An Integral Equation for Riemann’s Zeta Function and its Approximate Solution

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Abstract

Two obscure identities extracted from the literature are coupled to obtain an integral equation for Riemann’s \( \xi(s) \) function, and thus \( \zeta(s) \) indirectly. The equation has a number of simple properties from which useful derivations flow, the most notable of which relates \( \zeta(s) \) anywhere in the critical strip to its values on a line anywhere else in the complex plane. From this, I obtain both an analytic expression for \( \zeta(\sigma + it) \) everywhere inside the asymptotic \( (t \to \infty) \) critical strip, and an approximate solution, within the confines of which the Riemann Hypothesis is shown to be true. The approximate solution predicts a simple, but strong correlation between the real and imaginary components of \( \zeta(\sigma + it) \) for different values of \( \sigma \) and equal values of \( t \); this is illustrated in a number of figures. Finally, arguments are presented to show that that the proffered “approximate” solution is less approximate and more rigorous than meets the eye.

1 Introduction

The Riemann Zeta function \( \zeta(s) \) is well-known to satisfy a functional equation, and many representations, both integral and series, have been developed over the years for both \( \zeta(s) \) and its avatar \( \xi(s) \). Additionally, and more significantly, at least two independent contour integral representations are known, [1 Section (1.4)] and [2 Eq.(7)] either of which could be utilized as the primary definition, from which many of the properties of \( \zeta(s) \) can be derived. In contrast, it has been long-ago proven that \( \zeta(s) \) does not satisfy a differential equation of quite general form [3 (and citations therein)] and I am aware of three integral equations which \( \zeta(s) \) does satisfy [4 Eq.(1.6)], [5 Eq.(3.29)], and [6 Eq.(1.5) with \( \alpha = 0 \)]. In this work, an obscure series representation of \( \xi(s) \) in terms of generalized Integro-exponential functions \( (E_s(z)) \), is coupled with an equally obscure contour integral representation of these functions, to obtain what I believe to be a new integral equation satisfied by \( \xi(s) \) and, equivalently, \( \zeta(s) \). This equation has several remarkable properties:

- Through the action of an integral operator, the value of \( \zeta(s_1) \), (the “dependent”) anywhere in the complex plane can be determined with respect to the properties of \( \zeta(s_2) \), (the “master”) on a line \( (\Re(s_2) = \text{constant}) \) anywhere else in the punctured \( (s_1 \neq s_2) \) complex plane;
- The transfer function that mediates the aforementioned action is a simple, at most sixth order, real rational polynomial function of \( s_1 \), and is therefore quite amenable to analysis;

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The only $s_1$ dependence resides in the transfer function.

In this work, after initially listing some definitions and lemmas (Section 2), I first give a derivation of the integral equation over a bounded region of the $s_2$-plane (Section 3), and obtain its analytic continuation over the remainder of the punctured ($s_1 \neq s_2$) complex plane (Section 4). In several subsections of Section 5, the various representations are then used to obtain a few simple integrals involving $\zeta(s_2)$ corresponding to special values of $s_1$; it is also demonstrated that the simple integrals under consideration are convergent.

Following these preliminaries, the general form of the transfer function is carefully presented in a series of Appendices. Based on a particularly useful property of the transfer function presented in Appendix 4 - it closely mimics a Dirac delta function on the $s_2$ line - a reasonably accurate model for $\zeta(s_1)$ is established, which yields reasonably accurate estimates of the asymptotic nature of $\zeta(s_1)$ inside the critical strip (Section 6). In consequence, it is shown that, asymptotically

$$|\zeta(\sigma + it)| \to t^{1/2-\sigma/2} \times (\log(\log(t))/\log(t) + ...),$$

and, within the confines of the model, it is both proven that $\zeta(s_1) \neq 0$ if $\Re(s_1) \neq 1/2$, and explained, in transparent terms (Section 7), why this happens (does not happen?). Graphical examples and comparisons are given, demonstrating the accuracy of a predicted universality between the real and imaginary components of $\xi(\sigma + it)$ at equal values of $t$ but different $\sigma$.

Sections 8 includes a demonstration that a predicted property of the approximation is satisfied by its first few terms. Together with Section 9 these Sections discuss the requirements to improve the rigour of the model developed here.

## 2 Preamble

### 2.1 Notation

Throughout, I use $s = \sigma + it$ to denote the independent variable defining $\zeta(s)$. In those instances where dependence on, for example, $t$ is the focus, I will sometimes just shorten, for example $M(c, s, v) \Rightarrow M(t, v)$ for typographical brevity and clarity. Throughout, subscripts 'R' and 'I' refer to the Real and Imaginary components of whatever they are attached to. Much use is made of Riemann’s $\xi$ function, defined by

$$\xi(s) \equiv (s - 1) \pi^{-s/2} \zeta(s) \Gamma(1 + s/2)$$

and

$$\Upsilon(s) \equiv \zeta(s) \Gamma(s/2) \pi^{-s/2}$$

both of which satisfy

$$\xi(s) = \xi(1 - s)$$

and

$$\Upsilon(s) = \Upsilon(1 - s).$$

### 2.2 Definitions and Lemmas

**Definition:** Unit box operator $\Box(v - x)$

means an operator that is zero everywhere except near $v = x$ where it has magnitude unity and unit width. This means that, contrary to the Dirac delta function, the width never vanishes and the area enclosed by an integrand can be approximated by multiplying $\Box(v - x)$ by an appropriate width and height centred at $v = x$ to create a box of equal area. Symbolically, this allows an approximation to be simply expressed, but has no other profound mathematical significance.

**Definition:** polar form

(2.6)
means that a complex function \( f(s) \) is written as \( \exp(i\theta(s))|f(s)| \) where \( \theta(s) \equiv \arg(f(s)) \).

The following asymptotic limits \([7, \text{Eq. (5.6.9)}]\) will be required:

\[
\lim_{t \to \infty} |\Gamma(5/4 + it/2)| = 2^{-1/4}t^{3/4}\sqrt{\pi}e^{-\pi t/4},
\]

\[
\lim_{t \to \infty} |\Gamma(\sigma/2 + it/2)| \approx \sqrt{2\pi} \left( \sigma^2/4 + t^2/4 \right)^{(\sigma-1)/4} \exp(-\pi t/4),
\]

and the special case

\[
\lim_{t \to \infty} |\Gamma(1/2 \pm it/2)| \approx \sqrt{2\pi} \exp(-\pi t/4).
\]

(Remark: I have tested the approximation \((2.8)\) numerically, and find that it is remarkably accurate for even modest values of \( t \) such that \( 0 < \sigma < 1 \ll t \).)

The correct Riemann-Siegel identity \([8, \text{Eq. (4.17.2)}]\) is

\[
\arg(\zeta(1/2 + it)) = -\arg(\Gamma(1/4 + it/2)) + t/2 \log(\pi) - k\pi
\]

where \( k \) is a positive integer. Define

\[
\phi(s) \equiv \arg(\zeta(s)) + \arg(\Gamma(s/2)) - (1/2)t \log(\pi) = \arg(\Upsilon(s))
\]

and

\[
\Phi(1 + it) \equiv \arg(\zeta(1 + it)) + \arg(\Gamma(1/2 + it/2)) - (1/2)t \log(\pi)
\]

but, for clarity, I occasionally abbreviate expressions such as the following:

\[
\cos(\Phi(1 + it))) := \cos(\Phi).
\]

3 LeClair’s representation

In a recent work, LeClair \([9, \text{Eq. (15)}]\) has obtained the following series representation of Riemann’s \( \xi \) function

\[
\xi(s) = \pi (s - 1) \sum_{n=1}^{\infty} n^2 E_{-s/2} (\pi n^2) - \pi s \sum_{n=1}^{\infty} n^2 E_{(s-1)/2} (\pi n^2) + 4\pi \sum_{n=1}^{\infty} n^2 e^{-\pi n^2}
\]

where generically, \( E_s(z) \) is the (generalized) “Exponential Integral”, a limiting case of what is elsewhere \([10]\) referred to as the “Generalized Integro-Exponential Function”. Further to the above, it has been shown (\([11, \text{Eq. (3)}]\)) that the infinite sum in \((3.1)\) can be readily evaluated:

\[
4\pi \sum_{n=1}^{\infty} n^2 e^{-\pi n^2} = \frac{\pi^{1/4}}{2\Gamma(3/4)}.
\]

In his work, LeClair truncates the sum(s) at \( N \) terms, refers to the result as an “approximation” and proceeds to obtain approximations to the location of the zeros on the critical line on that basis. Here, I treat the sums as an infinite series representation of \( \xi(s) \), and hence an identity because the series is easily shown to be convergent due to the asymptotic property of \( E_s(z) \) (see \([10, \text{Eq. (2.25)}]\)). Similar, but inequivalent series representations will be found in Paris \([12, \text{Eq. (1.1)}]\), Patkowski \([4, \text{Eq. (1.20)}]\) and elsewhere. From \([10]\), some useful integral and contour integral representations of the function \( E_s(z) \) are

\[
E_s(z) = z^{s-1} \Gamma(1 - s, z)
\]

\[
= \int_1^\infty v^{-s} \exp(-zv)dv
\]
\[ E_s(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-v) z^v}{s-1-v} dv \]  
\[ (3.5) \]
\[ E_s(z) = \frac{e^{-z}}{2\pi i \Gamma(s)} \int_{c-i\infty}^{c+i\infty} \Gamma(-v) \Gamma(s-1-v) \Gamma(1+v) z^v dv. \]  
\[ (3.6) \]

In (3.5), \( \Gamma(1-s, z) \) is the incomplete Gamma function, and (3.4) provides the fundamental definition of \( E_s(z) \). The results (3.5) and (3.6) are given in [10] Eqs. (2.6a) and Eq. (2.17), where the latter is equivalent to the contour integral representation of a special case of Meijer’s G-function. Here, the integration contour in (3.5), originally defined to enclose the real axis \( v \geq 0 \) as well as the singularity at \( v = s - 1 \) in a clockwise direction, has been converted into the line \( c < \Re(s) - 1 \), because the integrand vanishes as \( v \to \pm i\infty \). In (3.6), the original contour enclosed poles at \( v = 0, 1, \ldots \) and \( v = s - 1 + n, n \geq 0 \), but avoided poles at \( v = -1, -2, \ldots \). Therefore, in (3.6), when opened, for \( \Re(s) > 0 \) we require \( -1 < c < \Re(s) - 1 \). This paper investigates the application of (3.5) to (3.1).

### 3.1 A novel Integral Representation

In the following, I focus on the limited range \( 0 \leq \sigma \leq 1 \), where \( s = \sigma + it \), in which case, applying (3.5) to the first term in (3.1) yields

\[ \pi (s-1) \sum_{n=1}^{\infty} n^2 E_{s/2} \left( \pi n^2 \right) = \frac{(s-1)}{2i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(-v) \pi^v \sum_{n=1}^{\infty} n^{2+2v}}{-s/2-1-v} dv \]  
\[ (3.7) \]
\[ = \frac{(s-1)}{2i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(-v) \pi^v \zeta(-2v-2)}{-s/2-1-v} dv \]  
\[ (3.8) \]

where the interchange of integration and summation in (3.7) is justified if \( c_1 < -3/2 \) since the sum converges under this condition. The general requirement that \( c_1 < -\sigma/2 - 1 \) imposes the effective constraint \( c_1 < -3/2 \) for \( 0 \leq \sigma \leq 1 \); the contour may be shifted leftwards with impunity since there are no singularities in that direction. Similarly, under the transformation \( s \to 1-s \) we find the following expression for the second term of (3.1)

\[ -\pi s \sum_{n=1}^{\infty} n^2 E_{(s-1)/2} \left( \pi n^2 \right) = -\frac{s}{2i} \int_{c_2-i\infty}^{c_2+i\infty} \frac{\Gamma(-v) \pi^v \zeta(-2v-2)}{-3/2+s/2-v} dv \]  
\[ (3.9) \]

again valid for \( c_2 < -3/2 \) if \( \sigma > 0 \). In the case that \( c_1 = c_2 = c \) where \( c < -3/2 \), and focussing on \( 0 \leq \sigma \leq 1 \), we find the integral equation

\[ \xi(s) = \frac{is(s-1/2)}{\pi} \int_{c-i\infty}^{c+i\infty} \frac{\xi(v+3)}{(v+3)(3-s+v)(s+2+v)} dv + \frac{i}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{\xi(v+3)}{(v+3)(s+2+v)} dv \]  
\[ \frac{1}{2} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-v) \pi^v \zeta(-2v-2)}{-3+s-2v} dv \]  
\[ + \frac{1}{2\Gamma(3/4)} \]  
\[ (3.10) \]

which, in terms of \( \zeta(s) \) can be rewritten

\[ (s-1)\zeta(s) \Gamma(s/2+1) \pi^{-s/2} = 2i s(s-1/2) \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-v) \pi^v \zeta(-2v-2)}{-3+s-2v} dv \]  
\[ - i/2 \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-v) \pi^v \zeta(-2v-2)}{s/2+1+v} dv + \frac{1}{2\Gamma(3/4)}. \]  
\[ (3.11) \]

Notice that with a simple application of the recursion formula for \( \Gamma(-v) \), the numerator in (3.11) can be written in terms of \( \Gamma(-2v-2) \) – see (2.2). Alternatively, under a simple change of variables, with the same condition \( c < -3/2 \) and the same range of \( \sigma \), (3.11) becomes
\[ \xi(s) = \frac{\pi^{1/4}}{2\Gamma(3/4)} - J(s, c), \]  

(3.12)

where

\[ J(s, c) = \int_{-\infty}^{\infty} \pi e^{-iv} (2c + 2iv - 2) \Gamma (-c + iv) M(c, s, v) \, dv \]  

(3.13)

and

\[ M(c, s, v) = \frac{(-2s - 1)^2/2 + 2iv - 2(c + 5/4)}{-(2s - 1)^2/4 + 2i(2iv - 4(c + 5/4))v + 4(c + 5/4)^2}. \]  

(3.14)

This result is worthy of a few comments:

- Its form is almost (exception: see [4, Eq.(1.18)]) unique among representations of \( \xi(s) \). Usually \( s \)-dependence that is formally embedded inside an integral or series representation appears, either as an exponent, or buried inside the argument of a transcendental function; here \( s \)-dependence exists only in the form of a coefficient in a simple rational (polynomial) function. This augers well for further analysis, and leads to some surprising predictions;

- \((3.10)\) and \((3.11)\) present a prescription, in the form of an integral transform, for the value of \( \xi(s) \) or \( \zeta(s) \) anywhere in the complex strip \( 0 \leq \Re(s) \leq 1 \), that depends only on its values on a vertical line in the complex plane corresponding to \( c < -3/2 \). The region in which \((3.12)\) is valid is labelled “I” in Figure (1b), and delineated as everything to the left of the point at \( v = -1.5 \) in Figure (1a);

- In the form \((3.12)\) and \((3.13)\) are written, the function \( M(c, s, v) \) acts as a transfer function between \( \zeta(v) \) on the line \( \Re(v) = -2c - 2 \) and \( \zeta(s) \) elsewhere in the complex plane, through the medium of the integral operator \((3.13)\). Its properties will be of interest in studying \((3.12)\) for varying values of the parameters \( c \) and \( s \).

In the following sections simple choices of \( s \) and \( c, c_1, \) and \( c_2 \) will be applied to \((3.12)\).

4 Analytic Continuations

In the following, let \( c_1 = c_2 = c \). By shifting the contour \((3.11)\) into the region \( c > -3/2 \), (see Figure (1a)) various representations are obtained for each of the regions labelled in Figure (1b), by subtracting \((2\pi i \times) \) the residues of the singularities that are transited in the complex \( v \)-plane.

To enter the region labelled “II” in Figure (1b), the contour in Figure (1a) must pass to the right of the pole at \( v = -3/2 \), whose residue is given by

\[ \text{Residue}_{\text{blue}} = 1/(2\pi i) \]  

(4.1)

giving, for region II,

\[ \xi(s) = \frac{\pi^{1/4}}{2\Gamma(3/4)} - 1 - J(s, c). \]  

(4.2)

To enter the region labelled III from region II, (crossing the red line in Figure (1b)) the corresponding residue (red in Figure (1a))

\[ \text{Residue}_{\text{red}} = i\xi(s)/(2\pi) \]  

(4.3)

must be subtracted. The result, valid for Region III is

\[ 0 = \frac{\pi^{1/4}}{2\Gamma(3/4)} - 1 - J(s, c). \]  

(4.4)
This figure shows the location of the contour of integration (black arrow at $\Re(v) < -3/2$ extending from $-\infty$ to $\infty$ as well as the various poles. The fixed pole at $v = -3/2$ corresponds to a singularity belonging to $\zeta(-2v - 2)$. The green and red poles correspond to singularities of the integrand that depend on the value of $s$ acting as a parameter. The coloured arrows show the motion of corresponding poles as $\Re(s)$ increases from zero to one. The colours correspond to colours used in Figure 1b.

(b) This figure defines the various regions where different continuations of (3.12) apply projected onto the $t = 0$ plane. Five distinct regions are bounded by (coded and coloured) lines $c = -3/2$ (blue, solid), $c = -\sigma/2 - 1$ (red, dash) and $c = \sigma/2 - 3/2$ (green, dash-dot) when $0 \leq \sigma \leq 1$. The bounding regions extend vertically out of the plane of the figure when $t \neq 0$ because the contour used in (3.5) is chosen to be a straight line, vertical in the complex $v-$plane ($c$ is a constant). The dotted line corresponds to the interesting case $c = -5/4$.

Figure (1) This Figure illustrates the various regions of validity of the analytic continuations developed in Section 4 both (a) in the complex $v-$plane and (b) in the cross section of a complex plane cut by the real $(\sigma, c)$ plane at $t = 0$.

To cross the green line, either from Region II, entering Region V, or from Region III, entering Region IV, the residue

$$\text{Residue}_{\text{green}} = \frac{i\pi^{(s-3)/2} \zeta (1 - s) \Gamma (-s/2 + 1/2) (s - 1)}{4}$$

(4.5)

corresponding to the green pole in Figure 1a must be subtracted. A similar argument applies for a transition from region V to region IV - the residue (4.3) must be subtracted. This gives, for Region IV

$$-\xi(s) = \frac{\pi^{1/4}}{2 \Gamma (3/4)} - 1 - J(s, c).$$

(4.6)

In particular, this analysis relates the results applicable to region IV to that for region I, including regions where both representations are unconstrained with respect to $0 \leq \sigma \leq 1$, and equates the result for region V to that for region III. It is emphasized that the regional definitions in Figure 1b do not include the lines, which correspond to the punctured regions ($s_1 = s_2$) alluded to in the Introduction. Of particular interest, is the value at the point $(s = 1/2, c = -5/4)$, as well as the values along the horizontal (dotted) line $s = \sigma + it |_{c = -5/4}$ and a vertical line $c |_{s = 1/2 + it}$. To reiterate, Figure 1b is intended to illustrate the projection of a (complex) space $(s = \sigma + it)$ defined by a third axis extending out of the plane of the Figure, corresponding to $t \neq 0$, onto the (real) plane $(\sigma, c)$ where $t = 0$.

5 Crossing the line at $c=-3/2$

Remark: Several of the identities in this Section are (redundantly) declared to be real and
marked $\Re$. This is done as a reminder that unless so-declared, an attempted numerical computer evaluation of that expression will likely fail because the Imaginary part of the identity contains a singularity (e.g. \textcolor{red}{\textbf{(5.5)}}).

5.1 The general case $c=-3/2$

Disregarding the contour integral representations from which they are obtained, \textcolor{red}{\textbf{(3.12)}} , \textcolor{red}{\textbf{(4.2)}} and \textcolor{red}{\textbf{(4.4)}} simply involve several complex functions of two real variables (if $t = 0$), and ought to be amenable to analysis from that viewpoint. Compare \textcolor{red}{\textbf{(3.12)}} and \textcolor{red}{\textbf{(4.2)}} respectively below and above the horizontal line demarking regions I and II of Figure 1b by setting $c = -3/2 - \eta$ in \textcolor{red}{\textbf{(3.12)}} and $c = -3/2 + \eta$ in \textcolor{red}{\textbf{(4.2)}} where $0 \leq \eta \approx 0$. Note that $\eta$ is a real variable (there is no cut); since none of the terms in either equation (other than $J(s,c)$) depend on $c$, we are free to choose a convenient value for $c$, subject to the (regional) conditions under which each equation is valid. We have in Region I

$$\xi(s) = \frac{\pi^{1/4}}{2 \Gamma(3/4)} - J(s, -3/2 - \eta)$$

and in Region II

$$\xi(s) = \frac{\pi^{1/4}}{2 \Gamma(3/4)} - 1 - J(s, -3/2 + \eta).$$

Simple comparison between \textcolor{red}{\textbf{(5.1)}} and \textcolor{red}{\textbf{(5.2)}} suggests that

$$J(s, -3/2 - \eta) - J(s, -3/2 + \eta) = 1$$

and this may be independently verified by a change of variables $v \to \eta v$ in both integrals, followed by a series expansion about $\eta = 0$, leading to

$$J(s, -3/2 - \eta) - J(s, -3/2 + \eta) \to \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{1 + v^2} dv = 1.$$

As is often done where discontinuities arise in the theory of functions of a real variable, when $\eta = 0$ it is conventional to assign half the discontinuity at that point ($c = -3/2$). This is equivalent to deforming the contour in \textcolor{red}{\textbf{(3.11)}} to avoid the fixed pole belonging to $\zeta(-2v - 2)$ at $v = -3/2$ by including only half the residue at that point.

Remark: This cannot be done for the other poles since they are defined by $s$ (complex), rather then $v = -3/2$ (fixed).

5.1.1 The case $\sigma = 1/2$

The result from \textcolor{red}{\textbf{(5.1)}} and \textcolor{red}{\textbf{(5.2)}} after setting $\eta = 0$ (see Section 5.1) when $s = 1/2$ is

$$\int_{-\infty}^{\infty} \Re \left( \frac{\pi^{-3/2 - iv} \zeta(1 + 2 iv)}{2 iv + 1/2} \right) dv = \frac{\zeta(1/2)}{8 \Gamma(3/4)} \left( \frac{\pi^{3/4}}{2 \Gamma(3/4)} - 1/2 \right).$$

This result (compare with \textcolor{red}{\textbf{[8, page 204, Lemma $\beta$]}}) can be verified numerically. The need to specify that only the real part of the integrand in \textcolor{red}{\textbf{(5.5)}} is to be used, is twofold:

- the right-hand side is real and so must be the left-hand side;
- ostensibly, the integrand of \textcolor{red}{\textbf{(5.5)}} appears to be singular at $v = 0$ unless one notes that

$$\zeta(1 \pm 2iv) \approx \mp i/(2v) + \gamma + \ldots$$

and $\gamma$ is Euler’s constant. Therefore, the real part of the integrand converges at $v = 0$ and the imaginary part, singular at the origin, integrates to zero by anti-symmetry.
Of interest, the left-hand side of (5.5) can also be written
\[
\int_{-\infty}^{\infty} \Re \left( \frac{\pi^{-3/2-iv}\zeta(1+2iv)\Gamma(3/2+iv)}{2iv+1/2} \right) dv
= \frac{2}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{\Gamma(3/2+iv)|\zeta(1+2iv)|\left(4v\sin(\theta(v)) + \cos(\theta(v))\right)}{16v^2+1} dv
\]
where
\[
\theta(v) = \arg(\Gamma(3/2+iv)) + \arg(\zeta(1+2iv)) - v \ln(\pi).
\]
(5.8)
The integrand in (5.7) is symmetric about \( v = 0 \) so there is no simplification due to possible anti-symmetry. As noted, although the real part of (5.5) does not vanish, the imaginary part must. Written in the form (5.7), the imaginary part yields the result
\[
\int_{-\infty}^{\infty} \frac{\Gamma(3/2+iv)|\zeta(1+2iv)|\left(4v\cos(\theta(v)) - \sin(\theta(v))\right)}{16v^2+1} dv = 0.
\]
(5.9)
Near \( v = 0 \), the cosine term in the integrand does not diverge, and the term containing that term vanishes by anti-symmetry about \( v = 0 \). Because the right- and left-hand limits of \( \sin(\theta(v)) \) are of opposite sign (see (5.6)), the term containing that factor also vanishes by anti-symmetry, although it is singular at the point \( v = 0 \).

In general, when \( s = 1/2 + it \), the left-hand side of both (5.1) and (5.2) are real, and so the equivalent of (5.5) applies. Written in terms of \( t \), (5.5) generalizes to
\[
\xi(1/2 + it)
= -\frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} \Re \left( \frac{\pi^{-iv}\zeta(1+2iv)\left(4t^2 + 4iv + 1\right)\Gamma(3/2+iv)}{t^2 + 2iv - 4v^2 + 1/4} \right) dv - \frac{1}{2} + \frac{\pi^{1/4}}{2\Gamma(3/4)}.
\]
(5.10)

5.2 The case \( c=-5/4 \)

In regions III and V, writing (4.4) in full gives
\[
\pi^c \int_{-\infty}^{\infty} \pi^{-iv}\zeta(-2c+2iv-2)\Gamma(-c+iv)(2iv - 2s^2 - 2c + 2s - 3)\left(-2c + 2iv - s - 2\right) dv = \frac{\pi^{1/4}}{2\Gamma(3/4)} - 1
\]
(5.11)
or, equivalently
\[
\int_{-\infty}^{\infty} \frac{(2iv - 2s^2 - 2c + 2s - 3)\xi(-2c + 2iv - 2)}{(-2c + 2iv + s - 3)(-2c + 2iv - s - 2)(-2c + 2iv - 3)} dv = \frac{\pi^{1/4}}{2\Gamma(3/4)} - 1
\]
(5.12)
Setting \( c = -5/4 \) in (5.11) leads to
\[
\int_{-\infty}^{\infty} \zeta(1/2 + 2iv)\pi^{-5/4-iv}\Gamma(5/4+iv)\left(iv - (2s - 1)^2/4\right) dv = 1 - \frac{\pi^{1/4}}{2\Gamma(3/4)},
\]
(5.13)
a result that is valid for all \( s \) (see the dotted line in Figure 13). Since the left-hand side is a function of \( s \) and the right-hand side is not, it must be true that the derivative of the left-hand side with respect to \( s \) vanishes. Performing this calculation (the integral is convergent, so the derivative and integral operator can be interchanged) yields
\[
(s - 1/2) \int_{-\infty}^{\infty} \frac{\pi^{-5/4-iv}\left(2s - 1)^2/4\right)\zeta(1/2 + 2iv)\Gamma(5/4+iv)}{(2s - 1)^2/8 + 2v^2} dv = 0
\]
(5.14)
Compare (5.13) with Patkowski [1], Theorem 1.
\[5.2.1 \quad \sigma = 1/2\]

Consider (5.13) in the case \(s = 1/2\), which gives

\[
\int_{-\infty}^{\infty} \frac{i \pi^{-5/4-i v}}{2 v} \zeta (1/2 + 2 iv) \Gamma (5/4 + iv) \, dv = 1 - \frac{\pi^{1/4}}{2 \Gamma (3/4)},
\]

and the integrand apparently has a singularity at \(v = 0\). However, a simple expansion of the integrand about that point shows that

\[
\Re \left( \frac{i \pi^{-5/4-i v}}{2 v} \zeta (1/2 + 2 iv) \Gamma (5/4 + iv) \right) \approx -\frac{\zeta (1/2)}{\sqrt{2} \pi^{1/4} \Gamma (3/4)} + O(v^2),
\]

demonstrating that the real part of the integrand in (5.13) is non-singular at \(v = 0\). However, near \(v = 0\) it is similarly shown that the imaginary part of the integrand diverges like \(v^{-1}\) and the imaginary part of the integral vanishes by anti-symmetry about \(v = 0\).

In the case \(s = 1/2 + it\), (5.13) can be written in terms of \(\xi(1/2 + 2iv)\) as

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi (1/2 + 2 iv) (iv + t^2)}{(-t^2/2 + 2 v^2) (-1/2 + 2 iv)} \, dv = 1 - \frac{\pi^{1/4}}{2 \Gamma (3/4)}
\]

whose integrand appears to become singular at \(v = \pm t/2\). Writing the integrand in terms of its real and imaginary parts, and noting that \(\xi(1/2 + 2iv)\) is real, we find, for the real part

\[
\frac{4}{\pi} \int_{-\infty}^{\infty} \frac{\xi (1/2 + 2 iv)}{16 v^2 + 1} \, dv = 1 - \frac{\pi^{1/4}}{2 \Gamma (3/4)}
\]

and, for the imaginary part,

\[
\int_{-\infty}^{\infty} \frac{v \xi (1/2 + 2 iv)}{(4 v^2 - t^2) (16 v^2 + 1)} \, dv = 0.
\]

Notice that in the former case there is no \(t\) dependence, so in (5.17), the variable \(t\) is in reality a free parameter, consistent with the argument applied to obtain (5.14). In the latter case although a singularity exists, the integrand is anti-symmetric about \(v = 0\) and so the singularities at \(\pm v\) cancel.

6 Asymptotics

6.1 An asymptotic approximation to \(\zeta(\sigma + it)\)

Consider (4.10) - region IV - in the general case \(s = \sigma + it\), written in terms of the transfer function \(M(c, s = \sigma + it, v)\). The case \(c = -1\) is of interest since the integral will span the 0-line, which connects to the 1-line by reflection, and, acting in its capacity as a master function, \(\zeta(1 + it)\) has many well-known properties (e.g. [13]). In addition, according to Figure (11), (4.10) is valid for all values of \(\sigma\) spanning the critical strip 0 < \(\sigma < 1\). In this case the transfer function \(M_x(-1, \sigma + it, v)\) and its real and imaginary components, shortened, except where necessary, to \(M_{R}(t, v)\) and \(M_{I}(t, v)\), are respectively:

\[
M_x(-1, \sigma + it, v) = \frac{(2 \sigma + 2 it - 1)(\sigma + it)}{(2 iv - 1 + \sigma + it)(2 iv - \sigma - it)^{-1}} - (2 iv - \sigma - it)^{-1}
\]

\[
M_R(t, v) = \frac{8 v^3 (t^2 - \sigma^2 + \sigma) + 2 t v (1 - 2 \sigma) + 2 \sigma^3 (2 - \sigma) - (4 t^2 + 3) \sigma^2 + (4 t^2 + 1) \sigma - 2 t^4 - t^2}{(t^2 + 4 v t + \sigma^2 + 4 v^2 - 2 \sigma + 1)(t^2 - 4 v t + \sigma^2 + 4 v^2)}
\]

\[
M_I(t, v) = \frac{8 v^3 (-16 \sigma + 8) v^2 + (-6 t^2 + 6 \sigma^2 - 6 \sigma + 2) v + (2 \sigma - 1) t}{(t^2 + 4 v t + \sigma^2 + 4 v^2 - 2 \sigma + 1)(t^2 - 4 v t + \sigma^2 + 4 v^2)}.
\]

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Furthermore, these functions possess the following symmetry properties:

\[ M_R(\sigma + it, v) = M_R(1 - \sigma + it, -v) \]  \tag{6.4} \\
\[ M_I(\sigma + it, v) = -M_I(1 - \sigma + it, -v). \]  \tag{6.5} \\

which symmetry also holds true for all values of \( c \). A complete description of these functions is given in Appendices A and B. From (6.6), the basic equation in Region IV using \( c = -1 \) is

\[ 1 - \frac{\pi^{1/4}}{2\Gamma(3/4)} - \xi (\sigma + it) = \frac{i}{\pi} \int_{-\infty}^{\infty} (M_R(t, v) + iM_I(t, v)) \Upsilon(2iv) v \, dv. \]  \tag{6.6} \\

It is now convenient to rewrite (6.6) as an integral over the range \([0, \ldots, \infty)\) by first splitting the integral, setting \( v \to -v \) and combining the two halves. The expression obtained is straightforward, although rather lengthy. A second lengthy equation can be obtained by first replacing \( \sigma \to 1 - \sigma \) in (6.6) and similarly reducing the range to \([0, \ldots, \infty)\). After splitting \( \Upsilon(2iv) \) into its real and imaginary components, these two equations can be added and subtracted, and with the help of (6.4) and (6.5) two new fundamental equations emerge:

\[ \frac{\pi^{1/4}}{2\Gamma(3/4)} - 1 = -\xi_R(\sigma + it) + \frac{1}{\pi} \int_{0}^{\infty} (T_1(s, v) \Upsilon_R(2iv) + (T_2(s, v) - 4) \Upsilon_I(2iv)) v \, dv \]  \tag{6.7} \\
and

\[ \xi_I(s) = \frac{2}{\pi} \int_{0}^{\infty} (T_3(s, v) \Upsilon_R(2iv) + T_4(s, v) \Upsilon_I(2iv)) v \, dv, \]  \tag{6.8} \\

where

\[ T_1(s, v) = -M_I(s, v) + M_I(s, -v) \]  \tag{6.9} \\
\[ T_2(s, v) = -M_R(s, v) - M_R(s, -v) + 4 \]  \tag{6.10} \\
\[ T_3(s, v) = M_R(s, v) - M_R(s, -v) \]  \tag{6.11} \\
\[ T_4(s, v) = -M_I(s, v) - M_I(s, -v). \]  \tag{6.12} \\

In this manner, because \( \xi(s) \) is self-conjugate, the real and imaginary components of \( \xi(s) \) are isolated, and expressed in terms of convergent integrals. A simple calculation also demonstrates that, with the exception of the neighbourhood \( v = t/2 \) (see Appendix C), each of the \( T_{1,2,3,4} \) defined above asymptotically approaches zero with order \( t^{-2} \) or faster, as \( t \to \infty \) (see Section 8).

As written, the left-hand side of (6.7), is independent of both \( \sigma \) and \( t \), which allows significant simplification. In particular, it is convenient to evaluate the right-hand side at the special value of \( t \) corresponding to the point \( \infty \) (not approaching!) infinity. As will shortly be discussed in more detail, in the immediate vicinity of \( v = t/2 \), considerable structure exists in each of the \( T_{1,2,3,4} \), but as \( t \to \infty \), no elements of this structure increase faster than \( O(t) \). Since each of the \( T_{1,2,3,4} \) is multiplied by either \( \Upsilon_R(2iv) \) or \( \Upsilon_I(2iv) \), each of which decreases exponentially, in the asymptotic limit corresponding to \( v = t/2 \), all of the terms containing any of the \( T_{1,2,3,4} \) vanish when \( t \) attains the point at infinity, leaving behind the identity

\[ 1 - \frac{\pi^{1/4}}{2\Gamma(3/4)} = \frac{4}{\pi} \int_{0}^{\infty} \Upsilon_I(2iv) v \, dv. \]  \tag{6.13} \\

(Remark1: I have verified this identity numerically to sixty significant digits. 

Remark2: See (8.7) for an alternate derivation of (6.13).)

After subtracting (6.13) from (6.7) a more useful, and slightly more stable form of (6.7) arises:

\[ \xi_R(s) = \frac{1}{\pi} \int_{0}^{\infty} (T_1(s, v) \Upsilon_R(2iv) + T_2(s, v) \Upsilon_I(2iv)) v \, dv \]  \tag{6.14}
For the case $t = 0, 0 < \sigma < 1$, (6.14) reduces to a relation between $\xi(\sigma)$ on a section of the real line, and $\Upsilon(2iv)$ on the complex line $\Re(v) = 0$, specifically

$$\xi_R(\sigma) = \frac{1}{\pi} \int_0^\infty \frac{1}{(\sigma^2 + 4v^2)\left((\sigma - 1)^2 + 4v^2\right)} \left(4v^2(3\sigma(\sigma - 1) + 4v^2 + 1)\Upsilon_R(2iv) + v(64v^4 + 16(\sigma^2 - \sigma + 1)v^2 - 2\sigma(\sigma - 1))\Upsilon_I(2iv)\right) dv. \tag{6.15}$$

(6.14) can now be used to obtain an approximation to $\xi_R(\sigma + it)$. From Appendix C notice that $T_2(s,v)$ changes sign near $v = t/2$, and therefore will tend to integrate to zero relative to $T_1(s,v)$, which possesses a peak at the same location, of magnitude $-t/(\sigma(1 - \sigma))$ - see (C.1). Therefore, in (6.14), approximate $T_2(s,v) \approx 0$, on the assumption that the other multiplicative factors (namely $\Upsilon_I(2iv)$) do not vary significantly over the small range corresponding to the width between the opposing peaks in $T_2(s,v)$. Then, replace the section of the integral belonging to the peak of $T_1(s,v)$, with a box of equal area, symbolized by the “unit box operator function” discussed previously (see (2.3)) as follows:

$$\int_0^\infty T_1(s,v)\Upsilon_R(2iv)dv \approx -\frac{w_1\mathcal{B}(v - t/2)v\Upsilon_R(2iv)}{\sigma(1 - \sigma)}. \tag{6.16}$$

The fact that only values at $v = t/2$ are being selected, is compensated for by introducing a parameter $w_1$ which models the width of $T_1(s,v)$ near its peak - effectively putting a box of equal area around the peak and thereby approximating parts of the integrand near $v = t/2$ which dominate $T_1(s,v)$ and thereby, the integral itself. According to the Mean Value Theorem for Integrals, this can always be done and $w_1 \neq 0$. The parameter $w_1$, (actually $w_1(\sigma)$), could be chosen to equal the full width of the peak at half-height; other possibilities exist and should be considered in a future study, including the possibility that it is a (weak) function of $t$. Here, I use $w_1 = \sqrt{\sigma(1 - \sigma)}$ based on numerical experimentation, and later, $w_2 = w_1$.

Thus, by replacing $T_1(s,v)$ by the value of its magnitude in the vicinity of its extremum, noting that all other components of the integrand vanish as $t^{-2}$ or faster (see Section 8), evaluating major components of the integrand at $v = t/2$ and replacing $\Upsilon_R(2iv)$ by its reflection $\Upsilon_R(1 - 2iv)$, (see (2.4)) we arrive at

$$\xi_R(\sigma + it) \approx -\frac{w_1t^2\Upsilon_R(1 - it)}{2\pi\sigma(1 - \sigma)}. \tag{6.17}$$

To obtain an asymptotic approximation of $\zeta(\sigma + it) = t \to \infty$, first write the left-hand side of (6.17) in polar form and employ (2.9), to obtain

$$\xi_R(\sigma + it) \approx \left(-1/2 \cos(\phi) t^2 + (1/2 - \sigma) \sin(\phi) t + 1/2 \sigma (\sigma - 1) \cos(\phi)\right) \times \pi^{-\sigma/2} |\zeta(\sigma + it)| \sqrt{2\pi} \left(\sigma^2/4 + t^2/4\right)^{(\sigma - 1)/4} e^{-\pi t/4}, \tag{6.18}$$

and, on the assumption that $t \gg \sigma$, and $\cos(\phi) \neq 0$, (6.18) reduces to

$$\xi_R(\sigma + it) \approx -\cos(\phi) \pi^{1/2} |\zeta(\sigma + it)| t^{(\sigma - 1)/2} e^{-\pi t/4}. \tag{6.19}$$

Now, consider the right-hand side of (6.17) by identifying $\Upsilon_R(1 - it)$ in terms of its components, apply the same approximations used above and cancel common non-zero factors on both sides. Finally, reunite both sides of the equation, and eventually arrive at

$$|\zeta(\sigma + it)| \cos(\phi) = \frac{w_1 \cos(\Phi) |\zeta(1 - it)| 2^{(\sigma - 1)/2} t^{1/2} \sigma^{\sigma/2} e^{-\pi t/4}}{\sigma(1 - \sigma)}. \tag{6.20}$$

Starting from (6.8), a similar set of approximations can be applied. As in the model for $\xi_R(s)$, and with reference to Figure 11, the function $T_3(s,v)$ approximately integrates to zero, so the main contribution to $\xi(s)$ arises from $T_4(s,v)$. Thus the equivalent of (6.16) is
\[
\int_0^\infty T_4(s, v) Y_R(2 iv) v \, dv \approx w_4 B(v - t/2) v T_4(\min) \, Y_I(1 - 2 iv), 
\]
from (C.8), the equivalent of (6.17) is
\[
\xi_I(\sigma + it) \approx \frac{2 w_4 t^2 (1/2 - \sigma) Y_I(1 - it)}{\pi \sigma (1 - \sigma)},
\]
and the equivalent of (6.18) is
\[
\xi_I(\sigma + it) = \left((\sigma - 1) t \cos(\phi) + 1/2 (\sigma - 1 - t^2) \sin(\phi)\right) \pi^{-\frac{\Phi}{2}} |\zeta(\sigma + it)| |\Gamma(\sigma/2 + it/2)|.
\]
In the case that \(\sin(\phi) \neq 0\) and \(0 < \sigma < 1 \ll t\), follow the exact steps used previously and ultimately arrive at
\[
|\zeta(\sigma + it)| \sin(\phi) = \frac{w_4 \left(1 + 2 \sigma\right) \sin(\Phi) \left(1 - it\right) \left(1/2 - \sigma\right) \Gamma(1/2 - \sigma/2)}{\frac{3}{2} \Gamma(1/2 - \sigma/2)}.
\]

Remark: If \(\sigma = 1/2\) in (6.23), \sin(\phi) on the left-hand side vanishes. See Section 6.2.

Further, it is possible to extract approximations for \(\zeta_R(\sigma + it)\) and \(\zeta_I(\sigma + it)\) by solving (6.20) and (6.21) simultaneously, using a simple trigonometric expansion (because \(\arg(\zeta(\sigma + it))\) is embedded within the definition of \(\phi\) (see (2.11)), by noting that \(\cos(\arg(\zeta(s)))|\zeta(s)| = \zeta_R(s)\) and similarly for \(\zeta_I(s)\)). The result is
\[
\zeta_R(\sigma + it) = t^{1/2 - \sigma/2} \pi^{\sigma/2 - 3/2}
\]
\[
\times \left(\frac{2^{\sigma/2 + 1/2} (2 \sigma - 1) \sin(\beta) w_4 \sin(\Phi)}{\sigma \sqrt{1 - \sigma}} - \frac{2^{\sigma/2 - 1/2} \cos(\Phi) \cos(\beta) w_4}{\sigma (1 - \sigma)}\right) |\zeta(1 - it)|.
\]
and
\[
\zeta_I(\sigma + it) = t^{1/2 - \sigma/2} \pi^{\sigma/2 - 3/2}
\]
\[
\times \left(\frac{2^{\sigma/2 + 1/2} (2 \sigma - 1) \cos(\beta) w_4 \sin(\Phi)}{\sigma \sqrt{1 - \sigma}} + \frac{2^{\sigma/2 - 1/2} \sin(\beta) w_4 \cos(\Phi)}{\sigma (1 - \sigma)}\right) |\zeta(1 - it)|
\]
where
\[
\beta = \arg(\Gamma(\sigma/2 + it/2)) - \frac{t}{2} \log(\pi).
\]
Notice that the right-hand sides of both (6.25) and (6.26) do not contain any components of \(\zeta(\sigma + it)\) and therefore the left-hand sides are expressed in terms of independent functions. Figure 2 illustrates both these approximations. Compare with [14].

6.1.1 Comments and Discussion

From (6.17), it is seen that any structure in \(\xi_R(\sigma + it)\) must originate from the function \(\Upsilon(1 - it)\) on the right-hand side, and this function lacks \(\sigma\) dependence. Thus (6.17) predicts that the structure of \(\xi_R(\sigma + it)\), particularly the location of the zeros, must be relatively independent of \(\sigma\), since, as was noted in Section 6.1, \(\sigma\) dependence only resides in the transfer functions. Thus if \(\Upsilon_R(1 - 2iv)\) happens to vanish at \(v = t/2\), then the peak associated with \(T_1(t, v)\) is multiplied by a vanishing factor and contributions to the integral from both functions \(T_1(t, v = t/2)\) and \(T_2(t, v = t/2)\) vanish - independently of \(\sigma\)!. So, the location of the zeros is approximately governed by \(\cos(\Phi) = 0\). See the discussion surrounding (7.3). This prediction can be checked - see Figure (3a) which demonstrates that this property appears to be approximately true, for two randomly chosen, disparate values of \(\sigma\).
Figure (2) This figure illustrates the approximations (6.25) and (6.26) to $\zeta_R(\sigma+it)$ and $\zeta_I(\sigma+it)$ respectively near $t=500$ using $w_1 = w_4 = \sqrt{\sigma(1-\sigma)}$. In both cases, the functions have been scaled by a factor $t^2 \exp(\pi t/4)$.

From this observation, arises a constraint on the approximation. Consider two instances of (6.17) with different choices $\sigma = \sigma_1$ and $\sigma = \sigma_2$. The approximation predicts that

$$\frac{\xi_R(\sigma_1+it)}{\xi_R(\sigma_2+it)} = \frac{w_1(\sigma_1)\sigma_2(1-\sigma_2)}{w_1(\sigma_2)\sigma_1(1-\sigma_1)}$$

since the model suggests that $w_1$ is only a function of $\sigma$. Now suppose that $\sigma_2 = 1 - \sigma_1$, in which case the left hand side of (6.28) becomes unity (see (2.1)); for consistency, this imposes a constraint on the parameter $w_1$, specifically

$$w_1(\sigma) = w_1(1-\sigma).$$

As noted previously, based on numerical experiments, for this work, I use

$$w_1 = \sqrt{\sigma(1-\sigma)}.$$

and recommend that this choice be refined in the future. All this suggests that it is of interest to investigate $\xi(\sigma+it)$ for two different choices of $\sigma$. (6.28) and (6.30) predict that

$$\sqrt{\sigma_1(1-\sigma_1)}\xi_R(\sigma_1+it) \approx \xi_R(\sigma_2+it)\sqrt{\sigma_2(1-\sigma_2)},$$

an example of which is shown in Figure (3b). Finally, it is of interest to consider approximations to $|\zeta(\sigma+it)|$ as given in (6.20). See Figure (4).

### 6.2 The case $\sigma = 1/2$

The case (6.20) is of particular interest when $\sigma = 1/2$. By straight substitution, using $w_1 = \sqrt{(1-\sigma)\sigma}$, we obtain

$$|\zeta(1/2+it)|\cos(\phi) = \frac{2^{3/4}\cos(\Phi)|\zeta(1-it)| t^{1/4}}{\pi^{5/4}}.$$  

For an example, see Figure (5). In (6.32), note that the factor $\cos(\phi) = (-1)^k$ because of (2.11), where $k$ counts the number of zeros of $\zeta(1/2+it)$ relative to $t=0$. Since the basis of the
This is a comparison of $\xi_R(\sigma + it)$ for two very different values of $\sigma$ near $t = 500$.

This is a comparison of the left- and right-hand sides of (6.31) for two very different values of $\sigma$ near $t = 500$.

Figure (3) This is a comparison of $\xi_R(\sigma_1 + it)$ and $\xi_R(\sigma_2 + it)$ as they exist (left), and as they are predicted to exist by the approximation (right) - see (6.31). In both cases, the functions have been scaled by a factor $t^2 \exp(\pi t/4)$.

approximations used here are specific with respect to the leading asymptotic behaviour (it is only details of the $\sigma$ dependence that are approximate), one concludes that $|\zeta(1/2 + it)| \sim t^{1/4}$ with further logarithmic dependence embedded in the factor $|\zeta(1 - it)|$ (see Titchmarsh and Heath-Brown [8, Theorem 5.16], Edwards [1, Section (9.2) - Lindelöf’s Estimate] and, for more recent references, [13] and [15].

In the case of (6.22) with $\sigma = 1/2$, both sides vanish, as they must because it is known that $\xi_t(1/2 + it) = 0$. This is also true of (6.24) because in the limit $\sigma = 1/2$, we have $\phi = k\pi$ because of (2.10) and (2.11). However, further analysis may be possible, because $\phi = k\pi$ represents the exceptional case in the derivation that led to (6.24) - see (6.23). Thus, given that both sides of (6.24) are analytic functions, it should be possible to carefully evaluate that particular limit, by comparing the coefficients of the leading terms in $1/2 - \sigma$. From the left-hand side of (6.23), a straightforward series expansion about $\sigma = 1/2$, including the application of (2.10), eventually yields

$$\frac{\partial}{\partial \sigma} \phi(\sigma + it) \bigg|_{\sigma=1/2} = 8 w_4 \frac{\sin(\Phi) t^{1/4} \zeta(1 - it)}{\pi^{5/4} |\zeta(1/2 + it)| \cos(\phi)} - 2 t^{-1}. \quad (6.33)$$

Further, taking note of (6.32), (6.33) asymptotically simplifies, giving

$$\frac{\partial}{\partial \sigma} \phi(\sigma + it) \bigg|_{\sigma=1/2} = 4 \frac{w_4 \sin(\Phi)}{w_3 \cos(\Phi)}. \quad (6.34)$$

Unfortunately, (6.34) does not result in a reasonable comparison when it is tested numerically, for a large number of choices of $w_1$ and $w_4$. This should be investigated in the future.

7 Zeros of $\zeta(\sigma + it)$

As noted earlier, $c = -1$ was chosen in (6.6) because it would lead to an expression involving a master function $\zeta(1 - it)$ whose properties are fairly well-known, and whose validity spans the critical strip. In particular, it is known that $|\zeta(1 - it)| \neq 0$, (see Ivic, [16, Theorem (6.1)]), although both $\zeta_R(1 - it)$ and $\zeta_I(1 - it)$ do have many (non-coincident) zeros, and their location is the subject of some interest (e.g. [17]). Further, it is well-known that $\xi(s)$ shares the zeros of $\zeta(s)$ and $\Upsilon(1 - it)$ shares the zeros of $\zeta(1 - it)$ so any properties of the zeros belonging to the first of
Figure (4) This is a comparison of the left- and right-hand sides of (6.20) for two very different values of \( \sigma \) near \( t = 300 \). The parameter \( w_1 = \sqrt{\sigma(1-\sigma)} \) in both cases.

Therefore consider (6.17) and (6.22). The first of these relates \( \xi_R(s) \) to \( \Upsilon_R(1-it) \) and predicts that the zeros of these functions should coincide, and, in fact the two should be proportional to one another if \( \sigma \neq 1 \) and \( \sigma \neq 1 \) where the model (I) breaks down anyway. The second of these relates \( \xi_I(s) \) to \( \Upsilon_I(1-it) \) and predicts a further coincidence between the zeros of \( \xi_I(\sigma+it) \) and \( \Upsilon_I(1-it) \) unless \( \sigma = 1/2 \), in which case \( \xi_I(1/2+it) \) is always zero [1, Section 6.5]; \( \Upsilon_I(1-it) \) is therefore independent if \( \sigma = 1/2 \). But, as noted, because \( |\zeta(1-it)| \neq 0 \) then no zeros of \( \Upsilon_R(1-it) \) and \( \Upsilon_I(1-it) \) coincide. Consequently, neither do the zeros of \( \xi_R(\sigma+it) \) and \( \xi_I(\sigma+it) \). Further, if \( \xi(\sigma+it) = 0 \) and \( \sigma \neq 1/2 \), this would require the right-hand sides of (6.17) and (6.22) to vanish simultaneously, a contradiction with the known properties of \( \zeta(1-it) \) because neither of the two parameters \( w_1 \) and \( w_4 \) vanish.

Therefore, if \( \sigma \neq 1/2 \), it is impossible for \( \xi(\sigma+it) \) to vanish because it is overly constrained by the properties of \( \zeta(1-it) \): if \( \sigma = 1/2 \), one of the constraints vanishes, and \( \xi(1/2+it) = 0 \) becomes possible. Thus, within the confines of this approximation, the Riemann Hypothesis is true. However, none of the above predicts that \( \zeta(1/2+it) = 0 \) anywhere. That is a separate issue. A comparison between the right- and left-hand sides of (6.17) and (6.22) is given in Figure 6 demonstarting that the discussion here closely approximates reality.

It is also worthwhile to consider how (6.25) and (6.26) allow us to reach the same conclusion. Suppose that \( \zeta_R(\sigma+it) = 0 \) on the left-hand side of (6.25) and simultaneously \( \zeta_I(\sigma+it) = 0 \) on the left-hand side of (6.26). Since it is known that \( |\zeta(1-it)| \neq 0 \) it must be that each of the terms in parenthesis in those two equations vanish simultaneously. That is, after removing all non-zero common factors, and noting that the equations are only valid for \( 0 < \sigma < 1 \) we have

\[
\sqrt{2} (1+2\sigma) \sin(\beta) w_L \sin(\Phi) \sqrt{1-\sigma} + \sqrt{2} \cos(\Phi) \cos(\beta) w_L \frac{2(1-\sigma)}{2(1-\sigma)} = 0 \\
\sqrt{2} (1+2\sigma) \cos(\beta) w_L \sin(\Phi) \sqrt{1-\sigma} - \sqrt{2} \sin(\beta) w_L \cos(\Phi) \frac{2(1-\sigma)}{2(1-\sigma)} = 0
\]

(7.1)

If \( \sigma \neq 1/2 \), if \( \cos(\Phi) \neq 0 \), the only solution to (7.1) requires that

\[
\cos^2(\beta) + \sin^2(\beta) = 0
\]

(7.2)
which is obviously impossible. If \( \cos(\Phi) = 0 \), a second formal solution exists; it requires that either \( \sin(\Phi) = 0 \) also, or that \( \sin(\beta) \) and \( \cos(\beta) \) vanish simultaneously. Both of these conditions are incompatible. Therefore \( \zeta(s) \neq 0 \) if \( \sigma \neq 1/2 \). However in the case that \( \sigma = 1/2 \), the only solution to (7.1) is

\[
\cos(\Phi) = 0 \quad (7.3)
\]

which is entirely possible and consistent with everything else that has been deduced so far.

Finally, it needs to be mentioned that the above discussion did not consider any functional dependence of \( w_1 \) and \( w_4 \). Both these parameters were introduced to model the width of the peaks in \( T_1(s,v) \) and \( T_4(s,v) \), and it has been shown in the Appendices that these widths never vanish. Therefore neither do \( w_1 \) and \( w_4 \), and so neither affects the discussion regarding the solution of (7.1) and the existence of the zeros. This suggests that the approximate solution discussed in this paper has more general validity than presented. Keeping this in mind, within the confines of a more general interpretation of the model discussed here, Riemann’s Hypothesis appears to be true.

8 More about (6.8) and (6.14)

Both (6.8) and (6.14) were rigorously derived and are exact. Their representations at \( t = 0 \) are easily obtained both numerically and analytically (e.g. (6.15)). However, it will be observed that any attempt to evaluate these results numerically as \( t \) increases from zero quickly runs into trouble, in the form of severe loss-of-significant digits. Since these equations are fundamental to the model developed here, it is important to understand why this must be so.

Consider the left-hand side of either of these equations. As shown in Section 6, the function \( \xi(\sigma + it) \) decreases exponentially with increasing \( t \). Therefore, so must the right-hand side(s) since both equations are rigorous equalities. In fact, it is the extraction of an exponentially decreasing needle embedded in a haystack of much larger terms that leads to the main results reported here. Specifically, as \( t \) increases, although the terms, associated with the peak(s) discussed in the Appendices, dominate the transfer function within the integrand, other terms exist that decrease...
(a) This is a comparison of the left- and right-hand sides of (6.17) near \( t = 800 \). The parameter \( w_1 = \sqrt{\sigma(1-\sigma)} \) and \( \sigma = 1/2 \).

(b) This is a comparison of the left- and right-hand sides of (6.22) near \( t = 800 \). The parameter \( w_4 = \sqrt{\sigma(1-\sigma)} \) and \( \sigma = 0.22 \).

Figure (6) This Figure gives a comparison of the predicted relationship between \( \xi_R(\sigma + it) \) (left) and \( \xi_I(\sigma + it) \) (right) versus the corresponding real and imaginary components of \( \Upsilon(1 - it) \) at two different choices of \( \sigma \) near \( t = 800 \). The figures are scaled by \( e^{t\pi/4}/t^2 \).

algebraically (of order \( t^{-2}, t^{-3} \ldots \)) and these dominate the integral (and interfere with) any numerical attempt to extract any trace of (the exponentially vanishing) \( \xi(\sigma + it) \) from the right-hand sides of (6.8) or (6.14) when the integral is evaluated numerically. These are the “background” terms discussed in Appendix C.

Since it is impossible for a series of the form \( \sum_{k=2}^{\infty} a_k t^{-k} \) to add up to an exponential of the form \( \exp(-t\pi/4) \), (e.g. see (6.19)), each of the integrals (represented by \( a_k \)) associated with each of these background terms must vanish, independently of \( \sigma \) and \( t \), since each must be true for all values of both variables. Otherwise the sum of such terms cannot vanish and (6.8) or (6.14) would not be true. This is a difficult thesis to develop analytically in general, although for reasons given it must be true. Thus, a numerical evaluation of (6.8) or (6.14) will possibly succeed, but only if a very large number of significant digits are brought to bear, consistent with the value of \( t \). The numerical instability reflects the analytic issue. In the following, the truth of this inference will be demonstrated, both numerically and analytically for the first few coefficients of the series.

Consider the large \( t \) limit of (6.14), by evaluating the asymptotic limit of \( T_1(t, v) \) and \( T_2(t, v) \). To isolate the background terms, omit consideration of the effects occurring at \( v = t/2 \), giving

\[
T_1(t, v) \sim -\frac{12 v}{t^2} + \frac{4 v (9 \sigma^2 - 20 v^2 - 9 \sigma + 4)}{t^4} + O(t^{-6}) \quad (8.1)
\]

and

\[
T_2(t, v) \sim -\frac{16 v^2 + 2}{t^2} + \frac{-64 v^4 + (48 \sigma^2 - 48 \sigma + 48) v^2 - 6 \sigma^2 + 6 \sigma - 2}{t^4} + O(t^{-6}) . \quad (8.2)
\]

Based on the above discussion, the integrals involving \( t^{-2} \) must vanish, that is

\[
\frac{1}{t^2} \int_0^\infty \left( -12 v^2 \Upsilon_R(2iv) + v (-16 v^2 + 2) \Upsilon_I(2iv) \right) dv = 0 \quad (8.3)
\]

which appears to be true (numerically) to 30 significant digits. Note that this result is independent of \( \sigma \), as predicted. Now consider the coefficients of \( t^{-4} \) in (8.1) and (8.2). Arguing
that (8.3) must be true for reasons given, substitute these terms into (6.14) also applying (8.3), to obtain a second level (recursive) equation, which must vanish, that is
\[
\frac{1}{t^3} \int_0^\infty \left( \Upsilon_R(2iv) v^4 + \frac{v(96v^4 - 40v^2 - 1)}{120} \Upsilon_I(2iv) \right) dv = 0.
\] (8.4)

This identity can also be verified numerically true to the same level of significance. Again, \(\sigma\) dependence has cancelled from (8.4), as predicted. This sequence can be continued to the next recursive level, producing the identity
\[
\frac{1}{t^6} \int_0^\infty \left( 224 \Upsilon_R(2iv) v^6 - \left(-128v^7 + 112v^5 + \frac{28v^3}{3} + v^3/3 \right) \Upsilon_I(2iv) \right) dv = 0.
\] (8.5)

Again, \(\sigma\) dependence has cancelled of its own volition; numerically, this result appears to be true to 10 significant digits - computer evaluations start to have problems as inaccuracies due to loss of significant figures arise as predicted. With respect to (6.8), a similar series of demonstrations can be performed, and equivalent identities arise with minor changes, those being that the integrals associated with \(T_3\) and \(T_4\) lead to a series in \(t^{-3}, t^{-5}, \ldots\) and each term is prefaced by an overall coefficient \((2\sigma - 1)\). The above predictions will now be proven analytically.

Consider the classical result \([8, \text{Eq.}(2.15.6)]\) relating the inverse Mellin transform of \(\Upsilon(c + iv)\) to the Jacobi \(\Theta_4\) function:
\[
\int_{-\infty}^\infty \zeta(c + iv) \Gamma(c/2 + iv/2) x^{-c/2 - iv/2} dv = 4\pi \sum_{n=1}^\infty e^{-n^2x} = 2\pi(\Theta_3(0, \exp(-\pi x)) - 1)
\] (8.6)
valid for \(c > 1/2\). We are interested in the case \(c = 0\), so shift the contour by subtracting the residue at \(v = (1 - c)/i\) and half the residue at \(v = ic\) resulting in the identity
\[
\pi \left( 2 \sum_{n=1}^\infty e^{-n^2x} - \frac{\sqrt{\pi}}{\sqrt{x}} \frac{1}{2} \right) = \int_0^\infty (\pi iv x^{-iv} + \pi iv x^{iv}) \Upsilon_R(2iv) dv + i \int_0^\infty (\pi iv x^{-iv} - \pi iv x^{iv}) \Upsilon_I(2iv) dv
\] (8.7)
for the case \(c = 0\), after converting the integration limits to \((0, \infty)\) and writing \(\Upsilon(2iv)\) as the sum of its real and imaginary parts. By an increasing sequence of higher order derivatives, evaluated at \(x = \pi\), it is possible to obtain relevant sums using identities given in Romik’s paper \([11]\). The first seven such sums are listed in Appendix \([1]\) from which it is possible to obtain the first seven even moments of \(\Upsilon_R(2iv)\) and odd moments of \(\Upsilon_I(2iv)\), listed also in Appendix \([1]\). Substituting these moments \((D.8)\) to \((D.14)\) into \((S.3)\), \((S.4)\) and \((S.5)\) proves that they are all true as predicted. Note that \((D.8)\) provides an alternate derivation of \((6.13)\).

Finally, it is also worth mentioning that as the number of (demonstrably vanishing) terms labelled by the variable \(t^{-k}\) increases, the credibility of \((6.8)\) and \((6.14)\) for smaller and smaller values of \(t\) also increases. The label “asymptotic” becomes more tenous and the generality of the approximate solution developed here increases. If the reader is willing to accept that the results discussed in this section are true in general, then the approximation symbols employed in \((6.17)\) and \((6.22)\) become more exact and less approximate; in fact the entire model then devolves into a search for a suitable approximation for the width functions \(w_1\) and \(w_4\), and secondary effects due to the neglected terms \(T_2(t, v)\) and \(T_3(t, v)\).

9 Summary

Remark: Throughout this work, it should be considered that the Figures presented were all obtained from the computer program Maple \([19]\) the accuracy of whose numerical evaluation of
functions that do not lie in its mainstream is open to question. I have performed spot checks comparisons between Figures from Maple and Mathematica [20] and find no grounds for concern.)

Although it is known in the literature that $\xi(s)$ satisfies integral equations as cited in Section [1] only the one presented by Patkowski [11 Theorem (1.1)] comes close to sharing the desirable properties uncovered here. In particular, because the $s$–dependence is isolated within the transfer function, it became possible to associate properties of $\xi(s)$ anywhere in the critical strip with its properties on the 1-line, which are relatively well known. One surprising prediction is that the zeros of $\xi(\sigma + it)$ and those of $\Upsilon(1 - it)$ are universally correlated, as demonstrated by a number of Figures. Although the correlation shown in the Figures is not exact, the prediction is based on a model that has two unknowns:

- What is a reasonable model to employ for the width functions?
- How will the two terms $T_2(t, v)$ and $T_3(t, v)$ affect (6.17) and (6.22) once they are incorporated into the model?

These two questions implore further investigation.

It is the author’s opinion that the functional dependence of the width functions will affect the fidelity of the correlation between the location of the zeros, but neither of the above will affect the structure of the results (6.17) and (6.22). If that is the case, then the structure of (7.1) will persist, and since the width functions cancel from the solution to those equations, the range of validity of the statement, demonstrated here, that $\zeta(s) \neq 0$ if $s \neq 1/2$ will gain validity. Future work along these lines is recommended.

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Figure (7) This Figure summarizes the properties of the transfer function $M_I(\sigma + it, v)$ using $\sigma = 0.1$ over a range of $v$ with $t = 100$.

Figure (8) This Figure details the properties of $M_I(\sigma + it, v)$ when $t = 100$, $\sigma = 0.1$, and $v < 0$ (left) and $v > 0$ (right). Shown is the value of the extremum (dotted) as well as the location of one of its half-width intercepts (dashed) on the $v$–axis to first (asymptotic) order in $t^{-1}$. Note the difference in vertical scales for the same values of $t$ and $\sigma$ between these two Figures, as illustrated in Figure 7 where both are shown at the same scale.
Appendices

A Appendix: Properties of $M_I(t, v)$

This appendix gives a detailed description of the properties of $M_I(c, s, v)$ in the case $c = -1$. For simplicity, $c$ dependence is omitted from the notation. Because the interest is in the properties as a function of $t$, in this Appendix, I use $M_I(\sigma + it, v) \equiv M_I(t, v)$ and $M_R(\sigma + it, v) \equiv M_R(t, v)$ except in those cases where $\sigma$ dependence is considered. Because of symmetry properties about the line $c = -5/4$ (see Figure 1b), the cases $c = -1$ and $c = -3/2$ (see Section 5.1) are related by the functional equation of $\zeta(s)$. With reference to its definition (3.14), equating $\partial M_I(t, v)/\partial v = 0$ and solving for $v$, identifies, as $t \to \infty$, the fact that $M_I(t, v)$ has only three extrema, those being

- an extremum near $v=0$, consistent with the illustrations in Figures 7 and 8
- two maxima, one at $v^+ = t/\sigma$ (A.1)
  the other at $v^- = -t/\sigma$. (A.2)

At the positive maximum, we find

$$M_I(t \to \infty, v^+) = \frac{t}{\sigma}$$ (A.3)

and for $v < 0$ we find

$$M_I(t \to \infty, v^-) = \frac{t}{(1 - \sigma)}.$$ (A.4)

These are the characteristics of two sharp “pulses” - see Figure 7, in which case it is of further interest to determine the width. Solving $M_I(t \to \infty, v) = M_I(t \to \infty, v^+)/2$ identifies the abscissa of $v$ at half-height, that being

$$v_h = \frac{t}{2} \pm \frac{\sigma}{2}.$$ (A.5)

Thus the full-width at half-height of the function $M_I(t \to \infty, v)$ is $\sigma$, independent of $t$ in the asymptotic limit and thereby mimics, but does not approach, a representation of a Dirac delta function whose width vanishes. Figure 8, which illustrates these estimates in specific numerical form, suggests the following approximation

$$M_I(t \to \infty, v > 0) \approx -(t/\sigma) \times w_+ \times \mathcal{B}(v - t/2)$$ (A.6)
$$M_I(t \to \infty, v < 0) \approx (t/(1 - \sigma)) \times w_- \times \mathcal{B}(v + t/2)$$ (A.7)

where $\mathcal{B}$ is the “unit” box operator (see (2.5)), intended to indicate that the function values of any integrand containing $M_I$ (e.g. see (6.6)) are to be evaluated at $v = \pm t/2$ corresponding to the sign of $v$. Embedded in this model are the factors $(t/\sigma)$ and $t/(1 - \sigma)$ reflecting the magnitudes of the respective peaks that depend on the sign of $v$. Included also is a width factor $w_{\pm}$ intended to capture the fact that non-zero width exists in the asymptotic limit. For the purpose of this work, the estimates $w_+ = \sqrt{\sigma}$ and $w_- = \sqrt{(1 - \sigma)}$ appear to be reasonable based on numerical experimentation (see main text), but there is little doubt that these estimates could benefit from future refinement.

B Properties of $M_R(\sigma + it, v)$

(Definition: “In that same limit” as used here means that if any approximate solution of an equation is substituted back into the equation from which it was obtained, the resultant will
Figure (9) This Figure summarizes the properties of the transfer function $M_R(\sigma + it, v)$ using $\sigma = 0.9$ over a range of $v$ with $t = 10^9$.

Figure (10) This Figure details the properties of the transfer function $M_R(\sigma + it, v)$ when $t = 100$, $\sigma = 0.1$ and $v < 0$ (left) and $v > 0$ (right). Shown are values of the extrema (dotted) as well as the location of their abscissa (dashed) on the $v$–axis to first (asymptotic) order in $t^{-1}$. See (B.1) to (B.5). Note the difference in vertical scales for the same value of $t$ and $\sigma$ between these two Figures.

always be of equal or greater (negative) order relative to the order of the putative solution - effectively verifying a solution at the same level of approximation.)

The function $M_R(s, v)$, slightly more complicated, can be analyzed in a similar fashion. First of all, ignoring any extrema, in the asymptotic limit $t \to \infty$, it is easily shown that $M_R \approx -2$ if $|v| << t/2$. An overview of $M_R(\sigma + it, v)$ can be seen in Figures 9 and 10. Details follow below.

To locate the extrema, a solution of $\frac{\partial}{\partial t} M_R(\sigma + it, v) = 0$ leads to a sixth order polynomial equation which does not appear to be amenable to direct analysis except in the case $\sigma = 1/2$. To solve the general case $\sigma \neq 1/2$, we expect that any extremum of the function $M_R(\sigma + it, v)$ will approximately coincide with a zero of the derivative of the denominator, consistent with the presence of a nearby pole in the complex $v$–plane (see (6.1)). Equate the derivative of this denominator to zero in the asymptotic limit $t \to \infty$, solve, then substitute the result into the expression for $\frac{\partial}{\partial v} M_R(\sigma + it, v)$. With a bit of experimentation using the result so-obtained as a guide, the location of three extrema in that same limit, can be found, those being to first order in $t^{-1}$, in order of decreasing $v$: 
In order to determine the explicit values of the extrema and their locations. However, in the limit $T \to 0$ involve $6^{th}$ order rational polynomials which do not lend themselves to analytic solution in order to determine the explicit values of the extrema and their locations. However, in the limit

$$V_1 = \frac{t}{2} + \frac{\sigma}{2} - \frac{\sigma(1-\sigma)}{2t}$$  \hspace{1cm} (B.1)$$
$$V_2 = \frac{t}{2} - \frac{\sigma}{2} - \frac{\sigma(1-\sigma)}{2t}$$  \hspace{1cm} (B.2)$$
$$V_3 = \frac{(1/2 - \sigma)}{4t}$$  \hspace{1cm} (B.3)$$
$$V_4 = -\frac{t}{2} + \frac{(1-\sigma)}{2} + \frac{\sigma(1-\sigma)}{2t}$$  \hspace{1cm} (B.4)$$
$$V_5 = -\frac{t}{2} - \frac{(1-\sigma)}{2} - \frac{\sigma(1-\sigma)}{2t}$$  \hspace{1cm} (B.5)$$

Notice that the sets $\{V_1, V_5\}$ and $\{V_2, V_4\}$ are related by the interchange $\sigma \leftrightarrow (1 - \sigma)$ together with an overall change of sign. The result $V_3$ corresponds to an inflection near the origin.

The extreme values at each of these extremum locations can be found by substitution into $M_R(\sigma + it, v)$ in the limit $t \to \infty$. For the case $v > 0$ we obtain

$$M_R(t \to \infty, \{V_1, V_2\}) \approx \pm \frac{\sigma}{2\sigma} + \frac{(1/2 - \sigma)}{\sigma} \pm \frac{(2\sigma^2 - 2\sigma + 1)}{4\sigma t}$$  \hspace{1cm} (B.6)$$

and for the case $v < 0$ the extreme values are

$$M_R(t \to \infty, \{V_4, V_5\}) \approx \mp \frac{\sigma}{2(1-\sigma)} - \frac{(1/2 - \sigma)}{(1-\sigma)} \mp \frac{(2\sigma^2 - 2\sigma + 1)}{4(1-\sigma)t}$$  \hspace{1cm} (B.7)$$

all again in order, and, as in the previous case, the extrema are separated by a constant distance $\Delta v = \sigma$ or $\Delta v = 1 - \sigma$ according as $v > 0$ or $v < 0$. Using the same methods, the two midpoints can be more accurately obtained; in the asymptotic limit for $v > 0$ and $v < 0$ respectively, they are

$$V_{\text{mid}}^+ = t/2 + 3(\sigma - 1)^2/(4t)$$  \hspace{1cm} (B.8)$$
$$V_{\text{mid}}^- = -t/2 - 3\sigma^2/(4t).$$  \hspace{1cm} (B.9)$$

Figure 9 illustrates the above in general. $M_R(t, v)$ fundamentally differs from $M_I(t, v)$ because, in the asymptotic limit, it vanishes with slope $-2t/(\sigma - 1)^2$ near either of the midpoints $V_{\text{mid}}^+$ of the extrema rather than rising to an extreme value near that point, although it spans $2t$ between maximum and minimum ordinates as does $M_I(t, v)$. It can also be shown that, to a reasonable degree of approximation, $M_R(t, v)$ is nearly symmetric (in the asymptotic limit) about the midpoint.

**Remark:** In the physics lexicon, sharp peaks of the form shown in Figure 9 are labelled “Breit-Wigner” or Lorentzian. The shape is a reflection of the presence of a nearby pole in the complex $v-$plane, the closer, the sharper the peak. Choosing $c = -1$ with $0 < \sigma < 1$ guarantees that the peak shapes will be sharp, optimizing the approximation discussed in Section 6. As can be seen from Figure 1a, when the two parametrized moving poles coalesce, they create a pole of one higher order, and this occurs only when $\sigma = 1/2$, providing one reason why the critical line differs from all other locations in the critical strip. The locations of the poles can be read from the denominator parameters of (6.2) and (6.3).

### C Properties of the functions $T_{1,2,3,4}(s, v)$

The functions $T_{1,2,3,4}(s, v)$ were defined previously (see (6.9) through (6.12)). Since they are combinations of $M_{R,I}$ they share similar properties which can be visualized by examining Figure 11. As before, the formal equations defining these functions, available from (6.2), (6.3), (6.9) - (6.12) involve $6^{th}$ order rational polynomials which do not lend themselves to analytic solution in order to determine the explicit values of the extrema and their locations. However, in the limit
$t \to \infty$, it is possible to obtain reasonable approximations by experimentation. It is likely that these estimates could be improved in future; they are listed below, employing to the following notation:

- $V_j(\text{min/max})$ refers to the abscissa of an extremum projected onto the $v$–axis;
- $T_j$ gives the magnitude (amplitude) of the extremum;
- $Z_j$ locates any points where the function crosses the $v$–axis;
- $H_j$ locates points corresponding to the half-height, and
- $W_j$ gives the full-width at half-height,

all with reference to function $T_j(s, v), \ j = 1, 2, 3, 4$. Note that in some cases, the minimum and maximum reverse or vanish according to whether $\sigma > 1/2, \sigma < 1/2$ or $\sigma = 1/2$.

\[
T_1(\text{Min}) = \frac{t}{\sigma(\sigma - 1)} + \frac{3}{4t} \quad \text{(C.1)}
\]

\[
H_1 = t/2 \pm 1/4 \sqrt{-2 - 8\sigma^2 + 2\sqrt{4\sigma(\sigma - 1)(5\sigma^2 - 5\sigma + 2) + 1 + 8\sigma}} \quad \text{(C.2)}
\]

\[
W_1 = 1/2 \sqrt{-2 - 8\sigma^2 + 2\sqrt{4\sigma(\sigma - 1)(5\sigma^2 - 5\sigma + 2) + 1 + 8\sigma}} \quad \text{(C.3)}
\]

\[
V_2(\text{Min/Max}) = t/2 \mp (1/4 - (1/2 - \sigma)^2) \quad \text{(C.4)}
\]

\[
T_2(\text{Min/Max}) = \mp \frac{t \left( -4\sigma(1 - \sigma)(2\sigma - 1)^2 + 2 \right)}{\sigma(1 - \sigma)(4\sigma^2 + 1)(4(1 - \sigma)^2 + 1)} \quad \text{(C.5)}
\]

\[
T_3(\text{Min/Max}) = \mp \frac{2t(\sigma - 1/2)}{\sigma(\sigma^2 - 2\sigma + 2)(1 - \sigma)(\sigma^2 + 1)} \quad \text{(C.6)}
\]

\[
V_3(\text{Min/Max}) = t/2 \mp \sigma (1 - \sigma)/2 \quad \text{(C.7)}
\]

\[
T_4(\text{Min}) = 2t \frac{(\sigma - 1/2)}{\sigma(1 - \sigma)} \quad \text{(C.8)}
\]

\[
T_4(\text{Max}) = -2t \frac{(\sigma - 1/2)}{1 + 2\sqrt{\sigma(1 - \sigma)}} \quad \text{(C.9)}
\]

\[
V_4(\text{Min/Max}) = t/2 \mp 1/2 \sqrt{\sigma(1 - \sigma) + \sqrt{\sigma(1 - \sigma)}} \quad \text{(C.10)}
\]

\[
Z_4(\text{Min/Max}) = t/2 \mp 1/2 \sqrt{\sigma(1 - \sigma)} \quad \text{(C.11)}
\]

\[
H_4 = t/2 \pm 1/4 \sqrt{-2 + 2\sqrt{4(\sigma - 1)^2\sigma^2 + 1}} \quad \text{(C.12)}
\]

\[
W_4 = 1/2 \sqrt{-2 + 2\sqrt{4(\sigma - 1)^2\sigma^2 + 1}} \quad \text{(C.13)}
\]

Results obtained in the main text, are based on these functions. It is important to realize that each can be split into two parts - a pole term which will be denoted by the symbol $P_{k,x}$ and a background term denoted by $B_{k,x}$, where $k = 1, 2, 3, 4$ and $x = a, b$ specifying whether that particular term derives from the first ($a$) or second ($b$) term in $T_k$. As an example, consider the first term of $T_1(\sigma + it, v)$ (i.e. $-M_1(\sigma + it, v)$, calculable from (6.9). By a partial fraction decomposition of the denominator, we have

\[
\frac{1}{((t - 2v)^2 + \sigma^2)((t + 2v)^2 + (-1 + \sigma)^2)} = P_{1a}(\sigma, v) N(\sigma, t, v) + B_{1a}(\sigma, v) N(\sigma, t, v)
\]
Figure (11) This Figure illustrates the nature of the functions $T_1(s, v)$ to $T_4(s, v)$ using $\sigma = 1/3$, $t = 2000$. The extremum values are shown to demonstrate the accuracy of the estimates (C.1), (C.5), (C.6), (C.8) and (C.9). The location points (C.4), (C.7) and (C.10) are also shown (dashdot). The arrows correspond to (C.11) and the vertical dotted lines show the location ((C.2) and (C.12)) of the half-height line.

where the background term, corresponding to a distant pole at $v < 0$, is defined by

$$B_{1,a}(\sigma, v) \equiv -\frac{8 t^2 + 8 t v + 2 \sigma - 1}{(t + 2v)^2 + (-1 + \sigma)^2} \frac{1}{(4t^2 + (2\sigma - 1)^2)}$$

and the pole term is defined by

$$P_{1,a}(\sigma, v) \equiv \frac{8 t^2 - 8 t v - 2 \sigma + 1}{(t - 2v)^2 + \sigma^2} \frac{1}{(4t^2 + (2\sigma - 1)^2)}$$

multiplied by an overall factor $N(\sigma, t, v)$ belonging to the numerator of the first term in $T_1(\sigma + it, v)$, given by

$$N_1(\sigma, t, v) = 8 v^3 - (16 \sigma - 8) t v^2 - (6 t^2 - 6 \sigma^2 + 6 \sigma - 2) v - (2 \sigma + 1) t$$

By examination we find

$$P_{1,a}(\sigma, v) = -B_{1,a}(1 - \sigma, -v)$$

along with

$$P_{1,b}(\sigma, v) = P_{1,a}(\sigma \to 1 - \sigma, v)$$

and each of $P_{1,b}$ and $B_{1,b}$ is multiplied by the numerator of the second term in $T_1$, namely $N_2(\sigma, t, v) = N_1(\sigma \to 1 - \sigma, t, v)$. The extension of this decomposition to $T_{2,3,4}$ is obvious.

As $t \to \infty$, each of $P_{1,x}$ and $B_{1,x}$ vanishes as $-3v/t^2$ except near $v = t/2$. However, at the point $v = t/2$, the sum of the pole terms, influenced by their proximity to the two nearby poles in the complex $v$–plane, increases as $t/(\sigma(1 - \sigma))$, whereas all of the background terms vanish as $-3/4/t$. Insofar as the integral itself is concerned, in the main text we deal with integrands of the form $v T_1(\sigma + it, v) Y_R(2iv)$, which product vanishes exponentially at large values of $v$, so the integral itself exists, although its magnitude is governed by the $t$ dependence of its dominant term. This is
There are several important points to note about these functions:

- To at least first order in $t$ each is more-or-less symmetric near the point $v = t/2$;
- Other than near $v = t/2$ the functions otherwise vanish as $O(t^{-2})$ or $O(t^{-3})$ - for an overview see Figures 4 and 9. Therefore the properties of $\xi(s)$ are determined to a large extent by the properties of the integrand of (6.8) and (6.14) near the point $v = t/2$, although other non-vanishing, but ultimately cancelling terms obfuscate this result numerically;
- The function extrema approximate a unit box function in the limit $t \rightarrow \infty$ - see (2.5);
- The magnitude estimates given here are all expected to be accurate to $O(t)$; the locations (and therefore widths) less so; in Section 6.1 both are used to obtain an asymptotic estimate of $\zeta(s)$ which is therefore expected to be reasonably accurate, but not exact because the widths are only estimated. It is the magnitude(s) of the pole terms embedded in $T_1(s)$ and $T_4(s)$ that govern the leading order in $t$; since these are exact to leading order in $t$, so will be the accuracy of $\xi(s)$ (and hence $\zeta(s)$) to the leading order in $t$. The widths govern the accuracy of the approximation details shown in various Figures in the main text, but do not appear to affect the derivation of the location of the zeros.

D Listing of Sums and Moments

The following sums were calculated by applying higher order derivative with respect to $x$ to Romik’s Theorem 1 11 at $x = \pi$. They are needed to obtain the moments of $\Upsilon(2iv)$ below, for use in Section 8. Note: $\Gamma(3/4)^n$ means $(\Gamma(3/4))^n$.

\[
\sum_{n=1}^{\infty} -n^2 e^{-\pi n^2} = -\frac{1}{8 \pi^{3/4}\Gamma(3/4)} \quad (D.1)
\]
\[
\sum_{n=1}^{\infty} n^4 e^{-\pi n^2} = 1/32 \frac{3 + 1/2 \pi^4}{\pi^{7/4}\Gamma(3/4)} \quad (D.2)
\]
\[
\sum_{n=1}^{\infty} -n^6 e^{-\pi n^2} = -\frac{15}{128 \pi^{11/4}\Gamma(3/4)} - \frac{15 \pi^{5/4}}{256 \Gamma(3/4)^9} \quad (D.3)
\]
\[
\sum_{n=1}^{\infty} n^8 e^{-\pi n^2} = \frac{105}{512 \pi^{29/4}\Gamma(3/4)} + \frac{105 \sqrt{\pi}}{512 \Gamma(3/4)^9} - \frac{\pi^{13/4}}{2048 \Gamma(3/4)^{19}} \quad (D.4)
\]
\[
\sum_{n=1}^{\infty} -n^{10} e^{-\pi n^2} = -\frac{945}{2048 \pi^{37/4}\Gamma(3/4)} - \frac{1575}{2048 \pi^{3/4}\Gamma(3/4)^9} + \frac{45 \pi^{17/4}}{8192 \Gamma(3/4)^{17}} \quad (D.5)
\]
\[
\sum_{n=1}^{\infty} n^{12} e^{-\pi n^2} = \frac{10395}{8192 \pi^{25/4}\Gamma(3/4)} + \frac{51975}{16384 \pi^{7/4}\Gamma(3/4)^9} - \frac{1485 \pi^{9/4}}{32768 \Gamma(3/4)^{17}} \quad (D.6)
\]
\[
\quad + \frac{51 \pi^{21/4}}{65536 \Gamma(3/4)^{25}} \quad (D.7)
\]

From the above sums, it is possible to obtain the moments of $\Upsilon(2iv)$ by evaluating the derivatives of (5.7) at $x = \pi$ as follows:
\[ \int_0^\infty t \mathcal{Y}_I(2it) \, dt = -\frac{\pi}{4} \left( \frac{1}{2} \sqrt{\frac{\pi}{\Gamma(3/4)}} - 1 \right) \quad (D.8) \]

\[ \int_0^\infty t^2 \mathcal{Y}_R(2it) \, dt = \frac{\pi}{8} + \frac{\pi^{5/4}}{32 \Gamma(3/4)} - \frac{\pi^{21/4}}{64 \Gamma(3/4)^9} \quad (D.9) \]

\[ \int_0^\infty t^3 \mathcal{Y}_I(2it) \, dt = \frac{3 \pi^{3/4} - 10 \pi^{5/4} \Gamma(3/4)^3 - 16 \pi \Gamma(3/4)^9}{256 \Gamma(3/4)^9} \quad (D.10) \]

\[ \int_0^\infty t^4 \mathcal{Y}_R(2it) \, dt = \frac{-\pi^{11/4} - 76 \pi^{21/4} \Gamma(3/4)^3 + 68 \pi^{5/4} \Gamma(3/4)^{16} - 64 \pi \Gamma(3/4)^{17}}{2048 \Gamma(3/4)^{17}} \quad (D.11) \]

\[ \int_0^\infty t^5 \mathcal{Y}_I(2it) \, dt = \frac{5 \pi^{5/4} + 420 \pi^{21/4} \Gamma(3/4)^3 - 484 \pi^{5/4} \Gamma(3/4)^{16} + 128 \pi \Gamma(3/4)^{17}}{8192 \Gamma(3/4)^{17}} \quad (D.12) \]

\[ \int_0^\infty t^6 \mathcal{Y}_R(2it) \, dt = \frac{1}{65536 \Gamma(3/4)^{25}} \left( -51 \pi^{21/4} - 350 \pi^{25/4} \Gamma(3/4)^3 - 11644 \pi^{21/4} \Gamma(3/4)^{16} + 5768 \pi^{5/4} \Gamma(3/4)^{24} + 512 \pi \Gamma(3/4)^{25} \right) \quad (D.13) \]

\[ \int_0^\infty t^7 \mathcal{Y}_I(2it) \, dt = \frac{1}{262144 \Gamma(3/4)^{25}} \times \left( 357 \pi^{21/4} + 2590 \pi^{25/4} \Gamma(3/4)^3 + 93492 \pi^{21/4} \Gamma(3/4)^{16} - 54760 \pi^{5/4} \Gamma(3/4)^{24} - 1024 \pi \Gamma(3/4)^{25} \right) \quad (D.14) \]