Multivalued backward stochastic differential equations with oblique subgradients

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October 4, 2013

Abstract

We study the existence and uniqueness of the solution for the following backward stochastic variational inequality with oblique reflection (for short, $BSVI (H(t,y), \varphi, F)$), written under differential form

$$\begin{cases}
-dY_t + H(t,Y_t) \partial \varphi (Y_t) \, (dt) \ni F(t,Y_t,Z_t) \, dt - Z_t \, dB_t, & t \in [0,T], \\
Y_T = \eta,
\end{cases}$$

where $H$ is a bounded symmetric smooth matrix and $\varphi$ is a proper convex lower semicontinuous function, with $\partial \varphi$ being its subdifferential operator. The presence of the product $H \partial \varphi$ does not permit the use of standard techniques because it does conserve neither the Lipschitz property of the matrix nor the monotonicity property of the subdifferential operator. We prove that, if we consider the dependence of $H$ only on the time, the equation admits a unique strong solution and, allowing the dependence also on the state of the system, the above $BSVI (H(t,y), \varphi, F)$ admits a weak solution in the sense of the Meyer-Zheng topology. However, for that purpose we must renounce at the dependence on $Z$ for the generator function and we situate our problem in a Markovian framework.

Keywords and phrases: multivalued backward stochastic differential equations, oblique reflection, subdifferential operators, Meyer-Zheng topology

2010 AMS Subject Classification: 60H10, 60H30, 93E03

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$^*$The work for this paper was supported by founds from the Grant PN-II-ID-PCE-2011-3-0843, Deterministic and stochastic systems with state constraints, no. 241/05.10.2011.

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1 Introduction

Backward stochastic differential equations (in short BSDE’s) were first introduced by Bismut in 1973 in the paper [2] as equation for the adjoint process in the stochastic version of Pontryagin maximum principle. In 1990, Pardoux and Peng [18] generalized and consecrated the well known now notion of nonlinear backward stochastic differential equation and they provided existence and uniqueness results for the solution of this kind of equation. Starting with the paper of Pardoux and Peng [19], a stochastic approach to the existence problem of a solution for many types of deterministic partial differential equations has been developed. Since then the interest in BSDEs has kept growing, both in the direction of generalization of the emerging equations and construction of approximation schemes for them. BSDEs have been widely used as a very useful instrument for modelling various physical phenomena, in stochastic control and especially in mathematical finance, as any pricing problem, by replication, can be written in terms of linear BSDEs, or non-linear BSDEs with portfolios constraints. Pardoux and Răşcanu [20] proved, using a probabilistic interpretation, the existence of the viscosity solution for a multivalued PDE (with subdifferential operator) of parabolic and elliptic type.

Backward stochastic variational inequalities (for short, BSVIs) were first analyzed by Pardoux and Răşcanu in [20] and [21] (the extension for Hilbert spaces case), by using a method that consisted of a penalizing scheme, followed by its convergence. Even though this type of penalization approach is very useful when dealing with multivalued backward stochastic dynamical systems governed by a subdifferential operator, it fails when dealing with a general maximal monotone operator. This motivated a new approach for the later case of equations, via convex analysis instruments. In [24], Răşcanu and Rotenstein established, using the Fitzpatrick function, a one-to-one correspondence between the solutions of those types of equations and the minimum points of some proper, convex, lower semicontinuous functions, defined on well-chosen Banach spaces.

Multi-dimensional backward stochastic differential equations with oblique reflection (in fact BSDEs reflected on the boundary of a special unbounded convex domain along an oblique direction), which arises naturally in the study of optimal switching problem were recently studied by Hu and Tang in [9]. As applications, the authors apply the results to solve the optimal switching problem for stochastic differential equations of functional type, and they give also a probabilistic interpretation of the viscosity solution to a system of variational inequalities.

It worth mentioning that, until now, even for quite complex problems like the ones analyzed by Maticiuc and Răşcanu in [15] or [16], when dealing with BSVIs, the reflection was made upon the normal direction at the frontier of the domain and it was caused by the presence of the subdifferential operator of a convex lower semicontinuous function. As the main achievement of this paper we prove the existence and uniqueness of the solution for the more general BSVI with oblique subgradients

\[
\begin{cases}
-dY_t + H(t, Y_t) \partial \varphi(Y_t) (dt) \ni F(t, Y_t, Z_t) dt - Z_t dB_t, & t \in [0, T], \\
Y_T = \eta,
\end{cases}
\]

where \(B\) is a standard Brownian motion defined on a complete probability space, \(F\) is the generator function and the random variable \(\eta\) is the terminal data. The term \(H(X)\) acts
on the set of subgradients, fact which will determine a oblique reflection for the feedback process. A similar setup was constructed and studied for forward stochastic variational inequalities by Gassous, Răşcanu and Rotenstein in [8] by considering first a (deterministic) generalized Skorokhod problem with oblique subgradients, prior to the general stochastic case. In the current paper the problems also rise when we operate with the product \( H(t, Y_t) \partial \varphi(Y_t) \), which does not inherit neither the monotonicity of the subdifferential operator nor the Lipschitz property of the matrix involved, problems which will be overcome by using different methods compared to the ones used for subgradients reflected upon the normal direction. We will split our problem into two new ones. For the situation when we have only a time dependence for the matrix \( H \) we obtain the existence of a strong solution, together with the existence of a absolutely continuous feedback-subgradient process. However, for the general case of a state dependence for \( H \) we will use tightness criteria in order to get a solution for the equation. In [5], Buckdahn, Engelbert and Răşcanu discussed the concept of weak solutions of a certain type of backward stochastic differential equations (not multivalued). Using weak convergence in the Meyer–Zheng topology, they provided a general existence result. We will put also our problem into a Markovian framework. The problem consists in answering in which sense can we take the limit in the sequence \( \{(Y^n, Z^n, U^n)\}_n \), given by the solutions of the approximating equations. We have to prove that it is tight in a certain topology. Even the \( S-\)topology introduced by Jakubowski in [11] (and used for similar setups by Boufoussi and Casteren [3] or LeJai [12]) seems suitable for our context, the regularity of the subgradient process given by the approximating equation as part of its solution permits us to show a convergence in the sense of the Meyer-Zheng topology, that is the laws converge weakly if we equip the space of paths with the topology of convergence in \( dt-\)measure. The tightness of \( \{Z^n\}_n \) is hard to get, therefore we renounce at the dependence on \( Z \) for the generator function of the multivalued backward equation. This framework permits also to analyze the existence of viscosity solutions for systems of parabolic variational inequalities driven by generalized subgradients.

The article is organized as follows. Section 2 presents the framework of our study, the assumptions and the hypotheses on the data, the notions of weak and strong solution for the equations and it closes with the enunciations of the main results of the paper, the complete proofs representing the core of Sections 4 and 5. Section 3 is dedicated to some useful a priori estimates for the solutions of the approximating equations. Section 4 proves the strong existence and uniqueness of the solution when the matrix \( H \) does not depend on the state of the system, while Section 5 deals with the existence of a weak solution for the general case of \( H = H(t, y) \). For the clarity of the presentation, the last part of the paper groups together, under the form of an Annex with three subsections, some useful results that are used throughout this article.

2 Setting the problem

This section is dedicated to the construction of the problem that we will study in the sequel. We present the hypothesis imposed on the coefficients and we formulate the main results of this article. The proofs will be detailed in the next three sections.

Let \( T > 0 \) be fixed and consider the backward stochastic variational inequality with oblique reflection (for short, we will write \( BSVI(H(t, y), \varphi, F), BSVI(H(t), \varphi, F) \) or, re-
respectively, BSVI \((H(y), \varphi, F)\) if the matrix \(H\) depends only on time or, respectively, on the state of the system, \(\mathbb{P} - a.s. \omega \in \Omega\),

\[
\begin{align*}
Y_t + \int_t^T H(s, Y_s) dK_s &= \eta + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \\
dK_s &\in \partial \varphi (Y_s)(ds),
\end{align*}
\]

where

\((H_1)\) \(\langle \Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0} \rangle\) is a stochastic basis and \(\{B_t : t \geq 0\}\) is a \(\mathbb{R}^k\)-valued Brownian motion. Moreover, \(\mathcal{F}_t = \mathcal{F}_t^B = \sigma(\{B_s : 0 \leq s \leq t\}) \cup \mathcal{N}\).

\((H_2)\) \(H(\cdot, \cdot, y) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{d \times d}\) is progressively measurable for every \(y \in \mathbb{R}^d\); there exists \(\Lambda, b > 0\) such that \(\mathbb{P}-a.s. \omega \in \Omega, H = (h_{i,j})_{d \times d} \in \mathcal{C}^{1,2} (\mathbb{R}_+ \times \mathbb{R}^d ; \mathbb{R}^{d \times d})\) and, for all \(t \in [0, T]\) and \(y, \tilde{y} \in \mathbb{R}^d, \mathbb{P}-a.s. \omega \in \Omega,\)

\[
\begin{align*}
(i) & \quad h_{i,j}(t, y) = h_{j,i}(t, y), \quad \forall i, j \in \overline{1,d}, \\
(ii) & \quad \langle H(t, y) u, u \rangle \geq a |u|^2, \quad \forall u \in \mathbb{R}^d \text{ (for some } a \geq 1), \\
(iii) & \quad |H(t, \tilde{y}) - H(t, y)| + |H(t, \tilde{y})|^{-1} - |H(t, y)|^{-1} \leq \Lambda |\tilde{y} - y|, \\
(iv) & \quad |H(t, y)| + |H(t, y)|^{-1} \leq b,
\end{align*}
\]

where \(|H(x)| = \left( \sum_{i,j=1}^d |h_{i,j}(x)|^2 \right)^{1/2}\). We denoted by \([H(t, y)]^{-1}\) the inverse matrix of \(H(t, y)\). Therefore, \([H(t, y)]^{-1}\) has the same properties \((2- (i), (ii))\) as \(H(t, y)\).

\((H_3)\) the function

\(\varphi : \mathbb{R}^d \to []-\infty, +\infty]\) is a proper lower semicontinuous convex function.

The generator function \(F(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^d\) is progressively measurable for every \((y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}\) and there exist \(L, \ell, \rho \in L^2(0, T; \mathbb{R}_+)\) such that

\[
\begin{align*}
(i) & \quad \text{Lipschitz conditions: for all } y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times k}, d\mathbb{P} \otimes dt - a.e.: \\
& \quad |F(t, y', z) - F(t, y, z)| \leq L(t) |y' - y|, \\
\end{align*}
\]

\((H_4)\) \(\text{Boundedness condition:}

\[
\begin{align*}
& \quad |F(t, y, z') - F(t, y, z)| \leq \ell(t) |z' - z|;
\end{align*}
\]

Denote by \(\partial \varphi\) the subdifferential operator of \(\varphi:\)

\[
\partial \varphi (x) \overset{def}{=} \{ \hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle + \varphi(x) \leq \varphi(y) \}, \text{ for all } y \in \mathbb{R}^d \}
\]

and \(\text{Dom}(\partial \varphi) = \{ x \in \mathbb{R}^d : \partial \varphi(x) \neq \emptyset \}\). We will use the notation \((x, \hat{x}) \in \partial \varphi\) in order to express that \(x \in \text{Dom}(\partial \varphi)\) and \(\hat{x} \in \partial \varphi(x)\). The vector given by the quantity \(H(x) \hat{x}\), with \(\hat{x} \in \partial \varphi(x)\) will be called in what follows \(oblique\ \text{subgradient}\).
Remark 1 If $E$ is a closed convex subset of $\mathbb{R}^d$ then the convex indicator function
\[
\varphi(x) = I_E(x) = \begin{cases} 
0, & \text{if } x \in E, \\
\infty, & \text{if } x \notin E,
\end{cases}
\]
is a convex l.s.c. function and, for $x \in E$,
\[
\partial I_E(x) = \{ \hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle \leq 0, \forall y \in E \} = N_E(x),
\]
where $N_E(x)$ is the closed external normal cone to $E$ at $x$. We have $N_E(x) = \emptyset$ if $x \notin E$ and $N_E(x) = \{0\}$ if $x \in \text{int}(E)$ (we denote by $\text{int}(E)$ the interior of the set $E$).

We shall call oblique reflection directions at time $t$ the vectors given by
\[
\nu_{t,x} = H(t, x) n_x, \quad x \in Bd(E),
\]
where $n_x \in N_E(x)$ (we denote by $Bd(E)$ the boundary of the set $E$).

Let $k : [t, T] \to \mathbb{R}^d$, where $0 \leq t \leq T$. We denote, $\|k\|_{[t,T]} \overset{def}{=} \sup \{ |k(s)| : t \leq s \leq T \}$ and, for $t = 0$, $\|k\|_T \overset{def}{=} \|k\|_{[0,T]}$. Considering $\mathcal{D}[t,T]$ the set of the partitions of the time interval $[t,T]$, of the form $\Delta = (t = t_0 < t_1 < \ldots < t_n = T)$, let
\[
S_\Delta(k) = \sum_{i=0}^{n-1} \|k(t_{i+1}) - k(t_i)\|
\]
and $\uparrow k_{[t,T]} \overset{def}{=} \sup_{\Delta \in \mathcal{D}} S_\Delta(k)$; if $t = 0$, denote $\uparrow k_{[0,T]} \overset{def}{=} \uparrow k_{[0,T]}$. We consider the space of bounded variation functions $BV([0,T];\mathbb{R}^d) = \{ k : [0,T] \to \mathbb{R}^d, \|k\|_{[0,T]} < \infty \}$. Taking on the space of continuous functions $C([0,T];\mathbb{R}^d)$ the usual supremum norm, we have the duality connection $(C([0,T];\mathbb{R}^d))^* = \{ k \in BV([0,T];\mathbb{R}^d) \mid k(0) = 0 \}$, with the duality between these spaces given by the Riemann-Stieltjes integral $(y, k) \mapsto \int_0^T \langle y(t), dk(t) \rangle$. We will say that a function $k \in BV_{loc}(\mathbb{R}_+;\mathbb{R}^d)$ if, for every $T > 0$, $k \in BV([0,T];\mathbb{R}^d)$.

Definition 2 Given two functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ we say that $dk(t) \in \partial \varphi(x(t))(dt)$ if
\[
(a) \quad x : \mathbb{R}_+ \to \mathbb{R}^d \text{ is continuous}, \\
(b) \quad \int_0^T \varphi(x(t)) \, dt < \infty, \text{ for all } T \geq 0, \\
(c) \quad k \in BV_{loc}(\mathbb{R}_+;\mathbb{R}^d), \quad k(0) = 0, \\
(d) \quad \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r)) \, dr \leq \int_s^t \varphi(y(r)) \, dr,
\]
for all $0 \leq s \leq t \leq T$ and $y \in C([0,T];\mathbb{R}^d)$.

We introduce now the notion of solution for Eq.(1). We will study two types of solution, given by the following Definitions. For the case $H_t(y) \equiv H(t)$ we obtain the existence of a strong solution while, for $H(t,y)$ we obtain a weak solution for Eq.(1).
Definition 3  Given \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\) a fixed stochastic basis and \(\{B_t : t \geq 0\}\) a \(\mathbb{R}^k\)-valued Brownian motion, we state that a triplet \((Y, Z, K)\) is a strong solution of the BSVI \((H(t), \varphi, F)\) if \((Y, Z, K) : \Omega \times [0, T] \to \mathbb{R}^d \times \mathbb{R}^{d \times k} \times \mathbb{R}^d\) are progressively measurable continuous stochastic processes and \(\mathbb{P} - a.s. \omega \in \Omega\),

\[
\begin{align*}
Y_t + \int_t^T H(s) \, dK_s &= \eta + \int_t^T F(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s, \quad \forall t \in [0, T], \\
dK_s &\in \partial \varphi(Y_s) \, (ds),
\end{align*}
\]

Consider now the case when the matrix \(H\) depends on the state of the system. We can reconsider the backward stochastic variational inequality with oblique reflection in the following manner, \(\mathbb{P} - a.s. \omega \in \Omega\),

\[
\begin{align*}
Y_t + \int_t^T H(s, Y_s) \, dK_s &= \eta + \int_t^T F(s, Y_s, Z_s) \, ds - (M_T - M_t), \quad \forall t \in [0, T], \\
dK_s &\in \partial \varphi(Y_s) \, (ds),
\end{align*}
\]

where \(M\) is a continuous martingale (possible with respect to its natural filtration if not any other filtration available). If

\[
H(\omega, t, y) \equiv H(t, y) \quad \text{and} \quad F(\omega, t, y, z) \equiv F(t, y, z)
\]

we introduce the notion of weak solution of the equation.

Definition 4  If there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a triplet \((Y, M, K) : \Omega \times [0, T] \to (\mathbb{R}^d)^3\) such that

\((a)\) \(M\) is a continuous martingale with respect to the filtration given, for \(\forall t \in [0, T]\), by

\[
\mathcal{F}_t \overset{\text{def}}{=} \mathcal{F}_t^Y,M = \sigma(\{Y_s, M_s : 0 \leq s \leq t\}) \lor \mathcal{N},
\]

\((b)\) \(Y, K\) are càdlàg stochastic processes, adapted to \(\{\mathcal{F}_t\}_{t \geq 0}\),

\((c)\) relation (3) is verified for every \(t \in [0, T]\), \(\mathbb{P} - a.s. \omega \in \Omega\),

the collection \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, Y_t, M_t, K_t)_{t \in [0, T]}\) is called a weak solution of the BSVI \((H(y), \varphi, F)\).

In both cases given by Definition 3 or Definition 4 we will say that \((Y, Z, K)\) or \((Y, M, K)\) is a solution of the considered oblique reflected backward stochastic variational inequality.

Now we are able to formulate the main results of this article. Denote

\[
\nu_t = \int_0^t L(s) \left[ \mathbb{E}^{\mathcal{F}_s} |\eta|_p^p \right]^{1/p} \quad \text{and} \quad \theta = \sup_{t \in [0, T]} \left( \mathbb{E}^{\mathcal{F}_t} |\eta|_p^p \right)^{1/p}.
\]

Theorem 5  Let \(p > 1\) and the assumptions \((H_1 - H_4)\) be satisfied, with \(l(t) \equiv l < \sqrt{a}\). If

\[
\mathbb{E}e^{\delta \theta} + \mathbb{E}|\varphi(\eta)| < \infty
\]

for all \(\delta > 0\) then the BSVI \((H(t), \varphi, F)\) admits a unique strong solution \((Y, Z, K) \in S^p_{\delta} [0, T] \times \Lambda^0_{d \times k} (0, T) \times S^p_{\delta} [0, T]\), such that, for all \(\delta > 0\),

\[
\mathbb{E} \sup_{s \in [0, T]} e^{\delta \nu_s} |Y_s|^p + \mathbb{E} \left( \int_0^T e^{2\delta \nu_s} |Z_s|^2 \, ds \right)^{p/2} < \infty.
\]

6
Moreover, there exists a positive constant, independent of the terminal time \( T \), \( C = C(a, b, \Lambda) \) such that, \( \mathbb{P} - \text{a.s.} \ \omega \in \Omega \),

\[
|Y_t| \leq C \left( 1 + \mathbb{E}^{\mathcal{T}_T} \left| \eta \right|^p \right)^{1/p}, \quad \text{for all } t \in [0, T]
\]

and the process \( K \) can be represented as

\[
K_t = \int_0^t U_s ds,
\]

where

\[
\mathbb{E} \int_0^T |U_t|^2 dt + \mathbb{E} \int_0^T |Z_t|^2 dt \leq C \left( \mathbb{E} |\eta|^2 + \mathbb{E} |\varphi(\eta)| + \mathbb{E} \int_0^T |F(t, 0, 0)|^2 dt \right).
\]

**Remark 6** The boundedness conditions imposed to the exponential moments from (4) is not a very restrictive one. For example, it takes place if we consider \( k = 1 \) and \( \eta = B_{\alpha}^{c}, \) with \( 0 < \alpha < 2 \).

**Theorem 7** Let the assumptions \((H_2 - H_4)\) be satisfied. Then the BSVI \((H(t, y), \varphi, F)\) (1) admits a unique weak solution \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, Y_t, M_t, K_t)_{t \in [0, T]}\).

The proofs of the above results are detailed in the next sections. Section 4 deals with a sequence of approximating equations and apriori estimates of their solutions. The estimates will be valid for both cases covered by Theorem 5 and Theorem 7. After this, the proof is split in Section 5 and Section 6, each one being dedicated to the particularities brought by Theorem 5 and Theorem 7.

### 3 Approximating problems and apriori estimates

In order to prove the existence of the solution (strong or weak) we can assume, without loosing the generality, that

\[
\varphi(y) \geq \varphi(0) = 0
\]

because, otherwise, we can change the functions \( \varphi, F \) and \( H \) as follows

\[
\begin{align*}
\tilde{\varphi}(y) &= \varphi(y + u_0) - \varphi(u_0) - \langle \hat{u}_0, y \rangle \geq 0, \\
\tilde{F}(t, y, z) &= F(t, y + u_0, z) - H(t, y + u_0) \hat{u}_0, \\
\tilde{H}(t, y) &= H(t, y + u_0),
\end{align*}
\]

with \( u_0 \in \text{Dom} (\partial \varphi) \) and \( \hat{u}_0 \in \partial \varphi(u_0) \). The solution is now given by \((Y, Z, K) = (\tilde{Y} + u_0, \tilde{Z}, \tilde{K})\), where

\[
\begin{align*}
\begin{cases}
\tilde{Y}_t + \int_t^T \tilde{H}(s, \tilde{Y}_s) d\tilde{K}_s = (\eta - u_0) + \int_t^T F(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dB_s, \ \forall t \in [0, T], \\
\frac{d\tilde{K}_s(\omega)}{d\tilde{\varphi}(\tilde{Y}_s(\omega))} (ds), \ \forall s, \ \mathbb{P} - \text{a.s.} \ \omega \in \Omega.
\end{cases}
\end{align*}
\]

We start simultaneously the proofs of Theorem 5 and Theorem 7 by obtaining some apriori estimates for the solutions of the approximating equations.
Proof. Let $p > 1$.

Step 1. Boundedness under the assumption

$$0 \leq \ell (t) \equiv \ell < \sqrt{a}.$$ 

Let $0 < \varepsilon \leq 1$. Consider the approximating BSDE

$$Y^\varepsilon_t + \int_t^T H (s, Y^\varepsilon_s, \nabla \varphi^\varepsilon (Y^\varepsilon_s)) \, ds = \eta + \int_t^T F (s, Y^\varepsilon_s, Z^\varepsilon_s) \, ds - \int_t^T Z^\varepsilon_s dB_s, \quad \forall t \in [0, T].$$

Let $\tilde{F} (t, y) = F (t, y, z) - H (t, y) \nabla \varphi^\varepsilon (y)$. Using the Lipschitz and boundedness hypothesis imposed on $F$ and $H$, we have, for all $t \in [0, T], y, y', z, z' \in \mathbb{R}^d$,

$$|\tilde{F}(t, y', z') - \tilde{F}(t, y, z)| \leq |F(t, y', z') - F(t, y, z)| + |H(t, y) - H(t, y')| \nabla \varphi^\varepsilon(y)| + |H(t, y') \left[ \nabla \varphi^\varepsilon(y) - \nabla \varphi^\varepsilon(y') \right]| \leq L(t)|y' - y| + \ell|z' - z| + \frac{\Lambda}{\varepsilon}|y' - y| + \frac{b}{\varepsilon}|y' - y| \leq \left( L(t) + \frac{\Lambda}{\varepsilon} + \frac{b}{\varepsilon} \right) (1 + |y| + |y'|)|y' - y| + \ell|z' - z|

and

$$|\tilde{F}(t, y, 0)| \leq \rho(t) + L(t)|y| + \frac{b}{\varepsilon}|y|.$$ 

By Theorem 13 (see Annex 6.1), the BSDE (6) has a unique solution $(Y^\varepsilon, Z^\varepsilon) \in S^0 [0, T] \times \Lambda^0_{d \times k} (0, T)$ such that for all $\delta > 0$,

$$\mathbb{E} \sup_{s \in [0, T]} e^{\delta \nu_s} |Y^\varepsilon|^p + \mathbb{E} \left( \int_0^T e^{2\delta \nu_s} |Z^\varepsilon|^2 \, ds \right)^{p/2} < \infty.$$ 

By Energy Equality we obtain

$$|Y_t^\varepsilon|^2 + 2 \int_t^s \langle Y_r^\varepsilon, H(r, Y_r^\varepsilon) \nabla \varphi^\varepsilon(Y^\varepsilon_r) \rangle \, dr + \int_t^s |Z^\varepsilon_r|^2 \, dr = |Y_s^\varepsilon|^2 + 2 \int_t^s \langle Y_r^\varepsilon, F(r, Y_r^\varepsilon, Z_r^\varepsilon) \rangle \, dr - 2 \int_t^s \langle Y_r^\varepsilon, Z_r^\varepsilon d B_r \rangle.$$ 

Since $y \mapsto \varphi^\varepsilon(y) : \mathbb{R}^d \to \mathbb{R}$ is a convex $C^1$--function, then by the subdifferential inequality (38) (see Annex 6.3)

$$\varphi^\varepsilon(Y^\varepsilon_t) + \int_t^s \langle \nabla \varphi^\varepsilon(Y^\varepsilon_r), H(r, Y_r^\varepsilon) \nabla \varphi^\varepsilon(Y^\varepsilon_r) \rangle \, dr \leq \varphi^\varepsilon(Y^\varepsilon_s) + \int_t^s \langle \nabla \varphi^\varepsilon(Y^\varepsilon_r), F(r, Y_r^\varepsilon, Z_r^\varepsilon) \rangle \, dr - \int_t^s \langle \nabla \varphi^\varepsilon(Y^\varepsilon_r), Z_r^\varepsilon d B_r \rangle.$$ 

As consequence, combining the previous two inequalities, we obtain

$$|Y_t^\varepsilon|^2 + \varphi^\varepsilon(Y^\varepsilon_t) + \int_t^s \langle \nabla \varphi^\varepsilon(Y^\varepsilon_r), H(r, Y_r^\varepsilon) \nabla \varphi^\varepsilon(Y^\varepsilon_r) \rangle \, dr + \int_t^s |Z^\varepsilon_r|^2 \, dr \leq |Y_s^\varepsilon|^2 + \varphi^\varepsilon(Y^\varepsilon_s) + 2 \int_t^s \langle Y_r^\varepsilon, F(r, Y_r^\varepsilon, Z_r^\varepsilon) \rangle \, dr + \int_t^s \langle \nabla \varphi^\varepsilon(Y^\varepsilon_r), Z_r^\varepsilon d B_r \rangle.$$
Let $\lambda > 0$. In the sequel we denote by $C$ a generic positive constant, independent of $\varepsilon, \delta \in (0, 1]$, constant which can change from one line to another, without affecting the result. The assumptions $(H_2)$ and $(H_4)$ lead to the following estimates:

- $\langle \nabla \varphi_\varepsilon (Y_\varepsilon^s), H (r, Y_\varepsilon^s) \rangle \nabla \varepsilon (Y_\varepsilon^s) \rangle \geq a \| \nabla \varphi_\varepsilon (Y_\varepsilon^s) \|^2$

- $2 \langle Y_\varepsilon^s, F (r, Y_\varepsilon^s, Z_\varepsilon^s) \rangle \leq 2 \ell |Y_\varepsilon^s| |Z_\varepsilon^s| + 2L (r) |Y_\varepsilon^s|^2 + 2 |Y_\varepsilon^s| |F (r, 0, 0)|$

- $\langle \nabla \varphi_\varepsilon (Y_\varepsilon^s), F (r, Y_\varepsilon^s, Z_\varepsilon^s) \rangle - 2H (r, Y_\varepsilon^s) Y_\varepsilon^s$

Inserting the above estimates in (8), we obtain, $\mathbb{P}$-a.s., for all $0 \leq t \leq s \leq T$,

$$|Y_t^\varepsilon|^2 + \varphi_\varepsilon (Y_t^\varepsilon) + \left( a - a + \frac{\ell^2}{4 \lambda} \right) \int_t^s \| \nabla \varphi_\varepsilon (Y_{r}^\varepsilon) \|^2 dr + (1 - 2 \lambda) \int_t^s |Z_r^\varepsilon|^2 dr$$

$$\leq |Y_s^\varepsilon|^2 + \varphi_\varepsilon (Y_s^\varepsilon) + \int_t^s \left( 1 + \frac{2 \lambda \lambda}{a} \right) |F (r, 0, 0)|^2 dr$$

$$+ \int_t^s \left( 2L (r) + \frac{\ell^2}{\lambda} + 1 + \frac{2 \lambda}{a} (L (r) + 2b)^2 \right) |Y_r^\varepsilon|^2 dr - \int_t^s \langle 2Y_r^\varepsilon + \nabla \varphi_\varepsilon (Y_r^\varepsilon), Z_r^\varepsilon dB_r \rangle .$$

Denote

$$K_t^\lambda = \int_0^t \left[ \left( 1 + \frac{2 \lambda \lambda}{a} \right) |F (r, 0, 0)|^2 - \left( a - a + \frac{\ell^2}{4 \lambda} \right) |\nabla \varphi_\varepsilon (Y_r^\varepsilon)|^2 - (1 - 2 \lambda) |Z_r^\varepsilon|^2 \right] dr$$

and

$$A (t) = \int_0^t \left( 2L (r) + \frac{\ell^2}{\lambda} + 1 + \frac{2 \lambda}{a} (L (r) + 2b)^2 \right) dr .$$

Since $\varphi_\varepsilon (y) \geq \varphi_\varepsilon (0) = 0$ we have

$$|Y_t^\varepsilon|^2 + \varphi_\varepsilon (Y_t^\varepsilon) \leq |Y_s^\varepsilon|^2 + \varphi_\varepsilon (Y_s^\varepsilon) + \int_t^s \left[ dK_r^\lambda + \left[ |Y_r^\varepsilon|^2 + \varphi_\varepsilon (Y_r^\varepsilon) \right] dA (r) \right]$$

$$- \int_t^s \langle 2Y_r^\varepsilon + \nabla \varphi_\varepsilon (Y_r^\varepsilon), Z_r^\varepsilon dB_r \rangle$$

and, by Proposition 17 (see Annex 6.3), we infer

$$e^{A(t)} \left( |Y_t^\varepsilon|^2 + \varphi_\varepsilon (Y_t^\varepsilon) \right) \leq e^{A(s)} \left[ |Y_s^\varepsilon|^2 + \varphi_\varepsilon (Y_s^\varepsilon) \right] + \int_t^s e^{A(r)} dK_r^\lambda$$

(9)

$$- \int_t^s e^{A(r)} \langle 2Y_r^\varepsilon + \nabla \varphi_\varepsilon (Y_r^\varepsilon), Z_r^\varepsilon dB_r \rangle .$$
Let $\lambda = \frac{1}{2} \left( \frac{a + \ell^2}{4a} + \frac{1}{2} \right)$ be fixed. It follows that, for all $0 \leq t \leq s \leq T$,

$$
|Y_t^\varepsilon|^2 + \varphi_\varepsilon (Y_t^\varepsilon) + \mathbb{E}^F \int_t^s |\nabla \varphi_\varepsilon (Y_r^\varepsilon)|^2 \, dr + \mathbb{E}^F \int_t^s |Z_r^\varepsilon|^2 \, dr \\
\leq C \mathbb{E}^F |Y_s^\varepsilon|^2 + \mathbb{E}^F \varphi_\varepsilon (Y_s^\varepsilon) + C \mathbb{E}^F \int_t^s |F (r, 0, 0)|^2 \, dr.
$$

(10)

In particular, we consider $s = T$ and, since $0 \leq \varphi_\varepsilon (\eta) \leq \varphi (\eta)$,

$$
\mathbb{E} \int_0^T |\nabla \varphi_\varepsilon (Y_r^\varepsilon)|^2 \, dr + \mathbb{E} \int_0^T |Z_r^\varepsilon|^2 \, dr \leq C \left[ \mathbb{E} |\eta|^2 + \mathbb{E} \varphi (\eta) + \mathbb{E} \int_0^T |F (r, 0, 0)|^2 \, dr \right] = \tilde{C}.
$$

(11)

Using the definition of $\nabla \varphi_\varepsilon$ we also obtain

$$
\mathbb{E} \int_0^T |Y_r^\varepsilon - J_\varepsilon (Y_r^\varepsilon)|^2 \, dr \leq \tilde{C} \varepsilon^2.
$$

(12)

We write the approximating BSDE (6) under the form

$$
Y_t^\varepsilon = \eta + \int_t^T dK_s^\varepsilon - \int_t^T Z_s^\varepsilon dB_s,
$$

where

$$
dK_s^\varepsilon = [F (s, Y_s^\varepsilon, Z_s^\varepsilon) - H (s, Y_s^\varepsilon) \nabla \varphi_\varepsilon (Y_s^\varepsilon)] \, ds.
$$

If we denote

$$
N_t = \int_0^t \left[ |F (s, 0, 0)| + b |\nabla \varphi_\varepsilon (Y_s^\varepsilon)| \right] \, ds \quad \text{and} \quad V (t) = \int_0^t (L (s) + \ell^2) \, ds,
$$

then

$$
\langle Y_t^\varepsilon, dK_t^\varepsilon \rangle \leq |Y_t^\varepsilon| \, dN_t + \left[ |F (t, 0, 0)| + b |\nabla \varphi_\varepsilon (Y_t^\varepsilon)| \right] \, dt + |Y_t^\varepsilon|^2 \, dV (t) + \frac{1}{4} |Z_t^\varepsilon|^2 \, dt.
$$

We apply Proposition 15 (see Annex 6.3) and it infers, for $p = 2$,

$$
\mathbb{E}^F \sup_{s \in [0, T]} |e^{V (s)} Y_s^\varepsilon |^2 + \mathbb{E}^F \left( \int_t^T e^{2V (s)} |Z_s^\varepsilon|^2 \, ds \right) \\
\leq C \mathbb{E}^F \left[ |e^{V (T)} \eta|^2 + \left( \int_t^T e^{V (s)} \left[ |F (s, 0, 0)| + b |\nabla \varphi_\varepsilon (Y_s^\varepsilon)| \right] \, ds \right]^2 \right].
$$

Taking into account (11) it follows

$$
|Y_0^\varepsilon|^2 \leq \mathbb{E} \sup_{s \in [0, T]} |Y_s^\varepsilon|^2 \leq C \left[ \mathbb{E} |\eta|^2 + \mathbb{E} \varphi (\eta) + \mathbb{E} \int_0^T |F (r, 0, 0)|^2 \, dr \right].
$$

(13)
The Lipschitz and the boundedness hypotheses \((H_4)\) imposed on \(F\) lead, due to the fact that \(l\) is constant, \(L \in L^2(0, T; \mathbb{R}_+)\) and \(\rho \in L^1(0, T; \mathbb{R}_+)\), to
\[
\mathbb{E} \int_0^T |F(r, Y^\varepsilon, Z^\varepsilon)|^2 dr \leq 2\mathbb{E} \int_0^T |F(r, Y^\varepsilon, Z^\varepsilon) - F(r, Y^\varepsilon, 0)|^2 dr + 2\mathbb{E} \int_0^T |F(r, Y^\varepsilon, 0)|^2 dr \\
\leq 2l^2 \mathbb{E} \int_0^T |Z^\varepsilon|^2 dr + 4\mathbb{E} \int_0^T L^2(r) |Y^\varepsilon|^2 dr + 4\mathbb{E} \int_0^T |F(r, 0, 0)|^2 dr \leq C,
\]
(14) \[
\mathbb{E} \int_0^T |F(r, Y^\varepsilon, Z^\varepsilon)| dr \leq \mathbb{E} \int_0^T [L(r) |Y^\varepsilon| + l |Z^\varepsilon| + |F(r, 0, 0)|] dr \leq C
\]
For the convenience of the reader, we will group together, under the form of a Lemma, some useful estimations on the solution of the approximating equation, estimation that we just obtained in Step 1.

**Lemma 8** Consider the approximating BSDE (6), with its solution \((Y^\varepsilon, Z^\varepsilon)\) and denote \(U^\varepsilon = \nabla \varphi_\varepsilon(Y^\varepsilon)\). There exists a positive constant \(C = C(a, b, \Lambda, l, L(\cdot))\), independent of \(\varepsilon\), such that
\[
\mathbb{E} \sup_{s \in [0, T]} |Y^\varepsilon_s|^2 + \mathbb{E} \int_0^T (|U^\varepsilon_r|^2 + |Z^\varepsilon_r|^2) dr \leq C \left[ \mathbb{E} |\eta|^2 + \mathbb{E} \varphi(\eta) + \mathbb{E} \int_0^T |F(r, 0, 0)|^2 dr \right].
\]
(15)

**Step 2. Convergences under the assumption**

\[0 \leq \ell(t) \equiv \ell < \sqrt{a}.
\]
The estimations of Step 1 imply that there exist a sequence \(\{\varepsilon_n : n \in \mathbb{N}^*\}, \varepsilon_n \rightarrow 0\) as \(n \rightarrow \infty\), and six progressively measurable stochastic processes \(Y, Z, U, F, \chi, h\) such that
\[
Y^\varepsilon_n \rightarrow Y, \quad \text{in } \mathbb{R}^d, \\
Z^\varepsilon_n \rightharpoonup Z, \quad \text{weakly in } L^2(\Omega \times (0, T); \mathbb{R}^{d \times k}),
\]
and, weakly in \(L^2(\Omega \times (0, T); \mathbb{R}^d),\)
\[
Y^\varepsilon_n \rightharpoonup Y, \quad \nabla \varphi_{\varepsilon_n}(Y^\varepsilon_n) \rightharpoonup U, \quad H(\cdot, Y^\varepsilon_n) \rightharpoonup h, \\
H(\cdot, Y^\varepsilon_n) \nabla \varphi_{\varepsilon_n}(Y^\varepsilon_n) \rightharpoonup \chi \quad \text{and} \quad F(\cdot, Y^\varepsilon_n, Z^\varepsilon_n) \rightharpoonup F.
\]
The convergence \(Y^\varepsilon_n \rightharpoonup Y\) and the inequality (12), written for \(\varepsilon = \varepsilon_n\), imply that, on the sequence \(\{\varepsilon_n : n \in \mathbb{N}^*\},\)
\[
J_{\varepsilon_n}(Y^\varepsilon_n) \rightharpoonup Y, \quad \text{weakly in } L^2(\Omega \times (0, T); \mathbb{R}^d).
\]
We write (7) for \(\varepsilon = \varepsilon_n\) and, passing to \(\lim \inf_{n \rightarrow +\infty}\), we obtain
\[
\mathbb{E} \sup_{s \in [0, T]} e^{\delta p\nu s} |Y_s|^p + \mathbb{E} \left( \int_0^T e^{2\delta p\nu s} |Z_s|^2 ds \right)^{p/2} < \infty.
\]
From the approximating BSDE (6) we have that, at the limit,
\[ Y_t + \int_t^T \chi_s ds = \eta + \int_t^T F_s ds - \int_t^T Z_s dB_s. \]
The continuity of the three integrals from the above equation imply also the continuity of the process \( Y \), but the previous convergences are not yet sufficient to conclude that \( (Y, Z) \) is a solution of the considered equation. The remaining problems consist in proving that, for every \( s \in [0,T] \), \( \mathbb{P} \)-a.s. \( \omega \in \Omega \),
\[ \chi_s = h_s U_s, \quad h_s = H(s,Y_s), \quad U_s \in \partial \varphi(Y_s) \quad \text{and} \quad F_s = F(s,Y_s,Z_s). \]

**Step 3. Boundedness under the assumptions**
\[ 0 \leq \ell(t) \equiv \ell < \sqrt{a} \quad \text{and} \quad |\eta|^2 + |\varphi(\eta)| \leq c, \quad \mathbb{P} \text{-a.s. } \omega \in \Omega. \]

From inequality (10), written for \( s = T \) it follows, \( \mathbb{P} \)-a.s.,
\[ |Y_T^\varepsilon|^2 + \varphi(\varepsilon Y_T) \leq C \left( c + \int_0^T \rho(r) dr \right) = C', \quad \text{for all } t \in [0, T]. \]

Starting with this point, the proofs of Theorem 5 and Theorem 7 will take two separate paths.

### 4 Strong existence and uniqueness for \( H(t,y) \equiv H_t \)

We will continue in this section the proof of Theorem 5.

**Proof.** We continue the proof of the existence of a solution. Under the assumptions of Step 3 (Section 3) we prove that \( \{Y^\varepsilon : 0 < \varepsilon \leq 1\} \) is a Cauchy sequence. To simplify the presentation of this task we assume \( k = 1 \).

The form of the matrix \( H \) leads to
\[ H \nabla \varphi_{\varepsilon_n}(Y^\varepsilon_n) \to HU, \quad \text{weakly in } L^2(\Omega \times (0,T) ; \mathbb{R}^d), \]
that is
\[ \lim_{n \to \infty} \mathbb{E} \int_0^T H_r \nabla \varphi_{\varepsilon_n}(Y^\varepsilon_n) dr = \mathbb{E} \int_0^T H_r U_r dr. \]

Starting from here, by the symmetric and strictly positive matrix \( H_s^{-1} \) we will understand the inverse of the matrix \( H_s \) and not the inverse of the function \( H \).

We have
\[ H_{t-1/2} = H_{T-1/2} + \int_t^T D_s ds \quad \text{and} \quad H^{-1}_t = H^{-1}_T + \int_t^T \tilde{D}_s ds, \]
where \( D_s = -\frac{1}{2} H_s^{-3/2} \frac{d}{ds} H_s \) and \( \tilde{D}_s = -\frac{d}{ds} H_s^{-1} \) are \( \mathbb{R}^{d \times d} \)-valued progressively measurable stochastic processes such that, \( \mathbb{P} \)-a.s., \( |D_s| \leq C = \frac{1}{2} b^{3/2} \Lambda \) and \( |\tilde{D}_s| \leq \Lambda. \)
Denote $\Delta^\varepsilon_\delta = H_s^{-1/2} (Y^\varepsilon - Y^\delta_s)$. We have
\[
\Delta^\varepsilon_\delta = -\int_t^T dH_s^{-1/2} (Y^\varepsilon - Y^\delta_s) - \int_t^T H_s^{-1/2} d(Y^\varepsilon - Y^\delta_s)
= \int_t^T dK^\varepsilon_\delta - \int_t^T Z^\varepsilon_\delta dB_s,
\]
where
\[
dK^\varepsilon_\delta = D_s (Y^\varepsilon_s - Y^\delta_s) ds + H_s^{-1/2} [F (s, Y^\varepsilon_s, Z^\varepsilon_s) - F (s, Y^\delta_s, Z^\delta_s)] ds
- H_s^{-1/2} \left( H_s \nabla \varphi^\varepsilon (Y^\varepsilon_s) - H_s \nabla \varphi^\delta (Y^\delta_s) \right) ds
= D_s (Y^\varepsilon_s - Y^\delta_s) ds + H_s^{-1/2} \left( F (s, Y^\varepsilon_s, Z^\varepsilon_s) - F (s, Y^\delta_s, Z^\delta_s) \right) ds
- H_s^{-1/2} \left( \nabla \varphi^\varepsilon (Y^\varepsilon_s) - \nabla \varphi^\delta (Y^\delta_s) \right) ds
\]
and $Z^\varepsilon_\delta = H_s^{-1/2} (Z^\varepsilon_s - Z^\delta_s)$. By denoting with $C$ a generic positive constant independent of $\varepsilon$ and $\delta$ that can change from one line to another we obtain that
\[
\left\langle \Delta^\varepsilon_\delta, dK^\varepsilon_\delta \right\rangle \leq C \left( |D_s| + L(s) \right) |Y^\varepsilon_s - Y^\delta_s|^2 ds + C \left| Y^\varepsilon_s - Y^\delta_s \right| |Z^\varepsilon_s - Z^\delta_s| ds
- \left( \nabla \varphi^\varepsilon (Y^\varepsilon_s) - \nabla \varphi^\delta (Y^\delta_s), Y^\varepsilon_s - Y^\delta_s \right) ds
\leq C \left( |D_s| + L(s) \right) |Y^\varepsilon_s - Y^\delta_s|^2 ds + C \left| Y^\varepsilon_s - Y^\delta_s \right| |Z^\varepsilon_s - Z^\delta_s| ds
+ (\varepsilon + \delta) |\nabla \varphi^\varepsilon (Y^\varepsilon_s)| |\nabla \varphi^\delta (Y^\delta_s)| ds.
\]
Therefore, from the formula of $\Delta^\varepsilon_\delta$ we have
\[
\left\langle \Delta^\varepsilon_\delta, dK^\varepsilon_\delta \right\rangle \leq (\varepsilon + \delta) |\nabla \varphi^\varepsilon (Y^\varepsilon_s)| |\nabla \varphi^\delta (Y^\delta_s)| ds + |\Delta^\varepsilon_\delta|^2 dv_s + \frac{1}{4} |Z^\varepsilon_\delta|^2,
\]
where, for $\bar{C} = \bar{C}(l, a, b, \Lambda) > 0$, $V_t = \bar{C} \int_t^T (|D_s| + L(s)) ds$. We apply now Proposition 15 (see Annex 6.3) with $p \geq 2$, $\lambda = 1/2$, $D = N \equiv 0$ and we obtain, for a positive constant $C = C(l, a, b, p)$ and for $C_1 > 0$ well chosen,
\[
C_1 \mathbb{E} \sup_{s \in [0, T]} |Y^\varepsilon_s - Y^\delta|^p + \mathbb{E} \int_0^T |Z^\varepsilon_s - Z^\delta|^2 ds
\leq \mathbb{E} \sup_{s \in [0, T]} e^{p V_s} |\Delta^\varepsilon_\delta|^p + \mathbb{E} \left( \int_0^T e^{2V_s} |Z^\varepsilon_\delta|^2 ds \right)^{p/2}
\leq C(\varepsilon + \delta) \mathbb{E} \left( \int_0^T e^{2V_s} |\nabla \varphi^\varepsilon (Y^\varepsilon_s)| |\nabla \varphi^\delta (Y^\delta_s)| ds \right)^{p/2}
\leq C(\varepsilon + \delta) \left( \mathbb{E} \left( \int_0^T |\nabla \varphi^\varepsilon (Y^\varepsilon_s)|^2 ds \right)^{p/2} + \mathbb{E} \left( \int_0^T |\nabla \varphi^\delta (Y^\delta_s)|^2 ds \right)^{p/2} \right),
\]
which implies, according to (11), that $\{Y^\varepsilon : 0 < \varepsilon \leq 1\}$ is a Cauchy sequence.

With standard arguments, passing to the limit in the approximating equation (6) we infer that
\[
Y_t + \int_t^T H_s U_s ds = \eta + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad \forall t \in [0, T].
\]
From (7), by Fatou’s Lemma, (5) easily follows. Moreover, since $\nabla \varphi_\varepsilon(x) \in \partial \varphi(J_\varepsilon x)$ we have, on the subsequence $\varepsilon_n$,

$$E \int_0^T \langle \nabla \varphi_{\varepsilon_n}(Y_{t}^{\varepsilon_n}), v_t - Y_{t}^{\varepsilon_n} \rangle dt + E \int_0^T \varphi(J_{\varepsilon_n}(Y_{t}^{\varepsilon_n}))dt \leq E \int_0^T \varphi(v_t)dt,$$

for every progressively measurable continuous stochastic process $v$. Hence $U_s \in \partial \varphi(Y_s)$ for every $s \in [0,T]$, $P$-a.s. $\omega \in \Omega$ and we can conclude that the triplet $(Y, Z, K)$ is a strong solution of the BSVI $(H(t), \varphi, F)$.

**Uniqueness.** Suppose that the BSVI $(H(t), \varphi, F)$ admits two strong solutions, denoted by $(Y, Z, K)$ and respectively $(\tilde{Y}, \tilde{Z}, \tilde{K})$, with the processes $K$ and $\tilde{K}$ represented as

$$K_t = \int_0^t U_s ds \quad \text{and} \quad \tilde{K}_t = \int_0^t \tilde{U}_s ds.$$

Following the same arguments found in the existence part of the theorem, denoting $\Delta_s = H_s^{-\frac{1}{2}}(Y_s - \tilde{Y}_s)$, we have

$$\Delta_t = \int_t^T dK_s - \int_t^T Z_s dB_s,$$

where

$$dK_s = D_s(Y_s - \tilde{Y}_s)ds + H_s^{-\frac{1}{2}}[F(s, Y_s, Z_s) - F(s, \tilde{Y}_s, \tilde{Z}_s)]ds - H_s^{\frac{1}{2}}(U_s - \tilde{U}_s)ds$$

and $Z_s = H_s^{-\frac{1}{2}}(Z_s - \tilde{Z}_s)$.

Since $Y$ and $\tilde{Y}$ are two solutions of the equation, $U_s \in \partial \varphi(Y_s)$ and $\tilde{U}_s \in \partial \varphi(\tilde{Y}_s)$, $\forall s \in [0, T], P$-a.s. $\omega \in \Omega$,

$$\langle Y_s - \tilde{Y}_s, U_s - \tilde{U}_s \rangle \geq 0$$

and we obtain, for a positive constant $\overline{C} = \overline{C}(l, a, b)$,

$$\langle \Delta_s, dK_s \rangle \leq C ((D_s + L(s)) |Y_s - \tilde{Y}_s|^2 ds + Cl |Y_s - \tilde{Y}_s| |Z_s| ds$$

$$\leq \overline{C} |\Delta_s|^2 ((D_s + L(s)) ds + \frac{1}{4} |Z_s|^2.$$}

Since

$$E \sup_{t \in [0,T]} (e^{pV_t}|\Delta_t|^p) \leq C E \sup_{t \in [0,T]} |Y_t - \tilde{Y}_t|^p < +\infty$$

we obtain by Proposition 15 (see Annex 6.3) that

$$e^{pV_t}|\Delta_t|^p \leq E^{F_t}e^{pV_T}|\Delta_T|^p = 0$$

and the uniqueness of a strong solution for BSVI $(H(t), \varphi, F)$ easily follows. ■

**Remark 9** Inequality (8) permits us to derive some more estimations regarding the limit processes. We write (8) for $s = T$ and, since $\varphi(J_\varepsilon(x)) \leq \varphi(\varepsilon(x)) \leq \varphi(x)$, by passing to $\lim_{\varepsilon \to 0}$ in (8), we have for all $t \in [0, T], P$-a.s. $\omega \in \Omega$,

$$\langle Y_t \rangle^2 + \varphi(Y_t) + a \int_t^T |U_r|^2 dr + \int_t^T |Z_r|^2 dr \leq \langle \eta \rangle^2 + \varphi(\eta) + 2 \int_t^T \langle Y_r, F(v, Y_r, Z_r) \rangle dr$$

$$+ \int_t^T \langle U_r, F(v, Y_r, Z_r) - 2H_r Y_r \rangle dr - \int_t^T \langle 2Y_r + U_r, Z_r dB_r \rangle.$$
5 Weak existence for $H(t,x)$

We will continue in this section the proof of Theorem 7. All the apriori estimates obtained in Section 3 remain valid. In Section 4 we proved that the approximating sequence given by BSDE (6) is a Cauchy sequence when the matrix $H$ does not depend on the state of the system and, as a consequence, we derived the existence and uniqueness of a strong solution for $BSVI\ (H(t),\varphi,F)$. In the current setup, allowing the dependence on $Y$ we will situate ourselves in a Markovian framework and we will use tightness criteria in order to prove the existence of a weak solution for $BSVI\ (H(t,y),\varphi,F)$.

First let $b: [0,T] \times \mathbb{R}^k \rightarrow \mathbb{R}^k$, $\sigma: [0,T] \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$ be two continuous functions satisfying the classical Lipschitz conditions, which imply the existence of a non-exploding solution for the following SDE

$$X_{s,x}^{t,x} = x + \int_t^s b(r,X_{r,x}^{t,x})dr + \int_t^s \sigma(r,X_{r,x}^{t,x})dB_r, \quad t \leq s \leq T.$$  \hfill (17)

According to Friedmann [7] it follows that, for every $(t,x) \in [0,T] \times \mathbb{R}^k$, the equation (17) admits a unique solution $X_s^{t,x}$. Moreover, for $p \geq 1$, there exists a positive constant $C_{p,T}$ such that

$$\mathbb{E}\sup_{s \in [0,T]}|X_s^{t,x}|^p \leq C_{p,T}(1+|x|^p) \quad \text{and}$$

$$\mathbb{E}\sup_{s \in [0,T]}|X_s^{t,x} - X_s^{t,x'}|^p \leq C_{p,T}(1+|x|^p)(|t-t'|^{p/2} + |x-x'|^p),$$

for all $x, x' \in \mathbb{R}^k$ and $t, t' \in [0,T]$.

Let now consider the continuous generator function $F: [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and assume there exist $L \in L^2(0,T;\mathbb{R}_+)$ such that, for all $t \in [0,T]$ and $x \in \mathbb{R}^k$,

$$(H'_4) \quad |F(t,x,y') - F(t,x,y)| \leq L(t)|y' - y|, \quad \text{for all } y, y' \in \mathbb{R}^d.$$  \hfill (H'_4)

Given a continuous function $g: \mathbb{R}^k \rightarrow \mathbb{R}^d$, satisfying a sublinear growth condition, consider now the $BSVI\ (H(t,y),\varphi,F)$

$$Y_s^{t,x} + \int_t^T H(r,Y_r^{t,x})dK_r^{t,x} = g(X_T^{t,x}) + \int_t^T F(r,X_r^{t,x},Y_r^{t,x})dr - \int_s^T Z_r^{t,x}dB_r, \quad t \leq s \leq T;$$

$$dK_r^{t,x} \in \partial \varphi(Y_r^{t,x})(dr), \quad \text{for every } r.$$  \hfill (19)

**Remark 10** The utility of studying the notion of weak solution for our problem is justified by the non-linear Feynman-Kac representation formula. Following the same arguments as the one from [20], for $k = 1$, it can easily be proven that $u(t,x) = Y_s^{t,x}$ is a continuous function and it represents a viscosity solution for the following semilinear parabolic PDE:

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(t,x) + A_t u(t,x) + F(t,x,u(t,x)) \in H(t,u(t,x))\varphi(u(t,x)), \\
(t,x) \in [0,T) \times \mathbb{R}^k \quad \text{and} \quad u(T,x) = g(x), \quad \forall x \in \mathbb{R}^k,
\end{array} \right.$$

where the operator $A_t$ is the infinitesimal generator of the Markov process $\{X_s^{t,x}, t \leq s \leq T\}$ and it is given by

$$A_t v(x) = \frac{1}{2} \text{Tr}[(\sigma \sigma^*)(t,x)D^2 v(x)] + \langle b(t,x), \nabla v(x) \rangle.$$
However, for the multi-dimensional case, the situation changes and the proof of the existence and uniqueness of a viscosity solution for the above system of parabolic variational inequalities must follow the approach from Maticiuc, Pardoux, Răşcanu and Zălinescu [14].

More details concerning the restriction to the case when the generator function does not depend on $Z$ can be found in the comments from Pardoux [17], Section 6, page 535. Assume also that all hypothesis given by $(H_2)$ still hold for the deterministic matrix $H : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$. For the clarity of the presentation we will omit writing the superscript $t$, $x$, especially when dealing with sequences of approximating equations and solutions.

Consider now the Skorokhod space $\mathcal{D}([0, T]; \mathbb{R}^m)$ of càdlàg functions $y : [0, T] \to \mathbb{R}^m$ (i.e. right continuous and with left-hand side limit). It can be shown (see Billingsley [1]) that, although $\mathcal{D}([0, T]; \mathbb{R}^m)$ is not a complete space with respect to the Skorokhod metric, there exists a topologically equivalent metric with respect to which it is complete and that the Skorokhod space is a Polish space. The space of continuous functions $C([0, T]; \mathbb{R}^m)$, equipped with the supremum norm topology is a subspace of $\mathcal{D}([0, T]; \mathbb{R}^m)$; the Skorokhod topology restricted to $C([0, T]; \mathbb{R}^m)$ coincides with the uniform topology. We will use on $\mathcal{D}([0, T]; \mathbb{R}^m)$ the Meyer-Zheng topology, which is the topology of convergence in measure on $[0, T]$, weaker than the Skorokhod topology. The Borel $\sigma$–field for the Meyer-Zheng topology is the canonical $\sigma$–field as for Skorokhod topology. Note that for the Meyer-Zheng topology, $\mathcal{D}([0, T]; \mathbb{R}^m)$ is a metric space but not a Polish space. Contrary to the Skorokhod topology, the Meyer-Zheng topology on the product space is the product topology.

We continue now the proof of Theorem 7.

**Proof.** For any fixed $n \geq 1$ consider the following approximating equation, which is in fact BSDE (6) from Section 3, adapted to our new setup. We have, $\mathbb{P}$ – a.s. $\omega \in \Omega$,

$$Y^n_t + \int_t^T H(s, Y^n_s) \nabla \varphi_{1/n}(Y^n_s) \, ds = g(X^n_T) + \int_t^T F(s, X^n_s, Y^n_s) \, ds - \int_t^T Z^n_s dB_s, \quad \forall t \in [0, T]. \tag{20}$$

The estimations obtained in Section 3, Lemma 8 apply also to the triplet $(Y^n, Z^n, U^n) = (Y^n, Z^n, \nabla \varphi_{1/n}(Y^n))$, which satisfies the uniform boundedness condition given by (15) with the positive constant $C = C(a, b, \Lambda, L(\cdot))$ now independent of $n$. We will prove a weakly convergence in the sense of the Meyer-Zheng topology, that is the laws converge weakly if we equip the space of paths with the topology of convergence in $dt$–measure.

In the sequel we will employ the following notations:

$$M^n_t = \int_0^t Z^n_s dB_s \quad \text{and} \quad K^n_t = \int_0^t \nabla \varphi_{1/n}(Y^n_s) \, ds. \tag{21}$$

Our goal is to prove the tightness of the sequence $\{Y^n, M^n\}_n$ with respect to the Meyer-Zheng topology. For doing this we must prove the uniform boundedness (with respect to $n$) for quantities of the type

$$\text{CV}_T(\Psi) + \mathbb{E} \sup_{s \in [0,T]} |\Psi_s|,$$

where the conditional variation $\text{CV}_T$ is defined for any adapted process $\Psi$ with paths a.s. in $\mathcal{D}([0, T]; \mathbb{R}^m)$ and with $\Psi_t$ a integrable random variable, for all $t \in [0, T]$. The conditional
variation of $\Psi$ is given by

$$
CV_T(\Psi) \overset{df}{=} \sup_{\pi} \sum_{i=0}^{m-1} \mathbb{E} \left[ \left| \mathbb{E}^{\mathcal{F}_i} [\Psi_{t_{i+1}} - \Psi_{t_i}] \right| \right],
$$

where the supremum is taken over all the partitions $\pi : t = t_0 < t_1 < \cdots < t_m = T$. If $CV_T(\Psi) < \infty$ then the process $\Psi$ is called a quasi-martingale. It is clear that if $\Psi$ is a martingale then $CV_T(\Psi) = 0$.

We will denote by $C$ a generic constant that can vary from one line to another, but which remains independent of $n$. Since $M^n$ is a $\mathcal{F}_t^B$-martingale, we have, by using the hypothesis on $F$ and the boundedness of $H$,

$$
CV_T(Y^n) = \sup_{\pi} \sum_{i=0}^{m-1} \mathbb{E} \left[ \left| \mathbb{E}^{\mathcal{F}_i} [Y^n_{t_{i+1}} - Y^n_{t_i}] \right| \right] \leq \mathbb{E} \int_0^T |F(s, X_s, Y^n_s)| ds + \int_0^T |H(Y^n_s)| d\mathbb{P}^{K^n_{\uparrow T}}.
$$

Since $d\mathbb{P}^{K^n_{\uparrow T}} = \int_0^T |U^n_s|^2 ds \leq \sqrt{T} \left( \int_0^T |U^n_s|^2 ds \right)^{1/2} \leq C$ it infers, along with the uniform boundedness condition given by (15) that

$$
\sup_{n \geq 1} \left( CV_T(Y^n) + \mathbb{E} \sup_{s \in [0,T]} |Y^n_s| \right) < +\infty.
$$

For the rest of the quantities, by standard calculus and using (15) we have the following estimations.

$$
CV_T(M^n) = 0 \text{ because } M^n \text{ is a } \mathcal{F}_t \text{-martingale. Using the Burkholder-Davis-Gundy inequality we obtain the second boundedness which involves } M^n.
$$

$$
\mathbb{E} \sup_{t \in [0,T]} |M^n_t| = \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t Z^n_s dB_s \right| \leq 3 \mathbb{E} \left( \int_0^T |Z^n_s|^2 ds \right)^{1/2} \leq 3 \mathbb{E} \left( \int_0^T |Z^n_s|^2 ds \right)^{1/2} \leq C.
$$

Therefore, taking the supremum over $n \geq 1$ we obtain that the conditions from the tightness criteria in $\mathcal{D}([0,T]; \mathbb{R}^d) \times \mathcal{D}([0,T]; \mathbb{R}^d) \equiv \mathcal{D}([0,T]; \mathbb{R}^{d+d})$ for the sequence $\{(Y^n, M^n)\}_n$ are verified. Using the Prohorov theorem, we have that there exists a subsequence, still denoted with $n$, such that, as $n \to \infty$,

$$
(X, B, Y^n, M^n) \longrightarrow (X, B, Y, M), \quad \text{in law}
$$

in $C([0,T]; \mathbb{R}^{k+k+1}) \times \mathcal{D}([0,T]; \mathbb{R}^{d+d})$. We equipped the previous space with the product of the topology of uniform convergence on the first factor and the topology of convergence in measure on the second factor. For each $0 \leq s \leq t$, the mapping $(x, y) \to \int_s^t F(x(r), y(r)) dr$ is continuous from $C([0,T]; \mathbb{R}^k) \times \mathcal{D}([0,T]; \mathbb{R}^d)$ topologically equipped in the same manner, into $\mathbb{R}$. By the Skorokhod theorem, we can choose now a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (it is in fact $([0,1], \mathcal{B}_{[0,1]}, \mu)$) on which we define the processes

$$
\{(\tilde{X}^n, \tilde{B}^n, \tilde{Y}^n, \tilde{M}^n)\}_n \quad \text{and} \quad (\tilde{X}, \tilde{B}, \tilde{Y}, \tilde{M}),
$$
having the same law as \( \{(X, B, Y^n, M^n)\}_n \) and \((X, B, Y, M)\), respectively, such that, in the product space \( C([0, T]; \mathbb{R}^{k+k}) \times D([0, T]; \mathbb{R}^{d+d}) \), as \( n \to \infty \),

\[
(X^n, B^n, Y^n, M^n) \overset{\mathbb{P}-a.s.}{\to} (X, B, Y, M).
\]

Moreover, for each \( n \in \mathbb{N}^* \), \((X^n, Y^n)\) satisfy, for \( t \in [0, T] \), \( \mathbb{P} - a.s. \omega \in \tilde{\Omega} \),

\[
d\tilde{X}^n_s = b(s, \tilde{X}^n_s)ds + \sigma(s, \tilde{X}^n_s)dB^s_s, \quad t \leq s \leq T, \quad \tilde{X}^n_t = x \quad \text{and}
\]

\[
\tilde{Y}^n_t + \int_t^T H(s, \tilde{Y}^n_s) \nabla \varphi_{1/n}(\tilde{Y}^n_s)ds = g(\tilde{X}^n_T) + \int_t^T F(s, \tilde{X}^n_s, \tilde{Y}^n_s)ds - (\tilde{M}^n_t - \tilde{M}^n_s).
\]

We focus now to the issue of passing to the limit and to the identification of a solution for our problem. Since \( dK^n_s = \nabla \varphi_{1/n}(Y^n_s)ds \in \partial \varphi(J_n(Y^n_s))(ds) \) we have, for all \( v \in \mathbb{R}^d \) and \( 0 \leq t \leq s_1 \leq s_2 \),

\[
\int_{s_1}^{s_2} \varphi(J_n(Y^n_s))ds \leq \int_{s_1}^{s_2} (J_n(Y^n_s) - v)\nabla \varphi_{1/n}(Y^n_s)ds + \int_{s_1}^{s_2} \varphi(v)ds.
\]

Using similar arguments to the ones found in Pardoux and Răşcanu [22], Proposition 1.19, it easily follows that, also for all \( v \in \mathbb{R}^d, 0 \leq t \leq s_1 \leq s_2 \) and every \( A \in \mathcal{F} \),

\[
\mathbb{E} \int_{s_1}^{s_2} 1_A \varphi(J_n(Y^n_s))ds \leq \mathbb{E} \int_{s_1}^{s_2} 1_A (J_n(Y^n_s) - v)\nabla \varphi_{1/n}(Y^n_s)ds + \mathbb{E} \int_{s_1}^{s_2} 1_A \varphi(v)ds,
\]

that is, \( \mathbb{P} - a.s. \omega \in \tilde{\Omega} \), \( \nabla \varphi_{1/n}(\tilde{Y}^n_s) \in \partial \varphi(J_n(\tilde{Y}^n_s)) \), for all \( s \in [t, T] \). We write (15) for \( \tilde{Y}^n \) and, by using the definition of the Yosida approximation, we obtain that there exists a positive constant \( C \), independent of \( n \), such that \( \mathbb{E} \int_0^T |Y^n_s - J_n(Y^n_s)|^2 ds \leq \frac{1}{n^2} C \). The fact that \( Y^n \overset{\mathcal{L}}{\rightarrow} \tilde{Y}^n \) yields

\[
\mathbb{E} \int_0^T |\tilde{Y}^n_s - J_n(\tilde{Y}^n_s)|^2 ds \leq \frac{1}{n^2} C.
\]

Consequently, \( \tilde{Y}^n_n - J_n(\tilde{Y}^n_n) \to 0 \) as \( n \to \infty \) in \( L^2(\Omega \times (0, T); \mathbb{R}^d) \). Therefore, \( J_n(\tilde{Y}^n) \) converges also in \( L^2(\Omega \times (0, T); \mathbb{R}^d) \) to \( \tilde{Y} \) when \( n \to \infty \). The boundedness (15) also implies the existence of a process \( \tilde{U} \) such that

\[
\nabla \varphi_{1/n}(\tilde{Y}^n) \to \tilde{U} \quad \text{as} \quad n \to \infty, \quad \text{in} \quad L^2(\Omega \times (0, T); \mathbb{R}^d).
\]

In addition, passing to lim inf \( n \to \infty \) in (24), due to the lower-semicontinuity of \( \varphi \) we obtain, for all \( v \in \mathbb{R}^d \) and all \( 0 \leq t \leq s_1 \leq s_2 \), \( \mathbb{P} - a.s. \omega \in \Omega \),

\[
\int_{s_1}^{s_2} \varphi(\tilde{Y}_s)ds \leq \int_{s_1}^{s_2} (\tilde{Y}_s - v)\tilde{U}_s ds + \int_{s_1}^{s_2} \varphi(v)ds,
\]

which means \( dK^n_s \overset{\text{def}}{=} \tilde{U}_s ds \in \partial \varphi(\tilde{Y}_s)(ds) \).

Finally, we pass to the limit, as \( n \to \infty \), in the equations (22) and (23). The convergence of \((X^n, B^n, \tilde{Y}^n, M^n)\) to \((X, B, \tilde{Y}, M)\) implies, \( \mathbb{P} - a.s. \omega \in \Omega \),

\[
\tilde{X}_s = x + \int_t^s b(r, \tilde{X}_r)dr + \int_t^s \sigma(r, \tilde{X}_r)d\tilde{B}_r, \quad t \leq s \leq T
\]
and
\[ Y_t + \int_t^T H(s, Y_s) \, ds = g(X_T) + \int_t^T F(s, X_s, Y_s) \, ds - (M_T - M_t). \]

Since the processes \( \bar{Y} \) and \( \bar{M} \) are càdlàg, the above equality takes place for any \( t \in [0, T] \).

Summarizing, we obtained that the collection \( (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathcal{F}}^\bar{Y}, \bar{Y}, \bar{M}, \bar{K})_{t \in [0, T]} \) is a weak solution of Eq. (19), in the sense of Definition (4), and the proof is now complete.

\[ \blacksquare \]

**Remark 11** Alternatively, one can use another approximating equation instead of (20) to prove the existence of a weak solution. This new approach comes with additional benefits from the perspective of constructing numerical approximating schemes for our stochastic variational inequality. For \( n \in \mathbb{N}^* \) we consider a partition of the time interval \( [0, T] \) of the form \( 0 = t_0 < t_1 < \ldots < t_n = T \) with \( t_i = \frac{it}{n} \) for every \( i = 0, n - 1 \) and define

\[
\begin{align*}
Y^n_{t_i} &= \eta, \\
Y^n_t + \int_t^{t_{i+1}} H^n_s dK^n_s &= Y^n_{t_{i+1}} + \int_t^{t_{i+1}} F(s, X_s, Y^n_s) \, ds - \int_t^{t_{i+1}} Z^n_s \, dB_s, \; \forall t \in [t_i, t_{i+1}), \\
dK^n_s &= U^n_s \, ds \in \partial \varphi(Y^n_s)(ds),
\end{align*}
\]

where, for \( s \in \left[ \frac{T}{n}, \frac{(i+1)t}{n} \right) \),

\[ H^n_s \overset{\text{def}}{=} \frac{n}{T} \int_{s - \frac{T}{n}}^{s} \mathbb{E}^{\mathcal{F}_r} \left( H \left( r, Y^n_{r+\frac{T}{n}} \right) \right) \, dr. \]

For the consistence of (25) we must extend \( Y^n_t = \eta, U^n_t = 0 \) for \( t \notin [0, T] \) and, \( \mathbb{P} - \text{a.s. } \omega \in \Omega, U^n_t \in \partial \varphi(Y^n_t) \) a.e. \( t \in (0, T) \). The application \( s \to H^n_s \) is a bounded \( C^1 \) progressively measurable matrix on each interval \((t_i, t_{i+1})\); \( H^n \) and its inverse \( [H^n]^{-1} \) satisfy (2). We highlight that all the constants that appear in (2) remain independent of \( n \). Also, it is clear that, for any continuous process \( V \),

\[
\frac{n}{T} \int_{s - \frac{T}{n}}^{s} \mathbb{E}^{\mathcal{F}_r} \left( H \left( r, V_{r+\frac{T}{n}} \right) \right) \, dr \to H(s, V_s).
\]

By Theorem 5 the triplet \((Y^n, Z^n, U^n)\) is uniquely defined by Eq. (25) as its strong solution. One can rewrite Eq. (25) under a global form on the entire time interval \([0, T]\). We have, \( \mathbb{P} - \text{a.s. } \omega \in \Omega, \)

\[
\begin{align*}
Y^n_{t_i} &= \eta, \\
Y^n_t + \int_t^T H^n_s \, dK^n_s &= \eta + \int_t^T F(s, X_s, Y^n_s) \, ds - \int_t^T Z^n_s \, dB_s, \; \forall t \in [0, T], \\
dK^n_s &= U^n_s \, ds \in \partial \varphi(Y^n_s)(ds)
\end{align*}
\]

and we obtain that the triplet \((Y^n, Z^n, U^n)\) satisfies a boundedness property similar to (15). This permits us to prove, in the same manner as in Theorem 7, the tightness criteria followed by the existence of a weak solution.
6 Annex

For the clarity of the proofs from the main body of this article we will group in this section some useful results that are used throughout this paper. For more details the interested reader can consult the monograph of Pardoux and Răşcanu [22].

6.1 BSDEs with Lipschitz coefficient

We first introduce the spaces that will appear in the next results. Denote by $S^p_d[0,T], p \geq 0$, the space of progressively measurable continuous stochastic processes $X : \Omega \times [0,T] \to \mathbb{R}^d$, such that

$$
\|X\|_{S^p_d} = \begin{cases} 
\left( \mathbb{E} \|X\|^p_T \right)^{\frac{1}{p}} < \infty, & \text{if } p > 0, \\
\mathbb{E} [1 \wedge \|X\|_T], & \text{if } p = 0,
\end{cases}
$$

where $\|X\|_T = \sup_{t \in [0,T]} |X_t|$. The space $(S^p_d[0,T], \|\cdot\|_{S^p_d}), p \geq 1$, is a Banach space and $S^p_d[0,T], 0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{S^p_d}$ (when $p = 0$ the metric convergence coincides with the probability convergence).

Denote by $\Lambda^p_{d \times k}(0,T), p \in [0,\infty)$, the space of progressively measurable stochastic processes $Z : \Omega \times (0,T) \to \mathbb{R}^d \times k$ such that

$$
\|Z\|_{\Lambda^p_{d \times k}} = \begin{cases} 
\left[ \mathbb{E} \left( \int_0^T \|Z_s\|^2 ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \wedge 1, & \text{if } p > 0, \\
\mathbb{E} \left( \int_0^T \|Z_s\|^2 ds \right)^{\frac{1}{2}}, & \text{if } p = 0.
\end{cases}
$$

The space $(\Lambda^p_{d \times k}(0,T), \|\cdot\|_{\Lambda^p_{d \times k}}), p \geq 1$, is a Banach space and $\Lambda^p_{d \times k}(0,T), 0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{\Lambda^p_{d \times k}}$.

Let consider the following generalized BSDE

$$
Y_t = \eta + \int_t^T \Phi(s, Y_s, Z_s) dQ_s - \int_t^T Z_s dB_s, \ t \in [0,T], \mathbb{P} - a.s. \omega \in \Omega,
$$

where

- $\eta : \Omega \to \mathbb{R}^d$ is a $\mathcal{F}_T$-measurable random vector;
- $Q$ is a progressively measurable increasing continuous stochastic process such that $Q_0 = 0$;
- $\Phi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ for which we denote $\Phi^\rho (t) \overset{d}{=} \sup_{|y| \leq \rho} |\Phi(t, y, 0)|$.

We shall assume that:

(BSDE-LH):
Theorem 12 (See Theorem 5.29 from Pardoux and Răşcanu [22])

(i) for all $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^{d \times k}$ the function $\Phi (\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^d$ is progressively measurable;

(ii) there exist the progressively measurable stochastic processes $L, \ell, \alpha : \Omega \times [0, T] \to \mathbb{R}_+$ such that

$$\alpha_t dQ_t = dt \quad \text{and} \quad \int_0^T \left( L_t dQ_t + \ell_t^2 dt \right) < \infty, \ P - a.s. \omega \in \Omega$$

and, for all $t \in [0, T], y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}^{d \times k}$, $P - a.s. \omega \in \Omega$

$$|\Phi (t, y, z) - \Phi (t, y', z)| \leq L_t |y' - y|,$$

$$|\Phi (t, y, z') - \Phi (t, y, z)| \leq \alpha_t \ell_t |z' - z|,$$

$$\int_0^T \Phi_p^n (t) dQ_t < \infty, \ \forall \rho \geq 0.$$ (28)

Lipschitz conditions:

(a) $|\Phi (t, y', z) - \Phi (t, y, z)| \leq L_t |y' - y|,$

(b) $|\Phi (t, y, z') - \Phi (t, y, z)| \leq \alpha_t \ell_t |z' - z|,$

Boundedness condition:

(c) $\int_0^T \Phi_p^n (t) dQ_t < \infty, \ \forall \rho \geq 0.$

Remark that condition $\alpha_t dQ_t = dt$ implies

$$\Phi (t, Y_t, Z_t) dQ_t = F (t, Y_t, Z_t) dt + G (t, Y_t) dA_t,$$

where $G$ does not depend on the $z$ variable.

Let $p > 1$ and $n_p \overset{def}{=} 1 \wedge (p - 1)$. The following existence and uniqueness result takes place.

Theorem 12 (See Theorem 5.29 from Pardoux and Răşcanu [22]) Suppose that the assumptions (BSDE-LH) are satisfied. Consider

$$V_t = \int_0^t L_s dQ_s + \frac{1}{n_p} \int_0^t \ell_s^2 ds.$$ (29)

If, for all $\delta > 1$,

$$E \left[ e^{\delta V_T} \eta \right]^p + E \left( \int_0^T e^{\delta V_s} |\Phi (t, 0, 0)| dQ_t \right)^p < \infty$$

then the BSDE (27) admits a unique solution $(Y, Z) \in S^0_d [0, T] \times \Lambda^0_{d \times k} (0, T)$ such that

$$E \sup_{s \in [0, T]} e^{\delta V_s} |Y_s|^p + E \left( \int_0^T e^{\delta V_s} |Y_s|^2 L_s dQ_s \right)^{p/2} + E \left( \int_0^T e^{\delta V_s} |Z_s|^2 ds \right)^{p/2} < \infty.$$ (30)

Consider now the BSDE

$$Y_t = \eta + \int_0^T F (s, Y_s, Z_s) ds - \int_0^T Z_s dB_s, \ t \in [0, T], \ P - a.s. \omega \in \Omega.$$ (30)

where for all $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d \times k}$, the function $F (\cdot, y, z) : [0, T] \to \mathbb{R}^d$ is measurable and there exist some measurable deterministic functions $L, \kappa, \rho \in L^1 (0, T; \mathbb{R}_+)$ and $\ell \in L^2 (0, T; \mathbb{R}_+)$ such that, for all $y, y' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d \times k}$, $dt - a.e.,$

$$|F (t, y', z) - F (t, y, z)| \leq L (t) (1 + |y| \vee |y'|) |y' - y|,$$

$$|F (t, y, z') - F (t, y, z)| \leq \ell (t) |z' - z|,$$

$$|F (t, y, 0)| \leq \rho (t) + \kappa (t) |y|.$$ (31)
Letting $\gamma(t) = \kappa(t) + \frac{1}{n_p}t^2(t)$ and $\tilde{\gamma}(t) = \int_0^t \left( \kappa(s) + \frac{1}{n_p}t^2(s) \right) ds$, consider the stochastic process $\beta \in S^0_1 [0, T]$ given by

$$
\beta_t = \mathbb{C}' \left( 1 + \mathbb{E}^{\mathcal{F}_t} |\eta|^p \right)^{1/p} \geq (C_p)^{1/p} e^{-\gamma(t)} \mathbb{E}^{\mathcal{F}_t} \left[ |e^{\gamma(T)}\eta|^p + \left( \int_0^T e^{\gamma(s)} \rho(s) ds \right)^p \right]^{1/p},
$$

where $\mathbb{C}' = \mathbb{C}'(p, \tilde{\gamma}(T), \int_0^T \rho(s) ds)$.

Denote

$$
\nu_t = \int_0^t L(s) \left[ \mathbb{E}^{\mathcal{F}_s} |\eta|^p \right]^{1/p} \quad \text{and} \quad \theta = \sup_{t \in [0, T]} \left( \mathbb{E}^{\mathcal{F}_t} |\eta|^p \right)^{1/p}.
$$

**Theorem 13** Let $p > 1$ and the assumptions (31) be satisfied. If $\mathbb{E} e^{\delta \eta} < \infty$, for all $\delta > 0$, then the BSDE (30) admits a unique solution $(Y, Z) \in S^0_1 [0, T] \times \Lambda_{d \times k}^0 (0, T)$ such that, for all $\delta > 0$,

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\frac{\delta}{2} \nu_s} |Y_s|^p + \mathbb{E} \left( \int_0^T e^{2\delta \nu_s} |Z_s|^2 ds \right)^{p/2} < \infty.
$$

Moreover, $\mathbb{P}$ a.s. $\omega \in \Omega$,

$$
|Y_t| \leq \mathbb{C}' \left( 1 + \left( \mathbb{E}^{\mathcal{F}_t} |\eta|^p \right)^{1/p} \right), \quad \text{for all } t \in [0, T].
$$

**Proof.** Consider the projector operator $\pi : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$
\pi_t(\omega, y) = \pi(\omega, t, y) = y \left[ 1 - \left( 1 - \frac{\beta_t(\omega)}{|y|} \right)^+ \right] = \begin{cases} 
  y, & \text{if } |y| \leq \beta_t(\omega), \\
  \frac{y}{|y|} \beta_t(\omega), & \text{if } |y| > \beta_t(\omega).
\end{cases}
$$

Remark that, for all $y, y' \in \mathbb{R}^d$, $\pi(\cdot, \cdot, y)$ is a progressively measurable stochastic process, $|\pi_t(y)| \leq \beta_t$ and

$$
|\pi_t(y) - \pi_t(y')| \leq |y - y'|.
$$

Let $\tilde{\Phi}(s, y, z) = \Phi(s, \pi_s(y), z)$. The function is globally Lipschitz with respect to $(y, z)$:

$$
|\tilde{\Phi}(s, y, z) - \tilde{\Phi}(s, y', z)| = |\Phi(s, \pi_s(y), z) - \Phi(s, \pi_s(y'), z)|
\leq L(s) \left( 1 + |\pi_s(y)| \vee |\pi_s(y')| \right) |\pi_s(y) - \pi_s(y')|
\leq L(s) \left( 1 + \beta_s \right) |y - y'|
$$

and

$$
|\tilde{\Phi}(s, y, z) - \tilde{\Phi}(s, y', z)| = |\Phi(s, \pi_s(y), z) - \Phi(s, \pi_s(y), z')| \leq \alpha_s \ell(s) |z - z'|.
$$

Then, according to Theorem 12, the BSDE

$$
Y_t = \eta + \int_t^T \tilde{\Phi}(s, Y_s, Z_s) dQ_s - \int_t^T Z_s dB_s, \quad t \in [0, T].
$$
admits a unique solution \((Y, Z) \in S^0_d [0, T] \times L^0_{d \times k} (0, T)\) satisfying
\[
E \sup_{s \in [0, T]} e^{\delta p V_s} |Y_s|^p + E \left( \int_0^T e^{2 \delta V_s} |Z_s|^2 \, ds \right)^{p/2} < \infty,
\]
where
\[
V_t = \int_0^t \left[ \kappa(s) + L(s) (1 + \beta_s) + \frac{1}{n_p} \ell^2 (s) \right] \, ds \leq C + C \int_0^t L(s) \left[ E^{F_s} [n^p] \right]^{1/p}.
\]
Since we have
\[
\langle Y_t, \Phi (t, Y_t, Z_t) \rangle = \langle Y_t, \Phi (t, \pi_t (Y_t), Z_t) \rangle dQ_t \leq |Y_t| \rho(t) dQ_t + |Y_t|^2 \gamma(t) dQ_t + \frac{n_p}{4} |Z_t|^2 \, dt
\]
then \(|Y_t| \leq \beta_t\) and, consequently, \(\Phi (t, Y_t, Z_t) = \Phi (t, Y_t, Z_t)\), that is \((Y, Z)\) is the unique solution of BSDE (30).

6.2 Moreau-Yosida regularization of a convex function

By \(\nabla \varphi_{\varepsilon}\) we denote the gradient of the Yosida’s regularization \(\varphi_{\varepsilon}\) of the function \(\varphi\). More precisely (see Brézis [4]),
\[
\varphi_{\varepsilon}(x) = \inf \left\{ \frac{1}{2\varepsilon} |z - x|^2 + \varphi(z) : z \in \mathbb{R}^d \right\} = \frac{1}{2\varepsilon} |x - J_{\varepsilon} x|^2 + \varphi(J_{\varepsilon} x),
\]
where \(J_{\varepsilon} x = x - \varepsilon \nabla \varphi_{\varepsilon}(x)\). The function \(\varphi_{\varepsilon} : \mathbb{R}^d \to \mathbb{R}\) is a convex and differentiable one and it has the following main properties. For all \(x, y \in \mathbb{R}^d, \varepsilon > 0:\)
\[
\begin{align*}
& a) \quad \nabla \varphi_{\varepsilon}(x) = \partial \varphi_{\varepsilon}(x) \in \partial \varphi(J_{\varepsilon} x), \text{ and } \varphi(J_{\varepsilon} x) \leq \varphi_{\varepsilon}(x) \leq \varphi(x), \\
& b) \quad |\nabla \varphi_{\varepsilon}(x) - \nabla \varphi_{\varepsilon}(y)| \leq \frac{1}{\varepsilon} |x - y|, \\
& c) \quad \langle \nabla \varphi_{\varepsilon}(x) - \nabla \varphi_{\varepsilon}(y), x - y \rangle \geq 0, \\
& d) \quad \langle \nabla \varphi_{\varepsilon}(x) - \nabla \varphi_{\varepsilon}(y), x - y \rangle \geq -(\varepsilon + \delta) \langle \nabla \varphi_{\varepsilon}(x), \nabla \varphi_{\varepsilon}(y) \rangle.
\end{align*}
\]
If \(0 = \varphi (0) \leq \varphi (x)\) for all \(x \in \mathbb{R}^d\) then
\[
\begin{align*}
(a) \quad & 0 = \varphi_{\varepsilon}(0) \leq \varphi_{\varepsilon}(x) \quad \text{and} \quad J_{\varepsilon} (0) = \nabla \varphi_{\varepsilon}(0) = 0, \\
(b) \quad & \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon}(x)|^2 \leq \varphi_{\varepsilon}(x) \leq \langle \nabla \varphi_{\varepsilon}(x), x \rangle, \quad \forall x \in \mathbb{R}^d.
\end{align*}
\]

Proposition 14 Let \(\varphi : \mathbb{R}^d \to \mathbb{R}_{+} = [0, \infty] \) be a proper convex lower semicontinuous function such that \(\text{int} (\text{Dom} (\varphi)) \neq \emptyset\). Let \((u_0, \bar{u}_0) \in \partial \varphi, r_0 \geq 0\) and
\[
\varphi^\#_{u_0, r_0} \overset{def}{=} \sup \{ \varphi (u_0 + r_0 v) : |v| \leq 1 \}.
\]
Then, for all \( 0 \leq s \leq t \) and \( dk(t) \in \partial \varphi (x(t)) (dt) \),

\[
 r_0 (\mathcal{KL}_t - \mathcal{KL}_s) + \int_s^t \varphi(x(r)) dr \leq \int_s^t (x(r) - u_0, dk(r)) + (t - s) \varphi^\#_{u_0,r_0} \tag{35}
\]

and, moreover,

\[
 r_0 (\mathcal{KL}_t - \mathcal{KL}_s) + \int_s^t \varphi(x(r)) - \varphi(u_0) | dr \leq \int_s^t (x(r) - u_0, dk(r)) + \int_s^t (2 |u_0| |x(r) - u_0| + \varphi^\#_{u_0,r_0} - \varphi(u_0)) dr. \tag{36}
\]

### 6.3 Basic inequalities

We shall derive some important estimations on the stochastic processes \((Y, Z) \in S_d^0 [0, T] \times \Lambda^0_{d \times k} (0, T)\) satisfying for all \( t \in [0, T], \mathbb{P} - a.s. \omega \in \Omega, \)

\[
 Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s,
\]

with \( K \in S_d^0 \) be such that \( K(\omega) \in BV_{loc} ([0, \infty [; \mathbb{R}^d], \mathbb{P} - a.s. \omega \in \Omega. \) For more details concerning the results found in this subsection one can consult Section 6.3.4 from Pardoux and Răşcanu [22].

**Backward Itô’s formula.** If \( \psi \in C^{1,2} ([0, T] \times \mathbb{R}^d), \) then \( \mathbb{P} - a.s. \omega \in \Omega, \) for all \( t \in [0, T], \)

\[
 \psi(t, Y_t) + \int_t^T \left\{ \frac{\partial \psi}{\partial t} (s, Y_s) + \frac{1}{2} \text{Tr} \left[ Z_s Z_s^* \psi'_{xx} (s, Y_s) \right] \right\} ds = \psi(T, Y_T) + \int_t^T \langle \psi'_x (s, Y_s), dK_s \rangle - \int_t^T \langle \psi'_x (s, Y_s), Z_s dB_s \rangle \tag{37}
\]

According to Lemma 2.35 from [22], if \( \psi : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is a \( C^1 \)-class function, convex in the second argument, then, \( \mathbb{P} - a.s. \omega \in \Omega, \) for every \( t \in [0, T], \) the following stochastic subdifferential inequality takes place:

\[
 \psi(t, Y_t) + \int_t^T \frac{\partial \psi}{\partial t} (s, Y_s) ds \leq \psi(T, Y_T) + \int_t^T \langle \nabla \psi(s, Y_s), dK_s \rangle - \int_t^T \langle \nabla \psi(s, Y_s), Z_s dB_s \rangle. \tag{38}
\]

### A fundamental inequality

Let \((Y, Z) \in S_d^0 [0, T] \times \Lambda^0_{d \times k} (0, T)\) satisfying an identity of the form

\[
 Y_t = Y_T + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \mathbb{P} - a.s. \omega \in \Omega,
\]

where \( K \in S_d^0 ([0, T]) \) and \( K(\omega) \in BV ([0, T] ; \mathbb{R}^d), \mathbb{P} - a.s. \omega \in \Omega. \)

Assume there exist
• $D, R, N$ - three progressively measurable increasing continuous stochastic processes with $D_0 = R_0 = N_0 = 0$,

• $V$ - a progressively measurable bounded variation continuous stochastic process with $V_0 = 0$,

• $0 \leq \lambda < 1 < p$,

such that, as measures on $[0, T], \mathbb{P} - a.s. \omega \in \Omega$,

\begin{equation}
D_t + \langle Y_t, dK_t \rangle \leq \left[ 1_{p \geq 2} dR_t + |Y_t| dN_t + |Y_t|^2 dV_t \right] + \frac{n_p \lambda}{2} |Z_t|^2 dt,
\end{equation}

where \( n_p \overset{\text{def}}{=} 1 \land (p - 1) \).

Proposition 6.80 from Pardoux and Răşcanu [22] yields the following important result.

**Proposition 15** If (39) and (40) hold, and moreover \( \mathbb{E} \left[ \left| Y_T \right|^p \right] < \infty \),

then there exists a positive constant \( C_{p, \lambda} \), depending only upon \( p, \lambda \), such that, \( \mathbb{P} - a.s. \omega \in \Omega \),

\begin{align}
\mathbb{E}^F_t \sup_{s \in [t, T]} |Y_s|^p + \mathbb{E}^F_t \left( \int_t^T e^{2V_s} dD_s \right)^{p/2} \\
+ \mathbb{E}^F_t \int_t^T e^{pV_s} |Y_s|^{p-2} 1_{Y_s \neq 0} dD_s + \mathbb{E}^F_t \int_t^T e^{pV_s} |Y_s|^{p-2} 1_{Y_s \neq 0} |Z_s|^2 ds \\
\leq C_{p, \lambda} \mathbb{E}^F_t \left[ |Y_T|^p + \left( \int_t^T e^{2V_s} 1_{p \geq 2} dR_s \right)^{p/2} + \left( \int_t^T e^{V_s} dN_s \right)^p \right].
\end{align}

In addition, if $R = N = 0$, then, for all $t \in [0, T]$,

\begin{equation}
e^{pV_t} |Y_t|^p \leq \mathbb{E}^F_t e^{pV_T} |Y_T|^p, \quad \mathbb{P} - a.s. \omega \in \Omega.
\end{equation}

**Corollary 16** Under the assumptions of Proposition 15, if $V$ is a determinist process and $\sup_{s \geq 0} |V_s| \leq c$ then, $\mathbb{P} - a.s. \omega \in \Omega$, for all $t \in [0, T]$,

\begin{align}
\mathbb{E}^F_t \sup_{s \in [t, T]} |Y_s|^p + \mathbb{E}^F_t \left( \int_t^T |Z_s|^2 ds \right)^{p/2} \\
\leq C_{p, \lambda} e^{2c} \mathbb{E}^F_t \left[ |Y_T|^p + \left( \int_t^T 1_{p \geq 2} dR_s \right)^{p/2} + \left( \int_t^T dN_s \right)^p \right].
\end{align}
Proposition 17 (See Proposition 6.69 from [22]) Let $\delta \in \{-1, 1\}$ and consider $Y, K, A : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $G : \Omega \times \mathbb{R}_+ \to \mathbb{R}^k$ four progressively measurable stochastic processes such that

i) $Y, K, A$ are continuous stochastic processes,

ii) $A_r, K_r \in BV_{loc}([0, \infty[; \mathbb{R})$, $A_0 = K_0 = 0$, $\mathbb{P}$-a.s. $\omega \in \Omega$,

iii) $\int_0^t |G_r|^2 \, dr < \infty$, $\mathbb{P}$-a.s. $\omega \in \Omega$, $\forall 0 \leq t \leq s$.

If, for all $0 \leq t \leq s$,

$$\delta (Y_t - Y_s) \leq \int_t^s (dK_r + Y_r \, dA_r) + \int_t^s \langle G_r, dB_r \rangle, \quad \mathbb{P}$-a.s. $\omega \in \Omega,$

then

$$\delta \left( Y_t e^{\delta A_t} - Y_s e^{\delta A_s} \right) \leq \int_t^s e^{\delta A_r} \, dK_r + \int_t^s e^{\delta A_r} \langle G_r, dB_r \rangle, \quad \mathbb{P}$-a.s. $\omega \in \Omega.$

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