Minimal generating set of planar moves for surfaces embedded in the four-space

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Abstract

We derive a minimal generating set of planar moves for diagrams of surfaces embedded in the four-space. These diagrams appear as the bonded classical unlink diagrams.

1 Introduction

There is a set of ten planar moves \( \{ \Omega_1, \ldots, \Omega_8, \Omega_4', \Omega_6' \} \) for surface-links introduced by K. Yoshikawa [Yos94], and proven by F.J. Swenton, C. Kearton and V. Kurlin [Swe01, KeaKur08] to be a generating set of moves between any diagrams of equivalent surface-links. However, it is still an open problem whether this set is minimal, in particular it is not known if any move from the set \( \{ \Omega_4, \Omega_4', \Omega_5 \} \) is independent from the other nine moves, see [JKL15] for more details.

In this paper we introduce planar moves for surface bonded link diagrams that generates moves between any surface bonded link diagrams of equivalent surface-links, and prove the minimality of this set.

**Theorem 1.1.** Two surface bonded link diagrams are related by a planar isotopy and a finite sequence of moves from the set \( \mathcal{M} = \{ M_1, \ldots, M_{12} \} \) depicted in Fig. 1 if and only if they represent equivalent surface-links. Moreover, any move from \( \mathcal{M} \) is independent from the other moves in \( \mathcal{M} \).

Toward the end of this paper we show two examples of known unknotted surface-link diagrams and transform them to the simple closed curves without using the M12 move. Minimal generating set of moves in three-space (in terms of links with bands) for surfaces embedded in the four-space was obtained by the author in [Jab20]. Minimal generating set of moves in three-space (in terms of broken surface diagrams) for surfaces embedded in the four-space was obtained by K. Kawamura.
in [Kaw15]. For transformations of some diagrams we use F. Swenton’s Kirby calculator [KLO19].

2 Preliminaries

For the case of surfaces in manifolds $S^4$ and $\mathbb{R}^4$, we will work in the standard smooth category (with maps of class $C^\infty$). An embedding (or its image when no confusion arises) of a closed (i.e. compact, without boundary) surface $F$ into the Euclidean $\mathbb{R}^4$ (or into the $S^4 = \mathbb{R}^4 \cup \{\infty\}$) is called a surface-link (or surface-knot if it is connected). A surface-knot homeomorphic to the $S^2$ is called a 2-knot. When it is homeomorphic to a torus or a projective plane, it is called a $T^2$-knot or a $P^2$-knot, respectively.

Two surface-links are equivalent (or have the same type denoted also by $\cong$) if there exists an orientation preserving homeomorphism of the four-space $\mathbb{R}^4$ to itself (or equivalently auto-homeomorphism of the four-sphere $S^4$), mapping one of those surfaces onto the other. We will use a word classical referring to the theory of embeddings of circles $S^1 \sqcup \ldots \sqcup S^1 \hookrightarrow \mathbb{R}^3$ modulo ambient isotopy in $\mathbb{R}^3$ with their planar or spherical regular projections.

To describe surface-links in $\mathbb{R}^4$, we will use hyperplane cross-sections $\mathbb{R}^3 \times \{t\} \subset \mathbb{R}^4$ for $t \in \mathbb{R}$, denoted by $\mathbb{R}^3_t$. This method (called motion picture method) introduced by Fox and Milnor was presented in [Fox62]. By a general position argument the intersection of $\mathbb{R}^3_t$ and a surface-link $F$ can (except in the finite cases) be either empty or a classical link. In the finite singular cases the intersection can be a single point or a four-valent embedded graph, where each vertex corresponds to a saddle point. For more introductory material on this topic refer to [CKS04].
2.1 Hyperbolic splitting, marked graph diagrams and Yoshikawa moves

**Theorem 2.1** ([Lom81], [KSS82], [Kam89]). Any surface-link $F$ admits a hyperbolic splitting, i.e. there exists a surface-link $F'$ satisfying the following: $F'$ is equivalent to $F$ and has only finitely many Morse’s critical points, all maximal points of $F'$ lie in $\mathbb{R}^3_1$, all minimal points of $F'$ lie in $\mathbb{R}^3_{-1}$, all saddle points of $F'$ lie in $\mathbb{R}^3_0$.

**Example 2.2.** An example of hyperbolic splitting and its zero cross-section is presented in Fig. 2. It is obtained by a rotation of the standard embedding of a trivial torus.

The zero cross-section $\mathbb{R}^3_0 \cap F'$ of the surface $F'$ in the hyperbolic splitting described above gives us then a 4-regular graph. We assign to each vertex a marker that informs us about one of the two possible types of saddle points (see Fig. 3) depending on the shape of the cross-section $\mathbb{R}^3_\epsilon \cap F'$ or $\mathbb{R}^3_{-\epsilon} \cap F'$ for a small real number $\epsilon > 0$. The resulting (rigid-vertex) graph is called a marked graph presenting $F$.

Making a projection in general position of a marked graph to $\mathbb{R}^2 \times \{0\} \times \{0\} \subset \mathbb{R}^4$ and assigning types of classical crossings between regular arcs, we obtain a marked graph diagram. For a marked graph diagram $D$, we denote by $L_+(D)$ and $L_-(D)$ the classical link diagrams obtained from $D$ by smoothing every vertex as presented in Fig. 3 for $+\epsilon$ and $-\epsilon$ case respectively. We call $L_+(D)$ and $L_-(D)$ the positive resolution and the negative resolution of $D$, respectively.

Any abstractly created marked graph diagram is an admissible diagram if and only if both its resolutions are trivial classical link diagrams.

In [Yos94] Yoshikawa introduced local moves on admissible marked graph diagrams that do not change corresponding surface-link types and conjectured that the converse is also true. It was resolved as follows.

**Theorem 2.3** ([Swe01], [KeaKur08]). Any two marked graph diagrams representing the same type of surface-link are related by a finite sequence of Yoshikawa local moves presented in Fig. 4 and a planar isotopy of the diagram.
Theorem 2.4 ([JKL13], [JKL15]). Any Yoshikawa move from the set
\{Ω₁, Ω₂, Ω₃, Ω₆, Ω₆′, Ω₇\} is independent from the other nine types.

Theorem 2.5 ([Jab20]). The Yoshikawa move Ω₈ is independent from the other nine types.

2.2 Links with bands

A band on a link L is an image of an embedding \( b : I \times I \to \mathbb{R}^3 \) intersecting the link L precisely in the subset \( b(\partial I \times I) \), where \( I \) the closed unit interval. A link with bands \( LB \) in \( \mathbb{R}^3 \) is a pair \( (L, B) \) consisting of a link \( L \) in \( \mathbb{R}^3 \) and a finite set \( B = \{b_1, \ldots, b_n\} \) of pairwise disjoint \( n \) bands spanning \( L \).

By an ambient isotopy of \( \mathbb{R}^3 \), we shorten the bands of a link with bands \( LB \) so that each band is contained in a small 2-disk. Replacing the neighborhood of each band with the neighborhood of a marked vertex as in Fig. 5, we obtain a marked graph, called a marked graph associated with \( LB \).

Conversely, when a marked graph \( G \) in \( \mathbb{R}^3 \) is given, by replacing each marked vertex with a band as in Fig. 5, we obtain a link with bands \( LB(G) \), called a link with bands associated with \( G \).

Let \( D \) be an admissible diagram with associated link with bands \( LB(D) = (L, B) \), \( L = L_-(D) \), \( B = \{b_1, \ldots, b_n\} \) and \( \Delta_1, \ldots, \Delta_n \subset \mathbb{R}^3 \) be mutually disjoint 2-disks with \( \partial(\cup_{j=1}^n \Delta_j) = L_+(D) \), and let \( \Delta_1', \ldots, \Delta_b' \subset \mathbb{R}^3 \) be mutually disjoint 2-disks with

![Figure 4: Yoshikawa moves.](image)

![Figure 5: A band corresponding to a marked vertex.](image)
\[ \partial(\bigcup_{k=1}^{n} \Delta_k) = L_-(D). \] We define \( S(D) \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4 \) a surface-link corresponding to a diagram \( D \) by the following cross-sections.

\[
(\mathbb{R}^3_t, S(D) \cap \mathbb{R}^3) = \begin{cases} 
(\mathbb{R}^3, \emptyset) & \text{for } t > 1, \\
(\mathbb{R}^3, L_+(D) \cup (\bigcup_{j=1}^{n} \Delta_j)) & \text{for } t = 1, \\
(\mathbb{R}^3, L_+(D)) & \text{for } 0 < t < 1, \\
(\mathbb{R}^3, L_-^1(D) \cup (\bigcup_{i=1}^{n} b_i)) & \text{for } t = 0, \\
(\mathbb{R}^3, L_-^1(D)) & \text{for } -1 < t < 0, \\
(\mathbb{R}^3, L_-(D) \cup (\bigcup_{k=1}^{n} \Delta'_k)) & \text{for } t = -1, \\
(\mathbb{R}^3, \emptyset) & \text{for } t < -1.
\]

It is known that the surface-link type of \( S(D) \) does not depend on choices of trivial disks (cf. [KSS82]). It is straightforward from the construction of \( S(D) \) that \( D \) is a marked graph diagram presenting \( S(D) \). For more material on this topic refer to [Kam17].

3 Surface bonded link diagrams

Let \( L \) be an oriented link in \( \mathbb{R}^3 \), let \( B = \{b_1, b_2, \ldots, b_n\} \) be a set of bonds (closed intervals) properly embedded into \( \mathbb{R}^3 \setminus L \) and let \( \chi : B \to \mathbb{Z} \) be any function, called here a coloring function. A bonded link diagram is a regular projection of \( L \) and the bonds to a plane with information of over/under-crossings and the coloring (i.e. the value of the coloring function). For more on bonded link diagrams see [Gab19] and [Kau89].

A surface bonded link diagram \( D = (L, B) \) is a bonded link diagram such that replacing each bond with \( k \)-times half-twisted band (see Fig. 6), both links \( L_+(D') \) and \( L_-(D') \) are unknotted and unlinked classical diagrams, where \( D' \) is a marked graph associated with \( LB \). So the coloring function here values a bond with the half-twisting of the corresponding band. We call this replacement a bandaging.

The reverse transformation we call an unbandaging (when there are negative half-twists in a band we count each of them as \(-1\) half-twisting).

3.1 Flat forms of surface bonded link diagrams

By analogy to the flat forms of links with bands \( LB \) defined in [Jab16], we can define a flat forms of surface bonded link diagrams as a diagrams where the components of the
link $L$ are embedded circles (without crossings between them) in the plane of the diagram.

The flat form of surface bonded link diagrams for a surface-link $F$ is especially useful for reading a presentation of the surface-link group, i.e. $\pi_1(\mathbb{R}^4\setminus\text{int}(N(F)))$ where $N(F)$ is a tubular neighborhood of $F$. It is because we neither have relations from crossing between links (i.e. link-link crossings), as we do not have them, nor we have relations from crossings between bonds (i.e. bond-bond crossings) as they do not contribute to new relations. Therefore, the interesting here are only tree-valent vertices and crossings between links and bonds (i.e. link-bond crossings). In Table 2 we derive flat forms of surface bonded link diagrams of every nontrivial surface-link from Yoshikawa table [Yos94].

3.2 Proof of Theorem 1.1

Proof. First, notice that bandaging all bonds in the moves from the set $\mathcal{M} = \{M_1, \ldots, M_{12}\}$ (with appropriate twisting) and allowing the diagrams to isotope in $\mathbb{R}^3$ we obtain a set of four moves: cup move, cap move, band-slide, band-pass on a link with bands (see [Jab20] for more details and proof of their minimality). Therefore, our set $\mathcal{M}$ contains only those moves that do not change the corresponding surface-link type.

Now we prove that the moves from the set $\mathcal{M}$ on surface bonded planar diagrams generates Yoshikawa moves on marked graph diagrams. It is sufficient to derive all moves from the set $\Omega = \{\Omega_1, \ldots, \Omega_6, \Omega'_4, \Omega'_6\}$ by the moves from $\mathcal{M}$ (and performing bandaging/unbandaging operations). But first we have to make sure that at any time we can make a surface bonded link diagram prepared to make a Yoshikawa move. We do this by moves $M_1, \ldots, M_8$ making all bonds do not intersect any other bond or link (except for their ends) and have coloring zero. Then contract the bond to a four-valent crossing with marker.

The moves $\Omega_1, \Omega_2, \Omega_3$ are equivalent to the moves $M_1, M_2, M_3$ (see also [Pol10]). The moves $\Omega_6, \Omega'_6, \Omega'_3$ are easily obtained by the moves $M_9, M_{10}, M_{11}$ respectively, simply by operations of exchanging markers with bands and bandaging/unbandaging operations. The remaining moves $\Omega_4, \Omega'_4, \Omega_5, \Omega_8$ are obtained as shown in Fig. 7.

Now we prove the minimality of elements of the set $\mathcal{M}$. To obtain this task it is sufficient to construct twelve semi-invariants $f^k$ such that they preserve their values after performing each move from the set $\mathcal{M}\setminus\{k\}$, where $k \in \mathcal{M}$; and construct twelve pairs of diagrams $D^k_1, D^k_2$ of equivalent surface-links such that $f^k(D^k_1) \neq f^k(D^k_2)$.

In the case where $k \in \{M_3, M_9, M_{10}, M_{11}, M_{12}\}$ the semi-invariant $f^k$ can be picked the same as in [JKL15] and [Jab20] after making bandaging on their zero-colored bonds. Recall here the shortest two functions to define: function $f^{M_9}$ counts the number of link components after positive resolution of each band. Function $f^{M_{10}}$ counts the number of link components after negative resolution of each band. Function $f^{M_{11}}$ counts the number of link components after adding one to all bond
We now define the remaining seven functions. Define $f_{M1}$ as a function counting the parity of the sum of classical crossings and colors of bonds. Define $f_{M2}$ as a function that counts the number of connected components of the planar graph (with valency 3 or 4) obtained from a surface bonded link diagram by omitting over-under information of all link-link, link-bond and bond-bond crossings. Define $f_{M4}$ as a function counting the parity of the number of crossings between bonds and classical links such that a bond is higher than a link. Define $f_{M5}$ as a function counting the parity of the number of crossings between bonds and classical links such that a bond is lower than a link. Define $f_{M8}$ as a function counting the sum of colors of every bond. Cases $M6$ and $M7$ are more
complicated.

For each bond \( b_i \), if we travel along this bond and meet two crossings (possibly non-consecutive) such that in both crossings the bond \( b_i \) goes over other bondstrands \( b_j, b_k \) (possibly \( j = k \)) define the two crossings to be a bond under-crossing pair for \( b_i \). When moreover, traveling along two under-crossing stands of \( b_j, b_k \) the mentioned crossings are between a bond under-crossing pair for both \( b_j, b_k \) define the two crossings to be a blocked bond under-crossing pair of \( b_i \).

Similarly we define a blocked bond over-crossing pair, switching words “over” with “under” in the above definition. Define \( f^{M_6} \) to be the number of bonds in the diagram that has blocked bond under-crossing pair. Define \( f^{M_7} \) to be the number of bonds in the diagram that have at least one blocked bond over-crossing pair.

It is straightforward to check that the above functions are well-defined and have the desired property of being semi-invariants in respect to appropriate moves.

To finish the proof we show in Table 1 twelve pairs of diagrams such that to transform the diagram \( D_{k_1} \) to the diagram \( D_{k_2} \) by a planar isotopy and moves from \( M \) one have to use the move of type \( k \).

\[ \square \]

From the moves in \( M \) we can easily derive useful moves with a general colors of bonds as in Fig.8.

![Figure 8: Derived moves on surface bonded links.](image-url)
3.3 Unknotted surface-links

An orientable surface-link in $\mathbb{R}^4$ is *unknotted* (or *trivial*) if it is equivalent to a surface embedded in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. A surface bonded link diagram for an unknotted standard 2-knot is shown in Fig. 9(a), an unknotted standard $T^2$-knot is in Fig. 9(d).

A $P^2$-knot in $\mathbb{R}^4$ is *unknotted* if it is equivalent to a surface whose surface bonded link diagram is an unknotted *standard projective plane*, which looks like in Fig. 9(b) that is a *positive* $P^2_+$ or looks like in Fig. 9(c) that is a *negative* $P^2_-$. A non-orientable surface-knot is *unknotted* if it is equivalent to some finite connected sum of unknotted $P^2$-knot (see for example Klein bottle $Kb^2 = P^2_+ \# P^2_-$ in Fig. 9(e)). A non-orientable surface-link is *unknotted* if it is equivalent to some split unions of finitely many unknotted non-orientable surface-knots and (possibly empty) set of orientable surface-links. Diagrams in Table 1 are all diagrams on unknotted surface-links.

E.C. Zeeman in [Zee65] generalized E. Artin spinning construction to the twist-spinning construction creating a smooth 2-knot in $\mathbb{R}^4$ from a given smooth classical knot $K$. A marked graph diagram for any $n$-twist spun knot $K$ is given in [Mon86].

In Fig. 10 we see transformations between a diagram of the 1-twist spun trefoil (defined as a closure of a braid $a_2c_1^2b_2c_1^{-3} \Delta^2$ see [Jab13]) and the trivial sphere diagram (we do not show moves $M_1, \ldots, M_8$ as they can be easily obtained in $\mathbb{R}^3$).

In Fig. 11 we see transformations between the minimal hard marked sphere diagram (defined as a diagram $9_{[2,38]}^{[1,2,0,1]}$ in [Jab19]) and the trivial sphere diagram. (we again do not show moves $M_1, \ldots, M_8$). It is natural then to consider the following.
Figure 11: Unknotting the minimal hard prime surface-unlink diagrams without using $M_{12}$ type move.

**Question 3.1.** Are every two diagrams of the standard 2-knot related by a planar isotopy and moves $M_1, \ldots, M_{11}$ (i.e. do not require $M_{12}$ move in a transformation)?

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Table 1: Diagrams for showing independence of moves $M_1, \ldots, M_{12}$.

| $D_1^{M_1}$ | $D_2^{M_1}$ | $D_1^{M_2}$ | $D_2^{M_2}$ | $D_1^{M_3}$ | $D_2^{M_3}$ |
|---|---|---|---|---|---|
| $D_1^{M_4}$ | $D_2^{M_4}$ | $D_1^{M_5}$ | $D_2^{M_5}$ | $D_1^{M_6}$ | $D_2^{M_6}$ |
| $D_1^{M_7}$ | $D_2^{M_7}$ | $D_1^{M_8}$ | $D_2^{M_8}$ | $D_1^{M_9}$ | $D_2^{M_9}$ |
| $D_1^{M_{10}}$ | $D_2^{M_{10}}$ | $D_1^{M_{11}}$ | $D_2^{M_{11}}$ | $D_1^{M_{12}}$ | $D_2^{M_{12}}$ |
Table 2: Nontrivial surface-links in flat form with ch-index $\leq 10$. 

| Link     | Diagram |
|----------|---------|
| $6_{1}^{0,1}$ | ![Diagram](image1) |
| $7_{1}^{0,-2}$ | ![Diagram](image2) |
| $8_{1}$ | ![Diagram](image3) |
| $8_{1}^{1,1}$ | ![Diagram](image4) |
| $8_{1}^{-1,-1}$ | ![Diagram](image5) |
| $9_{1}$ | ![Diagram](image6) |
| $9_{1}^{0,1}$ | ![Diagram](image7) |
| $9_{1}^{1,-2}$ | ![Diagram](image8) |
| $10_{1}$ | ![Diagram](image9) |
| $10_{2}$ | ![Diagram](image10) |
| $10_{3}$ | ![Diagram](image11) |
| $10_{1}^{1}$ | ![Diagram](image12) |
| $10_{1}^{0,1}$ | ![Diagram](image13) |
| $10_{2}^{0,1}$ | ![Diagram](image14) |
| $10_{1}^{1,1}$ | ![Diagram](image15) |
| $10_{1}^{0,0,1}$ | ![Diagram](image16) |
| $10_{1}^{0,-2}$ | ![Diagram](image17) |
| $10_{2}^{0,-2}$ | ![Diagram](image18) |
| $10_{1}^{-1,-1}$ | ![Diagram](image19) |
| $10_{1}^{-2,-2}$ | ![Diagram](image20) |
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