Generating Anisotropic Fluids From Vacuum Ernst Equations

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Abstract

Starting with any stationary axisymmetric vacuum metric, we build anisotropic fluids. With the help of the Ernst method, the basic equations are derived together with the expression for the energy-momentum tensor and with the equation of state compatible with the field equations. The method is presented by using different coordinate systems: the cylindrical coordinates $\rho, z$ and the oblate spheroidal ones. A class of interior solutions matching with stationary axisymmetric asymptotically flat vacuum solutions is found in oblate spheroidal coordinates. The solutions presented satisfy the three energy conditions.

Keywords : Anisotropic pressure; Ernst equations; Interior solutions.
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Introduction

In literature, many solutions generating technique exist. Ehlers [1] first showed how it is possible to construct new stationary exterior solutions and interior ones starting from static vacuum solutions and applying certain conformal transformations to auxiliary metrics defined on three-dimensional manifolds. These are one-parameter family of solutions. Geroch [2] showed that one can obtain an infinite-parameter family. Further, Xanthopoulos [3] gave a technique for generating, from any one-parameter family of vacuum solutions, a two-parameter family. Physically, it is very important to find interior solutions describing isolated rotating fluids. Methods have been
developed [4] [11] to build a physically admissible source for a rotating body, but a complete physically reasonable space-time for an isolated body is still lacking. Particularly interesting is the technique described in [12] to generate perfect fluid solutions by using a method similar to the vacuum generating method present in [13] [14]. The equations of state compatible with this method are \( \epsilon = p \), \( \epsilon + 3p = 0 \). This technique has been also applied in [15] and generalized in [16] to anisotropic fluids. Anisotropic fluids are having an increasing interest since they are considered physically reasonable and appropriate in systems with higher density and therefore for very compact objects as the core of neutron stars. Anisotropic fluids have been studied, for example, in [17] [19].

In this paper we present a simple technique to obtain anisotropic fluids from any vacuum solution of Ernst [20] equations. A technique similar to the one outlined in this paper has been used [21] in a cosmological context, with two spacelike Killing vectors, to generate inhomogeneous cosmological solutions. In particular, we consider solutions representing isolated bodies together with a study of the matching conditions with any stationary asymptotically flat vacuum metric. All the solutions presented satisfy the three energy conditions [22] [23].

In sections 1 and 2 we write down the basic equations. In section 3 our method is presented and the energy-momentum tensor compatible with the field equations is studied together with a discussion of the energy conditions. In section 4 we analyze a class of solutions available with our method. Section 5 collects some final remarks.

1 Basic Equations

Our starting point is the line element for a stationary axisymmetric space-time:

\[
\text{d}s^2 = e^v \left[ (\text{d}x^1)^2 + (\text{d}x^2)^2 \right] + \text{d}\phi^2 + 2m\text{d}\phi\text{d}t - f\text{d}t^2,
\]

with:

\[
v = v(x^1, x^2), \quad l = l(x^1, x^2), \quad m = m(x^1, x^2), \quad f = f(x^1, x^2),
\]

where \( x^1, x^2 \) are spatial coordinates, \( x^3 = \phi \) is an angular coordinate and \( x^4 = t \) is a time coordinate. Further, we have [24] [26]:

\[
fl + m^2 = \rho^2,
\]

where \( \rho \) is the radius in a cylindrical coordinate system. With the only non-vanishing components of the energy-momentum tensor \( T_{\mu\nu} \) given by
$T_{11}, T_{22}, T_{33}, T_{34}, T_{44}$, the field equations $R_{\mu\nu} = -(T_{\mu\nu} - \frac{g_{\mu\nu}T}{2})$, where $T = g_{\mu\nu}T_{\mu\nu}$, are:

\[ R_{11} = \frac{1}{2}(v_{11} + v_{22}) + \frac{\rho_{11}}{\rho} - \frac{v_{11} \rho_{11}}{2\rho} + \frac{v_{22} \rho_{22}}{2\rho} - \frac{1}{2\rho^2}(f_{11}l_1 + m_1^2) = \]

\[ = \frac{e^v}{2}T - T_{11}, \quad (4) \]

\[ R_{22} = \frac{1}{2}(v_{11} + v_{22}) + \frac{\rho_{22}}{\rho} + \frac{v_{11} \rho_{11}}{2\rho} - \frac{v_{22} \rho_{22}}{2\rho} - \frac{1}{2\rho^2}(f_{22}l_2 + m_2^2) = \]

\[ = \frac{e^v}{2}T - T_{22}, \quad (5) \]

\[ R_{12} = \frac{\rho_{12}}{\rho} - \frac{v_{21} \rho_{11}}{2\rho} - \frac{v_{12} \rho_{22}}{2\rho} - \frac{1}{4\rho^2}\Pi' = 0, \quad (6) \]

\[ R_{33} = \frac{e^{-v}}{2} \left[ \tilde{\nabla}^2 l + \frac{f}{\rho^2}(f_{11}l_1 + f_{22}l_2 + m_1^2 + m_2^2) \right] = \frac{1}{2}T - T_{33}, \quad (7) \]

\[ R_{34} = \frac{e^{-v}}{2} \left[ \tilde{\nabla}^2 m + \frac{m}{\rho^2}(f_{11}l_1 + f_{22}l_2 + m_1^2 + m_2^2) \right] = \frac{m}{2}T - T_{34}, \quad (8) \]

\[ R_{44} = \frac{e^{-v}}{2} \left[ -\tilde{\nabla}^2 f - \frac{f}{\rho^2}(f_{11}l_1 + f_{22}l_2 + m_1^2 + m_2^2) \right] = \]

\[ = -\frac{f}{2}T - T_{44}, \quad (9) \]

where:

\[ \Pi' = f_{11}l_2 + f_{22}l_1 + 2m_1m_2, \]

\[ \tilde{\nabla}^2 = \partial_{\alpha\alpha} - \frac{\rho_\alpha}{\rho} \partial_\alpha. \quad (10) \]

A summation over $\alpha$ is implicit in (10) with $\alpha = 1, 2$, i.e. $x^1, x^2$ and subindices denote partial derivatives. From (4) and (5) we obtain:

\[ -\frac{\rho_{11}}{\rho} + \frac{\rho_{22}}{\rho} + \frac{v_{11} \rho_{11}}{\rho} - \frac{v_{22} \rho_{22}}{\rho} + \frac{1}{2\rho^2}\Sigma' = T_{11} - T_{22}, \quad (11) \]

where:

\[ \Sigma' = f_{11}l_1 - f_{22}l_2 + m_1^2 - m_2^2 \quad (12) \]

From the equations (4) and (11) we can obtain a first order differential system for $v$. In the vacuum (4), (5), (6) reduce to two independent equations. In fact, when (3), (4), (11) and (7)-(9) are used, the equation $R_{11} + R_{22}$ becomes an identity. Therefore, the relevant equations for the vacuum are (6) and (11) together with (7)-(9) (with $T_{\mu\nu} = 0$).
In the vacuum, the equations (7)-(9) permit to know $f, l, m$ and, as a result, the equations (6) and (11) can be completely solved. Conversely, when matter is present, equations (4)-(6) do not reduce to two independent equations. Therefore, an equation can be obtained by adding (4) with (5). Further, the vacuum equations, i.e. (4)-(9) with $T_{\mu\nu} = 0$, imply that:

$$\rho_{\alpha\alpha} = 0.$$  \hspace{1cm} (13)

Condition (13) is retained also in the presence of matter since it is an assumption greatly simplifying the calculations.

Finally, we get:

$$v_1 = c - \frac{\rho_1 \Sigma' + \rho_2 \Pi'}{2 \rho (\rho_1^2 + \rho_2^2)} + \frac{\rho \rho_1 (T_{11} - T_{22})}{\rho_1^2 + \rho_2^2},$$  \hspace{1cm} (14)

$$v_2 = d + \frac{\rho_2 \Sigma' - \Pi' \rho_1}{2 \rho (\rho_1^2 + \rho_2^2)} - \frac{\rho \rho_2 (T_{11} - T_{22})}{\rho_1^2 + \rho_2^2},$$  \hspace{1cm} (15)

$$v_{11} + v_{22} - \frac{1}{2 \rho^2} (f_\alpha l_\alpha + m_\alpha^2) = T e^v - T_{11} - T_{22},$$  \hspace{1cm} (16)

$$c = \frac{2 \rho_1 \rho_2 + \rho_1 (\rho_{11} - \rho_{22})}{\rho_1^2 + \rho_2^2}, \quad d = \frac{2 \rho_1 \rho_2 - \rho_2 (\rho_{11} - \rho_{22})}{\rho_1^2 + \rho_2^2},$$  \hspace{1cm} (17)

$$e^{-v}[\nabla^2 f + \frac{f}{\rho^2} (f_\alpha l_\alpha + m_\alpha^2)] = f T + 2 T_{44},$$  \hspace{1cm} (18)

$$e^{-v}[\nabla^2 l + \frac{l}{\rho^2} (f_\alpha l_\alpha + m_\alpha^2)] = l T - 2 T_{33},$$  \hspace{1cm} (19)

$$e^{-v}[\nabla^2 m + \frac{m}{\rho^2} (f_\alpha l_\alpha + m_\alpha^2)] = m T - 2 T_{34}.$$  \hspace{1cm} (20)

### 2 Ernst-like form for the field equations

In this section we write down the relevant field equations in a form similar to the one of vacuum Ernst equations.

First of all, thanks to (3), we can eliminate $l$ from the field equations and so the equation (19), with the help of (18) and (20), becomes an identity. Further, we introduce the functions $\gamma, \omega$ with $e^{2\gamma} = fe^v, m = f \omega$. After made these simplifications, we can introduce the Ernst potential $\Phi$, where:

$$\Phi_1 = \frac{f^2}{\rho} \omega_2, \quad \Phi_2 = -\frac{f^2}{\rho} \omega_1.$$  \hspace{1cm} (21)

When (21) is used, the equation (20) is an identity. To obtain another field equation, we impose the integrability condition for (21), i.e. $\omega_{12} = \omega_{21}$.
Further, to take advantage of the Ernst method [20] for the vacuum, the simplest assumption is:

\begin{align*}
2T_{44} + fT &= 0, \quad \text{(22)} \\
\omega fT - 2T_{34} &= 0, \quad \text{(23)} \\
\frac{(p^2 - \omega^2 f^2)}{f}T - 2T_{33} &= 0. \quad \text{(24)}
\end{align*}

The conditions (22)-(24) permit us to set to zero the right hand side of the equations (18)-(20). In practice, the equations involving the functions $f, \Phi$ are the same of the vacuum case. As a result, the field equations (14)-(15) and (18)-(20) become

\begin{align*}
f \nabla^2 f + \Phi f_\alpha - f_\alpha^2 &= 0, \quad \text{(25)} \\
f \nabla^2 \Phi - 2f_\alpha \Phi_\alpha &= 0, \quad \text{(26)}
\end{align*}

\begin{align*}
\gamma_1 &= -\frac{\rho_1 \Sigma + \rho_2 \Pi}{4\rho (\rho_1^2 + \rho_2^2)} + \frac{c}{2} + \frac{\rho \rho_1 (T_{11} - T_{22})}{2(\rho_1^2 + \rho_2^2)}, \quad \text{(27)} \\
\gamma_2 &= \frac{\rho_2 \Sigma - \Pi \rho_1}{4\rho (\rho_1^2 + \rho_2^2)} + \frac{d}{2} - \frac{\rho \rho_2 (T_{11} - T_{22})}{2(\rho_1^2 + \rho_2^2)}, \quad \text{(28)}
\end{align*}

\begin{align*}
\Sigma &= \frac{\rho^2}{f^2}(f_1^2 - f_2^2) + f^2[\omega_1^2 - \omega_2^2], \quad \text{(29)} \\
\Pi &= -2\frac{\rho^2}{f^2} f_1 f_2 + 2f^2 \omega_1 \omega_2, \quad \text{(30)}
\end{align*}

where:

\begin{align*}
\nabla^2 &= \partial^2_{\alpha\alpha} + \frac{\rho_\alpha}{\rho} \partial_\alpha, \quad \text{(31)}
\end{align*}

with the line element:

\begin{align*}
ds^2 &= f^{-1} \left[ e^{2\gamma} \left( (dx^1)^2 + (dx^2)^2 \right) + \rho^2 d\phi^2 \right] - f(dt - \omega d\phi)^2. \quad \text{(32)}
\end{align*}

The line element (32) is written in the so called Papapetrou gauge [24]. Finally, with the help of (14) and (15), the equation (16) looks as follows:

\begin{align*}
\frac{e^{2\gamma}}{f} T_{11} - T_{22} &= \frac{(\rho_1^2 - \rho_2^2)(T_{11} - T_{22}) + \rho \rho_1 (T_{11} - T_{22})_1 - \rho \rho_2 (T_{11} - T_{22})_2}{(\rho_1^2 + \rho_2^2)}. \quad \text{(33)}
\end{align*}

In the next section we present our method.
3 The method

3.1 Integrability and energy conditions

Thanks to equations (22)-(24), the equations (25)-(26) have the same structure of the vacuum Ernst equations with the Ernst potential Φ given by equations (21). Obviously, since the equations (27)-(28) contain terms involving the energy-momentum tensor, to integrate our system of equations we need a further integrability condition.

We get this integrability condition by setting \( \gamma_{12} = \gamma_{21} \). We read:

\[
2\rho_2\rho_1(T_{11} - T_{22}) + \rho\rho_1(T_{11} - T_{22})_2 + \rho\rho_2(T_{11} - T_{22})_1 = 0.
\]

(34)

To obtain the equation (34), we have used the equations (25)-(26). In this way, we can obtain an interior solution with the same two-metric, spanned by the Killing vectors \( \partial_t \) and \( \partial_\phi \), of the vacuum seed metric. In practice, we obtain anisotropic fluids by “gauging” the metric function \( \gamma \) of the seed vacuum solution. Obviously, is always possible to build anisotropic spacetimes by varying the metric coefficients. But, in this manner, we obtain an energy-momentum tensor with a non-appealing expression. In fact, since \( g_{12} = 0 \), if \( R_{12} \neq 0 \) then \( T_{\mu\nu} \) has the component \( T_{12} \neq 0 \), and this does not allow to write a simple expression for \( T_{\mu\nu} \). Further, if \( f, \omega \) are not solutions of the vacuum Ernst equations, then equation (36) is not easy to solve.

The next step is to study the form of the energy-momentum tensor allowed by the field equations. To this purpose, we assume that the four-velocity \( V^\mu \) of the fluid is \( V^\mu = \delta^\mu_4 / \sqrt{f} \), i.e. we assume that our coordinates are co-rotating with the fluid (see (25)). With the line element (32), we can write \( T_{\mu\nu} \) in the form:

\[
T_{\mu\nu} = (\epsilon + p_1)V_\mu V_\nu + p_1 g_{\mu\nu} + (p_3 - p_1)K_\mu K_\nu + (p_2 - p_1)S_\mu S_\nu
\]

(35)

where \( \epsilon \) is the mass-energy density, \( p_1, p_2, p_3 \) are the principal stresses, \( V_\mu, S_\mu, K_\mu \) are four-vectors satisfying:

\[
V^\mu V_\mu = -1, \quad K^\mu K_\mu = S^\mu S_\mu = 1, \quad V^\mu K_\mu = V^\mu S_\mu = K^\mu S_\mu = 0,
\]

(36)

being, with respect to (32):

\[
K_\mu = \begin{bmatrix} 0, 0, \frac{\rho}{\sqrt{f}}, 0 \end{bmatrix},
\]

\[
S_\mu = \begin{bmatrix} 0, \frac{e^\gamma}{\sqrt{f}}, 0, 0 \end{bmatrix},
\]

\[
V_\mu = \begin{bmatrix} 0, 0, \omega\sqrt{f}, -\sqrt{f} \end{bmatrix}.
\]

(37)
With respect to the tensor (35), the eigenvalues $\lambda$ (see [23]) are given by the roots of the equation:

$$|T_{\mu\nu} - \lambda g_{\mu\nu}| = 0,$$

(38)

i.e. $(p_1 - \lambda)(p_2 - \lambda)(\epsilon + \lambda)(p_3 - \lambda) = 0$. Therefore, we obtain: $\lambda_1 = p_1, \lambda_2 = p_2, \lambda_3 = p_3, \lambda_4 = -\epsilon$.

Furthermore, we must impose the energy conditions [22, 23], that in our notations are, for the weak energy condition ($i = 1, 2, 3$):

$$-\lambda_4 \geq 0, \quad -\lambda_4 + \lambda_i \geq 0;$$

(39)

for the strong energy condition:

$$-\lambda_4 + \sum \lambda_i \geq 0, \quad -\lambda_4 + \lambda_i \geq 0;$$

(40)

and for the dominant energy condition:

$$-\lambda_4 \geq 0, \quad \lambda_4 \leq \lambda_i \leq -\lambda_4.$$

(41)

We can write the equations (22)-(24) in terms of the principal stresses and of the mass-energy density. We obtain:

$$ (\epsilon + p_3)(\rho_f^2 + f^2 \omega^2) - (p_1 + p_2)(\rho_1^2 - f^2 \omega^2) = 0, $$

(42)

$$ \omega[\epsilon + p_1 + p_2 + p_3] = 0, $$

(43)

$$ \epsilon + p_1 + p_2 + p_3 = 0. $$

(44)

Equations (43) and (44) are equivalent. When (44) is put in (42), we have $p_1 = -p_2, \epsilon = -p_3$.

### 3.2 Starting steps

Let us summarize all the relevant equations we need:

$$ p_1 = -p_2 = p, \quad \epsilon = -p_3, $$

(45)

$$ T_{11} - T_{22} = \frac{2p}{f}\varepsilon^2, $$

(46)

$$ (2\rho_2 \rho_1 + \rho p_1 \partial_2 + \rho \rho_2 \partial_1) [T_{11} - T_{22}] = 0, $$

(47)

$$ \frac{(\rho_1^2 - \rho_2^2 + \rho p_1 \partial_1 - \rho \rho_2 \partial_2) [T_{11} - T_{22}]}{(\rho_1^2 + \rho_2^2)} = \frac{\varepsilon^2}{f}(p_3 - \epsilon). $$

(48)

together with the equations (27) and (28). A simple step by step procedure to solve the system (45)-(48) is the following.
1. First of all, we choose a solution of the vacuum Ernst equations by means of the functions \((f, \omega)\).

2. We specify the initial coordinates by the relation with \(\rho, z\).

3. We integrate the equation \((47)\) to find \(T_{11} - T_{22}\) as a function of the choosen coordinates.

4. We calculate, with the obtained function \(T_{11} - T_{22}\) and by means of equation \((45)\), the function \(p\) as a function of the unknown function \(\gamma\) and of the known function \(f\).

5. We put the solution obtained for \(T_{11} - T_{22}\) in \((48)\) to calculate, by means of the second equation of \((45)\), the function \(\epsilon\), and therefore all the principal stresses together with the mass-energy density \(\epsilon\) are found in terms of the unknown function \(\gamma\) and of the known function \(f\).

6. As a final step, we can substitute the expression for \(T_{11} - T_{22}\) in \((27)\) and \((28)\) and calculate the metric function \(\gamma\). The condition \((47)\) guaranties the integrability of \((27)\) and \((28)\).

7. As a result, starting with choosen functions \(f, \omega\) as known solutions of the vacuum Ernst equations, we generate the solution with \(\gamma, f, \omega\) with the line element given by \((32)\) and with a non-vanishing energy-momentum tensor given by \((35)\).

Note that, since the equation \((47)\) does not depend on \(\gamma\), our method is self-consistent. Hence, the equations for \(\gamma\) can be solved without ambiguity. Our method cannot describe perfect fluid solutions. In fact, for a perfect fluid solution \(p_1 = p_2 = p_3\) and therefore \(T_{11} - T_{22} = 0\). In this case, the field equations imply that \(p_1 = p_2 = p_3 = \epsilon = 0\). Therefore the only perfect fluid solution allowed with our method is the vacuum one.

Further, note that no restrictions are made on \(f\), and \(\omega\): they are only solutions of the vacuum Ernst equations. Furthermore, in the static limit \(\omega = 0\), we obtain interior solutions with equation of state \((45)\), \((46)\), \((48)\). In the next section we present our method with some physically interesting examples.
4 Application of the method

4.1 First example: Cylindrical coordinates

Starting with the line element (32) with $x^1 = \rho, x^2 = z$, we have $\rho_1 = 1, \rho_2 = 0$. In the chosen coordinates, the most general solution for (47) is:

$$T_{11} - T_{22} = F(\rho), \quad (49)$$

where $F(\rho)$ is an arbitrary regular function.

Expressions (46) and (48) become:

$$\epsilon = -\frac{f}{2e^{2\gamma}}(F + \rho F_\rho), \quad (50)$$

$$p = \frac{f}{2e^{2\gamma}}F. \quad (51)$$

It is a simple matter to verify that the only regular class of functions for $F(\rho)$ satisfying all the energy conditions are the non-positive ones that are monotonically decreasing. As an example, we assume $F(\rho) = c(-\rho - \rho^2)$, and as a result the generated solution is:

$$\epsilon = \frac{cf}{2e^{2\gamma}}[2\rho + 3\rho^2]$$

$$2\gamma = 2\gamma_0 + c\rho^3 \left[-\frac{\rho}{4} - \frac{1}{3}\right] + c\alpha, \quad (52)$$

with $c, \alpha$ constant and $c > 0$.

If we search for interior solutions to match with exterior vacuum ones, then we need solutions with a static surface and with vanishing hydrostatic pressure. To this purpose, the continuity of the first and the second fundamental form [10, 11] on a surface with $\rho = R = \text{const.}$ with vanishing pressure requires that:

$$\gamma_0(R) = \gamma(R), \quad \gamma_{0\rho}(R) = \gamma_\rho(R). \quad (53)$$

It is easy to see that, in order to satisfy the energy conditions and (53), $F(\rho)$ must be positive and singular on the axis at $\rho = 0$. To see this, we start from expressions (50), (51). First, suppose that $F(0) < 0$. Then, the energy conditions are satisfied only if

$$- F - \rho F_\rho \geq -F. \quad (54)$$

Expression (54) implies that $F_\rho \leq 0$, and thus $F$ is decreasing in a neighbourhood of $\rho = 0$. But, in this way, $F$ cannot be zero at some radius
\( \rho = R. \)

Conversely, suppose that \( F(0) \geq 0. \) To satisfy the energy conditions, we must have:

\[-\rho F'_\rho \geq 2F. \tag{55}\]

Thus, from (55) we deduce that \( F \) is decreasing in a neighbourhood of \( \rho = 0. \)

But, thanks to (55), if \( F \) is regular for \( 0 \leq \rho \leq R \), then we conclude that \( F(0) = 0 \) and therefore \( F \) cannot be zero at some radius \( R. \)

Concluding, the only way to satisfy both energy conditions and (55) is to choose \( F \) to be positive and irregular on the axis. Equation (55) implies that, in a neighbourhood of \( \rho = 0, \) \( F \) must show the following behaviour:

\( F(\rho) \geq \frac{1}{\rho}. \)

Perhaps the most simple class of solutions that we can consider is:

\[ F(\rho) = \frac{c}{\rho^{k^2+1}} (R - \rho)^{s^2+1}, \tag{56}\]

where \( c > 0 \) and \( |k| \geq 1, |s| \geq 1. \) It is easy to see that the solution (56) satisfies all the energy conditions and (53). After integrating the equations for \( \gamma \) we get:

\[ 2\gamma = 2\gamma_0 + c \int \frac{(R - \rho)^{s^2+1}}{\rho^{k^2}} d\rho + z(k, s), \tag{57}\]

where \( z(k, s) \) is an integration constant chosen to satisfy the first of equations (53). Generally, the integral (57) involves expressions in terms of hypergeometric functions.

Expression (57) for \( \gamma \) can be potentially singular on the axis. However, we have to choose a seed vacuum metric. Thus, we could choose an expression for \( \gamma_0 \) such that, in a neighbourhood of \( \rho = 0, \) the expression (57) be regular. This can be accomplished by taking a vacuum seed solution with an appropriate singular expression for \( \gamma_0 \) on the axis. We do not enter into such a discussion, but only mention the fact that the Lewis [26] class of solutions are not appropriate.

Concerning the solution (56), the most simple example we can consider is:

\[ F(\rho) = \frac{c}{\rho^2}(R - \rho)^2. \tag{58}\]

With (58), after integrating the field equations, our interior solutions, matching smoothly on \( \rho = R \) with any vacuum solution and satisfying all the

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energy conditions are:

\[ p = -p_z = p , \quad \epsilon = -p_\phi, \]

\[ 2\gamma = 2\gamma_0 + c \left[ \frac{1}{2} \rho^2 - 2\rho R + R^2 \ln(\rho) + \frac{3}{2} R^2 - R^2 \ln(R) \right], \]

\[ p = \frac{cf}{2\rho(2+cR^2)e^{(2\gamma_0)}}(R - \rho)^2 e^{-c[\frac{1}{2} \rho^2 - 2\rho R + \frac{3}{2} R^2 - R^2 \ln(R)]}, \]

\[ \epsilon = \frac{cf}{2\rho(2+cR^2)e^{(2\gamma_0)}}(R^2 - \rho^2)e^{-c[\frac{1}{2} \rho^2 - 2\rho R + \frac{3}{2} R^2 - R^2 \ln(R)]}. \] (59)

As an example, if we choose for \( \gamma_0 \) an expression that, for \( \rho \to 0 \), looks as follows:

\[ e^{(2\gamma_0)} \approx \frac{H(z)}{\rho(2e^{2\gamma})} + \text{higher orders}, \] (60)

where \( H(z) \) is a regular positive non-vanishing function and \( f \to \rho^2 \) when \( \rho \to 0 \), then all the expressions in (59) for \( \gamma, p \) and \( \epsilon \) are regular on the axis. To conclude this subsection, we write down a class of interior solutions matching with vacuum ones in such a way that the matter is in the region \( R \leq \rho < \infty \), while the region \( 0 \leq \rho < R \) is filled with vacuum. In this case it is a simple matter to see that, to satisfy the energy conditions, \( F(\rho) \) must be a regular negative decreasing function when \( \rho \geq R \). As a simple class of solutions, we have:

\[ F = -c(\rho - R)^\alpha , \quad \alpha \geq 1 , \quad c > 0 \]

\[ \epsilon = \frac{cf}{2e^{2\gamma}}(\rho - R)^{\alpha-1}(\rho - R + \rho\alpha) \]

\[ 2\gamma = 2\gamma_0 - c \int \rho(\rho - R)^\alpha d\rho. \] (61)

When \( \alpha \) is an integer, the solution (61) has a simple expression.

### 4.2 Second example: Oblate spheroidal coordinates

We consider oblate spheroidal coordinates defined in terms of the cylindrical ones \( \rho, z \) as: \( \rho = \cosh \mu \cos \theta \), \( z = \sinh \mu \sin \theta \), where \( 0 \leq \mu < \infty \) and \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).

Looking for metrics describing isolated objects with a vanishing hydrostatic pressure surface, we have the following solution:

\[ T_{11} - T_{22} = \frac{c(s^2 \cosh^2 \mu - k\cos^2 \theta - ks^2 \cos^2 \theta)}{\cosh^4 \mu}, \] (62)
where $k, s, c$ are arbitrary constants ($s > 0$).

Thus, we get:

$$ p = \frac{f}{2e^{2\gamma}} \frac{1}{\cosh^4 \mu} (T_{11} - T_{22}), \quad p_\mu = -p_\theta = p, \quad \epsilon = -p_\phi, \quad (63) $$

$$ \epsilon = \frac{fc}{2e^{2\gamma}} \frac{1}{\cosh^4 \mu} \left[ s^2 \cosh^2 \mu - 3k(1 + s^2) \cos^2 \theta \right]. \quad (64) $$

The energy conditions can be fulfilled if $c > 0, k < 0$. But, in this way, $p$ cannot vanish on some boundary surface. Remember that it is not necessary for $p_\phi$ to vanish at some surface to identify the boundary of the source region. To this purpose, we must take in (63) and (64) $c < 0, k > 0$. Hence, the energy conditions are satisfied in the region where

$$ \cosh \mu \leq \frac{\cos \theta}{s} \sqrt{2k(1 + s^2)}. \quad (65) $$

The surface of vanishing hydrostatic pressure is:

$$ \cosh \mu = \frac{\cos \theta}{s} \sqrt{k(1 + s^2)} = A \cos \theta. \quad (66) $$

Expression (66) represents effectively the equation of a boundary surface only if:

$$ \frac{s^2}{k(1 + s^2)} < 1, \quad (67) $$

or $A > 1$. In fact, by expressing the equation (66) in terms of cylindrical coordinates, we read

$$ z^2 = \frac{1}{A} (A \rho - 1)(A - \rho). \quad (68) $$

The surface (68) is a toroid (genus=1) with inner and outer radii $r_1, r_2$ given by $r_1 = 1/A, r_1 r_2 = 1$, with $\rho \in [r_1, r_2]$ and $z \in [(1 - A^2)/(2A), (A^2 - 1)/(2A)]$. When $A = 1$ the surface degenerates to a circle of radius $\rho = 1$ with $z = 0$.

As a consequence, thanks to (65) and (67), in the region enclosed by the surface (66), the energy conditions follow. Integrating the equations for $\gamma$, we get:

$$ \gamma = \frac{c}{2} \left[ (s^2 - k(1 + s^2)) \ln(\frac{\sqrt{\cosh^2 \mu - \cos^2 \theta}}{\cosh \mu}) - \frac{1}{2} k(1 + s^2) \frac{\cos^2 \theta}{\cosh^2 \mu} \right] + $$

$$ + \gamma_0 + \frac{c}{2} \alpha, \quad (69) $$
with $\alpha$ a constant. Expression (69) has a ring singularity at $\mu = 0, \theta = 0$, i.e. $\rho = 1, z = 0$. Thanks to (67), this singularity lies in the interior of the surface (66) ($\cosh \mu < \cos \theta \sqrt{k(1 + s^2)}$). Consequently, the metric can be potentially singular at that place. However, to understand the nature of this singularity, a seed vacuum metric must be specified. The situation is similar to the one for cylindrical coordinate. Also in this case, we must find an expression for $\gamma_0$ such that the expression (69) is regular, but we do not enter into the search.

It is a simple matter to verify that by choosing

$$\alpha = \frac{s^2}{2} + \frac{1}{2}(ks^2 + k - s^2) \ln \left(\frac{k + ks^2 - s^2}{k(1 + s^2)}\right),$$

we have, on the boundary $S$ of (66):

$$\gamma_0(S) = \gamma(S), \quad \gamma_0(iS) = \gamma(iS), \quad i = \mu, \theta.$$  

(71)

Therefore, our interior metric is $C^1$ on the boundary surface $S$ and thus it can be matched smoothly to any stationary axisymmetric asymptotically flat solution with $f, \omega, \gamma_0$. Note that, thanks to (67), expression (70) is real. Further, the principal stress $p_\mu$ is always positive.

### 4.3 Third example: The Kerr metric

We choose now, as seed metric, the Kerr one. After writing the solution (69) in the Boyer-Lindquist coordinates (see [27, 28]), we get:

$$ds^2 = \Sigma \left(d\theta^2 + \frac{dr^2}{\Delta}\right)e^F + (r^2 + a^2)\sin^2\theta d\phi^2 - dt^2 +$$

$$+ 2mr \frac{(dt + as\sin^2\theta d\phi)^2}{\Sigma},$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2mr,$$

$$F = c \left[ (s^2 - k(1 + s^2)) \ln \frac{\sqrt{\Delta - \sin^2\theta}}{\sqrt{\Delta}} - \frac{1}{2}k(1 + s^2)\frac{\sin^2\theta}{\Delta} + \alpha \right],$$

with $\alpha$ given by (70). The interior metric written in the Boyer-Lindquist coordinates can be extended to all the values of the parameters $a^2, m^2$ allowed by the Kerr solution. In particular, when $a^2 \leq m^2$, the surface of zero pressure $\Delta = \frac{k(1 + s^2)}{s^2}\sin^2\theta$ generally does not describe a toroidal surface, but a closed surface passing through the $z$ axis. Further, solution (72) can be
defined when $\Delta - \sin^2 \theta < 0$ by setting, for example, $\frac{c(s^2 - k(1+s^2))}{2} = 2n$, where $n$ is a positive integer.

When $a^2 > m^2$, the surface $\Delta = \frac{k(1+s^2)}{s^2} \sin^2 \theta$ becomes a toroidal rotational surface as (66) (see [28]).

It is interesting to note that, with the coordinates used in (72), when $a^2 > 1 + m^2$ ($\Delta > 1$), the ring singularity of (69) disappears. In this case, the only singularity for the global spacetime (both interior and exterior metric) is the ring of the vacuum Kerr solution. For

$$m^2 + 1 < a^2 < \frac{k(1+s^2)}{s^2}$$

(73)

the Kerr ring lies in the matter region. Otherwise, the Kerr ring belongs to the vacuum exterior Kerr solution.

Finally, note that $\Sigma e^F = \frac{e^{2\gamma}}{r}\quad > 0$, in such a way that energy conditions follow within the surface (66) ($c < 0, k > 0$).

5 Conclusions

In this paper we have presented a simple technique to obtain anisotropic fluids starting from any vacuum solution of the Ernst equations. The equation of state compatible with our method is: $p_1 = -p_2, \quad \epsilon = -p_3$.

In [16], anisotropic solutions with $\epsilon + p_1 + p_2 + p_3 = 0$ have been obtained with the help of Geroch [14] transformations applied to matter space-times. In this way, starting with matter space-times endowed with almost a Killing vector and with the appropriate equation of state, we can obtain new solutions adding twist to the seed space-time. For the solutions so obtained, the equation of state is the one of the seed metric with the matter parameters scaled by a common factor.

The equations of state compatible with the method of Krisch and Glass are: $3\epsilon + p_2 = 0, \quad p_1 = p_3 = -\epsilon$ for a space-time admitting a space-like Killing vector, and $\epsilon = p_2, \quad p_2 = -p_1 = -p_3$ for a space-time with a time-like Killing vector. However, to apply our method, no seed matter space-time is needed, but only vacuum stationary axially symmetric solutions. Moreover, note that our equation of state $p_1 = -p_2, \epsilon = -p_3$ contains the one of Krisch and Glass given by $\epsilon = p_2, p_2 = -p_1 = -p_3$ as a limiting case.

It is worth noticing that with our method we can always obtain isolated sources matching with asymptotically flat solutions.

We have analyzed the problem of joining the generating solutions with exterior vacuum ones. By using cylindrical coordinates, we are able to match
our anisotropic metrics, on the boundary surfaces \( \rho = R \), with all vacuum solutions in such a way that all the energy conditions are satisfied. Further, an interior solution is obtained representing an isolated body with an “unusual” but physically acceptable equation of state matching with any stationary axisymmetric asymptotically flat solution.

In both cases, the regularity of the solutions so obtained are discussed. Finally, it can be noticed that the solution (72) could be used to describe extreme astrophysical situations where a Kerr black hole is surrounded by non usual matter.

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