Parameterized summation relations for the Stieltjes constants

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Abstract

The Stieltjes constants $\gamma_k(a)$ appear in the regular part of the Laurent expansion of the Hurwitz zeta function about its only polar singularity at $s = 1$. We present multi-parameter summation relations for these constants that result from identities for the Hurwitz zeta function. We also present multi-parameter summation relations for functions $A_k(x)$ that may be expressed as sums over the Stieltjes constants. Integral representations, especially including Mellin transforms, play an important role. As a byproduct, reciprocity and other summatory relations for polygamma functions and Bernoulli polynomials may be obtained.

Key words and phrases
Hurwitz zeta function, summation relation, Stieltjes constants, Mellin transform

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Statement of results

Let \(\zeta(s,a)\) denote the Hurwitz zeta function and \(\gamma_k(a)\) the Stieltjes constant. We present multi-parameter summation relations for these constants. For background on the Stieltjes constants, one may see [2, 8, 9, 10, 11, 12, 14, 15]. For known summation relations [3, 4, 5] may be consulted. In the Laurent series about \(s = 1\),

\[
\zeta(s,a) = \frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a)(s - 1)^k,
\]

(1.1)

\(\gamma_0(a) = -\psi(a)\), where \(\psi = \Gamma'/\Gamma\) is the digamma function and \(\Gamma\) is the Gamma function. For the coefficients corresponding to the Laurent expansion for the Riemann zeta function \(\zeta(s)\), one denotes \(\gamma_k(1) = \gamma_k\). The quantities \(\gamma_k(a)\) are of interest in analytic number theory, asymptotic analyses, and other areas.

We let \(\mathcal{P}\) denote the set of prime numbers.

We have

**Proposition 1.** Let \(p \geq 1\) and \(q \geq 1\) be integers, \(b \geq 0\), and \(\min(p/q, q/p) > b\). Then we have for integers \(k \geq 0\)

\[
\sum_{r=1}^{q} \gamma_k \left( \frac{pr}{q} - b \right) = q(-1)^k \frac{k \ln^{k+1}(q/p)}{k + 1} \gamma_k(a) + \frac{q}{p} \sum_{\ell=0}^{p-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln^j \left( \frac{q}{p} \right) \gamma_{k-j} \left[ 1 + \left( \ell - b \right) q/p \right].
\]

(1.2)

**Proposition 2.** Let \(p \in \mathcal{P}\) and \(m, N \geq 0\) be integers. We have

\[
(p - 1)\gamma_m + \frac{\ln^{m+1}}{m+1} \frac{p^m}{m+1} \left( \sum_{k=0}^{m-1} \binom{m}{k} \ln^{m-k} p \gamma_k \right)
= (1 - p) \frac{\ln^{m+1}}{m+1} \frac{p^{N+1}}{m+1} + \frac{1}{p^N} \sum_{k=0}^{m} \binom{m}{k} \ln^{m-k} p^{N+1} \sum_{1 \leq j < p^{N+1}} \gamma_k \left( \frac{j}{p^{N+1}} \right).
\]

(1.3)
Proposition 3. Let $p \in \mathcal{P}$ and $m, N \geq 0$. Then for any positive integer $k_p$ coprime to $p$ and nonnegative integers $\alpha$ and $\beta$ such that $\alpha + jk_p = p\beta$ for some $j$ with $0 \leq j \leq p - 1$, we have

$$(p - 1)\gamma_m \left( \frac{\alpha}{k_p} \right) + \frac{\ln^{m+1} p}{m + 1} - \sum_{k=0}^{m-1} \binom{m}{k} \ln^{m-k} p \gamma_k \left( \frac{\beta}{k_p} \right)$$

$$= (1 - p)\frac{\ln^{m+1} p^{N+1}}{m + 1} + \frac{1}{p^N} \sum_{k=0}^{m} \binom{m}{k} \ln^{m-k} p^{N+1} \sum_{j=\alpha(k \mod k_p) \atop (j,p)=1}^{\alpha + k_p \ell} \gamma_k \left( \frac{j}{k_p p^{N+1}} \right).$$

(1.4)

On the right side, the sum is over all integers $j$ of the form $\alpha + k_p \ell$ and $\alpha \leq j < \alpha + k_p p^{N+1}$.

From these Propositions we obtain many Corollaries. As an illustration we give some of those resulting from Proposition 1. For this purpose, we let $B_n(x)$ be the Bernoulli polynomial of degree $n$ (e.g., [1], Ch. 23.1) and $\psi^{(j)}$ the polygamma function (e.g., [1], Ch. 6.4). We have the following.

**Corollary 1.** For $p, q \geq 1$ integers, $b \geq 0$, and $\min(p/q, q/p) > b$, we have

$$\ln q + \frac{1}{q} \sum_{r=0}^{q-1} \psi \left( \frac{pr}{q} - b \right) = \ln p + \frac{1}{p} \sum_{\ell=0}^{p-1} \psi \left( \ell - b \right) \frac{q}{p}. \quad (1.5)$$

**Corollary 2.** For $p, q \geq 1$, $n > 1$ integers, $b \geq 0$, and $\min(p/q, q/p) > b$, we have

$$\frac{1}{q} \sum_{r=0}^{q-1} \psi^{(n-1)} \left( \frac{pr}{q} - b \right) = \frac{1}{p} \left( \frac{q}{p} \right)^{n-1} \sum_{\ell=0}^{p-1} \psi^{(n-1)} \left( \ell - b \right) \frac{q}{p}. \quad (1.6)$$

**Corollary 3.** For $p, q \geq 1$, $m \geq 0$ integers, $b \geq 0$, and $\min(p/q, q/p) > b$, we have

$$\sum_{r=1}^{q} B_m \left( \frac{pr}{q} - b \right) = \left( \frac{q}{p} \right)^{1-m} \sum_{\ell=0}^{p-1} B_m \left[ 1 + (\ell - b) \frac{q}{p} \right]. \quad (1.7)$$
The functions for integers $k$

\[ A_k(q) \equiv k \frac{\partial}{\partial z} \zeta(z, q) \bigg|_{z=1-k}, \]  \hspace{1cm} (1.8)

are very useful in evaluating integrals over the Hurwitz zeta function $\zeta$. They may be written in terms of the Stieltjes constants as

\[ A_k(q) = -\frac{1}{k} - k \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(q)}{n!} k^n. \]  \hspace{1cm} (1.9)

We present representative summation relations for the functions $A_k(q)$. We have

**Proposition 4.** Let $p \geq 1$ and $q \geq 1$ be integers, $b \geq 0$, and $\min(p/q, q/p) > b$. Then we have

\[ \sum_{r=1}^{q} \left[ A_k \left( \frac{pr}{q} - b \right) + \ln \left( \frac{q}{p} \right) B_k \left( \frac{pr}{q} - b \right) \right] = \left( \frac{q}{p} \right)^{1-k-1} \sum_{\ell=0}^{p-1} A_k \left( 1 + \frac{(\ell - b)q}{p} \right). \]  \hspace{1cm} (1.10)

**Proposition 5.** For $p \in \mathcal{P}$ and integer $N \geq 0$ we have

\[ (1 - p^{k-1})A_k(1) - (-1)^k (\ln p) B_k[N(1 - p^{k-1}) + 1] = p^{(N+1)(k-1)} \sum_{\frac{1}{p^{N+1}} \leq j < \frac{1}{p^{N+1}}} A_k \left( \frac{j}{p^{N+1}} \right). \]  \hspace{1cm} (1.11)

Here, $B_k = B_k(0) = (-1)^k B_k(1)$, $k \geq 0$ are the Bernoulli numbers.

**Proof of Propositions**

**Proposition 1.** We apply Lemma 1 and then proceed as in the proof of Proposition 5 of [5].

We first have the following.

**Lemma 1.** Let $p \geq 1$ and $q \geq 1$ be integers, $b \geq 0$, and $\min(p/q, q/p) > b$. Then we
have
\[ \sum_{r=1}^{q} \zeta \left( s, \frac{pr}{q} - b \right) = \left( \frac{q}{p} \right)^{s} \sum_{\ell=0}^{p-1} \zeta \left( s, \frac{\ell q + p - qb}{p} \right). \]  
\hspace{1cm} (2.1)

**Proof.** The Lemma may be proved in three different ways: (i) interchange of a
double sum, (ii) use of an integral representation for the Hurwitz zeta function, and
(iii) evaluating the zeta functions associated with a certain rational function. The
Lemma is first demonstrated for \( \text{Re} \ s > 1 \), and then by analytic continuation it holds
for all \( s \in C \).

First method. We have for \( \text{Re} \ s > 1 \),
\[ \sum_{r=1}^{q} \zeta \left( s, \frac{pr}{q} - b \right) = \sum_{r=1}^{q} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{pr}{q} - b)^{s}} = \left( \frac{q}{p} \right)^{s} \sum_{n=0}^{\infty} \left[ \zeta \left( s, \frac{(n-b)q}{p} + 1 \right) - \zeta \left( s, \frac{(n+p-b)q}{p} + 1 \right) \right]. \]  
\hspace{1cm} (2.2)
Successive terms of the summand in blocks of length \( p \) are then taken to find (2.1).
Indeed, (2.1) follows as the \( u \to \infty \) limit of the relation
\[ \sum_{n=0}^{u} \left[ \zeta \left( s, \frac{(n-b)q}{p} + 1 \right) - \zeta \left( s, \frac{(n+p-b)q}{p} + 1 \right) \right] = \sum_{\ell=0}^{p-1} \left[ \zeta \left( s, \frac{\ell q + p - qb}{p} \right) - \zeta \left( s, \frac{uq + (\ell + 1)q + p - qb}{p} \right) \right]. \]  
\hspace{1cm} (2.3)

Second method. We use a standard integral representation for \( \text{Re} \ s > 1 \) and \( \text{Re} \ a > 0 \),
\[ \zeta(s, a) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{x^{s-1} e^{-(a-1)x}}{e^{x} - 1} \, dx, \]  
\hspace{1cm} (2.4)

and, to obtain
\[ \sum_{r=1}^{q} \zeta \left( s, \frac{pr}{q} - b \right) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} \frac{e^{(b+1-p)x}(e^{px} - 1)}{(e^{x} - 1)(e^{px/q} - 1)} \, dx. \]  
\hspace{1cm} (2.5)
Similarly, we have
\[
\left( \frac{q}{p} \right)^{s-1} \sum_{\ell=0}^{p-1} \zeta \left( s, \frac{\ell q + p}{p} - \frac{qb}{p} \right) = \left( \frac{q}{p} \right)^s \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{(e^{qx} - 1)e^{(b+1-p)qx/p}}{(e^x - 1)(e^{qx/p} - 1)} \, dx
\]
\[
= \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{e^{(b+1-p)u}(e^{pu} - 1)}{(e^u - 1)(e^{pu/q} - 1)} \, du,
\]
(2.6)
where we used the scaling \( u = qx/p \).

Third method. We use the method of finding a zeta function \( Z_f \) associated with a rational function, (see the Appendix for a brief description), and express \( Z_f \) in two different ways. We employ the rational function
\[
f(T) = \sum_{r=1}^q \frac{T^{pr-bq}}{1 - T^q} = \sum_{r=1}^q \sum_{n=0}^\infty T^{pr+(n-b)q}.
\]
(2.7)
By relation (A.3) of the Appendix, we have
\[
Z_f(s)\Gamma(s) = \Gamma(s)q^{-s} \sum_{r=1}^q \zeta \left( s, \frac{pr}{q} - b \right).
\]
(2.8)
We also recognize that
\[
f(T) = \sum_{\ell=0}^{p-1} \frac{T^{p-bq}}{1 - T^p} T^{\ell q} = \sum_{\ell=0}^{p-1} T^\ell \frac{T^{(\ell-b)q}}{1 - T^p}.
\]
(2.9)
The Mellin transform representation corresponding to this equation is given by
\[
Z_f(s)\Gamma(s) = 1 \left( \frac{1}{p^s} \sum_{\ell=0}^{p-1} \int_0^\infty v^{s-1} \frac{e^{-(\ell-b)v/p}e^{-v}}{e^v - 1} \, dv = p^{-s}\Gamma(s)\sum_{\ell=0}^{p-1} \zeta \left( s, 1 + (\ell - b)\frac{q}{p} \right) \right.
\]
(2.10)
Comparing (2.8) and (2.10), we have the Proposition.

Corollary 1 follows by putting \( k = 0 \) in Proposition 1, and using the functional equation of the digamma function \( \psi(x+1) = \psi(x) + 1/x \). For instance, an intermediate
form is
\[ \ln q + \frac{1}{q} \sum_{r=1}^{q} \psi \left( \frac{pr}{q} - b \right) = \ln p + \frac{1}{q} \sum_{\ell=0}^{p-1} \frac{1}{\ell - b} + \frac{1}{p} \sum_{\ell=0}^{p-1} \psi \left[ (\ell - b) \frac{q}{p} \right]. \] (2.11)

Corollary 2 follows by differentiating (2.11) with respect to \(-b\) \((n-1)\) times. Alternatively, it follows by taking \(s = n\), \(n > 1\) an integer, in Lemma 1, using the relation
\[ \psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x), \] (2.12)
and manipulating with the functional equation of the polygamma function, \(\psi^{(n-1)}(x+1) = \psi^{(n-1)}(x) + (-1)^{n-1}(n-1)!/x^n\).

Corollary 3 obtains from Lemma 1 since we have the relation \(B_n(x) = -n\zeta(1-n, x)\) for \(n > 0\).

**Remarks.** Equation (2.1) reduces properly at \(p = 1\).

In relation (2.1), there is cancellation of polar \(q/(s-1)\) terms.

Of course, the \(j = k\) term on the right side of (1.2) may be separated.

In [5] (5.1) and (5.2), the summand factor \((-1)^k\) should read \((-1)^j\).

We see that (2.6) corresponds to the following form of the function \(f(T)\) used in the third method:
\[ f(T) = \sum_{n=0}^{\infty} T^p \frac{1 - T^{pq}}{1 - T^p} T^{(n-b)q} = \frac{T^{p-bq}(1 - T^{pq})}{(1 - T^q)(1 - T^p)}. \] (2.13)

Corollaries 1 and 2 correspond to successively differentiating Schobloch’s relatively little known reciprocity formula of 1884 for the function \(\ln \Gamma\). For a proof of this formula, see Theorem 3.7 in [13].
Owing to the many functional properties of the Bernoulli polynomials (e.g., [1], Ch. 23.1), such as
\[ B_m \left( 1 - q \frac{b}{p} \right) = (-1)^m \frac{b}{p} \right) = (-1)^m q^{m-1} \sum_{k=0}^{q-1} B_m \left( \frac{b}{p} + \frac{k}{q} \right), \] (2.14)

Corollary 3 may be written in many equivalent ways. Corollary 3 is probably a new reciprocity relation for the Bernoulli polynomials.

**Proposition 2.** We use

**Lemma 2.** For \( p \in \mathcal{P} \) and \( N \geq 0 \), we have
\[ (1 - p^{-s})\zeta(s) = p^{-(N+1)s} \sum_{1 \leq j < \frac{p^{N+1}}{p}} \zeta \left( s, \frac{j}{p^{N+1}} \right). \] (2.15)

Lemma 2 initially holds for \( \text{Re} \ s > 1 \), and then by analytic continuation for all of \( \mathbb{C} \). We have found that Lemma 2 and Lemma 3 below overlap with Proposition 1 in [6] as used in connection with a study of congruences of Bernoulli numbers. As shown above for Lemma 1, there are alternative means of determining such identities.

**Proof.** First method. We use the method of finding a zeta function \( Z_f \) associated with a rational function, (see the Appendix), expressing \( Z_f \) in two different ways. Associated with the function
\[ f(T) = \frac{1}{1 - T} - \frac{1}{1 - T^p}, \] (2.16)
we first have for \( \text{Re} \ s > 1 \),
\[ Z_f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(pn)^s} = (1 - p^{-s})\zeta(s). \] (2.17)
Alternatively, we have

\[ f(T) = \frac{T + T^2 + \ldots + T^{p-1}}{1 - T^p} = \frac{(T + T^2 + \ldots + T^{p-1})(1 + T^p + T^{2p} + \ldots + T^{p(p^N-1)})}{1 - T^{pN+1}} \]

\[ = \sum_{1 \leq j < p^{N+1}} \sum_{k=0}^{\infty} T^{j+kp^{N+1}}. \]  \hfill (2.18)

Then for Re \( s > 1 \),

\[ Z_f(s) = p^{-(N+1)s} \sum_{1 \leq j < p^{N+1}} \zeta \left( s, \frac{j}{p^{N+1}} \right), \]  \hfill (2.19)

and the Lemma follows.

Second method for Lemma 2, using the integral representation (2.4). We have

\[ p^{-(N+1)s} \Gamma(s) \sum_{1 \leq j < p^{N+1}} \frac{1}{(j,p) = 1} \zeta \left( s, \frac{j}{p^{N+1}} \right) = p^{-(N+1)s} \sum_{1 \leq j < p^{N+1}} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} e^{-(j/p^{N+1}-1)t} dt \]

\[ = p^{-(N+1)s} \sum_{k=1}^{p-1} \sum_{\ell=0}^{p^{N-1}} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} e^{-(\ell\ell+k)/p^{N+1}-1]t} dt \]

\[ = p^{-(N+1)s} \int_0^{\infty} \frac{t^{s-1}}{(e^{p^{N}t} - e^{p^{N}-1})} dt \]

\[ = p^{-(N+1)s} \int_0^{\infty} \frac{u^{s-1}}{(e^{u/p} - 1)(e^u - 1)} du \]

\[ = p^{-(N+1)s} \int_0^{\infty} u^{s-1} \left[ \frac{1}{e^{u/p} - 1} - \frac{1}{e^u - 1} \right] du \]

\[ = \Gamma(s)(1 - p^{-s})\zeta(s). \]  \hfill (2.20)

In Lemma 2, the polar term that cancels on each side is \((1 - 1/p)/(s - 1)\). This is confirmed by the sum from the right side,

\[ \sum_{1 \leq j < p^{N+1}} 1 = \varphi(p^{N+1}) = p^{N+1} - p^N, \]  \hfill (2.21)
where \( \varphi \) is the Euler totient function. By using the expansion (1.1), the left side is
\[
(1 - p^{-s})\zeta(s) = \left[1 - \frac{1}{p} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \ln^j p \ (s - 1)^j\right] \left[\frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a)(s - 1)^k \right],
\]
and the right side is given by
\[
p^{-(N+1)s} \sum_{\substack{j \leq pN+1 \ \text{(mod \ } kp)} \ \gamma_k \left(\frac{j}{p^{N+1}}\right)(s-1)^k.
\]
Manipulating the series in (2.22) and (2.23), identifying the coefficients of \((s - 1)^m\) on both sides, and multiplying by \(p\) gives Proposition 2.

Remark. From Proposition 2 with \(m = 0\) we have

**Corollary 4.** For \(p \in \mathcal{P}\) and \(N \geq 0\) an integer, we have
\[
(p-1)\gamma + \ln p = (1-p)\ln p^{N+1} - \frac{1}{p^N} \sum_{\substack{1 \leq j < pN+1 \ \text{(mod \ } kp)} \ \gamma_k \left(\frac{j}{p^{N+1}}\right)}.
\]

**Proposition 3.** We use

**Lemma 3.** For \(p \in \mathcal{P}, \ N \geq 0, \ \alpha \geq 0, \ \beta \geq 0, \) and \((kp, p) = 1\) as in the Proposition, we have
\[
\zeta\left(s, \frac{\alpha}{kp}\right) - p^{-s} \zeta\left(s, \frac{\beta}{kp}\right) = p^{-(N+1)s} \sum_{\substack{j \equiv \alpha \text{ (mod } kp)} \ \text{ (mod } kp)} \zeta\left(s, \frac{j}{kp^{N+1}}\right).
\]

**Proof.** We find zeta functions \(Z_f\) associated with the rational function
\[
f(T) = \frac{T^\alpha}{1 - T^{kp}} - \frac{T^{p\beta}}{1 - T^{kp}}.
\]
We have the series for $|T| < 1$,

$$f(T) = \sum_{n=0}^{\infty} T^{\alpha+k_pn} - \sum_{n=0}^{\infty} T^{p\beta+k_pn}. \tag{2.26}$$

By (A.3) of the Appendix, we have

$$Z_f(s)\Gamma(s) = \int_0^{\infty} t^{s-1} f(e^{-t}) dt = \Gamma(s) \left[ k_p^{-s} \zeta \left( s, \frac{\alpha}{k_p} \right) - (k_p)^{-s} \zeta \left( s, \frac{\beta}{k_p} \right) \right]. \tag{2.27}$$

Also expressing $f(T)$ as a rational function with denominator $1 - T^{k_pN+1}$, we find

$$f(T) = \sum_{j=\alpha \mod k_p} \sum_{n=0}^{\infty} T^{j+nk_pN+1}. \tag{2.28}$$

Then we also have for $\Re s > 1$,

$$Z_f(s)\Gamma(s) = \int_0^{\infty} t^{s-1} f(e^{-t}) dt = \Gamma(s) (k_pN+1)^{-s} \sum_{j=\alpha \mod k_p} \zeta \left( s, \frac{j}{k_pN+1} \right). \tag{2.29}$$

Equating (2.27) and (2.29), we obtain the Lemma.

The result of the Lemma extends to all of $C$ by analytic continuation. There is again cancellation of polar terms $(1 - 1/p)/(s - 1)$. Proceeding as in the proof of Proposition 2, we obtain Proposition 3.

**Proposition 4.** We first differentiate (2.1) with respect to $s$,

$$k \sum_{r=1}^{q} \zeta' \left( s, \frac{pr}{q} - b \right) = \left( \frac{q}{p} \right)^s k \ln \left( \frac{q}{p} \right) \sum_{\ell=0}^{p-1} \zeta \left( s, \frac{\ell q + p}{p} - \frac{q b}{p} \right) + \left( \frac{q}{p} \right)^s k \sum_{\ell=0}^{p-1} \zeta' \left( s, \frac{\ell q + p}{p} - \frac{q b}{p} \right). \tag{2.30}$$

We then put $s = 1 - k$ and use the definition (1.8), so that

$$\sum_{r=1}^{q} A_k \left( \frac{pr}{q} - b \right) = -\left( \frac{q}{p} \right)^{1-k} k \ln \left( \frac{q}{p} \right) \sum_{\ell=0}^{p-1} B_k \left( \frac{\ell q + p}{p} - \frac{q b}{p} \right) + \left( \frac{q}{p} \right)^{1-k} k \sum_{\ell=0}^{p-1} A_k \left( \frac{\ell q + p}{p} - \frac{q b}{p} \right). \tag{2.31}$$
We now apply Corollary 3 for the Bernoulli polynomials to arrive at the Proposition.

**Proposition 5.** By evaluating (2.15) at $s = 1 - k$ we have

**Corollary 5.** We have

$$(-1)^k (1 - p^{k-1}) B_k = p^{(N+1)(k-1)} \sum_{1 \leq j < p^{N+1}} B_k \left( \frac{j}{p^{N+1}} \right). \quad (2.32)$$

We now differentiate (2.15) with respect to $s$,

$$k(1 - p^{-s})(\ln p)\zeta(s) + k(1 - p^{-s})\zeta'(s) = -p^{-(N+1)s} (N + 1)(\ln p)k \sum_{1 \leq j < p^{N+1}} \zeta \left( s, \frac{j}{p^{N+1}} \right)$$

$$+ p^{-(N+1)s} k \sum_{1 \leq j < p^{N+1}} \zeta' \left( s, \frac{j}{p^{N+1}} \right). \quad (2.33)$$

Putting $s = 1 - k$, using (1.8), we have

$$-p^{k-1}(\ln p)B_k(1) + (1 - p^{k-1})A_k(1) = (N + 1)p^{(N+1)(k-1)}(\ln p) \sum_{1 \leq j < p^{N+1}} B_k \left( \frac{j}{p^{N+1}} \right)$$

$$+ p^{(N+1)(k-1)} \sum_{1 \leq j < p^{N+1}} A_k \left( \frac{j}{p^{N+1}} \right). \quad (2.34)$$

Applying Corollary 5 leads to (1.11).

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Appendix: Zeta functions associated with a rational function

Suppose that \( p(T) \) is a polynomial in \( T \) and \( n_1, n_2, \ldots, n_r \) are positive integers.

We consider the rational function

\[
f(T) = \frac{p(T)}{(1 - T^{n_1})(1 - T^{n_2}) \cdots (1 - T^{n_r})}.
\]

We assume that about \( T = 0 \) there is a power series expansion \( f(T) = \sum_{n=0}^\infty a_n T^n, \) \(|T| < 1\). Then the zeta function associated with \( f(T) \) is given by

\[
Z_f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}, \quad \text{Re} \ s > r.
\]

In addition, \( Z_f \) is given by the Mellin transform

\[
Z_f(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}[f(e^{-t}) - f(0)]dt.
\]

So \( Z_f \) is initially defined in a right half plane and has an analytic continuation.
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