Eigenvalues of the Neumann Laplacian in symmetric regions

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Abstract

In this work we are concerned with the multiplicity of the eigenvalues of the Neumann Laplacian in regions of $\mathbb{R}^n$ which are invariant under the natural action of a compact subgroup $G$ of $O(n)$. We give a partial positive answer (in the Neumann case) to a conjecture of V. Arnold on the transversality of the transformation given by the Dirichlet integral to the stratification in the space of quadratic forms according to the multiplicities of the eigenvalues. We show, for some classes of subgroups of $O(N)$ that, generically in the set of $G$–invariant, $C^2$–regions, the action is irreducible in each eigenspace $\text{Ker}(\Delta + \lambda)$. These classes include finite subgroups with irreducible representations of dimension not greater than 2 and, in the case $n = 2$, any compact subgroup of $O(2)$. We also obtain some partial results for general compact subgroups of $O(n)$.

Keywords: Laplacian, Neumann boundary condition, symmetric regions, perturbation of the boundary.

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1 Introduction

Perturbation of the boundary in boundary value problems have been considered by many authors, from various points of view since the classical works of J. Hadamard [5] and J.W.S. Rayleigh [20]. We also mention the more recent works [6, 11, 22, 24, 25]. In particular, generic properties for the solutions of boundary value problems have been proved in [14, 15, 23].

In [6], D. Henry developed a kind of differential calculus where the independent variable is the domain of definition of the differential equation. In this way, he was able to use standard analytic tools such as the Implicit Function Theorem and the Lyapunov-Schmidt method. In the same work, he proved a generalized version of the Transversality Theorem of Thom and Abraham and applied it to obtain generic properties for the solution of elliptic equations with various boundary conditions.

Generic properties for the eigenvalues and eigenfunctions of elliptic problems have also been investigated by many authors, among which we mention [4, 15, 17, 18, 19, 26, 27]. The generic situation for the eigenvalues of elliptic problems in symmetric regions has been specifically considered in [14, 15, 23].

One can find at least two approaches in the literature to deal with the problem of simplicity of the eigenvalues for elliptic problems: using the expression of the derivatives of the eigenvalues as functions of the domain or the Transversality Theorem. The first method is used, for instance, in [4, 14, 17, 18, 19, 26]. A combinations of the two methods is used in [14, 15, 19, 27].

If $G$ is a compact subgroup of $O(n)$, we say that a region $\Omega \subset \mathbb{R}^n$ is $G$-symmetric if it is invariant under the natural action of $G$. In [14], V. Arnold conjectured that the transversality of the transformation given by the Dirichlet integral to the stratification...
in the space of quadratic forms according to the multiplicities of the eigenvalues should be the generic situation for the eigenvalues of the Dirichlet Laplacian in symmetric regions. Equivalently, in the generic situation, the representation $\Gamma : G \to L^2(\Omega)$ given by $\Gamma_g u = u \circ g^{-1}$ should be irreducible in the set of regular bounded $G$-symmetric regions, when restricted to the eigenspaces of the Neumann Laplacian.

The first partial answer to Arnold’s conjecture was given in [26], for $\mathbb{Z}_3$ symmetric regions. In this particular case, there are only two possibilities for the eigenfunctions, they are either symmetric: $u \circ g^{-1} = u$, or “anti-symmetric”: $u + u \circ g^{-1} + u \circ (g^2)^{-1} = 0$, where $g \in O(n)$ is a generator of $\mathbb{Z}_3$. Theorem 1.1 of [26] states that, generically in the set of $\mathbb{Z}_3$ symmetric regions, the symmetric eigenvalues (that is, whose associated eigenfunctions are all symmetric) of the Dirichlet Laplacian are all double. However, the author does not take into account the possibility of the existence of eigenvalues with both symmetric and “anti-symmetric” eigenfunctions.

The complete answer to the question of the genericity of the eigenvalues of the Dirichlet Laplacian in planar $\mathbb{Z}_3$-symmetric regions was given in [17]. In the same work, the author also considered planar regions with $\mathbb{Z}_p$ symmetry for $p = 2, 3, 4$.

A detailed investigation of the generic situation of the eigenvalues of the Dirichlet Laplacian in symmetric regions is done in [18] or [17]. In particular, conditions for the existence of multiple eigenvalues on $G$-symmetric regions are established for arbitrary compact subgroups of $O(n)$. More precisely, it is shown there that, if $G < O(n)$ is compact and $\Omega$ has a free point under the action $G$, then there always exist multiple eigenvalues, except in the exceptional case $G = \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$, (see corollary [3]). The presence of a free point under the action $G$ guarantees the existence of irreducible sub-representations of $\Gamma$ for each possible class. As a consequence, it follows that for each irreducible representation of $\Gamma$ there exists an eigenvalue with multiplicity at least equal to the dimension of the sub-representation (see theorem [8]). Therefore, the best we can hope for is for the sub-representation $\Gamma_{\ker(\Delta+\lambda)}$ to be irreducible for any eigenvalue $\lambda$ in a generic set of bounded regular $G$-symmetric regions of $\mathbb{R}^n$.

Indeed, it is shown in [17] that this is true for some classes of finite groups, namely commutative groups and non commutative groups whose irreducible representations have at most dimension 2 (see theorem 7.1 of [17]). Though not explicitly stated in [17], the genericity property follows then for planar regions and arbitrary subgroups of $O(2)$ (see remark [3]).

In [17], [19] the theory developed by Henry in [6] is also used to prove some generic properties for the eigenvalues of the Dirichlet Laplacian and Bilaplacian on symmetric domains, using Henry’s Transversality theorem as the main tool.

Here, we obtain some partial answers to the Arnold’s conjecture for the Neumann Laplacian on symmetric regions. More precisely, we consider the problem

\begin{equation}
\begin{cases}
(\Delta + \lambda)u = 0, & \text{in } \Omega; \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Following the formulation of [17], we call an eigenvalue $G$-simple if the action $\Gamma|_{\ker(\Delta+\lambda)}$ is irreducible and investigate the validity of the following

**Conjecture 1.** Let $G$ be a compact subgroup of $O(n)$. Then, in a residual set of bounded, regular $G$-symmetric regions of $\mathbb{R}^n$ the eigenvalues of the Neumann Laplacian are all $G$-simple.

The representation $\Gamma$, which will be called here the quasi-regular representation of $G$ in $L^2(\Omega)$, induces an orthogonal decomposition in the space $L^2(\Omega)$ (see theorem [7]), that is

$$L^2(\Omega) = \bigoplus_{\sigma \in \hat{G}} M_\sigma,$$

where each subspace $M_\sigma$ is invariant by the Laplacian operator (see proposition [1]). These spaces will be called symmetry spaces.

The conjecture [1] can be split in two sub-conjectures:

(I) In a residual set of $G$-symmetric regions of $\mathbb{R}^n$, the representation $\Gamma$ of $G$ in $\text{Ker}(\Delta+\lambda) \cap M_\sigma$ is irreducible, for each $\sigma \in \hat{G}$.

(II) In a residual set of $G$-symmetric regions of $\mathbb{R}^n$, there are no eigenvalues with eigenfunctions belonging to two different symmetry spaces.

In fact we analyze here the validity of conjecture [1] only for finite groups. The case of infinite groups presents additional technical difficulties and will be considered in a forthcoming paper.

In what follows, we will say that an eigenvalue $\lambda$ of the Laplacian restricted to $M_\sigma$ is $G_\sigma$-simple if the quasi-regular representation $\Gamma$ of $G$ in $\text{Ker}(\Delta+\lambda) \cap M_\sigma$ is irreducible.

Theorem 1 of [26] proves then that, generically in the set of bounded $\mathbb{Z}_3$-symmetric regions of $\mathbb{R}^n$ the eigenvalues of the Dirichlet Laplacian are all $G_\sigma$-simple.

We show the validity of sub-conjecture I, for any finite subgroup of $O(n)$ (see corollary [3]) that is all eigenvalues of the Neumann Laplacian are $G_\sigma$-simple. The main result of this work is that [1] is true for finite subgroups with irreducible representations of dimension at most 2. As a corollary, we obtain a proof of the conjecture for arbitrary subgroups of $O(2)$ in planar regions.

## 2 Preliminaries

In this section we present some results on boundary perturbations that will be needed in the sequel. More details and proofs can be found in [6].
2.1 Definitions and preliminary results

We represent a point $x \in \mathbb{R}^n$ as a $n$-uple of real numbers $x = (x_1, ..., x_n)$ and use the multi-index notation for the partial derivatives.

$$\partial_x^\alpha = \left( \frac{\partial}{\partial x} \right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. Partial derivatives will also be denoted by

$$D_i = \frac{\partial}{\partial x_i} e \ D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is $m$-times differentiable at a point $x$, its $m$-th derivative may be considered as a $m$-linear symmetric form in $\mathbb{R}^n$

$$h \mapsto D^m f(x) h^m$$

with norm

$$|D^m f(x)| = \max_{|h| \leq 1} |D^m f(x) h^m|.$$  

We denote the boundary of an open subset $\Omega$ of $\mathbb{R}^n$ by $\partial \Omega$ and its closure by $\overline{\Omega}$. Given a normed vector space $E$ we denote by $C^m(\Omega, E)$ the space of $m$-times continuously and bounded differentiable functions $f : \Omega \to E$ whose derivatives extend continuously to $\overline{\Omega}$, with norm

$$\|f\|_{C^m(\Omega, E)} = \max_{0 \leq j \leq m} \sup_{x \in \Omega} |D^j f(x)|.$$  

If $E = \mathbb{R}$, we denote $C^m(\Omega, E)$ simply by $C^m(\Omega)$. We also define the subspaces

- $C^m_0(\Omega, E)$, the subspace of $m$-th continuously differentiable functions with compact support in $\Omega$.
- $C^m_{unif}(\Omega, E)$ is the closed subspace of functions in $C^m(\Omega, E)$ with $m$-th derivative uniformly continuous.
- $C^{m, \alpha}(\Omega, E)$ is the closed subspace of functions in $C^m(\Omega, E)$ with Hölder continuous $m$-th derivative and norm

$$\|f\|_{C^{m, \alpha}(\Omega, E)} = \max \left\{ \|f\|_{C^m(\Omega, E)}, H^\Omega_\alpha(D^m f) \right\}$$

where

$$H^\Omega_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1 + \alpha}} : x \neq y \in \Omega \right\}.$$  

We say that an open set $\Omega \subset \mathbb{R}^n$ is $C^m$-regular or has $C^m$-regular boundary if there exists $\phi \in C^m(\mathbb{R}^n, \mathbb{R})$, $m \geq 2$ or at least $C^1_{unif}$, such that

$$\Omega = \{ x ; \phi(x) > 0 \}$$

and $\phi(x) = 0$ implies $|\nabla \phi(x)| \geq 1$.

It is proved in [3] that, for bounded open sets, the above definition is equivalent to the ones in [2] and [3].

Besides these spaces of smooth functions, we will frequently work on Sobolev spaces, of which we present some basic definitions below.

Let $m$ be a non negative integer, $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ an open bounded set. If $u \in C^m(\Omega)$ we define the norm

$$\|u\| = \left( \int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u|^{p} dx \right)^{\frac{1}{p}}.$$  

The completion of $C^m(\Omega)$ with respect to this norm is denoted by $H^{m, p}(\Omega)$. We also consider $W^{m, p}(\Omega)$, the space of functions $m$-th weakly differentiable, whose weak derivatives up to order $m$ belong to $L^p(\Omega)$. It can be proved that $W^{p, m}(\Omega) = C^{p, m}(\Omega)$ when $\Omega$ is $C^{m}$-regular. If $p = 2$, we use the notation $H^{m, 2}(\Omega) = H^{m}(\Omega)$.

We also define $H_0^{m, p}(\Omega)$ as the completion of $C^0_0(\Omega)$ and $W^{m, p}_0(\Omega)$ the space of functions in $W^{m, p}(\Omega)$ satisfying $D^\alpha u = 0$ on $\partial \Omega$ for $|\alpha| \leq m$.

For functions $\phi$ defined in $\partial \Omega$, we can introduce the class of functions $W^{m - \frac{1}{p}, p}(\partial \Omega)$ in such a way that $\phi \in W^{m - \frac{1}{p}, p}(\partial \Omega)$ if and only if it is the boundary value of functions in $W^{m}(\Omega)$ with norm

$$\|\phi\| = \inf \|v\|_{W^{m, p}(\partial \Omega)}$$

where the infimum is taken over all $v \in W^{m}(\Omega)$ such that $v|_{\partial \Omega} = \phi$, where $v|_{\partial \Omega}$ is the trace of $v$ on $\partial \Omega$ (see [12]).

We also frequently encounter differential operators on hypersurfaces of $\mathbb{R}^n$.

Let $S$ be a $C^1$ hypersurface in $\mathbb{R}^n$ and $\phi : S \to \mathbb{R}$ a $C^1$ functions. The tangential gradient of $\phi$ is the tangent vector field in $S$ such that, for any (sufficiently smooth) curve $x(t)$ in $S$, we have

$$\frac{d}{dt}\phi(x(t)) = \nabla_S \phi(x(t)) \cdot \dot{x}(t).$$
If $S$ is of class $C^2$ and $\vec{a}$ is a $C^1$ vector field on $S$, we define its tangential divergent $\text{div}_S \vec{a} : S \to \mathbb{R}$ as the unique continuous function in $S$ such that, for any $\phi : S \to \mathbb{R}$ of $C^1$ with compact support in $S$

$$\int_S \phi \text{div}_S \vec{a} = -\int_S \vec{a} \cdot \nabla \phi.$$ 

If $u : S \to \mathbb{R}$ is of class $C^2$ then its tangential Laplacian is defined by $\Delta_S u = \text{div}_S \nabla_S u$.

**Theorem 1.**

1. If $S$ is a $C^1$ hypersurface in $\mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ is $C^1$ in a neighborhood of $S$, then $\nabla_S \phi(x)$ is the component $\nabla \phi$ tangent to $S$ at the point $x$, that is

$$\nabla_S \phi = \nabla \phi - N \frac{\partial \phi}{\partial N},$$

where $N$ is an unit normal field on $S$.

2. If $S$ is a $C^2$ hypersurface, $\vec{a} : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ in a neighborhood of $S$, $N : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ unit-vector field on a neighborhood of $S$, which is a normal field at points of $S$ near $x_0 \in S$, and $H = \text{div}N$ is the mean curvature of $S$ (near $x_0$), then

$$\text{div}_S \vec{a} = \text{div} \vec{a} - H(x) \vec{a} \cdot N - \frac{\partial}{\partial N}(\vec{a} \cdot N)$$

$S$ (near $x_0$).

3. If $S$ is $C^2$ hypersurface $u : \mathbb{R}^n \to \mathbb{R}$ is $C^2$ on a neighborhood of $S$, and $N$ is as in 2) above, then

$$\Delta_S u = \Delta u - \text{div}N \frac{\partial u}{\partial N} - \frac{\partial^2 u}{\partial N^2} + \text{div}_S u \cdot \frac{\partial N}{\partial u}$$

on $S$ near $x_0$. We may choose $N$ so that $\frac{\partial N}{\partial u} = 0$ and then the final term is omitted. $\Delta_S u$ depends only on the values of $u$ on $S$.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ be a $C^2$-regular domain $h(t, \ldots)$ a family of diffeomorphisms such that $\frac{\partial h}{\partial t}(t,x)$, $\frac{\partial^2 h}{\partial t^2}$, $\frac{\partial^2 h}{\partial x^2}$ are continuous and $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is $C^1$ then, for small $t \mapsto \int_{\partial \Omega(t)} f(t,x) dA_x$ is $C^1$ and

$$\frac{d}{dt} \int_{\partial \Omega(t)} f(t,x) dA_x = \int_{\partial \Omega(t)} \left( \frac{\partial f}{\partial t} + V \cdot N \frac{\partial f}{\partial N} + HV \cdot N f \right) dA_x,$$

where $N$ is the unit outward normal $\partial \Omega(t)$ and $H = \text{div}N$.

The following uniqueness result will be frequently needed.

**Theorem 3.** Uniqueness in the Cauchy Problem Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded $C^2$-regular region, and $B$ an open ball in $\mathbb{R}^n$ such that $B \cap \partial \Omega C^2$ hypersurface. Suppose that $u \in H^2(\Omega)$ satisfies

$$|\Delta u| \leq C(|\nabla u| + |u|) \text{ a.e in } \Omega$$

for some positive constant $C$ and

$$u = \frac{\partial u}{\partial N} = 0 \text{ on } B \cap \partial \Omega.$$

Then $u$ vanishes in $\Omega$.

### 2.2 Perturbation of domains

Given an open, bounded, $C^m$ region $\Omega_0 \subset \mathbb{R}^n$, consider the following open subset of $C^m(\Omega, \mathbb{R}^n)$

$$\text{Diff}^m(\Omega) = \{ h \in C^m(\Omega, \mathbb{R}^n) \mid h \text{ is injective and } \frac{1}{|\text{deth}(h(x))|} \text{ is bounded in } \Omega \}.$$

and the collection of regions $\{ h(\Omega_0) \mid h \in \text{Diff}^m(\Omega_0) \}$. We introduce a topology in this set by defining a (sub-basis of) the neighborhoods of a given $\Omega$ by

$$\{ h(\Omega_0) \mid h \in \text{Diff}^m(\Omega_0) \},$$

where $i_\Omega : \Omega \hookrightarrow \mathbb{R}^n$ is the inclusion. When $\| h - i_\Omega \|_{C^m(\Omega, \mathbb{R}^n)}$ is small, $h$ is a $C^m$ embedding of $\Omega$ in $\mathbb{R}^n$, a $C^m$ diffeomorphism to its image $h(\Omega)$. Michleletti [10] shows this topology is metrizable, and the set of regions $C^m$-diffeomorphic to $\Omega$ may be considered a complete and separable metric space which we denote by $\mathcal{M}_m(\Omega) = \mathcal{M}_m$. We say that a function $F$ defined in the space $\mathcal{M}_m$ with values in a Banach space is $C^m$ or analytic if $h \mapsto F(h(\Omega))$ is $C^m$ or analytic as a map of Banach spaces ($h$ near $i_\Omega$ in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.
Consider the formal linear differential operator

\[ Lu(x) = \left( u(x), \frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x), \frac{\partial^2 u}{\partial x_1^2}(x), \ldots, \frac{\partial^2 u}{\partial x_n \partial x_1}(x), \ldots \right), \quad x \in \mathbb{R}^n, \]

where \( Lu(x) \in \mathbb{R}^p \). Given a function \( f : O \subset \mathbb{R}^n \times \mathbb{R}^p \), where \( O \) is open, writing

\[ v(x) = f(x, Lu(x)), \]

one can define, for any open set \( \Omega \in \mathbb{R}^n \), the nonlinear differential operator \( F_\Omega \) by

\[ F_\Omega = f(x, Lu(x)), \quad x \in \Omega, \]

for sufficiently smooth functions defined in \( \Omega \), with \((x, Lu(x)) \in O\), for any \( x \in \Omega \). If \( f \) is continuous, \( \Omega \) is bounded and the differential operator \( L \) is of order less or equal than \( m \), the domain of \( F_\Omega \) is a non empty open subset of \( C^m(\Omega) \) with values in \( C^0(\Omega) \), that is

\[ F_\Omega : D_{F_\Omega} \subset C^m(\Omega) \to C^0(\Omega) \]

\[ u \mapsto f(x, Lu(x)). \]

Let \( h : \Omega \to \mathbb{R}^n \) be a \( C^m \) embedding. If \( u \) is defined in \( h(\Omega) \), we define the composition or "pull-back" map by

\[ h^* : C^m(h(\Omega)) \to C^m(\Omega) \]

\[ u \mapsto u \circ h \]

which is then an isomorphism with inverse \( h^{*-1} = (h^{-1})^* \). We use the same notation for the pull-back in other function spaces.

If \( h \) is such an embedding we can consider the differential operator acting on the perturbed region \( h(\Omega) \)

\[ F_{h(\Omega)} : D_{F_{h(\Omega)}} \subset C^m(h(\Omega)) \to C^0(h(\Omega)) \]

which is termed the Eulerian form of the formal nonlinear differential operator \( v \mapsto f(\cdot, Lv(\cdot)), \) \( x \) on \( h(\Omega) \), while

\[ h^* F_{h(\Omega)} h^{*-1} : h^* D_{F_{h(\Omega)}} \subset C^m(\Omega) \to C^0(\Omega) \]

is called its Lagrangean form.

We also treat boundary conditions in the same way. The Neumann problem requires \( N_{\Omega(t)}(y) \cdot \nabla u = 0 \) on \( \partial \Omega(t) \) in this case the particular extension of \( N_{\Omega(t)} \) away from the boundary is irrelevant. We choose some extension of \( N_{\Omega} \) in the reference region and then define \( N_{\Omega(t)} = N_{h(t, \Omega)} \) by

\[ \tag{2.1} h^* N_{h(\Omega)}(x) = N_{h(\Omega)}(h(x)) = T_h x^{-1} N_{\Omega}(x) \frac{1}{||T_h x^{-1} N_{\Omega}(x)||}, \]

for \( x \in \partial \Omega \), where \( T_h x^{-1} \) is the inverse-transpose of the Jacobian matrix \( h_x = [\frac{\partial h_i}{\partial x_j}]_{i,j=1}^n \) and \( ||.|| \) is the Euclidean norm.

The Eulerian form is more natural and, usually, more convenient for computations (see, for example, Corollary 1) while the Lagrangean form is more appropriate to prove results (see section 3).

The advantage of the Lagrangean form is to act in spaces which don’t depend on \( h \), which facilitates (for example) the use of the Implicit Function Theorem. However, we then need to know the smoothness of

\[ \tag{2.2} (u, h) \mapsto h^* F_{h(\Omega)} h^{*-1}, \]

and we need to be able to compute derivatives with respect to \( h \). It is shown by Henry in [6] that the map \([\Omega] \times C^{m}(\Omega) \to C^{0}(\Omega) \) is as regular as the function \( f \) (other function spaces can also be used, with similar results).

The next result is used throughout the paper.

**Lemma 1.** Let \( \Omega \) a \( C^2 \)-regular region, \( N_{\Omega(\cdot)} \) a \( C^1 \) unit-vector field defined on a neighborhood of \( \partial \Omega \) which is the outward normal on \( \partial \Omega \), and for \( C^2 \) embeddings \( h : \Omega \to \mathbb{R}^n \) define \( N_{\Omega(\cdot)} \) on a neighborhood of \( h(\partial \Omega) = \partial h(\Omega) \) by \( (2.1) \) above. Suppose \( h(t, \cdot) \) is an embedding for each \( t \), defined by

\[ \frac{\partial}{\partial t} h(t, x) = V(t, h(t, x)), \quad x \in \Omega, \quad h(0, x) = x, \]

\( (t, x) \to V(t, x) \) is \( C^2 \) and \( \Omega(t) = h(t, \Omega) \), \( N_{\Omega(t)} = N_{h(t, \Omega)} \). Then for \( x \) near \( \partial \Omega \), \( y = h(t, x) \) near \( \partial \Omega(t) \),

\[ \left( \frac{\partial}{\partial t} \right) N_{\Omega(t)}(y) = - (\nabla \cdot N_{\Omega(t)}) \sigma + \sigma \left( \frac{\partial N_{\Omega(t)}}{\partial N_{\Omega(t)}} \right), \]

\( \sigma = V \cdot N_{\Omega(t)} \) is the normal velocity and \( \nabla \cdot N_{\Omega(t)} = \nabla - N_{\Omega(t)} \frac{\partial \sigma}{\partial N_{\Omega(t)}} \) is the component of the gradient tangent to \( \partial \Omega(t) \).
3 Continuity and analiticity of curves of eigenvalues

In this section, we present some results on the continuity of the eigenvalues of the Neumann Laplacian with respect to $C^2$ perturbations of the domain and in the case of parametrized families of $C^2$ domains we prove the existence of analytic curves of eigenvalues and eigenfunctions. Although these results could probably be obtained adapting results in [10], we found it easier to follow the approach of Henry (see examples 4.1 and 4.4 of [6]) which relies on a careful use of the Lyapunov-Schimdt method.

We also obtain expressions for the first and second derivatives of the eigenvalues in this case.

3.1 Continuity

We consider here the slightly more general case of the Laplace problem with Robin boundary conditions in a regular bounded open region $\Omega \subset \mathbb{R}^n$.

\begin{equation}
\begin{cases}
(L + \lambda)u = 0, & \text{in } \Omega; \\
\left( \frac{\partial}{\partial N} + \beta(x) \right)u = 0, & \text{on } \partial \Omega;
\end{cases}
\end{equation}

where $L = \Delta + c(x)$ and $c$ and $\beta$ are of class $C^2$.

The associated Lagrangean form is then

\begin{equation}
\begin{cases}
h^*(L + \lambda)h^{*-1}u = 0, & \text{in } \Omega; \\
h^*\left( \frac{\partial}{\partial N} + \beta(x) \right)h^{*-1}u = 0, & \text{on } \partial \Omega;
\end{cases}
\end{equation}

where $h \in Diff^2(\Omega)$. The regularity of the perturbed problem with respect to $h$ depends on the regularity of the functions $c$ and $\beta$. More precisely, if $\Omega, h \in Diff^m(\Omega)$, $c \in C^{r+m-2}$ and $\beta \in C^{r+m-1}$, then for $u \in H^m(\Omega)$

\[(h,u) \mapsto h^*(\Delta + c)h^{*-1}u \in H^{m-2}(\Omega),\]

is of class $C^r$ and

\[(h,u) \mapsto h^*\left( \frac{\partial}{\partial N} + \beta \right)h^{*-1}u \in H^{m-\frac{3}{2}}(\partial \Omega),\]

is of class $C^r$ since $(h,u) \mapsto (c \circ h)u \in H^{m-r}(\Omega)$ is of class $C^r$ and $(h,u) \mapsto (\beta \circ h)u \in H^{m-r-1}(\Omega)$ is of class $C^r$. (in the purely Neumann case, we obtain that both maps are of class $C^1$ requiring $h$ of class $C^2$) (see [6], Example 3.2).

Theorem 4. Suppose $\lambda_0$ is the unique eigenvalue of $\left(\Delta + L\right)$ in the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. If $\lambda_0$ has multiplicity $m$ then there exists $\delta > 0$ such that, for all $h \in Diff^2(\Omega)$, $\|h - i\epsilon\|_{C^2} < \delta$, there exist exactly $m$ eigenvalues (counting multiplicity) of the problem $\left(3.2\right)$ in $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Proof. Let $\{\phi_j\}_{j=1}^m$ be an orthonormal basis for the eigenspace associated to $\lambda_0$ and $Pu = \sum_j \phi_j \int_\Omega \phi_j u$ the orthogonal projection into it. We write an arbitrary function $u \in L^2(\Omega)$ in a unique way as $u = \phi + \psi$, where $\phi \in \mathcal{R}(P) = \mathcal{N}(L + \lambda_0)$ and $\psi \in \mathcal{N}(P) = \mathcal{R}(L + \lambda_0)$. The perturbed problem $\left(3.2\right)$ is then equivalent to the equations

\begin{equation}
\begin{cases}
(P(h^*(L + \lambda_0)h^{*-1}(\phi + \psi))) = 0, & \text{in } \Omega; \\
(I - P)(h^*(L + \lambda_0)h^{*-1}(\phi + \psi)) = 0, & \text{in } \Omega; \\
h^*\left( \frac{\partial}{\partial N} + \beta(x) \right)h^{*-1}(\psi + \phi) = 0, & \text{on } \partial \Omega;
\end{cases}
\end{equation}

We first solve the second and third equations. The boundary term can be rewritten as

\[
\left( \frac{\partial}{\partial N} + \beta \right) \psi + \left( h^* \left( \frac{\partial}{\partial N} + \beta \right) h^{*-1} - \frac{\partial}{\partial N} + \beta \right) \psi = 0.
\]

Now, summing and subtracting the term $(L + \lambda)\psi$ in the second equation and observing that

\begin{align*}
PL\psi &= P(L + \lambda)\psi \\
&= \sum_{j=1}^m \phi_j \int_\Omega \phi_j (L + \lambda)\psi \\
&= \sum_{j=1}^m \phi_j \left( \int_\Omega \phi_j (L + \lambda)\psi - \psi(L + \lambda)\phi_j \right) \\
&= \sum_{j=1}^m \phi_j \left( \int_{\partial \Omega} \phi_j \frac{\partial \psi}{\partial N} - \psi \frac{\partial \phi_j}{\partial N} \right) \\
&= \sum_{j=1}^m \phi_j \left( \int_{\partial \Omega} \phi_j \left( \frac{\partial}{\partial N} + \beta \right) \psi - \psi \left( \frac{\partial}{\partial N} + \beta \right) \phi_j \right) \\
&= \sum_{j=1}^m \phi_j \int_{\partial \Omega} \phi_j \left( \frac{\partial}{\partial N} + \beta \right) \psi.
\end{align*}
and

\[(L + \lambda)\psi = (I - P)[(L + \lambda)\psi] + \sum_{j=1}^{m} \phi_j \int_{\Omega} \phi_j (L + \lambda)\psi,\]

we obtain

\[(L + \lambda)\psi + (I - P)(h^* L h^{-1} - L)(\psi + \phi) - \sum_{j=1}^{m} \phi_j \int_{\partial\Omega} \phi_j \left( \frac{\partial}{\partial N} + \beta \right) \psi = 0.\]

Therefore, the second and third equations are equivalent to \(F(h, \lambda, \phi, \psi) = 0\), where

\[F : Diff^2(\Omega) \times \mathbb{R} \times \mathcal{R}(P) \times H^2(\Omega) \cap \mathcal{N}(P) \rightarrow \mathcal{N}(P) \times H^{1/2}(\Omega)\]

and

\[
\begin{align*}
F_1 &= (L + \lambda)\psi + (I - P)(h^* L h^{-1} - L)(\psi + \phi) - \sum_{j=1}^{m} \phi_j \int_{\partial\Omega} \phi_j \left( \frac{\partial}{\partial N} + \beta \right) \psi, \\
F_2 &= (\frac{\partial}{\partial N} + \beta(x))\psi + (h^* (\frac{\partial}{\partial N} + \beta) h^{-1} - (\frac{\partial}{\partial N} + \beta)) (\psi + \phi).
\end{align*}
\]

Now, since the map

\[
\frac{\partial F}{\partial \lambda}(i\Omega, \lambda_0, 0, 0) \psi = ((L + \lambda_0)\psi - \sum_{j=1}^{m} \phi_j \int_{\partial\Omega} \phi_j \left( \frac{\partial}{\partial N} + \beta \right) \psi)
\]

is an isomorphism from \(H^2(\Omega) \cap \mathcal{N}(P)\) into \(\mathcal{N}(P) \times H^{1/2}(\Omega)\). It follows from the Implicit Function Theorem, that the equation \(F(h, \lambda, \phi, \psi) = (0, 0)\) can be solved for \(\psi\) as a function of \(\lambda, h\) and \(\phi\). More precisely, there exist neighborhoods \(\mathcal{V}\) in \(C^2(\mathbb{R}^n, \mathbb{R}^n)\) of \(i\Omega, (\lambda_0 - \epsilon, \lambda_0 + \epsilon)\) of \(\lambda_0\) and a \(C^1\) function \(\psi = \psi(h, \lambda, \phi)\) which gives the unique solution of \(F(h, \lambda, \phi, \psi) = 0\), with \(h \in \mathcal{V}\) and \(\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)\). Furthermore, \(S(h, \lambda, \phi)\) is analytic \(\lambda\) and linear in \(\phi\).

Now, to solve the first equation in \((3.3)\), observe that, since \(\phi \in \mathcal{R}(P)\), there exist real numbers \(c_1, c_2, \ldots, c_m\) not all equal to zero, such that \(\psi = \sum_{j=1}^{m} c_j \phi_j\) and, therefore, the equation \((3.3)\) is equivalent to the system in the variables \(c_1, \ldots, c_j\)

\[
\sum_{j=1}^{m} c_j \int_{\Omega} \phi_k h^*(L + \lambda)h^{-1} \phi_j + S(h, \lambda)\phi_j = 0
\]

for \(k = 1, 2, \ldots, m\). Thus, \(\lambda\) is an eigenvalue of \((3.2)\) if, and only if \(DetM(h, \lambda) = 0\), where

\[M_{k,j}(h, \lambda) = \int_{\Omega} \phi_k h^*(L + \lambda)h^{-1} \phi_j + S(h, \lambda)\phi_j.
\]

and, in this case, the associated eigenfunctions are given by

\[u = \sum_{j=1}^{m} c_j (\phi_j + S(h, \lambda)\phi_j),\]

where \(c = (c_1, \ldots, c_m)\) satisfies \(M(h, \lambda)c = 0\).

Finally, we observe that the equation \(DetM(h, \lambda) = 0\) has exactly \(m\) roots in a neighborhood \(\mathcal{V} \times B_\delta(\lambda_0)\) of \((i\Omega, \lambda_0)\), by Rouche’s theorem since \(h = i\Omega\), \(DetM(i\Omega, \lambda) = (\lambda - \lambda_0)^m\) if \(h = i\Omega\).

**3.2 Existence of analytic curves**

The next result ensures the existence of analytic curves of eigenvalues and eigenvectors for the problem \((3.2)\) near \(\lambda_0\) and its associated eigenfunctions.

**Theorem 5.** Suppose \(\lambda_0\) is an eigenvalue of multiplicity \(m\) for the problem \((3.7)\) with \(c \equiv 0\) and \(\beta \equiv 0\), and let \(h(t, \cdot)\) be an analytic curve of diffeomorphisms of class \(C^3\) such that \(h(0, x) = x\). Then, there exist \(m\) analytic curves \(\mu_1(t), \mu_2(t), \ldots, \mu_m(t)\) and \(m\) analytic curves \(\phi_1(t), \phi_2(t), \ldots, \phi_m(t)\), giving the eigenvalues and eigenfunctions of \((3.2)\) near \(\lambda_0\) and its associated eigenfunctions.

**Proof.** Let \(\{\phi_j\}_{j=1}^{m}\) be an orthonormal basis of eigenfunctions of \((3.1)\) associated to \(\lambda_0\). For each \(j = 1, \ldots, m\), consider the problem

\[
(3.4)
\]

Consider the map

\[F^j : Diff^2(\Omega) \times H^2(\Omega) \rightarrow [\phi_1, \phi_2, \ldots, \phi_m] \times \mathcal{R}(P) \times H^{1/2}(\partial\Omega) : \]

\[F^j(h, \omega) = ((L + \lambda_0)\omega, P\omega, h^* \frac{\partial}{\partial N} h^{-1} \phi_j + \omega),\]
where $[\varphi_1, \varphi_2, \cdots, \varphi_m]^\perp$ is the orthogonal complement to $\mathcal{N}(L + \lambda_0)$ (with homogeneous Neumann boundary condition) in $L^2(\Omega)$. Since $\frac{\partial}{\partial t}(t_\Omega, 0)$ is an isomorphism, the Implicit Function Theorem ensures the existence of a neighborhood $\mathcal{V}$ of $i_\Omega$ in $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and an analytic function $\omega_j(h)$ on $\mathcal{V}$ such that $\omega_j(h)$ is the unique solution of $F^i(h, \omega) = 0$, for $h \in \mathcal{V}$.

In this way we obtain, for each $h$ in $\mathcal{V}$, a set $\{\varphi_j(h)\}_{j=1}^m$, $\varphi_j(h) = \varphi_j + \omega_j(h)$, of linearly independent solutions of (3.4). Using the Gram-Schmidt method, we can produce a new set of solutions $\{\hat{\varphi}_j(h)\}_{j=1}^m$ which is orthonormal with respect to the inner product $(u,v)_h = \int_{\Omega} uv \det_h dx$. We observe that the $\hat{\varphi}_j(h)$ belong to the domain of the operator $h^*Lh^{*-1}$, $D_h = \{u \in H^2(\Omega), h^* \frac{\partial}{\partial n} h^{*-1} u = 0\}$. Furthermore, since with this inner product this operator is self-adjoint, it follows that the matrix given by $\int_{\Omega} \hat{\varphi}_j(h^*Lh^{*-1}) \hat{\varphi}_k(deth_h)dx$ is symmetric.

Consider now the family of diffeomorphisms $h(t, x) = x + tV(x)$ for some $V \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R}^n)$ and the family of projections

$$P(t)u = \sum_{j=1}^m \hat{\varphi}_j(t) \int_{\Omega} u \hat{\varphi}_j(t)(deth_h(t, .))dx.$$ 

Define the map

$$G_j = (G_{1,j}, G_{2,j}, G_{3,j}) : (-\epsilon, \epsilon) \times \mathbb{R} \times H^2(\Omega) \longrightarrow L^2(\Omega) \times H^\frac{2}{2}(\Omega) \times L^2(\Omega)$$

where

$$\begin{cases} G_{1,j} = (I - P(t))(h^*(t, .)(L + \lambda)Lh^{*-1}(t, .))(\omega + \hat{\varphi}_j(t)) \\ G_{2,j} = h^* \frac{\partial}{\partial n} h^{*-1} \omega; \\ G_{3,j} = P(t)\omega, \end{cases}$$

(3.5)

Again by the Implicit Function Theorem, there exists a neighborhood $\mathcal{U}$ of $(0, \lambda_0)$ and an application $\omega_j(t, \lambda)$ which gives the unique solution of $G_j(t, \lambda, \omega) = (0, 0, 0)$ in $\mathcal{U}$. Since, for small $t$ and $\lambda$ near $\lambda_0$, the operator $(I - P(t))(h^*(t, .)(L + \lambda)Lh^{*-1}(t, .))(\omega + \hat{\varphi}_j(t))$ with $h^* \frac{\partial}{\partial n} h^{*-1} \omega = 0$ has an $m$ dimensional kernel, the solutions of the first and second equations will be of the form $\sum_{j=1}^m c_j (\hat{\varphi}_j(t) + \omega_j(t, \lambda))$. Therefore, a number $\lambda$ will be an eigenvalue of (3.2) $n(t, .)$ with eigenfunction $\sum_{j=1}^m c_j (\hat{\varphi}_j(t) + \omega_j(t, \lambda))$ if, and only if $c = (c_1, \cdots, c_m)$ is a nonzero vector such that $M(t, \lambda)c = 0$, where

$$M_{ij}(t, \lambda) = \int_{\Omega} (\hat{\varphi}_i(t)(h^*(t, .)(L + \lambda)Lh^{*-1}(t, .))(\hat{\varphi}_j(t) + \omega_j(t, \lambda))deth_h(t, .).$$

that is, $\lambda$ is an eigenvalue if and only if Det $M(t, \lambda) = 0$. Now

$$M(t, \lambda) = \int_{\Omega} (\hat{\varphi}_i(t)(\omega_i(t, \lambda))h^*(t, .)(L + \lambda)Lh^{*-1}(t, .))(\hat{\varphi}_j(t) + \omega_j(t, \lambda))deth_h(t, .) - \int_{\Omega} \omega_i(t, \lambda)(L + \lambda)Lh^{*-1}(t, .)(\hat{\varphi}_j(t) + \omega_j(t, \lambda))deth_h(t, .).$$

and the last term is zero by the first and third equations in (3.5). It follows that $M$ is symmetric and Puiseux theorem [23] then ensures the existence of $m$ analytic curves $\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t)$ giving the $m$ (not necessarily distinct) solutions of $Det M(t, \lambda) = 0$. Since $M$ is symmetric for each curve $\lambda_i(t)$, there also exists an analytic curve $C^1(t) \in \mathbb{R}^m$ of solutions of $M(t, \lambda_i)C(t) = 0$, with $C^1(t), C^2(t), \cdots, C^m(t)$ linearly independent. Therefore,

$$\psi(t) = \sum_{j=1}^m C^j(t)(\hat{\varphi}_j(t) + \omega_j(t, \lambda_i(t)))(l = 1, \cdots, m)$$

is an analytic curve of associated eigenfunctions.

\[\square\]

**Remark 1.** The above proof is similar to the argument in [22] example 4.4. However, here we needed to first construct solutions for the auxiliary problem (3.4) since, otherwise, we would have not obtained a symmetric matrix $M$. This is due to the fact that now the domain of the operator $h^*Lh^{*-1}$ varies with $h$.

Once we know the eigenvalues are analytic in the parameter $t$, its first and second derivatives can be obtained using the methods developed in [20].

**Corollary 1.** Let $\lambda_0$ be an eigenvalue of multiplicity $m$ of (3.4), with $\beta = c = 0$ and $h(t, .) + I + tV(\cdot)$, with $V$ of class $C^2$ a curve of diffeomorphisms. Then, if $\lambda(t)$ is one of the curves of eigenvalues given by theorem [25] the derivatives $\dot{\lambda} = \frac{d}{dt}\lambda|_{t=0}$, $\ddot{\lambda} = \frac{d^2}{dt^2}\lambda|_{t=0}$ satisfy

$$\begin{align*}
(\dot{\lambda}I + \dot{M})c &= 0, \\
(\ddot{\lambda}I + \ddot{M})c + 2(\dot{\lambda}I + \dot{M})c &= 0,
\end{align*}$$

where $M(t, \lambda) = \sum_{j=1}^m C^j(t)(\hat{\varphi}_j(t) + \omega_j(t, \lambda_i(t)))$, $l = 1, \cdots, m$. 
for some \( c \) and \( \tilde{c} \) in \( \mathbb{R}^n \). The matrices \( \breve{M}, \tilde{M} \) are given by

\[
\breve{M}_{k,j} = \int_{\partial \Omega} \sigma (\nabla_{\partial \Omega} \phi_k \cdot \nabla_{\partial \Omega} \phi_j - \lambda_0 \phi_k \phi_j) \\
\tilde{M}_{k,j} = \int_{\partial \Omega} 2\sigma Q_{jk} + \sigma^2 \frac{\partial}{\partial N} Q_{jk} + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + H \sigma^2 \right] Q_{jk},
\]

where \( \{\phi_j\}_{j=1}^n \) is an orthonormal basis for the eigenspace associated to \( \lambda_0 \) and \( \dot{\phi}_j \) satisfies \( \dot{\phi}_j \perp \text{span}[\phi_i]_1^n \),

\[
\begin{aligned}
\{ & (\Delta + \lambda_0)\dot{\phi}_j \in \text{span}[\phi_i]_1^n; \\
& \frac{\partial \dot{\phi}_j}{\partial N} = (\text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} \phi_j) + \lambda_0 \sigma \phi_j), & \text{on } \partial \Omega.
\end{aligned}
\]

**Proof.** We know that each eigenpair \( (\lambda(t), v(t)) \) satisfies

\[
\begin{aligned}
(\Delta + \lambda(t))v(t,.) = 0, & \quad \text{in } \Omega_t; \\
\frac{\partial v(t,.)}{\partial N_{\Omega_t}} = 0, & \quad \text{on } \partial \Omega_t;
\end{aligned}
\]

Differentiating the first equation with respect to \( t \), at \( y = h(t, x) \in \Omega_t \), we obtain

\[
(\Delta + \lambda(t)) (\frac{\partial v(t,.)}{\partial t} + \frac{d}{dt} \lambda(t)v(t, y)) = 0
\]

From now on, we use the notation \( \dot{v} \) for the derivative \( \frac{\partial}{\partial t} v(t,. \) and also for any derivative with respect to \( t \).

In the boundary we have, for each \( x \in \partial \Omega \)

\[
\frac{\partial v(t,h(t,x))}{\partial N_{\Omega_t}} = N_{\Omega_t}(t,h(t,x)) \cdot \nabla_y v(t,h(t,x)) = 0
\]

where \( \nabla_y \) is the derivative in the variable \( y = h(t,x) \). Differentiating with respect to \( t \), we obtain

\[
0 = \frac{d}{dt} \left[ \nabla_y v(t,h(t,x)) \right] = \frac{d}{dt} [N_{\Omega_t}(t, h(t,x))] \cdot \nabla_y v(t,h(t,x)) + N_{\Omega_t}(t,h(t,x)) \cdot \frac{d}{dt}[\nabla_y v(t,h(t,x))]
\]

\[
= \dot{N}_{\Omega_t}(t,y) \cdot \nabla_y v(t,y) + N_{\Omega_t}(t,y) \cdot \nabla_y v(t,y) + \nabla_y v(t,y) V + (\nabla_y N_{\Omega_t}(t,y) V) \cdot \nabla_y v(t,y)
\]

From lemma \( \Box \)

\[
\dot{N}_{\Omega_t} = -\nabla_{\partial \Omega} \sigma - \sigma \frac{\partial N_{\Omega_t}}{\partial N_{\Omega_t}}
\]

where \( \sigma = V(t,y) \cdot N_{\Omega_t} \). Since \( \frac{\partial v}{\partial N_{\Omega_t}} = 0 \) on \( \partial \Omega_t \), it follows that

\[
V \cdot \nabla_y \left( \frac{\partial v}{\partial N_{\Omega_t}} \right) = \sigma \frac{\partial}{\partial N_{\Omega_t}} \left( \frac{\partial v}{\partial N_{\Omega_t}} \right), \quad \text{on } \partial \Omega_t.
\]

Thus

\[
\frac{\partial \dot{v}}{\partial N_{\Omega_t}} - \nabla_{\partial \Omega} \sigma \cdot \nabla_{\partial \Omega} v + \sigma \frac{\partial}{\partial N_{\Omega_t}} \left( \frac{\partial v}{\partial N_{\Omega_t}} \right) = 0 \quad \text{on } \partial \Omega_t.
\]

Now, using Theorem \( \Box \) we obtain

\[
\frac{\partial \dot{v}}{\partial N_{\Omega_t}} = \text{div}_{\partial \Omega} (\sigma \nabla_{\partial \Omega} v) + \lambda(t) \sigma v, \quad \text{on } \partial \Omega_t.
\]
Therefore \( \dot{v} \) must satisfy the problem

\[
\begin{aligned}
(\Delta + \lambda(t))\dot{v} + \lambda(t)v = 0, & \quad \text{in } \Omega; \\
\frac{\partial \dot{v}}{\partial \sigma} = \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \lambda(t)\sigma v, & \quad \text{on } \partial \Omega;
\end{aligned}
\]

We know that \( v(0, \cdot) = \sum_{j=1}^{m} c_j \phi_j \) for some scalars \( c_j \), not all zero. Multiplying the equation \((3.6)\) with \( t = 0 \) by \( \phi_k \) and integrating, we obtain

\[
\lambda c_k = -\int_{\Omega} \phi_k (\Delta + \lambda_0) \dot{v}
\]

\[
= \int_{\Omega} \dot{v} (\Delta + \lambda_0) \phi_k - \phi_k (\Delta + \lambda_0) \dot{v}
\]

\[
= -\int_{\partial \Omega} \phi_k (\text{div}_{\partial \Omega} (\sigma \nabla_{\partial \Omega} v) + \lambda_0 \sigma v))
\]

\[
= \int_{\partial \Omega} \sigma (\nabla_{\partial \Omega} \phi_k \cdot \nabla_{\partial \Omega} v - \lambda_0 \dot{\phi}_k)
\]

\[
= \sum_{j=1}^{m} c_j \int_{\partial \Omega} \sigma (\nabla_{\partial \Omega} \phi_k \cdot \nabla_{\partial \Omega} \phi_j - \lambda_0 \dot{\phi}_k) \phi_j.
\]

Writing \( c = (c_1, c_2, \ldots, c_m) \) and

\[
\hat{M}_{k,j} = \int_{\partial \Omega} \sigma (\nabla_{\partial \Omega} \phi_k \cdot \nabla_{\partial \Omega} \phi_j - \lambda_0 \dot{\phi}_k) \phi_j
\]

we see that \((\hat{M} - \lambda) c = 0\) and, therefore, the derivative \( \lambda(t) \) is an eigenvalue of the matrix \( \hat{M} \).

Now, to compute \( \dot{\lambda} \), we need to differentiate \((3.6)\) once again. We start with the boundary condition

\[
\frac{\partial \dot{v}}{\partial \sigma} = \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \lambda(t)\sigma v = 0
\]

If \( f(t, h(t, x)) = 0, \forall x \in \partial \Omega, \) with \( f \), we obtain, differentiating with respect to \( t \)

\[
f(0, x) + \sigma \frac{\partial f}{\partial N}(0, x) = 0, \quad \text{on } \partial \Omega.
\]

Applying this formula in the equation \((3.7)\), it follows that

\[
\dot{f}(0, x) = \frac{\partial \dot{v}}{\partial N} - \nabla_{\partial \Omega} \sigma \cdot \nabla_{\partial \Omega} \dot{v} - \left[ \frac{\partial}{\partial t} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \frac{\partial}{\partial t}(\sigma \lambda v) \right]
\]

\[
= \sigma \frac{\partial^2 v}{\partial N^2} - \sigma \left[ \frac{\partial}{\partial N} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \frac{\partial}{\partial N}(\sigma \lambda v) \right].
\]

Thus

\[
\frac{\partial \dot{v}}{\partial N} = \nabla_{\partial \Omega} \sigma \cdot \nabla_{\partial \Omega} \dot{v} - \sigma \frac{\partial^2 v}{\partial N^2} + (\lambda v + \lambda \dot{v}) + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + \sigma^2 H \right] \lambda v +
\]

\[
+ \frac{\partial}{\partial t} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \sigma \frac{\partial}{\partial N} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v).
\]

Multiplying the equation \((3.7)\) by \(-\sigma H\) and summing with the above equation, we obtain the boundary condition

\[
\frac{\partial \dot{v}}{\partial N} = \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} \dot{v}) + 2 \lambda \dot{v} + \lambda \ddot{v} + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + \sigma^2 H \right] \lambda v +
\]

\[
\frac{\partial}{\partial t} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \sigma \frac{\partial}{\partial N} \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v) + \sigma \text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} v).
\]

Now, differentiating the equation in the interior, we obtain

\[
(\Delta + \lambda) \ddot{v} + 2 \lambda \dot{v} + \lambda v = 0.
\]

Thus, to compute the second derivative, we need know \( \dot{v} \). To this aim, we first observe that there is a unique \( \dot{\phi}_j \in H^2(\Omega) \), such that \( \dot{\phi}_j \perp [\phi_1, \phi_2, \ldots, \phi_m] \)

\[
\begin{aligned}
(\Delta + \lambda) \dot{\phi}_j \in \text{span}[\phi_1]^{\perp m} \\
\frac{\partial \dot{\phi}_j}{\partial N} = (\text{div}_{\partial \Omega}(\sigma \nabla_{\partial \Omega} \phi_j) + \lambda_0 \sigma \phi_j), & \quad \text{on } \partial \Omega.
\end{aligned}
\]
Thus \( \dot{a}_{i=0} = \sum_{j=1}^{m} c_j \dot{\phi}_j + \dot{\lambda} \phi_j \), where the \( \dot{c}_j \) are not all zero and the \( c_j \) as before. Multiplying the equation by \( \phi_k \) and integrating in \( \Omega \), we have

\[
\tilde{\lambda}_k + 2\lambda \dot{c}_k = - \int_{\partial \Omega} \phi_k \frac{\partial \dot{v}}{\partial N} = - \int_{\partial \Omega} \phi_k \left( \text{div}_\Omega (\sigma \nabla \dot{\phi}_k \dot{v}) + 2\sigma (\dot{\lambda} v + \lambda \dot{v}) + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + \sigma^2 H \right] \lambda v \right)
\]

Thus

\[
\int_{\partial \Omega} \phi_k \text{div}_\Omega (\sigma \nabla \dot{\phi}_k \dot{v}) + \phi_k \sigma \frac{\partial}{\partial N} \text{div}_\Omega (\sigma \nabla \dot{\phi}_k \dot{v}) + \sigma \text{H} \phi_k \text{div}_\Omega (\sigma \nabla \dot{\phi}_k \dot{v}).
\]

It is convenient to write this expression in a different form. We split the computation in two parts. We call \( I \) and \( II \) the first and second integrals and start with the second.

Extending \( \phi_k \) arbitrarily in a neighborhood of \( \Omega \), we observe that

\[
\frac{d}{dt} \left[ \int_{\partial \Omega} \phi_k \text{div}_\Omega (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k) \right]_{t=0} = \int_{\partial \Omega} \phi_k \frac{\partial}{\partial N} \text{div}_\Omega (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k)_{t=0} + \int_{\partial \Omega} \sigma \frac{\partial}{\partial N} (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k)_{t=0}.
\]

On the other hand

\[
\int_{\partial \Omega} \phi_k \text{div}_\Omega (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k) = - \int_{\partial \Omega} \sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k.
\]

Thus

\[
II = - \frac{d}{dt} \left( \int_{\partial \Omega} \sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k \right)_{t=0} = - \int_{\partial \Omega} \sigma \frac{\partial}{\partial N} (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k)_{t=0}.
\]

Now

\[
\frac{\partial}{\partial t} \left( \nabla_{\partial \Omega} (\nabla_{\partial \Omega} \phi_k \cdot \nabla \phi_k) \right)_{t=0} = \left( \frac{\partial}{\partial t} \nabla_{\partial \Omega} \phi_k \right)_{t=0} \cdot \nabla \phi_k + \left( \frac{\partial}{\partial t} \nabla_{\partial \Omega} \phi_k \right)_{t=0} \cdot \nabla \phi_k = \left[ \nabla_{\partial \Omega} \left( \frac{\partial \phi_k}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \phi_k}{\partial N} \right) N - \frac{\partial \phi_k}{\partial N} N \right] \cdot \nabla \phi_k + \left[ \nabla_{\partial \Omega} \left( \frac{\partial \phi_k}{\partial t} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \phi_k}{\partial N} \right) N - \frac{\partial \phi_k}{\partial N} N \right] \cdot \nabla \phi_k
\]

It follows that

\[
II = - \int_{\partial \Omega} \sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k - \int_{\partial \Omega} \sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k.
\]

For the first term in the integral \( I \), we have

\[
\int_{\partial \Omega} \phi_k \text{div}_\Omega (\sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k) = - \int_{\partial \Omega} \sigma \nabla \phi_k \cdot \nabla \dot{\phi}_k.
\]

Thus

\[
\int_{\partial \Omega} \phi_k \frac{\partial \dot{v}}{\partial N} = - \int_{\partial \Omega} 2\sigma (\nabla \phi_k \cdot \nabla \dot{\phi}_k - \lambda_0 \phi_k \dot{v} - \dot{\lambda} \phi_k v) + \int_{\partial \Omega} \sigma^2 \frac{\partial}{\partial N} (\nabla \phi_k \cdot \nabla \phi_k v - \lambda_0 \phi_k v) + \int_{\partial \Omega} \frac{\partial}{\partial t} \left[ \sigma + \frac{\partial \sigma}{\partial N} + \sigma^2 H \right] (\nabla \phi_k \cdot \nabla \phi_k v - \lambda_0 \phi_k v).
\]
Recalling that \( t_{t=0} = \sum_{j=1}^{m} c_j \hat{\phi}_j + \hat{\sigma} \hat{\phi}_j \), \( v = \sum_{j=1}^{m} c_j \hat{\phi}_j \) and
\[
\lambda c_k + 2 \lambda \hat{c}_k = - \int_{\partial \Omega} \phi_k \frac{\partial \hat{v}}{\partial N},
\]
we conclude that the possible values of \( \lambda \) are given by the following equations in \( \mathbb{R}^m \):
\[
(\lambda I + \hat{M})c + 2(\lambda I + \hat{M})\hat{c} = 0
\]
\[
(\lambda I + \hat{M})c = 0
\]
where \( \hat{M} \) was given above and
\[
\hat{M}_{j,k} = \int_{\partial \Omega} 2\sigma \hat{Q}_{jk} + \sigma^2 \frac{\partial}{\partial N} \hat{Q}_{jk} + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + H \sigma^2 \right] \hat{Q}_{jk},
\]
\[
\hat{Q}_{jk} = \nabla_{\partial \Omega} \phi_j \cdot \nabla_{\partial \Omega} \phi_k - \lambda_0 \phi_j \phi_k
\]
\[
\hat{Q}_{jk} = \nabla_{\partial \Omega} \phi_k \cdot \nabla_{\partial \Omega} \phi_j - \lambda_0 \phi_k \phi_j - \lambda \phi_k \phi_j.
\]

**Remark 2.** It is not difficult to see that the matrix \( \hat{M}_{j,k} \) is symmetric. This will be important in the sequel.

## 4 Multiplicity of the eigenvalues on symmetric domains

In this section, we discuss some consequences of the symmetry on the multiplicity of the eigenvalues of problem \( \text{[1]} \). If \( G \) is a compact subgroup of the orthogonal group \( O(n) \), we say that \( \Omega \) is \( G \)-symmetric (or, it is \( G \)-invariant, or it has symmetry \( G \)) if \( g\Omega = \Omega \) for all \( g \in G \). Let
\[
\text{Diff}_G^m(\Omega) = \{ h \in \text{Diff}(\Omega) \mid h \circ g = gh, \text{ for any } g \in G \}. 
\]
If \( \Omega \) is \( G \)-symmetric and \( h \in \text{Diff}_G^m(\Omega) \) then clearly \( h(\Omega) \) is also \( G \)-symmetric and we can then restrict the topology defined in section \( \text{[22]} \) to the set of \( G \)-symmetric regions.

### 4.1 Algebraic preliminaries

We now present some definitions and results from the Representation Theory of Compact Groups (see \[3\] chapter 3, section 27 for details and proofs) that will be used in the sequel.

Let \( G \) be a compact group. A representation of \( G \) in a Hilbert space \( H \) is a group homomorphism \( V : G \to GL(H) \), where \( GL(H) \) is the group (under composition) of invertible continuous linear operators in \( H \). If \( H \) is a complex (resp. real) Hilbert space the representation \( V \) is called unitary (resp. orthogonal) if it is an unitary (resp. orthogonal) operator, for any \( g \in G \).

**Definition 1.** A representation \( G \) is strongly continuous if \( \lim_{x \to e} V_x \xi = \xi \) for any \( \xi \in H \).

**Definition 2.** Let \( V : G \to GL(H) \) and \( V' : G \to GL(H') \) be continuous representations of \( G \). We say that

1. \( V \) and \( V' \) are equivalent if there exists a linear isometry \( T : H \to H' \) such that \( V'_x \circ T = T \circ V_x \), for any \( x \in G \).
2. \( V \) is finite dimensional if \( H \) is finite dimensional.
3. A closed subspace \( H_1 \subset H \) is invariant for \( V \) if \( V_x H_1 \subset H_1 \) for any \( x \in G \). The representation \( V' : G \to GL(H_1) \) is called a sub-representation of \( V \) and will be denoted by \( V|_{H_1} \).
4. \( V \) is irreducible if its only closed invariant subspaces are \( \{0\} \) and \( H \). Otherwise, \( V \) is called reducible.
5. If \( H = H_1 \oplus H_2 \oplus \ldots \oplus H_m \), where the \( H_i \) are invariant under \( V \), we write \( V = V|_{H_1} \oplus V|_{H_2} \oplus \ldots \oplus V|_{H_m} \) and say that \( V \) is a direct sum of the representations \( V_{|H_i} \).

**Theorem 6.** Any irreducible unitary (resp. orthogonal) representation of a compact group \( G \) is finite dimensional. \( G \) is abelian if and only if all its irreducible representations have dimension (complex) 1.

Let \( V \) be a finite dimensional representation of \( G \). The function \( \chi_V \) given by \( g \to tr V_g \), where \( tr \) is the trace of the operator \( V_g \), is called the character of \( V \). Clearly, two equivalent representations have the same character.

Let \( G \) be a compact subgroup of the orthogonal group \( O(n) \). The set of all equivalent classes of continuous irreducible representations of \( G \) is called the dual object of \( G \) and is denoted by \( \hat{G} \). We denote by \( \chi_{\sigma} \) the character of any representation in the class \( \sigma \in \hat{G} \) and by \( d_\sigma \) its dimension. If \( H \) is a Hilbert space and \( V : G \to L(H) \) is a continuous orthogonal representation of \( G \), we can define, for each \( \sigma \in \hat{G} \), the operator \( P_\sigma \) in \( H \) by
\[
\langle P_\sigma \xi , \eta \rangle = \int_G \langle V_g \xi , \eta \rangle d_\sigma \chi_{\sigma}(x) \ dx
\]
\( P_\sigma \) is a continuous projection (see \[13\]). We set \( M_\sigma := P_\sigma H \).

The following decomposition theorem will be important in the sequel. A proof for unitary representations can be found in \[3\]. For real spaces it can be obtained from this result by complexification (see \[13\]).
Theorem 7. Let $G$ be a compact subgroup of $O(n)$ and $V$ a continuous (unitary) orthogonal representation of $G$ in $H$. For every $\sigma \in G$, let $P_{\sigma}$ be the operator in $H$ defined by

$$\langle P_{\sigma} \xi, \eta \rangle = \int_G \langle V_g \xi, \eta \rangle d_{\sigma} \chi_g(x) \, dx.$$ 

Then $P_{\sigma}$ is a projection operator in $H$.

If $\sigma \neq \sigma'$ then $M_{\sigma}$ and $M_{\sigma'}$ are orthogonal subspaces of $H$, $H = \bigoplus_{\sigma \in G} M_{\sigma}$.

For each $\sigma \in G$, $M_{\sigma}$ is either $\{0\}$ or a direct sum of $m_{\sigma}$ pairwise orthogonal, $d_{\sigma}$-invariant subspaces $L_{\sigma,j}$, on each of which $V_{L_{\sigma,j}} \in \sigma$.

The cardinal number $m_{\sigma}$ may be finite or infinite.

The subspace $M_\sigma$ is the smallest closed subspace of $H$ containing all invariant subspaces of $H$ on which $V$ is in the class $\sigma$.

This direct sum decomposition of $V$ is unique in the following sense. If

$$H = \bigoplus_{\lambda \in \Lambda} N_\lambda,$$

where each $N_\lambda$ is an invariant subspace on which $V$ is irreducible, then

$$\{ \oplus N_\lambda | V|_{N_\lambda} \in \sigma \} = M_\sigma$$

and there are $m_{\sigma}$ subspaces $N_\lambda$ on each of which $V|_{N_\lambda} \in \sigma$.

4.2 Consequences of the symmetry

We now apply the abstract results of the previous section to derive some results on the multiplicity of the eigenvalues of $\Delta$. The main result was obtained in [17] and [18], for the Dirichlet Laplacian. The proof in the Neumann case is completely similar but is presented here for completeness. If $G$ is a compact subgroup of $O(n)$, the “natural” action of $G$ in $\mathbb{R}^n$ is given by $(g, x) \mapsto gx$. The subgroup $G_x = \{g \in G : gx = x\}$ is called the isotropy group of $x \in \mathbb{R}^n$ and $G(x) = \{g \in G : gx \in G\}$ is the orbit of $x$ under this action. A point $x \in \mathbb{R}^n$ such that $G_x = I_n$ is called a free point for the action.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, $G$-invariant and $\Gamma : G \to GL(L^2(\Omega))$ the quasi-regular representation of $G$

$$\Gamma_g u = u \circ g^{-1}, \forall g \in G, \forall u \in L^2(\Omega).$$

This representation is orthogonal and commutes with the Laplacian, that is

$$(\Gamma_g \circ \Delta)u = \Gamma_g(\Delta u) = (\Delta u) \circ g^{-1} = \Delta(u \circ g^{-1}) = (\Delta \circ \Gamma_g)u$$

for any $u \in H^2(\Omega)$, and $g \in G$. As an immediate consequence the eigenspaces are invariant under the representation $\Gamma$.

For any $\sigma \in G$, let $P_{\sigma}$ be the projection

$$\langle P_{\sigma} f, h \rangle = \int_G \langle \Gamma_g f, h \rangle d_{\sigma} \chi_g \, dg.$$

Theorem 4 asserts that

$$L^2(\Omega) = \bigoplus_{\sigma \in G} M_\sigma,$$

where $M_\sigma = P_{\sigma} L^2(\Omega)$.

The spaces $M_\sigma$ are invariant for the Laplacian. More precisely

Proposition 1. If $D_N = \{ u \in H^2(\Omega) | \frac{\partial u}{\partial N} = 0, \text{ on } \partial \Omega \}$, then the Laplacian is a linear transformation from $M_\sigma \cap D$ to $M_\sigma$.

Furthermore, we have

Proposition 2. Each symmetry space $M_\sigma$ can be decomposed as a direct sum of subspaces $M^j_\sigma$ satisfying

1. $M^j_\sigma$ is invariant under the representation $\Gamma$ and $\Gamma M^j_\sigma$ is an irreducible representation in the class $\sigma$.

2. $M^j_\sigma$ is invariant for the Laplacian and $\Delta M^j_\sigma$ is a multiple of the identity, that is, the elements of $M^j_\sigma$ are eigenfunctions associated to the same eigenvalue $\lambda$.

Proof. Consider the spectral decomposition of the Laplacian restricted to $M_\sigma$, that is, $M_\sigma = \oplus V^j_\lambda$, where $V^j_\lambda$ is the eigenspace associated to the eigenvalue $\lambda$. Since the Laplacian commutes with $\Gamma$, the eigenspaces $V^j_\lambda$ are invariant for the representation. From Theorem 7 we have the decomposition $V^j_\lambda = V^1_\lambda \oplus \ldots \oplus V^K_\lambda$, where each $V^K_\lambda$ is an irreducible space in the class $\sigma$. This proves the result.

Corollary 2. The multiplicity of each eigenvalue of the Laplacian restricted to $M_\sigma$ is a multiple of the irreducible representations in the class $\sigma$. □
Lemma 2. Then, at least one of those functions vanishes on perturbations preserving the symmetry. If $h \in G$ there is an eigenvalue $\lambda$ of the Laplacian, and a subspace $H$ of the associated eigenspace $V_\lambda$ such that $\Gamma |_H$ is in the class $\sigma$. In particular, for any $\sigma \in G$, there exist an infinite number of eigenvalues whose multiplicity is a multiple of the dimension $d_\sigma$.

Proof. The result follows immediately from Corollary 2 once it is known that the spaces $M_\sigma$ are all infinite dimensional. This is proved in [18] (Theorem 3.2).

As an immediate consequence, we also obtain the following result.

Corollary 3. If $G$ is not a direct sum of cyclic groups of order 2, $\Omega$ is $G$-symmetric and contains a free point under the action of $G$, then there always exist multiple eigenvalues of the Neumann Laplacian in $\Omega$.

5 Generic $G$-simplicity of the eigenvalues

In this section, we analyze the validity of Conjecture 1 for the Neumann Laplacian in the case of finite groups. We establish the validity of part I of the conjecture for arbitrary finite subgroups $G$ of $O(n)$. Part II of the conjecture will be proved under an additional assumption on the dimension of the irreducible representations of $G$.

An important step in our proof will be the analysis of the behavior of the eigenvalues in each symmetric space. Here, in contrast to the Dirichlet case analyzed in [18], the knowledge of the first derivative of the eigenvalues did not suffice to separate multiple eigenvalues and it became necessary to compute also the second derivative.

5.1 A special case

In this section we consider the very special case where the symmetry group $G$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$ ($m$ times).

We first prove a technical result due to Uhlenbeck.

Lemma 2. Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, $C^2$-regular domain $\lambda$ is a positive real number and $f, g$ are $C^2$ functions on $\partial \Omega$. If

$$\nabla_{\partial \Omega} f \cdot \nabla_{\partial \Omega} g - \lambda f g = 0, \text{ on } \partial \Omega.$$ 

Then, at least one of those functions vanishes on $\partial \Omega$.

Proof. Let $x(t)$ be a solution of the equation $\nabla_{\partial \Omega} f(x(t)) = \dot{x}(t)$, $x(0) = x_0 \in \partial \Omega$. Since $\partial \Omega$ is compact $x(t)$ is defined for $t$ and $\frac{d}{dt} f(x(t)) = \left| \nabla f(x(t)) \right|^2 \geq 0$. Now, the function $g(x(t))$ satisfies the equation $\dot{u}(t) = \lambda f(x(t)) u(t)$, $u(0) = g(x_0)$ and, thus, $g(x(t)) = g(x_0) \exp(\lambda \int_0^t f(x(s)) ds)$. Therefore, if $f(x_0) \neq 0$ and $g(x_0) \neq 0$, then $g(x(t))$ would be unbounded which cannot occur since $\partial \Omega$ is compact.

Theorem 9. Suppose $G$ be a subgroup of $O(n)$ which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$ and $\Omega \subset \mathbb{R}^n$ an open, bounded, $C^2$-regular, $G$-symmetric domain. If $\lambda_0$ is an eigenvalue of $\Gamma_{\Omega}$ with multiplicity $m > 1$ then, given $\epsilon > 0$ there exist $\delta > 0$ and $h \in Diff^2_+(\Omega)$, $\|h - h_\Omega\|_{C^3} < \epsilon$ such that the eigenvalues of $\Gamma_h$ in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ are all simple.

Proof. Suppose $\lambda_0$ is an eigenvalue of $\Gamma_{\Omega}$ with multiplicity $m > 1$. It is enough to show that it can be separated by small perturbations preserving the symmetry. If $h \in Diff^2_+(\Omega)$, the perturbed problem in the Lagrangean form is

$$\begin{cases}
\frac{d}{dt} h^*(\Delta + \lambda)h^{*-1} u = 0, & \text{in } \Omega; \\
\frac{d}{dt} h^* \nabla h^{*-1} u = 0, & \text{on } \partial \Omega;
\end{cases}$$

(5.1)

If we choose an analytic family of diffeomorphism $t \mapsto h(t, \cdot) \in C^3$, Theorem 3 guarantees the existence of $m$ corresponding analytic curves of eigenvalues with derivatives given by the eigenvalues of the matrix (see corollary 1)

$$\frac{d}{dt} \sigma(\nabla \phi_i \cdot \nabla \phi_j - \lambda_0 \phi_i \phi_j).$$

Suppose, by contradiction, that $\lambda_0$, cannot be split into eigenvalues of smaller multiplicity. Then $\dot{M}$ must be a multiple of the identity, that is

$$\int_{\partial \Omega} \sigma(|\nabla \phi_i|^2 - \lambda_0 \phi_i^2 - (|\nabla \phi_j|^2 - \lambda_0 \phi_j^2)) = 0$$

(5.2)

$$\int_{\partial \Omega} \sigma(\nabla \phi_i \cdot \nabla \phi_j - \lambda_0 \phi_i \phi_j) = 0, i \neq j.$$ 

(5.3)

Since the family of diffeomorphism can be arbitrarily chosen in $Diff^2_+(\Omega)$, the function $\sigma$ can be any $G$-invariant function on $\partial \Omega$. 

Proof. It follows immediately from proposition 2.
Let $L^2(\Omega) = \bigoplus_{\chi \in G} M_{\chi}$ be the decomposition given by Theorem\ref{thm:decomposition}. In the present case, $M_{\chi} = \{ f \in L^2(\Omega) : f \circ g = \chi(g) f, \forall g \in G \}$ and $\chi(g) \in \{-1, 1\}$ for all $g \in G$. Furthermore, we can choose an orthonormal basis of the eigenspace $V_{\lambda_0} \{ \phi_j \}_{j=1}^m$, with $\phi_j \in M_{\chi_j}$ (the spaces $M_{\chi_j}$ need not be all distinct). We need to analyze two situations case i) there exist more than one eigenfunction in the same symmetry space $M_{\chi_j}$. Thus, the expression $\nabla \phi_i \cdot \nabla \phi_j - \lambda_0 \phi_i \phi_j$ is a $G$-invariant function on $\partial \Omega$. From (5.3) it follows that $\nabla \phi_i \cdot \nabla \phi_j - \lambda_0 \phi_i \phi_j = 0$ on $\partial \Omega$, which cannot occur by Lemma\ref{lem:invariant_function}

case ii) happens, there exist two eigenfunctions $\phi_i, \phi_j$ belonging to distinct symmetry spaces. Since the functions $|\nabla \phi_i|^2 - \lambda_0 \phi_i^2$ are $G$-invariant for each $i$, it follows from (5.3), that
\[ \nabla (\phi_i + \phi_j) \cdot \nabla (\phi_i - \phi_j) - \lambda_0 (\phi_i + \phi_j) (\phi_i - \phi_j) = |\nabla \phi_i|^2 - \lambda_0 \phi_i^2 - (|\nabla \phi_j|^2 - \lambda_0 \phi_j^2) = 0 \]
on $\partial \Omega$. Writing $\psi^+ = \phi_i + \phi_j$ and $\psi^- = \phi_i - \phi_j$, we have $\nabla \psi^+ \cdot \nabla \psi^- - \lambda_0 \psi^+ \psi^- = 0$, where $\psi^+ e \psi^-$ are eigenfunctions associated to $\lambda_0$, again in contradiction with Lemma\ref{lem:invariant_function}.

**Corollary 4.** If $G$ is a finite subgroup of $O(n)$ isomorphic to $Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2$, then, for a residual set of open, bounded, connected $C^2$-regular $G$-symmetric regions $\Omega$ of $R^n$, the eigenvalues of (7.1) are all simple.

**Proof.** Let
\[ C_k = \{ h \in Diff^1_G(\Omega) : \text{the eigenvalues, } \lambda \text{ of } (\ref{eq:matrix}), \lambda < k, \text{ are all simple} \}. \]
$C_k$ is open by the continuity properties asserted by Theorem\ref{thm:continuity}. Theorem\ref{thm:continuity} guarantees that $C_k$ is also dense. The result then follows by taking intersection in $k$.

5.2 General finite groups

We now consider the problem for a general finite group $G$. As we will see, the first part of conjecture\ref{conj:main_conjecture} (sub-conjecture I) can be established in this general case (though the arguments are more involved than the Dirichlet case). However the second part is much more difficult and we have only been able to establish it in some special cases. In fact, even in the first step and supposing the eigenvalues do not split, the expression of the first derivative of the eigenvalues, given by the matrix $\tilde{M}$ does not suffice to obtain a contradiction. Therefore we are forced to compute the second derivative. Then, the hypothesis of non separability implies that a certain boundary operator is of finite range. At this point, we use the “Method of Rapidly Oscillating Solutions” (see section\ref{sect:method}) to obtain more information on the eigenfunctions, which finally lead to the searched contradiction.

**Theorem 10.** Let $G$ be a finite subgroup of $O(n)$ and $\Omega \subset R^n$ and open bounded, connected, $C^3$-regular $G$-invariant domain. Let also $\lambda_0$ be an eigenvalue with multiplicity $md_\sigma$, $m > 1$, which is the unique eigenvalue for the problem (7.1) restricted to $M_\sigma$ in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$. Given $\epsilon > 0$ there exists $h \in Diff^1_G(\Omega)$, $||h - i\Omega||_{C^3} < \epsilon$ such that the problem (7.1) restricted to $M_\sigma$ has exactly $m$ $G_\sigma$-simple eigenvalues in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$.

**Proof.** Let $\{ \phi_j \}$, $i = 1, \ldots, m; j = 1, \ldots, d_\sigma$ be an orthonormal basis for the eigenspace associated to $\lambda_0$ satisfying
\begin{equation}
(\phi^1_1) \\
\vdots \\
(\phi^1_{d_\sigma}) \\
\cdots \\
(\phi^m_1) \\
\vdots \\
(\phi^m_{d_\sigma})
\end{equation}
\begin{align}

\circ g = A_\sigma(g) \begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_{d_\sigma}
\end{pmatrix},
\end{align}
for all $g \in G$ where $g \mapsto A_\sigma(g)$ is an irreducible matrix representation of dimension $d_\sigma$ in the class $\sigma$. Consider the remnumbering of the functions $\phi_j$ given by, $\varphi_k = \phi_j^i$, where $k = (i - 1)d_\sigma + j$, that is
\[ \varphi_1 = \phi^1_1, \ldots, \varphi_{d_\sigma} = \phi^1_{d_\sigma}, \varphi_{d_\sigma + 1} = \phi^2_1, \ldots, \varphi_{2d_\sigma} = \phi^2_{d_\sigma}, \ldots, \varphi_{md_\sigma} = \phi^m_{d_\sigma}. \]
Suppose that the multiplicity of $\lambda_0$ cannot be reduced by small $G$-symmetric perturbations of $\Omega$. Then, the matrix $\tilde{M}$, given by
\[ \tilde{M}_{ik} = \int_{\partial \Omega} \sigma(\nabla \varphi_i \cdot \nabla \varphi_k - \lambda_0 \varphi_i \varphi_k) \]
is such that $\tilde{M} = \lambda I$, that is
\begin{align}

\int_{\partial \Omega} \sigma(|\nabla \varphi_k|^2 - \lambda_0 \varphi_k^2) = \int_{\partial \Omega} \sigma(|\nabla \varphi_i|^2 - \lambda_0 \varphi_i^2) \nabla \varphi_k \cdot \nabla \varphi_l - \lambda_0 \varphi_k \varphi_l = 0, 1 \leq k, l \leq md_\sigma.
\end{align}

It is difficult to obtain some information from these relations, since the integrands are not $G$-invariant. However, taking into account the remnumbering above, we see that the entries of the matrix $\tilde{M}$ contain the expressions
\[ \nabla \phi^i_j \cdot \nabla \phi^j_i - \lambda_0 \phi^i_j \phi^j_i \]
for $1 \leq i, l \leq m$. We can obtain some new information, if we show that their sum

$$
\sum_{j=1}^{d_\sigma} \nabla \phi_j^i \cdot \nabla \phi_j^l - \lambda_0 \phi_j^i \phi_j^l
$$

$$
\sum_{j=1}^{d_\sigma} |\nabla \phi_j^i|^2 - \lambda_0 (\phi_j^i)^2
$$

are $G$-invariant functions on $\partial \Omega$. To this aim, we show that the sum involving the gradient is $G$-invariant, since the other sum is clearly $G$-invariant. In fact,

$$
\phi_j^i \circ g^{-1}(x) = \sum_{k=1}^{d_\sigma} a_{j,k}(g) \phi_k^i,
$$

where $a_{j,k}(g)$ are the entries in the matrix representation $g \to A_g(g)$. It follows that

$$
\sum_{j=1}^{d_\sigma} (\nabla \phi_j^i \cdot \nabla \phi_j^l)(g^{-1}(x)) = \sum_{j,k,p} a_{j,k}(g)a_{jp}(g) \nabla \phi_k^i \cdot \nabla \phi_p^l(x)
$$

$$
= \sum_{k,p} \delta_{kp} \nabla \phi_k^i \cdot \nabla \phi_p^l(x)
$$

$$
= \sum_{j=1}^{d_\sigma} \nabla \phi_j^i \cdot \nabla \phi_j^l(x).
$$

The proof that $\sum_{j=1}^{d_\sigma} |\nabla \phi_j^i|^2 - \lambda_0 (\phi_j^i)^2$ is $G$-invariant in $\partial \Omega$ is analogous.

Therefore, observing that the function $\sigma$ can be chosen arbitrarily close to any $G$-invariant function on $\partial \Omega$, relations (5.6) and (5.8) give

$$
\sum_{j=1}^{d_\sigma} \nabla \phi_j^i \cdot \nabla \phi_j^l - \lambda_0 \phi_j^i \phi_j^l = 0, \quad \text{on} \quad \partial \Omega.
$$

Even with this new information about the eigenfunctions in the boundary, we could not obtain a contradiction. We thus calculated the second derivative of the curve of eigenvalues, using corollary 1.

$$
\tilde{M}_{k,j}^{\infty} = \int_{\partial \Omega} 2\sigma Q_{jk} + \sigma^2 \frac{\partial}{\partial N} Q_{jk} + \left[ \frac{\partial \sigma}{\partial t} + \sigma \frac{\partial \sigma}{\partial N} + H \sigma^2 \right] Q_{jk}
$$

where

$$
Q_{jk} = \nabla_{\partial \Omega} \varphi_j \cdot \nabla_{\partial \Omega} \varphi_k - \lambda_0 \varphi_j \varphi_k
$$

$$
\tilde{Q}_{jk} = \nabla_{\partial \Omega} \varphi_k \cdot \nabla_{\partial \Omega} \varphi_j - \lambda \varphi_k \varphi_j - \lambda \varphi_k \varphi_j,
$$

and $\varphi_j$ is the unique solution of

$$
\begin{cases}
(\Delta + \lambda_0) \varphi_j \in \text{span}\left[\varphi_i\right]_{1}^{md_\sigma}, \\
\frac{\partial \varphi_j}{\partial N} = \nabla_{\partial \Omega} \sigma \cdot \nabla_{\partial \Omega} \varphi_j - \sigma \frac{\partial \sigma}{\partial N} \varphi_j, \quad \text{on} \quad \partial \Omega \\
\varphi_j \perp \text{span}\left[\varphi_i\right]_{1}^{md_\sigma}.
\end{cases}
$$

In order to obtain $G$ invariant functions, we will again need to sum up some entries of the matrix $\tilde{M}^{\infty}$. Actually, we will see that the integrand of $\sum_{j=1}^{d_\sigma} \tilde{M}_{j,j+d_\sigma}$ is $G$-invariant. We know from (5.7) that, if the multiplicity cannot be reduced, then

$$
\sum_{j=1}^{d_\sigma} Q_{j,j+d_\sigma} = \sum_{j=1}^{d_\sigma} \nabla \phi_j^1 \cdot \nabla \phi_j^2 - \lambda_0 \phi_j^1 \phi_j^2 = 0.
$$

\footnote{One can obtain the expression of the matrix of the second derivative without appealing to corollary 1 since, supposing the non separability of the eigenvalues it is legitimate to take derivatives directly from the expression for the first derivative.}
From the $G$ invariance of $\sum_{j=1}^{d_\sigma} Q_{j,j+d_\sigma}$, it follows that $\sum_{j=1}^{d_\sigma} \frac{\partial}{\partial N} Q_{j,j+d_\sigma}$ is $G$-invariant. From the symmetry of $\hat{M}_{j,k}$, we obtain

$$\int_{\partial \Omega} 2\sigma \hat{Q}_{jk} = \int_{\partial \Omega} \sigma (\hat{Q}_{jk} + \hat{Q}_{kj}).$$

Therefore, to show that the integrand of the expression $\sum_{j=1}^{d_\sigma} \hat{M}_{j,j+d_\sigma}$ is also $G$-invariant it is enough to show that $\sum_{j=1}^{d_\sigma} \hat{Q}_{jj+d_\sigma} + \hat{Q}_{j+d_\sigma,j}$ is $G$-invariant. This follows from the fact that $t \rightarrow \sum_{j=1}^{d_\sigma} \hat{M}_{j,j+d_\sigma} (t)$ is a $C^1$ curve in the space of $G$-invariant functions.

From the non separability of the eigenvalues, it follows that $\sum_{j=1}^{d_\sigma} \hat{M}_{j,j+d_\sigma} = 0$ for any $G$-invariant $\sigma$ and, therefore

$$(5.9) \quad \sum_{j=1}^{d_\sigma} \hat{Q}_{jj+d_\sigma} + \hat{Q}_{j+d_\sigma,j} + \sigma \frac{\partial}{\partial N} \hat{Q}_{j,j+d_\sigma} = 0.$$  

To simplify the notation, we introduce the bilinear form $Q(u,v) = \nabla v \cdot \nabla u - \lambda_0 \nu u$. Then (5.9) can be rewritten as

$$(5.10) \quad \sum_{j=1}^{d_\sigma} \sigma \frac{\partial}{\partial N} Q(\phi_j^1, \phi_j^2) + Q(\phi_j^1, \phi_j^2) + Q(\phi_j^1, \phi_j^2) = \sum_{j=1}^{d_\sigma} \lambda (\phi_j^1 \phi_j^2).$$

The solutions $\phi_j^1 = \hat{\phi}_{(i-1)d_\sigma+j}$ of (5.8) as functions of $\sigma$ define a boundary operator which we denote by $C_j^1(\sigma)$. Then, equation (5.10) defines a boundary operator given by

$$(5.11) \quad \Xi(\sigma) = \sum_{j=1}^{d_\sigma} \sigma \frac{\partial}{\partial N} Q(\phi_j^1, \phi_j^2) + Q(\phi_j^1, C_j^1(\sigma)) + Q(\phi_j^2, C_j^1(\sigma))$$

where $\sigma$ is a $G$-invariant function on $\partial \Omega$. From (5.11), it follows that the operator $\Xi$ is of finite range. A necessary condition for this (Theorem 13) is that

$$\sum_{j=1}^{d_\sigma} \frac{\partial \phi_j^1}{\partial \tau} \frac{\partial \phi_j^2}{\partial \tau} = 0$$

for any $x \in \partial \Omega$ and $\tau \in T_x \partial \Omega$.

We can repeat the whole process substituting $\phi_j^1$ by $\phi_j^1 \circ g$. Looking at this relation as the inner product of vectors $v_1 = (\frac{\partial \phi_j^1}{\partial \tau}, \ldots, \frac{\partial \phi_j^1}{\partial \tau})$, $v_2 = (\frac{\partial \phi_j^2}{\partial \tau}, \ldots, \frac{\partial \phi_j^2}{\partial \tau})$ in $R^{d_\sigma}$, we have that $\langle A_\sigma (g) v_1, v_2 \rangle = 0$ for all $g \in G$. Since $A_\sigma$ are irreducible representation of the $G$, we have $\frac{\partial \phi_j^1}{\partial \tau} = 0$ for all $x \in \partial \Omega$ and $\tau \in T_x \partial \Omega$. It follows that $\nabla \phi_j^1 = 0$ on $\partial \Omega$. Therefore, using (5.11), we obtain $\sum_{j=1}^{d_\sigma} \phi_j^1 \phi_j^2 = 0$ on $\partial \Omega$. The process can be repeated again with $\phi_j^1 \circ g$ in the place of $\phi_j^1$ to obtain $\phi_j^2 = 0$ on $\partial \Omega$. Since $\phi_j^2$ also satisfies $\frac{\partial \phi_j^2}{\partial N} = 0$ on $\partial \Omega$, Cauchy Uniqueness Theorem assures that $\phi_j^2 \equiv 0$ on $\Omega$, which gives the desired contradiction.

\begin{corollary}
Let $G$ be a finite subgroup of $O(n)$ and $\sigma \in \hat{G}$. Then the set

$$C = \{ h \in Diff^2_G(\Omega) \mid \text{the eigenvalues of the problem (1.7) restricted to } M_\sigma \text{ are all } G_\sigma \text{-simple} \}$$

is residual in $Diff^2_G(\Omega)$.  

Proof. Let

$$C_k = \{ h \in Diff^2_G(\Omega) \mid \text{the eigenvalues of the problem (1.7) restricted to } M_\sigma \text{ are all } G_\sigma \text{-simple} \}$$

We prove that $C_k$ is open and dense and then take intersection for $k \in \mathbb{N}$. To prove openness it is enough to observe that the proof of continuity property of the eigenvalues given in Theorem 14 can be easily adapted to show the same properties for the problem restricted to each symmetry space $M_\sigma$. For the density part, we can assume more smoothness and then use Theorem 10 above.

\end{corollary}

We now consider the second part of conjecture [1] for finite groups. For this step, which involves the separation of eigenvalues in different spaces of symmetry we will need an additional hypotheses on the dimension of irreducible representations of $G$. We start with a technical auxiliary result.

\begin{lemma}
Let $M$ be a differentiable manifold and $F,G : M \rightarrow \mathbb{R}^2$ differentiable functions. If $|F(x)| = |G(x)|$ and $|\frac{\partial F}{\partial \tau}| = |\frac{\partial G}{\partial \tau}|$ for any $\tau \in T_x M$, then there exists an open set $V$ in $M$ and an orthogonal transformation $T$ in $\mathbb{R}^2$ such that $F(x) = TG(x)$ in $V$.

Proof. Using complex notation, we have $F(x) = e^{i\theta(x)}G(x)$. If $\theta(x)$ is constant in some open set, we are done. Suppose then that $\nabla F(x)$ does not vanish identically in any open subset of $M$. Choosing local coordinates $(x_1, \ldots, x_{n-1})$ in $M$ and $\tau = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$, it follows from the condition $|\frac{\partial E}{\partial \tau}| = |\frac{\partial G}{\partial \tau}|$ that

$|\frac{\partial F}{\partial \tau}| = |\frac{\partial G}{\partial \tau}|$ that
Therefore, the orthogonal transformation 

\[ T \]

is given by 

\[ T = \begin{bmatrix} \cos C & -\sin C \\ \sin C & \cos C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

**Theorem 11.** Let \( G \) be a finite subgroup of \( O(n) \) such that \( d_\sigma \leq 2 \) for any \( \sigma \in \hat{G} \) and \( \Omega \subset \mathbb{R}^n \) an open bounded connected \( C^3 \)-regular and \( G \)-symmetric domain. Suppose \( \lambda \) is the unique eigenvalue for the problem (5.11) restricted to the symmetry spaces \( M_{\sigma_1} \) and \( M_{\sigma_2} \) in the interval \( (\lambda - \delta, \lambda + \delta) \). Suppose also that the action of \( G \) in both \( \text{ker}(\Delta_{|M_{\sigma_1}} + \lambda) \) and \( \text{ker}(\Delta_{|M_{\sigma_2}} + \lambda) \) is irreducible. Then, for any \( \epsilon > 0 \), there exists \( h \in \text{Diff}_C^3(\Omega), \|h - i\sigma\|_{C^3} < \epsilon \) and \( \delta > 0 \) such that there are exactly two \( \lambda_1(h), \lambda_2(h) \) \( G \)-simple eigenvalues for the problem (5.11) restricted to the space \( M_{\sigma_1} \oplus M_{\sigma_2} \) in the interval \( (\lambda - \delta, \lambda + \delta) \). In other words, the natural action of \( G \) on \( \text{ker}(h^*\Delta h^{-1}|_{M_{\sigma_2} \oplus M_{\sigma_1}} + \lambda_1(h)) \) and \( \text{ker}(h^*\Delta h^{-1}|_{M_{\sigma_2} \oplus M_{\sigma_1}} + \lambda_2(h)) \) is irreducible.
Proof. Assume that the eigenvalue $\lambda$ cannot be separated by small $G$-symmetric perturbations. Then the matrix of the first derivatives $M_i$ given by the Corollary\textsuperscript{11} must be a multiple of the identity. Thus

$$(5.16) \quad \int_{\partial \Omega} \sigma (|\nabla \varphi_k|^2 - \lambda_0 \varphi_k^2) = \int_{\partial \Omega} \sigma (|\nabla \varphi|^2 - \lambda_0 \varphi^2)$$

where $\varphi_j = \phi_j^1$ if $1 \leq j \leq d_{\sigma_1}$, $\varphi_j = \phi_{j-d_{\sigma_1}}^2$ if $d_{\sigma_1} + 1 \leq j \leq d_{\sigma_1} + d_{\sigma_2}$, and the eigenfunctions $\{\phi_j^1\}_{j=1}^{d_{\sigma_1}}$ and $\{\phi_j^2\}_{j=1}^{d_{\sigma_2}}$ satisfy (6.4).

As in the proof of Theorem\textsuperscript{10} we build the $G$-invariant functions

$$\sum_{j=1}^{d_{\sigma_1}} |\nabla \varphi_j|^2 - \lambda(\varphi_j)^2 = \sum_{j=1}^{d_{\sigma_1}} |\nabla \phi_j^1|^2 - \lambda(\phi_j^1)^2,$$

$$\sum_{j=1+d_{\sigma_1}}^{d_{\sigma_1} + d_{\sigma_2}} |\nabla \varphi_j|^2 - \lambda(\varphi_j)^2 = \sum_{j=1}^{d_{\sigma_2}} |\nabla \phi_j^2|^2 - \lambda(\phi_j^2)^2.$$

It then follows from (5.10) that

$$(5.17) \quad \frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} |\nabla \phi_j^1|^2 - \lambda(\phi_j^1)^2 = \frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} |\nabla \phi_j^2|^2 - \lambda(\phi_j^2)^2.$$

Since we still cannot find a contradiction, we proceed by computing the second derivative. Arguing as in Theorem\textsuperscript{10} we conclude that the boundary operator

$$(5.18) \quad \Phi(\sigma) = \frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} \sigma \frac{\partial}{\partial N} Q(\phi_j^1, \phi_j^2) - 2(Q(\phi_j^1, C_1^1(\sigma))$$

is of finite range. It follows from Theorem\textsuperscript{11} that

$$(5.19) \quad \frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} \left( \frac{\partial \phi_j^1}{\partial \tau} \right)^2 = \frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} \left( \frac{\partial \phi_j^2}{\partial \tau} \right)^2$$

for any $\tau \in T_x(\partial \Omega)$. Thus $\frac{1}{d_{\sigma_1}} |\nabla \phi_j|^2 = \frac{1}{d_{\sigma_2}} |\nabla \phi_j|^2$.

Using (5.11), it follows that

$$(5.20) \quad \frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} (\phi_j^1)^2 = \frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} (\phi_j^2)^2.$$

Now, if $d_{\sigma_1} = 2$ for $i = 1, 2$, define

$$F(x) = (\phi_1, \ldots, \phi_{d_{\sigma_1}}) = (\phi_1^1, \phi_2^2),$$

$$G(x) = (\phi_1, \ldots, \phi_{d_{\sigma_2}}) = (\phi_1^2, \phi_2^1).$$

If one of the $d_{\sigma_i}$ is equal to 1 we just put the two coordinates equal to $\phi_1^1$.

It follows from (5.20) and (5.19) that $|F| = |G|$ e $|\frac{\partial F}{\partial \tau}| = |\frac{\partial G}{\partial \tau}|$, and then, from Lemma\textsuperscript{8} there is an orthogonal transformation $T$ such that $F(x) = TG(x)$ in an open set $V$ on $\partial \Omega$. Thus we have, in particular $\phi_1^1 = \alpha \phi_1^2 + \beta \phi_2^2$ on $V$ and

$$\begin{cases} (\Delta + \lambda)(\phi_1^1 - \alpha \phi_1^2 - \beta \phi_2^2) = 0 & \text{in } \Omega; \\ \frac{\partial}{\partial N}(\phi_1^1 - \alpha \phi_1^2 - \beta \phi_2^2) = 0 & \text{on } \partial \Omega; \\ \phi_1^1 - \alpha \phi_1^2 - \beta \phi_2^2 = 0 & \text{on } V \cap \partial \Omega. \end{cases}$$

From Cauchy uniqueness theorem\textsuperscript{9} $\phi_1^1 = \alpha \phi_1^2 + \beta \phi_2^2$ in $\Omega$, which is a contradiction since $M_{d_{\sigma_1}} \cap M_{d_{\sigma_2}} = 0$.

\[ \square \]

**Corollary 6.** If $G$ is a finite subgroup of $O(n)$ such that $d_\sigma \leq 2$ for all $\sigma \in \tilde{G}$, then, for a residual set set of open bounded connected $C^3$-regular and $G$-symmetric domains the eigenvalues of the problem (1.7) are all $G$-simple.

Proof. Let $C_k = \{ h \in Diff^1_0(\Omega) \mid \text{all eigenvalues } \lambda \text{ of } (1.1) \text{ with } \lambda < k \text{ are all } G \text{- simple} \}$.

It is enough to prove that $C_k$ is open and dense. The proof is completely analogous to the one of Corollary\textsuperscript{5} using Theorem\textsuperscript{11} instead of Theorem\textsuperscript{10}.

\[ \square \]
Remark 3. The results above give a complete answer in the particular case of compact subgroups of the $O(2)$. In fact, in this case, the irreducible representations must have dimension at most 2. This is well known, and also follows from corollary 4 since the eigenvalues of the Neumann Laplacian (for example in the disk of case, the irreducible representations must have dimension at most 2. This is well known, and also follows from corollary 2, since the case of finite groups. In the infinite case, the only invariant subgroups are $SO(2)$ and $O(2)$ itself. But then the only invariant regions are the disks, for which the result is well known.

The next result shows that the eigenvalues associated to subspaces $M_\sigma$ with $d_\sigma = 1$ are generically simple, that is, they can be separated from the eigenvalues in other symmetry spaces. In particular, generically in the set of $G$-symmetric regions there is an infinite number of simple eigenvalues for the Neumann Laplacian.

Theorem 12. Let $G$ be a finite subgroup of $O(n)$ and $\Omega \subset \mathbb{R}^n$ an open bounded connected $C^3$-regular and $G$-symmetric domain. Suppose that $d_\sigma_1 = 1$ and $\lambda$ is the unique eigenvalue for the problem (1.1) restricted to the symmetry spaces $M_{\sigma_1}$ and $M_{\sigma_2}$ in the interval $(\lambda - \delta, \lambda + \delta)$. Suppose also that the action of $G$ in both $\ker(\Delta|_{M_{\sigma_1}} + \lambda)$ and $\ker(\Delta|_{M_{\sigma_2}} + \lambda)$ is irreducible. Then $\lambda$ can be separated by small $G$-symmetric perturbations of $\Omega$ in two eigenvalues one of which is simple. More precisely, for any $\epsilon > 0$, there exists $h \in \text{Diff}_G^3(\Omega)$, $||h - \text{id}||_{C^3} < \epsilon$ and $\delta > 0$ such that there are exactly two eigenvalues $\lambda_1(h), \lambda_2(h)$ for the problem (5.1) restricted to the space $M_{\sigma_1} \oplus M_{\sigma_2}$ in the interval $(\lambda - \delta, \lambda + \delta)$, with $\lambda_1(h)$ simple. In other words, the natural action of $G$ on $\ker(h^*\Delta h^{*-1}|_{M_{\sigma_2}\oplus M_{\sigma_1}} + \lambda_1(h))$ and $\ker(h^*\Delta h^{*-1}|_{M_{\sigma_2}\oplus M_{\sigma_1}} + \lambda_2(h))$ is irreducible.

Proof. Assuming that the eigenvalues cannot be separated and following the arguments in the proof of Theorem 11, we obtain the functions in $\mathbb{R}^{d_{\sigma_2}}$

\[ F(x) = \phi_1^1(1, \ldots, 1) \]
\[ G(x) = (\phi_1^2, \ldots, \phi_{d_{\sigma_2}}^2) \]

satisfying the relations

\[ (G(x), G(x)) = (F(x), F(x)) = d_{\sigma_2}(\phi_1^1)^2 \]

and

\[ \left\langle \frac{\partial G}{\partial \tau}(x), \frac{\partial G}{\partial \tau}(x) \right\rangle = \left\langle \frac{\partial F}{\partial \tau}(x), \frac{\partial F}{\partial \tau}(x) \right\rangle = d_{\sigma_2} \left( \frac{\partial \phi_1^1}{\partial \tau} \right)^2, \]

for any $x \in \partial \Omega$, and $\tau \in T_x(\partial \Omega)$ Denoting $(1, 1, \ldots, 1) = 1$, we can write

\[ F(x) = \phi_1^1 A(x) 1, \]

where $A(x)$ is an orthogonal linear transformation $F$. Differentiating, we obtain

\[ \frac{\partial F}{\partial x_i} = \frac{\partial \phi_1^1}{\partial x_i} A(x) 1 + \phi_1^1 \frac{\partial}{\partial x_i} A(x) 1 \]

It follows from (5.21) that

\[ 2 \frac{\partial \phi_1^1}{\partial x_i} \phi_1^1 \left\langle A(x) 1, \frac{\partial}{\partial x_i} A(x) 1 \right\rangle + (\phi_1^1)^2 \left| \frac{\partial}{\partial x_i} A(x) 1 \right|^2 = 0. \]

Note that, since \( \left\langle A(x) 1, A(x) 1 \right\rangle = \left\langle 1, 1 \right\rangle \), it follows that

\[ (\phi_1^1)^2 \left| \frac{\partial}{\partial x_i} A(x) 1 \right|^2 = 0, \]

for $i = 1, 2, \ldots, n - 1$. Since $\phi_1^1 \neq 0$ in a dense set of $\partial \Omega$, it follows that $\nabla_{\partial \Omega}(A(x) 1) = 0$ and, therefore $A(x) 1$ is constant $\partial \Omega$. This implies that $\phi_1^1 = a_1 \phi_1^2$ on $\partial \Omega$ which cannot occur, since $\phi_1^1 \notin M_{\sigma_2}$.

Corollary 7. Suppose that $G$ is a finite subgroup of $O(n)$ and $d_\sigma = 1$. Then, for a residual set set of open bounded connected $C^3$-regular, $G$-symmetric domains the eigenvalues of the problem (1.1) in the symmetry space $M_\sigma$ are simple.

Proof. Let

\[ C = \{ h \in \text{Diff}_G^3(\Omega) \mid \text{the eigenvalues of the problem (1.1) with eigenfunctions in } M_\sigma \text{ are all } G_\sigma \text{-simple} \} \]

Openness follows from Theorem 4 and density from Theorem 12 above.

\[ \square \]

\[ ^2 \text{It is important to observe that from the fact the the action of } G \text{ on } \ker(\Delta|_{M_{\sigma_1}} + \lambda) \text{ is simple it does not follow that the action in } \ker(\Delta + \lambda) \text{ is also simple.} \]
We show here how the “Method of rapidly oscillating functions”, developed in [3] can be used to obtain necessary conditions for the operators $\Xi$, and $\Phi$, defined in (5.11), and (5.18) to be of finite range. We start with an auxiliary result.

**Lemma 4.** Suppose $S$ is a $C^1$ manifold; $A$ and $B \in L^2(S)$ with compact support; $\theta$ is a $C^1$ real valued function on $S$ with $\nabla \partial \theta \neq 0$ in the union of the supports of $A$ and $B$; $E$ is a finite dimensional subspace of $L^2(S)$ and $u(\omega) \in E$ for all large $\omega \in \mathbb{R}$ satisfying

$$u(\omega) = A \cos(\omega \theta) + B \sin(\omega \theta) + o(1) \text{ in } L^2(S)$$

as $\omega \to \infty$. Then $A = B = 0$.

**Proof.** See [6].

We do the computations in detail for the operator $\Xi$; the computations for $\Phi$ are completely analogous. Recall that $\Xi$ was defined in (5.11) by

$$\Xi(\gamma \cos(\omega \theta)) = \omega^\gamma \cos(\omega \theta) \sum_{j=1}^{d_x} \frac{\partial \phi_j^1}{\partial \theta} \frac{\partial \phi_j^2}{\partial \theta} + O(\omega)$$

as $\omega \to \infty$. Here $\frac{\partial}{\partial \theta} = \nabla \partial \theta \cdot \nabla \theta$ is the derivative in the direction of $\nabla \partial \theta$. If $\Xi$ is supposed to be of finite rank, we conclude from Lemma 4 that

$$\sum_{j=1}^{d_x} \frac{\partial \phi_j^1}{\partial \theta} \frac{\partial \phi_j^2}{\partial \theta} = 0 \text{ on } \partial \Omega.$$

Following the method presented in [6] we search first formal solutions $u = e^{i\omega S(x)} \sum_{k \geq 0} U_k(x)$ of

$$
\begin{cases}
(\Delta + \lambda)u = (2\omega)F & \text{in } \Omega; \\
\frac{\partial u}{\partial N} = 2\omega G(x) & \text{on } \partial \Omega;
\end{cases}
$$

where $F(x) = e^{i\omega S(x)} \sum_{k \geq 0} \frac{G_k(x)}{(2\omega)^k}$, $G(x) = e^{i\theta(x)} \sum_{k \geq 0} \frac{G_k(x)}{(2\omega)^k}$ is given, $S|_{\partial \Omega} = i\theta$, $Re \frac{\partial S}{\partial \theta}|_{\partial \Omega} > 0$ and $F_k, G_k$ are smooth functions with values in $\mathbb{C}$.

We choose the complex-valued $S$ so $\nabla S \cdot \nabla S = 0$ on a neighborhood of $\partial \Omega$ and the $U_k$ inductively, solving

$$
\begin{cases}
\Lambda U_k + (\Delta + \lambda) U_{k-1} = F_k & \text{in } \Omega; \\
\frac{\partial U_k}{\partial N} + 1 \frac{\partial S}{\partial N} U_k = G_k & \text{on } \partial \Omega;
\end{cases}
$$

with $U_{-1} = 0$, where $\Lambda = \nabla S \cdot \nabla + \frac{1}{2} \Delta S$. They are not ordinarily, exact solutions, but we only need that $\nabla S \cdot \nabla S$ and the $\Lambda U_k + (\Delta + \lambda) U_{k-1} - F_k$ tend to zero rapidly as $x \to \partial \Omega$, which is shown in [6] (for the Dirichlet case, but the argument also applies here).

Using the notation above, we have

$$C_j^i(\sigma) = e^{i\omega \theta} U_0^{i,j} + O(1).$$

Thus

$$\nabla \phi_j^1 \cdot \nabla - \lambda \phi_j^1)C_j^2(\sigma) = \nabla \partial \theta \phi_j^1 \cdot \nabla \theta (e^{i\omega \theta} U_0^{2,j} - \lambda e^{i\omega \theta} U_0^{2,j}) - \lambda e^{i\omega \theta} U_0^{2,j} \phi_j^1

= e^{i\omega \theta} \left\{ \frac{\partial \phi_j^1}{\partial \theta} U_0^{2,j} \omega + Q(\phi_j^1, U_0^{2,j}) \right\}

= ie^{i\omega \theta} \frac{\partial \phi_j^1}{\partial \theta} U_0^{2,j} \omega + O(1),$$

that is

$$Q(\phi_j^1, C_j^2(\sigma)) = i\omega e^{i\omega \theta} \frac{\partial \phi_j^1}{\partial \theta} U_0^{2,j} \omega + O(1).$$
Analogously
\[
Q(\phi_j^i, C_j^1(\sigma)) = i\omega e^{i\omega\theta} \frac{\partial \phi^2_j}{\partial \theta} U_{0}^{1,j}\omega + O(1).
\]
Therefore
\[
\Xi(\gamma e^{i\omega\theta}) = e^{i\omega\theta} \left\{ \sum_{j=1}^{d_x} -i\omega \left( \frac{\partial \phi^1_j}{\partial \theta} U_{0}^{2,j} + \frac{\partial \phi^2_j}{\partial \theta} U_{0}^{1,j} \right) + \gamma \frac{\partial}{\partial N} Q(\phi^1_j, \phi^2_j) \right\}.
\]

We want to determine the term \(U_{0}^{1,j}\) in the formal solution. To this end, using the notation \(M_j^i(\sigma) = \nabla_{\partial \Omega} \phi^i_j \cdot \nabla_{\partial \Omega} \sigma - \sigma \frac{\partial^2 \phi^i_j}{\partial N^2}\), we write
\[
M_j^i(\gamma e^{i\omega\theta}) = 2\omega e^{i\omega\theta} \sum_{k \geq 0} G_k \frac{2\omega^k}{(2\omega)^k}.
\]
We have
\[
\Xi(\gamma e^{i\omega\theta}) = \sum_{j=1}^{d_x} \frac{1}{2} \frac{\partial \phi^j_i}{\partial \theta} + \frac{1}{2\omega} M_j^i(\gamma),
\]
and then \(U_{0}^{1,j} = i\gamma e^{i\omega\theta} \frac{1}{4} \frac{\partial \phi^j_i}{\partial \theta} \). Therefore
\[
\Xi(\gamma e^{i\omega\theta}) = \gamma e^{i\omega\theta} \omega \sum_{j=1}^{d_x} \frac{\partial \phi^1_j}{\partial \theta} \frac{\partial \phi^2_j}{\partial \theta} + O(1).
\]
Observing that
\[
\Xi(\gamma \cos(\omega \theta)) = \frac{1}{2} Re \left\{ \Xi(\gamma e^{i\omega\theta}) + \Xi(\gamma e^{-i\omega\theta}) \right\},
\]
it follows that
\[
\Xi(\gamma \cos(\omega \theta)) = \omega \gamma \cos(\omega \theta) \sum_{j=1}^{d_x} \frac{\partial \phi^1_j}{\partial \theta} \frac{\partial \phi^2_j}{\partial \theta} + O(1).
\]
If \(\Xi\) is of finite range we obtain, from Lemma 1
\[
\sum_{j=1}^{d_x} \frac{\partial \phi^1_j}{\partial \theta} \frac{\partial \phi^2_j}{\partial \theta} = 0 \text{ em } \partial \Omega.
\]

**Theorem 13.** Let \(G\) be a compact subgroup of \(O(n)\); \(\Omega\) an open, bounded, connected \(C^1\)-regular and \(e\) \(G\)-symmetric region. Suppose the natural action of \(G\) on \(\partial \Omega\) has a free point \(x\) and \(\{\phi^i_j\}_{j=1}^{d_x}, i = 1, 2\) are eigenfunctions for the problem \((1.1)\) belonging to the symmetry space \(M_\sigma\), satisfying
\[
\sum_{j=1}^{d_x} Q(\phi^i_j, \phi^j_i) = 0
\]
on \(\partial \Omega\), where \(Q\) was given in \((6.2)\). If the operator \(\Xi\) given in \((6.7)\) is of finite range, then
\[
\sum_{j=1}^{d_x} \frac{\partial \phi^1_j}{\partial \tau} \frac{\partial \phi^2_j}{\partial \tau} = 0
\]
in a neighborhood \(V\) of \(x\) in \(\partial \Omega\), for all \(\tau \perp T_x(G(x))\). In particular, if \(G\) is finite, this is true for any \(\tau \in T_x(\partial \Omega)\).

**Proof.** Taking \((4)\) into account, it remains only to show that \(\nabla \theta\) can be any chosen to be any unit vector \(\tau \perp T_x(G(x))\). But this is guaranteed by Lemma 10.3 of \([17]\).
Similar arguments lead to similar results for the operator $\Phi$ defined in (6.18).

$$
\Phi(\sigma) = \frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} \sigma \frac{\partial}{\partial N} Q(\phi_j^1, \phi_j^1) - 2(Q(\phi_j^1, C_j^1(\sigma)))
$$

(6.3)

$$
-\frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} \sigma \frac{\partial}{\partial N} Q(\phi_j^2, \phi_j^2) - 2(Q(\phi_j^2, C_j^2(\sigma))).
$$

\textbf{Theorem 14.} Let $G$ be a compact subgroup of $O(n)$; $\Omega$ an open, bounded, connected $C^3$-regular and $e^G$-symmetric region. Suppose the natural action of $G$ on $\partial \Omega$ has a free point $x$ and $\{\phi_j^i\}_{j=1}^{d_i}, i = 1, 2$ are eigenfunctions for the problem (1.1) belonging to the symmetry space $M_\sigma$, satisfying

$$
\frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} Q(\phi_j^1, \phi_j^1) = \frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} Q(\phi_j^2, \phi_j^2)
$$

on $\partial \Omega$, where $Q$ was given (6.2). If the operator $\Phi$ given in (6.3) is of finite range, then

$$
\frac{1}{d_{\sigma_1}} \sum_{j=1}^{d_{\sigma_1}} \left( \frac{\partial \phi_j^1}{\partial \tau} \right)^2 = \frac{1}{d_{\sigma_2}} \sum_{j=1}^{d_{\sigma_2}} \left( \frac{\partial \phi_j^2}{\partial \tau} \right)^2
$$

in a neighborhood $V$ of $x$ in $\partial \Omega$, for all $\tau \perp T_x(G(x))$. In particular, if $G$ is finite this is true for any $\tau \in T_x(\partial \Omega)$.

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