Context-Free Grammars with Storage

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Abstract. Context-free $S$ grammars are introduced, for arbitrary (storage) type $S$, as a uniform framework for recursion-based grammars, automata, and transducers, viewed as programs. To each occurrence of a nonterminal of a context-free $S$ grammar an object of type $S$ is associated, that can be acted upon by tests and operations, as indicated in the rules of the grammar. Taking particular storage types gives particular formalisms, such as indexed grammars, top-down tree transducers, attribute grammars, etc. Context-free $S$ grammars are equivalent to pushdown $S$ automata. The context-free $S$ languages can be obtained from the deterministic one-way $S$ automaton languages by way of the delta operations on languages, introduced in this paper.

Foreword

This is a slightly revised version of a paper that appeared as Technical Report 86-11 of the Department of Computer Science of the University of Leiden, in July 1986. Small errors are corrected and large mistakes are repaired. Here and there the wording of the text is improved. The references to the literature are updated, and a few New Observations are added. However, I have made no effort to bring the paper up-to-date. New references to the literature are indicated by a *. The results of Section 8 were published in [*Eng11].

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Introduction

Context-free grammars, as formally defined by Chomsky, are a very particular type of rewriting system. However, the reason for their popularity is that they embody the idea of recursion, in its simplest form. A context-free grammar is really just a nondeterministic recursive program that generates or recognizes strings. For instance, the context-free grammar rules

\[ A \rightarrow aBbCD \]
\[ A \rightarrow b \]

can be understood as the following program piece (in case the grammar is viewed as a string generator)

```plaintext
procedure A;
begin
    write(a); call B; write(b); call C; call D
end
```

or

```plaintext
begin
    write(b)
end
```

where “write(a)” means “write a on the output tape”. If the grammar is viewed as a (non-deterministic) string recognizer or acceptor, as is usual in recursive descent parsing, “write” should be replaced by “read” (where “read(a)” means “read a from the input tape”). Thus, context-free grammars are recursive programs; the nonterminals \( A, B, \ldots \) are procedures (without parameters), and all the rules with left-hand side \( A \) constitute the body of procedure \( A \); the main program consists of a call of the main procedure, i.e., the initial nonterminal. Actually, it is quite funny that in formal language theory some programs are called grammars (as suggested here), and other programs are called automata: e.g., a program with one variable, of type pushdown, is called a pushdown automaton. Maybe the underlying idea is: nonrecursive program = automaton, recursive program = grammar.

Given that context-free grammars consist of recursive procedures without parameters, what would happen if we generalize the concept by allowing parameters? This is an idea that has turned up in several places in the literature. Since the resulting formalisms are usually still based on the idea of recursive procedures, such context-free grammars with parameters are easy to understand, construct, and prove correct, just as ordinary context-free grammars. Here, we will fix this idea as follows: each nonterminal of the context-free grammar will have one (input) parameter of a given type. Although this is a very simple case of the general idea, we will show that the resulting formalism of “context-free \( S \) grammars” (where \( S \) is the type of the parameter) has its links with several existing formalisms; this is due partly to the fact that such a generalized context-free grammar may be viewed as a grammar, an automaton, a program, or a transducer, and partly to the freedom in the choice of \( S \). In this way we will see that the following formalisms can be “explained”, each formalism corresponding to the context-free \( S \) grammars for a specific type \( S \): indexed grammars, top-down tree transducers, ETOL systems, attribute grammars, macro grammars, etc. Moreover, viewing the type \( S \) as the storage type of an automaton (e.g., \( S = \) pushdown, \( S = \) counter, etc.) context-free \( S \) grammars can be used to model all one-way \( S \) automata, where \( S \) is any type of storage, and all alternating \( S \) automata, where \( S \) is any type of [storage plus input]. In particular, it should be clear that right-linear \( S \) grammars correspond to one-way \( S \) automata (just as, classically, right-linear grammars correspond to one-way finite automata). To stress this link to automata,
context-free $S$ grammars are also called “grammars with storage” or even “recursive automata”, and the type $S$ is also called a storage type. Thus this programming paradigm strengthens the similarities between automata and grammars. Tree grammars and tree automata (such as the top-down tree transducer) can be obtained either by defining an appropriate type $S$ of trees, or by considering context-free $S$ grammars that generate trees, or both. Note that trees, in their intuitive form of expressions, are particular strings.

Two main results are the following.

(1) Context-free $S$ grammars correspond to pushdown $S$ automata. A pushdown $S$ automaton, introduced in [Gre], is an iterative program that manipulates a pushdown of which each pushdown cell contains a symbol and an object of type $S$. This is of course the obvious way to implement recursive procedures with one parameter, and so the result is not surprising. What is nice about it, however, is that it provides pushdown-like automata, in one stroke, for all formalisms that can be explained as context-free $S$ grammars (e.g., we will obtain pushdown$^2$ automata for indexed grammars, tree-walking pushdown transducers for top-down tree transducers, checking-stack/pushdown automata for ETOL systems, etc.). To deal with determinism (of the context-free $S$ grammar, viewed as a transducer) the notion of look-ahead (on the storage $S$) is introduced, as a generalization of both look-ahead on the input (in parsing, and for top-down tree transducers) and the predicting machines of [HopUll].

(2) Apart from this automaton characterization of context-free $S$ grammars we also give a characterization by means of operations on languages (cf. AFL/AFA theory [Gin]), but only for rather specific $S$ (including iterated pushdowns). We define a new class $\delta$ of “delta” operations on languages, such that the languages generated by context-free $S$ grammars can be obtained by applying the delta operations to the languages accepted by deterministic(!) one-way $S$ automata. A delta operation is quite different from the usual operations on languages; it takes a (string) language, views the strings as paths through labeled trees, constructs a tree language out of these paths, and then produces a (string) language again by taking the yields of these trees. As an example, the indexed languages are obtained by the delta operations from the deterministic context-free languages.

Thus, the aim of this paper is to provide a uniform framework for grammars and automata that are based on recursion, including the usual one-way automata as special case. The general theory that is built in this framework should give transparent (and possibly easier) proofs of results for particular formalisms, i.e., for particular $S$. In fact, this paper may be viewed as an extension of abstract automata theory to recursive automata, i.e., as a new branch of AFA/AFL theory [Gin, Gre]. The above two main results constitute a modest beginning of such a general theory for context-free $S$ grammars; more can be found in [EngVog2, EngVog3, EngVog4, DamGue, Vog1, Vog2, Vog3, Eng9]. Although these papers are based on the first, very rough, version of this paper ([Eng8]), we will now feel free to mention results from them.

How to read this paper

Since rather many formalisms will be discussed in this paper, the reader is not expected to know them all. However, this paper is not a tutorial, and so, whenever a formalism is discussed, it is assumed that the reader is more or less familiar with it. If he is not, he is therefore advised
to skip that part of the paper, or read it with the above in his mind.\(^1\) Hopefully the paper is written in such a way that skipping is easy.

The reader who is interested in the expressiveness of the context-free S grammar formalism only, should read Sections 1, 2, 3, 4, and 6 (after glancing at Section 5), or parts of them. The reader who is interested in the theory of context-free S grammars only, can restrict himself to Sections 1.1, 5, 7, and 8.

**Organization of the paper**

Context-free S grammars are defined in Section 1.1. They are compared with attribute grammars in Section 1.2, which can be skipped without problems. In Section 2 two particular cases are defined: regular grammars and regular tree grammars, both with storage S. It is argued that these correspond to one-way automata and top-down tree automata, both with storage S, respectively. The reader is advised to at least glance through this section. Section 3 is divided into 8 parts; in each part a specific storage type S is defined (e.g., S = Pushdown in the second part), and it is shown how the resulting context-free S grammars relate to existing formalisms. Although these parts are not completely independent, it should be easy to skip some of them. The relationship between context-free S grammars and alternating automata is contained in Section 4, which can easily be skipped (it is needed in Section 6(9) only). In Section 5 we start the theory and show the first main result mentioned above: the relation to pushdown S automata. Then, in Section 6, it is shown how this gives pushdown-like automata for all the formalisms discussed in Section 3. Thus, Section 6 is divided into the same 8 parts as Section 3, according to the storage type, with one additional part concerning alternating automata. Section 7 is a technical section devoted to determinism, as needed for Section 8. Section 8 contains the second main result mentioned above: the characterization of context-free S grammars by means of the delta operations.

**Notation**

Before we start, we mention some elementary notation. We assume the reader to be familiar with formal language theory, see [HopUll, Sal, Har, Ber], and, to a much lesser extent, with tree language theory, see [GécSte, Eng1]. We denote by REG, CF, DCF, Indexed, and RE, the classes of regular, context-free, deterministic context-free, indexed, and recursively enumerable languages, respectively. (RT denotes the class of regular tree languages, also called recognizable tree languages.)

For a set \(A\), \(A^*\) is the set of strings over \(A\). For \(w \in A^*\), \(|w|\) denotes the length of \(w\). The empty string is denoted \(\lambda\), and \(A^+ = A^* - \{\lambda\}\). (In ranked alphabets, \(\varepsilon\) is a symbol of rank 0 denoting \(\lambda\), in the sense that \(\text{yield}(\varepsilon) = \lambda\).)

For a relation \(\tau\), \(\tau^*\) is its reflexive, transitive closure, \(\text{dom}(\tau)\) is its domain, and \(\text{ran}(\tau)\) is its range. For a set \(A\), \(\text{id}(A)\) denotes the identity mapping \(A \to A\). An ordered pair \((\varphi, \psi)\) of objects \(\varphi\) and \(\psi\) will also be denoted \(\varphi(\psi)\), not to be confused with function application. For sets \(\Phi\) and \(\Psi\), both \(\Phi \times \Psi\) and \(\Phi(\Psi)\) will be used to denote their cartesian product \(\{\varphi(\psi) \mid \varphi \in \Phi, \psi \in \Psi\}\).

\(^1\)New Observation. The reader is asked to consider the word “he” to stand for “he/she”, and the word “his” for “his/her”. My personal ideal is to remove all female forms of words from the language, and to let the male forms refer to all human beings.
1 Context-free $S$ grammars

1.1 Examples and definitions

To give the reader an idea of what context-free $S$ grammars are, let us first discuss three simple examples.

In the first grammar, $G_1$, there is just one nonterminal $A$, with one parameter of type integer (i.e., $S = \text{Integer}$), and there is one terminal symbol $a$. The two rules of the grammar are

$$A(x) \rightarrow \text{if } x \neq 0 \text{ then } A(x-1)A(x-1)$$
$$A(x) \rightarrow \text{if } x = 0 \text{ then } a$$

where $x$ is a formal parameter. The meaning of the first rule is that, for any integer $n$, $A(n)$ may be rewritten as $A(n-1)A(n-1)$, provided $n \neq 0$; and similarly for the second rule: $A(0)$ may be rewritten by $a$. Thus, for $n \geq 0$, $A(n)$ generates $a^{2^n}$. We may view $G_1$

(1) as a grammar generating the language $L(G_1) = \{a^{2^n} \mid n \geq 0\}$ (the input $n$ is chosen nondeterministically),

(2) as a nondeterministic acceptor that recognizes $L(G_1)$ (cf. the Introduction; $n$ is again chosen nondeterministically),

(3) as a deterministic transducer that translates $n$ into $a^{2^n}$, and, finally,

(4) as a deterministic acceptor of all integers $n \geq 0$ (the domain of the translation).

These four points of view will be taken for all context-free $S$ grammars. The grammar $G_1$ is not such a good example for the 4th point of view; a better example will be given in Section 4 (viz., $G_6$). From all four points of view, $G_1$ (and any other context-free $S$ grammar) can be thought of as a program, similar to the one for the context-free grammar in the Introduction. As a transducer, $G_1$ corresponds intuitively to the program

```
procedure A(x: integer);
        begin if x \neq 0
              then call A(x-1); call A(x-1)
              else write(a)
        fi
{main program}
obtain n;
call A(n)
```

where “obtain n” means “read $n$ from an input device”. If $G_1$ is a generator of $L(G_1)$, “obtain n” means “choose an integer $n$”. If $G_1$ is an acceptor of $L(G_1)$, then “obtain n” again means “choose an integer $n$”, and, in the program, “write(a)” should be replaced by “read(a)”, as observed for the context-free grammar in the Introduction. If $G_1$ is an acceptor of the nonnegative integers, then “write(a)” can be replaced by “skip” (i.e., terminals do not matter).

The second grammar, $G_2$, generates the language $\{a^n b^n c^n \mid n \geq 0\}$. It has nonterminals $A_{in}, A, B, C$, each having a parameter of type pushdown (i.e., $S = \text{Pushdown}$); the pushdown
symbols are \( a \) and \( \# \) (the bottom marker). The derivations of \( G_2 \) start with \( A_{\text{in}}(\#) \); the initial call of the main procedure. The rules of the grammar are the following, where the tests and operations on the pushdown should be obvious (and \( \lambda \) denotes the empty string).

\[
\begin{align*}
A_{\text{in}}(x) & \rightarrow A(x) \\
A(x) & \rightarrow aA(\text{push } a \text{ on } x) \\
A(x) & \rightarrow B(x)C(x) \\
B(x) & \rightarrow \text{if } \text{top}(x) = a \text{ then } bB(\text{pop } x) \\
B(x) & \rightarrow \text{if } \text{top}(x) = \# \text{ then } \lambda \\
C(x) & \rightarrow \text{if } \text{top}(x) = a \text{ then } cC(\text{pop } x) \\
C(x) & \rightarrow \text{if } \text{top}(x) = \# \text{ then } \lambda 
\end{align*}
\]

The unconditional rules may always be used, the conditional rules only if their tests are true. From \( A_{\text{in}}(\#) \), \( G_2 \) generates nondeterministically \( a^nA(a^n\#) \), followed by \( a^nB(a^n\#)C(a^n\#) \), where the top of the pushdown is at the left. Then \( B(a^n\#) \) and \( C(a^n\#) \) generate deterministically \( b^n \) and \( c^n \), respectively. Thus, \( A_{\text{in}}(\#) \) generates all strings \( a^n b^n c^n \). Note that dropping \( C(x) \) from the third rule gives a right-linear grammar (with pushdown parameter), generating \( a^n b^n \); clearly, as an acceptor of \( \{ a^n b^n \mid n \geq 0 \} \), this grammar is just an ordinary nondeterministic pushdown automaton.

Another way to generate the language \( \{ a^n b^n c^n \mid n \geq 0 \} \) is by a grammar \( G_3 \) that is almost the same as \( G_2 \). The first three rules of \( G_2 \) should be replaced by

\[
\begin{align*}
A_{\text{in}}(x) & \rightarrow A(x)B(x)C(x) \\
A(x) & \rightarrow \text{if } \text{top}(x) = a \text{ then } aA(\text{pop } x) \\
A(x) & \rightarrow \text{if } \text{top}(x) = \# \text{ then } \lambda 
\end{align*}
\]

But now the idea is that any integer \( n \geq 0 \) can be taken as input, encoded as a pushdown \( a^n \# \). The grammar \( G_3 \) starts the derivation with \( A_{\text{in}}(a^n\#) \), and deterministically generates \( a^n b^n c^n \). Thus \( G_3 \) translates \( n \) into \( a^n b^n c^n \), and generates (or accepts) the language \( \{ a^n b^n c^n \mid n \geq 0 \} \).

Let us now turn to the formal definitions. They are inspired by Ginsburg and Greibach [Gin], who developed a general theory of automata that is based on the separation of storage and control. We will try to make our notation as readable as that of Scott, who also started such a theory [Sco]. We begin with the definition of type, i.e., of the possible types of the parameter. Because of the intuitive connection to automata we will talk about storage type rather than type. A storage type is a set of objects (called the storage configurations), together with the allowed tests and operations on these objects. Since each nonterminal has only one parameter, we may restrict ourselves to unary tests and operations. But that is not all. Since our context-free \( S \) grammars will also be viewed as transducers, it is necessary to specify a set of input elements, together with the possibility to encode them as storage configurations (e.g., in \( G_3 \), integer \( n \) is encoded as pushdown \( a^n \# \)). But, in general, different transducers may use different encodings (e.g., we should also have the freedom to encode \( n \) as \( b^n \infty \)). Thus, a set of possible encodings is specified.
Definition 1.1. A storage type $S$ is a tuple $S = (C, P, F, I, E, m)$, where $C$ is the set of configurations, $P$ is the set of predicate symbols, $F$ is the set of instruction symbols, $I$ is the set of input elements, $E$ is the set of encoding symbols, and $m$ is the meaning function that associates with every $p \in P$ a mapping $m(p) : C \to \{\text{true, false}\}$, with every $f \in F$ a partial function $m(f) : C \to C$, and with every $e \in E$ a partial function $m(e) : I \to C$.

We let $\text{BE}(P)$ denote the set of all boolean expressions over $P$, with the usual boolean operators and, or, not, true, and false. For $b \in \text{BE}(P)$, $m(b) : C \to \{\text{true, false}\}$ is defined in the obvious way. The elements of $\text{BE}(P)$ are also called tests.

We will also say “predicate $p$” instead of “predicate symbol $p$”, with the intention to talk about $p$ and $m(p)$ at the same time (when the distinction is not so important), and similarly for “instruction” and “encoding”.

Next we give our main definition: that of context-free $S$ grammar, for any storage type $S$. However, to remain as general as possible, we will call it a context-free $S$ transducer (but also grammar and acceptor, depending on the point of view).

First a remark on the notation of rules. Since all nonterminals, predicate symbols, and instruction symbols always have one formal argument, we will drop "(x)" from our formal notation. Thus, the rules of $G_1$ can first be written as

$$
A(x) \rightarrow \text{if not null(x) then } A(\text{dec(x)})A(\text{dec(x)}) \\
A(x) \rightarrow \text{if null(x) then } a
$$

where $\text{null(x)}$ and $\text{dec(x)}$ stand for $x = 0$ and $x - 1$, respectively ($\text{null}$ is a predicate symbol, and $\text{dec}$ is an instruction symbol of the storage type Integer). And then they can be written, formally, as

$$
A \rightarrow \text{if not null then } A(\text{dec})A(\text{dec}) \\
A \rightarrow \text{if null then } a
$$

The definition now follows. Recall that, for objects $\varphi$ and $\psi$, $\varphi(\psi)$ is just another notation for the ordered pair $(\varphi, \psi)$; similarly, $\Phi(\Psi)$ is another notation for $\Phi \times \Psi$. This is done to formalize $A(\text{dec})$ as an ordered pair $(A, \text{dec})$, but keep the old notation.

Definition 1.2. Let $S = (C, P, F, I, E, m)$ be a storage type. A context-free $S$ transducer, or CF($S$) transducer, is a tuple $G = (N, e, \Delta, A_{\text{in}}, R)$, where $N$ is the nonterminal alphabet, $e \in E$ is the encoding symbol, $\Delta$ is the terminal alphabet (disjoint with $N$), $A_{\text{in}} \in N$ is the initial nonterminal, and $R$ is the finite set of rules; every rule is of the form

$$
A \rightarrow \text{if } b \text{ then } \xi
$$

with $A \in N$, $b \in \text{BE}(P)$, and $\xi \in (N(F) \cup \Delta)^*$.

The set of total configurations (or, instantaneous descriptions) is $(N(C) \cup \Delta)^*$. The derivation relation of $G$, denoted by $\Rightarrow_G$ or just $\Rightarrow$, is a binary relation on the set of total configurations, defined as follows:
if \( A \rightarrow \text{if } b \text{ then } \xi \) is in \( R \), \( m(b)(c) = \text{true} \), and \( m(f)(c) \) is defined for all \( f \) that occur in \( \xi \), then \( \xi_1 A(c) \xi_2 \Rightarrow_G \xi_1 \xi' \xi_2 \) for all total configurations \( \xi_1 \) and \( \xi_2 \), where \( \xi' \) is obtained from \( \xi \) by replacing every \( B(f) \) by \( B(m(f)(c)) \).

The translation defined by \( G \) is
\[
T(G) = \{(u, w) \in I \times \Delta^* \mid A_{\text{in}}(m(e)(u)) \Rightarrow_G^* w\}.
\]

Note that \( T(G) \subseteq \text{dom}(m(e)) \times \Delta^* \).

The language generated (or, \( r \)-accepted) by \( G \) is
\[
L(G) = \text{ran}(T(G)) = \{w \in \Delta^* \mid A_{\text{in}}(m(e)(u)) \Rightarrow_G^* w \text{ for some } u \in I\}.
\]

The input set \( d \)-accepted by \( G \) is
\[
A(G) = \text{dom}(T(G)) = \{u \in I \mid A_{\text{in}}(m(e)(u)) \Rightarrow_G^* w \text{ for some } w \in \Delta^*\}.
\]

The \( \text{CF}(S) \) transducer \( G \) is (transducer) deterministic if for every \( c \in C \) and every two different rules \( A \rightarrow \text{if } b_1 \text{ then } \xi_1 \) and \( A \rightarrow \text{if } b_2 \text{ then } \xi_2 \), \( m(b_1 \text{ and } b_2)(c) = \text{false} \). \( \square \)

The corresponding classes of translations, languages, and input sets, are defined by \( \tau-\text{CF}(S) = \{T(G) \mid G \text{ is a } \text{CF}(S) \text{ transducer}\} \), \( \lambda-\text{CF}(S) = \{L(G) \mid G \text{ is a } \text{CF}(S) \text{ transducer}\} \), and \( \alpha-\text{CF}(S) = \{A(G) \mid G \text{ is a } \text{CF}(S) \text{ transducer}\} \). Moreover, \( \tau-\text{DCF}(S) = \{T(G) \mid G \text{ is a deterministic } \text{CF}(S) \text{ transducer}\} \), and similarly for \( \lambda-\text{DCF}(S) \) and \( \alpha-\text{DCF}(S) \). Note that this \( \lambda \) has nothing to do with the empty string.

Let us discuss some notational conventions concerning rules. A rule \( A \rightarrow \text{if true then } \xi \) will be abbreviated by \( A \rightarrow \xi \) (these are the unconditional rules we saw in \( G_2 \) and \( G_3 \)). A rule \( A \rightarrow \text{if false then } \xi \) may be omitted. Obviously, if \( b \) and \( b' \) are equivalent boolean expressions, then they are interchangeable as tests of rules. Thus, if \( S \) is a storage type with \( P = \emptyset \), we may assume that all rules are unconditional. We write \( A \rightarrow \text{if } b \text{ then } \xi_1 \text{ else } \xi_2 \) as an abbreviation of the two rules \( A \rightarrow \text{if } b \text{ then } \xi_1 \) and \( A \rightarrow \text{if } \text{not } b \text{ then } \xi_2 \) (cf. the program for \( G_1 \)); the two rules of \( G_1 \) could be written \( A \rightarrow \text{if } \text{null then } \text{a else } A(\text{dec})A(\text{dec}) \). A rule \( A \rightarrow \text{if } b_1 \text{ or } b_2 \text{ then } \xi \) may be replaced by the two rules \( A \rightarrow \text{if } b_1 \text{ then } \xi \) and \( A \rightarrow \text{if } b_2 \text{ then } \xi \). Thus, using the disjunctive normal form of boolean expressions, we may always assume that all tests in rules are conjunctions of predicate symbols and negated predicate symbols. We will allow rules \( A \rightarrow \text{if } b \text{ then } \xi \) with \( \xi \in (N(F^+) \cup \Delta)^* \), where concatenation in \( F^+ \) is denoted by a semicolon. A rule \( A \rightarrow \text{if } b \text{ then } \cdots B(f_1; f_2; \ldots; f_k) \cdots \) abbreviates the rules \( A \rightarrow \text{if } b \text{ then } \cdots B_1(f_1) \cdots, B_1 \rightarrow B_2(f_2), \ldots, B_{k-1} \rightarrow B_k(f_k), \) where \( B_1, \ldots, B_{k-1} \) are new nonterminals.

Let \( G \) be a \( \text{CF}(S) \) transducer. Whenever we view \( G \) in particular as a generator of \( L(G) \), we will call \( G \) a context-free \( S \) grammar, or \( \text{CF}(S) \) grammar. Similarly, when viewing it as a recognizer of \( L(G) \), as discussed before, we call it a context-free \( S \) \( r \)-acceptor, or \( \text{CF}(S) \) \( r \)-acceptor (where \( r \) abbreviates range). But, of course, \( G \) can also be viewed as an acceptor of \( A(G) \); in that case we call it a context-free \( S \) \( d \)-acceptor, or \( \text{CF}(S) \) \( d \)-acceptor (where \( d \) abbreviates domain). Note that in this case the terminal alphabet \( \Delta \) of \( G \) is superfluous, i.e., we may assume that \( \Delta = \emptyset \).

In the definition of \( \text{CF}(S) \) transducer we took the usual terminology for grammars. From the point of view of recursive automata, i.e., for transducers and acceptors, it would be more appropriate to call the elements of \( N \) states, \( A_{\text{in}} \) the initial state, \( \Delta \) the output alphabet (or the
input alphabet, for r-acceptors), and to formalize \( R \) as a transition function. The derivations of the transducer would then be called computations; these computations start with \( A_{\text{in}}(c) \) where \( c \) is an initial configuration, i.e., an element of \( \text{ran}(m(e)) \). As said before, from the point of view of programs, \( N \) is the set of procedure names, \( A_{\text{in}} \) is the main procedure, and \( R \) the program (consisting of the procedure declarations).

The notion of (transducer) determinism, as defined above, is what one would expect for transducers and d-acceptors (and, perhaps, grammars). Obviously, for a deterministic \( CF(S) \) transducer \( G, T(G) \) is a partial function from \( I \) to \( \Delta^* \). For r-acceptors, this notion is too strong, because the terminals should also be involved; r-acceptor determinism will be considered in Section 7.

As an example we now give a complete formal definition of \( G_1 \). First we define the storage type Integer = \((C, P, F, I, E, m)\), where \( C \) is the set of integers, \( P = \{\text{null}\} \), \( F = \{\text{dec}\} \), \( I = C \), \( E = \{\text{en}\} \); for every \( c \in C \), \( m(\text{null})(c) = (c = 0) \) and \( m(\text{dec})(c) = c - 1 \); and \( m(\text{en}) = \text{id}(C) \), the identity on \( C \). Second we define the \( CF(\text{Integer}) \) transducer \( G_1 = (N, e, \Delta, A_{\text{in}}, R) \) where \( N = \{A\} \), \( e = \text{en} \), \( \Delta = \{a\} \), \( A_{\text{in}} = A \), and

\[
R = \{ A \rightarrow \text{if null then } a, \ A \rightarrow \text{if not null then } A(\text{dec})A(\text{dec}) \}.
\]

A derivation of \( G_1 \) is

\[
A(2) \quad \Rightarrow \quad A(1)A(1) \\
\Rightarrow \quad A(0)A(0)A(1) \\
\Rightarrow \quad aA(0)A(1) \\
\Rightarrow \quad aA(0)A(0)A(0) \\
\Rightarrow^* \quad aaaa
\]

Hence, since \( m(\text{en})(2) = 2 \), we have \( (2, aaaa) \in T(G_1) \), \( aaaa \in L(G_1) \), and \( 2 \in A(G_1) \). Clearly, \( T(G_1) = \{ (n, a^{2n}) \mid n \geq 0 \} \), \( L(G_1) = \{ a^{2n} \mid n \geq 0 \} \), and \( A(G_1) = \{ n \mid n \geq 0 \} \).

1.2 Comparison with attribute grammars

As remarked in the Introduction, \( CF(S) \) grammars are a very special case of the general idea of adding parameters to context-free grammars. A much more powerful realization of this idea is the notion of attribute grammar [Knu], in particular in its formulation as affix grammar [Kos, Wat1]. In fact, a \( CF(S) \) grammar may be viewed as an attribute grammar with one, inherited, attribute. To explain this, let in particular \( S = (C, P, F, I, E, m) \) be a storage type such that \( I \) is a singleton \( \{i_0\} \), and let \( G = (N, e, \Delta, A_{\text{in}}, R) \) be a \( CF(S) \) grammar. Let us call the inherited attribute \( i \). It is an attribute of all nonterminals in \( N \), and it has type \( S \), i.e., \( C \) is the set of attribute values for \( i \), and \( P \) and \( F \) contain the possible tests and operations on these attribute values (\( S \) is also called the semantic domain, cf. [EngFil]). Every rule \( A \rightarrow \text{if} \ b \text{ then } w_0B_1(f_1)w_1B_2(f_2)w_2 \cdots B_n(f_n)w_n \) of \( R \) (with \( A, B_j \in N \) and \( w_j \in \Delta^* \)) determines a rule of the underlying context-free grammar \( \overline{G} = (N, \Delta, A_{\text{in}}, \overline{R}) \) of the attribute grammar, together with the semantic rules and semantic conditions for its attributes, as follows (for the notion of semantic condition, see [Wat2]). The rule \( A \rightarrow w_0B_1w_1B_2w_2 \cdots B_nw_n \) is in \( \overline{R} \), the semantic rules to compute the attributes (of the sons) are \( i(B_j) = f_j(i(A)) \), for \( 1 \leq j \leq n \),
and the semantic condition (on the father) is \( b(i(A)) = \text{true}. \)

Usually the initial nonterminal \( A_{\text{in}} \) is not allowed to have an inherited attribute; we allow this but fix its value \( i(A_{\text{in}}) \) to be \( m(e)(u_0) \). It should now be clear that \( L(G) \) is the set of all strings in \( L(G') \) that have derivation trees of which the values of the attributes satisfy all semantic conditions. This is the usual way in which attribute grammars define the context-sensitive syntax of languages. Note that to the (left-most) derivations of the CF(S) grammar \( G \) correspond derivation trees, in an obvious way; the nodes of these trees are labeled by pairs \( A(c) \) from \( N(C) \). These derivation trees correspond to the semantic trees of the attribute grammar \( G \), i.e., derivation trees of \( G' \) together with the values of their attributes. Thus, in this case, attribute evaluation can be defined by way of the derivations of the CF(S) grammar; in fact this holds in general, and it is precisely the way in which attribute evaluation is defined formally in affix grammars (see [Kos, Wat1]).

As an example, a variation \( G'_1 \) of \( G_1 \) might have the rules

\[
\begin{align*}
A_{\text{in}}(x) & \rightarrow B(x) \\
B(x) & \rightarrow B(x+1) \\
B(x) & \rightarrow A(x) \\
A(x) & \rightarrow \text{if } x \neq 0 \text{ then } A(x-1) A(x-1) \\
A(x) & \rightarrow \text{if } x = 0 \text{ then } a
\end{align*}
\]

with \( m(e)(u_0) = 0 \). This corresponds to an attribute grammar with \( i(A_{\text{in}}) = 0 \) and rules \( A_{\text{in}} \rightarrow B, B \rightarrow B, B \rightarrow A, A \rightarrow AA, \) and \( A \rightarrow a \), in \( \overline{F} \). The attribute values are defined as follows. Note that, as usual, subscripts denote different occurrences of the same nonterminal; we use \( \text{cond} \) to indicate a semantic condition.

\[
\begin{array}{ll}
syntactic \text{ rules} & \text{semantic \text{ rules}} \\
A_{\text{in}} & \rightarrow B & i(B) = i(A_{\text{in}}) \\
B_0 & \rightarrow B_1 & i(B_1) = i(B_0) + 1 \\
B & \rightarrow A & i(A) = i(B) \\
A_0 & \rightarrow A_1 A_2 & \text{cond } i(A_0) \neq 0 \\
A & \rightarrow a & \text{cond } i(A) = 0 \\
\end{array}
\]

The underlying context-free grammar generates all strings in \( a^+ \), but the attribute grammar \( G'_1 \) generates \( \{a^{2n} | n \geq 0\} \).

Although CF(S) transducers are a very particular case of attribute grammars, we will see in Section 3(7) how they can be used to model arbitrary attribute grammars!

\textbf{New Observation.} In this paper, the direct descendants of a node of a tree are called its “sons” and the node itself is then called the “father”; moreover, two such sons are “brothers” of each other. To avoid this patriarchate, many authors now use “children”, “parent” and “siblings”. That terminology is misleading, because every child has two parents.
2 Regular grammars

Two particular subcases of the context-free grammars are the regular (= right-linear) grammars and the regular tree grammars. Adding storage $S$, these can be used to model known classes of automata: regular $S$ grammars for one-way $S$ automata, and regular tree $S$ grammars for top-down tree automata with storage $S$. We now discuss these two subcases one by one.

**Definition 2.1.** A regular $S$ transducer, or REG($S$) transducer, is a context-free $S$ transducer $G = (N, e, \Delta, A_{in}, R)$ of which all rules in $R$ have one of the forms $A \rightarrow \text{if } b \text{ then } wB(f)$ or $A \rightarrow \text{if } b \text{ then } w$, where $A, B \in N$, $b \in \text{BE}(P)$, $w \in \Delta^*$, and $f \in F$. □

The corresponding classes of translations, languages, and input sets are denoted by $\tau$-REG($S$), $\lambda$-REG($S$), and $\alpha$-REG($S$), respectively, and similarly for the deterministic case by $\tau$-DREG($S$), $\lambda$-DREG($S$), and $\alpha$-DREG($S$).

From the programming point of view a regular $S$ transducer consists of recursive procedures that only call each other at the end of their bodies (tail recursion). It is well known that such recursion can easily be removed, replacing calls by goto’s, and keeping the actual parameter in the global state. Thus, we may view REG($S$) transducers as ordinary flowcharts with elements of BE($P$) in their diamonds and elements of $F \cup \{\text{write}(a) \mid a \in \Delta\}$ in their boxes. These flowcharts operate, in the usual way, on a global state consisting of an object of type $S$ (i.e., an $S$-configuration) and an output tape. If we consider the REG($S$) $r$-acceptors (with a one-way input tape instead of an output tape: replace “write” by “read”), it should be clear that these are precisely the usual nondeterministic one-way $S$ automata. Thus:

“REG($S$) $r$-acceptor = one-way $S$ automaton”,

and $\lambda$-REG($S$) is the class of languages accepted by one-way $S$ automata (but, as observed before, determinism does not carry over; see Section 7).

An informal example of a REG($S$) $r$-acceptor was given in the discussion of $G_2$ in Section 1.1: a one-way pushdown automaton accepting $\{a^n b^n \mid n \geq 0\}$. It should be clear that the type pushdown can be formalized as a storage type Pushdown, in such a way that the REG(Pushdown) $r$-acceptor corresponds to the usual one-way pushdown automaton, see Section 3(2). This can be done for all usual one-way automata, as shown successfully in AFA theory [Gin]. Thus AFA theory is the theory of REG($S$) $r$-acceptors.

Let us look more closely at our “notation” for one-way $S$ automata. As noted before in Section 1.1, for a REG($S$) $r$-acceptor $G = (N, e, \Delta, A_{in}, R)$, the elements of $N$ are its states, $A_{in}$ is the initial state, $\Delta$ is the input alphabet, ran($m(e)$) is the set of initial $S$-configurations with which $G$ may start its computations, and $R$ represents the transition function. A rule $A \rightarrow \text{if } b \text{ then } wB(f)$ should be interpreted as: “if the current state of the automaton is $A$, $b$ holds for its current storage configuration, and $w$ is a prefix of the (rest of the) input, then the automaton may read $w$, go into state $B$, and apply $f$ to its storage configuration.” A rule $A \rightarrow \text{if } b \text{ then } w$ should be interpreted as: “if ... (as before) ..., then the automaton may read $w$, and halt.” Thus, a string is accepted if the automaton reads it to its end, and then halts according to a rule of the second form. Note that in the total configurations of the REG($S$) $r$-acceptor the already processed part of the input appears, rather than the rest of the input. This is unusual, but may be viewed as a notational matter. Actually, REG($S$) transducers are as close to one-way $S$ automata as right-linear grammars are to finite automata (which is very close!).
However, intuitively, REG(S) r-acceptors only correspond to one-way S automata in case S has an identity: in general a one-way S automaton is not forced to transform its storage at each move.

**Definition 2.2.** A storage type \( S = (C, P, F, I, E, m) \) has an identity, if there is an instruction symbol \( \text{id} \in F \) such that \( m(\text{id}) = \text{id}(C) \).

Grammar \( G_2 \) in Section 1.1 uses an identity in the first and third rules.

Thus, when modelling particular well-known types of one-way automata, such as pushdown automata, we should see to it that the corresponding storage type has an identity. The reason that we also consider storage types without identity is that there exist devices, such as the top-down tree transducer, that have to transform their storage configuration at each step of their computation. Instead of formalizing this in the control (i.e., the form of the rules) of the device, it turns out to be useful to formalize it in the storage type, as we do now. In case we wish to consider both types of transducers, we define \( S \) without an identity, and then add one, as follows.

**Definition 2.3.** For a storage type \( S = (C, P, F, I, E, m) \), \( S \) with identity is the storage type \( S_{\text{id}} = (C, P, F \cup \{\text{id}\}, I, E, m') \), where \( \text{id} \) is a “new” instruction symbol, \( m' \) is the same as \( m \) on \( P \cup F \cup E \), and \( m'(\text{id}) = \text{id}(C) \).

This is useful in particular when (part of) the storage is viewed as input: then the identity constitutes a “\( \lambda \)-move” on this input.

Note that, in the above definition, \( S_{\text{id}} \) is also defined in case \( S \) already has an identity; this simplifies some technical definitions.

As an illustration of the use of an identity, for a REG(S) r-acceptor, we note the following. Some people may not like that the automaton can read a whole string from the input in one stroke. Let us say that a REG(S) r-acceptor is in normal form if \( w \in \Delta \cup \{\lambda\} \) in all its rules. Now let us assume that \( S \) has an identity \( \text{id} \). Then it is quite easy to see that every REG(S) r-acceptor can be put into normal form: replace a rule of the form \( A \rightarrow \textbf{if} \ b \ \textbf{then} \ a_1a_2\ldots a_nB(f) \), with \( n \geq 2 \), by the \( n \) rules \( A \rightarrow \textbf{if} \ b \ \textbf{then} \ a_1B_1(\text{id}) \), \( B_1 \rightarrow a_2B_2(\text{id}), \ldots, B_{n-1} \rightarrow a_nB(f) \), where \( B_1, \ldots, B_{n-1} \) are new states, and similarly for a rule of the form \( A \rightarrow \textbf{if} \ b \ \textbf{then} \ a_1a_2\ldots a_n \).

In the remaining part of this section we consider regular tree grammars and generalize them to regular tree \( S \) grammars, just as we did for context-free grammars. First we need some well-known terminology on trees (see, e.g., [GécSte, Eng1]).

A ranked set \( \Delta \) is a set together with a mapping \( \text{rank} : \Delta \rightarrow \{0, 1, 2, \ldots\} \). If \( \Delta \) is finite, it is called a ranked alphabet. For \( k \geq 0 \), \( \Delta_k = \{\sigma \in \Delta \mid \text{rank}(\sigma) = k\} \). The set of trees over \( \Delta \), denoted \( T_{\Delta} \), is the smallest subset of \( \Delta^* \) such that (1) for every \( \sigma \in \Delta_0 \), \( \sigma \) is in \( T_{\Delta} \), and (2) for every \( \sigma \in \Delta_k \) with \( k \geq 1 \), and every \( t_1, t_2, \ldots, t_k \in T_{\Delta} \), \( \sigma t_1t_2\ldots t_k \) is in \( T_{\Delta} \). For a set \( Y \) disjoint with \( \Delta \), \( T_{\Delta}[Y] \) denotes \( T_{\Delta \cup Y} \), where the elements of \( Y \) are given rank 0. Note that we write trees in prefix notation, without parentheses or commas; however, for the sake of clearness, we will sometimes write \( \sigma(t_1, t_2, \ldots, t_k) \) instead of \( \sigma t_1t_2\ldots t_k \). A language \( L \subseteq \Delta^* \) is called a tree language if \( L \subseteq T_{\Delta} \).

**Definition 2.4.** A regular tree \( S \) transducer, or RT(S) transducer, is a context-free \( S \) transducer \( G = (N, e, \Delta, A_{in}, R) \), such that \( \Delta \) is a ranked alphabet, and, for every rule \( A \rightarrow \textbf{if} \ b \ \textbf{then} \ \xi \) of \( R \), \( \xi \) is in \( T_{\Delta}[N(F)] \).
As usual, the corresponding classes of translations, languages, and input sets are denoted \( \tau\)-RT\((S)\), \( \lambda\)-RT\((S)\), and \( \alpha\)-RT\((S)\), respectively.

It is easy to see that, for an RT\((S)\) transducer \( G \), \( L(G) \subseteq T_\Delta \). Thus \( L(G) \) is a tree language, and \( T(G) \) translates input elements into trees.

As an example we consider the RT\((S)\) transducer \( G'_2 \), a variation of \( G_2 \); as for \( G_2 \), \( S = \text{Pushdown} \). The (informal) rules of \( G'_2 \) are the same as those of \( G_2 \), except that \( \lambda \) has to be replaced by \( \tau \)(of rank 0), and the rule \( A(x) \to B(x)C(x) \) should be replaced by the rule \( A(x) \to \sigma B(x)C(x) \), where \( \sigma \) has rank 2. Symbols \( a, b, \) and \( c \) have rank 1. Thus \( \Delta = \{a, b, c, \sigma, \tau\} \) with \( \Delta_0 = \{\tau\} \), \( \Delta_1 = \{a, b, c\} \), and \( \Delta_2 = \{\sigma\} \). The transducer \( G'_2 \) generates all trees of the form \( a^n \sigma (b^n \tau, c^n \tau) \), i.e., a chain of \( a \)'s that forks into a chain of \( b \)'s and a chain of \( c \)'s, cf. Fig. 1 for \( n = 3 \). This ends the example.

We will now discuss the fact that the regular tree \( S \) grammar may also be viewed as a generalization of the context-free \( S \) grammar, just as in the case without \( S \). A tree is an expression built up from the “operators” of the ranked alphabet \( \Delta \). When an interpretation of these operators is given, as operations on some set \( D \)(a so-called \( \Delta \)-algebra \( D \)), then the expressions of \( T_\Delta \) denote elements of \( D \), in the usual way (see [GogThaWagWri]). Thus an RT\((S)\) transducer \( G \) together with a \( \Delta \)-algebra \( D \) define a translation from the input set \( I \) to the output set \( D \). Note that from the programming point of view it is this time best to view an RT\((S)\) transducer as a set of recursive function procedures, with arguments of type \( S \) and results of type \( T_\Delta \)(or \( D \)). In this way the RT\((S)\) transducer (together with a \( \Delta \)-algebra) generalizes the CF\((S)\) transducer. In fact, as is well known, by taking the \( \Delta \)-algebra \( \Delta^*_0 \) with every \( \sigma \in \Delta \) of rank \( \geq 1 \) interpreted as concatenation, the RT\((S)\) transducer turns, as a special case, into the CF\((S)\) transducer, with terminal alphabet \( \Delta_0 \). In this case every tree denotes its yield, defined as follows.

Let \( \varepsilon \) be a special symbol of rank 0. Then (1) for \( \sigma \in \Delta_0 \) with \( \sigma \neq \varepsilon \), yield\((\sigma) = \sigma \), and yield\((\varepsilon) = \lambda \), and (2) for \( \sigma \in \Delta_k \), \( k \geq 1 \), yield\((\sigma t_1 t_2 \cdots t_k) = \text{yield}(t_1) \text{yield}(t_2) \cdots \text{yield}(t_k) \).

For a tree language \( L \subseteq T_\Delta \), yield\((L) = \{\text{yield}(t) \mid t \in L\} \). For a relation \( R \subseteq I \times T_\Delta \), we define yield\((R) = \{(u, \text{yield}(t)) \mid (u, t) \in R\} \). And for a class \( K \) of tree languages or relations, yield\((K) = \{\text{yield}(B) \mid B \in K\} \).
It is easy to see (and will be proved in Theorem 8.3(1)) that, for every $S$,

$$\lambda\text{-CF}(S) = \text{yield}(\lambda\text{-RT}(S))$$

and, in fact, $\tau\text{-CF}(S) = \text{yield}(\tau\text{-RT}(S))$, and so $\alpha\text{-RT}(S) = \alpha\text{-CF}(S)$. Of course also $\tau\text{-RT}(S) \subseteq \tau\text{-CF}(S)$, because an RT$(S)$ transducer is defined as a special type of CF$(S)$ transducer.

Just as we viewed REG$(S)$ r-acceptors as one-way $S$ automata, we can view RT$(S)$ r-acceptors as top-down $S$ tree automata, i.e., ordinary top-down tree automata such that to each occurrence of its states an $S$-configuration is associated. Such an automaton receives a tree from $T_\Delta$ as input, processes the tree from the root to its leaves (splitting at each node into as many copies as the node has sons), and accepts the tree if all parallel computations are successful. Let us say that an RT$(S)$ r-acceptor $G = (N, e, \Delta, A_{\text{in}}, R)$ is in normal form if all its rules are of one of the forms $A \rightarrow \text{if } b \text{ then } B(f)$ or $A \rightarrow \text{if } b \text{ then } \sigma B_1(f_1) \cdots B_k(f_k)$ with $\sigma \in \Delta_k$, $k \geq 0$. As for the regular case it is easy to show that if $S$ has an identity, then every RT$(S)$ r-acceptor can be transformed into normal form. A rule $A \rightarrow \text{if } b \text{ then } \sigma B_1(f_1) \cdots B_k(f_k)$, with $k \geq 1$, should be interpreted as: “if, at the current node, the state of the tree automaton is $A$, $b$ holds for its storage configuration, and the label of the node is $\sigma$, then the tree automaton splits into $k$ copies, one for each son of the node; at the $j$-th son, the automaton goes into state $B_j$ and applies $f_j$ to its storage configuration.” A rule $A \rightarrow \text{if } b \text{ then } \sigma$, with $\sigma \in \Delta_0$, should be interpreted as: “if ... (as before) ... , then the automaton halts at this node.” Finally, a rule $A \rightarrow \text{if } b \text{ then } B(f)$ should be interpreted as: “if ... (as before) ... , then the automaton stays at this node, goes into state $B$, and applies $f$ to its storage configuration.” Grammar $G_2'$ discussed above, is a top-down pushdown tree automaton, in this way.

Thus, if $S$ has an identity, then $\lambda\text{-RT}(S)$ is the class of tree languages accepted by top-down $S$ tree automata.

An example of an RT$(\text{Integer})$ transducer is $\tilde{G}_1 = (N, e, \Delta, A_{\text{in}}, R)$, where $N = \{A\}$, $e = \text{en}$, $\Delta = \{+, 1\}$ with rank$(+) = 2$ and rank$(1) = 0$, $A_{\text{in}} = A$, and $R$ contains the rules (written informally)

\[
\begin{align*}
A(x) & \rightarrow \text{if } x \neq 0 \text{ then } +A(x - 1) \ A(x - 1) \\
A(x) & \rightarrow \text{if } x = 0 \text{ then } 1
\end{align*}
\]

Note that $\tilde{G}_1$ is in normal form. It translates a nonnegative integer $n$ into an expression over $\{+, 1\}$ that, when interpreted over the integers, with $+$ as ordinary addition and 1 as the integer 1, denotes $2^n$. As a tree, this expression is the full binary tree of depth $n$. Thus, since

\[
\begin{align*}
A(2) & \Rightarrow +A(1) A(1) \\
& \Rightarrow ++A(0) A(0) A(1) \\
& \Rightarrow ++1 A(0) A(1) \\
& \Rightarrow ++1 A(0) + A(0) A(0) \\
& \Rightarrow^* ++11 + 11,
\end{align*}
\]

2 is translated into the tree $+(+(1,1),+(1,1))$ that denotes 4 when interpreted over the $\Delta$-algebra of integers. Note that, when interpreted over the $\Delta$-algebra $a^*$, with $+$ interpreted
as concatenation and 1 as $a$, this tree denotes $aaaa$ (and $\tilde{G}_1$ is really $G_1$). Viewed as a program, with $\{+,1\}$ interpreted over the integers, $\tilde{G}_1$ looks as follows (where $+$ is written infix, as usual):

```
function A(x: integer): integer;
    begin if $x \neq 0$
        then return(A(x-1) + A(x-1))
        else return(1)
    fi
end;
{main program}
obtain n;
deliver A(n).
```
3 Specific storage types

In this section we discuss several cases in which, taking $S$ to be a specific storage type, the $\text{CF}(S)$ transducer turns into a well-known device. In each of these cases we claim that the $\text{CF}(S)$ transducer is just a “definitional variation” of the known device. This means that their definitions are very close (differing in some technical details only), and that the equivalence of the two formalisms is easy to prove (sometimes using some nontrivial known property of the device). Moreover, we hope that the formulation of the device as a $\text{CF}(S)$ transducer gives more insight into “what it really is”, in other words, that the $\text{CF}(S)$ transducer captures the essence of the device. Thus, whenever the reader is not familiar with a certain device, he may safely consider the corresponding $\text{CF}(S)$ transducer as its definition (but he should be careful with determinism). In what follows we will usually say that the $\text{CF}(S)$ transducer “is” the device, in order to avoid the repeated use of phrases like “can be viewed as”, “corresponds to”, “is a definitional variation of”, etc.

(1) The first case is trivial: after adding a storage type to context-free grammars, we now drop it again.

**Definition 3.1.** The trivial storage type $S_0$ is defined by $S_0 = (C, P, F, I, E, m)$, where

- $C = \{c_0\}$ for some arbitrary, but fixed, object $c_0$,
- $P = \emptyset$,
- $F = \{\text{id}\}$,
- $I = C$,
- $E = \{\text{en}\}$, and
- $m(\text{id}) = m(\text{en}) = \text{id}(C)$. □

Obviously, the $\text{CF}(S_0)$ grammar is the context-free grammar. As argued in the Introduction, it may also be viewed as a recursive r-acceptor. Similarly, the $\text{REG}(S_0)$ grammar is the regular (or right-linear) grammar, and the $\text{RT}(S_0)$ grammar is the regular tree grammar. Moreover, the $\text{REG}(S_0)$ r-acceptor is the finite automaton, in particular when it is in normal form (note that $S_0$ has an identity). The $\text{RT}(S_0)$ r-acceptor is the top-down finite tree automaton, in particular, again, when it is in normal form. Note, however, that usually finite (tree) automata do not have $\lambda$-moves, i.e., rules $A \rightarrow B(\text{id})$; it is easy to see that these can be removed.

Thus $\lambda\text{-CF}(S_0) = \text{CF}$, $\lambda\text{-REG}(S_0) = \text{REG}$, and $\lambda\text{-RT}(S_0) = \text{RT}$ (the class of regular tree languages).

(2) Our first nontrivial case is to take $S$ to be the storage type pushdown. It is funny to attach pushdowns to the nonterminals of a context-free grammar, but let us see what happens.

**Definition 3.2.** The storage type $\text{Pushdown}$, abbreviated $P$ (not to be confused with the set of predicate symbols!), is defined by $\text{Pushdown} = (C, P, F, I, E, m)$, where
• \( C = \Gamma^+ \) for some fixed infinite set \( \Gamma \) of pushdown symbols,
• \( P = \{ \text{top} = \gamma \mid \gamma \in \Gamma \} \cup \{ \text{bottom} \} \),
• \( F = \{ \text{push}(\gamma) \mid \gamma \in \Gamma \} \cup \{ \text{pop} \} \cup \{ \text{stay}(\gamma) \mid \gamma \in \Gamma \} \cup \{ \text{stay} \} \),
• \( I = \{ u_0 \} \) for a fixed object \( u_0 \),
• \( E = \Gamma \), with \( m(\gamma)(u_0) = \gamma \) for every \( \gamma \in E \),

and for every \( c = \delta \beta \) with \( \delta \in \Gamma \) and \( \beta \in \Gamma^* \) (intuitively, \( \delta \) is the top of the pushdown \( \delta \beta \)),

• \( m(\text{top} = \gamma)(c) = \text{true} \) iff \( \delta = \gamma \),
• \( m(\text{bottom})(c) = \text{true} \) iff \( \beta = \lambda \),
• \( m(\text{push}(\gamma))(c) = \gamma \delta \beta \),
• \( m(\text{pop})(c) = \beta \) if \( \beta \neq \lambda \) and undefined otherwise,
• \( m(\text{stay}(\gamma))(c) = \gamma \beta \), and
• \( m(\text{stay})(c) = c \). □

It should be clear that this Pushdown corresponds to the usual storage type of pushdowns. Note however that there is no empty pushdown. In fact, ordinary pushdown automata halt in case the pushdown becomes empty, but (if Pushdown would have an empty pushdown) a CF(Pushdown) transducer could always continue on an empty pushdown with unconditional rules; this would cause some technical inconveniences.

Note that Pushdown has an identity, viz. stay. Note that, due to our use of a set of encodings, each CF(Pushdown) transducer \( G = (N, \gamma_0, \Delta, A_{\text{in}}, R) \) has its own initial bottom pushdown symbol \( \gamma_0 \). Note that the stay(\gamma) instructions are superfluous: if the pushdown does not consist of one (bottom) cell, then stay(\gamma) can be simulated by (pop; push(\gamma)); the pushdown symbol of the bottom cell can be kept in the finite control. The bottom predicate is also superfluous: one can always mark the pushdown symbol of the bottom cell (and keep it marked). Note finally that we may assume that each test in a rule of a CF(P) transducer consists of a single predicate symbol of the form top = \gamma. We state this as a lemma (see Lemma 3.30 of [EngVog2]).

**Lemma 3.3.** Every CF(P) transducer is equivalent to one in which all rules are of the form \( A \rightarrow \text{if} \ \text{top} = \gamma \ \text{then} \ \xi \).

**Proof.** Let \( G = (N, \gamma_0, \Delta, A_{\text{in}}, R) \) be a CF(P) transducer. We may assume that \( G \) does not use the bottom predicate (see above). Let \( \Gamma_G \) be the set of all pushdown symbols that occur in \( R \), together with \( \gamma_0 \) (these are all pushdown symbols \( G \) ever uses). First transform \( G \) so that all tests in rules are conjunctions of negated and nonnegated predicate symbols (through their disjunctive normal form); we may assume that every predicate symbol top = \gamma, with \( \gamma \in \Gamma_G \), occurs exactly once in such a conjunction. Now consider a rule \( A \rightarrow \text{if} \ b \ \text{then} \ \xi \). If \( b \) contains only negated predicate symbols, i.e., \( \not \text{top} = \gamma \), then throw the rule away. Do the same if \( b \) contains two nonnegated predicate symbols (they are mutually exclusive). In the remaining rules, erase all negated predicate symbols (because top = \( \gamma_1 \) and \( \not \text{top} = \gamma_2 \) is equivalent to top = \( \gamma_1 \)). Now all tests are of the form top = \gamma. Note that the construction preserves several special properties of CF(P) transducers (such as determinism and regularity). □

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3Throughout this paper we use “iff” as an abbreviation of “if and only if”. 

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Hence the REG(P) r-acceptor is the ordinary one-way pushdown automaton. And the RT(P) r-acceptor, in particular when in normal form, is the top-down pushdown tree automaton, recently defined by Guessarian ([Gue2]; in fact she calls it the restricted pushdown tree automaton, see (8) of this section). Actually this is the only example in the literature of a top-down S tree automaton.

Now we turn to a less predictable connection. The CF(P) grammar is the indexed grammar of [Aho1] (see [HopUll, Sal]) and so λ-CF(P) = Indexed. Viewing flags as pushdown symbols, the sequence of flags attached to each nonterminal in a sentential form of an indexed grammar behaves just as a pushdown. Flag production corresponds to pushing, and flag consumption to popping. A flag producing rule (called a production in [Aho1]) corresponds to an unconditional rule \( A \rightarrow w_0B_1\text{push}(\gamma_1)w_1 \cdots B_n\text{push}(\gamma_n)w_n \), where some of the push(\(\gamma_j\)) may also be stay (actually, in an indexed grammar, more than one symbol can be pushed; this can be done here by using elements of \( N(F^+) \)). A flag consuming rule (called an index production in [Aho1]) corresponds to a rule \( A \rightarrow \text{if top} = \gamma \text{then } w_0B_1\text{pop}w_1 \cdots B_n\text{pop}w_n \) (consumption of \(\gamma\)). Thus, in a CF(P) grammar both kinds of rules are present, and flag production and consumption can even be mixed in one rule (it is easy to see that this can be simulated by an indexed grammar). Consequently, a CF(P) grammar has a more uniform notation than an indexed grammar (see [ParDusSpe2] for a definition of indexed grammar closer to the CF(P) grammar). Also, personally, I must confess that I only understood indexed grammars when I found out they were just CF(P) grammars in disguise!

An example of a CF(P) grammar was given as \( G_2 \) of Section 1.1. The reader should now recognize it as an indexed grammar. Let us write down \( G_2 \) formally: \( G_2 = (N, e, \Delta, A_{in}, R) \), where \( N = \{ A_{in}, A, B, C \} \), \( e = \# \), \( \Delta = \{ a, b, c \} \), and \( R \) consists of the rules

\[
\begin{align*}
A_{in} & \rightarrow A(\text{stay}) \\
A & \rightarrow aA(\text{push}(a)) \\
A & \rightarrow B(\text{stay})C(\text{stay}) \\
B & \rightarrow \text{if } \text{top} = a \text{ then } BB(\text{pop}) \\
B & \rightarrow \text{if } \text{top} = \# \text{ then } \lambda \\
C & \rightarrow \text{if } \text{top} = a \text{ then } cC(\text{pop}) \\
C & \rightarrow \text{if } \text{top} = \# \text{ then } \lambda
\end{align*}
\]

Note that the pushdown alphabet is not explicitly mentioned in \( G_2 \); it can be obtained from \( R \) and \( \# \). As noted before, when dropping \( C(\text{stay}) \) from the third rule, \( G_2 \) turns into a REG(P) r-acceptor, i.e., a pushdown automaton. Also, changing \( \lambda \) into \( \tau \), and changing the third rule into \( A \rightarrow \sigma B(\text{stay})C(\text{stay}) \), we obtain the RT(P) r-acceptor \( G'_2 \) of Section 2, i.e., a top-down pushdown tree automaton.

Thus, CF(S) grammars may also be viewed as a generalization of indexed grammars: the nonterminals are indexed by S-configurations rather than sequences of flags.

Since CF(S) grammars are attribute grammars with one inherited attribute (Section 1.2), it follows that the indexed languages are generated by attribute grammars with one, inherited, attribute of type Pushdown; this was shown in [Döb].

The RT(P) grammar might be called the indexed tree grammar. It corresponds to the regular tree grammar in exactly the same way as the indexed grammar to the context-free grammar. Thus \( \lambda \)-RT(P) is both the class of tree languages accepted by pushdown tree au-
tomata and the class of tree languages generated by indexed tree grammars (in fact, as shown in [Gue2], these are the context-free tree languages [Rou]). Since, as noted in the previous section for arbitrary \( S \), yield(\( \lambda \)-RT(\( P \))) = \( \lambda \)-CF(\( P \)), the yields of these languages are the indexed languages (cf. Section 5 of [Gue2]).

One-turn pushdown automata are pushdown automata that can never push anymore after doing a pop move. This property can easily be incorporated in the storage, and this gives rise to the storage type One-turn Pushdown, abbreviated \( P_{1t} \), defined and studied in [Gin, Vog2, Vog3]. The REG(\( P_{1t} \)) r-acceptor is the one-way one-turn pushdown automaton [GinSpa], and the CF(\( P_{1t} \)) grammar is the restricted indexed grammar [Aho1] (defined in such a way that after flag consumption there may be no more flag production). The grammar \( G_2 \) of Section 1.1 can easily be turned into such a CF(\( P_{1t} \)) grammar, because its pushdowns make one turn only.

(3) The storage type Counter (of the usual one-way counter automaton) can be obtained by restricting Pushdown to have one pushdown symbol only: therefore we will call this storage type also Pure-pushdown.

**Definition 3.4.** The storage type Counter or Pure-pushdown is defined in exactly the same way as Pushdown, except that \( \Gamma = \{ \gamma_0 \} \) for some fixed symbol \( \gamma_0 \). □

Thus, the REG(Counter) r-acceptor is the one-way counter automaton. Note that a pushdown of \( k \) cells represents the fact that the counter, say \( x \), contains the number \( k - 1 \); push(\( \gamma_0 \)) corresponds to \( x := x + 1 \), pop to \( x := x - 1 \), and the bottom predicate to the predicate \( x = 0 \).

The CF(Counter) grammar is a special type of indexed grammar: the 1-block-indexed grammar (see [Ern]). Note that the grammar \( G_2 \) of Section 1.1 can easily be turned into a CF(Counter) grammar by using the predicate bottom rather than top = \# (and, dropping C(stay), it is a one-way counter automaton).

(4) The (new) storage type Count-down is the same as Integer, defined at the end of Section 1.1, except that we drop the negative integers from the set of configurations.

**Definition 3.5.** The storage type Count-down is \( (C,P,F,I,E,m) \), where

- \( C \) is the set of nonnegative integers,
- \( P = \{ \text{null} \} \),
- \( F = \{ \text{dec} \} \),
- \( I = C \),
- \( E = \{ \text{en} \} \), with \( m(\text{en}) = \text{id}(C) \),

and for every \( c \in C \),

- \( m(\text{null})(c) = \text{true} \) iff \( c = 0 \),
- \( m(\text{dec})(c + 1) = c \), and
- \( m(\text{dec})(0) \) is undefined. □
The CF(Count-down) grammar is the EOL system (see [RozSal]), and so \( \lambda \)-CF(Count-down) is the class of EOL languages. Rather than applying rules in parallel (as in a “real” EOL system), the CF(Count-down) grammar applies rules in the ordinary context-free way; but it chooses an integer \( n \) at the start of the derivation, and then sees to it, by counting down to zero, that the paths through the derivation tree do not become longer than \( n \). Note that Count-down has no identity.

Clearly, grammar \( G_1 \) of Section 1.1 is a CF(Count-down) grammar. It corresponds to the EOL system with the rule \( a \rightarrow aa \). In general, an EOL system with alphabet \( \Sigma \) and terminal alphabet \( \Delta \subseteq \Sigma \) corresponds to a CF(Count-down) grammar with terminals \( \Delta \) and nonterminals \( \{ \tilde{a} \mid a \in \Sigma \} \); an EOL rule \( a \rightarrow a_1a_2\cdots a_n \) corresponds to a rule \( \tilde{a} \rightarrow \text{if not null then } \tilde{a}_1(\text{dec})\tilde{a}_2(\text{dec})\cdots \tilde{a}_n(\text{dec}) \); and moreover the grammar has rules \( \tilde{a} \rightarrow \text{if null then } a \) for all \( a \in \Delta \). Vice versa, it is not very difficult to show that the CF(Count-down) grammar is not more powerful than the EOL system (cf. [Eng2]); the proof involves closure of EOL under homomorphisms.

Note that \( \lambda \)-REG(Count-down) is just the class of regular languages.

(5) Strings can be read from left to right, symbol by symbol. This is defined in the next storage type that we call One-way, because it corresponds to the input tape of one-way automata.

Definition 3.6. The storage type One-way is \((C,P,F,I,E,m)\), where

- \( C = \Omega^* \) for some fixed infinite set \( \Omega \) of input symbols,
- \( P = \{ \text{first} = a \mid a \in \Omega \} \cup \{ \text{empty} \} \),
- \( F = \{ \text{read} \} \),
- \( I = \Omega^* \),
- \( E = \{ \Sigma \mid \Sigma \text{ is a finite subset of } \Omega \} \), with \( m(\Sigma) = \text{id}(\Sigma^*) \) for every \( \Sigma \in E \),

and for every \( c = bw \in \Omega^* \) (with \( b \in \Omega, w \in \Omega^* \)),

- \( m(\text{first} = a)(c) = (b = a) \),
- \( m(\text{first} = a)(\lambda) = \text{false} \),
- \( m(\text{empty})(c) = \text{false} \),
- \( m(\text{empty})(\lambda) = \text{true} \),
- \( m(\text{read})(c) = w \), and
- \( m(\text{read})(\lambda) \) is undefined. \[\square\]

Note that One-way has no identity. Note that every alphabet in \( E \) is viewed as one encoding symbol. Note that a CF(One-way) transducer \((N,\Sigma,\Delta,A_{in},R)\) translates strings of \( \Sigma^* \) into strings of \( \Delta^* \). Thus the encoding of the transducer determines its input alphabet (another example of the usefulness of a set of encodings). Note finally that it may be assumed that all rules of a CF(One-way) transducer are of the form \( A \rightarrow \text{if first = } a \text{ then } \xi \) or of the form \( A \rightarrow \text{if empty then } \xi \) (cf. Lemma 3.3 for Pushdown).
As an example, the following CF(One-way) grammar $G_4$ translates every string over the alphabet $\Sigma$ into the sequence of its suffixes. We define $G_4 = (N, \Sigma, \Sigma \cup \{\#\}, A_{in}, R)$, where $N = \{A, C\}$, $A_{in} = A$, and $R$ consists of the rules

\[
\begin{align*}
A & \rightarrow \text{ if } \text{first} = a \text{ then } aC\text{(read)}\#A\text{(read)} \quad \text{for all } a \in \Sigma \\
C & \rightarrow \text{ if } \text{first} = a \text{ then } aC\text{(read)} \quad \text{for all } a \in \Sigma \\
A & \rightarrow \text{ if } \text{empty} \text{ then } \lambda \\
C & \rightarrow \text{ if } \text{empty} \text{ then } \lambda
\end{align*}
\]

Clearly, $G_4$ is a deterministic CF(One-way) transducer that translates $a_1a_2\cdots a_n$ (with $a_j \in \Sigma$) into $a_1a_2\cdots a_n\#a_2\cdots a_n\#a_3\cdots a_n\#\cdots a_n\#$. Note, by the way, that, viewing $G_4$ as a system of recursive (function) procedures, the correctness of $G_4$ is immediate.

The CF(One-way) grammar is the ETOL system, and the deterministic CF(One-way) grammar is the deterministic ETOL system, or EDTOL system [RozSal]. Thus $\lambda$-(D)CF(One-way) is the class of E(D)TOL languages. The sequence of tables applied during the derivation of the ETOL system corresponds to the input string of the CF(One-way) grammar (chosen nondeterministically). Otherwise the correspondence is exactly the same as for EOL systems and CF(Count-down) grammars. In fact, Count-down is the same as the restriction of One-way to one symbol $a$, i.e., $\Omega = \{a\}$: null corresponds to empty (and also to not first = a), and dec corresponds to read. See [EngRozShl] for a definition of ETOL system that is close to the CF(One-way) grammar.

Grammar $G_4$ corresponds to the EDTOL system with a table $a$ for each $a \in \Sigma$, containing the rules $A \rightarrow aC\#A$ and $C \rightarrow aC$ (and identity rules $b \rightarrow b$ for all other symbols $b$), plus an additional table that contains the rules $A \rightarrow \lambda$ and $C \rightarrow \lambda$ (and identity rules).

We note here that it is quite easy to show that $\lambda$-CF(One-way) is included in $\lambda$-CF(P) (and even in $\lambda$-CF($P_{1t}$)). Given a CF(One-way) grammar $G = (N, \Sigma, \Delta, A_{in}, R)$, construct the CF(P) grammar $G' = (N \cup \{Z\}, \#, \Delta, Z, R')$, where $R'$ consists of (1) all rules that are needed to build up an arbitrary string in the pushdown: $Z \rightarrow Z(\text{push}(a))$ and $Z \rightarrow A_{in}$, for all $a \in \Sigma$, and (2) all rules that simulate $G$ with this input string, i.e., all rules of $G$ in which first = $a$ is replaced by top = $a$, empty by bottom, and read by pop. It should be clear that $L(G') = L(G)$. Thus the ETOL languages are contained in the (restricted) indexed languages, as originally shown, in this way, in [Cul]. This result is generalized in [Vog1], showing that “the ETOL hierarchy” is included in “the OI-hierarchy”. As a particular case it can be shown that $\lambda$-CF(Count-down) $\subseteq \lambda$-CF(Counter), i.e., the EOL languages are contained in the 1-block-indexed languages.

We also note that the controlled ETOL systems [Asv, EngRozShl] can be modelled by an appropriate generalization of One-way, as shown in [Vog1]; as a special case the controlled linear context-free grammars are obtained, see [Vog2]. In fact, the other way around, context-free $S$ grammars may be viewed as “storage controlled” context-free grammars, i.e., context-free grammars of which the derivations are controlled by the storage configurations of $S$. This point of view is explained in the first chapter of [Vog4].

The REG(One-way$_{id}$) transducer is the finite-state transducer or a-transducer (see Definition 2.3 for the definition of $S_{id}$ for a storage type $S$). Thus, $\tau$-REG(One-way$_{id}$) is the class of a-transductions. A small difference is that, using empty, the REG(One-way$_{id}$) transducer can detect the end of the input string (and so the deterministic transducers define slightly
different classes of translations). Thus it would be better to say that it is the a-transducer with endmarker. Note also that the REG(One-way) transducer has a look-ahead of one input symbol.

The REG(One-way) transducer is slightly more powerful than the generalized sequential machine (gsm). A gsm can only translate the empty input string \( \lambda \) into itself, whereas a REG(One-way) transducer can translate it into any finite number of output strings, using rules of the form \( A_{in} \rightarrow \text{if empty then } w \). Disregarding the empty input string, \( \tau\text{-REG(One-way)} \) is the class of gsm mappings (with endmarker).

As an example, if we drop \( C \text{(read)} \) from the first set of rules of \( G_4 \), we obtain a gsm that translates every \( a_1a_2\ldots a_n \) into \( a_1\#a_2\#\ldots \#a_n\# \).

Note the asymmetric way in which input and output strings are modeled in REG(One-way) transducers: the input by a storage type, the output by a terminal alphabet. This asymmetry is of course inherent to the formalism of CF(S) transducers.

Note finally that the REG(One-way) d-acceptor is the finite automaton again.

(6) Next we generalize input strings to input trees. They can be read from top to bottom, node by node. See Section 2 for notation concerning trees.

**Definition 3.7.** The storage type Tree is \( (C,P,I,E,m) \), where
- \( C = T_\Omega \) for some fixed ranked set \( \Omega \), such that \( \Omega_k \) is infinite for every \( k \geq 0 \),
- \( P = \{ \text{root} = \sigma \mid \sigma \in \Omega \} \),
- \( F = \{ \text{sel}_i \mid i \in \{1,2,3,\ldots\} \} \),
- \( I = T_\Omega \),
- \( E = \{ \Sigma \mid \Sigma \text{ is a finite subset of } \Omega \} \), with \( m(\Sigma) = \text{id}(T_\Sigma) \) for every \( \Sigma \in E \),

and for every \( c = \tau t_1 \ldots t_k \in T_\Omega \) (with \( \tau \in \Omega_k \), \( k \geq 0 \), \( t_1,\ldots,t_k \in T_\Omega \)),
- \( m(\text{root} = \sigma)(c) = (\tau = \sigma) \),
- \( m(\text{sel}_i)(c) = t_i \) if \( 1 \leq i \leq k \), and
- \( m(\text{sel}_i)(c) \) is undefined if \( i > k \).

Thus \( m(\text{sel}_i) \) selects the \( i \)-th subtree of the given tree. Clearly, REG(Tree) transducers do not have much sense: they can only look at one path through the tree.

The RT(Tree) transducer is the top-down tree transducer, and similarly for determinism [Rou, Tha, EngRozSlu, GécSte]. Thus \( \tau\text{-}\text{RT(Tree)} \) is the class of (deterministic) top-down tree transductions. Hence, the CF(Tree) transducer is the top-down tree-to-string transducer (see, e.g., [EngRozSlu]), or the generalized syntax-directed translation scheme (GSDT, [AhoUll]). A rule of a top-down tree-to-string transducer has the form \( q(\sigma(x_1,\ldots,x_k)) \rightarrow w_0q_1(x_{j_1})w_1\cdots q_n(x_{j_n})w_n \) with \( n \geq 0 \); this corresponds to the CF(Tree) transducer rule \( q \rightarrow \text{if root} = \sigma \text{ then } w_0q_1(\text{sel}_{j_1})w_1\cdots q_n(\text{sel}_{j_n})w_n \).

As an example, let \( G = (N,\Delta,A_{in},R) \) be an ordinary context-free grammar, say, in Chomsky normal form. The following CF(Tree) transducer \( G' \) generates the language \( L(G') = \{ w\#w\#w \mid w \in L(G) \} \). Let \( G' = (N \cup \{Z\},\Sigma,\Delta \cup \{\#\},Z,R') \), where \( \Sigma = R \cup \{\$\} \), \( \Sigma_0 = \{ r \in R \mid r \text{ has the form } A \rightarrow a \} \), \( \Sigma_1 = \{\$\} \), \( \Sigma_2 = \{ r \in R \mid r \text{ has the form } A \rightarrow BC \} \), and \( R' \) is defined as follows.
• The rule $Z \rightarrow \text{if \ root = } $ then $A_{\text{in}}(\text{sel}_1)A_{\text{in}}(\text{sel}_1)A_{\text{in}}(\text{sel}_1)$ is in $R'$.

• If $r : A \rightarrow BC$ is in $R$, then $A \rightarrow \text{if \ root = } r \text{ then } B(\text{sel}_1)C(\text{sel}_2)$ is in $R'$.

• If $r : A \rightarrow a$ is in $R$, then $A \rightarrow \text{if \ root = } r \text{ then } a$ is in $R'$.

Clearly $G'$ translates a (rule labeled) derivation tree of $G$ (with extra root $\$$) into $w\#w\#w$ where $w$ is the “yield” of the tree.

As shown in [Eng2, EngRozSlu], the ETOL system is really the “monadic case” of the top-down tree transducer (i.e., all input symbols have rank 1 or 0). In fact, if all elements of $\Omega$ have rank 1 or 0, then Tree becomes (almost) the same as One-way.

The class $\lambda$-RT(Tree) is called the class of surface tree languages [Rou, Eng2]; $\alpha$-RT(Tree) is the class RT of regular tree languages (as shown in [Rou]). Note finally that the RT(Tree$_{\text{id}}$) transducer is a top-down tree transducer for which “$\lambda$-moves” are allowed (considered in [Eng6,*MalVog]).

(7) Input trees can of course also be read by walking from node to node along the edges of the tree. This is a generalization of the previous storage type, in which one can only walk downwards.

Informally, the storage configurations of Tree-walk are nodes of trees. We assume the reader to be familiar with the usual informal terminology concerning trees.

**Definition 3.8.** The storage type Tree-walk is $(C, P, I, E, m)$, where

- $C = \{(t, n) \mid t \in T_\Omega, n \text{ is a node of } t\}$, with $\Omega$ as in Tree,
- $P = \{\text{label} = \sigma \mid \sigma \in \Omega\} \cup \{\text{root}\} \cup \{\text{son} = i \mid i \in \{1, 2, 3, \ldots\}\}$,
- $F = \{\text{down}_i \mid i \in \{1, 2, 3, \ldots\}\} \cup \{\text{up, stay}\}$,
- $I = T_\Omega$,
- $E = \{\Sigma \mid \Sigma \text{ is a finite subset of } \Omega\}$,

and $m$ is defined as follows. For every $(t, n) \in C$,

- $m(\text{label} = \sigma)(t, n) = \text{true \ iff } n \text{ is labeled } \sigma$ in $t$,
- $m(\text{root})(t, n) = \text{true \ iff } n \text{ is the root of } t$,
- $m(\text{son} = i)(t, n) = \text{true \ iff } n \text{ is the } i\text{-th son of its father}$,
- $m(\text{down}_i)(t, n) = (t, i\text{-th son of } n)$, and
- $m(\text{up})(t, n) = (t, \text{father of } n)$.

Note that some of these may be undefined. Furthermore,

- $m(\text{stay}) = \text{id}(C)$,

and for every $\Sigma \in E$,

- $m(\Sigma)(t) = (t, n)$ for every $t \in T_\Sigma$, where $n$ is the root of $t$, and
- $m(\Sigma)(t)$ is undefined for $t \notin T_\Sigma$. □
The son = i predicates are needed because otherwise a REG(Tree-walk) transducer is not even able to do a depth-first left-to-right search of the input tree (as shown in [KamSlu]): when it returns to the father, it does not know which son to visit next. By the way, it is open whether the REG(Tree-walk) d-acceptor can accept all regular tree languages.

**New Observation 3.9.** This is not open anymore. The problem was solved by Bojańczyk and Colcombet in [*BojCol2*]: the REG(Tree-walk) d-acceptor cannot accept all regular tree languages, i.e., α-REG(Tree-walk) is properly included in RT. In [*BojCol1*] they proved that α-DREG(Tree-walk) is properly included in α-REG(Tree-walk).

The REG(Tree-walk) transducer is the tree walking automaton of [AhoUll], which is called checking tree transducer (CT transducer) in [EngRozSlu]. It translates trees into strings, and is equivalent to a subcase of the top-down tree-to-string transducer (cf. (7) of Section 6).

We now wish to convince the reader that the deterministic RT(Tree-walk) transducer can be viewed as the attribute grammar of [Knu]. Actually, it corresponds to the attribute grammar viewed as tree transducer, cf. [EngFil]. But if we also take into account the possibility of interpreting Δ (into a Δ-algebra, see Section 2), it really corresponds to the attribute grammar. However, in this case we do not claim that τ-DRT(Tree-walk) is equal to the tree transductions realized by attribute grammars; due to several (uninteresting) technical details this may not be true. Still, the formalisms are “very close”. To see this, let us consider an arbitrary deterministic RT(Tree-walk) transducer $G = (N, \Sigma, \Delta, A_m, R)$. Then $\Sigma$ determines, in some sense, the underlying context-free grammar of the attribute grammar; the input trees from $T_\Sigma$ are the derivation trees of the context-free grammar (this holds in particular in the “many-sorted case”, see [GogThaWagWri]). The elements of $N$ are the attributes. They are not partitioned into inherited and synthesized attributes (for the fact that this is not necessary, see [Tie]). The elements of $T_\Delta$ (or of $D$, in case a Δ-algebra $D$ is also given) are the attribute values. The attribute $A_m$ is a designated attribute of the root of the input tree, the value of which is the translation of the tree. Finally, $R$ contains the semantic rules for computing the attribute values. A rule of the form, say, $A \rightarrow \text{if label} = \sigma \text{ then } \cdots B(\text{up}) \cdots C(\text{down}) \cdots D(\text{stay}) \cdots$ with rank($\sigma$) ≥ 2, expresses the $A$-attribute of any node labeled $\sigma$ in terms of the $B$-attribute of its father, the $C$-attribute of its second son, and its own $D$-attribute. Thus, such a rule combines the inherited and synthesized features of the attributes; it just expresses the attribute of a node in terms of those of its neighbors (and itself). Note that semantic conditions cannot be used explicitly, but should be simulated by semantic rules. The son = i predicates are needed to express attributes of a node in terms of those of its brothers (see the next example).

This point of view on attribute grammars is actually one of the most obvious ones. Recall from Section 2 that an RT(S) transducer is a set of recursive function procedures with arguments of type $S$ (here: node of a tree) and results of type $T_\Delta$ (or $D$, where $D$ is a Δ-algebra). Now, it should be clear that the semantic rules of an attribute grammar are just a way of recursively programming functions of that type, one function for each attribute. The derivations of the RT(Tree-walk) transducer are precisely the computations of the recursive function procedures, and thus form a (inefficient) way of attribute evaluation. In [Fül, Kam] these derivations are used to formally define the translation realized by an attribute grammar, and in [Jal, Jou] they form the basis of more efficient attribute evaluation (cf. Section 4.1 of [Eng10]).

Note that if we drop “up” and “stay” from Tree-walk, we more or less reobtain Tree. This shows the well-known fact that attribute grammars with synthesized attributes only are
closely related to DRT(Tree) transducers, i.e., deterministic top-down tree transducers (see, e.g., [CouFra]).

Finally we note that attribute grammars with strings as values, and concatenation as only operation (see [DusParSedSpe, Kam, EngFil, Eng7]), are modeled by deterministic CF(Tree-walk) transducers.

To illustrate the use of the son = i predicates we give an example of a deterministic CF(Tree-walk) transducer $G_5$. It has two nonterminals, $L$ and $U$; for every node $(t, n), L(t, n)$ generates that part of the yield of $t$ that is to the left of $n$, and $U(t, n)$ generates that part of the yield of $t$ that is to the left of $n$ or below $n$, see Fig. 2. Intuitively, $L$ is an inherited attribute, and $U$ is synthesized. So, $G_5 = (\{L, U\}, \Sigma, \Sigma_0, U, R)$, where $R$ contains the following rules (recall the convention of using elements of $N(F^+)$, see Section 1.1).

\[
\begin{align*}
L & \rightarrow \text{ if root then } \lambda \\
L & \rightarrow \text{ if son = 1 then } L(\text{up}) \\
L & \rightarrow \text{ if son = } i \text{ then } U(\text{up; down}_{i-1}) \\
& \quad \text{ (for } 2 \leq i \leq m, \text{ where } m \text{ is the maximal rank in } \Sigma) \\
U & \rightarrow \text{ if label = } \sigma \text{ then } L(\text{stay})\sigma \quad \text{ for all } \sigma \in \Sigma_0 \\
U & \rightarrow \text{ if label = } \sigma \text{ then } U(\text{down}_{k}) \quad \text{ for all } \sigma \in \Sigma_k, k \geq 1
\end{align*}
\]

Note that the purpose of $G_5$ is not to define $T(G_5)$, because that could be done in a much easier way; instead, its purpose is to be able to compute $L(t, n)$ and $U(t, n)$ for arbitrary $(t, n)$.

(8) The last storage type we consider is the generalization of the pushdown to trees, introduced in [Gue2] and formalized in [DamGue]. The top of the tree-pushdown is the root of the tree. This root may be replaced by any piece of tree. Let $Y = \{y_1, y_2, \ldots\}$ be an infinite set of “variables”.

![Fig. 2: $l = L(t, n)$ and $u = U(t, n)$.](image-url)
Definition 3.10. The storage type Tree-pushdown is \((C, P, F, I, E, m)\), where

- \(C = T_\Omega\) (with \(\Omega\) as in Tree),
- \(P = \{\text{root} = \sigma \mid \sigma \in \Omega\}\),
- \(F = \{\text{expand}(\zeta) \mid \zeta \in T_\Omega[Y]\}\),
- \(I = \{u_0\}\) for a fixed object \(u_0\),
- \(E = \Omega_0\), with \(m(\sigma)(u_0) = \sigma\) for every \(\sigma \in E\),

and for every \(c = \tau t_1 \ldots t_k \in T_\Omega\) (with \(\tau \in \Omega_k\), \(k \geq 0\), \(t_1, \ldots, t_k \in T_\Omega\)),

- \(m(\text{root} = \sigma)(c) = (\tau = \sigma)\), and
- \(m(\text{expand}(\zeta))(c) = \) the result of substituting \(t_i\) for \(y_i\) in \(\zeta\), for all \(1 \leq i \leq k\) (undefined if this is not in \(T_\Omega\)).

Note that an \(\text{expand}(y_i)\) corresponds to a pop, whereas all other expands correspond to pushes.

The RT(Tree-pushdown) r-acceptor is the top-down tree-pushdown tree automaton (called pushdown tree automaton in [Gue2]). Although Tree-pushdown has no identity, it can easily be simulated.

As noted in [DamGue], the RT(Tree-pushdown) grammar is the creative dendrogrammar of [Rou]. The creative dendrogrammar with one state (= nonterminal) only, is the (outside-in) context-free tree grammar. Moreover, every creative dendrogrammar is equivalent to one with one state only ([Rou], the proof is nontrivial). Thus \(\lambda\)-RT(Tree-pushdown) is the class of context-free tree languages (see also [Gue2, EngVog2]). In the same way the CF(Tree-pushdown) grammar is related to the (outside-in) macro grammar of [Fis]. Recall that the CF(Pushdown) grammar is the indexed grammar; thus the equivalence of the macro grammar and the indexed grammar [Fis] can (partly) be explained by the equivalence of the storage types Tree-pushdown and Pushdown (see [EngVog2]), and similarly for the context-free tree grammar and the indexed tree grammar (cf. RT(P) of point (2)).

Inside-out context-free tree grammars and macro grammars do not seem to fit into the CF(S) formalism.
4 Alternating automata

Since alternation is close to recursion, context-free grammars may be viewed as the prototype of alternating automata. In fact, the CF(S) d-acceptor is the alternating S automaton (see [ChaKozSto, LadLipSto, Ruz] for alternating automata; and see [May] for the relationship between AND/OR programming and context-free grammars). Here, S is assumed to contain both the input and the internal storage of the automaton (as opposed to the treatment of one-way S automata in Section 2, using r-acceptors). In particular when S is a storage type with I = Ω* for some infinite alphabet Ω, the input set A(G) d-accepted by a CF(S) d-acceptor G is an ordinary language (because A(G) ⊆ I), and so α-CF(S) is the class of languages accepted by alternating S automata. Note that, as observed before, we may assume that the terminal alphabet is empty.

To explain the correspondence between CF(S) d-acceptors and alternation, recall that an alternating automaton has two kinds of states: existential and universal states. In an existential state, some possible next move has to lead to success, whereas in a universal state all possible next moves should lead to acceptance. In a CF(S) d-acceptor, there is no such difference between nonterminals, but existentiality is modeled by the choice between two possible rules with the same left-hand nonterminal, whereas universality is modeled by having several nonterminals in the right-hand side (as opposed to a REG(S) d-acceptor, where there is at most one). For instance, if all rules for A and b are A → if b then B(f1)C(f2) and A → if b then D(f3), then the alternating automaton, in state A and with b true for its storage configuration, either goes into state D (applying f3), or splits itself in two and goes into state B (applying f1) and into state C (applying f2). It is easy to see that, assuming that S has an identity, every CF(S) d-acceptor can be transformed into one that has existential states and universal states: for an existential state A, all rules with left-hand side A have the form of those of a REG(S) d-acceptor, and for a universal state A, all rules with left-hand side A have the form of those of a deterministic CF(S) d-acceptor.

From this it should also be clear that the deterministic CF(S) d-acceptor is the universal S automaton, i.e., the alternating S automaton with universal states only.

As a first example, the CF(One-way) d-acceptor is the alternating one-way finite automaton. Thus, α-CF(One-way) is the class of languages accepted by alternating one-way finite automata. Recall that the CF(One-way) grammar is the ETOL system; this relationship between the parallelism of L systems and that of alternating automata was pointed out in [Eng6]. Note that the REG(One-way) d-acceptor is the finite automaton. In general, the REG(S) d-acceptor is the nondeterministic S automaton (where S contains both the input and the internal storage of the automaton).

To be able to consider arbitrary alternating one-way S automata (i.e., alternating automata with a one-way input tape and internal storage S), we need the notion of product of storage types.

Definition 4.1. Let Si = (Ci, Pi, Fi, Ii, Ei, mi), for i = 1, 2, be two storage types, with P1 ∩ P2 = ∅. Their product S1 × S2 is the storage type

\[(C_1 \times C_2, P_1 \cup P_2, F_1 \times F_2, I_1 \times I_2, E_1 \times E_2, m),\]

where
\begin{itemize}
  \item \(m(p)(c_1, c_2) = m_i(p)(c_i)\) for \(p \in P_i\),
  \item \(m(f_1, f_2)(c_1, c_2) = (m_1(f_1)(c_1), m_2(f_2)(c_2))\), and
  \item \(m(e_1, e_2)(u_1, u_2) = (m_1(e_1)(u_1), m_2(e_2)(u_2))\).
\end{itemize}

\[\Box\]

In other words, everything is defined element-wise except the predicates, which can be combined in boolean expressions anyway. We can now say that the \(\text{CF}(\text{One-way}_{id} \times S)\) d-acceptor is the alternating one-way \(S\) automaton. As an example, the following \(\text{CF}(\text{One-way}_{id}\times P)\) d-acceptor \(G_6\), i.e., alternating one-way pushdown automaton, accepts the language \(A(G_6) = \{a^n b^n c^n \mid n \geq 1\}\). Since for \(P\) the input set \(I\) is a singleton, we identify the input set of \(\text{One-way}_{id}\times P\) with that of \(\text{One-way}_{id}\), i.e., \(\Omega^*\).

Let \(G_6 = (N, e, \emptyset, A_{in}, R)\), where \(N = \{A_{in}, A, B, C\}\), \(e = \{a, b, c, \#\}\), i.e., the input alphabet is \(\{a, b, c\}\), the bottom pushdown symbol is \#, and \(R\) contains the following rules

\[
\begin{align*}
A_{in} & \rightarrow A(\text{id, stay}) \\
A & \rightarrow \text{if first} = a \text{ then } A(\text{read, push}(a)) \\
A & \rightarrow \text{if first} = b \text{ and } \text{top} = a \text{ then } B(\text{read, pop})C(\text{read, stay}) \\
B & \rightarrow \text{if first} = b \text{ and } \text{top} = a \text{ then } B(\text{read, pop}) \\
B & \rightarrow \text{if first} = c \text{ and } \text{top} = \# \text{ then } \lambda \\
C & \rightarrow \text{if first} = b \text{ then } C(\text{read, stay}) \\
C & \rightarrow \text{if first} = c \text{ and } \text{top} = a \text{ then } C(\text{read, pop}) \\
C & \rightarrow \text{if empty and } \text{top} = \# \text{ then } \lambda.
\end{align*}
\]

Note that \(G_6\) is deterministic, i.e., universal. Note also that, informally, we may view the nonterminals of \(G_6\) to have two parameters, say, \(x\) and \(y\). The fourth rule of \(G_6\) may be written informally as \(B(x, y) \rightarrow \text{if first}(x) = b \text{ and } \text{top}(y) = a \text{ then } B(\text{read}(x), \text{pop}(y))\), and similarly for the other rules.

Replacing One-way by Tree or Tree-walk gives alternating tree automata [Slu]. Thus, the \(\text{CF(Tree)}\) d-acceptor is the alternating top-down finite tree automaton; note that it is just the domain of a top-down tree transducer. In general, the \(\text{CF(Tree}_{id} \times S)\) d-acceptor may be called the alternating top-down \(S\) tree automaton. Similarly, the \(\text{CF(Tree-walk)}\) d-acceptor is the alternating tree walking automaton; its deterministic (i.e., universal) version is just the domain of an attribute grammar.

Instead of One-way, it is possible to consider other ways of handling the input, such as allowing auxiliary Turing machine space. For every space-constructable function \(f\) on the nonnegative integers, the storage type \(\text{SPACE}(f)\) can be defined as \((C, P, F, I, E, m)\) where \(C\) consists of all 4-tuples of strings \((\#w_1, w_2\$, \#v_1, v_2\$) with \(|v_1| + |v_2| = f(|w_1| + |w_2|)\). Intuitively, \(w_1\) is the content of the input tape to the left of the reading head, and \(w_2\) is the remainder of the input tape. Similarly, \(v_1v_2\) is the content of the auxiliary space-restricted Turing machine tape, with the reading head on the first symbol of \(v_2\). The sets \(P\) and \(F\) can be defined so as to model the usual tests and operations on a two-way input tape and a Turing machine tape. Finally, \(I = \Omega^*\), and \(E\) consists of all functions \(e\) that encode a string \(w\) over some alphabet \(\Sigma \subseteq \Omega\) as \(m(e)(w) = (\#, w\$, \#, v\$), where \(v\) is the blank tape of length \(f(|w|)\). Then the \(\text{REG(\text{SPACE}(f)})\) d-acceptor is the nondeterministic \(\text{SPACE}(f)\) Turing machine, and the \(\text{CF(\text{SPACE}(f)})\) d-acceptor is the alternating \(\text{SPACE}(f)\) Turing machine. Also, e.g., the
CF(SPACE(f) × P) d-acceptor is the alternating SPACE(f) auxiliary pushdown automaton [LadLipSto]. Similarly TIME(f) can be defined as a storage type, such that C consists of pairs (u, t), where u codes an input tape and a sequence of Turing machine tapes, and t ∈ N. Initially $t = f(|w|)$ where w is the content of the input tape, and each instruction applied to (u, t) decreases t by 1 (cf. Count-down). In this way, the CF(TIME(f)) d-acceptor is the alternating TIME(f) Turing machine.

We conclude this section with two related, rather technical, observations.

First observation. In the regular case, both $\alpha$-REG(One-way_id × S) and $\lambda$-REG(S) are the class of languages accepted by nondeterministic one-way S automata. In the context-free case there does not seem to be any relationship between the alternating one-way S automaton and the context-free S grammar, i.e., between $\alpha$-CF(One-way_id × S) and $\lambda$-CF(S). Since the alternating finite automata accept the regular languages [ChaKozSto], taking $S = S_0$ gives

$$\alpha$-CF(One-way_id × $S_0$) = $\alpha$-CF(One-way_id) = REG ⊊ CF = $\lambda$-CF($S_0$).

But, since the alternating one-way pushdown automata accept $\bigcup\{\text{DTIME}(c^n) | c > 0\}$, see [ChaKozSto], taking $S = \text{Pushdown}$ gives

$$\lambda$-CF($P$) = Indexed ⊊ $\bigcup\{\text{DTIME}(c^n) | c > 0\} = \alpha$-CF(One-way_id × $P$),

because the indexed languages are context-sensitive and closed under homomorphisms. Thus there is no generally valid inclusion between these classes.

Second observation. However, there does exist a relationship between $\alpha$-RT(One-way_id × S) and $\lambda$-RT(S): it is stated in Theorem 6 of [Eng6] (see [DamGue] for a detailed proof) that

$$\alpha$-CF(One-way_id × S) = $\tau$-RT(One-way_id)⁻¹($\lambda$-RT(S))

where it is assumed that S has an identity and that I is a singleton (cf. Section 7). This establishes a formal link between the alternating one-way S automata and the top-down S tree automata (by way of monadic top-down tree transducers with $\lambda$-moves). In [DamGue], the same result is also proved with One-way_id replaced by Tree_id, establishing an analogous link between the alternating and the nondeterministic top-down S tree automata (by way of top-down tree transducers with $\lambda$-moves). Note that the RT(Tree×S) transducers studied in [DamGue] may be called top-down S tree transducers.
5 Pushdown $S$ automata

The recursion of the context-free grammar can be simulated by the (nonrecursive) pushdown automaton, and, vice versa, every pushdown automaton can be simulated by a context-free grammar. Of course, recursion can always be implemented on a pushdown. Thus, for the context-free grammar we now look for a pushdown-like automaton equivalent to it. Clearly, recursive procedures with one parameter (of type $S$) can be implemented on a pushdown of which each cell contains both a pushdown symbol (the name of a procedure) and an object of type $S$ (its actual parameter), see Fig. 3. Let us define this as a storage type (introduced in [Gre]).

![Diagram](image)

**Fig. 3:** Configuration $(\gamma_m, c_m) \cdots (\gamma_2, c_2)(\gamma_1, c_1)$ of Pushdown of $S$. Only $\gamma_m$ and $c_m$ are accessible.

**Definition 5.1.** Let $S = (C, P, F, I, E, m)$ be a storage type. The storage type *Pushdown of $S$*, abbreviated by $P(S)$, is $(C', P', F', I', E', m')$, where

- $C' = (\Gamma \times C)^+$ for the fixed infinite set $\Gamma$ of pushdown symbols,
- $P' = \{\text{top} = \gamma \mid \gamma \in \Gamma\} \cup \{\text{bottom}\} \cup \{\text{test}(p) \mid p \in P\}$,
- $F' = \{\text{push}(\gamma, f) \mid \gamma \in \Gamma, f \in F\} \cup \{\text{pop}\} \cup \{\text{stay}(\gamma) \mid \gamma \in \Gamma\} \cup \{\text{stay}\}$,
- $I' = I$,
- $E' = \Gamma \times E$, with $m'(\gamma, e)(u) = (\gamma, m(e)(u))$ for every $\gamma \in \Gamma$, $e \in E$, and $u \in I$,

and for every $c' = (\delta, c)\beta$ with $\delta \in \Gamma$, $c \in C$, and $\beta \in (\Gamma \times C)^*$ (intuitively, $(\delta, c)$ is the top of the pushdown $(\delta, c)\beta$),

- $m'(\text{top} = \gamma)(c') = \text{true}$ iff $\delta = \gamma$,
- $m'(\text{bottom})(c') = \text{true}$ iff $\beta = \lambda$,
- $m'(\text{test}(p))(c') = m(p)(c)$,
- $m'(\text{push}(\gamma, f))(c') = (\gamma, m(f)(c))c'$ if $m(f)$ is defined on $c$, and undefined otherwise,
- $m'(\text{pop})(c') = \beta$ if $\beta \neq \lambda$, and undefined otherwise,
• $m'(\text{stay}(\gamma))(c') = (\gamma, c)\beta$, and
• $m'(\text{stay})(c') = c'$.

For $b \in \text{BE}(P)$, we will use test$(b)$ to denote the element of $\text{BE}(P')$ that is obtained from $b$ by replacing every $p \in P$ by test$(p)$.

**Definition 5.2.** The storage type *Pure-pushdown* of $S$, abbreviated by $P_p(S)$, is defined in exactly the same way as $P(S)$, except that $\Gamma = \{ \gamma_0 \}$ for some fixed symbol $\gamma_0$.

Remarks: (1) The storage type *Pushdown* of $S_0$ is almost the same as *Pushdown*. The only difference is that it additionally has $c_0$ in all its pushdown cells. Therefore we may identify the two. Thus $P(S_0) = \text{P}$ (not to be confused with the set $P$ of predicates). Similarly, $P_p(S_0) = \text{Counter}$.

(2) As for $P$, it can be shown that the stay$(\gamma)$ instructions and the bottom predicate are superfluous, see Lemma 3.31 of [EngVog2].

(3) $P(S)$, or rather $P_p(S)$, was introduced, in a slightly different form, in [Gre], where it was called a *nested AFA*. For the reader familiar with [Gre], we now discuss the correspondence in more detail. Let us add instructions stay$(\gamma, f)$ to $P(S)$, with meaning $m'(\text{stay}(\gamma, f))(c') = (\gamma, m(f)(c))\beta$, where $c'$ is as in the definition of $P(S)$ (see Lemma 3.31 of [EngVog2] for the fact that this does not strengthen $P(S)$). Now, if $S$ contains an identity $\text{id}$, then stay$(\gamma)$ is the same as stay$(\gamma, \text{id})$. Moreover, it is easy to see that the push$(\gamma, f)$ instructions can then be restricted to the “duplicate” instructions push$(\gamma, \text{id})$. Thus the instructions now are: push$(\gamma, \text{id})$, pop, stay$(\gamma, f)$, and stay. In this way $P_p(S)$ is very close to the nested AFA. Formal equivalence between $P(S)$ and the nested AFA (in the sense that the corresponding classes of one-way automata are equivalent) holds under a few weak restrictions on $S$.

(4) In Section 1.2 we have seen that the CF$(S)$ grammar is an attribute grammar with one inherited attribute. In [LewRosSte] the *attributed pushdown machine* was defined, and shown to be equivalent to a particular type of attribute grammar (the L attribute grammar, including the case of one inherited attribute). Clearly, a pushdown cell $(\gamma, c)$ of $P(S)$ may be viewed as an attributed pushdown symbol. Thus, in the case of one inherited attribute, the attributed pushdown machine has storage type $P(S)$. To handle synthesized attributes, the attributed pushdown machine also has instructions of the form, say, pop$(f)$ that send attribute values downwards in the pushdown (as opposed to push$(\gamma, f)$ that send them upwards).

Our aim is now to show that the context-free $S$ grammar is equivalent to the one-way $P(S)$ automaton, or, more general, that the CF$(S)$ transducer is equivalent to the REG$(P(S))$ transducer. It turns out that this is true in case $S$ has an identity. However, in general the CF$(S)$ transducer can be simulated by the REG$(P(S))$ transducer but not vice versa. To obtain an equivalence anyway, there are two solutions: either restrict the REG$(P(S))$ transducer or extend the CF$(S)$ transducer. Here we consider the second solution (for the first see [EngVog2], where the so-called bounded excursion pushdown is defined). See Definition 2.3 for the definition of $S_{\text{id}}$.

**Definition 5.3.** Let $S = (C, P, F, I, E, m)$ be a storage type. An extended context-free $S$ transducer, or $\text{CF}_{\text{ext}}(S)$ transducer, is a CF$(S_{\text{id}})$ transducer $(N, e, \Delta, A_{\text{in}}, R)$ such that, for every nonterminal $A$, $A(\text{id})$ may only appear at the end of a rule. In other words: for every rule $A \rightarrow \text{if } b \text{ then } \xi$ of $R$, either $\xi \in (N(F) \cup \Delta)^*$ or $\xi \in (N(F) \cup \Delta)^*N(\{\text{id}\})$. □
The corresponding classes of translations, languages, and input sets are denoted as usual, with subscript \text{ext}.

It is easy to see that there is a \textit{normal form} for \( \text{CF}_{\text{ext}}(S) \) transducers (similar to Chomsky normal form): we may assume that all rules are of one of the forms

\[
A \rightarrow \text{if } b \text{ then } wB(id) \\
A \rightarrow \text{if } b \text{ then } C(f)B(id) \\
A \rightarrow \text{if } b \text{ then } w
\]

where \( A, B, C \in N \), \( b \in \text{BE}(P) \), \( f \in F \), and \( w \in \Delta^* \).

We now state the correspondence between \( \text{CF}(S) \) and \( \text{REG}(P(S)) \). Note that if two classes of translations are equal, then so are the classes of their ranges and of their domains. Thus, “if something holds for \( \tau \)-, then it also holds for \( \lambda \)- and \( \alpha \)-.”

\textbf{Theorem 5.4.} Let \( S \) be a storage type.

1. \( \tau\text{-CF}_{\text{ext}}(S) = \tau\text{-REG}(P(S)) \).
2. If \( S \) has an identity, then \( \tau\text{-CF}(S) = \tau\text{-REG}(P(S)) \).
3. The \( \subseteq \)-inclusions in (1) and (2) also hold for the corresponding deterministic transducers.

\textit{Proof.} If \( S \) has an identity, then that can be used instead of \text{id}; and so \( \tau\text{-CF}_{\text{ext}}(S) = \tau\text{-CF}(S) \). Hence (2) follows from (1). We now show (1) and (3).

“\( \tau\text{-CF}_{\text{ext}}(S) \subseteq \tau\text{-REG}(P(S)) \).” Let \( G = (N,e,\Delta,A_{\text{in}},R) \) be a \( \text{CF}_{\text{ext}}(S) \) transducer in normal form. The \( \text{REG}(P(S)) \) transducer \( G' \) to be constructed simulates, as usual, the left-most derivations of \( G \). To this end it uses the elements of \( N \) as pushdown symbols. The \( S \)-configuration associated to a nonterminal \( A \) in a derivation of \( G \), is stored in the same pushdown cell as \( A \). Thus, \( G' = (N',e',\Delta,A'_{\text{in}},R') \), where \( N' = \{\$\} \), \( e' = (A_{\text{in}},e) \), \( A'_{\text{in}} = \$ \), and \( R' \) is defined as follows (recall the convention concerning the use of \( N(F^+) \) in rules, see Section 1.1).

- If \( A \rightarrow \text{if } b \text{ then } wB(id) \) is in \( R \), then \( R' \) contains the rule
  \[
  \$ \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = A \text{ then } w\$\text{(stay}(B)).
  \]

- If \( A \rightarrow \text{if } b \text{ then } C(f)B(id) \) is in \( R \), then \( R' \) contains the rule
  \[
  \$ \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = A \text{ then } \$\text{(stay}(B); \text{push}(C,f)).
  \]

- If \( A \rightarrow \text{if } b \text{ then } w \) is in \( R \), then \( R' \) contains the rules
  \[
  \$ \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = A \text{ and not bottom then } w\$\text{(pop)}, \quad \text{and}
  \$ \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = A \text{ and bottom then } w.
  \]

This concludes the construction of \( G' \). It should be clear that \( T(G') = T(G) \). Note that if \( G \) is deterministic, then so is \( G' \) (the transformation into normal form also preserves determinism).

“\( \tau\text{-REG}(P(S)) \subseteq \tau\text{-CF}_{\text{ext}}(S) \).” Let \( G = (N,e,\Delta,A_{\text{in}},R) \) be a \( \text{REG}(P(S)) \) transducer, with \( e = (\gamma_0,e') \). An equivalent \( \text{CF}_{\text{ext}}(S) \) transducer will be obtained by the usual triple construction. To facilitate this construction we make three assumptions concerning \( G \). First,
we assume that “final” rules \( A \rightarrow \text{if } b \text{ then } w \) of \( G \) (with \( w \in \Delta^* \)) are only applied when the pushdown contains exactly one cell (clearly, using the bottom predicate, one can reduce the pushdown to one cell just before such a rule is applied). Second, we assume that the bottom predicate is not used by \( G \). Third, we assume that all rules of \( G \) are of the form \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } \xi \), with \( b \in \text{BE}(P) \) and \( \gamma \in \Gamma \). This can be achieved in a straightforward way (see Lemma 3.3, and see Lemma 3.30 of [EngVog2]).

Let \( \Gamma_G \) be the set of all pushdown symbols that occur in \( R \), together with \( \gamma_0 \). We now construct the \( \text{CF}_{\text{ext}}(S) \) transducer \( G' = (N', e', \Delta, A_{\text{in}}, R') \), where \( N' \) consists of all triples \( \langle A, \gamma, B \rangle \) with \( A \in N, \gamma \in \Gamma_G \), and \( B \in N \) or \( B = \omega; \omega \) is a new symbol, indicating the end of a derivation of \( G \); \( A_{\text{in}} = \langle A_{\text{in}}, \gamma_0, \omega \rangle \). Intuitively, \( G' \) has a derivation \( \langle A, \gamma, B \rangle(c) \Rightarrow^* w \) (with \( B \in N \)) iff \( G \), starting in state \( A \) and with a cell containing \( (\gamma, c) \) at the top of its pushdown, can derive \( w \) and \( wB \) that cell from the pushdown, ending in state \( B \) (i.e., \( G \) has a derivation \( A((\gamma, c)B) \Rightarrow^* wB(\beta) \) that does not test \( \beta \)). Similarly, \( \langle A, \gamma, \omega \rangle(c) \Rightarrow^* w \) indicates that \( G' \) has a derivation \( A((\gamma, c)) \Rightarrow^* w \). From this, the following construction of \( R' \) should be clear.

- If \( R \) contains \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } w \), then \( R' \) contains \( \langle A, \gamma, \omega \rangle \rightarrow \text{if } b \text{ then } w \).
- If \( R \) contains \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } wB(\text{pop}) \), then \( R' \) contains \( \langle A, \gamma, B \rangle \rightarrow \text{if } b \text{ then } w \).
- If \( R \) contains \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } wB(\text{push}(\delta, f)) \), then \( R' \) contains the rules \( \langle A, \gamma, C \rangle \rightarrow \text{if } b \text{ then } wB(\delta, E)(f)(E, \gamma, C)(\text{id}) \) for all \( C \in N \cup \{\omega\} \) and \( E \in N \).
- If \( R \) contains \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } wB(\text{stay}(\delta)) \), then \( R' \) contains the rules \( \langle A, \gamma, C \rangle \rightarrow \text{if } b \text{ then } wB(\delta, C)(\text{id}) \) for all \( C \in N \cup \{\omega\} \).
- If \( R \) contains \( A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } wB(\text{pop}) \), then \( R' \) contains the rules \( \langle A, \gamma, C \rangle \rightarrow \text{if } b \text{ then } wB(\gamma, C)(\text{id}) \) for all \( C \in N \cup \{\omega\} \).

This concludes the construction of \( G' \). It is left to the reader to prove formally that \( T(G') = T(G) \). Note that even if \( G \) is deterministic, \( G' \) is not deterministic, due to the choice of the “return nonterminal” \( E \) in the rules of \( G' \) corresponding to the push-rules of \( G \). \qed

It is quite easy to show, for all \( S \), that if \( S \) has an identity, \( \lambda-\text{REG}(S) \) is a full trio, i.e., it is closed under \( a \)-transductions (see [Gin], or [EngVog4]). Thus, using this for \( P(S) \) rather than \( S \), it follows from Theorem 5.4(2) that if \( S \) has an identity, then \( \lambda-\text{CF}(S) \) is a full trio. Probably \( \lambda-\text{CF}(S) \) is even a (full) super-AFL (see [Gre]), under suitable conditions on \( S \). It seems that AFL theory for \( \lambda-\text{RT}(S) \), i.e., for top-down \( S \) tree automata, does not yet exist.

In the remaining part of this section we will try to say more about the deterministic case, in the direction from \( \text{REG}(P(S)) \) to \( \text{CF}_{\text{ext}}(S) \). By the previous theorem we know of course that \( \tau-\text{DREG}(P(S)) \subseteq \tau-\text{CF}_{\text{ext}}(S) \), but actually the extension is not needed. Since this is only based on the fact that the translation defined by a deterministic transducer is a partial function, we state this as follows.

**Lemma 5.5.** Let \( S \) be a storage type, and let \( \text{PF} \) be the class of partial functions. Then \( \tau-\text{CF}_{\text{ext}}(S) \cap \text{PF} \subseteq \tau-\text{CF}(S) \).

**Proof.** Let \( G = (N, e, \Delta, A_{\text{in}}, R) \) be a \( \text{CF}_{\text{ext}}(S) \) transducer in normal form such that \( T(G) \) is a partial function. Construct a \( \text{CF}(S) \) transducer \( G' \) by repeatedly replacing the \( A(\text{id}) \) in the
right-hand side of a rule by the right-hand side of one of A’s rules, avoiding repetition of the same A. Formally, \( G' = (N, e, \Delta, A_{in}, R') \), where \( R' \) is defined as follows.

If \( R \) contains the rules

\[
\begin{align*}
A_1 &\rightarrow \text{if } b_1 \text{ then } \xi_1A_2(\text{id}), \\
A_2 &\rightarrow \text{if } b_2 \text{ then } \xi_2A_3(\text{id}), \\
&\vdots \\
A_{n-1} &\rightarrow \text{if } b_{n-1} \text{ then } \xi_{n-1}A_n(\text{id}), \\
A_n &\rightarrow \text{if } b_n \text{ then } \xi_n
\end{align*}
\]

where \( A_1, \ldots, A_n \) are different nonterminals of \( N \) \((n \geq 1)\), \( b_i \in \text{BE}(P) \), and \( \xi_i \in (N(F) \cup \Delta)^* \), then the rule \( A_1 \rightarrow \text{if } b_1 \text{ and } \ldots \text{ and } b_n \text{ then } \xi_1 \cdots \xi_n \) is in \( R' \).

If we would drop the condition that the \( A_1, \ldots, A_n \) are different, it would be clear that \( T(G') = T(G) \) (if you are not afraid of infinitely many rules). A repetition of, say, \( A \) in \( A_1, \ldots, A_n \) allows for derivations in \( G \) of the form \( A(c) \Rightarrow^* \xi A(c) \). But, derivations in \( G \) of the form \( A_{in}(m(e)(u)) \Rightarrow^* \alpha A(c) \beta \Rightarrow^* \alpha \xi A(c) \beta \Rightarrow^* wxyz \) (with \( \alpha \Rightarrow^* w \in \Delta^* \), etc.) are superfluous; in fact, for the derivation \( A_{in}(m(e)(u)) \Rightarrow^* \alpha A(c) \beta \Rightarrow^* wyz \), we must have \( wyz = wxyz \) because \( T(G) \) is a partial function. This shows that we can restrict ourselves to different \( A_1, \ldots, A_n \), and so \( T(G') = T(G) \). \( \square \)

Remarks: (1) Since the construction in the previous proof preserves determinism of the transducer, we conclude that \( \tau\text{-DCF}_{\text{ext}}(S) = \tau\text{-DCF}(S) \).

(2) Since every \( \text{CF}_{\text{ext}}(S) \) transducer with empty terminal alphabet defines a partial function, we conclude that \( \alpha\text{-CF}_{\text{ext}}(S) = \alpha\text{-CF}(S) \).

(3) An argument similar to the proof of this lemma would show, for \( S = \text{Tree} \), that the \( \text{CF}_{\text{ext}}(\text{Tree}) \) transducer is equivalent to the regularly extended top-down tree-to-string transducer of [EngRozSlu]. \( \square \)

We now know that \( \tau\text{-DREG}(P(S)) \) is in \( \tau\text{-CF}(S) \), and we would like to show that it is even in \( \tau\text{-DCF}(S) \). But in general this is not true. Take for instance \( S = \text{Tree} \). Let \( \sigma \) be of rank 2, and \( a, b \) of rank 0, and consider the tree-to-string translation \( T = \{(\sigma aa, a), (\sigma bb, b)\} \).

It is easy to see (and well known) that \( T \) cannot be defined by a deterministic top-down tree transducer (i.e., a \( \text{DCF(Tree)} \) transducer): then also \( \sigma ab \) and \( \sigma ba \) would be in its domain. On the other hand, it is also easy to see that \( T \) can be defined by a \( \text{DREG}(\text{P(Tree)}) \) transducer: use \( \text{push}(\gamma, \text{sel}_1) \) and \( \text{push}(\gamma, \text{sel}_2) \) to inspect the subtrees of the input tree. Thus \( \tau\text{-DCF}(\text{Tree}) \) is properly included in \( \tau\text{-DREG}(\text{P(Tree)}) \). To solve such problems, the top-down tree transducer was equipped with a look-ahead facility [Eng4, Eng5, EngRozSlu]. To define \( T \), the top-down tree transducer could look ahead at the subtrees of the input tree, to see whether they have the same label.

Let \( G \) be a deterministic \( \text{REG}(P(S)) \) transducer, and consider the equivalent \( \text{CF}_{\text{ext}}(S) \) transducer \( G' \) as defined in the second part of the proof of Theorem 5.4. As observed at the end of that proof, the only determinism of \( G' \) is due to the choice of the “return nonterminal” \( E \) in rules of the form

\[
(A, \gamma, C) \rightarrow \text{if } b \text{ then } w(B, \delta, E)(f)(E, \gamma, C)(\text{id})
\]

that correspond to the push-rules of \( G \). For a configuration \( c \), \( (B, \delta, E)(c) \) generates a terminal string iff \( G \) has a derivation \( B((\delta, c)\beta) \Rightarrow^* wE(\beta) \) that does not test the nonempty pushdown \( \beta \). Clearly, due to the determinism of \( G \), the “return state” \( E \) is uniquely determined by the
“state” $B$, the pushdown symbol $\delta$, and the configuration $c$. In other words, for given $B$, $\delta$ and $c$, there is at most one $E$ such that $\langle B, \delta, E \rangle(c)$ has a successful derivation in $G'$. If $G'$ could test the latter property (i.e., have some knowledge about its own future behavior), then it could pick, deterministically, the unique successful rule for $\langle A, \gamma, C \rangle$ (if it exists). Such tests will be called look-ahead tests (also because in case $S = \text{Tree}$ it corresponds to the above notion of look-ahead at subtrees). Formally, we define them as an extension $S_{LA}$ of a given storage type $S$. Thus, a transducer with look-ahead will not only be able to test its own future behavior, but also that of others.

**Definition 5.6.** Let $S = (C, P, F, I, E, m)$ be a storage type, and let $G = (N, e, \Delta, A_{in}, R)$ be a CF$(S)$ transducer. The set of configurations accepted by $G$ is $\text{Acc}(G) = \{ c \in C \mid A_{in}(c) \Rightarrow^* w$ for some $w \in \Delta^* \}$. □

**Definition 5.7.** For a storage type $S = (C, P, F, I, E, m)$, $S$ with look-ahead is the storage type $S_{LA} = (C, P \cup P', F, I, E, m')$, where $P' = \{ \text{acc}(G) \mid G$ is a CF$(S)$ transducer $\}$, $m'$ restricted to $P \cup F \cup E$ is equal to $m$, and, for every $c \in C$, $m'(\text{acc}(G))(c) = \text{id}(C) \Rightarrow^* c \in \text{Acc}(G)$. □

Note that $\text{Acc}(G) = A(G')$, where $G' = (N, e, \Delta, A_{in}, R)$ is a CF$(S')$ transducer for $S' = (C, P, F, I, \text{en}, \text{en}, m)$, with $m(\text{en}) = \text{id}(C)$. Thus the encoding $e$ of $G$ is irrelevant and its terminal alphabet can be taken empty. This also shows, by Lemma 5.5, that in Definition 5.7 we can equivalently put $P' = \{ \text{acc}(G) \mid G$ is a CF$_{ext}(S)$ transducer $\}$, see the second remark after Lemma 5.5.

Since the domain of a top-down tree(-to-string) transducer is a regular tree language [Rou], it follows that $\text{Acc}(G)$, restricted to some $T_{\Sigma}$, is a regular tree language for every CF$(\text{Tree})$ transducer $G$. Hence the RT$(\text{Tree}_{LA})$ transducer is the top-down tree transducer with regular look-ahead of [Eng4] (and the notions of determinism are the same).

We now show that a deterministic REG(P(S)) transducer can be simulated by a deterministic CF(S$_{LA}$) transducer.

**Theorem 5.8.** For every storage type $S$, $\tau$-DREG(P(S)) $\subseteq$ $\tau$-DCF(S$_{LA}$).

**Proof.** Let $G = (N, e, \Delta, A_{in}, R)$ be a deterministic REG(P(S)) transducer, and let $G' = (N', e', \Delta, A_{in}, R')$ be the CF$_{ext}(S)$ transducer constructed in the second part of the proof of Theorem 5.4, where $N'$ consists of all triples $\langle B, \delta, E \rangle$ with $B \in N$, $\delta \in \Gamma_{G}$, and $E \in N \cup \{ \omega \}$. Note that the assumptions on $G$, made in that proof, preserve determinism. For every triple $\langle B, \delta, E \rangle$ with $E \in N$, and every instruction symbol $f$, define the CF$_{ext}(S)$ transducer $G'((B, \delta, E)(f)) = (N' \cup \{ A \}, e', \Delta, A, R)$ where $A$ is a new nonterminal and $R$ consists of all rules of $R'$ plus the rule $A \rightarrow (B, \delta, E)(f)$. Note that $G'((B, \delta, E)(f))$ can be used as a look-ahead test, as observed after Definition 5.7. We now construct the CF$_{ext}(S_{LA})$ transducer $G''$ from $G'$ by changing every rule

$$\langle A, \gamma, C \rangle \rightarrow \text{if } b \text{ then } w(B, \delta, E)(f)(E, \gamma, C)(\text{id})$$

that corresponds to a push-rule

$$A \rightarrow \text{if } \text{test}(b) \text{ and } \text{top} = \gamma \text{ then } w(B, \delta, E)(\text{push}(\delta, f))$$

of $G$, into the rule

$$\langle A, \gamma, C \rangle \rightarrow \text{if } b \text{ and } \text{acc}(G'((B, \delta, E)(f))) \text{ then } w(B, \delta, E)(f)(E, \gamma, C)(\text{id}).$$
Since $G$ is deterministic, and hence the tests $\text{acc}(G'(\langle B, \delta, E \rangle(f)))$ are mutually disjoint (for fixed $B$, $\delta$, and $f$), it should now be clear that $G''$ is deterministic and equivalent to $G$.

This proves that $\tau\text{-DREG}(P(S)) \subseteq \tau\text{-DCF}_{ext}(S_{LA}) = \tau\text{-DCF}(S_{LA})$, see the first remark after Lemma 5.5. \hfill \Box

We now have $\tau\text{-DCF}(S) \subseteq \tau\text{-DREG}(P(S)) \subseteq \tau\text{-DCF}(S_{LA}) \subseteq \tau\text{-DREG}(P(S_{LA}))$, and in general not more can be said. For $S = \text{Tree}$ it can be shown that $P(\text{Tree}_{LA})$ is “equivalent” to $P(\text{Tree})$, see [EngVog2], and hence $\tau\text{-DREG}(P(\text{Tree})) = \tau\text{-DCF}(\text{Tree}_{LA})$, as shown in Theorem 4.7 of [EngRozSlu] (cf. next section). Also, there exist storage types $S$ that are “closed under look-ahead”, i.e., for which $S_{LA}$ is “equivalent” to $S$ (see [EngVog2, Eng9] for this notion of equivalence). For such storage types the nice equality $\tau\text{-DCF}(S) = \tau\text{-DREG}(P(S))$ holds. As proved in [EngVog2], an example of such a storage type is $P(\text{Tree})$; this was used in [EngVog2] to show that deterministic macro tree-to-string transducers [EngVog1, CouFra] are equivalent to $\tau\text{-DCF}(P(\text{Tree})) = \tau\text{-DREG}(P^2(\text{Tree}))$, i.e., to deterministic $\text{REG}(P(P(\text{Tree})))$ transducers.

Another storage type closed under look-ahead is Pushdown, i.e., $P_{LA}$ is “equivalent” to $P$. In fact, pushdown automata with look-ahead, i.e., $\text{REG}(P_{LA})$-acceptors, are similar to the predicting machines of (Section 10.3 of) [HopUll]. Note how funny the development of the notion of look-ahead is: pushdown automata with look-ahead on the input string (in parsing theory), top-down tree transducers with look-ahead on the input tree, $\text{CF}(S)$ transducers with look-ahead on the storage (note that, very often, the storage is also the input), pushdown automata with look-ahead on the pushdown (also useful in parsing theory, as shown in [EngVog4]).

Anyway, “look-ahead” seems to be a useful tool in several parts of formal language theory.

We have shown that $\tau\text{-DREG}(P(S))$ is included in $\tau\text{-CF}(S) \cap \text{PF}$, and that it is included in $\tau\text{-DCF}(S_{LA})$. This suggests that maybe even $\tau\text{-CF}(S) \cap \text{PF} \subseteq \tau\text{-DCF}(S_{LA})$. We show that this holds provided $S$ is noetherian, which means that there do not exist $c_i \in C$ and $f_i \in F$ for $i \in \mathbb{N}$ such that $m(f_i)(c_i) = c_{i+1}$ for every $i \in \mathbb{N}$. This property of $S$ ensures that a $\text{CF}(S)$ transducer has no infinite derivations. The storage types One-way and Tree are noetherian, but the storage types Pushdown and Tree-walk are not.

For a noetherian storage type $S$, consider a $\text{CF}(S)$ transducer $G$ such that $T(G)$ is a partial function. At each moment of time during a derivation of $G$, several rules may be applicable. Since $T(G)$ is a partial function, it really does not matter which of these rules is taken, as long as application of the rule leads to a successful derivation. This can be detected by a look-ahead test, which allows to pick one of the successful rules, deterministically. Thus, look-ahead can be used to turn semantic determinism into syntactic determinism.

**Theorem 5.9.** Let $S$ be a noetherian storage type, and $\text{PF}$ the class of all partial functions. Then $\tau\text{-CF}(S) \cap \text{PF} \subseteq \tau\text{-DCF}(S_{LA})$.

**Proof.** The proof is entirely analogous to the one for the case $S = \text{Tree}$ in [Eng5]. Let $G = (N, e, \Delta, A_{in}, R)$ be a $\text{CF}(S)$ transducer such that $T(G)$ is a partial function. For every rule $r: A \rightarrow \textbf{if } b \textbf{ then } \xi$, define the $\text{CF}(S)$ transducer $G(r) = (N \cup \{A\}, e, \Delta, A, R)$, where $R$ consists of all rules of $R$ plus the rule $A \rightarrow \textbf{if } b \textbf{ then } \xi$. The acceptor $G(r)$ will be used as a look-ahead test, to see whether $r$ leads to a successful derivation:

\[\text{Acc}(G(r)) = \{ c \in C \mid \text{Acc}(c) \Rightarrow^*_G \xi' \Rightarrow^*_G w \text{ for some } w \in \Delta^* \text{ and } \xi' \in (N(F) \cup \Delta)^* \},\]

where $\Rightarrow^*_G$ means that the derivation step consists of the application of $r$. We now construct the deterministic $\text{CF}(S_{LA})$ transducer $G' = (N, e, \Delta, A_{in}, R')$, where $R'$ is defined as follows.
For a given nonterminal $A$, let $r_1 : A \rightarrow \text{if } b_1 \text{ then } \xi_1, \ldots, r_k : A \rightarrow \text{if } b_k \text{ then } \xi_k$ be all rules in $R$ with left-hand side $A$, numbered arbitrarily from 1 to $k$. Let $p_i = \text{acc}(G(r_i))$, a predicate symbol of $S_{LA}$. For every test $d = q_1 \text{ and } q_2 \text{ and } \cdots \text{ and } q_k$ with $q_i \in \{ p_i, \text{ not } p_i \}$ for all $i$, the rule $A \rightarrow \text{if } d \text{ then } \xi_{i(d)}$ is in $R'$, where $i(d)$ is the first integer $i$, $1 \leq i \leq k$, such that $q_i = p_i$ (if there is such an $i$). For this $G'$, $T(G') = T(G)$.

**New Observation 5.10.** One of the most stupid mistakes that I have made in my entire life (I mean, mathematical mistakes), is to think that the proof of Theorem 5.9 works for arbitrary storage types: in the original version of this report the noetherian condition was not present in Theorem 5.9, and Theorem 5.8 was derived from Lemma 5.5 and Theorem 5.9. This stupid mistake was repeated for two-way pebble transducers in the proof of Theorem 3 of [*EngMan*], which is therefore wrong. Fortunately, Hendrik Jan Hoogeboom discovered the mistake in 2006. I keep wondering about all my mistakes that were not discovered yet . . . .
6 Specific pushdown machines

Applying the theorems of the previous section to the specific storage types discussed in Sections 3 and 4, immediately gives us several known pushdown machine characterizations of the corresponding formalisms. The idea is to convince the reader that these characterizations “trivially” follow from the fact that the formalisms are (or: can be viewed as) context-free $S$ grammars. Let us discuss them one by one (as numbered in Section 3).

1. $S = S_0$. Since $S_0$ has an identity, $\lambda$-$\text{CF}(S_0) = \lambda$-$\text{REG}(P(S_0))$. This shows that the context-free grammar and the one-way pushdown automaton are equivalent (surprise!).

2. $S = \text{Pushdown}$. Since Pushdown has an identity, it follows from Theorem 5.4(2) that $\lambda$-$\text{CF}(P) = \lambda$-$\text{REG}(P(P))$. In other words: the indexed grammar is equivalent to the one-way $P(P)$ automaton. Since a storage configuration of this automaton is a pushdown of pushdowns (see Fig. 4), we also call this the one-way pushdown^2 automaton (or $P^2$ automaton). This result was first mentioned in [Gre], then its proof was sketched in [Mas1, Mas2], and finally it was proved in [ParDusSpe1], where the $P^2$ automaton is defined as indexed pushdown automaton (which means that with each pushdown symbol a sequence of flags is associated). As mentioned in [Mas1, Mas2], and shown in [Eng9, EngVog2], the $P^2$ automaton is equivalent to (and rather close to) the nested stack automaton of [Aho2] (the nested stack may be viewed as a space optimization of the pushdown of pushdowns). Thus, this fits with the equivalence of the indexed grammar and the nested stack automaton, established in [Aho2].

![Fig. 4: A configuration of P(P): a pushdown of pushdowns. Each “small” pushdown $p_i$ is inside a cell of the “big” pushdown. The “small” pushdowns are drawn with their top to the right. Only $\gamma_m$ and the top-cell of $p_m$ are accessible. The division of cells in the “small” pushdowns is not shown.](image)

This storage type $P^2$ might give you the idea to consider $\text{CF}(P^2)$ grammars. By Theorem 5.4(2) again, these are equivalent with the one-way $P^3$ automata, etc. Define in general $P^{n+1}(S) = P(P^n(S))$ and $P^0(S) = S$. Abbreviate $P^n(S_0)$ by $P^n$. The $\text{CF}(P^n)$ grammar is rather close to the n-level indexed grammar of [Mas1, Mas2], restricted to left-most derivations. The one-way $P^n$ automaton is the n-level pushdown automaton of [Mas1, Mas2, DamGoe].
called \emph{n-iterated pushdown automaton} in \cite{Eng9, EngVog2, EngVog3, EngVog4}. Theorem 5.4(2) implies that \(\lambda\text{-CF}(P^n) = \lambda\text{-REG}(P^{n+1})\), as shown in \cite{Mas1, Mas2}. Thus, the \(n\)-level indexed grammars are equivalent to the \((n + 1)\)-iterated pushdown automata.

\(3\) \(S = \text{Counter}\). We get \(\lambda\text{-CF}(\text{Counter}) = \lambda\text{-REG}(P(\text{Counter}))\), a special case of the pushdown\(^2\) automaton.

\(4\) \(S = \text{Count-down}\). The one-way \(P(\text{Count-down})\) automaton is the \emph{preset pushdown automaton} of \cite{vLe1}. It is just like an ordinary pushdown automaton, except that at the beginning of each computation the length of the pushdown is preset to some number of cells. The automaton “knows” when it reaches this “ceiling”, and can react to it. In fact, the number of cells is chosen (nondeterministically) by the encoding, and is decreased by one by each \(\text{push}(\gamma, \text{dec})\) instruction. The “ceiling” can be detected by the predicate test(null).

As an example (taken from \cite{vLe1}) we give a regular \(P(\text{Count-down})\) transducer \(G_7\) with \(L(G_7) = \{a^n^2 \mid n \geq 1\}\). When the pushdown is preset to \(k\) cells (i.e., the encoding chooses \(k - 1\)), \(G_7\) first generates \(k\) \(a\)'s, by moving to the ceiling and back again. Then \(G_7\) pushes twice, and generates \(k - 2\) \(a\)'s, by moving to the ceiling, and back to the same square, marked for that purpose. Then \(G_7\) pushes twice, etc. In this way \(G_7\) generates \(k + (k - 2) + \cdots + 3 + 1\) \(a\)'s (assuming that \(k\) is odd), i.e., all squares. Formally \(G_7 = (N, e, \{a\}, A_{in}, R)\), where \(N = \{A, B, C\}\), \(A_{in} = A, e = (\#, \text{en})\), and \(R\) consists of the following rules (pushdown symbols \(\#\) and \$ are used; \# as marker, and \$ as a dummy symbol).

\[
\begin{align*}
A & \rightarrow \text{ if not test(null) then } aB(\text{push($\$, dec)}) \text{ else } a \\
B & \rightarrow \text{ if not test(null) then } aB(\text{push($\$, dec)}) \text{ else } aC(\text{pop}) \\
C & \rightarrow \text{ if not top = $\#$ then } C(\text{pop}) \text{ else } A(\text{push($\#$, dec); push($\#$, dec)})
\end{align*}
\]

This concludes the example.

By Theorem 5.4(1), \(\lambda\text{-REG}(P(\text{Count-down})) = \lambda\text{-CF}_{ext}(\text{Count-down})\). In other words, the preset pushdown automaton is equivalent to what might be called the extended EOL system. It is easy to see (cf. the proof of Lemma 5.5) that this extended EOL system can be viewed as an EOL system in which the right-hand sides of rules are regular languages: a so-called REG-iteration grammar, with one substitution (see \cite{RozSal, Asv, Eng3}; in \cite{vLe1} the corresponding class of languages is called the hyper-algebraic extension of REG). This equivalence of preset pushdown automata to REG-iteration grammars was proved in \cite{vLe1}. To obtain an automaton equivalent to the EOL system, the preset pushdown automaton should be restricted to have the bounded excursion property (see \cite{vLe1, EngVog2}).

\(5\) \(S = \text{One-way}\). By a rather easy argument it can be shown that \(\lambda\text{-CF}_{ext}(\text{One-way}) = \lambda\text{-CF}(\text{One-way})\), cf. Lemma 3.3.2 of \cite{EngRozSlu}. Thus Theorem 5.4(1) implies that the one-way \(P(\text{One-way})\) automaton is equivalent to the ETOL system. In fact, this automaton is the \emph{checking-stack/pushdown automaton} (CS-PD automaton), introduced in \cite{vLe2} where this equivalence was proved (see also \cite{EngSchvLe, EngRozSlu, RozSal}). To see that the \(\text{REG}(P(\text{One-way}))\) \(r\)-acceptor is the CS-PD automaton, note that the pushdown of a \(\text{REG}(P(\text{One-way}))\) \(r\)-acceptor \(G\) is always of the form \((\gamma_m, c_m) \cdots (\gamma_2, c_2)(\gamma_1, c_1)\)
with \[ c_1 = a_1 a_2 \ldots \ldots a_n \]
\[ c_2 = a_2 \ldots \ldots a_n \]
\[ \ldots \]
\[ c_m = a_m \ldots a_n \]

where \( a_1 a_2 \ldots a_n \) is the string “guessed” by the encoding of \( G \) at the beginning of its computation. Another, obviously equivalent, way of representing this storage configuration is by an ordinary pushdown \( \gamma_m \ldots \gamma_2 \gamma_1 \) and a checking stack \( a_1 a_2 \ldots a_n \), with their reading heads combined into one, pointing to \( \gamma_m \) and \( a_m \), see Fig. 5. The one reading head moves in a two-way fashion over the checking stack, synchronously with the movements of the top of the pushdown.

A push(\( \gamma \), read) instruction moves the reading head to the right, and a pop instruction moves it to the left. This is precisely the behavior of a CS-PD automaton.

It can be shown that \( P(\text{One-way}_{LA}) \) is “equivalent” to \( P(\text{One-way}) \): it is the monadic case of \( P(\text{Tree}_{LA}) = P(\text{Tree}) \), see [EngVog2]. Hence \( \lambda\text{-DREG}(P(\text{One-way})) = \lambda\text{-DCF}(\text{One-way}_{LA}) \), cf. the discussion after Theorem 5.8. It can also easily be shown that \( \lambda\text{-DCF}(\text{One-way}_{LA}) = \lambda\text{-DCF}(\text{One-way}) \), cf. Lemma 3.3.4 of [EngRozShu]. Hence \( \lambda\text{-DREG}(P(\text{One-way})) \) is the class of EDTOL languages. Note that the D stands for transducer determinism (thus EDTOL does not correspond to the deterministic CS-PD automaton), cf. Section 5 of [EngSchvLe].

Let us consider \( P(\text{One-way}) \) with just one pushdown symbol, i.e., \( P_p(\text{One-way}) \). It should be clear from the discussion above that \( P_p(\text{One-way}) \) could be called \textit{Two-way}: the storage type corresponding to a two-way read-only tape (with endmarkers). Thus, the \( \text{REG}(P_p(\text{One-way})) \) transducer is the \textit{two-way finite state transducer} or two-way gsm, the \( \text{CF}(P_p(\text{One-way})) \) d-acceptor is the \textit{alternating two-way finite automaton}, and the \( \text{REG}(P_p(\text{One-way})) \) r-acceptor is the one-way \textit{checking stack automaton}.

Fig. 5(a): Configuration of \( P(\text{One-way}) \).

\[\begin{array}{c|c|c|c}
\gamma_1 & c_1 & \gamma_2 & \ldots & \gamma_m & c_2 & \ldots & c_m \\
\hline
c_1 & \gamma_1 & c_2 & \ldots & c_m & \gamma_2 & \ldots & \gamma_m \\
\hline
\end{array}\]

with
\[
c_1 = a_1 \ a_2 \ | \ldots \ | \ a_n \\
c_2 = \ | \ldots \ | \ a_n \\
\vdots \\
c_m = \ a_m \ | \ldots \ | \ a_n
\]

\[\begin{array}{c|c|c|c}
\gamma_1 & \gamma_2 & \ldots & \gamma_m \\
\hline
a_1 & a_2 & \ldots & a_m & \ldots & a_n \\
\end{array}\]

Fig. 5(b): Configuration of CS-PD.
(6) $S = \text{Tree}$. Theorem 5.4(1) shows that $\tau$-\text{CF}_{\text{ext}}(\text{Tree}) = \tau$-\text{REG}(P(\text{Tree})); see Theorem 4.5 of [EngRozSlu]. The \text{CF}_{\text{ext}}(\text{Tree}) transducer is the \textit{regularly extended top-down tree transducer}. Moreover, the \text{REG}(P(\text{Tree})) transducer is the checking-tree/pushdown transducer (CT-PD transducer). To see this, note that, as for the previous case of $S = \text{One-way}$, the pushdown of a \text{REG}(P(\text{Tree})) transducer is always of the form $(\gamma_m, t_m) \cdots (\gamma_2, t_2)(\gamma_1, t_1)$ where $t_1$ is the input tree (as given by the encoding), and $t_{i+1}$ is a direct subtree of $t_i$, for all $i$. Therefore, this storage configuration can also be represented by an ordinary pushdown (viz. $\gamma_1 \cdots \gamma_2 \gamma_1$) and a tree (viz. $t_1$) with a pointer to one of its nodes (viz. the one corresponding to the subtree $t_m$ of $t_1$). Again, the pushdown pointer and the tree pointer are combined into one: a $\text{push}(\gamma, \text{sel}_i)$ moves the pointer down in the tree to the $i$-th son of the current node, and a $\text{pop}$ moves it up to its father. Thus, the length of the pushdown equals the length of the path from the root to the current node of the input tree. This is precisely the storage type of the CT-PD transducer, see Fig. 6. In programming terminology it is just the familiar fact that tree walking (CT-PD) can be implemented by a pushdown of pointers to the tree $(\text{P}(\text{Tree}))$.

![Fig. 6: To the left a configuration of $\text{P}(\text{Tree})$, and to the right a configuration of CT-PD, where $n_i$ is the root of the subtree $t_i$, and $n_{i+1}$ is a son of $n_i$.](image)

Note that a $\text{P}(\text{Tree})$ configuration can be represented by several CT-PD configurations: a sequence of consecutive direct subtrees does not uniquely determine the path in the tree (think, e.g., of a full binary tree over $\{\sigma, a\}$, with $\text{rank}(\sigma) = 2$ and $\text{rank}(a) = 0$). However, there is still a one-to-one correspondence between the instructions and predicates of $\text{P}(\text{Tree})$ and those of CT-PD. In fact, the \textit{storage type CT-PD} can easily be defined, as a generalization of Tree-walk. Its instructions are: $\text{down}_i(\gamma)$, to move down to the $i$-th son and push $\gamma$; $\text{up}$, to move up to the father and pop; $\text{stay}(\gamma)$, to stay at the same node and change the top of the pushdown to $\gamma$; and $\text{stay}$, the identity. Its predicates are: $\text{top} = \gamma$, to test the top of the pushdown; $\text{label} = \sigma$, to test the label of the current node; and $\text{root}$, to see whether the current node is the root of the input tree. The correspondence between the instructions and predicates of CT-PD and those of $\text{P}(\text{Tree})$ is as follows.

\begin{align*}
\text{CT-PD:} & \quad \text{down}_i(\gamma) \quad \text{up} \quad \text{stay}(\gamma) \quad \text{stay} \quad \text{top} = \gamma \quad \text{label} = \sigma \quad \text{root} \\
\text{P(Tree):} & \quad \text{push}(\gamma, \text{sel}_i) \quad \text{pop} \quad \text{stay}(\gamma) \quad \text{stay} \quad \text{top} = \gamma \quad \text{test(root} = \sigma) \quad \text{bottom}
\end{align*}

Thus, we may identify CT-PD and $\text{P}(\text{Tree})$. 
As noted in Section 5, the storage types \( P(\text{Tree}_{\text{LA}}) \) and \( P(\text{Tree}) \) are equivalent, and hence \( \tau\text{-DCF}(\text{Tree}_{\text{LA}}) = \tau\text{-DREG}(P(\text{Tree})) \), i.e., the deterministic top-down tree transducer with regular look-ahead is equivalent to the deterministic CT-PD transducer, as shown in Theorem 4.7 of [EngRozSlu].

The storage type \( P(\text{Tree}) \) might give the reader the idea to consider \( CF(P(\text{Tree})) \) transducers: by Theorem 5.4(2) they are equivalent to \( \text{REG}(P^2(\text{Tree})) \) transducers. Since \( P(\text{Tree}) \) is “closed under look-ahead”, this also holds for the deterministic transducers. In fact, both the \( RT(P(\text{Tree})) \) transducer and the \( \text{REG}(P^2(\text{Tree})) \) transducer are studied in [EngVog2], where they are called \textit{indexed tree transducer} (because of the similarity to the \( CF(P) \) grammar, i.e., the indexed grammar) and \textit{pushdown}\(^2\) \textit{tree-to-string transducer}, respectively. Let us point out (again) a correspondence between alternation and tree transducers. In (5) of this section we noted that the alternating two-way finite automaton is the \( \text{CF}(P_p(\text{One-way})) \) \( \alpha \)-acceptor. Since One-way is the monadic case of Tree, \( P_p \) is a special case of \( P, \) and \( CF \) is the yield of \( RT, \) it follows that every \( \text{CF}(P_p(\text{One-way})) \) \( \alpha \)-acceptor may be viewed as an \( RT(P(\text{Tree})) \) \( \alpha \)-acceptor: \( \alpha\text{-CF}(P_p(\text{One-way})) \subseteq \alpha\text{-RT}(P(\text{Tree})). \) Thus, the alternating two-way finite automaton languages are domains of indexed tree transducers. In Lemma 6.11 of [EngVog2] it is shown that \( \alpha\text{-RT}(P(\text{Tree})) = \alpha\text{-RT}(\text{Tree}), \) which equals \( RT. \) From this follows the fact, shown in [LadLipSto], that \( \alpha\text{-CF}(P_p(\text{One-way})) = \text{REG}. \) Thus, regularity of the domains of indexed tree transducers implies, “immediately”, regularity of the languages accepted by alternating two-way finite automata.

Since \( \alpha\text{-RT}(P(\text{Tree})) = \alpha\text{-RT}(\text{Tree}) = RT, \) it follows that also \( \alpha\text{-REG}(P(\text{Tree})) = RT \) (because \( \alpha\text{-CF}(\text{Tree}) \subseteq \alpha\text{-REG}(P(\text{Tree})) \subseteq \alpha\text{-CF}(P(\text{Tree})). \) Hence, the \( \text{REG}(P(\text{Tree})) \) \( \alpha \)-acceptor accepts the regular tree languages; it is a tree walking automaton with a synchronized pushdown. Thus, by New Observation 3.9, \( \alpha\text{-REG}(\text{Tree-walk}) \) is properly included in \( \alpha\text{-REG}(P(\text{Tree})). \) Note that the class \( \alpha\text{-CF}(P(\text{Tree})) \) of domains of indexed tree transducers may also be viewed as the class of tree languages accepted by the \( CF(P(\text{Tree})) \) \( \alpha \)-acceptor, i.e., the alternating \( P(\text{Tree}) \) automaton; from this point of view \( \alpha\text{-CF}(P(\text{Tree})) = RT \) was shown in [Slu].

(7) \( S = \text{Tree-walk}. \) From the previous discussion it should now be clear that, apart from the \( \text{son} = i \) predicates, Tree-walk is the same as \( P_p(\text{Tree}). \) In fact, using the natural numbers as pushdown symbols, Tree-walk can be simulated by \( P(\text{Tree}) \) as follows: \( \text{down}_i \) corresponds to \( \text{push}(i, \text{sel}_i), \) up to \( \text{pop}, \) stay to \( \text{stay}, \) \( \text{label} = \sigma \) to \( \text{test}((\text{root} = \sigma), \) \( \text{son} = i \) to \( \text{top} = i, \) and root to \( \text{bottom}. \) Thus, \( \tau\text{-DRT}(\text{Tree-walk}) \subseteq \tau\text{-DRT}(P(\text{Tree})), \) i.e., the attribute grammar is a special case of the deterministic indexed tree transducer. Similarly \( \tau\text{-REG}(\text{Tree-walk}) \subseteq \tau\text{-REG}(P(\text{Tree})), \) i.e., the tree walking automaton of \([\text{AhoUll}]\) is a special case of the CT-PD transducer. Note that Two-way = \( P_p(\text{One-way}) \) is the monadic case of Tree-walk (in the monadic case the \( \text{son} = i \) predicates are superfluous).

Theorem 5.4(2) shows that \( \tau\text{-CF}(\text{Tree-walk}) = \tau\text{-REG}(P(\text{Tree-walk})). \) Although the deterministic case of this equation also involves look-ahead, it should be clear that there is a close relationship between the attribute grammar (with strings as values) and the deterministic \( \text{REG}(P(\text{Tree-walk})) \) transducer. This relationship was pointed out in [Kam], where the \( \text{REG}(P(\text{Tree-walk})) \) transducer is called the \textit{tree-walking (synchronized) pushdown tree-to-string transducer}. Intuitively, it is a tree-walking automaton that can back-track on the path it has walked (cf. also the description of \( P(P(\text{Tree})) \) in [EngVog2]). Since \( P(\text{Tree-walk}) \) can be simulated by \( P^2(\text{Tree}), \) it is a special case of the pushdown\(^2\) tree-to-string transducer. The
alternating version of the corresponding d-acceptor was shown to accept the regular tree languages in [Slu]. This fits with the fact that domains of RT(P^2(Tree)) transducers are regular (cf.[EngVog2, EngVog3]).

Since trees are strings, \(\tau\)-DRT(Tree-walk) \(\subseteq\) \(\tau\)-CF(Tree-walk) = \(\tau\)-REG(P(Tree-walk)). In particular when we would write trees in postfix rather than prefix notation (i.e., \(\sigma t_1 \cdots t_k\) rather than \(\sigma t_1 \cdots t_k\sigma\)) it would be quite natural to output a tree symbol by symbol on the output tape of a REG(P(Tree-walk)) transducer: these symbols may be viewed as instructions to operate on a stack of (attribute) values, in the usual way. Note however, that this is still the same, very inefficient, way of evaluating attributes.

It may be worthwhile to see what happens if one extends attribute grammars (in their usual notation) to “attribute grammars with pushdown”, which are equivalent to DRT(P(Tree)) transducers (i.e., deterministic indexed tree transducers), in the same way as attribute grammars are equivalent to DRT(Tree-walk) transducers.

(8) Since Tree-pushdown is closely related to Pushdown, P(Tree-pushdown) is closely related to P^2. We note here that Tree-pushdown has been generalized to an operator Tree-pushdown of \(S\), abbreviated TP(S), in [EngVog2]. The RT(TP(Tree)) transducer with one state only is the macro tree transducer of [CouFra, Eng6, EngVog1], and it is closely related to the RT(P(Tree)) transducer, i.e., the indexed tree transducer (see [EngVog2]).

(9) The last storage type we consider is SPACE(\(f\)), as discussed in Section 4. By Theorem 5.4(2), \(\alpha\)-CF(SPACE(\(f\)) \(\times\) \(S\)) = \(\alpha\)-REG(P(SPACE(\(f\)))). We now note that it is quite easy to see that the storage type P(SPACE(\(f\)) can be simulated by SPACE(\(f\)) \(\times\) P (see Definition 4.1 for the product of storage types). In fact, a pushdown \((\gamma_m, c_m)(\gamma_{m-1}, c_{m-1}) \cdots (\gamma_1, c_1)\) where \(c_m, c_{m-1}, \ldots, c_1\) are SPACE(\(f\)) configurations, can be represented by the SPACE(\(f\)) configuration \(c_m\) and the ordinary pushdown \(\gamma_{m-1}c_{m-1} \cdots \gamma_1c_1\) (where it is understood that the \(c_i\) are put on the pushdown, coded as strings in the usual way); \(\gamma_m\) can be kept in the control of the involved transducer. Thus we obtain that \(\alpha\)-CF(SPACE(\(f\)) \(\subseteq\) \(\alpha\)-REG(SPACE(\(f\)) \(\times\) P)). In other words, the alternating SPACE(\(f\)) Turing machine can be simulated by the non-deterministic SPACE(\(f\)) auxiliary pushdown automaton [Coo]. In fact, the above inclusion is an equality: the well-known relationship between alternating Turing machines and auxiliary pushdown automata (see [Ruz]). The inclusion in the other direction requires a different construction.

More generally, \(\alpha\)-CF(SPACE(\(f\)) \(\times\) \(S\)) = \(\alpha\)-REG(SPACE(\(f\)) \(\times\) P(\(S\))) for every storage type \(S\) that has an identity, see Theorem 2 of [Eng9]. Note that, in this equation, SPACE(\(f\)) cannot be replaced by One-way_{id}. In fact, \(\alpha\)-REG(One-way_{id} \(\times\) P(\(S\)) = \(\lambda\)-REG(P(\(S\)) = \(\lambda\)-CF(S), and we have seen at the end of Section 4 (first observation) that, in general, \(\alpha\)-CF(One-way_{id} \(\times\) \(S\)) and \(\lambda\)-CF(S) are not equal.

New Observation 6.1. Another stupid mistake in the original version of this report, was the statement that it is quite easy to see that P(SPACE(\(f\)) and SPACE(\(f\)) \(\times\) P are equivalent storage types. This was a typical case of wishful thinking: it would have been so nice if Theorem 5.4 would have “explained” the relationship between alternating Turing machines and auxiliary pushdown automata. The mistake was discovered by Roland Bol in 1987, when he attended my lectures on the subject.
7 Deterministic r-acceptors

In the next section we will show that, for a restricted kind of storage type $S$, the languages generated by CF($S$) grammars can be obtained from the languages accepted by deterministic one-way $S$ automata, by a certain class of operations on languages. Therefore we give in this section the definition of determinism for REG($S$) r-acceptors (cf. Section 2). We call this $r$-acceptor determinism to distinguish it from the (transducer) determinism discussed up to now. It has to be admitted at this point that the deterministic $r$-acceptors do not fit so nicely in our framework of CF($S$) grammars, due to the different ways in which they end their computations.

What is a deterministic REG($S$) $r$-acceptor? First of all we have to require that the encoding cannot be used to “guess” an initial configuration; we formulate this as a property of the storage type (and since we consider one-way $S$ automata, we also require an identity; see Section 2).

**Definition 7.1.** A storage type $S = (C, P, F, I, E, m)$ is $r$-acceptor deterministic if $S$ has an identity, and $I$ is a singleton. □

Thus, for such a storage type, with, say, $I = \{u_0\}$, an encoding $e$ just determines a fixed initial configuration $m(e)(u_0)$. Most of the usual storage types for one-way $S$ automata are $r$-acceptor deterministic. The particular ones discussed in this paper are Counter and the iterated pushdown storage types $P^n$, for $n \geq 0$. The results of the next section will be applied to the one-way iterated pushdown automata.

We now define determinism of $r$-acceptors in the obvious way. Recall from Section 2 the intuitive interpretation of a rule $A \rightarrow \text{if } b \text{ then } wB(f)$ of a REG($S$) $r$-acceptor $G$. Note that such a rule is applicable if $G$ is in state $A$, $b$ holds for its current storage configuration, and $w$ is a prefix of the rest of the input. Determinism should ensure the applicability of at most one rule in every situation. Recall also the notion of normal form from Section 2: every rule is of the form $A \rightarrow \text{if } b \text{ then } \xi$, where $\xi = aB(f)$ or $\xi = B(f)$ or $\xi = a$ or $\xi = \lambda$ (with $a \in \Delta$).

**Definition 7.2.** Let $S$ be an $r$-acceptor deterministic storage type. A REG($S$) transducer $G = (N, e, \Delta, A_{in}, R)$ is $r$-acceptor deterministic if the following two conditions hold.

1. $G$ is in normal form.
2. If $A \rightarrow \text{if } b_1 \text{ then } a_1Q_1$ and $A \rightarrow \text{if } b_2 \text{ then } a_2Q_2$ are two different rules in $R$ (with $a_i \in \Delta \cup \{\lambda\}$ and $Q_i \in N(F) \cup \{\lambda\}$) such that $a_1 = a_2$ or $a_1 = \lambda$ or $a_2 = \lambda$, then $m(b_1 \text{ and } b_2)(c) = \text{false}$ for every $c \in C$. □

Rather than “$r$-acceptor deterministic REG($S$) transducer” we will also say “deterministic REG($S$) $r$-acceptor”.

An example of an $r$-acceptor deterministic REG(P) transducer is $G_2$, discussed in Sections 1.1 and 3(2), in which the third rule should be replaced by the rule $A \rightarrow bB(pop)$. This transducer is not deterministic. Thus, every deterministic REG($S$) transducer is $r$-acceptor deterministic, but not vice versa.

However, the deterministic REG($S$) $r$-acceptors do not yet correspond to the usual deterministic one-way $S$ automata. The reason is that they decide deterministically when the input string should end; hence they only accept prefix-free languages: a language $L \subseteq \Delta^*$ is prefix-free if $w \in L$ implies $wv \notin L$ for all $v \in \Delta^+$. Thus, they accept, so to say, by empty storage, and what we need is acceptance by final state.
Definition 7.3. Let $G = (N, e, \Delta, A_{\text{in}}, R)$ be a deterministic REG($S$) r-acceptor. Then $L(G)$ is called the language of $G$ accepted by empty store. For a set $N_H \subseteq N$ of final states, the language of $G$ and $N_H$ accepted by final state, denoted $L(G, N_H)$, is defined by $L(G, N_H) = \{ w \in \Delta^* | A_{\text{in}}(m(e)(u_0)) \Rightarrow_G^* wA(c) \text{ for some } A \in N_H \text{ and } c \in C \}$, where $I = \{ u_0 \}$.

We denote by $\lambda$-$D_r$-REG($S$) the class \{ $L(G) \mid G$ is a deterministic REG($S$) r-acceptor \}, and by $\lambda$-$D_f$-REG($S$) the class \{ $L(G, N_H) \mid G$ is a deterministic REG($S$) r-acceptor and $N_H$ a subset of its set of states \}. It should be clear to the reader that

“$\lambda$-$D_f$-REG($S$) = deterministic one-way $S$ automaton languages”.

Lemma 7.4. $\lambda$-$D_r$-REG($S$) $\subseteq$ $\lambda$-$D_f$-REG($S$) $\subseteq$ $\lambda$-REG($S$).

Proof. First inclusion: replace every rule $A \rightarrow \text{if } b \text{ then } w$ (with $w \in \Delta^*$) by the rule $A \rightarrow \text{if } b \text{ then } wQ(\text{id})$, where $Q$ is a (new) final state, and id is the identity of $S$. Second inclusion: remove all rules $A \rightarrow \text{if } b \text{ then } w$ with $A \in N_H$ (which are superfluous when accepting by final state) and add rules $A \rightarrow \lambda$ for all $A \in N_H$.

In fact, $\lambda$-$D_r$-REG($S$) is the class of prefix-free languages in $\lambda$-$D_f$-REG($S$), a situation that is familiar from the deterministic pushdown automata; proof: let $L(G, N_H)$ be prefix-free; for every $A \in N_H$, replace all rules with left-hand side $A$ by the one rule $A \rightarrow \lambda$.

In the next section we will also show that the tree languages accepted by deterministic top-down $S$ tree automata can be obtained from the languages accepted by deterministic one-way $S$ automata, by certain string-to-tree operations. Therefore we also define determinism for RT($S$) r-acceptors, in the obvious way. Recall the notion of normal form from Section 2: every rule is of the form $A \rightarrow \text{if } b \text{ then } \xi$, where $\xi = B(f)$ or $\xi = \sigma B_1(f_1) \cdots B_k(f_k)$ (with $\sigma \in \Delta_k$).

Definition 7.5. Let $S$ be an r-acceptor deterministic storage type. An RT($S$) transducer $G = (N, e, \Delta, A_{\text{in}}, R)$ is r-acceptor deterministic if the following two conditions hold.

1. $G$ is in normal form.
2. If $A \rightarrow \text{if } b_1 \text{ then } \sigma_1Q_1$ and $A \rightarrow \text{if } b_2 \text{ then } \sigma_2Q_2$ are two different rules in $R$ (with $\sigma_i \in \Delta \cup \{ \lambda \}$ and $Q_i \in N(F)^*$) such that $\sigma_1 = \sigma_2$ or $\sigma_1 = \lambda$ or $\sigma_2 = \lambda$, then $m(b_1 \text{ and } b_2)(c) = \text{false for every } c \in C$.

We denote by $\lambda$-$D_r$-RT($S$) the class \{ $L(G) \mid G$ is a deterministic RT($S$) r-acceptor \}. For $S = S_0$, this is the class of tree languages accepted by deterministic top-down finite tree automata (rules $A \rightarrow B(\text{id})$ can easily be removed). For $S = P$, it is the class of tree languages accepted by deterministic pushdown tree automata [Gue2].

Although tree languages are prefix-free “by nature” (viewing trees as strings), there is still a difference between $\lambda$-$D_r$-REG($S$) and $\lambda$-$D_r$-RT($S$), due to the fact that there is an empty string but no empty tree. In fact, a deterministic REG($S$) r-acceptor can read the last symbol of the input string, and then check some property of the storage configuration (reading the empty string). On the other hand, a deterministic RT($S$) r-acceptor has to decide whether to accept or reject as soon as it sees a symbol of rank 0. We now define RT($S$) r-acceptors that can “read beyond the leaves” by way of a trick, as follows (see [Gue1]).
**Definition 7.6.** For a ranked alphabet $\Delta$, and $\# \not\in \Delta$, $\Delta#$ is the ranked alphabet with $\Delta#_0 = \{\#\}$, $\Delta#_1 = \Delta_0 \cup \Delta_1$, and $\Delta#_k = \Delta_k$ for $k \geq 2$. The mapping \text{mark} : T_\Delta \to T_{\Delta#}$ is defined as follows: for $t \in T_\Delta$, \text{mark}(t) is the result of replacing, in $t$, every $\sigma$ by $\sigma#$, for all $\sigma \in \Delta_0$. For a tree language $L$, $\text{mark}(L) = \{\text{mark}(t) \mid t \in L\}$. \hfill $\square$

Thus every leaf $\sigma$ of a tree is made to be of rank 1, and a new leaf $\#$ is attached to it, i.e., it is replaced by the tree $\sigma(\#)$.

We denote by $\lambda$-$\text{D}_f\text{RT}(S)$ the class $\{L \mid \text{mark}(L) \in \lambda$-$\text{D}_f\text{RT}(S)\}$.

**Lemma 7.7.** $\lambda$-$\text{D}_f\text{RT}(S) \subseteq \lambda$-$\text{D}_e\text{RT}(S) \subseteq \lambda$-$\text{RT}(S)$.

**Proof.** First inclusion: replace every rule $A \to \text{if } b \text{ then } \sigma$ with $\sigma \in \Delta_0$ by the rule $A \to \text{if } b \text{ then } \sigma Q(\text{id})$, where $Q$ is a new nonterminal (and $\text{id}$ is the identity of $S$), and add the rule $Q \to \#$. Second inclusion: introduce new nonterminals $B_\sigma$ for every nonterminal $B$ and every $\sigma \in \Delta_0$; replace every rule $A \to \text{if } b \text{ then } \sigma B(f)$, with $\sigma \in \Delta_0$ (and so $\sigma \in (\Delta#)_1$), by the rule $A \to \text{if } b \text{ then } B_\sigma(f)$; if $A \to \text{if } b \text{ then } B(f)$ is a rule, then add the rules $A_\sigma \to \text{if } b \text{ then } B_\sigma(f)$, for every $\sigma \in \Delta_0$; replace every rule $A \to \text{if } b \text{ then } \#$ by the rules $A_\sigma \to \text{if } b \text{ then } \sigma$, for every $\sigma \in \Delta_0$. \hfill $\square$

In case $S$ is “closed under look-ahead” (e.g., $S = \text{P}$ or $S = S_0$), these two ways of acceptance are the same, i.e., $\lambda$-$\text{D}_e\text{RT}(S) = \lambda$-$\text{D}_f\text{RT}(S)$. To see this, we prove the following fact.

**Lemma 7.8.** $\lambda$-$\text{D}_f\text{RT}(S) \subseteq \lambda$-$\text{D}_e\text{RT}(S_{\text{LA}})$.

**Proof.** Let $L \in \lambda$-$\text{D}_f\text{RT}(S)$, and let $G = (N, e, \Delta#, A_{\text{in}}, R)$ be a deterministic $\text{RT}(S)$ $r$-acceptor such that $L(G) = \text{mark}(L)$. Delete every rule in which $\#$ occurs. Replace every rule $A \to \text{if } b \text{ then } \sigma B(f)$, with $\sigma \in \Delta_0$, by the rule $A \to \text{if } b \text{ and acc}(G(B, f)) \text{ then } \sigma$. In this rule, $G(B, f)$ is the $\text{RT}(S)$ transducer $(N \cup \{A\}, e, \Delta#, A, R)$, where $A$ is a new initial nonterminal, and $R$ is $R$ plus the rule $A \to B(f)$. \hfill $\square$

This concludes our discussion of determinism for $r$-acceptors.
8 A new operation on languages

One of the questions in formal language theory (particularly in AFL/AFA theory) is whether one class of languages can be obtained from another class by the application of certain operations on languages. As an example, the context-free languages can be obtained from the regular languages by the application of nested iterated substitutions, and the recursively enumerable languages can be obtained from the (deterministic) context-free languages by the application of intersections and homomorphisms (in both cases the resulting class of languages is also closed under these operations). One of the concrete questions in this area is whether it is possible to obtain the indexed languages from the context-free languages by a class of (natural) operations (see [Gre]). The adjective “natural” is important here: one can, e.g., always view an indexed grammar as an operation on languages. We give a partial answer to this question: we define a class $\delta$ of unary operations on languages, such that $\delta(\text{REG}) = \text{CF}$, and $\delta(\text{DCF}) = \text{Indexed}$, where DCF denotes the class of deterministic context-free languages, i.e., in our notation, $\lambda$-D$_1$REG(P). (But CF and Indexed are not closed under the $\delta$ operations; in fact, $\delta(\text{CF}) = \text{RE}$. ) These are two particular cases of our general result that $\lambda$-CF($S$) = $\delta(\lambda$-D$_1$REG($S$)), for r-acceptor deterministic $S$, shown in Theorem 8.3(3). Thus the languages generated by CF($S$) grammars can be obtained by the $\delta$ operations from the languages accepted by deterministic one-way $S$ automata.

The $\delta$ operations are defined by viewing strings as paths through a tree, and taking the yield of that tree; thus they incorporate the essence of the general philosophy in tree language theory (for obtaining higher level devices), as discussed in [Eng1, Eng6, Dam]. We start by defining paths through trees.

For a ranked alphabet $\Delta$, the path alphabet $\pi(\Delta)$ is the (nonranked) alphabet $\Delta_0 \cup \{ (\sigma, i) \mid \sigma \in \Delta_k$ and $1 \leq i \leq k$, for some $k \geq 1 \}$. We will also write $\sigma_i$ rather than $(\sigma, i)$. Intuitively, $\sigma_i$ denotes the fact that the path goes to the $i$-th son of a node labeled $\sigma$. For every $t \in T_\Delta$, the set of paths through $t$, denoted $\pi(t)$, is the finite subset of $\pi(\Delta)^*$ defined as follows: (1) for $\sigma \in \Delta_0$, $\pi(\sigma) = \{ \sigma \}$, and (2) for $\sigma \in \Delta_k$ ($k \geq 1$) and $t_1, \ldots, t_k \in T_\Delta$, $\pi(\sigma t_1 \cdots t_k) = \cup \{ (\sigma, i) \cdot \pi(t_i) \mid 1 \leq i \leq k \}$. Thus, $\pi(t)$ contains all paths from the root of $t$ to its leaves (coded as strings).

For a tree language $L$, $\pi(L) = \{ \pi(t) \mid t \in L \}$ is its path language.

As an example, let $\Delta_3 = \{ a \}$, $\Delta_2 = \{ b \}$, and $\Delta_0 = \{ c, d, \varepsilon \}$, and let $t = abcd\varepsilon\varepsilon = a(b(c, d))b(\varepsilon, c), \varepsilon)$. Then $\pi(t) = \{ a_1 b_1 c, a_1 b_2 d, a_2 b_1 \varepsilon, a_2 b_2 c, a_3 \varepsilon \}$. Note that yield($t$) = cdc (because yield($\varepsilon$) = $\lambda$, by convention).

**Definition 8.1.** Let $\Delta$ be a ranked alphabet. For a (string) language $L$, $\text{tree}_\Delta(L)$ is the tree language $\text{tree}_\Delta(L) = \{ t \in T_\Delta \mid \pi(t) \subseteq L \}$, and $\delta_\Delta(L)$ is the (string) language $\delta_\Delta(L) = \text{yield}(\text{tree}_\Delta(L)) = \{ w \in \Delta^*_0 \mid w = \text{yield}(t) \text{ for some } t \in T_\Delta \text{ such that } \pi(t) \subseteq L \}$. We call $\delta_\Delta$ a delta operation.

For a class $K$ of (string) languages $\text{tree}(K)$ denotes $\{ \text{tree}_\Delta(L) \mid L \in K, \Delta$ is a ranked alphabet $\}$, and $\delta(K)$ denotes $\text{yield}(\text{tree}(K))$, i.e., $\{ \delta_\Delta(L) \mid L \in K, \Delta$ is a ranked alphabet $\}$.

Thus, $\text{tree}_\Delta(L)$ is obtained by taking paths from the building kit $L$ and gluing them together to trees over $\Delta$. One argument for the “naturalness” of the delta operations is that they are continuous in the following sense: for every language $L$, $\delta_\Delta(L) = \cup \{ \delta_\Delta(F) \mid F$ is a finite subset of $L \}$. All the full AFL operations are also continuous in this sense (think for instance of Kleene $*$).

We use the symbol $\delta$ because its capital version $\Delta$ looks like a (mathematical) tree, see
Figs. 2, 6, and 8. Let us consider some examples of $\delta_\Delta(L)$.

Examples 8.2. (1) Let $\Delta_3 = \{c\}$ and $\Delta_0 = \{a,b,\varepsilon\}$. Consider the regular language $L = c_1^* (c_1 a \cup c_2 \varepsilon \cup c_3 b)$ over $\pi(\Delta)$. A simple argument shows that the trees that can be put together from the paths in $L$, are those that have a “spine” of $c$’s, ending in an $\varepsilon$, with one $a$ and one $b$ sticking out of each $c$, see Fig. 7(1). Thus $\text{tree}_\Delta(L)$ is the tree language generated by the regular tree grammar with rules $A \rightarrow caAb$, and $A \rightarrow ca\varepsilon b$. Hence $\delta_\Delta(L) = \text{yield(tree}_\Delta(L)) = \{a^n b^n \mid n \geq 1\}$. Thus $\delta_\Delta$ transforms a regular language into a (nonregular) context-free language.

(2) Let $\Delta_2 = \{b\}$, $\Delta_1 = \{c\}$, and $\Delta_0 = \{a\}$. Consider the (deterministic) context-free language $L = \{c^n \mid w \in \{b_1, b_2\}^n, n \geq 0\}$. The trees in $\text{tree}_\Delta(L)$ consist of a monadic “handle” of, say, $n c$’s, connected with a full binary tree of $b$’s of depth $n$ (with $a$’s at the leaves), see Fig. 7(2). This tree language can be generated by the RT(P) grammar with rules

\[
A \rightarrow cA \text{ (push(c))}, \quad A \rightarrow B \text{ (stay)}, \quad B \rightarrow \text{if top = c then } bB \text{ (pop) } B \text{ (pop) else } a.
\]

Hence $\delta_\Delta(L)$ is the indexed language $\{a^{2n} \mid n \geq 0\}$.

(3) This example is similar to the one in (2). Let $\Delta_2 = \{f, d, b\}$, $\Delta_1 = \{c\}$, and $\Delta_0 = \{p, q, r, s, a\}$. Consider the context-free language $L = \{f_1 c_1^* d_1 wp \mid w \in \{b_1, b_2\}^n, n \geq 0\}$ and $L_2 = \{f_1 c_1^* d_2 wx \mid w \in \{b_1, b_2\}^n, x \in \{q, r\}, n \geq 0\}$.

For a tree in $\text{tree}_\Delta(L)$, see Fig. 7(3). It should now be clear that $\delta_\Delta(L)$ is the indexed language $\{p^{2n} ws \mid w \in \{q, r\}^{2n}, n \geq 0\}$. Now suppose that $p, q, r, s$ really denote the symbols $c_1, b_1, b_2, a$, respectively. Then, $\delta_\Delta(\delta_\Delta(L)) = \{a^{d(n)} \mid n \geq 0\}$, where $d(n) = 2^{2n}$. This language can be generated by a CF(P^2) grammar. □
We now show the announced result. The basic idea involved is that a tree language that can be recognized by a deterministic top-down $S$ tree automaton, is completely determined by its path language.

**Theorem 8.3.** Let $S$ be an $r$-acceptor deterministic storage type.

1. $\lambda$-CF$(S) = \text{yield}(\lambda$-DT$_T(S)) = \text{yield}(\lambda$-RT$(S))$
2. $\lambda$-DT$_T(S) = \text{tree}(\lambda$-DT$_R\text{REG}(S))$
3. $\lambda$-CF$(S) = \delta(\lambda$-DT$_R\text{REG}(S))$.

**Proof.** (3) follows immediately from (1) and (2).

(1) First, yield(\lambda$-RT$(S)) \subseteq \lambda$-CF$(S)$: replace every rule $A \rightarrow \text{if } b \text{ then } t$ by the rule $A \rightarrow \text{if } b \text{ then } \text{yield}(t)$. Second, we show that $\lambda$-CF$(S) \subseteq \text{yield}(\lambda$-DT$_R(T_S))$, see Lemma 7.7. Let $G = (N, e, \Delta, A_{in}, R)$ be a CF$(S)$ grammar. We construct an RT$(S)$ r-acceptor $G'$ that recognizes (deterministically) the derivation trees of $G$: to make this possible, we assume that each internal node of a derivation tree is labeled with the rule applied at that node; leaves are labeled with terminals as usual (or with $\varepsilon$, to denote $\lambda$). Formally, $G' = (N', e, \Sigma, A_{in}, R')$, where $N' = N \cup \{A_\sigma \mid \sigma \in \Delta \cup \{\varepsilon\}\}$, and $\Sigma$ and $R'$ are determined as follows. The symbols of rank 0 in $\Sigma$ are $\Sigma_0 = \Delta \cup \{\varepsilon\}$, where $\varepsilon$ is the symbol with yield($\varepsilon$) = $\lambda$. For each rule $r : A \rightarrow \text{if } b \text{ then } \lambda$ of $R$, we put a symbol $\hat{r}$ of rank $|\lambda| \in \Sigma$, and we put the rule $A \rightarrow \text{if } b \text{ then } \hat{r}\xi'$ in $R'$, where $\xi'$ is the result of replacing, in $\xi$, every $\sigma \in \Delta$ by $A_{\sigma(id)}$. Similarly, for each rule $r : A \rightarrow \text{if } b \text{ then } \lambda$ of $R$, we put a symbol $\hat{r}$ of rank 1 in $\Sigma$, and put the rule $A \rightarrow \text{if } b \text{ then } \hat{r}A_\varepsilon(id)$ in $R'$. Finally, $R'$ contains all rules $A_\sigma \rightarrow \sigma$ with $\sigma \in \Delta \cup \{\varepsilon\}$.

Clearly, $G'$ is r-acceptor deterministic, and $L(G')$ is the set of (rule labeled) derivation trees of $G$, i.e., yield($L(G')$) = $L(G)$.

(2) First we show that $\lambda$-DT$_R(T_S) \subseteq \text{tree}(\lambda$-DT$_R\text{REG}(S))$, see Lemma 7.4. Let $L \subseteq T_\Delta$ be a tree language in $\lambda$-DT$_R(T_S)$. Thus there is a deterministic RT$_R(S)$ r-acceptor $G = (N, e, \Delta\#, A_{in}, R)$ such that $L(G) = \text{mark}(L)$. We construct a deterministic REG$(S)$ r-acceptor $G'$ such that $\text{tree}_\Delta(L(G')) = L$. The acceptor $G'$ just imitates the behavior of $G$, on the paths of the trees. Thus, $G' = (N \cup \overline{N}, e, \pi(\Delta), A_{in}, R')$, where $\overline{N} = \{\overline{A} \mid A \in N\}$, and the rules of $R'$ are determined as follows.

- If the rule $A \rightarrow \text{if } b \text{ then } \sigma B_1(f_1) \cdots B_k(f_k)$ is in $R$, with $k \geq 2$, or $k = 1$ and $\sigma \not\in \Delta_0$, then the rules $A \rightarrow \text{if } b \text{ then } \sigma_i B_i(f_i)$ are in $R'$, for every $i, 1 \leq i \leq k$.

- If the rule $A \rightarrow \text{if } b \text{ then } \sigma B(f)$, with $\sigma \in \Delta_0$, is in $R$, then the rule $A \rightarrow \text{if } b \text{ then } \sigma \overline{B}(f)$ is in $R'$.

- If the rule $A \rightarrow \text{if } b \text{ then } \#$ is in $R$, then the rule $\overline{A} \rightarrow \text{if } b \text{ then } \lambda$ is in $R'$.

- If the rule $A \rightarrow \text{if } b \text{ then } B(f)$ is in $R$, then this rule and the rule $\overline{A} \rightarrow \text{if } b \text{ then } \overline{B}(f)$ are in $R'$.

This concludes the construction of $G'$. Due to the use of bars, $G'$ is r-acceptor deterministic. It is obvious that if $t \in L$, then $\pi(t) \subseteq L(G')$, i.e., all paths of $t$ are accepted by $G'$. On the other hand, if $t$ is a tree over $\Delta$ such that all its paths are accepted by $G'$, then $\text{mark}(t)$ is accepted.
by $G$, due to the determinism of $G$, and the fact that all computations of $G'$ (on all paths) start in the same initial configuration $m(e)(u_0)$, where $I = \{ u_0 \}$. (We note that $G'$ does not accept the language $\pi(L)$; $L(G')$ may contain paths that are not in $\pi(L)$. Acceptance of $\pi(L)$ can be realized by look-ahead, i.e., by a deterministic $\text{REG}(S_{\lambda})$ r-acceptor, see Lemma 5.2 of [Vog3].)

Next we show that $\text{tree}(\lambda \text{-D}_1 \text{REG}(S)) \subseteq \lambda \text{-D}_1 \text{RT}(S)$. Let $L \subseteq \Sigma^*$ be a language in $\lambda \text{-D}_1 \text{REG}(S)$, and let $\Delta$ be a ranked alphabet. Consider $L' = L \cap (\Sigma - \Delta_0)^* \Delta_0$. Since $(\Sigma - \Delta_0)^* \Delta_0$ is a regular language, $L'$ is still in $\lambda \text{-D}_1 \text{REG}(S)$, by the usual product construction. Moreover $\text{tree}_\Delta(L') = \text{tree}_\Delta(L)$. Now note that $L'$ is prefix-free. Hence $L'$ is in $\lambda \text{-D}_e \text{REG}(S)$, cf. the remark after Lemma 7.4.

Thus, it suffices to show that $\text{tree}(\lambda \text{-D}_e \text{REG}(S)) \subseteq \lambda \text{-D}_1 \text{RT}(S)$. Let $G = (N, e, \Sigma, A_{\text{in}}, R)$ be a deterministic $\text{REG}(S)$ r-acceptor, and let $\Delta$ be a ranked alphabet. As shown above, we may assume that $L(G) \subseteq (\Sigma - \Delta_0)^* \Delta_0$. We want to construct a deterministic $\text{RT}(S)$ r-acceptor $G'$ such that $L(G') = \text{mark}(\text{tree}_\Delta(L(G)))$. The acceptor $G'$ just simulates $G$ on all paths of the input tree. Thus, $G' = (N \cup \{ Q \}, e, \Delta\#, A_{\text{in}}, R')$, where $Q$ is a new nonterminal, and $R'$ is defined as follows.

- If, for $\sigma \in \Delta_k$ with $k \geq 1$, the rules $A \rightarrow \text{if } b_i \text{ then } \sigma B_i(f_i)$ are in $R$ for all $i$, $1 \leq i \leq k$, then the rule $A \rightarrow \text{if } b_1 \text{ and } \cdots \text{ and } b_k \text{ then } \sigma B_1(f_1) \cdots B_k(f_k)$ is in $R'$.
- If, for $\sigma \in \Delta_0$, the rule $A \rightarrow \text{if } b \text{ then } \sigma$ is in $R$, then the rules $A \rightarrow \text{if } b \text{ then } \sigma Q(\text{id})$ and $Q \rightarrow \#$ are in $R'$.
- If, for $\sigma \in \Delta_0$, the rule $A \rightarrow \text{if } b \text{ then } \sigma B(f)$ is in $R$, then it is also in $R'$.
- If the rule $A \rightarrow \text{if } b \text{ then } \lambda$ is in $R$, then the rule $A \rightarrow \text{if } b \text{ then } \#$ is in $R'$.
- If the rule $A \rightarrow \text{if } b \text{ then } B(f)$ is in $R$, then it is also in $R'$.

This concludes the construction of $G'$. It is left to the reader to show that $G'$ is indeed r-acceptor deterministic, and that $L(G') = \text{mark}(\text{tree}_\Delta(L(G)))$.

The idea for this theorem came from the fact, proved by Magidor and Moran, that $\lambda \text{-D}_e \text{RT}(S_0)$, the class of tree languages accepted by deterministic top-down finite tree automata, is included in $\text{tree}(\text{REG})$, see [Tha]. See also [Cou] and Section II.11 of [GécSte]; a tree language $L$ over $\Delta$ is said to be closed if $L = \text{tree}_\Delta(\pi(L))$; it is easy to see that, for every string language $L$, $\text{tree}_\Delta(L)$ is closed; hence Theorem 8.3(2) shows that all tree languages in $\lambda \text{-D}_1 \text{RT}(S)$ are closed.

The main application of Theorem 8.3 is to the iterated pushdown languages: the class of languages accepted by the nondeterministic one-way $(n + 1)$-iterated pushdown automata can be obtained by the delta operations from the class of languages accepted by the deterministic $n$-iterated pushdown automata.

**Theorem 8.4.** For every $n \geq 0$, $\lambda \text{-REG}(P^{n+1}) = \delta(\lambda \text{-D}_1 \text{REG}(P^n))$.

In particular, $\text{CF} = \delta(\text{REG})$, and $\text{Indexed} = \delta(\text{DCF})$. 

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Proof. By Theorem 5.4(2), $\lambda$-REG($P^{n+1}$) = $\lambda$-CF($P^n$), and by Theorem 8.3(3) $\lambda$-CF($P^n$) = $\delta(\lambda$-D$_T$REG($P^n$)). Note that finite automata can be made deterministic; hence $\lambda$-D$_T$REG($S_0$) = REG.

Of course, we also get from Theorem 8.3(2) that $\lambda$-D$_T$RT($P^n$) = tree($\lambda$-D$_T$REG($P^n$)). Using (effective) closure of $P^n$ under look-ahead, one could rather easily prove that the equivalence problems for $\lambda$-D$_T$RT($P^n$) and for $\lambda$-D$_T$REG($P^n$) are equivalent (see [Cou]); note that the decidability of these problems is open.\footnote{New Observation. For $n = 1$ this problem (the dpda equivalence problem) was solved by Sénizergues in [*Sen1] (see also [*Sen2]).}

Another consequence of Theorem 8.3(2) is that the languages accepted by alternating one-way $S$ automata can be expressed in terms of the languages accepted by deterministic one-way $S$ automata. In fact, as observed at the end of Section 4, the class $S$-decidability of these problems is open.

4

To illustrate that determinism plays an essential role in Theorems 8.3 and 8.4, we show that $\delta$(LIN) = RE, where LIN is the class of linear context-free languages.

Theorem 8.5. $\delta$(LIN) = RE.

Proof. Obviously $\delta$(LIN) $\subseteq$ RE. We now prove that RE $\subseteq$ $\delta$(LIN). Every recursively enumerable language is of the form $h(L \cap M)$, where $L$ and $M$ are linear context-free languages, over some alphabet $\Sigma$, and $h$ is a homomorphism from $\Sigma$ to some alphabet $\Omega$ (see, e.g., [EngRoz]). We now construct a linear context-free language $K$ and a ranked alphabet $\Delta$, such that $\delta_\Delta(K) = h(L \cap M)$. In fact tree$_\Delta(K)$ consists of all trees of the form suggested in Fig. 8, with $a_1a_2\cdots a_n \in L \cap M$ (and $a_i \in \Sigma$). Let $\Delta$ be the ranked alphabet with $\Delta_0 = \Omega \cup \{\varepsilon\}$, $\Delta_1 = \{\#_1\}$, $\Delta_2 = \Sigma \cup \{$$$\#, \varepsilon$$\}$, and $\Delta_k = \{\#_k\}$ for $3 \leq k \leq m$, where $m = \max \{|h(a)| \mid a \in \Sigma\}$. We define $K = L_2(\varepsilon, 1)\varepsilon \cup M_2(\varepsilon, 2)\varepsilon \cup R$, where $L_2 = \{(a_1, 2)(a_2, 2)\cdots (a_n, 2) \mid a_1a_2\cdots a_n \in L\}$, and similarly for $M_2$. Moreover, $R$ is the regular language $\cup\{\Sigma_2^mF_a \mid a \in \Sigma\}$, where $\Sigma_2 = \{(a, 2) \mid a \in \Sigma\}$, and

- if $h(a) = b_1 \cdots b_k$ with $k \geq 1$ and $b_i \in \Omega$, then $F_a = \{(a, 1)(\#_k, 1) b_1, \ldots, (a, 1)(\#_k, k) b_k\}$,
- if $h(a) = \lambda$, then $F_a = \{(a, 1)(\#_1, 1)\varepsilon\}$.

Clearly $K$ is a linear context-free language, and $\delta_\Delta(K) = h(L \cap M)$.

This implies that the linear context-free languages are not contained in $\lambda$-D$_T$REG($P^n$), as shown, in a different way, in [EngVog4]. In fact, if LIN $\subseteq \lambda$-D$_T$REG($P^n$), then RE = $\delta$(LIN) $\subseteq \delta(\lambda$-D$_T$REG($P^n$)) = $\lambda$-REG($P^{n+1}$). However, the languages in $\lambda$-REG($P^n$) are known to be recursive [Dam, Eng9].

In [Gre] an example is given of a (r-acceptor deterministic) storage type $\tilde{S}$ such that $\lambda$-REG($\tilde{S}$) = CF and $\lambda$-REG($\tilde{P}(\tilde{S})$) = RE. In our notation, $\tilde{S} = (C, P, F, I, E, m)$ with $C =$
Ω∗ ∪ {⊥}, where Ω is a fixed infinite set of symbols and ⊥ ∉ Ω, I = {u0} and E = {e0}
with m(e0)(u0) = λ, P = ∅, F = {write(v) | v ∈ Ω∗} ∪ {test(L) | L ∈ CF}, and for w ∈ Ω∗,
m(write(v))(w) = wv, m(write(v))(⊥) = ⊥, m(test(L))(w) = ⊥ if w ∈ L and undefined otherwise. Thus the storage of S is that of an ordinary write-only output tape, and the content of that tape may be tested once for membership in a context-free language. Hence, for L ∈ CF, the REG(S) grammar with rules A → aA(write(a)) for all a, A → B(test(L)), B → λ, generates L, and so CF ⊆ λ-REG(Ñ). By standard AFA/AFL techniques it can be shown that λ-REG(S) ⊆ CF. The intersection L1 ∩ L2 of two context-free languages L1 and L2 can be generated by the CF(Ñ) grammar with rules A → aA(write(a)) for all a, A → B(test(L1))B(test(L2)), B → λ. Hence, since every recursively enumerable language is the homomorphic image of such an intersection, and λ-CF(Ñ) is closed under homomorphisms (it is a full trio), RE ⊆ λ-CF(Ñ) = λ-REG(P(Ñ)).

Thus, as noted in [Gre], the following desirable property of P(S), and thus of λ-CF(S), is not true:

(+ ) if λ-REG(S1) = λ-REG(S2) then λ-REG(P(S1)) = λ-REG(P(S2)).

Indeed, for S1 = Ñ and S2 = P, we get λ-REG(P(S1)) = RE and λ-REG(P(S2)) = λ-CF(P) = Indexed ⊆ the class of recursive languages. Property (+ ) holds in fact for most other operations O(S) on storage types, such as the well-nested AFA, because there exists a class F of operations on languages such that λ-REG(O(S)) = F(λ-REG(S)), see [Gin, Gre] (for well-nested AFA, F is the class of nested iterated substitutions). From Theorems 5.4(2) and 8.3(3) it follows, for an r-acceptor deterministic storage type S, that the analogous equality λ-REG(P(S)) = ¬(λ-Df REG(S)) holds. Hence the following property of P(S), for r-acceptor deterministic storage types, is true:

(∗∗ ) if λ-Df REG(S1) = λ-Df REG(S2) then λ-REG(P(S1)) = λ-REG(P(S2)).

It can probably even be shown that: if λ-Df REG(S1) = λ-Df REG(S2) then λ-Df REG(P(S1)) = λ-Df REG(P(S2)). This would suggest to define two storage types S1 and S2 to be equivalent if λ-Df REG(S1) = λ-Df REG(S2), and use this notion of equivalence, rather than the, more structural, one used in [Eng9, EngVog2, EngVog3], as the basic notion of indistinguishability.

Fig. 8: Illustration of δ(LIN) = RE.
of storage types. In this respect it would be nice to have a class $\delta'$ of operations such that $\lambda$-$D_f$-$REG(P(S)) = \delta'(\lambda$-$D_f$-$REG(S))$.

Some remaining questions in this section are the following.

- Is there a (natural) class $F$ of operations such that $F(\lambda$-$REG(P^n)) = \lambda$-$REG(P^{n+1})$?
- Is there a formal relationship between $\delta$ and YIELD? Note that repeated application of YIELD to RT gives the IO-hierarchy, whereas, apart from the restriction to determinism, repeated application of $\delta$ to RT gives the OI-hierarchy $\{\lambda$-$REG(P^n)\}_n$ (see [Dam, Eng6]).
- Is there a “delta theorem” (i.e., a theorem analogous to Theorem 8.3) for arbitrary storage types?
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In the following list, new references (i.e., references added in 2014) are starred.

References

[Aho1] A. V. Aho; Indexed grammars, an extension of context-free grammars, J. ACM 15 (1968), 647–671.

[Aho2] A. V. Aho; Nested stack automata, J. ACM 16 (1969), 383–406.

[AhoUll] A. V. Aho, J. D. Ullman; Translations on a context-free grammar, Inform. Control 19 (1971), 439–475.

[Asv] P. R. J. Asveld; Controlled iteration grammars and full hyper-AFL’s, Inform. Control 34 (1977), 248–269.

[Ber] J. Berstel; Transductions and Context-Free Languages, Teubner, Stuttgart, 1979.

[*BojCol1] M. Bojańczyk, T. Colcombet; Tree-walking automata cannot be determinized, Theor. Comput. Sci. 350 (2006), 164–173.

[*BojCol2] M. Bojańczyk, T. Colcombet; Tree-walking automata do not recognize all regular languages, SIAM J. Comput. 38 (2008), 658–701.

[ChaKozSto] A. K. Chandra, D. C. Kozen, L. J. Stockmeyer; Alternation, J. ACM 28 (1981), 114–133.

[Coo] S. A. Cook; Characterizations of pushdown machines in terms of time-bounded computers, J. ACM 18 (1971), 4–18.

[Con] B. Courcelle; A representation of trees by languages I,II, Theor. Comput. Sci. 6 (1978), 255–279, and Theor. Comput. Sci. 7 (1978), 25–55.

[CouFra] B. Courcelle, P. Franchi-Zannettacci; Attribute grammars and recursive program schemes I,II, Theor. Comput. Sci. 17 (1982), 163–191 and 235–257.

[Cul] K. Culik II; On some families of languages related to developmental systems, Internat. J. Comput. Math. 4 (1974), 31–42.

[Dam] W. Damm; The IO- and OI-hierarchies, Theor. Comput. Sci. 20 (1982), 95–207.

[DamGoe] W. Damm, A. Goerdt; An automata-theoretical characterization of the OI-hierarchy, Inform. Control 71 (1986), 1–32.
[DamGue] W. Damm, I. Guessarian; Implementation techniques for recursive tree transducers on higher-order data types, Report 83-16, Laboratoire Informatique Theorique et Programmation, Université Paris 7, 1983.

[Döb] K. Döbeling; Festlegung zweier Sprachklassen mit Hilfe attributierter Grammatiken, Diplomarbeit, T.U. Hannover, 1978.

[DusParSEDSpe] J. Duske, R. Parchmann, M. Sedello, J. Specht; IO-macro languages and attributed translations, Inform. Control 35 (1977), 87–105.

[Eng1] J. Engelfriet; Tree Automata and Tree Grammars, Lecture Notes, DAIMI FN-10, University of Aarhus, 1975.

[Eng2] J. Engelfriet; Surface tree languages and parallel derivation trees, Theor. Comput. Sci. 2 (1976), 9–27.

[Eng3] J. Engelfriet; Iterating iterated substitution, Theor. Comput. Sci. 5 (1977), 85–100.

[Eng4] J. Engelfriet; Top-down tree transducers with regular look-ahead, Math. Systems Theory 10 (1977), 289–303.

[Eng5] J. Engelfriet; On tree transducers for partial functions, Inform. Proc. Letters 7 (1978), 170–172.

[Eng6] J. Engelfriet; Some open questions and recent results on tree transducers and tree languages, in: Formal Language Theory – Perspectives and Open Problems (ed. R. V. Book), American Press, New York, 1980, pp. 241–286.

[Eng7] J. Engelfriet; The complexity of languages generated by attribute grammars, SIAM J. Comput. 15 (1986), 70–86.

[Eng8] J. Engelfriet; Recursive automata, unpublished notes, 1982.

[Eng9] J. Engelfriet; Iterated stack automata and complexity classes, Inform. Comput. 95 (1991), 21–75.

[Eng10] J. Engelfriet; Attribute grammars: attribute evaluation methods, in: Methods and Tools for Compiler Construction (ed. B. Lorho), Cambridge University Press, 1984, pp. 103–138.

[*Eng11] J. Engelfriet; The delta operation: from strings to trees to strings, in: Formal and Natural Computing – Essays Dedicated to Grzegorz Rozenberg (eds. W. Brauer, H. Ehrig, J. Karhumäki, A. Salomaa), Lecture Notes in Computer Science 2300, Springer-Verlag, 2002, pp. 39–56.

[EngFil] J. Engelfriet, G. File; The formal power of one-visit attribute grammars, Acta Informatica 16 (1981), 275–302.

[*EngMan] J. Engelfriet, S. Maneth; Two-way finite state transducers with nested pebbles, Proc. MFCS 2002, Lecture Notes in Computer Science 2420, Springer-Verlag, 2002, pp. 234–244.
[EngRoz] J. Engelfriet, G. Rozenberg; Fixed point languages, equality languages, and representation of recursively enumerable languages, *J. ACM* 27 (1980), 499–518.

[EngRozShu] J. Engelfriet, G. Rozenberg, G. Slutzki; Tree transducers, L systems, and two-way machines, *J. Comput. System Sci.* 20 (1980), 150–202.

[EngSchvLe] J. Engelfriet, E. Meineche Schmidt, J. van Leeuwen; Stack machines and classes of nonnested macro languages, *J. ACM* 27 (1980), 96–117.

[EngVog1] J. Engelfriet, H. Vogler; Macro tree transducers, *J. Comput. System Sci.* 31 (1985), 71–146.

[EngVog2] J. Engelfriet, H. Vogler; Pushdown machines for the macro tree transducer, *Theor. Comput. Sci.* 42 (1986), 251–368; correction: *Theor. Comput. Sci.* 48 (1986), 339.

[EngVog3] J. Engelfriet, H. Vogler; High level tree transducers and iterated pushdown machines, *Acta Informatica* 26 (1988), 309–315.

[EngVog4] J. Engelfriet, H. Vogler; Look-ahead on pushdowns, *Inform. Comput.* 73 (1987), 245–279.

[Ern] W. J. Erni; On the time and tape complexity of hyper(1)-AFL’s, Proc. 4-th ICALP, Lecture Notes in Computer Science 52, Springer-Verlag, 1977, pp. 230–243.

[Fis] M. J. Fischer; Grammars with macro-like productions, Ph.D. Thesis, Harvard University, 1968 (see also: Proc. 9-th SWAT, pp. 131–142).

[Fül] Z. Fülöp; On attributed tree transducers, *Acta Cybernetica* 5 (1981), 261–279.

[GécSte] F. Gécseg, M. Steinby; *Tree Automata*, Akademiai Kiado, Budapest, 1984.

[Gin] S. Ginsburg; *Algebraic and Automata-Theoretic Properties of Formal Languages*, North-Holland/American Elsevier, Amsterdam/New York, 1975.

[GinSpa] S. Ginsburg, E. H. Spanier; Finite-turn pushdown automata, *SIAM J. Control* 4 (1966), 429–453.

[GogThaWagWri] J. A. Goguen, J. W. Thatcher, E. G. Wagner, J. B. Wright; Initial algebra semantics and continuous algebras, *J. ACM* 24 (1977), 68–95.

[Gre] S. A. Greibach; Full AFLs and nested iterated substitution, *Inform. Control* 16 (1970), 7–35.

[Gue1] I. Guessarian; On pushdown tree automata, Proc. CAAP ’81, Lecture Notes in Computer Science 112, Springer-Verlag, 1981, pp. 211–223.

[Gue2] I. Guessarian; Pushdown tree automata, *Math. Systems Theory* 16 (1983), 237–263.

[Har] M. A. Harrison; *Introduction to Formal Language Theory*, Addison-Wesley, Reading, Mass., 1978.

[HopUll] J. E. Hopcroft, J. D. Ullman; *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, Reading, Mass., 1979.

56
[Jal] F. Jalili; A general incremental evaluator for attribute grammars, *Science of Computer Programming* 5 (1985), 83–96.

[Jou] M. Jourdan; Recursive evaluators for attribute grammars: an implementation, in: *Methods and Tools for Compiler Construction* (ed. B. Lorho), Cambridge University Press, 1984, pp. 139–164.

[Kam] T. Kamimura; Tree automata and attribute grammars, *Inform. Control* 57 (1983), 1–20.

[KamSlu] T. Kamimura, G. Slutzki; Parallel and two-way automata on directed ordered acyclic graphs, *Inform. Control* 49 (1981), 10–51.

[Knu] D. E. Knuth; Semantics of context-free grammars, *Math. Systems Theory* 2 (1968), 127–145; correction: *Math. Systems Theory* 5 (1971), 95–96.

[Kos] C. H. A. Koster; Affix-grammars, in: Proc. IFIP Working Conf. on Algol 68 implementation (ed. J. E. L. Peck), North-Holland, Amsterdam, 1971, pp. 95–109.

[LadLipSto] R. E. Ladner, R. J. Lipton, L. J. Stockmeyer; Alternating pushdown and stack automata, *SIAM J. Comput.* 13 (1984), 135–155.

[LewRosSte] P. M. Lewis, P. J. Rosenkrantz, R. E. Stearns; Attributed translations, *J. Comput. System Sci.* 9 (1974), 279–307.

[*MalVog] A. Maletti, H. Vogler; Compositions of top-down tree transducers with $\varepsilon$-rules, in: Proc. 8th FSMNLP, Lecture Notes in Artificial Intelligence 6062, Springer-Verlag, 2010, pp. 69–80.

[May] B. H. Mayoh; The meaning of logical programs, Report DAIMI PB-126, University of Aarhus, September 1980.

[Mas1] A. N. Maslov; The hierarchy of indexed languages of an arbitrary level, *Soviet. Math. Dokl.* 15 (1974), 1170–1174.

[Mas2] A. N. Maslov; Multilevel stack automata, *Problems of Inf. Transm.* 12 (1976), 38–43.

[ParDusSpe1] R. Parchmann, J. Duske, J. Specht; On deterministic indexed languages, *Inform. Control* 45 (1980), 48–67.

[ParDusSpe2] R. Parchmann, J. Duske, J. Specht; Indexed LL(k) grammars, *Acta Cybernetica* 7 (1984), 33–53.

[Rou] W. C. Rounds; Mappings and grammars on trees, *Math. Systems Theory* 4 (1970), 257–287.

[RozSal] G. Rozenberg, A. Salomaa; *The Mathematical Theory of L Systems*, Academic Press, New York, 1980.

[Ruz] W. L. Ruzzo; Tree-size bounded alternation, *J. Comput. System Sci.* 21 (1980), 218–235.

[Sal] A. Salomaa; *Formal Languages*, Academic Press, New York, 1973.
[Sco] D. Scott; Some definitional suggestions for automata theory, *J. Comput. System Sci.* 1 (1967), 187–212.

[*Sen1*] G. Sénizergues; L(A)=L(B)? decidability results from complete formal systems, *Theor. Comput. Sci.* 251 (2001), 1–166.

[*Sen2*] G. Sénizergues; L(A)=L(B)? A simplified decidability proof, *Theor. Comput. Sci.* 281 (2002), 555–608.

[Slu] G. Slutzki; Alternating tree automata, *Theor. Comput. Sci.* 41 (1985), 305–318.

[Tha] J. W. Thatcher; Tree automata: an informal survey, in: *Currents in the Theory of Computing* (ed. A. V. Aho), Prentice-Hall, Englewood Cliffs, 1973, pp. 143–172.

[Tie] M. Tienari; On the definition of attribute grammar, in: *Semantics-Directed Compiler Generation* (ed. N. D. Jones), Lecture Notes in Computer Science 94, Springer-Verlag, 1980, pp. 408–414.

[vLe1] J. van Leeuwen; Notes on pre-set pushdown automata, in *L Systems* (eds. G. Rozenberg, A. Salomaa), Lecture Notes in Computer Science 15, Springer-Verlag, 1974, pp. 177–188.

[vLe2] J. van Leeuwen; Variations of a new machine model, Proc. 17th FOCS, 1976, pp. 228–235.

[Vog1] H. Vogler; The OI-hierarchy is closed under control, *Inform. Comput.* 78 (1988), 187–204.

[Vog2] H. Vogler; Iterated linear control and iterated one-turn pushdowns, *Math. Systems Theory* 19 (1986), 117–133.

[Vog3] H. Vogler; Basic tree transducers, *J. Comput. System Sci.* 34 (1987), 87–128.

[Vog4] H. Vogler; *Tree transducers and pushdown machines*, Ph.D. Thesis, Twente University of Technology, March 1986.

[Wat1] D. A. Watt; The parsing problem for affix grammars, *Acta Informatica* 8 (1977), 1–20.

[Wat2] D. A. Watt; Contextual constraints, in: *Methods and Tools for Compiler Construction* (ed. B. Lorho), Cambridge University Press, 1984, pp. 45–80.