HYPERFIELDS FOR TROPICAL GEOMETRY I.
HYPERFIELDS AND DEQUANTIZATION

OLEG VIRO

Abstract. New hyperfields, that is fields in which addition is multivalued, are introduced and studied. In a separate paper these hyperfields are shown to provide a base for the tropical geometry. The main hyperfields considered here are classical number sets, such as the set $\mathbb{C}$ of complex numbers, the set $\mathbb{R}$ of real numbers, and the set $\mathbb{R}_+$ of real non-negative numbers, with the usual multiplications, but new, multivalued additions. The new hyperfields are related with the classical fields and each other by dequantisations. For example, the new complex tropical field $\mathcal{T}\mathbb{C}$ is a dequantization of the field $\mathbb{C}$ of complex numbers.

1. Introduction

1.1. Natural, but not well-known. This paper is devoted to hyperfields. The notion of hyperfield is an immediate generalization of the notion of field. A hyperfield is just a field, in which the addition is multivalued. Hyperfields are very natural and useful algebraic objects. However, still, they have to find their way to the mainstream mathematics: still, it is easier to re-invent them than to find out that they have been invented.

When I realized that objects of this kind would be very useful in my efforts to find an appropriate base for tropical geometry, it was not difficult to build up a basic theory around my examples, but it took much longer to find the theory in literature.

I gave several talks about this matter (in particular, a talk [18] at MSRI workshop Tropical Structures in Geometry and Physics on November 30, 2009). The talks were attended by many mathematicians, but nobody told me about acquaintance with generalized fields having a multivalued addition. I am very grateful to Anatoly Vershik: when he listened to my talk in mid January, 2010, he told me that there were many papers devoted to multigroups and the likes. I found by Google a paper [10] of 2006 by Murray Marshall, where multiring and multifields were defined.
In the introduction to [10] Marshall wrote: "The idea of a multiring is very natural, although there seems to be no reference to it in the literature. Some basic properties of multigroups and multirings are established in Sections 1 and 2." In Section 2 Marshall defined also a multifield.

I have changed a draft of my paper, replacing my term "tropical field" by Marshall’s "multifield" and uploaded it to arXiv as [20]. Of course, I contacted Marshall and expressed to him my excitement about multifields.

A couple of months later Marshall informed me that he learned from recent preprints [3], [4] of Alain Connes and Caterina Consani that our multifields under the name of hyperfields were introduced as early as in 1956 by Marc Krasner [6]. Krasner [6] introduced also hyperrings, but the notion of multiring introduced by Marshall [10] is more general than the notion of hyperring considered by Krasner, and the difference is essential for the applications that Marshall developed. Therefore both terms will be used. But Krasner’s hyperfield and Marshall’s multifield are absolutely the same, and the term hyperfield wins as it is older.

This paper is a new version of [20]. I insert the corresponding correction of references and terminology and make a few new remarks inspired by new information that I got from the papers [3], [4] of Alain Connes and Caterina Consani.

Krasner, Marshall, Connes and Consani and the author came to hyperfields for different reasons, motivated by different mathematical problems, but we came to the same conclusion: the hyperrings and hyperfields are great, very useful and very underdeveloped in the mathematical literature.

Probably, the main obstacle for hyperfields to become a mainstream notion is that a multivalued operation does not fit to the tradition of set-theoretic terminology, which forces to avoid multivalued maps at any cost.

I believe the taboo on multivalued maps has no real ground, and eventually will be removed. Hyperfields, as well as multigroups, hyperrings and multirings, are legitimate algebraic objects related in many ways to the classical core of mathematics. They provide elegant terminological and conceptual opportunities. In this paper I try to present new evidences for this.

I rediscovered hyperfields in an attempt [18] to find a true algebraic background of the tropical geometry. I believe hyperfields are to displace the tropical semifield in the tropical geometry. They suit the role better. In particular, with hyperfields the varieties are defined by equations, as in other branches of algebraic geometry.
1.2. Results. The main results of this paper are new examples of hyperfields. The underlying sets of these hyperfields are classical: the set \( \mathbb{C} \) of all complex numbers, the set \( \mathbb{R} \) of all real numbers, the set \( \mathbb{R}_+ \) of non-negative real numbers. Multiplication also is the usual multiplication of numbers. The additions are new multivalued operations.

These hyperfields are related to each other and the classical fields by hyperfield homomorphisms, and also via degenerations of the structures, similar to the Litvinov-Maslov dequantization \([8]\), which relates the semifield \((\mathbb{R}_+, +, \times)\) of non-negative real numbers with the usual arithmetic operations to the tropical semifield \( \mathbb{T} = (\mathbb{R} \cup \{0\}, \max, +) \).

In particular, the fields \( \mathbb{C} \) and \( \mathbb{R} \) are dequantized. I call the results complex tropical hyperfield \( \mathcal{T}_\mathbb{C} \) and real tropical hyperfield \( \mathcal{T}_\mathbb{R} \).

A new hyperfield that does not appear via dequantizing a field, is a **triangle hyperfield** \( \Delta \). Its underlying set is \( \mathbb{R}_+ \), and the addition is related to the triangle inequality: the sum of two non-negative numbers \( a \) and \( b \) is defined as the set of non-negative numbers \( c \) such that there exists an Euclidean triangle with sides of lengths \( a, b \) and \( c \). This hyperfield dequantizes to a similar hyperfield \( \mathcal{Y}_\times \) in which addition is related in the same way with the ultra-metric triangle inequality \( c \leq \max(a, b) \).

Applications of the hyperfields introduced in this paper to the tropical geometry will be presented in a separate papers \([21]\) and \([22]\), a preliminary exposition can be found in \([19]\).

1.3. Organization of the paper. Section 2 is devoted to the general multivalued algebra. It starts with a discussion of the terminology related to multivalued maps. Then multivalued binary operations are discussed.

In Section 3 the notions related to multivalued generalizations of groups are discussed. This discussion is not complete, due to long history and a huge number of various level of the generalizations. We concentrate mainly on the notions needed to what follows.

In Section 4 we turn to multirings, hyperrings and hyperfields, their examples and general properties. Section 4 finishes with a discussion of multiring homomorphisms, their examples and first applications.

In Section 5 a few hyperfields related to triangle inequalities are introduced (triangle, ultra-triangle, tropical and amoeba hyperfields).

In Section 6 we introduce tropical addition of complex numbers and discuss its properties. In Section 7 subhyperfields of the complex tropical hyperfield are considered.
In Section 9 the dequantization are considered. We start with the Litvinov-Maslov dequantization, then study dequantization of the triangle hyperfield to the ultratriangle one, and dequantization of the field $\mathbb{C}$ to the complex tropical hyperfield. All the dequantizations are related to each other at the end of Section 9.

1.4. Acknowledgements. The research presented in this paper was partly made when the author participated in a semester program on tropical geometry at MSRI. I am grateful to MSRI for an excellent environment and opportunity of direct contact with the leading specialists in tropical geometry. I am grateful to G. Mikhalkin, Ya. Eliashberg, V. Kharlamov, I. Itenberg, L. Katsarkov, I. Zharkov, A. M. Vershik, M. Marshall, A. Connes and C. Consani for useful discussions.
| Section | Title                                           | Page |
|---------|-------------------------------------------------|------|
| 4.7     | Hyperfields from a linearly ordered group       | 16   |
| 4.8     | Multiring homomorphisms                        | 16   |
| 4.9     | The sign homomorphism                          | 17   |
| 4.10    | Ideals of a multiring                           | 17   |
| 4.11    | Multiplicative kernel                           | 17   |
| 4.12    | Multiplicative factorization                    | 18   |
| 4.13    | Prime ideals and homomorphisms to \( K \)      | 18   |
| 5.      | Hyperfields from triangle inequalities          | 19   |
| 5.1     | Triangle hyperfield                            | 20   |
| 5.2     | Ultratriangle hyperfield                       | 21   |
| 5.3     | Tropical hyperfield                            | 21   |
| 5.4     | Amoeba hyperfield                              | 22   |
| 5.5     | Multiplicative seminorm                        | 22   |
| 5.6     | Non-archimedian multiplicative seminorm        | 22   |
| 6.      | Tropical addition of complex numbers           | 22   |
| 6.1     | Definition                                     | 22   |
| 6.2     | Obvious properties                              | 23   |
| 6.3     | Associativity                                   | 23   |
| 6.4     | Distributivity                                  | 23   |
| 6.5     | Complex tropical hyperfield                    | 24   |
| 6.6     | The tropical sum of several complex numbers    | 24   |
| 7.      | Relations of \( \mathcal{T} \mathbb{C} \) with other hyperfields | 24   |
| 7.1     | Submultirings and subhyperfields               | 24   |
| 7.2     | The tropical real hyperfield \( \mathcal{T} \mathbb{R} \) | 24   |
| 7.3     | The hyperfield of signs                        | 25   |
| 7.4     | The phase hyperfield                           | 25   |
| 7.5     | Embedding \( \mathbb{T} \subset \mathcal{T} \mathbb{C} \) | 26   |
| 7.6     | The absolute value and amoeba maps             | 27   |
| 7.7     | Complex polynomials and \( \mathcal{T} \mathbb{C} \) | 27   |
| 7.8     | Real exponents                                  | 28   |
| 8.      | Continuity                                     | 29   |
| 8.1     | Vietoris topologies                             | 30   |
| 8.2     | Continuity and semi-continuities               | 30   |
| 8.3     | Tropical additions                             | 30   |
| 8.4     | Continuity of triangle additions               | 31   |
| 8.5     | Properties of upper semi-continuous multimaps   | 32   |
| 9.      | Dequantizations                                | 33   |
| 9.1     | The Litvinov-Maslov dequantization             | 33   |
| 9.2     | Dequantization of the triangular hyperfield to the ultra-triangular | 34   |
| 9.3     | Dequantization of \( \mathbb{C} \) to \( \mathcal{T} \mathbb{C} \) | 34   |
| 9.4     | Dequantizations commute                         | 35   |
2. Multivalued maps and operations

Multivalued operations hardly belong to the mainstream of conventional mathematics, but they appear here and there. In this section the basic terminology related to multivalued maps is introduced and discussed.

2.1. Multivalued mappings. For a set $X$, the symbol $2^X$ denotes the set of all subsets of $X$. A multivalued map or multimap of a set $X$ to a set $Y$ is a map $X \to 2^Y$, which is treated for some reasons as a map $X \to Y$ that does not satisfy the usual requirement of being univalent (according to this requirement a map must take each element of $X$ to a single element of $Y$).

The reason for considering a multivalued map is usually a desire to emphasize an analogy to another situation, in which the corresponding map is univalued. In this paper we study a generalization of addition with the sum allowed to be multivalued. Usage of the modern set-theoretic terminology would make analogies with the usual addition more difficult to recognize. Cf., for example, [10], where a multivalued binary operation is introduced, according to the standards of set theory, as a subset of the Cartesian cube of the underlying set, but a couple of pages after that the multivalued notation take over, anyway. Therefore we dare to use less conventional terminology of multivalued maps.

The term set-valued is used as a synonym for multivalued. A multivalued map $f$ of $X$ to $Y$ is denoted by $f : X \rightsquigarrow Y$.

2.2. Adjustment of terminology. As in other cases of disrespect towards the standards of set-theoretic terminology, this one implies a whole chain of modifications of commonly accepted terminology and notation. Some of the modifications are straightforward and cannot lead to a confusion. For example, the value $f(a)$ at $a \in X$ is the subset of $Y$ which is the image of $a$ under the corresponding map $X \to 2^Y$. 

What happens to the notion of the \textit{image} of a set is less logical, but still easy to guess: for $A \subset X$, the symbol $f(A)$ denotes not the subset \{ $f(x) : x \in A$ \} of $2^Y$, but the subset $\bigcup_{x \in A} f(x)$ of $Y$.

In the same spirit: the \textit{composition} of multivalued maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is a multimap $g \circ f : X \rightarrow Z$ that takes $a \in X$ to $g(f(a)) = \bigcup_{y \in f(a)} g(y)$.

Other modifications may be quite confusing. For example, what is the preimage of a set $B \subset Y$ under a multivalued map $f : X \rightarrow Y$? The set $\{ a \in X : f(a) \subset B \}$ or $\{ a \in X : f(a) \cap B \neq \emptyset \}$? We see that the notion of the preimage of a set splits under the transition from univalued maps to multivalued ones. In cases of such ambiguity one needs to adjust the terminology. For example, the set $\{ a \in X : f(a) \subset B \}$ is called the \textit{upper preimage} of $B$ under $f$ and denoted by $f^+(B)$, while $\{ a \in X : f(a) \cap B \neq \emptyset \}$ is called the \textit{lower preimage} of $B$ under $f$ and denoted by $f^-(B)$. The names seems to be confusing because $f^+(B) \subset f^-(B)$, the upper preimage is smaller than the lower one.

In order to take refuge in the standard set-theoretic terminology, we will pass from a multivalued map $f : X \rightarrow Y$ to the corresponding univalued map $X \rightarrow 2^Y$. The latter will be denoted by $f^\uparrow$.

Sets, multimaps and their compositions form a category. Thus, although multivalued maps do not quite comply with the set-theoretic terminology, they fit comfortably to a more modern category-theoretic setup.

\section*{2.3. Multivalued binary operations.}

A multivalued map $X \times X \rightarrow X$ with non-empty values is called a \textit{binary multivalued operation} in $X$.

A binary multivalued operation $f : X \times X \rightarrow X$ is said to be \textit{commutative} if $f(a, b) = f(b, a)$ for any $a, b \in X$.

A binary multivalued operation $f : X \times X \rightarrow 2^X$ is said to be \textit{associative} if $f(f(a, b), c) = f(a, f(b, c))$ for any $a, b, c \in X$. Certainly, in the latter formula, by $f$ we mean not only $f$, but also its natural extension to all subsets of $X$, that is

$$2^X \times 2^X \rightarrow 2^X : (A, B) \mapsto \bigcup_{a \in A, b \in B} \{ f(a, b) \}.$$ 

Let $Y \subset X$ and $f : X \times X \rightarrow X$ be a multivalued binary operation. A multivalued binary operation $g : Y \times Y \rightarrow Y$ is said to be \textit{induced} by $f$, if $g(a, b) = f(a, b) \cap Y$ for any $a, b \in Y$. Of course, the induced operation is completely determined by the original one. It exists iff $f(a, b) \cap Y \neq \emptyset$ for any $a, b \in Y$ (recall that according to the definition of a multivalued operation the set $g(a, b)$ is not allowed to be empty).
3. Hypergroup-Multigroups-Polygroups

3.1. Definition of multigroup. A set $X$ with a multivalued binary operation $(a, b) \mapsto a \cdot b$ is called a multigroup if

1. the operation $(a, b) \mapsto a \cdot b$ is associative;
2. $X$ contains an element $1$ such that $1 \cdot a = a = a \cdot 1$ for any $a \in X$;
3. for each $a \in X$ there exists a unique $b \in X$ such that $1 \in a \cdot b$ and there exists a unique $c \in X$ such that $1 \in c \cdot a$. Furthermore, $b = c$. This element is denoted by $a^{-1}$.
4. $c \in a \cdot b$ iff $c^{-1} \in b^{-1} \cdot a^{-1}$ for any $a, b, c \in X$.

This is a straightforward generalization of the notion of group: any group is a multigroup and a multigroup, in which the group operation is univalued (i.e., $a \cdot b$ consists of a single element) is a group.

The axioms of multigroup presented above are not minimal. I have chosen them, because they give a good idea what is the structure and why multigroups generalize groups. To my taste, they are convenient if you know already that you deal with a multigroup and want to deduce something from axioms. For checking if something is a multigroup, a more minimalistic set of axioms, like the one provided by following theorem, may serve better.

3.A. Theorem (Cf. Marshall [11]). A set $X$ with a multivalued operation $(a, b) \mapsto a \cdot b$ is a multigroup iff there exist a map $X \to X : a \mapsto a^{-1}$ and an element $1 \in X$ such that:

1. $(a \cdot b) \cdot c \in a \cdot (b \cdot c)$ for any $a, b, c \in X$,
2. $a \cdot 1 = a$ for any $a \in X$,
3. Reversibility property. $c \in a \cdot b$ implies $a \in c \cdot b^{-1}$ and $b \in a^{-1} \cdot c$ for $a, b, c \in X$.

Proof. First, let us prove that statements $(1') - (3')$ hold true in any multigroup. Obviously, $(1')$ follows from axiom $(1)$, and $(2')$ follows from axiom $(2)$. Let us prove $(3')$.

If $c \in a \cdot b$, then $1 \in (a \cdot b) \cdot c^{-1} = a \cdot (b \cdot c^{-1})$. By axiom $(3)$, $a^{-1}$ is the unique element $x$ such that $1 \in a x$. Therefore $a^{-1} \in b \cdot c^{-1}$. By axiom $(4)$, $a^{-1} \in b \cdot c^{-1}$ iff $a \in c \cdot b^{-1}$. Thus, we proved that $c \in a \cdot b$ implies $a \in c \cdot b^{-1}$. The other implication is proved similarly: $c \in a \cdot b$ implies $1 \in c^{-1} \cdot (a \cdot b) = (c^{-1} \cdot a) \cdot b$, therefore $b^{-1} \in c^{-1} \cdot a$. Hence $b \in a^{-1} \cdot c$.

Now let us deduce the axioms $(1) - (4)$ from $(1') - (3')$.

First, observe that $1^{-1} = 1$. Indeed, $1 \cdot 1 = 1$ by $(2')$, hence $1 \in 1^{-1} \cdot 1$ by $(3')$, and $1^{-1} \cdot 1 = 1^{-1}$ by $(2')$.

Second, observe that $1 \in a^{-1} \cdot a$. Indeed, $a \in a \cdot 1$ by $(2')$, and by applying $(3')$ we get $1 \in a^{-1} \cdot a$. 
Now let us prove that the map $X \to X : a \mapsto a^{-1}$ is an involution, that is $(a^{-1})^{-1} = a$ for any $a \in X$. We have just proved that $1 \in a^{-1} \cdot a$.

By (3'), it follows $a \in (a^{-1})^{-1} \cdot 1$, but by (2') $(a^{-1})^{-1} \cdot 1 = (a^{-1})^{-1}$.

Now let us deduce axiom (4). Assume that $c \in a \cdot b$. By (3') this implies that $c \in c \cdot b^{-1}$. Again by (3'), this implies $b^{-1} \in c^{-1} \cdot a$. Finally, again by (3'), this implies $c^{-1} \in b^{-1} \cdot a^{-1}$. The opposite implication $c^{-1} \in b^{-1} \cdot a^{-1} \implies c \in a \cdot b$ follows from the one that we proved by substituting $a^{-1}$ for $a$, $b^{-1}$ for $b$ and $c^{-1}$ for $c$ and using the fact the $a \mapsto a^{-1}$ is an involution.

Now let us deduce (2). One of the two equalities constituting (2) is (2'). The other one follows from (2'), (4) and the fact that $a \mapsto a^{-1}$ is a bijection (as an involution).

In order to deduce (3), we need to prove that from (1') - (3') it follows that $1 \in a \cdot b$ implies $b = a^{-1}$ and $a = b^{-1}$. Apply (3') to $1 \in a \cdot b$. It gives $a \in 1 \cdot b^{-1}$ and $b \in a^{-1} \cdot 1$. By (2) this implies $b = a^{-1}$ and $a = b^{-1}$.

In order to prove (1), we need to prove the inclusion $(a \cdot b) \cdot c \supset a \cdot (b \cdot c)$. By (1'), $(c^{-1} \cdot b^{-1})^{-1} \cdot a^{-1} \in c^{-1} \cdot (b^{-1} \cdot a^{-1})$. Applying the involution $x \mapsto x^{-1}$ to both sides of this inclusion, we obtain $((c^{-1} \cdot b^{-1}) \cdot a^{-1})^{-1} \in (c^{-1} \cdot (b^{-1} \cdot a^{-1}))^{-1}$. Then, applying axiom (4), which has already been deduced above, we obtain $a \cdot (b \cdot c) \supset (a \cdot b) \cdot c$. \qed

Notice, that in the proof of Theorem 3.A we proved that in any multigroup $1^{-1} = 1$ and $(a^{-1})^{-1} = a$.

3.2. Notation. In what follows we meet mostly commutative multigroups. Then we will use an additive notation: the neutral element 1 will be denoted by 0, the element $a^{-1}$ will be denoted by $-a$, the multigroup operation will be denoted by various symbols such as $\tau$, $\vee$, $\triangleright$, $\triangleright$. We use these symbols (instead of commonly used +), because the multivalued operations will be considered below in an environment where the usual addition $(a, b) \mapsto a + b$ is also present, and, moreover, two multivalued additions may be considered simultaneously.

3.3. The smallest multigroup. The smallest multigroup which is not a group: in the set $\{0, 1\}$ define an operation $\tau$ by formulas: $0 \tau 0 = 0$, $0 \tau 1 = 1 = 1 \tau 0$, $1 \tau 1 = \{0, 1\}$. One can easily check that this is a multigroup. Following Marshall [10], we denote this multigroup by $Q_1$. This is the only multigroup of two elements that is not a group.

3.4. Multigroups of a linear order. $Q_1$ belongs to a family of multigroups defined by linearly ordered sets. Let $X$ be a linearly ordered set with order $<$ and an element 0 such that $0 < x$ for any $x \in X$.
different from 0. Define in $X$ a binary multivalued operation

\[(a, b) \mapsto a \triangleright b = \begin{cases} \max(a, b), & \text{if } a \neq b \\ \{x \in X : x \leq a\}, & \text{if } a = b. \end{cases}\]

It is easy to verify that $X$ with $\triangleright$ is a multigroup and $-a = a$ for any $a \in X$.

This construction gives $Q_1$ if $X = \{0, 1\}$ and $0 < 1$.

In the same situation $X$ can be turned into a different multigroup. For this, define a binary multivalued operation

\[(a, b) \mapsto a \widthhat{\triangleright} b = \begin{cases} \max(a, b), & \text{if } a \neq b \\ \{x \in X : x < a\}, & \text{if } a = b \neq 0 \\ 0, & \text{if } a = b = 0. \end{cases}\]

It is easy to verify that $X$ with $\widthhat{\triangleright}$ is a multigroup and $-a = a$ for any $a \in X$. For $X = \{0, 1\}$ and $0 < 1$ this construction gives a group. If $X$ consists of more than 2 elements, the operation $\widthhat{\triangleright}$ is truly multivalued.

We will call $(X, \triangleright)$ a linear order multigroup, and $(X, \widthhat{\triangleright})$ a strict linear order multigroup.

3.5. Three element multigroups. Define in a three element set $\{-1, 0, 1\}$ operation $\sim$ by formulas $0 \sim x = x \sim 0 = x$ and $x \sim x = x$ for any $x$, and $-1 \sim 1 = 1 \sim (-1) = \{-1, 0, 1\}$. One can easily check that this is a multigroup. Following Marshall [10], we denote this multigroup by $Q_2$.

Yet another multigroup of three elements can be defined as follows. In $\{0, 1, 2\}$ define operation $\triangleright$ by formulas $0 \triangleright x = x \triangleright 0 = x$ for any $x$, $1 \triangleright 1 = 2$, $1 \triangleright 2 = 2$, $1 = \{0, 1\}$, $2 \triangleright 2 = \{1, 2\}$. Denote this multigroup by $M$.

3.6. Multigroups of double cosets. Traditional examples of multigroups come from the group theory. Let $G$ be a group, and $H$ be a subgroup of $G$. Let $X$ be the set of double cosets, $X = \{HgH : g \in G\}$. Define a binary multivalued operation $(HaH)(HbH) = \{HaHbH : h \in H\}$. This is a multigroup, see Dresher and Ore [6].

3.7. Multigroup homomorphisms. Let $X$ and $Y$ be multigroups. A map $f : X \to Y$ is called a (multigroup) homomorphism if $f(e) = e$ and $f(a \cdot b) \subset f(a) \cdot f(b)$ for any $a, b \in X$.

A multigroup homomorphism $f : X \to Y$ is said to be strong if $f(a \cdot b) = f(a) \cdot f(b)$ for any $a, b \in X$. If $Y$ is a group, then any multigroup homomorphism $f : X \to Y$ is strong.

Example. If $X$ and $Y$ are linearly ordered sets with the smallest elements $0_X$ and $0_Y$, respectively, then any monotone map $X \to Y$ mapping $0_X$ to $0_Y$ is a multigroup homomorphism. Such a map is
a strong homomorphism iff it is injective on the complement of the preimage of 0Y.

3.8. Submultigroups. Let X be a multigroup with neutral element e, and Y ⊂ X. If e ∈ Y, the multigroup operation in X induces a binary multivalued operation in Y and together with any a ∈ Y the inverse element a⁻¹ is contained in Y, then Y with the induced operation is a multigroup. It is called a submultigroup of X. The inclusion Y ↪ X is a multigroup homomorphism. If this is a strong homomorphism, then Y is said to be a strong submultigroup of X.

A strong submultigroup Y of a multigroup X is said to be normal, if a⁻¹ · Y · a = Y for any a ∈ X. Observe, that a normal submultigroup of X contains the set a · a⁻¹ for any a ∈ X.

For a multigroup homomorphism f : X → Y, the set \{a ∈ X : f(a) = e\} is called the kernel of f and denoted by Ker f. Obviously, this is a normal submultigroup of X.

3.9. Factorization of a multigroup homomorphism. As in the group theory, for any normal submultigroup Y of a multigroup X one can construct the quotient X/Y, and a multigroup structure in X/Y such that the projection X → X/Y is a strong multigroup homomorphism. Any multigroup homomorphism f : X → Y admits a natural factorization X → X/Ker f → Y.

3.B. Theorem. If f is surjective and strong, then the induced multigroup homomorphism \(\overline{f} : X/Ker f \to Y\) is an isomorphism.

Proof. Let \(\alpha, \beta \in X/Ker f\) and their images \(\overline{f}(\alpha), \overline{f}(\beta)\) under \(\overline{f} : X/Ker f \to Y\) coincide. Take representative a, b ∈ X of \(\alpha\) and \(\beta\), respectively. Then \(f(a) = \overline{f}(\alpha) = \overline{f}(\beta) = f(b)\). Since \(f\) is strong, \(f(b^{-1}a) = f(b)^{-1}f(a) = f(a)^{-1}f(a) \supseteq 1\). Thus \(b^{-1}a \cap Ker f \neq \emptyset\). Therefore there exists \(c \in b^{-1}a \cap Ker f\). Then \(a \in bc \subset Ker f\) and \(\alpha = \beta\).

The assumption that \(f\) is strong is necessary here. Without this assumption, a multigroup homomorphism with a trivial kernel may be non injective. On the other hand, most of interesting multigroup homomorphisms are not strong. This is a major new phenomenon distinguishing multigroups from groups.

Here is the simplest example: \(Q_2 \to Q_1 : 1, -1 \mapsto 1, 0 \mapsto 0\). It is easy to see that \(f\) is a multigroup homomorphism with Ker \(f = \{0\}\), but \(f\) is not injective. In order to verify that \(f\) is not strong, consider \(f(1 \cdot 1) = f(1) = 0\), on the other hand, \(f(1) \cdot f(1) = 1 \cdot 1 = \{0, 1\}\).
3.10. Remarks on the history of multigroups. The notion of multigroup appeared in the literature in various contexts, sometimes under other names (such as hypergroup and polygroup). The earliest papers [12], [23] about them that I could find are dated by 1934. Some of the authors who introduced these objects apparently were not aware of their predecessors.

Often the terms multigroup and hypergroup were used for objects of wider classes. For example, Dresher and Ore [5] used the word multigroup for much wider class of object, while what is called multigroup above, Dresher and Ore [5] would call a regular reversible in itself multigroup with an absolute unit.

The definition given in Section 3.1 seems to be the narrowest and closest multivalued generalization of the notion of group. In comparatively recent literature exactly the same notion was considered by S. D. Comer [2] (under the name of polygroup) and M. Marshall [10]. A. Connes and C. Consani [4] consider the same notion under the name of hypergroup, but restrict themselves to commutative hypergroups.

There is another breed of multigroups in which the value of the operation contains a fixed number of elements some of which may coincide to each other. Thus the operation takes values in the $n$th symmetric power of the set rather than in the set of all its subsets. This kind of multigroups was considered by Wall [24] and, more recently, by Buchstaber and Rees [1]. The author is not aware about any construction which would allow to relate multigroups of this kind with multigroups defined in Section 3.1.

4. Multirings, hyperrings and hyperfields

4.1. Multirings. A set $X$ equipped with a binary multivalued operation $\tau$ and (univalued) multiplication $\cdot$ is called a multiring if

- $(X, \tau)$ is a commutative multigroup,
- $(X, \cdot)$ is a monoid with unity 1 (i.e., multiplication $(a, b) \mapsto a \cdot b$ is associative and $1 \cdot a = a = a \cdot 1$ for any $a \in X$),
- $0 \cdot a = 0$ for any $a \in X$.
- the multiplication is distributive over $\tau$ in the sense that for every $a \in X$ maps $X \to X$ defined by formulas $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are homomorphisms of multigroup $(X, \tau)$ to itself.

The distributivity means that $a \cdot (b \tau c) \subset a \cdot b \tau a \cdot c$ and $(b \tau c) \cdot a \subset (b \cdot a) \tau (c \cdot a)$ for any $a, b, c \in X$.

A multiring is said to be commutative if the multiplication is commutative.
4.2. Hyperrings. If in a multiring $X$ distributivity holds in a stronger form: $a \cdot (b \cdot c) = a \cdot b \cdot a \cdot c$ and $(b \cdot c) \cdot a = (b \cdot a) \cdot c \cdot a$ for any $a, b, c \in X$, then $X$ is called a hyperring.

An equivalent description of this strong form of distributivity: for every $a \in X$ maps $X \to X$ defined by formulas $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are strong homomorphisms of the multigroup $(X, \cdot)$ to itself.

Hyperrings were introduced by Krasner \[6\], see also \[7\] and \[3\]. Multirings were introduced by Marshall \[10\]. In Marshall’s work the extra generality of the notion of multirings is used: some of the multirings that he considers in \[10\] are not hyperrings.

4.3. Hyperfields. A multiring $X$ is called a hyperfield if $X \setminus 0$ is a commutative group under multiplication.

4.A. Theorem (See Marshall \[10\]). Any hyperfield is a hyperring.

Proof. This theorem claims that in a hyperfield distributivity holds in a strong form: $a(b \cdot c) = (ab) \cdot (ac)$. Indeed, the inclusion $a(b \cdot c) \subset (ab) \cdot (ac)$ that holds true in multirings implies the opposite inclusion if $a \neq 0$:

$$(ab) \cdot (ac) = a^{-1}a((ab) \cdot (ac)) \subset a(a^{-1}ab \cdot a^{-1}ac) = a(b \cdot b).$$

For $a = 0$, the equality $a(b \cdot c) = (ab) \cdot (ac)$ holds true since both sides are equal to 0.

The notion of hyperfield is a direct generalization of the notion of field: a field is a hyperfield, in which the addition is univalued.

4.4. Double distributivity. A multivalued addition creates various new phenomena, some of which may be quite unexpected.

For example, in a usual ring, distributivity implies that $(a+b)(x+y) = ax + ay + bx + by$. In a multiring and even in a hyperfield the proof fails. Moreover, the equality

$$(a \cdot b)(x \cdot y) = ax \cdot ay \cdot bx \cdot by$$

may be incorrect, see Sections \[5.1\] \[5.3\].

Let us analyze, why the arguments that deduce $(a + b)(x + y) = a x + a y + b x + b y$ from distributivity for univalued addition do not work for multivalued addition. In the univalued case, $x + y$ is just an element, and one can apply distributivity: $(a + b)(x + y) = a(x + y) + b(x + y)$. Then for each summand distributivity is applied again, giving the equality.

In the case of multivalued addition $\cdot \mathcal{r}$, $(x \cdot y)$ is not an element, but a set. Therefore the distributivity $(a \cdot b)c = ac \cdot bc$, in which $c$ is a single element (that is an axiom in a multiring) cannot be applied in the situation when $c$ is a set $x \cdot y$. 

4.B. **Theorem.** In any multiring, \((a \uparrow b)(x \uparrow y) \subset ax \uparrow ay \uparrow bx \uparrow by\).

**Proof.** For each element \(c \in (x \uparrow y)\) the distributivity gives \((a \uparrow b)c = ac \uparrow bc\), and we get \((a \uparrow b)(x \uparrow y) = \bigcup_{c \in (x \uparrow y)} (a \uparrow b)c = \bigcup_{c \in (x \uparrow y)} (ac \uparrow bc)\). On the other hand,

\[ax \uparrow ay \uparrow bx \uparrow by = (ax \uparrow ay) \uparrow (bx \uparrow by) =
\]

\[a(x \uparrow y) \uparrow b(x \uparrow y) \supset ac \uparrow bc\]

for any \(c \in (x \uparrow y)\), and therefore

\[ax \uparrow bx \uparrow ay \uparrow by \supset \bigcup_{c \in (x \uparrow y)} (ac \uparrow bc) = (a \uparrow b)(x \uparrow y)\].

\[\square\]

The opposite inclusion \((a \uparrow b)(x \uparrow y) \supset ax \uparrow ay \uparrow bx \uparrow by\) in some multirings does not hold true (see Section 6.4). However, there are multirings in which it is true. Such multirings will be called **doubly distributive**.

In a doubly distributive multiring, \((\tau_{i=1}^n a_i)(\tau_{j=1}^m b_j) = \tau_{i,j} a_i b_j\). This can be easily proved by induction over \(m\) and \(n\).

4.5. **The smallest hyperfields.** Multigroups \(Q_1\) and \(Q_2\) defined in Sections 3.3 and 3.5 above turn into hyperfields in a unique way. In \(Q_1 = \{0,1\}\) the multiplicative group is trivial and all the products are defined by axioms. In \(Q_2 = \{-1,0,1\}\) the multiplicative group is of order 2, therefore it is uniquely defined up to isomorphism.

Following Connes and Consani [3], we will call the two-element hyperfield \(Q_1\) the **Krasner hyperfield** and denote it by \(K\). It can be obtained from any field \(k\) with more than two elements by identifying all invertible elements. This is a multiplicative factorization (see Section 1.12 below) that was invented by Krasner [7]. To the best of my knowledge, \(K\) did not appear in Krasner’s papers.

The hyperfield \(Q_2\) is called the **sign hyperfield** and denoted by \(S\).

These two hyperfields are doubly distributive.

Multigroup \(M\) defined also in Section 3.5 above cannot be turned into a hyperfield, unless a multivalued multiplication would be allowed. In this paper I prefer to stay with univalued multiplications only. If a multiplication in a hyperfield was allowed to be multivalued, one could define \(1 \cdot x = x\) and \(0 \cdot x = 0\) for any \(x\) and \(2 \cdot 2 = \{1,2\}\). Then the multiplicative multigroup of \(M\) would be isomorphic to \(Q_1\).

4.6. **Characteristics.** The notion of characteristic of ring splits when we pass to multirings.
Recall that the characteristic of a ring is the smallest positive integer \( n \) such that the sum \( 1 + \cdots + 1 \) of \( n \) summands is 0, and zero if any sum \( 1 + \cdots + 1 \) does not vanish.

This definition can be reformulated as follows: an integer \( n \) is the characteristic of a ring if \( n \) is the smallest positive number such that the sum \( 1 + \cdots + 1 \) of \( n+1 \) summands equals 1; if \( 1 + \cdots + 1 \neq 1 \) for any number \( k > 1 \) of summands, than the characteristic is zero.

For multirings, straightforward generalizations of these two definitions are not equivalent. I propose to preserve the old term of characteristic for the number defined by a generalization of the first definition, and to call the second one \textbf{C-characteristic} in honor of A. Connes and C. Consani, who discovered the opportunity of speaking about hyperrings of characteristic one, see [3] and [4].

A natural number \( n \) is called the \textbf{characteristic} of a multiring if this is the smallest natural number such that \( 0 \in 1 \uparrow 1 \uparrow \cdots \uparrow 1 \) where the number of summands on the right hand side is \( n \). A multiring which has no finite characteristic \( n \geq 2 \) is said to be of characteristic 0. The characteristic of a multiring \( X \) is denoted by \( \text{char} X \).

A natural number \( n \) is called the \textbf{C-characteristic} of a multiring if \( n \) is the smallest positive number such that the sum \( 1 \uparrow \cdots \uparrow 1 \) of \( n+1 \) summands \textit{contains} 1; if \( 1 \not\in 1 \uparrow \cdots \uparrow 1 \) for any number \( k > 1 \) of summands, than the C-characteristic is zero. The C-characteristic of a multiring \( X \) is denoted by \( C\text{-char} X \).

Obviously, a multiring of characteristic \( p \neq 0 \) has C-characteristic \( \leq p \). On the other hand, \( \text{char} S = 0 \) and \( C\text{-char} S = 1 \), while \( \text{char} K = 2 \) and \( C\text{-char} K = 1 \).

A multiring is said to be \textit{idempotent}, if \( a \uparrow a = a \) for any \( a \) in it. A multiring is idempotent iff \( 1 \uparrow 1 = 1 \) in it.

An idempotent multiring has C-characteristic 1, but the converse is not true: in a multiring of C-characteristic 1 the set \( 1 \uparrow 1 \) may consist of more than one element. For example, \( S \) is idempotent, \( K \) is not idempotent (because \( 1 \uparrow 1 = \{0,1\} \) in \( K \)), but both have C-characteristic 1.

The characteristic of an idempotent multiring is 0, because in it \( 1 \uparrow \cdots \uparrow 1 = 1 \) for any number of summands. In particular, \( \text{char} S = 0 \).

A fundamental importance of the characteristic in the theory of rings comes from the fact that the characteristic determines the minimal subring of the ring. For multirings no structural theorem of this sort is known.

Moreover, there is no commonly accepted notion of submultiring or even subhyperfield. The point of disagreement is whether to require that the subset underlying a submultiring would be closed under the
multivalued addition, or just require that the intersection of the subset with the sum of any of its two elements would be non-empty. In the univalued situation there is no difference between these two requirements.

If we accept the alternative in which the subset contains the whole sum of any two of its elements, then there is no hope for a reasonable list of simple hyperfields.

Under the other alternative, I am not aware about any conjectural list of simple hyperfields. However, the following two simple results in this direction sounds inspiring.

4.C. **Theorem.** In any multiring \( R \) with \( \text{char} \ R = 0 \) and \( C \text{-char} \ R = 1 \) the set \( \{-1,0,1\} \) inherits from \( R \) operations identical to the hyperfield operations in the sign hyperfield \( S \). \( \square \)

4.D. **Theorem.** In any multiring \( R \) with \( \text{char} \ R = 2 \) the set \( \{0,1\} \) inherits from \( R \) operations identical to the operations either in the Krasner hyperfield \( K \), if \( C \text{-char} \ R = 1 \), or in the field \( \mathbb{F}_2 \), if \( C \text{-char} R = 2 \). \( \square \)

4.7. **Hyperfields from a linearly ordered group.** Let \( X \) be a multiplicative group with a linear order \( \prec \) such that if \( a \prec b \), then \( ac \prec bc \) for any \( a,b,c \in X \). Let \( Y = X \cup \{0\} \). Extend the order \( \prec \) from \( X \) to \( Y \) by setting \( 0 \prec x \) for any \( x \in X \).

The linear order \( \prec \) gives rise to two multigroup structures in \( Y \), with additions \( \gamma \) and \( \dagger \), defined in Section 3.4. Extend the multiplication in the group \( X \) to \( Y \) by defining \( x0 = 0 \) for each \( x \in Y \). It is easy to see that \( Y \) with any of the additions, either \( \gamma \) or \( \dagger \), and this multiplication is a doubly distributive hyperfield.

The Krasner hyperfield \( K \) can be obtained via this construction with \( \gamma \) applied to the trivial group. The construction with \( \dagger \) applied to the trivial group gives the field \( \mathbb{F}_2 \).

The sign hyperfield \( S \) cannot be presented as a hyperfield of a linear order, because \( S \) is idempotent, while any hyperfield of linear order is of characteristic 2: indeed, \( 0 \in 1 \dagger 1 \subseteq 1 \gamma 1 \).

4.8. **Multiring homomorphisms.** Let \( X \) and \( Y \) be multirings. A map \( f : X \rightarrow Y \) is called a (multiring) homomorphism if it is a multigroup homomorphism for the additive multigroups of \( X \) and \( Y \) and a multiplicative homomorphism for their multiplicative semi-groups (the latter means that \( f(ab) = f(a)f(b) \) for any \( a,b \in X \)). A multiring homomorphism is said to be strong if it is strong as a multigroup homomorphism for the additive multigroups.
There are many well known commonly used maps which are multiring homomorphisms. Below we consider a few examples.

4.9. The sign homomorphism. The sign function
\[ \mathbb{R} \to \{0, \pm 1\} : x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \]
is a multiring homomorphism of the field \( \mathbb{R} \) to the hyperfield \( \mathbb{S} \). For generalizations of this, see Marshall [10].

4.10. Ideals of a multiring. As in a ring, an ideal of a commutative multiring \( X \) is a non-empty subset \( I \subset X \) such that \( a \top b \in I \) for any \( a, b \in I \), and \( ab \in I \) if \( a \in X \) and \( b \in I \). For any multiring homomorphism \( f : X \to Y \), its kernel \( \text{Ker } F = \{ a \in X : f(a) = 0 \} \) is an ideal in \( X \).

As in the ring theory, for any ideal \( I \) of a multiring \( X \) one can construct the quotient \( X/I \), and a multiring structure in \( X/I \) such that the projection \( X \to X/I \) is a strong multiring homomorphism. Any multiring homomorphism \( f : X \to Y \) admits a natural factorization \( X \to X/\text{Ker } f \to Y \). If \( f \) is surjective and strong, then the induced multiring homomorphism is an isomorphism.

The assumption that \( f \) is strong is necessary here. Without this assumption, a multiring homomorphism with a trivial kernel may be non injective. On the other hand, most of interesting multiring homomorphisms are not strong. This is a major new phenomenon distinguishing multirings from rings, cf 3.9.

The example \( \mathbb{S} \to \mathbb{K} : 1, -1 \mapsto 1, 0 \mapsto 0 \) of a non-injective multigroup homomorphism with trivial kernel considered in Section 3.9 above, is in fact a multiring homomorphism of the sign hyperfield \( \mathbb{S} \) to the Krasner hyperfield \( \mathbb{K} \) with the hyperfield structures defined in Section 4.5.

The sign homomorphism \( \mathbb{R} \to \mathbb{S} \) defined in Section 4.9 above is also a non-injective multiring homomorphism with trivial kernel.

In a hyperfield \( X \), the only ideals are \( \{0\} \) and \( X \).

4.11. Multiplicative kernel. The kernel does not contain all the information about a multiring epimorphism, in contrast to the ring theory. On the other hand, there exists a multiring epimorphism that is not an isomorphism even if both the multirings involved are hyperfields.

If \( f : X \to Y \) is a multiring homomorphism, and \( X \) is hyperfields, then either \( \text{Ker } f = 0 \) or \( f = 0 \). Indeed, any ideal of a hyperfield \( X \) is either 0 or \( X \) exactly for the same reasons as if \( X \) was a field.

A hyperfield belongs to the traditional algebra at least in its multiplicative structure. In a hyperfield the complement of the zero is a commutative group. A non-trivial multiring homomorphism between
hyperfields is a group homomorphism of the multiplicative groups. As such, it has a kernel, the preimage of unity.

In the univalued algebra, preimages of any two elements under a ring homomorphism are cosets related by translations which map bijectively one of them onto another. In multivalued algebra this phenomenon has no analogue. The formula $x \mapsto x + a$ defining a translation by $a$ in a ring, in a multiring turns into $x \mapsto x \tau a$ which defines a multivalued map. This map restricted to a preimage $f^{-1}(b)$ of an element $b$ under a multiring homomorphism $f : X \to Y$ does not send it to the preimage of an element, but to the preimage of a set $b \tau f(a)$, and this restriction is not invertible. So, everything is broken.

If $X$ and $Y$ are hyperfields and $f : X \to Y$ is a multiring homomorphism, then nonempty preimages of any non-zero element $b \in Y$ is related via natural bijections, which are multiplicative translations, with $f^{-1}(1)$. For any $\beta \in f^{-1}(b)$ formula $x \mapsto \beta^{-1}x$ maps $f^{-1}(b)$ onto $f^{-1}(1)$, and this map has inverse $x \mapsto \beta x$. The set $f^{-1}(1)$ is the kernel of the group homomorphism $X \setminus 0 \to Y \setminus 0$ induced by $f$. Denote this kernel by $\text{Ker}_m f$ and call it the multiplicative kernel of $f$. Obviously, $\text{Ker}_m f$ is a subgroup of the multiplicative group of $X$.

Some fragments of this nice picture take place in a more general setup, when $X$ and $Y$ are multirings and $f : X \to Y$ is a multiring homomorphism. Still the multiplicative kernel $\text{Ker}_m f$ is defined as $f^{-1}(1)$. This set is obviously closed under multiplication, but may be not a subgroup. Let $b \in f(X)$ and $\beta \in X$ such that $f(\beta) = b$. Then multiplication by $\beta$ maps $\text{Ker}_m f$ to $f^{-1}(b)$. However, as $\beta$ may be non-invertible, the construction for the inverse map $f^{-1}(b) \to \text{Ker}_m f$ is not available. Moreover, simple examples show that the map $x \mapsto \beta x : \text{Ker}_m f \to f^{-1}(b)$ may be neither injective nor surjective.

Elements $\beta, \gamma \in X$ have the same image under a map $f$ with given $\text{Ker}_m f$ if there exist $s, t \in \text{Ker}_m f$ such that $s\beta = t\gamma$. This is the weakest sufficient conditions, which can be formulated solely in terms of $\text{Ker}_m f$. However, this is not a necessary condition.

### 4.12. Multiplicative factorization

Any subgroup $S$ of the multiplicative group of a hyperfield $X$ can be presented as the multiplicative kernel of a multiring homomorphism of $X$ to a hyperfield. A construction of this hyperfield was proposed by Krasner [7], see also Marshall [10] and [11].

The resulting hyperfield is denoted by $X/_{mS}$. As a set, this is $(X^*/S) \cup \{0\}$, a disjoint union the zero and the quotient of the multiplicative group $X^*$ by the subgroup $S$. The multiplication in $X/_{mS}$ is defined by the multiplication in the quotient group and the identity
$x_0 = 0$. The addition in $X/mS$ induced by the addition in $X$. For cosets $aS, bS \in X^{\times}/S$ the sum is $\{cS : c \in aS \tau bS\}$, where $\tau$ denotes the addition of subsets of $X$ induced by the addition in $X$.

The natural map $X \to X/mS$ is a multiring homomorphism with multiplicative kernel $S$.

**Examples.** 1. $X/m(X \setminus \{0\}) = K$ for any hyperfield $X \neq F_2$.

2. $\mathbb{R}/m\mathbb{R}_{>0} = S$.

Marshall [10], Example 2.6 introduced the multiplicative factorization for more general situation in which $X$ is an arbitrary multiring and $S$ an arbitrary subset of $X$ closed under multiplication. Then $X/mS$ is a multiring obtained as the set of equivalence classes for the following equivalence relation: $a \sim b$ if there exist $s, t \in S$ such that $sa = tb$. If $0 \in S$, then $X/mS = 0$.

Marshall’s papers [10], [11] contain numerous interesting applications of this construction. We restrict here to a simple elementary example that was not considered in these papers.

In a ring $\mathbb{Z}$ of integers, let $S$ be the set of all odd numbers. Then $\mathbb{Z}/mS$ can be identified with the set $\{2^n : n = 0, 1, 2, \ldots\}$ of powers of 2. The multiplication in this multiring is the usual multiplication of powers of 2 (i.e., addition of the exponents). The multivalued addition is the strict linear order operation $\lhd$ from Section 3.4 for the order opposite to the ordering $<$ (i.e., $2^p \lhd 2^q$ iff $p > q$). This operation addresses to the following question: given two powers of 2, what is the highest power of 2 that can divide a sum of two integers $m$ and $n$ for which the highest powers of 2 that divide $m$ and $n$ are the given powers of 2. Clearly, $\text{char} \mathbb{Z}/mS = 2$ and $\text{C-char} \mathbb{Z}/mS = 2$.

If in this example we would replace 2 by an odd prime number $p$, then the strict linear order operation would be replaced by a non-strict one, the characteristic of the multiplicative quotient would be still 2, and the C-characteristic would change to 1.

### 4.13. Prime ideals and homomorphisms to $K$.

Cf. [10] Section 2.8 and [3] Proposition 2.9. An ideal $I$ of a multiring is said to be prime if $1 \notin I$ and $ab \in I$ implies that either $a \in I$ or $b \in I$.

Notice that the kernel of any multiring homomorphism $f : X \to K$ is a prime ideal in $X$. Vice versa, any prime ideal can be presented in this way. Indeed, for any prime ideal $I$ of a multiring $X$, define

$$f_I : X \to K : x \mapsto \begin{cases} 0, & \text{if } x \in I, \\ 1, & \text{if } x \notin I. \end{cases}$$
This gives a multiring interpretation of prime ideals in usual rings. Thus, the prime ideal spectrum $\text{Spec}K$ of a multiring $K$ can be identified with the set of multiring homomorphisms $K \to K$.

5. Hyperfields from triangle inequalities

5.1. Triangle hyperfield. In the set $\mathbb{R}_+$ of non-negative real numbers, define a multivalued addition $\triangledown$ by formula

$$a \triangledown b = \{ c \in \mathbb{R}_+ : |a - b| \leq c \leq a + b \}.$$ 

In other words, $a \triangledown b$ is the set of all real numbers $c$ such that there exists an Euclidean triangle with sides of lengths $a, b, c$.

5.A. Theorem. The set $\mathbb{R}_+$ with the multivalued addition $\triangledown$ and usual multiplication is a hyperfield.

Proof. This addition is obviously commutative. It is also associative. In order to prove this, just observe that both $(a \triangledown b) \triangledown c$ and $a \triangledown (b \triangledown c)$ coincide with the set of real numbers $x$ such that there exists a Euclidean quadrilateral with sides of lengths $a, b, c, x$.

The usual multiplication is distributive over $\triangledown$. The role of zero is played by 0. The negation $a \mapsto -a$ for $\triangledown$ is identity, as for any $a \in \mathbb{R}_+$ the only real number $x$ such that $0 \in a \triangledown x$ is $a$. □

This hyperfield is called the triangle hyperfield and denoted by $\Delta$.

5.B. Theorem. Hyperfield $\Delta$ is not doubly distributive.

Proof. Indeed, $2 \triangledown 1 = [1, 3]$. Therefore $(2 \triangledown 1) \cdot (2 \triangledown 1) = [1, 3] \cdot [1, 3] = [1, 9]$. On the other hand,

$$2 \cdot 2 \triangledown 2 \cdot 1 \triangledown 1 \cdot 2 \triangledown 1 \cdot 1 = 4 \triangledown 2 \triangledown 2 \triangledown 1$$

contains 0, because there exists an isosceles trapezoid with sides 4, 2, 1, and 2. In fact, $4 \triangledown 2 \triangledown 2 \triangledown 1 = [0, 9]$. □

The operation $\triangledown$ appears in the representation theory. Denote by $V^{(a)}$ the $a$th irreducible representation of $\mathfrak{sl}_2 \mathbb{C}$ (i.e., the symmetric power $\text{Sym}^a V$ of the standard 2-dimensional representation $V$). Then
the set \( \{ a \land b \} \cap (2\mathbb{Z} + a + b) \) parametrizes the set of irreducible representations of \( \mathfrak{sl}_2\mathbb{C} \) which are the summands in \( V(a) \otimes V(b) \):

\[
V(a) \otimes V(b) = \bigoplus_{c \in (a+b) \cap (2\mathbb{Z}+a+b)} V(c)
\]

5.2. Ultratriangle hyperfield. The construction of Section 4.7 of hyperfield of linearly ordered group, when applied to the multiplicative group of positive real numbers equipped with the usual order \(<\), defines a structure of hyperfield in \( \mathbb{R}_+ \). Recall that the addition in this hyperfield is defined by formula

\[
(a, b) \mapsto a \triangleright b = \begin{cases} 
\max(a, b), & \text{if } a \neq b \\
\{ x \in \mathbb{R}_+ : x \leq a \}, & \text{if } a = b.
\end{cases}
\]

the multiplication is the usual multiplication of real numbers. As any hyperfield of a linear order, this one is doubly distributive, see Section 4.7.

There is another way to construct the same hyperfield. It is completely similar to the construction of the triangle hyperfield of Section 5.1, but with the triangle inequality replaced by the non-archimedian (or ultra) triangle inequality \(|c| \leq \max(|a|, |b|)\). This hyperfield is called the ultratriangle hyperfield and denoted by \( Y_x \).

5.3. Tropical hyperfield. The map \( \log : \mathbb{R}_+ \to \mathbb{R} \) is naturally extended by mapping 0 to \(-\infty\). The resulting map \( \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\} \) is denoted also by \( \log \). This is a bijection, and the hyperfield structure of \( Y_x \) can be transferred via \( \log \) to \( \mathbb{R} \cup \{-\infty\} \). Denote the resulting hyperfield by \( Y \), and call it the tropical hyperfield.

The hyperfield structure of \( Y \) can be obtained by the construction of Section 4.7 applied to the additive group of all real numbers with the usual order \(<\). The hyperfield addition here differs from the semifield addition \( (a, b) \mapsto \max(a, b) \) in \( \mathbb{T} \) only on the diagonal: \( \max(a, a) = a \neq a \land a = \{ x \in \mathbb{T} : x \leq a \} \), although \( \max(a, a) \in a \land a \).

Since \( Y \) will play an important role in what follows, let me describe it explicitly and independently of constructions above. The underlying set of \( Y \) is \( \mathbb{R} \cup \{-\infty\} \), the addition is

\[
(a, b) \mapsto a \triangleright b = \begin{cases} 
\max(a, b), & \text{if } a \neq b \\
\{ x \in Y : x \leq a \}, & \text{if } a = b.
\end{cases}
\]

the multiplication is the usual addition of real numbers extended in the obvious way to \(-\infty\), the hyperfield zero is \(-\infty\), the hyperfield unity is \( 0 \in \mathbb{R} \).
5.4. Amoeba hyperfield. Transfer via the same bijection \( \log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\} \) the structure of the triangle hyperfield \( \Delta \) defined above in Section 5.1 to \( \mathbb{R} \cup \{-\infty\} \). The resulting hyperfield is called the amoeba hyperfield and denoted by \( \Delta^{\log} \).

The addition in \( \Delta^{\log} \) is defined by formula

\[
a \triangleright b = \{ c \in \mathbb{R} : \log(|e^a - e^b|) \leq c \leq \log(e^a + e^b) \},
\]

while the multiplication in \( \Delta^{\log} \) is the usual addition.

5.5. Multiplicative seminorm. Let \( K \) be a ring. Recall that a map \( K \to \mathbb{R}_+ : x \mapsto |x| \) is a multiplicative seminorm if \( |x + y| \leq |x| + |y| \) and \( |xy| = |x||y| \) for any \( x, y \in K \). Obviously, a multiplicative seminorm is nothing but a multiring homomorphism of \( K \to \Delta \).

5.6. Non-archimedian multiplicative seminorm. Recall that a multiplicative seminorm \( K \to \mathbb{R}_+ : x \mapsto |x| \) is non-archimedian if \( |x + y| \leq \max(|x|, |y|) \). A non-archimedian multiplicative seminorm \( K \to \mathbb{R}_+ \) is a multiring homomorphism \( K \to \mathbb{Y}_x \). A non-archimedian valuation map, that is a composition of a non-archimedian multiplicative seminorm with \( \mathbb{R}_+ \to \mathbb{Y} : x \mapsto \log x \), is a multiring homomorphism \( K \to \mathbb{Y} \).

6. Tropical addition of complex numbers

6.1. Definition. The tropical sum \( a \sim b \) of arbitrary complex numbers \( a \) and \( b \) is defined as follows.

- If \( |a| > |b| \), then \( a \sim b = a \).
- If \( |a| < |b| \), then \( a \sim b = b \).
- If \( |a| = |b| \) and \( a + b \neq 0 \), then \( a \sim b \) is the set of all complex numbers which belong to the shortest arc connecting \( a \) with \( b \) on the circle of complex numbers with the same absolute value.

In formulas: if \( a = re^{\alpha i}, b = re^{\beta i} \) with \( |\beta - \alpha| < \pi \), then \( a \sim b = \{ re^{\varphi i} : |\alpha - \varphi| + |\varphi - \beta| = |\alpha - \beta| \} \).

- If \( a + b = 0 \), then \( a \sim b \) is the whole closed disk \( \{ c \in \mathbb{C} : |c| \leq |a| \} \).

6.2. Obvious properties. The tropical addition is commutative, \( a \sim b = b \sim a \) for any \( a, b \in \mathbb{C} \). This follows immediately from the definition.

The zero plays the same role of the neutral element as it plays for the usual addition: \( a \sim 0 = a \) for any \( a \in \mathbb{C} \).

Furthermore, for any complex number \( a \) there is a unique \( b \) such that \( 0 \in a \sim b \). This \( b \) is \(-a\).
6.3. Associativity.

**6.A. Theorem.** The tropical addition of complex numbers is associative.

A straightforward proof is elementary, but quite cumbersome. It is postponed till Appendix 1.

6.4. Distributivity.

**6.B. Theorem.** The usual multiplication of complex numbers is distributive over the tropical addition: \( a(b \sim c) = ab \sim ac \) for any complex numbers \( a, b \) and \( c \).

Indeed, all the constructions and characteristics of summands involved in the definition of tropical addition are invariant under multiplication by a complex number: the ratio of absolute values of two complex numbers is preserved, an arc of a circle centered at 0 is mapped to an arc of a circle centered at 0, a disk centered at 0 is mapped to a disk centered at 0.

**6.C. Theorem.** The multiplication of complex numbers is not doubly distributive over the tropical addition.

**Proof.** Compare \((1 \sim i)(1 \sim -i)\) with \(1 \sim 1 \sim 1 \sim -i \sim i \sim 1 \sim (-i) = 1 \sim -i \sim -i \sim 1\).
Since $1 \sim i$ is the arc of the unit circle connecting $1$ and $i$, and $1 \sim -i$ is the arc of the unit circle connecting $1$ and $-i$, their (pointwise) product is the arc of the unit circle connecting $i$ and $-i$. On the other hand, the tropical sum $1 \sim i \sim -i \sim 1$ is the whole unit disk. \hfill \square

6.5. **Complex tropical hyperfield.** Thus, the set $\mathbb{C}$ of complex numbers with the tropical addition and usual multiplication is a hyperfield. Denote it by $\mathcal{T}\mathbb{C}$ and call **complex tropical hyperfield.**

6.6. **The tropical sum of several complex numbers.** The tropical sum of several complex numbers is affected only by those summands which have the greatest absolute value. A summand whose absolute values is not maximal does not contribute at all.

6.D. **Theorem.** Let $a_1, \ldots, a_n$ be complex numbers with absolute values equal $r$. Then

- either $a_1 \sim \ldots \sim a_n$ is the closed disk with radius $r$ centered at $0$, it can be obtained as the sum of at most three of the summands $a_1, \ldots, a_n$ and $0 \in \text{Conv}(a_1, \ldots, a_n)$,
- or $a_1 \sim \ldots \sim a_n$ is contained in a half of the circle of radius $r$ centered at $0$ and is the tropical sum of at most two of the summands $a_1, \ldots, a_n$ (so, it is either a point or a closed arc).

The proof of Theorem 6.D is elementary and straightforward. See Appendix 2.

6.E. **Corollary.** The tropical sum of any finite set of complex numbers equals the tropical sum of a subset consisting at most of three summands. If the tropical sum does not contain the zero, then the number of summands can be reduced to two. \hfill \square

6.F. **Corollary.** The tropical sum of a finite set of complex numbers contains the zero iff the zero is contained in the convex hull of the summands having the greatest absolute value. \hfill \square

7. Relations of $\mathcal{T}\mathbb{C}$ with other hyperfields

7.1. Submultirings and subhyperfields.

7.A. **Theorem.** Any subset $A$ of $\mathbb{C}$ containing $0$, invariant under the involution $x \mapsto -x$ and closed with respect to multiplication inherits the structure of multiring from $\mathcal{T}\mathbb{C}$.

If, furthermore, $A \setminus 0$ is invariant under the involution $x \mapsto x^{-1}$, then $A$ with the inherited structure is a hyperfield. \hfill \square
In particular, any subfield of \( \mathbb{C} \) inherits structure of hyperfield from \( \mathcal{T} \mathbb{C} \).

**7.2. The tropical real hyperfield** \( \mathcal{T} \mathbb{R} \). For example, \( \mathbb{R} \) inherits the structure of hyperfield. The induced addition \((a, b) \mapsto a \sim_{\mathbb{R}} b = (a \sim b) \cap \mathbb{R}\) can be described directly as follows:

\[
a \sim_{\mathbb{R}} b = \begin{cases} 
\{a\}, & \text{if } |a| > |b|, \\
\{b\}, & \text{if } |a| < |b|, \\
\{a\}, & \text{if } a = b, \\
[-|a|, |a|], & \text{if } a = -b.
\end{cases}
\]

See also Figure 2.

![Figure 2. Tropical addition of real numbers.](image)

The operation \((a, b) \mapsto a \sim_{\mathbb{R}} b\) is called the **tropical real addition** or even just **tropical addition**, when there is no danger of confusion. The set \( \mathbb{R} \) with the tropical real addition and usual multiplication is called the **tropical real hyperfield** and denoted by \( \mathcal{T} \mathbb{R} \).

**7.B. Theorem.** \( \mathcal{T} \mathbb{R} \) is doubly distributive.

**Proof.** The only situation in which double distributivity

\[
(a \sim_{\mathbb{R}} b)(c \sim_{\mathbb{R}} d) = ac \sim_{\mathbb{R}} ad \sim_{\mathbb{R}} bc \sim_{\mathbb{R}} bd
\]

is not obvious, is when both factors in the left hand side consist of more than one element. Then \( a = -b \) and \( c = -d \) and both the left hand side and right hand side equal \([-|ac|, |ac|]\). □

**7.3. The hyperfield of signs.** A structure of hyperfield comes in the same way to subsets \( A \subset \mathbb{C} \) which are not subfields of \( \mathbb{C} \). For example, \( \{-1, 0, +1\} \subset \mathbb{C} \) satisfies the conditions of Theorem 7.A, and hence inherits a hyperfield structure from \( \mathcal{T} \mathbb{C} \). This hyperfield appeared above as \( \mathbb{S} \).
Recall that there is a multiring homomorphism
\[ \mathbb{R} \to \{0, \pm 1\} : x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \]
of the field \( \mathbb{R} \) to the hyperfield \( \mathbb{S} \).
This map is also a multiring homomorphism \( \mathcal{T}\mathbb{R} \to \mathbb{S} \).

### 7.4. The phase hyperfield.
Let \( \Phi \) be \( \{z \in \mathbb{C} : |z| = 1\} \cup \{0\} \), that is the unit circle in \( \mathbb{C} \) united with its center. This set satisfies the conditions of Theorem 7.4, and, by that theorem, \( \Phi \) inherits a hyperfield structure from \( \mathcal{T}\mathbb{C} \). It will be called the phase hyperfield and denoted by \( \Phi \). One can obtain \( \Phi \) also as \( \mathbb{C}/_{m}\mathbb{R}_{>0} \). Notice that \( \mathbb{S} = \Phi \cap \mathbb{R} \).

The map
\[ \mathbb{C} \to \Phi : z \mapsto \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases} \]
is called the phase map. This is a multiring homomorphism in two senses: \( \mathbb{C} \to \Phi \) and \( \mathcal{T}\mathbb{C} \to \Phi \).

### 7.5. Embedding \( \mathcal{T} \subset \mathcal{T}\mathbb{C} \).
Recall that a semifield is a set with two (univalued) operations, addition and multiplication, which satisfy all the axioms of field, except that there is no subtraction.

A classical example of a semifield is the set \( \mathbb{R}_{+} \) of non-negative real numbers with the usual addition and multiplication. Another semifield structure in the same set is defined by replacing the usual addition with the operation of taking the greatest of two numbers: \( (a, b) \mapsto \max(a, b) \).

There is an isomorphism of the tropical semifield \( \mathcal{T} \) onto the semifield \( \mathbb{R}_{\geq 0, \max, \times} \) mapping \( x \mapsto \exp x \) for \( x > 0 \), and \( -\infty \mapsto 0 \).

Observe that the semifield addition \( (a, b) \mapsto \max(a, b) \) in \( \mathbb{R}_{+} \) is induced from the addition in \( \mathcal{T}\mathbb{C} \) (or \( \mathcal{T}\mathbb{R} \), does not matter). Indeed, \( a \sim b = \max(a, b) \) for any \( a, b \in \mathbb{R}_{+} \).

Thus, the semifield \( \mathbb{R}_{\geq 0, \max, \times} \) is a subset of the hyperfield \( \mathcal{T}\mathbb{C} \) closed with respect to both binary operations of \( \mathcal{T}\mathbb{C} \), and the binary operations coincide with the operations of the semifield \( \mathbb{R}_{\geq 0, \max, \times} \). In particular, the inclusion \( \mathbb{R}_{\geq 0, \max, \times} \to \mathcal{T}\mathbb{C} \) and its composition \( \mathcal{T} \to \mathcal{T}\mathbb{C} \) with the isomorphism \( \mathcal{T} \to \mathbb{R}_{\geq 0, \max, \times} \) are homomorphisms.

**Warning.** There is a natural map in the opposite direction \( \mathcal{T}\mathbb{C} \to \mathbb{R}_{+} : z \mapsto |z| \). It is a right inverse for the inclusion. However, this is not a homomorphism for the tropical addition \( \sim \). Indeed, \( x \sim (-x) \cap \mathbb{R}_{+} = [0, |x|] \) for any \( x \in \mathbb{R} \), but \( |x| \sim -x = |x| \), which does not contain \( [0, |x|] \) for \( x \neq 0 \).
In order to make the map $\mathcal{T}C \to \mathbb{R}_+: z \mapsto |z|$ a homomorphism, one should consider a hyperfield structure in $\mathbb{R}_+$.

### 7.6. The absolute value and amoeba maps.

The map $\mathbb{C} \to \mathbb{R}_+: z \mapsto |z|$ is also a homomorphism from many points of view. This is

- a multiring homomorphism $\mathbb{C} \to \Delta$ from the field of complex numbers to the triangle hyperfield (see Section 5.1);
- a multiring homomorphism $\mathcal{T}C \to \Delta$ from the complex tropical hyperfield $\mathcal{T}C$ to the triangle hyperfield;
- a multiring homomorphism $\mathcal{T}C \to \mathcal{Y}_x$ from $\mathcal{T}C$ to the ultratriangle hyperfield (see Section 5.2);

The composition of this map with $\log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is a multiring homomorphism

- $\mathbb{C} \to \Delta^{\log}$;
- $\mathcal{T}C \to \Delta^{\log}$;
- $\mathcal{T}C \to \mathcal{Y}$.

### 7.7. Complex polynomials and $\mathcal{T}C$.

The map $w$ which is defined and discussed in this section and the next one, essentially was defined by Mikhalkin [13] and used him in his definition of complex tropical curves. However, the algebraic properties of $w$ were not considered, because the tropical addition of complex numbers was not available.

Let $p(X) \in \mathbb{C}[X]$ be a polynomial in one variable $X$ with complex coefficients, $p(X) = \sum_{k=0}^n a_k X^k$, where $a_k \in \mathbb{C}$, $a_n \neq 0$. Let $w(p) = \frac{a_n}{|a_n|} e^{|a_n|}$. Further, let $w(0) = 0$. This defines a map $\mathbb{C}[X] \to \mathbb{C}: p \mapsto w(p)$.

### 7.C. Theorem.

The map $w$ is a multiring homomorphism of the polynomial ring $\mathbb{C}[X]$ to the hyperfield $\mathcal{T}C$, that is $w(p+q) = w(p) \vee w(q)$ and $w(pq) = w(p)w(q)$ for any $p, q \in \mathbb{C}[X]$.

**Proof.** The value of $w$ on a polynomial $p$ is equal to the value of $w$ on the monomial of $p$ having the greatest degree. For a monomial $p(X) = aX^n$ the value of $w$ equals $\frac{p(e)}{|p(1)|}$. Obviously, the latter formula defines a multiplicative homomorphism.

Let us prove that $w(p+q) = w(p) \vee w(q)$ for any $p, q \in \mathbb{C}[X]$. Let the highest degree monomials of $p$ and $q$ are $aX^n$ and $bX^m$, respectively (so that $\deg p = n$, $\deg q = m$). If $n > m$, then the highest degree term of $p+q$ equals $aX^n$ and $w(p+q) = w(p) = w(p) \vee w(q)$. Similarly, if $n < m$, then $w(p+q) = w(q) = w(p) \vee w(q)$.

If the degrees of $p$ and $q$ are the same, and the coefficients $a$ and $b$ of their monomials of the highest degree are such that $\frac{a}{|a|} + \frac{b}{|b|} \neq 0$, then these monomials do not annihilate each other in the sum, and
the monomial of highest degree of $p + q$ is the sum of these monomials. Its degree equals $\deg p = \deg q$, the coefficient is $a + b$. However, the argument $\frac{a+b}{|a+b|}$ of this coefficient is not determined by $\frac{a}{|a|}$ and $\frac{b}{|b|}$. It can take any value in the open interval between the arguments of the summands. In particular, it takes values in the set of arguments of complex numbers belonging to $w(p) \cap w(q)$.

If $\deg p = \deg q$ and the coefficients $a$ and $b$ of the highest terms are such that $\frac{a}{|a|} + \frac{b}{|b|} = 0$, then the highest terms may annihilate under summation. Therefore the highest term of $p + q$ is either equal to the sum of the highest terms of $p$ and $q$, or come from terms of lower degrees and cannot be recovered from the terms of the highest degree. The only that we can say about it if we know only $w(p)$ and $w(q)$ (i.e., if we know only the arguments of the coefficients in the terms of the highest degrees and the degrees), is that its degree is not greater than the degree of the summands. This implies $w(p + q) \in w(p) \cap w(q)$. □

7.8. Real exponents. The image of $w$ consists of only those complex numbers whose absolute values are powers of $e$. However similar constructions are able to provide multiring homomorphisms onto the whole $tc$. For this, it is enough to replace usual polynomials by polynomials with arbitrary real exponents.

Let us replace $\mathbb{C}[X]$ by the group algebra $\mathbb{C}[\mathbb{R}]$ of the additive group $\mathbb{R}$. Elements of $\mathbb{C}[\mathbb{R}]$ can be thought of as $\sum_k a_k X^{r_k}$, where $a_k \in \mathbb{C}$, $r_k \in \mathbb{R}$. The formal variable $X$ symbolizes here the transition from additive notation for addition in $\mathbb{R}$ to multiplicative notation in $\mathbb{C}[\mathbb{R}]$, where additive notation is reserved for the formal sum.

Elements of $\mathbb{C}[\mathbb{R}]$ may be interpreted as functions $\mathbb{C} \rightarrow \mathbb{C}$. For this, let us turn $\sum_k a_k X^{r_k}$ into an exponential sum $\sum_k a_k e^{r_k T}$ by replacing $X$ with $e^T$.

The map $w : \mathbb{C}[X] \rightarrow \mathbb{C}$ extends to $\mathbb{C}[\mathbb{R}]$ as follows: choose from the sum $\sum_k a_k X^{r_k}$ the summand with the greatest exponent, say, $a_n X^{r_n}$ and apply the same formula to it $\frac{a_n}{|a_n|} e^{r_n}$. The map is a multiring homomorphism of the ring $\mathbb{C}[\mathbb{R}]$ onto the hyperfield $\mathcal{T} \mathbb{C}$. The proof that this is a multiring homomorphism is literally the same as the proof of Theorem [7.3] above.

A ring can be replaced here by an algebraically closed field real-power Puiseux series $\sum_{r \in I} a_r t^r$, where $I \subset \mathbb{R}$ is a well-ordered set. Cf. Mikhalkin [13], Section 6.

This construction demonstrates how one can obtain the tropical addition of complex numbers from the usual addition of polynomials. It is clear why it should be multivalued. For complex numbers $a$ and $b$ with $|a| = |b|$, but $a \neq -b$ any $c$ for the open arc $(a \cap b) \setminus \{a, b\}$, one can
find $A, B, C \in \mathbb{C}[\mathbb{R}]$ such that $w(A) = a$, $w(B) = b$ and $w(C) = c$, see Figure 3. Complex numbers $a, b \in \mathbb{C}$ with $a + b = 0$ are represented as the images under $w$ of polynomials $A, B \in \mathbb{C}[\mathbb{R}]$ with highest degree terms opposite to each other and annihilating under addition of the polynomials. The highest degree term of $A + B$ is not controlled by the highest degree terms of the summands $A$ and $B$, but its degree does not exceed the degree of the summands.

8. Continuity

In the set of all subsets of a topological space, there are various natural topological structures. However none of them is perfect. The most classical of them are three structures introduced by Vietoris [17] in 1922. The multivalued additions considered above are continuous with respect to one of them, the upper Vietoris topology, and this implies important properties of multivalued functions defined by polynomials over these hyperfields.

8.1. Vietoris topologies. The upper Vietoris topology in the set $2^X$ of all subsets of a topological space $X$ is the topology generated by the sets $2^U \subset 2^X$, where $U$ is open in $X$. A neighborhood of a set $A \subset X$ in the upper Vietoris topology should contain all subsets of a set $U$ that is open in $X$ and contains $A$.

This topology is quite odd. For example, it is far from being Hausdorff: sets with non-empty intersection cannot have disjoint neighborhoods in it. Therefore usually a limit in the upper Vietoris topology is not unique. By adding new points to a limit we would get a limit. Probably this is what motivates the word upper in the name of the topology.

The lower Vietoris topology in the set $2^X$ of all subsets of a topological space $X$ is the topology generated by the sets $2^X \setminus 2^C$, where $C$ is a
closed subset of \(X\). In other words, the lower Vietoris topology is generated by sets \(\{Y \subset X : Y \cap U \neq \emptyset\}\), where \(U\) is an open set of \(X\). In the lower Vietoris topology, closed sets are generated by closed sets of \(X\) in the most direct way: a closed set \(C \subset X\) gives rise to the set \(2^C \subset 2^X\) closed in the lower Vietoris topology. Recall that in the upper Vietoris topology open sets are generated similarly by open subsets of \(X\). A neighborhood of a set \(A \in 2^X\) in the lower Vietoris topology should contain all sets intersecting with open sets \(U_1, \ldots, U_n \subset X\) which meet \(A\). A limit in the lower Vietoris topology also usually is not unique, but for the opposite reason: it would stay a limit under removing of its points.

The topology generated by the upper and lower Vietoris topologies is called just the Vietoris topology.

### 8.2. Continuity and semi-continuities.

A multimap \(X \twoheadrightarrow Y\) is said to be

- **upper semi-continuous** if the corresponding map \(f^\uparrow : X \to 2^Y\) is continuous with respect to the upper Vietoris topology in \(2^Y\);
- **lower semi-continuous** if \(f^\downarrow : X \to 2^Y\) is continuous with respect to the lower Vietoris topology in \(2^Y\);
- **continuous** if \(f^\uparrow : X \to 2^Y\) is continuous with respect to the Vietoris topology in \(2^Y\) (i.e., \(X \twoheadrightarrow Y\) is both upper and lower semi-continuous).

Recall that the set \(\{a \in X : f(a) \subset B\}\) is called the **upper preimage** of \(B\) under \(f\), and the set \(\{a \in X : f(a) \cap B \neq \emptyset\}\) is called the **lower preimage** of \(B\) under \(f\).

It is easy to see that \(f : X \twoheadrightarrow Y\) is upper (respectively, lower) semi-continuous if and only if the upper (respectively, lower) preimage of any set open in \(Y\) is open in \(X\).

### 8.3. Tropical additions.

**8.A. Theorem.** The tropical addition \(\mathbb{C} \times \mathbb{C} \twoheadrightarrow \mathbb{C} : (a, b) \mapsto a \sim b\) is not lower semi-continuous (i.e., the corresponding map \(\mathbb{C} \times \mathbb{C} \to 2^\mathbb{C}\) is not continuous with respect to the classical topology in \(\mathbb{C}^2\) and the lower Vietoris topology in \(2^\mathbb{C}\)).

**Proof.** It would suffice to find a set such that it is open in the lower Vietoris topology and its preimage is not open in the classical topology of \(\mathbb{C}^2\). Take, for instance, the set \(H\) consisting of sets \(A\) meeting the open disk of radius 1 and center 0. Its preimage is the set of pairs \((a, b)\) of complex numbers such that \(a \sim b\) meets the disk. The preimage of \(H\) consists of pairs \((a, b)\) which satisfy one of the following two conditions: either \(|a| < 1\) and \(|b| < 1\), or \(a = -b\). Obviously, this set is not open. \(\square\)
Similarly one can prove that the additions in the ultratriangle hyperfield \( \mathbb{Y}_x \) (see Section 5.2), the tropical hyperfield \( \mathbb{Y} \) (Section 5.3) and the real tropical hyperfield \( T \mathbb{R} \) (Section 7.1) are not lower semi-continuous.

**8.B. Theorem.** The tropical addition \( \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} : (a, b) \mapsto a \bowtie b \) is upper semi-continuous (i.e., the corresponding map \( \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^2 \) is continuous with respect to the classical topology in \( \mathbb{C}^2 \) and the upper Vietoris topology in \( \mathbb{C}^2 \)).

**Proof.** Let us prove the corresponding local continuity, i.e., prove that for any neighborhood \( V \subset \mathbb{C}^2 \) of the image \( a \bowtie b \) of \( (a, b) \) there exists a neighborhood \( U \subset \mathbb{C}^2 \) of \( (a, b) \) such that the image of \( U \) is contained in \( V \). In the upper Vietoris topology, a base of neighborhoods of \( a \bowtie b \) is composed by sets \( 2^W \) where \( W \) runs over a base of neighborhoods of \( a \bowtie b \) in \( \mathbb{C} \). Thus, it would suffice for an arbitrarily small neighborhood \( W \supset a \bowtie b \) to find a neighborhood \( U \) of \( (a, b) \) in \( \mathbb{C}^2 \) such that \( x \bowtie y \in W \) for any \( (x, y) \in U \). Consider one by one each of the three kinds of \( (a, b) \).

If \( |a| > |b| \), then \( a \bowtie b = a \). Any neighborhood of \( a \) contains an open disk centered at \( a \). Diminish it if needed in order to ensure that its radius \( r \) is smaller than \( \frac{1}{2}(|a| - |b|) \). Choose for \( W \) an open disk \( B_r(a) \) of radius \( r \) centered at \( a \). Then for \( U \) one can take the neighborhood \( B_r(a) \times B_r(b) \) of \( (a, b) \). Obviously, \( B_r(a) \bowtie B_r(b) = B_r(a) \).

If \( |a| = |b| \) and \( a \bowtie b \neq 0 \), then \( a \bowtie b \) is the shortest arc \( C \) connecting \( a \) and \( b \) in the circle centered at 0. Let \( r \) be a positive real number, which so small that the disks \( B_r(a) \) and \( B_r(b) \) do not contain points symmetric to each other with respect to 0. Any neighborhood of \( C \) in \( \mathbb{C} \) contains \( W = B_{\rho}(a) \bowtie B_{\rho}(b) \) with some \( \rho \in (0, r) \). Let \( U = B_{\rho}(a) \times B_{\rho}(b) \).

If \( |a| = |b| \) and \( a + b = 0 \), then \( a \bowtie b \) is the closed disk centered at 0 with radius \( |a| \). Any neighborhood of this disk in \( \mathbb{C} \) contains a concentric open disk of some radius \( r > |a| \). Let \( U \) be this disk. The image of \( U \times U \) under the tropical addition is \( U \).

Similarly one can prove that the tropical addition of real numbers is upper semi-continuous.

**8.4. Continuity of triangle additions.** Recall that the triangle addition of non-negative real numbers is defined by formula \( a \triangledown b = \{ c \in \mathbb{R}_+ : |a - b| \leq c \leq a + b \} \).

**8.C. Lemma.** A multimap \( f : X \rightarrow \mathbb{R} \) is continuous if there exist continuous functions \( f_+: X \rightarrow \mathbb{R} \) with \( f(x) = [f_-(x), f_+(x)] \) for any \( x \in X \) and \( f_+(x) > f_-(x) \) on everywhere dense subset of \( X \).
The triangle addition satisfies the hypothesis of Lemma, hence it is continuous, (i.e., the corresponding map \( \mathbb{R}_+ \times \mathbb{R}_+ \to 2^{\mathbb{R}_+} \) is continuous with respect to the classical topology in \( \mathbb{R}_+ \times \mathbb{R}_+ \) the the Vietoris topology in \( 2^{\mathbb{R}_+} \)).

Similarly, from Lemma 8.C it follows that the addition in the amoeba hyperfield \( \Delta^{log} \) is continuous.

### 8.5. Properties of upper semi-continuous multimaps.

As we see above the additions in hyperfields \( \mathcal{T}C, \mathcal{T}R, \Delta, \mathcal{Y}_x, \mathcal{Y} \) and \( \Delta^{log} \) are upper semi-continuous. Let \( K \) denote one of these hyperfields.

First, notice, that for univalued maps upper semi-continuity is equivalent to continuity.

Second, obviously, a composition of upper semicontinuous maps is upper semi-continuous.

From these two statements it follows immediately that a multimap defined by a polynomial over \( K \) is upper semi-continuous.

**8.D. Theorem.** Let \( X, Y \) be topological spaces, \( f : X \to Y \) be an upper semi-continuous multimap and \( C \subset Y \) be a closed set. Then the set \( \{ a \in X : f(a) \cap C \neq \emptyset \} \) is closed.

**Proof.** The set \( \{ B \in 2^Y : B \subset X \setminus C \} \) is open in the upper Vietoris topology of \( 2^Y \). Therefore, due to upper semi-continuity of multimap \( f \), the preimage \( f^+ = \{ a \in X : f(a) \subset X \setminus C \} \) of this set under the \( f^+ : X \to 2^Y \) is open. Consequently, the set \( \{ a \in X : f(a) \cap C \neq \emptyset \} = X \setminus \{ a \in X : f(a) \subset X \setminus C \} \) is closed. \( \Box \)

**8.E. Corollary.** For any polynomial \( p \) over \( K \), the set defined by condition \( 0 \in p(x_1, \ldots, x_n) \) is closed in \( K^n \). \( \square \)

Finally, recall two well-known theorems about upper semi-continuous multimaps.

**8.F. Theorem.** The image of a connected set under an upper semi-continuous multimap is connected, if the image of each point is connected.

**8.G. Theorem.** The image of a compact set under an upper semi-continuous multimap is compact, if the image of each point is compact.

**8.H. Corollary.** A multimap defined by a polynomial \( p(x_1, \ldots, x_n) \) over \( K \) maps connected sets to connected and compact sets to compact. In particular, the graph of \( p \) is connected. \( \square \)
9. Dequantizations

9.1. The Litvinov-Maslov dequantization. Consider a family of semirings \(\{S_h\}_{h \in [0,\infty)}\) (recall that a semiring is a sort of ring, but without subtraction). As a set, each of \(S_h\) is \(\mathbb{R}\). The semiring operations \(+_h\) and \(\times_h\) in \(S_h\) are defined as follows:

\[
\begin{align*}
    a +_h b &= \begin{cases} 
    h \ln(e^{a/h} + e^{b/h}), & \text{if } h > 0 \\
    \max\{a, b\}, & \text{if } h = 0
    \end{cases} \\
    a \times_h b &= a + b
\end{align*}
\]

These operations depend continuously on \(h\). For each \(h > 0\) the map

\[
D_h : \mathbb{R}_{>0} \to S_h : x \mapsto h \ln x
\]

is a semiring isomorphism of \(\{\mathbb{R}_{>0}, +, \cdot\}\) onto \(\{S_h, +_h, \times_h\}\), that is

\[
D_h(a + b) = D_h(a) +_h D_h(b), \quad D_h(ab) = D_h(a) \times_h D_h(b).
\]

Thus \(S_h\) with \(h > 0\) can be considered as a copy of \(\mathbb{R}_{>0}\) with the usual operations of addition and multiplication. On the other hand, \(S_0\) is a copy \(\mathbb{R}_{\max,+}\) of \(\mathbb{R}\), where the operation of taking maximum is considered as an addition, and the usual addition, as a multiplication.

Applying the terminology of quantization to this deformation, we must call \(S_0\) a classical object, and \(S_h\) with \(h \neq 0\), quantum ones. The whole deformation is called the **Litvinov-Maslov dequantization** of positive real numbers. The addition in the resulting semiring \(\mathbb{R}_{\max,+}\), is idempotent in the sense that \(\max(a, a) = a\) for any \(a\).

The analogy with Quantum Mechanics motivated the following

**Correspondence principle** (Litvinov and Maslov [8]). “There exists a (heuristic) correspondence, in the spirit of the correspondence principle in Quantum Mechanics, between important, useful and interesting constructions and results over the field of real (or complex) numbers (or the semiring of all nonnegative numbers) and similar constructions and results over idempotent semirings.”

This principle proved to be very fruitful in a number of situations, see [8], [9]. The Litvinov-Maslov dequantization helps to relate the corresponding things.

Indeed, any valid formula involving only positive real numbers and only arithmetic operations survives under the limit and turns into a valid formula in \(\mathbb{R}_{\max,+}\).

The correspondence principle is formulated much wider, than this transition to limit allows: not only for the semirings of all positive real numbers and \(\mathbb{R}_{\max,+}\), but for any idempotent semiring, on one hand, and the fields \(\mathbb{R}\) and \(\mathbb{C}\), on the other hand.
One may expect that there are extra mathematical reasons for this heuristic correspondence. Below similar dequantization deformations are presented. However, the dequantized objects are not semifields, but rather multifields.

9.2. Dequantization of the triangular hyperfield to the ultra-triangular. For a positive real number $h$, let $R_h : \mathbb{R}_+ \to \mathbb{R}_+$ be the map defined by formula $x \mapsto x^h$. This map is invertible. Its inverse is defined by $R_h^{-1} : x \mapsto x^{1/h}$.

Obviously, $R_h$ is an isomorphism with respect to multiplication, and does not commute with the triangular addition

$$(a, b) \mapsto a \triangledown b = \{ c \in \mathbb{R}_+ : |a-b| \leq c \leq a+b \}.$$ 

In order to make $R_h$ a hyperfield isomorphism, pull back the triangular addition, that is define

$$a \triangledown_h b = R_h^{-1}(R_h(a) \triangledown R_h(b)) = \{ c \in \mathbb{R}_+ : |a^{1/h} - b^{1/h}|^h \leq c \leq (a^{1/h} + b^{1/h})^h \}.$$ 

Observe that if $a \neq b$, then

$$\lim_{h \to 0} |a^{1/h} - b^{1/h}|^h = \lim_{h \to 0} (a^{1/h} + b^{1/h})^h = \max(a, b),$$

and if $a = b$, then $|a^{1/h} - b^{1/h}|^h = 0$, while $\lim_{h \to 0} (a^{1/h} + b^{1/h})^h = a$. Thus the endpoints of the segment $a \triangledown_h b$ tend to the endpoints of the segment $a \triangledown b$ as $h \to 0$. Define $a \triangledown_0 b$ to be $a \triangledown b$.

For $h \geq 0$, denote by $\Delta_h$ the hyperfield with the underlying set $\mathbb{R}_+$, addition $\triangledown_h$ and multiplication coinciding with the usual multiplication of real numbers. For $h > 0$, the map $R_h$ is an isomorphism $\Delta_h \to \Delta$. The hyperfield $\Delta_0$ coincides with $\mathbb{Y}_x$.

Thus, $\Delta_h$ is a dequantization (degeneration) of $\Delta$ to $\mathbb{Y}_x$. The map $\log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ converts $\Delta_h$ into a dequantization of the amoeba hyperfield $\Delta_{\log}$ to the tropical hyperfield $\mathbb{Y}$.

9.3. Dequantization of $\mathbb{C}$ to $\mathbb{T}\mathbb{C}$. For a positive real number $h$, let $S_h : \mathbb{C} \to \mathbb{C}$ be the map defined by the formula

$$z \mapsto \begin{cases} |z|^{1/h} \frac{z}{|z|} & \text{for } z \neq 0; \\ 0 & \text{for } z = 0. \end{cases}$$

This map is invertible. Its inverse is defined by the formula

$$S_h^{-1} : z \mapsto \begin{cases} |z|^{1/h} \frac{z}{|z|} & \text{for } z \neq 0; \\ 0 & \text{for } z = 0. \end{cases}$$

Obviously, $S_h$ is an isomorphism with respect to multiplication, that is $S_h(ab) = S_h(a)S_h(b)$. However, it does not commute with addition.
In order to make $S_h$ an isomorphism with respect to addition, let us redefine the addition on the source of the map. In other words, induce a binary operation on the set of complex numbers:

$$a +_h b = S_h^{-1}(S_h(a) + S_h(b)).$$

In this way we get a field $\mathbb{C}_h = (\mathbb{C}, +_h, \times)$ (which is nothing but a copy of $\mathbb{C}$) and an isomorphism $S_h : \mathbb{C}_h \to \mathbb{C}$.

It is easy to see that $a +_h b$ converges as $h$ tends to zero. Namely:

- if $|a| > |b|$, then $\lim_{h \to 0}(a +_h b) = a$;
- if $|a| = |b|$ and $a + b \neq 0$, then $\lim_{h \to 0}(a +_h b) = \frac{a + b}{|a + b|}$;
- if $a + b = 0$, then $\lim_{h \to 0}(a +_h b) = 0$.

Denote $\lim_{h \to 0}(a +_h b)$ by $a +_0 b$. See figure 4.

![Figure 4](image)

**Figure 4.** The limit $a +_0 b$ of $a +_h b$ as $h \to 0$.

Some properties of the operation $(a, b) \mapsto a +_0 b$ are nice. It is commutative, distributive for the usual multiplication of complex numbers, the zero behaves appropriately: $a +_0 0 = a$ for any $a \in \mathbb{C}$. Furthermore, for any $a \in \mathbb{C}$ there exists a unique complex number $b$ such that $a +_0 b = 0$, and this $b$ is nothing but $-a$.

However, the operation $(a, b) \mapsto a +_0 b$ is far from being perfect. First, $a +_0 b$ is not continuous as a function of $a$ and $b$. Certainly, this happens because the convergence $a +_h b \to a +_0 b$ is not uniform. Second, it is not associative.
In order to see the latter, compare \((-1 +_0 i) +_0 1 = 1 +_0 (i +_0 1)\):

\[
(-1 +_0 i) +_0 1 = \left( \exp(\pi i) +_0 \exp\left(\frac{\pi i}{2}\right) \right) +_0 1 \\
= \exp\left(\frac{3\pi i}{4}\right) +_0 \exp(0) = \exp\left(\frac{3\pi i}{8}\right)
\]

On the other hand,

\[
1 +_0 (i +_0 1) = \exp(\pi i) +_0 \left( \exp\left(\frac{\pi i}{2}\right) +_0 \exp(0) \right) \\
= \exp(\pi i) +_0 \exp\left(\frac{\pi i}{4}\right) = \exp\left(\frac{5\pi i}{8}\right).
\]

The tropical addition \((a, b) \mapsto a \sim b\) introduced in Section 6.1 above does not have this defect. It is associative, see Appendix 1. Though, it is multivalued.

Another advantage of the tropical addition is that it is upper semi-continuous, see Section 8.3. The tropical addition is also a limit of \(+_h\) as \(h \to 0\), but not in the sense of pointwise convergence.

**9.A. Theorem.** Let \(\Gamma = \{(a, b, a +_h b, h) \in \mathbb{C}^3 \times \mathbb{R}_+ : h > 0, a \in \mathbb{C}, b \in \mathbb{C}\}\). Then the intersection of \(\mathbb{C}^3 \times \{0\}\) with the closure of \(\Gamma\) is the \(\Gamma_\sim \times \{0\}\), where \(\Gamma_\sim\) is the graph \(\{(a, b, a \sim b) : a \in \mathbb{C}, b \in \mathbb{C}\}\) of \((a, b) \mapsto a \sim b\).

Thus the tropical addition of complex numbers is a dequantization of the usual addition of complex numbers in the same way as taking maximum is a dequantization of the usual addition of positive real numbers.

**Proof of Theorem 9.A.** In order to prove the equality

\[
\Gamma_\sim \times \{0\} = \text{Cl}(\Gamma) \cap \mathbb{C}^3 \times \{0\},
\]

that is the content of Theorem 9.A, we will prove the corresponding two inclusions:

(3) \(\Gamma_\sim \times \{0\} \subseteq \text{Cl}(\Gamma) \cap \mathbb{C}^3 \times \{0\}\),

(4) \(\Gamma_\sim \times \{0\} \supset \text{Cl}(\Gamma) \cap \mathbb{C}^3 \times \{0\}\).

**Proof of (3).** There are three types of points in \(\Gamma_\sim \subseteq \mathbb{C}^3\):

1. \((a, b, a)\) with \(|a| > |b|\), or \((a, b, b)\) with \(|a| < |b|\);
2. \((a, b, c)\) with \(|a| = |b| = |c|\) and \(a + b \neq 0\);
3. \((a, -a, b)\) with \(|b| \leq |a|\).
In the first case, if \(|a| > |b|\), then \(a = a +_0 b\), hence \((a, b, a)\) belongs to the graph of \(+_0\), and therefore \((a, b, a, 0)\) belongs to the closure of \(\Gamma\). If \(|a| < |b|\), then \(b = a +_0 b\), hence \((a, b, b)\) belongs to the graph of \(+_0\), and therefore \((a, b, b, 0)\) belongs to the closure of \(\Gamma\).

In the second case, let \(|a| = |b| = |c| = r\). Recall that \(c\) belongs to the shortest arc connecting \(a\) and \(b\) on the circle \(|z| = r\). Therefore \(c = \lambda a + \mu b\) with \(\lambda, \mu \in [0, 1]\).

Assume that \(c \neq a, b\). From this assumption, it follows that \(0 < \lambda, \mu < 1\). Let us prove, first, that \(c = (\lambda h a) +_h (\mu h b)\).

Indeed, \(|S_h(\lambda h a)| = |\lambda h a|^\frac{1}{h} = \lambda |a|^\frac{1}{h}\) and \(S_h(\lambda h a) = \lambda |a|^\frac{1}{h} = \lambda a r^{\frac{1}{h}}\).

Similarly, \(S_h(\mu h b) = \mu |b|^\frac{1}{h} = \mu b r^{\frac{1}{h}}\) and \((\lambda h a) +_h (\mu h b) = S_h^{-1}(S_h(\lambda h a) + S_h(\mu h b)) = S_h^{-1}(r^{\frac{1}{h}}(c)) = c\).

Thus \((\lambda h a, \mu h b, c, h) \in \Gamma\). Since \(\lim_{h \to 0} x^h = 1\) for any \(x \in (0, 1)\),

\[
(a, b, c, 0) = \lim_{h \to 0}(\lambda h a, \mu h b, c, h) \in \text{Cl}(\Gamma).
\]

Thus each interior point of the arc \((a \sim b) \times 0\) belongs to the closure of \(\Gamma\). Therefore, its boundary points belong to the closure of \(\Gamma\), too.

Consider finally the last case, \((a, -a, b)\) with \(|b| \leq |a|\). It would suffice to prove that \((a, -a, b, 0)\) belongs to the closure of \(\Gamma\) for \(b\) with \(|b| < |a|\). Obviously, \((a +_h b, -a, b, h)\) belongs to \(\Gamma\). Indeed, \((a +_h b) +_h (-a) = a +_h (-a) +_h b = b\). Further, \(\lim_{h \to 0}(a +_h b) = a +_0 b = a\), since \(|b| < |a|\).

**Proof of (I).** The Inclusion (I) follows from the following lemmas.

**9.B. Lemma.** If \((a, b, c, 0) \in \text{Cl} \Gamma\), then \(|c| \leq \max(|a|, |b|)\).

**9.C. Lemma.** If \((a, b, c, 0) \in \text{Cl} \Gamma\) with \(|a| > |b|\), then \(c = a\).

**9.D. Lemma.** If \((a, b, c, 0) \in \text{Cl} \Gamma\) and \(|a| = |b|\), but \(a + b \neq 0\), then \(|c| \geq |a|\).

**9.E. Lemma.** If \((a, b, c, 0) \in \text{Cl} \Gamma\) and \(|a| = |b|\), but \(a + b \neq 0\), then \(c \in a\mathbb{R}_+ + b\mathbb{R}_+\).

**Proof of Lemma 9.B.**

\[
|a +_h b| = |S_h^{-1}(S_h(a) + S_h(b))| = |S_h(a) + S_h(b)|^h \\
\leq (|S_h(a)| + |S_h(b)|)^h \\
\leq (2 \max(S_h(a), S_h(b)))^h \\
= 2^h \left(\max(|a|^\frac{1}{h}, |b|^\frac{1}{h})\right)^h = 2^h \max(|a|, |b|)
\]
Since $2^h \to 1$, it follows that for any $C > \max(|a|,|b|)$ there exists neighborhoods $U$ and $V$ of $a$ and $b$, respectively, and a real number $\varepsilon > 0$ such that $\sup\{|x| : x \in U + h V\}$ is not greater than $C$ for any $h \in (0, \varepsilon)$.

**Proof of Lemma 9.C.** For any complex numbers $x, y$ with $|x| > |y|$, $$x + h y = S_h^{-1}(S_h(x) + S_h(y)) = S_h^{-1}\left(S_h(x)\left(1 + \frac{S_h(y)}{S_h(x)}\right)\right) = xS_h^{-1}\left(1 + \frac{S_h(y)}{S_h(x)}\right)$$ Further, $|\frac{S_h(y)}{S_h(x)}| = \left|\frac{y}{x}\right|^\frac{1}{h}$. Hence $\left|1 + \frac{S_h(y)}{S_h(x)}\right| \leq 1 + \left|\frac{y}{x}\right|^\frac{1}{h}$ and $$\left|S_h^{-1}\left(1 + \frac{S_h(y)}{S_h(x)}\right)\right| = \left|1 + \frac{S_h(y)}{S_h(x)}\right|^h \leq 1 + \left|\frac{y}{x}\right|^h.$$ The family $\left|1 + a^\frac{1}{h}\right|$ converges to 1 as $h \to 0$ uniformly for $a \in (0, r)$ if $r < 1$. Therefore $x + h y$ converges to $x$ as $h \to 0$ uniformly on the set $|x| \leq R$ and $|y| \leq r$ if $R$ and $r$ are positive real numbers with $0 < r < R$.

If $(a, b, c, 0) \in \text{CFL}$ and $|a| > |b|$, then for any neighborhoods $U$, $V$ and $W$ of $a$, $b$ and $c$, respectively, and any $\varepsilon > 0$ there exist $h \in (0, \varepsilon)$ and $(x, y) \in U \times V$ such that $x + h y \in W$. Let $R$ and $r$ be real numbers with $|a| > R > r > |b|$. We may take neighborhoods $U$ and $V$ such that $|y| < r$ ad $R < |x|$ for any $x \in U$ and $y \in V$. When $(x, y) \in U \times V$, $x + h y$ uniformly converges to $x$ as $h \to 0$. On the other hand we see that by shrinking $W$ towards $c$ and pushing $\varepsilon$ to 0, we force $x + h y$ converge to $c$, while by shrinking $U$ towards $a$, we force $x$ converge to $a$. Hence $c = a$.

**Proof of Lemma 9.D.** The numbers $a$ and $b$ can be related by formula $b = ae^{i\varphi}$ with $|\varphi| < \pi$. Then $S_h(b) = S_h(ae^{i\varphi}) = e^{i\varphi}S_h(a)$ and $|a + h b| = |S_h^{-1}(S_h(a) + S_h(b))| = |S_h^{-1}(S_h(a)(1 + e^{i\varphi}))| = |a||1 + e^{i\varphi}|^h$.

**Proof of Lemma 9.E.** Fix $h > 0$. The numbers $S_h(a)$ and $S_h(b)$ have the same arguments as $a$ and $b$. Therefore their sum $S_h(a) + S_h(b)$ belongs to $a\mathbb{R}_+ + b\mathbb{R}_+$. The number $a + h b = S_h^{-1}(S_h(a) + S_h(b))$ has the same argument as $S_h(a) + S_h(b)$. Hence, it also belongs to $a\mathbb{R}_+ + b\mathbb{R}_+$.

**9.4. Dequantizations commute.** We have constructed the following three 1-parameter families of hyperfields:
• $\Delta_h$ degenerating the triangle hyperfield $\Delta$ to the ultratriangle hyperfield $\mathbb{Y}_x$;
• $\Delta_h^{\log}$ degenerating the amoeba hyperfield $\Delta^{\log}$ to the tropical hyperfield $\mathbb{Y}$;
• $\mathbb{C}_h$ degenerating the field $\mathbb{C}$ of complex numbers to the complex tropical hyperfield $\mathcal{T}\mathbb{C}$.

These families are related. The map $\log : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ maps the first of them to the second one. This is how the second deformation was obtained. Furthermore, the map $\mathbb{C} \to \mathbb{R}_+ : z \mapsto |z|$ maps the third deformation onto the first one. The composition of these two maps, the amoeba map $\mathbb{C} \to \mathbb{R} \cup \{-\infty\} : z \mapsto \log |z|$, maps the third deformation to the second one.

\[
\begin{array}{c}
\mathbb{C} \cong \mathbb{C}_h \xrightarrow{h \to 0} \mathbb{C}_0 = \mathcal{T}\mathbb{C} \\
x \mapsto |x| \\
\Delta \cong \Delta_h \xrightarrow{h \to 0} \Delta_0 = \mathbb{Y}_x \\
x \mapsto \log x \\
\Delta^{\log} \cong \Delta_h^{\log} \xrightarrow{h \to 0} \mathbb{Y}
\end{array}
\]

All vertical arrows in this diagram are multiring homomorphisms discussed above. The horizontal arrows denote passing to limits.

Double distributivity of a hyperfield is not preserved under dequantization. In the first line of the diagram the original hyperfield is a field $\mathbb{C}$. It is doubly distributive. The complex tropical hyperfield is not (see Section 6.4). In the second and third lines the original hyperfields are not doubly distributive (see Section 5.1), while the dequantized hyperfields are (cf. Sections 4.7, 5.2 and 5.3).

Each of the hyperfields gives rise to its own algebraic geometry. The classical complex algebraic geometry corresponds to the left upper corner of the diagram. The left vertical arrows correspond to construction of amoeba for a complex algebraic variety. The bottom right corner of the diagram corresponds to the tropical geometry.

The least studied of these algebraic geometries is the one corresponding to the right upper corner of the diagram. This is the complex tropical geometry. It occupies an intermediate position between the complex algebraic geometry and tropical geometry, cf. [22].
Appendix 1. Proof of Theorem [6.A]

Let us prove that \( (a \sim b) \sim c = a \sim (b \sim c) \) for any complex numbers \( a, b, c \). The following list exhausts all possible triples of complex numbers:

1. the absolute value of one of the numbers, say \( a \), is greater than the absolute values of the other two numbers: \( |a| > |b|, |c| \);
2. \( |a| = |c| > |b| \);
3. \( |a| = |b| > |c| \) and
   a. either \( a \neq -b \),
   b. or \( a = -b \);
4. \( |a| = |b| = |c| \) and
   a. \( a + b \neq 0 \neq b + c \);
   b. either \( a + b = 0 \), or \( b + c = 0 \), but not both;
   c. \( a + b = 0 = b + c \), but \( a \neq 0 \);
   d. \( a = b = c = 0 \).

Let us prove that \( (a \sim b) \sim c = a \sim (b \sim c) \) in each of these cases. In the framework of the prove in the case when \( x \sim y \) is an arc (i.e., \( |x| = |y| \) and \( x + y \neq 0 \)), let us denote this arc by \( \sim (xy) \).

1. In the first case (i.e., if \( |a| > |b|, |c| \)) the tropical sum equals \( a \), that is the summand with the greatest absolute value independently on the order of operations. For any order this summand majorizes the others and eventually becomes the final result. \( \square \)

2. If \( |a| > |b| \) and \( |b| < |c| \), then \( a \sim b = a \) and \( b \sim c = c \). Hence \( (a \sim b) \sim c = a \sim c \) and \( a \sim (b \sim c) = a \sim c \).

   (3a) If \( |a| = |b| \) and \( a \neq -b \), then \( a \sim b = \sim (ab) \), and since \( |c| < |a| \), then \( c \sim x = x \) for any \( x \) with \( |x| = |a| \). Therefore \( (a \sim b) \sim c = (\sim (ab)) \sim c = \sim (ab) \). On the other hand, \( a \sim (b \sim c) = a \sim b = \sim (ab) \). \( \square \)

(3b)

\[
(a \sim -a) \sim c = \{ x : |x| \leq |a| \} \sim c = \left( \{ x : |c| < |x| \leq |a| \} \cup \{ x : |x| = |c|, x \neq -c \} \cup \{ -c \} \cup \{ x : |x| < |c| \} \right) \sim c = \left( \begin{array}{l} \{ x : |c| < |x| \leq |a| \} \cup \{ y : |y| = |c|, x \neq -c \} \cup \{ x : |x| \leq |c| \} \cup \{ c \} \end{array} \right) = \{ x : |x| \leq |a| \}
\]

On the other hand, \( a \sim ( -a \sim c) = a \sim ( -a) = \{ x : |x| \leq |a| \} \)

\( \square \)
(4a) \[(a \sim b) \sim c = (\sim(ab)) \sim c = \begin{cases} \{x : |x| \leq |a|\}, & \text{if } -c \in (\sim(ab)) \\ (\sim(ac)) \cup (\sim(bc)), & \text{if } -c \notin (\sim(ab)) \end{cases} \]

On the other hand,
\[a \sim (b \sim c) = a \sim (\sim(bc)) = \begin{cases} \{x : |x| \leq |a|\}, & \text{if } -a \in (\sim(bc)) \\ (\sim(ab)) \cup (\sim(ac)), & \text{if } -a \notin (\sim(bc)) \end{cases} \]

The statements \(-c \in (\sim(ab))\) and \(-a \in (\sim(bc))\) are equivalent. Indeed, each of them means that the convex hull of the set \(\{a, b, c\}\) contains 0. If the convex hull of \(\{a, b, c\}\) does not contain 0, then \(\{a, b, c\}\) is contained in a half of the circle \(\{x : |x| = |a|\}\) and then \((\sim(ac)) \cup (\sim(bc)) = (\sim(ab)) \cup (\sim(ac))\) is the shortest arc of the circle containing \(a, b, c\), that is it is a sort of convex hull of \(\{a, b, c\}\) in a half-circle.

(4b) If \(|a| = |b| = |c|, a + b = 0\), but \(b + c \neq 0\), then \((a \sim b) \sim c = (x : |x| \leq |a|) \sim c = \{-c\} \cup \{x : x \neq -c|x| \leq |a|\} = \{x : |x| \leq |a|\}\). On the other hand, \(a \sim (-a \sim c) = a \sim (-(-a, c) = \{x : |x| \leq |a|\}\).

(4c) If \(|a| = |b| = |c| \neq 0\) and \(a + b = 0 = b + c\), then \((a \sim b) \sim c = (a \sim -a) \sim a = \{x : |x| \leq |a|\}\) \sim a = \{x : |x| \leq |a|\}. On the other hand, \(a \sim (b \sim c) = a \sim (-a \sim c) = a \sim \{x : |x| \leq |a|\} = \{x : |x| \leq |a|\}\).

(4d) Does not require a proof.

**Appendix 2. Proof of Theorem 6.4**

For \(n = 2\) the statement of Theorem 6.4 follows immediately from the definition of tropical sum. Assume that for all \(n < k\) the statement is proved and prove it for \(n = k\).

By the assumption, the tropical sum of the first \(k - 1\) summands is either the whole closed disk, and then \(0 \in \text{Conv}(a_1, \ldots, a_{k-1})\), or \(a_1 \sim \ldots \sim a_{k-1}\) is a connected subset of a half of the circle. In the former case the sum of all \(k\) summands is the same disk, since \(-a_k \in a_1 \sim \ldots \sim a_{k-1}\), and \(0 \in \text{Conv}(a_1, \ldots, a_k)\), since \(0 \in \text{Conv}(a_1, \ldots, a_{k-1})\).

In the latter case there may happen one of the following two mutually exclusive situations: either \(-a_k \in a_1 \sim \ldots \sim a_{k-1}\), and then \(a_1 \sim \ldots \sim a_k\) is the disk, or \(-a_k \notin a_1 \sim \ldots \sim a_{k-1}\).

In the first situation, the diameter of the disk which connects \(a_k\) and \(-a_k\) meets the chord subtending the arc \(a_1 \sim \ldots \sim a_{k-1}\). (We do not exclude the case when \(a_1 \sim \ldots \sim a_{k-1}\) is a point, but just consider a point...
as a degenerated arc. All the arguments below have obvious versions for this case.) The center of the disk lies on the part of the diameter connecting \( a_k \) with the chord subtending the arc \( a_1 \ldots a_{k-1} \). The end points of the arc are some of the first \( k-1 \) summands by the induction assumption. Therefore, \( 0 \in \text{Conv}(a_1, \ldots, a_k) \).

In the second situation (that is if \( -a_k \notin a_1 \ldots a_{k-1} \)) either \( a_k \in a_1 \ldots a_{k-1} \) and then \( a_1 \ldots a_{k-1} = a_1 \ldots a_{k-1} \), so the second alternative takes place, or \( a_k \notin a_1 \ldots a_{k-1} \) and then \( a_1 \ldots a_{k-1} \) lies on one side of the diameter connecting \( a_k \) with \( -a_k \). In the latter case \( a_1 \ldots a_k \) is an arc one of the end points of which is \( a_k \), while the other end point is one of the end points of the arc \( a_1 \ldots a_{k-1} \). \qed

### Appendix 3. Other tropical additions

A3.1 Tropical addition of quaternions. Denote by \( \mathbb{H} \) the skew field of quaternions \( \{x + y\mathbf{i} + z\mathbf{j} + tk : x, y, z, t \in \mathbb{R} \} \). Let \( a, b \in \mathbb{H} \). Like in Section 6.1 define

\[
a \sim b = \begin{cases} 
\{a\}, & \text{if } |a| > |b|; \\
\{b\}, & \text{if } |a| < |b|; \\
\text{the set of points on the shortest geodesic arc connecting } a \text{ and } b & \text{if } |a| = |b|, a + b \neq 0 \\
\text{in the sphere } \{c \in \mathbb{H} : |c| = |a|\}, & \text{if } a + b = 0. \\
\text{the ball } \{c \in \mathbb{H} : |c| \leq |a|\}, & \text{if } a + b = 0.
\end{cases}
\]

Let us call the set \( a \sim b \) the tropical sum of quaternions \( a \) and \( b \).

9.F. Theorem. The set \( \mathbb{H} \) equipped with the tropical addition is a commutative multigroup.

The proof reproduces almost literally the proof of Theorem 6.1A. \qed

It is easy to verify that the quaternion multiplication is distributive over the tropical addition. Thus we have a skew hyperfield.

A3.2 Vector spaces over \( \mathcal{T} \mathbb{C} \). The construction of tropical addition of quaternions is a special case of a more general construction. In an arbitrary normed vector space \( V \) over \( \mathbb{C} \), define multivalued operation \((a, b) \mapsto a \sim b)\):

\[
a \sim b = \begin{cases} 
\{a\}, & \text{if } |a| > |b|; \\
\{b\}, & \text{if } |a| < |b|; \\
\text{Cl} \left\{ \frac{|a|}{|aa + \beta b|} (\alpha a + \beta b) \in V : \alpha, \beta \in \mathbb{R}_{>0} \right\}, & \text{if } |a| = |b|, a + b \neq 0 \\
\{c \in V : |c| \leq |a|\}, & \text{if } a + b = 0.
\end{cases}
\]
This operation turns \( V \) into a multigroup and satisfies two kinds of
distributivity: 
\[
a(v \triangleright w) = av \sim aw \quad \text{and} 
abla b v = (a \oplus b)v \quad \text{where} \quad a, b \in \mathbb{C} 
\]
and \( v, w \in V \). In other words, \( V \) becomes a vector space over \( \mathbb{T}\mathbb{C} \) in
the sense of the following definition.

Let \( F \) be a hyperfield. A set \( V \) with a multivalued binary operation
\( (v, w) \mapsto v \triangleright w \) and with an action \((a, v) \mapsto av, a \in F, v \in V \) of the
multiplicative group of \( F \) is called a vector space over \( F \) if

- \( \triangleright \) defines in \( V \) a structure of commutative multigroup;
- \( (ab)v = a(bv) \) for any \( a, b \in F \) and \( v \in V \);
- \( 1v = v \) for any \( v \in V \);
- \( a(v \triangleright w) = av \triangleright aw \) for any \( a \in F \) and \( v, w \in V \);
- \( (a \oplus b)v = av \oplus bv \) for any \( a, b \in F \) and \( v \in V \).

Of course, any hyperfield is a vector space over itself. Copies of
this vector space are contained in any vector space over a hyperfield.
Indeed, if \( V \) is a vector space over a hyperfield \( F \) and \( w \in V \), then
the subset \( W = \{aw : a \in F\} \) is a vector subspace of \( V \) in the obvious
sense, the map \( F \to V : a \mapsto aw \) maps \( F \) onto \( W \) and this map is an
isomorphism of vector spaces.

As in a category of vector spaces over a field, the Cartesian produ ct
\( V \times W \) of vector spaces \( V, W \) over a hyperfield \( F \) is naturally equipped
with structure of vector space over \( F \):

\[
(v_1, w_1) \triangleright (v_2, w_2) = \{(v, w) : v \in v_1 \triangleright v_2, w \in w_1 \oplus w_2\}
\]

\[
a(v, w) = (av, aw).
\]

Notice, however that, in contrast to vector spaces over a field, a
vector space over a hyperfield generated by a finite set of its elements
is not necessarily isomorphic to the Cartesian product of its vector
subspaces each of which is generated by a single element. Indeed, a
vector space over \( \mathbb{T}\mathbb{C} \) constructed in the way described above starting
from a two-dimensional Hilbert space over \( \mathbb{C} \), is not isomorphic to
\( \mathbb{T}\mathbb{C} \times \mathbb{T}\mathbb{C} \).

### A3.3 Hyperfields of monomials.

The next example was inspired
by Brett Parker’s paper [15], which was also motivated by a desire to
understand tropical degenerations of complex structures.

What if one would apply the construction of Section 7.8, but taking
into account the absolute value of the coefficient in the monomial of
the highest degree?

Consider the set of monomials \( at^r \) with complex coefficient \( a \neq 0 \) and
real exponent \( r \). Adjoin zero to this set. As a set, this is \((\mathbb{C}\setminus 0) \times \mathbb{R} \cup \{0\}\).
Denote it by \( P \) and define in it arithmetic operations.
Define multiplication as the usual multiplication of monomials. The set of non-zero monomials is an abelian group with respect to the multiplication. This group is naturally isomorphic to the product of the multiplicative group of non-zero complex numbers by the additive group of all real numbers.

Define multivalued addition by the following formulas:

\[
 at^r \rightarrow bt^s = \begin{cases} 
 at^r, & \text{if } r > s \\
 bt^s, & \text{if } s > r \\
 (a + b)t^r, & \text{if } s = r, a + b \neq 0 \\
 \{ct^u : u < r\} \cup \{0\} & \text{if } s = r, a + b = 0,
\end{cases}
\]

This addition is obviously commutative. The multiplication is distributive over it. There is neutral element 0 and for each monomial \( x \) there is a unique \( y \) such that \( x \rightarrow y \ni 0 \). Let us verify associativity.

If one of the summands is zero, then associativity takes place and the proof is obvious: \((x \rightarrow 0) \rightarrow y = x \rightarrow y = x \rightarrow (0 \rightarrow y)\).

Consider three non-zero monomials, \( at^u \), \( bt^v \) and \( ct^w \). The following list represent all possibilities:

1. the exponents of one of the monomials is greater than the exponents of the other two monomials, say, \( u > v, w \);
2. two exponents, say \( u \) and \( v \), are equal, while the third one is less, and \( a + b \neq 0 \);
3. two exponents, say \( u \) and \( v \), are equal, the third is less, and \( a + b = 0 \);
4. all the three exponents are equal and either
   (a) none of the sums \( a + b, b + c, a + b + c \) vanishes;
   (b) or the sum of two coefficients vanishes, say \( a + b = 0 \), (but \( a + b + c \neq 0 \));
   (c) or \( a + b + c = 0 \).

Let us prove associativity in each of these cases.

(1) The sum equals the summand with the greatest exponent independently on the order of operations. For any order this summand is the final result. □

(2) \((at^u \rightarrow bt^u) \rightarrow ct^w = (a + b)t^u \rightarrow ct^w = (a + b)t^u\), on the other hand, \( at^u \rightarrow (bt^u \rightarrow ct^w) = at^u \rightarrow bt^w = (a + b)t^w \). □
(3) 
\[(at^u - at^w) \tau ct^w = (\{xt^r : r < u\} \cup \{0\}) \tau ct^w =\]
\[
\begin{pmatrix}
\{xt^r : w < r < u\} \cup \\
\{xt^r : r = w, x \neq -c\} \cup \\
\{-ct^w\} \cup \\
\{xt^r : r < w\} \cup \{0\}
\end{pmatrix} \tau ct^w = \begin{pmatrix}
\{xt^r : w < r < u\} \cup \\
\{yt^w : y \neq 0, y \neq c\} \cup \\
\{xt^r : r < w\} \cup \{0\} \cup \\
\{ct^w\}
\end{pmatrix} = \{xt^r : r < u\} \cup \{0\}
\]

On the other hand,
\[at^u \tau (-at^u \tau ct^w) = at^u \tau (-at^u) = \{xt^r : r < u\} \cup \{0\}\]
\[
\begin{array}{l}
(4a) \ (at^u \tau bt^u) \tau ct^u = (a+b)t^u \tau ct^u = (a+b+c)t^u \quad \text{and} \quad at^u \tau (bt^u \tau ct^u) = at^u \tau (b+c)t^u = (a+b+c)t^u. \\
(4b) \ If \ a+b=0, \ and \ none \ of \ the \ sums \ b+c, \ a+b+c \ \text{vanishes}, \ then \ \ (at^u \tau -at^u) \tau ct^u = (\{xt^r : r < u\} \cup \{0\}) \tau ct^u = ct^u. \quad \text{On the other hand,} \\
at^u \tau (-at^u \tau ct^u) = at^u \tau (-a+c)t^u = ct^u. \\
(4c) \ If \ all \ three \ exponents \ equal \ and \ a+b+c=0, \ then \\
(at^u \tau bt^u) \tau ct^u = (a+b)t^u \tau ct^u = (-c)t^u \tau ct^u = \{xt^r : r < u\} \cup \{0\} \\
, \ on \ the \ other \ hand, \\
at^u \tau (bt^u \tau ct^u) = at^u \tau (b+c)t^u = at^u \tau (-a)t^u = \{xt^r : r < u\} \cup \{0\}.
\end{array}
\]

**Remark.** There are numerous variants of this construction. For example, in the definition of the addition of monomials all the inequalities can be reverted. Another opportunity for modification: restrict consideration to monomials whose exponents take only rational or integer values. More generally, exponents can be taken from any linearly ordered abelian group.

**A3.4 Tropical addition of p-adic numbers.** Construction of Section 7.8 admits a modification applicable to any field with a non-archimedian norm. In any such field one can define a multivalued addition which together with the original multiplication form a structure of hyperfield. Below this scheme is realized only in the case of field of p-adic numbers. The general case will be considered elsewhere.

Recall that a p-adic number can be defined as series
\[\sum_{n=-v(a)}^{\infty} a_n p^n,\]
where $a_n$ takes values in the set of integers from the interval $[0, p - 1]$ and $a_{-v(a)} \neq 0$. Define a multivalued sum of $p$-adic numbers $a = \sum_{n=-v(a)}^\infty a_n p^n$ and $b = \sum_{n=-v(b)}^\infty b_n p^n$ via the following formula:

\[
a \triangledown b = \begin{cases} 
a, & \text{if } v(a) > v(b); \\
b, & \text{if } v(b) > v(a); \\
a + b, & \text{if } v(a) = v(b), a_{-v(a)} + b_{-v(b)} \neq p; \\
\{x : v(x) < v(a)\}, & \text{if } v(a) = v(b), a_{-v(a)} + b_{-v(b)} = p. \\
\end{cases}
\]

Exactly as in the preceding Section, one can prove that this binary multivalued operation is associative and, together with the usual multiplication, gives rise to a structure of multivalued field in the set of $p$-adic numbers.

References

[1] V. M. Buchstaber and E. G. Rees, Multivalued groups, their transformations and Hopf algebras, Transform. Groups 2 (1997), 325-349.
[2] S. D. Comer, Combinatorial aspects of relations, Algebra Universalis, 18 (1984) 77-94.
[3] Alain Connes and Caterina Consani, The hyperring of adele classes, arXiv: 1001.4260 [mathAG,NT].
[4] Alain Connes and Caterina Consani, From monoids to hyperstructures: in search of an absolute arithmetic. arXiv:1006.4810v1 [math.AG].
[5] M. Dresher, O. Ore, Theory of Multigroups, Amer.J.Math. 60 (1938), 705-733.
[6] Marc Krasner, Approximation des corps valued complets de caracteristique $p \neq 0$ par ceux de caracteristique 0, (French) 1957 Colloque d’algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956 pp. 129-206 Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris.
[7] M. Krasner, A class of hyperrings and hyperfields, Internat. J. Math. Math. Sci. 6 (1983), no. 2, 307-311.
[8] G. L. Litvinov and V. P. Maslov, Correspondence principle for idempotent calculus and some computer applications, (IHES/M/95/33), Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette. 1995. Also in book Idempotency, J. Gunawardena (Editor), Cambridge University Press, Cambridge, 1998, p.420-443 and arXiv:math.GM/0101021.
[9] G. L. Litvinov, V. P. Maslov, A. N. Sobolevskii, Idempotent Mathematics and Interval Analisys, Preprint math.SC/9911126, (1999).
[10] M. Marshall, Real reduced multirings and multifields J. Pure and Applied Algebra 205 (2006), 452-468.
[11] M. Marshall, Review of a book Valuations, orderings and Milnor K-theory, by Ido Efrat, Mathematical Surveys and Monographs, vol. 124, American Mathematical Society, Providence, RI, 2006, xiv+288 pp., ISBN 978-0-8218-4041-2 Bulletin of the American Mathematical Society 45:3 (2008) 439-444.
[12] F. Marty, *Sur une généralisation de la notion de groupe*, Särtryck ur Förhandlingar vid Åttonde Skandinaviska Matematiker­kongressen i Stockholm (1934), p. 45-49.

[13] G. Mikhalkin, *Enumerative tropical algebraic geometry in \( \mathbb{R}^2 \)*, J. Amer. Math. Soc. 18 (2005), no. 2, 313-377. arXiv: math.AG/0312530.

[14] G. Mikhalkin, *Tropical geometry and its applications*, International Congress of Mathematicians, Vol. II, 827-852, Eur. Math. Soc., Zürich, 2006.

[15] Brett Parker, *Exploded fibrations*, Proceedings of 13th Gökova, Geometry-Topology Conference pp. 1-39, arXiv:0705.2408 [math.SG].

[16] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, AMS Providence, RI 2002 (Chapter 9).

[17] L. Vietoris, *Bereiche zweiter Ordnung*, Monatsh. f. Math. 32 (1922), 258-280.

[18] Oleg Viro, *Complex Tropical Geometry*, Lecture in the workshop *Tropical Structures in Geometry and Physics* at MSRI, November 30, 2009, http://198.129.64.244/13933//13933-13933-Quicktime.mov

[19] Oleg Viro, *On basic notions of the tropical geometry*, to appear in Trudy MIAN (Russian).

[20] Oleg Viro, *Multifields for Tropical Geometry I. Multifields and dequantization* arXiv:1006.3034v1.

[21] Oleg Viro, *Hyperfields for Tropical Geometry II. Equations in a hyperfield*, in preparation.

[22] Oleg Viro, *Hyperfields for Tropical Geometry III. Three tropical geometries*, in preparation.

[23] H. S. Wall, *Hypergroups*, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 36. [Presented at the annual meeting of the American Mathematical Society, Pittsburgh, December 27-31, 1934.]

[24] H. S. Wall, *Hypergroups*, American Journal of Mathematics, vol. 59 (1937) 705-733.