Notions, Stability, Existence, and Robustness of Limit Cycles in Hybrid Systems

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Abstract—This paper deals with existence and robust stability of hybrid limit cycles for a class of hybrid systems given by the combination of continuous dynamics on a flow set and discrete dynamics on a jump set. For this purpose, the notion of Zhukovskii stability, typically stated for continuous-time systems, is extended to the hybrid systems. Necessary conditions, particularly, a condition using a forward invariance notion, for existence of hybrid limit cycles are first presented. In addition, a sufficient condition, related to Zhukovskii stability, for the existence of (or lack of) hybrid limit cycles is established. Furthermore, under mild assumptions, we show that asymptotic stability of such hybrid limit cycles is not only equivalent to asymptotic stability of a fixed point of the associated Poincaré map but also robust to perturbations. Specifically, robustness to generic perturbations, which capture state noise and unmodeled dynamics, and to inflations of the flow and jump sets are established in terms of $K_v$ bounds. Furthermore, results establishing relationships between the properties of a computed Poincaré map, which is necessarily affected by computational error, and the actual asymptotic stability properties of a hybrid limit cycle are proposed. In particular, it is shown that asymptotic stability of the exact Poincaré map is preserved when computed with enough precision. Several examples, including a congestion control system and spiking neurons, are presented to illustrate the notions and results throughout the paper.

I. INTRODUCTION

A. Motivation

Due to recent technological advances requiring advanced mathematical models, hybrid systems have drawn considerable attention in recent years. Hybrid systems have state variables that can evolve continuously (flow) and/or discretely (jump), leading to trajectories of many types, such as those with finitely many jumps and infinite amount of flow, with an infinite number of intervals of flow with finite (nonzero) length followed by a jump – potentially defining a limit cycle with a jump – and with an infinite number of intervals of flow with finite (but decreasing) length followed by a jump, which are the so-called Zeno solutions. There exist several frameworks capable of modeling such systems as well as tools for their analysis and design [1], [2], [3], [4]. Recent progress in the development of a robust stability theory for hybrid dynamical systems has led to a new framework, known as hybrid inclusions. These developments appear in [5] and include results to assure existence of solutions, as well as to certify asymptotic stability of closed sets and robustness to perturbations. In spite of these advances, the study of existence and robust stability of limit cycles for such systems has not received much attention, even though numerous applications in robotics [6], mechanical systems [7], genetic regulatory networks [8], and neuroscience [9] would benefit from results guaranteeing such properties.

B. Related Work

Nonlinear dynamical systems with periodic solutions are found in many areas, including biological dynamics [10], neuronal systems [11], and population dynamics [12], to name just a few. In recent years, the study of limit cycles in hybrid systems has received renewed attention, mainly due to the existence of hybrid limit cycles in many engineering applications, such as walking robots [6], genetic regulatory networks [8], holonomic mechanical systems subject to impacts [13], among others. Theory for the study of such periodic behavior dates back to the work Andronov et al. in 1966 [14], where self-oscillations (limit cycles) and discontinuous oscillations were studied. Limit cycles has been studied within the impulsive differential equations framework [15], [16], [17], for example in strongly nonlinear systems [18], [19], in slowly impulsive systems [20], in the Van der Pol equation [21], in a holonomic mechanical system subject to impacts [13], and in a weakly nonlinear two-dimensional impulsive system [22]. These early developments pertain to nominal systems given in the form of impulsive differential equations, leaving the question of whether it is possible to handle more general models, such as hybrid system models, and guarantee robustness to generic perturbations wide open.

As a difference to general continuous-time systems, for which the Poincaré-Bendixson theorem uses the topology of $\mathbb{R}^2$ to rule out chaos and offers criteria for existence of limit cycles/periodic orbits, the problem of identifying the existence of limit cycles for hybrid systems has been studied for specific classes of hybrid systems. Specific results for existence of hybrid limit cycles include [6], [23], [30]. In particular, Grizzle et al. establish the existence and stability properties of a periodic orbit of nonlinear systems with impulsive effects via the method of Poincaré sections [6]. Using the transverse contraction framework, the existence and orbital stability of nonlinear hybrid limit cycles are analyzed for a class of autonomous hybrid dynamical systems with impulse in [25]. In [26], the stability and existence of limit cycles in reset control systems are investigated via techniques that rely on the linearization of the Poincaré map about its fixed point. In [27], we analyze the existence of hybrid limit cycles in hybrid...
dynamical systems and establish necessary conditions for the existence of hybrid limit cycles. Clark et al. prove a version of the Poincaré-Bendixson theorem for planar hybrid dynamical systems with empty intersection between the flow set and the jump set [28], and extend the results to the case of an arbitrary number of state spaces (each of which is a subset of $\mathbb{R}^2$) and impacts in [29]. More recently, Goodman and Colombo propose necessary conditions for existence of a periodic orbit related to the Poincaré map and sufficient conditions for local conjugacy between two Poincaré maps in systems with prespecified jump times evolving on a differentiable manifold [30]. We believe that conditions for existence of hybrid limit cycles in general hybrid systems should play a more prominent role in analysis and control of hybrid limit cycles. To the best of our knowledge, tools for the analysis of existence or nonexistence of hybrid limit cycles for the class of hybrid systems in [5], [31] are still not available in the literature.

Stability issues of hybrid limit cycles are currently a major focus in studying hybrid systems for their practical value in applications. Due to the complicated behavior caused by interaction between continuous change and instantaneous change, the study of stability of limit cycles in hybrid systems is more difficult than the study in continuous systems or discrete systems, and so becomes a challenging issue. In this respect, the Poincaré map and its variations or generalizations still play a dominating role; see, e.g., [32], [41]. For instance, Nersesov et al. generalize the Poincaré method to analyze limit cycles for left-continuous hybrid impulsive dynamical systems [32]. Gonçalves analytically develops the local stability of limit cycles in a class of switched linear systems when a limit cycle exists [35]. The authors in [36] analyze local stability of a predefined limit cycle for switched affine systems and design switching surfaces by computing eigenvalues of the Jacobian of the Poincaré map. The trajectory sensitivity approach in [37] is employed to develop sufficient conditions for stability of limit cycles in switched differential-algebraic systems. Motivated by robotics applications, the authors in [38], [39], [40], [41] analyze the stabilization of periodic orbits in systems with impulsive effects using the Jacobian linearization of the Poincaré return map and the relationship between the stability of the return map and the stability of the hybrid zero dynamics. To the best of our knowledge, all of the aforementioned results about limit cycles are only suitable for hybrid systems that have jumps on switching surfaces and under nominal/noise-free conditions. In fact, the results therein do not characterize the robustness properties to perturbations of stable hybrid limit cycles, which is a very challenging problem due to the impulsive behavior in such systems.

Besides our preliminary results in [27], [42], [43], results for the study of existence and robustness of limit cycles in hybrid systems are currently missing from the literature, being perhaps the main reason that a robust stability theory for such systems has only been recently developed in [5], [31]. In fact, all of the aforementioned results about limit cycles are formulated for hybrid systems operating in nominal/noise-free conditions. The development of tools that characterize the existence of hybrid limit cycles and the robustness properties to perturbations of stable hybrid limit cycles is very challenging and demands a modeling framework that properly handles time and the complex combination of continuous and discrete dynamics.

C. Contributions

Tools for the analysis of existence of limit cycles and robustness of asymptotic stability of limit cycles in hybrid systems are not yet available in the literature. In this paper, we propose such tools for hybrid systems given as hybrid inclusions [5], which is a broad modeling framework for hybrid systems as it subsumes hybrid automata, impulsive systems, reset systems, among others; see [5], [31] for more details. We introduce a notion of hybrid limit cycle for hybrid systems modeled as hybrid equations, which are given by

$$\mathcal{H} \left\{ \begin{array}{ll}
\dot{x} = f(x) & x \in C, \\
x^+ = g(x) & x \in D,
\end{array} \right.$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ denotes the state of the system, $\dot{x}$ denotes its derivative with respect to time, and $x^+$ denotes its value after a jump. The state $x$ may have components that correspond to physical states, logic variables, timers, memory states, etc. The map $f$ and the set $C$ define the continuous dynamics (or flows), and the map $g$ and the set $D$ define the discrete dynamics (or jumps). In particular, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ (respectively, $g : \mathbb{R}^n \to \mathbb{R}^n$) is a single-valued map describing the continuous evolution (respectively, the discrete evolution) while $C \subset \mathbb{R}^n$ (respectively, $D \subset \mathbb{R}^n$) is the set on which the flow map $f$ is effective (respectively, from which jumps can occur).

For this hybrid systems framework, we develop tools for characterizing existence of hybrid limit cycles and robustness properties to perturbations of stable hybrid limit cycles. The contributions of this paper include the following:

- We introduce a notion of hybrid limit cycle (with one jump per period)\(^1\) for the class of hybrid systems in (1). Also, we define the notion of flow periodic solution and asymptotic stability of the hybrid limit cycle for such hybrid systems\(^2\).
- We present necessary conditions for existence of hybrid limit cycles, including compactness, finite-time convergence of the jump set, and transversality of the limit cycle, as well as a continuity of the so-called time-to-impact function. Particularly, a condition using a forward invariance notion for existence of hybrid limit cycles is first presented.
- Motivated by the use of Zhukovskii stability methods for periodic orbits in continuous-time systems, as done in [45], [46], [47], we introduce this notion for the class of hybrid systems introduced in (1) and provide a sufficient condition for Zhukovskii stability that involves the incremental stability notion introduced in [48].
- By assuming that the state space contains no isolated equilibrium point for the flow dynamics, we establish

\(^1\)Here, we mainly focus on hybrid limit cycles with “one jump per period.” The case of multiple jumps per period can be treated similarly; see [45], [49].

\(^2\)In this work, a hybrid limit cycle is given by a closed set, while the limit cycle defined in [6], [38], [44] is given by an open set due to the right continuity assumption in the definition of solutions.
a sufficient condition for the existence of hybrid limit cycles based on Zhukovskii stability. In addition, based on an incremental graphical stability notion introduced in [43], an approach to rule out existence of hybrid limit cycles in some cases is proposed.

• We establish sufficient and necessary conditions for guaranteeing (local and global) asymptotic stability of hybrid limit cycles for a class of hybrid systems. In the process of deriving these results, we construct time-to-impact functions and Poincaré maps that cope with one jump per period of a hybrid limit cycle.

• Via perturbation analysis for hybrid systems, we propose a result on robustness to generic perturbations of asymptotically stable hybrid limit cycles, which allows for state noise and unmodeled dynamics, in terms of $KL$ bounds.

• Due to the wide applicability of the Poincaré section method, we present results that relate the properties of a computed Poincaré map, which is necessarily affected by computational error, to the actual asymptotic stability properties of hybrid limit cycles.

D. Organization and Notation
The organization of the paper is as follows.

• Section II presents two motivational examples.

• Section III gives some preliminaries on hybrid systems and basic properties of hybrid limit cycles.

• Section IV presents several necessary conditions for existence of hybrid limit cycles.

• In Section V-A, the Zhukovskii stability notion and incremental graphical stability notion are introduced. Moreover, the relationship between these two notions is studied.

• In Section V-B a sufficient condition for existence of hybrid limit cycles is established including a result on nonexistence of hybrid limit cycles.

• With the hybrid limit cycle definition, Section VI establishes sufficient conditions for stability of hybrid limit cycles.

• Section VII provides results on general robustness of stability to perturbations. In addition, several examples are presented throughout the paper, including spiking neurons, which exhibit regular spiking, and a congestion control system, which exhibits periodic solutions.

• Section VIII concludes the paper.

Notation. The set $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} := [0, +\infty)$, and $\mathbb{N}$ denotes the set of natural numbers including 0, i.e., $\mathbb{N} := \{0, 1, 2, \ldots\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set $S$, $S^n$ denotes $n$ cross products of $S$, namely $S^n = S \times S \times \cdots \times S$. Given a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the Lie derivative of $h$ at $x$ in the direction of $f$ is denoted by $L_fh(x) := \langle \nabla h(x), f(x) \rangle$. Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, its domain of definition is denoted by $\text{dom} f$, i.e., $\text{dom} f := \{x \in \mathbb{R}^n : f(x) \text{ is defined}\}$. The range of $f$ is denoted by $\text{rge} f$, i.e., $\text{rge} f := \{f(x) : x \in \text{dom} f\}$. Given a closed set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_A := \inf_{y \in A} |x-y|$.

Given a set $A \subset \mathbb{R}^n$, $\overline{A}$ (respectively, $\text{int} A$) denotes its closure (respectively, its closed convex hull) and $A^0$ denotes its interior. Given an open set $\mathcal{X} \subset \mathbb{R}^n$ containing a compact set $A$, a function $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a proper indicator for $A$ on $\mathcal{X}$ if $\omega$ is continuous, $\omega(x) = 0$ if and only if $x \in A$, and $\omega(x) \to \infty$ as $x$ approaches the boundary of $\mathcal{X}$ or as $|x| \to \infty$. Given a sequence of set $\mathcal{X}_i$, $\limsup_{i \to \infty} \mathcal{X}_i$ denotes the outer limit of $\mathcal{X}_i$. The set $\mathcal{B}$ denotes a closed unit ball in Euclidean space (of appropriate dimension) centered at zero. Given $\delta > 0$ and $x \in \mathbb{R}^n$, $x + \delta \mathcal{B}$ denotes a closed ball centered at $x$ with radius $\delta$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$KL$ ($\alpha \in KL$) if it is continuous, zero at zero, and strictly increasing; it belongs to class-$KL$ ($\alpha \in KL$) if, in addition, is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$KL$ ($\beta \in KL$) if, for each $t \geq 0$, $\beta(t, \cdot)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s, t) = 0$ and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to +\infty} \beta(s, t) = 0$.

II. Motivational Examples
The following examples motivate the need of tools for the study of existence and robust stability of limit cycles in hybrid systems.

Example 2.1: Consider the hybrid model for a congestion control mechanism in TCP proposed in [49]. The hybrid model in congestion avoidance mode can be described as follows:

• when $q \in [0, q_{\text{max}}]$

\[
\begin{aligned}
\dot{q} &= \begin{cases}
\max\{0, r - B\} & \text{if } q = 0 \\
\frac{r}{a} - B & \text{if } q > 0
\end{cases} \\
\dot{r} &= \begin{cases}
a & \text{if } q = 0 \\
r - B & \text{if } q > 0
\end{cases}
\end{aligned}
\] (2a)

• when $q = q_{\text{max}}, r \geq B$

\[
\begin{aligned}
q^+ &= \begin{cases}
q_{\text{max}} & \text{if } q = 0 \\
\frac{r}{m} & \text{if } q > 0
\end{cases} \\
r^+ &= \begin{cases}
q_{\text{max}} & \text{if } q = 0 \\
r - B & \text{if } q > 0
\end{cases}
\end{aligned}
\] (2b)

where $q \in [0, q_{\text{max}}]$ denotes the current queue size, $q_{\text{max}}$ is the maximum queue size, $r \geq 0$ is the rate of incoming data packets, and $B \geq 0$ is the rate of outgoing packets. The constant $a \geq 1$ reflects the rate of growth of incoming data packets $r$ while $m \in (0, 1)$ reflects the factor that makes the rate of incoming packets decrease; see [49] for details. The model in (2) reduces the rate of incoming packets $r$ by the factor $m$ if the queue size $q$ equals the maximum value $q_{\text{max}}$ with rate larger than or equal to $B$.

We are interested in the hybrid system (2) restricted to the region

\[
M_T := \{(q, r) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : q \leq q_{\text{max}}, aq \geq \frac{1}{2} r^2 - Br + \frac{B^2}{2}\}
\] (3)

for given parameters $a$, $m$, $q_{\text{max}}$ and $B$ (later, the set $M_T$ will be part of our analysis); see Fig. I. From the first piece in the definition in (2) with $a > 0$, for any maximal solution with initial condition with zero $q$ and $r$ less than $B$, $q$ remains at zero until $r > B$. Fig. I is shown to analyze how we get a region from which a limit cycle with one jump exists. The points in the curve $P_3 \to P_4 \to P_5$ satisfy $aq = \frac{1}{2} r^2 - Br + \frac{B^2}{2}$. Solutions from the region $M_T$ result in solutions such that $q$ reaches zero and remains at zero until $r = B$ (point $P_4$).
The open set $M_1 := \{(q_{\text{max}}, B)\} + \varepsilon B^0$ with $\varepsilon > 0$ small enough, will be part of our analysis in Example 4.4 and be ruled out to ensure the transversality of the limit cycle. We are not interested in the region $M_2$ with gray filled pattern as it leads to a complex hybrid model which might be hard to be analyzed. The compact set $M_T$ is marked by the region with light green filled pattern. Hence, the set $M_T \setminus M_1$ (the region surrounded by blue line) is the region of the state space that we are interested in. Note that if the value of $r$ after a jump from the point $P_3$ is larger than $B$ (for instance, point $P_5$ jumps to point $P_1$), a consecutive jump will happen. Therefore, to avoid this case, we impose the condition $m(B + \sqrt{2a}q_{\text{max}}) < B$.

From points in the set $M_T$, solutions approach a limit cycle. On $M_T$ and for parameters satisfying the conditions above, the resulting system with $(q, r) \in M_T$ can be described as a hybrid system $\mathcal{H}_{\text{TCP}}$ on $M_T$ with data

$$
\mathcal{H}_{\text{TCP}} \left\{ \begin{align*}
\dot{x} &= f_{\text{TCP}}(x) := \left[ \frac{r - B}{a} \right] x \in \mathcal{C}_{\text{TCP}}, \\
q^+ &= g_{\text{TCP}}(x) := q_{\text{max}} m r \quad x \in \mathcal{D}_{\text{TCP}},
\end{align*} \right.
$$

where $x = (q, r)$, $\mathcal{C}_{\text{TCP}} = \{x \in \mathbb{R}^2 : q \leq q_{\text{max}}\}$, $\mathcal{D}_{\text{TCP}} = \{x \in \mathbb{R}^2 : q = q_{\text{max}}, r \geq B\}$.

A limit cycle of the system in (4) with parameters $B = 1, a = 0.25$, and $q_{\text{max}} = 1$ is depicted in Fig. 2. This figure shows in blue a limit cycle denoted as $\mathcal{O}$ and defined by the solution to the congestion control system with initial condition $P_2 = \{(1, 0.4)\}$. This solution flows to the point $P_1$, jumps to the point $P_2$, and then flows back to $P_1$. The interest in this paper is to find conditions under which such limit cycles may exist.

**Example 2.2:** Consider the Izhikevich neuron model given by

$$
\begin{align*}
\dot{v} &= 0.04v^2 + 5v + 140 - w + I_{\text{ext}}, \\
w &= a(bv - w),
\end{align*}
$$

where $v$ is the membrane potential, $w$ is the recovery variable, and $I_{\text{ext}}$ represents the synaptic (injected) DC current. When the membrane voltage of a neuron increases and reaches a threshold (which in [9] is equal to 30 millivolts), the membrane voltage and the recovery variable are instantaneously reset via the following rule:

$$
\text{when } v \geq 30, \quad \left\{ \begin{align*}
v^+ &= c, \\
w^+ &= w + d.
\end{align*} \right.
$$

The value of the input $I_{\text{ext}}$ and the model parameters $a, b, c$, and $d$ are used to determine the neuron type. In fact, the model in (5) can exhibit a specific firing pattern (of all known types) of cortical neurons when these parameters are appropriately chosen [9]. For instance, when the parameters are chosen as

$$a = 0.02, \quad b = 0.2, \quad c = -55, \quad d = 4, \quad I_{\text{ext}} = 10,$$

the neuron model exhibits intrinsic bursting behavior. This corresponds to a limit cycle, which is denoted as $\mathcal{O}$, and is defined by the solution to (5) that jumps from point $P_1$ to point $P_2$ and then flows back to $P_1$; see Fig. 3.

As suggested in Fig. 3, the limit cycle $\mathcal{O}$ is asymptotically stable. In particular, solutions initialized close to the set $\mathcal{O}$ stay close for all time and converge to the set $\mathcal{O}$ as time gets large. For instance, the trajectory (black line) of a solution starting from $(-54.76, -3.5)$ (the point $P_3$ in the subfigure), which is close to the point $P_2$, remains close to the limit cycle $\mathcal{O}$ and approaches it eventually. However, solutions initialized relatively far away from the set $\mathcal{O}$ may not stay close for all time. For instance, as shown in Fig. 3 the trajectory (red line) of a solution starting from $(-54.5, -3.5)$ (the point $P_3$ in the subfigure) that is close to the point $P_2$ first goes far away from the limit cycle $\mathcal{O}$ and approaches it eventually.

Interestingly, solutions to the neuron model with state perturbations, in particular, solutions to (5) with an admissible state perturbation may not be always close to the nominal solutions. For instance, an additive perturbation $e = (0.24, 0)$ (or $e = (0.5, 0)$, respectively) to $(v^+, w^+)$ after a jump from the point $P_1$ would result in a state value equal to the point $P_3$ (or to the point $P_4$, respectively) instead of the point $P_2$. As shown in Fig. 3 the trajectory (black line) from the point $P_3$ remains close to the limit cycle $\mathcal{O}$ and approaches it eventually, while the trajectory (red line) from the point $P_3$ does not stay close to the one from the point $P_2$. Since the points $P_3$ and $P_4$ are close to each other but the trajectories from those points are not, the stability property of the limit cycle $\mathcal{O}$ has a small margin of robustness to perturbations.

A mapping $e$ is an admissible state perturbation if $\text{dom } e$ is a hybrid time domain and the function $t \mapsto e(t, j)$ is measurable on $\text{dom } e \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. See [5, Definition 4.5] for more details.
points $x \in \mathbb{R}^n$ for which there exists a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ of points $(t_i, j_i) \in \text{dom} \phi$ with $\lim_{i \to \infty} t_i + j_i = \infty$ and $\lim_{i \to \infty} \phi(t_i, j_i) = x$. Every such point $x$ is an $\omega$-limit point of $\phi$.

For more details about this hybrid systems framework, we refer the readers to [5].

B. Hybrid Limit Cycles

Before revealing their basic properties, we define hybrid limit cycles. For this purpose, we consider the following notion of flow periodic solutions.

**Definition 3.2:** (flow periodic solution) A complete solution $\phi^*$ to $\mathcal{H}$ is flow periodic with period $T^*$ and one jump in each period if $T^* \in (0, \infty)$ is the smallest number such that $\phi^*(t + T^*, j + 1) = \phi^*(t, j)$ for all $(t, j) \in \text{dom} \phi^*$.

The definition of a flow periodic solution $\phi^*$ with period $T^* > 0$ above implies that if $(t, j) \in \text{dom} \phi^*$, then $(t + T^*, j + 1) \in \text{dom} \phi^*$. For a notion allowing for multiple jumps in a period, see [43], [50]. A flow periodic solution to $\mathcal{H}$ as in Definition 3.2 generates a hybrid limit cycle.

**Definition 3.3:** (hybrid limit cycle) A flow periodic solution $\phi^*$ with period $T^* \in (0, \infty)$ and one jump in each period defines a hybrid limit cycle $\mathcal{O} := \{x \in \mathbb{R}^n : x = \phi^*(t, j), (t, j) \in \text{dom} \phi^*\}$.

Perhaps the simplest hybrid system with a hybrid limit cycle is the scalar system capturing the dynamics of a timer with resets, namely

$$
\mathcal{H}_T \quad \begin{cases}
\dot{x} = 1 & \chi \in [0, 1), \\
\dot{x} = 0 & \chi = 1,
\end{cases}
$$

where $\chi \in [0, 1]$. Its unique maximal solution from $(0, 0)$ is given by $\phi(t, j) = \xi + t - j$ for each $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $t \in [\max\{0, j - \xi\}, j + 1 - \xi]$. The hybrid limit cycle generated by $\phi$ is $\mathcal{O} = \{\xi \in [0, 1] : \phi(t, 1), t \in [1 - \xi, 2 - \xi]\} = [0, 1]$.

**Remark 3.4:** The definition of a hybrid limit cycle $\mathcal{O}$ with period $T^* > 0$ implies that $\mathcal{O}$ is nonempty and contains more than two points; in particular, a hybrid arc that generates $\mathcal{O}$ cannot be discrete. A hybrid limit cycle $\mathcal{O}$ is restricted to have one jump per period, but extensions to more complex cases are possible [43], [50].

Next, the more advanced examples in Section III are revisited to further illustrate the hybrid limit cycle notion in Definition 3.3.

**Example 3.5:** Consider the hybrid congestion control system $\mathcal{H}_{TCP}$ on $M_T$ in Example 2.1. As suggested by Fig. 2, system 3 has a flow periodic solution $\phi^*$ with period $T^*$. In fact, a solution $\phi^*$ to the system 2 starting from the point $P_2$ in Fig. 2, i.e., $\phi^*(0, 0) = \xi_1$ and $\phi^*(2, 0) = \xi_2$ with

$$
(\xi_1, \xi_2) = \left(\frac{q_{\max}}{2}, \frac{2Bm}{(1 + m)}\right) \in M_T \cap (C_{TCP} \cup D_{TCP}),
$$

is given by

$$
\phi^*(t, j) = \xi_1 + (\xi_2 - B)(t - jT^*) + \frac{a(t - jT^*)^2}{2},
$$

$$
\phi^*(t, j) = \xi_2 + a(t - jT^*),
$$

Alternatively, the hybrid limit cycle $\mathcal{O}$ can be written as $\{x \in \mathbb{R}^n : x = \phi^*(t, j), t \in [t_s, t_s + T^*], (t, j) \in \text{dom} \phi^*\}$ for some $t_s \in \mathbb{R}_{\geq 0}$.
for all \((t, j) \in \text{dom} \phi^*\) with \(t \in [T^*, (j + 1)T^*)\) and \(j \geq 0\), where \(T^* = 2B(1 - m)/(a + ma)\). From (8) and (9), when \(t = T^*\) and \(j = 0\), it is easy to obtain that \(\phi_1^*(T^*, 0) = \xi_1 + (\xi_2 - B)T^* + Bm/2 = q_{\text{max}}\) and \(\phi_2^*(T^*, 0) = \xi_2 + aT^* = 2Bm/(1 + m) + 2B(1 - m)/(1 + m) = 2B/(1 + m) > B\) (using the fact \(m \in (0, 1)\)). Therefore, \(\phi^*(T^*, 0) = D_{\text{cr}}\) and the first jump happens. Then, according to the jump map, the state is updated with \(\phi_1^*(T^*, 1) = q_{\text{max}} = \xi_1\) and \(\phi_2^*(T^*, 1) = m\phi_2^*(T^*, 0) = 2Bm/(1 + m) = \xi_2\), which equals the initial value. In addition, since \(m \in (0, 1)\), we have \(2Bm/(1 + m) < B\) implying that there is only one jump after every interval of flow.

Using (8) and (9), for solutions initialized at \((\xi_1, \xi_2) = (q_{\text{max}}, 2Bm/(1 + m))\), a hybrid limit cycle can be characterized as \(\mathcal{O} = \{x \in \mathbb{R}^2 : x = \phi(t, 1), t \in [T_1, T_1 + T^*)\}\). Furthermore, any solution \(\bar{\phi}\) to \(\mathcal{H}_{\text{cr}}\) from \(\bar{\phi}(0, 0) \in \mathcal{O}\) satisfies \(|\bar{\phi}(t, J)| = 0\) for all \((t, J) \in \text{dom} \bar{\phi}\). In fact, using the parameters given in Example 2.2 the solution to \(\mathcal{H}_{\text{cr}}\) from \(P_2 = (1, 0, 4)\) generates a hybrid limit cycle as depicted in red in Fig. 4.

Example 3.6: Consider the Izhikevich neuron system in Example 2.2. This neuron system is slightly modified and written as a hybrid system \(\mathcal{H}_1\) in (10), which we denote as \(\mathcal{H}_1\) and is given by

\[
\begin{align*}
\dot{x} &= f_1(x) := \begin{cases} f_1(x) = \frac{1}{a(bv - w)} x \in C_1, \\ g_1(x) = \frac{c}{w + d} \quad x \in D_1, \end{cases} 
\end{align*}
\]

where \(x = (v, w)\), \(f_1(x) = 0.04v^2 + 5v + 140 - w + I_{\text{ext}}\), \(C_1 = \{x \in \mathbb{R}^2 : v \leq 30\}\), \(D_1 = \{x \in \mathbb{R}^2 : v = 30, f_1(x) \geq 0\}\), where \(f_1(x) \geq 0\) models the fact that the spikes occur when the membrane potential \(v\) grows to the 30mV threshold.

The neuron system has a flow periodic solution with \(T^*\). However, due to the nonlinear form of the flow map, an analytic expression of the solution is not available. For parameters of \(\mathcal{H}_1\) given by (6), a numerical approximation of this solution is shown in Fig. 4 which is given by the solution \(\phi^*\) to \(\mathcal{H}_1\) from \(\phi^*(0, 0) = (-55, -3.5)\). This solution is flow periodic with \(T^* \approx 31.24\text{ms}\).

IV. NECESSARY CONDITIONS

Next, we study necessary conditions for existence of hybrid limit cycles defined in Section III in Section IV-A under mild assumptions, we derive necessary conditions that guarantee compactness and transversality of the limit cycle, finite-time convergence of the jump set, as well as a continuity of the time-to-impact function. In Section IV-B we present a necessary condition for existence of a hybrid limit cycle with period \(T^*\) using a forward invariance notion.

A. NECESSARY CONDITIONS FOR EXISTENCE OF HYBRID LIMIT CYCLES IN A CLASS OF HYBRID SYSTEMS

In this subsection, we derive several necessary conditions for the existence of hybrid limit cycles for a class of hybrid systems \(\mathcal{H}\) as in (1) satisfying the following properties.

Assumption 4.1: For a hybrid system \(\mathcal{H} = (C, f, D, g)\) on \(\mathbb{R}^n\) and a compact set \(M \subseteq \mathbb{R}^n\), there exists a continuously differentiable function \(h : \mathbb{R}^n \to \mathbb{R}\) such that

1) the flow set can be written as \(C = \{x \in \mathbb{R}^n : h(x) \geq 0\}\) and the jump set as \(D = \{x \in \mathbb{R}^n : h(x) = 0, L_fh(x) < 0\}\);
2) the flow map \(f\) is continuously differentiable on an open neighborhood of \(M \cap C\), and the jump map \(g\) is continuous on \(M \cap D\);
3) \(L_fh(x) < 0\) for all \(x \in M \cap D\), and \(g(M \cap D) \cap (M \cap D) = \emptyset\).

Remark 4.2: Item 1) in Assumption 4.1 implies that flows occur when \(h\) is nonnegative while jumps only occur at points in the zero level set of \(h\). Note that since \(h\) is continuous and \(f\) is continuously differentiable, the flow set and the jump set are closed. The state \(x\) may include logic variables, counters, timers, etc. The continuity property of \(f\) in item 2) of Assumption 4.1 is further required for the existence of solutions to \(\dot{x} = f(x)\) according to [5, Proposition 2.10]. Moreover, item 2) also guarantees that solutions to \(\dot{x} = f(x)\) depend continuously on initial conditions. In the upcoming results, item 3) in Assumption 4.1 allows us to establish a transversality property and restrict the analysis of a hybrid system \(\mathcal{H}\) to a region of a state space \(M \subseteq \mathbb{R}^n\), leading to the restriction of \(\mathcal{H}\) given by \(\mathcal{H}|_M := (M \cap C, f, M \cap D, g)\). The condition \(g(M \cap D) \cap (M \cap D) = \emptyset\) is assumed to exclude discrete solutions. As we will show later, the set \(M\) is appropriately chosen for each specific system such that it guarantees completeness of maximal solutions to \(\mathcal{H}|_M\) and the existence of flow periodic solutions. This is illustrated in Example 2.2 with a set \(M_T\). See also the forthcoming Example 4.7 and Example 4.16.

Remark 4.3: By items 1) and 2) of Assumption 4.1, the data of \(\mathcal{H}|_M\) satisfies the hybrid basic conditions [5, Assumption 6.5]. Then, using item 3) of Assumption 4.1 [5, Lemma 2.7] implies that for any bounded and complete solution \(\phi\) to \(\mathcal{H}|_M\) there exists \(r > 0\) such that \(t_{j+1} - t_j \geq r\) for all \(j \geq 0, t_j = \min I_j^\prime, t_{j+1} = \max I_j^\prime\); i.e., the elapsed time between two consecutive jumps is uniformly bounded below by a positive constant.

It can be shown that a hybrid limit cycle generated by periodic solutions as in Definition 3.3 is closed and bounded, as established in the following result.

Lemma 4.4: Given a hybrid system \(\mathcal{H} = (C, f, D, g)\) on \(\mathbb{R}^n\) and a closed set \(M \subseteq \mathbb{R}^n\) satisfying Assumption 4.1...
suppose that $\mathcal{H}$ has a hybrid limit cycle $O$. Then, $O$ is compact and forward invariant.\footnote{Every $\phi \in \mathcal{S}_H(\mathcal{O})$ is complete and satisfies rge $\phi \subset \mathcal{O}$; see Definition 3.3.}

Proof: We show that any hybrid limit cycle $O$ is closed and bounded. First, to prove closedness of $O$, consider a flow periodic solution $\phi^*$ to $\mathcal{H}$ from $\phi^*(0,0)$ associated with a hybrid limit cycle $O$ with period $T^*$. Then, $O = \{x \in \mathbb{R}^n : x = \phi^*(t,j), t \in [0,T^*], j \in \{0,1\}, (t,j) \in \text{dom } \phi^*\}$. We have the following cases for the first jump of $\phi^*$. Let $t^*$ be such that $(t^*,0) \in \text{dom } \phi^*$, $(t^*,1) \in \text{dom } \phi^*$.

- If $t^* = 0$, i.e., $(0,1) \in \text{dom } \phi^*$, then $O = \{x \in \mathbb{R}^n : x = \phi^*(0,0), t \in [0,T^*], (t,1) \in \text{dom } \phi^*\}$. Since $\phi^*(0,0) \in D$ and $D \subset \text{dom } g$, $\phi^*(0,1) = g(\phi^*(0,0))$ is bounded. Since $t \mapsto \phi^*(t,1)$ is continuous on $[0,T^*]$ due to the continuity of $f$ from item 2) of Assumption 4.1 and $[0,t^*]$ is a closed interval, the set $\{x \in \mathbb{R}^n : x = \phi^*(t,1), t \in [0,T^*], (t,1) \in \text{dom } \phi^*\}$ is closed. Then, $O$ is closed as it is the union of two closed sets;

- If $0 < t^* < T^*$, then $O = \{x \in \mathbb{R}^n : x = \phi^*(t,0), t \in [0,t^*], (t,0) \in \text{dom } \phi^*\} \cup \{x \in \mathbb{R}^n : x = \phi^*(t,1), t \in [t^*,T^*], (t,1) \in \text{dom } \phi^*\}$. Using the continuity of $f$ from item 2) of Assumption 4.1, since $t \mapsto \phi^*(t,0)$ is continuous on $[0,t^*]$ and the set $[0,t^*]$ is a closed interval, the set $\{x \in \mathbb{R}^n : x = \phi^*(t,0), t \in [0,t^*], (t,0) \in \text{dom } \phi^*\}$ is closed. Similarly, since $t \mapsto \phi^*(t,1)$ is continuous on the closed interval $[t^*,T^*]$, the set $\{x \in \mathbb{R}^n : x = \phi^*(t,1), t \in [t^*,T^*], (t,1) \in \text{dom } \phi^*\}$ is closed. Therefore, $O$ is closed;

- If $t^* = T^*$, the argument follows similarly as in the first case above.

To prove its boundedness, we proceed by contradiction. Suppose that $O$ is unbounded. Then, due to $T^*$ being finite and $D \subset \text{dom } g$, it follows that the flow periodic solution $\phi^*$ can only escape to infinity in finite time during flows. This further implies that $\phi^*$ is not complete and that its domain is not closed, which leads to a contradiction with the definition of flow periodic solution with period $T^*$.

Next, we prove that $O$ is forward invariant. First, each $\phi^* \in \mathcal{S}_H(\mathcal{O})$ is complete, which directly follows from the definition of flow periodic solution in Definition 5.2. Then, it remains to prove that each $\phi^* \in \mathcal{S}_H(\mathcal{O})$ stays in $O$, that is, $\text{rge } \phi^* \subset O$. By contradiction, suppose that $\text{rge } \phi^* \not\subset O$. Then, by Definition 5.3, the flow periodic solution $\phi^*$ defines another hybrid limit cycle $O'$, which contradicts with the uniqueness of the hybrid limit cycle $O$. Therefore, for each $\phi^* \in \mathcal{S}_H(\mathcal{O})$, we have $\text{rge } \phi^* \subset O$. Hence, $O$ is forward invariant.

Remark 4.5: Since a hybrid limit cycle $O$ to $\mathcal{H}|_M$ is compact, for any solution $\phi$ to $\mathcal{H}|_M$, the distance $|\phi(t,j)|_O$ is well-defined for all $(t,j) \in \text{dom } \phi$.

We revisit the previous examples to illustrate the properties of a hybrid system $\mathcal{H}$ satisfying Assumption 4.1.

Example 4.6: Consider the congestion control system in Example 2.1, see also Example 3.5. By definition, the sets $C_{TCP}$ and $D_{TCP}$ of the model in (4) are closed. Moreover, $f_{TCP}$ and $g_{TCP}$ are continuously differentiable. Define the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $h(x) = q_{max} - q$. Then, $C_{TCP}$ and $D_{TCP}$ can be written as $C_{TCP} = \{x \in \mathbb{R}^2 : h(x) \geq 0\}$ and $D_{TCP} = \{x \in \mathbb{R}^2 : h(x) = 0, L_{TCP}(x) \leq 0\}$, respectively. Consider the compact set $M_{TCP} := (M_T \cap C_{TCP}) \cap M_I$, where $M_T$ is defined in (4) and $M_I = \{(q_{max},B) + C\mathcal{B}^{\omega} | \epsilon > 0\}$ small enough; see Fig. 1. We obtain that $M_{TCP} \cap D_{TCP} = \{x \in \mathbb{R}^2 : q = q_{max}, \tau \in [B + \epsilon, B + \sqrt{2q_{max}}]\}$ and for each $x \in M_{TCP} \cap D_{TCP}, f_{TCP}(x) = h(x) = B - r < 0$. Moreover, due to the condition on parameters $m(B + \sqrt{2q_{max}}) < B$ (see Example 2.1), it can be verified that $g_{TCP}(M_{TCP} \cap D_{TCP}) \cap (M_{TCP} \cap D_{TCP}) = \emptyset$ and $g_{TCP}(M_{TCP} \cap D_{TCP}) \subset M_{TCP} \cap C_{TCP}$. Furthermore, for any point $x \in M_{TCP} \cap C_{TCP}$, since the $r$ component of the flow map $f_{TCP}$, i.e., $r \rightarrow a$, is positive, $M_{TCP} \cap C_{TCP} \cap f_{TCP}(x) \neq \emptyset$. Let $x \in M_{TCP} \cap D_{TCP}$, we have $q = q_{max}$ and $r \geq B + \epsilon$ with $\epsilon > 0$ small enough, which implies that $r - B > 0$ and solutions from $x$ cannot be extended via flow. By Proposition 6.10, every maximal solution to $\mathcal{H}_{TCP}|_{M_{TCP}} = (C_{TCP} \cap M_{TCP}, \{f_{TCP}, M_{TCP} \cap D_{TCP}, g_{TCP}\})$ is complete. Therefore, Assumption 4.1 holds. Moreover, a solution $\phi^*$ to $\mathcal{H}_{TCP}|_{M_{TCP}}$ from $\phi^*(0,0) = (q_{max}, 2Bm/(1 + m)) \in M_{TCP} \cap C_{TCP}$ is a flow periodic solution with $T^* = 2B(1/m)/(a + ma)$.

Example 4.7: Consider the Izhikevich neuron system introduced in Example 3.5. By design, the sets $C_I$ and $D_I$ of the model in (10) are closed. Moreover, $f_I$ and $g_I$ are continuously differentiable. Define the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $h(x) = 30 - v$. Then, $C_I$ and $D_I$ can be written as $C_I = \{x \in \mathbb{R}^2 : h(x) \geq 0\}$ and $D_I = \{x \in \mathbb{R}^2 : h(x) = 0, L_f(x) \leq 0\}$, respectively. Consider the closed set $M_I := \{x \in \mathbb{R}^2 : v \leq 325 + I_{ext}\}$. Then, for each $x \in M_I \cap D_I$ we have $L_f(x) = -f_I(x) = -0.04v^2 + 5v + 140 - w + I_{ext} \leq -326 < 0$. Hence, $H_I|_{M_I}$ is complete. Therefore, the neuron system $H_I$ on $\mathbb{R}^2$ and $M_I$ satisfy Assumption 4.1 and, as pointed out in Example 3.6, $H_I|_{M_I}$ has a flow periodic solution $\phi^*$ with period $T^*$, which defines a unique hybrid limit cycle $O \subset M_I \cap C_I$.

Example 4.8: Consider a compass gait biped in (54), (55) which consists of a double pendulum with point masses $m_h$ and $m$ concentrated at the hip and legs (a stance leg and a swing leg), respectively, as shown in Fig. 5. The two legs are modeled as rigid bars without knees and feet, and with a frictionless hip. Given adequate initial conditions, the compass gait biped robot is powered only by gravity and performs a
passive walk for a given constant slope $\phi$ without any external intervention. The movement of the compass gait biped is mainly composed of two phases: a swing phase and an impact phase. In the first case, the biped is modeled as a double pendulum. The latter case occurs when the swing leg strikes the ground and the stance leg leaves the ground. Therefore, the compass gait biped can be modeled as a hybrid system to describe the continuous and discrete dynamics of the system.

The movement of the legs and hip during each step is described by the continuous dynamics of the model, while the discrete dynamics describe the instantaneous change that occurs upon the impact at the end of each step. At all times, one of the legs is the stance leg while the other is the swing leg, and they switch roles upon each step. The state component vector $x$ is comprised of the angle vector $\theta$, which contains the non-support (swing) leg angle $\theta_n$ and the stance leg angle $\theta_s$; the velocity vector $\dot{\theta}$, which contains the non-support (swing) leg angular velocity $\dot{\theta}_n$ and the stance leg angular velocity $\dot{\theta}_s$.

A hybrid system model of the compass gait biped, denoted $H_B$, is defined as

\[
H_B : \begin{cases} 
\dot{x} = f_B(x) := -\mathcal{M}(q)^{-1}(\mathcal{N}(q, \dot{q}) + \mathcal{G}(q)) & x \in C_B \\
\dot{x} = g_B(x) := \begin{bmatrix} \Lambda q \\ Q_p(\alpha) + Q_m(\alpha)q \end{bmatrix} & x \in D_B 
\end{cases}
\]

where $x = (q, \dot{q}) = (\theta_n, \theta_s, \dot{\theta}_n, \dot{\theta}_s) \in \mathbb{R}^4$ is the state, $\mathcal{M}(q)$ is the inertia matrix, the matrix with centrifugal coefficients, and the vector of gravitational torques, which is given as

\[
\mathcal{M}(q) = \begin{bmatrix} m_b^2 & -m_b \cos(\theta_s - \theta_n) \\
-m_b \cos(\theta_s - \theta_n) & m_b^2 \\
m_b \cos(\theta_s - \theta_n) \\
\cos(\theta_s - \theta_n) \\
-m_b \cos(\theta_s - \theta_n) & m_b^2 & -m_b \cos(\theta_s - \theta_n) \\
-m_b \cos(\theta_s - \theta_n) & m_b^2 & -m_b \cos(\theta_s - \theta_n) \\
\end{bmatrix}
\]

\[
\mathcal{N}(q, \dot{q}) := \begin{bmatrix} m_b \gamma \sin(\theta_n) \\
-m_b \gamma \sin(\theta_n) \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\mathcal{G}(q) := \begin{bmatrix} -m_b \dot{\theta}_s \sin(\theta_s) \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

The open set $M_0 := \{(q, \dot{q}) \in \mathbb{R}^4 : \dot{\theta}_n > -\epsilon\}$ with $\epsilon > 0$ small enough, is ruled out to ensure that the compass gait biped walks down the slope. Under adequate initial conditions, the compass gait biped system has a flow periodic solution with $T^*$. However, due to the nonlinear form of the flow map, an analytic expression of the solution is not available. For instance, when the initial condition is taken as $(q(0, 0), \dot{q}(0, 0)) = (0, 0, 2, -0.4)$ and the parameters are chosen as

\[
\gamma = 9.81 \text{ m/s}^2, \quad m_b = 12 \text{ kg}, \quad m = 5 \text{ kg}, 
\]

\[
a = b = 0.5 \text{ m}, \quad \phi = 0.0524 \text{ rad}, 
\]

a numerical approximation of this solution is shown in Fig. 6 which is given by the solution $\phi^*$ to $H_B$ from $\phi^*(0, 0) = (-0.33, 0.22, -0.38, -1.09)$. This solution is flow periodic with $T^* \approx 0.385 \text{ s}$.

By design, the sets $C_B$ and $D_B$ are closed. Moreover, $f_B$ and $g_B$ are continuously differentiable. The open sets $M_1 := \{q, \dot{q}) \in C_B : \theta_n + \theta_s + 2\phi \in (-\epsilon, \epsilon), \sin(\theta_n + \phi)(\theta_n + \dot{\theta}_n) > -\epsilon\}$, $M_2 := \{(q, \dot{q}) \in C_B : \theta_n + \theta_s + 2\phi \in (-\epsilon, \epsilon), \theta_n > -\epsilon\}$, and $M_3 := \{(q, \dot{q}) \in C_B : \theta_n - \theta_s \in (-\epsilon, \epsilon), \sin(\theta_n(\theta_n - \theta_s) < \epsilon\}$ with $\epsilon > 0$ small enough, are ruled out to ensure the transversality of the limit cycle, ensure that the non-support leg walks down the slope at jumps (this is natural, since the non-support leg switches its role to the stance leg when it is in contact with the slope), and ensure that a jump happens only when two legs are in contact with the slope, respectively. Consider the closed set $M_B := M_\text{c} \setminus (M_1 \cup M_2 \cup M_3)$. Then, for each $x \in M_B \setminus D_B$, we have $\cos(\theta_n + \phi) = \cos(\theta_n + \phi)$ and $L_{f_B} h(x) = \sin(\theta_n + \phi)\theta_s - \sin(\theta_n + \phi)\theta_n \leq 0$. Hence, at this time, we have two cases: $\theta_n = \theta_s$, or $\theta_n + \phi = -\theta_n + \phi$. By the definition of $M_3$, since $L_{f_B} h(x) = \sin(\theta_n + \phi)(\theta_n - \theta_s) \geq \epsilon > 0$ if $\theta_n = \theta_s$, only the case $\theta_n + \phi = -\theta_n + \phi$ will happen for each $x \in M_B \setminus D_B$, which implies that $\sin(\theta_n + \phi) = -\sin(\theta_n + \phi)$. In addition, by the definition of $M_1$, we have

\[
L_{f_B} h(x) = \sin(\theta_n + \phi)(\theta_n + \theta_s) \leq -\epsilon < 0. 
\]

Moreover, by the jump map $\eta^+ = \Delta q$ for each $x \in M_B \cap D_B$, we have $\theta_n + \phi = -\theta_n + \phi$ and $\theta_n^+ + \phi = \theta_n + \phi = -\theta_n + \phi = -\theta_n + \phi$. In addition, by the definitions of $M_0$ and $M_2$, we have for each $x \in M_B \cap D_B$, $\theta_n + \theta_s < 0$ and $\theta_n^+ + \theta_s^+ < 0$, which together with (13) imply $\sin(\theta_n^+ + $
\( \phi = -\sin(\theta^+_n + \phi) > 0 \). Then, for each \( x \in M_B \cap D_B \), we have that \( L_{f_B} h(x^+) = \sin(\theta^+_n + \phi)(\theta^+_n + \theta^+_s) > 0 \). Therefore, we have \( g_B(M_B \cap D_B) \cap (M_B \cap D_B) = \emptyset \) and \( g_B(M_B \cap D_B) \subset M_B \cap C_B \). Furthermore, for any point \( x \in M_B \cap C_B \), since the \( \theta_n \) component of the flow map \( f_B \), i.e., \( \theta_n \leq -\varepsilon < 0 \), is negative, \( T_{M_B \cap C_B}(x) \cap \{ f_B(x) \} \neq \emptyset \) for each \( x \in (M_B \cap C_B) \cap D_B \). When \( x \in M_B \cap D_B \), we have \( \cos(\theta_n + \phi) = \cos(\theta_n + \phi) \) and \( L_{f_B} h(x) \leq -\varepsilon \) with \( \varepsilon > 0 \) small enough, which implies that \( L_{f_B} h(x) < 0 \) and solutions from \( x \) cannot be extended via flow. By [5], Proposition 6.10], every maximal solution to \( H_{MB} \) is complete. Therefore, the compass gait biped system \( H_{MB} \) on \( \mathbb{R}^4 \) and \( M_B \) satisfy Assumption 4.1 and \( H_{MB} \) has a flow periodic solution \( \phi^* \) with period \( T^* \), which defines a hybrid limit cycle \( \mathcal{O} \subset M_B \cap C_B \).

The following result establishes a transversality property of any hybrid limit cycle for \( H \) restricted to \( M_B \).

**Lemma 4.9:** Given a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1, suppose that \( \mathcal{H}_{|M} = (M \cap C, f, M \cap D, g) \) has a hybrid limit cycle \( O \subset M \cap C \). Then, \( O \) is transversal to \( M \cap D \).

**Proof:** We proceed by contradiction. Consider the flow periodic solution \( \phi^* \) with period \( T^* \) that generates the hybrid limit cycle \( O \) for \( \mathcal{H}_{|M} \). By definition, there exists \( x^0 \in O \) such that \( x^0 \in O \cap (M \cap D) \) and \( \phi^*(t^*, j^*) = x^0 \) for some \( (t^*, j^*) \in dom \phi^* \). Suppose that \( O \) intersects \( M \cap D \) at another point \( x \neq x^0 \), i.e., \( x \in O \cap (M \cap D) \) and \( \phi^*(t^*, j^*) = x \) for some \( (t^*, j^*) \in dom \phi^* \). Then, by items 1) and 3) of Assumption 4.1, it follows that \( h(x^0) = 0 \) and \( L_f h(x^0) < 0 \). Since \( h \) is continuously differentiable and \( f \) is continuous, \( x \rightarrow L_f h(x) \) is continuous. Then, there exists \( \delta > 0 \) such that \( L_f h(x) < 0 \) for all \( x \in x^0 + \delta B \). Therefore, the solution \( \phi^* \) to \( \mathcal{H}_{|M} \) cannot be extended through flow at \( x \). In fact, since \( x^0 \in M \cap D \), \( \phi^* \) will jump immediately when it reaches \( x^0 \). This contradicts the fact that \( \phi^* \) has only one jump in its period \( T^* \).

Asymptotic stability of \( O \) implies a finite-time convergence and recurrence property of the jump set from points in its basin of attraction, which is denoted as \( B_O \). The following result states this property.

**Lemma 4.10:** Consider a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1. Suppose that \( \mathcal{H}_{|M} = (M \cap C, f, M \cap D, g) \) has a flow periodic solution \( \phi^* \) with period \( T^* > 0 \) that defines a locally asymptotically stable hybrid limit cycle \( O \subset M \cap C \). Then, the set \( M \cap D \) is finite time attractive and recurrent from the basin of attraction of \( O \) in the sense that for each solution \( \phi \) to \( \mathcal{H}_{|M} \) with \( \phi(0, 0) \) in \( B_O \), there exists \( \{ (t_i, j_i) \} \subset \mathbb{N} \), \( j_0 = 0 \), \( (t_i, j_i) \in dom \phi \) with \( t_i \) and \( j_i \) strictly increasing and unbounded such that \( |\phi(t_i, j_i)|_{M \cap D} = 0 \) for all \( i \in \mathbb{N} \).

**Proof:** First, we show that \( M \cap D \) is finite time attractive with basin of attraction of \( O \), namely, \( B_O \). We proceed by contradiction. Suppose that there exists \( \phi \in S_{\mathcal{H}_{|M}} \) with \( \phi(0, 0) \) in \( B_O \subset M \cap C \) such that for any \( (t, j) \in dom \phi \),

\( \phi(t, j) \notin M \cap D \). From the local attractivity of \( \phi \), there exists \( \mu > 0 \) such that the solution \( \phi \) to \( \mathcal{H}_{|M} \) starting from \( |\phi(0, 0)|_\infty < \mu \) is complete and satisfies \( \lim_{t \to +\infty} |\phi(t, j)|_\infty = 0 \).

Therefore, at least one jump will happen in finite time flow, namely, there exists \( (t^*, j^*) \in dom \phi \) such that \( \phi(t^*, j^*) \in D \). Recalling that \( \phi(0, 0) \) in \( M \cap C \) and \( \phi \in S_{\mathcal{H}_{|M}} \), by definition of solution we have \( \phi(t^*, j^*) \in M \cap D \), which leads to a contradiction. Therefore, the set \( M \cap D \) is finite time attractive from \( B_O \).

Next, we prove that the set \( M \cap D \) is recurrent from \( B_O \). Suppose that there exists \( \phi \in S_{\mathcal{H}_{|M}} \) with \( \phi(0, 0) \) in \( B_O \cap (M \cap D) \). By item 3) of Assumption 4.1, \( g(M \cap D) \cap (M \cap D) = \emptyset \). Then, \( \phi(0, 1) = g(\phi(0, 0)) \notin M \cap D \). In addition, from the local asymptotic stability of \( O \), \( \phi(0, 1) \in B_O \). We claim that there exists \( t^* \) such that \( \phi(t^*, 1) \in M \cap D \). Proceeding by contradiction, suppose that \( \phi(t^*, 1) \notin M \cap D \). Recurrence follows by repeating the same argument and using completeness of solutions from \( B_O \).

**Remark 4.11:** The finite-time convergence and recurrence property of the jump set in Lemma 4.10 are necessary conditions for the existence of asymptotically stable hybrid limit cycles. This provides a way to verify the existence of an asymptotically stable hybrid limit cycle in a hybrid system.

To state our next result, let us introduce the time-to-impact function for hybrid dynamical systems as in \( \mathcal{H} \). Alternative equivalent definitions can be found in [6] and [58]. Let \( \mathcal{H} = (C, f, D, g) \), the time-to-impact function with respect to \( D \) is defined by \( T_I : \mathbb{C} \cap D \to \mathbb{R}_{\geq 0} \cup \{ \infty \} \), where

\[
T_I(x) := \inf \{ t \geq 0 : \phi(t, j) \in D, \phi \in S_{\mathcal{H}(x)} \}
\]  

for each \( x \in \mathbb{C} \cap D \).

Inspired by [6] Lemma 3, we show that the function \( x \rightarrow T_I(x) \) is continuous on a subset of \( M \cap (\mathbb{C} \cap D) \), as specified next.

**Lemma 4.12:** Suppose a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfy Assumption 4.1. Then, \( T_I \) is continuous at points in \( X := \{ x \in M \cap C : 0 \leq T_I(x) < \infty \} \). From the definition of \( T_I \) and the condition \( 0 < T_I(x^*) < \infty \), \( \phi^* \) remains in \( M \cap C \) over \( [0, T_I(x^*)] \). Denote \( \bar{x} := \phi^*(T_I(x^*), x^*) \). Then, \( \bar{x} \in M \cap D \). By the definition of \( T_I \) and items 1) and 3) in Assumption 4.1, we have \( h(\phi(t, x^*), 0) > 0 \) for all \( t \in [0, T_I(x^*)] \). Let \( \bar{e} := \frac{1}{2} \min \{ T_I(x^*), e \} \) and \( \bar{t} := T_I(x^*) - \bar{e} \). Using item 2) of Assumption 4.1, \( \phi^* \) can be

\[
\phi^*(t, j) \notin M \cap D
\]

In particular, when there does not exist \( t \geq 0 \) such that \( \phi^*(t, x) \in D \), we have \( \{ t \geq 0 : \phi^*(t, x) \in D \} = \emptyset \), which gives \( T_I(x) = \infty \).
extended to the interval \([T_1(x^*), T_2(x^*) + \varepsilon]\) if needed, by decreasing \(\varepsilon\). Then, by using the property that \(L_1(h(\tilde{x})) < 0\) and \(h(\tilde{x}) = 0\), we obtain \(h(x) < 0\), where \(\tilde{x} = \phi^f(t_2, x^*)\) and \(t_2 = T_1(x^*) + \varepsilon\). From item 2) of Assumption 4.11, \(f\) is continuous on \(M \cap C\) and differentiable on a neighborhood of \(M \cap C\), which implies Lipschitz continuity of \(f\). Then, by [57, Theorem 3.5], it follows that solutions to \(\dot{x} = f(x)\) depend continuously on the initial conditions, that is, for every \(\varepsilon > 0\) there exist \(\delta(\varepsilon) > 0\) such that \(|x - x'| < \delta\), then \(|\phi^f(t, x) - \phi^f(t, x')| < \varepsilon\) for all \(t \in \text{dom}\phi^f\). Therefore, given \(\varepsilon = \min\{|\phi^f(t_1, x^*)| : M \cap D, |\phi^f(t_2, x^*)| : M \cap D\}\), there exists \(\delta > 0\) such that for all \(x \in M \cap C\) satisfying \(|x - x'| < \delta\), \(|\phi^f(t, x) - \phi^f(t, x')| < \varepsilon\) for all \(t \in [0, T_1(x^*) + \varepsilon]\). Then, it follows that \(h(\phi^f(t, x)) > 0, h(\phi^f(t, x)) < 0\), and by continuity of \(h, t_1 < T_1(x) < t_2\). Therefore, \(T_1(x) - T_1(x^*) < \varepsilon\), which implies \(T_1\) is continuous at any \(x^*\) such that \(x^* \in M \cap C\) and \(0 < T_1(x^*) < \infty\).

Next, we show that the function \(x \mapsto T_1(x)\) is also continuous on a subset of \(O\).

**Lemma 4.13:** Given a hybrid system \(\mathcal{H} = (C, f, D, g)\) on \(\mathbb{R}^n\) and a closed set \(M \subset \mathbb{R}^n\) satisfying Assumption 4.1, suppose that \(\mathcal{H}_{|M} = (M \cap C, f, M \cap D, g)\) has a unique hybrid limit cycle \(\mathcal{O} \subset M \cap C\) defined by the flow periodic solution \(\phi^*\). Then, \(T_1\) is continuous on \(O \setminus \{\phi^*(t^*, 0)\}\), where \(t^*\) is such that \((t^*, 0), (t^*, 1) \in \text{dom}\phi^*,\) namely, \((t^*, 0)\) is a jump time of \(\phi^*\) and \(\phi^*(t^*, 0)\) is the point in \(M \cap D\) at which \(\phi^*\) jumps.

**Proof:** Consider a hybrid cycle \(O \subset M \cap C\) defined by the flow periodic solution \(\phi^*\). For \((t^*, 0), (t^*, 1) \in \text{dom}\phi^*,\) we have \(\phi^*(t^*, 0) \in M \cap D\). By Lemma 4.3 since \(O\) is forward invariant, for all \(x \in O \setminus \{\phi^*(t^*, 0)\}\), there exists \(t > t^*\) such that \(\phi^*(t, 1)\) has a jump, which implies that \(0 < T_1(x) < \infty\). By Lemma 4.12, \(T_1\) is continuous at points in \(\mathcal{X}' := \{x \in M \cap C : 0 < T_1(x) < \infty\}\). Then, \(T_1\) is continuous on \(O \setminus \{\phi^*(t^*, 0)\}\). \qed

**B. A Necessary Condition via Forward Invariance**

Following the spirit of the necessary condition for existence of limit cycles in nonlinear continuous-time systems in [57], we have the following necessary condition for general hybrid systems with a hybrid limit cycle given by the zero-level set of a smooth enough function.

**Proposition 4.14:** Consider a hybrid system \(\mathcal{H} = (C, f, D, g)\) on \(\mathbb{R}^n\) satisfying the hybrid basic conditions with \(f\) continuously differentiable. Suppose every solution \(\phi \in \mathcal{S}_H\) is unique and there exists a hybrid limit cycle \(\mathcal{O} \subset H\) with period \(T^* > 0\) satisfying

\[
\mathcal{O} \subseteq \{x \in \mathbb{R}^n : p(x) = 0\},
\]

where \(p : \mathbb{R}^n \to \mathbb{R}\) is twice continuously differentiable on an open neighborhood \(U\) of \(O\). Then, there exists a function \(W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\) that is twice continuously differentiable on \(O\) and

\[
W(x) \geq 0, \quad \forall x \in O, \quad (15)
\]

\(
\langle \nabla W(x), f(x) \rangle = 0 \quad \forall x \in O \cap C, \quad (16)
\)

\[
\langle \nabla(\nabla W(x), f(x)), f(x) \rangle = 0 \quad \forall x \in O \cap C, \quad (17)
\]

\[
W(g(x)) - W(x) = 0 \quad \forall x \in O \cap D. \quad (18)
\]

Furthermore, if \(p\) is such that \(p(\tilde{x}) \neq 0\) for some \(\tilde{x} \in C \cup D\), then \(W\) is such that [57] holds with strict inequality.

**Proof:** Using forward invariance of \(O\) (see Lemma 4.12), under the assumption that \(p(x_0) = 0\) for all \(x_0 \in O\) and \(p\) is continuously differentiable on \(U\), we have that the solution \(t \mapsto \phi^f(t, x_0)\) to \(\dot{x} = f(x)\) with \(x_0 \in g(D) \cap O\) satisfies

\[
\frac{\partial p \circ \phi^f}{\partial t}(t, x_0) = 0 \quad \forall t \in (0, T^*), \quad (19)
\]

Since forward invariance of \(O\) implies that \(\text{rg} \phi^f \subset O\), this property leads to

\[
\langle \nabla p(x), f(x) \rangle = 0 \quad \forall x \in O \cap C. \quad (20)
\]

Similarly, we have that the solution \(t \mapsto \phi^f(t, x_0)\) to \(\dot{x} = f(x)\) with \(x_0 \in g(D) \cap O\) satisfies

\[
\frac{\partial^2 p \circ \phi^f}{\partial t^2}(t, x_0) = 0 \quad \forall t \in (0, T^*), \quad (21)
\]

which, since \(\frac{\partial p \circ \phi^f}{\partial t}(t, x_0) = 0\), leads to

\[
\langle \nabla(\nabla p(x), f(x)), f(x) \rangle = 0 \quad \forall x \in O \cap C. \quad (22)
\]

In fact, if there exists \(\mathcal{X} \in O \cap C\) such that

\[
\langle \nabla(\nabla p(x), f(x)), f(x) \rangle \neq 0,
\]

then \(\langle \nabla p(x), f(x) \rangle \neq 0\), which contradicts (20).

Since \(O\) is forward invariant and every maximal solution to \(\mathcal{H}\) is unique, for each \(x \in O \cap D\) we have \(g(x) \in O\). In addition, since \(p\) vanishes on \(O\), we obtain

\[
p(g(x)) - p(x) = 0 \quad \forall x \in O \cap D. \quad (23)
\]

Now, we use the properties above to construct a function \(W\) such that it satisfies the properties stated in the result. By assumption, since there exists \(\tilde{x} \in C \cup D\) such that \(p(\tilde{x}) \neq 0\), following [57], define the function \(W\) as

\[
W(x) = (p(x) - p(\tilde{x}))^{\tilde{n}},
\]

where \(\tilde{n} \in \mathbb{N} \setminus \{0\}\) is an arbitrary positive even integer. Then, \(W(x) \geq 0\) and \(W\) is continuously differentiable on \(U\). In particular, since \(p(x) = 0\) for all \(x \in O\) and \(p(\tilde{x}) \neq 0\), we have \(W(x) > 0\) for all \(x \in O\). Using (20) and (22), we have, for all \(x \in O \cap C\),

\[
\langle \nabla(\nabla W(x), f(x)), f(x) \rangle = \tilde{n}(\tilde{n} - 1)(p(x) - p(\tilde{x}))^{\tilde{n} - 1}\langle \nabla p(x), f(x) \rangle = 0,
\]

and

\[
\langle \nabla(\nabla(\nabla W(x), f(x)), f(x)) \rangle = \tilde{n} \langle \nabla p(x), f(x) \rangle^2 + \tilde{n} (p(x) - p(\tilde{x}))^{\tilde{n} - 1}\langle \nabla(\nabla p(x), f(x)), f(x) \rangle = 0.
\]

Finally, using (23), we have, for all \(x \in O \cap D\),

\[
W(g(x)) - W(x) = (p(g(x)) - p(x))^{\tilde{n}} - (p(x) - p(\tilde{x}))^{\tilde{n}} = 0.
\]
flow and jump maps, it is verified that $O \cap O^{\infty}$ is forward invariant.

To validate Proposition 4.14, define the continuously differentiable function $W : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ as

$$W(x) = \left( (q - \frac{(r-B)^2}{2a} - R) - (\frac{-B^2}{2a} - R) \right)^2 = \left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)^2 > 0 \quad \forall x \in O.$$

This function satisfies (16)-(18) since $\langle \nabla p(x), f_{TCP}(x) \rangle = [\frac{1}{a} \frac{B-c}{a}] f_{TCP}(x) = r - B - (r - B) = 0$ for all $x \in C_{TCP}$.

Then, for all $x \in O \cap M_{TCP} \cap C_{TCP}$,

$$\langle \nabla W(x), f_{TCP}(x) \rangle = 2\left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)(r - B - r + B) = 0$$

and

$$\langle \nabla \langle \nabla W(x), f_{TCP}(x) \rangle, f_{TCP}(x) \rangle = \langle \nabla \langle \nabla p(x), f_{TCP}(x) \rangle, f_{TCP}(x) \rangle = 0.$$

Moreover, for all $x \in O \cap M_{TCP} \cap D_{TCP}$, using the fact that $q = q_{\max}$ and $r = 2B/(m+1)$, we have

$$W(g_{TCP}(x)) - W(x) = \left[ q - \frac{m(r-B)^2}{2a} + \frac{B^2}{2a} \right]^2 - \left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)^2 = 0.$$

The following example illustrates the result in Proposition 4.14.

**Example 4.15:** Consider the hybrid congestion control system in Example 4.6. The set defined by points $(q, r)$ such that $q - \frac{(r-B)^2}{2a} = R$ with $R = q_{\max} - \frac{B^2(m-1)}{2a(m+1)}$ represents a hybrid limit cycle for $H_{TCP}$, namely,

$$O := \left\{ (q, r) \in M_{TCP} : q - \frac{(r-B)^2}{2a} = R \right\},$$

is a hybrid limit cycle. In particular, the state vector $x = (q, r)$ moves clockwise within $O$ as depicted in Fig. 2. Using the flow and jump maps, it is verified that $O$ is forward invariant.

To validate Proposition 4.14, define the continuously differentiable functions $p(x) := q - \frac{(r-B)^2}{2a} - R$, pick the point $\bar{x} := (0, 0)$, which satisfies $p(\bar{x}) = -\frac{B^2}{2a} - R = q_{\max} - \frac{B^2}{2a(m+1)} < 0$, and define a continuously differentiable function $W : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ as

$$W(x) = \left( (q - \frac{(r-B)^2}{2a} - R) - (\frac{-B^2}{2a} - R) \right)^2 = \left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)^2 > 0 \quad \forall x \in O.$$

This function satisfies (16)-(18) since $\langle \nabla p(x), f_{TCP}(x) \rangle = [\frac{1}{a} \frac{B-c}{a}] f_{TCP}(x) = r - B - (r - B) = 0$ for all $x \in C_{TCP}$.

Then, for all $x \in O \cap M_{TCP} \cap C_{TCP}$,

$$\langle \nabla W(x), f_{TCP}(x) \rangle = 2\left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)(r - B - r + B) = 0$$

and

$$\langle \nabla \langle \nabla W(x), f_{TCP}(x) \rangle, f_{TCP}(x) \rangle = \langle \nabla \langle \nabla p(x), f_{TCP}(x) \rangle, f_{TCP}(x) \rangle = 0.$$

Moreover, for all $x \in O \cap M_{TCP} \cap D_{TCP}$, using the fact that $q = q_{\max}$ and $r = 2B/(m+1)$, we have

$$W(g_{TCP}(x)) - W(x) = \left[ q - \frac{m(r-B)^2}{2a} + \frac{B^2}{2a} \right]^2 - \left( q - \frac{(r-B)^2}{2a} + \frac{B^2}{2a} \right)^2 = 0.$$

The following example illustrates the result in Lemma 4.3, Lemma 4.9, Lemma 4.12 as well as Proposition 4.14.

**Example 4.16:** Consider a hybrid system $H_S = (C_s, f_s, D_s, g_s)$ with state $x = (x_1, x_2)$ and data

$$H_S \begin{cases} \dot{x} = f_s(x) := b \frac{x_2}{-x_1} & x \in C_s, \\ x^+ = g_s(x) := \begin{bmatrix} c \\ 0 \end{bmatrix} & x \in D_s, \end{cases}$$

where $C_s := \{ x \in \mathbb{R}^2 : x_1 \geq 0 \}$ and $D_s := \{ x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0 \}$. The two parameters $b$ and $c$ satisfy $b \geq 1$ and $c > 0$. Since $C_s$ and $D_s$ are closed, and the flow and jump maps are continuous with $f_s$ continuously differentiable, the hybrid system $H_S$ satisfies the hybrid basic conditions. Note that every solution $\phi \in S_{H_S}$ is unique. The flow dynamics characterizes an oscillatory behavior. In fact, a maximal solution $\phi^* \to H_S$ from $\phi^*(0,0) = (c,0)$ is a unique flow periodic solution with period $T^* = \frac{1}{c}$. As depicted in Fig. 7 a solution $\phi$ starting from the point $P_0 = \{(5,0)\}$ flows to the point $P_1$, jumps to the point $P_2$, flows to the point $P_3$, and jumps back to the point $P_2$.

![Fig. 7. Phase plot of a solution to $H_S$ from $\phi(0,0) = (5,0)$ where $P_0 = \{(5,0)\}$, $P_1 = \{(0,-5)\}$, $P_2 = \{(c,0)\}$ and $P_3 = \{(5,0)\}$. Here, we set $b = 0.8$ and $c = 3.5$.](image)
\{(x_1, x_2) \in M_\triangle \cap C_\triangle : x_1^2 + x_2^2 = \epsilon^2\} is bounded, otherwise a flow periodic solution \(\phi^*\) will escape to infinity in finite time which leads to a contradiction with the definition of a flow periodic solution. Moreover, due to the closeness of \(M_\triangle \cap C_\triangle\), \(O\) is closed; hence, \(O\) is compact. By the data of \(H_\triangle\) in (23), one can verify that each \(\phi \in S_H(O)\) is complete and satisfies rge \(\phi \subseteq O\). Then, \(O\) is compact and forward invariant, which illustrates Lemma 4.4. In addition, since for all \(x \in M_\triangle \cap D_\triangle\), \(L_{C\triangle} b(x) = b(x, \epsilon) < 0\) and \(\tilde{x} = O \cap (M_\triangle \cap D_\triangle) = (\epsilon, 0)\), \(O\) is transversal to \(M_\triangle \cap D_\triangle\), which illustrates Lemma 4.5. Finally, for the hybrid limit cycle \(O\) defined by a flow periodic solution \(\phi^*\) and for any \(x \in \bar{O} \setminus \{\phi^*(t, 0)\}\), \(T_1(x) \in [0, \frac{1}{\epsilon})\) is continuous which illustrates Lemma 4.12.

V. Existence of Hybrid Limit Cycles

In this section, we introduce a stability notion that relates a solution to nearby solutions, which enables us to provide sufficient conditions for the existence of hybrid limit cycles for the class of hybrid systems in [1].

A. Zhukovskii Stability for Hybrid Systems

Zhukovskii stability for a continuous-time system consists of the property that, with a suitable reparametrization of perturbed trajectories, Zhukovskii stability implies Lyapunov stability; see, e.g., [46, 47]. We extend this notion to hybrid systems and establish links to the existence of hybrid limit cycles. To this end, inspired by [43, 46, 47], we employ the family of maps \(T\) defined by

\[ T = \{\tau(\cdot): \tau: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\} \text{ is a homeomorphism, } \tau(0) = 0. \]

A map \(\tau\) in the family \(T\) is employed to reparametrize ordinary time for the trajectories of the hybrid system in [1] and formulate stability and attractivity notions involving the reparametrized trajectories, as formulated next.

Definition 5.1: Consider a hybrid system \(H\) on \(\mathbb{R}^n\) as in [1]. A maximal solution \(\phi_1\) to \(H\) is said to be

1) Zhukovskii stable (ZS) if for each \(\epsilon > 0\) there exists \(\delta > 0\) such that for each \(\phi_2 \in S_H(\phi_1(0, 0) + \delta B)\) there exists \(\tau \in T\) such that for each \((t, j) \in \text{ dom } \phi_2\) we have 
\[ (\tau(t), j) \in \text{ dom } \phi_2 \text{ and } \left| \phi_1(t, j) - \phi_2(\tau(t), j) \right| \leq \epsilon; \]

2) Zhukovskii locally attractive (ZLA) if there exists \(\mu > 0\) such that for each \(\phi_2 \in S_H(\phi_1(0, 0) + \mu B)\) there exists \(\tau \in T\) such that for each \(\epsilon > 0\) there exists \(T > 0\) for which we have that \((t, j) \in \text{ dom } \phi_1\) and \(t + j > T\) imply 
\[ (\tau(t), j) \in \text{ dom } \phi_2 \text{ and } \left| \phi_1(t, j) - \phi_2(\tau(t), j) \right| \leq \epsilon; \]

3) Zhukovskii locally asymptotically stable (ZLAS) if it is both ZS and ZLA.

Remark 5.2: The map \(\tau\) in Definition 5.1 reparametrizes the flow time of the solution \(\phi_2\). In particular, the ZS notion only requires that the solution \(\phi_2\) stays close to the solution \(\phi_1\) for the same value of the jump counter \(j\) but potentially at different flow times \(t\). Note that \(\tau\) in the ZS and ZLA notions may depend on the initial conditions of \(\phi_1\) and \(\phi_2\). For simplicity and for the purposes of this work, the ZLA notion is written as a uniform property, in the sense of hybrid time and over the compact set of initial conditions defined by \(\mu\). When \(\phi_1\) and each \(\phi_2\) are complete, the nonuniform version of that property would require

\[ \lim_{(t, j) \in \text{ dom } \phi_1, t + j \to \infty} \| \phi_1(t, j) - \phi_2(\tau(t), j) \| = 0, \]

which resembles the notion defined in the literature of continuous-time systems; see [46 Definition 4.1] and [47 Definition 2].

The ZLAS notion will be related to existence of hybrid limit cycles by analyzing the properties of a Poincaré map in Section V.B (within the proof of Theorem 5.12 and the \(\omega\)-limit set of a hybrid arc. Next, the ZLAS notion in Definition 5.1 is illustrated in two examples with a hybrid limit cycle.

Example 5.3: Consider the timer system in [7]. Note that every maximal solution to the timer system is unique and complete. To verify the ZS notion, consider \(\phi_1 \in S_{HT}\). Given \(\epsilon > 0\), let \(0 < \delta < \epsilon\). Then, for each \(\phi_2 \in S_{HT}(\phi_1(0, 0) + \delta B)\), \(T_1(\phi_1(0, 0)) = 1 - \phi_1(0, 0)\) and \(T_1(\phi_2(0, 0)) = 1 - \phi_2(0, 0)\). Without loss of generality, we further suppose \(\phi_1(0, 0) > \phi_2(0, 0)\). Then, the solution \(\phi_1\) jumps before \(\phi_2\) since jumps occur when the timer reaches one. Denote \(\tau_\Delta = T_1(\phi_2(0, 0) - T_1(\phi_1(0, 0)) = \phi_1(0, 0) - \phi_2(0, 0) > 0\). Now construct the map \(\tau\) as

\[ \tau(t) = \begin{cases} T_1(\phi_1(0, 0)) & t \in [0, T_1(\phi_1(0, 0))], \\ t + \tau_\Delta & t > T_1(\phi_1(0, 0)). \end{cases} \]

Note that \(\tau\) is a homeomorphism and satisfies \(\tau(0) = 0\), hence it belongs to \(T\), and, in addition, is continuous. Then, for \(j = 0\), for each \(t \in [0, T_1(\phi_1(0, 0))\) we have \(\| \phi_1(t, 0) - \phi_2(\tau(t), 0) \| \leq |\phi_1(0, 0) - \phi_2(0, 0)| \leq \delta < \epsilon. \)

Consider \(\phi_1(0, 0) > \phi_2(0, 0)\). Then, for each \(j \in \mathbb{N}\setminus\{0\}\) and each \(t \geq T_1(\phi_1(0, 0))\) such that \((t, j) \in \text{ dom } \phi_1, we have \(\tau(t) = t + \tau_\Delta\), which satisfies \(\tau(t), j \in \text{ dom } \phi_2\) and \(\| \phi_1(t, j) - \phi_2(\tau(t), j) \| = 0 < \epsilon. \)

Therefore, the solution \(\phi_1\) is ZS. In fact, any solution \(\phi_1 \in S_{HT}\) is ZS.

To verify the ZLA notion, let \(\mu > 0\). Let \(\phi_1\) be a maximal solution to \(H_T\). Then, for each \(\epsilon > 0\) and for each \(\phi_2 \in S_{HT}(\phi_1(0, 0) + \mu B)\), we have \(T_1(\phi_2(0, 0)) = 1 - \phi_2(0, 0)\). Similar to the above steps showing ZS of \(\phi_1\), without loss of generality, assume \(\phi_1(0, 0) > \phi_2(0, 0)\). Then, the solution \(\phi_1\) jumps before \(\phi_2\). Note that \(\phi_1(0, 0) + \mu B)\).
is ZLA. Hence, every maximal solution to $\mathcal{H}_T$ is ZLAS.

(a) The projections of two solutions from $\phi_1(0,0) = 0.8$ and $\phi_2(0,0) = 0$ on the $t$ direction.

(b) Comparison between Euclidean distance $d(\phi_1, \phi_2)$ (top) and the distance $d_2(\phi_1, \phi_2)$ (bottom).

Fig. 8. Two solutions $\phi_1$ and $\phi_2$ to the timer system in Example 5.3. Unlike the Euclidean distance, which is $d(\phi_1(t, j), \phi_2(t, j)) := |\phi_1(t, j) - \phi_2(t, j)|$ for all $(t, j) \in \text{dom } \phi_1$ and $(t, j) \in \text{dom } \phi_2$, which assumes the value 0.2 for 0.8 seconds after every 0.2 seconds, the distance satisfies $d_2(\phi_1(t, j), \phi_2(t, j)) := |\phi_1(t, j) - \phi_2(t, j)|$ for all $(t, j) \in \text{dom } \phi_1$ and $(t, j) \in \text{dom } \phi_2$, which decreases to zero after 0.2 seconds. Here, $\tau(t) = 5t$ for $t \in [0, 0.2]$ and $\tau(t) = t + t_\Delta$ for $t > 0.2$ with $t_\Delta = (\phi_1(0, 0) - \phi_2(0, 0)) = 0.8$.

Fig. 8 shows two solutions to the timer system in Eq. (29) and the distance between them. Fig. 8(a) shows that no matter how close the two maximal solutions are initialized, the peak always exists for the Euclidean distance between them. However, the distance between solution $\phi_1$ and solution $\phi_2$ with parameterization $\tau(t)$, denoted $d_2$ and shown in Fig. 8(b), is zero a short time.

Example 5.4: Consider the academic system $\mathcal{H}_A = (C_A, f_A, D_A, g_A)$ with scalar state $x$ and data

$$\mathcal{H}_A \left\{ \begin{array}{ll} \dot{x} = f_A(x) := -ax + b & x \in C_A, \\ x^+ = g_A(x) := b_2 & x \in D_A, \end{array} \right. \quad (28)$$

where $C_A := [0, b_1]$ and $D_A := \{ x \in [0, b_1] : x = b_1 \}$. The parameters $a$, $b$, $b_1$, and $b_2$ satisfy $a > 0$ and $b > ab_1 > ab_2 > 0$. Define the compact set $M_A := [0, b_1]$ and define a continuously differentiable function $h : M_A \to \mathbb{R}$ as $h(x) := b_1 - x$. Then, $C_A$ and $D_A$ can be rewritten as $C_A = \{ x \in M_A : h(x) \geq 0 \}$ and $D_A = \{ x \in C_A : h(x) = 0, L_{f_A}h(x) \leq 0 \}$, respectively. We used the property $L_{f_A}h(x) = -(ax + b) = ab_1 - b < 0$ for all $x \in M_A \cap D_A$. By design, the sets $C_A$ and $D_A$ are closed. Moreover, the function $f_A$ is continuously differentiable and the function $g_A$ is continuous. Furthermore, it can be verified that $g_A(M_A \cap D_A) \cap (M_A \cap D_A) = 0$. Therefore, Assumption 4.1 holds. Note that every maximal solution $\phi$ to $\mathcal{H}_A|_{M_A} = (M_A \cap C_A, f_A, M_A \cap D_A, g_A)$ is unique via [5, Proposition 2.11].

To verify the ZS notion, let us consider a maximal solution $\phi_1$ to $\mathcal{H}_A|_{M_A}$. For a given $\varepsilon$, let $0 < \delta < \varepsilon$. Then, for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_A|_{M_A}}(\phi_1(0, 0) + \delta B)$, we have $T_{\tau}(\phi_1(0, 0)) = \frac{1}{a} \ln \frac{\phi_1(0, 0)}{b_2} - \frac{b}{a}$ and $T_{\tau}(\phi_2(0, 0)) = \frac{1}{a} \ln \frac{\phi_2(0, 0) - b}{ab_2 - b}$. Without loss of generality, assume $\phi_1(0, 0) > \phi_2(0, 0)$. Then, the solution $\phi_1$ jumps before $\phi_2$ since jumps occur when $x$ reaches $b_1$. Denote $t_\Delta = T_{\tau}(\phi_2(0, 0)) - T_{\tau}(\phi_1(0, 0)) = \frac{1}{a} \ln \frac{\phi_2(0, 0) - b}{ab_2 - b} - \frac{b}{a} \ln \frac{\phi_2(0, 0) - b}{ab_2 - b} > 0$. Let us construct $\tau$ as

$$\tau(t) = \begin{cases} T_{\tau}(\phi_2(0, 0)) & t \in [0, T_{\tau}(\phi_1(0, 0))], \\ t + t_\Delta & t > T_{\tau}(\phi_1(0, 0)). \end{cases} \quad (29)$$

Note that $\tau$ is a homeomorphic satisfies $\tau(0) = 0$, hence it belongs to $\mathcal{T}$, and, in addition, is continuous. Then, for $j = 0$, for each $t \in [0, T_{\tau}(\phi_1(0, 0))]$, we have $T_{\tau}(\phi_j(0, 0))$, which satisfies $(\tau(t), 0) \in \text{dom } \phi_2$ and

$$|\phi_1(t, 0) - \phi_2(\tau(t), 0)| = \left| (\frac{\phi_1(0, 0) - b}{a}) e^{-at} + \frac{b}{a} \right| - \left| (\phi_2(0, 0) - \frac{b}{a}) e^{-at} + \frac{b}{a} \right|.$$
φ₂. Note that \( \phi_1(T_t(\phi_1(0,0)), 1) = \phi_2(\tau(T_t(\phi_2(0,0))), 1 = b_2 \). Then, for \( j = 1 \) and for each \( t \geq T_t(\phi_1(0,0)) \), we have \( \tau(t) = t + t_\Delta \), which satisfies \( (\tau(t), 1) \in \text{dom} \phi_2 \) and \( |\phi_1(t, 1) - \phi_2(\tau(t), 1)| = 0 < \varepsilon \). In fact, for each \( j \in \mathbb{N}\setminus\{0\} \) and each \( t \geq T_t(\phi_1(0,0)) \), we have that \( \tau(t) = t + t_\Delta \) and 

\[
(t, j) \in \text{dom} \phi_1, \quad t + j > T = T_I(\phi_1(0,0)) + 1
\]

imply that \( (\tau(t), j) \in \text{dom} \phi_2 \) and \( |\phi_1(t, j) - \phi_2(\tau(t), j)| = 0 < \varepsilon \). Therefore, \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M_A} \) is ZLA. In fact, any solution \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M_A} \) is ZLAS. Hence, every maximal solution to \( \mathcal{H}|M_A \) is ZLAS.

Next, we establish a link between the Zhukovskii stability notion in Definition 5.5 and incremental graphical stability as introduced in [43]. The latter notion is presented next for self-containedness.

**Definition 5.5**: [43, Definition 3.2] Consider a hybrid system \( \mathcal{H} \) on \( \mathbb{R}^n \) as in (1). The hybrid system \( \mathcal{H} \) is said to be

1) **incrementally graphically stable (δS)** if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any two maximal solutions \( \phi_1, \phi_2 \) to \( \mathcal{H}, \phi_1(0,0) - \phi_2(0,0) | \leq \delta \) implies that, for each \( (t, j) \in \text{dom} \phi_1 \), there exists \( (s, j) \in \text{dom} \phi_2 \) satisfying \(|t - s| \leq \varepsilon \) and

\[
|\phi_1(t, j) - \phi_2(s, j)| \leq \varepsilon;
\]

2) **incrementally graphically locally attractive (δLA)** if there exists \( \mu > 0 \) such that for every \( \varepsilon > 0 \) and for any two maximal solutions \( \phi_1, \phi_2 \) to \( \mathcal{H}, \phi_1(0,0) - \phi_2(0,0) | \leq \mu \) implies that there exists \( T \) such that for each \( (t, j) \in \text{dom} \phi_1 \), such that \( t + j > T \), there exists \( (s, j) \in \text{dom} \phi_2 \) satisfying \(|t - s| \leq \varepsilon \) and

\[
|\phi_1(t, j) - \phi_2(s, j)| \leq \varepsilon;
\]

3) **incrementally graphically locally asymptotically stable (δLAS)** if it is both δS and δLA.

The following theorem establishes a sufficient condition for ZS and ZLA.

**Theorem 5.6**: Consider a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.4. Suppose every maximal solution \( \phi \) to \( \mathcal{H}|M = (M \cap C, f, M \cap D, g) \) is complete. The following hold:

a) If the hybrid system \( \mathcal{H}|M \) is δS, then each \( \phi \in \mathcal{S}_{\mathcal{H}|M} \) is ZS;

b) If the hybrid system \( \mathcal{H}|M \) is δLA, then each \( \phi \in \mathcal{S}_{\mathcal{H}|M} \) is ZLA.

**Proof**: Consider any solution \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M} \).

a) To show ZS, given \( \varepsilon > 0 \), let \( 0 < \delta < \varepsilon \) and consider any solution \( \phi_2 \in \mathcal{S}_{\mathcal{H}|M}(\phi_1(0,0) + \delta B) \). Since \( \mathcal{H}|M \) is δS, then, for each \( (t, j) \in \text{dom} \phi_1 \), there exists \( s \) satisfying \( |s, j| \in \text{dom} \phi_2 \), \(|t - s| \leq \varepsilon \), and (32). Therefore, we simply take \( s := t \) and let the function \( \tau \in \mathcal{T} \) be \( \tau(t) := s \) with \( s \) satisfying the above conditions. Therefore, each \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M} \) is ZS. In fact, from the construction above, for a solution \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M} \) and for any given \( \varepsilon > 0 \), there exists \( 0 < \delta < \varepsilon \) such that, for each \( \phi_2 \in \mathcal{S}_{\mathcal{H}|M}(\phi_1(0,0) + \delta B) \) there exists \( (t, j) \in \text{dom} \phi_1 \) such that for each \( (t, j) \in \text{dom} \phi_2 \), we have that \( (\tau(t), j) \in \text{dom} \phi_2 \) and \( |\phi_1(t, j) - \phi_2(\tau(t), j)| \leq \varepsilon \).

b) To show ZLA, given \( \varepsilon > 0 \), let \( 0 < \mu < \varepsilon \) and consider any solution \( \phi_2 \in \mathcal{S}_{\mathcal{H}|M}(\phi_1(0,0) + \mu B) \). Since \( \mathcal{H}|M \) is δLA, then, there exists \( T > 0 \) and for, each \( (t, j) \in \text{dom} \phi_1 \) such that \( t + j > T \), there exists \( (s, j) \in \text{dom} \phi_2 \) satisfying \(|t - s| \leq \varepsilon \) and (33). Therefore, we simply take \( s := t \) and let the function \( \tau \in \mathcal{T} \) be \( \tau(t) := s \) with \( s \) satisfying the above conditions. Therefore, each \( \phi_1 \in \mathcal{S}_{\mathcal{H}|M} \) is ZLA. In fact, from the construction above, for any given \( \varepsilon > 0 \), there exists \( 0 < \mu < \varepsilon \) such that, for each \( \phi_2 \in \mathcal{S}_{\mathcal{H}|M}(\phi_1(0,0) + \mu B) \) there exists \( (t, j) \in \text{dom} \phi_1 \) such that \( (\tau(t), j) \in \text{dom} \phi_2 \) and \( |\phi_1(t, j) - \phi_2(\tau(t), j)| \leq \varepsilon \). □

The following example is provided to illustrate the sufficient condition for ZS in Theorem 5.6.

**Example 5.7**: Consider the hybrid system \( \mathcal{H}|M_A \) as \( (M \cap C, f, M \cap D, g) \) in Example 4.16 with \( M_A \) given therein. We verify δS for \( \mathcal{H}|M_A \) (Items 1) and 3) of Assumption 4.1 and have been shown to hold in Example 4.16. Moreover, \( f_3 \) and \( g_3 \) are continuously differentiable, and \( g_2(M \cap D_A) \cap (M \cap D_A) = \emptyset \). Furthermore, due to the definition of \( g_3 \) in (25), for each \( \phi \in \mathcal{S}_{\mathcal{H}|M_A}(M \cap C_A), (\phi(t), j) \in \mathcal{S}_{\mathcal{T}} \) converges to zero in finite time, namely, \( (\phi(t), j) \in \mathcal{T} = 0 \) for all \( t + j > 1 + \varepsilon \), \( (t, j) \in \text{dom} \phi \). Note that every \( \phi \in \mathcal{S}_{\mathcal{H}|M_A} \) is complete. Next, we show that \( \mathcal{H}|M_A \) is δS by definition.
δ = \sqrt{\left|\phi_1(0,0)\right|^2 + \left|\phi_2(0,0)\right|^2 - 2|\phi_1(0,0)||\phi_2(0,0)|\left(1 - \frac{\varepsilon^2}{2c^2}\right)}.

Without loss of generality, assume \(\phi_1\) jumps first. For each \(j \in \mathbb{N}\setminus\{0\}\), let \(t_j = \max\{t_j - 1\} \in \text{dom } \phi_1 \cap \text{dom } \phi_2\) and \(\bar{t}_j = \min\{t_j - 1\} \in \text{dom } \phi_1 \cap \text{dom } \phi_2\). Then, we have that for each \(t \in [0, \bar{t}_j]\), \((s,0) = (t,0) \in \text{dom } \phi_2\) and \(\left|\phi_1(t,0) - \phi_2(s,0)\right| = \left|e^{A t} \phi_1(0,0) - e^{A t} \phi_2(0,0)\right| \leq \delta < \varepsilon\), where \(A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}\).

For each \(t \in \bar{t}_j, \bar{t}_j\), we have that \((s,0) = (t,0) \in \text{dom } \phi_2\) and
\[
\left|\phi_1(t,0) - \phi_2(s,0)\right| = \left|e^{A t} \phi_1(0,0) - e^{A t} \phi_2(0,0)\right| \leq \varepsilon.
\]

Note that by the form of system (35),
\[
\bar{t}_j - \bar{t}_j = T_j(\phi_2(0,0)) - T_j(\phi_2(0,0)) = \frac{1}{2} \arccos \frac{\left|\phi_2(0,0) - 2(\phi_2(0,0) - \phi_2(0,0))\right|^2}{2|\phi_2(0,0)||\phi_2(0,0)|} \leq \frac{1}{2} \arccos \left|\phi_2(0,0)\right|^2 - \delta^2 \leq \varepsilon.
\]

Then, for each \(t \in [\bar{t}_j, \bar{t}_j]\), \(t - \bar{t}_j \leq |t - \bar{t}_j| \leq \varepsilon\). By definition of \(\phi_2\), we have \(\phi_2(t,1) = (c,0)\). For each \(t \in \bar{t}_j, \bar{t}_j\), \(\phi_1(t,1) = e^{A(t - \bar{t}_j)}(c,0)\). Therefore, using (35), we have
\[
\left|\phi_1(t,1) - \phi_2(t,1)\right| = \left|c \left(1 - \cos(b(t - \bar{t}_j)), \sin(b(t - \bar{t}_j))\right)\right| = \sqrt{2c}\sqrt{1 - \cos(b(t - \bar{t}_j))} \leq \varepsilon.
\]

In fact, for each \(t \in [\bar{t}_j, \bar{t}_j]\), where \(j \in \mathbb{N}\setminus\{0\}\), \(|t - \bar{t}_j| \leq |t - \bar{t}_j| \leq \varepsilon\). Moreover, for each \(t \in [\bar{t}_j, \bar{t}_j]\), where \(j \in \mathbb{N}\setminus\{0\}\), we have that \((s,j) = (t,j) \in \text{dom } \phi_2\) and
\[
\left|\phi_1(t,j) - \phi_2(s,j)\right| = \left|e^{A(t - \bar{t}_j)} \phi_1(\bar{t}_j, j) - e^{A(t - \bar{t}_j)} \phi_2(\bar{t}_j, j)\right| = \left|\phi_1(\bar{t}_j, j) - \phi_2(\bar{t}_j, j)\right| \leq \varepsilon.
\]

For each \(t \in [\bar{t}_j, \bar{t}_j]\), where \(j \in \mathbb{N}\setminus\{0\}\), we have that \((s,j) = (t,j) \in \text{dom } \phi_2\) and
\[
\left|\phi_1(t,j) - \phi_2(s,j)\right| = \left|e^{A(t - \bar{t}_j)} \phi_1(\bar{t}_j, j) - e^{A(t - \bar{t}_j)} \phi_2(\bar{t}_j, j)\right| \leq \varepsilon.
\]

Therefore, the system is \(\delta S\). By applying Theorem (5.6) every \(\phi \in \mathcal{S}_{\mathbb{R}^n}|_{\mathcal{M}_g}\) is \(\mathcal{Z}\).

B. Existence of Hybrid Limit Cycles via Zhukovskii and Incremental Graphical Stability

In this section, we present conditions for the existence of a hybrid limit cycle for hybrid systems that are ZLAS. The existence of such a hybrid limit cycle is related to nonemptiness of an \(\omega\)-limit set and continuity of a Poincaré map \(\Gamma\) on a closed set \(\Sigma\) near an \(\omega\)-limit point.

The importance of such a Poincaré map is that it allows one to determine the existence of a hybrid limit cycle, as we show in this section. Before presenting that result, inspired by [56 Chapter V, Definition 2.13], the following notion is introduced in a sufficiently “short” tube \(\Phi_t(U) := \{\phi_x(t,0) : t \mapsto \phi_x(t,0)\) is a solution to \(\dot{x} = f(x)\) \(x \in \mathbb{R}^n\) from \(\phi_x(0,0) \in U\), \(t \in [0, \bar{t}], (t,0) \in \text{dom } \phi_x\}, \) where \(U \subset \mathbb{R}^n\) and \(\bar{t} \geq 0\).

**Definition 5.8:** (forward local section) Consider a dynamical system \(\dot{x} = f(x) \in \mathbb{R}^n\). Given \(U \subset \mathbb{R}^n\) and \(\bar{t} \geq 0\), a closed set \(\Sigma \subset \Phi_t(U)\) is called a local section if for each solution \(\phi_x\) to \(\dot{x} = f(x) \in \mathbb{R}^n\) starting from \(\phi_x(0,0) \in U\), there exists a unique \(t_u \in [0, \bar{t}]\) such that \(t_u \in \text{dom } \phi_x\) and \(\phi_x(t_u) \in \Sigma\).

To guarantee the existence of a forward local section, inspired by [56 Chapter V, Theorem 2.14], we present the following result, which is different from [56 Chapter V, Theorem 2.14] as it only allows for forward times.

**Lemma 5.9:** Consider the dynamical system \(\dot{x} = f(x) \in \mathbb{R}^n\). If \(f\) is continuously differentiable and \(v\) is not an equilibrium point of the dynamical system, then, for any sufficiently small \(\bar{t} > 0\), there exists \(\sigma > 0\) such that there exists a forward local section \(\Sigma \subset \Phi_t(v + \sigma B)\).

**Proof:** Consider a maximal flow solution \(t \mapsto \phi^f(t, x_0)\) to \(\dot{x} = f(x)\) from \(x_0 \in \mathbb{R}^n\). Obviously, \(\phi^f(0, x_0) = x_0\).

By assumption, \(f\) is continuously differentiable on \(\mathbb{R}^n\), which implies Lipschitz continuity of \(f\). Then, by [59 Theorem 3.2], for each initial condition, there exists a unique solution to the dynamical system \(\dot{x} = f(x) \in \mathbb{R}^n\) for all \(t \geq 0\), and, by [59 Theorem 3.5], it follows that solutions to \(\dot{x} = f(x)\) depend continuously on the initial conditions. In addition, since \(v\) is not an equilibrium point of the dynamical system, there exists \(t_0 > 0\) such that \(|v - \phi^f(t_0, v)| > 0\).

Define a function \(\varphi\) as
\[
\varphi(x, t) = \int_t^{t+t_0} |v - \phi^f(\tau, x)|d\tau,
\]
which is continuous with respect to \(t\) and has the partial derivative
\[
\frac{\partial \varphi(x, t)}{\partial t} = |v - \phi^f(t + t_0, x)| - |v - \phi^f(t, x)|.
\]
Moreover, since each solution to \(\dot{x} = f(x) \in \mathbb{R}^n\) depends continuously on the initial condition, the function \(\varphi\) is also continuous with respect to \(x\).

Using the fact that \(\phi^f(0, v) = v\), we have
\[
\left|\frac{\partial \varphi(v, t)}{\partial t}\right| = |v - \phi^f(t_0, v)| > 0.
\]

Here, for the system \(\dot{x} = f(x) \in \mathbb{R}^n\), since it is a continuous-time system, we have \(\text{dom } \phi_x \subset \mathbb{R}_0^\infty\). Then, we write \(\phi_x(t)\) instead of \(\phi_x(0, t)\).
Then, there exists $\varepsilon > 0$ such that for every $p \in v + \varepsilon B^o$
$$\frac{\partial \varphi(t, v)}{\partial t} \big|_{t=0} = |v - \phi'(t_0, p)| > 0.$$ (39)

We next prove that i) there exists $\tilde{t} > 0$ and $\sigma > 0$ such that for all $t \in [0, \tilde{t}]$, $\phi^f(t, v + \sigma B^o) \subset v + \varepsilon B$, and that ii) for each $q \in v + \varepsilon B$, there exists a unique $t_v \in [0, \tilde{t}]$ such that $\varphi(q, t_v) = \varphi(v, \tilde{t}/2)$.

We prove the first assertion. First, due to the continuous dependence on initial conditions, we can choose $\tilde{t} > 0$ such that for all $t \in [0, \tilde{t}]$
$$\phi^f(t, v) \in v + \varepsilon B^o.$$ Denote $\tilde{t}_v := \frac{\tilde{t}}{t}$. Then, using the fact that for each $q \in v + \varepsilon B$, $\varphi(q, t)$ is an increasing, continuous function of $t$, and using (39), we have $\varphi(v, \tilde{t}_v) > \varphi(v, \tilde{t}_1) > \varphi(v, 0)$.

Next, one can choose $\eta > 0$ such that
$$\phi^f_0(\tilde{t}, v) + \eta B \subset v + \varepsilon B^o,$$
$$\phi^f(0, v) + \eta B \subset v + \varepsilon B^o,$$
and such that for $q \in \phi^f_0(\tilde{t}, v) + \eta B^o$ there holds $\varphi(q, 0) > \varphi(v, \tilde{t}_1)$ and for $q \in \phi^f(0, v) + \eta B^o$ there holds $\varphi(q, 0) < \varphi(v, \tilde{t}_1)$.

Finally, we choose a number $\sigma > 0$ such that
$$\phi^f(\tilde{t}, v + \sigma B^o) \subset \phi^f_0(\tilde{t}, v) + \eta B^o,$$
$$\phi^f(0, v + \sigma B^o) \subset \phi^f_0(0, v) + \eta B^o,$$
and such that for all $t \in [0, \tilde{t}]$
$$\phi^f(t, v + \sigma B^o) \subset v + \varepsilon B.$$ (42)

We next prove the second assertion. For each $q \in v + \sigma B^o$, using (40) and (41), we have
$$\phi^f(\tilde{t}, q) \in \phi^f_0(\tilde{t}, v) + \eta B^o \subset v + \varepsilon B^o,$$
$$\phi^f(0, q) \in \phi^f_0(0, v) + \eta B^o \subset v + \varepsilon B^o.$$ Then, we obtain $\varphi(q, \tilde{t}_v) > \varphi(v, \tilde{t}_1) > \varphi(q, 0)$.

Hence, we have that for each $q \in v + \sigma B^o$, there exists a unique $t_v \in [0, \tilde{t}]$ such that $\varphi(q, t_v) = \varphi(v, \tilde{t}_v)$. Therefore, for the chosen $\sigma$, the unique time $t_v \in T_v := \{ t \in [0, \tilde{t}] : q \in v + \sigma B, \varphi(q, t) = \varphi(q, t_v) \}$ can be used to construct the desired forward local section $\Sigma$. The visualized idea of the above proof can be seen in Fig. 10.

Finally, by the definition of $\Phi_t$, since $\phi^f_0(\tilde{t}_1, v) \in \Phi_t(v + \sigma B^o)$, we have $\varphi(q, t_v) \in \Phi_t(v + \sigma B^o)$, and the desired forward local section $\Sigma$ can be constructed as
$$\Sigma := \{ \phi^f(t_v, v) : q \in v + \sigma B \}$$
where $t_v \in T_v$. This completes the proof.

The following result reveals the behavior of the solutions to the flow dynamics of the hybrid system $H | M = (M \cap C, f, M \cap D, g)$ in some neighborhood of any point in $M \cap C$ and ensures the existence of a forward local section $\Sigma$ in the tube $\Phi_t$.

Lemma 5.10: Consider a hybrid system $H = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that for the hybrid system $H | M = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no equilibrium set for the flow dynamics
$$\dot{x} = f(x) \quad x \in M \cap C.$$ (43)

For each $v \in (M \cap C)^o$ and a sufficiently small $\tilde{t} > 0$, there exists $\sigma > 0$ such that each solution $\phi_x$ to (43) starting from $\phi_x(0, 0) \in \Phi_t(v + \sigma B^o)$ has the following properties: i) $\Phi_t(v + \sigma B^o) \subset (M \cap C)^o$; ii) there exists a forward local section $\Sigma \subset \Phi_t(v + \sigma B^o)$.

Proof: The first property, namely, $\Phi_t(v + \sigma B^o) \subset (M \cap C)^o$, follows directly from the definition of $\Phi_t(v + \sigma B^o)$, since $\Phi_t(v + \sigma B^o)$ are truncated solutions to (43) starting from $v + \sigma B$. Next, we apply Lemma 5.9 to prove the second property. Since $M \cap C$ contains no equilibrium set for the flow dynamics (43), each $x \in M \cap C$ is not an equilibrium point. Moreover, from item 2) of Assumption 4.1 $f$ is continuous on $M \cap C$ and differentiable on a neighborhood of $M \cap C$. Then, by Lemma 5.9 for each point $v \in (M \cap C)^o$ and sufficiently small $\tilde{t} > 0$, there exists $\sigma > 0$ such that each solution $\phi_x$ to (43) starting from $\phi_x(0, 0) \in v + \sigma B$ satisfies the second property in the theorem. The proof is complete.

The following result is derived via an application of the tubular flow theorem (Theorem 5.1 in Appendix) to the flow dynamics (43).

Lemma 5.11: Consider a hybrid system $H = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 4.1. Suppose that for the hybrid system $H | M = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no critical point of the map $f$. For any open set $U \subset M \cap C$ and for each point $v \in U$, there exists an open neighborhood $N_v \subset U$ of $v$ such that solutions to (43) from $N_v$ are diffeomorphic to the solutions to the system
$$\dot{x} = f(x) \quad x \in \{ 2, 3, \ldots, n \}$$ (44)
on $(-1,1)^n$.

Proof: We use the tubular flow theorem (see Theorem 9.1 in Appendix) to prove the result. First, we verify the conditions of the tubular flow theorem. Since $M \cap C$ contains no critical points for the map $f$ in (43), each $x \in M \cap C$ is a regular point of $f$. Moreover, since $M \cap C$ has a nonempty interior and $f$ is continuously differentiable by item 2) of Assumption 4.1, $f$ is a vector field of class $C^r, r \geq 1$, on any open set $U \subset M \cap C$. Therefore, all conditions in the tubular flow theorem are verified.

Now, by the tubular flow theorem, letting $v \in U$ be a regular point of $f$, there exists an open neighborhood $N_{v} \subset U$ of $v$ such that solutions to (43) from $N_{v}$ are diffeomorphic to the solutions to the system (44) on $(-1,1)^n$. □

The following result provides sufficient conditions for the existence of a hybrid limit cycle of a hybrid system. In addition to technical conditions, ZLAS would serve as a sufficient condition for the existence of a hybrid limit cycle, which is motivated by the use of ZLAS for continuous-time systems in [45].

**Theorem 5.12:** Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfying Assumption 7.7. Suppose that for the hybrid system $\mathcal{H}_{|M} = (M \cap C, f, M \cap D, g)$, $M \cap C$ has a nonempty interior and contains no critical points for the map $f$, and contains no equilibrium set for the flow dynamics (42), and for each $x \in M \cap C$, each maximal solution to $\mathcal{H}_{|M}$ is complete with its hybrid time domain unbounded in the $t$ direction, and each solution to (43) is not complete and ends at a point in $M \cap C$. Then, for each solution $\phi \in \mathcal{S}_{\mathcal{H}_{|M}}(M \cap C), \mathcal{H}_{|M}$ has a nonempty $\omega$-limit set $\Omega(\phi)$. In addition, if the solution $\phi$ is ZLAS and $\Omega(\phi) \cap (M \cap C)^g$ is nonempty, then $\Omega(\phi)$ is a hybrid limit cycle for $\mathcal{H}_{|M}$ with period given by some $T^* > 0$ and multiple jumps per period.

Proof: First, we prove nonemptyness and forward invariance of $\Omega(\phi)$. Since the hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a compact set $M \subset \mathbb{R}^n$ satisfy Assumption 4.1, every maximal solution to $\mathcal{H}_{|M}$ is unique via [5] Proposition 2.11] and $\mathcal{H}_{|M}$ satisfies the hybrid basic conditions. By [5] Theorem 6.8], $\mathcal{H}_{|M}$ is nominally well-posed. Since each solution $\phi \in \mathcal{S}_{\mathcal{H}_{|M}}(M \cap C)$ is unique and complete, the set $M \cap C$ is forward invariant for $\mathcal{H}_{|M}$. Note that completeness of each solution $\phi$ and the compactness of $M$ imply that each $\phi$ is bounded. Then, it follows from [52] Lemma 3.3] that the $\omega$-limit set $\Omega(\phi)$ is a nonempty, compact, and weakly invariant subset of $M$.

Next, we prove the existence of a forward local section $\Sigma$. By assumption, let the solution $\phi \in \mathcal{S}_{\mathcal{H}_{|M}}(M \cap C)$ be ZLAS. Since $\Omega(\phi) \cap (M \cap C)^g$ is nonempty, we can choose a point $p \in \Omega(\phi) \cap (M \cap C)^g$. By Definition 5.1 one can choose a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} t_i + j_i = \infty$.

15$C^r$ denotes the differentiability class of mappings having $r$ continuous derivatives.

16Here, we establish sufficient conditions for the existence of a hybrid limit cycle with multiple jumps in each period. A hybrid limit cycle notion allowing for multiple jumps in a period can be defined similarly; see [43, 50]. For specific systems with one jump as will be illustrated in next example, the result is also applicable.

17A point $q$ is a fixed point of a Poincaré map $\Gamma: \Sigma \to \Sigma$ if $q = \Gamma(q)$.

18If this conclusion holds for $j_m = 1$, the remaining proofs will show that $\mathcal{H}_{|M}$ has a hybrid limit cycle with one jump in each period.

19Since $\phi(0,0) \in \mathcal{S} \cap (p + \sigma B)$, there always exists $(t_m, j_m)$ such that $\phi(t_m, j_m) \in \mathcal{S} \cap (p + \sigma B)$ in $(M \cap C)^g$. In fact, if that were not the case, for each $(t_m, j_m) \in dom \phi$ satisfying $t_m > 0, j_m > 1$ and $t_m + j_m > T$, we would have $\phi(t_m, j_m) \not\in \mathcal{S} \cap (p + \sigma B)$ and $|\phi(t_m, j_m) - \phi(t_m, j_m)\| \leq 2/\sigma$, since $\phi$ is ZLAS. Since $\phi(0,0) \in \mathcal{S} \cap (p + \sigma B)$ and $\phi(t_m, j_m) \not\in \mathcal{S} \cap (p + \sigma B)$, we contradict the fact $p \in \Omega(\phi) \cap (M \cap C)^g$. Therefore, there exist positive constants $0 < \sigma < \delta$ such that $p + \delta B \subset \phi(t, j) + \delta B$ and for sufficiently small $\bar{t} > 0$, $\Phi_p(p + \sigma B) \subset \phi(t, j) + \delta B$ for some $(t, j) \in \{(t_i, j_i)\}_{i=1}^{\infty}$ with $(t_i, j_i) \in dom \phi$. Then, with the picked constants $\sigma$ and $\bar{t}$ (which can be chosen smaller if necessary), by Lemma 5.10, we have the following properties: i) $\Phi_p(p + \sigma B) \subset (M \cap C)^g$; ii) there exists a forward local section $\Sigma \subset \mathcal{F}_p(p + \sigma B)$.

Now, to show the existence of a hybrid limit cycle, let us introduce a Poincaré map for local structure of hybrid systems. Given the forward local section $\Sigma \subset \mathcal{F}_p(p + \sigma B)$, we denote the Poincaré map as $\Gamma: \Sigma \to \Sigma$ and define it as

$$\Gamma(x) := \{ \psi(t, j) \in \Sigma: \psi \in \mathcal{S}_{\mathcal{H}_{|M}}(x), t > 0, (t, j) \in dom \psi \} \quad \forall x \in \Sigma.$$  \tag{45}

We next prove that i) for each solution $\psi$ to $\mathcal{H}_{|M}$ starting from $\psi(0,0) \in \Sigma$, there exists $(t, j) \in dom \psi$ with $t > 0$ such that $\psi(t, j) \in \Sigma$, and that ii) the Poincaré map $\Gamma$ has a fixed point $q \in \Sigma$.

We prove the first assertion. By the definition of $\Omega(\phi)$ and from the analysis above, we have the following claim.

**Claim 1:** With the solution $\phi$ and $\sigma$ above, there exists $(t_k, j_k) \in \{(t_i, j_i)\}_{i=1}^{\infty}$ such that $|\phi(t_k, j_k) - p| \leq \sigma/2$ and $|\phi(t_m, j_m) - p| \leq \sigma/2$ for each $t_m \geq t_k$ and each $j_m \geq j_k$.

Let $\phi_1(0,0) := \phi(t_k, j_k)$ as above and define $\phi_1$ as the translation of $\phi$ by $(t_k, j_k)$, which leads to a complete solution $\phi_1$ due to completeness of $\phi$. By assumption, the solution $\phi_1$ to $\mathcal{H}_{|M}$ is ZLAS. Then, we have the following claim by Definition 5.1.

**Claim 2:** With $\sigma$ above, for each $\phi_2 \in \mathcal{S}_{\mathcal{H}_{|M}}(\phi_1(0,0) + \delta B)$ there exists a function $\tau \in \mathcal{T}$ such that for $\varepsilon = \sigma/2 > 0$ there exists $T > 0$ for which we have $(t, j) \in dom \phi_1, t + j > T$ implies that $(\tau(t), j) \in dom \phi_2$ and $|\phi_1(t, j) - \phi_2(\tau(t), j)| \leq \varepsilon = \sigma/2$.

Since $\Sigma \subset \mathcal{F}_p(p + \sigma B) \subset \phi_1(0,0) + \delta B$, each solution $\phi_3$ to $\mathcal{H}_{|M}$ from $\phi_3(0,0) \in \Sigma$ also satisfies Claim 2. In addition, by assumption, since each solution to (43) is not complete and ends at a point in $M \cap C$, we have recurrent jumps. Therefore, from Claim 1 and Claim 2, for each $\phi_3 \in \mathcal{S}_{\mathcal{H}_{|M}}(\Sigma)$, there exist $\tau \in \mathcal{T}$ and $(t_m, j_m) \in dom \phi_3$ satisfying $t_m \geq 0, j_m \geq 1$, and $t_m + j_m > T$ such that $(\tau(t_m), j_m) \in dom \phi_3, |\phi_3(t_m, j_m) - p| \leq \sigma/2$, and

$|\phi_3(t_m, j_m) - \phi_3(\tau(t_m), j_m)| \leq \sigma/2$, which leads to $\phi_3(\tau(t_m), j_m) \in \mathcal{F}_p(p + \sigma B)$. In addition, there exists $(t_m, j_m) \in dom \phi_3$ as above such that $\phi_3(\tau(t_m), j_m) \in (p + \sigma B) \cap \mathcal{T}$. Let $\phi_3(0,0) = \phi_3(\tau(t_m), j_m)$ and define $\phi_3$ as the translation of $\phi_3$ by $(\tau(t_m), j_m)$, which leads to a complete solution $\phi_3$ due to completeness of $\phi_3$.

Then, $\phi_3(0,0) \in (p + \sigma B) \cap \mathcal{T}$. By the second property of
Lemma 5.10 and the definition of forward local section in Definition 5.8, we see that the solution $\phi$ to (43) reaches the forward local section set $\Sigma \subset \Phi_1(p + \sigma B)$ at a unique time $t_p \in [0, t]$, that is, $(t_p, 0) \in \text{dom} \phi$ and $\phi(t_p, 0) \in \Sigma$, which implies $\phi_3(t_m + t_p, j_m) \in \Sigma$. Therefore, the first assertion is proved.

To prove the second assertion, that is, the Poincaré map $\Gamma$ has a fixed point $q \in \Sigma$, first we show continuity of $\Gamma$ on $\Sigma$. Since $\phi_3(0, 0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$, we have that $\Gamma(\phi_3(0, 0)) = \phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$. Since $\Sigma \subset \Phi_1(p + \sigma B)$ and $t$ can be sufficiently small, we have $\phi_3(0, 0) \in p + \sigma B$ and $\Gamma(\phi_3(0, 0)) \in p + \sigma B$. Then, it follows that $|\Gamma(\phi_3(0, 0)) - \phi_3(0, 0)| \leq |\Gamma(\phi_3(0, 0))| - p| + |p - \phi_3(0, 0)| \leq 2 \sigma$.

Therefore, since the chosen $\sigma$ can be small enough, we have that the map $\Gamma$ in (45) is continuous on $\Sigma$.

Next, by applying the Brouwer’s fixed point theorem (see Theorem 9.2 in Appendix), we show that the map $\Gamma$ has a fixed point $q \in \Sigma$. With the point $p$ and $\sigma$ as chosen above, since $\Sigma \subset \Phi_1(p + \sigma B)$, there exists some neighborhood $N_p \subset \Phi_1(p + \sigma B)$ of $p$ such that $\Sigma \subset N_p$. Note that by assumption, each $x \in M \cap C$ is a regular point of $f$ and all conditions in Lemma 5.11 are satisfied. Therefore, $p$ is a regular point of $f$, and by using Lemma 5.11 solutions to (43) from $N_p$ are diffeomorphic to the solutions to the system (44) on $(-1, 1)^n$. In other words, there exists a $C^r$ diffeomorphism $H : (-1, 1)^n \to N_p$ such that for any solution $\phi_x$ to (44), $\phi_x = H(\phi_3(x))$ is a solution to $x = f(x) x \in N_p$. Note that given an initial condition $(s_1, s_2, \ldots, s_n) \in (-1, 1)^n$, a solution to (44) is given by $\phi_x(s_1, s_2, \ldots, s_n) = (s_1 + ts_2, \ldots, s_n)$ for all $t \in [0, 1 - s_1)$, and the function $t \mapsto H(\phi_x(t)) = x = f(x) x \in N_p$. Denote $\Sigma_x = H^{-1}(\Sigma) = \{\phi_x(0, 0) \in (-1, 1)^n : t \rightarrow \phi_x(t, 0)\}$ is a solution to (44) from $\phi_3(0, 0) \in (-1, 1)^n$, $t \in [0, t_p]$. Denote note that $\Sigma_x$ is convex and bounded. In addition, since $\Sigma$ is closed and $H$ is a diffeomorphism, $\Sigma_x$ is also closed and thus $\Sigma_x$ is a convex compact set. Define a map $\Gamma_x = H^{-1} \circ \Gamma \circ H$ as $\Gamma_x : \Sigma_x \to \Sigma_x$. Due to $H$ being a diffeomorphism and continuity of $\Gamma$, $\Gamma_x$ is continuous. Therefore, by Brouwer’s fixed point theorem (see Theorem 9.2 in Appendix), $\Gamma_x$ has a fixed point $q^* \in \Sigma_x$, i.e., $\Gamma_x(q^*) = q^*$. Since $\Gamma_x = H^{-1} \circ \Gamma \circ H$, we have $H^{-1} \circ \Gamma \circ H(q^*) = q^*$, which implies that $\Gamma \circ H(q^*) = H(q^*) \in \Sigma$. Let $q = H(q^*)$. Therefore, we have that $\Gamma$ has a fixed point $q \in \Sigma$, i.e., $\Gamma(q) = q$.

From the existence of a fixed point for $\Gamma$ and the fact that $\phi_3(0, 0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$, there is a flow periodic solution $\phi^\ast \subset H_\mid M$ with period $T^\ast = \tau(t_m) + t_p$ and $j_m$ jumps in each period. Therefore, $H_\mid M$ has a hybrid limit cycle $O$ with $j_m$ jumps in each period. Note that for the solution $\phi \in S_{H_\mid M}(q)$, every point $\zeta^\ast$ in the hybrid limit cycle $O$ is in $\Omega(\phi)$ since there exists a sequence $\{(t_i, j_i)\}_{i=1}^\infty$ of points $(t_i, j_i) \in \text{dom} \phi$ such that $\lim_{i \to \infty} \phi(t_i, j_i) = \zeta^\ast$ with $\lim_{i \to \infty} t_i + j_i = \infty$. To prove that every point in $\Omega(\phi)$ is also in the hybrid limit cycle $O$, we proceed by contradiction. Suppose that $q \in \Omega(\phi)$ and $q \notin O$. Since from the analysis above, there is a fixed point $q \in \Omega(\phi) \cap \Sigma$ for any chosen $\sigma \in (0, \delta)$ we have $\phi_3(0, 0) \in \Sigma$ and $\phi_3(\tau(t_m) + t_p, j_m) \in \Sigma$. Then, $\Gamma(q) = q$, which leads to a contradiction with $q \notin O$. Thus, every point in $\Omega(\phi)$ is also in the hybrid limit cycle $O$. Therefore, we have that $\Omega(\phi)$ is a hybrid limit cycle. □

Remark 5.13: In Theorem 5.12 there are several ways to guarantee that $M \cap C$ contains no equilibrium set for the flow dynamics (43). One way to assure that is to check if for each $x \in M \cap C$, $f^\ast(x) f(x) > 0$.

The following examples are provided to illustrate Theorem 5.12.

Example 5.14: Consider the academic system $H_{A \mid M}$ in Example 5.4. We will verify the existence of a hybrid limit cycle via Theorem 5.12. First, items 1-3) of Assumption 4.1 have been illustrated in Example 5.4. Since the Jacobian of the map $f_A$ given by $j_{f_A}(x) = -\alpha$ with $\alpha > 0$ has the maximal rank 1, $M_A \cap C_A$ contains no critical points for the map $f_A$.

By the definition of $M_A$, for all $x \in M_A \cap C_A$, $x \leq b_1 < b/a$, which implies that $f_A^3(x) f_A(x) = (-ax + b)^2 \geq (b - ab_1)^2 > 0$. Then, by Remark 5.13, $M_A \cap C_A$ contains no equilibrium set for the flow dynamics $x = f_A(x) x \in M_A \cap C_A$, where $(M_A \cap C_A)^\circ = (0, b_1)$ is nonempty. By the definitions of $f_A$ and $g_A$, the set $M_A$ is forward invariant and each $\phi \in S_{H_{A \mid M}}(M_A \cap C_A)$ is unique and complete with $\text{dom} \phi$ unbounded in the $t$ direction. Next, from the data of $H_{A \mid M}$, each solution $\phi \in S_{H_{A \mid M}}(M_A \cap C_A)$ to $\dot{x} = f_A(x) x \in M_A \cap C_A$ is not complete and ends at a point in $M_A \cap C_A$. Therefore, for each maximal solution $\phi$ from $\xi \in [b_2, b_1]$ given by, for each $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, $\phi(t, j) = \begin{cases} \left( x - \frac{b}{a} \right) e^{-a(t - jT^\ast) + \frac{b}{a}} + \frac{b}{a} & t \in [jT^\ast, jT^\ast + t_1] \\ (b_2 - \frac{b}{a}) e^{-a(t - t_1) + \frac{b}{a}} + \frac{b}{a} & t \in [t_1, jT^\ast + t_1] \end{cases}$

where $t_1 = (j - 1)T^\ast + t_1$, $t_1 = \frac{1}{a} \ln \frac{ab_2 - b}{ab_1 - b}$, and $T^\ast = \frac{1}{a} \ln \frac{ab_2 - b}{ab_1 - b}$ by Theorem 5.12. $H_{A \mid M}$ has a nonempty $\omega$-limit set $\Omega(\phi) = \{ x \in [b_2, b_1] : x = \phi(t, 1), t \in [t_1, t_1 + T^\ast] \}$. Finally, the ZLAS property of each $\phi \in S_{H_{A \mid M}}(M_A \cap C_A)$ has been verified in Example 5.4. In addition, from the construction of $\Omega(\phi)$ and the condition $b_1 > b_2 > 0$, $\Omega(\phi) \cap (M_A \cap C_A)^\circ = [b_2, b_1]$ is nonempty. Therefore, by Theorem 5.12, $\Omega(\phi)$ is the hybrid limit cycle for $H_{A \mid M}$ with period $T^\ast = \frac{1}{a} \ln \frac{ab_2 - b}{ab_1 - b}$ and one jump per period as shown in Fig. 11.

Fig. 11. State trajectories of a solution to $H_{A \mid M}$ from $b_2$. Parameters used in the plot are $a = 2$, $b = 6$, $b_1 = 2$ and $b_2 = 1$. The solution is flow periodic with $T^\ast = \ln 2/2$.

Example 5.15: Consider the hybrid system $H_{S \mid M} = (M_s \cap C_s, f_s, M_s \cap D_s, g_s)$ in Example 5.7. We will verify
the existence of a hybrid limit cycle via Theorem 5.12. First, items 1)-3) of Assumption 4.1 have been verified in Example 5.7. Since the Jacobian of the map \( f_\delta \) given by
\[
J_{f_\delta}(x) = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}
\]
with \( b \geq 1 \) has the maximal rank 2, \( M_\delta \cap C_\delta \) contains no critical points for the map \( f_\delta \). By the definition of \( M_\delta \) in Example 4.16, for all \( x \in M_\delta \cap C_\delta \), we have
\[
f_\delta(x) = b^2(x_1^2 + x_2^2) + b^2(e-c^2) > 0.
\]
Then, by Remark 5.11, \( M_\delta \cap C_\delta \) contains no equilibrium set for the flow dynamics \( \dot{x} = f_\delta(x) \). Thus, the hybrid limit cycle is defined by the solution to \( \dot{x} = f_\delta(x) \) with \( t \in [0, \infty) \). The existence of a hybrid limit cycle via Theorem 5.12. Fig. 7, the hybrid limit cycle is defined by the solution to \( \dot{x} = f_\delta(x) \) with \( t \in [0, \infty) \). Theorem 5.16: For a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1, consider the hybrid system \( \mathcal{H}_M = (M \cap C, f, M \cap D, g) \) and assume that each solution \( \phi \in \mathcal{S}_{\mathcal{H}_M} \) is complete with \( \phi \) unbounded in the direction. If the hybrid system \( \mathcal{H}_M \) is ZLAS, then \( \mathcal{H}_M \) has no hybrid limit cycles for \( \mathcal{H}_M \) with period given by some \( T^* > 0 \).

Proof: We proceed by contradiction. Suppose \( \mathcal{H}_M \) has a hybrid limit cycle \( \mathcal{O} \) defined by a flow periodic solution \( \phi^* \) with period \( T^* > 0 \). By Assumption 4.1, every maximal solution to \( \mathcal{H}_M \) is unique via [5] Proposition 2.11. Consider two maximal solutions \( \phi_1, \phi_2 \) in \( \mathcal{S}_{\mathcal{H}_M} \) where \( \phi_1, \phi_2 \) are two flow periodic solutions with \( \phi_1(t, j) = \phi_2(t + T^*, j + 1) \) for each \( (t, j) \in \text{dom} \phi_1, i = 1, 2 \). Without loss of generality, for any \( \mu > 0 \), we can pick \( \phi_1(0, 0) \in M \cap C_\delta \) and \( \phi_2(0, 0) \in M \cap C_\delta \) satisfying \( |\phi_1(0, 0) - \phi_2(0, 0)| < \mu \). Then, \( \phi_1 \) reaches the jump set before \( \phi_2 \), as \( \phi_1(0, 0) \) already belongs to the jump set. For each \( j \in \mathbb{N} \), let \( t_j = \max_{(t, j) \in \text{dom} \phi_1} \delta(t, j) \) and \( t_j = \min_{(t, j) \in \text{dom} \phi_2} \delta(t, j) \). Then, for each \( j \in \mathbb{N} \), \( t_j \). The hybrid time domain of \( \phi_1 \) is equal to the union of intervals \((t_j, t_{j+1}], j + 1)\) and the hybrid time domain of \( \phi_2 \) is equal to the union of intervals \((t_j, t_{j+1}], j + 1)\). In addition, since for each \( j \in \mathbb{N} \), \( t_{j+1} > t_j \), it follows that \((t_j, t_{j+1}] \in \text{dom} \phi_1 \) and \((t_j, t_{j+1}] \in \text{dom} \phi_2 \). Moreover, since \( \phi_1 \) and \( \phi_2 \) are two flow periodic solutions that share the same hybrid limit cycle with period \( T^* \), we have \( t_{j+1} - t_j = t_{j+1} - t_1 \) for each \( j \in \mathbb{N} \). Let \( \varepsilon > 0 \) and for any \( T > 0 \), \( p' = \frac{1}{2}(t_j + t_{j+1}) \) at some \( j \in \mathbb{N} \) such that \( (t_{j+1}, t_j) \) is in \( \phi_1 \) and \( (t_{j+1}, t_j) \) is in \( \phi_2 \). Then, it is impossible to find \( (s, j+1) \in \text{dom} \phi_2 \) such that \( |\phi_1(t_{j+1}, j+1) - \phi_2(s, j+1)| < \varepsilon \). This contradicts the assumption that the hybrid system \( \mathcal{H}_M \) is ZLAS. \( \square \)

VI. SUFFICIENT CONDITIONS FOR ASYMPTOTIC STABILITY OF HYBRID LIMIT CYCLES

In this section, we present stability properties of hybrid limit cycles for \( \mathcal{H} \).

A. NOTIONS

Following the stability notion introduced in [5] Definition 3.6, we employ the following notion for stability of hybrid limit cycles.

**Definition 6.1:** Consider a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a compact hybrid limit cycle \( \mathcal{O} \). Then, the hybrid limit cycle \( \mathcal{O} \) is said to be
- **stable** for \( \mathcal{H} \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every solution \( \phi \in \mathcal{H} \) with \( |\phi(0, 0)|_\mathcal{O} \leq \delta \) satisfies \( |\phi(t, j)|_\mathcal{O} \leq \varepsilon \) for each \( (t, j) \in \text{dom} \phi \).
• globally attractive for $H$ if every maximal solution $\phi$ to $H$ from $\bar{C} \cup D$ is complete and satisfies $\lim_{t \to +\infty} |\phi(t, j)|_\sigma = 0$;
• globally asymptotically stable for $H$ if it is both stable and globally attractive;
• locally attractive for $H$ if there exists $\mu > 0$ such that every maximal solution $\phi$ to $H$ starting from $|\phi(0, 0)|_\sigma \leq \mu$ is complete and satisfies $\lim_{t \to +\infty} |\phi(t, j)|_\sigma = 0$;
• locally asymptotically stable for $H$ if it is both stable and locally attractive.

Given $M \subset \mathbb{R}^n$ and $H = (C, f, D, g)$, for $x \in M \cap (C \cup D)$, define the “distance” function $d : M \cap (C \cup D) \to \mathbb{R}_{\geq 0}$ as

$$d(x) := \sup_{t \in [0, T_x(x)], (t, j) \in \text{dom } \phi, \phi \in S_{H|M}(x)} |\phi(t, j)|_\sigma$$

when $0 \leq T_x(x) < \infty$, and

$$d(x) := \sup_{(t, j) \in \text{dom } \phi, \phi \in S_{H|M}(x)} |\phi(t, j)|_\sigma$$

if $T_x(x) = \infty$, where $T_x$ is the time-to-impact function defined in (44). Now that $d$ vanishes on $O$. Then, following the ideas in [6] Lemma 4, the following property of the function $d$ can be established.

**Lemma 6.2:** Consider a hybrid system $H = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.7. Suppose that every maximal solution to $H|M = (M \cap C, f, M \cap D, g)$ is complete and $H|M$ has a flow periodic solution $\phi^*$ with period $T^* > 0$ that defines a hybrid limit cycle $O \subset M \cap C$. Then, the function $d : M \cap C \to \mathbb{R}_{\geq 0}$ is well-defined and continuous on $O$.

**Proof:** Given $x_0$ such that $x_0 \in M \cap C$ and $0 < T_x(x_0) < \infty$, the solution $t \to \phi^*(t, x_0)$ to $\dot{x} = f(x)$ is well-defined on $t \in [0, T(x_0)]$. Therefore, $d(x_0)$ is also well-defined at $x_0$.

To show continuity of $d$ on $O$, consider a point $x \in O$. As argued in the proof of Lemma 4.12, the solution $t \to \phi_f(t, x_0)$ to $\dot{x} = f(x)$ depends continuously on the initial condition. Then, by Lemma 4.12 given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in O$ and $x' \in x + \delta B$, via the triangle inequality, we have that

$$|\phi_f(t, x')|_\sigma \leq |\phi_f(t, x) - \phi_f(t, x')| + |\phi_f(t, x)|_\sigma \quad (47)$$

for each $t \in [0, \min \{T_f(x), T_f(x')\}]$, where $|T_f(x) - T_f(x')| < \varepsilon$ and $|\phi_f(t, x) - \phi_f(t, x')| < \varepsilon$ for each $t \in [0, \min \{T_f(x), T_f(x')\}]$. Then, for the solution $\phi \in S_H(x')$, we have $\phi(t, 0) = \phi_f(t, x')$ for each $0 \leq t < T_f(x')$. Note that by properties of $f$, $|\phi_f(t, x)|_\sigma = 0$. Then, it follows that

$$|\phi_f(t, x')|_\sigma \leq \varepsilon$$

and

$$d(x') = \sup_{t \in [0, T_f(x')]} |\phi(t, 0)|_\sigma = \sup_{t \in [0, T_f(x')]} |\phi_f(t, x')|_\sigma.$$

\[\Box\]

**B. Asymptotic Stability Properties of $O$**

To establish conditions for asymptotic stability of a hybrid limit cycle, let us introduce a Poincaré map for hybrid systems. Referred to as the **hybrid Poincaré map**, given a maximal solution $\phi$ to $H|M$, we denote it as $P : M \cap D \to M \cap D$ and define it as

$$P(x) := \begin{cases} \phi(T_f(g(x)), j) : \phi \in S_{H|M}(g(x)), & \text{if } T_f(g(x)) < +\infty, \\ (T_f(g(x)), j) \in \text{dom } \phi & \forall x \in M \cap D, \end{cases}$$

where $T_f$ is the time-to-impact function defined in (44).

The importance of the hybrid Poincaré map in (48) is that it allows one to determine the stability of hybrid limit cycles. Before revealing the stability properties of a hybrid limit cycle, we introduce the following stability notions for the hybrid Poincaré map $P$ in (48). Let $P^k$ denote $k$ compositions of the hybrid Poincaré map $P$ with itself; namely, $P^k(x) = \circ \cdots \circ P \circ P(x)$.

**Definition 6.3:** A fixed point $x^*$ of a hybrid Poincaré map $P : M \cap D \to M \cap D$ defined in (48) is said to

• be **stable** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in M \cap D$, $|x - x^*| \leq \delta$ implies $|P^k(x) - x^*| \leq \varepsilon$ for all $k \in \mathbb{N}$;
• be **globally attractive** if for each $x \in M \cap D$, $\lim_{k \to +\infty} P^k(x) = x^*$;
• be **locally asymptotically stable** if it is both stable and globally attractive;
• be **locally attractive** if there exists $\mu > 0$ such that for each $x \in M \cap D$, $|x - x^*| \leq \mu$ implies $\lim_{k \to +\infty} P^k(x) = x^*$;
• be **locally asymptotically stable** if it is both stable and locally attractive.

A relationship between stability of fixed points of hybrid Poincaré maps and stability of the corresponding hybrid limit cycles is established next.

**Theorem 6.4:** Consider a hybrid system $H = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.7. Suppose that every maximal solution to $H|M = (M \cap C, f, M \cap D, g)$ is complete and $H|M$ has a flow periodic solution $\phi^*$ with period $T^* > 0$ that defines a hybrid limit cycle $O \subset M \cap C$. Then, the following statements hold:

1. $x^* \in M \cap D$ is a stable fixed point of the hybrid Poincaré map $P$ in (48) if and only if the hybrid limit cycle $O$ of $H|M$ generated by a flow periodic solution $\phi^*$ with period $T^*$ from $\phi^*(0, 0) = x^*$ is stable for $H|M$.
2. $x^* \in M \cap D$ is a globally asymptotically stable fixed point of the hybrid Poincaré map $P$ if and only if $H|M$ has a unique hybrid limit cycle $O$ generated by a flow periodic solution $\phi^*$ with period $T^*$ from $\phi^*(0, 0) = x^*$ that is globally asymptotically stable for $H|M$ with basin of attraction containing every point $\mu_2 M \cap C$.

**Proof:** We first prove the sufficiency of item 1). By Assumption 4.1, every maximal solution to $H|M$ is unique via [5] Proposition 2.11. Consider the hybrid limit cycle $O$ generated by a flow periodic solution to $H|M$ from $x^*$ with $x^* \in M \cap D$. Since $O$ is stable for $H|M$, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $\phi$ to $H|M$.

\[\text{The hybrid Poincaré map } P \text{ in (48) is different from the Poincaré map } \Gamma : \Sigma \to \Sigma \text{ in (45). The map } \Gamma \text{ in (45) maps } M \cap D \to M \cap D \text{ within one jump, while the map } P \text{ in (48) maps a closed set } \Sigma \subset (M \cap C)^2 \to \Sigma \text{ and allows for multiple jumps.}\]

\[\text{A "global" property for } H|M \text{ implies a "global" property of the original system } \text{ only when } M \text{ contains } \Sigma. \text{ For tools to establish the asymptotic stability property, see [5].}\]
all the operator implies \( |\phi(t,j)| \leq \varepsilon \) for each \((t,j) \in \text{dom } \phi\). Since \( \phi \) is complete and \( P^k(x^*) = \phi(T_j(g(x^*)), j) \) for some \( j \), in particular, we have that \( |P^k(x^*)| \leq \varepsilon \) for each \( k \in \mathbb{N} \). Therefore, \( x^* \in M \cap D \) is a stable fixed point of the hybrid Poincaré map \( P \).

Next, we prove the necessity of item 1) as in the proof of [6] Theorem 1]. Suppose that \( x^* \in M \cap D \) is a stable fixed point of \( P \). Then, \( P(x^*) = x^* \) due to the continuity of \( P \) in \( \mathbb{R}^n \) and, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\tilde{x} \in (x^* + \delta B) \cap (M \cap D)
\]
implies
\[
P^k(\tilde{x}) \in (x^* + \varepsilon B) \cap (M \cap D) \quad \forall k \in \mathbb{N}.
\]
Moreover, by assumption, every maximal solution \( \phi \) to \( H|_M \) from \( \tilde{x} \in (x^* + \delta B) \cap (M \cap D) \) is complete. Since solutions are guaranteed to exist from \( M \cap D \), there exists a complete solution \( \phi \) from every such point \( \tilde{x} \). Furthermore, the distance between \( \phi \) and the hybrid limit cycle \( O \) satisfies
\[
\sup_{(t,j) \in \text{dom } \phi} |\phi(t,j)| \leq \sup_{x \in (x^* + \varepsilon B) \cap (M \cap D)} d \circ g(x).
\]
By Lemma 6.2, \( d \) is continuous at \( x^* \). Since \( O \) is transversal to \( M \cap D \), \( O \cap (M \cap D) \) is a singleton, \( g(x^*) \in O \), and \( g \) is continuous, we have that \( d \circ g \) is continuous at \( x^* \). Moreover, since \( d \circ g(x^*) = 0 \), it follows by continuity that given any \( \varepsilon > 0 \), we can pick \( \bar{\varepsilon} \) and \( \bar{\delta} \) such that \( 0 < \bar{\varepsilon} < \varepsilon \) and
\[
\sup_{x \in (x^* + \bar{\varepsilon} B) \cap (M \cap D)} d \circ g(x) \leq \varepsilon.
\]
Therefore, an open neighborhood of \( O \) given by \( V := \{ x \in \mathbb{R}^n : d(x) \in [0, \bar{\varepsilon}] \} \) is such that any solution \( \phi \) to \( H|_M \) from \( \phi(0,0) \in V \) satisfies \( |\phi(t,j)| \leq \varepsilon \) for each \((t,j) \in \text{dom } \phi \). Thus, the necessity of item 1) follows immediately.

The stability part of item 2) follows similarly. Sufficiency of the global attractivity part in item 2) is proved as follows. Suppose the hybrid limit cycle \( O \) generated by a flow periodic solution \( \phi \) is globally attractive for \( H|_M \) with basin of attraction containing every point in \( M \cap C \) if and only if \( \phi \) is complete and \( \phi(0,0) = x^* \) is locally attracting fixed point of the hybrid Poincaré map \( P \) generated by a flow periodic solution \( \phi \) with period \( T^* > 0 \) that defines a hybrid limit cycle \( O \subset M \cap C \). Then, \( x^* \in M \cap D \) is a locally asymptotically stable fixed point of the hybrid Poincaré map \( P \).

Proof: The sufficiency and necessity of the stability part have been proven in Theorem 6.4. The sufficiency of the claim is proved as follows. Suppose the hybrid limit cycle \( O \) generated by a flow periodic solution \( \phi \) is complete and \( \phi(0,0) = x^* \) is locally attracting fixed point of the hybrid Poincaré map \( P \). Then, \( x^* \in M \cap D \) is a locally attracting fixed point of the hybrid Poincaré map \( P \).

Finally, we prove the necessity of the global attractivity property in item 2). Assume that \( x^* \in M \cap D \) is a locally attracting fixed point. Then, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( k \in \mathbb{N} \),
\[
\tilde{x} \in (x^* + \delta B) \cap (M \cap D)
\]
implies \( \lim_{k \to \infty} P^k(\tilde{x}) = x^* \). Moreover, following from Definition 6.3, it is implied that a maximal solution \( \phi \) to \( H|_M \) from \( x^* \) is complete. Then, by continuity of \( d \) and \( g \),
\[
\lim_{k \to \infty} d \circ g(P^k(x)) = d \circ g(x^*) = 0,
\]
from which it follows that
\[
\lim_{t+j \to \infty} |\phi(t,j)| \leq \lim_{k \to \infty} \sup_{\tilde{x} \in (x^* + \delta B) \cap (M \cap D)} d \circ g(P^k(\tilde{x})) \leq d \circ g(x^*) = 0,
\]
and the set \( O \) is a locally attractive fixed point of the hybrid Poincaré map \( P \).

Remark 6.5: In [6], sufficient and necessary conditions for stability properties of periodic orbits in impulsive systems are established using properties of the fixed points of the corresponding Poincaré maps. Compared to [6], Theorem 6.4 enables the use of the Lyapunov stability tools in [5] to certify asymptotic stability of a fixed point without even computing the hybrid Poincaré map.

At times, one might be interested only on local asymptotic stability of the fixed point of the hybrid Poincaré map. Such case is handled by the following result.

Corollary 6.6: Consider a hybrid system \( H = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.7. Suppose that every maximal solution \( x^* \) of \( H|_M \) is complete and \( \phi(0,0) = x^* \) is locally attracting fixed point of the hybrid Poincaré map \( P \) if and only if \( H|_M \) has a unique hybrid limit cycle \( O \) generated by a flow periodic solution \( \phi \) with period \( T^* > 0 \) that defines a hybrid limit cycle \( O \subset M \cap C \). Then, \( x^* \in M \cap D \) is a locally asymptotically stable fixed point of the hybrid Poincaré map \( P \).

Proof: The necessity and sufficiency of the stability part have been proven in Theorem 6.4. The sufficiency of the claim is proved as follows. Suppose the hybrid limit cycle \( O \) generated by a flow periodic solution \( \phi \) is complete and \( \phi(0,0) = x^* \) is locally attracting fixed point of the hybrid Poincaré map \( P \). Then, \( x^* \in M \cap D \) is a locally attracting fixed point of the hybrid Poincaré map \( P \).

Finally, we prove the necessity of the global attractivity property in item 2). Assume that \( x^* \in M \cap D \) is a locally attracting fixed point. Then, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( k \in \mathbb{N} \),
\[
\tilde{x} \in (x^* + \delta B) \cap (M \cap D)
\]
implies \( \lim_{k \to \infty} P^k(\tilde{x}) = x^* \). Moreover, following from Definition 6.3, it is implied that a maximal solution \( \phi \) to \( H|_M \) from \( x^* \) is complete. Then, by continuity of \( d \) and \( g \),
\[
\lim_{k \to \infty} d \circ g(P^k(x)) = d \circ g(x^*) = 0,
\]
from which it follows that
\[
\lim_{t+j \to \infty} |\phi(t,j)| \leq \lim_{k \to \infty} \sup_{\tilde{x} \in (x^* + \delta B) \cap (M \cap D)} d \circ g(P^k(\tilde{x})) \leq d \circ g(x^*) = 0,
\]
and the set \( O \) is a locally attractive fixed point of the hybrid Poincaré map.
Due to the quadratic form in the flow map of \( \mathcal{H} \), using Corollary 6.6, we know that the hybrid limit cycle is difficult to obtain. Instead, we compute the Jacobian based on a numerically approximated Poincaré map. We consider the case of intrinsic bursting behavior with parameters given in [25], and obtain a fixed point \( x^* \approx (30, -7.50) \) and limit cycle period \( T^* \approx 31.22 \text{ s} \). The Jacobian of the hybrid Poincaré map at the fixed point is

\[
\mathbb{J}_{P_1}(x^*) = \begin{bmatrix}
0 & 0 \\
0 & 0.025
\end{bmatrix}.
\]

The eigenvalues of \( \mathbb{J}_{P_1} \) are \( \lambda_1 = 0 \) and \( \lambda_2 = -0.025 \), which are inside the unit circle. Corollary 6.6 implies that the hybrid limit cycle \( \mathcal{O} \) of the Izhikevich neuron model is locally asymptotically stable. The properties of the hybrid limit cycle \( \mathcal{O} \) are illustrated numerically in Fig. 6, where the hybrid limit cycle is defined by the solution from the point \( P_2 \) that jumps at the point \( P_1 \). Note that this hybrid limit cycle \( \mathcal{O} \) is locally asymptotically stable, but not globally asymptotically stable with basin of attraction containing every point in \( M_1 \cap C_1 \).

Example 6.10: Consider the compass gait biped system analyzed in Example 4.7. Suppose the hybrid Poincaré map for \( \mathcal{H}_\mathcal{B} \) is given by \( P_1 \) with a fixed point \( x^* \). Using Corollary 6.6, it suffices to check the eigenvalues of the Jacobian matrix of the hybrid Poincaré map at the fixed point. Due to the nonlinear form in the flow map of \( \mathcal{H}_\mathcal{B} \), an analytical expression of the Jacobian of the hybrid Poincaré map is difficult to obtain. Instead, we compute the Jacobian based on a numerically approximated Poincaré map. We consider the parameters given in (12), and obtain a fixed point \( x^* \approx (0.22, -0.32, -1.79, -1.49) \) and limit cycle period \( T^* \approx 0.734 \text{ s} \). The Jacobian of the hybrid Poincaré map at the fixed point is

\[
\mathbb{J}_{P_1}(x^*) = \begin{bmatrix}
0.0000 & 0.0000 & -0.0000 & -0.0000 \\
4.8611 & -1.7156 & -0.0948 & 0.8187 \\
5.3550 & -1.9180 & -0.1241 & 1.2845 \\
9.3471 & -3.2933 & -0.2036 & 1.9851
\end{bmatrix}.
\]

The eigenvalues of \( \mathbb{J}_{P_1} \) are \( \lambda_1 = 0.8897, \lambda_2 = -0.7456, \lambda_3 = -0.0000, \lambda_4 = 0.0013 \) with one eigenvalue located at zero and the other one locates inside the unit circle. Corollary 6.6 implies that the hybrid limit cycle \( \mathcal{O} \) of the compass gait biped system is locally asymptotically stable. The properties of the hybrid limit cycle \( \mathcal{O} \) are illustrated numerically in Fig. 6, where the hybrid limit cycle is defined by the solution from the point \( P_2 \) that jumps at the point \( P_1 \). Note that this hybrid limit cycle \( \mathcal{O} \) is locally asymptotically stable, but not globally asymptotically stable with basin of attraction containing every point in \( M_1 \cap C_1 \).
A. Robustness to General Perturbations

First, we present results guaranteeing robustness to generic perturbations of asymptotically stable hybrid limit cycles. More precisely, we consider the perturbed continuous dynamics of the hybrid system \( \mathcal{H}_M = (M \cap C, f, M \cap D, g) \) given by

\[
x^+ = f(x + d_1) + d_2 \quad x + d_3 \in M \cap C,
\]

where \( d_1 \) corresponds to state noise (e.g., measurement noise), \( d_2 \) captures unmodeled dynamics or additive perturbations, and \( d_3 \) captures generic disturbances on the state when checking if the state belongs to the constraint. Similarly, we consider the perturbed discrete dynamics

\[
x^+ = g(x + d_1) + d_2 \quad x + d_4 \in M \cap D,
\]

where \( d_4 \) captures generic disturbances on the state when checking if the state belongs to the constraint \( M \cap D \). The hybrid system \( \mathcal{H}_M \) with such perturbations results in the perturbed hybrid system

\[
\mathcal{H}^\rho_M \equiv \left\{ \begin{array}{l}
x^+ = f(x + d_1) + d_2 \quad x + d_3 \in M \cap C, \\
x^+ = g(x + d_1) + d_2 \quad x + d_4 \in M \cap D.
\end{array} \right.
\]

The perturbations \( d_i \) (\( i = 1, 2, 3, 4 \)) might be state or hybrid time dependent, but are assumed to have Euclidean norm bounded by \( M_i \geq 0 \) (\( i = 1, 2, 3, 4 \)), and to be admissible, namely, \( \text{dom} \; d_i \) (\( i = 1, 2, 3, 4 \)) is a hybrid time domain and the function \( t \mapsto d_i(t, j) \) is measurable on \( d_i \cap (\mathbb{R}_{\geq 0} \times \{j\}) \) for each \( j \in \mathbb{N} \).

Next, we recall the following stability notion from [5] Definition 7.10).

**Definition 7.1:** (KL asymptotic stability) Let \( \mathcal{H} \) be a hybrid system on \( \mathbb{R}^n \), \( A \subset \mathbb{R}^n \) be a compact set, and \( B_A \) be the basin of attraction of the set \( A \) [23]. The set \( A \) is KL asymptotically stable on \( B_A \) for \( \mathcal{H} \) if for every proper indicator \( \omega \) of \( A \) on \( B_A \), there exists a function \( \beta \in \mathcal{K} \) such that every solution \( \phi \in \mathcal{S}_H(B_A) \) satisfies

\[
\omega(\phi(t, j)) \leq \beta(\omega(\phi(0, 0)), t + j) \quad \forall (t, j) \in \text{dom} \; \phi.
\]

The next result establishes that local asymptotic stability of \( \mathcal{O} \) and Assumption 4.1 guarantee a KL bound as in (52); namely, local asymptotic stability of \( \mathcal{O} \) is uniform.

**Theorem 7.2:** Consider a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1.

If \( \mathcal{O} \) is a locally asymptotically stable hybrid limit cycle for \( \mathcal{H}_M \) with basin of attraction \( B_\mathcal{O} \), then \( \mathcal{O} \) is KL asymptotically stable on the basin of attraction \( B_\mathcal{O} \) of the set \( \mathcal{O} \).

**Proof:** First, it is proved in Lemma 4.4 that \( \mathcal{O} \) is a compact set. Second, note that for a hybrid system \( \mathcal{H} \) on \( \mathbb{R}^n \)

24The \( B_A \) is the set of points \( x \in \mathbb{R}^n \) such that every complete solution \( \phi \) to \( \mathcal{H}_M \) with \( \phi(0, 0) = x \) is bounded and \( \lim_{t \to \infty} |\phi(t, j)|_A = 0 \)

and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1 \( \mathcal{H}_M \) is well-posed [5] Definition 6.29]. Then, it is also nominally well-posed. Therefore, according to [5] Theorem 7.12, \( B_\mathcal{O} \) is open and \( \mathcal{O} \) is KL asymptotically stable on \( B_\mathcal{O} \).

The following result establishes that the stability of \( \mathcal{O} \) for \( \mathcal{H}_M \) is robust to the class of perturbations defined above.

**Theorem 7.3:** Consider a hybrid system \( \mathcal{H} = (C, f, D, g) \) on \( \mathbb{R}^n \) and a closed set \( M \subset \mathbb{R}^n \) satisfying Assumption 4.1.

If \( \mathcal{O} \) is a locally asymptotically stable compact set for \( \mathcal{H}_M \) with basin of attraction \( B_\mathcal{O} \), then for every proper indicator \( \omega \) of \( \mathcal{O} \) on \( B_\mathcal{O} \) there exists \( \beta \in \mathcal{K} \) such that for every \( \epsilon > 0 \) and every compact set \( K \subset B_\mathcal{O} \), there exist \( M_i > 0 \), \( i \in \{1, 2, 3, 4\} \), such that for any admissible perturbations \( d_i \), \( i \in \{1, 2, 3, 4\} \), with Euclidean norm bounded by \( M_i \), respectively, every solution \( \phi \) to \( \mathcal{H}_M \) with \( \phi(0, 0) \in K \) satisfies

\[
\omega(\phi(t, j)) \leq \beta(\omega(\phi(0, 0)), t + j) + \epsilon \quad \forall (t, j) \in \text{dom} \; \phi.
\]

**Proof:** Following [5] Section 6.4, we introduce the following perturbed hybrid system \( \mathcal{H}^\rho_M \) with constant \( \rho > 0 \):

\[
\mathcal{H}^\rho_M \equiv \left\{ \begin{array}{l}
x^+ = f(x + d_1) + d_2 \quad x + d_3 \in M \cap C, \\
x^+ = g(x + d_1) + d_2 \quad x + d_4 \in M \cap D.
\end{array} \right.
\]

Then, every solution to \( \mathcal{H}^\rho_M \) with admissible perturbations \( d_i \) having Euclidean norm bounded by \( M_i \), \( i \in \{1, 2, 3, 4\} \), respectively, is a solution to the hybrid system \( \mathcal{H}_M \) with \( \rho \geq \max\{M_1, M_2, M_3, M_4\} \), which corresponds to an outer perturbation of \( \mathcal{H}_M \) and satisfies the convergence property [31] Assumption 3.25]. Then, the claim follows by [31] Theorem 3.26 and the fact that every solution to \( \mathcal{H}_M \) is a solution to \( \mathcal{H}^\rho_M \). In fact, using [31] Theorem 3.26, for every proper indicator \( \omega \) of \( \mathcal{O} \) on \( B_\mathcal{O} \) there exists \( \beta \in \mathcal{K} \) such that for each compact set \( K \subset B_\mathcal{O} \) and each \( \epsilon > 0 \), there exists \( \rho^* > 0 \) such that for each \( \rho \in (0, \rho^*) \), every solution \( \phi_\rho \) from \( K \) satisfies \( \omega(\phi_\rho(t, j)) \leq \beta(\omega(\phi_\rho(0, 0)), t + j) + \epsilon \) for each \( (t, j) \in \text{dom} \; \phi_\rho \). The proof concludes using the relationship between the solutions to \( \mathcal{H}_M \) and \( \mathcal{H}^\rho_M \), and picking \( M_i \), \( i \in \{1, 2, 3, 4\} \), such that \( \max\{M_1, M_2, M_3, M_4\} \in (0, \rho^*] \).

**Remark 7.4:** Robustness results of stability of compact sets for general hybrid systems are available in [5]. Since \( \mathcal{O} \) is an asymptotically stable compact set for \( \mathcal{H}_M \), Theorem 7.3 is novel in the context of the literature of Poincaré maps. In particular, if one was to write the systems in [6] and [32] within the framework of [5], then one would not be able to apply the results on robustness for hybrid systems in [5] since the hybrid basic conditions would not be satisfied and the hybrid limit cycle may not be given by a compact set. Furthermore, through an application of [5] Lemma 7.19, it can be shown that the hybrid limit cycle is KL asymptotically stable on \( B_\mathcal{O} \).

**Remark 7.5:** Recently, the authors in [44] and [60] present static or dynamic decentralized (event-based) controllers for...
robust stabilization of hybrid periodic orbits against possible disturbances and established results on $H_2/H_\infty$ optimal decentralized event-based control design. In contrast to our work, they use input-to-state stability for robust stability properties of hybrid periodic orbits with respect to disturbance inputs in the discrete dynamics. Note that the results in [44], [60] consider possible disturbances only on the discrete dynamics and are only suitable for nonlinear impulsive systems that have jumps on switching surfaces. On the other hand, in this paper, we establish conditions for robustness of hybrid limit cycles that allow disturbances in the continuous/discrete dynamics and are applicable for hybrid dynamical systems with nonempty intersection between the flow set and the jump set.

**Remark 7.6:** Very recently, the authors in [51] propose a reachability-based approach to compute regions-of-attraction for hybrid limit cycles in a class of hybrid systems with a switching surface and bounded disturbance. Note that the approach in [51] deals with bounded disturbance only on the continuous dynamics and is only suitable for hybrid systems that have jumps on switching surfaces.

**Example 7.7:** Consider the Izhikevich neuron system in Example 3.6. We illustrate Theorem 6.4 for the hybrid system $\mathcal{H}_1$ by plotting the solutions from the initial condition $(-55, -6)$, when an additive perturbation $d_2 = (d_{2u}, d_{2v})$ affects the jump map. The noise is injected as unmodeled dynamics on the jump map as $d_2 = (d_{2u}, d_{2v}) = (\rho \sin(t), 0)$, where a variety values for $\rho$ are used to verify robustness. Fig. 12 shows the phase plots for the perturbed solution (red line) and normal solution (blue line). It is found that the hybrid limit cycle $\Omega$ is robust to the additive perturbation $d_2$ when $\rho \in (0, 0.4)$ as shown in Fig. 12(a) while $\Omega$ is not robust to the additive perturbation $d_2$ when $\rho = 0.42$ as shown in Fig. 12(b).

Next, more simulations are performed to quantify the relationship between $\rho^*$ (the maximal value of the perturbation parameter $\rho$) and $\varepsilon$ (the desired level of closeness to $\Omega$). Given a compact set $K := [-55, -53] \times [-6.20, -5.80]$, and different desired region radiuses $\varepsilon \in \{0.3, 0.6, 0.9, 1.2, 1.5\}$, the simulation results are shown in Table I which indicates that the relationship between $\rho^*$ and $\varepsilon$ can be approximated as $\rho^* \approx 0.004\varepsilon$. As it can be seen, the larger admissible convergence error the larger the perturbation parameter $\rho^*$ can be. This study validates Theorem 7.3.

| $\rho^*$ | $\varepsilon$ | $\rho^*/\varepsilon$ |
|----------|---------------|---------------------|
| 0.0010   | 0.30          | 0.0031              |
| 0.0019   | 0.60          | 0.0030              |
| 0.0032   | 0.90          | 0.0035              |
| 0.0042   | 1.20          | 0.0034              |
| 0.0058   | 1.50          | 0.0039              |

**B. Robustness to Inflations of $C$ and $D$**

We consider the following specific parametric perturbation on $h$, in both the flow and jump sets, with $\varepsilon > 0$ denoting the parameter: the perturbed flow set is an inflation of the original flow set while the condition $h(x) = 0$ in the jump set is replaced by $h(x) \in [-\varepsilon, \varepsilon]$. The resulting system is denoted as $\mathcal{H}'_{\varepsilon, M}$ and is given by

$$
\mathcal{H}'_{\varepsilon, M} = \{ \dot{x} = f(x), \quad x \in C_{\varepsilon} \cap M, \\
\quad x^+ = g(x), \quad x \in D_{\varepsilon} \cap M, \}
$$

where the flow set and the jump set are replaced by $C_{\varepsilon} = \{x \in \mathbb{R}^n : h(x) \geq -\varepsilon\}$ and $D_{\varepsilon} = \{x \in \mathbb{R}^n : h(x) \leq -\varepsilon, L_f h(x) \leq 0\}$, respectively, while the flow map and jump map are the same as for $\mathcal{H}_{|M}$. We have the following result, whose proof follows from the proof of Theorem 7.3.

**Theorem 7.8:** Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subset \mathbb{R}^n$ satisfying Assumption 4.7. If $\Omega$ is a locally asymptotically stable compact set for $\mathcal{H}_{|M}$ with basin of attraction $B_\Omega$, then there exists $\tilde{\beta} \in K_{\mathcal{L}}$ such that, for every $\varepsilon > 0$ and each compact set $K \subset B_\Omega$, there exists $\tilde{\beta} > 0$ such that for each $\varepsilon \in (0, \varepsilon)$ every solution $\phi$ to $\mathcal{H}'_{\varepsilon, M}$ in (54) with $\phi(0, 0) \in K$ satisfies

$$
|\phi(t, j)|_{|\Omega} \leq \tilde{\beta}(|\phi(0, 0)|_{|\Omega}, t + j) + \varepsilon \quad \forall (t, j) \in \text{dom } \phi. \tag{55}
$$

Theorem 7.8 implies that the asymptotic stability property of the hybrid limit cycle $\Omega$ is robust to a parametric perturbation on $h$. Note that the $K_{\mathcal{L}}$ bound (55) is obtained when the parametrically perturbed system $\mathcal{H}'_{\varepsilon, M}$ in (54) should also exhibit a hybrid limit cycle. At times, a relationship between the maximum value $\varepsilon$ of the perturbation and the factor $\varepsilon$ in the semiglobal and practical $K_{\mathcal{L}}$ bound in (55) can be established numerically. Next, Theorem 7.8 and this relationship are illustrated in the TCP congestion control example.
**Example 7.9:** Let us revisit the hybrid congestion control system \([4]\) in Example 3.8, where, now, the flow set \(C_{\text{HCP}}\) and the jump set \(D_{\text{HCP}}\) are replaced by \(C'_{\text{HCP}} = \{ x \in \mathbb{R}^2 : q_{\text{max}} - q \geq -\epsilon \}, \ D'_{\text{HCP}} = \{ x \in \mathbb{R}^2 : q_{\text{max}} - q \in [-\epsilon, \epsilon], r \geq B \}\), respectively. To validate Theorem 7.8 multiple simulations are performed to show a relationship between \(\epsilon\), the maximal value of the perturbation parameter \(\epsilon\), and \(\epsilon\), the desired level of closeness to the hybrid limit cycle \(\mathcal{O}\). Given the compact set \(K = [0.68, 0.72] \times [0.58, 0.64]\) and different desired level \(\epsilon \in \{0.01, 0.02, 0.03, 0.04\}\) of closeness to the hybrid limit cycle, the simulation results are shown in Table II which indicates that the relationship between \(\bar{\epsilon}\) and \(\epsilon\) can be approximated as \(\bar{\epsilon} \approx 2.8\epsilon\).

| \(\bar{\epsilon}\) | \(\epsilon\) | \(\bar{\epsilon}/\epsilon\) |
|---|---|---|
| 0.022 | 0.01 | 2.20 |
| 0.056 | 0.02 | 2.80 |
| 0.078 | 0.03 | 2.60 |
| 0.107 | 0.04 | 2.68 |

**C. Robustness to Computation Error of The Hybrid Poincaré Map**

The hybrid Poincaré map defined in \([48]\) indicates the evolution of a trajectory of a hybrid system from a point on the jump set \(M \cap D\) to another point in the same set \(M \cap D\). As stated in Theorem 6.4 and Corollary 6.6 stability of hybrid limit cycles can be verified by checking the eigenvalues of the Jacobian of the hybrid Poincaré map at its fixed point. However, errors in the computation of the hybrid Poincaré map may influence the statements made about asymptotic stability. Typically, the hybrid Poincaré map is computed numerically by discretizing the flows, using integration schemes such as Euler, Runge-Kutta, and multi-step methods \([62]\), which unavoidably lead to an approximation of Poincaré maps.

Following the ideas in \([62, 63]\) about perturbations introduced by computations, the discrete-time system associated with the (exact) hybrid Poincaré map \(P\) in \([48]\) is given by \(P\)

\[
\mathcal{H}_P : x^+ = P(x) \quad x \in M \cap D, \quad (56)
\]

which we treat as a hybrid system without flows. As argued above, due to unavoidable errors in computations and computer implementations, only approximations of the map \(P\) and of the sets \(M\) and \(D\) are available. In particular, given a point \(x \in M \cap D\), the value of the step size, denoted \(s > 0\), used in the computation of \(P\) at a point \(x\) affects the precision of the resulting approximation, which, in turn, may prevent the solution to (56) to remain in \(M \cap D\) and be complete. Due to this, we denote by \(P_s\) the results of computing \(P\) and by \(M_s\) and \(D_s\), the approximations of \(M\) and \(D\), respectively. With some abuse of notation, the discrete-time system associated with \(P_s, M_s,\) and \(D_s\) is defined as

\[
\mathcal{H}_{P_s} : x^+ = P_s(x) \quad x \in M_s \cap D_s, \quad (57)
\]

The approximations of \(P_s, M_s,\) and \(D_s\) are assumed to satisfy the following properties.

**Assumption 7.10:** Given \(M \subset \mathbb{R}^n\) and \(H = (C, f, D, g)\), the function \(P_s : \mathbb{R}^n \to \mathbb{R}^n\) parameterized by \(s > 0\) is such that, for some continuous function \(\varphi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\), there exists \(s^* > 0\) such that, for all \(x \in M \cap D\),

\[
P_s(x) \in P(x) \quad \forall s \in (0, s^*] \quad (58)
\]

where

\[
P_s(x) := \{ v \in \mathbb{R}^n : v \in g + \varphi(\mathcal{B}) \}, \ g \in P(x + g(\mathcal{B}))
\]

and the set \(M_s \cap D_s\) satisfies, for any positive sequence \(\{s_i\}_{i=1}^\infty\) such that \(s_i \to 0\),

\[
\lim \sup_{i \to \infty} M_{s_i} \cap D_{s_i} \subset M \cap D. \quad (59)
\]

**Remark 7.11:** The property in (58) is a consistency condition on the integration scheme used to compute the flows involved in (48). For instance, when the forward Euler method is used to approximate those flows, the numerical values of \(\phi\) are generated using the scheme \(x + sf(x)\), which, under Lipschitzness of \(f\) and boundedness of solutions (and its derivatives) to \(\dot{x} = f(x)\), is convergent of order 1; in particular, the error between \(P_s\) and \(P\) is \(O(s)\), which implies that (58) holds for some function \(\varphi\), see, e.g., \([64, \text{Chapter 3.2}]\). Vaguely, the property in (59) holds when a distance between \(M_s \cap D_s\) and \(M \cap D\) approaches zero as the step size vanishes, which is an expected property as precision improves with a decreasing step size. Condition (59) is satisfied when, for small enough \(s > 0\), \(M_s \cap D_s\) is contained in an outer perturbation of \(M \cap D\).

Very often, the jump set \(M \cap D\) can be implemented accurately in the computation of the hybrid Poincaré map, i.e., it may be possible to take \(M_s = M\) and \(D_s = D\), as illustrated in Example 7.15 below.

The following closeness result between solutions to \(\mathcal{H}_P\) and \(\mathcal{H}_{P_s}\) holds.

**Theorem 7.12:** closeness between solutions and approximations on compact domains Consider a hybrid system \(H = (C, f, D, g)\) on \(\mathbb{R}^n\) and a closed set \(M \subset \mathbb{R}^n\) satisfying Assumption 4.4. Assume the computed Poincaré map \(P_s\) approximating \(P\) and the sets \(M_s\) and \(D_s\), approximating \(M\) and \(D\), respectively, satisfy Assumption 7.10. Then, for every compact set \(K \subset M \cap D\), every \(\epsilon > 0\), and every simulation horizon \(s \in \mathbb{N}\), there exists \(s^* > 0\), such that for each \(\delta \in (0, \delta^*]\), for each \(s \in (0, s^*]\), and any solution \(\phi_P \in S_{H_P}(K + \delta \mathcal{B})\) there exists a solution \(\phi_{P_s} \in S_{H_{P_s}}(K)\) with \(\text{dom} \phi_{P_s} \subset \mathbb{N}\) such that \(\phi_{P_s}\) and \(\phi_P\) are \((J, \epsilon, \delta^-)\)-close.

**Proof:** By Assumption 4.4 and the definition of \(P\) in (48), \(P\) is well-defined and, by Lemma 4.12, continuous on the closed set \(M \cap D\). Then, the hybrid system \(\mathcal{H}_P\) without flows satisfies the hybrid basic conditions A1) and A2) in Section III-A. Next, to show the convergence property between \(\mathcal{H}_P\) and \(\mathcal{H}_{P_s}\), let us first consider an outer perturbation of \(\mathcal{H}_P\). Given \(\delta > 0\), let \(s \in (0, \delta]\). Obviously, the step size \(s\)

\[25\]See \([63, \text{Definition 3.2}]\) for a definition of \((T, \lambda, \epsilon, \delta^-)\)-close to quantify the distance between hybrid arcs (and solutions). Here, it is just the hybrid case but with \(t = 0\).
explicitly depends on $\delta$ and approaches zero as $\delta \searrow 0$. The outer perturbation (see [61, Example 5.3] for more details) of $\mathcal{H}_P$ for a state dependent perturbation determined by the constant $\delta$ and a continuous function $\varrho: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is given by

$$\mathcal{H}_{P_{\delta}}: x^+ \in P_{\delta}(x) \quad x \in D_{\delta},$$

(60)

where $P_{\delta}(x) := \{ v \in \mathbb{R}^n : v \in g + \delta \varrho(g)B, \ g \in P(x+\delta \varrho(x)B) \}$, $D_{\delta} := \{ x \in \mathbb{R}^n : x + \delta \varrho(x)B \cap (M \cap D) \neq \emptyset \}$.

Then, by [62, Lemma 5.1], the outer perturbation $\mathcal{H}_{P_{\delta}}$ of $\mathcal{H}_P$ has the convergence property. Consequently, the closeness between solutions to $\mathcal{H}_P$ and $\mathcal{H}_{P_{\delta}}$ follows from [62, Theorem 3.4]. Using Assumption 7.10 for every compact set $K \subseteq \mathbb{R}^n$, the solutions $\phi_{P_{\delta}} \in S_{\mathcal{H}_{P_{\delta}}}(K + \delta B)$ are solutions to the perturbed hybrid system $\mathcal{H}_{P_{\delta}}$. Hence, the properties of solutions to the perturbed hybrid system $\mathcal{H}_{P_{\delta}}$ also hold for those of $\mathcal{H}_P$. The proof concludes by exploiting the closeness property between solutions to $\mathcal{H}_P$ and $\mathcal{H}_{P_{\delta}}$.

Inspired by [62, Theorem 5.3], the following stability result shows that when Assumption 7.10 holds, asymptotic stability of the fixed point of $P$ (assumed to be unique) is preserved under the computation of $P$.

**Theorem 7.13:** (stability preservation under computation error of $P$) Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subseteq \mathbb{R}^n$ satisfying Assumption 4.1. Assume that the computed Poincaré map $P$ approximating $P$ and the sets $M$ and $D$, approximating $M$ and $D$, respectively, satisfy Assumption 7.10 and that $x^*$ is a unique globally asymptotically stable fixed point of $P$. Then, $x^*$ is a unique semiglobally practically asymptotically stable fixed point of $P_{\delta}$ with basin of attraction containing every point in $M \cap D$, i.e., there exists $\beta \in K_{\mathcal{L}}$ such that, for every $\delta > 0$, each compact set $K \subseteq M \cap D$, and every simulation horizon $J \in \mathbb{N}$, there exists $s^* > 0$ such that for each $s \in (0, s^*)$, every solution $\phi_{P_{\delta}} \in S_{\mathcal{H}_{P_{\delta}}}(K)$ to $\mathcal{H}_{P_{\delta}}$, satisfies for each $j \in \text{dom} \phi_{P_{\delta}}$,

$$|\phi_{P_{\delta}}(j) - x^*| \leq \beta(|\phi_{P_{\delta}}(0) - x^*|, j) + \epsilon.$$

**Proof:** Since the hybrid system $\mathcal{H}_P$ without flows satisfies the hybrid basic conditions A1) and (A2) in Section III.A and $x^*$ is a unique globally asymptotically stable fixed point of $P$, by [62, Theorem 3.1], there exists $\beta \in K_{\mathcal{L}}$ such that each solution $\phi_P \in S_{\mathcal{H}_P}(M \cap D)$ to $\mathcal{H}_P$ satisfies

$$|\phi_P(j) - x^*| \leq \beta(|\phi_P(0) - x^*|, j) \quad \forall j \in \text{dom} \phi_P.$$

Given a compact set $K \subseteq M \cap D$ and a simulation horizon $J \in \mathbb{N}$, by the assumptions, [62, Lemma 5.1] implies that the outer perturbation $\mathcal{H}_{P_{\delta}}$ of $\mathcal{H}_P$ in (60) satisfies the convergence property in [62, Definition 3.3]. Then, using $K$ above, [62, Theorem 3.5] implies that for each $\delta > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*)$, every solution $\phi_{P_{\delta}} \in S_{\mathcal{H}_{P_{\delta}}}(K + \delta B)$ to $\mathcal{H}_{P_{\delta}}$ satisfies for each $j \in \text{dom} \phi_{P_{\delta}}$

$$|\phi_{P_{\delta}}(j) - x^*| \leq \beta(|\phi_{P_{\delta}}(0) - x^*|, j) + \epsilon.$$

By Assumption 7.10, the properties of solutions to the perturbed hybrid system $\mathcal{H}_{P_{\delta}}$ also hold for solutions $\phi_{P_{\delta}}$. The result follows by this preservation and the $K_{\mathcal{L}}$ bound of solutions to $\mathcal{H}_P$.

Note that the property in Theorem 7.13 holds for small enough step size $s$. The step size bound $s^*$ decreases with the desired level of closeness to $x^*$, which is given by $\epsilon$.

The next result shows that the computed Poincaré map $P_{\delta}$ has a semiglobally asymptotically stable compact set $A_s$ with basin of attraction containing every point in $M \cap D$ that reduces to a singleton $\{x^*\}$ as $s$ approaches zero.

**Theorem 7.14:** (continuity of asymptotically stable fixed points) Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ on $\mathbb{R}^n$ and a closed set $M \subseteq \mathbb{R}^n$ satisfying Assumption 4.1. Assume that $x^*$ is a unique globally asymptotically stable fixed point of the hybrid Poincaré map $P$ and the computed Poincaré map $P_{\delta}$, approximating $P$ and the sets $M_{\delta}$, and $D_{\delta}$ approximating $M$ and $D$, respectively, satisfy Assumption 7.10. Then, there exists $s^* > 0$ such that for each $s \in (0, s^*)$, the computed Poincaré map $P_{\delta}$ has a semiglobally asymptotically stable compact set $A_s$ with basin of attraction containing every point in $M \cap D$ satisfying

$$\lim_{s^* \to 0} A_s = x^*.$$

**Proof:** Let $K$ be any compact set such that for some $\epsilon > 0$, $x^* + 2\epsilon B \subseteq K \subseteq \mathbb{R}^n$. Using $K$ as above and an arbitrary simulation horizon $J \in \mathbb{N}$, consider the perturbed system $\mathcal{H}_{P_{\delta}}$ in (60) and define $\tilde{\mathcal{H}}_{P_{\delta}}$ with $\tilde{T}_{\delta}(x) := \{ P_{\delta}(x) \cup \{x^*\} \ x \in D_{\delta} \} \cup \{ x^* \} \ x \in \mathbb{R}^n \setminus D_{\delta}$ and $D_{\delta} = \mathbb{R}^n$. Using $K$ and $\epsilon$ as above, [62, Theorem 3.5] implies that for each $\epsilon > 0$ there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*)$, every solution $\tilde{T}_{\delta}(x) \in S_{\mathcal{H}_{P_{\delta}}}(K)$ to $\tilde{\mathcal{H}}_{P_{\delta}}$, satisfies for each $j \in \text{dom} \tilde{T}_{\delta}$,

$$|\tilde{T}_{\delta}(j) - x^*| \leq \beta(|\tilde{T}_{\delta}(0) - x^*|, j) + \epsilon.$$

(61)

For a simulation horizon $J \in \mathbb{N}$, let $\text{Reach}_{J,\tilde{\mathcal{H}}_{P_{\delta}}}(x^* + 2\epsilon B)$ be the reachable set of $\tilde{\mathcal{H}}_{P_{\delta}}$ from $x^* + 2\epsilon B$ up to $J$, i.e.,

$$\text{Reach}_{J,\tilde{\mathcal{H}}_{P_{\delta}}}(x^* + 2\epsilon B) := \{ \phi_{P_{\delta}}(j) : \phi_{P_{\delta}} \text{ is a solution to } \tilde{\mathcal{H}}_{P_{\delta}}, \phi_{P_{\delta}}(0) \in x^* + 2\epsilon B, j \in \text{dom} \phi, j \leq J \}.$$

Now, following a similar step as in the proof of [62, Theorem 5.4], let

$$B_{\epsilon} := \text{Reach}_{\infty,\tilde{\mathcal{H}}_{P_{\delta}}}(x^* + 2\epsilon B).$$

By [61, B_{\epsilon}] is bounded. Moreover, since $B_{\epsilon}$ is closed by definition, it follows that it is compact. Next, we show that it is forward invariant. Consider a solution $\phi_{P_{\delta}} \in S_{\tilde{\mathcal{H}}_{P_{\delta}}}(B_{\epsilon})$ to $\tilde{\mathcal{H}}_{P_{\delta}}$. Assume that there exists $j' \in \text{dom} \phi_{P_{\delta}}$ for which $\phi_{P_{\delta}}(j') \notin B_{\epsilon}$. By definition of $B_{\epsilon}$, since $\phi_{P_{\delta}}(0) \in B_{\epsilon}$, the solution $\phi_{P_{\delta}}$ belongs to $B_{\epsilon}$ for each $j \in \text{dom} \phi_{P_{\delta}}$. This is a contradiction. Next, we show that solutions to $\tilde{\mathcal{H}}_{P_{\delta}}$, starting from $K$ converge to $B_{\epsilon}$ uniformly. [61] implies that for the given $K$ and $\epsilon$, there exists $N > 0$ such that for every $\tilde{T}_{\delta}(x) \in S_{\mathcal{H}_{P_{\delta}}}(K)$ to $\tilde{\mathcal{H}}_{P_{\delta}}$, and for each $j \in \text{dom} \tilde{T}_{\delta}$, $j \geq N$:

$$|\tilde{T}_{\delta}(j) - x^*| \leq 2\epsilon.$$
and Assumption 7.10 semiglobal asymptotic stability of $B_\varepsilon$ for $\mathcal{H}_{P}$, with basin of attraction containing every point in $M \cap D$ follows.

Finally, note that $B_0 = \{x^*\}$ and that as $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} B_\varepsilon = x^*$. By (61), $\varepsilon \searrow 0$ implies $\delta \searrow 0$. Moreover, from the proof of Theorem 7.12 we have $s \searrow 0$ as $\delta \searrow 0$. It follows that $s \searrow 0$ as $\varepsilon \searrow 0$. Therefore, the result follows by $A_k = B_2$.

The following example illustrates that the Euler integration scheme for differential equations satisfies the continuity property in Theorem 7.14.

Example 7.15: Consider the hybrid congestion control system in Example 4.6. To approximate the hybrid Poincaré map at $x = (q, r) \in D_{TCP}$, we numerically compute solutions from $g_{TCP}(x) = (q_1, q_2) = (q_{\text{max}}, m r)$ by discretizing the flows of the hybrid system using Euler integration scheme with step size $s$. From the definition of the hybrid Poincaré map and the flow from $g_{TCP}(x)$, it follows that, using (49), the computed Poincaré map after $k$ steps of size $s$ with $k s \geq T^*$ and $(k - 1) s < T^*$ is

$$\mathcal{H}_{P_{TCP}}: x^+ = P_{TCP}(x) = \left[ \begin{array}{c} g_{rs} \\ m r + a k s \end{array} \right], \quad x \in M_{TCP} \cap D_{TCP},$$

where $g_{rs} := q_{\text{max}} + (m r - B) k s + \frac{1}{2} m^2 a s^2 = 0$ and $m r + a k s = r$. By solving these two equations, we obtain $r = 2 B/(m + 1)$. Therefore, when $k s = 2(B - m r)/a$, the fixed point $x^*_s$ of $P_{TCP}$ can be computed as $x^*_s = (q_{\text{max}}, 2 B/(m + 1))$, which is equivalent to the fixed point $x^*$ of the hybrid Poincaré map $P_{TCP}$ in (50). In this case, using the fact that $r = 2 B/(m + 1)$, we have $k s = 2(B - m r)/a = 2 B(1 - m)/a_m = T^*$ satisfying the conditions $k s \geq T^*$ and $(k - 1) s < T^*$ for each $k \in \{1, 2, 3, \ldots\}$. In addition, we have $|x^*_s - x^*| \to 0$ as $s \searrow 0$, which illustrates Theorem 7.14.

VIII. CONCLUSION

Notions and tools for the analysis of existence and stability of hybrid limit cycles in hybrid dynamical systems were proposed. Necessary conditions were established for the existence of hybrid limit cycles. The Zhukovskii stability notion for hybrid systems was introduced and a relationship between Zhukovskii stability and the incremental graphical stability was presented. A sufficient condition relying on Zhukovskii stability of the hybrid system was established for the existence of hybrid limit cycles. In addition to nominal results, the key novel contributions included an approach relying on incremental graphical stability for the nonexistence of hybrid limit cycles. To investigate the stability properties of the hybrid limit cycles, we also constructed a time-to-impact function inspired by those introduced 6, 58, 44. Based on these constructions, sufficient and necessary conditions for the stability of hybrid limit cycles were presented. Moreover, comparing to previous results in the literature, we established conditions for robustness of hybrid limit cycles with respect to small perturbations and to computation error of the hybrid Poincaré map, which is a very challenging problem in systems with impulsive effects. An extension effort that characterizes the robust stability properties for the situation where a hybrid limit cycle may contain multiple jumps within a period can be found in 43, 50. Examples were included to aid the reading and illustrate the concepts and the methodology of applying the new results. Future work includes exercising the presented conditions on systems of higher dimension and more intricate dynamics, and hybrid control design for asymptotic stabilization of limit cycles as well as their robust implementation.

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IX. APPENDIX

The following theorems are used in the proof of Theorem 5.12 and we present them here for completeness.

**Theorem 9.1:** (Tubular Flow Theorem, [65, Chapter 2, Theorem 1.1]) Let $f$ be a vector field of class $C^r$, $r \geq 1$, on $U \subset \mathbb{R}^n$ and let $v \in U$ be a regular point of $f$. Let $\Xi =: \{ (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : |\xi_i| < 1, i = 1, 2, \cdots, n \}$ and let $f_\Xi$ be the vector field on $\Xi$ defined by $f_\Xi(\xi) = (1, 0, \cdots, 0)$. Then there exists a $C^r$ diffeomorphism $H : N_v \to \Xi$, for some neighborhood $N_v$ of $v$ in $U$, taking trajectories of $\dot{x} = f(x)$ to trajectories of $\dot{\xi} = f_\Xi(\xi)$.

**Theorem 9.2:** (Brouwer’s Fixed Point Theorem, [66, Corollary 1.1.1]) Let $X$ be a nonempty compact convex subset of $\mathbb{R}^n$ and $P : X \to X$ a continuous (single-valued) mapping. Then there exists a $q \in X$ such that $P(q) = q$. 
