NONLINEAR SCHRÖDINGER EQUATION ON FOUR-DIMENSIONAL COMPACT MANIFOLDS

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Abstract. We prove two new results about the Cauchy problem in the energy space for nonlinear Schrödinger equations on four-dimensional compact manifolds. The first one concerns global wellposedness for Hartree-type nonlinearities and includes approximations of cubic NLS on the sphere. The second one provides, in the case of zonal data on the sphere, local wellposedness for quadratic nonlinearities as well as global wellposedness for small energy data in the Hamiltonian case. Both results are based on new multilinear Strichartz-type estimates for the Schrödinger group.

1. Introduction

In a recent series of papers ([5], [7], [8], see also [9], [12]), Burq-Gérard-Tzvetkov investigated the Cauchy problem for nonlinear Schrödinger equations (NLS) on Riemannian compact manifolds, generalizing the work of Bourgain on tori ([1], [2]). In [5], Strichartz estimates with fractional loss of derivatives were established for the Schrödinger group. They led to global wellposedness of NLS on surfaces with any defocusing polynomial nonlinearity. On three-manifolds, these estimates also provided global existence and uniqueness for cubic defocusing NLS, but they failed to prove the Lipschitz continuity of the flow map on the energy space. These results were improved in [7], [8] for specific manifolds such as spheres, taking advantage of new multilinear Strichartz inequalities for the Schrödinger group (see also [6]). In particular, on such three-manifolds the Lipschitz continuity and the smoothness of the flow map on the energy space were established for cubic NLS, as well as global existence on the energy space for every defocusing subquintic NLS.

However, none of the above methods provided global wellposedness results in the energy space for NLS on four-dimensional manifolds. This is in strong contrast with the Euclidean case (see [14], [16], [10], [22]). The only available global existence result on a compact four-manifold seems to be the one of Bourgain in [2], which concerns defocusing nonlinearities of the type $|u|u$ and Cauchy data in $H^s(T^4)$, $s > 1$. Let us discuss briefly the reasons of this difficulty. On the one hand, Strichartz estimates of [5] involve a too large loss of derivative in four space dimension; typically, for cubic NLS, they lead to local wellposedness in $H^s$ for $s > 3/2$, which is not sufficient in view of the energy and $L^2$ conservation laws. Moreover, these estimates are
restricted to $L^p_t L^q_x$ norms with $p \geq 2$ and the admissibility condition
\[ \frac{1}{p} + \frac{2}{q} = 1, \]
so that the analysis does not improve when the nonlinearity becomes subcubic. On the other hand, the analysis based on bilinear Strichartz estimates is currently restricted to nonlinearities of cubic type, and on $S^4$ it only yields local wellposedness in $H^s$ for $s > 1$. In fact, this obstruction can be made more precise by combining two results from [5] and [7]. Indeed, from Theorem 4 in [5], we know that the estimate
\[ \int_0^{2\pi} \int_{S^4} |e^{it\Delta} f(x)|^4 dt dx \lesssim \|f\|^4_{H^{1/2}(S^4)} \]
is wrong, which, by Remark 2.12 in [7], implies that the flow map of cubic NLS cannot be $C^3$ near the Cauchy data $u_0 = 0$ in $H^1(S^4)$. Moreover, notice that this phenomenon occurs for zonal data, namely functions depending only on the distance to a fixed point.

The purpose of this paper is to provide further results on four-dimensional manifolds. We shall study two types of NLS equations. In section 2, we study NLS with the following nonlocal nonlinearity,
\[ \begin{cases} 
  i\partial_t u + \Delta u = (1 - \Delta)^{-\alpha} |u|^2 u, \\ 
  u(0, x) = u_0(x)
\end{cases} \tag{1} \]
where $\alpha > 0$. Notice that the homogeneous version of this nonlinearity on the Euclidean space $\mathbb{R}^d$ reads
\[ \left( \frac{1}{|x|^{d-2\alpha}} \ast |u|^2 \right) u \]
so that (1) can be seen as a variant of Hartree’s equation on a compact manifold. We obtain the following result.

**Theorem 1.** Let $(M, g)$ be a compact Riemannian manifold of dimension 4 and let $\alpha > \frac{1}{2}$. There exists a subspace $X$ of $\mathcal{C}(\mathbb{R}, H^1(M))$ such that, for every $u_0 \in H^1(M)$, the Cauchy problem (1) has a unique global solution $u \in X$. Moreover, in the special case $M$ is the four-dimensional standard sphere $M = S^4$, the same result holds for all values $\alpha > 0$ of the parameter.

The proof of Theorem 1 relies on the combination of conservation laws for equation (1) with the following quadrilinear estimates,

\[ \sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{M} \chi(t) e^{ist} (1 - \Delta)^{-\alpha} (u_1 \overline{u}_2) u_3 \overline{u}_4 dx dt \right| \leq C(m(N_1, \cdots, N_4))^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \|f_3\|_{L^2(M)} \|f_4\|_{L^2(M)}, \]

for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, for every $s_0 < 1$ and for $f_1, f_2, f_3, f_4$ satisfying
\[ 1^{\sqrt{1-\Delta \in [N_j, 2N_j]}}(f_j) = f_j, \quad j = 1, 2, 3, 4. \]
Here and in the sequel $m(N_1, \cdots, N_4)$ denotes the product of the smallest two numbers among $N_1, N_2, N_3, N_4$. Moreover $u_j$ and $f_j$ are linked by
\[ u_j(t, x) = S(t)f_j(x), \quad j = 1, 2, 3, 4, \]
where $S(t) = e^{it\Delta}$. Notice that, compared to the multilinear estimates used in [8], a frequency variable $\tau$ is added to the left hand side of the estimate. It would be interesting to know if the smallest value of $\alpha$ for which these estimates (and hence Theorem 1) are valid depends or not on the geometry of $M$.

In Section 3, we come back to power nonlinearities. Since we want to go below the cubic powers and at the same time we want to use multilinear estimates, we are led to deal with quadratic nonlinearities. In other words, we study the following equations,

\begin{equation}
  i\partial_t u + \Delta u = q(u),
\end{equation}

where $q(u)$ is a homogeneous quadratic polynomial in $u, \overline{u}$

\[ q(u) = au^2 + b|u|^2. \]

We start with wellposedness results.

**Theorem 2.** If $(M, g)$ is the four-dimensional standard sphere, then the Cauchy problem (2) is (locally in time) uniformly well-posed in $H^{s\text{zonal}}(S^4)$ for every $s > 1/2$, where $H^{s\text{zonal}}(S^4)$ denotes the $H^s$ space of zonal functions relative to some pole $\omega \in S^4 : f(x) = \tilde{f}((x, \omega))$.

The main tool in the proof of Theorem 2 is the following trilinear estimate on linear solutions $u_j(t) = S(t)f_j$,

\begin{equation}
  \sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{S^4} \chi(t) e^{it\tau} T(u_1(t, x), u_2(t, x), u_3(t, x)) \, dx \, dt \right| \\
  \leq C \left( \min(N_1, N_2, N_3) \right)^{s_0} \|f_1\|_{L^2(S^4)} \|f_2\|_{L^2(S^4)} \|f_3\|_{L^2(S^4)},
\end{equation}

for every $\mathbb{R}$-trilinear expression $T$ on $\mathbb{C}^3$, for every $\chi \in C_0^\infty(\mathbb{R})$, for every $s_0 > 1/2$ and for zonal functions $f_1, f_2, f_3$ satisfying

\[ \chi_{[N_j, 2N_j]}(f_j) = f_j, \quad j = 1, 2, 3. \]

It would be interesting to know whether the above estimate holds with non zonal functions for some $s_0 < 1$; this would extend the above theorem to any finite energy Cauchy data.

Notice that a subclass of these equations consists of Hamiltonian equations

\[ q(u) = \frac{\partial V}{\partial \overline{u}} \]

where $V$ is a real-valued homogeneous polynomial of degree 3 in $u, \overline{u}$; with the above expression of $q(u)$, this corresponds to $c = 2\overline{a}$. In this case, the following energy is conserved,

\[ E = \int_M |\nabla u|^2 + V(u) \, dx. \]

A typical example is

\[ V(u) = \frac{1}{2} |u|^2 (u + \overline{u}), \quad q(u) = |u|^2 + \frac{1}{2} |u|^2. \]

This Hamiltonian structure does not prevent solutions from blowing up in general. In the above example, for instance, a purely imaginary constant as
Cauchy data leads to a blow up solution. However it is possible to give a classification of all the Hamiltonian quadratic nonlinearities for which the Cauchy problem associated to (2) has a unique global solution for small initial data in $H^1_{zonal}(S^4)$.

**Corollary 1.** Assume $(M,g)$ is the four-dimensional standard sphere and $c = 2\pi$. Then the following assertions are equivalent.

i) There exists a subspace $X$ of $\mathcal{C}(\mathbb{R}, H^1_{zonal}(S^4))$ such that, for every small initial data $\|u_0\|_{H^1_{zonal}(S^4)} \leq \varepsilon$, the Cauchy problem (2) has a unique global solution $u \in X$.

ii) The parameters $a,b$ satisfy

$$\frac{a^2}{a} = b.$$  

It would be interesting to know whether blowing up solutions exist for non small data under property (4).

When property (4) is not satisfied, our blowing up solutions are particularly simple, since they are solutions of the ordinary differential equation deduced from (2) for space-independent solutions. Another open problem is of course to find a wider variety of blowing up solutions for equation (2) in this case.

**Acknowledgements.** This paper was written while the second author visited the Laboratoire de Mathématiques d’Orsay, supported by the HYKE network. She is grateful to these two institutions for their hospitality and support.

2. **Wellposedness via multilinear estimates**

The main step of this section is to prove a result of local existence in time for initial data in $H^1(M)$ using some multilinear estimates associated to the nonlinear Schrödinger equation, that we will establish in Section 3 with a special attention to the case of the sphere. For that purpose we follow closely the ideas of Burq, Gérard and Tzvetkov ([9], [7]). In those papers, the authors extended to general compact manifolds the nonlinear methods introduced by Bourgain ([1], [2], [4]) in the context of tori $\mathbb{R}^d/\mathbb{Z}^d$. Finally, we achieve the global wellposedness thanks to the conservation laws.

2.1. **Well-posedness in Sobolev spaces for the Hartree nonlinearity.**

In this subsection we prove that the uniform wellposedness of (1) on $M$ can be deduced from quadrilinear estimates on solutions of the linear equation. Firstly, we recall the notion of wellposedness we are going to address.

**Definition 1.** Let $s \in \mathbb{R}$. We shall say that the nonlinear Schrödinger equation (1) is (locally in time) uniformly well-posed on $H^s(M)$ if, for any bounded subset $B$ of $H^s(M)$, there exists $T > 0$ and a Banach space $X_T$ continuously contained into $\mathcal{C}([-T,T], H^s(M))$, such that

i: For every Cauchy data $u_0 \in B$, (1) has a unique solution $u \in X_T$.

ii: If $u_0 \in H^\sigma(M)$ for $\sigma > s$, then $u \in \mathcal{C}([-T,T], H^\sigma(M))$.

iii: The map $u_0 \in B \mapsto u \in X_T$ is uniformly continuous.
The following theorem stresses the general relationship between uniform wellposedness for equation (1) and a certain type of quadrilinear estimates.

**Theorem 3.** Suppose that there exists $C > 0$ and $s_0 \geq 0$ such that for any $f_1, f_2, f_3, f_4 \in L^2(M)$ satisfying
\[(5)\quad \frac{1}{\sqrt{1 - \Delta}} \in [N_j, 2N_j](f_j) = f_j, \quad j = 1, 2, 3, 4,
\]one has the following quadrilinear estimates
\[(6)\quad \sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{M} \chi(t) e^{i\tau t} (1 - \Delta)^{-\alpha} (u_1 \overline{w}_2) u_3 \overline{w}_4 dx dt \right| \leq C(m(N_1, \cdots, N_4))^{s_0} \| f_1 \|_{L^2(M)} \| f_2 \|_{L^2(M)} \| f_3 \|_{L^2(M)} \| f_4 \|_{L^2(M)},
\]where $\chi \in C_0^{\infty}(\mathbb{R})$ is arbitrary, and $m(N_1, \cdots, N_4)$ denotes the product of the smallest two numbers among $N_1, N_2, N_3, N_4$. Then the Cauchy problem (1) is uniformly well-posed in $H^{s}(M)$ for any $s > s_0$.

**Proof.** The proof follows essentially the same lines as the one of Theorem 3 in [7] and relies on the use of a suitable class $X^{s,b}$ of Bourgain-type spaces. We shall sketch it for the commodity of the reader. We first show that (6) is equivalent to a quadrilinear estimate in the spaces $X^{s,b}$. Then we prove the crucial nonlinear estimate, from which uniform wellposedness can be obtained by a contraction argument in $X^{s,b}$. Since this space is continuously embedded in $C([-T, T], H^{s}(M))$ provided $b > \frac{1}{2}$, this concludes the proof of the local well posedness result.

Following the definition in Bourgain [1] and Burq, Gérard and Tzvetkov [9], we introduce the family of Hilbert spaces
\[(7)\quad X^{s,b}(\mathbb{R} \times M) = \{ v \in S'(\mathbb{R} \times M) : (1 + |i\partial_t + \Delta|^2)^{\frac{b}{2}} (1 - \Delta)^{\frac{s}{2}} v \in L^2(\mathbb{R} \times M) \}
\]for $s, b \in \mathbb{R}$. More precisely, with the notation
\[\langle x \rangle = \sqrt{1 + |x|^2},\]
we have the following definition:

**Definition 2.** Let $(M, g)$ be a compact Riemannian manifold, and consider the Laplace operator $-\Delta$ on $M$. Denote by $(e_k)$ an $L^2$ orthonormal basis of eigenfunctions of $-\Delta$, with eigenvalues $\mu_k$, by $\Pi_k$ the orthogonal projector along $e_k$, and for $s \geq 0$ by $H^s(M)$ the natural Sobolev space generated by $(I - \Delta)^{\frac{s}{2}}$, equipped with the following norm
\[(8)\quad \| u \|_{H^s(M)}^2 = \sum_k \langle \mu_k \rangle^s \| \Pi_k u \|_{L^2(M)}^2.
\]Then, the space $X^{s,b}(\mathbb{R} \times M)$ is defined as the completion of $C_0^{\infty}(\mathbb{R} \times H^s(M))$ for the norm
\[(9)\quad \| u \|_{X^{s,b}(\mathbb{R} \times M)}^2 = \sum_k \| \langle \tau + \mu_k \rangle^b \langle \mu_k \rangle^s \Pi_k u(\tau) \|_{L^2(\mathbb{R} \times L^2(M))}^2 = \| S(-t) u(t, \cdot) \|_{H^b(\mathbb{R} \times H^s(M))}^2,
\]where $\widehat{\Pi_k u(\tau)}$ denotes the Fourier transform of $\Pi_k u$ with respect to the time variable.
Denoting by $X^{s,b}_T$ the space of restrictions of elements of $X^{s,b}(\mathbb{R} \times M)$ to $]-T,T[\times M$, it is easy to prove the embedding
\begin{equation}
\forall b > \frac{1}{2}, \quad X^{s,b}_T \subset C([-T,T], H^s(M)).
\end{equation}
Moreover, we have the elementary property
\begin{equation}
\forall f \in H^s(M), \quad \forall b > 0, \quad (t, x) \mapsto S(t)f(x) \in X^{s,b}_T.
\end{equation}

We next reformulate the quadrilinear estimates (6) in the context of $X^{s,b}$ spaces.

**Lemma 1.** Let $s \in \mathbb{R}$. The following two statements are equivalent:

i) For any $f_j \in L^2(M), \ j = 1, 2, 3, 4$, satisfying (5), estimate (6) holds;

ii) For any $b > \frac{1}{2}$ and any $u_j \in X^{0,b}(\mathbb{R} \times M), \ j = 1, 2, 3, 4$, satisfying
\[ \frac{1}{\sqrt{1-\Delta}} \in [N_j, 2N_j] \{ u_j \} = u_j, \]
one has
\begin{equation}
\left| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2)u_3 \overline{u}_4 dxdt \right| \leq C(m(N_1, \cdots, N_4))^{s_0} \prod_{j=1}^{4} \| u_j \|_{X^{0,b}(\mathbb{R} \times M)}.
\end{equation}

**Proof.** We sketch only the essential steps of the proof of ii) assuming i), since we follow closely the argument of Lemma 2.3 in [9]. The reverse implication is easier and will not be used in this paper.

Suppose first that $u_j$ are supported in time in the interval $(0, 1)$ and we select $\chi \in C^\infty_0(\mathbb{R})$ such that $\chi = 1$ on $[0, 1]$; then writing $u_j^\#(t) = S(-t)u_j(t)$ we have easily
\[ ((1-\Delta)^{-\alpha}(u_1 \overline{u}_2)u_3 \overline{u}_4) (t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it(\tau_1 - \tau_2 + \tau_3 - \tau_4)} \]
\[ \times (1-\Delta)^{-\alpha}(S(t)\hat{u}_1^\#(\tau_1)S(t)\hat{u}_2^\#(\tau_2))S(t)\hat{u}_3^\#(\tau_3)S(t)\hat{u}_4^\#(\tau_4)) d\tau_1 d\tau_2 d\tau_3 d\tau_4, \]
where $\hat{u}_j^\#$ denotes the Fourier transform of $u_j^\#$ with respect to time. Using i) and the Cauchy-Schwarz inequality in $(\tau_1, \tau_2, \tau_3, \tau_4)$ (here the assumption $b > \frac{1}{2}$ is used, in order to get the necessary integrability) yields
\begin{align*}
\left| \int_{\mathbb{R} \times M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2)u_3 \overline{u}_4 dxdt \right| &\lesssim m(N_1, \cdots, N_4)^{s_0} \prod_{j=1}^{4} \| (\tau)^{b-\alpha} \hat{u}_j^\# \|_{L^2(\mathbb{R} \times M)} \\
&\lesssim m(N_1, \cdots, N_4)^{s_0} \prod_{j=1}^{4} \| u_j \|_{X^{0,b}(\mathbb{R} \times M)}.
\end{align*}

Finally, by decomposing $u_j(t) = \sum_{n \in \mathbb{Z}} \psi(t - \frac{n}{2})u_j(t)$ with a suitable $\psi \in C^\infty_0(\mathbb{R})$ supported in $(0, 1)$, the general case for $u_j$ follows from the special case of $u_j$ supported in the time interval $(0, 1)$. \qed

Returning to the proof of Theorem 3, there is another way of estimating the $L^1$ norm of the product $((1-\Delta)^{-\alpha}(u_1 \overline{u}_2)u_3 \overline{u}_4)$. 

Lemma 2. Assume $\alpha$ as in Theorem 1 and that $u_1, u_2, u_3, u_4$ satisfy
\begin{equation}
1_{\sqrt{1-\Delta} \in [N, 2N]}(u_j) = u_j.
\end{equation}

Then, for every $s' > s_0$ there exists $b' \in [0, \frac{1}{2}]$ such that
\begin{equation}
\left\| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) u_3 \overline{u}_4 \, dx \, dt \right\| \leq Cm(N_1, \cdots, N_4)^{s'} \prod_{j=1}^{4} \|u_j\|_{X_0,b'}. 
\end{equation}

Proof. We split the proof in several steps.

First of all we prove that, for $\alpha > 0$,
\begin{equation}
\left\| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) u_3 \overline{u}_4 \, dx \, dt \right\| \leq Cm(N_1, \cdots, N_4)^{2} \prod_{j=1}^{4} \|u_j\|_{X_0,1/4}. 
\end{equation}

By symmetry we have to consider the following three cases: $m(N_1, \cdots, N_4) = N_1 N_2$, $m(N_1, \cdots, N_4) = N_3 N_4$, $m(N_1, \cdots, N_4) = N_1 N_3$.

In the first case, by a repeated use of Hölder’s inequality, we obtain
\begin{align*}
\left\| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) u_3 \overline{u}_4 \, dx \, dt \right\| & \leq C\left\| (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) \right\|_{L^2(\mathbb{R}, L^\infty(M))} \left\| u_3 \overline{u}_4 \right\|_{L^2(\mathbb{R}, L^1(M))} \\
& \leq C\left\| u_1 \overline{u}_2 \right\|_{L^2(\mathbb{R}, L^\infty(M))} \left\| u_3 \overline{u}_4 \right\|_{L^2(\mathbb{R}, L^1(M))} \\
& \leq C\left\| u_1 \right\|_{L^4(\mathbb{R}, L^\infty(M))} \left\| u_2 \right\|_{L^4(\mathbb{R}, L^\infty(M))} \left\| u_3 \right\|_{L^4(\mathbb{R}, L^2(M))} \left\| u_4 \right\|_{L^4(\mathbb{R}, L^2(M))},
\end{align*}
where we also used that $(1-\Delta)^{-\alpha}$ is a pseudodifferential operator of negative order, hence acts on $L^\infty(M)$. By Sobolev inequality, we infer
\begin{equation}
\left\| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) u_3 \overline{u}_4 \, dx \, dt \right\| \leq C(N_1 N_2)^{2} \prod_{j=1}^{4} \|u_j\|_{L^4(\mathbb{R}, L^2(M))}.
\end{equation}

By the Sobolev embedding in the time variable for the function $v(t) = S(-t)u(t)$, we have $X^{0,1/4} \subset L^4(\mathbb{R}, L^2(M))$, and this conclude the proof of the first case.

In the second case $m(N_1, \cdots, N_4) = N_3 N_4$ we can proceed in the same way by writing the integral in the form
\begin{equation}
\left\| \int_{\mathbb{R}} \int_{M} u_1 \overline{u}_2 (1-\Delta)^{-\alpha}(u_3 \overline{u}_4) \, dx \, dt \right\|.
\end{equation}

Finally, when $m(N_1, \cdots, N_4) = N_1 N_3$, we write the integral as follows
\begin{equation}
\left\| \int_{\mathbb{R}} \int_{M} (1-\Delta)^{-\frac{\alpha}{2}}(u_1 \overline{u}_2)(1-\Delta)^{-\frac{\alpha}{2}}(u_3 \overline{u}_4) \, dx \, dt \right\|,
\end{equation}
and by Cauchy-Schwarz and Hölder’s inequalities we estimate it by
\begin{align*}
& \leq \|(1-\Delta)^{-\frac{\alpha}{2}}(u_1 \overline{u}_2)\|_{L^2(\mathbb{R}, L^2(M))} \|(1-\Delta)^{-\frac{\alpha}{2}}(u_3 \overline{u}_4)\|_{L^2(\mathbb{R}, L^2(M))} \\
& \leq C\left\| u_1 \overline{u}_2 \right\|_{L^2(\mathbb{R}, L^2(M))} \left\| u_3 \overline{u}_4 \right\|_{L^2(\mathbb{R}, L^2(M))} \\
& \leq C\left\| u_1 \right\|_{L^4(\mathbb{R}, L^\infty(M))} \left\| u_2 \right\|_{L^4(\mathbb{R}, L^\infty(M))} \left\| u_3 \right\|_{L^4(\mathbb{R}, L^2(M))} \left\| u_4 \right\|_{L^4(\mathbb{R}, L^2(M))}.
\end{align*}

Finally we conclude the proof of (15) by means of Sobolev’s inequality in both space and time variables as above.
The second step consists in interpolating between (12) and (15) in order to get the estimate (14). To this end we decompose each $u_j$ as follows

$$u_j = \sum_{K_j} u_{j,K_j}, \quad u_{j,K_j} = 1_{K_j \leq (\partial \theta + \Delta) < 2K_j}(u_j),$$

where $K_j$ denotes the sequence of dyadic integers. Notice that

$$\|u_j\|_{X_0, b}^2 \simeq \sum_{K_j} K_j^{2b} \|u_{j,K_j}\|_{L^2(\mathbb{R} \times M)}^2 \simeq \sum_{K_j} \|u_{j,K_j}\|_{X_0, b}^2.$$

We then write the integral in the left hand side of (14) as a sum of the following elementary integrals,

$$I(K_1, \cdots, K_4) = \int_M \int_{\mathbb{R}} (1 - \Delta)^{-\alpha} (u_{1,K_1} \overline{u}_{2,K_2} u_{3,K_3} \overline{u}_{4,K_4}) dxdt.$$

Using successively (12) and (15), we estimate these integrals as (16)

$$|I(K_1, \cdots, K_4)| \leq Cm(N_1, \cdots, N_4)^{\sigma} \sum_{K_1,K_2,K_3} (K_1 K_2 K_3 K_4)^{\beta} \prod_{j=1}^4 \|u_{j,K_j}\|_{L^2},$$

where either $(\sigma, \beta) = (s_0, b)$ for every $b > 1/2$, or $(\sigma, \beta) = (2, 1/4)$. Therefore, for every $s' > s_0$, there exists $b_1 < 1/2$ such that (16) holds for $(\sigma, \beta) = (s', b_1)$. Choosing $b' \in ]b_1, 1/2[$, this yields

$$\left| \int_M \int_{\mathbb{R}} (1 - \Delta)^{-\alpha} (u_{1,K_1} \overline{u}_{2,K_2} u_{3,K_3} \overline{u}_{4,K_4}) dxdt \right|$$

$$\leq C m(N_1, \cdots, N_4)^{s'} \sum_{K_1, \cdots, K_4} (K_1 K_2 K_3 K_4)^{b_1 - b'} \prod_{j=1}^4 \|u_{j,K_j}\|_{X_0, s'} ,$$

which completes the proof, since the right hand side is a convergent series.

We are finally in position to prove Theorem 3. We can write the solution of the Cauchy problem (1) using the Duhamel formula

$$u(t) = S(t)u_0 - i \int_0^t S(t - \tau) ((1 - \Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)) d\tau .$$

The next lemma contains the basic linear estimate.

**Lemma 3.** Let $b, b'$ such that $0 \leq b' < \frac{1}{2}$, $0 \leq b < 1 - b'$. There exists $C > 0$ such that, if $T \in [0, 1]$, $w(t) = \int_0^t S(t - \tau)f(\tau)d\tau$, then

$$\|w\|_{X_T^{s,b}} \leq CT^{1-b-b'}\|f\|_{X_T^{-s,-b'}}.$$

We refer to [15] for a simple proof of this lemma.

The last integral equation (17) can be handled by means of these spaces $X_T^{s,b}$ using Lemma 3 as follows

$$\left\| \int_0^t S(t - \tau) ((1 - \Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)) d\tau \right\|_{X_T^{s,b}}$$

$$\leq CT^{1-b-b'}\left\| ((1 - \Delta)^{-\alpha}(|u(\tau)|^2)u(\tau)) \right\|_{X_T^{-s,-b'}}.$$
Thus to construct the contraction $\Phi : X^{s,b}_{T} \to X^{s,b}_{T}$, $\Phi(v_{i}) = u_{i}, i = 1, 2$ and to prove the propagation of regularity ii) in Definition 1, it is enough to prove the following result.

**Lemma 4.** Let $s > s_{0}$. There exists $(b,b') \in \mathbb{R}^{2}$ satisfying

\begin{equation}
0 < b' < \frac{1}{2} < b, \quad b + b' < 1,
\end{equation}

and $C > 0$ such that for every triple $(u_{j})$, $j = 1, 2, 3$ in $X^{s,b}(\mathbb{R} \times M)$,

\begin{equation}
\|(1 - \Delta)^{-\alpha}(u_{1}\overline{u}_{2})u_{3}\|_{X^{s,-b'}} \leq C\|u_{1}\|_{X^{s,b}}\|u_{2}\|_{X^{s,b}}\|u_{3}\|_{X^{s,b}}.
\end{equation}

Moreover, for every $\sigma > s$, there exists $C_{\sigma}$ such that

\begin{equation}
\|(1 - \Delta)^{-\alpha}(u^{2})u\|_{X^{\sigma,-b'}} \leq C_{\sigma}\|u\|_{X^{\sigma,b}}^{2}\|u\|_{X^{\sigma,b}}.
\end{equation}

**Proof.** We only sketch the proof of (21). The proof of (22) is similar. Thanks to a duality argument it is sufficient to show the following

\begin{equation}
\left|\int_{\mathbb{R}}\int_{M}(1 - \Delta)^{-\alpha}(u_{1}\overline{u}_{2})u_{3}\overline{u}_{4}dxdt\right| \leq C\left(\prod_{j=1}^{3}\|u_{j}\|_{X^{s,b}}\right)\|u_{4}\|_{X^{-s,b'}}.
\end{equation}

The next step is to perform a dyadic expansion in the integral of the left hand-side of (23), this time in the space variable. We decompose $u_{1}, u_{2}, u_{3}, u_{4}$ as follows:

\[u_{j} = \sum_{N_{j}}u_{j,N_{j}}, \quad u_{j,N_{j}} = 1_{\sqrt{1 - \Delta} \in [N_{j}, 2N_{j}]}(u_{j}).\]

In this decomposition we have

\[\|u_{j}\|_{X^{s,b}}^{2} \approx \sum_{N_{j}}N_{j}^{2s}\|u_{j,N_{j}}\|_{X^{0,b}}^{2} \approx \sum_{N_{j}}\|u_{j,N_{j}}\|_{X^{s,b}}^{2}.
\]

We introduce now this decomposition in the left hand side of (23), and we are left with estimating each term

\[J(N_{1}, \cdots, N_{4}) = \int_{\mathbb{R}}\int_{M}(1 - \Delta)^{-\alpha}(u_{1,N_{1}}\overline{u}_{2,N_{2}})u_{3,N_{3}}\overline{u}_{4,N_{4}}dxdt.
\]

Consider the terms with $N_{1} \leq N_{2} \leq N_{3}$ (the other cases are completely similar by symmetry). Choose $s'$ such that $s > s' > s_{0}$. By Lemma 2 we can find $b'$ such that $0 < b' < \frac{1}{2}$ and

\begin{equation}
|J(N_{1}, \cdots, N_{4})| \leq C\sum_{N_{j}}(N_{1}N_{2})^{s'}\prod_{j=1}^{4}\|u_{j,N_{j}}\|_{X^{0,b'}}.
\end{equation}

This is equivalent to

\[|J(N_{1}, \cdots, N_{4})| \leq C\sum_{N_{j}}(N_{1}N_{2})^{s'-s}\left(\frac{N_{4}}{N_{3}}\right)^{s}\prod_{j=1}^{3}\|u_{j,N_{j}}\|_{X^{s,b'}}\|u_{4,N_{4}}\|_{X^{-s,b'}}.
\]

In this series we separate the terms in which $N_{4} \leq CN_{3}$ from the others. For the first ones the series converges thanks to a simple argument of summation of geometric series and Cauchy-Schwarz inequality. To perform the summation of the other terms, it is sufficient to apply the following lemma, which is a simple variant of Lemma 2.6 in [7].
Lemma 5. Let \( \alpha \) a positive number. There exists \( C > 0 \) such that, if for any \( j = 1, 2, 3, \) \( C \mu_{k_j} \leq \mu_{k_4}, \) then for every \( p > 0 \) there exists \( C_p > 0 \) such that for every \( w_j \in L^2(M), \) \( j = 1, 2, 3, 4, \)

\[
\int_M (1 - \Delta)^{-\alpha} (\Pi_{k_1} w_1 \Pi_{k_2} w_2) \Pi_{k_3} w_3 \Pi_{k_4} w_4 dx \leq C_p \mu_{k_4}^{-p} \prod_{j=1}^4 \| w_j \|_{L^2}.
\]

Remark. Notice that if \( M = S^4 \) the above lemma is trivial since in that case, by an elementary observation on the degree of the corresponding spherical harmonics, we obtain that if \( k_4 > k_1 + k_2 + k_3 \) then the integral (24) is zero.

Finally, the proof of Lemma 4 is achieved by choosing \( b \) such that \( \frac{1}{2} < b < 1 - b' \) and by merely observing that

\[
\| u_j \|_{X^{s,b}} \leq \| u_j \|_{X^{s,b}}, \quad j = 1, 2, 3.
\]

\[\square\]

2.2. Local wellposedness for the quadratic nonlinearity. In this subsection, we study the wellposedness theory of the quadratic nonlinear Schrödinger equation posed on \( S^4 \)

\[
i \partial_t u + \Delta u = q(u), \quad q(u) = a u^2 + b |\nabla|^2 + c |u|^2,
\]

with zonal initial data \( u(0, x) = u_0(x). \)

In fact we shall prove Theorem 2 on every four-manifold satisfying the trilinear estimates (3). This is a result of independent interest that we state below.

Theorem 4. Let \( M \) be a Riemannian manifold, let \( G \) be a subgroup of isometries of \( M. \) Assuming that there exists \( C > 0 \) and \( \rho_0 \) such that for any \( u_1, u_2, u_3 \in L^2(S^4) \) \( G \)-invariant functions on \( M \) satisfying

\[
1_{\sqrt{3} \leq |N_j, 2N_j|}(t) = f_j, \quad j = 1, 2, 3,
\]

one has the trilinear estimates

\[
\sup_{\tau \in \mathbb{R}} \left| \int_M \chi(t) e^{i \tau \cdot T(u_1, u_2, u_3) dx dt} \right| \leq C (\min(N_1, N_2, N_3))^{\rho_0} \prod_{j=1}^3 \| f_j \|_{L^2},
\]

where \( T(u_1, u_2, u_3) = u_1 u_2 u_3 \) or \( T(u_1, u_2, u_3) = u_1 u_2 \bar{u}_3 \) and \( \chi \in C^\infty_0(\mathbb{R}) \) is arbitrary. Then, for every \( s > \rho_0, \) the Cauchy problem (25) is uniformly well-posed on the subspace of \( H^s(M) \) which consists of \( G \)-invariant functions.

Proof. It is close to the one of Theorem 3 above, so we shall just survey it. We denote by \( L^2_v(M), H^s_v(G)(M), X_v^{s,b}(\mathbb{R} \times M) \) the subspaces of \( L^2(M), \)

\( H^s(M), X_v^{s,b}(\mathbb{R} \times M) \) which consist of \( G \)-invariant functions. For the sake of simplicity, we shall focus on the case

\[
q(u) = |u|^2 + \frac{1}{2} u^2.
\]
The general case follows from straightforward modifications. As in the proof of Theorem 3, it is enough, for every \( s > s_0 \), to show that there exists \( b, b' \) such that

\[
0 < b' < \frac{1}{2} < b < 1 - b'
\]

with the following estimates,

\[
\|u_1 u_2\|_{X^{s,-b'}} \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}}, \quad \|u_1 \overline{u}_2\|_{X^{s,-b'}} \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}},
\]

\[
\|u_2\|_{X^{s,-b'}} \leq C_\sigma \|u\|_{X^{s,b}} \|u\|_{X^{s,b}}, \quad \|u(2)\|_{X^{s,-b'}} \leq C_\sigma \|u\|_{X^{s,b}} \|u\|_{X^{s,b}}, \quad \sigma > s,
\]

where \( u_1, u_2, u \) are \( G \)-invariant. As before, we focus on the first set of estimates. Thanks to a duality argument, these estimates are equivalent to

\[
\left| \int_{\mathbb{R}} \int_M u_1 u_2 \overline{u}_3 dx dt \right| \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} \|u_3\|_{X^{-s,b}},
\]

\[
\left| \int_{\mathbb{R}} \int_M \overline{u}_1 u_2 u_3 dx dt \right| \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} \|u_3\|_{X^{-s,b}},
\]

In this way, writing the solution of the Cauchy problem (25) using the Duhamel formula

\[
u(t) = S(t)u_0 - i \int_0^t S(t - \tau) (|u(\tau)|^2 + \frac{1}{2} u^2(\tau)) d\tau,
\]

and applying Lemma 3, we obtain a contraction on \( X^{s,b}_T \) proving a result of local existence of the solution to (25) on \( H^s(M) \), \( s > s_0 \). Thus the proof of this theorem is reduced to establishing the trilinear estimates (28) for suitable \( s, b, b' \). We just prove the first inequality in (28). The proof of the second one is similar.

First we reformulate trilinear estimates (27) in the context of Bourgain spaces.

**Lemma 6.** Let \( s_0 \in \mathbb{R} \). The following two statements are equivalent:
- For any \( f_1, f_2, f_3 \in L^2_T(M) \) satisfying (26), estimate (27) holds.
- For any \( b > \frac{1}{2} \) and any \( u_1, u_2, u_3 \in X^{0,b}_T(\mathbb{R} \times M) \) satisfying

\[
1_{\sqrt{T-M} \in [N_j, 2N_j]}(u_j) = u_j, \quad j = 1, 2, 3,
\]

one has

\[
\left| \int_{\mathbb{R}} \int_M (u_1 u_2 \overline{u}_3) dx dt \right| \leq C (\min(N_1, N_2, N_3))^{s_0} \prod_{j=1}^3 \|u_j\|_{X^{0,b}}.
\]

**Proof.** The proof of this lemma follows lines of Lemma 1 above. First we assume that \( u_1, u_2, u_3 \) are supported for \( t \in [0, 1] \), and we select \( \chi \in C_0^\infty(\mathbb{R}) \) such that \( \chi = 1 \) on \( [0, 1] \). We set \( u_j^\tau(t) = S(-t)u_j(t) \). Using the Fourier transform, we can write

\[
\left| \int_{\mathbb{R}} \int_M u_1 u_2 \overline{u}_3 dx dt \right|
\]

\[
\leq C \int_{\tau_1} \int_{\tau_2} \int_{\tau_3} \left| \int_{\mathbb{R}} \int_M \chi(t) e^{it\tau} \prod_{j=1}^3 S(t)u_j^\tau(\tau_j) dx dt \right| d\tau_1 d\tau_2 d\tau_3,
\]
where \( \tau = (\tau_1 + \tau_2 - \tau_3) \). Supposing for instance \( N_1 \leq N_2 \leq N_3 \) and applying (27) we obtain that the right hand side is bounded by

\[
\leq C N_1^{s_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \| \hat{u}_1^2(\tau_1)\|_{L^2(M)} \| \hat{u}_2^2(\tau_2)\|_{L^2(M)} \| \hat{u}_3^2(\tau_3)\|_{L^2(M)} d\tau_1 d\tau_2 d\tau_3.
\]

We conclude the proof as in the proof of Lemma 1 in section 2, using the Cauchy-Schwarz inequality in \((\tau_1, \tau_2, \tau_3)\), and finally decomposing each \( u_j \) by means of the partition of unity

\[
1 = \sum_{n \in \mathbb{Z}} \psi \left( t - \frac{n}{2} \right),
\]

where \( \psi \in C^\infty_0([0,1]) \). \( \square \)

**Lemma 7.** For every \( s' > s_0 \) there exist \( b' \) such that \( 0 < b' < \frac{1}{2} \) and, for every \( G \)-invariant functions \( u_1, u_2, u_3 \) satisfying (30),

\[
(32) \quad \left| \int_{\mathbb{R}} \int_{M} (u_1 u_2 \bar{\tau}_3) dxdt \right| \leq C \min(N_1, N_2, N_3)^{s'} \prod_{j=1}^{3} \| u_j \|_{X^{s,b'}}.
\]

**Proof.** Following the same lines of the proof of Lemma 2, it is enough to establish

\[
(33) \quad \left| \int_{\mathbb{R}} \int_{M} (u_1 u_2 \bar{\tau}_3) dxdt \right| \leq C \min(N_1, N_2, N_3)^{2} \prod_{j=1}^{3} \| u_j \|_{X^{s,b'}(\mathbb{R} \times M)}.
\]

Then the lemma follows by interpolation with (31). Indeed, assuming for instance \( N_1 \leq N_2 \leq N_3 \), we apply the Hölder inequality as follows,

\[
\left| \int_{\mathbb{R}} \int_{M} (u_1 u_2 \bar{\tau}_3) dxdt \right| \leq C \| u_1 \|_{L^{3}(\mathbb{R}, L^{s}(M))} \| u_2 \|_{L^{3}(\mathbb{R}, L^{s}(M))} \| u_3 \|_{L^{3}(\mathbb{R}, L^{s}(M))}
\]

and using the Sobolev embedding we obtain

\[
\leq C (N_1)^{2} \| u_1 \|_{L^{3}(\mathbb{R}, L^{s}(M))} \| u_2 \|_{L^{3}(\mathbb{R}, L^{s}(M))} \| u_3 \|_{L^{3}(\mathbb{R}, L^{s}(M))}.
\]

By the Sobolev embedding in the time variable for function \( v(t) = S(-t)u(t) \), we know that

\[
\| u \|_{L^{3}(\mathbb{R}, L^{s}(M))} \leq \| u \|_{X^{s,b'}(\mathbb{R} \times M)}
\]

and this completes the proof. \( \square \)

Let us sketch the last part of the proof of Theorem 4. We decompose \( u_1, u_2, u_3 \) as follows:

\[
u_j = \sum_{N_j} u_{j,N_j}, \quad u_{j,N_j} = 1 \chi_{\{ N_j \leq |N_j, 2N_j| \}}(u_j).
\]

We introduce this decomposition in the left hand side of (28) and we use Lemma 7. Supposing now for simplicity that \( N_1 \leq N_2 \), we obtain that for any \( s' > s_0 \) we can find \( b' \) such that \( 0 < b' < \frac{1}{2} \) and

\[
(34) \quad \left| \int_{\mathbb{R}} \int_{M} u_1 u_2 \bar{\tau}_3 dxdt \right| \leq C \sum_{N_j} (N_1)^{s'-s} \left( \frac{N_3}{N_2} \right)^{s} \| u_1 \|_{X^{s,b'}} \| u_2 \|_{X^{s,b'}} \| u_3 \|_{X^{-s,b'}}
\]
for any \( s > s' > s_0 \). Notice that the summation over \( N_1 \) can be performed via a crude argument of summation of geometric series. As for the summation over \( N_2, N_3 \), following the same proof as in Section 2.1, we conclude by observing that the main part of the series corresponds to the constraint \( N_3 \lesssim N_2 \). □

2.3. Conservation laws and global existence for the Hartree nonlinearity. Next we prove that for an initial datum \( u_0 \in H^1(M) \), the local solution of the Cauchy problem (1) obtained above can be extended to a global solution \( u \in C(\mathbb{R}, H^1(M)) \).

By the definition of uniform wellposedness, the lifespan \( T \) of the local solution \( u \in C([0, T), H^1(M)) \) depends only on the \( H^1 \) norm of the initial datum. Thus, in order to prove that the solution can be extended to a global one, it is sufficient to show that the \( H^1 \) norm of \( u \) remains bounded on any finite interval \([0, T)\). This is a consequence of the following conservation laws, which can be proved by means of the multipliers \( \overline{\pi} \) and \( \overline{\pi}_t \),

\[
\int_M |u(t, x)|^2 \, dx = Q_0 ;
\]

\[
\int_M |\nabla u(t, x)|^2 + \frac{1}{2} |(1 - \Delta)^{-\alpha/2}(|u|^2)(t, x)|^2 \, dx = E_0 .
\]

Remark. Notice that a similar argument can be applied in the case of an attractive Hartree nonlinearity, at least when \( \alpha > 1 \). Indeed, consider the focusing Schrödinger equation

\[
iu_t + \Delta u = -(1 - \Delta)^{-\alpha/2}(|u|^2)u,
\]

where the nonlinear term has the opposite sign. Computing as above, we obtain the conservation of energy

\[
\|\nabla u\|^2_{L^2(M)} - \frac{1}{2} \|\nabla u\|^2_{L^2} \leq C + C\|\nabla u\|^2_{L^2} ;
\]

but now the energy \( E(t) \) does not control the \( H^1 \) norm of \( u \). However, we can write

\[
\|(1 - \Delta)^{-\alpha/2}(|u|^2)\|^2_{L^2} \leq C\|u\|^2_{L^2} \equiv C\|u\|^4_{L^2} ,
\]

and by Sobolev embedding we have

\[
\|u\|^4_{L^q} \leq C\|u\|^2_{L^2} \equiv C\|u\|^4_{L^2} ,
\]

so that we obtain, with \( p = 2q \),

\[
\|\nabla u\|^2_{L^2} \leq C + C\|u\|^2_{L^p} , \quad \frac{1}{p} = \frac{1}{4} + \frac{\alpha}{8} .
\]

We now use the Gagliardo-Nirenberg inequality (for \( d = 4 \))

\[
\|w\|^p_{L^p} \leq C(\|w\|^{p-2}_{L^2})^{\frac{4}{2}}(\|\nabla w\|^{p-2}_{L^2} + \|w\|^{p}_{L^2})
\]

and we obtain

\[
\|\nabla u\|_{L^2} \leq C(1 + \|u\|^{2}_{L^2}) + C\|u\|^{2-4(p-2)/p}_{L^2} \|\nabla u\|_{L^2}^{4(p-2)/p} .
\]

Notice that, as in the defocusing case above, the \( L^2 \) norm of \( u \) is a conserved quantity. If the power \( 4(p-2)/p \) is strictly smaller than 1, we infer that the
$H^1$ norm of $u$ must remain bounded. In other words, we have proved global existence provided
\[ 4 \cdot \frac{p - 2}{p} < 1 \quad \iff \quad \alpha > 1. \]

\[ \square \]

2.4. Studying the global existence for the quadratic nonlinearity.

**Proposition 1.** Let $(M, g)$ be a four-dimensional Riemannian manifold satisfying the assumptions of Theorem 4. There exists $\varepsilon > 0$ and a subspace $X$ of $C(\mathbb{R}, H^1_G(M))$ such that, for every initial data $u_0 \in H^1_G(M)$ satisfying $\|u_0\|_{H^1} \leq \varepsilon$, the Cauchy problem (2), where $q(u) = (\text{Re } u)^2$, has a unique global solution $u \in X$.

**Proof.** By Theorem 4, we obtain that for an initial datum $u_0 \in H^1_G(M)$, there exists a local solution of the Cauchy problem
\[
\begin{aligned}
&i\partial_t u + \Delta u = (\text{Re } u)^2, \\
u(0, x) = u_0(x).
\end{aligned}
\]

By the definition of uniform wellposedness, the lifespan $T$ of the local solution $u \in C([0, T), H^1_G(M))$ only depends on a bound of the $H^1$ norm of the initial datum. Thus, in order to prove that the solution can be extended to a global one, it is sufficient to show that the $H^1$ norm of $u$ remains bounded on any finite interval $[0, T)$. This is a consequence of the following conservation laws and of a suitable assumption of smallness on the initial data. Notice that
\[
\partial_t \left( \int_M u(t, x) \, dx \right) = -i \int_M (\text{Re } u)^2 \, dx,
\]
from which
\[ (36) \quad \int_M \text{Re } u(t, x) \, dx = \text{const.} \]

Moreover the following energy is conserved,
\[ (37) \quad \int_M |\nabla u(t, x)|^2 + \frac{2}{3}(\text{Re } u(t, x))^3 \, dx = E_0. \]

Consequently we can write
\[ \|\nabla u\|_{L^2}^2 \leq E_0 + C \left| \int_M (\text{Re } u)^3 \right|. \]

Since by Gagliardo-Nirenberg inequality we have
\[ \left| \int_M (\text{Re } u)^3 \, dx \right| \leq C \|\text{Re } u\|_{L^2} \|\nabla (\text{Re } u)\|_{L^2}^2 + \|\text{Re } u\|_{L^2}^3, \]
and by the following inequality
\[ \|\text{Re } u\|_{L^2} \leq C \left| \int_M \text{Re } u \, dx \right| + \|\nabla \text{Re } u\|_{L^2}, \]
we deduce that
\[ \|\nabla u\|_{L^2}^2 \leq E_0 + C \left( \left| \int_M \text{Re } u \, dx \right| + \|\nabla u\|_{L^2} \right) \|\nabla u\|_{L^2}^2. \]
Thanks to (36) we know that
\[
\left| \int_M \text{Re } u \, dx \right| \leq \|u_0\|_{L^1(M)} \leq C\|u_0\|_{H^1(M)},
\]
thus we obtain
\[
\|\nabla u\|_{L^2}^2 \leq E_0 + C (\|u_0\|_{H^1} + \|\nabla u\|_{L^2}) \|\nabla u\|_{L^2}^2.
\]
Assuming that
\[
\|u_0\|_{H^1} \leq \varepsilon,
\]
we infer, by a classical bootstrap argument, that \(\|\nabla u\|\) cannot blow up, as well as \(\|\text{Re } u\|_{L^2}\). Using again the evolution law of the integral of \(u\), this implies that this integral cannot blow up, and completes the proof of the proposition.

Notice that the proof above extends without difficulty to \(q(u) = c(\text{Re } u)^2\), for any real number \(c\). If \((M, g)\) satisfies the assumptions of Theorem 4, we can now prove that the conclusions of Corollary 1 hold on \(M\).

**Proof.** Let \(q(u) = au^2 + b|u|^2 + 2|\omega|u^2\). The idea is to transform the equation into an equivalent one using the change of unknown \(u = \omega v\), with \(|\omega| = 1\), and then impose conditions on \(a, b\) such that the transformed equation is of the special type corresponding to \(q(u) = c(\text{Re } u)^2\) for which, thanks to Proposition 1, we know that the solution is global. Thus we try to impose
\[
q(\omega v) = c\omega (\text{Re } v)^2
\]
for some \(c \in \mathbb{R}\) and some \(\omega\) with \(|\omega| = 1\), and we obtain the polynomial identity
\[
a\omega^2 v^2 + b\overline{\omega}^2 v^2 + 2|\omega|v^2 = \frac{c\omega}{4} (v + \overline{v})^2.
\]
Equating the coefficients of the two polynomials we obtain
\[
a = c\frac{\omega}{4}, \quad b = c\frac{\omega^3}{4}
\]
and this is equivalent to
\[
\frac{b}{a} = c.
\]
Conversely, we prove that if this condition is not satisfied, it is always possible to construct small energy solutions which blow up in a finite time. We take as initial datum a constant in the form
\[
u_0(x) = \omega y_0, \quad y_0 \in \mathbb{R} \setminus \{0\}, \quad |\omega| = 1,
\]
and then the equation reduces to the ordinary differential equation
\[
iu_t = q(u), \quad u(0) = \omega y_0.
\]
Defining \(y(t) = u(t)/\omega\), we see that \(y(t)\) is a solution of the equation
\[
i\omega y'(t) = q(u) = y^2 q(\omega)
\]
which can be written
\[
y'(t) = -iq(\omega)\overline{\sigma} y^2, \quad y(0) = y_0 \in \mathbb{R}
\]
The solution can be written explicitly as
\[ y(t) = \frac{1}{y_0^{-1} + iq(\omega)\omega t} \]
and is not global if and only if \( q(\omega)\omega \) is purely imaginary. Thus to conclude the proof it is sufficient to show that we can find an \( \omega \) such that
\[ q(\omega)\omega = a\omega + b\omega^3 + 2\omega \]
is purely imaginary (and not 0).
Writing \( a = Ae^{i\alpha}, \ b = Be^{i\beta}, \ \omega = e^{i\theta} \) with \( A, B \geq 0 \), this is equivalent to finding a simple zero for the following function
\[ f(\theta) = 3A\cos(\alpha + \theta) + B\cos(\beta - 3\theta). \]
Observe that the average of \( f \) vanishes. A point where the sign of \( f \) changes cannot be a double zero unless it is a triple zero, and a straightforward calculation shows that this corresponds exactly to the case \( A = B \) and \( 3\alpha + \beta = 2k\pi \), namely \( \frac{\omega^2}{\alpha} = b \). Hence, if this condition is not satisfied, \( f \) has a simple zero. This completes the proof.

\[ \square \]

3. Multilinear estimates

In this section we establish multilinear estimates, which, combined with Theorems 3 and 4, yield Theorems 1 and 2. We recall that \( S(t) = e^{it\Delta} \).

3.1. Quadrilinear estimates. This subsection is devoted to the proof of quadrilinear estimates (6) with \( s_0 < 1 \) on arbitrary four-manifolds with \( \alpha > 1/2 \), and on the sphere \( S^4 \) with \( \alpha > 0 \). In view of subsections 2.1 and 2.3, this will complete the proof of Theorem 1.

**Lemma 8.** Let \( \alpha > \frac{1}{2}, \ s_0 = \left(\frac{3}{2} - \alpha\right) \) and let \( (M, g) \) a compact four-dimensional Riemannian manifold. Then there exists \( C > 0 \) such that for any \( f_1, f_2 \in L^2(M) \) satisfying
\[ \left(1 - \Delta\right)^{-\frac{q}{2}}(u_1 u_2) \in L^2((0,1) \times M) \]
\[ \min(\alpha, 1) \leq \frac{\alpha}{4}, \quad \frac{1}{q} = \frac{1}{2} + \frac{\alpha}{4}, \]
\[ \|\left(1 - \Delta\right)^{-\frac{q}{2}}(u_1 u_2)\|_{L^2((0,1) \times M)} \leq C\|u_1 u_2\|_{L^2((0,1), L^4(M))}, \]
and applying the Hölder inequality we obtain
\[ \|\left(1 - \Delta\right)^{-\frac{q}{2}}(u_1 u_2)\|_{L^2((0,1) \times M)} \leq C\|u_1\|_{L^2((0,1), L^6(M))}\|u_2\|_{L^\infty((0,1), L^2(M))}. \]
Thanks to the conservation of the \( L^2 \) norm we can bound the last factor with the \( L^2 \) norm of \( f_2 \); on the other hand, the \( L^2 L^{4/\alpha} \) term can be bounded using the Strichartz inequality on compact manifolds established by Burq, Gérard, Tzvetkov in [5] (see Theorem 1), which reads, in this particular case,
\[ \|u_1\|_{L^2((0,1), L^4(M))} \leq C N^{1/2}\|f_1\|_{L^2(M)}. \]
Combining this estimate with the Sobolev inequality, we obtain (39) as claimed.

\[\]

**Proposition 2.** Let \( \alpha > \frac{1}{2} \), \( s_0 > \left( \frac{3}{2} - \alpha \right) \) and let \((M,g)\) a compact four dimensional Riemannian manifold. Then there exists \( C > 0 \) such that for any \( f_1, f_2, f_3, f_4 \in L^2(M) \) satisfying

\[
1_{\{1 - \Delta \in [N_j,2N_j]\}}(f_j) = f_j, \quad j = 1,2,3,4,
\]

one has the following quadrilinear estimate for \( u_j(t) = S(t)f_j \):

\[
\begin{align*}
\sup_{\tau \in \mathbb{R}} & \left| \int_M \int_{\mathbb{R}} \chi(t) e^{i\tau t} (1 - \Delta)^{-\alpha} (u_1 \overline{u}_2) u_3 \overline{u}_4 dx \right| \\
& \leq C \| (1 - \Delta)^{-\alpha} (u_1 \overline{u}_2) \|_{L^2((0,1) \times M)} \| (1 - \Delta)^{-\alpha} (u_3 \overline{u}_4) \|_{L^2((0,1) \times M)} \\
& \leq C \| (m(N_1,\ldots,N_4))^{s_0} \|_{L^2(M)} \| f_1 \|_{L^2(M)} \| f_2 \|_{L^2(M)} \| f_3 \|_{L^2(M)} \| f_4 \|_{L^2(M)},
\end{align*}
\]

where \( \chi \in C_c^\infty(\mathbb{R}) \) is arbitrary and \( m(N_1,\ldots,N_4) \) is the product of the smallest two numbers among \( N_1,N_2,N_3,N_4 \).

**Proof.** The proof of our quadrilinear estimate (40) when \( m(N_1,\ldots,N_4) = N_1N_2 \) follows directly by the Cauchy-Schwarz inequality and Lemma 8. In fact, assuming for instance that \( \chi \) is supported into \([0,1]\), we have

\[
I \equiv \sup_{\tau \in \mathbb{R}} \left| \int_M \int_{\mathbb{R}} \chi(t) e^{i\tau t} (1 - \Delta)^{-\alpha} (u_1 \overline{u}_2) u_3 \overline{u}_4 dx \right| \\
\leq C \| (1 - \Delta)^{-\alpha} (u_1 \overline{u}_2) \|_{L^2((0,1) \times M)} \| (1 - \Delta)^{-\alpha} (u_3 \overline{u}_4) \|_{L^2((0,1) \times M)} \\
\leq C \| (m(N_1,\ldots,N_4))^{s_0} \|_{L^2(M)} \| f_1 \|_{L^2(M)} \| f_2 \|_{L^2(M)} \| f_3 \|_{L^2(M)} \| f_4 \|_{L^2(M)},
\]

by applying (39). By symmetry, it remains to consider only the case \( m(N_1,\ldots,N_4) = N_1N_2 \).

By the self-adjointness of \( (1 - \Delta) \), Hölder’s inequality and Sobolev’s inequality we have

\[
I \leq C \| u_1 \overline{u}_2 \|_{L^1((0,1),L^{q'}(M))} \| (1 - \Delta)^{-\alpha} (u_3 \overline{u}_4) \|_{L^\infty((0,1),L^q(M))} \\
\leq C \| u_1 \overline{u}_2 \|_{L^1((0,1),L^{q'}(M))} \| u_3 \overline{u}_4 \|_{L^\infty((0,1),L^q(M))},
\]

provided \( \frac{1}{q} > 1 - \frac{\alpha}{2} \). Using again Hölder’s inequality, we infer

\[
I \leq C \prod_{j=1,2} \| u_j \|_{L^2((0,1),L^{2q'}(M))} \prod_{k=3,4} \| u_k \|_{L^\infty((0,1),L^2(M))}.
\]

Conservation of energy implies that \( \| u_k \|_{L^\infty((0,1),L^2(M))} = \| f_k \|_{L^2(M)} \). On the other hand by Sobolev embedding we have

\[
\| u_j \|_{L^2((0,1),L^{2q'}(M))} \leq CN_j^{\frac{q-1}{q}} \| u_j \|_{L^2((0,1),L^4(M))}.
\]

Now we can apply the above-mentioned Strichartz estimate of [5] to obtain

\[
\| u_j \|_{L^2((0,1),L^{2q'}(M))} \leq CN_j^{\frac{2}{q} - \frac{1}{2}} \| f_j \|_{L^2(M)}.
\]

Since

\[
s_0 = \frac{2}{q} - \frac{1}{2} > \frac{3}{2} - \alpha,
\]

and \( s_0 \) can be arbitrarily close to \( \frac{3}{2} - \alpha \), the proof is complete. \( \square \)
Remark. In this case, an iteration scheme for solving can be performed as in [5], avoiding the use of Bourgain spaces, making in \( X_T = C([0, T], H^1) \cap L^2([0, T], H^2_0) \).

On the four dimensional sphere, endowed with its standard metric, the precise knowledge of the spectrum \( \mu_k = k(k + 3), k \in \mathbb{N} \) makes it possible to improve our quadrilinear estimate. We proceed in several steps, starting with an estimate on the product of two spherical harmonics.

**Lemma 9.** Let \( \alpha \in [0, \frac{1}{2}] \) and let \( s_0 = 1 - \frac{2\alpha}{T} \). There exists \( C > 0 \) such that for any \( H_n, \tilde{H}_l \) spherical harmonics on \( \mathbb{S}^4 \) of degree \( n, l \) respectively, the following bilinear estimate holds:

\[
\|(1 - \Delta)^{-\alpha/2}(H_n \tilde{H}_l)\|_{L^2(\mathbb{S}^4)} \leq C(1 + \min((n, l))^{s_0})\|H_n\|_{L^2(\mathbb{S}^4)}\|\tilde{H}_l\|_{L^2(\mathbb{S}^4)}.
\]

**Proof.** It is not restrictive to assume that \( 1 \leq n \leq l \). We shall adapt the proof of multilinear estimates in [6],[8], using the approach described in [9].

Writing

\[
h = (n(n+3))^{-1/2}, \quad \tilde{h} = (l(l+3))^{-1/2},
\]

the equations satisfied by the eigenfunctions \( H_n, \tilde{H}_l \) read

\[
h^2 \Delta H_n + H_n = 0, \quad \tilde{h}^2 \Delta \tilde{H}_l + \tilde{H}_l = 0.
\]

In local coordinates, these are semiclassical equations, with principal symbol

\[
p(x, \xi) = 1 - g_x(\xi, \xi).
\]

We now decompose \( H_n \) and \( \tilde{H}_l \) using a microlocal partition of unity with semi-classical cut-off of the form \( \chi(x, hD), \tilde{\chi}(x, \tilde{h}D) \) respectively. When

\[
\text{supp} \chi(x, \xi) \cap \{ g_x(\xi, \xi) = 1 \} = \emptyset,
\]
i.e. in the "elliptic" case, the estimates are quite strong : we have, for all \( s, p, \)

\[
\|(1 - \Delta)^{-\alpha/2}(H_n \tilde{H}_l)\|_{L^2(\mathbb{S}^4)} \leq C_{s, p} h^p \|H_n\|_{L^2(\mathbb{S}^4)},
\]

with similar estimates for \( \tilde{H}_l \). Consequently, it is sufficient to estimate

\[
\|(1 - \Delta)^{-\alpha/2}(\chi(x, hDx)H_n \tilde{\chi}(x, \tilde{h}Dx)\tilde{H}_l)\|_{L^2(\mathbb{S}^4)}
\]

when cut-off functions \( \chi, \tilde{\chi} \) are localized near the characteristic set

\[
\{ g_x(\xi, \xi) = 1 \}.
\]

Refining the partition of unity, we may assume that the supports of \( \chi, \tilde{\chi} \) are contained in small neighborhoods of \( (m, \omega), (m, \tilde{\omega}) \) where \( m \in M \) and \( \omega, \tilde{\omega} \) are covectors such that

\[
g_m(\omega, \omega) = g_m(\tilde{\omega}, \tilde{\omega}) = 1.
\]

Notice that functions \( u = \chi(x, hDx)H_n, \tilde{u} = \tilde{\chi}(x, \tilde{h}Dx)\tilde{H}_l \) are compactly supported and satisfy

\[
p^u(x, hD)u = hF, \quad p^u(x, \tilde{h}D)\tilde{u} = \tilde{h}\tilde{F},
\]

where \( \|F\|_{L^2} \lesssim \|H_n\|_{L^2} \) and \( \|\tilde{F}\|_{L^2} \lesssim \|\tilde{H}_l\|_{L^2} \).
Set \( g_x(x, \xi) = \langle A(x)\xi, \xi \rangle \). Choose any system \((x_1, \ldots, x_4)\) of linear coordinates on \( \mathbb{R}^4 \) such that
\[
\langle A(m)\omega, dx_1 \rangle \neq 0 \quad \text{and} \quad \langle A(m)\tilde{\omega}, dx_1 \rangle \neq 0.
\]

Then, on the supports of \( \chi \) and \( \tilde{\chi} \), one can factorize the symbol of the equation as
\[
p(x, \xi) = e(x, \xi)(\xi_1 - q(x, \xi')) , \quad p(x, \xi) = \tilde{e}(x, \xi)(\xi_1 - \tilde{q}(x, \xi')),
\]
where \( e, \tilde{e} \) are elliptic symbol while \( q, \tilde{q} \) are real valued symbols. In other words, we can reduce the equations for \( u, \tilde{u} \) to evolution equations with respect to the variable \( x_1 \). Notice that \( \xi' \in \mathbb{R}^{d-1} = \mathbb{R}^3 \), i.e., the spatial dimension of these evolution equations is 3. Moreover, since the second fundamental form of the characteristic ellipsoid \( \{ \xi : g_m(\xi, \xi) = 1 \} \) is non-degenerate, the Hessian of \( q, \tilde{q} \) with respect to the \( \xi' \) variables does not vanish on the supports of \( \chi, \tilde{\chi} \) respectively.

Therefore we can apply to this equation the (local) three-dimensional Strichartz estimates (see Corollary 2.2 of [9] for more details). We conclude that \( u \) satisfies the 3-dimensional semiclassical Strichartz estimates in the following form:

\[
\|u\|_{L^p_{x_1}L^q_{x'}} \leq C h^{-\frac{3}{p}} \|H_n\|_{L^2} \lesssim n^{\frac{3}{p}} \|H_n\|_{L^2},
\]

for all \((p, q)\) satisfying the admissibility condition

\[
\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad p \geq 2.
\]

An identical argument is valid for \( \tilde{u} \). In fact, for \( \tilde{u} \) we shall only need the energy estimate

\[
\|\tilde{u}\|_{L^\infty_{x_1}L^2_{x'}} \leq C \|\tilde{H}_1\|_{L^2}.
\]

Finally, we estimate the product \( u\tilde{u} \) as follows. By the Sobolev inequality,

\[
\| (1 - \Delta)^{-\frac{\alpha}{2}} (u\tilde{u}) \|_{L^2} \leq C \|u\tilde{u}\|_{L^4}, \quad \frac{1}{q} = \frac{1}{2} + \frac{\alpha}{4}.
\]

Applying the H"older inequality we obtain

\[
\| (1 - \Delta)^{-\frac{\alpha}{q}} (u\tilde{u}) \|_{L^2} \leq C \|u\|_{L^q_{x_1}L^4_{x'}} \|\tilde{u}\|_{L^\infty_{x_1}L^2_{x'}}.
\]

Noticing that \( q < 2 \) and using the compactness of the support of \( u \), we have

\[
\|u\|_{L^q_{x_1}L^4_{x'}} \leq C \|u\|_{L^2_{x_1}L^4_{x'}}.
\]

Applying the Strichartz estimate (44) with \( p = 2 \) and the Sobolev embedding in the \( x' \) variables, we obtain

\[
\|u\|_{L^2_{x_1}(L^4_{x'})} \leq C n^{-\frac{1}{2} - \frac{\alpha}{4}} \|u\|_{L^2_{x_1}L^6_{x'}} \leq C n^{-1 - \frac{\alpha}{4}} \|H_n\|_{L^2}.
\]

Combining with the \( L^\infty L^2 \) estimate (45) on \( \tilde{u} \), this completes the proof.

\[\square\]

We now come to a quadrilinear estimate on spherical harmonics.
Lemma 10. Let $\alpha \in [0, \frac{1}{2}]$ and $s_0 = 1 - \frac{3\alpha}{4}$. There exists $C > 0$ such that for any $H_{n_j}^{(j)}, j = 1, \cdots, 4$, spherical harmonics on $\mathbb{S}^4$ of degree $n_j$ respectively, the following quadrilinear estimate holds:

$$
\int_{\mathbb{S}^4} (1 - \Delta)^{-\alpha} (H_{n_1}^{(1)} H_{n_2}^{(2)} H_{n_3}^{(3)} H_{n_4}^{(4)}) \, dx \leq C (1 + m((n_j)))^{s_0} \prod_{j=1}^{4} \| H_{n_j}^{(j)} \|_{L^2(\mathbb{S}^4)}.
$$

Proof. By symmetry, it is sufficient to consider the two cases

$$
m(n_1, \cdots, n_4) = n_1 n_3 ; \quad m(n_1, \cdots, n_4) = n_1 n_2 .
$$

In the first case, the proof follows directly by the Cauchy-Schwarz inequality and Lemma 9. It remains to consider only the case $m(n_1, \cdots, n_4) = n_1 n_2$.

We use the same idea as in Lemma 9 to decompose, if $n_j \geq 1$, each $H_{n_j}^{(j)}$ into a sum of terms of the form

$$
u_j = \chi_j(x, h_j D_x) H_{n_j}^{(j)}, \quad h_j = (n_j(n_j + 3))^{-1/2}, \quad j = 1, 2, 3, 4.
$$

As before, each $\nu_j$ may be microlocalized either into the elliptic zone, in which case we have much stronger semiclassical estimate (42), in particular an $L^\infty$ bound, or near the characteristic set, and for these terms we can use the Strichartz type estimate (44). Notice that the very special case $n_j = 0$ can be included into the elliptic case. Thus we have several possibilities to consider.

If at least two $\nu_j$’s are microlocalized in the elliptic zone, then the quadrilinear estimate holds trivially (with $s_0 = 0$) by a simple application of the Cauchy-Schwarz inequality.

If $\nu_3$ or $\nu_4$ is microlocalized in the elliptic zone, then, again by the Cauchy-Schwarz inequality, the quadrilinear estimate is a consequence of estimate (41) of Lemma 9, with $\alpha$ replaced by $2\alpha$.

It remains to deal with the cases when only $\nu_1$ or $\nu_2$ is microlocalized in the elliptic zone, and when all the $\nu_j$’s are microlocalized near the characteristic set. In both cases, we shall make use of the following variant of the Sobolev inequality.

Lemma 11. Let $A$ be a pseudodifferential operator of order $-2\alpha$ on $\mathbb{R}^4$, and let $B$ be a bounded subset of $\mathbb{R}^4$. For any smooth function $F$ on $\mathbb{R}^4$ with support in $B$, we have the estimate

$$
\| A(F) \|_{L_x^\infty(L^q_r)} \leq C \| F \|_{L_x^\infty(L^1_r)}
$$

provided $\frac{1}{q} > 1 - 2\alpha/3$.

Proof. The kernel $K(x, y)$ of $A$ admits an estimate like

$$
|K(x, y)| \leq \frac{C}{(|x - y|)^{4-2\alpha}} \leq \frac{C}{(|x_1 - y_1| + |x'' - y''|)^{4-2\alpha}}.
$$

The claim is then a consequence of Young’s inequality in variables $x''$. \qed

By the self-adjointness of $(1 - \Delta)$ the terms to estimate can be written as follows:

$$
I = \left| \int_{\mathbb{S}^4} (u_1 u_2) \times (1 - \Delta)^{-\alpha}(u_3 u_4) \, dx \right|.
$$
As in the proof of Lemma 9 we select a splitting \( x = (x_1, x') \) of the local coordinates such that \( u_2, u_3, u_4 \) are solutions of semiclassical evolution equations, and therefore satisfy Strichartz estimates (44). Using the \( L^\infty \) bound on \( u_1 \), we have

\[
I \leq C \left\| H_{n_1}^{(1)} \right\|_{L^2(\mathbb{S}^4)} \left\| u_2 \right\|_{L^1_{x_1}(L^q_{x'})} ((1 - \Delta)^{-\alpha}(u_3u_4)) \left\| L^\infty_{x_1}(L^3_{x'}) \right\|
\]

and by Lemma 11 we obtain

\[
I \leq C \left\| H_{n_1}^{(1)} \right\|_{L^2(\mathbb{S}^4)} \left\| u_2 \right\|_{L^1_{x_1}(L^q_{x'})} \left\| u_3u_4 \right\|_{L^\infty_{x_1}(L^3_{x'})}
\]

provided \( \frac{1}{q} > 1 - \frac{3\alpha}{4} \). Hölder’s inequality gives

\[
I \leq C \left\| H_{n_1}^{(1)} \right\|_{L^2(\mathbb{S}^4)} \left\| u_2 \right\|_{L^2_{x_1}(L^q_{x'})} \left\| u_3 \right\|_{L^\infty_{x_1}(L^2_{x'})} \left\| u_4 \right\|_{L^\infty_{x_1}(L^2_{x'})}
\]

and, applying estimate (45) on \( u_3, u_4 \) and estimate (44) with \( p = 2 \) on \( u_2 \), we obtain

\[
I \leq C n_{n_2}^4 \prod_{j=1}^4 \left\| H_{n_j}^{(4)} \right\|_{L^2(\mathbb{S}^4)},
\]

with

\[
s = \max \left( \frac{1}{2}, 1 - \frac{3}{4q} \right) < s_0,
\]

since \( q' \) is arbitrary with \( \frac{1}{q} < \frac{2\alpha}{3} \).

Finally, we treat the case when all the factors are microlocalized near the characteristic set. Once again, we select a splitting \( x = (x_1, x') \) of the local coordinates for which Strichartz estimates (44) are valid for each \( u_j \). By Hölder’s inequality and Lemma 11 we have

\[
I \leq C \left\| u_1u_2 \right\|_{L^1_{x_1}(L^q_{x'})} \left\| u_3u_4 \right\|_{L^\infty_{x_1}(L^3_{x'})}
\]

\[
\leq C \left\| u_1 \right\|_{L^2_{x_1}(L^{2q'}_{x'})} \left\| u_2 \right\|_{L^2_{x_1}(L^{2q'}_{x'})} \left\| u_3 \right\|_{L^\infty_{x_1}(L^2_{x'})} \left\| u_4 \right\|_{L^\infty_{x_1}(L^2_{x'})}.
\]

By estimates (44) with \( p = 2 \) on \( u_1, u_2 \) and (45) on \( u_3, u_4 \), we conclude

\[
I \leq C (n_1n_2)^4 \prod_{j=1}^4 \left\| H_{n_j}^{(4)} \right\|_{L^2(\mathbb{S}^4)},
\]

with

\[
s = \max \left( \frac{1}{2}, 1 - \frac{3}{2q} \right) < s_0,
\]

since \( q' \) is arbitrary with \( \frac{1}{q} < \frac{2\alpha}{3} \). This completes the proof.

**Remark.** It is clear that Lemma 9 and Lemma 10 extend to Laplace eigenfunctions on arbitrary compact four-manifolds. Moreover, a refinement of the study of the elliptic case shows that, as in [6], [8], eigenfunctions can be replaced by functions belonging to the range of spectral projectors of the type \( 1_{[s, s+1]}(\sqrt{-\Delta}) \).

We now come to the main result of this subsection.

**Proposition 3.** For every \( \alpha > 0 \), for every \( s_0 > 1 - \frac{3\alpha}{4} \), the quadrilinear estimate (6) holds on \( \mathbb{S}^4 \).
Proof. Let \( f_1, \ldots, f_4 \) be functions on \( S^4 \) satisfying the spectral localization property
\[
1_{\sqrt{1-\Delta} \in [N_j, 2N_j]}(f_j) = f_j, \quad j = 1, 2, 3, 4.
\]
This implies that one can expand
\[
f_j = \sum_{n_j} H_{n_j}^{(j)},
\]
where \( H_{n_j}^{(j)} \) are spherical harmonics of degree \( n_j \), and where the sum on \( n_j \) bears on the domain
\[
N_j/2 \leq 1 + n_j \leq 2N_j.
\]
Consequently, the corresponding solutions of the linear Schrödinger equation are given by
\[
u_j(t) = S(t) f_j = \sum_{n_j} e^{-itn_j(n_j+3)} H_{n_j}^{(j)},
\]
and we have to estimate the expression
\[
Q(f_1, \ldots, f_4, \tau) = \int_{\mathbb{R}} \int_{S^4} \chi(t) e^{it\tau} (1-\Delta)^{-\alpha}(u_1 \overline{u}_2) u_3 \overline{u}_4 dt dx
\]
with \( \epsilon_j = (-1)^{j-1} \) and
\[
I(H_{n_1}^{(1)}, \ldots, H_{n_4}^{(4)}) = \int_{S^4} (1-\Delta)^{-\alpha}(H_{n_1}^{(1)} \overline{H}_{n_2}^{(2)}) H_{n_3}^{(3)} \overline{H}_{n_4}^{(4)} dx .
\]
Appealing to Lemma 10, we have, with \( s = 1 - 3\alpha/4 \),
\[
|I(H_{n_1}^{(1)}, \ldots, H_{n_4}^{(4)})| \leq C m(N_1, \ldots, N_4)^s \prod_{j=1}^4 \| H_{n_j}^{(j)} \|_{L^2} .
\]
Using the fast decay of \( \hat{\chi} \) at infinity, we infer
\[
|Q(f_1, \ldots, f_4, \tau)| \leq C m(N_1, \ldots, N_4)^s \sum_{\ell \in \mathbb{Z}} (1+|\ell|^2)^{-1} \sum_{A(|\tau|+\ell)} \prod_{j=1}^4 \| H_{n_j}^{(j)} \|_{L^2}
\]
\[
\lesssim m(N_1, \ldots, N_4)^s \sup_{k \in \mathbb{Z}} \prod_{A(k)} \| H_{n_j}^{(j)} \|_{L^2} ,
\]
where \( A(k) \) denotes the set of \( (n_1, \ldots, n_4) \) satisfying (52) for \( j = 1, 2, 3, 4 \) and
\[
\sum_{j=1}^4 \epsilon_j n_j(n_j+3) = k .
\]
Now we write
\[
\{1, 2, 3, 4\} = \{\alpha, \beta, \gamma, \delta\}
\]
with \( m(N_1, \ldots, N_4) = N_\alpha N_\beta \), and we split the sum on \( A(k) \) as
\[
|Q(f_1, \ldots, f_4, \tau)| \lesssim m(N_1, \ldots, N_4)^s \sup_{k \in \mathbb{Z}} S(a) S'(k-a)
\]
where

\[ S(a) = \sum_{\Gamma(a)} \|H_{n_\alpha}^{(\alpha)}\|_{L^2} \|H_{n_\gamma}^{(\gamma)}\|_{L^2} ; \quad S'(a') = \sum_{\Gamma'(a')} \|H_{n_\beta}^{(\beta)}\|_{L^2} \|H_{n_\delta}^{(\delta)}\|_{L^2} , \]

\[ \Gamma(a) = \{(n_\alpha, n_\gamma) : (52) \text{ holds for } j = \alpha, \gamma, \sum_{j \in \{\alpha, \gamma\}} \varepsilon_j n_j (n_j + 3) = a \}, \]

\[ \Gamma'(a') = \{(n_\beta, n_\delta) : (52) \text{ holds for } j = \beta, \delta, \sum_{j \in \{\beta, \delta\}} \varepsilon_j n_j (n_j + 3) = a' \}. \]

Now we appeal to the following elementary result of number theory (see e.g. Lemma 3.2 in [7]).

**Lemma 12.** Let \( \sigma \in \{\pm 1\} \). For every \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that, given \( M \in \mathbb{Z} \) and a positive integer \( N \),

\[ \#\{(k_1, k_2) \in \mathbb{N}^2 : N \leq k_1 \leq 2N , k_1^2 + \sigma k_2^2 = M \} \leq C_\varepsilon N^{\varepsilon} . \]

A simple application of Lemma 12 implies, for every \( \varepsilon > 0 \),

\[ \sup_a \#\Gamma(a) \leq C_\varepsilon N_\alpha \varepsilon ; \quad \sup_{a'} \#\Gamma'(a') \leq C_\varepsilon N_\beta \varepsilon , \]

and consequently, by a repeated use of the Cauchy-Schwarz inequality,

\[ \sum_a S(a) S'(k - a) \leq C_\varepsilon (N_\alpha N_\beta)^\varepsilon \times \]

\[ \left( \sum_a \sum_{\Gamma(a)} \|H_{n_\alpha}^{(\alpha)}\|_{L^2} \|H_{n_\gamma}^{(\gamma)}\|_{L^2} \right)^{1/2} \left( \sum_a \sum_{\Gamma'(k - a)} \|H_{n_\beta}^{(\beta)}\|_{L^2} \|H_{n_\delta}^{(\delta)}\|_{L^2} \right)^{1/2} \]

\[ \leq C_\varepsilon (N_\alpha N_\beta)^\varepsilon \prod_{j=1}^4 \|f_j\|_{L^2} , \]

where, in the last estimate, we used the orthogonality of the \( H_{n_j}^{(j)} \)'s as \( n_j \) varies. Coming back to (53), this completes the proof. \( \square \)

**Remark.** Using the remark before the statement of this proposition, the proof above extends easily to any compact four-dimensional Zoll manifold (see [7] for more details).

### 3.2. Trilinear estimates on the sphere.

In this subsection, we prove trilinear estimates (27) on \( S^4 \), for every \( s_0 > 1/2 \), for zonal solutions of the Schrödinger equation. In view of subsections 2.2 and 2.4, this will complete the proof of Theorem 2 and of Corollary 1, by choosing for \( G \) the group of rotations which leave invariant a given pole on \( S^4 \).

First we recall the definition of zonal functions.

**Definition 3.** Let \( d \geq 2 \), and let us fix a pole on \( S^d \). We shall say that a function on \( S^d \) is a zonal function if it depends only on the geodesic distance to the pole.

The zonal functions can be expressed in terms of zonal spherical harmonics which in their turn can be expressed in terms of classical polynomials (see
e.g. [17]). As in [8], we can represent the normalized zonal spherical harmonic $Z_p$ in the coordinate $\theta$ (the geodesic distance of the point $x$ to our fixed pole) as follows:

$$Z_p(x) = C (\sin \theta)^{-\frac{d+1}{2}} \left\{ \cos[(p + \alpha)\theta + \beta] + \frac{\mathcal{O}(1)}{p \sin \theta} \right\}, \quad \frac{c}{p} \leq \theta \leq \pi - \frac{c}{p}$$

with $\alpha, \beta$ independent of $p$, and $C$ uniformly bounded in $p$. On the other hand, near the concentration points $\theta = 0, \pi$ we can write

$$|Z_p(x)| \leq C_p \frac{d-1}{2}, \quad \theta \notin [c/p, \pi - c/p].$$

and $\|Z_p\|_{L^2(S^d)} = 1$.

With this notation, we have the following trilinear eigenfunction estimates.

**Lemma 13.** There exists a constant $C > 0$ such that the following trilinear estimate holds:

$$\|Z_p Z_q Z_l\|_{L^1(S^d)} \leq C (\min(p, q, l))^{1/2}. \quad (56)$$

**Proof.** It is not restrictive to assume that $p \leq q \leq l$. Moreover, by Cauchy-Schwarz inequality it is sufficient to prove (56) in the special case $q = l$. Then we have

$$\|Z_p Z_q^2\|_{L^1(S^d)} = c \int_0^\pi |Z_p(\theta)|Z_q(\theta)^2 (\sin \theta)^3 d\theta,$$

where $c$ is some universal constant. We split the interval $[0, \pi]$ into the intervals $I_1 = [0, c/q], I_2 = [c/q, c/p], I_3 = [c/p, \pi/2]$ and $I_4 = [\pi, 2, \pi - c/p], I_5 = [\pi - c/p, \pi - c/q], I_6 = [\pi - c/p, \pi]$. Clearly, by symmetry, it is sufficient to estimate the integral on the first three intervals $I_1, I_2, I_3$.

On $I_1$ we can use (55) for both harmonics $Z_p, Z_q$ and the simple estimate $\sin \theta \leq \theta$, and we obtain

$$\int_0^{c/q} |Z_p|Z_q^2 (\sin \theta)^3 d\theta \leq C p^{3/2} q^{-1} \int_0^{c/q} \theta^3 d\theta \leq C p^{3/2} q^{-3/2} \leq C p^{1/2}$$

since $q \geq p$.

On the second interval $I_2$ we use (54) for $Z_p$ and (55) for $Z_q$:

$$\int_{c/q}^{c/p} |Z_p|Z_q^2 (\sin \theta)^3 d\theta \leq C p^{3/2} \int_{c/q}^{c/p} \left(1 + \frac{1}{q \sin \theta} \right)^2 d\theta$$

and by the elementary inequality

$$\left(1 + \frac{1}{q \sin \theta} \right)^2 \leq C + \frac{C}{q^2 \theta^2} \quad (57)$$

we have immediately

$$\int_{c/q}^{c/p} |Z_p|Z_q^2 (\sin \theta)^3 d\theta \leq C p^{3/2} \left(\frac{c}{p} - \frac{c}{q} + \frac{C}{q^2} (q/c - p/c) \right) \leq C p^{1/2}.$$ 

Finally, in the interval $I_3$ we must use (54) for both harmonics:

$$\int_{c/p}^{\pi/2} |Z_p|Z_q^2 (\sin \theta)^3 d\theta \leq C \int_{c/p}^{\pi/2} \left(1 + \frac{1}{p \sin \theta} \right) \left(1 + \frac{1}{q \sin \theta} \right)^2 (\sin \theta)^{-3/2} d\theta.$$
Using again (57), the inequality \( \sin \theta \geq C \theta \) on \([0, \pi/2] \), and the fact that \( q \geq p \), we have easily
\[
\left( 1 + \frac{1}{p \sin \theta} \right) \left( 1 + \frac{1}{q \sin \theta} \right)^2 (\sin \theta)^{-3/2} \leq C \theta^{-3/2} + C p^{-3} \theta^{-9/2}.
\]
Then integrating on \( I_3 \) we obtain
\[
\int_{c/p}^{\pi/2} |Z_p Z_q^2 (\sin \theta)^3| \, d\theta \leq C p^{1/2}
\]
and this concludes the proof. \( \square \)

We now come to the main result of this subsection, which asserts that trilinear estimates (27) hold for every \( s_0 > 1/2 \) on \( M = S^4 \) in the particular case of zonal Cauchy data.

**Proposition 4.** Let \( s_0 > \frac{1}{2} \) and \( \chi \in C_c^\infty (\mathbb{R}) \). There exists \( C > 0 \) such that for any \( f_1, f_2, f_3 \in L^2 (S^4) \) are zonal functions and satisfying
\[
1 \sqrt{1 - x^2 e^{2N \epsilon_j}} (f_j) = f_j, \quad j = 1, 2, 3,
\]
one has the following trilinear estimate for \( u_j (t) = S(t) f_j \).
\[
\sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_{S^4} \chi(t) e^{it\tau} u_1 u_2 u_3 dx dt \right| \leq C (\min(N_1, N_2, N_3))^{s_0} \| f_1 \|_{L^2 (S^4)} \| f_2 \|_{L^2 (S^4)} \| f_3 \|_{L^2 (S^4)}.
\]

**Proof.** The proof is very similar to the one of Proposition 3. We write
\[
u_j (t) = \sum_{n_j} e^{-i t n_j (n_j + 3)} c_j (n_j) Z_{n_j},
\]
where \( n_j \) is subject to the condition (52) and
\[
\sum_{n_j} |c_j (n_j)|^2 \sim \| f_j \|_{L^2}^2.
\]
Thus we can write the integral of the left hand-side of (59) as
\[
J = \sum_{n_1, n_2, n_3} \hat{\chi}^3 \left( \sum_{j=1}^3 \epsilon_j n_j (n_j + 3) - \tau \right) c_1 (n_1) c_2 (n_2) c_3 (n_3) \int_{S^4} Z_{n_1} Z_{n_2} Z_{n_3} dx,
\]
where \( \epsilon_1 = \epsilon_2 = 1 \) and \( \epsilon_3 = -1 \). Using the fast decay of the Fourier transform \( \hat{\chi} \) and the estimate of Lemma 13, we obtain
\[
|J| \leq C (\min(N_1, N_2, N_3))^{s_0} \sum_{\ell \in \mathbb{Z}} \frac{1}{1 + \ell^2} \sum_{n_j \in \Lambda_{|n_j| + \ell}} |c_1 (n_1) c_2 (n_2) c_3 (n_3)|,
\]
\[
\lesssim (\min(N_1, N_2, N_3))^{s_0} \sup_{k \in \mathbb{Z}} \sum_{n_j \in \Lambda_k} |c_1 (n_1) c_2 (n_2) c_3 (n_3)|,
\]
where
\[
\Lambda_k = \{(n_1, n_2, n_3) : (52) holds for j = 1, 2, 3; \sum_{j=1}^3 \epsilon_j n_j (n_j + 3) = k \}.
\]

Suppose for instance that \( \min(N_1, N_2, N_3) is N_1 or N_2 \). Introducing
\[
\Lambda_k (n_3) = \{(n_1, n_2) : (n_1, n_2, n_3) \in \Lambda_k \},
\]
we specialize index $n_3$ in the above sum as

$$J \leq C \sup_k \sum_{n_3} |c_3(n_3)| \left( \sum_{(n_1, n_2) \in \Lambda_k(n_3)} |c_1(n_1)c_2(n_2)| \right)$$

$$\leq C \sup_k \left( \sum_{n_3} |c_3(n_3)|^2 \right)^{\frac{1}{2}} \left( \sum_{n_3} \left( \sum_{(n_1, n_2) \in \Lambda_k(n_3)} |c_1(n_1)c_2(n_2)| \right)^2 \right)^{\frac{1}{2}}$$

$$\leq C \left( \sum_{n_3} |c_3(n_3)|^2 \right)^{\frac{1}{2}} \sup_k \left( \sum_{n_3} \left\lbrack \#\Lambda_k(n_3) \sum_{(n_1, n_2) \in \Lambda_k(n_3)} |c_1(n_1)|^2|c_2(n_2)|^2 \right\rbrack \right)^{\frac{1}{2}}$$

To complete the proof, it remains to appeal once again to Lemma 12, which yields the estimate

$$\#\Lambda_{r,\ell}(n_3) \leq C_\delta (\min(N_1, N_2))^\delta$$

for every $\delta > 0$. If $N_3$ is $\min(N_1, N_2, N_3)$, the proof is similar, by specializing the sum with respect to $n_1$, say.

□

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