LOG-SCALE EQUIDISTRIBUTION OF ZEROS OF QUANTUM ERGODIC EIGENSECTIONS

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Abstract. Under suitable hypotheses, a symplectic map can be quantized as a sequence of unitary operators acting on the $N$th powers of a positive line bundle over a Kähler manifold. We show that, if in addition the symplectic map satisfies an exponential growth condition and has exponential decay of correlations, then there exists a density one subsequence of eigensections whose masses and zeros become equidistributed in balls of logarithmically shrinking radii of lengths $|\log N|^{-\gamma}$ for some constant $\gamma > 0$ independent of $N$.

1. Introduction

This article is concerned with the equidistribution of masses and of zeros of holomorphic eigensections at the logarithmic scale. Let $(L, h) \to (M, \omega)$ be a prequantum line bundle over a Kähler manifold of complex dimension $m$. In other words, $(L, h)$ is a positive Hermitian line bundle with $c_1(h) = \omega$. Let $(L^N, h^N)$ denote the $N$th tensor power. Under certain quantization conditions (discussed in §2.3 and [22]), a symplectic map

$$\chi: (M, \omega) \to (M, \omega), \quad \chi^* \omega = \omega$$
on

on the base manifold can be quantized as a sequence $\{U_{\chi,N}\}_{N=1}^\infty$ of unitary Fourier integral Toeplitz operators

$$U_{\chi,N}: H^0(M, L^N) \to H^0(M, L^N)$$
on

acting on the spaces $H^0(M, L^N)$ of holomorphic sections of $L^N$ with the inner product induced by $h$ (see [2]).

The eigensections $s_j^N \in H^0(M, L^N)$ of the operators $U_{\chi,N}$ are characterized by

$$U_{\chi,N}s_j^N = e^{i\theta_{N,j}}s_j^N, \quad 1 \leq j \leq d_N,$$

where $e^{i\theta_{N,j}}$ are eigenphases and $d_N = \dim H^0(M, L^N)$. We write

$$Z_{s_j^N} = \{z \in M: s_j^N(z) = 0\} \quad \text{and} \quad [Z_{s_j^N}] = \frac{i}{\pi} \int_M |s_j^N(z)|^2 h$$

for the zero set of $s_j^N$ and the current of integration over the zero set of $s_j^N$, respectively. Assuming $\chi$ is ergodic, [NV] and [ShZ1] (see also [R] for the modular surface setting) proved that the eigensections of the quantum maps $U_{\chi,N}$ are quantum ergodic: There exists a subsequence $\Gamma \subset \{(N, j): N \geq 1, j = 1, \ldots, d_N\}$ of density one for which

$$\lim_{(N,j) \to \Gamma} \int_M f(z) \left[ \frac{1}{N} Z_{s_j^N} \right] \wedge \omega^{m-1} = \int_M f \frac{\omega^m}{m!}.$$
In other words, the zero sets of $s_j^N$ become equidistributed with respect to the Kähler volume form.

1.1. **Statement of main results.** Recall that $m = \dim_{\mathbb{C}} M$. We fix a logarithmic scale $\varepsilon_N$ depending on parameter $\gamma$:

$$\varepsilon_N := |\log N|^{-\gamma}$$

for some constant $0 < \gamma < \frac{1}{6m}$ independent of $N$.

The main purpose of this paper is to show (with additional assumptions on $\chi$, described below) that the equidistribution result (1) holds with the domain of integration $M$ replaced by any ball $B(p, \varepsilon_N)$ centered at $p \in M$ with radius $\varepsilon_N = |\log N|^{-\gamma}$ for any $\gamma < (6m)^{-1}$. This is what is meant by “equidistribution of zeros at the logarithmic scale.”

To obtain this log-scale improvement, $\chi$ is assumed throughout to satisfy the following two dynamical conditions.

- For $T \in \mathbb{Z}$, let $\chi^T$ denote the $T$-fold iterate of $\chi$ (or of its inverse $\chi^{-1}$, depending on the sign of $T$). We assume $\chi$ satisfies the exponential growth estimate

$$\|\chi^T\|_{C^2} = O(e^{T\delta_0})$$

for some fixed constant $\delta_0 > 0$ independent of $T$.

- We further assume that $\chi$ has exponential decay of correlations. That is, for each $0 < \beta < 1$, there exist constants $c_1, c_2 > 0$ depending only on $\beta$ such that

$$\left| \int_M (g \circ \chi^T) f \, dV - \int_M f \, dV \int_M g \, dV \right| \leq c_1 e^{-c_2 |T|} \|f\|_{C^{0,\beta}} \|g\|_{C^{0,\beta}}$$

for all $f, g \in C^{\beta}(M)$. Note that, in particular, $\chi$ is ergodic.

The explicit error estimate in Egorov’s theorem for Toeplitz operators (Proposition 3.1 proved in Appendix A) relies on assumption (3). Assumption (4) is used to apply the variance estimate of [L] in the proof of logarithmic decay of quantum variances (Theorem 4) in §3.

1.1.1. **Log-scale equidistribution of zeros.** The log-scale equidistribution of zeros states that zeros in balls of radii $\varepsilon_N$ are uniformly distributed with respect to the volume form $dV_\omega$. It is simplest to state the result by dilating such shrinking balls by $\varepsilon_N^{-1}$ back to a fixed reference ball of radius 1. In a local Kähler normal coordinate chart $(U, z)$ with $z = 0$ at $p$, define local dilation maps

$$D^p_{\varepsilon} : B(p, 1) \to B(p, \varepsilon), \quad D_{\varepsilon}z = \varepsilon z.$$

Here we abuse notation by writing $B(p, 1)$ when we mean the image of the metric unit ball centered at $p$ in the local coordinate chart based at $p$. The inverse dilation is defined by

$$(D^p_{\varepsilon})^{-1} : B(p, \varepsilon) \to B(p, 1).$$

Let $D^p_{\varepsilon*}$ be the corresponding pullback operator on forms. For simplicity of notation we denote the pullback $(D^p_{\varepsilon*})^{s-1}$ of the inverse dilation by $D^p_{\varepsilon*}$ so that

$$D^p_{\varepsilon*} : \mathcal{D}^{m-1,m-1}(B(p, 1)) \to \mathcal{D}^{m-1,m-1}(B(p, \varepsilon)).$$
where $D^{m-1, m-1}$ denotes the space of compactly supported smooth $(m - 1, m - 1)$ test forms. In particular, for $\eta \in D^{m-1, m-1}(B(p, 1))$, we have

$$\int_{B(p, \varepsilon)} D^p_{\varepsilon} \eta \wedge \frac{1}{N} [Z_{s_j^N}] = \int_{B(p, 1)} \left( \eta \wedge \frac{1}{N} D^p_{\varepsilon} [Z_{s_j^N}] \right).$$

**Theorem 1.** Let $(L, h) \to (M, \omega)$ be a prequantum line bundle. Let $\chi$ satisfy (3) and (4). Let \{s_1^N, \ldots, s_d_N^N\} be an orthonormal basis of eigensections of $U_{\chi, N} : H^0(M, L^N) \to H^0(M, L^N)$. Then, for every $0 < \gamma < (6m)^{-1}$ and $\varepsilon_N = |\log N|^{-\gamma}$, there exists a full density subsequence $\Gamma \subset \{(N, j) : j = 1, \ldots, d_N\}$ such that for every $p \in M$,

$$\frac{1}{N \varepsilon_N^2} D^p_{\varepsilon} [Z_{s_j^N}] \rightarrow_{\Gamma \ni (N, j) \to \infty} \omega_0^p$$

in the weak sense of currents on $B(p, 1)$,

where $\omega_0^p = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |z|^2$ is the flat Kähler form in Kähler normal coordinates based at $p$.

**Remark 1.1.** The weak convergence statement in Theorem 1 means that for every test form $\eta \in D^{m-1, m-1}(B(p, 1))$, one has

$$\int_{B(p, 1)} \left( \eta \wedge \frac{1}{N \varepsilon_N} D^p_{\varepsilon} [Z_{s_j^N}] \right) = \int_{B(p, 1)} \eta \wedge \omega_0^p + o(1).$$

The key ingredients of the proof are the log-scale mass comparison result (Theorem 2), the Poincaré-Lelong formula (7) and compactness results on logarithms of scaled sections. This equidistribution result should be compared to Lester-Matomäki-Radziwill [LMR, Theorem 1.1] for a sequence \{f_k\} of Hecke modular cusp forms of weight $k$. They proved that for a certain $\delta > 0$,

$$\# \{z \in B(z_0, r) : f_k(z) = 0\} = \frac{3}{\pi} \int_{B(z_0, r)} \frac{dx_1 dy}{y^2} + O \left( r (\log k)^{-\delta + \varepsilon} \right)$$

when $r \geq (\log k)^{-\delta/2 + \varepsilon}$. This is a quantum unique ergodicity result in that it holds for the entire orthonormal basis of Hecke eigenforms, whereas we discard a density zero subsequence of eigensections because we work in the more general setting of dynamical Toeplitz quantizations of quantizable ergodic symplectic maps.

1.1.2. Log-scale equidistribution of mass. The equidistribution result of Theorem 1 is based on log-scale volume comparison theorems similar to those of [HR, Lemma 3.1] and [Ha, Corollary 1.9].

**Theorem 2** (Log-scale equidistribution of masses). Assume the hypotheses of Theorem 1. Then, given any $0 < \gamma' < (6m)^{-1}$ and $\varepsilon_N' = |\log N|^{-\gamma'}$ as defined by (2), there exist a full density subsequence $\Gamma$ and constants $C_1, C_2$ uniform in $p \in M$ and independent of $N$ such that

$$C_1 \frac{\text{Vol}(B(p, \varepsilon_N'))}{\text{Vol}(M)} \leq \int_{B(p, \varepsilon_N')} ||s_j^N||^2_{h} dV \leq C_2 \frac{\text{Vol}(B(p, \varepsilon_N'))}{\text{Vol}(M)}$$

as $\Gamma \ni (N, j) \to \infty$.

There is no need to put primes on $\gamma$ or $\varepsilon_N$ in the statement above, but we do so to foreshadow that in the proof of Theorem 1 the result of Theorem 2 is applied with $\gamma < \gamma'$ and $\varepsilon_N' < \varepsilon_N$. The comparison (as opposed to asymptotic) result on log-scale mass equidistribution is sufficient for deriving equidistribution of zeros at a slightly larger logarithmic scale.
In fact, only the lower bound is used, and the bound itself is much stronger than necessary for the proof.

Theorem 2 is based on a quantitative quantum variance estimate (Theorem 4) in the holomorphic setting. Before stating the estimate, we record here another one of its corollaries, which is analogous to [Ha, Corollary 1.8].

**Proposition 3.** Assume the hypotheses of Theorem 1. Fix \( z_0 \in M \). Then, given any \( 0 < \gamma < (4m)^{-1} \) and \( \varepsilon_N \) as defined by (2), there exists a subsequence \( \Gamma_{z_0} \subset \{(N,j)\} \) of density one such that

\[
\int_{B(z_0,\varepsilon N)} \|s_j^N\|_{L^2}^2 \, dV = \frac{\text{Vol}(B(z_0,\varepsilon N))}{\text{Vol}(M)} + o(|\log N|^{-2m\gamma}).
\]

Recall \( \dim_C M = m \), so \( \frac{\text{Vol}(B(z_0,\varepsilon N))}{\text{Vol}(M)} = C(M,g)\varepsilon_{2m}^N = C(M,g)|\log N|^{-2m\gamma} \). The differences between Proposition 3 and Theorem 2 are that the former is an asymptotic result for a fixed base point, whereas the latter is a comparison result that holds for all points in \( M \). Moreover, in the former case the range of values that \( \gamma \) can take is improved. Proposition 3 is not used in proving Theorem 1 or Theorem 2.

1.1.3. Log-scale quantum ergodicity. By the quantum variance associated to \( f \) we mean the quantity

\[
\mathcal{V}_N(f) := \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \int_M f(z)\|s_j^N\|_{L^2}^2 \, dV - \int_M f \, dV \right|^2 \quad \text{for } f \in C^\infty(M).
\]

Thanks to Egorov’s theorem for Toeplitz operators (Proposition 3.1 proved in Appendix A) and the decay of correlations assumption [1], we show the quantum variance has a logarithmic decay rate when \( f \in C^\infty(M) \):

**Theorem 4 (Logarithmic decay of quantum variances).** Assumes the hypotheses of Theorem 1. Then, there exists a constant \( \kappa_0 > 0 \) independent of \( N \) such that for every \( 0 < \beta < 1 \) and for every \( f \in C^{0,\beta}(M) \),

\[
\mathcal{V}_N(f) = O \left( \frac{\|f\|_{C^{0,\beta}}^2}{\log N} \right) + O(\|f\|_{L^2}^{2N^{-\kappa_0}}),
\]

where \( \| \cdot \|_{C^{0,\beta}} \) is the Hölder norm.

We specialize to the following logarithmically dilated symbols. In Kähler normal coordinates, let \( f_{z_0} \in C^\infty_0(B(z_0,2),\mathbb{R}) \) be a smooth cut-off function that is equal to 1 on \( B(z_0,1) \), vanishes outside of \( B(z_0,2) \) and satisfies \( 0 \leq f_{z_0} \leq 1 \). For “small-scale quantum ergodicity,” we work with locally dilated symbols (cf. [4]) of the form

\[
f_{z_0,\varepsilon}(z) := D_{z_0}^\varepsilon f_{z_0}(z) = f\left( \frac{z}{\varepsilon} \right) \in C^\infty_0(B(z_0,2\varepsilon),\mathbb{R}), \quad \text{where } z_0 \in M \text{ and } \varepsilon > 0.
\]
Then set $\varepsilon = \varepsilon_N$. It follows from Theorem 4 that the quantum variance associated to such symbols have the estimate
\[
\mathcal{V}_N(f_{z_0,\varepsilon_N}) = \mathcal{O}\left(\frac{\|f_{z_0,\varepsilon_N}\|^2_{C^0,\beta}}{\log N}\right) + \mathcal{O}(\|f_{z_0,\varepsilon_N}\|_{C^2} N^{-\kappa_0})
\]
\[
= \mathcal{O}(\|f_{z_0}\|^2_{C^0,\beta} \log N|2^{\gamma \beta - 1} N - \kappa_0|) + \mathcal{O}(\|f_{z_0}\|_{C^2} \log N|2^{2\gamma N - \kappa_0}|).
\]
Since $0 < \beta < 1$ and $\gamma < \frac{1}{6m}$, we have $2\gamma \beta - 1 < 0$. Since the second term is smaller than the first, we obtain:

**Corollary 5** (Log-scale quantum variance estimates). Let $\varepsilon_N$ be as defined in (2). Under the same hypotheses as in Theorem 1, we have
\[
\mathcal{V}_N(f_{z_0,\varepsilon_N}) = \mathcal{O}(\|f_{z_0}\|^2_{C^0,\beta} \log N|2^{\gamma \beta - 1}|),
\]
where the error estimate is uniform in $z_0$.

Following the arguments of [HR] and [Ha], an application of Corollary 5 and a covering argument together imply Theorem 2.

**1.2. Further results.** The results of this paper are the Kähler analog of the small-scale quantum ergodicity results in the Riemannian setting proved in [HR, Ha]. For Hecke modular eigenforms, it is proved in Theorem 1.2 of [LMR] that (in the notation defined above),
\[
\sup_{F \subset \mathcal{F}} \left| \int_{\mathcal{R}} y^k |f_k(z)|^2 \frac{dxdy}{y^2} - \frac{3}{\pi} \int_{\mathcal{R}} \frac{dxdy}{y^2} \right| \leq C \varepsilon_k \varepsilon (\log k)^{\gamma - \varepsilon},
\]
where the supremum is taken over all rectangles $\mathcal{R}$ with sides parallel to the $x-$ and $y-$axis. This is a stronger result since it is valid for all Hecke eigenforms and since the supremum is taken over rectangles of any size rather than over rectangles (rather than balls) of ‘radius’ $\varepsilon_k = |\log k|^{-\gamma}$.

In the Kähler setting, [ShZ1] proves equidistribution of zeros (not at the logarithmic scale) for random orthonormal bases of $H^0(M, L^N)$ as well as for eigensections of quantized ergodic symplectic maps. It is probable that Theorem 1 can also be generalized to random orthonormal bases. This is work in progress of the first author.

**1.3. Existence of quantizable ergodic symplectic diffeomorphisms.** An obvious question is whether quantizable ergodic symplectic diffeomorphisms satisfying (3) and (4) exist on a given Kähler manifold. There are few studies to date of existence of ergodic quantizable symplectic maps on general Kähler manifolds. We list the examples known to us.

Hyperbolic symplectic toral automorphisms satisfying certain parity conditions on the matrix elements are known to be quantizable and to satisfy the two assumptions (3) and (4) (see [Z1, Ke]). Indeed, any hyperbolic or Anosov map satisfies the assumptions. As explained in [FT], the parity conditions can be removed if one tensors with a flat line bundle and modifies the contact form.

There are further partially hyperbolic examples obtained by perturbation. As explained to the authors by A. Wilkinson, a symplectic toral automorphism induced by an element of $Sp(2n, \mathbb{Z})$ (or any partially hyperbolic symplectic diffeomorphism) can be perturbed to produce a symplectic diffeomorphism which is stably accessible (see [DW]). Moreover, if the original map is “center bunched,” then the perturbed map is stably ergodic (see [BW]). This
gives examples on symplectic tori other than linear maps. We refer to these articles for the definitions and further discussion. See also [Ma].

Further, there exist ergodic (indeed, Bernoulli) symplectic diffeomorphisms of surfaces of any genus (see [Ka] and Theorem 1.26 of [BP]). As mentioned above, they are quantizable. We also mention without any claim to precision that there also exist symplectic pseudo-Anosov diffeomorphisms, i.e. homeomorphisms away from a finite number of singular points which act hyperbolically with respect to two transverse (singular) measured foliations. Since they are singular, our techniques do not apply directly but it is plausible that they can be modified by suitably cutting off singular points. These examples may turn out to be the most explicitly computable ones on surfaces other than tori.

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2. Background

2.1. Complex geometry. We follow the notation used in [ShZ1, Z2, Z3] and refer there for further discussion. Let \((M, \omega)\) be a compact Kähler manifold of dimension \(\dim C M = m\). Let \((L, h) \to (M, \omega)\) be a pre-quantum line bundle. In other words, \(L\) is an ample Hermitian line bundle endowed with a smooth metric \(h\) whose curvature form \(c_1(h)\) is strictly positive with \(c_1(h) = \omega\), that is, for \(e_L\) a local holomorphic frame for \(L\) over an open set \(U \subset M\) one has

\[
c_1(h) = -\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \| e_L \|_h.
\]

We work with spaces \(H^0(M, L^N)\) of holomorphic sections \(s^N\) of \(L^N\). (The superscript on \(s\) indexes the degree and does not mean the \(N\)th power.) These are finite dimensional Hilbert spaces of dimensions

\[
d_N := \dim H^0(M, L^N) \sim \frac{c_1(L)^m}{m!} N^m \quad \text{as } N \to \infty.
\]

The Hermitian metric \(h^N\) and the inner product structure on \(H^0(M, L^N)\) are tensor powers of the metric \(h\) on \(L\):

\[
\begin{align*}
\| s^N(z) \|_{h^N} := & \| s(z) \|_{h}^N, & s \in H^0(M, L), \\
\langle s_1^N, s_2^N \rangle := & \int_M h^N(s_1^N(z), s_2^N(z)) dV, & s_1^N, s_2^N \in H^0(M, L^N).
\end{align*}
\]

Given a holomorphic section \(s^N \in H^0(M, L^N)\), we denote by \([Z_{s^N}]\) its current of integration over the zero divisor of \(s^N\). In a local frame \(e^N_L\) for \(L^N\), we can write \(s = f e^N_L\) with \(f\) a holomorphic function. Let \(g(z) := \| e^N_L(z) \|_{h}^2 = e^{-\varphi(z)}\) where \(\varphi\) is the Kähler potential, then \(\| e^N_L(z) \|_{h^N}^2 = g(z)^N\) and \(\| s^N \|_{h^N}^2 = |f^{(N)}|^2 g^N\). The Poincaré-Lelong formula states that

\[
[Z_{s^N}] = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |f| = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log \| s^N \|_{h^N} + N \omega.
\]
2.2. Hardy space of CR holomorphic functions. Let \((L^*, h^*)\) be the dual line bundle to \(L \rightarrow M\). Thanks to the positivity of \(c_1(h)\), the unit co-disk bundle \(D^* \subset L^*\) relative to dual metric \(h^*\) is a strictly pseudoconvex domain whose boundary

\[ X := \partial D^* = \{ v \in L^*: h^*(v, v) = 1 \} \subset L^* \]

is a CR manifold. The Hardy space \(H^2(X)\) is the space of square integrable CR functions on \(X\), or equivalently the space of boundary values of holomorphic functions on the unit disk bundle with finite \(L^2(X)\) norm.

We introduce a defining function \(\rho\) for \(X\), which will be featured in the Boutet de Monvel-Sjöstrand parametrix. We write points in the co-disk bundle as \(x = (z, \lambda e_L^*(z))\), where \(\lambda \leq 1\) and \(e_L^*(z)\) is a normalized dual frame centered at \(z \in M\). Define

\[ \rho: D^* \rightarrow \mathbb{R}, \quad \rho(z, \lambda e_L^*(z)) = 1 - |\lambda|^2 e^{\varphi(z)} \]  

where \(\varphi\) is the Kähler potential.

Then \(\rho\) is a defining function for \(X\) satisfying

- \(\rho\) is defined in a neighborhood of \(X\);
- \(\rho > 0\) in \(D^*\);
- \(\rho = 0\) on \(X\);
- \(d\rho \neq 0\) near \(X\).

We define the contact form

\[ \alpha = d^c\rho|_X. \]

Let \(r_\theta\) be the natural circle action on \(X\), that is, \(r_\theta x = e^{i\theta} x\) for \(x \in X\). Note that a section \(s \in H^0(M, L)\) determines an equivariant function \(\hat{s}\) on \(L^*\) by the rule

\[ \hat{s}(z, \lambda) = (\lambda, s(z)), \quad z \in M, \lambda \in L^*_z. \]

It is easy to verify restricting \(\hat{s}\) to \(X\) yields \(\hat{s}(r_\theta x) = e^{i\theta} \hat{s}(x)\). Conversely, a section \(s^{N} \in H^0(M, L^N)\) determines an equivariant function \(\hat{s}^N\) on \(L^*\) whose restriction to \(X\) satisfies \(\hat{s}(N)(r_\theta x) = e^{iN\theta} \hat{s}^N(x)\). The map \(s^{N} \mapsto \hat{s}^N\) is in fact a unitary equivalence between the space \(H^0(M, L^N)\) of holomorphic sections and the weight spaces

\[ H^2_N(X) := \{ F \in H^2(X): F(r_\theta x) = e^{iN\theta} F(x) \} \quad \text{with} \quad H^2(X) = \bigoplus_{N \geq 0} H^2_N(X). \]

The Szegő projector is the orthogonal projection

\[ \Pi: L^2(X) \rightarrow H^2(X) \]

and its Fourier components are denoted by

\[ \Pi_{h^N}: L^2(X) \rightarrow H^2_N(X). \]

2.3. Quantization of symplectic maps. We use the dynamical Toeplitz quantization method of [Z2]. A symplectic map \(\chi: M \rightarrow M\) is quantizable if and only if it lifts to a connection-preserving contact transformation \(\tilde{\chi}: X \rightarrow X\), that is, \(\tilde{\chi}^* \alpha = \alpha\). Denote by

\[ T_{\tilde{\chi}}: L^2(X) \rightarrow L^2(X), \quad T_{\tilde{\chi}} F = F \circ \tilde{\chi} \]

the pre-composition by the lift \(\tilde{\chi}\). Note that \(\tilde{\chi}\) commutes with the natural circle action \(r_\theta\) on \(X\), and \(\| \tilde{\chi} \|_{C^2(X)} = c \cdot \| \chi \|_{C^2(M)}\) for some constant \(c\).

The quantization of a quantizable map \(\chi\) is defined to be a unitary Fourier integral operator

\[ U_{\chi} := \Pi \sigma T_{\tilde{\chi}} \Pi: H^2(X) \rightarrow H^2(X). \]
Here, $\sigma$ is a semi-classical symbol that makes the operator $U_\chi$ defined by (10) unitary. Its existence is guaranteed by the construction in [Z2]. We emphasize again that $T_\chi$ denotes translation by the lifted map; such translation is not well-defined on the base because it does not preserve the line bundle.

Under the identification $H^2(X) = \bigoplus_{N \geq 0} H^2_N(X)$, $U_\chi$ decomposes into a sequence of unitary Fourier integral operators $U_{\chi,N}$ defined by

$$
U_{\chi,N} := \Pi_{h^N} \sigma_N T_\chi \Pi_{h^N} : H^2_N(X) \to H^2_N(X).
$$

The Fourier coefficients $\Pi_{h^N}$ have an explicit parametrix given in (12).

2.4. Boutet de Monvel-Sj" ostrand parametrix for the Szeg" o projector. In preparation for the proof of Egorov’s theorem for Toeplitz operators (Proposition 3.1), we briefly recall the Boutet de Monvel-Sj" ostrand parametrix for the Szeg" o kernel. Let $\Pi(x,y)$ denote the kernel of the Szeg" o projector $\Pi$ in (9), that is,

$$
\Pi F(x) = \int_X \Pi(x,y) F(y) \, dV(y) \quad \text{for all } F \in L^2(X).
$$

It is proved in [BS] that $\Pi$ is a complex Fourier integral operator of positive type. Near the diagonal, there is a parametrix of the form

$$
\Pi(x,y) \sim \int_0^\infty e^{it\psi(x,y)} s(x,y,t) \, dt,
$$

where

$$
s(x,y,t) \sim \sum_{n=0}^\infty t^{m-n} s_n(x,y)
$$

belongs to the symbol class $S^m(X \times X \times \mathbb{R}_{\geq 0})$ and $\psi \in C^\infty(D^* \times D^*)$ is a complex phase of positive type. (Recall that $D^*$ stands for the unit co-disk bundle, of which $X$ is the boundary.)

The phase function $\psi$ is obtained as the almost-analytic continuation of the defining function $\rho$ in [S]. Explicitly, for $x_j = (z_j, \lambda_j e_L^*(z_j)) \in D^*$, we have

$$
\psi(x_1,x_2) = \frac{1}{i} \left( 1 - \lambda_1 \lambda_2 e^{\frac{\varphi(z_1)}{2} - \frac{\varphi(z_2)}{2} + \varphi(z_1,z_2)} \right),
$$

where $\varphi(z_1, \bar{z}_2)$ is obtained from the Kähler potential $\varphi$ by writing $\varphi(z_1) = \varphi(z_1, \bar{z}_1)$ on the diagonal of $M \times \overline{M}$ and extending to a neighborhood of the diagonal. When the metric is real analytic the extension is analytic; in the general $C^\infty$ case it is almost-analytic. If we assume in addition that $x_j \in X$ lie on the co-circle bundle, then $\lambda_j = e^{i\tau_j}$ is uni-modular, whence $x_j = (z_j, \tau_j)$ and

$$
\psi(x_1,x_2) = \psi(z_1, \tau_1, z_2, \tau_2) = \frac{1}{i} \left( 1 - e^{-\frac{\varphi(z_1)}{2} - \frac{\varphi(z_2)}{2} + \varphi(z_1,z_2)} e^{i(\tau_1 - \tau_2)} \right) \quad \text{on } X \times X.
$$
The kernels of the partial Szegő projectors $\Pi_{h,N}$ in (10) are the Fourier coefficients of $\Pi(x,y)$:

$$\Pi_{h,N}(x,y) = \int_0^\infty \int_{S^1} e^{-iN\theta} e^{i\psi(\theta x,y)} s(r_\theta x,y,t) \, d\theta dt$$

$$= N \int_0^\infty \int_{S^1} e^{iN[-\theta + i\psi(\theta x,y)]} s(r_\theta x,y,Nt) \, d\theta dt,$$

where the second line follows from a change of variable $t \mapsto Nt$.

2.5. Off-diagonal estimates and scaling asymptotics. We will be using two off-diagonal estimates for the lifted Szegő kernel on $X \times X$. Again, write $x_j = (z_j, \tau_j)$ for points in the co-circle bundle $X$. Let $d(z,w)$ be the distance with respect to the Kähler metric on $M$.

The first is an Agmon-type estimate giving global off-diagonal bounds:

$$|\Pi_h(x_1, x_2)| \leq A_1 N^m e^{-A_2 \sqrt{N} d(z_1, z_2)}$$

for constants $A_1, A_2$ independent of $N, x_1, x_2$ due to Lindholm, Delin and others. The second is a near diagonal Gaussian decay estimate: There exists $A_3 < 1$ independent of $N, x_1, x_2$ such that

$$|\Pi_h(x_1, x_2)| \leq \left( \frac{1}{\pi^m} + o(1) \right) N^m e^{-\frac{1-A_3}{2} N d(z_1, z_2)^2} + O(N^{-\infty}) \quad \text{whenever} \quad d(z, w) \leq N^{-\frac{1}{3}}.$$

We refer to [ShZ2, ShZ3, MM] for background and references.

We further use near off-diagonal scaling asymptotics from [ShZ2, LuSh]. At each $z \in M$ there is an osculating Bargmann-Fock or Heisenberg model associated to $(T_z M, J_z, h_z)$. Let $(u, \theta_1, v, \theta_2)$ be linear coordinates on $T_z M \times S^1 \times T_z M \times S^1$. The model Heisenberg Szegő kernel on the tangent space is denoted by

$$\Pi_{h_z,J_z}^{T_zM} (u, \theta_1, v, \theta_2) : L^2(T_z M) \to \mathcal{H}(T_z M, J_z, h_z) = \mathcal{H}_f.$$

We recall that the semi-classical Szegő kernels of the Heisenberg group have the form

$$\Pi_N^{H}(x_1, x_2) = \frac{1}{\pi^m} N^m e^{iN(\tau_1 - \tau_2)} e^{N(z_1 \cdot z_2 - \frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2)}.$$

In [LuSh] the notion of K-coordinates is introduced, refining the notion of Heisenberg coordinates in [ShZ2]. These are Kähler-type coordinates in which (14) equals (15) to leading order (up to rescaling):

$$\Pi_{h_z,J_z}^{T_zM} (u, \theta_1, v, \theta_2) = \pi^{-m} e^{i(\theta_1 - \theta_2)} e^{u \cdot v - \frac{1}{4}(|u|^2+|v|^2)} = \pi^{-m} e^{i(\theta_1 - \theta_2)} e^{i^2(u \cdot v) - \frac{1}{2}|u-v|^2}.$$

The lifted Szegő kernel is shown in [ShZ2] and in Theorem 2.3 of [LuSh] to have the following scaling asymptotics.

**Theorem 2.1.** Fix $P_0 \in M$ and choose a $K$-frame centered at $P_0$. Then, identifying coordinates $(z_1, \tau_1, z_2, \tau_2)$ on $X^2$ with coordinates $(u, \theta_1, v, \theta_2)$ on $(T_z M \times S^1)^2$, we have

$$k^{-m} \Pi_{h_z \times J_z} (u, \theta_1, v, \theta_2) = \Pi_{h_z,J_z}^{T_zM} (u, \theta_1, v, \theta_2) \left( 1 + k^{-1} A_1(u, \theta_1, \theta_2) + \cdots \right),$$

where $\Pi_{h_z,J_z}^{T_zM}$ is the osculating Bargman-Fock Szegő kernel for the tangent space $T_z M \simeq \mathbb{C}^n$ equipped with the complex structure $J_z$ and Hermitian metric $h_z$. 
3. Proof of Logarithmic Decay of Variances (Theorem 4)

The variance estimate is similar to the ones given in [Z1, Sc1, Sc2, HR, Ha]. A key ingredient is Egorov’s theorem in the Kähler setting, whose proof is deferred to Appendix A.

**Proposition 3.1** (Egorov’s theorem with remainder). Let \( M_f \) denote multiplication by a smooth function \( f \in C^\infty(M) \). Let \( T \in \mathbb{Z} \) be an integer. Then

\[
U_{\chi,N}^T(\Pi_{h^N} M_f \Pi_{h^N})(U_{\chi,N}^*)^T = \Pi_{h^N} M_f \Pi_{h^N} + R_{T,N}^T,
\]

where \( f \circ \chi^T \) denotes the \( T \)-fold composition of \( f \) with \( \chi \), and \( R_{T,N}^T \) is a Toeplitz operator with

\[
\frac{1}{d_N} \text{Tr}[(R_{T,N}^T)^* R_{T,N}^T] = O\left( \frac{\|f \circ \chi^T\|_{C^2}}{N} \right).
\]

In particular, at the level of matrix elements one has

\[
\langle U_{\chi,N}^T \Pi_{h^N} M_f \Pi_{h^N} (U_{\chi,N}^*)^T s_j^N, s_j^N \rangle = \langle \Pi_{h^N} M_f \Pi_{h^N} s_j^N, s_j^N \rangle + O\left( \frac{\|f \circ \chi^T\|_{C^2}}{N} \right).
\]

Taking Proposition 3.1 for granted, we proceed to prove Theorem 4. We write each integral in the Cesàro sum (3) as a matrix element:

\[
\int_M f(z) \|s_j^N\|_{h^N}^2 dV = \langle \Pi_{h^N} M_f \Pi_{h^N} s_j^N, s_j^N \rangle.
\]

Let

\[
[\Pi_{h^N} M_f \Pi_{h^N}]_T := \frac{1}{2T} \sum_{n=-T}^{T} U_{\chi,N}^n (\Pi_{h^N} M_f \Pi_{h^N}) U_{\chi,N}^{*n}
\]

denote the time average of \( \Pi_{h^N} M_f \Pi_{h^N} \) up to time \( T \in \mathbb{N} \) relative to the quantum map \( U_{\chi,N} \). Since \( s_j^N \) are eigensections of \( U_{\chi,N} \), we may replace \( \Pi_{h^N} M_f \Pi_{h^N} \) in (16) by its time average defined in (17):

\[
\int_M f(z) \|s_j^N\|_{h^N}^2 dV = \langle [\Pi_{h^N} M_f \Pi_{h^N}]_T s_j^N, s_j^N \rangle.
\]

Proposition 3.1 that is Egorov’s theorem, gives

\[
[\Pi_{h^N} M_f \Pi_{h^N}]_T = \Pi_{h^N} [M_f]_T \Pi_{h^N} + R_{T,N}^{(T)},
\]

with the remainder term satisfying the error estimate

\[
\frac{1}{d_N} \text{Tr}[(R_{T,N}^{(T)})^* R_{T,N}^{(T)}] = O\left( \frac{T\|f \circ \chi^T\|_{C^2}}{N} \right) = O\left( \frac{T\|f\|_{C^2 e^{\delta_0 T}}}{N} \right).
\]

Here the exponential growth condition (3) on \( \chi \) is used.

By substituting (19) into (18), the quantum variance (3) can be rewritten as

\[
\mathcal{V}_N(f) = \frac{1}{d_N} \sum_{j=1}^{d_N} \left| \langle [M_f]_{T(N)} s_j^N, s_j^N \rangle + \langle R_{T,N}^{(N)} s_j^N, s_j^N \rangle - \int_M f dV \right|^2
\]

\[
\leq \frac{2}{d_N} \sum_{j=1}^{d_N} \left| \langle [M_f]_{T(N)} s_j^N, s_j^N \rangle - \int_M f dV \right|^2 + \frac{2}{d_N} \sum_{j=1}^{d_N} \left| \langle R_{T,N}^{(N)} s_j^N, s_j^N \rangle \right|^2.
\]
Introduce the shorthand
\[ [M_f]_T := M_f(T) \quad \text{with} \quad [f]_T := \frac{1}{2T} \sum_{n=-T}^{T} f \circ \chi^n. \]

Applying the Cauchy-Schwarz inequality to the first term and the error estimate (20) to the second term, we find
\[ \mathcal{V}_N(f) \leq \frac{2}{d_N} \sum_{j=1}^{d_N} \int_M \left| [f]_T \| s_j^N \|_{k_N}^2 - \int_M f \, dV \right|^2 \, dV + \mathcal{O}\left( \frac{T \| f \|_{C^2} e^{\delta_0 T}}{N} \right). \]

Then, applying the local Weyl law for semi-classical Toeplitz operators with remainder estimate (20) and from [BG, Z2], we get
\[ \mathcal{V}_N(f) \leq \int_M \left| [f]_T - \int_M f \, dV \right|^2 \, dV + \mathcal{O}\left( \frac{T \| f \|_{C^2} e^{\delta_0 T}}{N} \right). \]

Thanks to the decay of correlations assumption (4), the classical variance estimate of Liverani [L] says (for all \( 0 < \beta < 1 \))
\[ \mathcal{V}_N(f) = \mathcal{O}\left( \frac{\| f \|_{C^{0,\beta}}^2}{T} \right) + \mathcal{O}\left( \frac{T \| f \|_{C^2} e^{\delta_0 T}}{N} \right). \]

Finally, set
\[ T = T(N) = \kappa |\log N| \quad \text{with} \quad \kappa > \delta_0^{-1}, \]
which gives
\[ \mathcal{V}_N(f) = \mathcal{O}\left( \frac{\| f \|_{C^{0,\beta}}^2}{|\log N|} \right) + \mathcal{O}(\| f \|_{C^2} N^{-\delta_0^{-1}} |\log N|), \]
proving Theorem 4 with some \( \kappa_0 \) satisfying \( 1 - \delta_0 \kappa > \kappa_0 > 0 \).

4. PROOFS OF LOG-SCALE MASS EQUIDISTRIBUTION (PROPOSITION 3 AND THEOREM 2)

4.1. Proof of Proposition 3. We begin by defining constants \( \kappa_1, \kappa_2 \) that will appear in the proof. Let \( \kappa_1 \) be any constant satisfying
\[ 0 < \kappa_1 < 1 - 4m\gamma. \]
It follows that
\[ \kappa_1 \leq 1 - 4\gamma(m + \beta) \quad \text{for some} \quad 0 < \beta < 1, \]
whence
\[ |\log N|^{2\gamma \beta - 1} \leq |\log N|^{-4m\gamma - \kappa_1}. \]
We also let \( \kappa_2 \) be any constant satisfying
\[ 0 < \kappa_2 < \frac{\kappa_1}{2}. \]
Now fix $z_0 \in M$. Define symbols $\rho_N \in C_0^\infty(B(z_0, 1 + \log N|^{-\frac{\kappa_2}{\kappa + 1}}, [0, 1])$ by

$$\rho_N(z) := \begin{cases} 1 & \text{for } z \in B(z_0, 1 + \log N|^{-\frac{\kappa_2}{\kappa + 1}}), \\ 0 & \text{for } z \notin B(z_0, 1 + 2\log N|^{-\frac{\kappa_2}{\kappa + 1}}). \end{cases}$$

Note that the support of $\rho_N$ depends on $N$. We perform a further rescaling

$$(D_{\epsilon N}^{-1})^* \rho_N(z) = \rho_N(\epsilon N^{-1} z).$$

The statement of Corollary (5) (which follows easily from Theorem 4 as discussed in (1.1.3) with $f_{z_0, \epsilon N}$ replaced by $\rho_N(\epsilon N^{-1} z)$ becomes

$$\frac{1}{d_N} \sum_{j=1}^{d_N} \left| \langle M(D_{\epsilon N}^{-1})^* \rho_N s_j^N, s_j^N \rangle - \int_M \rho_N(\epsilon N^{-1} z) \ dV \right|^2 = \mathcal{O}(\|\rho_N\|_{C^{0, \beta}}^2 \log N|^{-1-2\gamma})$$

(25)

$$\leq \mathcal{O}(\|\rho_N\|_{C^{0, \beta}}^2 \log N|^{-4m\gamma} \log N|^{-\kappa_1})$$

for any $\kappa_1$ satisfying (21). In the last line we used (22).

By Markov’s inequality $\mathbb{P}(X \geq a) \leq a^{-1} \mathbb{E}X$, with $a = \log N|^{-\epsilon}$ (with a small $\epsilon > 0$) times the right side of (24), for any constant $\kappa_2$ satisfying (23) there exists a full density subsequence $\Gamma'_{z_0} \subset \{(N, j)\}$ such that the corresponding eigensections satisfy

$$\int_{B(z_0, 2)} \rho_N(\epsilon N^{-1} z) \|s_j^N\|^2 \ dV \leq \int_{B(z_0, 2)} C\|\rho_N\|_{C^{0, \beta}} \log N|^{-2m\gamma} \log N|^{-\kappa_2}$$

(26)

for $(N, j) \in \Gamma'_{z_0}$. In other words, almost all the terms in the averaged sum (26) each satisfies the slightly worse than the average upper bound $\mathcal{C}\|\rho_N\|_{C^{0, \beta}} \log N|^{-4m\gamma} \log N|^{-\kappa_2}$. (For more details, see [JJZ] Proposition 3.5.)

We then have

$$\int_{B(z_0, \epsilon N)} \|s_j^N\|^2 \ dV \leq \int_{B(z_0, 2)} \rho_N(\epsilon N^{-1} z) \|s_j^N\|^2 \ dV$$

$$\leq \frac{1}{\Vol(M)} \int_{B(z_0, 2)} \rho_N(\epsilon N^{-1} z) \ dV + \mathcal{C}\|\rho_N\|_{C^{0, \beta}} \log N|^{-2m\gamma} \log N|^{-\kappa_2}$$

for $(N, j) \in \Gamma'_{z_0}$. The first inequality follows from the definition (24) of $\rho_N$. The second inequality follows from the estimate (25). The third inequality follows from the support condition of (24) and from the volume of spherical shells (the “thickness” of the shell being $2\log N|^{-\frac{\kappa_2}{\kappa + 1}}$):

$$\int_{B(z_0, 2)} \rho_N(\epsilon N^{-1} z) \ dV = \int_{B(z_0, 1 + 2\log N|^{-\frac{\kappa_2}{\kappa + 1}})\setminus B(z_0, 1)} \rho_N(\epsilon N^{-1} z) \ dV + \int_{B(z_0, 1)} \rho_N(\epsilon N^{-1} z) \ dV$$

$$\leq \epsilon N\int_{B(z_0, 1 + 2\log N|^{-\frac{\kappa_2}{\kappa + 1}})\setminus B(z_0, 1)} \ dV + \int_{B(z_0, \epsilon N)} \ dV$$

$$\leq \mathcal{C}\epsilon N^2 \log N|^{-\frac{\kappa_2}{\kappa + 1}} + \Vol(B(z_0, \epsilon N)),$$

where $\mathcal{C}$ depends only on $(M, \omega)$ and the choice of $\rho$. 
Note that \( \|\rho_N\|_{C^{0,\beta}} \leq C(|\log N|^{\frac{\alpha}{\beta+1}})^{-\beta} \), which gives
\[
\int_{B(z_0,\varepsilon N)} \|s_j^N\|^2_{h_N} \, dV \leq \frac{\text{Vol}(B(z_0,\varepsilon N))}{\text{Vol}(M)} + C|\log N|^{-2m\gamma} \left(|\log N|^{-\frac{\alpha}{\beta+1}} + |\log N|^{-\frac{\alpha}{\beta+1}}\right)
\]
(27)
\[
= \frac{\text{Vol}(B(z_0,\varepsilon N))}{\text{Vol}(M)} + o(|\log N|^{-2m\gamma}).
\]
(From (21) and (23) of how \( \kappa_1, \kappa_2 \) are defined, we have \( 0 < \beta \kappa_2 / (\beta + 1) < 1 \).)

A similar argument using appropriately chosen \( \tilde{\rho}_N \) gives the opposite inequality
\[
\int_{B(z_0,\varepsilon N)} \|s_j^N\|^2_{h_N} \, dV \geq \frac{\text{Vol}(B(z_0,\varepsilon N))}{\text{Vol}(M)} + o(|\log N|^{-2m\gamma})
\]
for a full density subsequence \( \Gamma'' \) of eigensections. The intersection \( \Gamma''_{z_0} \cap \Gamma''_{z_0} =: \Gamma_{z_0} \) indexes a full density subsequence of eigensections for which (27) and (28) hold simultaneously. This completes the proof of Proposition 3.

4.2. Proof of Theorem 2. Note that one must first fix a single base point \( z_0 \in M \) for the statement of Corollary 3 to hold. To move towards global statements that hold for all \( z \in M \) simultaneously, we introduce the concept of log-good covers, whose existence is proved in [Hä].

**Definition 4.1.** Let \( \varepsilon_N = |\log N|^{-\gamma} \) for any fixed \( 0 < \gamma < (6m)^{-1} \) as before. A log-good cover \( \mathcal{U}_N \) is a cover of \( M \) by geodesic balls \( \{B(z_N,\varepsilon_N)\}_{\alpha=1}^{R(\varepsilon_N)} \) with the following properties:

- The number \( R(\varepsilon_N) \) of balls in the cover is bounded above
  \[ R(\varepsilon_N) \leq c_1 \varepsilon_N^{-2m} \quad (\dim \, M = 2m) \]
  by some constant (independent of \( N \)) multiple of \( \varepsilon_N^{-2m} \).

- Every ball \( B(z_N,\varepsilon_N) \subset M \) is covered by at most \( c_2 \) (independent of \( N \)) number of balls from the cover.

- Every ball \( B(z_N,\varepsilon_N) \subset M \) contains at least one of the shrunken balls \( B(z_N,\varepsilon_N) \).

We now proceed with the proof of Theorem 2 suppressing the prime notation on \( \gamma \) and \( \varepsilon_N \). Let \( 0 < \gamma < (6m)^{-1} \) be given and set \( \varepsilon_N = |\log N|^{-\gamma} \). For each \( N \), fix a log-good cover \( \mathcal{U}_N \) as defined above. As before, let \( 0 \leq f_{z_a} \leq 1 \) be a smooth cut-off function that is equal to 1 on \( B(z_N,\alpha, 1) \), and vanishes outside \( B(z_N,\alpha, 2) \). Let \( f_{z_a,\varepsilon_N} = f_{z_a}(\varepsilon_N \cdot) \). (This is a slight abuse of notation, where we mean balls in Kähler normal coordinate charts centered at \( z_N,\alpha \).)

In what follows, \( \kappa_3 > 0 \) is a parameter independent of \( N, j, \alpha \). Let us define random variables
\[
X_{N,j,\alpha} := \left| \int_M f_{z_a,\varepsilon_N} \|s_j^N\|^2_{h_N} \, dV - \int_M f_{z_a,\varepsilon_N} \, dV \right|^2.
\]

Thanks to Corollary 3 and (22), the expected value is
\[
\mathbb{E}X_{N,j,\alpha} = O(|\log N|^{-(1-2\gamma\beta)}) = O(|\log N|^{-(4m\gamma + \kappa_1)}) \quad \text{for any } \kappa_1 \text{ satisfying (21)}.
\]
(The error is uniform in \( z_a \).) In particular, we may choose \( \kappa_1 \) to equal
\[
0 < \kappa_1 := 1 - 4m(\gamma + \beta) < 1 \quad \text{for some } 0 < \beta < \frac{1 - 6m\gamma}{4m} < 1.
\]
It follows from an application of Markov’s inequality with $X = X_{N,j,\alpha}$; with the normalized counting measure on $\{1, \ldots, d_N\}$; and with $a = |\log N|^{-(4m\gamma - \kappa_3)}$, that the ‘exceptional sets’

$$\Lambda_\alpha(N) := \left\{ j = 1, \ldots, d_N : \int_M f_{z_\alpha, \varepsilon N} \|s_j^N\|_{h_N}^2 dV - \int_M f_{z_\alpha} dV \geq |\log N|^{-(4m\gamma - \kappa_3)} \right\}$$

satisfy

$$\frac{\#\Lambda_\alpha(N)}{d_N} \leq C|\log N|^{4m\gamma - \kappa_3}|\log N|^{-(4m\gamma + \kappa_1)} = C|\log N|^{-(1-4m(\gamma + \beta) - \kappa_3)}.$$

Now define ‘generic sets’

$$\Sigma_\alpha(N) := \{ j : 1 \leq j \leq d_N \} \setminus \Lambda_\alpha(N) \quad \text{and} \quad \Sigma(N) := \bigcap_{\alpha : B(z_{N,\alpha}, \varepsilon N) \in U_N} \Lambda_\alpha(N).$$

The number of elements in the cover $U_N$ is of order $\varepsilon^{-2m} = |\log N|^{2m\gamma}$, whence

$$\frac{\#\Sigma(N)}{d_N} \geq 1 - C\sum_{\alpha} \frac{\#\Sigma_\alpha(N)}{d_N}$$

$$= 1 - C|\log N|^{2m\gamma} |\log N|^{-(1-4m(\gamma + \beta) - \kappa_3)}$$

$$= 1 - C|\log N|^{-(1-6m\gamma - 4m\beta - \kappa_3)}$$

$$\to 1 \quad \text{by choosing } \beta, \kappa_3 > 0 \text{ sufficiently small.} \quad (30)$$

Indeed, by choice (29) of $\beta$, we have $1 - 6m\gamma - 4m\beta > 0$, so $\kappa_3$ can always be chosen to ensure (30) holds. This is analogous to the estimate in [HR] preceding Lemma 3.1 or in [Ha p.3263].

The construction of indexing sets $\Sigma(N)$ yields a full density subsequence

$$\Sigma := \bigcup_{N \geq 1} \Sigma(N)$$

such that, for every $B(z_{N,\alpha}, \varepsilon N) \in U_N$, we have

$$\int_{B(z_{N,\alpha}, \varepsilon N)} \|s_j^N\|_{h_N}^2 dV \leq \int_{B(0,2)} f_{z_\alpha, \varepsilon N} \|s_j^N\|_{h_N}^2 dV$$

$$\leq \frac{1}{\text{Vol}(M)} \int_{B(0,2)} f_{z_\alpha, \varepsilon N} dV + C|\log N|^{-(2m\gamma + \kappa_1/2)}$$

$$\leq \frac{\text{Vol}(B(z_{N,\alpha}, 2\varepsilon N))}{\text{Vol}(M)} + o(|\log N|^{-2m\gamma})$$

$$\leq C\text{Vol}(B(z_{N,\alpha}, \varepsilon N))$$

simultaneously for all $\alpha = 1, \ldots, R(\varepsilon N)$ as $\Sigma \ni (N, j) \to \infty$. The constant $C$ is independent of $\alpha$.

Now let $p \in M$ be arbitrary. By construction, the ball $B(p, \varepsilon N)$ is contained in at most $c_2$ number (independent of $N$) of elements of the log-good cover $U_N$. Thus,

$$\int_{B(p, \varepsilon N)} \|s_j^N\|_{h_N}^2 dV \leq \sum_{i=1}^{c_2} \frac{1}{\text{Vol}(M)} \int_{B(0,2)} f_{z_\alpha, \varepsilon N} dV + o(|\log N|^{-2m\gamma}) \leq C\text{Vol}(B(p, \varepsilon N))$$
for every $p \in M$ as $\Sigma \ni (N, j) \to \infty$. The constant $C$ is independent of $p$. This is the statement of the volume upper bound.

It remains to repeat the same construction by dilating the symbol $0 \leq g_{z, \alpha} \leq 1$ that is a smooth cut-off function supported in $B(z, 1/3)$ and equals to 1 in $B(0, 1/6)$. There exists a full density subsequence $\Sigma'$ such that

$$
\int_{B(z_{N, \alpha}, \varepsilon_N/3)} \|s^N_j\|^2_{h^N} \, dV \geq \int_{B(z_{N, \alpha}, 1/3)} g_{z, \alpha, \varepsilon_N/3} \|s^N_j\|^2_{h^N} \, dV
$$

$$
\geq \frac{1}{\text{Vol}(M)} \int_{B(z_{N, \alpha}, 1/3)} g_{z, \alpha, \varepsilon_N/3} \, dV - C|\log N|^{-(2m\gamma + \kappa_3/2)}
$$

$$
\geq \frac{\text{Vol}(B(z_{N, \alpha}, \varepsilon_N/6))}{\text{Vol}(M)} - o(|\log N|^{-2m\gamma})
$$

$$
\geq c\text{Vol}(B(z_{N, \alpha}, \varepsilon_N))
$$

simultaneously for all $\alpha = 1, \ldots, R(\varepsilon_N)$ as $\Sigma \ni (N, j) \to \infty$. Now let $p \in M$ be arbitrary. Every ball $B(p, \varepsilon_N)$ contains at least one element $B(z_{N, \alpha}, \varepsilon_N/3) \in U_N$ of the log-good cover, whence

$$
\int_{B(p, \varepsilon_N)} \|s^N_j\|^2_{h^N} \, dV \geq c\text{Vol}(B(p, \varepsilon_N))
$$

for every $p \in M$ as $\Sigma \ni (N, j) \to \infty$. This is the statement of the volume lower bound.

The intersection $\Gamma = \Sigma \cap \Sigma'$ is again a full density subsequence. By construction, the eigensections indexed by $\Gamma$ satisfy the two-sided bound: for all $p \in M$,

$$
c\text{Vol}(B(p, \varepsilon_N)) \leq \int_{B(p, \varepsilon_N)} \|s^N_j\|^2_{h^N} \, dV \leq C\text{Vol}(B(p, \varepsilon_N)) \quad \text{as } \Gamma \ni (N, j) \to \infty.
$$

This completes the proof of Theorem 2.

4.3. Proof of log-scale equidistribution of zeros (Theorem 1). Let $0 < \gamma < (6m)^{-1}$ from the statement of Theorem 1 be given. We distinguish two logarithmic scales by fixing another parameter $\gamma'$:

$$
0 < \gamma < \gamma' < \frac{1}{6m} \quad \text{so that} \quad |\log N|^{-\gamma'} = \varepsilon'_N < \varepsilon_N = |\log N|^{-\gamma}.
$$

Let $\Gamma$ be the full density subsequence corresponding to scale $\varepsilon'$ as guaranteed by Theorem 2

We show that the same $\Gamma$ satisfies the statement of Theorem 1 at the scale $\varepsilon'_N > \varepsilon'_N$.

In the notation of (2.1) relative to a local frame we write the eigensections locally as

$$
s^N_j = f^{(N)}_j e^N_L, \quad f^{(N)}_j \text{ a local holomorphic function}.
$$

The Poincaré-Lelong formula (7) reduces the growth rate of zeros to the growth rate of the local plurisubharmonic function $N^{-1} \log |f^{(N)}_j|^2$ or to the global quasi-plurisubharmonic function $u^{(N)}_j(z) = N^{-1} \log \|s^N_j(z)\|^2_{h^N}$. Fix $p \in M$ and consider the dilated function

$$
u^{(N)}_j(z) := \frac{1}{N} \log \|s^N_j(\varepsilon_N z)\|^2_{h^N} = D_{\varepsilon_N} \left[ \frac{1}{N} \log \|s^N_j(z)\|^2_{h^N} \right] \quad \text{on } B(p, 1),
$$

11'quasi' means p.s.h. up to a fixed continuous term, here the potential $\log g$ where $g(z) := \|e_L(z)\|^2_{h_L}$.
where $D^p_{\varepsilon_N}$ is the local dilation defined by (5) in Kähler normal coordinates centered at $p = 0$. Since $D^p_{\varepsilon_N}$ is a local holomorphic map, (31) remains quasi-plurisubharmonic. We state a key lemma:

**Lemma 4.2.** Let $\Gamma$ be the subsequence of density one for the finer scale $\varepsilon_N$ of Theorem 2. For $(N, j) \in \Gamma$, the logarithmically dilated potential (31) satisfies

$$\|u_j^{(N)}\|_{L^1(B(p, 1))} = o(\varepsilon_N^2),$$

where the remainder is at a coarser scale $\varepsilon_N$.

**Remark 4.1.** We emphasize that we are assuming the eigensections indexed by $\Gamma$ satisfy

$$C_1 \frac{\text{Vol}(B(p, \varepsilon_N))}{\text{Vol}(M)} \leq \int_{B(p, \varepsilon_N)} \|s_j^N\|^2_{h_N} dV \leq C_2 \frac{\text{Vol}(B(p, \varepsilon'_N))}{\text{Vol}(M)}$$

and then inverse dilating $B(p, \varepsilon_N)$ to $B(p, 1)$, so that any ball $B(q, \varepsilon'_N) \subset B(p, \varepsilon)$ gets inverse dilated to (slightly deformed) by $(D^p_{\varepsilon_N})^{-1}$ to (slightly deformed) balls of radius $\varepsilon^{-1} \varepsilon'_N \simeq |\log N|^{-\gamma + \eta}$ in $B(p, 1)$.

Let’s assume Lemma 4.2 for now and proceed to finish the proof of Theorem 1. Using the Poincaré-Lelong formula and the fact that the holomorphic rescaling $D^p_{\varepsilon}$ commutes with $\partial \bar{\partial}$, we obtain

$$\frac{1}{N} D^{p^*}_{\varepsilon_N} [Z_{s_j^N}] = \frac{-1}{2\pi N} \partial \bar{\partial} \log |f_j^{(N)}(\varepsilon_N z)|^2 = \frac{-1}{2\pi N} \partial \bar{\partial} \log \|s_j^N(\varepsilon_N z)\|^2_{h_N} + D^{p^*}_{\varepsilon_N} \omega.$$

For every test form $\eta \in \mathcal{D}^{m-1, m-1}(B(p, 1))$ and $\Gamma \ni (N, j) \to \infty$, integration by parts and Lemma 4.2 give

$$\int_{B(p, 1)} \left( \eta \wedge \frac{1}{N} D^{p^*}_{\varepsilon_N} [Z_{s_j^N}] \right) = \int_{B(p, 1)} \eta \wedge D^{p^*}_{\varepsilon_N} \omega + \int_{B(p, 1)} \frac{-1}{2\pi N} \log \|s_j^N(\varepsilon_N z)\|^2_{h_N} \partial \bar{\partial} \eta(z)$$

$$= \int_{B(p, 1)} \eta \wedge D^{p^*}_{\varepsilon_N} \omega + o(\varepsilon_N^2).$$

Locally at $p = 0$, the Kähler potential can be written as $\varphi(z) = |z|^2 + O(|z|^4)$, so

$$D^{p^*}_{\varepsilon_N} \omega = \frac{-1}{2\pi} D^{p^*}_{\varepsilon_N} \partial \bar{\partial} \varphi = \varepsilon_N^2 \frac{-1}{2\pi} \partial \bar{\partial} |z|^2 + O(\varepsilon_N^4) = \varepsilon_N^2 \omega_0^p + O(\varepsilon_N^4),$$

with $\omega_0^p$ the flat Kähler form. Combining (33) and (34) (and dividing by $\varepsilon_N^2$) yields

$$\int_{B(p, 1)} \left( \eta \wedge \frac{1}{N} D^{p^*}_{\varepsilon_N} [Z_{s_j^N}] \right) = \int_{B(p, 1)} \eta \wedge \omega_0^p + o(\varepsilon_N^2)$$

as $\Gamma \ni (N, j) \to \infty$, which is equivalent to the statement of Theorem 1.

**Proof of Lemma 4.2.** The argument is similar to the one in [ShZ1] except for the dilation of the pluriharmonic functions. The log-scale quantum ergodicity successfully replaces unscaled quantum ergodicity in the key step of the argument due to the fact that the local dilation is holomorphic. But we need to use two logarithmic scales and for later applications we need the remainder estimate.
Let $N_0$ be sufficiently large so that for all $N \geq N_0$, $e_L$ is a local frame for $L$ over an open subset $U$ containing $B(p, 1)$ and $e_L^N$ is the corresponding frame for $L^N$. Since $g(z) = \|e_L(z)\|_{h^N}$, we have so that
\[
\|e_L^N(z)\|_{h^N}^2 = g^N \quad \text{and} \quad \|s_j^N(z)\|_{h^N}^2 = |f_j^N(z)|^2 g^N(z).
\]

We first show that $\|u_j^{(N)}\|_{L^1} \to 0$, and then indicate how the argument can be adapted to yield the $o(\varepsilon_N^2)$ improvement.

Observe that any $L^2$-normalized section satisfies
\[
\|s_N(z)\|_{h^N}^2 \leq \Pi_h^N(z, z) = \left( \frac{c_1(L)^m}{m!} + O\left(\frac{1}{N}\right) \right) N^m.
\]

Hence $\|s_N(z)\|_{h^N} \leq C N^m/2$ for some $C < \infty$ and taking the logarithm gives

(i) The functions $u^{(N)}$ are uniformly bounded above on $M$;
(ii) $\limsup_{N \to \infty} u_N \leq 0$.

Now consider the plurisubharmonic function
\[
v_j^{(N)}(z) := \frac{1}{N} \log |f_j^{(N)}(\varepsilon_N z)|^2 = u_j^{(N)}(z) - \log g(\varepsilon_N z) \in \text{PH}(B(p, 1)).
\]

It is clear that $v_j^{(N)}$ are uniformly upper bounded. A standard result on plurisubharmonic functions (see [Ha, Theorem 4.1.9]) then implies a subsequence $v_j^{(N_k)}$ either converges uniformly to $-\infty$ on $B(p, 1)$ or else has a subsequence that is convergent in $L^1_{\text{loc}}(B(p, 1))$.

Let us rule out the first possibility. If it occurred, there would exist $K > 0$ such that
\[
\frac{1}{N_k} \log \|s_j^{N_k}(\varepsilon_{N_k} z)\|_{h^{N_k}}^2 \leq -1 \iff \|s_j^{N_k}(\varepsilon_{N_k} z)\|_{h^{N_k}} \leq e^{-N_k} \quad \text{on} \ B(p, 1) \quad \text{for all} \ k \geq K.
\]

Equivalently, the same exponential decay estimate holds on $B(p, \varepsilon_{N_k})$ for the undilated sections. But this contradicts the lower bound of \([32]\).

Therefore the sequence $v_j^{(N)}$ is pre-compact in $L^1(B(p, 1))$, and every sequence contains a subsequence, which we continue to denote by $\{v_j^{(N_k)}\}$, that converges in $L^1(B(p, 1))$ to some $v \in L^1(B(p, 1))$. By passing if necessary to a further subsequence, we may assume that $\{v_j^{(N_k)}\}$ converges pointwise almost everywhere in $B(p, 1)$ to $v$, and hence by observation (ii),
\[
v(z) = \limsup_{(N_k, j) \to \infty} \left( u_j^{(N_k)}(z) - \log g(\varepsilon_{N_k} z) \right) \leq 0 \quad \text{a.e. on} \ B(p, 1).
\]

Let
\[
v^*(z) := \limsup_{w \to z} v(w) \leq 0
\]
be the upper-semicontinuous regularization of $v$. Then $v^*$ is plurisubharmonic on $B(p, 1)$ and $v^* = v$ almost everywhere. We claim that $v^* = 0$. To this end, we use the second scale $\varepsilon'_N$. If $v^* \neq 0$, then
\[
\|v_j^{(N_k)} + D_{\varepsilon_{N_k}}^* \log g\|_{L^1(B(p, 1))} = \|u_j^{(N_k)}\|_{L^1(B(p, 1))} \geq \delta > 0.
\]
Hence, for some $c > 0$, the open set $U_c = \{z \in B(p, 1) : v^*(z) < -c\}$ is nonempty. For sufficiently large $k$, this set contains a ball $B(q, \varepsilon_N^{-1})$. By Hartogs’ Lemma, there exists a positive integer $K$ such that $v_j^N(z) \leq -c/2$ for $z \in B(q, \varepsilon_N^{-1})$ and $k \geq K$, that is
\[
\|s_j^N(z)\|^2_{h_Nk} \leq e^{-cK/2} \quad \text{on } B(q, \varepsilon_N^{-1}) \quad \text{for all } k \geq K.
\]
But this again contradicts the lower bound in Theorem 2 on $B(q, \varepsilon_N^{-1})$. We have therefore proved $\|u_j^{(N)}\|_{L^1(B(p, 1))} = o(1)$.

We now exploit the exponential decay to prove the sharper result $\|u_j^{(N)}\|_{L^1(B(p, 1))} = o(e^{N/2})$. Consider the renormalized sequence
\[
\varepsilon_N^{-2} u_j^{(N)} = \frac{1}{N \varepsilon_N^2} D_{\varepsilon_N} \log \|s_j^N(z)\|^2_{h_Nk}.
\]
Note that this is still an upper-bounded sequence of plurisubharmonic functions because of the exact cancellation between dilating by $D_{\varepsilon_N}$ and dividing by $\varepsilon_N^2$. Indeed, $\log g = |z|^2 + O(|z|^4)$ as $|z| \to p = 0$ in local coordinates, so $\varepsilon_N^{-2} D_{\varepsilon_N} \log g$ remains bounded.

We now run through the previous argument again with this re-normalized sequence. If $\varepsilon_N^{-2} u_j^{Nk} \to -\infty$ uniformly on compact subsets of $B(p, 1)$, then
\[
\frac{1}{N \varepsilon_N^2} \|s_j^N(z)\|^2_{h_Nk} \leq -1 \iff \|s_j^N(z)\|^2_{h_Nk} \leq e^{-\varepsilon_N^2 Nk} \quad \text{on } B(p, 1),
\]
a contradiction to (52) as before. The alternative (namely $\varepsilon_N^{-2} u_j^{Nk}$ being pre-compact) leads to the estimate
\[
\|s_j^N(z)\|^2_{h_Nk} \leq e^{-c\varepsilon_N^2 Nk/2} \quad \text{on } B(q, \varepsilon_N^{-1}) \quad \text{for all } k \geq K,
\]
again a contradiction. This completes the proof of Lemma 42. □

APPENDIX A. Egorov’s theorem

The purpose of this section is to prove a long time Egorov’s theorem with remainder as stated in Proposition 3.1. It is convenient to work on the contact manifold $(X, \alpha)$ by lifting $\chi$ on $M$ to the contact transformation $\tilde{\chi}$ on $X$ and viewing sections $s_j^N \in H^0(M, L^N)$ as equivariant functions $s_j^N \in L^2(X)$ as discussed in Section 2.

We recall the setting. Let $\chi$ be a quantizable symplectic map (whose quantization $U_{\chi,N}$ is defined in (10)) satisfying the exponential growth condition (3) and decay of correlations condition (4). Let $M_F$ denote multiplication by $F \in C^\infty(X)$ and $F \circ \tilde{\chi}^T$ the composition of $F$ with the $T$-fold iterate of $\tilde{\chi}$ (or $\tilde{\chi}^{-1}$, depending on the sign of $T$). Proposition 3.1 which is a statement on the base manifold $M$, is equivalent to the following statement on the co-circle bundle $X$.

**Proposition A.1.** Let $\chi$ be a quantizable symplectic map on $M$ satisfying conditions (3) and (4). Let $\tilde{\chi}$ denote its lift to $(X, \alpha)$ as a contact transformation. Let $F \in C^\infty(X)$ and $T \in \mathbb{N}$. Then
\[
U_{\chi,N}(\Pi_{hN} M_F \Pi_{hN})(U_{\chi,N}^*)^T = \Pi_{hN} M_{\chi^T} \Pi_{hN} + P_N(T),
\]
where $R_N^{(T)}$ is a Toeplitz operator with
\[
\frac{1}{d_N} \| R_N^{(T)} \|_{\text{HS}}^2 = \frac{1}{d_N} \text{Tr}[(R_N^{(T)})^* R_N^{(T)}] = O \left( \frac{T \| F \circ \hat{\chi}^T \| C^2}{N} \right).
\]

The proposition is the analogue for Toeplitz operators of the estimate of the Egorov remainder in Lemma 2.14 of [Z1], except that the remainder is stated in terms of the normalized Hilbert-Schmidt norm rather than the operator norm.\footnote{The more difficult norm estimate of the remainder will be presented elsewhere.} The Hilbert-Schmidt norm is simpler to estimate since it is defined by a trace, and the remainder estimate is simply the standard one in the stationary phase expansion [He]. Sharper remainder estimates have been proved for quantizations of Hamiltonian flows on $T^*\mathbb{R}^n$ in Theorems 1.4 and 1.8 of [BouR]. Subsequently, there are many articles proving related results for $T^*\mathbb{M}$. But there do not seem to exist parallel results for Toeplitz operators in the Kähler setting, in particular for powers of a map rather than for Hamiltonian flows. In special cases such as symplectic toral automorphisms and their perturbations, Egorov’s theorem with remainder have been proved (see [Sc1, Sc2]) but the proofs use special properties of the metaplectic representation and do not generalize to our setting. Egorov’s theorem without estimate of the time-dependence of the remainder may be obtained from the composition theorem for Toeplitz operators in [BG].

A.1. Reduction to $T = 1$ case. In this section we reduce the proof of Proposition A.1 to the proof of the following lemma.

**Lemma A.2.** Under the same assumption as Proposition A.1, we have
\[
(35) \quad U_{\chi,N} \Pi_{hN} M_F \Pi_{hN} U_{\chi,N}^* = \Pi_{hN} M_{F \circ \hat{\chi}} \Pi_{hN} + R_N,
\]
where $R_N$ is a Toeplitz operator with
\[
\frac{1}{d_N} \| R_N \|_{\text{HS}}^2 = \frac{1}{d_N} \text{Tr}[R_N^* R_N] = O \left( \frac{\| F \circ \hat{\chi} \| C^2}{N} \right).
\]

We now indicate how Lemma A.2 implies the statement of Egorov’s theorem. The rest of the appendix is then devoted to proving Lemma A.2.

**Proof of Proposition A.1 given Lemma A.2.** Given $T \in \mathbb{N}$ and two operators $U$ and $A$, we introduce the shorthand
\[
\text{Ad}^T(U)(A) = U^T A (U^*)^T
\]
for the $T$-fold conjugation of $A$ by $U$. To keep track of the remainders we henceforth denote $R_N$ in the statement of Lemma A.2 by $R_N^{(1)}$. Invoking the assumption (3) that $\| \hat{\chi}^T \| C^2 = O(e^{T \delta_0})$, Lemma A.2 reads
\[
\text{Ad}(U_{\chi,N}) \Pi_{hN} M_F \Pi_{hN} = \Pi_{hN} M_{F \circ \hat{\chi}} \Pi_{hN} + R_N,
\]
\[
\frac{1}{d_N} \text{Tr}[R_N^* R_N] = O \left( \frac{\| F \circ \hat{\chi} \| C^2}{N} \right) = O(N^{-1} e^{T \delta_0}).
\]
We now iterate the conjugation. Conjugating a second time by $U_{\chi,N}$ yields two terms:

\[
(36) \quad \text{Ad}^2(U_{\chi,N})\Pi_{hN} M_F \Pi_{hN} \equiv \text{Ad}(U_{\chi,N})\Pi_{hN} M_{F\tilde{\chi}} \Pi_{hN} + \text{Ad}(U_{\chi,N})R_{N}^{(1)}.
\]

It follows from Lemma A.2 (with $M_F$ replaced by $M_{F\tilde{\chi}}$) that the first term on the right-hand side of (36) equals

\[
(37) \quad \begin{cases}
\text{Ad}(U_{\chi,N})\Pi_{hN} M_{F\tilde{\chi}} \Pi_{hN} = \Pi_{hN} M_{F\tilde{\chi}}^2 \Pi_{hN} + \tilde{R}_{N}^{(2)}, \\
\frac{1}{d_N} \text{Tr}[\tilde{R}_{N}^{(2)}*\tilde{R}_{N}^{(2)}] = O\left(\frac{\|F \circ \tilde{\chi} \circ \chi\|c^2}{N}\right) = O(N^{-1}e^{2\delta_0}).
\end{cases}
\]

In the error estimate we again made use of the exponential growth assumption (3).

The unitarity of $U_{\chi,N}$ implies that the second term Ad$(U_{\chi,N})R_{N}^{(1)}$ in (37) satisfies

\[
(38) \quad \text{Tr}[(\text{Ad}(U_{\chi,N})R_{N}^{(1)})*\text{Ad}(U_{\chi,N})R_{N}^{(1)}] = \text{Tr}[(R_{N}^{(1)})*R_{N}^{(1)}] = O\left(\frac{\|F \circ \chi\|c^2}{N}\right) = O(N^{-1}e^{\delta_0}).
\]

Combining (36) and (37) gives

\[
(39) \quad \text{Ad}^2(U_{\chi,N})\Pi_{hN} M_F \Pi_{hN} = \Pi_{hN} M_{F\tilde{\chi}}^2 \Pi_{hN} + \tilde{R}_{N}^{(2)} + \text{Ad}(U_{\chi,N})R_{N}^{(1)}.
\]

Set

\[
(40) \quad R_{N}^{(2)} := \tilde{R}_{N}^{(2)} + \text{Ad}(U_{\chi,N})R_{N}^{(1)},
\]

then (37) and (38) imply

\[
(41) \quad \frac{1}{d_N} \text{Tr}[(R_{N}^{(2)})*R_{N}^{(2)}] \leq \frac{2}{d_N} \text{Tr}[(\tilde{R}_{N}^{(2)})*\tilde{R}_{N}^{(2)} + (R_{N}^{(1)})*R_{N}^{(1)}]
\]

\[
= 2\left(O(N^{-1}e^{2\delta_0}) + O(N^{-1}e^{\delta_0})\right)
\]

\[
= 3O(N^{-1}e^{2\delta_0}).
\]

The statement of Proposition A.1 with $T = 2$ is proved thanks to (39), (40) and (41). Note that the factor of 2 in the error estimate $2O(e^{2\delta_0})$ equals the number of conjugations of $M_F$ by $U_{\chi,N}$.

The calculation is similar when Ad$(U_{\chi,N})$ is iterated $T$ times. On the $T$th iterate, we pick up the leading order term

\[
\begin{cases}
\text{Ad}(U_{\chi,N})\Pi_{hN} M_{F\tilde{\chi}_{T-1}} \Pi_{hN} = \Pi_{hN} M_{F\tilde{\chi}_{T}} \Pi_{hN} + \tilde{R}_{N}^{(T)}, \\
\frac{1}{d_N} \text{Tr}[\tilde{R}_{N}^{(T)}*\tilde{R}_{N}^{(T)}] = O\left(\frac{\|F \circ \tilde{\chi}_{T}\|c^2}{N}\right) = O(N^{-1}e^{T\delta_0})
\end{cases}
\]

by the same stationary phase computation. We also have to conjugate the $(T - 1)$ ‘old’ remainders from the $(T - 1)$st iterate:

\[
\text{Ad}(U_{\chi,N})\tilde{R}_{N}^{(T-1)} + \text{Ad}^2(U_{\chi,N})\tilde{R}_{N}^{(T-2)} + \text{Ad}^3(U_{\chi,N})\tilde{R}_{N}^{(T-3)} + \cdots + \text{Ad}^{T-1}(U_{\chi,N})\tilde{R}_{N}^{(1)}.
\]

The Hilbert-Schmidt norm of $\tilde{R}_{N}^{(T)}$ does not change under conjugation by $U_{\chi,N}$. Therefore, the combined remainder term

\[
R_{N}^{(T)} := \tilde{R}_{N}^{(T)} + \text{Ad}(U_{\chi,N})\tilde{R}_{N}^{(T-1)} + \text{Ad}^2(U_{\chi,N})\tilde{R}_{N}^{(T-2)} + \cdots + \text{Ad}^{T-1}(U_{\chi,N})\tilde{R}_{N}^{(1)}
\]
at the $T$th stage of the iterate has the estimate
\[
\frac{1}{d_N} \text{Tr}[(R_N^{(T)})^* R_N^{(T)}] \leq \frac{T}{d_N} \sum_{t=1}^T \text{Tr}[(R_N^{(t)})^* R_N^{(t)}] = T \sum_{t=1}^T O\left(\frac{\|F \circ \tilde{\chi}^t\|C_2^2}{N}\right) \leq C T e^{T \delta_0},
\]
where $C$ is independent of $N,T$. This completes the proof of Proposition A.1 assuming Lemma A.2.

**Remark A.1.** This estimate is not very efficient, since one could have used that $\sum_{t=1}^T \|F \circ \tilde{\chi}^t\|C_2^2 \leq C \sum_{t=1}^T e^{t \delta_0} \leq C_0 e^{T \delta_0}$ to eliminate the factor $T$. But the estimate is sufficient for our purposes.

**A.2. Proof of Lemma A.2 via stationary phase computation.** Let 
\[
\tilde{L}_N := U_{\chi,N} \Pi_{hN} M_F \Pi_{hN} U_{\chi,N}^* \quad \text{and} \quad L_N := \Pi_{hN} M_{F_{0\chi}} \Pi_{hN},
\]
From the definition (110) of Toeplitz quantization, the conjugated operator has the form
\[
\tilde{L}_N = \Pi_{hN} \sigma_N T_{\tilde{\chi}} \Pi_{hN} M_F \Pi_{hN} T_{\tilde{\chi}}^{-1} \sigma_N \Pi_{hN}.
\]
Next, insert the identity operator $\text{Id} = T_{\tilde{\chi}}^{-1} T_{\chi}$ between the operators $\Pi_{hN}$ and $M_F$ in the above expression. Note that $T_{\tilde{\chi}} F T_{\tilde{\chi}}^{-1} = F \circ \tilde{\chi}$. Hence, the expression becomes
\[
(42) \quad \tilde{L}_N = \Pi_{hN} \sigma_N \Pi_{hN}^{\tilde{x}} M_{F_{0\chi}} \Pi_{hN}^{\tilde{x}} \sigma_N \Pi_{hN}.
\]
where $\Pi_{hN}^{\tilde{x}} := T_{\tilde{\chi}} \Pi_{hN} T_{\tilde{\chi}}^{-1}$ is the operator with Schwartz kernel $\Pi_{hN}^{\tilde{x}}(x,y) = \Pi_{hN}(\chi(x), \chi(y))$.

In the notation (35),
\[
R_N = \tilde{L}_N - L_N = \Pi_{hN} (\sigma_N \Pi_{hN}^{\tilde{x}} M_{F_{0\chi}} \Pi_{hN}^{\tilde{x}} - M_{F_{0\chi}}) \Pi_{hN}.
\]
Evidently,
\[
(43) \quad \text{Tr}[R_N^* R_N] = \text{Tr}[\tilde{L}_N^* \tilde{L}_N] - 2 \text{Tr}[\tilde{L}_N L_N] + \text{Tr}[L_N^* L_N].
\]
We evaluate each term asymptotically by stationary phase with remainder and add the terms. Lemma A.2 follows from:

**Lemma A.3.** We have
\[
(44) \quad \frac{1}{d_N} \text{Tr}[L_N^* L_N] = \int_M |F \circ \tilde{\chi}|^2 dV + O\left(\frac{\|F \circ \tilde{\chi}\|C_2}{N}\right).
\]
Moreover,
\[
\frac{1}{d_N} \text{Tr}[\tilde{L}_N^* \tilde{L}_N] = \frac{1}{d_N} \text{Tr}[\tilde{L}_N L_N] + O\left(\frac{\|F \circ \tilde{\chi}\|C_2}{N}\right) \quad \frac{1}{d_N} \text{Tr}[L_N^* L_N] + O\left(\frac{\|F \circ \tilde{\chi}\|C_2}{N}\right).
\]
In particular, thanks to (43) we have
\[
\frac{1}{d_N} \text{Tr}[R_N^* R_N] = O\left(\frac{\|F \circ \tilde{\chi}\|C_2}{N}\right).
\]

The first statement (44) is the well-known Szegő limit formula with remainder. Since $\tilde{\chi}$ is symplectic it may be removed from $f \circ \tilde{\chi}$ in the integral. The leading order term is calculated in [BG] using the homogeneous calculus of Toeplitz operators. The semi-classical calculation and the remainder estimate may be calculated by the method below.
For the rest of the Appendix, we calculate the most difficult of the three terms, namely \( d_N^{-1} \text{Tr}[\tilde{L}^*_N \tilde{L}_N] \), asymptotically to leading order by the method of stationary phase for oscillatory integrals with complex phases of positive type (\cite{16}, Theorem 7.7.5). We use the remainder estimate from that theorem. The calculations of the other two terms are similar and therefore omitted.

All three traces in (13) have the same leading order term (14), and so the leading term cancels when taking the sum (13). The cancellation between the ‘symbols’ \( \sigma_N \) and the Hessian determinants in the calculation of the leading order terms (14) is guaranteed by unitarity of \( U_{\chi,N} \) (see also \cite{22} for explicit calculation of the symbol).

From (12), we have

\[
\frac{1}{d_N} \text{Tr}[\tilde{L}^*_N \tilde{L}_N] = \frac{1}{d_N} \text{Tr} \left[ \Pi_{h,N} \bar{\sigma}_N \Pi^\dagger_{h,N} M_{F_0 \chi} \Pi^\dagger_{h,N} \sigma_N \Pi_{h,N} \bar{\sigma}_N \right].
\]

Note that we may drop the factor of \( \Pi_{h,N} \) at the end when computing the trace. We use the shorthand

\[
\tilde{y}_j := \tilde{\chi}(y_j), \quad y_j \in X.
\]

Recall that \( \sigma_N \) denotes multiplication by the symbol \( \sigma_N \), and the Szegö projectors have Schwartz kernels

\[
\Pi^\dagger_{h,N}(y_1, y_2) = \Pi_{h,N}(\tilde{y}_1, \tilde{y}_2), \quad \Pi_{h,N}(y_1, y_2) = N \int_0^\infty \int_S e^{iN[-\theta+\psi(y_1,y_2)]} s(r_{\theta y_1}, y_2, Nt) d\theta dt.
\]

The last equality is the Boutet de Monvel-Sjöstrand parametrix introduced in Section 2.4.

Using Schwartz kernels, the trace (15) can be written as the following oscillatory integral

\[
\frac{1}{d_N} \text{Tr}[\tilde{L}^*_N \tilde{L}_N] = \frac{1}{d_N} \int_X (\tilde{L}^*_N \tilde{L}_N)(x, x) dx = \frac{1}{d_N} \int_X \left( N^6 \int_{X^5 \times (S^1)^6} A(x, y, \theta, t) e^{iN\psi(x,y,\theta,t)} dt d\theta dy \right) dx,
\]

where

\[
y = (y_1, \ldots, y_5) \in X^5, \quad \theta = (\theta_1, \ldots, \theta_6) \in (S^1)^6, \quad t = (t_1, \ldots, t_6) \in (\mathbb{R}_+)^6
\]

and the amplitude and phase function are given by

\[
A = s(r_{\theta_1,y_1}, \ldots, t_1 N) \bar{\sigma}_N(y_1) s(r_{\theta_2,y_2}) s(r_{\theta_3,y_3}) s(r_{\theta_4,y_4}) s(r_{\theta_5,y_5}) s(r_{\theta_6,y_6}) \sigma_N(x),
\]

\[
\Psi = t_1 \psi(r_{\theta_1,y_1}) - \theta_1 + t_2 \psi(r_{\theta_2,y_2}) - \theta_2 + t_3 \psi(r_{\theta_3,y_3}) - \theta_3 + t_4 \psi(r_{\theta_4,y_4}) - \theta_4 + t_5 \psi(r_{\theta_5,y_5}) - \theta_5 + t_6 \psi(r_{\theta_6,y_6}).
\]

The functions \( s \) and \( \psi \) come from the Boutet de Monvel-Sjöstrand parametrix (12), and \( \sigma_N \) comes from the quantization formula (10).

The method of stationary phase is used to compute the inner integral. The off-diagonal exponential decay estimate (13) for the phase function \( \psi \) allows us to localize the \( X^5 \)-space integral to the region \( \{ d(y_j, y_k) < N^{-1/3} \} \) and absorb the error in the remainder estimate for
$R_N$. To locate the critical points of the phase function $\Psi$, recall from (11) that the function $\psi$ has the form

$$\psi(x, y) = \frac{1}{i} \left( 1 - \Lambda(x, y) \right)$$

with

$$\Lambda(x, y) := e^{-i \varphi(x, y)} e^{i \varphi(z_1, \bar{z}_2)} e^{i (\tau_1 - \tau_2)},$$

from which it follows

$$\psi(r_0 x, y) = \frac{1}{i} \left( 1 - e^{i \theta} \Lambda(x, y) \right).$$

Therefore,

$$D_{\theta_1} \Psi = \psi(r_0, x, y_1) = 0 \iff 1 = e^{i \theta_1} \Lambda(x, y_1).$$

The Schwarz inequality shows that a real critical point exists if and only if $x = y_1$. Similar computations for $D_{\theta_j} \Psi$ demand that $\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3$, $y_3 = y_4$, and $\tilde{y}_4 = \tilde{y}_5 = \tilde{x}$. The real critical point of $\Psi$ must therefore satisfy

(46)

$$x = y_1 = y_2 = y_3 = y_4 = y_5.$$

Consider now the $\theta_1$ derivative:

$$D_{\theta_1} \Psi = -t_1 e^{i \theta_1} \Lambda(x, y_1) - 1 = 0 \iff 1 = -t_1 e^{i \theta_1} \Lambda(x, y_1).$$

From the constraint (46), we must have $x = (z_1, \tau_1) = (z_2, \tau_2) = y_1$, so $\Lambda(x, y_1) = 1$. It follows that $t_1 = -1$ and $\theta_1 = 0$. Similar computations for $D_{\theta_j} \Psi$ show that the real critical point of $\Psi$ satisfies

(47)

$$\theta_1 = \cdots = \theta_6 = 0 \quad \text{and} \quad t_1 = \cdots = t_6 = -1.$$

Finally, we claim that $D_{y_j} \Psi$ automatically vanishes at the points satisfying (46) and (47). Indeed, at the critical point we have

$$D_{y_j} \Psi \Big|_{x = y_1 = \cdots = y_5} = -D_{y_j} \psi(x, y_1) \Big|_{y_1 = x} - D_{y_j} \psi(\tilde{y}_1, \tilde{y}_2) \Big|_{y_2 = y_1 = x}.$$  

Recall, however, that along the diagonal of $X \times X$ we have

$$d_1 \psi = -d_2 \psi = \frac{1}{i} d\rho|_X = \alpha,$$

where $\alpha$ is the contact form. Here $d_j$ refers to the derivative with respect to the $j$th slot of $\psi(\cdot, \cdot)$. The assumption that $\chi$ lifts to a contact transformation, that is, $\chi^* \alpha = \alpha$, implies

$$-D_{y_j} \psi(x, y_1) \big|_{y_1 = x} - D_{y_j} \psi(\tilde{y}_1, \tilde{y}_2) \big|_{y_2 = y_1 = x} = \alpha(x) - \frac{1}{i} d\rho(\chi(x)) = \alpha(x) - \chi^* \left( \frac{1}{i} d\rho \right)(x) = 0.$$  

Similar computations for $D_{y_j} \Psi$ show that the real critical points of $\Psi$ are completely given by (46) and (47).

It is straightforward to verify that the Hessian at the critical point is a block matrix of the form

$$\text{Hess} \, \Psi(x) = \begin{bmatrix}
D_{tt} \Psi & D_{t\theta} \Psi & D_{t1} \Psi & D_{t2} \Psi \\
D_{\theta t} \Psi & D_{\theta \theta} \Psi & D_{\theta 1} \Psi & D_{\theta 2} \Psi \\
D_{11} \Psi & D_{12} \Psi & D_{11} \Psi & D_{12} \Psi \\
D_{21} \Psi & D_{22} \Psi & D_{21} \Psi & D_{22} \Psi
\end{bmatrix}.$$
with

\[
D_{t1}\Psi = -D_{t2}\Psi = \begin{bmatrix}
\alpha(x) & 0 & 0 & 0 & 0 \\
-\alpha(x) & \alpha(x) & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & -\alpha(x) & \alpha(x) \\
0 & 0 & 0 & 0 & -\alpha(x)
\end{bmatrix} = -(D_{2t}\Psi)^t = (D_{1t}\Psi)^t,
\]

\[
D_{\theta1}\Psi = -D_{\theta2}\Psi = \begin{bmatrix}
-i\alpha(x) & 0 & 0 & 0 & 0 \\
i\alpha(x) & -i\alpha(x) & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & i\alpha(x) & -i\alpha(x) \\
0 & 0 & 0 & 0 & i\alpha(x)
\end{bmatrix} = -(D_{2\theta})^t = (D_{2\theta}\Psi)^t,
\]

\[
D_{11}\Psi = \begin{bmatrix}
-d\alpha(x) & d\alpha(x) & 0 & 0 & 0 \\
d\alpha(x) & -d\alpha(x) & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & d\alpha(x) & -d\alpha(x)
\end{bmatrix} = D_{22}\Psi.
\]

This Hessian matrix is invertible by the Schur complement formula (recall that \(-id\rho = \alpha\) is non-vanishing in a neighborhood of \(X\)). The method of stationary phase shows that the Schwartz kernel \((\tilde{L}_N^* \tilde{L}_N)(x,x)\) along the diagonal has the expansion

\[
(\tilde{L}_N^* \tilde{L}_N)(x,x) \sim \frac{N^6}{(N^{12+10m} \det \text{Hess} \Psi(x))^{1/2}} \sum_{j,k,\ell,p,q,u,v \geq 0} N^{6m-j-k-\ell-p-q-u-v} L_j(s_k(x,x)s_\ell(x,x)s_p(\tilde{x},\tilde{x})s_q(\tilde{x},\tilde{x})s_u(\tilde{x},\tilde{x})s_v(\tilde{x},\tilde{x})|\sigma_N(x)|^4|F(\tilde{x})|^2),
\]

where \(L_j\) are differential operators of order at most 2\(j\) that can be explicitly expressed in terms of \(s_k\) and the Hessian (see [HC]).

Observe that the leading order term (obtained from the above expression by setting \(j = k = \cdots = v = 0\)) is of order \(N^6(N^{12+10m})^{-1/2}N^{6m} = N^m\). Therefore, after dividing by \(d_N \sim N^m\) (for \(N\) large enough), the leading order term of \(\tilde{d}_N \text{Tr}[\tilde{L}_N^* \tilde{L}_N]\) is of order 0. It is calculated in (14). From the stationary phase computation, the second order term of \(\tilde{L}_N^* \tilde{L}_N(x,x)\) is bounded above in sup norm by

\[
C' \sum_{\alpha \leq 2} \left\| D^\alpha \left( (\det \text{Hess} \Psi(x))^{-\frac{1}{2}} s_0(x,x)^2 s_0(\tilde{x},\tilde{x})^4 |\sigma_N(x)|^4 |F \circ \tilde{x}|^2 \right) \right\|_\infty \leq C' \| F \circ \tilde{x} \|_{C^2}
\]

for some constant \(C'\) independent of \(F\) and \(\tilde{x}\). Dividing through by \(d_N \sim N^m\) yields the desired error estimate \(O(N^{-1}\| F \circ \tilde{x} \|_{C^2})\). This completes the computation for \(\tilde{L}_N\).
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