Abstract: We study globularily generated double categories. The condition of a double category being globularily generated is a finiteness condition generalizing the condition of a double category being trivial. We establish analogies between the way trivial double categories and globularily generated double categories relate to general double categories. We organize globularily generated double categories into a 2-category and we prove, among other things, that 2-category of globularily generated double categories is a strictly 2-reflective sub 2-category of an appropriate sub 2-category of 2-category of double categories.

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1 Introduction

There exists, in the mathematical literature, a variety of competing ideas of what a higher order categorical structure should be. This set of ideas reduces, in the case of categorical structures of order 2, to the concepts of bicategory and double category, both types of structures introduced by Ehresmann, in [3] and [4]. These two concepts are related in different ways. The most obvious being that every bicategory can be considered as a ‘trivial’ double category. Less obviously, every double category admits an adjacent trivial double category, its horizontal bicategory. These relations admit first order categorical extensions. These relations can be summed up by the following statement: Category adjacent to 3-category of bicategories, bifunctors, lax natural transformations, and deformations is a reflective subcategory of category adjacent to 2-category of double categories, double functors, and
double natural transformations, with horizontalization functor as reflector. Second order extensions to this statement are studied in [7].

We study globularily generated double categories. We regard the condition of a double category being globularily generated as the condition of double category being 'almost trivial.' Our aim is to formally articulate this idea and to establish analogies between the way trivial double categories and globularily generated double categories relate to the general concept of double category. More precisely, we organize globularily generated double categories into a sub 2-category of 2-category of double categories, and we prove this sub 2-category is 2-reflective. We explicitly construct a 2-reflector which we regard as a globularily generated analog of horizontalization functor defined in the trivial case [7]. We call this 2-functor the globularily generated piece 2-functor. We compute the globularily generated piece of classic examples of double categories. We now sketch the contents of this paper.

In section 2 we recall some of the basic concepts related to the theory of double categories. We present relevant examples and set notation used in the rest of the paper. In section 3 we define and study the concept of globularily generated double category. We establish the technical framework needed for results in subsequent sections. We construct the vertical filtration of category of morphisms of a globularily generated double category and we define the notion of vertical length of a globularily generated 2-morphism. With the aid of these concepts we establish technical results describing relations between different types of 2-morphisms in globularily generated double categories. In section 4 we define and study the concept of globularily generated piece of a double category. We extend this construction to a 2-categorical setting. We define the globularily generated piece 2-functor and the diagram of vertical functors. We generalize the fact that sequence of vertical categories of a double category defines a filtration of its category of morphisms to a categorical setting by proving that morphism category of globularily generated piece 2-functor is the limit of sequence of vertical functors. Moreover, we prove that globularily generated piece 2-functor is a strict 2-reflector, thus generalizing the relation between bicategories and double categories established by horizontalization functor. Finally, in sections 5 we perform computations of the globularily generated piece of double categories presented in section 2. Precisely, we compute globularily generated piece of double category of algebras and bimodules and we compute globularily generated piece of double category of closed manifolds and cobordisms.

The exposition of the paper will be elementary. We assume nevertheless that the reader is familiar with the basic notions of the theory of bicategories,
bifunctors, and transformations. We refer the reader to [5] for basic notation and definitions. We follow notational conventions in the theory of double categories for the most part. We refer the reader to [7] for a survey on notational conventions in the theory of double categories.

2 Double categories

In this first section we establish the theoretical and notational framework needed for the rest of the paper. We recall the concepts of double category, double functor, and double natural transformation. We present relevant examples and set notational conventions for subsequent sections.

Definition 2.1. We understand, for a double category, a category weakly internal to 2-category \( \text{Cat} \) of categories, functors, and natural transformations. More precisely, a double category \( C \) consists of the following data:

1. **Objects and morphisms**: Categories \( C_0, C_1 \). We call \( C_0 \) the category of objects of \( C \) and \( C_1 \) the category of morphism of \( C \).

2. **Source and target**: Functors \( s, t : C_1 \rightarrow C_0 \). We call \( s \) and \( t \) the source and target functors of \( C \) respectively.

3. **Horizontal identity**: Functor \( i : C_0 \rightarrow C_1 \). We call \( i \) the horizontal identity functor of \( C \).

4. **Horizontal composition**: Bifunctor \( * : C_1 \times C_0 C_1 \rightarrow C_1 \), where fibration in \( C_1 \times C_0 C_1 \) is taken with respect to pair \( s, t \). We call \( * \) the horizontal composition bifunctor of \( C \).

5. **Identity transformations**: Natural isomorphisms \( \lambda : * (i \times \text{id}_{C_1}) \rightarrow \text{id}_{C_1} \) and \( \rho : * (\text{id}_{C_1} \times i t) \rightarrow \text{id}_{C_1} \). We call \( \lambda \) and \( \rho \) the left and right identity transformations of \( C \) respectively.

6. **Associator**: Natural isomorphism \( \xi : *(\star \times \text{id}_{C_1}) \rightarrow *(\text{id}_{C_1} \times \star) \). We call \( \xi \) the associator of \( C \).

We require left and right identity transformations and associator of \( C \) to be related by McLane’s triangular and pentagonal axioms [6]. Moreover, we require that source and target of each component of left and right identity transformations and associator of \( C \) be identity endomorphisms in \( C_0 \).
We call objects and morphisms of category of objects $C_0$ of a double category $C$ objects and vertical morphisms of $C$ respectively. We call objects and morphisms of category of morphisms of double category $C$ horizontal morphisms and 2-morphisms of $C$ respectively. We write $\circ$ for composition in category of morphisms $C_1$ of $C$ and we call $\circ$ the vertical composition operation in $C$. We call the image, under source and target functors $s$ and $t$ of $C$, of any 2-morphism in $C$, its source and its target respectively. We will call the image, under horizontal identity functor $i$ of $C$, of any object or any vertical morphism of $C$, its horizontal identity. Finally, we will call the image, under horizontal composition bifunctor $*$ of $C$, of a compatible pair of horizontal or 2-morphisms the horizontal composition of the pair. The existence of left and right identity transformations for $C$ can be interpreted by saying that horizontal identity functor $i$ of $C$ acts as a left and right identity for horizontal composition up to natural isomorphisms. The existence of associator of $C$ can be interpreted by saying that horizontal composition in $C$ is associative up to a natural isomorphism. We say that a 2-morphism in double category $C$ is globular if its source and target are vertical identity endomorphisms. The last condition in the definition of a double category can be interpreted by saying that components of left and right identity transformations and of associator of a double category are globular. In the case in which horizontal identity transformations and associator of double category $C$ are identity natural transformations we say that double category $C$ is strict. The following are the main examples of double categories that will be used throughout the paper.

**Bicategories:** Let $B$ be a bicategory. Denote by $\overline{B}$ pair formed by discrete category generated by collection of 0-cells $B_0$ of $B$ and category whose collection of objects and whose collection of morphisms are collection of 1-cells of $B$ and collection of 2-cells of $B$ respectively. Denote by $i$ functor generated by function associating, to every 0-cell in $B$ its identity 1-cell in $B$ and denote by $*$ bifunctor generated by horizontal composition of 1- and 2-cells in $B$. With this structure $\overline{B}$ is a double category. Identity transformations and associator in $\overline{B}$ are defined in terms of those defining the structure of bicategory in $B$. Double category $\overline{B}$ is strict if and only if bicategory $B$ is a 2-category. We call double categories arising from bicategories in this way trivial double categories. Observe that every 2-morphism in a trivial double category is globular. Every double category such that all its 2-morphisms are globular is trivial.

**Algebras:** Let $\text{Alg}_0$ denote category whose collection of objects is collection
of complex algebras and whose collection of morphisms is collection of unital algebra morphisms. Given algebras $A, B, C$ and $D$, left-right $A$-$B$ bimodule $M$ and left-right $C$-$D$-bimodule $N$, we say that a triple $(f, \Phi, g)$, where $f$ is a unital algebra morphism from $A$ to $C$, $g$ is a unital algebra morphism from $B$ to $D$ and $\Phi$ is a linear transformation from $M$ to $N$; is an equivariant bimodule morphism from $M$ to $N$; if for every $a \in A$, $b \in B$, and $x \in M$ equation $\Phi(axb) = f(a)\Phi(x)g(b)$ holds. Composition of equivariant bimodule morphisms is performed entry-wise. Let $\text{Alg}_1$ denote category whose collection of objects is collection of bimodules over complex algebras and whose collection of morphisms is collection of equivariant bimodule morphisms. Denote by $\text{Alg}_0$ pair formed by categories $\text{Alg}_0$ and $\text{Alg}_1$. Denote by $i$ functor from $\text{Alg}_0$ to $\text{Alg}_1$, associating algebra $A$ as a left-right $A$-bimodule to every algebra $A$, and associating equivariant bimodule morphism $(f, f, f)$ to every unital algebra morphism $f$. Given algebras $A, B$ and $C$, left $A$-$B$ bimodule $M$ and left-right $B$-$C$ bimodule $N$ write $N \ast M$ for tensor product $M \otimes_{B} N$ relative to $B$. Given equivariant bimodule morphisms $(f, \Phi, g)$ and $(g, \Psi, h)$ we write $(g, \Psi, h) \ast (f, \Phi, g)$ for tensor product $(f, \Phi \otimes \Psi, h)$. Denote by $\ast$ bifunctor, from $\text{Alg}_1 \times \text{Alg}_0$ to $\text{Alg}_1$ defined by operations $\ast$ defined above. This structure provides pair $\text{Alg}$ of categories $\text{Alg}_0$ and $\text{Alg}_1$ with the structure of a double category. Left and right identity transformations and associator in $\text{Alg}$ are defined by those of relative tensor product bifunctor. An analogous structure is defined in [1] where algebras are replaced by von Neumann algebras with finite dimensional center. Denote by $vN^f_0$ category whose objects are von Neumann algebras with finite dimensional center and whose morphisms are finite index morphisms. Denote by $vN^f_1$ category whose objects are bimodules over von Neumann algebras with finite dimensional center and whose morphisms are bounded intertwiners. Denote by $vN^f$ pair formed by $vN^f_0$ and $vN^f_1$. Denote by $i$ and $\ast$ the Haagerup standard form functor and the Connes fusion operation bifunctor. Both functors defined in [1]. In analogy with the case of bicategory $\text{Alg}$ this pair of functors provides pair $vN^f$ with the structure of a bicategory.

Cobordisms: Let $n$ be a positive integer. Let $\text{Cob}(n)_0$ denote category whose collection of objects is collection of closed $n$-dimensional smooth manifolds and whose collection of morphisms is collection of diffeomorphisms between manifolds. Given closed $n$-manifolds $X, Y, Z$ and $W$, a cobordism $M$ from $X$ to $Y$, and a cobordism $N$ from $Z$ to $W$, we will say that a triple $(f, \Phi, g)$, where $f$ is a diffeomorphism from $X$ to $Z$, $g$ is a diffeomorphism from $Y$ to $W$, and where $\Phi$ is a diffeomorphism from $M$ to $N$; is an equiv-
ariant diffeomorphism from $M$ to $N$ if restriction of $\Phi$ to $X$ equals $f$ and restriction of $\Phi$ to $Y$ equals $g$. Composition of equivariant morphisms between cobordisms is performed entry-wise. Let $\text{Cob}(n)_1$ denote category whose collection of objects is collection of cobordisms between closed $n$-dimensional manifolds and whose collection of morphisms is collection of equivariant diffeomorphisms between cobordisms. Given an $n$-dimensional manifold $X$ we write $i_X$ for cobordism $X \times [0,1]$. Given a diffeomorphism $f : X \rightarrow Y$ between closed $n$-dimensional manifolds $X$ and $Y$, we denote by $i_f$ equivariant diffeomorphism $(f, f \times [0,1], f)$ from cobordism $i_X$ to cobordism $i_Y$. These two functions define a functor from $\text{Cob}(n)_0$ to $\text{Cob}(n)_1$. Denote this functor by $i$. Given compatible cobordisms $M$ and $N$ with respect to manifold $X$ we write $N \star M$ for joint union $M \cup_X Y$ with respect to $X$. Finally, given equivariant diffeomorphisms $(f, \Phi, g)$ and $(g, \Psi, h)$, compatible with respect to diffeomorphism $g$ denote by $(g, \Psi, h) \star (f, \Phi, g)$ joint union $(f, \Phi \cup_g \Psi, h)$ of $(f, \Phi, g)$ and $(g, \Psi, h)$ with respect to $g$. The two operations defined above form a bifunctor which we denote by $\star$. With this structure pair $\text{Cob}(n)$ is a double category, where identity transformations and associator come from the obvious diffeomorphisms from cobordism theory.

We now describe how to organize double categories into a 2-category. We begin with the following definition.

**Definition 2.2.** Let $C, D$ be double categories. We understand for a double functor from $C$ to $D$, a functor from $C$ to $D$, internal to 2-category $\text{Cat}$ of categories, functors, and natural transformations. More precisely, a double functor $F : C \rightarrow D$ from $C$ to $D$ consists of the following data:

1. **Components:** Functors $F_0 : C_0 \rightarrow D_0$ and $F_1 : C_1 \rightarrow D_1$. We call $F_0$ the object functor of $F$ and we call $F_1$ the morphism functor of $F$.

2. **Unit transformation:** Natural isomorphism $\mu^F : F_1 i \rightarrow i F_0$. We call $\mu$ the unit transformation of $F$.

3. **Composition:** Natural isomorphism $\nu^G : \star(F_1 \times F_1) \rightarrow F_1 \star$. We call $\nu$ the horizontal composition transformation of $F$.

We require object and morphism functors of $F$ to intertwine source and target functors of $C$ and $D$. We require components of unit and horizontal composition transformations of $F$ to be globular isomorphisms. Finally, we require unit and horizontal composition transformations of $F$ to satisfy McLane’s coherence conditions for a monoidal functor [6].
Composition of double functors is defined component-wise. This composition operation is associative and unital. When unit and horizontal composition transformations of a double functor $F$ are identity natural isomorphisms we say that $F$ is a strict double functor. We now describe cells between double functors.

**Definition 2.3.** Let $C, D$ be double categories. Let $F, G : C \to D$ be double functors from $C$ to $D$. We will understand for a double natural transformation from $F$ to $G$ a natural transformation from $F$ to $G$ internal to 2-category $\textbf{Cat}$ of categories, functors, and natural transformations. More precisely, a double natural transformation $\eta : F \to G$ consists of a pair of natural transformations $\eta_0 : F_0 \to G_0$ and $\eta_1 : F_1 \to G_1$. We call $\eta_0$ the object natural transformation of $\eta$ and $\eta_1$ the morphism natural transformation of $\eta$. We require $\eta_0$ and $\eta_1$ to satisfy the following conditions:

1. **Source and target:** Let $f$ be a horizontal morphism of $C$ with domain and codomain $a$ and $b$ respectively. In that case the following equations hold.

   $$s\eta_f = \eta_x \text{ and } t\eta_f = \eta_y$$

2. **Horizontal identity:** Let $x$ be an object in $C$. In that case the following equation holds.

   $$\eta_i x \mu^F = \mu^G \eta_x$$

3. **Horizontal composition:** Let $f, g$ be a composable pair of horizontal morphisms of $C$. In that case the following equation holds.

   $$\eta_g \ast \eta_f ^G \nu_{gsf} = \nu_{GgsGf}^G(\eta_g \ast \eta_f)$$

Vertical and horizontal compositions of compatible double natural transformations is performed component-wise. With these operations collection of double categories, collection of double natural transformations, and collection of double natural transformations form collection of 0-, 1-, and 2-cells of a 2-category respectively. We denote this 2-category by $d\textbf{Cat}$. 

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3 Globularily generated double categories

In this section we introduce the condition of a double category being globularily generated. We develop the technical tools necessary to obtain results on the structure of globularily generated double categories.

We will say that a pair $D$ of categories $D_0, D_1$ is a sub-double category of a double category $C$ if $D_0$ is a subcategory of object category $C_0$ of $C$, $D_1$ is a subcategory of morphism category $C_1$ of $C$, source and target functors $s, t$ of $C$ restrict to functors from $D_1$ to $D_0$, horizontal identity functor $i$ of $C$ restricts to a functor from $D_0$ to $D_1$, horizontal composition bifunctor $*$ restricts to a bifunctor from $D_1 \times_{D_0} D_1$ to $D_1$, components of left and right identity transformations of $C$, associated to horizontal morphisms in $D_1$, are morphisms in $D_1$, and if components of associator of $C$, associated to composable triples of horizontal morphisms of $C$ in $D_1$, are morphisms in $D_1$. Observe that in this case pair $D$ together with restrictions of structure functors $s, t, i$, and $*$ of $C$, left and right identity natural isomorphisms, and associator of $C$, is itself a double category. We say that a sub-double category $D$ of a double category $C$ is complete, if collections of objects, vertical morphisms, and horizontal morphisms of $D$ are equal to collections of objects, vertical morphisms, and horizontal morphisms of $C$ respectively.

Given a collection of sub-double categories $D_\alpha, \alpha \in A$, of a double category $C$, pair formed by intersection $\cap_{\alpha \in A} D_\alpha$ of object categories of double categories $D_\alpha, \alpha \in A$, and intersection $\cap_{\alpha \in A} D_\alpha$ of morphism categories of double categories $D_\alpha, \alpha \in A$, is again a sub-double category of $C$. Given a collection of 2-morphisms $X$ of double category $C$, we call the intersection of all complete sub-double categories $D$, of $C$, such that collection $X$ is contained in collection of 2-morphisms of $D$, the complete sub-double category of $C$ generated by $X$. We say that a collection of 2-morphisms $X$ of double category $C$ generates $C$ if $C$ is equal to complete sub-double category of $C$ generated by $X$. The following is the main definition of this section.

Definition 3.1. Let $C$ be a double category. We say that $C$ is a globularily generated double category if $C$ is generated by its collection of globular 2-morphisms.

We now proceed to the introduction of our main technical tool in proving results concerning globularily generated double categories, namely, the vertical length of 2-morphisms. We begin by recursively associating, to every globularily generated double category, a filtration of its category of morphisms as follows:
Let $C$ be a globularily generated double category. Denote by $H_1^C$ the union of collection of globular 2-morphisms of $C$ and collection of horizontal identities of vertical morphisms of $C$. Write $V_1^C$ for subcategory of category of morphisms $C_1$ of $C$, generated by collection $H^C_1$, that is, $V_1^C$ denotes subcategory of $C_1$ whose morphisms are vertical compositions of globular 2-morphisms and horizontal identities of $C$. Let $n$ be an integer strictly greater than 1. Suppose category $V_{n-1}^C$ has been defined. We now define category $V_n^C$.

First denote by $H_n^C$ collection of all possible horizontal compositions of 2-morphisms in category $V_{n-1}^C$. We make, in that case, category $V_n^C$ to be subcategory of category of morphisms $C_1$ of $C$, generated by collection $H_n^C$. That is, category $V_n^C$ is subcategory of $C_1$ whose collection of morphisms is collection of vertical compositions of elements of collection $H_n^C$.

We have thus associated, to every double category $C$, a sequence of subcategories $\{V_n^C\}$ of category of morphisms $C_1$ of $C$. We call, for every $n$, category $V_n^C$ the $n$-th vertical category associated to double category $C$. We have used, in the above construction, for every $n$, an auxiliary collection of 2-morphisms $H_n^C$ of $C$. Observe that for each $n$ collection $H_n^C$ both contains the horizontal identity of every vertical morphism in $C$ and is closed under horizontal composition. If double category $C$ is strict, then, for every $n$, collection $H_n^C$ is collection of morphisms of a category whose collection of objects is collection of vertical morphisms of $C$. In that case we call category $H_n^C$ the $n$-th horizontal category associated to double category $C$. By the way sequence of vertical categories $\{V_n^C\}$ associated to $C$ was constructed it is easily seen that for every $n$, inclusions $\text{Hom}V_n^C \subseteq H_{n+1}^C \subseteq \text{Hom}V_{n+1}^C$ hold. This implies that $n$-th vertical category $V_n^C$ associated to double category $C$ is a subcategory of $n + 1$-th vertical category $V_{n+1}^C$ associated to $C$ for every $n$. Moreover, in the case in which double category $C$ is strict, $n$-th horizontal category $H_n^C$ associated to $C$ is a subcategory of $n + 1$-th horizontal category $H_{n+1}^C$ associated to $C$ for every $n$. The following lemma says that sequence of vertical categories of a globularily generated double category forms a filtration of its category of morphisms.

**Lemma 3.2.** Let $C$ be a globularily generated double category. Morphism category $C_1$ of $C$ is equal to the limit $\lim \rightarrow V_n^C$ in category adjacent to $\textbf{Cat}$ of sequence $\{V_n^C\}$ of vertical categories associated to $C$.

**Proof.** Let $C$ be a globularily generated double category. We wish to prove that morphism category $C_1$ of $C$ is equal to limit $\lim \rightarrow V_n^C$ of sequence of vertical categories associated to $C$. 

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By the way sequence of vertical categories associated to $C$ was constructed, it is easily seen that the union $\bigcup_{n=1}^{\infty} \text{Hom}_{V_n}^C$ of sequence of its collections of morphisms is closed under the operations of taking vertical and horizontal compositions in $C$, and that it contains collection of horizontal identities of vertical morphisms of $C$. It follows, from this and from the requirement that components of identity transformations and associator of $C$ are globular, that pair formed by category of objects $C_0$ of $C$ and limit $\varinjlim V_n^C$ of sequence of vertical categories associated to $C$ is a sub-double category of $C$. Collection of objects of $n$-th vertical category $V_n^C$ associated to $C$ is equal to collection of horizontal morphisms of $C$ for every $n$. Thus collection of objects of category $\varinjlim V_n^C$ is equal to collection of horizontal morphisms of $C$. Pair formed by category of objects $C_0$ of $C$ and limit $\varinjlim V_n^C$ of sequence of vertical categories associated to $C$ is thus a complete sub-double category of $C$. Moreover, collection of morphisms $\bigcup_{n=1}^{\infty} \text{Hom}_{V_n}^C$ of category $\varinjlim V_n^C$ contains collection of globular 2-morphisms of $C$. We conclude, from this, and from the fact that double category $C$ is globularily generated, that pair formed by category of objects $C_0$ of $C$ and limit $\varinjlim V_n^C$ of sequence of horizontal categories associated to $C$ is equal to the limit $\varinjlim V_n^C$ of sequence of vertical categories associated to $C$. This concludes the proof.

Given a strict double category $C$ pair $\tau C$ formed by collection of vertical morphisms of $C$ and collection of 2-morphisms of $C$ is a category. Composition operation in pair $\tau C$ is horizontal composition in $C$. We call category $\tau C$ associated to a strict double category $C$ the transversal category associated to $C$.

**Corollary 3.3.** Let $C$ be a globularily generated double category. If $C$ is a strict double category, then transversal category $\tau C$ associated to $C$ is equal to the limit $\varinjlim H_n^C$, in category adjacent to $\text{Cat}$, of sequence of horizontal categories associated to $C$.

**Proof.** Let $C$ be a strict globularily generated double category. We wish to prove, in this case, that transversal category $\tau C$ associated to $C$ is equal to the limit $\varinjlim H_n^C$ of sequence of horizontal categories associated to $C$.

By the way sequence of horizontal categories associated to globularily generated double category $C$ was constructed, it is easily seen that collection of objects of $n$-th horizontal category $H_n^C$ associated to $C$ is equal to collection of vertical morphisms of $C$ for every $n$. It follows, from this, that collection of objects of limit $\varinjlim H_n^C$ of sequence of horizontal categories associated to $C$ is equal to collection of vertical morphisms of $C$ and thus is
equal to collection of objects of transversal category $\tau C$ of $C$. Collection of morphisms of limit $\lim H_n^C$ is equal, to the union $\bigcup_{n=1}^{\infty} \text{Hom}H_n^C$ of collections of morphisms of horizontal categories associated to $C$. This union is equal to union $\bigcup_{n=1}^{\infty} \text{Hom}V_n^C$ of collections of morphisms of vertical categories associated to $C$, which by lemma 3.2 is equal to collection of 2-morphisms of $C$. This concludes the proof.

**Definition 3.4.** Let $C$ be a globularily generated double category. Let $\Phi$ be a 2-morphism in $C$. We call the minimal integer $n$ such that $\Phi$ is a morphism of $n$-th vertical category $V_n^C$ associated to $C$ the vertical length of $\Phi$.

We now apply the concept of vertical length to the proof of results concerning the structure of globularily generated double categories. We first establish some notational conventions.

Assuming a double category $C$ is strict, horizontal composition $\Phi_k \ast \ldots \ast \Phi_1$ of any composable sequence $\Phi_1, \ldots, \Phi_k$ of 2-morphisms in $C$, is unambiguously defined. This is not the case in general. If double category $C$ is not assumed to be strict, then horizontal compositions of a composable sequence $\Phi_1, \ldots, \Phi_k$ of 2-morphisms in $C$, following different parentheses patterns, might yield different 2-morphisms. If 2-morphism $\Phi$ in double category $C$ can be obtained as horizontal composition, following a certain parentheses pattern, of composable sequence of 2-morphisms $\Phi_1, \ldots, \Phi_k$, we will write $\Phi \equiv \Phi_k \ast \ldots \ast \Phi_1$.

Given 2-morphisms $\Phi$ and $\Psi$ in a double category $C$, we say that $\Phi$ and $\Psi$ are globularily equivalent if there exist globular 2-isomorphisms $\Theta_1, \Theta_2$ in $C$ such that equation $\Phi = \Theta_1 \Psi \Theta_2^{-1}$ holds. From the fact that associators in double categories satisfy McLane’s pentagon axiom and from the easy observation that collection of globular 2-morphisms of any double category is closed under the operations of taking vertical and horizontal composition, it follows that if two 2-morphisms $\Phi$ and $\Psi$ satisfy equation $\Phi \equiv \Phi_k \ast \ldots \ast \Phi_1$ for a composable sequence of 2-morphisms $\Phi_1, \ldots, \Phi_k$ in $C$, then $\Phi$ and $\Psi$ are globularily equivalent. Finally, we will say that a 2-morphism $\Phi$ in a double category $C$ is a horizontal endomorphism, if source and target $s\Phi, t\Phi$ of $\Phi$, are equal. Horizontal identities are examples of horizontal endomorphisms.

The next proposition says that a 2-morphism in a globular double category is either globular or a horizontal endomorphism.

**Proposition 3.5.** Let $C$ be a globularily generated double category. Let $\Phi$ be a 2-morphism in $C$. If $\Phi$ is not globular then $\Phi$ is a horizontal endomorphism.

**Proof.** Let $C$ be a globularily generated double category. Let $\Phi$ be a non-
globular 2-morphism in $C$. We wish to prove that $\Phi$ is a horizontal endomorphism.

We proceed by induction on the vertical length of $\Phi$. Suppose first that $\Phi$ is an element of $H_1^C$. In that case, by the assumption that $\Phi$ is non-globular, $\Phi$ must be the horizontal identity of a vertical morphism in $C$ and thus must be a horizontal endomorphism. Suppose now that $\Phi$ is a general element of first vertical category $V_1^C$ associated to $C$. Write $\Phi$ as a vertical composition $\Phi = \Phi_k \circ \ldots \circ \Phi_1$ where $\Phi_i$ is an element of $H_1^C$ for every $k$. Moreover, assume that the length $k$ of this decomposition is minimal. We prove by induction on $k$ that $\Phi$ must be a horizontal endomorphism. Suppose first that $k = 1$. In that case $\Phi$ is an element of $H_1^C$ and thus a horizontal endomorphism. Suppose now that $k$ is strictly greater than 1. Let $\Psi$ be a composition $\Phi_k \circ \ldots \circ \Phi_1$. In this case equation $\Phi = \Psi \circ \Phi_1$ holds. Now, since collection of globular 2-morphisms of $C$ is closed under the operation of taking vertical composition, one of $\Psi$ and $\Phi_1$ is not globular. If both $\Psi$ and $\Phi_1$ are not globular, then by induction hypothesis both $\Phi$ and $\Psi$ are horizontal endomorphisms and thus their vertical composition $\Phi$ is a horizontal endomorphism. Suppose now that $\Psi$ is globular. In that case $\Phi_1$ is a horizontal endomorphism. Now, from the fact that source and target of $\Psi$ are in this case vertical identities and from the fact that source and target are functorial, equations $s\Phi = s\Phi_1$ and $t\Phi = t\Psi_1$ follow and thus $\Phi$ is a horizontal endomorphism. The case in which $\Phi_1$ is globular is handled analogously. This concludes the base of the induction.

Let $n$ be strictly greater than 1. Assume now that every non-globular 2-morphism in $C$ of vertical length strictly less than $n$ is a horizontal endomorphism. Suppose first that $\Phi$ is an element of $H_n^C$. Let $\Phi_k \ast \ldots \ast \Phi_1$ represent a horizontal composition in $C$ such that $\Phi_i$ is an element of $V_{n-1}^C$ for each $k$ and such that $\Phi \equiv \Phi_k \ast \ldots \ast \Phi_1$. Suppose the length $k$ of this decomposition is minimal. We proceed by induction over $k$. If $k = 1$ then $\Phi$ is an element of $V_{n-1}^C$ and is thus horizontal endomorphism by induction hypothesis. Suppose now that $k$ is strictly greater than 1 and that the result is true for every non-globular 2-morphism in $H_n^C$ that can be written as a horizontal composition of strictly less than $k$ 2-morphisms in $V_{n-1}^C$. Choose $\Psi$ such that $\Psi \equiv \Phi_k \ast \ldots \ast \Phi_2$. In this case $\Phi$ and $\Psi \ast \Phi_1$ are globularly equivalent and thus have the same source and target. Now, if both $\Psi$ and $\Phi_1$ are globular, then their horizontal composition, and every 2-morphism globularly equivalent to it, is globular. We thus assume that one of $\Psi$ and $\Phi_1$ is non-globular. If $\Psi$ is globular, then equation $t\Phi_1 = s\Psi$ together with
induction hypothesis implies that $\Phi_1$ is globular. An identical argument implies that if $\Phi_1$ is globular $\Psi$ is globular. We conclude that both $\Psi$ and $\Phi_1$ are non-globular and thus by induction hypothesis are horizontal endomorphisms. This and equation $t\Phi_1 = s\Psi$ implies that $\Psi \ast \Phi_1$ and thus $\Phi$ is a horizontal endomorphism. Assume now that $\Phi$ is a general element of $V^n_C$. Write $\Phi$ as a vertical composition $\Phi_k \circ \ldots \circ \Phi_1$ where $\Phi$ is an element of $H^n_C$ for every $k$. Moreover, assume again that the length $k$ of this decomposition is minimal. An induction argument over $k$ together with an argument analogous to that presented in the base of the induction proves that $\Phi$ is a horizontal endomorphism. This concludes the proof. 

The following corollary follows immediately from the previous proposition.

**Corollary 3.6.** Let $C$ be a globularily generated double category. Let $\Phi$ and $\Psi$ be 2-morphisms in $C$. Suppose $\Phi$ and $\Psi$ are composable. In that case horizontal composition $\Psi \ast \Phi$ is a globular if and only if $\Phi$ and $\Psi$ are both globular.

We conclude this section with the following technical lemma.

**Lemma 3.7.** Let $C$ be a globularily generated double category. Let $\Phi$ be a 2-morphism in $C$. If vertical length of $\Phi$ is equal to 1 then $\Phi$ can be written as a vertical composition of the form

$$\Psi_k \circ \Phi_k \circ \ldots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

where $\Phi_i$ is a horizontal identity for every $1 \leq i \leq k$ and $\Psi_i$ is globular for every $0 \leq i \leq k$.

**Proof.** Let $C$ be a globularily generated double category. Let $\Phi$ be a 2-morphism in $C$. Suppose that the vertical length of $\Phi$ is equal to 1. We wish to prove, in this case, that $\Phi$ admits a decomposition as described in the statement of the lemma.

Suppose first that $\Phi$ is an element of $H^C_1$. In that case $\Phi$ is either globular or $\Phi$ is the horizontal identity of a vertical morphism in $C$. Suppose first that $\Phi$ is globular. In that case make $k = 0$ and $\Psi_0 = \Phi$. Suppose now that $\Phi$ is the horizontal identity of a vertical morphism $\alpha$ in $C$, with domain and codomain $x$ and $y$ respectively. In that case make $k = 1$, make $\Psi_0$ to equal to identity 2-morphism of horizontal identity of $x$, make $\Phi_1$ to be equal to $\Phi$ and make $\Psi_1$ to be equal to identity 2-morphism of horizontal identity of $y$.

Suppose now that $\Phi$ is a general element of first vertical category $V^C_1$ associated to $C$. Write $\Phi$ as the vertical composition $\Phi = \Theta_m \circ \ldots \circ \Theta_1$,
where $\Theta_i$ is an element of $H_1^C$ for every $i$. Choose this decomposition in such a way that its length $m$ is minimal. We proceed by induction on $m$. In the case in which $m$ is equal to 1 $\Phi$ is an element of $H_1^C$. Suppose now that $m$ is strictly greater than 1 and that the result is true for every 2-morphism in $V_1^C$ that can be written as a vertical composition of strictly less than $m$ elements of $H_1^C$. Write $\Psi$ for vertical composition $\Theta_{m-1} \circ \ldots \circ \Theta_1$. In that case $\Psi$ admits a decomposition as

$$\Psi = \Psi_k \circ \Phi_k \circ \ldots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

for some $k$, where $\Phi_i$ is a horizontal identity for every $1 \leq i \leq k$ and $\Psi_i$ is globular for every $0 \leq i \leq k$. Since $\Theta_m$ is an element of $H_1^C$ then it is either globular or it is the horizontal identity of a vertical morphism in $C$. Suppose first that $\Theta_m$ is globular. In that case write $\Psi_k'$ for vertical composition $\Theta_m \circ \Psi_k$. In that case decomposition

$$\Phi = \Psi_k' \circ \Phi_k \circ \ldots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. Suppose now that $\Theta_m$ is the vertical identity of a vertical morphism $\alpha$, with domain and codomain $x$ and $y$ respectively. In that case write $\Phi_{k+1}$ for $\Theta_m$ and write $\Psi_{k+1}$ for the identity 2-endomorphism of identity horizontal endomorphism of $x$. In that case decomposition

$$\Phi = \Psi_{k+1} \circ \Phi_{k+1} \circ \ldots \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. This concludes the proof.

\section{4 Globularily generated piece}

In this section we explain how to associate, to every double category, a globularily generated double category, its globularily generated piece. We furnish the the globularily generated piece construction with the structure of a 2-functor and we prove that globularily generated piece 2-functor is a strict reflector. We regard the globularily generated piece construction as a globularily generated analog of horizontalization functor [7]. Finally, we categorize the vertical filtration construction to a filtration of globularily generated piece 2-functor.

Given a double category $C$, we write $\gamma^C$ for sub-double category of $C$ generated by collection of globular 2-morphisms in $C$. We call double category $\gamma^C$
the globularily generated piece of double category $C$. Globularily generated piece $\gamma C$ of double category $C$ is globularily generated. Moreover, globularily generated piece $\gamma C$ of double category $C$ is equal to both the maximal globularily generated sub-double category of $C$ and to the minimal complete sub-double category of $C$ containing collection of globular 2-morphism of $C$. This last condition is equivalent to the following universal property characterizing globularily generated piece $\gamma C$ of $C$ up to double isomorphisms: Given a double functor $F : D \to C$ from a globularily generated double category $D$ to $C$, there exists a unique double functor $\tilde{F} : D \to \gamma C$ from $D$ to globularily generated piece $\gamma C$ of $C$ such that equation $\epsilon \tilde{F} = F$ holds, where $\epsilon$ denotes the inclusion double functor of globularily generated piece $\gamma C$ in $C$.

Given a double category $C$ we call 2-morphisms in $C$ lying in globularily generated piece $\gamma C$ of $C$ globularily generated. We show how to extend the definition of the globularily generated piece of a double category to a 2-functor. Given double functors $F, G : C \to D$ from double category $C$ to a double category $D$ we say that a double natural transformation $\eta : F \to G$, from $F$ to $G$, is globularily generated, if every component of morphism part $\eta_1$ of $\eta$ is globularily generated. Identity natural transformations are examples of globularily generated double natural transformations. Further, collection of globularily generated double natural transformations is closed under the operations of taking vertical and horizontal compositions. It follows, from this, that triple formed by collection of double categories, collection of double functors, and collection of globularily generated natural transformations forms a sub 2-category of 2-category $\mathcal{dCat}$. We denote this 2-category by $\mathcal{dCat}^g$. Further, we denote by $\mathcal{gCat}$ full sub 2-category of $\mathcal{dCat}^g$ generated by globularily generated double categories. 2-category $\mathcal{gCat}$ thus has collection of globularily generated double categories, collection of double functors between globularily generated double categories, and collection of globularily generated double natural transformations as collections of 0-, 1-, and 2-cells respectively. Observe that collection of globular 2-morphisms of a double category is invariant under the application of double functors. It follows, from this, that function associating, to every double category $C$, globularily generated piece $\gamma C$ of $C$, extends to a 2-functor from 2-category $\mathcal{dCat}^g$ to 2-category $\mathcal{gCat}$. We call this 2-functor the globularily generated piece 2-functor. We denote the globularily generated piece 2-functor by $\gamma$. We consider 2-functor $\gamma$ as a categorification of the globularily generated piece construction. Given 2-categories $B$ and $B'$, such that $B$ is a full sub 2-category of $B'$, we will say that $B$ is strictly 2-reflective in $B'$ if inclusion 2-functor of $B$ in $B'$ admits an left adjoint 2-functor with counit and unit.
being strict 2-natural transformations [5]. In that case we will say that any 2-functor, left adjoint to inclusion 2-functor of $B$ in $B'$, is a 2-reflector of $B'$ on $B$. The next proposition says that 2-category $g\text{Cat}$ as a sub 2-category of 2-category $d\text{Cat}^g$ is strictly 2-reflective and that globularily generated piece 2-functor $\gamma$ serves as a 2-reflector.

**Proposition 4.1.** 2-category $g\text{Cat}$ is a strictly 2-reflective sub 2-category of 2-category $d\text{Cat}^g$ with globularily generated piece 2-functor $\gamma$ as 2-reflector.

**Proof.** Let $i$ denote inclusion 2-functor of 2-category $g\text{Cat}$ in 2-category $d\text{Cat}^g$. We wish to provide pair $(\gamma, i)$ formed by globularily generated piece 2-functor $\gamma$ and inclusion 2-functor $i$ with the structure of an adjoint pair. We associate, to pair $(\gamma, i)$ a counit-unit pair $(\epsilon, \eta)$.

Let $C$ be a double category. Write $\epsilon_C$ for inclusion double functor of globularily generated piece $\gamma_C$ associated to $C$ in $C$. We write $\epsilon$ for collection of inclusions $\epsilon_C$ where $C$ runs through collection of all double categories. We prove that thus defined $\epsilon$ is a 2-natural transformation (see [5]) from composition $i\gamma$ of globularily generated piece 2-functor $\gamma$ and inclusion $i$ to identity 2-endofunctor $id_{d\text{Cat}^g}$ of 2-category $d\text{Cat}^g$. Let $C$ and $D$ be double categories. Let $F : C \to D$ be a double functor from double category $C$ to double category $D$. Since double functor $\gamma$ is defined by restriction on 1-cells, the following square

$$
\begin{array}{ccc}
  i\gamma C & \xrightarrow{i\gamma F} & i\gamma D \\
  \epsilon_C \downarrow & & \downarrow \epsilon_D \\
  C & \xrightarrow{F} & D
\end{array}
$$

commutes. Now, let $F, G : C \to D$ be double functors from double category $C$ to double category $D$ and let $\mu : F \to G$ be a globularily generated natural transformation from $F$ to $G$. We wish to prove that equation

$$
\epsilon_D \mu = \mu \epsilon_C
$$

holds. The above equation is equivalent to the following pair of equations

$$
\epsilon_{D_0} \mu_0 = \mu_0 \epsilon_{C_0} \text{ and } \epsilon_{D_1} \mu_1 = \mu_1 \epsilon_{C_1}
$$

Observe that since globularily generated piece 2-functor acts as the identity 2-functor on object categories, object functors, and object natural transformations, first equation above is trivial. We thus need only to prove that
second equation holds. Let $\alpha$ be a horizontal morphism in $C$. In that case $\mu_\alpha$ is a globularily generated $2$-morphism and thus $\epsilon_D\mu_\alpha$ is equal to $\mu_\alpha$. Now, $\epsilon_C\alpha$ is equal to $\alpha$ and thus $\mu_C\epsilon_C\alpha$ is equal to $\mu_\alpha$. We conclude that both equations above hold and thus $\epsilon$ is a strict $2$-natural transformation from composition $i\gamma$ of globularily generated piece $2$-functor $\gamma$ and inclusion $i$ to identity $2$-endofunctor $id_{d\text{Cat}^g}$ of $2$-category $d\text{Cat}^g$.

Now, since globularily generated piece of a globularily generated double category is equal to original globularily generated category and globularily generated piece double functor $\gamma$ acts by restriction on double functors and double natural transformations, composition $\gamma i$ of inclusion $i$ of $2$-category $g\text{Cat}$ in $2$-category $d\text{Cat}^g$ and globularily generated piece $2$-functor $\gamma$ is equal to identity $2$-endofunctor of $2$-category $g\text{Cat}$. Denote by $\eta$ identity double natural transformation of identity $2$-endofunctor $id_{g\text{Cat}}$ of $g\text{Cat}$ as a natural transformation from $id_{g\text{Cat}}$ to composition $\gamma i$. Thus defined $\eta$ is a strict natural transformation. Finally, observe that from the way $\eta$ was defined pair of natural transformations $(\epsilon, \eta)$ clearly satisfy the counit-unit triangle equations and it is thus a counit-unit pair for pair $(\gamma, i)$. We conclude that $2$-category $g\text{Cat}$ is a reflective $2$-subcategory of $2$-category $d\text{Cat}^g$ and that $2$-functor $\gamma$ acts as a reflector. ■

We interpret proposition 4.1 by considering globularily generated piece $2$-functor as a globularily generated analog of horizontalization functor. We now categorify the construction of filtration of vertical categories of a globularily generated double category introduced in section 3. We begin with the following lemma.

**Lemma 4.2.** Let $C$ and $D$ be globularily generated double categories. Let $F : C \to D$ be a double functor from $C$ to $D$. Let $n$ be a positive integer. The image of $n$-th vertical category $V_n^C$ associated to $C$, under morphism functor $F_1$ of $F$, is contained in $n$-th vertical category $V_n^D$ associated to $D$. Moreover, in the case in which $C, D,$ and $F$ are strict, the image of $n$-th horizontal category $H_n^C$ associated to $C$, under morphism functor $F_1$ of $F$, is contained in $n$-th horizontal category $H_n^D$ associated to $D$.

**Proof.** Let $C$ and $D$ be globularily generated double categories. Let $F : C \to D$ be a double functor from $C$ to $D$. Let $n$ be a positive integer. We wish to prove that the image of $n$-th vertical category $V_n^C$ associated to $C$, under morphism functor $F_1$ of $F$, is a subcategory of $n$-th vertical category $V_n^D$ associated to $D$. Moreover, we wish to prove that if $C, D,$ and $F$ are all strict then the image of $n$-th horizontal category $H_n^C$ associated to $C$, under
morphism functor $F_1$ of $F$, is a subcategory of $n$-th horizontal category $H^D_n$ associated to $D$.

We proceed by induction on $n$. Let $\Phi$ be a 2-morphism in first vertical category $V^C_1$ associated to $C$. We wish to prove, in this case that $F_1\Phi$ is a morphism in first vertical category $V^D_1$ associated to $D$. Suppose first that $\Phi$ is an element of $H^C_1$. In that case $\Phi$ is either globular or the horizontal identity of a vertical morphism in $C$. Suppose first that $\Phi$ is the horizontal identity of a vertical morphism $\alpha$ in $C$. In that case the image $F_1\Phi$ of $\Phi$ under functor $F_1$ is globularly conjugate to horizontal identity of the image $F_0\alpha$ of $\alpha$ under functor $F_0$ and is thus a morphism in category $V^D_1$. Observe that in the case in which double functor $F$ is strict $F_1\Phi$ is precisely horizontal identity of vertical morphism $F_0\alpha$ and is thus an element of $H^D_0$. From this and from the fact that collection of globular 2-morphisms of a double category is invariant under the application of double functors it follows that the image of collection $H^C_1$, under morphism functor $F_1$, is contained in collection of morphisms of first vertical category $V^D_1$ of $D$. Moreover, in the case in which $F$ is strict, the image of $H^C_1$ under $F_1$ is contained in $H^D_1$. Suppose now that $\Phi$ is a general element of first vertical category $V^C_1$ associated to $C$. Write $\Phi$ as a vertical composition

$$\Phi = \Phi_k \circ \ldots \circ \Phi_1$$

where $\Phi_i$ is an element of $H^C_1$ for every $1 \leq i \leq k$. In that case the image of $\Phi$ under morphism functor $F_1$ of $F$ is equal to vertical composition

$$F_1\Phi_k \circ \ldots \circ F_1\Phi_1$$

which is a morphism of first vertical category $V^D_1$ associated to $D$. Thus the image of first vertical category $V^C_1$ associated to $C$, under morphism functor $F_1$ of $F$ is a subcategory of first vertical category $V^D_1$ associated to $D$. Moreover, if we assume that $C, D$, and $F$ are strict, $H^C_1$ and $H^D_1$ are categories, and the image of $H^C_1$ under morphism functor $F_1$ of $F$ is a subcategory of $H^D_1$.

Let $n$ now be strictly greater than 1. Suppose that the conclusions of the proposition are true for every $m < n$. Let $\Phi$ now be a morphism in $n$-th vertical category $V^C_n$ associated to $C$. We wish to prove in this case that the image $F_1\Phi$ of $\Phi$ under morphism functor $F_1$ of $F$ is a morphism in $n$-th vertical category $V^D_n$ associated to $D$. Suppose first that $\Phi$ is a morphism in $H^C_n$. Write $\Phi$, up to globular equivalences, as a horizontal composition of the form

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\[ \Phi \equiv \Phi_k \ast \ldots \ast \Phi_1 \]

where \( \Phi_i \) is an element of \( n - 1 \)-th vertical category \( V^C_{n-1} \) associated to \( C \) for every \( 1 \leq i \leq k \). In that case the image \( F_1 \Phi \) under functor \( F_1 \) is globularly equivalent to any possible interpretation of horizontal composition

\[ F_1 \Phi_k \ast \ldots \ast F_1 \Phi_1 \]

in \( D \). By the induction hypothesis \( F_1 \Phi_i \) is a morphism of \( n - 1 \)-th vertical category \( V^D_{n-1} \) associated to \( D \) for every \( 1 \leq i \leq k \) and thus any interpretation of horizontal composition above is a morphism of \( n \)-th horizontal category \( V^D_n \) associated to \( D \). We conclude that the image \( F_1 \Phi \) of \( 2 \)-morphism \( \Phi \) under functor \( F_1 \) is a morphism in \( n \)-th vertical category \( V^D_n \) associated to \( D \). Moreover, if \( F \) is strict then image \( F_1 \Phi \) of \( \Phi \) under functor \( F_1 \) is an element of \( H^D_n \). Suppose now that \( \Phi \) is a general morphism of \( n \)-th vertical category \( V^C_n \) associated to \( C \). Write \( \Phi \) as a vertical composition of the form

\[ \Phi = \Phi_k \circ \ldots \circ \Phi_1 \]

where \( \Phi_i \) is an element of \( H^C_n \) for every \( 1 \leq i \leq k \). In that case the image \( F_1 \Phi \) of \( \Phi \) under morphism functor \( F_1 \) of \( F \) is equal to vertical composition

\[ F_1 \Phi_k \circ \ldots \circ F_1 \Phi_1 \]

in \( D \) and thus is an element of \( n \)-th vertical category \( V^D_n \) of \( D \). We conclude that the image of \( n \)-th vertical category \( V^C_n \) associated to \( C \), under morphism functor of double functor \( F \), is a subcategory of \( n \)-th vertical category \( V^D_n \) associated to \( D \) and that if \( C, D, \) and \( F \) are strict then moreover the image of \( n \)-th horizontal category \( H^C_n \) associated to \( C \), under morphism functor \( F_1 \) of \( F \), is a subcategory of \( n \)-th horizontal category \( H^D_n \) associated to \( D \). This concludes the proof.

Let \( C \) and \( D \) be globularly generated double categories. Let \( F : C \to D \) be a double functor from \( C \) to \( D \). Let \( n \) be a positive integer. We write \( V^F_n \) for restriction, to \( n \)-th vertical category \( V^C_n \) associated to \( C \) of morphism functor \( F_1 \) of \( F \). Thus defined \( V^F_n \) is, by lemma 4.2, a functor from \( n \)-th vertical category \( V^C_n \) associated to \( C \) to \( n \)-th vertical category \( V^D_n \) associated to \( D \). We call \( V^F_n \) the \( n \)-th vertical functor associated to \( F \). Pair formed by function associating \( n \)-th vertical category \( V^C_n \) associated to \( C \) to every double category \( C \) and \( n \)-th vertical double functor \( V^F_n \) to every double functor...
$F$ forms a functor from category adjacent to 2-category $\mathbf{gCat}$ of globularily generated double categories, double functors, and globularily generated double natural transformations to category adjacent to 2-category $\mathbf{Cat}$ of categories, functors, and natural transformations. We call functor $V_n$ the $n$-th vertical functor.

Denote now by $\pi_0$ and $\pi_1$ 2-functors from 2-category $\mathbf{dCat}$ of double categories, double functors, and double natural transformations to 2-category $\mathbf{Cat}$ of categories, functors, and natural transformations such that for every double category $C$, $\pi_0 C$ and $\pi_1 C$ are equal to object category $C_0$ of $C$ and morphism category $C_1$ of $C$ respectively, such that for every double functor $F$, $\pi_0 F$ and $\pi_1 F$ are equal to object functor $F_0$ of $F$ and morphism functor $F_1$ of $F$ respectively, and finally, such that for every double natural transformation $\eta$, $\pi_0 \eta$ and $\pi_1 \eta$ are equal to object natural transformation $\eta_0$ associated to $\eta$ and morphism natural transformation $\eta_1$ associated to $\eta$. We call $\pi_0$ and $\pi_1$ object and morphism projections of $\mathbf{dCat}$ respectively. We keep denoting by $\pi_0$ and $\pi_1$ restrictions of object and morphism projections of 2-category $\mathbf{dCat}$, to sub 2-category $\mathbf{gCat}$ of globularily generated double categories, double functors, and globularily generated double natural transformations. We write $\gamma_1$ for composition $\pi_1 \gamma$ of globularily generated piece 2-functor $\gamma$ and morphism projection $\pi_1$. Given a double category $C$ and positive integers $m$ and $n$ such that $n \geq m$, $m$-th vertical category $V_m^C$ associated to globularily generated piece $\gamma C$ of $C$ is a subcategory of $n$-th vertical category $V_n^C$ associated to $\gamma C$. We write $\eta_{m,n}^C$ for the inclusion functor of category $V_m^C$ in $V_n^C$. Observe that given a double functor $F : C \to D$ from double category $C$ to a double category $D$, the fact that vertical functors $V_m^F$ and $V_n^F$ associated to double functor $\gamma F$ are restrictions of morphism functor $\gamma F_1$ of $\gamma F$, implies that square:

$$
\begin{array}{ccc}
V_m^C & \xrightarrow{V_m^F} & V_m^D \\
\downarrow^{\eta_{m,n}^C} & & \downarrow^{\eta_{m,n}^D} \\
V_n^C & \xrightarrow{V_n^F} & V_n^D
\end{array}
$$

commutes. That is, if we denote by $\eta_{m,n}$ collection of inclusions $\eta_{m,n}^C$ with $C$ running through collection of double categories, then $\eta_{m,n}$ is a natural transformation from composition $V_m \gamma$ of globularily generated piece 2-functor $\gamma$ and $m$-th vertical functor $V_m$ to composition $V_n \gamma$ of globularily generated
piece 2-functor $\gamma$ and $n$-th vertical functor $V_n$. Sequence formed by functors $V_n \gamma$ together with collection formed by natural transformations $\eta_{m,n}$ forms a diagram in category adjacent to 2-category $\text{Cat}$ with base in category adjacent to 2-category $\text{dCat}$. The following proposition says that the limit of this diagram is functor $\gamma_1$. Its proof follows directly from lemma 3.2.

**Proposition 4.3.** Functor $\gamma_1$ defined above is equal to the limit $\lim_{\to} V_n \gamma$ of diagram formed by sequence of vertical functors $V_n \gamma$ and collection natural transformations $\eta_{m,n}$.

Let now $C$ and $D$ be strict globularily generated double categories and let $F : C \to D$ be a strict double functor from $C$ to $D$. If $n$ is a positive integer, then from the assumption that $C, D,$ and $F$ are strict, and from lemma 4.3, it follows that pair formed by morphism function of object functor $F_0$ of $F$ and morphism function of morphism functor $F_1$ of $F$ restrict to a functor from $n$-th horizontal category $H^n_C$ associated to $C$ to $n$-th horizontal category $H^n_D$ associated to $D$. We denote this functor by $H^n_F$ and we call it the $n$-th horizontal functor associated to double functor $F$. Denote by $\text{dCat}$ sub 2-category of $\text{dCat}$ generated by collection of strict double categories and collection of strict double functors between them and denote by $\text{gCat}$ sub 2-category of $\text{dCat}$ generated by collection of strict globularily generated double categories. Given a positive integer $n$, pair of functions associating, for every strict globularily generated double category $C$ $n$-th horizontal category $H^n_C$ associated to $C$ and to every strict double functor $F$ $n$-th horizontal functor $H^n_F$ associated to $F$ is a functor from category adjacent to 2-category $\text{gCat}$ of strict globularily generated double categories, strict double functors, and double natural transformations, to category adjacent to 2-category $\text{Cat}$ of categories, functors, and natural transformations.

Given a strict double category $C$, we denoted, in section 3, by $\tau C$ category whose collection of objects is collection of vertical morphisms of $C$ and whose collection of morphisms is collection of 2-morphisms of $C$. We called category $\tau C$ the transversal category associated to $C$. Given a strict double functor $F : C \to D$ from strict double category $C$ to a strict double category $D$, we denote by $\tau F$ functor from transversal category $\tau C$ associated to $C$ to transversal category $\tau D$ associated to $D$ such that object and morphism functions of $\tau F$ are morphism function of object functor $F_0$ associated to $F$ and morphism function of morphism functor $F_1$ associated to $F$ respectively. We call $\tau F$ transversal functor associated to strict double functor $F$. Pair of functions associating transversal category $\tau C$ to a double category $C$ and transversal functor $\tau F$ to double functor $F$ forms a functor from category adjacent to 2-category $\text{dCat}$ of strict double categories, strict
double functors, and double natural transformations, to category adjacent to 2-category \textbf{Cat} of categories, functors, and natural transformations. We denote this functor by $\tau$. We call $\tau$ the transversal category functor. We keep denoting by $\tau$ restriction of transversal functor to category adjacent to 2-category $\textbf{gCat}$. We will denote composition $\tau \gamma$ of adjacent functor of globularily generated piece double functor $\gamma$ and transversal functor $\tau$ by $\gamma^\tau$.

Given a strict globularily generated double category $C$ and positive integers $m$ and $n$ such that $m \leq n$, $m$-th horizontal category $H^C_m$ associated to $C$ is a subcategory of $n$-th horizontal category $H^C_n$ associated to $C$. In this case denote by $\nu_{m, n}^C$ inclusion functor of $H^C_m$ in $H^C_n$.

Given a strict double functor $F : C \to D$, from a strict double category $C$ to a strict double category $D$, for every pair of positive integers $m$ and $n$ such that $m \leq n$, diagram

\[
\begin{array}{ccc}
H^C_m & \xrightarrow{H^F_m} & H^D_m \\
\downarrow \nu_{m, n}^C & & \downarrow \nu_{m, n}^D \\
H^C_n & \xrightarrow{H^F_n} & H^D_n
\end{array}
\]

commutes. That is, if in this case we denote by $\nu_{m, n}$ collection formed by inclusions $\nu_{m, n}^C$ with $C$ running through collection of strict double categories, then $\nu_{m, n}$ is a natural transformation from composition $H^C_m \gamma$ of functor adjacent to globularily generated piece 2-functor $\gamma$ and $m$-th horizontal functor to composition $H^C_n \gamma$ of globularily generated piece $\gamma$ and $n$-th globularily generated piece functor $H_n$. Sequence formed by functors $H_n$ together with collection of natural transformations $\nu_{m, n}$ forms a diagram in category adjacent to \textbf{Cat}, with base in category adjacent to $\textbf{dCat}$. The following proposition says that the limit of this diagram is functor $\gamma^\tau$. Its proof now follows directly from the second part of lemma 3.3.

\textbf{Proposition 4.4.} Functor $\gamma^\tau$ defined above is the limit $\lim\limits_{\longrightarrow} H_n \gamma$ formed by sequence of horizontal functors $H_n \gamma$ and collection of natural transformations $\nu_{m, n}$.

\section{5 Examples}

In this final section we present explicit computations of globularily generated piece of double categories introduced in section 2. We begin with the
computation of globularily generated piece $\gamma \text{Alg}$ of double category $\text{Alg}$ of algebras, algebra morphisms, bimodules, and equivariant bimodule morphisms.

We will write horizontal equivariant endomorphism $(f, \Phi, f)$ in double category $\text{Alg}$ simply as $(f, \Phi)$. If an equivariant morphism in $\text{Alg}$ is written in this way it will be assumed it is a horizontal endomorphism. Given a complex algebra $A$ and a left-right $A$-bimodule $M$, we say that $M$ is an $A$-cyclic bimodule if $M$ is generated, as an $A$-bimodule, by a single element intertwining left and right actions of $A$ on $M$. Equivalently, bimodule $M$ is $A$-cyclic if there exists a bimodule epimorphism $A \to M$. Algebra $A$ considered as a left-right bimodule over itself is an example of a cyclic bimodule. Given algebras $A$ and $B$ and $f : A \to B$ a unital algebra morphism, morphism $f$ induces, on every left-right $B$-bimodule $M$, the structure of a left-right $A$-bimodule through equation $axa' = f(a)x f(a')$ for every $x \in M$ and $a, a' \in A$. We will call this left-right $A$-bimodule structure on left-right $B$-bimodule $M$ the $A$-bimodule structure on $M$ induced by morphism $f$. Given algebras $A$ and $B$, a left-right $A$-bimodule $M$ and a left-right $B$-bimodule $N$, we say that an equivariant morphism $(f, \varphi) : M \to N$ from $M$ to $N$, is 2-subcyclic if there exists a cyclic $A$-submodule $L$ of $N$ and a cyclic $B$-submodule $K$ of $N$ such that inclusions $\text{Im} \varphi \subseteq L \subseteq K$ hold. Equivariant morphisms between algebras are examples of 2-subcyclic equivariant morphisms. In order to explicitly compute globularily generated piece $\gamma \text{Alg}$ of double category $\text{Alg}$, by proposition 3.5 we need only to compute collection of non-globular, globularily generated 2-morphisms between horizontal endomorphisms in $\text{Alg}$.

We begin with the following lemma.

**Lemma 5.1.** Let $A$ and $B$ be algebras. Let $M$ and $M'$ be left-right $A$-bimodules and let $N$ and $N'$ be left-right $B$-bimodules. Let $(f, \varphi) : M \to N$ be an equivariant morphism from $M$ to $N$ and let $(f, \varphi') : M' \to N'$ be equivariant morphism from $M'$ to $N'$. If both $(f, \varphi)$ and $(f, \varphi')$ are 2-subcyclic then relative tensor product $(f, \varphi \otimes_{f} \varphi')$ is 2-subcyclic.

**Proof.** Let $A$ and $B$ be algebras. Let $M$ and $M'$ be left-right $A$-bimodules and let $N$ and $N'$ be left-right $B$-bimodules. Let $(f, \varphi) : M \to N$ and $(f, \varphi') : M' \to N'$ be equivariant morphisms from $M$ to $N$ and from $M'$ to $N'$ respectively. Suppose both $(f, \varphi)$ and $(f, \varphi')$ are 2-subcyclic. We wish to prove in this case that relative tensor product $(f, \varphi \otimes_{f} \varphi')$ is 2-subcyclic.

Let $L, L'$ and $K, K'$ be bimodules such that $L$ and $L'$ are $A$-cyclic submodules of $N$ and $N'$ respectively and such that $K$ and $K'$ are $B$-cyclic.
submodules of $N$ and $N'$ respectively. Moreover, let $L, L'$ and $K, K'$ satisfy inclusions $\text{Im}\varphi \subseteq L \subseteq K$ and $\text{Im}\varphi' \subseteq L' \subseteq K'$ respectively. Relative tensor product $L \otimes_A L'$ is an $A$-cyclic submodule of $N \otimes_A N'$, relative tensor product $K \otimes_B K'$ is a $B$-cyclic submodule of $N \otimes_B N'$, $L \otimes_A L'$ is contained in $K \otimes_B K'$, and finally $\text{Im}\varphi \otimes f \varphi'$ is contained in $L \otimes_A L'$. This concludes the proof. ■

Proposition 5.2. Let $A$ and $B$ be algebras. Let $M$ be left-right $A$-bimodule and let $N$ be a left-right $B$-bimodule. In that case collection of non-globular globularily generated equivariant morphisms from $M$ to $N$ is precisely collection of non-globular 2-subcyclic equivariant morphisms from $M$ to $N$. Moreover, every globularily generated equivariant morphism from $M$ to $N$ has vertical length equal to 1.

Proof. Let $A$ and $B$ be algebras. Let $M$ be a left-right $A$-bimodule and let $N$ be a left-right $B$-bimodule. We wish to prove that collection of non-globular globularily generated equivariant morphisms from $M$ to $N$ is precisely collection of non-globular 2-subcyclic equivariant morphisms from $M$ to $N$. Moreover, we wish to prove that every globularily generated equivariant morphism from $M$ to $N$ has vertical length equal to 1.

We prove first that every non-globular 2-subcyclic equivariant morphism from $M$ to $N$ is globularily generated. Let $(f, \varphi) : M \to N$ be non-globular and 2-subcyclic. Let $K$ be a $B$-cyclic submodule of $N$ and let $L$ be an $A$-cyclic submodule of $K$ such that inclusions $\text{Im}\varphi \subseteq L \subseteq K$ hold. Let $j$ denote the inclusion of $K$ in $N$. Let $\overline{\varphi}$ denote the codomain restriction of $\varphi$ to $K$. Thus defined $j$ is a globular and equivariant morphism $(f, \overline{\varphi})$ makes the following triangle:

$$
\begin{array}{ccc}
M & \xrightarrow{(f, \varphi)} & N \\
\downarrow{(f, \overline{\varphi})} & & \downarrow{(id_B, j)} \\
K & & 
\end{array}
$$

commute. Denote now by $j'$ inclusion of $L$ in $K$ and denote by $\overline{\varphi}$ codomain restriction of $\varphi$ to $L$. Thus defined $j'$ is globular, and equivariant morphism $(f, \overline{\varphi})$ makes the following triangle:
commute. The following square:

\[
\begin{array}{ccc}
M & \xrightarrow{(f, \varphi)} & K \\
\downarrow{(f, \tilde{\varphi})} & & \downarrow{(id_A, j')} \\
L & \xrightarrow{(id_A, j')} & K
\end{array}
\]

is thus commutative. Commutativity of this square is clearly equivalent to commutativity of square:

\[
\begin{array}{ccc}
M & \xrightarrow{(f, \varphi)} & N \\
\downarrow{(f, \tilde{\varphi})} & & \downarrow{(id_B, j)} \\
L & \xrightarrow{(id_A, j')} & K
\end{array}
\]

Left and right hand sides of this last square are Globular. Finally, triangle:

\[
\begin{array}{ccc}
L & \xrightarrow{(f, j')} & K \\
\downarrow{(f, f)} & & \downarrow{(id_B, j')} \\
K
\end{array}
\]

commutes, which proves that equivariant morphism \((f, j')\) is a morphism in first vertical category \(V^\text{Alg}_1\) associated to \(\text{Alg}\). We conclude that 2-subcyclic equivariant morphism \((f, \varphi)\) is globularily generated and has vertical length equal to 1.
We now prove that every non-globular globularily generated equivariant morphism from $M$ to $N$ is 2-subcyclic. Let $(f, \varphi) : M \to N$ be non-globular and globularily generated. Assume first that $(f, \varphi)$ is an element of $H_1^{Alg}$. From the assumption that $(f, \varphi)$ is non-globular it follows that $(f, \varphi)$ is the horizontal identity of an algebra morphism and thus is 2-subcyclic. Suppose now that $(f, \varphi)$ is a general morphism in first vertical category $V_1^{Alg}$ associated to $Alg$. We wish to find, in this case, an $A$-cyclic submodule $L$ of $N$ and a $B$-cyclic submodule $K$ of $N$ such that inclusions $\text{Im} \varphi \subseteq L \subseteq K$ hold.

Write $(f, \Phi)$ for composition $(f_k, \phi_k) \circ ... \circ (f_1, \phi_1)$. Thus defined $(f, \Phi)$ is an equivariant morphism from left-right $A$-bimodule $A$ to left-right $B$-bimodule $BB$. Now make $K$ to be equal to image $\text{Im} \psi_1$ of $\psi_1$ and make $L$ to be equal to image $\text{Im} \Phi \psi$ of composition $\Phi \psi$. Thus defined $K$ and $L$ satisfy the conditions required. We conclude that every equivariant morphism in first vertical category $V_1^{Alg}$ associated to $Alg$ is 2-subcyclic. From this and from lemma 5.1 it follows that every globularily generated equivariant morphism between $M$ and $N$ is 2-subcyclic. The fact that every non-globular globularily generated 2-morphism in $Alg$ has vertical length equal to 1 follows from this and from the first part of the proof. This concludes the proof. □

We consider, due to proposition 3.5 that proposition 5.2 provides an explicit description of globularily generated piece $\gamma_{Alg}$ of double category $Alg$. A similar computation provides a complete description of globularily generated piece $\gamma_{vN^f}$ of double category $vN^f$ of von Neumann algebras with finite dimensional center, finite algebra morphisms, bimodules, and equivariant bimodule morphisms. We now compute, using the same procedure used to compute globularily generated piece $\gamma_{Alg}$ of double category $Alg$, globularily generated piece $\gamma_{Cob(n)}$ of double category $Cob(n)$ of $n$-dimensional manifolds, diffeomorphisms, cobordisms, and equivariant diffeomorphisms, for every positive integer $n$. We make the same considerations regarding horizontal equivariant endomorphisms in $Cob(n)$ as we did with horizontal equivariant endomorphisms in $Alg$. We will say that cobordisms $M$ and $N$ from a closed manifold $X$ to itself are globularily diffeomorphic if $M$ and $N$ are diffeomorphic relative to $X$. We begin with the following lemma.
Lemma 5.3. Let $n$ be a positive integer. Let $X$ and $Y$ be closed $n$-dimensional manifolds. Let $M$ be a cobordism from $X$ to $X$ and let $N$ be a cobordism from $Y$ to $Y$. If there exist non-globular globularily generated diffeomorphisms from $M$ to $N$ then $M$ and $N$ are globularily diffeomorphic to identity cobordisms $i_X$ and $i_Y$ respectively.

Proof. Let $n$ be a positive integer. Let $X$ and $Y$ be closed $n$-dimensional manifolds. Let $M$ be a cobordism from $X$ to $X$ and let $N$ be a cobordism from $Y$ to $Y$. Suppose there exists a non-globular globularily generated diffeomorphism from $M$ to $N$. In that case we wish to prove that $M$ and $N$ are globularily diffeomorphic to identity cobordisms $i_X$ and $i_Y$ respectively.

Let $(f, \Phi) : M \to N$ be a non-globular globularily generated diffeomorphism from $M$ to $N$. We proceed by induction on the vertical length of $(f, \Phi)$ to prove that the existence of $(f, \Phi)$ implies that $M$ and $N$ are globularily diffeomorphic to horizontal identities $i_X$ and $i_Y$ respectively. Suppose first that vertical length of $(f, \Phi)$ is equal to 1. By lemma 3.7 there exists a decomposition of $(f, \Phi)$ as a vertical composition of the form

$$(id_{X_k}, \Psi_k) \circ (f_k, f_k \times id_{[0,1]}) \circ ... \circ (id_{X_1}, \Psi_1) \circ (f_1, f_1 \times id_{[0,1]}) \circ (id_{X_0}, \Psi_0)$$

where $X_0, ..., X_k$ are $n$-dimensional manifolds, $X_0$ and $X_k$ are equal to $X$ and $Y$ respectively, $f_i : X_i \to X_{i+1}$ is a diffeomorphism from $X_i$ to $X_{i+1}$ for all $i \leq k - 1$, and where $\Psi_i$ is a globular diffeomorphism from $X_i$ to $X_i$ for all $i$. Since we assume that $(f, \Phi)$ is not globular then the length $k$ of this decomposition is greater than or equal to 1. Domain of horizontal identity of $\Phi_1$ is equal to horizontal identity $i_X$ of manifold $X$ and codomain of horizontal identity $\Phi_k$ is equal to horizontal identity $i_Y$ of manifold $Y$. Thus $\Psi_0$ and $\Psi_k$ define globular diffeomorphisms between $M$ and $N$ and horizontal identities $i_X$ and $i_Y$ respectively.

Let $m$ be a positive integer strictly greater than 1. Assume now that the result is true for every pair of cobordisms admitting a non-globular globularily generated diffeomorphism of vertical length strictly less than $m$. Assume first that non-globular globularily generated diffeomorphism $(f, \Phi)$ is an element of $H^\text{Cob(n)}_m$. Write, in this case $(f, \Phi)$ as a horizontal composition

$$(f, \Phi) \equiv (f, \Phi_k) \ast \ast (f, \Phi_1)$$

where $(f, \Phi_i)$ is a morphism in $m-1$-th vertical category $V^\text{Cob(n)}_{m-1}$ associated to $\text{Cob}(n)$ for every $i \leq k$. Moreover, assume that length $k$ of this
decomposition is minimal. We proceed by induction on $k$ to prove that in this case the existence of $(f, \Phi)$ implies that $M$ and $N$ satisfy the conditions of the lemma. If $k = 1$ then $(f, \Phi)$ is an element of $m - 1$-th vertical category $V_{m-1}^{\text{Cob}(n)}$ associated to $\text{Cob}(n)$ and by induction hypothesis its existence implies that $M$ and $N$ satisfy the conditions of the lemma. Suppose now that $k$ is strictly greater than 1. Write $(f, \Psi)$ for any representative of $(f, \Phi_k) \ast \ldots \ast (f, \Phi_2)$. In that case horizontal composition $(f, \Psi) \ast (f, \Phi_1)$ is equivalent to $(f, \Phi)$. From the assumption that $(f, \Phi)$ is not globular and from corollary 3.6 it follows that non of its conjugate morphisms is globular. Thus horizontal composition $(f, \Psi) \ast (f, \Phi_1)$ is not globular and thus, again by corollary 3.6 neither of $(f, \Psi)$ or $(f, \Phi_1)$ is globular. Both $(f, \Psi)$ and $(f, \Phi_1)$ are globularily equivalent to the horizontal composition of strictly less than $k$ morphisms in $m - 1$-th vertical category $V_{m-1}^{\text{Cob}(n)}$ associated to $\text{Cob}(n)$. Let $M_1$ and $N_1$ be domain and codomain of $(f, \Phi_1)$ and let $M_2$ and $N_2$ be domain and codomain of $(f, \Psi)$. By induction hypothesis $M_1$ and $M_2$ are both globularily diffeomorphic to horizontal identity $i_X$ of $X$ and both $N_1$ and $N_2$ are globularily diffeomorphic to horizontal identity $i_Y$ of $Y$. It follows that $M_2 \ast M_1$ is globularily diffeomorphic to horizontal identity $i_X$ of $X$ and that $N_2 \ast N_1$ is globularily diffeomorphic to horizontal identity $i_Y$ of $Y$. Finally, by the exchange property in $\text{Cob}(n)$ we conclude that $M$ and $N$ are globularily diffeomorphic to horizontal identities $i_X$ and $i_Y$ of $X$ and $Y$ respectively.

Suppose now that $(f, \Phi)$ is a general element of $m$-th vertical category $V_m^{\text{Cob}(n)}$ associated to $\text{Cob}(n)$. In that case write $(f, \Phi)$ as a vertical composition

$$(f, \Phi) = (f_k, \Phi_k) \circ \ldots \circ (f_1, \Phi_1)$$

where $(f_i, \Phi_i)$ is an element of $H_m^{\text{Cob}(n)}$ for every $i$. Moreover, assume that length $k$ of this decomposition is minimal. We again proceed by induction on $k$. If $k = 1$ then $(f, \Phi)$ is an element of $H_m^{\text{Cob}(n)}$. Suppose now that $k$ is strictly greater than 1 and that the existence of a non-globular globularily generated diffeomorphism in $m$-th vertical category $V_m^{\text{Cob}(n)}$ associated to $\text{Cob}(n)$, between manifolds $X$ and $Y$, that can be written as a vertical composition of strictly less than $k$ diffeomorphisms in $H_m^{\text{Cob}(n)}$ implies the conclusion of the lemma for $X$ and $Y$. Write $(g, \Psi)$ for composition $(f_k, \Phi_k) \circ \ldots \circ (f_2, \Phi_2)$. In that case $(f, \Phi)$ is equal to vertical composition $(g, \Psi) \circ (f_1, \Phi_1)$. Moreover, from the assumption that $(f, \Phi)$ is not globular it follows that one of $(g, \Psi)$ or $(f_1, \Phi_1)$ is non-globular. Assume first that
$(g, \Psi)$ is globular. In that case source and taget of $(f_1, \Phi_1)$ are both equal to $f$. By induction hypothesis domain and codomain of $(f, \Phi)$ are globularly diffeomorphic to horizontal identity $i_X$ of $X$ and horizontal identity $i_Y$ of $Y$ respectively. Domain of $(f, \Phi)$ is equal to codomain of $(f_1, \Phi_1)$ and $(g, \Psi)$ defines a globular diffeomorphism between codomain of $(f, \Phi)$ and codomain of $(f_1, \Phi_1)$. We conclude that in this case, the existence of non-globular globularly generated diffeomorphism $(f, \Phi)$ implies the existence of a globular diffeomorphism between $M$ and horizontal identity $i_X$ of $X$ and between $N$ and horizontal identity $i_Y$ of $Y$. The case in which it is assumed that $(f_1, \Phi_1)$ is globular is handled analogously. Suppose now that neither $(g, \Psi)$ nor $(f_1, \Phi_1)$ are globular. In that case, induction hypothesis implies that there exists a globular diffeomorphism between $M$, which is the domain of $(f_1, \Phi_1)$, and horizontal identity $i_X$ of $X$ and that there exists a globular diffeomorphism between $N$, which is the codomain of $(g, \Psi)$, and horizontal identity $i_Y$ of $Y$. This concludes the proof.

As a consequence of lemma 3.5 and proposition 5.3, in order to compute globularly generated piece $\gamma\text{Cob}(n)$ of double category $\text{Cob}(n)$ it is enough to compute collection of non globular globally generated diffeomorphisms between horizontal endomorphisms globularly diffeomorphic to horizontal identities of closed $n$-dimensional manifolds. This is achieved in the following proposition.

**Proposition 5.4.** Let $n$ be a positive integer. Let $X$ and $Y$ be closed $n$-dimensional manifolds. Let $M$ be a cobordism from $X$ to $X$ and let $N$ be a cobordism from $Y$ to $Y$. Suppose that $N$ is globularly diffeomorphic to identity cobordism $i_X$ associated to $X$ and that $N$ is globularly diffeomorphic to identity cobordism $i_Y$ associated to $Y$. In that case every horizontal endomorphism from $M$ to $N$, in double category $\text{Cob}(n)$, is globularly generated and has vertical length equal to 1.

**Proof.** Let $n$ be a positive integer. Let $X$ and $Y$ be closed $n$-dimensional manifolds. Let $M$ be a cobordism from $X$ to $X$, globularly diffeomorphic to horizontal identity $i_X$ associated to manifold $X$ and let $N$ be a cobordism from $Y$ to $Y$, globularly diffeomorphic to horizontal identity $i_Y$ associated to $Y$. We wish to prove, in this case, that every 2-morphism, in double category $\text{Cob}(n)$, from $M$ to $N$, is globularly generated and has vertical length equal to 1.

We first prove the proposition for the case in which $M$ and $N$ are equal to horizontal identity cobordisms $i_X$ and $i_Y$ respectively. Let $(f, \Phi) : i_X \to i_Y$ be a 2-morphism, in $\text{Cob}(n)$, from $i_X$ to $i_Y$. In that case equivariant
morphism \((id_X, (f^{-1} \times id_{[0,1]})\Phi)\) is a globular endomorphism of horizontal identity \(i_X\) of \(X\) making the following triangle commute. Since \((id_X, (f^{-1} \times id_{[0,1]})\Phi)\) is globular and \((f, f \times id_{[0,1]})\) is horizontal identity \(i_f\) of diffeomorphism \(f\) of \(X\), we conclude that \((f, \Phi)\) is globularily generated and that its vertical length is equal to 1.

Suppose now \(M\) is a general cobordism from \(X\) to \(X\) globularily diffeomorphic to horizontal identity \(i_X\) of \(X\) and that \(N\) is a general cobordism from \(Y\) to \(Y\) globularily diffeomorphic to horizontal identity \(i_Y\) of \(Y\). Let \((f, \Phi) : M \to N\) be a general 2-morphism, in \(\text{Cob}(n)\), from \(M\) to \(N\). Let \((id_X, \varphi) : M \to i_X\) be a globular diffeomorphism from \(M\) to horizontal identity \(i_X\) of \(X\) and let \((id_Y, \phi) : N \to i_Y\) be a globular diffeomorphism from \(N\) to horizontal identity \(i_Y\) of \(Y\). In that case composition \((f, \Psi) = (id_Y, \phi)(f, \Phi)(id_X, \varphi^{-1})\) is a 2-morphism from horizontal identity \(i_X\) of \(X\) to horizontal identity \(i_Y\) of \(Y\) and is thus a morphism in first vertical category \(V_1^{\text{Cob}(n)}\) associated to \(\text{Cob}(n)\). We conclude that \((f, \Phi) = (id_Y, \phi^{-1})(f, \Psi)(id_X, \varphi)\) is also a morphism in first vertical category \(V_1^{\text{Cob}(n)}\) associated to \(\text{Cob}(n)\). This concludes the proof.

By lemma 3.5 proposition 5.4 provides an explicit description of globularily generated piece of double categories of the form \(\text{Cob}(n)\). Examples of globularily generated double categories having 2-morphisms of vertical length strictly greater than 1 will be studied in subsequent papers.

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