Modular and fractional $L$-intersecting families of vector spaces

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Abstract

This paper is divided into two logical parts. In the first part of this paper, we prove the following theorem which is the $q$-analogue of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon, Babai, Suzuki, J. Combin. Theory Series A, 1991]. It is also a generalization of the main theorem in [Frankl and Graham, European J. Combin. 1985] under certain circumstances.

Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0,1,\ldots, b-1\}$ with $k_1 < \cdots < k_r$. Let $F = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that (a)

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for every $i \in [m]$, $\dim(V_i) \mod b = k_i$, for some $k_i \in K$, and (b) for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \mod b = \mu_j$, for some $\mu_j \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of 2, and $b = 2$

(ii) $q = 2, b = 6$.

Then,

$$|\mathcal{F}| \leq \begin{cases} N(n, s, r, q), & \text{if } (s + k_r \leq n \text{ and } r(s - r + 1) \leq b - 1) \text{ or } (s < k_1 + r) \\ N(n, s, r, q) + \sum_{k \in [r]} \binom{n}{k}, & \text{otherwise,} \end{cases}$$

where $N(n, s, r, q) := \binom{n}{s} + \binom{n}{s - 1} + \cdots + \binom{n}{s - r + 1}$.

In the second part of this paper, we prove $q$-analogues of results on a recent notion called fractional $L$-intersecting family of sets for families of subspaces of a given vector space over a finite field of size $q$. We use the above theorem to obtain a general upper bound to the cardinality of such families. We give an improvement to this general upper bound in certain special cases.

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### 1 Introduction

Let $[n]$ be the set of all natural numbers from 1 to $n$. A family $\mathcal{F}$ of subsets of $[n]$ is called *intersecting* if every set in $\mathcal{F}$ has a non-empty intersection with every other set in $\mathcal{F}$. One of the earliest studies on intersecting families dates back to the famous Erdős-Ko-Rado Theorem [Erdős et al., 1961] about maximal uniform intersecting families. Ray-Chaudhuri and Wilson [Ray-Chaudhuri and Wilson, 1975] introduced the notion of $L$-intersecting families. Let $L = \{l_1, \ldots, l_s\}$ be a set of non-negative integers. A family $\mathcal{F}$ of subsets of $[n]$ is said to be $L$-intersecting if for every distinct $F_i, F_j$ in $\mathcal{F}$, $|F_i \cap F_j| \in L$. The Ray-Chaudhuri-Wilson Theorem states that if $\mathcal{F}$ is $t$-uniform (that is, every set in $\mathcal{F}$ is $t$-sized), then $|\mathcal{F}| \leq \binom{n}{s}$. This bound is tight as shown by the set of all $s$-sized subsets of $[n]$ with $L = \{0, \ldots, s - 1\}$. Frankl-Wilson Theorem [Frankl and Wilson, 1981a] extends this to non-uniform families by showing that $|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$, where $\mathcal{F}$ is any family of subsets of $[n]$ that is $L$-intersecting. The collection of all the subsets of $[n]$ of size at most $s$ with $L = \{0, \ldots, s - 1\}$ is a tight example to this bound. The first proofs of these theorems were based on the technique of higher incidence matrices. Alon, Babai, and Suzuki in [Alon et al., 1991] generalized the Frankl-Wilson Theorem using a proof that operated on spaces of multilinear polynomials. They showed that if the sizes of the sets in $\mathcal{F}$ belong to $K = \{k_1, \ldots, k_r\}$ with each $k_i > s - r$, then $|\mathcal{F}| \leq \binom{n}{s} + \cdots + \binom{n}{s - r + 1}$. A modular version of the Ray-Chaudhuri-Wilson Theorem was shown in [Frankl and Wilson, 1981b]. This result was generalized in [Alon et al., 1991]. See [Liu and Yang, 2014] for a survey on $L$-intersecting families.
Researchers have also been working on similar intersection theorems for subspaces of a given vector space over a finite field. Hsieh [Hsieh, 1975], and Deza and Frankl [Deza and Frankl, 1983] showed Erdős-Ko-Rado type theorems for subspaces. Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. The number of $d$-dimensional subspaces of $V$ is given by the $q$-binomial coefficient (also known as Gaussian binomial coefficient) $\binom{n}{d}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)}$. The following theorem which is a $q$-analog of the Ray-Chaudhuri-Wilson Theorem by considering families of subspaces instead of subsets is due to [Frankl and Graham, 1985].

**Theorem 1.** ([Theorem 1.1 in [Frankl and Graham, 1985]] Let $V$ be a vector space over of dimension $n$ over a finite field of size $q$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that $\dim(V_i) = k$, for every $i \in [m]$. Let $0 \leq \mu_1 < \mu_2 < \cdots < \mu_s < b$ be integers such that $k \not\equiv \mu_t \pmod{b}$, for any $t$. For every $1 \leq i < j \leq m$, $\dim(V_i \cap V_j) \equiv \mu_t \pmod{b}$, for some $t$. Then, $|\mathcal{F}| \leq \binom{n}{s}_q$

except possibly for $q = 2, b = 6, s \in \{3, 4\}$.

**Example 2** (Remark 3.2 in [Frankl and Graham, 1985]). Let $n = k + s$. Let $\mathcal{F}$ be the family of all the $k$-dimensional subspaces of $V$, where $V$ is an $n$-dimensional vector space over a finite field of size $q$. Observe that, for any two distinct $V_i, V_j \in \mathcal{F}$, $k - s \leq \dim(V_i \cap V_j) \leq k - 1$. This is a tight example for Theorem 1.

Alon et al. in [Alon et al., 1991] proved a generalization of the non-modular version of the above theorem. This result was subsequently strengthened in [Liu et al., 2018].

Our paper is divided into two logical parts. In the first part (i.e., Section 2), we prove the following theorem which is a generalization of Theorem 1 due to Frankl and Graham under certain circumstances. It is also the $q$-analog of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon et al., 1991]. We assume that $\binom{a}{b}_q = 0$, when $b < 0$ or $b > a$. Let

$$N(n, s, r, q) := \binom{n}{s}_q + \binom{n}{s-1}_q + \cdots + \binom{n}{s-r+1}_q.$$ 

**Theorem 3.** Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0, 1, \ldots, b-1\}$ with $k_1 < \cdots < k_r$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that (a) for every $i \in [m]$, $\dim(V_i) \mod{b} = k_t$, for some $k_t \in K$, and (b) for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \mod{b} = \mu_t$, for some $\mu_t \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of 2, and $b = 2$
(ii) \( q = 2, b = 6 \)

Then,

\[
|\mathcal{F}| \leq \begin{cases} 
N(n, s, r, q), & \text{if } (s + k_r \leq \text{nand}(s - r + 1) \leq b - 1) \text{ or } (s < k_1 + r) \\
N(n, s, r, q) + \sum_{i \in [q]} \left[ \frac{n}{k_i} \right], & \text{otherwise}.
\end{cases}
\]

In the second part (i.e., Section 3), we study a notion of fractional \( L \)-intersecting families which was introduced in [Balachandran et al., 2019]. We say a family \( \mathcal{F} = \{F_1, F_2, \ldots, F_m\} \) of subsets of \([n]\) is a fractional \( L \)-intersecting family, where \( L \) is a set of irreducible fractions between 0 and 1, if for every distinct \( i, j \in [m], \frac{|F_i \cap F_j|}{|F_i|} \in L \) or \( \frac{|F_i \cap F_j|}{|F_j|} \in L \). In this paper, we extend this notion from subsets to subspaces of a vector space over a finite field.

**Definition 4.** Let \( L = \{\frac{a_1}{b_1}, \ldots, \frac{a_t}{b_t}\} \) be a set of positive irreducible fractions, where every \( \frac{a_i}{b_i} < 1 \). Let \( \mathcal{F} = \{V_1, \ldots, V_m\} \) be a family of subspaces of a vector space \( V \) over a finite field. We say \( \mathcal{F} \) is a fractional \( L \)-intersecting family of subspaces if for every two distinct \( i, j \in [m], \frac{\dim(V_i \cap V_j)}{\dim(V_i)} \in L \) or \( \frac{\dim(V_i \cap V_j)}{\dim(V_j)} \in L \).

When every subspace in \( \mathcal{F} \) is of dimension exactly \( k \), it is an \( L' \)-intersecting family where \( L' = \{\frac{a_{ik}}{b_i}, \ldots, \frac{a_{bk}}{b_i}\} \). Applying Theorem 1, we get \( |\mathcal{F}| \leq \left[ \frac{n}{s} \right]_q \). A tight example to this is the collection of all \( k \)-dimensional subspaces of \( V \) with \( L = \{\frac{k}{k}, \ldots, \frac{k-1}{k}\} \). However, the problem of bounding the cardinality of a fractional \( L \)-intersecting family of subspaces becomes more interesting when \( \mathcal{F} \) contains subspaces of various dimensions. In Section 3, we obtain upper bounds for the cardinality of a fractional \( L \)-intersecting family of subspaces that are \( q \)-analog of the results in [Balachandran et al., 2019]. With the help of Theorem 3 that we prove in Section 2, we obtain the following result in Section 3.

**Theorem 5.** Let \( L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_t}{b_t}\} \) be a collection of positive irreducible fractions, where every \( \frac{a_i}{b_i} < 1 \). Let \( \mathcal{F} \) be a fractional \( L \)-intersecting family of subspaces of a vector space \( V \) of dimension \( n \) over a finite field of size \( q \). Let \( t = \max_{i \in [s]} b_i \), \( g(t, n) = \frac{2(2t+1)n}{\ln(2t+1)n} \), and \( h(t, n) = \min(g(t, n), \frac{\ln n}{\ln t}) \). Then,

\[
|\mathcal{F}| \leq 2g(t, n)h(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right]_q + h(t, n) \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_q.
\]

Further, if \( 2g(t, n) \ln(g(t, n)) \leq n + 2 \), then

\[
|\mathcal{F}| \leq 2g(t, n)h(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right]_q.
\]

**Example 6.** Let \( s \) be a constant, \( L = \{\frac{0}{1}, \frac{1}{s}, \ldots, \frac{s-1}{s}\} \), and \( \mathcal{F} \) be the family of all the \( s \)-sized subspaces of \( V \). Clearly, \( \mathcal{F} \) is a fractional \( L \)-intersecting family showing that the bound in Theorem 5 is asymptotically tight up to a multiplicative factor of \( \frac{\ln^2 n}{\ln \ln n} \).
We improve the bound obtained in Theorem 5 for the special case when \( L = \{\frac{a}{b}\} \), where \( b \) is a prime.

**Theorem 7.** Let \( L = \{\frac{a}{b}\} \), where \( \frac{a}{b} \) is a positive irreducible fraction less than 1 and \( b \) is a prime. Let \( \mathcal{F} \) be a fractional \( L \)-intersecting family of subspaces of a vector space \( V \) of dimension \( n \) over a finite field of size \( q \). Then, we have \(|\mathcal{F}| \leq (b - 1)(\binom{n - 1}{1}_q + 1)\lceil \frac{\ln n}{\ln b} \rceil + 2\).

**Example 8.** Let \( L = \{\frac{1}{2}\} \). Let \( V \) be a vector space of dimension \( n \) over a finite field of size \( q \). Let \( \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \). Let \( V' := \text{span}(\{v_2, \ldots, v_n\}) \) be an \((n - 1)\)-dimensional subspace of \( V \). Let \( \mathcal{F} \) be the set of all \( \binom{n - 1}{1}_q \) 2-dimensional subspaces of \( V \) each of which is obtained by a span of \( v_1 \) and each of the \( \binom{n - 1}{1}_q \) 1-dimensional subspaces of \( V' \). This example shows that when \( b \) and \( q \) are constants, the bound in Theorem 7 is asymptotically tight up to a multiplicative factor of \( \ln n \).

## 2 Generalized modular RW Theorem for subspaces

As mentioned before, in this part we prove Theorem 3. The approach followed here is similar to the approach used in proving Theorem 1.5, a generalized modular Ray-Chaudhuri-Wilson Theorem for subsets, in [Alon et al., 1991]. We start by stating the Zsigmondy’s Theorem which will be used in the proof of Theorem 3.

**Theorem 9 ([Zsigmondy, 1892]).** For any \( q, b \in \mathbb{N} \), there exists a prime \( p \) such that \( q^b \equiv 1 \pmod{p} \), \( q^i \not\equiv 1 \pmod{p} \) \( \forall i \), 0 < \( i < b \), except when (i) \( q + 1 \) is a power of 2, \( b = 2 \), or (ii) \( q = 2, b = 6 \).

### 2.1 Notations used in Section 2

Unless defined explicitly, in the rest of this section, the symbols \( K = \{k_1, \ldots, k_r\} \), \( r \), \( L = \{\mu_1, \ldots, \mu_s\} \), \( s, q, V, \mathcal{F}, n, b, m \), and \( V_1, \ldots, V_m \) are defined as they are defined in Theorem 3. We shall use \( U \subseteq V \) to denote that \( U \) is a subspace of \( V \). Using Zsigmondy’s Theorem, we find a prime \( p \) so that \( q^i \not\equiv 1 \pmod{p} \) for 0 < \( i < b \) and \( q^b \equiv 1 \pmod{p} \). This is possible except in the two cases specified in Theorem 9. We ignore these two cases from now on in the rest of Section 2.

### 2.2 Möbius inversion over the subspace poset

Consider the partial order defined on the set of subspaces of the vector space \( V \) over a finite field of size \( q \) under the ‘containment’ relation. Let \( \alpha \) be a function from the set of subspaces of \( V \) to \( \mathbb{F}_p \). A function \( \beta \) from the set of subspaces of \( V \) to \( \mathbb{F}_p \) is the zeta transform of \( \alpha \) if for every \( W \subseteq V \), \( \beta(W) = \sum_{U \subseteq W} \alpha(U) \). Then, applying the Möbius inversion formula we get for all \( W \subseteq V \), \( \alpha(W) = \sum_{U \subseteq W} \mu(U, W) \beta(U) \), where \( \alpha \) is called
the Möbius transform of $\beta$ and $\mu(U,W)$ is the Möbius function for the subspace poset. In the proposition below, we show that the Möbius function for the subspace poset is defined as

$$\mu(X,Y) = \begin{cases} (-1)^{q(d)}, & \text{if } X \subseteq Y \\ 0, & \text{otherwise,} \end{cases}$$

$\forall X, Y \subseteq V$ with $d = \text{dim}(Y) - \text{dim}(X)$. The following proposition gives the Möbius inversion formula for the subspace lattice. See [Mathew et al., 2020] for a proof.

**Proposition 10.** Let $\alpha$ and $\beta$ be functions from the set of subspaces of $V$ to $\mathbb{F}_p$. Then, $\forall W \subseteq V$,

$$\beta(W) = \sum_{U \subseteq W} \alpha(U) \iff \alpha(W) = \sum_{U \subseteq W} (-1)^d q^{\frac{d(d-1)}{2}} \beta(U).$$

**Definition 11.** Given two subspaces $U$ and $W$ of the vector space $V$, we define their union space $U \cup W$ as the span of union of sets of vectors in $U$ and $W$.

The proposition below follows from the definitions of $\alpha$ and $\beta$. See [Mathew et al., 2020] for a proof.

**Proposition 12.** Let $\alpha$ and $\beta$ be functions as defined in Proposition 10. Then, $\forall W, Y$ such that $W \subseteq Y \subseteq V$,

$$\sum_{T: W \subseteq T \subseteq Y} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \sum_{U: U \cup W = Y} \alpha(U).$$

**Corollary 13.** For any non-negative integer $g$, the following are equivalent for functions $\alpha$ and $\beta$ defined in Proposition 10:

(i) $\alpha(U) = 0$, $\forall U \subseteq V$ with $\text{dim}(U) \geq g$.

(ii) $\sum_{W \subseteq T \subseteq Y} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0$, $\forall W, Y \subseteq V$ with $\text{dim}(Y) - \text{dim}(W) \geq g$.

**Definition 14.** Let $H = \{h_1, h_2, \ldots, h_t\}$ be a subset of $\{0,1,\ldots,n\}$ where $h_1 < h_2 < \cdots < h_t$. We say $H$ has a gap of size $\geq g$ if either $h_1 \geq g - 1$, $n - h_t \geq g - 1$, or $h_{i+1} - h_i \geq g$ for some $i \in [t-1]$.

**Lemma 15.** Let $\alpha$ and $\beta$ be functions as in Proposition 10. Let $H \subseteq \{0,1,\ldots,n\}$ be a set of integers and $g$ an integer, $0 \leq g \leq n$. Suppose we have the following conditions:

(i) $\forall U \subseteq V$, we have $\alpha(U) = 0$ whenever $\text{dim}(U) \geq g$.

(ii) $\forall T \subseteq V$, we have $\beta(T) = 0$ whenever $\text{dim}(T) \notin H$. 

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(iii) \( H \) has a gap \( \geq g + 1 \).

Then, \( \alpha = \beta = 0 \).

Proof. Let \( H = \{ h_1, h_2, \ldots, h_{|H|} \} \). Suppose, for some \( i \in [|H|] \), \( h_i - h_{i-1} \geq g \) or \( h_1 \geq g \), then we have \( h_i \in H \) and \( h_i - j \notin H \) for \( 1 \leq j \leq g \) and \( h_i - g \geq 0 \). Choose any two subspaces, say \( U \) and \( W \), of \( V \) of dimensions \( h_i \) and \( h_i - g \), respectively. Since \( \dim(U) \geq g \), \( \alpha(U) = 0 \). We know from Corollary 13 that

\[
\sum_{W \subseteq T \subseteq U \atop d = \dim(U) - \dim(T)} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0
\]

But whenever \( \dim(T) < h_i \), it lies between \( h_i - g \) and \( h_i - 1 \), and hence \( \beta(T) = 0 \). Then,

\[
\sum_{W \subseteq T \subseteq U \atop d = \dim(U) - \dim(T)} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \beta(U) = 0
\]

Since our choice of \( U \) was arbitrary, we may conclude that \( \beta(U) = 0 \), for all \( U \subseteq V \) with \( \dim(U) = h_i \). Thus, we can remove \( h_i \) from the set \( H \), and then use the same procedure to further reduce the size of \( H \) till it is an empty set. If \( H \) is empty, \( \beta(U) = 0 \), for all \( U \subseteq V \), giving \( \alpha(U) = \beta(U) = 0 \) as required.

Now suppose \( n - h_{|H|} \geq g \). In this case, we take \( U \) of dimension \( h_{|H|} \) and \( W \) of dimension \( h_{|H|} + g \) to show that \( \beta(U) = 0 \), and remove \( h_{|H|} \) from \( H \). Note that removing a number from the set \( H \) can never reduce the gap. \( \square \)

2.3 Defining functions \( f^{x,y} \) and \( g^{x,y} \)

Consider all the subspaces of the vector space \( V \). We can impose an ordering on the subspaces of same dimension, and use the natural ordering across dimensions, so that every subspace can be uniquely represented by a pair of integers \( (d,e) \), indicating that it is the \( e \)th subspace of dimension \( d \), \( 0 \leq d \leq n \), \( 1 \leq e \leq \left[ \frac{n}{d} \right] \). Let us call that subspace \( V_{d,e} \). Let \( S \) be the number of subspaces of \( V \) of dimension at most \( s \), that is, \( S = \sum_{t=0}^{s} \left[ \frac{n}{t} \right] \). Let each subspace \( V_{d,e} \) of dimension at most \( s \) be represented as a 0-1 containment vector \( v_{d,e} \) of \( S \) entries, each entry of the vector denoting whether a particular subspace of dimension \( \leq s \) is contained in \( V_{d,e} \) or not.

\[
v_{d,e}^{x,y} = \begin{cases} 1, & \text{if } V_{x,y} \text{ is a subspace of } V_{d,e} \\ 0, & \text{otherwise} \end{cases}
\]

The vector \( v_{d,e} \) consists of \( v_{d,e}^{x,y} \) values for \( 0 \leq x \leq s \), \( 1 \leq y \leq \left[ \frac{n}{d} \right] \), making it a vector of size \( S \). Thus, \( v_{d,e}^{x,y} \) is simply the indicator function of whether \( V_{x,y} \) is a subspace of \( V_{d,e} \).
For $0 \leq x \leq s, 1 \leq y \leq \left[\frac{n}{x}\right]_q$, we define functions $f^{x,y} : \mathbb{F}_2^n \rightarrow \mathbb{F}_p$ as

$$f^{x,y}(v) = f^{x,y}(v^{0,1}, v^{1,1}, \ldots, v^{1,\left[\frac{n}{x}\right]_q}, \ldots, v^{s,1}, \ldots, v^{s,\left[\frac{n}{x}\right]_q}) := v^{x,y}.$$ 

For $0 \leq x \leq s - r, 1 \leq y \leq \left[\frac{n}{x}\right]_q$, we define functions $g^{x,y} : \mathbb{F}_2^n \rightarrow \mathbb{F}_p$ as

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\left[\frac{n}{x}\right]_q} v^{1,j} - \left[\frac{k_t}{1}\right]_q \right).$$

Let $\Omega$ denote $\mathbb{F}_2^n$. The functions $f^{x,y}$ and $g^{x,y}$ reside in the space $\mathbb{F}_p^n$. Note that the functions $g^{x,y}$ do not exist if $s < r$.

### 2.4 Swallowing trick: linear independence of functions $f^{x,y}$ and $g^{x,y}$

**Lemma 16.** Let $s + k_r \leq n$ and $r(s - r + 1) \leq b - 1$. The functions $g^{x,y}$, $0 \leq x \leq s - r, 1 \leq y \leq \left[\frac{n}{x}\right]_q$, are linearly independent in the function space $\mathbb{F}_p^n$ over $\mathbb{F}_p$.

**Proof.** If $s < r$, then the statement of the lemma is vacuously true. Assume $s \geq r$. We wish to show that the only solution to $\sum_{0 \leq x \leq s-r, 1 \leq y \leq \left[\frac{n}{x}\right]_q} \alpha^{x,y}g^{x,y} = 0$ is the trivial solution $\alpha^{x,y} = 0, \forall x,y$. We define function $\alpha$ from the set of all subspaces of $V$ to $\mathbb{F}_p$ as:

$$\alpha(V_{d,e}) = \begin{cases} \alpha^{d,e}, & \text{if } 0 \leq d \leq s - r \\ 0, & \text{if } d > s - r \end{cases}$$

We show that functions $\alpha$ and $\beta(U) := \sum_{T \subseteq U} \alpha(T)$ satisfy the conditions of Lemma 15, thereby implying $\alpha(U) = 0$, for all $U \subseteq V$, including $\alpha(V_{d,e}) = \alpha^{d,e} = 0$ for $0 \leq d \leq s - r$, which will in turn imply that the functions $g^{x,y}$ above are linearly independent.

Let $H = \{x : 0 \leq x \leq n, x \equiv k_t \pmod{b}, t \in [r]\}$. We claim that $H$ has a gap of size at least $s - r + 2$. Suppose $n \geq b + k_1$. Then, $k_1 \leq s - r + 1 \leq b - 1$, by pigeonhole principle, there is a gap of at least $s - r + 2$ between some $k_i$ and $k_{i+1}$, $i \in [r - 1]$, or between $k_r$ and $b + k_1$. Suppose $s + k_r \leq n < b + k_1$. Then, there is a gap of at least $s + 1$ right above $k_r$. This proves the claim. We now need to show that for $T \subseteq V$, $\beta(T) = 0$ whenever $\dim(T) \notin H$, or whenever $\dim(T) \equiv k_t \pmod{b}$, for any $t \in [r]$. Suppose $v_T$ is the $S$-sized containment vector for $T$. When $\dim(T) \neq k_t \pmod{b}$ for any $t \in [r]$, it follows from the property of
the prime $p$ given by Theorem 9 that \[ \sum_{1 \leq j \leq [r]} v_{i,j}^{1,j} - \left[ \begin{array}{c} k_i \\ 1 \end{array} \right] \neq 0 \text{ in } \mathbb{F}_p, \text{ for every } t \in [r]. \]

\[ \beta(T) = \sum_{U \subseteq T} \alpha(U) = \sum_{\dim(U) \leq s-r} \alpha(U) = \sum_{1 \leq e \leq [s]} \alpha(V^{d,e}_i f^{d,e}(v_T)) \]

Since \[ \sum_{1 \leq j \leq [r]} v_{i,j}^{1,j} - \left[ \begin{array}{c} k_i \\ 1 \end{array} \right] \neq 0 \text{ in } \mathbb{F}_p, \text{ for every } t \in [r], \] \[ f^{d,e}(v_T) = c(T)g^{d,e}(v_T) \text{ where } c(T) \neq 0. \]

Then, \[ \beta(T) = c(T) \sum_{0 \leq d \leq s-r} \alpha(V^{d,e}_i g^{d,e}(v_T)) = c(T) \sum_{0 \leq d \leq s-r} \alpha^{d,e}g^{d,e}(v_T) = c(T) \cdot 0 = 0. \]

Since the set $H$ and the functions $\alpha$ and $\beta$ satisfy the conditions of Lemma 15, we have $\alpha = 0$. This proves the lemma. □

Recall that we are given a family $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ of subspaces of $V$ such that for every $i \in [m]$, $\dim(V_i) \mod b = k_i$, for some $k_i \in K$. Further, $\dim(V_i \cap V_j) \mod b = \mu_t$, for some $\mu_t \in L$ and $K$ and $L$ are disjoint subsets of $\{0, 1, \ldots, b-1\}$. Let $v_i$ be the containment vector of size $S$ corresponding to subspace $V_i \in \mathcal{F}$. We define the following functions from $\mathbb{F}_p^S \to \mathbb{F}_p$.

\[ g^i(v) = g^i(v^{0,1}, v^{1,1}, \ldots, v^1, \ldots, v^{s,1}, \ldots, v^s) \]

\[ := \prod_{j=1}^s \left( \sum_{1 \leq y \leq [r]} (v_i^{1,y}v^{1,y}) - \left[ \begin{array}{c} \mu_j \\ 1 \end{array} \right] \right) \]

Let $v = v_j$. Then, \[ \sum_{1 \leq y \leq [r]} (v_i^{1,y}v^{1,y}) \] counts the number of 1-dimensional subspaces common to $V_i$ and $V_j$. That is, \[ \sum_{1 \leq y \leq [r]} (v_i^{1,y}v^{1,y}) = \left[ \dim(V_i \cap V_j) \right] \] in $\mathbb{F}_p$, \[ \left[ \dim(V_i \cap V_j) \right] \neq \left[ \begin{array}{c} \mu_t \\ 1 \end{array} \right] \]

for any $1 \leq t \leq s$, if $i = j$, and \[ \left[ \dim(V_i \cap V_j) \right] = \left[ \begin{array}{c} \mu_t \\ 1 \end{array} \right] \] for some $1 \leq t \leq s$ if $i \neq j$.

Accordingly, \[ g^i(v_j) = \begin{cases} 0, & \text{if } i = j \\ \neq 0, & \text{if } i \neq j \end{cases} \]

**Lemma 17** (Swallowing trick 1). Let $s + k_r \leq n$ and $r(s - r + 1) \leq b - 1$. The collection of functions $g^i$, $1 \leq i \leq m$ together with the functions $g^{x,y}$, $0 \leq x \leq s - r$, $1 \leq y \leq [n]_q$, are linearly independent in $\mathbb{F}_p^q$ over $\mathbb{F}_p$. 

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Lemma 18. The collection of functions $f^{x,y}$, $0 \leq x \leq s$, $1 \leq y \leq \left[\frac{n}{x}\right]_q$, spans all the functions $g^{x,y}$, $0 \leq x \leq s - r$, $1 \leq y \leq \left[\frac{n}{x}\right]_q$ as well as the functions $g^i$, $1 \leq i \leq m$.

**Proof.** Let $v \in \mathbb{F}_p^S$. The key observation here is that the product $f^{x,y}(v)f^{1,z}(v)$, $0 \leq x \leq s - 1, 1 \leq y \leq \left[\frac{n}{x}\right]_q$, may be replaced by the function $f^{x',w}(v)$, where $x \leq x' \leq x + 1, 1 \leq w \leq \left[\frac{n}{x'}\right]_q$. If $V_{1,z} \subseteq V_{x,y}$, it is trivial that $f^{x,y}(v)f^{1,z}(v) = f^{x,y}(v)$, since $f^{x,y}(v) = 1$ only if $f^{1,z}(v) = 1$. If $V_{1,z} \nsubseteq V_{x,y}$, we let $V_{x',w}$ be the span of union of vectors of $V_{1,z}$ and $V_{x,y}$. Suppose, a vector space $U$ contains both $V_{1,z}$ and $V_{x,y}$. Then, it is clear that it must contain the span of their union as well. Similarly, a vector space $U$ that does not contain either $V_{1,z}$ or $V_{x,y}$, cannot contain $V_{x',w}$. Thus, $f^{x,y}(v)f^{1,z}(v) = f^{x',w}(v)$. To see why $x' = x + 1$ (in case $V_{1,z} \nsubseteq V_{x,y}$), the space $V_{x',w}$ may be obtained by taking any (non-zero) vector of $V_{1,z}$ and introducing it into the basis of $V_{x,y}$. The space spanned by this extended basis is exactly $V_{x',w}$ by definition, and the size of basis has increased by exactly 1.

By induction, it follows that,

$$f^{y_1}(v)f^{y_2}(v)\ldots f^{y_l}(v) = f^{x,y}(v)$$

for some $x, y$ where, $1 \leq x \leq l, 1 \leq y \leq \left[\frac{n}{x}\right]_q$. That is, a product of $l$ functions of the form $f^{1,y}$ may be replaced by a single function $f^{x,y}$ where $x$ is at most $l$. 

2.5 Proof of Theorem 3: in the case when $s + k_r \leq nand(s - r + 1) \leq b - 1$
Now consider functions
\[ g^i(v) = g^i(v^{0,1}, v^{1,1}, \ldots, v^{1,[n]}, \ldots, v^{s,1}, \ldots, v^{s,[n]}) \]
\[ = \prod_{j=1}^{s} \left( \sum_{1 \leq y \leq [n]} (v_1^{1,y} v_1^{1,y}) - [\mu_j^1]_q \right) \]
\[ = \prod_{j=1}^{s} \left( \sum_{1 \leq y \leq [n]} (v_1^{1,y} f^{1,y}(v)) - [\mu_j^1]_q \right) \]

Since the functions \( f^{x,y} \) only take 0/1 values, we can reduce any exponent of 2 or more on the function after expanding the product to 1. Moreover, the terms will all be products of the form \( f^{1,y_1} f^{1,y_2} \ldots f^{1,y_l}(v), 1 \leq l \leq s \). These are replaced according to the observation above by single function of the form \( f^{x,y}(v) \), and thus the set of functions \( f^{x,y}, 0 \leq x \leq s, 1 \leq y \leq \left\lfloor \frac{n}{x} \right\rfloor_q \) span all functions \( g^i(v) \). Note that \( f^{0,1}(v) \) is the constant function 1.

Similarly, for \( 0 \leq x \leq s - r, 1 \leq y \leq \left\lfloor \frac{n}{x} \right\rfloor_q \),
\[ g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \left\lfloor \frac{n}{x} \right\rfloor_q - \left\lfloor k_t^1 \right\rfloor_1 \right) \]
\[ = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\left\lfloor \frac{n}{x} \right\rfloor_q} f^{1,j}(v) - \left\lfloor k_t^1 \right\rfloor_1 \right) \]
\[ = f^{x,y}(v) \left( \sum_{x'=0}^{r} \sum_{y'=1}^{y_x} c_{x',y'} f^{x',y'}(v) \right) \quad (c_{x',y'} \text{ are constants}) \]
\[ = \sum_{x'=0}^{s} \sum_{y'=1}^{y_x} c_{x',y'} f^{x',y'}(v) \quad (c_{x',y'} \text{ are constants}) \]

Thus, the set of function \( f^{x,y}, 0 \leq x \leq s, 1 \leq y \leq \left\lfloor \frac{n}{x} \right\rfloor_q \) span all functions \( g^{x,y}(v), 0 \leq x \leq
This means that the above functions $g^{x,y}$ and $g^i$ belong to the span of functions $f^{x,y}$ which is a function space of dimension at most $S$. From Lemma 17, we know that $g^{x,y}$ and $g^i$ are together linearly independent. Thus,

$$\sum_{j=0}^{s-r} \begin{bmatrix} n \\ j \end{bmatrix}_q + m \leq S = \sum_{j=0}^{s} \begin{bmatrix} n \\ j \end{bmatrix}_q .$$

$$\Rightarrow |\mathcal{F}| = m \leq \begin{bmatrix} n \\ s \end{bmatrix}_q + \begin{bmatrix} n \\ s-1 \end{bmatrix}_q + \cdots + \begin{bmatrix} n \\ s-r+1 \end{bmatrix}_q .$$

### 2.6 Proof of Theorem 3

Let $X \subseteq \{0, \ldots, s-r\}$ be the set of those integers that are not congruent to any $k \in K$. The, in the following lemma, we show that the family $g^{x,y}$ with $x \in X$ is linearly independent.

**Lemma 19.** The collection of functions

$$\{g^{x,y} \mid 0 \leq x \leq s-r, 1 \leq y \leq \begin{bmatrix} n \\ x \end{bmatrix}_q , \text{ and for all } t \in [r], x \not\equiv k_t \pmod{b}\}$$

are linearly independent in the function space $F_p^\Omega$ over $F_p$.

**Proof.** Recall that

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} v^{1j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right).$$

The statement of the lemma is vacuously true, if $s < r$. Assume $s \geq r$. Assume, for the sake of contradiction, $\sum_{x \not\equiv k_t \pmod{b}, \forall t \in [r]} \alpha^{x,y} g^{x,y} = 0$ with at least one $\alpha^{x,y}$ as non-zero.

Let $(x_0, y_0)$ be the first subspace, based on the ordering of subspaces defined in Section 2.3, such that $\alpha^{x_0,y_0}$ is non-zero. Evaluating both sides on $v_{x_0,y_0}$, we see that all $f^{x,y}$ (and therefore $g^{x,y}$) with $(x, y)$ higher in the ordering than $(x_0, y_0)$ will vanish (due to the virtue of our ordering), and so we get $\alpha^{x_0,y_0} = 0$ which is a contradiction. Here we have crucially used the fact that by ignoring $x \equiv k_t \pmod{b}$ cases, for any $t \in [r]$, we make sure that $v_{x_0,y_0}$ used above always has $x_0 \not\equiv k_t \pmod{b}$ and therefore

$$\sum_{j=1}^{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} v_{x_0,y_0}^{1j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \not\equiv 0 \pmod{p}, \forall t \in [r].$$
Lemma 20 (Swallowing trick 2). The collection of functions \( g^i, 1 \leq i \leq m \) together with the functions \( g^{r,y}, 0 \leq x \leq s-r, x \not\equiv k_i \pmod{b} \), for all \( t \in [r] \), \( 1 \leq y \leq \left[ \frac{n}{x} \right] \) are linearly independent in \( \mathbb{F}_p^\Omega \) over \( \mathbb{F}_p \).

**Proof.** Proof is similar to the proof of Lemma 17.

Since \( s < b \), for any \( 0 \leq x \leq s-r \) and for any \( t \in [r] \), \( x \not\equiv k_t \pmod{b} \) is equivalent to \( x \neq k_t \). Combining Lemmas 19, 20 and 18, we have

\[
\sum_{\substack{0 \leq j \leq s-r, \\ j \neq k_t, t \in [r]}} \left[ \frac{n}{j} \right] q + m \leq \sum_{j=0}^s \left[ \frac{n}{j} \right] q.
\]

This implies,

\[
|\mathcal{F}| = m \leq \begin{cases} 
N(n,s,r,q), & \text{if } s < k_1 + r \\
N(n,s,r,q) + \sum_{t \in [r]} \left[ \frac{n}{k_t} \right] q, & \text{otherwise.}
\end{cases}
\]

We thus have the following theorem which combined with the result in Section 2.5 yields Theorem 3.

**Theorem 21.** Let \( V \) be a vector space of dimension \( n \) over a finite field of size \( q \). Let \( K = \{k_1, \ldots, k_r\} \), \( L = \{\mu_1, \ldots, \mu_s\} \) be two disjoint subsets of \( \{0, 1, \ldots, b-1\} \) with \( k_1 < \cdots < k_r \). Let \( \mathcal{F} = \{V_1, V_2, \ldots, V_m\} \) be a family of subspaces of \( V \) such that for all \( i \in [m] \), \( \dim(V_i) \equiv k_i \pmod{b} \), for some \( k_i \in K \); for every distinct \( i, j \in [m] \), \( \dim(V_i \cap V_j) \equiv \mu_t \pmod{b} \), for some \( \mu_t \in L \). Moreover, it is given that neither of the following two conditions hold:

(i) \( q+1 \) is a power of 2, and \( b = 2 \)

(ii) \( q = 2, b = 6 \)

Then,

\[
|\mathcal{F}| \leq \begin{cases} 
N(n,s,r,q), & \text{if } s < k_1 + r \\
N(n,s,r,q) + \sum_{t \in [r]} \left[ \frac{n}{k_t} \right] q, & \text{otherwise.}
\end{cases}
\]

### 3 Fractional \( L \)-intersecting families of subspaces

Let \( L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\} \) be a collection of positive irreducible fractions, each strictly less than 1. Let \( V \) be a vector space of dimension \( n \) over a finite field of size \( q \). Let \( \mathcal{F} \) be a family of subspaces of \( V \). Recall that, we call \( \mathcal{F} \) a fractional \( L \)-intersecting family of subspaces if for all distinct \( A, B \in \mathcal{F} \), \( \dim(A \cap B) \in \{\frac{a_1}{b_1} \dim(A), \frac{a_2}{b_2} \dim(B)\} \), for some \( \frac{a_i}{b_i} \in L \). In Section 3.1, we prove a general upper bound for the size of a fractional \( L \)-intersecting family using Theorem 3 proved in Section 2. In Section 3.2, we improve this upper bound for the special case when \( L = \{\frac{a}{b}\} \) is a singleton set with \( b \) being a prime number.
3.1 A general upper bound

The key idea we use here is to split the fractional $L$ intersecting family $\mathcal{F}$ into subfamilies and then use Theorem 3 to bound each of them.

Lemma 22. Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $k > 0$ and $p > \max(b_1, b_2, \ldots, b_s)$. Let $\mathcal{F}_k^p$ denote subspaces in $\mathcal{F}$ whose dimensions leave a remainder $k \pmod{p}$, where $p$ is a prime number. That is, $\mathcal{F}_k^p := \{W \in \mathcal{F} \mid \dim(W) \equiv k \pmod{p}\}$.

Then,

$$|\mathcal{F}_k^p| \leq \begin{cases} \left[ \frac{n}{s} \right]_q, & \text{if } (2p \leq n + 2) \text{ or } (s < k + 1) \\ \left[ \frac{n}{s} \right]_q + \left[ \frac{n}{k} \right]_q, & \text{otherwise.} \end{cases}$$

Proof. Apply Theorem 3 with family $\mathcal{F}$ replaced by $\mathcal{F}_k^p$, $K = \{k\}$, $r = 1$, $b$ replaced by $p$, and each $\mu_i$ replaced by $(\frac{a_i}{b_i}) \pmod{p} = (b_i^{-1}a_i) \pmod{p}$ where $b_i^{-1}$ is the multiplicative inverse of $b_i$ in $\mathbb{F}_p$. Let $s' \leq s$ be the number of distinct $\mu_i$'s. Notice that $k > 0$, and $p > b_i > \max(b_i, b_s)$ ensure that $k \not\equiv \frac{a_i}{b_i} \pmod{p}$ or $k \not\equiv \mu_i$. Thus $\mathcal{F}_k^p$ is a family of subspaces of $V$ such that (a) for every $W \in \mathcal{F}_k^p$, $\dim(W) \pmod{p} = k$, and (b) for every distinct $U, W \in \mathcal{F}_k^p$, $\dim(U \cap W) \pmod{p} = L$, where $L = \{\mu_1, \ldots, \mu_{s'}\}$ and $k \not\in L$. Moreover, since $s' \leq p - 1$ and $k \leq p - 1$, we have $s' + k \leq n$ if $2p \leq n + 2$. Since $p > b_i$ and every $b_i > 2$, we have $p > 2$. This avoids bad case (i) of Theorem 3. That $p$ is a prime avoids bad case (ii) of Theorem 3. Thus, we satisfy the premise of Theorem 3 and the conclusion follows.

Suppose $2p \leq n + 2$. The above lemma immediately gives us a bound of $|\mathcal{F}| \leq |\mathcal{F}_0^p| + (p - 1) \left[ \frac{n}{s} \right]_q$. But it could be that most subspaces belong to $\mathcal{F}_0^p$. To overcome this problem, we instead choose a set of primes $P$ such that no subspace can belong to $\mathcal{F}_0^p$ for every $p \in P$. A natural choice is to take just enough primes in increasing order so that the product of these primes exceeds $n$, because then any subspace with dimension divisible by all primes in $P$ will have a dimension greater than $n$, which is not possible.

Lemma 23. Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $t := \max(b_1, b_2, \ldots, b_s)$ and $g(t, n) := \frac{2^{2t+\ln n}}{\ln(2t+\ln n)}$. Suppose $2g(t, n) \ln(g(t, n)) \leq n + 2$. Then,

$$|\mathcal{F}| \leq 2g^2(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right]_q$$
Proof. For some $\beta$ to be chosen later, choose $P$ to be the set $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta}\}$ where $p_l$ denotes the $l^{th}$ prime number and $p_{\alpha} \leq t < p_{\alpha+1} < p_{\alpha+2} < \cdots < p_{\beta}$. Let $l\#$ denote the product of all primes less than or equal to $l$. Thus, $p_l\#$ which is known as the primorial function, is the product of the first $l$ primes. It is known that $p_l\# = e^{(1+o(1))l\ln l}$ and $l\# = e^{(1+o(1))l}$. We require the following condition for the set $P$:

$$\frac{p_{\beta}\#}{l\#} > n$$

Using the bounds for $p_l\#$ and $l\#$ discussed above, we find that it is sufficient to choose $\beta \geq 2(2t + \ln n)\ln(2t + \ln n)$.

Next, apply Lemma 22 on $F^p_{0\alpha+1}$ with $p = p_{\alpha+2}$ and so on. As argued above, no subspace is left uncovered after we reach $p_{\beta}$. This means,

$$|F| \leq |F^p_{0\alpha+1}| + (p_{\alpha+1} - 1) \left[ \frac{n}{s} \right]q$$

Lemma 24. Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $F = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $t := \max(b_1, b_2, \ldots, b_s)$ and $g(t, n) := \frac{2(2t + \ln n)}{\ln(2t + \ln n)}$. Then,

$$|F| \leq 2g^2(t, n)\ln(g(t, n)) \left[ \frac{n}{s} \right]q + g(t, n) \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]q$$

Proof. Let $P = \{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta}\}$, where $\beta = g(t, n)$ and $p_{\beta} \leq 2g(t, n)\ln(g(t, n))$. The proof is similar to the proof of Lemma 23. We apply Lemma 22 with $p = p_{\alpha+1}$ to show that

$$|F| \leq |F^p_{0\alpha+1}| + (p_{\alpha+1} - 1) \left[ \frac{n}{s} \right]q + \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]q$$
Next, we apply Lemma 22 on $\mathcal{F}_{0}^{p_{a+1}}$ with $p = p_{a+2}$ and so on as shown in the proof of Lemma 23 to get the desired bound.

$$|\mathcal{F}| \leq (p_{a+1} + p_{a+2} + \cdots + p_{\beta} - (\beta - \alpha)) \left[ \frac{n}{s} \right]_{q} + (\beta - \alpha) \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_{q}$$

\[ < (\beta - \alpha) \left( p_{\beta} \left[ \frac{n}{s} \right]_{q} + \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_{q} \right) \]

\[ < \beta \left( p_{\beta} \left[ \frac{n}{s} \right]_{q} + \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_{q} \right) \]

\[ \leq 2g^{2}(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right]_{q} + g(t, n) \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_{q} \]

Since $p_{a+1} > t$, we have $p_{a+1}p_{a+2} \cdots p_{\beta} > t^{\beta-\alpha}$. This implies that, if $t^{\beta-\alpha} \geq n$, then the product of the primes in $P$ will be greater than $n$ as desired. Substituting $\beta - \alpha$ with $\frac{\ln n}{\ln t}$ (and $p_{\beta}$ with $2g(t, n) \ln(g(t, n)))$ in the second inequality above, we get another upper bound of $|\mathcal{F}| \leq 2g(t, n) \frac{\ln(n) \ln(g(t, n))}{\ln t} \left[ \frac{n}{s} \right]_{q} + \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right]_{q}$. We can do a similar substitution for $\beta - \alpha$ in the calculations done at the end of the proof of Lemma 23 to get a similar bound.

Combining all the results in this section, we get Theorem 5.

### 3.2 An improved bound for singleton $L$

In this section, we improve the upper bound for the size of a fractional $L$-intersecting family obtained in Theorem 5 for the special case $L = \{\frac{a}{b}\}$, where $b$ is a constant prime. Before we give the proof, below we restate the the statement of Theorem 7.

**Statement of Theorem 7:** Let $L = \{\frac{a}{b}\}$, where $\frac{a}{b}$ is a positive irreducible fraction less than 1 and $b$ is a prime. Let $\mathcal{F}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Then, we have $|\mathcal{F}| \leq (b-1) \left[ \frac{n}{1} \right]_{q} + 1 + \left[ \frac{\ln n}{\ln b} \right] + 2$.

**Proof.** We assume that all the subspaces in the family except possibly one subspace, say $W$, have a dimension divisible by $b$. Otherwise, $\mathcal{F}$ cannot satisfy the property of a fractional $\frac{a}{b}$-intersecting family. Let us ignore $W$ in the discussion to follow. For any subspace $V_{i}$ that is not the zero subspace, let $k$ be the largest power of $b$ that divides $\dim(V_{i})$. Then, $\dim(V_{i}) = rb^{k+1} + jb^{k}$, for some $0 \leq j < b, r \geq 0$. Consider the subfamily, $\mathcal{F}^{j,k} = \{V_{i} : b^{k} \mid \dim(V_{i}), b^{k+1} \mid \dim(V_{i}), \dim(V_{i}) = rb^{k+1} + jb^{k} \text{ for some } r \geq 0, j \in [b-1] \}$. The subfamily $\mathcal{F}^{j,k}, 1 \leq k \leq \left[ \frac{\ln n}{\ln b} \right], 1 \leq j < b$, cover each and every subspace (except the zero subspace and the subspace $W$) of $\mathcal{F}$ exactly once. We will show that $|\mathcal{F}^{j,k}| \leq \left[ \frac{n}{1} \right]_{q} + 1$. 


which when multiplied with the number of values $j$ and $k$ can take will immediately imply the theorem.

Let $m^{j,k} = |\mathcal{F}^{j,k}|$. Let $M^{j,k}$ be an $m^{j,k} \times \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ 0-1 matrix whose rows correspond to the subspaces of $\mathcal{F}^{j,k}$ in any given order, whose columns correspond to the 1-dimensional subspaces of $V$ in any given order, and the $(i,l)^{th}$ entry is 1 if and only if the $i^{th}$ subspace of $\mathcal{F}^{j,k}$ contains the $l^{th}$ 1-dimensional subspace. Let $N^{j,k} = M^{j,k} \cdot (M^{j,k})^T$. Any diagonal entry $N^{j,k}_{i,i}$ is the number of 1-dimensional subspaces in the $i^{th}$ subspace in $\mathcal{F}^{j,k}$, and an off-diagonal entry $N^{j,k}_{j,k}$ is number of 1-dimensional subspaces common to the $i^{th}$ and $l^{th}$ subspaces of $\mathcal{F}^{j,k}$. In the rest of the proof, to reduce notational clutter, we shall use $G(x,y,z)$ to denote the Gaussian binomial coefficient $\binom{x}{y}_z$. We have

\[
N^{j,k}_{i,i} = G(r_1b^{k+1} + jb^k, 1, q) = G(b^{k-1}, 1, q)G(r_1b^2 + jb, 1, q^{b-1}),
\]

\[
N^{j,k}_{j,k} = G(r_2ab^k + ja b^{k-1}, 1, q) = G(b^{k-1}, 1, q)G(r_2ab + ja, 1, q^{b-1}),
\]

for some $r_1, r_2$ (may be different for different values of $i, l$). Let $P^{j,k}$ be the matrix over $\mathbb{R}$ obtained by dividing each entry of $N^{j,k}$ by $G(b^{k-1}, 1, q)$.

\[
\det(N^{j,k}) = G(b^{k-1}, 1, q)^{m^{j,k}} \det(P^{j,k})
\]

We will show that $\det(P^{j,k})$ is non-zero, thereby implying $\det(N^{j,k})$ is also non-zero. Consider $\det(P^{j,k}) \pmod{G(b, 1, q^{b-1})}$.

\[
P^{j,k}_{i,i} \equiv G(r_1b^2 + jb, 1, q^{b-1}) \equiv 0 \pmod{G(b, 1, q^{b-1})},
\]

\[
P^{j,k}_{j,k} \equiv G(r_2ab + ja, 1, q^{b-1}) \equiv G(r_3, 1, q^{b-1}) \pmod{G(b, 1, q^{b-1})},
\]

where $r_3 = ja \mod b$ and $1 \leq r_3 \leq b - 1$ (since $j < b, a < b$, and $b$ is a prime, we have $1 \leq r_3 \leq b - 1$). We know that the determinant of an $r \times r$ matrix where diagonal entries are 0 and off-diagonal entries are all 1 is $(-1)^{r-1}(r-1)$.

\[
\det(P^{j,k}) \equiv (G(r_3, 1, q^{b-1}))^{m^{j,k}}(-1)^{m^{j,k}-1}(m^{j,k} - 1) \pmod{G(b, 1, q^{b-1})}
\]

Let $Q^{j,k}$ be the matrix formed by taking all but the last row and the last column of $P^{j,k}$.

\[
\det(Q^{j,k}) \equiv (G(r_3, 1, q^{b-1}))^{m^{j,k}-1}(-1)^{m^{j,k}-2}(m^{j,k} - 2) \pmod{G(b, 1, q^{b-1})}
\]

We will now show that one of $\det(P^{j,k})$ or $\det(Q^{j,k})$ is non-zero \pmod{G(b, 1, q^{b-1})} and therefore non-zero in $\mathbb{R}$. First, we show that $G(r_3, 1, q^{b-1})^{m^{j,k}}$ is not divisible by $G(b, 1, q^{b-1})$. Suppose $s_3 \equiv r_3^{-1} \pmod{b}$.

\[
G(r_3, 1, q^{b-1})^{m^{j,k}} G(s_3, 1, q^{r_3 b-1})^{m^{j,k}} = G(r_3 s_3, 1, q^{b-1})^{m^{j,k}}
\]

\[
G(r_3 s_3, 1, q^{b-1})^{m^{j,k}} \equiv G(1, 1, q^{b-1})^{m^{j,k}} \pmod{G(b, 1, q^{b-1})} \equiv 1 \pmod{G(b, 1, q^{b-1})}
\]
Therefore, $G(r_3, 1, q^{bk-1})$ is invertible modulo $G(b, 1, q^{bk-1})$, and hence the former is not divisible by the latter. Suppose $G(r_3, 1, q^{bk-1})m^{j,k}(−1)^{m^{j,k}−1}(m^{j,k} − 1)$ is divisible by $G(b, 1, q^{bk-1})$. We may ignore $(-1)^{m^{j,k}−1}$ for divisibility purpose. Then, there must be a product of prime powers that is equal to $(m^{j,k} − 1)$ multiplied by $G(r_3, 1, q^{bk-1})m^{j,k}$ such that this product is divisible by $G(b, 1, q^{bk-1})$. Observe that, $G(r_3, 1, q^{bk-1})m^{j,k} - 1$ has only lesser powers of the same primes, and $m^{j,k} - 1$ and $m^{j,k} - 2$ cannot have any prime in common. So, the product $G(r_3, 1, q^{bk-1})m^{j,k}−1(m^{j,k}−2)$ cannot be divisible by $G(b, 1, q^{bk-1})$, which is what we wanted to prove.

Therefore, either $P^{j,k}$ or $Q^{j,k}$ is a full rank matrix, or $\text{rank}(P^{j,k}) \geq m^{j,k} - 1$. Being a non-zero multiple of $P^{j,k}$, $\text{rank}(N^{j,k}) \geq m^{j,k} - 1$. But we know that $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$, for any two matrices $A, B$.

$$m^{j,k} - 1 \leq \text{rank}(N^{j,k}) \leq \min(\text{rank}(M^{j,k}), \text{rank}((M^{j,k})^T)) = \text{rank}(M^{j,k}) \leq G(n, 1, q)$$

Or, $m^{j,k} \leq G(n, 1, q) + 1$, as required. It follows that,

$$|F| = m \leq 2 + \sum_{1 \leq k \leq \left\lfloor \frac{\ln n}{\ln b} \right\rfloor \atop 1 \leq j < b} m^{j,k} \leq (b - 1)(G(n, 1, q) + 1) \left\lceil \frac{\ln n}{\ln b} \right\rceil + 2. \quad \square$$

4 Concluding remarks

In Theorem 3, for $|F|$ to be at most $N(n, s, r, q)$, one of the necessary conditions is $r(s - r + 1) \leq b - 1$. When $r = 1$, this condition is always true as $L \subseteq \{0, 1, \ldots, b - 1\}$. However, when $r \geq 2$, it is not the case. Would it be possible to get the same upper bound for $|F|$ without having to satisfy such a strong necessary condition? Another interesting question concerning Theorem 3 is regarding its tightness. From Example 2, we know that Theorem 3 is tight when $r = 1$. However, since Theorem 4 requires the sets $K$ and $L$ to be disjoint it is not possible to extend the construction in Example 2 to obtain a tight example for the case $r \geq 2$. Further, we know of no other tight example for this case. Therefore, we are not clear whether Theorem 3 is tight when $r \geq 2$.

We believe that the upper bounds given by Theorems 5 and 7 are not tight. Proving tight upper bounds in both the scenarios is a question that is obviously interesting. One possible approach to try would be to answer the following simpler question. Consider the case when $L = \{\frac{1}{2}\}$. We call such a family a bisection-closed family of subspaces. Let $F$ be a bisection closed family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. From Theorem 7, we know that $|F| \leq \left(\binom{n}{1}_q\right) + 1)$ log$_2 n + 2$. We believe that $|F| \leq c \left(\binom{n}{1}_q\right)$, where $c$ is a constant. Example 8 gives a ‘trivial’ bisection-closed family of
size $\left[ \begin{array}{c} n - 1 \\ 1 \end{array} \right]_q$ where every subspace contains the vector $v_1$. It would be interesting to look for non-trivial examples of large bisection-closed families.

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