Multiplayer Rock-Paper-Scissors

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We will view the game of RPS as a magma. We let $A := \{r, p, s\}$ and define a binary operation $f: A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

|     | r | p | s |
|-----|---|---|---|
| r   | r | p | r |
| p   | p | p | s |
| s   | r | s | s |
A selection game is a game consisting of a collection of items $A$, from which a fixed number of players $n$ each choose one, resulting in a tuple $a \in A^n$, following which the round’s winners are those who chose $f(a)$ for some fixed rule $f : A^n \to A$. RPS is a selection game, and we can identify each such game with an $n$-ary magma $A := (A, f)$. 
Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.
Properties of RPS: Conservativity

We say that an operation $f: A^n \rightarrow A$ is conservative when for any $a_1, \ldots , a_n \in A$ we have that $f(a_1, \ldots , a_n) \in \{a_1, \ldots , a_n\}$. We say that $A$ is conservative when each round has at least one winning player.
We say that an operation $f : A^n \rightarrow A$ is *essentially polyadic* when there exists some $g : \text{Sb}(A) \rightarrow A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$. We say that $A$ is essentially polyadic when a round’s winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item.
Let $A_k$ denote the members of $A^n$ which have $k$ distinct components for some $k \in \mathbb{N}$. We say that $f$ is strongly fair when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have
\[
|f^{-1}(a) \cap A_k| = |f^{-1}(b) \cap A_k|.
\]
We say that $A$ is strongly fair when each item has the same chance of being the winning item when exactly $k$ distinct items are chosen for any $k \in \mathbb{N}$.
We say that $f$ is *nondegenerate* when $|A| > n$. In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components. If $A$ is essentially polyadic with $|A| \leq n$ it is impossible for $A$ to be strongly fair unless $|A| = 1$. 
The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic. The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.
The only “valid” RPS variants for two players use an odd number of items.

**Theorem**

Let $A$ be a selection game with $n = 2$ which is essentially polyadic, strongly fair, and nondegenerate and let $m := |A|$. We have that $m \neq 1$ is odd. Conversely, for each odd $m \neq 1$ there exists such a selection game.
**Definition (RPS magma)**

Let $A := (A, f)$ be an $n$-ary magma. When $A$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $A$ is an RPS *magma*. When $A$ is an $n$-magma of order $m$ with these properties we say that $A$ is an RPS($m, n$) *magma*. We also use RPS and RPS($m, n$) to indicate the classes of such magmas.
Theorem

Let $A$ be a selection game with $n$ players and $m$ items which is essentially polyadic, strongly fair, and nondegenerate. For all primes $p \leq n$ we have that $p \nmid m$. Conversely, for each pair $(m, n)$ with $m \neq 1$ such that for all primes $p \leq n$ we have that $p \nmid m$ there exists such a selection game.
Since \( A \) is nondegenerate we must have that \( m > n \).
Since \( A \) is strongly fair we must have that
\[
|f^{-1}(a) \cap A_k| = |f^{-1}(b) \cap A_k| \quad \text{for all } k \in \mathbb{N}.
\]
As the \( m \) distinct sets \( f^{-1}(a) \cap A_k \) for \( a \in A \) partition \( A_k \) and are all the same size we require that \( m \mid |A_k| \).
When \( k > n \) we have that \( A_k = \emptyset \) and obtain no constraint on \( m \).
Proof (Forward Direction)

When $k \leq n$ we have that $A_k$ is nonempty. As we take $A$ to be essentially polyadic we have that $f(x) = f(y)$ for all $x, y \in A_k$ such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. Let $B_k$ denote the collection of unordered sets of $k$ distinct elements of $A$. Note that the size of the collection of all members $x \in B_k$ such that $\{x_1, \ldots, x_n\} = \{z_1, \ldots, z_k\}$ for distinct $z_i \in A$ does not depend on the choice of distinct $z_i$. This implies that for a fixed $k \leq n$ each of the $m$ items must be the winner among the same number of unordered sets of $k$ distinct elements in $A$. We have that $|B_k| = \binom{m}{k}$ so we require that $m \mid |B_k| = \binom{m}{k}$ for all $k \leq n$. 
Proof (Forward Direction)

Let

\[ d(m, n) := \gcd \left( \left\{ \binom{m}{k} \mid 1 \leq k \leq n \right\} \right). \]

Since \( m \mid \binom{m}{k} \) for all \( k \leq n \) we must have that \( m \mid d(m, n) \). Joris, Oestreicher, and Steinig showed that when \( m > n \) we have

\[ d(m, n) = \frac{m}{\text{lcm}(\{ k^{\varepsilon_k(m)} \mid 1 \leq k \leq n \})} \]

where \( \varepsilon_k(m) = 1 \) when \( k \mid m \) and \( \varepsilon_k(m) = 0 \) otherwise. Since we have that \( m \mid d(m, n) \) and \( d(m, n) \mid m \) it must be that \( m = d(m, n) \) and hence

\[ \text{lcm} \left( \left\{ k^{\varepsilon_k(m)} \mid 1 \leq k \leq n \right\} \right) = 1. \]

This implies that \( \varepsilon_k(m) = 0 \) for all \( 2 \leq k \leq n \). That is, no \( k \) between 2 and \( n \) inclusive divides \( m \). This is equivalent to having that no prime \( p \leq n \) divides \( m \), as desired.
Our numerical condition also allows us to fix the number of items $m$ and ask how many players $n$ may use that number of items.

**Theorem**

*Given a fixed $m$ there exists an RPS($m, n$) magma if and only if $n < t(m)$ where $t(m)$ is the least prime dividing $m$.***
The class RPS is not closed under taking subalgebras. The French variant is a subalgebra of Rock-Paper-Scissors-Spock-Lizard. The class of RPS magmas is as far from being closed under products as possible.

**Theorem**

Let $A$ and $B$ be nontrivial RPS $n$-magmas with $n > 1$. The magma $A \times B$ is not an RPS magma.

This can be done by showing that the product $A \times B$ is not conservative.
Current Directions

1. Geometric interpretation as in tournaments.
2. Asymptotics on conservativity.
3. Properties of clones. Note the connection with cyclic/symmetric groups.
Thank you.