Dark gap solitons in periodic nonlinear media with competing cubic-quintic nonlinearity

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Abstract Solitons are nonlinear self-sustained wave excitations and probably among the most interesting and exciting emergent nonlinear phenomenon in the corresponding theoretical settings. Bright solitons with sharp peak and dark solitons with central notch have been well known and observed in various nonlinear systems. The interplay of periodic potentials, like photonic crystals and lattices in optics and optical lattices in ultracold atoms, with the dispersion has brought about gap solitons within the finite band gaps of the underlying linear Bloch-wave spectrum and, particularly, the bright gap solitons have been experimentally observed in these nonlinear periodic systems, while little is known about the underlying physics of dark gap solitons. Here, we theoretically and numerically investigate the existence, property and stability of one-dimensional gap solitons and soliton clusters in periodic nonlinear media with competing cubic-quintic nonlinearity, the higher-order of which is self-defocusing and the lower-order (cubic) one is chosen as self-defocusing or focusing nonlinearities. By means of the conventional linear-stability analysis and direct numerical calculations with initial perturbations, we identify the stability and instability areas of the corresponding dark gap solitons and clusters ones.

Keywords Dark gap solitons · Cubic-quintic nonlinearity · Nonlinear Schrödinger equation · Bose-Einstein condensate

1 Introduction

It is common knowledge in nonlinear science that the propagation of waves (optical waves, plasma waves, matter waves, etc) presents dispersion or diffraction which makes the waves cannot keep the waveforms unchanged, while the introduction of nonlinearity (of the medium) can do that by balancing the dispersion or diffraction and, accordingly, results into the formation of self-sustained nonlinear waves that what we called solitons [1]. Solitons are widespread nonlinear excitations of the underlying theoretical model in many subjects ranging from optical waves in nonlinear optics, matter waves in ultracold atoms (condensed matters), and water waves in fluids to plasmas waves in plasma, etc [2,3,4]. In the context of one-dimensional (1D) physical settings, the cubic (Kerr) nonlinearity could reach a mutual suppression agreement with dispersion or diffraction, leading to exact soliton solution of the system [2,3,4]. Particularly, the self-focusing nonlinearity is responsible for bright soliton having the form of hyperbolic secant, and the self-defocusing nonlinear term is for dark soliton appearing as hyperbolic tangent. The theoretical frameworks of both bright and dark solitons have been well established [2,3,4,5] and, noteworthy, the ultracold bosonic atoms condensed as Bose-Einstein condensates (BECs) provide a clean and unique platform for their observations [6], manifesting in deliberated experimental results of dark [7,8] and bright matter-wave solitons [9,10,11,12,13] in various ultracold atoms in recent years.

High-order nonlinearity like the quintic nonlinear term [14,15,16,17] also exists in dense media or for high density wave propagation, the combination of which with cubic nonlinearity forms the cubic-quintic (nonlinearity) media [18,19,20,21,22,23], where the high-order nonlinear term takes the self-defocusing, while the cubic term can be either self-focusing or self-defocusing [24,25,26,27,28]. It
should be pointed out that, in two- and three-dimensional models, the competing cubic-quintic nonlinearity of the media can stabilize more complicated wave structures including the multidimensional solitons and their vortical ones (the solitons with embedded vorticity) [25, 29, 30], ring-shaped soliton clusters [31]. In addition, theoretical predictions [32, 33] and experimental confirmations [34, 35, 36] have demonstrated that the competing nonlinear terms, arising from mean-field term (atom-atom collisions) and beyond-mean-field term (quantum fluctuations) named by Lee-Huang-Yang quantum corrections [37, 38, 39], in the context of two-component Bose-Einstein condensates (BECs) could give rise to stable matter wave states called quantum droplets from one to three dimensions.

Periodic potentials, such as photonic crystals and lattices in the context of optics and optical lattices in BECs, have demonstrated as a versatile toolbox for operating and controlling the dynamics of optical and matter waves [3, 4, 5, 6, 40, 41, 42, 43]. Worthwhile mentioning is the generation of bright gap solitons [44, 45, 46, 47, 48] under the condition of self-defocusing nonlinearity which, as mentioned above, admits only the dark solitons and cannot allow the existence of bright solitons in uniform media [49]. The underlying nonlinear physics is that the periodic potentials, counterintuitively, can invert the sign of the effective dispersion of the media, and thus support bright gap solitons [50, 51, 52]. In last decades, the bright gap solitons have been observed in soliton experiments in diverse nonlinear periodic physical systems [53, 54, 55, 56, 57, 58]. Very recently, the formation of solitons has also been proposed and proved experimentally in other periodic potential like photonic moiré lattices that featured by flat-band physics in photorefractive nonlinear media [59, 60].

Although the bright gap solitons have been previously reported in the physical systems of cubic-quintic nonlinearity and optical lattices [46, 47], their opposite nonlinear excitations—the dark gap solitons—however, are yet to be explored in such systems. Furthermore, the dark gap soliton clusters in periodic nonlinear media are currently not being surveyed extensively [61, 62, 63, 64] and, therefore, the underlying soliton physics is yet to be disclosed. Our recent two literatures have respectively examined in detailed the existence and stability of dark gap solitons in the periodic nonlinear media with either cubic nonlinearity [63] or quintic one [64], and striking results were obtained. It is necessary to emphasize that the nonlocal dark solitons have recently been revealed in optical nonlocal nonlinear media with nonlocal cubic-local quintic nonlinearities [65, 66], while their gap counterparts localized in periodic potential are lacking.

This paper is devoted to the theoretical and numerical investigations of the formation and property of 1D dark gap solitons in nonlinear periodic media with competing cubic-quintic nonlinearities, based on the linear-stability analysis and direct perturbed simulations. In contrast to our recent prediction of dark gap solitons in self-defocusing quintic nonlinearity, where they cannot be stabilized within the second finite band gap [64], we demonstrate that the self-defocusing strength in both cubic and quintic nonlinear terms can liberate such constraint, leading to the formation of robust dark gap solitons populated in the second gap of the adopted model. Owning to the fact that the focusing nonlinearity is a disastrous factor for constructing dark solitons, as aforementioned, we conjecture that the focusing-defocusing cubic-quintic model may allow the existence of dark gap solitons while their stability regions within the finite band gaps will reduce quickly; our numerical results verify this fact. Besides the fundamental dark gap solitons, their cluster ones, dark gap soliton clusters composed of several dark gap solitons [63, 67], are also under detailed investigations. The cubic-quintic model applies to the description of propagation of light waves in photonic crystals, and the dynamics of dense BECs trapped onto an optical lattice and under two- and three-body collisions.

2 The model and its theoretical methods

Our physical model describing the evolution of wave function (order parameter) of ultracold atomic gases (BECs of bosonic atoms) having simultaneously the two- and three-body interactions and trapped on an optical lattice induced by counterpropagating laser beams is based on the Gross-Pitaevskii equation (alias nonlinear Schrödinger equation):

$$i \hbar \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V_{\text{trap}}(x)\psi + \frac{g_1}{2\pi a_\perp^2} |\psi|^2 \psi + \frac{g_2}{3\pi^2 a_\perp^4} |\psi|^4 \psi,$$

where the sign and strength of the cubic and quintic nonlinear terms are defined by coefficients $g_1$ and $g_2$, respectively; $a_\perp$ characterizes the s-wave scattering length of two-body (atom-atom) collisions and $a_\perp^4$ is for three-body interaction.

The linear optical trap $V_{\text{trap}}(x)$ will be given below. For the sake of discussion, the above equation can be generalized to a dimensionless version by substituting with the variables $g_1 = 4\pi \hbar^2 a_s/m$, $t = \hbar \tau/m$, $a_s = ga_{so}$, $U = \psi \sqrt{a_\perp^2 / 2a_{so}}$, $V(x) = V_{\text{trap}}(x)m/\hbar^2$, and $\gamma = mg_2/12\pi^2 \hbar^2 a_{so}^2$, then the dimensionless model for wave function $\psi$ yields:

$$i \frac{d\psi}{dt} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(\psi) \psi + g |\psi|^2 \psi + \gamma |\psi|^4 \psi,$$

where the linear periodic potential is taken as an optical lattice having the expression

$$V(x) = V_0 \sin^2 (x).$$
In the following analysis, the five-order (quintic) nonlinear strength in Eq. (1) is taken as a defocusing one $\gamma = 1$, and the three-order (cubic) nonlinear coefficient can be chosen as either focusing ($g < 0$) or defocusing ($g > 0$). It should be noted that, as previously described, besides the BECs with two- and three-body collisions, the cubic-quintic model can be applied to other contexts including nonlinear optics too.

The wave function at definite chemical potential $\mu$ can be rewritten as $\psi = \phi \exp(-i\mu t)$, with the substitution of it in Eq. (1) leads to the stationary equation for the stationary solution $\phi$:

$$\mu \phi = \frac{1}{2} \frac{d^2}{dx^2} \phi + V \phi + g |\phi|^2 \phi + |\phi|^4 \phi.$$  

(3)

To investigate the localized dark gap modes in the finite gaps of the underlying model Eq. (1) with an optical lattice $V(x) = V_0 \sin^2(x)$ [Eq. (2)], the linear Bloch-wave spectrum should be firstly given, which is implemented by linearizing the Eq. (1) or Eq. (3) and by using the Floquet-Bloch theory. The corresponding structures of Bloch bands and gaps are portrayed in Fig. 1(a), from which one can observe the widening feature of first and second finite gaps with an increase of lattice depth $V_0$; and at a given value of depth to be considered ($V_0 = 4$), there is a wide first gap, as shown in Fig. 1(b) [and from a marked dashed line in Fig. 1(a)].

The stability of the soliton solution in [Eq. (3)] is usually done by means of linear-stability analysis, which can be executed by analyzing the influence of initial small perturbations to the stationary state. To this aim, the wave function should be taken as perturbed one

$$\psi(x, t) = \phi(x) + u(x) e^{i\mu t} + v^*(x) e^{i\chi t},$$

(4)

with $u(x)$ and $v(x)$ being tiny perturbations at a certain eigenvalue $\lambda$. Following from such expression, the linear eigenvalue problem of the solution can be gained through Eq. (1), which reads as:

$$i\lambda u = -\frac{1}{2} \frac{d^2}{dx^2} u + (V - \mu) u + g \chi_3(u, v) \phi^2 + \chi_5(u, v) \phi^4,$$

(5)

$$i\lambda v = -\frac{1}{2} \frac{d^2}{dx^2} v - (V - \mu) v - g \chi_3(v, u) \phi^2 - \chi_5(v, u) \phi^4.$$  

(6)

In the above eigenvalue equations, the parameters in front of cubic and quintic nonlinear terms yield $\chi_3(m, n) = n + 2m$ and $\chi_5(m, n) = 2n + 3m$. It is then the stability property of the soliton solution against the initial perturbations is defined by the eigenvalue $\lambda$: the criterion is that the soliton is stable provided that $\text{Re}(\lambda) = 0$, and is unstable otherwise.

To proceed with the numerical results, we would like to outline the numerical recipes we used. First, the stationary soliton solution is acquired in Eq. (3) using the Newton-Rapson iteration; then the stability of the dark gap soliton is judged by linear-stability eigenvalue Eqs. (5) and (6) by means of the conventional finite-difference method, and is double-checked in direct simulation of the perturbed soliton in evolutional Eq. (1) with fourth-order Runge-Kutta method.

3 Numerical results and discussion

3.1 Fundamental dark gap solitons

As has been said above, the defocusing nonlinearity is responsible for the generation of ordinary dark solitons; in order to comply with this principle, we first consider the defocusing Kerr (cubic) nonlinear strength $g > 0$ in the cubic-quintic model proposed here [Eq. (1)], recall that the quintic term has been set to defocusing throughout.

Typical shapes of fundamental dark gap solitons (featured by single notch in the center) under defocusing cubic ($g = 1$) and defocusing quintic nonlinearities and an optical lattice are shown in Fig. 1(c), where the case of second finite band gap is also included. From such panel, we can see that, compared to its counterpart in first finite gap, the dark gap soliton in higher band gap has a larger amplitude and nonlinear Bloch wave background on account of larger Bragg reflections existed in the second gap, resembling those reported in the previous studies in defocusing cubic-or quintic-only model. For a bigger cubic term, e.g., $g = 2$, the soliton’s height decreases, exemplified by their profiles within both the first and second gaps in Fig. 1(d).
Although the dark gap solitons can only be limited to the nonlinear periodic media with defocusing nonlinearity, in the cubic-quintic model considered here, it is natural to ask whether such localized gap modes can be stable object in the system with combined focusing cubic \((g < 0)\) and defocusing quintic nonlinearities. Our answer is affirmative, as demonstrated by their cases at focusing cubic nonlinear strength \((g = -1)\) in Fig. 1(e). The underlying soliton physics is natural and understandable, taking into consideration in such circumstance, the overall nonlinear term behaves as defocusing, since the focusing cubic nonlinearity is always overwhelmed by the defocusing higher-order (quintic) nonlinearity. Otherwise, the dark gap solitons cannot be formed at all, required by the phase transition at the central notch for the dark gap solitons. On the contrary, there is no such red tape for their bright counterparts—bright gap solitons—because not phase transition happens for them, as previous literatures have proved the existence of stable bright gap solitons in nonlinear periodic media under the focusing cubic nonlinearity. For a focusing cubic strength, the overall defocusing nonlinear effect of combined cubic-quintic nonlinearities reduces, making the relevant dark gap solitons have larger amplitudes, comparing the Fig. 1(e) to Figs. 1(c) and 1(d).

A detailed insight into the relationship between the number of atoms (norm) \(N\) and chemical potentials \(\mu\) of the dark gap solitons can be built, which is accumulated in Fig. 2 where, three classes of cubic nonlinear strength \((g)\) given by two different defocusing cases \((g = 1\) and \(g = 2\)) and one focusing case are respectively shown in Figs. 2(a), 2(b) and 2(c). Showings are also for the stability and instability regions of dark gap solitons, on the basis of linear stability analysis [Eqs. (5) and (6)] and direct perturbed evolutions [Eq. (1)] of the soliton solutions; the instability appears only when \(\mu\) approaches the edges of the first and second finite gaps [Figs. 2(a) and 2(b)], and when the defocusing cubic nonlinear term switches to the focusing one [Fig. 2(c)]. For the two defocusing cases, our cubic-quintic model is a complete defocusing one, the noteworthy result is the existence of stable dark gap solitons in the second finite band gap of the underlying linear Bloch-wave spectrum; opposing to this, the dark gap solitons can hardly be stabilized in second gap, as reported very recently in defocusing quintic-only model. For the model with competing focusing cubic and defocusing quintic nonlinearities, as conjugated above, shortens the corresponding stability regions of dark gap solitons within the first gap, and does not allow the formation of stable dark gap modes in the second gap, as can be observed from Fig. 2(c). One important feature of dark gap solitons in the various cubic-quintic model deserved to be pointed out is that, as from the dependencies \(N(\mu)\) in Fig. 2, the stability criterion for localized gap modes in diverse nonlinear periodic optical and matter-wave media, the ‘anti-Vakhitov–Kolokolov’ (anti-VK) criterion, \(dN/d\mu > 0\) still prevails [46, 63, 64, 68, 69, 70].

Two examples of dark gap solitons in the circumstances of double defocusing cubic-quintic model and focusing cubic-quintic (defocusing) model are depicted in Figs. 3(a) and 3(b) respectively. Our linear stability analysis of these two soliton solutions based on Eqs. (5) and (6) reveals that one in first band gap is stable and another in second gap is unstable, verified by their direct perturbed simulations in the bottom panels of Fig. 3, where shows respectively the good coherence of stable dark gap solitons and the quickly decoherence of unstable one in the course of their evolutions.

### 3.2 Higher-order localized modes: dark gap soliton clusters

Next, we proceed to survey the possibility of generating 1D higher-order localized modes consisting of several dark gap solitons with equal spacing between each individual, such higher-order modes are usually named dark gap soliton clusters [63, 67], which we are adopted here. As a sample and an example of such higher-order modes, we take our focus on five-soliton clusters configured as five identical fundamental
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4 Conclusion

We have addressed the existence, property and stability of 1D dark gap solitons and soliton clusters in periodic nonlinear media with competing cubic-quintic nonlinearities in the form of double defocusing nonlinear terms and focusing-quintic nonlinear terms. The stability of these dark gap solitons is constrained within the first and second finite band gaps of the underlying linear Bloch-wave spectrum, with stability regions that are determined by the signs and relative strengths of the nonlinear terms. The stability of larger clusters is constrained in a similar manner, with stability regions appearing at first and second gaps.

The existence and stability of these dark gap solitons has important applications in areas such as optical communications and nonlinear optics, where they can be used to transmit information in a robust manner. These solitons can be used to improve the efficiency and reliability of optical communication systems by offering a way to transmit data over long distances with minimal distortion.

The properties of these dark gap solitons, such as their stability and interaction with other solitons, are further explored in the following sections, with implications for the design and implementation of new communication technologies. The results presented here provide a foundation for future research in this area, including the development of new methods for controlling and manipulating these solitons for practical applications.
defocusing nonlinear arrangement. Linear stability analysis method is used to judge the stability and instability of the dark gap localized modes against small initial background perturbations and, particularly, which can be balanced with the direct perturbed simulations of the localized modes. We find that the stable fundamental dark gap solitons, produced in the double defocusing cubic-quintic model, can exist in the second finite band gap of the underlying linear spectrum; on the contrary, our recent theoretical work predicted that they can hardly be stabilized in second gap under the quintic-only model. In terms of the model with competing self-focusing cubic and defocusing quintic nonlinearities, the stability regions of dark gap solitons shrink and are only limited to the first finite band gap. For all cases of competing cubic-quintic nonlinearities, the dark gap soliton clusters can be stable physical objects in first gap. The model may be realized in the context of nonlinear optics with periodic cubic-quintic nonlinear medium and in BECs trapped by an optical lattice.

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Conflict of interest

The authors declare no conflicts of interest.

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