On explicit birational geometry for minimal $n$-folds of canonical dimension $n-1$

Meng Chen$^1$ | Louis Esser$^2$ | Chengxi Wang$^3$

$^1$School of Mathematical Sciences, Fudan University, Shanghai, China
$^2$Princeton Mathematics Department, Fine Hall, Princeton, NJ, USA
$^3$UCLA Mathematics Department, Los Angeles, CA, USA

Correspondence
Chengxi Wang, UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, United States.
Email: chwang@math.ucla.edu

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Abstract
Let $n \geq 2$ be any integer. We study the optimal lower bound of the canonical volume and the optimal upper bound of the canonical stability index for minimal projective $n$-folds of general type, which are canonically fibered by $i$-folds ($i = 0, 1$). The results for $i = 0$ are known to experts. In this article, we compute sharp bounds for $i = 1$ and give some characterization of optimal examples. The machinery introduced is applicable to all canonical dimensions $n-i$.

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1 | INTRODUCTION

In birational geometry, minimal varieties of general type form one of the basic building blocks of the minimal model program. A crucial step toward classifying these varieties is to study the distribution of their birational invariants. Since every smooth variety of general type has a minimal model, this will give the distribution of invariants in the smooth case as well. Throughout, we will work over any algebraically closed field of characteristic zero.

Let $X$ be a minimal projective $n$-fold of general type, so that $X$ has at worst $\mathbb{Q}$-factorial terminal singularities and the canonical divisor $K_X$ is nef. The volume of $X$ is a measure of the asymptotic growth of the plurigenera $P_m(X) := \dim(H^0(X, mK_X))$, defined as

$$\text{Vol}(X) := \lim_{m \to \infty} \frac{n! P_m(X)}{m^n},$$

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which is a positive rational number. Since $X$ is minimal, so that $K_X$ is nef, the Riemann–Roch formula shows that $\text{Vol}(X) = K^n_X$, the top intersection number of $K_X$.

A remarkable theorem of Hacon–McKernan [18], Takayama [24], and Tsuji [25] shows that for each positive integer $n$, there is a positive integer $r_n$ so that the $n$th pluricanonical map $\phi_{m,X}$ induced by the linear system $|mK_X|$ is birational onto its image for every $m \geq r_n$ and every minimal $n$-fold $X$ of general type. As a consequence, we also have the positive lower bound $\frac{1}{(r_n)^n}$ for the volume of any such $X$.

In this paper, we compute exactly the optimal bounds for an important class of minimal $n$-folds of general type. We consider varieties $X$ for which the image of the map $\phi_{1,X}$ induced by the canonical class has large dimension such as $n$ or $n-1$. The dimension of this image is called the canonical dimension of $X$. The same bounds below also hold for smooth projective varieties $X$ of general type with the given canonical dimension.

**Theorem 1.1.** Let $n \geq 2$ be any integer.

1. If $X$ is a minimal projective $n$-fold of general type and canonical dimension $n$, then: $\text{Vol}(X) \geq 2$ and $\phi_{mK_X}$ is birational for $m \geq n + 2$. These two bounds are optimal.
2. If $X$ is a minimal projective $n$-fold of general type and canonical dimension $n - 1$, then: $\text{Vol}(X) \geq \frac{6}{2n+\left(\frac{n}{3}\right)}$ and $\phi_{mK_X}$ is birational for $m \geq \frac{1}{3}(5n + 3 + (n \mod 3))$. These two bounds are optimal.

Here, $(n \mod 3) \in \{0, 1, 2\}$ is the residue of $n$ modulo 3. For each $n$, there are varieties of general type achieving equality in each of the bounds.

In each dimension, the optimal examples we exhibit are weighted projective hypersurfaces of general type. For an introduction to weighted projective hypersurfaces, see [16, Section 2].

The key idea is that when $X$ has canonical dimension $n - i$ for small $i$, we can analyze it using its canonical fibration by $i$-folds. These fibrations have long been studied with applications to the classification of algebraic varieties, beginning with surfaces.

## 2 NOTATION AND KNOWN RESULTS

Throughout the paper, we will utilize the following notation. For a given minimal variety $X$ of general type, the canonical stability index of $X$ is

$$r_s(X) := \min\{p \in \mathbb{Z}_{>0} | \phi_{m,X} \text{ is birational for all } m \geq p\},$$

where $\phi_{m,X}$ denotes the $m$-canonical map of $X$. For any dimension $n$, we let $r_n$, the $n$th canonical stability index, be the smallest integer satisfying the condition in the theorem of Hacon–McKernan, Takayama [24], and Tsuji [25]:

$$r_n := \max\{r_s(X) | X \text{ is any minimal projective } n\text{-fold of general type}\}.$$

Similarly, we define

$$v_n := \inf\{\text{Vol}(X) | X \text{ is any minimal projective } n\text{-fold of general type}\}.$$
For $1 \leq i < n$, we define $r_{n,n-i}$ and $v_{n,n-i}$ to be the maximal canonical stability index and the infimum of volumes, respectively, among all minimal $n$-folds of general type with canonical dimension $n-i$. Similarly, $r_{n,0}$ and $v_{n,0}$ (resp. $r_{n,-\infty}$ and $v_{n,-\infty}$) are the corresponding values for minimal $n$-folds with geometric genus $p_g = 1$ (resp. $p_g = 0$). With these definitions,

\[
 r_n = \max\{r_{n,j} | -\infty \leq j \leq n\} \quad \text{and} \quad v_n = \min\{v_{n,j} | -\infty \leq j \leq n\}.
\]

Using this notation, we state Theorem 1.1 equivalently as

1. $v_{n,n} = 2, r_{n,n} = n + 2$,
2. $v_{n,n-1} = \frac{6}{2n + (n \mod 3)}$, and $r_{n,n-1} = \frac{1}{3}(5n + 3 + (n \mod 3))$

for all dimensions $n \geq 2$. We list some previously known results concerning these constants:

- The surface case is due to Bombieri [4]; in particular, $r_2 = 5$ and $v_2 = 1$.
- By Iano-Fletcher [20], $r_3 \geq 27$ and $v_3 \leq \frac{1}{420}$; by Chen–Chen [9–11] and Chen [8], $r_3 \leq 57$ and $v_3 \geq \frac{1}{1680}$.
- By Esser–Totaro–Wang [16], for $n \geq 3$, $r_n > 2^{2(n-2)/2}$ and $v_n < \frac{1}{2^{n/2}}$.
- The values of $v_{3,j}$ for $j > 0$ are obtained by one of the authors (Chen) [6, 7].
- The value of $v_{n,n}$ is originally due to Kobayashi [22].
- The values of $v_{4,3}, v_{4,2}, v_{5,4}$ and $v_{5,3}$ are already obtained in Chen–Jiang–Li [13].

Finally, we use the following conventions for divisors and linear systems:

- For two $\mathbb{Q}$-divisors $D_1$ and $D_2$, $D_1 \sim \mathbb{Q} D_2$ (resp. $D_1 \equiv D_2$) means that $D_1$ is $\mathbb{Q}$-linearly equivalent (resp., numerically equivalent) to $D_2$. Similarly, $D_1 \geq_{\text{num}} D_2$ (resp., $D_1 \geq_{\mathbb{Q}} D_2$) means that $D_1 - D_2$ is numerically (resp., $\mathbb{Q}$-linearly) equivalent to an effective $\mathbb{Q}$-divisor.
- For two linear systems $|A|$ and $|B|$ on a variety, we write $|A| \geq |B|$ (or symmetrically, $|B| \leq |A|$) if there exists an effective divisor $F$ such that $|A| \geq |B| + F$.

\section{A Key Technical Theorem}

\textbf{Theorem 3.1.} Let $X$ be a minimal projective $n$-fold of general type. Let $X'$ be a nonsingular projective variety and $\pi : X' \to X$ be a birational morphism. Assume that there exists a chain of smooth subvarieties of general type

\[Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n = X'\]

with $\dim Z_j = j$ for $j = 1, 2, \ldots, n-1$. Suppose that the following conditions hold:

\begin{enumerate}
  \item $\pi^*(K_X)|_{Z_i}$ is big for each $i = 2, \ldots, n-1$;
  \item $\pi^*(K_X)|_{Z_i} \equiv \beta_i Z_{i-1} + \Delta_{i-1}$ where $\beta_i$ is a positive rational number, $\Delta_{i-1}$ is an effective $\mathbb{Q}$-divisor on $Z_i$, and
\end{enumerate}
\[ Z_{i-1} \notin \text{Supp} \left( \Delta_{i-1} + \sum_{j \geq i} \Delta_j |_{Z_i} \right) \]

for each \( i = 2, \ldots, n; \)

(iii) the number \( \xi := (\pi^*(K_X) \cdot Z_1) > 0. \)

Then each of the following statements holds.

(1) For any integer \( m \) with \( \alpha_m := (m - 1 - \sum_{i=2}^{n} \frac{1}{\beta_i})\xi > 1, \) the inequality

\[ m\xi \geq (2g(Z_1) - 2) + \lfloor \alpha_m \rfloor \]  

(3.1)

holds. In particular, under this situation, one has

\[ \xi \geq \frac{2g(Z_1) - 2}{1 + \sum_{i=2}^{n} \frac{1}{\beta_i}}. \]  

(3.2)

(2) The canonical volume of \( X \) has the lower bound:

\[ K^n_X \geq \beta_2 \beta_3 \cdots \beta_n \xi. \]  

(3.3)

**Proof.** First of all, if we replace \( X' \) by any nonsingular birational model (which dominates \( X' \)) for which no \( Z_i \) is contained in the image of the exceptional locus, then we may replace each \( Z_i \) by its strict transform. After performing this replacement, the assumption still holds for the same values of \( \beta_i \) and the number \( \xi \) remains unchanged by the projection formula. Hence, fixing an effective divisor \( K_1 \sim K_X \) at the beginning, we may and do assume that, on \( X' \), the union of the divisor \( \pi^*(K_1)|_{Z_i}, \Delta_{i-1} \) (for all \( i = 2, \ldots, n \)) and exceptional divisors has simple normal crossing supports.

Suppose \( m > 1 + \sum_{i=2}^{n} \frac{1}{\beta_i} \). Since the \( \mathbb{Q} \)-divisor

\[ (m - 1)\pi^*(K_X)|_{Z_n} - \frac{1}{\beta_n} \Delta_{n-1} - Z_{n-1} \equiv \left( m - 1 - \frac{1}{\beta_n} \right) \pi^*(K_X)|_{Z_n} \]

is nef and big, and the fractional part has snc supports, the Kawamata–Viehweg vanishing theorem [21,26] implies:

\[ |mK_X'|_{Z_{n-1}} \geq |K_{X'} + [(m - 1)\pi^*(K_X) - \frac{1}{\beta_n} \Delta_{n-1}]||_{Z_{n-1}} \]

\[ \geq |K_{Z_{n-1}} + [(m - 1)\pi^*(K_X) - Z_{n-1} - \frac{1}{\beta_n} \Delta_{n-1}]||_{Z_{n-1}}|. \]  

(3.4)

By induction, one clearly gets, for \( i = 2, \ldots, n - 1, \) that

\[ |mK_X'|_{Z_{i-1}} \geq |K_{Z_i} + [(m - 1)\pi^*(K_X)]_{Z_i} - \sum_{l=i+1}^{n} \left( Z_{l-1} + \frac{1}{\beta_l} \Delta_{l-1} \right)||_{Z_{i-1}} \]

\[ \geq |K_{Z_{i-1}} + [(m - 1)\pi^*(K_X)]_{Z_{i-1}} - \sum_{l=i}^{n} \left( Z_{l-1} + \frac{1}{\beta_l} \Delta_{l-1} \right)||_{Z_{i-1}}|. \]  

(3.5)
By (3.4) and (3.5) while repeatedly using [5, Lemma 2.7], we have

\[ M_m|Z_1 \geq \text{Mov}|K_{Z_1} + [Q_m]|, \tag{3.6} \]

where \( Q_m := (m - 1)\pi^*(K_X)|_{Z_1} - \sum_{i=2}^{n}(Z_{i-1} + \frac{1}{\beta_i} \triangle_{i-1})|_{Z_1} \) and \( M_m := \text{Mov}|mK_{X'}| \). Note that, by Condition (ii), the divisor \( Z_{i-1}|_{Z_{i-1}} \) never appears in fractional part of all above \( \mathbb{Q} \)-divisors.

Whenever \( \alpha_m = \text{deg}(Q_m) > 1 \), \( Q_m \equiv (m - 1 - \sum_{i=2}^{n}\frac{1}{\beta_i})\pi^*(K_X)|_{Z_1} \) is nef and big, and \( \text{Mov}|K_{Z_1} + [Q_m]| = K_{Z_1} + [Q_m] \), because divisors of degree at least \( 2g(Z_1) \) are base point free. Since \( m\pi^*(K_X) \geq M_m \) and \( \text{deg}(\lceil Q_m \rceil) \geq \text{deg}(Q_m) \), taking degrees on both sides of (3.6) proves (3.1) in (1).

Taking a sufficiently large integer \( m' \) so that \( \alpha_{m'} > 1 \), (3.2) is a consequence of (3.1). Statement (2) then directly follows from the assumptions and the fact that \( \pi^*(K_X) \) is nef.

**Definition 3.2** [14, Definition 2.3]. A generic irreducible element \( S \) of a movable linear system \( |N| \) on a variety \( Z \) is a generic irreducible component in a general member of \( |N| \). Thus, it is a general member of \( |N| \) whenever \( \dim \varphi|_N(V) \geq 2 \). If \( |N| \) is composed with a pencil, that is, \( \dim \varphi|_N(V) = 1 \), one has \( N \equiv tS \) for some integer \( t \geq 1 \).

**Definition 3.3** [11, Definition 2.6]. Let \( |N| \) be a movable linear system on a variety \( Z \). Pick two different generic irreducible elements \( S', S'' \) in \( |N| \). We say that a linear system \( |M| \) (resp. a rational map \( \varphi \) corresponding to a linear system) distinguishes \( S' \) and \( S'' \) if \( \varphi|_M(S') \neq \varphi|_M(S'') \) (resp. \( \varphi(S') \neq \varphi(S'') \)).

In proving birationality of the rational map \( \varphi_{\Lambda} \), where \( \Lambda \subset |L| \) and \( L \) a divisor on a projective variety \( Z \), we tacitly use the following rule:

Let \( |M| \) be a base point free linear system on \( Z \) and denote by \( S \) a generic irreducible element of \( |M| \). If \( \varphi_{\Lambda} \) distinguishes different generic irreducible elements of \( |M| \) and \( \varphi_{\Lambda}|_S \) is birational, then \( \varphi_{\Lambda} \) is birational. Conversely, if \( \varphi_{\Lambda} \) is birational, it is clear that, for any base point free linear system \( |M| \), \( \varphi_{\Lambda} \) distinguishes different generic irreducible elements of \( |M| \) and that \( \varphi_{\Lambda}|_S \) is birational.

We will also make use of the following [10, Section 2.7].

**Proposition 3.4** (Birationality principle). Let \( D \) and \( M \) be two divisors on a smooth projective variety \( Z \). Assume that \( |M| \) is base point free. Take the Stein factorization of \( \varphi|_M| : Z \rightarrow W \rightarrow \mathbb{P}^{h^0(Z,M)-1} \), where \( f \) is a fibration onto a normal variety \( W \). The rational map \( \varphi|_D| \) is birational onto its image if one of the following conditions is satisfied:

- \( \dim \varphi|_M|(Z) \geq 2 \), \( |D - M| \neq \emptyset \) and \( \varphi|_D||_S \) is birational for a general member \( S \) of \( |M| \),
- \( \dim \varphi|_M|(Z) = 1 \), \( \varphi|_D| \) distinguishes general fibers of \( f \) and \( \varphi|_D||_S \) is birational for a general fiber \( S \) of \( f \).

In practice, in order to prove the birationality of \( \varphi|_D| \), it is sufficient to find two sublinear systems \( \Lambda_i \subset |D| \) \((i = 1, 2)\) such that \( \varphi_{\Lambda_i} \) distinguishes different generic irreducible elements of \( |M| \), whereas \( \varphi_{\Lambda_2}|_S \) is birational for a generic irreducible element \( S \) of \( |M| \).
Corollary 3.5. Keep the same assumptions and notation as in Theorem 3.1. Let \( m \) be a positive integer. Assume that the following conditions are satisfied.

For each \( i = 2, \ldots, n \),

1. there exists a base point free linear system \( |N_i| \) on \( Z_i \) such that \( Z_{i-1} \) is the generic irreducible element of \( |N_i| \);
2. the linear system \( |mK_{X'}||Z_i| \) distinguishes different generic irreducible elements of \( |N_i| \);
3. \( \alpha_m > 2 \).

Then \( \varphi_{m,X} \) is birational onto its image.

Proof. We refer to the proof of Theorem 3.1. Applying the birationality principle repeatedly, by Assumptions (1)–(2), Relation (3.4), and Relation (3.5), \( \varphi_{m,X} \) is birational if and only if so is \( \varphi_{m,X'}|Z_{n-1}, \ldots, if and only if so is \( \varphi_{m,X'}|Z_1 \). Now we know that \( \varphi_{KZ_1+|Q_m||Z_1} \) is birational as \( \alpha_m > 2 \). From the proof of Theorem 3.1, we have seen that \( |KZ_1+|Q_m||Z_1 \). Hence, \( \varphi_{m,X'}|Z_1 \) is birational. We are done. \( \square \)

4 \ | \ THE CANONICAL MAP OF VARIETIES OF GENERAL TYPE

4.1 \ | \ Setup for \( \varphi_{1,X} \)

Let \( X \) be a minimal projective \( n \)-fold (\( n \geq 2 \)) of general type with \( p_g(X) \geq 2 \) and canonical dimension \( d_1 \). Clearly, we have \( 1 \leq d_1 \leq n \). Let \( \pi : X' \rightarrow X \) be a succession of blowups along nonsingular centers such that \( |M_1| \), the linear system of Mov\( |K_{X'}| \), is base point free and that the union of the fixed part of \( |K_{X'}| \) and exceptional divisors of \( \pi \) has simple normal crossings supports. Take \( g_1 := \varphi_{1,X} \circ \pi \), which is a projective morphism. Denote by \( f : X' \rightarrow \Gamma \) be the induced fibration after taking the Stein factorization of \( g_1 \). So, \( \dim \Gamma = d_1 \).

Set \( |N_n| := |M_1| \) and \( Z_n := X' \). For any integer \( k \) with \( n - d_1 + 2 \leq k \leq n \), inductively, take \( Z_{k-1} \) to be the generic irreducible element of \( |N_k| \) and set \( |N_{k-1}| := |M_1|_{Z_{k-1}} \). By Bertini’s theorem, \( |N_{k-1}| \) is base point free and is not composed with a pencil for each \( k \). We have defined the following chain of nonsingular subvarieties:

\[
Z_{n-d_1+1} \subset \cdots \subset Z_{n-1} \subset Z_n = X'.
\]

Let us review some established inequalities in Chen–Jiang–Li [13, Step 0, Proof of Theorem 5.1]. Remodify \( \pi \) (if necessary) such that, for \( 1 \leq j \leq d_1 - 1 \), \( Z_{n-j} \) dominates a minimal model \( Z_{n-j,0} \) and, in particular, there is a birational morphism \( \pi_{Z_{n-j,0}} : Z_{n-j} \rightarrow Z_{n-j,0} \). By induction, for \( 1 \leq j \leq d_1 - 1 \), one has

\[
h^0(X', M_1|Z_{n-j}) \geq p_g(X') - j. \tag{4.1}
\]

Similar to Chen–Jiang–Li [13, Inequality (5.4)], we have

\[
\pi^*(K_{X'})|Z_{n-j} \geq \frac{1}{j+1}\pi^*_Z(K_{Z_{n-j,0}}) \tag{4.2}
\]
for \( 1 \leq j \leq d_1 - 1 \). In particular, we have

\[
\pi^*(K_{X'})|_{Z\cap Z_{n-d_1+1}} \geq \frac{1}{d_1} \pi^*_n(Z_{n-d_1+1})(K_{Z_{n-d_1+1}}^1).
\]

### 4.2 Proof of Theorem 1.1: The case \( d_1 = n \)

Theorem 3.1 and Corollary 3.5 naturally apply to this very special case. We have already defined the chain of subvarieties:

\[
Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n = X'.
\]

By construction, we have \( \beta_i \geq 1 \) for each \( i \) with \( 2 \leq i \leq n \). Since \( |M_1|_{Z_2} \) is base point free and is not composed with a pencil, we have

\[
\xi \geq (M_1 \cdot Z_1)^{X'} \geq (Z_1^2). \]

We know \( (Z_1^2) \) equals the degree of \( \mathcal{O}(M_1|_{Z_1}) \), which is the degree of the image curve of the morphism from \( Z_1 \) to a projective space induced by the base point free linear system \( |M_1|_{Z_1} \). If the projective space has dimension larger than 1, then the degree of the curve is larger than 1 because it is nondegenerate. If the projective space has dimension 1, then the degree of the curve is larger than 1 since the genus of \( Z_1 \) is larger than 1. Thus, \( (Z_1^2) \geq 2 \). Hence, by (3.3), \( v_{n,n} \geq 2 \).

Take any integer \( m > n + 1 \). Since \( p_g(X) > 0 \), the proof of Theorem 3.1 simply implies

\[
|mK_{X'}|_{Z_1} \geq |N_i| \text{ for } 2 \leq i \leq n.
\]

Because each linear system \( |N_i| \) is not composed with a pencil, the linear system \( |mK_{X'}|_{Z_1} \) naturally distinguishes generic irreducible elements of \( |N_i| \). Conditions of Corollary 3.5 (1) and (2) are therefore automatically satisfied. Since \( \alpha_m > \xi \geq 2 \), we have proved that \( r_s(X) \leq n + 2 \) by Corollary 3.5. Hence, \( r_{n,n} \leq n + 2 \).

**Example 4.1.** For any integer \( n \geq 2 \), the general hypersurface

\[
X = V_{2(n^2+2)} \subset \mathbb{P}^{(n+1,n+2)}
\]

(of degree \( 2(n+2) \)) is a smooth canonical \( n \)-fold with \( \omega_X \cong \mathcal{O}_X(1) \), \( \text{Vol}(X) = 2 \), and \( r_s(X) = n + 2 \). Clearly, \( X \) has canonical dimension \( d_1 = n \).

So, we conclude that Theorem 1.1 (1) is true, that is, \( v_{n,n} = 2 \) and \( r_{n,n} = n + 2 \). Conversely, if \( X \) has canonical dimension \( n \) and \( \text{Vol}(X) = v_{n,n} = 2 \), \( X \) is a double cover of \( \mathbb{P}^n \) [22, Proposition 2.5].

### 4.3 Further setting for the case \( d_1 < n \)

We take \( Z_{n-d_1} \) to be the generic irreducible element of \( |N_{n-d_1+1}| : = |M_1|_{Z_{n-d_1+1}} \). Since \( |N_{n-d_1+1}| \) is composed with a pencil and by (4.1), we may define the constant \( t \) by

\[
N_{n-d_1+1} \equiv tZ_{n-d_1},
\]
where \( t \geq p_g(X) - d_1 \). Indeed, the image in \( \mathbb{P}^{p_g(X)-1} \) of the morphism induced by the base point free linear system \( |M_1|_{Z_{n-d_1+1}} \) is nondegenerate. Thus, the degree of image is not less than \( p_g(X) - 1 - d_1 + 1 \). Since \( t \) is not less than this degree, we get \( t \geq p_g(X) - d_1 \). We have that the conditions of Theorem 3.1 are satisfied for some \( \beta_{n-d_1+1} \geq t \geq p_g(X) - d_1 \) and \( \beta_i \geq 1 \) for each \( i \) with \( n - d_1 + 2 \leq i \leq n \).

5 | **PROOF OF THEOREM 1.1: THE CASE \( d_1 = n - 1 \)**

We have defined the chain \( \{Z_i | i = 1, \ldots, n\} \) according to Subsections 4.1 and 4.3. Note that \( |N_2| \) is composed with a pencil of curves on the smooth surface \( Z_2 \). By our definition, \( C := Z_1 \) is the generic irreducible element of \( |N_2| \). Furthermore, we have \( \beta_i \geq 1 \) for \( 3 \leq i \leq n \) and \( \beta_2 \geq p_g(X) - n + 1 \geq 1 \). Since \( Z_1 \) moves in an algebraic family and \( \pi^*(K_X)|_{Z_2} \) is nef and big, we have \( \xi > 0 \) by the Hodge index theorem. Indeed, the self-intersection of the nef and big divisor \( \pi^*(K_X) \) is strictly positive [23, Theorem 2.2.16]. Thus, \( \xi = \pi^*(K_X)|_{Z_1} \cdot Z_1 = 0 \) would imply \( Z_1^2 < 0 \) because \( Z_1 \) is not numerically trivial [3, Corollary IV.2.16]. However, this contradicts the fact that \( Z_1 \) moves.

5.1 | **Volume estimation**

Noting that \( g(Z_1) \geq 2 \), by (3.2), we have \( \xi \geq \frac{2}{n} \). We optimize the estimation by distinguishing three exclusive cases.

**Case 5.1.1.** \( n = 3k + 2, k \geq 0 \).

Take \( m_1 = \lfloor \frac{9}{2} k + 4 \rfloor \). Then, since \( \alpha_{m_1} > 1 \), we have

\[
\xi \geq \frac{8}{9k + 8} = \frac{2 \cdot 4}{(2 \cdot 4 + 1)k + 2 \cdot 4}
\]

by (3.1). We will repeatedly use (3.1) to optimize the estimation of \( \xi \). In fact, suppose that we already know

\[
\xi \geq \frac{2 \cdot 4^l}{(2 \cdot 4^l + 1)k + 2 \cdot 4^l}
\]

for an integer \( l > 0 \). Take

\[
m_l = \left\lfloor \frac{(2 \cdot 4^l + 1)k}{2 \cdot 4^l} \right\rfloor + 3k + 4.
\]

Then, we see \( \alpha_{m_l} > 1 \) and (3.1) implies

\[
\xi \geq \frac{2 \cdot 4^{l+1}}{(2 \cdot 4^{l+1} + 1)k + 2 \cdot 4^{l+1}}.
\]
Taking the limit, as $l \to +\infty$, we have

$$\xi \geq \frac{1}{k+1} = \frac{3}{n+1}.$$  \hfill (5.1)

\textbf{Case 5.1.2.} $n = 3k + 1$, $k \geq 1$.

Take $m_1 = \lfloor \frac{9k+1}{2} + 2 \rfloor$. Then, since $\alpha_{m_1} > 1$, we have

$$\xi \geq \frac{8}{9k+5} = \frac{2 \cdot 4}{(2 \cdot 4 + 1)k + 4 + 1}$$

by (3.1). Assume that we have shown

$$\xi \geq \frac{2 \cdot 4^l}{(2 \cdot 4^l + 1)k + (4^l + 4^{l-1} + \ldots + 1)}$$

for some integer $l > 0$. Take $m_l = \lfloor \frac{(2 \cdot 4^l + 1)k + (4^l + 4^{l-1} + \ldots + 1)}{2 \cdot 4^l} \rfloor + 3k + 2$. Then, since $\alpha_{m_l} > 1$, we get

$$\xi \geq \frac{2 \cdot 4^{l+1}}{(2 \cdot 4^{l+1} + 1)k + (4^{l+1} + 4^{l} + \ldots + 1)}.$$

Taking the limit, as $l \to +\infty$, we have $\xi \geq \frac{1}{k+2/3}$. Finally, taking $m = 4k + 2$, then we get

$$\xi \geq \frac{2}{2k+1} = \frac{6}{2n+1}.$$  \hfill (5.2)

\textbf{Case 5.1.3.} $n = 3k$, $k \geq 1$.

Take $m_1 = \lfloor \frac{9k}{2} + 1 \rfloor$. Then, since $\alpha_{m_1} > 1$, we get

$$\xi \geq \frac{8}{9k+2} > \frac{2 \cdot 4}{(2 \cdot 4 + 1)k + 4}$$

by (3.1). Suppose that, for some $l > 0$, we have already shown

$$\xi \geq \frac{2 \cdot 4^l}{(2 \cdot 4^l + 1)k + (4^l + \ldots + 4)}.$$

Take

$$m_l = \left\lfloor \frac{(2 \cdot 4^l + 1)k + (4^l + \ldots + 4)}{2 \cdot 4^l} \right\rfloor + 3k + 2.$$

Then, since $\alpha_{m_l} > 1$, we have

$$\xi \geq \frac{2 \cdot 4^{l+1}}{(2 \cdot 4^{l+1} + 1)k + (4^{l+1} + \ldots + 4)}$$
by (3.1). Taking the limit, as $l \to +\infty$, we have $\xi \geq \frac{1}{k+2/3}$. Take $m = 4k + 1$. Then $\xi \geq \frac{4}{4k+1} = \frac{1}{k+1/4}$.

We continue the optimization for the lower bound of $\xi$. Assume that, for some integer $t \geq 4$, we already know $\xi \geq \frac{1}{k+1/t}$. Take $m_t = (t+1)k + 1$. Then, since $\alpha_{m_t} > t - 2 \geq 2$, we have

$$\xi \geq \frac{t+1}{(t+1)k+1} = \frac{1}{k+1/(t+1)}$$

by (3.1). Taking the limit, as $t \to +\infty$, then we have

$$\xi \geq \frac{1}{k} = \frac{3}{n}. \quad (5.3)$$

By (5.1), (5.2), and (5.3), we have proved the following.

**Corollary 5.1.** For any integer $n \geq 2$, $v_{n,n-1} \geq \frac{6}{2n+(n \mod 3)}$.

### 5.2 The canonical stability index

As we have seen, for $3 \leq i \leq n$, $|N_i|$ is not composed with a pencil. Applying the Kawamata–Viehweg vanishing theorem in the similar way to that of (3.4) and (3.5), for any $m > n - 2$ and any $j$ with $3 \leq j \leq n - 1$, we have

$$|mK_X'|_{|Z_j} \geq |K_X' + [(m - n + 2)\pi^*(K_X)] + (n - 3)M_1|_{|Z_j} \quad (5.4)$$

$$\geq |M_1|_{|Z_j}. \quad (5.5)$$

Indeed, we have

$$mK_X' = K_X' + (m - n + 2)K_X' + (n - 3)K_X'$$

$$\geq K_X' + [(m - n + 2)\pi^*(K_X)] + (n - 3)M_1,$$

which gives (5.4). Since $Z_{n-1} \in |M_1|$ and $K_{Z_{n-1}} \geq M_1|_{Z_{n-1}}$, we get

$$(K_X' + (n - 3)M_1)|_{Z_{n-1}} = (n - 4)M_1|_{Z_{n-1}} + (Z_{n-1} + K_X')|_{Z_{n-1}}$$

$$= (n - 4)M_1|_{Z_{n-1}} + K_{Z_{n-1}}$$

$$\geq M_1|_{Z_{n-1}}.$$ 

Then, by $\pi^*(K_X) \geq M_1$, we get (5.5) when $j = n - 1$. Similarly, we get

$$(n - 4)M_1|_{Z_{n-1}} + K_{Z_{n-2}}|_{Z_{n-2}} = (n - 5)M_1|_{Z_{n-2}} + K_{Z_{n-2}}$$

$$\geq M_1|_{Z_{n-2}}.$$
Since $\pi^* (K_X) \geq M_1$, we get (5.5) when $j = n - 2$. The derivation for (5.5) is similar when $j = n - 3, \ldots, 4$. We repeat the processes to finally get

$$(M_1|Z_4 + K_{Z_4})|Z_3 = (Z_4 + K_{Z_4})|Z_3 = K_{Z_3} \geq M_1|Z_3.$$  

Since $\pi^* (K_X) \geq M_1$, (5.5) holds also for $j = 3$.

Hence, $|mK_X'||Z_j|$ distinguishes different generic irreducible elements of $|N_j|$ whenever $m > n - 2$ and $3 \leq j \leq n$.

We study the situation on $|N_2|$, which is composed with a pencil of curves. If $|N_2|$ is composed with a rational pencil of curves, as long as $m > n - 1$, $|mK_X'||Z_2|$ distinguishes different generic irreducible elements of $|N_2|$ since $|mK_X'||Z_2 \geq |N_2|$. If $|N_2|$ is composed with a nonrational pencil of curves, picking two different generic irreducible elements $C_1, C_2$ of $|N_2|$, then $(M_1|Z_2 - C_1 - C_2)$ is nef and

$$(M_1|Z_2 - C_i)|C_i \sim 0$$

for $i = 1, 2$. Hence, by the vanishing theorem and for any $m > n$, one has the relation:

$$|mK_{X'}|Z_2 \geq |K_{X'} + [(m-n)\pi^*(K_X)] + (n-1)M_1||Z_2$$

$$\geq |K_{Z_2} + [(m-n)\pi^*(K_X)]|Z_2| + M_1|Z_2|$$

and the surjective map:

$$H^0(Z_2, K_{Z_2} + [(m-n)\pi^*(K_X)]|Z_2| + M_1|Z_2|) \longrightarrow H^0(C_1, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2),$$

where

$$D_i := [((m-n)\pi^*(K_X)|Z_2| + M_1|Z_2 - C_i)|C_i] = [(m-n)\pi^*(K_X)|Z_2]||C_i$$

for $i = 1, 2$. Note that $H^0(C_i, K_{C_i} + D_i) \neq 0$ since $\text{deg}(D_i) > 0$. Therefore, $|mK_{X'}|$ can distinguish different generic irreducible elements of $|N_2|$ in this case. In summary, for any $m > n$, Conditions (1) and (2) of Corollary 3.5 are satisfied.

We have proved that $\xi \geq \frac{6}{2n + (n \text{mod} \ 3)}$. For any $m \geq \frac{5n + 3 + (n \text{mod} \ 3)}{3}$, we have $\alpha_m > 2$. By Corollary 3.5, whenever $m \geq \frac{5n + 3 + (n \text{mod} \ 3)}{3}$, $\varphi_{mX}$ is birational onto its image. We have proved the following.

**Corollary 5.2.** For any integer $n \geq 2$, $r_{n,n-1} \leq \frac{5n + 3 + (n \text{mod} \ 3)}{3}$.

### 5.3 Examples

**Example 5.3** (cf. [13, Example 6.6(1)]). Let $n = 3k + 2$. The general hypersurface $X = V_{10k+10} \subset \mathbb{P}(1^{(3k+2)}, 2(k + 1), 5(k + 1))$ of degree $10(k + 1)$ has at worst canonical singularities and is a minimal $n$-fold with $\omega_X = \mathcal{O}_X(1)$, canonical dimension $n - 1$, and volume $\text{Vol}(X) = \frac{1}{k+1} = \frac{3}{n+1} = \frac{6}{2n + (n \text{mod} \ 3)}$. Furthermore, it is clear that $r_s(X) = 5k + 5 = \frac{5n + 3 + (n \text{mod} \ 3)}{3}$.
Example 5.4 (cf. [13, Example 6.6(2)]). Let $n = 3k + 1$. The general hypersurface $X = V_{10k+6} \subset \mathbb{P}(1^{3k+1}, 2k + 1, 5k + 3)$ of degree $10k + 6$ has at worst canonical singularities and is a minimal $n$-fold with $\omega_X = \mathcal{O}_X(1)$, canonical dimension $n - 1$, and volume $\text{Vol}(X) = \frac{6}{2n+1} = \frac{6}{2n+(n \mod 3)}$. Furthermore, it is clear that $r_s(X) = 5k + 3 = \frac{5n+3+(n \mod 3)}{3}$.

Example 5.5. Let $n = 3k$. The general hypersurface $X = V_{20k+2} \subset \mathbb{P}(1, 2^{(3k-1)}, 4k, 10k+1)$ of degree $20k+2$ has at worst canonical singularities and is a minimal $n$-fold with $\omega_X = \mathcal{O}_X(2)$, canonical dimension $n - 1$, and volume $\text{Vol}(X) = \frac{3}{n} = \frac{6n}{2n+(n \mod 3)}$. Furthermore, it is clear that $r_s(X) = 5k + 1 = \frac{5n+3+(n \mod 3)}{3}$.

Then, Theorem 1.1 (2), that is,

$$v_{n,n-1} = \frac{6}{2n + (n \mod 3)} \quad \text{and} \quad r_{n,n-1} = \frac{1}{3}(5n + 3 + (n \mod 3)),$$

follows directly from Corollary 5.1, Corollary 5.2, Example 5.3, Example 5.4, and Example 5.5.

6 | GEOMETRY OF OPTIMAL EXAMPLES

In this section, we summarize an alternate approach proving the volume bound in Theorem 1.1 (2). This method also shows that any $X$ realizing the minimal value of $v_{n,n-1}$ must strongly resemble one of the weighted projective hypersurfaces in Section 5.3. We omit some of the proof details and continue to use the notation of Section 4.1.

Claim 6.1. Let $X$ be a minimal variety of general type with dimension $n \geq 3$ and canonical dimension $n - 1$ such that

$$\text{Vol}(X) = \frac{6}{2n + (n \mod 3)}.$$  \hfill (6.1)

Then, the canonical map of $X$ is a dominant rational map to $\mathbb{P}^{n-1}$ with general fiber a genus 2 curve. Furthermore, the general surface $Z_2$ has minimal volume in the sense of Proposition 6.2 below. Its minimal model $Z_{2,0}$ contains a tree of rational curves contracted by the morphism from $X'$ to its canonical model, described in Cases 6.1, 6.2, and 6.3.

Proof. Let $X$ be a minimal variety realizing (6.1) as above. We immediately see that $g(Z_1) = 2$, or else $\xi \geq \frac{4}{n}$ by (3.2). The results of Section 4.3 imply that $\beta_2 \geq p_g(X) - n + 1$ and $\beta_i \geq 1$ for $3 \leq i \leq n$. Therefore, $p_g(X) = n$, or else $\beta_2 \geq 2$, so that $K^n_X \geq 2 \cdot \frac{2}{n} = \frac{4}{n}$ by (3.3). The image $\varphi_{1,X}(X)$ of the canonical map is then a subvariety of $\mathbb{P}^{n-1}$ of dimension $n - 1$, so $\varphi_{1,X}(X) = \mathbb{P}^{n-1}$. Therefore, the canonical map is a dominant rational map to $\mathbb{P}^{n-1}$ whose general fiber is a smooth genus 2 curve.

We can also say a great deal about $Z_2$ and the properties of the birational contraction $\pi : X' \to X_{\text{can}}$, restricted to $Z_2$. Here, $X_{\text{can}} = \text{Proj} \bigoplus_{m=1}^{\infty} H^0(X, mK_X)$ is the canonical model of the variety $X$ (equivalently, of $X'$) and $K^n_{X_{\text{can}}} = K^n_X$.
The key idea is to find a commutative diagram

\[ \begin{array}{ccc}
X' & \xrightarrow{\pi_{\text{can}}} & X_{\text{can}} \\
\uparrow & & \uparrow \\
Z_2 & \xrightarrow{g} & Z_{2,0} \\
\downarrow & & \downarrow \\
& \xrightarrow{f} & S,
\end{array} \]

where \( g : Z_2 \to Z_{2,0} \) is the minimal model, \( S \) is a projective normal surface with \( K_S \) Q-Cartier, and \( f : Z_{2,0} \to S \) is a birational morphism whose exceptional divisors have nonpositive discrepancies.

By inequality \((4.3)\), \((n-1)\pi(Z_2, (K_X^\text{can})) \geq g^*(K_{Z_{2,0}}^\text{can})\). Taking the pushforward to \( S \) gives \((n-1)^2\psi^*(K_X^\text{can}) \geq K_S^2\) by the commutativity of the diagram and because \( K_{Z_{2,0}} \) and \( f^*(K_S) \) differ by exceptional divisors contracted by \( f \). Both \( \psi^*(K_X^\text{can}) \) and \( K_S \) are nef divisors, the latter by the assumption on discrepancies. Therefore,

\[ ((n-1)^2\psi^*(K_X^\text{can}))^2 \geq (n-1)^2\psi^*(K_X^\text{can}) \cdot K_S \geq K_S^2. \tag{6.2} \]

Combining the inequality \((6.2)\) with \( K_n^X \geq (\pi^\text{can}_Z(K_X^\text{can})|_{Z_2})^2 \), we get

\[ (n-1)^2K_n^X \geq ((n-1)^2\pi^\text{can}_Z(K_X^\text{can})|_{Z_2})^2 = ((n-1)^2\psi^*(K_X^\text{can}))^2 \geq K_S^2. \]

Moreover, the intersection number \( K_S^2 \) is determined by the volume of the minimal surface \( Z_{2,0} \) and the divisors are contracted by \( f \). We may obtain sharper bounds on \( K_n^X \) via this method than from only considering smooth models of \( Z_2 \) because often, \( K_S^2 > \text{Vol}(Z_2) \). Therefore, we seek to factor \( \pi^\text{can}_{Z_2} : Z_2 \to X_{\text{can}} \) through contractions to certain singular surfaces \( S \).

The minimal model \( Z_{2,0} \) of \( Z_2 \) is a fibration of genus 2 curves over \( \mathbb{P}^1 \). The general fiber is a genus 2 curve \( Z_1 \). We know by the adjunction formula that \( Z_2 \) satisfies \( K_{Z_2} = (K_{Z_1} + Z_3)|_{Z_2} = \cdots = (K_X + (n-2)Z_{n-1})|_{Z_2} = (n-1)Z_1 \) by the setup of Section 4.3. Supposing \( n \geq 3 \), this gives \( p_g(Z_2) \geq h^0(Z_2, (n-1)Z_1) \geq n \geq 3 \). This is because \( Z_1 \) varies in (at least) a pencil on \( Z_2 \), so \( h^0(Z_2, Z_1) \geq 2 \); polynomials of degree \( n-1 \) in these two linearly independent sections give at least \( n \) independent sections of \( (n-1)Z_1 \). Under the assumptions of this paragraph, we have the following properties (cf. proof of [12, Proposition 2.9]).

**Proposition 6.2.**

\[ \text{Vol}(Z_2) = (K_{Z_{2,0}}^2) \geq \begin{cases} 
\frac{8n-16}{3}, & n \equiv 2 \text{ mod } 3 \\
\frac{8n-14}{3}, & n \equiv 1 \text{ mod } 3 \\
\frac{8n-12}{3}, & n \equiv 0 \text{ mod } 3.
\end{cases} \tag{6.3} \]

The morphism \( Z_2 \to \mathbb{P}^1 \) factors through \( Z_{2,0} \) and \( K_{Z_{2,0}} = (n-1)Z_1 + G \), where \( G \) is an effective divisor.

If we have equality in \((6.3)\), then \( G = V + 2E \), where \( V \) is a vertical divisor, \( E \cong \mathbb{P}^1 \) is a section of the fibration, and \( E^2 = -\frac{2}{3}(n+1+V \cdot E) \). The divisor \( V \) satisfies \( K_{Z_{2,0}} \cdot V = 0 \) so that all irreducible components of \( V \) are \((-2)-\)curves contained in fibers. Finally, \( V \cdot E = 0, 1, 2 \), in the cases \( n = 3k + 2 \), \( n = 3k + 1 \), and \( n = 3k \), respectively.

One can show that if \( K_n^X \) is small enough to compete with the examples in Section 5.3, then \( \text{Vol}(Z_2) \) also must be equal to the minimum in \((6.3)\). Further, \( \pi^\text{can}_{Z_2} : Z_2 \to X_{\text{can}} \) factors
through the minimal model \( g : Z_2 \rightarrow Z_{2,0} \). Using Proposition 6.2, we may identify some additional curves on \( Z_{2,0} \) that are contracted by \( \pi_{\text{can}} \). In each case, we will contract a connected curve of arithmetic genus 0 with negative-definite self-intersection matrix. This results in a projective surface with rational singularities (see [1, Theorem 2.3] and [2, Proposition 1]), which is therefore \( \mathbb{Q} \)-Gorenstein.

Case 6.1. \( n = 3k + 2, k \geq 1 \).

This is the easiest case, because we can define the appropriate \( f : Z_{2,0} \rightarrow S \) by only contracting the curve \( E \) of Proposition 6.2 on \( Z_{2,0} \). Using the adjunction formula and \( E^2 = -\frac{1}{3}(n + 1) \), this gives \( K_{Z_{2,0}} = f^*K_S + aE \) with \( a = \frac{-k+1}{k+1} < 0 \). (We have used \( n \geq 3 \) here because when \( n = 2 \), so \( k = 0 \), the curve \( E \) would have self-intersection \(-1\) and \( Z_{2,0} \) would not be minimal. The fibration of genus 2 curves does not factor through the minimal model of \( X = Z_2 \) in this special case.) Assuming that \( K_{Z_{2,0}}^2 \) is its minimum of \( \frac{8n-16}{3} \), we may compute

\[
(n-1)^2K^n_X \geq K^2_S = \frac{8n-16}{3} - (aE)^2 = \frac{3(n-1)^2}{n+1}.
\]

Therefore, \( K^n_X \geq \frac{3}{n+1} \) and we recover the same volume bound.

We can see this curve \( E \) explicitly in Example 5.3: the general weighted projective hypersurface

\[
X_{10(k+1)} \subset \mathbb{P}^{n+1}(1(1), 2(k + 1), 5(k + 1))
\]

has canonical dimension \( n - 1 \), with the canonical map a rational map \( X \rightarrow \mathbb{P}^{n-1} \) having generic fiber a curve of genus 2.

Now, consider the intersection of \( n - 2 \) general sections of the reflexive sheaf \( K_X \). This is a general surface \( S = S_{10(k+1)} \subset \mathbb{P}^3(1, 1, 2(k + 1), 5(k + 1)) \) with one singular point \( p \), which is a quotient singularity of type \( \frac{1}{k+1}(1, 1) \). This is a cone over the rational normal curve of degree \( k + 1 \), so we can explicitly resolve it with a single blowup at \( p \). Denote this blowup as \( f : Z_{2,0} \rightarrow S \). This has the same numerical properties as in the general case above. From the description as a weighted projective hypersurface, we know \( K_S^2 = \frac{(3k+1)^2}{k+1} \). Therefore,

\[
K^2_{Z_{2,0}} = \frac{8n-16}{3},
\]

so that the smooth model has volume matching the minimum in Proposition 6.2. As the notation suggests, this is the same \( Z_{2,0} \) obtained by desingularizing \( X \) with a blowup and applying the machinery of Section 4.1.

Case 6.2. \( n = 3k + 1, k \geq 1 \). If \( K^2_{Z_{2,0}} \) achieves the minimum of \( \frac{8n-14}{3} \), then by Proposition 6.2, \( V \cdot E = 1 \). In general, \( V \) could have several \((-2)\)-curves as irreducible components, but a unique one, say \( V_1 \), intersects \( E \) in a single point. There is a birational morphism \( f : Z_{2,0} \rightarrow S \) contracting exactly the connected curve \( V_1 \cup E \).

The morphism \( \pi_{\text{can}} : Z_2 \rightarrow X_{\text{can}} \) contracts both these curves on the minimal model, so it factors through \( f : Z_{2,0} \rightarrow S \) and \( K_{Z_{2,0}} = f^*K_S + aV_1 + bE \). The adjunction formula and the values \( E^2 = -\frac{1}{3}(n + 2) \) and \( V^2_1 = -2 \) yield \( a = \frac{1-k}{2k+1} \) and \( b = \frac{2-2k}{2k+1} \); these are both nonpositive. One can
calculate
\[(n - 1)^2 K^n_X \geq K^2_S = \frac{8n - 14}{3} - (aV_1 + bE)^2 = \frac{6(n - 1)^2}{2n + 1},\]
so that \(K^n_X \geq \frac{6}{2n+1}\), in agreement with Corollary 5.1.

Once again, these exceptional curves appear in Example 5.4. The general weighted projective hypersurface
\[X_{10k+6} \subset \mathbb{P}^{n+1}(1^{(n)}, 2k + 1, 5k + 3)\]
has canonical dimension \(n - 1\) with the canonical map a rational map \(X \rightarrow \mathbb{P}^{n-1}\) having generic fiber a curve of genus 2. As above, we consider the intersection of \(n - 2\) general sections of \(K_X\) to obtain \(S = S_{10k+6} \subset \mathbb{P}^3(1, 1, 2k + 1, 5k + 3)\). Then \(K^2_S = \frac{6(n-1)^2}{2n+1}\) and \(S\) has only one singular point, at \([0 : 0 : 1 : 0]\), which is a quotient singularity of type \(\frac{1}{2k+1}(1, 5k + 3)\). The resolution of any surface quotient singularity of the form \(\frac{P}{Q}(1, Q)\) is related to the Hirzebruch–Jung continued fraction of \(\frac{P}{Q}\) (see, e.g., [17, Section 2]). In this case,
\[\frac{2k + 1}{k + 1} = 2 - \frac{1}{k + 1},\]
so \(\frac{P}{Q} = [2, k + 1]\) is the continued fraction representation. This means that the exceptional locus has two rational curves as irreducible components, with self-intersections \(-2\) and \(-(k + 1) = -\frac{1}{2}(n + 2)\), corresponding to \(V_1\) and \(E\) above, respectively. It is straightforward to verify (e.g., using [17, Section 2]) that the discrepancies also agree with the above, so that
\[K^2_{Z_{2,0}} = \frac{8n - 14}{3},\]
as expected. For \(n = 4\), the singularity \(\frac{1}{3}(1, 2)\) is canonical (in particular, an \(A_2\) singularity) and the resolution is crepant. For \(n \geq 7\), \(S\) has worse than canonical singularities, so \(K^2_{Z_{2,0}} < K^2_S\).

**Case 6.3.** \(n = 3k, k \geq 1\). If \(K^2_{Z_{2,0}}\) achieves the minimum of \(\frac{8n-12}{3}\), then by Proposition 6.2, \(V \cdot E = 2\). Unlike in the previous two cases, these intersection numbers do not pin down a unique component graph for the connected component of \(V_{\text{red}} \cup E\) containing \(E\). The simplest configuration is that there are two irreducible components \(V_1\) and \(V_2\) of \(V\) intersecting \(E\), each in a single point, and no other components of \(V\) intersecting these. Fortunately, one can verify that the other possible configurations yield the same volume bound; suppose for clarity that we have the simple configuration. Once again, there is a birational morphism \(f : Z_{2,0} \rightarrow S\) contracting exactly \(E \cup V_1 \cup V_2\) and \(\pi_{\text{can}}\) factors through this morphism. Using \(E^2 = -\frac{1}{3}(n + 3)\) and \(V_1^2 = V_2^2 = -2\), we may calculate as above that \(K_{Z_{2,0}} = f^*K_S + aV_1 + bE + cV_2\) with \(a = c = \frac{1-k}{2k}\) and \(b = \frac{1-k}{k}\) (both nonpositive). Then
\[(n - 1)^2 K^n_X \geq K^2_S = \frac{8n - 12}{3} - (aV_1 + bE + cV_2)^2 = \frac{3(n - 1)^2}{n}.
\]
This gives \(K^n_X \geq \frac{3}{n}\), as desired.
Example 5.5 again displays the same behavior. For any positive integer \( n = 3k, k \geq 1 \), the general weighted projective hypersurface

\[
X_{20k+2} \subset \mathbb{P}^{n+1}(1, 2(\!n\!-\!1), 4k, 10k+1)
\]

has canonical dimension \( n-1 \). It has \( K_X = \mathcal{O}_X(2) \) and volume \( \frac{1}{k} = \frac{3}{n} \). The rational map \( |K_X| = |\mathcal{O}_X(2)| \) has image \( \mathbb{P}^{2(n-1)} \cong \mathbb{P}^{n-1} \). The intersection of \( n-2 \) general sections of \( K_X \) can be identified with a general \( S_{20k+2} \subset \mathbb{P}^{3}(1, 2, 4k, 10k+1) \). To see this, let \( x_1 \) have degree 1 and \( x_2, \ldots, x_n \) have degree 2, so that a section of \( \mathcal{O}_X(2) \) is a polynomial of the form \( a_1x_1^2 + a_2x_2 + \cdots + a_nx_n \). After a suitable change of coordinates within \( x_2, \ldots, x_n \), we may assume that the common zero set of \( n-2 \) general sections is given by equations \( x_2^2 = x_3, x_2^2 = x_4, \ldots, x_2^2 = x_n \), which yields the result.

This surface \( S \) satisfies \( K_S^2 = \frac{3(n-1)^2}{n} \) and is smooth away from the weighted \( \mathbb{P}^1 \) with weights \( 2 \) and \( 4k \). The general \( S \) only intersects this stratum at the coordinate point of \( 4k \). At this base point, there is a singularity of type \( \frac{1}{4k}(1, 2k+1) \). The corresponding Hirzebruch–Jung continued fraction is

\[
\frac{4k}{2k+1} = 2 - \frac{1}{(k+1) - \frac{1}{2}}
\]

also represented as \( \frac{4k}{2k+1} = [2, k + 1, 2] \). Therefore, the minimal resolution of this singularity is a chain of three exceptional rational curves with self-intersections \(-2, -(k + 1) = -\frac{1}{3}(n + 3), \) and \(-2 \). These play the role of \( V_1, E, \) and \( V_2 \) above, respectively. Let \( f : Z_{2,0} \rightarrow S \) be the resolution of singularities. Then

\[
K_{Z_{2,0}}^2 = \frac{8n - 12}{3}.
\]

This \( Z_{2,0} \) is a minimal surface and admits a genus 2 fibration to \( \mathbb{P}^1 \) as above. When \( n = 3, S \) is canonical, with a singularity of type \( A_3 \). For \( n \geq 6, S \) has worse-than-canonical singularities and \( K_S^2 > K_{Z_{2,0}}^2 \). Table 1 summarizes the exceptional curves in each of the three cases.

The results of this section notwithstanding, not all minimal varieties satisfying (6.1) are necessarily quasi-smooth weighted projective hypersurfaces. This is illustrated by the following example.

### Table 1

| Example | Exceptional curves of \( Z_{2,0} \rightarrow S \) for optimal examples in the three cases. |
|---------|------------------------------------------------------------------|
| \( n = 3k + 2 \) | \( V_1 \) | \( V_1 \) | \( V_2 \) |
| \( n = 3k + 1 \) | \( V_1 \) | \( V_1 \) | \( V_2 \) |
| \( n = 3k \) | \( V_1 \) | \( V_1 \) | \( V_2 \) |
**Example 6.3.** Let $n = 3k$ for $k \geq 1$. The general hypersurface

$$X_{10k+1} \subset \mathbb{P}(1^{(3k)}, 2k, 5k)$$

is quasi-smooth and has only one noncanonical singularity of type $\frac{1}{5k}(1^{(3k-1)}, 2k)$, which satisfies the “nefness criterion” of Chen–Jiang–Li [13, Theorem 1.3]. So, after performing a weighted blowup at this point, by [13, Proposition 2.9], one gets a minimal variety $Y$ of dimension $n$ with canonical volume $\frac{1}{k} = \frac{3}{n} = \frac{6}{2n+(n \mod 3)}$. This example does not have Pic($Y$) $\cong \mathbb{Z}$, so it is not a quasi-smooth weighted projective hypersurface (see [15, Theorem 3.2.4]).

We study the canonical stability index of this example. Denote by $x_i$ ($i = 1, \ldots, 3k$) the coordinates of weight 1, by $w$ the coordinate with weight 2$k$, and by $t$ the coordinate with weight 5$k$ in $\mathbb{P}(1^{(3k)}, 2k, 5k)$. The defining equation can be written:

$$f_0(x_1, \ldots, x_{3k}, w) + f_1(x_1, \ldots, x_{3k}, w)t + (a_1x_1 + \cdots + a_{3k}x_{3k})t^2 = 0.$$  

Denote by $\vartheta : Y \to X := X_{10k+1}$ the weighted blowup with weights $\frac{1}{5k}(1^{(3k-1)}, 2k)$. Then we have $5kK_Y = \vartheta^*(5kK_X) - E$ with $E$ the exceptional divisor and $\vartheta(E) = Q \notin (t = 0)$. This means that there are the following injective maps:

$$H^0(Y, 5kK_Y) \hookrightarrow \Lambda \hookrightarrow H^0(X, 5kK_X)$$

where $\Lambda$ is the vector space spanned by products of $x_1, \ldots, x_{3k}, w$ of weighted degree 5$k$. As $\Lambda$ does not give a birational map of $X$ (since $t$ is missing), $\varphi_{5k,Y}$ is nonbirational. Hence, $r_s(Y) = 5k + 1 = \frac{5n+3}{3} = r_{n,n-1}$.

We can also compute the geometric genus and volume of the minimal surface $Z_{2,0}$ in this example. By Claim 6.1, since $X$ achieves the minimal volume, $K_{Z_{2,0}}^2 = \frac{8(3k)-12}{3} = 8k - 4$. To find $p_g(Z_{2,0})$, note that $Z_{2,0}$ admits a contraction onto a quasi-smooth hypersurface $S$ of degree $10k + 1$ in $\mathbb{P}^3(1, 1, 2k, 5k)$. If $K_S$ is the canonical sheaf of $S$, then $p_g(Z_{2,0}) = p_g(S)$ since $S$ is a klt birational contraction of $Z_{2,0}$. One readily computes $h^0(K_S) = h^0(\mathcal{O}_S(3k - 1)) = 4k$.

**Remark 6.4.** In the explicit examples in Section 5.3 and Example 6.3, the smooth models $Z_{2,0}$ of surfaces that appear are Horikawa surfaces, that is, they are on the Noether line $K_{Z_{2,0}}^2 = 2p_g(Z_{2,0}) - 4$. For such varieties, $|K_{Z_{2,0}}|$ gives a ramified double cover to a ruled surface and the types of ruled surfaces and branch loci that appear are completely classified (see [19]). This suggests the following question.

**Question 6.5.** If $X$ is a minimal variety of general type of dimension $n$, canonical dimension $n - 1$, and smallest possible volume, is the minimal model of the associated general surface $Z_2$ a Horikawa surface?

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