The isoperimetric profile of a noncompact Riemannian manifold for small volumes

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Abstract In the main theorem of this paper we treat the problem of existence of minimizers of the isoperimetric problem in a noncompact Riemannian manifold $M$, under the assumption of small volumes. We use a new approach to the lack of compactness of this problem. Thanks to compactness theorems of the theory of the convergence of Riemannian manifolds we are able to prove, under suitable bounded geometry assumptions, that isoperimetric regions for small volumes always exist in a larger manifold obtained by attaching to the original one a pointed Gromov-Hausdorff limit of a sequence $(M, g, p_i)$ for a diverging sequence of points $p_i \in M$. Applications of the main theorem to asymptotic expansions of the isoperimetric problem are given.

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1 Introduction

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. We deal mainly with the problem of finding a relatively compact domain $D \subset M$ of small volume $Vol(D)$ that minimizes boundary area $Area(\partial D)$ among all domains of the same volume. In the language of currents from geometric measure theory, the problem can be restated thusly: given $0 < v < Vol(M)$, consider all integral currents $T$ in $M$ with volume $v$, and denoting the mass of the boundary as $Area(\partial T)$, find a minimizing current with volume $v$. This problem is referred to as the isoperimetric problem throughout the paper. Generally we will not mention the metric explicitly when referring to the concepts of area and volume, but when it is necessary to do so for clarity, we may write them as $Area_g$ and $Vol_g$. 
The principal achievements of this paper concern the link between the theory of pseudo-bubbles and the isoperimetric problem for small volumes in a complete Riemannian manifold with some kind of “boundedness at infinity” on the metric and its fourth derivatives. This task was carried out by the same author for manifolds in which minimizers exist for all volumes, including manifolds with co-compact isometry group and those with finite volume (compare with [15]). In this paper, we consider the same questions but using entirely new techniques, ones capable of surmounting the difficulties arising from an absence of minimizers. Namely, we isometrically embed the manifold $M$ into a metric space composed of a disjoint union of pieces $(M_{\infty}, p_{\infty}, g_{\infty})$. These pieces are limit manifolds of sequences $(M, p_{j}, g)$, with $p_{j} \in M$, in some suitable pointed $C^{k,\alpha}$ topology. The arguments presented here are useful because they allow us to prove non-trivial phenomena for complete, non-compact manifolds which possibly lack minimizers, provided that sufficiently many sequences $(M, p_{j}, g)$ have a limit in a $C^{k,\alpha}$ topology. For a self-contained exposition and the convenience of the reader, we repeat the relevant material from [2,11–14] without proofs.

First we recall the definition of a pseudo-bubble. Let $Q = id - P$, where $P$ is orthogonal projection of $L^{2}(T_{p}^{1}M)$ on the first eigenspace of the Laplacian, where $T_{p}^{1}M$ is the fiber over $p$ of the unit tangent bundle of the Riemannian manifold $M$.

**Definition 1.1** [11] A pseudo-bubble is a hypersurface $N$ embedded in $M$ such that there exists a point $p \in M$ and a function $u$ belonging to $C^{2,\alpha}(T_{p}^{1}M \simeq S^{n-1}, \mathbb{R})$, such that $N$ is the graph of $u$ in normal polar coordinates centered at $p$, i.e., $N = \{ \exp_{p}(u(\theta)\theta), \theta \in T_{p}^{1}M \}$, and $Q(H(u))$ is a real constant, where $H$ is the mean curvature operator.

To state a uniqueness theorem for pseudo-bubbles we need the notion of center of mass.

**Definition 1.2** Let $(\Omega, \mu)$ be a probability space and $f : \Omega \to M$ a measurable function. We consider the following function $E : M \to [0, +\infty[$:

$$E(x) := \frac{1}{2} \int_{\Omega} d^{2}(x, f(y))d\mu(y).$$

The center of mass of $f$ with respect to the measure $\mu$ is the minimum of $E$ on $M$, provided that it exists and is unique.

In particular, we can speak about the center of mass of a hypersurface of small diameter (we apply this definition to the $(n-1)$-dimensional measure of the boundary). The main result on pseudo-bubbles is the following theorem.

**Theorem 1.1** [11, Theorem 1] Let $M$ be a complete Riemannian manifold. Let $F^{k,\alpha}$ be the bundle on $M$ whose fiber over $p$ is the space of $C^{k,\alpha}$ functions on the unit tangent sphere $T_{p}^{1}M$. There exists a smooth map, $\beta : M \times 0, \text{Vol}(M) \to F^{2,\alpha}$ such that for all $p \in M$ and all $v > 0$ sufficiently small, the hypersurface $\exp_{p}(\beta(p, v)(\theta)\theta)$ is the unique pseudo-bubble with center of mass $p$ enclosing volume $v$.

**Remark** In general the volume $v$ depends on the point $p$, but it can be chosen to depend continuously on the point $p$. In the case of manifolds with $C^{2,\alpha}$-bounded geometry, there is a uniform upper bound $\bar{v} = \bar{v}(n, k, \alpha, Q)$ such that the conclusion of the preceding theorem is true for every $p \in M$ and every $0 < v < \bar{v}$. For the precise definition of $C^{m,\alpha}$-bounded geometry, see the definition below.

**Remark** If $g$ is an isometry of $M$, $g$ sends pseudo-bubbles to pseudo-bubbles and $g \circ \beta = \beta \circ g$ ($g$ acts only on the first factor $M$).
1.1 Main results

According to [10], small solutions of the isoperimetric problem in compact Riemannian manifolds, or noncompact manifolds with cocompact isometry group, are close to geodesic balls. Namely, they are graphs in normal coordinates of $C^{2,\alpha}$ small functions. This holds also for non-compact manifolds under a $C^4$-bounded geometry assumption, which will be proven in Sect. 3. In any case, it follows that these small isoperimetric domains are pseudo-bubbles.

**Remark** $C^4$-boundedness is due only to the technical limits of the methods employed for proving Theorem 3.2.

The main result of this paper is Theorem 1, which provides a criterion for existence of minimizers having sufficiently small volume. To state this theorem correctly, let us recall the basic definitions from the theory of convergence of manifolds, as articulated in Petersen [13].

**Definition 1.3** (Petersen [13]) A sequence of pointed complete Riemannian manifolds is said to converge in the pointed $C^{m,\alpha}$ topology $(M_i, p_i, g_i) \to (M, p, g)$ if for every $R > 0$ we can find a domain $\Omega_R$ with $B(p, R) \subseteq \Omega_R \subseteq M$, a natural number $v_R \in \mathbb{N}$, and embeddings $F_{i,R} : \Omega_R \to M_i$ for $i \geq v_R$ such that $B(p_i, R) \subseteq F_{i,R}(\Omega_R)$ and $F_{i,R}(g_i) \to g$ on $\Omega_R$ in the $C^{m,\alpha}$ topology.

It is easy to see that this type of convergence implies pointed Gromov–Hausdorff convergence, because $C^{0,\alpha}$ convergence implies Lipschitz convergence, which implies Gromov–Hausdorff (see [8], Theorem 3.7 p. 74). When all manifolds in question are closed, the maps $F_i$ are diffeomorphisms. So for closed manifolds we can speak about unpointed convergence. In this case, convergence can only happen if all the manifolds in the tail end of the sequence are diffeomorphic. In particular, classes of closed Riemannian manifolds that are precompact in some $C^{m,\alpha}$ topology contain at most finitely many diffeomorphism types.

**Definition 1.4** (Petersen [13]) Suppose $A$ is a subset of a Riemannian $n$-manifold $(M, g)$. We say that the $C^{m,\alpha}$-norm on the scale of $r$ of $A \subseteq (M, g)$ satisfies $||A||_{C^{m,\alpha}} \leq Q$ if we can find charts $\psi_s : \mathbb{R}^n \supseteq B(0, r) \to U_s \subseteq M$ such that

(i) For all $p \in A$ there exists $U_s$ such that $B(p, \frac{1}{10}e^{-Q} r) \subseteq U_s$.

(ii) $|D\psi| \leq e^Q$ on $B(0, r)$ and $|D\psi^{-1}| \leq e^Q$ on $U_s$.

(iii) $r^{(|j|+\alpha)}||D^j g_s||_{C^{m,\alpha}} \leq Q$ for all multi indices $j$ with $0 \leq |j| \leq m$, where $g_s$ is the matrix of functions of metric coefficients in the $\psi_s$ coordinates regarded as a matrix on $B(0, r)$.

**Definition 1.5** For fixed $Q > 0, n \geq 2, m \geq 0, \alpha \in [0, 1]$, and $r > 0$, define $\mathcal{M}^{m,\alpha}(n, Q, r)$ as the class of complete, pointed Riemannian $n$-manifolds $(M, p, g)$ with $||M||_{C^{m,\alpha}} \leq Q$.

In the sequel, $n \geq 2, r, Q > 0, m \geq 4, \alpha \in [0, 1]$.

**Theorem 1** There exists $0 < v^* = v^*(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$ and $0 < v < v^*$,

(I) The following two statements are equivalent:

(a) the function $p \mapsto f_M(p, v)$ attains its minimum,

(b) there exists solutions of the isoperimetric problem at volume $v$,

(II) $I_M(v) = \min\{f_{M_{\infty}}(p_{\infty}, v) : (M, p, g) \to (M_{\infty}, p_{\infty}, g) \text{ for some } (p, j)\}$.

Here $p_j \in M$ and the function $p \mapsto f_M(p, v)$ gives the area of pseudo-bubbles contained in a given manifold $M$, with center of mass $p \in M$ and enclosed volume $v$. Moreover, every
solution $D$ of the isoperimetric problem has $\partial D = \{ \exp_{p_0}(u(\theta)\theta), \theta \in T^1_{p_0} M \}$, where $p_0$ is a minimum of $p \mapsto f_M(p, v)$, and conversely. With $\beta$ obtained in Theorem 1.1, $f_M$ is invariant and $\beta$ equivariant under the group of isometries of $M$.

The proof of Theorem 1 will be achieved at the end of Sect. 3.

**Remark** The interest in Theorem 1 is the reduction of the infinite-dimensional problem of finding a minimizer to the finite-dimensional problem of finding the minimum of a smooth function on the manifold $M$.

Let us mention one important consequence (Theorem 2) of the isoperimetric profile defined below.

**Definition 1.6** Let $M$ be a Riemannian manifold of dimension $n$ (with possibly infinite volume). Denote by $\tau_M$ the set of relatively compact open subsets of $M$ with smooth boundary. The function $I : [0, \text{Vol}(M)] \to [0, +\infty]$ defined by

$$I(v) = \inf_{\Omega \in \tau_M} \left\{ \frac{\text{Area}(\partial \Omega)}{\text{Vol}(\Omega)} = v \right\}$$

is called the *isoperimetric profile function* (or simply *isoperimetric profile*) of the manifold $M$.

We wish to compute an asymptotic expansion of the function $v \mapsto f(p, v)$ using results of [14]. Note that any term denoted $O(r^k)$ here is a smooth function on $S^{n-1}$ which is bounded by a constant independent of $p$ times $r^k$ in the $C^2$ topology.

**Definition 1.7** Denote by $c_n := \frac{\text{Area}(S^{n-1})}{\text{Vol}(B^n)^{\frac{n}{2n-1}}}$ the constant in the Euclidean isoperimetric profile.

The following lemma uses a calculation partially performed in Pacard and Xu [14]. Denote by $Sc$ the scalar curvature function of $M$.

**Lemma 1.1** [11] The asymptotic expansion of the area of pseudo-bubbles as a function of the enclosed volume is

$$f(p, v) = c_n v^{\frac{n-1}{2}} \left( 1 + a_p \left( \frac{v}{\omega_n} \right)^2 + O(v^2) \right), \quad (1)$$

with $a_p := -\frac{1}{2n(n+2)} Sc(p)$.

**Theorem 2** For all $M \in \mathcal{M}^{m,0}(n, Q, r)$, let

$$S = \sup_{p \in M} \{ Sc(p) \}.$$ 

Then the isoperimetric profile $I_M(v)$ has the following asymptotic expansion in a neighborhood of the origin

$$I_M(v) = c_n v^{\frac{n-1}{2}} \left( 1 - \frac{S}{2n(n+2)} \left( \frac{v}{\omega_n} \right)^2 + o(v^2) \right), \quad (2)$$

where $o(t^\alpha)$ indicates a function $g : ]-\varepsilon, \varepsilon[ \to \mathbb{R}$, with $\varepsilon > 0$, such that $\lim_{t \to 0} \frac{g(t)}{t^\alpha} = 0$.

In Theorem 2 and Lemma 1.1, $O(t^\alpha)$ and $o(t^\alpha)$ are functions that depend only on $t$. The asymptotic expansion of the volume of pseudo-bubbles and the volume of their boundary can be computed with Lemma 1.1, which yields an expansion for the profile.
1.2 Plan of the article

1. Section 2 describes why and in what sense approximate solutions of the isoperimetric problem, in the case of small volumes, are close to Euclidean balls, providing a decomposition theorem for domains belonging to an almost minimizing sequences in small volumes.

2. In Sect. 3 we prove Theorem 1, generalizing to the case of $C^4$-bounded geometry manifolds some results of [11], in particular Corollary 3.1 which constitutes the only known proof to my knowledge of the fact that for small volume, minimizers are invariant under the action of the groups of isometries of $M$ that fix their barycenters.

3. In Sect. 4 the results of preceding sections and those of [10,12,14] are applied to obtain the first two non-zero coefficients in the asymptotic expansion of the isoperimetric profile in the non-compact case under $C^4$-bounded geometry assumption on $M$.

2 Partitions of domains

2.1 Introduction

In this section it is assumed that

1. $M$ has bounded geometry, $|\mathcal{K}| \leq \Lambda$ and $\text{inj}_M \geq \varepsilon > 0$, where $\text{inj}_M$ is the injectivity radius of $M$,

2. the domains $D_j \in \tau_M$ are approximate solutions i.e., $\frac{\text{Area}(\partial D_j)}{\text{Vol}_g(D_j)} \to 1$ for $j \to +\infty$.

We prove in this section the following theorem.

**Theorem 3** Let $(M, g)$ be a Riemannian manifold with bounded geometry and $D_j$ a sequence of approximate solutions of the isoperimetric problem such that $\text{Vol}_g(D_j) \to 0$. Then there exist points $p_j \in M$ and radii $R_j \to 0$ such that

$$\lim_{j \to +\infty} \frac{\text{Vol}(D_j \Delta B(p_j, R_j))}{\text{Vol}(D_j)} \to 0.$$ (3)

2.2 Euclidean version of Theorem 3

Roughly speaking, in $\mathbb{R}^n$ we have that approximate solutions of the isoperimetric problem are close to balls in the mass norm, as stated in the following theorem. A good reference for this result is Leonardi and Rigot [9].

**Theorem 2.1** If $\{T_j\} \subset \mathbb{I}_n(\mathbb{R}^n)$ is a sequence of integral currents satisfying

$$\lim_{j \to +\infty} \frac{\text{M}(\partial T_j)}{\text{M}(T_j \frac{\partial T_j}{\text{M}(T_j)})} = c_n,$$

then there exist balls $W_j$ such that up to a subsequence

$$\frac{\text{M}(T_j \Delta W_j)}{\text{M}(W_j)} \to 0.$$

**Sketch of proof** We can use here the theory of $BV$ functions and that of finite perimeter sets as described in Giusti [5] because for all polyhedral chains $P$, $\|\chi_{\text{Sp}P}\|_{BV(\mathbb{R}^n)} < +\infty$. In what follows we translate our problem into the language of $BV$ functions.
Let $|\cdot|$ be the Lebesgue measure on $\mathbb{R}^n$. The following argument regarding minimizing sequences will be useful in the sequel. Let $(E_k)_{k \geq 1}$ be a minimizing sequence of domains for the functional $H_{n-1}(\partial(\cdot))$ such that $|E_k| = 1$.

1. A compactness theorem stated in Giusti [5, p. 17] ensures that there exists a set $E$ such that a subsequence

$$\chi_{E_k} \to \chi_E$$

in $L^1_{loc}(\mathbb{R}^n)$. By lower semicontinuity of Lebesgue measure and of the perimeter function, it follows that

$$|E| \leq \liminf_{k \to +\infty} |E_k| \leq 1,$$

$$P(E, \mathbb{R}^n) \leq \liminf_{k \to +\infty} P(E_k, \mathbb{R}^n) \leq c_n.$$ 

Now if we show that $|E| = 1$, then we finish the proof, because Euclidean isoperimetric domains are round balls, so $E$ is the Euclidean ball of volume 1. This together with $L^1(B(0, 2))$ convergence ensure that the mass outside this Euclidean ball goes to zero and that the volume of the set-theoretic symmetric difference $|E \Delta E_k|$ goes to zero.

A clear proof that $|E| = 1$ for Carnot–Caratheodory groups is given in Leonardi and Rigot [9], and for this reason we will not repeat it here. It occurs in two steps:

- to show that there exist translates of $E_k$ having an intersection with the ball of radius 1 of mass not less than a constant $m_0 > 0$ (Lemma 4.1 of [9]),
- to argue that we cannot find a non-negligible subset of $E_k$ far away from this radius 1 ball because $E_k$ is almost perimeter-minimizing among all sets of measure 1 (Lemma 4.2, [9]).

To prove the theorem it is sufficient to apply the preceding argument to sets $E_j$ obtained from dilating $\text{supp} |T_j|$ by a factor of $\frac{1}{M(T_j)^\frac{1}{n}}$ and setting $W_j$ equal to $M(T_j)^{\frac{1}{n}} E$.

2.3 Lebesgue numbers

Let $(M, g)$ be a Riemannian manifold with bounded geometry. We can construct a good covering of $M$ by balls of equal radius.

Lemma 2.1 Let $(M, g)$ be a Riemannian manifold with bounded geometry. There exist an integer $N$, constants $C$ and $\epsilon > 0$, and a covering $\mathcal{U}$ of $M$ by balls of radius $3\epsilon$ with the following properties.

1. $\epsilon$ is a Lebesgue number for $\mathcal{U}$, i.e., every ball of radius $\epsilon$ is entirely contained in at least one element of $\mathcal{U}$ and meets at most $N$ elements of $\mathcal{U}$.
2. For every ball $B$ in this covering, there exists a C-bi-Lipschitz diffeomorphism of $B$ with a Euclidean ball of the same radius.

Proof Let $\epsilon = \frac{\text{inj}_M}{3}$. Let $B = \{B(p, \epsilon)\}$ be a maximal family of balls of $M$ of radius $\epsilon$ with the property that any pair of distinct members of $B$ have empty intersection. Then the family $2B := \{B(p, 2\epsilon)\}$ is a covering of $M$. Furthermore, for all $y \in M$ there exist $B(p, \epsilon) \in B$ such that $y \in B(p, 2\epsilon)$, and thus $B(y, \epsilon) \subseteq B(p, 3\epsilon)$. Hence $\epsilon$ is a Lebesgue number for the covering $3B$. Let $B(p, 3\epsilon)$ and $B(p', 3\epsilon)$ be two balls of $3B$ having nonempty intersection. Then $d(p, p') < 6\epsilon$, hence $B(p', \epsilon) \subseteq B(p, 7\epsilon)$. The ratios $\text{Vol}(B(p, 7\epsilon))/\text{Vol}(B(p, \epsilon))$ are uniformly bounded because the Ricci curvature of $M$ is bounded from below, and hence the
Bishop–Gromov inequality applies. The number \( N \) of disjoint balls of radius \( \epsilon \), contained in \( B(p, 7\epsilon) \), is bounded and does not depend on \( p \). Thus the number of balls of \( 3B \) that intersect one of them is uniformly bounded by \( N \). We conclude the proof by taking \( U := 3B \). In fact by Rauch’s comparison theorem, see Cheeger-Ebin [3, p. 29], for every ball \( B(p, \epsilon) \), the exponential map is \( C \) bi-Lipschitz with a constant \( C \) that depends only on \( \epsilon \) and on upper bounds for the sectional curvature \( K \).

2.4 Partition domains in small diameter subdomains

This section is inspired by the article of Bérard and Meyer [2], Lemma II.15 and the theorem in appendix C, p. 531.

**Proposition 2.1** Let \( I \) be the isoperimetric profile of \( M \). Then

\[
\limsup_{a \to 0} \frac{I(a)}{a^{n-1}} \leq c_n.
\]

**Proof** Fix a point \( p \in M \).

\[
\limsup_{a \to 0} \frac{I(a)}{a^{n-1}} \leq \limsup_{a \to 0} \frac{\text{Area}(\partial B(p, r(a)))}{\text{Vol}(B(p, r(a)))} \frac{n-1}{n}
\]

with \( r(a) \) such that \( \text{Vol}(B(p, r(a))) = a \). Changing variables in the limits, we find

\[
\begin{align*}
&\limsup_{r \to 0} \frac{\text{Area}(\partial B(p, r(a)))}{\text{Vol}(B(p, r(a)))} = \limsup_{r \to 0} \frac{\text{Area}(\partial B(p, r))}{\text{Vol}(B(p, r))} \frac{n-1}{n}, \\
&\limsup_{r \to 0} \frac{r^{n-1} \text{Area}(\mathbb{S}^{n-1}) + \cdots}{r^n \text{Vol}(\mathbb{B}^n) + \cdots} \frac{n-1}{n} = c_n.
\end{align*}
\]

\( \square \)

**Definition 2.1** Let \( r > 0 \). We define the unit grid \( G_1 \) of \( \mathbb{R}^n \) as the set of points which have at least one integer coordinate. We call \( G \) a grid of mesh \( r \) if \( G \) is of the form \( x + rG_1 \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). We denote by \( \mathcal{G}_r := ([0, r)^n, \mathcal{L}^n) \) the set of all grids of mesh \( r \), endowed with the natural Lebesgue measure given by the bijection \( \Phi : [0, r)^n \to \mathcal{G}_r \), \( \Phi : x \to x + rG_1 \), on \( ([0, r)^n, dx_1 \ldots dx_n) \).

**Proposition 2.2** Let \( D \) be an open subset of \( \mathbb{R}^n \).

\[
\frac{1}{r^n} \int_{\mathcal{G}_r} \text{Area}(D \cap G) \mathcal{L}^n(dG) = \frac{n}{r} \text{Vol}(D).
\]

**Proof** We observe that every grid \( G \) of mesh \( r \) decomposes as a union of \( n \) sets \( G^{(i)} \) of the type \( x + rG_1^{(i)} \) where \( G_1^{(i)} \) is the set of points with integer \( i \)th coordinate. Moreover \( G^{(i)} \cap G^{(j)} \) has \((n - 1)\)-dimensional Hausdorff measure equal to zero for every \( i \neq j \). This latter fact allows us to ensure that

\[
\text{Area}(D \cap G) = \sum_{i=1}^{n} \text{Area}(D \cap G^{(i)}).
\]

which implies that

\[
\frac{1}{r^n} \int_{\mathcal{G}_r} \text{Area}(D \cap G) \mathcal{L}^n(dG) = \frac{1}{r^n} \sum_{i=1}^{n} \int_{(0,r)^n} \text{Area}(D \cap G^{(i)}) \mathcal{L}^n(dG).
\]
But
\[
\int_{[0,r]^n} \text{Area}(D \cap G^{(i)}_x) \, dG = \int_{[0,r]^n} \text{Area}(D \cap G^{(i)}_x) \, dx_1 \ldots dx_n, \tag{6}
\]
by the identification \( \Phi \). Letting
\[
F^{(i)}(x_i) := \int_{[0,r]^{n-1}} \text{Area}(D \cap G^{(i)}_x) \, dx_1 \ldots \hat{dx_i} \ldots dx_n,
\]
we have
\[
\int_{[0,r]^n} \text{Area}(D \cap G^{(i)}_x) \, dx_1 \ldots dx_n = \int_0^r F^{(i)}(x_i) \, dx_i \tag{7}
\]
by Fubini theorem and
\[
F^{(i)}(x_i) = r^{n-1} \text{Area}(D \cap G^{(i)}_{(0,\ldots,x_i,\ldots,0)}), \tag{8}
\]
by domain invariance. It follows that
\[
\int_{[0,r]^n} \text{Area}(D \cap G^{(i)}_x) \, dx_1 \ldots dx_n = \int_0^r r^{n-1} \text{Area}(D \cap G^{(i)}_{(0,\ldots,x_i,\ldots,0)}) \, dx_i, \tag{9}
\]
and
\[
\frac{1}{r^n} \sum_{i=1}^n \int_0^r r^{n-1} \text{Area}(D \cap G^{(i)}_x) \, dG = \frac{n}{r} \text{Vol}(D), \tag{10}
\]
which finally gives
\[
\frac{1}{r^n} \int_{G_r} \text{Area}(D \cap G) \, dG = \frac{n}{r} \text{Vol}(D). \tag{11}
\]

\begin{corollary}
Let \( r > 0 \). Let \( D \) be an open set of \( \mathbb{R}^n \). There exists a grid \( G \) of mesh \( r \) such that
\[
\text{Area}(D \cap G) \leq \frac{n}{r} \text{Vol}(D). \tag{12}
\]
\end{corollary}

\begin{proposition}
We denote \( D_{G,k} \) the connected components of \( D \setminus G \). Then
\[
\sum_k \text{Area}(\partial D_{G,k}) - \text{Area}(\partial D) = 2\text{Area}(D \cap G).
\]
\end{proposition}

\begin{proof}
For every grid \( G \),
\[
\sum_k \text{Area}(\partial D_{G,k}) - \text{Area}(\partial D) = 2\text{Area}(D \cap G).
\]
\end{proof}
By Corollary 2.2, there exists a grid $G$ such that $\frac{\text{Area}(\partial D \cap G)}{\text{Vol}(D)} \leq \frac{n}{r} \text{Vol}(D)$. We deduce that

$$0 \leq \frac{\sum_k \text{Area}(\partial D_{G,k}) - \text{Area}(\partial D)}{\text{Vol}(D)^{\frac{n-1}{n}}} \leq \frac{2n \text{Vol}(D)}{r} \text{Vol}(D)^{\frac{1}{n}} = \frac{2n \text{Vol}(D)}{r}.$$ 

Thus if $r$ is very large with respect to $\text{Vol}(D)^{\frac{1}{n}}$ then

$$\frac{\sum_k \text{Area}(\partial D_{G,k}) - \text{Area}(\partial D)}{\text{Vol}(D)^{\frac{n-1}{n}}}$$

is close to 0. \hfill \Box

**Proposition 2.4** Let $M$ be a Riemannian manifold with bounded geometry. Let $D_j$ be a sequence of domains of $M$ so that

1. $\text{Vol}(D_j) \to 0$.
2. $\limsup_{j \to +\infty} \frac{\text{Area}(\partial D_j)}{\text{Vol}(D_j)^{\frac{n-1}{n}}} \leq c_n$.

For any sequence $(r_j)$ of positive real numbers that tends to zero $(r_j \to 0)$ and $\frac{\text{Vol}(D_j)^{\frac{1}{n}}}{r_j} \to 0$, there exists a partition $D_j = \bigcup_k D_{j,k}$ of $D_j$ in domains $D_{j,k}$ with $\text{Diam}(D_{j,k}) \leq \text{const}_M \cdot r_j$ such that

$$\limsup_{j \to +\infty} \frac{\sum_k \text{Area}(\partial D_{j,k})}{\text{Vol}(D_{j,k})^{\frac{n-1}{n}}} \leq c_n.$$ 

**Proof** Take a covering $\{U\}$ of $M$ by balls of radius $3\epsilon$ of multiplicity $N$, i.e., $N$ is as in Lemma 2.1 and Lebesgue number $\epsilon > 0$. For every ball $B(p, 3\epsilon)$ of this family, we fix a diffeomorphism $\phi_p : B(p, 3\epsilon) \to B_{\mathbb{R}^n}(0, 3\epsilon)$ of Lipschitz constant $C$. Observe here that by the bounded geometry assumption, $C$ can be chosen independently of $p$. For every $j$ we also fix a radius $r_j \gg \text{Vol}(D_j)^{\frac{1}{n}}$ and map the grids of mesh $r_j$ of $\mathbb{R}^n$ into $B(p, 3\epsilon)$ via $\phi_p$, i.e., for $G \in \mathcal{G}_{r_j}$, we have

$$G_p = \phi_p^{-1}(G).$$

Let us denote by $D_{j,k}$ the connected boundary components of $D_j \setminus (\bigcup_p G_p)$. We are looking for an estimate on the additional boundary area introduced by the partition in this $D_{j,k}$,

$$\sum_k \text{Area}(\partial D_{j,k}) - \text{Area}(\partial D_j) = 2\text{Area}(D_j \cap (\bigcup G_i)).$$

First estimate the average $m = \frac{1}{r_j} \int_{\mathcal{G}_{r_j}} \text{Area}(D_j \cap (\bigcup G_i)) \mathcal{L}^n(dG)$ of this volume over all possible choices of the grids $G \in \mathcal{G}_{r_j}$ as

$$m \leq \frac{1}{r_j} \sum_p \int_{\mathcal{G}_{r_j}} \text{Area}(D_j \cap G_p) \mathcal{L}^n(dG) \leq \frac{1}{r_j} \sum_p \int_{\mathcal{G}_{r_j}} \text{Area}(\mathbb{R}^n, \phi_p^{-1}(g))(\phi_p(D_j) \cap G) \mathcal{L}^n(dG) \leq \frac{C}{r_j} \sum_p \int_{\mathcal{G}_{r_j}} \text{Area}(\mathbb{R}^n, \text{can})(\phi_p(D_j) \cap \mathcal{U}_p) \mathcal{L}^n(dG).$$
\[ \leq C \frac{n}{r_j} \sum_p \text{Vol}(\phi_p(D_j \cap B(p, 3\epsilon))) \]
\[ \leq C^2 \frac{n}{r_j} \sum_p \text{Vol}(D_j \cap B(p, 3\epsilon)) \]
\[ \leq C^2 \frac{n}{r_j} N \text{Vol}(D_j), \]

where we have used that every point of \( M \) is contained in at most \( N \) balls \( B(p, 3\epsilon) \). Then there exists \( G \) in \( \mathcal{G}_{r_j} \) such that
\[ \text{Area}(D_j \cap (\cup_p G_p)) \leq C^2 \frac{n}{r_j} N \text{Vol}(D_j), \]
and so
\[ 0 \leq \sum_k \frac{\text{Area}(\partial D_{j,k}) - \text{Area}(\partial D_j)}{\text{Vol}(D_j)^{\frac{n-1}{n}}} \leq 2C^2 \frac{n}{r_j} N \text{Vol}(D_j)^{\frac{1}{n}}. \]

From the last inequality we obtain
\[ \limsup_{j \to +\infty} \frac{\sum_k \text{Area}^M(\partial D_{j,k})}{\left( \sum_k \text{Vol}^M(D_{j,k}) \right)^{\frac{n-1}{n}}} \leq \limsup_{j \to 0} \frac{\text{Area}^M(\partial D_j)}{\text{Vol}^M(D_j)^{\frac{n-1}{n}}} \leq c_n. \]

Now fix \( x \in D_j \setminus (\cup_p G_p) \). By construction there exists a ball \( B(p, 3\epsilon) \) that contains \( B(x, \epsilon) \). Let \( D_{j,k} \) denote the connected component of \( D_j \setminus (\cup_p G_p) \) containing \( x \), and \( D'_{j,k} \) the connected component of \( \phi_p(B(p, 3\epsilon)) \setminus G \) containing \( \phi_p(x) \). We observe that \( D'_{j,k} \) is contained in a cube of edge length \( r_j \); if \( j \) is large enough so that \( r_j \leq \epsilon / C \sqrt{n} \), then \( D'_{j,k} \) is contained in \( \phi_p(B(x, \epsilon)) \), hence \( D_{j,k} \) is contained in \( \phi_p^{-1} D'_{j,k} \), which has diameter at most \( C r_j \).

2.5 Selecting a large subdomain

We first show that an almost Euclidean isoperimetric inequality can be applied to small domains.

**Lemma 2.2** Let \( M \) be a Riemannian manifold with bounded geometry. Then
\[ \frac{\text{Area}(\partial D)}{\text{Vol}(D)^{\frac{n-1}{n}}} \geq c_n (1 - \eta(diam(D))) \] (13)
with \( \eta \to 0 \) as \( diam(D) \to 0 \).

**Proof** In a ball of radius \( r < inj(M) \), we reduce to the Euclidian isoperimetric inequality via the exponential map, which is a \( C \)-bi-Lipschitz diffeomorphism with \( C = 1 + O(r^2) \). This implies for all domains of diameter less than \( r \),
\[ \frac{\text{Area}(\partial D)}{\text{Vol}(D)^{\frac{n-1}{n}}} \geq c_n C^{-2n+2} = c_n (1 - O(r^2)). \]

Second, we have a combinatorial lemma that tells us that the largest domain in a partition contains almost all the volume.
Lemma 2.3 Let $f_{j,k} \in [0, 1]$ be numbers such that for all $j$, $\sum_k f_{j,k} = 1$. Then
\[
\limsup_{j \to +\infty} \sum_k f_{j,k}^{n-1} \leq 1
\]
implies that
\[
\lim_{j \to +\infty} \max_k f_{j,k} = 1.
\]

Proof We argue by contradiction. Suppose there exists $\varepsilon > 0$ and $j_\varepsilon \in \mathbb{N}$ such that for all $j \geq j_\varepsilon$, we have $\max_k \{f_{j,k}\} \leq 1 - \varepsilon$. Then for all $j \geq j_\varepsilon$, we have $f_{j,k} \leq 1 - \varepsilon$. From this inequality,
\[
\sum_k f_{j,k}^{\frac{n-1}{n}} = \sum_k f_{j,k} f_{j,k}^{\frac{-1}{n}} \geq \frac{\sum_k f_{j,k}}{(1 - \varepsilon)^{\frac{1}{n}}} \geq \frac{1}{(1 - \varepsilon)^{\frac{1}{n}}},
\]
hence
\[
\limsup_{j \to +\infty} \sum_k f_{j,k}^{\frac{n-1}{n}} \geq \frac{1}{(1 - \varepsilon)^{\frac{1}{n}}} > 1,
\]
which is a contradiction. \(\square\)

Proposition 2.5 Let $M$ be a Riemannian manifold with bounded geometry. Let $D_j$ be a sequence of approximate solutions in $M$ with volumes that tend to zero. Let $r_j$ be a sequence of positive real numbers such that $r_j \to 0$ and $\frac{\text{Vol}(D_j)}{r_j} \to 0$. There exist $p_j \in M$ and $\varepsilon_j \leq \text{const}_M \cdot r_j$ and subdomains $D'_j \subset D_j$ such that

1. $D'_j \subseteq B(p_j, \varepsilon_j)$,
2. $\frac{\text{Area}(\partial D'_j)}{\text{Vol}(D'_j)^{\frac{n-1}{n}}} \to c_n$,
3. $\lim_{j \to +\infty} \frac{\text{Vol}(D'_j)}{\text{Vol}(D_j)} = 1$.

Proof We apply Proposition 2.4. By the definition of isoperimetric profile and Lemma 2.2 we have
\[
\text{Area}(\partial D_{j,k}) \geq I(\text{Vol}(D_{j,k})) \geq c_n \text{Vol}(D_{j,k})^{\frac{n-1}{n}} (1 - \eta_j)
\]
where $\eta_j \to 0$. Since
\[
\limsup_{j \to +\infty} \sum_k c_n \text{Vol}(D_{j,k})^{\frac{n-1}{n}} (1 - \eta_j) \leq \limsup_{j \to +\infty} \sum_k \frac{\text{Area}(\partial D_{j,k})}{\text{Vol}(D_j)^{\frac{n-1}{n}}} \leq c_n,
\]
\[
\limsup_{j \to +\infty} \sum_k \frac{\text{Vol}(D_{j,k})^{\frac{n-1}{n}}}{\text{Vol}(D_j)^{\frac{n-1}{n}}} \leq \limsup_{j \to +\infty} \frac{1}{1 - \eta_j} = 1.
\]
Now set $f_{j,k} = \frac{\text{Vol}(D_{j,k})}{\text{Vol}(D_j)}$. We can suppose that $f_{j,1} = \max_k \{f_{j,k}\}$. We can apply Lemma 2.3 to deduce that
\[
\frac{\text{Vol}(D_{j,1})}{\text{Vol}(D_j)} \to 1.
\]
But by construction $D_{j,1} \subset B_M(p_j, \text{const} M r_j)$ for some sequence of points $p_j$ in $M$. Finally, proposition 2.4 gives
\[
\limsup \frac{\text{Area}(\partial D_j, 1)}{\text{Vol}(D_j)\frac{n-1}{n}} \leq \limsup \leq c_n.
\]
Indeed, since $\{D_{j,k}\}$ is a partition of $D_j$ we have $\text{Vol}(D_j) = \sum_k \text{Vol}(D_{j,k})$. Thus one can take $D_j' = D_{j,1}$. \qed

2.6 End of the proof of Theorem 3

In this subsection we terminate the proof of Theorem 3.

Proof Let $D_j$ be a sequence of approximate solutions with $\text{Vol}(D_j) \to 0$. According to proposition 2.5 there exist subdomains $D_{j}' \subseteq D_j$, points $p_j \in M$ and radii $\epsilon_j \to 0$ such that

(i) $D_{j}' \subseteq B(p_j, \epsilon_j)$,
(ii) $\frac{\text{Vol}(D_{j}')}{\text{Vol}(D_j)} \to 1$,
(iii) $\frac{\text{Area}(\partial D_{j}')}{\text{Vol}(D_{j}')\frac{n-1}{n}} \to c_n$.

We identify all tangent spaces $T_{p_j} M$ with a fixed Euclidean space $\mathbb{R}^n$ and consider the domains $D_{j}'' = \exp^{-1}(D_{j}')$ in $\mathbb{R}^n$. Since the pulled back metrics $\tilde{g}_j = \exp^*_{p_j}(g_M)$ converge to the Euclidean metric,
\[
\frac{\text{Area}(\partial D_{j}'')}{\text{Vol}(D_{j}'')\frac{n-1}{n}} \to c_n.
\]
According to Theorem 2.1, there exist Euclidean balls $W_j = B_{\text{eucl.}}(\tilde{q}_j, R_j)$ in $\mathbb{R}^n$ such that
\[
\frac{\text{Vol}_{\text{eucl.}}(D_{j}'' \Delta W_j)}{\text{Vol}_{\text{eucl.}}(D_{j}'')} \to 0.
\]
Note that $\tilde{g}_j$-balls are close to Euclidean balls, thus
\[
\frac{\text{Vol}_{\text{eucl.}}(D_{j}'' \Delta B_{\tilde{g}_j}(\tilde{q}_j, R_j))}{\text{Vol}_{\text{eucl.}}(D_{j}'')} \to 0,
\]
where $B_{\tilde{g}_j}(\tilde{q}_j, R)$ is the geodesic ball of radius $R$ in $(\mathbb{R}^n, \tilde{g}_j)$, and so for $q_j = \exp_{p_j}(\tilde{q}_j)$,
\[
\frac{\text{Vol}_{\text{eucl.}}(D_{j}' \Delta B_{\tilde{g}_j}(\tilde{q}_j, R_j))}{\text{Vol}_{\text{eucl.}}(D_{j}')} = \frac{\text{Vol}_{\text{eucl.}}(D_{j}'' \Delta B_{\tilde{g}_j}(\tilde{q}_j, R_j))}{\text{Vol}_{\tilde{g}}(W_j)} \to 0.
\]
Finally, since $\frac{\text{Vol}(D_{j} \Delta D_{j}')}{\text{Vol}(D_{j})} \to 0$ and $\frac{\text{Vol}_{\tilde{g}}(D_{j} \Delta B(q_j, R_j))}{\text{Vol}_{\tilde{g}}(D_{j})} \to 0$, Theorem 3 follows easily. \qed

2.7 Case of exact solutions

Remark When we consider the solutions of the isoperimetric problem (this is the case treated in Morgan and Johnson [10]), and not approximate solutions, the conclusion is stronger. In fact we can prove directly by the monotonicity formula that $D_j$ is of small diameter, which simplifies the argument that they are close to round balls in flat norm.

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Lemma 2.4 Assume $D_j$ is a sequence of solutions of the isoperimetric problem. The dilated domains $D_j'' := \exp_{g_j}^{-1}(D_j) \frac{\text{Vol}_{g_j}(D_j)^{\frac{1}{n}}}{\text{Vol}_{g_j}(D_j)}$ are of bounded diameter and hence we can find a positive constant $R > 0$ as in the proof of the preceding theorem so that for all $j \in \mathbb{N}$ we have

$$D_j'' \subseteq B(0, R).$$

Proof For the domains $D_j''$, the mean curvature of the boundary in $(\mathbb{R}^n, \text{eucl})$ is bounded $h_j^{\text{eucl}} \leq \lambda = \text{const.}$ for all $j$. This follows from an application of the Lévy-Gromov isoperimetric inequality [6, 7] analogous to that in the proof of Theorem 2.2 of [10]). Thus the monotonicity formula of Morgan and Johnson [1, 5.1 (3) p. 446] gives

$$||\partial D_j''||_{(B(a_j, r_0))} \geq e^{-\lambda r_0} \Theta^{n-1} (||\partial D_j'||, a_j) \omega_{n-1} r_0^{n-1}$$

for a fixed $r_0$ and all $j$, where $a_j \in \text{spt}(||\partial D_j'||)$. Arguing that $\text{const} \geq \text{Area}_{g_{\text{sc}}} (\partial D_j'') \geq \left[ \frac{\text{Diam}_{g_{\text{sc}}}(D_j'')}{2r_0} \right] \omega_{n-1} r_0^{n-1}$, we can conclude $\text{Diam}_{g_{\text{sc}}}(D_j'')$ is uniformly bounded.

3 Existence for small volumes

For compact manifolds, the regularity theorem of Morgan and Johnson [10] applies and there is no need to use the more general Theorem 3.2. For noncompact manifolds the situation is quite involved.

3.1 Minimizers are pseudo-bubbles

When $M$ is noncompact, the regularity theorem of Morgan and Johnson [10] has to be replaced by a more general statement, for the following reasons:

1. Solutions of the isoperimetric problem need not exist in $M$.
2. Minimizing sequences may escape to infinity, therefore varying ambient metrics cannot be avoided.

Recall the basic result from the theory of convergence of manifolds, as exposed in Petersen [13].

Theorem 3.1 (Fundamental Theorem of Convergence Theory [13, Theorem 72]) $\mathcal{M}^{m, \alpha}(n, Q, r)$ is compact in the pointed $C^m, \beta$ topology for all $\beta < \alpha$.

In subsequent arguments, we will need a regularity theorem in the presence of variable metrics.

Theorem 3.2 [12] Let $M^n$ be a compact Riemannian manifold and $g_j$ a sequence of Riemannian metrics on $M^n$ converging to a fixed metric $g_\infty$ in the $C^4$ topology. Assume that $B$ is a domain of $M$ with smooth boundary $\partial B$ and $T_j$ is a sequence of $n$-currents in $(M^n, g_j)$ which minimize boundary area under volume constraints and satisfying

$$(*) : \text{Vol}_{g_\infty}(B \Delta T_j) \to 0.$$  

Then $\partial T_j$ is the graph in normal exponential coordinates of a function $u_j$ on $\partial B$. Furthermore, for all $\alpha \in ]0, 1]$, $u_j \in C^{2,\alpha}(\partial B)$ and $||u_j||_{C^{2,\alpha}(\partial B)} \to 0$ as $j \to +\infty$. 

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Theorems 3.1 and 3.2 are the main sources of the $C^4$-bounded geometry hypotheses appearing in this paper.

In the sequel we often use the following classical isoperimetric inequality due to Pierre Berard and Daniel Meyer.

**Theorem 3.3** [2, Appendix C]. Let $M^{n+1}$ be a smooth, complete Riemannian manifold, possibly with boundary, of bounded geometry (bounded sectional curvature and positive injectivity radius). Then given $0 < \delta < 1$, there exists $v_0 > 0$ such that any open set $U$ of volume $0 < v < v_0$ satisfies

$$\text{Area}(\partial U) \geq \delta c_n v^{\frac{n-1}{n}}. \quad (15)$$

**Remark** The preceding theorem implies in particular that for a complete Riemannian manifold with bounded geometry, $I_M(v) \sim c_n v^{\frac{n-1}{n}}$ as $v \to 0$.

**Lemma 3.1** Let $M \in \mathcal{M}^{n,\alpha}(n, Q, r)$ and $(D_j)$ a sequence of solutions of the isoperimetric problem with $\text{Vol}_g(D_j) \to 0$. Then possibly extracting a subsequence, there exist points $p_j \in M$ such that the domains $D_j$ are graphs in polar normal coordinates of functions $u_j$ of class $C^{2,\alpha}$ on the unit sphere of $T_{p_j}M$ of the form $u_j = r_j(1 + v_j)$ with $\|v_j\|_{C^{2,\alpha}(\mathbb{B}_{r_j}(0,1))} \to 0$ and radii $r_j \to 0$.

**Proof** By identifying the tangent spaces $T_{p_j}M$ with a fixed copy of $\mathbb{R}^n$, we can carry out an analysis almost identical to that performed in Nardulli [11, Lemma 3.1], the difference being that we must consider a sequence of points $p_j$ instead of just a single point. To this end, take domains $T_j = \frac{1}{r_j} \exp^{-1}_{p_j}(D_j)$ in this fixed copy of $\mathbb{R}^n$. This $T_j$ is a solution of the isoperimetric problem for the rescaled pulled-back metric $g_j = \frac{1}{r_j^2} \exp^{-2}_{p}(g)$, and the sequence converges volume-wise to a unit ball. Since the $g_j$ converge at least $C^4$ to the Euclidean metric, the same argument as in the preceding lemma applies, by the $C^4$-bounded geometry assumption on $g$. \qed

**Lemma 3.2** For all $n, r, Q, m \geq 4, \alpha$, there exists $0 < v_1 = v_1(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{n,\alpha}(n, Q, r)$ and every solution $D$ of the isoperimetric problem with $0 < \text{Vol}(D) \leq v_1$, there exists a point $p_D \in M$ (depending on $D$) such that $D$ is the normal graph of the function $u_D = r_D(1 + v_D) \in C^{2,\alpha}(\mathbb{S}^{n-1})$, where $\|v_D\|_{C^{2,\alpha}(\mathbb{S}^{n-1})} \to 0$ as $\text{Vol}(D) \to 0$.

**Proof** The contrary, that there exists a sequence $D_j$ of solutions of the isoperimetric problem with volumes $\text{Vol}(D_j) \to 0$ for which $\partial D_j$ is not the graph over the sphere $\mathbb{S}^{n-1}$ in $T_pM$ of the function $u_j = r_j(1 + v_j)$ where $\|v_j\|_{C^{2,\alpha}}$ goes to 0, would contradict Lemma 3.1. \qed

**Theorem 3.4** For all $n, r, Q, m, \alpha$, there exists $0 < v_2 = v_2(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{n,\alpha}(n, Q, r)$ and $0 < v < v_2$, if $D \subseteq M$ has volume $v$ and $I_M(v) = \text{Area}(\partial D)$, then $\partial D$ is a pseudo-bubble.

**Proof** An analysis of the proof of Theorem 1 of [11] shows how an application of the implicit function theorem gives a constant, say $C_0$, depending on $n, r, Q, m, \alpha$, such that the normal graph of a function $u$ on the unit tangent sphere centered at $p \in M$ with $\|u\|_{C^{2,\alpha}} \leq C_0$, a solution of the pseudo-bubble equation, is of the form $\beta(p, r)$ for $r < r_0$. Then the argument given in Theorem 3.1 of [11] applies. \qed
Corollary 3.1 For $0 < v < v_2$, suppose that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$ there exist a minimizing current $T$ for the isoperimetric problem with small enclosed volume $v$, with center of mass $p \in M$. If $\text{St}_p \leq \text{Isom}(M)$ denotes the stabilizer of $p$ for the canonical action of the group of isometries $\text{Isom}(M)$ on $M$, then $k(T) = T$ for all $k \in \text{St}_p$.

Proof Following Theorem 1.1, $\partial T$ is the pseudo-bubble $\beta(p, r)$, where $\omega_n \rho^n = \text{Vol}(T)$. If $k \in \text{St}_p$, then $k(\beta(p, r)) = \beta(k(p), r\ast)$ for some small $r\ast$. For small volumes, the parameter $v$ determines the parameter $r$ uniquely, but volume $v$ does not change under isometric action, so by uniqueness of pseudo-bubbles we have $r\ast = r$, hence $\beta(k(p), r) = \beta(p, r)$ and $k(T) = T$. $\square$

3.2 Proof of Theorem 1

For what follows it will be useful to give the following definitions.

Definition 3.1 We say that $(D_j)_j \subseteq \tau_M$ is an almost minimizing sequence in volume $v > 0$ if

(i) $\text{Vol}(D_j) \to v$,  
(ii) $\text{Area}(\partial D_j) \to I_M(v)$.

Definition 3.2 Given $\phi : M \to N$ a diffeomorphism between two Riemannian manifolds and $\varepsilon > 0$, we say that $\phi$ is a $(1 + \varepsilon)$-isometry if $\frac{1}{1+\varepsilon}d_M(x, y) \leq d_N(\phi(x), \phi(y)) \leq (1 + \varepsilon)d_M(x, y)$ for every $x, y \in M$.

For the convenience of the reader we have divided the proof into a sequence of lemmas.

3.2.1 Existence of a minimizer in a $C^{m,\alpha}$ limit manifold

Lemma 3.3 Let $M$ be with bounded sectional curvature and positive injectivity radius. $(M, p_j) \to (M_\infty, p_\infty, g_\infty)$ in $C^{m,\alpha}$ topology, $m \geq 1$. Then

\[ I_{M_\infty} \geq I_M. \]  \hspace{1cm} (16)

Proof Fix $0 < v < \text{Vol}(M)$. Let $D_\infty \subseteq M_\infty$ an arbitrary domain of volume $v = \text{Vol}_{g_\infty}(D_\infty)$. Put $r := d_H(D_\infty, p_\infty)$, where $d_H$ denotes the Hausdorff distance. Consider the sequence $\varphi_j : B(p_\infty, r + 1) \to M$ of $(1 + \varepsilon_j)$-isometries given by the convergence of pointed manifolds, where $\varepsilon_j \searrow 0$. Setting $D_j := \varphi_j(D_\infty)$ and $v_j := \text{Vol}(D_j)$, then since $\varphi_j$ is a $1 + \varepsilon_j$ isometry, it is easy to see that

(i) $v_j \to v$,  
(ii) $\text{Area}_{g}(\partial D_j) \to \text{Area}_{g_\infty}(\partial D_\infty)$.

With this very general preliminary construction in place (which requires only the existence of a limit manifold along a sequence and not the assumption of bounded geometry), we proceed to the proof of (16) by contradiction. Suppose that there exist a volume $0 < v < \text{Vol}(M)$ satisfying

\[ I_{M_\infty}(v) < I_M(v). \]  \hspace{1cm} (17)

Then there is a domain $D_\infty \subseteq M_\infty$ such that

\[ I_{M_\infty}(v) \leq A_{g_\infty}(\partial D_\infty) < I_M(v). \]
Take domains $D_j \subset M$ satisfying (i)–(ii) as above. The volumes $v_j$ in general are not exactly equal to $v$, so to get the desired contradiction, we will adjust the domains $D_j$ to achieve $v_j = v$ while preserving the property $A_\varepsilon (\partial D_j) \to A_\varepsilon (\partial D_\infty )$ as $j \to +\infty$. This can be done with the following construction, which will be used repeatedly in the sequel. Looking carefully at the proof of the deformation lemma of [4] and the compensation lemma of [12], one can convince oneself that it is possible construct small perturbations of $D_\infty$, domains denoted $D_j^\infty \subseteq B(p_\infty, r + 1) \subseteq M_\infty$, such that

$$A_\varepsilon (\partial D_j^\infty ) \leq A_\varepsilon (\partial D_\infty ) + c_v j, \quad j \to \infty .$$

and

$$\text{Vol}_\varepsilon (\varphi_j(D_j^\infty )) = v.$$  

(19)

The preceding discussion implies the existence of bounded finite perimeter sets (in fact, smooth domains) $D_j := \varphi_j(D_j^\infty ) \subset M$ satisfying

$$\text{Vol}_\varepsilon (D_j) = v,$$

(20)

$$|A_\varepsilon (\partial D_j) - A_\varepsilon (\partial D_\infty )| \to 0,$$

(21)

(22)

(where the last equation comes from the fact that $\varphi_j$ is a $1 + \varepsilon_j$ isometry). Thus we obtain a sequence of domains $D_j$, each with volume $v$, such that

$$A_\varepsilon (\partial D_j) \to A_\varepsilon (\partial D_\infty ) < I_M(v).$$

(23)

The last equation is the desired contradiction, and the theorem follows from the arbitrariness of $v$. $\square$

The next lemma is simply a restatement of Theorem 3.

**Lemma 3.4** For all $n, r, Q, m, \alpha$ and $\varepsilon > 0$, there exists $0 < v_3 = v_3(n, r, Q, m, \alpha, \varepsilon)$ such that for each $M \in \mathcal{M}^{m, \alpha}(n, Q, r)$, there is a positive number $\eta = \eta(\varepsilon, M) > 0$ with the following properties

if $0 < v = \text{Vol}(D) < v_3$, then there exists $p = p_D \in M$ and $R = C(n, r, Q, m, \alpha) v^\frac{1}{2}$ such that

$$\frac{\text{Vol}(D \Delta B(p, R))}{\text{Vol}(D)} \leq \varepsilon.$$  

(24)

**Proof** Observing that the constant $C$ used in the proof of Lemma 2.4 depends only on $n, r, Q, m, \alpha$, it is easy to check that this lemma is a restatement of Theorem 3 in $\varepsilon$-$\delta$ language with an additional uniformity property due to the fact that $M \in \mathcal{M}^{m, \alpha}(n, Q, r)$. $\square$

**Definition 3.3** Let $M$ be a Riemannian manifold and $0 < v < \text{Vol}(M)$. We say that $I_M(v)$ is achieved if there exists an integral current $D \subseteq M$ such that $\text{Vol}(D) = v$ and $\text{Area}(\partial D) = I_M(v)$.

**Lemma 3.5** For all $n, r, Q, m, \alpha$, there exist $0 < v_4 = v_4(n, r, Q, m, \alpha)$ and $C_1 = C_1(n, r, Q, m, \alpha) > 0$ such that for all $M \in \mathcal{M}^{m, \alpha}(n, Q, r)$ and $0 < v < v_4$ with $I_M(v)$ achieved, we have

$$I_M(v + h) \leq I_M(v) + C_1 h v^{-\frac{1}{2}},$$  

(25)

provided that $v + h < v_4$. $\square$
By Theorem 3.1 applied to the sequence of pointed manifolds $M$ and enclosing volume $\tilde{v}$, then $\tilde{v} \mapsto \psi_{M,p}(\tilde{v})$ is $C^1$ and $\|\psi_{M,p}\|_{C^1([0, v])} \leq C$ uniformly with respect to $M$ and $p$, i.e., $C = C(n, r, Q, m, \alpha)$ (a nontrivial consequence of the proof of the existence of pseudo-bubbles in Nardulli [11]). When $v + h < v_4$,

$$\psi_{M,p}(v + h) \leq \psi_{M,p}(v) + Ch.$$  

$$I_M(v + h) \leq \psi_{M,p}(v + h)^{\frac{n-1}{n}}\left(1 + \frac{Ch}{\psi_{M,p}(v)}\right)^{\frac{n-1}{n}} \leq \psi_{M,p}(v)^{\frac{n-1}{n}}(1 + \frac{n-1}{n} C'h) \leq \psi_{M,p}(v)^{\frac{n-1}{n}} + C_1 hv^{-\frac{1}{n}} \leq I_M(v) + C_1 hv^{-\frac{1}{n}}.$$  

Now we want to apply a suitable mixture of the theory of manifold convergence and geometric measure theory to the isoperimetric problem for small volumes. Some parts of the proof are inspired from [15, Theorem 2.1].

**Lemma 3.6** For any $n, r, Q, m, \alpha$, there exists $0 < v_6 = v_6(n, r, Q, m, \alpha)$ such that for all $M \in \mathcal{M}^{m,\alpha}(n, Q, r)$ and $0 < v < v_6$ there is a sequence of points $p_j$, a limit manifold $(M_\infty, p_\infty, g_\infty) \in \mathcal{M}^{m,\alpha}(n, Q, r)$, and a domain $D_\infty \subset M_\infty$ such that

(I) $(M, p_j, g) \rightarrow (M_\infty, p_\infty, g_\infty)$ in $C^{m,\beta}$ topology for $\beta < \alpha$,

(II) $I_{M_\infty}(v) = \text{Area}_{g_\infty}(\partial D_\infty)$, so $I_{M_\infty}(v)$ is achieved,

(III) $\partial D_\infty$ is a pseudo-bubble,

(IV) $I_M(v) = I_{M_\infty}(v)$.

**Proof** Fix $1 > \delta > 0$, and $\varepsilon > 0$ such that

$$\frac{1}{2} \delta \frac{c_n}{C_1} > \gamma'(\varepsilon) \frac{1}{\varepsilon} > 0,$$

with $\gamma = \gamma(\varepsilon) = \frac{\varepsilon}{1 - \varepsilon}$. Observe that this is possible because $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Set $v_6 = \text{Min}\{v_0, v_2, v_3, v_4\}$ as obtained respectively in Lemmas 3.3, 3.4, 3.5 and Theorem 3.4, and let $0 < v < v_6$. If $D_j$ is a minimizing sequence in volume $v$ (i.e., $\text{Vol}(D_j) = v$ and $\text{Area}(\partial D_j) \rightarrow I_M(v)$), we can take $j$ sufficiently large that $\frac{\text{Area}(\partial D_j)}{I_M(v)} < 1 + \eta_\varepsilon$, with $\eta_\varepsilon > 0$ as in Theorem 3.4. There exist $p_j$ and $R$ such that

$$\frac{\text{Vol}(D_j \Delta B(p_j, R))}{\text{Vol}(D_j)} \leq \varepsilon.$$

By Theorem 3.1 applied to the sequence of pointed manifolds $(M, p_j, g) \rightarrow (M_\infty, p_\infty, g_\infty)$, we obtain the existence of a pointed manifold $(M_\infty, p_\infty, g_\infty)$ such that $(M, p_j) \rightarrow (M_\infty, p_\infty, g_\infty)$ in the $C^{4,\beta}$ topology.

Using the diffeomorphisms $F_j$ of $C^{4,\beta}$ convergence, the next step is to define domains $\tilde{D}_j \subseteq M_\infty$ as $F_j$-images of a suitable intersection $D'_j$ of $D_j$ with balls whose radii $t_j$ are given by the coarea formula (needed to control the amount of area added in the procedure), to obtain an integral current $D_\infty \subseteq M_\infty$ such that $\tilde{D}_j \rightarrow D_\infty$ in the $\mathcal{F}_{loc}(M_\infty)$-topology. This will be accomplished by taking an exhaustion of $M_\infty$ by geodesic balls, applying a standard
compactness argument from geometric measure theory in each ball, and using a diagonal process.

Take a sequence of scalars \( (r_i) \) satisfying \( r_0 \geq R \) and \( r_{i+1} \geq r_i + 2i \), and consider an exhaustion of \( M_\infty \) by balls of center \( p_\infty \) and radius \( r_i \), i.e., \( M_\infty = \bigcup B(p_\infty, r_i) \). Then for every \( i \), convergence in \( C^{4,\beta} \)-topology gives \( v_{r_i} > 0 \) and diffeomorphisms \( F_{j,r_i} : B(p_\infty, r_i) \to B(p_j, r_i) \) for all \( j \geq v_{r_i} \) which are \((1 + \epsilon_j)\)-isometries for some sequence \( 0 \leq \epsilon_j \to 0 \).

For the diagonal process, we determine a suitable double sequence of cutting radii \( t_{i,j} \) with \( i \geq 1 \) and \( j \in S_1 \subseteq \mathbb{N} \) for some inductively defined sequence of infinite sets \( S_1 \supseteq \cdots \supseteq S_{i-1} \supseteq S_i \supseteq S_{i+1} \supseteq \ldots \). First recall the coarea argument: For every domain \( D \subseteq M \), point \( p \in M \), and interval \( J \subseteq \mathbb{R} \), there exists \( t \in J \) such that
\[
\text{Area}(D \cap (\partial B(p, t))) = \frac{1}{|J|} \int_J \text{Area}((\partial B(p, s)) \cap D) ds \leq \frac{\text{Vol}(D)}{|J|}.
\] (28)

Cutting by coarea with radii \( t_{1,j} = t_1, r_1 + j \) for \( j \geq v_{r_2} \), we get domains \( D_1' = D_j \cap B(p_j, t_{1,j}) \), \( D_1'' = D_j - D_1' \) which for \( j \geq v_{r_1} \) satisfying
\[
\left| \text{Area}(\partial D_1') + \text{Area}(\partial D_1'') - \text{Area}(\partial D_j) \right| \leq \frac{v}{1}.
\] (29)

Note that (29) is equivalent to
\[
2\text{Area}(\partial D \cap B(p_j; t_{1,j})) \leq v.
\]
Consider the sequence of domains \( \left( \tilde{D}_1, j = F_{j,r_2}^{-1} (D_1', j) \right) \) such that for \( j \geq v_{r_2} \), we have
1. \( \text{Area}(\partial D_1') \leq \text{Area}(\partial D_j) + 2\frac{v}{1} \leq I_\infty(v) + 2\frac{v}{1} \).
2. \( \text{Vol}(D_1') \leq v \).

Thus the volume and boundary area of the sequence of domains are bounded by a constant. A standard argument of geometric measure theory then allows us to extract a subsequence \( D_1' \) with \( j \in S_1 \subseteq \mathbb{N} \) converging on \( B(p_\infty, r_2) \) to a domain \( D_\infty, 1 \) in \( \mathcal{F}_B(p_\infty, r_2) \). Now look at the subsequence \( D_j \) with \( j \in S_1 \) and repeat the preceding argument to obtain cutting radii \( t_{2,j} \in ]r_2, r_3[ \) and a subsequence \( D_2' \subseteq D_j - B(p_j, t_{2,j}) \) for \( j \in S_1 \) and \( j \geq v_{r_3} \) such that
\[
\left| \text{Area}(\partial D_2') + \text{Area}(\partial D_2'') - \text{Area}(\partial D_j) \right| \leq \frac{v}{2}.
\] (30)

Again the sequence \( \left( \tilde{D}_2, j = F_{j,r_3}^{-1} (D_2', j) \right) \) for \( j \in S_1 \) has bounded volume and boundary area, so there is a convergent subsequence \( \left( \tilde{D}_2, j \right) \) defined on some subset \( S_2 \subseteq S_1 \) converging on \( B(p_\infty, r_3) \) to a domain \( D_\infty, 2 \) in \( \mathcal{F}_B(p_\infty, r_3) \). Continuing in this way, we obtain \( S_1 \supseteq \cdots \supseteq S_{i-1} \supseteq S_i \supseteq S_{i+1} \supseteq \ldots \), and domains \( D_i', j = D_j \cap B(p_j, t_{i,j}) \), \( D_i'' = D_j - D_i' \), satisfying
\[
\left| \text{Area}(\partial D_i') + \text{Area}(\partial D_i'') - \text{Area}(\partial D_j) \right| \leq \frac{v}{k}.
\] (31)

for all \( 1 \leq k \leq i \), \( j \in S_k \), and \( i \geq 1 \). Moreover, putting \( \tilde{D}_k, j = F_{j,r_{k+1}}^{-1} (D_k', j) \) for all \( 1 \leq k \leq i \) and \( j \in S_k \), we have convergence of \( (\tilde{D}_k, j) \) in \( \mathcal{F}_B(p_\infty, r_{k+1}) \) to a domain \( D_\infty, k \) in \( \mathcal{F}_B(p_\infty, r_{k+1}) \) for all \( i \geq 1 \) and \( k \leq i \). Let \( j_i \) be chosen inductively so that

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$$j_i < j_{i+1} \quad (32)$$

$$\text{Vol}(\tilde{D}_{i,\sigma(j_i)} \Delta D_{\infty,i}) \leq \frac{1}{t}, \quad (33)$$

and define $\sigma(i) = \sigma(j_i)$. Then the sequence $D_j^c := F_{\sigma(i),r_{i+1}}^{-1}(D'_{i,\sigma(i)})$ converges to $D_{\infty} = \bigcup_i D_{\infty,i}$ in the $\mathcal{F}_{loc}(M_{\infty})$-topology. Observe that $|t_{i+1} - t_i| > i$. From now on we restrict our attention to the sequences $\tilde{D}_i = D_{\sigma i}$, $\tilde{D}'_i = D'_{\sigma i}$, $\tilde{D}''_i = D''_{\sigma i}$, which we will call $D_i$, $D'_i$, and $D''_i$ by abuse of notation. Put $F_i = F_{\sigma(i),r_{i+1}}$ and replace $i$ by $j$. Possibly passing to a subsequence, one can build a minimizing sequence $D_j$ with the following properties

(i) $Area(\partial D_j') + Area(\partial D_j'') - Area(\partial D_j) \leq \frac{v}{j}$, for all $j$,

(ii) $\lim_{j \to +\infty} Area_{\gamma}(\partial D_j') = \lim_{j \to +\infty} Area_{\gamma}(\partial \tilde{D}_j)'$,

(iii) $\text{Vol}(\tilde{D}_j'') \to \text{Vol}(D_{\infty}) = v_{\infty}$,

(iv) $Area(\partial D_{\infty}) \leq \liminf Area(\partial \tilde{D}_j)'$,

(v) $v \geq v_{\infty} \geq (1-\varepsilon)v > 0$,

(vi) $\frac{w_{\infty}}{v_{\infty}} \leq \gamma$ with $w_{\infty} = v - v_{\infty}$,

(vii) $I_{\mathcal{M}_{\infty}}(v_{\infty}) = Area(\partial D_{\infty})$,

(viii) $Area(\partial D_{\infty}) = \liminf Area(\partial \tilde{D}_j)'$.

Property (i) follows directly by the construction of the sequences $(D_j')$, and (ii) from the fact that the diffeomorphisms given by $C^{4,\beta}$ convergence are $(1+\varepsilon_j)$-isometries for some sequence $0 \leq \varepsilon_j \to 0$. Writing $B_{r_{j+1}} = B(p_{\infty}; r_{j+1})$, (iii) comes from the observation that

$$|\text{Vol}(\tilde{D}_j') - \text{Vol}(D_{\infty})| \leq |\text{Vol}(\tilde{D}_j') - \text{Vol}(D_{\infty} \cap B_{r_{j+1}})| + \text{Vol}(D_{\infty} - B_{r_{j+1}})$$

$$\leq \text{Vol}((\tilde{D}_j' \Delta D_{\infty}) \cap B_{r_{j+1}}) + \text{Vol}(D_{\infty} - B_{r_{j+1}}),$$

where $B_{r_{j+1}}$ is the ball $B(p_{\infty}, r_{j+1})$ in $M_{\infty}$. It follows by (33) that

$$\lim_{j \to +\infty} \text{Vol}(\tilde{D}_j') = \text{Vol}(D_{\infty}).$$

On the other hand, the definition of the sets $\tilde{D}_j$ give us $\{D_j'\} \to D$ in $\mathcal{F}_{loc}(M)$. Hence $Area(\partial D) \leq \liminf_{j \to +\infty} Area(\partial \tilde{D}_j')$ by the lower semicontinuity of boundary area with respect to the flat norm in $\mathcal{F}_{loc}(M)$, which proves (iv). In (v), the first inequality holds because every $D_{\infty,i}$ is a limit in flat norm of a sequence of currents having volume less than $v$, and the second because the radii $r_i$ are greater than $R$, so $\text{Vol}(D_{\infty,i}) \geq (1-\varepsilon)v$, and (vi) follows easily by (v). To show (vii), we proceed by contradiction. Suppose that there exists a domain $\tilde{E} \in \tau_{M_{\infty}}$ having $\text{Vol}(\tilde{E}) = v_{\infty}$ and $Area(\partial \tilde{E}) < Area(\partial D_{\infty})$. Take the sequence of radii $s_j \in [t_j, t_{j+1}]$ and cut $\tilde{E}$ by coarea to obtain $\tilde{E}_j := \tilde{E} \cap B(p_{\infty}, s_j)$ in such a way that

$$Area_{\gamma}(\tilde{E}_j \cap \partial B(p_{\infty}, s_j)) \leq \frac{v_{\infty}}{j}, \quad (34)$$

Of course $\text{Vol}_{\gamma}(\tilde{E}_j) \to v_{\infty},$ since $s_j \nearrow +\infty$. Now fix a point $x_0 \in \partial \tilde{E}$ and a small neighborhood $U$ of $x_0$. For $j$ large enough, $U \subseteq B(p_{\infty}, r_j)$. Pushing forward $\tilde{E}$ in $M$, we get $E_j := F_j(\tilde{E}_j) \subseteq B(p_j, r_{j+1})$, and readjusting volumes by modifying slightly $E_i$ in $F_i(U)$ we obtain domains $E'_j \subseteq B(p_j, r_{j+1})$ with the properties
\[ E'_j \cap D''_j = \emptyset, \quad (35) \]
\[ \text{Vol}_g(E'_j \cup D''_j) = v, \quad (36) \]
\[ \text{Area}(\partial E'_j) \leq \text{Area}(\partial E_j) + c \Delta v_j, \quad (37) \]

with \( \Delta v_j = \text{Vol}_g(E'_j) - \text{Vol}_g(E_j) \) satisfying \( \Delta v_j \to 0 \) as \( j \to +\infty \), by virtue of \( \text{Vol}(\tilde{E}_j) \to v_\infty \) and \( \text{Vol}(D''_j) \to v - v_\infty \). Note that \( c = c(n, Q) \) is a constant independent of \( j \). Defining \( D^*_j := E'_j \cup D''_j \), we have \( \text{Area}(\partial D^*_j) \leq \text{Area}(\partial E'_j) + \text{Area}(\partial D''_j) \leq (1 + \varepsilon_j)^{n-1} \left( \text{Area}(\partial \tilde{E}) + \frac{v_\infty}{j} \right) + c \Delta v_j \), so that

\[ \liminf_{j \to +\infty} \text{Area}(\partial D^*_j) \leq \text{Area}(\partial \tilde{E}) + \liminf_{j \to +\infty} \text{Area}(\partial D''_j) \leq \text{Area}(D_\infty) + \liminf_{j \to +\infty} \text{Area}(\partial D''_j) \leq I_M(v). \]

This means that the sequence of domains \( D^*_j \) do better than the minimizing sequence \( D_j \), which is a contradiction, proving (vii). The proof of (viii) is similar. In fact, we can work directly with \( D_\infty \) instead of \( \tilde{E} \), because the set of regular points in \( \partial D_\infty \cap M_\infty \) is open, so inside it we can perform the suitable small deformation as in the Compensation Lemma of [12] and the Deformation Lemma of [4]. Roughly speaking, inside the regular part it is possible to make a smooth deformation of the domain at constant volume to produce a competitor with controlled area variation.

Letting \( i \to +\infty \) in (i), taking into account (ii), (iv) and (vii), and applying the Berard–Meyer inequality yields

\[ I_{M_\infty}(v_\infty) + \delta c_n \omega_{n-1} v^{\frac{n-1}{n}} \leq I_M(v). \quad (38) \]

It remains to prove that \( v_\infty \) cannot be strictly less than \( v \). By Corollary 3.4, we know that for \( v \leq v_4 \leq v_2 \), \( D_\infty \) is a pseudo-bubble. Then following estimate a direct consequence of Lemma 3.5:

\[ I_{M_\infty}(v) = I_{M_\infty}(v_\infty + w_\infty) \leq I_{M_\infty}(v_\infty) + C_1 v^{\frac{1}{n}} w_\infty. \quad (39) \]

Assuming \( w_\infty > 0 \), we deduce from (38), (39) and Lemma 3.3 that

\[ I_{M_\infty}(v_\infty) + \delta c_n \omega_{n-1} v^{\frac{n-1}{n}} \leq I_M(v) \leq I_{M_\infty}(v_\infty) + C_1 v^{\frac{1}{n}} w_\infty, \quad (40) \]
\[ \delta c_n \omega_{n-1} v^{\frac{n-1}{n}} \leq C_1 v^{\frac{1}{n}} w_\infty. \quad (41) \]

Dividing the above inequalities by \( w_\infty^{\frac{n-1}{n}} \) and combining with (vi), we obtain

\[ \gamma(\varepsilon)^{\frac{1}{n}} \geq \frac{c_n}{C_1}. \quad (42) \]
which by our choice of $\epsilon > 0$ contradicts (27). So $w_\infty = 0$, which means $v_\infty = v$ and $I_{M_\infty}(v) = I_{M_\infty}(v_\infty)$, which proves (3.6) and (3.6). To finish the proof, observe that

$$I_M(v) = \liminf Area(\partial D_j) + \liminf Area(\partial D_j')$$

$$= I_{M_\infty}(v_\infty) + \liminf Area(\partial D_j')$$

$$= I_{M_\infty}(v) + \liminf Area(\partial D_j')$$

$$\geq I_{M_\infty}(v).$$

which combined with $I_M(v) \leq I_{M_\infty}(v)$ gives $I_M(v) = I_{M_\infty}(v)$, and this is exactly (3.6).

$\square$

**Remark** It is easy to check that $\lim \inf Area(\partial D_j') = 0$.

End of the proof of Theorem 1.

**Proof** Take $v^* \leq v_6$ and suppose $0 < v < v^*$. First we show (1) implies (1). Let $p_0$ be a point where $p \mapsto f(p,v)$ attains its minimum. We show by contradiction that $\beta(p_0, v)$ is a solution of the isoperimetric problem. Assume that there is no isoperimetric domain having volume $v$. Let $D_j$ be a minimizing sequence with $Vol(D_j) = v$,

$$Area(\partial D_j) \to I_M(v) < f_M(p_0,v).$$

(43)

The choice of $v^*$ ensures the existence of a pseudo-bubble $D_\infty \subseteq M_\infty$, and points $p_j$ satisfying (I)–(IV) of Lemma 3.6. Hence $I_M(v) = I_{M_\infty}(v) = Area(\partial D_\infty) = f_{M_\infty}(p_\infty, v)$. A continuity argument with respect to $C^{4,\beta}$-convergence applies, giving $f_{M_\infty}(p_\infty, v) = \lim f_M(p_j,v)$. Furthermore, since $p_0$ is a minimum point, so that $\forall j f_M(p_j,v) \geq f_M(p_0,v)$, from which one can argue finally that $f_{M_\infty}(p_\infty, v) \geq f_M(p_0,v)$, contradicting (43).

Next we show that (1) implies (1). If $D$ is an isoperimetric domain of sufficiently small volume, it follows from Theorem 3.4 that $D = \beta(p_0, v)$ for some point $p$ and small $v$. This ensures that $p \mapsto f(p,v)$ attains its minimum at $p_0$.

Finally, (1) is a straightforward consequence of Lemma 3.6, noticing that for small volumes $I_M(v) = I_{M_\infty}(v)$ for some limit manifold $(M, \tilde{p}_j, g)$, obtained as the limit of the sequence $(M, p_j, g)$ for some sequence of points $p_j$. Furthermore, $I_{M_\infty}(v) = f_{M_\infty}(p_\infty, v)$ for some point $p_\infty$ possibly different from $\tilde{p}_\infty$. Now adjust the sequence of points $p_j$ to get a sequence $p_j \in M$ such that $(M, p_j, g) \to (M, p_\infty, g)$ with the same $M_\infty$ as above (which can be achieved by taking $p_j = F_j(p_\infty) = F_{B_{M_\infty}(\tilde{p}_\infty, R), j}(p_\infty)$ for large $j$, where $R = d_{M_\infty}(\tilde{p}_\infty, p_\infty) + 1$).

$\square$

4 Asymptotic expansion of the isoperimetric profile

We now prove Theorem 2 stated in the introduction.

**Proof** Let us just recall here the definition of $S = \text{Sup}_{p \in M} \{Sc(p)\}$. Let $(p_j)_j$ be such that $Sc(p_j) \not\to S$ and take the sequence $(M, p_j, g)$. Applying Theorem 3.1 then we get the existence of $(M'_\infty, p'_\infty, g)$ such that (passing to a subsequence if needed) $(M, p_j, g) \to (M'_\infty, p'_\infty, g)$ in the $C^{m,\beta}$-topology for $0 < \beta < \alpha$. A continuity argument shows that

$$Sc_{M_\infty}(p'_\infty) = S.$$ 

(44)

From the definition of isoperimetric profile and Lemma 3.3 we have

$$f_{M_\infty}(p'_\infty, v) \geq I_{M_\infty}(v) \geq I_M(v).$$

(45)
Taking an arbitrary sequence of volumes \( v_k \to 0 \) and looking at the corresponding \( D v_k \), we conclude that

\[
I_M(v_k) = I_{M_{\infty,k}}(v_k) = f_{M_{\infty,k}}(p_{\infty,k}, v_k).
\]

An application of the Fundamental Theorem of Convergence of Manifolds shows that the sequence \( (M_{\infty,k}) \) belongs again to \( M^{4,\alpha}(n, Q, r) \), and a second application implies the existence of a \( C^{4,\beta} \)-convergent subsequence \( (M_{\infty,k}, p_{\infty,k}) \to (M_\infty, p_\infty) \) for every \( 0 < \beta < \alpha \). From the latter construction, it follows that

\[
I_M(v_k) \sim f_{M_\infty}(p_\infty, v_k), \quad k \to +\infty. \tag{46}
\]

Combining (44), (45), (46), and (1) yields

\[
\frac{f_{M_\infty'}(p_\infty', v_k) - c_n v_k^{n-1}}{v_k^{n+1}} \leq \frac{I_M(v_k) - c_n v_k^{n-1}}{v_k^{n+1}}. \tag{47}
\]

From the asymptotic relation (46), we let \( k \to +\infty \) and conclude that

\[
-Sc_{M_{\infty'}}(p_\infty') \geq -Sc_{M_\infty}(p_\infty), \tag{48}
\]

which immediately gives

\[
S \leq Sc_{M_\infty}(p_\infty). \tag{49}
\]

Since the construction of \( M_\infty \) provides a sequence of points \( p_j'' \in M \) with \( Sc_{M}(p_j'') \to Sc_{M_\infty}(p_\infty) \), we obtain

\[
Sc_{M_\infty}(p_\infty) \leq S. \tag{50}
\]

Finally, (49), (50), and the arbitrariness of the sequence \( v_k \) gives (2). \( \square \)

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