NEW CONFORMAL MODELS WITH $c < 2/5$

M. YU. LASHKEVICH*
Landau Institute for Theoretical Physics,
Kosygina 2, GSP-1, 117940 Moscow V-334, Russia

ABSTRACT
The zoo of two-dimensional conformal models has been supplemented by a series of nonunitary conformal models obtained by cosetting minimal models. Some of them coincide with minimal models, some do not have even Kac spectrum of conformal dimensions.

* E-mail: lashkevi@cpd.landau.free.msk.su
In this paper we continue to explore coset constructions of minimal models. Let us designate as $M_{PQ}$ the minimal model with the central charge of the Virasoro algebra $c_{P,Q} = 1 - 6\frac{(Q-P)^2}{PQ}$.

Recall that the monodromy properties of $M_{PQ}$ are described by braiding irreducible representations of the quantum group $U_q(P,Q)(sl(2)) \times U_q(Q,P)(sl(2))$, where

$$q(P,Q) = \exp\left(2\pi i \frac{Q}{P}\right).$$

The minimal model $M_{PQ}$ is described by vertex operators

$$\phi_{mn}(z) : \mathcal{H}_{(p,q)} \to \mathcal{H}_{(p+1-2m,q+1-2n)},$$

with conformal dimensions

$$\Delta_{(p,q)} = \frac{(Qp - Pq)^2 - (Q - P)^2}{4PQ}.$$ 

Here $\mathcal{H}_{(p,q)}$ is an irreducible representation of the Virasoro algebra over the state $\phi_{(p,q)}(0)|\text{vacuum}\rangle$, $\mathcal{H}_{Q-P-q} \sim \mathcal{H}_{(p,q)}$. In the bosonic representation the indices $m$ and $n$ mean numbers of screenings. In terms of quantum group, the pairs $\left(\frac{1}{2}(p-1), \frac{1}{2}(p-1) - m\right)$ and $\left(\frac{1}{2}(q-1), \frac{1}{2}(q-1) - n\right)$ are pairs (highest weight, weight) or (“moment”, “projection of moment”) of the representation of respective $U_x(sl(2))$ quantum group.

Monodromy invariant fields can be constructed as

$$\phi_{(p,q)}(z,z) = \sum_{m,n} X_p(m; q(P,Q)) X_q(n; q(Q,P)) \phi_{mn}^{(p,q)}(z) \phi_{mn}^{(p,q)}(\bar{z}),$$

where coefficients $X_p(m, x)$ are expressed in terms of braiding matrices of conformal blocks or $R$-matrix of the quantum group.

Consider two models $M_{PS}$ and $M_{SQ}$ with vertices $\phi_{(m,r)}^{(1)}(z)$ and $\phi_{(n,s)}^{(2)}(z)$ respectively. If

$$q(S,P) = q(S,Q),$$

we can consider a convolution of two models $M_{PS}M_{SQ}$ generated by vertices

$$\phi_{(m,n)}^{(p,s,q)}(z) = \sum_{r} X_{s}(r; q(S,P)) \phi_{(m,r)}^{(1)}(z) \phi_{(n,s)}^{(2)}(z).$$
We shall designate them as

$$\phi_{(p,s,q)}(z) = \phi_{(p,s)}^{(1)}(z)\phi_{(s,q)}^{(2)}(z)$$

and call them convolutions of vertex operators. Monodromy properties of such convolutions are described by the quantum group $U_{q(P,S)}(sl(2)) \times U_{q(Q,S)}(sl(2))$. The multipliers $U_{q(S,P)}(sl(2))$ and $U_{q(S,Q)}(sl(2))$ connected to indices $s$ and $r$ drop out.

Condition (2) holds, if

$$P + Q = NS, \quad N \in Z. \quad (4)$$

If we want to consider a coset construction $M_{PS}M_{SQ}/(something)$, we must construct the energy-momentum tensor of the denominator in terms of fields of the numerator. The vertices $\phi_{(1,s_0,1)}(z)$ possess trivial monodromy properties and can be considered as chiral currents. Thus, we shall look for the energy-momentum tensor of the denominator, $T_H(z)$, and that of the coset construction, $T_C(z)$, in the form

$$T_H(z) = A T_1(z) + B T_2(z) + C\phi_{(1,s_0,1)}(z),$$
$$T_C(z) = (1 - A)T_1(z) + (1 - B)T_2(z) - C\phi_{(1,s_0,1)}(z), \quad (5)$$

where $A$, $B$ and $C$ are constants, $T_1(z)$ and $T_2(z)$ are the energy-momentum tensors of $M_{PS}$ and $M_{SQ}$ respectively. The third term in (5) must be of conformal dimension 2:

$$\Delta^{(1)}_{(1,s_0)} + \Delta^{(2)}_{(s_0,1)} \equiv \frac{s_0 - 1}{4S}[(P + Q)(s_0 + 1) - 4S] = 2. \quad (6)$$

Both conditions (4) and (6) are satisfied only for $s_0 = 2, N = 4$ and $s_0 = 3, N = 2$. The case $s_0 = 3, N = 2$ for unitary models was considered earlier,\textsuperscript{1,2} and its generalization to nonunitary models is nearly straightforward. In this paper we shall concentrate on the other case

$$s_0 = 2, \quad P + Q = 4S. \quad (7)$$

Using bosonic representation we obtain the operator product expansion (OPE) for the chiral current $\phi_{(1,2,1)}(z)$

$$\phi_{(1,2,1)}(z')\phi_{(1,2,1)}(z) = \frac{1}{(z' - z)^4} + \frac{2\theta(z)}{(z' - z)^2} + \frac{\partial\theta(z)}{z' - z} + O(1),$$

$$\theta(z) = \frac{2P}{3Q - 5P}T_1(z) + \frac{2Q}{3P - 5Q}T_2(z), \quad (8)$$
where \( \partial \equiv \partial/\partial z \), \( O(1) \) designates the terms regular at \( z' \rightarrow z \). Now it is easy to check that the currents

\[
\begin{align*}
T_H(z) &= -\frac{2}{5} \frac{P}{Q-P} T_1(z) + 2 \frac{Q}{5Q-P} T_2(z) \\
&\quad + i \frac{\sqrt{2(3Q-5P)(3P-5Q)}}{5(Q-P)} \phi_{(1,2,1)}(z), \\
T_C(z) &= \frac{1}{5} \frac{3Q-5P}{Q-P} T_1(z) + 2 \frac{3Q-5P}{5Q-P} T_2(z) \\
&\quad - i \frac{\sqrt{2(3Q-5P)(3P-5Q)}}{5(Q-P)} \phi_{(1,2,1)}(z)
\end{align*}
\]

(9)

obey the OPE’s

\[
T_i(z') T_i(z) = \frac{\frac{1}{2} c_i}{(z'-z)^4} + \frac{2T_i(z)}{(z'-z)^2} + \frac{\partial T_i(z)}{z'-z} + O(1), \quad i = H, C,
\]

\[
T_H(z') T_C(z) = O(1),
\]

where the central charges are given by

\[
\begin{align*}
c_H &= -\frac{22}{5}, \\
c_C &= \frac{(3Q-5P)(3P-5Q)}{10PQ} < \frac{2}{5}. \\
\end{align*}
\]

(10)

\( c_H \) is the central charge of the minimal model \( M_{2,5} \). Thus, we shall consider the coset construction

\[
\frac{M_{PS}M_{SQ}}{M_{2,5}}, \quad S = \frac{P + Q}{4} \in \mathbb{Z}.
\]

(11)

Now we direct our attention to primary fields of the coset model. Consider the OPE

\[
\begin{align*}
\phi_{(1,2,1)}(z') \phi_{(p,s,q)}(z) \sim (z' - z)^{-1-2(s-\frac{p+2}{4})} \left[ \phi_{(p,s-1,q)}(z) \right] \\
&\quad + (z' - z)^{-1+2(s-\frac{p+2}{4})} \left[ \phi_{(p,s+1,q)}(z) \right].
\end{align*}
\]

(12)

We write down clearly the factors of the kind \((z' - z)^\alpha\) at the fields of the lowest dimensions in conformal families. If

\[
\frac{1}{4}(p + q - 2) \leq s \leq \frac{1}{4}(p + q + 2),
\]

there are no poles of the power > 2 in the expansion (12), and the field \( \phi_{(p,s,q)} \) can be primary with respect to the coset energy-momentum tensor \( T_C(z) \) from (9).
We shall discuss all cases in sequence.

1. $p + q \in 4\mathbb{Z}$, $s = \frac{1}{4}(p + q)$. In this case
   \[ T_C(z')\phi_{(p,s,q)}(z) \sim (z'-z)^{-2}\left[\phi_{(p,s,q)}\right] \]
   \[ + \left((z'-z)^{-1}\left\{[\phi_{(p,s-1,q)}] + [\phi_{(p,s+1,q)}]\right\}\right). \]
   The conformal dimension of the field $\phi_{(p,s,q)}$ with respect to $T_C(z)$ is given by
   \[ \Delta^0_{p,q} = \frac{(Qp - Pq)^2 - (Q - P)^2}{16PQ} - \frac{1}{20}, \quad (13) \]
   and the conformal dimension with respect to $T_H(z)$ is $-\frac{1}{5}$. It means that
   \[ \phi'_{(1,2)}(z)\phi_{p,q}^0(z) = \phi_{(p,s)}(z)\phi_{(s,q)}(z), \quad p + q \in 4\mathbb{Z}, \quad s = \frac{1}{4}(p + q), \quad (14) \]
   where $\phi'_{(1,2)}(z)$ is the primary field of the conformal dimension $-\frac{1}{5}$ in the model $M_{2,5}$, and $\phi_{p,q}^0(z)$ are vertices of the coset model (11). There is a convolution of $\phi'_{(1,2)}(z)$ and $\phi_{p,q}^0(z)$ in the left-hand side of (14). Monodromy properties of the coset model are described by the quantum group $U_q(P,S)(sl(2)) \times U_q(S,Q)(sl(2)) \times U_q(2,5)(sl(2))$.

2. $p + q \pm 1 \in 4\mathbb{Z}$, $s = \frac{1}{4}(p + q + 1)$. In this case
   \[ \phi_{(1,2,1)}(z')\phi_{(p,s,q)}(z) \sim (z'-z)^{-\frac{3}{4}} \cdot \text{(something)}. \]
   Therefore, the product $T_C(z')\phi_{(p,s,q)}(z)$ contains in its decomposition half-integer powers of $(z'-z)$ as well as integer ones. It means that $T_C(z)$ is no longer a chiral current. Fortunately, one can eliminate this sector, because there are no fields $\phi_{(p,s,q)}$ with odd $p + q$ in fusions of fields with even $p + q$.

3. $p + q \pm 2 \in 4\mathbb{Z}$, $s_\pm = \frac{1}{4}(p + q \pm 2)$, $s_+ - s_- = 1$. The fields $\phi_{(p,s_+,q)}(z)$ and $\phi_{(p,s_-,q)}(z)$ have the same conformal dimensions with respect to $T_1(z) + T_2(z)$. In other words,
   \[ \phi_{(1,2,1)}(z')\phi_{(p,s_+,q)}(z) \sim (z'-z)^{-2}\left[\phi_{(p,s_-,q)}\right] + O(1), \]
   \[ \phi_{(1,2,1)}(z')\phi_{(p,s_-,q)}(z) \sim (z'-z)^{-2}\left[\phi_{(p,s_+,q)}\right] + O(1). \]
   The operator $L_0^\phi = \oint \frac{du}{2\pi i} (u-z)T_C(u)$ mixes fields $\phi_{(p,s_+,q)}(z)$ and $\phi_{(p,s_-,q)}(z)$. Conformal dimensions in the coset model are eigenvalues of this operator. Diagonalizing it we obtain two fields
   \[ \phi_{p,q}^-(z) = \sqrt{y + \frac{1}{2} \phi_{(p,s_+,q)}(z) \phi_{(s_+,q)}^{(2)}(z)} + i \sqrt{y - \frac{1}{2} \phi_{(p,s_-,q)}(z) \phi_{(s_-,q)}^{(2)}(z)}, \]
   \[ (15a) \]
   \[ \phi_{p,q}^+(z) = -i \sqrt{y - \frac{1}{2} \phi_{(p,s_+,q)}(z) \phi_{(s_+,q)}^{(2)}(z)} + \sqrt{y + \frac{1}{2} \phi_{(p,s_-,q)}(z) \phi_{(s_-,q)}^{(2)}(z)}, \]
   \[ (15b) \]
   \[ y = \frac{Qp - Pq}{2(Q - P)}, \quad p + q - 2 \in 4\mathbb{Z}, \quad s_\pm = \frac{1}{2}(p + q \pm 2) \quad (15c) \]
with conformal dimensions
\[
\Delta_{\rho, q}^- = \frac{(Q\rho - Pq)^2 - (Q - P)^2}{16PQ}, \quad (16a)
\]
\[
\Delta_{\rho, q}^+ = \frac{(Q\rho - Pq)^2 - (Q - P)^2}{16PQ} + \frac{1}{5}. \quad (16b)
\]

Other primary fields can appear in such models too, but at present there is no simple method to find them.

Consider some examples. The first example is \(M_{2,3}M_{3,10}/M_{2,5}\). The central charge \(c_C = -22/5\) coincides with that of the minimal model \(M_{2,5}\). The conformal dimensions of the coset primary fields
\[
\Delta_{1,1}^- = \Delta_{1,9}^- = \Delta_{1,5}^+ = 0, \quad \Delta_{1,3}^0 = \Delta_{1,7}^0 = \Delta_{1,5}^- = -\frac{1}{5}
\]
confirm the identification
\[
\frac{M_{2,3}M_{3,10}}{M_{2,5}} \sim M_{2,5}.
\]

For \(M_{2,5}M_{5,18}/M_{2,5}\), \(c = -154/15\), the conformal dimensions are given by
\[
\Delta_{1,1}^- = 0, \quad \Delta_{1,3}^0 = \Delta_{1,9}^+ = -\frac{11}{45}, \quad \Delta_{1,5}^- = -\frac{1}{3},
\]
\[
\Delta_{1,5}^+ = -\frac{2}{15}, \quad \Delta_{1,7}^0 = -\frac{7}{15}, \quad \Delta_{1,9}^- = -\frac{4}{9}.
\]
We can identify this model at least with some sector in \(M_{5,18}\).

For \(M_{5,4}M_{4,11}/M_{2,5}\) the central charge \(c = -32/55\) corresponds to an irrational conformal model. The conformal dimensions
\[
\Delta_{1,1}^- = 0, \quad \Delta_{1,3}^0 = -\frac{4}{55}, \quad \Delta_{2,2}^0 = \frac{4}{55}, \quad \Delta_{3,1}^0 = \frac{4}{5},
\]
\[
\Delta_{1,5}^- = \frac{2}{11}, \quad \Delta_{2,4}^- = -\frac{2}{55}, \quad \Delta_{3,3}^- = \frac{18}{55}, \quad \Delta_{4,2}^- = \frac{14}{11},
\]
\[
\Delta_{1,5}^+ = \frac{21}{55}, \quad \Delta_{2,4}^+ = \frac{9}{55}, \quad \Delta_{3,3}^+ = \frac{29}{55}, \quad \Delta_{4,2}^+ = \frac{81}{55},
\]
\[
\Delta_{1,7}^0 = \frac{31}{55}, \quad \Delta_{2,6}^0 = -\frac{1}{55},
\]
do not generally coincide with any Kac conformal dimensions.

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