Entanglement Entropy as a Probe to the Proximity Effect in Holographic Superconductors

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We study the entanglement entropy as a probe to the proximity effect of a superconducting system by using the gauge/gravity duality in a fully back-reacted gravity system. While the entanglement entropy in the superconducting phase is less than the entanglement entropy in the normal phase, we find that near the contact interface of the superconducting to normal phase the entanglement entropy has a different behavior due to the proximity effect. We verify this behavior by calculating the conductivity near the boundary interface.

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The standard Ginzburg-Landau theory is an important phenomenological theory in understanding conventional superconductivity. Recently a remarkable idea was put forward by constructing a holographic dual of superconductor which has a close similarity to the standard Ginzburg-Landau theory. This was archived by employing the gauge/gravity duality by which strongly coupled phenomena can be studied using dual gravitational systems in weak coupling\textsuperscript{1}.

The simplest holographic superconductor model which is extensively studied is described by an Einstein-Maxwell-scalar field theory with a negative cosmological constant\textsuperscript{2}. Its gravity sector is described by an Abelian-Higgs model with a stationary black hole metric which below a certain critical temperature acquires scalar hair. The holographic principle was further developed and applied to many other condensed matter systems like the conventional and unconventional superfluids and superconductors\textsuperscript{3}, Fermi liquid behavior\textsuperscript{4}, non-linear hydrodynamics\textsuperscript{5}, quantum phase transitions\textsuperscript{6} and transport\textsuperscript{7}.

In all these studies it is crucial to understand the low temperature limit. In the gravity sector this requires to go beyond the probe limit and to consider fully back-reacted gravitational systems. In the boundary theory the $T\to 0$ limit leads to strongly coupled systems in the quantum physics regime. This limit is interesting because new superfluid phenomena arise like the inhomogeneous FFLO states a holographic dual of which was discussed in\textsuperscript{8, 9}. Recently it was proposed\textsuperscript{10} that this low temperature regime can be probed by the entanglement entropy (EE).

The EE can be considered as a measure of how a given quantum system is strongly correlated (entangled). The EE is directly related to the degrees of freedom of a quantum system and is introduced as a tool to describe different phases at low temperature\textsuperscript{11}. The EE was used to study various properties of holographic superconductors at low temperatures. It was found that the EE is lower in the superconducting phase than in the normal phase\textsuperscript{10}. This behavior was attributed to some kind of reorganization of the degrees of freedom of the system: because electrons are bounded in the superconducting phase to form Cooper pairs so that less degrees of freedom remain comparable to the normal phase. The study of EE was also extended to other holographic superconductors’ applications\textsuperscript{12, 14}.

In this work we will use the EE as a probe to the proximity effect in a holographic superconductor. We will show that EE plays the role of an order parameter and it can give important information on the dynamics of system in the interface of superconducting/normal phase of a holographic superconductor in low temperatures. This behavior is also confirmed by calculating the conductivity near the boundary interface.

In condensed matter physics the proximity effect describes the dynamics of a system near the superconductor-normal metal interface where the superconducting electrons (Cooper pairs) may penetrate from the superconducting to normal phase. The leakage of the Cooper pairs weakens the superconductivity near the interface with a normal metal. This phenomenon can appear even in the absence of a magnetic field\textsuperscript{13}. One of our motivations here is to construct a computationally tractable gravity model and show that it can reproduce basic properties of proximity effect in superconductivity.

The phenomenological analysis of the proximity effect was given by the generalized Ginzburg-Landau theory\textsuperscript{15}. In this theory a complex scalar field $\Psi$ is considered as the order parameter which contributes to the free energy in the function $F_G = a(T)|\Psi|^2 + \frac{a(T)}{2}\Gamma(\Psi)|\Psi|^4 + \gamma(T)|\nabla\Psi|^2$, where the coefficient $a$ is negative at $T < T_c$ and a minimum of $F_G$ occurs for a uniform superconducting state. One defines the correlation length $\xi(T) = \gamma(\alpha)$ to characterize the length on which the order parameter $\Psi$ changes. In the superconducting/normal interface the correlation length $\xi$ measures how fast the order parameter decays.

Our aim is to build a holographic superconductor with a dual boundary field described by a theory similar to the generalized Ginzburg-Landau theory. To keep similarities of the holographic superconductor to the generalized Ginzburg-Landau theory which contains higher
order spatial derivatives, we can add a potential term which depends on the scalar field $\Psi$ as well as its spatial derivatives along the boundary. However, spatial derivatives are problematic in our case because they signify the breakdown of Lorentz invariance. To obtain an expression in a covariant way we consider $V(\Psi) = m^2|\Psi|^2 + \eta F_{\mu\nu} D_\mu \Psi D^\nu \Psi$, where $F = dA$ is the strength of a $U(1)$ gauge field $A_\mu$, $\Psi$ is a charged complex scalar field of charge $q$ and mass $m$ and $D_\mu = \nabla_\mu - i q A_\mu$. In this case the coefficient $\eta$ plays the role of $\gamma$ of the boundary Ginzburg-Landau theory.

The high order derivative terms can arise in string theory. Indeed, there are models based on exact solutions of $D = 11$ and type IIB supergravity in which after consistent Kaluza-Klein truncations to four dimensions, fully back-reacted solutions describing holographic superconductors have been found [16, 17]. These models contain a large number of scalar, gauge fields and high order derivatives of them [18, 19].

We will consider a scalar field coupled to a $U(1)$ gauge field in the action discussed in [9]

$$S = \int d^4x \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |D_\mu \Psi|^2 - V(\Psi) \right].$$

We set $L = 1$, $8\pi G = 1$, $q = 1$. We will generalize the probe limit discussion [9] to a full back-reacted formalism by taking the ansatz of metric and matter fields as

$$\begin{align*}
0 &= \chi' - \frac{z(1 + 2e^\chi z^4 \eta A t^2)}{f^2} \left(e^{\chi^2} A t^2 \psi^2 + f^2 \psi'^2\right), \\
0 &= f' + \frac{3(1 - f)}{z} - \frac{m^2 z^2}{2z} - \frac{e^\chi z^2 A_t^2}{4} - \frac{e^\chi z^2 q^2 A_t^2 (2 + 5e^\chi z^4 \eta A_t^2) \psi^2}{4 f} - \frac{z f (2 + 3e^\chi z^4 A_t^2) \psi^2}{4}, \\
0 &= A_t' + \left[ \frac{2f^2 \chi' + 4 e^\chi z^3 \psi \psi' A_t^2 (z \psi' f - 2 f (\psi + z \psi') + \frac{1}{2} z \chi'))}{2a_0 f} + \frac{2q^2 \psi'^2 (z \psi' f - 2 f (\psi' (1 + \frac{1}{2} z \chi') + z \psi'))}{a_0 f} \right] A_t - \frac{2q^2 \psi'^2 (1 + e^\chi z^4 A_t^2)}{a_0 f} A_t, \\
0 &= \psi'' + \left[ \frac{f'}{f} + \frac{(4 + z \chi')(1 + e^\chi z^4 A_t^2) + 4 e^\chi z^4 A_t A_t''}{2z (e^\chi z^4 A_t^2 - 1)} \right] \psi' + \frac{e^\chi z^4 A_t^2}{2 z f^2 + \frac{m^2}{z f^2 (e^\chi z^4 A_t^2 - 1)}}, 
\end{align*}$$

with $a_0 = z^2 f (1 + 2\eta z^2 f \psi'^2) - 2 e^\chi z^4 \psi'^2 q^2 A_t^2$.

Near the boundary $z \to 0$, we require $\chi(z = 0) = 0$ to recover the pure AdS boundary. The matter fields behave as $\psi = \psi^{(1)} z^{\Delta_{-}} + \psi^{(2)} z^{\Delta_{+}} + \ldots, A_t = \mu - \rho z + \ldots$ where according to the AdS/CFT dictionary, $\psi^{(i)} = \langle O_i \rangle / \sqrt{\gamma}$, $i = 1, 2$ and $O_i$ with the conformal dimensions $\Delta_{\pm} = \frac{\Delta_{-}}{2} \pm \frac{1}{2} \sqrt{9 + 4m^2}$ are the corresponding dual operators on the field theory side. $\mu$ and $\rho$ are the corresponding chemical potential and charge density in the dual boundary field theory, respectively. Here we focus on the case $m^2 = -2$ and set $\psi^{(2)} = 0$ to consider $\psi^{(1)}$ as the vacuum expectation value of the operator $\langle O_1 \rangle$.

At the horizon $z = z_h$, the regularity condition implies $A_t(z_h) = 0$ and $f(z_h) = 0$. We can expand all the fields near the horizon and use the scaling symmetries to set $z_h = 1$.

At high temperature, there is no condensation of the scalar field. The solution in the gravity side is the Reissner-Nordström (RN) AdS black hole. When the temperature decreases below a critical value, the black hole acquires scalar hair. The dependence of the critical temperature for the onset of the scalar field condensation on the coupling $\eta$ agrees to the result in the probe limit [4], which increases as the coupling $\eta$ becomes larger. Examining the dual operator $O_1$ of the scalar field, we observe that as the coupling increases, the condensation gap is lower. This is consistent with the observation in the probe limit and agrees well with the property of the critical temperature, which shows that the greater strength of the coupling $\eta$ makes the condensation easier to form.

Having the solution of normal phase and the hairy phase, we are going to study the holographic entanglement entropy (HEE) to reveal the quantum properties of the superconducting system. Consider a strongly coupled field theory with gravity dual, the EE of subsystem $A$ with its complement is given by searching for the minimal area surface $\gamma_A$ extended into the bulk with the same boundary of $A$. Then the EE of $A$ with its complement
is given by the area law $S_A = \frac{\text{Area}(\gamma_A)}{4G_{d+2}}$, where the constant $G_{d+2}$ is the Newton constant in $\text{AdS}_{d+2}$.

We consider the subsystem $A$ with a straight strip geometry described by $-\frac{1}{2} \leq x \leq \frac{1}{2}, 0 \leq y \leq L$, where $l$ is defined as the size of region $A$ and $L$ is a regulator which can set to infinity. The induced metric on the hypersurface $\gamma_A$ whose boundary is the same as the stripe and has a profile like (2) reads $ds^2_{\text{induced}} = \frac{1}{z^2} \left( \frac{1}{z} + x'(z)dz^2 + dy^2 \right)$. The HEE connecting with the area of the surface is given by

$$4G_4S = \text{Area}(\gamma_A) = L \int_{-1/2}^{1/2} \frac{dx}{z^2} \sqrt{1 + z'(x)^2/f}. \quad (7)$$

Minimizing the above expression, we are interested in the case that the surface is smooth at $z_A = z^2\sqrt{1 + z'(x)^2/f}$ with $z_A$ satisfying the condition $dz/dx|_{z=z_A} = 0$. The width $l$ of the subsystem $A$ and $z$ are connected by the relation $l/2 = \int_{-\epsilon}^{\epsilon} dz/z^2/\sqrt{(z_A^2 - z^2)f}$. Finally, we obtain the EE as

$$4G_4S = 2L \int_{-\epsilon}^{\epsilon} dzz_A^2 \frac{1}{\sqrt{(z_A^2 - z^2)f}} = 2L(s + \frac{1}{\epsilon}). \quad (8)$$

Here the UV cutoff has been taken into consideration. $s$ is a finite term, which is physically important. We can calculate the EE $s$ based on the solution discussed above.

We plot the HEE in Fig. 1 in the dimensionless quantities $T/\sqrt{\rho}$, $\sqrt{\rho}l$ and $s/\sqrt{\rho}$.

Fixing the temperature, we see how the HEE changes as the width of stripe $l$ changes (left panel of Fig. 1). The green line is for the normal phase with the RN-AdS black hole background. We see that below the critical temperature when the scalar field starts to condensate, the EE becomes smaller and it drops when the temperature becomes lower. This is consistent with the expectation that in the superconducting phase the degrees of freedom decrease due to the formation of Cooper pairs [10]. This property holds for different values of the coupling $\eta$.

To illustrate the influence of the $\eta$ coupling, we present the HEE in change of temperature for a fixed $l$ in Fig. 1 (middle panel). The light green line is the HEE for the normal state with RN-AdS black hole background. As the temperature decreases, the slope of HEE presents a discontinuous change at a critical temperatures $T_c$ denoted by vertical dashed lines in the figure for different strength of the coupling $\eta$. The discontinuous change of the HEE marks the phase transition point from the normal state to the superconducting state. We again observe that the superconducting phase always have smaller HEE after the phase transition.

At low temperature, the HEE for the smaller $\eta$ superconductor is smaller. Physically this is because that at low temperature the condensation becomes stronger with higher condensation gap for smaller coupling $\eta$ so that the number of Cooper pairs is increased, which results in the less degree of freedom left.

However this property does not hold when the temperature increases near the critical value marked in the ellipse in the middle plot of Fig. 1, which is enlarged in the right panel in Fig. 1. When the temperature is increased above $T/\sqrt{\rho} = 0.1793$, we see that the HEE for $\eta = -0.3$ becomes higher than the case with $\eta = 0$. When the temperature is above $T/\sqrt{\rho} = 0.188$, the HEE for $\eta = -0.3$ surpasses the value for $\eta = 0.3$. The sharp change of the HEE relating to the coupling $\eta$ in the contact interface is not trivial. This can be attributed to the proximity effect. For the smaller coupling holographic superconductor, it is possible that the Cooper pairs can penetrate to the normal state, which effectively results in the increase of the HEE in the superconductor phase. While for more positive $\eta$ coupling, Cooper pairs remain in the superconductor phase, thus leads the HEE relatively smaller.

The coupling $\eta$ corresponds in the boundary theory to the phenomenological coefficient $\gamma$ which defines the correlation length $\xi^2 = \gamma/|\alpha|$. A bigger $\gamma$ for a fixed $\alpha$, gives a correlation length that suppresses the order parameter $\Psi$ of the superconducting material [13], and for this reason the entanglement entropy is smaller.

In the following we are going to discuss the conductivity. In the superconductor, the conductivity possesses certain distinguishing properties which are largely dependent of microscopic details. We consider the perturbation of metric and $U(1)$ field as $\delta g_{\mu\nu} = g_{tx}(z)e^{-i\omega t}$ and $\delta A_\mu = A_\mu(z)e^{-i\omega t}$. Then, the first order perturbation equation of $g_{tx}$ can be deduced as

$$g_{tx}' - 2g_{tx}/z - A_x A'_x [1 - z^2\eta(f\psi^2 - q^2e^\chi A_x^2\psi^2/f)] = 0. \quad (9)$$

Having this equation we can derive the linearised perturbative Maxwell equation which decouples from $g_{tx}$

\begin{equation}
(1 + 2z^2\eta f\psi'^2)A_x'' + (f' - \chi'/2 + a_1\eta)A_x' + ([\omega^2/f^2 - z^2A_x^2/f])e^\chi - 2q^2\psi^2/(z^2f) + a_2\eta + a_3\eta^2]A_x = 0 \quad (10)
\end{equation}

with

\begin{align}
a_1 &= z\psi'[z\psi'(4f' - f\chi') + 4f(\psi' + z\psi'')], \\
a_2 &= e^\chi q^2z^2A_x^2\psi^2(2\omega^2/f) + e^\chi[2q^2z^2A_x^2 A_x' + z A_x^2(2q^2\psi^2 + z^2\psi'f) + q^2z A_x^2 A_x'(4\psi + 4z\psi' + z^2\chi')]/f, \\
a_3 &= 2e^\chi z A_x^2 f/\psi'(4\psi^2(4qA_x^2\psi f))-2e^\chi[qA_x^2\psi^2/f]^2 + e^\chi(4qA_x^2\psi f)]/f. \quad (11)
\end{align}
The above equation can be solved by imposing the ingoing boundary condition \( A_{n}(z \rightarrow z_{h}) \propto f \xi \) with the temperature \( T \) near the horizon. Near the asymptotic AdS boundary, the perturbation reads \( A_{n}(z \rightarrow 0) = A_{n}^{(0)} + zA_{n}^{(1)} \). Considering the AdS/CFT duality, the conductivity of holographic superconductor can be expressed as \( \sigma = \frac{\Gamma A_{n}(1)}{\omega A_{n}} \). After solving the equation with the boundary condition near the horizon, we can numerically extract the asymptotical value of \( A_{n} \) to calculate the conductivity of our system.

We concentrate on the real part of the conductivity, since it is the dissipative part of the conductivity and measures the presence of charged states as a function of energy. At the low temperature, for example when \( T/\sqrt{s} = 0.0574 \), we see in the left panel of Fig. 2 that at low frequencies the holographic superconducting system with smaller coupling \( \eta \) has lower real part of conductivity and a bigger frequency gap. The gap in the frequency dependent conductivity depends on the condensation, \( \omega \eta \sim \langle O \rangle \), which indicates a gap in the spectrum of charged excitations. For the smaller coupling, the higher condensation leads to the larger superconducting gap so as to the smaller real part of the conductivity. The drop in the real part of the conductivity corresponds to a drop in the density of excitations at energies below the chemical potential. Thus for smaller \( \eta \), more 'electrons' are bounded in Cooper pairs which explains the smaller HEE at low temperature.

At high temperature near the critical point, where the dependence of HEE on the coupling \( \eta \) is reversed compared with the low temperature case, the behavior of the real part of the conductivity is plotted in the middle panel with \( T/\sqrt{s} = 0.20407 \) for example. In the low frequency, we see that the real part of conductivity is bigger when the coupling is smaller. Fixing the frequency, we show the dependence of the real part of the conductivity on the temperature in the right panel of Fig. 2. The sharp different dependences of the real part of the conductivity on the coupling \( \eta \) at low and high temperatures are clearly shown. In the contact interface, we see that for the smaller \( \eta \), the real part of the conductivity is higher. The bigger real part of the conductivity in the contact interface supports the proximity effect argument that less Cooper pairs are left in the superconducting phase with smaller \( \eta \) coupling.

In summary, we have studied the behavior of the EE of a superconducting system. Motivated by the general Ginzburg-Landau theory we have introduced a higher-derivative coupling between the \( U(1) \) gauge field and the scalar field with coupling constant \( \eta \). We have solved numerically the fully back-reacted gravitational system and found that the system with larger coupling is easier to enter the superconducting phase.

We have calculated the EE using a stripe geometry and found that the EE for the superconducting phase is less than that of the normal phase as expected. Further we observed that at low temperature, a larger coupling gives a larger EE. But near the boundary superconducting/normal interface the EE has reversed behavior. Smaller coupling gives higher EE near the critical temperature. This can be explained because of the proximity effect. Cooper pairs have leaked to the normal phase which results in the higher value of the EE.

In addition, we have calculated the real part of conductivity for two characteristic temperatures. For low fre-
quency and low temperature we have found that smaller coupling has lower conductivity. This happens because smaller values of the real part of the conductivity corresponds to less density of excitations, and therefore lower EE. On the contrary, for the temperature near the critical temperature, the behavior of the conductivity relative to the strength of the coupling is reversed due to the proximity effect.

In conclusion we have built a holographic superconductor and present a holographic description of the proximity effect in superconductivity. It would be interesting to extend this study in the presence of an external magnetic field. For an inhomogeneous magnetic field the EE can give us more information on the phase structure of the holographic superconducting system at low temperatures.

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