FIRST EIGENVALUE OF THE $p$-LAPLACIAN UNDER INTEGRAL CURVATURE CONDITION

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Abstract. We give various estimates of the first eigenvalue of the $p$-Laplace operator on closed Riemannian manifold with integral curvature conditions.

1. Introduction

On a compact Riemannian manifold $(M^n, g)$, for $1 < p < \infty$, the $p$-Laplacian is defined by

\[
\Delta_p(f) := \text{div}(|\nabla f|^{p-2}\nabla f).
\]

It is a second order quasilinear elliptic operator and when $p = 2$ it is the usual Laplacian. The $p$-Laplacian has applications in many different contexts from game theory to mechanics and image processing. Corresponding to the $p$-Laplacian, we have the eigenvalue equation

\[
\begin{cases}
\Delta_p(f) = -\lambda |f|^{p-2}f & \text{on } M \\
\nabla\nu f \equiv 0 \text{ (Neumann) or } f \equiv 0 \text{ (Dirichlet) on } \partial M
\end{cases}
\]

where $\nu$ is the outward normal on $\partial M$. The first nontrivial Neumann eigenvalue for $M$ is given by

\[
\mu_{1,p}(M) = \inf \left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2}f = 0 \right\},
\]

and the first Dirichlet eigenvalue of $M$ is given by

\[
\lambda_{1,p}(M) = \inf \left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W^{1,p}_c(M) \setminus \{0\} \right\}.
\]

Though the regularity theory of the $p$-Laplacian is very different from the usual Laplacian, many of the estimates for the first eigenvalue of the Laplacian (when $p = 2$) can be generalized to general $p$. Matei [11] generalized Cheng’s first Dirichlet eigenvalue comparison of balls [5] to the $p$-Laplacian. For closed manifolds with Ricci curvature bounded below by $(n - 1)K$, Matei for $K > 0$ [11], Valtora for $K = 0$ [17] and Naber-Valtora for general $K \in \mathbb{R}$ [12] give a sharp lower bound for the first nontrivial eigenvalue. Andrews-Clutterbuck [1],[2] also gave a proof using modulus of continuity argument. L.F. Wang [18] considered the case when the Bakry-Emery curvature has a positive lower bound for weighted $p$-Laplacians. Recently Y.-Z. Wang and H.-Q. Li [19] extended the estimates to smooth metric measure space and Cavalletti-Mondino [4] to general metric measure space. For a general reference on the $p$-Laplace equation, see [10]. See also [20] and references in the paper for related lower bound estimates.

In this paper, we extend the first eigenvalue estimates for $p$-Laplacian given in [11] to the integral Ricci curvature setting.
For each \( x \in M^n \) let \( \rho(x) \) denote the smallest eigenvalue for the Ricci tensor \( \text{Ric} : T_x M \to T_x M \), and \( \text{Ric}^K(x) = ((n-1)K - \rho(x))_+ = \max\{0, (n-1)K - \rho(x)\} \), the amount of Ricci curvature lying below \((n-1)K\). Let

\[
\|\text{Ric}^K\|_{q,R} = \sup_{x \in M} \left( \int_{B(x,R)} (\text{Ric}^K)^q \, d\text{vol} \right)^{\frac{1}{q}}.
\]

Then \( \|\text{Ric}^K\|_{q,R} \) measures the amount of Ricci curvature lying below a given bound, in this case, \((n-1)K\), in the \(L^q\) sense. Clearly \( \|\text{Ric}^K\|_{q,R} = 0 \) iff \( \text{Ric}_M \geq (n-1)K \). Denote the limit as \( R \to \infty \) by \( \|\text{Ric}^K\|_q \), which is a global curvature invariant. The Laplace and volume comparison, the basic tools for manifolds with pointwise Ricci curvature lower bound, have been extended to integral Ricci curvature bound \([14]\), see Theorem 2.1.

We denote \( \|f\|_{q,\Omega}^* \) the normalized \( q \)-norm on the domain \( \Omega \). Namely

\[
\|f\|_{q,\Omega}^* = \left( \frac{1}{\text{vol}(\Omega)} \int_{\Omega} |f|^q \right)^{\frac{1}{q}}.
\]

Under the assumption that the integral Ricci curvature is controlled (\( \|\text{Ric}^K\|_q^* \) is small), we give the following first eigenvalue estimates:

**Theorem 1.1** (Cheng-type estimate). Let \( (M^n, g) \) be a complete Riemannian manifold. For any \( x_0 \in M, \ K \in \mathbb{R}, \ r > 0, \ p > 1, \ q > \frac{n}{2} \), denote \( \bar{q} = \max\{q, \frac{n}{2}\} \), there exists an \( \varepsilon = \varepsilon(n, p, \bar{q}, K, r) \) such that if \( \partial B(x_0, r) \neq \emptyset \) and \( \|\text{Ric}^K\|_{q,B(x_0,r)}^* < \varepsilon \), then

\[
\lambda_{1,p}(B(x_0, r)) \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left( \|\text{Ric}^K\|_{q,B(x_0,r)}^* \right)^{\frac{1}{2}},
\]

where \( M^n_K \) is the complete simply connected space of constant curvature \( K \), \( B_K(r) \subset M^n_K \) is the ball of radius \( r \) and \( \bar{\lambda}_{1,p} \) is the first Dirichlet eigenvalue of the \( p \)-Laplacian in the model space \( M^n_K \).

This generalizes the Dirichlet \( p \)-Laplacian first eigenvalue comparison in \([11]\). When \( p = 2 \), this is proved in \([13]\).

**Theorem 1.2** (Lichnerowicz-type estimate). Let \( (M^n, g) \) be a complete Riemannian manifold. For \( q > \frac{n}{2}, \ p \geq 2 \) and \( K > 0 \), there exists \( \varepsilon = \varepsilon(n, p, q, K) \) such that if \( \|\text{Ric}^K\|_q \leq \varepsilon \), then

\[
\mu_{1,p}^2 \geq \frac{\sqrt{n(p-2) + n}}{(p-1)(\sqrt{n(p-2) + n} - 1)} \frac{(n-1)K - 2\|\text{Ric}^K\|_q}{n-1}.
\]

In particular, when \( \text{Ric} \geq (n-1)K \), we have

\[
\mu_{1,p}^2 \geq \frac{\sqrt{n(p-2) + n}}{\sqrt{n(p-2) + n} - 1} \frac{(n-1)K}{p-1} \geq \frac{(n-1)K}{p-1}.
\]

Under these assumption, Aubry’s diameter estimate implies that \( M \) is closed \([3]\). That paper also has the proof for \( p = 2 \).

The explicit estimate \((1.7)\) improves the estimate in \([11, \text{Theorem 3.2}]\), where it is shown that \( (\mu_{1,p})^2 \geq \frac{(n-1)K}{p-1} \). When \( p = 2 \), the estimate \((1.7)\) recovers the Lichnerowicz estimate that \( \mu_{1,2} \geq nK \). The explicit estimate \((1.6)\) is optimal when \( p = 2 \), but not optimal when \( p > 2 \). For optimal estimate we have the following Lichnerowicz-Obata-type estimate.
**Theorem 1.3** (Lichnerowicz-Obata-type estimate). Let $M^n$ be a complete Riemannian manifold. Then for any $\alpha > 1$, $K > 0$, $q > \frac{n}{2}$ and any $p > 1$, there is an $\varepsilon = \varepsilon(n, p, q, \alpha, K) > 0$ such that if $\|\text{Ric}^{K}_n\|_{1}^{\ast} \leq \varepsilon$, then

$$\alpha \mu_{1,p}(M) \geq \mu_{1,p}(M^n_K).$$

When $\|\text{Ric}^{K}_n\|_{1}^{\ast} = 0$, we can take $\alpha = 1$ and this gives Theorem 3.1 in [11].

This result is obtained from the following Faber-Krahn type estimate. Recall the classical Faber-Krahn inequality asserts that in $\mathbb{R}^n$ balls (uniquely) minimize the first eigenvalue of the Dirichlet-Laplacian among sets with given volume.

**Theorem 1.4** (Faber-Krahn-type estimate). Under the same set up as in Theorem 1.3, let $\Omega \subset M$ be a domain and $B_K \subset M^n_K$ be a geodesic ball in the model space such that $\text{vol}(\Omega) = \text{vol}(B_K) / \text{vol}(M^n_K)$.

Then

$$\alpha^p \lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_K).$$

Again when $\|\text{Ric}^{K}_n\|_{1}^{\ast} = 0$, we can take $\alpha = 1$ and this gives Theorem 2.1 in [11].

To prove these results, since we do not have pointwise Ricci curvature lower bound, one key is to control the error terms.

We now give a quick overview of the paper. In §2 we prove the Cheng-type upper bound using the first eigenfunction of $\Delta_p$ for the model case as a test function in the $L^p$-Rayleigh quotient and using the Laplacian comparison and volume doubling for integral curvature (Theorem 2.1) to control the error. In §3, we prove the Lichnerowicz-type lower bound by using the $p$-Bochner formula and the Sobolev inequality. In §4, to prove a Faber-Krahn-type lower bound, a necessary tool we need is an integral curvature version of the Gromov-Levy isoperimetric inequality, which we first show. The proof of the eigenvalue estimate then follows from an argument using the co-area formula.

2. **Proof of Theorem 1.1**

First we recall the Laplace and volume comparison for integral Ricci curvature proved by the second author joint with Petersen [14, 15].

Let $M^n$ be a complete Riemannian manifold of dimension $n$. Given $x_0 \in M$, let $r(x) = d(x_0, x)$ be the distance function and $\psi(x) = (\Delta r - \bar{\Delta}^K r)_+$, where $\bar{\Delta}^K$ is the Laplacian on the model space $M^n_K$. The classical Laplace comparison states that if $\text{Ric}_M \geq (n - 1)K$, then $\Delta r \leq \bar{\Delta}^K r$, i.e., if $\text{Ric}^K_n \equiv 0$, then $\psi \equiv 0$. In [14] this is generalized to integral Ricci lower bound.

**Theorem 2.1** (Laplace and Volume Comparison [14, 15]). Let $M^n$ be a complete Riemannian manifold of dimension $n$. If $q > \frac{n}{2}$, then

$$\|\psi\|_{2q,B(x,r)}^n \leq C(n, q) \left(\|\text{Ric}^K_n\|_{q,B(x,r)}^{\ast}\right)^{\frac{1}{2}}.$$  

There exists $\varepsilon = \varepsilon(n, q, K, r) > 0$ such that, if $\|\text{Ric}^K_n\|_{q,B(x,r)}^{\ast} \leq \varepsilon$, then

$$\frac{\text{vol}(B(x, r))}{\text{vol}(B(x, r_0))} \leq 2 \frac{\text{vol} B_K(r)}{\text{vol} B_K(r_0)}, \quad \forall r_0 \leq r.$$  

For $p$-Laplacian of radial function, we have the following comparison.
Proposition 2.1 \((p\text{-Laplace comparison})\). If \(f\) is a radial function such that \(f' \leq 0\), then

\[
\Delta_p f \geq \bar{\Delta}_p^K f + f' |f'|^{p-2} \psi.
\]  

\textit{Proof.} From the definition of the \(p\text{-Laplacian} \ (1.1),

\[
\Delta_p f = \text{div}(|\nabla f|^{p-2} \nabla f) = (\nabla |\nabla f|^{p-2}, \nabla f) + |\nabla f|^{p-2} \Delta f
\]

\[
= (p-2)|\nabla f|^{p-4} \text{Hess} f(\nabla f, \nabla f) + |\nabla f|^{p-2} \Delta f.
\]

Hence when \(f = f(r)\) is a radial function

\[
\Delta_p f = (p-2)|f'|^{p-2} f'' + |f'|^{p-2} (f'' + \Delta r f')
\]

\[
= (p-1)|f'|^{p-2} f'' + \Delta r f'|f'|^{p-2}
\]

\[
= (p-1)|f'|^{p-2} f'' + \bar{\Delta}_p^K f' |f'|^{p-2} + (\Delta r - \bar{\Delta}_p^K) f' |f'|^{p-2}
\]

\[
= \bar{\Delta}_p^K r + (\Delta r - \bar{\Delta}_p^K) f' |f'|^{p-2}.
\]

When \(f' \leq 0\), \((\Delta r - \bar{\Delta}_p^K) f' |f'|^{p-2} \geq \psi f' |f'|^{p-2}\), which gives the estimate. \(\square\)

Let \(\bar{f} > 0\) be the first eigenfunction for the Dirichlet problem for \(\Delta_p\) in \(B_K(r) \subset \mathbb{M}_K^n\). By \([7]\) \(\bar{f}\) is radial. Below we show that \(\bar{f}\) is a decreasing function of the radius. For \(p \geq 2\), this was shown in \([11]\). Our proof is much shorter.

Lemma 2.1. For \(t \in (0, r)\) and \(p > 1\), \(\bar{f}'(t) \leq 0\).

\textit{Proof.} Write the volume element of \(\mathbb{M}_K^n\) in geodesic polar coordinate \(d\text{vol} = A(t)dtd\theta_{S^{n-1}}\). As the first eigenfunction \(\bar{f}\) is radial, by (2.5) it satisfies the ODE

\[
(A |f'|^{p-2} f')' = -\lambda_1 A |f'|^{p-2} f.
\]

As \(A(0) = 0\) and \(p > 1\), integrating both sides from \(0\) to \(t\) we get

\[
A |f'|^{p-2} f'(t) = -\lambda_1 \int_0^t |f'|^{p-2} f A \leq 0.
\]

\(\square\)

Now we are ready to prove Theorem 1.1.

\textit{Proof.} Let \(\bar{f}\) be a first eigenfunction for the Dirichlet problem for \(\Delta_p\) in \(B_K(r) \subset \mathbb{M}_K^n\) with \(\bar{f}(0) = 1\). Hence \(0 \leq \bar{f} \leq 1\). Let \(r = r(x) = d(x_0, x)\) be the distance function on \(M\) centered at the point \(x_0\). Then \(\bar{f}(r) \in W_0^{1,p}(B(x_0, r))\). Denote \(Q = \frac{\int_B |\nabla \bar{f}|^p}{\int_B |\bar{f}|^p}\), where \(B := B(x_0, r)\). By (1.4) we have,

\[
\lambda_{1,p}(B(x_0, r)) \leq Q.
\]
Using integration by part, $\bar{f}' \leq 0$, $0 \leq \bar{f} \leq 1$ and the p-Laplacian comparison (2.3), we have
\[
Q = -\frac{\int_B \Delta_p \bar{f} \cdot \bar{f}}{\int_B |\bar{f}|^p} \\
\leq -\frac{\int_B \Delta \bar{f} \cdot \bar{f} + \psi \bar{f}'|\bar{f}'|^{p-2} \bar{f}}{\int_B |\bar{f}|^p} \\
= \bar{\lambda}_{1,p}(B_K(r)) - \frac{\int_B \psi \bar{f}'|\bar{f}'|^{p-2} \bar{f}}{\int_B |\bar{f}|^p} \\
\leq \bar{\lambda}_{1,p}(B_K(r)) + \frac{\int_B \psi |\bar{f}'|^{p-1}}{\int_B |\bar{f}|^p}.
\]

By Hölder inequality
\[
\int_B \psi|\bar{f}'|^{p-1} \leq \left( \int_B \psi^p \right)^{\frac{1}{p}} \left( \int_B |\bar{f}'|^p \right)^{\frac{1}{p}}.
\]

Let $r_0 = r_0(n, K, r) \in (0, r)$ such that $\bar{f}(r_0) = \frac{1}{2}$. Then $\bar{f} \geq \frac{1}{2}$ on $[0, r_0]$, and
\[
\int_B |\bar{f}'|^p \geq \left( \int_{B(x_0, r_0)} |\bar{f}'|^p \right)^{\frac{1}{p}} \geq \left( \int_B |\bar{f}|^p \right)^{1-\frac{1}{p}} \cdot (\text{vol } B(x_0, r_0))^{2-p} \cdot (\text{vol } B(x_0, r_0))^{\frac{1}{p}}.
\]

Hence the error term
\[
\frac{\int_B \psi|\bar{f}'|^{p-1}}{\int_B |\bar{f}|^p} \leq 2 \left( \frac{\int_B \psi^p}{\text{vol } B(x_0, r_0)} \right)^{\frac{1}{p}} \left( \frac{\int_B |\bar{f}'|^p}{\int_B |\bar{f}|^p} \right)^{1-\frac{1}{p}} \\
= 2 Q^{1-\frac{1}{p}} \left( \frac{\int_B \psi^p}{\text{vol } B(x_0, r_0)} \right)^{\frac{1}{p}} \left( \frac{\text{vol } B(x_0, r)}{\text{vol } B(x_0, r_0)} \right)^{\frac{1}{p}} \\
\leq 2 Q^{1-\frac{1}{p}} \|\psi\|_{2q, B(x_0, r)}^* \left( \frac{\text{vol } B(x_0, r)}{\text{vol } B(x_0, r_0)} \right)^{\frac{1}{p}}.
\]

Choose $\varepsilon \leq \varepsilon_0$ in Theorem 2.1, using (2.1) and (2.2), and combining above, we have
\[
Q \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left( \|\text{Ric}^{-}_\bar{q} \|_{q, B(x_0, r)}^* \right)^{\frac{1}{p}} Q^{1-\frac{1}{p}}.
\]

Applying Young’s inequality to the last term, we have
\[
Q \leq \bar{\lambda}_{1,p}(B_K(r)) + \frac{1}{p} C(n, p, \bar{q}, K, r) \left( \|\text{Ric}^{-}_\bar{q} \|_{q, B(x_0, r)}^* \right)^{\frac{1}{p}} + \frac{p-1}{p} Q.
\]

Moving $Q$ to the left hand side, we obtain
\[
Q \leq p \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left( \|\text{Ric}^{-}_\bar{q} \|_{q, B(x_0, r)}^* \right)^{\frac{1}{p}}.
\]

Applying this to (2.8) so that the $Q^{1-\frac{1}{p}}$ can be bounded in terms of the fixed quantities, we obtain
\[
Q \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left( \|\text{Ric}^{-}_\bar{q} \|_{q, B(x_0, r)}^* \right)^{\frac{1}{p}}.
\]

□
3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following Bochner formula for $p$ power.

**Lemma 3.1 (p-Bochner).**

\[
\frac{1}{p} \Delta (|\nabla f|^p) = (p - 2)|\nabla f|^{p-2} |\nabla \nabla f| + \frac{1}{2} |\nabla f|^{p-2} \left\{ |\text{Hess } f|^2 + \langle \nabla f, \nabla \Delta f \rangle + \text{Ric}(\nabla f, \nabla f) \right\}.
\]

One can find this implicitly in the literature, see e.g. [6, 12, 16]. The proof is very simple, for completeness, we present it here.

**Proof.** One computes

\[
\frac{1}{p} \Delta (|\nabla f|^p) = \frac{1}{p} \Delta (|\nabla f|^2)^{p/2} = (p - 2)|\nabla f|^{p-2} |\nabla \nabla f| + \frac{1}{2} |\nabla f|^{p-2} \Delta (|\nabla f|^2).
\]

Recall the Bochner formula

\[
\frac{1}{2} \Delta (|\nabla f|^2) = |\text{Hess } f|^2 + \langle \nabla f, \nabla \Delta f \rangle + \text{Ric}(\nabla f, \nabla f),
\]

Plugging this into (3.2) gives (3.1). \qed

We also need the following Sobolev inequality, which follows from Gallot’s isoperimetric constant estimate for integral curvature [8] and Aubry’s diameter estimate [3].

**Proposition 3.1.** Given $q > \frac{n}{2}$ and $K > 0$, there exists $\varepsilon = \varepsilon(n, q, K)$ such that if $M^n$ is a complete manifold with $\|\text{Ric}^K\|^s_q \leq \varepsilon$, then there is a constant $C_s(n, q, K)$ such that

\[
\left( \int_M f^{2q} \right)^{2q-2} \leq C_s(n, q, K) \int_M |\nabla f|^2 + 2 \int_M f^2
\]

for all functions $f \in W^{1,2}$. \[ \]

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** When $p = 2$ the result is proved in [3]. In the rest we assume $p > 2$.

By Aubry’s diameter estimate [3], $M$ is closed. Integrating (3.1) on $M$ we have

\[
0 = \int_M |\nabla f|^{p-2} \left\{ (p - 2)|\nabla \nabla f| + |\text{Hess } f|^2 + \langle \nabla f, \nabla \Delta f \rangle + \text{Ric}(\nabla f, \nabla f) \right\}.
\]

For the Hessian term, using the Cauchy-Schwarz inequalities

\[
|\text{Hess}(\nabla f, \nabla f)| \leq |\nabla f|^4 |\text{Hess } f|^2;
\]

\[
|\Delta f|^2 \leq n |\text{Hess } f|^2
\]
and the formula for $p$-Laplacian (2.4), we have

$$
\int_M |\nabla f|^{p-2} |\text{Hess } f|^2 \geq \int_M |\nabla f|^{p-2} \frac{(\Delta f)^2}{n}
$$

$$
= \frac{1}{n} \int_M \Delta f \Delta_p f - \frac{p-2}{n} \int_M \Delta f |\nabla f|^{p-4} \text{Hess } f(\nabla f, \nabla f)
$$

$$
\geq \frac{1}{n} \int_M \Delta f \Delta_p f - \frac{p-2}{n} \int_M |\Delta f||\nabla f|^{p-2} |\text{Hess } f|
$$

$$
\geq \frac{1}{n} \int_M \Delta f \Delta_p f - \frac{p-2}{\sqrt{n}} \int_M |\nabla f|^{p-2} |\text{Hess } f|^2.
$$

Hence

(3.5) $$
\int_M |\nabla f|^{p-2} |\text{Hess } f|^2 \geq \frac{1}{n + \sqrt{n(p-2)}} \int_M \Delta f \Delta_p f.
$$

For the third term,

$$
\int_M |\nabla f|^{p-2} (\nabla f, \nabla (\Delta f)) = -\int_M \Delta_p f \Delta f.
$$

For the curvature term,

$$
\int_M |\nabla f|^{p-2} \text{Ric}(\nabla f, \nabla f) \geq (n-1)K \int_M |\nabla f|^p - \int_M |\text{Ric}^K| |\nabla f|^p
$$

$$
\geq (n-1)K \int_M |\nabla f|^p - \|\text{Ric}^K\|_q^* \left(\int_M |\nabla f|^{\frac{p}{q}}\right)^{\frac{q-1}{q}}.
$$

Apply the Sobolev inequality (3.3) to the function $|\nabla f|^\frac{p}{q}$ gives

$$
\left(\int_M |\nabla f|^\frac{p}{2}\right)^{\frac{2}{p}} \leq C_s \int_M |\nabla |\nabla f|^{\frac{p}{2}}|^2 + 2 \int_M |\nabla f|^p
$$

$$
= C_s \frac{p^2}{4} \int_M |\nabla f|^{p-2} |\nabla |\nabla f||^2 + 2 \int_M |\nabla f|^p.
$$

Plug these to (3.4), we have

$$
0 \geq -\frac{n-1 + \sqrt{n(p-2)}}{n + \sqrt{n(p-2)}} \int_M \Delta_p f \Delta f + ((n-1)K - 2\|\text{Ric}^K\|_q^*) \int_M |\nabla f|^p
$$

$$
+ \left((p-2) - C_s\|\text{Ric}^K\|_q^* C_s \|\nabla f||^2\right) \int_M |\nabla f|^{p-2} |\nabla |\nabla f||^2.
$$

Choosing $\|\text{Ric}^K\|_q^*$ small so that $\left((p-2) - C_s\|\text{Ric}^K\|_q^* C_s \|\nabla f||^2\right) \geq 0$. Then we can throw the last term away and get

(3.6) $$
0 \geq -\frac{n-1 + \sqrt{n(p-2)}}{n + \sqrt{n(p-2)}} \int_M \Delta_p f \Delta f + ((n-1)K - 2\|\text{Ric}^K\|_q^*) \int_M |\nabla f|^p.
$$
Let $f$ be the first eigenfunction for $\Delta_p$, that is, $\Delta_p f = -\mu |f|^{p-2} f$. Then
\[
\int_M \Delta_p f \Delta f = -\mu \int_M |f|^{p-2} f \Delta f = \mu \int_M \langle \nabla(|f|^{p-2} f), \nabla f \rangle = \mu (p-1) \int_M |f|^{p-2} |\nabla f|^2 
\leq (p-1) \mu \left( \int_M |f|^p \right)^{1-\frac{2}{p}} \left( \int_M |\nabla f|^p \right)^{\frac{2}{p}} = (p-1) \mu \int_M |\nabla f|^p,
\]
where we use the fact that $f$ is the first eigenfunction, so we have
\[
\int_M |f|^p = \frac{1}{\mu} \int_M |\nabla f|^p.
\]
This gives
\[
(\mu)^{\frac{2}{p}} \left( p-1 \right) \frac{n-1 + \sqrt{n(p-2)}}{n + \sqrt{n(p-2)}} \geq (n-1)K - 2 \| \text{Ric}^K \|_q^*,
\]
which is (1.6).

\section{4. Proof of Theorem 1.4}

To prove the Faber-Krahn-type estimate, we will need a version of the Gromov-Levy isoperimetric inequality for integral curvature. The inequality follows from the following volume comparison for tubular neighborhood of hypersurface of Petersen-Sprouse.

\begin{proposition}[\cite{13}, Lemma 4.1 ]
Suppose that $H \subset M$ is a hypersurface with constant mean curvature $\eta \geq 0$, and that $H$ divides $M$ into two domains $\Omega_\pm$, where $\Omega_+$ is the domain in which mean curvature is positive. Furthermore, let $d_+ > 0$ such that $d_+ + d_- \leq \text{diam}(M) \leq D$ and $\Omega_\pm \subset B(H, d_\pm)$. Let $\tilde{H} = S(x_0, r_0) \subset \mathbb{M}_K^p$, a sphere of constant positive mean curvature $\eta$, and let $\Omega_+ = B(x_0, D) - B(x_0, r_0)$, $\Omega_- = B(x_0, r_0)$. Finally assume that $d_+ \leq D - r_0$ and $d_- \leq r_0$. Then for any $\alpha > 1$, there is an $\varepsilon(n, p, \alpha, K) > 0$ such that if $\| \text{Ric}_K \|_q^* \leq \varepsilon$, then
\[
\text{vol}(\Omega_\pm) \leq \alpha \frac{\text{area}(H)}{\text{vol}(\Omega_\pm)}.
\]

Using this, the Gromov-Levy isoperimetric inequality for the integral curvature case can be shown by following the original proof given in [9] page 522 and keeping track of the error term coming from the integral curvature.

\begin{proposition}
Let $\Omega \subset M$ be a domain. Then for any $\alpha > 1$, there is an $\varepsilon = \varepsilon(n, p, \alpha, K) > 0$ such that if $\| \text{Ric}_K \|_q^* \leq \varepsilon$, then
\[
\text{area}(\partial B_K(r_0)) \leq \alpha \frac{\text{area}(\partial \Omega)}{\text{vol}(B_K(r_0))} \frac{\text{vol}(B_K(r_0))}{\text{vol}(\Omega)}
\]
where $B_K(r_0) \subset \mathbb{M}_K^p$ is the ball of radius $r_0$ in constant curvature $K$ space. When $\| \text{Ric}_K \|_q^* = 0$, we can take $\alpha = 1$.

Now we prove Theorem 1.4, the Faber-Krahn inequality.
Proof. Without loss of generality, we can suppose that our test functions are Morse functions to ensure that the level sets of $f$ are closed regular hypersurfaces for almost all values. Let \( \Omega_t := \{ x \in \Omega \mid f > t \} \) and consider the decreasing rearrangement of $f$ defined by
\[
\bar{f}(s) = \inf \{ t \geq 0 \mid |\Omega_t| < s \}
\]
It is a non-increasing function on \([0, |\Omega|]\). We define the spherical rearrangement $\bar{\Omega}$ of $\Omega$ as the ball in $\mathbb{M}^n_K$ centered at some fixed point such that $\beta|\bar{\Omega}| = |\Omega|$, where $\beta := \frac{\text{vol}(M)}{\text{vol}(M^n_K)}$. By abuse of notation, we define the spherical decreasing rearrangement $\bar{f} : \bar{\Omega} \to \mathbb{R}$ to be
\[
\bar{f}(x) = \bar{f}(C_n|x|^n)
\]
for $x \in \bar{\Omega}$, where $|x|$ is the distance from the center of $\bar{\Omega}$ and $C_n$ is the volume $S^n_K$. Note that
\[
(4.1) \quad \text{vol}(\{ f > t \}) = \text{vol}(\{ \bar{f} > t \}).
\]
Now by construction, we have
\[
\hat{\Omega} f_p = \hat{\Omega} (\bar{f}(s))^p ds = \beta \hat{\bar{\Omega}}. \hat{\bar{f}}^p.
\]
Next we want to compare the $L^p$ norm of $\nabla f$ and $\nabla \bar{f}$. Now $\partial \Omega_t = \{ x \in \Omega \mid f = t \}$ and since $\bar{f}$ is a radial function, we have
\[
|\nabla \bar{f}| = \left| \frac{\partial \bar{f}}{\partial r} \right|
\]
which is a constant on $\partial \Omega_t$. By Hölder’s inequality, we have
\[
\text{vol}(\{ f = t \}) = \int_{\{ f = t \}} 1 = \int_{\{ f = t \}} |\nabla f|^{\frac{p-1}{p}} = \int_{\{ f = t \}} \frac{1}{|\nabla \bar{f}|} \left( \int_{\{ f = t \}} |\nabla f|^{p-1} \right)^{\frac{1}{p}}.
\]
By Proposition 4.2
\[
\alpha \text{vol}(\{ f = t \}) \geq \text{vol}(\{ \bar{f} = t \})
\]
for some $\alpha > 1$. We have
\[
\text{vol}(\{ \bar{f} = t \}) = \int_{\{ \bar{f} = t \}} 1 = \left( \int_{\{ \bar{f} = t \}} \frac{1}{|\nabla \bar{f}|} \right)^{\frac{p-1}{p}} \left( \int_{\{ \bar{f} = t \}} |\nabla \bar{f}|^{p-1} \right)^{\frac{1}{p}}.
\]
By the co-area formula, we have
\[
\frac{\partial}{\partial t} \text{vol}(\{ f > t \}) = \frac{\partial}{\partial t} \int_{\{ f > t \}} 1 = \int_{t}^{\infty} \left( \int_{\{ f = c \}} \frac{1}{|\nabla f|} \right) dc = -\int_{\{ f = t \}} \frac{1}{|\nabla f|}.
\]
and similarly for $\bar{f}$ with (4.1) so that
\[ \int_{\{f=t\}} \frac{1}{|\nabla f|} = \int_{\{\bar{f}=t\}} \frac{1}{|\nabla \bar{f}|}. \]
Combining and applying the co-area formula once more to integrate over $\Omega$, we obtain
\[ \alpha^p \int_\Omega |\nabla f|^p \geq \int_\Omega |\nabla \bar{f}|^p \]
and by the Rayleigh quotient, we have
\[ \frac{\int_\Omega |\nabla f|^p}{\int_\Omega |f|^p} \geq \frac{1}{\alpha^p} \frac{\int_\Omega |\nabla \bar{f}|^p}{\int_\Omega |\bar{f}|^p} \geq \frac{1}{\alpha^p} \lambda_{1,p}(\Omega). \]

To get Theorem 1.3 from Theorem 1.4, one follows the argument given in [11]. One first shows the relation between the first non-trivial Neumann eigenvalue and the first Dirichlet eigenvalue of its nodal domain. Namely, let $f$ be a first nontrivial Neumann eigenfunction of $\Delta_p$ on $M$ with $p > 1$, let $A_+ = f^{-1}(\mathbb{R}_+)$ and $A_- = f^{-1}(\mathbb{R}_-)$ be the nodal domains of $f$. Then
\[ \mu_{1,p}(M) = \lambda_{1,p}(A_+) = \lambda_{1,p}(A_-). \]
Using the fact that the nodal domains of $\Delta_p$ for the first nontrivial Neumann eigenfunction on spheres $M^n_K$ are hemispheres $S^n_{K,\pm}$, in particular we have
\[ \mu_{1,p}(M^n_K) = \lambda_{1,p}(S^n_{K,\pm}). \]
Applying the Faber-Krahn-type estimate (Theorem 1.4) to the nodal domain, we get Theorem 1.3.

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