Ultrametric Root Counting

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Abstract

Let $K$ be a complete non-archimedean field with a discrete valuation, $f \in K[X]$ a polynomial with non-vanishing discriminant, $A$ the valuation ring of $K$, and $\mathfrak{M}$ the maximal ideal of $A$. The first main result of this paper is a reformulation of Hensel’s lemma that connects the number of roots of $f$ with the number of roots of its reduction modulo a power of $\mathfrak{M}$. We then define a condition — regularity — that yields a simple method to compute the exact number of roots of $f$ in $K$. In particular, we show that regularity implies that the number of roots of $f$ equals the sum of the numbers of roots of certain binomials derived from the Newton polygon.

1 Introduction

Let $f$ be a univariate polynomial with real coefficients. Sturm’s Theorem [9] allows us to determine the exact number of real roots of $f$ in a given interval $[a, b]$. This is done by computing the difference between the number of sign changes of two sequences of real numbers called Sturm sequences [9, 7].

We are interested in the analogue of Sturm’s Theorem over $K$, where $K$ is a field, complete with respect to a non-archimedean discrete valuation. More precisely, we give an algorithmic method to compute the exact number of roots in $K$ (total or with a given valuation) for a large class of polynomials in $K[X]$ called regular polynomials (see definition 2.2).

A classical construction associated to any polynomial $f \in K[X]$ is the Newton polygon (see section 2 below), which also associates monomials of $f$ to points in $\mathbb{Q}^2$. For any lower edge $S$ of the Newton polygon of a regular polynomial $f \in K[X]$, containing only 2 points associated to monomials of $f$, the binomial containing the corresponding two terms of $f$ is called a lower binomial of $f$. We prove in Theorem 4.6 that the number of roots in $K^* = K \setminus \{0\}$ of a regular $f$ is the sum of the numbers of roots in $K^*$ of all its lower binomials. A simple explicit formula for the number of roots of each lower binomial appears in Theorem 4.5.

On the other hand, Descartes’ rule of signs implies that any univariate polynomial $f \in \mathbb{R}[X]$ with exactly $t + 1$ monomial terms has at most $2t$ non-zero real roots, counted with multiplicity. Note that Descartes’ bound over the reals doesn’t depend on the degree of the polynomial, and is linear in the number of monomial terms. In [5], H. W. Lenstra gave an analogue of Descartes’ bound over the $p$-adic numbers: if $f \in K[X]$ has exactly $t + 1$ monomial terms and $K$ is a finite extension of the $p$-adic rationals $\mathbb{Q}_p$, then the number of roots of $f$ in $K$ counted with multiplicity is $O(t^2(q - 1) \log t)$ where $q$ is the cardinality of the residue field of $K$. As a consequence of our root count from Theorem 4.6 we can improve Lenstra’s bound to $t(q - 1)$ for regular polynomials. We also prove that our bound for regular polynomials is sharp. For fields of non-zero characteristic, our improvement is
even greater: B. Poonen showed in [6, Thm. 1] that when \( p = \text{char}(K) \), the number of roots of a sparse polynomial with \( t + 1 \) terms is at most \( q^t \), and that there are explicit polynomials attaining this bound. Our bound is linear in \( t \) (for regular polynomials) in Poonen’s setting as well. All this work is done in sections 2 and 3.

In Theorem 3.9 of section 3, we obtain a reformulation of the classical construction of Hensel lifting. Let \( f \in A[X] \) be a monic polynomial with coefficients in the valuation ring \( A \) of \( K \), and \( \bar{f} \) the reduction of \( f \) modulo \( \mathfrak{M}^N \) for a sufficiently large integer \( N \). We give a bijection between the set of roots of \( f \) in \( K \) and the set of classes of roots of \( \bar{f} \) in the ring \( A/\mathfrak{M}^N \) under a particular equivalence relation. As a consequence, for any polynomial in \( K[X] \) with non-vanishing discriminant, the number of roots in \( K \) depends only on the first few “digits” of the coefficients (see Corollary 3.10).

2 Newton Polygon and Regularity

Let \( K \) be a field that is complete with respect to a non-archimedean discrete valuation \( v \). We denote by \( A = \{ x \in K : v(x) \geq 0 \} \) the valuation ring of \( K \), \( \mathfrak{M} = \{ x \in K : v(x) > 0 \} \) the maximal ideal of \( A \), \( \pi \in \mathfrak{M} \) a generator of the principal ideal \( \mathfrak{M} \) of \( A \), and \( \kappa = A/\mathfrak{M} \) the residue field of \( K \) with respect to \( v \). We assume that \( \kappa \) is finite with \( q \) elements and characteristic \( p \) and that \( v(\pi) = 1 \). We also denote by \( \nu \) the unique extension of the valuation of \( K \) to its algebraic closure \( \overline{K} \).

Let \( f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X] \). The Newton polygon of \( f \) is the convex hull of the set of points \( \{(i, v(a_i)) : i \in \{0, 1, \ldots, n\}\} \). An edge of a polygon in \( \mathbb{R}^2 \) is said to be a lower edge if it has an inner normal vector with positive second coordinate. For instance, the hexagon that is the convex hull of \( \{(-3, 1), (-1, 0), (1, 0), (3, 1), (-1, 2), (1, 2)\} \) has exactly 3 lower edges.

**Theorem 2.1.** Let \( f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X] \) be such that \( n \geq 1 \) and \( a_0a_n \neq 0 \). Let \( S \) be a lower edge of the Newton polygon of \( f \) with vertices \( (s, v(a_s)) \) and \( (s', v(a_{s'})) \) with \( s > s' \). Then \( f \) has exactly \( s - s' \) roots in \( \overline{K} \), counted with multiplicities, with valuation \( m \) where \( -m \) is the slope of \( S \). Moreover, \( f \) can be factored as

\[
    f(X) = a_n \prod_{m=\nu(\zeta)} \frac{f_m(X)}{f(\zeta)=0}, \zeta \in \overline{K}
\]

where, for each \( m \), \( f_m \) is a non-constant monic polynomial in \( K[X] \) with all roots of valuation \( m \).

**Proof.** See [10, Prop. 3.1.1].

If \( S \) is a lower edge of the Newton polygon of \( f \) then we will abuse notation slightly by also calling \( S \) a lower edge of \( f \).

**Definition 2.2.** A polynomial \( f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X] \) is regular if for any lower edge \( S \) of \( f \) with vertices \( (s, v(a_s)) \) and \( (s', v(a_{s'})) \) with \( s > s' \) we have:

1. \( S \) contains exactly two points in the set \( \{(i, v(a_i)) : i = 1, \ldots, n\} \).
2. \( \text{char}(\kappa) \nmid (s - s') \).

The polynomial \( a_{s'}X^{s'} + a_sX^s \) is called the lower binomial of \( f \) corresponding to the lower edge \( S \).
Remark: The notion of regularity introduced in the previous definition is not generic in the sense of algebraic geometry, i.e., regularity does not hold for all polynomials of degree $n$ with coefficients in a non-empty Zariski open set in $K^{n+1}$. Nevertheless, regularity has already proved quite useful in certain algorithmic questions \cite{1} and, for any choice of exponents, is satisfied by infinitely many polynomials. A complete discussion of how likely a given $f \in K[X]$ is to be regular would have to include a discussion of probability measures on $\mathbb{Q}_p$ and $\mathbb{Q}_p[X]$, and how they compare with the current notions of “natural” measures on $\mathbb{R}[X]$. These questions are actually far from settled (see, e.g., \cite{2,3}) and are thus beyond the scope of this paper.

Theorem 2.3. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X]$ be a regular polynomial. Then all factors $f_m(X)$ in equation (1) are also regular.

Proof. Via Theorem 2.1, the Newton polygon of the factor $f_m(X)$ has exactly 1 lower edge, lying in the first quadrant and intersecting both the coordinate axes, and its slope is $-m \leq 0$ since $f_m$ is monic. In particular, all factors $f_m(X)$ satisfy condition (2) of regularity. Therefore it is enough to show that they also satisfy condition (1) in definition 2.2.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \overline{K}$ be all the roots of $f$. Assume that

$$v(a_1) = \cdots = v(a_{s_1}) = m_1$$
$$v(a_{s_1+1}) = \cdots = v(a_{s_2}) = m_2$$
$$\vdots$$
$$v(a_{s_{t-1}+1}) = \cdots = v(a_n) = m_t$$

where $m_1 < m_2 < \cdots < m_t$. In order to keep consistent notation we set $s_0 = 0$ and $s_t = n$. Let $g$ be the factor $f_{m_{j+1}}$ of $f$ and let $n_j = s_{j+1} - s_j$ be the degree of $g$. Then

$$g(X) = (X - \alpha_{s_j+1})(X - \alpha_{s_j+2}) \cdots (X - \alpha_{s_{j+1}}) = X^{n_j} + b_{n_j-1}X^{n_j-1} + \cdots + b_1X + b_0.$$ 

The coefficients $b_{n_j-k}$ and $a_{n-s_j-k}$, with $0 \leq k \leq n_j$, can be written in terms of the roots of $f$ as

$$b_{n_j-k} = (-1)^k \sum_{I \subseteq \{s_{j+1}, \ldots, s_{j+1}\}} \prod_{i \in I} \alpha_i$$

$$a_{n-s_j-k} = (-1)^{s_j+k} \sum_{I \subseteq \{1, \ldots, n\}} \prod_{i \in I} \alpha_i$$

where, as usual, an empty product is defined as 1. Note that in the case $k = 0$, the term $\delta = (-1)^{s_j} \alpha_1 \alpha_2 \cdots \alpha_{s_j}$ appears in the sum corresponding to $a_{n-s_j}$ and has (strictly) the minimum possible valuation. This means that $v(\delta) = v(a_{n-s_j}) = n_0m_1 + n_1m_2 + \cdots + n_{j-1}m_j$. When
Proof. Assume that \( \alpha \in \mathbb{K} \) is such that \( f(\alpha) = 0 \), we have that \( v(\alpha) \geq 0 \).

Proof. Assume that \( v(\alpha) < 0 \). Since \( f(\alpha) = 0 \), we have that

\[
\begin{align*}

v(\alpha) &= v(\alpha^n) = v(a_{n-1}a^{n-1}_1 + \cdots + a_0) \\
&\geq \min\{v(a_i\alpha^i) : 0 \leq i < n\} \\
&\geq \min\{v(\alpha^i) : 0 \leq i < n\} = (n-1)v(\alpha)
\end{align*}
\]

which implies \( v(\alpha) \geq 0 \), a contradiction.

The following lemma gives a lower bound for the distance between roots in terms of the valuation \( r \) of the discriminant.

Lemma 3.2. If \( f(X) = \prod_{i=1}^{n}(X - \alpha_i) \) with \( \alpha_i \in \mathbb{K} \) for \( i = 1, \ldots, n \), then

\[
v(\alpha_i - \alpha_j) \leq \frac{r}{2} \quad \forall i \neq j.
\]

Proof. From the formula of the discriminant \( \Delta = \prod_{1 \leq i < j \leq n}(\alpha_i - \alpha_j)^2 \) we get \( r = 2 \sum_{1 \leq i < j \leq n} v(\alpha_i - \alpha_j) \). Since all the roots satisfy \( v(\alpha_i) \geq 0 \), all the terms in this sum are non-negative. Therefore \( v(\alpha_i - \alpha_j) \) can not exceed \( r/2 \) for any \( i \neq j \).

The bound of Lemma 3.2 is sharp. For instance, the polynomial \( f = x(x - p) \in \mathbb{Q}_p[X] \) has discriminant \( \Delta(f) = p^2 \) of valuation \( v_p(\Delta(f)) = 2 \), and the valuation of the difference of the roots is 1. We can also use Lemma 3.2 to derive an upper bound for the number of roots of \( f \) in \( \mathbb{K} \).

Corollary 3.3. The number of roots of \( f \) in \( \mathbb{K} \) is not greater than \( q^{[r/2]+1} \).

Proof. Otherwise we would have two roots \( x, y \in A \) with \( x \equiv y \mod \pi^{[r/2]+1} \), that is, \( v(x - y) \geq [r/2] + 1 > r/2 \) in contradiction with Lemma 3.2.
Let \( f_N \in (A/\pi^N A)[X] \) denote the reduction of the polynomial \( f \) modulo \( \pi^N \). We denote by \( \beta_1, \ldots, \beta_l \in A \) the roots of \( f \) in \( K \) (by Lemma 3.1 we know that they are in \( A \)). It is clear that the reduction of any of these roots modulo \( \pi^N \) is a root of \( f_N \). Unfortunately, the reduction modulo \( \pi^N \) does not give a bijection between the set of roots of \( f \) in \( K \) and the set of roots of \( f_N \) in \( A/\pi^N A \) in general. However, we will show that the reduction homomorphism is a bijection between the roots of \( f \) and classes of roots of \( f_N \) under a particular equivalence relation. The inverse of the reduction homomorphism is given by a reformulation of the standard Hensel’s lemma.

We denote by \( \overline{\pi} \) the reduction modulo \( \pi^N A \) of \( x \in A \).

**Definition 3.4.** Let \( S_N \subseteq A/\pi^N A \) be the set of roots of \( f_N \). Two roots \( x, y \in S_N \) are in the same equivalence class (denoted by \( x \approx y \)) if and only if either \( x = y \) or \( x \equiv y \mod \pi^{r+1} \) and \( N > r \). The class containing a root \( x \in S_N \) is written \( [x] \) and the set of classes is written \( S_N/\approx \).

**Lemma 3.5.** If \( N > r \) then the number of roots of \( f \) in \( K \) is not greater than \( |S_N/\approx| \).

*Proof.* Write \( f \) of the form \( f(X) = (X - \beta_1)(X - \beta_2)\ldots(X - \beta_l)g(X) \) where \( g \) has no roots in \( K \). Let \( \overline{\beta_i} = \beta_i \mod \pi^N A \) be the reduction of \( \beta_i \) modulo \( \pi^N A \). Since this reduction is a ring homomorphism, \( \overline{\beta_i} \) is a root of \( f_N \). Take \( 1 \leq i < j \leq l \). By Lemma 3.1 we have \( v(\beta_i - \beta_j) \leq r/2 \leq r \), i.e., \( \overline{\beta_i} \neq \overline{\beta_j} \mod \pi^{r+1} \). This implies that \( \beta_i \neq \beta_j \mod \pi^{r+1} \). This implies that \( \beta_i \neq \beta_j \mod \pi^{r+1} \). Hence \( \beta_i \neq \beta_j \mod \pi^{r+1} \).

**Lemma 3.6.** Let \( \gamma \in A \) be such that \( v(f(\gamma)) > r \). Then \( v(f'(\gamma)) \leq r \).

*Proof.* Write \( f \) of the form \( f(X) = (X - \beta_1)(X - \beta_2)\ldots(X - \beta_l)g(X) \) with \( a, b \in A[X] \) and evaluate at \( X = \gamma \). Since \( v(a(\gamma)) \geq 0 \), we have that \( v(a(\gamma)f(\gamma)) \geq r \), and therefore \( v(b(\gamma)f'(\gamma)) = v(\Delta - a(\gamma)f(\gamma)) = r \). We conclude that \( v(f'(\gamma)) \leq r \) because \( v(b(\gamma)) \geq 0 \).

In order to proceed, we need the following version of Hensel’s lemma. This lemma allows us to lift an approximate root of \( f \) to an exact root.

**Lemma 3.7** (Hensel). If \( \gamma \in A \) satisfies \( v(f(\gamma)/f'(\gamma)^2) > 0 \) then there exists a root \( \xi \in A \) of \( f \) such that \( v(\xi - \gamma) = v(f(\gamma)/f'(\gamma)) \).

*Proof.* See [3] Sec. 1.5, Ch. 2.

**Lemma 3.8.** If \( N > 2r \) then the number of roots of \( f \) in \( K \) is not less than \( |S_N/\approx| \).

*Proof.* Take \( \overline{\beta} \in S_N/\approx \) and take some \( \gamma \in A \) such that \( \beta = \overline{\gamma} \). Since \( f(\overline{\gamma}) = f_N(\beta) = 0 \), we have that \( v(f(\gamma)) > N > 2r \geq r \). By Lemma 3.6 we have that \( v(f(\gamma)) \leq 0 \) and then \( v(f(\gamma)/f'(\gamma)^2) > 0 \). By Hensel’s lemma, there exists \( \xi \in A \) such that \( f(\xi) = 0 \) and \( \xi \equiv \gamma \mod \pi^{N-r} \) because \( v(f(\gamma)/f'(\gamma)) \geq N - r \). Since \( N - r > r \) we have that \( \xi \equiv \gamma \mod \pi^{r+1} \) and also \( \overline{\xi} \equiv \overline{\gamma} \mod \pi^{r+1} \) because \( N > r \). This means that \( \beta = \overline{\xi} \).

Note that if \( \xi \) and \( \xi' \) are two different roots of \( f \) in \( A \), then \( v(\xi - \xi') \leq r/2 \leq r \) by Lemma 3.2. This implies that \( \xi \neq \xi' \mod \pi^{r+1} \) because \( \overline{\xi} \neq \overline{\xi'} \mod \pi^{r+1} \) and \( \overline{\xi} \neq \overline{\xi'} \). We conclude from here that the procedure described above gives a well defined map from the set \( S_N/\approx \) to the set of roots of \( f \) in \( K \) (we can not lift the same class to two different roots). Moreover, this map is injective, because it is possible to reconstruct the equivalence class from the lifted root.
As an immediate consequence of Lemmas 3.5 and 3.8, we obtain a bijection between the number of roots of \( f \) in \( K \) and the number of equivalence classes. The following theorem is the main result of this section.

**Theorem 3.9.** For any \( N > 2r \), the number of roots of \( f \) in \( K \) is equal to \( |S_N/\approx| \). More precisely, the map \( x \mapsto [x] \) is a bijection between the set of roots of \( f \) in \( A \) (or in \( K \)) and \( S_N/\approx \).

**Corollary 3.10.** Let \( g = X^n + b_{n-1}X^{n-1} + \cdots + b_0 \in A[X] \) be a polynomial such that \( v(a_i - b_i) > 2r \). Then \( f \) and \( g \) have the same number of roots in \( K \).

**Proof.** Since \( a_i \equiv b_i \mod \pi^N \), then \( \text{Res}_X(g, g') \equiv \text{Res}_X(f, f') \equiv \Delta \mod \pi^{2r+1} \).

Therefore the discriminant of \( g \) has also valuation \( r \). We conclude by applying Theorem 3.9 to \( f \) and \( g \) with \( N = 2r + 1 \).

It is important to note that the proofs of Lemmas 3.5 and 3.8 remain valid if we change our equivalence relation \( \approx \) by the (apparently finer) relation \( \sim \) defined by \( x \sim y \mod \pi^{N-r} \). Therefore Theorem 3.9 remains true with this new equivalence relation. Denote by \( [[x]] \) the equivalence class of roots with respect to \( \sim \) that contains \( x \). It is clear that \( [[x]] \subseteq [x] \) for all \( x \). On the other hand, the number of classes with respect to \( \sim \) or \( \approx \) must be the same (they coincide with the number of roots of \( f \) in \( K \)), thus \( [[x]] = [x] \) for all roots \( x \in A/\pi N A \) of \( f_N \). We derive several corollaries from this remark.

**Corollary 3.11.** For any \( N > 2r \), the number of roots of \( f_N \) in \( A/\pi N A \) is less than or equal to \( q^r \) times the number of roots of \( f \) in \( K \).

**Proof.** Any class \( [[x]] \) contains at most \( q^r \) elements and the number of classes is the number of roots of \( f \) in \( K \).

**Corollary 3.12.** For any \( N > 2r \), the number of roots of \( f_N \) in \( A/\pi N A \) is not greater than \( q^r + [r/2] + 1 \).

**Proof.** Apply Corollaries 3.11 \( \square \) and 3.13 \( \square \)

**Corollary 3.13.** If \( r = 0 \) then the number of roots of \( f_N \) in \( A/\pi N A \) coincide with the number of roots of \( f \) in \( K \) for all \( N \geq 1 \).

**Proof.** Apply Corollary 3.11 \( \square \) and Lemma 3.5 \( \square \)

### 4 Roots of Regular Polynomials

The goal of this section is to give a procedure to count the exact the number of roots in \( K^* \) of regular polynomials. This is done in Theorems 4.5 and 4.6. The following corollary is just a special case of Theorem 3.9 when \( r = 0 \) and \( N = 1 \), but we are going to use it in this section, so we would like to state it as a separate result. It should also be pointed out the both Corollary 4.11 and Lemma 4.12 are standard results.
Corollary 4.1. If \( v(\Delta) = 0 \) then the number of roots of \( f \) in \( K^* \) is equal to the number of roots of \( f_1 \) in \( \kappa^* \) where \( f_1 \) is the reduction of \( f \) modulo \( \pi A \).

Lemma 4.2. If \( f(X) = X^n + a_0 \) then the discriminant of \( f \) is

\[
\Delta(f) = (-1)^{n(n-1)/2} n^n a_0^{n-1}.
\]

Proof. Write \( f(X) = X^n + a_0 = \prod_{i=1}^{n} (X - \alpha_i) \) with \( \alpha_i \in \overline{K} \). Then

\[
\Delta(f) = (-1)^{n(n-1)/2} \text{Res}(f, f') = (-1)^{n(n-1)/2} \prod_{i=1}^{n} f'(\alpha_i) = (-1)^{n(n-1)/2} \prod_{i=1}^{n} n\alpha_i^{n-1} = (-1)^{n(n-1)/2} n^n (-1)^{n-1} a_0^{n-1}.
\]

Lemma 4.3. If \( g(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in A[X] \) satisfies \( v(a_0) = 0, v(a_i) > 0 \) for all \( 1 \leq i < n \) and \( p \nmid n \) then the number of roots of \( g \) in \( K^* \) is equal to the number of roots of the lower binomial \( X^n + a_0 \) of \( g \) in \( K^* \).

Proof. By Lemma 4.2 the discriminant of \( X^n + a_0 \) has valuation 0. On the other hand, the polynomial \( g \) satisfies the hypothesis of Corollary 3.10 with respect to \( f = X^n + a_0 \). Then both \( g \) and its lower binomial \( f \) have the same number of roots in \( K \).

Definition 4.4. Let \( a \in K^* \) be an element with valuation \( v(a) = l \). The first non-zero digit of \( a \) is \( \delta(a) = a/l^l \in \kappa^* \).

The following result gives a procedure to count the number of roots of a regular polynomial when its Newton polygon has only one lower edge.

Theorem 4.5. Let \( f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in A[X] \) with \( p \nmid n \) and \( a_0 \neq 0 \). Write \( l = v(a_0) \) and assume that \( v(a_{n-i}) > il/n \) for all \( i = 1, \ldots, n-1 \). Then the number \( R \) of roots of \( f \) in \( K^* \) is equal to the number of roots of the lower binomial \( X^n + a_0 \) in \( K^* \). Moreover, if \( n \nmid l \) we have \( R = 0 \), and if \( n \mid l \) then

\[
R = \begin{cases} 
gcd(n, q - 1) & \text{if} -\delta(a) \text{ is an } n^{th} \text{ power in } \kappa, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. By Theorem 2.1 all the roots of both \( f \) and \( \tilde{f} = X^n + a_0 \) have valuation \( e = l/n \). It is clear that if \( n \nmid l \), then neither \( f \) nor \( \tilde{f} \) have a root in \( K \), because all the elements in \( K \) have integer valuation. Therefore, we only need to consider the case \( n \mid l \). Define \( h(X) = \pi^{-l} f(\pi^l X) \). It is clear that \( f \) and \( h \) have the same number of roots in \( K \). Our assumptions on the coefficients of \( f \) guarantee that \( h \) is a monic polynomial in \( A[X] \). Moreover, if \( h = X^n + b_{n-1}X^{n-1} + \cdots + b_0 \), then \( v(b_0) = 0 \) and \( v(b_{n-i}) > 0 \) for all \( 1 \leq i < n \). By Lemma 4.3 the number of roots of \( h \) in \( K \) coincides with the number of roots of its lower binomial \( \tilde{h} = X^n + \pi^{-l}a_0 \)
in \( K \). Since \( \tilde{h}(X) = \pi^{-l} \overline{f}(\pi^{s} X) \), then \( \tilde{f} \) and \( \tilde{h} \) have the same number of roots in \( K \). We conclude that \( f, \tilde{f}, h \) and \( \tilde{h} \) have all the same number \( R \) of roots in \( K \).

It only remains to prove the formula for \( R \). By Lemma 4.2, the discriminant of \( \tilde{h} \) has valuation 0 (since \( p \nmid n \) and \( v(b_0) = 0 \)). Therefore, by Corollary 4.1, the number of roots \( R \) of \( \tilde{h} \) in \( K \) equals the number of roots of \( \kappa \) of the reduction \( \tilde{h}_1 = X^n + \delta(a_0) \) of \( \tilde{h} \) modulo \( \mathfrak{M} \). If \( -\delta(a_0) \) is not an \( n^{th} \) power in \( \kappa \), then \( \tilde{h} \) has no roots. Otherwise, the number of roots of \( \tilde{h} \) in \( \kappa \) coincides with the number of \( n^{th} \) roots of the unity in \( \kappa \). Since \( \kappa^* \) is a cyclic group with \( q - 1 \) elements, \( R = \gcd(q - 1, n) \) in this case.

**Theorem 4.6.** Let \( f = a_n X^n + \cdots + a_0 \in K[X] \) be a regular polynomial. Then the number of roots of \( f \) in \( K^* \) is equal to the sum of the number of roots in \( K^* \) of all its lower binomials.

**Proof.** By Theorem 2.1, we can write \( f = a_n \prod_{j=0}^{t} f_j \) where \( f_0, \ldots, f_t \in K[X] \) are monic polynomials and all the roots of each \( f_j \) have the same valuation \( m_{j+1} \). Here \( t + 1 \) is the number of lower edges of the Newton polygon of \( f \) and \( -m_1 > \cdots > -m_{t+1} \) are the slopes of the lower edges. Following the notation of Theorem 2.3, we define \( n_{j+1} = \deg(f_j) \) and \( s_j = |\{ \alpha \in K^* : f(\alpha) = 0 \text{ and } v(\alpha) \leq m_j \}| \). Setting \( s_0 = 0 \) we have \( n_j = s_{j+1} - s_j \). The lower binomials of \( f \) are the polynomials \( g_j = a_{n-s_j} X^{n-s_j} + a_{n-s_{j+1}} X^{n-s_{j+1}} \). Let \( R \) and \( R_j \) denote the number of roots in \( K^* \) of \( f \) and \( f_j \) respectively. It is clear that \( R = R_0 + \cdots + R_t \). By Theorem 2.3, the polynomials \( f_j \) are regular, and then, by Theorem 4.6, its number \( R_j \) of roots in \( K^* \) depends only on its degree and the first digit of its constant term. In order to conclude we only need to prove that \( R_j \) coincides with the number of roots of \( g_j \) in \( K^* \). The number of roots of the lower binomial \( g_j = a_{n-s_j} X^{n-s_j+1} (X^{s_j+1} - 1) + a_{n-s_{j+1}} X^{n-s_{j+1}} \) in \( K^* \) coincide with the number of roots of the regular monic polynomial \( X^{s_j+1} - 1 + a_{n-s_{j+1}} X^{n-s_{j+1}} \) in \( K^* \). The degree of this polynomial is \( n_j = \deg(f_j) \) and by the equation 2 (with \( k = n_j \)) in the proof of Theorem 2.3 the first digit of \( a_{n-s_{j+1}} / a_{n-s_j} \) is equal to the first digit of the constant term of \( f_j \). Therefore \( R_j \) is also the number of roots of \( g_j \) in \( K^* \).

**Corollary 4.7.** Let \( f \in K[X] \) be a regular polynomial with \( t + 1 \) terms. Then the number of roots of \( f \) in \( K^* \) is at most \( t(q - 1) \), and all the roots of \( f \) in \( K^* \) are simple.

**Proof.** The number of lower binomials (i.e., the number of lower edges of the Newton polygon) of \( f \) is bounded above by \( t \). By Theorem 2.3, the number of non-zero roots of each lower binomial is at most \( q - 1 \). Using Theorem 4.6 we conclude that \( f \) has at most \( t(q - 1) \) roots in \( K^* \).

We conclude this section by showing that the bound of Corollary 4.7 is sharp. Consider the polynomial

\[
    f = \sum_{i=0}^{t} (-1)^i \pi^{2(q-1)} X^i(q-1) \in K[X].
\]

It is then easily verified that the Newton polygon of \( f \) has exactly \( t \) lower edges, and their vertices consists of pairs of the form

\[
    \{(i(q-1), i^2(q-1)), ((i+1)(q-1), (i+1)^2(q-1))\}
\]

for all \( i \in \{0, \ldots, t-1\} \). The polynomial \( f \) is regular: \( f \) satisfies the first item of definition 2.2 because all the coefficients of \( f \) correspond to vertices of the lower hull of the Newton polygon, and
f satisfies the second item since \( p = \text{char}(\kappa) \) is coprime to \((i + 1)(q - 1) - i(q - 1) = q - 1\). The lower binomials of \( f \) are

\[
f_i = (-1)^{i+1} \pi^{(i+1)(q-1)} X^{(i+1)(q-1)} + (-1)^i \pi^i (q-1) X^{i(q-1)} \in K[X]
\]

for all \( i \in \{0, \ldots, t-1\} \). The number of roots of \( f_i \) in \( K^* \) coincides with the number of roots of \( X^{q-1}-\pi^{(2i+1)(q-1)} \), which is \( q-1 \) according to Theorem 4.5. Moreover, by Theorem 4.6, the number of roots of \( f \) in \( K^* \) is the sum of the number of roots of the lower binomials \( f_i \) in \( K^* \). This proves that \( f \) has exactly \( t(q - 1) \) roots in \( K^* \).

5 Conclusion

Our root counting method, given in Theorems 4.5 and 4.6, works only with regular polynomials. Is it possible to give a similar procedure for general polynomials? We believe that the result in Theorem 3.9 could be a first step in that direction. We also ask whether the upper bound in Corollary 4.7 can be extended to a larger class of polynomials. Finally, we point out that an extension of regularity to the multivariate case was initiated in [4].

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References

[1] M. Avendaño, A. Ibrahim, J. M. Rojas, and K. Rusek, “Randomized NP-Completeness for \( p \)-adic Rational Roots of Sparse Polynomials in One Variable,” proceedings of ISSAC 2010, ACM Press, to appear.

[2] A. Edelman and E. Kostlan, “How Many Zeros of a Random Polynomial are Real?,” Bull. Amer. Math. Soc., 32, January (1995), pp. 1–37.

[3] Steven N. Evans, “The expected number of zeros of a random system of \( p \)-adic polynomials,” Electron. Comm. Probab. 11 (2006), pp. 278-290.

[4] A. Ibrahim, Ultrametric Fewnomial Theory, Mathematics Ph.D. thesis, Texas A&M University (Dec. 2009), University Microfilms, Ann Arbor, Michigan.

[5] H. W. Lenstra, On the Factorization of Lacunary Polynomials, Number Theory in Progress, vol. 1, Berlin 1999, pp. 277–291.

[6] B. Poonen, Zeros of sparse polynomials over local fields of characteristic \( p \), Math. Res. Lett. 5(3), pp. 273–279, 1998.

[7] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Clarendon Press, London Mathematical Society Monographs 26, 2002.
[8] A. Robert, *A course in p-adic Analysis*, GTM, Vol.198, Springer-verlag, 2000.

[9] C. Sturm, “Mémoire sur la résolution des équations numériques,” Inst. France Sc. Math. Phys., 6 (1835).

[10] E. Weiss, *Algebraic Number Theory*, McGraw-Hill, New York 1963.