ON LOG MINIMALITY OF WEAK K-MODULI
COMPACTIFICATIONS OF CALABI-YAU VARIETIES

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Abstract. For moduli of polarized smooth K-trivial a.k.a., Calabi-Yau varieties in a general sense, we revisit a classical problem of constructing its “weak K-moduli” compactifications which parametrizes K-semistable (i.e., semi-log-canonical K-trivial) degenerations. Although weak K-moduli is not unique in general, they always contain a unique partial compactification (K-moduli).

Our main theorem is the log minimality of their normalizations, under some conditions. Partially to confirm that known examples satisfy the conditions, we also include an appendix on the algebro-geometric reconstruction of Kulikov models via the MMP, which has been folklore at least but we somewhat strengthen.

1. Introduction

In this paper, we focus on K-trivial variety (often called Calabi-Yau variety in e.g. Kähler geometric or birational geometric context) which means projective variety whose singularity is mild so that the canonical divisor makes sense and is linearly equivalent to 0. The problem of compactifying moduli, say $M^o \subset \overline{M}$, of K-trivial varieties is classical and rich topic with many important connections with other fields. The renowned existence theorem of Ricci-flat Kähler metrics [Yau78] and important roles they play in the context of string theory give one such aspect. Rather than expanding such broad but widely well-known backgrounds somewhat, we refer to our previous review [Od20, §1.2] or the references cited below. Also, some connection with Kähler geometry via a compactification of moduli is explored in [OO21] but note that the compactification therein is not even a variety. Nevertheless, we expect it to parametrize the collapsing of Ricci-flat metrics.

In a purely algebro-geometric side, [Od20, §1.2], we introduced the terminology of weak K-moduli compactification for K-trivial varieties (see Definition 2.1), which roughly means compactification $\overline{M}$ of moduli $M^o$ of polarized smooth Calabi-Yau varieties whose boundary parametrizes semi-log-canonical K-trivial degenerations. The idea had been implicitly existed in the field, as indeed the problem of constructing such compactification has

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been well-pursued by many experts and hence classical topic. Therefore this may be regarded as a matter of words. The terminology “K-moduli” comes from the notion of K-stability ([Tia97, Don02]) which is the algebrao-
geometric counterpart of existence of constant scalar curvature Kähler metrics, and is introduced in [Od10] with its existence speculation, after various background works of both algebraic geometry and differential geometry. However we need reformulation and more precise version of its existence conjecture in general. For anticanonically polarized Q-Fano varieties case, the precise reformulation is introduced in [OSS16, 3.13, 6.2] after [Spo12, 1.3.1] and the existence conjecture is now solved as a combination of (at least) a dozen of papers.

In turn, for the K-trivial/Calabi-Yau case, K-semistability (resp., K-stability) is simply equivalent to semi-log-canonicity (resp., log terminality) [Od13a, Od12]. A very important caution here is that, even for a fixed moduli $M^o$, weak K-moduli compactifications are not unique (e.g., [Sha80, AET19]). Recall that sometimes (normalization of) weak K-moduli is shown or expected to be (one of) the so-called toroidal compactifications [AMRT] (cf., [Mum77, §3, §4]), and note that they are log minimal with appropriate natural boundary divisor. Some other examples of weak K-moduli compactification such as [AET19] is normalized to be semi-toric compactification by Looijenga [Looi03a, Looi03b] which are similarly log minimal.

One aim of this paper is to show such log minimality of weak K-moduli under certain condition as a general theorem.

**Theorem 1.1** (Log minimality - see Theorem 3.2 for details). Under certain assumptions, the normalization of weak K-moduli compactification of moduli spaces of polarized K-trivial varieties, is log minimal.

The point is that this log minimality gives a fairly strong restriction on the possible compactifications, which we plan to explore in the next work. For the proof, we use variation of mixed Hodge structures or smooth mixed Hodge modules, as well as recent refinement of cone theorems [Sva19, Ejn21]. The result itself also matches to concrete examples of [Nam76, Sha80, Looi03b, AN99, Nak2, Ale02, Zhu18, AET19, AE21] (see also [AB19, ABE20, HLL20]) among others, but a very different point is that our proof in this notes do not rely on any Torelli type theorem nor the structure of Shimura varieties but rather of more general Hodge theoretic flavor.

As an auxiliary result, we also write a proof that, although weak K-moduli $\overline{M}$ is not unique for fixed $M^o$, it must contain a particular canonical (unique) partial compactification $M$, which coincides with the Weil-Petersson metric completion of $M^o$, and is quasi-projective. Actually this follows from a simple combination of known results [Vie95, Wan97, DS14, Tos15, Zha16].
and \cite{Bir20} and is essentially their corollary. Last missing piece for a while had been a boundedness result (although doubted as \cite{Zha16}, 1.2) which is now a big result of Birkar \cite{Bir20}, 1.6]. We acknowledge appreciation to Chenyang Xu for pointing it to me. Further, the set of parametrized objects in $M$ are characterised by (strict) K-stability or the minimization of the degree of CM line bundles as we recall in Lemma 2.6, Corollary 2.7 (also see the references).

For the case of polarized K3 surfaces, this partial compactification is explicit and coincides with an orthogonal modular variety. This follows from the semi-classical fact that allowing polarized ADE singular K3 surfaces (or equivalently, quasi-polarized smooth K3 surfaces) fills Heegner divisors. The same statements also generalizes to the case of irreducible symplectic varieties, once we replace ADE singularities by symplectic singularities in the sense of Beauville, as we showed in \cite{OO21} §8.3, Theorem 8.3, Corollary 8.4]. See \textit{loc.cit} for details.

In this paper, after clarifying the existence of K-moduli and basic properties of weak K-moduli in §2 in §3 we discuss weak K-moduli and its log minimality. The proof depends on analysis of variation of mixed Hodge structures in §4. Finally, to fill the lack of reference, we discuss an algebraic reconstruction of Kulikov models for K-trivial surfaces which is indeed substantially related to the weak K-moduli problem for K-trivial surfaces.

Except for appendix, we work over an arbitrary algebraically closed field $k$ of characteristics 0 but §3, §4 requires $k = \mathbb{C}$ since we use Hodge theory.

2. PREPARATION - SETUP AND K-MODULI

We make a general setup by fixing a connected Deligne-Mumford moduli stack $M^o$ of polarized smooth K-trivial varieties i.e., there is a universal family $\pi^o: (U^o, L^o) \to M^o$. We define weak K-moduli compactification and also K-moduli (partial compactification) below.

\textbf{Definition 2.1} (Weak K-moduli cf., \cite{Od20}). We follow the above notation. We call a proper Deligne-Mumford moduli stack $\overline{M}$ compactifying $M$, together with a $\mathbb{Q}$-Gorenstein family of polarized semi-log-canonical, or equivalently, K-semistable\footnote{the equivalence follows from \cite{Od12,Od13a}} Calabi-Yau varieties $\bar{\pi}: (\bar{U}, \bar{L}) \to \overline{M}$, a weak K-moduli stack if it satisfies the following two conditions:

(i) $\bar{\pi}$ extends $\pi: (U^o, L^o) \to M^o$,

(ii) underlying family of varieties $\bar{U} \to \overline{M}$ is effective (i.e., no isomorphic varieties occur as fibers at different $k$-rational points).
We note that when you apply the theory of proper moduli of “stable pairs”, or equivalently log (i.e., attaching divisors) KSBA theory, the second assertion (ii) is a priori quite nontrivial if it holds. Indeed, the theory a priori only gives a moduli of *log pairs* (encoding the additional information of \(\mathbb{Q}\)-or \(\mathbb{R}\)-divisors) so that there may be locus in which the underlying varieties do not deform and only divisors deform. Nevertheless, if one takes a general section of very ample \((L \otimes \pi^*N)^{\otimes\alpha}\) for \(N \gg 0\) and ample \(N\) on \(\overline{M}\), it immediately follows that in general weak K-moduli should be an algebraic substack of some “log KSBA”-type moduli stack (if it exists) by taking a general element of relatively very ample linear system on the universal family. See [Od20, Section §1.2] for more detailed discussions.

The following is announced in [OO21, Remark 9.1].

**Theorem 2.2 (K-moduli).** For fixed moduli algebraic stack \(\mathcal{M}^o\) of polarized smooth K-trivial varieties as above, we have a finite type Deligne-Mumford algebraic stack \(\mathcal{M}\) which includes \(\mathcal{M}^o\) as an open substack together with a Gorenstein family of polarized K-trivial varieties with only canonical singularities \((\mathcal{U}, \mathcal{L}) \to \mathcal{M}\) such that the following holds.

(i) different \(k\)-valued points \(p_1\) and \(p_2\) of \(\mathcal{M}\) have non-isomorphic polarized fiber \(\pi^{-1}(p_i)(i = 1, 2)\).

(ii) \(\mathcal{M}\) is maximum with respect to the inclusion relation among those which satisfies above

(iii) the coarse moduli space \(\mathcal{M}\) of \(\mathcal{M}\) is quasi-projective. If \(k = \mathbb{C}\), its analytification coincides with the completion of \(\mathcal{M}^o\) with respect to its generalized Weil-Petersson Kähler orbiflaries.

We call this \(\mathcal{M}\) the K-moduli (partial compactification) of polarized K-trivial varieties.

We emphasize that this \(\mathcal{M}\) is not proper in general so is still a “partial compactification” of \(\mathcal{M}^o\). Nevertheless, since the parametrized polarized varieties are all K-stable (not only K-semistable) by [Od12], and also characterized by K-polystability, we would call it the K-moduli of K-trivial varieties.\(^\dagger\) One reason of the name is that in the K-trivial case, the K-polystability and K-stability are equivalent (cf., [Od13a, Od12]).

**Example 2.3.** In the case if \(\mathcal{M}^o\) is a moduli of polarized smooth irreducible symplectic manifolds, the above Theorem 2.2 is proved with a more refined statement by [OO21, §8.3 Theorem 8.3, Corollary 8.4] and [Sch85, Theorem 4.8], which shows further that \(\mathcal{M}\) is nothing but the locally Hermitian symmetric space whose Weil-Petersson metric is the Bergman metric. Note

\(^2\)The definition is different from the “over-ambitious” version long ago [Od10, Od13b], although many later works show the version works for anticanonically polarized \(\mathbb{Q}\)-Fano varieties case.
that this refinement crucially depends on the Torelli type theorem due to Verbitsky [Ver13, Ver20].

**Proof of Theorem 2.2.** From [Zha16, Theorem 1.1] or [Vie95, Theorem 8.23 (and §8.3, §8.6)], it only remains to show the boundedness of the possibly klt K-trivial projective limit of the members of $\mathcal{M}^o$. The following abstract proof of such boundedness is a corollary to a combination of known results with recent big input due to Birkar [Bir20]. Here we expand the details of the proof.

We replace $\mathcal{L}^o$ by $(\mathcal{L}^o)^{\otimes m}$ for fixed $m \gg 0$ such that it is relatively very ample and some effective relatively Cartier and relatively smooth divisor $D$ exists so that $D \sim_{\mathcal{M}^o} (\mathcal{L}^o)^{\otimes m}$. For the sake of simplicity, we can and do assume $m = 1$.

For the remained boundedness problem, we take any $k$-valued point $s \in \mathcal{M}(k)$ and consider corresponding polarized variety $\pi^{-1}(s)$. We can take a smooth curve $(C, s) \subset \mathcal{M}$ passing through $s$ and $\mathcal{M}^o$, take $D|_{(C\setminus s)}$ and its restriction $(D|_{(C\setminus s)})|_s$ as a Weil divisor.

An important caution here is that it is a priori a hard problem if one can take $m$ uniformly so that $L|_{\pi^{-1}(s)}$ is line bundle (not only $\mathbb{Q}$-line bundle) for all $s$, or equivalently $(D|_{(C\setminus s)})|_s$ is Cartier for all $s$. Note that whether the latter holds or not does depend on the choice of $D$ due to the equivalence.

Note that by the same arguments as [OSS16, (proof of) Lemma 2.4], it follows that $N$ is $\mathbb{Q}$-Cartier and ample although not necessarily Cartier. Now we apply the big result [Bir20, Corollary 1.6] to the setup $X = \pi^{-1}(s)$, $B = 0$, $N := (D|_{(C\setminus s)})|_s$ in the notation of loc.cit. From the boundedness assertion, one can in particular assume that there is a uniform $l \in \mathbb{Z}_{>0}$ such that $lN$ is Cartier, which does not depend on $s$.

Therefore, for instance, if we apply the effective basepoint freeness [Kol93, Theorem 1.1] to $(\pi^{-1}(s), l(D|_{(C\setminus s)})|_s)$ and then the very ampleness lemma [Fjn17a, Lemma 7.1] together with the uniform existence of the Castelnuovo-Mumford regularity, we obtain the finite typeness of $\mathcal{M}$.

Further differential geometric fact that the hyperK"ahler metrics parametrized by $\mathcal{M}$ is Gromov-Hausdorff continuous with respect to the Gromov-Hausdorff topology and the fact that $\mathcal{M}$ the completion of $\mathcal{M}^o$ with respect to the Weil-Petersson metric also follows from [Zha16] which crucially depends on [DST14] (see also [Tos15] and related algebro-geometric issue [Od13b, §4]).

Next we show that weak K-moduli always contains the above K-moduli, as our terminology may suggest.
Theorem 2.4. For a fixed $M^o$ and the family on it, any weak $K$-moduli $M$ contains the $K$-moduli $M'$ as an open algebraic substack with the compatible family on it.

Remark 2.5. The following proof also shows that existence of weak $K$-moduli compactification implies the existence of finite type $K$-moduli (Theorem 2.2) without using Birkar’s result [Bir20, Corollary 1.6] as above proof of Theorem 2.2.

Proof. By [Zha16, Theorem 1.1] or [Vie95, Theorem 8.23 (and §8.3, §8.6)], there is an increasing (and exhausting) sequence of moduli substacks of $K$-moduli stack (which may be a priori locally finite type)

$M_1 \subset M_2 \subset \cdots$. We prove that for any $i \in \mathbb{Z}_{\geq 0}$, $M_i$ has open immersion $\iota_i$ to $M$ preserving the ($\mathbb{Q}$-)polarized family. Then by the Noetherian argument, the assertion follows. We prove the existence of such $\iota_i$ by contradiction. Assuming the contrary, there should be a pointed curve $C \ni p$ together with a morphism $\varphi^o: (C \setminus \{p\}) \to M^o$ such that it extends to both $\varphi_1: C \to M_1$ and also $\varphi_2: C \to M$ but not extends to $C \to M$. If we take the polarized flat projective family $(X^o, L^o) \to (C \setminus \{p\})$ which corresponds to $\varphi^o$ and its two extensions to $C$ which corresponds to $\varphi_1$ and $\varphi_2$, then the following Corollary 2.7 (of Lemma 2.6) leads to the contradiction, which we reproduce for the sake of convenience.

Lemma 2.6 (cf., [Od13b, 4.2 (i)] [Od18, 2.14(1)]). Suppose $\pi: \mathcal{X} \to C \ni p$ is a $\mathbb{Q}$-Gorenstein flat projective family of $n$-dimensional semi-log-canonical $K$-trivial varieties (resp., semi-log-canonical $K$-trivial varieties such that $\pi^{-1}(p)$ is log terminal) over a projective curve $C$ and a relatively ample line bundle $L$ on $\mathcal{X}$. Take any other non-isomorphic flat projective family $\mathcal{X}' \to C$, together with a relatively ample line bundle $L'$, which is isomorphic to $(\mathcal{X}, L) \rightarrow C$ away from the fiber $\pi^{-1}(p)$. Then, we have

$$\deg(\lambda_{CM}(\mathcal{X}, L)) \leq (\text{resp. } <)\deg(\lambda_{CM}(\mathcal{X}', L')).$$

Here, $\lambda_{CM}$ stands for the CM line bundles ([FS90, PT06, FR06]) which values at Pic$(C)$ in above situations.

Corollary 2.7 (same references and [Bou14]). Consider a punctured ($\mathbb{Q}$-Gorenstein) family of polarized log terminal $K$-trivial projective varieties $(\mathcal{X}^o, L^o) \to (C \setminus \{p\})$, where $C$ is a smooth curve and $p$ a closed point, and suppose the existence of its completion $(\mathcal{X}, L) \to C$ with a log terminal $K$-trivial fiber $\pi^{-1}(p)$. Then, there is no other completion $(\mathcal{X}, L) \to C$ with semi-log-canonical $K$-trivial $\pi^{-1}(p)$.

We conclude the proof of Theorem 2.4. □
3. Log Minimality of Weak K-moduli

3.1. Statements. We now proceed to the log minimality discussion. Our setup and the list of assumptions is as follows which we believe to be ubiquitous. Indeed, examples include those recently constructed in [AET19, AE21] (also cf., [ABE20]). In this section and next section, we assume $k = \mathbb{C}$.

3.1.1. Notation and Assumptions. In this section, we work on the normalization of weak K-moduli and put the following natural assumption and notation. Although we take normalization, we still denote it as $M^o$ for simplicity. The following conditions are designed to fit to various known explicit examples of the compactifications, as we explain later.

- $G$ is a finite group, which could be a priori trivial.
- $S$ is a klt normal projective variety, $S^o$ is its Zariski open subset, $\Delta = S \setminus S^o$ is purely codimension 1 such that $G$ acts on $S$ preserving $\Delta$ such that $(S, \Delta)$ is log canonical. For instance, any smooth $S$ with normal crossing divisor $\Delta$ is allowed.
- K-trivial varieties are parametrized by: $\pi: \mathcal{X} \to S$ a $G$-equivariant flat proper morphism from an algebraic space $\mathcal{X}$, with the relative dimension $n$, such that
  (i) For any $s \in S^o$, $\pi^{-1}(s)$ is K-trivial projective variety with only canonical singularities,
  (ii) there is a $\pi|_{S^o}$-relatively ample line bundle $L^o$ on $\pi^{-1}(S^o)$,
  (iii) for any closed point $s \in \Delta = S \setminus S^o$, the $\pi$-fiber $\pi^{-1}(s)$ is again a K-trivial variety with only semi-log-canonical singularities.

Remark 3.1. Some technical remarks are in order here. Note that we are not assuming $\mathcal{X}$ itself is a variety while only fiberwise algebraicity is assumed. Compare with the classical Kulikov model situation (cf., [Kul77], [PP81], Theorem [A2] in the appendix). From our singularities assumption, note that any $\pi^{-1}(s)$ for $s \in S^o$ (resp., $s \in \Delta$) is K-stable (resp., K-semistable) for any polarization by [Od12].

Also $\pi$ is automatically a ($\mathbb{Q}$-)Gorenstein family and locally stable in the sense of Kollár [Kol].

- On each log canonical center $T$ of $(S, \Delta)$, whose support is automatically inside inside $\Delta$, write $\pi^{-1}(T)$ as $\mathcal{X}_T$. Then, the following hold: there is a birational proper modification $\mathcal{X}_T' \to \mathcal{X}_T$ such that the composite $\mathcal{X}_T' \to T$ is an analytically locally trivial fibration with snc K-trivial fibers. We assume either the relative dimension $n$ is 2 or that for general points $s$ of $S$, $\mathcal{X}_s$ is irreducible holomorphic
symplectic manifold and the fibers of $X'_T$ occur as good degenerations in the sense of [Nag08, 4.2].

(For $n = 2$, imagining with Brieskorn simultaneous resolution of deforming ADE singularities may help understanding. See also our Appendix.)

- We assume $\overline{M} := [S/G]$ is the normalization of some moduli stack $[S'/G]$ i.e., closed points $s, s' \in S'$ parametrize the same projective surface if and only if $Gs = Gs'$. The naturality of taking normalization can be seen e.g. in [ABE20].
- By Keel-Mori theorem [KeM97], $M$ has a coarse moduli algebraic space $\overline{M} \to M$ which we assume to be projective.
- Denote the irreducible decomposition of the branch divisor of $S \to M$ as $\bigcup_i D_i$ such that the branch degree along $D_i$ is $m_i$. We set $D := \sum_i \frac{m_i - 1}{m_i} D_i$.

**Theorem 3.2** (Log minimality). Under the above assumptions, $\overline{M}$ is log minimal with the natural boundary $\mathbb{Q}$-divisor i.e., $K_{\overline{M}} + D + \Delta$ is nef so that $(\overline{M}, D + \Delta)$ is a (log canonical) log minimal model. Equivalently, $K_S + \Delta$ is nef.

**Remark 3.3.** A related very recent result for the log smooth case under local Torelli type assumption, whose proof uses curvature consideration appears in [GGR21, Theorem 3.1(=1.17)].

**Remark 3.4.** One can also regard this theorem as a variant of another recent result [AE21, Theorem 1.2] which claims certain geometric compactifications under similar conditions are semi-toric in the sense of Looijenga [Looi03b], which are log minimal as easily follows from [AE21, 7.18]. On the other hand, the construction of $X_T$ in [AET19, ABE20, AE21] starts with $X'_T$ and then use contraction to obtain $X_T$ as their key idea. As it follows from the proof, Theorem 3.2 holds also if we weaken the maximal variation-ness of $X'_T$ over $T$ for each (fixed) $T$.

The following provides another partial reconstruction of Satake-Baily-Borel compactification or its analogue a priori.

**Corollary 3.5** (Alternative construction of Baily-Borel compactification). We work under the same setup as above §3.1.1, Theorem 3.2. If $(S, \Delta)$ is dlt and $\dim(S) \leq 4$, the normalized weak K-moduli $\overline{M} = S/G$ has a birational morphism to its log canonical model $\overline{M}^{lc}$ which exists. If $S^o$ is a locally Hermitian symmetric space, the lc model $\overline{M}^{lc}$ coincides with the Satake-Baily-Borel compactification.

**Proof.** The existence of log canonical model $\overline{M}^{lc}$ follows from [Fjn10] and the proof that $\overline{M} \dashrightarrow \overline{M}^{lc}$ is a morphism follows from Theorem 3.2. □
This gives a rather partial confirmation of [Od20, Conjecture B.1 (ii)], but it also seems to be the first arguments, which do not logically use the theory of Shimura varieties.

3.2. **Admissible variation of mixed Hodge structures.** Our approach to Theorem 3.2 is partially algebraic and partially analytic. The algebraic part is connected to a logarithmic refinement of the cone theorem. In particular, we use the log minimality criterion [Sva19, Fjn21] via Mori hyperbolicity which was introduced in [LZ17]. The analytic part involves variation of mixed Hodge structures. Indeed, by using *loc.cit*, we can and do reduce Theorem 3.2 to the following claim; Theorem 3.6 on the isotriviality of certain singular families and then prove it by crucially using variation of mixed Hodge structures. Its relatively smooth case is known by [VZ02, Theorem 1.4 (iii)] as its Corollary, but the point below is that we generalize it to allow even non-normal fibers so that the same results as smooth case ([VZ03, VZ02]) do not literally hold. Also note that taking normalization does not reduce to normal case, except for the case (i), since the variation of glueing data is usually important here.

**Theorem 3.6.** Consider any projective flat family $f: \mathcal{X} \to \mathbb{A}^1$ of relative dimension $n$, such that $\mathcal{X}$ is simple normal crossing variety whose strata maps smoothly over $\mathbb{A}^1$ and (relative) dualizing sheaf is trivial i.e., $\omega_{\mathcal{X}/\mathbb{A}^1} \sim_{\mathbb{A}^1} \mathcal{O}_\mathcal{X}$. Then it follows that for the irreducible decomposition $\mathcal{X} = \bigcup_i \mathcal{X}_i$, at least one strata $S_I := \bigcap_{i \in I} \mathcal{X}_i$, with its natural boundary divisor $D(S_I) := (\bigcup_{j \not\in I} \mathcal{X}_j) \cap S_I$ is isotrivial i.e., the isomorphic class of the pair

$$(S_I \cap \pi^{-1}(t), D(S_I) \cap \pi^{-1}(t))$$

does not depend on $t$.

Furthermore, assume that at least one of the following holds.

(i) $n = 1$ (in this case, $\omega_{\mathcal{X}/\mathbb{A}^1} \sim_{\mathbb{A}^1} \mathcal{O}_\mathcal{X}$ is also allowed) or

(ii) $n = 2$ (not assuming d-semistability of the fibers i.e., they may not be smoothable), or

(iii) $n$ is even and the fibers $\mathcal{X}_t (t \in \mathbb{A}^1)$ occur as good degenerations of holomorphic symplectic varieties in the sense of [Nag08, Definition 4.2] and its mixed Hodge structures on $H^2(\mathcal{X}_t, \mathbb{Z})$ [Del74] are not constant i.e., local (mixed) Torelli theorem holds for $\mathcal{X}/\mathbb{A}^1$.

Then $f$ is isotrivial i.e., the closed fibers $\mathcal{X}_t$ of $f$ are isomorphic to each other.

Here are some remarks especially on the latter assertion of isotriviality of $f$ in case (ii).

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4in particular, it is basic slc-trivial fibration in the sense of Fujino
Remark 3.7 (On the projectivity assumption). We expand the details of this remark at our Appendix §A. Recall that for any projective semistable family \( \mathcal{Y} \) over a curve whose general fibers are K3 surfaces, the result of Kulikov-Pinkham-Persson’s degeneration \([\text{Kul}77, \text{PP}81]\) (reviewed as Theorem [A.1]) is not necessarily a projective family. However from our Theorem [A.2] and its proof using MMP, it follows that the closed fibers \( \tilde{X}_0 \) are all projective even in such a case i.e., fiberwise projectivity holds. This explains the naturality of our projectivity assumption. Indeed, various part of Theorem [A.2] naturally extend over higher dimensional base case.

Remark 3.8. Recall that, for instance if \( n = 2 \) with nonempty 0-dimensional strata i.e., such as family of maximally degenerated K3 surfaces or abelian surfaces, the period spaces of such simple normal crossing K-trivial surfaces are known to be often algebraic torus (cf., e.g., \([\text{Carl}87, \text{FS}86]\)). Such arguments may look implying the above Theorem 3.6. However, for instance, there is still a exponential map from \( \mathbb{C} \) to algebraic torus which could be a priori the (mixed) period map, we have to exclude such possibility. We need more generalized version of such claim for the application to moduli compactifications, which is the point of above theorem and its formulation.

Remark 3.9. One can slightly weaken the locally triviality assumption of Theorem 3.6.

For instance, suppose \( n = 2 \) and neighborhood of normal crossing singular locus are topologically locally trivial over \( \mathbb{A}^1 \). Further, suppose \( f \) extends to \( f: \overline{\mathcal{X}} \to \mathbb{P}^1 \) and the fibers \( f^{-1}(t) \) only normal crossing singularities and also finite ADE singularities for at most one \( t(\neq \infty) \). Now we employ the Brieskorn simultaneous resolution for the relatively ADE locus to reduce to analytically locally trivial case. (Compare with our Appendix.) Note that for the simultaneous resolution, we do not need base change of \( \mathbb{A}^1 \) in such case, since the monodromy around \( \mathcal{X}_t \) with possibly ADE singularities is trivial. The triviality follows from its unipotency because \( \overline{\mathcal{X}}_{\infty} \) is reduced normal crossing by \([\text{Land}73]\) and also of finite order.

For the actual proof of Theorem 3.6 in the above generality, we analyze admissible variation of mixed Hodge structures (also known as smooth mixed Hodge modules) associated to our singular family.

Proof of Theorem 3.6 The proof of the former assertion follows from \([\text{Den}19]\) which we apply to the deforming strata (as a log pair). Indeed, if we consider the normalization \( S'_{I,t} \) of the fibers \( S_{I,t} \to \mathbb{A}^1 \), denoted as \( \nu_{I,t}: S'_{I,t} \to S_{I,t} \), the log pairs \( (S'_{I,t}, \nu_{I,t}^{-1}(\text{Sing}(S_{I,t}) + D(S_{I,t}))) \) becomes log Calabi-Yau pairs discussed in \([\text{Den}19]\) by a simple subadjunction. Hence we can apply \([\text{Den}19, \text{Theorem A}]\) to show our former assertion of Theorem 3.6.
For the latter assertion of the isotriviality of $f$, we first prove the cases (ii) and (iii), and later the case (i) for better comparison.

Consider $V := R^2 f_* f^{-1} O_{\mathbb{A}^1}$ which admits an admissible variation of mixed hodge structure $(V, F^\cdot, W^\cdot)$ due to [SZ85], [ElZ86], [Kas86], [FF14, 4.15]. For simplicity, by taking base change with respect to $m$-th power map $\mathbb{A}^1 \to \mathbb{A}^1$ sending $t$ to $t^m$ for certain $m \in \mathbb{Z}_{>0}$, we can and do assume that the monodromy of $W$ is unipotent. The above admissibility is in the sense of [SZ85, Kas86], which in particular asserts that for the Deligne canonical extension $V$ of $V$, we can take subbundles integral $W^\cdot$ as the canonical extensions of $W$ and locally free subbundles of $F^\cdot$. We consider the following Higgs field

$$
F_1^0/F_1^1 \xrightarrow{\theta_1} F^1/F^2 \otimes \Omega^1_{\mathbb{P}^1}(\log([\infty])),
$$

which comes from the Gauss-Manin connection, or in other words, cup product with the Kodaira-Spencer section. To make it explicit,

$$
F_1^1/F_2^2 \simeq R^1 f_* \tilde{\Omega}^1_{\mathcal{X}/\mathbb{A}^1},
$$

$$
F_0^0/F_1^1 \simeq R^2 f_* O_{\mathcal{X}},
$$

where $\tilde{\Omega}^1_{\mathcal{X}/\mathbb{A}^1} := \Omega^1_{\mathcal{X}/\mathbb{A}^1}/\tau^1_{\mathcal{X}}, \tau^1_{\mathcal{X}} := \text{tors}(\Omega^1_{\mathcal{X}/\mathbb{A}^1})$ (torsion subsheaf, supported on the singular locus of $\mathcal{X}$). After usual notation, we put $\mathcal{T}^o_{\mathcal{X}/\mathbb{A}^1} := (\Omega^1_{\mathcal{X}/\mathbb{A}^1})^\vee = (\Omega^1_{\mathcal{X}/\mathbb{A}^1}/\tau^1)^\vee$. Then the Kodaira-Spencer class of $f$ gives rise to a section $s$ of $R^1 f_* \mathcal{T}^o_{\mathcal{X}/\mathbb{A}^1}$ and $\theta_1$ is the cup product with $s$.

Suppose that $f$ is not isotrivial so that $s$ is nontrivial by the locally trivial version of deformation theory. In our case, the locally free sheaf $F^1/F^2$ is Grothendieck dual to $R^1 f_* \mathcal{T}^o_{\mathcal{X}/\mathbb{A}^1}$ hence the existence of nonzero $s$ asserts that $F^1/F^2$ is also not vanishing. On the other hand, as [FF14] shows for instance, $F^0/F^1 \simeq (f_* \omega_{\mathcal{X}/\mathbb{A}^1})^\vee$ which is an invertible sheaf.

Note that in (ii) case, $n = 2$ assumption implies a natural inclusion $F^2 \subset f_* \omega_{\mathcal{X}^\nu/k^1}$ from the description via standard Hodge type spectral sequence (cf., e.g., [Fri83, 1.5]) where $\mathcal{X}^\nu$ stands for the normalization of $\mathcal{X}$ which is smooth over $\mathbb{A}^1$. We can and do assume that $\mathcal{X}$ is not smooth since otherwise [VZ02, Theorem 1.4 (iii)] applies. Then, since each component of closed fibers of $\mathcal{X}^\nu$ should have nontrivial effective anticanonical divisor, we have $f_* \omega_{\mathcal{X}^\nu/k^1} = 0$ hence $F^2 = 0$.

Similarly, in (iii) case, we have $F^2 \subset f_* \Omega^2_{\mathcal{X}^\nu/k^1}$ whose right hand side vanishes by the assumption. Also, if $\mathcal{X}$ is non-isotrivial which we assume here, the local Torelli type assumption for (iii) implies that the Higgs field $F^1 \to F^0/F^1$, induced by the Gauss-Manin connection, is automatically nontrivial.
In both cases, either (ii) and (iii), the seminegativity results [FF14, 1.3] following the method of the curvature formula of Hodge metrics due to Griffiths [Gri70] [Zuc82], show \( F_1 = F^1 / F^2 \) and \( F_0 / F_1 \) are both seminegative. This contradicts with (i) since \( \Omega_{\text{log}}^2(\mathbb{P}^1) \) has negative degree \((-1)\). This completes the proof of Theorem 3.6 cases (ii), (iii).

Now we move on to the (relatively) 1-dimensional case (i). In this case, we do not need to assume \( X / \mathbb{A}^1 \) is K-trivial. We consider the \( V := R^1 f_* f^{-1} O_{\mathbb{A}^1} \) which again admits an admissible variation of mixed hodge structure \((V, F \cdot, W \cdot)\).

Then similarly as above, \( F_1 = f_* (\Omega_{X / \mathbb{A}^1} / \tau_{X / \mathbb{A}^1}) \), where \( \tau_{X / \mathbb{A}^1} \) denotes the torsion of \( \Omega_{X / \mathbb{A}^1} \). \( F_1 \) is also canonically isomorphic to \((f \circ \nu)_* \omega_{X / \mathbb{A}^1} \), where \( \nu: X^\nu \to X \) denotes the normalization of \( X \). Also, \( F_0 / F_1 \simeq R^1 f_* O_X \simeq (f_* \omega_{X / \mathbb{A}^1})^\vee \).

Suppose the contrary of the assertion i.e., non-isotriviality of \( X \) over \( \mathbb{A}^1 \). Note that since the base \( C \) is simply connected, the combinatorial structure i.e., dual graph of the fibers are canonically identified to each other. Then the fibers of \( X^\nu \) with natural marking on \( \nu^{-1} \text{Sing}(X / \mathbb{A}^1) \) where \( \text{Sing}(X / \mathbb{A}^1) \) denotes the (relatively) non-smooth locus, should contain non-isotrivial family of marked smooth curves. For every rational components of the fibers of \( X^\nu \) over \( \mathbb{A}^1 \), since any four marked points gives rise to a nonvanishing cross ratio, we conclude that \( \mathbb{P}^1 \) in the normalization together with the marked points are (iso)trivial. On the other hand, for any non-rational components \( Z_t \) of the fibers of \( X^\nu \), the same arguments to the above proof of cases (ii), (iii) show \( Z_t \) itself is isotrivial since \( F_1 \) deforms nontrivially in \( R^1 f_* f^{-1} O_{\mathbb{A}^1} \) otherwise, to lead to the contradiction. (Or apply [VZ03, VZ02].) Furthermore, since the base is rational it also follows that the natural marked points in the components \( Z_t \) are again (iso)trivial. Therefore, combining these, we again conclude that \( X \) is isotrivial and hence the proof of the latter assertion of Theorem 3.6 for (i) case.

\[ \square \]

Remark 3.10. The above proof applies rather partially for \( n > 2 \) under “artificial” assumptions (e.g., \( n = 3 \) and all components \( V_i \) satisfying \( h^{2,1}(V_i) = 0 \) such as \( \mathbb{P}^3 \)) but we omit it here as we have no particular application.

Now we are ready to combine our results to confirm the desired log minimality.

**proof of Theorem 3.2** We use [Sva19, 1.1, 6.9] or [Fjn21, 1.4, 9.1] to reduce the proof to the claim that each log canonical center of \((S, \Delta)\) does not allow any non-constant morphism from the affine line \( \mathbb{A}^1 \). On the other hand, our theorem 3.6 using admissible variation of mixed Hodge structures implies it by our fourth assumption in the Notation 3.1.1 This completes the proof.

\[ \square \]
Remark 3.11. As an analogue of Theorem [3.2] Theorem [3.6] combined with \[\text{Sva19, Fjn21}\] it follows that \((\mathcal{M}_g^{\text{DM}}, \mathcal{M}_g^{\text{DM}} \setminus \mathcal{M}_g)\) is at least log minimal. Here, \(\mathcal{M}_g\) refers to the moduli stack of smooth projective curve of genus \(g \geq 2\) and \(\mathcal{M}_g^{\text{DM}}\) refers to its Deligne-Mumford compactification. What we meant by log minimality at the stacky level above can be also rephrased as follows: if we take a coarse moduli scheme \(\mathcal{M}_g \to \mathcal{M}_g\) branches at \(D_i\) with degree \(b_i\), the above log minimality is equivalent to that of \((\mathcal{M}_g^{\text{DM}}, \sum_i b_i^{-1}D_i + (\mathcal{M}_g^{\text{DM}} \setminus \mathcal{M}_g))\). This partially recover [CH88, Theorem 1.3] by a fairly different method.

Appendix A. Algebraic construction of Kulikov models

A.1. Review of history and original statements. The following classical theorem is our topic of this appendix, which is indirectly related to the problem of weak K-moduli (see e.g., [AE21]).

**Theorem A.1 ([Kul77, Theorem I], [PP81, Theorem]).** \(\pi : \mathcal{X} \to \Delta = \{ t \in \mathbb{C} \mid |t| < 1 \}\) be a proper flat family of complex analytic surfaces such that

(i) general fibers \(\mathcal{X}_t\) are smooth with trivial canonical line bundles

(ii) central fiber \(\mathcal{X}_0\) is (reduced) simple normal crossing with all irreducible components algebraic

Then there is another model \(\tilde{\mathcal{X}}\) which coincides away from \(\mathcal{X}_0\) which satisfies both two above conditions and further that \(K_{\tilde{\mathcal{X}}/\Delta} \sim 0\).

Recall their construction was highly topological with respect to complex analytic topology. In [Kul77, Theorem I], an additional assumption that the family is projective is put but the structures of the degenerations are also determined. Such determination of degeneration types is also similarly done in arithmetic setting (cf., e.g., [Nakk00, 3.4]).

The purpose here is to recover and generalize the above model construction, by means of the minimal model program, which originally emerged shortly after [Kul77, PP81] and much developed during these several decades.

Hence the idea is fairly natural and simple. The author once believed such arguments had been naturally expected as folklore or perhaps even known to some experts of birational geometry as he indeed felt during several conversations with experts. Nevertheless, the only literature the author has been able to find so far is the nice series of works by Yuya Matsumoto and Christian Liedke in a more arithmetic context. See the proofs of [Mat15, 3.1], [LM18, 3.1], which we review at A.7 later. Their papers [Mat15, LM18] indeed went beyond; establishing good reduction criteria for arithmetic K3 surfaces as in abelian varieties case, rather than just (re)constructing Kulikov model in general. This notes only mean to fill the apparent lack of (more)
complete reference on this matter, as a slight refined version disallowing a base change. See Theorem A.2 below and their proofs for the details.

A.2. **Proof of algebraic case.** We first (re)construct the Kulikov model via the minimal model program, with the base field an algebraically closed field $k$ of any characteristics.

**Theorem A.2 (Kulikov model - algebraic reconstruction).** Suppose the base field $k$ is algebraically closed. Take any strictly semistable proper scheme of dimension 3 over a smooth $k$-curve $C$, denoted as $\pi: X \to C$, whose general fibers have trivial canonical divisors. Then, there is a birational map from an algebraic space $\tilde{X}/\pi$ such that

(i) strictly semistable
(ii) $K_{\tilde{X}/C} \sim_C 0$.
(iii) all closed fibers of $\tilde{X} \to C$ are projective algebraic.

Note that the above statements do not involve base change of $C$. Also note that the point (iii) gives slight improvements of the component-wise algebraicity in original A.1. For extensions over more general base with similar arguments, we refer to §A.3.

**proof of Theorem A.2.** First we replace $X$ by its blow up which is projective over $C$ and its further projective log resolution so that one can assume $X$ is projective over $C$. Then we replace $X$ by its minimal model by either [Fjn11] (for characteristic 0) or [HX15, HW19] (for positive characteristics). Our proof depends on analysis of the possible singularities on a relative minimal model of $X$ over $C$, which we denote $\pi_{\text{min}}: X_{\text{min}} \to C$. Our first step is as follows, which is not discussed in [Kawa94, Mat15, LM18] because [Kawa94] assumes a weaker version of $\mathbb{Q}$-factoriality which excludes most subtle singularities.

In this notes, dlt minimal model $\pi_{\text{min}}: X_{\text{min}} \to C$ only means that $(X_{\text{min}}, X_{\text{min},0})$ is dlt, $K_{X_{\text{min}}}(+, X_{\text{min},0}) \sim_{C, \mathbb{Q}} 0$, without $\mathbb{Q}$-factoriality for generalization (e.g. to allow all nodes). The next proposition is a first step, which also aims at understanding the local strutures of all non-necessarily $\mathbb{Q}$-factorial dlt models.

**Proposition A.3 (Small resolution as algebraic space).** For any dlt minimal model $\pi_{\text{min}}: X_{\text{min}} \to C$, there is a small resolution from relatively proper (over $C$) algebraic space $Y \to X_{\text{min}}$ such that any closed point $p \in Y$ satisfies one of the followings. Here, the composite $Y \to C$ is denoted as $\pi_Y$.

(i) $Y$ is regular at $p$ and the fiber $\pi_Y^{-1}(p)$ is normal crossing around $p$ or
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(ii) \( \pi_Y^{-1}(p) \ni p \) is a rational double point.

**proof of Proposition A.2** We first analyze the local structure of \( X_{\text{min}} \). While we make some arguments partially self-contained, they are essentially in [Rei80], [Mor85] as their special cases. Since singular fibers lie over only finite closed points in \( C \), we can focus on one of them which we denote as \( 0 \in C \). We denote its fiber as \( X_0 \).

If one takes the germ of any (but automatically terminal) singularity \( p \in \mathcal{X} \), \( X_{\text{min},0} \) is Gorenstein by the upper semicontinuity of \( \dim(H^0(\omega_{\mathcal{X}_{\text{min}}/C}|_{\mathcal{X}_{\text{min},t}})) \) (cf., [Fjn11]). Therefore, from the adjunction of \( X_{\text{min}} \) to \( X_{\text{min},0} \), it also follows that \( X_{\text{min}} \ni p \) is also Gorenstein, not only \( \mathbb{Q} \)-Gorenstein. Also note that \( X_{\text{min},0} \) is semi-dlt in the sense of [Kol13, 5.19] which is stronger than that of [Fjn00, 1.1, 1.2].

Its general hyperplane passing through \( p \) obtains DuVal singularity at \( p \), by the classics of M.Reid, S.Mori in characteristics 0 (cf., e.g., [Rei80], [Mor85], [KM98 §5.3]) which also holds in positive characteristics due to recent [ST17]. Hence \( p \in X_{\text{min}} \) itself is locally a hypersurface singularity, so that we naturally wish to analyze the local equation in the completion of \( \mathcal{O}_{x,X_{\text{min}}} \).

We take a uniformizer of \( 0 \in C \) as \( u \), which we can regard as an element of \( \mathcal{O}_{x,X_{\text{min}}} \) or its completion \( \widehat{\mathcal{O}}_{x,X_{\text{min}}} \) via \( \pi_{\text{min}} \).

In our situation, we know that \( p \in X_{\text{min}} \) is Gorenstein. If \( p \in (u = 0) \cap \mathcal{X} \) passed through by three smooth irreducible components, then from sdlt condition it is locally strictly simple normal crossing thus in the situation of (i). Also, if \( p \in (u = 0) \cap \mathcal{X} \) is normal then it is in the situation of (ii). If \( p \in \pi_Y^{-1}(0) \) is an irreducible germ, it is klt and Gorenstein hence rational double point i.e., in the situation of (ii).

Suppose otherwise - there would be exactly two irreducible components \( V_i \ni p(i = 1, 2) \) passing through \( p \in (u = 0) \cap \mathcal{X} \), with the double locus \( D = V_1 \cap V_2 \). Then, the germ \( (V_i, D) \) is plt, hence its formal germ is same as the cyclic quotient singularity \( \frac{1}{n}(1, r) \) in \( \mathbb{A}^2_{x,y} \) where \( D \) is the vanishing locus of the first coordinate \( x \) and \( \gcd(r, n) = 1 \) (cf., e.g., [Kol13 3.31, 3.32]). A direct self-contained explanation to confirm it is to take the index 1 covering of \( V_1 \), which is a cyclic covering, to make the pullback of \( V_1 \cap V_2 \) Cartier and regular again. Thus the cyclic cover itself is smooth and we observe the above description.

As the Gorensteinness of \( X_{\text{min},0} \) implies, \( \text{Res}_{(x=0)}(\frac{dx}{x} \wedge dy)|_{y=0} = dy|_{y=0} \) needs to be \( \mu_n \)-invariant, hence \( n = 1 \) so that \( V_i \ni p(i = 1, 2) \) are both smooth at \( p \). This implies that

\[
\widehat{\mathcal{O}}_{p,X_{\text{min}}}(u) \simeq k[[x, y, z]]/(xy).
\]
Since the embedded dimension of $p \in X_{\min}$ is 4, $u$ gives a regular element of $\tilde{O}_{p,X_{\min}}$ and any lift of $x, y, z$ denoted by the same letters complements the system of parameters of $\tilde{O}_{p,X_{\min}}$ as $x, y, z, u$. Hence, there is a $F \in k[[x, y, z, u]]$ such that

$$\tilde{O}_{p,X_{\min}} \simeq k[[x, y, z, u]]/F(x, y, z, u).$$

From (2),

$$F(x, y, z, u) = xy + uf(x, y, z, u).$$

We note that the above type singularity is (even algebraically) non-$\mathbb{Q}$-factorial, as it follows when we consider the blow up of the prime divisor which is formally locally $(x, u)$ in the above. Hence, in $\mathbb{Q}$-factorial assumption, we can shorten the following arguments. Note in particular that, in the case $X_{\min}$ is obtained by running relative MMP over $C$ from regular semistable model $\mathcal{Y}$, it is $\mathbb{Q}$-factorial (see [KM98, 3.18, 3.37] whose proofs work over any field).

We continue the analysis of the above type (non-$\mathbb{Q}$-factorial) singularity, If $f \equiv aux + bzu (mod m_{x,X_{\min}}^2)$ for some $a, b \in k$, by replacing $x, y$ by $x + bu, y + au$ (which we still denote by the same letters, avoiding complications) the equation becomes

$$xy + uh_x + uy_h_y + ug(z, u),$$

where $h_x, h_y, g \in k[[x, y, z, u]]$ and $g$ involves the variables only $z, u$. By further replacing $x, y$ by $x + uh_y, y + uh_x$, we can and do assume $h_x = h_y = 0$. Hence, we obtain a normal form of $F$ as

$$F(x, y, z, u) = xy + ug(z, u),$$

where $g(z, u)$ does not contain non-zero constant i.e., not unit.

Note that this is essentially proven as a special case in [Mor85, Theorem 12 (also cf., Theorem 3, Corollary 4)]. Also note that without Gorenstein property, a priori it would possibly have been $\mu_m$-quotient of the above equation in general but comparing with [KSB88, 6.8(vi)] anyhow implies $m = 1$. Now we decompose $ug(u, z) \in k[[x, y]]$ into the finite product of irreducible formal power series $g_i(u, z)$s for $i = 1, \cdots, l$. We can and do suppose $g_1 = u$.

Now we consider the formal blow up of the formal scheme

$$\text{Spf}(\tilde{O}_{p,X_{\min}} \simeq k[[x, y, z, u]]/F(x, y, z, u))$$

along the ideal sheaf $(x, g_i)$. Clearly, it is covered by a regular open subset and

$$\text{Spf}(k[[x, y, z, u]]/\frac{F(x, y, z, u)}{u}),$$
where $F(x,y,z,u) = g(z,u)$. Repeating the same procedure for $g,s$ inductively, we obtain a formal $k$-scheme $\tilde{X}$ properly mapping to $\text{Spf}(\hat{O}_p)$ whose singularities are all formally isomorphic to $\text{Spf}(k[[x,y,z,u]]/xy-g_i(z,u))$, where one of the formal coordinate $x$ is replaced during each formal blow up. Note that $\text{Spf}(k[[x,y,z,u]]/xy-g_i(z,u))$ is either smooth or the completion of isolated singularity at $(x,y,z,u) = (0,0,0,0)$ which restricts to a type rational double points along $u = 0$ hyperplane: hence satisfying the desired property of Proposition A.3 at the formal (and local) level. We do the same procedures at all $p \in \text{Sing}(X_{\text{min}})$.

Finally, using [Art70, Theorem 3.2, §5], there is an algebraic $k$-space $\tilde{X} \to X$ which gives rise to the above $\tilde{X} \to \text{Spf}(\hat{O}_p)$ for any $p \in X_{\text{min}}$. We complete the proof of Proposition A.3.

Remark A.4. The above type (5) terminal singularity, with possibly nonunit $g(z,u)$ are not contained in the classification list of the singularities [Kawa94, 4.4] due to the assumption in loc.cit 1.1 (3) which excludes it. Indeed, if you take étale presentation of such algebraic space, and pull back the ideal we blow up, we observe that above type singularity violates the condition 1.1(3) of [Kawa94] (or only the $\mathbb{Q}$-Cartierness of $V_i$s in our setting). This arguments adds another explanation to the proof of [Kawa94, 4.4].

Now we come back to continue the proof of Theorem A.2. Our second step is the following, a wellknown procedure due to [Bri70, Slo80] (see also recent [SB01, SB21, Mat15, LM18]). We write here just for clarity and self-containedness.

Lemma A.5 (Simultaneous resolution). Consider any proper algebraic space $\pi_Y: Y \to C$ over a smooth $k$-curve $C$ whose generic fiber is smooth with trivial canonical divisor, such that any closed point $y \in Y$ satisfies one of the followings:

(i) $Y$ is regular at $y$ and the fiber $\pi_Y^{-1}(\pi_Y(y))$ is normal crossing around $y$ or

(ii) $\pi_Y^{-1}(\pi_Y(y)) \ni y$ is a rational double point.

Further, suppose that there is a birational strictly semistable model $X \to C$ with the same generic fiber as that of $Y \to C$.

Then there is a simultaneous (small) resolution $\tilde{X} \to Y$ in the category of algebraic spaces.

proof of Lemma A.5. Fix a rational double point $y$ of fiber as (ii), take its neighborhood $U \subset Y$ and denotes its complement as $Z$. \footnote{this allows any field or excellent Dedekind scheme as a base}
The strict semistability of \( X^o := X \setminus (Z \setminus \pi_Y^{-1}(y)) \) implies that action of the Galois group \( \text{Gal}(K^{sep}/K) \) on \( H^2_{\text{ét}}(X^o_{K(\eta)}, \mathbb{Q}_l) \) is unipotent ([RZ82]) and of finite index hence trivial action on it. Then, Brieskorn simultaneous resolution due to [Bri70, Slo80] and also [SB21, Corollary 2.13], [SB01, 4.7] shows the existence of simultaneous resolution of \( y \) without base change, at the category of algebraic spaces. Indeed, note that the assumption made at the first paragraph of [SB21, §2] automatically holds since we work over an algebraically closed field. We do the same for all fiberwise rational double points \( y \). If \( k = \mathbb{C} \), we could replace the use of above [SB21] by more classical [Land73].

Combining Proposition A.3 and A.5, we conclude the proof of Theorem A.2.

A.3. For generalizations. We discuss towards generalization of the above approach. Firstly, it is natural to expect to recover the analytic version Theorem A.1 fully from the MMP method.

Remark A.6 (Complex analytic version). For more complex analytic statement Theorem A.1 the above arguments with following slight verbatim modifications give an alternative partial proof when we further assume that \( X \to C \) is proper and projective away from singular fibers.

(i) For relative MMP, use [KNX18] (cf., also [HP16]) instead of the version with algebraic base [Fin11, HX15, HW19],
(ii) use original [Land73, Bri70, Slo80] rather than [SB01, SB21],
(iii) prove the analytic version of the Proposition A.3 by replacing our use of formal blow up by direct small analytic blow up

To recover Theorem A.1 fully by this method, we (only) need relative/semistable extension of Kähler (absolute) MMP after [HP16].

On the other hand, also recall that Theorem A.1 cannot be generalized to the case when the central fiber is allowed to contain non-Kähler components \( V \), as the counterexamples found by Nishiguchi [Nis88, Theorem 4.4, §5 (compare with §2)] show. In his counterexamples, \( V \) are certain VII surfaces which he calls CB surfaces.

We also review the following result due to [Mat15, LM18] for convenience. That is, over more general Dedekind schemes with arbitrary (possibly mixed) characteristics and non-closed residue fields, at least we know the following version which allow finite base change. The reason we only discuss such weak version comes from the additional implicit assumptions

\footnote{not only the inertia group around \( \pi_Y(y) \). By the way, this difference becomes crucial at least over general CDVR cf., [Mat15 5.3], [LM18, §7]}
made in the proofs of [SB21, Corollary 2.13], [SB01, 4.7]. Below, we do not use [SB01, SB21].

**Proposition A.7** ([Mat15, 3.1], [LM18, 3.1]). *Take any projective scheme over an excellent Dedekind scheme \( C \), denoted as \( \pi : X \to C \), whose generic fiber has a trivial canonical divisor. We assume it has a strictly semistable model.*

Then, possibly after a finite base change of \( C \), is a birational map from an algebraic space \( \tilde{X} \) such that

1. \( \tilde{X} \) is strictly semistable over \( C \)
2. \( K_{\tilde{X}/C} \sim_{C} 0 \).

**Proof.** This is essentially proven in the arguments during the proofs of [Mat15, 3.1] and [LM18, 3.1], although the latter forgets the following process (i).

We just review that the differences with the above proof of Theorem [A.2] are:

1. do natural toroidal resolution (cf., e.g., [Sai04, 2.9.2]) after a priori nontrivial base change, corresponding to a regular subdivision of a cone complex
2. replace the partial use of [SB01, SB21] by more classical [Art74] for simultaneous resolution (in the proof of Proposition [A.5]),
3. replace the use of [Fjn11, HX15, HW19] by [Kawa94, TY20, BMPSTWW20]).

□

Again, it seems reasonable to expect refinement of above at the level of Theorem [A.2] may be possible by similar method (see [SB21] again) but we do not pursue further in this notes.

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