Aharonov-Bohm Effect in Cyclotron and Synchrotron Radiations

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Abstract

We study the impact of Aharonov-Bohm solenoid on the radiation of a charged particle moving in a constant uniform magnetic field. With this aim in view, exact solutions of Klein-Gordon and Dirac equations are found in the magnetic-solenoid field. Using such solutions, we calculate exactly all the characteristics of one-photon spontaneous radiation both for spinless and spinning particle. Considering non-relativistic and relativistic approximations, we analyze cyclotron and synchrotron radiations in detail. Radiation peculiarities caused by the presence of the solenoid may be considered as a manifestation of Aharonov-Bohm effect in the radiation. In particular, it is shown that new spectral lines appear in the radiation spectrum. Due to angular distribution peculiarities of the radiation intensity, these lines can in principle be isolated from basic cyclotron and synchrotron radiation spectra.

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Aharonov-Bohm (AB) effect [1] plays an important role in quantum theory refining the status of electromagnetic potentials in this theory. First this effect was discussed in relation to a study of interaction between a non-relativistic charged particle and an infinitely long and infinitesimally thin magnetic solenoid field\(^1\) (further AB field). It was discovered that particle wave functions vanish at the solenoid line. In spite of the fact that the magnetic field vanishes out of the solenoid, the phase shift in the wave functions is proportional to the corresponding magnetic flux [3]. A non-trivial particle scattering by the solenoid is interpreted as a possibility for quantum particles to ”feel” potentials of the corresponding electromagnetic field. Indeed, potentials of AB field do not vanish out of the solenoid. A number of theoretical works and convinced experiments was done to clarify and prove the existence of AB effect. The detailed exposition of this activity can be encountered, for example, in [4–7]. In particular, it was shown [8–10] that AB scattering is accompanied by an electromagnetic radiation. Pair creation by a photon in the presence of AB field was calculated in [10]. The interaction between electron spin and AB field leads to Dirac wave functions that do not vanish at the solenoid line. Thus, the issue of spin changes slightly the interpretation of AB effect. Theoretical study of AB scattering for spinning particles was presented in many papers, see for example [11] and [12]. AB effect was also discussed in connection with fractional spin and statistics [13] and with cosmic strings [14–15]. AB effect in anyon scattering was considered in [16]; radiative corrections to the effect were calculated in [17]. AB scattering within the Chern-Simons theory of scalar particles was studied in [18]. There exist impressive applications of AB effect in solid state physics [19,21].

A splitting of Landau levels in a superposition of parallel uniform magnetic field and AB field (further magnetic-solenoid field) gives an example of AB effect for bound states. First, exact solutions of Schrödinger equation in the magnetic-solenoid field (non-relativistic case)

\(^1\)A similar effect was discussed earlier by Ehrenberg and Siday [2].
were studied in [20]. Then these solutions were used in [22–24] to discuss AB effect.

It is well-known that a charged particle irradiates moving in a uniform magnetic field. The corresponding radiation is called cyclotron one (CR) in the non-relativistic case; it is called synchrotron radiation (SR) in the relativistic case. In the present article we study how the presence of AB field affects CR and SR. It is clear that classical trajectories do not feel the presence of AB field whenever they do not intersect the solenoid. Thus, from the classical point of view, CR and SR are not affected by the presence of AB field. However, the latter field changes quantum trajectories, thus we expect that in the framework of quantum theory characteristics of CR and SR may be affected by such a field. We calculate spontaneous one-photon radiation of a particle (both spinless and spinning) in the magnetic-solenoid field in the framework of quantum theory. We consider from the beginning quantum relativistic problem in order to analyze consistently both relativistic (SR) and non-relativistic (CR) cases. One ought to mention that conventional CR and SR were studied in detail in numerous works (see [25] and Refs. there). The analysis of the radiation in the magnetic-solenoid field is much more complicated and contains many new aspects and technical details.

The article is organized in the following way: In Sect.II we present exact solutions of Klein-Gordon and Dirac equations in the magnetic-solenoid field and analyze the energy spectrum of particles in such a field. In Sect.III matrix elements of transitions (both for spinless and spinning particles) with one-photon radiation are calculated exactly. In Sect.IV we analyze frequencies of the radiation. Spinless particle radiation is studied in detail in Sect.V. Here an exact expression for the radiation intensity is obtained. The non-relativistic approximation, semiclassical approximation, and weak magnetic field limit are considered. Besides, we reveal some important peculiarities of angular distribution of the radiation. Results that were obtained for spinless particle radiation are generalized to spinning particle case in Sect.VI. Particular emphasis is given to electron transitions that cause low frequency (less that the basic synchrotron frequency) radiation. In the end, we summarize results focusing our attention on manifestations of AB effect in CR and SR.
II. RELATIVISTIC PARTICLE IN MAGNETIC-SOLENOID FIELD

As was mentioned above, the magnetic-solenoid field is a superposition of a constant uniform magnetic field of strength $H$ directed along the axis $z$ and a solenoid field (AB field). The latter field is created by an infinitely long and infinitesimally thin solenoid situated along the same axis $z$. The solenoid creates a finite magnetic flux $\Phi$ along the axis $z$. The magnetic-solenoid field is given by electromagnetic potentials of the form

$$A_1 = x^2 \left( \frac{\Phi}{2\pi r^2} + \frac{H}{2} \right), \quad A_2 = -x^1 \left( \frac{\Phi}{2\pi r^2} + \frac{H}{2} \right), \quad A_0 = A_3 = 0,$$

$$r^2 = (x^1)^2 + (x^2)^2, \quad x^0 = ct, \quad x = (x^i), \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \quad (2.1)$$

The potentials (2.1) define the magnetic field $\mathbf{H}$ of the form

$$\mathbf{H} = (0, 0, H_z), \quad H_z = H + \Phi \delta(x^1) \delta(x^2). \quad (2.2)$$

It is convenient to present the magnetic flux $\Phi$ as

$$\Phi = (l_0 + \mu)\Phi_0, \quad \Phi_0 = 2\pi c\hbar/|e|, \quad 0 \leq \mu < 1, \quad l_0 \in \mathbb{Z}. \quad (2.3)$$

The integer $l_0$ gives a number of quanta $\Phi_0$ in the total flux $\Phi$. The quantity $\mu$ will be called the mantissa of the magnetic flux $\Phi$. In the cylindrical coordinates $r, \varphi,$

$$x^1 = r \cos \varphi, \quad x^2 = r \sin \varphi, \quad \rho = \frac{\gamma r^2}{2}, \quad \gamma = \frac{|eH|}{c\hbar} > 0, \quad (2.4)$$

the non zero potentials have the form

$$\frac{|e|}{c\hbar}A_1 = \frac{l_0 + \mu + \rho}{r} \sin \varphi, \quad \frac{|e|}{c\hbar}A_2 = -\frac{l_0 + \mu + \rho}{r} \cos \varphi. \quad (2.5)$$

Doing a transformation of relativistic wave functions $\Psi(x) = e^{-il_0\varphi}\tilde{\Psi}(x)$, we can eliminate $l_0$ dependence from the potentials. Indeed, electromagnetic potentials enter in relativistic wave equations via operators of momenta $\hat{P}_\mu = i\hbar \partial_\mu - \frac{|e|}{c} A_\mu$. Thus, equations for $\tilde{\Psi}(x)$ contain momentum operators of the form

$$e^{il_0\varphi} P_\mu e^{-il_0\varphi} = \hbar \left( i \partial_\mu + \tilde{A}_\mu \right), \quad \tilde{A}_0 = \tilde{A}_3 = 0,$$

$$\tilde{A}_1 = \frac{\mu + \rho}{r} \sin \varphi, \quad \tilde{A}_2 = -\frac{\mu + \rho}{r} \cos \varphi. \quad (2.6)$$
Therefore, the functions $\tilde{\Psi}(x)$ depend on the mantissa of the magnetic flux only.

Consider first solutions of Klein-Gordon equation

$$\left(\hat{P}^2 - m_0^2c^2\right)\Psi(x) = 0, \quad \hat{P}_\mu = i\hbar\partial_\mu - \frac{e}{c}A_\mu$$  \hfill (2.7)

in the solenoid-magnetic field. The operators $\hat{P}_0, \hat{P}_3,$ and $\hat{L}_z = x^2p_1 - x^1p_2 = -i\hbar\partial_\varphi$ are integrals of motion in the case under consideration ( $\hat{L}$ is angular momentum operator). We are looking for solutions of (2.7) that are eigenvectors for these operators,

$$\hat{P}_0\Psi = \hbar k_0\Psi, \quad \hat{P}_3\Psi = \hbar k_3\Psi, \quad \hat{L}_z\Psi = \hbar (l - l_0)\Psi, \quad l \in \mathbb{Z}.$$  \hfill (2.8)

The integer $l$ is called the azimuthal quantum number. As a consequence of (2.8) and (2.7), we have

$$\hat{P}^2_r\Psi = \hbar^2 k^2\Psi, \quad \hat{P}^2_r = \hat{P}^2_1 + \hat{P}^2_2, \quad k_0^2 = m^2 + k_3^2 + k^2, \quad m = \frac{m_0c}{\hbar}.$$  \hfill (2.9)

Solutions of the equations (2.7), (2.8) can be written as

$$\Psi(x) = e^{-i\varphi}\psi(\rho), \quad \Gamma = k_0x^0 + k_3x^3 + (l_0 - l)\varphi,$$  \hfill (2.10)

where the functions $\psi(\rho)$ obey the equation

$$\rho\psi'' + \psi' + \left[\bar{n} + 1 + \frac{(\bar{l} + \rho)^2}{4\rho}\right]\psi = 0, \quad k^2 = 2\gamma\left(\bar{n} + 1 + \frac{1}{2}\right), \quad \bar{l} = l + \mu.$$  \hfill (2.11)

Solutions of this equation can be expressed via Laguerre functions $I_{n,m}(x)$. The latter functions are defined (for any complex $n, m, x, \ n \neq -1, -2, -3, \ldots$) by the relation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}}\frac{\exp(-x/2)}{\Gamma(1 + n - m)}x^{\frac{n-m}{2}}\Phi(-m, n - m + 1; x),$$  \hfill (2.12)

where $\Phi(\alpha, \beta; x)$ is the confluent hypergeometric function (\cite{34}, 9.210). The Laguerre functions $I_{n+m,m}(x)$ are quadratically integrable on the interval $x \geq 0$ whenever $m = 0, 1, 2, \ldots$, and $\Re \alpha > -1$. These functions form a complete and orthonormal set on the interval $x \geq 0$ whenever $m = 0, 1, 2, \ldots$, and $\Im \alpha = 0, \ \alpha > -1$. Namely,
\[
\sum_{n=0}^{\infty} I_{\alpha+n,n}(x)I_{\alpha+n,n}(y) = \delta(x-y), \ x, y > 0, \tag{2.13}
\]
\[
\int_{0}^{\infty} I_{\alpha+n,n}(x)I_{\alpha+m,m}(x)dx = \delta_{m,n}, \quad \alpha > -1, \ n, m = 0, 1, 2, \ldots . \tag{2.14}
\]

It is a matter of direct verification (using an equation for the confluent hypergeometric functions) to prove that a general solution of the differential equation
\[
4x^2 I'' + 4xI' - \left[x^2 - 2x(1 + s + n) + (s - n)^2\right]I = 0 \tag{2.15}
\]
has the form \(I = c_1 I_{s,n} + c_2 I_{n,s}\). The functions \(I_{s,n}\) and \(I_{n,s}\) are linearly independent for \(s - n \notin Z\). Otherwise we have
\[
I_{n,s} = (-1)^{n-s} I_{s,n}, \quad s - n \in Z. \tag{2.16}
\]

Let \(m\) be an integer and non-negative; then Laguerre functions are connected to the Laguerre polynomials \(L_{m}^{\alpha}(x)\) by the relation (30), 8.970
\[
I_{\alpha+m,m}(x) = \sqrt{\frac{\Gamma(1+m)}{\Gamma(1+\alpha+m)}} \exp(-x/2)x^\alpha L_{m}^{\alpha}(x), \quad m = 0, 1, 2, \ldots . \tag{2.17}
\]

Taking the above information into account, we can see that bounded and quadratically integrable solutions of Eq.(2.11) are divided in two types, \(\psi^{(j)}_{n,l}(r), \ j = 1, 2,\)
\[
\psi^{(1)}_{n,l}(r) = I_{\bar{n},\bar{n}-l}(\rho), \ \bar{n} = n + \mu, \ 0 \leq l \leq n,
\]
\[
\psi^{(2)}_{n,l}(r) = I_{\bar{n},-\bar{n}}(\rho), \ \bar{n} = n, \ l < 0, \ n \in Z. \tag{2.18}
\]

The states of the first type \((j = 1)\) correspond to the energy spectrum of the form
\[
k_0^2 = m^2 + k_3^2 + 2\gamma(n + \mu + \frac{1}{2}), \ 0 \leq l \leq n, \tag{2.19}
\]
and ones of the second type \((j = 2)\) correspond to the following spectrum
\[
k_0^2 = m^2 + k_3^2 + 2\gamma(n + \mu + \frac{1}{2}), \ l < 0. \tag{2.20}
\]

The integer \(n \geq 0\) is referred to as the principle quantum number. Note that the spectrum (2.20) of the second type states corresponds exactly to the spectrum of spinless particles in
a uniform magnetic field. The spectrum (2.19) is deformed by the presence of the solenoid field whenever \( \mu \neq 0 \). Thus, the solenoid field partially lifts a degeneracy of the magnetic field spectrum with respect to the quantum number \( l \) whenever \( \mu \neq 0 \). Namely, in the general case, the particle energy spectrum in the magnetic-solenoid field depends on sign \( l \).

In accordance with Eq. (2.18), it is convenient to define an effective quantum number \( \bar{n} \) by the relation

\[
\bar{n} = n + \mu (2 - j) = \begin{cases} 
  n + \mu, & j = 1, \\
  n, & j = 2,
\end{cases} \quad n = 0, 1, 2, \ldots 
\]  

(2.21)

Then Eqs. (2.19), (2.20) can be integrated into a single formula

\[
k_0^2 = m^2 + k_3^2 + 2\gamma (\bar{n} + \frac{1}{2}), \quad \bar{l} \leq \bar{n}.
\]  

(2.22)

We stress that the solutions (2.18) vanish at \( r = 0 \). That allows us to speak about AB effect in the case under consideration whenever \( \mu \neq 0 \).

Similar to Klein-Gordon equation, the Dirac one (in the magnetic-solenoid field)

\[
(\gamma^\mu \hat{P}_\mu - m_0 c)\Psi(x) = 0
\]  

(2.23)

admits \( \hat{P}_0, \hat{P}_3 \) to be integrals of motion. Besides, \( \hat{J}_z = \hat{L}_z + \frac{\hbar}{2} \Sigma_3 \) (\( \hat{J} \) is the total angular momentum operator and \( \Sigma = \text{diag}(\sigma, \sigma) \)) is an integral of motion as well. Thus, we are looking for solutions of (2.23) that are eigenvectors for these integrals of motion,

\[
\hat{P}_0 \Psi = \hbar k_0 \Psi, \quad \hat{P}_3 \Psi = \hbar k_3 \Psi, \quad \hat{J}_z \Psi = \hbar (l - l_0 - \frac{1}{2}) \Psi.
\]  

(2.24)

Solutions of Eqs. (2.23), (2.24) can be written in the form

\[
\Psi(x) = N_D \exp(-i\Gamma) \begin{pmatrix}
  e^{-i\varphi} c_1 \psi_{n-1,l-1}^{(j)}(\rho) \\
  ic_2 \psi_{n,l}^{(j)}(\rho) \\
  e^{-i\varphi} c_3 \psi_{n-1,l-1}^{(j)}(\rho) \\
  ic_4 \psi_{n,l}^{(j)}(\rho)
\end{pmatrix},
\]  

(2.25)
where $N_D$ is a normalization factor and the constant bispinor $C = (c_a, a = 1, 2, 3, 4)$ is subjected to the following algebraic system of equations (we use the standard representation for $\gamma$-matrices)

$$AC = 0, \quad A = \gamma^0 k_0 + \gamma^3 k_3 - \sqrt{2} \gamma \bar{n} \gamma^1 - m.$$ (2.26)

The system (2.26) has a nontrivial solution whenever

$$\det A = \left(k_0^2 - m^2 - k_3^2 - 2\gamma \bar{n}\right)^2 = 0.$$ (2.27)

It follows from (2.27) that the rank of the matrix $A$ equals 2. Thus, a nontrivial general solution of Eqs. (2.26) contains two arbitrary constants and can be written in the following block form via an arbitrary spinor $v$,

$$C = \begin{pmatrix} (k_0 + m) v \\ \left(\sqrt{2} \gamma \bar{n} \sigma_1 - k_3 \sigma_3\right) v \end{pmatrix}, \quad C^+ C = 2k_0 (k_0 + m) v^+ v.$$ (2.28)

As in the spinless particle case, we have here two types of states ($j = 1, 2$). The energy spectrum of spinning particles in the magnetic-solenoid field follows from (2.27),

$$k_0^2 = m^2 + k_3^2 + 2\gamma \bar{n}.$$ (2.29)

States of the second type (with $j = 2$) have one spin orientation only whenever $n = 0$. Indeed, in such a case we must set $c_1 = c_3 = 0$. Thus,

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n = 0, \quad j = 2.$$ (2.30)

In this case, the wave functions (2.25) are eigenvectors for the operator $\Sigma_3$ ($\Sigma_3 \Psi = -\Psi$) with the eigenvalue $-1$ (the electron spin is always opposite to the magnetic field). That fact is well-known [25,26] in the absence of the solenoid field.

The states of the first type ($j = 1$) vanish at $r = 0$ whenever $l \neq 0$. For $l = 0, \mu \neq 0$, these states become singular at $r = 0$. However they still can be normalized to a $\delta$–function. The states of the second type ($j = 2$) vanish at $r = 0$ for any $l$. 

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The arbitrary constant spinor $\nu$ from (2.28) can be specified by an appropriate choice of spin integrals of motion [26]. In what follows we are going to write $\nu$ as

$$\nu = \frac{1}{2} \begin{pmatrix} 1 + \zeta \\ 1 - \zeta \end{pmatrix}, \ \zeta = \pm 1. \quad (2.31)$$

In this case, $\zeta = +1$ corresponds to the spin along the magnetic field, and $\zeta = -1$ corresponds to the spin opposite to the magnetic field.

Finally, we briefly review classical motion of a charged particle in the magnetic solenoid field. That is useful for an interpretation of quantum numbers in the problem under consideration. Suppose we consider classical trajectories that do not intersect $z$ axis. Such trajectories are not affected by the solenoid field and have the form

$$x^0 = \frac{k_0}{m} \tau, \ x^1 = R \cos \kappa + x^1_{(0)}, \ x^2 = R \sin \kappa + x^2_{(0)}, \ x^3 = -\frac{k_3}{m} (\tau - \tau_0),$$

$$\kappa = \omega_0 \tau + \varphi_0, \ \omega_0 = \frac{\gamma}{m}, \ k_0^2 = m^2 + k_3^2 + \gamma^2 R^2. \quad (2.32)$$

Here $\tau$ is the relativistic interval and $R, \varphi_0, x^1_{(0)}, x^2_{(0)}, \tau_0, k_0, k_3$ are integration constants. Classical analogs of quantum operators $\hat{P}_\mu$ and $\hat{L}_z$ read

$$P_0 = \hbar k_0, \ P_1 = \hbar \gamma R \sin \kappa, \ P_2 = -\hbar R \cos \kappa, \ P_3 = \hbar k_3,$$

$$L_z = \frac{\hbar \gamma}{2} \left( R^2 - R_0^2 \right) - \hbar (l_0 + \mu), \ R_0^2 = \left( x^1_{(0)} \right)^2 + \left( x^2_{(0)} \right)^2. \quad (2.33)$$

On the plane $z = \text{const}$, the trajectories (2.32) are circles $\left( x^1 - x^1_{(0)} \right)^2 + \left( x^2 - x^2_{(0)} \right)^2 = R^2$ of radius $R$. The motion along the axis $z$ is uniform with the velocity $v_3 = c k_3 / k_0$. Comparing the classical radial momentum $P_r^2 = P_1^2 + P_2^2$ with the corresponding quantum expressions (2.9), (2.11), we get

$$R^2 = \frac{2 \bar{n} + 1}{\gamma}. \quad (2.34)$$

This equation relates the principal quantum number to the radius $R$ of the classical motion. Comparing $L_z$ from (2.33) with the corresponding quantum expression (2.8), we find

$$\bar{l} = l + \mu = \frac{\gamma}{2} \left( R^2 - R_0^2 \right). \quad (2.35)$$
Thus, we can conclude that classical trajectories with $l \geq -\mu$ embrace the solenoid ($R^2 > R_0^2$) and ones with $l < -\mu$ do not. In quantum theory these conditions are $l \geq 0$ and $l < 0$ respectively. A minimal distance $\Delta R$ between a classical trajectory and the solenoid is related to $l + \mu$ as follows

$$\Delta R = |R - R_0| = \frac{2|l + \mu|}{\gamma (R + R_0)} . \quad (2.36)$$

Thus, in fact, the absolute value of $l$ specifies the above distance.

Trajectories with $l = 0$ and $l = -1$ pass most close to the solenoid. In the first case they embrace the solenoid and in the second one do not. As was already mentioned above, Dirac wave functions with $l = 0$ are singular at $r = 0$. Bearing in mind the classical interpretation of such trajectories, we may treat the existence of the singularity as a result of a superstrong interaction between the electron spin and the solenoid.

The wave functions (2.23) with $N_D = [8\pi Lk_0(k_0 + m)/\gamma]^{-1/2}$ obey the following orthonormality relations ($-L < z < L, L \to \infty$)

$$\langle \Psi_{n',l',k_3'}, \Psi_{n,l,k_3} \rangle = \int \Psi_{n',l',k_3'}^+ \Psi_{n,l,k_3} d\mathbf{r} = \delta_{n,n'} \delta_{l,l'} \delta_{k_3,k_3'} (v^+ v) . \quad (2.37)$$

III. MATRIX ELEMENTS OF ELECTRON TRANSITIONS WITH ONE PHOTON RADIATION

In QED, one-photon radiation intensity caused by electron transitions is given by the expression

$$W_\lambda = \frac{ce^2}{2\pi} \int d\mathbf{\kappa} \delta \left( \kappa + k_0^a - k_0^b \right) |\mathbf{e}_\lambda|^2 ;$$

$$\kappa = \kappa (\sin \theta \cos \varphi', \sin \theta \sin \varphi', \cos \theta) , \quad (3.1)$$

where $\kappa$ is photon wave vector [23]. Spherical angles $\theta, \varphi'$ define angular distribution of the emitted photons and $\kappa = |\kappa|$ defines the frequency $\omega = c\kappa$ and the energy $E_{ph} = \hbar c \kappa$ of a photon. The quantities $k_0^a, k_0^b$ are related to electron energies $E^a, E^b$ in initial and final
states as \( E^{a,b} = chk_0^{a,b} \). Unit vectors \( e_\lambda \) characterize radiation polarization, see for example [25]. \( \alpha \) denotes a matrix element of the operator \( \alpha = (\alpha^i = \gamma^0\gamma^i) \),

\[
\alpha = \int dx \Psi_a^+(x) e^{-iKx} \alpha \Psi_b(x). \tag{3.2}
\]

For spinless particle case, one has to replace \( \bar{\alpha} \) by \( \bar{\mathbf{P}} \),

\[
\bar{\mathbf{P}} = \int dx \Psi_a^+(x) e^{-iKx} \bar{\mathbf{P}} \Psi_b(x), \tag{3.3}
\]

where \( \bar{\mathbf{P}} = (\hat{P}^\mu) \). To get total intensity of the polarized radiation, we have to sum (3.1) over all the final states of the electron. In the SR theory [25], a linear polarization of the radiation is described by \( \sigma \) and \( \pi \) components of the operator \( \bar{\mathbf{P}} \),

\[
\hat{P}_\sigma = \hat{P}_1 \sin \varphi' - \hat{P}_2 \cos \varphi', \quad \hat{P}_\pi = -(\hat{P}_1 \cos \varphi' + \hat{P}_2 \sin \varphi') \cos \theta + \hat{P}_3 \sin \theta. \tag{3.4}
\]

Using Eq. (2.5), these components can be written as

\[
\hat{P}_\sigma = i\hbar \sqrt{\frac{\gamma \rho}{2}} (A - B), \quad \hat{P}_\pi = i\hbar \sqrt{\frac{\gamma \rho}{2}} (A + B) + i\hbar \sin \theta \frac{\partial}{\partial x^3},
\]

\[
A = e^{i(\varphi' - \varphi)} \left( \frac{\rho + l_0 + \mu - i\partial_\varphi}{2\rho} - \partial_\rho \right), \quad B = e^{-i(\varphi' - \varphi)} \left( \frac{\rho + l_0 + \mu - i\partial_\varphi}{2\rho} + \partial_\rho \right). \tag{3.5}
\]

Consider first the spinless particle case. Thus, we have to substitute (3.4) and functions (2.10), (2.18) into (3.3). We are going to mark off quantum numbers of final states by primes. The integration over \( x^3 \) in (3.3) leads to a conservation low for \( z \)-component of the momentum

\[
k_3 - k_3' = \kappa \cos \theta. \tag{3.6}
\]

Integrating over \( \kappa \) in (3.1), we get a conservation low for the energy,

\[
k_0 - k_0' - \kappa = 0. \tag{3.7}
\]

Doing integration over \( \varphi \), we meet integrals of the form

\[
J = \frac{1}{2\pi} \int_0^{2\pi} \exp[i(l - l')\varphi - ikr \sin \theta \cos(\varphi - \varphi')] d\varphi
\]

\[
= J_{l - l'} (2\sqrt{qr}) \exp \left[ i (l - l') \left( \varphi' + \frac{\pi}{2} \right) \right], \tag{3.8}
\]

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where
\[ q = \frac{\kappa^2 \sin^2 \theta}{2\gamma}. \] (3.9)

To make sure that (3.8) is correct, one can use an integral representation for Bessel functions ([30], 8.411.1). Integrating over \( \rho \), we meet two integrals containing the Laguerre functions. These integrals can be done exactly as well,

\[ \int_0^\infty I_{\alpha+m,m}(x)I_{\beta+n,n}(x)J_{\alpha+\beta}(2\sqrt{qx})dx = (-1)^{n+m} I_{\beta+n,m}(q)I_{\alpha+m,n}(q), \]
\[ 0 \leq n, m \in \mathbb{Z}, \Re(\alpha + \beta + 1) > 0; \tag{3.10} \]

\[ \int_0^\infty I_{\alpha+m,m}(x)I_{\beta+n,n}(x)J_{\alpha-\beta}(2\sqrt{qx})dx = (-1)^{n+m} I_{n+m}(q)I_{\alpha+m,\beta+n}(q), \]
\[ 0 \leq n, m \in \mathbb{Z}, \Re(\alpha + 1) > 0. \tag{3.11} \]

Similar integrals can be encountered in ([30], 7.422.2), however there the calculation was in error.

Spinning particle case can be analyzed in the same manner. Thus, as in the conventional SR theory, matrix elements of electron transitions in the magnetic-solenoid field can be calculated exactly.

IV. ANALYSIS OF RADIATION FREQUENCIES

The relations (3.6) and (3.7) together with ones (2.22) or (2.29) define the frequency of the radiation \( \kappa \) as a function of initial and final quantum numbers and of the angle \( \theta \). Due to the axial symmetry of the problem, \( \kappa \) does not depend on \( \varphi' \). Similar to the conventional SR theory, we introduce a number \( \nu \) of emitted harmonic as

\[ \nu = n - n'. \] (4.1)

For \( \mu = 0 \), the frequency \( \kappa \) is a function of the principle quantum number \( n \), of \( \nu \), and of the angle \( \theta \) (see [25]). This frequency does not depend on the azimuthal quantum numbers \( l, l' \). For \( \mu \neq 0 \), this degeneracy is partially lifted. In such a case, the frequency \( \kappa \) depends
on the type of initial and final states. Namely, it depends on the quantum numbers $j, j'$ in accordance with Eq. (3.7). Thus $\kappa = \kappa_{jj'}$. Introducing an effective number $\bar{\nu} = \bar{\nu}_{jj'}$ of emitted harmonic as

$$\bar{\nu} = \bar{n} - \bar{n}' = \nu + \mu (j' - j) = \begin{cases} \nu, j = j' \\ \nu + \mu, j = 1, j' = 2, \bar{\nu} > 0 \\ \nu - \mu, j = 2, j' = 1 \end{cases}$$

one can easily get for spinless particle case

$$\kappa_{jj'} = \frac{k_{0j}}{\sin^2 \theta} \left( 1 - \sqrt{1 - \beta_{j}^2 \frac{2\bar{\nu}}{2\bar{n} + 1} \sin^2 \theta} \right), \quad (4.3)$$

where

$$\beta_{j}^2 = 1 - \left( \frac{m}{k_{0j}} \right)^2 = 1 - \left( \frac{m_0 c^2}{E_j} \right)^2 \quad (4.4)$$

Similar formula takes place for spinning particle case,

$$\kappa_{jj'} = \frac{k_{0j}}{\sin^2 \theta} \left( 1 - \sqrt{1 - \beta_{j}^2 \frac{\bar{\nu}}{\bar{n}} \sin^2 \theta} \right). \quad (4.5)$$

The expressions (4.3) and (4.5) are obtained for initial states with $k_3 = 0$ (next we use the same supposition). Both expressions can be written in the form

$$\kappa_{jj'} \equiv \frac{2\gamma \bar{\nu}}{k_{0j} + \sqrt{k_{0j}^2 - 2\gamma \bar{\nu} \sin^2 \theta}}. \quad (4.6)$$

We can also get the following formulas

$$\kappa = \frac{\gamma}{k_{0j}} (\bar{\nu} + q), \quad \sin \theta = \sqrt{\frac{2}{\gamma k_{0j} \sqrt{\bar{\nu} + q}}}$$

$$\sqrt{k_{0j}^2 - 2\gamma \bar{\nu} \sin^2 \theta} = k_{0j} \frac{\bar{\nu} - q}{\bar{\nu} + q}, \quad (4.7)$$

where the quantity $q$ was defined by Eq. (3.9).

Thus, for $\mu \neq 0$, there appear two spectral series: one results from transitions without any change of the quantum number $j$ and another one results from transitions with the change of $j$. In the former case $\bar{\nu} = \nu$ whereas in the latter case $\bar{\nu} = \nu \pm \mu$ (effective numbers
of emitted harmonics are not integer anymore). Whenever \( \nu > 0 \) and \( n \) are fixed, we get an inequality

\[
\kappa_{21} < \kappa_{11} < \kappa_{22} < \kappa_{12},
\]

which becomes the equality for \( \mu = 0 \). The difference between the frequencies \( \kappa_{11} \) and \( \kappa_{22} \) can be easily estimated for laboratory magnetic fields \( H \ll H_0 \), where \( H_0 = m_0^2 c^3/e\hbar \approx 4.41 \times 10^{13} \) gauss is a critical field. This difference is proportional to \( \mu \),

\[
\kappa_{22} - \kappa_{11} \approx \mu \kappa_{22} \delta, \quad \delta = \frac{\gamma}{k_0^2} = \frac{\gamma}{m^2 k_0^2} = \left( \frac{H}{H_0} \right)^2 \left( \frac{m_0 c^2 \gamma}{E_1} \right)^2.
\]

One can see that \( \delta < 10^{-9} \) for not very high electron energies and for typical (those which are realized in accelerators) magnetic fields \( H \approx 10^4 \) gauss. In such a case, the frequency difference reads

\[
\kappa_{12} - \kappa_{21} \approx 2\kappa_{12} \mu \frac{\mu}{\nu} = 2\omega \mu, \quad \omega = \frac{|e c H|}{E_j},
\]

where \( \omega \) is the synchrotron frequency. Thus, this difference becomes quite noticeable for harmonics with small numbers.

For \( \mu \neq 0 \), there exist a radiation of a harmonic \( \nu = 0 \) due to \( j = 1 \rightarrow j' = 2 \) transitions. For magnetic fields \( H \ll H_0 \) the frequency of such a harmonic is

\[
\omega_{12} = \mu \omega = \mu \frac{e c H}{E_1}.
\]

\( j = 2 \rightarrow j' = 1 \) transitions cause a radiation of a harmonic \( \nu = 1 \). For the above magnetic fields the corresponding frequency reads

\[
\omega_{21} = (1 - \mu) \omega.
\]

Such a radiation does not exist in pure magnetic field. Both frequencies \( \omega_{12}, \omega_{12} \) are less than \( \omega \) (\( \omega \) is the least radiated frequency in pure magnetic field).
V. RADIATION OF SPINLESS PARTICLE

A. Exact expression for radiation intensity

As was discussed above, transition matrix elements that define one-photon radiation in the magnetic-solenoid field can be calculated exactly. For spinless particle case, the differential (with respect to the polarization) radiation intensity has the form:

\[ W_j = W_0 \frac{H}{H_0} \left( \frac{\gamma}{k_{0j}^2} \right)^2 \sum_{\nu,j'} \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \frac{1}{2} \int_0^\pi d\theta \sin \theta \frac{(\bar{\nu} + q)^3}{\nu - q} Q_{jj'} |F_{jj'}|^2 , \tag{5.1} \]

where

\[ W_0 = \frac{e^2 m_0^2 c^3}{\hbar^2} , \quad F_{jj'} = 2 l_2 \sqrt{q} I_{jj'}(q) + l_3 \cot \theta \sqrt{\frac{2k_{0j}^2}{\gamma}} I_{jj'}(q) , \]
\[ I_{1j'}(q) = I_{\bar{n},\bar{n}'}(q), \quad I_{2j'}(q) = I_{\bar{n}',\bar{n}}(q) . \tag{5.2} \]

The quantities \( F_{jj'} \) do not depend on the azimuthal quantum number \( l \). These quantities are completely defined by the quantum numbers \( \bar{n}, \bar{\nu}, j, j' \) and by the polarization of the radiation. The polarization is characterized by quantities \( l_2 \) and \( l_3 \) (see [23]). For \( l_2 = 1, l_3 = 0 \), we get so called \( \sigma \)-component of the linear polarization; for \( l_2 = 0, l_3 = 1 \) we get so called \( \pi \)-component of the linear polarization; for \( l_2 = \pm l_3 = 1/\sqrt{2} \), we get right (left) circular polarization, and, finally, for \( l_2^2 = l_3^2 = 1, l_2 \cdot l_3 = 0 \), we get total intensity of non-polarized radiation. The quantities \( Q_{jj'} \) depend on initial quantum number \( l \) only,

\[ Q_{1j'} = \sum_{\nu} I_{\bar{n}' - \bar{\nu}, \bar{n} - \nu}(q), \quad Q_{2j'} = \sum_{\nu} I_{\bar{n} - \bar{l}, \bar{n}' - \bar{\nu}}(q) . \tag{5.3} \]

Limits of the summation over final quantum numbers \( l' \) depend on \( j' \). Namely, \( 0 \leq l' \leq n - \nu \) whenever \( j' = 1 \), and \( -\infty < l' \leq -1 \) whenever \( j' = 2 \).

The integrand in (5.1) does not depend on \( \varphi' \). Thus, the integration over this angle is trivial. The corresponding factor will be taken into account in following expressions.

Integrating over \( \theta \), we get zero for total circular polarization. The reason is that dominant circular polarizations in the upper \( (0 \leq \theta \leq \pi/2) \) and in the lower \( (\pi/2 \leq \theta \leq \pi) \) half-planes have opposite signs and compensate each other exactly. If we are interested in linear polarization only, then we can always set \( l_2 \cdot l_3 = 0 \) in Eq. (5.1).
Now we are going to fulfil summation in the intensity of the radiation over final azimuthal quantum numbers.

Consider first the case $\mu = 0$. Here the quantity $|F|^2$ in Eq. (5.1) does not depend on the type of the final state since the property (2.16) is valid in this case. Effective quantum numbers coincide with ordinary ones, $\bar{n} = n$, $\bar{l} = l$. Thus, taking into account (5.3), we get

$$\sum_{j'} Q_{jj'} = \sum_{k=0}^{\infty} I_{k,s}^2(q) = 1, \ s = n - l.$$  \hspace{1cm} (5.4)

That is a well-known result in SR theory [25]. In other words, the radiation intensity does not depend on the initial azimuthal quantum number $l$.

In magnetic-solenoid field with $\mu \neq 0$, the quantities $Q_{jj'}$ depend on $l, n, \nu$. Thus, the degeneracy with respect to $l$ is lifted completely. That may be interpreted as follows: According to Eq. (2.36), the quantum numbers $l$ define distances between classical trajectories and the solenoid. At the same time, $l$ defines the type of trajectories. Clearly, that the radiation intensity depends on the distances as well as on the type of states. For $\mu = 0$, the origin is not fixed anymore by the presence of the solenoid. Thus, the $l$ dependence of the radiation intensity dies out.

Let us return to Eq. (5.3), which define $Q_{jj'}$. Using properties of the Laguerre functions, we can get the following expression for a derivative of $Q_{jj'}$

$$\frac{d}{dq} Q_{jj'}(q) = (-1)^{1+j+j'} \sqrt{\frac{k+1}{q}} \left[ (2-j) I_{k+1,s}(q) I_{k,s}(q) 
+ (j-1) I_{s,k+1}(y) I_{s,k}(y) \right], \ k = \bar{n}' - \mu, \ s = \bar{n} - \bar{l}. \hspace{1cm} (5.5)$$

Then, taking into account the behavior of $Q_{jj'}$ at $q = 0$ and at $q = \infty$, we obtain

$$Q_{jj'}(q) = j' - 1 + (-1)^{j'-1} \left[ (2-j) \int_{q}^{\infty} \sqrt{\frac{k+1}{y}} I_{k+1,s}(y) I_{k,s}(y) dy 
+ (j-1) \int_{0}^{q} \sqrt{\frac{k+1}{y}} I_{s,k+1}(y) I_{s,k}(y) dy \right]. \hspace{1cm} (5.6)$$

The result (5.4) follows from (5.6) as $\mu \to 0$. 

B. Radiation in weak magnetic field approximation

Consider here the magnetic-solenoid field with \( H \) obeying the condition \( H \ll H_0 \). Besides, we suppose that

\[
2\gamma\bar{\nu}k_0^2 = 2\frac{H}{H_0}\left(\frac{m_0c^2}{E_j}\right)^2 \bar{\nu} \ll 1. \tag{5.7}
\]

It is known that the only \( \bar{\nu} \sim (E_j/m_0c^2)^3 \) harmonics are effectively emitted in the relativistic case. For such harmonics Eq. (5.7) results in

\[
2\frac{H}{H_0} \frac{E_j}{m_0c^2} \ll 1. \tag{5.8}
\]

It was demonstrated in \[23\] that the condition (5.8) implies insignificance of quantum corrections in the relativistic case. That may be not true in the non-relativistic approximation since the only harmonic \( \bar{\nu} \sim 1 \) is emitted effectively and the condition (5.7) always holds for \( H \ll H_0 \). Practically, the condition (5.8) always holds for real laboratory magnetic fields and electron energies. In the above suppositions, the quantity (3.9) reads

\[
q = \frac{1}{2} \frac{H}{H_0} \left(\frac{m_0c^2}{E_j}\right)^2 \bar{\nu}^2 \sin^2 \theta. \tag{5.9}
\]

Suppose that the numbers of harmonics are not very big, then we can expect that

\[
q \ll 1. \tag{5.10}
\]

For the ultra-relativistic case \( \nu \sim (E_j/m_0c^2)^3 \), we find

\[
q \sim \frac{H}{H_0} \left(\frac{E_j}{m_0c^2}\right)^4.
\]

That means that in the latter case \( q \) can be not small. Thus, namely the condition (5.10) defines the non-relativistic case in the weak magnetic field approximation. In other words, the condition (5.10) corresponds to CR in weak magnetic field approximation. Namely such a radiation is of concern to us in this Section. Below we suppose that (5.10) takes place.

In the case under consideration, we can present the radiation intensity in the following form
\[ W_j = W_j^{cl} \bar{W}_j, \quad W_j^{cl} = \frac{2 e^4 H^2 \beta_j^2 (1 - \beta_j^2)}{m_0^2 c^3}, \]  
(5.11)

\[ \bar{W}_j = \sum_{j'} \bar{W}_{jj'}, \quad \bar{W}_{jj'} = \frac{3}{4(2n + 1)} \int_0^\pi \sin \theta d\theta S_{jj'}^2 \sum_\nu \tilde{\nu}^2 R_{jj'}, \]  
(5.12)

\[ S_{11} = S_{22} = S_{12} = l_2 + l_3 \cos \theta = S, \quad S_{21} = l_2 - l_3 \cos \theta = \bar{S}. \]  
(5.13)

The quantity \( W_j^{cl} \) is the radiation intensity of a first harmonic in the semiclassical approximation (see [25]). The radiation polarization is characterized by the factor \( S_{jj'} \). In particular, one can see that for transitions \( j = 2 \rightarrow j' = 1 \) the sign of the radiation circular polarization is opposite to the sign of the circular polarization for all other transitions. That observation can be useful to identify the radiation related to \( j = 2 \rightarrow j' = 1 \) transitions. The quantities \( R_{jj'} \) can be calculated in lowest order of \( q \) using exact expressions (5.1), (5.2), and (5.6).

1. For transitions \( j = 1 \rightarrow j' = 1 \), we find:

\[ R_{11} = \frac{\Gamma(n + \mu + 1)q^{\nu-1}}{\Gamma(n + \mu + 1 - \nu)\Gamma^2(\nu)} , \quad 1 \leq \nu \leq l , \]
\[ R_{11} = \frac{\Gamma(n - l + 1)\Gamma(n + \mu + 1)q^{2\nu-l-1}}{\Gamma(n - \nu + 1)\Gamma(n - \nu + \mu + 1)\Gamma^2(\nu)\Gamma^2(\nu - l + 1)} , \quad l \leq \nu \leq n . \]  
(5.14)

One can see that the only harmonic \( \nu = 1 \) is effectively emitted in these transitions. At the same time, the radiation intensity does not depend on \( l \) whenever \( 1 \leq l \leq n \). The quantity \( \bar{W}_{11} \) can be easily calculated,

\[ \bar{W}_{11} = \frac{2(n + \mu)}{2(n + \mu) + 1} \bar{S}^2 , \quad \bar{S}^2 = \frac{3}{4} l_2^2 + \frac{1}{4} l_3^2 . \]  
(5.15)

Thus, the radiation is polarized similarly to the conventional (\( \mu = 0 \)) SR case [25]. The quantity \( \bar{W}_{11} \) increases as \( n \rightarrow \infty \), in particular, \( \lim_{n \rightarrow \infty} \bar{W}_{11} = \bar{S}^2 \).

For initial states with \( l = 0 \), transition probabilities are of order \( q \). Let \( \nu = 1 \), then we find for such transitions

\[ \bar{W}_{11} = \frac{H}{H_0} \left( \frac{m_0 c^2}{E_1} \right)^2 \frac{3n(n + \mu)}{5(2n + 2\mu + 1)} \left( \frac{5}{6} l_2^2 + \frac{1}{6} l_3^2 \right) . \]  
(5.16)

Here the linear polarization of the radiation is greater than in (5.13). However, these transitions contribute insignificantly to the radiation compared to all other transitions.
2. For transitions $j = 2 \rightarrow j' = 2$, we find

$$R_{22} = \frac{\Gamma(n + 1)\nu^{-1}}{\Gamma(n + 1 - \nu)\Gamma^2(\nu)}. \quad (5.17)$$

As before, we see that the only first harmonic is effectively emitted. For this harmonic $R_{22} = n$ and

$$\bar{W}_{22} = \frac{2n}{2n + 1} \bar{S}^2. \quad (5.18)$$

(Eq. (5.18) follows from (5.15) as $\mu \to 0$.) The radiation intensity does not depend on $l < 0$ in the least order of $q$.

3. For transitions $j = 2 \rightarrow j' = 1$, we find

$$R_{21} = \frac{\Gamma(n + |l| + 1 - \mu)\Gamma(n - \nu + 1 + \mu)\Gamma^2(1 + \nu - \mu)\mu^2(1 - \mu)^2q^{|l|-1}}{\Gamma(n + 1 - \nu)\Gamma(n + 1)\Gamma^2(|l| + \nu + 1 - \mu)} f^2(\mu). \quad (5.19)$$

We have introduced here a function $f(\mu)$, $0 \leq \mu \leq 1$,

$$f(\mu) = \frac{\sin \mu \pi}{\mu(1 - \mu)\pi}, \quad f(\mu) = f(1 - \mu), \quad f(0) = f(1) = 1,$n

$$f_{\text{max}}(\mu = 1/2) = 4/\pi > 1, \quad 1 \leq f(\mu) \leq 4/\pi, \quad (5.20)$$

which differs insignificantly from the unit whenever $\mu \neq 0$.

In the transitions under consideration, we meet a situation, which is completely different from the one considered before. Here the only transitions from states with $l = -1$ really contribute to the radiation. That fact has a natural physical explanation: For $l = -1$, $j = 2$, classical trajectories do not embrace the solenoid but pass maximally close to the latter. A transition to trajectories embracing the solenoid is more likely namely from such states. It is important to stress that no restrictions exist on numbers of emitted harmonics. For $l = -1$, we get

$$R_{21} = \frac{\Gamma(n + 2 - \mu)\Gamma(n - \nu + 1 + \mu)\mu^2(1 - \mu)^2}{\Gamma(n + 1)\Gamma(n - \nu + 1)(\nu + 1 - \mu)^2} f^2(\mu), \quad (5.21)$$

and
\[
W_{21} = \frac{2\Gamma(n + 2 - \mu)\mu^2(1 - \mu)^2 M_{21} f^2(\mu)}{(2n + 1)\Gamma(n + 1)} S^2,
\]
\[
M_{21} = \sum_{\nu=1}^{n} M_{21}^{\nu}, \quad M_{21}^{\nu} = \frac{\Gamma(n - \nu + 1 + \mu)}{\Gamma(n - \nu + 1)} \left(\frac{\nu - \mu}{\nu - \mu + 1}\right)^2.
\] (5.22)

For example, for \(n = 1\) we obtain
\[
W_{21}(n = 1) = \frac{2\mu^2(1 - \mu)^4}{3(2 - \mu)} f(\mu) S^2.
\] (5.23)

Expressions for big \(n\) can be calculated approximately. Let us demonstrate how one can get an estimation for a typical sum. The sum can be written as
\[
\sum_{\nu=0}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n - \nu + 1)} \left(\frac{\nu + \mu}{\nu + 1}\right)^2 = \frac{\Gamma(n + 2 - \mu)\mu^2}{\Gamma(n + 1)(1 - \mu)^2} + \frac{\Gamma(n + 1 - \mu)n(1 + \mu)^2}{\Gamma(n + 1)\mu^2} + \frac{\Gamma(n - \mu)n(n - 1)}{\Gamma(n + 1)} \left(\frac{2 + \mu}{1 + \mu}\right)^2
\]
\[
+ \sum_{\nu=3}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n - \nu + 1)} \left(\frac{\nu + \mu}{\nu + 1}\right)^2.
\] (5.24)

For \(\nu = 3\) we have an inequality
\[
1 < \left(\frac{\nu + \mu}{\nu + 1}\right)^2 < \left(\frac{3 + \mu}{2 + \mu}\right)^2,
\] (5.25)
which allows us to write a relation
\[
\sum_{\nu=3}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n - \nu + 1)} \left(\frac{\nu + \mu}{\nu + 1}\right)^2 = \delta \sum_{\nu=3}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n - \nu + 1)},
\] (5.26)
where \(\delta\) can be estimated as
\[
1 < \delta < \left(\frac{3 + \mu}{2 + \mu}\right)^2.
\] (5.27)

The latter sum can be calculated exactly using the following well-known relation
\[
\sum_{\nu=1}^{n} \frac{\Gamma(n + 1 + \mu - \nu)}{\Gamma(n - \nu + 1)} = \frac{\Gamma(n + 1 + \mu)}{(1 + \mu)\Gamma(n)}.
\] (5.28)

This estimation can be improved if we write separately four or more terms in (5.24).

In the same manner, we get the following expression for the radiation intensity
\[ \bar{W}_{21} = \frac{2n}{2n + 1} R_n(\mu) \mu^2 (1 - \mu)^2 \left[ \left( \frac{1 - \mu}{2 - \mu} \right)^2 + (n - 1) \delta \right] \bar{S}^2, \quad \frac{1}{2} < \delta < 1, \quad (5.29) \]

where

\[ R_n(\mu) = \frac{\Gamma(n + \mu)\Gamma(n + 2 - \mu)}{\Gamma^2(n + 1)} f^2(\mu), \quad R_0(\mu) = \frac{f(\mu)}{\mu}, \]

\[ R_1(\mu) = (2 - \mu) f(\mu), \quad R_2(\mu) = \frac{1}{4} (1 + \mu) (2 - \mu) (3 - \mu) f(\mu), \ldots, \]

\[ \frac{n + 1}{n} f^2(\mu) \geq R_n(\mu) \geq R_n(1) = 1, \quad \lim_{n \to \infty} R_n(\mu) = f^2(\mu). \quad (5.30) \]

One can see that the quantities \( M^{21} \) from (5.22) change slightly as \( \nu \) changes. Thus, at least a whole succession of first harmonics has equal probabilities of the radiation. In this approximation, such harmonics are not emitted for \( \mu = 0 \) (they appear only in higher orders of \( q \)). For big \( n \), one can see that \( M^{21} \approx n \). The can serve as an additional argument in the favor of the above observation.

The case \( \nu = 1 \) deserves to be considered especially. As was already remarked before, the corresponding radiation frequency (4.12) is less than the cyclotron one. It follows from (5.22) that

\[ \bar{W}_{21}(\nu = 1) = \frac{2n}{2n + 1} R_n(\mu) \left[ \frac{\mu(1 - \mu)^2}{2 - \mu} \right]^2 \bar{S}^2. \quad (5.31) \]

Thus, in this case, the radiation intensity is approximately equal to the classical one multiplied by the factor

\[ \left[ \frac{\mu(1 - \mu)^2 f(\mu)}{2 - \mu} \right]^2. \quad (5.32) \]

4. Finally, consider transitions \( j = 1 \to j' = 2 \). In this case we get

\[ R_{12} = \frac{\Gamma(n - \nu - \mu + 2)\Gamma(n + \mu + 1)(\nu + \mu)^2 q}{\Gamma(n - l + 1)\Gamma(n - \nu + 1)\Gamma^2(l - \nu + 2 - \mu)\Gamma^2(\nu + 1 + \mu)}. \quad (5.33) \]

We see that the only transitions from states with \( l = 0 \) contribute effectively to the radiation. Classical trajectories, which pass maximally close to the solenoid (embracing it), correspond to such initial states. Then, a possible physical interpretation is similar to the one given above. Thus, for \( l = 0 \) we get
\[
\bar{W}_{12} = \frac{2\Gamma(n + 1 + \mu)\mu^2(1 - \mu)^2M_{12}^{12}f^2(\mu)}{(2n + 2\mu + 1)\Gamma(n + 1)} S^2, \\
M_{12}^{12} = \sum_{\nu=0}^{n} M_{\nu}^{12}, \quad M_{\nu}^{12} = \frac{\Gamma(n - \nu + 2 - \mu)}{\Gamma(n - \nu + 1)} \left( \frac{\nu + \mu}{\nu + \mu - 1} \right)^2.
\]

(5.34)

As before, we have here a radiation of \(\nu = 0\) harmonic (even for \(n = 0\) in the initial state). Such a radiation is forbidden for \(\mu = 0\). The frequency of the corresponding radiation is given by the expression (4.11). For \(\nu = 0\), one finds

\[
\bar{W}_{12}(\nu = 0) = \mu^4 \frac{2(n + \mu)}{2(n + \mu) + 1} R_n(\mu) S^2.
\]

(5.35)

In particular, for \(n = 0\) we find

\[
\bar{W}_{12}(\nu = n = 0) = \frac{2\mu^4 f(\mu)}{2\mu + 1} S^2.
\]

(5.36)

Thus, the radiation intensity of such a harmonic is approximately equal to the classical intensity reduced by the factor \(\mu^4 f(\mu)\).

In the transitions \(j = 1 \rightarrow j' = 2\), all the harmonics with different numbers \(\nu\) contribute almost equally to the radiation intensity since \(M_{\nu}^{12}\) from (5.34) does not change significantly as \(\nu\) varies. One can find the following estimation for \(\bar{W}_{12}\) (taking into account the estimation (5.29) for \(\delta\))

\[
\bar{W}_{12} = \frac{2(n + \mu)}{2(n + \mu) + 1} R_n(\mu) \left[ \mu^4 + \frac{n (1 - \mu)^2}{n + 1 - \mu} + \frac{n (n - 1) \mu^2 (1 - \mu^2)^2 \delta}{n + 1 - \mu} \right] S^2.
\]

(5.37)

The existence of the transitions under consideration may be treated as a manifestation of the AB effect. Indeed, in the absence of the solenoid field (more exactly for \(\mu = 0\)) and in the approximation under consideration, the only \(\nu = 1\) harmonic survives.

C. Peculiarities of radiation angular distribution

As it is known [25], in the relativistic case the intensity of the conventional SR is concentrated in the vicinity of the orbit plane within a small angular interval

\[
\Delta\theta \approx \frac{m_0c^2}{E} = \sqrt{1 - \beta^2}.
\]

(5.38)
In such a case, the radiation intensity is maximal for harmonics with big numbers

$$\nu \sim \left(\frac{E}{m_0c^2}\right)^3$$

(5.39)

Thus, it is widely believed that low number harmonics cannot practically be isolated against the background of intensive high frequency radiation.

However, there exist one exclusion from this rule. Indeed, we can easily see that in the conventional SR the intensity of all the harmonics with $\nu \geq 2$ is exactly zero in the directions $\theta = 0, \pi$ (along the magnetic field). Besides, the radiation intensity of the first harmonic ($\nu = 1$) is maximal along the magnetic field for any particle energy. Moreover, the latter radiation has total circular polarization and, thus, can be easily identified.

The presence of the solenoid field modifies both the spectrum and angular distribution of SR. Consider, for example, the intensity of SR in the magnetic-solenoid field in the directions $\theta = 0, \pi$ and within the infinitesimal solid angle $d\Omega = \sin\theta d\theta d\varphi'$. The expressions (5.1), (5.2), and (5.3) allow us to get the following exact result

$$4\pi \left. \frac{dW_{jj'}}{d\Omega} \right|_{\theta=0,\pi} = W^{cl}_{jj'} (l, n, \nu; \mu).$$

(5.40)

The quantity $W^{cl}$ is defined by Eq. (5.11) and

$$G_{11} = \frac{3(n + \mu)(1 - \delta_{l,0}) \delta_{\nu,1}}{2n + 2\mu + 1},$$

$$G_{12} = \frac{3(n + \mu) R_n (\mu) \delta_{l,0}}{2n + 2\mu + 1} \left[ \mu^4 \delta_{\nu,0} + \frac{n(1 - \mu^2)^2 \delta_{\nu,1}}{n - \mu + 1} \right. + \frac{\mu^2(1 - \mu)^2 \Gamma(n + 1) \sum_{\nu=2}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n + 1 - \nu)} \left( \frac{\nu + \mu}{\nu + \mu - 1} \right)^2}{\Gamma(n + 2 - \mu)} \left. \right],$$

$$G_{21} = \frac{3\mu^2(1 - \mu)^2 R_n (\mu) \Gamma(n) \delta_{l,-1}}{(2n + 1) \Gamma(n + \mu)} \sum_{\nu=1}^{n} \frac{\Gamma(n + 1 + \mu - \nu)}{\Gamma(n + 1 - \nu)} \left( \frac{\nu - \mu}{\nu - \mu + 1} \right)^2.$$ 

(5.41)

The function $R_n (\mu)$ is given by Eq. (5.30).

Let us briefly run through some of consequences of the above expression.

Transitions without a change of the type of the initial state (without a change of $j$) cause the only first harmonic ($\nu = 1$) radiation in the directions $\theta = 0, \pi$ whenever $l \neq 0$. This fact does not depend on particle energies. One can see that the quantities $G_{jj}$ grow slightly and
tend to finite constant values as \( n \to \infty \). Transitions from initial states with \( l = 0 \) without a change of \( j \) do not cause any radiation in \( \theta = 0, \pi \) directions.

Transitions with a change of the type of the initial state (with a change of \( j \)) cause a radiation in the directions \( \theta = 0, \pi \) solely for \( l = 0, -1 \) (the solenoid is situated maximally close to a classical trajectory). In such cases all possible harmonics \( (0 \leq \nu \leq n) \) are emitted with approximately equal intensities since the quantities \( G_{12}, G_{21} \) grow proportionally to \( n \). For \( \mu = 0 \), the only first harmonic radiation survives.

Expressions (5.41) allow us to conclude that all the transitions cause totally circular polarized radiation in the directions \( \theta = 0, \pi \). Moreover, as it follows from (5.13), the sign of the circular polarization for \( j = 2 \to j' = 1 \) transitions is opposite to the one for all other transitions.

We believe that the peculiarities of the angular distribution of the radiation open up possibilities for experimental observation of superlow frequencies (4.11), (4.12) and of frequencies that are not multiple of the synchrotron one.

Note, that the expressions (5.41) (and the above mentioned consequences from them) were not known before even in the absence of the solenoid field (for \( \mu = 0 \)).

D. Semiclassical approximation

It was shown in the conventional SR theory [25] that a semiclassical expansion of the radiation intensity can be done in terms of a small parameter \( \nu/n \). Practically, to this end the formula

\[
\lim_{\nu \to \infty} I_{p, \alpha, p + \beta} \left( \frac{x^2}{4p} \right) = J_{\alpha - \beta}(x) \tag{5.42}
\]

was used. Here \( J_{\alpha}(x) \) are Bessel functions. It is natural to believe that for the case \( \mu \neq 0 \) we can use the same parameter to perform the semiclassical expansion. Thus, we get a classical part of the intensity

\[
W_{\alpha j}^{\text{cl}} = \frac{e^4 H^2 (1 - \beta_{j}^2)}{m_0^2 c^3} \sum_{\nu, \nu'} \int_{0}^{\nu} \sin \theta \, d\theta p^2 \bar{Q}_{j j'}^{\text{cl}} |F_{\text{cl}}|^2, \quad F_{\text{cl}} = l_2 \beta_{j} J_{j j'}^{\text{cl}}(z) + l_3 \cot \theta J_{j j'}^{\text{cl}}(z),
\]
\[ z = \bar{v} \beta_j \sin \theta, \quad J_{11} = J_{22} = J_{12} = J_{\bar{\nu}}(z), \quad J_{21} = J_{-\bar{\nu}}(z), \]
\[ Q_{jj'}^{\text{cl}} = \frac{1}{2} + (-1)^{j+j'} \int_z^{\infty} dy(2-j)J_{l-\bar{\nu}+1}(y)J_{l-\bar{\nu}}(y) + (j-1)J_{|l|+\bar{\nu}}(y)J_{|l|+\bar{\nu}-1}(y). \] (5.43)

The components of \( Q_{jj'}^{\text{cl}} \) have the form
\[ Q_{11}^{\text{cl}} = \begin{cases} 1 - f_0^z J_{l-\nu+1}(y)J_{l-\nu}(y)dy, l \geq \nu \\ f_0^z J_{l-\nu-1}(y)J_{l-\nu}(y)dy, l < \nu, \end{cases} \]
\[ Q_{22}^{\text{cl}} = 1 - \int_0^z J_{|l|+\nu}(y)J_{|l|+\nu-1}(y)dy, \]
\[ Q_{12}^{\text{cl}} = \begin{cases} f_0^z J_{l-\nu+1-\mu}(y)J_{l-\nu-\mu}(y)dy, l \geq \nu \\ \frac{1}{2} - f_z^\infty J_{l-\nu+1-\mu}(y)J_{l-\nu-\mu}(y)dy, l < \nu, \end{cases} \]
\[ Q_{21}^{\text{cl}} = \int_0^z J_{|l|+\nu-\mu}(y)J_{|l|+\nu-\mu}(y)dy. \] (5.44)

For \( \mu = 0 \), we see that \(|F^{\text{cl}}|^2\) does not depend on \( j' \) and \( \sum_{j'} Q_{jj'}^{\text{cl}} = 1 \). Then the expression (5.43) presents the well-known [25] classical SR differential intensity.

In the non-relativistic approximation, we get
\[ W_{jj}^{\text{cl}} = W_{jj'}^{\text{cl}} \sum_{j'} W_{jj'}^{\text{cl}} \text{, } W_{12}^{\text{cl}} = 1, \quad W_{12}^{\text{cl}} = \mu^4 f^2(\mu), \quad W_{21}^{\text{cl}} = \left[ \frac{\mu(1-\mu)^2 f(\mu)}{2-\mu} \right]^2. \] (5.45)

Here \( l = \nu = 0 \) for \( j = 1 \rightarrow j' = 2 \) transitions, and \(|l| = \nu = 1 \) for \( j = 2 \rightarrow j' = 1 \) transitions. Eqs. (5.45) follow from the exact quantum expressions (5.18), (5.32), and (5.34) as \( n \rightarrow \infty \).

Semiclassical expressions (5.44) depend essentially on the initial azimuthal quantum number \( l \) whenever \( \mu \neq 0 \). Taking into account Eq. (2.33), we can express \( l \) in terms of the pure classical quantity \( R^2 - R_0^2 \). If \( R^2 - R_0^2 \) is fixed, we get \(|l| \sim 1/h \rightarrow \infty \) as \( h \rightarrow 0 \). Then, it follows from (5.44) that
\[ Q_{jj'}^{\text{cl}} = \delta_{jj'}. \] (5.46)

Such a result seems to be natural. In classical theory of radiation trajectories of particles are fixed (there is no back reaction from emitted photons) and transitions with a change of initial states are not considered.

From Eqs. (5.44), we find the following relations
\[ Q_{11}^{\text{cl}} = \sum_{k=-\infty}^{l-\nu} J_k^2(z), \quad Q_{12}^{\text{cl}} = \sum_{k=l-\nu+1}^{\infty} J_k^2(z), \quad Q_{22}^{\text{cl}} = \sum_{k=-\infty}^{l+\nu-1} J_k^2(z), \quad Q_{21}^{\text{cl}} = \sum_{k=|l|+\nu}^{\infty} J_k^2(z). \] (5.47)
It is clear that \( Q_{jj}^{cl} \) are monotonically increasing functions and \( Q_{jj'}^{cl} \) \((j \neq j')\) are monotonically decreasing functions of |\( l \)| such that

\[
\lim_{|l| \to \infty} Q_{jj'}^{cl} = \delta_{jj'}.
\] (5.48)

Thus, manifestations of the AB effect in SR are maximal for initial states with \( l = 0, -1 \).

All the angular distribution peculiarities, which were noted for the general case in the previous Section, take place in the approximation under consideration as well. Calculating the quantity \( 4\pi \left\{ dW_{jj'} \right\}_{\theta=0, \pi} \) by the help of Eqs. (5.43) and (5.47), we can see that the representation (5.40) holds provided

\[
G_{11} = \frac{3}{2} (1 - \delta_{l,0}) \delta_{\nu,1}, \quad G_{22} = \frac{3}{2} \delta_{\nu,1}, \quad G_{12} = \frac{3}{2} f^2 (\mu) \delta_{\nu,0} + \left( 1 - \mu^2 \right)^2 \delta_{\nu,1} + \mu^2 (1 - \mu)^2 \sum_{\nu=2}^{n} \left( \frac{\nu + \mu}{\nu + \mu - 1} \right)^2 \delta_{\nu,1}, \quad G_{21} = \frac{3}{2} f^2 (\mu) \mu^2 (1 - \mu)^2 \delta_{l,-1} \sum_{\nu=1}^{n} \left( \frac{\nu - \mu}{\nu - \mu + 1} \right)^2.
\] (5.49)

The latter quantities can be obtained from (5.41) in the limit \( n \to \infty \).

VI. RADIATION OF SPINNING PARTICLE

A. Exact expression for radiation intensity

The analysis of radiation frequencies presented in Sect.IV was done for spinless particle case. However, all the qualitative results of this analysis remain valid for the spinning particle case. That can be seen from the corresponding exact formula (4.5).

In the spinning particle case, we get the following exact expression for the differential radiation intensity:

\[
W_{j} = W_{0} \left( \frac{H}{H_0} \right)^2 \varepsilon_{j} \int_{0}^{\pi} d\theta \sin \theta \sum_{\nu, j', \zeta'} Q_{jj'} \left[ 1 - 2p_j (\bar{\nu} + q) \right]^{-1} \frac{(\bar{\nu} + q)^3}{\bar{\nu} - q} |F_{jj'}|^2.
\] (6.1)

Here

\[
\varepsilon_{j} = 1 - \beta_{j}^2 = \left( \frac{m}{k_0} \right)^2 = \left( \frac{m_0 c^2}{E_j} \right)^2, \quad p_{j} = \frac{1}{2} \frac{H}{H_0} \frac{\varepsilon_{j}}{1 + \sqrt{\varepsilon_{j}}},
\] (6.2)
and the constant $W_0$ was introduced in (5.2). Remarkable that the quantities $Q_{jj'}$ are given by the same expressions (5.3) as in spinless particle case. Thus, all the previous conclusions related to these quantities are applied here as well.

In the spinning particle case, the quantities $F_{jj'}$ have the form

$$F_{jj'} = l_2 F_{jj'}^{(2)} + l_3 F_{jj'}^{(3)};$$

$$F_{jj'}^{(2)} = \sqrt{\frac{H}{\varepsilon_j H_0}} 2q \left[ \delta_{\zeta,\zeta'} (-1)^{j-1} \left( \frac{1 + \zeta}{2} A^i_+ + \frac{1 - \zeta}{2} A^i_- \right) + \delta_{\xi_{-\xi'}} q \cot \theta \left( \frac{1 + \zeta}{2} \chi^i_+ + \frac{1 - \zeta}{2} \chi^i_- \right) \right],$$

$$F_{jj'}^{(3)} = \delta_{\zeta,\zeta'} (-1)^{j-1} \cot \theta \times \left( \frac{1 + \zeta}{2} B^i_+ + \frac{1 - \zeta}{2} B^i_- \right) + \delta_{\xi_{-\xi'}} P_j \left[ (-1)^{j-1} \frac{1 + \zeta}{2} C^i_+ + \frac{1 - \zeta}{2} C^i_- \right],$$

where

$$A^i_\pm = 2q \left[ 1 - p_j (\bar{\nu} + q) \right] \frac{d^2 \varphi^i_\pm}{dq} + \left[ q - p_j (\bar{\nu} + q)^2 \right] \varphi^i_\pm,$$

$$B^i_\pm = \left[ 1 - p_j (\bar{\nu} + q) \right] \varphi^i_\pm + 2qp_j \frac{d \varphi^i_\pm}{dq},$$

$$C^i_\pm = \left[ 1 \pm \sqrt{\varepsilon_j} (\bar{\nu} + q) \right] \chi^i_\pm + 2q \frac{d \chi^i_\pm}{dq},$$

$$\varphi^1_+ = I_{n-1,n'-1}(q), \quad \varphi^2_+ = I_{n'-1,n-1}(q), \quad \varphi^1_- = I_{n,n'}(q), \quad \varphi^2_- = I_{n',n}(q),$$

$$\chi^1_+ = I_{n-1,n'}(q), \quad \chi^2_+ = I_{n',n-1}(q), \quad \chi^1_- = I_{n,n'}(q), \quad \chi^2_- = I_{n',n-1}(q),$$

and $I_{n,n'}(x)$ are the Laguerre functions. All the final quantum numbers are primed here. The quantities $l_2$ and $l_3$ characterize the radiation polarization. Contributions from transitions without ($\sim \delta_{\zeta,\zeta'}$) and with ($\sim \delta_{\xi_{-\xi'}}$) spin-flip are separated.

The states with $n = 0$ are a special case. For $j = 2$, there exist the only one (opposite to the magnetic field) spin orientation. Thus, all the transitions from any states with $\zeta = -1$ to $n = 0, \ j = 2$ states do not cause a spin-flip ($A^i_+ = B^i_+ = \varphi^i_+ = 0$), and all the transitions from any states with $\zeta = 1$ to $n = 0, \ j = 2$ states do cause a spin-flip ($C^i_- = \chi^i_- = 0$). One of such transitions is studied below. States with $n = 0, \ j = 1$ are singular at $r = 0$. However, they still can be normalized to a $\delta-$function.

As in the spinless particle case, we can conclude that the radiation intensity depends on the mantissa $\mu$ only but not on the total solenoid magnetic flux $\Phi$.

The radiation in question has not a preferential circular polarization. Total (integrated over all angles) intensities of the right and left circular polarizations are equal as in the scalar
particle case. However, for transitions \( j = 2 \to j' = 1 \) and \( j = 2, \zeta = 1 \to j' = 2, \zeta = -1 \), the sign of the circular polarization is opposite to the one for all other transitions. Further, we are going to analyze the linear polarization only.

The angular distribution of the electron radiation intensity in the magnetic-solenoid field is quite similar to the one for spinless particle. The corresponding exact formula reads

\[
4\pi \left. \frac{dW_{jj'}}{d\Omega} \right|_{\theta=0,\pi} = W_{\text{cl}} \frac{3}{2} \left( \frac{1+\zeta}{2} G^+_{jj'} + \frac{1-\zeta}{2} G^-_{jj'} \right),
\]

where

\[
\begin{align*}
G^+_{11} &= \frac{(n + \mu - 1)(1 - \delta_{l,0})}{n + \mu} \delta_{\nu,1}, \quad G^-_{11} = (1 - \delta_{l,0}) \delta_{\nu,1}, \quad G^+_{22} = \frac{n-1}{n} \delta_{\nu,1}, \quad G^-_{22} = \delta_{\nu,1}, \\
G^+_{12} &= \frac{nR_n(\mu)}{n + \mu} \left[ \frac{n}{n - \mu + 1} \right] \left\{ \frac{\mu^4 \delta_{\nu,0} + \frac{(n - 1)(1 - \mu^2)}{n - \mu + 1} \delta_{\nu,1}}{n - \mu + 1} + \frac{\mu^2 (1 - \mu)^2 \Gamma(n)}{\Gamma(n + 2 - \mu)} \right. \\
&\quad \times \left. \sum_{\nu=2}^{n} \frac{\Gamma(n + 2 - \mu - \nu)}{\Gamma(n - \nu)} \left( \frac{\nu + \mu}{\nu + \mu - 1} \right)^2 \right\}, \quad G^-_{12} = R_n(\mu) \delta_{l,0} \left[ \frac{\mu^4 \delta_{\nu,0} + \frac{n(1 - \mu^2)}{n - \mu + 1} \delta_{\nu,1}}{n - \mu + 1} \right] \\
G^+_{21} &= \frac{R_n(\mu) \mu^2 (1 - \mu)^2 \Gamma(n + 1)}{\Gamma(n + \mu)} \sum_{\nu=1}^{n} \frac{\Gamma(n + \mu - \nu)}{\Gamma(n + 1 - \nu)} \left( \frac{\nu - \mu}{\nu - \mu + 1} \right)^2, \\
G^-_{21} &= \frac{R_n(\mu) \mu^2 (1 - \mu)^2 \Gamma(n)}{\Gamma(n + \mu)} \sum_{\nu=1}^{n} \frac{\Gamma(n + 1 + \mu - \nu)}{\Gamma(n + 1 - \nu)} \left( \frac{\nu - \mu}{\nu - \mu + 1} \right)^2.
\end{align*}
\]

\[ (6.4) \]

B. Radiation in weak magnetic field approximation

Here we suppose that the magnetic field is weak, i.e. \( H \ll H_0 \) (more exactly \( q \ll 1 \)), and that initial quantum numbers are not very big (thus the particle remains non-relativistic). In this approximation, the main contributions to the radiation are due to transitions without a spin-flip. Below we present the only first terms in the \( H/H_0 \) decomposition for the radiation intensity.

First consider the radiation caused by transitions without a change of the state type \((j = j')\). Here the transitions \( \nu = 1, l' = l - 1 \) play the main role. For initial states with \( l \neq 0 \) the radiation intensity reads
\[ W_j = W_j^{\text{cl}} \left\{ \delta_{\zeta,\zeta'} \left( \frac{1 + \zeta \bar{n} - 1}{2} \frac{n}{\bar{n}} + \frac{1 + \zeta}{2} \right) S_0 \\
\quad + \delta_{\zeta,-\zeta'} \left[ \frac{1 + \zeta}{2} \frac{H S_1}{H_0 2 \bar{n}} + \frac{1 - \zeta}{2} \left( \frac{H}{H_0} \right)^3 \frac{n - 1}{70} S_2 \right] \right\}. \] (6.6)

The quantity \( W_j^{\text{cl}} \) is the radiation intensity of the first harmonic in the semiclassical approximation (see (5.11)). The linear radiation polarization is characterized by the factors

\[ S_0 = \frac{3}{4} l_2^2 + \frac{1}{4} l_3^2, \quad S_1 = \frac{1}{4} l_2^2 + \frac{3}{4} l_3^2, \quad S_2 = \frac{1}{8} l_2^2 + \frac{7}{8} l_3^2. \] (6.7)

Whenever the initial state has the spin along the field (\( \zeta = 1 \)), the ratio between the transitions with and without a spin-flip is of the order \( H/H_0 \). The same ratio is of the order \( (H/H_0)^3 \) for \( \zeta = -1 \). Thus, states with \( \zeta = -1 \) are more stable than ones with \( \zeta = 1 \). That is the reason of the self-polarization effect [25] in SR. The presence of the solenoid affects the only effective quantum numbers \( \bar{n} \), the latter are not always integer, for example, \( \bar{n} = n + \mu \) for \( j = 1 \). The radiation has a preferential linear polarization. For \( \zeta = 1 \) initial states, transitions with and without a spin-flip cause radiation intensities of almost (with the interchange of \( l_2 \) and \( l_3 \)) the same form. For \( \zeta = -1 \) initial states, transitions with a spin-flip cause almost (with the interchange of \( \sigma \) and \( \pi \) components) the same linear polarization of the radiation intensity as SR has in the relativistic case [23]. For \( \mu = 0 \), the expression (6.6) (without the polarization specification) coincides with a corresponding expression presented in [31,32].

For \( l = 0 \) in initial states, transitions without any change of \( j \) are suppressed; these transitions contribute to the radiation intensity in higher orders of \( H/H_0 \) only. For example, for such transitions with a spin-flip, we find

\[ W = W^{\text{cl}} \frac{H}{H_0} \frac{3n (1 - \beta^2)}{10} \left( \frac{1 + \zeta}{2} \frac{n + \mu - 1}{n + \mu} + \frac{1 - \zeta}{2} \right) \left( \frac{5}{6} l_2^2 + \frac{1}{6} l_3^2 \right). \] (6.8)

We see that for the latter transitions, the polarization is distinctive and may serve to isolate such transitions.

Transitions with a change of the state type (with a change of \( j \)) are of particular interest from the AB effect point of view. Leading (with respect to \( H/H_0 \)) contributions correspond
to \(j = 1, l = 0 \rightarrow j' = 2, l' = -1\) and to \(j = 2, l = -1 \rightarrow j' = 1, l' = 0\) transitions. In such transitions a whole set of successive harmonics is emitted, all these harmonics have approximately equal probabilities. The same situation was discovered by us in the spinless particle case.

As an example, we considered the radiation intensity for \(j = 1 \rightarrow j' = 2\) transitions without a spin-flip in detail. In such a case, this intensity has the form

\[
W = W^{cl} S_0 M_{12} \delta_{\zeta,\zeta'},
\]

(6.9)

where \(S_0\) is defined by (6.7) and the quantity \(M_{12}\) is a function of initial quantum numbers \(n\) and \(\zeta\),

\[
M_{12} = \mu^2 (1 - \mu)^2 f^2(\mu) \frac{\Gamma(n+\mu)}{\Gamma(n+1)}
\times \sum_{\nu=0}^{n} \frac{\Gamma(n+2-\mu-\nu)}{\Gamma(n+1-\nu)} \left(\frac{\mu+\nu}{\mu+\nu-1}\right)^2 \left(\frac{1+\zeta n-\nu}{2n+\mu+1-\zeta}\right).
\]

(6.10)

(\(f(\mu)\) was defined by (5.20).) In particular, here there is a possibility for \(\nu = 0\) transition with the emission of a superlow frequency (4.11). For the latter transition,

\[
M_{12}(\nu = 0) = \mu^4 R_n(\mu) \left(\frac{1+\zeta}{2} \frac{n}{n+\mu} + \frac{1-\zeta}{2}\right),
\]

(6.11)

where \(R_n(\mu)\) is given by (5.30). Similar transition is possible even from \(n = 0\) states. Then

\[
M_{12}(n = 0, \zeta, \mu) = \frac{1-\zeta}{2} \mu^3 f(\mu).
\]

(6.12)

Taking into account that \(\beta^2 = 2\mu H/H_0\) for a state \(j = 1, n = 0\), we find from (6.11)

\[
W(n = 0) = \frac{41-\zeta}{3} W_0 \left(\frac{H}{H_0}\right)^3 \mu^4 f(\mu) S_0.
\]

(6.13)

This results fitted well with Eq. (5.36) since in the spinless particle case

\[
\beta^2 = (1 + 2\mu) \frac{H}{H_0}
\]

(6.14)

for the state under consideration.
As before, for big \( n \), one can easily obtain estimations for emerged sums. Thus, for the radiation intensity caused by transitions with a change of \( j \) and without any spin-flip, we get the following expression:

\[
W = W^{cl} R_n(\mu) \left( \frac{1 + \zeta}{2} M^+ + \frac{1 - \zeta}{2} M^- \right) S_0,
\]

where

\[
M^+_{12} = \frac{n}{n+\mu} \left[ \mu^4 + \frac{(n-1)(1-\mu^2)^2}{n+1-\mu} + \frac{(n-1)(n-2)\mu^2(1-\mu)^2}{n+1-\mu} \delta_1 \right],
\]

\[
M^-_{12} = \mu^4 + \frac{n(1-\mu^2)^2}{n+1-\mu} + \frac{n(n-1)\mu^2(1-\mu)^2}{n+1-\mu} \delta_1 ,
\]

\[
M^+_{21} = \frac{\mu(1-\mu)^2 n}{n+\mu-1} \times \left[ \mu \left( \frac{1-\mu}{2-\mu} \right)^2 + (n-1) \delta_2 \right],
\]

\[
M^-_{21} = \mu^2 (1-\mu)^2 \left[ \left( \frac{1-\mu}{2-\mu} \right)^2 + (n-1) \delta_2 \right].
\]

For \( \delta_k, k = 1, 2 \), we have the estimation \( 1 < \delta_1 < 2, \ 1/2 < \delta_2 < 1 \). The quantities \( M^\pm_{jj'} \) are linearly increasing functions of \( n \) whenever \( \mu \neq 0 \) (the same property takes places in spinless particle case). For \( \mu = 0 \), we get another behavior

\[
\lim_{n \to \infty} M^\pm_{12}(\mu = 0) = \delta_{\nu,1}, \ M^\pm_{21}(\mu = 0, 1) = 0.
\]

Consider transitions that cause non zero contributions to \( M^\pm_{21} \) for \( \mu \neq 0 \). We know that the contribution of such transitions to the radiation intensity is of higher order of \( H/H_0 \) whenever \( \mu = 0 \). Thus, only in the presence of the solenoid (with \( \mu \neq 0 \)) a whole set of successive harmonics is emitted with approximately equal probabilities. The number of harmonics in the set is comparable with the number of the energy level.

For the radiation intensity caused by transitions with a change of the state type and with the spin-flip, we get the following results:

For \( j = 1 \to j' = 2 \) transitions

\[
W_{12} = W^{cl} R_n(\mu) \left[ \frac{1 + \zeta}{2} \frac{H}{H_0} \frac{S_1}{2(n+\mu)} N^+_{12} + \frac{1 - \zeta}{2} \frac{H}{H_0} \frac{3}{35} N^-_{12} \right],
\]

\[
N^+_{12} = \mu^6 + \frac{n(1+\mu^2)(1-\mu^2)^2}{n+1-\mu} + \frac{n(n-1)(n+2)\mu^2(1-\mu)^2}{n+1-\mu} \delta_1 ,
\]

\[
N^-_{12} = \mu^6 \frac{1}{(1+\mu)^2} \left[ \frac{\mu^2}{8} l_2^2 + \frac{7}{8} l_3^2 \right] + \frac{(n-1)(1-\mu^2)^2(1+\mu)^2}{(n+1-\mu)(2+\mu)} \left[ \frac{(1+\mu)^2}{8} l_2^2 + \frac{7}{8} l_3^2 \right].
\]
\[\frac{(n-1)(n-2)\mu^2(1-\mu)^2\delta_1}{n+1-\mu}\left[\frac{(n+2)^2}{8}l_2^2 + \frac{7}{8}l_3^2\right].\] (6.17)

For \(j = 2 \rightarrow j' = 1\) transitions

\[W_{21} = W^{cl}R_n(\mu)\left[\frac{1+\zeta}{2}\left(\frac{H}{H_0}\right)^3\frac{2n}{35}N_2^1 + \frac{1-\zeta}{2}\frac{H\mu(1-\mu)^2S_1}{H_0(2(n+\mu-1))}N_2^-\right],\]

\[N_2^+ = \frac{(1-\mu)^6}{(2-\mu)^2}\left[\frac{(1-\mu)^2}{8}l_2^2 + \frac{7}{8}l_3^2\right] + \frac{(n-1)^2(2-\mu)^4}{(n+1-\mu)(3-\mu)^2}\left[\frac{(2-\mu)^2}{8}l_2^2 + \frac{7}{8}l_3^2\right],\]

\[N_2^- = \frac{\mu(1-\mu)^4}{(2-\mu)^2} + (n-1)(n+2)^2\delta_2.\] (6.18)

Here the radiation intensity grows as \(n^4\) whenever \(\mu \neq 0\), and the radiation polarization depends essentially on \(\mu\) and \(n\).

Of special note is the loss of spin \(\zeta = -1\) stability in transitions \(j = 2 \rightarrow j' = 1\). It follows from (6.18), for

\[\delta_3 < \mu < 1 - \delta_3, \quad \delta_3 = \frac{nH}{3H_0},\] (6.19)

that the spin \(\zeta = 1\) is more stable in the transitions under considerations. Let an initial state be of second \((j = 2)\) type and the condition (6.19) holds, then the radiation creates a two-phase system of final electron states. Final electron states of second type have in the main negative spin orientation and final electron states of first type have in the main positive spin orientation. Thus, the presence of the solenoid field with \(\mu \neq 0\) plays a role of a depolarization factor in the above mentioned self-polarization effect.

**C. Semiclassical approximation**

Consider here the radiation intensity in the semiclassical approximation. From the previous discussion, we know that such an approximation corresponds to the condition \(v/n \ll 1\). Similar to the spinless particle case, we can approximate the Laguerre functions by the Bessel ones to get the following expression for the radiation intensity
\[ W_j = W_0 \left( \frac{H}{H_0} \right)^2 (1 - \beta_j^2) \sum_{\nu, j'} \int_0^{\pi} Q_{jj'}^1 |F_{jj'}^\text{cl}|^2 \sin \theta d\theta. \]  

(6.20)

The quantities \( Q_{jj'}^1 \) are defined by Eq. (5.44) and \( F_{jj'}^\text{cl} \) have the form

\[ F_{jj'}^\text{cl} = \beta \delta \zeta \nu F_{jj'}^{(0)\text{cl}} + \delta \zeta \nu \frac{H}{H_0} \frac{1 - \beta^2}{2} \nu F_{jj'}^{(1)\text{cl}}, \quad F_{jj'}^{(0)\text{cl}} = l_2 I_{jj'}(x) + l_3 \cos \theta \frac{I_{jj'}(x)}{\beta \sin \theta}, \]

\[ F_{jj'}^{(1)\text{cl}} = (-\zeta)^j l_2 \cos \theta \left[ \frac{I_{jj'}(x)}{\beta \sin \theta} + \zeta I_{jj'}'(x) \right] - l_3 \left[ \frac{a I_{jj'}(x)}{\beta \sin \theta} + \zeta I_{jj'}'(x) \right], \]

\[ I_{11}(x) = I_{22}(x) = I_{12}(x) = J_\nu(\nu \beta \sin \theta), \quad I_{21}(x) = J_{-\nu}(\nu \beta \sin \theta), \]

\[ x = \nu \beta \sin \theta, \quad a = \cos^2 \theta + \sqrt{1 - \beta^2 \sin^2 \theta}. \]  

(6.21)

In the non-relativistic approximation \( \beta^2 = 2 \bar{n} H / H_0 \), then results of the previous Section follow from (6.20).

It follows from (6.21) that the solenoid field with \( \mu \neq 0 \) suppresses the electron self-polarization effect due to transitions \( j = 2 \rightarrow j' = 1 \). This suppression can be considered as a manifestation of AB effect in SR. For \( \mu = 0 \) such a manifestation disappears due to the property

\[ J_{-\nu}(x) = (-1)^\nu J_\nu(x), \]

which takes place whenever \( \nu \) are integer.

Similarly to the spinless particle case, the degeneracy of the radiation intensity with respect to the azimuthal quantum number is lifted here completely. That can be also considered as one of manifestations of AB effect in SR.

D. Electron transitions from zero energy levels with a change of state type

Consider here the radiation intensity caused by electron transitions from \( n = 0 \) energy level with a change of the type of state (namely \( n = 0, j = 1 \rightarrow j' = 2 \) transitions). In this case a superlow frequency (4.11) is emitted. One can get an exact expression for the quantity \( Q_{12} \),

\[ Q_{12} = \frac{q^{1-\mu} \exp(-q) \Phi(1, 2 - \mu; q)}{\Gamma(2 - \mu)}, \quad q = \mu \frac{1 - \sqrt{p}}{1 + \sqrt{p}}, \quad p = 1 - \alpha(1 - x^2), \]  

(6.22)
where $\Phi(\alpha, \gamma; x)$ is the confluent hypergeometric function. In the case under consideration, we can express $\Phi(\alpha, \gamma; x)$ via the incomplete $\Gamma$–function and get the following expression

$$
\Phi\left(1, 2 - \mu; x\right) = (1 - \mu)x^{\mu-1}e^x \int_0^x e^{-y} y^{-\mu} dy, \quad \mu < 1.
$$

(6.23)

For the transitions under consideration, the radiation intensity has the form

$$
W = W_0 \frac{H}{H_0} (\alpha \mu)^2 f(\mu) G(\alpha, \mu), \quad G(\alpha, \mu) = \int_0^1 \frac{\sqrt{p} + \sqrt{1 - \alpha}}{\sqrt{p}(1 + \sqrt{p})} e^{-2q\Phi(1, 2 - \mu; q)} F(x) dx,
$$

$$
F(x) = \delta_{\zeta, \zeta'} \frac{1 - \zeta}{2} \left[l_2^2 + l_3^2 \psi(x)\right] + \delta_{\zeta, -\zeta'} \frac{\alpha}{2} \left[\frac{1 + \zeta}{1 - \alpha} \right] \frac{1}{2} [l_2^2 \psi(x) + l_3^2],
$$

(6.24)

$$
\alpha = \frac{2\mu H}{H_0 + 2\mu H}, \quad \psi(x) = \frac{x^2 (1 + \sqrt{1 - \alpha})^2}{(\sqrt{p} + \sqrt{1 - \alpha})^2}
$$

( $f(\mu)$ was defined in (6.20)). The function $G(\alpha, \mu)$ depends on the magnetic field via the quantity $\alpha$,

$$
0 < \alpha < 1, \quad \alpha \approx 2\mu \frac{H}{H_0} \left(\frac{H}{H_0} \ll 1\right), \quad \lim_{H \to \infty} \alpha = 1.
$$

(6.25)

It is easy to see that

$$
\alpha = 1 - (m/k_0)^2 = \beta^2.
$$

(6.26)

However, for such quantum states ($n = 0$), we cannot use a classical interpretation for $\beta$.

It follows from (6.24) that transitions with and without spin-flip have almost (with the interchange of $\sigma$ and $\pi$ components) the same linear polarization of the radiation intensity.

Doing summation over photon polarization states, over final electron spin states, and averaging over initial spin states, we get total radiation intensity for a non-polarized electron

$$
\bar{W} = 2W_0 \frac{H}{H_0} \alpha \mu^2 f(\mu) \int_0^1 \frac{\sqrt{p} + \alpha - 1}{\sqrt{p}(1 + \sqrt{p})} e^{-2q\Phi(1, 2 - \mu; q)} dx.
$$

(6.27)

In the weak magnetic field approximation ($\alpha \ll 1$), we obtain from (6.24)

$$
W = \frac{1}{3} W_0 \frac{H}{H_0} (\alpha \mu)^2 f(\mu) \left(\delta_{\zeta, \zeta'} \frac{1 - \zeta}{2} S_0 + \delta_{\zeta, -\zeta'} \frac{\alpha}{4} \frac{1 + \zeta}{2} S_1\right).
$$

(6.28)
Finally consider the case of superstrong magnetic fields ($H \gg H_0$, $\alpha = 1$). Here $\psi(x) = 1$ and the radiation intensity has the form

$$W = \frac{1}{2} \bar{W}\left(l_2^2 + l_3^2\right)\left(\delta_{\zeta,\zeta'} \frac{1 - \zeta}{2} + \delta_{\zeta,-\zeta'} \frac{1 + \zeta}{2}\right),$$

$$\bar{W} = W_0 \frac{H}{H_0} \mu^2 f(\mu) J(\mu), \quad J(\mu) = \int_0^1 (1 + x)e^{-2\mu x} \Phi (1, 2 - \mu; \mu x) \, dx.$$ (6.29)

$J(\mu)$ is a monotonically decreasing function of $\mu$. In particular, $J(0) = 1, 5$; $J(1) = 2 - \frac{3}{e} \approx 0.896$. Thus, in the superstrong magnetic fields, transitions with and without spin-flip have equal probabilities, the radiation is completely depolarized, and the radiation intensity is linearly increasing function of the magnetic field.

VII. SUMMARY

We have obtained exact solutions of Klein-Gordon and Dirac equations in the magnetic-solenoid field. Employing these solutions, we succeeded to calculate various characteristics of one-photon radiation in such a field. Namely, peculiarities of the radiation related to the presence of the AB solenoid are considered by us as manifestations of AB effect in CR and SR. Below we list the most important results obtained.

1. It is demonstrated that all the peculiarities of the radiation related to the presence of AB solenoid depend on the mantissa $\mu$ of the solenoid flux only. For the fluxes with $\mu = 0$, these peculiarities disappear.

2. The energy spectrum of charge particles in the magnetic-solenoid field differs essentially from the one in pure magnetic field. In particular, the degeneracy with respect to the azimuthal quantum number is partially lifted. Each magnetic field energy level splits in two ones in the magnetic-solenoid field. In turn, this complicates the radiation spectrum. In particular, the degeneracy of the radiation intensity with respect to the azimuthal quantum number is lifted completely.

3. New lines in the radiation spectrum appear, they do not have an analog in the pure magnetic field case. These lines consist of two series of harmonics (the latter are not
multiple of the basic synchrotron frequency) and of two superlow frequency harmonics (their frequencies are less than the basic synchrotron frequency).

4. It is shown that the only one basic synchrotron harmonic and the new frequencies are irradiated along the magnetic field. We stress important peculiarities of the radiation along the magnetic field. The basic synchrotron harmonic has total circular polarization; the radiation intensity of superlow harmonics has maximum in the magnetic field direction; all the harmonics from the two above mentioned series have approximately equal radiation intensities. The latter property of the radiation is not typical for the conventional CR and SR. We believe that a considerable relative shift between new harmonics and the basic synchrotron one as well as the peculiarities of the angular distribution of the radiation intensity open up possibilities for experimental observation of AB effect in CR and SR.

5. It is discovered that the presence of the solenoid field can suppress the well-known in SR electron self-polarization effect.

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