Closing the proximity gap in a metallic Josephson junction between three superconductors

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We describe the proximity effect in a short disordered metallic junction between three superconducting leads. Andreev bound states in the multi-terminal junction may cross the Fermi level. We reveal that for a quasi-continuous metallic density of states, crossings at the Fermi level manifest as closing of the proximity-induced gap. We calculate the local density of states for a wide range of transport parameters using quantum circuit theory. The gap closes inside an area of the space spanned by the superconducting phase differences. We derive an approximate analytic expression for the boundary of the area and compare it to the full numerical solution. The size of the area increases with the transparency of the junction and is sensitive to asymmetry. The finite density of states at zero energy is unaffected by electron-hole decoherence present in the junction, although decoherence is important at higher energies. Our predictions can be tested using tunneling transport spectroscopy. To encourage experiments, we calculate the current-voltage characteristic in a typical measurement setup. We show how the structure of the local density of states can be mapped out from the measurement.

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I. INTRODUCTION

The density of states in a normal metal is strongly modified by contact to one or multiple superconductors placed in its proximity. In junctions with two superconductors, the development of a minigap in the density of states has been in the focus of research for many years.1 Recent attention has been given to the possibility of engineering states inside the minigap in the context of Majorana zero energy states.2–4 To this end, it would be advantageous if the Andreev states in a Josephson junction could be brought to zero energy, i.e. at the Fermi level, by controlling only the superconducting phase of the junction. Zero energy Andreev states together with spin-orbit coupling provide the opportunity to manipulate single fermionic quasiparticles.5 The motivation is to increase understanding of quasiparticle dynamics and facilitate designs of superconducting spin quantum memory bits.6 Unfortunately, in two terminal Josephson junctions Andreev states do not cross the Fermi level except at a singular phase difference, , and only in systems with reflectionless transport channels.

Recent work shows that the situation is different in a Josephson junction with multiple terminals ( ). The Andreev states have been shown to cross the Fermi level if the superconducting phases wind by 2 around the junction. The crossing points have been termed Weyl points, as they are analogous to Weyl singularities studied in 3D solids, with Andreev bound states corresponding to energy bands and the superconducting phase differences corresponding to quasi-momenta. The presence of Weyl points has been illustrated theoretically by using scattering theory to model junctions consisting of a quantum dot connected to three superconductors. In particular, in three-terminal junctions, Weyl points appear along a closed curve in the space spanned by two independent superconducting phases. The area enclosed by the curve is shown to be maximal for a fully transparent quantum dot.

Weyl points have been found for some choices of the scattering matrix describing the junction, but not for all. The existence or absence of Weyl points in the spectrum is determined by the details of the junction scattering matrix. The requirements for presence of Weyl points are not yet well understood, but are very important in view of their experimental observation. From the viewpoint of the experimental realization, it is advantageous to study metallic junctions, that do not require the strict electrical confinement of quantum dots. Metallic junctions have a quasi-continuous spectrum of states, in contrast to the discrete states in a quantum dot. The larger number of states discourages use of the scattering theory approach.

FIG. 1. (Color online.) Left. Two superconducting rings can be used to control the two independent superconducting phase differences in a three terminal superconducting junction via fluxes . Right. Quantum circuit theory model of a three terminal superconducting device using a single node.
To model metallic junctions, quasiclassical methods are better suited.

In this paper, we study the local density of states in a disordered metallic junction between three superconductors (see Fig. 1). The dimensions of the junction are assumed short at the scale of the superconducting coherence length, but much larger than the mean free path of electrons in the metallic island, in the normal state. We show that the junction exhibits two regimes. One where the density of states has a finite minigap, similar to the metallic island, in the normal state. We assume the same for all three superconductors, $\phi_S$ takes the value of the superconducting phase $\phi$ corresponding to superconductor $i = \{1, 2, 3\}$, and $\theta_S$ has the following energy dependence,

$$\theta_S = \begin{cases} -i\pi + \frac{1}{2}\ln \frac{1 + i\omega}{1 - i\omega}, & \omega < 1, \\ \frac{1}{2}\ln \frac{\omega^2 + 1}{\omega^2 - 1}, & \omega > 1, \end{cases}$$

where $\omega = E/|\Delta|$. In the limit of low energy $\omega \ll 1$, the spectral vector in superconductor $i$ has the following simple form, $\tilde{n}_i = (\sin(\phi_i), \cos(\phi_i), -i\omega)$. In terms of the spectral vector of the island $\tilde{n}$ and of those of the superconductors $\tilde{n}_i$, the quasiclassical action of the junction takes the form

$$S = 2\text{Re} \left\{ \sum_i \sum_p \ln \left[ 1 + \frac{T_{p,i}}{2} \left( (\tilde{n} \cdot \tilde{n}_i) - 1 \right) \right] \right\},$$

(1)

where $p$ labels the open transport channels in each of the junction contacts and $T_{p,i}$ represents the corresponding transmission coefficient. The spectral vector in the metallic island $\tilde{n}(\theta, \phi)$ is generally parametrized by two complex numbers $\theta$ and $\phi$, with $n_x = \sin(i\theta)\sin(\phi)$, $n_y = \sin(i\theta)\cos(\phi)$, and $n_z = \cos(i\theta)$. In the superconducting bulk with energy gap $|\Delta|$, assumed the same for all three superconductors, $\phi_S$ takes the value of the superconducting phase $\phi$ corresponding to superconductor $i = \{1, 2, 3\}$, and $\theta_S$ has the following energy dependence,

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where $p$ labels the open transport channels in each of the junction contacts and $T_{p,i}$ represents the corresponding transmission coefficient. The spectral vector in the metallic island $\tilde{n}(\theta, \phi)$ is obtained by extremization of the action under the constraint $\tilde{n}^2 = 1$. We introduce a complex Lagrange multiplier $\lambda$ such that the extremization problem reduces to a set of four complex equations

$$\tilde{\nabla}_\tilde{n} A(\tilde{n}, \lambda) = 0, \quad A(\tilde{n}, \lambda) = S(\tilde{n}) + \lambda \tilde{n}^2.$$ 

(2)

In general, the equations are strongly non-linear in terms of the unknowns, $\tilde{n}$ and $\lambda$, and are solved numerically. The local density of states in the metallic island is obtained from $\tilde{n}$ by $\rho(\omega)/\rho_N = \text{Re}\{\tilde{n}_z\} = \text{Re}\{\cos(i\theta)\}$, where $\rho_N$ is the density of states in the normal state.

### III. ANALYTICAL RESULTS

Before we discuss the full numerical solution, we present analytic results in the tunnel limit, i.e. the transmission coefficients of all channels are small, $T_{p,i} \ll 1$. In...
the lowest order, the action takes the form
\[ S^{(1)} = 2 \text{Re} \{ (\mathbf{n} \cdot \mathbf{n}_s) \}, \]  \hspace{1cm} (3)
where \( \mathbf{n}_s = \sum_i g_i \mathbf{n}_i \) is a spectral vector that depends on the superconducting phase differences and \( g_i = \sum_p (T_{p,i}/2) \) is proportional to the normal state conductance of contact \( i \) in units of the conductance quantum. We find two types of solutions in terms of \( \mathbf{n}_s \) that differ in the structure of the local density of states. i. If \( \mathbf{n}_s \) is non-vanishing, the extrema of the action \( S^{(1)} \) occur when the vectors \( \mathbf{n} \) and \( \mathbf{n}_s \) are aligned. In this case, we find that the z-component of \( \mathbf{n} \) is purely imaginary in the limit of low energy \( \omega \ll 1 \), \( n_z = -i\omega \), meaning that the local density of states in the island has a finite gap. ii. In the opposite case, when \( \mathbf{n}_s = 0 \), the action vanishes in the lowest order. To study this situation we include the
where we define the coefficients $a_i$ of the equation is of order $\omega$
$a_j = \sum_i (g_i + q_i) \sin(\phi_i)$, $a_2 = \sum_i \cos(\phi_i)$, $a_4 = \sum_i g_i \cos(\phi_i)$, and $a_5 = \sum_i q_i \cos^2(\phi_i)$. Components $n_x$ and $n_y$ are real, while component $n_z$ is real if $n_x^2 + n_y^2 \leq 1$ and purely imaginary otherwise. The
relation $n_x^2 + n_y^2 \leq 1$ defines an area in the space of superconducting phases where the local density of states in the
is finite at small energies, i.e. the gap in the density of states is closed. Estimations of $n_x$ and $n_y$ yield $n_x, y \propto T^{-1}$, meaning that $n_x^2 + n_y^2 \leq 1$ defines a smaller
area for smaller transmission coefficients. We have compared the analytic results for the boundary of this area with our full numerical solution in Figs. 3, 4 and 5 and have found excellent agreement up to large transmission coefficients.

IV. NUMERICAL RESULTS

The numerical method used to obtain the exact solution is presented in Appendix. To restrict the number of parameters, we assume in all calculations that the number of open channels is the same for all contacts and that the channels in each contact have equal transmission coefficients, i.e. they are independent of channel label $p$. We stress that our results are not restricted to this situation; the method can be applied for any type of contacts. We have checked that the conclusions we present are robust with respect to change of the details of the contacts, as well as to change of the relative size of the superconducting gap of the three superconducting leads.

The details of the contacts determine the shape of the area corresponding to the gapless regime. For a symmetric junction this area diminishes, but does not vanish, at low transparency (see Fig. 3). As the transparency is increased the area reaches a maximum at $T = 1$. It is notable that the maximal area resembles the full parameter space where the superconducting phases wind by $2\pi$ around the junction, i.e. the necessary condition for presence of Weyl point [4] but does not fully cover it.

For the asymmetric junction with two fully transparent contacts, described in Fig. 4, the case of low transparency corresponds to a symmetric junction with only two, rather than three, superconductors. The symmetric two terminal junction exhibits reflectionless transport...
recent extensive studies have so far been restricted to superconductors. However, in conventional junctions with two superconductors, the coherence transfer of multiple Cooper pairs between the leads has received relatively little attention until now. Recent experimental work on Andreev reflection in junctions with three superconducting phases involved.

The fundamental process at the origin of non-local transport is the non-local Andreev reflection, also termed crossed Andreev reflection. Crossed Andreev reflection has been studied for years in the context of junctions formed between a superconductor and two normal leads. On the contrary, the physics of crossed Andreev reflection in junctions with three superconducting leads has received relatively little attention until now. Recent extensive studies have so far been restricted to transport through a single level dot. The rigorous analysis of non-local processes in metallic junctions with three superconductors is beyond the scope of this paper and will be reported elsewhere.

V. DECOHERENCE

Let us turn our attention to the effect of electron-hole decoherence that appears as a result of quasiparticles spending a finite dwell time, $\tau_d$, in the junction. Within quantum circuit theory, decoherence is modeled by adding the following term to the action that corresponds to a fictitious circuit element where electron-hole coherence can be dissipated (see Fig. 1).

$$S_f = 2 Re \left\{ \sum_i \sum_p \ln \left[ 1 + \frac{T_p}{2 \tau_d} \frac{(\hat{\vec{n}} \cdot \vec{n}_f - 1)}{1 - \omega^2 - i \omega \tau_d} \right] \right\},$$

(7)

where $\vec{n}_f = (|\Delta|/\hbar)(0,0,-i\omega)$. This additional term modifies only the $z$-component of the vector equation $\vec{\nabla}_{\vec{r}} \Delta(\vec{r},\lambda) = \vec{0}$ that can be eliminated at low energy $\omega \ll 1, (\hbar/|\Delta|\tau_d)$. It does not modify the $x$- and $y$-components, leaving the low energy density of states unchanged. Thus, the boundary of the area in the space of superconducting phases where the gap is absent remains unaffected by decoherence.

On the contrary, away from the limit of low energy, $\omega \gtrsim \min(1, (\hbar/|\Delta|\tau_d))$, decoherence plays an important role in the structure of the density of states, as illustrated in Fig. 6. In the gapless regime, the minigap is replaced by a dip in the density of states. As illustrated in Fig. 6a, the effect of decoherence is to decrease the width of this dip. The dip narrows more rapidly when $\hbar/(|\Delta|\tau_d) \lesssim 1$. In the gapped regime, we recover a similar effect of decoherence as in the two terminal junctions, where the size of the minigap depends strongly on $\hbar/(|\Delta|\tau_d)$ (see Fig. 6c).

Near $\phi = \varphi_1 = -\varphi_3 = \pi$, the density of states exhibits a smile-shaped structure located just below the superconducting gap (see Fig. 6b). The rigorous analysis of non-local processes in metallic junctions with three superconductors is beyond the scope of this paper and will be reported elsewhere.
two-terminal junctions. We note that the structure is present at low decoherence, but vanishes for intermediate decoherence, when \( h/(|\Delta|\tau_d) \approx 1 \) (see Fig. 8b). Further investigation of this structure will be presented elsewhere.

VI. MEASUREMENT

Our predictions can be observed experimentally by means of tunneling spectroscopy. The minimal setup has an additional normal metal lead brought in tunnel contact to the junction, as in Fig. 7a. The differential conductance of the tunnel contact is proportional to the local density of states, \( \rho(eV) \), measured at the applied voltage, provided the contact is sufficiently weak. In this case, the local density of states is not significantly perturbed by the presence of the tunnel probe.

To check this requirement for our three-terminal superconducting junction, we have used the quantum circuit model in Fig. 7a to calculate numerically the tunnel current as a function of bias voltage. The results obtained are presented such that the normalized quantities are independent of the contact transmission coefficient, chosen to \( t_p = 0.01 \) in the numerics.

In Fig. 7c we show that the differential conductance of the tunnel contact plotted as a function of bias voltage reproduces the energy dependence of the density of states plotted in Fig. 2 for the same transport parameters. The signature of the gapless regime can also be observed by measuring the low bias current flowing through the tunnel contact as a function of superconducting phases. Fig. 7d shows that the low bias current matches the dependence depicted in Fig. 2 mapping out the local density of states at zero energy as a function of superconducting phase differences.

The plots in Fig. 7 have been obtained assuming the limit of vanishing temperature. Our theoretical approach is also valid for finite temperatures, assuming the temperature is sufficiently small to permit a well developed proximity effect, \( k_BT \ll \Delta \tau_d^{-1} \). In this temperature window, the local density of states in the junction is independent of temperature.

However, from the viewpoint of the experimental realization, at finite temperature, the features presented in Fig. 7 are blurred by thermal broadening of electronic transport in the normal contact at the scale \( eV \approx k_BT \). If \( \Delta \tau_d^{-1} \gg k_BT \), the thermal broadening does not conceal the transition between the gapped and gapless regimes. The measurement simulated in Fig. 7c can be used as a signature of this transition. The measurement simulated in Fig. 7d does not depend on temperature if the tunnel current can be measured at a voltage bias chosen such that \( eV \ll \min\{|\Delta|,\hbar\tau_d^{-1}\} \) and \( eV \gg k_BT \).

VII. CONCLUSION

In conclusion, we have shown that the minigap in the density of states in a three terminal metallic junction can be closed by varying the superconducting phases. The area in the space of phase differences where the minigap is closed increases with the junction transparency and its shape depends strongly on the asymmetry of the three contacts. These conclusions characterize the low energy physics and are unaffected by electron-hole decoherence in the junction. At large energies, the density of states depends strongly on decoherence, as in two terminal junctions. Our most important result is the possibility to efficiently switch between a regime where the spectrum of states in the junction shows a separation that can be as large as \( |\Delta| \), to a regime where the neighboring states are separated by the smallest energy scale of the junction, the level spacing in the normal metal. This opens interesting opportunities for designs of single fermion manipulation schemes and for the realization of Majorana bound states in multi-terminal superconducting junctions. We propose a tunneling transport experiment that can observe the predicted results and calculate its measurement output.

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Appendix: Exact numerical method

For the exact numerical solution we have used the matrix representation of Green’s functions in the full Keldysh-Nambu space, rather than spectral vectors as in the main text. The Green’s functions of the superconductors are

\[ \hat{G}_i = \left( \begin{array}{cc} \hat{G}_i^R & \hat{G}_i^K \\ 0 & \hat{G}_i^A \end{array} \right) \]  \hspace{0.5cm} (A.1)

\[ \hat{G}_i^R = \frac{1}{\xi} \left( \begin{array}{cc} \Delta_i & \epsilon \\ -\Delta_i^* & -\epsilon \end{array} \right) \]  \hspace{0.5cm} (A.2)

\[ \hat{G}_i^A = -\frac{1}{\xi} \left( \begin{array}{cc} \epsilon^* & \Delta_i \\ -\Delta_i^* & -\epsilon^* \end{array} \right) \]  \hspace{0.5cm} (A.3)

\[ \hat{G}_i^K = \left( \hat{G}_i^A - \hat{G}_i^K \right) \tanh(E/2k_B T). \]  \hspace{0.5cm} (A.4)

where complex energies have been introduced \( \epsilon = E + i\eta^* \) and \( \xi = \sqrt{\epsilon + i\Delta/\sqrt{\epsilon - i\Delta}} \).

The transport properties are encoded into the current matrices flowing from reservoir \( i \),

\[ I_i = \frac{2e^2}{\pi\hbar} \sum_p \frac{T_{p,i}[\hat{G}_i, \hat{G}]_{p,i} - 2}{4 + T_{p,i}([\hat{G}_i, \hat{G}]_{p,i} - 2)}. \]  \hspace{0.5cm} (A.5)

where \( \hat{G} \) denotes the Keldysh-Nambu Green’s function of the node.

Decoherence is described by a matrix current flowing into a fictitious terminal,

\[ I_f = \frac{2e^2}{\pi\hbar} \sum_{i,p} T_{p,i}[\hat{G}_f, \hat{G}_c]_{p,i}. \]  \hspace{0.5cm} (A.6)

\[ \hat{G}_f = \hat{G}_f + \frac{E\tau_3}{\hbar} \left( \begin{array}{cc} 3 & 0 \\ 0 & -3 \end{array} \right). \]  \hspace{0.5cm} (A.7)

where \( \tau \) denotes the vector of Pauli matrices in Nambu space. The choice of \( \hat{G}_f \) is such that \( I_f \) carries no particle or heat currents, but only the information about loss of electron-hole coherence.

The unknown Green’s function \( \hat{G} \) is determined by the balance of current matrices, equivalent to the extremization of the action \( I \) defined in the main text.

\[ \sum_i I_i + I_f = 0. \]  \hspace{0.5cm} (A.8)

It is convenient to express the current balance equation as a commutator between matrix \( \hat{G} \) and a matrix \( \hat{M} \) that depends on all \( \hat{G}_i \) and on the unknown matrix \( \hat{G} \),

\[ [\hat{G}, \hat{M}] = 0, \]  \hspace{0.5cm} (A.9)

\[ \hat{M} = \sum_{i=1,2,3} \sum_p T_{p,i} \frac{\hat{G}_i}{4 + T_{p,i}([\hat{G}_i, \hat{G}] - 2)} + \sum_{i=1,2,3} \sum_p T_{p,i}\hat{G}_f. \]  \hspace{0.5cm} (A.10)
The above equations can be solved numerically using the following iterative algorithm:

1. Start with an initial guess for $\tilde G$.
2. Find the matrix $\tilde P$ that brings both $\tilde G$ and $\tilde M(\tilde G)$ in diagonal form, i.e. $\tilde P$ contains the eigenvectors of $\tilde M$ as its columns.
3. Diagonalize the matrix $\tilde M$, $\tilde M' = \tilde P^{-1} \tilde M \tilde P$.
4. Update the Green’s function of the node $\tilde G^{\text{new}} = \tilde P \, \text{sgn}[\text{Re}(\tilde M')] \, \tilde P^{-1}$.
5. Repeat iteratively until convergence, i.e. matrices $\tilde G$ and $\tilde G^{\text{new}}$ are within a predefined accuracy.
6. Repeat the iterative procedure for every point in energy space.

The solution at a given point in energy space provides a convenient initial guess for the iterative procedure at the next step in energy.