SPECTRAL SEQUENCES OF COLORED JONES POLYNOMIALS, COLORED RASMUSSEN INVARIANTS AND NANOPHRASES

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Abstract. We introduce three spectral sequences which give some expressions of colored Jones polynomials. Each spectral sequence contains a Khovanov-type homology groups. Two of them are derived from a bicomplex of the colored Jones polynomial. The other is the spectral sequence that deduces a colored Rasmussen invariant of links. We also introduce three functors between categories of nanophrases, generalizations of links, and obtain their applications using colored Jones polynomials and their categorifications.

1. Three spectral sequences for the colored Jones polynomial.

In the article [5], we gave the existence of a bicomplex with respect to two grades $k$ and $i$ preserving $j$ where $k$ is a homological degree of the cabling, $i$ is another homological degree derived from the Khovanov homology of the Jones polynomial and $j$ is $q$-degree. This implies the existence of spectral sequences of the bicomplex. These sequences contain the Khovanov homology groups. On the other hand, in link homology theory, there is the well-known spectral sequence that produce the Rasmussen invariant of knots. In this paper, we consider a “colored” version of it for links. Then, first, we introduce these spectral sequences. By using the symbols in [5, Section 2], the total complex for the bicomplex with two grades $k$ and $i$ is denoted by $\{C_n^{k,i}(D), d^{k,i}, d'^{k,i}\}$, then the differential is $\sum_{n=k+i} (d^{k,i} + d'^{k,i})$ by the definition. The symbol $d (=\delta_{0,0}$ in [5]) stands for the differential of the Khovanov homology of the Jones polynomial.

Theorem 1. There exist three spectral sequences (1)–(3) for the diagram $D$ of an arbitrary link $L$.

1. $\{E_r^{k,i}\}_{r=0}^{\infty}$ preserving $j$, satisfying $E_0^{k,i} \simeq C_n^{k,i,j}(D)$, $E_1^{k,i} \simeq \mathcal{H}^i(C_n^{k,j}(D), d^{mk,i})$, $E_2^{k,i} \simeq \mathcal{H}^k(\mathcal{H}^i(C_n^{k,j}(D), d^{mk,i}), d'^{k,i})$ and converging to $\mathcal{H}^i(\text{Total}(C_n^{k,j}(D)), \Phi + d)$;

2. $\{E_r^{i,k}\}_{r=0}^{\infty}$ preserving $j$, satisfying $E_0^{i,k} \simeq C_n^{k,i,j}(D)$, $E_1^{i,k} \simeq \mathcal{H}^k(C_n^{k,i,j}(D), d^{mk,i})$, $E_2^{i,k} \simeq \mathcal{H}^i(\mathcal{H}^k(C_n^{k,i,j}(D), d^{mk,i}), d'^{k,i})$ and converging to $\mathcal{H}^i(\text{Total}(C_n^{k,j}(D)), \Phi + d)$;

3. $\{E_r^{i,j}\}_{r=0}^{\infty}$ preserving $k$, satisfying $E_0^{i,j} \simeq C_n^{k,i,j}(D)$, $E_1^{i,j} \simeq \mathcal{H}^i(C_n^{k,j}(D), \Phi + d)$

where the coefficient is $\mathbb{Z}$ for the first two cases and is $\mathbb{Q}$ in the last sequence.

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Proof. The existences of the first two sequences are directly deduced by [3, Theorem 2]. Third spectral sequence is constructed by the direct summation of the original Rasmussen’s case as follows. Let the complex $C_n^{k,i}(D)$ be $\oplus_{s \in I_k, |k|=k} C^i(D^s)$ where $D^s$ is isotopic to $D^{n-2k}$ and $C^i(D)$ of a link diagram $D$ is the direct summation of the original Khovanov complex $\oplus_j C^{i-j}(D)$ of the Jones polynomial. By the definition, the number of $\{s \in I_k\} = \prod_i (n_{k_i}^{k})$. For each of copies, we can obtain the spectral sequence to define the Rasmussen invariant [13, 20]. Every differential $d_r$ of the spectral sequence preserves $s$. Then the spectral sequence starting at $C_n^{k,i}(D)$ converge to the direct summation for $s$ of $H^i(C(D), \Phi + d)$ ("Kh" in [13, 20]). Therefore we conclude the third statement [3].

Corollary 1. \{$E_r\}_{r=1}^{\infty}$, \{$''E_r\}_{r=2}^{\infty}$ and \{$E_r\}_{r=1}^{\infty}$ are link invariants.

Proof. For the first and third sequences, the statements are directly deduced by the second and third Reidemeister invariances of the Khovanov homology of the Jones polynomial. The proof of the statement for the second sequence is given as follows. First, consider the group $''E_1 \simeq H^k(C^k,i,j(D), d^{mk,i})$. This group is represented as the quotient group $\bar{C}_n(D)/\sim \simeq H^k(C^k,i,j, d^{mk,i})$ since this group is finitely generated. Then, since the retractions and chain homotopy maps [3] giving the invariance under each of the second and third Reidemeister moves $D \simeq D'$ of $H^i(C^{i,j}(D))$ can also be used for that of $C_n^{k,i,j}(D)$ [4], the isomorphism $H^i(C_n^{k,i,j}(D)) \simeq H^i(C_n^{k,i,j}(D'))$ implies $H^i(C_n^{k,i,j}(D)/\sim) \simeq H^i(C_n^{k,i,j}(D')/\sim)$. Then we have the invariance of $''E_2$.

Corollary 2.

\[
J_n(L) = \sum_j q^j \sum_{i,k} (-1)^{j+k} \text{rk}''E_2^{i,k} \\
= \sum_j q^j \sum_{i,k} (-1)^{j+k} \text{rk}''E_2^{i,j} \\
= \sum_j q^j \sum_{i,k} (-1)^{j+k} \text{rk} E_2^{i,j}.
\]

2. A colored Rasmussen invariant of links.

Let us recall the definition of the Rasmussen invariant of knots. Rasmussen considered the spectral sequence with the coefficient $\mathbb{Q}$ beginning with the Khovanov complex $C^{i,j}(D)$ of the Jones polynomial and converging to the Lee homology group $Kh'(K)$ of a knot $K$ [13]. This spectral sequence is derived from the filtered spaces by $q$-gradings.

\[
\{0\} = V_n \subset V_{n-1} \subset \ldots \subset V_m = Kh'(K)
\]

where each $V_p$ is the subspace of $Kh'(K)$ generated by enhanced Kauffman states \{$S \mid j(S) \leq p$\}.

For an element of $x$ of the Lee homology group, $q$-grading of $x$ is $p$ if $x \in V_p/V_{p+1}$. By using the well-known fact $Kh'(K) \simeq \mathbb{Q} \oplus \mathbb{Q}$ and more facts, $s(K)$ is defined as $s_{\text{min}}(K) + 1$ where $s_{\text{min}}(K) = \min\{q\text{-grading } s(x) \mid x \in Kh', x \neq 0\}$. A. Beliakova and S. Wehrli extend the original Rasmussen invariant of knots to that of links [1].
Let $s$ and $s_\bar{o}$ be canonical generators of the Lee homology corresponding to the orientation $o$ of a given oriented link $L$ and to opposite orientation $\bar{o}$, respectively [9, 13, 19]. The Rasmussen invariant of a oriented link $L$ is defined as follows.

**Definition 1** (Beliakova, Wehrli).

\[
s(L) = \frac{\deg(s_o + s_{\bar{o}}) - \deg(s_o - s_{\bar{o}})}{2}
\]

where “deg(x)” means $q$-grading of $x$.

In this section, we denote $s(L)$ for a oriented link $L$ with $o$ by $s(L, o)$. For a $l$-component framed link $L$, the $(m_1, m_2, \ldots, m_l)$-cable of the diagram $D_L$ of $L$ is defined by replacing $i$-th component of $D_L$ by $m_i$ parallel strands, pushed off in the direction of the normal vector at every point of $D_L$. Let $m$ be $(m_1, m_2, \ldots, m_l)$. The $m$-cable of a $l$-component framed link $L$ is given by taking each $m_i$-cable of the diagram of $L$ using the blackboard framing.

Now we define a colored Rasmussen invariant of links.

**Definition 2.** For $l$-component unoriented framed link $L$, let $D_L$ be a 0-framing link diagram of $L$. We define a colored Rasmussen invariant by the set \( \{ s(D_L^{n_1, \ldots, n_l, k_1, \ldots, k_l}) \} \) where $n$ and $k$ corresponding to the colored Jones polynomial with a fixed orientation $o_0$ as follows.

\[
J_n(L) = \lfloor \frac{n}{2} \rfloor \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{|k|} \binom{n-k}{k} J(L^{n-2k})
\]

where $n = (n_1, n_2, \ldots, n_l)$, $k = (k_1, k_2, \ldots, k_l)$ with $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $|k| = \sum_i k_i$, $\binom{n-k}{k} = \prod_{i=1}^{l} (\binom{n_i-k_i}{k_i})$, $J(L)$ is the Jones polynomial and the orientation of $n_i - 2k_i$ strands are oriented by alternating the original and opposite direction starting with original direction. We oriented these strands in the direction of the normal vector.

**Remark 1.** In the aspect of the spectral sequences derived from filtered spaces such as (2), we should consider the filtered spaces (3)

\[
\{0\} = V_0 \subset V_{-1} \subset \cdots \subset V_m = Kh'(L)
\]

for a link $L$. Since the spectral sequence (3) converges to $Kh'(L^{n, k}) \oplus \binom{n, k}{k}$, it is natural to define a colored Rasmussen invariant of links by $\binom{n, k}{k} s(D_{L}^{n, k})$. But for getting simply, we defined it as above.

### 3. A GENERALIZATION TO A THEORY OF NANOPHRASES

V. Turaev [15–17] introduced a universal category\(^1\), which objects and morphisms are called nanophrases and homotopies. This category is a generalization of that of links and curves [15] (See also [3]). In this section, we use the same terminology and symbols as in [2].

A functor $\mathcal{U}_L$ from the category of (nanophrases over any $\alpha$, $\Delta_\alpha$) to (nanophrases over $\alpha_0$, $S_0$) is introduced by [2]. In order to treat more general or various invariants\(^2\),

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\(^1\)T. Kadokami introduced an explanation of the term “nanophrases” using categories in his talk.

\(^2\)A. Gibson pointed out the possibility of a variation of $\mathcal{U}_L$ for a talk about [2] by the author.
we introduce three functors \( \mathcal{V}_{L_{1}\subseteq L} \), \( \mathcal{V}_{L}^{1} \) and \( \mathcal{V}_{L}^{2} \) in this section. To do this, first, we extend \( \Delta_{a} \)-homotopy to a “knotlike” \( S_{2} \)-homotopy.

**Definition 3.** Assume that \( \nu\tau = \tau\nu \) where \( \nu \) is an involution \( \alpha \rightarrow \alpha \) for \( \nu \)-shift. The \( S_{2} \)-homotopy is defined by \( S_{2} \subseteq \alpha^{3} \) such that

\[
S_{2} = \{(a, a, a), (a, a, b), (b, a, a), (b, a, b), (a, b, b), (b, b, b)| \nu\tau(b) = a\}.
\]

Such a homotopy data is denoted by \( \mathcal{P}(\alpha, S_{2}, \tau, \nu) \).

In the rest of paper, we consider \( S_{2} \) which has a mapping from \( S_{2} \) onto \( S_{1} \) (The definition of \( S_{1} \) is in [15]). We denote such a \( S_{2} \) by the same symbol “\( S_{2} \)” below.

**Remark 2.** \( \tau\nu \) becomes an involution.

**Remark 3.** Usually, we omit \( \tau \) for a category of nanophrases such as “\( \mathcal{P}(\alpha, S, \nu) \)” (See [15]).

**Example 1.** \( (\alpha_{x}, S_{x}, \nu), (\alpha_{0}, S_{0}, \nu_{0}) \) where \( \nu_{0} \) is an involution; \( a \mapsto b \) in [15] are examples of \( (\alpha, S_{2}, \tau, \nu) \).

Second, recall [2, Section 6]. For an arbitrary alphabet \( \alpha \) and an arbitrary nonempty subset \( L \) of \( \text{sign}_{L} \), we take a nonempty subset \( L_{1} \subseteq \text{crs}(L/\nu) \). By replacing \( \tau \) with \( \nu \) and \( \alpha \) with \( L \) on [2, Section 6, Definition 6.1–6.3], we get \( \text{sign}_{L_{1}} \). In the same way, for \( L_{1} \), \( \text{sign}_{\tau(L_{1})} \) is defined. We define the second sign denoted by \( \text{sign}(\cdot; L_{1} \subseteq L) \) with involutions \( \tau_{i} (i = 2, 3, \ldots, i) \) as above.

Using the couple \((., \cdot) := (\text{sign}_{L_{1}}(\cdot), \text{sign}(\cdot; L_{1} \subseteq L))\), we define a functor \( \mathcal{V}_{L_{1}\subseteq L} \) in the same manner as in [2, Definition 6.3]. To eliminate ambiguity, \( \mathcal{P}(\alpha_{x}, S_{x}, \nu) \) in [15] is denoted by \( \mathcal{P}(\alpha_{x}, S_{x}, \tau_{x}, \nu_{x}) \) in Definition 5. The index \( k \) of \( \mathcal{P}(\alpha, S_{1}, \tau, \nu) \) stands for the length \( k \) of nanophrases in this set.

**Definition 5.** For an arbitrary \( (\alpha, \tau) \) and an arbitrary nonempty subset \( L \subseteq \text{crs}(\alpha/\tau) \), we take an arbitrary nonempty subset \( L_{1} \) using \( \nu \) as above. \( \mathcal{V}_{L_{1}\subseteq L} : \mathcal{P}(\alpha, S_{2}, \tau, \nu) \rightarrow \mathcal{P}(\alpha_{x}, S_{x}, \tau_{x}, \nu_{x}) \) is defined by the following two steps. First, remove \( A \in \mathcal{A} \) such that \( (., \cdot) = (0, \ast) \) or \( (\ast, 0) \) from \( (\mathcal{A}, P) \in \mathcal{P}(\alpha, S_{2}, \tau, \nu) \) and then we get \( (\mathcal{A}', P') \). Second, consider an \( \alpha_{x} \)-alphabet \( \mathcal{B} \) such that \( \text{card}\mathcal{B} = \text{card}\mathcal{A}' \) and \( \mathcal{A}' \cap \mathcal{B} \) is the empty set. Transpose each letter \( A \) of \( (\mathcal{A}', P') \) and a letter \( B \in \mathcal{B} \) as follows. If \( A \) satisfies \( (., \cdot) = (1, 1) \), \( A \mapsto B \) with \( |B| = a_{+} \) and if \( (., \cdot) = (1, -1) \), \( A \mapsto B \) with \( |B| = b_{+} \). Similarly, if \( (., \cdot) = (-1, 1) \) (resp. \( (-1, -1) \)), \( A \mapsto B \) with \( |B| = a_{-} \) (resp. \( b_{-} \)). By two steps above, we have the nanophrase over \( \alpha_{x} \) projected from \( (\mathcal{A}, P) \). We denote it by \( \mathcal{V}_{L_{1}\subseteq L} \) and regard \( \mathcal{V}_{L_{1}\subseteq L} \) as a functor.
Remark 6. This definition is similar to that of $U_L$ [2].

Theorem 2. If $(A_1, P_1) \simeq_{S_1} (A_2, P_2)$, then $V_{L_{1\subset L}}((A_1, P_1)) \simeq_S V_{L_{1\subset L}}((A_2, P_2))$.

Proof. This proof is similar to [2] Theorem 6.1. In fact, for the isomorphism and the first move, the proof of [2] Theorem 6.1 can be used. But in the other moves, we have to handle not only two signs but also at least two letters. Then to give proof simply, we use the symbol $(\cdot, \cdot)(A) := (\text{sign}_L(A), \text{sign}(A; L_1, L))$ below. It is sufficient to show that two (resp. three) letters survive synchronously for the second (resp. third) move. More precisely, for the second (resp. third) move, it is sufficient to prove that $(\cdot, \cdot)(A) = (\pm 1, \pm 1) \leftrightarrow (\cdot, \cdot)(B) = (\pm 1, \pm 1)$ (and also $(\cdot, \cdot)(C) = (\pm 1, \pm 1)$ for the third move) under the condition $|A| = \tau(|B|)$ (resp. $(|A|, |B|, |C|) \in S_2$). We will give a proof for the second move. It is easy to prove $(\Rightarrow)$ by checking four cases of $(\pm 1, \pm 1)$. For example, if $(\cdot, \cdot)(A) = (1, 1)$, $|A| \in L_1$, then $|B| \in \tau(L_1)$ by using $|A| = \tau(|B|)$. The other three cases can be treated similarly. The proof of $(\Leftarrow)$ consists of two parts. First, we assume that $(\cdot, \cdot)(A) = (\ast, \ast)$. By the assumption, $\nu(|A|) = |A|$. Therefore $\nu(|B|) = \nu(\tau(A)) = \tau(\nu(|A|)) = \tau(|A|) = |B|$. Then $(\cdot, \cdot)(B) = (\ast, \ast)$. By assumption, $\tau(|A|) | A$. Therefore $\tau(|B|) = \tau(\tau(|A|)) = \tau(|A|) = |B|$. Then $(\cdot, \cdot)(B) = (\ast, \ast)$. Next, we will give a proof for the third move. Assume that $(\cdot, \cdot)(A) = (\pm 1, \pm 1)$. Then $|A| = |B| = |C|$. If $|A| = |B| = |C|$, then $|B| = \nu(\nu(|A|)) = |A| = \nu(\nu(|A|)) = |B| = \nu(\nu(|A|))$. If $(\cdot, \cdot)(B), (\cdot, \cdot)(C) \in \{(\ast, \ast), (\ast, \ast)\}$, each of two values $(\cdot, \cdot)(B), (\cdot, \cdot)(C)$ is equal to $(\ast, \ast)$ or $(\ast, \ast)$. Next, we consider the case $|A| = |B| = \nu(\nu(|C|)).$ If $(\cdot, \cdot)(A) = (\ast, \ast)$, $|A| = \nu(|A|).$ Then $\nu(|C|) = \nu(\nu(|A|)) = \nu(\nu(|A|)) = |C|$. Therefore $(\cdot, \cdot)(B), (\cdot, \cdot)(C) \in \{(\ast, \ast), (\ast, \ast)\}$. If $|A| = \nu(|A|), \tau(|A|) = |A|.$ Then $\tau(|C|) = \tau(\nu(|A|)) = \nu(|A|) = |C|.$ Therefore $(\cdot, \cdot)(B) = (\ast, \ast)$. Finally we consider $|A| = \nu(|B|) = \nu(|B|).$ But we have considered the case $|A| = \nu(|C|)$ above, then the proof is similar to that case.

The above functor produces general results for links but deletes every letter $|A|$ with $\nu(|A|) = |A|$. Then we will consider the case that such a letter survives. Let us denote the category of phrases $P(\alpha, S_2, \tau, \nu)$ with $\nu = \id$ by $P(\alpha, S_2, \tau, \id)$.

Theorem 3. If $(A_1, P_1) \simeq_{S_1} (A_2, P_2)$, then $V^1_L((A_1, P_1)) \simeq_S V^1_L((A_2, P_2))$.

Proof. The proof of the statement for $U_L$ [2] Theorem 6.1 can be used except for some replacements. First, we should replace $U$ with $V$. For the isomorphisms, the first move and the second move, the same proof should be done. For the third move, the proof can be available by replacing the condition $|A| = |B| = |C|$ with $(|A|, |B|, |C|) \in S_2$.

Remark 7. The symbol “sign” was often used to replace “signL” in [2] Proof of Theorem 6.1.
Let $S_{12}$ be $S_2$ with $\tau = \text{id}$. For $\mathcal{V}_L^2$, we can consider a functor $\mathcal{V}_L^2$ from $\mathcal{P}(\alpha, S_{12}, \text{id}, \nu)$ to $\mathcal{P}(\alpha_2, S_2, \text{id}, \nu_2)$ by exchanging the role of $\tau$ for that of $\nu$ and replacing $\alpha_1$ with $\alpha_2$. For the values of $\text{sign}_\tau$ and $\alpha_2 = \{c, d\}$, we identify $1$ with $c$ and $-1$ and $d$. This functor $\mathcal{V}_L^2$ also deduces the next similar result.

**Theorem 4.** If $(A_1, P_1) \simeq_{S_{12}} (A_2, P_2)$, then $\mathcal{V}_L^2((A_1, P_1)) \simeq_{S_2} \mathcal{V}_L^2((A_2, P_2))$.

**Proof.** The proof is similar to that of Theorem 3, then we omit it. \qed

**Remark 8.** The category $\mathcal{P}(\alpha_2, S_2, \text{id}, \nu_2)$ produces a quandle theory–see [15]. But we do not discuss this here.

We will consider a nanoword theory of the colored Jones polynomial. When we discuss the colored Jones polynomial and its categorification of nanophrases, we have at least two choice, one is $\mathcal{V}_{L_1 \subset L}$ and another is $\mathcal{V}_L^1$. First, we use Manturov’s theory of Khovanov homology and virtual knots. Using the bijection from the set $S_2$-homotopy nanophrases over $\alpha_s$ to that of pointed virtual links, we give the following result.

**Theorem 5.** There exists a cohomology group $H_n^{k,i,j}(\mathcal{V}_{L_1 \subset L}(P))$ with the coefficient $\mathbb{Z}$ and a colored Jones polynomial $J_n(\mathcal{V}_{L_1 \subset L}(P))$ which are $S_2$-homotopy invariants of an arbitrary nanophrase $P$ over an arbitrary $\alpha$ satisfying

$$J_n(\mathcal{V}_{L_1 \subset L}(P)) = \sum_j q^j \sum_{i,k} (-1)^{i+k} \text{rk} H_n^{k,i,j}(\mathcal{V}_{L_1 \subset L}(P)).$$

**Proof.** V. O. Manturov gave the Khovanov homology $KH^{i,j}$ of virtual links with the coefficient $\mathbb{Z}$ [12]. This Khovanov homology of Manturov is invariant under virtualization. An equivalence class of virtual links by virtualization is interpreted as pseudolinks. On the other hand, there are a bijection from virtual links to nanophrases over $\alpha_s$ and the natural projection $p$ from nanophrases over $\alpha_s$ to nanophrases over $\alpha_1$ defined by $a_+ \mapsto 1$ and $a_-, b_- \mapsto -1$ [13]. Then the fact $KH^{i,j}(P) \simeq KH^{i,j}(p(P))$. Using the definition of the cabling of pseudolinks [5], Theorem 2 implies the statement above. \qed

**Corollary 3.** We have the bicomplex and three spectral sequences for nanophrases over $\alpha$ which higher terms are $S_2$-homotopy invariants corresponding to the results: Theorem 7, Corollary 7 and Corollary 2.

Second, we use the functor $\mathcal{V}_L^1(P)$. The following result is similar to the previous one but different from that since we get invariants under $\nu$-shift [13]. Note also that Manturov’s Khovanov homology [12] is invariant under virtualization, that is, it is $S_1$-invariant of pseudolinks.

**Theorem 6.** There exists a cohomology group $H_n^{k,i,j}(\mathcal{V}_L^1(P))$ with the coefficient $\mathbb{Z}$ and a colored Jones polynomial $J_n(\mathcal{V}_L^1(P))$ which are $S_2$-homotopy invariants of an arbitrary nanophrase $P$ over an arbitrary $\alpha$ satisfying

$$J_n(\mathcal{V}_L^1(P)) = \sum_j q^j \sum_{i,k} (-1)^{i+k} \text{rk} H_n^{k,i,j}(\mathcal{V}_L^1(P)).$$
Proof. Manturov’s Khovanov homology above is also $S_1$-homotopy invariant of pseudolinks [12]. Then Theorem 3 implies the statement above. □

Corollary 4. We have the bicomplex and three spectral sequences for nanophrases over $\alpha$ which higher terms are $S_{\sharp 1}$-homotopy invariants corresponding to the results: Theorem 4, Corollary 7 and Corollary 2.

Remark 9. By using the bypass via pseudolinks as above, we get the invariance under $\nu$-shift, that is, we eliminate a fixed starting letter of each component of nanophrases (The definition of “components” of nanophrases is obtained in [2, Section 2.1]). On the other hand, the Khovanov homology group by Manturov for virtual knots is also invariant of pseudolinks. The category introduced by Turaev [15] is really suitable for the Jones polynomial.

Remark 10. By using $V_{L, L} \subset L$ (resp. $V_{L, 1}$), we have the Khovanov and Lee homology of the Jones polynomial for nanophrases over $\alpha$ up to $S_{\sharp}$ (resp. $S_{\sharp 1}$) homotopy with the coefficient $\mathbb{Z}$.

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