On the integer \( \{k\} \)-domination number of circulant graphs

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Abstract

Let \( G = (V, E) \) be a simple undirected graph. \( G \) is a circulant graph defined on \( V = \mathbb{Z}_n \) with difference set \( D \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \} \) provided two vertices \( i \) and \( j \) in \( \mathbb{Z}_n \) are adjacent if and only if \( \min\{|i-j|, n-|i-j|\} \in D \). For convenience, we use \( G(n; D) \) to denote such a circulant graph.

A function \( f : V(G) \rightarrow \mathbb{N} \cup \{0\} \) is an integer \( \{k\} \)-domination function if for each \( v \in V(G) \), \( \sum_{u \in N_G[v]} f(u) \geq k \). By considering all \( \{k\} \)-domination functions \( f \), the minimum value of \( \sum_{v \in V(G)} f(v) \) is the \( \{k\} \)-domination number of \( G \), denoted by \( \gamma_k(G) \). In this paper, we prove that if \( D = \{1, 2, \ldots, t\}, 1 \leq t \leq n-1 \), then the integer \( \{k\} \)-domination number of \( G(n; D) \) is \( \lceil \frac{k n}{2t+1} \rceil \).

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1 Introduction and preliminaries

The study of domination number of a graph \( G \) has been around for quite a long time. Due to its importance in applications, there are various versions of extension study, see [3] for reference.

The idea of integer \( \{k\} \)-domination was proposed by Domke et al. in [2]. It can be dealt as a labeling problem. The vertices of the graph \( G \) are labeled by integers in \( \mathbb{N} \cup \{0\} \) such that for each vertex \( v \), the total (sum) values in its closed neighborhood \( N_G[v] \) must be at least \( k \). The problem is asking for finding the minimum total value labeled on \( G \). Finally, we say that \( f : V(G) \rightarrow \mathbb{N} \cup \{0\} \)
is an integer \( \{k\}\)-domination function if for each \( v \in V(G) \), \( \sum_{u \in N_G[v]} f(u) \geq k \).

Among all such functions \( f \), the minimum value of \( \sum_{v \in V(G)} f(v) \) is called the integer \( \{k\}\)-domination number of \( G \), denoted by \( \gamma_k(G) \).

It is not difficult to see that the original domination number of a graph \( G \), \( \gamma(G) \), can be recognized as \( \gamma_1(G) \) since the vertices with label ‘1’ gives a dominating set. For more information about domination problem, the readers may refer to \( \{1, 4, 5, 6, 8\} \). Hence, the integer \( \{k\}\)-domination problem is also an NP-hard problem. So far, results obtained are all on special classes of graphs, see \( \{1, 3, 7, 9\} \).

In this paper, we shall consider the class of circulant graph \( G = G(n; D) \) where \( D = \{1, 2, \ldots, t\}, \) \( 1 \leq t \leq \frac{2n-1}{3} \), i.e., \( V(G) = \mathbb{Z}_n \) and two vertices \( i \) and \( j \) are adjacent if and only if \( d(i,j) := \min\{|i-j|, n-|i-j|\} \in D \). Since \( D = \{1, 2, \ldots, t\} \), \( G(n; D) \) is exactly the power graph \( C_n^t \) where \( C_n \) is a cycle of order \( n \).

The following results are obtained by Lin \[10\]. For clearness, we also outline its proof in which basic linear algebra is applied.

**Proposition 1.1 (III).** Let \( G \) be the circulant graph \( G(n; D) \) where \( D = \{1, 2, \ldots, t\} \). Then, \( \gamma_k(G) \geq \lfloor \frac{k n}{2t+1} \rfloor \).

**Proof.** Let \( A \) be the adjacency matrix of \( G \) and \( f \) be an \( \{k\}\)-domination function of \( G \). Let \( I_n \) denote the all 1 column vector of length \( n \). Then, we have

\[
(2t+1) \sum_{v \in V(G)} f(v) = (f(v_1), f(v_2), \ldots, f(v_n))(A + I_n)1_n \geq 1_n^T \cdot k \cdot 1_n = nk,
\]

which implies the inequality. \( \Box \)

By the aid of an algorithm, Lin was able to show the following.

**Proposition 1.2 (III).** For \( t \leq 5 \), \( \gamma_k(G(n; \{1, 2, \ldots, t\})) = \lfloor \frac{k n}{2t+1} \rfloor \).

But, for larger \( t \), it remains unsettled. Our main result of this paper shows that the equality holds for all \( 1 \leq t \leq \frac{4n-4}{3} \).

## 2 The main result

By Proposition 1.1, in order to determine \( \gamma_k(G) \), it suffices to show that \( \gamma_k(G) \leq \lfloor \frac{n k}{2t+1} \rfloor \). That is, we need a proper distribution of values for \( f(v_1), f(v_2), \ldots, f(v_n) \) such that for each \( v_i \), \( \sum_{u \in N_G[v_i]} f(u) \geq k \) and \( \sum_{i=1}^n f(v_i) \leq \lfloor \frac{k n}{2t+1} \rfloor \). Since we are dealing with circulant graphs, \( \sum_{u \in N_G[v_i]} f(u) \) is in fact the sum of \( 2t + 1 \) consecutive labels assigned to the circle \( C_n = (v_1, v_2, \ldots, v_n) \). Therefore, we turn our focus on providing suitable labels to meet the condition.

For example, let \( n = 8 \) and \( t = 2 \). Then, the following labeling of \( (v_1, v_2, \ldots, v_8) \), \((x_4, x_3, x_1, x_2, x_0, x_4, x_3, x_1)\) will satisfy the requirement, where \( x_i = \lfloor \frac{4i}{9} \rfloor \), \( i = 0, 1, 2, 3, 4 \).

We are considering \( 1 \leq t \leq \frac{4n-4}{3} \) in what follows. First, we need an estimation of the sum of rational numbers which take its floor or ceiling values.
**Lemma 2.1.** For positive integers \(a, b\) and nonnegative integer \(k\), we have the following.

1. \(\left\lfloor \frac{k+x}{a} \right\rfloor = \left\lfloor \frac{k+x}{a} \right\rfloor\) for any real number \(x\),
2. \(k = \sum_{i=0}^{a-1} \left\lfloor \frac{k+i}{a} \right\rfloor\), and
3. \(\left\lceil \frac{ak}{b} \right\rceil = \sum_{i=1}^{a} \left\lfloor \frac{k+\lfloor (i+1)b/a \rfloor - 1}{b} \right\rfloor\).

**Proof.** (1) and (2) are easy to check, we prove (3).

\[
\left\lceil \frac{ak}{b} \right\rceil = \sum_{i=0}^{a-1} \left\lfloor \frac{k+i}{b} \right\rfloor = \sum_{i=0}^{a-1} \frac{(ak+b-1)/b+i}{a} = \sum_{i=0}^{a-1} \frac{k + ((i+1)b-1)/a}{b} = \sum_{i=0}^{a-1} \frac{k + [((i+1)b-1)/a]}{b} = \sum_{i=0}^{a-1} \frac{k + [((i+1)b-1)/a] - 1}{b}.
\]

Since the variables \(a, b\) and \(k\) are all integers, the uniqueness of the formula in Lemma 2.1(3) can be confirmed.

**Corollary 2.2.** If integers \(0 \leq s_0 \leq s_1 \leq \cdots \leq s_{a-1} < b\) satisfy

\[
\left\lceil \frac{ak}{b} \right\rceil = \sum_{i=0}^{a-1} \frac{k+s_i}{b}
\]

for positive integers \(a, b\) and nonnegative integer \(k\), then \(s_i = \lceil (i+1)b/a \rceil - 1\) for \(i = 0, 1, \ldots, a - 1\).}

According to the \(s_i\)'s given above, we split \([b] := \{0, 1, \ldots, b-1\}\) into subintervals with maximal elements \(s_i\)'s. For positive integers \(a < b\), let \([b]\) be partitioned into \(a\) subsets such that

\[
S_i = \left\{ \left\lceil \frac{b_i}{a} \right\rceil, \left\lceil \frac{b_i}{a} \right\rceil + 1, \ldots, \left\lfloor \frac{b(i+1)}{a} \right\rfloor - 1 \right\}
\]

for \(i = 0, 1, \ldots, a - 1\). It is clear that \(|S_i| - |S_j|\leq 1\) for all \(i, j\). We analyze the subsets containing more elements in the following.

**Lemma 2.3.** Let \(q\) and \(r\) be the quotient and remainder of \(b\) divided by \(a\), respectively. Then the cardinality \(|S_i| = q + 1\) if \(i = \left\lfloor \frac{a}{r} \right\rfloor\) for \(j = 0, 1, \ldots, r - 1\).
Proof. By definition, \( |S_i| = [(i + 1)b/a] - [ib/a] = q + [(i + 1)r/a] - [ir/a] \). Therefore, \( |S_i| = q + 1 \) if and only if there exists some integer \( 0 \leq j \leq r - 1 \) such that \( ir/a \leq j < (i + 1)r/a \). The above inequality can be rewritten as \( i \leq ja/r < i + 1 \), and hence \( i = \lfloor ja/r \rfloor \).

Example 2.4. Let \( a = 3 \), \( b = 8 \), and \( r = 2 \) be the remainder of \( b \) divided by \( a \). Then

\[
\left\lfloor \frac{3k}{8} \right\rfloor = \sum_{i=1}^{3} \left\lfloor \frac{k + \left\lfloor \frac{8i}{3} - 1 \right\rfloor}{8} \right\rfloor = \left\lfloor \frac{k + 2}{8} \right\rfloor + \left\lfloor \frac{k + 5}{8} \right\rfloor + \left\lfloor \frac{k + 7}{8} \right\rfloor,
\]

and \( \{8\} = \{0,1,\ldots,7\} \) can be partitioned into 3 subsets such that

\[
S_0 = \{0,1,2\}, \ S_1 = \{3,4,5\} \text{ and } S_2 = \{6,7\},
\]

where the subsets numbered with \( \lfloor \frac{a_j}{r} \rfloor = 0 \) and 1 as \( j = 0 \) and 1, respectively, have more than 1 elements. Note that the maximal elements 2, 5, 7 of subsets \( S_i \)'s are the integers in \( \{1\} \) that construct \( \lfloor \frac{3k}{8} \rfloor \).

Additionally, we need a result of the comparison between two sequences. For two real finite non-decreasing sequences \( A = (a_i), A' = (a'_i) \) of the same length \( n \), we say that \( A \leq A' \) if \( a_i \leq a'_i \) for \( i = 0,1,\ldots,n-1 \).

Lemma 2.5. Let \( A \) and \( A' \) be two subsequences of a real finite non-decreasing sequence \( B \) which have equal length \( 0 < |A| = |A'| < |B| \). Then \( A \leq A' \) if and only if \( B \setminus A' \leq B \setminus A \).

Proof. Because of the symmetry, we prove \( A \leq A' \) implies \( B \setminus A' \leq B \setminus A \) by induction on \( |A| \) in the following. It is clearly true when \( |A| = 1 \). Suppose the statement is correct for \( |A| < m < |B| \). Assume that \( A = (a_i)_{i=0}^{m-1} \) and \( A' = (a'_i)_{i=0}^{m-1} \) satisfying \( A \leq A' \). From induction hypothesis, \( B \setminus (a'_i)_{i=0}^{m-1} \leq B \setminus (a_i)_{i=0}^{m-1} \). It is clear that the non-decreasing sequence obtained by exchanging an entry \( a \) of the original sequence into \( \tilde{a} \geq a \) (and inserting \( \tilde{a} \) to the appropriate position) is not less than the original sequence. Thus, we have

\[
B \setminus A' \leq B \setminus \tilde{A} \leq B \setminus A,
\]

where \( \tilde{A} \) is obtained from \( A \) by deleting \( a_0 \) and adding \( a'_0 \). The result follows.

Now, we are ready to find the desired integer \( \{k\} \)-domination function \( f \). Let \( [a] := \{0,1,\ldots,a-1\} \) for each positive integer \( a \). For a sequence \( A \) of length \( a \), let the entries of \( A \) indexed by \( [a] \) and \( A(i) \) be the \( i \)-th entry of \( A \). For \( 0 \leq i < j \leq a \), the subsequence \( A[i : j] := [A(i), A(i+1), \ldots, A(j-1)] \). If \( A \) is a permutation of \( [a] \), then the complement of \( A \) is a sequence \( \overline{A} \) of length \( a \) defined as \( \overline{A}(i) = a - 1 - A(i) \) for \( 0 \leq i \leq a - 1 \). The concatenation \( A \circ B \) of two sequences \( A \) and \( B \) of lengths \( a \) and \( b \), respectively, is a sequence of length
In this case, the remainders of \( j \) case 2:

\( a \) result is straightforward by the definition of extension sequence.

Let \( A \) be a permutation of \([a]\) and thus a sequence of length \( a \). For positive integers \( a < b \), we call \( B \) the extension sequence of the pair \((A, b)\) if \( B \) is a permutation of \([b]\) satisfying \( B(i) < B(j) \) if and only if \( A(i_0) < A(j_0) \) or \( A(i_0) = A(j_0) \) with \( i < j \), where \( i_0 \) and \( j_0 \) are the remainders of \( i \) and \( j \) divided by \( a \), respectively. For example, when \((a, b) = (3, 7)\) and \( A = [0, 1, 2] \), the extension sequence of \((A, b)\) is \( B = [0, 3, 5, 1, 4, 6, 2] \), which is attained by extending \( A \) to the sequence \([0, 1, 2, 0, 1, 2, 0]\) of length 7 and renumbering it with \([0, 1, \ldots, 6]\).

A permutation \( A \) of \([a]\) is said to be nice corresponding to some \( b > a \) with \( a \nmid b \) if

\[
A(i) < A(i + r) \quad \text{for } 0 \leq i \leq a - r - 1
\]

and

\[
A(j) < A(j - a + r) \quad \text{for } a - r \leq j \leq a - d - 1,
\]

where \( r \) is the remainder of \( b \) divided by \( a \) and \( d = \gcd(a, b) \). Note that if \( r = d \) then the condition \( \mathbf{[3]} \) can be ignored. For example, \([1, 3, 0, 2, 4]\) is nice corresponding to 8 (or any larger integer congruent to 3 modulo 5) and \([4, 1, 6, 3, 0, 5, 2, 7]\) is nice corresponding to 13.

The following properties will carry out the recursive constructions.

**Proposition 2.6.** Suppose that \( R \) is a nice permutation of \([r]\) corresponding to some \( a > r \) with \( r \nmid a \). Let \( \overline{R} \) be the complement of \( R \). Then the extension sequence of \((\overline{R}, a)\) is also nice corresponding to some \( b > a \) with \( b \equiv r \pmod{a} \).

**Proof.** Let \( A \) be the extension sequence of \((\overline{R}, a)\). Note that \( A(i) < A(i + r) \) for \( 0 \leq i \leq a - r - 1 \) can be verified directly by the definition of extension sequences. Assume that \( s \) is the remainder of \( a \) divided by \( r \). It’s left to consider the case \( a - r \leq j \leq a - d - 1 \) where \( r \) is the remainder of \( b \) divided by \( a \) and \( d = \gcd(a, b) \).

Assume that \( s \) is the remainder of \( a \) divided by \( r \). By Euclidean algorithm, \( s \geq d \).

**Case 1:** \( a - r \leq j \leq a - s - 1 \).

\( a - r \leq j \leq a - s - 1 \) in order to show \( A(j) < A(j - a + r) \), we observe that the remainders of \( j \) and \( j - a + r \) divided by \( r \) are \( j' + s \) and \( j' \), respectively, where \( j' = j - a + r \). Moreover, since \( R \) is nice, we have \( \overline{R}(i + s) < \overline{R}(i) \) for \( 0 \leq i \leq r - s - 1 \). The result is straightforward by the definition of extension sequences.

**Case 2:** \( a - s \leq j \leq a - d - 1 \).

In this case, the remainders of \( j \) and \( j - a + r \) divided by \( r \) become \( j' - r + s \) and \( j' \), respectively, where \( j' = j - a + r \). Once again, since \( R \) is nice, \( \overline{R}(i - r + s) < \overline{R}(i) \) for \( r - s \leq i \leq r - d - 1 \). We have the proof. \( \square \)
Proposition 2.7. Let positive integers $a < b$ with $r > 0$ the remainder of $b$ divided by $a$ and $R$ a permutation of $[r]$ with complement $\overline{R}$. If the extension sequence $A$ of $(R, a)$ satisfies

$$\left\lfloor \frac{rk}{a} \right\rfloor = \sum_{i=a-r}^{a-1} \left\lfloor \frac{k + A(i)}{a} \right\rfloor,$$

then the extension sequence $B$ of $(A', b)$ satisfies

$$\left\lfloor \frac{ak}{b} \right\rfloor = \sum_{i=b-a}^{b-1} \left\lfloor \frac{k + B(i)}{b} \right\rfloor$$

where $A'$ is the extension sequence of $(\overline{R}, a)$.

Proof. By Corollary 2.2

$$\{ A(i) \mid a - r \leq i \leq a - 1 \} = \left\{ \left\lfloor \frac{aj}{r} \right\rfloor - 1 \mid 1 \leq j \leq r \right\}.$$

Let $q$ and $s$ be the quotient and remainder of $a$ divided by $r$, respectively. Claim that the set of $A'[0 : r]$ equals the set of $a - 1 - A[a - r : a - 1]$. It is clear for $s = 0$. If $s > 0$, we have

$$A(i) = a - 1 - A'(a - s + i) \quad \text{for } i = 0, 1, \ldots, s - 1,$$

since entries in $A$ that larger than $A(i)$ become smaller than $A'(a - s + i)$ in $A'$, and vice versa. Therefore, the set of $A[s+j r : s + (j + 1)r]$ equals to the set of $a - 1 - A'[a - s - (j + 1)r : a - s - jr]$ for $j = 0, 1, \ldots, q$. The claim follows by taking $j = q - 1$. Moreover, since

$$a - 1 - (\left\lfloor \frac{ai}{r} \right\rfloor - 1) = a + \left\lfloor \frac{-ai}{r} \right\rfloor = \left\lfloor \frac{a(r-i)}{r} \right\rfloor$$

for $1 \leq i \leq r$, we have

$$\{ A'(i) \mid 0 \leq i \leq r - 1 \} = \left\{ \left\lfloor \frac{aj}{r} \right\rfloor \mid 0 \leq j \leq r - 1 \right\}$$

which exactly indicates the indices of subsets defined in Lemma 2.3. Hence the set of $B[b - a : b]$ gives the maximal elements in each of the subsets $S_0, S_1, \ldots, S_{a-1}$, and this fact completes the proof.

For each pair of positive integers $(a, b)$ with $a < b$, define two codes $C_1$ and $C_2$ as follows. If $a$ divides $b$, then

$$C_1(a, b) := [0, 1, \ldots, a - 1].$$

If the remainder $r$ of $b$ divided by $a$ is positive, then

$$C_1(a, b) := \text{the extension sequence of } (C_1(r, a), a)$$
where $C_1(r, a)$ is the complement of $C_1(r, a)$. Now $C_2$ can be constructed subsequently. Let $C_2(a, b)$ be the extension sequence of $(C_1(a, b), b)$. It is clear that $C_1(a, b)$ and $C_2(a, b)$ are permutations of $[a]$ and $[b]$, respectively. Suppose that each entry $a$ in $C_2(a, b)$ is corresponding to $\lfloor \frac{k + a}{b} \rfloor$. Then the following result can be obtained by proving that

$$C := B[b - a : b] \circ B \circ \cdots \circ B$$

is a feasible distribution of the circulant graph $G$, where $b = 2t + 1$, $n = qb + a$, and $B = C_2(a, b)$.

**Theorem 2.8.**

$$\gamma_k(G) = \left[ \frac{kn}{2t + 1} \right].$$

**Proof.** By Proposition 1.1, it suffices to show that $\gamma_k \leq \left[ \frac{kn}{2t + 1} \right]$. Let $G = G(n; \{1, 2, \ldots, t\})$, $n = qb + a$ and $b = 2t + 1$. First, we construct $B[b - a : b]$. If $a$ divides $b$ such that $b = \ell a$, then

$$B[b - a : b] = [\ell - 1, 2\ell - 1, \ldots, a\ell - 1]$$

which collects the numbers $\lfloor ib/a \rfloor - 1$ for $1 \leq i \leq a$ given in Lemma 2.1.

Therefore, any substring of $C$ of length $a$ is not larger than $B[b - a : b]$. By Lemma 2.5 every length $b$ string of $B \circ B[b - a : b]$ or $B[b - a : b] \circ B$ is not less than $[0, 1, \ldots, b - 1]$, so does $C$. Furthermore, the sequence $[0, 1, \ldots, b - 1]$ is of sum

$$\sum_{i=0}^{b-1} \left\lfloor \frac{k + i}{b} \right\rfloor = k,$$

which confirms the case for $a$ divides $b$.

On the other hand, let $a > \gcd(a, b)$ and $r$ be the remainder of $b$ divided by $a$. Since the initial case is examined above, by Proposition 2.7 $B[b - a : b]$ collects the elements $\{\lfloor ib/a \rfloor - 1\}_{i=1}^{a}$. For the initial case, if $r$ divides $a$ then $C_1(r, a) = [0, 1, \ldots, r - 1]$. Moreover, by Proposition 2.6 $C_1(a, b)$ is always nice, and hence

$$C[i : i + a] \leq B[b - a : b] \quad \text{for} \quad 0 \leq i \leq a - d - 1,$$

where $d = \gcd(a, b)$. Moreover, we also have $C[i : i + a] \leq B[b - a : b]$ for $a - d \leq i \leq qb - 1$ immediately from the construction of $C$. The result follows. \(\square\)

**Example 2.9.** Assume that $G(n; D)$ is a circulant graph on $n = 8$ vertices with $D = \{1, 2\}$ (i.e., $t = 2$). Let $b = 2t + 1 = 5$ and $a = 3$ be the remainder of $n$ divided by $b$. First of all, we obtain $C_1(3, 5)$ by the process of Euclidean algorithm. Since the initial condition $C_1(1, 2) = [0], C_1(2, 3) = [0, 1]$ is directly
the extension code of \([0\). Next, the complement of \(C_1(2, 3)\) is \(\overline{C_1(2, 3)} = [1, 0]\). Thus, \(C_1(3, 5) = [1, 0, 2]\), the extension code of \([1, 0, 3]\). Hence,
\[
C_2(3, 5) = [2, 0, 4, 3, 1],
\]
the extension code of \((C_1(3, 5), 5)\). Attach the last 3 entries in front of \(C_2(3, 5)\), we attain the distribution \([4, 3, 1, 2, 0, 4, 3, 1]\). Then the circular sequence \(f(v) : v \in V(G)\) is given by
\[
\begin{align*}
\left\lfloor \frac{k}{5} \right\rfloor, & \left\lfloor \frac{k + 3}{5} \right\rfloor, \left\lfloor \frac{k + 1}{5} \right\rfloor, \left\lfloor \frac{k + 2}{5} \right\rfloor, \left\lfloor \frac{k}{5} \right\rfloor, \left\lfloor \frac{k + 4}{5} \right\rfloor, \left\lfloor \frac{k + 3}{5} \right\rfloor, \left\lfloor \frac{k + 1}{5} \right\rfloor
\end{align*}
\]
which satisfies \(\sum_{v \in V(G)} f(v) = \lceil 8k/5 \rceil\) and \(\sum_{u \in N_G[v]} f(u) \geq k\) for each \(v \in V(G)\).

3 Concluding remark

We remark finally that the construction of code \(C_2\) can be obtained by giving an algorithm with inputs \(a\) and \(b\).

\begin{verbatim}
Data: Positive integers \(a < b\).
Result: \(C_2(a, b)\).
\end{verbatim}

\begin{verbatim}
C_1(a, b) \text{ if } a = \gcd(a, b) \text{ then }
| \quad \text{return } [0, 1, \ldots, a - 1];
| \text{end}
\text{else}
\quad r \leftarrow \text{the remainder of } b \text{ divided by } a;
\quad R \leftarrow C_1(r, a);
\quad \text{return the extension sequence of } (\overline{R}, a);
\text{end}
\end{verbatim}

Main(\(a, b\) return the extension sequence of \((C_1(a, b), b)\);

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