Entanglement and Sensitivity in Precision Measurements with States of a Fluctuating Number of Particles
Philipp Hyllus, Luca Pezzé, Augusto Smerzi

To cite this version:
Philipp Hyllus, Luca Pezzé, Augusto Smerzi. Entanglement and Sensitivity in Precision Measurements with States of a Fluctuating Number of Particles. Physical Review Letters, American Physical Society, 2010, 105 (12), pp.120501. 10.1103/PhysRevLett.105.120501 . hal-00746823

HAL Id: hal-00746823
https://hal-iogs.archives-ouvertes.fr/hal-00746823
Submitted on 7 Dec 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Entanglement and Sensitivity in Precision Measurements with States of a Fluctuating Number of Particles

P. Hyllus, L. Pezzé, and A. Smerzi

INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, I-38123 Povo, Italy

Laboratoire Charles Fabry de l’Institut d’Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique, RD 128, F-91127 Palaiseau cedex, France

(Received 31 March 2010; revised manuscript received 23 July 2010; published 15 September 2010)

The concepts of separability, entanglement, spin squeezing, and the Heisenberg limit are central in the theory of quantum-enhanced metrology. In the current literature, these are well established only in the case of linear interferometers operating with input quantum states of a known fixed number of particles. This manuscript generalizes these concepts and extends the quantum phase estimation theory by taking into account classical and quantum fluctuations of the particle number. Our analysis concerns most of the current experiments on precision measurements where the number of particles is known only on average.

 DOI: 10.1103/PhysRevLett.105.120501

PACS numbers: 03.67.Bg, 06.20.Dk, 42.50.Dv, 42.50.St

Introduction.—After the pioneer works of the 1980s and early 1990s [1–4], the field of quantum-enhanced metrology is witnessing a revival of interest due to the possible implications on fundamental questions of quantum information and technological applications. Recent theoretical analyses are mainly focusing on the interplay between spin squeezing, entanglement and phase sensitivity [5,6], the set of fundamental sensitivity bounds [7–9], and the quest for optimal phase estimation protocols [10,11]. This activity has come along with breakthrough experimental investigations with photons [12], ions [13], cold atoms [14], and Bose-Einstein condensates (BECs) [15,16]. The interplay between theory and experiments plays a crucial role for the development of the field. It should be noticed, however, that while the theoretical investigations have been developed in the context of systems having a fixed, known number of particles, most experiments have been performed with a fluctuating number of particles. The consequences of classical and quantum fluctuations have been generally disregarded.

The goal of this manuscript is to extend along this direction the quantum metrology theory, and, in particular, to discuss the relation between separability, entanglement, spin squeezing, and sub-shot-noise (SSN) sensitivity, and to settle the concept of the Heisenberg limit. In the current literature, these are well established only in the case of linear interferometers operating with input quantum states of a known fixed number of particles. This manuscript generalizes these concepts and extends the quantum phase estimation theory by taking into account classical and quantum fluctuations of the particle number. Our analysis concerns most of the current experiments on precision measurements where the number of particles is known only on average.

\[ \hat{\rho}_{\text{inc}} = \sum_{N=0}^{\infty} Q_N \hat{\rho}^{(N)}, \]

where \( \hat{\rho}^{(N)} \) is a state of \( N \) particles. We extend the usual definition of separability of states of \( N \) particles [20] by considering such incoherent mixtures as separable if they can be written as

\[ \hat{\rho}_{\text{sep}} = \sum_{N=0}^{\infty} Q_N \hat{\rho}_{\text{sep}}^{(N)}, \]

where \( \hat{\rho}_{\text{sep}}^{(N)} = \sum_k P_{k,N} |\phi_{k,N}^{(1)}\rangle\langle \phi_{k,N}^{(1)}| \otimes \cdots \otimes |\phi_{k,N}^{(N)}\rangle\langle \phi_{k,N}^{(N)}| \), \( \{Q_N\} \) and \( \{P_{k,N}\} \) are probability distributions and \( |\phi_{k,N}^{(j)}\rangle \) is a two-mode pure state of a single particle [21]. The existence of a SSR is the consequence of the lack of a suitable phase reference frame (RF) [19]. However, the possibility that a suitable RF can be established in principle cannot be excluded [19]. If SSRs are lifted, then states \( \hat{\rho}_{\text{coh}} \) containing coherent superpositions of different numbers of particles would become physically relevant. These will be called separable if they are separable in every fixed-\( N \) subspace, i.e., if the state \( \hat{\rho}_{\text{coh}} = \sum_N \hat{\rho}_{\text{coh}}^{(N)} \) without such coherences is of the form of Eq. (2) [21]. States which are not separable according to this definition are entangled [22]. In this manuscript, it will be shown that entanglement is a necessary resource for sub-shot-noise sensitivity in linear interferometers.

Phase sensitivity with a linear interferometer.—We consider a linear two-mode interferometer where the density matrix \( \hat{\rho} \) evolves as \( \hat{\rho}(\theta) = e^{-i\theta \hat{J}_z} \hat{\rho} e^{+i\theta \hat{J}_z} \), where \( \theta \) is a real number, \( \vec{n} \) is an arbitrary direction in the three-dimensional space, and \( \hat{J}_z = \hat{\vec{J}} \cdot \vec{n} \) is a collective spin operator. For a nonfixed number of particles \( \hat{\vec{J}} \) is defined as \( \hat{\vec{J}} = \Theta^{+\infty}_{N=1} \vec{J}^{(N)} \), where \( \vec{J}^{(N)} = \frac{1}{2} \sum_{l=1}^{N} \vec{\sigma}^{(l)} \) and \( \vec{\sigma}^{(l)} \) is the vector of Pauli matrices acting on the \( l \)th particle [24]. Note that linear interferometers preserve the number of particles,
\[ [\hat{J}_{\theta}, \hat{N}] = 0. \] The goal is to estimate the parameter \( \theta \) with the maximum possible sensitivity \( \Delta \theta \). The quantum Cramer-Rao theorem ensures that for arbitrary, unbiased, phase estimation protocols, the phase sensitivity is bounded by [25, 26]

\[ \Delta \theta_{\text{QCR}} = \frac{1}{\sqrt{mF_Q[\hat{J}_{\theta}, \hat{\rho}]}}. \tag{3} \]

Here \( m \) is the number of independent measurements done with identical input states \( \hat{\rho} \) and \( F_Q[\hat{J}_{\theta}, \hat{\rho}] \) is the quantum Fisher information (QFI) [26]. Equation (3) is a bound on \( \Delta \theta \) given the state \( \hat{\rho} \) and the number of measurements \( m \) and can be saturated in the central limit.

In the following, we set the fundamental sensitivity bounds of the phase estimation problem by taking, as a constraint, the average total number of particles, \( \bar{N}_{\text{tot}} = m\langle \hat{N} \rangle \), without fixing \( m \) or \( \langle \hat{N} \rangle \) separately. This correctly accounts for the finite available resources [10, 27]. Averages are computed with respect to the input state of the interferometer.

\textbf{Shot-noise limit.}—We define the shot-noise limit as the maximal phase sensitivity attainable with separable states. For separable incoherent mixtures, Eq. (2), \( F_Q[\hat{\rho}_{\text{sep}}, \hat{J}_{\theta}] = \sum_{N}Q_N F_Q[\hat{\rho}^{(\text{inc})}, \hat{J}_{\theta}^{(\text{inc})}] \leq \sum_{N}Q_N N = \langle \hat{N} \rangle \). The first equality is due to the block-diagonal form of \( \hat{J}_{\theta} \) and \( \hat{\rho}_{\text{sep}} \) [28] and the second inequality was proved in Ref. [6]. By using this result and Eq. (3), the shot-noise limit is thus given by

\[ \Delta \theta_{\text{SN}} = \frac{1}{\sqrt{\bar{N}_{\text{tot}}}}, \tag{4} \]

which agrees with the common definition of the shot noise or standard quantum limit. As shown in Appendix A, Eq. (4) holds also when considering separable states with coherences. This brings us to the following results.

\textbf{Entanglement and sub-shot-noise sensitivity.}—An arbitrary state with a nonfixed number of particles is entangled if it fulfills the inequality

\[ \chi^2 = \frac{\langle \hat{N} \rangle}{F_Q[\hat{J}_{\theta}, \hat{\rho}]} < 1, \tag{5} \]

for some direction \( \hat{\theta} \). States satisfying Eq. (5) are useful in a linear interferometer implemented by the transformation \( \hat{J}_{\theta} \), since, according to Eq. (3), they provide a sub-shot-noise phase sensitivity. We further introduce the spin-squeezing condition

\[ \xi^2 = \frac{\langle \hat{N} \rangle (\Delta \hat{J}_{\theta})^2}{(\hat{J}_{\theta})^2 + (\Delta \hat{J}_{\theta})^2} < 1, \tag{6} \]

where the vectors \( \vec{n}_1, \vec{n}_2, \) and \( \vec{n}_3 \) define a right-handed coordinate system [29]. In Appendix B, we prove that \( \chi^2 \leq \xi^2 \) holds. Hence, spin-squeezed states (\( \xi < 1 \)) are entangled and useful for SSN interferometry. Equations (5) and (6) generalize the conditions for entanglement and SSN discussed in [3, 5, 6] to states of a nonfixed number of particles and only require the replacement of \( N \) with \( \langle \hat{N} \rangle \). This result has been previously used without justification in several experimental works.

\textbf{The Heisenberg limit.}—While it is well known that the shot-noise limit Eq. (4) can be overcome, finding the Heisenberg limit (HL), i.e., the ultimate scaling of the phase sensitivity imposed by quantum mechanics for states with nonfixed \( N \), has generated a vivid debate [8, 27, 30–32]. For instance, it has been suggested that \( \Delta \theta \sim 1/\langle \hat{N} \rangle \) is the fundamental sensitivity bound [30]. Recently it was argued that the HL is given by \( \Delta \theta = 1/\sqrt{\langle \hat{N}^2 \rangle} \) [8]. Not only does this limit overcome the bound \( \Delta \theta = 1/\langle \hat{N} \rangle \), but also, since the quantity \( \langle \hat{N}^2 \rangle \) can grow arbitrarily fast with \( \langle \hat{N} \rangle \), the phase sensitivity can be arbitrarily high when considering wildly fluctuating number distributions [33].

The definition of the HL depends on the specific constraints imposed on the resources. In the presence of the SSR, and by taking \( \langle \hat{N} \rangle, \langle \hat{N}^2 \rangle \) and \( m \) as constraints, the phase sensitivity is bounded by

\[ \Delta \theta \geq \max \left[ \frac{1}{\sqrt{m\langle \hat{N}^2 \rangle}}, \frac{1}{m\langle \hat{N} \rangle} \right]. \tag{7} \]

The first bound is derived from Eq. (3) by noticing that, for incoherent mixtures (1), \( F_Q[\hat{\rho}_{\text{coh}}, \hat{J}_{\theta}] = \sum_{N}Q_N F_Q[\hat{\rho}^{(\text{inc})}, \hat{J}_{\theta}^{(\text{inc})}] \leq \sum_{N}Q_N N^2 = \langle \hat{N}^2 \rangle \). It agrees with the bound of [8], except for the factor \( m \), accounting for repeated independent measurements. Note that in the central limit, the bound scales as \( 1/\sqrt{m} \). The second bound is proven in Appendix C. When the number of particles is fixed and equal to \( N \), Eq. (7) recovers the definition of the HL \( \Delta \theta = 1/\sqrt{mN} \) discussed in [7] which takes, as constraints, \( N \) and \( m \), separately. Note, however, that from this limit one does not obtain Eq. (7) by naively replacing \( N \rightarrow \langle \hat{N} \rangle \).

Here, we define the HL as the maximum phase sensitivity attainable with fixed \( \bar{N}_{\text{tot}} \). If coherences can neither be generated nor measured, as in current experimental interferometric protocols because of the SSR, the HL is

\[ \Delta \theta_{\text{HL}} = \frac{1}{\bar{N}_{\text{tot}}}. \tag{8} \]

If the state contains coherences but only POVMs without number coherences are applied, the HL is still given by Eq. (8) cf. Appendix C. In particular, this is true for the Mach-Zehnder interferometer, even if only the relative number of particles is measured or the parity at one exit port. The latter measurement was considered in Ref. [31]. By adapting the proof in Appendix C it can also be shown that the HL Eq. (8) holds for incoherent mixtures even if POVMs with coherences are available.

If the state contains coherences and POVMs with coherences are used, then Eq. (8) is valid in the central limit only [34]. Outside the central limit the bound

\[ \Delta \theta \geq \frac{1}{\sqrt{m\langle \hat{N}^2 \rangle}}, \tag{9} \]
holds, as shown in Appendix C. However, there are arguments suggesting that the HL may be given by Eq. (8) also in this case [27,35].

Examples.—We consider a Mach-Zehnder interferometer, corresponding to a collective rotation around the $y$ axis, i.e., $\vec{n} = \vec{y}$, with a coherent state $|\alpha\rangle$, in the input mode $a$ and a squeezed state $|\zeta\rangle_b$ in the input $b$ as suggested by Caves [1,10,36]. Here $\alpha = |\alpha|e^{i\phi_\alpha}$ and $\zeta = re^{i\phi_\zeta}$. When fixing the relative phase $2\phi_\alpha = \theta$, Eq. (5) gives

$$\chi^2 = \frac{(|\alpha|^2 + \sinh^2 r)}{(|\alpha|^2 - \sinh^2 r)^2}.$$  

For $r = 0$ we have a coherent state entering one of the two ports and vacuum the other port. We obtain $\chi^2 = 1$ and hence shot noise, as expected. In contrast, we obtain $\chi^2 < 1$ when $r > 0$ and $|\alpha|^2 \neq 0$. Spin squeezing is lost for $|\alpha|^2 = \sinh^2 r$, even though the state remains entangled. Equations (10) and (11) hold even for the state $\rho_{red} = \sum_{n} |n\rangle\langle n| \times \rho_{\alpha,\zeta}$, $\alpha, \zeta \in \{0\}$ obtained in the presence of the SSR in [19,37]. Finally, we notice that the state $|\alpha\rangle \otimes |\zeta\rangle$ reaches the HL $\Delta \theta = 1/N_{opt}$ when $|\alpha|^2 = \sinh^2 r$ [10].

Conclusions.—In this Letter we have extended the quantum phase estimation theory to the case of states with fluctuating number of particles. In particular, we have discussed entanglement and spin squeezing and provided the class of entangled states useful for sub-shot-noise sensitivity. The Heisenberg limit has been defined by taking into account the finite available resources. Our results are relevant for most current experiments on the creation, detection, and application of entangled states with a fluctuating number of particles for quantum metrology. We thank J. I. Cirac, K. Mølmer, A. S. Sørensen, and T. Rudolph for stimulating discussions.

Appendix A: Shot-noise limit.—Here we demonstrate that $F_Q(\hat{n}_{cs}) \leq \langle N \rangle$ for states of the form $\rho_{cs} = \sum_k p_k |\psi_{cs}^{(k)}\rangle \langle \psi_{cs}^{(k)}|$ which contain coherent states between different numbers of particles and are separable in every fixed-$N$ subspace. We obtain $F_Q(\hat{n}_{cs}) \leq \max x \sum_k p_k (\Delta \hat{n}_k)^2 |\psi_{cs}^{(k)}\rangle \langle \psi_{cs}^{(k)}| \leq 4\sum_k p_k \sum_{N} (\Delta \hat{n}_k)^2 |\psi_{cs}^{(k)}\rangle \langle \psi_{cs}^{(k)}| \sim \langle N \rangle$, by using the convexity of the quantum Fisher information [38]. We expanded $|\psi_{cs}^{(k)}\rangle = \sum_{N} \left(\sum_{n} |n\rangle_{cs}\langle n|_{cs}^{(k)}\right) \langle \psi_{cs}^{(k)}|$ and used that for pure separable states $\langle (|\psi_{cs}^{(N,k)}\rangle)^2 |\psi_{cs}^{(N,k)}\rangle \leq \frac{N}{4} \langle N \rangle$. By inserting $F_Q(\hat{n}_{cs}) \leq \langle N \rangle$ into Eq. (3), we recover Eq. (4).

Appendix B: Spin-squeezing inequality.—We consider, without loss of generality, a coordinate system such that $\langle \hat{n}_z \rangle = 0$. From the inequalities $F_Q[\hat{\mathcal{H}}, \hat{J}_z] \geq F_{E_{\mathcal{H}}}(\hat{\mathcal{H}})$ [26] and $F_{E_{\mathcal{H}}}(\hat{\mathcal{H}}) \leq 1/\langle \Delta \hat{J}_z \rangle^2$ (obtained by extending the proof in [39] to the case of nonfixed number of particles), it follows that $\chi^2 = \langle N \rangle(\Delta \hat{A}_x^2/(\Delta \hat{A}_y^2)$. By choosing $\tilde{n} = n_2$, $\hat{A} = \hat{J}_z$ and using the commutation relations of the $\hat{J}_z$ operators [29], we obtain $\frac{\sqrt{\chi}}{\Delta \hat{J}_z} = i\text{Tr}[[\hat{J}_z, \hat{\mathcal{H}}], \hat{\mathcal{H}}] = \langle \hat{J}_z \rangle$. Hence $\chi^2 = \langle \Delta \hat{J}_z \rangle^2 / \langle \Delta \hat{J}_z \rangle^2 = \chi^2$.

Appendix C: Heisenberg limit.—In the presence of the SSR, only POVMs without coherences are available. A POVM is of this form if we can write its elements as $\hat{E}(\lambda) = \sum_{M,N} |M\rangle\langle N| \times \mathbb{E}_{M,N}^\lambda$, where $\mathbb{E}_{M,N} = \frac{P(M/N)}{P(_{M/N})}$. For unbiased estimators [7]. For $m = 1$, the conditional probabilities are $P(\lambda) = \sum_{M,N} P(M/N) |\lambda\rangle$ for unbiased estimators [7]. For $m = 1$, the conditional probabilities are $P(\lambda) = \sum_{M,N} P(M/N) |\lambda\rangle$. This can be done in analogy for $m = 1$. The last inequality we used that $\langle \Delta \theta_{\text{est}} \rangle^2 \geq 1/N^2$ holds for unbiased estimators [7]. For $m = 1$, the conditional probabilities are $P(\lambda) = \sum_{M,N} P(M/N) |\lambda\rangle$. This can be done in analogy for $m = 1$. This proves that $\Delta \theta_{\text{est}} \geq 1/m(N)$. The bound (9) can be proven as in Appendix A by using general states and $F_Q[^{\psi^{(N)}, \hat{J}_z}] \leq N^2$. 

[1] C. M. Caves, Phys. Rev. D 23, 1693 (1981).
[2] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A 33, 4033 (1986); M. Xiao, L. A. Wu, and H. J. Kimble, Phys. Rev. Lett. 59, 278 (1987); P. Grangier, R. E. Slusher, B. Yurke, and A. LaPorta, Phys. Rev. Lett. 59, 2153 (1987); M. J. Holland and K. Burnett, Phys. Rev. Lett. 71, 1355 (1993).
[3] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Phys. Rev. A 50, 67 (1994).
[4] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[5] A. Sørensen et al., Nature (London) 409, 63 (2001).
[6] L. Pezzè and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. 96, 010401 (2006).
[8] H. F. Hofmann, Phys. Rev. A 79, 033822 (2009).
[9] Á. Rivas and A. Luis, Phys. Rev. Lett. 105, 010403 (2010).
[10] L. Pezzè and A. Smerzi, Phys. Rev. Lett. 100, 073601 (2008).
For bosons, the operators \( \hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b} \) are then given by Eq. (2) with \( \langle \phi_{kN}^i | \phi_{kN}^f \rangle = 0 \) for all \( i \). It has been argued that particle entanglement due to (anti-)symmetrization in first quantization is unphysical since the individual particles cannot be addressed [23]. However, in interferometers using only collective operations, it can be a useful resource for sub-shot-noise sensitivity.

Density matrices \( \hat{\varrho}^{(N)} \) of \( N \) particles with 2 degrees of freedom act on a Hilbert space \( \mathcal{H}_N = \bigotimes_{k=1}^N \mathcal{H}_N^{(i)} \), where \( \mathcal{H}_N^{(i)} \cong \mathbb{C}^2 \). The operator \( \mathbb{1}_N \) is the projector onto \( \mathcal{H}_N \).

The definition of separability can also be applied formally when treating indistinguishable particles in first quantization, where the set of admissible quantum states is reduced to the symmetric (antisymmetric) subspace for bosons (fermions). Separable states of bosons without coherences are then given by Eq. (2) with \( \langle \phi_{kN}^i | \phi_{kN}^f \rangle = 0 \) for all \( i \). It has been argued that particle entanglement due to (anti-)symmetrization in first quantization is unphysical since the individual particles cannot be addressed [23]. However, in interferometers using only collective operations, it can be a useful resource for sub-shot-noise sensitivity.

Density matrices \( \hat{\varrho}^{(N)} \) of \( N \) particles with 2 degrees of freedom act on a Hilbert space \( \mathcal{H}_N = \bigotimes_{k=1}^N \mathcal{H}_N^{(i)} \), where \( \mathcal{H}_N^{(i)} \cong \mathbb{C}^2 \). The operator \( \mathbb{1}_N \) is the projector onto \( \mathcal{H}_N \).

The definition of separability can also be applied formally when treating indistinguishable particles in first quantization, where the set of admissible quantum states is reduced to the symmetric (antisymmetric) subspace for bosons (fermions). Separable states of bosons without coherences are then given by Eq. (2) with \( \langle \phi_{kN}^i | \phi_{kN}^f \rangle = 0 \) for all \( i \). It has been argued that particle entanglement due to (anti-)symmetrization in first quantization is unphysical since the individual particles cannot be addressed [23]. However, in interferometers using only collective operations, it can be a useful resource for sub-shot-noise sensitivity.