FINITE CHEVALLEY GROUPS AND LOOP GROUPS

MASAKI KAMEKO

Abstract. Let \( p, \ell \) be distinct primes and let \( q \) be a power of \( p \). Let \( G \) be a connected compact Lie group. We show that there exists an integer \( b \) such that the mod \( \ell \) cohomology of the classifying space of a finite Chevalley group \( G(F_q) \) is isomorphic to the mod \( \ell \) cohomology of the classifying space of the loop group \( LG \) for \( q = p^a b, a \geq 1 \).

1. Introduction

Let \( p, \ell \) be distinct primes and let \( q \) be a power of \( p \). We denote by \( F_q \) the finite field with \( q \)-elements. Let \( G \) be a connected compact Lie group. There exists a reductive complex linear algebraic group \( G(\mathbb{C}) \) associated with \( G \), called the complexification of \( G \). One may consider \( G(\mathbb{C}) \) as \( \mathbb{C} \)-rational points of a group scheme over \( \mathbb{C} \) obtained by the base-change of a reductive integral affine group scheme \( G_{\mathbb{Z}} \), so-called Chevalley group scheme, with the complex analytic topology.

For a field \( k \), taking the \( k \)-rational points of the group scheme \( G_k = G_{\mathbb{Z}} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k) \) over \( k \), we have the (possibly infinite) Chevalley group

\[
G(k) = \text{Hom}_{\text{Sch}/k}(\text{Spec}(k), G_k),
\]

where \( \text{Sch}/k \) is the category of schemes over \( k \). We consider the Chevalley group \( G(k) \) as a discrete group unless otherwise is clear from the context. Denote by \( \overline{F}_p \) the algebraic closure of the finite field \( F_q \). We may consider the finite Chevalley group \( G(\overline{F}_q) \) as the fixed point set \( G(\overline{F}_p)_{\phi^q} \) where

\[
\phi^q : G(\overline{F}_p) \to G(\overline{F}_p)
\]

is the Frobenius map induced by the Frobenius homomorphism \( \phi^q : \overline{F}_p \to \overline{F}_p \) sending \( x \) to \( x^q \).

In [9], Quillen computed the mod \( \ell \) cohomology of a finite general linear group \( GL_n(F_q) \). The finite general linear group \( GL_n(F_q) \) is a finite Chevalley group associated with the unitary group \( U(n) \). We recall Quillen’s computation from the viewpoint of the the following Theorem 1.1 due to Friedlander, Friedlander-Mislin [3, Theorem 1.4].

Throughout the rest of this paper, we fix a connected compact Lie group \( G \) and associated reductive integral affine group scheme \( G_{\mathbb{Z}} \). Let \( BG^\wedge \) be the Bousfield-Kan \( \mathbb{Z}/\ell \)-completion of the classifying space \( BG \) of the connected compact Lie group \( G \). We write \( H^*(X) \), \( \tilde{H}^*(X) \) for the mod \( \ell \) cohomology, reduced mod \( \ell \) cohomology of a space \( X \), respectively. We also write \( H_*(X) \), \( \tilde{H}_*(X) \) for the mod \( \ell \) homology, reduced mod \( \ell \) homology of \( X \), respectively. We denote by \( \text{fib}(\alpha), \pi_0 : P_\alpha \to X \) the
homotopy fibre, mapping track of a map \( \alpha : A \to X \). That is, 

\[ P_\alpha = \{(a, \lambda) \in A \times X^I \mid \alpha(a) = \lambda(1)\}, \]

\( \pi_0((a, \lambda)) = \lambda(0) \) and \( \text{fib}(\alpha) = \pi_0^{-1}(*) \), where \( I = [0, 1] \) is the unit interval, \( X^I \) is the set of continuous maps from \( I \) to \( X \), \(* \) is the base-point of \( X \).

**Theorem 1.1** (Friedlander-Mislin). There exist maps

\[ D : BG(F_p) \to BG^\wedge \]

and

\[ \phi^q : BG^\wedge \to BG^\wedge \]

satisfying the following three conditions:

1. The induced homomorphism \( D^* : H^\ast(BG^\wedge) \to H^\ast(BG(F_p)) \) is an isomorphism.
2. \( \phi^q \circ D = D \circ \phi^q \) where \( \phi^q : BG(F_p) \to BG(F_p) \) is the Frobenius map induced by the Frobenius homomorphism \( \phi^q : F_p \to F_p \).
3. There exists a map \( \text{fib}(D_q) \to \text{fib}(\Delta) \) induces an isomorphism \( H^\ast(\text{fib}(\Delta)) \to H^\ast(\text{fib}(D_q)) \), where the above map is obtained from the following homotopy commutative diagram by choosing a suitable homotopy:

\[ BG(F_q) \xrightarrow{\phi^q \circ \Delta} BG^\wedge \xrightarrow{\Delta} \]

where \( D_q = D \circ i_q, i_q = BG(F_q) \to BG(F_p) \) is the map induced by the inclusion of \( F_q \) into \( F_p \) and \( \Delta : BG^\wedge \to BG^\wedge \times BG^\wedge \) is the diagonal map.

**Remark 1.2.** In [2, Proposition 8.8], Friedlander constructed a chain of maps between simplicial sets \( \text{holim} \left\langle (\mathbb{Z}/\ell)_{\infty}(BG_{\mathbb{F}_p})_{\text{et}} \right\rangle \) and \((\mathbb{Z}/\ell)_{\infty} \text{Sing}_b(BG(\mathbb{C}))\), where \( G(\mathbb{C}) \) is given the complex analytic topology. He showed that these maps are weak homotopy equivalences. We take \( BG^\wedge \) to be the geometric realization of \( \text{holim} \left\langle (\mathbb{Z}/\ell)_{\infty}(BG_{\mathbb{F}_p})_{\text{et}} \right\rangle \), so that the Forbenius map \( \phi^q : BG^\wedge \to BG^\wedge \) is induced by the map defined on \( G_{\mathbb{F}_p} \). Therefore, the map \( \phi^q \) is an automorphism and there holds

\[ \phi^q \circ \cdots \circ \phi^q = \phi^q^e. \]

We emphasize here that the equality holds in the category of sets and maps, not in the homotopy category.

**Remark 1.3.** For a discrete group \( H \), we may identify the classifying space \( BH \) with the geometric realization of \( \text{holim} \left\langle BH_k \right\rangle_{\text{et}} \), where \( k \) is an algebraically closed field and \( H_k \) is a group scheme \( H \otimes \text{Spec}(k) \). The map \( D \) in Theorem 1.1 is induced by the obvious homomorphism of group schemes

\[ G(k)_k = \text{Hom}_{\text{sch}}(\text{Spec}(k), G_k) \otimes \text{Spec}(k) \to G_k. \]

See Friedlander-Mislin [3, Section 2] for detail. Thus, we have the equality \( \phi^q \circ D = D \circ \phi^q \) in Theorem 1.1.
Now, we recall Quillen’s computation of the mod $\ell$ cohomology of $GL_n(\mathbb{F}_q)$. The first part of Quillen’s computation is the homotopy theoretical interpretation of the problem. For a map $f : X \to X$, let us define a space $L_f X$ by

$$L_f X = \{ \lambda \in X^I \mid \lambda(1) = f(\lambda(0)) \}.$$ 

We call this space the twisted loop space of $f$ following the terminology of [5]. Let $\pi_0 : L_f X \to X$ be the evaluation map at 0. Let $\pi_0 : P_\Delta \to X \times X$ be the mapping track of the diagonal map $\Delta : X \to X \times X$. Associated with the diagram in Theorem 1.1 (3), we have the following fibre square:

$$
\begin{array}{ccc}
L_f X & \xrightarrow{g} & P_\Delta \\
\downarrow{\pi_0} & & \downarrow{\pi_0} \\
X & \xrightarrow{(1 \times f) \circ \Delta} & X \times X,
\end{array}
$$

where $g$ is given by $g(\lambda) = (\lambda(0), \lambda') \in X \times (X \times X)^I$, and $\lambda'(t) = (\lambda(\frac{t}{2}), \lambda(1 - \frac{t}{2}))$. Theorem 1.1 (3) implies that $H^*(L_f X)$ is isomorphic to $H^*(BG(\mathbb{F}_q))$ for $X = BG^\wedge$, $f = \phi^\wedge$. Thus, the computation of the mod $\ell$ cohomology of a finite Chevalley group is nothing but the computation of the mod $\ell$ cohomology of the twisted loop space $L_f X$.

The second part of Quillen’s computation is the computation using the Eilenberg-Moore spectral sequence. For a twisted loop space, there exists the Eilenberg-Moore spectral sequence converging to the associated graded algebra of the mod $\ell$ cohomology of the twisted loop space $L_f X$. Let us write $A$ for $H^*(X)$. The $E_2$-term of the Eilenberg-Moore spectral sequence is given by $\text{Tor}_{A \otimes A}(A, A)$. If the induced homomorphism $f^* : A \to A$ is the identity homomorphism and if $A$ is a polynomial algebra, then the above $E_2$-term is a polynomial tensor exterior algebra $A \otimes V$ where $V = \text{Tor}_A(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$, and since as an algebra over $\mathbb{Z}/\ell$, it is generated by $\text{Tor}_{A \otimes A}^0(A, A)$ and $\text{Tor}_{A \otimes A}^{-1}(A, A)$, the spectral sequence collapses at the $E_2$-level.

On the other hand, it is well-known that there exists a homotopy equivalence between the classifying space of the loop group $L G$ and the free loop space $L BG$, where

$$L X = \{ \lambda \in X^I \mid \lambda(1) = \lambda(0) \}.$$ 

See Proposition 2.4 in [1]. For the free loop space $L BG$, we have the following fibre square:

$$
\begin{array}{ccc}
L X & \xrightarrow{\Delta} & P_\Delta \\
\downarrow{\pi_0} & & \downarrow{\pi_0} \\
X & \xrightarrow{\Delta} & X \times X,
\end{array}
$$

where $\pi_0$ is the evaluation map at 0, so that $\pi_0(\lambda) = \lambda(0)$, and $\pi_0 : P_\Delta \to X \times X$ is the mapping track of the diagonal map $\Delta : X \to X \times X$. As in the case of finite Chevalley groups, there exists the Eilenberg-Moore spectral sequence

$$\text{Tor}_{A \otimes A}(A, A) \Rightarrow \text{gr} H^*(L BG).$$
Thus, it is easy to see that if $A = H^*(BG)$ is a polynomial algebra, the $E_2$-term of the spectral sequence is equal to the polynomial tensor exterior algebra $A \otimes V = A \otimes \text{Tor}_A(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$, the spectral sequence collapses at the $E_2$-level as in the case of finite Chevalley groups. Therefore, if $H^*(BG)$ is a polynomial algebra, the mod $\ell$ cohomology of the free loop space of the classifying space $BG$ is isomorphic to the mod $\ell$ cohomology of the finite Chevalley group $G(\mathbb{F}_q)$ as a graded $\mathbb{Z}/\ell$-module.

Even if $H^*(BG)$ is not a polynomial algebra over $\mathbb{Z}/\ell$, if the induced homomorphism $\phi^{**} : A \to A$ is the identity homomorphism, $E_2$-terms of the above Eilenberg-Moore spectral sequences are the same. Observing this phenomenon, Tezuka in [10] asked the following:

**Conjecture 1.4.** If $\ell | q - 1$ (resp. $4 | q - 1$) when $\ell$ is odd (resp. even), there exists a ring isomorphism between $H^*(BG(\mathbb{F}_q))$ and $H^*(\mathcal{L}BG)$.

In conjunction with this conjecture, in this paper, we prove the following result:

**Theorem 1.5.** There exists an integer $b$ such that, for $q = p^a$ where $a$ is an arbitrary positive integer, there exists an isomorphism of graded $\mathbb{Z}/\ell$-modules

$$H^*(BG(\mathbb{F}_q)) = H^*(\mathcal{L}BG).$$

**Remark 1.6.** Although we give an example of the integer $b$ in Theorem 1.5 in Section 2 as a function of the graded $\mathbb{Z}/\ell$-module $H^*(G)$, it is not at all the best possible.

**Remark 1.7.** If $H^*(BG)$ is not a polynomial algebra, it is not easy to compute the mod $\ell$ cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$. The only computational results in the literature are the computation of the mod 2 cohomology of $B\text{Spin}(10)(\mathbb{F}_q)$ and $\mathcal{L}B\text{Spin}(10)$ in [6] and [7] for $\ell = 2$ and the mod 3 cohomology of $\mathcal{L}B\text{PU}(3)$ for $\ell = 3$ in [7].

When we want to show that the cohomology of a space $X$ is isomorphic to the cohomology of another space $Y$, we usually try to construct a chain of maps

$$X = X_0 \xleftarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\ldots} X_n \xrightarrow{f_n} X_{n+1} = Y$$

such that maps $f_k$’s induce isomorphisms in mod $\ell$ cohomology. For example, Theorem 1.1 is proved by this method. However, when we try to prove Theorem 1.5 or Conjecture 1.4, we can not construct such a chain of maps. Consider the case $G = S^1$. Then, we have $G(\mathbb{F}_q) = \mathbb{Z}/(q - 1)$, $G(\mathbb{C}) = \mathbb{C}\setminus\{0\}$. So, we have $H^*(BG(\mathbb{F}_q); \mathbb{Q}) = \mathbb{Q}$, $H^*(\mathcal{L}BG; \mathbb{Q}) = \mathbb{Q}[y] \otimes \Lambda(x)$. If there exists a chain of maps such as above, then they also induce isomorphisms of Bockstein spectral sequences. This contradicts the above observation on the rational (and integral) cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$.

Thus, in the proof of Theorem 1.5, we construct maps which induce monomorphisms among Leray-Serre spectral sequences associated with fibrations $\pi_0 : \mathcal{L}f X \to X$, $\pi_0 : \mathcal{L}X \to X$, $\pi_0 : \mathcal{L}_f X \times_X P_\alpha \to X$, where $X = BG^\alpha$, $f = \phi^\alpha$ and $\alpha : A \to X$ is a certain map we define in Section 3. By comparing the image of Leray-Serre spectral sequences, we construct an isomorphism between Leray-Serre spectral sequences associated with fibrations $\pi_0 : \mathcal{L}f X \to X$ and $\pi_0 : \mathcal{L}X \to X$. This isomorphism could not be realized by a chain of maps. We announced and outlined the proof of Theorem 1.5 in [4]. By choosing $\phi^\alpha$ and $D$ as in Remarks 1.2, 1.3, in this paper, we can give a simpler proof for Theorem 1.5.
In Section 2, we define the integer $b$ as a function of a graded $\mathbb{Z}/\ell$-module $H^*(G)$. In Section 3, we give a proof of Theorem 1.5 assuming Lemma 3.2. In Section 4, we prove Lemma 3.2.

Since there exists no map realizing the isomorphism between $H^*(BG(F_q))$ and $H^*(LBG)$, it is difficult to believe the existence of such isomorphism for arbitrary connected compact Lie group $G$. It is my pleasure to thank M. Tezuka for informing me of Conjecture 1.4. The author is partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) 19540105.

2. THE INTEGER $b$

In this section, we define the integer $b$ in Theorem 1.5. We define the integer $b$ as

$$b = e_1 \dim G e_2$$

and we define $e_1$, $e_2$ as follows: For the sake of notational simplicity, let $V = H^*(G)$.

We have isomorphisms

$$V = H^*_\text{et}(G_{\mathbb{F}_q}; \mathbb{Z}/\ell) = H^*(\text{fib}(D_q)) = H^*(\text{fib}(\Delta)) = H^*(\Omega BG')\times.$$  

Denote by $\text{GL}(V)$ be the group of automorphisms of $V$ and we also denote by $|\text{GL}(V)|$ the order of the finite group $\text{GL}(V)$. Let

$$e_1 = (\ell |\text{GL}(V)|)^{2 \dim G} \text{ and } e_2 = |\text{GL}(V \otimes V)|.$$

Before we proceed to lemmas, we set up some notations. Let us consider a commutative diagram.

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

We write $f : X^I \to Y^I$ for the map induced by $f : X \to Y$, so that

$$f(\lambda)(t) = f(\lambda(t)).$$

We also write $g \times f$ for the map $A \times X^I \to B \times Y^I$ induced by $f$, $g$ and the restriction of $g \times f : A \times X^I \to B \times Y^I$ to the mapping tracks $P_\alpha \to P_\beta$ and the homotopy fibres $\text{fib}(\alpha) \to \text{fib}(\beta)$.

Let $q' = q^e$ and $q$ is a power of $p$. The inclusion of $\mathbb{F}_q$ into $\mathbb{F}_{q'}$ induces maps

$$i : BG(\mathbb{F}_q) \to BG(\mathbb{F}_{q'})$$

and

$$i \times 1 : \text{fib}(D_q) \to \text{fib}(D_{q'}).$$

As for $e_1$, we have the following lemma. A variant of this lemma is used in the proof of Theorem 1.4 in [3].

**Lemma 2.1.** Suppose that $e$ is divisible by $e_1$. Then, the induced homomorphism

$$(i \times 1)^* : \check{H}^*(\text{fib}(D_{q'})) \to \check{H}^*(\text{fib}(D_q))$$

is zero.
The isomorphism between
\( (i \times 1)_*(x) \in \Delta_*((i \times 1)_*(x)) = 1 \otimes (i \times 1)_*(x) + \sum (i \times 1)_*(y') \otimes (i \times 1)_*(y'') + (i \times 1)_*(x) \otimes 1, \)
where \( \deg y' < \deg x \) or \( \deg y'' < \deg x \). Hence, if \( (i \times 1)_*(y) = 0 \) for \( \deg y < \deg x \), then we have
\[ \Delta_*((i \times 1)_*(x)) = 1 \otimes (i \times 1)_*(x) + (i \times 1)_*(x) \otimes 1. \]
So, if \( (i \times 1)_*(y) = 0 \) for \( \deg y < \deg x \), \( (i \times 1)_*(x) \) is primitive.

The Frobenius map \( \phi^q \) is an element of the Galois group \( \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \), the induced homomorphism \( \phi^q \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) is given by (1) and the Frobenius map \( \phi^q : \mathbb{F}_p \rightarrow \mathbb{F}_p \) defined by \( (1/\phi^q)(y) = g \cdot (\phi^q(g))^{-1} \). Thus, the map \( (i \times 1)_* \) corresponds to a homomorphism \( \theta^q/\phi^q : G(\mathbb{F}_p) \rightarrow G(\mathbb{F}_p) \) given by the diagram
\[ \begin{array}{ccc} G(\mathbb{F}_p) / G(\mathbb{F}_q) & \xrightarrow{\pi} & G(\mathbb{F}_p) / G(\mathbb{F}_q) \\ 1/\phi^q \downarrow & & \downarrow 1/\phi^q' \\ G(\mathbb{F}_p) & \xrightarrow{\theta^q/\phi^q} & G(\mathbb{F}_p), \end{array} \]
where \( \pi \) is the obvious projection. In other words, \( \theta^q/\phi^q \) is given by
\[ \theta^q/\phi^q(g) = g \cdot (\phi^q(g)) \cdots (\phi^q)^{-1}(g). \]
Thus, \( (i \times 1)_*(x) \) is given by
\[ (i \times 1)_*(x) = (\mu \circ (1 \times \phi^q \times \cdots \times \phi^{q_n-1}) \circ \Delta)_*(x) \]
If \( x \) is primitive, we have
\[ (i \times 1)_*(x) = \sum_{t=0}^{e-1} (\theta^q)^t_* (x) = (e/m) \cdot \left( \sum_{t=0}^{m-1} (\theta^q)^t_* (x) \right) = 0. \]
Thus, if \( e \) is divisible by \( (\ell \cdot m)^2 \) and if \( (i \times 1)_*(y) = 0 \) for \( \deg y < j \), then \( (i \times 1)_*(x) = 0 \) for \( \deg x \leq j \). Therefore, if \( G \) is connected and if \( e \) is divisible by \( (\ell \cdot m)^2 \), the induced homomorphism
\[ \tilde{H}^*(\text{fib}(D_q)) \rightarrow \tilde{H}^*(\text{fib}(D_q)) \]
is zero up to degree \( k \). Let \( k = \dim G \). Since, by definition, \( H^j(\text{fib}(D_q)) = \{0\} \) for \( j > k \), we have Lemma 2.1. \( \square \)

As for the integer \( e_2 \), we prove the following:
Lemma 2.2. Suppose that $e$ is divisible by $e_2$. Then, the induced homomorphism
\[(1 \times (1 \circ \phi^q))^* : H^*(\text{fib}(\Delta \circ D_q)) \to H^*(\text{fib}(\Delta \circ D_q))\]
is the identity homomorphism.

Proof. Let $p_i : BG^\wedge \times BG^\wedge \to BG^\wedge$ be the projections onto the first and second factors for $i = 1, 2$. Consider the diagram

\[
\begin{array}{ccccccc}
\Omega BG^\wedge & \xrightarrow{=} & \Omega BG^\wedge & \xrightarrow{\Omega \Delta} & \Omega(BG^\wedge \times BG^\wedge) \\
\downarrow & & \downarrow & & \downarrow \\
\text{fib}(\Delta \circ D_q) & \xrightarrow{1 \times p_2} & \text{fib}(D_q) & \xrightarrow{D_q \times \Delta} & \text{fib}(\Delta) \\
\downarrow & & \downarrow & & \downarrow \\
\text{fib}(D_q) & \xrightarrow{p_1} & BG(F_q) & \xrightarrow{D_q} & BG^\wedge \\
\end{array}
\]

The induced homomorphism
\[(\Omega \Delta)_* : H_*(\Omega BG^\wedge) \to H_*(\Omega(BG^\wedge \times BG^\wedge))\]
is a monomorphism and $\pi_1(BG^\wedge) = \{0\}$. So, $\pi_1(BG(F_q))$ acts trivially on $H_*(\Omega BG^\wedge)$. Therefore, $\pi_1(\text{fib}(D_q))$ also acts trivially on $H_*(\Omega BG^\wedge)$. Thus, the local coefficient of the induced fibre sequence
\[\Omega BG^\wedge \to \text{fib}(\Delta \circ D_q) \xrightarrow{1 \times p_2} \text{fib}(D_q)\]
is trivial. Hence, the $E_2$-term of the Leray-Serre spectral sequence for the cohomology of $\text{fib}(\Delta \circ D_q)$ is given by
\[V \otimes V = H^*(\text{fib}(D_q)) \otimes H^*(\Omega BG^\wedge).\]

Therefore, we have that
\[\dim_{\mathbb{Z}/\ell} H^*(\text{fib}(\Delta \circ D_q)) \leq \dim_{\mathbb{Z}/\ell}(V \otimes V).\]

Since the map $\phi^q : BG^\wedge \to BG^\wedge$ is an automorphism, the induced map
\[1 \times (1 \circ \phi^q) : \text{fib}(\Delta \circ D_q) \to \text{fib}(\Delta \circ D_q)\]
is also an automorphism. Since
\[\dim_{\mathbb{Z}/\ell} H^*(\text{fib}(\Delta \circ D_q)) \leq \dim_{\mathbb{Z}/\ell}(V \otimes V),\]
$e_2$ is divisible by the order of
\[(1 \times (1 \circ \phi^q))^* : H^*(\text{fib}(\Delta \circ D_q)) \to H^*(\text{fib}(\Delta \circ D_q)).\]
Hence, if $e$ is divisible by $e_2$, we have
\[(1 \times (1 \circ \phi^q))^* = ((1 \times (1 \circ \phi^q))^e = 1.\]
3. Proof of Theorem 1.5

Let $X$ be a space and let $f : X \to X$ be a self-map of $X$ with a non-empty fixed point set. Let $\alpha : A \to X$ be a map such that

$$f \circ \alpha = \alpha.$$ 

We choose a base-point $*$ in $A, X$, so that both $f, \alpha$ are base-point preserving.

Firstly, we define a map

$$\varphi : \mathcal{L}_f X \times X \mathcal{L}_f X \to \mathcal{L}X,$$

where

$$\mathcal{L}_f X \times X \mathcal{L}_f X = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times \mathcal{L}_f X \mid \lambda_1(0) = \lambda_2(0)\}.$$  

The map $\varphi$ is defined by

$$\varphi(\lambda_1, \lambda_2)(t) = \begin{cases} 
\lambda_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\
\lambda_2(2 - 2t) & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Since $\lambda_1(1) = f(\lambda_1(0)), \lambda_2(1) = f(\lambda_2(0))$ and $\lambda_1(0) = \lambda_2(0)$, this map is well-defined.

Next, we define a map from $P_\alpha$ to $\mathcal{L}_f X$, say $\psi : P_\alpha \to \mathcal{L}_f X$, by

$$\psi((a, \lambda))(t) = \begin{cases} 
\lambda(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\
f(\lambda(2 - 2t)) & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Since $\lambda(1) = f(\lambda(1))$, this map is also well-defined.

Now, we consider the following diagram:

$$\xymatrix{ 
\mathcal{L}_f X \ar[r]^{p_1} \ar[d]_{1 \times \psi} & \mathcal{L}_f X \times X \mathcal{L}_f X \ar[r]^{\varphi} & \mathcal{L}X \\
\mathcal{L}_f X \times X P_\alpha \ar[ru]_{\psi^*} & & \mathcal{L}X \ar[lu]_{\psi^*}
}$$

where

$$\mathcal{L}_f X \times X P_\alpha = \{(\lambda_1, (a, \lambda_2)) \in \mathcal{L}_f X \times P_\alpha \mid \lambda_1(0) = \lambda_2(0), \alpha(a) = \lambda_2(1)\},$$

$p_1$ is the projection onto the first factor and $\pi_0 \circ p_1 = \pi_0, \pi_0 \circ \varphi = \pi_0, \pi_0 \circ (1 \times \psi) = \pi_0$. Let us denote by $E_r(Y)$ the Leray-Serre spectral sequence associated with a fibration $\xi : Y \to X$. Then we have the following diagram of spectral sequences:

$$\xymatrix{ 
E_r(\mathcal{L}_f X) \ar[r]^{p_1^*} \ar[d]_{1 \times \psi^*} & E_r(\mathcal{L}_f X \times X \mathcal{L}_f X) \ar[r]^{\varphi^*} & E_r(\mathcal{L}X) \\
E_r(\mathcal{L}_f X \times X P_\alpha) & & \mathcal{L}X \ar[lu]_{\psi^*}
}$$

By abuse of notation, we denote by $\psi : \text{fib}(\alpha) \to \Omega X$ the restriction of $\psi : P_\alpha \to \mathcal{L}_f X$ to fibres. Let us consider a sufficient condition for the induced homomorphism

$$\psi^* : \tilde{H}^*(\Omega X) \to \tilde{H}^*(\text{fib}(\alpha))$$
to be zero. Again, by abuse of notation, we denote by \(\varphi : \Omega X \times \Omega X \to \Omega X\) the restriction of \(\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \to \mathcal{L}X\) to fibres.

**Lemma 3.1.** If the induced homomorphism

\[(1 \times (1 \times f))^* : H^*(\text{fib}(\Delta \circ \alpha)) \to H^*(\text{fib}(\Delta \circ \alpha))\]

is the identity homomorphism, then the induced homomorphism

\[\psi^* : \tilde{H}^*(\Omega X) \to \tilde{H}^*(\text{fib}(\alpha))\]

is zero.

**Proof.** The map \(\psi : \text{fib}(\alpha) \to \Omega X\) factors through

\[\text{fib}(\alpha) \xrightarrow{1 \times \Delta} \text{fib}(\Delta \circ \alpha) \xrightarrow{1 \times (1 \times f)} \text{fib}(\Delta \circ \alpha) \xrightarrow{\varphi} \Omega X.\]

It is clear that the composition \(\varphi \circ \Delta\) is null homotopic since an obvious null homotopy \(h_s\) is given by

\[h_s((a, \lambda))(t) = \begin{cases} 
\lambda(2st) & \text{for } 0 \leq t \leq \frac{1}{2}, \\
\lambda(2s - 2st) & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}\]

Therefore, we have

\[(1 \times \Delta)^*((1 \times (1 \times f))^* (\varphi^*(x))) = (1 \times \Delta)^*(\varphi^*(x)) = 0.\]

for \(x \in \tilde{H}^*(\Omega X)\).

We also need the following lemmas in the proof of Theorem 1.5.

**Lemma 3.2.** Suppose that \(X\) is simply connected, that \(H^i(\text{fib}(\alpha)) = 0\) for \(i > k\) and that there exists a sequence of maps

\[A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \longrightarrow \cdots \longrightarrow A_k \xrightarrow{i_k} A \xrightarrow{\alpha} X\]

such that the induced homomorphism \(\tilde{H}^*(\text{fib}(\alpha)) \to \tilde{H}^*(\text{fib}(\alpha_{j-1}))\) is zero for \(j = 1, 2, \ldots, k\). Then the projection on the first factor \(p_1 : Y \times_X P_\alpha \to Y\) induces a monomorphism \(p_1^* : E_r^e(Y) \to E_r(\text{fib}(\alpha))\) of Leray-Serre spectral sequences for arbitrary fibration \(\xi : Y \to X\).

We need the following lemma to compare the spectral sequences.

**Lemma 3.3.** Let

\[E_r' \xrightarrow{\rho'_r} E_r \xleftarrow{\rho''_r} E_r''\]

be homomorphisms of spectral sequences. Suppose that

1. \(\text{Im} \rho'_2 = \text{Im} \rho''_2\),
2. \(\rho'_r\) is a monomorphism for \(r \geq 2\).

Then, there exists a unique homomorphism of spectral sequences

\[\{\tau_r : E_r'' \to E_r' \mid r \geq 2\}\]

such that \(\rho'_r \circ \tau_r = \rho''_r\) for \(r \geq 2\). In particular, if \(\rho''_2\) is also a monomorphism, then \(\rho'_r\) is a monomorphism and \(\tau_r\) is an isomorphism for \(r \geq 2\).
Proof. We define $\tau_2(x'')$ by

$$\rho'_2(\tau_2(x'')) = \rho''_2(x'').$$

Since $\mathrm{Im} \rho'_2 = \mathrm{Im} \rho''_2$ and $\rho'_2$ is a monomorphism, it is well-defined and we have

$$\rho'_2 \circ \tau_2 = \rho''_2$$

at the $E_2$-level. Suppose that we have

$$\rho' \circ \tau_r = \rho''_r.$$

Then, we want to show that

$$d'_r(\tau_r(x'')) = \tau_r(d''_r(x'')).$$

Since $\rho'_r$ is a monomorphism, it suffices to show that

$$\rho'_r(d'_r(\tau_r(x''))) = \rho'_r(\tau_r(d'_r(x''))).$$

It is easily verified as follows:

$$\rho'_r(d'_r(\tau_r(x''))) = d_r(\rho'_r(\tau_r(x'')))$$

$$= d_r(\rho''_r(x''))$$

$$= \rho''_r(d'_r(x''))$$

$$= \rho'_r(\tau_r(d'_r(x''))).$$

Then, $\tau_r$ induces a homomorphism

$$\tau_{r+1} : E''_{r+1} \rightarrow E'_{r+1}$$

such that

$$\rho'_{r+1} \circ \tau_{r+1} = \rho''_{r+1}.$$ 

Continue this process, we have a homomorphism of spectral sequence

$$\tau_r : E''_r \rightarrow E'_{r}$$

for $r \geq 2$. It is clear that if $\rho''_2$ is a monomorphism, then $\tau_2$ is an isomorphism. It is also clear from the construction that $\rho''_r$ is a monomorphism for $r \geq 2$ and $\tau_r$ is an isomorphism for $r \geq 2$. \hfill \Box

Now, we prove Theorem 1.5 assuming Lemma 3.2.

Proof of Theorem 1.5. Let $k = \dim G$. Let $q_j = p^{\sigma_j}$ for $j = 0, \ldots, k$. Let $q = p^{ab} = q_k^{2\sigma_2} (a \geq 1)$. Let $X = BG^n$, $A = BG(\mathbb{F}_{q_k})$, $\alpha = D_{q_k}$, $f = \phi_\alpha$, $A_j = BG(\mathbb{F}_{q_j})$ and $\alpha_j = D_{q_j}$ for $j = 0, 1, \ldots, k$. In order to prove Theorem 1.5, we consider the Leray-Serre spectral sequence $E_\cdot(L_f X)$, $E_\cdot(L.X)$ and establish an isomorphism of spectral sequences $\tau : E_\cdot(L.X) \rightarrow E_\cdot(L_f X)$.

By Lemma 2.1, we have that the induced homomorphism

$$\tilde{H}^\cdot(\text{fib}(\alpha_j)) \rightarrow \tilde{H}^\cdot(\text{fib}(\alpha_{j-1}))$$

is zero for $j = 1, \ldots, k$. By Lemma 3.2, we have a monomorphism

$$(1 \times \psi)^* \circ \rho^*_\alpha : E_\cdot(L_f X) \rightarrow E_\cdot(L_f X \times_X P_\alpha).$$

The fibres of fibrations $\pi_0 : L_f X \rightarrow X$, $\pi_0 : L.X \rightarrow X$, $\pi_0 : L_f X \times_X P_\alpha \rightarrow X$ are $\Omega X$, $\Omega X$, respectively. Identifying the $E_2$-terms $E_2(L_f X)$,
Theorem 2.1. Let \( X \) be a CW complex and denote its \( n \)-skeleton by \( X^{(n)} \). We denote \( \eta^{-1}(X^{(n)}) \subset Y \) by \( F_nY \) for \( n < 0 \). For the sake of notational simplicity, we let
\[
\begin{align*}
M_{m,n}(Y) &= H^*(F_mY, F_nY), \\
M_{m,n}(Z) &= H^*(F_mZ, F_nZ),
\end{align*}
\]
respectively. We denote by \( E_r(Y), E_r(Z) \) the Leray-Serre spectral sequences associated with fibrations \( \xi, \xi \circ \eta \), respectively.

Lemma 4.1. Suppose that for \( m \geq n \geq 0 \), the induced homomorphism
\[
\eta^* : M_{m,n}(Y) \longrightarrow M_{m,n}(Z)
\]
is a monomorphism. Then, the induced homomorphism
\[
\eta^* : E_r(Y) \longrightarrow E_r(Z)
\]
is also a monomorphism for \( r \geq 2 \).

Proof. Let us consider the following diagram:

\[
\begin{array}{ccc}
H^*(Y, F_{s+r}Y) & \xrightarrow{j_r} & M_{s+r,s}(Y) \\
\downarrow & & \uparrow \delta_r \\
H^*(Y, F_{s-1}Y) & \xrightarrow{i_1} & M_{s-1,s}(Y) \\
\downarrow \delta_{s-1} & & \downarrow \delta_1 \\
M_{s-1, s-r}(Y) & \xrightarrow{j_r} & H^*(Y, F_{s+r}Y).
\end{array}
\]

Let \( Z_r(Y) = \text{Ker} \delta_r \) and \( B_r(Y) = \text{Im} \delta_r \). Then, there holds \( E_r(Y) = Z_r(Y)/B_r(Y) \) for \( r \geq 2 \). See standard text books, for instance, McLeary’s book [8], for detail. We consider the same diagram and \( Z_r(Z), B_r(Z) \) for \( Z \) in the same manner. Then, we
have $E_r(Z) = Z_r(Z)/B_r(Z)$. Thus, in order to prove the injectivity of the induced homomorphism $\eta^*: E_r(Y) \to E_r(Z)$, it suffices to show that

$$\eta^*(Z_r(Y)) \cap B_r(Z) = \eta^*(B_r(Y)),$$

that is, if $\eta^*(x) \in \text{Im} \delta^r_\eta$ in $M_{s,s-1}(Z)$, then $x \in \text{Im} \delta^r_\eta$ in $M_{s,s-1}(Y)$. We consider the following diagram:

$$
\begin{array}{ccc}
M_{s-1,s-r}(Z) & \xrightarrow{\delta^r_\eta} & M_{s,s-1}(Z) \\
\downarrow\eta & & \downarrow\eta \\
M_{s-1,s-r}(Y) & \xrightarrow{\delta^r_\eta} & M_{s,s-1}(Y) \\
\end{array}
$$

where horizontal sequences are cohomology long exact sequences associated with triples $(F_sZ, F_{s-1}Z, F_{s-r}Z)$, $(F_sY, F_{s-1}Y, F_{s-r}Y)$. Suppose $\eta^*(x) \in \text{Im} \delta^r_\eta$, then $\delta^r_\eta(\eta^*(x)) = 0$. So, we have that $\eta^*(\delta^r_\eta(x)) = 0$. Since $\eta^*$ is a monomorphism, we have $\eta^*(x) = 0$. Hence, we have $x \in \text{Im} \delta^r_\eta$. This completes the proof. \hfill $\square$

Now, we complete the proof of Lemma 3.2. Recall that $P_{\alpha_j} \to X$ is the fibration with the fibre $\text{fib}(\alpha_j)$. Let $Y_j = Y \times_X P_{\alpha_j}$ for $j = 0, \ldots, k$. There is a sequence of fibrations and fibre maps over $X$,

$$
Y_0 \to Y_1 \to \cdots \to Y_k \to Y \times_X P_{\alpha_j} \to Y,
$$

where $i_j = 1 \times i_j \times 1$ for $j = 0, \ldots, k$. We denote the projection onto the first factor from $Y_j$ to $Y$ by $\eta_j$. The following diagram is a fibre square and the fibre of $\eta_j$ is also $\text{fib}(\alpha_j)$.

$$
\begin{array}{ccc}
Y_j & \xrightarrow{\eta_j} & Y \\
\downarrow{\xi} & & \downarrow{\pi_0} \\
Y & \xrightarrow{\pi} & X.
\end{array}
$$

**Proposition 4.2.** Let $Y'' \subset Y' \subset Y$ be subspaces of $Y$ and let $Y''_j = \eta_j^{-1}(Y'')$, $Y'_j = \eta_j^{-1}(Y')$ be subspaces of $Y_j$. Suppose that $X$ is simply connected and that $H^i(\text{fib}(\alpha)) = \{0\}$ for $i > k$. The induced homomorphism

$$
H^*(Y', Y'') \to H^*(Y' \times_X P_{\alpha}, Y'' \times_X P_{\alpha})
$$

is a monomorphism.

**Proof.** We have relative fibrations

$$
\text{fib}(\alpha) \to (Y' \times_X P_{\alpha}, Y'' \times_X P_{\alpha}) \to (Y', Y'') \quad \text{and} \quad \text{fib}(\alpha_j) \to (Y'_j, Y''_j) \to (Y', Y'')
$$

for $j = 0, \ldots, k$.

There exist associated Leray-Serre spectral sequences

$$
E_r(Y' \times_X P_{\alpha}, Y'' \times_X P_{\alpha}) \quad \text{and} \quad E_r(Y'_j, Y''_j),
$$

converging to

$$
\text{gr} H^*(Y' \times_X P_{\alpha}, Y'' \times_X P_{\alpha}) \quad \text{and} \quad \text{gr} H^*(Y'_j, Y''_j),
$$

for $j = 0, \ldots, k$, respectively. The fundamental group $\pi_1(Y)$ acts on $H_*(\text{fib}(\alpha_j))$. This action factors through $\pi_1(X) = \{0\}$. Therefore, the action of $\pi_1(Y)$ on
Suppose that \( H_*(\text{fib}(\alpha_j)) \) is trivial. Hence, its action on \( H^*(\text{fib}(\alpha_j)) \) is also trivial. Thus, the \( E_2 \)-term of the Leray-Serre spectral sequence associated with the relative fibration

\[ \text{fib}(\alpha_j) \to (Y'_j, Y''_j) \to (Y', Y''), \]

is

\[ E_2(Y'_j, Y''_j) = H^*(Y', Y'') \otimes H^*(\text{fib}(\alpha_j)). \]

Suppose that \( d_{r_1}(y) = y \) in \( E^{s,0}_{r_1}(Y' \times_X P_\alpha, Y'' \times_X P_\alpha) \) for some \( y \in E^{s-r_1,r_1-1}_{r_1}(Y' \times_X P_\alpha, Y'' \times_X P_\alpha) \). Let \( z_1 = i'^*_{k,1}(z) \) and \( y_k = i'^*_{k,1}(y) \). Since \( r_k-1 > 0 \), we have \( i'_{k-1,*}(y_k) = 0 \). Therefore, \( i'_{k-1,*}(z_1) \) in \( E^{s,0}_{r_k}(Y'_{k-1}, Y''_{k-1}) \) is also zero. So, for some \( r_{k-1} < r_k, i'^*_{k-1,*}(z_k) \) in \( E^{s,0}_{r_{k-1}}(Y'_{k-1}, Y''_{k-1}) \) must be hit, that is, there exists \( y_{k-1} \) in \( E^{s-r_{k-1},r_{k-1}-1}_{r_{k-1}}(Y'_{k-1}, Y''_{k-1}) \) such that \( d_{r_{k-1}}(y_{k-1}) = i'^*_{k-1,*}(z_k) \) in \( E^{s,0}_{r_{k-1}}(Y'_{k-1}, Y''_{k-1}) \).

Continuing this process, we have a sequence of integers

\[ 2 \leq r_0 < r_1 < \cdots < r_k. \]

Hence, we have \( r_k \geq k+2 \). However, \( d_r = 0 \) for \( r \geq k+2 \) in

\[ E_r(Y' \times_X P_\alpha, Y'' \times_X P_\alpha). \]

It is a contradiction. So, each element in \( H^*(Y', Y'') \) is not hit in \( E_r(Y' \times_X P_\alpha, Y'' \times_X P_\alpha) \) for \( r \geq 2 \). In other words, it is a permanent cocycle. Therefore, the induced homomorphism

\[ H^*(Y', Y'') \to H^*(Y' \times_X P_\alpha, Y'' \times_X P_\alpha) \]

is a monomorphism.

By Proposition 4.2, we have that the induced homomorphism

\[ \eta^*: M_{m,n}(Y) \to M_{m,n}(Z) \]

is a monomorphism for \( Z = Y \times_X P_\alpha, \eta = p_1: Y \times_X P_\alpha \to Y \). Thus, Lemma 4.1 completes the proof of Lemma 3.2.

References

[1] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615.
[2] E. M. Friedlander, Étale homotopy of simplicial schemes, Ann. of Math. Stud., 104, Princeton Univ. Press, Princeton, N.J., 1982.
[3] E. M. Friedlander and G. Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups, Comment. Math. Helv. 59 (1984), no. 3, 347–361.
[4] M. Kameko, On the cohomology of finite Chevalley groups and free loop spaces, Sūrikaisekikyōshō Kōkyūroku No. 1057 (1998), 54–55.
[5] D. Kishimoto, Cohomology of twisted loop spaces, preprint.
[6] S. N. Kleinerman, The cohomology of Chevalley groups of exceptional Lie type, Mem. Amer. Math. Soc. 39 (1982), no. 268, viii+82 pp.
[7] K. Kuribayashi, M. Mimura and T. Nishimoto, Twisted tensor products related to the cohomology of the classifying spaces of loop groups, Mem. Amer. Math. Soc. 180 (2006), no. 849, vi+85 pp.
[8] J. McCleary, A user’s guide to spectral sequences, Second edition, Cambridge Univ. Press, Cambridge, 2001.
[9] D. Quillen, On the cohomology and \( K \)-theory of the general linear groups over a finite field, Ann. of Math. (2) 96 (1972), 552–586.
[10] M. Tezuka, On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces, Sūrikaisekikyōshō Kōkyūroku No. 1057 (1998), 54–55.
Department of Mathematics, Faculty of Contemporary Society, Toyama University of International Studies, 65-1 Higashikuromaki, Toyama, 930-1292, Japan

E-mail address: kameko@tuins.ac.jp