Spectra of Heavy Quarkonia in a Magnetized-Hot Medium in the Framework of Fractional Non-relativistic Quark Model

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Abstract

In the fractional nonrelativistic potential model, the decomposition of heavy quarkonium in a hot magnetized medium is investigated. The analytical solution of the fractional radial Schrödinger equation for the hot-magnetized interaction potential is displayed by using the conformable fractional Nikiforov-Uvarov method. Analytical expressions for the energy eigenvalues and the radial wave function are obtained for arbitrary quantum numbers. Next, we study the charmonium and bottomonium binding energies for different magnetic field values in the thermal medium. The effect of the fractional parameter on the decomposition temperature is also analyzed for charmonium and bottomonium in the presence of hot magnetized media. We conclude that the dissociation of heavy quarkonium in the fractional nonrelativistic potential model is more practical than the classical nonrelativistic potential model.

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I. INTRODUCTION

Quantum chromodynamics theory calculates that at sufficiently high temperatures and densities, the gluons and quarks confined inside the hadrons are freed into a medium of gluons and quarks. Recent works have focused on producing and identifying this new state of matter theoretically [1-7] and experimentally in ultra-relativistic heavy-ion collisions (URHIC) with the increasing center of mass energies in the BNL AGS, CERN SPS, BNL RHIC, and CERN LHC experiments. However, for the noncentral events in URHICs, the powerful magnetic field is generated at the collisions’ initial stages due to very high relative velocities of the spectator quarks concerning the fireball [8,9].

There are numerous research studies to investigate the properties of quarkonium in the magnetic field [10], in which the three-dimensional Schrödinger equation (SE) numerically solved with the Cornell and the QCD Coulomb potentials. In Ref. [11], the properties of quarkonium states have been studied in the presence of strong magnetic field. Two methods were used to calculate the critical value of the magnetic field for both charmonium and bottomonium states. Bagchi et. al. inferred that in the presence of a magnetic field, the bound states $J/\psi$ and $Y(1S)$ become more firmly bound than in a pure thermal QGP owing to the alteration of the heavy quark potential [12]. In Ref. [13], the authors studied the effect of a strong external magnetic field on quarkonium states $c\bar{c}$ and $b\bar{b}$ in the framework of a non-relativistic quark model. Furthermore, the authors included in their calculation anisotropies through static quark-antiquark potential in agreement with recent lattice studies. In Ref. [14], the dissociation of heavy quarks in hot QCD plasma in the presence of a strong magnetic field is studied by using Nikiforov-Uvarov (NU) method. In Ref. [15], one can get more recent review about heavy quarkonium in extreme condition.

In recent years, there has been considerable interest in fractional calculus of several branches of physics [16-21]. The analytical-exact iteration method was extended to the conformable fractional form to obtain the analytical solutions of the N-dimensional radial SE with its applications on heavy mesons [16]. The generalized NU method was also extended to the fractional domain of high-energy physics by using the radial SE [17]. Abdeljawad used the fractional concept of NU to solve fractional radial SE for different interaction potentials such as the oscillator potential, Woods-Saxon potential, and Hulthén potential [18]. Furthermore, Herrmann applied the derivative Caputo fractional Schrödinger wave equa-
tion by using quantitative of the classical nonrelativistic Hamiltonian [19]. The conformable fractional form was extended to a finite temperature medium to study the binding energy and dissociation of temperature [20].

The aim of the present work is to find the binding energy and the dissociation temperature of quarkonia by using fractional non-relativistic quark model. It shows that the fractional model is more practical to study like that problems. In addition, heavy quarkonium has been thoroughly studied in a hot-magnetized medium. This paper is organized as follows: In Sec. II we provide the theoretical method. In Sec. III the method is given in detail to solve the N-dimensional SE. In Sec. IV we discuss the obtained results. The conclusion is given in Sec. V.

II. THEORETICAL MODEL

Fractional derivative plays an important role in the applied science. Riemann-Liouville and Riesz and Caputo gave an elegant formula that allows to apply boundary and initial conditions as in Ref. [21].

\[ D_t^\alpha f(r) = \int_{r_0}^{r} K_a(r-s)f^{(n)}(s)d(s), r > r_0 \]

with

\[ K_a(r-s) = \frac{(r-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}, \]

where, \( f^{(n)} \) is the \( n \)th derivative of the function \( f(r) \), and \( K_a(r-s) \) is the kernel, which is fixed for a given real number \( \alpha \). The kernel \( K_a(r-s) \) has singularity at \( r = s \). Caputo and Fabrizio[22] suggested a new formula of the fractional derivative with smooth exponential kernel of the form to avoid the difficulties that found in Eq. (2.1)

\[ D_r^\alpha = \frac{M(a)}{1-\alpha} \int_{r_0}^{r} \exp \left( \frac{\alpha(t-s)}{1-\alpha} \right) \dot{y}(s) d(s), \]

where \( M(a) \) is a normalization function with \( M(0) = M(1) = 1 \).

A new formula of fractional derivative called conformable fractional derivative (CFD) was proposed by Khalil et al [23].

\[ D_t^\alpha f(r) = \lim_{\varepsilon \to 0} \frac{f(r-\varepsilon r^{1-\alpha}) - f(r)}{\varepsilon}, r > 0 \]

\[ f(0) = \lim_{\varepsilon \to 0} f(r), \]
where
\[
D^\alpha [ f_{nl}(r)] = r^{1-\alpha} \hat{f}_{nl}(r),
\]
(2.6)
\[
D^\alpha [ D^\alpha f(r)] = (1-\alpha)r^{1-2\alpha} \hat{f}_{nl}(r) + r^{2-2\alpha} f_{nl}''(r),
\]
(2.7)
with \(0 < \alpha \leq 1\). This a new definition is simple and provides a natural extension of differentiation with integer order \(n \in \mathbb{Z}\) to fractional order \(\alpha \in \mathbb{C}\). Moreover, the CFD operator is linear and satisfies the interesting properties that traditional fractional derivatives do not, such as the formula of the derivative of the product or quotient of two functions and the chain rule [24].

### III. THE SOLUTION OF THE RADIAL SCHRÖDINGER EQUATION IN THE PRESENCE OF A STRONG MAGNETIC FIELD

As in Refs. [16, 25], in the \(N\)-dimensional space, the SE for two particles which interact with symmetrical potentials takes following form
\[
\left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} + 2\mu(E - V(r)) \right] \Psi(r) = 0,
\]
(3.1)
where \(l, N\) and \(\mu\) are the angular momentum quantum number, the dimensional number, and the reduced mass of the system. The following radial SE is obtained by applying the wave function \(\Psi(r) = r^{\frac{1-N}{2}} R(r)\),
\[
\frac{d^2 R(r)}{dr^2} - 2\mu \left( E - V(r) - \frac{(l + \frac{N-2}{2})^2 - \frac{\mu}{4}}{2\mu r^2} \right) R(r) = 0.
\]
(3.2)
The potential takes the form as in Ref.[25]:
\[
V(r) = -\frac{4}{3} \alpha \left( \frac{e^{-m_D r}}{r} + m_D \right) + \frac{4}{3} \frac{\sigma}{m_D} (1-e^{-m_D r})
\]
(3.3)
where, the string tension \(\sigma = 0.18\) GeV\(^2\) and
\[
\alpha = \frac{12\pi}{11N_c \ln(\frac{\mu_0^2 + M_B^2}{\Lambda_V^2})}
\]
(3.4)
where \(N_c\) is the number of colors, \(M_B\) \((\sim 1\) GeV\) is an infrared mass interpreted as the ground state mass of the two gluons bound to by the basic string, \(\mu_0 = 1.1\) (GeV), \(\Lambda_V = 0.385\) (GeV) as in Refs. [27-29] and the Debye mass [29] becomes as:
\[
m_D^2 = g^2 T^2 + \frac{g^2}{4\pi^2 T} \sum_f |q_f B| \int_0^\infty \frac{e^{\beta \sqrt{p_z^2 + m_f^2}}}{(1 + e^{\beta \sqrt{p_z^2 + m_f^2}})^2} dp_z,
\]
(3.5)
where the first term is the contribution from the gluon loops and dependent on temperature and the magnetic field does not affect it. The second term is this term strongly depends on the magnetic field $eB$ and is not much sensitive to the temperature $T$ of the medium. In the first term, where $\hat{g}$ is the running strong coupling constant and is given by

$$\hat{g} = 4\pi \alpha_s(T), \quad (3.6)$$

where, $\alpha_s(T)$ is the usual temperature-dependent running coupling constant. It is given by

$$\alpha_s(T) = \frac{2\pi}{(11 - \frac{2}{3}N_f) \ln(\Lambda_{QCD}/\Lambda)} , \quad (3.7)$$

where, $N_f$ is the number of flavors, $\Lambda$ is the renormalization scale is taken as $2\pi T$ and $\Lambda_{QCD} \sim 0.2$ (GeV) as in Ref. [30].

The second term is $g = 3.3$, $q_f$ is the quark flavor $f = u$ and $d$, $B$ is the magnetic field, $\beta$ is the inverse of temperature and quark mass massive $m_f = 0.307$ (GeV) as in Ref. [31]. In Eq. (3.5), $e^{-m_D r}$ is extend if $m_D r \ll 1$ is considered in Ref.[32]. We rewrite Eq. (3.3) as follows:

$$V(r) = a_1 r^2 + a_2 r + \frac{a_3}{r} \quad (3.8)$$

where,

$$a_1 = -\frac{2}{3} \sigma m_D , \quad (3.9)$$

$$a_2 = -\frac{4}{3} \alpha m_D^2 + \frac{4}{3} \sigma , \quad (3.10)$$

$$a_3 = -\frac{4}{3} \alpha . \quad (3.11)$$

The fractional of Eq. (3.1) takes the following form as in Ref. [24]

$$D^\alpha [D^\alpha \Psi(r^\alpha)] + \left[ 2\mu (E - V(r^\alpha)) - \left( \frac{l + \frac{N-2}{2}}{2\mu r^{2\alpha}} \right)^2 - \frac{1}{4} \right] \Psi(r^\alpha) = 0 \quad (3.12)$$

the interaction potential in Eq. (3.8) is written in the fractional form as in Refs. [16,17]

$$V(r^\alpha) = a_1 r^{2\alpha} + a_2 r^\alpha + \frac{a_3}{r^\alpha} \quad (3.13)$$

By applying NU method (For detail, see Refs. [17,18,20]), we obtain the spectrum of energy
\[ E = \frac{6a_1}{\delta^2} + \frac{3a_2}{\delta} - \frac{2\mu(\frac{8a_1}{\delta^4} + \frac{3a_2}{\delta^2} - a_3)^2}{(2n + 1) \pm \sqrt{w + 8\mu \left( \frac{3a_1}{\delta^4} + \frac{a_2}{\delta^2} + \frac{(l + N - 2)^2}{2\mu} - \frac{1}{4} \right)}}, \]  

(3.14)

where

\[ w = (2n\alpha)^2 - 4(n(3\alpha - \alpha^2) + \frac{1}{2}n(n - 1)\alpha(\alpha + 1) + \alpha - 1) \]  

(3.15)

The radial wave function takes the following form:

\[ R_{nl}(r^\alpha) = C_{nL}r^{(-\frac{B_1}{\sqrt{2A_1}}-1)\alpha}e^{\sqrt{2A_1}r^\alpha}(-r^{2n}D)^n(r^{(-2n+\frac{B_1}{\sqrt{2A_1}})\alpha}e^{-2\sqrt{2A_1}r^{2\alpha}}) \]  

(3.16)

IV. RESULTS AND DISCUSSION

The Fig. 1 indicates that the real-part is more screened by increasing fractional parameter. Besides, the fractional parameter has an effect on the linear term of potential. In addition, the potential becomes more attractive by increasing temperature from \( T_c \) to \( 2T_c \).

In Fig. 2(a), the real part of potential is plotted as a function of the temperature ratio and magnetic field with dependent on the fractional parameters for the fixed value of \( r = 0.2 \) fm. By taking the temperature range \( T = 0.17 - 0.3 \ \text{GeV} \), the potential is more attractive with increasing fractional parameter than temperature.

In Fig. 2(b), we see that the potential interaction is more screened by increasing magnetic field and fractional parameter. Thus, we deduce that the fractional parameter plays a role in the hot medium at fixed magnetic field and the magnetized medium at fixed temperature.

The QGP is distinguished by color screening: the range of interaction between heavy quarks becomes inversely proportional to temperature. At sufficiently high temperatures, forming a bound state with a heavy quark (\( c \) or \( b \)) and its anti-quark becomes impossible. With increasing temperature, the range of interaction decreases. For temperatures above
FIG. 1. (Color online) The potential interaction as a function of \( r \) with dependent on different parameters \( \alpha \) at (a) \( T = T_c \) and (b) \( T = 2T_c \).

the transition temperature, \( T_c \), the heavy quark interaction range becomes comparable to the charmonium radius. Based on this general observation, one would expect that the charmonium states, as well as the bottomonium states, do not exist above the deconfinement transition.

**A. Binding energy**

By solving the radial SE, we obtain the binding energies \( E_b \) of \( c\bar{c} \) and \( b\bar{b} \). In following, we see the change of the binding energy under the effect of fractional parameter in the hot-magnetized medium. Charmonium binding energy \( E_b \) is plotted in Fig. 3. The binding energy of charmonium decreases with increasing temperature as well as the binding energy shifts to lower values by increasing magnetic field at fixed \( \alpha = 1 \). (See Fig 3(a)) The binding energy go to zero energy faster by decreasing fractional parameter \( \alpha \). It indicates that the dissociation of temperature are affected with depending on fractional parameter. Similarly, the binding energy decreases when the magnetic field increases. (See the Fig 3(c,d)) The binding energy shifts slightly to the lower values by increasing temperature of medium. Hence, the binding temperature tends to zero when temperature increases. Furthermore, the binding energy is faster to tends to zero by decreasing fractional parameter from \( \alpha = 1 \) to \( \alpha = 0.5 \). As seen from Fig. 4 the binding energy of 1S bottomonium decreases by increasing temperature. Besides, the binding energy decreases by increasing magnetic field. While
FIG. 2. (Color online) The potential interaction as (a) a function of temperature ratio and (b) magnetic field with dependent on fractional parameter.

Comparing Fig. 4(a,b), it is seen that the binding energy tends to zero rapidly by decreasing fractional parameter from \( \alpha = 1 \) to \( \alpha = 0.5 \). Furthermore, while comparing Fig. 4(c,d), it is seen that the binding energy \( E_b \) decreases with increasing temperature \( T \) and magnetic field \( eB \). The binding energy tends to zero faster the left panel by decreasing fractional parameter from \( \alpha = 1 \) to \( \alpha = 0.5 \), too.
FIG. 3. (Color online) The binding energy $E_b$ of charmonium as a function of the $T$ in the thermal medium with dependent on the different $eB$ magnetic field values at (a) $N_f = 2; \alpha = 1$ and (b) $N_f = 2; \alpha = 0.5$. The binding energy $E_b$ of charmonium as a function of the magnetic field $eB$ in the thermal medium with dependent on different values of the temperature at (c) $N_f = 2; \alpha = 1$ and (d) $N_f = 2; \alpha = 0.5$.

B. Dissociation temperature with fractional parameter

In the present work, we obtain the dissociation temperature at $E_b \simeq 0$, the approximation provides good accuracy in calculating the dissociation temperature. In the current analysis, we also study influence of the fractional parameter on the dissociation temperature in the presence of hot-magnetized medium for charmonium and bottomonium, using the calculated binding energies.

Table II presents the effect fractional parameter on the dissociation of temperature for different values of magnetic field. It is seen that the dissociation of temperature decreases
FIG. 4. (Color online) The binding energy $E_b$ of bottomonium as a function of the $T$ in the thermal medium with dependent on the different $eB$ magnetic field values at (a) $N_f = 2; \alpha = 1$ and (b) $N_f = 2; \alpha = 0.5$. The binding energy $E_b$ of bottomonium as a function of the magnetic field $eB$ in the thermal medium with dependent on different values of the temperature at (c) $N_f = 2; \alpha = 1$ and (d) $N_f = 2; \alpha = 0.5$.

by increasing magnetic field at $\alpha = 1$. By taking $\alpha = 0.5$, the dissociation of temperature gets lower values than the values at $\alpha = 1$. Similarly, the dissociation of bottomonium decreases by decreasing fractional parameter. As a result, the dissociation of bottomonium is lower than the dissociation of charmonium.

C. Dissociation of heavy quarkonia in a magnetic field

We calculate the dissociation of charmonium and bottomonium at fixed temperature and fractional parameter when $E_b \simeq 0$. By taking thermal medium at $T = 2.5T_c$, the binding
energy of charmonium dissociates while the magnetic field increases up to $eB = 22.5m^2$. By decreasing the temperature of the medium up to $T = 2.490T_c$, the binding energy dissociated at $eB = 24m^2$. A similar situation is also observed for the dissociation of bottomonium. However, the dissociation of bottomonium is larger than the dissociation of charmonium. This conclusion is in good agreement with following works in Ref:[13, 14,25].

V. CONCLUSION

The SE is analytically solved by conformable fractional of the NU method, where the real fractional potential includes temperature $T$ and $eB$. The eigenvalues of energy and corresponding wave functions were obtained, in which they depend on the fractional parameter $0 < \alpha \leq 1$. The study showed the effect of fractional parameter on the effective interaction potential, the binding energy, dissociation of quarkonium in which the interaction potential is screened by increasing the fractional parameter. The binding energy and the dissociation of temperature in the fractional quark model were found to the lower than the classical quark model at $\alpha=1$. We have also observed that the magnetic field is largely affected by

TABLE I. Dissociation temperature ($T_D$) for charmonium

| State | $eB = 5m^2$ | $eB = 25m^2$ | $eB = 50m^2$ |
|-------|-------------|-------------|-------------|
| $\alpha = 1$ | 2.50 $T_c$ | 2.00 $T_c$ | 1.72 $T_c$ |
| $\alpha = 0.5$ | 1.91 $T_c$ | 1.86 $T_c$ | 1.69 $T_c$ |

TABLE II. Dissociation temperature ($T_D$) for bottomonium

| State | $eB = 5m^2$ | $eB = 25m^2$ | $eB = 50m^2$ |
|-------|-------------|-------------|-------------|
| $\alpha = 1$ | 1.99$T_c$ | 1.85$T_c$ | 1.61$T_c$ |
| $\alpha = 0.5$ | 1.82$T_c$ | 1.78$T_c$ | 1.55$T_c$ |

TABLE III. Dissociation of charmonium in the magnetic field

| $\alpha$ | $eB = 22.5m^2$ | $eB = 23m^2$ | $eB = 24m^2$ |
|----------|---------------|---------------|---------------|
| $\alpha = 1$ | $2.5T_c$ | $2.495T_c$ | $2.490T_c$ |
| $\alpha = 0.5$ | $2.0m^2$ | $2.1m^2$ | $2.2m^2$ |
large-distance interaction, as a result of which the real part of potential is more attractive. The sound representation of the fraction solution provides an efficient and elegant way to solve the specific problems on the physics of interest. Consequently, studying of analytical solution of the modified fractional radial Schrödinger equation for the hot-magnetized interaction potential within the framework conformable fractional the Nikiforov-Uvarov method could provide valuable information on the quantum mechanical dynamics at nuclear, atomic and molecule physics and opens new window.

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