Metric Algebroid and Poisson-Lie T-duality in DFT

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Received: 30 January 2023 / Accepted: 15 May 2023
Published online: 1 August 2023 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract: In this article we investigate the gauge invariance and duality properties of DFT based on a metric algebroid formulation given previously in Carow-Watamura et al (Metric algebroid and Dirac generating operator in Double Field Theory, 2020. https://doi.org/10.1007/JHEP10(2020)192). The derivation of the general action given in this paper does not employ the section condition. Instead, the action is determined by requiring a pre-Bianchi identity on the structure functions of the metric algebroid and also for the dilaton flux. The pre-Bianchi identity is also a sufficient condition for a generalized Lichnerowicz formula to hold. The reduction to the \( D \)-dimensional space is achieved by a dimensional reduction of the fluctuations. The result contains the theory on the group manifold, or the theory extending to the GSE, depending on the chosen background. As an explicit example we apply our formulation to the Poisson-Lie T-duality in the effective theory on a group manifold. It is formulated as a 2\( D \)-dimensional diffeomorphism including the fluctuations.

1. Introduction

In the effort to find a geometrical understanding of supergravity and string theory, algebroid structures have received increasing attention in the recent decades. Supergravity can be formulated in various ways, depending on how one treats the gauge symmetry. In generalized geometry [2,3] the diffeomorphisms and the gauge symmetry of the \( B \)-field are treated in a geometrical way as generalized diffeomorphisms. Since the \( T \)-duality in string theory interchanges the metric and the \( B \)-field, this unification is very natural. The algebraic structure of the generalized diffeomorphisms is known to be a Courant algebroid [4]. In the generalized geometry formulation of supergravity [5,6], the parameter of an infinitesimal transformation of a generalized diffeomorphism is a generalized vector, and the commutator of two infinitesimal transformations defines the Courant bracket. For reviews see [7,8].
Double Field Theory (DFT) [9–11] has been proposed as a T-duality invariant formulation of supergravity. In DFT the theory is formulated on a doubled spacetime, i.e., the dimension is twice of that of the usual supergravity or string theory. This is natural from the T-duality point of view. The states of the string on the torus have two quantum numbers, momentum and winding number, the momentum corresponding to the Fourier modes of the torus and the winding numbers corresponding to those on the dual torus. The doubled spacetime is motivated by this property. See [12,13] for review and further references on DFT.

In the literature DFT is usually formulated on flat space with an $O(D,D)$ metric and a section condition, i.e. a condition on the space of the field configuration. The theory is constructed with the requirement that it becomes a gauge theory after the section condition is applied. Here, we denote this formulation of Double Field Theory by DFT$_{sec}$. The section condition has often been criticized since it is not formulated at a Lagrangian level, such as a constraint imposed by a Lagrange multiplier or a constraint by a gauge symmetry. First efforts to address the problem of the section condition have been given in [13–15].

From the string theory point of view the section condition is easy to accommodate conceptually, i.e. it can be considered as an equivalent to the matching condition. In DFT, on the other hand, it is the condition to obtain the field theory of the half-dimensional spacetime. However, the section condition does not completely exclude the dual coordinate dependence. This causes an ambiguity in the counting of the degrees of freedom of the resulting theory and the meaning of the doubled space becomes obscure. In this paper, we extend the formulation of the DFT using the metric algebroid structure [16]. And for this purpose we are forced to be strict with the configuration space of the resulting field theory. Technically, we will apply a certain dimensional reduction on the field to obtain the field theory of the half-dimensional spacetime. We will see that then it becomes possible to analyze the mapping of the fields induced by the Poisson-Lie T-duality in the effective field theory.

In the formulation of an algebroid, the use of a graded symplectic manifold, the so-called QP-manifold [17–19] is very convenient, (see also [20,21] and reference therein), and its application to DFT has been discussed in [22–24]. Later, it was shown that a pre-QP manifold is very useful to clarify the algebraic structure of the generalized diffeomorphisms in DFT [25].

However, in the QP-manifold approach, the dilaton is not included a priori due to the grading. To include it, we use a Clifford bundle where the grading is $Z_2$ which allows us to include the dilaton into the Dirac generating operator (DGO) [26]. The DGO plays the same role as the homological function in the QP-manifold, i.e., it defines the algebroid structure using the derived bracket. With the use of the DGO the dilaton can be included naturally as shown in [1], due to an ambiguity in the DGO which is not related to the algebroid structure of the generalized vectors. From the point of view of physics we need a spin structure for supergravity to formulate the fermionic sector and the R–R sector, therefore, the use of the Dirac operator is natural.

As mentioned above, as an algebraic structure of "gauge transformation", we employ the metric algebroid and define a generalized DFT action without referring to the section condition. We achieve this by means of a generalized Bianchi identity, called pre-Bianchi identity, which gives a condition on the structure functions of the metric algebroid [1].

In this paper we show that with a metric algebroid structure as an underlying symmetry, we have an appropriate setup to investigate the properties of a stringy geometry and get a clearer idea about T-duality. First, we analyse the properties of gauge symmetry and
in the second part of this paper we focus on the formulation of the Poisson-Lie T-duality. We use the property that the metric algebroid can be reduced to a Courant algebroid by a dimensional reduction of the fluctuations in metric and dilaton. Technically, this is achieved by taking the metric algebroid such that the structure functions coincide with the structure functions of a Lie algebroid in a certain frame. Together with the pre-Bianchi identity, the bracket in the metric algebroid reduces to the Courant bracket after the dimensional reduction of the fluctuations. Since our action is constructed in terms of the structure functions of the metric algebroid, after dimensional reduction the action is given in terms of the structure functions of the Courant algebroid. This Courant algebroid is the algebraic structure of the generalized diffeomorphisms in the sense of generalized geometry. Thus, it is guaranteed that the resulting theory is gauge invariant under the generalized diffeomorphisms on the reduced spacetime.

By taking appropriate structure functions of the metric algebroid, we show that the generalized supergravity equation (GSE) \[27, 28\] can be derived.

In the literature \[29, 30\], the GSE is derived from DFT by adding a generalized vector to the dilaton flux. When the generalized vector is constant, it is possible to include a linear dual-coordinate dependence in the dilaton while keeping the section condition into the DFT. To include a generalized vector which is not constant, a modification of DFT, denoted here as DFT\(_{\text{mod}}\), has been proposed where a generalized vector is added directly to the dilaton flux. However, the justification of this modification, i.e. the consistency with the algebroid structure, is not addressed.

As we will show, in the formulation given here the form of the dilaton flux is defined geometrically and the modified DFT\(_{\text{mod}}\) is included as a special case of the background, thus filling in the missing algebraic background for the modified DFT action.

To construct an action scalar from the Dirac operator, we have previously established a generalized Lichnerowicz formula which follows when the pre-Bianchi identity is satisfied \[1\]. Originally, the Lichnerowicz formula is based on the fact that the square of the Dirac operator contains the scalar curvature, which is the Einstein-Hilbert action of gravity. Here, we show the generalized Lichnerowicz formula using the DGO and formulate a general action scalar for DFT from the metric algebroid which possesses one free parameter. The supergravity action, the DFT action including GSE and the DFT\(_{\text{WZW}}\) action can be derived by particular choices of this parameter.

In the second part of this paper, as an explicit application we investigate the Poisson-Lie T-duality on a Drinfel’d Double as a 2D dimensional diffeomorphism.

The organization of this paper is the following:

In Sect. 2, we summarize definitions of the algebroid structures involved and introduce notations. We also introduce a new viewpoint concerning the algebraic structure for DFT and list some results obtained in \[1\] relevant for the present paper.

In Sect. 3, we recall the formulation of the Dirac generating operator of the metric algebroid.

In Sect. 4, the generalized Lichnerowicz formula is given. Then, its decomposition into the subspaces of positive/negative modes is formulated, and details on the dilaton and integration measure are explained.

While in our previous paper we concentrated on the derivation of the gauge invariant DFT action, here we give a more general approach in Sect. 5, starting from the gauge transformations of the corresponding fields. Based on this, the formulation of a general gauge invariant action for DFT is derived. In the derivation of this general gauge invariant action no reference to the section condition is made.
Then, we give the relation of this general DFT action and the covariant curvature tensor of the MA, Ricci tensor and scalar curvature which hold as usual. The action for Supergravity, the DFT action including the GSE and the action for the DFT Wess-Zumino-Witten model are obtained as particular choices of a free parameter in the general DFT action. In the last part of this section we show how the dimensional reduction of the fluctuations in metric and dilaton works out.

In Sect. 6, after a brief summary of the structure of a Drinfel’d Double and the Poisson-Lie T-duality for the case of the sigma model, we give a derivation of Poisson-Lie T-duality for the case of DFT on a Drinfel’d double. In order to establish this duality, the fluctuations on the full doubled space have to be considered in order to establish Poisson-Lie T-duality. This result shows another merit of the metric algebroid formulation.

In Sect. 7, we give the construction of the R–R sector for the DFT action, show its reduction to the DFT sec, as well as its properties under local Lorentz transformation and Poisson-Lie T-duality.

Discussion and conclusions are given in Sect. 8.

2. Algebroid Structures

We briefly recall here the definitions and introduce notations of Lie algebroid, Courant algebroid [4] and metric algebroid [16] for convenience. Then, we give an improved description of the algebroid structure for DFT. For further aspects of algebroid structures in DFT see also [31,32] and references therein.

2.1. Lie algebroid. A Lie algebroid consists of a vector bundle $E \rightarrow M$ over a base manifold $M$, an anti-symmetric bracket $[\cdot, \cdot]_L : E \times E \rightarrow E$ and a bundle map (anchor) $\rho : E \rightarrow TM$ to the tangent bundle $TM$, satisfying the following relations for $\forall a, b, c \in E$:

\[
[a, f b]_L = \rho(a)(f)b + f[a, b]_L , \quad (2.1)
\]
\[
[a, [b, c]_L]_L = [[a, b]_L, c]_L + [b, [a, c]_L]_L . \quad (2.2)
\]

The anchor maps the algebroid bracket to the standard Lie bracket $[-,-]_{TM}$ on the tangent bundle $TM$ as

\[
\rho([a, b]_L) = [\rho(a), \rho(b)]_{TM} . \quad (2.3)
\]

2.2. Courant algebroid. A Courant algebroid (CA) is introduced as a kind of double of a Lie bialgebroid [33], given by a vector bundle $E \rightarrow M$, endowed with a bracket $[-,-] : E \times E \rightarrow E$, a bundle map (anchor) $\rho : E \rightarrow TM$ to the tangent bundle $TM$, and a non-degenerate symmetric fiber metric $\langle -,- \rangle$ satisfying the three conditions:

\[
\rho(a)\langle b, c \rangle = \langle [a, b], c \rangle + \langle b, [a, c] \rangle , \quad (2.4)
\]
\[
[a, a] = \frac{1}{2} \partial \langle a, a \rangle , \quad (2.5)
\]
\[
[a, [b, c]] = [[a, b], c] + [b, [a, c]] , \quad (2.6)
\]

for $a, b, c \in E$ and $f \in C^\infty(M)$, where a differential $\partial$ is defined as $\langle \partial f, a \rangle = \rho(a)f$. 

The bracket is not necessarily anti-symmetric up to a derivative term, and the Jacobi identity in Leibniz form of the bracket holds. Some useful properties of the bracket can be derived from the above defining equations:

\[
[a, fb] = (\rho(a)f)b + f[a, b], \tag{2.7}
\]

\[
[fa, b] = - (\rho(b)f)a + (\partial f)[a, b] + f[a, b], \tag{2.8}
\]

\[
[\partial f, a] = 0, \tag{2.9}
\]

\[
\rho(\partial f) = 0, \tag{2.10}
\]

\[
\rho([a, b]) = [\rho(a), \rho(b)]_{TM}, \tag{2.11}
\]

where the bracket \([−, −]_{TM}\) is the standard Lie bracket on \(TM\). The bracket on \(E = TM \oplus T^*M\) is called the Dorfman bracket.

2.3. Metric algebroid. A metric algebroid (MA) \((E, [−, −], \langle−, −\rangle, \rho)\) [34] is a wider structure than a CA with the Leibniz rule relaxed. It consists of a vector bundle \(E \to M\) endowed with a bracket \([−, −] : E \times E \to E\), an inner product \(\langle−, −\rangle : E \times E \to C^\infty(M)\), a bundle map (anchor) \(\rho : E \to TM\), and a differential \(\partial\) such that \(\langle\partial f, a\rangle = \rho(a)f\) satisfying

\[
(a) \quad \rho(a)(b, c) = \langle[a, b], c\rangle + \langle b, [a, c]\rangle, \tag{2.12}
\]

\[
(b) \quad [a, a] = \frac{1}{2} \partial \langle a, a\rangle, \tag{2.13}
\]

with \(a, b, c \in \Gamma(E)\). Note the absence of the Jacobi identity of the bracket, i.e., the compatibility of the anchor with the bracket (2.11) does not hold in general.

2.4. Algebroid of DFT revisited. We construct DFT as a geometry on the vector bundle \(E\). We consider a vector bundle \(E\) over a 2D-dimensional manifold \(M\), called the doubled space, which is an idea inspired by string theory [9,35]. We endow \(E\) with a metric algebroid structure as well as a Lie algebroid structure. The maps and brackets are the following:

- vector bundle \(\pi : E \to \mathbb{M}\),
- LA-bracket (Lie algebroid) \([·, ·]_L : E \times E \to E\),
- MA-bracket (Metric algebroid) \([·, ·] : E \times E \to E\),
- inner product \(\langle·, ·\rangle : E \times E \to C^\infty(M)\),
- anchor map \(\rho : E \to TM\).

The anchor map \(\rho\) is common for both, Lie algebroid and metric algebroid. We denote fiber metric by \(\eta_{AB}\), and the structure functions \(F_{ABC}, F'_{ABC}\) and \(\phi'_{ABC}\) in a basis \(E_A \in E\) are given by

\[
\eta_{AB} := \langle E_A, E_B\rangle, \tag{2.15}
\]

\[
F_{ABC} := \langle [E_A, E_B], E_C\rangle, \tag{2.16}
\]

\[
F'_{ABC} := \langle [E_A, E_B]_L, E_C\rangle, \tag{2.17}
\]

\[
\phi'_{ABC} := F_{ABC} - F'_{ABC}. \tag{2.18}
\]
For DFT we require the $O(D, D)$ structure and thus the fiber metric is a constant $O(D, D)$ metric. For later purpose we defined $\phi'$ which measures the difference of the MA bracket and the LA bracket. The Jacobi identity in the LA gives a relation on the structure functions (Bianchi identity) as follows

$$J_{ABC}^D = \rho(E_A)(F^D_{BC}) + F^D_{[AB}E_{C]E} = 0.$$  (2.19)

Taking the trace w.r.t. the indices $A$ and $D$, we obtain the relation

$$J_{ABC}^A = 2\rho(E_B)(\phi'_{|AC})^A + (\rho(E_A) - \phi'_{DA})F_{BE}^A - (\rho(E_A) - \phi'_{DA})F_{BC}^A = 0.$$  (2.20)

This condition will play a role when we introduce the dilaton in the formulation.

### 2.4.1. Jacobiator in metric algebroid and Jacobi identity

In the MA we define a quantity which traces the augmented properties compared to the CA, by defining the following maps $\mathcal{L} : E \times E \times E \to E$ and $\mathcal{L}' : E \times E \to E$:

$$\mathcal{L}(a, b, c) = [a, [b, c]] - [[a, b], c] - [b, [a, c]] ,$$  (2.21)

$$\mathcal{L}'(a, b) = [a, b] - [a, b]_L .$$  (2.22)

The map $\mathcal{L}$ in (2.21) is a Jacobiator in Leibniz-like form. Note that $\rho \circ \mathcal{L}'$ does not vanish in general, while in the Courant algebroid, both quantities are zero. These quantities satisfy the following relations [1]:

$$\mathcal{L}(a, b, c) + \mathcal{L}(b, a, c) = -[\partial \langle a, b \rangle, c] ,$$  (2.23)

$$\mathcal{L}'(a, b) + \mathcal{L}'(b, a) = \partial \langle a, b \rangle .$$  (2.24)

The above maps are not $C^\infty(M)$-linear in all arguments, however, in a previous investigation [1] we showed how to obtain a tensor by using a map $\phi$

$$\phi(a, b, c, d) = (\mathcal{L}(a, b, c), d) ,$$  (2.25)

the tensorial property of which is

$$\Delta \phi := \phi(f a, b, c, d) + \phi(a, gb, c, d) + \phi(a, b, hc, d) + \phi(a, b, c, kd)$$

$$-(f + g + h + k)\phi(a, b, c, d)$$

$$= -(\rho \circ \mathcal{L}'(b, c) f)(a, d) + (\rho \circ \mathcal{L}'(a, c) g)(b, d) - (\rho \circ \mathcal{L}'(a, b) h)(c, d)$$

$$- \langle a, b \rangle(\rho \circ \mathcal{L}'(c, d) f) + \langle a, c \rangle(\rho \circ \mathcal{L}'(b, d) f) - \langle b, c \rangle(\rho \circ \mathcal{L}'(a, d) g) .$$  (2.26)

Note that the map $\mathcal{L}'$ given in [1] corresponds to the map $\rho \circ \mathcal{L}'$ in the present paper, since

$$\rho \circ \mathcal{L}'(a, b) = \rho([a, b] - [a, b]_L) = \rho([a, b] - [\rho(a), \rho(b)])_{TM} .$$  (2.27)

Therefore, in the CA the map $\mathcal{L} = 0$, from which $\rho \circ \mathcal{L}' f = 0$ follows. This means that the MA reduces to a CA if $\mathcal{L} = 0$. From Eq. (2.18) it follows that

$$\mathcal{L}'(E_A, E_B) = \phi'_{AB}^C E_C .$$  (2.28)

Note that the pre-Courant algebroid [34,36] is characterized by the condition on the MA $\mathcal{L} \neq 0$, but $\rho \circ \mathcal{L}' f = 0$.
3. Dirac Generating Operator

A Dirac generating operator is a Dirac operator which generates the algebroid structure. Similarly as done in the generalized geometry [26,37,38] we can formulate the MA structure by using a Dirac generating operator. For this end, we consider a Clifford bundle $\text{Cl}(E)$ endowed with a graded bracket $\{\cdot,\cdot\}$. The generators $\gamma_A$ of the Clifford bundle satisfy

\[ \{\gamma_A, \gamma_B\} = 2\eta_{AB}, \quad (3.1) \]

where $\eta_{AB}$ is the $O(D, D)$ metric. The basis of $\text{Cl}(E)$ is given by the products

\[ \{1, \gamma_{A_1}, \gamma_{A_1A_2}, \cdots, \gamma_{A_1A_2\cdots A_D}\} \in \text{Cl}(E), \quad (3.2) \]

where $\gamma_{A_1A_2\cdots A_n}$ is

\[ \gamma_{A_1A_2\cdots A_n} = [\gamma_{A_1}, \gamma_{A_2}, \cdots, \gamma_{A_n}] . \quad (3.3) \]

$\gamma_A$ is related to $E_A$ by a linear map $\gamma : E \rightarrow \text{Cl}(E)$,

\[ \gamma(E_A) = \gamma_A , \quad (3.4) \]

and for $\gamma a, b \in E$,

\[ \{\gamma(a), \gamma(b)\} = 2\langle a, b \rangle . \quad (3.5) \]

In the following we do not explicitly write the map $\gamma$ to avoid cumbersome notation, i.e., $\gamma_A = E_A$ and $E \subset \text{Cl}(E)$. A representation space of $\text{Cl}(E)$ is given by a $O(D, D)$ spin bundle $\mathbb{S}$. For the spin bundle we introduce a basis $\gamma a = (\gamma^a, \gamma_a)$ which satisfies

\[ \{\gamma^a, \gamma_b\} = 2\delta^a_b . \quad (3.6) \]

The vacuum state of $\mathbb{S}$ is defined in the standard way,

\[ \gamma_a|0\rangle = 0 , \quad (3.7) \]

and a 2D-dimensional spinor is defined by multiplying $\gamma^a$ onto the vacuum. We define a derivative $\partial_A$ acting on $\mathbb{S}$ by

\[ \{\partial_A, \gamma_B\} = 0 , \quad (3.8) \]

\[ \{\partial_A, f\} = \rho(E_A)(f) , \quad (3.9) \]

\[ \partial_A|0\rangle = 0 . \quad (3.10) \]

The operations of the metric algebroid are generated by the Dirac generating operator via the derived bracket [39] as

\[ \{[\mathcal{D}, a], b\} = [a, b] , \quad (3.11) \]

\[ \{\mathcal{D}, f\} = \frac{1}{2}\partial f , \quad (3.12) \]

\[ \{[\mathcal{D}, a], f\} = \rho(a)(f) . \quad (3.13) \]

A concrete form of $\mathcal{D}$ for the MA has been given in [1] as

\[ \mathcal{D} = \frac{1}{2}\gamma^A\partial_A - \frac{1}{24}\gamma^{ABC}F_{ABC} - \frac{1}{4}\gamma^AF_A , \quad (3.14) \]
where $\gamma^{ABC} = \gamma^{[A} \gamma^{B} \gamma^{C]}$ and $F_A$ is an ambiguity of $\mathcal{D}$ which is not determined by the axioms. A priori, the flux $F_A$ has nothing to do with a dilaton in this level. To obtain a condition which connects $F_A$ to the dilaton we need to require further structure, e.g., scale invariance or introduction of a line bundle or a determinant bundle [26,40].

**Compatible connection** We can represent the Dirac generating operator $\mathcal{D}$ using a covariant derivative on $\mathbb{S}$. The covariant derivative on $\mathbb{S}$ is given by a spin connection $W_{ABC}$,

$$\nabla^\mathbb{S}_A = \partial_A - \frac{1}{4} \gamma^{BC} W_{ABC}. \quad (3.15)$$

We can choose the spin connection such that

$$\mathcal{D} = \frac{1}{2} \gamma^A \nabla^\mathbb{S}_A, \quad (3.16)$$

becomes the Dirac generating operator by the following condition

$$T_{ABC} := 3W_{[ABC]} - F_{ABC} = 0, \quad (3.17)$$
$$W_{BA} = F_A, \quad (3.18)$$

where $T_{ABC}$ is a generalized torsion, i.e., we obtain a torsion-free connection [41].

### 4. Generalized Lichnerowicz Formula and Generalized Dilaton

#### 4.1. Generalized Lichnerowicz formula

In a previous paper [1] we have shown that in the MA formulation DFT belongs to a class which is specified by a generalized Bianchi identity, which we called pre-Bianchi identity, for the structure functions $F_{ABC}$. We also found that the pre-Bianchi identity for $F_{ABC}$ and for $F_A$ are the necessary and sufficient conditions for a generalized Lichnerowicz formula to hold. We defined a generalized Lichnerowicz formula for the MA by using the Dirac generating operator.

In generalized geometry, an action has been constructed by the square of the Dirac operator $\mathcal{D}^2$ [5,6,40,44].

In DFT on the other hand, $\mathcal{D}^2$ contains second order and first order derivative terms. In order to derive a scalar from $\mathcal{D}^2$, we subtract a particular Laplacian using $\phi'$ as follows.

We observe that $\phi'_{ABC}$ changes under the local $O(D, D)$ transformation in the same way as the spin connection $W_{CAB}$, i.e., under the local $O(D, D)$ transformation of the basis

$$\delta_A E_A = \Lambda_A^B E_B, \quad (4.1)$$

they transform as

$$\delta_A W_{CAB} = \rho(E_C)(\Lambda_{AB}) + \Lambda \triangleright W_{CAB}, \quad (4.2)$$
$$\delta_A \phi'_{ABC} = \rho(E_C)(\Lambda_{AB}) + \Lambda \triangleright \phi'_{ABC}, \quad (4.3)$$

where $\Lambda \triangleright$ means the linear term of the transformation, e.g.,

$$\Lambda \triangleright \phi'_{ABC} = \Lambda_A^{A'} \phi'_{A'BC} + \Lambda_B^B \phi'_{AB'C} + \Lambda_C^C \phi'_{ABC'}. \quad (4.4)$$

\footnote{For the original Lichnerowicz formula see [42,43]. An application to M-theory has been formulated in [44].}
Proof. Recall that in the present formulation $\phi'_{ABC}$ is introduced as the difference of MA and LA bracket. The tensorial property of $\phi'_{ABC}$ is then given by

$$\Delta\langle[a, b] - [a, b]_L, c\rangle := \langle [f a, b] - [f a, b]_L, c\rangle + \langle [a, gb] - [a, gb]_L, c\rangle$$
$$+\langle [a, b] - [a, b]_L, hc\rangle - (f + g + h)\langle[a, b] - [a, b]_L, c\rangle = \langle a, b \rangle \rho(c)f \quad (4.5)$$

Thus, in the basis $E_A$ this gives

$$\Delta\langle[E_A, E_B] - [E_A, E_B]_L, E_C\rangle = \eta_{AB}\rho(E_C)f \quad (4.6)$$

and we obtain (4.3).

We use this fact to define a second covariant derivative $\nabla_A\phi'$ as

$$\nabla_A\phi' = \partial_A - \frac{1}{4}\gamma^{BC}\phi'_B\phi'_{CA} \quad (4.7)$$

from which we construct a Laplacian $\Delta\phi'$

$$\Delta\phi' = \eta^{AC}(\nabla^A\phi' - \phi'_B B - U_A)\nabla^C\phi' = div U_{\nabla\phi'} \quad (4.8)$$

The quantity $U = U^A E_A$ is an arbitrary generalized vector $U$ appearing as an ambiguity in the divergence $div U_{\nabla\phi'}$. See also [40].

Then, the generalized Lichnerowicz formula can be obtained by requiring that

$$4\mathcal{D}^2 - \Delta\phi' \in C^\infty(M) \quad (4.9)$$

i.e., this difference should yield a function. In the general case,

$$4\mathcal{D}^2 - \Delta\phi' = -\frac{1}{24} F_{ABC} F^{ABC} - \frac{1}{2}(\rho(E^A) F_A) + (-F^{A} + \phi^{A}_{E} E^{A} + U^{A})\partial_A$$
$$+\frac{1}{4} F_A F^A + \frac{1}{8}\phi'_{BCA} \phi'_{B}\phi'_{C}$$
$$+\frac{1}{4}\left(-J_{BCD} D + 2(\rho(E[I])(-F[1] + \phi[D]_{C[D)})\right) \gamma^{BC}$$
$$-\frac{1}{48}(4(\rho(E[I])(F_{BCD})) - 3F[AB] F_{CD}[E] + 3\phi'_[A] E \phi'_{CD}[E]) \gamma^{BCB'}C' \quad (4.10)$$

where $J_{BCD} D$ is the tensor defined in (2.20) which is zero and thus, the expression simplifies correspondingly.

From the requirement (4.9), i.e., the difference gives a scalar, the terms proportional to $\partial_A, \gamma^{AB}$ and $\gamma^{ABCD}$ must vanish. This requirement gives a condition for $F_A$ and $U_A$:

$$(F_A - \phi^{B}_{BA} B - U_A)\rho(E^A) = 0 \quad (4.11)$$

Together with the condition from the term proportional to $\gamma^{AB}$,

$$\rho(U^A E^A) = 2\rho(E_A)(d) \quad (4.12)$$
This defines $F_A$ with an ambiguity $U'_A$
\[
F_A = 2\rho(E_A)d + \phi'_{BA} B + U'_A \tag{4.13}
\]
where
\[
\rho(U'_A E^A) = 0, \tag{4.14}
\]
\[
2\rho(E_B(U'_A C)) = F_{BC} A U'_{A} \tag{4.15}
\]
and $d \in C^\infty(M)$ is an arbitrary scalar, which can be interpreted as a generalized dilaton if we introduce further structure. When $E_A M$ is invertible, $U'_A = 0$.\footnote{In DFT$_{sec}$, the vector bundle is identified with the tangent bundle itself, thus $E_A M$ is invertible.} If $E \neq T M$, $U'_A$ is not automatically zero. The non-invertible case opens a vast number of possibilities, meaning that we have to use some physical requirements in order to narrow down the number of choices. We choose that the degree of freedom of $F_A$ is given by the scalar $d$ only, since from the physics point of view the dilaton is a unique (up to a shift) object, i.e.,
\[
U'_A = 0. \tag{4.16}
\]
From this discussion we obtain that
\[
F_A = \phi'_{BA} B + 2\rho(E_A)(d), \tag{4.17}
\]
Using this solution of $F_A$, Eq. (4.10) simplifies as
\[
4\mathcal{D}^2 - \Delta \phi' = -\frac{1}{24} F_{ABC} F^{ABC} - \frac{1}{2} \rho(E^A)(F_A) + \frac{1}{4} F_A F^A + \frac{1}{8} \phi'_{BCA} \phi'^{BCA}
\]
\[
-\frac{1}{4} \left( 2\rho(E_B(F_C)) + \rho(E_A)(F_{BC} A) - F_A F_{BC} A + \rho(E_A)(\phi'_{BC} A) - F_A \phi'_{BC} A \right) \gamma_{BC}
\]
\[
-\frac{1}{48} \left( 4\rho(E_A)(F_{BCD}) - 3 F_{[AB} E_{FCD]|E} + 3 \phi'_{[AB} E \phi'_{CD]|E} \right) \gamma^{ABCD}. \tag{4.18}
\]
The terms proportional to $\gamma^{AB}$ give the pre-Bianchi identity including the flux $F_A$,
\[
\mathcal{B}_{BC} := 2\rho(E_B(F_C)) + \rho(E_A)(F_{BC} A) - F_A F_{BC} A + \rho(E_A)(\phi'_{BC} A) - F_A \phi'_{BC} A = 0, \tag{4.19}
\]
This pre-Bianchi identity includes Eq. (4.17) as a solution, meaning that the standard DFT is contained in this generalization of DFT to the curved space.

The terms proportional to $\gamma^{ABCD}$ give the pre-Bianchi identity
\[
\mathcal{B}_{ABCD} := 4\rho(E_A)(F_{BCD}) - 3 F_{[AB} E_{FCD]|E} + 3 \phi'_{[AB} E \phi'_{CD}|E] = 0. \tag{4.20}
\]
for the flux $F_{ABC}$ (which corresponds to the tensor $\tilde{\phi}$ in [1]). This is a condition on the structure functions of the metric algebroid.

Therefore, with the 3 conditions (4.11), (4.19) and (4.20), the requirement that (4.9) is a scalar is satisfied and the scalar part of the generalized Lichnerowicz formula is simply given by
\[
4\mathcal{D}^2 - \Delta \phi' = -\frac{1}{24} F_{ABC} F^{ABC} - \frac{1}{2} \rho(E^A)(F_A) + \frac{1}{4} F_A F^A + \frac{1}{8} \phi'_{BCA} \phi'^{BCA}. \tag{4.21}
\]
This is the desired scalar which is $O(D, D)$ invariant. However, it is not the action of DFT since we have not yet used the data associated with the generalized metric $H_{AB}$. To obtain the action for DFT, we have to construct an $O(1, D - 1) \times O(D - 1, 1)$ invariant scalar.
4.2. Projected Lichnerowicz formula. After the introduction of the generalized metric \( H_{AB} \) on the fiber, the local \( O(D, D) \) symmetry reduces to \( O(1, D - 1) \times O(D - 1, 1) \) in such a way that the invariance of the action under generalized diffeomorphism is preserved at least up to a closure constraint. The projected Lichnerowicz formula gives \( O(1, D - 1) \otimes O(D - 1, 1) \) invariant Lagrangians.

The generalized metric on the fiber \( H_{AB} \) is an \( O(D, D) \) symmetric tensor and satisfies

\[
H_{AB} \eta^{BC} H_{CD} = \eta_{AD}. \tag{4.22}
\]

As in the generalized geometry, the generalized metric gives the split of the tangent bundle to the positive/negative subbundle as \( E = V^+ \oplus V^- \):

\[
V^+ = \{ V \in TM | H^A_B V^B = V^A \}, \\
V^- = \{ V \in TM | H^A_B V^B = -V^B \} \tag{4.23}
\]

where \( H^A_B = \eta^{AC} H_{CB} \). Since \( H^A_B H^B_C = \delta^A_B \), we have corresponding projectors given by

\[
P^{\pm A}_B = \frac{1}{2}(\delta^A_B \pm H^A_B), \tag{4.24}
\]

which satisfy

\[
P^+ A_B + P^- A_B = \delta^A_C, \quad P^{\pm A}_B P^{\pm B}_C = P^{\pm A}_C, \quad P^{\pm A}_B P^{\mp B}_C = 0. \tag{4.25}
\]

Correspondingly, we consider the projected spin bundle \( \Gamma(\mathbb{S}^\pm) \) which is a module over \( Cl(V^\pm) \), constructed on \( |0\rangle \) by multiplying the elements of \( \Gamma(V^\pm)[1] \). The Dirac generating operator is also split by using the covariant derivatives \( \nabla^\pm : \Gamma(\mathbb{S}^\pm) \rightarrow \Gamma(E^*) \otimes \Gamma(\mathbb{S}^\pm) \) and denoted as \( \mathcal{D}^\pm \).

The projected Lagrangians \( \mathcal{L}^\pm \) are given by

\[
\mathcal{L}^\pm = 4(\mathcal{D}^\pm)^2 + div \nabla^\pm \Delta \phi^\pm , \tag{4.26}
\]

where \( \Delta \phi^\pm \) is the divergence of the projected covariant derivative \( \nabla^\pm \phi^\pm \). For details see [1]. The explicit form of \( \mathcal{L}^\pm \) is given as:

\[
\mathcal{L}^+ = -\frac{1}{24} F_{abc} F^{abc} - \frac{1}{8} F^a_{abc} F^{abc} - \frac{1}{2} \rho(E_a) F^a + \frac{1}{4} F_a F^a + \frac{1}{8} \phi^a_b \phi^b_c + \frac{1}{8} \phi^a_{bc} \phi^{ab} \tag{4.27}
\]

and

\[
\mathcal{L}^- = -\frac{1}{24} F_{\bar{a}bc} F^{\bar{a}bc} - \frac{1}{8} F^{\bar{a}b\bar{c}} F_{\bar{a}b\bar{c}} - \frac{1}{2} \rho(E_{\bar{a}}) F^{\bar{a}} + \frac{1}{4} F_{\bar{a}} F^{\bar{a}} + \frac{1}{8} \phi^\bar{a}_{b\bar{c}} \phi^{b\bar{c}} + \frac{1}{8} \phi^\bar{a}_{bc} \phi^{b\bar{c}} \tag{4.28}
\]

Here, the bases of the positive/negative subspaces are denoted as \( (E_a, E_{\bar{a}}) \) i.e., the indices of the corresponding fluxes are \( a/\bar{a} \), respectively.
4.3. Dilaton and measure. The generalized Lie derivative $L_a$ is the generator of the generalized diffeomorphisms in $M$. On the spin bundle $\mathbb{S}$ using a Dirac generating operator $\mathcal{D}$ the Lie derivative can be formulated for all $a \in E$ and all $\chi \in \mathbb{S}$ as

$$L_a \chi = (\mathcal{D}, a) \chi,$$

which is compatible with the generalized Lie derivative on $E$, i.e., the Leibniz rule holds for all $a, b \in E, \chi \in \mathbb{S}$

$$L_a b \chi = (\mathcal{D}, a) b \chi = b (\mathcal{D}, a) \chi + (L_a b) \chi + b L_a \chi.$$

This gives a gauge transformation of a spinor with weight $\frac{1}{2}$ [1]. We define an inner product $(\chi_1, \chi_2)_A$ on $\mathbb{S}$ which is invariant under the infinitesimal $O(D, D)$ transformation, and satisfies for all $\chi_1, \chi_2 \in \Gamma(\mathbb{S})$:

$$(\gamma_A \chi_1, \chi_2)_A = (\chi_1, \gamma_A \chi_2)_A$$

(4.31)

$$(\mathcal{D} \chi_1, \chi_2)_A = (\chi_1, -\mathcal{D} \chi_2)_A.$$

(4.32)

In this paper, we call $(\cdot, \cdot)_A$ simply an $A$-product. (See the appendix for details). The $A$-product is an inner product such that the anti-hermiticity of the Dirac operator holds.

Then, we can compute

$$((\mathcal{D}, \gamma_A) f_1 | 0 \rangle, f_2 K | 0 \rangle)_A = (f_1 | 0 \rangle, -(\mathcal{D}, \gamma_A) f_2 K | 0 \rangle)_A,$$

(4.33)

for all $f_1, f_2 \in C^\infty(M)$. The $K$ in front of the vacuum is introduced in (A.17) to define the dual vacuum as (A.33). By a straightforward calculation, we get

$$((\partial_A - \frac{1}{2} F_A) f_1 | 0 \rangle, f_2 K | 0 \rangle)_A = (f_1 | 0 \rangle, -(\partial_A - \frac{1}{2} F_A) f_2 K | 0 \rangle)_A.$$

(4.34)

This is a necessary condition when we define the integration measure of the $A$-product.

A concrete form of the $A$-product can be derived by assigning for $f \in C(M)$ and vacuum $(|0\rangle, K | 0 \rangle)_A$

$$((0 \rangle, f K | 0 \rangle)_A = \int d\mu f,$$

(4.35)

$$d\mu = dX \tilde{h}.$$

(4.36)

Then, the measure $\tilde{h}$ has to satisfy

$$\tilde{h}^{-1} \rho(E_B) (\tilde{h}) = -\partial_N E_B^N - \phi_A^B A - 2 \rho(E_B) d = -\partial_N E_B^N + F_{AB}^C - 2 \rho(E_B) d,$$

(4.37)

where $\rho(E_A) = E_A^N \partial_N$. Thus, we factor out the function $d$ as $\tilde{h} = e^{-2d} h$. To obtain an explicit form for $h$ we require that the trace of the flux on the LA satisfies

$$F_{AB}^C (\delta_C^A - E_C^N \eta_{NM} E^{AM}) = 0,$$

(4.38)

where $\eta_{MN}$ is an induced metric on $TM$,

$$\eta^{MN} := E_A^M \eta^{AB} E_B^N.$$

(4.39)
\[ \eta_{LM} \eta^{MN} = \delta_L^N. \] (4.40)

Then, we can compute \( h \) as follows

\begin{align*}
    h^{-1} \rho(E_B)(h) &= -\partial_N E_B^N + F'_{AB}^A \\
    &= -\partial_N E_B^N + F'_{AB}^C E_C^N \eta_{NM} E_D^M \eta^{DA} \\
    &= -\partial_N E_B^N + (E_A^L \partial_L E_B^N - E_B^L \partial_L E_A^N) \eta_{NM} E_D^M \eta^{DA} \\
    &= -\partial_N E_B^N + \partial_N E_B^N - E_B^L \partial_L E_A^N \eta_{NM} E_D^M \eta^{DA} \\
    &= -\frac{1}{2} E_B^L \partial_L E_A^N \eta_{NM} E_D^M \eta^{DA} - \frac{1}{2} E_B^L \partial_L E_D^N \eta_{NM} E_A^M \eta^{DA} \\
    &= \frac{1}{2} E_A^N \rho(E_B)(\eta_{NM}) E_D^M \eta^{DA} \\
    &= \frac{1}{2} \eta^{NM} \rho(E_B)(\eta_{NM}) \\
    &= \sqrt{\det \eta_{NM}}^{-1} \rho(E_B)(\sqrt{\det \eta_{NM}}). \quad (4.41)
\end{align*}

This yields a concrete form for the measure in \((|0\rangle, K|0\rangle)_A\) as

\begin{align*}
    h &= c_0 \sqrt{\det \eta_{NM}}, \quad (4.42) \\
    \tilde{h} &= c_0 e^{-2d} \sqrt{\det \eta_{NM}}, \quad (4.43) \\
    (|0\rangle, K|0\rangle)_A &= c_0 \int dX e^{-2d} \sqrt{\det \eta_{MN}}. \quad (4.44)
\end{align*}

where \(c_0\) is a constant.

The transformation of the scalar \( d \) under the generalized Lie derivative is obtained as follows. We consider the \( A \)-product of the vacuum \(|0\rangle\) and its dual \( K|0\rangle\),

\begin{align*}
    \delta_a (|0\rangle, f K|0\rangle)_A &= (L_a |0\rangle, f K|0\rangle)_A + (|0\rangle, f L_a K|0\rangle)_A \\
    &= ((f \mathcal{D}, a)|0\rangle, f K|0\rangle)_A + (|0\rangle, f \{\mathcal{D}, a\} K|0\rangle)_A \\
    &= (|0\rangle, K|0\rangle)_A (f (\rho(E_A) - F_A)(a^A), \quad (4.45)
\end{align*}

\begin{align*}
    \delta_a \int dX \tilde{h} f &= \int dX c_0 \delta_a e^{-2d} \sqrt{\det \eta_{MN}} f \\
    &= \int dX c_0 (-2 \delta_a d) e^{-2d} \sqrt{\det \eta_{MN}} f \\
    &= (|0\rangle, K|0\rangle)_A (-2 \delta_a d) f. \quad (4.46)
\end{align*}

Therefore, the transformation rule of the scalar \( d \) is determined as

\begin{align*}
    \delta_a d &= -\frac{1}{2} (\rho(E_A) - F_A)(a^A), \quad (4.47) \\
    \delta_a e^{-2d} &= (\rho(E_A) - F_A)(a^A) e^{-2d}. \quad (4.48)
\end{align*}

The last equation reflects the gauge transformation of the dilaton. Therefore, we identify \( d \) with the dilaton.
5. DFT Actions and Gauge Symmetry

First, we define a NS–NS field $U^B_A$ as the fluctuation of the generalized vielbein. We assume that all physical degrees of freedom of the generalized vielbein are carried by the fluctuation field $U^B_A$. Thus, the degrees of freedom are equal to those of the generalized metric $H_{MN} = E^A_ME^B_NH_{AB}$. This means that $\eta_{MN} = E^A_ME^B_N\eta_{AB}$ does not carry dynamical degrees of freedom but is just a parameter of the theory.

To extract the physical data from this structure, we define for a given $\eta_{MN}$ a background vielbein $\bar{E}^M_A$ such that the $O(D, D)$ metric in the background basis $\bar{E}^N_A\bar{\partial}_N = \bar{E}_A$ is given by

$$\bar{\eta}_{AB} := \bar{E}^M_A\bar{E}^N_B\eta_{MN} = \left(\delta_a^b\right).$$

(5.1)

Then, we can separate the generalized vielbein $E^M_A$ into a fluctuation $U^B_A$ and the background $\bar{E}^N_A$ as

$$E^N_A = U^B_A\bar{E}^N_B, \quad (5.2)$$

As we will show, $\bar{E}^N_A$ and $U^B_A$ relate to $\eta_{MN}$ and $H_{MN}$, respectively. This $O(D, D)$ metric $\bar{\eta}_{AB}$ and the $O(D, D)$ metric $\eta_{AB}$ in the local Lorentz basis are connected by $U^B_A$ as

$$U^A_{A'}U^B_{B'}\bar{\eta}_{A'B'} = \eta_{AB}. \quad (5.3)$$

Since $\eta_{AB} = \bar{\eta}_{AB}$, $U^B_A \in O(D, D)$. Moreover, considering that those degrees of freedom of $U^B_A$ which do not change $H_{MN}$ are not physical, the space of fluctuations $U^B_A$ is given by

$$U^B_A \in (O(1, D - 1) \times O(D - 1, 1))\backslash O(D, D). \quad (5.4)$$

The concrete form of $U^B_A$ can be written by

$$E_A = U^B_A\bar{E}_B, \quad U^B_A = \begin{pmatrix} e^{-Ta}_{a\,b} & 0 \\ -e^{-}_{a\,c}B^{cb}e^c_{\,b} & \end{pmatrix}, \quad (5.5)$$

where $e^c_{\,b}$ has $O(1, D - 1)$ symmetry and $B_{ab}$ is an anti-symmetric tensor. In DFT$_{sec}$, the background $\bar{E}^M_A$ is flat, i.e., $\bar{E}^N_A\bar{\partial}_N$ can be chosen as $\delta^N_A$.

5.1. Action in MA formalism. The most general $O(1, D - 1) \otimes O(D - 1, 1)$ invariant action is given by a linear combination of $\mathcal{L}^+$ and $\mathcal{L}^-$ as

$$S = \mathcal{I}(\beta_+, \beta_-) = \beta_+ \left(4(\mathcal{D}^+|0\rangle, \mathcal{D}^+K|0\rangle)_A - (\nabla^{S+}_{E_{a}}|0\rangle, \nabla^{S+}_{E_{b}}K|0\rangle)_A + (\nabla^{(\phi^+)}_{E_{a}}|0\rangle, \nabla^{(\phi^+)}_{E_{b}}K|0\rangle)_A \right)$$

$$+ \beta_- \left(4(\mathcal{D}^-|0\rangle, \mathcal{D}^-K|0\rangle)_A - (\nabla^{S-}_{E_{a}}|0\rangle, \nabla^{S-}_{E_{b}}K|0\rangle)_A + (\nabla^{(\phi^-)}_{E_{a}}|0\rangle, \nabla^{(\phi^-)}_{E_{b}}K|0\rangle)_A \right),$$

$$= (|0\rangle, K|0\rangle)_A(\beta_+\mathcal{L}^+ + \beta_-\mathcal{L}^-),$$

$$= c_0 \int dX\sqrt{\det \eta}e^{-2d(\beta_+\mathcal{L}^+ + \beta_-\mathcal{L}^-)}. \quad (5.6)$$

The coefficients $\beta_+$ and $\beta_-$ are free parameters.
Our requirement is that after the dimensional reduction we obtain the supergravity action. This requirement leads to a condition on the parameters $\beta_\pm$ as

$$- \beta_+ + \beta_- = 8c_0^{-1} ,$$

where the overall constant $c_0$ is fixed such that we obtain the standard normalization of the Einstein-Hilbert action.

To discuss the Poisson-Lie T-duality in DFT, we choose the action with the following parameters

$$S_{DFT} = \mathcal{I}(0, 8c_0^{-1}) .$$

As we shall see, the GSE is naturally included in this parametrization.

From the action (5.8) the DFT$_{sec}$ action can also be produced by requiring that the MA-bracket on the coordinate basis vanishes,

$$[\partial_L, \partial_M] = 0 .$$

The action of the DFT Wess-Zumino-Witten model $S_{DFTWZW}$ discussed in [45–47] can be constructed when the MA-bracket equals to the LA-bracket in the basis $\bar{E}_A$:

$$[\bar{E}_A, \bar{E}_B] = [\bar{E}_A, \bar{E}_B]_{L} .$$

The action of the DFT$_{WZW}$ model is given by

$$S_{DFTWZW} = \mathcal{I}(-4c_0^{-1}, 4c_0^{-1}) .$$

**Riemann tensor and DFT action** In a previous paper we formulated the generalized curvature $\mathcal{R}(a, b, c, d)$ on the metric algebroid. The result is

$$\mathcal{R}(a, b, c, d) = \mathcal{R}^\nabla(a, b, c, d) + \mathcal{R}^\nabla(c, d, a, b) + \langle \mathcal{L}'(a, b), \mathcal{L}'(c, d) \rangle$$

where

$$\mathcal{R}^\nabla(a, b, c, d) := \langle (\nabla^E_a \nabla^E_b - \nabla^E_b \nabla^E_a)c - \nabla^E_{[a,b]}c, d \rangle + \frac{1}{2} \langle \nabla^E_{[a}a, b \rangle \langle \nabla^E_{a]}c, d \rangle .$$

Note that the first two terms

$$\mathcal{R}^\nabla(a, b, c, d) + \mathcal{R}^\nabla(c, d, a, b)$$

is the curvature defined by Hohm and Zwiebach [48]. In the basis $E_A$ the generalized Riemann tensor is given as

$$\mathcal{R}^{ABCD} := \mathcal{R}(E_A, E_B, E_C, E_D) = \mathcal{R}^\nabla_{ABCD} + \mathcal{R}^\nabla_{CDAB} + \phi'_A E \phi'_B E ,$$

where

$$\mathcal{R}^\nabla_{ABCD} := 2 \rho(E_{[A}W_{B]CD} - 2W_{[A}|c E W_{B]}E_D - F_{AB}E W_{ECD} + \frac{1}{2} W_{EAB} W_{ECD} .$$
From the generalized Riemann tensor $\mathcal{R}_{ABCD}$, we can construct the various generalized Ricci tensors and curvature scalar by using the projection operators made by the $O(D, D)$ metric $\eta_{AB}$ and the $O(1, D - 1) \times O(D - 1, 1)$ invariant generalized metric $H_{AB}$. The above action can be expressed by using the projected generalized Riemann scalar as

$$L^+ = \frac{1}{8} R_{ab}^{\ ab},$$

$$L^- = \frac{1}{8} R_{\ \hat{a}\hat{b}}^{\ \hat{a}\hat{b}}. \tag{5.19}$$

Thus the Lagrangian of the most general action is

$$L = \beta_+ L^+ + \beta_- L^- = \frac{1}{8} \beta_+ R_{ab}^{\ ab} + \frac{1}{8} \beta_- R_{\ \hat{a}\hat{b}}^{\ \hat{a}\hat{b}}. \tag{5.20}$$

5.2. Variation of the action. The variation of the action due to the fluctuation yields the equations of motion. Here, we discuss the general properties of the DFT action (5.6) under this fluctuation. We will use the results later to obtain the equation of motion for DFT on a Drinfeld double.

**Variation with respect to the vielbein** In this subsection, we consider a variation of the DFT action (5.6) by an infinitesimal $O(D, D)$ rotation. The variation of the vielbein $\delta(E)$ is defined by

$$\delta(E)_{EA} = \Lambda_{A}^{\ B} E_{B} \tag{5.21}$$

where $\Lambda_{AB} = -\Lambda_{BA}$. The variation of the fluxes yields

$$\delta^{(E)} F_{ABC} = \rho(E_{A})(\Lambda_{BC}) \tag{5.22}$$

$$\delta^{(E)} \phi'_{ABC} = \rho(E_{C})(\Lambda_{AB}) + \Lambda_{A'}^{\ A} \phi'_{A'BC} + \Lambda_{B}^{\ B} \phi'_{AB'C} + \Lambda_{C}^{\ C} \phi'_{ABC'}, \tag{5.23}$$

$$\delta^{(E)} F_{A} = \rho(E_{B})(\Lambda_{A}^{\ B}) + \Lambda_{A}^{\ B} F_{B}. \tag{5.24}$$

The measure is invariant

$$\delta^{(E)} \left(dX \sqrt{\det \eta e^{-2d}} \right) = 0. \tag{5.25}$$

Since the action is given by the projection corresponding to positive/negative generalized tangent vectors, we also split the infinitesimal transformation parameter accordingly as

$$\Lambda_{AB} \rightarrow \{\Lambda_{ab}, \Lambda_{\hat{a}\hat{b}}, \Lambda_{\hat{a}\hat{b}}\}. \tag{5.26}$$

Then, the variation of the $L^+$ part of the action is

$$\delta^{(E)} \left(c_{0} \int dX \sqrt{\det \eta e^{-2d} L^+} \right)$$

$$= c_{0} \int dX \sqrt{\det \eta e^{-2d}}$$

$$\left[ -\frac{1}{4} \Lambda^{ab} \left( \left(E_{C}^{N} \partial_{N} - F_{C} \right) F_{ab}^{\ \ C} + 2 E_{[a}^{N} \partial_{N} F_{b]} - \left( E_{C}^{N} \partial_{N} - F_{C} \right) \phi'_{ab}^{\ \ C} \right) \right]$$

$$\tag{5.27}$$
Here we have used that the term proportional to $\lambda^{ab}$ is exactly the pre-Bianchi identity $B_{AB}$ in Eq. (4.19) and thus vanishes. Therefore, the nontrivial variation is given by the terms proportional to $\lambda_{\bar{a}b}$ only, which gives the above result.

For $\mathcal{L}^-$ we compute correspondingly:

$$\delta^{(E)}(c_0 \int dX \sqrt{\det \eta} e^{-2d} \mathcal{L}^-)$$

$$= -c_0 \int dX \sqrt{\det \eta} e^{-2d}$$

$$\frac{1}{2} \Lambda^{\text{ad}} \left( F_{\bar{d} \bar{b} \bar{c}} F_{abc} - (E_{\bar{c}d} \partial_N - F_{\bar{c}d}) F_{a \bar{d} \bar{c}} + F_{\bar{a}d} \partial_N F_{\bar{b}c} - \phi'_{a \bar{b}c} \phi'_{\bar{b} \bar{c}} - \phi'_{a \bar{b} \bar{c}} \right).$$

Again, the term proportional to $\Lambda_{\bar{a}b}$ vanishes due to the pre-Bianchi identity and therefore, the nontrivial part of the variation is proportional to $\Lambda_{\bar{a}b}$ only. We shall see that the above variation by the $O(D, D)$ transformation gives the equation of motion of the generalized vielbein for DFT on a Drinfeld double.

**Variation with respect to the dilaton** We also need the variation of the generalized dilaton $d \rightarrow d + \delta d$, which gives for the fluxes

$$\delta^{(d)} F_{ABC} = 0,$$

$$\delta^{(d)} \phi'_{ABC} = 0,$$

$$\delta^{(d)} F_a = 2 \rho (E_A)(\delta d)$$

and for the measure

$$\delta^{(d)} (dX \sqrt{\det \eta} e^{-2d})(-2\delta d) = \sqrt{\det \eta} e^{-2d} (-2\delta d).$$

Thus, the variation of the action with respect to the generalized dilaton is

$$\delta^{(d)} S = c_0 \int dX \sqrt{\det \eta} e^{-2d} \delta d (-2\beta_+ \mathcal{L}^+ - 2\beta_- \mathcal{L}^-).$$

### 5.3. Variation and Ricci tensors.

The terms proportional to $\Lambda^{\text{ad}}$ in Eqs. (5.27) and (5.28) are related to the Ricci tensor also in the metric algebroid approach. From the generalized Riemann tensor (5.16), we can formulate two types of generalized Ricci tensor by taking the trace with the projection, namely:

$$\mathcal{R}_{a\bar{a}}^+ = 2 \mathcal{R}_{a\bar{a}}^b,$$

$$\mathcal{R}_{\bar{a}a}^- = 2 \mathcal{R}_{\bar{a}a}^\bar{b}.$$
Explicitly, they are
\[
\mathcal{R}^+_{a\bar{a}} = \mathcal{R}^+_{\bar{a}a} = -2(F_a^\bar{c} F_{\bar{a}\bar{b}c} - (\partial_c - F_c) F_{a\bar{a}c} + \partial_a F_{\bar{a}} - \phi_{a}^{\bar{b}c} \phi'_{\bar{b}c}) \tag{5.34}
\]
\[
\mathcal{R}^-_{a\bar{a}} = \mathcal{R}^-_{\bar{a}a} = -2(F_a^\bar{c} F_{\bar{a}\bar{b}c} - (\partial_c - F_c) F_{a\bar{a}c} + \partial_a F_{\bar{a}} - \phi_{a}^{\bar{b}c} \phi'_{\bar{b}c}) \tag{5.35}
\]
Comparing with (5.27) and (5.28), we can write
\[
\delta (E) \int dX h L^+ = \frac{1}{4} \int dX h \Lambda^{a\bar{a}} \mathcal{R}^+_{a\bar{a}} \tag{5.36}
\]
\[
\delta (E) \int dX h L^- = -\frac{1}{4} \int dX h \Lambda^{a\bar{a}} \mathcal{R}^-_{a\bar{a}} \tag{5.37}
\]
Thus, for the most general action we get
\[
\delta^{(E)} I(\beta_+, \beta_-) = \delta^{(E)} \int dX h (\beta_+ L^+ + \beta_- L^-) = 2 \int dX h \Lambda^{a\bar{a}} \left( \frac{1}{8} \beta_+ \mathcal{R}^+_{a\bar{a}} - \frac{1}{8} \beta_- \mathcal{R}^-_{a\bar{a}} \right) \tag{5.38}
\]

5.4. Local gauge symmetry. As in generalized geometry, the gauge transformation of vielbein is generated by the generalized Lie derivative with weight. Here, we denote the gauge parameter, which is a generalized vector, by \( V \in T_M \)
\[
\delta_V E_A = \mathcal{L}_V E_A = [V, E_A] = \Lambda_A^B E_B , \tag{5.39}
\]
where
\[
\Lambda_{AB} = \rho(E_B) V_A - \rho(E_A) V_B - V^C F_{CBA} . \tag{5.40}
\]
Therefore, we can use the result for the general variation. The gauge transformation of the generalized dilaton \( d \) is given in (4.47).

Thus, we obtain the gauge variation of the action as
\[
\delta_V S_{DFT} = c_0 \int dX \sqrt{\det \eta} e^{-2d} \left( \frac{\beta_+}{8} \left( -2(\partial_a V_{\bar{b}} - \partial_{\bar{b}} V_a - V^c F_{ca\bar{b}} - V^\bar{c} F_{\bar{c}a\bar{b}}) \mathcal{R}^{a\bar{b}} + (\partial_A - F_A) V^A \mathcal{R}_{a\bar{b}} \right) \right) + \frac{\beta_-}{8} \left( \left( 2(\partial_a V_{\bar{b}} - \partial_{\bar{b}} V_a - V^c F_{ca\bar{b}} - V^\bar{c} F_{\bar{c}a\bar{b}}) \mathcal{R}^{-a\bar{b}} + (\partial_A - F_A) V^A \mathcal{R}_{\bar{a}\bar{b}} \right) \right) \tag{5.41}
\]
For the gauge invariance, we require the condition
\[
\delta_V S_{DFT} = 0 , \tag{5.42}
\]
after the dimensional reduction. We find that the corresponding sufficient conditions are given by
\[
\delta_V F_{BCD} = \rho(V) F_{BCD} , \tag{5.43}
\]
\[
\delta_V F_A = \rho(V) F_A , \tag{5.44}
\]
\[ \phi'_{ABC} \phi'_{DE}^C = 0 . \] (5.45)

On the other hand, the gauge transformations of the structure functions with gauge parameter \( V^A \) are

\[
\delta_V F_{BCD} = V^A \partial_A F_{BCD} + \mathcal{Z}_{VBCD} ,
\]

(5.46)

\[
\delta_V F_B = V^A \partial_A F_B + \mathcal{Z}_{VB} ,
\]

(5.47)

where \( \mathcal{Z}_{VBCD} \) and \( \mathcal{Z}_{VA} \) are obtained from the commutator of \( V^A \) with the square of the Dirac generating operator as

\[
\{ 4 \mathcal{D}^2, V \} = -\gamma^A \mathcal{Z}_{VA} + 2 \gamma^A \left[ (\partial^B V_A) + V^C \phi'_{CA}^B \right] \partial_B - \frac{1}{6} \mathcal{Z}_{VBCD} \gamma^{BCD} ,
\]

(5.48)

and are defined by

\[
\mathcal{Z}_{VBCD} = \frac{1}{2} \{ \{ \mathcal{D}^2, V \}, E_B, E_C, E_D \} ,
\]

(5.49)

\[
\mathcal{Z}_{VB} = - (\partial^A - F^A) \partial_A V_B - \phi'_{CB} A \partial_A V_C - V^A (\partial_C - F_C) \phi'_{AB} C .
\]

(5.50)

Therefore, conditions (5.43) and (5.44) imply that

\[
\mathcal{Z}_{VBCD} = 0 ,
\]

(5.51)

\[
\mathcal{Z}_{VA} = 0 .
\]

(5.52)

Thus (5.45), (5.51) and (5.52) are the conditions for the gauge invariance.  3

In DFT, we require the gauge invariance only in the \( D \)-dimensional theory, and here we realize it by the dimensional reduction of the fluctuation.

As a demonstration, we construct a gauge invariant DFT which includes DFT_{sec} and DFT_{WZW} as follows. First, we separate the vielbein into a fluctuation \( U^B_A \in O(D, D) \) and a background \( \bar{E}_A \) as in (5.2).

For a reduction to the \( D \)-dimensional field theory, we require that the fluctuation \( U^B_A \) depends only on the \( D \)-dimensional coordinate \( x^m \), where the polarization of the \( D \)-dimensional coordinate should be chosen appropriately for a given 2D-dimensional manifold. We will show an example for the case of the Drinfel’d double in the following section. The same reduction condition must be applied for the dilaton \( d \) and the gauge parameter \( V^A \).

Furthermore, we have to require that the derivative \( \rho(E_A) \) on \( U^B_A \) and \( d \) in the action also depends on \( x^m \) only, meaning that \( \bar{E}_A^m \partial_m \) is required to depend only on \( x^m \).

With the above requirement for \( U^B_A \), \( d \) and \( \bar{E}_A^m \), we get the action on the \( D \)-dimensional space. To restore the gauge invariance of the D-dimensional action originated from the metric algebroid, we consider (5.43) and (5.44), as well as the condition on the \( \phi'_{ABC} \) in (5.45).

\[ ^3 \text{Note that using the pre-Bianchi identity, in the local Lorentz basis the above conditions reduce to the Bianchi identities}
\]

\[
(\partial_C - F_C) F_{AB}^C + 2 \partial_{[A} F_{B]} = (\partial_C - F_C) \phi'_{AB}^C = -\mathcal{Z}_{AB} \]

(5.53)

\[
4 \partial_{[A} F_{BCD]} - 3 F_{[AB} F_{CDE]} = -3 \phi'_{[AB} E \phi'_{CD]E} = -\mathcal{Z}_{ABCD}
\]

(5.54)

However, since \( \mathcal{Z}_{AB} \) and \( \mathcal{Z}_{ABCD} \) are not tensors, this does not mean that the conditions (5.51) and (5.52) hold.
We start with the condition (5.45) on the structure function $\phi'_{ABC}$ which is given by

$$\phi'_{ABC} = U_A^{A'} U_B^{B'} U_C^{C'} \tilde{\phi}'_{A'B'C'} + \rho(E_C)(U_A^{A'}) U_B^{B'} \eta_{A'B'} .$$  

(5.55)

By choosing the MA such that in the $\bar{E}_A$ basis

$$\langle \mathcal{L}'(\bar{E}_A, \bar{E}_B), \bar{E}_C \rangle = \bar{\phi}'_{ABC} = 0 ,$$  

(5.56)

the first term can be dropped. Then, (5.45) holds if the $O(D, D)$ metric satisfies

$$\eta^{CC'} \rho(E_C)(U_A^{A'}) \rho(E_C')(U_D^{D'}) = 0 .$$  

(5.57)

This means that $\eta^{-1}_{MN}$ in a local coordinate satisfies

$$\eta^{-1}_{mn} = 0 .$$  

(5.58)

i.e., this block of spacetime metric becomes zero. Note that the condition (5.57) is not the section condition. It is a condition on the 2D-dimensional manifold. The condition (5.58) depends on the choice of the local coordinate. We require that we can find such a polarization of $x^m$ and a local coordinate system satisfying (5.58) by choosing an appropriate coordinate system. If we can not find such a coordinate system, we can not apply the dimensional reduction and thus we exclude such a doubled space from our consideration.

The condition (5.58) holds of course in the flat case. We shall see that in the case of a Drinfel’d double, this condition is also satisfied naturally.

To restore the gauge invariance of the D-dimensional action, we must still consider (5.43) and (5.44). Since the pre-Bianchi identities in the basis $\bar{E}_A$ reduce to the Bianchi identities by the condition $\bar{\phi}'_{ABC} = 0$, which we have already required by the choice of the MA, the conditions (5.56), (5.51) and (5.52) are satisfied after the dimensional reduction. Therefore, (5.43) and (5.44) also hold.

Since in the usual discussion of the section condition the basis where the closure holds is not clearly specified, its meaning is obscure even in the flat case. On the other hand, in our approach the condition for the gauge invariance is formulated in the local Lorentz frame, thus it is covariant. The dimensional reduction is performed with respect to the fluctuation in a specific local coordinate and therefore, the function space of the dynamical field is well defined. It means that a dual coordinate dependence is not allowed. This is different from the section condition which allows a dual coordinate dependence in general.

To summarize, by taking a MA satisfying $\bar{\phi}'_{ABC} = 0$ and requiring that the 2D-dimensional metric in a local coordinate satisfies $\eta^{-1}_{mn} = 0$, the gauge invariance is guaranteed by the above dimensional reduction of the fluctuations.  

---

4 Note that taking the MA such that $\bar{\phi}'_{ABC} = 0$ in the $\bar{E}_A$ frame means

$$\bar{F}_{ABC} = \bar{F}'_{ABC} .$$  

(5.59)

Since $\bar{F}'_{ABC}$ satisfies the closure condition of the Lie algebroid, from the pre-Bianchi identity (4.20) for $\bar{F}_{ABC}$, we conclude that

$$\rho(\bar{E}_{[A}) \bar{F}_{BCD]} = 0 .$$  

(5.60)

Thus, we assume that $\bar{F}_{ABC} = \bar{F}'_{ABC}$ is a constant flux in the following.
6. Derivation of Poisson-Lie T-duality by DFT Action

Poisson-Lie T-duality was first introduced in [49] as a generalization of non-abelian T-duality in the non-linear sigma model. In the algebroid context it is discussed in [40,50–52]. In DFT it has been discussed in [45,53–56]. Here, we apply our results of the MA formulation of DFT to the Poisson-Lie T-duality. We briefly recall the Poisson-Lie T-duality for convenience and to introduce notation.

6.1. Poisson-Lie T-duality. Poisson-Lie T-duality is a generalization of T-duality based on a Drinfel’d double. The Poisson-Lie T-duality is formulated in the most clear form by using the $E$ model [57], which is a special case of a sigma model on a Drinfel’d double. The Drinfel’d double $D = \mathcal{G} \bowtie \bar{\mathcal{G}}$ is a 2$D$-dimensional group constructed by two $D$-dimensional groups $\mathcal{G}$, $\bar{\mathcal{G}}$. The corresponding Lie algebras are denoted by $\mathfrak{d}$ and $\mathfrak{g}$, $\bar{\mathfrak{g}}$, respectively. Their generators and brackets are denoted as

$$t_a \in \mathfrak{g}, \quad \tilde{t}^a \in \bar{\mathfrak{g}}.$$  \hspace{1cm} (6.1)

$$[t_a, t_b]_{\mathfrak{g}} = f_{ab}^\ c t_c, \quad [\tilde{t}^a, \tilde{t}^b]_{\bar{\mathfrak{g}}} = \tilde{f}^{ab} c \tilde{t}^c,$$ \hspace{1cm} (6.2)

while the generators of $\mathfrak{d}$ are given by

$$\mathfrak{d} \ni T_A := (t_a, \tilde{t}^a).$$ \hspace{1cm} (6.3)

The Lie algebra $\mathfrak{d}$ is expressed in terms of the structure constants of $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ as

$$[t_a, t_b]_{\mathfrak{d}} = f_{ab}^\ c t_c,$$ \hspace{1cm} (6.4)

$$[\tilde{t}^a, \tilde{t}^b]_{\mathfrak{d}} = \tilde{f}^{ab} c \tilde{t}^c.$$ \hspace{1cm} (6.5)

$$[t_a, \tilde{t}^b]_{\mathfrak{d}} = \tilde{f}^{bc}_a t_c - f_a^\ bc \tilde{t}^c.$$ \hspace{1cm} (6.6)

The third bracket is due to the $O(D, D)$ structure: $\mathfrak{d}$ has an inner product defined as

$$\langle T_A, T_B \rangle = \eta_{AB},$$ \hspace{1cm} (6.7)

with the compatibility condition

$$\langle T_A, [T_B, T_C]_{\mathfrak{d}} \rangle = \langle T_B, [T_C, T_A]_{\mathfrak{d}} \rangle.$$ \hspace{1cm} (6.8)

This condition yields the third equation containing both structure constants. Finally, from the Jacobi identity

$$[T_A, [T_B, T_C]_{\mathfrak{d}}]_{\mathfrak{d}} + [T_B, [T_C, T_A]_{\mathfrak{d}}]_{\mathfrak{d}} + [T_C, [T_A, T_B]_{\mathfrak{d}}]_{\mathfrak{d}} = 0,$$ \hspace{1cm} (6.9)

the following condition for the structure constants is obtained

$$f_{ab}^\ e \tilde{f}^{cd}_e = 4 f_a^\ [c \bar{f}^d]_{b]}.$$ \hspace{1cm} (6.10)
The Poisson-Lie T-duality is the equivalence of two non-linear sigma models in different backgrounds specified by the metric $G$ and the $B$-field,

$$S[X] \sim \int [G_{mn} dX^m \wedge \ast dX^n + B_{mn} dX^m \wedge dX^n] , \quad (6.11)$$

where the field $X$ is an embedding into the target space. Formulated in the $\mathcal{E}$-model, the field is an element $l \in \mathcal{D}$ of a Drinfel’d double $[58–60]$. Its action is given by

$$S[l] = \frac{1}{2} \int d^2 \sigma \left[ (l^{-1} \partial_{\sigma} l, l^{-1} \partial_{\tau} l) - (l^{-1} \partial_{\sigma} l, \hat{\mathcal{H}}_0(l^{-1} \partial_{\sigma} l)) \right] + \frac{1}{12} \int (l^{-1} dl \wedge [l^{-1} dl \wedge l^{-1} dl]) , \quad (6.12)$$

where $\hat{\mathcal{H}}_0$ is a linear map $\mathcal{D} \rightarrow \mathcal{D}$, defined by

$$(T_A, \hat{\mathcal{H}}_0(T_B)) = \left( \begin{array}{cc} (G^{-1}_0)^{ab} & (G^{-1}_0 B_0)^{a b} \\ -(B_0 G^{-1}_0)^{a b} & (G_0 - B_0 G^{-1}_0 B_0)^{a b} \end{array} \right) . \quad (6.13)$$

$G_0$ and $B_0$ in the above equation are constant matrices. $l$ can be parameterized by $g \in \mathcal{G}$ and $\bar{g} \in \bar{\mathcal{G}}$ as

$$l = \bar{g}(\bar{x}) g(x) . \quad (6.14)$$

We denote the local coordinates of $\mathcal{D} = \bar{\mathcal{G}} \times \mathcal{G}$ as $X^M = (\bar{x}_m, x^m)$, respectively. Since $\bar{g}$ does not carry physical degrees of freedom in this parameterization, it can be integrated out. Then, we obtain a non-linear sigma model $\hat{S}[g]$ with a metric $\hat{G}$ and a $B$-field $\hat{B}$ as:

$$\hat{G} + \hat{B} = L^{-1} \frac{1}{G_0 + B_0 + \Pi} L^{-T} , \quad (6.15)$$

where $L^{-1}$ and $\Pi$ are defined by

$$g^{-1} dg = L^{-1} m^a t_a dX^m , \quad (6.16)$$

$$(g^{-1} l^a g, g^{-1} t_a g) = \left( \begin{array}{ccc} a^{-T} a_{b(g)} & -a^{-T} c_{b(g)} & a^{b(c)}(g) \\ 0 & a^{b(c)}(g) & \Pi^{b(c)}(g) \end{array} \right) \left( \begin{array}{c} \tilde{t}^b \\ l_b \end{array} \right) \quad (6.17)$$

The other hand, using a coordinate transformation $X \rightarrow X'$, the field $l \in \mathcal{D}$ can also be parameterized as

$$l = g(x') \bar{g}(\bar{x}') . \quad (6.18)$$

with the local coordinates $X'^M = (\bar{x}'_m, x'^m)$. In this case, $g$ can be integrated out and we obtain an action $\hat{S}[\bar{g}]$ with a metric and a $B$-field given by

$$\hat{G} + \hat{B} = \bar{L}^{-1} \frac{1}{G_0 + B_0 + \Pi} \bar{L}^{-T} , \quad (6.19)$$
where
\[ \tilde{g}^{-1} d \tilde{g} = L^{-1 m a} \tilde{r}^a d \tilde{x}^m, \tag{6.20} \]
\[ \tilde{g}^{-1} t_a \tilde{g} = \left( - \tilde{a}^{-T} a_c (\tilde{g}) \tilde{g}^{-1} \right) (t_b). \tag{6.21} \]

Since the two actions \( S[g(x)] \) and \( \tilde{S} [\tilde{g}(\tilde{x}')] \) are equivalent at classical level, the nonlinear sigma models on the backgrounds \( \hat{G} + \hat{B} \) and \( \tilde{G} + \tilde{B} \) are classically equivalent. This equivalence is called Poisson-Lie T-duality.

### 6.2. DFT on Drinfel’d Double

In this section, we define the DFT on a Drinfel’d double.

We define a vielbein \( E_A^N \) at a point \( X^M \) on \( D \) corresponding to an element \( l \in D \) and a fluctuation \( U^B_A \) in a local basis
\[ E_A = E_A^N (l) \partial_N = U^B_A (X) \tilde{E}_B^N (X) \partial_N, \tag{6.22} \]
where \( \tilde{E}_A^N (l) \) is defined by the left invariant current of \( D \):
\[ l^{-1} dl = \tilde{E}^{-1} N_A T_A d X^N = \tilde{E}^{-1} N_a t_a d X^N + \tilde{E}^{-1} N_a \tilde{r}^a d X^N. \tag{6.23} \]

The Lie algebra of the left invariant vector \( \tilde{E}_A = \tilde{E}_A^N (l) \partial_N \) is given by
\[ [\tilde{E}_A, \tilde{E}_B]_L = \tilde{F}^{C}_{AB} \tilde{E}_C \tag{6.24} \]
where \( \tilde{F}^{C}_{AB} \) are the structure constants of \( D \). Splitting the basis \( \tilde{E}_A \) into \( \tilde{E}^a \) and \( \tilde{E}_a \), we obtain
\[ [\tilde{E}^a, \tilde{E}^b]_L = f^{ab} c \tilde{E}_c, \tag{6.25} \]
\[ [\tilde{E}_a, \tilde{E}^b]_L = f_{ac} b \tilde{E}^c - f_{ac} b \tilde{E}_c, \tag{6.26} \]
\[ [\tilde{E}^a, \tilde{E}_b]_L = \tilde{f}^{ab} c \tilde{E}_c. \tag{6.27} \]

Since in the MA the flux is not determined uniquely by the vielbein, we have a freedom to fix the flux \( \tilde{F}^{C}_{AB} \) of the MA in the basis \( \tilde{E}_A \) as \(^5\)
\[ [\tilde{E}_A, \tilde{E}_B] = \tilde{F}^{C}_{AB} \tilde{E}_C = \tilde{F}^{C}_{AB} \tilde{E}_C. \tag{6.28} \]

Consequently, for this choice the structure function \( \tilde{\phi}^{C}_{ABC} \) vanishes in the \( \tilde{E}_A \) basis
\[ \tilde{\phi}^{C}_{ABC} := \langle [\tilde{E}_A, \tilde{E}_B] - [\tilde{E}_A, \tilde{E}_B]_L, \tilde{E}_C \rangle = 0. \tag{6.29} \]

The \( O(D, D) \) metric is defined by
\[ \tilde{\eta}_{AB} = \langle \tilde{E}_A, \tilde{E}_B \rangle = \begin{pmatrix} 0 & \delta^a b \\ \delta_a b & 0 \end{pmatrix}. \tag{6.30} \]

With the above \( O(D, D) \) metric we have
\[ \langle \tilde{E}_A, [\tilde{E}_B, \tilde{E}_C]_L \rangle = \langle \tilde{E}_B, [\tilde{E}_C, \tilde{E}_A]_L \rangle. \tag{6.31} \]

\(^5\) This means that we are considering a class which allows us to take this choice. As we shall see this class includes the case on the Drinfeld double.
which is compatible with the MA, i.e., since in the \( \tilde{E}_A \) basis \( \phi' = 0 \) the above cyclicity generalizes to the relation on the MA consistently:

\[
- \langle \tilde{E}_A, [\tilde{E}_C, \tilde{E}_B] \rangle = \langle \tilde{E}_A, [\tilde{E}_B, \tilde{E}_C] \rangle = \langle \tilde{E}_B, [\tilde{E}_C, \tilde{E}_A] \rangle.
\]

(6.32)

Note that the generalized vielbein \( E_A^N \) defined in (6.22) contains the fluctuation \( U_A^B \in O(1, D - 1) \times O(D - 1, 1) \setminus O(D, D) \) around the basis \( \tilde{E}_A \). We parametrize \( U_A^B \) as

\[
U_A^B = \begin{pmatrix} e^{-T} & 0 \\ eB & e \end{pmatrix},
\]

(6.33)

where \( e \in GL(D) \) and \( B \) is a \( D \)-dimensional antisymmetric matrix.

In order to derive a \( D \)-dimensional action from the DFT action (5.8), we apply the parametrization of an element of \( D \) described in the previous section, \( l(X) = \tilde{g}(\tilde{x})g(x) \) or \( l(X') = g(x')\tilde{g}(\tilde{x}') \).

**Case 1: \( l = \tilde{g}(\tilde{x})g(x) \)** For the case \( l = \tilde{g}(\tilde{x})g(x) \) the left invariant current is given by

\[
(l^{-1}dl) = \tilde{E}_M^{\bar{A}} T_A dX^M = g^{-1} \tilde{g}^{-1} dg + g^{-1} dg.
\]

(6.34)

From this expression we can read off \( \tilde{E}_M^{-1B} \) as

\[
\tilde{E}_M^{-1B} = \begin{pmatrix} \tilde{L}^{-1} & \tilde{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^T & 0 \\ 1 & 1 \end{pmatrix} B,
\]

(6.35)

where \( L \) is given in (6.16), \( \tilde{L} \) is given in (6.20), and the right invariant vector field \( R \) of \( \mathcal{G} \) is defined by

\[
dg^{-1} = R^{-1a} dX^a.
\]

(6.36)

The generalized metric is given by

\[
H_{MN} = E_M^{-1A} H_{AB} E_N^{-1B} = (\tilde{E}_M^{-1} U H U^{-T} \tilde{E}_N^{-T})_{MN}
\]

\[
= \begin{pmatrix} \tilde{L}^{-1} & \tilde{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^T & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}
\]

\[
= \begin{pmatrix} \tilde{L}^{-1} & \tilde{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1} \hat{B} \\ -B\hat{G}^{-1} & \hat{G} - B\hat{G}^{-1} \hat{B} \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix}.
\]

(6.37)

Here, \( \hat{G} \) and \( \hat{B} \) are defined by

\[
\hat{G}_{mn} + \hat{B}_{mn} = L^{-1}_{m a} \left( \frac{1}{G + B + \Pi} \right)_{a b} L^{-T} b_n.
\]

(6.38)

It is convenient to introduce a new basis \( \hat{E}_M \), called 'hat-basis' in the following, such that

\[
\hat{E}_N := \hat{A}_N^M \partial_M = \hat{A}_N^M \hat{E}_M^A \hat{E}_A
\]
The relation between the local basis $E_A$ and the hat-basis $\hat{E}_M$ is given by a matrix
$$\hat{E}_M^A \in O(D, D)$$
as
$$E_A = \hat{E}_M^A \hat{E}^M.$$

Then, we also define $\hat{H}_{MN}$ and $\hat{\eta}_{MN}$ in this basis. In terms of $\hat{G}$ and $\hat{B}$, they are given
as
$$\hat{H}_{MN} := \hat{E}^{-1} M^A H_{AB} \hat{E}^{-T} B^N = \left( \begin{array}{cc} \hat{G}^{-1} & \hat{G}^{-1} \hat{B} \\ -\hat{B} \hat{G}^{-1} \hat{G} & -\hat{B} \hat{G}^{-1} \hat{B} \end{array} \right),$$
$$\hat{\eta}_{MN} := \hat{E}^{-1} M^A \eta_{AB} \hat{E}^{-T} B^N = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).$$

The relation between these bases is therefore
$$\partial_M \hat{\nu}^N \mapsto -\hat{E}^N \hat{E}^M.$$

The flux in the basis $\hat{E}_M$ can be calculated by using the flux in $\bar{E}_A$ as follows. Using (6.39) we obtain
$$\hat{F}^I_{LMN} := \langle [\hat{E}_L, \hat{E}_M], \hat{E}_N \rangle,$$
The non-zero components are
$$\hat{F}^I_{lmn} = \partial_I L^m c L^{-1} n^c - L^{-1} l^a L^m b L^{-1} n^c,$$
$$\hat{F}^{ilm} = \hat{f}^{ilm} + 2 f_{np}^{l} [l \Pi^m p].$$
For $\hat{\phi}'$ given by
$$\hat{\phi}'_{LMN} := \langle [\hat{E}_L, \hat{E}_M] - [\hat{E}_L, \hat{E}_M]_L, \hat{E}_N \rangle,$$
the non-zero components are
$$\hat{\phi}'_{lmn} = \partial_n L^{-1} l a L^m a,$$
$$\hat{\phi}'_{ilm} = -(\hat{f}^{ilm} + 2 f_{np}^{l} [l \Pi^m p]),$$
$$= -L^l a L^m b \partial_n \Pi^{ab}.$$
Then we obtain the relation
$$\hat{F}_{LMN} := \langle [\hat{E}_L, \hat{E}_M], \hat{E}_N \rangle = \hat{F}'_{LMN} + \hat{\phi}'_{LMN} = 0.$$
For the flux with one index we get
$$\hat{F}_M := \hat{\phi}'_{LM} + 2 \rho(\hat{E}_M)(d),$$
and in components
$$\hat{F}_m = 2 \partial_m (d + \frac{1}{2} \log \det(R)).$$
\[ \hat{F}^m = -R_b^m \tilde{f}^{ab} \hat{a} + 2\rho(\hat{E}^m)(d). \] (6.50)

Note that \( \hat{F}_M \) has contributions from \( \hat{\phi}'_{NM} \) besides the term \( 2\partial_M d \). As we will show, the Poisson-Lie T-duality of the dilaton is automatically achieved due to these contributions from \( \hat{\phi}'_{NM} \) without the use of a linear dilation.

We write the DFT action in the hat-basis using the field \( \hat{E}^A_N \) given in (6.40) as

\[ F_{ABC} = 3\hat{\Omega}_{(ABC)} + \hat{E}^L_A \hat{E}^M_B \hat{E}^N_C \hat{F}_{LMN} \]
\[ = 3\hat{\Omega}_{(ABC)}, \] (6.51)

\[ F_A = \hat{\Omega}^{BA}_E + \hat{E}^A_N \hat{F}_N, \] (6.52)

\[ \phi'_{ABC} = \hat{\Omega}_{CAB} + \hat{E}^L_A \hat{E}^M_B \hat{E}^N_C \hat{\phi}'_{LMN}. \] (6.53)

\( \hat{\Omega}_{ABC} \) is the Weizenböck connection

\[ \hat{\Omega}_{ABC} = \hat{E}^L_A \rho(\hat{E}^A_L)(\hat{E}^B_M \hat{E}^C_N \hat{\eta}_{MN}). \] (6.54)

For the measure we obtain

\[ c_0 \int dX e^{-2d} \sqrt{\det \eta_{MN}} = c_0 \int dX e^{-2d} \sqrt{(\det \hat{E}^{-1}_A L^A)(\det \hat{A}^{-1}_M N)^2} \]
\[ = c_0 \int dX e^{-2d} \det R^{-1} \det \tilde{L}^{-1} \]
\[ = c_0 \int dX e^{-2(d + \frac{1}{2} \log \det R)} \det \tilde{L}^{-1}. \] (6.55)

We denote the DFT action (5.8) with the measure and flux written in the hat-basis by \( \hat{S}_{DFT}[U, d] \).

What we can immediately see is the following: When \( U \) and \( d \) depend on \( x^m \) only, the \( \tilde{x}_m \) integration factorizes (with the factor \( \det \tilde{L}^{-1} \)) and \( \hat{S}_{DFT}[U(x), d(x)] \) reduces to the D-dimensional action up to a normalization. We find that the e.o.m. of the action \( \hat{S}_{DFT}[U(x), d(x)] \) is, in general, the Generalized Supergravity Equation (GSE). Note that this procedure does not mean that we use the section condition, where it is allowed to introduce a linear dilaton depending on the dual coordinate. Here, we apply a dimensional reduction of the fluctuation and thus, there is no dual coordinate dependence.

Denoting a solution of the e.o.m. of the action \( \hat{S}_{DFT}[U(x), d(x)] \) by \( U_0(x), d_0(x) \), where \( U_0(x) \) is determined by \( G_0 \) and \( B_0 \) via (6.33), a solution of the GSE is given by

\[ \hat{G} + \hat{B} = L^{-1} \frac{1}{U_0 + B_0} + \Pi \]
\[ \hat{G} = d_0 + \frac{1}{2} \log \det R + \frac{1}{4} \log \det \hat{G}, \] (6.57)

\[ \hat{I}^m = R_b^m(x) \tilde{f}^{ab} \hat{a}. \] (6.58)

In the equations above \( \hat{\phi} \) is the dilaton and \( \hat{I}^m \) is a Killing vector in the GSE.

---

\(^6\) When the generalized dilaton depends on \( x^m \) only and we consider the unimodular case, \( d + \frac{1}{2} \log \det(R) \) corresponds to the generalized dilaton in DFT\text{sec}.
The easiest way to see this is as follows. Let us denote the modified DFT action given in [30] as $\hat{S}_{DFT}^{mod}[E_{A}^{N}(x), d(x), \hat{I}^{m}(x)]$, for details see Appendix B. We find that $\hat{S}_{DFT}^{mod}[E_{A}^{N}(x), d(x), \hat{I}^{m}(x)]$ and $\hat{S}[U(x), d(x)]$ are related as

$$\hat{S}_{DFT}[U(x), d(x)] = \left( \int d\bar{x} \det \bar{L}^{-1} \right) \hat{S}_{DFT}^{mod}[\hat{E}_{A}^{N}(x), d(x)]$$

$$+ \frac{1}{2} \log \det R(x), R_{b}^{m}(x) \bar{f}_{ab}^{a} \right]. \quad (6.59)$$

The $x_{m}$ dependence in this action exists only in the factor $\int d\bar{x} \det \bar{L}^{-1}$. Thus, when $U$ and $d$ depend only on $x^{m}$, $\hat{S}[U(x), d(x)]$ reduces to the GSE.

**Case 2: $l = g(x') \tilde{g}(\bar{x})$**  Next, we consider the case where $l = g(x') \tilde{g}(\bar{x}')$. This case is connected to the previous one by a coordinate transformation. The generalized metric is

$$H_{MN} = E_{M}^{A}H_{AB}E_{B}^{N}$$

$$= \left( \begin{array}{cc} 1 & \bar{L}^{-1} L^{-T} \\ \bar{L}^{-1} & \bar{L}^{-T} \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ -\bar{\Pi} & 1 \end{array} \right) \left( \begin{array}{cc} e^{T} & e^{-1} \\ e & e^{-T} \end{array} \right) s^{-1} \right)$$

$$= \left( \begin{array}{cc} e^{T} & e^{-1} \\ e & e^{-T} \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ -\bar{\Pi} & 1 \end{array} \right) \left( \begin{array}{cc} \bar{L}^{-T} & \bar{L}^{-1} L^{-T} \\ \bar{L}^{-1} & \bar{L}^{-T} \end{array} \right)$$

$$= \left( \begin{array}{cc} \tilde{G} - \tilde{B} \tilde{G}^{-1} \tilde{B} - \tilde{B} \tilde{G}^{-1} & 1 \\ \tilde{G}^{-1} \tilde{B} & \tilde{G}^{-1} \tilde{B} \tilde{G}^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & \bar{L}^{-1} L^{-T} \end{array} \right), \quad (6.60)$$

where $\tilde{G}^{mn} + \tilde{B}^{mn}$ is defined by

$$\tilde{G}^{mn} + \tilde{B}^{mn} = \bar{L}^{-1} a \left( \frac{1}{G + B + \bar{\Pi}} \right) a b \bar{L}^{-T} b^{n}. \quad (6.61)$$

We separate $E_{A}^{M}$ into an $O(D, D)$ part $\bar{E}_{A}^{N}$ and a part denoted by $\tilde{A}_{N}^{M}$.

$$E_{A}^{M} = \bar{E}_{A}^{N} \tilde{A}_{N}^{M}, \quad \tilde{A}_{M}^{N} = \left( \begin{array}{c} 1 \\ \bar{R}^{T} L \end{array} \right). \quad (6.62)$$

Similarly as before, we have now introduced the ’check-basis’ $\bar{E}_{A}^{N}$ such that

$$\bar{E}_{N} = \bar{E}^{-1}_{N} A E_{A} = \bar{A}_{N}^{M} \partial_{M} = \left( \begin{array}{c} 1 \\ \bar{R}^{T} L \end{array} \right) N^{M} \partial_{M}. \quad (6.63)$$

and

$$\tilde{E}_{N} = \left( \begin{array}{c} \tilde{L}^{-1} \\ -\tilde{L}^{T} \bar{\Pi} L^{T} \end{array} \right) N^{A} \tilde{E}_{A}. \quad (6.64)$$

The relation between the bases is

$$\partial_{M} \tilde{A}_{N}^{M} \mapsto \tilde{E}_{N} \mapsto E_{A}. \quad (6.65)$$
The metrics in the check-basis $\check{E}_M$ are given by
\[
\check{H}_{MN} = \check{E}^{-1}_M H_{AB} \check{E}^{-T} B^N = \begin{pmatrix} \check{G} - \check{B} \check{G}^{-1} \check{B} & -\check{B} \check{G}^{-1} \\ \check{G}^{-1} \check{B} & \check{G}^{-1} \end{pmatrix},
\]
(6.66)
\[
\check{\eta}_{MN} = \check{E}^{-1}_M \check{\eta}_{AB} \check{E}^{-T} B^N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
(6.67)

The fluxes in the check-basis $\check{E}_M$ can be calculated from the corresponding quantities in the $\check{E}_A$ basis. They are related as:
\[
\check{F}'_{LMN} := \langle [\check{E}_L, \check{E}_M]_L, \check{E}_N \rangle,
\]
(6.68)
with non-zero components
\[
\check{F}'_{lmn} = \partial_l \check{L} c m \check{L}^{-1} n c - \check{L}^{-1} l a \check{L} b m \check{f} a c b \check{L}^{-1} n c, \\
\check{F}'_{lmn} = f_{lmn} + 2 \check{f}^m n p [l \check{\Pi}_m] p.
\]
(6.69)

For $\check{\phi}'_{LMN}$
\[
\check{\phi}'_{LMN} := \langle [\check{E}_L, \check{E}_M] - [\check{E}_L, \check{E}_M]_L, \check{E}_N \rangle,
\]
(6.70)
with non-zero components
\[
\check{\phi}'_{lmn} = \partial^m \check{L}^{-1} l a \check{L} c m, \\
\check{\phi}'_{lmn} = -(f_{lmn} + 2 \check{f}^m n p [l \check{\Pi}_m] n') \\
= -\check{L}^a l \check{L} b m \partial^m \check{\Pi}_{ab}.
\]
(6.71)

Thus, we get the relation
\[
\check{F}_{LMN} := \langle [\check{E}_L, \check{E}_M]_L, \check{E}_N \rangle = \check{F}'_{LMN} + \check{\phi}'_{LMN} = 0.
\]
(6.72)

For the flux with one index
\[
\check{F}_M := \check{\phi}'_{LM} + 2 \check{\rho}(\check{E}_M)(d), \\
\check{F}^m = 2 \partial^m (d + \frac{1}{2} \log \det(\check{R})) , \\
\check{F}_m = -\check{R}^b_m f a b + 2 \check{\rho}(\check{E}_m)(d).
\]
(6.73)

To obtain the DFT action in the check-basis, we rewrite the fluxes as
\[
F_{ABC} = 3 \check{\Omega}_{[ABC]} + \check{E}_A L \check{E}_B M \check{E}_C \check{H}_{MN} \\
= 3 \check{\Omega}_{[ABC]} ,
\]
(6.74)
\[
F_A = \check{\Omega}^B_{BA} + \check{E}_A N \check{E}_N ,
\]
(6.75)
\[
\check{\phi}'_{ABC} = \check{\Omega}_{ABC} + \check{E}_A L \check{E}_B M \check{E}_C \check{\eta}_{MN},
\]
(6.76)

where $\check{\Omega}_{ABC}$ is the corresponding Weizenböck connection
\[
\check{\Omega}_{ABC} = \check{E}_A L \check{\rho}(\check{E}_L) (\check{E}_B M) \check{E}_C N \check{\eta}_{MN}.
\]
(6.77)
The measure of the DFT action in the original basis is related to the one in the check-basis as

\[ c_0 \int dX e^{-2d} \sqrt{\det \eta_{MN}} = c_0 \int dX e^{-2(d + \frac{1}{2} \log \det \hat{R})} \det L^{-1}. \]  

(6.78)

Similar to the previous case, we denote the DFT action in the check-basis by \( \hat{S}_{DFT}[U, d] \).

The relation between \( \hat{S}_{DFT}^{\text{mod}}[E_A^N(x'), d(x'), \hat{I}^m(x')] \) generating the GSE and \( \hat{S}_{DFT}[U(x'), d(x')] \) is

\[ \hat{S}_{DFT}[U(x'), d(x')] = \left( \int dx' \det L^{-1} \right) \hat{S}_{DFT}^{\text{mod}}[\hat{E}_A^N(x'), d(x') + \frac{1}{2} \log \det \hat{R}(x') \hat{R}_m(x') f_{ab} \]  

(6.79)

On the r.h.s. only the integration measure depends on \( x'^m \) giving an overall factor \( \int dx' \det L^{-1} \). When \( U \) and \( d \) depend only on \( x'_m \), the action \( \hat{S}_{DFT}[U(x'), d(x')] \) reduces to the GSE.

### 6.3. SUGRA solution and DFT solution.

In the previous section, we have assumed that \( U_0(x) \) and \( d_0(x) \) are solutions of the e.o.m. of the D-dimensional action \( \hat{S}[U(x), d(x)] \), i.e.,

\[ \hat{S}[U_0(x) + \delta U(x), d_0(x) + \delta d(x)] - \hat{S}[U_0(x), d_0(x)] = 0. \]  

(6.80)

This means that there exist the e.o.m. for \( U_{AB}(x) \) and \( d(x) \) denoted by \( C_{AB}(x, U_0, d_0), C(x, U_0, d_0) \) such that,

\[ 0 = \hat{S}[U_0(x) + \delta U(x), d_0(x) + \delta d(x)] - \hat{S}[U_0(x), d_0(x)] \\
= \int dX \left( \delta U_{AB}(x) C_{AB}(x, U_0, d_0) + \delta d(x) C(x, U_0, d_0) \right). \]  

(6.81)

Now, we extend the fluctuation over the full doubled space \( X \) and consider

\[ \hat{S}[U_0(x) + \delta U(x), d_0(x) + \delta d(x)] - \hat{S}[U_0(x), d_0(x)]. \]  

(6.82)

Note that \( \delta U(X) \) and \( \delta d(X) \) depend on all coordinates in general. Compared to (6.80), this variation has new contributions denoted as \( C_{mAB}(x, U_0, d_0), C_m(x, U_0, d_0) \in \mathcal{C}^\infty(M) \) via the Weizenböck connection \( \hat{\Omega}_{ABC} \) and the flux \( F_A \), and can be written as

\[ \hat{S}[U_0 + \delta U(X), d_0 + \delta d(X)] - \hat{S}[U_0, d_0] \\
= \int dX \left( \delta U_{AB}(X) C_{AB}(x, U_0, d_0) + \delta d(X) C(x, U_0, d_0) \right) \\
+ c_0 \int dX e^{-2d_0} \sqrt{\det \eta_{MN}} \left( \rho(\hat{E}^m)(\delta U_{AB}(X)) C_{mAB}(x, U_0, d_0) \right) \\
+ \rho(\hat{E}^m)(\delta d(X)) C_m(x, U_0, d_0) \\
= c_0 \int dX e^{-2d_0} \sqrt{\det \eta_{MN}} \left( \delta U_{AB}(X)(-\rho(\hat{E}^m) + \hat{F}^m(d_0)) C_{mAB}(x, U_0, d_0) \right) \\
= c_0 \int dX e^{-2d_0} \sqrt{\det \eta_{MN}} \left( \delta U_{AB}(X)(-\rho(\hat{E}^m) + \hat{F}^m(d_0)) C_{mAB}(x, U_0, d_0) \right). \]
\[ + \delta d(X) \left( - \rho(\hat{E}^m) + \hat{F}^m(d_0) \right) (C_m(x, U_0, d_0)) \]
\[ = c_0 \int dX e^{-2d_0} \sqrt{\det \eta_{MN}} \left( \delta U^{AB}(X) \hat{F}^m(d_0) C_{mAB}(x, U_0, d_0) \right. \]
\[ + \delta d(X) \hat{F}^m(d_0) C_m(x, U_0, d_0) \right) . \]  \( (6.83) \)

Since \( d_0 \) depends only on \( x^m \), \( \hat{F}^m(d_0) \) is given by

\[ \hat{F}^m(d_0) = - R_b^m \bar{f}_{ab} a . \]  \( (6.84) \)

Thus, if \( \bar{G} \) is unimodular, \( \bar{f}_{ab} a = 0 \), then \( U_0 \) and \( d_0 \) are solutions of the 2D-dimensional DFT, i.e.,

\[ \hat{S}[U_0 + \delta U(X), d_0 + \delta d(X)] - \hat{S}[U_0, d_0] = 0 . \]  \( (6.85) \)

Now, we are ready to derive the Poisson-Lie T-duality using the above considerations as follows.

1. Given a solution of the GSE

\[ \hat{G} + \hat{B} = L^{-1} \frac{1}{G_0 + B_0} L^{-T} , \]  \( (6.86) \)

\[ \hat{\phi} = d + \frac{1}{2} \log \det R + \frac{1}{4} \log \det \hat{G} , \]  \( (6.87) \)

\[ \hat{I}^m = R_b^m(x) \bar{f}_{ab} a , \]  \( (6.88) \)

where \( G_0 \) and \( B_0 \) depend only on \( x^m \). This means that \( U_0 \) and \( d_0 \) satisfy

\[ \hat{S}[U_0(x) + \delta U(x), d_0(x) + \delta d(x)] - \hat{S}[U_0(x), d_0(x)] = 0 . \]  \( (6.89) \)

2. If \( \bar{G} \) is unimodular (\( \bar{f}_{ab} a = 0 \)), \( U_0 \) and \( d_0 \) are a solution of the 2D-dimensional DFT action, i.e.,

\[ \hat{S}[U_0(x) + \delta U(X), d_0(x) + \delta d(X)] - \hat{S}[U_0(x), d_0(x)] = 0 . \]  \( (6.90) \)

Note that in this case \( \hat{I}^m = 0 \), meaning that \( (\hat{G}, \hat{B}, \hat{\phi}) \) is a solution for SUGRA.

3. Then, we perform a coordinate transformation such that \( \bar{g}(\tilde{x}) g(x) = g(x') \bar{g}(\tilde{x'}) \), we obtain

\[ \hat{S}[U_0(x(x')) + \delta U(X'), d_0(x(x')) + \delta d(X')] - \hat{S}[U_0(x(x')), d_0(x(x'))] = 0 . \]  \( (6.91) \)

4. Since we performed the coordinate transformation of the fluctuation in full space, it holds also in the restricted space, \( \delta U(\tilde{x}_m'), \delta d(\tilde{x}_m') \),

\[ \hat{S}[U_0(x(x')) + \delta U(\tilde{x'}), d_0(x(x')) + \delta d(\tilde{x'})] - \hat{S}[U_0(x(x')), d_0(x(x'))] = 0 . \]  \( (6.92) \)
5. In general, \( U_0(x(X')) \) depends on both \( \vec{x}'_m \) and \( x'^m \). However, in such a case, \( U_0 \) cannot generate a solution on a \( D \)-dimensional space. Therefore, in order to obtain \( \tilde{G} \) and \( \tilde{B} \) as \( D \)-dimensional background, we require that \( U_0(x(X')) \) depends only on \( \vec{x}'_m \).

\[
\frac{\partial}{\partial x'^m} U_0(x(X')) = 0 .
\] (6.93)

6. On the other hand, the condition for \( d_0 \) can be relaxed, since the \( x'^m \)-dependence of \( d_0 \) can vanish by a redefinition of \( d_0 \) and \( \tilde{I}_m \), as shown in the following. We require

\[
d_0(x(X')) = d_1(x') + d_2(\vec{x}') ,
\] (6.94)

\[
\frac{\partial}{\partial x'^m} (\rho(\tilde{E}_m)(d_1)) = 0 .
\] (6.95)

Then, we can define \( d'_0 \) and \( \tilde{I}'_m \) which depend only on \( \vec{x}'_m \). Using \( \tilde{E}_m = \tilde{R}_m^T a L_a \partial \eta \), the condition (6.95) means

\[
L_a \partial \eta d_1 = J_a , \quad (J_a : \text{constant}) .
\] (6.96)

\( d'_0 \) and \( \tilde{I}'_m \) are given by

\[
d'_0(\vec{x}') = d_2(\vec{x}') ,
\] (6.97)

\[
\tilde{I}'_m(\vec{x}') = \tilde{I}_m(\vec{x}') - 2 \rho(\tilde{E}_m)(d_1)
= \tilde{I}_m(\vec{x}') - 2 \tilde{R}_m^T a (\vec{x}') J_a .
\] (6.98)

\( d'_0 \) and \( \tilde{I}'_m \) do not depend on \( x' \).

7. Finally, we obtain a solution of the GSE in the dual space.

\[
\tilde{G}'_{mn} + \tilde{B}'_{mn} = \tilde{L}^{-1} \frac{1}{G_0 + B_0 + \Pi} \tilde{L}^{-T} ,
\] (6.99)

\[
\tilde{\phi} = d'_0 + \frac{1}{2} \log \det \tilde{R} + \frac{1}{4} \log \det \tilde{G}
= d_2 + \frac{1}{2} \log \det \tilde{R} + \frac{1}{4} \log \det \tilde{G} ,
\] (6.100)

\[
\tilde{I}'_m = \tilde{R}_m^b f_{ab}^a - 2 \rho(\tilde{E}_m)(d_1) .
\] (6.101)

In the following we show how the redefinition in step 6 is obtained. First, we consider

\( d'_0 \) and \( \tilde{I}'_m \) to depend on \( \vec{x}'_m \) only, i.e.,

\[
\tilde{F}'^m = 2 \partial^m (d_0(x(X'))) + \frac{1}{2} \log \det(\tilde{R})
= 2 \partial^m (d'_0(\vec{x}')) + \frac{1}{2} \log \det(\tilde{R}) ,
\] (6.102)

\[
\tilde{F}_m = -\tilde{I}_m(\vec{x}') + 2 \rho(\tilde{E}_m)(d_0(x(X')))
= -\tilde{I}_m' + 2 \rho(\tilde{E}_m)(d'_0) .
\] (6.103)

To obtain a \( D \)-dimensional solution, \( d'_0 \) and \( \tilde{I}'_m \) have to depend on \( \vec{x}'_m \) only, i.e.,

\[
\tilde{F}'^m = 2 \partial^m (d_0(x(X'))) + \frac{1}{2} \log \det(\tilde{R})
= 2 \partial^m (d'_0(\vec{x}')) + \frac{1}{2} \log \det(\tilde{R}) ,
\] (6.104)

\footnote{Note that originally \( U_0 \) and \( d_0 \) are assumed to be constant, meaning that the conditions given in (6.93), (6.94) and (6.95) are satisfied. Then it follows that the shift of \( I_m \) in (6.101) vanishes.}
Thus, the conditions (6.94), (6.95) for \( d_0(x(X')) \) are sufficient. From (6.105) and (6.105) we obtain \( d'_0 \) and \( \tilde{I}_m \) given in (6.97) and (6.98), respectively.

The redefinition \( d' = d - d_1, \tilde{I}'_m = \tilde{I}_m - 2\rho(\tilde{E}_m)(d_1) \) changes the measure of the action as

\[
c_0 \int dX \sqrt{\det \eta_{MN} e^{-2d}} = c_0 \int dX \sqrt{\det \eta_{MN} e^{-2d'} e^{-2d_1}}.
\]

Using these equations, (6.92) becomes

\[
0 = \tilde{S}[U_0(x(X')) + \delta U(x(X')), d_0(x(X')) + \delta d(x')] - \tilde{S}[U_0(x(X')), d_0(x(X'))] = \left( \int dx' \det L^{-1} e^{-2d_1} \right) \left( \tilde{S}^{\text{mod}}_{\text{DFTsec}}[\tilde{E}_0^N, d'_0 + \delta d', \tilde{I}'_m] - \tilde{S}^{\text{mod}}_{\text{DFTsec}}[\tilde{E}_0^N, d'_0, \tilde{I}_m] \right),
\]

where \( \tilde{E}_0^N(x') \) is defined by

\[
\tilde{E}_0^{-1} M^A H_{AB} \tilde{E}_0^{-T} B N = \begin{pmatrix} \tilde{G}_0 - \tilde{B}_0 \tilde{G}_0^{-1} \tilde{B}_0 - \tilde{B}_0 \tilde{G}_0^{-1} \\ \tilde{G}_0^{-1} \tilde{B}_0 \tilde{G}_0^{-1} \end{pmatrix},
\]

Using (6.93), (6.97) and (6.98), we find that \( (\tilde{E}_0^N, d'_0, \tilde{I}'_m) \) is a \( D \)-dimensional solution generating a solution for the GSE.

We summarize the Poisson-Lie T-duality derived in this section:

**Poisson-Lie T-duality in DFT**

We consider a solution for SUGRA given by

\[
\tilde{G} + \tilde{B} = L^{-1} \frac{1}{\tilde{G}_0 + \tilde{B}_0 + \Pi} L^{-T},
\]
\[ \hat{\phi} = d_0 + \frac{1}{2} \log \det R + \frac{1}{4} \log \det \hat{G}, \]  

(6.114)

where \( \hat{G} \) is unimodular, and \( G_0, B_0 \) and \( d_0 \) satisfy

\[ \frac{\partial}{\partial x'^m} (G_0(x(X')) + B_0(x(X'))) = 0, \]  

(6.116)

\[ d_0(x(X')) = d_1(x') + d_2(\bar{x}'), \]  

(6.117)

\[ \frac{\partial}{\partial x'^n} (L_a^m \delta_m(d_1)) = 0. \]  

(6.118)

In this case, another solution of the GSE is constructed by

\[ \tilde{g}^{mn} + \tilde{B}^{mn} = \tilde{L}^{-1} \frac{1}{G_0 + B_0 + \tilde{\Pi}} \tilde{L}^{-T}, \]  

(6.119)

\[ \hat{\phi} = d_2 + \frac{1}{2} \log \det \tilde{R} + \frac{1}{4} \log \det \tilde{G}, \]  

(6.120)

\[ \tilde{I}'_m = \tilde{R}_m f_a^b a - 2 \rho(\tilde{E}_m)(d_1). \]  

(6.121)

For constant \( d_0 \), (6.120) and (6.121) simplify to

\[ \hat{\phi} = d_0 + \frac{1}{2} \log \det R + \frac{1}{4} \log \det \hat{G}, \]  

(6.122)

\[ \tilde{I}'_m = \tilde{R}_m f_a^b a. \]  

(6.123)

In this case, we obtain a compact formula of the transformation rule for the dilaton without \( d_0 \) as

\[ e^{-2\hat{\phi}} = e^{-2\tilde{\phi}} \frac{\det(1 + (G_0 + B_0)\tilde{\Pi})}{\det(1 + (G_0 + B_0)\Pi)} \det a^{-1}. \]  

(6.124)

This T-duality of the dilaton is consistent with [51,61,62], (See also [55]).

Since our discussion starts with a SUGRA solution, we require that \( G \) is unimodular to get a corresponding solution for DFT. On the other hand, as in the usual discussion, we can also start with a DFT solution, then we obtain the T-duality where \( G \) and \( \hat{G} \) are not necessarily unimodular. In that case, \( \tilde{a} \) defined in (6.21) is not equal to 1 in general, and the transformation formula is given by

\[ e^{-2\hat{\phi}} = e^{-2\tilde{\phi}} \frac{\det(1 + (G_0 + B_0)\Pi) \det \tilde{a}}{\det(1 + (G_0 + B_0)\tilde{\Pi}) \det a}. \]  

(6.125)

Note that for a given solution of GSE, a solution of DFT may not exist as discussed in sec. 6.3.
7. R–R Sector

The fields of the R–R sector can be assembled into a spinor $\chi$ [63,64]. See also [65]. To construct an action of the R–R sector, we define in the appendix a product called AK-product $(-,-)_{AK}$ for the spinors $\chi_1, \chi_2 \in S$ as

$$(\chi_1, \chi_2)_{AK} = \chi_1^A K \chi_2 .$$

(7.1)

The AK-product is invariant under an infinitesimal $O(1, D - 1) \times O(D - 1, 1)$ transformation, i.e., for $\Lambda_{AB} = -\Lambda_{BA}$, $\Lambda_{AB} = -H_{B'B'} A'A' \Lambda_{B'A'}$ we obtain

$$(\chi_1, \chi_2)_{AK} = \left( \exp \left( \frac{1}{4} \Lambda_{AB} \gamma^{AB} \right) \chi_1, \exp \left( \frac{1}{4} \Lambda_{A'B'} \gamma^{A'B'} \right) \chi_2 \right)_{AK} .$$

(7.2)

The DFT action for the R–R sector $S_{RR}$ can be defined by

$$S_{RR} = \beta_{RR}(\mathcal{D}\chi, \mathcal{D}\chi)_{AK} ,$$

(7.3)

where $\chi \in S$ is the R–R potential of DFT and $\beta_{RR}$ is a constant. The constant $\beta_{RR}$ is determined by reducing $S_{RR}$ to the case of a flat background, and then comparing with the action in DFT$_{sec}$. The result is

$$\beta_{RR} = -\frac{1}{2c_0} .$$

(7.4)

For details, see Appendix B.

Note that a self-duality condition must be imposed on $\chi$ at the level of the equation of motion. Usually, the R–R field is defined in the local coordinate. Here, we construct the R–R field in the local Lorentz frame $E_A$ in the similar way as for the NS–NS sector.

7.1. $O(1, D - 1) \times O(D - 1, 1)$ transformation of R–R sector. An $O(1, D - 1) \times O(D - 1, 1)$ transformation of the R–R flux $F := \mathcal{D}\chi$ is given by

$$\chi \mapsto S_O \chi ,$$

(7.5)

with $S_O$ being defined by

$$S_O \gamma_A S_O^{-1} = O_A^B \gamma_B .$$

(7.6)

where the transformation $O_A^B$ is $O(1, D - 1) \times O(D - 1, 1)$:

$$O_A^{A'} O_B^{B'} \eta_{A'B'} = \eta_{AB} .$$

(7.7)

$$O_A^{A'} O_B^{B'} H_{A'B'} = H_{AB} .$$

(7.8)

The spin representation $S_O$ used in this paper satisfies

$$(S_O^{-1})^\dagger A_K S_O^{-1} = (-1)^{#O} A_K ,$$

(7.9)

$$(S_O^{-1} K)^\dagger A_K S_O^{-1} K = -(1)^{#O} A_K ,$$

(7.10)

where $#O$ is defined as the number of the transformations $(\Gamma_0 \mapsto -\Gamma_0, \bar{\Gamma}_0 \mapsto \bar{\Gamma}_0)$ and $(\Gamma_0 \mapsto \Gamma_0, \bar{\Gamma}_0 \mapsto -\bar{\Gamma}_0)$ in $O$. Therefore, together with (7.5) under the $O(1, D - 1) \times O(D - 1, 1)$ transformation the fields transform as follows:

$$E_A' N = O_A^B E_B^N ,$$

(7.11)
The action of the NS–NS sector is invariant under this transformation. On the other hand, the action $S_{RR}$ of the R–R sector is not invariant in general, i.e.,

$$S'_{RR} = \beta_{RR}(S^{-1}_O F, S^{-1}_O F)_{AK} = (-1)^{#O} S_{RR}. \quad (7.12)$$

When $#O = \text{odd}$, the $O(1, D - 1) \times O(D - 1, 1)$ transformation is not a symmetry of DFT. On the other hand, we can consider a local transformation

$$E''_A = O_A B E_B^N, \quad d'' = d, \quad F'' = S^{-1}_O K F. \quad (7.13)$$

for $#O = \text{odd}$. With this transformation the DFT action is invariant, i.e.,

$$S''_{RR} = \beta_{RR}(S^{-1}_O K F, S^{-1}_O K F)_{AK} = (-1)^{#O} S_{RR} = S_{RR}. \quad (7.14)$$

Therefore, there exists a symmetry for $\forall O \in O(1, D - 1) \times O(D - 1, 1)$:

$$(E_A^N, d, F) \mapsto \begin{cases} (O_A B E_B^N, d, S^{-1}_O F), & #O = \text{even}, \\ (O_A B E_B^N, d, S^{-1}_O K F), & #O = \text{odd}. \end{cases} \quad (7.15)$$

This invariance guarantees the $O(1, D - 1) \times O(D - 1, 1)$ covariance of the equation of motion after the self-duality condition.

To see this mechanism, we consider an $O(1, D - 1) \times O(D - 1, 1)$ transformation of a solution of DFT given by

$$E_A^N = E_{0A}^N, \quad d = d_0, \quad F = F_0. \quad (7.16)$$

From the above discussion we can rotate solutions, i.e.,

$$E_A^N = O_A B E_{0B}^N, \quad d = d_0, \quad F = S^{-1}_O F_0, \quad (#O : \text{even}), \quad (7.17)$$

$$E_A^N = O_A B E_{0B}^N, \quad d = d_0, \quad F = S^{-1}_O K F_0, \quad (#O : \text{odd}). \quad (7.18)$$

However, for the e.o.m. we require the self-duality for the R–R field $F$

$$KF = F, \quad (7.19)$$

meaning that $#O$ does not change the form of the solution and therefore (7.17) and (7.18) become the same equation

$$E_A^N = O_A B E_{0B}^N, \quad d = d_0, \quad F = S^{-1}_O F_0. \quad (7.20)$$

This shows that the $O(1, D - 1) \times O(D - 1, 1)$ transformation is not a symmetry of the action, in general, but the e.o.m. is covariant and gives a mapping of a solution into another solution.
7.2. Poisson-Lie T-duality of R–R sector. In the previous section we have derived the Poisson-Lie T-duality of the NS–NS sector. Here, we show the Poisson-Lie T-duality of the R–R sector.

Similar as in the discussion of NS–NS sector, we consider the case where $l = g(\bar{x})g(x)$.

\[
\hat{E}_A^N = \begin{pmatrix} e^{-T} & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^{-T} & 0 \\ 0 & L \end{pmatrix}.
\] (7.21)

The vielbein $\hat{E}_A^N$ generates the D-dimensional metric and B-field (6.41). Then, using a representation $\hat{\mathcal{C}} \in O(1, D - 1) \times O(D - 1, 1)$, we write $\hat{E}_A^N$ as

\[
\hat{E}_A^N = \hat{\mathcal{C}} \begin{pmatrix} e^{-T} & 0 \\ 0 & \hat{\mathcal{C}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (7.22)

As in the previous section, the $GL(D)$ part and the B-transformation part are denoted by

\[
\hat{E}^{(e)N}_A = \begin{pmatrix} e^{-Ta} & 0 \\ 0 & e^{Ta} \end{pmatrix},
\] (7.23)

\[
\hat{E}^{(B)N}_M = \begin{pmatrix} m_n & 0 \\ 0 & B_{mn} \end{pmatrix}.
\] (7.24)

To obtain the D-dimensional action, we define a rotated R–R flux $\hat{F}$ as

\[
\hat{F} = \frac{1}{\sqrt{2^{p+1}p!}} \hat{F}_{m_1 \ldots m_p \delta a_1 \ldots \delta a_p \gamma^{a_1 \ldots a_p}} |0\rangle = e^{-\hat{d}} \mathcal{D}_{(e)} \mathcal{D}_X,
\] (7.25)

where $\hat{d}$ is defined by

\[
\hat{d} = d + \frac{1}{2} \log \det R.
\] (7.26)

Using this $\hat{F}$, the action of the R–R sector is given by

\[
S_{RR} = \beta_{RR}(\mathcal{D}_X, \mathcal{D}_X)_{AK}
\]

\[
= \beta_{RR} \left( e^{d} S_{\hat{E}^{(e)}_A}^{-1} \hat{F}, e^{d} S_{\hat{E}^{(e)}_A}^{-1} \hat{F} \right)_{AK}
\]

\[
= \beta_{RR} \sum_{p,q} \left( e^{d} S_{\hat{E}^{(e)}_A}^{-1} \hat{F}_{m_1 \ldots m_p \delta a_1 \ldots \delta a_p \gamma^{a_1 \ldots a_p}} |0\rangle, e^{d} S_{\hat{E}^{(e)}_A}^{-1} \hat{F}_{n_1 \ldots n_q \delta b_1 \ldots \delta b_q \gamma^{b_1 \ldots b_q}} |0\rangle \right)_{AK}
\]

\[
= \beta_{RR} \frac{1}{\sqrt{2^{p+1}p!} \sqrt{2^{q+1}q!}} \sum_{p,q} \left( S_{\hat{E}^{(e)}_A}^{-1} \hat{F}_{m_1 \ldots m_p \delta a_1 \ldots \delta a_p \gamma^{a_1 \ldots a_p}} |0\rangle, e^{2d} S_{\hat{E}^{(e)}_A}^{-1} \hat{F}_{n_1 \ldots n_q \delta b_1 \ldots \delta b_q \gamma^{b_1 \ldots b_q}} |0\rangle \right)_{AK}
\]

\[
= \frac{1}{2} \beta_{RR} \left( S_{\hat{E}^{(e)}_A}^{-1} |0\rangle, S_{\hat{E}^{(e)}_A}^{-1} |0\rangle \right)_{AK} e^{2d} \hat{G}_{m_1 n_1} \ldots \hat{G}_{m_p n_p} \hat{F}_{m_1 \ldots m_p} \hat{F}_{n_1 \ldots n_p}
\]

\[
= \frac{1}{2} \beta_{RR} \left( |0\rangle, |0\rangle \right)_{AK} \sqrt{\det \hat{G}_{\mu} e^{2d} \hat{G}_{m_1 n_1} \ldots \hat{G}_{m_p n_p} \hat{F}_{m_1 \ldots m_p} \hat{F}_{n_1 \ldots n_p}}
\]
Then, we can write the R–R flux
\[ \hat{\det} \hat{\bar{g}} \]
Since the R–R flux
\[ F \]
Note that
\[ U \]
When
\[ x_n \]
integration simply gives a factor
\[ \det \hat{L}^{-1} \] as in (6.59).
Let \( F^{(0)}(x) \) be a solution of the e.o.m. for the action \( S_{RR} \):
\[ \hat{\mathcal{D}} X = F^{(0)}(x) \].
(7.28)
Note that \( F_0 \) depends only on \( x^m \), and the D-dimensional R–R flux \( \hat{F} \) is given by
\[ \sum_p \frac{1}{\sqrt{2^{p+1} p!}} \hat{F}_{n_1 \ldots n_p} \delta_{a_1}^{n_1} \ldots \delta_{a_p}^{n_p} \gamma^{a_1 \ldots a_p} |0\rangle = e^{-\hat{d}} S_{E(0)} F^{(0)}. \]
(7.29)
With \( \hat{C}_0 \in O(1, D - 1) \times O(D - 1, 1) \), the solution of the vielbein \( \hat{E}_{0A}^N \) is given by
\[ E_{0A}^N = U_0 \left( \begin{array}{cc} 1 & \Pi \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} L^{-T} & 0 \\ 0 & L \end{array} \right) = \hat{C}_0 \left( \hat{e}^{-T} & 0 \\ 0 & \hat{e} \end{array} \right) \left( \begin{array}{cc} 1 \\ \hat{B} \\ 1 \end{array} \right). \]
(7.30)
Then, we can write the R–R flux \( \hat{F} \) by the algebraic structures \( L, \Pi \) as
\[ \sum_p \frac{1}{\sqrt{2^{p+1} p!}} \hat{F}_{n_1 \ldots n_p} \delta_{a_1}^{n_1} \ldots \delta_{a_p}^{n_p} \gamma^{a_1 \ldots a_p} |0\rangle = e^{-\hat{d}} S_{E(0)} L S_{\Pi} S_{U_0} S_{\hat{C}_0} F^{(0)}, \]
(7.31)
where the spin operators \( S_L, S_{\Pi}, S_{U_0} \) and \( S_{\hat{C}_0} \) are defined by
\[ S_L \gamma A S_L^{-1} = \left( L^{-T} a_m \right) \delta_M^B \gamma_B, \]
(7.32)
\[ S_{\Pi} \gamma A S_{\Pi}^{-1} = \left( \begin{array}{cc} \Pi^{a b} \\ \gamma^a_b \end{array} \right), \]
(7.33)
\[ S_{U_0} \gamma A S_{U_0}^{-1} = U_{0A}^B \gamma_B, \]
(7.34)
\[ S_{\hat{C}_0} \gamma A S_{\hat{C}_0}^{-1} = \hat{C}_0^A B \gamma_B. \]
(7.35)
Since the R–R flux \( F \) has \( O(1, D - 1) \times O(D - 1, 1) \) invariance as mentioned in (7.20), the contribution of \( \hat{C}_0 \) in (7.31) vanishes. Thus, we obtain the solution of the R–R flux \( \hat{F} \) as
\[ \hat{F} = e^{-\hat{d}} S_{E(0)} L S_{\Pi} S_{U_0} F^{(0)}. \]
(7.36)
In this section, we have assumed that \( F^{(0)} \) is a solution of DFT. On the other hand, considering \( F^{(0)} \) to be a solution of the DFT action under a fluctuation which depends on \( x^m \) only, \( F^{(0)} \) is, in general, not a solution of the action for a 'full' fluctuation depending
on both $x^m$ and $\bar{x}_m$. For the case that $\hat{G}$ is unimodular, $F^{(0)}$ is a solution of the action for full fluctuations as discussed in (6.90).

As in the discussion above, a solution of the $R$–$R$ flux $\hat{F}$ in the dual space can be derived:

$$
\sum_{p} \frac{1}{\sqrt{2^{p+1} p!}} \hat{F}^{m_1 \cdots m_p} \delta^{a_1}_{m_1} \cdots \delta^{a_p}_{m_p} \gamma_{a_1 \cdots a_p} \hat{K} |0\rangle = e^{-\hat{d}} S_{\bar{E}(b)}^{-1} S_{L} S_{\bar{\Pi}} S U_0^0 F^{(0)} .
$$

(7.37)

Therefore, we can derive the Poisson-Lie T-duality for the $R$–$R$ sector along the line given in Sect. 6.3.

1. Let a solution of SUGRA be given by

$$
\hat{G} + \hat{B} = L^{-1} \frac{1}{G_0 + B_0} + \Pi^{-T} ,
$$

(7.38)

$\hat{\phi} = d_0 + \frac{1}{2} \log \det R + \frac{1}{4} \log \det \hat{G}$ ,

(7.39)

$$
\hat{F} = e^{-\hat{d}} \frac{1}{S_{\bar{E}(b)}} S_{L} S_{\bar{\Pi}} S U_0^0 F^{(0)} ,
$$

(7.40)

where $G_0, B_0, d_0$ and $F^{(0)}_{m_1 \cdots m_p}$ depend only on $x^m$. Note that $\hat{G}$ is unimodular. This means that a solution of DFT is given by

$$
U = U_0 , \ d = d_0 , \ F = F^{(0)} .
$$

(7.41)

2. Transform the coordinates so that $\hat{g}(\bar{x}) g(x) = g(x') \hat{g}(\bar{x}')$.

3. Then, we assume that the transformed solution $U_0(x(X'))$ and $F^{(0)}_{m_1 \cdots m_p}(x(X'))$ depend only on $\bar{x}'_m$, i.e.,

$$
\frac{\partial}{\partial x'^m} U_0(x(X')) = 0 , \ \frac{\partial}{\partial x'^m} F^{(0)}_{m_1 \cdots m_p} = 0 .
$$

(7.42)

and $d_0$ satisfies

$$
d_0(x(X')) = d_1(x') + d_2(\bar{x}') ,
$$

(7.43)

$$
\frac{\partial}{\partial x'^m} (\rho(\check{E}_m)(d_1)) = 0 .
$$

(7.44)

4. Using $\check{E}_m = \check{R}^T_{m} \check{R} L_a^l \partial_l$, the condition (7.44) means

$$
L_a^l \partial_l d_1 = J_a , \ (J_a : \text{constant}) .
$$

(7.45)

Define $d'_0$ and $\check{I}'_m$ by

$$
d'_0(\bar{x}') = d_2(\bar{x}') ,
$$

(7.46)

$$
\check{I}'_m(\bar{x}') = \check{I}_m(\bar{x}') - 2 \rho(\check{E}_m)(d_1)
= \check{I}_m(\bar{x}') - 2 \check{R}^T_{m} (\bar{x}') J_a ,
$$

(7.47)

$$
\check{F}' = e^{-d_2 - \frac{1}{2} \log \det \check{R}} S_{\bar{E}(b)}^{-1} S_{L} S_{\bar{\Pi}} S U_0^0 F^{(0)} .
$$

(7.48)

Then, $d'_0$ and $\check{I}'_m$ do not depend on $x'$. 

5. Finally, we obtain a solution of the GSE in the dual space
\[
\dot{G}^{mn} + \dot{B}^{mn} = \tilde{L}^{-1} \frac{1}{\tilde{G}_0 + B_0 + \Pi} \tilde{L}^{-T},
\]
(7.49)
\[
\dot{\phi} = d_0' + \frac{1}{2} \log \det \tilde{R} + \frac{1}{4} \log \det \tilde{G},
\]
\[
= d_2 + \frac{1}{2} \log \det \tilde{R} + \frac{1}{4} \log \det \tilde{G},
\]
(7.50)
\[
\dot{\varphi}_m = \tilde{R}_m f_{ab} \varphi^a - 2 \tilde{\rho}(\tilde{E}_m)(d_1),
\]
(7.51)
\[
\sum_p \frac{1}{\sqrt{2^{p+1} p!}} \tilde{F}_m^{m_1 \ldots m_p} \delta^{a_1}_{m_1} \ldots \delta^{a_p}_{m_p} \gamma_{a_1 \ldots a_p} K|0\rangle
\]
\[
= e^{-d_2 - \frac{1}{2} \log \det \tilde{S}} S_{E(\phi)}^{-1} S_L S_{\tilde{\eta}} S_{U_0} F^{(0)}.\]
(7.52)

Note that we use the dual vacuum $K|0\rangle$ to define the R–R flux $\tilde{F}^{m_1 \ldots m_p}$ on the dual space. In the action of the R–R sector, $\tilde{F}^{m_1 \ldots m_p}$ is contracted by the metric $\tilde{G}_{mn}$ of the dual space.

To see how the concrete form of the R–R flux $\tilde{F}'$ in the dual space is determined, we show the dual action of the R–R sector explicitly. First, we define the R–R flux in the dual space as in the original space by
\[
\tilde{F} = \sum_p \frac{1}{\sqrt{2^{p+1} p!}} \tilde{F}_m^{m_1 \ldots m_p} \delta^{a_1}_{m_1} \ldots \delta^{a_p}_{m_p} \gamma_{a_1 \ldots a_p} K|0\rangle = e^{-\dot{d}} S_{E(\phi)}^{-1} K \dot{\psi} \chi.
\]
(7.53)

Using this R–R flux $\tilde{F}$, the action of the R–R sector is given by
\[
S_{RR} = \beta_{RR}(\dot{\psi}, \dot{\chi})_{AK}
\]
\[
= \beta_{RR}(e^{\dot{d}} K^{-1} S_{E(\phi)}^{-1} \tilde{F}, e^{\dot{d}} K^{-1} S_{E(\phi)}^{-1} \tilde{F})_{AK}
\]
\[
= -\beta_{RR}(e^{\dot{d}} S_{E(\phi)}^{-1} \tilde{F}, e^{\dot{d}} S_{E(\phi)}^{-1} \tilde{F})_{AK}
\]
\[
= -\beta_{RR} \sum_{p, q} \left( e^{\dot{d}} S_{E(\phi)}^{-1} \frac{1}{\sqrt{2^{p+1} p!}} \tilde{F}_m^{m_1 \ldots m_p} \delta^{a_1}_{m_1} \ldots \delta^{a_p}_{m_p} \gamma_{a_1 \ldots a_p} K|0\rangle, \right.
\]
\[
\left. e^{\dot{d}} S_{E(\phi)}^{-1} \frac{1}{\sqrt{2^{q+1} q!}} \tilde{F}_n^{n_1 \ldots n_q} \delta^{b_1}_{n_1} \ldots \delta^{b_q}_{n_q} \gamma_{b_1 \ldots b_q} K|0\rangle \right)_{AK}
\]
\[
= -\frac{1}{2} \beta_{RR} \sum_p e^{2\dot{d}} \tilde{F}_m^{m_1 \ldots m_p} \tilde{F}_n^{n_1 \ldots n_q} \tilde{G}_{m_1 n_1} \ldots \tilde{G}_{m_p n_p} \left( S_{E(\phi)}^{-1} K|0\rangle, S_{E(\phi)}^{-1} K|0\rangle \right)_{AK}
\]
\[
= \frac{1}{2} \beta_{RR} \sum_p \int dX \sqrt{\det \eta_{MN}} e^{-2d} e^{2\dot{d}} \tilde{F}_m^{m_1 \ldots m_p} \tilde{F}_n^{n_1 \ldots n_q} \tilde{G}_{m_1 n_1} \ldots \tilde{G}_{m_p n_p} \sqrt{\det \tilde{G}}\]
\[
= \frac{1}{2} \beta_{RR} \sum_p \int dX \det L^{-1} \sqrt{\det \tilde{G}} \tilde{F}_m^{m_1 \ldots m_p} \tilde{F}_n^{n_1 \ldots n_q} \tilde{G}_{m_1 n_1} \ldots \tilde{G}_{m_p n_p}. \]
On the other hand, the action of the NS–NS sector is given by (6.109), in which the \( x' \)-dependence is given by a factor \( \int dx' \det L^{-1} e^{-2d_1} \). To extract this factor for the full bosonic action \( S + S_{RR} \), we rewrite the R–R flux as

\[
\tilde{F}^{m_1 \cdots m_p} = e^{d_1} \tilde{F}^{m_1 \cdots m_p} .
\] (7.55)

Using this \( \tilde{F}^{m_1 \cdots m_p} \), the action of the R–R sector is given by

\[
S_{RR} = \frac{1}{2} \beta_{RR} \sum_p \int dx' \det L^{-1} e^{-2d_1} \int dx' \sqrt{\det \tilde{G}} \tilde{F}^{m_1 \cdots m_p} \tilde{F}^{m_1 \cdots m_q} \tilde{G}_{m_1n_1} \cdots \tilde{G}_{m_pn_p} .
\] (7.56)

Thus, the concrete form of the R–R flux \( \tilde{F}^{m_1 \cdots m_p} \) in the dual space is determined as

\[
\sum_p \frac{1}{\sqrt{2p+1} p!} \tilde{F}^{m_1 \cdots m_p} \delta_{m_1}^{a_1} \cdots \delta_{m_p}^{a_p} \gamma_{a_1 \cdots a_p} K |0 \rangle = e^{d_1} e^{-d} S_{E(e)} K \Phi \chi
\]
\[
= e^{-d_2 - \frac{1}{2} \log \det R S_{E(e)} K \Phi \chi .}
\] (7.57)

To obtain the R–R flux from the algebraic structure, we consider the relation between the spin operators:

\[
S_{E(b)} S_{E(e)} S_{\tilde{C}_0} = S_L S_{\tilde{f}i} S_{U_0} ,
\] (7.58)

where the spin operators \( S_L, S_{\tilde{f}i} \) and \( S_{\tilde{C}_0} \) are defined by

\[
S_L \gamma_A S_L^{-1} = \left( \tilde{L}_a^m \tilde{L}_a^{-T} \right) \delta_M^B \gamma_B ,
\] (7.59)

\[
S_{\tilde{f}i} \gamma_A S_{\tilde{f}i}^{-1} = \left( \tilde{T}^{ab}_{1a} \tilde{T}^{ab} \right) \gamma_B ,
\] (7.60)

\[
S_{\tilde{C}_0} \gamma_A S_{C_0}^{-1} = \tilde{C}_0 A B \gamma_B , (\tilde{C}_0 \in O(1, D - 1) \times O(D - 1, 1)).
\] (7.61)

Using this relation, the solution of the R–R flux is given by

\[
\sum_p \frac{1}{\sqrt{2p+1} p!} \tilde{F}^{m_1 \cdots m_p} \delta_{m_1}^{a_1} \cdots \delta_{m_p}^{a_p} \gamma_{a_1 \cdots a_p} K |0 \rangle
\]
\[
= e^{-d_2 - \frac{1}{2} \log \det R S_{E(b)}^{-1} S_L S_{\tilde{f}i} S_{U_0} S_{\tilde{C}_0}^{-1} K F_0
\]
\[
= e^{-d_2 - \frac{1}{2} \log \det R S_{E(b)}^{-1} S_L S_{\tilde{f}i} S_{U_0} S_{\tilde{C}_0}^{-1} F_0 ,}
\] (7.62)

where we used the self-duality condition: \( K F_0 = F_0 \). Finally, since \( \tilde{C}_0 \in O(1, D - 1) \times O(D - 1, 1) \) is a map of a solution into another solution as discussed in the previous section, the spin operator \( S_{\tilde{C}_0}^{-1} \) vanishes. Thus, we obtain the solution of the R–R flux as

\[
\sum_p \frac{1}{\sqrt{2p+1} p!} \tilde{F}^{m_1 \cdots m_p} \delta_{m_1}^{a_1} \cdots \delta_{m_p}^{a_p} \gamma_{a_1 \cdots a_p} K |0 \rangle = e^{-d_2 - \frac{1}{2} \log \det R S_{E(b)}^{-1} S_L S_{\tilde{f}i} S_{U_0} F_0 .}
\] (7.63)
8. Discussion and Outlook

In this paper we have shown that several open questions of DFT can be clarified by using a Metric Algebroid as an underlying symmetry structure. The most important result is, probably, that there is no need to imply the section condition. Instead the more geometrical pre-Bianchi identity gives us the necessary structure. From the metric algebroid we can deduce relations between various DFT models. Also the role of the doubled space, an idea inspired by string theory, where we have the spaces of momentum and of winding numbers, becomes more transparent from the metric algebroid viewpoint. We gave a formulation of a general action for DFT obtained from a class of metric algebroid, the structure functions of which satisfy a pre-Bianchi identity, and analyzed how the gauge symmetry in $D$-dimensional theory is obtained. The action is formulated by a Lichnerowicz formula using the DGO without referring to the section condition. The general action contains a parameter $\beta_+$, which takes different values when we apply this formulation to the DFT$_{sec}$ or to the DFT$_{WZW}$ case. As a concrete example, we applied our formalism to the Poisson-Lie T-duality of the effective action on the group manifold.

We have two types of the Lichnerowicz formula, a generalized formula (4.21) and a projected one (4.26).\footnote{Note that the r.h.s. of (4.21) and (4.26) can be written by the corresponding curvatures as shown in Sect. 5. For details see [1].} To obtain the generalized Lichnerowicz formula from the DGO, we need to require conditions on the structure functions $F_{ABC}$ and $\phi_{ABC}$ of the metric algebroid, and also on the ambiguity of the DGO given by the flux $F_A$. These conditions are in fact the pre-Bianchi identity and a corresponding identity for $F_A$. Thus, the existence of the generalized Lichnerowicz formula is equivalent to a restriction on the metric algebroid and a restriction on the structure of the spin bundle related to the DGO by these identities.

The generalized Lichnerowicz formula can be interpreted as a sufficient condition to generate an $O(1, D - 1) \times O(D - 1, 1)$ invariant scalar from which we obtain the projected Lichnerowicz formula given in terms of $\mathcal{L}^+$, $\mathcal{L}^-$. It is not a necessary condition, since for the projected Lichnerowicz formula we do not need the full pre-Bianchi identities, meaning that the parts with mixed indices $\bar{a}$ and $\tilde{a}$, for example $\tilde{\phi}_{abcd}$, $B_{ab\tilde{c}}$ etc., do not appear. On the other hand, for the gauge invariance after the dimensional reduction the metric algebroid structure is nevertheless necessary, i.e., we need the pre-Bianchi identity and therefore, we require the existence of the generalized Lichnerowicz formula for the 2D-dimensional space.

To recover physical models in $D$ dimensions, we use a dimensional reduction on the fluctuations. From the point of view of physics, it is natural to require that the fluctuations should only occur in the $D$ directions which are of physical relevance. On the other hand, the geometry of the 2D dimensional space gives a restriction on the directions of the fluctuations. In this sense, a solution of the 2D-dimensional background is a condition on the parameter of the $D$-dimensional field theory.

The Poisson-Lie T-duality of the $DFT_{WZW}$ has also been discussed by Hassler [54]. The coordinate transformation of the Drinfel’d Double $\mathcal{D} = \mathcal{G} \bowtie \tilde{\mathcal{G}}$ and the fluctuation considered by Hassler are given by $\left(l(X) = \tilde{g}(\tilde{x})g(x), \delta U(x)\right) \mapsto \left(l(X(X')) = g(x')\tilde{g}(\tilde{x}'), \delta U(\tilde{x}')\right)$ in terms of our notation. However, this coordinate transformation does not exist in general, since the transformation is determined by $l(X) \mapsto l(X(X'))$. This does not necessarily mean that there exists a transformation of the fluctuation $\delta U$.
satisfying the condition $\frac{\partial}{\partial x} \delta U(x(X')) = 0$. Therefore, the 'coordinate transformation' in [54] is not exactly a coordinate transformation of the DFT action.

On the other hand, our procedure discussed in this paper uses a coordinate transformation
\[ l(X) = \bar{g}(\bar{x}) g(x), \delta U(X) \mapsto l(X(X')) = g(x') \bar{g}(\bar{x}'), \delta U(X') \]
after extending the fluctuations to a 2D-dimensional function, which is possible since we do not use a section condition. We only require the metric algebroid structure in 2D-dimensions, and that the background is a solution of DFT. Thus, the Poisson-Lie T-duality is understood as a coordinate transformation including the fluctuations.

Finally, in this paper we have also shown that starting from our general action with parametrization (5.8) we can derive the GSE of $S^\text{mod}_{\text{DFT sec}}[E_A^M, d, I^m]$, which coincides the action defined in [29], thus giving the missing algebraic background for the modification of the action of the Drinfel’d Double case.

Acknowledgements The authors would like to thank G. Aldazabal, N. Ikeda, C. Klimčík, Y. Sakatani, P. Ševera and K. Yoshida for stimulating discussions and lectures. We also thank T. Kaneko, S. Sekiya, S. Takezawa, and T. Yano for valuable discussions. S.W. is supported by the JSPS Grant-in-Aid for Scientific Research (B) No.18H01214.

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Appendix

A. Spin Representation

Here, we construct a spin representation of $SO(10, 10)$ using the basis $\Gamma_A = (\Gamma_a, \bar{\Gamma}_{\bar{a}})$. $\Gamma_A$ is defined by

\[
\Gamma_A = \begin{pmatrix} \Gamma_a \\ \bar{\Gamma}_{\bar{a}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_{ab} & \delta_{\bar{a}}^b \\ -s_{\bar{a}b} & \delta_a^b \end{pmatrix} \begin{pmatrix} \gamma^b \\ \bar{\gamma}^\bar{b} \end{pmatrix}.
\] (A.1)

Note that $s = \text{diag}(-1, 1, 1, \cdots, 1)$ for all suffices $a, \bar{a}, a, \bar{a}$, i.e.,

\[
\Gamma_\bar{a} = \eta^{\bar{a}\bar{b}} \Gamma_\bar{b} = -s_{\bar{a}b} \Gamma_b.
\] (A.2)

The inner product of the basis is diagonalized as

\[
\{ \Gamma_A, \Gamma_B \} = 2 \begin{pmatrix} s_{ab} & 0 \\ 0 & -s_{\bar{a}b} \end{pmatrix}.
\] (A.3)

The component of the generalized metric using this basis is equal to that using $\gamma_A$:

\[
H_{AB} = \begin{pmatrix} s_{ab} & 0 \\ 0 & s_{\bar{a}b} \end{pmatrix}.
\] (A.4)

In this paper, we use the following spin representation for the basis $\Gamma_A$:

\[
\Gamma_A = \{ \Gamma_a, \bar{\Gamma}_{\bar{a}} \} = \{ \Gamma_0, \cdots, \Gamma_9, \bar{\Gamma}_{\bar{0}}, \cdots, \bar{\Gamma}_{\bar{9}} \}.
\] (A.5)
which we give here explicitly,

\[
\begin{align*}
\Gamma_0 &= i\sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_1 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_2 &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_3 &= \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_4 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 \\
\Gamma_6 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \\
\Gamma_7 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \\
\Gamma_8 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \\
\Gamma_9 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \\
\Gamma_\delta &= (\sigma_3)^{\otimes 5} \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1
\end{align*}
\] (A.6)

where \(\sigma_1, \sigma_2\) and \(\sigma_3\) are the Pauli matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (A.7)

\(A_+\) is defined as a generator of the Hermitian conjugate as

\[
A_+ \Gamma_A A_+^{-1} = \Gamma_A^\dagger.
\] (A.8)

\(A_+\) is the charge conjugate matrix in this representation given by

\[
A_+ = i\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3.
\] (A.9)

To obtain the Majorana representation, we define the generator of the complex conjugate as

\[
B_+ \Gamma_A B_+^{-1} = \Gamma_A^*.
\] (A.10)

The representation of \(B_+\) is given by

\[
B_+ = \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1.
\] (A.11)
The spinor basis $e_\alpha$ of the Majorana representation is defined by

$$B_+^{-1}e_\alpha^n = e_\alpha .$$  \hspace{1cm} (A.12)

A Majorana spinor $\varphi \in \mathbb{S}$ is denoted by

$$\varphi = \varphi^\alpha e_\alpha .$$  \hspace{1cm} (A.13)

where $\varphi^\alpha \in \mathbb{R}$. $\Gamma_\chi$ is defined as

$$\{\Gamma_\chi, \Gamma_A\} = 0 .$$  \hspace{1cm} (A.14)

The explicit form of $\Gamma_\chi$ is given by

$$\Gamma_\chi = \sigma_3^{\otimes 5} \otimes \sigma_3^{\otimes 5} .$$  \hspace{1cm} (A.15)

Finally, the spin representation of the generalized metric is given by

$$K \Gamma_A K^{-1} = H_A B \Gamma_B ,$$  \hspace{1cm} (A.16)

$$K = 1^{\otimes 5} \otimes \sigma_3^{\otimes 5} .$$  \hspace{1cm} (A.17)

### A.1 Majorana representation.

We list the properties under Hermitian conjugation and taking the transpose for the operators in the Majorana representation:

$$J_{\alpha\beta} = e_\beta^\dagger e_\alpha ,$$  \hspace{1cm} (A.18)

$$\Gamma_{\alpha\beta} = e_\beta^\dagger \Gamma_A e_\beta ,$$  \hspace{1cm} (A.19)

$$A_{+\alpha\beta} = e_\beta^\dagger A_{+} e_\beta ,$$  \hspace{1cm} (A.20)

$$K_{\alpha\beta} = e_\beta^\dagger K e_\beta ,$$  \hspace{1cm} (A.21)

where $J_{\alpha\beta}$ is the metric on a Majorana spinor, i.e.,

$$e^\alpha = J^{-1\alpha\beta} e_\beta ,$$  \hspace{1cm} (A.22)

$$\varphi_\alpha = J_{\alpha\beta} \varphi^\beta .$$  \hspace{1cm} (A.23)

$J_{\alpha\beta}$, $A_{+\alpha\beta}$ and $K_{\alpha\beta}$ are (anti-)symmetric and (anti-)Hermitian, and $\Gamma_A$ is in the Majorana representation. Then, they are real matrices as follows,

$$J_{\alpha\beta}^* = J_{\beta\alpha} ,$$  \hspace{1cm} (A.24)

$$J_{\alpha\beta} = J_{\beta\alpha} ,$$  \hspace{1cm} (A.25)

$$\Gamma_A e_\alpha = \Gamma_A^\dagger e_\alpha ,$$  \hspace{1cm} (A.26)

$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}^* ,$$  \hspace{1cm} (A.27)

$$A_{+\alpha\beta}^* = -A_{+\beta\alpha} ,$$  \hspace{1cm} (A.28)

$$A_{+\alpha\beta} = -A_{+\beta\alpha} ,$$  \hspace{1cm} (A.29)

$$K_{\alpha\beta}^* = K_{\beta\alpha} ,$$  \hspace{1cm} (A.30)

$$K_{\alpha\beta} = K_{\beta\alpha} .$$  \hspace{1cm} (A.31)
A.2 Vacuum. The vacuum of the spin space $S$ is defined by

$$B_k^{-1}|0\rangle^* = |0\rangle, \quad \gamma_a |0\rangle = 0. \quad (A.32)$$

$\gamma_a$ is a suffix of standard $O(D, D)$. On the other hand, $K|0\rangle$ is a dual vacuum and satisfies

$$\gamma^a K |0\rangle = 0. \quad (A.33)$$

A.3 A-product. An $O(D, D)$ transformation of a spinor is given by

$$\varphi \mapsto \exp \left( \frac{1}{4} \Lambda_{AB} \Gamma^{AB} \right) \varphi, \quad (A.34)$$

where $\Lambda_{AB} = -\Lambda_{BA}$. We define the A-product $(\cdot, \cdot)_A$ as an $O(D, D)$ invariant product of any two spinors $\varphi_1, \varphi_2$:

$$(\varphi_1, \varphi_2)_A = \varphi_1^+ A \varphi_2. \quad (A.35)$$

We show the $O(D, D)$ invariance of the A-product as follows,

$$(\varphi_1, \varphi_2)_A \mapsto \left( \exp \left( \frac{1}{4} \Lambda_{AB} \Gamma^{AB} \right) \varphi_1, \exp \left( \frac{1}{4} \Lambda_{A'B'} \Gamma^{A'B'} \right) \varphi_2 \right)_A$$

$$= \varphi_1^+ \exp \left( \frac{1}{4} \Lambda_{AB} \Gamma^{AB} \right) A_+ \exp \left( \frac{1}{4} \Lambda_{A'B'} \Gamma^{A'B'} \right) \varphi_2$$

$$= \varphi_1^+ A_+ \varphi_2$$

$$= (\varphi_1, \varphi_2)_A. \quad (A.36)$$

The A-product of two Majorana spinors $\varphi_1, \varphi_2$ is real:

$$(\varphi_1, \varphi_2)_A = \varphi_1^\alpha e_\alpha \varphi_2^\beta e_\beta = \varphi_1^\alpha A_{\alpha \beta} \varphi_2^\beta \in \mathbb{R}. \quad (A.37)$$

As we see from (Eq. 4.42), $(|0\rangle, K|0\rangle)_A$ includes the measure. The representation of $(|0\rangle, K|0\rangle)_A$ is given by

$$(|0\rangle, K|0\rangle)_A = \langle 0| A_+ K |0\rangle. \quad (A.38)$$

Since $A_+$ and $K$ are constant matrices, $|0\rangle$ has to be a half density.

$$|0\rangle = \sqrt{dX e^{-2d}} \sqrt{\det \eta_{NM}} |0\rangle, \quad (A.39)$$

where $|0\rangle$ is a constant spinor and satisfies

$$\langle 0| A_+ K |0\rangle = c_0. \quad (A.40)$$
A.4 AK-product. We can define the AK-product \((- , -)_{AK}\) as the \(O(1, D-1) \times O(D-1, 1)\) invariant product of any two spinors \(\varphi_1, \varphi_2\):

\[
(\varphi_1, \varphi_2)_{AK} = \varphi_1^{\dagger} A K \varphi_2 .
\]

We can see its \(O(1, D-1) \times O(D-1, 1)\) invariance as:

\[
(\varphi_1, \varphi_2)_{AK} \mapsto (\exp \left( \frac{1}{4} \Lambda_{AB} \Gamma^{AB} \right) \varphi_1, \exp \left( \frac{1}{4} \Lambda_{A'B'} \Gamma^{A'B'} \right) \varphi_2)_{AK} = \varphi_1^{\dagger} A_+ K \varphi_2
\]

\[
= \varphi_1^{\dagger} \exp \left( - \frac{1}{4} \Lambda_{AB} \Gamma^{AB} \right) A_+ \exp \left( \frac{1}{4} \Lambda_{A'B'} \Gamma^{A'B'} \right) \varphi_2
\]

\[
= \varphi_1^{\dagger} \Lambda_{AB} \Gamma^{AB} \exp \left( \frac{1}{4} \Lambda_{A'B'} \Gamma^{A'B'} \right) \varphi_2
\]

\[
= (\varphi_1, \varphi_2)_{AK} ,
\]

where \(\Lambda_{AB}\) satisfies

\[
\Lambda_{AB} = - \Lambda_{BA} , \quad \Lambda_{AB} = H_{A'}^{\ A} H_{B'}^{\ B} \Lambda_{A'B'} .
\]

As an A-product, the AK-product is real.

\[
(\varphi_1, \varphi_2)_{AK} \in \mathbb{R} .
\]

B. Reduction of \(S_{RR}\) to DFT\(_{sec}\)

In this section, we show that for the case of a flat background \(S_{RR}\) reduces to DFT\(_{sec}\). For this end we consider the case where the metric and the structure function \(\phi'\) are given by

\[
\eta_{MN} = \begin{pmatrix} 0 & \delta^m_n \\ \delta_m^n & 0 \end{pmatrix} , \quad \phi'_{LM} = 0 .
\]

Then, the vielbein \(E_A^N\) is given by

\[
E_A^N = \begin{pmatrix} e^{-T_a} & 0 \\ 0 & e^m_a \end{pmatrix} \begin{pmatrix} \delta^m_n & 0 \\ -B_{mn} & \delta_m^n \end{pmatrix} .
\]

For the purpose to define the B-transformation later, we separate \(E_A^N\) into a \(GL(D)\) part \(E_A^{(e) M}\) and a B-field part \(E_A^{(B) N}\) as
Using these spin operators, we define the spin operator $S$ related to $E^{(c)}$ and $E^{(B)}$, respectively, can be defined as

$$S_{E^{(c)}} = S_{E^{(B)}} = e^{-\frac{1}{2} \beta_a \gamma^a_b} = e_{a}^{m} \delta_{m}^{b},\quad (\gamma_{(c)}^{a})_{b} = e_{a}^{m} \delta_{m}^{b},$$

$$S_{E^{(B)}} = e^{\frac{1}{2} B_{mn} \delta_{m}^{a} \delta_{n}^{b} \gamma^{ab}},$$

Using these spin operators, we define the spin operator $S_E$ related to $E_A^N$ as

$$S_E = S_{E^{(B)}} S_{E^{(c)}} = \delta_A^M E^N_M \delta_B^N \gamma_B,$$

With this $S_E$, the corresponding Dirac generating operator $\mathcal{D}$ denoted by $\partial_N := E^{-1}_N \partial_A$ is

$$e^{-d} S_{E} \mathcal{D}(e^{-d} S_{E})^{-1} = \frac{1}{2} \gamma^A \delta_A^N \partial_N =: \mathcal{D}_0.$$ 

$\mathcal{D}_0$ is a Dirac operator in the local coordinate basis. To see the relation between $S_{RR}$ and DFT$_{sec}$, we rewrite the action of the R–R sector $S_{RR}$ by $\mathcal{D}_0$:

$$S_{RR} = \beta_{RR}(\mathcal{D}_{\chi}, \mathcal{D}_{\chi})_{\chi} \chi$$

$$= \beta_{RR}(\mathcal{D}_{\chi}) \chi = \beta_{RR}(e^{-d} S_{E})^{-1} \mathcal{D}_0 (e^{-d} S_{E}) \chi$$

$$= \beta_{RR}(e^{-d} S_{E})^{-1} \mathcal{D}_0 (e^{-d} S_{E}) \chi$$

$$= \beta_{RR}(S_{E^{(B)}} \mathcal{D}_0 \chi) \chi = e^{2d} A_k S_{E^{(c)}} K S_{E^{(c)}}^{-1} S_{E^{(B)}}^{-1} \mathcal{D}_0 \chi,$$

where $\chi$ is defined by

$$\chi = e^{-d} S_{E} \chi.$$ 

The coefficient of the R–R flux is defined by

$$\sum_{p} \frac{1}{\sqrt{p}} F_{m_1 \cdots m_p}^{RR} \delta_{a_1}^{m_1} \cdots \delta_{a_p}^{m_p} \gamma^{a_1 \cdots a_p} |0\rangle := \sqrt{2} S_{E^{(B)}}^{-1} \mathcal{D}_0 \chi$$

$$= e^{-\frac{1}{2} B_{mn} \delta_{m}^{a} \delta_{n}^{b} \gamma^{ab}} \frac{1}{\sqrt{2}} \gamma^C \delta_C^N \partial_N \chi.$$ 

(B.15)
This defines the RR field $\tilde{F}^{RR}$ from the spinor $\tilde{\chi}$. It is the equivalent to the relation used in DFT$_{sec}$ [63]. Using $\tilde{F}^{RR}$, the action of the R–R sector becomes

$$S^{RR}_{DFT} = \beta^{RR} (S^{-1}_{E(c)} \delta_0 \tilde{\chi}) + e^{2d} A_S S^{E(c)} K S^{-1}_{E(c)} (S^{-1}_{E(c)} \delta_0 \tilde{\chi})$$

$$= \frac{1}{2} \beta^{RR} \left( \sum_p \frac{1}{\sqrt{2^p p!}} \tilde{F}^{RR}_{m_1 \ldots m_p} \gamma^{a_1 \ldots a_p} |0\rangle \right) + e^{2d} A_S S^{E(c)} K S^{-1}_{E(c)}$$

$$= \frac{1}{2} \beta^{RR} \left( \sum_{m_1 \ldots m_p, q_1 \ldots q_l} \frac{1}{\sqrt{2^{m_1 + \ldots + m_p + q_1 + \ldots + q_l} (m_1 + \ldots + m_p)! (q_1 + \ldots + q_l)!}} \tilde{F}^{RR}_{m_1 \ldots m_p} \tilde{F}^{RR}_{q_1 \ldots q_l} \gamma^{a_1 \ldots a_p} \gamma^{b_1 \ldots b_q} |0\rangle \right)$$

$$= \frac{1}{2} \beta^{RR} \left( \sum_{m_1 \ldots m_p, q_1 \ldots q_l} \frac{1}{\sqrt{2^{m_1 + \ldots + m_p + q_1 + \ldots + q_l} (m_1 + \ldots + m_p)! (q_1 + \ldots + q_l)!}} \tilde{F}^{RR}_{m_1 \ldots m_p} \tilde{F}^{RR}_{q_1 \ldots q_l} \gamma^{a_1 \ldots a_p} \gamma^{b_1 \ldots b_q} |0\rangle \right)$$

$$= \frac{1}{2} \beta^{RR} \left( \sum_{m_1 \ldots m_p, q_1 \ldots q_l} \frac{1}{\sqrt{2^{m_1 + \ldots + m_p + q_1 + \ldots + q_l} (m_1 + \ldots + m_p)! (q_1 + \ldots + q_l)!}} \tilde{F}^{RR}_{m_1 \ldots m_p} \tilde{F}^{RR}_{q_1 \ldots q_l} \gamma^{a_1 \ldots a_p} \gamma^{b_1 \ldots b_q} |0\rangle \right)$$

where we used

$$S^{-1}_{E(c)} \langle 0 | = (\det g_{mn})^{\frac{1}{2}} |0\rangle.$$  \hspace{1cm} (B.17)

To compare with the action in DFT$_{sec}$, we determine the constant $\beta^{RR}$ as

$$\beta^{RR} = -\frac{1}{2c_0},$$  \hspace{1cm} (B.18)

which yields the action of R–R sector in a flat space as:

$$S^{RR} = -\frac{1}{4} \int dX \sum_p \sqrt{\det G_{ll'}} \frac{1}{p!} \tilde{F}^{RR}_{m_1 \ldots m_p} \tilde{F}^{RR}_{n_1 \ldots n_p} G^{m_1 n_1} \ldots G^{m_p n_p}.$$  \hspace{1cm} (B.19)
Thus, the action of the R–R sector (7.3) reduced to a flat background is consistent with the results given in the literature for DFT_{sec}.

### C. GSE and DFT

The Generalized Supergravity Equations (GSE) are defined by

\[
R^{(e)} + 4 \nabla^m \partial_m \phi - 4 |\partial \phi|^2 - \frac{1}{2} |H|^2 - 4 (I^m I_m + U^m U_m + 2 U^m \partial_m \phi - \nabla_m U^m) = 0,
\]

\[
R_{mn}^{(e)} - \frac{1}{4} H_{mpq} H_{n}^{pq} + 2 \nabla_m \partial_n \phi + \nabla_m U_n + \nabla_n U_m = 0,
\]

\[
- \frac{1}{2} \nabla^k H_{kmn} + \partial_k \phi H^k_{mn} + U^k H_{kmn} + \nabla_m I_n - \nabla_n I_m = 0.
\]

(C.1)

Here \( R^{(e)} \) is the Ricci scalar given by the \( D \)-dimensional vielbein and \( I = I^m \partial_m \) is a constant Killing vector which satisfies

\[
L_I G = 0, \quad L_I B = 0, \quad L_I \phi = 0.
\]

(C.2)

\( U_m \) is defined by

\[
U_m = I^n B_{nm}.
\]

(C.3)

It was shown in [29] that the GSE can be derived from the DFT_{sec} by taking an ansatz

\[
H_{MN} = \begin{pmatrix} G^{-1} & -G^{-1} B \\ B G^{-1} & G - B G^{-1} B \end{pmatrix}, \quad d = \phi - \frac{1}{4} \log(\det G) + I^m \tilde{x}_m.
\]

(C.4)

On the other hand, we would like to use a different ansatz, to obtain the Poisson–Lie T-duality in which the sign of \( B_0 \) equals to that of \( \Pi \) and \( \tilde{\Pi} \) as in section 6.1 Since a change of the ansatz \((B, I) \rightarrow (-B, -I)\) respects the GSE (C.1), we use here the ansatz

\[
H_{MN} = \begin{pmatrix} G^{-1} & G^{-1} B \\ -B G^{-1} & G - B G^{-1} B \end{pmatrix}, \quad (C.5)
\]

\[
d = \phi - \frac{1}{4} \log(\det G) - I^m \tilde{x}_m.
\]

(C.6)

Moreover, when \( I^m \) is not constant, we use the following redefinition of \( F_A \):

\[
d = \phi - \frac{1}{4} \log(\det G),
\]

\[
F_M = 2 \partial_M d - 2 I_M.
\]

(C.7)

where \( I_M = (I^m, I_m) \) as in the modified DFT.

Thus, in this paper, we denote the DFT action \( \mathcal{I}(0, 8e_{0}^{-1}) \) using the ansätze (C.5), (C.7) which derives the GSE of \( \mathcal{S}^{\text{DFT}_{sec}}_{\text{mod}}[E_A^M, d, I^m] \). The resulting action coincides with the modified DFT action defined in [30], and thus is giving the missing algebraic background of their modification for the Drinfel’d double case.
References

1. Carow-Watamura, U., Miura, K., Watamura, S., Yano, T.: Metric algebroid and Dirac generating operator in Double Field Theory. https://doi.org/10.1007/JHEP10(2020)192
2. Hitchin, N.: Brackets, forms and invariant functionals. Asian J. Math 10(3), 541–560 (2006). arXiv:arXiv:math/0508618 [math.DG]
3. Gualtieri, M.: Generalized complex geometry. arXiv:math/0703298 [mathDG]
4. Courant, T.J.: Dirac manifolds. Trans. Am. Math. Soc. 319(2), 631–661 (1990). https://doi.org/10.1090/S0002-9947-1990-0998124-1
5. Coimbra, A., Strickland-Constable, C., Waldram, D.: Supergravity as generalized geometry I: type II theories. J. High Energy Phys. 11(11), 091 (2011). https://doi.org/10.1007/JHEP11(2011)091. arXiv:1107.1733
6. Coimbra, A., Strickland-Constable, C., Waldram, D.: Generalised Geometry and type II Supergravity. Fortschr. Phys. 60(9–10), 982–986 (2012). arXiv:1202.3170
7. Hitchin, N.: Lectures on generalized geometry. Surv. Differ. Geom. 16(1), 79–124 (2011). https://doi.org/10.4310/SDG.2011.v16.n1.a3
8. Bowkknegt, P.: Lectures on cohomology, T-duality, and generalized geometry. Lect. Notes Phys. 807, 261–311 (2010). https://doi.org/10.1007/978-3-642-11897-5_5
9. Siegel, W.: Superspace duality in low-energy superstrings. Phys. Rev. D 48(6), 2826–2837 (1993). https://doi.org/10.1103/PhysRevD.48.2826
10. Hull, C., Zwiebach, B.: The gauge algebra of double field theory and Courant brackets. JHEP 09, 090 (2009). arXiv:0908.1792
11. Hohm, O., Hull, C., Zwiebach, B.: Generalized metric formulation of double field theory. J. High Energy Phys. 08(8), 008 (2010). https://doi.org/10.1007/JHEP08(2010)008. arXiv:1006.4823
12. Zwiebach, B.: Doubled field theory, T-duality and courant-brackets. Lect. Notes Phys. 851, 265–291 (2012). https://doi.org/10.1007/978-3-642-25947-0_7
13. Aldazabal, G., Marqués, D., Núñez, C.: Double field theory: a pedagogical review. Class. Quantum Gravity 30(16), 163001 (2013). https://doi.org/10.1088/0264-9381/30/16/163001. arXiv:1305.1907
14. Deser, A., Stasheff, J.: Even symplectic supermanifolds, derived brackets and even symplectic supermanifolds, Dissertation (1999). arXiv:math/9910078
15. Cattaneo, A.S., Felder, G.: On the AKSZ formulation of the Poisson sigma model. Lett. Math. Phys. 56(2), 163–179 (2001). https://doi.org/10.1023/A:1010963926853
16. Ikeda, N.: Lectures on AKSZ sigma models for physicists. Noncomm. Geometry Phys. 4, 79 (2017). https://doi.org/10.1142/9789813144613_0003. arXiv:1204.3714
17. Deser, A., Stasheff, J.: Even symplectic supermanifolds and double field theory. Commun. Math. Phys. 339(3), 1003–1020 (2015). https://doi.org/10.1007/s00220-015-2443-4
18. Carow-Watamura, U., Ikeda, N., Kaneko, T., Watamura, S.: Higher gauge theories from Lie n-algebras and off-shell covariantization. J. High Energy Phys. 07(7), 125 (2016). https://doi.org/10.1007/JHEP07(2016)125. arXiv:1606.03861
19. Deser, A., Sámann, C.: Extended Riemannian geometry I: local double field theory. Ann. Henri Poincaré 19(8), 2927–2946 (2018). https://doi.org/10.1007/s00023-018-0694-2
20. Carow-Watamura, U., Ikeda, N., Kaneko, T., Watamura, S.: DFT in supermanifold formulation and group manifold as background geometry. J. High Energy Phys. 04, 002 (2019). https://doi.org/10.1007/JHEP04(2019)002. arXiv:1812.03464
21. Alekseev, A., Xu, P.: Derived brackets and Courant algebroids, unpublished (2001). Available at http://www.math.psu.edu/ping/anton-final.pdf
22. Tseytlin, A.A., Wulff, L.: Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations. J. High Energy Phys. 06(6), 174 (2016). https://doi.org/10.1007/JHEP06(2016)174. arXiv:1605.04884
23. Arutyunov, G., Frolov, S., Hoare, B., Roiban, R., Tseytlin, A.: Scale invariance of the η-deformed AdS5 × S5 superstring, T-duality and modified type II equations. Nuclear Phys. B 903, 262–303 (2016)
29. Sakamoto, J., Sakatani, Y., Yoshida, K.: Weyl invariance for generalized supergravity backgrounds from the doubled formalism. Prog. Theor. Exp. Phys. 2017(5), 053B07 (2017). https://doi.org/10.1093/ptep/ptx067
30. Y. Sakatani, S. Uehara, K. Yoshida, "Generalized gravity from modified DFT. J. High Energy Phys. 2017(4), (2017). https://doi.org/10.1007%2Fjhep04%282017%292123
31. Chatzistavrakis, A., Jonke, L., Khoo, F.S., Szabo, R.J.: Double field theory and membrane sigma-models. J. High Energy Phys. 2018(7), (2018). https://doi.org/10.1007/JHEP07(2018)015
32. Mori, H., Sasaki, S.: More on double aspects of algebroids in double field theory. J. Math. Phys. 61, 123504 (2020). arXiv:2008.00402
33. Liu, Z.-J., Weinstein, A., Xu, P.: Manin triples for lie bialgebroids. J. Differ. Geometry 45(3), (1997). https://doi.org/10.4310/jdg/1214447806
34. Vaisman, I.: Transitive Courant algebroids. Int. J. Math. Math. Sci. 2005(11), 1737–1758 (2005). https://doi.org/10.1155/ijmms.2005.1737
35. Hull, C., Zwiebach, B.: Double Field Theory. JHEP 09, 099 (2009). arXiv:0904.4664 [hep-th]
36. Bruce, A.J., Grabowski, J.: Pre-Courant algebroids. J. Geometry Phys. 142, 254–273 (2019)
37. Garcia-Fernandez, M.: Ricci flow, Killing spinors, and T-duality in generalized geometry. Adv. Math. 350, 1059–1108 (2019). arXiv:1611.08926
64. Jeon, I., Lee, K., Park, J.-H.: Ramond–Ramond Cohomology and O(D, D) T-duality. JHEP 09, 079 (2012). arXiv:1206.3478

65. Fukuma, M., Oota, T., Tanaka, H.: Comments on T dualities of Ramond–Ramond potentials on tori. Prog. Theor. Phys. 103, 425–446 (2000). arXiv:hep-th/9907132 [hep-th]

Communicated by Y. Kawahigashi