Numerical solution of fourth order boundary value problem using sixth degree spline functions

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Abstract. In this communication, we developed sixth degree spline functions by using Bickley’s method for obtaining the numerical solution of linear fourth order differential equations of the form $y^{(4)}(x) + f(x)y(x) = r(x)$ with the given boundary conditions where $f(x)$ and $r(x)$ are given functions. Numerical illustrations are tabulated to demonstrate the practical usefulness of the method.

1. Introduction
Spline functions play an important role in finding numerical solution of boundary value problems. Splines are types of curves, originally developed for shipbuilding in the days before computer modeling. Bickley[1] has suggested a different approach of solving linear two-point boundary value problem. To model the solution curve he used cubic spline interpolation and applied the differential equation as well as the boundary conditions for obtaining unknown constants. Khan [2] has considered the applications of cubic spline functions for the solution of two point boundary value problems. Some of the books which discuss splines include Ahlberget al.[3], deBoor [4], Prenter[5], Schumaker [6], Shikin and Plis [7] and Spath [8]. Chawla and Katti [9] employed finite difference method for a class of singular two point BVPs. A class of BVPs has been solved by Rama Chandra Rao[10] using numerical integration. Caglar et al. [11] have solved third order linear and nonlinear BVPs using fourth degree B-spline. Usmani and Sakai [12] solved the special case of linear order BVPs using the quartic spline.Usmani [13] considered the problem of bending a rectangular clamped beam resting on an elastic foundation. He derived numerical techniques of order 2, 4 and 6 for the solution of a fourth order linear boundary value problem. Papamichael and Worsey [14] have developed a cubic spline method for the solution of a linear fourth order two point boundary value problem. Siddiqi and Akram [15] described a quintic spline method for the solution of fourth order boundary value problems and derived end conditions for quintic spline interpolation, at equally spaced knots. We used Bickley’s method for obtaining the numerical solution of linear fourth order boundary value problems by constructing sixth degree spline function. Numerical illustrations are summarized in tabular form and shown graphically.
2. Cubic spline-Bickley’s method

Suppose the interval \([x_0, x_n]\) is divided into \(n\) subintervals with knots \(x_0, x_1, x_2, \ldots, x_n\) starting at \(x_0\), the function \(u(x)\) in the interval \([x_0, x_1]\) is represented by a cubic spline in the form

\[
s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3.
\]  

(1)

For the next interval \([x_1, x_2]\) the spline function \(s(x)\) is supposed in the form

\[
s(x) = s(x) \text{ on } [x_0, x_1] + d_1(x - x_1)^3,
\]

proceeding in to the next interval \([x_2, x_3]\) we add another term \(d_2(x - x_2)^3\) and so until we reach \(x_n\). Thus the function \(s(x)\) is represented in the form

\[
s(x) = a + b(x - x_0) + c(x - x_0)^2 + \sum_{i=0}^{n-1} d_i(x - x_i)^3,
\]  

(2)

\[
s'(x) = b + 2c(x - x_0) + 3 \sum_{i=0}^{n-1} d_i(x - x_i)^2,
\]  

(3)

\[
s''(x) = 2c + 6 \sum_{i=0}^{n-1} d_i(x - x_i).
\]  

(4)

2.1. The two-point second order boundary value problem

First, we consider the linear differential equation

\[
p(x)u'' + q(x)u' + r(x)u = v(x),
\]  

(5)

with the boundary conditions

\[
\alpha_0 u + \beta_0 u' = \gamma_0 \text{ at } x = x_0, \alpha_n u + \beta_n u' = \gamma_n \text{ at } x = x_n.
\]  

(6)

The number of coefficients in (2) is \((n+3)\), the satisfaction of the differential equation by the spline function at \((n+1)\) nodes gives \((n+1)\) equations in the \((n+3)\) unknowns. Also the boundary conditions (6) give two more equations in the unknowns. Thus we get \((n+3)\) equations in \((n+3)\) unknowns \(a, b, c, d_0, d_1, \ldots, d_{(n-1)}\), after determining these unknowns we substitute them in (2) and thus we get the cubic spline approximation of \(u(x)\). Putting \(x = x_0, x_1, x_2, \ldots, x_n\) in the spline function thus determined, we get the solution at the nodes. The system of equations to be satisfied by the constants \(a, b, c, d_0, d_1, \ldots, d_{(n-1)}\) is derived below. Substituting (2), (3), (4) in (5), at \(x = x_m\) we get

\[
ar_m + b[r_m(x_m - x_0) + q_m] + c[r_m(x_m - x_0)^2 + 2q_m(x_m - x_0) + 2p_m] + \sum_{i=0}^{m-1} d_i[r_m(x_m - x_i)^3 + 3q_m(x_m - x_i)^2 + 6p_m(x_m - x_i)] = s_m, m = 0, 1, 2, \ldots, n
\]  

(7)

where \(p_m = p(x_m), q_m = q(x_m), r_m = r(x_m)\) and \(s_m = s(x_m)\). Applying boundary conditions (6), we get

\[
\alpha_0 a + \beta_0 b = \gamma_0; \alpha_n a + [\alpha_n(x_n - x_0) - \beta_n]b + [\alpha_n(x_n - x_0)^2 - 2\beta_n(x_n - x_0)]c + \sum_{m=0}^{n-1} (\alpha_n(x_n - x_m)^3 - 3\beta_n(x_n - x_m)^2)d_m = \gamma_n.
\]  

(8)

If these equations are taken in the order (8), (7) with \(m = n, n - 1, \ldots, 0\) the coefficient matrix of unknowns \(d_{(n-1)}, d_{(n-2)}, d_1, d_0, c, b, a\) is of the Hessenberg form, namely an upper triangle with a single lower sub-diagonal. The forward elimination is then simple with only one multiplier at each step and the back substitution is correspondingly easy.
3. Construction of sixth degree spline
Suppose the interval [x_0, x_n] is divided into subintervals with grid points x_0, x_1, x_2, ..., x_n. Starting at x_0, the function y(x) in the interval [x_0, x_1] is represented by a sixth degree spline.

\[ s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + h_0(x - x_0)^6. \]

Proceeding in to the next interval [x_1, x_2], we add a term h_1(x - x_1)^6, proceeding in to the next interval [x_2, x_3] we add another term h_2(x - x_2)^6 and so until we reach x_n. Thus the function y(x) is represented in the form

\[ s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + \sum_{i=0}^{n-1} h_i(x - x_i)^6. \]

(9)

It can be shown that s(x) and its first five derivatives are continuous across nodes.

3.1. Method of obtaining the solution of fourth order boundary value problem by sixth degree spline
Consider the linear fourth order differential equation

\[ y^{(4)}(x) + f(x)y(x) = r(x), \]

(10)

with the boundary conditions

\[ y(x_0) = \alpha, y(x_n) = \beta, y'(x_0) = \alpha', y'(x_n) = \beta'. \]

(11)

From (11, and taking spline approximation in (10) at x = x_i for i = 0, 1, 2, ..., n, we get (n + 5) equations in (n + 6) unknowns a, b, c, d, e, g, h_0, h_1, h_2, ..., h_{n-1}. To have the solution for the unknowns, one more equation is required. So we assume that h_{n-1} = h_{n-2}. After determining these unknowns we substitute them in (9) and thus we get the sixth degree spline approximation of y(x). Putting x = x_1, x_2, x_3, ..., x_n in the spline function thus determined, we get the solution at the nodes.

4. Numerical illustrations
In this section we consider a linear boundary value problem. The numerical solution and absolute errors are obtained by sixth degree spline function when h = 0.2. The approximate solution, exact solutions and absolute errors at the grid points are given in the tabular form. Further the approximate solution and exact solution are shown graphically.

Example 1
Consider the following fourth order linear boundary value problem

\[ y^{(4)}(x) + xy(x) = -(8 + 7x + x^3)e(x), 0 \leq x \leq 1, \]

(12)

With the boundary conditions

\[ y(0) = y(1) = 0, y'(0) = 1, y'(1) = -e. \]

(13)

Solution of BVP (12)-(13) when h = 0.2:
The interval [0,1] is divided in to 5 equal subintervals , we denote the knots by x_0, x_1, x_2, x_3, x_4
and \( x_5 \) where \( x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1 \) The sixth order spline \( s(x) \) which approximate \( y(x) \) is given by

\[
s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + \sum_{i=0}^{4} h_i(x - x_i)^6,
\]

There are 11 unknowns in \( s(x) \) which are to be determined from the following conditions by assuming \( h_3 = h_4 \)

\[
s(x_0) = 0, s(x_5) = 0, s'(x_0) = 1, s'(x_5) = -e, \\
s^{(4)}(x_i) + x_is(x_i) = -(8 + 7x_i + x_i^3)e^{(x_i)}, \text{ for } i = 0, 1, 2, 3, 4, 5.
\]

From the conditions \( s(x_0) = 0, s'(x_0) = 1 \), we have \( a = 0, b=1 \), hence the spline \( s(x) \) reduces to the form

\[
s(x) = (x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + e(x - x_0)^4 + g(x - x_0)^5 + \sum_{i=0}^{4} h_i(x - x_i)^6, \quad (15)
\]

From (16)

\[
s'(x) = 1 + 2c(x - x_0) + 3d(x - x_0)^2 + 4e(x - x_0)^3 + 5g(x - x_0)^4 + 6 \sum_{i=0}^{4} h_i(x - x_i)^5, \quad (16)
\]

and

\[
s^{(4)}(x) = 24e + 120g(x - x_0) + 360 \sum_{i=0}^{4} h_i(x - x_i)^2. \quad (17)
\]

Substituting (16), (17) in (15) we get a system of equations and solving those equations we get

\[
c = -0.000025, \quad d = -0.499905, \quad e = -0.033333, \\
g = -0.1250311, \quad h_0 = -0.03676, \quad h_1 = -0.016080, \\
h_2 = -0.013420, \quad h_3 = -0.025155, \quad h_4 = -0.025155.
\]

Substituting these values in (16) we get the spline approximation \( s(x) \) of \( y(x) \). The values of \( s(x), y(x) \) and the corresponding absolute errors at \( x_1, x_2, x_3, x_4, x_5 \) have been given in the Table-1 and the comparison has been shown in figure 1.

**Table 1.** Approximate solution \( s(x) \), exact solution \( y(x) \) and absolute errors

| x    | \( s(x) \)   | \( y(x) \)   | Absolute error |
|------|--------------|--------------|----------------|
| 0.2  | 0.19542406   | 0.19542440   | 3.7715E-07     |
| 0.4  | 0.35803683   | 0.35803793   | 1.0954E-06     |
| 0.6  | 0.43730732   | 0.43730851   | 1.1955E-06     |
| 0.8  | 0.35608594   | 0.35608655   | 6.0648E-07     |
5. Conclusion
The numerical method is developed to obtain the solution of fourth order boundary value problems using sixth degree spline. Approximate solution, exact solution and absolute errors with $h = 0.2$ are summarized in the table. The comparison has been shown graphically. It is observed that the approximate solutions are in good agreement with the exact solution. It is also observed that the approximate solution is more close to the exact solution when $h$ is small. Further it is also noted that the numerical solutions obtained are remarkably accurate and have negligible absolute errors even for the values of $h$ as large as 0.2.

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