ON THE ISOMORPHISM PROBLEM FOR $C^*$-ALGEBRAS OF NILPOTENT LIE GROUPS

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Abstract. We investigate to what extent a nilpotent Lie group is determined by its $C^*$-algebra. We prove that, within the class of exponential Lie groups, direct products of Heisenberg groups with abelian Lie groups are uniquely determined even by their unitary dual, while nilpotent Lie groups of dimension ≤ 5 are uniquely determined by the Morita equivalence class of their $C^*$-algebras. We also find that this last property is shared by the filiform Lie groups and the 6-dimensional free two-step nilpotent Lie group.

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1. INTRODUCTION

The main results of the present paper are related to one of the central problems in noncommutative harmonic analysis: To what extent a locally compact group is determined by its representation theory? If $G$ is a locally compact group, then important information on its representation theory is encoded by the natural topology of its unitary dual $\hat{G}$, that is, the set of all equivalence classes of unitary irreducible representations of $G$. When $G$ is type I, additional information is encoded by the group $C^*$-algebra of $G$, denoted by $C^*(G)$, whose space of primitive ideals is canonically homeomorphic to the dual space $C^*(\hat{G})$, which is further homeomorphic to $\hat{G}$.

The isomorphism problem referred to in the title of the present paper is to find conditions for $G$ to be uniquely determined by $C^*(G)$.

Locally compact groups are not in general determined by their $C^*$-algebras. For instance, for all compact Lie groups, their unitary dual spaces are countable discrete topological spaces, hence are homeomorphic to each other. As commutative $C^*$-algebras having homeomorphic spaces of primitive ideals are $*$-isomorphic, it then follows that the $C^*$-algebras of all compact abelian Lie groups of dimension ≥ 1 are mutually $*$-isomorphic, hence for this class of Lie groups even the dimension of a group cannot be read off its $C^*$-algebra. This phenomenon is not confined to compact groups, as several examples of non-isomorphic exponential Lie groups having isomorphic $C^*$-algebras were pointed out in [24] and [15].

However, examples of groups are known which are uniquely determined by their $C^*$-algebras, or even by their unitary dual space, within reasonably large classes of groups. Results of this type were recently obtained for discrete nilpotent groups (see for instance [25]). As regards the non-discrete groups, if $G$ is a connected simply connected abelian Lie group, then $G = (\mathcal{V}, +)$ for some finite-dimensional real vector
space $V$, and then $\hat{G}$ is homeomorphic to the dual vector space $V^\ast$. By Brouwer’s theorem on the invariance of domain, any two homeomorphic vector spaces have equal dimensions, and this shows that the unitary dual does distinguish the Lie groups from each other within the class of the connected simply connected abelian Lie groups. A weaker result of this type for Heisenberg groups was established in [16], which uses however some structures that cannot be encoded by the group $C^\ast$-algebras. Using a completely different method we establish this result on its natural level of generality in Theorem 1.2 below.

One of the main motivations of this paper is to show that the above simple observation carries over to several noncommutative Lie groups, with the notion of dimension of vector spaces replaced by notions of dimension that are suitable for the study of noncommutative Lie groups and their $C^\ast$-algebras. In particular we address the following question:

If $G_1$ and $G_2$ are nilpotent Lie groups whose $C^\ast$-algebras are isomorphic, are $G_1$ and $G_2$ necessarily isomorphic as Lie groups?

This open problem was also stated in [27]. In this paper we indicate the first examples of noncommutative nilpotent Lie groups that are determined by the Morita equivalence class of their group $C^\ast$-algebras. This type of information on representation theory is in-between the topology of the unitary dual and the $\ast$-isomorphism class of these $C^\ast$-algebras, and our results suggest that the natural framework of the above question might be offered by Morita equivalence of $C^\ast$-algebras. To emphasize this point it is convenient to make the following definition.

**Definition 1.1.** An exponential Lie group $G$ is called stably $C^\ast$-rigid if for every other exponential Lie group $H$ one has

$$G \simeq H \iff C^\ast(G), C^\ast(H) \text{ are Morita equivalent.}$$

**Main results and structure of this paper.** The first result of the paper proves that, within the class of exponential Lie groups, the groups that are direct product of a Heisenberg group with a vector space group are uniquely determined by their spectrum, which is a stronger property than stably $C^\ast$-rigidity.

**Theorem 1.2.** Let $G_1$ a nilpotent Lie group with Lie algebra $\mathfrak{g}_1$ and assume that $\dim[\mathfrak{g}_1, \mathfrak{g}_1] \leq 1$. Then for any $G_2$ exponential Lie group, $\hat{G}_1$ is homeomorphic to $\hat{G}_2$ if and only if $G_1$ is isomorphic to $G_2$.

We then analyze the nilpotent Lie groups of dimension $\leq 5$. In this case we need more than only the unitary dual to distinguish a group, namely we need to pinpoint a special open dense subset given by the spectrum of the largest bounded trace ideal in the $C^\ast$-algebra of the group. The result in this case is the following theorem.

**Theorem 1.3.** Any nilpotent Lie group of dimension $\leq 5$ is stably $C^\ast$-rigid.

Similar properties are also established for the filiform (threadlike) Lie groups (Theorem 6.3) and for the free two-step Lie group of dimension 6 (Proposition 6.6). The unitary dual of filiform Lie groups has been used to illustrate the intricacy of the group $C^\ast$-algebras in [2], [3], [28], and the references therein.

The proof of Theorem 1.2 is given in Section 4 and is based on detailed study of set of characters (Section 3). Theorem 1.3 is proved in the last subsection of
Section 5 and relies on some special properties of nilpotent Lie groups whose non-trivial coadjoint orbits have the same dimension. Finally, in Section 6 we study the filiform Lie groups and some other examples.

2. Preliminaries

2.1. General notation and terminology. We denote the Lie groups by upper case Roman letters and their corresponding Lie algebras by the corresponding lower case Gothic letters. By nilpotent Lie group we always mean a connected simply connected nilpotent Lie group. By exponential Lie group we mean a Lie group $G$ whose exponential map $\exp_G : \mathfrak{g} \to G$ is bijective.

For an integer $k \geq 1$, we denote by $a_k$ the commutative nilpotent Lie algebra of dimension $k$, which is nothing else than a real vector space of dimension $k$. Then $A_k$ is the corresponding nilpotent Lie group. Also, we denote by $H_{2k+1}$ the Heisenberg group of dimension $2k + 1$, and by $h_{2k+1}$ its Lie algebra.

For a Lie group $G$ with Lie algebra $\mathfrak{g}$, and Lie bracket $\{\cdot, \cdot\}$, we denote by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the corresponding duality. For any subset $s \subseteq \mathfrak{g}$, we set

$$s^\perp = \{ \xi \in \mathfrak{g}^* \mid \langle \xi, s \rangle = 0 \}.$$ 

We denote its corresponding coadjoint action by $\text{Ad}^*_G : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. The space of all coadjoint orbits is denoted by $\mathfrak{g}^*/G$, seen as a topological space with its natural quotient topology. The corresponding quotient map is denoted by $q : \mathfrak{g}^* \to \mathfrak{g}^*/G$, $\xi \mapsto O_\xi$, hence for every $\xi \in \mathfrak{g}^*$, $O_\xi = \text{Ad}^*_G(G)\xi$ is its corresponding coadjoint orbit. The coadjoint isotropy subalgebra at $\xi \in \mathfrak{g}^*$ is

$$\mathfrak{g}(\xi) = \{ X \in \mathfrak{g} \mid \langle \xi, [X, \mathfrak{g}] \rangle = 0 \}.$$ 

We recall that for any locally compact group $G$, $\widehat{G}$ stands for the set of unitary equivalence classes of unitary irreducible representations of $G$ with the natural topology defined in terms of the group $C^*$-algebra $C^*(G)$. Then, in the case of exponential Lie groups, we may tacitly identify

$$\mathfrak{g}^*/G \simeq \widehat{G} \simeq \widehat{C^*(G)},$$

via canonical homeomorphism (see [18], [26], and [22]).

Two separable $C^*$-algebras $A_1$ and $A_2$ are Morita equivalent if and only if they are stably isomorphic, in the sense that there is a $*$-isomorphism $A_1 \otimes \mathcal{K} \simeq A_2 \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators on a separable infinite-dimensional complex Hilbert space. See If two $C^*$-algebras $A_1$ and $A_2$ are Morita equivalent, then there is a homeomorphism $\widehat{A}_1 \simeq \widehat{A}_2$ (see [31]).

We use the notion of real rank $\text{RR}(A)$ for a $C^*$-algebra $A$ and results on the real rank for $C^*$-algebras of exponential Lie groups. See [12] and the references therein.

Finally, we denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. We also denote $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $T := \{ z \in \mathbb{C} \mid |z| = 1 \}$, and both these sets are usually regarded as 1-dimensional Lie groups with respect to the group operation given by multiplication.

2.2. Special $\mathbb{R}$-spaces. The notions introduced in Definitions 2.1 and 2.5 were suggested by the special topological properties of unitary dual spaces of nilpotent Lie groups established in [16].
**Definition 2.1.** A *special $\mathbb{R}$-space* is a topological space $X$ endowed with a continuous map $\mathbb{R} \times X \to X$, $(t,x) \mapsto t \cdot x$, called structural map, and with a distinguished point $x_0 \in X$ satisfying the following conditions:

1. For every $x \in X$ and $t \in \mathbb{R}$ one has $0 \cdot x = t \cdot x_0 = x_0$ and $1 \cdot x = x$.
2. For all $t, s \in \mathbb{R}$ and $x \in X$ one has $t \cdot (s \cdot x) = ts \cdot x$.
3. For every $x \in X \setminus \{x_0\}$ the map $\psi_x : \mathbb{R} \to X$, $t \mapsto t \cdot x$ is a homeomorphism onto its image.

An $\mathbb{R}$-subspace of the special $\mathbb{R}$-space $X$ is any subset $\Gamma \subseteq X$ such that $\mathbb{R}^\times \cdot \Gamma \subseteq \Gamma$. If this is the case, then $\Gamma \cup \{x_0\}$ is a special $\mathbb{R}$-space on its own.

If $Y$ is another special $\mathbb{R}$-space with its structural map $\mathbb{R} \times X \to X$, $(t,x) \mapsto t \cdot x$ and its distinguished point $y_0 \in Y$, then a map $\psi : X \to Y$ is called an isomorphism of special $\mathbb{R}$-spaces if $\psi$ is a homeomorphism and $\psi(t \cdot x) = t \cdot \psi(x)$ for all $t \in \mathbb{R}$ and $x \in X$.

In the notation above, a function $\varphi : X \to \mathbb{R}$ is called homogeneous if there exists $r \in [0,\infty)$ such that $\varphi(t \cdot x) = t^r \varphi(x)$ for all $t \in \mathbb{R}$ and $x \in X$.

**Remark 2.2.** If $X$ is a special $\mathbb{R}$-space, then one has:

(i) If $t \in \mathbb{R}^\times$ and $x \in X \setminus \{x_0\}$, then $t \cdot x = x_0$ if and only if $t = 0$.

(ii) If $\psi : X \to Y$ is an isomorphism of special $\mathbb{R}$-spaces then $\psi(x_0) = y_0$.

**Example 2.3.** If $X$ is a special $\mathbb{R}$-space and $\Gamma \subseteq X$ is an $\mathbb{R}$-subspace of $X$, then $X \setminus \Gamma$ is also an $\mathbb{R}$-subspace of $X$.

**Example 2.4.** If $X$ is a special $\mathbb{R}$-space and $X$ is homeomorphic to $\mathbb{R}$, then one can check that for every $x \in X \setminus \{x_0\}$ the map $\psi_x : \mathbb{R} \to X$, $t \mapsto t \cdot x$, is a homeomorphism, and moreover $\psi_x$ is an isomorphism of $\mathbb{R}$-spaces between $X$ and the $\mathbb{R}$-space $\mathbb{R}$ regarded as a 1-dimensional real vector space.

**Definition 2.5.** A topological space $X$ is a $\mathbb{R}$-space of finite length if it is second countable, is locally quasi-compact, has the property $T_1$, and there exists an increasing finite family of open subsets,

$$\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = X,$$

such that $V_j \setminus V_{j-1}$ is Hausdorff in its relative topology and dense in $X \setminus V_{j-1}$, for $j = 1,\ldots,n$, and satisfy the following additional properties:

1. $X$ has the structure of a special $\mathbb{R}$-space and $\Gamma_j := V_j \setminus V_{j-1} \subseteq X$ is an $\mathbb{R}$-subspace for $j = 1,\ldots,n$.
2. $\Gamma_n := X \setminus V_{n-1}$ is isomorphic as a special $\mathbb{R}$-space to a finite-dimensional vector space, whose origin is the distinguished point $x_0$ of $X$.
3. For $j = 1,\ldots,n-1$ the points of $\Gamma_{j+1}$ are closed and separated in $X \setminus V_j$.
4. For $j = 1,\ldots,n$, $\Gamma_j$ is isomorphic as a special $\mathbb{R}$-space to a cone $C_j$ in a finite-dimensional vector space. In addition, $C_1$ is assumed to be a semi-algebraic Zariski open set, and the dimension of the corresponding ambient vector space is called the index of $X$ and is denoted by $\text{ind } X$.

If $G$ is a nilpotent Lie group with Lie algebra $\mathfrak{g}$, then its unitary dual $\hat{G}$ is a $\mathbb{R}$-space of finite length by [10, Thm. 4.11], and we define $\text{ind } G = \text{ind } \mathfrak{g} := \text{ind } \hat{G}$.

More details on the above definition could be found in [13].

**Lemma 2.6.** If $G$ is a nilpotent Lie group and $r \in \mathbb{N}$, then the following conditions are equivalent:

1. $G$ is $\mathbb{R}$-space of finite length.
2. $\text{ind } G = r$.
3. $\text{ind } \mathfrak{g} = r$.
4. $\text{ind } \hat{G} = r$.

...
countable base of open neighborhoods of $x$ which is an open subset of $x$. We first assume

Proof. Use [16] Prop. 4.9].

Lemma 2.7. If $G_1$ and $G_2$ are nilpotent Lie groups such that $\hat{G}_1$ is homeomorphic to $\hat{G}_2$, then $\text{ind } G_1 = \text{ind } G_2$.

Proof. By Lemma 2.6 we obtain $\text{ind } G_1 = \text{ind } G_2$.

3. Topological characterization of the set of characters

For any locally compact group $G$ one defines

$$K(G) := \{ K \mid K \text{ closed subgroup } \subseteq G \}.$$  

It is well known that $K(G)$ is a compact topological space with its Fell topology, for which a basis consists of the sets

$$U(C, S) := \{ K \in K(G) \mid K \cap C = \emptyset; \ (\forall A \in S) \ K \cap A \neq \emptyset \} \quad (3.1)$$

for all compact sets $C \subseteq G$ and all finite sets $S$ of open subsets of $G$.

Lemma 3.1. Let $G$ be a locally compact group which is first countable. For every $(K_j)_j, K \in K(G)$ with $\lim_{j \to \infty} K_j = K$ in $K(G)$, then for every finite family $x^{(1)}, \ldots, x^{(m)} \in K$ there exist a subsequence $\{K_{j_i}\}_{i \geq 1}$ and $x^{(1)}_i, \ldots, x^{(m)}_i \in K_{j_i}$ for every $i \geq 1$ with $\lim_{i \to \infty} x^{(s)}_i = x^{(s)}$ in $G$ for $s = 1, \ldots, m$.

Proof. We first assume $m = 1$ and denote $x := x^{(1)}$. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a countable base of open neighborhoods of $x$ in $G$. For every $i \geq 1$ one has

$$U(\emptyset, \{A_i\}) := \{ H \in K(G) \mid H \cap A_i \neq \emptyset \}$$

which is an open subset of $K(G)$, cf. (3.1). Since $x \in K \cap A_i$, one has $K \in U(\emptyset, \{A_i\})$, hence there exists $j_i \geq 1$ with $K_{j_i} \in U(\emptyset, \{A_i\})$, that is, $K_{j_i} \cap A_i \neq \emptyset$, for every $j \geq j_i$, by the hypothesis $\lim_{j \to \infty} K_j = K$ in $K(G)$. We may assume $j_1 < j_2 < \cdots$. Selecting any $x_i \in K_{j_i} \cap A_i$ and using the fact that $A_1 \supseteq A_2 \supseteq \cdots$ is a base of neighborhoods of $x$, one can check that $\lim_{i \to \infty} x_i = x$ in $G$.

The proof in the general case is done by induction. If $m \geq 2$ and one has a subsequence $\{K_{j_i}\}_{i \geq 1}$ and $x^{(1)}_i, \ldots, x^{(m-1)}_i \in K_{j_i}$ for every $i \geq 1$ with $\lim_{i \to \infty} x^{(s)}_i = x^{(s)}$ in $G$ for $s = 1, \ldots, m-1$, then one defines $x := x^{(m)}$ and one can select by the above method a suitable subsequence $\{K_{j_r}\}_{r \geq 1}$ of $\{K_{j_i}\}_{i \geq 1}$ for which there exists $x^{(m)}_r \in K_{j_r}$ for all $r \geq 1$ with $\lim_{r \to \infty} x^{(m)}_r = x^{(m)}$ in $G$. This completes the induction step and the proof.
Remark 3.2. The proof of Lemma 3.1 involves neither the group structures of $G$ and $K_j$, nor the fact that $G$ is locally compact. Therefore its reasoning leads to a more general result: Let $X$ be a topological space with its corresponding space $X(X)$ of all closed subsets of $X$ endowed with its Fell topology. If $X$ is first countable and $\lim_{j \to \infty} K_j = K$ in $X(X)$, then for every finite family $x^{(1)}, \ldots, x^{(m)} \in K$ there exist a subsequence $\{K_{j_i}\}_{i \ge 1}$ and $x_i^{(1)}, \ldots, x_i^{(m)} \in K_{j_i}$ for every $i \ge 1$ with $\lim_{i \to \infty} x_i^{(s)} = x^{(s)}$ in $X$ for $s = 1, \ldots, m$.

The following lemma uses the weak containment of subgroup representations in the sense of [25].

Lemma 3.3. Let $G$ be any locally compact group which is first countable, and assume that $\lim_{j \to \infty} K_j = K$ in $K(G)$. Also let $\varphi \in C(G)$ whose restriction $\varphi|K_j$ is a character of $K_j$ for every $j \ge 1$. Then the restriction $\varphi|K$ is a character of $K$ and the subgroup representation $(K_\varphi|K)$ is weakly contained in the family of subgroup representations $\{(K_j, \varphi|K_j)\}_{j \ge 1}$.

Proof. To check that $\varphi|K$ is a character of $K$ we must prove that $\varphi(K) \subseteq \mathbb{T}$ and $\varphi|K : K \to \mathbb{T}$ is a group morphism, since $\varphi|K$ is continuous by the hypothesis $\varphi \in C(G)$. To this end let $x, y \in K$ arbitrary. It follows by Lemma 3.1 that, after replacing $\{K_j\}_{j \ge 1}$ by a suitable subsequence, we may assume that there exist $x_j, y_j \in K_j$ for every $j \ge 1$ with $\lim_{j \to \infty} x_j = x$ and $\lim_{j \to \infty} y_j = y$. Since $\varphi \in C(G)$, it then follows that $\varphi(x) = \lim_{j \to \infty} \varphi(x_j)$ and $\varphi(y) = \lim_{j \to \infty} \varphi(y_j)$. Now, using the hypothesis that $\varphi|K_j$ is a character of $K_j$ for every $j \ge 1$, it is straightforward to check that $\varphi|K$ is a character of $K$.

The weak containment assertion follows by [25, Lemma 3.2 and Thm. 3.1]. □

Notation 3.4. For every nilpotent Lie group $G$ we denote by $Q(G)$ the set of all closed connected subsets $S \subseteq \mathfrak{g}^*/G$ for which the relative topology of $S$ is Hausdorff, and endow $Q(G)$ with the partial ordering given by inclusion.

The set of characters $[\mathfrak{g}, \mathfrak{g}]^\perp$, regarded as a subset of $\mathfrak{g}^*/G$, is closed, thus $[\mathfrak{g}, \mathfrak{g}]^\perp \in Q(G)$.

For every even integer $d \ge 0$ we denote

$$(\mathfrak{g}^*/G)_d := \{O \in \mathfrak{g}^*/G \mid \dim O = d\}.$$ 

In particular $(\mathfrak{g}^*/G)_0 = [\mathfrak{g}, \mathfrak{g}]^\perp$.

We also recall that any nilpotent Lie group $G$ has the distinguished coadjoint orbit $\mathcal{O}_0^G := \{0\}$.

Lemma 3.5. Let $G$ be a connected simply connected nilpotent Lie group with its corresponding Lie algebra $\mathfrak{g}$. If $q: \mathfrak{g}^* \to \mathfrak{g}^*/G$, $\xi \mapsto O_\xi$, is the quotient map onto the set of coadjoint orbits, then the following assertions hold:

(i) The complement $\tilde{G} \setminus [\mathfrak{g}, \mathfrak{g}]^\perp$ of the set of characters is either empty (if $G$ is commutative) or dense in $\tilde{G}$ (if $G$ is noncommutative).

(ii) For any dense open set $D \subseteq \mathfrak{g}^*/G$, the set $q^{-1}(D) \subseteq \mathfrak{g}^*$ is also open and dense.

Proof. Straightforward: Use the fact that $q$ is an open continuous map. See [13, Lemma 4.5] for more detail. □
The following result is a version of [10] Lemma 6.8(2) for coadjoint orbits that are not necessarily flat.

**Lemma 3.6.** Let $G$ be a nilpotent Lie group. Let $S \subseteq g^*/G$ be a closed subset such that the relative topology of $S$ is Hausdorff. Then the set $S \cap [g, g]^\perp$ is both closed and open in $S$.

**Proof.** It suffices to prove the set $S \cap [g, g]^\perp$ is simultaneously closed and open with respect to the relative topology of $S$. The set $[g, g]^\perp$ is a closed subset of $g^*/G$ therefore $S \cap [g, g]^\perp$ is a relatively closed subset of $S$. Thus it remains to prove that $S \setminus [g, g]^\perp$ is a relatively closed subset of $S$.

Assume that $S \setminus [g, g]^\perp$ is not a relatively closed subset of $S$, hence there exist a sequence $\{\xi_j\}_{j \in \mathbb{N}}$ in $g^*$ and $\eta \in g^*$ with $\lim_{j \to \infty} \xi_j = \eta$ and $\mathcal{O}_{\xi_j} \subseteq S \setminus [g, g]^\perp$ for every $j \in \mathbb{N}$ while $\mathcal{O}_\eta = \{\eta\} \subseteq S \cap [g, g]^\perp$.

Selecting a suitable subsequence, we may assume that there exists an integer $d > 0$ with $\dim \mathcal{O}_{\xi_j} = d$ for every $j \in \mathbb{N}$. Selecting again a suitable subsequence, using that the Grassmann manifold $\text{Gr}(g)$ is compact, we may assume that for every $j \in \mathbb{N}$ there exists a polarization $p_j \subseteq g$ at $\xi_j \in g^*$ such that the limit $m := \lim_{j \to \infty} p_j$ exists in $\text{Gr}(g)$. The subspace $m$ is a subalgebra of $g$ subordinated to $\eta$. If we denote by $P_j$ and $M$ the corresponding closed subgroups of $G$ with Lie algebras $p_j$ and $m$ respectively, we have that $\lim_{j \to \infty} P_j = M$ in $\mathcal{K}(G)$. (See [14].) It follows by Lemma 3.3 and [25] Thm. 4.3 that the set $L$ of limit points of the sequence $\{\mathcal{O}_{\xi_j}\}_{j \in \mathbb{N}}$ contains the coadjoint orbits of all irreducible representations of $G$ that occur in the disintegration of the representation $\text{Ind}_M^G(e^{i\eta})$.

By [19] Thm. 2 and p. 556 we thus get that $L \supseteq L_0 := \{\eta + m^+\}$ and $\mathcal{O}_\eta \subseteq L_0$. Since $\dim \mathcal{O}_\eta = 0$, the affine subspace $\eta + m^+$ is not contained in the coadjoint orbit of $\eta$, hence $L_0$ is a connected set that contains more than one point. Thus $\eta$ is not an isolated point in $L$. On the other hand, since $S$ is a closed subset of $\hat{G}$, it follows that $L \subseteq S$. We thus obtained a contradiction with the assumption that the relative topology of $S$ is Hausdorff, and this completes the proof.

**Proposition 3.7.** For every nilpotent Lie group $G$, the set of characters $[g, g]^\perp$ is a maximal element of $Q(G)$.

**Proof.** If $S \subseteq Q(G)$ and $[g, g]^\perp \subseteq S$, then $[g, g]^\perp$ is a closed-open subset of $S$ by Lemma 3.6 since $[g, g]^\perp = (g^*/G)_d$ for $d = 0$. Using the fact that $S$ is connected and $[g, g]^\perp \neq \emptyset$ we then obtain $[g, g]^\perp = S$. Hence $[g, g]^\perp$ is a maximal element of $Q(G)$.

**Remark 3.8.** The set $Q(G)$ is not inductively ordered, as could be seen for instance in the case when $G$ is a Heisenberg group. Thus not every element of $Q(G)$ is necessarily included in a maximal element.

For the nilpotent Lie groups for which all coadjoint orbits are flat, the result of Proposition 3.7 can be improved as follows.

**Proposition 3.9.** Let $G$ be a nilpotent Lie group for which all coadjoint orbits are flat. Then $[g, g]^\perp$ is the unique maximal element of $Q(G)$.

**Proof.** It follows by Proposition 3.7 that $[g, g]^\perp$ is a maximal element of $Q(G)$. Let $S$ be an arbitrary maximal element of $Q(G)$. We prove that $S \subseteq [g, g]^\perp$ and then, the maximality of $S$ implies that $S = [g, g]^\perp$. 

Since $S$ is connected, it follows by [16] Lemma 6.8(2)] that there exists an even integer $d \geq 0$ such that $S$ is contained in the set $(\mathfrak{g}^*/G)_d$ of all $d$-dimensional coadjoint orbits. We may assume $S \neq \emptyset$ and then we may select $\xi \in \mathfrak{g}^*$ with $O_\xi \subseteq S$. Let us define the continuous path $\gamma : \mathbb{R} \to \mathfrak{g}^*/G$, $\gamma(t) := O_{t\xi}$. One has $\gamma(\mathbb{R}^+) \subseteq (\mathfrak{g}^*/G)_d$.

Since $S$ is closed, it follows that $A := \{ t \in (0, \infty) \mid \gamma(t) \in S \}$ is a closed subset of $(0, \infty)$. For $t_0 := \inf A$ one has $t_0 \leq 1$ since $\gamma(1) = O_\xi \in S$.

If $t_0 > 0$ then we define $S_0 := \gamma([t_0/2, 1])$, which is a compact subset of $(\mathfrak{g}^*/G)_d$, since the relative topology of $(\mathfrak{g}^*/G)_d$ is Hausdorff by [16] Lemma 6.8(1)]. In particular $S_0$ is a closed subset of $\mathfrak{g}^*/G$, and it is clear that $S_0$ is connected. Moreover, $\gamma(1) \in S_0 \cap S$, hence $S_0 \cup S$ is a closed connected subset of $\mathfrak{g}^*/G$.

Since $S_0 \cup S \subseteq (\mathfrak{g}^*/G)_d$, the relative topology of $S_0 \cup S$ is Hausdorff, and thus $S_0 \cup S \subseteq Q(S)$. On the other hand, by the definition of $t_0$ one has $S_0 \not\subseteq S$, hence $S \not\subseteq S_0 \cup S$, which is a contradiction with the hypothesis that $S$ is a maximal element of $Q(G)$.

Consequently $t_0 = 0$, and it follows that there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $(0, \infty)$ with $\lim_{n \to \infty} t_n = 0$ and $O_{t_n\xi} = \gamma(t_n) \in S$ for every $n \in \mathbb{N}$. Since $\lim_{n \to \infty} t_n = 0$, the coadjoint orbit $O_0 = \{0\}$ is a limit point of the sequence $\{O_{t_n\xi}\}_{n \in \mathbb{N}}$. The set $S$ is a closed subset of $\mathfrak{g}^*/G$, and thus $O_0 \in S$. We noted above that $S \subseteq (\mathfrak{g}^*/G)_d$, hence $d = 0$, that is, $S \subseteq [\mathfrak{g}, \mathfrak{g}]^\perp$. This concludes the proof.

**Corollary 3.10.** Let $G_1$ be nilpotent Lie group and assume that all its coadjoint orbits are flat. If $G_2$ is a nilpotent Lie group such that there is a homeomorphism $\psi : \tilde{G}_1 \to \tilde{G}_2$, then $\psi([g_1, g_1]^\perp) = [g_2, g_2]^\perp$ and $\dim[g_1, g_1]^\perp = \dim[g_2, g_2]^\perp$.

**Proof.** The mapping $Q(G_1) \to Q(G_2)$, $S \mapsto \psi(S)$, is an isomorphism of partially ordered sets, since $\psi : \tilde{G}_1 \to \tilde{G}_2$ is a homeomorphism. Hence $S \subseteq Q(G_1)$ is a maximal element if and only if $\psi(S)$ is a maximal element of $Q(G_2)$.

Since all the coadjoint orbits of $G_1$ are flat, it follows by Proposition 3.9 that the set $Q(G_1)$ has exactly one maximal element, namely $[g_1, g_1]^\perp$. Therefore the set $Q(G_2)$ in turn has exactly one maximal element, namely $\psi([g_1, g_1]^\perp)$. On the other hand, we know from Proposition 3.7 that $[g_2, g_2]^\perp$ is a maximal element of $Q(G_2)$, hence $\psi([g_1, g_1]^\perp) = [g_2, g_2]^\perp$. Along with the fact that $\psi$ is a homeomorphism, this further implies that $\psi([g_1, g_1]^\perp) : [g_1, g_1]^\perp \to [g_2, g_2]^\perp$ is a homeomorphism. Moreover, $[g_1, g_1]^\perp$ and $[g_2, g_2]^\perp$ are vector spaces, hence it follows by Brouwer’s theorem on invariance of domain that $\dim[g_1, g_1]^\perp = \dim[g_2, g_2]^\perp$. This concludes the proof.

For the case of nilpotent Lie groups with no assumption on the coadjoint orbits, the above results have the following versions.

**Proposition 3.11.** Let $G$ be a nilpotent Lie group and define

$$Q_0(G) := \{ S \in Q(G) \mid O_0^G \subseteq S \}.$$ 

Then $[\mathfrak{g}, \mathfrak{g}]^\perp$ is the greatest element of the partially ordered set $Q_0(G)$, in the sense that $S \subseteq [\mathfrak{g}, \mathfrak{g}]^\perp$ for every $S \in Q_0(G)$.

**Proof.** Let $S \in Q_0(G)$ arbitrary. Thus $O_0^G \subseteq S$, hence $S \cap [\mathfrak{g}, \mathfrak{g}]^\perp \neq \emptyset$, and moreover $S \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ is a relatively closed and open subset of $S$, by Proposition 3.6. Since $S$ is connected, we obtain $S \cap [\mathfrak{g}, \mathfrak{g}]^\perp = S$, that is, $S \subseteq [\mathfrak{g}, \mathfrak{g}]^\perp$, and we are done.
Corollary 3.12. Let $G_1$ and $G_2$ be nilpotent Lie groups, such that there is a homeomorphism $\psi: \hat{G}_1 \to \hat{G}_2$ with $\psi(\mathcal{O}^G_0) = \mathcal{O}^G_0$. Then $\psi([g_1, g_1]) = [g_2, g_2]$ and $\dim [g_1, g_1] = \dim [g_2, g_2]$.

Proof. The corollary follows by using the method of proof of Corollary 3.10 based on Proposition 3.11. □

Corollary 3.13. Let $G_1$ and $G_2$ be nilpotent Lie groups. If $C^*(G_1)$ is Morita equivalent to $C^*(G_2)$, then $\dim [g_1, g_1] = \dim [g_2, g_2]$.

Proof. Since $C^*(G_1)$ and $C^*(G_2)$ are Morita equivalent and are type I $C^*$-algebras, it follows by [31, Cor. 3.33] that any $C^*(G_1)$-Morita equivalence bimodule defines a Rieffel homeomorphism $\psi: \hat{G}_1 \to \hat{G}_2$. It follows by the construction of the Rieffel induction (see [31, Prop. 2.66]) that it takes the 0-dimensional representation of $C^*(G_1)$ to the 0-dimensional representation of $C^*(G_2)$, hence $\psi(\mathcal{O}^G_0) = \mathcal{O}^G_0$. Now the assertion follows by Corollary 3.12. □

It follows by Corollary 3.13 above and [12, Thm. 3.5] that real rank of $C^*$-algebras of nilpotent Lie groups is preserved by Morita equivalence. As it is well known, this is not the case for arbitrary $C^*$-algebras (see for instance [12, Lemma 3.1 and Rem. 2.2]).

4. Proof of Theorem 1.2

Lemma 4.1. Let $G$ be an exponential solvable Lie group. Then the following assertions are equivalent:

(i) $G$ is a liminary group.
(ii) Every coadjoint orbit of $G$ is a closed subset of $\mathfrak{g}^*$.
(iii) $G$ is a nilpotent Lie group.

Proof. The group $G$ is exponential, hence it is a connected, simply connected, solvable Lie group of type I (see [7, Sect. 0, Rem. 3]). Then Glimm’s characterization of separable $C^*$-algebras of type I (see [22, §9.1]) implies that $G$ is postliminary. By [8, Ch. V, Thm. 1–2] we get that the first assertion is equivalent with the fact that $G$ is of type R, that is, for every $x \in \mathfrak{g}$, all the eigenvalues of the operator $\text{ad}_x: \mathfrak{g} \to \mathfrak{g}$ are purely imaginary.

On the other hand since $G$ is an exponential Lie group, it follows that for every $x \in \mathfrak{g}$, the operator $\text{ad}_x: \mathfrak{g} \to \mathfrak{g}$ has no nonzero purely imaginary eigenvalues (see [26, Prop. 5.2.13, Thm. 5.2.16]), hence the first and third assertion are equivalent. Finally, the equivalence between the first and second assertion follows from [26, Thm. 5.3.31] and [22, Ex. 9.5.3]. □

The next lemma treats the case of abelian Lie groups.

Lemma 4.2. Let $G_1$ be an abelian Lie group. If $G_2$ is an exponential Lie group with $\hat{G}_1$ homeomorphic to $\hat{G}_2$, than the Lie groups $G_1$ and $G_2$ are isomorphic.

Proof. Since $\hat{G}_1$ homeomorphic to $\hat{G}_2$, it follows that $\hat{G}_2$ must be Hausdorff. Therefore, by [9, Thm. 1], the group $G_2$ must be a compact extension of vector group. Since $G_2$ is assumed to be exponential, it cannot have non-trivial compact subgroups, hence it is a vector group.
Thus, the duals of $G_1$ and $G_2$ are vector spaces of dimensions $\dim G_1$ and $\dim G_2$, respectively. By Brouwer’s theorem on the invariance of domain we then obtain that $\dim G_1 = \dim G_2$, hence $G_1$ and $G_2$ are isomorphic. □

Proof of Theorem 1.2. We may assume by Lemma 1.2 that $G_1$ is not abelian, therefore $\dim [g_1, g_1] = 1$.

Assume that $\psi: \hat{G}_1 \to \hat{G}_2$ is a homeomorphism. Since $G_1$ is a nilpotent Lie group, the singleton subsets of $\hat{G}_1$ are closed, hence the singleton subsets of $\hat{G}_2$ are also closed. The group $G_2$ is an exponential Lie group, thus Lemma 1.1 implies now that $G_2$ is a nilpotent Lie group.

Since $\dim [g_1, g_1] = 1$ it follows that there exist some integers $d_1 \geq 1$ and $k_1 \geq 0$ with $G_1 = A_{d_1} \times H_{2d_1+1}$. Then all the coadjoint orbits of $G_1$ are flat and we then obtain by Corollary 3.10 that $\psi([g_1, g_1]) = [g_2, g_2]$. In particular, for $\Gamma_1^{(j)} := \hat{G}_j \setminus [g_j, g_j]$ we also obtain $\psi(\Gamma_1^{(1)}) = \Gamma_1^{(2)}$. Moreover, $\hat{G}_1 = \hat{A}_{k_1} \times \hat{H}_{2d_1+1}$, and $\Gamma_1^{(1)} = A_{k_1} \times \mathbb{R}^\times$ (up to a homeomorphism), which is a disconnected topological space. Therefore the set $\Gamma_1^{(2)}$ must be disconnected as well.

Assume now that $\dim [g_2, g_2] \geq 2$. Then $g_2^* \setminus [g_2, g_2]$ is connected, hence its image through the quotient map $q: g_2^* \to g_2^*/G_2$ is connected. That is, the set $\Gamma_1^{(2)}$ is connected, which is a contradiction. Consequently $\dim [g_2, g_2] \leq 1$, and then there exist some integers $k_2, d_2 \geq 0$ with $G_2 = A_{k_2} \times H_{2d_2+1}$ (up to a group isomorphism). One actually has $d_2 \geq 1$ since $\Gamma_1^{(2)} = \psi(\Gamma_1^{(1)}) \neq \emptyset$. One then has $\Gamma_1^{(2)} = \hat{A}_{k_2} \times \mathbb{R}^\times$ and this set is homeomorphic to $A_{k_2} \times \mathbb{R}^\times$. Thus we obtain $k_1 = k_2$ by the theorem on invariance of domain. Since $\dim [g_j, g_j] = 2d_j + k_j$ we then also obtain $d_1 = d_2$, and this completes the proof. □

5. Application to nilpotent Lie groups of dimension $\leq 5$

5.1. A weaker version of Theorem 1.3. In this subsection we prove that two nilpotent Lie groups of dimension $\leq 5$ are isomorphic if and only if their $C^*$-algebras are Morita equivalent, that is, a weaker version of Theorem 1.3.

We first recall the classification of nilpotent Lie algebras of dimension $\leq 5$ over $\mathbb{R}$, and to this end we need to introduce some notation. Unless otherwise mentioned, $X_1, \ldots, X_n$ is a basis of a nilpotent Lie algebra of dimension $n \leq 5$, and we give only the brackets $[X_j, X_k]$ that are different from zero.

We consider the following nilpotent Lie algebras:

(a) Case $n = 3$:
   - $\mathfrak{n}_3$: $[X_3, X_2] = X_1$

(b) Case $n = 4$:
   - $\mathfrak{n}_4$: $[X_4, X_3] = X_2$, $[X_4, X_2] = X_1$

(c) Case $n = 5$:
   - $\mathfrak{n}_{5,1}$: $[X_5, X_4] = X_1$, $[X_5, X_2] = X_1$
   - $\mathfrak{n}_{5,2}$: $[X_5, X_4] = X_2$, $[X_5, X_3] = X_1$
   - $\mathfrak{n}_{5,3}$: $[X_5, X_4] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$
   - $\mathfrak{n}_{5,4}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_4, X_3] = X_1$
   - $\mathfrak{n}_{5,5}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_5, X_2] = X_1$
   - $\mathfrak{n}_{5,6}$: $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$

With the above notation we can state the classification of nilpotent Lie algebras of dimension $\leq 5$. 
Proposition 5.1. Every nilpotent Lie algebra of dimension \( \leq 5 \) over \( \mathbb{R} \) is isomorphic to exactly one of the following Lie algebras:

- **Dimension 1**: \( a_1 \)
- **Dimension 2**: \( a_2 \)
- **Dimension 3**: \( a_3, n_3 \)
- **Dimension 4**: \( a_4, a_1 \times n_3, n_4 \)
- **Dimension 5**: \( a_5, a_2 \times n_3, a_1 \times n_4, n_{5,1}, n_{5,2}, n_{5,3}, n_{5,4}, n_{5,5}, n_{5,6} \)

Proof. See [20, Prop. 1]. \( \Box \)

Proposition 5.2. If \( G_1 \) and \( G_2 \) are nilpotent Lie groups with \( \max \{ \dim G_1, \dim G_2 \} \leq 5 \), then \( G_1 \) is isomorphic to \( G_2 \) if and only if \( C^*(G_1) \) is Morita equivalent to \( C^*(G_2) \).

Proof. Let \( G_1 \) and \( G_2 \) be connected simply connected nilpotent Lie groups with \( \max \{ \dim G_1, \dim G_2 \} \leq 5 \) such that \( C^*(G_1) \) is Morita equivalent to \( C^*(G_2) \).

Then \( g_1 \) and \( g_2 \) belong to the list of Lie algebras provided by Proposition 5.1. Using the detailed information that is available on the coarse partition of the spaces of coadjoint orbits (see for instance [30]), one easily obtains the following values for the real rank and the index of the connected simply connected nilpotent Lie groups corresponding to the Lie algebras in the above list. For any nilpotent Lie algebra \( g \) and \( G \) the corresponding connected simply connected nilpotent Lie group we denote here \( RR(C^*(G)) := RR(g) \). We recall that \( RR(g) = \dim(g/[g,g]) \).
(See [13].)

\[
\begin{array}{|c|c|c|}
\hline
\mathbf{g} & \mathbf{RR(g)} & \mathbf{ind(G)} \\
\hline
a_k \ (1 \leq k \leq 5) & k & k \\
n_3 (= h_3) & 2 & 1 \\
a_1 \times n_3 (= a_1 \times h_3) & 3 & 2 \\
n_4 (= f_4) & 2 & 2 \\
a_2 \times n_3 (= a_2 \times h_3) & 4 & 3 \\
a_1 \times n_4 (= a_1 \times f_4) & 3 & 3 \\
n_{5,1} (= h_5) & 4 & 1 \\
n_{5,2} & 3 & 3 \\
n_{5,3} & 3 & 1 \\
n_{5,4} & 2 & 3 \\
n_{5,5} (= f_5) & 2 & 3 \\
n_{5,6} & 2 & 1 \\
\hline
\end{array}
\]

If either \( G_1 \) or \( G_2 \) is a direct product of an abelian group with a Heisenberg group, then \( G_1 \) is isomorphic to \( G_2 \) by Theorem 1.2.

Because of these observations, it suffices to compare to each other the \( C^* \)-algebras of any two Lie groups from the above list from which one has removed the Heisenberg groups and the abelian groups. Comparing the values of \( RR \) and \( ind \) from the above table, one easily checks that only in dimension 5 one encounters pairs of distinct Lie algebras with groups having the same real rank and index, namely

1. \( a_1 \times n_4 (= a_1 \times f_4) \) and \( n_{5,2} \),
2. \( n_{5,4} \) and \( n_{5,5} (= f_5) \).
So, due to Lemma 2.7 and Corollary 3.13 it remains to compare to each other the
$C^*$-algebras of the above pairs, which can be done as follows:

(1) $C^*(A_1 \times N_4)(= C^*(A_1 \times F_4))$ is not Morita equivalent to $C^*(N_{5,2})$ since there
is no homeomorphism between the unitary duals of the groups $G_1 := A_1 \times F_4$ and $G_2 := N_{5,2}$ which should map to each other the spaces of
characters of these groups.

In fact, by [21] Prop. 4, the group $G_2$ has the property that the complement
of the characters $\hat{G}_2 \setminus [g_2, g_2]^\perp$ is Hausdorff in its relative topology. On
the other hand, the relative topology of the complement of the characters
$\hat{G}_1 \setminus [g_1, g_1]^\perp$ does not have the Hausdorff property as a direct consequence
of [21] Prop. 2.

(2) $C^*(N_{5,4})$ is not Morita equivalent to $C^*(N_{5,5})(= C^*(F_5))$ since there is no
homeomorphism between the unitary duals of the groups $N_{5,4}$ and $N_{5,5}(= F_5)$.

In fact, as noted in [23] §5, there exists a properly convergent sequence
in $\hat{F}_3$ that has exactly three limit points.

On the other hand [24] Thm. 5.6 (see also [29] Thm. 8.2) shows that
the set of limit points of any properly convergent sequence in $\hat{N}_{5,4}$ contains
either one point, or two points, or infinitely many points, as soon as we
proved the following assertion:

Assume that $\lim_{n \to \infty} s_n = 0$ in $\mathbb{R}$, $\lim_{n \to \infty} w_n = 0$ in $(0, \infty)$, and \{\theta_n\}_{n \geq 1} is a
sequence of real numbers. Then the set

$$\left\{ (u_1, u_2) \in \mathbb{R}^2 \mid \liminf_{n \to \infty} \left( \frac{s_n}{w_n} - u_1 \sin \theta_n - u_2 \cos \theta_n \right) \leq 0 \right\}$$

(5.1)

is either empty or infinite.

To prove this, denote by $E \subseteq \mathbb{R}^2$ the set in (5.1), and assume that $E \neq \emptyset$, so that we can select a point $(u_1, u_2) \in E$.

Restricting to a suitable subsequence, we may assume that all the real
numbers in the sequence \{sin $\theta_n\}_{n \geq 1} have the same sign $\varepsilon_1 \in \{\pm 1\}$, and
likewise all the real numbers in the sequence \{cos $\theta_n\}_{n \geq 1} have the same
sign $\varepsilon_2 \in \{\pm 1\}$.

Let $u_j' \in \mathbb{R}$ arbitrary with $0 \leq (u_j' - u_j) \varepsilon_j$ for $j = 1, 2$. Then for every
$n \geq 1$ one has $0 \leq (u_j' - u_j) \varepsilon_j \leq |u_j' - u_j| = (u_j' - u_j) \sin \theta_n$ and similarly
$0 \leq (u_2' - u_2) \cos \theta_n$. We then obtain

$$\forall n \geq 1 \quad \frac{s_n}{w_n} - u_1' \sin \theta_n - u_2' \cos \theta_n \leq \frac{s_n}{w_n} - u_1 \sin \theta_n - u_2 \cos \theta_n$$

and, since $(u_1, u_2) \in E$, it then follows that $(u_1', u_2') \in E$. As the set of
points $(u_1', u_2') \in \mathbb{R}^2$ with $0 \leq (u_j' - u_j) \varepsilon_j$ for $j = 1, 2$ is infinite, $E$ is infinite.

This completes the proof. □

5.2. Nilpotent Lie groups of whose all nontrivial coadjoint orbits have the
same dimension. In order to prepare for the proof of Theorem 1.3 we study here
a class of more tractable nilpotent Lie groups, namely those whose all nontrivial
coadjoint orbits have the same, maximal, dimension. We make first a definition.

Definition 5.3. A nilpotent Lie algebra $g$ is called of class $T$ if for all $\xi \in g^* \setminus [g, g]^\perp$ one has $\dim g(\xi) = \text{ind} g$. (See also [15] Rem. 4.8.) If this is the case, then the
nilpotent Lie group $G$ is called of class $T$. 
Lemma 5.4. Let \( \mathfrak{g} \) be a finite-dimensional real Lie algebra and let \( \mathcal{J}(G) \) be the set of its ideals, that is,
\[
\mathcal{J}(\mathfrak{g}) := \{ \mathfrak{h} \in \text{Gr}(\mathfrak{g}) \mid [\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h} \}.
\]
Then \( \mathcal{J}(\mathfrak{g}) \) is a closed subset of \( \text{Gr}(\mathfrak{g}) \).

Proof. We prove that for every sequence \( \{ \mathfrak{h}_n \}_{n \in \mathbb{N}} \) in \( \mathcal{J}(\mathfrak{g}) \) for which there exists \( \mathfrak{h} = \lim_{n \to \infty} \mathfrak{h}_n \) in \( \text{Gr}(\mathfrak{g}) \), one has \( \mathfrak{h} \in \mathcal{J}(\mathfrak{g}) \). To this end let \( \mathfrak{x} \in \mathfrak{g} \) and \( \mathfrak{y} \in \mathfrak{h} \) arbitrary. Then for every \( n \in \mathbb{N} \) there exists \( \mathfrak{h}_n \in \mathfrak{h}_n \) with \( \mathfrak{x} = \lim_{n \to \infty} \mathfrak{h}_n \), hence \( [\mathfrak{x}, \mathfrak{y}] = \lim_{n \to \infty} [\mathfrak{x}, \mathfrak{h}_n] \).
Then for every \( n \in \mathbb{N} \) one has \( \mathfrak{h}_n \in \mathcal{J}(\mathfrak{g}) \), hence \( [\mathfrak{x}, \mathfrak{y}] \in \mathcal{J}(\mathfrak{g}) \), and then \( [\mathfrak{x}, \mathfrak{y}] \in \lim_{n \to \infty} \mathfrak{h}_n = \mathfrak{h} \).
Thus \( [\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h} \), and we are done. \( \square \)

Lemma 5.5. Let \( \mathfrak{g} \) be a nilpotent Lie algebra. Assume that there exist two subsets \( A_0, A \subseteq \mathfrak{g}^* \) and an even natural number \( d \in \mathbb{N} \) satisfying the following conditions:
(a) For every \( \xi \in A \) one has \( \dim \mathcal{O}_\xi = d \).
(b) One has \( A \subseteq \bigcup_{\xi \in A_0} \mathcal{O}_\xi \).
(c) For every \( \xi \in A_0 \) one has \( \mathcal{O}_\xi = \xi + \mathfrak{g}(\xi) \perp \).

Then the following assertions hold:
(i) For every \( \xi \in A \) one has \( \mathcal{O}_\xi = \xi + \mathfrak{g}(\xi) \perp \).
(ii) If \( A = \mathfrak{g}^* \setminus [\mathfrak{g}, \mathfrak{g}] \perp \), then the Lie algebra \( \mathfrak{g} \) is 2-step nilpotent.

Proof. (i) It follows by \( \square \) along with \[14\] Lemma 5.3] that the mapping
\[
\psi: A \to \text{Gr}(\mathfrak{g}), \quad \xi \mapsto \mathfrak{g}(\xi),
\]
is continuous. Now let us denote \( B_0 := \bigcup_{\xi \in A_0} \mathcal{O}_\xi \subseteq \mathfrak{g}^* \). By \[18\] Thm. 3.2.3, the hypothesis \( \square \) is equivalent to \( \psi(A_0) \subseteq \mathcal{J}(\mathfrak{g}) \). This is easily seen to be further equivalent to \( \psi(B_0) \subseteq \mathcal{J}(\mathfrak{g}) \), which implies, by Lemma 5.4, \( \psi(B_0) \subseteq \mathcal{J}(\mathfrak{g}) \). Therefore, using hypothesis \( \square \) and the continuity of \( \psi \), one obtains
\[
\psi(A) \subseteq \psi(B_0) \subseteq \mathcal{J}(\mathfrak{g}).
\]
The assertion then follows by \[18\] Thm. 3.2.3] again.
(ii) Using the notion of cortex \( \text{Cor}(\mathfrak{g}) \perp \) with respect to the coadjoint action of \( G \) defined as in \[18\] Ch. III and \[19\], we first prove that
\[
\text{Cor}(\mathfrak{g}) \perp \subseteq [\mathfrak{g}, \mathfrak{g}] \perp.
\]
Indeed, the set \( \mathfrak{g}^* \setminus [\mathfrak{g}, \mathfrak{g}] \perp \) consists of flat coadjoint orbits of maximal dimension of \( G \), hence by \[19\] Thm. 2.2, they are separated points in \( \mathfrak{g}^*/G \). Thus \( \square \) follows.
It follows by \[19\] Thm. III-3.3] along with \( \square \) that for all \( \xi \in \mathfrak{g}^* \) and \( x \in \mathfrak{g} \) one has \( (\text{ad}_x^\mathfrak{g})\xi \in \text{Cor}(\mathfrak{g}) \perp \subseteq [\mathfrak{g}, \mathfrak{g}] \perp \). Therefore for all \( \xi \in \mathfrak{g}^* \) and \( x, y, z \in \mathfrak{g} \) one has \( \langle \xi, [x, [y, z]] \rangle = 0 \), which directly implies that the Lie algebra \( \mathfrak{g} \) is 2-step nilpotent. \( \square \)

In what follows, unless otherwise mentioned, for a nilpotent Lie algebra \( \mathfrak{g} \), we denote \( \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}] \).

Proposition 5.6. If \( G \) is a nilpotent Lie group of class \( \mathcal{T} \), and \( Z \) is the center of \( G \), then the following assertions hold.
(i) One has \( \text{ind} G = 1 \) if and only if \( G \) is a Heisenberg group.
(ii) If \( \text{ind} G = 2 \), then either \( G \) is 2-step nilpotent, or one has \( \dim \mathfrak{z} = 1 \), \( \dim \mathfrak{g}^1 = 2 \), and there exists \( \xi \in \mathfrak{g}^* \) with \( \mathfrak{g}(\xi) = \mathfrak{g}_1 \).

Proof. By Lemma 2.6 one obtains \( \dim \mathfrak{z} = 1 \) and there exists \( \xi_0 \in \mathfrak{g}^* \) with \( \mathfrak{g}(\xi_0) = \mathfrak{z} \) and for which \( \Gamma := \{ \mathcal{O}_{t\xi_0} \mid t \in \mathbb{R}^x \} \) is an open dense subset of \( \mathfrak{g}^*/G \). Then, defining \( \eta: \mathfrak{g}^* \to \mathfrak{g}^*/G, \xi \mapsto \mathcal{O}_{\xi} \), it follows by Lemma 5.9 that \( \eta^{-1}(\Gamma) \) is dense in \( \mathfrak{g}^* \), that is, \( \bigcup_{t \in \mathbb{R}^x} \mathcal{O}_{t\xi_0} \) is dense in \( \mathfrak{g}^* \). Moreover, since \( \mathfrak{g}(t\xi_0) = \mathfrak{z} \) for all \( t \in \mathbb{R}^x \), it follows by [18] Thm. 3.2.3 that \( \mathcal{O}_{t\xi_0} = t\xi_0 + \mathfrak{z} \). Hence we may use Lemma 5.5(ii) with \( A_0 = \{ t\xi_0 \mid t \in \mathbb{R}^x \} \) to obtain that the Lie algebra \( \mathfrak{g} \) is 2-step nilpotent. Since we have noted above that the center of \( \mathfrak{g} \) is 1-dimensional, it then follows that \( \mathfrak{g}_2 \) is a Heisenberg algebra.

Let us assume that \( \dim \mathfrak{g} = 2 \) and \( \mathfrak{g} \) is not 2-step nilpotent, that is, \( \mathfrak{g}^1 \not\subseteq \mathfrak{z} \). In particular \( \mathfrak{g}^1 \neq \{0\} \), and then there exists \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \) with \( \mathfrak{g}^1 \subseteq \mathfrak{g}(\xi) \), by Lemma 5.9. Since \( \xi \not\in (\mathfrak{g}^1)_+ \) and \( \mathfrak{g} \) is of class \( T \), one has \( \dim \mathfrak{g}(\xi) = \dim \mathfrak{g} = 2 \). One also has \( \mathfrak{z} \subseteq \mathfrak{g}(\xi) \) and \( \mathfrak{g}^1 \not\subseteq \mathfrak{z} \), hence necessarily \( \dim \mathfrak{z} = 1 \). Then \( \mathfrak{z} \subseteq \mathfrak{g}^1 \). If \( \mathfrak{z} = \mathfrak{g}^1 \), then \( \mathfrak{g} \) is a Heisenberg algebra, which is a contradiction with the assumption that \( \mathfrak{g} \) is not 2-step nilpotent. Consequently \( \mathfrak{z} \not\subseteq \mathfrak{g}^1 \subseteq \mathfrak{g}(\xi) \). Since \( \dim \mathfrak{g}(\xi) = 2 \), the assertion follows.

Example 5.7. There are many examples of 2-step nilpotent Lie algebras satisfying the hypotheses of Lemma 5.5(ii), completely different from the Heisenberg algebras. Here is a list of examples that illustrate this assertion.

1. Let \( \mathfrak{g} = \mathfrak{g}_D := \mathcal{V} \rtimes_{\alpha_D} \mathbb{R} \) for any finite-dimensional real vector space \( \mathcal{V} \) and any \( D \in \text{End}(\mathcal{V}) \) with \( D^2 \neq 0 \). Then \( \mathfrak{g} \) is 2-step nilpotent and \( \dim \mathcal{O}_{\xi} = 2 \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \). (See for instance [18] Thm. (iii).) This includes for instance the Lie algebra defined by a basis \( X_1, X_2, X_3, X_4, X_5 \) satisfying \( [X_5, X_4] = X_2, [X_5, X_3] = X_1 \), studied in [30] N5N2.

2. Let \( \mathfrak{g} \) be the so-called free 2-step nilpotent Lie algebra with 3 generators, defined by a basis \( X_1, X_2, X_3, X_4, X_5, X_6 \) satisfying \( [X_6, X_5] = X_3, [X_6, X_4] = X_1, [X_5, X_4] = X_2 \). This satisfies \( \dim \mathcal{O}_{\xi} = 2 \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \), and was studied in [18] Thm. (iv) and [30] N6N15.

3. Let \( \mathfrak{g} \) be the 2-step nilpotent Lie algebra with a basis \( X_1, X_2, X_3, X_4, X_5, X_6 \) satisfying \( [X_6, X_5] = X_2, [X_6, X_3] = X_1, [X_5, X_4] = X_1, [X_4, X_3] = X_2 \). Then one has \( \dim \mathcal{O}_{\xi} = 4 \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \). (See [30] N6N17.)

Example 5.8. Here we give some counterexamples related to Lemma 5.5(ii).

1. If \( \mathfrak{g} \) is a 2-step nilpotent Lie algebra, then there may exist no number \( d \in \mathbb{N} \) with \( \dim \mathcal{O}_{\xi} = d \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \). For instance, the 8-dimensional Lie algebra \( \mathfrak{g} = \mathfrak{h}_3 \times \mathfrak{h}_5 \) (where \( \mathfrak{h}_{2k+1} \) denotes the \( (2k+1) \)-dimensional Heisenberg algebra for any \( k \in \mathbb{N} \)) is 2-step nilpotent and yet it has both 2-dimensional and 4-dimensional coadjoint orbits. See also Lemmas 5.13 and 5.14.

2. If \( \mathfrak{g} \) be a nilpotent Lie algebra for which there exists \( d \in \mathbb{N} \) with \( \dim \mathcal{O}_{\xi} = d \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \), then it does not necessarily follow that \( \mathfrak{g} \) is 2-step nilpotent. This can be proved by several examples:
   - If \( \mathfrak{g} = \mathfrak{g}_D := \mathcal{V} \rtimes_{\alpha_D} \mathbb{R} \) for any finite-dimensional real vector space \( \mathcal{V} \) and any nilpotent operator \( D \in \text{End}(\mathcal{V}) \) with \( D^2 \neq 0 \), then \( \mathfrak{g} \) is not 2-step nilpotent and yet \( \dim \mathcal{O}_{\xi} = 2 \) for all \( \xi \in \mathfrak{g}^* \setminus (\mathfrak{g}^1)_+ \). (See for instance [18] Thm. (ii), which includes [30] N6N18.)
Lemma 5.9. Let \( g \) be a Lie algebra with \( g^1 \neq \{0\} \). Then there exists \( \xi \in g^* \setminus (g^1)^\perp \) with \( g^1 \subseteq g(\xi) \), and \( g(\xi) \leq g \).

**Proof.** Since \( g \) is nilpotent, one has \([g, g^1] \subsetneq g^1\). Then there exists \( \xi \in g^* \) with \([g, g^1] \subseteq \ker \xi \) and \( g^1 \not\subseteq \ker \xi \). That is, \( g^1 \subseteq g(\xi) \) and \( \xi \not\in (g^1)^\perp \).

Since \( g^1 \subseteq g(\xi) \), one has \( g(\xi) \leq g \), and this completes the proof. \( \square \)

**Lemma 5.10.** If the nilpotent Lie algebra \( g \) is of class \( T \) then

\[
\dim g \leq \text{ind} g + \dim [g, g]^\perp.
\]

**Proof.** One has \( \dim (g^1)^\perp = \dim g - \dim g^1 \), where we recall the notation \( g^1 := [g, g] \).

Hence the assertion is equivalent to

\[
\dim g^1 \leq \text{ind} g. \tag{5.3}
\]

To prove this inequality, we may assume \( g^1 \neq \{0\} \). Then, by Lemma 5.9, there exists \( \xi \in g^* \setminus (g^1)^\perp \) with \( g^1 \subseteq g(\xi) \), hence \( \dim g^1 \leq \dim g(\xi) \). Since \( \xi \not\in (g^1)^\perp \) and \( g \) is of class \( T \), one has \( \dim g(\xi) = \text{ind} g \), and thus (5.3) follows. \( \square \)

**Lemma 5.11.** Let \( G_1 \) and \( G_2 \) be nilpotent Lie groups such that \( C^*(G_1) \) and \( C^*(G_2) \) are Morita equivalent. Then \( G_1 \) is of class \( T \) if and only if \( G_2 \) is of class \( T \).

**Proof.** Assume that \( G_1 \) is of class \( T \), and let \( \psi: \hat{G}_1 \to \hat{G}_2 \) be a Rieffel homeomorphism as in the proof of Corollary 3.12. Then \( \psi([g_1, g_1]^\perp) = [g_2, g_2]^\perp \) by Corollary 5.12, hence also

\[
\psi(\hat{G}_1 \setminus [g_1, g_1]^\perp) = \hat{G}_2 \setminus [g_2, g_2]^\perp. \tag{5.4}
\]

On the other hand, since \( G_1 \) is of class \( T \), the open subset \( D_1 := \hat{G}_1 \setminus [g_1, g_1]^\perp \) of its unitary dual corresponds to the coadjoint orbits of \( G_1 \) having maximal dimension. It then follows by [3] Cor. 2.9 that \( D_1 \) is exactly the set of all points in \( \hat{G}_1 = C^*(G_1) \) that have finite upper multiplicities. By [1] Cor. 13(2)], the set \( \psi(D_1) \) is consists of the points of \( C^*(G_2) \) that have finite upper multiplicities. Then, by [3] Cor. 2.9 again, \( \psi(D_1) \) corresponds to the set of coadjoint orbits of \( G_1 \) having maximal dimension. It then follows by (5.4) that all the points of \( \hat{G}_2 \setminus [g_2, g_2]^\perp \) correspond to coadjoint orbits of \( G_2 \) having maximal dimension. That is, \( G_2 \) is of class \( T \), and this completes the proof. \( \square \)

**Proposition 5.12.** Let \( G_1 \) be a Lie group of class \( T \) with \( \dim G_1 \leq 5 \). If \( G_2 \) is a nilpotent Lie group for which \( C^*(G_1) \) is Morita equivalent to \( C^*(G_2) \), then \( G_1 \) is isomorphic to \( G_2 \).

The proof of this proposition requires several lemmas.

**Lemma 5.13.** If \( g \) is a nilpotent Lie algebra of class \( T \) and there exist Lie algebras \( g_1 \) and \( g_2 \) with \( g = g_1 \times g_2 \), then at least one of the Lie algebras \( g_1 \) and \( g_2 \) is abelian.

**Proof.** Assuming for \( j = 1, 2 \) that \( g_j \) is not abelian, it follows that there exists \( \xi_j \in g_j^* \) with \( \{0\} \subsetneq g_1(\xi_1) \subsetneq g_j \). For \( \xi := (\xi_1, \xi_2) \in g_1^* \times g_2^* = g^* \) one has \( g(\xi) = g_1(\xi_1) \times g_2(\xi_2) \). Similarly, for \( \eta := (0, \xi_2) \in g_1^* \times g_2^* = g^* \) one has \( g(\eta) = \ldots \)
\(g_1 \times g_2(\xi),\) hence \(g(\eta) \subseteq g(\xi) \subseteq g,\) and this shows that \(g\) is not of class \(T,\) which is a contradiction with the hypothesis. \(\square\)

**Lemma 5.14.** A Lie algebra \(g\) is of class \(T\) if and only if there exist an integer \(k \geq 0\) and an indecomposable Lie algebra \(g_0\) of class \(T\) with \(g = a_k \times g_0.\)

**Proof.** If \(g_0\) is of class \(T,\) then it is easily checked that \(a_k \times g_0\) is of class \(T.\)

We prove the converse assertion by induction on \(\dim g.\) If \(\dim g = 1,\) then \(g = a_1,\) and we are done.

For the induction step, if \(g\) is indecomposable, then we may set \(k := 0\) and \(g_0 := g.\) If \(g\) is not indecomposable, then there exist Lie algebras \(g_1\) and \(g_2\) with \(g = g_1 \times g_2\) and \(\dim g_j \geq 1\) for \(j = 1, 2.\) Since \(g\) is of class \(T,\) it follows by Lemma \(5.13\) that one of the Lie algebras \(g_1\) and \(g_2\) is abelian. We may assume that there exists an integer \(k_1 \geq 1\) with \(g_1 = a_{k_1}.\) Thus \(g = a_{k_1} \times g_2\) with \(\dim g_2 < \dim g.\) Since \(g\) is of class \(T,\) it is straightforward to check that \(g_2\) is of class \(T\) and then, by the induction hypothesis, there exist an integer \(k_2 \geq 0\) an an indecomposable Lie algebra \(g_0\) of class \(T\) with \(g_2 = a_{k_2} \times g_0.\) Thus \(a = a_{k_1} \times a_{k_2} \times g_0 = a_{k_1+k_2} \times g_0,\) and this completes the induction step. \(\square\)

**Proof of Proposition \(5.14\)** By a simple analysis using Proposition \(5.1\) we see that every Lie algebra of class \(T\) having dimension \(n \leq 5\) is isomorphic to precisely one of the following Lie algebras:

(a) Case \(n = 1:\) \(a_1\)

(b) Case \(n = 1:\) \(a_2\)

(c) Case \(n = 3:\) \(a_3, n_3\)

(d) Case \(n = 4:\) \(a_4, a_1 \times n_3, n_4\)

(e) Case \(n = 5:\) \(a_5, a_2 \times n_3, a_1 \times n_4, n_{5,1}, n_{5,2}, n_{5,4}, n_{5,5}\)

We may assume without loss of generality that \(G_1\) is nonabelian. One has

\[
\text{ind } G_1 = \text{ind } G_2 \text{ and } \dim [g_1, g_1]^{\perp} = \dim [g_2, g_2]^{\perp}
\]

(5.5)

by Corollaries \(2.7\) and Corollary \(5.13\) respectively. On the other hand, it follows by Lemma \(5.11\) that \(G_2\) is of class \(T\) and then, by Lemma \(5.10\) we obtain

\[
\dim g_2 \leq \text{ind } G_1 + \dim [g_1, g_1]^{\perp}.
\]

(5.6)

Therefore we need to discuss the cases below.

- \(\dim g_1 = 3.\) Then \(g_1 = n_3,\) hence \(\text{ind } G_1 = 1\) and \(\dim [g_1, g_1]^{\perp} = 2.\)

- \(\dim g_1 = 4.\) If \(g_1 = a_1 \times n_3,\) then \(\text{ind } G_1 = 2\) and \(\dim [g_1, g_1]^{\perp} = 3.\) If \(g_1 = n_4,\) then \(\text{ind } G_1 = 2\) and \(\dim [g_1, g_1]^{\perp} = 2.\)

Thus, if \(\dim G_1 \leq 4,\) then we obtain \(\dim G_2 \leq 5\) by (5.6), hence \(G_1\) is isomorphic to \(G_2\) by Proposition \(5.2\)

- \(\dim g_1 = 5.\) If \(g_1 = a_2 \times n_3,\) then Theorem \(1.2\) is applicable. If \(g_1\) is one of the Lie algebras \(a_1 \times n_4, n_{5,1}, n_{5,2}, n_{5,4},\) or \(n_{5,5},\) then an inspection of the table from the proof of Proposition \(5.2\) shows that \(\text{ind } G_1 + \dim [g_1, g_1]^{\perp} \leq 6.\)

Thus, if \(\dim G_1 = 5,\) then we obtain \(\dim G_2 \leq 6\) by (5.6). Moreover, Lemma \(6.7\) shows that \(\dim G_2 \neq 6,\) hence \(\dim G_2 \leq 5,\) and then \(G_1\) is isomorphic to \(G_2\) by Corollary \(1.3\)

\(\square\)

5.3. **Proof of Theorem \(1.3\)**

**Lemma 5.15.** Let \(G_1\) and \(G_2\) be nilpotent Lie groups for which \(C^{*}(G_1)\) is Morita equivalent to \(C^{*}(G_2),\) and denote by \(Z_j\) the center of \(G_j\) for \(j = 1, 2.\) If \(\text{ind } G_1 =\)
1, then \( \text{ind } G_2 = \dim Z_2 = \dim Z_1 = 1 \) and \( C^*(G_1/Z_1) \) is Morita equivalent to \( C^*(G_2/Z_2) \).

**Proof.** One has \( \text{ind } G_2 = \text{ind } G_1 = 1 \) by Corollary 2.7. Therefore, by Lemma 2.6 one has \( \dim Z_2 = \dim Z_1 = 1 \) and there exists \( \xi_j \in g^*_j \) with \( g^*_j(\xi_j) = 3 \) and moreover the mapping \( \mathbb{R}^x \to g^*_j/G_j, t \to \text{Ad}^*_g(\xi_j)(t^j) \), is a homeomorphism of \( \mathbb{R}^x \) onto an open dense subset \( D_j \) of \( g^*_j/G_j \). The set \( D_j \) is the set of all coadjoint orbits of \( G_j \) having maximal dimension, whose union is \( \{ \eta \in g^*_j \mid \eta \not\subseteq \text{Ker } \eta \} \). Let \( J_j \subseteq C^*(G_j) \) be the closed two-sided ideal with \( \widehat{J_j} = D_j \). Then there is a short exact sequence of \( C^* \)-algebras

\[
0 \to J_j \to C^*(G_j) \to C^*(G_j/Z_j) \to 0.
\]

Moreover, \( J_j \) is the largest bounded-trace ideal of \( C^*(G_j) \) (see [2, Sect. 2]).

It then follows by [11, Cor. 9] that for any fixed imprimitivity \( C^*(G_1) \)-\( C^*(G_2) \)-bimodule its corresponding Rieffel correspondence carries \( J_1 \) to \( J_2 \). Now, using [31 Prop. 3.25], we obtain that the quotients \( C^*(G_1)/J_1 \) and \( C^*(G_2)/J_2 \) are Morita equivalent. Then, taking into account the above short exact sequences, the \( C^* \)-algebras \( C^*(G_1/Z_1) \) and \( C^*(G_2/Z_2) \) are Morita equivalent. This completes the proof. \( \square \)

**Proof of Theorem 1.3.** Let \( G_1 \) be a nilpotent Lie group of dimension \( \leq 5 \) and \( G_2 \) an exponential Lie group such that \( C^*(G_1) \) is Morita equivalent to \( C^*(G_2) \). We must prove that the Lie groups \( G_1 \) and \( G_2 \) are isomorphic.

First, Lemma 1.1 implies that \( G_2 \) must be nilpotent.

Then, if \( G_1 \) is of class \( T \), the assertion follows by Proposition 5.12.

Now let us assume that \( G_1 \) is not of class \( T \). It follows by Proposition 5.1 and the list in the proof of Proposition 5.12 that the only 5-dimensional nilpotent Lie algebras which are not of class \( T \) are \( n_{5,3} \) and \( n_{5,6} \). Let us denote the center of \( g_j \) by \( z_j \) for \( j = 1, 2 \).

If either \( g_1 = n_{5,3} \) or \( g_1 = n_{5,6} \), then \( \text{ind } G_1 = 1 \), hence by Lemma 5.13 we obtain that \( C^*(G_1/Z_1) \) is Morita equivalent to \( C^*(G_2/Z_2) \) and \( \dim Z_2 = 1 \). Here \( G_1/Z_1 \) is isomorphic either to \( A_1 \times H_3 \) (if \( g_1 = n_{5,3} \)) or to \( N_4 \) (if \( g_1 = n_{5,6} \)). Both Lie groups \( A_1 \times H_3 \) and \( N_4 \) are 4-dimensional and are of class \( T \), hence by Proposition 5.12 we obtain that \( G_1/Z_1 \) is isomorphic to \( G_2/Z_2 \). In particular \( \dim G_2 = 5 \), and then \( G_1 \) is isomorphic to \( G_2 \) by Proposition 5.2. This completes the proof. \( \square \)

6. OTHER EXAMPLES

6.1. Filiform Lie groups. Lie algebras of the filiform Lie groups are defined as follows: For \( n := \dim g \geq 3 \), the nilpotent Lie algebra \( f_n \) has a basis \( X_1, \ldots, X_n \) with the commutation relations

\[
[X_n, X_j] = X_{j-1} \quad \text{for} \quad j = 1, \ldots, n-1,
\]

where \( X_0 := 0 \), and \( [X_k, X_j] = 0 \) if \( 1 \leq j \leq k \leq n-1 \).

**Proposition 6.1.** For any \( m, n \geq 3 \), \( \widehat{F_n} \) and \( \widehat{F_m} \) are homeomorphic if and only if \( n = m \).

**Proof.** For every \( n \geq 3 \) one has \( \text{ind } F_n = n - 2 \), and then the assertion follows by Lemma 2.7. \( \square \)

**Remark 6.2.** The stronger hypothesis that \( C^*(F_m) \) is Morita equivalent to \( C^*(F_n) \) implies that \( m = n \), by [2 Thms. 4.1 and 4.8], [5 Prop. 2.2] and [1 Thm. 10].
We introduce here the 6-dimensional free 2-step nilpotent Lie algebra, denoted \( \mathfrak{n}_{6,15} \) in [30], that is, the Lie algebra defined by a basis \( X_1, X_2, X_3, X_4, X_5, X_6 \) satisfying the commutation relations
\[
[X_6, X_5] = X_3, \quad [X_6, X_4] = X_1, \quad [X_5, X_4] = X_2.
\]
This will be needed in the proof of the next theorem, and also treated in Subsection 6.3.

**Theorem 6.3.** The filiform Lie group \( F_n \) is stably \( C^* \)-rigid for every \( n \geq 3 \).

We first prove the following lemma.

**Lemma 6.4.** If \( \mathfrak{g} \) is a nilpotent Lie algebra of class \( T \) with \( \text{RR}(\mathfrak{g}) \leq 3 \), then \( \text{ind} \mathfrak{g} = \dim \mathfrak{g} - 2 \).

**Proof.** Since \( \mathfrak{g} \) is class \( T \), it follows by Lemma 5.10 that there exists \( \xi \in \mathfrak{g}^* \) with
\[
[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}(\xi) \subseteq \mathfrak{g}.
\]
Here \( 2 \leq \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = \text{RR}(\mathfrak{g}) \leq 3 \) by hypothesis, hence \( 2 \leq \dim(\mathfrak{g}/\mathfrak{g}(\xi)) \leq 3 \). On the other hand \( \dim(\mathfrak{g}/\mathfrak{g}(\xi)) \) is an even integer, hence \( \dim(\mathfrak{g}/\mathfrak{g}(\xi)) = 2 \). Since \( \mathfrak{g} \) is class \( T \), we then obtain \( \dim \mathfrak{g} = \dim \mathfrak{g}(\xi) = \dim \mathfrak{g} - 2 \). \( \square \)

**Proof of Theorem 6.3.** We must prove that if \( G \) is an exponential Lie group for which \( C^*(G) \) is Morita equivalent to \( C^*(F_n) \), then \( G \) is isomorphic to the Lie group \( F_n \).

It follows by Lemma 4.11 that \( G \) is a nilpotent Lie group. Moreover, \( \text{RR}(\mathfrak{g}) = \text{RR}(\mathfrak{f}_n) = 2 \) by Corollary 3.13. On the other hand, \( G \) is class \( T \) by Lemma 5.11.

Therefore we may use Lemma 6.4 to obtain \( \dim \mathfrak{g} = \dim \mathfrak{g} - 2 \), that is, all the nontrivial coadjoint orbits of \( \mathfrak{g} \) have dimension 2. Since \( \mathfrak{g} \) is nilpotent, it then follows by [6] Thm. that one of the following cases may occur:

**Case 1:** There exists a hyperplane abelian ideal of \( \mathfrak{g} \).

**Case 2:** There exists an integer \( k \geq 0 \) with \( \mathfrak{g} = \mathfrak{a}_k \times \mathfrak{n}_{6,15} \).

**Case 3:** There exists an integer \( k \geq 0 \) with \( \mathfrak{g} = \mathfrak{a}_k \times \mathfrak{n}_{5,4} \).

In Case 2 one has \( 2 = \text{RR}(\mathfrak{g}) = \text{RR}(\mathfrak{a}_k \times \mathfrak{n}_{6,15}) = k + 3 \), hence \( k = -1 \), which is impossible. Similarly, in Case 3, one has \( 2 = \text{RR}(\mathfrak{g}) = \text{RR}(\mathfrak{a}_k \times \mathfrak{n}_{5,4}) = k + 2 \), hence \( k = 0 \). That is, \( \mathfrak{g} = \mathfrak{n}_{5,4} \), but this is impossible since Theorem 1.3 shows that \( C^*(N_{5,4}) \) is not Morita equivalent to \( C^*(F_n) \).

It follows by the above discussion that only Case 1 can occur, hence there exists an abelian ideal \( \mathfrak{a} \subset \mathfrak{g} \) with \( \dim(\mathfrak{g}/\mathfrak{a}) = 1 \). Let us denote \( m := \dim \mathfrak{g} \) and select any \( X \in \mathfrak{g} \setminus \mathfrak{a} \). Then \([\mathfrak{g}, \mathfrak{g}] = [X, \mathfrak{a}] \), since \([\mathfrak{a}, \mathfrak{a}] = \{0\}\). Since \( \dim(\mathfrak{g}, \mathfrak{g}) = \dim \mathfrak{g} - \text{RR}(\mathfrak{g}) = m - 2 \), it then follows that the operator \( D := (\text{ad}_X)|_{\mathfrak{a}} \colon \mathfrak{a} \to \mathfrak{a} \) is nilpotent and its range has codimension 1 in \( \mathfrak{a} \). This implies that the Jordan decomposition of \( D \) consists of exactly one Jordan cell, which further implies that \( \mathfrak{g} \) is isomorphic to the filiform Lie algebra \( \mathfrak{f}_n \). Since \( C^*(G) \) and \( C^*(F_n) \) are Morita equivalent, it then follows by Proposition 6.11 that \( m = n \), hence \( G \) is isomorphic to \( F_n \), and this completes the proof. \( \square \)

6.2. **The groups \( H_{m,n} \).** For any \( m, n \geq 1 \), \( H_{m,n} \) are the nilpotent Lie groups with Lie algebras \( \mathfrak{h}_{m,n} \) with a basis \( \{X_1, \ldots, X_m\} \cup \{Y_0, \ldots, Y_n\} \) and the bracket given by
\[
[X_i, Y_j] = Y_{i+j}
\]
for all \( i \in \{1, \ldots, m\} \) and \( j \in \{0, \ldots, n\} \) with \( i + j \leq n \). We recall that \( \dim H_{m,n} = m + n + 1 \) for all \( m \geq n \geq 1 \), and all the coadjoint orbits of \( H_{m,n} \) are flat. (See [16] Subsect. 6.2 and the references therein.) We also define \( H_{m,0} := A_{m+1} \), the \((m+1)\)-dimensional abelian Lie group.

For any \( C^* \)-algebra \( A \) we will denote by \( \mathcal{I}(A) \) the set of all closed 2-sided ideals of \( A \).

**Proposition 6.5.** If \( m_1 \geq n_1 \geq 1 \) and \( m_2 \geq n_2 \geq 1 \), then \( C^*(H_{m_1,n_1}) \) is Morita equivalent to \( C^*(H_{m_2,n_2}) \) if and only if \( m_1 = m_2 \) and \( n_1 = n_2 \).

**Proof.** Denote \( A_k := C^*(H_{m_k,n_k}) \) for \( k = 1, 2 \). Assume that \( A_1 \) is Morita equivalent to \( A_2 \), and let \( X \) be an \( A_1 - A_2 \)-imprimitivity bimodule with Rieffel correspondence \( \mathcal{I}(A_1) \to \mathcal{I}(A_2) \) and \( J_k \in \mathcal{I}(A_k) \) is the largest bounded-trace ideal of \( A_k \) for \( k = 1, 2 \) then, using [17] Thm. 2.8 and Cor. 2.9, one obtains the canonical homeomorphism \( \hat{J}_k \simeq \hat{\Gamma}_k \), where \( \hat{\Gamma}_k \subseteq \hat{\mathcal{I}}_k \) is the open subset corresponding to the coadjoint orbits of \( H_{m_k,n_k} \) having maximal dimension. The short exact sequence \([17]\) Eq. (6.3) then takes on the form

\[
0 \to J_k \to A_k \to C^*(H_{m_k,n_k-1}) \to 0.
\]

On the other hand, since \( J_k \) is the largest bounded-trace ideal of \( A_k \), one has \( \mathcal{I}(J_2) = J_1 \) by [17] Cor. 9, and it further follows by [31] Prop. 3.25 that \( A_1/J_1 \) is Morita equivalent to \( A_2/J_2 \) for \( k = 1, 2 \). The above short exact sequence then shows that \( C^*(H_{m_1,0}) \) is Morita equivalent to \( C^*(H_{m_2,n_2-1}) \).

Now let us assume that \( n_1 \leq n_2 \). Iterating the above reasoning, we obtain that \( C^*(H_{m_1,0}) \) is Morita equivalent to \( C^*(H_{m_2,n_2-1}) \). Since \( H_{m_1,0} \) is the abelian \((m_1 + 1)\)-dimensional Lie group, it then follows by the \( C^* \)-rigidity of abelian Lie groups with respect to the family of nilpotent Lie groups that \( H_{m_2,n_2-1} \) is abelian and in fact is isomorphic to \( H_{m_1,0} \), and then \( n_2 - n_1 = 0 \) and \( m_2 = m_1 \), which concludes the proof.

### 6.3. The 6-dimensional free 2-step nilpotent Lie group

We now turn our attention towards the 6-dimensional free 2-step nilpotent Lie algebra defined just before Theorem 6.3.

**Proposition 6.6.** If \( G \) is an exponential Lie group for which \( C^*(G) \) is Morita equivalent to \( C^*(N_{6,15}) \), then \( G \) is isomorphic to \( N_{6,15} \).

For the proof we need the following lemma.

**Lemma 6.7.** If \( G_1 \) and \( G_2 \) are simply connected solvable Lie groups for which \( C^*(G_1) \) and \( C^*(G_2) \) are Morita equivalent, then \( \dim G_1 - \dim G_2 \) is an even integer.

**Proof.** It follows by [17] Sect. V, Cor. 7 that for any simply connected solvable Lie group \( G \) there is a group isomorphism \( K_i(C^*(G)) \simeq K_{i+j_G}(\mathbb{C}) \) for any integer \( i \geq 0 \), where \( j_G = 0 \) if \( \dim G \) is an even integer, and \( j_G = 1 \) if \( \dim G \) is an odd integer. On the other hand, it is well known that \( K_0(\mathbb{C}) = \mathbb{Z} \) and \( K_1(\mathbb{C}) = \{0\} \) (cf. [32] §6.5), while the groups \( K_i(C^*(G)) \) and \( K_i(C^*(G) \otimes K) \) are isomorphic for \( i = 0, 1 \). (See for instance [32] Cor. 6.2.11 and 7.1.9.) Now the assertion follows at once.

**Proof of Proposition 6.6.** It is well known that the coadjoint orbits of \( N_{6,15} \) have dimensions \( \leq 2 \) and then we directly obtain \( \text{ind} N_{6,15} = 4 \). Thus \( \text{ind} G = \text{ind} N_{6,15} = 4 \).
by Lemma 3.13. Also, by Corollary 3.13
\[ \dim(g^1) = \dim([n_{6,15}, n_{6,15}]) = 3, \]
where we have used again the notation \( g^1 = [g, g] \). On the other hand, it follows by Lemma 5.11 that \( G \) is of class \( T \) and then, by Lemma 5.10, we obtain \( \dim g \leq 4 + 3 = 7 \). Now, by Lemma 6.7 it follows that \( \dim g \in \{2, 4, 6\} \). We discuss these cases separately below.

- Case \( \dim g = 2 \). Then \( g \) is abelian, hence \( \dim(g^1) = 2 \), which is a contradiction with the equality \( \dim(g^1) = 3 \) established above.

- Case \( \dim g = 4 \). If \( g \) is abelian, then \( \dim(g^1) = 4 \), which is a contradiction as above. If \( g \) is not abelian, then \( g = b_3 \times a_1 \) and then, by Theorem 1.22, we obtain that the group \( N_{6,15} \) is isomorphic to \( H_3 \times A_1 \), which is again a contradiction.

- Case \( \dim g = 6 \). There are two possible subcases.

Subcase 1: There exist Lie algebras \( g_1 \) and \( g_2 \) with \( g = g_1 \times g_2 \) and \( \dim g_j \geq 1 \) for \( j = 1, 2 \). We may assume \( \dim g_1 \leq \dim g_2 \) without loss of generality. Since \( \dim g_1 + \dim g_2 = \dim g = 6 \), we may have either \( \dim g_1 = 1 \), or \( \dim g_1 = 2 \), or \( \dim g_1 = 3 \).

If neither \( g_1 \) nor \( g_2 \) is abelian, then \( g \) is not of class \( T \) by Lemma 5.13 thus \( C^*(G) \) cannot be Morita equivalent to \( C^*(N_{6,15}) \), by Lemma 5.11.

If one of the Lie algebras \( g_1 \) and \( g_2 \) is abelian, then we have either \( \dim g_1 \leq 2 \), or \( \dim g_1 = 3 \) and \( g_1 \) is abelian, hence \( g_1 = a_k \) with \( k \in \{1, 2, 3\} \). Then
\[ 3 = \dim(g^1) = RR(g) = RR(a_k \times g_2) = k + RR(g_2) = k + \dim[g_2, g_2], \]

hence \( \dim[g_2, g_2] = 3 - k \). Since \( g_2 \) is nilpotent, it then follows that \( k \leq 1 \), hence \( k = 1 \), and then \( RR(g_2) = 2 \) and \( g = a_1 \times g_2 \). On the other hand, \( 4 = \text{ind} g = \text{ind}(a_1 \times g_2) = 1 + \text{ind} g_2 \), hence \( \text{ind} g_2 = 3 \). An inspection of the table from the proof of Proposition 5.2 shows that the only nilpotent Lie algebras \( g_2 \) with \( \dim g_2 = 5 \), \( RR(g_2) = 1 \), and \( \text{ind} g_2 = 3 \) are \( n_{5,4} \) and \( n_{5,5} \). But this is impossible: The group \( N_{6,15} \) is two-step nilpotent and of class \( T \), therefore the relative topology of \( N_{6,15} \setminus \{n_{6,15}, n_{6,15}\} \) is Hausdorff, while this property is not shared by the complement of characters of any of the groups \( A_1 \times N_{5,4} \) and \( A_1 \times N_{5,5} \), since suitable quotients of these groups are isomorphic to the filiform group \( F_4 = N_4 \).

Subcase 2: The Lie algebra \( g \) is indecomposable, that is, there exist no Lie algebras \( g_1 \) and \( g_2 \) with \( g = g_1 \times g_2 \) and \( \dim g_j \geq 1 \) for \( j = 1, 2 \). Since \( \dim g = 6 \), it then follows that \( g \) is one of the 24 Lie algebras labeled as \( N_{6,1}, N_{6,2}, \ldots, N_{6,24} \) in [30]. Since we already established that \( \text{ind} g = 4 \) and \( \dim(g^1) = 3 \), it then easily follows by a direct inspection that either \( g = n_{6,15} \) or \( g = n_{6,18} \), where \( n_{6,18} \) is the Lie algebra defined by a basis \( X_1, X_2, X_3, X_4, X_5, X_6 \) satisfying the commutation relations
\[ [X_6, X_5] = X_3, \ [X_6, X_4] = X_2, \ [X_6, X_3] = X_1. \]
To complete the proof, we must show that, assuming \( g = n_{6,18} \), one obtains a contradiction. In fact, if we define \( h := \mathbb{R}X_2 + \mathbb{R}X_4 \), then \( h \) is an ideal of \( n_{6,18} \) for which the quotient \( n_{6,18}/h \) is isomorphic to the 4-dimensional filiform Lie algebra \( n_4 \) that occurs in Proposition 5.1. Therefore the unitary dual \( \hat{N}_{6,18} \) is homeomorphic to a closed subset of \( N_{6,18} \) via a homeomorphism that takes the characters \( [n_4, n_4]^\perp \) of \( N_4 \) to characters \( [n_{6,18}, n_{6,18}]^\perp \) of \( N_{6,18} \). (See for instance [21 Lemme 3]). The
relative topology in \( \hat{N}_4 \) of \( \hat{n}_4 \) is not Hausdorff by \cite{21} Prop. 2.3. Hence the relative topology in \( \hat{N}_{6,18} \) of \( \hat{n}_{6,18} \) is not Hausdorff.

On the other hand, since \( N_{6,15} \) is a 2-step nilpotent Lie groups, it follows that all its coadjoint orbits are flat. Since the coadjoint orbits of \( N \) are flat, it then follows by \cite{16, Lemma 6.8(1)} that the relative topology in \( \hat{N}_{6,15} \) is Hausdorff. Taking into account the above remarks on \( N_{6,18} \), it then follows that \( C^*(N_{6,15}) \) and \( C^*(N_{6,18}) \) are not Morita equivalent. Indeed, if these two \( C^* \)-algebras were Morita equivalent then, as in the proof of Corollary \cite{5,13} one obtains a homeomorphism from \( \hat{N}_{6,15} \) onto \( \hat{N}_{6,18} \) that takes the characters of \( N_{6,15} \) onto the characters of \( N_{6,18} \). This homeomorphism will then map the complement of the characters of \( N_{6,15} \) homeomorphically onto the complement of the characters of \( N_{6,18} \), which is a contradiction with the fact that one of these spaces is Hausdorff while the other is not, as established above.

The assumption \( g = n_{6,18} \) thus leads to a contradiction, and then there remains the fact that \( g = n_{6,15} \), which completes the proof.

\begin{remark}
A by-product of the proof of Proposition \cite{6,6} is that the Lie group \( N_{6,18} \) is stably \( C^* \)-rigid as well.
\end{remark}

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