Special Issue Article

The effect of an axial mean temperature gradient on communication between one-dimensional acoustic and entropy waves

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Abstract

This work performs a theoretical and numerical analysis of the communication between one-dimensional acoustic and entropy waves in a duct with a mean temperature gradient. Such a situation is highly relevant to combustor flows where the mean temperature drops axially due to heat losses. A duct containing a compact heating element followed by an axial temperature gradient and choked end is considered. The proposed jump conditions linking acoustic and entropy waves on either side of the flame show that the generated entropy wave is generally proportional to the mean temperature ratio across the flame and the ratio \( \frac{F}{C_0^1} \), where \( F \) is the flame transfer function. It is inversely proportional to the Mach number immediately downstream of the flame \( M_2 \). The acoustic and entropy fields in the region of axial mean temperature gradient are calculated using four approaches: (1) using the full three linearised Euler equations as the reference; (2) using two linearised Euler equations in which the acoustic and entropy waves are assumed independent (thus allowing the extent of communication between the acoustic and entropy wave to be evaluated); (3) using a Helmholtz solver which neglects mean flow effects and (4) using a recently developed analytical solution. It is found that the communication between the acoustic and entropy waves is small at low Mach numbers; it rises with increasing Mach number and cannot be neglected when the mean Mach number downstream of the heating element exceeds 0.1. Predictions from the analytical method generally match those from the full three linearised Euler equations, and the Helmholtz solver accurately determines the acoustic field when \( M_2 \leq 0.1 \).

Keywords

Linearised Euler equations, acoustic waves, entropy waves, Helmholtz solver, choked boundary condition

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1. Introduction

Combustion chambers of gas turbines or aero-engines typically exhibit an axial mean temperature gradient due to heat losses or additional flow dilution.1–4 Many previous studies5–8 have shown that flows passing through regions with mean temperature (or density) gradient may produce acoustic waves and also entropy waves. Meanwhile, entropy disturbances entering these regions can also generate or diminish acoustic waves. As shown in Dowling5 and Bauerheim et al.,9 the entropy wave generated by a still compact flame or heater is generally inversely proportional to the mean flow velocity. For a very weak mean flow situation, which is common in a combustion chamber to stabilise the flame at a certain position,10 it is still unclear if this generated large entropy wave interacts with and greatly changes the acoustic field when it is convected through a region with a mean temperature gradient. Under this situation, it would be interesting to know whether approaches that only account for acoustic waves, e.g. Helmholtz solvers4,11,12 and analytical or semi-analytical methods13–16 based on linearised Euler equations (LEEs) which are functions of only pressure and velocity perturbations, are still reliable. Furthermore, when the outlet of the

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The generation of entropy waves by a still compact flame or heater is investigated in this section. The flame or heater is considered still such that any spatial movement of the flame in response to acoustic disturbances is beyond the scope of this work.\textsuperscript{20,21} The configuration is schematically presented in Figure 1. It is a uniform duct and contains a still compact flame or heater located at $x = x_1 = 0$. The duct has a sectional surface area $A$. The dominant acoustic wavelength is considered sufficiently larger than the sectional size of the duct and only longitudinal acoustic waves propagate, which in turn enables a one-dimensional (1-D) representation of the acoustic field. The sectional averaged pressure, axial velocity, density and temperature are represented by $p(x)$, $u(x)$, $\rho(x)$ and $T(x)$, respectively. The mean thermodynamic and flow properties in the upstream region are uniform and the mean temperature in the downstream region is taken to decrease with axial location, $x$, due to heat losses.

2.1 Jump conditions across a still compact flame or heater

The mass, momentum and energy conservation equations linking waves on either side of the still compact flame or heater (located at $x_1^-$ and $x_1^+$ respectively) are mathematically expressed as\textsuperscript{22}

\begin{equation}
\rho(x_1^+)u(x_1^+) = \rho(x_1^-)u(x_1^-)
\end{equation}

\begin{equation}
p(x_1^+) + \rho(x_1^+)u_2^2(x_1^+) = p(x_1^-) + \rho(x_1^-)u_2^2(x_1^-)
\end{equation}

\begin{equation}
\frac{\rho(x_1^+)u(x_1^+)}{c_p}T(x_1^+) + \frac{u_2^2(x_1^+)}{2} = \frac{\rho(x_1^-)u(x_1^-)}{c_p}T(x_1^-) + \frac{u_2^2(x_1^-)}{2} + \frac{\bar{Q}}{A}
\end{equation}

According to linear acoustic theory, all flow and thermodynamic variables (e.g. $p$) can be decomposed into a mean value (e.g. $\bar{p}$) and an acoustic perturbation (e.g. $p'$), which is assumed small compared to the corresponding mean value. One also assumes that all fluctuating quantities have a time dependence in the form $b' = \bar{b}e^{i\omega t}$, where $\omega$ is the complex angular frequency. By using the ideal gas law

\begin{equation}
p = \rho R_g T = \rho \frac{Y - 1}{\gamma} c_p T \quad \text{(4)}
\end{equation}

where $R_g$ is the gas constant and $\gamma$ is the ratio of specific heats, as well as the relation between the density disturbance $\rho'$ and the entropy perturbation $s'$\textsuperscript{5,9}

\begin{equation}
\rho' = \frac{\dot{\rho}}{c_s} - \frac{\rho}{c_p} \dot{s}
\end{equation}
it is possible to obtain the jump conditions connecting first-order perturbation terms

\[
\begin{bmatrix}
\hat{\rho}(x_1^+), \\
\hat{u}(x_1^+), \\
\hat{S}(x_1^+)
\end{bmatrix} = \begin{bmatrix}
\hat{\rho}(x_1^-), \\
\hat{u}(x_1^-), \\
\hat{S}(x_1^-)
\end{bmatrix} + \begin{bmatrix}
0, \\
0, \\
\frac{\gamma - 1}{\gamma M_1^2} \Delta \hat{Q}
\end{bmatrix}
\]

(6)

where

\[
\begin{bmatrix}
\hat{\rho}_1, \\
\hat{u}_1, \\
\hat{S}_1
\end{bmatrix} = \begin{bmatrix}
\xi M_1^{\frac{\rho_2}{\rho_1}}, \\
(1 + M_1^2)^{\frac{\rho_2}{\rho_1}}, \\
1 + \frac{\gamma - 1}{2} M_1^2 - \frac{\rho_1}{\rho_2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{\rho}_2, \\
\hat{u}_2, \\
\hat{S}_2\end{bmatrix} = \begin{bmatrix}
M_2, \\
1 + M_2^2, \\
\gamma M_2 + \frac{\gamma - 1}{2} M_2^2 - \frac{\rho_2}{\rho_1} M_2^2
\end{bmatrix}
\]

(7)

and

\[
\hat{\rho}_1 = \frac{\hat{\rho}_1}{\hat{p}_1}, \quad \hat{u}_1 = \frac{\hat{u}_1}{\hat{c}_1}, \quad \hat{T}_1 = \frac{\hat{T}_1}{\hat{c}_1}, \quad \hat{M}_1 = \frac{\hat{u}_1}{\hat{c}_1}, \quad \hat{c}_2 = \frac{\gamma R \hat{T}_2}{\hat{c}_1}, \quad \hat{M}_2 = \frac{\hat{u}_2}{\hat{c}_2}, \quad \hat{S}_1 = \frac{\hat{S}_1}{\hat{c}_1}, \quad \hat{S}_2 = \frac{\hat{S}_2}{\hat{c}_2}
\]

(8)

\[\hat{T}_1, \hat{p}_1, \hat{u}_1\] and \[\hat{c}_1\] represent the mean temperature, pressure, density and velocity in the region upstream of the flame or heater. \[\hat{T}_2, \hat{p}_2, \hat{u}_2, \hat{c}_2\] represent the mean temperature, pressure, density and velocity immediately downstream of the flame or heater. \(c^2 = \frac{T_2}{T_1}, \quad M_1 = \frac{u_1}{c_1}\) and \(M_2 = \frac{u_2}{c_2}\) are the mean temperature ratio, the Mach numbers either side of the flame or heater, respectively. \(\hat{c} = (\gamma R \hat{T}_2)\) is the speed of sound. \(\gamma\) and \(R\) are assumed constant; the ratio of speed of sound thus equals \(\frac{\hat{c}_2}{\hat{c}_1} = \xi\). Two normalised parameters \(\hat{u}(x) = \hat{\rho}_2 \hat{c}_2 \hat{u}(x)\) and \(\hat{S}(x) = \hat{\rho}_2 \hat{c}_2^2 \hat{S}(x) / \hat{c}_2\) are proposed as they generally have the same order of magnitude as the pressure perturbation \(\hat{\rho}(x)\).

As shown in equations (7) and (8), coefficients of acoustic terms \(\hat{\rho}\) or \(\hat{u}\) are generally of the order of unity \(O(1)\) in the three conservation equations. Coefficients of entropy terms \(\hat{S}(x_1^+)\) or \(\hat{S}(x_1^-)\) in the mass equation are first order in Mach number, while those in the momentum and energy equations are second and third orders in Mach number, respectively. For subsonic flow, contributions from entropy perturbations to acoustic waves (\(\hat{\rho}\) or \(\hat{u}\)) are thus more significant in the mass conservation equation.

The jump conditions for zero-mean flow, which neglect entropy terms, are generally derived based on the momentum and energy conservation equations.\(^{23}\) By neglecting Mach number terms from the mass conservation equation, these derivations may lead to big errors.\(^9\)

The heat release rate perturbation can be linked to the acoustic field using a flame transfer function\(^{24}\) (or a flame describing function for weakly nonlinear flame responses (Noiray et al.\(^{25}\))), which is expressed as

\[
\mathcal{F} = \frac{\hat{Q}}{\hat{Q}}\frac{\hat{Q}}{\hat{Q}_1}/\hat{u}_1\]

(9)

where \(\hat{Q}\) indicates the mean total heat release rate. Note that there is no entropy perturbation upstream of the flame or the heater; one thus has \(\hat{S}(x_1^+) = 0\). It is then possible to substitute the flame model (equation (9)) and the expression for \(\hat{Q}\) (see Dowling\(^5\)) into the jump conditions (equation (6)) to give

\[
\begin{bmatrix}
\hat{p}(x_1^+), \\
\hat{u}(x_1^+), \\
\hat{S}(x_1^+)
\end{bmatrix} = \begin{bmatrix}
\xi M_1, \\
1 + M_1^2, \\
\sigma_1
\end{bmatrix} \begin{bmatrix}
\hat{p}(x_1^-), \\
\hat{u}(x_1^-), \\
\hat{S}(x_1^-)
\end{bmatrix} + \begin{bmatrix}
\frac{\gamma - 1}{\gamma M_1^2}, \\
\frac{\gamma - 1}{\gamma M_1^2} \sigma_2, \\
\sigma_2 \frac{\rho_2}{\rho_1}
\end{bmatrix} \begin{bmatrix}
\hat{u}(x_1^-)
\end{bmatrix}
\]

(10)

where

\[
\sigma_1 = \frac{1}{\xi} (\gamma M_1 + \frac{\gamma - 1}{2} M_1^2)
\]

(11)

\[
\sigma_2 = 1 + \frac{3(\gamma - 1)M_1^2}{2} + \mathcal{F} \left(\xi^2 - 1 + \frac{\gamma - 1}{2} (\xi^2 M_1^2 - M_1^2)\right)
\]

(12)

2.2 Relations for low Mach number situations

When the Mach number of the flow across or in the vicinity of the flame is not very large, as is typically the case to stabilise the flame, the above equations can be simplified by neglecting Mach number terms with orders exceeding \(M\). This leads to the following simplified relations

\[
\hat{\rho} = \text{constant}
\]

(13)

\[
\hat{\rho}_1 \hat{c}_1^2 = \gamma \hat{\rho} = \hat{\rho}_2 \hat{c}_2^2
\]

(14)

\[
M_2 = \xi M_1
\]

(15)

By substituting the above equations into the jump conditions (equation (6)) and also neglecting Mach number terms with order exceeding \(M\), one obtains the simplified jump conditions

\[
\begin{bmatrix}
1, \\
-\gamma M_2 (1 - \frac{1}{\xi}), \\
-\gamma (1 - \frac{1}{\xi}) \mathcal{F}
\end{bmatrix}
\]

(16)
The first two simplified jump conditions agree with those in Polifke et al.\textsuperscript{26} It is interesting to see that there is also pressure perturbation jump in the presence of mean flow. This jump is proportional to the Mach number $M_2$, the mean temperature jump ratio $(\zeta^2 - 1)$ and the value $(\mathcal{F} + 1)$.\textsuperscript{27} The entropy generation $\dot{S}(x_1)$ is generally proportional to the mean temperature jump ratio $(\zeta^2 - 1)$ and the value $(\mathcal{F} - 1)$, and is inversely proportional to the Mach number $M_2$.

When $M_2$ (or $M_1$) tends to zero, the entropy perturbation generated at the flame cannot be convected downstream by the mean flow, accumulates there and goes to infinity, as consistent with findings in Bauerheim et al.\textsuperscript{9}

When the Mach number is very small, the jump conditions can be further simplified by neglecting terms containing $M_2$, yielding

\begin{align*}
\frac{\tilde{p}(x_1)}{\tilde{u}(x_1)} &= \frac{1}{\tilde{u}(x_1)} \\
\frac{\tilde{u}(x_1)}{\tilde{u}(x_1)} &= \frac{1}{\tilde{u}(x_1)} \\
\frac{\dot{S}(x_1)}{\dot{u}(x_1)} &= \frac{1}{\dot{u}(x_1)}
\end{align*}

\textbf{Figure 2.} Evolution of $\frac{\tilde{p}(x_1)}{\tilde{u}(x_1)}$ (top figure), $\frac{\tilde{u}(x_1)}{\tilde{u}(x_1)}$ (middle figure) and $\frac{\dot{S}(x_1)}{\dot{u}(x_1)}$ (bottom figure) with $M_1$ or $M_2$ using three methods. $\tilde{p}_1 = 1 \times 10^5$ Pa, $\tilde{T}_1 = 300$ K, $\zeta^2 = 4$ and $\mathcal{F} = 2$. (a) $Z_f = 0.1$. (b) $Z_f = 10$. (c) $Z_f = 1$. 
\[
B_1 = \begin{bmatrix}
1 & 0 \\
0 & 1 + (\zeta^2 - 1)\mathcal{F} \\
0 & \frac{\mathcal{F}}{M^2} (\zeta^2 - 1)
\end{bmatrix}
\]  

(17)

It should be noted that the term containing \(1/M_2\) is retained even though the expansion in Mach is truncated because of the trouble linearising about the singularity \(M_2 = 0\). The \(O(1)\) term, \(B_3(3,1)\), is assumed zero because it is very small compared to the \(O(1/M_2)\) term, \(B_3(3,2)\), when the Mach number is small. The first two jump conditions agree with those for zero-mean-flow condition.\(^{23,28}\)

In order to validate these simplifications, the pressure and velocity perturbations upstream of the flame or heater can be related using \(\tilde{p}(x_1^+) = Z_f^+ \tilde{u}(x_1^+)\), where \(Z_f^+\) is the acoustic impedance. This makes it possible to define the transfer functions from \(\tilde{u}(x_1^+)\) to \(\tilde{p}(x_1^+)\), \(\tilde{u}(x_1^+)\) and \(\tilde{S}(x_1^+)\) for comparison. Figure 2 shows the comparisons of the three transfer functions with inlet Mach number \(M_1\) (or \(M_2\)), using the full equations (equation (10)), the simplified equations retaining Mach number terms up to the first order (equation (16)) and the simplified equations assuming very small mean flow (equation (17)) for three acoustic impedances \(Z_f^+\). The flame transfer function is assumed to be \(\mathcal{F} = 2\), which is a typical value for laboratory scale\(^{28,29}\) or industrial\(^{30}\) combustors. Three cases have been accounted for: (a) \(Z_f^+ = 0.1\) close to the pressure perturbation node situation; (b) \(Z_f^+ = 10\) close to the velocity perturbation node situation; (c) \(Z_f^+ = 1\), the situation between (a) and (b). From Figure 2, it is noted that the simplified equations retaining Mach number terms up to the first order always perform well when the inlet Mach number is smaller than 0.1. The validity of relation equation (17) for the pressure and velocity perturbation is limited to very small Mach number flow conditions.

### 3. Heated duct target cases

One now investigates the convection of the generated entropy wave through the region with axial mean temperature gradient – this results in communication between the acoustic and entropy waves. The 1-D duct in Figure 1 is considered, with the flame or heater located at \(x = x_1 = 0\). The temperature before the flame or heater is \(\bar{T}_1\) and is assumed uniform, while a temperature jump gives \(\bar{T}_2\) just after it. A linear distributed mean temperature zone between the flame or heater and the outlet (located at \(x_2\)) is considered with the mean temperature profile defined by

\[
\bar{T}(x) = \bar{T}_2 + \frac{\bar{T}_3 - \bar{T}_2}{l_2} (x - x_1), \quad \text{where } x \in [x_1^+, x_2]
\]

(18)

Table 1. Parameters used in the analysis. They are fixed unless otherwise stated.

| Parameter | Value |
|-----------|-------|
| \(l_1\) (m) | 0.5 |
| \(l_2\) (m) | 0.5 |
| \(\hat{p}_1\) (Pa) | \(1 \times 10^3\) |
| \(\bar{T}_1\) (K) | 300 |
| \(\bar{T}_2\) (K) | 1200 |
| \(\bar{T}_3\) (K) | 600 |
| \(R_g\) (J/K/kg) | \((\gamma - 1) f_0/c_0\) |
| \(\nu\) (\((-\)) | 0.5 |
| \(\tau\) (\((-\)) | 0.5 |
| Inlet Mach number | 287 |

The monopole can be assumed completely transparent to any incident disturbances\(^{31}\) and can be considered to have no effect on the entropy waves.

The classical \(n - \tau\) model\(^{24}\) is used as the flame transfer function

\[
\mathcal{F} = n \rho e^{-\rho \tau}
\]

(21)

It is interesting to consider some experiments to estimate realistic heat losses (or axial mean temperature decreases) within combustion chambers. In the Pennsylvania State University experiment,\(^{2}\) the mean temperature decreases from \(\bar{T}_1 = 1591\) K in the flame zone \((x = 0)\) to \(\bar{T}_2 = 1193\) K at the combustor exit \((x = l_2 = 1.524)\). Assuming a linear mean temperature distribution,\(^{16}\) the mean temperature gradient equals \(dT/dx = (\bar{T}_3 - \bar{T}_2)/l_2 = -261.2\) K/m. In the MICCA combustor, strong heat losses occur — the mean temperature decreases from \(\bar{T}_2 = 1470\) K in the flame zone \((x = 0)\) to \(\bar{T}_3 = 1130\) K at the combustor exit \((x = l_2 = 0.241)\),\(^{4,32}\) giving a mean temperature gradient of \(-1410.8\) K/m. Based on these results, the parameters used in following analysis are chosen and listed in Table 1 unless otherwise stated. The mean temperature gradient equals \(-1200\) K/m in the present work.

For the boundary conditions, the inlet is always assumed anechoic, which means no acoustic waves are reflected. The entropy oscillation at the inlet is assumed to be zero. Combining these with equations (19) and (20) yields an equation linking the two transfer functions \(
\tilde{p}(x_1^+) / \hat{p}_L \) and \(\tilde{u}(x_1^+) / \hat{p}_L\)

\[
\frac{\tilde{p}(x_1^+)}{\hat{p}_L} + \frac{\mathcal{F}}{\hat{p}_L} \frac{\tilde{u}(x_1^+)}{\hat{p}_L} = e^{-i k_1^+ h_1}
\]

(22)
where \( \hat{p}_L = \hat{p}_1 \hat{c}_1 \hat{V}_L / A \) and \( k^+_i = \omega / (\hat{c}_1 + \hat{u}_1) \).

The outlet is assumed to be choked and the compact boundary condition from Marble and Candel\(^{17}\) is incorporated, written as

\[
\frac{2 \hat{u}(x_2)}{\hat{u}(x_2)} + \frac{\hat{p}(x_2)}{\hat{p}(x_2)} - \frac{\hat{p}(x_2)}{\hat{p}(x_2)} = 0
\]  

(23)

Replacing the parameters by their corresponding normalised coefficients leads to

\[
\frac{(\gamma - 1) \hat{p}_2}{\hat{p}(x_2)} \mathcal{M}(x_2) \hat{p}(x_2) - 2 \hat{U}(x_2) + \mathcal{M}(x_2) \hat{S}(x_2) = 0
\]

(24)

where \( \mathcal{M}(x) = \hat{u}(x) / \hat{c}_2 \) is the normalised mean velocity. From this equation, it can be seen that entropy waves also contribute to the acoustic reflection at the choked outlet, which is associated with the entropy noise.\(^{33,34}\)

The entropy-acoustic coupling (1) during their propagation from the flame or heater to the outlet and (2) at the choked outlet may both affect the acoustic response of the whole thermoacoustic system. In order to differentiate those two coupling effects, two cases are considered. In case 1, the entropy noise generated at the outlet is neglected, and the entropy and acoustic waves are decoupled at the outlet. In case 2, entropy noise generated at the outlet is taken into consideration.

### 3.1 Case 1: choked outlet without considering entropy noise

To neglect the entropy-generated noise at the choked outlet, the entropy term in equation (24) is assumed zero. This is the same as neglecting the acoustic reflection due to the entropy wave as used in Goh and Morgans\(^{35}\) and Yang and Morgans.\(^{36}\) It should also be noted that this boundary condition becomes a velocity perturbation node when the Mach number at the outlet tends to zero.

### 3.2 Case 2: full choked outlet boundary condition – considering entropy noise

In this case, the full expression for a choked outlet in equation (24) is used, so the entropy-acoustic coupling both during their propagations from the flame or heater to the outlet and at the choked outlet are considered.

Accounting for the acoustic-entropy communication due to the axial temperature gradient

### 3.3 Method 1: using three LEEs

One now considers the region \( (x \in [x_1, x_2]) \) downstream of the still compact flame or heater. It is possible to write out three LLEs as functions of \( \hat{p}, \hat{u} \) and \( \hat{s} \)

\[
\left( i \omega + \frac{\gamma \hat{u}}{\hat{p}} \right) \hat{p} + \hat{u} \frac{\partial \hat{p}}{\partial x} + \frac{\partial \hat{p}}{\partial x} \hat{u} + \frac{\gamma \hat{u}}{\hat{p}} \frac{\partial \hat{u}}{\partial x} = 0
\]  

(25)

\[
\left( i \omega + \frac{\partial \hat{u}}{\partial x} \right) \hat{u} + \hat{u} \frac{\partial \hat{u}}{\partial x} + \hat{u} \frac{\partial \hat{p}}{\partial x} + \frac{1}{\hat{p}} \frac{\partial \hat{p}}{\partial x} - \frac{\hat{u}}{c_p} \hat{s} = 0
\]

(26)

\[
i \omega \frac{\partial \hat{s}}{\partial x} + \hat{u} \frac{\partial \hat{u}}{\partial x} = c_p \hat{u} (1 - M^2) \alpha \left( \frac{\hat{p}}{\hat{p}} + \hat{u} \frac{\partial \hat{u}}{\partial x} \right)
\]

(27)

where \( \alpha = \frac{\gamma - 1}{\gamma + 1} \) is the normalised first-order differential of mean density. It should be noted herein that parameters \( \hat{b} \) and \( \hat{b} \) correspond to \( \hat{b}(x) \) and \( \hat{b}(x) \), respectively, in order to save space in the following equations. Substituting \( \hat{U} = \rho_2 \hat{c}_2 \hat{S}, \hat{S} = \rho_2 \hat{c}_2 \hat{s} / \hat{c}_p \) and the mean relations (see Li and Morgans\(^{3}\)) into the above three equations leads to

\[
(i k_2 - \gamma M \hat{p}) - M \frac{\partial \hat{p}}{\partial x} + \alpha M^2 \frac{\hat{p}}{\hat{p}} \hat{U} + \frac{\hat{p}}{\hat{p}} \frac{\partial \hat{U}}{\partial x} = 0
\]

(28)

\[
(i k_2 - \alpha M \hat{U} + M \frac{\partial \hat{U}}{\partial x} - \alpha M^2 \frac{\hat{p}}{\hat{p}} \hat{p} + \frac{\hat{p}}{\hat{p}} \frac{\partial \hat{p}}{\partial x} + \alpha M^2 \hat{S} = 0
\]

(29)

\[
i \omega \frac{\partial \hat{s}}{\partial x} + \frac{\partial \hat{U}}{\partial x} (1 - M^2) \alpha \left( \frac{\hat{p}}{\hat{p}} + \hat{u} \frac{\partial \hat{u}}{\partial x} \right)
\]

(30)

where \( k_2 = \omega / \hat{c}_2 \). The full prediction using these three LEEs is used as the reference to validate other methods.

### 3.4 Method 2: using two LEEs

To investigate the importance of capturing the interaction between the acoustic and entropy waves, two LEEs, which consider acoustic and entropy wave propagations to be independent, as in a uniform flow,\(^5\) are compared with the full three LEEs. The entropy term in equation (29) and acoustic terms in equation (30) thus vanish and one has the following two relations

\[
(i k_2 - \alpha M \hat{U} + M \frac{\partial \hat{U}}{\partial x} - \alpha M^2 \frac{\hat{p}}{\hat{p}} \hat{p} + \frac{\hat{p}}{\hat{p}} \frac{\partial \hat{p}}{\partial x} = 0
\]

(31)

and

\[
i \omega \frac{\partial \hat{s}}{\partial x} + \frac{\partial \hat{S}}{\partial x} = 0
\]

(32)
Equations (31) and (28) are used to calculate the acoustic field and are named “two LEEs” in the present work. The entropy wave can be evaluated from the 1-D convection equation (32), yielding

$$\tilde{S}(x) = \tilde{S}(x_1^t) e^{-i \omega \tau_c(x)}$$

where the convective time delay $\tau_c(x)$ equals

$$\tau_c(x) = \int_{x_1}^{x} \frac{dX}{u}$$

By comparing the acoustic and entropy wave fields predicted by the three LEEs and two LEEs, it is possible to evaluate the 1-D communication between them in the region with axial mean temperature gradient.

3.5 Method 3: using a Helmholtz solver

Acoustic propagation in an inhomogeneous medium (with a temperature gradient in this case) but without mean flows can be obtained by solving the Helmholtz equation. A Helmholtz solver is now used to predict the acoustic field, and by comparing the predictions to the reference result, the applicability of this approach can be investigated. Neglecting terms containing mean flow velocities (or Mach numbers) in equations (31) and (28) leads to two zero-mean-flow linearised equations

$$ik_2 \tilde{p} + \frac{\tilde{p}}{\rho} \frac{d \tilde{U}}{dx} = 0$$

$$ik_2 \tilde{U} + \frac{\tilde{p}}{\rho} \frac{dp}{dx} = 0$$

It should be noted that equations (35) and (36) can be combined to yield a Helmholtz equation as a function of pressure perturbation $\tilde{p}$

$$\frac{d^2 \tilde{p}}{dx^2} - \alpha \frac{dp}{dx} + \frac{\omega^2}{c^2} \tilde{p} = 0$$

which is consistent with that in Sujith et al.\textsuperscript{37} It should be noted that equations (35) and (36) are used in the present calculation, instead of equation (37). This method is combined with the entropy wave convection solution (equation (33)) in the following calculation, and is still termed “Helmholtz solver prediction”.

3.6 Method 4: using the analytical solutions

Recently, analytical solutions for the 1-D acoustic field with mean temperature gradient and mean flow were derived in the authors’ group.\textsuperscript{8} The analytical solution reproduces the acoustic field very accurately across a wide range of flow conditions which span both low and moderate-to-high subsonic Mach numbers. It always performs well when the frequency exceeds a certain value ($|\omega/c| > |\alpha|$ and $|\omega/c| > (|\beta|/2)^{1/2}$ for low and moderate-to-high subsonic Mach number flow, where $\beta = \frac{\rho c^2}{\gamma p}$ indicates the normalised second-order differential of mean density); when the mean temperature profile is linear, it also performs well at very low frequencies. The analytical solution of pressure perturbation, $\tilde{p}$, is expressed as

$$\tilde{p}(x, \omega) = C_1 \mathcal{P}_1(x, \omega) + C_2 \mathcal{P}_2(x, \omega)$$

where

$$\mathcal{P}_1 = \left( \frac{\rho c}{\gamma p} \right)^{1/2} \int_1^{\gamma p} \frac{\gamma p^2 - (x - x_1) \gamma p}{\gamma p^2 - (x - x_1) \gamma p}$$

$$\mathcal{P}_2 = \left( \frac{\rho c}{\gamma p} \right)^{1/2} \int_1^{\gamma p} \frac{\gamma p^2 - (x - x_1) \gamma p}{\gamma p^2 - (x - x_1) \gamma p}$$

$C_1$ and $C_2$ are two arbitrary coefficients which can be determined for given initial or boundary conditions. The analytical solution of velocity perturbation, $\tilde{U}$, is expressed as

$$\tilde{U}(x, \omega) = D_1(x, \omega) C_1 \mathcal{P}_1(x, \omega) - D_2(x, \omega) C_2 \mathcal{P}_2(x, \omega)$$

where

$$D_1 = \frac{\tilde{p} \bar{c} \omega \gamma p}{\gamma p \bar{c} \omega - \alpha M}$$

$$D_2 = \frac{\tilde{p} \bar{c} \omega \gamma p}{\gamma p \bar{c} \omega - \alpha M}$$

The entropy wave is also predicted using equation (33), which assumes that the acoustic wave does not affect the entropy wave convection.

4. Forcing technique

Instead of calculating the modal frequency and its corresponding growth rate of the system,\textsuperscript{38,39} a forcing technique is used in the present work to examine the performance of the four approaches and to investigate the interaction between the acoustic and entropy waves during their propagations through the region with axial mean temperature gradient. A pure harmonic pressure signal is generated by the actuator, to excite the system. As the inlet is an anechoic boundary, the
\[
\begin{bmatrix}
\frac{\xi^2 \bar{p}_i}{\rho_i} (M_1 - 1) & 0 & 0 & 0 \\
\frac{\xi^2 \bar{p}_i}{\rho_i} (1 - M_1) & \mathbf{B}_2 & \ldots & 0 \\
\frac{\xi^2 \bar{p}_i}{\rho_i} (\sigma_1 - \sigma_2) & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

three LEEs scheme: \(3N \times 3(N - 1)\) matrix

\[
\begin{bmatrix}
\mathbf{D}_1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{U}(x_1) / \bar{p}_L \\
\mathbf{T}_r(x(1)) \\
\mathbf{T}_u(x(1)) \\
\mathbf{T}_f(x(1)) \\
\vdots \\
\mathbf{T}_r(x(j)) \\
\mathbf{T}_u(x(j)) \\
\mathbf{T}_f(x(j)) \\
\vdots \\
\mathbf{T}_r(x(N)) \\
\mathbf{T}_u(x(N)) \\
\mathbf{T}_f(x(N)) \\
\end{bmatrix}
\times
\begin{bmatrix}
\zeta M_1 \\
1 + M_1^2 \\
\sigma_1 \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
\end{bmatrix}
= 
\begin{bmatrix}
e^{-\hat{k}_f^{1}t_i} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\xi^2 \bar{p}_i}{\rho_i} (M_1 - 1) & 0 & 0 \\
\frac{\xi^2 \bar{p}_i}{\rho_i} (1 - M_1) & \mathbf{B}_2 & \ldots & 0 \\
\frac{\xi^2 \bar{p}_i}{\rho_i} (\sigma_1 - \sigma_2) & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\frac{(\nu - 1) \bar{p}_i}{\rho(x_2)} M(x_2) & 0 & \frac{(\nu - 1) \bar{p}_i}{\rho(x_2)} M(x_2) & 0 \\
\end{bmatrix}
\]

two LEEs scheme: \((2N + 1) \times 2(N - 1)\) matrix

or Helmholtz solver scheme: \((2N + 1) \times 2(N - 1)\) matrix

\[
\begin{bmatrix}
\mathbf{D}_2 \\
\mathbf{D}_3 \\
\end{bmatrix}
\]
system is generally stable and self-excited thermoacoustic instability does not arise, except for the situation under which the intrinsic thermoacoustic instability occurs.\textsuperscript{41} For simulation of the LEEs, spatial discretisation is via a second-order finite difference scheme on a uniform grid containing \( N = 8001 \) points in the 1-D region \( x \in [x_1, x_2] \). The location of each point is indicated by \( x(j) \) where \( j = 1, 2, \ldots, N \). A linear system is then established, with the inlet boundary condition prescribed by equations (10) and (22) and corresponding outlet boundary conditions for Case 1 and Case 2. The transfer functions from \( \hat{p}_L \) to perturbations \( \hat{u} \) and \( \hat{S} \), shown below, can then be calculated using the four approaches.

\[
\begin{bmatrix}
\hat{u}(x_1)/\hat{p}_L \\
\hat{p}(x_1)/\hat{p}_L \\
\hat{u}(x_N)/\hat{p}_L \\
\hat{S}(x_1)/\hat{p}_L \\
\hat{c}_1/\hat{p}_L \\
\hat{c}_2/\hat{p}_L 
\end{bmatrix} = \begin{bmatrix}
\xi M_1 \\
\frac{1 + M_1^2}{\sigma_1} \\
\sigma_1 \\
0 \\
0 \\
\end{bmatrix} e^{-ik_1 h}
\]

(48)

\[
\begin{bmatrix}
\xi^2 \hat{p}_L (M_1 - 1) \\
\xi \hat{p}_L (1 - M_1)^2 \\
\xi \hat{p}_L (\sigma_1 - \sigma_2) \\
0 \\
0 \\
0 
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -p_1(x_1^+) & -p_2(x_1^+) \\
0 & 0 & -p_1(x_1^+) & p_2(x_1^+) \\
0 & 0 & aM(x_2)e^{-i\omega_1 T_s} & 0 \\
0 & 0 & 0 & aM(x_2)e^{-i\omega_1 T_s}
\end{bmatrix}
\]

(49)

\[
\begin{bmatrix}
\xi M_1 \\
\frac{1 + M_1^2}{\sigma_1} \\
\sigma_1 \\
0 \\
0 \\
\end{bmatrix} e^{-ik_1 h}
\]

(48)

\[
T_s = \hat{S}/\hat{p}_L
\]

(46)

where the two coefficients \( K_1 \) and \( K_2 \) are mathematically expressed as

\[
K_1 = \frac{(\gamma - 1)\hat{p}_2}{\hat{p}(x_2)} M(x_2) - 2D_1(x_2),
\]

\[
K_2 = \frac{(\gamma - 1)\hat{p}_2}{\hat{p}(x_2)} M(x_2) + 2D_2(x_2)
\]

Equation (47) is the linear system linking the three transfer functions \([T_p(x(j)), T_u(x(j)), T_s(x(j))]\) (where \( j = 1, 2, \ldots, N \)) at each grid node for the three LEEs. The first three rows of the matrix \( \mathbf{D}_1 \) and the array \( \mathbf{A}_1 \) correspond to the inlet boundary condition and the fourth row corresponds to the outlet boundary condition. It is noted that a coefficient \( a \) is multiplied to the coefficient corresponding to the transfer function.
The transfer functions \[ T_p(x|j), T_u(x|j), T_s(x|j) \] can then be obtained from \[ A_1/D_1 \], and are used as the reference. Equation (48) applies for the two LEEs and the Helmholtz solver algorithms. The entropy wave term at the outlet is not accounted for. This term is needed for the choked outlet boundary condition with generated entropy noise (equation (24) of Case 2). In this case, the delayed entropy coupled boundary condition is used, which assumes that the entropy wave generated from the still compact flame or the heater does not interact with the acoustic field before reaching the outlet. The entropy wave term thus can be evaluated using equation (33), which corresponds to the fourth term in the fourth row of matrix \[ D_2 \] (or \[ D_3 \]) in equation (48).

For the analytical algorithm, the two coefficients \( C_1 \) and \( C_2 \) are determined by equation (49). The delayed entropy coupled boundary condition is also used for Case 2. The transfer functions \[ T_p(x|j), T_u(x|j), T_s(x|j) \] can be reconstructed using equations (38)–(43).

In order to evaluate the contribution from the entropy wave to the acoustic wave in the region with the axial mean temperature gradient, one defines an error coefficient \( \varepsilon \), which is the superposition of the normalised pressure and velocity perturbation differences between predictions by the reference three LEEs (corresponding to subscript 1) and

---

**Figure 3.** Case 1. Evolution of transfer functions \( T_p, T_u \) and \( T_s \) with \( x/l_2, M_1 = 0.02, f_1/c_2 = 0.25, n_f = 0.5 \) and \( \tau_f = 1/(2f) \).

- (a) transfer function gain \( |T_p| \).
- (b) Transfer function phase \( \angle T_p \).
- (c) Transfer function gain \( |T_u| \).
- (d) Transfer function phase \( \angle T_u \).
- (e) Transfer function gain \( M_2^2 \times |T_s| \).
- (f) Transfer function phase \( \angle T_s \).
the two LEEs (corresponding to subscript 2), expressed as

\[ \varepsilon = \left( \frac{\sum_{j=1}^{N} |T_{p,1}(x(j)) - T_{p,2}(x(j))|^2}{\sum_{j=1}^{N} |T_{p,1}(x(j))|^2} \right)^{1/2} + \left( \frac{\sum_{j=1}^{N} |T_{u,1}(x(j)) - T_{u,2}(x(j))|^2}{\sum_{j=1}^{N} |T_{u,1}(x(j))|^2} \right)^{1/2} \]  

(50)

Replacing \( T_{p,2}(x(j)) \) and \( T_{u,2}(x(j)) \) by \( T_{p,3}(x(j)) \) and \( T_{u,3}(x(j)) \), respectively, one defines a similar coefficient \( \varepsilon_H \) to quantify the difference between the calculated transfer functions using the reference three LEEs and the Helmholtz solver (corresponding to subscript 3).

5. Results and discussions

5.1 Preliminary analysis

Before presenting the predicted results from the four methods, one first analyses equation (29), describing the contribution of the entropy wave to the acoustic wave in the region with mean temperature gradient. The first four terms on the LHS of equation (29) contain acoustic contribution and the fifth term...
corresponds to the entropy contribution. Assuming that the pressure and velocity perturbation feature a wave form and can be approximately expressed as

\[ \hat{b}(x) \approx \hat{b}e^{-ikx} = \hat{b} \exp\left(-\frac{i\omega x}{c}\right) \]  

equation (29) can be simplified to

\[ \left(ik\frac{\hat{p}_2}{\rho} + M^2 \frac{\hat{p}_2}{\rho} \alpha\right)\hat{p} - (ik_2 - (\alpha + ik)M)\hat{u} \approx \alpha M^2 S \]  

From this equation, it appears that only when \( \alpha M^2 S \) has the same order of magnitude as the acoustic wave (\( \hat{p} \) or \( \hat{U} \)) can the communication between the entropy wave and acoustic wave be significant.

It is known that the upstream propagating component of the entropy-induced noise associated with the isentropic acceleration in the choked nozzle may interact with the flame and change the thermoacoustic instability of the combustor. The present work extends the entropy-induced noise family. Although it would be interesting to compare the noise levels induced by the two mechanisms, this is difficult as the entropy-induced noise relies on the upstream and
downstream acoustic impedances, the ratio of the incident acoustic wave to the entropy perturbation, etc. The LEEs linking the acoustic and entropy waves in the isentropic choked nozzle (e.g. see equations (6)–(8) in Marble and Candel17) have similar forms as those (equations (28)–(30)) in the present work. When the normalised first-order differential of cross-surface area of the isentropic nozzle \( \frac{1}{A} \frac{dA}{dx} \) tends to infinity (or the nozzle dimensions are sufficiently small in comparison with the shortest wave length in the flow field,17 the LEEs can be simplified to the choked boundary condition (equation (23) or (24)). Similarly, when \( \sigma \to \infty \) (or the width of the zone with mean temperature gradient is sufficiently small compared to the shortest wave length), equation (29) in the present work can be approximately reduced to

\[
\mathcal{M} \frac{\hat{p}_2}{\hat{p}} \hat{\mathcal{U}} - \mathcal{M} \hat{S} = 0
\]

The contribution to the acoustic wave (\( \hat{p} \) or \( \hat{\mathcal{U}} \)) from the entropy perturbation \( \hat{S} \) is of the same order as that from the compact choked nozzle, as modelled in equation (24).

Due to the nonlinear response of the flame, the flame describing function may change during the establishment of limit cycle (the change rate in the gain is generally within 50\%,43,44), which in turn changes the entropy production from the flame (see equation (16)). Any entropy perturbations associated with

Figure 6. Case 1. Contour maps of \( \epsilon \) as functions of \( f \times \tau_f \) (corresponding to the \( x \) axis) and \( n_f \) (corresponding to the \( y \) axis) for operating under three inlet Mach number conditions: (a) \( M_1 = 0.02 \), (b) \( M_1 = 0.05 \) and (c) \( M_1 = 0.15 \). \( \beta_2/\varepsilon_2 = 0.25 \).
reactants downstream of the flame is not considered in the present work. Generally, their impact on the entropy-acoustic coupling depends on their strength. Given a good model for their generation, the present models could be used to study resulting linear entropy-acoustic coupling. For very large entropy perturbations, a nonlinear analysis in entropy perturbations would need to be performed, most likely building on the work of Huet and Giauque.45

5.2 Results for Case 1

One first analyses the results for Case 1, where the interaction between acoustic and entropy waves only occurs in the region with axial mean temperature gradient. This does not require any model for the entropy wave at the outlet when using the latter three calculation methods. Figure 3 shows the predicted gains and phases of the transfer functions $T_p$, $T_u$, and $T_s$ against axial distance in the case of a very low Mach number mean flow; the Mach number in the upstream region is $M_1 = 0.02$ and that in the entrance of the region with mean temperature gradient equals $M_2 \approx 0.04$. The normalised frequency equals $fL/c_2 = 0.25$ and the gain and time delay of the flame model equal $n_f = 0.5$ and $\tau_f = 1/(2f)$, respectively. When $M_1 \to 0$, the outlet boundary becomes a velocity perturbation node. In Figure 3(c), the gain $|T_u|$ at the outlet approximately equals 0; that agrees with the theoretical analysis. Predictions of $T_p$ and $T_u$ using the two LEEs, the
Helmholtz solver and the analytical solutions show excellent agreement with those predicted using the reference three LEEs. There is a small mismatch of phase (Figure 3(a)) between the Helmholtz solver and the reference three LEEs because the Helmholtz solver totally neglects the mean flow. The gains of predicted $T_s$ using the latter three methods exhibit small errors compared to those from the reference three LEEs (Figure 3(e)), because they neglect the communication between the acoustic and entropy waves (equations (30) and (32)). The same trend can also be found for a larger modulating frequency $f_l/\bar{c}_2 = 0.5$, presented in Figure 4. It should be further noticed that the value $M_2^2 |T_s|$ is generally much smaller than $|T_p|$ or $|T_u|$ for the small Mach number situations; the contribution from the entropy wave to the acoustic wave thus can be neglected, resulting in no difference between the predicted acoustic field using three and two LEEs.

Results for an increased inlet mean flow Mach number of $M_1 = 0.15$ ($M_2 = 0.39$) are shown in Figure 5. The predictions using the latter three methods now exhibit big errors, both in gains and phases. The value $M_2^2 |T_s|$ now is of the same order of magnitude as $|T_p|$ and $|T_u|$, leading to the contribution from the

Figure 8. Case 1. (a) Plots of $M_2^2 \times |\bar{S}(x_\perp)|/P(x_\perp)$ vs. $M_1$ (corresponding to the bottom x axis) or $M_2$ (corresponding to the top x axis) for four operating conditions. (b) Plots of $\varepsilon$ vs. $M_1$ or $M_2$. (c) Plots of $\varepsilon_{\bar{H}}$ vs. $M_1$ or $M_2$. $\tau_f = 0.7/f$.
entropy wave to the acoustic wave being significant. It is interesting to find that the predicted phase of $T_s$ using the latter three methods, which basically uses equations (33) and (34), matches that calculated using the reference three LEEs very well, even for the large Mach number: $M_2 = 0.39$. This is because the propagation of the entropy wave itself is generally affected little in these cases. The prediction using the analytical solution generally matches that from the two LEEs, because the analytical solutions are derived based on the same assumptions.$^8$

A parametric study is now conducted to investigate the interaction between the acoustic wave and the entropy wave at other frequencies $f l_2 / c_2$, gains $n_f$ and time delays $\tau_f$ of the $n - \tau$ model. Figure 6 shows the coefficient $\varepsilon$ as functions of the time delay $\tau_f = 1 / f$ (corresponding to the $x$ axis) and the gain $n_f$ (corresponding

Figure 9. Case 1. Plots of $\varepsilon$ vs. $T_3 / T_2$. $M_1 = 0.15$ and $\tau_f = 0.7 / f$.

Figure 10. Case 2. Evolution of transfer functions $T_p$, $T_u$ and $T_s$ with $x / l_2$, $M_1 = 0.02$, $l_2 / c_2 = 0.25$, $n_f = 0.5$ and $\tau_f = 1 / (2f)$. (a) Transfer function gain $|T_p|$. (b) Transfer function phase $\angle T_p$. (c) Transfer function gain $|T_u|$. (d) Transfer function phase $\angle T_u$. (e) Transfer function gain $M_2^2 \times |T_s|$. (f) Transfer function phase $\angle T_s$. 
to the y axis) for three inlet Mach numbers $M_1 = 0.02, 0.05$ and 0.15, when the normalised frequency equals $fl_2/\tilde{e}_2 = 0.25$. The maximum values of $\log_{10}(\varepsilon)$ for the three Mach numbers are $-1.91$, $-1.07$ and $-0.36$, respectively. Figure 7 shows the coefficient $\varepsilon$ as functions of the time delay $\tau_f = 1/f$ (corresponding to the x axis) and the normalised frequency $fl_2/\tilde{e}_2$ (corresponding to the y axis) for these three Mach number situations, when the gain equals $n_f = 0.5$. The maximum values of $\log_{10}(\varepsilon)$ are $-1.74$, $-0.56$ and $-0.12$, respectively. When the inlet Mach number of the region with the axial mean temperature gradient $M_2$ is smaller than 0.1 ($M_1 \leq 0.05$), the coefficient $\varepsilon$ generally remains small and the interaction between the acoustic wave and entropy wave can be neglected.

It is also useful to define a coefficient to evaluate the acoustic wave strength

$$\mathcal{P}(x) = (2\tilde{e}^2/\varepsilon)^{1/2}$$  \hspace{1cm} (54)

where $\tilde{e}$ is the acoustic energy density and is defined by

$$\tilde{e} = \frac{\bar{p}^2}{2\rho c^2} + \frac{\bar{u}^2}{2} + \frac{\bar{w}^2}{\tilde{e}}$$  \hspace{1cm} (55)
It should be noted that when $Z^{\frac{1}{2}} = 0$, $\tilde{U} = 0$ and $P_{f} = |\tilde{P}|$. When $Z = 0$, $\tilde{\rho} = 0$ and $P_{f} = \tilde{\rho} c^{2}/(\tilde{\rho}_{2} c_{2})$. One now examines the coefficients $M_{2}^{f} S(x_{f}^{+})/P(x_{f}^{+})$, $\epsilon$ and $\epsilon_{H}$ as the function of $M_{1}$ (corresponding to the bottom $x$ axis) or $M_{2}$ (corresponding to the top $x$ axis) for four cases. As shown in Figure 8(a), $M_{2}^{f} S(x_{f}^{+})/P(x_{f}^{+})$ increases with increasing $M_{1}$ or $M_{2}$, indicating more contribution from the entropy wave to the acoustic wave and larger differences between the predictions using three and two LEEs; this agrees with results shown in Figure 8(b). The evolution of $\epsilon_{H}$ with the inlet Mach number $M_{1}$ or $M_{2}$ is also investigated, with the results shown in Figure 8(c). When $M_{2} \leq 0.1$, the error coefficient $\epsilon_{H}$ remains small and the Helmholtz solver accurately predicts the acoustic field.

The effect of mean temperature ratio $\tilde{T}_{3}/\tilde{T}_{2}$ on the interaction between acoustic and entropy waves is also investigated, with results shown in Figure 9. As $\tilde{T}_{3}/\tilde{T}_{2}$ decreases, indicating more enthalpy losses and larger normalised mean temperature gradient $\alpha$, the error coefficient $\epsilon$ increases and the communication between these two waves increases, in agreement with the preliminary analysis.

5.3 Results for Case 2

One now considers the Case 2, in which entropy noise is also produced at the choked outlet. In equation (24), the coefficient of entropy wave is proportional to the Mach number; while in equation (29), the corresponding coefficient contains the square of the Mach number.
The contribution from the entropy wave to the acoustic wave at the choked outlet is thus much larger than that in the region with the mean temperature gradient. When using two LEEs or the Helmholtz solver, the error in the predicted entropy wave field (equation (32)) accumulates during the entropy wave convection, and may be amplified at the choked outlet and leads to larger errors in the acoustic field prediction. It is thus worth repeating the previous analyses for Case 2.

Figures 10 and 11 show the predicted gains and phases of the transfer functions $T_p$, $T_u$ and $T_s$ against axial distance in the case of a very small Mach number $M_1 = 0.02$ and a moderate Mach number $M_1 = 0.15$. By comparing Figure 10 to Figure 3, it is found that the evolution of acoustic wave with axial location are totally changed due to the presence of the entropy term $M(x_2)S(x_2)$, even for a very small Mach number $M_1 = 0.02$ mean flow. Predictions of the three transfer functions $T_p$, $T_u$ and $T_s$ using two LEEs, the Helmholtz solver and the analytical solution still show excellent agreement with those predicted using the three LEEs method for very small Mach numbers, e.g. $M_1 = 0.02$, but exhibit big errors when $M_2 \geq 0.1$ ($M_1 \geq 0.05$).

The error coefficient $\varepsilon$ for other gains $n_f$ of $n - \tau$ model, time delays $\tau_f$ and normalised frequency $f l_2/\omega_2$ are also examined, with results shown in Figures 12 and 13. The evolution of $M_2^2 S(x_1^+)/P(x_1^+)$, $\varepsilon$ and $\varepsilon_n$ with the inlet Mach number $M_1$ or $M_2$ is also investigated, with results shown in Figure 14. The trend generally matches that for the Case 1.

The effect of mean temperature ratio $T_3/T_2$ is also investigated, as shown in Figure 15. The trend also matches that for the Case 1.

Figure 13. Case 2. Contour maps of $\varepsilon$ as functions of $f \times \tau_f$ (corresponding to the x axis) and $f l_2/\omega_2$ (corresponding to the y axis) for operating under three inlet Mach number conditions: (a) $M_1 = 0.02$, (b) $M_1 = 0.05$ and (c) $M_1 = 0.15$, $n_f = 0.5$. 

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6. Conclusions

This work has presented theoretical and numerical analyses of the entropy wave generated by a still compact flame or a heater and its convection in a region with an axial mean temperature gradient. Jump conditions linking acoustic waves and entropy waves either sides of the flame have been constructed, and simplified to expressions of first order and of zero order in the mean flow Mach number. It has been shown that the generated entropy wave is generally proportional to the mean temperature jump across the flame ($\zeta^2 - 1$) and the ratio ($F - 1$), where $F$ is the flame transfer function.

It is inversely proportional to the Mach number immediately downstream of the flame, $M_2$.

A 1-D duct with a still compact flame or heater and a region with an axial mean temperature gradient between the flame and the outlet was considered to investigate the interaction between the entropy and acoustic waves during their propagations in this region. The inlet of the duct is an anechoic boundary and the outlet is choked and without (Case 1) and with (Case 2) entropy noise produced at the choked outlet. In order to identify the interaction between entropy and acoustic waves in this region, full three LEEs as functions of pressure, velocity and entropy perturbations.

Figure 14. Case 2. (a) Plots of $M_2^2 \times \frac{\dot{S}(x^*)}{P(x^*)}$ vs. $M_1$ (corresponding to the bottom x axis) or $M_2$ (corresponding to the top x axis) for four operating conditions. (b) Plots of $\epsilon$ vs. $M_1$ or $M_2$. (c) Plots of $\epsilon_H$ vs. $M_1$ or $M_2$. $\tau_f = 0.7/f$. 
and two LEEs as functions of only pressure and velocity perturbations have been used to reconstruct the acoustic and entropy wave fields. In the two LEEs, it was assumed that these two waves propagate independently. Numerical results have shown that differences between the predicted acoustic and entropy wave fields using the two methods are small and can be neglected for small Mach number mean flow. As the Mach number, $M_2$, increases, differences arise which cannot be neglected when $M_2/C_0 > 1$; the entropy wave greatly changes the acoustic field in mean temperature gradient region. The performance of a Helmholtz solver and a recently developed analytical method have also been investigated. Numerical simulations have shown that the prediction using the analytical method generally matches that calculated using two LEEs. It has also been shown that the Helmholtz solver can generally be used to determine the acoustic field when $M_2 \leq 0.1$. Similar results have been found for both the cases.

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Appendix

Notation

\( A \) the cross-sectional area of the duct
\( \bar{c}_1 \) the speed of sound upstream of the compact flame
\( \bar{c}_2 \) the speed of sound immediately downstream of the compact flame
\( c_p \) the heat capacity at constant pressure, considered constant in the present work
\( f \) the frequency
\( \mathcal{F} \) the flame transfer function or flame describing function
\( l_1 \) the distance between the actuator and the flame
\( l_2 \) the length of the zone with mean temperature gradient
\( M \) the Mach number: \( M = \bar{u}/\bar{c}_2 \)
\( M_j \) the Mach number: \( M_j = \bar{u}_j/\bar{c}_j \)
\( M^* \) the normalised mean velocity: \( M^* = \bar{u}_1/\bar{c}_2 \)
\( n_f \) the gain of the \( n - \tau \) flame model
\( \bar{p}_1 \) the mean pressure upstream of the compact flame
\( \bar{p}_2 \) the mean pressure immediately downstream of the compact flame
\( p_L \) the acoustic pressure from the actuator
\( \dot{Q} \) the heat release rate from the compact flame or heater
\( R_g \) the gas constant
\( s \) the entropy
\( \mathcal{S} \) the normalised entropy perturbation: \( \mathcal{S} = \bar{c}_2^2 \text{\( s \text{\!/} c_p \)} \)
\( \bar{T}_1 \) the mean temperature upstream of the compact flame
\( \bar{T}_2 \) the mean temperature immediately downstream of the compact flame
\( \bar{T}_3 \) the mean temperature at the outlet
\( T_p \) the transfer function \( T_p = \dot{\bar{p}}/\bar{p}_L \)
\( T_u \) the transfer function \( T_u = \dot{\bar{U}}/\bar{p}_L \)
\( T_s \) the transfer function \( T_s = \mathcal{S}/\bar{p}_L \)
\( \bar{u}_1 \) the mean velocity upstream of the compact flame
\( \bar{u}_2 \) the mean velocity immediately downstream of the compact flame
\( \dot{\bar{U}} \) the normalised velocity perturbation
\( \dot{\bar{V}}_L \) the volume flow rate of the actuator
\( x \) the axial location
\( x_0 \) the location of the actuator
\( x_1 \) the location of the still compact flame or heater
\( x_2 \) the location of combustor end
\( Z_f \) the acoustic impedance immediately upstream of the compact flame
\( \alpha \) the normalised first-order differential of mean density: \( \alpha = \frac{1}{\bar{p}_2} \frac{\partial \bar{p}}{\partial x} \)
\( \beta \) the normalised second-order differential of mean density: \( \beta = \frac{1}{\bar{p}_2} \frac{\partial^2 \bar{p}}{\partial x^2} \)
\( \gamma \) the ratio of specific heats
\( \bar{\rho}_1 \) the mean density upstream of the compact flame
\( \bar{\rho}_2 \) the mean density immediately downstream of the compact flame
\( \bar{\rho}_\omega \) the complex angular frequency
\( \bar{\rho}_\zeta \) the square root of the ratio of mean temperature across the compact flame: \( \zeta = (\bar{T}_2/\bar{T}_1)^{1/2} = \bar{c}_2/\bar{c}_1 \)
\( \tau_f \) the time delay of the \( n - \tau \) flame model
\( \varepsilon \) the error coefficient quantifying the difference between the transfer functions using the reference three LEEs and two LEEs
\( \epsilon_H \) the error coefficient quantifying the difference between the transfer functions using the reference three LEEs and the Helmholtz solver

\( \tilde{\cdot} \) the mean value
\( \dot{\cdot} \) the perturbation

Script