ABSTRACT

A representation theory of the quantized Poincaré (κ-Poincaré) algebra (QPA) is developed. We show that the representations of this algebra are closely connected with the representations of the non-deformed Poincaré algebra. A theory of tensor operators for QPA is considered in detail. Necessary and sufficient conditions are found in order for scalars to be invariants. Covariant components of the four-momenta and the Pauli-Lubanski vector are explicitly constructed. These results are used for the construction of some q-relativistic equations. The Wigner-Eckart theorem for QPA is proven.
1. INTRODUCTION

Recently a quantum deformation of nonsemisimple Lie algebras has been proposed, using the contraction procedure [1]. This led in particular to the quantum deformation of the universal enveloping algebra \( U(\mathcal{P}) \) of the four-dimensional Poincaré algebra [2], [3], [4], which we denoted by \( U_q(\mathcal{P}) \) or \( U_\kappa(\mathcal{P}) \) (\( q \) being replaced by the dimensionful \( \kappa \)). This result has been generalized to any dimension [5]. In this way one obtains real noncocommutative ∗-Hopf algebras [4,5].

This new invariance algebra is a non ad hoc way of introducing a minimal length or a minimal time scale into relativistic theories. The quantum deformation implies deformed Casimir operators (i.e. dispersion relations) and deformed wave equations (Klein-Gordon, Dirac equations, etc).

Comparison with experiment yields limits on a possible nonlocality [6,7,8,9]. On the other hand this nonlocality can serve as a cutoff in field theory.

Several steps have already been taken toward a representation theory of \( U_\kappa(\mathcal{P}) \) [3,4,7], [10,11]. In this paper we intend to make a systematic study of this important question. The tool which we propose to use takes full advantage of the Hopf algebra structure. This tool is the quantum adjoint action, which is a homomorphism of the universal enveloping algebra \( U_\kappa(\mathcal{P}) \).

The plan of the paper is the following : In Section 2 we recall the Hopf algebra structure of \( U_q(\mathcal{P}) \) (commutation relations, coproducts, counits, antipodes). In Section 3 we discuss the general relations between the representation of the quantized and non-quantized Poincaré algebra. In Section 4 the main properties of the adjoint action are given. Section 5 compares scalars and invariants in the quantized algebra. Section 6 gives the explicit form of the irreducible tensor operators of \( U_q(\mathcal{P}) \). In Section 7 the quantum deformed relativistic equations are constructed. The proof of the Wigner-Eckart theorem is given in section 8, followed by the conclusion.

2. THE QUANTIZED POINCARE ALGEBRA (QPA)

The quantum deformation of the universal enveloping algebra \( U(\mathcal{P}) \) of the Poincaré algebra \( \mathcal{P} \) is an unital associative algebra \( U_q(\mathcal{P}) \) with the following generators : the angular moments \( J_i, i = 1, 2, 3 \), the Lorentz boosts \( N_i, i = 1, 2, 3 \), and the four-momenta \( P_i, P_0, i = 1, 2, 3 \), which satisfy the following relations [\( \mu, \nu = 0, 1, 2, 3 \)]:

\[
\begin{align*}
[J_j, J_k] &= i\varepsilon_{jkl}J_l, \\
[J_j, N_k] &= i\varepsilon_{jkl}N_l, \\
[J_j, P_k] &= i\varepsilon_{jkl}P_l,
\end{align*}
\]

(2.1a)

(2.1b)

(2.1c)

\( \varepsilon_{jkl} \) is the Levi-Civita symbol. Everywhere we use the following standard conventions of relativistic physics : repeated indices mean a summation; latin indices run over 1, 2, 3, and greek ones run over 0, 1, 2, 3.
\[ [J_j, P_0] = 0 \] \hspace{1cm} (2.1d) \\
\[ [N_j, N_k] = -i \varepsilon_{jkl} \{ \frac{1}{2} J_l (q^{P_0} + q^{-P_0}) - \frac{1}{4} (\ln q)^2 P_l (\bar{P} \bar{J}) \} \] \hspace{1cm} (2.1e) \\
\[ [N_j, P_k] = \frac{i}{2} \delta_{jkl} (\ln q)^{-1} (q^{P_0} - q^{-P_0}) \] \hspace{1cm} (2.1f) \\
\[ [N_j, P_0] = i P_j \] \hspace{1cm} (2.1g) \\
\[ [P_\mu, P_\nu] = 0 \] \hspace{1cm} (2.1h) \\
where the parameter \( q \) is given by 
\[ q := \exp \frac{1}{\kappa} \] \hspace{1cm} (2.1i) \\
and \( \varepsilon_{jkl} \) is the totally antisymmetric tensor \((\varepsilon_{123} = 1); \delta_{jkl} \) is the Kronecker symbol; \( \kappa \) is a parameter with the same dimension as \( P_0 \).

The standard Hopf structure of \( U_q(\mathcal{P}) \) is defined by the following formulas for a co-
product \( \Delta_q' \), an antipode \( S_q' \) and a counit \( \varepsilon := \varepsilon_q' : 
\[ \Delta_q'(J_j) = J_j \otimes 1 + 1 \otimes J_j \] \hspace{1cm} (2.2a) \\
\[ \Delta_q'(N_j) = N_j \otimes q^{P_0/2} + q^{-P_0/2} \otimes N_j + \frac{1}{2} (\ln q') \varepsilon_{jkl} (P_l \otimes J_k q^{P_0/2} + q^{-P_0/2} J_l \otimes P_k) \] \hspace{1cm} (2.2b) \\
\[ \Delta_q'(P_j) = P_j \otimes q^{P_0/2} + q^{-P_0/2} \otimes P_j \] \hspace{1cm} (2.2c) \\
\[ \Delta_q'(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \] \hspace{1cm} (2.2d) \\
\[ S_q'(J_j) = -J_j \] \hspace{1cm} (2.3a) \\
\[ S_q'(N_j) = -N_j + \frac{3i}{2} (\ln q') P_j \] \hspace{1cm} (2.3b) \\
\[ S_q'(P_\mu) = -P_\mu \] \hspace{1cm} (2.3c) \\
\[ \varepsilon(J_j) = \varepsilon(N_j) = \varepsilon(P_\mu) = 0 \hspace{1cm} \varepsilon(1) = 1 \] \hspace{1cm} (2.4) \\
where \( j = 1, 2, 3; \mu = 0, 1, 2, 3; q' = q \) or \( q' = q^{-1} \). Thus there are two standard Hopf structures 
for \( q' = q \) and \( q' = \bar{q} := q^{-1} \). It should be noticed that 
\[ \Delta_q' = \Delta_{q'} \] \hspace{1cm} (2.5) \\
where by \( \Delta' \) we denote the opposite coproduct, i.e. if \( \Delta(a) = \sum_i a_i \otimes b_i \) then \( \Delta'(a) = \sum_i b_i \otimes a_i \).

Remarks. (i) The defining relations (2.1a-h) are invariant with respect to the exchange 
\( q \rightarrow q^{-1} \).

(ii) If the parameter \( \kappa \) in (2.1i) lives on the real axis then this deformation is called of 
hyperbolical type and if \( \kappa \) lives on the imaginary axis then this deformation is called of 
spherical type. The first type was obtained in [4] and the second one in [2] by the Wigner-
Inönü contraction of some real forms of the quantum algebra \( U_q(B_2) (U_q(so(5,C)) \).
(iii) The Hopf algebra \((U_q(\mathcal{P}), \Delta_q, S_q, \varepsilon)\) is called also the quantized Poincaré algebra \(\mathcal{P}\) (QPA). This terminology will be used by us too.

3. RELATION BETWEEN REPRESENTATIONS OF THE QUANTIZED AND NON-QUANTIZED POINCARE ALGEBRAS

As is well known, the representation theory of the quantized simple Lie algebras (for generic \(q\)) is closely connected with the representation theory of the non-quantized case. Namely, a theorem states that any irreducible representation (IR) (from the category O) of any non-quantized simple Lie algebra can be lifted to an IR of the corresponding quantized algebra. And also any IR (from the category O) of any quantized simple Lie algebra (for generic \(q\)) is converted, when \(q \rightarrow 1\), to an IR of a corresponding non-quantized Lie algebra.

An analogous theorem exists also for the quantized Poincaré algebra. Moreover, for the usual Poincaré algebra there is a simple and very important class of IR’s which are simultaneously IR’s of \(U_q(\mathcal{P})\). We start to describe some properties of \(U_q(\mathcal{P})\) IR’s from this simple class.

PROPOSITION 3.1. Any representation of \(U_q(\mathcal{P})\), where the operators of the four-momenta act trivially (i.e. as zero operators) coincides with a representation of the classical Poincaré algebra \(\mathcal{P}\).

Proof. The proposition affirms that when the four-momenta operators are equal to zero, \(P_\mu = 0\), then the operators \(J_j, N_j, j = 1, 2, 3\), have the same commutation relations as the corresponding operators of the usual Lorentz algebra. Indeed, putting \(P_\mu = 0\) in the relations (2.1a-h) we see this to be the case.

COROLLARY. Any representation of the Lorentz algebra is trivially extended to a representation of \(\mathcal{P}\) and \(U_q(\mathcal{P})\) if we put \(P_\mu = 0, \mu = 0, 1, 2, 3\).

This simple proposition is very important from the practical point of view because it allows us to use for such types of representations of \(U_q(\mathcal{P})\) the usual Clebsch-Gordan coefficients of the Lorentz algebra.

Now we formulate a general theorem about a relation between Hermitian representations of the quantized and non-quantized Poincaré algebras. We recall that a Hermitian IR of the quantized Poincaré algebra is (like in the non quantized case) characterized by the the eigenvalues of the two Casimir operators \(C_1(q)\) and \(C_2(q)\) (see Section 6.4 and the paper [11]).

THEOREM 3.1. (i) A Hermitian IR with \(c_1 \geq 0\) of the Poincaré algebra \(\mathcal{P}\) can be lifted to a Hermitian IR of the quantum Poincaré algebra \(U_q(\mathcal{P})\).
(ii) A Hermitian IR with \(c_1(q) \geq 0\) of \(U_q(\mathcal{P})\) in the limit \(q \rightarrow 1\) becomes a Hermitian IR
of the usual Poincaré algebra \( \mathcal{P} \).

Sketch of proof. The validity of this theorem follows immediately from the results of the paper [11] where explicit formulas of a connection between the generators of the quantized and non-quantized Poincaré algebras was found.

4. ADJOINT ACTION AND TENSOR OPERATOR FOR QPA

Now we would like to recall the general definition of the adjoint action for an arbitrary Hopf algebra \( A = (A, \Delta, S, \varepsilon) \).

Let \( X \) be an \( A \)-bimodule, i.e. \( X \) is a linear space of two representations of \( A \) which act on the left-side and on the right-side of elements \( x \in X \). Then the adjoint action "ad" is a homomorphism \( A \to \text{End} X \) defined by the formula:

\[
ad_a x = ((\text{id} \otimes S)\Delta(a)) \circ x
\]  

(4.1)

for any \( x \in X \) and any \( a \in A \), where the symbol "\( \circ \)" is defined by the rule

\[
(a \otimes b) \circ x = axb.
\]  

(4.2)

If \( \Delta(a) = \sum_i a_i \otimes b_i \) then an explicit form of the adjoint action is

\[
ad_a x = \sum_i a_i xS(b_i).
\]  

(4.3)

If the linear space \( X \) is irreducible with respect to the adjoint action then the representation is called irreducible too.

Remark. As the Hopf algebra \( A \) is an \( A \)-bimodule, by definition, then we have the canonical actions \( \text{ad}_A \) on \( A \).

Let us show that the notation of the adjoint action is natural in theoretical physics where the representation theory is used. Indeed, let \( R \) be a representation space of any Lie algebra \( \mathcal{L} \) or its universal enveloping algebra \( U(\mathcal{L}) \). Since the representation of \( \mathcal{L} \) in \( X \) is a homomorphism \( \mathcal{L} \to \text{End} R \) we have some left-side and right-side actions of \( \mathcal{L} \) in \( \text{End} R \), i.e. the representation \( \mathcal{L} \to \text{End} R \) induces simultaneously an \( \mathcal{L} \)-bimodule structure in the linear space \( \text{End} R \). For the case of a usual Lie algebra \( \mathcal{L} \) the adjoint action is reduced to the usual commutator, i.e.

\[
ad_a x = [a, x]
\]  

(4.4)

for any \( x \in \text{End} R \) and any \( a \in \mathcal{L} \).

If an \( A \)-bimodule \( X \), where \( A \) is an arbitrary Hopf algebra, is a space of linear operators then we shall call \( X \) a space of tensor operators. Therefore the words "tensor operators" mean that there is an adjoint action of the Hopf algebra \( A \) in the operator space \( X \). If
the representation of the adjoint action is irreducible then the tensor operator is called irreducible too.

Now we write down the explicit formulas for the adjoint action of the quantum Poincaré algebra $U_q(P)$:

\[
\text{ad}^{(q')}_{J_j} x = [J_j, x], \tag{4.5a}
\]
\[
\text{ad}^{(q')}_{P_j} x = [P_j, x] q^{-\frac{P_j}{2}} - [q^{-\frac{P_j}{2}}, x] P_j, \tag{4.5b}
\]
\[
\text{ad}^{(q')}_{N_j} x = [N_j, x] q^{-\frac{P_j}{2}} - [q^{-\frac{P_j}{2}}, x] N_j + \frac{3i}{2} (\ln q') [q^{-\frac{P_j}{2}}, x] P_j - \frac{1}{2} (\ln q') \varepsilon_{jkl} (\{P_l, x\} q^{-\frac{P_k}{2}} + q^{-\frac{P_k}{2}} \{J_l, x\} P_k + [q^{-\frac{P_k}{2}}, x] J_l P_k), \tag{4.5c}
\]

for $j = 1, 2, 3$. If the four-momenta $P_{\mu}, (\mu = 0, 1, 2, 3)$ commute with $x$, i.e. $[P_{\mu}, x] = 0$, then the formulas (4.5a-d) take the form

\[
\text{ad}^{(q')}_{P_{\mu}} x = 0, \tag{4.6a}
\]
\[
\text{ad}^{(q')}_{J_j} x = [J_j, x], \tag{4.6b}
\]
\[
\text{ad}^{(q')}_{N_j} x = [N_j, x] q^{-\frac{P_j}{2}} - \frac{1}{2} (\ln q') \varepsilon_{jkl} q^{-\frac{P_k}{2}} [J_l, x] P_k. \tag{4.6c}
\]

Now we want to extend the representation "ad" to a representation of $U_q(P)$ considered as a Hopf algebra. For the sake of convenience we shall use the following notations:

\[
\text{ad}^{(1)} := \text{ad}^{(q)}, \quad \text{ad}^{(2)} := \text{ad}^{(q)}. \tag{4.7}
\]

Let $X$ and $Y$ be two $U_q(P)$-bimodules then we put

\[
\text{ad}^{(s,l)}_{a \otimes b} (x \otimes y) := (\text{ad}^{(s)}_a \otimes \text{ad}^{(l)}_b)(x \otimes y) = \text{ad}^{(s)}_a x \otimes \text{ad}^{(l)}_b y \tag{4.8}
\]

for any $a, b \in U_q(P)$, $x \in X$, $y \in Y$, and $s, l = 1, 2$. It is not difficult to show that the mapping $a \mapsto \text{ad}^{(s,l)}_{\Delta_{q'}(a)}$ (for $\forall a \in U_q(P)$) is a homomorphism. Therefore putting

\[
\Delta_{q'}(\text{ad}_a) := \text{ad}^{(s,l)}_{\Delta_{q'}(a)} \tag{4.9}
\]

for any $a \in U_q(P)$ we define representations of the Hopf algebra $(U_q(P), \Delta_{q'}, S_{q'}, \varepsilon)$.

The explicit expressions of the adjoint actions (4.9) on any element $x \otimes y$ have the forms:

\[
\text{ad}^{(s,l)}_{\Delta_{q'}(a)} (x \otimes y) = \sum_i \text{ad}^{(s)}_{a_i} x \otimes \text{ad}^{(l)}_{b_i} y, \tag{4.10}
\]
\[
\text{ad}_{\Delta_q(a)}^{(s,l)}(x \otimes y) = \sum_i \text{ad}_{b_i}^{(s)} x \otimes \text{ad}_{a_i}^{(l)} y
\] (4.11)

for any \( a \in U_q(\mathcal{P}) \), where \( \Delta_q(a) = \sum_i a_i \otimes b_i \) and \( s, l = 1, 2 \).

**Remark.** The formulas (4.10) and (4.11) define representations for any Hopf algebra \((A, \Delta, S, \varepsilon)\) [12].

Now let the \( U_q(\mathcal{P}) \)-bimodule \( X \) be an associative algebra, then we can consider the actions of \( \text{ad}^{(s)} \), \( s = 1, 2 \), on the product of two elements \( x \) and \( y \) from \( X \). It is not difficult to verify that the following formulas are valid

\[
\text{ad}_{a}^{(1)} xy = \sum_i \text{ad}_{a_i}^{(1)} x \text{ ad}_{b_i}^{(1)} y,
\] (4.12)

\[
\text{ad}_{a}^{(2)} xy = \sum_i \text{ad}_{b_i}^{(2)} x \text{ ad}_{a_i}^{(2)} y,
\] (4.13)

if \( \Delta_q(a) = \sum_i a_i \otimes b_i \). These formulas say that the multiplication of any two elements \( x \) and \( y \) from \( X \) is transformed with respect to the adjoint actions \( \text{ad}^{(1)} \) and \( \text{ad}^{(2)} \) in the same way as the tensor product \( x \otimes y \) is transformed with respect to \( \text{ad}_{\Delta_q}^{(11)} \) and \( \text{ad}_{\Delta_q}^{(22)} \) correspondingly.

The formulas (4.10), (4.11) and also (4.12), (4.13) allow us to construct new tensors by coupling two and more tensors.

**Remark.** The formulas (4.12), (4.13) are valid for any Hopf algebra \((A, \Delta, S, \varepsilon)\) [12].

5. SCALARS AND INVARIANTS OF QPA

In the case of usual (non-quantized) Lie algebras, as a rule, the terms ”scalar” and ”invariant” are used as synonyms. In the quantum case, i.e. for noncocommutative Hopf algebras, we cannot use these terms as synonyms because they correspond to two different notions. Let us remember the exact definitions of these notions for our case \( U_q(\mathcal{P}) \).

(i) An element \( x \in X \) is called a scalar (or a tensor of zero rank) if it satisfies the equations

\[
\text{ad}_{g_s}^{(q')} x = 0
\] (5.1)

for any \( g_s \in g := \{J_l, N_k, P_\mu \mid l, k = 1, 2, 3; \mu = 0, 1, 2, 3 \} \).

(ii) An element \( x \in X \) is called an invariant if it satisfies the equation:

\[
[a, x] = 0
\] (5.2)

for any \( a \in U_q(\mathcal{P}) \).

From the relation (4.5 a-d) follows the simple theorem.
THEOREM 5.1. For some element \( x \in X \) and for any \( a \in U_q(\mathcal{P}) \) and for any \( g_s \in g \) we have

\[
[a, x] = 0 \iff \text{ad}^{(g')}_{g_s} x = 0 \ .
\] (5.3)

Remark. (i) The Theorem 5.1 is valid for any quantized contragredient Kac-Moody (super) algebra (in a broad sense [13]) [12].

(ii) For usual Lie algebras the property (5.3) is trivial (because, for example, \( \text{ad}_{g_s} x = [g_s, x] \)).

We shall call scalars and invariants of types (5.1) and (5.2) one-particle types. Now we want to consider some properties of scalars and invariants of two-particle types. The following theorem is valid.

THEOREM 5.2. (about two-particle scalars and invariants). Let \( X \) and \( Y \) be two \( U_q(\mathcal{P}) \)-bimodules (the spaces of tensor operators) and let \( \sum_m x_m \otimes y_m \) be some element of \( X \otimes Y \) then we have

\[
\text{ad}^{(2,1)}_{\Delta_q(g_s)}(\sum_m x_m \otimes y_m) = 0 \iff [\Delta_q(g_s), \sum_m x_m \otimes y_m] = 0
\] (5.4)

and also

\[
\text{ad}^{(1,2)}_{\Delta_q(g_s)}(\sum_m x_m \otimes y_m) = 0 \iff [\Delta_q'(g_s), \sum_m x_m \otimes y_m] = 0
\] (5.5)

simultaneously for all elements \( g_s \in g := \{J_l, N_k, P_{\mu} \mid l, k = 1, 2, 3; \mu = 0, 1, 2, 3\} \).

Proof. We prove the first part of the theorem (i.e. the property (5.4)) the second part (5.5) is proven in a similar way. First of all it is evident that

\[
[\Delta_q(J_j), \sum_m x_m \otimes y_m] \equiv \text{ad}_{\Delta_q(J_j)}(\sum_m x_m \otimes y_m),
\] (5.6)

\[
[\Delta_q(P_0), \sum_m x_m \otimes y_m] \equiv \text{ad}_{\Delta_q(P_0)}(\sum_m x_m \otimes y_m) .
\] (5.7)

Further it is easy to show that if \( \Delta_q(P_0) \) commutes with \( \sum_m x_m \otimes y_m \) then we have

\[
[\Delta_q(P_j), \sum_m x_m \otimes y_m] = (\text{ad}^{(2,1)}_{\Delta_q(P_j)}(\sum_m x_m \otimes y_m))(q^{-\frac{P_0}{2}} \otimes q^{\frac{P_0}{2}}) .
\] (5.8)

Analogously but by more cumbersome calculation we obtain that if the generators \( \Delta_q(J_j), (j = 1, 2, 3), \Delta_q(P_\mu), (\mu = 0, 1, 2, 3) \), commute with the tensor \( \sum_m x_m \otimes y_m \) then

\[
[\Delta_q(N_j), \sum_m x_m \otimes y_m] = (\text{ad}^{(2,1)}_{\Delta_q(N_j)}(\sum_m x_m \otimes y_m))(q^{-\frac{P_0}{2}} \otimes q^{\frac{P_0}{2}}) .
\] (5.9)

Thus if the left hand sides of the relations (5.6)-(5.9) are equal to zero simultaneously then the right hand sides of the relations are equal to zero too and vice versa. The second part (5.5) of the theorem is proven in a similar way.
Remark. An analog of such a theorem is also valid for any quantized Kac-Moody (super) algebra [12].

Theorem 5.2 about two-particle scalars and invariants and also Theorem 5.1 for the one-particle case are very important for the constructions of q-relativistic equations, i.e. the q-analogs of the relativistic equations which are invariant under the quantum Poincaré algebra $U_q(P)$. In Section 7 we shall demonstrate this in explicit form. Theorem 5.2 is also important for the construction of the Casimir operators for many-particle systems.

6. EXPLICIT FORMS OF SOME IMPORTANT IRREDUCIBLE TENSOR OPERATORS FOR QPA

In this section we find explicit forms of three covariant four-vectors for the quantum Poincaré algebra $U_q(P)$, i.e. we construct three four-dimensional irreducible representations with respect to the adjoint action of $U_q(P)$. The first representation is realized in the space $U_q(P_0, \vec{P})$ of smooth functions of the four-momenta $P_\mu$, $\mu = 0, 1, 2, 3$. The second representation is realized in the space $U_q(P_0, \vec{P}, W_0, \vec{W})$ which is an universal enveloping algebra generated by the four-momenta and the Pauli-Lubanski four-vector of ”classical” type. The third representation is realized in the space of the usual Dirac $\gamma$-matrices. The first two representations give us an answer to the question : what are the right (covariant) components of the four-momenta and the Pauli-Lubanski four-vector. At the end of the section we construct two Casimir operators of $U_q(P)$ by couplings of two covariant four-momenta and also two covariant Pauli-Lubanski four-vectors of ”quantum” type. We start with the four-momenta.

6.1. The covariant four-momenta

The following simple proposition is valid.

PROPOSITION 6.1. Let $U(P_0, \vec{P})$ be an universal enveloping algebra (algebra of smooth functions) of the four-momentum sector of $U_q(P)$. Then the linear space $U_q(P_0, \vec{P})$ is invariant with respect to the adjoint action of $U_q(P)$ and this representation is equivalent to a representation of the adjoint action of $P$ (or the Lorentz algebra $L$) on the space $U_{q=1}(P_0, \vec{P})$.

The proof follows immediately from the relation (4.6a-c) and (2.1a-h).

PROPOSITION 6.2. The new four-momenta $P_\mu(q')$, $\mu = 0, 1, 2, 3$, defined by

$$P_j(q') = P_j q'^{\frac{2\mu}{2}}, \quad P_0(q') = [P_0] + \frac{1}{2} (\ln q') \vec{P}^2$$

(6.1)

for $j = 1, 2, 3$, are transformed with respect to the adjoint action of $U_q(P)$ exactly as the usual (non-deformed) four-momenta are transformed with respect to the adjoint action of the non-deformed Poincaré algebra $P$.  

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In the relation (6.1) and everywhere we use the nearly standard notation:

\[ [P_0] := \frac{q^{P_0} - q^{-P_0}}{2(\ln q')} = \frac{q^{P_0} - q^{-P_0}}{2(\ln q')} \tag{6.2} \]

The components (6.1) are naturally called the covariant four-momenta.

**Proof.** Let \( J_{\pm} := J_1 \pm iJ_2, \ N_{\pm} := N_1 \pm iN_2 \) and

\[ K^{(s)}_{\pm} := J_{\pm} - (-1)^s iN_{\pm}, \quad K^{(s)}_0 := J_3 - (-1)^s iN_3 \tag{6.3} \]

for \( s = 1, 2 \). It is easy to verify that the operators \( \text{ad}^{(q')}_{K^{(1)}_{j'}} \) and \( \text{ad}^{(q')}_{K^{(2)}_{j'}} \) commute one with the other in the space \( U_q(P_0, \vec{P}) \), i.e.

\[ [\text{ad}^{(q')}_{K^{(1)}_{j'}}, \ \text{ad}^{(q')}_{K^{(2)}_{j'}}]U_q(P_0, \vec{P}) = 0 \tag{6.4} \]

for any \( j, j' = +, -, 0 \). The operators \( \text{ad}^{(q')}_{K^{(1)}_{j'}} \) and \( \text{ad}^{(q')}_{K^{(2)}_{j'}} \) generate the usual complex Lorentz algebra \( \text{so}(4, \mathbb{C}) = \text{so}(3, \mathbb{C}) \oplus \text{so}(3, \mathbb{C}) \). Let

\[ T_{\pm, \pm}(q') = \pm (P_1 \pm iP_2)q^P_0, \]
\[ T_{\pm, \mp}(q') = -P_3 q^{P_0} \pm \left([P_0] + \frac{1}{2}(\ln q')\vec{P}\vec{J}\right) \tag{6.5} \]

Then the proof of the Proposition 6.2 is reduced to checking of the following relations

\[ \text{ad}^{(q')}_{K^{(1)}_{j'}} T_{\pm, j}(q') = 0, \quad \text{ad}^{(q')}_{K^{(2)}_{j'}} T_{j, \pm}(q') = 0, \]
\[ \text{ad}^{(q')}_{K^{(1)}_0} T_{\pm, j}(q') = \pm \frac{1}{2} T_{\pm, j}(q'), \quad \text{ad}^{(q')}_{K^{(2)}_0} T_{j, \pm}(q') = \pm \frac{1}{2} T_{j, \pm}(q'), \]
\[ \text{ad}^{(q')}_{K^{(1)}_{\pm}} T_{\mp, j}(q') = T_{\pm, j}(q'), \quad \text{ad}^{(q')}_{K^{(2)}_{\pm}} T_{j, \mp}(q') = T_{j, \pm}(q'), \tag{6.6} \]

for \( j = +, - \). This is easy to verify by the formulas (4.6a-c).

### 6.2. The covariant Pauli-Lubanski four-vector

Some components of the q-Pauli-Lubanski four-vector were suggested in the papers [3,4]. However these components are not covariant because they have not the right transformation properties with respect to the adjoint action of \( U_q(P) \). The covariant components of the Pauli-Lubanski four-vector are described by the following proposition:

**PROPOSITION 6.3.** The four-vector \( W_{\mu}(q'), \ \mu = 0, 1, 2, 3 \), defined by

\[ W_{\mu}(q') = J_{\mu}P_0 + \varepsilon_{ijk}P_iN_k + \frac{1}{2}(\ln q')P_\mu(\vec{P}\vec{J}) \]
\[ W_0(q') = (\vec{P}, \vec{J})q' \frac{P_0}{2} \]  

for \( j = 1, 2, 3 \) is transformed with respect to the adjoint action of \( U_q(\mathcal{P}) \) exactly as the usual (non-deformed) Pauli-Lubanski four-vector is transformed with respect to the adjoint action of the non-deformed Poincaré algebra \( \mathcal{P} \).

It is naturally called the covariant Pauli-Lubanski four-vector. The Pauli-Lubanski four-vector introduced in [3,4] differs from (6.7) by the factor \( q' \frac{P_0}{2} \) and by the additional term \( \frac{1}{2} (\ln q') \vec{P}(\vec{P}, \vec{J}) \).

**Proof.** First of all we verify that the operators \( \text{ad}^{(q')}_{P_\mu}, \mu = 0, 1, 2, 3 \) act as the zero operator, i.e. \( \text{ad}^{(q')}_{P_\mu} = 0 \), for \( \mu, \nu = 0, 1, 2, 3 \). Further the proof is reduced to checking the relations (6.6) where the components \( T_{j,j'}(q'), (j, j' = +, -) \), have now the form

\[
T_{\pm, \pm}(q') := (W_1(q') \pm W_2(q')) ,
T_{\pm, \mp}(q') := -(W_3(q') \pm W_0(q')) .
\]

It is well-known that an algebra generated by components of the usual Pauli-Lubanski four-vector plays an important role for the classification of the irreducible representations of the Poincaré algebra \( \mathcal{P} \). The same is valid for \( U_q(\mathcal{P}) \). Therefore we write down the commutation relations for the components of the q-analog Pauli-Lubanski four-vector \( W_\mu(q'), \mu = 0, 1, 2, 3 \):

\[
[W_j(q'), W_l(q')] = i \varepsilon_{jlk} (P_0(q') W_k(q') - W_0(q') P_k(q')) ,
[W_0(q'), W_j(q')] = -i \varepsilon_{jlk} P_l(q') W_k(q')
\]

and also

\[
[P_\mu(q'), W_\nu(q')] = 0 ,
\]

\[
P_\mu(q') W_\mu(q') := g^{\mu\nu} P_\mu(q') W_\nu(q') = 0 ,
\]

where \( g_{\mu\nu} \) is the metric tensor:

\[
g_{\mu\nu} = g^{\mu\nu} := \text{diag} (1, -1, -1, -1) .
\]

6.3. **Realization of** \( U_q(\mathcal{P}) \) **by the usual Dirac γ-matrices**

It is well-known that the usual Lorentz algebra has a four-dimensional realization in terms of the Dirac γ-matrices \( \gamma_\mu, \mu = 0, 1, 2, 3 \), which have the following basic property

\[
\{ \gamma_\mu, \gamma_\nu \} := \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} .
\]

This representation can be considered simultaneously as a representation of \( \mathcal{P} \) and \( U_q(\mathcal{P}) \), for which the four-momenta generators are realized as the zero operator. For the sake of convenience we write down this realization in the explicit form

\[
\hat{J}_j = \frac{1}{4} i \varepsilon_{jlk} \gamma_l \gamma_k ,
\]
\[ N_j = \frac{1}{2} \gamma_0 \gamma_j , \quad (6.14b) \]
\[ \dot{P}_\mu = 0 , \quad (6.14c) \]

for \( j = 1, 2, 3 \) and \( \mu = 0, 1, 2, 3 \).

The \( \gamma \)-matrices \( \gamma_\mu, \mu = 0, 1, 2, 3 \), sandwiched between Dirac spinors, are transformed with respect to the adjoint actions of the operators (6.14a-c) exactly as the covariant components \( P_\mu(q') \) (or \( W_\mu(q') \)) are transformed with respect to \( \text{ad}_j \) and \( \text{ad}_{N_j} \), \( j = 1, 2, 3 \).

### 6.4. The Casimir operators

Using the covariant tensor components of the four-momenta and the Pauli-Lubanski four-vector we can easily construct the Casimir operators of \( U_q(\mathcal{P}) \) by scalar coupling in accordance with the relations (4.12), (4.13) and Theorem 5.1.

The first scalar invariant \( I_1(q') \) is constructed by the coupling of two four-momenta:

\[ I_1(q') = P_\mu(q')P^\mu(q') = \left( [P_0] + \frac{1}{2} (\ln q') \vec{P}^2 \right)^2 - \vec{P}^2 q R_0 \quad (6.15) \]

where \([P_0] = \frac{1}{2} (\ln q')^{-1} (q R_0 - q^- R_0)\). After simple manipulations we obtain that

\[ I_1(q') = C_1(q') \left( 1 + \frac{1}{4} (\ln q')^2 C_1(q') \right) \quad (6.16) \]

where

\[ C_1(q') = 4 \left[ \frac{P_0}{2} \right]^2 - \vec{P}^2 . \quad (6.17) \]

However we can easily verify that the operator \( C_1(q') \) is an invariant too [2]. Moreover this operator cannot be represented as a coupling of some tensor operators. This property is typical for all quantized simple Lie algebras (see, for example, \( U_q(\text{sl}(2, \mathbb{C})) \) [14-16]). The operator \( C_1(q') \) is naturally called the first Casimir operator of \( U_q(\mathcal{P}) \).

The second scalar invariant \( I_2(q') \) is obtained by coupling two Pauli-Lubanski four-vectors:

\[ I_2(q') = W_\mu(q') W^\mu(q') . \quad (6.18) \]

After a simple manipulation, the invariant takes the form

\[ I_2(q') = \left( \frac{q R_0 + q^- R_0}{2} - \frac{1}{4} (\ln q')^2 \vec{P}^2 \right) (\vec{P} \vec{J}) - \left( J_0 [P_0] + \vec{P} \times \vec{N} \right)^2 . \quad (6.19) \]

This expression is well-known [3,4] and is called the second Casimir operator of \( U_q(\mathcal{P}) \). Thus we see that the second Casimir operator of \( U_q(\mathcal{P}) \) is exactly given by the scalar coupling of two covariant Pauli-Lubanski four-vectors. This Casimir operator is usually denoted by \( C_2(q') \).
It should be noted that both scalar invariants \( I_1(q') \), \( I_2(q') \) and also the first Casimir operator \( C_1(q') \) do not change if we put \( q' = q \) or \( \bar{q} \). Therefore we can remove the index prime of the argument \( q' \).

7. CONSTRUCTION OF q-ANALOGS FOR SOME RELATIVISTIC EQUATIONS

For the construction of q-analogs of relativistic equations we need some minimal (spinless) differential realization of \( U_q(P) \) on the Minkovski space-time manifold. Such a realization was already considered in Refs [3,4] and has the following form :

\[
P_\mu = -i\partial_\mu , \quad J_j = -i\varepsilon_{jlk}x^l\partial_k , \quad N_j = -ix_0\partial_j + x_j[-i\partial_0]
\]  \hspace{1cm} (7.1)

where \( \partial_\mu := \partial/\partial x^\mu \).

We start our consideration with the Klein-Gordon equation. For a construction of a q-analog of this equation there exist two possibilities : either to use the first Casimir operator \( C_1(q) \) (6.17) or to use the invariant \( I_1(q) \) (6.16). Thus we have two q-analog variants of the Klein-Gordon equation, namely

\[
(4m^2_0^2 - \vec{P}^2)^2 \Psi = 4m^2_0^2 \Psi \tag{7.2}
\]

and

\[
P_\mu(q)P^\mu(q)\Psi = 4m^2_0^2 \left( 1 + (\ln q)^2 \right) \Psi . \tag{7.3}
\]

We want to stress that in the equation (7.2) (and also (7.3)) we use a connection of the Casimir operator eigenvalue with the mass parameter \( m \) given by the q-number (6.2). In the literature up to now the mass parameter \( m \) on the right hand side of (7.2) was used without brackets. There are at least two reasons for our choice :

(i) The mass \( m \) does not depend on \( q \) when three momentum components are equal to zero, \( p_1 = p_2 = p_3 = 0 \), because such a representation of \( U_q(P) \) coincides with a representation of the usual Poincaré algebra (see [17]).

(ii) There exists a similar connection between the highest weight and the Casimir operator eigenvalue for IR’s as is the case for quantized simple Lie algebras (see [14-16] for example, for \( U_q(sl(2, C)) \)).

Now we want to construct a q-analog of the Dirac operator. This q-analog has to be an invariant of a representation space for the tensor product representations \( \text{ad}_{\hat{g}_s} \) and \( \text{ad}_{g_s}^{(q)} \) where \( \hat{g}_s \) are the operators of the \( \gamma \)-matrix realization (6.14a-c) and \( g_s \) are the operators of the differential realization (7.1). A basis of this representation space consists of the elements \( \gamma_\mu \otimes P^\nu(q), \mu, \nu = 0, 1, 2, 3 \). Since matrix elements of the \( \gamma \)-matrices commute with the components of the four-momenta, we may remove here the symbol of the tensor product, i.e. we put

\[
\gamma_\mu P^\nu(q) := \gamma_\mu \otimes P^\nu(q) , \tag{7.4}
\]
for $\mu, \nu = 0, 1, 2, 3$. It should be noted that the $\gamma$-matrix basis does not depend on the parameter $q$. In accordance with Theorem 5.2 (about two-particle scalars and invariants) we have the q-analog of the Dirac operator

$$D(q) := \gamma_\mu P^\mu(q) = \gamma_0\left([P_0] + \frac{1}{2}(\ln q)\vec{P}^2\right) - \vec{\gamma}\vec{P}q^{\frac{\mu}{2}} . \tag{7.5}$$

It is not difficult to verify that the square of this operator is proportional to the invariant (6.15). More exactly we have

$$D^2(q) = I_1(q) . \tag{7.6}$$

We write down the explicit form of the global generators which commute with the q-Dirac operator (7.5):

$$\mathcal{J}_j = \hat{\mathcal{J}}_j + J_j , \tag{7.7a}$$

$$\mathcal{N}_j = \hat{\mathcal{N}}_j q^{\frac{\mu}{2}} + N_j + \frac{1}{2}(\ln q)\varepsilon_{jkl}\hat{J}_lP_k , \tag{7.7b}$$

$$\mathcal{P}_\mu = P_\mu \tag{7.7c}$$

where $\hat{J}_j, \hat{\mathcal{N}}_j, (j = 1, 2, 3)$ are the operators of the $\gamma$-matrix realization (6.14a-c) and $J_j, N_j, (j = 1, 2, 3), P_\mu, (\mu = 0, 1, 2, 3)$, are the operators of the differential realization (7.1). The expressions (7.7a-c) are obtained in the following simple way. We take the formulas for the coproduct (2.2a-d) and put the $\gamma$-matrix realization instead of the first components and the differential realization instead of the second components.

At last we write the q-analog of the Dirac equation

$$\left(\gamma_0([P_0] + \frac{1}{2}(\ln q)\vec{P}^2) - \vec{\gamma}\vec{P}q^{\frac{\mu}{2}}\right)\Psi = 2\left[\frac{m}{2}\right](1 + (\ln q)^2\left[\frac{m}{2}\right]^2)^{1/2}\Psi . \tag{7.8}$$

The dependence of the right hand side on the mass parameter $m$ is defined by the demand that the square of the expression coincides with the right hand side of the equation (7.3). The q-Dirac equation was given in ref [18,7] with $\frac{m}{2}$ instead of $[\frac{m}{2}]$, but the derivation given here is much simpler.

Thus we have shown that the right covariant components of the tensor operator give powerful tools for the construction of q-relativistic equations. In such a way we can construct the q-relativistic equations for spin $s = 1, 3/2$ and other equations. We believe a q-analog of the Maxwell equation has the form

$$P_{\mu}(q')P^{\mu}(q')A_\nu - P_{\nu}(q')P^{\mu}(q')A^{\mu} = j_\nu . \tag{7.9}$$

We shall discuss this problem in detail in another paper.

8. THE WIGNER-ECKART THEOREM FOR QPA

As is well known the Wigner-Eckart theorem plays a very important role for calculations of quantum mechanical one- and many-body systems, where group representation
theory is used. We believe that this theorem is important also for an arbitrary Hopf algebra in particularly for $U_q(\mathbb{P})$.

Let us come back again to an arbitrary Hopf algebra $A = (A, \Delta, S, \varepsilon)$ (see Section 4). Let $V$ be a vector space of a representation of a Hopf algebra $A$ and also its $A$-bimodule $X$, i.e. we have two homomorphisms $A \to \text{End } V$ and $X \to \text{End } V$ simultaneously. Let us consider vectors of the form $ax|\psi\rangle$, where $a \in A$, $x \in X$, $|\psi\rangle \in V$.

**PROPOSITION 8.1.** If $\Delta(a) = \sum a_i \otimes b_i$ then the following relations are valid

$$ax|\psi\rangle = \sum_i (\text{ad}^{(1)} a_i x) b_i |\psi\rangle \tag{8.1}$$

and

$$ax|\psi\rangle = \sum_i (\text{ad}^{(2)} b_i x) a_i |\psi\rangle \tag{8.2}$$

**Proof.** Writing the adjoint actions $\text{ad}^{(1)} a_i$ and $\text{ad}^{(2)} b_i$ for the right parts of (8.1) and (8.2) in the explicit form and using standard properties of the coproduct $\Delta$ and the antipode $S$ we obtain immediately the left parts of the relations (8.1) and (8.2).

The formulas (8.1) and (8.2) affirm that the vector $x|\psi\rangle$ is transformed by the element $a \in A$ exactly the same way as the vector $x \otimes |\psi\rangle$ is transformed by the operators $(\text{ad}^{(1)} \otimes \text{id}) \Delta(a) = \sum_i \text{ad}^{(1)} a_i \otimes b_i$ and $(\text{ad}^{(2)} \otimes \text{id}) \Delta'(a) = \sum_i \text{ad}^{(2)} b_i \otimes a_i$, i.e. the vector space $X V$ is transformed with respect to the Hopf algebra $A$ exactly as the vector space $X \otimes V$ is transformed with respect to $\Delta(A)$ and $\Delta'(A)$, where the first components of the coproduct act by adjoint action on $X$.

This proposition is the basis for the proof of the Wigner-Eckart theorem for an arbitrary Hopf algebra $A$. Let us formulate and prove this theorem for $U_q(\mathbb{P})$. Let

$$P_0, P_1, P_2, P_3, (\vec{P} \vec{J}), C_1(q), C_2(q) \tag{8.3}$$

be a complete set of commuting operators. Let $\{ |c, \vec{p}, \lambda\rangle \}$ be an orthogonal basis of some Hermitian IR, where $c = (c_1, c_2)$ are parameters connected with eigenvalues of the Casimir operators $(C_1(q), C_2(q))$; $\vec{p} = (p_0, p_1, p_2, p_3)$ are eigenvalues of the operators $P_\mu$, $\mu = 0, 1, 2, 3$, and $\lambda$ is a eigenvalue of the operator $(\vec{P} \vec{J})$, i.e.

$$C_s(q)|c, \vec{p}, \lambda\rangle = c_s |c, \vec{p}, \lambda\rangle \; , \tag{8.4a}$$

$$P_\mu|c, \vec{p}, \lambda\rangle = p_\mu |c, \vec{p}, \lambda\rangle \; , \tag{8.4b}$$

$$(\vec{P} \vec{J})|c, \vec{p}, \lambda\rangle = \lambda p |c, \vec{p}, \lambda\rangle \; , \; p = \sqrt{\vec{p}^2} \; . \tag{8.4c}$$

6This theorem for the case of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ was considered extensively in Refs [15,16] and for an arbitrary quantized simple Lie algebra in [14].
Further let \( \{ x^{c'}_{\tilde{p}', \lambda'} \} \) be an IR basis in the tensor operator space \( X \). The basis element \( x^{c'}_{\tilde{p}', \lambda'} \) is called a component of the tensor operator.

**WIGNER-ECKART THEOREM.** The matrix element \( \langle c'', \tilde{p}'', \lambda'' | x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle \) is proportional to a Clebsch-Gordan coefficient with a constant which does not depend on intrinsic quantum number \( (p, \lambda; p', \lambda'; p'', \lambda'') \), i.e.

\[
\langle c'', \tilde{p}'', \lambda'' | x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle = (c, \tilde{p}, \lambda; c', \tilde{p}', \lambda' | c'', \tilde{p}'', \lambda'')_q (c'' || x^{c'} || c)_q \tag{8.5}
\]

where \( (\ldots ; \ldots | \ldots)_q \) is a Clebsch-Gordan coefficient of \( U_q(\mathcal{P}) \); \( (c'' || x^{c'} || c)_q \) is called the reduced matrix element which depends on the quantum numbers \( c, c' \) and \( c'' \) only.

**Remark.** For the sake of simplicity we consider here in (8.5) a multiplicity free case, i.e. when the vector \( x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle \) belongs to some Hilbert space, the direct sum of subspaces carrying IRs of \( U_q(\mathcal{P}) \) and each equivalence class of IRs occurs once and only once.

**Proof.** As follows from the Proposition 8.1 the vector \( x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle \) is transformed by an element \( a \in U_q(\mathcal{P}) \) exactly as the vector \( x^{c'}_{\tilde{p}', \lambda'} \otimes | c, \tilde{p}, \lambda \rangle \) is transformed by the representation \( \Delta_q(U_q(\mathcal{P})) \), where the first components of this coproduct act by adjoint action on \( x^{c'}_{\tilde{p}', \lambda'} \). Therefore the vector of the form

\[
\sum_{\lambda, \lambda'} \int (c, \tilde{p}, \lambda; c', \tilde{p}', \lambda' | c'', \tilde{p}'', \lambda'')_q x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle d\tilde{p}d\tilde{p}' \tag{8.6}
\]

is transformed by \( \Delta_q(U_q(\mathcal{P})) \) as the IR \( c'' \), i.e. the vector (8.6) is proportional to the vector \( | c'', \tilde{p}'', \lambda'' \rangle \) :

\[
\sum_{\lambda, \lambda'} \int (c, \tilde{p}, \lambda; c', \tilde{p}', \lambda' | c'', \tilde{p}'', \lambda'')_q x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle d\tilde{p}d\tilde{p}' = (c'' || x^{c'} || c)_q | c'', \tilde{p}'', \lambda'' \rangle \tag{8.7}
\]

where the coefficient \( (c'' || x^{c'} || c)_q \) does not depend on the quantum numbers \( \tilde{p}, \lambda, \tilde{p}', \lambda', \tilde{p}'', \lambda'' \). Converting back this relation we obtain

\[
x^{c'}_{\tilde{p}', \lambda'} | c, \tilde{p}, \lambda \rangle = (c'' || x^{c'} || c)_q \sum_{\lambda''} \int (c, \tilde{p}, \lambda; c', \tilde{p}', \lambda' | c'', \tilde{p}'', \lambda'' \rangle)_q | c'', \tilde{p}'', \lambda'' \rangle d\tilde{p}'' . \tag{8.8}
\]

Taking matrix elements with the bra-vector \( \langle c'', \tilde{p}'', \lambda'' | \) we obtain the formula (8.5).

9. CONCLUSION

In this paper we considered only a part of the problems of the representation theory for the quantized Poincaré algebra. In this conclusion we want to elucidate the program of our recent and near future research :

(i) Develop the theory of induced representations for \( U_q(\mathcal{P}) \).

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(ii) Construct the explicit bases for the most important (from the point of view of q-relativistic physics) irreducible representations.

(iii) Apply the tensor operator theory to the construction of q-relativistic kinematics.

(iv) Develop the representation theory for the quantized super Poincaré and the Poincaré algebra in higher dimensions.

(v) Try to construct the q-analog of quantum electrodynamics.

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