Electromagnetic Duality Based on Axiomatic Maxwell Equations

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Abstract

No positive result has been obtained on the magnetic monopoles search. This allows to consider different theoretical approaches as the proposed in this paper, developed in the framework of the Einstein General Relativity. The properties of second rank skew-symmetrical fields are the basis of electromagnetic theories. In the space-time the Hodge duality of these fields is narrowly related with the rotations in the SO(2) group. An axiomatic approach to a dual electromagnetic theory is presented. The main result of this paper is that the stress-energy tensor can be decomposed on two parts: the parallel and the perpendicular. The parallel part is easily integrated on the Lagrangian approach, while some problems appears with the perpendicular part. A solution with the parallel part alone is found, it generates a non-standard model of magnetic monopoles neutral to the electric charges.

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1 Introduction

The concept of magnetic monopole extends the meaning of the Maxwell equations, by providing a symmetric view of electric and magnetic fields. The asymmetry of these equations has been for a long time a motivational phenomenon in both the theoretical\cite{1,2} and experimental\cite{3} aspects. According with Hawking and Ross\cite{4}, the belief that electrically and magnetically charged black holes have identical quantum properties provides a new interest in all questions related with electromagnetic duality\cite{5,6,7,8}. Additionally, as asserted by Lee\cite{5}, the magnetic monopole dynamic is related to strongly interacting elementary particles. Most of the works in electromagnetic duality are dealing with quantum aspect, while as appointed by Israelit\cite{9,10}, little attention is considered to construct a correct classical theory in the General Relativistic arena. The classical theory can be considered as a limit of quantum theory, therefore advances in the first can be useful to the second\cite{11}. The construction of a classical theory of electromagnetic duality is not a closed matter. A pending problem deals with the duality invariance of both the stress-energy tensor and the Lagrange action\cite{4,8,12}.

In classical field theory, if the electromagnetic field is constructed from a potential vector as: $F_{ab} = 2\nabla_{[a}A_{b]}$, the vanishing of the second set of Maxwell equations, $\nabla_{[a}F_{bc]} = 0$, is due to a property of the Riemann curvature tensor: $R^{d}_{[abc]} = 0$. The definition of the field or the affine properties of the space-time must be changed to obtain a full electromagnetic duality with symmetrical Maxwell equations. First type of theories can include different field definitions from the potential vector, as in the Yang-Mills model based on gauge theories\cite{6,13,14}. Second type of theories can include torsion properties in the space-time, as based on Weil geometry\cite{9,15}. Any theory which modifies some of these aspects can generate a non-vanishing second set of Maxwell equations, and consequently some type of magnetic monopolar current.

Electromagnetic duality is related with the symmetrical role of electric and magnetic fields in the Maxwell equations. The Hodge duality is an useful tool to deal with this equation symmetry. It becomes a cornerstone to connect the physical and the mathematical concepts of duality. The double duality transformation in four-dimensional space-time generates an identity with negative sign. From far, this
fact has been qualitatively associated with the rotation in a plane \[12\] (p 108). Recently an increased interest arises in the symplectic structure of the electromagnetic equations \[17, 18\] and also in the SO(2) symmetrical group embedded in the duality \[19, 20, 21, 22\]. Chan and Tsou \[23, 24\] show some of the problems with duality in the classical theory, and extend it to nonabelian gauge theory.

In the classical domain of General Relativity the electromagnetic field can be represented by the skew-symmetrical strength tensor \(F_{ab}\), while in a quantum domain it must be potential based due to the Bohm-Aharonov effect \[14, 24, 25\]. In this paper a non-quantum approach is presented, the electromagnetic duality is obtained from the formal properties of the skew-symmetrical tensors rather than induced from an a priori magnetic monopole model.

The Hodge duality is a cornerstone of electromagnetic duality. On a space-time with \(D = 4k\) dimensionality, it generates a \(\text{SO}(2)\) symmetrical group \[19, 21\]. The other cornerstone is the Poincaré Lemma. It defines the necessary and sufficient condition for the existence of solutions involving differential forms. In this paper the solution to the general symmetrical Maxwell equations is fragmented on pieces based on closed forms. This is achieved using a two potential approach also previously in electromagnetic duality \[28, 35\].

The plan of this paper is as follows. The Sec. 2 contains most of the mathematical tools used along the paper. It is being presented in a formal way comprising a set of definitions, propositions and theorems, including also most of the deductive machinery. This section is partially an experiment in order to find a reduced set of mathematical elements needed to construct the basis of an axiomatic electromagnetic theory. The Sec. 3 presents the equations of the electromagnetic field as are deduced from the axiomatic. The Sec. 4 deals with the duality invariance, including an invariant formulation for the Maxwell equations and the stress-energy tensor. The Sec. 5 presents a model of Lagrange function. Finally, The Sec. 6 is related with the particle-field and particle-particle interactions.

### 2 The Axiomatic of Class \(\mathcal{E}M\)

The tensor class \(\mathcal{E}M\) is described as a tensor family with a common property. A set of definitions, propositions and theorems are formally presented, all are related with equations involving second rank skew-symmetric tensors. This section is linearly presented, such as most of the propositions and theorem become immediately proved from previous assertions. As in General Relativity a \((M, g_{ab})\) model of the space-time is considered, where \(M\) is a four-dimensional \(C^\infty\) connected manifold and \(g_{ab}\) is a metric on \(M\) with signature: \(\{-1, 1, 1, 1\}\). A p-form and a skew-symmetric tensor of rank p are equivalent representations used along this section as:

\[
X = \frac{1}{p!} X_{a_1 \ldots a_p} w^{a_1} \wedge \cdots \wedge w^{a_p} \Leftrightarrow X_{[a_1 \ldots a_p]} \tag{1}
\]

in a general basis, while for a coordinate basis \(w^k = dx^k\). The wedge product \(\wedge\) of a p-form and a q-form is a \((p+q)\)-form which verifies: \(X \wedge Y = (-1)^p g^{pq} Y \wedge X\). The dual of a p-form, \(X\), is a \((n-p)\)-form, \(*X\), where \(n = 4\) and \(0 \leq p \leq n\). It is defined as \[16\] (p 88):

\[
*X = \frac{1}{(n-p)!} \left[ \frac{1}{p!} \epsilon_{a_1 \ldots a_p} X^{a_1 \ldots a_p} \right] w^{a_{p+1}} \wedge \cdots \wedge w^{a_n} \tag{2}
\]

where \(\epsilon_{abcd}\) is the Levi-Civita fourth rank completely skew-symmetric tensor, with \(\epsilon_{0123} = \sqrt{-g}\), and \(g\) is the metric determinant. Throughout this paper the dual of first, second and third rank tensors will be considered, based on the previous expression they are defined as \[16\] (p 88):

\[
*A_{abcd} = \epsilon_{abcd} A^a; \quad *B_{cd} = \frac{1}{2} \epsilon_{abcd} B^{ab}; \quad *C_d = \frac{1}{3!} \epsilon_{abcd} C^{abc} \tag{3}
\]

The double duality verifies that: \(**X = (-1)^{p-1} X. It is verified that: **A = A, **B = -B and **C = C.
2.1 Linear Expressions

For an 1-form \( U = U_\alpha w^\alpha \) the exterior derivative can be expressed from the tensorial derivative as:
\[
(d \wedge U)_{ab} = (d \otimes U)_{ab} - (d \otimes U)_{ba},
\]
so if: \( d \otimes U = (\nabla_a U_b - \nabla_b U_a)w^a \wedge w^b \), it is obtained that:
\[
d \wedge U = dU = (\nabla_a U_b - \nabla_b U_a)w^a \wedge w^b
\]  
(4)

For the general case, the derivative of a p-form is a \((p+1)\)-form expressed in terms of the covariant derivative:
\[
Y = dX \Leftrightarrow Y_{\alpha_1 \ldots \alpha_p} = \nabla_{\alpha_1} X_{\alpha_2 \ldots \alpha_p} - \nabla_{\alpha_2} X_{\alpha_1 \alpha_3 \ldots \alpha_p} + \cdots + (-1)^p \nabla_{\alpha_p} X_{\alpha_1 \ldots \alpha_{p-1}}
\]  
(5)

For a scalar \( \phi \), an 1-form \( U \) and a 2-form \( X \), it is verified that:
\[
d\phi \Leftrightarrow \nabla_a \phi
\]  
(6)
\[
dU \Leftrightarrow \nabla_a U_b - \nabla_b U_a = 2\nabla_{[a} U_{b]}  
\]  
(7)
\[
dX \Leftrightarrow \nabla_a X_{bc} - \nabla_b X_{ac} + \nabla_c X_{ab} = 3\nabla_{[a} X_{bc]}  
\]  
(8)

It is verified that \( d\delta = 0 \), that is \( dY = 0 \) in the equation \([3]\). The co-derivative \( \delta X \) of a p-form is a \((p-1)\)-form expressed as: \( \delta X = *d*X \) in the four-dimensional space-time \([3\, p\, 40] \). For an 1-form it is verified that:
\[
Z = \delta X \Leftrightarrow Z = -\nabla^a X_a
\]  
(9)

If \( \phi \) is a scalar: \( d\delta \phi \Leftrightarrow -\Box \phi \), where \( \Box = \nabla^a \nabla_a \) is the second order differential operator. Also for a 2-form:
\[
Z = \delta X \Leftrightarrow Z_a = -\nabla^b X_{ba}
\]  
(10)

If \( U \) is an 1-form in a curved space-time, it is verified that:
\[
\delta dU + d\delta U \Leftrightarrow -\Box U_a + R^b_a U_b
\]  
(11)

where \( R^b_a \) is the Ricci tensor. It is also verified that: \( \delta \delta = 0 \), that is \( \delta Z = 0 \) in the previous cases.

**Definition 1.** A second rank tensor \( X \in \mathcal{E}M_0 \) if it is skew-symmetric, it means that: \( X_{ab} + X_{ba} = 0 \).

**Definition 2.** A tensor \( X \in \mathcal{E}M_1 \), also called an \( \alpha \)-field, if \( X \in \mathcal{E}M_0 \) and also it verifies: \( dX = 0 \), equivalent to: \( \nabla_{[a} X_{bc]} = 0 \).

**Definition 3.** A tensor \( X \in \mathcal{E}M_2 \), also called a \( \beta \)-field, if \( X \in \mathcal{E}M_0 \) and also it verifies: \( \delta X = 0 \), equivalent to: \( \nabla_b X^{ab} = 0 \).

These classes verify that: \( \mathcal{E}M_1 \subset \mathcal{E}M_0 \) and \( \mathcal{E}M_2 \subset \mathcal{E}M_0 \). Two new classes can be introduced, the first as intersection of the previous, \( \mathcal{E}M_3 = \mathcal{E}M_1 \cap \mathcal{E}M_2 \), which verifies that: \( \nabla_{[a} X_{bc]} = 0 \) and \( \nabla_b X^{ab} = 0 \). The other class is: \( \mathcal{E}M_4 = \mathcal{E}M_0 - \mathcal{E}M_1 \cup \mathcal{E}M_2 \), which verifies that: \( \nabla_{[a} X_{bc]} \neq 0 \) and \( \nabla_b X^{ab} \neq 0 \).

**Proposition 1.** If \( X \in \mathcal{E}M_0 \), the third rank tensor \( 3\nabla_{[a} X_{bc]} \) and the vector \( (-1)\nabla_b X^{ab} \) are both dual. Reciprocally, the tensor \( 3\nabla_{[a} *X_{bc]} \) and the vector \( \nabla_b X^{ab} \) are also dual.

**Proposition 2.** If \( X \in \mathcal{E}M_1 \), then \( *X \in \mathcal{E}M_2 \) and reciprocally.

**Proof.** If \( X \in \mathcal{E}M_1 \) then \( dX = 0 \), therefore it is verified that: \( *dX = d**X = -*dX = 0 \). The reciprocal is also right, if \( X \in \mathcal{E}M_2 \) then \( \delta X = 0 \), therefore it is verified that: \( **d*X = (-1)^2d*X = 0 \).

**Definition 4.** If \( X \in \mathcal{E}M_0 \), and the vector \( Z \) is defined as: \( Z = \delta X \), that is: \( Z_a = \nabla^b X_{ab} \), it is called the \( \alpha \)-current of \( X \), and it verifies that: \( \delta Z = 0 \).
Definition 5. If $X \in \mathcal{E}M_0$, and the vector $Y$ is defined as: $Y = \delta^* X$, that is: $Y_a = \nabla^b \ast X_{ab}$, it is called the $\beta$-current of $X$, and it verifies that: $\delta Y = 0$.

Proposition 3. If $\{Z, Y\}$ are the $\alpha$ and $\beta$-current of $X$, then $\{Y, -Z\}$ are the $\alpha$ and $\beta$-current of $\ast X$.

Theorem 1 (Maxwell-like). Any tensor $X \in \mathcal{E}M_0$ verifies the next equations, where $Z$ and $Y$ are its $\alpha$ and $\beta$-current respectively.

$$\delta X = Z \quad \delta^* X = Y$$ (12)

verifying that $\delta Z = 0$ and $\delta Y = 0$.

From Proposition 3 these equations can be expressed in tensorial form as:

$$\nabla_b X^{ab} = Z^a \quad \nabla_b [X_{cd}] = -\frac{1}{3} \epsilon_{abcd} Y^a$$ (13)

where $\nabla_a Z^a = 0$ and $\nabla_a Y^a = 0$. For the dual $\ast X$, it is verified that:

$$\nabla_b \ast X^{ab} = Y^a \quad \nabla_b [\ast X_{cd}] = \frac{1}{3} \epsilon_{abcd} Z^a$$ (14)

these become the symmetrical Maxwell-like equations system and the current conservation laws which are the starting point for the construction of the electromagnetic duality. The $\mathcal{E}M_1$ or the $\mathcal{E}M_2$ classes are obtained if $Y = 0$ or $Z = 0$.

Proposition 4. If $X \in \mathcal{E}M_1$, it can be expressed from a vector field $U$ as:

$$X = dU \leftrightarrow X_{ab} = \nabla_a U_b - \nabla_b U_a$$ (15)

Proof. It follows from the Poincaré Lemma [13] (p 21), [26] (p 141). In this case $X$ is closed, $dX = 0$, then it must be exact, $X = dU$. However, the vector field $U$ is not univocally defined, being possible a transformation as: $U' = U + d\phi$, where $\phi$ is a scalar field. It is verified that: $dU' = dU$, due to: $dd\phi = 0$. To fix this scalar it is imposed an additional equation as the Lorentz gauge: $\delta U = 0$. This restriction becomes:

$$\delta d\phi = 0 \Leftrightarrow -\Box \phi = 0$$ (16) \qed

Proposition 5. If $Z$ is a vector field verifying $\delta Z = 0$, then exist solution for $X \in \mathcal{E}M_1$ expressed as:

$$\delta X = Z \quad dX = 0$$ (17)

Proof. From the double duality it is concluded that if a p-form is null also is null its dual, so from $\delta Z = \ast d^* Z = 0$ it is concluded that $\ast Z$ is closed. Based on the Poincaré Lemma it must be exact, this is: $d^* X = \ast Z$, therefore exist $X$. This is not univocally defined, it is verified that: $X' = X + V$, where $V \in \mathcal{E}M_3$. \qed

Theorem 2 (Split). Any field $X \in \mathcal{E}M_0$ can be expressed by means of two auxiliary $\alpha$-fields, $\alpha X_{ab}$ and $\beta X_{ab}$, such as:

$$X_{ab} = \alpha X_{ab} - \ast \beta X_{ab}$$ (18)
$$\ast X_{ab} = \beta X_{ab} + \ast \alpha X_{ab}$$ (19)

where each auxiliary field verifies the following expressions based on the $\alpha$ and $\beta$ currents of $X$:

$$\nabla \cdot \alpha X = Z \quad \nabla \cdot \ast \alpha X = 0$$ (20)
$$\nabla \cdot \beta X = Y \quad \nabla \cdot \ast \beta X = 0$$ (21)
Proof. Firstly if \( X \in \mathcal{E}M_1 \) or \( X \in \mathcal{E}M_2 \) the theorem is immediately proved by choosing null any of the two auxiliary fields. The other case is \( X \in \mathcal{E}M_4 \) with non-null sources. The auxiliary fields \( \alpha X \) and \( \beta X \) are chosen according with the equations (20) and (21). From the previous Proposition it is proved that there are solution for both fields, however these solutions are not univocally defined. Therefore these solutions can verify that:

\[
D_{ab} = X_{ab} - \alpha X_{ab} + \beta X_{ab} \neq 0. \tag{22}
\]

however it is obtained that \( D \in \mathcal{E}M_3 \). Due to \( \mathcal{E}M_3 \subset \mathcal{E}M_1 \), it can be introduced a new tensor: 

\[
\alpha'X_{ab} = (\alpha X_{ab} + D_{ab}) \in \mathcal{E}M_1,
\]

consequently verifying that:

\[
X_{ab} = \alpha'X_{ab} - \beta'X_{ab}. \tag{24}
\]

Proposition 6. If \( A \) and \( B \) belong to \( \mathcal{E}M_1 \) and not to \( \mathcal{E}M_3 \), the solution for the equation:

\[
\lambda^*B + \sigma A = 0 \tag{23}
\]

where \( \lambda \) and \( \sigma \) are constant, is that: \( \lambda = \sigma = 0 \), i.e. \( A \) and \( *B \) are lineally independent.

Proof. From the previous conditions must be: \( dA = dB = 0 \) and also \( \delta A \neq 0 \) and \( \delta B \neq 0 \). However it is verified that from: \( \lambda d^*B + \sigma dA = 0 \), it must be: \( \lambda = 0 \). Also from: \( \lambda^*B + \sigma \delta A = 0 \), and due that \( \delta^*B = -dB = 0 \), it must be: \( \sigma = 0 \). Therefore in the general case the \( \mathcal{E}M_2 \) class is non-reducible to \( \mathcal{E}M_1 \) and reciprocally because both are lineally independent.

Proposition 7. If \( X \in \mathcal{E}M_3 \), it can be expressed by two co-potential vectors, \( \{U_a, V_a\} \) as:

\[
X_{ab} = \nabla_a U_b - \nabla_b U_a \quad *X_{ab} = \nabla_a V_b - \nabla_b V_a \tag{24}
\]

2.2 Bilinear Expressions

Some useful properties of the tensor class \( \mathcal{E}M_0 \) can be obtained by using bilinear expressions. All are based on a basic property related with a partial double duality transformation.

Proposition 8. If the fourth rank tensor \( H \) verifies that:

\[
H_{abcd} = H_{[ab]cd} = H_{ab[cd]} \tag{25}
\]

It is verified that:

\[
(*) H_{ab} = \frac{1}{2}\epsilon_{acde}H_{deuv}\frac{1}{2}b_{acuv} = H_{ba} - \frac{1}{2}\delta^b_a H \tag{26}
\]

where: \( H_{ab} = H_{bc}^{ac} \) and \( H = H_{cd}^{cd} \).

Proof. It is based on the properties of the \( \epsilon \) and \( \delta \) tensors(13)(p 75).

Proposition 9. If \( A \) and \( B \in \mathcal{E}M_0 \), it is verified that:

\[
\begin{align*}
*A_{ac}B_{bc} & = A_{ac}B_{bc} - \frac{1}{2}\delta^b_a A_{cd}B^{cd} \tag{27} \\
A_{ac}B_{bc} & = -*A_{ac}B_{bc} + \frac{1}{2}\delta^b_a A_{cd}B^{cd} \tag{28} \\
*A_{cd}B_{cd} & = -A_{cd}B_{cd} \tag{29} \\
A_{cd}B_{cd} & = *A_{cd}B_{cd} \tag{30}
\end{align*}
\]

Proof. The first is obtained directly from the previous Proposition, the second is obtained by using the intermediate step: \( A_{ac}B_{bc} = -*A_{ac}B_{bc} \). The two last expressions are contraction of the previous.
Proposition 10. If \( A \) and \( B \in \mathcal{E}M_0 \), it is verified that:

\[
\nabla_b (A_{ac} B^{bc}) = A_{ca} (\nabla_b B^{cb}) + B_{ca} (\nabla_b A^{cb}) + \frac{1}{2} B^{cd} \nabla_a (A_{cd})
\]

(31)

**Proof.** It is deduced directly applying the Theorem [3].

Proposition 11. Any \( X \in \mathcal{E}M_0 \), its dual and currents verify that:

\[
\nabla_b (X_{ac} X^{bc}) = Z^c X_{ca} + Y^c X_{ca} + \frac{1}{4} \nabla_a (X_{cd} X^{cd})
\]

(32)

\[
\nabla_b (\ast X_{ac} \ast X^{bc}) = Y^c \ast X_{ca} + Z^c X_{ca} + \frac{1}{4} \nabla_a (\ast X_{cd} \ast X^{cd})
\]

(33)

Theorem 3 (Stress-energy-like). If \( X \in \mathcal{E}M_0 \), the three following expressions are equivalent:

\[
W^b_a = X_{ac} X^{bc} - \frac{1}{4} \delta^b_a X_{cd} X^{cd}
\]

(34)

\[
= X_{ac} X^{bc} - \frac{1}{4} \delta^b_a \ast X_{cd} \ast X^{cd}
\]

(35)

\[
= \frac{1}{2} (X_{ac} X^{bc} + \ast X_{ac} \ast X^{bc})
\]

(36)

And the symmetric second rank tensor \( W \) verifies that:

\[
\nabla^b W_{ab} = Z^b X_{ba} + Y^b X_{ba}
\]

(37)

Proposition 12. The tensor \( W \) can be decomposed in three terms:

\[
W = (\alpha) W + (\beta) W + (\alpha \beta) W
\]

(38)

where it is verified that:

\[
\nabla^b (\alpha) W_{ab} = Z^b \alpha X_{ba}
\]

(39)

\[
\nabla^b (\beta) W_{ab} = Y^b \beta X_{ba}
\]

(40)

\[
\nabla^b (\alpha \beta) W_{ab} = -Z^b \ast X_{ba} + Y^b \ast X_{ba}
\]

(41)

**Proof.** This result is based on the Theorem [3]; each tensor is expressed as:

\[
(\alpha) W^b_a = \frac{1}{2} (\alpha \lambda \rho \delta \varepsilon - \beta \lambda \rho \delta \varepsilon)
\]

(42)

\[
(\beta) W^b_a = \frac{1}{2} (\beta \lambda \rho \delta \varepsilon + \beta \lambda \rho \delta \varepsilon)
\]

(43)

\[
(\alpha \beta) W^b_a = \frac{1}{2} (\beta \lambda \rho \delta \varepsilon + \beta \lambda \rho \delta \varepsilon - \alpha \lambda \rho \delta \varepsilon - \alpha \lambda \rho \delta \varepsilon)
\]

(44)

The last tensor can be simplified by using: \( N^b_a = \beta \lambda \rho \delta \varepsilon - \alpha \lambda \rho \delta \varepsilon \ast X_{ba} \ast X^{bc} \). Based on Proposition [3], it is verified that: \( N_{ab} = N_{ba} \) and \( N^a a = 0 \). It is obtained that: \( (\alpha \beta) W_{ab} = (1/2)(N_{ab} + N_{ba}) = N_{ab} \).

3 The axiomatic equations of the electromagnetic field

The class \( \mathcal{E}M_0 \) can be considered as an abstract and formal representation of a dual electromagnetic field. Due to the existence of magnetic monopoles as an hypothesis, any theoretical approach to this concept must be based on some axiomatic. The only condition imposed in this paper is that the field can be represented by a skew-symmetrical second rank tensor.
The electromagnetic field can be formally represented by a second rank skew-symmetric tensor $\mathbf{F} \in \mathcal{E}M_0$. Based on Theorems 1, 2 and 3, this tensor has some associated tensors: an $\alpha$-current, $\alpha\mathbf{J}$, a $\beta$-current, $\beta\mathbf{J}$, and a stress-energy tensor $\mathbf{T}$. They verify that:

$$\nabla \cdot \mathbf{F} = 4\pi \alpha\mathbf{J}$$  \hspace{1cm}  (45)
$$\nabla \cdot \ast \mathbf{F} = 4\pi \beta\mathbf{J}$$  \hspace{1cm}  (46)
$$T^b_a = \frac{1}{8\pi} (F_{ac} F^{bc} + \ast F_{ac} \ast F^{bc})$$ \hspace{1cm}  (47)

These equations are duality invariant, specifically the expression of the stress-energy tensor match with a general invariant form\[27\]. One of both the $\alpha$-current or the $\beta$-current is an electric current and the other one is a monopolar magnetic current. Both choice are valid because provide the same formal symmetry to Maxwell equations. In this paper, $\alpha\mathbf{J}$ and $\beta\mathbf{J}$ are considered respectively as the electric and the magnetic monopolar currents.

A difference appears when this formal derivation is compared with the classical theory of the electromagnetic field. Instead of a potential vector, $\mathbf{A}$, two potential vectors, $\alpha\mathbf{A}$ and $\beta\mathbf{A}$, can be considered based on the Theorem 2. Also two auxiliary fields $\alpha\mathbf{F}$ and $\beta\mathbf{F} \in \mathcal{E}M_1$ are introduced. They verify that:

$$\mathbf{F} = \alpha\mathbf{F} - \beta\mathbf{F}$$  \hspace{1cm}  (48)
$$\alpha F_{ab} = \nabla_a \alpha A_b - \nabla_b \alpha A_a$$  \hspace{1cm}  (49)
$$\beta F_{ab} = \nabla_a \beta A_b - \nabla_b \beta A_a$$  \hspace{1cm}  (50)

Each potential vector is independently related with its current type as:

$$-\Box \alpha A_a + R^b_a \alpha A_b = 4\pi \alpha J_a$$  \hspace{1cm}  (51)
$$-\Box \beta A_a + R^b_a \beta A_b = 4\pi \beta J_a$$  \hspace{1cm}  (52)

In an electromagnetic duality hypothesis, two independent currents must be considered: the magnetic and the electric sources. Two approaches can be considered for mapping these sources in an electromagnetic field. In this paper, similarly to other works on electromagnetic duality\[28, 35\], two vector fields are considered, this choice preserves the radial vs axial properties of fields. The magnetic field generated by a magnetic charge is of radial type based on the gradient of a scalar, while the magnetic field generated by an electric charge is of axial type based on a curl. In a full electromagnetic duality two radial and two axial field are involved. Most works in electromagnetic duality fit the dual sources into a single vector field. This task needs of some heuristic as the Dirac string monopole, which is a topological construction\[36, 34\].

The existence of magnetic monopoles is an hypothesis that extended the symmetry of Maxwell equations towards the symmetry of charges\[29\]. The field generated by a singular electric charge is well mapped into a vector potential. While that, if the field generates by a magnetic monopole is mapped in the same vector potential, the expected symmetry is broken. It appears two different concepts: the point singular electric charge and the string singular magnetic monopole. The Dirac quantization arises as a condition imposed to avoid the physical effect of the string\[29\] \[13\](p 165).

If a full symmetrical duality is desired, the topological counterpart of a point singular electric charge must be other one. Based on the Theorem 2 a simple and symmetrical solution is found by using two potential vectors. This approach has some advantages because a switch between the fields $\mathbf{F} \rightarrow \ast \mathbf{F}$ is reduced, as shown equation (48), to a switch between the two auxiliary fields: $\alpha\mathbf{F} \rightarrow \beta\mathbf{F}$ and $\beta\mathbf{F} \rightarrow -\alpha\mathbf{F}$, which is more similar to the current switch: $\alpha\mathbf{J} \rightarrow \beta\mathbf{J}$ and $\beta\mathbf{J} \rightarrow -\alpha\mathbf{J}$ as is deduced from equations (45) and (46).

### 4 Duality Invariance

The duality invariance is related to how field properties are modified with the transposition: $\mathbf{F} \rightarrow \ast \mathbf{F}$. Some equation changes are expected with this transposition. Invariant equations are preferred because it is supposed that both representations, $\mathbf{F}$ and $\ast \mathbf{F}$ are equivalents. The main principle involved in duality
invariance is that both descriptions have the same information, therefore both descriptions must be valid representations of the physical fact. Non privileged reference system can be found in General Relativity to study the physical laws, consequently all reference system are equivalent for this propose. By analogy, any symbol which have the same information should be equivalent to study the physical laws.

Instead of invariance in a discrete transposition as \( \mathbf{F} \rightarrow \ast \mathbf{F} \), it is required a continuous invariance related with a rotation phase \( \phi \) between the two fields. A complex tensor can be introduced to deal with this rotations: \( \mathbf{F} + i^* \mathbf{F} \). In this case a rotation generates the complex tensor: \( \epsilon^{i\phi}(\mathbf{F} + i^* \mathbf{F}) \). Duality invariance must be considered as mathematical invariance to rotations. As well as in General Relativity the covariant representation is introduced to be independent of any coordinate systems, a representation invariant to phase \( \phi \) must be introduced. Equivalent to the representation in the complex plane, a continuous transformation is introduced as:

\[
\mathbf{F}' = \mathbf{F} \cos \phi + \ast \mathbf{F} \sin \phi
\]  

(53)

By taking its dual: \( \ast \mathbf{F}' = \mathbf{F} \cos \phi - \ast \mathbf{F} \sin \phi \), it is obtained a representation based on a rotation matrix, \( \mathbf{R}(\phi) \), similar to the used by Deser et al.[8, 21]:

\[
\begin{pmatrix}
\mathbf{F} \\
\ast \mathbf{F}
\end{pmatrix}' = \mathbf{R}(\phi)
\begin{pmatrix}
\mathbf{F} \\
\ast \mathbf{F}
\end{pmatrix}
\]

\[
\mathbf{R}(\phi) =
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\]

(54)

A duality plane, \( D = 2 \), is introduced in addition to the Riemannian space-time with \( D = 4 \). This approach clearly evidences the well known connection between the duality problem and the rotation invariance, in this case with the \( \text{SO}(2) \) group. The tensors having two parts, e.g. \( \alpha \) and \( \beta \), can be represented by extended tensors with uppercase index:

\[
L_{a, \ldots}^{A_{a, \ldots}} \quad M_{a, \ldots}^{A_{a, \ldots}} \quad A = 0, 1 \quad a, \ldots, i, \ldots = 0, 1, 2, 3
\]  

(55)

Any extended tensor, \( L_{a, \ldots}^{A_{a, \ldots}} \), can be expressed symbolically in short as: \( \mathbf{L}^A \), \( \mathbf{L}_a^a \) and also as \( \mathbf{L} \), by abstracting some or all the index types. The covariant and contravariant uppercase index are related with the symmetrical group while the lower case index are related with the space-time. E.g. the electromagnetic field tensors, which have a dual interpretation, are represented by the extended tensors: \( F_{a_b}^C, A_{a}^C \) and \( J_{a}^C \), such as: \( F_{a_b}^0 = F_{a_b} \) and \( F_{a_b}^1 = * F_{a_b} \). The Maxwell equations are:

\[
\delta \left( \begin{array}{c}
\mathbf{F} \\
\ast \mathbf{F}
\end{array} \right) = 4\pi \left( \begin{array}{c}
\alpha \mathbf{J} \\
\beta \mathbf{J}
\end{array} \right)
\]  

(56)

It is obtained that:

\[
\begin{pmatrix}
\alpha \mathbf{J} \\
\beta \mathbf{J}
\end{pmatrix}' = \mathbf{R}(\phi) \begin{pmatrix}
\alpha \mathbf{J} \\
\beta \mathbf{J}
\end{pmatrix}
\]

(57)

The concept of rotation in the \( \mathcal{E} \mathcal{M}_0 \) class is extended to all \( \alpha \) and \( \beta \) related tensors. They are transformed as follows:

\[
\mathbf{L}^{i' \ldots}_{a, \ldots} = \mathbf{R}(\phi) \mathbf{L}^{i \ldots}_{a, \ldots}
\]  

(58)

In the duality plane the symmetrical identity tensor, \( \eta_{AB} \), and the skew-symmetrical tensor, \( \epsilon_{AB} \) are considered. They are represented as:

\[
\eta = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \epsilon = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]  

(59)

If \( \epsilon^T \) is the transpose of \( \epsilon \), it is verified that: \( \epsilon^T \epsilon = \eta \), and also: \( \epsilon \epsilon = -\eta \). The metric tensor allows the use of covariant and contravariant components related to: \( A_{a_a}^{C_{a_b}} = \eta_{CD} A_{a_a}^{D_{a_b}} \). Two operators are introduced in the duality plane. The following expression are called the scalar and the vectorial products in the duality plane, which are operators based on the previous matrices:

\[
A_{a_a}^{i \ldots} \bullet B_{b_b}^{i j \ldots} = \eta_{CD} A_{a_a}^{C_{a_b}} B_{b_b}^{D_{b_j}} = < A_{a_a}^{i \ldots}, B_{b_b}^{i j \ldots} >
\]

(60)

\[
A_{a_a}^{i \ldots} \wedge B_{b_b}^{i \ldots} = \epsilon_{CD} A_{a_a}^{C_{a_b}} B_{b_b}^{D_{b_j}} = [ A_{a_a}^{i \ldots}, B_{b_b}^{i j \ldots} ]
\]

(61)
These bilinear expressions are duality invariant because if $R^T$ is the transpose of the rotation matrix, it is verified the following classical expressions of the rotations groups:

$$ R^T(\phi)\eta R(\phi) = \eta \quad R^T(\phi)\epsilon R(\phi) = \epsilon $$

(62)

Any matrix, $\lambda$, which generates a bilinear invariant operator must verify: $R^T(\phi)\lambda R(\phi) = \lambda$, but any solution of this equation can be expressed as a linear combination of the previous: $\lambda = a\eta + b\epsilon$, where $a$ and $b$ are constant. This is due to:

$$ R(\phi) = \eta \cos \phi + \epsilon \sin \phi \quad \text{and} \quad R^T(\phi) = \eta \cos \phi - \epsilon \sin \phi $$

(63)

Therefore, any bilinear invariant in the duality plane can be constructed from the defined scalar and vectorial operators. The scalar and vectorial products are symmetric and skew-symmetric respectively. Both are linear operators, symbolically expressed as:

$$ L \cdot (cM + dN) = c(L \cdot M) + d(L \cdot N) \quad (64) $$

$$ L \wedge (cM + dN) = c(L \wedge M) + d(L \wedge N) \quad (65) $$

where $(c, d)$ are ordinary tensors, which are constant for the duality plane operators. Extended tensors allows to express the electromagnetic equations by using more compact and invariant expressions. The equations (51) and (52) are expressed as:

$$ -\Box A^C + R^b_a A^C_b = 4\pi J^C_a \quad (66) $$

The $K$ field is introduced to deal more easily with the first derivative of the potential vector, it is defined as follows:

$$ K^C_{ab} = \nabla_a A^C_b - \nabla_b A^C_a $$

(67)

The equation (48) is represented as:

$$ F^C = K^C - \epsilon^{CD} \ast K_D $$

(68)

Remark that the fields $F^0$ and $F^1$ are dependent due to: $F^1 = \ast F^0$, while $K^0$ and $K^1$ are two independent fields. It is possible to express the Maxwell equations and the current conservation law in a compact and invariant form as:

$$ \nabla \cdot F^C = 4\pi J^C \quad \nabla \cdot J^C = 0 $$

(69)

Finally, the stress-energy tensor $T$ can be expressed in duality invariant form as:

$$ T^b_a = \frac{1}{8\pi}(F_{ac}^d F^{bc}) \quad \nabla \cdot T = F = J^d \ast F $$

(70)

Where $F$ is the force density produced by the field-current interaction. When a duality plane rotation is produced with $\phi = \pi/2$, a permutation between the electric and magnetic field is verified in the field $F$, but this rotation does a permutation between $\alpha$ and $\beta$ parts in the extended tensors: $J$, $A$ and $K$. This last belongs to $EM_1$ and verifies the following Maxwell equations:

$$ \nabla \cdot K^C = 4\pi J^C \quad \nabla \cdot \ast K^C = 0 $$

(71)

Based on Proposition 12 the stress-energy tensor can be expressed as: $T = (\alpha) T + (\beta) T + (\alpha\beta) T$. The two first terms can be merged in a more compact representation called the parallel stress-energy tensor: $T_{\|}$:

$$ T^b_{\|a} = (\alpha) T^b_a + (\beta) T^b_a = \frac{1}{8\pi}(K_{ac}^d K^{bc} + \ast K_{ac}^d \ast K^{bc}) $$

(72)

which verifies:

$$ \nabla \cdot T_{\|} = F_{\|} = J^d \ast K $$

(73)
where \( J^d \cdot K = J^0 \cdot K^0 + J^1 \cdot K^1 \) is the parallel force density. In this case, the electrical current is interacting with the electrical generated field and reciprocally for the magnetic current. This case is composed of two independent and separable electromagnetic fields. Each field is produced by its own current, which is neutral to the action of the other field. The term \((a,b)\) \( T \) is called the perpendicular stress-energy tensor. Based on the proposition \[12\], it can be expressed as:

\[
T^b_{\perp a} = -\frac{1}{8\pi}(K^d_{ac} \wedge *K^{bc} + K^{bc} \wedge *K_{ac}) = -\frac{1}{4\pi}K^d_{ac} \wedge *K^{bc}
\]

(74)

It verifies that:

\[
\nabla \cdot T_{\perp} = F_{\perp} = -J^d \wedge *K
\]

(75)

where \( J^d \wedge *K = J^0 \cdot *K^1 - J^1 \cdot *K^0 \) is the perpendicular force density. In this case the electric current is interacting with the magnetic generated field being neutral to the electrical generated field, and reciprocally for the magnetic current.

### 5 Lagrange function invariance

A complete electromagnetic theory must include a model of the Lagrange action function. An invariant action must be proposed to solve the variational problem \( \delta I = 0 \), where:

\[
I = \int \sqrt{-g}L(A, F)d^4x
\]

(76)

In classical theory the Lagrange function \( L \) is proportional to \( F_{ab}F^{ab} \), however this expression is not duality invariant. Unfortunately all the following expressions are null:

\[
F_{ab} \bullet F^{ab} = F_{ab}^d \bullet *F^{ab} = *F_{ab}^d \bullet F^{ab} = 0
\]

(77)

\[
F_{ab} \wedge F^{ab} = F_{ab}^d \wedge *F^{ab} = *F_{ab}^d \wedge F^{ab} = 0
\]

(78)

The Lagrange function must verify three imposed restrictions. Firstly it must be duality invariant. The second restriction is that the Maxwell equations must be obtained with variations of the potential vector \( \delta A \). Finally, the function must provide the correct expression of the stress-energy tensor with the metric variations \( \delta g \). A lot of works are recently concerning with the construction of a Lagrange function verifying these restrictions \[19, 21, 22, 23\].

In the approach of this paper two vector fields are considered, therefore the function must be expressed as: \( L(A^C, K^C) \). The Lagrange function related with the field is composed of two terms, the first is concerning with the electromagnetic field itself, and the second with the current-field interaction:

\[
L = L_F(K^C) + L_I(J^C, A^C)
\]

(79)

In order to generate the Maxwell equations, the interaction term is based on the expression: \( J^d \bullet A^a \). Due to \( T = T_{\parallel} + T_{\perp} \), the Lagrange function can be also expressed as: \( L_F = L_{\parallel} + L_{\perp} \). The field term can be constructed by using bilinear invariant expressions formed with the tensors \( K \) and \( *K \). However the following vectorial-based bilinear expressions are null.

\[
K_{ab} \wedge *K^{ab} = K_{ab}^d \wedge *K^{ab} = *K_{ab}^d \wedge *K^{ab} = 0
\]

(80)

The scalar-based bilinear expressions are:

\[
\sigma = K_{ab} \bullet *K^{ab} \quad \tau = K_{ab} \bullet *K^{ab} \quad *K_{ab} \bullet *K^{ab} = -K_{ab} \bullet K^{ab}
\]

(81)

The \( L_F \) term must be dependent of these expressions: \( L_F(\sigma, \tau) \). In a linear electromagnetic theory must be: \( \partial_t K = cK \), consequently an expression as: \( L_F = c_\sigma \sigma + c_\tau \tau \) must be considered. However, \( L_{\perp} \) cannot be obtained with \( \sigma \) and \( \tau \) expressions, and also the \( \tau \) expression can be expressed as a gradient and removed from the Lagrange function:

\[
\tau = K_{ab} \bullet *K^{ab} = 2\nabla_a (e^{abcd}A_b \bullet \nabla_c A_d)
\]

(82)
The term $\sigma = K_{ab} \cdot K^{ab}$ is the only non-null, duality invariant and valid bilinear expression that generates a linear electromagnetic theory. It generates the Maxwell equations and provides the parallel stress-energy tensor. The $L_F$ term can be expressed as:

$$L_F = L_\parallel + \gamma L_\perp \quad \gamma \in \{0, 1\}$$

(83)

where the $\gamma$ constant is related with the inclusion of a perpendicular term in the Lagrangian. A solution for the the Lagrange function including the interaction term $L_I$ is expressed as:

$$L_\parallel + L_I = -\frac{1}{16\pi} K_{ab} d \cdot K^{ab} + J_a \cdot A^a$$

(84)

The variation $\delta A^a$ generates the following Euler-Lagrange conditions:

$$\frac{\partial L}{\partial A^a} = 2\nabla^b \left( \frac{\partial L}{\partial K^{ba}} \right)$$

(85)

which provide the Maxwell equations: $4\pi J^C_a = \nabla^b K^C_{ab}$. Based on equations (88) and (71), it is verified that:

$$\nabla^b F^C_{ab} = \nabla^b K^C_{ab} - \epsilon^{CD} \nabla^b J_D = \nabla^b K^C_{ab} = 4\pi J^C_a$$

(86)

According to Einstein General Relativity, the stress-energy tensor can be obtained from the field action as:

$$T_{ab} = -\frac{1}{2\pi} \frac{\partial L_F}{\partial g^{ab}} + g_{ab} L_F$$

(87)

It is obtained the next result:

$$T^b_{|a} = \frac{1}{4\pi} K_{ac} d \cdot K^{bc} - \frac{1}{16\pi} \delta^b_a K_{cd} \cdot K^{cd}$$

(88)

Based on the Proposition 9, it can be expressed as in the equation (72). By using the Einstein equation, $G = 8\pi T$, this expression of the stress-energy tensor allows to obtain the metric and field associated to a charged central body as a black hole. In the outer space of the body, the field can be expressed as: $K^C_{ab} = Q^C K_{ab}$, where $Q$ is the body charge, and $K_{ab}$ is the solution for an unitary charge. The perpendicular stress-energy tensor is proportional to the term: $Q^a \wedge Q^b$ which is null. The stress-energy tensor is only based on the scalar product of the charge, being multiplied by $(Q^2)^2 = Q \cdot Q$, where $Q$ is the equivalent combined charge. The Reissner-Nordström metric[4, 16] arises as the solution for this problem based on the mass $M$ and the combined charge: $(Q^2)^2$. It is concluded that from the viewpoint of the electromagnetic duality presented in this paper, the metric of an electric or magnetic charged black hole are identical formally but different numerically.

6 Particle-Field Interaction

From an operational viewpoint, a field is the formal representation of a physical fact which determines an interaction with some type of particle. A trajectory $x^c(s)$ is associated to each virtual particle. If non-quantum theory is considered, the trajectory is the formal representation of the virtual particle. A vector field $p^a$ tangent to the trajectory is considered. If the particle is massive, $p = \|p\| \neq 0$, the trajectory has an unitary tangent vector $u$. The absolute derivation of $p$ along a field line can be expressed as follows:

$$\frac{dp}{ds} = Dp^a e_a = (\nabla_b p^a) u^b e_a$$

(89)

Due to $p_a p^a$ is constant, it can be obtained the next expression: $(\nabla_b p_a) u^b = 2\nabla_b p_a u^b$. It is concluded that the vector field $p$ with non-null and constant $p = \|p\|$ verifies the next equation along any of its field lines:

$$\frac{Dp^a}{ds} = f^a = p^a_b u^b$$

(90)
Axiomatic Electromagnetical Duality

Where \( p_{ab} \in \mathcal{EM}_1 \) can be expressed as: \( p_{ab} = 2\nabla_{[b}p_{a]} \). The vector \( f \) is the Lorentz force based on a coupling constant or particle charge, which is represented by a constant vector in the duality plane \([31]\). \( \mathcal{q} \equiv (\alpha \mathcal{q}, \beta \mathcal{q}) \). It is transformed by a rotation as \([31]\): \( \mathcal{q}' = R(\phi)\mathcal{q} \). When a phase rotation with \( \phi = \pi/2 \) is performed, the charge is changed from \((\alpha \mathcal{q}, \beta \mathcal{q})\) to \((\beta \mathcal{q}, -\alpha \mathcal{q})\). The term \( \mathcal{F} = J \cdot \mathcal{F} \) is the force density in a continuous case. It can be expressed as: \( \mathcal{F} = \mathcal{F}_\parallel + \gamma \mathcal{F}_\perp \), its expression is:

\[
\mathcal{F}_a = (J^b)_d \mathcal{K}_{ab} - \gamma (J^b)_d ^* \mathcal{K}_{ab}
\]  

(91)

By analogy, for a virtual particle, the term \( j^a = \mathcal{q}u^a \) can be considered as the discrete particle current, and the expression: \( f = (\mathcal{q} \cdot \mathcal{F}) \cdot u \) as the force of the field-particle interaction, its expression is:

\[
f_a = (\mathcal{q}u^b)_d \mathcal{K}_{ab} - \gamma (\mathcal{q}u^b)_d ^* \mathcal{K}_{ab}
\]  

(92)

which also is composed of the parallel and the perpendicular force. From equations \([31]\) and \([32]\), the next equation must be considered to solve the particle trajectory:

\[
p_{ab} = \mathcal{q}^d _d \mathcal{K}_{ab} - \gamma \mathcal{q}^d _d ^* \mathcal{K}_{ab}
\]  

(93)

which can be transformed as:

\[
\nabla_{[b}(p + \mathcal{q}^d _d \mathcal{A})_{a]} = -\gamma ^* (\mathcal{q}^d _d \mathcal{K}_{ab})
\]  

(94)

This equation is non-homogeneous because the lhs is in \( \mathcal{EM}_1 \) and the rhs is in \( \mathcal{EM}_2 \). Based on the Proposition \([3]\), there are not a general solution for this equation. A particular solution is possible if \( \gamma \mathcal{q}^d _d \mathcal{K}_{ab} \in \mathcal{EM}_3 \), but this implies that: \( \gamma \mathcal{q}^d _d \mathcal{J} = 0 \).

A particular case can be obtained by considering the interaction of two particles. One is acting as the field source, its field tensors are proportional to the charge \( q' \). The other particle is acting as a singular test particle with charge \( q \). If the electric and magnetic charges are supposed basic values multiplied by integer numbers, being \( e \) and \( \mu \) the basic electric and magnetic values respectively. For a particle pair, the scalar and the vectorial product of the charges are two interaction constants similar to the charge product in classical electrodynamics:

\[
\mathcal{q}^d _d \mathcal{q}' = (e \mu) m \quad \mathcal{q}^d _d \mathcal{q}' = (e^2 m + (\mu)^2 k) \quad n, m, k = 0, \pm 1, \pm 2, \ldots
\]  

(95)

The vectorial product \( \mathcal{q}^d _d \mathcal{q}' = 0 \) is in the classical theory the equivalent to the corresponding in the quantum theory known as the Zwanziger-Schwinger charge quantization \([33, 34, 35, 36, 37]\) which is an extension of the Dirac quantization. The field out of the source particle can be expressed as: \( \mathcal{K}_{ab} = q' \mathcal{K}_{ab} \), with \( \mathcal{K}_{ab} \in \mathcal{EM}_3 \). Based on Proposition \([3]\) it is obtained: \( K_{ab} = \nabla_a A_b - \nabla_b A_a \) and also: \( ^* \mathcal{K}_{ab} = \nabla_a U_b - \nabla_b U_a \). The equation \([91]\) becomes:

\[
\nabla_{[b}(p + \mathcal{q}^d _d \mathcal{q}' \mathcal{A} + \gamma \mathcal{q}^d _d \mathcal{q}' \mathcal{U})_{a]} = 0
\]  

(96)

This equation can be immediately solved by means of a scalar gradient as:

\[
p_a = \nabla_a S - (\mathcal{q}^d _d \mathcal{q}') A_a - \gamma (\mathcal{q}^d _d \mathcal{q}') U_a
\]  

(97)

Where \( S \) is the particle action. Consequently, the Hamilton-Jacobi approach is obtained for the dynamic of the test particle, based on two co-potential vector, \( A_a \) and \( U_a \). By using the two constants: \( \theta_\parallel = \mathcal{q} \cdot \mathcal{q}' \) and \( \theta_\perp = \mathcal{q} \wedge \mathcal{q}' \), it is obtained:

\[
(\nabla_a S - \theta_\parallel A_a - \gamma \theta_\perp U_a)(\nabla^a S - \theta_\parallel A^a - \gamma \theta_\perp U^a) = (p)^2
\]  

(98)

A solution with \( \gamma = 0 \) is valid, but in this case the magnetic monopole with charge: \( (0, \pm \mu) \) is neutral to the electric charge \( (\pm e, 0) \). In this case, due to equations \([31]\) and \([32]\), the operational field responsible of the current-field and particle-field interaction is the set of two independent \( \mathcal{EM}_1 \) fields \( \mathcal{K} \), instead of the \( \mathcal{EM}_0 \) field \( \mathcal{F} \).
7 Conclusions

A theory of electromagnetic duality has been presented. It has been developed in the framework of the Einstein General Relativity. The proposed theory carries out the symmetry of Maxwell equations, the invariance of the stress-energy tensor, and also the invariance of part of the Lagrange action. The theory has been developed using an axiomatic method to deduce the dual Maxwell equations from the properties of differential forms. Two potential vectors are needed to deal with the two source types involved in the electromagnetic duality.

The main result of this paper is that the stress-energy tensor can be decomposed on two parts: the parallel and the perpendicular. In the parallel part each current type is interacting with the field of the same type. In the perpendicular part the interaction becomes between currents and fields of different type. The perpendicular part cannot be easily integrated in a Lagrange approach. On a linear electromagnetic theory, constructed with duality invariant bilinear expressions, do not exist a suitable expressions for this function. A Lagrange function with the parallel part alone is a valid solution, it generates a non-standard model of magnetic monopoles neutral to the electric charges.

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