On the embedding of spacetime in higher-dimensional spaces with torsion

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Abstract

We revisit the Riemann-Cartan geometry in the context of recent higher-dimensional theories of spacetime. After introducing the concept of torsion in a modern geometrical language we present some results that represent extensions of Riemannian theorems. We consider the theory of local embeddings and submanifolds in the context of Riemann-Cartan geometries and show how a Riemannian spacetime may be locally and isometrically embedded in a bulk with torsion. As an application of this result, we discuss the problem of classical confinement and the stability of motion of particles and photons in the neighbourhood of branes for the case when the bulk has torsion. We illustrate our ideas considering the particular case when the embedding space has the geometry of a warped product space. We show how the confinement and stability properties of geodesics near the brane may be affected by the torsion of the embedding manifold. In this way we construct a classical analogue of quantum confinement inspired in theoretical-field models by replacing a scalar field with a torsion field.

1 Introduction

The idea that our spacetime may have more than four dimensions seems to be a recurrent theme in contemporary theoretical physics research. Such idea was first conjectured by G. Nordström [1], in 1914 (before the completion of general relativity), whose aim was to achieve unification of gravity with electromagnetism. Although Nordström’s interesting work, done in the context of a scalar gravity theory, was ignored for a long time, the same basic idea was taken up again, a few years later, by the mathematician T. Kaluza [2]. Kaluza, assuming the existence of a fifth dimension, was able to show that the basic equations of
the gravitational and electromagnetism fields could be derived from the Einstein equations written in a five-dimensional space. Kaluza's theory, later known as the Kaluza-Klein theory (after the contribution from physicist O. Klein) became the point of departure of many higher-dimensional theories. In the seventies, the Kaluza-Klein theory was generalized to include more general gauge fields [3], which required the introduction of additional extra dimensions. These new developments led to string theory in the eighties [4], while the nineties saw other higher-dimensional proposals come to light. Among these we should mention D-brane theory, the *brane world scenario* [5] and the so-called induced matter approach [6].

In almost all theories mentioned above it has been generally assumed that the underlying higher-dimensional space (often referred to as the *bulk* [7]) has a Riemannian character. Surely this is the more natural assumption to be made since the Riemannian theory is the geometrical setting of the well-established theory of general relativity. With very few exceptions, there has not been much discussion on whether the bulk could admit more general geometries. Nevertheless, attempts to broaden this scenario started to appear recently in the literature. Non-Riemannian geometries, such as Weyl geometry or Riemann-Cartan geometry, are taken into consideration as viable possibilities to describe the bulk [8, 9, 10].

The development of differential geometry in the last century led to the discovery of a vast number of non-Riemannian geometries. The richness these geometries possess in the form of new geometrical structures (in addition to the metric and affine connection) render them rather apt to the formulation of new physical theories insofar as they introduce extra degrees of freedom suitable, for instance, for the description of non-gravitational physical fields. Of course this is a well-known fact and was for a long time explored by A. Einstein and others in their pursuit of a unified field theory [12]. Nevertheless, we believe it is still of interest to investigate non-Riemannian geometries in a more modern context, namely, that of higher-dimensional spacetime theories. An illustrative example of development in this direction has appeared recently, in which it is assumed that, in the context of five-dimensional spacetime theory, the bulk has a Weylian geometry. One of the results of such an approach is that one is able to establish a classical analogue of quantum confinement [13] by purely geometrical means [14].

Following the ideas mentioned above, we consider, in the present article, another kind of non-Riemannian geometry, namely the Riemann-Cartan geometry. The latter represents one of the simplest generalizations of Riemannian geometry. As is well known, it constitutes the geometrical framework of a theory formulated by E. Cartan [15] in an attempt to extend general relativity when matter with spin is present. In spite of the limited interest it has arisen among theoretical physicists since its conception (perhaps due to the fact that it differs very little from general relativity), some authors believe that the Einstein-Cartan theory can have an important role in a future quantum theory of gravitation [16]. Moreover, torsion cosmology has been investigated recently in connection with the acceleration of the Universe [17].
The paper is organized as follows. We start in Section 2 with a brief review of Riemann-Cartan geometry and then prove some mathematical results that represent straightforward extensions of Riemannian theorems. We proceed in Section 3 to consider the theory of local embeddings and submanifolds in the context of geometries with torsion. Here we show that a Riemannian manifold may be embedded in a higher-dimensional space with torsion and this constitutes one of our main results. Section 4 contains an application of the formalism to the problem of classical confinement and the stability of motion of particles and photons in the neighbourhood of hypersurfaces. In Section 5 we show how the presence of a torsion field may affect both the confinement and/or stability of the particle’s motion. In Section 6 we give a simple application of the ideas developed previously. We conclude, in Section 7, with our final remarks.

2 Riemann-Cartan geometry

In this section we review some basic definitions and mathematical facts of Riemannian and Riemann-Cartan geometry. As we shall see, the latter may be viewed as a kind of generalization of the first, and some theorems that will be presented here are straightforward extensions of corresponding theorems of Riemannian geometry. However, these extensions present new features specially as far as geodesic motion is concerned. Let us start with the definition of affine connection [18].

**Definition.** Let $M$ be a $n$-dimensional differentiable manifold and $T(M)$ the set of all differentiable vector fields on $M$. An affine connection is a mapping $\nabla : T(M) \times T(M) \rightarrow T(M)$, which is denoted by $(U, V) \rightarrow \nabla_U V$, satisfying the following properties:

1. $\nabla_{fV + gW} = f\nabla_V W + g\nabla_V W$, 
2. $\nabla_V (U + W) = \nabla_V U + \nabla_V W$, 
3. $\nabla_V (fU) = V[f]U + f\nabla_V U$, 

where $V, U, W \in T(M)$, and $f, g$ are $C^\infty$ scalar functions defined on $M$. An important result comes immediately from the above definition and allows one to define a covariant derivative along a differentiable curve.

**Proposition.** Let $M$ be a differentiable manifold endowed with an affine connection $\nabla$, $V$ a vector field defined along a differentiable curve $\alpha : (a, b) \subset R \rightarrow M$. Then, there exists a unique rule which associates another vector field $\frac{DV}{d\lambda}$ along $\alpha$ to $V$, such that

1. $\frac{D(V + U)}{d\lambda} = \frac{DV}{d\lambda} + \frac{DU}{d\lambda}$, 
2. $\frac{D(fV)}{d\lambda} = \frac{df}{d\lambda}V + f\frac{DV}{d\lambda}$,
where \( \alpha = \alpha(\lambda) \) and \( \lambda \in (a, b) \).

iii) If the vector field \( U(\lambda) \) is induced by a vector field \( \hat{U} \in T(M) \), i.e., \( U(\lambda) = \hat{U}(\alpha(\lambda)) \), then \( \frac{DU}{d\lambda} = \nabla_V U \), where \( V \) is the tangent vector to the curve \( \alpha \), i.e., \( V = \frac{d\alpha}{d\lambda} \). For a proof of this proposition we refer the reader to [18].

We now introduce the concept of parallel transport of a vector along a given curve.

**Definition.** Let \( M \) be a differentiable manifold with an affine connection \( \nabla \), \( \alpha : (a, b) \to M \) a differentiable curve on \( M \), and \( V \) a vector field defined along \( \alpha = \alpha(\lambda) \). The vector field \( V \) is said to be parallel if \( \frac{dV}{d\lambda} = 0 \) for any value of the parameter \( \lambda \in (a, b) \).

A concept that is basic to the Riemann-Cartan geometry is that of **torsion**, which is given by the following definition:

**Definition.** Let \( \nabla \) be an affine connection defined on \( M \) and \( U, V \in T(M) \). We define the **torsion** \( T \) of \( M \) as the mapping \( T : T(M) \times T(M) \to T(M) \), such that

\[
T(U, V) = \nabla_U V - \nabla_V U - [U, V].
\]

If the torsion vanishes identically we say that the affine connection \( \nabla \) is symmetric (or, simply, **torsionless**).

To establish a link between the affine connection \( \nabla \) and the metric \( g \) we need a further definition.

**Definition.** Let \( M \) be a differentiable manifold endowed with an affine connection \( \nabla \) and a metric tensor \( g \) globally defined in \( M \). We say that \( \nabla \) is compatible with \( g \) if for any vector fields \( U, V, W \in T(M) \), the condition below is satisfied:

\[
V[g(U, W)] = g(\nabla_V U, W) + g(U, \nabla_V W). \tag{7}
\]

We now state an important result.

**Theorem (Levi-Civita extended).** In a given differentiable manifold \( M \) endowed with a metric \( g \) on \( M \), there exists only one affine connection \( \nabla \) such that \( \nabla \) is compatible with \( g \).

**Proof.** Let us first suppose that such \( \nabla \) exists. Then, from (7) we have the following three equations

\[
V[g(U, W)] = g(\nabla_V U, W) + g(U, \nabla_V W) \tag{8}
\]

\[
W[g(V, U)] = g(\nabla_W V, U) + g(V, \nabla_W U) \tag{9}
\]

\[
U[g(W, V)] = g(\nabla_U W, V) + g(W, \nabla_U V) \tag{10}
\]

Adding (8) and (9) and subtracting (10), and also taking into account the definition of torsion (6), we are left with

\[
g(\nabla_V W, U) = \frac{1}{2} \{V[g(W, U)] + W[g(V, U)] - U[g(W, V)] + g([V, W], U) \\
+ g([U, V], W) + g([U, W], V) - g(T(W, U), V) - g(T(V, U), W) \\
- g(T(W, V), U)\} \tag{11}
\]
If the affine connection $\nabla$ is symmetric the above equation has a simpler form, and in this case $\nabla$ reduces to the celebrated Levi-Civita connection [18]. The equation (11) shows that the affine connection $\nabla$, if it exists, is uniquely determined from the metric $g$ and the torsion $T$. (In the torsionless case, $\nabla$ is determined from $g$ alone). Finally, to prove the existence of such a connection we just define $\nabla U V$ by means of (11).

A tensor that is naturally associated with $T$ is the torsion tensor $T$, defined by the mapping $T^* : T^*(M) \times T(M) \times T(M) \rightarrow \mathbb{R}$, such that $T^*(\tilde{w}, U, V) = \tilde{w}(T(U, V))$, where $T^*(M)$ denotes the set of all differentiable one-form fields on $M$ and $\tilde{w} \in T^*(M)$. It is easy to see that the components of $T$ in a coordinate basis associated with a local coordinate system $\{x^a\}$, $a = 1, ..., n$, are simply given in terms of the connection coefficients, i.e., $T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}$, where $\Gamma^a_{bc} \equiv dx^a(\nabla_{\partial_b} \partial_c)$. A straightforward calculation shows that one can express the components of the affine connection as

$$\Gamma^a_{bc} = \Gamma^a_{bc} - K^a_{bc}$$

where $\Gamma^a_{bc} \equiv \frac{1}{2}g^{ad}[g_{db,c} + g_{dc,b} - g_{bc,d}]$ denotes the Christoffel symbols of second kind and $K^a_{bc} = \frac{1}{2}(T^a_{cb} + T^a_{cb} + T^a_{bc})$, represents the components of another tensor, called the contorsion tensor.

Thus we see then that what basically makes the geometry discovered by Cartan distinct from Riemannian geometry is simply the fact that in the latter the affine connection $\nabla$ is not supposed to be symmetric. As a consequence, the affine connection $\nabla$ is no longer a Levi-Civita connection and for this reason affine geodesics do not coincide in general with metrical geodesics.

Since we are primarily interested in the embedding problem in the context of spaces with torsion, in the next section we shall briefly examine the theory of submanifolds in Riemann-Cartan geometry. As we shall see, the basic mathematical facts are still simple extensions of the Riemannian case.

## 3 Submanifolds and isometric embeddings in spaces with torsion

We need first to review some basic concepts of the theory of Riemannian submanifolds.

**Definition.** Let $(M, g, \nabla)$ and $(\overline{M}, \overline{g}, \overline{\nabla})$ be Riemann-Cartan differentiable manifolds of dimensions $m$ and $n = m + k$, respectively. A differentiable map $f : M \rightarrow \overline{M}$ is called an immersion if the differential $f_* : T_p(M) \rightarrow T_{f(p)}\overline{M}$ is injective for any $P \in M$. The number $k$ is called the codimension of $f$. We say that the immersion $f : M \rightarrow \overline{M}$ is isometric at a point $P \in M$ if $g(U, V) = \overline{g}(f_*(U), f_*(V))$ for every $U, V$ in the tangent space $T_P(M)$. If, in addition, $f$ is a homeomorphism onto $f(M)$, then we say that $f$ is an embedding.

If $M \subset \overline{M}$ and the inclusion $i : M \subset \overline{M}$ is an embedding, then $M$ is called a submanifold. The components of the torsion are raised and lowered with $g^{ab}$ and $g_{ab}$, respectively.

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1 Note that the indices appearing in the components of the torsion are raised and lowered with $g^{ab}$ and $g_{ab}$, respectively.
a submanifold of $\overline{M}$.

Let $f : M \to \overline{M}$ be an embedding. We may, therefore, identify $M$ with its image under $f$, so that we can regard $M$ as a submanifold embedded in $\overline{M}$, with $f$ actually being the inclusion map. Thus, we shall identify each vector $V \in T_P(M)$ with $f_*(V) \in T_{f(P)}(\overline{M})$ and consider $T_P(M)$ as a subspace of $T_{f(P)}(\overline{M})$. In the vector space $T_P(\overline{M})$ the metric $g$ allows one to make the decomposition $T_P(\overline{M}) = T_P(M) \oplus T_P(M)^\perp$, where $T_P(M)^\perp$ is the orthogonal complement of $T_P(M) \subset T_P(\overline{M})$. That is, for any vector $\nabla \in T_P(\overline{M})$, with $P \in M$, we can decompose $\nabla$ into $\nabla = V + V^\perp$, $V \in T_P(M)$, $V^\perp \in T_P(M)^\perp$.

Let us denote the connection on $\overline{M}$ by $\nabla$. We now can prove the following proposition.

Proposition. If $V$ and $U$ are local vector fields on $M$, and $\nabla$ and $\nabla$ are local extensions of these fields to $\overline{M}$, then the connection $\nabla_U V$ compatible with the induced metric on $M$ will be given by

$$\nabla_U V = (\nabla_U U)^\top$$

where $(\nabla_U U)^\top$ is the tangential component of $\nabla_U U$.

Proof. It is not difficult to verify that $\nabla_U V$ as defined by (13) satisfies (1), (2) and (3); hence our definition makes sense. Now consider the equation that expresses the compatibility requirement between $\nabla$ and $g$:

$$\nabla [g(U, W)] = \nabla g(U, U) + \nabla g(U, W)$$

(14)

where $\nabla$, $U$, $W \in T(\overline{M})$. Now, suppose that $\nabla$, $U$, $W$ are local extensions of the vector fields $V, U, W$ to $\overline{M}$. Clearly, at a point $P \in M$, we have

$$\nabla [g(U, W)] = V [g(U, W)] = V [g(U, W)]$$

(15)

where we have taking into account that the inclusion of $M$ into $\overline{M}$ is isometric.

On the other hand, evaluating the first term of the right-hand side of (14) at $P$ yields

$$\nabla g(U, W) = g((\nabla_U U)^\top + (\nabla_U U)^\perp, W) = g((\nabla_U U)^\top, W) = g(U, (\nabla_U U)^\top, W)$$

(16)

with an analogous expression for $g(U, (\nabla_U W)^\top)$. From the above equations we finally obtain

$$V [g(U, W)] = g((\nabla_U U)^\top, W) + g(U, (\nabla_U W)^\top)$$

From the Levi-Civita theorem extended to Riemann-Cartan manifolds, which asserts the uniqueness of affine connection $\nabla$ in a Riemann-Cartan manifold we conclude that the tangential component of $\nabla_U U$, evaluated at points of $M$, is, in fact, the connection of $M$ compatible with the induced metric $g$ of $M$.

Since the embedding $f : M \to \overline{M}$ induces a connection $\nabla$ in $M$, given by (13), we now turn our attention to the torsion $T$ in $M$ defined by this induced connection. If $U$ and $V$ are local vector fields on $M$, then from (6) and (13) we get

$$T(U, V) = \nabla_U V - \nabla_V U - [U, V] = (\nabla_U V)^\top - (\nabla_V U)^\top - [U, V]^\top$$

(17)
where $\overline{U}, \overline{V}$ are local extensions of the the vector fields $U, V$ to $\overline{M}$, and we have used the fact that at any point of $M$ one has $[U, V] = [\overline{U}, \overline{V}] = [\overline{U}, \overline{V}]^T$. Thus at any point of $M$ the equation (17) becomes

$$T(U, V) = (\overline{\nabla}_U \overline{V} - \overline{\nabla}_V \overline{U} - [\overline{U}, \overline{V}])^T = T(\overline{U}, \overline{V})^T$$

We conclude therefore that the induced torsion $T$ on $M$ is nothing less than the tangential component of the torsion $T$ defined in $\overline{M}$.

At this point consider the following question: Is it possible to have a purely Riemannian submanifold $M$ isometrically embedded in a non-Riemannian space $\overline{M}$ with a non-vanishing torsion? The answer is clearly affirmative. Indeed, a submanifold $M$ embedded in space $\overline{M}$ with torsion $T$ will be purely Riemannian if and only if the torsion $T$ induced from $\overline{T}$ vanishes throughout $M$. From the above we see that the necessary and sufficient condition for that is $T(U, V)^T = 0$, i.e., that the tangential component of $\overline{T}$ of $\overline{M}$ vanishes identically.

To get further insight into the ideas developed above let us consider the case in which the manifold $\overline{M}$ is foliated by a family of submanifolds defined by $k$ equations $y^A = \text{constant}$, with the spacetime $M$ corresponding to one of these manifolds $y^A = y^A_0 = \text{constant}$. In local coordinates $\{y^a\}$ of $\overline{M}$ adapted to the embedding it is not difficult to verify that the condition $T(U, V) = T(\overline{U}, \overline{V})^T = 0$ implies $T^{\lambda}_{\cdot \alpha \beta} = 0$, where $\alpha, \beta, \lambda$ are tensorial indices with respect to $M$. Therefore, if the components $T^{\lambda}_{\cdot bc}$ of the torsion tensor $T$ vanishes on $M$, then the geometry of the submanifold $M$ embedded in the non-Riemannian bulk $\overline{M}$ is Riemannian.

It should be noted that the Riemannian character of spacetime $M$ embedded in a Riemann-Cartan bulk $\overline{M}$ does not prevent the former from being indirectly affected by the torsion of $\overline{M}$. A nice illustration of this point is given by the behaviour of geodesics near $M$. In the next section we shall examine how a torsion field may affect the geodesic motion in the case of a bulk with a warped product geometry. We shall be interested particularly in the problem of classical confinement and stability of the motion of particles and photons near the spacetime submanifold. [19, 20]

### 4 Geodesic motion in a Riemannian warped product space

In this section let us consider the case where the geometry of the bulk contains two special ingredients: a) It is a Riemannian manifold and b) its metric has the structure of a warped product space [21]. As is well known, the importance of warped product geometry is closely related to the so-called braneworld scenario

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2From now on lower case Latin indices take value in the range $(0, 1, ..., (n + 3))$, while Greek indices run over $(0, 1, 2, 3)$. The coordinates of a generic point $P$ of the manifold $\overline{M}$ will be denoted by $y^A = (x^\alpha, y^4, ..., y^{n+3})$, where $x^\alpha$ denotes the four-dimensional spacetime coordinates and $y^A (A > 3)$ refers to the $n$ extra coordinates of $P$. 

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Let us start with the investigation of geodesics in warped product spaces, firstly considering the Riemannian case.

We define a warped product space in the following way. Let \((M, g)\) and \((N, h)\) be two Riemannian manifolds of dimension \(m\) and \(r\), with metrics \(g\) and \(h\), respectively. Suppose we are given a smooth function \(f: N \rightarrow R\) (which will be called the warping function). We construct a new Riemannian manifold by setting \(M = M \times N\) and endow \(M\) with the metric \(g = e^{2f} g \oplus k\). We view future application, we shall take \(M = M^4\) and \(N = R\), where \(M^4\) denotes a four-dimensional (4D) Lorentzian manifold with signature (+−−−) (henceforth referred to as spacetime). In local coordinates \(\{y^a = (x^\alpha, y^4)\}\) the line element corresponding to the metric \(g\) will be written as

\[
\text{ds}^2 = \bar{g}_{ab} dy^a dy^b
\]

The equations of geodesics in the five-dimensional (5D) space \(\overline{M}\) will be given by

\[
\frac{d^2 y^a}{d\lambda^2} + (5) \Gamma^a_{bc} \frac{dy^b}{d\lambda} \frac{dy^c}{d\lambda} = 0,
\]

where \(\lambda\) is an affine parameter and \((5) \Gamma^a_{bc}\) denotes the 5D Christoffel symbols \((5) \Gamma^a_{bc} = \frac{1}{2} \bar{g}^{ad} \left( \frac{\partial \bar{g}_{bd}}{\partial x^c} + \frac{\partial \bar{g}_{cd}}{\partial x^b} - \frac{\partial \bar{g}_{bc}}{\partial x^d} \right)\). Denoting the fifth coordinate \(y^4\) by \(y\) and the remaining coordinates \(y^\alpha\) (the spacetime coordinates) by \(x^\alpha\), i.e. \(y^a = (x^\alpha, y^4)\), we can easily show that the ”4D part” of the geodesic equations (18) can be rewritten in the form

\[
\frac{d^2 x^\mu}{d\lambda^2} + (4) \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \xi^\mu,
\]

where

\[
\xi^\mu = - (5) \Gamma^\mu_{44} \left( \frac{dy}{d\lambda} \right)^2 - 2(5) \Gamma^\mu_{4\alpha} \frac{dx^\alpha}{d\lambda} \frac{dy}{d\lambda} - \frac{1}{2} \bar{g}^{\alpha\beta} \left( \bar{g}_{4\alpha,\beta} + \bar{g}_{4\beta,\alpha} - \bar{g}_{\alpha\beta,4} \right) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},
\]

and \((4) \Gamma^\mu_{\alpha\beta} = \frac{1}{2} \bar{g}^{\mu\nu} \left( \bar{g}_{\alpha\nu,\beta} + \bar{g}_{\nu\beta,\alpha} - \bar{g}_{\alpha\beta,\nu} \right)\).

Let us now turn our attention to the five-dimensional brane-world scenario, where the bulk corresponds to the five-dimensional manifold \(\overline{M}\), which, as in the previous section, is assumed to be foliated by a family of submanifolds (in this case, hypersurfaces) defined by the equation \(y = \text{constant}\).

It turns out that the geometry of a generic hypersurface \(\Sigma\), say \(y = y_0\), will be determined by the induced metric \(g_{\alpha\beta}(x) = \bar{g}_{\alpha\beta}(x, y_0)\). Thus, on the hypersurface we have

\[
\text{ds}^2 = \bar{g}_{\alpha\beta}(x, y_0) dx^\alpha dx^\beta.
\]

Throughout this section Latin indices take values in the range (0,1,...4) while Greek indices run from (0,1,2,3).
We see then that the quantities \( \Gamma^\mu_{\alpha\beta} \) which appear on the left-hand side of Eq. (19) are to be identified with the Christoffel symbols associated with the induced metric in the leaves of the foliation defined above.

Let us restrict ourselves to the class of warped geometries given by the following line element

\[
dS^2 = e^{2f} g_{\alpha\beta} dx^\alpha dx^\beta - dy^2,
\]

where \( f = f(y) \) and \( g_{\alpha\beta} = g_{\alpha\beta}(x) \). For this metric it is easy to see\(^4\) that \((5) \Gamma^\mu_{\alpha\beta} = 0 \) and \((5) \Gamma^\mu_{\alpha\nu} = \frac{1}{2} g^\mu_{\alpha\nu} g_{\beta\lambda} \Gamma^\beta_{\nu\lambda} = f' \delta^\mu_{\nu}, \) where prime denotes derivative with respect to \( y \). Thus in the case of the warped product space (21) the right-hand side of Eq. (19) reduces to \( \xi^\mu = -2 f' \Gamma^\mu_{\beta\nu} \frac{dx^\beta}{d\lambda} \frac{dy}{d\lambda} \) and the 4D part of the geodesic equations becomes

\[
\frac{d^2 x^\mu}{d\lambda^2} + (4) \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -2 f' \frac{dx^\mu}{d\lambda} \frac{dy}{d\lambda}.
\]

On the other hand the geodesic equation for the fifth coordinate \( y \) in the warped product space becomes

\[
\frac{d^2 y}{d\lambda^2} + f' e^{2f} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0.
\]

If the 5D geodesics are assumed to be timelike \( \left( \eta_{ab} \frac{dy^a}{d\lambda} \frac{dy^b}{d\lambda} = 1 \right) \), then we can decouple the above equation from the 4D spacetime coordinates to obtain

\[
\frac{d^2 y}{d\lambda^2} + f' \left( 1 + \left( \frac{dy}{d\lambda} \right)^2 \right) = 0.
\]

Similarly, to study the motion of photons in 5D, we must consider the null geodesics \( \left( \eta_{ab} \frac{dy^a}{d\lambda} \frac{dy^b}{d\lambda} = 0 \right) \), in which case Eq. (23) becomes

\[
\frac{d^2 y}{d\lambda^2} + f' \left( \frac{dy}{d\lambda} \right)^2 = 0.
\]

Equations (24) and (25) are ordinary differential equations of second-order which, in principle, can be solved if the function \( f' = f'(y) \) is known. A qualitative picture of the motion in the fifth dimension may be obtained without the need to solve (24) and (25) analytically \[20\]. This is done by defining the variable \( q = \frac{dy}{d\lambda} \) and then investigating the autonomous dynamical system \[22\]

\[
\frac{dy}{d\lambda} = q \quad (26)
\]

\[
\frac{dq}{d\lambda} = F(q, y) \quad (27)
\]

\(^4\)In the above calculation we have used the fact that the matrix \( g_{\alpha\beta} \) has an inverse \( g^{\alpha\beta} \), that is, \( g^{\mu\beta} g_{\beta\nu} = \delta^\mu_{\nu} \). This may be easily seen since by definition \( \det g = -\det \eta \neq 0 \).
with \( F(q, y) = -f'(\epsilon + q^2) \), where \( \epsilon = 1 \) in the case of (24) (corresponding to the motion of particles with nonzero rest mass) and \( \epsilon = 0 \) in the case of (25) (corresponding to the motion of photons). In the investigation of dynamical systems a crucial role is played by their \textit{equilibrium points}, which in the case of system (26) are given by \( \frac{dq}{d\lambda} = 0 \) and \( \frac{dy}{d\lambda} = 0 \). These solutions, corresponding to fixed points in the phase space of the system, represent curves that lie entirely in a certain hypersurface \( \Sigma \) of the foliation previously mentioned. The knowledge of these points together with their stability properties provides a great deal of information on the types of behaviour allowed by the system. 

An investigation of the qualitative behaviour of the solutions to the above system was carried out in the cases when the five-dimensional \( \mathcal{M} \) is Riemannian [20], and when \( \mathcal{M} \) is Weylian [14]. In the next section we shall turn our attention to the case when \( \mathcal{M} \) is a Riemann-Cartan manifold, i.e., when \( \mathcal{M} \) has torsion.

One of the motivations for studying the geodesic motion in the presence of torsion is the following. As is well known, in the brane-world scenario the stability of the confinement of matter fields at the quantum level is made possible by assuming an interaction of matter with a scalar field. An example of how this mechanism works is nicely illustrated by a field-theoretical model devised by Rubakov, in which fermions may be trapped to a brane by interacting with a scalar field that depends only on the extra dimension [13]. On the other hand, the kind of confinement we are concerned with is purely geometrical, and that means the only force acting on the particles is the gravitational force. In a purely classical (non-quantum) picture, one would like to have effective mechanisms, other than the presence of a quantum scalar field, to constrain massive particles to move on hypersurfaces in a stable way. Two possibilities of implement such a program have already been studied. One is to assume a direct interaction between the particles and a physical scalar field [25]. Following this approach it has been shown that stable confinement in a thick brane is possible by means of a direct interaction of the particles with a scalar field through a modification of the Lagrangian of the particle. A second approach would appeal to pure geometry: for instance, modelling the bulk with a Weyl geometrical structure. In this case it is the Weyl field that provides the mechanism necessary for confinement and stabilization of the motion of particles in the brane [14]. At this stage, we would like to know whether classical confinement of particles and photons could also be obtained by using a torsion field, that is, by allowing for the bulk to have a Riemann-Cartan geometry. This is the question we shall deal with in the next section.

5 Geodesic motion in the presence of torsion

When the five-dimensional embedding space \( \mathcal{M} = M^5 \) is a warped product space endowed with a torsion field it is not difficult to verify, by putting \((5)\Gamma^a_{bc} = \{_{ab}^c \} - K^a_{bc} \) into (18) and noting that \( K^a_{a4} = 0 \), that the motion of a massive
particle in the fifth dimension is given by the equation

\[
\frac{d^2 y}{d\lambda^2} + f' \left( 1 + \left( \frac{dy}{d\lambda} \right)^2 \right) = K^4_{(\alpha\beta)} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} + K^4_{4\alpha} \frac{dx^\alpha}{d\lambda} \frac{dy}{d\lambda},
\]

(28)

where \( K^4_{(\alpha\beta)} \) denotes the symmetric part of \( K^4_{\alpha\beta} \). This equation is the equivalent of (24) for a five-dimensional bulk is a Riemann-Cartan manifold, i.e., when \( M^5 \) has a non-vanishing torsion. In this form Eq. (28) does not allow us, in general, to study the motion of the particle along the extra dimension decoupled from its motion in the four-dimensional spacetime. Nevertheless, there are some particular cases in which the five-dimensional motion is independent of the remaining dimensions. One case is, of course, when both \( K^4_{(\alpha\beta)} \) and \( K^4_{4\alpha} = 0 \) vanish. In this situation the torsion field does not influence the motion of the particle along the fifth dimension. A more interesting case, however, is when the symmetric part of the components \( K^4_{(\alpha\beta)} \) of the contorsion tensor is chosen proportional to \( h_{\alpha\beta} = e^{2f}g_{\alpha\beta} \), the induced metric on \( M \). For instance, if we set \( K^4_{(\alpha\beta)} = \psi(y)e^{2f}g_{\alpha\beta} \) and \( K^4_{4\alpha} = 0 \), then the equation (28) becomes

\[
\frac{d^2 y}{d\lambda^2} + \left( 1 + \left( \frac{dy}{d\lambda} \right)^2 \right) (f'(y) - \psi(y)) = 0
\]

The presence of the function \( \psi(y) \) modifies the dynamical system (26) leading to a new picture of the phase plane, where the equilibrium points and the stability properties of the solutions may completely change. To give an illustration, let us consider that the five-dimensional Riemannian space \( \tilde{M} \) is endowed with a Mashhoon-Wesson-type metric [24]

\[
dS^2 = \Lambda^2 y^2 g_{\alpha\beta} dx^\alpha dx^\beta - dy^2.
\]

(29)

It has been shown [14] that in this case there is no confinement of particles in the hypersurfaces \( y = \text{const} \). Now let us introduce a torsion field such that it gives rise to a contorsion field \( K^\lambda_{\alpha\beta} \) given by

\[
K^4_{\alpha\beta} = \psi(y)g_{\alpha\beta} + L_{\alpha\beta}
\]

(30)

with \( L^4_{\alpha\beta} \) antisymmetric in the indices \( \alpha, \beta \). With this choice let us show that we can set up a confinement mechanism induced by the contorsion only. With this objective in mind let us set \( \psi(y) = \frac{1}{y} - a(y - y_0) \), where \( a \) is a constant. It is an easy task to show that with such choice all timelike geodesics of \( M = M^4 \) will remain confined in \( M^4 \). Moreover, the stability of the confinement in this case is entirely governed by the sign of the constant \( a \). All these results can be obtained from an analysis of the dynamical system (26) carried out in the neighbourhood of the equilibrium points, now with the function \( F(q, y) \) modified.
Figure 1: In the absence of torsion there are no equilibrium points. The particles go through the leaves without being trapped by the hypersurface $\Sigma$.

due to the presence of torsion. The new dynamical system to be studied is thus

$$\frac{dy}{d\lambda} = q$$  \hspace{1cm} (31)

$$\frac{dq}{d\lambda} = -(1 + q^2)(f'(y) - \psi(y))$$  \hspace{1cm} (32)

For the Mashhoon-Wesson metric $f'(y) = \frac{1}{y}$, so if the torsion field is not present, then it is clear that there are no equilibrium points (because $f'(y)$ has no roots). That means no confinement of massive particles is possible at the hypersurfaces $y = y_0 = \text{const}$. On the other hand, if the torsion field is "turned on", then (31) becomes

$$\frac{dy}{d\lambda} = q$$  \hspace{1cm} (33)

$$\frac{dq}{d\lambda} = a(y - y_0)(1 + q^2)$$  \hspace{1cm} (34)

In regard to the dynamical system above we now have an equilibrium point $E$ at $q = 0, y = y_0$. This is a solution that corresponds to the worldline of a massive particle trapped under the action of the torsion field at the hypersurface $y = y_0$. It is straightforward to verify that if $a > 0$, then the equilibrium point $E$ corresponds to a center. In other words, the solutions near $E$ have the topology of a circle in the phase portrait of the dynamical system (33). The closed curves thus describe the motion of particles oscillating about the hypersurface $y = y_0$. In this case we are in the presence of a kind of confinement where particles lying near the $y = y_0$ will oscillate about it, entering and leaving the hypersurface indefinitely. On the other hand, if $a < 0$, $E$ is a saddle point. In this case,
although the particle is still constrained by the torsion field to move on $\Sigma$, this sort of confinement is highly unstable: Almost any small perturbation on the fifth-dimensional motion of the particle will cause it to be expelled from $\Sigma$.

Before concluding this section, let us consider the question whether (30) represents a possible choice, i.e., a legitimate choice for the components of con
torsion field. In order to answer this question just choose any torsion tensor
having the following components:

$$T^\alpha_{\beta 4} = -\delta^\alpha_\beta \psi(y), \quad T^4_{\alpha \beta} = 2L_{\alpha \beta}. \quad (35)$$

A simple calculation is sufficient to convince ourselves that (35) will lead to
(30).

6 Final Remarks

In the recent years there has been a renewed interest in a certain class of higher-
dimensional spacetime theories which start from the following assumptions: a) our spacetime is viewed as four-dimensional Riemannian hypersurface embed-
ded in a five-dimensional Riemannian manifold (the bulk); b) the geometry of the higher-dimensional space is characterized by a warped product space; c) fermionic matter is confined to the hypersurface by means of an interaction of the fermions with a scalar field which depends only on the extra dimension. In this scenario we have considered the possibility of describing the five-dimensional space by a non-Riemannian geometry, namely a Riemann-Cartan geometry, in which a new degree of freedom, the torsion, appears. We have shown that for a class of torsion fields, the geometry induced on four-dimensional spacetime
has a Riemannian structure. This means that it is possible to embed isometrically a Riemannian spacetime into a Riemann-Cartan five-dimensional bulk with non-vanishing torsion. We also have shown that confinement and stability properties of geodesics near the brane may be affected by the torsion. As an illustration of this fact, we have considered the case of a five-dimensional warped product space and have constructed a classical analogue of the quantum confinement by considering a very special case of torsion field. In a certain sense, this purely geometrical field, which has a purely geometrical nature, is able to replace the quantum scalar field that is usually responsible for the confinement in field-theoretical models [13].

Another comment concerning the embedding of the spacetime in spaces with torsion is the following. In the induced-matter approach an energy-matter tensor describing macroscopic matter is generated geometrically from the Einstein field equations in vacuum. It is now well understood that the question whether any energy-momentum tensor $T_{\alpha\beta}$ can be generated in this way is equivalent to know whether any solution of the Einstein equations for a prescribed $T_{\alpha\beta}$ can be locally embedded in some five-dimensional Ricci-flat space. As it happens, the answer to this question is given by the Campell-Magaard theorem, which states that any $n$-dimensional Riemannian manifold can be locally and isometrically embedded in a $(n + 1)$-dimensional Ricci-flat Riemannian space [26]. Transposing these ideas to the Einstein-Cartan theory of gravity, one would naturally wonder whether spin could also be generated, or induced, in the same manner, from a higher-dimensional space. The results obtained in Sec. 3, allow us to draw some conclusions in this respect. One is that if the bulk $\mathcal{M}$ is a torsionless space (hence not sourced by matter with spin), then it is not possible to generate spin geometrically (through dimensional reduction) in the four-dimensional spacetime $\mathcal{M}$. A second conclusion is that, in general, spin in four dimensions may be generated from five dimensions, but in some particular cases the bulk does not transfer spin to four-dimensions.

In a certain sense, the present article is a follow up of previous work, where basically the same questions treated here were examined in the context of another non-Riemannian setting, namely that of the geometry of Weyl [14]. In fact, we may regard these works as part of more general program of research whose underlying idea is to highlight and explore the role non-Riemannian geometries may play in the development of novel frameworks for modern higher-dimensional spacetime theories.

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