Ground-state two-spinon bonds in the Hubbard model on a square lattice

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Abstract. In this paper the spin configurations of the ground state and one- and two-electron excited states of the Hubbard model on the square lattice are studied. We profit from a general rotated-electron description, which is consistent with the model global $SO(3) \times SO(3) \times U(1)$ symmetry. For rotated electrons, doubly and single occupancy are good quantum numbers for on-site repulsion $U > 0$. The above states are within that description generated by occupancy configurations of charge $c$ fermions and spin-singlet two-spinon $s_1$ bond particles. Those describe the charge and spin degrees of freedom, respectively, of the rotated electrons that singly occupy sites. While the $c$ fermions have no internal structure, that of the spin-neutral $s_1$ bond-particle occupancy configurations is here described in terms of spinon occupancies of a well-defined effective spin lattice. In reference [2] it is confirmed that our results contribute to the further understanding of the role of electronic correlations in the spin spectrum of the parent compound La$_2$CuO$_4$. They are also of interest for studies of ultra-cold fermionic atoms on an optical lattice.

PACS numbers: 71.10.Fd, 75.10.Jm, 75.10.Lp, 71.10.Pm
1. Introduction

The Hubbard model on a square lattice is the simplest toy model for the description of the effects of electronic correlations in the spin spectrum of the Mott-Hubbard insulators cuprates parent compounds [1, 2]. On the square and cubic lattices the model has no exact solution and many open questions about its properties remain unsolved. Fortunately, it can be experimentally realized with unprecedented precision in systems of correlated ultra-cold fermionic atoms on an optical lattice [3]. Recent experiments considered the Hubbard model on the cubic lattice [4]. They involved ultra-cold fermionic atoms on an optical cubic lattice formed by interfering laser fields. Similar experiments referring to the model on the square lattice are in progress. One may expect very detailed experimental results over a wide range of parameters to be available.

In this paper we study the spin occupancy configurations of the ground state of the Hubbard model on a square lattice with \( N_a^2 \gg 1 \) sites within the spin-singlet two-spinon \( s_1 \) bond-particle description also used in the investigations of Ref. [2]. Here \( N_a \) is the number of sites of an edge of length \( L = N_a a \) and spacing \( a \). Hence the total number of sites of the square lattice of area \( L^2 \) is \( N_a^2 \equiv N_a \times N_a \). That description involves the generalization to all finite values of the on-site repulsion \( U > 0 \) of the exact transformation for separation of spin-1/2 fermions without constraints introduced for large \( U \) values in Ref. [5]. It is consistent with the global \( SO(3) \times SO(3) \times U(1) = [SO(4) \times U(1)]/\mathbb{Z}_2 \) symmetry found recently in Ref. [6] for the model on any bipartite lattice. That global symmetry is an extension of the \( SO(4) \) symmetry known to occur for the model on such lattices [7]. The extended global symmetry is related to the rotated electrons that for \( U > 0 \) emerge from the electrons through a unitary transformation of the type considered in Ref. [8] and to the local symmetries and unitary transformations considered in Ref. [9]. For rotated electrons double and single occupancy are good quantum numbers for \( U > 0 \).

Here we study the spin configurations of the model ground states and excited states that span the subspace relevant for the one- and two-electron physics. It is spanned by an initial vanishing-spin-density ground state and the energy and momentum eigenstates contained in the excitations generated by application onto that ground state of one- and two-electron operators. Within the description used in the studies of Ref. [2], for \( N_a^2 \gg 1 \) such states can be generated by suitable occupancy configurations of two basic quantum objects: spin-less and \( \eta \)-spin-less charge \( c \) fermions and spin-1/2 spinons. There are also \( \eta \)-spin-1/2 \( \eta \)-spinons, yet in that subspace they refer to a single occupancy configuration and thus play no active role. The spinons describe the spin degrees of freedom of the rotated-electron singly occupied sites. Their spin projection is the same as that of the corresponding rotated electron. The \( \eta \)-spinons of \( \eta \)-spin projection down and up describe the \( \eta \)-spin degrees of freedom of the rotated-electron doubly occupied and unoccupied sites, respectively. In our subspace rotated-electron double occupancy vanishes and there are only \( \eta \)-spin-up \( \eta \)-spinons. The spinon spin-singlet configurations involve in
general the $2\nu$-spinon and $2\nu$-site $s\nu$ bond particles \cite{2}. Here $\nu = 1, 2, 3, \ldots$ refers to the number of spinon pairs in each $s\nu$ bond. Fortunately, the ground states and their one- and two-electron excited states considered here, except the spin-singlet states, involve only two-spinon and two-site $s1$ bond particles. Moreover, the latter states involve in addition to the $s1$ bond particles a single four-spinon and four-site $s2$ bond particle.

The $c$ fermions live on a $c$ effective lattice identical to the original lattice. The $c$ fermion occupied sites describe the charge degrees of freedom to those singly occupied by rotated electrons. The corresponding spin degrees of freedom are described by the spinons. In turn, the $c$ fermion unoccupied sites correspond to those doubly occupied or unoccupied by rotated electrons. The corresponding $\eta$-spin degrees of freedom are described by the $\eta$-spinons. Importantly, the spatial coordinates of the $c$ fermion occupied and unoccupied sites are also the spatial coordinates in the original lattice of the sites referring to spinons and $\eta$-spinons, respectively. Provided that $N_a^2 \gg 1$ this is behind the occupancies of the spin effective lattice and $\eta$-spin effective lattice being independent \cite{2}. Such effective lattices involve only the rotated-electron singly occupied sites and the rotated-electron doubly occupied and unoccupied sites, respectively.

The spinons that are not part of $s\nu$ bond particles are called independent spinons. They are invariant under the electron - rotated-electron unitary transformation. For the lowest-weight states whose spin $S_s$ and spin projection $S_s^z$ are such that $S_s = -S_s^z$ the independent spinons have up spin projection. For the subspace considered here all $\eta$-spinons have up $\eta$-spin projection so that the $\eta$-spinon effective lattice plays no active role. In turn, since the location in the original lattice of the sites of the spin effective lattice is recorded in the $c$ fermion occupancy configurations, it turns out that provided that $N_a^2 \gg 1$ it is a good approximation to consider that the spin effective lattice is a square lattice with the same length edge $L$ as the original lattice and a number of sites $N_s^D$ and spacing $a_s$ given by,

$$N_s^D = (1 - x) \frac{N_a^D}{(1 - x)^{1/D}}; \quad a_s = \frac{a}{(1 - x)^{1/D}}, \quad (1 - x) \geq 1/N_a^D. \quad (1)$$

Here $D = 2$ refers to the present square lattice, $D = 1$ corresponds to the one-dimensional (1D) lattice some times considered below, and $x = (N_a^D - N)/N_a^D$ is the hole concentration. For $N_a^2 \gg 1$, in spite of rotated-electron single occupancy being only conserved globally and not locally, the spinon occupancies of the spin effective lattice provide a good approximation of the ground state and excited states spin configurations \cite{2}. Furthermore, within the present description such configurations involve the occupancies of a $s1$ effective lattice. Its unoccupied sites are the up-spin independent spinons. Its occupied sites refer to the two-spinon $s1$ bond particles studied in this paper. Hence each occupied site of the $s1$ effective lattice corresponds to two sites of both the original lattice and spin effective lattice.

Bond states based on two-spinon and two-site bonds are in general not orthogonal and their basis is overcomplete. However, within the representation used in the studies of this paper such a problem does not occur. This is due to physical restrictions of the occupancy configurations of the well-defined set of two-site bonds contributing to
a given s1 bond particle. For instance, that they are centered at the same real-space coordinate of the spin effective lattice. For 1D the discrete momentum values of both the c fermions and s1 fermions generated from the s1 bond particles studied here through an extended Jordan-Wigner transformation [2, 10, 11, 12] are good quantum numbers whose occupancy configurations generate the energy eigenstates [13].

Evidence that for the one- and two-electron subspace the states generated by the occupancy configurations of such discrete momentum values are for the model on the square lattice energy eigenstates is given in the related investigations carried out in Ref. [2]. The square-lattice quantum liquid introduced in that reference contains the one- and two-electron excitations of the Hubbard model on a square lattice. At hole concentration \(x = (N^2_a - N)/N^2_a = 0\), \(U/4t \approx 1.525\), and \(t \approx 295\) meV it is found in that reference to quantitatively describing the spin-wave spectrum observed in the parent compound La_{2}CuO_{4} [1]. In turn, it is expected that the description of the role of electronic correlations in the unusual properties of the cuprate high-temperature superconductors [14, 15, 16, 17] requires as well accounting for the effects of three-dimensional uniaxial anisotropy and intrinsic disorder.

The spin-singlet two-spinon s1 bond particle configurations studied in this paper are an example of the general resonating-valence-bond pictures for spin-singlet occupancy configurations of ground states studied in Refs. [18, 19]. As mentioned above, the particular type of spin configurations based on two-spinon and two-site bonds considered in our studies lack both the non-orthogonality and overcompleteness problems. Such spin-singlet two-spinon s1 bond particle configurations refer to the c and s1 fermion description also used in the investigations of Ref. [2] on the Hubbard model on the square lattice. Some progress in understanding the physics of that model has been achieved for different limits, through a variety of methods. It corresponds to a non-perturbative quantum problem in terms of electron operators, so that rewriting the theory in terms of the standard formalism of many-electron physics is an extremely complex problem. A detailed and extensive discussion of the relation between the new results obtained both in Ref. [2] and this paper by means of the c and s1 fermion description and previously known results is presented in that reference.

For instance, it is shown that the predictions of the square-lattice quantum liquid theory concerning the spin spectrum at half filling as described in terms of the spin-singlet two-spinon configurations studied in this paper agree both with experiments on the parent compound La_{2}CuO_{4} and results obtained by the standard formalism of many-body physics. Furthermore, in Ref. [2] the relation of the description used in the studies of this paper and that reference to several other schemes is discussed.

The paper is organized as follows. The model, the basic rotated-electron representation, and the description in terms of the basic quantum objects, which emerge from the rotated-electron occupancy configurations, are the subjects of Section 2. The s1 effective lattice, corresponding change of gauge structure, and s1 operators of the \(N^b_{s1} = 0\) configuration state are the topics addressed in Section 3. (Here \(N^b_{s1}\) denotes the number of unoccupied sites of the s1 effective lattice.) Section 4 contains a study
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of the $N_{s_1}^d = 1, 2$ configuration states and kink-like and anti-kink-like link occupancies associated with the s1 effective lattice unoccupied sites. Finally, Section 5 contains the concluding remarks.

2. The Hubbard model on the square lattice, rotated electrons, and the description used in our studies

2.1. The Hubbard model on the square lattice and rotated electrons

The Hubbard model on the two-dimensional (2D) square lattice with torus periodic boundary conditions and the same model on the 1D lattice with periodic boundary conditions, spacing $a$, $N_a^D = [N_a]_D$ sites where $D = 1$ and $D = 2$ for the 1D and square lattices, respectively, $N_a^D \gg 1$ even, and lattice edge length $L = N_a a$ for 2D and chain length $L = N_a a$ for 1D is given by,

$$\hat{H} = -t \sum_{\langle \vec{r}_j, \vec{r}_j' \rangle, \sigma = \uparrow, \downarrow} [c_{\vec{r}_j, \sigma}^\dagger c_{\vec{r}_j', \sigma} + h.c.] + U [N_a^D - \hat{Q}]/2.$$  \hspace{1cm} (2)

Here the operator,

$$\hat{Q} = \sum_{j=1}^{N_a^D} \sum_{\sigma, \sigma'} n_{\vec{r}_j, \sigma} (1 - n_{\vec{r}_j, -\sigma}),$$  \hspace{1cm} (3)

where $n_{\vec{r}_j, \sigma} = c_{\vec{r}_j, \sigma}^\dagger c_{\vec{r}_j, \sigma}$ and $-\sigma = \uparrow$ (and $-\sigma = \downarrow$) for $\sigma = \downarrow$ (and $\sigma = \uparrow$) counts the number of electron singly occupied sites. Hence the operator $\hat{D} = [\hat{N} - \hat{Q}]/2$ counts the number of electron doubly occupied sites where $\hat{N} = \sum_{\sigma} \hat{N}_{\sigma}$ and $\hat{N}_{\sigma} = \sum_{j=1}^{N_a^D} n_{\vec{r}_j, \sigma}$.

The studies of Ref. [2] use a uniquely-defined electron - rotated-electron unitary $\hat{V} = \hat{V}(U/4t)$. It is such that states $|\Psi_{U/4t}\rangle = \hat{V}^\dagger |\Psi_{\infty}\rangle$ are for $U/4t > 0$ energy eigenstates. It corresponds to a suitable chosen set $\{|\Psi_{\infty}\rangle\}$ of $U/4t \rightarrow \infty$ energy eigenstates. The unitary transformation maps the electronic operators $c_{\vec{r}_j, \sigma}^\dagger$ and $c_{\vec{r}_j, \sigma}$ onto rotated-electron creation and annihilation operators $\tilde{c}_{\vec{r}_j, \sigma}^\dagger = \hat{V}^\dagger c_{\vec{r}_j, \sigma}^\dagger \hat{V}$ and $\tilde{c}_{\vec{r}_j, \sigma} = \hat{V}^\dagger c_{\vec{r}_j, \sigma} \hat{V}$, respectively.

The studies of Ref. [6] reveal that for $U/4t > 0$ the Hubbard model on a square lattice has a global $SO(3) \times SO(3) \times U(1)$ symmetry. The generator $\hat{S}_c$ of the hidden global $U(1)$ symmetry reads $\hat{S}_c = \hat{\tilde{V}}^\dagger \hat{S}_c \hat{V}$ where $\hat{S}_c = \hat{Q}/2$ and the operator $\hat{Q}$ is given in Eq. (3). Its eigenvalue $S_c$ is one-half the number of rotated-electron singly occupied sites $2S_c$. Indeed, the rotated electrons of Refs. [2, 6] are constructed in such a way that $S_c$ is a good quantum number in the rotated-electron picture. The subspaces spanned by states whose number $2S_c$ is constant play an important role. For hole concentrations $0 \leq x < 1$ and maximum spin density $m = (1 - x)$ there is a fully polarized vacuum, which remains invariant under the electron - rotated-electron unitary transformation,

$$|0_{\eta_{\sigma}}\rangle = |0_{\eta}; N_a^D\rangle \times |0_{\eta}; N_a^D\rangle \times |GS_c; 2S_c\rangle.$$  \hspace{1cm} (4)

Here the $\eta$-spin $SU(2)$ vacuum $|0_{\eta}; N_a^D\rangle$ associated with $N_a^D = [N_a^D - 2S_c]$ independent $+1/2$ $\eta$-spinons, the spin $SU(2)$ vacuum $|0_s; N_a^D\rangle$ with $N_a^D = 2S_c$ independent $+1/2$
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spinons, and the $c$ $U(1)$ vacuum $|GS_c; 2S_c\rangle$ with $N_c = 2S_c$ $c$ fermions remain invariant under the electron - rotated-electron unitary transformation. In turn, for states with a finite number of $s1$ bond particles the $c$ fermions are not invariant under that transformation.

The whole physics can be extracted from the model $^{[2]}$ in the LWS-subspace referring to values of $S_\alpha$ and $S_\beta$ such that $S_\alpha = -S_\beta$ for $\alpha = \eta, s$ $^{[2]}$. Within the LWS representation, the $c$ fermion creation operator can be expressed in terms of the rotated-electron operators as follows,

$$f_{\vec{r}_j,c}^\dagger = \tilde{c}_{\vec{r}_j,\uparrow}^\dagger (1 - \tilde{n}_{\vec{r}_j,\downarrow}) + e^{i\vec{r}_j \cdot \vec{\eta}} \tilde{c}_{\vec{r}_j,\uparrow} \tilde{n}_{\vec{r}_j,\downarrow},$$

where $\tilde{n}_{\vec{r}_j,\sigma} = \tilde{c}_{\vec{r}_j,\sigma}^\dagger \tilde{c}_{\vec{r}_j,\sigma}$ and $e^{i\vec{r}_j \cdot \vec{\eta}}$ is $\pm 1$ depending on which sublattice site $\vec{r}_j$ is on. The three spinon local operators $s_{\vec{r}_j}^l$ and three $\eta$-spinon local operators $p_{\vec{r}_j}^l$ such that $l = \pm, z$ and $s_{\vec{r}_j}^\pm = s_{\vec{r}_j}^x \pm i s_{\vec{r}_j}^y$ and $p_{\vec{r}_j}^\pm = p_{\vec{r}_j}^x \pm i p_{\vec{r}_j}^y$, respectively, where the Cartesian coordinates $x, y, z$ are often denoted in this paper by $x_1, x_2, x_3$, respectively, are given by,

$$s_{\vec{r}_j}^l = n_{\vec{r}_j,c} q_{\vec{r}_j}^l \; ; \; p_{\vec{r}_j}^l = (1 - n_{\vec{r}_j,c}) q_{\vec{r}_j}^l, \; l = \pm, z; \; n_{\vec{r}_j,c} = f_{\vec{r}_j,c}^\dagger f_{\vec{r}_j,c}.$$  

Here $n_{\vec{r}_j,c} = f_{\vec{r}_j,c}^\dagger f_{\vec{r}_j,c}$ is the $c$ fermion local density operator and the rotated quasi-spin operators read as follows in terms of rotated-electron creation and annihilation operators,

$$q_{\vec{r}_j}^\dagger = (\tilde{c}_{\vec{r}_j,\uparrow}^\dagger - e^{i\vec{r}_j \cdot \vec{\eta}} \tilde{c}_{\vec{r}_j,\uparrow}) \tilde{c}_{\vec{r}_j,\downarrow}; \; q_{\vec{r}_j} = (q_{\vec{r}_j}^\dagger)^\dagger; \; q_{\vec{r}_j}^z = \frac{1}{2} - \tilde{n}_{\vec{r}_j,\downarrow}. \tag{7}$$

Inversion of the relations provided in Eqs. $(5)$ and $(7)$ gives,

$$\tilde{c}_{\vec{r}_j,\uparrow}^\dagger = f_{\vec{r}_j,c}^\dagger \left( \frac{1}{2} + q_{\vec{r}_j}^z \right) + e^{i\vec{r}_j \cdot \vec{\eta}} f_{\vec{r}_j,c} \left( \frac{1}{2} - q_{\vec{r}_j}^z \right);$$

$$\tilde{c}_{\vec{r}_j,\downarrow}^\dagger = q_{\vec{r}_j} (f_{\vec{r}_j,c}^\dagger - e^{i\vec{r}_j \cdot \vec{\eta}} f_{\vec{r}_j,c}). \tag{8}$$

For the one- and two-electron subspace $N_a^D \gg 1$ and the number $N_{a_{s1}}^D$ of sites of the $s1$ effective lattice, $N_{s1}$ of $s1$ bond particles, and $N_{s1}^h$ of unoccupied sites read,

$$N_{a_{s1}}^D = N_{s1} + N_{s1}^h = N_{a_{s1}}^D/2 + S_s; \quad N_{s1} = N_{a_{s1}}^D/2 - [S_s + 2N_{s2}], \quad N_{s1}^h = [2S_s + 2N_{s2}] = 0, 1, 2; \tag{9}$$

whereas for the corresponding $c$ fermions such numbers are given by,

$$N_{a_c}^D = N_c + N_{c_{s1}}^D = N_{a_c}^D; \quad N_c = 2S_c = (1 - x) N_a^D; \quad N_{c_{s1}}^D = x N_a^D. \tag{10}$$

For that subspace the $s1$ effective lattice is either full with $N_{s1} = N_{a_{s1}}^D = N_{a_{s1}}^D/2$ or has one or two unoccupied sites. For it there is commensurability between the real-space distributions of the $N_{a_{s1}}^D \approx N_{s1}$ sites of the $s1$ effective lattice and $N_{a_{s1}}^D \approx 2N_{s1}$ sites of the spin effective lattice. For $(1 - x) \geq 1/N_a^D$ the spin effective lattice has $N_{a_{s1}}^D = (1 - x) N_a^D$ sites. The $N_{a_{s1}}^D$ expression given in Eq. $(9)$ implies that $a_{s1} = L/N_{a_{s1}}$ reads,

$$a_{s1} = 2^{1/D} \frac{a_s}{\left(1 + \frac{2S_s}{(1-x)N_a^D}\right)^{1/D}} \approx 2^{1/D} a_s \left(1 - \frac{2S_s}{D(1-x)N_a^D}\right) \approx 2^{1/D} a_s, \tag{11}$$
where \( 2S_z = 0, 1, 2 \) and the lattice constant \( a_s \) of the spin effective lattice is given in Eq. (1).

For \( N_{s_1}^h = 0 \) states the bipartite 1D and square spin effective lattices have two well-defined sub-lattices. For the square lattice the two spin effective sub-lattices have lattice constant \( a_{s_1} = \sqrt{2} a_s \). In turn, for 1D the sites of each spin effective sub-lattice are distributed alternately along the chain, the corresponding nearest-neighboring sites being separated by \( a_{s_1} = 2a_s \). The fundamental translation vectors of such sub-lattices read,

\[
\vec{a}_{s_1} = a_{s_1} \vec{e}_{x_1} , \quad [1D] ; \quad \vec{a}_{s_1} = \frac{a_{s_1}}{\sqrt{2}} (\vec{e}_{x_1} + \vec{e}_{x_2}) ; \quad \vec{b}_{s_1} = -\frac{a_{s_1}}{\sqrt{2}} (\vec{e}_{x_1} - \vec{e}_{x_2}) , \quad [2D] , \quad (12)
\]

where \( \vec{e}_{x_1} \) and \( \vec{e}_{x_2} \) are the unit vectors. The vectors given in Eq. (12) are the fundamental translation vectors of the \( s_1 \) effective lattice [2].

The \( \eta \) fermions are \( \eta \)-spinless and spinless fermions without internal structure and their effective lattice is identical to the original lattice. In contrast, the composite two-spinon \( s_1 \) bond particles have internal structure and expression of their \( s_1 \) effective lattice occupancies in terms of spinon occupancies of the spin effective lattice and rotated-electron occupancies of the original lattice is a more complex problem, which deserves and requires further studies. This is the main subject of the remaining of this paper.

3. The \( N_{s_1}^h = 0 \) configuration state: \( s_1 \) effective lattice, corresponding change of gauge structure, and \( s_1 \) bond-particle operators

For the square-lattice model in the one- and two-electron subspace the numbers \( N_{s_1}, N_{a_1}^D \), and \( N_{s_1}^h = [N_{a_1}^D - N_{s_1}] \) are good quantum numbers whose values are fully controlled by those of \( S_z^S, S_z^s, \) and \( S_c \). We call configuration states the spinon occupancy configurations in the spin effective lattice which generate the spin degrees of freedom of the energy eigenstates that span the one- and two-electron subspace. Such configuration states refer to the overall occupancy of the \( N_{s_1} \) \( s_1 \) bond particles over the \( N_{a_1}^D \) sites of the \( s_1 \) effective lattice. Each configuration state refers to well-defined positions of the \( N_{s_1}^h \) unoccupied sites. A \( x \geq 0 \) and \( m = 0 \) ground state is an example of an energy eigenstate whose spin degrees of freedom are described by the \( N_{s_1}^h = 0 \) configuration state studied in the following.

We start by considering the \( N_{s_1}^h = [N_{a_1}^D - N_{s_1}] = 0 \) configuration state with no unoccupied sites in the spin and \( s_1 \) effective lattices. It describes the spin degrees of freedom of \( x \geq 0 \) and \( m = 0 \) ground states and its charge excited states belonging to the one- and two-electron subspace whose \( N_{a_1}^D = N_{a_1}^D / 2 = (1 - x) N_a^D / 2 \) sites of the \( s_1 \) effective lattice are occupied. Such a configuration state is described by suitable spinon occupancy configurations of the \( N_{a_1}^D = (1 - x) N_a^D \) sites of the spin effective lattice.
3.1. Independent two-site one-link bonds

For simplicity we consider that $N_{as} = (1-x)^{1/D} N_a$ is an integer number yet if otherwise one reaches the same results in the $N_{as}^D \gg 1$ limit that our study refers to. For the model on the square lattice the spin effective lattice is then a square lattice with $N_{as} \times N_{as}$ sites. For the model on that lattice we consider torus periodic boundary conditions for the spin effective lattice, alike for the original lattice. That implies periodic boundary conditions for the $N_{as}$ rows and $N_{as}$ columns. Periodic boundary conditions are also used for the one-chain spin effective lattice of the 1D model.

The spin effective lattice has in the present case two sub-lattices. The real-space coordinates of the sites of each of such sub-lattices correspond to a possible choice of those of the $s1$ effective lattice. Indeed, there is for the $N_{s1}^h = 0$ configuration state a gauge “symmetry” between the representations in terms of the occupancies of the two alternative choices of real-space coordinates of the $s1$ effective lattice, which are found to refer to two alternative and equivalent representations of that state. We use the same notation as Xiao-Gang Wen in Ref. [22] and say that there is a gauge structure when we use simultaneously the real-space coordinates of the two corresponding choices of $s1$ effective lattices to label the $N_{s1}^h = 0$ configuration state. The real-space coordinates $\vec{r}_j$ of such two sub-lattices have $j = 1, ..., N_{as}^D$ sites, in either case the fundamental translation vectors being those given in Eq. (12). The real-space coordinates $\vec{r}_j$ of the $s1$ bond particles are chosen to correspond to those of one of these two sub-lattices. Throughout the remaining of this paper we call sub-lattice 1 and sub-lattice 2 the sub-lattice of the spin effective lattice of a $N_{s1}^h = 0$ configuration state whose real-space coordinates are and are not the same as those of the $s1$ effective lattice, respectively.

An one-link bond connects two sites of the spin effective lattice whose spinons have opposite spin projection and correspond to a two-site spin-singlet configuration defined below. Each $s1$ bond particle is a suitable superposition of a well-defined set of two-site one-link bonds. Such a spin-singlet two-spinon $s1$ bond particle is related to the resonating-valence-bond pictures for spin-singlet occupancy configurations of ground states studied in Ref. [18, 19]. However, the $s1$ bond particle is well defined for all values of $U/4t > 0$ and not only for $U/4t \gg 1$, consistently with its two spinons referring to spins of the sites singly occupied by rotated electrons. In contrast, most schemes used previously for the Hubbard model or related models involving singly-occupied-site spins refer in general to large values of $U/4t \gg 1$ only [16, 12, 18, 22]. Here the $s1$ bond particles have been constructed to inherently involving spinon occupancy configurations of sites of the spin effective lattice, which for $U/4t > 0$ refers only to the sites of the original lattice singly occupied by rotated electrons.

For simplicity we often call two-site link or just link a two-site one-link bond. For the $N_{s1}^h = 0$ configuration state studied here the above superposition includes $2D = 2, 4$ families of two-site links, each family having $N_{s1}/2D$ different types of such links: $N_{s1}/2D$ is the largest number of independent links with the same link centre that exist for the above-considered boundary conditions. (Above and in the remaining of this
paper we denote often the number of family links by $2D = 2, 4$ where two and four is
the number of such families for the model on the 1D and square lattice, respectively.)
Link independence means here that for a given link centre all links involve different pairs
of sites and each site belongs to one pair only.

The set of independent links with the same link centre belong to the same link
family. Each one-link bond of a given family has some weight which in some cases may
vanish. For a s1 bond particle of real-space coordinate $\vec{r}_j$ there are $2D = 2, 4$ families
of links. The links of each family are centered at one of the $2D = 2, 4$ points
$\vec{r}_j + \vec{r}^g_{d,l}$
where the indexes $d = 1, 2$ for $D = 2$, $d = 1$ for $D = 1$, and $l = \pm 1$ uniquely define
the link family. Here $\vec{r}^g_{d,l}$ is the primary link vector. It connects the site of real-space
coordinate $\vec{r}_j$ to the centre of the four (and two for 1D) links of real-space coordinate
$\vec{r}_j + \vec{r}^g_{d,l}$ in the spin effective lattice involving that site and its nearest-neighboring sites.
Note that the former site and the latter four (and two for 1D) sites belong to sub-lattice
1 and 2, respectively. On choosing one of the two sub-lattices of the spin effective lattice
to be sub-lattice 1 and thus playing the role of s1 effective lattice and representing the
states in terms of the occupancy configurations of the latter lattice we say that there is
a change of gauge structure [22]. By considering all $N_{s1}/2D$ independent links for each
of the $2D$ link centres needed to describe a s1 bond particle of real-space coordinate $\vec{r}_j$
we consider the most general situation, the exact configuration referring to some choice
of the one-link bond weights considered below, some of which may vanish.

A s1 bond particle of real-space coordinate $\vec{r}_j$ involves $N_{s1}$ two-site one-link bonds
consistently with each family having $N_{s1}/2D$ links of different type. The link type is
labeled by an index $g = 0, 1, ..., [N_{s1}/2D - 1]$ uniquely defined below. Each link of a s1
bond particle of real-space coordinate $\vec{r}_j$ involves two sites of coordinates
$\vec{r} - \vec{r}^g_{0}$ and $\vec{r} + \vec{r}^g_{0}$ where $\vec{r}_j = \vec{r} - \vec{r}^g_{0}$ so that the link centre $\vec{r} \equiv \vec{r}_j + \vec{r}^g_{0}$
is the middle point located half-way between the two sites and the link vector $\vec{r}^g_{d,l}$
is defined below. The real-space coordinates $\vec{r}_j = \vec{r} - \vec{r}^g_{d,l}$ and $\vec{r} + \vec{r}^g_{d,l}$
belong to the sub-lattice 1 and sub-lattice 2 of the spin effective lattice, respectively.
For each family there are $N_{s1}/2D$ link vectors $\vec{r}^g_{d,l}$
which for the square lattice read,

$$\vec{r}^g_{d,l} = \vec{r}^l_{d,l} + T^g_{d,l} ; \quad \vec{r}^l_{d,l} = l \frac{a_s}{2} \vec{e}_{x_d} ; \quad g = 0, 1, ..., [N_{s1}/2D - 1],$$

where $d = 1, 2$, $l = \pm 1$, and $T^g_{d,l}$ is a $T$ vector. It has Cartesian components
$T^g_{d,l} = [T^g_{d,l,1}, T^g_{d,l,2}]$ for the square lattice and $T^g_{1,l} = [T^g_{1,l,1}]$ for 1D. There are $N_{s1}/2D$
$T$ vectors $T^g_{d,l}$, one for each choice of the following Cartesian components,

$$T^g_{d,l,i} = l a_s N_i ; \quad i = 1, 2,$$

$$N_d = 0, 1, ..., N_a/4 - 1 ; \quad N_d = -N_a/4 + 1, ..., -1, 0, 1, ..., N_a/4 . \quad (14)$$

Here $d = 1, 2$, $l = 2$, $l = \pm 1$ and $N_d$ and $N_d$ are consecutive integer numbers.
The expressions provided in Eq. (13) apply to the 1D lattice as well provided that only
the index value $d = 1$ is considered. The 1D component of the $T$ vectors is given by
$T^g_{1,l,1} = l a_s N_1$ where $N_1 = 0, 1, ..., N_a/4 - 1$. 

\[ \]
The link-type index \( g = 0, 1, \ldots, \lfloor N_{s1}/2D - 1 \rfloor \) labels the \( N_{s1}/2D \) \( T \) vectors \( \vec{T}^g_{d,l} \). For the square lattice it is defined in terms of the numbers \( N_d \) and \( N_d \) given in Eq. (14) and reads,

\[
g = N_d + 2|N_d| \frac{N_a}{4} \quad ; \quad N_d \leq 0, \\
= N_d + 2(N_d - 1) \frac{N_a}{4} \quad ; \quad N_d > 0.
\]

(15)

For 1D one has that \( g = N_1 = 0, 1, \ldots, N_{s1}/2 - 1 \).

The values of the link-type index \( g \) are consecutive positive integers whose minimum value \( g = 0 \) corresponds to \( N_1 = N_2 = 0 \) so that,

\[
T^0_{d,l} = 0.
\]

(16)

For the model on the square lattice the maximum value is \( g = \lfloor N_{s1}/2D - 1 \rfloor \) and refers to \( N_d = N_a/4 - 1 \) and \( N_d = N_a/4 \).

Each pair of values (and value) of the Cartesian coordinates of the \( T \) vector \( \vec{T}^g_{d,l} = [T^g_{d,l,1}, T^g_{d,l,2}] \) for the square lattice (and \( \vec{T}^g_{l,l} = [T^g_{l,l,1}] \) for 1D) corresponds to exactly one of the values \( g = 0, 1, \ldots, \lfloor N_{s1}/2D - 1 \rfloor \) so that,

\[
\sum_{N_d=0}^{N_a/4-1} \sum_{N_{d}}^{N_a/4+1} \equiv \sum_{g=0}^{N_a/4-1} ; \quad D = 2; \quad \sum_{N_1=0}^{N_{s1}/2-1} \equiv \sum_{g=0}^{N_a/4} ; \quad D = 1.(17)
\]

Two-site links with the same \( g \) value and \( d \neq d' \) and/or \( l \neq l' \) are equivalent links. Those are of the same type but belong to different families. Furthermore, \( T \) vectors \( \vec{T}^g_{d,l} \) and \( \vec{T}^g_{d',l'} \) with the same value of \( g \) and \( d \neq d' \) and/or \( l \neq l' \) are related as follows,

\[
\vec{T}^g_{d,l} = l'l' [\delta_{d,d'} + \delta_{d,d} \sigma_\times] \vec{T}^g_{d',l'}; \quad \sigma_\times = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

(18)

where \( \sigma_\times \) is the usual Pauli matrix.

As described below in terms of suitable operators, a \( s1 \) bond particle of real-space coordinate \( \vec{r}_j \) is a superposition of \( N_{s1} \) two-site one-link bonds each being associated with a link vector \( \vec{r}^g_{d,l} \). For each site of the spin effective lattice there is exactly one other site of the same lattice such that the link connecting the two sites has centre at \( \vec{r}_j + \vec{r}^g_{d,l} \). Therefore, any link of the same family involves two sites of well-defined real-space coordinate \( \vec{r}_j + \vec{r}^{0}_{d,l} - \vec{r}^{g}_{d,l} \) and \( \vec{r}_j + \vec{r}^{0}_{d,l} + \vec{r}^{g}_{d,l} \), which do not contribute together to any other link of the same family.

An important quantity is the distance between the two sites of a link which we call link length or two-site bond length. It is independent of the real-space coordinate \( \vec{r} = \vec{r}_j + \vec{r}^{0}_{d,l} \) of the link centre and is fully determined by the link vector \( \vec{r}^g_{d,l} \) and thus depends on the link type associated with the index \( g \) only. For \( D = 2 \) and \( D = 1 \) it reads,

\[
\xi_g \equiv |2\vec{r}^g_{d,l}| = a_s \sqrt{(1 + 2N_d)^2 + (2N_{d})^2}; \quad \xi_g \equiv |2x^g_{l,l}| = a_s (1 + 2N_1), \quad \xi_g \equiv |2\vec{x}^g_{l,l}| = a_s (1 + 2N_1),
\]

(19) respectively. Its minimum and maximum values are,

\[
\min \xi_g = \xi_0 = a_s; \quad \max \xi_g = \sqrt{2} a_s (N_{a}/2 - 1) + \mathcal{O}(1/N_a),
\]

(20)
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for the square lattice and \(\min \xi_g = \xi_0 = a_s\) and \(\max \xi_g = a_s (N_{a_s}/2 - 1)\) for 1D.

For 1D, links with different \(g\) have different link length \(\xi_g\). In turn, for the square lattice there are links of different type and hence different \(g\) which have the same link length \(\xi_g\). Indeed, analysis of the link-length expression of Eq. (19) reveals that links with different \(g\) value and numbers \([N_d, N_d']\) and \([N'_d, N'_d]\), respectively, such that \(N_d = N'_d\) and \(N_j = -N'_d\) have the same link length.

The set of values of the numbers \(N_1\) and \(N_2\) given in Eq. (14) imply that the maximum value of the link length is \(\sqrt{D} a_s (N_{a_s}/2 - 1)\) rather than \(\sqrt{D} a_s (N_{a_s} - 1)\). Indeed and as mentioned above, the links contributing to a \(s1\) bond particle of the \(N_{a_s}^h = 0\) configuration state are independent. It is then required that each link involves two sites that participate simultaneously in exactly one of such links. Within the torus row and column periodic boundary conditions for the square spin effective lattice such a requirement is fulfilled provided that the range of the numbers \(N_1\) and \(N_2\) is that given in Eq. (14).

For each family of two-site links associated with a \(s1\) bond particle of real-space coordinate \(\vec{r}_j\) there is a primary link. It corresponds to \(g = 0\) and thus connects two nearest-neighboring sites of the spin effective lattice, one of them having the same real-space coordinate \(\vec{r}_j = \vec{r} - \vec{r}^0_{d,l}\) as the \(s1\) bond particle. For primary links the link vector \(\vec{r}^0_{d,l}\) reads \(\vec{r}^0_{d,l} = \vec{r}^0_{d,l}\) where the primary link vector \(\vec{r}^0_{d,l}\) is given in Eq. (13).

For the model on the square lattice there are four primary links, one per family. Their link vectors \(\vec{r}^0_{d,l}\) have components such that \(N_1 = N_2 = 0\) in Eqs. (13) and (14). Therefore, the primary links have minimum length \(\xi_{\vec{r}_{d,l}} = a_s\). Alike the remaining links of its family, the centre of a primary link is located at \(\vec{r} = \vec{r}_j + \vec{r}^0_{d,l}\). For the square lattice there are two horizontal primary links whose centers are located at \(\vec{r}_j + \vec{r}^0_{1,l}\) with \(l = \pm 1\) and two vertical primary links whose centers are located at \(\vec{r}_j + \vec{r}^0_{2,l}\) with \(l = \pm 1\). In the case of the 1D lattice there are two primary links whose centers are located at \(\vec{r}_j + \vec{r}^0_{1,l}\) with \(l = \pm 1\).

3.2. Partitions and \(g\)-primary partitions

The building blocks of the \(N_{a_s}^h = 0\) configuration state are singlet pairs of spinons on sites \(\vec{r}_j^-\) and \(\vec{r}_j^+\) of the spin effective lattice,

\[
|\vec{r}_j^-,\vec{r}_j^+\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_{\vec{r}_j^-}\downarrow_{\vec{r}_j^+}\rangle - |\downarrow_{\vec{r}_j^-}\uparrow_{\vec{r}_j^+}\rangle \right),
\]

\[
\vec{r}_j^\mp = \vec{r}_j + \vec{r}^0_{d,l} \mp \vec{r}^0_{d,l} ; \quad d = d(j), \quad l = l(j), \quad g = g(j),
\]

where the values of the integer indexes \(d, l\), and \(g\) are in the ranges \(d = 1, 2\), \(l = \pm 1\), and \(g \in (0, N_{a_s}/2D - 1)\), respectively, and are a function of the index \(j = 1, \ldots, N_{a_s}\) of the real-space coordinate \(\vec{r}_j\) in the sublattice 1 of each \(s1\) bond particle. Indeed, the two sites of such pairs of sites are connected by one-link bonds and each bond is associated with exactly one \(s1\) bond particle.

Each connection involving \(N_{a_s}\) different bonds determines a partition. A partition is a \(2N_{a_s}\)-spinon occupancy configuration where each site of the spin effective lattice is
Figure 1. Sub-domain of the spin effective lattice with a primary partition of $d = 1$ horizontal links for the model on the square lattice. The primary links are represented by the thick horizontal lines connecting two sites of the spin effective lattice. For a reference frame where the site located at the corner on the left-hand side and lower limit of the squared sub-domain has Cartesian coordinates $(0,0)$ the family indices read $d = 1$, $l = +1$ if that site belongs to sub-lattice 2, whereas $d = 1$, $l = -1$ if instead it belongs to sub-lattice 1.

linked to one site only and all $2N_{s1}$ sites then correspond to $N_{s1}$ well-defined two-site one-link bonds, each belonging to a different $s1$ bond particle.

We recall that a $s1$ bond particle involves the superposition of $N_{s1}$ such two-site one-link bonds whereas a partition involves one two-site one-link bond from each of the $N_{s1}$ s1 bond particles. Each s1 bond particle contributes with exactly one of its two-site bonds to a partition. In a partition any site of the spin effective lattice participates in one bond only and there is a single link attached to each site which connects it to some other site. And the latter site is attached to the former site only.

The $N_{s1}^{A} = 0$ configuration state is then represented as,

$$|\phi\rangle = \sum_{P} C_{P} \prod_{j=1}^{N_{s1}} |\vec{r}_{j}^{-}, \vec{r}_{j}^{+}\rangle ; \quad C_{P} = \prod_{j=1}^{N_{s1}} h_{g(j)}^{*} ;$$

where the coefficients $h_{g}$ are associated with the bond weights and appear in the bond operators defined below, the product of singlet states $\prod_{j=1}^{N_{s1}} |\vec{r}_{j}^{-}, \vec{r}_{j}^{+}\rangle$ refers to a bond state associated with a given partition, and the summation $\sum_{P}$ is over all partitions.

A particular type of partition involves $N_{s1}$ identical links. The indexes $d$, $l$, and $g$ of identical links have the same values but correspond to $s1$ bond particles with different real-space coordinates $\vec{r}_{j}$. Such a partition involves a set of $N_{s1}$ identical two-site one-link bonds whose links connect different sites of the spin effective lattice, each site being linked to exactly one site. In this case the two real-space coordinates of the $N_{s1}$ pairs of sites are connected by the same real-space vector $2\vec{r}_{d,l}^{qg}$ so that each bond link has the same length.

Each of the $N_{s1}$ bonds of a partition involves two sites of real-space coordinates $\vec{r}_{j'}$ and $\vec{r}_{j'} + 2\vec{r}_{d,l}^{qg}$ which belong to different sub-lattices where $j' = 1, ..., N_{s1}$. The relation to the notation used above for the real-space coordinates of the two sites of a link is as follows,

$$\vec{r}_{j'} = \vec{r}_{j} + \vec{r}_{d,l}^{0} - \vec{r}_{d,l}^{qg} ; \quad \vec{r}_{j'} + 2\vec{r}_{d,l}^{qg} = \vec{r}_{j} + \vec{r}_{d,l}^{0} + \vec{r}_{d,l}^{qg} ; \quad j, j' = 1, ..., N_{s1},$$

(23)
where both $\vec{r}_{j'}$ and $\vec{r}_j$ are real-space coordinates of the sub-lattice 1 and thus of the s1 effective lattice. Except for a primary link one has that $j \neq j'$. The site of real-space coordinate $\vec{r}_j$ is that of the corresponding s1 bond particle. It is closest to the link centre at $\vec{r}_j + \vec{r}_{d,l}^0$. In turn, $\vec{r}_{j'}$ is the real-space coordinate of one of the two sites of the spin effective lattice involved in the link.

When a partition is a set of $N_{s1}$ identical primary links, all site pairs involve nearest-neighboring sites of the spin effective lattice. It is then called a primary partition. The family of a primary partition is labeled by the indexes $d$ and $l$ of the corresponding identical links. Figures 1 and 2 represent primary partitions of $d = 1$ horizontal and $d = 2$ vertical links, respectively, for a sub-domain of the spin effective lattice of the model on the square lattice.

An useful concept is that of a g-primary partition. It is defined as the superposition of the $2D = 2, 4$ primary partitions. It follows that a g-primary partition contains $2D N_{s1}$ two-site primary links. In such a configuration each of the $2N_{s1}$ sites of the spin-effective lattice has $2D = 2, 4$ links attached to it. Figure 3 shows a sub-domain of the spin-effective lattice with the g-primary partition of the $N_{s1}^h = 0$ configuration state for the model on the square lattice.

3.3. The s1 bond-particle operators and the subspace they act onto

The spinon occupancy configurations considered above are similar to those associated with multi-spin wave functions of spin-singlet states used by several authors [18, 20, 21]. Such wave functions are often constructed having as building blocks two-site and two-spin spin-singlet configurations similar to that of Eq. (21) except that here the two spins refer to sites singly occupied by rotated electrons and thus correspond to $U/4t > 0$ rather than only to $U/4t \gg 1$. In such schemes one also connects pairs of lattice sites with bonds and each such a connection determines a partition. However, here bonds involve sites of the spin effective lattice whereas those of previous related studies refer to the sites of the original lattice of the corresponding quantum problems.

For a given partition one can define a valence bond state [18, 20, 21] as a product of
Figure 3. Sub-domain of the spin-effective lattice representing the $g$-primary partition of the $N_{h1}^b = 0$ configuration state for the model on the square lattice. The sites belonging to the sub-lattice 1 are represented by filled circles. The horizontal (and vertical) thin and thick lines refer to $d = 1$ (and $d = 2$) $l = +1$ and $l = -1$ primary links, respectively. For the square lattice each site of the spin effective lattice has in a $g$-primary partition four links attached to it.

singlet states and represent an arbitrary singlet by a superposition of valence bond states of general form similar to that of Eq. (22). That involves a sum over all partitions of the lattice into set of pairs. However, for general wave functions such a decomposition works in general very badly. Indeed, valence-bond states are not orthogonal and their basis is overcomplete. Fortunately, here each of the $N_h^1$ two-site bonds of a partition belongs to a different $s_1$ bond particle so that each of such particles contributes to a partition with exactly one bond. Such a restriction eliminates the unwanted and unphysical contributions and renders the bond states free of the overcompleteness problem.

For the one- and two-electron subspace and $N_a \to \infty$, the operators $g_{\vec{r}_j,s_1}$ (and $g_{\vec{r}_j,s_1}^\dagger$) which annihilate (and create) a $s_1$ bond particle at a site of the spin effective lattice of real-space coordinate $\vec{r}_j$ have the following general form both for the 1D and square lattices,

$$
g_{\vec{r}_j,s_1} = \sum_{g=0}^{N_{h1}^b/2D-1} h_g a_{\vec{r}_j,s_1,g} ; \quad g_{\vec{r}_j,s_1}^\dagger = (g_{\vec{r}_j,s_1})^\dagger ,
$$

$$
a_{\vec{r}_j,s_1,g} = \sum_{d=1}^{D} \sum_{l=\pm 1} b_{\vec{r}_j+d\vec{r}_{d,l},s_1,d,l,g} ; \quad D = 1, 2 ,
$$

so that the expression of $g_{\vec{r}_j,s_1}^\dagger$ involves the operators $a_{\vec{r}_j,s_1,g}^\dagger = (a_{\vec{r}_j,s_1,g})^\dagger$ and $b_{\vec{r}_j,s_1,d,l,g}^\dagger = (b_{\vec{r}_j,s_1,d,l,g})^\dagger$. The operators $a_{\vec{r}_j,s_1,g}^\dagger$ and $a_{\vec{r}_j,s_1,g}$ create and annihilate, respectively, a superposition of $2D = 2, 4$ bonds of the same type and $b_{\vec{r}_j,s_1,d,l,g}^\dagger$ and $b_{\vec{r}_j,s_1,d,l,g}$ are two-site one-bond operators whose expression is given below.

For the square (and 1D) lattice the four (and two) primary links associated with the operators $a_{\vec{r}_j,s_1,0}^\dagger$ and $a_{\vec{r}_j,s_1,0}$ are behind most of the spectral weight of a $s_1$ bond particle of real-space coordinate $\vec{r}_j$. Consistently, the absolute value $|h_g|$ of the coefficients $h_g$ appearing in the expressions of such operators given in Eq. (24) decreases for increasing
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link length $\xi_g$. These coefficients obey the normalization sum-rule,

$$\sum_{g=0}^{[N_{s1}/2D-1]} |h_g|^2 = \frac{1}{2D}; \quad D = 1, 2.$$  \hspace{1cm} \text{(25)}$$

The exact dependence of $|h_g|$ on the link length $\xi_g$, value of $U/4t$, and hole concentration $x$ remains for the Hubbard model an involved open problem. The suitable use of this sum-rule and related symmetries leads though to useful information, as discussed below.

By definition the real-space coordinates $\vec{r}_j$ of the $s_1$ bond-particle operators of Eq. (24) are those of the sub-lattice 1.

The two-site one-bond operators $\hat{b}_{\vec{r},s_1,d,l,g}^\dagger$ and $\hat{b}_{\vec{r},s_1,d,l,g}^\dagger$ appearing in Eq. (24) are associated with a well-defined link connecting the two sites of real-space coordinates $\vec{r} - \vec{r}_{d,l}$ and $\vec{r} + \vec{r}_{d,l}$, respectively. Their expression can be obtained by considering the following related operator,

$$\left( -1 \right)^{d-1} \frac{1}{\sqrt{2}} \left[ \left( 1 - \tilde{n}_{\vec{r} - \vec{r}_{d,l}^g} \right) c_{\vec{r} - \vec{r}_{d,l}^g}^\dagger \tilde{c}_{\vec{r} + \vec{r}_{d,l}^g}^\dagger \right]$$

$$= f_{\vec{r} - \vec{r}_{d,l}^g, c}^\dagger f_{\vec{r} + \vec{r}_{d,l}^g, c}^\dagger \hat{b}_{\vec{r},s_1,d,l,g}^\dagger,$$  \hspace{1cm} \text{(26)}$$

where in the second expression the operator $\hat{b}_{\vec{r},s_1,d,l,g}^\dagger$ reads,

$$\hat{b}_{\vec{r},s_1,d,l,g}^\dagger = \left( -1 \right)^{d-1} \frac{1}{\sqrt{2}} \left( \frac{1}{2} + s_{\vec{r} - \vec{r}_{d,l}^g} \right) \tilde{s}_{\vec{r} - \vec{r}_{d,l}^g} - \left( \frac{1}{2} + s_{\vec{r} + \vec{r}_{d,l}^g} \right) \tilde{s}_{\vec{r} + \vec{r}_{d,l}^g}^\dagger,$$  \hspace{1cm} \text{(27)}$$

and $\hat{b}_{\vec{r},s_1,d,l,g} = \left( \hat{b}_{\vec{r},s_1,d,l,g}^\dagger \right)^\dagger$. Here the spinon operators are those given in Eq. (3). The second expression of Eq. (26) is obtained by the use of Eq. (3).

The phase factor $\left( -1 \right)^{d-1}$ which appears in the operator of Eq. (27) is associated with the $d$-wave symmetry of the $s_1$ bond-particle two-spinon pairing of the model on the square lattice. The introduction of such a phase-factor refers to a self-consistent procedure which follows from the $d$-wave symmetry of the spinon energy dispersion found in Ref. [2] for the $s_1$ fermions. Such objects emerge from the $s_1$ bond particles studied here through a suitable extended Jordan-Wigner transformation and their energy-dispersion $d$-wave symmetry arises naturally from symmetries beyond the form of the operators introduced in Eqs. (26) and (27).

According to the above discussions, the real-space coordinates $\vec{r} - \vec{r}_{d,l}^g$ and $\vec{r} + \vec{r}_{d,l}^g$ involved in the operators of Eqs. (26) and (27) correspond to two sites that belong to different sub-lattices of the spin effective lattice, $\vec{r} = \vec{r}_j + \vec{r}_{d,l}^0$ is the link centre, and the primary link vector $\vec{r}_{d,l}^0$ and link vector $\vec{r}_{d,l}^g$ are given in Eqs. (13) and (14). In the configuration generated by the operator of Eq. (26) the two sites are singly-occupied by the rotated electrons associated with the operators appearing in the first expression of that equation.
The $N^h_{s_1} = 0$ configuration state (22) can be written in terms of $s_1$ bond-particle operators given in Eq. (24) as follows,

$$|\phi\rangle = \prod_{j=1}^{N_{s_1}} g_{\vec{r}_j,s_1}^\dagger |0_{s_1}; N^D_{a_1}\rangle,$$

where $|0_{s_1}; N^D_{a_1}\rangle$ is the spin $SU(2)$ vacuum with $N^D_{a_1} = 2N_{s_1}$ independent $+1/2$ spinons on the right-hand side of Eq. (1). It corresponds to a fully polarized spin-up configuration. The subspace where the operators of Eqs. (24)-(27) act onto is defined by imposing the equality of the general configuration states given in Eqs. (22) and (28), respectively. Such an equality implies several restrictions on the transitions generated by the two-site one-bond operators (27) which are summarized in four corresponding rules for exclusion of unphysical and unwanted spin configurations given below. Before introducing such rules let us discuss several properties of the $s_1$ bond-particle operators which follow from the algebra given in Eqs. (A.7)-(A.9) of Appendix A for the basic spinon operators of the two-site one-bond operators expressions provided in Eq. (27).

The $s_1$ bond-particle operators $g_{\vec{r}_j,s_1}^\dagger$ and $g_{\vec{r}_j,s_1}$ of Eq. (24) involve a sum of $N_{s_1}$ two-site one-bond operators of general form given in Eq. (27) with $N_{s_1}/2D$ of such operators per family. The number of unoccupied sites $N_{h_{s_1}}$ of Eq. (9) refers to a subspace with constant number $N_{s_1}$ of $s_1$ bond particles. In turn, the creation and annihilation of one $s_1$ bond particle by application of these operators, respectively, onto the ground state involves a superposition of $N_{s_1}$ elementary processes, which do not conserve the number of these objects. Each such an elementary process is generated by an operator $b_{\vec{r},s_1,d,l,g}^\dagger$ and $b_{\vec{r},s_1,d,l,g}$, respectively, whose expression is given in Eq. (27).

Within the present LWS representation, application of the rotated-electron operators of Eq. (26) onto two unoccupied sites of the original lattice generates two virtual processes. The first process involves creation of two $c$ fermions and two independent $+1/2$ spinons. The second process refers to creation of a spin-singlet two-site and two-spinon configuration upon annihilation of two independent $+1/2$ spinons. Indeed, from analysis of the expression provided in Eq. (27), one finds that application of the operator $b_{\vec{r},s_1,d,l,g}^\dagger$ onto the sites of real-space coordinates $\vec{r} - \vec{r}_{d,l}$ and $\vec{r} + \vec{r}_{d,l}$ gives zero except when those sites are both occupied by an independent $+1/2$ spinon. This is consistent with a ”unoccupied site” referring to two sites of the spin effective lattice with real-space coordinates $\vec{r} - \vec{r}_{d,l}$ and $\vec{r} + \vec{r}_{d,l}$, respectively, which are occupied by independent $+1/2$ spinons in the initial configuration state.

As confirmed in Section 4 and consistently with the studies of Ref. [2], for excitations involving transitions between configuration states where a $s_1$ bond particle moves around in the spin effective lattice by elementary processes which conserve the numbers $N_{s_1}$ and $N^h_{s_1}$, an independent $+1/2$ spinon plays the role of an unoccupied site of the $s_1$ and spin effective lattices. In contrast, in elementary processes involving the creation of a $s_1$ bond particle the two annihilated independent $+1/2$ spinons play the role of a ”unoccupied site”, which becomes occupied in the final occupancy configuration.

According to the operator expression provided in Eq. (27), upon acting onto the
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Figure 4. Small sub-domain of two-g-primary partitions associated with \( N_{s1}^h = 0 \) and \( N_{s1}^h = 2 \) configuration states for the model on the square lattice. In the configuration of the figure on the left-hand side the filled circle at the middle of the sub-domain corresponds to the site whose real-space coordinate \( \vec{r}_j \) is that of the \( s1 \) bond particle. Annihilation of that object leads to the configuration on the figure right-hand side. Note that for a primary \( g \)-basic partition this is equivalent to the suppression of the four links attached to the above site of real-space coordinate \( \vec{r}_j \).

Independent-spinon occupancies the operator \( b_{\vec{r},s1,d,l,g}^\dagger \) generates a superposition of two configurations. For one of these configurations the elementary process generated by that operator flips the spin of the spinon at site \( \vec{r} + \vec{r}_{d,l}^g \) and checks that the spin of the spinon at site \( \vec{r} - \vec{r}_{d,l}^g \) remains up. The elementary process generating the other configuration flips the spin of the spinon at site \( \vec{r} - \vec{r}_{d,l}^g \) and checks that the spin of the spinon at site \( \vec{r} + \vec{r}_{d,l}^g \) remains up. The relative phase factor \(-1\) of the two configurations insures that the \( s1 \) bond particle created by the operator \( g_{\vec{r}_j,s1}^\dagger \) of Eq. (24) is a suitable superposition of spin-singlet configurations.

A superficial analysis of the configurations shown in Fig. 4 seems to indicate that there is one unoccupied site in the sub-domain of the final \( N_{s1}^h = 2 \) configuration state. However, if instead of the \( g \)-primary partition one considers the corresponding four primary partitions one finds that there are two nearest-neighbor unoccupied sites. For the \( d = 1 \) and \( d = 2 \) primary partitions these two sites belong to the same row and column, respectively, as further discussed in Section 4.

Concerning the application onto spin configurations of two-site one-bond operators and four-site two-bond operators, the restrictions arising from imposing that the representations of the \( N_{s1}^h = 0 \) configuration state given in Eqs. (22) and (28), respectively, are identical correspond to the following four rules whose fulfillment prevents the generation of unwanted and unphysical spin configurations:

**First rule** according to which application onto a spin configuration of a two-site one-bond operator \( b_{\vec{r},s1,d,l,g}^\dagger \) or \( b_{\vec{r}_1,s1,d,l,g} \) generates a physical spin configuration provided that its sites of real-space coordinates \( \vec{r} - \vec{r}_{d,l}^g \) and \( \vec{r} + \vec{r}_{d,l}^g \), respectively, are in the initial
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spin configuration (i) occupied by independent $+1/2$ spinons or (ii) linked by a bond.

Second rule states that application onto a spin configuration of any of the elementary four-site two-bond operators $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}^\dagger$, $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}$, $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}$, and $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}^\dagger$ gives zero when one of the two sites of real-space coordinates $\vec{r} - \vec{r}_d^\dagger$ and $\vec{r} + \vec{r}_d^\dagger$ respectively, is the same as one of the two sites of real-space coordinates $\vec{r}' - \vec{r}_d^{g'}$ and $\vec{r}' + \vec{r}_d^{g'}$. Hence only when the two two-site one-bond operators of such four-site operators do not join sites or join both sites their application onto a spin configuration generates physical spin configurations.

Third rule according to which when the two two-site one-bond operators of an elementary four-site two-bond operator $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}^\dagger$, $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}$, $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}$, and $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}^\dagger$, correspond (i) to different $s_1$ bond particles so that in the real-space coordinates $\vec{r} = \vec{r}_j + \vec{r}_d^d$ and $\vec{r}' = \vec{r}_j + \vec{r}_d^{d'}$ of their link centres, respectively, one has that $j = j'$ or (ii) to the same $s_1$ bond particle and to the same sites so that $\vec{r} \pm \vec{r}_d^d = \vec{r}' \pm \vec{r}_d^{g'}$ the first rule applies independently to each of such two-site one-bond operators provided that in case (i) the second rule is obeyed.

Forth rule refers to when the real-space coordinates of the link centres of the two-site one-bond operators of any of the four elementary four-site two-bond operators considered in the third rule are given by $\vec{r} = \vec{r}_j + \vec{r}_d^d$ and $\vec{r}' = \vec{r}_j + \vec{r}_d^{d'}$, respectively, so that such two-site one-bond operators correspond to the same $s_1$ bond particle of real-space coordinate $\vec{r}_j$ but $\vec{r}_d^d \neq \vec{r}_d^{g'}$ and states that then there are two cases. When in the initial spin configuration where the operators $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}^\dagger$ and $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}$ (and $b_{\vec{r},s_1,d,l,g} b_{\vec{r}',s_1,d',l',g'}$ and $b_{\vec{r},s_1,d,l,g}^\dagger b_{\vec{r}',s_1,d',l',g'}^\dagger$) act onto the sites of the spin effective lattice of real-space coordinates $\vec{r}_j + \vec{r}_d^d - \vec{r}_d^{g'}$ and $\vec{r}_j + \vec{r}_d^{d'} + \vec{r}_d^{g'}$ are linked by a bond (and occupied by two independent $+1/2$ spinons) then a physical final spin configuration is generated provided that the sites of real-space coordinates $\vec{r}_j + \vec{r}_d^d - \vec{r}_d^{g'}$ and $\vec{r}_j + \vec{r}_d^{d'} + \vec{r}_d^{g'}$ are linked by a bond and (occupied by two independent $+1/2$ spinons) or (ii) linked by a bond. In turn, when in the initial spin configuration where these four-site two-bond operators act onto the sites of the spin effective lattice of real-space coordinates $\vec{r}_j + \vec{r}_d^d - \vec{r}_d^{g'}$ and $\vec{r}_j + \vec{r}_d^{d'} + \vec{r}_d^{g'}$ are occupied by two independent $+1/2$ spinons (and linked by a bond) then a physical final spin configuration is generated provided that their sites of real-space coordinates $\vec{r}_j + \vec{r}_d^d - \vec{r}_d^{g'}$ and $\vec{r}_j + \vec{r}_d^{d'} + \vec{r}_d^{g'}$ are linked by a bond (and occupied by two independent $+1/2$ spinons).

Such rules follow naturally from the definition of the subspace where the operators of Eqs. (21)-(27) act onto. The main criterion is that such operators have been constructed to inherently generating a faithful representation provided that the corresponding state $|N_s^h\rangle$ represents the $N_{s_1}^h = 0$ configuration state and hence is identical to that given in Eq. (22) with the same value of $N_s = N_{s_1}^D$. For instance, the second rule results from all partitions of the summation on the right-hand side of Eq. (22) a site of the spin effective lattice being linked to exactly only one site. Indeed, in a given partition
no two-site one bonds join the same site. Furthermore, the third rule refers to four-site two-bond operators whose two two-site one-bond operators belong to the same s1 bond-particle operator \( g^{\dagger}_{r_j,s1} \) or \( g_{r_j,s1} \), yet correspond to different pairs of sites of the spin effective lattice. Such four-site two-bond operators appear in the expressions of the s1 bond-particle operators \( [g^{\dagger}_{r_j,s1}]^2 \) or \( [g_{r_j,s1}]^2 \), respectively, which as confirmed in Appendix A give zero when acting onto the subspace which the operators of Eqs. (27) refer to. The point is that when in the initial spin configuration where the operators \( b^{\dagger}_{r,s1,d,l,g}g^{\dagger}_{r',s1,d',l',g'} \) and \( b_{r,s1,d,l,g}b^{\dagger}_{r',s1,d',l',g'} \) (and \( b^{\dagger}_{r,s1,d,l,g}b_{r',s1,d',l',g'} \) and \( b_{r,s1,d,l,g}b^{\dagger}_{r',s1,d',l',g'} \)) act onto the sites of the spin effective lattice of real-space coordinates \( \vec{r}_j + \vec{r}^0_{d',l'} - \vec{r}^0_{d,l'} \) and \( \vec{r}_j + \vec{r}^0_{d',l'} + \vec{r}^0_{d,l'} \) are occupied by two independent +1/2 spinons (and linked by a bond) the operator on the right-hand side of \( g^{\dagger}_{r_j,s1}g^{\dagger}_{r_j,s1} \) and \( g_{r_j,s1}g^{\dagger}_{r_j,s1} \) (and \( g^{\dagger}_{r_j,s1}g_{r_j,s1} \) and \( g^{\dagger}_{r_j,s1}g_{r_j,s1} \)) which such four-site two-bond operators belong to, respectively, has transformed that configuration into an intermediate virtual state where such sites are linked by a bond (and occupied by two independent +1/2 spinons).

The \( N^a_{s1} = 0 \) configuration state is given exactly by the same superposition of partitions when the real-space coordinates \( \vec{r}_j \) of the s1 bond-particle operators of Eq. (24) where \( j = 1, \ldots, N^D_{a,s1} \) are chosen to refer to any of the two sub-lattices of the spin effective lattice. That property is straightforwardly confirmed for the g-primary partition represented in Fig. 3 and is fulfilled provided that one sums over all possible partitions. Hence, whether the above real-space coordinates are those of one or the other of such sub-lattices, the occupancy configuration associated with the creation of the \( N_{s1} = N^D_{a,s1}/2 \) bond particles is exactly the same. This confirms that the two corresponding choices for the s1 effective lattice describe the same \( N^a_{s1} = 0 \) configuration state so that there is a gauge structure. This holds in the limit \( N_o \gg 1 \) that the present description refers to.

### 3.4. Two-site one-bond weights

The exact dependence of the absolute value \( |h_g| \) of the coefficients appearing in Eqs. (22) and (25) on the link length \( \xi_g \) remains an open problem. However, if one assumes that \( |h_g| \) has the following simple power-law dependence on that length,

\[
|h_g| = \frac{C}{\xi_g^{\alpha_{s1}}}, \tag{29}
\]

the link-length expression (19) and normalization condition (25) alone imply that \( C^2 \) be given by

\[
C^2 = \frac{\sum_{N_d} \sum_{N_{d1}} [(1 + 2N_d)^2 + (2N_{d1})^2]^{-\alpha_{s1}}}{D = 2},
\]

\[
C^2 = \frac{\sum_{N_{d1}} [(1 + 2N_{d1})^{-2\alpha_{s1}}]}{D = 1}. \tag{30}
\]

Here the summations are over the range of \( N_d \) and \( N_{d1} \) (and \( N_1 \)) values given in Eq. (17) for the square (and 1D) lattice. An expression of the general form (29) would
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Figure 5. The relative spectral weight of the primary links $R_{2D}$ of Eq. (32) for the square lattice as a function of the exponent $\alpha_{s1}$ for the range $\alpha_{s1} \geq 1$. That relative weight refers to the simple power-law expressions given in Eqs. (29) and (30) for the coefficients $|h_g|$ appearing in the summations of Eq. (24). The physical range of the exponent $\alpha_{s1}$ corresponds typically to $R_{2D} > 0.9$ so that the primary links are behind most of the $s1$ bond-particle spectral weight. For $\alpha_{s1} \approx 2.5$ the ratio $R_{2D}$ is larger than 0.9 and approaches quickly the unit upon further increasing $\alpha_{s1}$.

The ratios of Eq. (32) refer to the limit $N_a \to \infty$ and are plotted in Figs. 5 and 6 for the square and 1D lattice, respectively, as a function of the exponent $\alpha_{s1}$ for the range $\alpha_{s1} \geq 1$. For $\alpha_{s1}$ slightly larger than $5D/4$ where $D = 1, 2$ the ratios $R_{1D}$ ($D = 1$) and $R_{2D}$ ($D = 2$) are larger than 0.9.

For $N_a \to \infty$ the exact coefficients $|h_g|$ are decreasing functions of the link length whose expressions most likely are not, at least for the whole link-length range, of the simple form given in Eqs. (29) and (30). However the exact sum-rules (25) together with the coefficients $|h_g|$ being decreasing functions of the link length implies that at least for link-length not too small the exact coefficients $|h_g|$ fall off as in Eq. (29) and the corresponding ratios of Eq. (31) have a behaviour similar to that shown in Figs. 5 and 6. Hence for the square lattice one expects that $\alpha_{s1} > 2.5$ so that the ratio $R_{2D}$ is larger than 0.9 and the primary links are behind most of the spectral weight of the $s1$ bond particle, alike for the 1D lattice for $\alpha_{s1} > 1.25$.

Here we considered the $N_{s1}^{h} = 0$ configuration state where all sites of the $s1$ effective lattice are occupied. In the following we generalize our results to configuration states with a finite number of unoccupied sites.
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Figure 6. The relative spectral weight of the primary links $R_{1D}$ of Eq. (32) for the 1D lattice as a function of the exponent $\alpha_{s1}$ for the range $\alpha_{s1} \geq 1$. The ratio $R_{1D}$ approaches the unit upon increasing the value of $\alpha_{s1}$ more quickly than the ratio plotted in Fig. 5 for the square lattice. For $\alpha_{s1} = n_{s1} = 1, 2, 3, \ldots$ integer the relative weight reads $R_{1D} = 1/[1 - 2^{-2n_{s1}}\zeta(2n_{s1})]$ where $\zeta(x)$ is a Riemann's zeta function. For instance $R_{1D} = 8/\pi^4 \approx 0.811$ for $\alpha_{s1} = 1$ and $R_{1D} = 96/\pi^4 \approx 0.986$ for $\alpha_{s1} = 2$.

4. The $N_{s1}^h = 1, 2$ configuration states and the kink-like and anti-kink-like link occupancies associated with the unoccupied sites

The one- and two-electron subspace considered in this paper is spanned by states with none, one, and two $s1$ effective-lattice unoccupied sites. There are several configuration states with one or two unoccupied sites in that lattice so that such sites can move around in it. We recall that such a motion is independent of that of the rotated electrons that singly occupy sites of the original lattice relative to the rotated-electron unoccupied and doubly-occupied sites of that lattice. The latter are instead described by the motion of the $c$ fermions whose occupancy configurations correspond to the state representations of the charge global $U(1)$ symmetry. In turn, the configuration states associated with the motion around in the $s1$ and effective lattices of the $s1$ bond particles relative to their unoccupied sites refer to state representations of the global spin $SU(2)$ symmetry. The global $\eta$-spin $SU(2)$ symmetry does not play any role in the one- and two-electron subspace, since it corresponds to a single occupancy configuration ($x > 0$) or none occupancy configurations ($x = 0$) for all states which span that subspace. For $x > 0$ this is the $\eta$-spin vacuum on the right-hand side of Eq. (34), which is invariant under the electron - rotated-electron unitary transformation. For $x = 0$ one has that $N_{a_{q1}}^D = 0$ so that the $\eta$-spin effective lattice does not exist.

Analysis of the transformation laws of the objects whose occupancy configurations generate the energy eigenstates of the square-lattice model in the one- and two-electron subspace [2] reveals that the excited states generated by application onto $x \geq 0$ and $m = 0$ ground states of one- and two-electron operators have one unoccupied site and none or two unoccupied sites in the $s1$ effective lattice, respectively. The energy eigenstates that span such a subspace are generated in Ref. [2] by $c$ and $s1$ fermions momentum occupancy configurations. The $s1$ fermion occupancies in the $s1$ momentum band introduced in Ref. [2] that generate energy eigenstates with one and two $s1$ band holes can be expressed as suitable superpositions of the set of $N_{s1}^h = 1$ and $N_{s1}^h = 2$ configuration states, respectively, studied in the following.
The structure of the $N_{h1}^h = 0$ configuration states is simpler than that of the $N_{h2}^h = 1, 2$ configuration states. A key property simplifies the study of the latter states: There is an one-to-one correspondence between the occupancies of the $N_{s1}$ occupied sites of the $s1$ effective lattice of the partitions of a $N_{h1}^h = 0$ configuration state and those of the occupied sites of the $s1$ effective lattice of partitions of the $N_{h1}^h = 1, 2$ configuration states with the same number $N_{s1}$ of $s1$ bond particles. The point is that except for a change in the real-space coordinates of some of the sites, the link configurations of the sites of the spin effective lattice in one-to-one correspondence with each other remain unaltered.

4.1. $N_{h1}^h = 1$ $g$-primary partitions of the four sub-configuration states

We start our analysis by considering the configuration states with one unoccupied site. The $s1$ fermion momentum occupancy of excited $[N_{t}^h - N_{v}^h] = 1$ states of $x \geq 0$ and $m = 0$ ground states generated by application of one-electron operators onto the latter states is described by a suitable superposition of the set of $N_{h1}^h = 1$ configuration states studied in the following. For such configuration states the unoccupied site is associated with their single independent $+1/2$ spinon. For simplicity we have considered above that for the $N_{h1}^h = 0$ configuration state the number of sites $N_{a_s}^D = 2S_c = N_c$ of the spin effective lattice is such that $N_{a_s}$ is an integer so that the spin effective lattice is a square lattice. In turn, for configuration states with one unoccupied site the number $N_{a_s}^D$ of sites of the spin effective lattice is odd so that for $D = 2$ the designation $N_{a_s}^D$ is not to be understood to imply that the number $N_{a_s}$ is an integer.

We recall that the number of sites of the spin effective lattice is given by $N_{a_s}^D = 2S_c = N_c$ where $S_c$ is the eigenvalue of the generator of the global $U(1)$ symmetry $\hat{S}_c = \hat{V}^\dagger \hat{S}_c \hat{V}$, i.e., $\hat{S}_c = \hat{Q}/2$, the operator $\hat{Q}$ is given in Eq. (3), and $N_c$ is the number of $c$ fermions. Since $S_c$ is a half-off-integer, according to the studies of Ref. [6] the global $SO(3) \times SO(3) \times U(1)$ symmetry of the present quantum problem implies that the $\eta$-spin $S_\eta$ and spin $S_s$ are half-off-integers as well. It follows that the number of sites of the $s1$ effective lattice is according to Eq. (3) given by $N_{a_s}^{D_{s1}} = [N_{a_s}^D/2 + S_s]$ where $S_s = 1/2$. Out of those, $N_{s1} = [N_{a_s}^{D_{s1}} - S_s]$ are occupied by $s1$ bond particles and one is unoccupied, consistently with $N_{h1}^h = 2S_s = 1$. Hence such states have also one unoccupied site in the $s1$ effective lattice.

For the study of the $N_{h1}^h = 0$ configuration state we considered in Section 3 a change of gauge structure such that the real-space coordinates of the sites of the $s1$ effective lattice correspond to one of the two sub-lattices of the spin effective lattice. The results of Ref. [2] confirm that for the limit $N_a^D \gg 1$ and hole concentrations $x$ such that $(1 - x) \geq 1/N_a^D$ the two choices of $s1$ effective lattice lead to the same quantum numbers for the whole one- and two-electron subspace.

Let us consider $N_{h1}^h = 1$ configuration states whose spin effective lattice has $2N_{s1} + 1$ sites and for the model on the square lattice the spin effective lattice of the corresponding $N_{h1}^h = 0$ configuration state with the same number $N_{s1}$ of $s1$ bond particles is a square
lattice and has $N_a \times N_a = 2N_{s1}$ sites. For the present $N_a^D \to \infty$ limit the correct physics is then achieved if one considers that out of the $2N_{s1} + 1$ sites of the spin effective lattice of such $N_{s1}^h = 1$ configuration states, $2N_{s1}$ sites correspond to a square lattice and the position of the extra site is well defined and given below. The latter site does not belong to the square lattice formed by the remaining $2N_{s1}$ sites of the spin effective lattice and has suitable boundary conditions such that it is the unoccupied site of the $s1$ effective lattice. However, for the spin effective lattice the extra site and the unoccupied site are not always the same site, as discussed below.

For the square (and 1D) lattice the extra site boundary conditions are compatible and consistent with the periodic boundary conditions of each row and column (and chain). For the $N_{s1}^h = 0$ configuration state of the model on the square lattice such a row and column periodic boundary conditions imply torus periodic boundary conditions for the spin effective lattice. Here the presence of the extra site slightly affects the latter boundary conditions. Hence the spin effective lattice of $N_{s1}^h = 1$ configuration states has no pure torus periodic boundary conditions. Within the extra-site boundary conditions given below each $N_{s1}^h = 1$ configuration state has for the square (and 1D) lattice four (and two) sub-configuration states. The spin effective lattice is the same for all such sub-configuration states. The concepts of a partition and $g$-partition remain the same as for the corresponding $N_{s1}^h = 0$ configuration state, except that now there is an extra site.

For the model on the square lattice the extra site of the spin effective lattice belongs to the row or column that for a $d = 1$ and $d = 2$ primary partition specified below, respectively, the unoccupied site belongs to. The real-space coordinate of the extra site and its Cartesian coordinates are denoted by,

$$\vec{r}_{j0} = [x_{1}^{0}, x_{2}^{0}] . \quad (33)$$

In the present $N_a \to \infty$ limit the following periodic conditions yield the correct physics and assure that the extra site is uniquely defined and the same site for all partitions of a given $N_{s1}^h = 1$ configuration state,

$$x_{d}^{0} = x_{d}^{0} + l (L + a) ; \quad x_{d}^{0} = x_{d}^{0} ; \quad d = 1, 2 , \ l = \pm 1 , \quad (34)$$

where the index $\tilde{d} = 1, 2$ is defined as in Eq. (14). Such boundary conditions apply to the square lattice for $d = 1, 2$ and $l = \pm 1$ and to the 1D lattice for $d = 1$ and $l = \pm 1$. For the row ($d = 1$) [and column ($d = 2$) for the model on the square lattice] the extra site belongs to, they are equivalent to periodic boundary conditions. Indeed that row [and column] has $N_a + 1$ sites and hence its length is $L + a = (N_a + 1) a$.

For each sub-configuration state there is exactly one $g$-primary partition, which is a superposition of the corresponding four primary partitions. Hence instead of $2D = 2, 4$ primary partitions, for a $N_{s1}^h = 1$ configuration state there are $8D = 8, 16$ primary partitions. For all partitions of the summation on the right-hand side of Eq. (22) the extra site has the same Cartesian coordinates. Nonetheless, the extra site and the unoccupied site are not the same site of the spin effective lattice for all such partitions.
For instance, the extra site and the unoccupied site are the same site for three out of the four primary partitions of a primary $g$-basic partition, as confirmed below.

For the model on the square lattice the periodic boundary conditions of the sites belonging to rows and columns other than that of the extra site refer to a length $L$ rather than $L + a$. For the extra site the boundary conditions (31) apply to both the square and 1D lattice and are valid for the $2D = 2, 4$ sub-configuration states considered below. In turn, for the model on the square lattice and concerning the remaining sites of the extra-site row and column such periodic boundary conditions apply to those belonging to the row for horizontal sub-configuration states and to the column for vertical sub-configuration states, respectively. For horizontal (and vertical) sub-configuration states the sites of the extra-site column (and row) obey periodic boundary conditions associated instead with the length $L$, alike the sites that do not belong to the extra-site column and row. That the periodic boundary conditions of one row or column of the spin effective lattice refer to the length $L + a$ rather than $L$ is for the model on the square lattice behind the above deviation from the pure torus periodic boundary conditions.

Except for the unoccupied site, the remaining $2N_{s1}$ sites of the spin effective lattice are occupied by $N_{s1}$ $s1$ bond particles. Furthermore, out of the $2N_{s1} + 1$ sites of that lattice, for the $2N_{s1}$ sites other than the extra site, which for some partitions is not the unoccupied site, there are two well-defined sub-lattices, alike for the $N^{h}_{s1} = 0$ configuration state. Indeed, the extra site belongs to the spin effective lattice but does not belong to any of these two sub-lattices. Moreover, now sub-lattice 1 is that whose real-space coordinates are the same as those of the sites of the $s1$ effective lattice whose row or column are not that of the extra site. In turn, for horizontal (and vertical) sub-configuration states the occupied sites of the $s1$ effective lattice of the extra-site row (and column) belong instead to sub-lattice 2.

The $s1$ bond-particle operators act onto a subspace with constant values $N^{D}_{a_{s1}} = 2S_c = N_c$ of spin-effective-lattice sites. Therefore, suitable analysis of the general expressions of the number of sites and unoccupied sites of the $\alpha\nu$ effective lattices [2] reveal that for $\alpha\nu = s1$ one of these operators or products of such operators can generate transitions where the value of $N^{h}_{s1}$ either is conserved or changes by an even number. Indeed, transitions where such a value changes by an odd number imply that that of the number $N^{D}_{a_{s1}} = 2S_c = N_c$ of both spin-effective-lattice sites and $c$ fermions also changes. For instance, transitions from the $N^{h}_{s1} = 0$ configuration state to a $N^{h}_{s1} = 1$ configuration state involve creation or annihilation of one electron. Hence they involve creation or annihilation of one rotated electron as well. In terms of the objects of our description that process involves creation or annihilation, respectively, of a $c$ fermion plus creation of an independent spinon [2]. In the LWS representation the latter is an independent $+1/2$ spinon whose creation is equivalent to adding an extra site to the spin effective lattice.

Therefore, addition of the extra site is a process beyond the $s1$ bond-particle algebra whose operators are well defined for subspaces with constant values of $N^{D}_{a_{s1}} = 2S_c = N_c$ spin-effective-lattice sites only. Fortunately, our general description also accounts for
such a process, which involves both creation or annihilation of a fermion and creation of an independent +1/2 spinon. In turn, the generators of the transitions between different \( N_{s_1}^{h} = 1 \) configuration states involve \( s_1 \) bond-particle operators only. Such transitions are equivalent to the motion around in the spin effective lattice of \( s_1 \) bond particles. But since in the present case there is a single unoccupied site it is often more convenient to describe such a motion in terms of those of that site.

The unoccupied site has the same position in the \( s_1 \) effective lattice and \( g \)-primary partitions of the \( 2D = 2, 4 \) sub-configuration states. Such a position is the same as that of the extra site in the spin effective lattice. For each position of the extra site in that lattice there is a \( N_{s_1}^{h} = 1 \) configuration state. Specifically, there are \( N_{a_{s_1}}^{D} = [N_{a_{s_1}}^{D}/2 + S_s] = N_{s_1} + 1 \) such configuration states. The \( s_1 \) fermion occupancies of Ref. [2] of the above mentioned excited states generated by application onto \( x \geq 0 \) and \( m = 0 \) ground states of one-electron operators can be expressed as suitable superpositions of such \( N_{a_{s_1}}^{D} = [N_{a_{s_1}}^{D}/2 + S_s] = N_{s_1} + 1 \) configuration states. For the model on the square lattice the four sub-configuration states of each of such \( N_{s_1}^{h} = 1 \) configuration states are called horizontal-kink, horizontal-anti-kink, vertical-kink, and vertical-anti-kink sub-configuration state.

It useful for the description of the \( N_{s_1}^{h} = 2 \) configuration states of Subsection 4-5 to introduce the \( N_{s_1}^{h} = 1 \) \( g \)-primary partition of the horizontal-kink sub-configuration state as a limiting case of a general configuration whose unoccupied site and extra site do not have the same real-space coordinate in the \( s_1 \) effective lattice. The \( g \)-primary partition of the above sub-configuration state is then reached by considering the particular case when the unoccupied and extra site are the same site. As discussed below, the unoccupied site of a \( g \)-primary partition (and that of the underlying \( s_1 \) effective lattice) is identified with the unoccupied site of three out of its four primary partitions. A sub-domain of the spin-effective lattice with a part of such a general \( g \)-primary partition is shown in Fig. 7 for the square lattice. Its extra site is not shown in the sub-domain under consideration and is not the unoccupied site marked by the open circle. The occupied sites of the \( s_1 \) effective lattice are represented by filled circles.

The row kink-like link occupancy starting on the right-hand side of the unoccupied site ends at the extra site. We say that such a link occupancy and those of the surrounding rows have opposite signs. Indeed, the \( s_1 \) bond-particle sites of the row part associated with the former occupancy belong to sublattice 2 of the spin effective lattice whereas those of the \( s_1 \) bond particles on the surrounding rows belong to sublattice 1.

The \( g \)-primary partition of the horizontal-kink sub-configuration state corresponds to a particular case of that represented in Fig. 7 whose extra and unoccupied site are the same site so that the kink-like link occupancy extends over the whole unoccupied-site row. Figure 8 shows a sub-domain of the spin-effective lattice representing the \( g \)-primary partition of the \( N_{s_1}^{h} = 1 \) horizontal-kink sub-configuration state for the square lattice. Since one of the issues to be further discussed is the periodic boundary conditions of the extra-site row, for simplicity the \( N_{a_s} \gg 1 \) sites of a row are represented in Fig. 8 by a few sites. The last site of each row is marked by a \( X \) to indicate that it is the same site.
Figure 7. Sub-domain of the spin-effective lattice representing a general type of link occupancy for the model on the square lattice which when the extra and unoccupied sites have the same real-space coordinate corresponds to the \( g \)-primary partition of the \( N_{s1}^h = 1 \) horizontal-kink sub-configuration state. However, in the link configuration shown here the unoccupied site does not coincide with the extra site so that the former site plays the role of a mobile domain wall, alike for some \( g \)-primary partitions of \( N_{s1}^h = 2 \) configuration states considered in Subsection 4-5. The horizontal (and vertical) thin and thick lines refer to \( d = 1 \) (and \( d = 2 \)) \( l = +1 \) and \( l = -1 \) primary links, respectively. For the \( d = 1, l = +1 \) partition the unoccupied site is that attached to three links. In turn, for the \( d = 2, l = \pm 1 \) partitions and \( d = 1, l = -1 \) partition the unoccupied site is that which has a single link attached to it. The latter site is marked by an open circle in the figure and plays the role of unoccupied site of the \( s1 \) effective lattice. The occupied sites of that lattice are represented by filled circles. The extra site does not belong to the sub-domain shown here and marks the end of the horizontal kink-like link occupancy located on the right-hand side of the unoccupied site.

as the first site of the same row. In turn, for the vertical direction the figure represents a sub-domain only so that no equivalent sites are included, alike for the sub-domain shown in Fig. 7. Note that for the extra-site row of Fig. 8 the extra site is marked by a \( X \) and thus is the same as the unoccupied site marked by an open circle. Therefore and in contrast to Fig. 7, in Fig. 8 the kink-like link configuration extends over the whole extra-site row.

Both for the \( d = 2, l = \pm 1 \) primary partitions and \( d = 1, l = -1 \) primary partition of the \( g \)-primary partition shown in Fig. 8 and that of Fig. 7 the unoccupied site is the site attached to a single link, marked by the open circle in both figures. The \( d = 1, l = +1 \) primary partition is the only one for which the unoccupied site is that attached to three links in the figures. The site marked by the open circle plays the role of the unoccupied site of both the \( g \)-primary partition and \( s1 \) effective lattice of the four sub-configuration states.

For the \( g \)-primary partition of the horizontal-kink sub-configuration state we call kink row or 1,+1 line the link occupancy of the sites belonging to the same row as the unoccupied site. The notation \( d,l \) line where \( d = 1 \) and \( l = +1 \) stems from the \( d = 1, l = +1 \) primary partition being the only one out of the four such configurations of the corresponding \( g \)-primary partition for which the unoccupied site is attached to
Figure 8. Sub-domain of the spin-effective lattice representing the $g$-primary partition of the $N_{s_1}^h = 1$ horizontal-kink sub-configuration state for the model on the square lattice. The horizontal (and vertical) thin and thick lines refer to $d = 1$ (and $d = 2$) $l = +1$ and $l = −1$ primary links, respectively, the unoccupied site is marked by an open circle, and the sites corresponding to real-space coordinates of $s_1$ bond particles by filled circles. For the vertical direction only part of the configuration is shown. In turn, for the horizontal direction the $N_{a_s} \gg 1$ sites of a row are represented by a few sites. Indeed, one of the goals of the figure is to illustrate the row periodic boundary conditions, a site marked by a $X$ being the same as the first site on the left-hand side of the same row. For instance, the unoccupied site marked by an open circle is the same site as the last site on the right-hand side of the same row. Note that the extra-site row has one more site than the remaining rows and such a site is both the extra site and the unoccupied site so that the kink-like link configuration extends over the whole extra-site row, in contrast to the configuration shown in Fig. 7. On the other hand, the link configurations associated with the unoccupied site are the same as in Fig. 7. For the $d = 1$, $l = +1$ primary partition the unoccupied site is that attached to three links and for the $d = 2$, $l = ±1$ primary partitions and $d = 1$, $l = −1$ primary partition that which has a single link attached to it.

three links. For the $g$-primary partition of each of the four sub-configuration states of a $N_{s_1}^h = 1$ configuration state there is a $d,l$ line and in each case the corresponding $d,l$ primary partition is that for which the unoccupied site is attached to three links. Furthermore, for the remaining three primary partitions the unoccupied site is always that attached to a single link: It is the extra site, as illustrated in Fig. 8.

We call $d,l$ $g$-primary partition that which involves a $d,l$ line. Let $\vec{r}_{jun}$ denote the real-space coordinate of the unoccupied site in the spin effective lattice for each of the four primary partitions of the $d,l$ $g$-primary partition. For each of such primary partitions it is given in terms of the real-space coordinate $\vec{r}_{j0}$ of the extra site provided in Eq. (33) as follows,

$$\vec{r}_{jun} = \vec{r}_{j0} + \vec{r}_{0d}^0 \quad \vec{r}_{j0}^0$$

where the index $\vec{d}$ is defined as in Eq. (14).

Figure 9 shows four small sub-domains of the spin-effective lattice representing part of general $g$-primary partitions of the type represented in Fig. 7. As discussed below in
Figure 9. Four small sub-domains of the spin-effective lattice showing part of \(g\)-primary partitions. Alike in Fig. 7, each sub-domain includes an unoccupied site which plays the role of a mobile domain wall. The first two configurations of the figure refer to kink-like and anti-kink-like link configurations, respectively, of horizontal sub-configuration states. The two last configurations correspond to kink-like and anti-kink-like link configurations, respectively, of vertical sub-configuration states. Such mobile domain walls occur in some \(N_{s1}^h = 2\) sub-configuration states with two unoccupied sites on the same row or column rather than in \(N_{s1}^h = 1\) sub-configuration states. However, the link structure of the unoccupied site is the same as for the latter states.

Subsection 4-5, such sub-domains involve unoccupied-site mobile domains walls which occur in \(N_{s1}^h = 2\) sub-configuration states when the two unoccupied sites belong to the same row or column. However, the link structure associated with the unoccupied site at the center of each of such sub-domains is that also occurring in the \(g\)-primary partitions of the four \(N_{s1}^h = 1\) sub-configuration states, respectively. In turn, each of these sub-domains includes four primary links of a \(s1\) bond particle with a single link attached to the unoccupied site. As confirmed for the horizontal-kink sub-configuration state on comparing it with the link occupancy configurations of Fig. 8, the only difference relative to the \(N_{s1}^h = 1\) sub-configuration states refers to the structure and position of the remaining three primary links of that \(s1\) bond particle, the link-structure of the unoccupied site being the same as that shown in Fig. 9.

Specifically, the unoccupied-site link structure of the top configurations on the left-and right-hand side of Fig. 9 are the same as those of the \(g\)-primary partitions of \(N_{s1}^h = 1\) horizontal-kink and horizontal-anti-kink sub-configuration states, respectively. For the \(d = 2, l = \pm 1\) primary partitions and \(d = 1, l = -1\) (and \(d = 1, l = +1\)) primary partition on the left-hand side (and right-hand site) of the figure top the unoccupied site is that attached to a single link, marked by an open circle. In turn, for the \(d = 1, l = +1\) (and \(d = 1, l = -1\)) primary partition the unoccupied site is that attached to three links. Moreover, for the \(d = 1, l = \pm 1\) primary partitions and the \(d = 2, l = -1\) (and \(d = 2, l = +1\)) primary partition on the left-hand side (and right-hand side) of the figure lower-limit associated with the vertical-kink (and vertical-anti-kink) sub-configuration state the unoccupied site is that attached to a single link, marked by an open circle. On the other hand, for the corresponding \(d = 2\) and \(l = +1\) (and \(d = 2, l = -1\)) primary partition the unoccupied site is that attached to three links. Such four types of unoccupied-site link structures correspond to the \(g\)-primary partitions of the four \(N_{s1}^h = 1\) sub-configuration states, respectively.
The $g$-primary partitions of the horizontal-kink and horizontal-anti-kink (and vertical-kink and vertical-anti-kink) sub-configuration states include a kink row as that represented in Fig. 8 and a anti-kink row (and a kink column and anti-kink column), respectively. Such a row (and column) link occupancy configuration involves the set of sites belonging to the same row (and column) as the extra site. As mentioned above, an alternative designation for the kink row, anti-kink row, kink column, and anti-kink column is 1, +1 line, 1, −1 line, 2, +1 line, and 2, −1 line, respectively. The general $d, l$ line occurs in a primary $g$-basic partition for whose $d, l$ primary partition the unoccupied site is attached to three links.

There is an apparent contradiction concerning the following issue. On the one hand, that the extra-site periodic boundary conditions (34) refer to both a row and column of $N_{a_s} + 1$ sites is valid for the $2D = 2, 4$ sub-configuration states simultaneously. On the other hand, concerning the $g$-primary partitions of such states, $N_{a_s} + 1$-site periodic boundary conditions apply to either the sites of the spin effective lattice other than the extra site belonging to the extra-site row or extra-site column, respectively. Specifically, for $g$-primary partitions of horizontal (and vertical) sub-configuration states such periodic boundary conditions apply to the sites belonging to the 1, $l$ (and 2, $l$) line only, whereas for the sites belonging to the extra-site column (and row) other than the extra site the periodic boundary conditions refer to $N_{a_s}$ sites rather than to $N_{a_s} + 1$ sites.

The reason behind that apparent inconsistency is that the link occupancy configurations of each of the $2D = 2, 4$ $d, l$ lines of the $2D = 2, 4$ $g$-primary partitions, respectively, are different and thus refer to different sub-configuration states yet they all involve the same extra site. We recall that the extra site of Figs. 7-9 is linked to one site only. For $g$-primary partitions with a 1, $l$ (and 2, $l$) line the extra site is linked to a site belonging to that line only so that only the real-space coordinates of the remaining sites belonging to the 1, $l$ (and 2, $l$) line are shifted, whereas those of the sites belonging to the extra-site column (and row) remain unaltered.

In addition to the $d, l$ line of a $g$-primary partition, it is useful to consider its two nearest neighboring lines. For each such an occupancy configuration there are three $d, l, l'$ lines where $l' = -1, 0, +1$. Here $d, l, 0 \equiv d, l$ line is a more general notation for the $d, l$ line. For a kink or anti-kink row the 1, $l, l'$ line is the row above it for $l' = +1$ and below it for $l' = -1$. Moreover, for a kink or anti-kink column the 2, $l, l'$ line is the column on its right-hand side for $l' = +1$ and on its left-hand side for $l' = -1$.

\section*{4.2. Motion of the unoccupied site around in the spin effective lattice}

The motion of the unoccupied site around in the spin effective lattice discussed here can be described as well in terms of the motion of the corresponding unoccupied site of the $s_1$ effective lattice around in such a lattice. In the case of the $s_1$ fermion description of the square-lattice model investigated in Ref. \cite{2} the generators of the motion considered here are the two $s_1$ translation generators in the presence of the fictitious magnetic field
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$\vec{B}_{s1}$ associated with the Jordan-Wigner transformation that maps the $s1$ bond particles into $s1$ fermions. The components $q_{jx1}$ and $q_{jx2}$ of the discrete momentum values $\vec{q}_j$ of the $s1$ fermions are the eigenvalues of such two $s1$ translation generators. Since the $s1$ fermion creation and annihilation operators are defined in and act onto subspaces spanned by mutually neutral states [2] such operators commute and the two components $q_{jx1}$ and $q_{jx2}$ can be simultaneously specified.

It follows from the above analysis that for the $s1$ effective lattice and the $2D = 2, 4$ $g$-primary partitions of all sub-configuration states the unoccupied site has a well-defined position. For $N^h_{s1} = 1$ there are $N_{as} = N_{s1} + 1$ configuration states and this is also the number of different positions that the unoccupied site can have in the $s1$ effective lattice.

Transitions between configuration states where the unoccupied site has different positions in the $s1$ effective lattice result from a set of well-defined elementary steps which are behind its motion around in the spin and $s1$ effective lattices. The generators of such processes have simple and obvious expressions in terms of $s1$ bond-particle creation and annihilation operators. The above elementary steps correspond to changes in the occupancy configurations of rotated electrons and the unitary operator $\hat{V}$ preserves the occurrence of nearest hopping only for such objects. Therefore, in the case of the square lattice rotated electrons can hop vertically or horizontally only and given the relation between the original lattice and the spin effective lattice that leads to restrictions in the changes of the position of the link-site attachments. Indeed, the elementary steps behind the net motion of the unoccupied site involve horizontal or vertical shifts by $\pm a_s$ of some of such attachments where $a_s$ is the lattice constant of the spin effective lattice given in Eq. $11$. The net length passed by the unoccupied site in an elementary step is $2a_s$.

For $1D$ the unoccupied site moves by $\pm 2a_s$ in each elementary step. As for the square-lattice case, such elementary steps involve smaller virtual elementary steps where link-site attachments move by $\pm a_s$. For the square lattice the net motion of the unoccupied site is given by $\pm 2a_s \vec{e}_{x_i}$ where $i = 1$ and $i = 2$ for horizontal and vertical elementary processes, respectively. In such elementary steps the net length $2a_s$ passed by the $s1$ bond particle corresponds to horizontal or vertical motion. In addition, for the square lattice there are horizontal-vertical or vertical-horizontal elementary steps where the unoccupied site moves by $\pm a_s [\vec{e}_{x_i} + \vec{e}_{x_{i'}}]$ or $\pm a_s [\vec{e}_{x_i} - \vec{e}_{x_{i'}}]$ where $i = 1, 2$ and $i' \neq i$. For the latter elementary steps the net length $2a_s$ corresponds to both a horizontal and a vertical $\pm a_s$ shift, so that the initial and final positions distance is $\sqrt{2}a_s$.

For the $g$-primary partitions any elementary process involves the virtual annihilation of one of the four $s1$ bond particles surrounding the extra site. That virtual process involves the creation of two unoccupied sites and is followed by suitable horizontal and/or vertical $\pm a_s$ shifts of well-defined link-site attachments. Those shifts are equivalent to the unoccupied site moving by $\pm 2a_s \vec{e}_{x_i}$, $\pm a_s [\vec{e}_{x_i} + \vec{e}_{x_{i'}}]$ or $\pm a_s [\vec{e}_{x_i} - \vec{e}_{x_{i'}}]$ where $i = 1, 2$ and $i' \neq i$. Finally, a $s1$ bond particle is created onto the site of the $s1$ effective lattice unoccupied in the initial configuration so that two unoccupied sites of
the virtual state are occupied.

For some $g$-primary partitions of the model on the square lattice such elementary steps also involve collective processes where a whole kink row or anti-kink row (and kink column or anti-kink column) configuration moves from one row (and column) to the nearest-neighboring or second-nearest-neighboring row (and column). Such processes consist of a collective horizontal (and vertical) shift of $-a_s$ or $+a_s$ of all the sites of a kink or anti-kink row (and kink or anti-kink column), respectively, plus a collective horizontal (and vertical) opposite shift of $+a_s$ or $-a_s$ of all the sites of a nearest-neighboring or second-nearest-neighboring kink or anti-kink row (and kink or anti-kink column), respectively.

Specifically, for $g$-primary partitions of horizontal (and vertical) sub-configuration states, unoccupied-site horizontal (and vertical) elementary steps do not involve such kink or anti-kink row interchanges (and kink or anti-kink column interchanges). In contrast, for $g$-primary partitions of vertical (and horizontal) sub-configuration states, unoccupied-site horizontal (and vertical) elementary steps involve kink or anti-kink column interchanges (and kink or anti-kink row interchanges) of second-nearest-neighboring columns (and rows). On the other hand, horizontal-vertical or vertical-horizontal elementary steps involve kink or anti-kink row interchanges of nearest-neighboring rows for $g$-primary partitions of horizontal sub-configuration states and kink or anti-kink column interchanges of nearest-neighboring columns for primary $g$-primary partitions of vertical sub-configuration states.

Any motion of the unoccupied site around in the spin effective lattice can be generated by a set of the above elementary steps. The processes behind such steps are easiest to describe for the primary partitions. However, they are also well-defined for the remaining partitions.

4.3. Correspondence between the $N^h_{s1} = 1$ and $N^h_{s1} = 0$ configuration states

The partitions that span the $2D = 2, 4$ sub-configuration states of a $N^h_{s1} = 1$ configuration state are independent so that the number of partitions of the latter state is $2D = 2, 4$ times larger than that of the $N^h_{s1} = 0$ configuration state. Indeed, there is an one-to-one correspondence between the partitions of that state and those of a $N^h_{s1} = 1$ sub-configuration state with the same number of $s1$ bond particles. Concerning the primary partitions of $N^h_{s1} = 1$ sub-configuration states, except for the sites of the spin effective lattice belonging to the row and column which the unoccupied site belongs to, the real-space coordinates of the sites of that lattice have the same real-space coordinates as those of the corresponding primary partitions of the $N^h_{s1} = 0$ configuration state with the same number of $s1$ bond particles.

A $N^h_{s1} = 1$ configuration state is defined by the position of its extra site whose real-space coordinate in the $s1$ effective lattice is the same as that of the corresponding unoccupied site. Except for the sites belonging to the kink row and anti-kink row (and kink column and anti-kink column), respectively, for the horizontal-kink and horizontal-
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anti-kink (and vertical-kink and vertical-anti-kink) sub-configuration state the two sites of each of the \( N_{s1} \) links of the \( 2D = 2, 4 \) primary partitions have the same real-space coordinates as the corresponding primary partitions of the \( N_{s1}^h = 0 \) configuration state.

The shifts of the real-space coordinates of the sites of the primary partitions of the \( N_{s1}^h = 0 \) configuration state which become sites of a \( d, l \) line in the corresponding final sub-configuration state lead to the following real-space coordinates for the latter sites,

\[
\vec{r}_{d,l}^0 \rightarrow \vec{r}_{d,l} = \vec{r}_{d,l}^0 + 2\vec{r}_{d,l}^0 = \vec{r}_{d,l}^0 + l a_s \vec{e}_d.
\]

Here \( \vec{r}_{d,l}^0 \) denotes the corresponding real-space coordinate of the initial \( N_{s1}^h = 0 \) configuration state.

The procedure to construct the non-primary partitions of a \( N_{s1}^h = 1 \) sub-configuration state profits from all two-site links remaining exactly the same and only the real-space coordinates of the sites of the spin effective lattice of the \( N_{s1}^h = 1 \) sub-configuration state whose real-space coordinates are shifted for the primary partitions as given in Eq. (36) being shifted exactly by the same amount for the corresponding non-primary partitions. Such a procedure applies to any of the \( N_{s1} = 1 \) positions of the unoccupied site in the s1 effective lattice and hence to all corresponding \( N_{s1}^h = 1 \) configuration states. It corresponds to the elementary processes generated by application of an one-electron operator onto a \( x \geq 0 \) and \( m = 0 \) ground state.

4.4. \( s1 \) bond-particle operators of \( N_{1}^h = 1 \) configuration states

Each of the \( 2D = 2, 4 \) sub-configuration states of a \( N_{s1}^h = 1 \) configuration state has a \( d, l \) line in the corresponding \( d, l \) g-primary partition where for 1D the index \( d \) is given only by \( d = 1 \). We use a single index \( i \equiv d, l \) to denote the indices \( d, l \) which are used to label each sub-configuration state according to the type of \( d, l \) line in its g-primary partition. For a \( N_{s1}^h = 1 \) configuration state the operators given in Eq. (24) for the \( N_{s1}^h = 0 \) configuration state read instead,

\[
g_{\vec{r}_{d,l}^0, s1} = \frac{1}{\sqrt{2D}} \sum_i g_{\vec{r}_{d,l}^0, s1,i} ; \quad g_{\vec{r}_{d,l}^0, s1} = \left( g_{\vec{r}_{d,l}^0, s1} \right); \quad g_{\vec{r}_{d,l}^0, s1,i} = \sum_{g=0}^{N_{s1}/2D-1} h_{g,i} a_{\vec{r}_{d,l}^0, s1,i,g} ,
\]

\[
a_{\vec{r}_{d,l}^0, s1,i,g} = \sum_{d'=1}^{D} \sum_{l'=\pm1}^{\pm1} b_{\vec{r}_{d,l}^0, s1,i,d',l',g} ; \quad [N_{s1}/2D-1] \sum_{g=0}^{N_{s1}/2D-1} |h_{g,i}|^2 = \frac{1}{2D}, \quad (37)
\]

where \( D = 1, 2 \) and the operator \( g_{\vec{r}_{d,l}^0, s1,i} \) acts onto the spin and s1 effective lattice associated with the \( N_{s1}^h = 1 \) configuration state. For each sub-configuration state the real-space coordinates \( \vec{r}_{d,l}^0 \) of the s1 bond particles are the same as for the \( N_{s1}^h = 0 \) configuration state except for the corresponding \( d, l \) line. For that line they are shifted by \( 2\vec{r}_{d,l}^0 \) as given in Eq. (36). In the above equation we denote by \( d' \) and \( l' \) the summation indices in the expression for the operator \( a_{\vec{r}_{d,l}^0, s1,i,g} \) given in Eq. (37) to distinguish them from the constant \( d, l \) line indices \( d \) and \( l \).

The two-site one-bond operators \( b_{\vec{r}_{d,l}^0, s1,i,d',l',g} \) and \( b_{\vec{r}_{d,l}^0, s1,i,d',l',g}^\dagger \) involved in Eq. (37) have for s1 bond particles whose real-space coordinates do not belong to \( d, l, l' \) lines the
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same expression for all $2D = 2, 4$ sub-configuration states and read,

$$b_{r,s,i,d',i',g}^\dagger = b_{r,s,i,d',i',g}^\dagger,$$  \hspace{1cm} (38)

where $b_{r,s,i,d',i',g}^\dagger$ is given in Eq. (27). In turn, when $\vec{r}_j$ belongs to the $d, l, 0 \equiv d, l$ line such operators are given by,

$$b_{\vec{r},s,i,d''}^\dagger = \frac{(-1)^{d+d''}}{\sqrt{2}} \left( \frac{1}{2} + s_{\vec{r}-\vec{d}',i'} s_{\vec{r}+\vec{d}',i'} \right) \left( \frac{1}{2} + s_{\vec{r}+\vec{d}',i'} s_{\vec{r}-\vec{d}',i'} \right),$$

$$\vec{r}_i = -\delta_{d',d} \vec{2} \vec{r}_0^0,$$  \hspace{1cm} (39)

where we recall that $\vec{1} = 2$ and $\vec{2} = 1$. Finally, for the $s_1$ bond particles whose real-space coordinate $\vec{r}_j$ belongs to the $d, l, l'$ such that $l' = \pm 1$ the two-site one-bond operators read,

$$b_{\vec{r},s,i,d''}^\dagger = \frac{(-1)^{d-d''}}{\sqrt{2}} \left( \frac{1}{2} + s_{\vec{r}-\vec{d}',i'} s_{\vec{r}+\vec{d}',i'} \right) \left( \frac{1}{2} + s_{\vec{r}+\vec{d}',i'} s_{\vec{r}-\vec{d}',i'} \right),$$

$$\vec{r}_i = \delta_{d',d} \delta_{l',-l'} 2 \vec{r}_0^0.$$  \hspace{1cm} (40)

4.5. More about $N^{s_1}_{a_1}$, 0, 1, 2 configuration states

The $s_1$ fermion momentum occupancies of Ref. [2] of excited states generated by application onto $x \geq 0$ and $m = 0$ ground states of two-electron operators involving (i) creation or annihilation of two electrons with the same spin projection and (ii) spin-triplet and spin-singlet excitations are described by a superposition of the set of $[N_{a_1} + 2][N_{a_1} + 1]/2$ configuration states with $N^{s_1}_{a_1} = 2$ unoccupied sites studied in the following. Indeed, there are $[N_{a_1} + 2][N_{a_1} + 1]/2$ different $N^{s_1}_{a_1} = 2$ configuration states corresponding to the possible $[N_{a_1} + 2][N_{a_1} + 1]/2$ positions of the two unoccupied sites in the $s_1$ effective lattice. According to Eq. (3), here $N_{s_1} + 2 = [N_{a_1}^D/2 + S_s]$ where $S_s = 1$ for $N_{s_2} = 0$ and $S_s = 0$ for $N_{s_2} = 1$ is the number of sites $N_{a_1}^D$ of that lattice whose number of unoccupied sites reads $N^{s_1}_{a_1} = [2S_s + 2N_{s_2}] = 2$. For such states the number $N_{a_1}^D = [2N_{s_1} + 4 - 2S_s]$ of sites of the spin effective lattice is even. There are two sets of $[N_{a_1} + 2][N_{a_1} + 1]/2$ configuration states which contribute to different $N^{s_1}_{a_1} = 2$ energy eigenstates. Those are the $N^{h}_{a_1} = 2$, $S_s = 1$, $N_{s_2} = 0$ spin-triplet configuration states and $N^{h}_{a_1} = 2$, $S_s = 0$, $N_{s_2} = 1$ spin-singlet configuration states, respectively.

Since it is assumed that for the square lattice $[2N_{s_1}]^{1/2}$ is an integer number, the notations $N_{a_1}^{h} = 2$ and $N_{a_1}^{s} = 2$ do not mean that the numbers $N_{a_1}$ and $N_{a_2}$ are integers. $N_{a_1}^{h} = 2$ and $N_{a_2}^{s} = [2N_{s_1} + 4 - 2S_s]$ are integer numbers. In turn, the integer numbers closest to $N_{a_1}$ and $N_{a_2}$ are the mean value of the number of sites of each row and column of the $s_1$ and spin effective lattice, respectively. Our study takes into account the exact number of sites of such rows and columns.
Concerning the above two types of \( N_{s_1}^h = 2 \) configuration states, the two unoccupied sites can either correspond to two independent +1/2 spinons \((S_s = 1 \text{ and } N_{s_2} = 0)\) or two spinons out of the four which are part of the \( s^2 \) fermion \((S_s = 0 \text{ and } N_{s_2} = 1)\). We recall that alike an independent spinon, the spin-neutral four-spinon occupancy configuration of such a \( s^2 \) fermion remains invariant under the electron - rotated-electron unitary transformation \(^2\).

For the \( N_{s_2}^2 \to \infty \) limit considered here for the model on the square lattice the correct physics is achieved if one considers that out of the \([2N_{s_1} + 4 - 2S_s]\) sites of the spin effective lattice, \(2N_{s_1}\) sites correspond to a square lattice and the position of the two extra sites which contribute to the \( s^1 \) effective lattice is defined as for the \( N_{s_1}^h = 1 \) configuration states. For the model both on the 1D and square lattice and \( S_s = 0 \) and thus \( N_{s_2} = 1 \) the number of sites of the spin effective lattice is given by \( N_{s_2}^D = 2N_{s_1} + 4\). However, two of the four extra sites do not contribute to the \( s^1 \) effective lattice. The point is that for \( N_{s_1}^h = 2 \) configuration states with \( S_s = 0 \) and \( N_{s_2} = 1 \) when the \( s^1 \) bond particles moves around in that lattice, they use as unoccupied sites only two out of the four sites of the \( s^2 \) fermion. Two of these four sites of the spin effective lattice do not belong to the \( s^1 \) effective lattice, alike the sites of the \( \eta \)-spin effective lattice associated with the original rotated-electron unoccupied sites do not belong to the spin effective lattice.

The spin \( SU(2) \) symmetry imposes that the \( s^1 \) effective lattices of the \( N_{s_1}^h = 2 \) configuration states such that (i) \( S_s = 1 \) for \( N_{s_2} = 0 \) and (ii) \( S_s = 0 \) for \( N_{s_2} = 1 \), respectively, must be identical. In either case each of the two extra sites obeys the boundary conditions given in Eq. (34) which for the present \( N_{s_2} \to \infty \) limit yield the correct physics. For the square lattice the row and column periodic boundary conditions are similar to those of the \( N_{s_1}^h = 1 \) configuration states. For the 1D lattice periodic conditions are also used.

Symmetry imposes that for \( g \)-primary partitions of \( N_{s_1}^h = 2 \) configuration states the two unoccupied sites are associated with kink-like and anti-kink-like link configurations, respectively. Moreover, symmetry implies that only sub-configuration states for whose \( g \)-primary partitions both the kink and anti-kink link configurations associated with the two unoccupied sites, respectively, correspond to rows or columns are allowed. It follows that alike for \( N_{s_1}^h = 1 \) configuration states, a \( N_{s_1}^h = 2 \) configuration state involves \( 2D = 2, 4 \) sub-configuration states. In the present case there are two horizontal sub-configuration states for whose \( g \)-primary partitions the unoccupied sites of real-space coordinates \( \vec{r}_{j_0} \) and \( \vec{r}_{j_0}' \) are associated with row kink-like and anti-kink-like link configurations and row anti-kink-like and kink-like link configurations, respectively. Furthermore, there are two vertical sub-configuration states for whose \( g \)-primary partitions the unoccupied sites with such real-space coordinates are associated with column kink-like and anti-kink-like link configurations and column anti-kink-like and kink-like link configurations, respectively.

It is useful to classify the \( N_{s_1}^h = 2 \) configuration states into three main classes:
1) $N_{s1}^h = 2$ configuration states whose two extra sites do not belong to the same row and column. Such states occur for the model on the square lattice but do not exist for the 1D lattice. For the corresponding $g$-primary partitions the occupied sites are the extra sites, which correspond to kink-like or anti-kink-like link configurations of the same form as those of $N_{s1}^h = 1$ states. Indeed, each unoccupied site behaves independently and can move around in the spin effective lattice by means of the elementary steps discussed above for the $N_{s1}^h = 1$ configuration states. For all $g$-primary partitions one unoccupied site is associated with a kink-like link configuration and the other with an anti-kink-like link configuration, respectively. The $s1$ bond-particle operators are defined as for the $N_{s1}^h = 1$ configuration states by Eqs. (37)-(40) but with parallel $d,+1$ and $d,-1$ lines and the index $i$ corresponding to the four sub-configuration states defined above for the $N_{s1}^h = 2$ configuration states.

2) $N_{s1}^h = 2$ configuration states whose two extra sites belong to the same row or column but are not nearest neighboring sites. For the 1D lattice only states of this type and of type 3 exist. Let the two extra sites have real-space coordinates $\vec{r}_{j0}$ and $\vec{r}_{j0}'$. Then for the $g$-primary partition of one of the two horizontal (and vertical) sub-configuration states the link occupancy involving sites belonging to the row part (and column part) located between the sites of real-space coordinates $\vec{r}_{j0}$ and $\vec{r}_{j0}'$ has opposite sign relative to those of the surrounding rows (and columns) whereas the link occupancy involving sites belonging to the row part (and column part) located between the sites of real-space coordinates $\vec{r}_{j0}'$ and $\vec{r}_{j0}$ has the same sign as those of the surrounding rows. In contrast, for the $g$-primary partition of the other horizontal (and vertical) sub-configuration states the sign of the link occupancies on the two above row parts (and column parts) is the opposite. For $g$-primary partitions the two unoccupied sites of real-space coordinates $\vec{r}_{j0}$ and $\vec{r}_{j0}'$ behave as kink or anti-kink mobile domain walls whose link structure is shown in Figs. 7 and 9. The $s1$ bond-particle operators are defined as for the $N_{s1}^h = 1$ configuration states by Eqs. (37)-(40) but with the operators whose real-space coordinates correspond to link occupancy configurations with opposite sign relative to those of the surrounding rows or columns referring to the $d, j$ line parts defined above instead of to the whole $d, j$ line.

3) $N_{s1}^h = 2$ configuration states whose two extra sites belong to the same row and column and are nearest-neighboring sites in the spin effective lattice. In that case there is both a horizontal and a vertical sub-configuration state whose extra-site row and column link occupancies have opposite sign relative to those of the surrounding rows and columns, respectively. In turn, for the other horizontal and vertical sub-configuration states, respectively, all link occupancies have the same sign and the two-unoccupied-site link occupancy configuration of the corresponding $g$-primary partitions are identical. A small sub-domain of such a link configuration including the two-unoccupied sites of the spin effective lattice is shown on the right-hand side of Fig. 4. These configuration states can be generated from those of type 2. This is achieved by moving both unoccupied sites through their row or column, one against the other, until they become nearest-
neighbor sites. For the square (and 1D) lattice, the $g$-primary partitions of two (and one) out of the four (and two) sub-configuration states such a motion and resulting "collision" leads to the full cancellation of the corresponding kink-like and anti-kink-like link occupancy configurations. Such a unoccupied-site kink and anti-kink annihilation has similarities with a particle - anti-particle annihilation process. The corresponding "collisions" occur more often for a unoccupied site moving around in the spin effective lattice of the 1D lattice. Indeed, then there is a single chain. In contrast, for the model on the square lattice there are both horizontal and vertical elementary steps so that the two unoccupied sites can move independently around in the spin effective lattice without colliding provided that their motion involves configuration states of type 1, which do not exist in 1D. For the two above sub-configuration states of the model on the square lattice whose extra-site row or column link occupancies have opposite sign relative to those of the surrounding rows or columns, respectively, the $s_1$ bond-particle operators are defined as for the $N_{s_1}^h = 1$ configuration states by Eqs. (37)-(40). For the other two sub-configuration states whose extra-site row or column link occupancies have the same sign as those of the surrounding rows or columns, respectively, the $s_1$ bond-particle operators are defined alike for the $N_{s_1}^h = 0$ configuration states by Eq. (24).

The number $N_{s_1}^h$ of unoccupied sites is a good quantum number for the model on the square lattice in the one- and two-electron subspace so that the $s_1$ fermion momentum occupancies of Ref. 1 of the states that span such a subspace are described by a suitable superposition of configuration states with constant number of unoccupied sites in the $s_1$ effective lattice. For instance, the $s_1$ fermion momentum occupancy of a $N_{s_1}^h = 2$ spin-triplet (and spin-singlet) excited state is described by a suitable superposition of the $[N_{s_1} + 2][N_{s_1} + 1]/2 N_{s_1}^h = 2$ configuration states with $S_s = 1$ and $N_{s_2} = 0$ (and with $S_s = 0$ and $N_{s_2} = 1$.) Application of a $s_1$ bond-particle creation operator of real-space coordinate $\vec{r}_j$ onto such a ground state gives zero for configuration states of the above types 1 and 2: only the two sub-configuration states of a configuration state of type 3 whose link configurations have all the same sign and the real-space coordinate of the central unoccupied site on the right-hand side of Fig. 4 coincides with that of the applied operator are transformed onto the $N_{s_1}^h = 0$ configuration state with one more $s_1$ bond particle than the initial state.

Moreover, application of a $s_1$ bond-particle creation operator onto $N_{s_1}^h = 0,1$ configuration states gives always zero. In turn, application of a $s_1$ bond-particle annihilation operator whose real-space coordinate is that of an occupied site of the $s_1$ effective lattice of a $N_{s_1}^h = 0,1,2$ configuration state transforms it into a state with two more unoccupied sites than the initial state.

Finally, in Appendix A it is confirmed that within the description used in the studies of Ref. 1 one has that for $U/4t > 0$ and the subspaces where the $s_1$ bond-particle operators of the one- and two-electron and $N_{s_1}^h = 0,1,2$ subspace are defined, upon acting onto the $s_1$ effective lattice such operators anticommute on the same site.
Ground-state two-spinon bonds in the Hubbard model on a square lattice

of that lattice,
\[
\{ g_{\mathbf{r}_j,s}^\dagger, g_{\mathbf{r}_i,s} \} = 1; \quad \{ g_{\mathbf{r}_j,s}^\dagger, g_{\mathbf{r}_i,s}^\dagger \} = \{ g_{\mathbf{r}_j,s}^\dagger, g_{\mathbf{r}_i,s} \} = 0, \tag{41}
\]
and commute on different sites,
\[
[g_{\mathbf{r}_j,s}, g_{\mathbf{r}_i,s}^\dagger] = [g_{\mathbf{r}_j,s}, g_{\mathbf{r}_i,s}^\dagger] = [g_{\mathbf{r}_j,s}, g_{\mathbf{r}_i,s}] = 0, \tag{42}
\]
where \( j \neq j' \). That algebra confirms that the \( s_1 \) bond-particle operators are hard-core like.

5. Concluding remarks

For the Hubbard model on the square lattice in the one- and two-electron subspace the number \( N_{s_1}^h \) of unoccupied sites of the \( s_1 \) effective lattice is a good quantum number. Such a subspace is spanned by the states with \( N_{s_1}^h = 0, 1, 2 \), whose spin configurations are investigated in this paper. In the studies of Sections 3 and 4 a change of gauge structure \cite{22} is considered so that the real-space coordinates of the sites of the \( s_1 \) effective lattice correspond to one of the two sub-lattices of the spin effective lattice. That change is fulfilled for the \( N_{s_1}^h = 0 \) configuration state for which the gauge structure occurs. That leads to two alternative definitions of the \( s_1 \) effective lattice for both that state and the related \( N_{s_1}^h = 1, 2 \) configuration states: As discussed in this paper, the occupancy configurations of the latter states can be generated from those of the \( N_{s_1}^h = 0 \) configuration state with the same number of \( s_1 \) bond particles.

The \( N_{s_1}^h = 0 \) configuration states studied here generate the spin degrees of freedom of the \( x \geq 0 \) and \( m = 0 \) ground states and their two-electron charge excited states of the Hubbard model on the square lattice in the one- and two-electron subspace. That quantum problem refers to the square-lattice quantum liquid of \( c \) and \( s_1 \) fermions of Ref. \cite{2}. In turn, the spin configurations of the excited states generated from application onto these ground states of one-electron operators and two-electron operators other than the charge operator are suitable superpositions of the \( N_{s_1}^h = 1 \) and \( N_{s_1}^h = 0 \) or \( N_{s_1}^h = 2 \) configuration states, respectively, studied in this paper. For the \( N_{s_1}^h = 1, 2 \) states the unoccupied sites move around in the spin effective lattice through well-defined elementary steps. For \( N_{s_1}^h = 2 \) configuration states the two unoccupied sites may move through the same row or column, one against the other, until they become nearest-neighbor sites. For the model on the square (and 1D) lattice, for two out of four (and one out of two) \( g \)-primary partitions defined in this paper such a motion and resulting "collision" leads to the full cancellation of the corresponding kink-like and anti-kink-like link occupancy configurations, similarly to a particle - anti-particle annihilation process. Such collisions occur more often for an unoccupied site moving around in the spin effective lattice of the model on the 1D lattice, which corresponds to a single chain. As confirmed in Appendix A, the spin configurations obtained from such a cancellation play an important role in the \( s_1 \) bond-particle operator algebra.

Moreover, in this paper suitable \( s_1 \) bond-particle operators have been constructed, which upon acting onto the \( s_1 \) effective lattice obey a hard-core like algebra. Such a
property is valid for the subspaces in which the $s_1$ bond-particle operators act onto. It plays an important role in the related studies of Ref. [2]. There corresponding $s_1$ fermion operators are generated by means of an extended Jordan-Wigner transformation from the $s_1$ bond-particle operators introduced in this paper. Concerning previous studies on the large-$U$ Hubbard model and $t-J$ model on a square lattice involving for instance the slave particle formalism [16, 18, 22] or Jordan-Wigner transformations [12], the crucial requirement is to impose the single occupancy constraint. Here that constraint is naturally implemented for all values of $U/4t > 0$, since the spins associated with the spin $SU(2)$ state representations refer to the rotated electrons of the singly occupied sites. Moreover, for the above schemes the spinless fermions arise from individual spin-$1/2$ spins or spinons. In contrast, within the extended Jordan-Wigner transformation performed in Ref. [2] the $s_1$ fermions emerge from the spin-neutral two-spinon composite $s_1$ bond particles studied in this paper.

The studies of Ref. [2] confirm that the results obtained here concerning the Hubbard model on the square lattice in the one- and two-electron subspace are useful for the further understanding of the role of electronic correlations in the spin-wave spectrum observed in the parent compound La$_2$CuO$_4$ [1]. A system of weakly coupled planes, each described by the square-lattice quantum liquid of Ref. [2], perturbed by the effects of intrinsic disorder is expected to be the simplest realistic toy model for the description of the role of correlations effects in the unusual properties of the cuprate high-temperature superconductors [14, 15, 16, 17].

I thank Nuno M. R. Peres for discussions and support in the figures production. I also thank Miguel A. N. Araújo, Daniel Arovas, Pedro D. Sacramento, and Maria J. Sampaio for discussions and the support of the ESF Science Program INSTANS and grant PTDC/FIS/64926/2006.

Appendix A. Hard-core character of the $s_1$ bond-particle operators

The goal of this Appendix is to confirm the validity of the relations provided in Eqs. (41) and (42). In order to probe such relations, we consider without any loss of generality that the initial state is a $N^h_{s_1} = 4$ configuration state. Indeed and as discussed in Subsection 4-5, application of a $s_1$ bond-particle creation operator onto $N^h_{s_1} = 0, 1$ configuration states gives zero. Application of two $s_1$ bond-particle creation operators onto $N^h_{s_1} = 2$ configuration states gives zero as well. The studies on $N^h_{s_1} = 2$ configuration states of that subsection can be straightforwardly generalized to $N^h_{s_1} = 4$ configuration states. That assures that application onto those of four-site two-bond operator may not give zero. Fortunately, the final results reached here apply to any configuration state with a finite number $N^h_{s_1}$ of unoccupied sites in the $s_1$ effective lattice, yet are simpler to derive for the $N^h_{s_1} = 4$ configuration states.

Sub-configuration states of type 3 considered in Subsection 4-5 for $N^h_{s_1} = 2$ also exist for $N^h_{s_1} = 4$. Within those with $N^h_{s_1} = 4$ unoccupied sites, the importance of sub-configuration states whose link configurations have all the same sign stems from
those being the only ones for which application of s1 bond-particle creation operators does not give zero. For other $N_{\text{h}}^N_{\text{l}}$-finite sub-configuration states the s1 bond-particle operators have slightly more involved expressions given in Section 4. Fortunately, the use of the operator expressions suitable for sub-configuration states of type 3 leads to the same final results. Therefore, for simplicity we consider here the latter s1 bond-particle operators whose expressions are the same as for the $N_{\text{h}}^N_{\text{l}} = 0$ configuration state studied in Section 3.

All the results of this Appendix apply to the model on both the square and 1D lattices. The 1D expressions are readily obtained if one considers in the general expressions given below only $d = 1$ contributions and terms, together with the choice $D = 1$ in the $D$-dependent quantities. In order to reach our goal, let us recall that the rotated-electron operators are related to the original electron operators as follows,

$$\tilde{c}_{\vec{r}_j,\sigma} = \hat{V} c_{\vec{r}_j,\sigma} \hat{V}^\dagger,$$

where $\hat{V}$ is the electron - rotated-electron unitary operator. Straightforward manipulations based on Eqs. (5)-(7) then lead to the following algebra for the related c fermion operators [2],

$$\{f_{\vec{r}_j,\vec{c}}, f_{\vec{r}_j',\vec{c}}^\dagger\} = \delta_{j,j'}; \quad \{f_{\vec{r}_j,\vec{c}}^\dagger, f_{\vec{r}_j',\vec{c}}\} = \{f_{\vec{r}_j,\vec{c}}, f_{\vec{r}_j',\vec{c}}\} = 0,$$

(A.2)

and rotated-quasi-spin operators,

$$[q_{\vec{r}_j}^x, q_{\vec{r}_j'}^x] = [q_{\vec{r}_j}^x, q_{\vec{r}_j'}^y] = [q_{\vec{r}_j}^z, q_{\vec{r}_j'}^x] = 0,$$

(A.3)

for $j \neq j'$ and,

$$[q_{\vec{r}_j}^x, q_{\vec{r}_j'}^y] = \delta_{j,j'} 2q_{\vec{r}_j}^{2x}, \quad [q_{\vec{r}_j}^x, q_{\vec{r}_j'}^z] = 0.$$

(A.4)

Hence the rotated-quasi-spin operators $q_{\vec{r}_j}^\pm$ anticommute on the same site and commute on different sites.

Moreover, combining the expressions given in Eq. (6) with the algebraic relations provided in Eqs. (A.4)-(A.6) one readily finds that,

$$\{s_{\vec{r}_j}^-, s_{\vec{r}_j'}^+\} = 1, \quad \{s_{\vec{r}_j}^+, s_{\vec{r}_j'}^-\} = 0,$$

(A.7)

$$[s_{\vec{r}_j}^+, s_{\vec{r}_j'}^-] = [s_{\vec{r}_j}^-, s_{\vec{r}_j'}^+] = 0,$$

(A.8)

for $j \neq j'$.

It follows that the spinon operators $s_{\vec{r}_j}^\pm$ anticommute on the same site and commute on different sites. Consistently with the rotated-electron singly-occupied site projector $n_{\vec{r}_j,\vec{c}}$ appearing in the expression of the spinon operators $s_{\vec{r}_j}^\pm$ and $s_{\vec{r}_j}^z$ provided in Eq. (3), their real-space coordinates $\vec{r}_j$ can in the present $N_a^D \to \infty$ limit be identified with
those of the spin effective lattice. Therefore, the corresponding operator index values \( j = 1, ..., N_{\alpha} \) are chosen to be those of that lattice.

For the sub-configuration states of type 3 the operators \( g^{\dagger}_{j,s} \) (and \( g_{j,s} \)) which create (and annihilate) a \( s \) bond particle at a site of the spin effective lattice of real-space coordinate \( \bf r_j \) have the general form given in Eq. (24) both for the model on the 1D and square lattices, where the absolute value \( |h_g| \) of the coefficients \( h_g \) decreases for increasing link length \( \xi_g \) and obeys the normalization sum-rule (25). Hence the expression of the \( s \) bond particle operator \( g^{\dagger}_{j,s} \) involves the operators \( a^{\dagger}_{j,s} \) and \( a_{j,s} \) of Eq. (24), which create and annihilate, respectively, a superposition of \( 2D = 2, 4 \) two-site bonds of the same type and \( b^{\dagger}_{j,s} \) and \( b_{j,s} \) are two-site one-bond operators whose expression is given in Eq. (27).

In order to confirm the validity of Eqs. (11) and (12) let us use Eqs. (24)-(27) to rewrite the anti-commutation relations of Eq. (11) in terms of anti-commutators of two-site one-bond operators as follows,

\[
\{ g^{\dagger}_{j,s}, g_{j,s'} \} = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h_{g'} \left\{ b^{\dagger}_{j,s} b^{0}_{j',s'} a_{j,s} a^{\dagger}_{j',s'} \right\}, \quad (A.10)
\]

\[
\{ g^{\dagger}_{j,s}, g^{\dagger}_{j,s'} \} = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h^{*}_{g'} \left\{ b^{\dagger}_{j,s} b^{0}_{j',s'} a^{\dagger}_{j,s} a_{j',s'} \right\}, \quad (A.11)
\]

\[
\{ g_{j,s}, g_{j,s'} \} = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h_{g'} \left\{ b^{\dagger}_{j,s} b^{0}_{j',s'} a_{j,s} a^{\dagger}_{j',s'} \right\}, \quad (A.12)
\]

where for simplicity we used the abbreviated summation notation,

\[
\sum_{d,l,g} \equiv \sum_{d=1}^{D} \sum_{l=\pm 1} \sum_{g=0}^{N_{\alpha}/2D-1} . \quad (A.13)
\]

Moreover, by the use of the same equations the commutation relations of Eq. (12) can be expressed in terms commutators of two-site one-bond operators. That leads to,

\[
[g^{\dagger}_{j,s}, g_{j',s'}] = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h_{g'} \left\{ [b^{\dagger}_{j,s} b^{0}_{j',s'} a_{j,s} a^{\dagger}_{j',s'}] \right\}, \quad (A.14)
\]

\[
[g^{\dagger}_{j,s}, g^{\dagger}_{j',s'}] = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h^{*}_{g'} \left\{ [b^{\dagger}_{j,s} b^{0}_{j',s'} a^{\dagger}_{j,s} a_{j',s'}] \right\}, \quad (A.15)
\]

\[
[g_{j,s}, g_{j',s'}] = \sum_{d,l,g} \sum_{d',l',g'} h^{*}_g h_{g'} \left\{ [b^{\dagger}_{j,s} b^{0}_{j',s'} a_{j,s} a^{\dagger}_{j',s'}] \right\}, \quad (A.16)
\]

where \( j \neq j' \).

According to the studies of Subsection 3-3, four rules follow from the definition of the subspace where the operators of Eqs. (24)-(27) act onto. The evaluation of the anti-commutators and commutators of the two-site one-bond operators on the right-hand side of Eqs. (A.10)-(A.12) and (A.14)-(A.16), respectively, relies on both such rules and the algebra given in Eqs. (A.7)-(A.9) of the spinon operators \( s^{\pm}_{\bf r_j} \) and \( s_{\bf r_j} \) of Eq. (6), which are the building blocks of the two-site one-bond operators of Eq. (27).
Fortunately, according to Eqs. (A.7)- (A.9) the spinon operators $s^z_j$ obey the usual algebra: They anticommute on the same site of the spin effective lattice and commute on different sites.

The two-site one-bond operators of Eq. (27) can be rewritten as,

$$b_{\vec{r}_1, \vec{r}_2}^\dagger = \frac{(-1)^d-1}{\sqrt{2}} \left( \left[ \frac{1}{2} + s^z_{\vec{r}_1} \right] s^-_{\vec{r}_1} - \left[ \frac{1}{2} + s^z_{\vec{r}_2} \right] s^-_{\vec{r}_2} \right),$$  \hspace{1cm} (A.17)

and $b_{\vec{r}_1, \vec{r}_2} = \left( b_{\vec{r}_1, \vec{r}_2}^\dagger \right)^\dagger$ where recalling that the real-space coordinate of their link centre reads $\vec{r} = \vec{r}_2 + \vec{r}_{d,l}^0$ the real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ are given by,

$$\vec{r}_1 = \vec{r}_2 + \vec{r}_{d,l}^0 - \vec{r}_{d,l}^g; \hspace{0.5cm} \vec{r}_2 = \vec{r}_j + \vec{r}_{d,l}^0 + \vec{r}_{d,l}^{g'},$$  \hspace{1cm} (A.18)

The evaluation of the anti-commutators and commutators of the two-site one-bond operators on the right-hand side of Eqs. (A.10)-(A.12) and (A.14)-(A.16), respectively, then relies on straightforward manipulations based on Eqs. (A.7)-(A.9) and on the four rules given in Subsection 3.3, which define the subspace that the operator algebra under consideration refers to. For the two general anti-commutators needed to evaluate the two-site one-bond operators on the right-hand side of Eq. (A.10) we find the following expressions,

$$\{b_{\vec{r}_1, \vec{r}_2}, b_{\vec{r}_1, \vec{r}_2}^\dagger\} = \sum_{i=1,2} \left\{ \frac{1}{2} \left( \frac{1}{2} + s^z_{\vec{r}_1} \right)^2 - s^+_{\vec{r}_1} s^-_{\vec{r}_1} \left( \frac{1}{2} + s^z_{\vec{r}_1} \right) \left( \frac{1}{2} + s^z_{\vec{r}_1} \right) \right\},$$  \hspace{1cm} (A.19)

$$\{b_{\vec{r}_1, \vec{r}_2}, b_{\vec{r}_1', \vec{r}_2'}\} = (-1)^{d+d'} \sum_{i=1,2} \left\{ s^+_{\vec{r}_1, \vec{r}_2} s^-_{\vec{r}_1, \vec{r}_2} \left( \frac{1}{2} + s^z_{\vec{r}_1} \right) \left( \frac{1}{2} + s^z_{\vec{r}_1} \right) - s^+_{\vec{r}_1, \vec{r}_2} s^-_{\vec{r}_1, \vec{r}_2} \right\},$$  \hspace{1cm} (A.20)

where $1 \equiv 2, \tilde{2} \equiv 1$, $\vec{r}_1 \neq \vec{r}_1', \vec{r}_2'$, and $\vec{r}_2 \neq \vec{r}_1', \vec{r}_2'$. Indeed, according to the second rule for the subspace where the two-site one-bond operators act onto only general operators $b_{\vec{r}_1, \vec{r}_2}^\dagger b_{\vec{r}_1', \vec{r}_2'}$ which do not join sites or join both sites of the spin effective lattice lead to wanted and physical spin configurations.

Moreover, in the initial configuration that the four-site two-bond operators of the relation (A.19) act onto one has according to the first and third rules that the two sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ either (i) are occupied by two independent $+1/2$ spinons or (ii) are linked by a bond. In turn, the anti-commutator of Eq. (A.20) is of the form of those on the right-hand side of Eq. (A.10) so that in Eq. (A.18) one has that $j = j'$ yet $\vec{r}_{d,l}^g \neq \vec{r}_{d,l}^{g'}$ and the restrictions imposed by the forth rule must be taken into account. Since we find that $b_{\vec{r}_1, \vec{r}_2}^\dagger b_{\vec{r}_1', \vec{r}_2'} = b_{\vec{r}_1, \vec{r}_2'} b_{\vec{r}_1', \vec{r}_2}$ and each of such operators is given by one half the operator on the right-hand side of Eq. (A.20), when the two sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1'$ and $\vec{r}_2'$) are linked by a bond (and occupied by two independent $+1/2$ spinons) one must consider both initial configurations where
the sites of real-space coordinates $\vec{r}_1'$ and $\vec{r}_2'$ (and $\vec{r}_1$ and $\vec{r}_2$) are (i) linked by a bond and (ii) occupied by two independent $+1/2$ spinons. In turn, when the two sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1'$ and $\vec{r}_2'$) are occupied by two independent $+1/2$ spinons (and linked by a bond) one must consider only initial configurations where the sites of real-space coordinates $\vec{r}_1'$ and $\vec{r}_2'$ (and $\vec{r}_1$ and $\vec{r}_2$) are linked by a bond (and occupied by two independent $+1/2$ spinons).

It then follows from analysis of the operator expression on the right-hand side of Eq. (A.19) that when in the initial configuration the two sites $\vec{r}_1$ and $\vec{r}_2$ are occupied by two independent $+1/2$ spinons the operator term $\sum_{i=1,2}[1/2](1/2 + s_{\vec{r}_i}^z)^2$ transforms that configuration onto itself whereas the remaining operator terms give zero. In turn, when in the initial configuration the two sites $\vec{r}_1$ and $\vec{r}_2$ are linked and correspond to an one-bond configuration the operator terms $\sum_{i=1,2}[1/2]\{((1/2 + s_{\vec{r}_i}^z)^2 - s_{\vec{r}_i}^+ s_{\vec{r}_i}^- (1/2 + s_{\vec{r}_i}^z))\}$ transform that configuration onto itself whereas the remaining operator terms give zero.

On the other hand, when acting on the above initial configurations the operator on the right-hand side of Eq. (A.20) gives always zero so that when acting onto the subspace that the operators of Eqs. (24)-(27) refer to the anti-commutators provided in Eqs. (A.19) and (A.20) simplify and are given by,

$$\{b_{\vec{r}_1,\vec{r}_2}^\dagger, b_{\vec{r}_1,\vec{r}_2}\} = 1; \quad \{b_{\vec{r}_1,\vec{r}_2}^\dagger, b_{\vec{r}_1',\vec{r}_2'}\} = 0,$$

(A.21)

where $\vec{r}_1' \neq \vec{r}_1, \vec{r}_2'$ and $\vec{r}_2' \neq \vec{r}_1', \vec{r}_2$.

Next concerning the two general anti-commutators needed to evaluate the two-site one-bond operators on the right-hand side of Eq. (A.11) we find the following expressions,

$$\{b_{\vec{r}_1,\vec{r}_2}^\dagger, b_{\vec{r}_1',\vec{r}_2'}^\dagger\} = -2 \left( \frac{1}{2} + s_{\vec{r}_1}^z \right) \left( \frac{1}{2} + s_{\vec{r}_2}^z \right) s_{\vec{r}_1'}^- s_{\vec{r}_2'}^- - \sum_{i=1,2} \left( \frac{1}{2} + s_{\vec{r}_i}^z \right) s_{\vec{r}_i}^- s_{\vec{r}_i}^- \left( \frac{1}{2} + s_{\vec{r}_i}^z \right) \right), \quad (A.22)$$

$$\{b_{\vec{r}_1,\vec{r}_2}^\dagger, b_{\vec{r}_1',\vec{r}_2'}^\dagger\} = (-1)^{d+d'} \sum_{i=1,2} \left\{ \left( \frac{1}{2} + s_{\vec{r}_i}^z \right) \left( \frac{1}{2} + s_{\vec{r}_i'}^z \right) s_{\vec{r}_i}^- s_{\vec{r}_i}^- \right\}, \quad (A.23)$$

where as above $\bar{l} = 2, \bar{2} = 1, \vec{r}_1 \neq \vec{r}_1', \vec{r}_2'$, and $\vec{r}_2 \neq \vec{r}_1', \vec{r}_2'$.

It follows from analysis of the operator on the right-hand side of Eq. (A.22) that according to the first and third rules when in the initial spin configuration the sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ are both (i) occupied by independent $+1/2$ spinons and (ii) linked by a bond, application of that operator onto such a configuration gives zero. In turn the anti-commutator of Eq. (A.23) is of the form of those on the right-hand side of Eq. (A.11) so that in Eq. (A.13) one has that $j = j'$ yet $\vec{r}_{d,l}^g \neq \vec{r}_{d',l'}^g$ and then the forth rule applies. Since we find that $b_{\vec{r}_1,\vec{r}_2}^\dagger b_{\vec{r}_1',\vec{r}_2'}^\dagger = b_{\vec{r}_1',\vec{r}_2'}^\dagger b_{\vec{r}_1,\vec{r}_2}$ and each of such operators is given by one half the operator on the right-hand side of Eq. (A.23), concerning the latter operator when the sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1'$ and $\vec{r}_2'$) are linked by a bond one must consider both initial configurations where the sites of real-space
coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1$ and $\vec{r}_2$) are (i) linked by a bond and (ii) occupied by two independent +1/2 spinons. In turn, when the two sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1$ and $\vec{r}_2$) are occupied by two independent +1/2 spinons one must consider only initial configurations where the sites of real-space coordinates $\vec{r}_1'$ and $\vec{r}_2'$ (and $\vec{r}_1$ and $\vec{r}_2$) are linked by a bond \[2\]. It follows then from analysis of the operator on the right-hand side of Eq. (A.23) that application of it onto any of such spin configurations gives zero.

A similar analysis for the two general anti-commutators needed to evaluate the two-site one-bond operators on the right-hand side of Eq. (A.12) leads to,

$$\{b_{\vec{r}_1,\vec{r}_2}, b_{\vec{r}_1',\vec{r}_2'}\} = -2s_{\vec{r}_1}^+ s_{\vec{r}_2}^+ \left(\frac{1}{2} + s_{\vec{r}_1}^z\right) \left(\frac{1}{2} + s_{\vec{r}_2}^z\right) - \sum_{i=1,2} s_{\vec{r}_i}^+ s_{\vec{r}_i'}^+ \left(\frac{1}{2} + s_{\vec{r}_i}^z\right) , \quad (A.24)$$

$$\{b_{\vec{r}_1,\vec{r}_2}, b_{\vec{r}_1',\vec{r}_2'}\} = (-1)^{d+d'} \sum_{i=1,2} \left\{s_{\vec{r}_i}^+ s_{\vec{r}_i'}^+ \left(\frac{1}{2} + s_{\vec{r}_i}^z\right) \left(\frac{1}{2} + s_{\vec{r}_i'}^z\right) - s_{\vec{r}_i}^+ s_{\vec{r}_i'}^+ \left(\frac{1}{2} + s_{\vec{r}_i}^z\right) \left(\frac{1}{2} + s_{\vec{r}_i'}^z\right)\right\} , \quad (A.25)$$

where as above $\bar{I} = 2, \bar{2} = 1, \vec{r}_1 \neq \vec{r}_1', \vec{r}_2', \text{ and } \vec{r}_2 \neq \vec{r}_1', \vec{r}_2'$.

Again analysis of the operator on the right-hand side of Eq. (A.24) reveals that according to the first and third rules when in the initial spin configuration the sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ are both (i) occupied by independent +1/2 spinons and (ii) linked by a bond, application of that operator onto such a configuration gives zero. On the other hand, the anti-commutator of Eq. (A.25) is of the form of those on the right-hand side of Eq. (A.11) so that in Eq. (A.18) one has that $j = j'$ yet $\vec{r}_{d,l}^0 \neq \vec{r}_{d',l'}^0$ and then the forth rule applies. Since we find that $b_{\vec{r}_1,\vec{r}_2} b_{\vec{r}_1',\vec{r}_2'} = b_{\vec{r}_1',\vec{r}_2} b_{\vec{r}_1,\vec{r}_2}$ and each of such operators is given by one half the operator on the right-hand side of Eq. (A.23), concerning the latter operator when the sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1'$ and $\vec{r}_2'$) are occupied by two independent +1/2 spinons one must consider both initial configurations where the sites of real-space coordinates $\vec{r}_1'$ and $\vec{r}_2'$ (and $\vec{r}_1$ and $\vec{r}_2$) are (i) linked by a bond and (ii) occupied by two independent +1/2 spinons. In turn, when the two sites of real-space coordinates $\vec{r}_1$ and $\vec{r}_2$ (and $\vec{r}_1'$ and $\vec{r}_2'$) are linked by a bond one must consider only initial configurations where the sites of real-space coordinates $\vec{r}_1'$ and $\vec{r}_2'$ (and $\vec{r}_1$ and $\vec{r}_2$) are occupied by two independent +1/2 spinons \[2\]. Analysis of the operator on the right-hand side of Eq. (A.23) then reveals that application of it onto any of such spin configurations gives zero.

It then follows from the above results that when acting onto the subspace that the operators of Eqs. (24)-(27) (and (A.23) and Eqs. (A.24) and (A.25) simplify and read,

$$\{b^+_{\vec{r}_1,\vec{r}_2}, b^+_{\vec{r}_1',\vec{r}_2'}\} = \{b^+_{\vec{r}_1,\vec{r}_2}, b^+_{\vec{r}_1',\vec{r}_2'}\} = 0 ; \quad \{b_{\vec{r}_1,\vec{r}_2}, b_{\vec{r}_1',\vec{r}_2'}\} = \{b_{\vec{r}_1,\vec{r}_2}, b_{\vec{r}_1',\vec{r}_2'}\} = 0 , \quad (A.26)$$

where $\vec{r}_1 \neq \vec{r}_1', \vec{r}_2'$ and $\vec{r}_2 \neq \vec{r}_1', \vec{r}_2'$. 



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The use in Eq. (A.10) of the anti-commutators of Eq. (A.21) with the two-site one-bond operators related to those of Eq. (27) by the expressions provided in Eqs. (A.17) and (A.18) leads to,

$$\{g_{\vec{r}_j,s_1}^{\dagger}, g_{\vec{r}_j,s_1}\} = \sum_d \sum_l |h_d|^2 = 2D \sum_g |h_g|^2 = 1,$$

which is the first relation of Eq. (41). To perform the summation of Eq. (A.27) the sum-rule (25) was used. Furthermore, the use of the anti-commutators of Eq. (A.26) in Eqs. (A.11) and (A.12) leads to the remaining relations of Eq. (41).

The evaluation of the commutators of Eq. (42) by the use of the expressions given in Eqs. (A.14)-(A.16) is much simpler. First it is simplified by the property that two-site one-link bonds belonging to $s_1$ bond-particle operators with different real-space coordinates are always different. Second the evaluation of such commutators also relies on straightforward manipulations based on Eqs. (A.7)-(A.9), which lead directly to,

$$[b_{\vec{r}_1,\vec{r}_2}^{\dagger}, b_{\vec{r}_1,\vec{r}_2}^{\dagger}, b_{\vec{r}_1,\vec{r}_2}^{\dagger} b_{\vec{r}_1,\vec{r}_2}^{\dagger}] = [b_{\vec{r}_1,\vec{r}_2}^{\dagger}, b_{\vec{r}_1,\vec{r}_2}^{\dagger} b_{\vec{r}_1,\vec{r}_2}^{\dagger}] = [b_{\vec{r}_1,\vec{r}_2}^{\dagger}, b_{\vec{r}_1,\vec{r}_2}^{\dagger}] = 0,$$

for $\vec{r}_1 \neq \vec{r}_1', \vec{r}_2'$ and $\vec{r}_2 \neq \vec{r}_1', \vec{r}_2'$.

Finally, the use in the expressions of the $s_1$ bond-particle operators of Eq. (42) of the commutators of Eq. (A.28) with the two-site one-bond operators related to those of Eq. (27) by the expressions provided in Eqs. (A.17) and (A.18) in Eqs. (A.14)-(A.16) leads readily to the commutation relations provided in Eq. (42).

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