A new approach to the Cramer-Rao type bound
of the pure state model

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Abstract

In this paper, new methodology – direct approach – for the determination of the attainable CR type bound of the pure state model, is proposed and successfully applied to the wide variety of pure state models, for example, the 2-dimensional arbitrary model, the coherent model with arbitrary dimension. When the weight matrix is \( SLD \) Fisher information, the bound is determined for arbitrary pure state models. Manifestation of complex structure in the Cramer-Rao type bound is also discussed.

Keywords: quantum estimation theory, pure state model, Cramer-Rao type bound, complex structure

1 Introduction

The quantum estimation theory deals with determination of the density operator of the given physical system from the data obtained in the experiment. For simplicity, it is assumed that a state belongs to a certain subset \( \mathcal{M} = \{ \rho(\theta) | \theta \in \Theta \subset \mathbb{R}^m \} \) of the space of the states, which is called model, and that the true value of the finite dimensional parameter \( \theta \) is left to be estimated statistically. In this paper, we restrict ourselves to pure state model case, where \( \mathcal{M} \) is a subset of the space \( \mathcal{P}_1 \) of pure states in \( d \)-dimensional Hilbert space \( \mathcal{H} \) (\( d \leq \infty \)). For example, \( \mathcal{M} \) is a set of spin states with given wave function part and unknown spin part.

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In the classical estimation (throughout the paper, ‘classical estimation’ means the estimation theory of probability distribution), the mean square error is often used as a measure of error of the estimate, and the Cramer-Rao inequality assures that the inverse of so-called Fisher information matrix is the tight lower bound of covariance matrices of locally unbiased estimator (Ref. [9]).

Analogically, in the quantum estimation theory, in 1967, Helstrom showed that in the faithful state model, the covariance matrix is larger than or equal to the inverse of SLD Fisher information matrix, and that in the 1-dimensional faithful model, the bound is attainable [6][7].

On the other hand, in the multi-dimensional model, it is proved that there is no matrix which makes attainable lower bound of covariance matrix, because of non-commutative nature of quantum theory. Hence, the measure of the error of the estimate which is often used is TrGV_\theta[M], where V_\theta[M] denotes the covariance matrix of the locally unbiased measurement M at \theta and G is a weight matrix, or an arbitrary given m \times m positive symmetric real matrix. The infimum of is said to be attainable or achievable Cramer-Rao (CR) type bound of the model at \theta with weight matrix G, and to determine the attainable CR type bound long had been one of the main topics in this field, and is solved only for the several specific models, because TrGV_\theta[M] is a functional of probability valued measure, or pair of infinite number of operators in the infinite dimensional Hilbert space.

Yuen, Lax and Holevo found out the attainable CR type bound of the Gaussian state model, which is a faithful 2-dimensional model obtained by superposition of coherent states by Gaussian kernel [14][8]. Nagaoka and Hayashi calculated the attainable CR type bound of the faithful faithful spin-1/2 model [11][5]. Fujiwara and Nagaoka determined the bound for the 1-dimensional pure state model and the 2-dimensional coherent model, which is the pure-state-limit of the Gaussian model. [3][4].

All of their works are based on a methodology, which we call indirect approach hereafter; First one somehow find an auxiliary bound which is not generally attainable and then proves it to be attained in the specific cases.
In the approach in this paper, called direct approach in contrast with indirect approach, we reduce the problem to the minimization of the functional of the finite numbers of the finite dimensional vectors.

The methodology is successfully applied to the general 2-dimensional pure state model, and coherent model with arbitrary dimension. These are relatively general category in comparison with the cases treated by other authors. Also, when the weight matrix is SLD Fisher information matrix, which will be defined in somewhere in the paper, the bound is calculated for arbitrary pure state models.

As a by-product, we have rather paradoxical corollary, which asserts that even for ‘non-commutative cases’, simple measurement attains the lower bound.

The paper is organized as follows. In section 2 and 3, basic concepts of the quantum estimation theory are introduced. In section 4, the commuting theorem, which plays key role in the foundation of the direct approach, is presented and is applied to the characterization the quasi-classical model, in which non-commutative nature of the theory is not apparent. We formulate the problem in the non-quasi-classical models in section 5. Our new methodology, direct approach, is introduced in section 6 and 7, and is applied to the 2-dimensional pure state model and the coherent model in section 8 and 11 respectively. In section 9, we consider informational correlation between the parameters, and the attainable CR type bound for the direct sum of the models. The manifestation of the quantum structure, together with the minimization of the minimum of $\text{Tr} J^S(\theta)V_\theta[M]$, is discussed in section 10.

2 Locally unbiased measurement

Let $\sigma(\mathbb{R}^m)$ be a $\sigma$- field in the space $\mathbb{R}^m$. Whatever measuring apparatus is used to produce the estimate $\hat{\theta}$ of the true value of the parameter $\theta$, the probability that the estimate $\hat{\theta}$ lie in a particular measurable set $B$ in $\mathbb{R}^m$ will be given by

$$\Pr\{\hat{\theta} \in B|\theta\} = \text{tr}\rho(\theta)M(B)$$

(1)
when \( \theta \) represents the true value of parameter. Here \( M \) is a mapping of a measurable set \( B \in \sigma(\mathbb{R}^m) \) to non-negative Hermitian operators on \( \mathcal{H} \), such that

\[
M(\phi) = 0, \quad M(\mathbb{R}^m) = I,
\]

\[
M\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} M(B_i) \quad (B_i \cap B_j = \phi, i \neq j),
\]  \hspace{1cm} (2)

(see Ref.\[7\],p.53 and Ref.\[8\],p.50). \( M \) is called a \textit{generalized measurement} or \textit{measurement}, because there is a corresponding measuring apparatus to any \( M \) satisfying (2) \[12\][13]. A measurement \( E \) is said to be \textit{simple} if \( E \) is projection valued.

A generalized measurement \( M \) is called an \textit{unbiased measurement} in the model \( \mathcal{M} \), if \( E_{\theta}[M] = \theta \) holds for all \( \theta \in \Theta \), i.e.,

\[
\int \hat{\theta}^j \text{tr} \rho(\theta) M((d\hat{\theta}) = \theta^j, \quad (j = 1, \cdots, m). \]  \hspace{1cm} (3)

Differentiation yields

\[
\int \hat{\theta}^j \text{tr} \frac{\partial \rho(\theta)}{\partial \theta^k} M(d\hat{\theta}) = \delta^j_k, \quad (j, k = 1, \cdots, m). \]  \hspace{1cm} (4)

If (3) and (4) hold at a some \( \theta \), \( M \) is said to be \textit{locally unbiased} at \( \theta \). Obviously, \( M \) is unbiased iff \( M \) is locally unbiased at every \( \theta \in \Theta \).

As a measure of error of a locally unbiased measurement \( M \), we employ the covariance matrix with respect to \( M \) at the state \( \rho_{\theta} \), \( V_{\theta}[M] = [v_{\theta}^{jk}] \in \mathbb{R}^{m \times m} \), where

\[
v_{\theta}^{jk} = \int (\hat{\theta}^j - \theta^j)(\hat{\theta}^k - \theta^k) \text{tr} \rho(\theta) M(d\hat{\theta}). \]  \hspace{1cm} (5)

We often abbreviated notation \( V[M] \) for \( V_{\theta}[M] \) when it is not confusing. The problem treated in this note is to find a lower bound for \( V_{\theta}[M] \).

Only locally unbiased measurements are treated from now on, because of the following reason. Given \( N \) copies of the system, we apply a proper measurement to the first \( pN \) copies, and the true value of parameter is known to lie in certain \( \epsilon \)-ball centered at \( \theta_0 \) with the probability \( \sim 1 - \)
\[ e^{-a/\varepsilon^2 N}. \] Therefore, applying the ‘best’ locally unbiased measurement at \( \theta_0 \) to the \((1-p)N\) copies, we can achieve the efficiency arbitrarily close to that of the ‘best’ locally unbiased measurement at \( \theta \), in the sense of the first order asymptotics.

3 CR bound by SLD Fisher information matrix

In 1995, Fujiwara and Nagaoka [3] defined SLD Fisher information for pure state models. Here, we try another definition which is adequate for our direct approach.

Analogically to the classical estimation theory, in the quantum estimation theory, we have the following \textit{SLD CR inequality}, which is proved for the exact state model by Helstrom [6] [7], and is proved for the pure state model by Fujiwara and Nagaoka [3]:

\[ V_{\theta} [M] \geq (J^S(\theta))^{-1}, \]  

(6) i.e., \( V_{\theta} [M] - (J^S(\theta))^{-1} \) is non-negative definite. Here \( J^S(\theta) \), called \textit{SLD Fisher information matrix}, is defined by

\[ J^S(\theta) \equiv \text{Re}\langle l_i(\theta)|l_j(\theta)\rangle, \]

where the notations \( |l_i(\theta)\rangle \) (\( i = 1, ..., m \)) are defined afterward.

The inequality (6) is of special interest, because \( J^S(\theta) \) is the one of the best bounds in the sense of the following theorem, which will be proved in the section 5.

**Theorem 1** Letting \( A \) be a real hermitian matrix which is larger than \( J^{S-1} \), that is, \( A > J^{S-1} \), there exists such an unbiased estimator \( M \) that \( V[M] \) is not smaller than \( A \).

To define the notations \( |l_i(\theta)\rangle \) (\( i = 1, ..., m \)) and to prove the SLD CR inequality, we introduce some basic notations. \( \tilde{\mathcal{H}} \) is a set of vectors with unit length,

\[ \tilde{\mathcal{H}} = \{|\phi\rangle | |\phi\rangle \in \mathcal{H}, \langle \phi|\phi\rangle = 1\}. \]
\( \mathcal{P}_1 \) denotes the totality of density operators of pure states in \( \mathcal{H} \). A map \( \pi \) from \( \tilde{\mathcal{H}} \) to \( \mathcal{P}_1 \) is defined by
\[
\pi(|\phi\rangle) \equiv |\phi\rangle\langle \phi |.
\]
For the manifold \( \mathcal{N} = \{|\phi(\theta)\rangle \mid \theta \in \Theta \subset \mathbb{R}^m \} \) in \( \tilde{\mathcal{H}} \), \( \pi(\mathcal{N}) \) is defined to be a manifold in \( \mathcal{P}_1 \) such that
\[
\pi(\mathcal{N}) = \{\rho(\theta) \mid \rho(\theta) = \pi(|\phi(\theta)\rangle), |\phi(\theta)\rangle \in \mathcal{N}\}.
\]
Throughout the paper, we only treat with the pure state model \( \mathcal{M} \) which writes \( \mathcal{M} = \pi(\mathcal{N}) \) for a manifold \( \mathcal{N} \) in \( \mathcal{P}_1 \).

The horizontal lift \( |l_\mathcal{X}\rangle \) of a tangent vector \( \mathcal{X} \in T_{\rho(\theta)}(\mathcal{M}) \) to \( |\phi(\theta)\rangle \), is an element of \( \mathcal{H} \) which satisfies
\[
X\rho(\theta) = \frac{1}{2}(|l_\mathcal{X}\rangle\langle \phi(\theta)| + |\phi(\theta)\rangle\langle l_\mathcal{X}|),
\]
and
\[
\langle l_\mathcal{X}|\phi(\theta)\rangle = 0.
\]
Here, \( X \) in the left hand side of (7) of is to be understood as a differential operator. We use the symbol \( |l_i(\theta)\rangle \) to denote a horizontal lift of \( \partial_i \in T_{\rho(\theta)}(\mathcal{M}) \).

Notice that \( \text{span}_{\mathbb{R}}\{|l_i\rangle \mid i = 1,...,m \} \) is a representation of \( T_{\rho(\theta)}(\mathcal{M}) \) because of unique existence of the horizontal lift to \( |\phi(\theta)\rangle \) which is proved as follows. Application of a differential operator \( X \) to the both sides of \( \rho(\theta) = \rho^2(\theta) \) yields
\[
X\rho(\theta) = (X\rho(\theta))|\phi(\theta)\rangle\langle \phi(\theta)| + |\phi(\theta)\rangle\langle \phi(\theta)|(X\rho(\theta))
\]
and therefore \( |l_\mathcal{X}\rangle \) is given by \( \frac{1}{2}X\rho(\theta)|\phi(\theta)\rangle \). Actually, taking trace of both sides of (8), it is shown that \( (X\rho(\theta))|\phi(\theta)\rangle \) satisfies (8). To prove the uniqueness, it suffices to show that \( |l\rangle = 0 \) if \( \langle l|\phi(\theta)\rangle = 0 \) and
\[
0 = |l\rangle\langle \phi(\theta)| + |\phi(\theta)\rangle\langle l|
\]
holds true. Multiplication of $|\phi(\theta)\rangle$ to the both sides of (10) proves the statement.

Fujiwara and Nagaoka defined SLD Fisher information matrix $J^S(\theta)$ by using the symmetric logarithmic derivative (SLD) of the parameter $\theta^i$ is a hermitian matrix $L^S_i(\theta)$ which satisfies

$$
\partial_i \rho(\theta) = \frac{1}{2} (L^S_i(\theta) \rho(\theta) + \rho(\theta) L^S_i(\theta)).
$$

Using SLD, the horizontal lift of $\partial_i$ to $T|\phi(\theta)\rangle(\tilde{H})$ writes $|l^i(\theta)\rangle = L^S_i(\theta)|\phi(\theta)\rangle$.

$J^S(\theta)$ is called SLD Fisher information matrix because $J^S(\theta)$ writes

$$
J^S(\theta) = \text{Re tr} \rho(\theta) L^S_i(\theta) L^S_j(\theta).
$$

SLD defined by (11) has the arbitrariness which corresponds to the kernel of $\rho(\theta)$, and Fujiwara and Nagaoka [3] showed that $J^S(\theta)$ is uniquely defined regardless this arbitrariness. Notice that in our framework, uniqueness of SLD Fisher information matrix is trivial.

We define estimation vector $|x^i[M,|\phi(\theta)\rangle]\rangle$ of the parameter $\theta^i$ by a measurement $M$ at $|\phi(\theta)\rangle$, by

$$
|x^i[M,|\phi(\theta)\rangle]\rangle \equiv \int (\tilde{\theta}^i - \theta^i) M(\tilde{d}\theta)|\phi(\theta)\rangle.
$$

An estimation vector $|x^i[M,|\phi(\theta)\rangle]\rangle$ is said to be locally unbiased iff $M$ is locally unbiased. The local unbiasedness conditions for estimating vectors writes

$$
\langle x^i[M,|\phi(\theta)\rangle]|\phi(\theta)\rangle = 0,
$$

$$
\text{Re}\langle x^i[M,|\phi(\theta)\rangle]|l_j(\theta)\rangle = \delta^i_j (i, j = 1, \ldots, m).
$$

Often, we omit the argument $\theta$ in $|l_j(\theta)\rangle$, $|\phi(\theta)\rangle$, $\rho(\theta)$, and $J^S(\theta)$ and denote them simply by $|l_j\rangle$, $|\phi\rangle$, $\rho$, $J^S$. Also, $|x^i[M,|\phi(\theta)\rangle]\rangle$ is denoted simply by $|x^i\rangle$ with the arguments $M, |\phi(\theta)\rangle$ left out, so far as no confusion is caused.

We denote the ordered pair of vectors

$$
[[x^1], [x^2], \ldots, [x^m]]
$$
and

$$[|l'_1⟩, |l'_2⟩, ..., |l'_m⟩]$$

by $X$ and $L$ respectively. Then, the unbiasedness conditions (14) writes

$$\text{Re}X^*L \equiv \text{Re}[⟨x^i|l^j⟩] = I_m,$$

(15)

where $I_m$ is the $m \times m$ unit matrix. The SLD Fisher information matrix $J^S$ writes

$$J^S = \text{Re}L^*L.$$

The imaginary part of $L^*L$ is denoted by $\tilde{J}$. Now, we are in the position to prove SLD CR inequality.

**Lemma 1** Following two inequalities are valid:

$$V[M] \geq \text{Re}X^*X.$$  

(16)

$$V[M] \geq X^*X.$$  

(17)

**Lemma 2**

$$\text{Re}X^*X \geq J^{S-1}$$

(18)

holds. The equality is valid iff

$$|x^j⟩ = \sum_k (J^{S-1})^{j,k} |l_k⟩,$$

or, equivalently,

$$X = L J^{S-1} \equiv \left[ \sum_k (J^{S-1})^{j,k} |l_k⟩, \; j = 1, ..., m \right]$$

(19)
They are proved in almost the same manner as the strictly positive case (see Ref. 8 p.88 and p.274 respectively). Lemmas 1-2 lead to the SLD CR inequality (6).

**Theorem 2** (Fujiwara and Nagaoka 3) SLD Fisher information gives a lower bound of covariance matrix of an unbiased measurement, i.e., (6) holds true.

The SLD CR inequality (6) looks quite analogical to CR inequality in classical estimation theory. However, as is found out in the next section, the equality does not generally establish.

4 **The commuting theorem and the quasi-classical model**

In this section, the necessary and sufficient condition for the equality in the SLD CR inequality to establish is studied. Fujiwara has proved the following theorem 3.

**Theorem 3** (Fujiwara 3) The equality in the SLD CR inequality establishes iff SLDs \( \{L^S_i | i = 1, ..., m \} \) can be chosen so that

\[
[L^S_i, L^S_j] = 0, \ (i, j = 1..., m).
\]

We prove another necessary and sufficient condition which is much easier to check for given models, by use of the following commuting theorem, which plays key role in our direct approach.

**Theorem 4** If there exists a unbiased measurement \( M \) such that

\[
|x^i\rangle = \int (\hat{\theta}^i - \theta^i) M(\hat{d}\theta)|\phi\rangle, \\
V[M] = \text{Re}X^*X,
\]

(20)
then,

$$\text{Im} \mathbf{X}^* \mathbf{X} = 0$$  \hspace{1cm} (21)

holds true. On the other hand, if (20) holds true, then there exists such a simple, or projection valued, unbiased measurement \( E \) that (21) holds and

$$E(\{\hat{\theta}_\kappa\})E(\{\hat{\theta}_\kappa\}) = E(\{\hat{\theta}_\kappa\}),$$

$$E(\{\hat{\theta}_0\}) = E_0,$$

$$E \left( \mathbb{R}^m / \bigcup_{\kappa=0}^{m} \{\hat{\theta}_\kappa\} \right) = 0,$$  \hspace{1cm} (22)

for some \( \{\hat{\theta}_\kappa | \hat{\theta}_\kappa \in \mathbb{R}^m, \kappa = 0, \ldots, m+1\} \), where \( E_0 \) is a projection onto orthogonal complement subspace of \( \text{span}_C \{\mathbf{X}\} \).

**Proof**  If (20) holds, inequality (17) in lemma 1 leads to

$$\text{Re} \mathbf{X}^* \mathbf{X} \geq \mathbf{X}^* \mathbf{X},$$

or

$$0 \geq i \text{Im} \mathbf{X}^* \mathbf{X},$$

which implies \( \text{Im} \mathbf{X}^* \mathbf{X} = 0 \).

Conversely, Let us assume that (21) holds true. Applying Schmidt’s orthogonalization to \( \{|\phi\rangle, |x^1\rangle, \ldots, |x^m\rangle\} \) and normalizing the product of orthogonalization, we obtain the orthonormal system \( \{|b_i\rangle | i = 1, \ldots, m+1\} \) of vectors such that,

$$|x^i\rangle = \sum_{j=1}^{m+1} \lambda_{ij} |b_j\rangle, \ \exists \lambda_{ij} \in \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, m+1.$$

Letting \( O = [o_{ij}^j] \) be a \((m+1) \times (m+1)\) real orthogonal matrix such that

$$\langle \phi | \sum_{j=1}^{m+1} o_{ij}^j |b_j\rangle \neq 0,$$
and denoting $\sum_{j=1}^{m+1} o_j^i |b^i\rangle$ by $|b^i\rangle$, we have

$$
|x^i\rangle = \sum_{j=1}^{m+1} \lambda_j^i \sum_{k=1}^{m+1} o_j^k |b^k\rangle \\
= \sum_{k=1}^{m+1} \left( \sum_{j=1}^{m+1} \lambda_j^i o_j^k \right) |b^k\rangle \\
= \sum_{k=1}^{m+1} \sum_{j=1}^{m+1} \lambda_j^i o_j^k \langle b^k|\phi\rangle |b^k\rangle 
$$

Therefore, noticing that the system $\{ |b^i\rangle | i = 1, \ldots, m+1 \}$ of vectors is orthonormal, we obtain an unbiased measurement which satisfies (22) as follows:

$$
\hat{\theta}_\kappa = \frac{\sum_{j=1}^{m+1} \lambda_j^i o_j^\kappa}{\langle b^\kappa|\phi\rangle}, \quad \kappa = 1, \ldots, m+1, \\
\hat{\theta}_0 = 0, \\
E(\{ \hat{\theta}_\kappa \}) = |b^\kappa\rangle\langle b^\kappa|, \quad \kappa = 1, \ldots, m+1, \\
E(\hat{\theta}_0) = I_\mathcal{H} - \sum_{\kappa=1}^{m+1} |b^\kappa\rangle\langle b^\kappa|, \\
E(\mathcal{R}|\{ \hat{\theta}_0, \ldots, \hat{\theta}_{m+1} \}) = 0.
$$

Here, $I_\mathcal{H}$ is the identity in $\mathcal{H}$.

\[\square\]

**Theorem 5** The equality in the SLD CR inequality establishes iff

$$\text{Im} L^* L = 0 \tag{23}$$

$\langle l_j|l_i\rangle$ is real for any $i,j$. When the equality establishes, that bound is achieved by a simple measurement, i.e., a projection valued measurement.

**Proof** If the equality establishes, by virtue of lemma [12] we have (21) and (19), which lead directly to (23).

Conversely, if $\text{Im} \langle l_j|l_k\rangle = 0$ for any $j,k$, by virtue of commuting theorem, there exists such a simple measurement $E$ that

$$
\sum_k \langle J^{S-1} |j,k|l_k\rangle = \int (\hat{\theta} - \theta) E(d\hat{\theta}) |\phi\rangle.
$$
Elementary calculations show that the covariance matrix of this measurement equals $J^{S^{-1}}$.\qed

Our theorem is equivalent to Fujiwara’s one, because by virtue of commuting theorem, $\langle l_j | l_i \rangle$ is real iff there exist such SLDs that $L^S_i$ and $L^S_j$ commute for any $i, j$. However, our condition is much easier to be checked, because to check Fujiwara’s condition, you must calculate all the possible SLDs, for the SLD is not unique. In addition, SLD is much harder to calculate than horizontal lift. When the model has only one-dimensional, we have the following corollary of theorem 5

**Corollary 6** when a manifold $\mathcal{M}$ is one-dimensional, the inverse of SLD Fisher information matrix is always attainable by a simple measurement.

**Remark** Often, a model is defined by an initial state and generators,

\[
\mathcal{N} \equiv \{ |\phi(\theta)\rangle | \phi(\theta_0)\rangle = |\phi_0\rangle, \partial_i|\phi(\theta)\rangle = iH_i(\theta)|\phi(\theta)\rangle, \theta \in \Theta \subset \mathbb{R}^m \},
\]

\[
\mathcal{M} \equiv \pi(\mathcal{N}).
\]

Then, $\langle l_j | l_i \rangle$ is real iff $\langle \phi(\theta)||[H_i(\theta), H_j(\theta)]|\phi(\theta)\rangle = 0$, which is equivalent to the existence of generators which commute with each other, $[H_i(\theta), H_j(\theta)] = 0$ by virtue of the commuting theorem.

Putting the remark and the Fujiwara’s theorem together, we may metaphorically say that the equality in the inverse of SLD Fisher information matrix is attainable iff any two parameters ‘commute’ at $\theta$. Throughout the paper, we say that a manifold $\mathcal{M}$ is quasi-classical at $\theta$ iff $\langle l_j | l_i \rangle$ is real at $\theta$. The following remark describes another ‘classical’ aspect of the condition $\text{Im}(\langle l_j | l_i \rangle) = 0$.

**Example** When the model $\mathcal{M}$ is given by

\[
\mathcal{M} = \{ \rho(\theta) \ | \ \rho(\theta) = \pi(|\phi(\theta)\rangle), |\phi(\theta)\rangle \text{is an element of real Hilbert space} \},
\]

the model is quasi-classical at any point in $\mathcal{M}$.

As is illustrated in this example, when the model $\mathcal{M}$ is quasi-classical at $\theta_0$, a state vector $|\phi(\theta)\rangle$ behaves like an element of real Hilbert space around $\theta_0$, and the state vector’s phase parts don’t change around $\theta_0$ at all.
5 Non-quasi-classical cases

As was concluded, the equality in the SLD CR inequality establishes only when the model is quasi-classical, and there is not any better bound than the inverse of SLD Fisher information matrix, as in theorem 1, which is straightforwardly derived from the following lemma, which is proved in the appendix A.

Lemma 3 For any $i$,

$$\inf \{ [V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \} = [J_{\theta}^{-1}]_{ii}$$

In general case, therefore, we must give up to find a matrix which makes attainable lower bound of $V[M]$, and instead, we try to determine

$$\text{CR}(\theta, G, \mathcal{M}) \equiv \inf \{ \text{Tr}GV \mid V \in V_\theta(\mathcal{M}) \} \quad (24)$$

for an arbitrary nonnegative symmetric real matrix $G$, where $V_\theta(\mathcal{M})$ (or in short, $V_\theta$) is the region of the map $V_\theta[\bullet]$ from unbiased estimators to $m \times m$ real positive symmetric matrices. CR($\theta, G, \mathcal{M}$) is the attainable CR type bound, and we often use abbreviated notations such as CR($\theta, G$), CR($G$).

To make the estimational meaning of (24) clear, let us restrict ourselves to the case when $G$ is $\text{diag}(g_1, g_2, \ldots, g_m)$. Then, the attainable CR type bound is nothing but the weighed sum of the covariance of the estimation of $\theta_i$. If one needs to know, for example, $\theta_1$ more precisely than other parameters, then he set $g_1$ larger than any other $g_i$, and choose a measurement which achieves the attainable CR type bound.

Notice that

$$\inf \left\{ \sum_{i=1}^{m} g_i [V_\theta[M]]_{ii} \left\mid M \text{ is locally unbiased at } \theta \right. \right\}$$

$$\geq \sum_{i=1}^{m} \inf \{ g_i [V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \}$$

$$= \sum_{i} g_i \left[ J_{\theta}^{-1}(\theta) \right]_{ii},$$
holds true by virtue of the lemma 3, and that the equality in the first inequality does not always establish, implying that in the simultaneous estimation of different parameters, there is information losses because of non-commutative nature of the quantum mechanics.

Another proper alternative of the classical Fisher information matrix is a set \( \inf \mathcal{V}(M) \) of symmetric real matrices, where the notation \( \inf \) is defined as follows. Let us define

\[
\begin{align*}
\text{lb}\mathcal{X} & \equiv \{ A \mid A \text{ is real and symmetric}, \forall B < A, B \in \mathcal{X} \}, \\
\text{ub}\mathcal{X} & \equiv \{ A \mid A \text{ is real and symmetric}, \forall B > A, B \in \mathcal{X} \},
\end{align*}
\]

where \( \mathcal{X} \) is a set of real symmetric matrices, and we define \( \inf \mathcal{V} \) by

\[
\inf \mathcal{V} \equiv \text{lb}\mathcal{V} \cap \text{ub}(\text{lb}\mathcal{V}).
\]

Then, we have the following lemma.

**Lemma 4** \( \inf \mathcal{V} \) is a subset of the boundary \( \text{bd}\mathcal{V} \) of \( \mathcal{V} \).

This lemma is a straightforward consequence of the following lemma, which is proved in the appendix B.

**Lemma 5** If \( V \) is an element of \( \mathcal{V} \), then \( V + V_0 \) is also an element of \( \mathcal{V} \), where \( V_0 \) is an arbitrary real nonnegative symmetric matrix.

Because of lemma 4, it is of interest to determine the boundary \( \text{bd}\mathcal{V} \). \( \text{bd}\mathcal{V} \) is turned out to be a subset of \( \mathcal{V} \) such that \( V = \text{CR}(G) \) for a weight matrix \( G \), because of lemma 5, and lemmas 6-7.

**Lemma 6** \( \mathcal{V} \) is convex.

**Proof** Let \( M \) and \( N \) be an unbiased estimator. Because

\[
\lambda V[M] + (1 - \lambda)V[N] = V[\lambda M + (1 - \lambda)N]
\]

holds true and \( \lambda M + (1 - \lambda)N \) is an unbiased estimator, we have the lemma.

\( \Box \)
Lemma 7 \( \mathcal{V} \) is closed.

Lemma 6 will be proved in the appendix C.

If a model \( \mathcal{M} \) has smaller value of the attainable CR type bound at \( \theta \) than another model \( \mathcal{N} \) at \( \theta' \) has, the \( \mathcal{V}_\theta(\mathcal{M}) \) of is located in the ‘lower part’ of \( \text{Sym}(m) \) compared with that of \( \mathcal{V}_{\theta'}(\mathcal{N}) \).

6 The reduction theorem and the direct approach

Theorem 7 (Naimark’s theorem, see Ref. [8], pp. 64-68.) Any generalized measurement \( \mathcal{M} \) in \( \mathcal{H} \) can be dilated to a simple measurement \( \mathcal{E} \) in a larger Hilbert space \( \mathcal{H}' \supset \mathcal{H} \), so that

\[
M(B) = P E(B) P
\]

will hold, where \( P \) is the projection from \( \mathcal{H}' \) onto \( \mathcal{H} \).

Naimark’s theorem, mixed with commuting theorem, leads to the following reduction theorem, which is essential to our direct approach.

Theorem 8 Let \( \mathcal{M} \) be a \( m \)-dimensional manifold in \( \mathcal{P}_1 \), and \( \mathcal{B}_\theta \) be a system \( \{ |\phi_i'\rangle | l_i'\rangle, \ i = 1, \ldots, m \} \) of vectors in \( 2m + 1 \)-dimensional Hilbert space \( \mathcal{H}'_\theta \) such that

\[
\langle \phi' | l_j' \rangle = \langle \phi | l_j \rangle = 0,
\]

\[
\langle l_i' | l_j' \rangle = \langle l_i | l_j \rangle,
\]

for any \( i, j \). Then, for any locally unbiased estimator \( \mathcal{M} \) at \( \theta \) in \( \mathcal{H} \), there is a simple measurement \( \mathcal{E} \) in \( \mathcal{H}'_\theta \) such that ‘locally unbiasedness’ is satisfied,

\[
| x^i \rangle = \int (\hat{\theta}^i - \theta^i) E(d\theta) | \phi' \rangle \in \mathcal{H}'_\theta
\]

\[
\langle x^i | \phi' \rangle = 0,
\]

\[
\text{Re} \langle x^i | l_j' \rangle = \delta^i_j (i, j = 1, \ldots, m),
\]
and that the ‘covariance matrix’ $V[E]$ of $E$ equals $V[M]$,

$$V[M] = V[E] \equiv \left[ \int (\hat{\theta}^i - \theta^i) (\hat{\theta}^j - \theta^j) \text{tr} \rho E(d\hat{\theta}) \right].$$ (29)

**Proof** For any locally unbiased measurement $M$, there exists a Hilbert space $\mathcal{H}_M$ and a simple measurement $E_M$ in $\mathcal{H}_M$ which satisfies (25) by virtue of Naimark’s theorem. Note that $E_M$ is also locally unbiased. Let $|y^i\rangle \in \mathcal{H}_M$ denote the estimation vector of $\theta^i$ by $E_M$, that is,

$$|y^i\rangle \equiv \int (\hat{\theta}^i - \theta^i) E_M(d\hat{\theta}) |\phi\rangle.$$

Mapping $\text{span}_{\mathbb{C}} \{ |\phi\rangle, |l_i\rangle, |y^i\rangle \mid i = 1, \ldots, m \}$ isometrically onto $\mathcal{H}_\theta'$ so that $\{ |\phi\rangle, |l_i\rangle \mid i = 1\ldots m \}$ are mapped to $\{ |\phi'\rangle, |l'_i\rangle \mid i = 1, \ldots, m \}$, we denote the images of $\{ |y^i\rangle \mid i = 1, \ldots, m \}$ by $\{ |x^i\rangle \mid i = 1, \ldots, m \}$.

Then, by virtue of the commuting theorem, we can construct a simple measurement $E$ in $\mathcal{H}_\theta'$ satisfying the equations (26) - (29). $\square$

The reduction theorem shows that $\mathcal{V}$ is identical with the set of matrices

$$V = \text{Re} \mathbf{X}^* \mathbf{X}$$

such that

$$|x^i\rangle \in \mathcal{H}_\theta' \setminus \{ |\phi'\rangle \} \ (i = 1, \ldots, m),$$

where $\mathcal{H}_\theta' \setminus \{ |\phi'\rangle \}$ denotes the orthogonal complement subspace of $\mathcal{H}_\theta'$, and that (23) and (28) are satisfied. Now, the problem is simplified to the large extent, because we only need to treat with vectors $\{ |x^i\rangle \mid i = 1, \ldots, m \}$ in finite dimensional Hilbert space $\mathcal{H}_\theta'$ instead of measurements, or operator valued measures.

We conclude this section with a corollary of reduction theorem, which is rather counter-intuitive because historically, non-projection-valued measurement is introduced to describe simultaneous measurements of non-commuting observables.
Corollary 9 When the dimension of $\mathcal{H}$ is larger than or equal to $2m + 1$, for any unbiased measurement $M$ in $\mathcal{H}$, there is a simple measurement $E$ in $\mathcal{H}$ which has the same covariance matrix as that of $M$.

Proof Chose $\{|l'_i\rangle | i = 1, ..., m\}$ to be $\{|l_i\rangle | i = 1, ..., m\}$. $\square$ Especially, if $\mathcal{H}$ is infinite dimensional, as is the space of wave functions, the assumption of the corollary is always satisfied.

7 Lagrange’s method of indeterminate coefficients in the pure state estimation theory

Now, we apply our direct approach to the problem presented in the section 5, or the minimization of the functional $\text{Tr} G \text{Re} X^* X$ of vectors in $\mathcal{H}'_0$. One of most straightforward approaches to this problem is Langrange’s indeterminate coefficients method. First, denoting an ordered pair $\{|l'_i\rangle | i = 1, ..., m\}$ of vectors in $\mathcal{H}'_0$ also by $L$, the symbol which is used also for an ordered pair $\{|l_i\rangle | i = 1, ..., m\}$ of vectors in $\mathcal{H}$, we define a function $\text{Lag}(X)$ by

$$\text{Lag}(X) \equiv \text{Re} \text{Tr} X^* X G - 2 \text{Tr} ((\text{Re} X^* L - I_m)\Xi) - \text{Tr} \text{Im} X^* X \Lambda,$$

(30)

where $\Xi, \Lambda$ are real matrices whose components are Langrange’s indeterminate coefficients. Here, $\Lambda$ can be chosen to be antisymmetric, for

$$\text{Tr} \text{Im} X^* X \Lambda = \text{Tr} \text{Im} X^* X (\Lambda - \Lambda^T)/2$$

holds true and only antisymmetric part of $\Lambda$ appears in (30).

From here, we follow the routine of Langrange’s method of indeterminate coefficients. Differentiating $L(X + \epsilon \delta X)$ with respect to $\epsilon$ and substituting 0 into $\epsilon$ in the derivative, we get

$$\text{Re} \text{Tr} (\delta X^* (2XG - 2L\Xi - 2iX\Lambda)) = 0.$$

Because $\delta X$ is arbitrary,

$$X(G - i\Lambda) = L\Xi$$

(31)
is induced.

Multiplying $X^*$ to both sides of (31), the real part of the outcomming equation, together with (15), yields

$$\Xi = \text{Re}X^*XG = VG. \quad (32)$$

Substituting (32) into (31), we obtain

$$X(G - i\Lambda) = LVG. \quad (33)$$

In this paper, we solve (15), (21), (33) and $V = \text{Re}X^*X$ with respect to $X$, real symmetric matrix $V$ and real antisymmetric matrix $\Lambda$, for the variety of pure state models. However, the general solution is still far out of our reach.

### 8 The model with two parameters

In this section, we determine the boundary of the set $V$ in the case of the 2-dimensional model.

The equation (33), mixed with (21), leads to

$$(G - i\Lambda)V(G - i\Lambda) = GV\bar{L}V. \quad (34)$$

whose real part and imaginary part are

$$GVG - \Lambda V\Lambda = GVJ^SVG, \quad (35)$$

and

$$GVA + AVG = -GV\bar{J}VG, \quad (36)$$

where $\bar{J}$ denotes $\text{Im}L^*L$, respectively.

As is proved in the following, when the matrix $G$ is strictly positive, (34) is equivalent to the existence of $X$ which satisfies (15), (21), (34), and $V = \text{Re}X^*X$. If $V$ and $\Lambda$ satisfying (34) exist, $X$ which satisfies (33) and (21)
is given by $X = UV^{1/2}$, where $U$ is such a $2m + 1 \times m$ complex matrix that $U^*U = I_m$. $X = UV^{1/2}$ also satisfies (15), because

$$VG = \text{Re}X^*LVG$$

is obtained by multiplying $X^*$ to and taking real part of the both sides of (33).

Hence, if $G$ is strictly positive, our task is to solve (35) and (36) for real positive symmetric matrix $V$ and real antisymmetric matrix $\Lambda$. When $G$ is not strictly positive, after solving (35) and (36), we must check whether there exists such $X$ which satisfies (15), (21) and $V = \text{Re}X^*X$.

Throughout this section, we parameterize the model so that $J^S$ is equal to the identity matrix $I_m$. Given an arbitrary coordinate system $\{\theta_i | i = 1, ..., m\}$, such a coordinate system $\{\theta'^i | i = 1, ..., m\}$ is obtained by the following coordinate transform:

$$\theta'^i = \sum_{j=1}^{m} [(J^S)^{1/2}]_{ij} \theta^j (i = 1, ..., m).$$

(37)

By this coordinate transform, $V$ is transformed as:

$$V' = (J^S)^{1/2}V(J^S)^{1/2}.$$  

(38)

If the result in the originally given coordinate is needed, one only needs to transform the result in the coordinate system $\{\theta'^i | i = 1, ..., m\}$ using (38) in the converse way.

So far, we have not assumed dim $\mathcal{M} = 2$. When dim $\mathcal{M} = 2$, covariance matrices are included in the space $\text{Sym}(2)$ of $2 \times 2$ symmetric matrices which is parameterized by $x, y,$ and $z$, where

$$\text{Sym}(2) = \left\{ V \bigg| V = \begin{bmatrix} z+x & y \\ y & z-x \end{bmatrix} \right\}.$$  

Before tackling the equations (35) and (36), three useful facts about this parameterization are noted. First, letting $A$ is a symmetric real matrix
which is represented by \((A_x, A_y, A_z)\) in the \((x, y, z)\)-space, the set \(C_+(A)\) of all matrices larger than \(A\) is

\[
C_+(A) = \{(x, y, z) \mid (z - A_z)^2 - (x - A_x)^2 - (y - A_y)^2 \geq 0, \ z \geq A_z\},
\]

that is, inside of a upside-down corn with its vertex at \(A = (A_x, A_y, A_z)\). Hence, \(\mathcal{V}\) is a subset of \(C_+(I_m)\), or inside of a upside-down corn with its vertex at \((0,0,1)\) because of the SLD CR inequality. When the model \(\mathcal{M}\) is classical at \(\theta\), \(\mathcal{V}\) coincides with \(C_+(I_m)\).

Second, an action of rotation matrix \(R_\theta\) to \(\mathcal{V}\) such that \(R_\theta \mathcal{V} R_\theta^T\), where

\[
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix},
\]

corresponds to the rotation in the \((x, y, z)\)-space around \(z\)-axis by the angle \(2\theta\).

Third, we have the following lemma.

**Lemma 8** \(\mathcal{V}\) is rotationally symmetric around \(z\)-axis, if \(\mathcal{M}\) is parameterized so that \(J^S\) writes the unit matrix \(I_m\).

**Proof** The necessary and sufficient condition for \(\mathcal{V}\) to have rotational symmetry around \(z\)-axis is the existence of a \(2m+1\) by \(m\) complex matrix \(Y\) satisfying (15), (21) and

\[
\forall \theta \ R_\theta (\text{Re} X^* X) R_\theta^T = R_\theta X^* X R_\theta^T = Y^* Y, \tag{39}
\]

for any given ordered pair \(X\) of vectors which satisfies (15) and (21).

On the other hand, because of \(L^* L = I_m + i\tilde{J}\), elementary calculation shows

\[
L^* L = R_\theta L^* L R_\theta^T,
\]

or equivalently, for some unitary transform in \(\mathcal{H}_\theta' \backslash \{\phi\}\),

\[
L^* U = R_\theta L^* ,
\]
which leads, together with (15), to
\[ \text{Re} L^*UX = R_\theta. \]

Therefore,
\[ \mathcal{Y} = UXR_\theta^T \]
satisfies (39), and we have the lemma. \(\square\)

Now, the boundary of the intersection \(\tilde{\mathcal{V}}\) of \(\mathcal{V}\) and \(zx\)-plain is to be calculated, because \(\mathcal{V}\) is obtained by rotating \(\tilde{\mathcal{V}}\) around \(z\)-axis, by virtue of this lemma. \(bd\tilde{\mathcal{V}}\) is obtained as the totality of the matrix \(V = \text{Re}X^*X\) which satisfies (15), (21), and (33), for a diagonal real nonnegative matrix \(G\).

Let us begin with the case where a diagonal matrix \(G\) is positive definite. In this case, we only need to deal with (35) and (36). Let
\[ \tilde{J} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}, \]
and
\[ \Lambda = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}, \quad V = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \]
where \(g, v,\) and \(u\) are positive real numbers. Note that
\[ |\beta| \leq 1 \]
holds, because \(L^*L = I_m + \tilde{J}\) is nonnegative definite. Then, (35) and (36) writes
\[ u + v\lambda^2 - u^2 = 0, \]
\[ vg^2 + u\lambda^2 - v^2g^2 = 0, \]
\[ vg\lambda + u\lambda + uv\beta g = 0. \] (40)
The necessary and sufficient condition for \(\lambda\) and positive \(g\) to exist is, after some calculations,
\[ \sqrt{u-1} + \sqrt{v-1} - |\beta|\sqrt{uv} = 0. \] (41)
Note that $u$ and $v$ are larger than or equal to 1, because $V \geq J^{S-1} = I_m$. Substitution of $u = z + x$ and $v = z - x$ into (41) and some calculation leads to
\[ |\beta| \sqrt{(z + x - 1)(z - x - 1) \pm \sqrt{1 - \beta^2} \left( \sqrt{z + x - 1} + \sqrt{z - x - 1} \right)} = |\beta|. \]

It is easily shown that the lower sign in the equation corresponds to the set of stationary points, and
\[ |\beta| \sqrt{(z + x - 1)(z - x - 1) \pm \sqrt{1 - \beta^2} \left( \sqrt{z + x - 1} + \sqrt{z - x - 1} \right)} = |\beta| \]

(42) gives a part of $bd\hat{V}$. In (42), $x$ takes value ranging from $-\beta^2/(1 - \beta^2)$ to $\beta^2/(1 - \beta^2)$ if $|\beta|$ is smaller than 1. When $|\beta| = 1$, $x$ varies from $-\infty$ to $\infty$. This restriction on the range of $x$ comes from the positivity of $z - x - 1$ and $z + x - 1$.

When
\[ G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{or} \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \]

we must treat the case of $|\beta| = 1$ and the case of $|\beta| < 1$ differently. In the case of $|\beta| = 1$, there exists no $2m \times m$ complex matrix $X$ which satisfies $V = \Re X^*X$, \((14)\), \((21)\) and \((34)\). On the other hand, if $|\beta| < 1$, such complex matrix $X$ always exists and $V = \Re X^*X$ is given by, in terms of $(x, y, z)$,
\[ z = -x + 1, \quad x \leq -\frac{\beta^2}{1 - \beta^2} \]

or
\[ z = x + 1, \quad x \geq \frac{\beta^2}{1 - \beta^2} \]

(44)

Because any element on the line \((44)\), if $x \neq \pm \beta^2/(1 - \beta^2)$, has an element of $\mathcal{V}$ which is smaller than itself, \((42)\) the intersection of $\inf \mathcal{V}$ and $zx$-plane, where
\[ -\beta^2/(1 - \beta^2) \leq x \leq \beta^2/(1 - \beta^2). \]
Figure 1: Two stationary lines;

(i) \(|\beta| \sqrt{(z + x - 1)(z - x - 1)} + \sqrt{1 - \beta^2} \left(\sqrt{z + x - 1} + \sqrt{z - x - 1}\right) = |\beta|; \)

(ii) \(|\beta| \sqrt{(z + x - 1)(z - x - 1)} - \sqrt{1 - \beta^2} \left(\sqrt{z + x - 1} + \sqrt{z - x - 1}\right) = |\beta|; \)
The intersection of $z$-axis and $bdV$ gives

$$\text{CR}(J^S) = \frac{4}{1 + \sqrt{1 - |\beta|^2}}, \quad (45)$$

where the equality holds in any parameterization of the model $\mathcal{M}$.

In arbitrary parameterization of the model $\mathcal{M}$, with help of $\text{(42)}$ and $\text{(38)}$, $\inf(V)$ is obtained as, for $|\beta| > 0$,

$$\det \sqrt{\hat{V}(V)} + \sqrt{(1/\beta^2) - 1} \text{Tr} \sqrt{\hat{V}(V)} = 1 \quad (|\beta| > 0),$$

$$\text{Tr} \sqrt{\hat{V}(V)} = 0 \quad (\beta = 0), \quad (46)$$

where

$$\hat{V}(V) \equiv \sqrt{J^SV\sqrt{J^S} - I_m}.$$

Slight look at the equations $\text{(46)}$ leads to the following theorem.

**Theorem 10** In the 2-dimensional model, if

$$|\beta(\theta, \mathcal{M})| \geq |\beta(\theta', \mathcal{M}')|,$$

$$J^S(\theta, \mathcal{M}) = J^S(\theta', \mathcal{M}'),$$

then the $\mathcal{V}_\theta(\mathcal{M})$ is a subset of $\mathcal{V}_{\theta'}(\mathcal{M}')$.

The equations $\text{(46)}$ and tedious but elementary calculations shows the following theorem.

**Theorem 11** In the 2-dimensional model, if

$$|\beta(\theta, \mathcal{M})| = |\beta(\theta', \mathcal{M}')|,$$

$$J^S(\theta, \mathcal{M}) \leq J^S(\theta', \mathcal{M}'),$$

then the $\mathcal{V}_\theta(\mathcal{M})$ is a subset of $\mathcal{V}_{\theta'}(\mathcal{M}')$.

By virtue of these theorems, $|\beta|$ can be seen as a measure of ‘uncertainty’ between the two parameters. Two extreme cases are worthy of special attention; When $|\beta| = 0$, the model $\mathcal{M}$ is classical at $\theta$ and $\mathcal{V}$ is maximum. On the other hand, if $|\beta| = 1$, $\mathcal{V}$ is minimum and uncertainty between $\theta^1$ and $\theta^2$ is maximum. In the latter case, we say that the model is coherent at $\theta$. 24
Figure 2: (a) $|\beta| = 0$; (b) $0 < |\beta| < 1$; (c) $|\beta| = 1$. 
**Example** (spin rotation model) We define *spin rotation model* \([1]\) by

\[
\mathcal{M}_{s,m} = \pi(N), \\
\mathcal{N}_{s,m} = \left\{ |\phi(\theta)\rangle \mid |\phi(\theta)\rangle = T(\theta)|s,m\rangle, 0 \leq \theta^1 < \pi, 0 \leq \theta^2 < 2\pi \right\}, \\
T(\theta) = \exp\left(i\theta^1(\sin^2 S_x - \cos^2 S_y)\right),
\]

where \(S_x, S_y, S_z\) are spin operators, and \(|s,m\rangle\) is defined by,

\[
S_z|j,m\rangle = \hbar m|s,m\rangle, \\
\left(S_x^2 + S_y^2 + S_z^2\right)|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle.
\]

\(s\) takes value of half integers, and \(m\) is a half integer such that \(-j \leq m \leq j\).

Then after tedious calculations, we obtain

\[
\beta(\theta, \mathcal{M}_{s,m}) = \frac{m}{s^2 + s - m^2}.
\]

If \(m = \alpha s\), where \(\alpha < 1\) is a constant, \(\beta(\theta, \mathcal{M}_{s,m})\) tends to zero as \(s \to \infty\), and the model \(\mathcal{M}_{s,m}\) becomes quasi-classical. However, if \(m = s\), the model \(\mathcal{M}_{s,m}\) is coherent for any \(s\).

**Example** (shifted number state model) *shifted number state model*, which has four parameters, is defined by

\[
\mathcal{M}_n = \pi(N_n), \\
\mathcal{N}_n = \left\{ |\phi(\theta)\rangle \mid |\phi(\theta)\rangle = D(\theta)|n\rangle, \theta \in \mathbb{R}^2 \right\},
\]

where letting \(P, X\) be the momentum operator and the position operator respectively,

\[
D(\theta) \equiv \exp\frac{i}{\hbar}\left(-\theta^1 X + \theta^2 P\right),
\]

and \(|n\rangle\) is the \(n\)th eigenstate of the harmonic oscillator,

\[
H = -\frac{1}{2}P^2 + \frac{1}{2}X^2.
\]

After some calculations, we have

\[
\beta(\theta, \mathcal{M}_n) = \frac{-1}{2n+1}.
\]

As \(n\) tends to infinity, \(\beta(\theta, \mathcal{M}_n)\) goes to 0 and the model becomes quasi-classical.
9 Informational exclusiveness and independence, and direct sum of the models

In a \( m \)-dimensional model \( \mathcal{M} \), we say parameter \( \theta^i \) and \( \theta^j \) are informationally independent at \( \theta_0 \), iff

\[
\text{Re}\langle l_i | l_j \rangle|_{\theta=\theta_0} = \text{Im}\langle l_i | l_j \rangle|_{\theta=\theta_0} = 0,
\]

because, if the equation holds true, letting the submodels \( \mathcal{M}(1|\theta_0) \), \( \mathcal{M}(2|\theta_0) \) and \( \mathcal{M}(1,2|\theta_0) \) of \( \mathcal{M} \), be

\[
\begin{align*}
\mathcal{M}(1|\theta_0) &\equiv \{ \rho(\theta) \mid \theta = (\theta^1, \theta^2, \ldots, \theta^m), \theta^1 \in \mathbb{R} \}, \\
\mathcal{M}(2|\theta_0) &\equiv \{ \rho(\theta) \mid \theta = (\theta^1, \theta^2, \ldots, \theta^m), \theta^2 \in \mathbb{R} \}, \\
\mathcal{M}(1,2|\theta_0) &\equiv \{ \rho(\theta) \mid \theta = (\theta^1, \theta^2, \theta^3, \ldots, \theta^m), (\theta^1, \theta^2) \in \mathbb{R}^2 \},
\end{align*}
\]

the following equality establishes:

\[
\text{CR}(\theta_0, \text{diag}(g_1, g_2), \mathcal{M}(1,2|\theta_0)) = \text{CR}(\theta_0, \text{diag}(g_1), \mathcal{M}(1|\theta_0)) + \text{CR}(\theta_0, \text{diag}(g_2), \mathcal{M}(2|\theta_0)),
\]

which means that in the simultaneous estimation of the parameter \( (\theta^1, \theta^2) \), both of the parameters can be estimated without the loss of information compared with the estimation of each parameters.

On the other hand, iff

\[
\text{Re}\langle l_1 | l_2 \rangle|_{\theta=\theta_0} = 0,
\]

and \( \mathcal{M}(1,2|\theta_0) \) is coherent, or equivalently,

\[
\text{Im}\langle l_1 | l_2 \rangle|_{\theta=\theta_0} = (\langle l_1 | l_1 \rangle \langle l_2 | l_2 \rangle)^{1/2}\bigg|_{\theta=\theta_0}
\]

hold true, we say the parameters are informationally exclusive at \( \theta_0 \), because of the following theorem.

**Theorem 12** Let \( \theta^1 \) and \( \theta^2 \) be informationally exclusive parameters at \( \theta_0 \), and \( \mathcal{M}' \) a measurement which takes value in \( \mathbb{R}^2 \) and satisfies local unbiasedness condition about \( \theta^1 \) at \( \theta_0 \),

\[
\begin{align*}
\int \hat{\theta}^1 \text{Tr} \rho(\theta_0) \mathcal{M}'(d\hat{\theta}) &= \theta^1_0, \\
\int \hat{\theta}^1 \text{Tr} \left. \frac{\partial \rho}{\partial \theta^2} \right|_{\theta=\theta_0} \mathcal{M}'(d\hat{\theta}) &= 0
\end{align*}
\]
If the measurement $M'$ satisfies i.e.,
\[
\int (\hat{\theta}^1 - \theta_0^1)^2 \text{Tr}(\rho(\theta_0)M'(d\hat{\theta})) = \text{CR}(\text{diag}(1,0), \theta_0) = [J^{S-1}]^{11},
\]
$M'$ can extract no information about $\theta^2$ from the system, i.e.,
\[
\forall B \subset \mathbb{R}^2 \text{ Tr} \left( M'(B) \frac{\partial \rho}{\partial \theta^2} \bigg|_{\theta = \theta_0} \right) = 0,
\]
and vice versa.

**Proof** Let $E$ be a Naimark’s dilation of the measurement $M'$, and decompose the estimation vector $|x\rangle \equiv |x[E, |\phi(\theta_0)\rangle]\rangle$ of $E$ as
\[
|x\rangle = z|\phi(\theta_0)\rangle + w|l_1(\theta_0)\rangle + |\psi\rangle,
\]
where $|\psi\rangle$ is orthogonal to both of $|\phi(\theta_0)\rangle$ and $|l_1(\theta_0)\rangle$. Then, local unbiasedness condition (50) leads to $z = 0$ and $w = [J^{S-1}]^{11} |\psi\rangle$. $|\psi\rangle$ must be the zero vector for $M'$ to achieve the the equality (51), because the variance of $M'$ writes
\[
\int (\hat{\theta}^1 - \theta_0^1)^2 \text{Tr}(\rho(\theta_0)M'(d\hat{\theta})) = \langle x|x\rangle = [J^{S-1}]^{11} + \langle \psi|\psi\rangle.
\]

Using the fact that by virtue of informational exclusiveness, $|l_2(\theta_0)\rangle$ writes
\[
|l_2(\theta_0)\rangle = ia|l_1(\theta_0)\rangle,
\]
where $i$ is the imaginary unit and $a$ a real number, we can check the equality (52) by the following calculations:
\[
\text{Tr} \left( M'(B) \frac{\partial \rho}{\partial \theta^2} \bigg|_{\theta = \theta_0} \right) = \text{Re} \langle \phi(\theta_0)|E(B)|l_2\rangle = -a \text{Im} \langle \phi(\theta_0)|E(B)|l_1\rangle = -a [J^S]_{11} \text{ Im} \langle \phi(\theta_0)|E(B) \int (\hat{\theta}^1 - \theta_0^1)E(d\hat{\theta})|\phi(\theta_0)\rangle = 0.
\]
Fujiwara and Nagaoka showed that in the 2-dimensional model with the informationally exclusive parameters, the best strategy for the estimation is alternative application of the best measurement for each parameter to the system. This fact is quite natural in the light of theorem 12.

For the submodels

\[ M_1 \equiv M(1, 2, \ldots, m_1 | \theta_0), \quad M_2 \equiv M(m_1, m_1 + 1, \ldots, m | \theta_0) \]

of \( M \), which are defined almost in the same way as the definition (48) of \( M(1, 2 | \theta_0) \), we say that \( M \) is the sum of \( M_1 \) and \( M_2 \) at \( \theta_0 \), and express the fact by the notation,

\[ M|_{\theta_0} = M_1 \oplus M_2|_{\theta_0}. \]

Throughout the section, \( m - m_1 \) is denoted by \( m_2 \).

**Lemma 9** If any parameter of \( M_1 \) is informationally independent of any parameter of \( M_2 \) at \( \theta_0 \), and the weight matrix \( G \) writes

\[ G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \]

then

\[ \text{CR}(G, \theta_0, M) = \text{CR}(G_1, \theta_0, M_1) + \text{CR}(G_2, \theta_0, M_2) \]

When the premise of the lemma is satisfied, \( M_1 \) and \( M_2 \) are said to be informationally independent at \( \theta_0 \).

**Proof** Let \( M \) be a locally unbiased measurement in \( M \), and define the measurements \( M_j \) (\( j = 1, 2 \)) in \( \mathbb{R}^{m_j} \) (\( j = 1, 2 \)) by

\[ M_1(B) \equiv M(B \times \mathbb{R}^{m_2}) (B \in \mathbb{R}^{m_1}), \]
\[ M_2(B') \equiv M(\mathbb{R}^{m_1} \times B') (B' \in \mathbb{R}^{m_2}). \]
respectively. Then, the measurement $M_j (j = 1, 2)$ is locally unbiased in $\mathcal{M}_j (j = 1, 2)$, respectively.

Therefore, we have
\[
\inf \{ \text{Tr} GV[M] \mid M \text{ is locally unbiased in } \mathcal{M} \}
\]
\[
= \inf \{ \text{Tr} G_1 V[M_1] \mid M \text{ is locally unbiased in } \mathcal{M} \}
+ \inf \{ \text{Tr} G_2 V[M_2] \mid M \text{ is locally unbiased in } \mathcal{M} \}
\geq \inf \{ \text{Tr} G_1 V[M'] \mid M' \text{ is locally unbiased in } \mathcal{M}_1 \}
+ \inf \{ \text{Tr} G_2 V[M''] \mid M'' \text{ is locally unbiased in } \mathcal{M}_2 \}
\]
or its equivalence,
\[
\text{CR}(G, \mathcal{M}) \geq \text{CR}(G_1, \mathcal{M}_1) + \text{CR}(G_2, \mathcal{M}_2).
\] (53)

Because $\mathcal{M}_1$ and $\mathcal{M}_2$ are informationally independent, $L$ for $\mathcal{M}$ writes
\[
L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix},
\]
in the appropriate coordinate, where $L_1 = [l_1, ..., l_{m_1}]$, and $L_2 = [l_{m_1+1}, ..., l_m]$.

In that coordinate, let us write $X$ as
\[
X = \begin{bmatrix} X_1 & X_{12} \\ X_{21} & X_2 \end{bmatrix}.
\]

Then
\[
\text{Re} X_i^* L_i = I_{m_i}, \quad \text{Im} X_i^* X_i = 0 \quad (i = 1, 2), \quad X_{12} = X_{21} = 0,
\]
is a sufficient condition for the measurements corresponding to $X$ to be locally unbiased. Therefore, we have
\[
\text{CR}(G, \mathcal{M})
\]
\[
= \inf \{ \text{Tr} GX^* X \mid \text{Re} X^* L = I_m, \text{Im} X^* X = 0 \}
\]
\[
\leq \inf \left\{ \sum_{i=1}^{2} \text{Tr} G_i X_i^* X_i \mid \text{Re} X_i^* L_i = I_{m_i}, \text{Im} X_i^* X_i = 0, \quad (i = 1, 2) \right\}
\]
\[
= \text{CR}(G_1, \mathcal{M}_1) + \text{CR}(G_2, \mathcal{M}_2),
\]
which, mixed with (53) leads to the lemma. \qed
10 Manifestation of complex structure

It is worthy of notice that $|\beta|$, which was shown to be a good index of 'uncertainty' in the case of the 2-dimensional model, is deeply related to the natural complex structure in $\mathcal{P}_1$.

Let us define the linear transform $D$ in $T_\rho(\mathcal{M})$ as follows; First, multiply the imaginary unit $i$ to $|l_X\rangle$. In general, however, $i|l_X\rangle$ is not an element of $\text{span}_R L$, and does not represent any of vectors in $T_\rho(\mathcal{M})$. Hence, we project $i|l_X\rangle$ onto $\text{span}_R L$ with respect to the inner product $\text{Re}\langle\ast|\ast\rangle$, and the image by $\pi_*$ of the product of the projection is defined to be $DX \in T_\rho(\mathcal{M})$, where $\pi_*$ is the differential map of $\pi$.

By elementary linear algebra, it is shown that the matrix which corresponds to $D$ is $JS^{-1}J$, and that, in the 2-dimensional model, its eigenvalues are $\pm i\beta$.

The definition of the map $D$ naturally leads to the following theorems.

**Theorem 13** The absolute value of the eigenvalue of $D$, or equivalently, of $JS^{-1}J$, is smaller than or equal to 1.

Is the eigenvalues of the linear map $D$ a good measure of ‘uncertainty’ in the arbitrary dimensional model? If all of eigenvalues of $D$ vanish, as is shown in the section 4, the model is quasi-classical, and ‘uncertainty’ among parameters vanishes. When eigenvalues of $D$ do not vanish, we have the following theorem.
Theorem 14  For any pure state model,

\[
\inf \{ \text{Tr} J^{S^{-1}} V \mid V \in \mathcal{V} \} = \text{Tr} \left\{ \text{Re} \sqrt{I_m + i \sqrt{J^{S^{-1}}}, \tilde{J} \sqrt{J^{S^{-1}}}} \right\}^{-2} = \sum_{\beta: \text{eigenvalues of } D^2} \frac{2}{1 + \sqrt{1 - |\beta|^2}}.
\]

The estimation theoretical meaning of \( \min \{ \text{Tr} J^{S^{-1}} V \mid V \in \mathcal{V} \} \) is hard to verify. However, this value remains invariant under any transform of the coordinate in the model \( \mathcal{M} \), and can be an index of distance between \( \mathcal{V} \) and \( J^{S^{-1}} \).

Proof  Because \( \min \{ \text{Tr} J^{S^{-1}} V \mid V \in \mathcal{V} \} \) is invariant by any affine coordinate transform in the model \( \mathcal{M} \), we choose a coordinate in which \( J^{S^{-1}} \) writes \( I_m \) and \( \tilde{J} \) writes

\[
\tilde{J} = \begin{bmatrix}
0 & -\beta_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\beta_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & -\beta_l & 0 & \cdots \\
0 & 0 & \cdots & 0 & \beta_l & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Then, The model \( \mathcal{M} \) is decomposed into the direct sum of the submodels one or two dimensional \( \mathcal{M}_\kappa \),

\[
\mathcal{M} = \bigoplus_\kappa \mathcal{M}_\kappa,
\]

where any two submodels \( \mathcal{M}_\kappa \) and \( \mathcal{M}_{\kappa'} \) are informationally independent, and \( \tilde{J} \) of a two dimensional submodel \( \mathcal{M}_\kappa \) is

\[
\begin{bmatrix}
0 & -\beta_\kappa \\
\beta_\kappa & 0
\end{bmatrix}.
\]
Therefore, by virtue of lemma 9 and the equation (45), we have the theorem. □

11 The coherent model

As for the model with arbitrary dimensions, the model is said to be coherent at \( \theta \) iff all of the eigenvalues of \( (J^S)^{-1}\tilde{J} \) are \( \pm i \). When the model is 2-dimensional, this definition of coherency reduces to \( |\beta| = 1 \). The dimension of the coherent model is even, for the eigenvalues of \( J^S^{-1}\tilde{J} \) are of the form \( \pm i\beta_j \) or 0.

In this section, we determine the attainable CR type bound of the coherent model. The coherent model is worthy of attention firstly because the coherent model is ‘the maximal uncertainty’ model, secondly because there are several physically important models which are coherent.

The definition of the map \( D \) leads to the following theorem.

**Lemma 10** The model \( \mathcal{M} \) is coherent at \( \theta \) iff \( \text{span}_R \{iL\} \) is identical to \( \text{span}_R L \), or equivalently, iff \( \text{span}_R \{L, iL\} \) is identical to \( \text{span}_R L \).

This lemma leads to the following lemma.

**Lemma 11** The model \( \mathcal{M} \) is coherent iff the dimension of \( \text{span}_C L \) is \( m/2 \).

**Proof** First, we assume that

\[
\dim_C \text{span}_C L = m/2. \tag{54}
\]

Because \( \text{span}_R L \) is a \( m \)-dimensional subspace of \( \text{span}_R \{L, iL\} \) whose dimension is smaller than or equal to \( m \) because of (54), we have \( \text{span}_R \{L, iL\} = \text{span}_R L \), or coherency of the model.

Conversely, let us assume that the model is coherent. Taking an orthonormal basis \( \{e_j | j = 1, \ldots, m\} \) of \( \text{span}_R L \) such that \( e_{j+m/2} = De_j \), horizontal lifts \( \{|j| | j = 1, \ldots, m\} \) of \( \{e_j | j = 1, \ldots, m\} \) satisfy \( |j + m/2| = i|j| \), and
any element $|u\rangle$ of $\text{span}_{\mathbb{R}} L = \text{span}_{\mathbb{R}} \{ L, iL \}$ writes

$$|u\rangle = \sum_{j=1}^{m} a_j |j\rangle$$

$$= \sum_{j=1}^{m/2} (a_j + ia_j+(m/2)) |j\rangle,$$

implying that the dimension of $\text{span}_{\mathbb{C}} L$ is $m/2$. 

Fujiiwara and Nagaoka [4] determined the attainable CR type bound of the two parameter coherent model. In the following, more generally, we work on the bound of the coherent model with arbitrary dimension. Throughout the section, the weight matrix $G$ is assumed to be strictly positive.

When the model is coherent, $\text{Re} L^* X = I_m$ or equivalently $\text{Re} L^* (X - L J^{S-1}) = 0$, implies, by virtue of $\text{span}_{\mathbb{R}} L = \text{span}_{\mathbb{R}} \{ iL, L \}$,

$$L^* (X - L J^{S-1}) = 0,$$

or equivalently,

$$L^* X = L^* L J^{S-1} = I_m + i \tilde{J} J^{S-1}. \quad (55)$$

Multiplication of $L^*$ to the both sides of (33), together with the equation (55), yields

$$(I_m + i \tilde{J} J^{S-1})(G - i \Lambda) = (J^S + i \tilde{J}) VG. \quad (56)$$

By virtue of the coherency, both of the real part and the imaginary part of (56) give the same equation,

$$G + \tilde{J} J^{S-1} \Lambda = J^S VG,$$

or

$$\sqrt{G} V \sqrt{G} - \sqrt{G} J^{S-1} \sqrt{G} = \left( \sqrt{G} J^{S-1} \tilde{J} J^{S-1} \sqrt{G} \right) \left( G^{-1/2} \Lambda G^{-1/2} \right). \quad (57)$$

The antisymmetric part of the both hands of the equation yields

$$\left[ \sqrt{G} J^{S-1} \tilde{J} J^{S-1} \sqrt{G}; G^{-1/2} \Lambda G^{-1/2} \right] = 0.$$
Therefore, letting $a_i$ and $b_i$ denote the eigenvalues of $\sqrt{G}J^{S-1}\bar{J}J^{S-1}\sqrt{G}$ and $G^{-1/2}\Lambda G^{-1/2}$ respectively, we have

$$\text{Tr} \left( \sqrt{G}J^{S-1}\bar{J}J^{S-1}\sqrt{G} \right) \left( G^{-1/2}\Lambda G^{-1/2} \right)
= \sum_i a_i b_i = \sum_i |a_i||b_i|,$$

where the last equality is valid because the left hand side of the equation (57) is positive symmetricity virtue of the SLD CR inequality.

On the other hand, from (33) or its equivalence,

$$X\sqrt{G} \left( I_m - iG^{-1/2}\Lambda G^{-1/2} \right) = LV\sqrt{G}, \quad (58)$$

we can deduce $|b_i| = 1 \ (i = 1, \ldots, m)$ as in the follows.

The rank of the right hand side of (58) is equal to $m/2$ because $G$ is strictly positive and

$$\text{rank} L = \dim \text{span}_G L = m/2.$$

On the other hand,

$$\text{rank} X = \dim \text{span}_G X = \dim \text{span}_R X = m,$$

where the second equality comes from $\text{Im}X^*X = 0$ and the last equality comes from $\text{Re}X^*L = I_m$. Therefore, the rank of the matrix $I_m - iG^{-1/2}\Lambda G^{-1/2}$ must be $m/2$, and the eigenvalues of $G^{-1/2}\Lambda G^{-1/2}$ are $\pm i$.

After all, we have

$$\min_{V \in \mathcal{V}(M)} \text{Tr} GV = \text{Tr}GJ^{S-1} + \text{Trabs}GJ^{S-1}\bar{J}J^{S-1},$$

where $\text{Trabs}A$ means the sum of the absolute values of the eigenvalues of the matrix $A$. When the minimum is attained, the covariance matrix $V$ is given by

$$V = J^{S-1} + G^{-1/2}|G|^{1/2}J^{S-1}\bar{J}J^{S-1}G^{-1/2}G^{-1/2},$$

where $|A| = (AA^*)^{1/2}$.

To check the coherency of the model, the following theorem, which is deduced from theorem [13], is useful.
**Theorem 15** The model is coherent at $\theta$ iff

$$|\det J^S| = |\det \tilde{J}|.$$ 

**Example** (squeezed state model) Squeezed state model, which has four parameters, is defined by

$$\mathcal{M} = \{ \rho(z, \xi) \mid \rho(z, \xi) = |z, \xi\rangle\langle z, \xi|, z, \xi \in \mathbb{C} \},$$

where

$$|z, \xi\rangle = D(z)S(\xi)|0\rangle,$$

$$D(z) = \exp(za^\dagger - \overline{z}a),$$

$$S(\xi) = \exp \left( \frac{1}{2}(\xi a^\dagger a - \overline{\xi}a^2) \right).$$

Letting $z = (\theta^1 + i\theta^2)/2^{1/2}$, $Q = (a + a^\dagger)/2^{1/2}$, and $\xi = \theta^3 e^{-2i\theta^4}$ ($0 \leq \theta^3$, $0 \leq \theta^4 < 2\pi$), we have

$$J^S = \frac{1}{2} \begin{bmatrix} \cosh 2\theta^3 - \sinh 2\theta^3 \cos 2\theta^4 & \sinh 2\theta^3 \sin 2\theta^4 & 0 & 0 \\ \sinh 2\theta^3 \sin 2\theta^4 & \cosh 2\theta^3 + \sinh 2\theta^3 \cos 2\theta^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sinh^2 2\theta^3 \end{bmatrix},$$

$$\tilde{J} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sinh 2\theta^3 \\ 0 & 0 & \sinh 2\theta^3 & 0 \end{bmatrix}.$$ 

Coherency of this model is easily checked by theorem 15,

$$|\det J^S| = |\det \tilde{J}| = \frac{1}{4} \sinh^2 2\theta^3.$$ 

**Example** (spin coherent model) As is pointed out by Fujiwara [4], spin coherent model $\mathcal{M}_{s,s}$, which is a special case of spin rotation model (47), is coherent.
**Example** (total space model) The total space model is the space of all the pure state $P$ in finite dimensional Hilbert space $\mathcal{H}$. By virtue of theorem 10, the coherency of the model is proved by checking that span$_{\mathbb{R}}L$ is invariant by the multiplication of the imaginary unit $i$. Let $|l\rangle$ be a horizontal lift of a tangent vector at $|\phi\rangle$. Then, $i|l\rangle$ is also a horizontal lift of another tangent vector at $|\phi\rangle$, because $|\phi\rangle + i|l\rangle dt$ is an element of $\mathcal{H}$ with unit length.

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**A  proof of lemma 3**

**Proof** Let $E^{(i)}$ be a projection valued measure such that,

$$\int_{\mathbb{R}} x E^{(i)}(dx) = \sum_{j=1}^{m} \left[J^{S-1}(\theta)\right]^{ij} L_{j}^{S}(\theta),$$

and $M_p$ be an unbiased measurement at $\theta$ such that

$$M_p \left(\{\theta^1\} \times ... \times \left[\theta^i + \frac{x}{p_i}, \theta^i + \frac{x + \Delta x}{p_i}\right] \times ... \times \{\theta^m\}\right) = p_i E^{(i)}([x, x + \Delta x]),$$

$$M_p \left(\mathcal{B}_1 \times ... \times \left(\mathbb{R}/\{\theta^1\}\right) \times ... \times \mathcal{B}_m\right) = 0$$

where $p = [p_i]$ $(i = 1, ..., m)$ is a real vector such that $\sum p_i = 1$ and $p_i \geq 0$, and $\mathcal{B}_i (i = 1, ..., m)$ are arbitrary measurable subset of $\mathbb{R}$. Then, we have for any $p$,

$$[V_\theta[M_p]]_{ii} = \frac{1}{p_i} \left[J^{S-1}(\theta)\right]^{ii},$$

which leads to

$$\inf \{ [V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \}$$

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\[
\inf \left\{ [V_\theta[M_p]]_{ii} \left| \sum_i p_i = 1, p_i \geq 0 \right. \right\} \\
= \left[ J^{S-1}(\theta) \right]_{ii}.
\]

On the other hand, SLD CR inequality leads to
\[
\inf \{ [V_\theta[M]]_{ii} \mid M \text{ is locally unbiased at } \theta \} \geq \left[ J^{S-1}(\theta) \right]_{ii},
\]
and we have the lemma. \qed

\section{proof of lemma 5}

\begin{proof}
Let \( M \) be a locally unbiased measurement, and \( v_\alpha = (v_1^\alpha, \ldots, v_m^\alpha) \) denote \( \sqrt{V_0} \alpha \), where \( \alpha \) is a vector whose components are 1 or \(-1\). Then, the measurement \( M' \), which is defined by
\[
M' \left( \prod_{i=1}^m ([a_i - v_\alpha^i, b_i - v_\alpha^i] \cup [a_i + v_\alpha^i, b_i + v_\alpha^i]) \right) = \frac{1}{2m} M \left( \prod_{i=1}^m [a_i, b_i] \right)
\]
is also locally unbiased and its covariance matrix is,
\[
V[M'] = V[M] + V_0.
\]
\end{proof}

\section{proof of lemma 7}

\begin{proof}
The equation (21) and the equation (28) implies that, for any element \( V \) of \( \mathcal{V} \), there is a \( m \times 2m \) matrix \( U \) which satisfies
\[
V^{-1} = V^{-1}(U) \equiv (\text{Re}UL)^T \text{Re}UL,
\]
and
\[
U^*U = I_m.
\]
\end{proof}
Because the map $V^{-1}(\ast)$ is continuous and the totality of the $m \times 2m$ matrix $U$ satisfying (59) is compact, the region of $V^{-1}(U)$ is compact. Therefore, the intersection of $V$ and the set

$$\{ A \mid A \leq V_0 \}.$$ 

is compact for any real symmetric matrix $V_0$, for the map

$$V^{-1} \rightarrow V$$

is continuous on the intersection of the region of $V^{-1}(U)$ and the set

$$\{ A \mid A \geq V^{-1}_0 \},$$

both of which are compact. Because $V_0$ is an arbitrary real symmetric matrix, we have the lemma. \hfill \square

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