Modified Crout’s method for an LU decomposition of an interval matrix

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Abstract. In this paper, we propose an algorithm for computing LU decomposition of an interval matrix using modified Crout’s method based on generalized interval arithmetic on interval numbers. Numerical examples are also provided to show the efficiency of the proposed algorithm.

Keywords: Generalized interval arithmetic, interval numbers, modified Crout’s method

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1. Introduction
Matrix decomposition is the process of transformation of the given matrix into product of a lower triangular matrix and an upper triangular matrix. Matrix decomposition is a fundamental concept in linear algebra and applied statistics which has both scientific and engineering significance. Computational convenience and analytic simplicity are the two major aspects of matrix decomposition. In the real world it is not easy for most of the matrix computation to be calculated in an optimal explicit way such as matrix inversion, matrix determinant, solving linear systems and least square fitting. Thus to convert a difficult matrix computation problem into several easier problems such as solving triangular or diagonal systems will greatly simplify the calculations. Also decomposition methods provide an important tool for analyzing the numerical stability of a system. In literature there are several authors such as Alefeld and Herzberger[1], Hansen and Smith[7], Neumaier[12], Rohn[17] and Ganesan and Veeramani[5] etc have discussed interval matrices. Alexandre Goldsztejn and Gilles Chabert[2] have investigated generalized interval LU decomposition and its applications. Zhili Zhao, Wei Li, Chongyang Deng and Huping Wang[19] proposed new interval Cholesky decomposition method based on generalized intervals. In this paper we propose an algorithm for modified Crout’s method for an LU decomposition of an interval matrix.

The rest of this paper is organized as follows: In section 2, we recall the basic concepts and the ranking of generalized interval numbers and their arithmetic operations. In section 3, we recall the basic concepts of interval matrices and arithmetic operations on interval matrices. In section 4, we propose an algorithm for
modified Crout’s method for an LU decomposition of an interval matrices. In section 5, Numerical examples are also provided to show the efficiency of the proposed algorithm.

2. Preliminaries
Let $\mathbb{IR} = \{ \bar{a} = [a_1, a_2] : a_1 \leq a_2 \text{ and } a_1, a_2 \in \mathbb{R} \}$ be the set of all proper intervals and $\overline{\mathbb{IR}} = \{ \bar{a} = [a_1, a_2] : a_1 > a_2 \text{ and } a_1, a_2 \in \mathbb{R} \}$ be the set of all improper intervals on the real line $\mathbb{R}$. If $a_1 = a_2 = a$, then $\bar{a} = [a, a] = a$ is a real number (or a degenerate interval). We shall use the terms “interval” and “interval number” interchangeably. The mid-point and width (or half-width) of an interval number $\bar{a} = [a_1, a_2]$ are defined as $m(\bar{a}) = \frac{a_1 + a_2}{2}$ and $w(\bar{a}) = \frac{a_2 - a_1}{2}$. We denote the set of generalized intervals (proper and improper) by $D = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{ [a_1, a_2] : a_1, a_2 \in \mathbb{R} \}$. The set of generalized intervals $D$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.

The “dual” is an important monadic operator proposed by Kaucher [8] that reverses the end-points of the intervals expresses an element to element symmetry between proper and improper intervals in $D$. For $\bar{a} = [a_1, a_2] \in D$, its dual is defined by dual($\bar{a}$) = dual([a_1, a_2]) = [a_2, a_1]. The opposite of an interval $\bar{a} = [a_1, a_2]$ is opp([a_1, a_2]) = [-a_1, -a_2] which is the additive inverse of [a_1, a_2] and \[ \left[ \frac{1}{a_1}, \frac{1}{a_2} \right] \] is the multiplicative inverse of [a_1, a_2], provided $0 \notin [a_1, a_2]$. That is,

\[ \bar{a} + (-\text{dual } \bar{a}) = \bar{a} - \text{dual}(\bar{a}) = [a_1, a_2] - \text{dual}([a_1, a_2]) = [a_1, a_2] - [a_2, a_1] = \frac{a_1 - a_1}{a_2 - a_2} = [0, 0] \]

and \[ \bar{a} \times \left( \frac{1}{\text{dual}(\bar{a})} \right) = [a_1, a_2] \times \left( \frac{1}{\text{dual}([a_1, a_2])} \right) = [a_1, a_2] \times \frac{1}{[a_2, a_1]} \]

\[ = [a_1, a_2] \times \left[ \frac{1}{a_1}, \frac{1}{a_2} \right] = \left[ \frac{a_1}{a_1}, \frac{a_2}{a_2} \right] = [1, 1]. \]

2.1. Comparing interval numbers
Let $\preceq$ be an extended order relation between the interval numbers $\bar{a} = [a_1, a_2], \bar{b} = [b_1, b_2] \in D$, then for $m(\bar{a}) \leq m(\bar{b})$, we construct a premise ($\bar{a} \preceq \bar{b}$) which implies that $\bar{a}$ is inferior to $\bar{b}$ (or $\bar{b}$ is superior to $\bar{a}$).

An acceptability function $A_{\preceq} : D \times D \to \{0, \infty\}$ is defined as:

\[ A_{\preceq}(\bar{a}, \bar{b}) = A(\bar{a} \preceq \bar{b}) = \frac{(m(\bar{b}) - m(\bar{a}))}{(w(\bar{b}) + w(\bar{a}))}, \text{ where } w(\bar{b}) + w(\bar{a}) \neq 0. \]
$A_2$ may be interpreted as the grade of acceptability of the first interval number $a$ to be inferior to the second interval number $b$.

### 2.2. A new interval arithmetic

Ganesan and Veeramani[5] proposed new interval arithmetic on IR. We extend these arithmetic operations to the set of generalized interval numbers D and incorporating the concept of dual. For $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2] \in D$ and for $* \in \{+, -, \cdot, \div\}$, we define

$$\tilde{a} * \tilde{b} = [m(\tilde{a}) \cdot m(\tilde{b}) - k, \ m(\tilde{a}) \cdot m(\tilde{b}) + k],$$

where $k = \min \left\{ \left( (m(\tilde{a}) \cdot m(\tilde{b})) - \alpha, \ (m(\tilde{a}) \cdot m(\tilde{b})) - \beta \right) \right\}$, $\alpha$ and $\beta$ are the end points of the interval under the existing interval arithmetic. In particular for any two $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2] \in D$,

(i) **Addition**

$$\tilde{a} + \tilde{b} = [a_1, a_2] + [b_1, b_2] = [\{m(\tilde{a}) + m(\tilde{b})\} - k, \ {m(\tilde{a}) + m(\tilde{b})} + k], \ \text{where} \ k = \frac{(b_2 + a_2) - (b_1 + a_1)}{2}.$$

(ii) **Subtraction**

$$\tilde{a} - \tilde{b} = [a_1, a_2] - [b_1, b_2] = [\{m(\tilde{a}) - m(\tilde{b})\} - k, \ {m(\tilde{a}) - m(\tilde{b})} + k], \ \text{where} \ k = \frac{(b_2 + a_2) - (b_1 + a_1)}{2}.$$

Also if $\tilde{a} = \tilde{b}$ i.e. $[a_1, a_2] = [b_1, b_2]$, then $\tilde{a} - \tilde{b} = \tilde{a} - \text{dual}(\tilde{a}) = [a_1, a_2] - \text{dual}([a_1, a_2]) = [a_1, a_2] - [a_2, a_1] = [a_1 - a_1, a_2 - a_2] = [0, 0].$

(iii) **Multiplication**

$$\tilde{a} . \tilde{b} = [a_1, a_2] . [b_1, b_2] = [\{m(\tilde{a}) m(\tilde{b})\} - k, \ \{m(\tilde{a}) m(\tilde{b})\} + k], \ \text{where} \ k = \min \left\{ \left( m(\tilde{a}) m(\tilde{b}) \right) - \alpha, \ m(\tilde{a}) m(\tilde{b})\right\}.$$

$$\alpha = \min (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2) \ \text{and} \ \beta = \max (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2).$$

(iv) **Division:**

$$\frac{1}{\tilde{a}} = \frac{1}{[a_1, a_2]} = \left[ \frac{1}{m(\tilde{a})} - k, \ \frac{1}{m(\tilde{a})} + k \right], \ \text{where} \ k = \min \left\{ \frac{1}{a_2} (\frac{a_2 - a_1}{a_1 + a_2}), \ \frac{1}{a_1} (\frac{a_2 - a_1}{a_1 + a_2}) \right\}.$$

and $0 \notin [a_1, a_2].$

Also if $\tilde{a} = \tilde{b}$ i.e. $[a_1, a_2] = [b_1, b_2]$, then
\[
\tilde{a} = \frac{\tilde{a}}{\text{dual}(\tilde{a})} = [a_1, a_2] \times \frac{1}{\text{dual}([a_1, a_2])} = [a_1, a_2] \times \frac{1}{[a_2, a_1]} = [a_1, a_2] \times \left[ \frac{1}{a_1}, \frac{1}{a_2} \right] = [\tilde{a}_1, \tilde{a}_2] = [1, 1]
\]

From (iii), it is clear that \( \lambda \tilde{a} = \begin{cases} [\lambda a_1, \lambda a_2], & \text{for } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1], & \text{for } \lambda < 0 \end{cases} \)

3. Main results

An interval matrix \( \tilde{A} \) is a matrix whose elements are interval numbers. An interval matrix \( \tilde{A} \) will be written as

\[
\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \ldots & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}, \text{ where each } \tilde{a}_{ij} = [a_{ij}, \bar{a}_{ij}] \text{ (or) } \tilde{A} = [\underline{A}, \overline{A}]
\]

for some \( \underline{A}, \overline{A} \) satisfying \( \underline{A} \leq \overline{A} \). We use \( \mathbb{D}^{m \times n} \) to denote the set of all \((m \times n)\) interval matrices. The midpoint of an interval matrix \( \tilde{A} \) is the matrix of midpoints of its interval elements defined as

\[
m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & \ldots & m(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ m(\tilde{a}_{m1}) & \ldots & m(\tilde{a}_{mn}) \end{pmatrix}
\]

The width of an interval matrix \( \tilde{A} \) is the matrix of widths of its interval elements defined as \( w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & \ldots & w(\tilde{a}_{1n}) \\ \vdots & \ddots & \vdots \\ w(\tilde{a}_{m1}) & \ldots & w(\tilde{a}_{mn}) \end{pmatrix} \) which is always nonnegative. We use \( \mathbb{O} \) to denote the null matrix \( \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \) and \( \overline{\mathbb{O}} \) to denote the null interval matrix \( \begin{pmatrix} \tilde{0} & \ldots & \tilde{0} \\ \vdots & \ddots & \vdots \\ \tilde{0} & \ldots & \tilde{0} \end{pmatrix} \). Also we use \( \mathbb{I} \) to denote the identity matrix \( \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & 1 & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \) and \( \mathbb{I} \) to denote the identity interval matrix

\[
\begin{pmatrix} \tilde{1} & \ldots & \tilde{0} \\ \vdots & \tilde{1} & \vdots \\ \tilde{0} & \ldots & \tilde{1} \end{pmatrix}
\]

If \( m(\tilde{A}) = m(\tilde{B}) \), then the interval matrices \( \tilde{A} \) and \( \tilde{B} \) are said to be equivalent and is denoted by \( \tilde{A} \approx \tilde{B} \). In particular if \( m(\tilde{A}) = m(\tilde{B}) \) and \( w(\tilde{A}) = w(\tilde{B}) \), then \( \tilde{A} = \tilde{B} \). If \( m(\tilde{A}) = \mathbb{O} \), then we say that \( \tilde{A} \) is a zero interval matrix and is denoted by \( \overline{\mathbb{O}} \). In particular if \( m(\tilde{A}) = \mathbb{O} \) and
then
then

\[
\begin{pmatrix}
[0,0] & \cdots & [0,0] \\
\cdots & \cdots & \cdots \\
[0,0] & \cdots & [0,0]
\end{pmatrix}
\]

Also if \( m(\tilde{A}) = O \) and \( w(\tilde{A}) \neq O \), then

\[
\tilde{A} = \begin{pmatrix}
\tilde{0} & \cdots & \tilde{0} \\
\cdots & \cdots & \cdots \\
\tilde{0} & \cdots & \tilde{0}
\end{pmatrix} \approx \tilde{O}.
\]

If \( \tilde{A} \neq \tilde{O} \) (i.e. \( \tilde{A} \) is not equivalent to \( \tilde{O} \)), then \( \tilde{A} \) is said to be a non-zero interval matrix. If \( m(\tilde{A}) = I \) then we say that \( \tilde{A} \) is an identity interval matrix and is denoted by \( \tilde{I} \). In particular if \( m(\tilde{A}) = I \) and \( w(\tilde{A}) = O \), then

\[
\tilde{A} = \begin{pmatrix}
[1,1] & \cdots & [0,0] \\
\cdots & [1,1] & \cdots \\
[0,0] & \cdots & [1,1]
\end{pmatrix}
\]

Also, if \( m(\tilde{A}) = I \) and \( w(\tilde{A}) \neq O \), then

\[
\begin{pmatrix}
\tilde{I} & \cdots & \tilde{0} \\
\cdots & \tilde{I} & \cdots \\
\tilde{0} & \cdots & \tilde{I}
\end{pmatrix} \approx \tilde{I}.
\]

3.1 Arithmetic Operations on Interval Matrices

We define arithmetic operations on interval matrices as follows: If \( \tilde{A}, \tilde{B} \in \mathbb{D}^{m \times n}, \tilde{x} \in \mathbb{D}^n \) and \( \tilde{a} \in \mathbb{D} \), then

(i) \( \tilde{a} \tilde{A} \approx (\tilde{a}_i \tilde{A})_{1 \leq i \leq m, 1 \leq j \leq n} \)

(ii) \( (\tilde{A} + \tilde{B}) \approx (\tilde{A}_{ij} + \tilde{B}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \)

(iii) \( (\tilde{A} - \tilde{B}) \approx \begin{cases} 
(\tilde{a}_{ij} - \tilde{b}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}, & \text{if } \tilde{A} \neq \tilde{B} \\
\tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{0} = 0, & \text{if } \tilde{A} \approx \tilde{B}
\end{cases} \)

(iv) \( \tilde{A} \tilde{B} \approx \left( \sum_{k=1}^{n} \tilde{a}_{ik} \tilde{b}_{kj} \right)_{1 \leq i \leq m, 1 \leq j \leq n} \)

(v) \( \tilde{A} \tilde{x} \approx \left( \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j \right)_{1 \leq i \leq m} \)

4. Algorithm for modified Crout’s method for LU decomposition of an interval matrix

The following steps to write the given interval matrix \( \tilde{A} \) into lower triangular interval matrix \( \tilde{L} \) and upper triangular interval matrix \( \tilde{U} \). For simplicity, consider a four-by-four square interval matrix for Crout’s method as
\[
\bar{A} = \begin{pmatrix}
\bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
\bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \bar{a}_{24} \\
\bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} & \bar{a}_{34} \\
\bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & \bar{a}_{44}
\end{pmatrix}
= \begin{pmatrix}
[\bar{a}_{11}, \bar{a}_{11}] & [\bar{a}_{12}, \bar{a}_{12}] & [\bar{a}_{13}, \bar{a}_{13}] & [\bar{a}_{14}, \bar{a}_{14}] \\
[\bar{a}_{21}, \bar{a}_{21}] & [\bar{a}_{22}, \bar{a}_{22}] & [\bar{a}_{23}, \bar{a}_{23}] & [\bar{a}_{24}, \bar{a}_{24}] \\
[\bar{a}_{31}, \bar{a}_{31}] & [\bar{a}_{32}, \bar{a}_{32}] & [\bar{a}_{33}, \bar{a}_{33}] & [\bar{a}_{34}, \bar{a}_{34}] \\
[\bar{a}_{41}, \bar{a}_{41}] & [\bar{a}_{42}, \bar{a}_{42}] & [\bar{a}_{43}, \bar{a}_{43}] & [\bar{a}_{44}, \bar{a}_{44}]
\end{pmatrix}
= \begin{pmatrix}
\bar{L}_{11} & \bar{L}_{12} & \bar{L}_{13} & \bar{L}_{14} \\
\bar{L}_{21} & \bar{L}_{22} & \bar{L}_{23} & \bar{L}_{24} \\
\bar{L}_{31} & \bar{L}_{32} & \bar{L}_{33} & \bar{L}_{34} \\
\bar{L}_{41} & \bar{L}_{42} & \bar{L}_{43} & \bar{L}_{44}
\end{pmatrix}
\begin{pmatrix}
\bar{U}_{12} & \bar{U}_{13} & \bar{U}_{14} \\
\bar{U}_{23} & \bar{U}_{24} \\
\bar{U}_{34} \\
\bar{U}_{44}
\end{pmatrix}
\]

**Step: 1**

Make the pivot element as unity, i.e. \( \bar{a}_{11} = \bar{I} \). For this purpose, if required, use elementary matrix row operations. The reduced interval matrix is

\[
\bar{A} = \begin{pmatrix}
\bar{a}_{11} = \bar{I} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
\bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \bar{a}_{24} \\
\bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} & \bar{a}_{34} \\
\bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & \bar{a}_{44}
\end{pmatrix}
\]

**Step: 2**

The first column of the matrix \( \bar{L} = \bar{L}_0 \) below \( \bar{a}_{11} = \bar{I} \) is the same as that of the matrix \( \bar{A} \), i.e., \( \bar{L}_{i1} = \bar{a}_{i1} \), for \( i = 1, 2, 3, 4 \).

\[
\bar{L} = \begin{pmatrix}
\bar{L}_{11} & \bar{L}_{12} & \bar{L}_{13} & \bar{L}_{14} \\
\bar{L}_{21} & \bar{L}_{22} & \bar{L}_{23} & \bar{L}_{24} \\
\bar{L}_{31} & \bar{L}_{32} & \bar{L}_{33} & \bar{L}_{34} \\
\bar{L}_{41} & \bar{L}_{42} & \bar{L}_{43} & \bar{L}_{44}
\end{pmatrix}
\]

**Step: 3**

The first row of the matrix \( \bar{U} = \bar{U}_0 \) after \( \bar{a}_{11} = \bar{I} \) is the same as that of the matrix \( \bar{A} \), i.e., \( \bar{U}_{1j} = \bar{a}_{1j} \), for \( j = 1, 2, 3, 4 \).

\[
\bar{U} = \begin{pmatrix}
\bar{I} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
\bar{0} & \bar{I} & \bar{U}_{23} & \bar{U}_{24} \\
\bar{0} & \bar{0} & \bar{I} & \bar{U}_{34} \\
\bar{0} & \bar{0} & \bar{0} & \bar{I}
\end{pmatrix}
\]
Step: 4
The diagonal elements of the matrix $\bar{U} = U_{ij}$ are kept as unity i.e., $\bar{U}_{ii} = \bar{I}$, for $i = 1, 2, 3, 4$.

$$\bar{U} = \begin{pmatrix}
\bar{I} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
0 & \bar{I} & \bar{U}_{23} & \bar{U}_{24} \\
0 & 0 & \bar{I} & \bar{U}_{34} \\
0 & 0 & 0 & \bar{I}
\end{pmatrix}$$

That is, we decompose an interval matrix $\bar{A}$ into the equivalent interval triangular form $\bar{A} \approx \bar{L}\bar{U}$ as

$$\bar{A} = \begin{pmatrix}
\bar{a}_{11} = \bar{I} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
\bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \bar{a}_{24} \\
\bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} & \bar{a}_{34} \\
\bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & \bar{a}_{44}
\end{pmatrix} \approx \begin{pmatrix}
\bar{I} & 0 & 0 & 0 \\
\bar{a}_{21} & \bar{L}_{22} & 0 & 0 \\
\bar{a}_{31} & \bar{L}_{32} & \bar{L}_{33} & 0 \\
\bar{a}_{41} & \bar{L}_{42} & \bar{L}_{43} & \bar{L}_{44}
\end{pmatrix} \begin{pmatrix}
\bar{I} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} \\
0 & \bar{I} & \bar{U}_{23} & \bar{U}_{24} \\
0 & 0 & \bar{I} & \bar{U}_{34} \\
0 & 0 & 0 & \bar{I}
\end{pmatrix}.$$ 

Step: 5
Compute the unknowns in matrices $\bar{L}$ and $\bar{U}$ using $\bar{A} \approx \bar{L}\bar{U}$.

5. Numerical examples

Example: 5.1
Consider an example discussed by Zhili Zhao, Wei Li, Chongyang Deng and Huping Wang [19]

$$\bar{A} = \begin{pmatrix}
[1.2] & [2.5] \\
[2.5] & [6.15]
\end{pmatrix}$$

By applying the proposed algorithm, divide the first row of the given interval matrix $\bar{A}$ by the element $a_{11}$, i.e.

$$\bar{A} = \begin{pmatrix}
[1.2] & [2.5] \\
[1.2] & [2.5] \\
[2.5] & [6.15]
\end{pmatrix} \Rightarrow \begin{pmatrix}
[1.1] & [0.66735, 3.9996] \\
[2.5] & [6.15]
\end{pmatrix}.$$ 

The new interval matrix $\bar{A}$ can be expressed as lower and upper triangular interval matrices $\bar{L}$ and $\bar{U}$,

$$\begin{pmatrix}
\bar{a}_{11} = \bar{I} & \bar{a}_{12} \\
\bar{a}_{21} & \bar{a}_{22}
\end{pmatrix} \approx \begin{pmatrix}
\bar{a}_{11} = \bar{I} & 0 \\
\bar{a}_{21} & \bar{L}_{22}
\end{pmatrix} \begin{pmatrix}
\bar{a}_{12} \\
\bar{a}_{22}
\end{pmatrix}.$$
That is \[
\begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[2,5] & [6,15]
\end{pmatrix}
\approx
\begin{pmatrix}
[1,1] & [0,0] \\
[2,5] & L_{22}
\end{pmatrix}
\begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[0,0] & [1,1]
\end{pmatrix}
\]
\[
\Rightarrow
\begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[2,5] & [6,15]
\end{pmatrix}
\approx
\begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[2,5] & [2,5][0.66735, 3.9996] + L_{22}
\end{pmatrix}
\]
Equating we get, \[2,5][0.66735, 3.9996] + L_{22} = [6,15]
\[
[1.3347, 14.9999] + L_{22} = [6,15]
\]
\[
\tilde{L}_{22} = [6,15] - [1.3347, 14.9999]
\]
\[
\tilde{L}_{22} = [-8.9999, 13.6653]
\]
Therefore,
\[
\tilde{A} = \begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[2,5] & [6,15]
\end{pmatrix}
\approx
\begin{pmatrix}
[1,1] & [0,0] \\
[2,5] & [-8.9999, 13.6653]
\end{pmatrix}
\begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[0,0] & [1,1]
\end{pmatrix}
\]
We arrive \[
\tilde{L} = \begin{pmatrix}
[1,1] & [0,0] \\
[2,5] & [-8.9999, 13.6653]
\end{pmatrix}
\]
and \[
\tilde{U} = \begin{pmatrix}
[1,1] & [0.66735, 3.9996] \\
[0,0] & [1,1]
\end{pmatrix}
\]

Example: 5. 2
Consider another example discussed by Zhili Zhao, Wei Li, Chongyang Deng and Huping Wang [19]
\[
\tilde{A} = \begin{pmatrix}
[1,4] & [2,2] \\
[2,2] & [5,5]
\end{pmatrix}
\]
Applying the proposed method, we get
\[
\tilde{A} = \begin{pmatrix}
[1,1] & [-0.4, 2] \\
[2,2] & [5,5]
\end{pmatrix}
\approx
\begin{pmatrix}
[1,1] & [0,0] \\
[2,2] & [1.5, 8]
\end{pmatrix}
\begin{pmatrix}
[1,1] & [-0.4, 2] \\
[0,0] & [1,1]
\end{pmatrix}
\]
We have \[
\tilde{L} = \begin{pmatrix}
[1,1] & [0,0] \\
[2,2] & [1.5, 8]
\end{pmatrix}
\]
and \[
\tilde{U} = \begin{pmatrix}
[1,1] & [-0.4, 2] \\
[0,0] & [1,1]
\end{pmatrix}
\]

Example: 5. 3
Consider another example discussed by Alexandre Goldsztejn and Gilles Chabert [2]
\[
\tilde{A} = \begin{pmatrix}
[9,11] & [-1, -11] \\
[-11, 11] & [8, 12]
\end{pmatrix}
\begin{pmatrix}
[-1, -1] \\
[-2, 2]
\end{pmatrix}
\begin{pmatrix}
[-11, 11] & [-12, 12] \\
[-11, 11] & [7, 13]
\end{pmatrix}
\]
Divide the first row by \( \bar{a}_{11} \), the new interval matrix can be expressed as
The new interval matrix $\tilde{A}$ can be expressed as lower and upper triangular interval matrices $\tilde{L}$ and $\tilde{U}$.

$$\tilde{A} = \begin{bmatrix} [9,11] & [-1,1] & [-1,1] \\ [9,11] & [9,11] & [9,11] \\ [-11,11] & [8,12] & [-2,2] \\ [-11,11] & [-12,12] & [7,13] \end{bmatrix} \Rightarrow \tilde{A} = \begin{bmatrix} [1,1] & [-1,1] & [-1,1] \\ [1,1] & [9,11] & [9,11] \\ [-11,11] & [8,12] & [-2,2] \\ [-11,11] & [-12,12] & [7,13] \end{bmatrix}$$

By applying the proposed algorithm, the given interval matrix $A$ can be expressed as lower and upper triangular interval matrices $\tilde{L}$ and $\tilde{U}$ as

$$\tilde{L} = \begin{bmatrix} [1,1] & [0,0] & [0,0] \\ [-11,11] & [8,12] & [0,0] \\ [-11,11] & [-12,12] & [7,13] \end{bmatrix} \text{ and } \tilde{U} = \begin{bmatrix} [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [1,1] \end{bmatrix}.$$
\[
\begin{bmatrix}
[1,1] & [0,0] & [0,0] \\
[1,1] & [1.53,2.67] & [0,0] \\
[0,0] & [1,1] & [1.42,2.63]
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
[1,1] & [0.33,0.47] & [0,0] \\
[0,0] & [1,1] & [0.37,0.58] \\
[0,0] & [0,0] & [1,1]
\end{bmatrix}
\]

6. Conclusion

By using a new set of arithmetic operations on generalized interval numbers, we have proposed an algorithm for modified Crout’s method for decomposing an interval matrix \( \tilde{A} \) into a lower triangular interval matrix \( \tilde{L} \) and an upper triangular interval matrix \( \tilde{U} \). For simplicity, we have considered the algorithm for \((4 \times 4)\) interval matrices alone. But this algorithm is true for any \((n \times n)\) interval matrices. It is to be noted that for decomposing any \((n \times n)\) interval matrix \( \tilde{A} \) into a lower triangular interval matrix \( \tilde{L} \) and an upper triangular interval matrix \( \tilde{U} \), the existing decomposition methods requires evaluation of a total of \( n^2 \) number of unknown elements of the matrices \( \tilde{L} \) and \( \tilde{U} \). But the proposed modified Crout’s method requires evaluation of only \((n-1)^2\) number of such unknowns. This difference becomes significant for interval matrices of higher orders. Hence by applying the proposed method, a significant amount of computational time and efforts can be reduced.

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