ldots, x_V)$$  
(5.24)

which follows from (5.3) and (5.9), and

$$\delta g L_{tot}^{(1,1)}(x_1, \ldots, x_V) = d_M B_{tot}(x_1, \ldots, x_V) = 2 d_M B_{tot}^{(0,1)}(x_1, \ldots, x_V)$$  
(5.25)

which follows from (5.3), (5.20) and (5.21), and

$$d_M L_{tot}(x_1, \ldots, x_V) = -\delta g_{tot}(x_1, \ldots, x_V) + \pi_{tot}(x_1, \ldots, x_V)$$  
(5.26)

which follows from (5.3) and (5.12) with $\delta^g_{ab}(x, y) := \delta_{ab} \delta(x, y)$. In obtaining the fourth equality in (5.23) we have used Stoke’s theorem; this requires $L_{tot}(x_1, \ldots, x_V)$ to be a smooth form on $M^V$ which is not actually true since $L(x, y)$ diverges at $x = y$. 

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Thus our calculation is formal at this point\footnote{In \cite[§5]{Axelrod}, Axelrod and Singer gave a rigorous treatment of this problem (for $A^c$ acyclic). They found that a metric-dependent phase factor appears, with phase given by (minus) Witten’s geometric counterterm. We are unsure as to whether their argument for this continues to hold in our case.}, however at all other points (here and below) we are rigorous. To show the formal metric-independence of $I_V(A^c)$ we show that the 3 integrals in (5.23) all vanish. The first integral in (5.23) has the form of the one appearing in the calculation of Axelrod and Singer \cite[(5.83)]{Axelrod} and vanishes by the same argument which they gave. (This involves cancellations between Feynman diagrams). The second and third integrals in (5.23),

\begin{align}
I_V^{(2)} &= \int_{M^V} \text{TR}(\pi_{tot} B_{tot}(L_{tot})^{\frac{1}{2}V-2}) \\
I_V^{(3)} &= \int_{M^V} \text{TR}(\delta_{bg} L_{tot}^{(0,2)})(L_{tot})^{\frac{1}{2}V-1}
\end{align}

(5.27) (5.28)
did not arise in the calculation of Axelrod and Singer; they are new features of the more general situation which we are considering. To show that $I_V^{(2)}$ and $I_V^{(3)}$ vanish we begin with some general observations. The formula $d^*_q = (-1)^{nq} * d_{n-q}$ shows \(\text{Im}(d^*) = * \text{Im}(d)\) and using this we get

\[ \int_M \phi^a \wedge \psi^a = 0 \quad \text{for all } \phi = \phi^a \rho^a \in \ker(d^*) , \psi = \psi^a \rho^a \in \text{Im}(d^*) \] (5.29)
(with summation over repeated indices) by writing $\psi = *d\tilde{\psi}$ and calculating

\[-\lambda g \int_M \text{Tr}(\phi \wedge \psi) = -\lambda g \int_M \text{Tr}(\phi \wedge *d\tilde{\psi}) = <\phi, d\tilde{\psi} >=< d^* \phi, \tilde{\psi} >= 0 \]

Recall from §2 that the Lie bracket in $g$ gives a Lie bracket in $\Omega(M; g)$. The covariant derivative $d^Ae$ is a derivation w.r.t. this bracket, and we have

\[ [h, \psi] \in \text{Im}(d^*) \quad \text{for all } h \in \ker(d_0), \psi \in \text{Im}(d^*) \] (5.30)

since, writing $\psi = *d\tilde{\psi}$,

\[ [h, \psi] = [h, *d\tilde{\psi}] = *[h, d\tilde{\psi}] = *[h, d\tilde{\psi}] = *([dh, \tilde{\psi}] + [h, d\tilde{\psi}]) = *d[h, \tilde{\psi}] \in \text{Im}(d^*) \]

Combining (5.30) and (5.29) gives

\[ \int_M f_{abc} h^a \phi^b \wedge \psi^c = 0 \quad \text{for all } h \in \ker(d_0), \phi \in \ker(d^*), \psi \in \text{Im}(d^*) \] (5.31)
(Note that (5.31) holds in particular for $\phi \in \text{Im}(d^*)$ since $\text{Im}(d^*) \subseteq \ker(d^*)$). The operator $\hat{L}$ defining the propagator $L_{ab}(x,y)$ in (5.2) has $\text{Im}(\hat{L}) = \ker(d)^\perp = \text{Im}(d^*)$; it follows from this and (5.9) that

$$L_{ab}(x,y) \in \text{Im}(d^*_{M_x}) \quad , \quad L_{ab}(x,y) \in \text{Im}(d^*_{M_y})$$

(5.32)

(This is also easily derived from (5.7)).

We now use (5.31) and (5.32) to sketch how $I^{(3)}V$ given by (5.28) vanishes. Using (5.3), $I^{(3)}V$ can be expanded as a sum of terms where each term involves an integral of the form

$$\int_{M_y} f_{ace}(\delta_{bg}L^{(0,2)}_{ab}(y,x_i))L_{cd}(y,x_j)L_{ef}(y,x_k)$$

(5.33)

(There are also terms where $L_{cd}(y,x_j)L_{ef}(y,x_k)$ is replaced by $L_{ce}(y,y)$ in (5.33) but these vanish since the integrand contains no 3-forms in $y$ in this case). From (5.19) we have $\delta_{bg}L^{(0,2)}_{ab}(y,x_i) \in \ker((d_{M_y})_0)$; combining this with (5.32) we see from (5.31) that (5.33) vanishes so $I^{(3)}V$ vanishes.

Finally, we sketch how $I^{(2)}V$ given by (5.27) vanishes. Substituting the expression (5.14) for $\pi_{ab}(x,y)$ into $\pi_{tot}(x_1,\ldots,x_V)$ in (5.27) and expanding (5.27) using (5.3) leads to a sum of terms, each consisting of an integral over $M^V$. A number of these terms vanish for one of the following reasons:

(i) $\pi_{ab}(x,x) = 0$. (This follows from (5.14)).

(ii) The integrand in the integral over $M^V$ (a differential form on $M^V$) is not of degree 3 in $x_i$ for all $i = 1,\ldots,V$. (Then the integral over $M_x$ vanishes).

(iii) The term contains an integral of the form

$$\int_{M_y} f_{abd}h^a(y)L_{bc}(y,x_i)L_{de}(y,x_j)$$

(5.34)

which vanishes by (5.31) since $h(y) \in \ker((d_{M_y})_0)$ and $L(y,x), L(y,x_j) \in \text{Im}(d^*_{M_y})$.

By inspection it is straightforward to check that the only terms which do not vanish

\[\text{(More precisely, the argument leading to (5.31) generalises in an obvious way to show that (5.33) vanishes.)}\]
due to (i), (ii) or (iii) above are those of the form
\[
\int_{M_z \times M_x} \left\{ f_{acd} f_{bfp} h^a(y) B_{bc}^{(0,1)}(z, y) L_{de}^{(2,0)}(y, x_i) \right. \\
\left. \times L_{fg}(z, x_j) L_{pq}(z, x_k) \Psi_{egq}(x_i, x_j, x_k) \right\} 
\]
(5.35)
or
\[
\int_{M_z \times M_x} \left\{ f_{ade} f_{bcg} h^a(y) h^b(z) \text{vol}(z) B_{cd}^{(0,1)}(z, y) \right. \\
\left. \times L_{ef}^{(2,0)}(y, x_i) L_{gh}(z, x_j) \Phi_{fh}(x_i, x_j) \right\} 
\]
(5.36)
Therefore, to show that \( I_V^{(2)} \) vanishes it suffices to show that (5.35) and (5.36) vanishes. To show that (5.35) vanishes it suffices to show that
\[
\int_{M_y} f_{acd} h^a(y) B_{bc}^{(0,1)}(z, y) L_{de}^{(2,0)}(y, x_i) \in \ker((d_{M_z})_0) 
\]
(5.37)
because then the integral over \( M_z \) in (3.35) vanishes by (3.31). To show (5.37) we begin by noting that
\[
\int_{M_y} f_{acd} h^a(y) L_{bc}^{(1,1)}(z, y) L_{de}^{(2,0)}(y, x_i) = 0 
\]
(5.38)
for the same reason that (5.34) vanished in (iii) above. Taking the metric-variation of this gives
\[
0 = \int_{M_y} f_{acd} h^a(y)(\delta_{dg} L_{bc}^{(1,1)}(z, y)) L_{de}^{(2,0)}(y, x_i) + \int_{M_y} f_{acd} h^a(y) L_{bc}^{(1,1)}(z, y) \delta_{dg} L_{de}^{(2,0)}(y, x_i) \\
= d_{M_z} \int_{M_y} f_{acd} h^a(y) B_{bc}^{(0,1)}(z, y) L_{de}^{(2,0)}(y, x_i) + \int_{M_y} f_{acd} h^a(y) L_{bc}^{(1,1)}(z, y) \delta_{dg} L_{de}^{(2,0)}(y, x_i) 
\]
(5.39)
where we have used (5.20)–(5.21). The first term in (5.39) belongs to \( \text{Im}(d_{M_z}) \) while the second term belongs to \( \text{Im}(d_{M_z}^*) \) because of (5.32). Since \( \text{Im}(d^*) = \ker(d) \perp \text{Im}(d) \perp \) it follows that both terms in (5.39) vanish individually; the vanishing of the first term implies (5.37) so (5.35) vanishes. To show that (5.36) vanishes we note that \( h(z) \text{vol}(z) \in \ker(d_{M_z}^*) \) since \( d^*(h \cdot \text{vol}) = d^* h \cdot \text{vol} = *dh = 0 \). Combining this with (5.37) and (5.32) we see that (5.36) vanishes by (5.31). This completes the argument for the formal metric-independence of the coefficients \( I_V(A^c) \) in the perturbative expansion of the partition function.
6 Conclusion

We have described a method for carrying out a formal perturbative expansion of the functional integral \( I(\alpha; f, S) \) after expanding the action functional \( S \) about a critical point \( A^c \), with the perturbative expansion being infrared-finite when \( A^c \) is isolated modulo gauge transformations. The main problem that we have solved in doing this is to carry out a gauge-fixing procedure of Faddeev-Popov type in such a way that infrared divergences do not arise in the ghost propagator when \( A^c \) is reducible. This problem is particularly relevant in Chern-Simons gauge theory on compact 3-manifolds, since for a number of simple 3-manifolds such as \( S^3 \) and lens spaces all the flat gauge fields \( A^c \) are reducible. The usual Faddeev-Popov procedure (the BRS version of which was used in [6]) leads to the Faddeev-Popov determinant

\[
\det \left( (d_0^{A^c})^* d_0^{A^c + B} \right)
\]

(6.1)

(with gauge fields \( A = A^c + B \)). Writing this as an integral over ghost fields \( \bar{C}, C \) leads to the ghost term in the action functional given by

\[
< \bar{C}, (d_0^{A^c})^* d_0^{A^c + B} C > = < \bar{C}, \Delta_0^{A^c} C > + < \bar{C}, [B, C] >
\]

(6.2)

Infrared divergences in the ghost propagator correspond to zero-modes in the operator \( \Delta_0^{A^c} = (d_0^{A^c})^* d_0^{A^c} \) in the quadratic term in (6.2); these are present when \( A^c \) is reducible since \( \ker(\Delta_0^{A^c}) = \ker(d_0^{A^c}) = \text{Lie}(H_{A^c}) \). In our refinement of the Faddeev-Popov procedure, which takes account of the ambiguities in the gauge-fixing condition coming from the isotropy subgroup \( H_{A^c} \) of \( A^c \), we obtain

\[
V(H_{A^c})^{-1} \det \left( (d_0^{A^c})^* d_0^{A^c + B} \big|_{\ker(d_0^{A^c})^\perp} \right)
\]

(6.3)

instead of (6.1). In this case the ghost propagator is infrared-finite for all \( A^c \) since the operator \( \Delta_0^{A^c} \) in (6.2) is now restricted to the orthogonal complement of its zero-modes. The appearance of the volume factor \( V(H_{A^c}) \) is also crucial for a number of reasons. We saw in §5 ((5.5) and the subsequent discussion) that this factor is necessary for metric-independence of the of the overall factor multiplying the perturbation

\[11\text{More precisely, when } A^c \text{ is not weakly irreducible.}\]
series for the Chern-Simons partition function. It is also necessary for reproducing
the general formula [12, app. II (9)] for the weak coupling limit of $I(\alpha; f, S)$. Finally,
the factor $V(H_{A^c})$ is necessary for reproducing the numerical factors in the large $k$
limit of the expressions for the partition function obtained from the non-perturbative
prescription of [3] from the semiclassical approximation. This was shown in [23] where
the factors $V(H_{A^c})$ were put in by hand, see also [16, §4.2].

Our requirement that the spacetime manifold be compact riemannian without
boundary was important for avoiding infrared divergences in the perturbative expan-
sions because it ensures that the operators $D_{A^c}$ and $\Delta_{A^c}^0$ have discrete spectra (at
least for Yang-Mills- and Chern-Simons gauge theories). If the spectra were conti-
uous, with eigenvalues $\lambda(p)$ and $\mu(q)$ labelled by continuous parameters $p$ and $q$ (as
is the case e.g. in the usual flat spacetime setting where $p$ and $q$ are momentum
vectors) then the propagators, which are essentially given by $\frac{1}{\lambda(p)}$ and $\frac{1}{\mu(q)}$, can be
arbitrarily large for sufficiently small $p$ and $q$, even though the values of $p$ and $q$ for
which $\lambda(p) = \mu(q) = 0$ are excluded. This leads in general to infrared divergences in
the expressions for the Feynman diagrams.

A drawback with our method is that in order to explicitly evaluate the terms
in the perturbation series the eigenvectors and eigenvalues of $D_{A^c}$ and $\Delta_{A^c}^0$ must be
determined. This is a non-trivial problem in general (as opposed to the usual flat
spacetime setting where it is trivial). However, in Chern-Simons gauge theory with
spacetime $S^3$, or $S^3$ divided out by the action of a finite group (e.g. a lens space),
the eigenvectors and -values can be determined using the techniques of [32, §4].

There are a number of interesting issues which are left unresolved in this paper.
These are as follows:

1. We have seen that the gauge-fixing procedure can be carried out using only the
   subgroup $G_0$ of topologically trivial gauge transformations (this is essentially because
   $G_0$ and $G$ have the same Lie algebra), and that this avoids the usual problems that
   arise due to gauge-fixing ambiguities provided that our assumption in §4 holds. This
   assumption, that all ambiguities in the gauge-fixing come either from $H_{A^c}$ or from
topologically non-trivial gauge transformations, should be verified (or disproved). (2) The approach to perturbative expansion should be extended to obtain an infrared-finite expansion in the general case where the critical point $A^c$ is not isolated modulo gauge transformations. (We briefly discussed in §4 how this might be done although our argument was incomplete. As mentioned in the introduction S. Axelrod has recently announced a method for doing this in Chern-Simons gauge theory when $A^c$ belongs to a smooth component of the modulispaces of flat connections. This still leaves the “non-generic” case where $[A^c]$ is a singular point in the modulispaces; this case was discussed in the semiclassical approximation in §2.)

(3) The problem of ultraviolet divergences should be resolved (particularly for Yang-Mills theory) by extending the usual regularisation- and renormalisation procedures to the framework for perturbative expansion given in this paper. The Pauli-Villars procedure and method of higher covariant derivatives have a geometric nature which might make them suitable for this.

(4) The property of “momentum conservation at the vertices” of the Feynman diagrams in the usual flat spacetime setting should be generalised to our setting. More precisely, the problem is to formulate and prove a theorem which describes how invariance of the quadratic term $< B, D_A B >$ in the action functional under a group of isometries of the spacetime manifold implies simplifying conditions analogous to momentum conservation at the vertices of the diagrams (cf. remark (ii) in §2).

(5) Perturbative expansion in Chern-Simons gauge theory is ultraviolet-finite (after a point-splitting regularisation) due to a result in §4. In our method the expansions are also infrared-finite when $A^c$ is isolated modulo gauge transformations, so the terms in the perturbative expansion of the Chern-Simons partition function are completely finite for a number of simple 3-manifolds such as $S^3$ and lens spaces. It would be very interesting to explicitly calculate the terms in the expansions of the partition function for these manifolds (using e.g. the techniques of §4), determine whether the perturbation series converges and see to what extent it reproduces the expressions obtained from the non-perturbative prescription of §3.

The expressions for the Chern-Simons partition function obtained from the non-
perturbative prescription of $SU(2)$ have been shown to agree with the semiclassical approximation in the limit of large $k$ for wide classes of 3-manifolds. (However, no general proof of equality between the semiclassical- and non-perturbative expressions in this limit has been given so far.)

In showing this the non-perturbative expressions were rewritten as a sum of terms with each term corresponding to a flat gauge field $A_c$ (up to gauge equivalence), then in the large $k$ limit the term coincides with the lowest order term in the perturbative expansion determined by $A_c$. This leads us to speculate that the full perturbation series determined by $A_c$ may be equal to the non-perturbative term corresponding to $A_c$ for all values of the parameter $k$, in which case the complete non-perturbative expression is reproduced by evaluating the perturbative expansion determined by $A_c$ for all the flat gauge fields $A_c$ on the 3-manifold and adding these together. One detail to be dealt with before this could work out is the fact that the non-perturbative expressions are analytic functions in $\frac{1}{\sqrt{k+2}}$ (for gauge group $SU(2)$) rather than the coupling parameter $\frac{1}{\sqrt{k}}$. However, this does not represent a serious problem because the non-perturbative expressions can be rewritten as power series in $\frac{1}{\sqrt{k}}$. In fact, for the cases we have looked at the resulting powerseries turns out to have a surprisingly simple form [21].

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