A GIT PROOF OF WŁODARCZYK’S WEIGHTED FACTORIZATION THEOREM

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Here we give a short proof of a recent result of Włodarczyk, [Włodarczyk99a]:

1.0 Theorem (Włodarczyk 99). Let $f : X \to Y$ be a birational rational map between smooth projective varieties in characteristic zero. $f$ can be factored as a composition of weighted blowups and weighted blowdowns.

For the precise definition of weighted blowup see [Włodarczyk99]. Locally analytically they are the maps between quasi-smooth toric varieties given by barycentric subdivision of cones. See [Włodarzky97].

Our goal here is a simple GIT proof of (1.0). Our main result is:

1.1 Theorem. Let $f : X \to Y$ be a birational morphism between normal $\mathbb{Q}$-factorial varieties projective over a field. Then there exists a normal projective variety $Z$ with a $G = \mathbb{C}^*$ action, and two $G$-linearized ample line bundles $L_1, L_2$ such that $Z^{ss}(L_i) // G = X$, $Z^{ss}(L_2) // G = Y$ and the induced rational map $X \dashrightarrow Y$ is $f$. Moreover

1. $Z^{ss}(L_i) = Z^*(L_i)$, i.e. the GIT quotient $Z^{ss}(L_i) // G$ is geometric
2. $G$ acts freely on $Z^{ss}(L_i)$
3. The two linearizations have the same underlying ample line bundle (i.e. they differ by a character).

If $X$ and $Y$ are smooth, and the characteristic is zero, one can choose $Z$ smooth as well.

(1.0) is immediate from (1.1) and theory of wall crossings in the variation of GIT quotient which provides a factorization as in (1.0). See [Thaddeus96,5.6] or [DolgachevHu,0.2.5]. We need the VGIT theory (which applies to any reductive group) only in the simplest case of $G = \mathbb{C}^*$. For this the wall crossing maps are easily described using the Bialynicki-Birula decomposition, see e.g. [Thaddeus96,1.12].
We thank Prof. Włodarczyk for sending us his preprint, which we read before writing this note. We first learned of Włodarczyk’s work from a talk by Kenji Matsuki at the University of Illinois, where he discussed his extension, joint with Abramovich, Kalle, and Włodarczyk, of (1.0) to a factorization by blowups and blowdowns with smooth centers. Matsuki specifically noted in his talk that Włodarczyk’s main device, \(\mathbb{C}^*\) cobordism (a pair of quotients obtained by selecting two equivariant open subsets of a variety with \(\mathbb{C}^*\) action in a special way) admits a GIT formulation. The \(\mathbb{Q}\)-factorial case of (1.1) was known to us before. It follows from the proof of [HuKeel98,6.3], some of which is reproduced below – an application of Thaddeus’s Master Space construction, [Thaddeus96,3.1]. The smooth case follows by equivariant blowup, but we realized this, and more importantly its striking implication (1.0), only on reading Włodarczyk’s preprint.

We will mix notation of Weil divisors and line bundles. For a linearized line bundle \(L\) and a character \(v\) we let \(L_v\) indicate the new linearization obtained by twisting by \(-v\). We write \(L_v^n\) for \((L_v)^\otimes n\).

**Proof of (1.1).** By (2.1-2.2) below, the smooth case (in characteristic zero) follows from the general case –note if \(X\) and \(Y\) are smooth then \(\text{Sing}(Z)\) is not semi-stable for either linearization by (1.1.2).

Now consider the \(\mathbb{Q}\)-factorial case. Let \(f: X \to Y\) be as in the statement. Choose an ample Cartier divisor \(D\) on \(Y\). By Kodaira’s lemma, see the proof of (2.2) below, there is an effective divisor \(E\) whose support is exceptional such that \(B = f^*(D) = A + E\) with \(A\) ample. Let \(C\) be the image of the injection \(\mathbb{N}^2 \to N^1(X)\) given by \((a, b) \to aA + bE\). Let \(R\) be the graded ring

\[ R = \bigoplus_{v \in C} R_v = \bigoplus_{(a, b) \in \mathbb{N}^2} H^0(X, aA + bE). \]

The edge generated by \(B\) divides \(C\) into two closed chambers: the subcone generated by \(A, B\), and the subcone generated by \(B, E\).

**Lemma 1.** \(R\) is finitely generated and after replacing \(A, B, E\) by multiples, the canonical map

\[ R_{v_1}^{\otimes n_1} \otimes R_{v_2}^{\otimes n_2} \to R_{v_1+v_2}^{\otimes n_1+n_2} \]

is surjective, whenever \(v_1\) and \(v_2\) are in the same chamber.

**Proof.** Its enough to check finite generation for the graded subring corresponding to each chamber. By the projection formula we reduce to the case when the two edge are
semi-ample (either \(A\) and \(B\) or \(D\) and \(O\)) and thus to a familiar result of Zariski. A similar argument yields the second statement. □

Let \(V = \text{spec}(R)\). \(V\) is an instance of Thaddeus’s Master Space. Let \(H = (\mathbb{C}^*)^2\). \(H\) acts on \(R\), with weights \((a, b)\) on \(R_{(a, b)}\). We can identify the characters \(\chi(H)\) with \(\mathbb{N}^2 = N^1(X)\). The invariants are \(H^0(V, L_v)^H = R_v\) and the corresponding GIT quotient is

\[(1.3) \quad \text{Proj}(\bigoplus_n R_{nv}).\]

**Lemma 2.** Linearizations in the interior of the same chamber have the same semi-stable locus, which also agrees with their stable locus. \(G\) acts freely on this locus. The two quotients are \(X\) and \(Y\).

**Proof.** Let \(v\) be a linearization in the interior of a chamber. A point \(h \in V\), which we identify with a \(\mathbb{C}\) algebra map \(h : R \to \mathbb{C}\), is \(v\) semi-stable iff \(h(R_{nv}) \neq 0\) for some \(n > 0\). By Lemma 1, this is true iff \(h(R_{v'}) \neq 0\) for all \(v'\) in the chamber. \(h\) is fixed by \(g = (s, t) \in H\) iff \(s^{a} t^{b} = 1\) for all \(v' = (a, b)\) in the chamber. Since either chamber generates \(\mathbb{Z}^2\) as a group, this occurs iff \(g\) is trivial. The GIT quotients are \(X\) and \(Y\) by (1.3) and the projection formula. □

Now if we replace \(V\) by \(Z = \text{Proj}(R)\), with respect to total degree, i.e. \(d(a, b) = a + b\) then (1.1.1-3) follow from Lemma 2. Here are the details:

\(Z\) is itself a GIT quotient, for the diagonal subgroup \(\Delta = \mathbb{C}^* \subset H\) and line bundle \(L_v\) with \(d(v) > 0\) which descends to \(O_Z(d(v))\). The non semi-stable locus is the irrelevant prime \(p = \sum_{v \neq 0} R_v\), the quotient \(\pi : V \setminus p \to Z\) is geometric and the action on the semi-stable locus is free. Let \(G = \mathbb{C}^* \subset H\) be a complementary subgroup to \(\Delta\), e.g. \(\mathbb{C}^* \times \{1\}\). \(H\) acts naturally on \(Z\), and \(O_Z(1)\) has a natural \(H\)-linearization, such that \(\Delta\) acts trivially. Let \(a : \chi(H) \to \chi(H/\Delta)\) be the natural surjection. \(L_v\), with its \(H\) action, descends to \(O_Z(d(v))_{a(v)}\) (i.e. \(L_v|_{V \setminus p}\) with its \(H\) action is the pullback of \(O_Z(d(v))_{a(v)}\)). For any \(v \neq 0\): \(p\) is \(L_v\) non-stable, there are are natural identifications

\[H^0(V, L_v^n)^H = [H^0(V, L_v^n)^\Delta]^G = H^0(Z, O_Z(d(v))_{a(v)}^n)^G,\]

\[V^{ss}(L_v) = \pi^{-1}(Z^{ss}(L_{a(v)})\) and \[V^{ss}(L_v)/H = (V^{ss}(L_v)/\Delta)/G = Z^{ss}(L_{a(v)})/G.\] □

**Remark.** A slight modification of the proof of (1.1) applies to any rational birational map.
2.1 Lemma. Let \( f : W \to Z \) be a birational \( G \)-equivariant morphism between normal projective varieties, with \( G \) reductive. Let \( L \) be a \( G \)-linearized ample line bundle on \( W \). Suppose that \( f \) is an isomorphism over \( Z^{ss}(L) \), and there exists an effective linearized exceptional Cartier divisor \( E \subset W \) such that \(-E\) is relatively ample.

Then there is an \( n_0 > 0 \) so that for any \( n > 1 \) and \( M = h^*(L^{mn_0}) - E \) with the induced linearization the following hold for every \( m > 0 \):

1. Every \( G \)-invariant section of \( M^m \) vanishes along the exceptional locus, and
2. the canonical map

\[
H^0(W, M^m)^G \to H^0(W, h^*(L^{mn_0}))^G = H^0(Z, L^{mn_0})^G
\]

is an isomorphism, where \( \sigma \) is the canonical section of \( \mathcal{O}(E) \).

In particular \( W^{ss}(M) = Z^{ss}(L) \) and the GIT quotients are canonically identified.

Proof. We can obviously replace \( L \) by a positive power. Since

\[
\bigoplus_d H^0(Z, L^d)^G
\]

is finitely generated, after replacing \( L \) by a power the canonical map

\[
sym_m(H^0(Z, L)^G) \to H^0(Z, L^m)^G
\]

is surjective for \( m \geq 0 \). By assumption every invariant section of \( h^*(L) \) vanishes along the exceptional locus, so after again replacing \( L \) by a power, invariant sections of \( h^*(L^m) \) vanish to high order –high as compared with the coefficients of \( mE \). The result follows. \( \square \)

We expect the following is known, but we include a proof as we do not know a reference:

2.2 Proposition. Let \( Z \) be a normal projective variety in characteristic zero, and \( L \) an ample \( \mathbb{Q} \)-Cartier divisor on \( Z \). There exists a resolution of singularities \( f : W \to Z \), and an effective Cartier divisor \( E \) whose support is the full exceptional locus, such that \(-E\) is relatively ample.

If a connected uniruled group acts, this can be done equivariantly.

Proof. Suppose first \( Z \) is \( \mathbb{Q} \)-factorial, and let \( f : W \to Z \) be a projective birational map with \( W \) \( \mathbb{Q} \)-factorial. Let \( A \) be any ample divisor on \( W \). Write \( f^*(L) - A = f^*(D) + E \)
with $E$ exceptional. After adding a multiple of $L$ to each side, we can assume $D$ is ample, and thus absorb it into $A$. So we have $f^*(L) = A + E$. Obviously $-E$ is relatively ample. $E$ is effective, with support the full exceptional locus by negativity of contraction, [Kollár et al.92,2.19].

For the general case, begin with any resolution of singularities $f$, and arbitrary ample $A$ on $W$. Since $f^*(L)$ is big, after replacing $L$ by a multiple, $f^*(L) = A + D$ with $D$ effective. After adding $f^*(L)$ to both sides, the base locus of $D$ is contained in the exceptional locus. So we can write $f^*(L) = A + M + E$, with $E$ effective and exceptional and $M$ moving, with base locus contained in the exceptional locus. Let $g : W \dashrightarrow Y$ be the rational map given by $M$. Let $r : W' \rightarrow W$ be a resolution of the closure of the graph of $g$. Then the induced $g' : W' \rightarrow Y$ is regular, and $r^*(M) = M' + E'$, where $M'$ is globally generated, with associated map $h'$, and $E'$ is $r$ exceptional. For the composition $p : W' \rightarrow X$, we have $p^*(L) = r^*(A) + M' + E''$, with $M'$ nef and $E''$ exceptional. Now apply the argument of the first case to $r^*(A)$. So we have (adjusting notation) $p^*(L) = A + M' + E$, with $A$ ample, $M'$ nef and $E$ exceptional. $M'$ can be absorbed into $A$. Now apply negativity of contraction as in the first case.

Suppose connected uniruled $G$ acts, and $L$ is $G$-linearized. Note that a multiple of any line bundle on a smooth projective $G$-variety $W$ has a linearization by [GIT,1.5]. Thus all the maps in the previous paragraph can be taken to be $G$-equivariant, and the equivariant case follows. □

References

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