Laplace Dirichlet heat kernels in convex domains

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Abstract

We provide general lower and upper bounds for Laplace Dirichlet heat kernel of convex $C^{1,1}$ domains. The obtained estimates precisely describe the exponential behaviour of the kernels, which has been known only in a few special cases so far. Furthermore, we characterize a class of sets for which the estimates are sharp, i.e. the upper and lower bounds coincide up to a multiplicative constant. In particular, this includes sets of the form $\{x \in \mathbb{R}^n : x_n > a|(x_1, \ldots, x_{n-1})|^p\}$ where $p \geq 2$, $n \geq 2$ and $a > 0$.

1 Introduction

Heat kernels are basic objects in mathematical analysis, as fundamental solutions to parabolic differential equation (heat equations), as well as in the theory of stochastic processes, as their transition probability densities. They are also, or maybe primarily, important from the point of view of Physics, since they describe evolution of particles, temperature and other phenomena. Despite of a very long and rich history of research on heat kernels in various settings, it turns out that there are still many open questions even in the most classical case, i.e. the one involving the Laplace operator $\Delta$ (or, equivalently, the Brownian motion) in Euclidean space, which is the subject of this article.

Let $p(t, x, y) = (4\pi t)^{-n/2}e^{-|x-y|^2/4t}$ be the global Laplace heat kernel in $\mathbb{R}^n$, $n \in \mathbb{N}$. For a domain $D \subset \mathbb{R}^n$ we denote by $p_D(t, x, y)$ the Dirichlet heat kernel of $D$, which is the fundamental solution to the heat equation with the Dirichlet condition at the boundary. From probabilistic point of view, $p_D(t, x, y)$ is the transition probability density of Brownian motion killed when exiting the set $D$. Since applicable explicit formulae for $p_D(t, x, y)$ are known only in a few cases (half-lines, intervals and their products), estimates are strongly desired. They have been intensively studied for more than half a century and led to numerous significant results (see, among others, [7, 9, 11, 13, 18, 29, 30, 23, 25, 26]). In particular, it follows from general theory ([9, 10]) that for a bounded domain $D$ with boundary smooth enough the heat kernel $p_D(t, x, y)$ is comparable for large $t$ with $\delta_D(x)\delta_D(y)e^{-\lambda_1 t}$, $x, y \in D$, where $\lambda_1$ stands for the first eigenvalue of $-\Delta$ in $D$, $\delta_D(x)$ for distance of $x$ to the boundary of $D$. For this reason, we will focus on small times (but will be considering unbounded domains as well).

Most of the general heat kernel estimates in the literature share one common weakness: lower and upper bounds are not comparable and their ratio is usually of the form $e^{c|x-y|^2/ct}$.
for some constant $c > 0$. Such estimates are therefore very imprecise. Let us recall Zhang’s result [29], which provides the sharpest known bounds for the heat kernel of any set $D$ which is a bounded $C^{1,1}$ domain or a complement of a closure of a bounded $C^{1,1}$ set. Namely, there are constants $c_1, c_2, c_3, c_4 > 0$ such that for $x, y \in D$ and $t < T$ it holds

$$c_1 \left( \frac{\delta_D(x) \delta_D(y)}{t} \wedge 1 \right) e^{-c_2 |x-y|^2/t} \leq p_D(t, x, y) \leq c_3 \left( \frac{\delta_D(x) \delta_D(y)}{t} \wedge 1 \right) e^{-c_4 |x-y|^2/t}. \quad (1)$$

One may observe that incomparability of above-given bounds is caused by two different constants in exponents. Estimates with such a property are known as quantitatively sharp estimates. In fact, there are some results with correct exponents, but they completely fail in describing the boundary behaviour (see e.g. [24, 25, 26]). For instance, the main result of [25] (combined with [18]), for simplicity restricted to convex domains, states that

$$p_D(t, x, y) \geq c \left( 1 \wedge \frac{(\delta_D(x) \wedge \delta_D(y))^2}{t} \right) e^{-\lambda t / (\delta_D(x) \wedge \delta_D(y))^2} \left( 1 + \frac{1}{t} (\delta_D(x) \wedge \delta_D(y))^2 \right)^{(n+2)/2} p(t, x, y),$$

for some $c > 0$, where $\lambda$ stands for the first eigenvalue of $-\Delta$ in the unit ball. However, the article was focused on asymptotics of the heat kernels with fixed space arguments, where the boundary behaviour plays marginal role. Until recently, precise two-sided estimates for Dirichlet heat kernels have been known only in such basic cases as a half-line and an interval (and their multidimensional extensions) as they are given by simple explicit formulae. Even the case of such classical set as a ball turned out to require a more subtle approach and has been solved in [18]. Precisely, for a unit ball $B = B(0, 1)$ centered at the origin and for every $T > 0$ there exists a constant $C = C(n, T) > 1$ such that

$$\frac{1}{C} h(t, x, y) p(t, x, y) \leq p_B(t, x, y) \leq C h(t, x, y) p(t, x, y) \quad (2)$$

for every $x, y \in B$ and $t < T$, where

$$h(t, x, y) = \left( 1 \wedge \frac{\delta_B(x) \delta_B(y)}{t} \right) + \left( 1 \wedge \frac{\delta_B(x) |x-y|^2}{t} \right) \left( 1 \wedge \frac{\delta_B(y) |x-y|^2}{t} \right). \quad (3)$$

Note that proper description of the exponential behaviour imposed the appearance of a new non-exponential factor $h(t, x, y)$. Above estimates have been complemented with asymptotics in [22], which revealed that the behaviour of $p_B(t, x, y)$ is in fact driven by the expression $\delta \left( \frac{|x-y|^2}{t} \right) / \sqrt{t}$. A similar property will be observed in general lower bound (5). We refer the reader to [3, 4, 5, 12, 19, 20, 21] for some other recent articles focused on sharp estimates of heat kernels in other settings.

The goal of this paper is to derive heat kernel estimates with correct exponential behaviour for general $C^{1,1}$ convex domains $D$. The first main result is the following upper bound (see Theorem 3.4)

$$p_D(t, x, y) \leq C p(t, x, y) \left[ \left( 1 \wedge \frac{\delta_D(x) \delta_D(y)}{t} \right) + \left( 1 \wedge \frac{\delta_{H_x}(x) \delta_{H_y}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_y}(y) \delta_{H_y}(x)}{t} \right) \right], \quad (4)$$

where $x, y \in D$, $0 < t < T$, $C = C(D, T)$ and $H_x$ is any half-space such that $D \subset H_x$ and
\[ \delta_D(x) = \delta_{H_n}(x). \]

Next, we provide lower bounds of the form (see Theorem \[1.3\])

\[
p_D(t, x, y) \geq C p(t, x, y) \left( 1 - \delta_D(x) \left( \delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right) \right) \left( 1 - \delta_D(y) \left( \delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right) \right),
\]

(5)

\[
\approx p(t, x, y) \left[ \left( 1 - \delta_D(x) \delta_D(y) \right) + \left( 1 - \delta_D(x) \delta_D \left( \frac{x+y}{2} \right) \right) \left( 1 - \delta_D(y) \delta_D \left( \frac{x+y}{2} \right) \right) \right],
\]

(6)

where \( \approx \) means that the ratio of both sides is uniformly bounded and bounded away from zero. The exponential behaviour is indeed treated well in each of the above bounds, as expected. Let us now focus on the non-exponential factors. They are similar, but not identical. The reason is that they strongly depend on the shape of the boundary of \( D \) (see Example \[5.1\]). Note that in one dimensional case, where any convex set is just an interval with (at most) two-point boundary, all of the above bounds are equivalent. If there exist elementary sharp estimates in other dimensions, they are presumably more complex and involving more detailed geometrical features of the set \( D \). The main advantages of bounds \([4] - [6]\) are therefore not only the proper exponential behaviour but also a relatively simple form. Furthermore, they follow sharp estimates for a large class of domains. Indeed, we introduce a class \( \mathcal{S}_R \) such that the heat kernel of any \( D \in \mathcal{S}_R \) admits two-sided estimates of the form \([1]\). As an example, we show that \( \mathcal{S}_R \) contains sets of the form \( \{ x \in \mathbb{R}^n : x_n > a \langle x_1, \ldots, x_{n-1} \rangle \} \), where \( p \geq 2, n \geq 2 \) and \( a > 0 \). Such sets are usually difficult to study, as neither they are bounded nor their complements are bounded; see \([1, 17]\) for some result concerning the first exit time of such sets and \([13]\) for quantitatively sharp heat kernel estimates. Then, we fully characterize a class \( \mathcal{S}_Q \subset \mathcal{S}_R \) of sets whose heat kernels satisfy two-sided estimates of the forms \([5]\) and \([6]\). In particular, \( \mathcal{S}_Q \) contains balls (cf. \([2]\)). Finally, let us note that all the obtained bounds immediately imply estimates for the first exit time and place density \( q_D^x(t, z) \) of Brownian motion from a domain \( D \). Precisely, the well known representation \( q_D^x(t, z) = \frac{1}{2} \frac{\partial}{\partial y} p_D(t, x, y) \) and Dirichlet boundary condition give us

\[
q_D^x(t, z) = \frac{1}{2} \lim_{D \ni y \to z} \frac{p_D(t, x, y)}{\delta_D(y)},
\]

where \( n_z \) is the inward normal direction at \( z \in \partial D \), which allows us easily transform estimates of \( p_D(t, x, y) \) into estimates of \( q_D^x(t, z) \).

The assumption of smoothness of the boundary of \( D \) in the main results is very natural and common in the topic. On the other hand, convexity of the set \( D \) is necessary to obtain exponential behaviour of the same order as in the global heat kernel \( p(t, x, y) \). Namely, S. R. S. Varadhan showed (Corollary 4.7 in \([27]\)) that

\[
\lim_{t \to 0} t \ln (p_D(t, x, y)) = \frac{1}{4} d_D^2(x, y),
\]

where \( d_D(x, y) \) is the infimum of lengths of arcs included in \( D \) and connecting \( x \) and \( y \). If \( D \) was concave, there would be \( x, y \in D \) such that \( d_D(x, y) > |x - y| \) and consequently \( e^{-d_D^2(x, y)/4t} \ll e^{-|x-y|^2/4t} \) for \( t \) small enough. One could naturally try to obtain estimates with the term \(-d_D^2(x, y)/4t\) in the exponent, but this seem to be a much more challenging task and rather a material for further research, as there are expected some additional exponential terms related to Buslaev conjecture \([6, 15]\). The first attempts of describing heat kernels behavior at points satisfying convexity property, i.e. such that the whole interval...
connecting \( x \) and \( y \) is contained in \( D \), are associated with the property of not feeling the boundary, introduced by M. Kac in [16], which says that for such points \( x \) and \( y \) it holds
\[
\lim_{t \to 0} \frac{p_D(t, x, y)}{p(t, x, y)} = 1.
\]
In [24], the following rate of convergence was derived
\[
p(t, x, y) \geq p_D(t, x, y) \geq p(t, x, y) \left( 1 - e^{-\rho^2/t} \sum_{k=1}^{n} \frac{2^k}{(k-1)!} \left( \frac{\rho^2}{t} \right)^{k-1} \right),
\]
where \( \rho = \sup_{w \in \partial D} |w - z| \) denotes the distance of the interval \( \overline{xy} \) to the boundary of \( D \).

A simple observation is that for \( \rho < c \sqrt{t} \) the bound (7) induces sharp estimates of \( p_D(t, x, y) \). As mentioned before, all the general results in the literature fail to describe the proper exponential behaviour in the remaining case \( \rho < c \sqrt{t} < C \), and consequently provide bounds with a substantial error. For this reason, the results presented in the paper are first of their kind. One way to explain such enhancement is application of mixture of probabilistic and analytical methods, as purely analytical methods usually lead to estimates with different constants in exponents. Such approach has been already successfully adopted in e.g. [5, 18, 19]. Since also many arguments are geometrical, the methods presented in the paper seem to be adoptable in wider generality.

The paper is organized as follows. In Section 2 we gather notational details as well as preliminary information about the Brownian motion, including some inequalities related to semi-group property of \( p_D(t, x, y) \). Sections 3 and 4 are devoted to general lower and upper bounds, respectively, of the examined heat kernels. Section 5 deals with two-sided estimates and contains some supporting examples.

## 2 Preliminaries

### 2.1 Notation

In this paper we work on the Euclidean space \( \mathbb{R}^n \), \( n \in \mathbb{N} \), equipped with the standard inner product \( x \cdot y \) and corresponding Euclidean metric \( |x - y| \), \( x, y \in \mathbb{R}^n \). By \( B_k(a, r) \), \( k \in \mathbb{N} \), we denote the \( k \)-dimensional ball of radius \( r > 0 \) and center \( a \in \mathbb{R}^k \). If \( k = n \), we simply write \( B_n(a, r) = B(a, r) \). A set \( D \subset \mathbb{R}^n \) is called a domain, if it is open and connected. The \( C^{1,1} \) sets are sets whose boundary is locally a graph of a \( C^{1,1} \) function \( f : \mathbb{R}^{n-1} \to \mathbb{R} \). It is well known that \( C^{1,1} \) sets satisfy the inner and outer ball condition, which means that for any point \( z \) from the boundary \( \partial D \) of the set \( D \) there are two balls tangent to \( D \) at \( z \) such that one of them is completely included in \( D \), and the other one in \( D^c \). Furthermore, if \( D \) is bounded, then there exists a radius \( r > 0 \) such that for any \( z \in \partial D \) the condition is satisfied with balls of radius at least \( r \). We will denote the class of sets with such property by \( C_r^{1,1}(\mathbb{R}^n) \).

For a domain \( D \) and \( x \in D \) we denote the distance of \( x \) to the boundary \( \partial D \) of \( D \) by \( \delta_D(x) \). For a convex domain \( D \) and \( z \in \partial D \) from its boundary, \( P_z \) stands for a hyperplane such that \( P_z \subset D^c \) and \( \{z\} \subset P_z \). Note that \( P_z \) might be not unique, but if \( D \) is a \( C^{1,1} \) domain, then there is only one such hyperplane and it is tangent to \( D \) at \( z \). Any hyperplane \( P_z \) divides the whole space \( \mathbb{R}^n \) into two half-spaces. The one including \( D \) will be denoted by \( H_z \). In that case we have \( \partial H_z = P_z \). For \( x \in D \) and \( z \in \partial D \) realising the distance of \( x \) to the boundary of \( D \), i.e. \( \delta_D(x) = |x - z| \), we put \( P_x = P_z \) and \( H_x = H_z \). Again, there might be more than one such point \( z \), but this is irrelevant from our point of view, as all the results presented in the paper are valid for any choice of \( z \). For two half-spaces \( H_1 \) and \( H_2 \) we define the angle \( \angle(H_1, H_2) \) between them as the angle inside \( H_1 \cap H_2 \).
where $\sigma$ is a process equal to $W$. When $\angle(H_1, H_2) \leq \pi$ then the angle is equal to $\pi - \angle(v_1, v_2)$, where $v_1, v_2$ are normal vectors of $\partial H_1$ and $\partial H_2$, respectively, directed inside the set $H_1 \cap H_2$.

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $n \geq 2$, we denote
\[ \tilde{x} = (x_2, ..., x_n) \in \mathbb{R}^{n-1}, \quad \text{and} \quad \bar{x} = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}. \]
If $x \in \mathbb{R}$, then $\tilde{x}$ and $\bar{x}$ will be treated as 0 in calculations. Analogously we define $\tilde{\bar{x}}$.

To compare two positive functions $f, g$ we use notation $f \approx g$, which means that there exist constants $c_1, c_2 > 1$, possibly depending on $n$, such that $c_1 < f/g < c_2$ for a given range of arguments. If the constants depend on other parameters $p_1, ..., p_k, k \in \mathbb{N}$, we write $f \approx_{p_1, ..., p_k} g$.

2.2 Brownian motion

Let us consider $n$-dimensional Brownian motion $W = (W(t))_{t \geq 0} = (W_1(t), ..., W_n(t))_{t \geq 0}$ starting from $x \in \mathbb{R}^n$. The global heat kernel $p(t, x, y)$ represents its transition probability density. By $\mathbb{P}^x$ and $\mathbb{E}^x$ we denote the corresponding probability law and the expected value, respectively.

For a domain $D \subset \mathbb{R}^n$ we define the first exit time $\tau_D$ of $W$ from $D$ by
\[ \tau_D := \inf\{t > 0 : W(t) \notin D\}. \]
Then by $W^D = (W^D(t))_{t \geq 0}$ we denote the Brownian motion killed upon leaving a set $D$, which is a process equal to $W_t$ before time $\tau_D$ and at that time it is moved to an additional state called cemetery. For sufficiently regular domains $D$ (e.g. Lipschitz domains) the transition density function of $W^D$ is given by the Dirichlet heat kernel $p_D(t, x, y)$. The relation between $p_D(t, x, y)$ and $p(t, x, y)$ is described by the Hunt formula
\[ p_D(t, x, y) = p(t, x, y) - \int_0^t \int_{\partial D} p(t-s, z, y) q^x_D(s, z) ds d\sigma(z), \]
where $\sigma(z)$ is the surface measure on $\partial D$ and $q^x_D(t, z)$ denotes the density function of the joint distribution $(\tau_D, W_{\tau_D})$ for the process $W$ starting from $x \in D$. Note that the function $p_D(t, x, y)$ is symmetric in space arguments and satisfies the Chapman-Kolmogorov identity, known also as the semi-group property, (see Theorem 2.4 in [8])
\[ p_D(t, x, y) = \int_D p_D(\alpha t, z) p_D((1-\alpha) t, z, y) dz, \quad x, y \in D, \quad t > 0, \quad \alpha \in (0, 1). \]

For any half-space $H \subset \mathbb{R}^n$ the Dirichlet heat kernel $p_H(t, x, y)$ takes especially simple form. Precisely, reflection principle gives us
\[ p_H(t, x, y) = p(t, x, y) - p(t, x, \bar{y}) = p(t, x, y) \left(1 - e^{-\delta_H(x)\delta_H(y)}\right), \]
where $\bar{y}$ is a symmetric reflection of $y$ with respect to the boundary $\partial H$ of $H$. This immediately implies
\[ p_H(t, x, y) \approx \left(1 \wedge \frac{\delta_H(x)\delta_H(y)}{t}\right) p(t, x, y). \]
Another important property of Dirichlet heat kernels, which follows e.g. from the Hunt formula, is their monotonicity with respect to inclusion of domains. Namely, if $D_1 \subset D_2$ then

$$p_{D_1}(t, x, y) \leq p_{D_2}(t, x, y), \quad x, y \in D_1. \quad (11)$$

To see this, let us observe that $\tau_{D_1} \leq \tau_{D_2}$ and consequently $\{\tau_{D_1} > t\} \subset \{\tau_{D_2} > t\}$. Hence, for any borel $A \subset D_1$ we have

$$\int_A p_{D_1}(t, x, y) dy = \mathbb{P}^x [W_t \in A, \tau_{D_1} > t] \leq \mathbb{P}^x [W_t \in A, \tau_{D_2} > t] = \int_A p_{D_2}(t, x, y) dy.$$

### 2.3 Chapman-Kolmogorov-like inequalities

Below we collect some inequalities related to the Chapman-Kolmogorov identity (semi-group property) for the global heat kernel $p(t, x, y)$, that help us with dealing with analogous identities in case of the Dirichlet heat kernels and provide some intuitions about typical trajectories of Brownian motion. The first proposition may be interpreted by saying that Brownian motion going from $x$ to $y$ in time $t$ is mostly at time $\alpha t$, $\alpha \in (0, 1)$, passing through a neighbourhood of the point $(1 - \alpha)x + \alpha y$ of a size comparable to $\sqrt{\alpha(1 - \alpha)t}$. In fact, we can move away from $(1 - \alpha)x + \alpha y$ by a distance comparable to $\sqrt{\alpha(1 - \alpha)t}$.

**Proposition 2.1.** For every $r, d \geq 0$ and $\alpha \in (0, 1)$ we have

$$\int_{B(a,r)} p(\alpha t, x, z)p((1 - \alpha)t, z, y) dz \geq e^{-\frac{d^2}{2\alpha(1 - \alpha)t}} \frac{1}{2^n \Gamma \left( \frac{n+2}{2} \right)} \left( 1 + \frac{r^2}{\alpha(1 - \alpha)t} \right)^{n/2} p(t, x, y),$$

where $x, y \in \mathbb{R}^n$, $t > 0$ and $a \in \mathbb{R}^n$ such that $|a - ((1 - \alpha)x + \alpha y)| = d$.

**Proof.** Without loss of generality we assume $x = (-\alpha|x - y|, 0, ..., 0)$, $y = ((1 - \alpha)|x - y|, 0, ..., 0)$. This follows

$$\frac{1}{\alpha}|x - z|^2 + \frac{1}{1 - \alpha}|y - z|^2 = |x - y|^2 + \frac{1}{\alpha(1 - \alpha)}|z|^2, \quad z \in \mathbb{R}^n,$$

and consequently

$$p(\alpha t, x, z)p((1 - \alpha)t, z, y) = \frac{\exp \left( -\frac{|x - y|^2}{4t} \right) \exp \left( -\frac{|z|^2}{4\alpha(1 - \alpha)t} \right)}{(4\pi t)^{n/2}(4\alpha(1 - \alpha)\pi t)^{n/2}} = p(t, x, y) \exp \left( -\frac{|z|^2}{4\alpha(1 - \alpha)\pi t} \right) \left( \frac{1}{4\alpha(1 - \alpha)\pi t} \right)^{n/2}. \quad (12)$$

Furthermore, the special form of $x$ and $y$ gives us $(1 - \alpha)x + \alpha y = 0$, which implies

$$|z|^2 \leq (|z - a| + |a|)^2 \leq 2(|z - a|^2 + |a|^2) = 2(|z - a|^2 + d^2).$$


Thus, we obtain

\[
\int_{B(a,r)} p(\alpha t, x, z) p((1-\alpha)t, z, y) dz \\
\geq \frac{p(t, x, y)}{(4\alpha(1-\alpha)\pi t)^n/2} \int_{B(a,r)} \exp \left(-\frac{|z-a|^2 + d^2}{2\alpha(1-\alpha)t}\right) dz \\
= \frac{p(t, x, y)}{(2\pi)^{n/2}} e^{-\frac{d^2}{2\alpha(1-\alpha)t}} \int_{B(0,r/\sqrt{2\alpha(1-\alpha)t})} \exp \left(-|z|^2\right) dz \\
\geq \frac{p(t, x, y)}{(2\pi)^{n/2}} e^{-\frac{d^2}{2\alpha(1-\alpha)t}} e^{-1/2} |B(0, 1 \land (r/\sqrt{2\alpha(1-\alpha)t}))| \\
= \frac{p(t, x, y)}{2^{n/2} \Gamma \left(\frac{n+2}{2}\right)} e^{-\frac{d^2}{2\alpha(1-\alpha)t}} \left(1 \land \frac{r}{\sqrt{2\alpha(1-\alpha)t}}\right)^n \\
\geq \frac{p(t, x, y)}{2^n \Gamma \left(\frac{n+2}{2}\right)} e^{-\frac{d^2}{2\alpha(1-\alpha)t}} \left(1 \land \frac{r^2}{\alpha(1-\alpha)t}\right)^{n/2},
\]

as required.

The next proposition deals with an integral over the intersection of the Dirichlet heat kernels of the half-spaces (cf. (10)).

**Proposition 2.2.** Let \( \alpha, \beta \geq 0 \) and \( H_1, H_2 \subset \mathbb{R}^n \) be two half-spaces. For \( D = H_1 \cap H_2 \) it holds

\[
\int_D p(t/2, x, z) p(t/2, z, y) (\delta_{H_1}(z))^\alpha (\delta_{H_2}(z))^\beta \, dz \\
\lesssim p(t, x, y) \left(\sqrt{t} + \delta_{H_1} \left(\frac{x+y}{2}\right)\right)^\alpha \left(\sqrt{t} + \delta_{H_2} \left(\frac{x+y}{2}\right)\right)^\beta.
\]

**Proof.** Similarly as in the proof of Proposition 2.1, we assume \( x = (\frac{1}{2}|x-y|, 0, \ldots, 0) \), \( y = (\frac{1}{2}|x-y|, 0, \ldots, 0) \). Then we have \( B((x+y)/2, r) = B(0, r) \) and

\[
\delta_{H_1}(z) \leq |z| + \delta_{H_1} \left(\frac{x+y}{2}\right), \quad \delta_{H_2}(z) \leq |z| + \delta_{H_2} \left(\frac{x+y}{2}\right),
\]

as well as

\[
p(t/2, x, z) p(t/2, z, y) = p(t, x, y) e^{-|z|^2/t} \left(\frac{\pi t}{2}\right)^{n/2}.
\]

This follows

\[
\int_D p(t/2, x, z) p(t/2, z, y) (\delta_{H_1}(z))^\alpha (\delta_{H_2}(z))^\beta \, dz \\
\lesssim p(t, x, y) \int_{\mathbb{R}^n} e^{-|z|^2/t} \left(\frac{\pi t}{2}\right)^{n/2} \left(|z| + \delta_{H_1} \left(\frac{x+y}{2}\right)\right)^\alpha \left(|z| + \delta_{H_2} \left(\frac{x+y}{2}\right)\right)^\beta \, dz \\
= p(t, x, y) \int_{\mathbb{R}^n} e^{-|z|^2/t} e^{-|\sqrt{t} + \delta_{H_1} \left(\frac{x+y}{2}\right)\alpha} \left(|\sqrt{t} + \delta_{H_2} \left(\frac{x+y}{2}\right)\right)\beta \, dz \\
\lesssim p(t, x, y) \left(\sqrt{t} + \delta_{H_1} \left(\frac{x+y}{2}\right)\right)^\alpha \left(\sqrt{t} + \delta_{H_2} \left(\frac{x+y}{2}\right)\right)^\beta,
\]

as required.
where the last estimate may be justified by the inequalities
\[
\frac{1}{2} (a^\gamma + b^\gamma) \leq (a + b)^\gamma \leq 2^\gamma (a^\gamma + b^\gamma),
\]
for \(a, b, \gamma > 0\).

\section{Upper bounds}

Our general idea of finding upper bounds is to circumscribe sets of the form \(H_1 \cap H_2\), where \(H_1\) and \(H_2\) are two half-spaces, on the domain \(D\) and to use the monotonicity property \((\text{III})\). Due to independence of coordinates of Brownian motion, the heat kernel of \(H_1 \cap H_2\) is just a product of a heat kernel of a two-dimensional cone and the \(n - 2\)-dimensional global heat kernel. However, in the literature there are neither estimates of heat kernels in cones that describe properly the exponential behaviour nor ones that are uniform with respect to the angle between the half-spaces (see \([2]\) for some formulae and properties of Brownian motion in cones). For this reason we provide some upper bounds that are sufficient for applications to estimation of heat kernels in \(C^{1,1}\) domains. We deal separately with cases when the angle between the half-spaces is obtuse or acute.

\textbf{Theorem 3.1.} Let \(H_1\) and \(H_2\) be two half-spaces such that \(\angle(H_1, H_2) \geq \pi/2\) and denote \(D = H_1 \cap H_2\). Then there is an absolute constant \(C > 0\) such that for \(x, y \in D\) and \(t > 0\) it holds
\[
p_D(t, x, y) \leq C p(t, x, y) \times 
\left[ 1 + \frac{\delta_D(x)\delta_D(y)}{t} \right] + \left( 1 + \frac{\delta_{H_1}(x)\delta_{H_1}(y)}{t} \right) \left( 1 + \frac{\delta_{H_2}(x)\delta_{H_2}(y)}{t} \right).
\]

\textit{Proof.} Without loss of the generality we assume \(\delta_{H_1}(x) \leq \delta_{H_2}(x)\). Since \(\delta_D(x) = \delta_{H_1}(x) \wedge \delta_{H_2}(x)\), we have \(\delta_D(x) = \delta_{H_1}(x)\).

If \(\delta_{H_1}(y) \leq 2\delta_{H_2}(y)\), then \(\delta_D(y) \geq \frac{1}{2} \delta_{H_1}(y)\), and consequently
\[
\delta_{H_1}(x)\delta_{H_1}(y) \leq 2\delta_D(x)\delta_D(y).
\]

Hence, the inequality \(p_D(t, x, y) \leq p_{H_1}(t, x, y)\) together with the estimate \((\text{III})\) finish the proof in this case.

Consider now \(\delta_{H_1}(y) > 2\delta_{H_2}(y)\) and let \(\bar{x}, \bar{y}\) be reflections of \(x, y\) with respect to hyperplanes \(\partial H_1, \partial H_2\), respectively. Since \(|y - \bar{y}| = 2\delta_{H_2}(y)\), we have \(\bar{y} \in H_1\). We may therefore repeat the argumentation of the formula \((3.2)\) in \([18]\) and get
\[
p_D(t, x, y) \leq p_{H_1 \cap H_2}(t, x, y) 
\leq p(t, x, y) - p(t, \bar{x}, y) - p(t, x, \bar{y}) + p(t, \bar{x}, \bar{y}).
\]

Note that for \(\angle(H_1, H_2) \leq \frac{1}{2} \pi\) the last inequality is expected to be opposite. Next, the right-hand side may be rewritten as
\[
p(t, x, y) \left( 1 - \frac{p(t, \bar{x}, y)}{p(t, x, y)} \right) \left( 1 - \frac{p(t, x, \bar{y})}{p(t, x, y)} \right)
+ p(t, \bar{x}, y) - p(t, x, y)
= \frac{p_{H_1}(t, x, y)p_{H_2}(t, x, y)}{p(t, x, y)} + \left( p(t, \bar{x}, y) - \frac{p(t, x, \bar{y})}{p(t, x, y)} \right).
\]
By (10), we clearly have
\[
\frac{p_{H_x}(t, x, y)p_{H_y}(t, x, y)}{p(t, x, y)} \approx p(t, x, y) \left( 1 \wedge \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_y}(x)\delta_{H_x}(y)}{t} \right).
\]

In order to deal with the other component, let us observe
\[
|\bar{x} - y|^2 = |(x - y) + (\bar{x} - x)|^2
= |x - y|^2 + (\bar{x} - x) \cdot ((\bar{x} - x) + 2(x - y)),
\]
\[
|\bar{x} - \bar{y}|^2 = |(x - y) + (\bar{x} - x) - (\bar{y} - y)|^2
= |x - y|^2 - 2(\bar{x} - x) \cdot (\bar{y} - y)
+ (\bar{x} - x) \cdot ((\bar{x} - x) + 2(x - y)) + (\bar{y} - y) \cdot ((\bar{y} - y) + 2(y - x)),
\]
which follows
\[
p(t, \bar{x}, \bar{y}) - \frac{p(t, \bar{x}, y)p(t, x, \bar{y})}{p(t, x, y)} = p(t, \bar{x}, \bar{y}) \left( 1 - e^{-(\bar{x} - x)(\bar{y} - y)/2t} \right).
\]

Since, by Cauchy-Schwarz inequality,
\[
(\bar{x} - x) \cdot (\bar{y} - y) \leq |\bar{x} - x||\bar{y} - y| = 4\delta_{H_x}(x)\delta_{H_y}(y) = 4\delta_D(x)\delta_D(y),
\]
we finally obtain
\[
p(t, \bar{x}, \bar{y}) \left( 1 - e^{-(\bar{x} - x)(\bar{y} - y)/t} \right) \leq p(t, x, y) \left( 1 - e^{-2\delta_D(x)\delta_D(y)/t} \right)
\]
\[
\lesssim p(t, x, y) \left( 1 \wedge \frac{\delta_D(x)\delta_D(y)}{t} \right),
\]
as required. The proof is complete. \(\square\)

In all the subsequent results \(H_x, H_y, x, y \in D\) stand for two half-spaces such that \(D \subset H_x \cap H_y\) and \(\delta_{H_x}(x) = \delta_D(x), \delta_{H_y}(y) = \delta_D(y)\).

**Corollary 3.2.** Let \(D \subset \mathbb{R}^n\) be any convex domain. There is an absolute constant \(C > 0\) such that
\[
p_D(t, x, y) \leq Cp(t, x, y) \left[ \left( 1 \wedge \frac{\delta(x)\delta(y)}{t} \right) + \left( 1 \wedge \frac{\delta_{H_x}(x)\delta_{H_x}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_y}(x)\delta_{H_y}(y)}{t} \right) \right]
\]
holds whenever \(\angle(H_x, H_y) \geq \frac{1}{2}\pi\).

Let us pass to the latter main result of this section.

**Theorem 3.3.** Let \(D \subset \mathbb{R}^n\) be a \(C_r^{1,1}\), \(r > 0\), domain. For \(0 < t < T\) we have
\[
p_D(t, x, y) \lesssim \left( 1 + \frac{T}{r^2} \right) p(t, x, y) \left( 1 \wedge \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_y}(x)\delta_{H_x}(y)}{t} \right). \tag{14}
\]

**Proof.** Let \(x', y' \in \partial D\) be tangent points of \(D\) to \(H_x\) and \(H_y\), respectively, i.e. such that \(\delta_D(x) = \delta_{H_x}(x) = |x - x'|\) and \(\delta_D(y) = \delta_{H_y}(y) = |y - y'|\). Since \(D\) is \(C_r^{1,1}\), there are balls of radius \(r\) inside \(D\) which are tangent at \(x'\) and \(y'\) to \(D\) (and consequently to \(H_x\) and \(H_y\), respectively). If we combine this with the assumption \(\angle(H_x, H_y) < \pi/2\), simple geometry shows that
\[
\delta_{H_x}(y), \delta_{H_y}(x) \geq r. \tag{15}
\]
Due to symmetry of $p_D(t, x, y)$, we may assume $\delta_D(x) \leq \delta_D(y)$. If $\delta_D(y) = \delta_{H_y}(y) > r$, then it holds $\left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t}\right) \geq \left(1 + \frac{x^2}{T}\right)$, and consequently, by (10) and (11),

$$p_D(t, x, y) \leq p_{H_x}(t, x, y) \lesssim p(t, x, y) \left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t}\right) \leq \left(1 + \frac{T}{r^2}\right) p(t, x, y) \left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t}\right) \left(1 + \frac{\delta_{H_y}(x)\delta_{H_y}(y)}{t}\right),$$

as required.

Consider now $\delta_D(y) \leq r$. By the assumption $\delta_D(x) \leq \delta_D(y)$ we also have $\delta_D(x) \leq r$. Combining this with (15) we get $\delta_{H_x}(x) \leq \delta_{H_y}(y)$ and $\delta_{H_y}(y) \leq \delta_{H_x}(x)$. Thus, equality $\delta_{H_x}\left(\frac{x+y}{2}\right) = \frac{1}{T}(\delta_{H_x}(x) + \delta_{H_y}(y))$ gives us

$$\frac{r}{2} \leq \delta_{H_x}\left(\frac{x+y}{2}\right) \leq \delta_{H_y}(y), \quad \frac{r}{2} \leq \delta_{H_y}\left(\frac{x+y}{2}\right) \leq \delta_{H_x}(x). \tag{16}$$

Next, using Chapman-Kolmogorov identity and estimating $p_D(t/2, x, z) \leq p_{H_x}(t/2, x, z)$, $p_D(t/2, z, y) \leq p_{H_y}(t/2, z, y)$ we obtain for every $i, j \in \{0, 1\}$

$$p_D(t, x, y) \lesssim \int_D \left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(z)}{t}\right) \left(1 + \frac{\delta_{H_y}(y)\delta_{H_y}(z)}{t}\right) p(t/2, x, z)p(t/2, x, y)dz \lesssim I_{i,j},$$

where

$$I_{i,j} := \int_{H_x \cap H_y} \left(\delta_{H_x}(x)\delta_{H_y}(z)\right)^i \left(\delta_{H_y}(y)\delta_{H_y}(z)\right)^j p(t/2, x, z)p(t/2, x, y)dz.$$

Applying Proposition 2.2 and the formula (16) we get

$$I_{i,j} \lesssim p(t, x, y) \left(\delta_{H_x}(x)\left(\sqrt{T} + \delta_{H_x}\left(\frac{x+y}{2}\right)\right)\right)^i \left(\delta_{H_y}(y)\left(\sqrt{T} + \delta_{H_x}\left(\frac{x+y}{2}\right)\right)\right)^j \lesssim \left(1 + \frac{T}{r^2}\right) p(t, x, y) \left(\delta_{H_x}(x)\delta_{H_x}(y)\right)^i \left(\delta_{H_y}(y)\delta_{H_y}(x)\right)^j,$$

which follows

$$p_D(t, x, y) \lesssim \left(1 + \frac{T}{r^2}\right) p(t, x, y) \min_{i,j \in \{0, 1\}} \left\{ \left(\delta_{H_x}(x)\delta_{H_y}(y)\right)^i \left(\delta_{H_y}(y)\delta_{H_y}(x)\right)^j \right\} = \left(1 + \frac{T}{r^2}\right) p(t, x, y) \left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t}\right) \left(1 + \frac{\delta_{H_y}(y)\delta_{H_x}(x)}{t}\right).$$

This ends the proof. \hfill \Box

Theorem 3.4 follows now directly from Corollary 3.2 and Theorem 3.3.

**Theorem 3.4.** Let $D \subset \mathbb{R}^n$ be a $C_r^{1,1}$, $r > 0$, domain. There is a constant $C = C(T, n, r) > 0$ such that

$$p_D(t, x, y) \leq Cp(t, x, y) \left[\left(1 + \frac{\delta(x)\delta(y)}{t}\right) + \left(1 + \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t}\right) \left(1 + \frac{\delta_{H_y}(y)\delta_{H_x}(x)}{t}\right)\right],$$

where $x, y \in D$, $t < T$. 

10
Using the inequality \( \delta_{H^x}(y) \leq (\delta_{H^x}(x) + \delta_{H^x}(y)) \leq 2\delta_{H^x}(\frac{x+y}{2}) \) we obtain another bound, which will be used in the sequel.

**Corollary 3.5.** Let \( D \subset \mathbb{R}^n \) be a \( C^1 \), \( r > 0 \), domain. There is a constant \( C = C(T, n, r) > 0 \) such that

\[
p_D(t, x, y) \leq Cp(t, x, y) \left[ \left( 1 + \frac{\delta(x)\delta(y)}{t} \right) + \left( 1 + \frac{\delta_{H^x}(x)\delta_{H^x}(\frac{x+y}{2})}{t} \right) \right],
\]

where \( x, y \in D \), \( t < T \).

### 4 Lower bounds

We start this section with a lower bound for heat kernels of a very specific set, which is so-called half-capsule. Namely, for \( L \geq R > 0 \) we define

\[
J_{R,L} = B_n(0, R) \cup ((0, L) \times B_{n-1}(0, R)).
\]

(17)

Roughly speaking, \( J_{R,L} \) is a cylinder of radius \( R \) and height \( L \) with a hemisphere of radius \( R \) attached to one on the bases of the cylinder.

**Lemma 4.1.** Let \( L > 3\sqrt{t} \) and \( x = (L - \sqrt{t}, 0, 0, ..., 0) \). There is a constant \( C \) depending only on \( n \) such that for \( 0 < s \leq t \) and \( y \in J_{\sqrt{t},L} \) such that \( y_1 \leq 0 \) we have

\[
p_{J_{\sqrt{t},L}}(s, x, y) \geq C \left( 1 + \frac{\delta_{J_{\sqrt{t},L}}(y)\sqrt{t}}{s} \right) p(s, x, y).
\]

(18)

**Proof.** The scaling property of Brownian motion gives us

\[
p_{J_{\sqrt{t},L}}(s, x, y) = p_{J_{1,\sqrt{t}/L}} \left( \frac{s}{t}, \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right),
\]

which, together with the equality \( \delta_{J_{\sqrt{t},L}}(y) = \sqrt{t}\delta_{J_{1,\sqrt{t}/L}} \left( \frac{y}{\sqrt{t}} \right) \), allows us to consider only the case \( t = 1 \). Without loss of generality we also assume \( y = (y_1, y_2, 0, ..., 0) \) with \( y_1 \leq 0 \) and \( y_2 \in [0, 1) \), which is justified by the rotational invariance of the set \( J_{1,L} \) with respect to the \( Ox_1 \) axis.

Recall the notation \( \mathbb{R}^n \ni x \rightarrow \tilde{x} = (x_2, x_3, ..., x_n) \in \mathbb{R}^{n-1} \) and let

\[
T = \left\{ x \in \mathbb{R}^n : \left( \frac{3}{2} \right)^2 + y_1^2 < \left( \frac{1}{4} \right)^2 \right\} \subset J_{1,L}
\]

be a torus tangent to \( J_{1,L} \) at \( \{0\} \times \partial B_{n-1}(0, 1) \), and let \( \tilde{T} := T \cup (\left( -\frac{1}{4}, \frac{1}{4} \right) \times B_{n-1}(0, \frac{3}{4})) \) be the set of all convex combinations of points from \( T \). Alternatively, we may define \( \tilde{T} \) as

\[
\tilde{T} = \bigcup_{z \in \{0\} \times B_{n-1}(0, \frac{3}{4})} B_n(z, \frac{1}{4}),
\]

which may better depict properties of this set.
The main step of the proof is to show that the assertion holds for \( y \in \mathbb{T} \cap \{ y_1 < 0 \} \). For \( y \in \mathbb{T} \setminus \mathbb{T} \subset (\frac{-1}{4}, \frac{1}{4}) \times B_{n-1}(0, \frac{3}{4}) \) it holds \( \delta_{J_{1,L}}(y) > \sqrt{5/8} \). Combining this with \( \delta_{J_{1,L}}(x) = 1 \) and convexity of \( J_{1,L} \), one may conclude (15) from (7).

For \( y \in \mathbb{T} \cap \{ y_1 < 0 \} \) the argument is more complicated. Let \( \tau_1 \) be the first time of hitting the hyperplane \( \{ z \in \mathbb{R}^n : z_1 = 0 \} \) by the Brownian motion \( W \) starting from \( x \), i.e.

\[
\tau_1 := \inf \{ u > 0 : W_1(u) = 0 \}.
\]

Then, for any Borel set \( A \subset \{ z \in J_{1,L} : z_1 \leq 0 \} \) the inclusion \( \{ W(s) \in A \} \subset \{ s \geq \tau_1 \} \), \( s > 0 \), and Strong Markov property give us

\[
\int_A p_{J_{1,L}}(s, x, z)dz = \mathbb{P}^x (W(s) \in A, s < \tau_{J_{1,L}})
= \mathbb{P}^x (W(s) \in A, s < \tau_{J_{1,L}}; s \geq \tau_1)
= \mathbb{E}^x \left[ s < \tau_{J_{1,L}}; s \geq \tau_1; \mathbb{E}^{W(u)} [s - u < \tau_{J_{1,L}}; W(s - u) \in A]_{u=\tau_1} \right]
= \mathbb{E}^x \left[ s < \tau_{J_{1,L}}; s \geq \tau_1; \int_A p_{J_{1,L}}(s - \tau_1, W(\tau_1), z)dz \right],
\]

which implies

\[
p_{J_{1,L}}(s, x, y) = \mathbb{E}^x \left[ s < \tau_{J_{1,L}}; s \geq \tau_1; p_{J_{1,L}}(s - \tau_1, W(\tau_1), z) \right],
\]

where \( y \in \{ z \in J_{1,L} : z_1 \leq 0 \} \). Let us now introduce another few sets. First, we denote

\[
B_0 = B((0, \frac{1}{4}, 0, ..., 0), \frac{3}{4}),
I_0 = B_{n-1}(\frac{1}{4}, 0, ..., 0, \frac{3}{4}).
\]

The relation between \( B_0 \) and \( I_0 \) is that \( B_0 \cap \{ x_1 = 0 \} = \{ 0 \} \times I_0 \). Furthermore, we define

\[
R_{L}^{(1)} := (-\infty, L) \times B_{n-1}(0, 1),
R_{L}^{(2)} := (-\infty, L) \times I_0,
H := \{ x \in \mathbb{R}^n : x_2 > 1 \}.
\]

The set \( R_{L}^{(1)} \) is an extension of \( J_{1,L} \) into a half-infinite cylinder, \( R_{L}^{(2)} \) is another half-infinite cylinder which is contained in \( R_{L}^{(1)} \) and \( H \) is a half-space tangent to both of the cylinders at \( (\infty, L) \times \{(1, 0, ..., 0)\} \subset \mathbb{R}^n \). The crucial properties of these sets are

\[
\delta_{R_{L}^{(1)}}(x) \approx \delta_{R_{L}^{(2)}}(x) \approx \delta_H(x) \approx 1,
\]

and

\[
\delta_{B(0,1)}(z) \approx \delta_{R_{L}^{(1)}}(z) \approx \delta_H(z), \quad \text{for } z \in B_0.
\]

Bounds in (21) are clear while bounds in (22) follow from the below-given calculations:

\[
\begin{align*}
\delta_H(z) & \geq \delta_{R_{L}^{(1)}}(z) \geq \delta_{B(0,1)}(y) = 1 - |z| \geq \frac{1}{2} \left( 1 - |z|^2 \right) \\
& = \frac{1}{2} \left[ \frac{1}{2} (1 - z_2) - \left( z_1^2 + (z_2 - \frac{1}{4})^2 + ... + z_n^2 - \left( \frac{3}{4} \right)^2 \right) \right]
= \frac{1}{4} (1 - z_2) + \frac{1}{2} \left[ \left( \frac{3}{4} \right)^2 - |z - (0, \frac{1}{4}, 0, ..., 0)|^2 \right]
> \frac{1}{4} (1 - z_2) = \frac{1}{4} \delta_H(z).
\end{align*}
\]

(23)
Our goal in this part of the proof is to show that \( p_{J_{1,L}}(s, x, y) \gtrsim p_{R_L^{(2)}}(s, x, y) \) holds for \( y \in \mathbb{T} \cap \{ y_1 \leq 0 \} \) (keeping in mind the special form of \( y = (y_1, y_2, 0, ..., 0) \)) and \( 0 < s < 1 \).

Since \( J_{1,L} \cap \{ x \in \mathbb{R}^n : x_1 > 0 \} = R_L^{(1)} \cap \{ x \in \mathbb{R}^n : x_1 > 0 \} \), then \( \{ s < \tau_{J_{1,L}}; s \geq \tau_1 \} = \{ s < \tau_{R_L^{(1)}}; s \geq \tau_1 \} \) so we may change the condition \( s < \tau_{J_{1,L}} \) into \( s < \tau_{R_L^{(1)}} \) in (20). Furthermore, adding also the condition \( W(\tau_1) \in \{ 0 \} \times I_0 \) under the expectation we arrive at

\[
p_{J_{1,L}}(s, x, y) \gtrsim \mathbb{E}^x \left[ s < \tau_{R_L^{(1)}}; s \geq \tau_1; W(\tau_1) \in \{ 0 \} \times I_0; p_{J_{1,L}}(s - \tau_1, W(\tau_1), y) \right]. \tag{24}
\]

Observe now that the assumption \( y = (y_1, y_2, 0, ..., 0) \in \mathbb{T} \) and the condition \( W(\tau_1) \in \{ 0 \} \times I_0 \) imply \( y, W(\tau_1) \in B_0 \). Consequently, by (11), (2), (22) and (10) we get for \( 0 < s \leq 1 \)

\[
p_{J_{1,L}}(s, W(\tau_1), y) \geq p_{B(0,1)}(s, W(\tau_1), y) \geq \left( 1 \wedge \frac{\delta_H(W(\tau_1))\delta_H(y)}{s} \right) p(s, W(\tau_1), y) \approx p_H(s, W(\tau_1), y) \geq p_{R_L^{(1)}}(s, W(\tau_1), y).
\]

Applying this to (24), we obtain

\[
p_{J_{1,L}}(s, x, y) \gtrsim \mathbb{E}^x \left[ s < \tau_{R_L^{(1)}}; s \geq \tau_1; W(\tau_1) \in \{ 0 \} \times I_0; p_{R_L^{(1)}}(s - \tau_1, W(\tau_1), y) \right].
\]

The inclusion \( R_L^{(2)} \subset R_L^{(1)} \) implies \( \{ s < \tau_{R_L^{(2)}} \} \subset \{ s < \tau_{R_L^{(1)}} \} \) and \( p_{R_L^{(1)}}(s - \tau_1, W(\tau_1), y) \geq p_{R_L^{(2)}}(s - \tau_1, W(\tau_1), y) \). Hence

\[
p_{J_{1,L}}(s, x, y) \gtrsim \mathbb{E}^x \left[ s < \tau_{R_L^{(2)}}; s \geq \tau_1; W(\tau_1) \in \{ 0 \} \times I_0; p_{R_L^{(2)}}(s - \tau_1, W(\tau_1), y) \right].
\]

Furthermore, the condition \( W(\tau_1) \in \{ 0 \} \times I_0 \) is always satisfied on the set \( \{ s < \tau_{R_L^{(2)}} \} \), so it may be removed. Thus, repeating argument from (19), we conclude

\[
p_{J_{1,L}}(s, x, y) \gtrsim \mathbb{E}^x \left[ s < \tau_{R_L^{(2)}}; s \geq \tau_1; p_{R_L^{(2)}}(s - \tau_1, W(\tau_1), y) \right] = p_{R_L^{(2)}}(s, x, y),
\]

as required. Next we will show that \( p_{R_L^{(2)}}(s, x, y) \) admits lower estimate from (18). Cylindrical form of \( R_L^{(2)} \) combined with (10) and (2) give us

\[
p_{R_L^{(2)}}(s, x, y) = p_{(-\infty,\cdot)}(s, x_1, y_1)p_{B_0}(s, \bar{x}, \bar{y}) \gtrsim \left( 1 \wedge \frac{(L-x_1)(L-y_1)}{s} \right) \left( 1 \wedge \frac{\delta_{I_0}(\bar{x})\delta_{I_0}(\bar{y})}{s} \right) p(s, x, y) = \left( 1 \wedge \frac{\delta_{I_0}(\bar{y})}{s} \right) p(s, x, y).
\]

Since \( y = (y_1, y_2, 0, ..., 0) \in \mathbb{T} \), we have \( y \in B_m((0, \frac{3}{4}, 0, ..., 0), \frac{1}{4}) \), and, analogously as in (23) (or by rescaling), one can show that \( \delta_{I_0}(\bar{y}) \approx \delta_{J_{1,L}}(y) \), which completes the proof for \( y \in \mathbb{T} \).

Finally, let us consider any \( y \in J_{1,L} \) such that \( y_1 \leq 0 \), as in the assertion. Denote by \( m = (m_1, ..., m_n) \) the point on the interval \( \mathbb{T} \) such that \( m_1 = 0 \). Due to the special form of \( x \) and \( y \) we have \( m = (0, m_2, 0, ..., 0) \) with \( m_2 \in [0, 1) \). Next, we put \( \alpha_{xy} = \frac{\lvert m-y \rvert}{\lvert x-y \rvert} \) and denote

\[
m' = m - \frac{1}{2} \sqrt{\alpha_{xy}}s(0, 1, 0, ..., 0) = (0, m_2 - \frac{1}{2} \sqrt{\alpha_{xy}}, 0, ..., 0).
\]
Since $\alpha_{xy} < \frac{1}{L-1} < \frac{1}{2}$, we have $B(m', \frac{1}{4} \sqrt{\alpha_{xy}s}) \subset \tilde{T}$, and for all $z \in B(m', \frac{1}{4} \sqrt{\alpha_{xy}s})$ it holds

$$
\delta_{J_{1,L}}(z) \geq \delta_{J_{1,L}}(m') - \frac{1}{4} \sqrt{\alpha_{xy}s} \\
\geq (1 - (m'_2 + \frac{1}{4} \sqrt{\alpha_{xy}s})) \wedge ((m'_2 - \frac{1}{4} \sqrt{\alpha_{xy}s}) - (-1)) \\
\geq (1 - m_2 + \frac{1}{4} \sqrt{\alpha_{xy}s}) \wedge (1 - \frac{3}{4} \sqrt{\alpha_{xy}s}) \\
\geq (\delta_{J_{1,L}}(m) + \frac{1}{4} \sqrt{\alpha_{xy}s}) \wedge \left( \frac{1}{4} \right) \\
\geq \frac{1}{8} \left( \delta_{J_{1,L}}(m) + \frac{1}{4} \sqrt{\alpha_{xy}s} \right).
$$

Furthermore, we clearly have $\delta_{J_{1,L}}(m) \geq \delta_{J_{1,L}}(y)$ and, by intercept theorem, $\delta_{J_{1,L}}(m) \geq \alpha_{xy} \delta_{J_{1,L}}(x) = \alpha_{xy}$, which eventually gives us

$$
\delta_{J_{1,L}}(z) \gtrsim \alpha_{xy} w(x, y) := \sqrt{\alpha_{xy}s} + \delta_{J_{1,L}}(y) + \alpha_{xy}, \quad z \in B(m', \frac{1}{4} \sqrt{\alpha_{xy}s}).
$$

Consequently, from previous case we have for $z \in \tilde{T} \cap \{z_1 < 0\}$

$$
p_{J_{1,L}}((1 - \alpha_{xy})s, x, z) \gtrsim p(t, x, z) \left( 1 \wedge \frac{\delta_{J_{1,L}}(z)}{(1 - \alpha_{xy})s} \right) \gtrsim p(t, x, z) \left( 1 \wedge \frac{w(x, y)}{s} \right).
$$

Additionally, for the same range of $z$, (2) implies

$$
p_{J_{1,L}}(\alpha_{xy}s, z, y) \geq p_{B(0,1)}(\alpha_{xy}s, z, y) \gtrsim p(\alpha_{xy}s, z, y) \left( 1 \wedge \frac{\delta_{J_{1,L}}(y) w(x, y)}{\alpha_{xy}s} \right).
$$

Then, by Chapman-Kolmogorov identity and Corollary 2.1 we get

$$
p_{J_{1,L}}(s, x, y) \gtrsim \int_{B(m', \frac{1}{4} \sqrt{\alpha_{xy}s}) \cap \{z_1 < 0\}} p_{J_{1,L}}((1 - \alpha_{xy})s, x, z) p_{J_{1,L}}(\alpha_{xy}s, z, y) dz \\
\gtrsim \int_{B(m', \frac{1}{4} \sqrt{\alpha_{xy}s})} p((1 - \alpha_{xy})s, x, z) p(\alpha_{xy}s, z, y) dz \\
\gtrsim \left( 1 \wedge \frac{w(x, y)}{s} \right) \left( 1 \wedge \frac{\delta_{J_{1,L}}(y) w(x, y)}{\alpha_{xy}s} \right) p(s, x, y),
$$

where

$$
m'' = m' - \frac{1}{2} \sqrt{\alpha_{xy}s}(1, 0, ..., 0) = (-\frac{1}{2} \sqrt{\alpha_{xy}s}, m_2 - \frac{1}{2} \sqrt{\alpha_{xy}s}, 0, ..., 0).
$$

In order to finish the proof, we need to show that the above product of two minima is greater (up to a constant factor) than $\left( 1 \wedge \frac{\delta_{J_{1,L}}(y)}{s} \right)$. If the right-hand side minimum is equal to 1, then it is enough to use the bound $w(x, y) > \delta_{J_{1,L}}(y)$. If both of them are smaller than 1, one can use $w(x, y) > \sqrt{\alpha_{xy}s}$. Finally, if only the right-hand side one is smaller than 1, then we need to employ the bound $w(x, y) > \alpha_{xy}$. The proof is complete.

The next theorem provides a general lower bound of the form as in (1) but with proper exponential behavior. Nevertheless, both results concern different classes of sets (with non-empty intersection).

14
Theorem 4.2. For any convex set $D \in C_{r,1}^1$ and $T > 0$ there is $C = C(n, r, T)$ such that

$$p_D(t, x, y) \geq C \left( 1 \wedge \frac{\delta_D(x)\delta_D(y)}{t} \right) p(t, x, y).$$

Proof. Assume $t < 1$. First, consider the case $\delta_D(x) > 10\sqrt{t}$. Due to the bound (7), we may additionally assume $\delta_D(y) \leq \sqrt{t}$. Put $\alpha_{xy} = 5\sqrt{t}/\delta_D(x) < 1/2$ and $m = \alpha_{xy}x + (1 - \alpha_{xy})y$. It follows from intercept theorem that $\delta_D(m) > 5\sqrt{t}$. Thus, for every $z \in B(m, \sqrt{\alpha_{xy}t})$ we have $\delta_D(z) \geq 4\sqrt{t}$ and $|z - y| \geq \delta_D(z) - \delta_D(y) \geq 3\sqrt{t}$, which lets us employ Lemma 4.1 precisely, for a given $z$ one can transform isometrically the set $J_{\sqrt{t}, L_z}$ into $J'_{\sqrt{t}, L_z}$, for suitably chosen $L = L(z) > 3\sqrt{t}$, such that $\delta_D(y) = \delta_{J'_{\sqrt{t}, L_z}}(y)$ and the point $(L_z - \sqrt{t}, 0, 0, ..., 0)$ is transformed into $z$. Then, Lemma 4.1 gives us for $z \in B(m, \sqrt{\alpha_{xy}t})$

$$p_D(\alpha_{xy}t, z, y) \geq p_{J'_{\sqrt{t}, L_z}}(\alpha_{xy}t, z, y) \geq \left( 1 \wedge \frac{\delta_D(y)\sqrt{t}}{\alpha_{xy}t} \right) p(\alpha_{xy}t, z, y) = \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{6t} \right) p(\alpha_{xy}t, z, y),$$

while, by (7), we have

$$p_D((1 - \alpha_{xy})t, x, z) \geq p((1 - \alpha_{xy})t, x, z).$$

Consequently, by Chapman-Kolmogorov identity and Corollary 2.1 we get

$$p_D(t, x, y) \geq \int_{B(m, \sqrt{\alpha_{xy}t})} p_D((1 - \alpha_{xy})t, x, z)p_D(\alpha_{xy}t, z, y)dz \geq \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{t} \right) \int_{B(m, \sqrt{\alpha_{xy}t})} p((1 - \alpha_{xy})t, x, z)p(\alpha_{xy}t, z, y)dz \geq \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{t} \right) p(t, x, y).$$

Consider now $\delta_D(x), \delta_D(y) \leq 10\sqrt{t}$. Let $p_{xy}$ be a point such that $|p_{xy} - \frac{x + y}{2}| = 11\sqrt{t}/2$ and $\delta_D(p_{xy}) \geq 11\sqrt{t}/2$. Such point exists for sufficiently small $T$. Then, for $z \in B(p_{xy}, \sqrt{t}/2)$, we have $\delta_D(z) \geq 10\sqrt{t}/2$ and, from the previous case, assertion is true for $p_D(t/2, x, z)$ and $p_D(t/2, z, y)$. Thus, by Corollary 2.1 we get

$$p_D(t, x, y) \geq \int_{B(p_{xy}, \sqrt{t}/2)} p_D(t/2, x, z)p_D(t/2, z, y)dz \geq \frac{\delta_D(x)\sqrt{t}}{t} \delta_D(y)\sqrt{t} \int_{B(p_{xy}, \sqrt{t}/2)} p(t/2, x, z)p(t/2, z, y)dz \approx \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{t} \right) p(t, x, y),$$

where we used $\delta_D(x) \approx \delta_D(y) \approx \sqrt{t}$. Finally, let us observe that the range of $T$ may be easily extended, but the cost we pay is decrease of the constant $C$. Indeed, for $T \leq t < 2T$
and \( x, y \in D \) there is a point \( q_{xy} \in D \) such that \(|q_{st} - \frac{x+y}{2}| \leq \sqrt{t \frac{r}{2\sqrt{T}}} \) and \( \delta_D(q_{xy}) \geq \sqrt{t \frac{r}{2\sqrt{T}}} \).

Then, repeating previous arguments, we obtain

\[
p_D(t, x, y) \geq \int_{B(q_{xy}, \sqrt{t \frac{r}{2\sqrt{T}}})} p_D(t/2, x, z) p_D(t/2, z, y) dz
\]

\[
\geq \left(1 \wedge \frac{\delta_D(x) \sqrt{t \frac{r}{2\sqrt{T}}}}{t}\right) \left(1 \wedge \frac{\delta_D(y) \sqrt{t \frac{r}{2\sqrt{T}}}}{t}\right) \int_{B(q_{xy}, \sqrt{t \frac{r}{2\sqrt{T}}})} p(t/2, x, z) p(t/2, z, y) dz
\]

\[
\geq c \left(1 \wedge \frac{\delta_D(y) \delta_D(x)}{t}\right) p(t, x, y),
\]

for some constant \( c = c(n, r, T) \).

The lower bound from Theorem 4.2 may be improved by suitable application of Chapman-Kolmogorov identity, as presented below. In fact, the procedure could be iterated, however, further iterations lead to much more complicated forms, which do not seem to be a relevant enhancement in the general case.

**Theorem 4.3.** For any convex set \( D \in C_{r;1} \) and \( T > 0 \) there is \( C = C(n, r, T) \) such that

\[
p_D(t, x, y)
\]

\[
\geq C p(t, x, y) \left(1 \wedge \frac{\delta_D(x) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right) \left(1 \wedge \frac{\delta_D(y) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right)
\]

\[
\approx C p(t, x, y) \left[\left(1 \wedge \frac{\delta_D(x) \delta_D(y)}{t}\right) + \left(1 \wedge \frac{\delta_D(x) \delta_D\left(\frac{x+y}{2}\right)}{t}\right) \left(1 \wedge \frac{\delta_D(y) \delta_D\left(\frac{x+y}{2}\right)}{t}\right)\right].
\]

**Proof.** For \( \delta_D\left(\frac{x+y}{2}\right) \leq 2\sqrt{t} \) the inequality (25) follows directly from Theorem 4.2. We therefore assume \( \delta_D\left(\frac{x+y}{2}\right) \geq 2\sqrt{t} \). Then, for \( z \in B\left(\frac{x+y}{2}, \sqrt{t}\right) \subset D \) it holds

\[
\delta_D(z) \geq \frac{1}{2} \delta_D\left(\frac{x+y}{2}\right) \geq \frac{1}{4} \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right).
\]

Hence, by Chapman-Kolmogorov equality, Theorem 4.2 and Corollary 2.1, we get

\[
p_D(t, x, y)
\]

\[
= \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz
\]

\[
\geq C \int_{B\left(\frac{x+y}{2}, \sqrt{t}\right)} p(t/2, x, z) \left(1 \wedge \frac{\delta_D(x) \delta_D(z)}{t}\right) p(t/2, z, y) \left(1 \wedge \frac{\delta_D(z) \delta_D(y)}{t}\right) dz
\]

\[
\geq C \left(1 \wedge \frac{\delta_D(x) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right) \left(1 \wedge \frac{\delta_D(y) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right)
\]

\[
\times \int_{B\left(\frac{x+y}{2}, \sqrt{t}\right)} p(t/2, x, z) p(t/2, z, y) dz
\]

\[
\geq C \left(1 \wedge \frac{\delta_D(x) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right) \left(1 \wedge \frac{\delta_D(y) \left(\delta_D\left(\frac{x+y}{2}\right) + \sqrt{t}\right)}{t}\right) p(t, x, y).
\]
as required. In order to show (26) we consider two cases as well. If \( \delta_D \left( \frac{x+y}{2} \right) \geq \sqrt{t} \),

\[
\left( 1 \wedge \frac{\delta_D(x)\left(\delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right)}{t} \right) \left( 1 \wedge \frac{\delta_D(y)\left(\delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right)}{t} \right)
\]

\( \approx \left( 1 \wedge \frac{\delta_D(x)\delta_D \left( \frac{x+y}{2} \right)}{t} \right) \left( 1 \wedge \frac{\delta_D(y)\delta_D \left( \frac{x+y}{2} \right)}{t} \right)
\]

\( \geq \left( 1 \wedge \frac{\delta_D(x)\delta_D(y)}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{\sqrt{t}} \right) \),

which means that the right-hand side term in brackets in (26) is dominating and comparable with the factor from (25). Similarly, for \( \delta_D \left( \frac{x+y}{2} \right) \leq \sqrt{t} \) we conclude \( \delta_D(x), \delta_D(y) \leq 2\sqrt{t} \), and hence

\[
\left( 1 \wedge \frac{\delta_D(x)\left(\delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right)}{t^2} \right) \left( 1 \wedge \frac{\delta_D(y)\left(\delta_D \left( \frac{x+y}{2} \right) + \sqrt{t} \right)}{t^2} \right)
\]

\( \approx \frac{\delta_D(x)\delta_D(y)}{t^2} \approx \left( 1 \wedge \frac{\delta_D(x)\delta_D(y)}{t^2} \right) \left( 1 \wedge \frac{\delta_D(y)\delta_D(x)}{t^2} \right) \),

which ends the proof. \( \square \)

5 Two-sided estimates

5.1 General results

For a strictly convex \( C^{1,1} \) domain \( D \) we define

\[
Q_D := \inf_{w,z \in \partial D, w \neq z} \frac{\delta_D \left( \frac{w+z}{2} \right)}{\delta_{H_w} \left( \frac{w+z}{2} \right)},
\]

\[
R_D := \min \left\{ \inf_{w,z \in \partial D, w \neq z} \frac{\delta_D \left( \frac{w+z}{2} \right)}{\delta_{H_w} \left( \frac{w+z}{2} \right)}, \inf_{w,z \in \partial D, w \neq z} \sup_{m \in \bar{w}z} \frac{\delta_D(m)}{\delta_{H_w}(m)} \right\}.
\]

Note that since \( D \) is a \( C^{1,1} \) domain, the half-space \( H_w \) is well defined for any \( w \in \partial D \). It is clear that \( 0 \leq Q_D, R_D \leq 1 \). Furthermore, since \( \frac{w+z}{2} \) is a possible value of \( m \) in the supremum in the definition of \( R_D \), it holds \( Q_D \leq R_D \). In general, we will be expecting \( Q_D, R_D > 0 \). The condition \( Q_D > 0 \) means that for any \( w, z \in \partial D \) the distance from the midpoint \( \frac{w+z}{2} \) to the boundary \( \partial D \) is comparable with the distances to \( P_w \) and \( P_z \). In case \( R_D > 0 \) the condition is weaker whenever \( \delta_D \left( \frac{w+z}{2} \right) > 1 \), as we only require existence of a point at the interval \( wz \) whose distance to \( \partial D \) is greater than 1 and comparable to distance to \( P_w \). Let us introduce the following two class of sets corresponding to the characteristics \( Q_D \) and \( R_D \)

\[
S_Q := \{ D \in C^{1,1}(\mathbb{R}^n) : D \text{ is strictly convex, } Q_D > 0 \},
\]

\[
S_R := \{ D \in C^{1,1}(\mathbb{R}^n) : D \text{ is strictly convex, } R_D > 0 \}.
\]
In the definition of $S_Q$ we do not require the sets to be in $C^{1,1}(\mathbb{R}^n)$, since it turns out that every $D \in S_Q$ is bounded (see Lemma 5.2) and it is well known that every bounded $C^{1,1}$ set belongs to $C^{1,1}(\mathbb{R}^n)$ for some $r > 0$. Both of the classes $S_Q$ and $S_R$ contain nontrivial and important examples (see Propositions 5.6 and 5.7). It seems also not easy to construct a strictly convex $C^{1,1}$ set which does not belong to $S_R$.

The following monotonicity property will be needed in the sequel:

**Proposition 5.1.** Let $D$ be a $C^{1,1}$ convex domain in $\mathbb{R}^n$, $n \geq 2$, and let $w, z \in \partial D$. Then the function 
$$\alpha \mapsto \frac{\delta_D ((1 - \alpha)w + \alpha z)}{\delta_{H_w} ((1 - \alpha)w + \alpha z)}$$
is non-increasing on $[0, 1]$.

**Proof.** Consider first $n = 2$. Let $p, q \in \overline{wz}$ be such that $0 < |w - p| < |w - q| \leq |w - z|$ and let $p' \in \partial D$ be a point realizing the distance of $p$ to $\partial D$. Then $P_p := P_{p'}$ is a line tangent to $D$ at $p'$, and $H_p$ is the related half-plane. Thus, intercept theorem gives us
$$\frac{\delta_{H_w}(p)}{|w - p|} = \frac{\delta_{H_w}(q)}{|w - q|}, \quad \text{and} \quad \frac{\delta_D(p)}{|w - p|} \geq \frac{\delta_{H_p}(q)}{|w - q|}.$$

This implies
$$\frac{\delta_D(p)}{\delta_{H_w}(p)} \geq \frac{\delta_{H_p}(q)}{\delta_{H_w}(q)},$$
as required. For $n \geq 3$ let $P$ be the 2-dimensional plane containing the interval $\overline{wz}$ and the point $p' \in \partial D$ realizing the distance of $p$ to $\partial D$. Since $\delta_{D \cap P}(p) = \delta_D(p)$, $\delta_{D \cap P}(q) \geq \delta_D(q)$ and, by intercept theorem, $\frac{\delta_{H_w \cap P}(p)}{\delta_{H_w}(p)} = \frac{\delta_{H_w}(p)}{\delta_{H_w}(q)}$, we may apply the result for $n = 2$ and get
$$\frac{\delta_D(p)}{\delta_{H_w}(p)} = \frac{\delta_{D \cap P}(p)}{\delta_{H_w}(p)} \geq \frac{\delta_{D \cap P}(q)}{\delta_{H_w}(p)} \geq \frac{\delta_D(q)}{\delta_{H_w}(p)} \geq \frac{\delta_D(q)}{\delta_{H_w}(q)},$$

which ends the proof. \hfill \Box

The next lemma shows that if $Q_D > 0$, then $D$ is bounded and the the infimum from the definition of $Q_D$ taken over all $z, w \in \overline{D}$ is positive as well, and consequently $\delta_D\left(\frac{w + z}{2}\right)$ and $\delta_{H_z}\left(\frac{w + z}{2}\right)$ (for any choice of $H_w$) are comparable for any $z, w \in \overline{D}$.

**Lemma 5.2.** Let $D \in S_Q$. Then

i) $D$ is bounded,

ii) for every $x, y \in D$ we have
$$\delta_D\left(\frac{x + y}{2}\right) \leq \delta_{H_x}\left(\frac{x + y}{2}\right) \leq \frac{3}{Q_D} \delta_D\left(\frac{x + y}{2}\right), \quad (27)$$

where $H_x$ is any half-space such that $\delta_D(x) = \delta_{H_x}(x)$ and $D \subset H_x$.

**Proof.** i) Assume $D$ is an unbounded strictly convex $C^{1,1}$ set with $Q_D > 0$. There exists a half-line $l$ starting at some point $w \in \partial D$ and contained in $\overline{D}$. Let $P$ be a (2-dimensional) plane containing $l$ and equipped with coordinate system of axes $Ox$ and $Oy$ such that $l$ is the nonnegative half-line of $Oy$. The intersection $P \cap \partial D$ is then a graph of a strictly convex $C^{1,1}$ function $f(x)$. Without loss of the generality we may assume that $f$ is increasing.
on \([0, \infty)\). The distance of any point from \(P\) of the form \((x, f(x))\) to the hyper-plane \(H_w\) is proportional to its distance to a line in \(P\) tangent to \(f\) at \((0,0)\) (i.e. the point \(w\)). Hence, since \(f\) is strictly convex and increasing on \([0, \infty)\), there is a constant \(c_1 > 0\) such that \(\delta_{H_w}(x, f(x)) > c_1 f(x)\) for \(x > 1\). Additionally, for \(y > f(x)\) we clearly have \(\delta_D((x, y)) \leq \delta_{D\cap P}((x, y)) \leq (y - f(x)) \land (f^{-1}(y) - x)\), where \(f^{-1}\) is inverse of \(f\) on \([0, \infty)\) and \((x, y)\) is a point on \(P \subset \mathbb{R}^n\) in coordinatized introduced on \(P\). Taking \(z = (x, f(x)) \in \partial D\), \(x > 1\), we get

\[
\frac{\delta_D \left(\frac{x+y}{2}\right)}{\delta_{H_w}(\frac{x+y}{2})} \leq \left(\frac{1}{2} f(x) - f \left(\frac{x+y}{2}\right)\right) \land \left(\frac{1}{2} f(x)\right) =: g(x).
\]

We will show that \(g(x)\) tends to zero as \(x \to \infty\), which contradicts the assumption \(Q_D > 0\). From strict convexity of \(f\) we have \(\lim_{x \to \infty} f(x) = \infty\) and \(f'(x), \frac{1}{2} f(x) - f \left(\frac{x+y}{2}\right)\) are increasing for \(x > 0\). If \(\frac{1}{2} f(x) - f \left(\frac{x+y}{2}\right)\) is bounded, then \(g(x)\) clearly tends to zero. In the other case, it is when \(\frac{1}{2} f(x) - f \left(\frac{x+y}{2}\right)\) tends to infinity, we employ L'Hôpital’s rule and obtain

\[
\lim_{x \to \infty} \frac{\delta_D(\frac{x+y}{2})}{\delta_{H_w}(\frac{x+y}{2})} \leq \lim_{x \to \infty} \frac{(f'(x) - f' \left(\frac{x+y}{2}\right)) \land 1}{c_1 f'(x)}.
\]

Since \(f'(x)\) tends either to a constant or to infinity, the last limit equals zero, as required.

\(\text{ii) The first inequality follows simply from the inclusion } D \subset H_x. \text{ Let } x', y' \in \partial D \text{ be points realising distances of } x \text{ and } y, \text{ respectively, to the boundary } \partial D, \text{ i.e. such that } |x - x'| = \delta_D(x) \text{ and } |y - y'| = \delta_D(y), \text{ and denote } H_x = H_{x'}, H_y = H_{y'}. \text{ If } x' = y', \text{ then } \delta_{H_x} \left(\frac{x+y}{2}\right) = \delta_D \left(\frac{x+y}{2}\right) \text{ and } (27) \text{ holds since } Q_D \leq 1. \text{ In case } x' \neq y' \text{ we observe}

\[
\frac{|x+y|}{2} - \frac{x'+y'}{2} \leq \frac{1}{2} |x - x'| + \frac{1}{2} |y - y'| = \frac{1}{2} (\delta_D(x) + \delta_D(y)) \leq \delta_D \left(\frac{x+y}{2}\right),
\]

where the last inequality follows from convexity of \(D\). Consequently,

\[
\delta_{H_x} \left(\frac{x+y}{2}\right) \leq \delta_{H_x} \left(\frac{x'+y'}{2}\right) + \frac{x+y}{2} - \frac{x'+y'}{2} \leq \frac{1}{Q_D} \delta_D \left(\frac{x+y}{2}\right) + \delta_D \left(\frac{x+y}{2}\right)
\]

\[
= \frac{1}{Q_D} \left(\delta_D \left(\frac{x+y}{2}\right) + \frac{|x+y|}{2} - \frac{|x'+y'|}{2}\right) + \delta_D \left(\frac{x+y}{2}\right)
\]

\[
\leq \left(\frac{2}{Q_D} + 1\right) \delta_D \left(\frac{x+y}{2}\right) \leq \frac{3}{Q_D} \delta_D \left(\frac{x+y}{2}\right),
\]

where we used the inequality \(Q_D \leq 1. \)

\(\text{Theorem } 4.3 \text{ and Lemma } 5.2 \text{ (i) applied to Corollary } 3.5 \text{ follow directly}

\(\textbf{Corollary 5.3. If } D \in S_Q \text{ then}

\[
p_D(t, x, y)
\]

\[
\overset{\tau, Q_D, T}{\approx} p(t, x, y) \left[1 \wedge \frac{\delta_D(x) \delta_D(y)}{t}\right] + \left[1 \wedge \frac{\delta_D(x) \delta_D \left(\frac{x+y}{2}\right)}{t}\right] 
\]

\[
\overset{Q_D}{\approx} p(t, x, y) \left[1 \wedge \frac{\delta_H(x) \delta_H(y)}{t}\right] + \left[1 \wedge \frac{\delta_H(x) \delta_H \left(\frac{x+y}{2}\right)}{t}\right].
\]

holds for \(x, y \in D, 0 < t < T. \)
Moreover, it turns out that \( S_Q \) is the exact subclass of \( C^{1,1} \) domains for which the lower bound from Theorem 4.3 is equivalent (up to a multiplicative constant) to the upper bound.

**Theorem 5.4.** Let \( D \) be a strictly convex \( C^{1,1} \) set. Then \( D \in S_Q \) if and only if

\[
p_D(t, x, y) \geq \frac{D(t, x, y)}{p(t, x, y)} \left[ \left( 1 \wedge \frac{\delta_D(x)\delta_D(y)}{t} \right) + \left( 1 \wedge \frac{\delta_D(x)\delta_D\left( \frac{x+y}{2} \right)}{t} \right) \left( 1 \wedge \frac{\delta_D(y)\delta_D\left( \frac{x+y}{2} \right)}{t} \right) \right].
\]  

(28)

holds for \( x, y \in D, 0 < t < T \).

**Proof.** If \( Q_D > 0 \), the estimate (28) follows from Corollary 5.3.

Let us assume that (28) holds for all \( x, y \in D \) and \( 0 < t < T \) for some \( T > 0 \), and consider \( w, z \in \partial D \) such that \( w \neq z \). Since \( D \) is a \( C^{1,1} \) set, there is a ball of radius \( r > 0 \) contained in \( D \) and tangent to it at \( w \), which ensures existence of a point \( m \) from the interval \( wz \) such that \( |m - w| < |w - z|/4 \) and

\[
\delta_D(m) > \frac{1}{2}\delta_{H_w}(m).
\]  

(29)

Set

\[
\sqrt{t} = \min \left\{ r, \frac{1}{6}\delta_D(m), \sqrt{T} \right\},
\]  

(30)

and let \( m' \) be a point at the interval \( wm \) such that \( \delta_D(m') = 6\sqrt{t} \). Then, from (29) and Proposition 5.1 we have

\[
\delta_D(m') \approx \delta_{H_w}(m') \approx \sqrt{t}.
\]  

(31)

Since we are going to apply estimates from (28) and \( w \notin D \), we approximate \( w \) by a point from \( D \). Precisely, let \( x \) be a point from \( w\overline{m}m' \) such that \( |w - x| < \sqrt{t} \) and let us put

\[
\alpha := \frac{|x - m'|}{|x - \frac{w + z}{2}|} < \frac{1}{2}.
\]

Then, Chapman-Kolmogorov identity gives us

\[
p_D(t, x, y) \geq \int_{B(m', \sqrt{at})} p_D(\alpha t, x, v)p_D\left( (1 - \alpha)t, v, \frac{w + z}{2} \right) dv.
\]  

(32)

For \( v \in B(m', \sqrt{at}) \) we have \( |x - v| \geq 4\sqrt{t} \) and \( \delta_D(v) \geq \frac{5}{6}\delta_D(m') \geq 5\sqrt{t} \), which allows us to employ Lemma 1.1 (in the same manner as in the proof of Theorem 4.2) and get

\[
p_D(\alpha t, x, v) \gtrsim \left( 1 \wedge \frac{\delta_D(x)\sqrt{t}}{\alpha t} \right) p(\alpha t, x, v) \approx \left( 1 \wedge \frac{\delta_D(x)\delta_D(m')}{\alpha t} \right) p(\alpha t, x, v).
\]

(33)

Note that we could have not used Theorem 4.2 since we assume \( D \) to be any strictly convex \( C^{1,1} \) set, so it may not belong to any \( C^{r,1} \), \( r > 0 \). Next, intercept theorem and the inequality (29) give us

\[
\delta_D(m') \gtrsim \delta_{H_w}(m') = 2\frac{|w - m'|}{|w - z|}\delta_{H_w}\left( \frac{w + z}{2} \right)
\]

\[
\gtrsim \frac{1}{2}\alpha\delta_{H_w}\left( \frac{w + z}{2} \right),
\]

(34)

\[
\gtrsim \frac{1}{2}\alpha\delta_{H_w}\left( \frac{w + z}{2} \right),
\]  

(35)
which follows
\[ p_D(\alpha t, x, w) \gtrsim \left( 1 \wedge \frac{\delta_D(x)\delta_{H_w}(\frac{w+z}{2})}{t} \right) p(\alpha t, x, w). \]

Furthermore, by (7), we have
\[ p_D \left( (1 - \alpha)t, v, \frac{w + z}{2} \right) \gtrsim p \left( (1 - \alpha)t, v, \frac{w + z}{2} \right). \]

Applying the last two bounds to (32) and using Corollary 2.1, we conclude
\[ p_D \left( t, x, \frac{w + z}{2} \right) \geq c_1 \left( 1 \wedge \frac{\delta_D(x)\delta_{H_w}(\frac{w+z}{2})}{t} \right) p \left( t, x, \frac{w + z}{2} \right), \tag{33} \]
for some \( c_1 > 0 \). On the other hand, convexity of \( D \) implies
\[ \delta_D \left( \frac{w + z}{2} \right) \geq \frac{1}{2} \sup_{v \in \partial D} \left\{ \delta_D(v) \right\}, \tag{34} \]
and consequently, by (28),
\[
p_D \left( t, x, \frac{w + z}{2} \right) \leq c_2 p \left( t, x, \frac{w + z}{2} \right) \times \left[ \left( 1 \wedge \frac{\delta_D(x)\delta_D(\frac{w+z}{2})}{t} \right) + \left( 1 \wedge \frac{\delta_D(x)\delta_D(\frac{1}{4}x + \frac{1}{4}w + \frac{1}{4}z)}{t} \right) \right] \leq 3c_2 \left( 1 \wedge \frac{\delta_D(x)\delta_D(\frac{w+z}{2})}{t} \right) p \left( t, x, \frac{w + z}{2} \right), \tag{35} \]
for some \( c_2(n, r, T) > 0 \). Finally, comparing (33) with (35) and taking \( \delta(x) \) sufficiently small, we arrive at
\[ \frac{\delta_D(\frac{w+z}{2})}{\delta_{H_w}(\frac{w+z}{2})} \geq \frac{c_1}{3c_2}, \]
valid for any \( w, z \in \partial D \), which is equivalent to \( Q_D \geq \frac{c_1}{3c_2} > 0 \).

After relaxing the condition \( Q_D > 0 \) into \( R_D > 0 \), the heat kernel \( p_D(t, x, y) \) keeps admitting two-sided estimates of the form of the upper bound from Theorem 3.4.

**Theorem 5.5.** If \( D \in S_R \), then
\[ p_D(t, x, y) \overset{r, T, R_D}{\approx} p(t, x, y) \left[ \left( 1 \wedge \frac{\delta(x)\delta(y)}{t} \right) + \left( 1 \wedge \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_x}(y)\delta_{H_y}(x)}{t} \right) \right]. \]
holds for \( x, y, 0 < t < T \), where \( H_x, H_y \) are any half-spaces such that \( D \subset H_x, H_y \) and \( \delta_D(x) = \delta_{H_x}(x), \delta_D(y) = \delta_{H_y}(y) \).

**Proof.** Due to Theorems 3.4 and 1.2 it is enough to show
\[ p_D(t, x, y) \overset{r, T, R_D}{\gtrsim} p(t, x, y) \left( 1 \wedge \frac{\delta_{H_x}(x)\delta_{H_y}(y)}{t} \right) \left( 1 \wedge \frac{\delta_{H_y}(y)\delta_{H_x}(x)}{t} \right). \tag{36} \]
Let \( D \in C_1^{r,1}(\mathbb{R}^n) \), \( r > 0 \), be a strictly convex domain with \( R_D > 0 \). If \( \delta_D(x), \delta_D(y) \geq \sqrt{t} \), the assertion follows from (7).
Consider $\delta_D(x) \leq \sqrt{t}$ and $\delta_D(y) \geq 6\sqrt{t}$. Let $m$ be a point on the interval $x \left(\frac{x+y}{2}\right)$ such that $\delta_D(m) = 2\sqrt{t}$ and denote $\alpha = \frac{|x-m|}{|x-y|} \leq \frac{1}{3}$. In order to take advantage of the assumption $D \in \mathcal{S}_R$, we need to choose suitably some points from the boundary. Indeed, let $x' \in \partial D$ be a point realising the distance of $x$ to $\partial D$ and let $y' \in \partial D$ be the other intersection point of $\partial D$ and the line containing $x'$ and $y$. Furthermore, denote $m' = (1-\alpha)x + \alpha y$. Since $|m' - x'| = \alpha|x' - y| \leq |x' - \frac{x+y}{2}|$ and $\delta_D(m') \leq \delta_D(m) + \delta_D(x) \leq 3\sqrt{t}$ we get for $t \leq \frac{1}{9}$

$$|m' - x'| \leq \left|x' - \frac{x+y}{2}\right| \wedge \min \left\{|x' - z| : z \in \overline{xy'}, \delta_D(z) \geq 1\right\},$$

and therefore Proposition 5.1 gives us for $H_x = H_x'$

$$\delta_D(m') \geq R_D \delta_H(x)(m') = \alpha R_D \delta_H(y).$$

Then, for $z \in B(m, \sqrt{t})$ we have

$$\delta_D(z) \geq \frac{1}{2} \delta_D(m) \geq \frac{1}{3} \delta_D(m') \geq \frac{1}{3} \alpha R_D \delta_H(y).$$

Consequently, by Theorem 4.2 for such $z$ it holds

$$p_D(\alpha t, x, z) \overset{r.RD}{\gtrsim} \left(1 \wedge \frac{\delta_H(x) \delta_H(y)}{t}\right) p(\alpha t, z, y), \quad t < \frac{1}{9},$$

and, by (7),

$$p_D((1-\alpha) t, z, y) \approx p((1-\alpha)t, z, y).$$

Thus, by virtue of Proposition 2.1 we get

$$p_D(t, z, y) = \int_{B(m, \sqrt{t})} p_D(\alpha t, x, z) p_D((1-\alpha)t, z, y) \, dz$$

$$\overset{r.RD}{\gtrsim} \left(1 \wedge \frac{\delta_H(x) \delta_H(y)}{t}\right) \int_{B(m, \sqrt{t})} p(\alpha t, x, z) p((1-\alpha)t, z, y) \, dz$$

$$\gtrsim \left(1 \wedge \frac{\delta_H(x) \delta_H(y)}{t}\right) p(t, x, y), \quad \text{(37)}$$

where $x, y \in D$ with $\delta_D(y) \geq 6\sqrt{t}$ and $t < \frac{1}{9}$.

Consider now any $x, y \in D$. For $T \leq \frac{1}{9}$ small enough there exists a point $p$ such that $|p - \frac{x+y}{2}| = 7\sqrt{t}$ and $\delta_D(p) \geq 7\sqrt{t}$. Then, for $z \in B(p, \sqrt{t})$ we have $\delta_D(p) \geq 6\sqrt{t}$ and

$$\delta_H(x) \approx \delta_H(p) \approx \delta_H \left(\frac{x+y}{2}\right) \geq \frac{1}{2} \delta_H(y), \quad \delta_H(z) \approx \delta_H(p) \approx \delta_H \left(\frac{x+y}{2}\right) \geq \frac{1}{2} \delta_H(y).$$

Hence, by Chapman-Kolmogorov identity, the estimate (37) and Proposition 2.1

$$p_D(t, z, y) = \int_{B(p, \sqrt{t})} p_D(t/2, x, z) p_D(t/2, z, y) \, dz$$

$$\overset{r.T,RD}{\gtrsim} \left(1 \wedge \frac{\delta_H(x) \delta_H(y)}{t}\right) \left(1 \wedge \frac{\delta_H(y) \delta_H(x)}{t}\right) p(t, x, y)$$

as required. In order to extend the range of $T$ into any positive number, we proceed analogously as in the proof of Theorem 4.2.

$\square$
5.2 Examples

In this section we present some examples of sets/classes of sets and discuss the behaviour of their heat kernels.

**Proposition 5.6.** Consider a domain \( U = \{ x \in \mathbb{R}^n : x_n > a |(x_1, ..., x_{n-1})|^p \} \), where \( p \geq 2, n \geq 2 \) and \( a > 0 \). Then \( U \in \mathcal{S}_R \).

As a consequence, the heat kernel \( p_U(t,x,y) \) admits estimates from Theorem [5.5] with constants depending on \( n, T, a, p \).

**Proof.** Since changing \( a \) only rescales distances in the definition of \( R_D \), we may assume \( a = 1 \). Consider \( w, z \in \partial U \) and let us denote \( w = (w_1, ..., w_{n-1}) \). Additionally, we assume, without loss of the generality, that \( w = (w_1, 0, ..., 0, w_n) \) with \( w_1, w_n \geq 0 \). The hyperplane \( P_w \) tangent to \( U \) at \( w \) is then given by the equation

\[
x_n = pw_1^{p-1}(x_1 - w_1) + w_1^p.
\]

First, we will show that for \( w, z \) satisfying \( |w - z| < \frac{1}{2}|w| \) it holds

\[
\frac{\delta_U(w + z)}{\delta_{H_w}(w + z)} > c_p,
\]

for some constant \( c_p > 0 \). Let \( \delta_{H_w}^1(z) \), \( z \in \partial U \), denote the distance between \( z \) and its "vertical" projection onto \( P_w \), i.e. projection along the vector \((0, ..., 0, -1)\), given by

\[
\delta_{H_w}^1(z) = z_n - (pw_1^{p-1}(z_1 - w_1) + w_1^p).
\]

Using the elementary formula for the distance between a point and a hyperplane, we get

\[
\delta_{H_w}(z) = \frac{|z_n - (pw_1^{p-1}(z_1 - w_1) + w_1^p)|}{\sqrt{(pw_1^{p-1})^2 + 1}} = \frac{\delta_{H_w}^1(z)}{\sqrt{(pw_1^{p-1})^2 + 1}},
\]

which implies

\[
\delta_{H_w}(w + z) = \frac{1}{2}\delta_{H_w}(z) \approx \frac{\delta_{H_w}^1(z)}{\frac{w_1^{p-1}}{1} + 1}.
\]

Furthermore, by two-dimensional Taylor’s formula applied to \( f(x, y) = ((w_1 + x)^2 + y^2)^{p/2} \) we may write for \( y = z - w \)

\[
\delta_{H_w}^1(z) = \left( (w_1 + v_1)^2 + |\tilde{v}|^2 \right)^{p/2} - (pw_1^{p-1}v_1 + w_1^p)
\]

\[
= v_1^2 \left( (w_1 + \xi_1)^2 + \xi_2^2 \right)^{p-1} + (p - 2)(w_1 + \xi_1)^2((w_1 + \xi_1)^2 + \xi_2^2)^{\frac{p-2}{2}}
\]

\[
+ |\tilde{v}|^2 \frac{p}{2} \left( ((w_1 + \xi_1)^2 + \xi_2^2) \right)^{p-1} + (p - 2)\xi_2^2((w_1 + \xi_1)^2 + \xi_2^2)^{\frac{p-2}{2}},
\]

where \( |\xi_1|, |\xi_2| \leq |\tilde{v}|, \xi_1, \xi_2 \in \mathbb{R} \). Hence, for \( |\tilde{v}|/|w| < \frac{1}{2} \) we have \( \delta_{H_w}^1(z) \approx w_1^{p-2}|\tilde{v}|^2 \), and consequently

\[
\delta_{H_w}(w + z) \approx \frac{w_1^{p-2}}{\frac{w_1^{p-1}}{1} + 1}|w - z|^2.
\]
Next we will estimate the distance from \( \frac{w+n}{2} \) to the boundary of \( U \). One can see that if \( \frac{w+n}{2} \leq 1 \), then \( \delta_U \left( \frac{w+n}{2} \right) \approx \delta_U (\frac{w}{2}) \), and if \( \frac{w+n}{2} > 1 \), then \( \delta_U \left( \frac{w+n}{2} \right) \approx \delta_U (\frac{w}{2}) \), where, analogously as previously, \( \delta_U (u) \) denotes the distance between \( u \) and its projection onto \( \partial U \) along the vector \((0, ..., 0, -1) \) and \( \delta_U (u) \) denotes the distance between \( z \) and its projection onto \( \partial U \) along the vector \( y \). In the case \( \frac{w+n}{2} \leq 1 \) convexity of the function \( \mathbb{R} \ni x \rightarrow x^{p/2} \) and the estimate \( a^q - b^q \approx q(a - b)a^{q-1} \) valid for \( 0 \leq b < a \) and \( q \geq 0 \), give us for \( v = z - w \) and \( |v|/|w| < \frac{1}{2} \)

\[
\delta_U \left( \frac{w + z}{2} \right) \approx \delta_U (\frac{w}{2}) = \frac{1}{2} (w_n + z_n) - \left| \frac{w + z}{2} \right|^p
\]

\[
= \left( \frac{1}{2} (w_1^2 + \cdots + 1) (w_1 + v_1)^2 + \frac{1}{2} (v_1)^2 \right)^{p/2} - \left( (w_1 + 1) v_1 + \frac{1}{2} (\bar{v}_1)^2 \right)^{p/2}
\]

\[
\geq \left( \frac{1}{4} v_1^2 + \frac{1}{2} \left| \frac{v_1}{2} \right|^2 \right) \left( \frac{1}{2} w_1^2 + (w_1 + v_1)^2 + \frac{1}{2} (\bar{v}_1)^2 \right)^{q-1}
\]

\[
\approx |v|^2 w_1^{p-2}.
\]

Furthermore, using the above bound, we get for \( \frac{w+n}{2} > 1 \)

\[
\delta_U \left( \frac{w + z}{2} \right) \approx \delta_U (\frac{w}{2}) = \left( \frac{w_n + z_n}{2} \right)^{1/p} - \left| \frac{w + z}{2} \right|^p
\]

\[
p \approx \frac{1}{2} (w_n + z_n) - \left| \frac{w + z}{2} \right|^p \left( \frac{w_n + z_n}{2} \right)^{1/p - 1}
\]

\[
p \approx \delta_U (\frac{w + z}{2}) \frac{w_1^{p-1}}{w_1^{p-1} + 1} |w - z|^2 \approx \delta_{H_w} (\frac{w + z}{2}),
\]

which ends the proof of (38).

Let us pass to the main part of the proof. Assume that \( U \notin \mathcal{S}_R \), i.e. \( \mathcal{R}_U = 0 \). Then, there is a sequence of pairs \((w^{(k)}, z^{(k)})\) from \( \partial U \times \partial U \) such that \( \lim_{k \to \infty} f_U (w^{(k)}, z^{(k)}) = 0 \), where

\[
f_U (w, z) = \begin{cases} 
\delta_U \left( \frac{w + z}{2} \right) & \text{if } \delta_U \left( \frac{w + z}{2} \right) \leq 1, \\
\sup_{m \in \mathbb{R}^2 \atop \delta_U (m) > 1} \frac{\delta_U (m)}{\delta_{H_w} (m)} & \text{if } \delta_U \left( \frac{w + z}{2} \right) > 1.
\end{cases}
\]

Furthermore, there is a subsequence \((w^{(k)}_l, z^{(k)}_l)\) such that \( |w^{(k)}_l| \to \infty \) or \( w^{(k)}_l \to w^{(0)} \) for some \( w^{(0)} \in \partial U \). Analogously, in each case there is a subsequence \((w^{(k)}_{l_m}, z^{(k)}_{l_m})\) such that \( |z^{(k)}_{l_m}| \to \infty \) or \( z^{(k)}_{l_m} \to z^{(0)} \) for some \( z^{(0)} \in \partial U \). We will show that in any case \( \lim_{m \to \infty} f_U (w^{(k)}_{l_m}, z^{(k)}_{l_m}) > 0 \), which will finish the proof due to contradiction with the assumption \( \mathcal{R}_U = 0 \).
Assume first $w^{(k_i)} \to w^{(0)}$. If $z^{(k_i)} \to z^{(0)} \neq w^{(0)}$, then $\liminf_{m \to \infty} f_U(w^{(k_i)}, z^{(k_i)}) \geq \delta_U\left(\frac{w^{(0)} + z^{(0)}}{2}\right)/\delta_{H^{(0)}}\left(\frac{w^{(0)} + z^{(0)}}{2}\right) > 0$. In case $z^{(k_i)} \to z^{(0)} = w^{(0)}$ there are two options: for $|w - z| < \frac{1}{2}|y|$ the inequality ensures that $f_U(w^{(k_i)}, z^{(k_i)}) > c_p$, while for $|w - z| > \frac{1}{2}|w|$, which may happen only if $w^{(0)} = z^{(0)} = 0$, we have $\delta_U\left(\frac{w^{(k_i)} + z^{(k_i)}}{2}\right) \approx \delta_{H^{(0)}}\left(\frac{w^{(k_i)} + z^{(k_i)}}{2}\right) \approx \max\{|w|, |z|\}$. Eventually, for $|z^{(k_i)}| \to \infty$ let us denote $m = (w^{(0)}, \ldots, w^{n-1}, m_n)$ such that $m_n > w^{(0)}$ and $\delta_U(m) = 2$ and let $m^{(k_i)} \in U(w^{(k_i)}z^{(k_i)})$ be such that $m^{(k_i)} = m_n$. We clearly have $m^{(k_i)} \to m$ and therefore $\delta_U(m^{(k_i)}) \to 2$, which implies $\liminf_{m \to \infty} f_U(w^{(k_i)}, z^{(k_i)}) \geq \frac{2}{\delta_{H^{(0)}}(m)} > 0$.

Assume now $|w^{(k_i)}| \to \infty$. The bound implies that if $|w^{(k_i)} - z^{(k_i)}|, |\tilde{w}^{(k_i)} - \tilde{z}^{(k_i)}| < \frac{1}{4}|w^{(k_i)}|$ then $f_U(w^{(k_i)}, z^{(k_i)}) > c_p$. Furthermore, one can verify that for $|\tilde{w}^{(k_i)} - \tilde{z}^{(k_i)}| > \frac{1}{4}|w^{(k_i)}|$ we have $\delta_U\left(\frac{w^{(k_i)} + z^{(k_i)}}{2}\right) \approx \delta_{H^{(0)}}\left(\frac{w^{(k_i)} + z^{(k_i)}}{2}\right) \approx \max\{|w|, |z|\}$, so the remaining case is $|w^{(k_i)} - z^{(k_i)}| > \frac{1}{4}|w^{(k_i)}| = \frac{1}{4}|w^{(k_i)}|$. Choosing $m^{(k_i)} \in U(w^{(k_i)}z^{(k_i)})$ such that $|m^{(k_i)}| - w^{(k_i)}| = \frac{1}{8}w^{(k_i)}$ we have $\delta_U(m^{(k_i)}) \approx \delta_{H^{(0)}}(m^{(k_i)}) \approx w^{(k_i)} \to \infty$, which may be observed, for instance, considering $z = \left(\frac{5}{4}w_1, 0, \ldots, 0, (\frac{5}{4}w_1)^p\right)$ and $z = \left(\frac{3}{4}w_1, 0, \ldots, 0, (\frac{3}{4}w_1)^p\right)$. This ends the proof.

\begin{proposition}
For $n = 2$ the class $S_Q$ contains strictly convex bounded domains with analytical boundary. As a consequence, the heat kernels of such sets admit estimates from both: Theorem 5.2 and Theorem 5.4.
\end{proposition}

\begin{proof}
The idea of the proof is similar as in the case of previous proposition. Let $D$ be a strictly convex bounded domain with analytical boundary and suppose there is a sequence $(w^{(k)}, z^{(k)})_{k \geq 1}$, $w^{(k)}, z^{(k)} \in \partial D$, such that

$$\liminf_{k \to \infty} \frac{\delta_D\left(\frac{w^{(k)} + z^{(k)}}{2}\right)}{\delta_{H^{(k)}}\left(\frac{w^{(k)} + z^{(k)}}{2}\right)} = 0. \quad (40)$$

Compactness of $\partial D$ allows us assume that $\lim_{k \to \infty} w^{(k)} = w^{(0)}$ and $\lim_{k \to \infty} z^{(k)} = z^{(0)}$ for some $w^{(0)}, z^{(0)} \in \partial D$. Since $\left[10\right]$ is clearly not satisfied for $w^{(0)} \neq z^{(0)}$, we therefore assume also $w^{(0)} = z^{(0)}$. Next, we rotate and translate $D$ such that $w^{(0)} = (0, 0)$ and $D \subset \mathbb{R} \times (0, \infty)$. Since $\partial D$ is analytical, there exists a neighbourhood $G$ of $(0, 0)$ such that $\partial D \cap G$ is a graph of a function $y = f(x) = \sum_{i=1}^{\infty} a_i x_i$ where $i_0 \geq 2$, $a_{i_0} > 0$, $(x, y) \in G$. In particular, this implies $\lim_{x \to 0} \frac{f(x)}{a_{i_0} x_0} = 1$. Hence, for $|w^{(k)} - z^{(k)}| > \frac{1}{2}|w^{(k)}|$ we have $\delta_D\left(\frac{w^{(k)} + z^{(k)}}{2}\right) \approx \delta_{H^{(k)}}\left(\frac{w^{(k)} + z^{(k)}}{2}\right) \approx \max\{|w^{(k)}|, |z^{(k)}|\}$, $k \geq k_0$ for some $k_0 \in \mathbb{N}$. Furthermore, denoting $u = w^{(k)}$ and $v =
$$z_1^{(k)} - w_1^{(k)}$$, Taylor’s formula gives us for $$|w_1^{(k)} - z_1^{(k)}| \leq \frac{1}{2} |w_1^{(k)}|$$

$$\delta_{H_{w^{(k)}}} \left( \frac{w^{(k)} + z^{(k)}}{2} \right) = \frac{1}{2} \delta_{H_{w^{(k)}}} (z^{(k)})$$

$$= \cos (\arctan (f'(u + v))) \left( f(u + v) - (f(u) + vf'(u)) \right)$$

$$\approx \sum_{i=i_0}^{\infty} a_i ((u + v)^i - (u^i + iu^{i-1}v))$$

$$= \sum_{i=i_0}^{\infty} a_i \frac{i(i - 1)}{2} (u + \xi)^i v^2 \approx |u|^{i_0-2} v^2,$$ \hspace{1cm} (41)

for some $$|\xi_i| \leq \frac{1}{2} |u|$$. Similarly,

$$\delta_{D} \left( \frac{w^{(k)} + z^{(k)}}{2} \right) \approx \frac{1}{2} \left( f(w_1^{(k)}) + f(z_1^{(k)}) \right) - f \left( \frac{w_1^{(k)} + z_1^{(k)}}{2} \right)$$

$$= \sum_{i=i_0}^{\infty} a_i \left( \frac{1}{2} u^i + \frac{1}{2} (u + v)^i - \left( u + \frac{1}{2} v \right)^i \right)$$

$$\geq a_{i_0} \left( \frac{1}{2} u^{i_0} + \frac{1}{2} (u + v)^{i_0} - \left( u + \frac{1}{2} v \right)^{i_0} \right)$$

$$- \sum_{i=i_0+1}^{\infty} |a_i| \left( \frac{1}{2} u^i + \frac{1}{2} (u + v)^i - \left( u + \frac{1}{2} v \right)^i \right).$$

Taking $$|\tilde{x}| = 0$$ in (39), we get

$$a_{i_0} \left( \frac{1}{2} u^{i_0} + \frac{1}{2} (u + v)^{i_0} - \left( u + \frac{1}{2} v \right)^{i_0} \right) \geq |u|^{i_0-2} v^2.$$

Furthermore, Taylor’s formula used twice implies

$$\left| \sum_{i=i_0+1}^{\infty} |a_i| \left( \frac{1}{2} u^i + \frac{1}{2} (u + v)^i - \left( u + \frac{1}{2} v \right)^i \right) \right|$$

$$\leq \sum_{i=i_0+1}^{\infty} |a_i| i(i - 1) (|u| + |v|)^{i-2} v^2 \approx |u|^{i_0-1} v^2,$$

and we conclude $$\delta_{D} \left( \frac{w^{(k)} + z^{(k)}}{2} \right) \geq |w_1^{(k)}|^{i_0-2} v^2$$ for $$|w^{(k)}|$$ small enough. Thus, in view of (41), for such $$w^{(k)}$$ it holds $$\delta_{H_{w^{(k)}}} \left( \frac{w^{(k)} + z^{(k)}}{2} \right) / \delta_{D} \left( \frac{w^{(k)} + z^{(k)}}{2} \right) \geq c(i_0) > 0$$, which contradicts (40).

\[\square\]

Let us define the following set

$$S = (B_2((-1, 0), 1)) \cup ((-1, 1) \times (-1, 1)) \cup (B_2((1, 0), 1)) \subset \mathbb{R}^2,$$

which is a square $$(-1, 1) \times (-1, 1)$$ with two semicircles added to its left and right sides. It is known as a stadium. The next example shows that for some range of arguments the heat
kernel \( p_s(t, x, y) \) is comparable neither to the bound from Theorem 3.3 nor to the one from Theorem 4.3. Note that the space arguments realising the indicated behaviour of \( p_s(t, x, y) \) are located at opposite ends of the 'flat' part of the boundary, which strongly suggests that non-strict convexity is indeed the property that impacts on the incomparability of the bounds.

**Example 5.1.** Let \( x, y \in S \) such that \( x_1 < -1, \ y_1 > 1, \ x_2, y_2 = 1 - t^\gamma, \ \gamma > 0, \) and \( \delta_S(x), \delta_S(y) < t^{1+\gamma} \) with \( t < 1. \) For \( 0 < \gamma \leq \frac{1}{2} \) we have

\[
p_s(t, x, y) \approx p(t, x, y) \left[ \left(1 - \frac{\delta(x)\delta(y)}{t}\right) + \left(1 - \frac{\delta_H(x)\delta_H(y)}{t}\right) \right],
\]

and for \( \gamma \geq \frac{2}{3} \) it holds

\[
p_s(t, x, y) \approx p(t, x, y) \left[ \left(1 - \frac{\delta_S(x)\delta_S(y)}{t}\right) + \left(1 - \frac{\delta_S(x)\delta_S(\frac{x+y}{2})}{t}\right) \right].
\]

However, for \( \frac{1}{2} < \gamma < \frac{2}{3} \) we have

\[
p_s(t, x, y) \approx \frac{\delta_S(x)\delta_S(y)}{t^3(1-\gamma)} p(t, x, y),
\]

while

\[
\left(1 - \frac{\delta_S(x)\delta_S(y)}{t}\right) + \left(1 - \frac{\delta_S(x)\delta_S(\frac{x+y}{2})}{t}\right) \approx \frac{\delta_S(x)\delta_S(y)}{t^2-\gamma},
\]

and

\[
\left(1 - \frac{\delta(x)\delta(y)}{t}\right) + \left(1 - \frac{\delta_H(x)\delta_H(y)}{t}\right) \approx \frac{\delta_S(x)\delta_S(y)}{t^{2-\gamma}} = \frac{\delta_S(x)\delta_S(y)}{t^{3(1-\gamma)+2(\gamma-\frac{1}{2})}}.
\]

**Proof.** First, let us observe

\[
\delta_S\left(\frac{x+y}{2}\right) = t^\gamma, \quad \text{and} \quad \delta_H(y) \approx \delta_H(x) \approx t^{\gamma/2},
\]

which immediately follows the last two approximations in the assertion.

Assume \( \gamma \leq \frac{2}{3} \) and put \( \alpha = \frac{|x-(1-x)|}{|x+y|}. \) We have \( \alpha \approx t^{\gamma/2} \) and therefore \( \sqrt{\alpha t} < ct^{(2+\gamma)/4} \leq ct^\gamma, \) \( t < t_0, \) for some \( c, t_0 > 0. \) Chapman-Kolmogorov identity, Theorem 4.2 and Proposition 2.1 give us for some \( \varepsilon > 0 \)

\[
p_s(t, x, y) \geq \int_{B_2((-1,x_2),\frac{1}{2}t^{\gamma})} \int_{B_2((1,x_2),\frac{1}{2}t^{\gamma})} p_s(\alpha t, x, w)p_s((1-2\alpha)t, w, z)p_s(\alpha t, z, y)dzdw
\]

\[
\approx \left(1 - \frac{\delta_S(x)\delta_S(y)}{t}\right) \left(1 - \frac{t^{2\gamma}}{\alpha t}\right) \left(1 - \frac{\delta_S(x)\delta_S(y)}{\alpha t}\right)
\]

\[
\int_{B_2((-1,x_2),\frac{1}{2}\sqrt{\alpha t})} \int_{B_2((1,x_2),\frac{1}{2}\sqrt{\alpha t})} p(\alpha t, x, w)p((1-2\alpha)t, w, z)p(\alpha t, z, y)dzdw
\]

\[
\approx \varepsilon \left(1 - \frac{\delta_S(x)\delta_H(y)}{t}\right) \left(1 - t^{2\gamma-1}\right) \left(1 - \frac{\delta_S(x)\delta_H(x)}{t}\right) p(t, x, y)
\]

\[
\approx \frac{\delta_S(x)\delta_S(y)}{t^{2-\gamma}} \left(1 - t^{2\gamma-1}\right) p(t, x, y).
\]
This implies the lower bound in (42) and, together with Theorem 3.4 and (43), the first estimate in the example. Furthermore, the lower bound in the latter estimate follows directly from (26). In order to obtain remaining upper bounds for $\gamma \geq \frac{1}{2}$, we denote $H = \{ x \in \mathbb{R}^2 : x_2 < 1 \}$ and estimate $\delta_{H_x}(z) \delta_{H_y}(z), \lesssim \delta_H(z) + t^{\gamma/2}$ for $z \in S$. Consequently, by Theorem 3.4,

$$p_S(t/2, x, z) \lesssim \left( 1 \wedge \frac{\delta_S(x) \delta_S(z)}{t} \right) \left( 1 \wedge \frac{\delta_H(z) + t^{\gamma/2}}{t} \right) p(t/2, x, z)$$

$$\lesssim \frac{\delta_S(x) \delta_H(z)}{t} \left( 1 + t^{\gamma - 1} \left( \delta_H(z) + t^{\gamma/2} \right) \right) p(t/2, x, z).$$

Similarly, $p_S(t/2, z, y) \lesssim \frac{\delta_S(x) \delta_S(y)}{t^2} \left( 1 + t^{\gamma - 1} \left( \delta_H(z) + t^{\gamma/2} \right) \right) p(t/2, z, y)$, hence

$$I_1 \lesssim \frac{\delta_S(x) \delta_S(y)}{t^2} \int_H (\delta_H(z))^2 \left( 1 + t^{2\gamma - 2} \left( \delta_H(z) + t^{\gamma} \right) \right) p(t/2, x, z)p(t/2, z, y)dz$$

$$\lesssim \frac{\delta_S(x) \delta_S(y)}{t} \left( 1 + t^{2\gamma - 1} + t^{3\gamma - 2} \right) p(t, x, y),$$

where the last inequality follows from Proposition 2.2 with $H_1 = H_2 = H$. For $\gamma \geq \frac{2}{3}$, we get $p_S(t, x, t) \lesssim \frac{\delta_S(x) \delta_S(y)}{t} p(t, x, y)$, which, combined with (26), implies the second estimate in the example. Finally, for $\frac{1}{2} < \gamma < \frac{2}{3}$, we have $(1 + t^{2\gamma - 1} + t^{3\gamma - 2}) \approx t^{3\gamma - 2}$, which follows the upper bound in (42). The proof is complete. 

\section*{Acknowledgements}

The author would like to thank Jacek Małecki for inspiration and some fruitful discussion. This work was supported by by the National Science Centre, Poland, grant no. 2015/18/E/ST1/00239

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