Three-Dimensional CFTs and RG Flow from Squashing M2-Brane Horizon

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abstract

Utilizing AdS/CFT correspondence in M-theory, an example of interacting $d = 3$ conformal field theories and renormalization group flow between them is presented. Near-horizon geometry of $N$ coincident M2-branes located on a conical singularity on eight-dimensional hyperkähler manifold or manifold with Spin(7) holonomy is, in large-$N$ limit, $AdS_4 \times X_7$, where $X_7$ is seven-sphere with squashing. Deformation from round $S_7$ to squashed one is known to lead to spontaneous breaking of $\mathcal{N} = 8$ local supersymmetry in gauged $AdS_4$ supergravity to $\mathcal{N} = 1, 0$. Via AdS/CFT correspondence, it is interpreted as renormalization group flow from $SO(5) \times SO(3)$ symmetric UV fixed point to $SO(8)$ symmetric IR fixed point. Evidences for the interpretation are found both from supergravity scalar potential and existence of interpolating static domain-wall thereof, and from conformal dimensions of relevant chiral primary operator at each fixed point.

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1 Introduction

In spacetime of dimensions more than two, few examples are known for interacting conformal field theories or for renormalization group flows among themselves. This is especially so in three dimensions, as the infrared limit corresponds typically to a strong coupling limit. Known examples include non-Gaussian fixed point of $O(N)$ vector model in large $N$ limit, infrared limit of (non)abelian gauge theories with $\mathcal{N} = 8, 4$ or 2 supersymmetries. The latter can be obtained by dimensional reduction of four-dimensional gauge theories with $\mathcal{N} = 4, 2, 1$ supersymmetries. As such, R-symmetries, holomorphy and supersymmetry non-renormalization theorems put strong constraints to the moduli space of these theories. Combining these analysis, a relation called ‘mirror symmetry’ between infrared limit of different gauge theories has been discovered [1] and rederived using brane configurations [2]: Kähler (complex) structure on the Coulomb branch of an $\mathcal{N} = 4$ ($\mathcal{N} = 2$) gauge theory is mapped to that on the Higgs branch of its mirror and vice versa.

In contrast, three-dimensional $\mathcal{N} = 1$ superconformal field theories do not have any continuous R-symmetries, holomorphy constraints or any known non-renormalization theorems. As such, it would be quite difficult to draw any useful information regarding these theories. The difficulty may be exemplified, for instance, from the infrared limit of a $\mathcal{N} = 1$ brane configuration consisting of $N_c$ D3-branes between a pair of rotated NS5-branes and $N_f$ D5-branes: the Higgs branch can be reached from any point on the Coulomb branch and hence, classically, every point of the moduli space defines an interacting conformal field theory. The situation suggests that one would really need to follow technically a rather different route in order to understand three-dimensional $\mathcal{N} = 1$ superconformal field theories.

In this paper, utilizing dual correspondence between $(d + 1)$-dimensional anti-de Sitter supergravity and $d$-dimensional conformal field theory (AdS/CFT correspondence) [3, 4, 5], we will present an example of interacting $\mathcal{N} = 1$ superconformal field theory (with some global symmetries) and renormalization group flow thereof. The example is based on the well-known result in Kaluza-Klein supergravity [6]: compactification of eleven-dimensional supergravity to four-dimensional anti-de Sitter spacetime on round-$S^7$ with $SO(8)$ isometry or squashed-$S^7$ with $SO(5) \times SO(3)$ isometry [7]. For the Freund-Rubin compactification [8] of eleven-dimensional supergravity on a seven-dimensional compact manifold $X_7$ that is topologically $S_7$, it has been known that round and squashed seven-spheres are the only possibilities, and that both are stable, at least, at perturbative level. More appropriately, the compactifications correspond to M-theory vacua describing near-horizon geometry of two distinct configurations (one spherically symmetric and another non-spherical) of $N$ coincident M2-branes.

In doing so, we will also discover another closely related, nonsupersymmetric interacting conformal field theory, which flows along the same renormalization group trajectory as the $\mathcal{N} =$
1 counterpart. This comes about as follows. An important aspect of the \( S_7 \) compactification is that, depending on the choice of the orientation, one and the same squashing deformation leads to two distinct vacua. Denoting chirality of supercharges on \( AdS_4 \) (which has descended from Killing spinors on \( X_7 \)) as \((N_L, N_R)\), the round and the squashed seven-spheres are known to preserve \((8,8)\) and \((1,0)\) supersymmetries, respectively. Because of triality of \( SO(8) \), squashing deformations can proceed in two different embeddings (so-called ‘skew-whipping’ \([3]\)) of residual isometries: \( SO(8) \to [SO(5) \times SO(3)]_c \) for left-handed orientation or \([SO(5) \times SO(3)]_s \) for right-handed orientation. Thus, for the squashing with left-handed orientation, the renormalization group flow interpolates between conformal field theories with \( \mathcal{N} = 8 \) and \( \mathcal{N} = 1 \) supersymmetry, while for the squashing with right-handed orientation, the flow interpolates between conformal field theories with \( \mathcal{N} = 8 \) and \( \mathcal{N} = 0 \). Along the deformation, the so-called ‘space invader scenario’ – complicated level-crossing phenomena among massless and massive Kaluza-Klein states – then implies that the supersymmetry is broken completely (except the two endpoints) and the scaling dimension of operators will become renormalized in a highly nontrivial manner. For both choices of orientation, however, the squashing deformation turns out to be governed by one and the same Kaluza-Klein mode. Hence, for both \( \mathcal{N} = 1 \) and 0 conformal field theories, the renormalization group flow ought to be governed by the same scaling operator.

We will begin our analysis in section 2 by recapitulating relevant aspects of round and squashed \( S_7 \) compactification vacua in eleven-dimensional supergravity, but rephrased in terms of M2-brane parameters. In section 3, we will investigate ‘squashing’ deformation of each vacua, and find that the deformation is described by an irrelevant operator at the \( \mathcal{N} = 8 \) conformal fixed point, but by a relevant operator at the \( \mathcal{N} = 1 \) or 0 conformal fixed points. The renormalization group flows along the squashing deformation trajectory would then interpolate between \( \mathcal{N} = 8 \) fixed point at the infrared and \( \mathcal{N} = 1 \) or 0 at the ultraviolet. As suggested in \([10]\), the renormalization group flow is described in \( AdS_4 \) supergravity by a static ‘domain wall’ interpolating between the round- and the squashed-\( S_7 \) vacua. In section 4, developing first a general argument regarding nonperturbative stability, we will reproduce the renormalization group flow along the squashing trajectory as a static domain wall.

Throughout this paper, we will be using the metric convention \((-,+,...,+))\). The eleven-dimensional Planck scale is denoted by \( \ell_p \). Our notation is that the \( d = 11 \) coordinates with indices \( A, B, \cdots \) are decomposed into \( d = 4 \) spacetime coordinates \( x \) with indices \( \alpha, \beta, \cdots \) and \( d = 7 \) internal space coordinates \( y \) with indices \( a, b, \cdots \). Denoting the \( d = 11 \) metric as \( g_{AB} \) and the antisymmetric tensor field as \( F_{ABCD} = 4 \nabla_{[A} C_{BCD]} \), the bosonic field equations are:

\[
\frac{1}{\ell_p^2} R^A_B = \frac{1}{3} F^{APQR} F_{BPQR} - \frac{1}{36} \delta^A_B F^{PQRS} F_{PQRS}
\]

\[
\nabla_A F^{ABCD} = -\frac{1}{4!} \epsilon^{BCDM_1 \cdots M_8} F_{M_1 \cdots M_4} F_{M_5 \cdots M_8}.
\]

(1)
2 AdS$_4$ Supergravity Vacua: Round and Squashed S$^7$

In this section, we will be recapitulating some of the relevant results regarding spontaneous compactification of $d = 11$ supergravity on AdS$_4 \times X_7$, where $X_7$ denotes a seven-dimensional compact Einstein manifold. The $X_7$ refers to the near-horizon geometry of M2-branes. For $X_7$ diffeomorphic to round seven-sphere, it is well-known that there exist only two possible Einstein manifolds [7]: round and squashed seven-spheres. We will denote them by $S^7$ and $\tilde{S}^7$, respectively. The latter is a homogeneous space $[SO(5) \times SO(3)]/[SO(3) \times SO(3)]$ with a weak $G_2$ holonomy. In fact, generically, M2-branes on a (noncompact) eight-dimensional hyperkähler manifold or manifold with $Spin(7)$ holonomy, the near-horizon geometry is expected to vary from spherical to squashed seven-sphere as the branes are placed away or at a conical singularity of the manifold [12, 11].

Embedded into $d = 11$, the interpolating metric between round and squashed seven-spheres may be written as

$$ds^2 = R^2 \left[ e^{-7u} g_{\alpha\beta} dx^\alpha dx^\beta + e^{2u+3v} \left( \frac{1}{4} d\mu^2 + \frac{1}{16} \omega^2 \sin^2 \mu \right) + e^{2u-4v} \frac{1}{16} (\tilde{\nu} + \tilde{\omega} \cos \mu)^2 \right].$$ (2)

Here, $\tilde{\nu}, \tilde{\omega}$ are diagonal linear combinations of one-forms, each satisfying $SU(2)$ algebra:

$$\tilde{\nu} = \tilde{\sigma} + \tilde{\Sigma}, \quad \tilde{\omega} = \tilde{\sigma} - \tilde{\Sigma}$$

$$d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$$

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The parameter $R$ measures the overall radius of curvature. The scalar fields $u(x), v(x)$ parametrize ‘size’ and ‘squashing’ deformation of $S^7$ over $d = 4$ spacetime: $\text{Vol}(S^7) = \frac{1}{3} \pi^4 e^{7u} R^7$, and squashing is parametrized by $\lambda^2 \equiv e^{-7u}$.

Spontaneous compactification of M-theory to AdS$_4 \times S^7$ is obtained from near-horizon geometry of $N$ coincident M2-branes. Geometry of the horizon can be deformed continuously but, as mentioned above, the resulting horizon is an Einstein manifold only for round- or squashed-$S^7$’s. Through the seven-sphere, the M2-branes thread nonvanishing flux of four-form field strength of the Freund-Rubin form:

$$F_{\alpha\beta\gamma\delta} = Q e^{-7u} \epsilon_{\alpha\beta\gamma\delta} = Q e^{-21u} \epsilon_{\alpha\beta\gamma\delta}. \quad \text{(3)}$$

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2 By placing M2-branes at conical singularities defined by

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^{6k-1} = 0, \quad k = \text{integer}$$

it is also possible to get exotic non-spherical horizons that are topologically equivalent to $S^7$ but not diffeomorphic to it [11]. Isometry group of the exotic seven-spheres is $SO(2)$ or $SO(3)$ and hence may correspond to $\mathcal{N} = 2, 3$ superconformal field theories.
The parameter \( Q \) refers to so-called ‘Page’ charge \( Q \equiv \pi^{-4} \int_X (\ast F + C \wedge F) \), and is related to the total number of M2-branes, \( N \), as

\[
Q = 96\pi^2 N \ell_p^6.
\]  

(4)

The \( d = 4 \) field equations resulting from insertion of the ansatz Eqs.(2, 3) into Eq.(1) can be compactly summarized by the following effective Lagrangian [13]:

\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{63}{2} (\partial u)^2 - 21 (\partial v)^2 - V(u, v) \right),
\]  

(5)

where

\[
V(u, v) = e^{-9u} \left[ -6e^{+4v} - 48e^{-3v} + 12e^{-10v} + 2Q^2 e^{-12u} \right].
\]  

(6)

The field equations are

\[
\begin{align*}
\partial^2 u &= \frac{6}{7}e^{-9u+4v} + \frac{48}{7}e^{-9u-3v} - \frac{12}{7}e^{-9u-10v} - \frac{2}{3}Q^2 e^{-21u}, \\
\partial^2 v &= -\frac{4}{7}e^{-9u+4v} + \frac{24}{7}e^{-9u-3v} - \frac{20}{7}e^{-9u-10v},
\end{align*}
\]  

(7)

and

\[
R_{\alpha\beta} = \frac{63}{2} \partial_\alpha u \partial_\beta u + 21 \partial_\alpha v \partial_\beta v + g_{\alpha\beta} e^{-9u} \left[ -3e^{+4v} - 24e^{-3v} + 6e^{-10v} + Q^2 e^{-12u} \right].
\]  

(8)

The AdS-invariant ground-states correspond to \( u, v \) taking constant values and the space-time curvature maximally symmetric, \( R_{\alpha\beta\gamma\delta} = \frac{1}{3} \Lambda (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \), \( R_{\alpha\beta} = \Lambda g_{\alpha\beta} \). The aforementioned two vacua of Eqs.(7,8) are:

\[
S_7 : \quad u = u_1 = \frac{1}{12} \ln(3^{-2}Q^2), \quad v = v_1 = 0 \quad (\lambda^2 = 1) \\
\Lambda_1 = -12 \left| \frac{Q}{3} \right|^{-3/2}
\]  

(9)

and

\[
\tilde{S}_7 : \quad u = u_2 = \frac{1}{12} \ln(3^{-45}10^7 Q^2), \quad v = v_2 = \frac{1}{7} \ln 5 \quad \left( \lambda^2 = \frac{1}{5} \right), \\
\Lambda_2 = -12 \cdot 3^{7/2} 5^{-5/2} \left| \frac{Q}{3} \right|^{-3/2}
\]  

(10)

respectively.

The scalar potential \( V(u, v) \) depicted in Figure 1 displays the two critical points: \( S_7 \) is a saddle point and corresponds to a minimum (nearly) along \( v \)-direction, while the \( \tilde{S}_7 \) is a
Figure 1: Scalar potential $V(u,v)$ for $Q = 3$. The local minimum point and the saddle point correspond to the $S_7$ and the $\tilde{S}_7$, respectively. Note that the steepest descent is nearly along the $v$ (squashing parameter) direction. Both ground-states are stable against $u$ (breathing) direction deformation.

maximum $[3]$. Recall that $u, v$ fields parametrize all possible deformation of $S^7$, but restricted to retain at least $SO(5) \times SO(3)$ isometry subgroup. The $S_7$ critical point is actually invariant under the full $SO(8)$ isometry group, while the $\tilde{S}_7$ is invariant only under the minimal $SO(5) \times SO(3)$ subgroup.

The $S_7$ has a trivial holonomy group that it gives rise to $(N_L, N_R) = (8, 8)$ Killing spinors. Therefore, for either choice of the orientation of seven-sphere, the near-horizon geometry preserves maximal $\mathcal{N} = 8$ supersymmetry. On the other hand, having a weak $G_2$ holonomy, it turns out that the $\tilde{S}_7$ gives rise to $(1, 0)$ Killing spinors. As such, near-horizon geometry preserves minimal $\mathcal{N} = 1$ supersymmetry for left-handed orientation of the seven-sphere and no supersymmetry at all for right-handed orientation. The two alternative choices of the seven-sphere orientation may be viewed as reversal of Page charge, which is identical to the M2-brane charge (up to numerical factors). This means that, when approaching a conical singularity of an eight-dimensional manifold with Spin(7) holonomy, M2- and $\overline{M}2$-branes behave differently. In particular, there will generically be a net attractive force among $\overline{M}2$-branes $[4]$.

\[3\text{ Numerically, } \sqrt{-\Lambda_2} = 0.91..\sqrt{-\Lambda_1}.\]

\[4\text{ In fact, conical singularities of hyperkähler or Calabi-Yau fourfolds exhibit the same phenomena. The near-horizon geometry } X_7 \text{ is, for each of them, 3-Sasaki manifold with Killing spinors (3, 0) or Sasaki-Einstein manifold with Killing spinors (2, 0), respectively.}\]
3 Three-Dimensional Conformal Field Theories

In the investigation of the Kaluza-Klein compactification on the seven-sphere, the phenomenon of ‘squashing’ has been interpreted as triggering spontaneous (super)symmetry breaking via the (super) Higgs-Kibble mechanism [3]. Two important features discovered regarding the compactification and ‘squashing’ deformation thereof are the ‘space invader scenario’ and ‘skew-whipping’ orientation reversal. According to the ‘space invader scenario’, along the continuous ‘squashing’ deformation, Kaluza-Klein spectrum for spin 2, 1, and 0 do not change under the ‘skew-whipping’. In particular, the ‘squashing’ deformation is part of 0 scalar modes and hence it can be studied on equal footing for both orientation of the seven-sphere.

In this section, utilizing the above aspects and the results of section 2 on Kaluza-Klein spectrum under squashing deformation, we will be identifying an operator that gives rise to a renormalization group flow associated with the symmetry breaking $SO(8) \to SO(5) \times SO(3)$ and find that the operator is relevant at the $\tilde{S}_7$ fixed point but becomes irrelevant at the $S_7$ fixed point.

3.1 $SO(8)$ Invariant Conformal Fixed Point

Let us begin with $X_7 = S_7$. Associated with it is the $d = 3$ conformal field theory with $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry. This theory describes the conformal origin of the moduli space of three-dimensional $\mathcal{N} = 8$ supersymmetric Yang-Mills theory with gauge group $SU(N)$, at which the gauge coupling becomes infinitely strong. Away from the origin, the gauge group is broken to $U(1)^{N-1}$ and the moduli space is given by $\mathbb{R}^{8N}/S_N$ [14].

To identify conformal field theory operator corresponding to the ‘squashing’ perturbation (while preserving $SO(5) \times SO(3)$ isometry), we shall be considering harmonic fluctuations of spacetime metric and $u(x), v(x)$ scalar fields around $AdS_4 \times S_7$. Inferring from Eq.(5), fluctuation of these fields is compactly summarized by the Lagrangian:

$$L_{S_7} = \sqrt{-g} \left( R - 2\Lambda_1 - \frac{63}{2}(\partial u)^2 - 21(\partial v)^2 - 2V_1(u, v) \right),$$

where $\Lambda_1 = \frac{1}{2}V(u_1, v_1)$ is the $AdS_4$ cosmological constant and the scalar potential $V_1(u, v)$ is given by

$$V_1(u, v) = -\Lambda_1 \left( 1 - \frac{1}{4}e^{-9(u-u_1)}(e^{4v} + 8e^{-3v} - 2e^{-10v}) + \frac{3}{4}e^{-21(u-u_1)} \right).$$

(12)

For our present purpose, following [13], it turns out more convenient to rewrite Eq.(11) in terms of the un-rescaled, M-theory metric $\tilde{g}_{\alpha\beta} = e^{-7u}g_{\alpha\beta}$:

$$L_{S_7} = \sqrt{-\tilde{g}e^{7u}} \left( R - 105(\partial u)^2 - 21(\partial v)^2 - e^{7u}V(u, v) \right)$$

$$= \sqrt{-\tilde{g}e^{7u}} \left( R - 2\Lambda_1 - 105(\partial u)^2 - 21(\partial v)^2 - 2\tilde{V}_1(u, v) \right).$$

(13)
Here, the scalar potential is

\[ V_1(u, v) = -\Lambda_1 \left(1 - \frac{1}{4} e^{-2(u-u_1)}(e^{4v} + 8e^{-3v} - 2e^{-10u}) + \frac{3}{4} e^{-14(u-u_1)}\right), \]  

in which the un-rescaled cosmological constant \( \Lambda_1 = e^{7u_1} \Lambda_1 \) is given by

\[ \Lambda_1 \equiv -12m_1^2 \frac{1}{\ell_p^2} = -12 \left(\frac{|Q|}{3}\right)^{-1/3} \frac{1}{\ell_p^2} \quad \text{where} \quad m_1 = \frac{1}{r_{\text{IR}}}. \]  

We have denoted the curvature radius as \( r_{\text{IR}} \) since, as we shall see shortly, the round \( S_7 \) ground-state is stable against ‘squashing’ perturbation and hence corresponds to an infrared stable fixed point. Moreover, comparing the cosmological constant with that of spherical, near-horizon geometry of \( N \) coincident M2-branes, one finds that \( r_{\text{IR}} \) is related to \( N \) and Planck scale \( \ell_p \) as

\[ r_{\text{IR}} = \ell_p \left(\frac{32\pi^2 N}{2}\right)^{1/6}. \]  

Conformal dimension of the perturbation operator that represents the ‘squashing’ is determined by fluctuation spectrum of the scalar fields. After rescaling the scalar fields as \( \sqrt{2}u \equiv u, \sqrt{4}v \equiv v \), one finds that the (correctly normalized) fluctuation spectrum for \( v \)-field around the \( S_7 \) takes a positive value:

\[ M_{vv}^2(S_7) = \left[ \frac{\partial^2}{\partial u^2} 2V_1 \right]_{u=v=0} = -\frac{4}{3} \Lambda_1 \ell_p^2 = +16m_1^2. \]  

The \( v \)-field represents ‘squashing’ of \( S_7 \) and hence, under \( SO(8) \) isometry group, ought to correspond to \( 300 \), the lowest mode of the transverse, traceless symmetric tensor representation. Recall that, on \( S_7 \), mass spectrum of the representation corresponding to \( SO(8) \) Dynkin label \( (n-2, 2, 0, 0) \) is given by

\[ M^2 = \left((n+3)^2 - 9\right) m^2, \]  

where \( m^2 \) is mass-squared parameter of a given \( AdS_4 \) spacetime and mass of a scalar field \( S \) is defined according to \( (\Delta_{AdS} + \widetilde{M}^2)S = 0 \). This follows easily from the known mass formula \[ M^2 = \Delta_L - 4m^2, \] where the Lichnerowicz operator \( \Delta_L \) has eigenvalues \( \Delta_L = [n(n+6) + 12]m^2 \) for \( O^{+}(2) \), and the fact that \( M^2 \) is traditionally defined according to \( (\Delta_{AdS} - 8m^2 + M^2)S = 0 \).

For \( 300 \) (corresponding to \( n = 2 \)), \( M^2_{300} = 16m_1^2 \) and this ought to equal to Eq.\( (16) \).

Indeed, recalling that \( r_{\text{IR}} = \ell_p \left(\frac{32\pi^2 N}{2}\right)^{1/6} \), one finds that Eq.\( (16) \) fits perfectly with \( 300 \) spectrum:

\[ M_{vv}^2(S_7) = 16 \left(\frac{|Q|}{3}\right)^{-1/3} \ell_p^2 = \frac{16}{r_{\text{IR}}^2} = \widetilde{M}_{300}^2. \]  

Via AdS/CFT correspondence, one thus concludes that, in \( d = 3 \) conformal field theory with \( \mathcal{N} = 8 \) supersymmetry, the \( SO(5) \times SO(3) \) symmetric ‘squashing’ ought to be an irrelevant
perturbation of conformal dimension $\Delta = 4$. Note that this is the same for either choice of the seven-sphere orientation.

An important point is that the ‘squashing’ deformation arises not at the lowest level but at the second of the $S_7$ Kaluza-Klein tower. As such, the deformation probes M-theory beyond the so-called consistent truncation of eleven-dimensional supergravity, viz. four-dimensional $\mathcal{N} = 8$ gauged supergravity.

3.2 $SO(5) \times SO(3)$ Invariant Conformal Fixed Point

Consider next the conformal fixed point corresponding to the $X_7 = \tilde{S}_7$. As mentioned already, due to the possibility of ‘skew-whipping’, the fixed point could be either left-squashed $\tilde{S}_7^L$ with $\mathcal{N} = 1$ supersymmetry or right-squashed $\tilde{S}_7^R$ with no supersymmetry.

From Eq.(5), harmonic fluctuation of $u(x), v(x)$ scalar fields around $X_7 = \tilde{S}_7$ is compactly summarized by the Lagrangian:

$$L_{\tilde{S}_7} = \sqrt{-\tilde{g}} \left( R - 2\Lambda_2 - \frac{63}{2} (\partial u)^2 - 21(\partial v)^2 - 2V_2(u, v) \right), \quad (19)$$

where the scalar potential $V_2(u, v)$ is given by

$$V_2(u, v) = -\Lambda_2 \left( 1 - \frac{1}{36} e^{-9(u-u_2)} (25e^{4(v-v_2)} + 40e^{-3(v-v_2)} - 2e^{-10(v-v_2)}) \right) + \frac{3}{4} e^{-21(u-u_2)}. \quad (20)$$

Again, in terms of the un-rescaled M-theory metric, the Lagrangian Eq.(19) may be reorganized as

$$L_{\tilde{S}_7} = \sqrt{-\tilde{g}} \tilde{e}^7 \left( R - 2\tilde{\Lambda}_2 - 105(\partial u)^2 - 21(\partial v)^2 - 2\tilde{V}_2(u, v) \right), \quad (21)$$

where

$$\tilde{V}_2(u, v) = -\tilde{\Lambda}_2 \left( 1 - \frac{1}{36} e^{-2(u-u_2)} (25e^{4(v-v_2)} + 40e^{-3(v-v_2)} - 2e^{-10(v-v_2)}) \right) + \frac{3}{4} e^{-14(u-u_2)}. \quad (22)$$

and the un-rescaled cosmological constant $\tilde{\Lambda}_2 = e^{7u_2} \Lambda_2$ is given by

$$\tilde{\Lambda}_2 \equiv -12m_2^2 \frac{1}{\ell_p^2} = -12 \cdot 3^{7/3} 5^{-5/3} \left( \frac{|Q|}{3} \right)^{-1/3} \frac{1}{\ell_p^2}, \quad \text{where} \quad m_2 = \frac{1}{r_{UV}}. \quad (23)$$

Once again, the scalar field $v(x)$ parametrize ‘squashing’ perturbation around $\tilde{S}_7$. Mass spectrum of the $v(x)$ field is calculated straightforwardly:

$$M_{vv}[\tilde{S}_7] \equiv \left[ \left. \frac{\partial^2}{\partial v^2} 2\tilde{V}_2 \right|_{v=v_{\pi}, \pi=v} \right]_{\pi=\pi_{\tilde{S}_7}, \pi=\pi_{\tilde{S}_7}} = \frac{20}{27} \tilde{\Lambda}_2 \ell_p^2 = -\frac{80}{9} m_2^2. \quad (24)$$

The spectrum is tachyonic but stays above the Breitenlohner-Freedman bound and hence, in the corresponding dual, unitary conformal field theory, the ‘squashing’ deformation ought to be described by a relevant operator.
The tachyonic spectrum Eq.(24) can be understood as follows. Under \(SO(8) \to SO(5) \times SO(3)\), the branching rule of a \(SO(8)\) Dynkin label \((0,2,0,0)\) (corresponding to the representation \(300\)) in terms of \(SO(5)\) Dynkin label \((p,q)\) and \(SO(3)\) Dynkin label \((r)\) (or in terms of their respective representations) is given as follows [17]:

\[
(0, 2, 0, 0) = (02)(4) \oplus (21)(2) \oplus (40)(0) \oplus (01)(4) \oplus (02)(0)
\oplus (20)(2) \oplus (01)(2) \oplus (00)(4) \oplus (00)(0).
\]

In terms of their representations,

\[
300 = (14, 5) \oplus (35, 3) \oplus (35, 1) \oplus (5, 5) \oplus (14, 1)
\oplus (10, 3) \oplus (5, 3) \oplus (1, 5) \oplus (1, 1).
\]

Since the ‘squashing’ preserves \(SO(5) \times SO(3)\) isometry group, the spectrum Eq.(24) ought to correspond to that of the singlet (the last state in the branching rule, Eq.(26)). One can check this explicitly. The eigenfunctions of the Lichnerowicz operator \(\Delta_L\) appear in the \(SO(5) \times SO(3)\) representations found by decomposing the \((0, 2, 0, 0)\) of \(SO(8)\). Depending on the particular representations of \(SO(5) \times SO(3)\), eigenvalues are given by [18]:

\[
\Delta_L = \frac{20}{9} m^2 \left(C_G + \frac{9}{5}\right) \quad \text{or} \quad \frac{20}{9} m^2 \left(C_G + \frac{8}{5} \pm \frac{2}{\sqrt{5}} \sqrt{C_G + \frac{1}{20}}\right)
\]

for \(8_v, 8_s\) and \(8_e\), respectively. Here, \(C_G\) denotes the second-order Casimir operator for the isometry group and is given by \(C_G = C_{SO(5)} + 3C_{SO(3)}\). Recall that, in our notation, mass-squared spectrum of a \(O^{+2}\) scalar field \(S\) satisfying \(\Delta_{AdS} + \tilde{M}^2)S = 0\) is given by \(\tilde{M}^2 = \Delta_L - 12m^2\). For the singlet \((1, 1)\), \(C_G = 0\) and one obtains \(\tilde{M}^{(1,1)}_2 = -8m^2\) or \(-\frac{80}{3}m^2\). Henceforth, identifying Eq.(24) with the lower, one identifies

\[
M^2_{\nu v}(\tilde{S}_7) = \frac{80}{27} (3^{-11/5} |Q|)^{-1/3} = -\frac{80}{9} \frac{1}{r_{UV}} = \tilde{M}^{(1,1)}_2.
\]

One thus finds that the perturbation that corresponds to ‘squashing’ around \(X_7 = \tilde{S}_7\) has a scaling dimension either \(\Delta = 4/3\) or \(5/3\) and hence corresponds to a relevant operator.

Note that, of the two possible scaling dimensions, the lower one is below the naive unitarity bound \(3/2\) in three-dimensional conformal field theories. The two-fold ambiguity of the scaling dimension arises for relevant operators whose scaling dimension is below the naive unitarity bound and, in \(AdS_4\) spacetime, originates from the fact that a scalar field whose mass is in the range \((-9/4, -5/4)\) (in units of \(4m^2\)) can be quantized in two inequivalent boundary conditions [15, 19]. Klebanov and Witten [24] have argued that the two possible choices of the scaling dimension are related to two distinct dual conformal field theories and have confirmed it.
explicitly in several examples in which R-symmetry is continuous. In the present situation, by the same argument, the two scaling dimensions of ‘squashing’ deformation operator ought to correspond to each of the two $\mathcal{N} = 1, 0$ conformal field theories, but, as there is no continuous R-symmetry, we were not able to resolve the twofold ambiguity completely.

4 Squashing Domain Wall and RG Flows

In the previous section, round and squashed seven-spheres have been identified as two possible near-horizon geometry of $N$ coincident M2-branes. Both seven-spheres then represent stable vacua of eleven-dimensional supergravity, at least in perturbative expansion in powers of $1/N$ and $\ell_p$, as there is no tachyon mode equal to or below the Breitenlohner-Freedman bound \[13\]. Are the two vacua stable also nonperturbatively? We will find that, surprisingly enough, an answer to this question turns out to be intimately connected to the holography of anti-de Sitter spacetime, existence of static domain-wall, and renormalization group flow between fixed points.

We have seen that, in Eqs.(9, 10), since $\Lambda_1 < \Lambda_2$, the $\tilde{S}_7$ ground-states have higher potential energy than the $S_7$ ground-state. The aforementioned perturbative stability states that, starting from the $\tilde{S}_7$ ground-state, there cannot be any smooth roll-over along the steepest descent to the $S_7$ one. On the other hand, nonperturbatively, one might suspect that there may be a potential source of instability: the $\tilde{S}_7$ ground-state, which is apparently a false vacuum of the gauged supergravity potential, may ‘tunnel’ (without barrier), as a result of false vacuum decay \[22\], to the $S_7$ ground-state.

Semiclassically, such a tunnelling process would be described by a gravitational instanton. However, utilizing the result of \[23, 24\], it is straightforward to see that the relevant gravitational instanton has an infinite action, leading to a complete suppression of the false vacuum decay. Consider an ansatz of the gravitational instanton in $AdS_4$ that preserves, at the least, the $SO(3)$ rotational symmetry. It then follows that unique instanton configuration is a time-translation invariant bubble configuration – a bubble of the $S_7$ vacuum surrounded by the $\tilde{S}_7$ vacuum. As the instanton is translationally invariant along time direction, the semiclassical instanton amplitude is zero. Analytically continuing to Minkowski spacetime (i.e. $AdS_4$), one now obtains a static $SO(3)$ symmetric domain-wall interpolating between the $S_7$ and $\tilde{S}_7$ vacua. We will now analyze the configuration.

Consider the equations of motion Eq.(4) for $g_{\alpha\beta}, u, v$. Let us take the following boost-
Figure 2: Schematic view of static domain wall interpolating the two vacua of $\text{AdS}_4$ supergravity. Note that interpolating scalar fields are critically damped between the two vacua.

invariant (viz. manifestly $\text{Poin}(2, 1)$ invariant along the domain-wall) ansatz:

$$u = \frac{1}{\sqrt{63}} \tilde{u}(z), \quad v = \frac{1}{\sqrt{42}} \tilde{v}(z),$$

$$ds_4^2 = \frac{r^2}{z^2} \left( dz^2 + e^{2h(z)} \eta_{\alpha\beta} dx^\alpha dx^\beta \right), \quad \eta_{\alpha\beta} = (- + +).$$

(29)

The ansatz is subject to boundary conditions:

$$ds_4^2 \rightarrow \frac{r_{\text{IR}}^2}{z^2} \left( dz^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta \right), \quad u \rightarrow u_1, \quad v \rightarrow v_1 \quad (z \rightarrow \infty)$$

$$ds_4^2 \rightarrow \frac{r_{\text{UV}}^2}{z^2} \left( dz^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta \right), \quad u \rightarrow u_2, \quad v \rightarrow v_2 \quad (z \rightarrow 0)$$

(30)
after appropriate change of variables is made. Note that the boundary conditions are manifestly boost invariant.

Components of Ricci tensor and Ricci scalar in the background Eq.(29) are given by

$$R_{00} = \frac{3}{t^2} \left( -1 + th' - t^2 h'^2 - t^2 h'' \right),$$

$$R_{11} = -\frac{e^{2h}}{t^2} \left( -3 + 5th' - 3t^2 h'^2 - t^2 h'' \right), \quad R_{22} = R_{33} = -R_{11}$$

$$R = -\frac{6}{r^2} \left( 2 - 3th' + 2t^2 h'^2 + t^2 h'' \right).$$

(31)

Denoting $\partial_z ='$, the field equations of motion are given by

$$\tilde{u}'' + \left( 3h' - \frac{2}{z} \right) \tilde{u}' - \frac{r^2}{z^2} \frac{\partial V}{\partial \tilde{u}} = 0,$$

(32)
\[
\ddot{v}'' + \left(3h' - \frac{2}{z}\right)\ddot{v}' - \frac{r^2 \partial V}{z^2 \partial \tilde{v}} = 0,
\]
\[
2h'' + 2\frac{h'}{z} + \frac{1}{2} \left(\ddot{u}'^2 + \ddot{v}'^2\right) = 0,
\]
\[
6 \left(\frac{h' - 1}{z}\right)^2 - \frac{1}{2} \left(\ddot{u}'^2 + \ddot{v}'^2\right) + \frac{r^2}{z^2} V = 0.
\]

One immediately notes that Eqs.(32, 33) may be interpreted as equations of motion of an ‘analog’ particle (of unit mass) in two dimensions, whose coordinates are parametrized by \(\tilde{u}\) and \(\tilde{v}\), under the influence of an ‘analog’ potential \(-V(\tilde{u}, \tilde{v})\). The particle is also subject to a frictional force. The ‘dynamic friction coefficient’ \((3h' - 2/z)\) turns out negative-definite always. Intuitively, this follows from the fact that \(h' = 0\) in the limit the gravity is turned off \(\ell_p \to 0\) and that the physics ought to be analytic in powers of \(\ell_p^2\). Because of the anti-friction, the ‘analog’ particle can start from the minima of \(-V\) initially (viz. at \(z = 0\)) and then creep up to the maxima of \(-V\) finally (viz. at \(z = \infty\)). What is less clear is that the anti-frictional energy injected to the ‘analog’ particle equals precisely to the potential energy difference \(2(\Lambda_1 - \Lambda_2)\).

Answer to the question is provided by the fact that, from Eqs.(32-33), \(\tilde{u}, \tilde{v} \to 0\) is satisfied at critical points of the scalar potential \(V\) and the fact that the resulting Minkowski domain-wall is an extremal configuration, saturating domain-wall energy density. An important point is that this also applies to the nonsupersymmetric \(\tilde{S}_7\) vacuum, as the field equations Eqs.(32 - 35) govern stability for both \(\tilde{S}_7^L\) and \(\tilde{S}_7^R\). The extremality condition implies that energy gain by creating a region of true vacuum around \(z = 0\) inside false vacuum is balanced exactly by mass of the \(SO(3)\) symmetric domain-wall, viz. saturation of the Coleman-DeLuccia bound and hence is a posteriori consistent with the static domain-wall ansatz. While an exact solution of the domain-wall configuration is not possible in analytic form, asymptotics of the domain-wall can be studied straightforwardly. Using the asymptotics, we will now check consistency of the \(SO(3)\) symmetric domain-wall configuration. In the IR region, \(z \to \infty\), asymptotic solutions of \(h(z), \tilde{u}(z)\) and \(\tilde{v}(z)\) are given by

\[
\begin{align*}
    h(z) & \sim 0 + \frac{h_\infty}{z^{2a}} + \cdots, \\
    \tilde{u}(z) & \sim \sqrt{\frac{7}{16}} \ln(3^{-2} Q^2) + \frac{\tilde{u}_\infty}{z^a} + \cdots \\
    \tilde{v}(z) & \sim 0 + \frac{\tilde{v}_\infty}{z^a} + \cdots
\end{align*}
\]

Solving the equations of motion Eqs.(32 - 35), one obtains

\[
16h_\infty + \left(\tilde{u}_\infty^2 + \tilde{v}_\infty^2\right) = 0, \\
3 + r^2 \Lambda_1 = 0.
\]
The first equation implies that $h_\infty < 0$, viz. the scale factor $h(z)$ asymptotes to $0^-$. Then, using this fact and the rescaling relation $\overline{r_{\text{IR}}} = e^{7u_1} r_{\text{IR}}^2$, one finds that the second equation implies that $r = r_{\text{IR}}$.

In the UV region, $z \to 0$, asymptotic solutions of $h(z)$, $\tilde{u}(z)$, and $\tilde{v}(z)$ are given by

$$
\begin{align*}
    h(z) &\sim (1 - \lambda) \ln z + h_0 z^{2b} + \cdots \\
    \tilde{u}(z) &\sim \sqrt{\frac{7}{16}} \ln \left(3^{-4} 5^{10/7} Q^2\right) + \tilde{u}_0 z^b + \cdots \\
    \tilde{v}(z) &\sim \sqrt{\frac{42}{7}} \ln 5 + \tilde{v}_0 z^b + \cdots.
\end{align*}
$$

One again finds that

$$
\lambda^2 = \frac{V(u_2, v_2)}{V(u_1, v_1)} = \frac{\Lambda_2}{\Lambda_1} = \frac{3^{7/2}}{5^{5/2}} < 1. \tag{36}
$$

Change of variables $w = z^c$ and trivial rescaling of $x^\alpha$ coordinates bring the asymptotic metric back into the $AdS_4$ form, but with a scaled radius of curvature $r_{\text{IR}}/\lambda$. This is precisely the radius of curvature $r_{\text{UV}}$ of the $S_{7L,R}$ vacua, and hence confirms the consistency of the asymptotic solution. From the equations of motion, one also finds conditions $h_0 < 0$ and $b^2 + \left(\frac{3}{2} \lambda - 2\right) b - \frac{5}{20} \lambda^2 = 0$. Choosing positive root of $b$, one obtains regular asymptotote of Eq.(36) with monotonic rescaling of the curvature of radius.

Another question is during the interpolation between the two vacua, whether the scalar fields $u, v$ are critically damped (as depicted in Figure 2) or under-damped (in which case the scalar fields will execute finite number of oscillations before settling down to the maxima of $-V$). In the ‘analog particle’ interpretation, mode decomposition (obtained from Laplace transform of Eqs.(33, 34) around minimum of $-V$) takes precisely the same form as the Fourier transform of their equations of motion in $AdS_4$. Suppose that the scalar field is tachyonic below the Breitenlohner-Freedman bound. In this case, two eigen-frequencies take a complex-value and corresponds to an under-damped interpolation. Hence, the interpolation is always critically damped for scalar fields whose masses are above the Breitenlohner-Freedman bound. Restated, renormalization group flows between unitary conformal field theories ought to be monotonic always and never exhibits an oscillatory behavior.

The monotonic radial behavior of the static $SO(3)$ domain-wall configuration is, as suggested by [10], the holographic representation of the renormalization group flow and a version of $c$-theorem thereof. We have shown that well-defined renormalization-group flow and $c$-theorem exist only if the supergravity vacua are stable both perturbatively and nonperturbatively. Non-perturbative instability of a perturbatively stable vacuum is signalled by an existence of a gravitational instanton with a finite Euclidean action. Analytically continuing to Minkowski
spacetime, the instanton corresponds to an expanding (or contracting) domain-wall instead of being a static one. Under the AdS/CFT correspondence, such a time-dependent process does not admit interpretation as a renormalization group flow.

So far, we have considered a particular one-parameter renormalization group flow between conformal fixed points of M2-brane worldvolume theory. Geometrically, the flow is induced from varying the position of M2-brane when placed near a conical singularity of an eight-dimensional manifold with $\text{Spin}(7)$ holonomy. In the infrared limit, the conical singularity and hence squashing of M2-brane horizon are washed out completely. At the infrared fixed point with $SO(8)$ symmetry, one may also flow further into another fixed points by turning on a set of relevant operators. It includes scalar operators of Dynkin label $(n,0,0,0)$, $n \geq 2$ and pseudoscalar operators of Dynkin label $(n,0,2,0)$, $n \geq 0$. Among them are 70 scalar fields $\mathbf{35}_v \oplus \mathbf{35}_c$ of $SO(8)$ in the massless gravity supermultiplet, parametrizing the coset space $E_{7(7)}/SU(8)$. Decomposing them under $SO(5) \times SO(3) \subset SO(8)$, $\mathbf{35}_v + \mathbf{35}_c \to 2[(\mathbf{5},1) + (\mathbf{10},3)]$. Turning on the two relevant operators $(\mathbf{5},1)$ breaks $SO(8) \to SO(3) \times SO(3)$. Utilizing on the known result [25], Distler and Zamora [26] have studied renormalization group flow to a nonsupersymmetric vacuum with $SO(3) \times SO(3)$ global symmetry (See Figure 3). Unlike the ‘skew-whipping’ nonsupersymmetric vacua as we have studied, however, their analysis does not shed any light on stability of the vacua. It is a logical possibility that, along a direction in coset space $E_{7(7)}/SU(8)$ (orthogonal to $SO(3) \times SO(3)$ orbit), perturbations violate the

Figure 3: The RG flow along the $SO(5) \times SO(3)$ invariant direction for I: $\tilde{S}^7 \to S^7$ and along the $SO(3) \times SO(3)$ invariant direction for II: $SO(8) \to SO(3) \times SO(3)$.
Breitenlohner-Freedman bound and the corresponding conformal field theory is nonunitary.

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