SPHERICAL FUNCTIONS FOR
THE QUANTUM GROUP $\text{su}_q(2)$

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Abstract
The representation theory of the quantum group $\text{su}_q(2)$ is used to introduce $q$-analogues
of the Wigner rotation matrices, spherical functions, and Legendre polynomials. The method
amounts to an extension of variable separation from Laplace equations to certain differential-
dilation equations.

1. Introduction
One of the present authors (P. W.) had the privilege of working for his Ph. D. degree under
the guidance of Professor Yakov Abramovich Smorodinsky in Dubna, starting 30 years ago. The
first article he was given to read was one by N. Ya. Vilenkin and Ya. A. Smorodinsky on invariant
expansions of relativistic scattering amplitudes [1]. This paper introduced a variety of different
types of “spherical functions” as basis functions for irreducible representations of the Lorentz
group, realized on $O(3,1)$ hyperboloids. This turned out to be a pathbreaking article that lead
to many interesting developments in the representation theory of noncompact groups and their
physical applications.

In addition to the original idea of two-variable expansion of scattering amplitudes [1-5], this
includes a group theoretical approach to the separation of variables in partial differential equations
[2,6-9]. Other scientific programs influenced by Ref. 1 are those of systematically classifying
subgroups of Lie groups [2,10,11], or of generating completely integrable Hamiltonian systems in
various spaces [2,12-14].

A much more recent development is the current interest in quantum groups [15-19] With it
came the realization that the so called $q$-hypergeometric series and other $q$-special functions have
a similar relation to quantum groups [20-24] as the classical special functions have to Lie groups
[1,25,26].

The purpose of this presentation and an accompanying article [27] is to use the theory of
irreducible representations of the quantum group $\text{su}_q(2)$ to construct $q$-spherical functions on an
ordinary (commutative) sphere $S^2$, i.e. the homogeneous space $SU(2)/O(2)$.

The motivation for this study is two-fold. First of all, if quantum groups are to play a role in
physics, then the corresponding $q$-special functions should occur in physics as wave functions, or
in some similar guise. Secondly, a systematic use of quantum group representation theory should
provide methods for introducing new special functions and obtaining new properties of known
functions.

1Invited talk presented by P. Winternitz at International Workshop “Symmetry Methods in Physics” in memory
of Ya. A. Smorodinsky (Dubna, Russia, July 1993).
We hope that this article demonstrates how strong the influence of one aspect of Ya. A. Smorodinsky’s work is on research being conducted now, almost 30 years later.

2. The Quantum Algebra \( su_q(2) \) and its Representations

2.1 Realization of the Quantum Algebra by Differential-Dilation Operators

The algebra \( su_q(2) \) is a deformation of the Lie algebra \( su(2) \) and is characterized by the commutation relations

\[
[H_3, H_+] = H_+, \quad [H_3, H_-] = -H_-, \quad [H_+, H_-] = \frac{q^{2H_3} - q^{-2H_3}}{q - q^{-1}},
\]

where \( q \) is some real number. For \( q \to 1 \) Eq. (1) reduce to the usual \( su(2) \) commutation relations

\[
[H_3, H_+] = H_+, \quad [H_3, H_-] = -H_-, \quad [H_+, H_-] = 2H_3.
\]

Finite-dimensional irreducible representations of \( su_q(2) \) are characterized by an integer or half-integer number \( J \). Basis functions for these representations can be denoted \( |JMq⟩ \) and satisfy

\[
H_3 |JMq⟩ = M |JMq⟩
\]

\[
H_+ |JMq⟩ = \alpha^J_M q_{M,q} |JM + 1q⟩
\]

\[
H_- |JMq⟩ = \alpha^J_{M,q} |JM − 1q⟩
\]

with

\[
\alpha^J_M q_{M,q} = \left[ \frac{q^{J-M+1} - q^{-J-M-1}}{q - q^{-1}} \right] \frac{q^{J+M} - q^{-J-M}}{q - q^{-1}} \right]^{1/2}.
\]

For \( q = 1 \) Eqs. (3) and (4) reduce to standard \( su(2) \) formulas with

\[
\alpha^J_{M1} = \alpha^J_{M} = [(J + M)(J − M + 1)]^{1/2}.
\]

Thus, in Eq. (3) we have chosen a basis of eigenfunctions of the operator \( H_3 \), corresponding to a nondeformed \( U(1) \) subalgebra of \( su_q(2) \). We have \( \alpha^J_{M+1,q} = \alpha^J_{M,q} = 0 \), hence there is a highest and lowest weight \( N = ±J \) and the representations are finite-dimensional.

We shall now pursue an analogy with the construction of spherical functions \( Y_{J,M}(\theta, \phi) \) for \( su(2) \). These functions can be viewed as being defined on a sphere \( S_2 \) defined by the relations:

\[
x_0 = \frac{1}{2} \sin \theta \cos \phi, \quad y_0 = \frac{1}{2} \sin \theta \sin \phi, \quad z_0 = \frac{1}{2} \cos \theta.
\]

Using a stereographic projection \( S^2 \to \mathbb{R}^2 \)

\[
x = \frac{x_0}{1/2 - z_0}, \quad y = \frac{y_0}{1/2 - z_0},
\]

we can see the spherical functions as being defined on the real plane \((x, y)\) with

\[
x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad \rho = \cot \frac{\theta}{2}
\]

\[
0 \leq \theta \leq \pi, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \rho < \infty.
\]

We shall also use the complex variable

\[
z = x + iy = \rho e^{i\phi}.
\]
We now need to construct operators \( H_3, H_+, \) and \( H_- \), acting on functions \( f(\theta, \phi) \) on \( S_2 \), or equivalently, on functions \( f(z, \bar{z}) \) on the plane \((z, \bar{z})\). The operators should satisfy the relations (2).

An “inspired guess” yields the following relations

\[
H_3 = -z\partial_z + \bar{z}\partial_{\bar{z}} - N
\]

\[
H_+ = -\frac{1}{z} \frac{q^{2\partial_z} - q^{-z\partial_z}}{q - q^{-1}} q^{\bar{z}\partial_{\bar{z}} - N/2} - q^{z\partial_z + N/2} \frac{q^{\bar{z}\partial_{\bar{z}} - N} - q^{-\bar{z}\partial_{\bar{z}} + N}}{q - q^{-1}}
\]

\[
H_- = \frac{q^{z\partial_z + N} - q^{-z\partial_z - N}}{q - q^{-1}} \frac{q^{\bar{z}\partial_{\bar{z}} - N/2} + q^{z\partial_z + N/2}}{z} \frac{1}{q - q^{-1}},
\]

where \( N \) is an integer or half integer parameter. The operators \( q^{z\partial_z} \) and \( q^{\bar{z}\partial_{\bar{z}}} \) act like dilations:

\[
q^{z\partial_z} f(z, \bar{z}) = f(qz, \bar{z})
\]

\[
q^{\bar{z}\partial_{\bar{z}}} f(z, \bar{z}) = f(z, q\bar{z})
\]

and it is easy to verify that \( H_\pm \) and \( H_3 \) satisfy the commutation relations (1). For \( q = 1 \) relations (10) reduce to \( su(2) \) expressions

\[
H_3 = -z\partial_z + \bar{z}\partial_{\bar{z}} - N, \quad H_+ = -\partial_z - \bar{z}^2\partial_{\bar{z}} + N\bar{z}, \quad H_- = z^2\partial_z + \partial_{\bar{z}} + Nz.
\]

For \( q = 1 \) we see that \( H_\pm \), as well as \( H_3 \) are first order differential operators. For \( q \neq 1 \) the operators become nonlocal: in addition to derivatives, they involve dilations of the independent variables.

### 2.2 Basis Functions for Irreducible Representations

We will now look for a realization of the basis functions

\[
|JMq\rangle = \Psi_{MNq}(z, \bar{z})
\]

satisfying Eq. (3) with \( H_\mu \) as in Eq. (10). The Casimir operator of \( su_q(2) \), commuting with \( H_\mu, \mu = 3, \pm, \) is

\[
C_q = H_+H_- + \left( \frac{q^{H_3-1/2} - q^{-H_3+1/2}}{q - q^{-1}} \right)^2 - \frac{1}{4},
\]

From Eq. (3) we deduce that the basis functions of an irreducible representation satisfy

\[
C_q \Psi_{MNq}^J = \left[ \left( \frac{q^{J+1/2} - q^{-J-1/2}}{q - q^{-1}} \right)^2 - \frac{1}{4} \right] \Psi_{MNq}^J.
\]

For \( q \to 1 \) Eq. (15) reduces to the standard angular momentum relation

\[
C \Psi_{MN}^J = J(J + 1) \Psi_{MN}^J.
\]

We are after explicit expressions for the basis functions (13) that for \( q = 1, N = 0 \) reduce to \( su(2) \) spherical functions (and for \( N \neq 0 \) to Jacobi polynomials). To do this we put:

\[
\Psi_{MNq}^J(z, \bar{z}) = N_{MNq}^J q^{-N/2} Q_J q(\eta) R_{NMq}(\eta) \bar{z}^M, \quad \eta = z\bar{z} = \cot^2 \left( \frac{\theta}{2} \right).
\]
To get a finite-dimensional representation (of dimension $2J + 1$) we request the existence of a highest and lowest weight, i.e.

$$H_+ \Psi^J_{JNq} = 0, \quad H_- \Psi^J_{-JNq} = 0$$

and by analogy with the $q = 1$ case, we put

$$R^J_{JNq} = \text{const.}$$

The first of relations (18) implies a functional relation for $Q_J(q)$, namely

$$Q_J(q^2\eta)(1 + \eta) = Q_J(\eta)(1 + q^{-2J} \eta).$$

(20)

Its solution in terms of Exton’s $q$-binomial function is

$$Q_J(q)(\eta) = 1_{\phi_0}(J; -; q^2; -\eta q^{-2J})$$

(21)

which for $q \to 1$ reduces to $Q_{J1}(\eta) = (1 + \eta)^{-J}$.

For $q = 1$ the expression $R^J_{MN1}(\eta)$ are polynomials related to the Jacobi polynomials. For general $q$ they are also polynomials, satisfying certain relations, following from Eq. (3). They are

$$R^J_{J+1,N,q}(\eta) = q^M \frac{[J + M, q]!}{[J - M, q]![2J, q]!}.$$ 

(24)

where $[a, q]!$ denotes a $q$-factorial and $[a, q]$ a $q$-number

$$[a, q] = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [k, q]! = \prod_{p=1}^{k} [p, q],$$

$$[0, q]! = 1, \quad \frac{1}{[k, q]!} = 0, \quad k \in \mathbb{Z}^<.$$

(25)

Relations (22) and (23) are nonlocal, in that the functions $R^J_{MNq}(q^2 \eta)$ are evaluated at $\eta$ and $q^2 \eta$, We can eliminate $R^J_{MNq}(q^2 \eta)$ from these two “difference-dilation” equations and obtain a recursion relation, namely

$$R^J_{J+1,N,q}(\eta) = 1_{\phi_0}(J; -; q^2; -\eta q^{-2J})$$

(26)

3. The $q$-Vilenkin-Wigner Functions and $q$-Spherical Functions

The recursion relation (26), as well as the relations (22) and (23) are solved by the following polynomial in $\eta$:

$$R^J_{MNq}(\eta) = [J - N, q][J - M, q]! \times$$

$$\sum_{k} \frac{(-1)^k \eta^k}{[k, q][J - M - k, q][J - N - k, q][M + N + k, q]!}$$

(27)
and thus the basis functions (17) are completely determined. With an appropriate choice of the normalization constant \( C_{JNq} \) in Eq. (24) we can rewrite them as

\[
\Psi_{MNq}^J(\theta, \phi) = \frac{1}{\sqrt{2\pi}} (2J + 1, q))^{1/2} e^{-i(2J + M + N)} P^J_{MNq}(\cos \theta) e^{-i(M + N)\phi}.
\] (28)

Here we have introduced the “\( q \)-Vilenkin-Wigner functions”

\[
P^J_{MNq}(\xi) = i^{J-M-N} \left( \frac{J + M, q ![J + N, q !]}{J - M, q ![J - N, q !]} \right)^{1/2} \eta^{(M+N)/2} Q_J q(\eta) R^J_{MNq}(\eta)
\]

\[
\eta = z \bar{z} = \frac{1 + \xi}{1 - \xi}, \quad \xi = \cos \theta.
\] (29)

For \( q = 1 \) these functions reduce to the functions \( P^J_{MN}(\cos \theta) \) extensively studied by Vilenkin\(^25\) and directly related to the Wigner rotation matrices\(^30\) \( d^I_{MN}(\theta) \). They are also related to Jacobi polynomials

\[
P^J_{MN}(\xi) = 2^{-M} (i)^{N+M} \left( \frac{(J - M)! (J + M)!}{(J - N)! (J + N)!} \right)^{1/2} \times
\]

\[
(1 - \xi)^{-N+M}/2 (1 + \xi)^{(N+M)/2} P^J_{k,p,q}(\xi)
\]

\[
k = J - M, \quad p = M - N, \quad q = M + N.
\] (30)

The functions \( P^J_{MN}(\xi) \) are usually introduced as rotation matrices, but they can just as well arise as basis functions for representations of \( su(2) \). It is this second role, that of basis functions for irreducible representations, that has been generalized to the quantum group \( su_q(2) \).

Most properties of the ordinary \((q = 1)\) Vilenkin-Wigner functions have their \( q \)-analogues. We shall just list some of them and refer to a related article\(^27\) for proofs

Recursion formula

\[
i ([M + N, q] \eta^{-1/2} - [M - N, q] \eta^{1/2}) P^J_{MNQ} =
\]

\[
= -([J - M, q] [J + M + 1, q])^{1/2} P^J_{M+1,N,q} +
\]

\[
+([J - M + 1, q] [J + M, q])^{1/2} P^J_{M-1,N,q}.
\] (31)

Generating function, defined as

\[
F^J_{N,q}(\omega) = \sum_{M=-J}^J \frac{P^J_{MNq}(\eta)}{([J - M, q] [J + M, q] !)^{1/2}} \omega^{J-M},
\] (32)

has the form

\[
F^J_{N,q}(\omega) = i^{J+N} \frac{1}{([J + N, q] ! [J - N, q] !)^{1/2}} \eta^{(J-N)/2} Q_J q(\eta)
\]

\[
\prod_{p=0}^{J-N-1} (\omega q^{-J+N+1+2p} + i \eta^{-1/2}) \prod_{p=0}^{J+N-1} (\omega q^{-J-N+1+2p} - i \eta^{1/2}).
\] (33)

For \( q = 1 \) this simplifies to the well-known generating function

\[
F^J_{N1}(\omega) \equiv F(\omega) = \frac{1}{([J - N] ! [J + N] !)^{1/2}} \left( \cos \frac{\theta}{2} + i \omega \sin \frac{\theta}{2} \right)^{J+N}
\]

\[
\times
\]

\[
\]
\[ \times \left( \omega \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)^{J-N}. \]

**Symmetry relations**

\[ P_{MNq}^{J}(\xi) = P_{NMq}^{J}(\xi) \]  
\[ P_{MNq}^{J}(\xi) = P_{-M,-N,q}^{J}(\xi) \]

\[ P_{MNq}^{J}(-\xi) = i^{2J-2M-2N} \sigma_{Jq} \frac{Q_{Jq}(\eta q^{2J+2})}{Q_{Jq}(\eta)} P_{-M,-N,q}^{J}(\xi) \]  

where

\[ \sigma_{Jq} = q^{J(J+1)} \quad \text{for } J \text{ integer} \]

\[ \sigma_{L+1/2,q} = q^{(L+1)^{2}} \sigma_{1/2,q} \quad \text{for } J \text{ half-odd integer} \]

\[ \sigma_{1/2,q} = \frac{1 + q}{1 + q^{-1}} \frac{\theta_{2}(0)}{\sqrt{q} \theta_{3}(0)}, \]

(\( \theta_{2}(u) \) and \( \theta_{3}(u) \) are ordinary theta functions).

It is now quite natural to introduce \( q \)-spherical harmonics in the same manner as ordinary harmonics, namely

\[ Y_{JMq}(\theta, \phi) = \frac{1}{(J - M, q)!} P_{M0q}^{J}(\cos \theta) e^{-iM\phi}. \]  

Similarly, the \( q \)-analogue of Legendre polynomials is

\[ P_{Jq}(\cos \theta) = P_{00q}^{J}(\cos \theta) = i^{2J} Q_{Jq}(\eta) R_{00q}^{J}(\eta). \]

Notice that \( P_{Jq}(\cos \theta) \) for \( q \neq 1 \) is not a polynomial in \( \cos \theta \) in view of the properties of \( Q_{Jq}(\eta) \).

Finally we mention the relations between the \( q \)-Vilenkin-Wigner functions and other \( q \)-functions in the literature. These relations are best written in terms of the polynomials \( R_{JMq}^{J}(\eta) \) of Eq. (27).

For instance, in terms of the basic hypergeometric series\(^{28,29}\)

\[ _{2}F_{1}(a, b; c; z) = \sum_{k=0}^{\infty} \frac{[a, k; q][b, k; q]}{[c, k; q][k, q]!} z^{k} \]  

\[ [a, k; q] = [a, q][a + 1, q] \cdots [a + k - 1, q] \]

we have, for \( M + N \geq 0 \)

\[ R_{MNq}^{J}(\eta) = \frac{1}{[M + N, q]!} _{2}F_{1}(M - J, N - J; M + N + 1; q; -\eta). \]  

Using the “little \( q \)-Jacobi functions” \( p_{n}(x; a, b; \eta) \) given e.g. by Koornwinder\(^{31}\), we have

\[ R_{MNq}^{J} = \frac{1}{[M + N, q]!} P_{J-M}^{J^{2}-1}(q^{2J+1}; q^{2(M+N)}, q^{-2(2J+1)}; q^{2}) \]  

(for \( J - M, J - N \)).

**4. Conclusions**

The full power of Lie group theory in its application to special functions only becomes apparent, when applied to partial differential equations and combined with the separation of variables. One way of viewing the results presented above is that we have extended the Lie algebraic treatment of variable separation to \( q \)-special functions and to quantum groups. The separation occurs in differential-difference equations of type (14) and (15), rather than in Laplace-Beltrami equations.
In the future we plan to apply similar techniques to other quantum groups and hence to other types of q-special functions.

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References

[1] N. Ya. Vilenkin and Y. A. Smorodinsky, Zh. Eksp. Teor. Fiz. 46 (1964) 1793.
[2] P. Winternitz and I. Friš, Yad. Fiz. 1 (1965) 889.
[3] P. Winternitz, Y. A. Smorodinsky, and M. B. Sheftel, Yad. Fiz. 7 (1968) 1325.
[4] M. Daumens and P. Winternitz, Phys. Rev. D21 (1980) 1919.
[5] J. Bystricky, P. LaFrance, F. Lehar, F. Perrot, and P. Winternitz, Phys. Rev. D32 (1985) 575.
[6] E. G. Kalnins, W. Miller Jr., and P. Winternitz, SIAM J. Appl. 30 (1976) 630.
[7] W. Miller Jr., J. Patera, and P. Winternitz, J. Math. Phys. 22 (1981) 251.
[8] W. Miller Jr., Symmetry and Separation of Variables (Addison Wesley, New York, 1977).
[9] E. G. Kalnins, Separation of Variables for Riemannian Symmetric Spaces of Constant Curvature (Longmans, Essex, England, 1986).
[10] J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 15 (1974) 1378 and 1932; 16 (1975) 1597 and 1615; 17 (1976) 717.
[11] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 17 (1976) 977 and 986; 18 (1977) 2259.
[12] I. Friš, V. Mandrosov, J. Smorodinsky, M. Uhliř, and P. Winternitz, Phys. Lett. 16 (1965) 354.
[13] A. Makarov, J. Smorodinsky, Kh. Valiev, and P. Winternitz, Nuovo Cim. A52 (1967) 1061
[14] M. A. del Olmo, M. A. Rodriguez, and P. Winternitz, J. Math. Phys. 34 (1993) 5118.
[15] V. G. Drinfeld, in Proc. Int. Congress Math. Vol 1 (Amer. Math. Soc., Providence, RI, 1986).
[16] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; 11 (1986) 247.
[17] S. L. Woronwicz, Commun. Math. Phys. 111 (1987) 613.
[18] Yu. I. Manin, Quantum Groups and Noncommutative Geometry (Centre de recherches mathématiques, Montréal, 1988).
[19] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Taktajan, Leningrad. Math. J. 1 (1990) 193.
[20] R. Floreanini and L. Vinet, *Lett. Math. Phys.* **22** (1991) 45; *J. Phys. A Math. Gen.* **23** (1990) L1019; *J. Math. Phys.* **33** (1992) 1358.

[21] E. G. Kalnins, H. L. Manocha, and W. Miller Jr., *J. Math. Phys.* **33**, (1992) 2365.

[22] L. L. Vaksman and Ya. S. Soibelman, *Funct. Anal. Pril.* **22** (1988) 1.

[23] N. M. Atakishiyev and S. K. Suslov, *Teor. Mat. Fiz.* **85** (1990) 64.

[24] S. K. Suslov, *Russian Math. Surveys* **44** (1989) 227.

[25] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations* (Amer. Math. Soc., Providence, RI, 1968).

[26] W. Miller Jr., *Lie Theory and Special Functions* (Academic Press, New York, 1986).

[27] G. Rideau and P. Winternitz, *J. Math. Phys.* **34** (1993) 6030.

[28] H. Exton, *q-Hypergeometric Functions and Applications* (Ellis Harwood, Chichester, 1983).

[29] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge Univ. Press, Cambridge, 1990).

[30] E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (Academic Press, New York, 1959).

[31] T. H. Koornwinder, *Proc. Nederl. Akad. Wetensch.* **A92** (1989) 97; *SIAM J. Math. Anal.* **22** (1991) 295.