Set-valued Itô’s formula with an application to the general set-valued backward stochastic differential equation

Yao-jia Zhang, Zhun Gou and Nan-jing Huang†

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China

Abstract. The overarching goal of this paper is to establish a set-valued Itô’s formula. As an application, we obtain the existence and uniqueness of solutions for the general set-valued backward stochastic differential equation which gives an answer to an open question proposed by Ararat et al. (C. Ararat, J. Ma and W.Q. Wu, Set-valued backward stochastic differential equation, arXiv:2007.15073).

Keywords: Set-valued stochastic integral; Set-valued Itô’s formula; Set-valued backward stochastic differential equation; Hukuhara difference; Picard iteration.

2020 Mathematics Subject Classification: 26E25, 28B20, 60G07, 60H05, 60H10.

1 Introduction

Set-valued differential equations, both deterministic and stochastic, have attracted the attention of many scholars due to the wide applications of set-valued functions (mappings) in practical problems [6,7,13,24,35]. Recently, many applications about set-valued deterministic/stochastic differential equations are found in computer science [37], economics and finance [9,17,24,30].

Itô’s formula, established firstly by Itô [20], plays an important role in stochastic analysis [21,22]. In order to study some new stochastic differential equations, the Itô’s formula has been extended in several directions. For example, Applebaum and Hudson [5] proved an Itô product formula for stochastic integrals against fermion Brownian motion. Al-Hussaini and Elliott [4] showed an Itô’s formula for a continuous semimartingale $X_t$ with a local time $L_\alpha^t$ at $\alpha$. Gradinaru et al. [15] gave an Itô’s formula for nonsemimartingales. Catuogno and Olivera [12] provided a fresh Itô’s formula for time dependent tempered generalized functions. Recently, a generalised Itô’s formula for Lévy-driven Volterra Processes was established by Bender et al. [10]. However, to the best of our knowledge, there is no set-valued Itô’s formula. The first goal of this paper is to give a set-valued Itô’s formula.

It is well known that backward stochastic differential equations (BSDEs), introduced by Pardoux and Peng [32], have been studied extensively in the literature. For instance, we refer the reader to [18,19,23,29,31]. Recently, various examples have been given in the literature [8,14,36,39] to motivate the study of BSDEs. Very recently, Ararat et al. [8] provided some sufficient conditions to ensure the existence and uniqueness of

*This work was supported by the National Natural Science Foundation of China (11471230, 11671282).
†Corresponding author. E-mail addresses: nanjinghuang@hotmail.com; njhuang@scu.edu.cn
solutions for the following set-valued backward stochastic differential equation:

\[ Y_t = \xi + \int_t^T f(t, Y_s) ds \ominus \int_t^T Z_s dW_s, \]

where \((W_s, s \geq 0)\) is a standard \(m\)-dimension Brownian movement with \(dW_i dW_j = 0 (i \neq j)\).

However, as pointed out by Ararat et al. [8], the systematic study of the following general set-valued backward stochastic differential equation (GSVBSDE):

\[ Y_t = \xi + \int_t^T f(t, Y_s, Z_s) ds \ominus \int_t^T Z_s dW_s, \tag{1.1} \]

is still widely open. It is worth mentioning that (1.1) is a modelling tool used to capture the risk measure problem arising in finance [8]. The second purpose of this paper is to study the existence and uniqueness of (1.1) by using the set-valued Itô’s formula.

The rest of this paper is structured as follows. The second section recalls some necessary preliminaries, including some properties of Hukuhara difference, set-valued stochastic processes and stochastic integrals. After that in Section 3, we obtain the set-valued Itô’s formula by employing some properties of set-valued stochastic integrals. Finally, we show the existence and uniqueness of the solutions to (1.1) as the application of the set-valued Itô’s formula.

2 Preliminaries

In this section, we recall some necessary notations and definitions.

2.1 Hukuhara difference

For a Hilbert space \(X\), let \(\mathcal{P}(X)\) be the set of all nonempty subsets of \(X\). Let \(\mathcal{L}(X)\) be the set of all closed sets in \(\mathcal{P}(X)\) and \(\mathcal{K}(X)\) be the set of all compact convex sets in \(\mathcal{P}(X)\).

For any \(A, B \in \mathcal{K}(X)\), the Hausdorff distance between \(A\) and \(B\) is defined by

\[ h(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y) \right\} \]

and the mapping \(\| \cdot \| : \mathcal{K}(X) \rightarrow [0, \infty)\) is defined by

\[ \|A\| := h(A, \{0\}) = \sup_{a \in A} |a|, \quad \forall A \in \mathcal{K}(X). \tag{2.1} \]

Moreover, for any \(A, B \in \mathcal{K}(\mathbb{R}^n)\) and \(\alpha \in \mathbb{R}\), define

\[ A + B := \{a + b : a \in A, b \in B\}; \quad \alpha A := \{\alpha a : a \in A\}. \]

Clearly, for \(A, B, C \in \mathcal{K}(\mathbb{R}^n)\), the following cancellation law holds:

\[ A + C = B + C \iff A = B. \]

Assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability measure space. \(A \in \mathcal{F}\) is called an atomic set, if \(\mathbb{P}(A) > 0\) implies that \(\mathbb{P}(B) = 0\) or \(\mathbb{P}(A \setminus B) = 0\) for any Borel subset \(B \subset A\). If there is no atomic sets under the measure \(\mathbb{P}\), then \(\mathbb{P}\) is called a nonatomic probability measure. In this paper, we always assume \(\mathbb{P}\) is nonatomic. In the sequel, we assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a filtered probability space satisfying the usual conditions.

Now we recall the Hukuhara difference, which defines the “subtraction” in the space \(\mathcal{K}(X)\).
Definition 2.1. The Hukuhara difference is defined as follows:

\[ A \ominus B = C \iff A = B + C, \quad A, B \in \mathcal{K}(\mathbb{R}). \]

For our main results, we need the following lemmas.

Lemma 2.1. Let \( A, B, A_1, A_2, B_1, B_2, C \in \mathcal{K}(\mathbb{R}^n) \). The following identities hold if all the Hukuhara differences involved exist.

\begin{enumerate}[(i)]
  \item \( A \ominus A = \{0\}, \ A \ominus \{0\} = A; \)
  \item \( (A_1 + B_1) \ominus (A_2 + B_2) = (A_1 \ominus A_2) + (B_1 \ominus B_2); \)
  \item \( (A_1 + B_1) \ominus B_2 = A_1 + (B_1 \ominus B_2) = (A_1 \ominus B_2) + B_1; \)
  \item \( A_1 + (B_1 \ominus B_2) = (A_1 \ominus B_2) + B_1; \)
  \item \( A = B + (A \ominus B). \)
\end{enumerate}

Lemma 2.2. Let \( A, B, C \in \mathcal{K}(\mathbb{R}^n) \). Then

\begin{enumerate}[(i)]
  \item If \( A \ominus C \) exists, then \( (A + B) \ominus C \) exists for any \( B \in \mathcal{K}(\mathbb{R}^n) \);
  \item If \( A \ominus C \) and \( C \ominus B \) exist, then \( A \ominus B \) exists and \( A \ominus B = (A \ominus C) + (C \ominus B) \).
\end{enumerate}

Proof. (i) Set \( M = A \ominus C \). Then \( A = C + M \) and so \( (A + B) \ominus C = (C + M + B) \ominus C = M + B \). (ii) Let \( M = A \ominus C, \ N = C \ominus B \), then \( A = C + M, \ C = B + N \). Taking \( C = B + N \) in \( A = C + M \), we have \( A = B + N + M \), which implies \( A \ominus B = M + N = (A \ominus C) + (C \ominus B) \). \( \square \)

Lemma 2.3. \[ \begin{align*}
  \text{(i) The mapping } & \| \cdot \| : \mathcal{K}(\mathbb{R}) \to [0, \infty) \text{ defined by (2.1) satisfies the properties of a norm;} \\
  \text{(ii) If } & A, B \in \mathcal{K}(\mathbb{R}) \text{ and } A \ominus B \text{ exists, then } h(A, B) = \| A \ominus B \|; \\
  \text{(iii) For any } & A, B \in \mathcal{K}(\mathbb{R}^n), \text{ both } A \ominus B \text{ and } B \ominus A \text{ exist if and only if } A \text{ is a translation of } B, \ i.e., \\
  & A = B + \{c\}, \text{ where } c \in \mathbb{R}^n. 
\end{align*} \]

2.2 Set-valued stochastic processes

In this subsection, we recall the set-valued stochastic processes and give a limit convergence theorem for set-valued processes.

For a sub-\( \sigma \)-filed \( \mathcal{G} \subset \mathcal{F} \), let \( L^2_{\mathcal{G}}(\Omega, \mathcal{K}(\mathbb{R}^n)) \) denote the set of all \( \mathcal{G} \)-measurable random variables valued in \( \mathcal{K}(\mathbb{R}^n) \). For any \( X \in \mathcal{K}(\mathbb{R}^n) \), let \( S_\mathcal{G}(X) \) be the set of all \( \mathcal{G} \)-measurable selection of \( X \). Moreover, let \( L^2_{\mathcal{G}}(\Omega, \mathcal{K}(\mathbb{R}^n)) \) denote the set of all \( \mathcal{G} \)-measurable square-integrably bounded random variables valued in \( \mathcal{K}(\mathbb{R}^n) \). For simplicity, let \( S^2_{\mathcal{G}}(X) = S_\mathcal{G}(X) \cap L^2_{\mathcal{G}}(\Omega, \mathcal{K}(\mathbb{R}^n)) \).

Definition 2.2. \[ \begin{align*}
  \text{(i) } & L^2_{\text{ad}}(\Omega, \mathbb{R}^n) = L^2_{\text{ad}}(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{R}^n), \text{ the set of all the } n\text{-dimensional measurable square-integrably random variables, is a Hilbert space equipped with the norm } \| \cdot \|_a = (E| \cdot |^2)^{\frac{1}{2}}. 
\end{align*} \]
A set-valued process is called measurable if, for any closed set $B \subset \mathbb{X}$, it holds that \{ω ∈ Ω : F(ω) ∩ B ≠ ∅\} ∈ F.

A measurable set-valued mapping $X : Ω → \mathcal{L}(\mathbb{R}^n)$ is called a set-valued random variable.

A set-valued stochastic process $f = \{f_t\}_{t ∈ [0,T]}$ is a family of set-valued random variables taking values in $\mathcal{L}(\mathbb{R}^n)$.

A set-valued stochastic process is called measurable if it is $\mathcal{B}([0,T]) ⊗ \mathcal{F}$-measurable as a single function on $[0,T] × Ω$.

$\mathbb{L}^0(Ω, \mathcal{L}(\mathbb{R}^n)) = \mathbb{L}^0(Ω, \mathcal{F}_T, \mathbb{P}, \mathcal{L}(\mathbb{R}^n))$, is the space of all measurable set-valued mappings $F : E → \mathcal{L}(\mathbb{R}^n)$ distinguished up to $\mathbb{P}$-almost everywhere equality.

$\mathbb{L}^p_t(Ω, \mathcal{L}(\mathbb{R}^n))$ is the set of all $\mathcal{F}_t$-measurable random variables valued in $\mathcal{L}(\mathbb{R}^n)$ and $\mathbb{L}^p(Ω, \mathcal{L}(\mathbb{R}^n))$ is the set of all $\mathcal{F}_t$-measurable $p$-integrably bounded random variables valued in $\mathcal{L}(\mathbb{R}^n)$ for any given $t ∈ [0,T]$.

$\mathcal{A}^2(Ω, \mathcal{L}(\mathbb{R}^d)) := \{F ∈ \mathbb{L}^0(Ω, \mathcal{L}(\mathbb{R}^n)) : S^{*2}(F) ≠ 0\}$ and

$\mathcal{A}^2_t(Ω, \mathcal{L}(\mathbb{R}^d)) := \{F ∈ \mathbb{L}^0(Ω, \mathcal{L}(\mathbb{R}^n)) : S^{*2}(F) ≠ 0\}$.

For any $X ∈ \mathcal{A}^2(Ω, \mathcal{L}(\mathbb{R}^d))$ and any $t ∈ [0,T]$, $S^*_t(X)$ is the set of all $\mathcal{F}_t$-measurable selections of $X$.

For any $X ∈ \mathbb{L}^0(Ω, \mathcal{L}(\mathbb{R}^n))$ the collection of all the selections of $L^2_{ad}(Ω, \mathbb{R}^n)$ is denoted by

$S^{*2}(X) = \{f ∈ L^2_{ad}(Ω, \mathbb{R}^n) : f(ω) ∈ F(ω) \quad \mathbb{P}\text{-a.s.}\}$.

$S^{2}_t(X) = S^*_t(X) ∩ L^2_{ad}(Ω, \mathbb{R}^n)$.

A set-valued random variable $X(ω)$ is called $p$-integrably bounded if there exists $m ∈ L^2_{ad}(Ω, \mathbb{R}^+) \text{ such that } \|X\| ≤ m(ω) \quad \mathbb{P}\text{-a.s.}.$

$L^p(E, \mathcal{L}(\mathbb{R}^n)) = \{F ∈ L^p(E, \mathcal{L}(\mathbb{R}^n)) : \|F\| ≤ m(ω) \quad \mathbb{P}\text{-a.s.}, \text{ where } m(ω) ∈ L^p(Ω, \mathbb{R}^+)\}$.

A set-valued stochastic process $F$ is called $p$-integrably bounded if there exists $m ∈ L^2_{ad}([0,T] × Ω, \mathbb{R}^+) \text{ such that } \|F\| ≤ m(t, ω) \quad \mathbb{P}\text{-a.s.}$.

Denote the subtrajectory integrals of a set-valued stochastic process $F : T × Ω → \mathcal{L}(\mathbb{R}^n)$ by $S(F)$, which is the set of all measurable and $dt × P$-integrable selectors of $F$.

Denote the subset

$S^2(F) := S(F) ∩ L^2_{ad}([0,T] × Ω, \mathbb{R}^n)$

and

$S^2_t(F) = \{f ∈ S^2(F) : f \text{ is } F\text{-measurable}\}$.

**Definition 2.3.** A set-valued process is called $\mathcal{F}$-nonanticipative if it is $\Sigma_\mathcal{F}$-measurable, where

$\Sigma_\mathcal{F} = \{A ∈ \beta_\mathcal{F} ⊗ \mathcal{F} : A_t ∈ \mathcal{F}_t \quad \forall t ∈ T\},$

and $A_t$ denotes the $t$-section of a set $A ⊂ T × Ω$, i.e., $A_t = \{ω ∈ \beta_\mathcal{F} : A(t,ω) ∈ \beta_\mathcal{F} ⊗ \mathcal{F}\}$. The set of all set-valued $\mathcal{F}$-nonanticipative measurable square integrably bounded stochastic processes taking values in $\mathcal{K}(\mathbb{R})$ with $S^2_t(\cdot) ≠ \emptyset$ is denoted by $L^2_{ad}([0,T] × Ω, \mathcal{K}(\mathbb{R}^n))$. 

---

4
Lemma 2.4. [24] Let $\Phi_t, \Psi_t \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{L}(\mathbb{R}^n))$. Then $\mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{L}(\mathbb{R}^n))$ is a complete metric space with the metric $d_H(\Phi_t, \Psi_t) = \left[ \mathbb{E} \int_0^T h^2(\Phi_t, \Psi_t) dt \right]^{\frac{1}{2}}$. Specially, $\mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ is a Banach space with the norm $\|Z(t)\|_s = \left[ \mathbb{E} \int_0^T \|Z(t)\|^2 dt \right]^{\frac{1}{2}}$ for $Z(t) \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$.

Lemma 2.5. (Kuratowski and Ryll-Nardzewski) [20] Assume that $(E, \mathcal{E})$ is a Polish space and $(T, \mathcal{F})$ is a measurable space. If $F : T \to \mathcal{E}$ is measurable, then $F$ admits a measurable selector.

The set-valued conditional expectation and the set-valued martingale can be defined as follows.

Definition 2.4. [24] For a sub-$\sigma$-field $\mathcal{G} \subset \mathcal{F}$ and $X \in \mathbb{L}^2(\Omega, \mathcal{K}(\mathbb{R}^n))$, the conditional expectation of $X$ with respect $\mathcal{G}$ is defined as the unique set-valued random variable $\mathbb{E}[X|\mathcal{G}] \in \mathbb{L}^2(\Omega, \mathcal{K}(\mathbb{R}^n))$ such that

$$\int_G \mathbb{E}[X|\mathcal{G}]d\mathbb{P} = \int_G Xd\mathbb{P}, \quad \forall G \in \mathcal{G}.$$ 

Definition 2.5. [24] A set-valued process $M = \{M_t\}_{t \in [0, T]}$ is said to be a set-valued square-integrable $\mathcal{F}$-martingale if

(i) $M \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{L}(\mathbb{R}^n))$;

(ii) $M_t \in \mathcal{A}^2(\Omega, \mathcal{L}(\mathbb{R}^n))$;

(iii) $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$ for all $0 \leq s \leq t$.

Lemma 2.6. [8] Let $F_1, F_2 \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$. Then, $F_1 + F_2 \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ and

$$S^2_F(F_1 + F_2) = S^2_F(F_1) + S^2_F(F_2).$$

Furthermore, if $F_1 \oplus F_2$ exists, then $F_1 \oplus F_2 \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ and

$$S^2_F(F_1 \oplus F_2) = S^2_F(F_1) \oplus S^2_F(F_2).$$

Lemma 2.7. Suppose that $\{X_p(t, \omega)\}^\infty_{p=1} \subset \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ is a sequence of set-valued stochastic processes satisfying

(i) $X_q \ominus X_p$ exists for $q \geq p$;

(ii) For each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|X_q \ominus X_p\|_s < \varepsilon$ when $p, q \geq N$.

Then there exists a unique $X(t, \omega) \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ such that $\|X_p \ominus X\|_s \to 0$ as $p \to +\infty$.

Proof. It follows from the assumption (i) that the potential of the $X_p$ is increasing as $p$ becoming larger. Thus, the assumption (ii) shows that, for each $\varepsilon > 0$, there exist an $N' \in \mathbb{N}$ and a $\xi \in \mathbb{R}^p$ such that $X_q \ominus X_p = \xi$ with $|\xi| < \varepsilon$ when $n, m \geq N'$. Since $X_q \ominus X_p$ exists, we have

$$\|X_q \ominus X_p\|_s = \left[ \mathbb{E} \int_0^T \|X_q \ominus X_p\|^2 ds \right]^{\frac{1}{2}} = \left[ \mathbb{E} \int_0^T h^2(X_q, X_p) ds \right]^{\frac{1}{2}}$$

when $p, q \geq N'$. By Lemma 2.4 and the fact $\{X_p(t, \omega)\}^\infty_{p=1} \subset \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$, there exists $X^* \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ such that $\|X^* \ominus X_p\|_s < \varepsilon$ for all $t \in [0, T]$ and a.s $\omega \in \Omega$ when $p \geq N'$.

Next we show the uniqueness. If $X_1, X_2 \in \mathcal{L}_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$ such that $\|X_p \ominus X_1\|_s \to 0$ and $\|X_p \ominus X_2\|_s \to 0$ with $p \to +\infty$, then it follows from Lemma 2.2 that

$$\|X_1 \ominus X_2\|_s = \|X_1 \ominus X_p + X_p \ominus X_2\|_s \leq \|X_p \ominus X_1\|_s + \|X_p \ominus X_2\|_s \to 0$$

as $p \to +\infty$. This ends the proof. \qed
2.3 Set-valued stochastic integrals

In this subsection, we recall the the notions of set-valued stochastic integrals.

Definition 2.6. (\cite{27})

- For \( F \in \mathcal{L}_ad^2([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) , \)
  \( \mathcal{M}_F([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) := \{ F \in \mathcal{L}_ad^2([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) : S_F^2(F) \neq \emptyset \} . \)

- A set \( V \subset \mathcal{L}_ad^2(\Omega, \mathbb{R}^n) \) is said to be decomposable with respect to \( \mathcal{F} \) if for any \( f_1, f_2 \in V \) and \( D \in \mathcal{F} \), it holds that \( 1_D f_1 + 1_D f_2 \in V . \)

- Denote the decomposable hull of \( V \subset \mathcal{L}_ad^2(\Omega, \mathbb{R}^n) \) by \( \text{dec}(V) \), which is the smallest decomposable subset of \( \mathcal{L}_ad^2(\Omega, \mathbb{R}^n) \) contain \( V . \) Denote the closure of \( \text{dec}(V) \) in \( \mathcal{L}_ad^2(\Omega, \mathbb{R}^n) \) by \( \overline{\text{dec}}(V) \).

- \( \mathcal{K}_w(\mathcal{L}_ad^2([0,T] \times \Omega, \mathbb{R}^{n \times m})) \) is the set of weakly compact convex subset of \( \mathcal{L}_ad^2([0,T] \times \Omega, \mathbb{R}^{n \times m}) . \)

Lemma 2.8. (\cite{27}) Suppose \( f \in \mathcal{L}_ad^2([0,T] \times \Omega, \mathbb{R}^m) . \) Then the Itô integral \( I(f) = \int_0^T f(t)dW(t) \) is a random variable with \( \mathbb{E}[I(f)] = 0 \) and

\[ \mathbb{E}[I(f)]^2 = \mathbb{E}\left[ \int_0^T f^2(t)dt \right] . \]

Definition 2.7. (\cite{27})

- Denote the \( J : \mathcal{L}_ad^2([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^n) \rightarrow \mathcal{L}_ad^2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \) by

\[ J(\phi)(\omega) = \left( \int_0^T \phi(t)dt \right)(\omega) \quad \mathbb{P} \text{-a.s..} \]

where \( \phi \in \mathcal{L}_ad^2([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^n) , \beta_T \) and \( \mathcal{F}_T \) are the \( \sigma \)-fields on \( T \) and \( \Omega \), respectively.

- Denote the \( J : \mathcal{L}_ad^2([0,T] \times \Omega, \Sigma_T, \mathbb{R}^{n \times m}) \rightarrow \mathcal{L}_ad^2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \) by

\[ J(\psi)(\omega) = \left( \int_0^T \psi(t)dW_t \right)(\omega) \quad \mathbb{P} \text{-a.s..} \]

where \( \psi \in \mathcal{L}_a([0,T] \times \Omega, \Sigma_T, \mathbb{R}^{n \times m}) . \)

- Denote the sets \( J[1_{[s,t]}S(F)] \) and \( J[1_{[s,t]}S(F)] \) by \( J_{st}[S_F(F)] \) and \( J_{st}[S_F(G)] \), respectively, which are said to be the functional set-valued stochastic integral of \( F \) and \( G \) on the interval \([s,t] \), respectively.

Definition 2.8. (\cite{27}) Let \( F \in \mathcal{L}_ad^2([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^n)) \) and \( G \in \mathcal{L}_ad^2([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^{n \times m})) \) such that \( S_F^2(F) \neq \emptyset \) and \( S_{G}^2(G) \neq \emptyset \). Then there exist unique set-valued random variables \( \int_0^t F(t)dW_t \) and \( \int_0^t G(t)dW_t \) in \( \mathcal{A}_T^2(E, \mathcal{L}(\mathbb{R}^d)) \) such that \( S_F^2(\int_0^T F(t)dW_t) = \overline{\text{dec}}(J[S_F^2(F)]) \) and \( S_{G}^2(\int_0^T G(t)dW_t) = \overline{\text{dec}}(J[S_{G}^2(G)]) \). The set-valued random variables \( \int_0^t F(t)dW_t \) and \( \int_0^t G(t)dW_t \) are, respectively, called the set-valued integrals of \( F(t) \) and \( G(t) \) with respect to \( t \) and \( W_t \) on \([0,T] \). Moreover, for any \( t \in [0,T] \), define

\[ \int_0^t F(s)ds := \int_0^T 1_{[0,t]}(s)F(s)ds, \quad \int_0^t G(s)dW_s := \int_0^T 1_{[0,t]}(s)G(s)dW_s \]

and

\[ \int_t^T F(s)ds := \int_0^T 1_{(t,T]}(s)F(s)ds, \quad \int_t^T G(s)dW_s := \int_0^T 1_{(t,T]}(s)G(s)dW_s . \]

In the sequel, we always suppose that the set-valued integrals exist.
Lemma 2.9. Let $F_1, F_2 \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^n))$ and $G_1, G_2 \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^{n \times m}))$. Then, for every $t \in [0, T]$,

$$\int_0^t (F_1(s) + F_2(s))ds = \int_0^t F_1(s)ds + \int_0^t F_2(s)ds$$

and

$$\int_0^t (G_1(s) + G_2(s))dW_s = \int_0^t G_1(s)dW_s + \int_0^t G_2(s)dW_s$$

hold $\mathbb{P}$-a.s.. If $F_1 \ominus F_2$ and $G_1 \ominus G_2$ exist, then $F_1 \ominus F_2 \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^n))$ and $G_1 \ominus G_2 \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^{n \times m}))$ for every $t \in [0, T]$. Moreover,

$$\int_0^t F_1(s) \ominus F_2(s)ds = \int_0^t F_1(s)ds \ominus \int_0^t F_2(s)ds$$

and

$$\int_0^t G_1(s) \ominus G_2(s)dW_s = \int_0^t G_1(s)dW_s \ominus \int_0^t G_2(s)dW_s$$

hold $\mathbb{P}$-a.s..

Lemma 2.10. Let $F \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^n))$ and $G \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^{n \times m}))$. Then for each $t \in [0, T]$,

$$\int_0^t F(s)ds = \int_0^t F(s)ds + \int_t^T F(s)ds, \quad \int_0^t G(s)dW_s = \int_0^t G(s)dW_s + \int_t^T G(s)dW_s$$

and

$$\int_0^t F(s)ds = \int_0^t F(s)ds \ominus \int_0^t F(s)ds, \quad \int_0^t G(s)dW_s = \int_0^t G(s)dW_s \ominus \int_0^t G(s)dW_s$$

hold $\mathbb{P}$-a.s..

Lemma 2.11. For any $Z, Z_1, Z_2 \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m}))$, the following statements hold:

(i) $Z_1 + Z_2 \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m}))$ and for every $t \in [0, T]$,

$$\int_0^t (Z_1 + Z_2)dW_s = \int_0^t Z_1dW_s + \int_0^t Z_2dW_s, \quad \mathbb{P}$-a.s.;$$

(ii) If $Z_1 \ominus Z_2$ exists, then $Z_1 \ominus Z_2 \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m}))$ and for every $t \in [0, T]$,

$$\int_0^t (Z_1 \ominus Z_2)dW_s = \int_0^t Z_1dW_s \ominus \int_0^t Z_2dW_s, \quad \mathbb{P}$-a.s.;$$

(iii) If $Z_1 \ominus Z_2$ exists and $\int_0^t Z_1dW_s = \int_0^t Z_2dW_s$, $\mathbb{P}$-a.s. for all $t \in [0, T]$, then $Z_1 = Z_2 \mathbb{P}$-a.s.;

(iv) $\int_0^T ZdW_s = \int_0^T ZdW_s + \int_t^T ZdW_s, \quad \int_0^T ZdW_s = \int_0^T ZdW_s \ominus \int_0^T ZdW_s.$

Lemma 2.12. For every convex-valued square-integrable set-valued martingale $M = \{M_t\}_{t \in [0, T]}$, there exists $\mathcal{G}^M \in \mathcal{P}(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m}))$ such that $M_t = M_0 + \int_0^t \mathcal{G}^M dB_s$, $\mathbb{P}$-a.s. for all $t \in [0, T]$. Moreover, if $M$ is uniformly square-integrably bounded, then $\mathcal{G}^M$ is a convex weakly compact set, i.e., $\mathcal{G}^M \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m}))$.

Denote $\mathcal{G}$ by

$$\mathcal{G} := \left\{ Z(t) \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathbb{R}^{n \times m})) : \left\{ \int_0^t Z(s)dW_s \right\}_{t \in [0, T]} \text{ is an } \mathcal{F}_t \text{ martingale} \right\}.$$
Remark 2.1. Since $K_\omega(L^2_{ad}([0,T] \times \Omega, R^{n \times m}))$ is weakly closed and bounded, $\mathcal{G}$ is closed.

Lemma 2.13. [3] Let $X_1, X_2 \in L^2_\mathcal{F}(\Omega, \mathcal{K}(\mathbb{R}^n))$ and $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-field. If $X_1 \oplus X_2$ exists, then $E[X_1 \odot X_2 | \mathcal{G}]$ exists in $L^2_\mathcal{G}(\Omega, \mathcal{K}(\mathbb{R}^n))$ and

$$E[X_1 \odot X_2 | \mathcal{G}] = E[X_1 | \mathcal{G}] \otimes E[X_2 | \mathcal{G}]$$

3 Set-valued Itô’s formula

In this section, we extend the classical Itô’s formula to the set-valued case. To this end, we need the following lemma.

Lemma 3.1. [24]

(i) If $(\Omega, \mathcal{F}, \mathbb{P})$ is separable and $\Phi_t \in L^2_{ad}([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^{n \times m}))$, then there exists a sequence $\{\phi^n\}_{n=1}^\infty \subset S^2_\Phi(\Phi)$ such that

$$\left( \int_0^T \Phi_s dW_s \right)(\omega) = cl_L \left\{ \left( \int_0^T \phi^n_s dW_s \right)(\omega) : n \geq 1 \right\} \quad \mathbb{P}\text{-a.s.},$$

where $cl_L$ means the closure taken in the norm topology of $L^2_{ad}([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^n))$.

(ii) If $(\Omega, \mathcal{F}, \mathbb{P})$ is separable and $\Psi_t \in L^2_{ad}([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^n))$, then there exists a sequence $\{\psi^n\}_{n=1}^\infty \subset S^2_\Psi(\Psi)$ such that

$$\left( \int_0^T \Psi_s ds \right)(\omega) = cl_L \left\{ \left( \int_0^T \psi^n_s ds \right)(\omega) : n \geq 1 \right\} \quad \mathbb{P}\text{-a.s.}$$

We define a set-valued Itô’s process as follows:

$$X_t = x_0 + \int_0^t f(s) dW_s + \int_0^t g(s) ds, \quad 0 \leq t \leq T,$$

(3.1)

where $x_0 = X_0 \in L^2_{ad}(\Omega, \mathbb{R}^n)$, $f(t) \in L^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^{n \times m}))$ and $g(t) \in L^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$.

Clearly, Lemma 3.1 shows that the set-valued Itô’s process defined by (3.1) is a set-valued stochastic process and (3.1) can be rewritten as follows:

$$X_i(t) = x_{0i} + \int_0^t f_i(s) dW_s + \int_0^t g_i(s) ds, \quad i = 1, \ldots, n,$$

(3.2)

where $f_i^T(t) \in L^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^m))$ and $g_i(t) \in L^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}))$ are the $i$-th component of $f(t)$ and $g(t)$, respectively.

Lemma 3.2. Let $A = \left\{ \left( \int_0^t \phi^n_s dW_s \right)(\omega) : n \geq 1 \right\}$ and $B = \left\{ \left( \int_0^t \psi^n_s ds \right)(\omega) : m \geq 1 \right\}$ $\mathbb{P}$-a.s., where $\{\phi^n\}_{n=1}^\infty \subset S^2_\Phi(\Phi)$ and $\{\psi^n\}_{n=1}^\infty \subset S^2_\Psi(\Psi)$. Then $cl_L(A) + cl_L(B) = cl_L(A + B)$.

Proof. Let $\overline{A} = cl_L(A)$ and $\overline{B} = cl_L(B)$. Then we need to show that $\overline{A} + \overline{B} = \overline{A + B}$.

Step 1. For any given $x \in \overline{A} + \overline{B}$, there exists $a \in \overline{A}$ and $b \in \overline{B}$ such that $x = a + b$. Moreover, there exist $\{a_n\} \subset A$ and $\{b_n\} \subset B$ such that $a_n \to a$ and $b_n \to b$. Thus, $a_n + b_n \to a + b$. Since $a_n + b_n \in A + B$, we have $x = a + b \in \overline{A + B}$, which implies $\overline{A + B} \subset \overline{A} + \overline{B}$. 

8
Step 2. For any given \( x \in \overline{A} + \overline{B} \), there exists \( \{x_n\} \subset A + B \) such that \( x_n \to x \). This means that, for any given \( \varepsilon > 0 \), there exists a positive integer \( N_0 > 0 \) such that \( \|x_n - x_m\| < \varepsilon \) when \( n, m > N_0 \). Since \( x_n \in A + B \), there exist \( a_n = \int_0^t f^n(s) dW_s \in A \) and \( b_n = \int_0^t g^n(s) ds \in B \) such that \( x_n = a_n + b_n \) such that

\[
\int_0^t f^n(s) dW_s + \int_0^t g^n(s) ds \to x.
\]

If there is no limit for \( \{a_n\} \), then that exists a \( \varepsilon_0 > 0 \) such that, for any \( N > 1 \), there exists \( n, m > N \) satisfying \( \|a_n - a_m\| \geq \varepsilon_0 \). This means that

\[
E \left[ \int_0^T \left( \int_0^t (f^n - f^m) dW_s \right)^2 dt \right] \geq \varepsilon_0^2.
\]

Thus, when \( n, m > N_0 \), it follows from Lemma 2.8 that

\[
\|x_n - x_m\|^2 = E \left[ \int_0^T \left( \int_0^t (f^n - f^m) dW_s + \int_0^t (g^n - g^m) ds \right)^2 dt \right]
= E \left[ \left( \int_0^T \left( \int_0^t (f^n - f^m) dW_s \right)^2 + \left( \int_0^t (g^n - g^m) ds \right)^2 dt \right) \right] \geq \varepsilon_0^2,
\]

which is a contradiction with the fact \( \|x_n - x_m\| < \varepsilon \). Therefore, we know that there exists an \( a \in \overline{A} \) such that \( a_n \to a \). Similarly, there exists a \( b \in \overline{B} \) such that \( b_n \to b \). Thus, we have \( a + b = x \in \overline{A} + \overline{B} \) and so \( \overline{A} + \overline{B} \subset \overline{A + B} \).

Combining Steps 1 and 2, we obtain \( \overline{A + B} = \overline{A} + \overline{B} \).

**Remark 3.1.** Let \( X \) be a Banach space. Then \( \overline{A + B} \subset \overline{A} + \overline{B} \) always holds for any \( A, B \subset X \). Especially, \( \overline{A + B} = \overline{A} + \overline{B} \) holds for any \( a \in X \) and \( B \subset X \).

In order to obtain the set-valued Itô’s formula, we need the following assumption.

**Assumption 3.1.** Assume that \( Y_1, Y_2 \) are two topological spaces and \( \varphi : Y_1 \to Y_2 \) is a continuous mapping such that, for any bounded subset \( A \) in \( Y_1 \),

\[
\varphi(c_1 Y_1(A)) = c Y_2(\varphi(A)),
\]

where \( \varphi(A) = \cup_{a \in A} \varphi(a) \).

**Remark 3.2.** (i) If \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) is continuous, then it is easy to check that \( \varphi \) satisfies Assumption 3.1.

(ii) Any continuous closed mapping from a Banach space to another one satisfies Assumption 3.1.

(iii) The mapping \( \varphi : L^2_{ad}([0, T] \times \Omega, \mathbb{R}^m) \to L^2_{ad}([0, T] \times \Omega, \mathbb{R}^n) \) defined by

\[
\varphi(x) = (x_1^2, x_2^2, \cdots, x_n^2), \quad \forall x = (x_1, x_2, \cdots, x_n) \in L^2_{ad}([0, T] \times \Omega, \mathbb{R}^n)
\]

satisfies Assumption 3.1.

Now we are able to derive the set-valued Itô’s formula.

**Theorem 3.1.** Let \( X_t \) be the set-valued Itô process defined by (3.1). If \( \phi(t, x) \) is a continuous function satisfying Assumption 3.1 with continuous partial derivatives \( \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x} \) and \( \frac{\partial^2 \phi}{\partial x^2} \) for every \( x \in X_t \), then \( \phi(t, X_t) \)
is a set-valued Itô process and
\[
\phi(t, X(t))_k = \phi(0, x_0)_k + \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, X(s)) f_i(s) dW_s + \int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, X(s)) + \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, X(s)) g_i(s) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi_k}{\partial x_i^2}(s, X(s))(f_i^2) \right] ds,
\]
for \(k = 1, \cdots, n\) \(\mathbb{P}\text{-a.s.}\),

where \(\phi(t, x) : T \times L^2_d([0, T] \times \Omega, \mathbb{R}^n) \to L^2_d([0, T] \times \Omega, \mathbb{R}^n)\), \(\phi(t, X(t)) = \{\phi(t, x(t)) : x(t) \in X(t)\}\), \(\frac{\partial \phi_k}{\partial t}(s, X(t)) = \{\frac{\partial \phi_k}{\partial t}(s, x(t)) : x(t) \in X(t)\}\), \(\frac{\partial^2 \phi_k}{\partial x_i^2}(s, X(t)) = \{\frac{\partial^2 \phi_k}{\partial x_i^2}(s, x(t)) : x(t) \in X(t)\}\) and \(f_i^2 = \{f_i^2 \in f_i\}\).

**Proof.** Since \(f(t)\) and \(g(t)\) are convex-valued, square integrably bounded, by Lemma 3.1 there exist two sequences \(\{f^{d_1}(s)\}_{d_1=1}^\infty \subset S^2(f)\) and \(\{g^{d_2}(s)\}_{d_2=1}^\infty \subset S^2(g)\) such that

\[
\int_0^t f(s) dW_s = \text{cl}_L \left\{ \left( \int_0^t f^{d_1}(s) dW_s \right) : d_1 \geq 1 \right\} \text{ a.e. } \mathbb{P}\text{-a.s..} \tag{3.3}
\]

and

\[
\int_0^t g(s) ds = \text{cl}_L \left\{ \left( \int_0^t g^{d_2}(s) ds \right) : d_2 \geq 1 \right\} \text{ a.e. } \mathbb{P}\text{-a.s..}
\]

It follows from (3.1) and Lemma 3.2 that

\[
X(t) = x_0 + \text{cl}_L \left\{ \left( \int_0^t f^{d_1}(s) dW_s \right) + \left( \int_0^t g^{d_2}(s) ds \right) : d_1, d_2 \geq 1 \right\} \text{ a.e. } \mathbb{P}\text{-a.s..}
\]

For any given \(d_1, d_2 \geq 1\), let

\[
x^{d_1,d_2}(t) = x_0 + \int_0^t f^{d_1}(s) dW_s + \int_0^t g^{d_2}(s) ds \text{ a.e. } \mathbb{P}\text{-a.s..}
\]

Then \(x^{d_1,d_2}(t)\) is a single-valued Itô’s process and so the classical Itô’s formula shows that

\[
\phi(t, x^{d_1,d_2}(t))_k = \phi(0, x_0)_k + \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_1,d_2}(s)) f_i^{d_1}(s) dW_s + \int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, x^{d_1,d_2}(s)) + \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_1,d_2}(s)) g_i^{d_2}(s) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi_k}{\partial x_i^2}(s, x^{d_1,d_2}(s))(f_i^{d_1})^2 \right] ds \tag{3.4}
\]

for \(k = 1, \cdots, n\) \(\mathbb{P}\text{-a.s.}\). It follows from Lemma 3.2 and Remark 3.1 that

\[
\text{cl}_L \left\{ x^{d_1,d_2}(t) : d_1, d_2 \geq 1 \right\} = x_0 + \text{cl}_L \left\{ \int_0^t f^{d_1}(s) dW_s + \int_0^t g^{d_2}(s) ds : d_1, d_2 \geq 1 \right\} = x_0 + \text{cl}_L \left\{ \int_0^t f^{d_1}(s) dW_s : d_1 \geq 1 \right\} + \text{cl}_L \left\{ \int_0^t g^{d_2}(s) ds : d_2 \geq 1 \right\} = X.
\]

Let \(A = \{ x^{d_1,d_2}(t) : d_1, d_2 \geq 1 \}\). Then \(\overline{A} = \text{cl}_L \left\{ x^{d_1,d_2}(t) : d_1, d_2 \geq 1 \right\}\) and so Assumption 3.1 yields

\[
\phi(t, X(t)) = \phi(t, \overline{A}) = \overline{\phi(t, A)}. \tag{3.5}
\]
From (3.4), Lemma 3.2 and Remark 3.1, one has

\[
\phi(t, X(t))_k = x_0 + cL \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_1d_2}(s)) f_i^{d_1}(s) dW_s : d_1, d_2 \geq 1 \right\} + cL \left\{ \int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, x^{d_1d_2}(s)) + \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_1d_2}(s)) g_i(s) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi_k}{\partial x_i^2}(s, x^{d_1d_2}(s))(f_i^{d_1})^2 \right] ds : d_1, d_2 \geq 1 \right\}.
\]

(3.6)

For fixed \( f^{d_1} \in S^2_\Phi(f) \), by Lemma 3.1 there exists a sequence \( \{x^{d_3}\} \subset S^2_\Phi(X) \) such that

\[
\int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s = cL \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_3 \geq 1 \right\}.
\]

(3.7)

For fixed \( x^{d_3} \in S^2_\Phi(X) \), it follows from Lemma 3.1 and (3.3) that there exists a sequence \( \{f^{d_1}\} \subset S^2_\Phi(f) \) such that

\[
\int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i(s) dW_s = cL \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_1 \geq 1 \right\}.
\]

(3.8)

Combining (3.7) and (3.8), we have

\[
\int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i(s) dW_s = cL \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_1, d_3 \geq 1 \right\}.
\]

(3.9)

Let

\[
M = \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_1, d_3 \geq 1 \right\}
\]

and

\[
N = \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_1, d_2 \geq 1 \right\}.
\]

Since \( x^{d_3} \in S^2_\Phi(X) = S^2_\Phi(cL A) \), we can choose \( d_3 = d_1 d_2 \) and so \( cL N \subset cL M \). On the other hand, for any \( a_0 \in cL M \), there exists a sequence \( \{a_q\} \subset M \) such that \( a_q \to a_0 \). Since \( a_q \in M \), there exist \( x_q \in \{x^{d_3}_q\}_{d_3=1}^\infty \subset S^2_\Phi(cL A) \) and \( f_q \in \{f_i^{d_1}\}_{d_1=1}^\infty \subset S^2_\Phi(f) \) such that

\[
a_q = \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x_q(s))(f_q)_i(s) dW_s.
\]

For fixed \( x_q \in S^2_\Phi(cL A) \), there exists a sequence \( \{x_{ql}\} \subset A \) such that \( x_{ql} \to x_q \). Letting \( f_{ql} = f_q \), we have

\[
\lim_{ql \to +\infty} \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x_{ql}(s))(f_{ql})_i(s) dW_s = a_0,
\]

which implies \( cL N \subset cL M \). Thus, we have \( cL M = cL N \) and so

\[
\int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, X(s)) f_i(s) dW_s = cL \left\{ \int_0^t \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_3}(s)) f_i^{d_1}(s) dW_s, d_1, d_2 \geq 1 \right\}.
\]

(10.1)

Similarly, we can obtain the following equality

\[
\int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, X(s)) + \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, X(s)) g_i(s) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi_k}{\partial x_i^2}(s, X(s)) f_i^{d_1} ight] ds = cL \left\{ \int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, x^{d_1d_2}) + \sum_{i=1}^n \frac{\partial \phi_k}{\partial x_i}(s, x^{d_1d_2}) g_i(s) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \phi_k}{\partial x_i^2}(s, x^{d_1d_2})(f_i^{d_1})^2 \right] ds, d_1, d_2 \geq 1 \right\}.
\]

(11.1)
Combining (3.5), (3.10), (3.11), we have

\[
\phi(t, X(t)) = \phi(0, X(0)) + \int_0^t \sum \frac{\partial \phi_k}{\partial x_i}(s, X(s)) f_i(s) dW_s
\]

\[
+ \int_0^t \left[ \frac{\partial \phi_k}{\partial t}(s, X(s)) + \sum \frac{\partial \phi_k}{\partial x_i}(s, X(s)) g_i(s) + \frac{1}{2} \sum \frac{\partial^2 \phi_k}{\partial x_i^2}(s, X(s)) f_i^2 \right] ds \quad \mathbb{P}\text{-a.s.}
\]

for \( k = 1, \ldots, n \). This completes the proof.

\[ \square \]

**Remark 3.3.** Theorem 3.7 is a set-valued version of the classical Itô’s formula.

**Theorem 3.2.** Let \( X(t) = x_T + \int_t^T f(s, Z(s)) ds \oplus \int_t^T Z(s) dW_s \) and \( Z \in \mathbb{G} \). Then

\[
X_k^2 + \int_t^T Z_k^2(s) ds \subset (x_T^2)_k + 2 \int_t^T X_k f_k(s, Z(s)) ds - 2 \int_t^T X_k Z_k(s) dW_s, \quad k = 1, \ldots, n \quad \mathbb{P}\text{-a.s.}
\]

**Proof.** From Lemma 3.1 and the property of Hukuhara difference, there exist \( \{ z^d_i \} \in S^2(Z) \) such that

\[
\int_t^T f(s, Z) ds \oplus \int_t^T Z_s dW_s \subset \int_t^T f(s, Z(s)) ds - \int_t^T Z(s) dW_s = \mathcal{C}_L \left\{ \int_t^T f(s, z^d) ds - \int_t^T z^d dW_s : d_1 \geq 1 \right\}.
\]

Let

\[
X^*(t) = x_T + \int_t^T f(s, Z(s)) ds - \int_t^T Z(s) dW_s,
\]

\[
x^d_1 = x_T + \int_t^T f(s, z^d_1) ds - \int_t^T z^d_1 dW_s.
\]

Then \( \mathcal{C}_L \{ x^d_1 : d_1 \geq 1 \} = X^* \) and \( x^d_1(t) \) is a single-valued Itô’s process. Let \( \phi(t, x) = (x_1^2, x_2^2, \ldots, x_n^2) \). By the classical Itô’s formula, we have

\[
(x^d_k^2)^2 = (x_T^2)_k + 2 \int_t^T x^d_k f_k(s, z^d_1) - (z^d_k^2) ds - 2 \int_t^T x^d_1 z^d_k(s) dW_s.
\]

Similar to the proof of Theorem 3.1 we can obtain

\[
X^*_k^2(t) = \mathcal{C}_L \left\{ (x^d_k^2)^2 : d_1 \geq 1 \right\}
\]

\[
=(x_T^2)_k + \mathcal{C}_L \left\{ 2 \int_t^T x^d_k f_k(s, z^d_1) - (z^d_k^2) ds : d_1 \geq 1 \right\} + \mathcal{C}_L \left\{ -2 \int_t^T x^d_1 z^d_k(s) dW_s : d_1 \geq 1 \right\}
\]

and so

\[
X_k^2(t) \subset X^*_k^2 = x_T^2 + 2 \int_t^T X_k f_k(s, Z(s)) - Z_k^2(s) ds - 2 \int_t^T X_k Z_k(s) dW_s, \quad k = 1, \ldots, n \quad \mathbb{P}\text{-a.s.}
\]

For any given \( y \in X_k^2 + \int_t^T Z_k^2(s) ds \), there exist \( m \in X_k^2 \) and \( n \in \int_t^T Z_k^2(s) ds \) such that \( y = m + n \). Since \( m \in X_k^2 \subset \mathcal{C}_L \{ (x^d_1^2)^2 : d_1 \geq 1 \} \) and \( n \in \int_t^T Z_k^2(s) ds = \mathcal{C}_L \left\{ \int_t^T (z^d_k^2) dW_s : d_1 \geq 1 \right\} \), we know that there exist \( \{ m_j \} \subset \{ (x^d_1^2)^2 : d_1 \geq 1 \} \) and \( \{ n_j \} \subset \left\{ \int_t^T (z^d_k^2) dW_s : d_1 \geq 1 \right\} \) such that \( m_j + n_j \rightarrow m + n \). It follows Lemma 3.2 and Remark 3.1 that

\[
m + n \in \mathcal{C}_L \left\{ (x_T^2)_k + 2 \int_t^T x^d_k f_k(s, z^d_1) ds - 2 \int_t^T x^d_1 z^d_k(s) dW_s : d_1 \geq 1 \right\}
\]

\[
=(x_T^2)_k + \mathcal{C}_L \left\{ 2 \int_t^T x^d_k f_k(s, z^d_1) ds : d_1 \geq 1 \right\} + \mathcal{C}_L \left\{ 2 \int_t^T x^d_1 z^d_k(s) dW_s : d_1 \geq 1 \right\}
\]

\[
= (x_T^2)_k + 2 \int_t^T X_k f_k(s, Z(s)) ds - 2 \int_t^T X_k Z_k(s) dW_s.
\]
This shows that
\[
X_k^2 + \int_0^T Z_k^2(s)ds \subset (x_T^2)_k + 2 \int_0^T X_k f_k(s, Z(s))ds - 2 \int_0^T X_k Z_k(s)dW_s, \quad k = 1, \ldots, n \quad \mathbb{P}\text{-a.s.}
\]
and so the proof is completed. \hfill \Box

Similar as the proof of Theorem 3.2 we can get the following Corollary.

**Corollary 3.1.** Let \( X_t = x_T + \int_0^T f(s, X_s, Z(s))ds \oplus \int_0^T Z(s)dW_s \) and \( Z \in G \). Then
\[
(X_k^2)_k + \int_0^T Z_k^2(s)ds \subset (x_T^2)_k + 2 \int_0^T (X_k)f_k(s, X_s, Z(s))ds - 2 \int_0^T (X_k)Z_k(s)dW_s, \quad k = 1, \ldots, n, \quad \mathbb{P}\text{-a.s.}
\]

## 4 An application to GSVBSDE

In this section, we apply the set-valued Itô’s formula to obtain the existence and uniqueness of solutions for GSVBSDE [11]. To this end, we first show the following lemma, which is a set-valued version of the Itô isometry.

**Lemma 4.1.** Suppose \( f \in L^2_{ad}([0, T] \times \Omega, \mathcal{P}(\mathbb{R}^m)) \). Then the integral \( I(f) = \int_0^T f(t)dW(t) \) is a set-valued random variable with \( \mathbb{E}[I(f)] = 0 \) and
\[
\mathbb{E}[I(t)]^2 = \mathbb{E} \left[ \int_0^T f^2(t)dt \right]. \quad (4.1)
\]

**Proof.** By Lemma 3.1 there exists a sequence \( \{f^d(t)\}_{d=1}^\infty \subset S^2(f) \) such that
\[
\int_0^T f(s)dW_s = \mathcal{cl}_L \left\{ \int_0^T f^d(s)dW_s : d \geq 1 \right\} \quad \mathbb{P}\text{-a.s.} \quad (4.2)
\]
and
\[
\int_0^T f^2(s)ds = \mathcal{cl}_L \left\{ \int_0^T (f^d(s))^2ds : d \geq 1 \right\} \quad \mathbb{P}\text{-a.s.} \quad (4.3)
\]
It follows from (4.2) that
\[
\mathbb{E} \left[ \int_0^T f(s)dW_s \right] = \mathbb{E} \left[ \mathcal{cl}_L \left\{ \int_0^T f^d(s)dW_s : d \geq 1 \right\} \right].
\]
For any given \( a_0 \in \mathcal{cl}_L \left\{ \int_0^T f^d(s)dW_s : d \geq 1 \right\} \), there exists a sequence \( \{f^n(t)\}_{n=1}^\infty \subset \{f^d(t)\}_{d=1}^\infty \) such that
\[
\int_0^T f^n(s)ds \to a_0.
\]
Thus, by Lemma 2.8 we know that \( \mathbb{E} \int_0^T f^n(s)ds = 0 \to \mathbb{E}a_0 \). This implies that \( \mathbb{E}a_0 = 0 \) and so
\[
\mathbb{E}[I(f)] = \mathbb{E} \left[ \int_0^T f(s)dW_s \right] = \mathbb{E} \left[ \mathcal{cl}_L \left\{ \int_0^T f^d(s)dW_s : d \geq 1 \right\} \right] = 0.
\]
Next we show that (4.1) holds. In fact, by (4.2), we have
\[
\mathbb{E}[I(f)]^2 = \mathbb{E} \left[ \mathcal{cl}_L^2 \left\{ \int_0^T f^d(s)dW_s : d \geq 1 \right\} \right] \quad \mathbb{P}\text{-a.s.},
\]
Then, we denote $x \in A$

Lemma 4.3.

Lemma 4.2. Let $b \in \text{cl}_L \left\{ \int_0^T f^d(s) dW_s : d \geq 1 \right\}$, there exists a sequence $\left\{ f^k(t) \right\}_{k=1}^\infty \subset \left\{ f^d(t) \right\}_{d=1}^\infty$ such that, for any $t$, one has $f^k(t) = f^d(t)$ for $d \geq k$. Thus, by (4.3), we know that (4.1) holds.

By Lemma 4.1, one has $X(s) \in L^2([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^{n \times m}))$ such that, for any $t$, one has $X(t) = \int_0^t Y(s) dW_s$, $\forall t \in [0,T]$. Then it follows from Lemma 4.2 that $X(t) \in Y(t)$ exists and $X(t) = Y(t) \, \mathbb{P}$-a.s.

Proof. Assume that $X(t) \not\subset Y(t)$ does not exist. Then Lemma 4.2 implies that there exists $a_0(t) \in \text{ext}(X(t))$ such that, for any $x(t) \in L^2([0,T] \times \Omega, \mathbb{R}^{n \times m})$, $a_0(t) \not\approx x(t) + Y(t)$ or $x(t) + Y(t) \not\subseteq X(t)$. Taking $x(t) = 0$, one has $a_0(t) \not\approx Y(t)$ or $Y(t) \not\subseteq X(t)$. Suppose that $a_0(t) \not\approx Y(t)$ holds. Then it follows from $a_0(t) \in \text{ext}(X(t))$ and Lemma 4.3 that

\[
\int_0^t a_0(s) dW_s \in \text{cl}_L \left\{ \int_0^T Y(s) dW_s : d \geq 1 \right\}, \, \forall t \in [0,T].
\]

From Lemma 3.1 there exists a sequence $\left\{ y^d(t) \right\}_{d=1}^\infty \subset S^2_T(Y(t))$ such that

\[
\int_0^T Y(s) dW_s = \text{cl}_L \left\{ \int_0^T y^d(s) dW_s : d \geq 1 \right\} \, \mathbb{P}$-a.s.
\]

This leads that there exists a sequence $\left\{ y^d(t) \right\}_{d=1}^\infty \subset \left\{ y^d(t) \right\}_{d=1}^\infty$ such that $\int_0^t (y^d(t) - a_0) dW_s \to 0$. It follows from Lemma 2.3 that $\mathbb{E} \left[ \int_0^T (y^d(t) - a_0)^2 dW_s \right] \to 0$, which implies $y^d(s) \to a_0 \, \mathbb{P}$-a.s.. Since $S^2_T(Y(t))$ is closed, we have $a_0 \in S^2_T(Y(t)) \subset Y(t)$. This in contradiction with $a_0(t) \not\approx Y(t)$. Similarly, when $Y(t) \not\subseteq X(t)$, we can obtain a contradiction. Thus, we know that $X(t) \subseteq Y(t)$ exists and so (4.4) implies that

\[
\int_0^t X(s) \subseteq Y(s) dW_s = 0, \, \forall t \in [0,T].
\]

By Lemma 4.4 one has

\[
\mathbb{E} \left[ \int_0^t (X(s) \subseteq Y(s))^2 dW_s \right] = 0, \, \forall t \in [0,T].
\]
Since $X(t), Y(t) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{L}(\mathbb{R}^{n \times m}))$, let $x^0(t) \in X(t) \ominus Y(t)$ such that $x_0(t) = \|X(t) \ominus Y(t)\|$, by taking $t = T$, we have
\[
\mathbb{E} \left[ \int_0^T \|X(s) \ominus Y(s)\|^2 \, ds \right] = \mathbb{E} \left[ \int_0^T x_0^2(t) \, ds \right] = \mathbb{E} \left[ \int_0^T (X(s) \ominus Y(s))^2 \, ds \right] = 0,
\]
which implies that
\[
\mathbb{E} \left[ \int_0^T \|X(s) \ominus Y(s)\|^2 \, ds \right] = \|X(s) \ominus Y(s)\|_a = 0
\]
and so $X(t) = Y(t)$ $\mathbb{P}$-a.s.. \(\square\)

We also need the following lemma.

**Lemma 4.4.** [5] Let $\xi \in L^2_T(\Omega, \mathcal{F}(\mathbb{R}^n))$ and $f(t, \omega) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$. Then there exists a unique pair $(X, Z) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathcal{G}$ such that
\[
X(t) = \xi + \int_t^T f(s, \omega) \, ds \ominus \int_t^T Z(s) \, dW_s, \quad \mathbb{P}\text{-a.s. } t \in [0,T].
\]

We first consider the following set-valued equation
\[
X(t) + \int_t^T Z(s) \, dW_s = \xi + \int_t^T f(s, Z(s)) \, ds, \quad \mathbb{P}\text{-a.s. } t \in [0,T],
\]
where $\xi \in L^2_T(\Omega, \mathcal{F}(\mathbb{R}^n))$, $Z_t \in \mathcal{G}$ and $f : [0,T] \times \Omega \times \mathcal{P}(\mathbb{R}^{n \times m}) \to \mathcal{K}(\mathbb{R}^n)$.

**Assumption 4.1.** Assume that $f : [0,T] \times \Omega \times \mathcal{P}(\mathbb{R}^{n \times m}) \to \mathcal{K}(\mathbb{R}^n)$ satisfies the following conditions:

(i) for any given $A \in \mathcal{P}(\mathbb{R}^{n \times m})$, $f(\cdot, \cdot, A) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$;

(ii) for any fixed $(t, \omega) \in [0,T] \times \Omega$ and $A, B \in \mathcal{P}(\mathbb{R}^{n \times m})$, the Hukuhara difference $f(t, \omega, A) \ominus f(t, \omega, B)$ exists whenever $A \ominus B$ exists;

(iii) for any given $A, B \in \mathcal{P}(\mathbb{R}^{n \times m})$ with $A \ominus B$ existing, there is a constant $c > 0$ such that
\[
\|f(t, \omega, A) \ominus f(t, \omega, B)\| \leq c\|A \ominus B\|, \quad \forall (t, \omega) \in [0,T] \times \Omega.
\]

**Theorem 4.1.** Let $\xi \in L^2_T(\Omega, \mathcal{F}(\mathbb{R}^n))$ and $f(t, \omega, Z(t))$ satisfy Assumption 4.1. Then there exists a pair $(X, Z) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathcal{G}$ such that
\[
X(t) + \int_t^T Z(s) \, dW_s = \xi + \int_t^T f(s, \omega, Z(s)) \, ds, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.} \tag{4.6}
\]
Moreover, if $(X_1(t), Z_1(t))$ and $(X_2(t), Z_2(t))$ are two solutions of \(4.6\) and $Z_1(t) \ominus Z_2(t)$ exists, then $X_1(t) = X_2(t)$ and $Z_1(t) = Z_2(t)$ $\mathbb{P}$-a.s..

**Proof.** We first show the existence of solutions for \(4.6\). Let $X^0 = \{0\}$ and $Z^0 = \{0\}$. By Assumption 4.1(i), we know that $f(t, Z^0) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n))$. It follows from Lemma 4.4 that there is a pair $(X^1(t), Z^1(t)) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathcal{G}$ such that
\[
X^1(t) + \int_t^T Z^1(s) \, dW_s = \xi + \int_t^T f(s, \omega, Z^0(s)) \, ds, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.}
\]
In the same way, we can obtain a sequence \(\{(X^p(t), Z^p(t))\} \subset \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathcal{G}\) such that
\[
X^p(t) + \int_t^T Z^p(s) \, dW_s = \xi + \int_t^T f(s, \omega, Z^{p-1}(s)) \, ds, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.}, \quad p = 1, 2, \ldots \tag{4.7}
\]
Now we conclude that for each $t \in [0, T]$, $X^p(t) \ominus X^{p-1}(t)$ and $Z^p(t) \ominus Z^{p-1}(t)$ exist $\mathbb{P}$-a.s. In fact, for $p = 1$, it is clear that $X^1(t) \ominus X^0$ and $Z^1(t) \ominus Z^0$ exist $\mathbb{P}$-a.s. since $X^0 = \{0\}$ and $Z^0 = \{0\}$. Assume that the assertion is true for $p - 1$ ($p > 1$). Then $X^{p-1}(t) \ominus X^{p-2}(t)$ and $Z^{p-1}(t) \ominus Z^{p-2}(t)$ exist $\mathbb{P}$-a.s. It follows from Assumption 4.1 (ii) that $f(t, \omega, Z^{p-1}(t)) \ominus f(t, \omega, Z^{p-2}(t))$ exists for all $t \in [0, T]$ and $\mathbb{P}$-a.s. Since $f(t, \omega, Z^{p-1}(t)), f(t, \omega, Z^{p-2}(t)) \in L^2_{ad}([0, T] \times \Omega, \mathcal{F}(\mathbb{R}^n))$, Lemma 2.10 shows that $f(t, \omega, Z^{p-1}(t)) \ominus f(t, \omega, Z^{p-2}(t)) \in L^2_{ad}([0, T] \times \Omega, \mathcal{F}(\mathbb{R}^n))$ and

$$
\int_t^T f(s, \omega, Z^{p-1}(s))ds \ominus \int_t^T f(s, \omega, Z^{p-2}(s))ds = \int_t^T \left[ f(s, \omega, Z^{p-1}(s)) \ominus f(s, \omega, Z^{p-2}(s)) \right]ds, \quad \forall t \in [0, T], \; \mathbb{P}$-a.s.
$$

For fixed $t \in [0, T]$, we know that $\int_t^T f(s, \omega, Z^{p-1}(s))ds, \; \int_t^T f(s, \omega, Z^{p-2}(s))ds \in L^2_T(\Omega, \mathcal{F}(\mathbb{R}^n))$ and so Lemma 2.13 implies that

$$
\mathbb{E}\left[ \int_t^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right] \ominus \mathbb{E}\left[ \int_t^T f(s, \omega, Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \int_t^T f(s, \omega, Z^{p-1}(s)) \ominus f(s, \omega, Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right].
$$

By (4.7), we have

$$
\mathbb{E}\left[ X^p(t) + \int_t^T Z^p(s)dW_s \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \mathcal{X} + \int_t^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right].
$$

It follows from Definition 2.5 and Lemma 4.1 that, $p = 1, 2, \cdots$,

$$
X^p(t) = X^p(t) + \mathbb{E}\left[ \int_t^T Z^p(s)ds \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \mathcal{X} + \int_t^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \; \mathbb{P}$-a.s. \hspace{1cm} (4.9)

Now from (4.8) and (4.9), one has

$$
X^p(t) \ominus X^{p-1}(t) = \mathbb{E}\left[ \int_t^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right] \ominus \mathbb{E}\left[ \int_t^T f(s, \omega, Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \; \mathbb{P}$-a.s.
$$

with $p = 1, 2, \cdots$. This means $X^p \ominus X^{p-1}$ exists $\mathbb{P}$-a.s.. Next we show that $Z^p \ominus Z^{p-1}$ exists $\mathbb{P}$-a.s.. To this end, let

$$
M^p(t) = \mathbb{E}\left[ \mathcal{X} + \int_0^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \; p = 1, 2, \cdots.
$$

Then, by Lemma 2.11 (ii) and Lemma 2.13, we have

$$
M^p(t) \ominus M^{p-1}(t) = \mathbb{E}\left[ \int_0^T f(s, \omega, Z^{p-1}(s)) \ominus f(s, \omega, Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \; p = 1, 2, \cdots.
$$

By (4.7) with $t = 0$, one has

$$
\mathbb{E}\left[ X^p(0) + \int_0^T Z^p(s)dW_s \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \mathcal{X} + \int_0^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right]
$$

and so Definition 2.5 shows that

$$
X^p(0) + \int_0^T Z^p(s)dW_s = \mathbb{E}\left[ \mathcal{X} + \int_0^T f(s, \omega, Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right] = M^p(t). \hspace{1cm} (4.10)
$$
Since \( M^p(t) \ominus M^{p-1}(t) \) is a uniformly square-integrable set-valued martingale, it follows from Lemma 4.12 that there exists \( \tilde{Z}^p(t) \in \mathbb{G} \) such that

\[
M^p(t) \ominus M^{p-1}(t) = M^p(0) \ominus M^{p-1}(0) + \int_0^t \tilde{Z}^p(s)dW_s = X^p(0) \ominus X^{p-1}(0) + \int_0^t \tilde{Z}^p(s)dW_s.
\]

Moreover, by (4.10) and Lemma 2.11 one has

\[
X^p(0) + \int_0^t Z^p(s)dW_s = M^{p-1}(t) + M^p(t) \ominus M^{p-1}(t)
\]

\[
= X^{p-1}(0) + \int_0^t Z^{p-1}(s)dW_s + X^p(0) \ominus X^{p-1}(0) + \int_0^t \tilde{Z}^p(s)dW_s
\]

\[
= X^{p-1}(0) + X^p(0) \ominus X^{p-1}(0) + \int_0^t [Z^{p-1}(s) + \tilde{Z}^p(s)]dW_s, \forall t \in [0,T], \mathbb{P}\text{-a.s.}
\]

with \( p = 1, 2, \cdots \). From Lemma 4.3 this yields that \( Z^p(t) = Z^{p-1}(t) + \tilde{Z}^p(t) \in \mathbb{G} \) and so \( Z^p(t) \ominus Z^{p-1}(t) = \tilde{Z}^p(t) \) exist \( \mathbb{P}\text{-a.s.} \). Thus, by the induction, we know that for each \( t \in [0,T] \), \( X^p(t) \ominus X^{p-1}(t) \) and \( Z^p(t) \ominus Z^{p-1}(t) \) exist \( \mathbb{P}\text{-a.s.} \).

Next we show that there is a pair \( (X(t), Z(t)) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathbb{G} \) such that

\[
\|X(s) \ominus X^p(s)\|_s \to 0, \quad \|Z(s) \ominus Z^p(s)\|_s \to 0.
\]

Indeed, it follows from (4.7) that

\[
X^{p+1}(t) \ominus X^p(t) + \int_t^T Z^{p+1}(s) \ominus Z^p(s)dW_s = \int_t^T f_i(s, Z^p(s)) \ominus f_i(s, Z^{p-1}(s))ds, \forall t \in [0,T], \mathbb{P}\text{-a.s.}, p = 1, 2, \cdots,
\]

which is equal to

\[
X^{p+1}_i(t) \ominus X^p_i(t) + \int_t^T Z^{p+1}_i(s) \ominus Z^p_i(s)dW_s = \int_t^T f_i(s, Z^p(s)) \ominus f_i(s, Z^{p-1}(s))ds, \quad t \in [0,T], \quad i = 1, \cdots, n,
\]

where

\[
X^{p+1}_i(t) \ominus X^p_i(t), f_i(t, Z^p(t)) \ominus f_i(t, Z^{p-1}(t)) \in \mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}))
\]

and

\[
(Z^{p+1}_i(t) \ominus Z^p_i(t))^T \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0,T] \times \Omega, \mathcal{K}(\mathbb{R}^m)))
\]

are the components of \( X^{p+1}(t) \ominus X^p(t), f(s, Z^p(s)) \ominus f(t, Z^{p-1}(t)) \) and \( Z^{p+1}(t) \ominus Z^p(t), \) respectively. It follows from Theorem 3.2 that

\[
\langle X^{p+1}(t) \ominus X^p(t) \rangle^2 + \int_t^T (Z^{p+1}(s) \ominus Z^p(s))^2 ds < 2 \int_t^T (X^{p+1}_i(s) \ominus X^p_i(s))(f_i(s, Z^p(s)) \ominus f_i(s, Z^{p-1}(s)))ds - 2 \int_t^T (X^{p+1}_i(s) \ominus X^p_i(s))(Z^{p+1}_i(s) \ominus Z^p_i(s))dW_s,
\]

\[
i = 1, \cdots, n.
\]

From Lemma 4.1 Assumption 4.1 and the basic inequality \( \frac{1}{2}a^2 + \rho b^2 \geq 2ab \) with \( \rho = 2c^2 \), we have

\[
\mathbb{E} \langle X^{p+1}(t) \ominus X^p(t) \rangle^2 + \mathbb{E} \int_t^T \|Z^{p+1}(s) \ominus Z^p(s)\|^2 ds \leq \frac{1}{2} \mathbb{E} \int_t^T \|Z^p(s) \ominus Z^{p-1}(s)\|^2 ds + 2c^2 \mathbb{E} \int_t^T \|X^{p+1}(s) \ominus X^p(s)\|^2 ds.
\]

(4.12)
Denote
\[ u_p(t) = E \int_t^T \|X^p(s) \ominus X^{p-1}(s)\|^2 ds, \quad v_p(t) = E \int_t^T \|Z^p(s) \ominus Z^{p-1}(s)\|^2 ds, \quad p = 1, 2, \ldots. \]

Then (4.12) leads to
\[ -\frac{d}{dt}(u_{p+1}(t)e^{2c^2 t}) + e^{2c^2 t}v_{p+1}(t) \leq \frac{1}{2}e^{2c^2 t}v_p(t). \quad (4.13) \]

Integrating from \( t \) to \( T \) for two sides of (4.13), we obtain
\[ u_{p+1}(t) + \int_t^T e^{2c^2(s-t)}v_{p+1}(s)ds \leq \frac{1}{2} \int_t^T e^{2c^2(s-t)}v_p(s)ds. \quad (4.14) \]

Noting \( u_{p+1}(t), v_{p+1}(t) \geq 0 \), it follows from (4.14) that
\[ \int_t^T e^{2c^2(s-t)}v_{p+1}(s)ds \leq \frac{1}{2} \int_t^T e^{2c^2(s-t)}v_p(s)ds. \]

Iterating the inequality and taking \( t = 0 \) in above inequality, we have
\[ \int_0^T e^{2c^2 t}v_{p+1}(t)dt \leq 2^{-p}c e^{2c^2}, \]

where
\[ \bar{c} = \sup_{0 \leq t \leq T} v_1(t) = E \int_0^T \|Z^1(t)\|dt. \]

Moreover, it follows from (4.14) that
\[ u_{p+1}(t) \leq \frac{1}{2} \int_t^T e^{2c^2(s-t)}v_p(s)ds \]

and so
\[ u_{p+1}(0) \leq 2^{-p}c e^{2c^2}. \quad (4.15) \]

However, from (4.13), (4.15) and the fact that \( \frac{d}{dt}u_{p+1}(t) \leq 0 \), we have
\[ v_{p+1}(0) \leq 2c^2u_{p+1}(0) + \frac{1}{2}v_p(0) \leq 2^{-p+1}c^2 e^{2c^2} + \frac{1}{2}v_p(0). \]

It is easy to check that
\[ v_{p+1}(0) \leq 2^{-p} \left( p\bar{c}e^{2c^2} + v_1(0) \right). \quad (4.16) \]

For any \( q > p \), from Lemma 2.2 we know that \( X^q(t) \ominus X^p(t) \) and \( Z^q(t) \ominus Z^p(t) \) exist and
\[ X^q(t) \ominus X^p(t) = X^q(t) \ominus X^{q-1}(t) + X^{q-1}(t) \ominus X^{q-2}(t) + \cdots + X^{q+1}(t) \ominus X^p(t), \]
\[ Z^q(t) \ominus Z^p(t) = Z^q(t) \ominus Z^{q-1}(t) + Z^{q-1}(t) \ominus Z^{q-2}(t) + \cdots + Z^{q+1}(t) \ominus Z^p(t). \]

It follows from (4.15), (4.16) and the triangle inequality that
\[ \|X^q(t) \ominus X^p(t)\| \leq (q-p)2^{-p}c e^{2c^2} \]

and
\[ \|Z^q(t) \ominus Z^p(t)\| \leq (q-p)2^{-p} \left( p\bar{c}e^{2c^2} + v_1(0) \right). \]
Thus, Theorem 2.7 and Remark 2.4 show that there exists a pair $(X, Z) \in \mathcal{L}^2_{ad}([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathcal{G}$ such that

$$
\|X(t) \ominus X^p(t)\|_s \to 0, \quad \|Z(t) \ominus Z^p(t)\|_s \to 0.
$$

Now it follows from (4.7) that

$$
X(t) + \int_t^T Z(s) dW_s = \xi + \int_t^T f(s, \omega, Z(s)) ds, \quad \forall t \in [0, T], \; \mathbb{P}\text{-a.s.}
$$

Next we show the uniqueness. Assume that $(X_1(t), Z_1(t))$ and $(X_2(t), Z_2(t))$ are two solutions of (4.6) and $Z_1(t) \ominus Z_2(t)$ exists. Then

$$
X_1(t) + \int_t^T Z_1(s) dW_s = \xi + \int_t^T f(s, Z_1(s)) ds, \quad X_2(t) + \int_t^T Z_2(s) dW_s = \xi + \int_t^T f(s, Z_2(s)) ds.
$$

It follows from Assumption 4.1 (ii) that $f(s, Z_1(s)) \ominus f(s, Z_2(s))$ exists. By Lemma 2.10 and Lemma 2.13 we have

$$
X_1(t) \ominus X_2(t) = \mathbb{E} \left[ \xi + \int_t^T f(s, Z_1(s)) ds \bigg| \mathcal{F}_t \right] \ominus \mathbb{E} \left[ \xi + \int_t^T f(s, Z_2(s)) ds \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \int_t^T f(s, Z_1(s)) \ominus f(s, Z_2(s)) ds \bigg| \mathcal{F}_t \right],
$$

which implies that $X_1(t) \ominus X_2(t)$ exists. This yields that

$$
X_1(t) \ominus X_2(t) + \int_t^T Z_1(s) \ominus Z_2(s) dW_s = \int_t^T f(s, Z_1(s)) \ominus f(s, Z_2(s)) ds,
$$

which is equal to

$$
X_{1i}(t) \ominus X_{2i}(t) + \int_t^T Z_{1i}(s) \ominus Z_{2i}(s) dW_s = \int_t^T f_i(s, Z_1(s)) \ominus f_i(s, Z_2(s)) ds, \quad i = 1, \ldots, n,
$$

where $X_{1i}(t) \ominus X_{2i}(t), f_i(t, Z_1(t)) \ominus f_i(t, Z_2(t)) \in \mathcal{L}^2_{ad}([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n)), (Z_{1i}(t) \ominus Z_{2i}(t))^T \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n)))$ are the component of $X_1(t) \ominus X_2(t), f(s, Z_1(t)) \ominus f(s, Z_2(t))$ and $Z_1(t) \ominus Z_2(t)$ respectively. It follows from Corollary 5.2 that

$$
\mathbb{E} \|X(t) \ominus X^p(t)\|^2 + \mathbb{E} \int_t^T \|Z_1(s) \ominus Z_2(s)\|^2 ds
\leq 2\mathbb{E} \int_t^T \|X_1(s) \ominus X_2(s)\| f_1(s, Z_1(s)) \ominus f_1(s, Z_2(s)) ds
\leq \frac{1}{2} \mathbb{E} \int_t^T \|Z_1(s) \ominus Z_2(s)\|^2 ds + 2c^2 \mathbb{E} \int_t^T \|X_1(s) \ominus X_2(s)\|^2 ds. \tag{4.17}
$$

Denote

$$
u_1(t) = \mathbb{E} \int_t^T \|X_1(s) \ominus X_2(s)\|^2 ds, \quad \nu_2(t) = \mathbb{E} \int_t^T \|Z_1(s) \ominus Z_2(s)\|^2 ds.
$$

Then (4.17) leads to

$$
- \frac{d}{dt} \left( u_1(t) e^{2c^2 t} \right) + e^{2c^2 t} \nu_1(t) \leq \frac{1}{2} e^{2c^2 t} \nu_1(t). \tag{4.18}
$$

Integrating from $t$ to $T$ for two sides of (4.18), we obtain

$$
u_1(t) + \int_t^T e^{2c^2 (s-t)} \nu_1(s) ds \leq \frac{1}{2} \int_t^T e^{2c^2 (s-t)} \nu_1(s) ds.
$$

This implies that

$$
\nu_1(t) \leq \int_t^T e^{2c^2 (s-t)} \nu_1(s) ds, \quad \nu_2(t) \leq \frac{1}{2} \nu_1(t).
$$

Thus, $\nu_1(0) = 0$ and $u_1(0) = 0$.

\[ \Box \]
We now consider the following set-valued equation
\[ X(t) + \int_t^T Z(s)dW_s = \xi + \int_t^T f(s, X(s), Z(s))ds, \quad \mathbb{P}\text{-a.s. } t \in [0, T]. \]
where \( \xi \in L_2^2(\Omega, \mathcal{K}(\mathbb{R}^n)), \) \( Z_t \in \mathbb{G} \) and \( f : [0, T] \times \Omega \times \mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^{n \times m}) \rightarrow \mathcal{K}(\mathbb{R}^n) \).

**Assumption 4.2.** Assume \( f : [0, T] \times \Omega \times \mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^{n \times m}) \rightarrow \mathcal{K}(\mathbb{R}^n) \) satisfies the following conditions:

(i) for any given \( A \in \mathcal{K}(\mathbb{R}^n) \) and \( B \in \mathcal{P}(\mathbb{R}^{n \times m}), f(\cdot, \cdot, A, B) \in L_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n)); \)

(ii) for any fixed \( (t, \omega) \in [0, T] \times \Omega, A, B \in \mathcal{K}(\mathbb{R}^n) \) and \( C, D \in \mathcal{P}(\mathbb{R}^{n \times m}), \) the Hukuhara difference \( f(t, \omega, A, C) \oplus f(t, \omega, B, C) \) exists whenever \( A \oplus B \) exists and the Hukuhara difference \( f(t, \omega, B, D) \) exists whenever \( C \ominus D \) exists;

(iii) there exists a constant \( c > 0 \) such that, for any \( A, B \in \mathcal{K}(\mathbb{R}^n) \) and \( C, D \in \mathcal{P}(\mathbb{R}^{n \times m}) \) with \( A \oplus B \) and \( C \ominus D \) existing,
\[ \| f(t, \omega, A, C) \ominus f(t, \omega, B, D) \| \leq c(\| A \oplus B \| + \| C \ominus D \|), \quad \forall t \in [0, T]. \]

**Theorem 4.2.** Let \( \xi \in L_2^2(\Omega, \mathcal{K}(\mathbb{R}^n)) \) and \( f \) satisfy the Assumption 4.2. Then, there exists a pair \((X, Z) \in L_{ad}^2([0, T] \times \Omega, \mathcal{K}(\mathbb{R}^n)) \times \mathbb{G}\) such that
\[ X(t) + \int_t^T Z(s)dW_s = \xi + \int_t^T f(s, X(s), Z(s))ds, \quad \mathbb{P}\text{-a.s. } t \in [0, T]. \quad (4.19) \]
Moreover, if \((X_1(t), Z_1(t)) \) and \((X_2(t), Z_2(t)) \) are two solutions to (4.19) with \( X_1(t) \ominus X_2(t) \) and \( Z_1(t) \ominus Z_2(t) \) existing, then \( X_1(t) = X_2(t), \ Z_1(t) = Z_2(t) \) \( \mathbb{P}\text{-a.s.} \).

**Proof.** We first show the existence of solutions for (4.19). For any given \( X^0 = \{0\} \) and \( Z^0 = \{0\} \), by Lemma [4.3], we can obtain a sequence \( \{(X^p(t), Z^p(t))\} \) satisfying
\[ X^p(t) + \int_t^T Z^p(s)dW_s = \xi + \int_t^T f(s, X^{p-1}(s), Z^{p-1}(s))ds, \quad t \in [0, T], \quad p = 1, 2, \ldots. \quad (4.20) \]
We conclude that \( X^p(t) \ominus X^{p-1}(t), \ Z^p(t) \ominus Z^{p-1}(t), \ f(t, X^p(t), Z^p(t)) \ominus f(t, X^{p-1}(t), Z^{p-1}(t)) \) exist. In fact, for \( p = 1 \), it is trivial since \( X^0 = \{0\} \) and \( Z^0 = \{0\} \). Assume that the assertion is true for \( p - 1 \) with \( p > 1 \). Then \( X^{p-1}(t) \ominus X^{p-2}(t) \) and \( Z^{p-1}(t) \ominus Z^{p-2}(t) \) exist. From Assumption 4.2 (ii), \( f(t, X^{p-1}(t), Z^{p-1}(t)) \ominus f(t, X^{p-2}(t), Z^{p-1}(t)) \) and \( f(t, X^{p-2}(t), Z^{p-1}(t)) \ominus f(t, X^{p-2}(t), Z^{p-2}(t)) \) exist for any \( t \in [0, T] \). It follows from Lemma 2.13 that \( f(t, X^{p-1}(t), Z^{p-1}(t)) \ominus f(t, X^{p-2}(t), Z^{p-2}(t)) \) exists and
\[ f(t, X^{p-1}(t), Z^{p-1}(t)) \ominus f(t, X^{p-2}(t), Z^{p-2}(t)) = f(t, X^{p-1}(t), Z^{p-1}(t)) - f(t, X^{p-2}(t), Z^{p-1}(t)) + f(t, X^{p-2}(t), Z^{p-1}(t)) - f(t, X^{p-2}(t), Z^{p-2}(t)). \]
Thus, Lemma 2.13 yields that
\[ \mathbb{E} \left[ \int_t^T f(s, X^{p-1}(s), Z^{p-1}(s)) - f(s, X^{p-2}(s), Z^{p-2}(s))ds \right] \bigg| \mathcal{F}_t \]
\[ = \mathbb{E} \left[ \int_t^T f(s, X^{p-1}(s), Z^{p-1}(s))ds \right] \bigg| \mathcal{F}_t \right] \oplus \mathbb{E} \left[ \int_t^T f(s, X^{p-2}(s), Z^{p-2}(s))ds \right] \bigg| \mathcal{F}_t \right]. \quad (4.21) \]
By (4.20), one has
\[ \mathbb{E} \left[ X^p(t) + \int_t^T Z^p(s)dW_s \right] \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \xi + \int_t^T f(s, \omega, X^{p-1}(s), Z^{p-1}(s))ds \right] \bigg| \mathcal{F}_t \right]. \]
It follows from Definition 2.20 and Lemma 2.11 that
\begin{align*}
X^p(t) &= X^p(t) + \mathbb{E}\left[ \int_t^T Z^p(s)dW_s \right] \\
&= \mathbb{E}\left[ \xi + \int_t^T f(s, \omega, X^{-1}(s), Z^{-1}(s))ds \bigg| \mathcal{F}_t \right], \quad \forall t \in [0, T], \text{P-a.s., } p = 1, 2, \cdots.
\end{align*}

By (4.21) and (4.22), \( X^n(t) \cap X^{n-1}(t) \) exists and
\begin{align*}
X^p(t) \cap X^{p-1}(t) &= \mathbb{E}\left[ \xi + \int_t^T f(s, X^{p-1}(s), Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right] \\
& \quad \oplus \mathbb{E}\left[ \xi + \int_t^T f(s, X^{p-2}(s), Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right].
\end{align*}

Let
\begin{align*}
M^p(t) &= \mathbb{E}\left[ \xi + \int_0^T f(s, X^{p-1}(s), Z^{p-1}(s))ds \bigg| \mathcal{F}_t \right].
\end{align*}

Then by the same argument above, we can show that \( M^p(t) \cap M^{p-1}(t) \) exists and
\begin{align*}
M^p(t) \cap M^{p-1}(t) &= \mathbb{E}\left[ \xi + \int_0^T f(s, X^{p-1}(s), Z^{p-1}(s)) \cap f(s, X^{p-2}(s), Z^{p-2}(s))ds \bigg| \mathcal{F}_t \right].
\end{align*}

By 4.20 with \( t = 0 \), we have
\begin{align*}
\mathbb{E}\left[ X^p(0) + \int_0^T Z^p(s)dW_s \bigg| \mathcal{F}_t \right] &= \mathbb{E}\left[ \xi + \int_0^T f(s, \omega, X^{-1}(s), Z^{-1}(s))ds \bigg| \mathcal{F}_t \right].
\end{align*}

It follows from Definition 2.20 that
\begin{align*}
M^p(t) &= X^p(0) + \int_0^t Z^p(s)dW_s = \mathbb{E}\left[ \xi + \int_0^T f(s, \omega, X^{-1}(s), Z^{-1}(s))ds \bigg| \mathcal{F}_t \right].
\end{align*}

Noting that \( M^p(t) \cap M^{p-1}(t) \) is a uniformly square-integrable set-valued martingale, by Lemma 2.12 there exists \( \tilde{Z}^p(t) \in \mathcal{G} \) such that
\begin{align*}
M^p(t) \cap M^{p-1}(t) &= M^p(0) \cap M^{p-1}(0) + \int_0^T \tilde{Z}^p(s)dW_s = X^p(0) \cap X^{p-1}(0) + \int_0^T \tilde{Z}^p(s)dW_s.
\end{align*}

Moreover, by 4.21 and Lemma 2.11 one has
\begin{align*}
X^p(0) + \int_0^t Z^p(s)dW_s &= M^{p-1}(t) \cap M^{p-1}(t) \\
&= X^{p-1}(0) + X^p(0) \cap X^{-1}(0) + \int_0^t Z^{p-1}(s)dW_s + \int_0^t \tilde{Z}^p(s)dW_s \\
&= X^{p-1}(0) + X^p(0) \cap X^{-1}(0) + \int_0^t [Z^{p-1}(s) + \tilde{Z}^p(s)]dW_s, \forall t \in [0, T], \text{P-a.s.}
\end{align*}

with \( p = 1, 2, \cdots \). From Lemma 4.3, this implies \( Z^p(t) = Z^{p-1}(t) + \tilde{Z}^p(t) \in \mathcal{G} \) and so \( Z^p(t) \cap Z^{p-1}(t) = \tilde{Z}^p(t) \) exists. Thus, it follows from (4.22) that
\begin{align*}
X^{p+1}(t) \cap X^p(t) + \int_t^T Z^{p+1}(s) \cap Z^p(s)dW_s &= \int_t^T f(s, X^{p+1}(s), Z^{p+1}(s)) \cap f(s, X^p(s), Z^p(s))ds, \quad t \in [0, T],
\end{align*}
which is equal to
\begin{align*}
X^{p+1}(t) \cap X^p(t) + \int_t^T Z^{p+1}(s) \cap Z^p(s)dW_s &= \int_t^T f_i(s, X^{p+1}(s), Z^{p+1}(s)) \cap f_i(s, X^p(s), Z^p(s))ds, \quad t \in [0, T],
\end{align*}
where
\[ X^{p+1}(t) \ominus X^p(t), \quad f_i(t, X^p(t), Z^{n+1}(t)) \ominus f_i(t, X^{p-1}(t), Z^p(t)) \in L^2_{ad}([0, T] \times \Omega, K(\mathbb{R})) \]

and
\[ (Z^{p+1}(t) \ominus Z^p(t))^T \in K_w(L^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^m))) \]

are the components of \(X^{p+1}(t) \ominus X^p(t), f(t, X^p(t), Z^{p+1}(t)) \ominus f(t, X^{p-1}(t), Z^p(t))\) and \(Z^{p+1}(t) \ominus Z^p(t)\), respectively, \(i = 1 \cdots, n\).

Similar to the proof of Theorem 4.1 from Corollary 3.1, Lemma 4.1 and Assumption 4.2 (ii), we have
\[ E\|X^{p+1}(t) \ominus X^p(t)\|^2 + E \int_t^T \|Z^{p+1}(s) \ominus Z^p(s)\|^2 ds \leq 4c^2 E \int_t^T \|X^{p+1}(s) \ominus X^p(s)\|^2 ds + \frac{1}{2} E \int_t^T \|X^p(s) \ominus X^{p-1}(s)\|^2 ds + \frac{1}{2} E \int_t^T \|Z^p(s) \ominus Z^{p-1}(s)\|^2 ds, \quad i = 1, \ldots, n. \]

Denote
\[ u_p(t) = E \int_t^T \|X^p(s) \ominus X^{p-1}(s)\|^2 ds, \quad v_p(t) = E \int_t^T \|Z^p(s) \ominus Z^{p-1}(s)\|^2 ds, \quad p = 1, 2, \ldots. \]

Then it follows that
\[ -\frac{d(u_{p+1}(t)e^{4c^2t})}{dt} + e^{4c^2t}v_{p+1}(t) \leq \frac{1}{2} e^{4c^2t}(u_p(t) + v_p(t)), \quad \forall t \in [0, T], \quad u_{n+1}(T) = 0, \quad p = 1, 2, \ldots. \]

Integrating from \(t\) to \(T\) for the two sides of the above inequality, one has
\[ u_{p+1}(t) + \int_t^T e^{4c^2(s-t)}v_{p+1}(s)ds \leq \frac{1}{2} \int_t^T e^{4c^2(s-t)}u_p(s)ds + \frac{1}{2} \int_t^T e^{4c^2(s-t)}v_p(s)ds, \]

which implies that
\[ u_{p+1}(t) \leq \frac{1}{2} \int_t^T e^{4c^2(s-t)}u_p(s)ds + \frac{1}{2} \int_t^T e^{4c^2(s-t)}v_p(s)ds \quad (4.24) \]

and
\[ \int_t^T e^{4c^2(s-t)}v_{p+1}(s)ds \leq \frac{1}{2} \int_t^T e^{4c^2(s-t)}u_p(s)ds + \frac{1}{2} \int_t^T e^{4c^2(s-t)}v_p(s)ds. \quad (4.25) \]

Let
\[ c_1 = \int_0^T \|Z^1\|ds = \sup_{0 \leq t \leq T} v_1(t), \quad c_2 = \int_0^T \|X^1\|ds = \sup_{0 \leq t \leq T} u_1(t). \]

Iterating the inequalities \(4.24\) and \(4.25\), and taking \(t = 0\), we have
\[ u_{p+1}(0) \leq \frac{T(c_1 + c_2)}{2^{p-1}} \sum_{k=1}^p \frac{(e^{4c^2T})^k}{k!}, \quad \int_0^T e^{4c^2s}v_{p+1}(s)ds \leq \frac{T(c_1 + c_2)}{2^{p-1}} \sum_{k=1}^p \frac{(e^{4c^2T})^k}{k!}. \]

Thus, \(u_p(0) \to 0\) and \(v_p(0) \to 0\). For \(q > p\), from Lemma 2.2, \(X^q(t) \ominus X^p(t)\) and \(Z^q(t) \ominus Z^p(t)\) exist and
\[ X^q(t) \ominus X^p(t) = X^q(t) \ominus X^{q-1}(t) + X^{q-1}(t) \ominus X^{q-2}(t) + \cdots + X^{p+1}(t) \ominus X^p(t), \]
\[ Z^q(t) \ominus Z^p(t) = Z^q(t) \ominus Z^{q-1}(t) + Z^{q-1}(t) \ominus Z^{q-2}(t) + \cdots + Z^{p+1}(t) \ominus Z^p(t). \]

From \(4.24\), \(4.25\) and the triangle inequality, we have
\[ \|X^q(t) \ominus X^p(t)\| \leq (q-p) \frac{T(c_1 + c_2)}{2^{p-1}} \sum_{k=1}^p \frac{(e^{4c^2T})^k}{k!} \]

22
and
\[ \|Z^q(t) \otimes Z^p(t)\|_s \leq (q-p) \frac{(c_1 + c_2)p}{2^{p-1}} \frac{p}{k!} (e^{4c^2T})^k \]
Thus, Theorem 2.7 and Remark 2.1 show that there exists a pair \((X, Z) \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^n)) \times \mathbb{G}\) such that
\[ \|X(t) \otimes X^p(t)\|_s \to 0, \quad \|Z(t) \otimes Z^p(t)\|_s \to 0. \]
Now (4.20) leads to
\[ X(t) + \int_t^T Z(s)dW_s = \xi + \int_t^T f(s, \omega, X(s), Z(s))ds, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \]
Next we prove the uniqueness of solutions for (4.19). Let \((X_1(t), Z_1(t))\) and \((X_2(t), Z_2(t))\) be two solutions. Then
\[ X_1(t) + \int_t^T Z_1(s)dW_s = \xi + \int_t^T f(s, X_1(s), Z_1(s))ds \]
and
\[ X_2(t) + \int_t^T Z_2(s)dW_s = \xi + \int_t^T f(s, X_2(s), Z_2(s))ds. \]
Since \(X_1(t) \otimes X_2(t), Z_1(t) \otimes Z_2(t)\) exist, by Assumption 4.2 (ii) and Lemma 2.2 we know that \(f(t, X_1(t), Z_1(t)) \otimes f(t, X_2(t), Z_2(t))\) exists and
\[ f(t, X_1(t), Z_1(t)) \otimes f(t, X_2(t), Z_2(t)) = f(t, X_1(t), Z_1(t)) \otimes f(t, X_1(t), Z_2(t)) + f(t, X_1(t), Z_2(t)) \otimes f(t, X_2(t), Z_2(t)) \]
This implies that
\[ X_1(t) \otimes X_2(t) + \int_t^T Z_1(s) \otimes Z_2(s)dW_s = \int_t^T f(s, X_1(s), Z_1(s)) \otimes f(s, X_2(s), Z_2(s))ds, \]
which equal to
\[ X_{1i}(t) \otimes X_{2i}(t) + \int_t^T Z_{1i}(s) \otimes Z_{2i}(s)dW_s = \int_t^T f_i(s, X_1(s), Z_1(s)) \otimes f_i(s, X_2(s), Z_2(s))ds, \quad i = 1, \cdots, n, \]
where
\[ X_{1i}(t) \otimes X_{2i}(t), f_i(t, X_1(t), Z_1(t)) \otimes f_i(t, X_2(t), Z_2(t)) \in \mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R})) \]
and
\[ (Z_{1i}(t) \otimes Z_{2i}(t))^T \in \mathcal{K}_w(\mathcal{L}^2_{ad}([0, T] \times \Omega, K(\mathbb{R}^m))) \]
are the component of \(X_1(t) \otimes X_2(t), f(t, X_1(t), Z_1(t)) \otimes f(t, X_2(t), Z_2(t))\) and \(Z_1(t) \otimes Z_2(t)\), respectively. It follows from Corollary 3.1, Lemma 2.8 and Assumption 4.2 (ii) that
\[ E\|X_1(t) \otimes X_2(t)\|^2 + E\int_t^T \|Z_1(s) \otimes Z_2(s)\|^2ds \leq \frac{1}{2}E\int_t^T \|Z_1(s) \otimes Z_2(s)\|^2ds + (4c^2 + \frac{1}{2})E\int_t^T \|X_1(s) \otimes X_2(s)\|^2ds, \]
Denote
\[ u_1(t) = E\int_t^T \|X_1(s) \otimes X_2(s)\|^2ds, \quad v_1(t) = E\int_t^T \|Z_1(s) \otimes Z_2(s)\|^2ds. \]
Then it follows that
\[ -\frac{d(u_1(t)e^{4(c^2 + \frac{1}{2})t})}{dt} + e^{4c^2t}v_1(t) \leq \frac{1}{2}e^{4(c^2 + \frac{1}{2})t}v_1(t), \quad \forall t \in [0, T]. \]
Integrating from $t$ to $T$ for the two sides of the above inequality, we have

$$u_1(t) + \int_t^T e^{(4c^2+\frac{1}{2})(s-t)}v_1(s)ds \leq \frac{1}{2} \int_t^T e^{(4c^2+\frac{1}{2})(s-t)}v_1(s)ds$$

This implies that $v_1(t) \leq \frac{1}{2}v_1(t)$ and $u_1(t) \leq \frac{1}{2} \int_t^T e^{(4c^2+\frac{1}{2})(s-t)}v_1(s)ds$. Thus, we have $v_1(0) = 0$ and $u_1(0) = 0$. \hfill \Box

**Remark 4.1.**

(i) Theorem 4.2 reduces to Theorem 5.9 of [8] when $f(t, X(t), Z(t)) \equiv f(t, X(t))$;

(ii) Theorem 4.2 gives an answer to an open problem proposed by Ararat et al. [8].

**References**

[1] B. Ahmad, S. Sivasundaram. Dynamics and stability of impulsive hybrid setvalued integro-differential equations with delay. *Nonlinear Anal. TMA*, 2006, 65(11): 2082-2093.

[2] B. Ahmad, S. Sivasundaram. The monotone iterative technique for impulsive hybrid setvalued integro-differential equations. *Nonlinear Anal. TMA*, 2006, 65(12): 2260-2276.

[3] A. Alexander. Mean-field type games between two players driven by backward stochastic differential equations. *Games*, 2018, 9(4): 88.

[4] A.N. Al-Hussaini, R.L. Elliott. An extension of Itô’s differentiation formula. *Nagoya Math. J.*, 1987, 105: 9-18.

[5] D.B. Applebaum, R.L. Hudson. Fermion Itô’s formula and stochastic evolutions. *Communications in Mathematical Physics*, 1984, 96(4): 473-496.

[6] J.P. Aubin, A. Cellina. *Differential Inclusions*. Springer, Berlin, 1984.

[7] J.P. Aubin, H. Frankowska. *Set-valued Analysis*. Birkhäuser, Boston, 1990.

[8] C. Ararat, J. Ma, W.Q. Wu. Set-valued backward stochastic differential equation. 2020, [arXiv:2007.15073](https://arxiv.org/abs/2007.15073).

[9] A. Aswani. Statistics with set-valued functions applications to inverse approximate optimization. *Math. Prog.*, 2019, 174(1): 225-251.

[10] Bender, Christian, R. Knobloch, P. Oberacker. A generalised Itô’s formula for Lévy-driven Volterra processes. *Proces. Appl.*, 2015, 125(8): 2989-3022.

[11] D. Borkowski, Forward and backward filtering based on backward stochastic differential equations. *Inver. Probl. Imag.*, 2017, 10(2): 305-325.

[12] P. Catuogno, C. Olivera. Time-dependent tempered generalized functions and Itô’s formula. *Appl. Anal.*, 2014, 93(3): 539-550.

[13] K. Deimling. *Multivalued Differential Equations*. Gruyter, Berlin, 1992.

[14] R.J. Elliott, T.K. Siu. Markovian forward-backward stochastic differential equations and stochastic flows. *Sys. Control Lett.*, 2012, 61(10):1017-1022.
[15] M. Gradinaru, I. Nourdin, F. Russo, et al. m-order integrals and generalized Itô’s formula: the case of a fractional Brownian motion with any Hurst index. *Anna. L’I.H.P. Probab. Statist.*, 2005, 41(4): 781-806.

[16] M. Hukuhara. Integration des applications measurables dont la valeur est un compact convexe. *Funkcionalaj Ekvacioj.*, 1967, 10: 205-223.

[17] A.H. Hamel, F. Heyde, B. Rudloff. Set-valued risk measures for conical market models. *Math. Finan. Econ.*, 2011, 5(1): 1-28.

[18] M.S. Hu, S.L. Ji, S.G. Peng, Y.S. Song. Backward stochastic differential equations driven by G-Brownian motion. *Stoch. Proces. Appl.*, 2012, 124(1):759-784.

[19] S. Hamadene, J.P. Lepeltier. Zero-sum stochastic differential games and backward equations. *Sys. Control Lett.*, 1995, 24(4): 259-263.

[20] K. Itô. Stochastic integral. *Proc. Imp. Acad. Tokyo*, 1944, 22: 519-524.

[21] K. Itô. *On Stochastic Differential Equations*. Amer. Math. Soc., Providence, 1951.

[22] K. Itô. On a formula concerning stochastic differentials. *Nagoya Math. J.*, 1951, 3:55-65.

[23] N.E. Karoui, S.G. Peng, M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 1997, 7(1):1-71.

[24] M. Kisielewicz. *Stochastic Differential Inclusions and Applications*. Springer, Berlin, 2013.

[25] M. Kisielewicz. Martingale representation theorem for set-valued martingales. *J. Math. Anal. Appl.*, 2014, 409(1): 111-118.

[26] K. Kuratowski, C. Ryll-Nardzewski. A general theorem on selectors. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 1965, 13: 397-403.

[27] V. Lakshmikantham, A.A. Tolstonogov. Existence and interrelation between set and fuzzy differential equations. *Nonlinear Anal. TMA*, 2003, 55(3): 255-268.

[28] V. Lakshmikantham, T.G. Bhaskar, J.V. Devi. *Theory of Set Differential Equations in a Metric Space*. Cambridge Scientific, Cambridge, 2006.

[29] Z. Li, J.W. Luo. Mean-field reflected backward stochastic differential equations. *Stoch. Proces. Appl.*, 2012, 82(11):1961-1968.

[30] J. Li, S. Li, Y. Ogura. Strong solution of Itô type set-valued stochastic differential equation. *Acta Math. Sinica*, 2010, 26(9): 1739-1748.

[31] S.G. Peng. Backward stochastic differential equation driven by fractional Brown motion. *SIAM J. Control Optim.*, 2009, 48(3): 1675-1700.

[32] E. Pardoux, S.G. Peng. Adapted solution of a backward stochastic differential equation. *Sys. Control Lett.*, 1990, 14: 55-61.

[33] N.D. Phu, L.T. Quang, T.T. Tung. Stability criteria for set control differential equations. *Nonlinear Anal. TMA*, 2008, 69(11): 3715-3721.
[34] D. Revuz. M. Yor. *Continuous Martingales and Brownian Motion*. Springer, Berlin, 2008.

[35] A. Tolstonogov. *Differential Inclusions in a Banach Space*. Springer, Dordrecht, 2000.

[36] G.C. Wang, H. Xiao. Linear quadratic non-zero sum differential games of backward stochastic differential equations with asymmetric information. [arXiv:1407.0430](arXiv:1407.0430), 2014.

[37] C. Xu, Q. Chen, H.B. Hu, J.L. Xu, X.J. Hei. Authenticating aggregate queries over set-valued data with confidentiality. *IEEE Trans. Knowl. Data Eng.*, 2018, 30(3): 630-644.

[38] J.A. Zaslavski. Turnpike theorem for a class of set-value mappings. *Numer. Func. Anal. Optim.*, 1996, 17(1-2): 215-240.

[39] J.F. Zhang. *Backward Stochastic Differential Equations*. Springer, New York, 2017.