THE EQUIVALENT CLASSICAL METRICS ON THE CARTAN-HARTOGS DOMAINS

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Abstract. In this paper we study the complete invariant metrics on Cartan-Hartogs domains which are the special types of Hua domains. Firstly, we introduce a class of new complete invariant metrics on these domains, and prove that these metrics are equivalent to the Bergman metric. Secondly, the Ricci curvatures under these new metrics are bounded from above and below by the negative constants. Thirdly, we estimate the holomorphic sectional curvatures of the new metrics, we prove that the holomorphic sectional curvatures are bounded from above and below by the negative constants. Finally, by using these new metrics and Yau’s Schwarz lemma we prove that the Bergman metric is equivalent to the Einstein-Kähler metric. That means the Yau’s conjecture is true on Cartan-Hartogs domain.

The concept of Hua domain was introduced by Weiping Yin in 1998. Since then, many good results have been obtained. The Bergman kernel functions are given in explicit forms[1-17]. The comparison theorems for Bergman metric and Kobayashi metric are proved on Cartan-Hartogs domains[18-21]. The explicit form of the Einstein-Kähler metric is got on non-symmetric domain which is the first time in the world[22-26], etc. In this paper we will study the equivalence between the classical metrics. There are many deep results on this subject. Let $\omega_B(D)$, $\omega_C(D)$, $\omega_K(D)$, $\omega_{EK}(D)$ be the Bergman metric, Carathéodory metric, Kobayashi metric and Einstein-Kähler metric on bounded domain $D$ in $\mathbb{C}^n$ respectively. Then we have $\omega_C(D) \leq 2\omega_B(D)$[27,28], $\omega_C(D) \leq \omega_K(D)[29]$, $\omega_C(D) = \omega_K(D)$ if $D$ is the convex domain[30], $\omega_B(D) = \omega_{EK}(D)$ if $D$ is the bounded homogeneous domain in $\mathbb{C}^n$[31,P.300]. For the $\omega_B(D)$ and $\omega_K(D)$, no relationship is known. People had hoped that the inequality $\omega_B(D) \leq C\omega_K(D)$ for some universal constant $C$ would hold, but in 1980 Diederich and Foraess [32] showed that there exist pseudoconvex domain in $\mathbb{C}^3$ where the quotient $\omega_B(D)/\omega_K(D)$ is unbounded. If the inequality $\omega_B(D) \leq C\omega_K(D)$ holds then we say that the comparison theorem for Bergman

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metric and Kobayashi metric on $D$ holds. The above comparison theorem [18-21] are in this sense. Recently, Kefeng Liu, Xiaofeng Sun and Shing-Tung Yau study the equivalence between the classical metrics on Teichmüller spaces and moduli spaces[33-35]. They proved that on Teichmüller spaces and moduli spaces the four classical metrics $\omega_B(D), \omega_C(D), \omega_K(D), \omega_{EK}(D)$ are equivalent. Especially, they proved the conjectures of Yau about the equivalence between the Einstein-Kähler metric and the Teichmüller metric and also its equivalence with the Bergman metric.

In this paper we study the complete invariant metrics on Cartan-Hartogs which is the special types of Hua domains. We prove that the Yau’s conjecture is true on the Cartan-Hartogs domains. This paper is organized as follows. We introduce a class of new complete invariant metrics on Cartan-Hartogs and prove that these metrics are equivalent to the Bergman metrics of the Cartan-Hartogs domains in the first section. In the second section we prove that the Ricci curvatures of these new metrics are bounded from above and below by the negative constants. We prove also that the holomorphic sectional curvatures of these new metrics on Cartan-Hartogs domains are bounded from above and below by the negative constants in the third section. In the fourth section, by using these new metrics and Yau’s Schwarz lemma[36] we prove that the Bergman metric is equivalent to the Einstein-Kähler metric. That means the Yau’s conjecture[37] is true on Cartan-Hartogs domain.

The Cartan-Hartogs domains are defined as follows:

$$Y_I = \{W \in \mathbb{C}^N, Z \in R_I(m, n) : |W|^{2K} < \det(I - ZZ^t), K > 0\},$$

$$Y_{II} = \{W \in \mathbb{C}^N, Z \in R_{II}(p) : |W|^{2K} < \det(I - ZZ^t), K > 0\},$$

$$Y_{III} = \{W \in \mathbb{C}^N, Z \in R_{III}(q) : |W|^{2K} < \det(I - ZZ^t), K > 0\},$$

$$Y_{IV} = \{W \in \mathbb{C}^N, Z \in R_{IV}(n) : |W|^{2K} < (1 - 2ZZ^t + |ZZ^t|^2), K > 0\}.$$

Where det is the abbreviation of determinant; $Z^t$ indicates the transpose of $Z$, $\overline{Z}$ denotes the conjugate of $Z$; and the $R_I(m, n)$, $R_{II}(p)$, $R_{III}(q)$, $R_{IV}(n)$ denote the classical domains in the sense of Hua[38](they are also called Cartan domains).

1. New complete invariant metrics on $Y_I$

In this section we will introduce the new complete invariant metrics on $Y_I$, and prove that these new metrics are equivalent to the Bergman metric on $Y_I$.

1.1. New invariant metrics
1.1.1. Suppose \((Z, W) \in Y_I\), \(Z = (z_{ij})\), \(W = (w_1, w_2, \ldots, w_N)\) and let

\[
Z_1 = (z_1, z_2, \ldots, z_{mn}) = (z_{11}, z_{12}, \ldots, z_{1m}, \ldots, z_{mn}),
\]

\[
Z_2 = (z_{mn+1}, z_{mn+2}, \ldots, z_{mn+N}) = (w_1, w_2, \ldots, w_N),
\]

then the point \((Z, W)\) can be denoted by a vector \(z\) with \(mn + N\) entries, that is

\[
z = (Z_1, Z_2) = (z_1, z_2, \ldots, z_{mn}, z_{mn+1}, z_{mn+2}, \ldots, z_{mn+N}).
\]

1.1.2. Let

\[
G_\lambda = G_\lambda(Z, W) = Y^\lambda[\det(I - Z Z^t)]^{-(m+n+N)}, \lambda > 0,
\]

\[
T_\lambda(Z, W; \overline{Z}, \overline{W}) = (g_{ij}) = \left(\frac{\partial^2 \log G_\lambda}{\partial z_i \partial \overline{z}_j}\right),
\]

where

\[
Y = (1 - X)^{-1}, X = |W|^2[\det(I - ZZ^t)]^{-\frac{1}{2}}.
\]

1.1.3. The following mappings are the holomorphic automorphism of \(Y_I\), which map the point \((Z, W)\) onto point \((0, W^*)\):

\[
\begin{align*}
W^* &= W \det(I - Z_0 \overline{Z}_0)^{\frac{i}{n}} \det(I - Z \overline{Z}_0)^{-\frac{i}{n}} \\
Z^* &= A(Z - Z_0)(I - \overline{Z}_0 Z)^{-1} \overline{D}^{-1}
\end{align*}
\]

Where \(A = (I - Z_0 \overline{Z}_0)^{-1}, \overline{D}^t D = (I - \overline{Z}_0 Z_0)^{-1}\). The set of these mappings is denoted by \(\text{Aut}(Y_I)\).

1.1.4. Let \(f \in \text{Aut}(Y_I)\) and \(Z_0 = Z\), then one has

\[
T_\lambda(Z, W; \overline{Z}, \overline{W}) = J_f|_{Z_0=Z} T_\lambda(0, W^*; 0, \overline{W^*}) J_f^t|_{Z_0=Z},
\]

the Jacobian matrix of \(f\) is equal to

\[
J_f = \begin{pmatrix}
\frac{\partial Z^*}{\partial Z_1} & \frac{\partial W^*}{\partial Z_1} \\
\frac{\partial Z^*}{\partial Z_2} & \frac{\partial W^*}{\partial Z_2}
\end{pmatrix}
\]

Let

\[
J_f|_{Z_0=Z} = \begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix}
\]

Then one has

\[
J_{11} = A^t \cdot \overline{D}^{-1},
\]

\[
J_{12} = \frac{1}{\lambda} \det(I - ZZ^t)^{-\frac{i}{n}} E(Z)^t W;
\]

\[
J_{21} = 0,
\]

\[
J_{22} = \det(I - ZZ^t)^{-\frac{i}{n}} I.
\]
where

\[ E(Z) = \left( \text{tr}[I - ZZ^t]^{-1}I_{11}, \text{tr}[I - ZZ^t]^{-1}I_{12}, \cdots, \text{tr}[I - ZZ^t]^{-1}I_{mn} \right) \]

is the \(1 \times mn\) matrix. And \(I_{\alpha\beta}\) is \(m \times n\) matrix, the \((\alpha\beta)\)-th entry of \(I_{\alpha\beta}\), i.e. the entry located at the junction of the \(\alpha\)-th row and \(\beta\)-th column of \(I_{\alpha\beta}\) is 1, and its others entries are zero. The meaning of \([A \times A]\) can be found in [39] or in 1.1.6 below.

1.1.5. By computations, one has

\[
T_{\lambda I}(0, W^*; 0, W^*) = \begin{pmatrix}
\frac{\lambda}{N} Y X + m + n + \frac{N}{K} I & 0 \\
0 & \lambda Y I + \lambda Y^2 W^* W^*
\end{pmatrix}.
\]

Where \(Y, X\) see 1.1.2.

1.1.6. Let

\[
T_{\lambda I}(Z, W; Z, W) = J_f |_{Z_0 = Z} T_{\lambda I}(0, W^*; 0, W^*) J_f |_{Z_0 = Z} = \begin{pmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{pmatrix},
\]

one has

\[
T_{11} = \left( \frac{\lambda}{N} Y X + m + n + \frac{N}{K} \right) A D D^t A^t + \frac{\lambda}{N} Y^2 X E(Z)^t E(Z),
\]

\[
T_{12} = \frac{\lambda}{K} \det(I - ZZ^t)^{-\frac{1}{2}} \lambda Y^2 E(Z)^t W,
\]

\[
T_{21} = T_{12},
\]

\[
T_{22} = \lambda Y \det(I - ZZ^t)^{-\frac{1}{2}} + \det(I - ZZ^t)^{-\frac{1}{2}} \lambda Y^2 W^* W.
\]

The computations form 1.1.3 to 1.1.6 can be found in [19] for detail. The definition of \(\times\) is the following (see [39]).

Definition: For the \(r \times s\) matrix

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1s} \\
\vdots & \ddots & \vdots \\
a_{r1} & \cdots & a_{rs}
\end{pmatrix},
\]

and \(p \times q\) matrix

\[
B = \begin{pmatrix}
b_{11} & \cdots & b_{1q} \\
\vdots & \ddots & \vdots \\
b_{p1} & \cdots & b_{pq}
\end{pmatrix}.
\]

The \(\times\) of these two matrices is as the follows:

\[
A \times B = \begin{pmatrix}
a_{11} B & \cdots & a_{1q} B \\
\vdots & \ddots & \vdots \\
a_{r1} B & \cdots & a_{rs} B
\end{pmatrix}.
\]

Which is a \(rp \times sq\) matrix.

1.1.7. From 1.1.5 and 1.1.6, one has \(T_{\lambda I}(Z, W; \overline{Z}, \overline{W}) > 0\), and then from the definition of \(G_\lambda\) in 1.1.2, the \(G_\lambda\) generates an invariant metric \(\omega_{G_\lambda}(Y_I)\) of \(Y_I\).

1.2. New invariant metrics are equivalent to Bergman metric
Definition: Two metrics $\mathcal{B}$ and $\mathcal{E}$ of domain $\Omega$ in $\mathbb{C}^n$ are called equivalent, if they are quasi-isometric to each other in the sense that

$$b \leq \frac{\mathcal{B}}{\mathcal{E}} \leq a.$$ 

for two positive constants $a$ and $b$. We will write this as $\mathcal{B} \sim \mathcal{E}$.

The Bergman kernel function of $Y_t$ has the following form:

$$K_{Y_t} = K^{-mn}e^{-mn+\lambda}G(X)\det(I-Z\overline{Z'})^{-(m+n+\frac{m}{2})}.$$  

Where $G(X) = \sum_{j=0}^{mn+1} b_j \Gamma(N+j)(1-X)^{-(N+j)}$, let

$$P(x) = (x+1)\left((x+1+Kn)(x+1+K(n-1))\ldots(x+1+K)\right)$$

Then $b_0 = P(-1) = 0$. And the other $b_j (j = 1, 2, \ldots, mn+1)$ is determined by the following form:

$$b_j = \frac{P(-j-1) - \sum_{k=0}^{j-1} b_k(-1)^k \frac{\Gamma(j+1)}{\Gamma(j-k+1)}}{(-1)^j \Gamma(j+1)},$$

The Bergman kernel function of $Y_t$ generates the Bergman metric $\omega_B(Y_t)$.

1.2.1. By calculations, the metric matrix of Bergman metric $\omega_B(Y_t)$,

$$T_B(Z,W;\overline{Z},\overline{W}) = J_f|_{Z_0=Z}T_{B}(Z^*,W^*;\overline{Z}^*,\overline{W}^*)|_{Z^*=0,J_f^t|_{Z_0=Z}}$$

$$= J_f|_{Z_0=Z} \left( \begin{array}{cc} [\frac{M'}{K}M'X + m + n + \frac{N}{K}]I & 0 \\ 0 & M'I + M'\overline{W}\overline{X} W^* \end{array} \right) J_f^t|_{Z_0=Z}.$$

Where

$$\log G(X) = M, \quad \frac{\partial \log G(X)}{\partial X} = M', \quad \frac{\partial^2 \log G(X)}{\partial X^2} = M''.$$  

And $J_f|_{Z_0=Z}$ is same as that in 1.1.4. For the details please see [03]. Hence

$$(\omega_B(Y_t))^2 = dz J_f|_{Z_0=Z}T_{B}(0,W^*;0,\overline{W}^*)J_f^t|_{Z_0=Z}dz^t$$

$$= dz J_f|_{Z_0=Z} \left( \begin{array}{cc} [\frac{1}{K}M'X + m + n + \frac{N}{K}]I & 0 \\ 0 & M'I + M'\overline{W}\overline{X} W^* \end{array} \right) J_f^t|_{Z_0=Z}dz^t.$$ 

1.2.2. Due to 1.1.5., one has

$$(\omega_{G_X}(Y_t))^2 = dz J_f|_{Z_0=Z}T_{\lambda X}(0,W^*;0,\overline{W}^*)J_f^t|_{Z_0=Z}dz^t$$

$$= dz J_f|_{Z_0=Z} \left( \begin{array}{cc} [\frac{1}{K}\lambda Y X + m + n + \frac{N}{K}]I & 0 \\ 0 & \lambda Y I + \lambda Y^2\overline{W}\overline{X} W^* \end{array} \right) J_f^t|_{Z_0=Z}dz^t.$$ 

Where $Y, X$ are same as that in 1.1.2.
1.2.3. Let $dz J_{|z_0 = z} = (d\tilde{z}, d\tilde{\Im})$, one has

$$\omega_{\mathcal{G}_h}(Y_t) = \left(\frac{1}{K} + m + n + \frac{N}{K}\right)|d\tilde{z}|^2 + d\tilde{\Im} \left(\lambda Y I + \lambda Y^2 W^* W^*\right) d\tilde{\Im},$$

$$\omega_B(Y_t) = \left(\frac{1}{K} M' X + m + n + \frac{N}{K}\right)|d\tilde{z}|^2 + d\tilde{\Im} \left(\lambda Y I + \lambda Y^2 W^* W^*\right) d\tilde{\Im}.$$

1.2.4. From [39], the vector $W^* = (w_1^*, w_2^*, \ldots, w_N^*)$ can be written as

$$W^* = e^{i\theta}(\mu, 0, \ldots, 0) U, \quad \mu \geq 0,$$

where $U$ is the unitary matrix, hence

$$\tilde{W}^* W^* = U^t \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} U,$$

$$M' I^{(N)} + M'' \tilde{W}^* W^* = U^t \begin{pmatrix} M' + M'' \mu^2 & 0 \\ 0 & M'I^{(N-1)} \end{pmatrix} U,$$

$$\lambda Y I^{(N)} + \lambda Y^2 \tilde{W}^* W^* = U^t \begin{pmatrix} \lambda Y + \lambda Y^2 \mu^2 & 0 \\ 0 & \lambda I^{(N-1)} \end{pmatrix} U > 0.$$

1.2.5. Let $d\tilde{\Im}^t = (dv, d\tilde{W})$, then $\omega_{\mathcal{G}_h}(Y_t)$ and $\omega_B(Y_t)$ can be written as:

$$\omega_{\mathcal{G}_h}(Y_t) = \left(\frac{1}{K} + m + n + \frac{N}{K}\right)|dv|^2 + (\lambda Y + \lambda Y^2 \mu^2)|d\tilde{W}|^2 + \lambda Y^2 |d\tilde{W}|^2,$$

$$\omega_B(Y_t) = \left(\frac{1}{K} M' X + m + n + \frac{N}{K}\right)|dv|^2 + (M' + M'' \mu^2)|d\tilde{W}|^2 + M' |d\tilde{W}|^2.$$

1.2.6. $T_M(Z, W; \tilde{Z}, \tilde{W}) > 0$ and $T_{BI}(Z, W; \tilde{Z}, \tilde{W}) > 0$, these imply

$$\frac{1}{K} M' X + m + n + \frac{N}{K} > 0, \quad M' + M'' \mu^2 > 0, \quad M' > 0, \quad Y > 0.$$

1.2.7. Let

$$\Phi(X) = \frac{1}{K} M' X + m + n + \frac{N}{K}, \quad \Psi(X) = \frac{M' + M'' \mu^2}{\lambda Y + \lambda Y^2 \mu^2}, \quad \Upsilon(X) = \frac{M'}{\lambda Y},$$

then all of $\Phi(X), \Psi(X), \Upsilon(X)$ is positive continuous function of $X$ on the interval $[0, 1]$. If

$$\lim_{X \to 1} \Phi(X), \quad \lim_{X \to 1} \Psi(X), \quad \lim_{X \to 1} \Upsilon(X)$$

are existent and positive, then all of $\Phi(X), \Psi(X), \Upsilon(X)$ have the positive maximum and the positive minimum on $[0, 1]$.

1.2.8. Because

$$G(X) = \sum_{j=0}^{mn+1} b_j (N + j)(1 - X)^{-(N+j)} = \sum_{j=0}^{mn+1} b_j (N + j) Y^{(N+j)},$$

therefore

$$\frac{dG(Y)}{dX} = G'(X) = \sum_{j=0}^{mn+1} b_j (N + j + 1) Y^{(N+j+1)},$$

$$\frac{d^2G(Y)}{dX^2} = G''(X) = \sum_{j=0}^{mn+1} b_j (N + j + 2) Y^{(N+j+2)}.$$

Let

$$M' = G'(X) G^{-1}(X),$$
\[ M'' = G''(X)G^{-1}(X) - G'(X)^2G^{-2}(X). \]

1.2.9. We will compute the limits of \( \Phi(X) \), \( \Psi(X) \) and \( \Upsilon(X) \) when \( X \) tends to 1. By calculations, one has

\[
\lim_{X \to 1} \Phi(X) = \lim_{Y \to \infty} \Phi(X) = \frac{mn + N + 1}{\lambda}.
\]

Hence there exists \( 0 < \nu < \delta \) such that \( 0 < \nu \leq \Phi(X) \leq \delta \).

1.2.10. Similarly, one has

\[
\lim_{X \to 1} \Psi(X) = \lim_{Y \to \infty} \Psi(X) = \frac{mn + N + 1}{\lambda},
\]

and

\[
\lim_{X \to 1} \Upsilon(X) = \lim_{Y \to \infty} \Upsilon(X) = \frac{mn + N + 1}{\lambda}.
\]

Therefore there exist \( \zeta, \eta, \rho \) and \( \varrho \) such that \( 0 < \zeta \leq \Psi(X) \leq \eta, 0 < \rho \leq \Upsilon(X) \leq \varrho \).

1.2.11. Let \( a^2 = \max\{\mu, \eta, \varrho\} \) and \( b^2 = \min\{\nu, \zeta, \rho\} \), then one has \( 0 < b \leq \omega_B(Y) \omega_G(Y) \leq a \). Therefore the following theorem is proved

Theorem: All of the above new complete invariant metrics is equivalent to the Bergman metric on \( Y \). That is

\[ \omega_B(Y) \sim \omega_G(Y). \]

Because the Bergman metric \( \omega_B(Y) \) is complete ([40]), hence the new metric \( \omega_G(Y) \) is also complete.

1.2.12. By using the same idea and method. If we introduce the following functions

\[
G_\lambda = Y^{\lambda} \beta(Z, \overline{Z})^{-\left(p+\frac{N}{2}\right)}, \lambda > 0; \\
(Y = (1 - X)^{-1}, X = |W|^2[\det(I - ZZ')]^{-\frac{p}{2}}, (Z, W) \in Y_{II})
\]

\[
G_\lambda = Y^{\lambda} \beta(Z, \overline{Z})^{-\left(q+\frac{N}{2}\right)}, \lambda > 0; \\
(Y = (1 - X)^{-1}, X = |W|^2[\det(I - ZZ')]^{-\frac{q}{2}}, (Z, W) \in Y_{III})
\]

\[
G_\lambda = Y^{\lambda} \beta(Z, \overline{Z})^{-\left(n+\frac{N}{2}\right)}, \lambda > 0; \\
(Y = (1 - X)^{-1}, X = |W|^2[\det(I - ZZ')]^{-\frac{n}{2}}, (Z, W) \in Y_{IV})
\]

for \( Y_{II}, Y_{III}, Y_{IV} \) respectively, then above functions generate the complete invariant metrics and equivalent to the Bergman metrics on \( Y_{II}, Y_{III}, Y_{IV} \) respectively.
2. Ricci curvatures of new complete invariant metrics

In this section we will prove that the Ricci curvatures of new complete invariant metrics are bounded from above and below by the negative constants on the Cartan-Hartogs domains.

2.1. Due to the definition, the Ricci curvature $\text{Ric}_{\lambda I}$ of $\omega_{G_{\lambda}}(Y_I)$ on $Y_I$ has the following form

$$
\text{Ric}_{\lambda I} = - \frac{dz \left( \frac{\partial^2 \log|\det T_{\lambda I}(Z,W;\overline{Z},\overline{W})|}{\partial z_i \partial \overline{z}_j} \right) dz}{dz T_{\lambda I}(Z,W;\overline{Z},\overline{W})}.
$$

2.2. Let

$$
G_I(X) = \left( \frac{\lambda Y}{K} + m + n + \frac{N - \lambda}{K} \right)^{mn} \lambda^N Y^{N+1},
$$

then from 1.1.4 and 1.1.5., one has

$$
\det T_{\lambda I}(Z,W;\overline{Z},\overline{W}) = G_I(X) \left[ \det(I - ZZ^t) \right]^{-(m+n+\frac{N}{K})}.
$$

2.3. Similar to 1.2.1, one has

$$
dz \left( \frac{\partial^2 \log|\det T_{\lambda I}(Z,W;\overline{Z},\overline{W})|}{\partial z_i \partial \overline{z}_j} \right) dz := (\omega_{\det(Y_I)})^2 = dz J_f|_{Z_0=Z} dz.
$$

Where

$$
\log G_I(X) = M_I, \quad \frac{\partial \log G_I(X)}{\partial X} = M_I', \quad \frac{\partial^2 \log G_I(X)}{\partial X^2} = M_I''.
$$

Here $J_f|_{Z_0=Z}$ is same as that in 1.1.4. In the following we will prove

$$
\left( \frac{\partial^2 \log|\det T_{\lambda I}(Z,W;\overline{Z},\overline{W})|}{\partial z_i \partial \overline{z}_j} \right) > 0.
$$

That is the

$$
\begin{pmatrix}
\frac{1}{K} M_I' X + m + n + \frac{N}{K} I & 0 \\
0 & M_I' I + M_I'' W^* W^*
\end{pmatrix}
$$

is positive definite matrix. By calculations one has

$$
M_I' = \frac{mn\lambda Y^2}{\lambda Y + K(m+n) + N - \lambda} + (N+1)Y > 0,
$$

hence

$$
\frac{1}{K} M_I' X + m + n + \frac{N}{K} = \frac{mn\lambda Y(Y-1)}{K(\lambda Y + K(m+n) + N - \lambda)} + \frac{(N+1)(Y-1)}{K} + m + n + \frac{N}{K} > 0.
$$

By calculations, one has

$$
M_I'' = \frac{mn\lambda Y^3(\lambda Y + 2Km + 2Kn + 2N - 2\lambda)}{\left(\lambda Y + K(m+n) + N - \lambda\right)^2} + (N+1)Y^2.
$$
Because $W^* = (w_1^*, w_2^*, \ldots, w_N^*)$ can be denoted by

$$W^* = e^{i\theta}(\mu, 0, \ldots, 0)U, \quad \mu \geq 0,$$

where $U$ is $(N, N)$ unitary matrix, therefore

$$W^* = U^t \begin{pmatrix} \mu^2 & 0 \\ 0 & 0 \end{pmatrix} U,$$

$$M'_I^{(N)} + M'' \overrightarrow{W^*} = U^t \begin{pmatrix} M'_I & M'' \mu^2 \\ 0 & M'_I(N-1) \end{pmatrix} U,$$

but $W^* \overrightarrow{W^*} = X = \mu^2$. By calculations, one has

$$M'_I + M'' \mu^2 = M'_I + M'' X = (N + 1)Y^2 + \frac{mn\alpha^2}{(\lambda Y + K(m+n) + N - \lambda)^2} [(\lambda Y + K(m+n) + N - \lambda)^2 + K(m+n) + \alpha Y + (K(m+n) + N)] > 0.$$ 

Therefore we proved that

$$\begin{pmatrix} \frac{1}{\lambda} M'_I X + m + n + \frac{N}{\lambda} I & 0 \\ 0 & M'_I M'' \overrightarrow{W^*} \end{pmatrix}$$

is positive matrix. That is

$$\left( \frac{\partial^2 \log|\det T_{\lambda}(Z, W; \overline{Z}, \overline{W})|}{\partial z_i \partial \overline{z}_j} \right) > 0.$$ 

2.4. From 1.2.2., we know that

$$dzT_{\lambda}(Z, W; \overline{Z}, \overline{W}) = (\omega_{\lambda}(Y_1))^2$$

$$= dz J_f|z_0 = \frac{1}{\lambda} \left( \frac{\lambda Y X + m + n + \frac{N}{\lambda} I}{\lambda Y + \lambda Y^2 \overrightarrow{W^*}} \right) T_f|z_0 = \frac{1}{\lambda} \left( \frac{\lambda Y X + m + n + \frac{N}{\lambda} I}{\lambda Y + \lambda Y^2 \overrightarrow{W^*}} \right) T_f$$

Where $Y, X$ are same as that in 1.1.2.

2.5. Let $dz J_f|z_0 = z = (dz, d\overline{z}, d\overline{W})$, then similar from 1.2.4 to 1.2.5, one has

$$(\omega_{\lambda}(Y_1))^2 = \left( \frac{\lambda Y X + m + n + \frac{N}{\lambda} I}{\lambda Y + \lambda Y^2 \overrightarrow{W^*}} \right) \left( d\overline{z} \right)^2 + (\lambda Y + \lambda Y^2 \mu^2) |d\overline{W}|^2,$$

$$(\omega_{det}(Y_1))^2 = \left( \frac{\lambda Y X + m + n + \frac{N}{\lambda} I}{\lambda Y + \lambda Y^2 \overrightarrow{W^*}} \right) \left( d\overline{z} \right)^2 + (M'_I + M'' \mu^2) |d\overline{W}|^2.$$ 

2.6. Because $T_{\lambda}(Z, W; \overline{Z}, \overline{W}) > 0$ and $\left( \frac{\partial^2 \log|\det T_{\lambda}(Z, W; \overline{Z}, \overline{W})|}{\partial z_i \partial \overline{z}_j} \right) > 0$, hence

$$\frac{1}{\lambda} M'_I X + m + n + \frac{N}{\lambda} > 0, \quad M'_I + M'' \mu^2 > 0, \quad M'_I > 0, \quad Y > 0.$$ 

2.7. Let

$$\Phi_f(X) = \frac{1}{\lambda} M'_I X + m + n + \frac{N}{\lambda}, \quad \Psi_f(X) = \frac{M'_I + M'' \mu^2}{\lambda Y + \lambda Y^2 \mu^2}, \quad Y_f(X) = \frac{M'_I}{\lambda Y},$$

then $\Phi_f(X), \Psi_f(X)$ and $Y_f(X)$ are the positive and continuous functions of $X$ on $[0, 1)$. If the

$$\lim_{X \to 1} \Phi_f(X), \lim_{X \to 1} \Psi_f(X), \lim_{X \to 1} Y_f(X)$$

exist and positive, then $\Phi_f(X), \Psi_f(X), Y_f(X)$ have the positive maximum and positive minimum on $[0, 1)$. 

2.8. We know the values of $M'_I, M''_I$ in 2.3., therefore one can calculate the limits of $\Phi_I(X), \Psi_I(X)$ and $\Upsilon_I(X)$ as $X \to 1$.

2.9. It is easy to show that
\[
\lim_{X \to 1} \Phi_I(X) = \lim_{Y \to \infty} \Phi_I(X) = \frac{mn + N + 1}{\lambda},
\]
then there exists $0 < \nu < \delta$ such that
\[
0 < \nu \leq \Phi(X) \leq \delta.
\]

2.10. By the same method, one has
\[
\lim_{X \to 1} \Psi_I(X) = \lim_{Y \to \infty} \Psi_I(X) = \frac{mn + N + 1}{\lambda},
\]
and
\[
\lim_{X \to 1} \Upsilon_I(X) = \lim_{Y \to \infty} \Upsilon_I(X) = \frac{mn + N + 1}{\lambda}.
\]
Therefore there exist $\zeta, \eta, \rho$ and $\varrho$ such that
\[
0 < \zeta \leq \Psi(X) \leq \eta, \quad 0 < \rho \leq \Upsilon(X) \leq \varrho.
\]

2.11. Let $a^2 = \max\{\mu, \eta, \varrho\}$ and $b^2 = \min\{\nu, \zeta, \rho\}$, then one has
\[
0 < b \leq \frac{\omega_{det}(Y_I)}{\omega_{G_\lambda}(Y_I)} \leq a.
\]

2.12. Up to now, the following theorem is proved.

Theorem: The Ricci curvature of $\omega_\lambda(Y_I)$ on $Y_I$ is bounded from above and below by the negative constants, that is
\[
-a \leq Ric_{\lambda_I} = -\frac{\omega_{det}(Y_I)}{\omega_{G_\lambda}(Y_I)} \leq -b.
\]

2.13. By using the same method, the Ricci curvature of $\omega_{G_{\lambda I I I}}(\omega_{G_{\lambda I I I}}, \omega_{G_{\lambda I V}})$ on $Y_{I I I}(Y_{I I I}, Y_{I V})$ is also bounded from above and below by the negative constants.

3. Holomorphic sectional curvature of the new metrics

In this section, the estimate of the holomorphic sectional curvatures of new complete invariant metrics on Cartan-Hartogs domains will be given. They are bounded form above and below by the negative constants.

3.1. By the definition the holomorphic sectional curvature $\omega_{\lambda I}(z, dz)$ of $\omega_{G_\lambda}(Y_I)$ on $Y_I$ has the following form:
\[
\omega_{\lambda I}(z, dz) = \frac{dz(-dtdT + dt T^{-1} dtd')dz'}{(dz T' dz')^2},
\]
where $T = T_M(Z, W; \overline{Z}, \overline{W})$. Because the holomorphic sectional curvature is invariant under the mapping of $\text{Aut}(Y_I)$. And for any $(Z, W) \in Y_I$, there exists a
f ∈ Aut(Y) such that f(Z, W) = (0, W*). Therefore it is sufficient to compute the value of ωλf(z, dz) at point (0, W*). In the following computation, W stands for W* for the sake of convenience. And if Z* = 0 then |W*|^2 = X. Where the X can be found in 1.1.2.

3.2. Because

\[ dT = \begin{pmatrix} dT_{11} & dT_{12} \\ dT_{21} & dT_{22} \end{pmatrix}, \quad d\bar{T} = \begin{pmatrix} d\bar{dT}_{11} & d\bar{dT}_{12} \\ d\bar{dT}_{21} & d\bar{dT}_{22} \end{pmatrix}. \]

Where the T_{11}, T_{12}, T_{21}, T_{22} can be found in 1.1.6.

3.3. We will compute the value of T, \( T^{-1}, dT, \bar{dT} \) at point (0, W). Where

\[ G(X) = Y^A, \quad \log G(X) = M = \log Y^A, \quad \frac{\partial \log G(X)}{\partial X} = M' = \lambda Y, \]

\[ \frac{\partial^2 \log G(X)}{\partial X^2} = M'' = \lambda Y^2. \quad M''' = 2\lambda Y^3, \quad M^{(4)} = 6\lambda Y^4. \]

Remark: Where the M, M', M'' are different from that in 1.2.1 to 1.2.8. By complicate calculations, one has

\[ T|_{Z=0} = \begin{pmatrix} (\frac{1}{K} M' X + m + n + \frac{N}{K}) I & 0 \\ 0 & M' + M'' W \end{pmatrix}. \]

\[ T^{-1}|_{Z=0} = \begin{pmatrix} \frac{1}{M'} (M' X + m + n + \frac{N}{K})^{-1} I & 0 \\ 0 & \frac{1}{M'} (I - (M' + M'' X)^{-1} W M'') \end{pmatrix}. \]

\[ dT|_{Z=0} = \begin{pmatrix} \frac{X M'' + M'}{K} W dW' I \\ \frac{X M'' + M'}{K} dZ_1 \\ \frac{M'' (W dW' I) W + M'' (W dW' I + W' dW)}{K} \end{pmatrix}. \]

\[ d\bar{dT}|_{Z=0} = \frac{X}{K} (X M'' + M') |dZ|^2 I + \frac{X}{K} (X M'' + M') |d\bar{Z}|^2 I. \]

\[ d\bar{dT}_1|_{Z=0} = \frac{1}{K} (X M'' + 2M') |W dW|^2 I + \frac{1}{K} (X M'' + M') |dW|^2 I. \]

\[ d\bar{dT}_2|_{Z=0} = \frac{1}{K} (X M'' + 2M'') (W dW') (dZ_1 W) + \frac{1}{K} (X M'' + M') (dW' dZ_1). \]

\[ d\bar{dT}_2|_{Z=0} = \frac{1}{K} (X M'' + M') |dZ|^2 I + \frac{1}{K} (X M'' + 2M'') |dZ_1|^2 W' W \]

\[ + (M'' I + M'' W W') |dW|^2 I + M'' |dW|^2 dW + M^{(4)} |W dW'|^2 |W dW|^2 |W dW'|^2 \]

\[ + M'' |W dW'|^2 I + (W dW') (dW' W) + (W dW') (W' dW'). \]

3.4. Let

\[ -\bar{d}dT + dTT^{-1} d\bar{T}|_{Z=0} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \]
then one has
\begin{align*}
R_{11} & = \left( \frac{1}{\kappa^2} (XM'' + M')^2 (\frac{k}{\kappa} M'X + m + n + \frac{N}{\kappa}) \right)^{-1} - \\
& \left( \frac{1}{\kappa} (XM'' + 2M'') \right) I|W| dW^1|dZ_1|^2 I \\
& - \left( \frac{1}{\kappa} (XM'' + M') |dW|^2 I \right) - \left( \frac{k}{\kappa} (XM'' + M') \right) dZ dZ \\
& - \left( \frac{1}{\kappa} M'X + m + n + \frac{N}{\kappa} \right) (dZ dZ \times I + I \times dZ dZ),
\end{align*}

\begin{align*}
R_{12} & = \left( \frac{1}{\kappa} (XM'' + M')^2 (\frac{k}{\kappa} M'X + m + n + \frac{N}{\kappa}) \right)^{-1} \\
& - \left( \frac{1}{\kappa} (XM'' + 2M'') \right) |W| dW^1 dZ_1 W - \left( \frac{1}{\kappa} (XM'' + M') \right) (dZ dZ dW),
\end{align*}

\begin{align*}
R_{21} & = \overline{R}_{12},
\end{align*}

\begin{align*}
R_{22} & = (M')^{-1} [M' (XM'' + 4M'') - (XM'' + M')^{-1} M'' (XM'' + 2M'')^2] \\
& |W| dW^1 |dW| W + [(M')^2 (M')^{-1} - M''] (|W| dW^1 I + W dW) \\
& (dW^1 I + dW dW) - \left( \frac{1}{\kappa} (XM'' + M') \right) |dZ_1|^2 I - M'' |dW|^2 I \\
& + \left( \frac{1}{\kappa} (XM'' + M')^2 (\frac{k}{\kappa} M'X + m + n + \frac{N}{\kappa}) \right)^{-1} \\
& - \left( \frac{1}{\kappa} (XM'' + 2M'') \right) |dZ_1|^2 |W| dW^1 dW - M'' |dW|^2 dW - M (4) |W| dW^1 |dW|^2 W.
\end{align*}

3.5. Because

\begin{align*}
dz \left( -\overline{dT} + dTT^{-1} \overline{dT} \right) dZ |_{Z=0} \\
= (dZ_1, dW) \left( \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right) (dZ_1, \overline{dW}) \\
= dZ_1 R_{11} \overline{dZ_1} + dW R_{21} \overline{dZ_1} + dZ_1 R_{12} dW dW + dW R_{22} dW dW.
\end{align*}

By calculations one has

\begin{align*}
dz \left[ -\overline{dT} + dTT^{-1} \overline{dT} \right] dZ |_{Z=0} \\
= P_1 |W| dW^1 |dW|^4 + P_{12} |W| dW |dW|^2 + P_2 |dW|^4 \\
+ Q_1 |dW|^2 |dZ_1|^2 + Q_2 |W| dW |dZ_1|^2 \\
+ R_1 |dZ_1|^4 - \left( \frac{k}{\kappa} M'X + m + n + \frac{N}{\kappa} \right) dZ_1 (dZ dZ \times I + I \times dZ dZ) dZ_1.
\end{align*}
Where
\[ P_1 = \frac{1}{M'}[M'''(XM'' + 4M'') - (XM'' + M')^{-1}M''(XM'' + 2M'')] - M^{(4)} \]
\[ = -2\lambda Y^4, \]
\[ P_{12} = 4[(M'')^2(M')^{-1} - M''] = -4\lambda Y^3, \]
\[ P_2 = -2M'' = -2\lambda Y^2, \]
\[ Q_1 = -\frac{4}{K}(XM'' + M') = -\frac{4\lambda Y^2}{K}, \]
\[ Q_2 = 4\left[\frac{1}{K}(XM'' + M')^2\left(\frac{1}{K}M'X + m + n + \frac{N}{K}\right)^{-1} - \frac{1}{K}(XM'' + 2M'')\right] \]
\[ = \frac{4}{K}\lambda^2 Y^4\left(\frac{1}{K} + m + n + \frac{N}{K}\right)^{-1} - \frac{8}{K}\lambda Y^3 \]
\[ = \frac{4}{K}\lambda^2 Y^4(\lambda Y + M_1)^{-1} - \frac{8}{K}\lambda Y^3, \]
\[ R = -\frac{2}{K}((XM'' + M')X = -\frac{2}{K}\lambda(Y^2 - Y). \]

where \(M_1 = (m + n)K + N - \lambda.\)

3.6. By calculations one has
\[ dz_1(d\overline{z}dZ^t \cdot I + I \cdot d\overline{Z}^t dz) = 2tr(d\overline{Z}d\overline{Z}^t dZ dZ^t). \]

Let
\[ \Omega_1 = P_1|Wd\overline{W}|^4 + P_{12}|Wd\overline{W}|^2|dW|^2 + P_2|dW|^4 + Q_1|dW|^2|dz_1|^2 \]
\[ + Q_2|Wd\overline{W}|^2|dz_1|^2 + R|dz_1|^4 - 2K^{-1}(\lambda Y + M_1)tr(d\overline{Z}dZ dZ d\overline{Z}^t), \]
and
\[ \Omega_2 = [K^{-1}(\lambda Y + M_1)|dz_1|^2 + \lambda Y|dW|^2 + \lambda Y^2|Wd\overline{W}|^2]^2. \]

Hence
\[ \omega_{\lambda t}(z, dz)|_{z=0} = \frac{\Omega_1}{\Omega_2}. \]

3.7. Let \(Z\) be the \((m, n)\) complex matrix, then it is easy to get
\[ tr(Z\overline{Z}Z\overline{Z}^t) \leq tr(Z\overline{Z}Z\overline{Z}^t)tr(Z\overline{Z}Z\overline{Z}^t) \leq mtr(Z\overline{Z}Z\overline{Z}^t). \]

Therefore one has
\[ \frac{2}{mK}(\lambda Y + M_1)|dz_1|^4 \leq \frac{2}{K}(\lambda Y + M_1)tr(d\overline{Z}dZ dZ d\overline{Z}^t) \leq \frac{2}{K}(\lambda Y + M_1)|dz_1|^4. \]

3.8. The lower bound of holomorphic sectional curvature

3.8.1. From 3.6., we know that the \(P_1\) and \(\lambda^2 Y^4\) are the coefficients of \(|Wd\overline{W}|^4\)
in \(\Omega_1\) and \(\Omega_2\) respectively, let
\[ \Phi_1 = \frac{P_1}{\lambda^2 Y^4}, \]
the $P_{12}$ and $2\lambda^2Y^3$ are the coefficients of $|WdW|^2|dW|^2$ in $\Omega_1$ and $\Omega_2$ respectively, let

$$\Phi_2 = \frac{P_{12}}{2\lambda^2Y^3},$$

the $P_2$ and $\lambda^2Y^2$ are the coefficients of $|dW|^4$ in $\Omega_1$ and $\Omega_2$ respectively, let

$$\Phi_3 = \frac{P_2}{\lambda^2Y^2},$$

the $Q_1$ and $2K^{-1}\lambda Y(\lambda Y + M_1)$ are the coefficients of $|dW|^2|dZ_1|^2$ in $\Omega_1$ and $\Omega_2$ respectively, let

$$\Phi_4 = \frac{Q_1}{2K^{-1}\lambda Y(\lambda Y + M_1)},$$

the $Q_2$ and $2K^{-1}(\lambda Y + M_1)\lambda Y^2$ are the coefficients of $|WdW|_t^2|dZ_1|^2$ in $\Omega_1$ and $\Omega_2$ respectively, let

$$\Phi_5 = \frac{Q_2}{2K^{-1}(\lambda Y + M_1)\lambda Y^2}$$

and let

$$\Phi_6 = \frac{R|dZ_1|^4 - 2K^{-1}(\lambda Y + M_1)|\text{tr}(dZdZ^t)dZdZ^t|}{K^{-2}(\lambda Y + M_1)^2|dZ_1|^4}.$$

Then one has

$$\Phi_1 = -\frac{2}{\lambda}, \quad \Phi_2 = -\frac{2}{\lambda}, \quad \Phi_3 = -\frac{2}{\lambda}, \quad \Phi_4 = -\frac{2Y}{\lambda Y + M_1}.$$

3.8.2. Because

$$\Phi_5 = -\frac{2Y(\lambda Y + 2M_1)}{(\lambda Y + M_1)^2} \geq -\frac{2Y(\lambda Y + 2(m + n)K + 2N)}{(\lambda Y + M_1)^2} := \Phi_{51},$$

and by using the inequality in 3.7., one has

$$\Phi_6 \geq \frac{R|dZ_1|^4 - 2K^{-1}(\lambda Y + M_1)|dZ_1|^4}{K^{-2}(\lambda Y + M_1)^2|dZ_1|^4} = \frac{-2\lambda(Y^2 - Y) - 2K(\lambda Y + M_1)}{(\lambda Y + M_1)^2} := \Phi_{61}.$$

3.8.3. It is easy to see that the $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Phi_4$, $\Phi_{51}$, $\Phi_{61}$ are the negative continues functions of $Y$ on the interval $[1, \infty)$. If $Y \to \infty$, then their limits are existent and are the negative numbers. Hence all of $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Phi_4$, $\Phi_{51}$, $\Phi_{61}$ have the negative minimums on $[1, \infty)$ respectively. Let $-a$ be the smallest one of them. Then it is easy to show that $\Omega_1 \geq -a\Omega_2$, that is

$$\omega_{\lambda t}(z, dz)|_{z=0} \geq -a.$$

3.9. The upper bound of holomorphic sectional curvature

The $\omega_{\lambda t}(z, dz)|_{z=0}$ can be rewritten as

$$\omega_{\lambda t}(z, dz)|_{z=0} = -C + \frac{\Omega_3}{\Omega_4}, \quad C > 0.$$

Where

$$\Omega_3 = P_1^*|\overline{WdW}|^4 + P_{12}^*|Wd\overline{W}|^2|dW|^2 + P_2^*|dW|^4 + Q_1^*|dW|^2|dZ_1|^2$$
\[
+ Q_2^* |WdW|^2 |dZ_1|^2 + R^* |dZ_1|^4 - 2K^{-1}(\lambda Y + M_1) \text{tr}(dZdZ' dZdZ'),
\]
\[
\Omega_1 = [K^{-1}(\lambda Y + M_1)|dZ_1|^2 + \lambda Y |dW|^2 + \lambda Y^2 |WdW|^2 ]^2,
\]
\[
P_1^* = P_1 + C\lambda^2 Y^4 = -\lambda Y^4 (2 - C\lambda),
\]
\[
P_{12}^* = P_{12} + aC\lambda^2 Y^3 = -2\lambda Y^3 (2 - C\lambda),
\]
\[
Q_1^* = Q_1 + 2CK^{-1}\lambda Y (\lambda Y + M_1) = -4K^{-1}\lambda Y^2 + 2CK^{-1}\lambda Y (\lambda Y + M_1),
\]
\[
Q_2^* = Q_2 + 2CK^{-1}\lambda Y^2 (\lambda Y + M_1)
\]
\[
= 4K^{-1}\lambda^2 Y^4 (\lambda Y + M_1)^{-1} - 8K^{-1}\lambda Y^3 + 2CK^{-1}\lambda Y^2 (\lambda Y + M_1),
\]
\[
R^* = R + CK^{-2}(\lambda Y + M_1)^2 = -2\lambda K^{-2}(Y^2 - Y) + CK^{-2}(\lambda Y + M_1)^2.
\]

If
\[
P_1^* \leq 0, \quad P_{12}^* \leq 0, \quad P_2^* \leq 0,
\]
and
\[
Q_1^* |dW|^2 |dZ_1|^2 + Q_2^* |WdW|^2 |dZ_1|^2 \leq 0,
\]
\[
R^* |dZ_1|^4 - 2K^{-1}(\lambda Y + M_1) \text{tr}(dZdZ' dZdZ') \leq 0,
\]
then
\[
\omega_M (z, dz)|_{z=0} \leq -C.
\]

3.9.1. It is easy to see that if \( C \leq \frac{2}{K} \), then \( P_1^* \leq 0, \quad P_{12}^* \leq 0, \quad P_2^* \leq 0. \)

3.9.2. Because \(|WdW|^2 \leq |W|^2 |dW|^2 = X|dW|^2 = (1 - Y^{-1}) |dW|^2\), and if \( C \leq \frac{2\lambda Y}{\lambda Y + M_1} := \Phi_{42}\), then \( Q_1^* \leq 0. \) At this time one has
\[
Q_1^* |dW|^2 |dZ_1|^2 + Q_2^* |WdW|^2 |dZ_1|^2 \leq (Q_1^* X^{-1} + Q_2^*) |WdW|^2 |dZ_1|^2.
\]

Then by calculations, one has
\[
Q_1^* X^{-1} + Q_2^* = \frac{2C\lambda Y^3 (\lambda Y + M_1)}{K(Y - 1)} - \frac{4\lambda Y^3 [\lambda(Y - 1)^2 + (M_1 + \lambda)(2Y - 1)]}{K(Y - 1)(\lambda Y + M_1)}.
\]

Therefore if
\[
C \leq \frac{2[\lambda(Y - 1)^2 + (M_1 + \lambda)(2Y - 1)]}{(\lambda Y + M_1)^2} := \Phi_{52},
\]
and
\[
C \leq \frac{2Y}{(\lambda Y + M_1)} := \Phi_{42},
\]
where \( M_1 = (m + n)K + N - \lambda \), then
\[
Q_1^* |dW|^2 |dZ_1|^2 + Q_2^* |WdW|^2 |dZ_1|^2 \leq 0.
\]

3.9.3. By using the inequality in 3.7., one has
\[
R^* |dZ_1|^4 - \frac{2(\lambda Y + M_1) \text{tr}(dZdZ' dZdZ')}{K} \leq R^* |dZ_1|^4 - \frac{2(\lambda Y + M_1)}{mK} |dZ_1|^4.
\]
By calculations, if
\[
C \leq \frac{2[\lambda Y^2 + Km^{-1}(\lambda Y + M_1)]}{(\lambda Y + M_1)^2} := \Phi_{62},
\]
then
\[
R^*|dZ_1|^4 - 2K^{-1}(\lambda Y + M_1)\text{tr}(dZ\overline{dZ}^t dZ\overline{dZ}^t) \leq 0.
\]

3.9.4. Because \(\Phi_{42}, \Phi_{52}, \Phi_{62}\) are the positive continues functions of \(Y\) on the interval \([1, \infty)\). It is easy to show that, when \(Y \to \infty\), the limits of \(\Phi_{42}, \Phi_{52}, \Phi_{62}\) are existent and are equal to the positive numbers \(\frac{2}{\lambda}\). Then \(\Phi_{42}, \Phi_{52}, \Phi_{62}\) have the positive minimums on \([1, \infty)\) respectively. Let \(b\) be the smallest one of them. Then if \(C \leq b\), one has
\[
Q_1^*|dW|^2|dZ_1|^2 + Q_2^*|W d\overline{W}^t|^2|dZ_1|^2 \leq 0,
\]
\[
R^*|dZ_1|^4 - 2K^{-1}(\lambda Y + M_1)\text{tr}(dZ\overline{dZ}^t dZ\overline{dZ}^t) \leq 0.
\]

3.9.5. From 3.9.1 to 3.9.4, there exists
\[
C \leq \min\{b, \frac{2}{\lambda}\},
\]
and \(C > 0\) such that
\[
\omega_M(z, dz) \leq -C.
\]

By the 3.8.3 and 3.9.5., one has the following theorem.

3.10. Theorem: There exists positive constant \(a, C\) dependent on \(Y_I, \lambda\) such that the holomorphic sectional curvature \(\omega_M(z, dz)\) of metric \(\omega_{G_{\lambda}}(Y_I)\) on \(Y_I\) satisfies
\[
-a \leq \omega_M(z, dz) \leq -C.
\]

This theorem is also true for the other Cartan-Hartogs domains.

4. BERGMAN METRIC IS EQUIVALENT TO THE EINSTEIN-KÄHLER METRIC

We proved that the new complete invariant metrics are equivalent to the Bergman metric on Cartan-Hartogs domains. We will prove that these new metrics are also equivalent to the Einstein-Kähler metric on Cartan-Hartogs. Therefore the Bergman metric is equivalent to the Einstein-Kähler metric on Cartan-Hartogs domain. By using the Yau’s Schwarz lemma and the theorem 3.10, we can prove that the Bergman metric is equivalent to the Einstein-Kähler metric on Cartan-Hartogs domain.

4.1. Yau’s Schwarz lemma[33]: Let \(f : (M^m, g) \to (N^n, h)\) be a holomorphic map between Kähler manifolds where \(M\) is complete and \(\text{Ric}(g) \geq -cg\) with \(c \geq 0\).
(1) if the holomorphic sectional curvature of $N$ is bounded above by a negative constant, then $f^*h \leq \tilde{c}g$ for some constant $\tilde{c}$.

(2) If $m = n$ and the Ricci curvature of $N$ is bounded above by a negative constant, then $f^*\omega^n_h \leq \tilde{c}\omega^n_g$ for some constant $\tilde{c}$.

Where $\omega^n_h$ and $\omega^n_g$ are the volume element for $(M^m, g)$ and $(N^n, h)$ respectively.

4.2. Consider the identity map

$$id : (Y_I, \omega_{EK}(Y_I)) \to (Y_I, \omega_{G\lambda}(Y_I)),$$

Because the holomorphic sectional curvature of $\omega_{G\lambda}(Y_I)$ is bounded above by a negative constant. Yau’s Schwarz lemma (1) implies

$$\omega_{G\lambda}(Y_I) \leq C_1 \omega_{EK}(Y_I).$$

Consider the identity map again

$$id : (Y_I, \omega_{G\lambda}(Y_I)) \to (Y_I, \omega_{EK}(Y_I)).$$

Because the Ricci curvature of $\omega_{G\lambda}(Y_I)$ is bounded below by a negative constant. Yau’s Schwarz lemma (2) implies

$$\omega^{mn+N}_{EK}(Y_I) \leq C_0 \omega^{mn+N}_{G\lambda}(Y_I).$$

Where $mn + N$ is the dimension of $Y_I$. This inequality implies

$$\det[T_{EKI}(Z, W; \overline{Z}, \overline{W})] \leq C_0 \det[T_{MI}(Z, W; \overline{Z}, \overline{W})].$$

Because $T_{MI}(Z, W; \overline{Z}, \overline{W}) > 0$, $T_{EKI}(Z, W; \overline{Z}, \overline{W}) > 0$. Then from the following proposition, one has

$$\omega_{EK}(Y_I) \leq C_2 \omega_{G\lambda}(Y_I).$$

Proposition: Let $A$ and $B$ be positive definite $n \times n$ Hermitian matrices and let $\alpha$, $\beta$ be positive constants such that $B \geq \alpha A$ and $\det(B) \leq \beta \det(A)$. Then there is a constant $\gamma > 0$ depending on $\alpha$, $\beta$ and $n$ such that $B \leq \gamma A$.

Up to now we proved that

Theorem: The Bergman metric is equivalent to the Einstein-Kähler metric on $Y_I$.

This theorem are also true for the other Cartan-Hartogs domains. Thus the Yau’s conjecture is true for the Cartan-Hartogs domains.

4.3. If $Y_I$ is convex, then $\omega_B(Y_I)$, $\omega_C(Y_I)$, $\omega_K(Y_I)$, $\omega_{EK}(Y_I)$, $\omega_{G\lambda}(Y_I)$ are equivalence on $Y_I$. This fact is also true for the other convex Cartan-Hartogs domains.
4.4. Because the holomorphic sectional curvature of $\omega_{G_\lambda}(Y_I)$ is bounded above by negative constant, then by the ref. [41, p.136], one has

$$\omega_{G_\lambda}(Y_I) \leq \beta \omega_K(Y_I)$$

Hence

$$\omega_B(Y_I) \leq \beta_1 \omega_K(Y_I)$$

and

$$\omega_{EK}(Y_I) \leq \beta_2 \omega_K(Y_I).$$

4.5. Because $\omega_C(Y_I) \leq 2 \omega_B(Y_I)$, then

$$\omega_C(Y_I) \leq \beta_3 \omega_{EK}(Y_I).$$

Where $\beta$, $\beta_1$, $\beta_2$, $\beta_3$ are the positive constants. The facts in 4.4 and 4.5 are also true for the other Cartan-Hartogs domains.

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