Some exact non-vacuum Bianchi VI$_0$ and VII$_0$ instantons

John D. Barrow$^1$, Yves Gaspar$^{1†}$ & P. M. Saffin$^2‡$

$^1$Astronomy Centre, University of Sussex, Brighton BN1 9QJ, UK
$^2$DAMTP, Silver Street, Cambridge CB3 9EW, UK

July 6, 2018

Abstract

We report some new exact instantons in general relativity. These solutions are Kähler and fall into the symmetry classes of Bianchi types VI$_0$ and VII$_0$, with matter content of a stiff fluid. The qualitative behaviour of the solutions is presented, and we compare it to the known results of the corresponding self-dual Bianchi solutions. We also give axisymmetric Bianchi VII$_0$ solutions with an electromagnetic field.

1 Introduction

Solutions to field equations in imaginary time have been of interest for many years and early work formalised the relevance of such systems to tunnelling problems [1] [4]. Much of the current interest in instantons dates from the work of Belavin, Polyakov, Shvarts, and Tyupkin (BPST) [3], where an exact

$^*\text{E-mail: J.D.Barrow@sussex.ac.uk}$
$^†\text{E-mail: yfj-mg@star.cpes.susx.ac.uk}$
$^‡\text{E-mail: P.M.Saffin@damtp.cam.ac.uk}$
solution to the SU(2) Yang Mills equations in imaginary time was found. The relevance of these solutions was then clarified by t’Hooft [4] and by Jackiw and Rebbi [5] who showed that the BPST instanton solution is a mediator for the transition between different vacua of the gauge theory.

The BPST instanton was found by looking for solutions with (anti) self-dual field strength. This led to the suggestion that analogous solutions may exist in gravity [6], where the duality governs the curvature two-form. The usefulness of such a duality is that it not only makes the curvature automatically solve Einstein’s vacuum equations [7], but also reduces the equations to first order. A geometry which possesses a symmetry also simplifies the problem. An important class of symmetries is the Bianchi classification of spatially homogeneous 3-geometries[8]. Some solutions of this type with the property of self-duality are known and can be found in reference [9]; we shall extend the collection of known solutions by finding metrics with the symmetry of Bianchi types VI₀ and VII₀ that satisfy Einstein’s equations with a stiff fluid, that is where the pressure equals the energy density. This means that we cannot use the simplification of self-duality, however we do use it as a guide to find an initial ansatz. We shall also see that this ansatz allows a natural complex structure such that the metric is Kähler.

The importance of gravitational instantons emerges in the path-integral formalism of quantum gravity. In this approach one considers a weighted sum over all Euclidean metrics, where the weight is the exponential of the Euclidean action. Instantons, as solutions to the classical field equations, are expected to provide the dominant contribution to such a sum, and this allows a saddle-point approximation to be used in the neighbourhood of these solutions.

The organisation of the paper is as follows. First we define the system of equations to be solved, and describe the self-dual vacuum solution. We solve the Bianchi VII₀ Einstein equations in the presence of a stiff fluid. Next, we follow Misner [10] by writing the action in the familiar form of a particle in a potential, in order to describe the qualitative nature of our solution. The Bianchi VI₀ solution follows from the Bianchi VII₀ results.
2 The Bianchi VII\(_0\) equations

The Bianchi universes are four-manifolds with embedded three geometries which are spatially homogeneous but anisotropic; they each possess a three-dimensional, simply-transitive, isometry group \([8]\). To exploit this isometry we write the metric using the left-invariant one-forms of the symmetry group, \(\sigma^a\); as we are interested in instantons, we use a Riemannian signature,

\[
ds^2 = d\bar{\tau}^2 + a^2(\bar{\tau})(\sigma^1)^2 + b^2(\bar{\tau})(\sigma^2)^2 + c^2(\bar{\tau})(\sigma^3)^2.
\] (1)

The left-invariant one-forms satisfy the structure equations,

\[
d\sigma^a = \frac{1}{2}C^a_{\ bc}\sigma^b \wedge \sigma^c,
\] (2)

For the case of Bianchi VII\(_0\), the structure constants take the values

\[
C^1_{\ 32} = -C^1_{\ 23} = C^2_{\ 13} = -C^2_{\ 31} = 1.
\] (3)

These metrics support Killing vectors (right-invariant vector fields) with the same structure constants as the Euclidean group in two dimensions. The self-dual solution will be of use to us in constructing an ansatz, so we shall calculate the torsion-free connection forms, \(\Theta^{ab}\), assuming they are metric connections, \(\Theta_{ab} = -\Theta_{ba}\). Thus, we rewrite the metric in terms of a new time variable \(d\bar{\tau} = abcd\tau\) and introduce an orthonormal basis,

\[
ds^2 = a^2(\tau)b^2(\tau)c^2(\tau)d\tau^2 + a^2(\tau)(\sigma^1)^2 + b^2(\tau)(\sigma^2)^2 + c^2(\tau)(\sigma^3)^2
\] (4)

\[
\omega^0 = a(\tau)b(\tau)c(\tau)d\tau, \ \omega^1 = a(\tau)\sigma^1, \ \omega^2 = b(\tau)\sigma^2, \ \omega^3 = c(\tau)\sigma^3.
\]

We find the following non-vanishing connection forms:

\[
\theta^0_{\ 1} = -\alpha'/(abc)\omega^1, \ \theta^1_{\ 2} = -(a^2 + b^2)/(2abc)\omega^3
\]

\[
\theta^0_{\ 2} = -\beta'/(abc)\omega^2, \ \theta^1_{\ 3} = (a^2 - b^2)/(2abc)\omega^2
\]

\[
\theta^0_{\ 3} = -\gamma'/(abc)\omega^3, \ \theta^2_{\ 3} = -(a^2 - b^2)/(2abc)\omega^1,
\] (5)

where we have defined \(\alpha = \ln(a), \beta = \ln(b), \gamma = \ln(c)\) and ' refers to differentiation with respect to \(\tau\). We now assume the connection to be self dual, which in turn implies that the curvature is self dual \([7]\), so \(\Theta_{01} = \Theta_{23}\) plus permutations.

\[
\alpha' = -\frac{1}{2}(a^2 - b^2), \ \beta' = \frac{1}{2}(a^2 - b^2), \ \gamma' = \frac{1}{2}(a^2 + b^2).
\] (6)
These equations show that $a(\tau)b(\tau)$ is a constant, and give a metric,

$$ds^2 = \frac{\lambda^2}{2} \sinh(2\tau) \left( d\tau^2 + (\sigma^3)^2 \right) + \coth(\tau)(\sigma^1)^2 + \tanh(\tau)(\sigma^2)^2,$$

(7)

where $\lambda$ is an integration constant.

We now derive a non-vacuum solution, (which requires that self duality is lost). However, we use the self-dual solution to lead us to the ansatz of $a(\tau) = 1/b(\tau)$. In order to show that this solution does retain the Kähler property, the metric is rewritten by introducing the complex forms $\zeta$ defined by

$$\zeta^1 = \omega^0 + i\omega^3$$

(8)

$$\zeta^2 = \omega^1 + i\omega^2,$$

(9)

where the $\omega^\alpha$ are those defined in (3). The metric now takes the form

$$ds^2 = \zeta^1 \zeta^\bar{1} + \zeta^2 \zeta^\bar{2}. $$

(10)

The Kähler form is then found to be

$$\Omega_K = ig_{\alpha\bar{\beta}} \zeta^\alpha \wedge \zeta^\bar{\beta}$$

$$= 4abc^2 d\tau \wedge \sigma^3 + 4ab\sigma^1 \wedge \sigma^2$$

(11)

(12)

The closure of the Kähler form, $d\Omega_K = 0$, then requires $(ab)' = 0$, so our ansatz leads to a Kähler manifold.

We now want to consider the full Bianchi VII\textsubscript{0} equations, without requiring self duality. These can be found in refs. [11], [12] where we are now using a Lorentzian signature,

$$ds^2 = a^2(t)b^2(t)c^2(t)dt^2$$

$$-a^2(t)(\sigma^1)^2 - b^2(t)(\sigma^2)^2 - c^2(t)(\sigma^3)^2 = 0$$

(13)

(14)

(15)

(16)

$$\ln((abc)^2) = 4 ((\ln c)(\ln ab) + (\ln a)(\ln b)) + 2\epsilon (abc)^2.$$ (17)
where $\epsilon \geq 0$ is the energy density, and $\frac{\partial}{\partial t}$ denotes differentiation with respect to $t$. Now we take the hint from the self-dual solution and assume $a(t) = 1/b(t)$, but we do not take $\epsilon$ to be zero. Then equations (14,15,16) become,

\[
\frac{d^2 y}{dx^2} = -\sinh(y) \tag{18}
\]

and

\[
\frac{d^2 h}{dx^2} = \sinh^2(y/2), \tag{19}
\]

where we have used $x = 2t$, $a^2 = \exp(y/2)$ and $c^2 = \exp(h)$. Equation (18) can be solved to give,

\[
\cosh(y(x)) = 1 + 2k^2 k'^2 \operatorname{cn}^2(x/k' + \alpha \mid k^2), \tag{20}
\]

where $k$ is the modulus of the elliptic function, $k'$ is the complementary modulus, and $\alpha$ is an arbitrary constant. Our notation conforms to that of [13]. With (20) substituted into (18) we find

\[
\frac{d^2 h}{dx^2} = k^2 k'^2 \operatorname{cn}^2(x/k' + \alpha \mid k^2). \tag{21}
\]

We can integrate this equation once by noting that the elliptic integral $E$ satisfies,

\[
\frac{1}{k'} \frac{d}{dx} E \left( \text{am}(x/k' + \alpha \mid k^2) \right) = 1 + \left( \frac{k}{k'} \right)^2 \operatorname{cn}^2(x/k' + \alpha \mid k^2), \tag{22}
\]

allowing us to write (21) as

\[
\frac{dh}{dx} = \frac{1}{k'} E \left( \text{am}(x/k' + \alpha \mid k^2) \right) - x + C_1. \tag{23}
\]

A change of variables to $\xi(x)$, and the introduction of a new function $h(x)$ defined by,

\[
h(x) = h_1(x) + \int dx (-x + C_1), \quad \xi = \text{am}(x/k' + \alpha \mid k^2) \tag{24}
\]

means that we have

\[
\frac{dh_1}{d\xi} = \frac{E(\xi \mid k^2)}{\sqrt{1 - k^2 \sin^2(\xi)}} \tag{25}
\]

and

\[
h_1(\xi) = F^2(\xi \mid k^2) E(k)/(2K(k)) + \ln \left( \Theta[F(\xi \mid k^2)]/\Theta[0] \right). \tag{26}
\]
The integration has been done using equation (630.02) of [14]. This further simplifies when we realise $F(\xi | k^2) = x/k' + \alpha$. The conservation of the energy-momentum tensor, $T^{\mu\nu} = 0$, gives the energy density $\epsilon$ as

$$\epsilon = (\beta/(abc))^2 = (\beta/c)^2,$$

where $\beta$ is a constant. The fourth Einstein equation [14] reduces to

$$\ddot{h} = - \left( (\dot{y})^2/4 + 4\beta^2 \right).$$

(28)

But we already know $y(t)$ is from (20), and the left-hand side comes from (21), so we have the two consistency relations

$$k'^2 + k^2 = 1, \quad k^2/k'^2 = -\beta^2.$$  

(29)

In what follows, we shall use the value $0 < \beta < 1$, with $k = i\beta/\sqrt{1 - \beta^2}$ and $k' = 1/\sqrt{1 - \beta^2}$. Now that we have a relation between $\beta$ and $k$ we can rewrite (20) using some elliptic function identities. We also make use of the fact that $\alpha$ is an arbitrary integration constant which just shifts the origin of the solution. It proves convenient to choose $\alpha = \sqrt{1 - \beta^2} K(\beta)$, turning (20) into (see appendix A)

$$\cosh(y(t)) = 1 - 2\beta^2 \text{sn}^2(2t | \beta^2).$$

(30)

This is where we make a connection with the instanton solution. It is seen that if we take $t$ to be an imaginary parameter, $t = i\tau$, then the right-hand side of (30) becomes greater than or equal to one, meaning that $y(\tau)$ is real.

$$\cosh(y(\tau)) = 1 + 2\beta^2 \text{sn}^2(2\tau | 1 - \beta^2) / \text{cn}^2(2\tau | 1 - \beta^2).$$

(31)

To proceed then, we are required to do an analytic continuation on (20) from $t$ to $\tau$ (see appendix A).

$$h(\tau) = 2\tau^2(1 - E(\beta)/K(\beta)) - \pi\tau^2 / \left( K(\beta) K(\sqrt{1 - \beta^2}) \right) + \ln[\Theta(2\tau | 1 - \beta^2)] + C_1\tau + C_2$$

(32)

This completes the solution for the Bianchi VII_0 instanton with a stiff fluid. The solution is defined in the range $K(\sqrt{1 - \beta^2}) < 2\tau < K(\sqrt{1 - \beta^2})$, with $c(\tau)$ going to zero at these limits. At these points then we see that the volume of the homogeneous 3-surfaces goes to zero, defining the end-points of the instanton. It can also be seen that the curvature of the instanton diverges at these points.
3 Qualitative nature of the solution

Let us now examine the system of equations (14-17) in order to obtain an effective action. Using imaginary time, \( \tau \), we see that these equations can be derived by extremising the action,

\[
S_{\text{eff}} = \int d\tau \left( \alpha' \beta' + \beta' \gamma' + \alpha' \gamma' - \frac{1}{4}(a^2 - b^2)^2 \right).
\]

Again, we are using \( \alpha = \ln(a) \), \( \beta = \ln(b) \), \( \gamma = \ln(c) \) and \( ' \) for differentiation with respect to \( \tau \). The form of this action coincides with the Einstein-Hilbert action for the Bianchi VII\( _0 \) metric, found by calculating the curvature of the connection forms (3) so that this effective action is proportional to the gravitational action, as expected,

\[
I_{\text{grav}} = -\frac{1}{16\pi} \int_M \sqrt{g} R d^4 x + \text{boundary terms.}
\]

The boundary terms are designed to cancel the second derivatives coming from the Ricci scalar term, (15), (16), in order that the equations of motion derive from the variational principle even on a boundary. The next step is to put the effective action (33) into a familiar form, so we can visualise out solution. This is achieved by using the following Misner variables (17),

\[
\begin{align*}
a(\tau) &= \exp(\Omega(\tau) + \beta_+(\tau) + \sqrt{3} \beta_-(\tau)) \\
b(\tau) &= \exp(\Omega(\tau) + \beta_+(\tau) - \sqrt{3} \beta_-(\tau)) \\
c(\tau) &= \exp(\Omega(\tau) - 2 \beta_+(\tau))
\end{align*}
\]

We then find that the effective action, in terms of the new functions, takes the familiar form for a particle moving in a potential well,

\[
S_{\text{eff}} = -3 \int d\tau \left( \beta_+^2 + \beta_-^2 - \Omega^2 + \frac{1}{3} \exp(4\Omega) \exp(4\beta_+) \sinh^2(2\sqrt{3} \beta_-) \right)
\]

The \( \beta_\pm \) functions evolve to create the time-dependent potential \( \mathcal{V}(\Omega, \beta_+, \beta_-) = \exp(4\Omega) \mathcal{U}(\beta_+, \beta_-) \). Fig. 4 illustrates the \( \Omega \)-independent part of the potential, with a ridge along the \( \beta_+ \) axis. The behaviour of the two solutions outlined above, the self-dual and stiff fluid instantons, are given in Fig. 5. The figure shows the self-dual solution (paths I_1 and I_2) evolves along the ridge and then falls off at some point determined by the arbitrary parameter \( \lambda \) of the solution, (4). The solution for the stiff fluid contains three parameters, \( k, C_1 \) and \( C_2 \). The general behaviour is shown in Fig. 6, which shows the solution evolving toward and then rolling over the ridge.
4 Bianchi $VI_0$ solution

The Bianchi $VI_0$ structure constants differ from Bianchi $VII_0$ (3) by a single minus sign:

$C_{32}^1 = -C_{23}^1 = -C_{13}^2 = C_{31}^2 = 1$.  \hspace{1cm} (37)

This changes the symmetry group from the two dimensional Euclidean group, $E(2)$, to $E(1,1)$, and also changes one of the equations of motion, (16). Using the same metric as (4), but with the left-invariant one forms of $E(1,1)$, we find the new equation,

$$\left(\ln(c^2)\right)' - (a^2 - b^2)^2 = 0. \hspace{1cm} (38)$$

Again we make the ansatz $a(t) = 1/b(t)$ giving

$$\frac{d^2 h}{dx^2} = 1 + \sinh^2(y/2), \hspace{1cm} (39)$$

where we have again taken $x = 2t$, $a^2 = \exp(y/2)$ and $c^2 = \exp(h)$. Comparing equations (19) and (39), we see that the solution for $h(x)$ in the Bianchi $VI_0$ case is the same as for the Bianchi $VII_0$ case, except that there is an addition of $1/2 x^2$. However, the Bianchi $VI_0$ solution differs in other respects. By looking at the fourth Einstein equation, (14), we find a different relation between $k$, $k'$ and the parameter $\beta$ of (27). Using (28) and the new $h(t)$ we may derive,

$$k'^2 + k^2 = 1, \hspace{0.5cm} k^2/k'^2 = -(1 + \beta^2). \hspace{1cm} (40)$$

Using methods similar to the previous analysis one may then derive,

$$\cosh(y(\tau)) = 1 + 2(1 + \beta^2)\mathrm{cs}^2 \left(2\sqrt{1 + \beta^2} \tau \mid \frac{\beta^2}{1 + \beta^2}\right) \hspace{1cm} (41)$$

$$h(\tau) = -2 \left(1 - E(\beta/\sqrt{1 + \beta^2})/K(\beta/\sqrt{1 + \beta^2})\right) \tau^2 \hspace{1cm} (42)$$

$$+ \ln \vartheta_1 \left(\frac{\sqrt{1 + \beta^2} \pi}{K(\beta/\sqrt{1 + \beta^2})} \tau \mid \frac{\beta^2}{1 + \beta^2}\right) + C_1 \tau + C_2$$
This instanton is defined in the range $0 < \tau < E(\beta/\sqrt{1 + \beta^2})/\sqrt{1 + \beta^2}$. Following the procedure outlined for the Bianchi VII$_0$ case, we may also derive the self-dual solution,

$$ds^2 = \frac{\lambda^2}{2} \sin(2\tau) \left( d\tau^2 + (\sigma^3)^2 \right) + \cot(\tau)(\sigma^1)^2 + \tan(\tau)(\sigma^2)^2,$$

(43)

The Hamiltonian method used to describe the qualitative behaviour may also be adapted. On doing so we find that the solution is described by the effective action as (36), only now the $\sinh^2(2\sqrt{3}\beta_-)$ gets replaced by $\cosh^2(2\sqrt{3}\beta_-)$, using the same allocations as (35). The effective potential for the $\beta_{\pm}$ is thus qualitatively the same as Fig. 1. The paths that this solution describes in the $\beta_{\pm}$ plane are shown in Fig. 3, where the examples of self-dual solutions are labelled $I_1$ and $I_2$, with $a$, $b$ and $c$ being stiff fluid examples.

5 Bianchi VII$_0$ solution with electromagnetic field

In this section we give an analytic solution of Bianchi type VII$_0$ in the presence of an electromagnetic two-form satisfying Maxwell’s relations. Using the orthonormal basis for the metric we write Maxwell’s relations as,

$$dF = 0, \quad d^*F = 0$$

(44)

which are valid here because there are no charged sources. We consider here a two-form field strength of the form

$$F = E\omega^0 \wedge \omega^3 + B\omega^1 \wedge \omega^2,$$

(45)

representing an electric and a magnetic field in the ‘3’ direction. One quickly sees then that (44) leads to both $E$ and $B$ being constant for the Bianchi VII$_0$ metric. When it comes to the field equations for the metric coefficients we have only a minor change. The right-hand side of (14-16) no longer vanishes but becomes $\eta Uc^2$, where $\eta$ is +1 for (14), (15) and −1 for (16). The constant $U$ that appears is given by $U = (E)^2 + (B)^2$ (see [18]). There is a clear choice of ansatz for these equations, namely the biaxial ansatz with $a = b$. Taking this ansatz leads to

$$(\ln a^2)'' = -Uc^2$$

(46)

$$(\ln c^2)'' = Uc^2,$$

(47)
and with the definition \( c^2 = \exp(\theta) \) the equation for \( c \) gives,

\[
\theta'' = U \exp(\theta),
\]

(48)

taking note that we are now using Euclidean time, \( \tau = it \). We can trivially perform the first integral of (48) by multiplying both sides by \( \theta' \) giving,

\[
\int \frac{d\theta}{\sqrt{D^2 + 2U \exp(\theta)}} = \int d\tau.
\]

(49)

Integrating this and using (46) we find

\[
c^2(\tau) = \frac{(x^2 - D^2)}{2U}
\]

(50)

\[
a^2(\tau) = \exp(C_1 \tau + C_2) \frac{2U}{x^2 - D^2}
\]

(51)

\[
x = D\frac{1 + \exp(\pm D(\tau + \tau_0))}{1 - \exp(\pm D(\tau + \tau_0))},
\]

(52)

with \( C_1, C_2, D \) and \( \tau_0 \) all constants of integration. The 4th Einstein field equation (eq.(15) in [18]) gives the constraint \( C_1 = D \). The qualitative behaviour of this electromagnetic Bianchi VII\(_0\) model follows from the previous discussion. The extra terms \( \eta U c^2 \) in the equations of motion correspond to the addition of an extra term \( \frac{1}{2} U c^2 \) in the effective action (33), which has the effect of tilting the potential in (36) in the \( \beta_+ \) direction. The trajectory in \((\beta_+, \beta_-)\) space is clear in this biaxial example, \( \beta_- = 0 \) so the solution rolls along the axis.

### 6 Conclusions

Exact solutions of the Einstein equations are only possible in the presence of symmetries or structural specialisations. When the metric signature is Riemannian it is possible to find vacuum solutions when self duality is combined with three-dimensional homogeneity. We have used the structure of these solutions to find a number of non-vacuum solutions with stiff matter and electromagnetic field sources when self duality is necessarily relaxed. These solutions are of Bianchi types VI and VII.
Acknowledgements

We would like to acknowledge helpful discussions with Gary Gibbons and Stephen Siklos. JDB was supported by a PPARC Senior Fellowship, and PMS was partially funded by PPARC.

A Appendix

In order to transform (20) to (31),
\[
\cosh(y) = 1 - 2\beta^2\text{cn}^2 \left( \frac{2t}{k'} + \alpha \mid -\frac{\beta^2}{1 - \beta^2} \right)
\] (53)
we use the transformation of elliptic functions for negative parameter, as found in [13].

\[
\cosh(y) = 1 - 2\beta^2\text{cd}^2 \left( \frac{2t}{k'} + \alpha \right) \sqrt{1 - \beta^2} \mid \beta^2 \right). \] (54)

We use the fact that \( \alpha \) is arbitrary and define, \( \alpha/\sqrt{1 - \beta^2} = K(\beta) \), and find
\[
\cosh(y) = 1 - 2\beta^2\text{sn}^2(2t \mid \beta^2). \] (55)

The final step is to perform the continuation \( t = i\tau \) which gives us (31).

Here we continue equation (25) to imaginary time \( t = i\tau \),
\[
h(t) = \left( \frac{2t}{k'} + \alpha \right)^2 E(k)/(2K(k)) + \ln \left( \frac{\Theta[2t/k' + \alpha \mid k^2]/\Theta[0]}{2t^2 + C_1 t + C_2} \right). \] (56)

We use the definition given by Cayley of the theta function, [19],
\[
\frac{\Theta(u \mid k^2)}{\Theta(0 \mid k^2)} = \exp \left[ \frac{1}{2}u^2 \left( 1 - E(k)/K(k) \right) - k^2 \int_0^u dz \int_0^z d\text{ysn}^2(y \mid k^2) \right]. \] (57)

\[
\Rightarrow h(t) = \frac{1}{2} \left( \frac{2t}{k'} + \alpha \right)^2 - 2t^2 + C_1 t + C_2 \] (58)
\[-k^2 \int_0^{2t/k' + \alpha} dz \int_0^z d\text{ysn}^2(y \mid k^2) \]

Define the double integral in (57) to be,
\[
I_1 = \int_0^{2t/k' + \alpha} dz \int_0^z d\text{ysn}^2(y \mid k^2). \] (59)
Change \( t \) to \( i \tau \) and make two changes of variables, \( z = 2iv/k' + \alpha \) and \( y = 2i\xi/k' + \alpha \) to obtain

\[
I_1(\tau) = A\tau + B - 4/k'^2 \int_0^\tau dv \int_0^v d\xi \frac{1}{\mathrm{cn}^2(2\xi | 1 - \beta^2)}.
\]  \hspace{1cm} (60)

In deriving this we have used the relation between \( k, k' \) and \( \beta \) \(^{(29)}\); note that \( k \) is imaginary. The double integral can be rewritten in terms of theta functions by making the following change of variables, \( \xi = iu/2 \) and \( v = iy/2 \). This leaves us with a double integral which may be related to a theta function with imaginary argument. This can be expressed in terms of a theta function with real argument using a result in \(^{(19)}\), and hence we obtain \(^{(32)}\).

References

[1] D. McLaughlin, *J. Math. Phys.* \textbf{13}, 1099 (1972).

[2] S. Coleman, *Phys. Rev.* \textbf{D 15} 2929 (1977).

[3] A.A Belavin, A.M. Polyakov, A.S. Shvarts, Yu.S. Tyupkin, *Phys.Lett. B* \textbf{59} (1975).

[4] G. t’Hooft, *Phys. Rev.* \textbf{D14} 3432 (1976), erratum \textit{ibid} \textbf{D18} 2199 (1978), *Phys. Rev. Lett.* \textbf{37} 172 (1976).

[5] R. Jackiw and C. Rebbi, *Phys. Rev. Lett.* \textbf{37} 8 (1976).

[6] V. Belinskii, G. Gibbons, D. Page and C. Pope, *Phys. Lett. B* \textbf{76} 433 (1978).

[7] T. Eguchi, P. Gilkey and A. Hanson, *Phys. Repts.* \textbf{66}, 213, (1980).

[8] M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmologies*, Princeton UP, Princeton, (1975).

[9] D. Lorenz-Petzold, *Prog. Theor. Phys.* \textbf{81}, 17 (1989).

[10] C. Misner, *Magic without Magic*, ed. J. Klauder, W.H. Freeman, San Francisco (1972).
[11] V. Lukash, *Sov. Astron.* 18, 164 (1974).

[12] D. Lorenz-Petzold, *Acta Phys. Pol. B* 15, 117 (1984).

[13] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, NY, (1984).

[14] P.F. Byrd and M. Friedmann, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer, NY (1971).

[15] G. Gibbons and S. Hawking, *Phys. Rev. D* 15, 2752 (1977).

[16] J.D. Barrow and M. Madsen, *Nucl. Phys. B* 323, 242 (1989).

[17] G. Gibbons and C. Pope, *Commun. Math. Phys.* 66, 267 (1979).

[18] B.L. Spokoiny, *Gen. Relativity and Gravitation*, 14, 279 (1982).

[19] A. Cayley, *An Elementary Treatise on Elliptic Functions*, Dover, NY, (1961).
Figure 1: $\mathcal{U}(\beta_+, \beta_-)$
Figure 2: Various Bianchi VII\(b\) instanton solutions in the \(\beta_+ - \beta_-\) plane. The \(I_{1,2}\) are examples of self dual solutions, with curves a, b, c being three stiff fluid instantons.
Figure 3: Various Bianchi VI\(_0\) instanton solutions in the \(\beta_+ - \beta_-\) plane. The \(I_1, 2\) are examples of self dual solutions, with curves a, b, c being three stiff fluid instantons.