GLOBAL $W^{2,p}$ REGULARITY ON THE LINEARIZED MONGE-AMPÈRE EQUATION WITH VMO TYPE COEFFICIENTS

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Abstract. In this paper, we establish global $W^{2,p}$ estimates for solutions of the linearized Monge-Ampère equation

$$L_{\phi}u := \text{tr}[\Phi D^2u] = f,$$

where the density of the Monge-Ampère measure $g := \det D^2\phi$ satisfies a VMO-type condition, and $\Phi := (\det D^2\phi)(D^2\phi)^{-1}$ is the cofactor matrix of $D^2\phi$.

1. Introduction

This paper is concerned with global $W^{2,p}$ estimates for solutions of the linearized Monge-Ampère equation

$$L_{\phi}u := \text{tr}[\Phi D^2u] = f \quad \text{in } \Omega,$$

where $\phi$ is a solution of

$$\det D^2\phi = g, \quad 0 < \lambda \leq g \leq \Lambda \quad \text{in } \Omega$$

for some constants $0 < \lambda \leq \Lambda < \infty$.

When the density $g = \det D^2\phi$ is continuous, the regularity of the Monge-Ampère equation (1.2) and the linearized Monge-Ampère equation (1.1) has been well studied. Caffarelli proved in [1] that if $g$ is continuous (resp. $C^{1,\alpha}$), then an interior $W^{2,p}$ estimate for any $p > 1$ (resp. $C^{2,\alpha}$ interior estimate) holds for strictly convex solutions of (1.2). Corresponding global $W^{2,p}$ and $C^{2,\alpha}$ regularity under further assumptions on $\Omega$ and the boundary data were established by Savin [9, 10] and Trudinger and Wang [11]. For the linearized equation (1.1), Gutiérrez and Tourin derived interior $W^{2,\alpha}$ estimates for small $\delta$ in [5], and interior $W^{2,p}$ regularity for general $1 < p < q$, $f \in L^q(q > n)$ was obtained in [4]. By using Savin’s localization theorem, Nam Le and Nguyen derived in [7] corresponding global $W^{2,p}$ estimates for the Dirichlet problem

$$(1.3) \quad \begin{cases} L_{\phi}u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial \Omega. \end{cases}$$

On the other hand, in the case that $g$ is discontinuous, Huang [6] proved interior $W^{2,p}$ estimates for solutions of (1.2) when $g$ belongs to a VMO-type space $\text{VMO}_{\text{loc}}(\Omega, \phi)$.

In this paper, we first study the interior $W^{2,p}$ estimates for solutions of (1.1) when $g$ belongs to the VMO-type space defined in [6]. Next, we define a global VMO-type space $\text{VMO}(\Omega, \phi)$ and establish global $W^{2,p}$ estimates for solutions of (1.1) when $g$ belongs to $\text{VMO}(\Omega, \phi)$. Hence, in the interior case, we generalize the result in [4] from the case that $g$ is continuous to the case that $g$ belongs to $\text{VMO}_{\text{loc}}(\Omega, \phi)$. And in the global case, we extend [7] from the case that $g$ is continuous to the case that $g$ belongs to $\text{VMO}(\Omega, \phi)$.

Our first main theorem concerns interior $W^{2,p}$ estimates for solutions of (1.1) and is stated as follows.

Theorem 1. Let $B_{\kappa} \subset \Omega \subset B_1$ be a normalized convex domain and $\phi \in C(\Omega)$ be a solution of (1.2) with $\phi = 0$ on $\partial\Omega$, where $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$. Let $p > 1$, $\max\{n, p\} < q < \infty$. Let $u \in C^{2}(\Omega)$ be a solution of (1.1). Let $\Omega' \Subset \Omega$, then there exists $C > 0$ depending only on $n, \lambda, \Lambda, p, q, \Omega'$ and $g$ such that

$$\|D^2u\|_{L^p(\Omega')} \leq C (\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}) .$$

The next theorem gives global $W^{2,p}$ estimates for solutions of (1.1) when $g$ satisfies a VMO-type small oscillation.

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Theorem 2. Assume $\Omega$ is a bounded, uniform convex domain with $\partial \Omega \in C^3$. Let $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of (1.2) with $\phi = 0$ on $\partial \Omega$. Let $u \in C^2(\Omega)$ be a solution of (1.3) with $\varphi = 0$, where $f \in L^q(\Omega)$ with $n < q < \infty$. Then, for any $1 < p < q$, there exist $0 < \epsilon < 1$ and $C > 0$ depending only on $n, \lambda, \Lambda, \Omega, p, q$ such that

$$\sup_{x \in \Omega, r > 0} \text{mosc}_{S_{\phi}(0,r)} g \leq \epsilon,$$

then

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^q(\Omega)}.$$

Theorem 1 implies the following result concerning global $W^{2,p}$ estimates for solutions of (1.1) when $g$ belongs to a global VMO-type space.

Theorem 3. Assume $\Omega$ is a bounded, uniform convex domain with $\partial \Omega \in C^3$. Let $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of (1.2) with $\phi = 0$ on $\partial \Omega$, where $g \in \text{VMO}(\Omega, \phi)$. Let $u \in C^2(\Omega) \cap C(\Omega)$ be a solution of (1.3), where $f \in L^s(\Omega)$, $\varphi \in C^2(\Omega) \cap W^{2,s}(\Omega)$ with $n < q < s < \infty$. Then, for any $1 < p < q$, there exists $C > 0$ depending only on $n, \lambda, \Lambda, \Omega, p, q, s, g$ such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left( \|\varphi\|_{W^{2,q}(\Omega)} + \|f\|_{L^s(\Omega)} \right).$$

The spaces $\text{VMO}_{\phi}(\Omega, \phi)$ and $\text{VMO}(\Omega, \phi)$ above are defined in Section 2.

We now indicate the main steps in the proof of the above theorems.

In the interior case, the main ingredient is the stability of the cofactor matrix of $D^2\phi$ under a VMO-type condition of $g = \det D^2\phi$. For this, we use the interior $W^{2,p}$ estimates for $\phi$ in [6]. Once we have this stability result, we can use similar arguments as in the case that $g$ is continuous in [4] to prove power decay estimates for the distribution function of $D^2u$, using the power decay estimates of the distribution function of $D^2\phi$ under a VMO-type condition of $g$ in [6] instead.

For the global case, correspondingly we first establish the stability of the cofactor matrices at the boundary. However, due to the VMO-type property of $g$, we need to modify the boundary data of the good Monge-Ampère equation so that the maximal principle can be applied. Next we give global power decay of the distribution function of $D^2 \phi$, using the interior case in [6] and the covering lemma in [7]. The main difficulty in proving Theorem 2 is a localized power decay estimate for the distribution function of $D^2 \phi$ at the boundary. Since $g$ is in VMO-type spaces, there is a difficulty in using convex envelopes of functions to help estimate the density of good sets where $D^2 \phi$ is bounded. Instead, we use the idea in corresponding interior case [6]. Namely, we compare $\phi$ with the solution $w$ of the good Monge-Ampère equation. We apply the localized covering lemma at the boundary and cover a neighborhood of a point at the boundary by sections of $w$. Then we use the one-sided small power decay estimate in [4] in each of these sections to the linearized operator $L_w$ to estimate the set where $\phi$ can be touched from above by a quadratic polynomial. With this localized power decay estimate and arguing as in [7], it is straightforward to prove the power decay estimate for the distribution function of $D^2 u$.

The paper is organised as follows. Sections 2 and 3 are devoted to the interior $W^{2,p}$ estimates for solutions $u$ of (1.1) when $g = \det D^2 \phi$ satisfies a VMO-type condition. In Section 2, we first introduce some notation and give the $W^{2,p}$ estimates for $\phi$ in [6]. Using these results, we then establish the stability of cofactor matrix of $D^2 \phi$ under a VMO-type condition of $g = \det D^2 \phi$. In Section 3, we give the power decay estimates for the distribution functions of $D^2 u$. Then we give the proof of Theorem 1. Sections 4-6 are devoted to the global $W^{2,p}$ estimates for solutions $u$ of (1.1). In Section 4, we prove a stability result at the boundary for the cofactor matrices of $D^2 \phi$ and the global power decay estimates for the distribution functions of $D^2 \phi$. In Section 5, we prove a localized power decay estimates for the distribution function of $D^2 \phi$ at the boundary. Then we use this result to establish the power decay estimates for the distribution function of $D^2 u$. In the last section, we give the proof of Theorems 2 and 3.

2. Interior estimates-Stability of cofactor matrices

In this section, we first introduce some notation and give the interior $W^{2,p}$ estimates of $\phi$ in [6]. Then we establish the stability of cofactor matrices of $D^2 \phi$ under a VMO-type condition of $g = \det D^2 \phi$.

Let $\Omega$ be a convex domain in $\mathbb{R}^n$ and $\phi \in C(\overline{\Omega})$ is a strictly convex function. A section of $\phi$ centered at $x_0 \in \overline{\Omega}$ with height $h$ is defined by

$$S_{\phi}(x_0, h) := \{ x \in \overline{\Omega} : \phi(x) < \phi(x_0) + \nabla \phi(x_0) \cdot (x - x_0) + h \}.$$
Lemma 2.1. (See [6, Lemma 2.1].) Assume that condition (H) holds. Then for any $\Omega' \in \Omega$, there exist positive constants $h_0$, $C$ and $q$ such that for $x_0 \in \Omega'$, and $0 < h \leq h_0$,

$$B_{C^{-1}h}(x_0) \subset S_{q}(x_0, h) \subset B_{C h}(x_0),$$

where $q = q(n, \lambda, \Lambda)$ and $h_0, C$ depend only on $n, \lambda, \Lambda$ and $\text{dist}(\Omega', \partial \Omega)$.

The maximum principle below is used to compare solutions $\phi$ of (1.2) and $w$ of (2.1), where $g = \det D^2\phi \in \text{VMO}_{\text{loc}}(\Omega, \phi)$ defined above.
Lemma 2.2. (See [6, Lemma 3.1].) Let \( \phi \) and \( w \) be the weak solutions to \( \det D^2 \phi = g^1 \geq 0 \) and \( \det D^2 w = g^2 \geq 0 \) in \( \Omega \), respectively. Assume that \( g_1, g_2 \in L^p(\Omega) \). Then
\[
\max_{\Omega}(\phi - w) \leq \max_{\partial \Omega}(\phi - w) + C_n \text{diam}(\Omega) \left( \int_{\Omega} (g_2 - g_1)^{\frac{p}{n}} \, dx \right)^{\frac{1}{p}}.
\]
Next we give the \( W^{2,p} \) estimates of solutions \( \phi \) of (1.2) in [6].

Theorem 2.1. (See [6, Theorem A(i)].) Assume that condition (H) holds.

(i) Let \( 0 < \alpha < 1, 1 \leq p < \infty \) and denote \( \alpha_0 := \frac{\alpha}{\alpha - 1} \). There exist constants \( 0 < \epsilon < 1 \) and \( C > 0 \) depending only on \( n, \lambda, \Lambda, p \) and \( \alpha \) such that if \( \text{mosc}_W g \leq \epsilon \) for any \( S = S_\phi(x_0, h) \subset \Omega_{\eta + 1} \), then
\[
\|D^2 \phi\|_{L^p(\Omega_{\eta})} \leq C.
\]

(ii) If \( g \in \text{VMO}_{\text{loc}}(\Omega, \phi) \), then \( D^2 \phi \in L^p_{\text{loc}}(\Omega) \) for any \( 1 \leq p < \infty \).

Using Lemma 2.1 we are ready to prove stability of cofactor matrices of \( D^2 \phi \) under a VMO-type small oscillation of \( g \).

Lemma 2.3. Let \( B_\delta \subset \Omega^k \subset B_n \) be a sequence of normalized convex domain converging in the Hausdorff metric to a normalized convex domain \( B_\delta \subset \Omega \subset B_n \). For each \( k \in \mathbb{N} \), let \( \phi_k \in C(\Omega^k) \) be a convex function satisfying
\[
\left\{ \begin{array}{ll}
det D^2 \phi_k = g_k & \text{in } \Omega^k, \\
\phi_k = 0 & \text{on } \partial \Omega^k,
\end{array} \right.
\]
where \( 0 < \lambda \leq g_k = (g^1_k)^n \leq \Lambda \) in \( \Omega^k \), \( \text{mosc}_{\Omega^k} g_k \leq \frac{1}{k} \) and
\[
\sup_{S_{\phi_k}(x,h) \subset \Omega^k} \text{mosc}_{S_{\phi_k}(x,h)} g_k \leq \frac{1}{k}.
\]

Suppose that \( \phi_k \) converges uniformly on compact subsets of \( \Omega \) to a convex function \( \phi \in C(\overline{\Omega}) \) which is a solution of
\[
\left\{ \begin{array}{ll}
det D^2 \phi = 1 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]
Then there exists a subsequence which we still denote by \( \phi_k \) such that for any \( 1 \leq p < \infty \),
\[
\lim_{k \to \infty} \|D^2 \phi_k - (g^1_k)_{B_1} D^2 \phi\|_{L^p(B_1)} = 0,
\]
and
\[
\lim_{k \to \infty} \|\Phi_k - (g^1_k)_{B_1}^{-1} \Phi\|_{L^p(B_1)} = 0,
\]
where \( \Phi_k \) and \( \Phi \) are the cofactor matrices of \( D^2 \phi_k \) and \( D^2 \phi \) respectively.

Proof. First we note that since \( \text{dist}(B_1, \partial \Omega^k) \geq \text{dist}(B_1, B_\delta) = \frac{1}{k} \), then \( B_1 \subset \Omega^k \) for some \( \alpha \) depending only on \( n, \lambda, \Lambda \). For any \( 1 \leq p < \infty \), let \( \epsilon(p) = \epsilon(n, \lambda, \Lambda, p, \alpha) = \epsilon(n, \lambda, \Lambda, p) \) be the constant in Theorem 2.1 (i), then for any \( k \geq k_{\epsilon(p)} := \left[ \frac{1}{\epsilon(p)} \right] + 1 \) we have
\[
\sup_{S_{\phi_k}(x,h) \subset \Omega^k} \text{mosc}_{S_{\phi_k}(x,h)} g_k \leq \epsilon(p).
\]
Thus Theorem 2.1 (i) implies that
\[
\|D^2 \phi_k\|_{L^p(B_1)} \leq \|D^2 \phi\|_{L^p(\Omega_{\eta})} \leq C(n, \lambda, \Lambda, p) \quad \forall k \geq k_{\epsilon(p)}.
\]
Let \( \delta > 0 \) be an arbitrary small constant, and let \( \Omega(\delta) := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \). Then there exists \( k_\delta \in \mathbb{N} \) such that for all \( k \geq k_\delta \),
\[
\text{dist}(x, \partial \Omega^k) \leq 2\delta, \quad \forall x \in \partial(\Omega(\delta)).
\]
Then Aleksandrov’s estimate ([2, Theorem 1.4.2]) implies that
\[
|\phi_k(x) - (g^1_k)_{B_1} \phi(x)| \leq C(n, \lambda, \Lambda) \delta^2 \quad \forall x \in \partial(\Omega(\delta)), \forall k \geq k_\delta.
\]
By choosing $k_\delta$ even larger, we have $\Omega(\delta) \subset \Omega^k$ for $k \geq k_\delta$. Then by Proposition 2.1,

$$
\left(\int_{\Omega(\delta)} |g_k^1 - (g_k^1)|_{B_1}^m dx \right)^{\frac{1}{m}} \leq \left(\int_{\Omega} |g_k^1 - (g_k^1)|_{B_1}^m dx \right)^{\frac{1}{m}} \leq C(n) \left(\int_{\Omega} |g_k^1 - (g_k^1)|_{B_2}^m dx \right)^{\frac{1}{m}} 
$$

(2.5)

$$
\leq C(n) \left(\int_{\Omega_k} |g_k - (g_k)|_{B_1} dx \right)^{\frac{1}{m}} \leq \frac{C(n)}{k^\frac{\delta}{\pi}} \quad \forall k \geq k_\delta.
$$

Using the above two estimates and applying Lemma 2.2 with $\phi \mapsto \phi_k$, $w \mapsto (g_k^1)_{B_1} \phi$, we get

$$
\max_{\Omega(\delta)} |\phi_k - (g_k^1)_{B_1}| \leq \max_{\Omega(\delta)} |\phi_k - (g_k^1)_{B_1}| + C_n \text{diam}(\Omega(\delta)) \left(\int_{\Omega(\delta)} |g_k^1 - (g_k^1)|_{B_1}^m dx \right)^{\frac{1}{m}}
$$

(2.6)

$$
\leq C(n, \lambda, \Lambda) \delta^{\frac{1}{m}} + \frac{C(n)}{k^\frac{\delta}{\pi}} \quad \forall k \geq k_\delta.
$$

Using (2.6), (2.4) and similar arguments to the proof of [3, Lemma 3.4], we obtain the first conclusion of the lemma. For the second conclusion, we write

$$
\Phi_k - (g_k^1)^{m-1} \Phi = \left[1 - \frac{(g_k^1)_{B_1}^{m-1}}{\det D^2 \phi_k}\right] \Phi_k - \frac{(g_k^1)_{B_1}^{m-1}}{\det D^2 \phi_k} \Phi_k \left(D^2 \phi_k - (g_k^1)_{B_1} D^2 \phi \right) \Phi.
$$

For any $1 \leq q, r < \infty$, if $qr \leq n$, then we have by (2.5)

$$
\left(\int_{B_1} |g_k^1 - (g_k^1)|_{B_1}^{qr} dx \right)^{\frac{1}{qr}} \leq C(q, r, \delta) \left(\int_{B_1} |g_k^1 - (g_k^1)|_{B_1} dx \right)^{\frac{1}{m}} \leq \frac{C(n, q, r)}{k^\frac{\delta}{\pi}}
$$

by Hölder inequality. On the other hand, if $qr > n$ then

$$
\left(\int_{B_1} |g_k^1 - (g_k^1)|_{B_1}^{qr} dx \right)^{\frac{1}{qr}} \leq \left(\int_{B_1} |g_k^1 - (g_k^1)|_{B_1} dx \right)^{\frac{1}{m}} \leq \frac{C(n, q, r, \lambda, \Lambda)}{k^\frac{\delta}{\pi}}.
$$

Note that

$$
|(g_k^1)^m - (g_k^1)_{B_1}| \leq C(n, \lambda, \Lambda)|g_k^1 - (g_k^1)_{B_1}|
$$

Therefore,

$$
\left\|1 - \frac{(g_k^1)_{B_1}^{m-1}}{\det D^2 \phi_k}\right\|_{L^q(B_1)} \to 0 \quad \text{as } k \to \infty.
$$

The rest of the proof is similar to [3, Lemma 3.5], using (2.4) and the first conclusion of the lemma instead.

Using Lemma 2.3 and Blaschke selection theorem, we can obtain:

**Corollary 2.1.** Given $0 < \epsilon_0 < 1$ and $0 < \alpha < 1$. Then there exists $\epsilon > 0$ depending only on $\epsilon_0, n, \lambda, \Lambda, \alpha$ such that if the assumption (H) holds and $w \in C(\overline{\Omega})$ is the solution of (2.1), where

$$
mosc_{\Omega} g \leq \epsilon \quad \text{and} \quad \sup_{S_{\alpha, \lambda}(x, h) \in \Omega} \m osc_{S_{\alpha, \lambda}(x, h)} g \leq \epsilon,
$$

then

$$
||\Phi - (g^1)^{m-1} W||_{L^\infty(\Omega_\alpha)} \leq \epsilon_0,
$$

where $\Phi$ and $W$ are the cofactor matrices of $D^2 \phi$ and $D^2 w$ respectively.

Next we begin to prove the power decay of the distribution function of $D^2 u$. The following lemma was given in [4].

**Lemma 2.4.** (See [4, Lemma 2.7].) Assume that condition (H) holds. Let $u \in C^2(\Omega)$. Then for $k > 1$, we have

$$
\{x \in \Omega_\alpha : |D_j u(x)| > \beta^\alpha\} \subset \left(\Omega_\alpha \setminus \left(n D^\alpha \frac{(\beta^\alpha)^{\frac{1}{\alpha}}}{\beta^\alpha}\right)^{\frac{1}{\alpha}}\right) \cup \left(\Omega_\alpha \setminus G(\beta, \Omega)\right)
$$

(2.7)
for any $\beta > 0$ satisfying $(c\beta^{\frac{2}{n+1}})^{\frac{1}{2}} \geq \frac{\text{diam} \Omega}{\sqrt{h}}$, where $c > 0$ is a constant depending only on $n, \lambda, \Lambda$.

We recall the related definitions in the above lemma.

Let $\Omega$ be a bounded convex set in $\mathbb{R}^n$ and $\phi \in C^1(\Omega)$ be a convex function. For $u \in C(\Omega)$ and $M > 0$, define the sets

$$G^*_M(u, \Omega) = \{ \bar{x} \in \Omega : u \text{ is differentiable at } \bar{x} \text{ and } u(\bar{x}) \leq u_{\phi, x}(\bar{x}) + \frac{M}{2} d(x, \bar{x})^2, \forall x \in \Omega \};$$

$$G^*_M(u, \Omega) = \{ \bar{x} \in \Omega : u \text{ is differentiable at } \bar{x} \text{ and } u(\bar{x}) \geq u_{\phi, x}(\bar{x}) - \frac{M}{2} d(x, \bar{x})^2, \forall x \in \Omega \};$$

and $G_M(u, \Omega) := G^*_M(u, \Omega) \cap G^*_M(u, \Omega)$, where for any $x \in \Omega$ and $x_0 \in \Omega$,

$$d(x, x_0)^2 := \phi(x) - \phi(x_0).$$

Under the assumption (H), for $0 < \alpha < 1$, let $\eta_\alpha$ be the constant such that the conclusion in Lemma 2.1 holds for $\Omega' \rightsquigarrow \Omega_\alpha$ and $S_\phi(x, h) \subset \Omega_{\alpha/2}$ for $x \in \Omega_\alpha$ and $h \leq \eta_\alpha$. Denote

$$D^\alpha_{\beta} = \{ x \in \Omega_\alpha : S_\phi(x, h) \subset B_{\beta \sqrt{n}}(x) \text{ for } h \leq \eta_\alpha \}$$

and

(2.8) $$A^\alpha_\beta = \{ \bar{x} \in \Omega : \phi(\bar{x}) \geq \sigma|x - \bar{x}|^2, \forall x \in \Omega \}$$

By [2, Lemma 6.2.2], $D^\alpha_{\beta} = \Omega_\alpha \cap A_{\beta^{-2}}$ if $\beta \geq \frac{\text{diam} \Omega}{\sqrt{n}}$.

The estimate of the first term on the right-hand side of (2.7) is given by the following lemma. It is an easy corollary of Theorem 6.3 and Equation (6.12) in [6].

**Lemma 2.5.** Let $0 < \alpha < 1$ and denote $\alpha_0 := \frac{\alpha}{2^{n+1}}$. Assume that condition (H) holds, where $\text{mosc}_S g \leq \epsilon$ with small $\epsilon$ for any section $S = S_\phi(x, h) \subset \Omega_{\alpha/2}$. Then there exist constants $M > 0$ depending only on $n, \lambda, \Lambda, \alpha$ and $0 < \alpha_1 < 1$ depending only on $n, \lambda, \Lambda, \alpha$ such that

$$|\Omega_\alpha \setminus D^\alpha_{\beta}| \leq \frac{|\Omega|}{(2e^{\alpha_1/2})^2} s^{\frac{2(\alpha_1^2/2)}{2n}}$$

for each $s > 0$.

3. INTERIOR ESTIMATES-Power decay results

In this section, we give the estimate of the second term on the right-hand side of (2.7) and the proof of Theorem 1. We first include the following result from [6]. Recall the definition of $A^\alpha_\beta$ in (2.8).

**Lemma 3.1.** (See [6, Lemma 6.1].) Assume that condition (H) holds, where $\text{mosc}_S g \leq \epsilon$. Let $0 < \alpha < 1$. Then there exist $0 < \alpha_1 < 1$ and $\sigma > 0$ depending only on $n, \lambda, \Lambda$ and $\alpha$ such that

$$|\Omega_\alpha \setminus A^\alpha_\beta| \leq e^{\alpha_1}|\Omega_\alpha|.$$

Next, we compare explicitly two solutions originating from two different linearized Monge-Ampère equations. The following lemma is a slight modification of [4, Lemma 4.1].

**Lemma 3.2.** Assume that condition (H) holds and $w$ is the solution of (2.1) with $\Omega \rightsquigarrow U$ and denote $g^1 := g^\frac{1}{2}$. Let $u \in W^{2, n}_{loc}(U) \cap C(\overline{U})$ be a solution of $L_\phi u = f$ in $U$ with $|u| \leq 1$ in $U$. Assume $0 < \alpha_1 < 1$ and $h \in W^{2, n}_{loc}(U_{\alpha_1}) \cap C(\overline{U_{\alpha_1}})$ is a solution of

$$\begin{cases} L_{\nu} h = 0 \quad &\text{in } U_{\alpha_1}, \\ h = u \quad &\text{on } \partial U_{\alpha_1}. \end{cases}$$

Then there exists $0 < \gamma < 1$ depending only on $n, \lambda, \Lambda$ such that for any $0 < \alpha_2 < \alpha_1$, we have

$$||u - h||_{L^n(U_{\alpha_2})} + ||f - \nu ||_{L^n(U_{\alpha_2})} \leq C \left( ||\Phi - (g^1)^{-1}W||_{L^n(U_{\alpha_1})} + ||f||_{L^n(U_{\alpha_1})} \right)$$

provided that $||\Phi - (g^1)^{-1}W||_{L^n(U_{\alpha_1})} \leq (\alpha_1 - \alpha_2)^{\frac{2}{(n+1)(\gamma - 1)}}$, where $C = C(n, \lambda, \Lambda, \alpha_1, \alpha_2)$.

Using Lemma 3.1 and following similar lines as in the proof of [4, Lemma 4.2], we obtain the density estimate below.
Lemma 3.3. Let $0 < \varepsilon < 1$, $0 < \alpha_0 < 1$. Assume $\Omega$ is a bounded convex set, $U \subset \Omega$ and the condition (H) holds with $\Omega \rightsquigarrow U$, where $\phi \in C^1(\Omega) \cap W^{2,n}_\text{loc}(U)$ and $\text{moc}_U g \leq \varepsilon$. Denote $g^1 := g^{1/2}.$ Let $u \in C(\Omega) \cap W^{2,n}_\text{loc}(U) \cap C^1(U)$ be a solution of $L_{\phi}u = f$ in $U$ with $|u| \leq 1$ in $U$ and

$$|u(x)| \leq C\left[1 + d(x, x_0)^2\right] \quad \text{in } \Omega \setminus U$$

for some $x_0 \in U_{\alpha_0}$. Then for any $0 < \alpha < \alpha_0$, there exist constants $C, \tau > 0$ and $0 < \delta_1 < 1$ depending only on $n, \lambda, \Lambda, \alpha_0, \alpha$ such that

$$|G_N(u, \Omega) \setminus U_{\alpha}| \geq \left\{1 - C\left(N^{-\tau} \delta_1^2 + \varepsilon^2\right)\right\}|U_{\alpha}|$$

for any $N \geq N_0 = N_0(n, \lambda, \Lambda, \alpha_0, \alpha, C^*)$ and provided that $\|\Phi - (g^1)^{n-1}_{U_{\alpha_0+1}} W\|_{L^\tau(U_{\alpha_0+1})} \leq \left(\frac{1 - \alpha_0}{\alpha_0}\right)^{\frac{n-1}{2}}$. Here $W, \gamma$ are from Lemma 3.2 and

$$\delta_0 := \left\{\int_{U_{\alpha_0+1}} \|\Phi - (g^1)^{n-1}_{U_{\alpha_0+1}} W\|_{L^\tau}^\tau dx\right\}^{\frac{1}{\tau}} + \left(\int_{U} |f|^\tau dx\right)^{\frac{1}{\tau}}.$$

By using Lemma 3.3, a localization process and stability of cofactor matrices (Corollary 2.1), we obtain the following two lemmas.

Lemma 3.4. Let $0 < \varepsilon_0 < 1$, $0 < \alpha_0 < 1$. Let $B_{\alpha_0} \subset \Omega \subset B_1$ be a normalized convex domain and $\Omega_1$ be a bounded convex set such that $\Omega \subset \Omega_1$. Assume $\phi \in C^2(\Omega_1)$ is a convex function satisfying

$$\begin{cases}
\text{det} D^2 \phi = g & \text{in } \Omega_1, \\
\phi = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $0 < \lambda \leq g = (g^1)^{\phi} \leq \Lambda$ in $\Omega_1$. Let $u \in C(\Omega_1) \cap C^1(\Omega) \cap W^{2,n}_\text{loc}(\Omega)$ be a solution of $L_{\phi}u = f$ in $\Omega$ with $\|u\|_{L^\infty(\Omega)} \leq 1$. There exists $\varepsilon > 0$ depending only on $n, \lambda, \Lambda, \varepsilon_0, \alpha_0$ such that if $S_{\phi}(x_0, t_0) \subset \Omega_\alpha_0$ and

$$\sup_{S_{\phi}(x,y)\subset S_{\phi}(x_0, t_0)} g \leq \varepsilon,$$

then

$$|G_N(u, \Omega_1) \cap S_{\phi}(x_0, t_0)| \geq \left\{1 - \varepsilon_0 - C\left(\frac{t_0}{N}\right)^{\tau} \left(\int_{S_{\phi}(x_0, t_0)} |f|^\tau dx\right)^{\frac{1}{\tau}}\right\}|S_{\phi}(x_0, t_0)|$$

for any $N \geq N_0$. Here $C, \tau, N_0$ are positive constants depending only on $n, \lambda, \Lambda, \alpha_0$.

Proof. Let $T$ be an affine map such that

$$B_{\alpha_0} \subset T\left(S_{\phi}(x_0, \frac{t_0}{\alpha_0})\right) \subset B_1.$$

Let $U := T\left(S_{\phi}(x_0, \frac{t_0}{\alpha_0})\right).$ For each $y \in T(\Omega_1)$, define

$$\tilde{\phi}(y) = \text{det } T^{\frac{1}{\tau}} \left[\phi(T^{-1}y) - l_{\phi, x_0}(T^{-1}y) - \frac{t_0}{\alpha_0}\right], \text{ and } \tilde{u}(y) = u(T^{-1}y).$$

Define $\tilde{g}(y) := g(T^{-1}y)$, and then

$$\text{moc}_U \tilde{g} = \text{moc}_{S_{\phi}(x_0, t_0)} g \leq \varepsilon$$

and

$$\sup_{S_{\phi}(T^{-1}y) \subset U} \tilde{g} \leq \sup_{S_{\phi}(T^{-1}y) \subset S_{\phi}(x_0, \frac{t_0}{\alpha_0})} \text{moc}_{S_{\phi}(T^{-1}y) \subset S_{\phi}(x_0, \frac{t_0}{\alpha_0})} g \leq \varepsilon.$$

We apply Lemma 3.3 to $\tilde{u}$. Then by Corollary 2.1 and arguing similarly as in [4, Lemma 4.3], we obtain (3.1).

Similarly, using Lemma 3.3 and Corollary 2.1, we can obtain the variant of [4, Lemma 4.5].
Lemma 3.5. Let $0 < \epsilon_0 < 1$, $0 < \alpha_0 < 1$. Let $B_{a_0} \subset \Omega \subset B_1$ be a normalized convex domain and $\Omega_1$ be a bounded convex set such that $\Omega \subset \Omega_1$. Assume $\phi \in C^2(\Omega_1)$ is a convex function satisfying

\[
\begin{cases}
\det D^2\phi = g & \text{in } \Omega_1, \\
\phi = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $0 < \lambda \leq g = (g^1)^p \leq \Lambda$ in $\Omega_1$. Let $u \in C(\Omega_1) \cap C^1(\Omega_1) \cap W^{2,p}_{\text{loc}}(\Omega)$ be a solution of $L_\phi u = f$ in $\Omega$. There exists $\epsilon > 0$ depending only on $n, \lambda, \Lambda, \epsilon_0, \alpha_0$ such that if $S_\phi(x_0, t_0) \subset \Omega_{a_0}$ with $t_0 \leq \eta_{a_0}$,

\[
\sup_{S_\phi(x_0, t_0) \subset S_\phi(x, h)} \mosc_{S_\phi(x, h)} g \leq \epsilon,
\]

and $S_\phi(x_0, t_0) \cap G_f(u, \Omega_1) \neq \emptyset$, then we have

\[
|G_{N_Y}(u, \Omega_1) \cap S_\phi(x_0, t_0)| \geq \left(1 - \epsilon_0 - C(N_Y)^{-r} \left(\int_{S_\phi(x_0, t_0)} |f|^p dx\right)^{\frac{1}{p'}} \right) |S_\phi(x_0, t_0)|
\]

for any $N \geq N_0$. Here $C, r, N_0, \eta_{a_0}$ are positive constants depending only on $n, \lambda, \Lambda, \alpha_0$.

Using Lemmas 3.4, 3.5, 2.4 and 2.5, we can obtain

Theorem 3.1. Given $0 < \lambda \leq \Lambda$. Let $p > 1$, $\max\{n, p\} < q < \infty$ and $0 < \alpha < 1$. Denote $\alpha_0 := \frac{q + 1}{2}$. Assume the condition (H) holds. Let $u \in C(\Omega_1)$ be a solution of $L_\phi u = f$ in $\Omega$. There exist $0 < \epsilon < 1$ and $C > 0$ depending only on $n, \lambda, \Lambda, \alpha, p, q$ such that if

\[
\sup_{S_\phi(x, h) \subset \Omega_{a_0}} \mosc_{S_\phi(x, h)} g \leq \epsilon,
\]

then we have

\[
\|D^2u\|_{L^p(\Omega_1)} \leq C (\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)})
\]

Now we are ready to prove the interior $W^{2,p}$ estimates for solutions to (1.1).

Proof of Theorem 1 : By Lemma 2.1, there exist positive constants $h_0, C$ and $g$ depending only on $n, \lambda, \Lambda$ and $\text{dist}(\Omega', \partial \Omega)$ such that

\[
B_{C^{-1}h_0}(x_0) \subset S_\phi(x_0, 2h_0) \subset B_{C(2h_0)^p}(x_0).
\]

Choose $h_0$ smaller and we can assume $S_\phi(x_0, 2h_0) \subset B_{C(2h_0)^p}(x_0) \subset \Omega'' \subset \Omega$. Since $g \in \text{VMO}_{\text{loc}}(\Omega, \phi)$, we have

\[
\eta_g(r, \Omega'') := \sup_{S_\phi(x, h) \subset \Omega''} \text{mosc}_{S_\phi(x, h)} g \to 0, \quad \text{as } r \to 0.
\]

Let $\epsilon > 0$ be the constant in Theorem 3.1 corresponding to $\alpha = \frac{1}{2}$, there exists $0 < r_0 < 1$ such that $\eta_g(r_0, \Omega'') < \epsilon$. Take $h_0$ smaller such that $\text{diam}(B_{C(2h_0)^p}(x_0)) \leq r_0$, then for any $S_\phi(x, h) \subset S_\phi(x_0, 2h_0)$, we have $S_\phi(x, h) \subset \Omega''$ and $\text{diam}(S_\phi(x, h)) \leq r_0$, thus,

\[
\text{mosc}_{S_\phi(x, h)} g \leq \eta_g(r_0, \Omega'') < \epsilon.
\]

Let $T$ be an affine map such that

\[
B_{a_0} \subset T \left( S_\phi(x_0, 2h_0) \right) \subset B_1.
\]

By (3.2) we have

\[
\|T\| \leq Ch_0^{-1}, \quad \|T^{-1}\| \leq Ch_0^q \leq 1
\]

if $h_0$ is even smaller. Let $\tilde{\Omega} := T \left( S_\phi(x_0, 2h_0) \right)$. For each $y \in \tilde{\Omega}$, define

\[
\tilde{\phi}(y) = |\det T|^{\frac{1}{2}} \left[ \phi(T^{-1}y) - l_{\phi,x_0}(T^{-1}y) - 2h_0 \right], \quad \text{and } \tilde{u}(y) = u(T^{-1}y).
\]

We have

\[
\det D^2\tilde{\phi} = \tilde{g}(y) = (g^1)^p \text{ in } \tilde{\Omega}, \quad \tilde{\phi} = 0 \text{ on } \partial \tilde{\Omega},
\]

\[
\lambda \leq \tilde{g}(y) = g(T^{-1}y) \leq \Lambda \text{ in } \tilde{\Omega},
\]

\[
L_{\tilde{g}} \tilde{u}(y) = \tilde{f}(y) := |\det T|^{\frac{2}{n}} f(T^{-1}y) \text{ in } \tilde{\Omega}.
\]
Moreover, choose $\alpha = 1/2$ and denote $\alpha_0 := \frac{d+1}{2}$, then we have
\[
\sup_{S_\rho(x,y) \subset \Omega_{\rho(x,y)}} \text{mose}S_\rho(x,y) \bar{g} \leq \sup_{S_\rho(x,y) \subset \Omega} \text{mose}S_\rho(x,y) \bar{g} \leq \sup_{S_\rho(T^{-1}y, \det T^{-1}x)} \text{mose}S_\rho(x,y,2h) \bar{g} \leq \epsilon.
\]
Applying Theorem 3.1 we have
\[
\|D^2\bar{u}\|_{L^p(\Omega_{\rho(\cdot)/2})} \leq C \left( \|\bar{u}\|_{L^p(\Omega_{\rho})} + \|\bar{f}\|_{L^p(\Omega_{\rho})} \right).
\]
Back to $u$, note (3.3),
\[
\|D^2u\|_{L^p(\Omega_{\rho(x,y,h_0)})} \leq C \left( \|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right).
\]
From this, by a standard covering argument, we obtain the conclusion.

4. Boundary estimate-A stability result at the boundary for the cofactor matrices

In this section, we prove a stability result at the boundary for the cofactor matrices of $D^2\phi$ and then establish the global power decay estimate for the distribution function of $D^2\phi$. We fix constants $0 < \lambda \leq \Lambda < \infty, \rho > 0$ and refer to all positive constants depending only $n, \lambda, \Lambda$ and $\rho$ as universal constants.

Assume
\[
B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_\rho.
\]
(4.2) $\Omega$ contains an interior ball of radius $\rho$ tangent to $\partial \Omega$ at each point on $\partial \Omega \cap B_\rho$.
Let $\phi : \Omega \rightarrow \mathbb{R}, \phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying
\[
\det D^2\phi = g, \quad 0 < \lambda \leq g \leq \Lambda \quad \text{in } \Omega.
\]
(4.3) Assume further that on $\partial \Omega \cap B_\rho, \phi$ separates quadratically from its tangent planes on $\partial \Omega$, namely, for any $x_0 \in \partial \Omega \cap B_\rho$ we have
\[
\rho|x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla \phi(x_0) \cdot (x - x_0) \leq \rho^{-1}|x - x_0|^2, \quad \forall x \in \partial \Omega.
\]
(4.4)

**Theorem 4.1.** (Localization Theorem [8, 9]) Assume $\Omega$ satisfies (4.1) and $\phi$ satisfies (4.3), and $\phi(0) = \nabla \phi(0) = 0$, $\rho|x|^2 \leq \phi(x) \leq \rho^{-1}|x|^2$, on $\partial \Omega \cap \{x_n \leq \rho\}$.

Then there exists a universal constant $k > 0$ such that for each $h \leq k$, there is an ellipsoid $E_h$ of volume $|B_1|n/2$ satisfying
\[
kE_h \cap \Omega \subset S_\rho(0, h) \subset k^{-1}E_h \cap \Omega.
\]
Moreover, the ellipsoid $E_h$ is obtained from the ball of radius $\rho h^2$ by a linear transformation $A_h^{-1}$ (sliding along the $x_n = 0$ plane)
\[
det A_h = 1, \quad A_h x = x - \tau_h e_n, \quad \tau_h \cdot e_n = 0, \quad h^{-1}A_h E_h = B_1, \quad |\tau_h| \leq k^{-1}|\log h|.
\]

**Proposition 4.1.** (see [8, Proposition 3.2]) Let $\phi$ and $\Omega$ satisfy the hypotheses of the Localization Theorem 4.1. Assume that for some $y \in \Omega$ the section $S_\rho(y, h) \subset \Omega$ is tangent to $\partial \Omega$ at 0 for some $h \leq c$ with $c$ universal. Then
\[
\nabla \phi(y) = ae_n \quad \text{for some } a \in [k_0h^2, k_0^{-1}h^2],
\]
\[
k_0E_h \subset S_\rho(y, h) - y \subset k_0^{-1}E_h, \quad k_0h^2 \leq \text{dist}(y, \partial \Omega) \leq k_0^{-1}h^2,
\]
where $E_h$ is the ellipsoid defined in the Localization Theorem 4.1, and $k_0 > 0$ a universal constant.

Under the assumptions of Theorem 4.1, we have for all $h$ small,
\[
\Omega \cap B_h^+ \subset \Omega \cap B_{c_1h^2/|\log h|} \subset S_\rho(0, h) \subset B_{c_1h^2/|\log h|} \subset B_h^+,
\]
(4.5) and
\[
c_1|x|^2 |\log |x||^{-2} \leq \phi(x) \leq C_1|x|^2 |\log |x||^2
\]
(4.6) for any $x \in B_{c_1}$, where $c_1, C_1$ are universal (see Equation (4.3) in [8]).
Denote the rescaled function of \( \phi \) and the rescaled domain of \( \Omega \) by

\[
(4.7) \quad \phi_h(x) := \frac{\phi(h^{1/2}A^{-1/2}h x)}{h} \quad \text{and} \quad \Omega_h := h^{-1}A_h \Omega.
\]

Denote \( U_h := S_{\phi_h}(0,1) = \{ x \in \overline{\Omega_h} : \phi_h(x) < 1 \} \), then

\[
\det D^2 \phi_h = g_h = (g^1_h)^n, \quad 0 < \lambda \leq g_h(x) := g(h^{1/2}A^{-1/2}h x) \leq \Lambda \quad \text{in} \ \Omega_h,
\]

\[
(4.8) \quad \Omega_h \cap B_k < h^{-1}A_h S_\phi(0,h) = U_h \subset B_{k+1}^+.
\]

To establish the stability of the cofactor matrix of \( D^2 \phi \) at the boundary we compare \( \phi_h \) and the solution \( w \) of \( \det D^2 w = 1 \). Due to the VMO-type condition, we change the boundary data of \( w \) as follows:

Given a constant \( A > 0 \) such that \( c(n, \lambda, \Lambda) \leq A \leq C(n, \lambda, \Lambda) \). Let \( w \in C(\overline{U_h}) \) be the convex solution of

\[
(4.9) \quad \begin{cases}
\det D^2 w = 1 & \text{in} \ U_h, \\
w = A \cdot \phi & \text{on} \ \partial U_h.
\end{cases}
\]

Correspondingly, we modify the class \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa} \) in [7]. Fix \( n, \rho, \lambda, \Lambda, \kappa \) and \( \alpha \), the class \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa} \) and \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha} \) consist of the quadruples \( (\Omega, \phi, g, U) \) satisfying the following conditions (i)-(vii) and (i)-(vi) respectively:

(i) \( 0 \in \partial \Omega, \ U \subset \Omega \subset \mathbb{R}^n \) are bounded convex domains such that

\[ B_k^+ \cap \overline{\Omega} \subset U \subset B_{k+1}^+ \cap \overline{\Omega}. \]

(ii) \( \phi : \overline{\Omega} \rightarrow \mathbb{R}^+ \) is convex satisfying \( \phi = 1 \) on \( \partial U \cap \Omega \) and

\[ \det D^2 \phi = g, \quad \lambda \leq g = (g^1)^n \leq \Lambda \ \text{in}\ \Omega, \quad \phi(0) = 0, \quad \nabla \phi(0) = 0, \quad \partial \Omega \cap \{ \phi < 1 \} = \partial U \cap \{ \phi < 1 \}. \]

(iii) (quadratic separation)

\[
\frac{\rho}{4} |x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla \phi(x_0) \cdot (x - x_0) \leq \frac{4}{\rho} |x - x_0|^2, \quad \forall x, x_0 \in \partial \Omega \cap B_{\frac{1}{2}k},
\]

(iv) (flatness)

\[ \partial \Omega \cap \{ \phi < 1 \} \subset G \subset \{ x_n \leq \kappa \}, \]

where \( G \subset B_{2k} \) is a graph in the \( e_n \) direction and its \( C^{1,1} \) norm is bounded by \( \kappa \).

(v) (localization and gradient estimates) \( \phi \) satisfies in \( U \) the hypotheses of the Localization Theorem in 4.1 at all points on \( \partial U \cap B_{\varepsilon} \) and

\[ |\nabla \phi| \leq C_0 \quad \text{in} \ U \cap B_{\varepsilon}. \]

(vi) (maximal sections around the origin) If \( y \in U \cap B_{\varepsilon} \) then the maximal interior section of \( \phi \) in \( U \) satisfies

\[ k_0^2 \text{dist}^2(y, \partial U) \leq \bar{h}(y) \leq c \quad \text{and} \quad S_\phi(y, \bar{h}(y)) \subset U \cap B_{\varepsilon}. \]

(vii) (Pogorelov estimates)

\[ ||\partial U \cap B_{\varepsilon}||_{C^{2,\alpha}} \leq c_0^{-1} \]

and if \( w \) is the convex solution to

\[
(4.10) \quad \begin{cases}
\det D^2 w = 1 & \text{in} \ U, \\
w = A \cdot \phi & \text{on} \ \partial U.
\end{cases}
\]

for some constant \( A > 0 \) such that \( c(n, \lambda, \Lambda) \leq A \leq C(n, \lambda, \Lambda) \), then

\[ ||w||_{C^{2,\alpha}(U \cap B_{\varepsilon})} \leq c_0^{-1} \quad \text{and} \quad c_0 I_n \leq D^2 w \leq c_0^{-1} I_n \quad \text{in} \ U \cap B_{\varepsilon}. \]

The constants \( k, k_0, c, C_0 \) above depend only on \( n, \lambda, \Lambda \) and \( \rho \) and \( c_0 \) depends also on \( \alpha \).

The class \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha} \) is the same as that in [7], while \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha} \) is slightly different from that in [7]. Let \( \Omega, \phi, g \) satisfy \( (4.1)-(4.4) \). It follows from [8, Lemma 4.2] and [7, Lemma 2.5] that if \( h \leq k \), then \( (\Omega_h, \phi_h, g_h, S_{\phi_h}(0,1)) \in \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha} \). It is easily seen that [7, Lemma 2.7] (quadratic separation on \( \partial U_h \cap B_{\varepsilon} \)) and [7, Proposition 2.8] (\( C^{2,\alpha} \) estimates in \( U_h \cap B_{\varepsilon} \)) hold for the solution \( w \) of \( (4.9) \). Hence we have the following version of [7, Proposition 2.12] corresponding to our new definition of \( \mathcal{P}_{\lambda,\Lambda,\rho,\kappa,\alpha} \).
Proposition 4.2. Let $\Omega$ and $\phi$ satisfy (4.1)-(4.4). Assume in addition that $\partial \Omega \cap B_p$ is $C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial \Omega \cap B_p)$ for some $\alpha \in (0,1)$. Then there exists $h_0 > 0$ depending only on $n, \Lambda, \rho, \alpha, \|\partial \Omega \cap B_p\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial \Omega \cap B_p)}$ such that for $h \leq h_0$, we have

$$\left(\Omega_h, \phi_h, g_h, S_{\phi_h}(0,1)\right) \in \mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$$

and $\|\partial \Omega_h \cap B_{1/h}\|_{C^{2,\alpha}} \leq C h^{1/2}$.

Here $k, C$ depend only on $n, \Lambda$ and $\rho$. $C'$ depends only on $n, \Lambda, \rho, \|\partial \Omega \cap B_p\|_{C^{2,\alpha}}$ and $\|\phi\|_{C^{2,\alpha}(\partial \Omega \cap B_p)}$.

The covering lemma [7, Lemma 3.13] holds for the newly defined class $\mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$.

Lemma 4.1. Assume $(\Omega, \phi, g, U) \in \mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$. Let $c$ be as in (vi) in the definition of $\mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$ and $w$ be the solution to (4.10) for some constant $A > 0$ such that $c(n, \Lambda, \rho) \leq A \leq C(n, \Lambda, \rho)$. Let $\psi$ denote one of the functions $\phi$ and $w$. Then there exists a sequence of disjoint sections $\{\psi(y_i, \delta_0(h(y_i)))\}_{i=1}^\infty$, where $\delta_0 = \delta_0(n, \Lambda, \rho), y_i \in U \cap B_{r^2}$ and $S_{\psi}(y_i, \delta_0(h(y_i)))$ is the maximal interior section of $\psi$ in $U$, such that

$$U \cap B_{r^2} \subset \bigcup_{i=1}^\infty S_{\psi}(y_i, \delta_0(h(y_i))/2).$$

Moreover, we have

$$S_{\psi}(y_i, \delta_0(h(y_i))) \subset U \cap B_{r^2}, \quad \delta_0(h(y_i)) \leq c.$$  

Let $M^\text{loc}$ denote the number of sections $S_{\psi}(y_i, \delta_0(h(y_i))/2)$ such that $d/2 < \delta_0(h(y_i)) \leq d \leq c$, then

$$M^\text{loc} \leq C_d d^{\frac{n-2}{2}}$$

for some constant $C_d$ depending only on $n, \Lambda, \rho$ and $\kappa$.

Next we prove a stability result at the boundary for the cofactor matrices of $D^2\phi$.

Proposition 4.3. Given $0 < \epsilon < 1$. Assume $(\Omega, \phi, g, U) \in \mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$. Let $c$ be as in (vi) in the definition of $\mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$ and $\bar{c}$ be a constant depending only on $n, \Lambda, \rho$ such that $\bar{c} \leq c^2$. Assume $w$ is the solution to

$$\left\{ \begin{array}{ll}
\det D^2w = 1 & \text{in } U, \\
\frac{\bar{c}}{\phi} & \text{on } \partial U.
\end{array} \right.$$  

For any $1 < p < \infty$, there exists $\epsilon = \epsilon(n, \Lambda, \rho, \kappa, p) > 0$ such that if

$$\text{mosc}_{U \cap B_r} \leq \epsilon \quad \text{and} \quad \sup_{S_{\phi}(x, \delta_0(h(x)))} \text{mosc}_{U \cap B_r} \leq \epsilon,$$

then the following statements hold:

(i)  

$$\|D^2\phi - (g^1)_{U \cap B_r} D^2w\|_{L^p(U \cap B_{r/2})} \leq C(n, \Lambda, \rho, \kappa, p, \epsilon)\epsilon^{\frac{1}{p-1}}.$$ 

(ii) Assume in addition that $(\Omega, \phi, g, U) \in \mathcal{P}_{\Lambda, \Lambda, \rho, \Lambda, \rho, \alpha}$. Then

$$\|\Phi - (g^1)_{U \cap B_r} W\|_{L^p(U \cap B_{r/2})} \leq C(n, \Lambda, \rho, \kappa, p, \alpha)\epsilon^{\frac{1}{p-1}}.$$ 

Here $\delta = \delta(n, \Lambda, \rho, p) \in (0, 1/2)$. $\Phi$ and $W$ are the cofactor matrices if $D^2\phi$ and $D^2w$ respectively.

Proof. The proof is similar to that of [7, Proposition 3.14], using Lemma 4.1 and Theorem 2.1 instead due to the VMO-type condition of $g$. Note that if we take $A : = [g^1_{U \cap B_r}]^{-1}$, then $c(n, \Lambda, \rho) \leq A \leq C(n, \Lambda, \rho)$, as required in Lemma 4.1. We sketch the proof.

(i) As in [7], the conclusion follows from the two claims below and the interpolation inequality.

Claim 1. For any $1 \leq p < \infty$, there exists $C_0 > 0$ depending only on $n, \Lambda, \rho, \kappa, p$ such that

$$\|D^2\phi\|_{L^p(U \cap B_{r/2})} \leq C_0$$

if $\epsilon = \epsilon(n, \Lambda, \rho, p) > 0$ is small.

Claim 2. There exist $\delta = \delta(n, \Lambda, \rho, \kappa) \in (0, 1/2)$ and $\epsilon = \epsilon(n, \Lambda, \rho, \kappa) > 0$ such that

$$\|D^2\phi - (g^1)_{U \cap B_r} D^2w\|_{L^p(U \cap B_{r/2})} \leq C\epsilon^{1/p}$$

for any $0 < \epsilon < 1$. 


Let \( \psi \) denote one of the functions \( \phi \) and \( w \). By Lemma 4.1, Claim 1 follows from this result: there exists \( C = C(n, \lambda, \Lambda, \rho, p) > 0 \) such that

\[
(4.14) \quad \int_{S_\phi(y, \frac{h(y)}{2})} |D^2 \psi|^p \leq C(h(y))^2 |\log \tilde{h}(y)|^2 \quad \forall y \in U \cap B_2.
\]

To prove the above inequality, we denote \( h := \tilde{h}(y) \leq c \). Apply Proposition 4.1 to \( S_\phi(y, h) \) and we obtain

\[
k_0 E_h \subset S_\phi(y, h) - y \subset k_0^{-1} E_h,
\]

where \( E_h := h^\frac{1}{p} A_h^{-1} B_1 \) with \( \det A_h = 1 \) and \( ||A_h||, ||A_h^{-1}|| \leq C |\log h| \). Define

\[
\tilde{g}_h(x) := \frac{1}{h} [\psi(y + h^\frac{1}{p} A_h^{-1} x) - \psi(y)], \quad x \in \tilde{\Omega}_h := h^{-\frac{1}{p}} A_h(\Omega - y)
\]

If \( \psi = \phi \), we denote \( \tilde{g}_h(x) := g(y + h^\frac{1}{p} A_h^{-1} x) \). Denote \( T_h x := h^{-\frac{1}{p}} A_h(x - y) \), then

\[
\sup_{S_\phi(x, \frac{h}{2})} \text{mosc}_{S_\phi(x, \frac{h}{2})} \tilde{g}_h \leq \sup_{S_\phi(T_h x, \frac{h}{2})} \text{mosc}_{S_\phi(T_h x, \frac{h}{2})} g \leq \sup_{S_\phi(T_h x, \frac{h}{2}) \cap U \cap B_2} \text{mosc}_{S_\phi(T_h x, \frac{h}{2})} g \leq \epsilon.
\]

Therefore by Theorem 2.1 (i) we have

\[
\int_{S_\phi(x, \frac{h}{2})} ||D^2 \tilde{\phi}_h||^p \leq C(n, \lambda, \Lambda, \rho, p)
\]

if \( \epsilon = e(n, \lambda, \Lambda, \rho, p) \) is small.

If \( \psi = w \) then a similar inequality holds for \( w \) because of Pogorelov’s estimate [2, (4.2.6)]. Changing variables as in Page 662 in [7], we obtain (4.14) and then Claim 1 follows.

The proof of Claim 2 is similar to that of Lemma 2.3. Let \( \nu := \phi - (g^1)_{U \cap B_2} \), then \( \nu = 0 \) on \( \partial U \). Note that \( |U \cap B_2| \geq c(n, \lambda, \Lambda, \rho) \), then by Lemma 2.2 and Proposition 2.1,

\[
\max_U |\phi - (g^1)_{U \cap B_2}| \leq C n \text{diam}(U) \left( \int_U |g^1 - (g^1)_{U \cap B_2}|^\frac{n}{n-1} dx \right)^\frac{n-1}{n} \leq C(n, \lambda, \Lambda, \rho, p) \left( \int_U |g^1 - (g^1)_{U \cap B_2}|^\frac{n}{n-1} dx \right)^\frac{n-1}{n} \leq C(n, \lambda, \Lambda, \rho, p) \epsilon^\frac{n}{n-1}.
\]

Using this and similar arguments to the proof of [7, Proposition 3.14], Claim 2 is proved and the conclusion (i) follows.

(ii) Write

\[
\Phi - (g^1)_{U \cap B_2} w = \left[ 1 - \frac{(g^1)_{U \cap B_2}}{\det D^2 \phi} \right] \Phi - \frac{(g^1)_{U \cap B_2}}{\det D^2 \phi} \Phi \left( D^2 \phi - (g^1)_{U \cap B_2} D^2 w \right) W.
\]

Arguing as in the proof of Lemma 2.3, for any \( 1 \leq q, r < \infty \) we have,

\[
\| \Phi - (g^1)_{U \cap B_2} W \|_{L^q(U \cap B_2)} \leq C(n, \lambda, \Lambda, \rho, q, r) \left( \epsilon^\frac{n}{n-1} + |\phi - (g^1)_{U \cap B_2} w|_{L^q(U \cap B_2)} \right)^\frac{n-1}{n} \| D^2 \phi - (g^1)_{U \cap B_2} D^2 w \|_{L^r(U \cap B_2)}.
\]

Choose \( r = n \) and then \( r' = \frac{n}{n-1} \), if \( \epsilon = e(n, \lambda, \Lambda, \rho, q) \) is small, then

\[
\| \Phi - (g^1)_{U \cap B_2} \|_{L^q(U \cap B_2)} \leq C(n, \lambda, \Lambda, \rho, q, k) \left( \epsilon^\frac{n}{n-1} + |\phi - (g^1)_{U \cap B_2} w|_{L^q(U \cap B_2)} \right)^\frac{n-1}{n} \| D^2 \phi - (g^1)_{U \cap B_2} D^2 w \|_{L^r(U \cap B_2)}\epsilon^\frac{n}{n-1}
\]

by (i) and Claim 1 in the proof of (i). By the definition of \( \mathcal{P}_{\lambda, \Lambda, \rho, k, \alpha} \), we have

\[
\| D^2 w \|_{L^r(U \cap B_2)} \leq C(n, \lambda, \Lambda, \rho, \alpha).
\]

Hence the conclusion follows. \( \square \)
We next list the global version of Lemma 2.4. Recall Section 2 for the definition of \(d(x, x_0)\) and \(G_M(u, \Omega)\). Assume
\[
\Omega \subset B_{1/\rho} \quad \text{and for each } y \in \partial \Omega \text{ there is a ball } B_{\rho}(y) \subset \Omega \text{ that is tangent to } \partial \Omega \text{ at } y.
\]
Let \(\phi : \bar{\Omega} \to \mathbb{R}, \phi \in C^{0,1}(\bar{\Omega}) \cap C^2(\Omega)\) be a convex function satisfying
\[
det D^2 \phi = g, \quad 0 < \lambda \leq g \leq \Lambda \quad \text{in } \Omega.
\]
Assume further that on \(\partial \Omega\), \(\phi\) separates quadratically from its tangent planes, namely,
\[
\rho |x - x_0|^2 \leq \phi(x) - \phi(x_0) - \nabla \phi(x_0) \cdot (x - x_0) \leq \rho^{-1} |x - x_0|^2, \quad \forall x, x_0 \in \partial \Omega.
\]

**Lemma 4.2.** (See [7, Lemma 3.4].) Assume \(\Omega\) satisfies (4.15) and \(\phi \in C(\bar{\Omega})\) is a solution of (1.2). Define
\[
A^{\text{loc}}_{\phi} := \{x_0 \in \Omega : d(x, x_0)^2 \geq \sigma |x - x_0|^2, \text{ for all } x \text{ in some neighborhood of } x_0\}.
\]
Let \(u \in C^2(\Omega)\). Then for \(\kappa > 1\), we have
\[
\{x \in \Omega : |D_j u(x)| > \beta^k\} \subset \left(\Omega \backslash A^{\text{loc}}_{\phi} \right) \cup \left(\Omega \backslash G_{\rho}(u, \Omega)\right)
\]
for any \(\beta > 0\), where \(c > 0\) is a universal constant.

The estimate of the first term on the right-hand side of (4.18) is given by the following theorem.

**Theorem 4.2.** Assume \(\Omega, \phi\) satisfy (4.15)-(4.17) and \(\partial \Omega \in C^{1,1}\). Let \(0 < \epsilon < 1\). Suppose
\[
\sup_{S_\phi(z, h) \in \Omega} \text{mosc}_{S_\phi(z, h)} g \leq \epsilon.
\]
Then there exist universal constants \(M > 0, 0 < \delta_1 < 1\) such that
\[
|\Omega \backslash A^{\text{loc}}_{\phi} | \leq C'(\epsilon, n, \lambda, \rho, \|\partial \Omega\|_{C^{1,1}}, s) s^{\ln(2\epsilon^{1/2})^{\omega M}} \quad \forall s > 0.
\]
In particular, for \(s = (\epsilon^{-\delta_1})^{\frac{1}{\omega M}}\) we have
\[
|\Omega \backslash A^{\text{loc}}_{\phi} | \leq C'(\epsilon, n, \lambda, \rho, \|\partial \Omega\|_{C^{1,1}}, \beta) s^{\ln(2\epsilon^{1/2})^{\omega M}} \quad \forall \beta > 0.
\]

**Proof.** The proof is similar to that of [7, Theorem 3.5], using Lemma 2.5 instead. Namely, we cover \(\Omega\) by sections of \(\phi\) given by [7, Lemma 3.12] and obtain
\[
|\Omega \backslash A^{\text{loc}}_{\phi} | \leq \sum_{h_1 \in \mathbb{C}} |S_\phi(y_1, h_1/2) \backslash A^{\text{loc}}_{\phi} | + \sum_{h_2 \in \mathbb{C}} |S_\phi(y_2, h_2/2) \backslash A^{\text{loc}}_{\phi} | = I + II,
\]
where \(h_1 := \tilde{h}(y_1)\). For the summation \(I\) corresponding to \(h \leq c\), let \(T x := h^{-\frac{1}{2}} A_h(x - y)\) where \(A_h\) is given by Proposition 4.1. Define the rescaled domain \(\tilde{U}_h := T(S_\phi(y, h))\) and function
\[
\tilde{\phi}_h(x) := \frac{1}{h} [\phi(T^{-1} x) - I_{\phi, y}(T^{-1} y) - h].
\]
Denote \(\tilde{g}_h = \det D^2 \tilde{\phi}_h\) and
\[
\tilde{D}_h^\perp := \{x \in S_{\phi_h}(0, 1/2) : S_{\phi_h}(x, t) \subset B(x, s \sqrt{t}), \quad \forall t \leq \eta\}.
\]

We apply Lemma 2.5 with \(\Omega \rightsquigarrow \tilde{U}_h, \phi \rightsquigarrow \tilde{\phi}_h, g \rightsquigarrow \tilde{g}_h, \alpha \rightsquigarrow \frac{1}{2}\). Denote \(\alpha_0 := \frac{\alpha + 1}{2} = \frac{3}{4}\). Note that for any \(S_{\phi_h}(x, t) \subset (\tilde{U}_h)(\alpha_0, 1) \subset \tilde{U}_h\), we have
\[
\text{mosc}_{S_{\phi_h}(x, t)} \tilde{g}_h = \text{mosc}_{S_{\phi}(T^{-1} x, h)} g \leq \epsilon.
\]
Thus by Lemma 2.5, there exist universal constants \(M > 0, 0 < \delta_1 < 1\) such that
\[
|S_{\phi_h}(0, 1/2) \backslash \tilde{D}_h^\perp | \leq \left(\frac{\tilde{U}_h}{(2\epsilon^{1/2})^{\omega M}}\right)^s .
\]
Then we obtain
\[
I \leq C(n, \lambda, \rho, \epsilon, \|\partial \Omega\|_{C^{1,1}}) s^{-p_\epsilon}, \quad p_\epsilon = -\frac{\ln(2\epsilon^{1/2})^{\omega M}}{\ln M}.
\]
The estimate of summation \(II\) is also similar to that in [7] by using standard normalization for interior sections. Again, we use Lemma 2.5 in each of these sections. \(\square\)
5. Boundary estimate-Power decay estimates for the solutions of (1.1)

In this section, we prove a localized estimate at the boundary of the density of good sets where $D^2\phi$ is bounded. Different from [7, Lemma 4.6], since $g$ is in VMO-type spaces, there is a difficulty in using convex envelopes of functions to help estimate the density of good sets. Instead, we compare $\phi$ with the solution $w$ of $\det D^2 w = 1$. We apply the localized covering lemma at the boundary and cover $U \cap B_{\bar{c}}$ by sections of $w$. Then we use the one-sided small power decay estimate [4, Proposition 3.4] in each of these sections to the linearized operator $L_w$ to estimate the set where $\phi$ can be touched from above by a quadratic polynomial. The precise statement is as follows.

**Lemma 5.1.** Assume $(\Omega, \phi, g, U) \in \mathcal{P}_{\lambda, \Lambda, \rho, \kappa, \alpha}$ with $0 < \kappa < 1$, $\phi \in C^1(\Omega)$ and $\max_{x, y} g \leq \varepsilon$. Let $c$ be the constant in the definition of the class $\mathcal{P}$. Then there exist $\sigma = \sigma(n, \lambda, \Lambda, \rho, \alpha, \kappa) > 0$, $0 < \bar{c} = \bar{c}(n, \lambda, \Lambda, \rho) \leq c^4$ and $0 < r' = r'(n) < 1/2$ such that

\[|(U \cap B_{\bar{c}}) \setminus A_v| \leq \varepsilon^\frac{\bar{c}}{4}|U \cap B_{\bar{c}}|,
\]

and for any $S_\phi(0, r)$ with $r = r(n, \lambda, \Lambda, \rho) \leq \bar{c}^3$, we have

\[|S_\phi(0, r) \setminus A_v| \leq \varepsilon^\frac{\bar{c}}{4}|S_\phi(0, r)|,
\]

where

\[(5.1) \quad A_v := \{\bar{x} \in U \cap B_{\bar{c}} : \phi(x) \geq l_{\phi, x}(x) + \sigma|x - \bar{x}|^2, \quad \forall x \in U \cap B_{\bar{c}}\}.
\]

**Proof.** Let $w$ be the convex solution of

\[
\begin{cases}
\det D^2 w = 1 & \text{in } U, \\
w = \frac{\phi}{(g^1)_{U}} & \text{on } \partial U.
\end{cases}
\]

By Lemma 2.2 and Proposition 2.1,

\[
\max_{U} |\phi - (g^1)_{U} w| \leq C_{\pi} \text{diam}(U) \left( \int_{U} |g^1 - (g^1)_{U}|^n dx \right)^{\frac{1}{n}} \leq C(n, \lambda, \Lambda, \rho) \varepsilon^\frac{c}{4},
\]

(5.2)

Consider the operator $Mu := (\det D^2 u)^{\frac{1}{2}}$ and its linearized operator

\[
\hat{L}_u v := \frac{1}{n} (\det D^2 u)^{\frac{1}{2}} \text{tr}[(D^2 u)^{-1} D^2 v],
\]

Denote $v := \frac{\phi}{(g^1)_{U}} - w$. Since $M$ is concave, we have

\[(5.3) \quad F := \frac{g^1}{(g^1)_{U}} - 1 = M(\frac{\phi}{(g^1)_{U}}) - Mw \leq \hat{L}_u v = \frac{1}{n} \text{tr}[(D^2 w)^{-1} D^2 v] \quad \text{in } U.
\]

Let $c$ be the constant in the definition of the class $\mathcal{P}$. For $u \in C^1(U \cap B_{\bar{c}})$ and $\beta > 0$ we define

\[(5.4) \quad G_{\beta}^+(u, w) := \left\{\bar{x} \in U \cap B_{\bar{c}} : u(x) - l_{u, x}(x) \leq \frac{\beta}{2} d_w(x, \bar{x})^2, \quad \forall x \in U \cap B_{\bar{c}}\right\},
\]

where $d_w(x, \bar{x})^2 := w(x) - l_{w, x}(x)$ for any $x, \bar{x} \in U \cap B_{\bar{c}}$.

By Lemma 4.1, there exists a sequence of disjoint sections $\{S_{u}(y_i, \delta_0 \bar{h}(y_i))\}_{i=1}^{\infty}$, where $\delta_0 = \delta_0(n, \lambda, \Lambda, \rho)$, $y_i \in U \cap B_{\bar{c}}$ and $S_{u}(y_i, \bar{h}(y_i))$ is the maximal interior section of $w$ in $U$, such that

\[U \cap B_{\bar{c}} \subset \bigcup_{i=1}^{\infty} S_{w}(y_i, \frac{\bar{h}(y_i)}{2})
\]

and

\[S_{u}(y_i, \bar{h}(y_i)) \subset U \cap B_{\bar{c}}, \quad \bar{h}(y_i) \leq c.
\]

Moreover, let $M^\text{loc}_d$ denote the number of sections $S_{w}(y_i, \bar{h}(y_i))$ such that $d/2 < \bar{h}(y_i) \leq d$ ≤ $c$, then

\[(5.5) \quad M^\text{loc}_d \leq C_{\beta} d^\frac{\beta}{2}
\]

for some constant $C_{\beta}$ depending only on $n, \lambda, \Lambda, \rho$ and $k$.

Let $h := \bar{h}(y)$ with $\bar{h}(y) \leq c$. Let $T$ be an affine map such that

\[B_{\alpha} \subset T(S_{u}(y, \bar{h})) \subset B_{1},
\]

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Let $\tilde{U} := T(S_w(y, h))$ and $\tilde{\Omega} := T(U \cap B_c)$. For each $\tilde{x} \in \tilde{\Omega}$, define

$$\tilde{w}(\tilde{x}) = |\det T|^{-\frac{1}{2}} \left[w(T^{-1} \tilde{x}) - l_{w, y}(T^{-1} \tilde{x}) - h\right],$$

and $\tilde{v}(\tilde{x}) = v(T^{-1} \tilde{x})$.

Then by (5.3) we have

$$\det D^2 \tilde{w} = 1 \text{ in } \tilde{\Omega}, \quad \tilde{w} = 0 \text{ on } \partial \tilde{U},$$

$$L_{\tilde{w}} \tilde{v}(\tilde{x}) \geq F(\tilde{x}) := n|\det T|^{-\frac{1}{2}} F(T^{-1} \tilde{x}) \text{ in } \tilde{\Omega}.$$  

Applying [4, Proposition 3.4] (a straightforward modification of the proof gives rise to one-sided estimates) with $U \rightsquigarrow \tilde{U}, \Omega \rightsquigarrow \tilde{\Omega}, \phi \rightsquigarrow \tilde{w}, u \rightsquigarrow \tilde{v}, \alpha \rightsquigarrow \frac{1}{2}$, we obtain

(5.6) \[ \left| \tilde{U} \frac{1}{2} G^+_\beta(\tilde{v}, \tilde{\Omega}, \tilde{w}) \right| \leq C \left[ \frac{\|v\|_{L^\infty(\tilde{\Omega})} + \|F\|_{L^r(\tilde{\Omega})}}{\beta^r} \right]^{\nu} \quad \forall \beta > 0, \]

where $C, r' > 0$ depending only on $n$ and

$$G^+_\beta(\tilde{v}, \tilde{\Omega}, \tilde{w}) := \left\{ \tilde{x} \in \tilde{\Omega} : \tilde{v}(\tilde{x}) - l_{\tilde{v}, \tilde{\Omega}}(\tilde{x}) \leq \frac{\beta}{2} |\tilde{w}(\tilde{x}) - l_{\tilde{w}, \tilde{\Omega}}(\tilde{x})|, \quad \forall \tilde{x} \in \tilde{\Omega} \right\}.$$  

Note that $\tilde{U} \frac{1}{2} = T(S_w(\frac{1}{2} S_w(y, h)))$ and

$$G^+_\beta(\tilde{v}, \tilde{\Omega}, \tilde{w}) = T(G^+_{\beta|\det T|^{\frac{1}{2}}}(v, w)),$$

where we recall (5.4) for the definition of $G^+_{\beta|\det T|^{\frac{1}{2}}}(v, w)$. Moreover,

$$||F||_{L^r(\tilde{U})} = \frac{n}{|\det T|^\frac{1}{2}} ||F||_{L^r(S_w(y, h))}.$$  

Thus (5.6) implies that

(5.7) \[ \left| S_w(\frac{1}{2} S_w(y, h)) G^+_{\beta|\det T|^{\frac{1}{2}}}(v, w) \right| \leq C \left[ \frac{\|v\|_{L^\infty(U \cap B_c)} + \|n\|_{\det T}^{\frac{1}{2}} ||F||_{L^r(S_w(y, h))}}{\beta^r} \right]^{\nu} \quad \forall \beta > 0. \]

By (vii) in the definition of $P$,

(5.8) \[ \frac{c_0}{2} |x - x_0|^2 \leq w(x) - l_{w, x_0}(x) \leq \frac{c_0^{-1}}{2} |x - x_0|^2 \quad \forall x, x_0 \in U \cap B_c, \]

where $c_0 = c_0(n, \lambda, \Lambda, \rho, \alpha)$. Hence for any $\tilde{x} \in G^+_{\beta|\det T|^{\frac{1}{2}}}(v, w)$ and $x \in U \cap B_c$, we have

$$v(x) - l_{v, \tilde{x}}(x) \leq \frac{C_0 \beta h^{-1}}{2} |x - \tilde{x}|^2,$$

where $C_0 = C_0(n, \lambda, \Lambda, \rho, \alpha)$. Therefore, by (5.7) and (5.2),

$$\left| S_w(\frac{1}{2} S_w(y, h)) \left\{ \tilde{x} \in U \cap B_c : v(x) - l_{v, \tilde{x}}(x) \leq \frac{\beta'}{2} |x - \tilde{x}|^2, \quad \forall x \in U \cap B_c \right\} \right| \leq \frac{C(n, \lambda, \Lambda, \rho, \alpha)}{\beta^r} h^{\frac{\beta'}{2} - r'} e^{\nu}, \quad \forall \beta' > 0,$$

which implies

$$\left| (U \cap B_{\frac{1}{2}}) \left\{ \tilde{x} \in U \cap B_c : v(x) - l_{v, \tilde{x}}(x) \leq \frac{\beta'}{2} |x - \tilde{x}|^2, \quad \forall x \in U \cap B_c \right\} \right| \leq \frac{C e^{\nu}}{\beta^{\nu'}} \sum_{k=0}^{\infty} \sum_{t \in T_{c+\lambda}} h(t)^{\frac{\beta'}{2} - r'}$$

$$\leq \frac{C e^{\nu}}{\beta^{\nu'}} \sum_{k=0}^{\infty} (c^{-k} t)^{\frac{\beta'}{2} - r'} M_{c+\lambda}^{loc} \leq \frac{C e^{\nu}}{\beta^{\nu'}} \quad \forall \beta' > 0,$$

if we choose $0 < \tau' < \frac{1}{2}$, where we use (5.5) and $C = C(n, \lambda, \Lambda, \rho, \alpha, \kappa)$.

For $x, \tilde{x} \in U \cap B_c$ such that $v(x) - l_{v, \tilde{x}}(x) \leq \frac{\beta'}{2} |x - \tilde{x}|^2$, we have by the definition of $v$ and (5.8)

$$\frac{1}{(g^1)^U}[\phi(x) - l_{\phi, \tilde{x}}(x)] \leq \frac{\beta'}{2} |x - \tilde{x}|^2 + w(x) - l_{w, \tilde{x}}(x) \leq \frac{\beta'}{2} |x - \tilde{x}|^2.$$
It follows that
\[
\left| (U \cap B_{2\varepsilon}) \setminus \{ \bar{x} \in U \cap B_{\varepsilon} : \phi(x) - l_{\phi,\lambda}(x) \leq \Lambda \frac{\varepsilon^2 + C_0^{-1}}{2} |x - \bar{x}|^2, \quad \forall x \in U \cap B_{\varepsilon} \right| \\
\leq \frac{C \varepsilon^2}{\beta \varepsilon^2} \leq \frac{C(n, \lambda, \Lambda, \rho, \alpha, \kappa) \varepsilon^2}{\beta \varepsilon^2} |U \cap B_{2\varepsilon}|, \quad \forall \beta > 0.
\]
(5.9)

For any fixed $\beta > 0$, we denote $M_1 = M_1(\beta^2) := \Lambda \frac{\varepsilon^2 + C_0^{-1}}{2}$. Let $0 < c \leq c^2$ be a constant to be chosen later. Fix $\bar{x} \in U \cap B_{\varepsilon}$ such that $\phi(x) - l_{\phi,\lambda}(x) \leq M_1 |x - \bar{x}|^2$, $\forall x \in U \cap B_{\varepsilon}$. Then we have
\[
(5.10)
(U \cap B_{\varepsilon}) \cap B \sqrt{\frac{\pi}{h^2}}(\bar{x}) \subset S_{\phi}(\bar{x}, h), \quad \forall h > 0.
\]

Fix any $h \leq c^4$, then we have $B \sqrt{\frac{\pi}{h^2}}(\bar{x}) \subset B_{2e^2} \subset B_{c/2}$. Let $T x = Ax + b$ be an affine transformation such that
\[
B_{2\varepsilon} \subset T(S_{\phi}(\bar{x}, h)) \subset B_1.
\]
By the volume estimate for boundary sections we have
\[
|\det T| \leq C(n, \lambda, \Lambda, \rho) h^{-\frac{\varepsilon}{2}}.
\]
(5.12)

Let $x_0 \in \partial U \cap B_{\varepsilon}$ such that $|x_0 - x_0| = \text{dist}(\bar{x}, \partial U)$. Denote $\tilde{y} := \bar{x} + \frac{1}{2} \sqrt{\frac{\pi}{h^2}} v_{x_0}$, then $B \sqrt{\frac{\pi}{h^2}}(\tilde{y}) \subset B \sqrt{\frac{\pi}{h^2}}(\bar{x})$.
We now prove that
\[
\partial B \sqrt{\frac{\pi}{h^2}}(\tilde{y}) \cap ((\partial U \cap B_{\varepsilon}) \setminus \{x_0\}) = \emptyset.
\]
(5.13)

In fact, for any $x \in \partial B \sqrt{\frac{\pi}{h^2}}(\tilde{y})$ we have
\[
(x - x_0) \cdot v_{x_0} \geq \sqrt{\frac{h}{M_1}} |(x - x_0) \cdot v_{x_0} - |x - x_0|| \geq |(x - x_0) - [(x - x_0) \cdot v_{x_0}] v_{x_0}| |v_{x_0}|^2.
\]
(5.14)
This together with (vi) in the definition of $P$ implies (5.13) and it follows that
\[
B \sqrt{\frac{\pi}{h^2}}(\tilde{y}) \subset (U \cap B_{\varepsilon}).
\]
This together with (5.10) yields
\[
B \sqrt{\frac{\pi}{h^2}}(\tilde{y}) \subset S_{\phi}(\bar{x}, h).
\]
(5.15)

By (5.15) and the second inclusion in (5.11) we have $|A| \leq 4 \sqrt{M_1}$. This together with (5.12) implies $|A^{-1}| \leq C \sqrt{M_1^{-1}} h$, where $C = C(n, \lambda, \Lambda, \rho)$. Using the second conclusion in (5.11) we obtain diam$(S_{\phi}(\bar{x}, h)) \leq C \sqrt{M_1^{-1}} h$. Therefore,
\[
(5.16)
S_{\phi}(\bar{x}, h) \subset B_{C \sqrt{M_1^{-1}} h}(\bar{x}).
\]
For any $x \in U \cap B_{\varepsilon}$, by (v) in the definition of $P$, there is a constant $C_0 = C_0(n, \lambda, \Lambda, \rho) > 0$ such that
\[
h := \phi(x) - l_{\phi,\lambda}(x) \leq 2C_0 |x - \bar{x}| \leq 4C_0 \bar{c} \leq c^4,
\]
where we choose $\bar{c} := \frac{c^2}{4C_0}$. Thus (5.16) gives
\[
\phi(x) - l_{\phi,\lambda} \geq \frac{1}{C^2 M_1^{-1}} |x - \bar{x}|^2.
\]
(5.17)
Since $0 < \bar{c} \leq c^2$, the estimate (5.9) also holds if we replace $U \cap B_{\varepsilon}$ by $U \cap B_{\varepsilon}$, hence by (5.17)
\[
\left| (U \cap B_{\varepsilon}) \setminus \{ \bar{x} \in U \cap B_{\varepsilon} : \phi(x) - l_{\phi,\lambda} \geq \frac{1}{C^2 M_1^{-1}} |x - \bar{x}|^2, \quad \forall x \in U \cap B_{\varepsilon} \right| \\
\leq \frac{C(n, \lambda, \Lambda, \rho, \alpha, \kappa) \varepsilon^2}{\beta \varepsilon^2} |U \cap B_{\varepsilon}|, \quad \forall \beta > 0,
\]
(5.18)
and for any \( S_\phi(0, r) \) with \( r = r(n, \lambda, \Lambda, \rho) \leq \tilde{c}_3 \),
\[
\begin{aligned}
S_\phi(0, r)\setminus \{ x \in U \cap B_{\tilde{c}} : \phi(x) - l_\phi(x) \geq \frac{1}{C^2 M_1^{n-1}}|x - \tilde{x}|^2, \; \forall x \in U \cap B_{\tilde{c}} \}
\leq \frac{c(n, \lambda, \Lambda, \rho, \alpha, \kappa)\sigma}{\beta^2} |S_\phi(0, r)|, \; \forall \beta > 0,
\end{aligned}
\]
where \( M_1 = M_1(\beta') := \lambda^{\frac{1}{\beta' + 2}} \) and \( C = C(n, \lambda, \Lambda, \rho) \). Fix \( \beta' \) large such that \( \frac{c(n, \lambda, \Lambda, \rho, \alpha, \kappa)}{\beta'^2} \leq 1 \) and denote \( \sigma := \frac{1}{C^2 M_1^{n-1}} \), and we obtain the conclusion. \( \square \)

The rest part of this section is devoted to the estimate of the second term on the right-hand side of (4.18). Firstly, a straightforward modification of [7, Lemma 4.5] gives the lemma below.

**Lemma 5.2.** Assume \((\Omega, \phi, g, U) \in \mathcal{P}_{\lambda, \Lambda, \rho, \kappa} \). Let \( c \) be the constant in the definition of the class \( \mathcal{P} \) and \( \tilde{c} \leq c^2/2 \) be the constant given by Lemma 5.1. Suppose \( u \in C(\bar{U}) \cap W^{2,n}_{\text{loc}}(U) \) is a solution of \( L_\phi u = f \) in \( U \cap B_{\tilde{c}} \) with
\[
\begin{aligned}
||u||_{L^\infty(U \cap B_{2\tilde{c}})} + ||u||_{L^2(\partial U \cap B_{2\tilde{c}})} \leq 1.
\end{aligned}
\]
Let \( w \) be defined as in (vii) of the definition of \( \mathcal{P} \) with \( A := [(g^1)_{U \cap B_{\tilde{c}}}]^{-1} \) and \( W \) be the cofactor matrix of \( D^2w \). Assume \( h \) is a solution of
\[
\begin{aligned}
\{ L_\phi h = 0 & \quad \text{in } U \cap B_{\tilde{c}}, \\
& \quad h = u \quad \text{on } \partial(U \cap B_{\tilde{c}}).
\end{aligned}
\]
Then there exist \( C > 0 \) and \( 0 < \gamma < 1 \) depending only on \( n, \lambda, \Lambda, \rho, \alpha \) such that
\[
(5.18) \quad ||h||_{C^{1, \gamma}(U \cap B_{\tilde{c}})} \leq C,
\]
and if \( ||\Phi - (g^1)^{r-1}_{U \cap B_{\tilde{c}}} W||_{L^2(U \cap B_{\tilde{c}})} \leq \left( \frac{\tilde{c}}{2} \right)^4 \) then
\[
\begin{aligned}
||u - h||_{L^\infty(U \cap B_{2\tilde{c}})} + \left| \int f - \text{tr}\left((\Phi - (g^1)^{r-1}_{U \cap B_{\tilde{c}}} W)D^2h\right)\right|_{L^p(U \cap B_{2\tilde{c}})}
\leq C \left( 1 + ||u||_{C^{1, \gamma}(\partial(U \cap B_{2\tilde{c}}))} \right) ||\Phi - (g^1)^{r-1}_{U \cap B_{\tilde{c}}} W||_{L^2(U \cap B_{2\tilde{c}})}^2 + ||f||_{L^p(U \cap B_{2\tilde{c}})}^2 \right) \right).
\end{aligned}
\]

With the above two lemmas, we next prove the basic power decay estimate for the distribution function of \( D^2u \), arguing as in the proof of [7, Lemma 5.1].

**Lemma 5.3.** Assume \((\Omega, \phi, g, U) \in \mathcal{P}_{\lambda, \Lambda, \rho, \kappa} \) with \( 0 < \kappa < 1, \phi \in C^1(\Omega) \) and \( \text{mosc}_U g \leq \epsilon \). Let \( c \) be the constant in the definition of the class \( \mathcal{P} \) and \( \tilde{c} \leq 3/2 \) be the constant given by Lemma 5.1. Assume \( u \in C(\Omega) \cap C^1(U) \cap W^{2,n}_{\text{loc}}(U) \) is a solution of \( L_\phi u = f \) in \( U \) such that
\[
(5.19) \quad ||u||_{L^\infty(U)} + ||u||_{C^{1, \gamma}(\partial(U \cap B_{2\tilde{c}}))} \leq 1
\]
and
\[
(5.19) \quad |u(x)| \leq C^*[1 + d(x, x_0)^2] \quad \text{in } \Omega \setminus U \text{ for some } x_0 \in U \cap B_{\tilde{c}/4}.
\]
Then there exist \( C = C(n, \lambda, \Lambda, \rho, \alpha, \kappa) > 0 \), \( \tau = \tau(n, \lambda, \Lambda, \rho) > 0 \) and \( N_0 = N_0(n, \lambda, \Lambda, \rho, \alpha, \kappa, C^*) > 0 \) such that
\[
|S_\phi(0, (\tilde{c}/2)^6) \cap G_N(u, \Omega)| \geq 1 - C \left( \frac{\delta_0}{N} \right)^\tau - e^{-\tau} |S_\phi(0, (\tilde{c}/2)^6)|, \; \forall N \geq N_0
\]
provided that \( ||\Phi - (g^1)^{r-1}_{U \cap B_{\tilde{c}}} W||_{L^2(U \cap B_{\tilde{c}})} \leq \left( \frac{\tilde{c}}{4} \right)^4 \). Here \( \tau' = \tau(n) \) is from Lemma 5.1, \( w \) is defined in (vii) of the definition of \( \mathcal{P} \) with \( A := [(g^1)_{U \cap B_{\tilde{c}}}]^{-1} \) and \( W \) is the cofactor matrix of \( D^2w \), \( \gamma = \gamma(n, \lambda, \Lambda, \rho, \alpha) \) is from Lemma 5.2, and
\[
\begin{aligned}
\delta_0 := \left( 1 + ||u||_{C^{1, \gamma}(\partial(U \cap B_{2\tilde{c}}))} \right) ||\Phi - (g^1)^{r-1}_{U \cap B_{\tilde{c}}} W||_{L^2(U \cap B_{\tilde{c}})}^2 + \left( \int_U |f|^\nu dx \right)^\frac{2}{\nu} \right).
\end{aligned}
\]
Using Lemma 5.3, the Localization Theorem, the interior power decay estimate in Lemmas 3.4, 3.5 and the stability of cofactor matrices at the boundary in Proposition 4.3, we can prove power decay estimates when $\det D^2\phi$ satisfies a VMO-type condition, following similar lines as in [7, Lemma 5.2, Lemma 5.3].

**Lemma 5.4.** Given $0 < \epsilon_0 < 1$ and $0 < \alpha < 1$. Assume $\Omega$ and $\phi$ satisfy (4.15)-(4.17). Assume in addition that $\partial \Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial \Omega)$. Let $u \in C^1(\Omega) \cap W^{2,p}_0(\Omega)$ be a solution of $\mathcal{L}_\phi u = f$ in $\Omega$ with $u = 0$ on $\partial \Omega$ and $||u||_{L^\infty(\Omega)} \leq 1$. Then there exist $\epsilon = \epsilon(\epsilon_0, n, \lambda, \Lambda, \rho, \alpha) > 0$, $c_1 = c_1(n, \lambda, \Lambda, \rho, \alpha, ||\partial \Omega||_{C^{2,\alpha}}, ||\phi||_{C^{2,\alpha}(\partial \Omega)}) > 0$ such that if

\[
\sup_{x \in \Omega, t \geq 0} \text{mosc}_{S_\phi(x,t)} g \leq \epsilon,
\]

then for any $x \in \Omega$ and $t \leq c_1$, we have

\[
|G_{\tau}(u, \Omega) \cap S_\phi(x,t)| \geq \left\{ 1 - \epsilon_0 - C \left( \frac{\sqrt{N}}{N} \right)^\tau \right\} |S_\phi(x,t)|, \quad \forall N \geq N_1,
\]

where $\tau = \tau(n, \lambda, \Lambda, \rho)$, $C, N_1 > 0$ depend only on $n, \lambda, \Lambda, \rho, \alpha$.

**Lemma 5.5.** Given $0 < \epsilon_0 < 1$ and $0 < \alpha < 1$. Assume $\Omega$ and $\phi$ satisfy (4.15)-(4.17). Assume in addition that $\Omega$ is uniformly convex, $\partial \Omega \in C^{2,\alpha}$ and $\phi \in C^{2,\alpha}(\partial \Omega)$. Let $u \in C^1(\Omega) \cap W^{2,p}_0(\Omega)$ be a solution of $\mathcal{L}_\phi u = f$ in $\Omega$ with $u = 0$ on $\partial \Omega$. Then there exist $\epsilon = \epsilon(\epsilon_0, n, \lambda, \Lambda, \rho, \alpha) > 0$, $c_2 = c_2(n, \lambda, \Lambda, \rho, \alpha, ||\partial \Omega||_{C^{2,\alpha}}, ||\phi||_{C^{2,\alpha}(\partial \Omega)}) > 0$ such that if

\[
\sup_{x \in \Omega, t \geq 0} \text{mosc}_{S_\phi(x,t)} g \leq \epsilon,
\]

then for any $x \in \Omega$, $t \leq c_2$ and $S_\phi(x,t) \cap G_{\gamma}(u, \Omega) \neq \emptyset$, we have

\[
|G_{\gamma}(u, \Omega) \cap S_\phi(x,t)| \geq \left\{ 1 - \epsilon_0 - C(N\gamma)^\tau \left( \int_{S_\phi(x,\Theta \Omega)} |f|^p dx \right)^\frac{1}{p} \right\} |S_\phi(x,t)|, \quad \forall N \geq N_2, \hat{x} \in S_\phi(x,t),
\]

where $\tau, \Theta$ depend only on $n, \lambda, \Lambda, \rho$, $C, N_2 > 0$ depend only on $n, \lambda, \Lambda, \rho, \alpha, ||\partial \Omega||_{C^{2,\alpha}}, ||\phi||_{C^{2,\alpha}(\partial \Omega)}$ and the uniform convexity of $\Omega$.

**6. Boundary estimate—Proof of the main theorems**

**Proof of Theorem 2** : We assume that $||u||_{L^\infty(\Omega)} \leq 1$ and $||f||_{L^p(\Omega)} \leq \epsilon$, and we only need to prove that

\[
|D^2 u|_{L^p(\Omega)} \leq C(n, \lambda, \Lambda, \Omega, p, q).
\]

Let $N_1, N_2$ be the large constants depending only on $n, \lambda, \Lambda, \Omega$ given by Lemmas 5.4 and 5.5 respectively. Denote $N_* := \max(N_1, N_2)$. Let $c_1, c_2$ be the small constants depending only on $n, \lambda, \Lambda, \Omega$ given by Lemmas 5.4 and 5.5 respectively and denote $\check{c} := \min(c_1, c_2)$. Fix $M \geq N_*$ such that $1/M < \check{c}$. Choose $0 < \epsilon_0 < 1$ small such that

\[
M^{\frac{q}{2}} \sqrt{2\epsilon_0} = \frac{1}{2}.
\]

Let $\epsilon = \epsilon(\epsilon_0, n, \lambda, \Lambda, \Omega) = \epsilon(n, \lambda, \Lambda, \Omega, q)$ be the smallest of the constants in Lemmas 5.4 and 5.5. As in the proof of [7, Theorem 1.1], to obtain (6.1), we only need to estimate the term

\[
\sum_{k=1}^{\infty} M^k |\Omega \setminus A_{\Omega}^{loc} \left( \frac{1}{cM^{\frac{1}{p-1}+\frac{n}{p}}} \right)^\frac{1}{\check{c}}|.
\]

For this, we apply Theorem 4.2 with $\beta \sim M^k, k \sim \frac{2}{p} > 1$ and find that

\[
\sum_{k=1}^{\infty} M^k |\Omega \setminus A_{\Omega}^{loc} \left( \frac{1}{cM^{\frac{1}{p-1}+\frac{n}{p}}} \right)^\frac{1}{\check{c}}| \leq C(\epsilon, n, \lambda, \Lambda, \Omega) \sum_{k=1}^{\infty} M^{\frac{1}{\check{c}}(\frac{2}{p} - \frac{n}{p} - \frac{1}{p})} < \infty
\]

if we choose $\epsilon = \epsilon(n, \lambda, \Lambda, \Omega, p, q)$ is small. Here $C, \check{c}_1 > 0$ are constants depending only on $n, \lambda, \Lambda, \Omega$.  


Proof of Theorem 3: Let $0 < \epsilon_0 = \epsilon_0(n, \lambda, \Lambda, p, q, \Omega) < 1$ and $\epsilon = (\epsilon_0, n, \lambda, \Lambda, p, q, \Omega) > 0$ be constants to be chosen later. We can assume that $\varphi = 0, \|u\|_{L^p(\Omega)} \leq 1$ and $\|f\|_{L^q(\Omega)} \leq \epsilon$, and we only need to prove that

$$
\|D^2 u\|_{L^p(\Omega)} \leq C(n, \lambda, \Lambda, p, q, \Omega, \epsilon) .
$$

Denote

$$
\eta_\epsilon(r, \Omega) = \sup_{x \in \Omega, \text{diam}(S_\epsilon(x)) \leq r} \text{mosc}_{S_\epsilon(x,t)} g .
$$

Since $g \in \text{VMO}(\Omega, \phi)$, then there exists $0 < m < 1$ depending only on $\epsilon$ and $g$ such that $\eta_\epsilon(r, \Omega) < \epsilon$ for any $0 < r \leq m$. It follows that

$$
\sup_{x \in \Omega, \text{diam}(S_\epsilon(x,t)) \leq m} \text{mosc}_{S_\epsilon(x,t)} g \leq \epsilon .
$$

Consider a point $y \in \partial \Omega$ and assume for simplicity that $y = 0$. Assume that $\Omega$ satisfies (4.1), and $\phi(0) = 0$, $\nabla \phi(0) = 0$. By the Localization Theorem 4.1, we have

$$
\Omega \cap B_{\epsilon/3}^+ \subset \Omega(0, \epsilon) \subset B_{\epsilon/3}^+ .
$$

if $s \leq c_1 = c_1(n, \lambda, \Lambda, \Omega)$. Hence if we choose $s$ such that $s \leq \min \{c_1, (\frac{\epsilon}{2})^3\}$, then (6.3) implies that

$$
\sup_{S_\epsilon(x,t) \leq \Omega(0, \epsilon)} \text{mosc}_{S_\epsilon(x,t)} g \leq \epsilon .
$$

Denote the rescaled functions and domains as in (4.7) and (4.8),

$$
\phi_s(x) = \frac{\phi(T_s^{-1} x)}{s} \quad \text{and} \quad \Omega_s := T_s(\Omega),
$$

$$
u_s(x) = u(T_s^{-1} x), \quad x \in \Omega_s,
$$

where $T_s := s^{-\tau}A_s$ and $A_s$ is the sliding given by Theorem 4.1. Then

$$
det D^2 \phi_s = g_s = (g_s^1)^n, \quad 0 < \lambda \leq g_s(x) := g(T_s^{-1} x) \leq \Lambda \quad \text{in} \quad \Omega_s,
$$

$$
\Omega_s \cap B_k \subset \Omega_s := T_s(\Omega_s) = \Omega(s, 0) \subset B_k^+, \text{ and by Proposition 4.2, for any } 0 < \alpha < 1, \text{ we have}
$$

$$
(\Omega_s, \phi_s, g_s, U_s) \in \mathcal{P}_{\lambda, \Lambda, \rho, C, 1/2, \alpha} \subset \mathcal{P}_{\lambda, \Lambda, \rho, 1/2, \alpha}
$$

if $s \leq c_0$, where $c_0$ is a small constant depending only on $n, \lambda, \Lambda, \rho, \alpha, ||\partial \Omega||_{C^{3,\alpha}}$ and $C = C(n, \lambda, \Lambda, \rho, \alpha)$. Moreover, by (6.5), we have

$$
\sup_{S_\epsilon(x,t) \leq \Omega(0, \epsilon)} \text{mosc}_{S_\epsilon(x,t)} g \leq \epsilon .
$$

Let $c = c(n, \lambda, \Lambda, \rho)$ be the small constant in (vi) in the definition of $(\Omega_s, \phi_s, g_s, U_s) \in \mathcal{P}_{\lambda, \Lambda, \rho, 1/2, \alpha}$. We claim that

$$
\|D^2 u_s\|_{L^p(S_\epsilon(0, c^2 \Omega))} \leq C,
$$

where $C > 0$ depends only on $n, \lambda, \Lambda, p, q, \Omega$. Then back to $u$ and using a covering argument and the interior estimate Theorem 1, the conclusion of Theorem 3 follows.

To prove (6.7) we establish the following two estimates which are the local versions of Lemmas 5.4 and 5.5 at the boundary respectively.

For any $x \in U_s \cap B_{\epsilon/2}$, $t \leq c_1$ and $N \geq N_1$, we have

$$
|S_{\phi_s}(x, t) \cap G_{\Omega_s}(u_s, U_s, \phi_s)| \geq \left\{ 1 - \epsilon_0 - C \left( \frac{\sqrt{N}}{N} \right)^\tau \|f_s\|_{L^q(\Omega_s)} \right\} |S_{\phi_s}(x, t)| .
$$

For any $x \in \bar{U_s} \cap B_{\epsilon/2}$, $t \leq c_2$ and $N \geq N_2$, if $S_{\phi_s}(x, t) \cap G_{\Omega_s}(u_s, U_s, \phi_s) \neq \emptyset$, then, for any $\bar{x} \in S_{\phi_s}(x, t)$, we have

$$
|S_{\phi_s}(x, t) \cap G_N(u_s, U_s, \phi_s)| \geq \left\{ 1 - \epsilon_0 - C \left( \frac{1}{N^\gamma} \right)^\tau \left( \int_{S_{\phi_s}(x, t)} |f_s|^\gamma dx \right)^\frac{\tau}{\gamma} \right\} |S_{\phi_s}(x, t)| .
$$

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Here $c_1, c_2, N_1, N_2, \Theta$ are constants depending only on $n, \lambda, \Lambda, \Omega$.

The proof of these two results are similar to that of Lemmas 5.4, 5.5, but we use localized versions of geometric properties of sections at the boundary (Dichotomy, Engulfing property, Volume estimates) instead of the global ones. We omit the proof.

Using (6.8) and (6.9), we are ready to prove (6.7). The proof of this is similar to that of [7, Theorem 1.1], but we use the localized versions of covering theorem Lemma and strong $p$-$p$ estimates for maximal functions instead of the global ones. We omit the proof.

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