Universal conductance fluctuations in non-integer dimensions

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Abstract

We propose an Ansatz for universal conductance fluctuations in continuous dimensions from 0 up to 4. The Ansatz agrees with known formulas for integer dimensions 1, 2 and 3, both for hard wall and periodic boundary conditions. The method is based solely on the knowledge of energy spectrum and standard assumptions. We also study numerically the conductance fluctuations in 4D Anderson model, depending on system size L and disorder W. We find a small plateau with a value diverging logarithmically with increasing L. Universality gets lost just in 4D.

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Disordered systems usually possess the metallic regime, where Ohm’s law for mean conductance \( g \) works well at least for cubic samples and the distribution of \( g \) for various realizations of the same disorder is Gaussian with constant (disorder, mean free path \( l_c \) and \( g \)) and independent) width, called universal conductance fluctuations (UCF). Higher cumulants of \( g \) should disappear as some power of \( 1/g \) [1], though recent experiment on gold wire did not confirm this in quasi-1D [2]. Recently [3] we analysed the statistical properties of conductance on bifractal lattices [4]. It became clear, that \( g \) and \( \var g \) depend (besides the spectral dimension \( d_s \)) on lattice topology. Simply speaking, bifractals are no hypercubes. But we hope that by changing other parameters, say anisotropy, we can tune the systems to cubic-like. If the UCF of anisotropic bifractals becomes those of non-integer dimensional hypercubes, other parameters of these systems may be comparable. The main goal of this work is to find a way to calculate the UCF also for non-integer dimensions. Rewriting (2) as a simple integral, we will propose its analytical continuation to real dimensions.

Two (on first sight) different expressions were given for UCF in literature [5] and [6]. Let us comment this seeming ambiguity. In classical papers of Lee, Stone and Fukuyama [5] - Appendix, formulas for UCF in 3D (2D, 1D) were given as a sum of three diagrammatic terms \( F_a, F_b \) and \( F_c \). They can be written in terms of convolutions, e. g. (the simple one-loop “Meeron” diagram):

\[
F_a = \frac{2}{L^4} \int \Pi(r, r') \Pi(r', r) \, dr \, dr'
\]

where \( \Pi(r, r') \) is standard propagator in a box, \( F_b \) and \( F_c \) contain 3 and 4 propagators (cyclically in one loop), respectively. In 1D it is possible to show analytically, that \( F_b = -F_a \) and \( F_c = 3/4 \, F_a \), with some tricky cancelling of divergences, Ref. [7]. Numerical calculations show, that the same holds for 2D and 3D. Thus we arrive at the formula, given in Ref. [6] and representing 3/4 \( F_a \) from Ref. [5]:

\[
(g^2)_c = \frac{12}{\pi^4} \sum_{i_z=1}^{\infty} \sum_{i_x=0}^{\infty} \sum_{i_y=0}^{\infty} (s^2 + i_x^2 + i_y^2)^{-2}
\]

where the number of sums defines the dimensionality, hard wall boundary conditions are applied and we limit ourselves to cubes \((L_x = L_y = L_z)\). This relation was derived using the technique of Hikami boxes. It gives in one dimension

\[
(g^2)_c^{1D} = \frac{12}{\pi^4 \zeta(4)} = \frac{2}{15}
\]

\( \zeta(4) \) is Riemann’s function. No such simple formulas were given for higher dimensions yet. It is also clear, that a fourfold sum, i. e. 4D case of Eq. (2) diverges and \( d = 4 \) plays a role of some kind “critical dimension”. According to the remark 6 of Ref. [8], it is possible that also ergodicity gets lost at that point.

Dealing with diffusive part of conductivity itself, the following formula for diagonal part of the propagator in 3D was given [9] and [10]:

\[
\Pi(r, r) = -\frac{\delta g}{g} = \frac{2}{g y^2} S(y)
\]

where \( \delta g \) is the diffusive part of conductivity and a function \( S(y) \) was specified as follows:

\[
S_{\text{HW}}(y) = \sum_{i_z=1}^{\infty} \sum_{i_x=0}^{\infty} \sum_{i_y=0}^{\infty} \exp[-\pi(i_x^2 + i_y^2 + i_z^2)y]
\]

\[
S_{\text{PBC}}(y) = \sum_{i_z=1}^{\infty} \sum_{i_x=-\infty}^{\infty} \sum_{i_y=-\infty}^{\infty} \exp[-\pi(4i_x^2 + 4i_y^2 + i_z^2)y]
\]

Here \( y = \pi \phi_d (l_c/L)^2 \), \( \phi_d \) is dimension specific, e. g. 1/3 in 3D, but for our purposes insignificant constant. HW stands for hard wall and PBC for periodic boundary conditions in directions perpendicular to the current, flowing in the \( z \) direction. Later we will give an alternative meaning to the variable \( y \). Contrary to Ref. [9] we include
zero modes in our Eq. (6), \( i_x = 0 \) and \( i_y = 0 \), similarly to Ref. [11]. Our normalization of \( y \)-zero modes in our Eq. (6), \( i \) one gets the Meeron diagram value

\[
\vartheta_3(u, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nu) \quad (7)
\]

where we set \( q = \exp(-\pi y) \) and \( u = 0 \), making (7) related to modular elliptic functions. It is worth mention that all \( \vartheta \) functions fulfil the 1D diffusion equation, \( \psi'' + c\vartheta = 0 \).

Let us recall the time \( t \) dependent retarded Green’s function (propagator):

\[
G(r, r', t) = \Theta(t) \sum_n \exp(-iE_nt)\Psi_n(r)\Psi_n^*(r') \quad (8)
\]

where \( \Theta(t) \) is Heavside step function, \( E_n \) are eigenvalues and \( \Psi_n \) orthonormal complete eigenfunctions of the proper Hamiltonian and \( \hbar = 1 \). In our scaling, e. g. 3D HW box case, \( E_n = \pi(i_1^2 + i_2^2 + i_3^2) \). One can go over to spectral representation performing a Fourier transformation

\[
G(r, r', E) = -i \int_0^{\infty} \exp(iEt)G(r, t, r', 0)dt = \sum_n \frac{\Psi_n(r)\Psi_n^*(r')}{E - E_n} \quad (9)
\]

Inserting this into (1) and exploiting the orthonormality one gets the Meeron diagram value

\[
F_a(E) = C \sum_n \frac{1}{(E - E_n)^2} \quad (10)
\]

as it was done in [5] and [6] at \( E = 0 \), see Eq. (2). The constant \( C \), including appropriate combinatoric factor, will be specified later.

We have an alternative way. Let us integrate out the space variables in advance

\[
Z(t) = \int d\mathbf{r} d\mathbf{r}' G(\mathbf{r}, 0, \mathbf{r}', 0)G(\mathbf{r}, t, \mathbf{r}', 0) = \sum_n \exp(-iE_nt) \quad (11)
\]

This simplifies the calculation, we will not have problems with the divergence of \( \Pi(\mathbf{r}, \mathbf{r}') \propto S(y) \) in more than one dimension, or with non-uniformly convergent, \( r \)-dependent series, Ref. [13]. Now we either first perform the Fourier transformation to get \( Z(E) = \sum(E - E_n)^{-1} \) and then apply a partial derivation, or we take the derivative of the Fourier integral to get

\[
F_a(0) = -C \frac{\partial Z(E)}{\partial E} \bigg|_{E=0} = C \int_0^{\infty} yZ(y)dy \quad (12)
\]

where \( y \) now means imaginary time \( y = it \), thus going over from (zero potential within a box) Schrödinger equation to the diffusion equation, or from Fourier to Laplace transformation. The quantity \( Z(y) \) strongly resembles a partition function, with \( y \) playing the role of inverse temperature.

Imry [14] also stated that variance of \( g \) over random matrix ensembles can be calculated by a similar integral formula. Our version of (1) and thus (2) with proper constant now reads:

\[
\langle g^2 \rangle_c = \frac{3F_a(0)}{4} = \frac{12}{\pi^2} \int_0^{\infty} yZ(y)dy. \quad (13)
\]

Noticing that \( Z(y) = \sum \exp(-Ey) = -S'(y)/\pi \), after per parts we get an interesting, though for calculations not quite practical formula

\[
\langle g^2 \rangle_c = \frac{12}{\pi^2} \int_0^{\infty} S(y)dy. \quad (14)
\]

Specifying \( Z(y) \) in (13) with help of \( \vartheta_3 \) function (7) we can rewrite (2) for small integer dimensions \( d \) as follows:

\[
\langle g^2 \rangle^{HW}_c = \frac{12}{2\pi^2} \int_{0}^{\infty} y[\vartheta_3(0, q) - 1]dE \quad (15)
\]

This is easy to verify: each term with \( \vartheta_3 \) creates one summation index in (5) for dimensions \( d = 1, 2 \) and \( 3 \). After per parts and integrating term by term we get (2).

In 1D case, one can make use of the formula [12]

\[
\int_{0}^{\infty} x^{s-1} \vartheta_3(0, e^{-\pi x^2}) - 1)dx = \pi^{-s/2} \Gamma(s/2)\zeta(s) \quad (16)
\]

If we put \( y = x^2 \) in (15), specifying \( s = 4 \) yields (3).

Similarly for PBC

\[
\langle g^2 \rangle^{PBC}_c = \frac{6}{\pi^2} \int_{0}^{\infty} y[\vartheta_3(0, q) - 1]dE \quad (17)
\]

Note that \( q^4 = \exp(-4\pi y) \). Now we can calculate the UCF with high numerical precision for small integer dimensions, see the Table 1. But the main generalization consists in regarding (15) and (17) as an analytical continuation in \( d \), i. e. being valid also for non-integer dimensions [3] and even for those ones below 1D, where it is hard to imagine any "perpendicular" direction.

The Equations (15) and (17) represent the completed Ansatz. It reproduces the UCF for any integer dimension, i. e. any number of sums in (2), which should make the continuation unambiguous. But this argument is rather naive, as the sums diverge for \( d \geq 4 \); some cut-off (e. g. as in [9]) would be necessary. Note that the integrals (15) and (17) also diverge for \( d \geq 4 \). The dependence of UCF on dimension is plotted in Fig. 1. The common statement, that this dependence is weak [6], is true in the region \( 0 \leq d \leq 3 \).

We shall now comment the limiting cases. For \( d = 0 \) one gets sound values (see Table 1) both slightly below the universal constant \( 1/8 \), which corresponds to ballistic transport in quantum dots [15]. For \( d \) approaching 4, \( d = 4 - \varepsilon \) the leading contribution to integrals (15) and (17)
comes from small $y$ values and we can make use of the known alternative expansion:

$$\vartheta_3(u, \exp(-\pi x^2)) = \frac{1}{x} \sum_{n=-\infty}^{\infty} \exp\left[\frac{(u-nx)^2}{\pi x^2}\right]$$  (18)

For small $x = \sqrt{y}$ and $u = 0$ only the $n = 0$ term is important (see Ref. [9])

$$\vartheta_3(0, \exp(-\pi x^2)) \approx \frac{1}{x}$$  (19)

and we get

$$\langle g^2 \rangle_c = \frac{3}{2\pi^2} + O(1)$$  (20)

both for HW and PBC. These expansions remain valid even for anti-periodic BC, if one simply replaces the second $\vartheta_3$ function in (17) and that one in (19) by $\vartheta_2$. The independence of the leading divergent term on boundary conditions seems to support our Ansatz. Similar statement was made in 3D for small $y$ expansion of $S_3(y)$, whose leading divergence also proved BC independent, see [9].

Unfortunately we cannot use formulas (15) and (17) to get the exact $L$-dependence of 4D conductance fluctuations directly. Introducing $1/L$ cut-off, e.g. by changing the lower limit of integration to small non-zero value, corresponds to making upper limits for summation indices in (2) finite. This would influence the propagator used in one-loop diagrams so that the delta function on rhs of diffusion equation in spectral representation (Refs. [5], [6]) would become just a peak of finite, $L$ dependent width. Thus the summation of diagrams, mentioned in the beginning, would be no more simple. Anyway, the too simple cut-off would give logarithmic $L$ dependence in 4D.

In [16] numerical calculations of UCF for 2D and 3D Anderson model were compared to theoretical values, computed also numerically with cca 3 digit precision from more complicated version of (2), given in [5] and its PBC counterpart. A plateau of $\text{var } g = \langle g^2 \rangle - \langle g \rangle^2 = \langle g^2 \rangle_c$ as a function of disorder $W$ was found with values reasonably close to theoretical predictions. The disorder had to be large enough to overcome the ballistic peak region, but not too close to the metal-insulator transition (if $d > 2$). The plateau grew broader with increasing system size $L$.

We performed these calculations for 4D Anderson model with HW BC. Up to now only the region of metal-insulator transition was addressed in 4D, Refs. [3], [4] and [17]. In diffusive regime, Ohm’s law for cubic samples applies in the sense of leading contribution in $L$:

$$\langle g \rangle = \sigma L^{d-2}$$  (21)

This serves as a test of diffusive kind transport, we expect $\langle g \rangle \propto L^2$, see Fig. 2. Here we find, that this regime is well pronounced around $W = 11$.

Results for $\text{var } g$ are shown in Fig. 3. A small plateau appears for larger $L$ and disorders cca within $10 < W < 11$. Typical statistical ensembles are $N_{\text{stat}} = 10^5$ for $L \leq 6$, $\approx 20000$ for $L = 7$ and $\approx 3000$ for $L = 8$. The value of the plateau diverges logarithmically (Fig. 4).

We conclude that a simple conjecture enables a high precision calculation of UCF in any real dimension $0 \leq d < 4$, with reasonable behavior in both limiting regions. Numerical calculations of 4D Anderson model show logarithmic divergence of CF with increasing system size.

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There is a misprint in last row of Eq. (9).

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Table 1: Numerical values of UCF, calculated by (15) and (17).

| d | $\langle g^2 \rangle_{HW}$ | $\langle g^2 \rangle_{PBC}$ |
|---|----------------|----------------|
| 0 | 0.104493901 | 0.12291543 |
| 1 | 0.133333333 | 0.13333333 |
| 2 | 0.185613444 | 0.15407842 |
| 3 | 0.314054080 | 0.21939280 |
| 4 | $\infty$ | $\infty$ |

Figure 1: Universal conductance fluctuations as a function of dimension for HW (open symbols) and PBC (full symbols). Dashed line is the asymptotic term from (20).

Figure 2: Mean conductance in 4D as a function of system size. Fitting parameter $n$ from $\langle g \rangle \propto L^n$ (for $L \geq 5$) shown in the Figure.

Figure 3: Conductance fluctuations in 4D as a function of disorder. System size $L$ is described in the Figure.

Figure 4: Conductance fluctuations in 4D as a function of $L$. The slope varies from 0.36 to 0.39.