Harnack Type Inequalities and Applications for SDE Driven by Fractional Brownian Motion

Xi-Liang Fan
Department of Mathematics, Anhui Normal University, Wuhu 241003, China

Abstract. For stochastic differential equation driven by fractional Brownian motion with Hurst parameter $H > 1/2$, Harnack type inequalities are established by constructing a coupling with unbounded time-dependent drift. These inequalities are applied to the study of existence and uniqueness of invariant measure for a discrete Markov semigroup constructed in terms of the distribution of the solution. Furthermore, we show that entropy-cost inequality holds for the invariant measure.

Mathematics Subject Classifications (2000): Primary 60H15

Key words and phrases: Fractional Brownian motion, Harnack inequality, coupling.

1 Introduction

The dimensional-free Harnack inequality with powers introduced in [27] and the log-Harnack inequality introduced in [25, 30] have been intensively investigated in the context of Markov processes, see, for example, [12, 14, 20, 21, 29, 31, 33, 36] and references within. Harnack type inequalities have become a useful tool in stochastic analysis. One can see, for instance, [23, 24, 28] for strong Feller property and contractivity properties; [1, 2, 13] for short times behaviors of infinite dimensional diffusions; [5, 11] for heat kernel estimates, entropy-cost inequalities and transportation cost inequalities.

In this note, we are concerned with stochastic differential equations (SDEs for short) driven by fractional Brownian motion, whose noise is not Markovian and even more not semimartingale. Based on the theory of rough path analysis introduced in [15], Coutin and Qian [6] presented an existence and uniqueness result with Hurst parameter $H \in (1/4, 1/2)$. Following the approach of [35], Nualart and Răşcanu [19] derived the existence and uniqueness result with $H > 1/2$. In the previous papers [9] and [8, 10], by using the method of derivative formulae we have established Harnack type inequalities for SDEs with fractional noises for $H < 1/2$ and $H > 1/2$, respectively. Motivated by the work [31], where a new technique is applied to construct the coupling for a diffusion process with multiplicative noise, we will establish directly Harnack type

1Supported by the Research project of Natural Science Foundation of Anhui Provincial Universities (Grant No. KJ2013A134).
inequalities for SDEs driven by fractional motion with Hurst parameter \( H > 1/2 \). That is the main purpose of this paper.

This paper is organized as follows. In the next section, we recall some basic results about fractional integrals and derivatives and fractional Brownian motion. In section 3, by means of the coupling and Girsanov transformation argument, the dimension-free Harnack type inequalities and the strong Feller property are shown for SDEs driven by fractional Brownian motion with \( H > 1/2 \). In terms of the distribution of the solution, we construct a discrete Markov semigroup. As applications of the inequalities, the existence and uniqueness of invariant probability measure for the corresponding semigroup is proved and its entropy-cost inequality is established.

2 Preliminaries

2.1 Fractional integrals and derivatives

Let \( a, b \in \mathbb{R} \) with \( a < b \). For \( f \in L^1(a,b) \) and \( \alpha > 0 \), the left- and right-sided fractional Riemann-Liouville integral of \( f \) of order \( \alpha \) on \([a,b]\) is given by

\[
I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} \, dy
\]

and

\[
I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} \, dy,
\]

where \( x \in (a,b) \) a.e., \((-1)^{-\alpha} = e^{-i\alpha\pi}, \Gamma \) denotes the Euler function. They extend the usual \( n \)-order iterated integrals of \( f \) for \( \alpha = n \in \mathbb{N} \).

Let \( I_{a+}^\alpha(L^p) \) (resp. \( I_{b-}^\alpha(L^p) \)) be the image of \( L^p(a,b) \) by the operator \( I_{a+}^\alpha \) (resp. \( I_{b-}^\alpha \)). If \( f \in I_{a+}^\alpha(L^p) \) (resp. \( I_{b-}^\alpha(L^p) \)) and \( 0 < \alpha < 1 \), the function \( \phi \) satisfying \( f = I_{a+}^\alpha \phi \) (resp. \( f = I_{b-}^\alpha \phi \)) is unique in \( L^p(a,b) \) and it coincides with the left-sided (resp. right-sided) Riemann-Liouville derivative of \( f \) of order \( \alpha \) defined by

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} \, dy \quad \text{resp.} \quad D_{b-}^\alpha f(x) = \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^\alpha} \, dy.
\]

The corresponding Weyl representation reads as follow

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} \, dy \right)
\]

\[
\text{resp.} \quad D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} \, dy \right),
\]

where the convergence of the integrals at the singularity \( y = x \) holds pointwise for almost all \( x \) if \( p = 1 \) and in \( L^p \)-sense if \( 1 < p < \infty \).

By definition, we have the following inversion formulas

\[
I_{a+}^\alpha(D_{a+}^\alpha f) = f, \quad \forall f \in I_{a+}^\alpha(L^p);
\]

\[
I_{b-}^\alpha(D_{b-}^\alpha f) = f, \quad \forall f \in I_{b-}^\alpha(L^p);
\]

\[
D_{a+}^\alpha(I_{a+}^\alpha f) = f, \quad D_{b-}^\alpha(I_{b-}^\alpha f) = f, \quad \forall f \in L^1(a,b).
\]
For any $\lambda \in (0,1)$, we denote by $C^\lambda(a,b)$ the set of $\lambda$-Hölder continuous functions on $[a,b]$. Recall from [26] that we have

(i) if $\alpha < 1/p$ and $q = p/(1-\alpha p)$, then $I^\alpha_{a+}(L^p) = I^\alpha_{b-}(L^p) \subset L^q(a,b)$;

(ii) if $\alpha > 1/p$, then $I^\alpha_{a+}(L^p) \cup I^\alpha_{b-}(L^p) \subset C^{1-1/p}(a,b)$.

Suppose that $f \in C^\lambda(a,b)$ and $g \in C^\mu(a,b)$ with $\lambda + \mu > 1$. By [34], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. In [35], Zähle provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivative. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann-Stieltjes integral can be expressed as

$$\int_a^b f dg = (-1)^\alpha \int_a^b D^\alpha_{a+} f(t) D^{1-\alpha}_{b-} g_{b-}(t) dt,$$

where $g_{b-}(t) = g(t) - g(b)$.

The relation (2.1) can be regarded as fractional integration by parts formula.

## 2.2 Fractional Brownian motion

In this subsection, we will recall some results about fractional Brownian motion. The main references for all these results are [3], [4], [7] and [17].

Fix a time interval $[0,T]$. Let $H \in (0,1)$. The $d$-dimensional fractional Brownian motion with Hurst parameter $H$ on the probability space $(\Omega, \mathcal{F}, P)$ can be defined as the centered Gauss process $B^H = \{B^H_t, t \in [0,T]\}$ with covariance function $\mathbb{E}B^H_t B^H_s = R_H(t,s)\delta_{i,j}$, where

$$R_H(t,s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

In particular, if $H = 1/2$, $B^H$ is a $d$-dimensional Brownian motion. Besides, one can show that $\mathbb{E}|B^H_t - B^H_s|^p = C(p)|t-s|^pH, \forall p \geq 1.$ Consequently, $B^H$ have $(H-\epsilon)$-Hölder continuous paths for all $\epsilon > 0, \ i = 1, \cdots, d$.

For each $t \in [0,T]$, we denote by $\mathcal{F}_t$ the $\sigma$-algebra generated by the random variables $\{B^H_s : s \in [0,t]\}$ and the $\mathbb{P}$-null sets.

We denote by $\mathcal{D}$ the set of step functions defined on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{D}$ with respect to the scalar product

$$\langle (I_{[0,t_1]}, \cdots, I_{[0,t_d]}), (I_{[0,s_1]}, \cdots, I_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^d R_H(t_i, s_i).$$

The mapping $(I_{[0,t_1]}, \cdots, I_{[0,t_d]}) \mapsto \sum_{i=1}^d B^H_{t_i,j}$ can be extended to an isometry between $\mathcal{H}$ and the Gauss space $\mathcal{H}_1$ associated with $B^H$. We denote the isometry between $\mathcal{H}$ and $\mathcal{H}_1$ by $\phi \mapsto B^H(\phi)$. On the other hand, the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr,$$

where $K_H$ is a square integrable kernel given by

$$K_H(t,s) = \Gamma \left( H + \frac{1}{2} \right)^{-1} (t-s)^{H-\frac{1}{2}} I (H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}).$$
in which \( F(\cdot, \cdot, \cdot) \) is the Gauss hypergeometric function (for details see [16]).

Now, we define the linear operator \( K^*_H : \mathcal{E} \rightarrow L^2([0, T], \mathbb{R}^d) \) by

\[
(K^*_H \phi)(s) = K_H(T, s) \phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K_H}{\partial t}(r, s) dr.
\]

It can be shown (see [3]) that, for all \( \phi, \psi \in \mathcal{E} \),

\[
(K^*_H \phi, K^*_H \psi)_{L^2([0, T], \mathbb{R}^d)} = \langle \phi, \psi \rangle_H,
\]

and therefore \( K^*_H \) is an isometry between \( H \) and \( L^2([0, T], \mathbb{R}^d) \). Consequently, the fractional Brownian motion \( B^H \) has the following integral representation

\[
B^H_t = \int_0^t K_H(t, s) dW_s,
\]

where \( \{W_t = B^H((K^*_H)^{-1}1_{[0, t]}), t \in [0, T]\} \) is a Wiener process.

Consider the operator \( K_H : L^2([0, T], \mathbb{R}^d) \rightarrow I^{H+1/2}_0(L^2([0, T], \mathbb{R}^d)) \) associated with the integrable kernel \( K_H(\cdot, \cdot) \)

\[
(K_H f^i)(t) = \int_0^t K_H(t, s) f^i(s) ds, \ i = 1, \ldots, d.
\]

It can be proved (see [7]) that \( K_H \) is an isomorphism and moreover, for each \( f \in L^2([0, T], \mathbb{R}^d), \)

\[
(K_H f)(s) = \begin{cases} 
I^H_0 s^{1/2-H} f^{1/2-H} H^{-1/2} f, & H \leq 1/2, \\
I^H_0 s^{-1/2-H} H^{-1/2} f, & H \geq 1/2. 
\end{cases}
\]

Consequently, for each \( h \in I^{H+1/2}_0(L^2([0, T], \mathbb{R}^d)) \), the inverse operator \( K^{-1}_H \) is of the form

\[
(K^{-1}_H h)(s) = s^{H-1/2} D^{H-1/2}_0 + s^{1/2-H} H^{1/2} h, \ H > 1/2, \quad (2.2)
\]

\[
(K^{-1}_H h)(s) = s^{-1/2-H} D^{1/2-H}_0 + s^{H-1/2} D^{2H}_0 h, \ H < 1/2. \quad (2.3)
\]

In particular, if \( h \) is absolutely continuous, we can write for \( H < 1/2 \)

\[
(K^{-1}_H h)(s) = s^{-1/2-H} I^{1/2-H}_0 + s^{H-1/2} H^{1/2} h. \quad (2.4)
\]

In the paper, we are interested in the equation driven by fractional Brownian motion with Hurst parameter \( H > 1/2 \)

\[
dX_t = b(t, X_t) dt + \sigma(t) dB^H_t, \ X_0 = x \in \mathbb{R}^d, \ t \in [0, T], \quad (2.5)
\]

where \( b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \ \sigma : [0, T] \rightarrow \mathbb{R}^d \).

Our aim is to establish Harnack type inequalities for (2.5), and moreover present some applications. We will also need some notations. Define \( P_t f(x) = E f(X_t^x), \ t \in [0, T], \ f \in \mathcal{B}_b(\mathbb{R}^d), \) where \( X_t^x \) is the solution of (2.5) with the initial point \( x \) and \( \mathcal{B}_b(\mathbb{R}^d) \) is the set of all bounded measurable functions on \( \mathbb{R}^d \). For all \( f \in C^x(0, T) \), let \( \|f\|_\infty = \sup_{0 \leq t \leq T} |f(t)| \) and \( \|f\|_\lambda = \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t-s|^{\lambda}} \).
3 Harnack type inequalities and their applications

We begin with the assumption (H1)

(i) \( b \) is Lipschitz continuous with non-negative constant \( K \):
\[
|b(t, x) - b(t, y)| \leq K|x - y|, \; \forall t \in [0, T], \; x, y \in \mathbb{R}^d,
\]
and \( b(\cdot, x) \) is Lipschitz continuous;

(ii) \( \sigma^{-1} \) is Hölder continuous of order \( H - 1/2 < \alpha_0 \leq 1 \) with non-negative constant \( \bar{K} \):
\[
|\sigma^{-1}(t) - \sigma^{-1}(s)| \leq \bar{K}|t - s|^\alpha_0, \; \forall t, s \in [0, T],
\]
and \( \sigma \) is bounded.

It has been shown in [19] that under the above assumption, there exists a unique adapted solution to equation (2.5) whose trajectories are Hölder continuous of order \( H - \epsilon \) for any \( \epsilon > 0 \).

For this kind of equation, main result reads as follow.

**Theorem 3.1** Assume (H1). Then there exist positive constants \( C, C' \) and \( C'' \) such that

1. the log-Harnack inequality
\[
\log P_T f(y) \leq \log P_T f(x) + (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3}|x - y|^2, \; x, y \in \mathbb{R}^d
\]
holds for any positive \( f \in \mathcal{B}_b(\mathbb{R}^d) \);

2. the Harnack inequality
\[
(P_T f)^p(y) \leq P_T f^p(x) \exp \left[ \frac{p}{p-1} (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3}|x - y|^2 \right], \; x, y \in \mathbb{R}^d
\]
holds for all non-negative \( f \in \mathcal{B}_b(\mathbb{R}^d) \).

To prove the theorem, we first construct a coupling equation.

Let \( x, y \in \mathbb{R}^d \) such that \( x \neq y \). For \( \theta_0 \in (0, 2) \), let
\[
\zeta(t) = \frac{2 - \theta_0}{2K}(1 - e^{\frac{2}{3}K(t-T)}), \; t \in [0, T].
\]
Then \( \zeta \) is smooth, nonincreasing and strictly positive on \([0, T)\) satisfying
\[
3\zeta'(t) - 2K\zeta(t) + 2 = \theta_0, \; t \in [0, T].
\]

Let \( X_t \) solve the equation (2.5) and introduce the coupling equation as follows
\[
dY_t = b(t, Y_t)dt + \sigma(t)dB_t^H + \frac{X_t - Y_t}{\zeta(t)}dt, \; Y_0 = y \in \mathbb{R}^d.
\]
By [19, Theorem 2.1], (2.5) and (3.3) have a unique solution \((X_t, Y_t)\) for \(t \in [0, T]\). That is, the fact that the additional drift in (3.3) is singular at time \(T\) leads to \(Y_t\) is well solved only before time \(T\). To solve \(Y_t\) for all \(t \in [0, T]\), we need to reformulate the equation by using a new fractional Brownian motion. To this end, let

\[
\tilde{B}_t^H = B_t^H + \int_0^t \sigma^{-1}(s) \frac{X_s - Y_s}{\zeta(s)} ds
\]

As a consequence, multiplying by \(1/\vartheta\)

\[
\int_0^T \frac{X_t - Y_t}{\zeta(t)} dt
\]

By [19, Theorem 2.1], (2.5) and (3.3) have a unique solution \((X_t, Y_t)\) for \(t \in [0, T]\). That is, the fact that the additional drift in (3.3) is singular at time \(T\) leads to \(Y_t\) is well solved only before time \(T\). To solve \(Y_t\) for all \(t \in [0, T]\), we need to reformulate the equation by using a new fractional Brownian motion. To this end, let

\[
\tilde{B}_t^H = B_t^H + \int_0^t \sigma^{-1}(s) \frac{X_s - Y_s}{\zeta(s)} ds
\]

\[
= \int_0^t K_H(t, s) \left( dW_s + K_H^{-1} \left( \int_0^t \sigma^{-1}(\theta) \frac{X_\theta - Y_\theta}{\zeta(\theta)} d\theta \right) (s) ds \right)
\]

\[
= \int_0^t K_H(t, s) d\tilde{W}_s, t \in [0, T].
\]

Now, for \(t \in [0, T)\) we set

\[
R(t) = \exp \left[ - \int_0^t \left( \int_0^t \sigma^{-1}(\theta) \frac{X_\theta - Y_\theta}{\zeta(\theta)} d\theta \right) (r) dr \right] + \left( \frac{1}{2} \right) \int_0^t K_H^{-1} \left( \int_0^t \sigma^{-1}(\theta) \frac{X_\theta - Y_\theta}{\zeta(\theta)} d\theta \right) (r)^2 dr.
\]

Lemma 3.2 Assume (H1). Then, there exist positive constants \(C, C' \) and \(C''\) such that

\[
\mathbb{E}(R(t) \log R(t)) \leq (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{\vartheta} K T})^3} |x - y|^2, \quad \forall t \in [0, T).
\]

Moreover, \(R(T) := \lim_{t \to T} R(t)\) exists and \(\{R(t)\}_{t \in [0, T]}\) is a uniformly integrable martingale.

Proof. Fixed \(s_0 \in [0, T)\). Note that

\[
d(X_t - Y_t) = (b(t, X_t) - b(t, Y_t)) dt - \frac{X_t - Y_t}{\zeta(t)} dt,
\]

then we obtain

\[
d|X_t - Y_t|^2 = 2(X_t - Y_t, b(t, X_t) - b(t, Y_t)) dt - 2\frac{|X_t - Y_t|^2}{\zeta(t)} dt
\]

\[
\leq 2 \left( K - \frac{1}{\zeta(t)} \right) |X_t - Y_t|^2 dt, \quad t \leq s_0.
\]

This, together with (3.2), leads to

\[
d\frac{|X_t - Y_t|^2}{\zeta^3(t)} = \frac{d|X_t - Y_t|^2}{\zeta^3(t)} - \frac{3\zeta(t)}{\zeta^4(t)} |X_t - Y_t|^2 dt
\]

\[
\leq - \frac{3\zeta(t) - 2K\zeta(t) + 2}{\zeta^4(t)} |X_t - Y_t|^2 dt
\]

\[
= - \frac{\theta_0}{\zeta^4(t)} |X_t - Y_t|^2 dt, \quad t \leq s_0.
\]

As a consequence, multiplying by \(1/\theta_0\) and integrating from 0 to \(s_0\) yield

\[
\int_0^{s_0} \frac{|X_t - Y_t|^2}{\zeta^3(t)} dt + \frac{|X_{s_0} - Y_{s_0}|^2}{\theta_0\zeta^3(s_0)} \leq \frac{|x - y|^2}{\theta_0\zeta^3(0)}, \quad s_0 \in [0, T).
\]

(3.4)
On the other hand, it follows from (2.2) that

\[ \int_0^s |K_H^{-1} \left( \int_0^r \frac{1}{\zeta(\theta)} \sigma^{-1}(\theta) X_{\theta} - Y_{\theta} d\theta \right)(r) |^2 dr \]

\[ = \frac{1}{\Gamma(3/2 - H)^2} \int_0^s \left[ r^{1/2 - H} \sigma^{-1}(r) \frac{X_r - Y_r}{\zeta(r)} \right]^2 dr + \left( H - \frac{1}{2} \right) \int_0^r \frac{r^{1/2 - H} - \sigma^{-1}(\theta) X_{\theta} - Y_{\theta} d\theta}{(r - \theta)^{1/2 + H}} \]

\[ + \left( H - \frac{1}{2} \right) \int_0^r \sigma^{-1}(r) \frac{X_r - Y_r - \sigma^{-1}(\theta) X_{\theta} - Y_{\theta}}{(r - \theta)^{1/2 + H}} \]

\[ \leq \int_0^s |K_H^{-1} \left( \int_0^r \frac{1}{\zeta(\theta)} \sigma^{-1}(\theta) X_{\theta} - Y_{\theta} d\theta \right)(r) |^2 dr \]

Next, we are to estimate (3.5).

Note that, by (H1) and (3.4), we conclude that

\[ \int_0^s \left| r^{1/2 - H} \sigma^{-1}(r) \frac{X_r - Y_r}{\zeta(r)} \right|^2 dr \leq \frac{||\sigma^{-1}||_\infty^2 \|\zeta\|_\infty T^{2(1-H)} |x - y|^2}{(1 - H)\zeta^3(0)} \]

and

\[ \int_0^s \left| r^{1/2 - H} - \sigma^{-1}(\theta) X_{\theta} - Y_{\theta} \right|^2 \]

\[ \leq \frac{\sup_{\theta \in [0, s_0]} |X_\theta - Y_\theta|^2 \|\zeta\|_\infty T^{2(1-H)} |x - y|^2}{2(1 - H)\zeta^3(0)} \]

where we use the relation

\[ \int_0^r \frac{r^{1/2 - H} - \theta^{1/2 - H}}{(r - \theta)^{1/2 + H}} d\theta = \int_0^1 \frac{s^{1/2 - H} - 1}{(1 - s)^{1/2 + H}} ds \cdot r^{1 - 2H} = C_0 r^{1 - 2H} \]

Besides, by (2.5) and (3.3), we have

\[ \int_0^r \frac{\sigma^{-1}(r) X_r - Y_r - \sigma^{-1}(\theta) X_{\theta} - Y_{\theta}}{(r - \theta)^{1/2 + H}} \]

\[ = \frac{\sigma^{-1}(r) (X_r - Y_r)}{\zeta(r)} \int_0^r \frac{1}{(r - \theta)^{1/2 + H}} \zeta(\theta) - \zeta(r) d\theta \]

\[ + \frac{\sigma^{-1}(r) (X_r - Y_r)}{\zeta(r)} \int_0^r \frac{1}{(r - \theta)^{1/2 + H}} \zeta(\theta) - \zeta(r) d\theta \]

\[ + \frac{\sigma^{-1}(r) (X_r - Y_r)}{\zeta(r)} \int_0^r \frac{1}{(r - \theta)^{1/2 + H}} \zeta(\theta) - \zeta(r) d\theta \]

\[ + \sigma^{-1}(r) \int_0^r \frac{1}{(r - \theta)^{1/2 + H} \zeta(\theta)} \int_\theta^r |b(s, X_s) - b(s, Y_s)| ds d\theta \]
\[-\sigma^{-1}(r) \int_0^r \frac{1}{(r - \theta)\frac{1}{2} + H} \int_\theta^r \frac{X_s - Y_s}{\zeta(s)} ds d\theta =: J_1(r) + J_2(r) + J_3(r) + J_4(r).\]

Observe that, by the definition of \(\zeta\) and [3.4], we get

\[
\int_0^{s_0} |J_1(r)|^2 dr \leq \|\sigma^{-1}\|_\infty^2 \int_0^{s_0} \frac{|X_r - Y_r|^2}{\zeta^4(r)} \left( \int_0^r \frac{\zeta(\theta) - \zeta(r)}{(r - \theta)^{\frac{1}{2} + H}} d\theta \right)^2 dr
\]

\[
= \left[ \frac{(2 - \theta_0)\|\sigma^{-1}\|_\infty e^{-\frac{2K}{T}}}{2K} \right]^2 \int_0^{s_0} \frac{|X_r - Y_r|^2}{\zeta^4(r)} \left( \int_0^r \frac{e^{\frac{2K}{T} - e^{\frac{2K}{T}}}}{(r - \theta)^{\frac{1}{2} + H}} d\theta \right)^2 dr
\]

\[
= \left[ \frac{(2 - \theta_0)\|\sigma^{-1}\|_\infty e^{-\frac{2K}{T}}}{3} \right]^2 \int_0^{s_0} \frac{|X_r - Y_r|^2}{\zeta^4(r)} \left( \int_0^r \frac{2K}{s} ds \int_0^s \frac{1}{(r - \theta)^{\frac{1}{2} + H}} d\theta \right)^2 dr
\]

\[
\leq \left[ \frac{(2 - \theta_0)\|\sigma^{-1}\|_\infty}{3(H - 1/2)(3/2 - H)} \right]^2 \int_0^{s_0} \frac{|X_r - Y_r|^2}{\zeta^4(r)} \left[ \frac{1}{\theta_0 \zeta^3(0)} \right] T^{3-2H} |x - y|^2. \tag{3.8}
\]

and with the help of (H1) and [3.4], it follows

\[
\int_0^{s_0} |J_2(r)|^2 dr \leq \sup_{0 \leq \theta \leq \theta_0} \frac{|X_\theta - Y_\theta|^2}{\zeta^2(\theta)} \int_0^{s_0} \frac{\left| \int_0^r e^{-\frac{1}{2} + H} d\theta \right|^2}{\zeta^4(r)} dr
\]

\[
\leq \frac{K^2\|\zeta\|_\infty}{2(\alpha_0 - H + 1)(\alpha_0 - H + 1/2)^2 \zeta^3(0)} T^{2(\alpha_0 - H + 1)} |x - y|^2. \tag{3.9}
\]

In view of (H1), the Fubini theorem, the Cauchy-Schwarz inequality and [3.4], we get

\[
\int_0^{s_0} |J_3(r)|^2 dr \leq (K\|\sigma^{-1}\|_\infty)^2 \int_0^{s_0} \left( \int_0^r \frac{|X_s - Y_s| ds}{\zeta(\theta)(r - \theta)^{\frac{1}{2} + H}} \right)^2 dr
\]

\[
\leq \left( \frac{K\|\sigma^{-1}\|_\infty}{H - 1/2} \right)^2 \int_0^{s_0} \left( \int_0^r \frac{|X_s - Y_s|}{\zeta(s)} (r - s)^{\frac{1}{2} - H} ds \right)^2 dr
\]

\[
\leq \left( \frac{K\|\sigma^{-1}\|_\infty}{H - 1/2} \right)^2 \int_0^{s_0} \int_0^r \frac{|X_s - Y_s|^2}{\zeta^4(s)} ds \int_0^r (r - s)^{1 - 2H} ds dr
\]

\[
\leq \left( \frac{K\|\sigma^{-1}\|_\infty}{H - 1/2} \right)^2 \int_0^{s_0} \frac{1}{(1 - H)(3 - 2H)\theta_0 \zeta^3(0)} T^{3-2H} |x - y|^2. \tag{3.10}
\]

As for \(J_4(r)\), similar to \(J_3(r)\) we have

\[
\int_0^{s_0} |J_4(r)|^2 dr \leq \left( \frac{\|\sigma^{-1}\|_\infty}{H - 1/2} \right)^2 \frac{1}{2(1 - H)(3 - 2H)\theta_0 \zeta^3(0)} T^{3-2H} |x - y|^2. \tag{3.11}
\]

Substituting [3.6], [3.7], [3.8], [3.9], [3.10], [3.11] into [3.5], we conclude that

\[
\frac{1}{2} \int_0^{s_0} \left| K_H^{-1} \left( \int_0^r e^{-\frac{1}{2} + H} d\theta \right) \frac{X_\theta - Y_\theta}{\zeta(\theta)} d\theta \right|^2 dr
\]

\[
\leq \frac{3}{\Gamma(3/2 - H)^2 \zeta^3(0)} \left\{ \frac{\|\sigma^{-1}\|_\infty^2 \|\zeta\|_\infty}{2(1 - H)} + \frac{|C_0 H - 1/2)\|\sigma^{-1}\|_\infty^2 \|\zeta\|_\infty}{2(1 - H)} \right\}
\]

\[8\]
Consequently,\[
\frac{(2 - \theta_0)\|\sigma^{-1}\|_{\infty}}{3(3/2 - H)} + \frac{\|K(H - 1/2)\|_{\infty}}{2(\alpha_0 - H + 1/2)} T^{2\alpha_0} = (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3}|x - y|^2.
\]
(3.12)

Then it follows that
\[
\mathbb{E} \left[ \frac{1}{2} \int_{0}^{s_0} \left| K_H^{-1} \left( \int_{0}^{r} \sigma^{-1}(\theta) \left( X_y - Y_y \right) \left( \zeta(\theta) \right) \right) (r) \right|^2 dr \right] < \infty.
\]

Consequently, \( \{ R_t \}_{t \in [0, s_0]} \) is a martingale and \( \{ \tilde{W}_t \}_{t \in [0, s_0]} \) is a \( d \)-dimensional Brownian motion under \( R(s_0) d\mathbb{P} \). Note that by the definition of \( \tilde{W} \), we have
\[
\log R(t) = - \int_{0}^{t} \left< K_H^{-1} \left( \int_{0}^{r} \sigma^{-1}(\theta) \left( X_y - Y_y \right) \left( \zeta(\theta) \right) \right) (r), d\tilde{W}_r \right> + \frac{1}{2} \int_{0}^{t} \left| K_H^{-1} \left( \int_{0}^{r} \sigma^{-1}(\theta) \left( X_y - Y_y \right) \left( \zeta(\theta) \right) \right) (r) \right|^2 dr.
\]

Combining this with (3.12) we obtain
\[
\mathbb{E}(R(s_0) \log R(s_0)) = \mathbb{E}_{s_0} \log R(s_0) \leq (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3}|x - y|^2, \quad s_0 \in [0, T),
\]
where \( \mathbb{E}_{s_0} \) denotes the expectation under the probability \( R(s_0) d\mathbb{P} \). Hence, \( \{ R_t \}_{t \in [0, T]} \) is a uniformly integrable martingale. As a consequence, by the martingale convergence theorem, we know that \( R(T) := \lim_{t \uparrow T} R(t) \) exists and \( \{ R(t) \}_{t \in [0, T]} \) is a uniformly integrable martingale. This completes the proof. \( \square \)

Lemma 3.2 ensures that \( \{ \tilde{B}_t \}_{t \in [0, T]} \) is a \( d \)-dimensional fractional Brownian motion under the probability \( R(T) d\mathbb{P} \) by the Girsanov theorem for the fractional Brownian motion (see e.g. [7, Theorem 4.9] or [18, Theorem 2]), and together with the Fatou lemma, there holds
\[
\mathbb{E}(R(T) \log R(T)) \leq (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3}|x - y|^2.
\]
(3.13)

Rewrite (2.5) and (3.3) as follow
\[
dX_t = b(t, X_t)dt + \sigma(t)d\tilde{B}_t^H - \frac{X_t - Y_t}{\zeta(t)} dt, \quad X_0 = x,
\]
(3.14)
\[
dY_t = b(t, Y_t)dt + \sigma(t)d\tilde{B}_t^H, \quad Y_0 = y.
\]
(3.15)

As a consequence, \( Y_t \) can be solved from the equation (3.14) up to time \( T \). Note that the relation \( \int_{0}^{T} 1/\zeta(t)dt = \infty \) holds, we shall see that the coupling \( (X_t, Y_t) \) is successful up to time \( T \). Thus, \( X_T = Y_T \) holds \( R(T) d\mathbb{P} \) a.s. and for the same initial points the distribution of \( Y_T \) under \( R(T) d\mathbb{P} \) coincides with that of \( X_T \) under \( \mathbb{P} \). Therefore, we will derive the desired Harnack type inequalities.
Proof of Theorem 3.1. Letting \( s_0 \uparrow T \) in (3.4), we conclude that

\[
\int_0^T \frac{|X_t - Y_t|^2}{\zeta^4(t)} \, dt \leq \frac{|x - y|^2}{\theta_0 \zeta^2(0)}.
\]  
(3.16)

This implies that the coupling time \( \tau := \inf\{ t \in [0, T] : X_t = Y_t \} \leq T \) and so, \( X_T = Y_T \) holds, where we set \( \inf \emptyset = \infty \) by convention. Indeed, if there exists \( \omega \in \Omega \) such that \( \tau(\omega) > T \), then the continuity of the processes \( X \) and \( Y \) yields

\[
\inf_{t \in [0, T]} |X_t(\omega) - Y_t(\omega)| > 0.
\]

As a consequence, we get

\[
\int_0^T \frac{|X_t(\omega) - Y_t(\omega)|^2}{\zeta^4(t)} \, dt \geq \inf_{t \in [0, T]} |X_t(\omega) - Y_t(\omega)|^2 \int_0^T \frac{1}{\zeta^4(t)} \, dt
\]

\[
\geq \inf_{t \in [0, T]} |X_t(\omega) - Y_t(\omega)|^2 \frac{1}{T^3} \left( \int_0^T \frac{1}{\zeta(t)} \, dt \right)^4 = \infty.
\]

This contradicts with (3.16). Therefore, by the Young inequality and the Hölder inequality we obtain

\[
P_T \log f(y) = \mathbb{E}(R(T) \log f(Y_T^y)) = \mathbb{E}(R(T) \log f(X_T^x)) \leq \mathbb{E}(R(T) \log R(T)) + \log P_T f(x) \quad (3.17)
\]

\[
(P_T f)^p(y) = \mathbb{E}(R(T) f(Y_T^y))^p = \mathbb{E}(R(T) f(X_T^x))^p \leq P_T f^p(x) \left( \mathbb{E} R(T) \right)^{p-1}, \quad (3.18)
\]

where the superscripts \( x \) and \( y \) stand for the initial points of corresponding equations, respectively.

Combining (3.17) and (3.13) implies the desired log-Harnack inequality. As for the Harnack inequality, by the definition of \( R(t) \) and (3.12), we have, for \( t \in [0, T) \),

\[
\mathbb{E} R(t) \frac{x}{p} = \mathbb{E}_t R(t) \frac{x}{p}
\]

\[
= \mathbb{E}_t \exp \left[ - \frac{1}{p - 1} \int_0^t \left( \int_0^{\sigma^{-1}(\theta)} \frac{X_{\theta} - Y_{\theta}}{\zeta(\theta)} \, d\theta \right) (r) \, d\tilde{W}_r \right]
\]

\[
+ \frac{1}{2p - 1} \int_0^t \left[ \int_0^{\sigma^{-1}(\theta)} \frac{X_{\theta} - Y_{\theta}}{\zeta(\theta)} \, d\theta \right] (r) \, d\tilde{W}_r
\]

\[
= \mathbb{E}_t \exp \left[ - \frac{1}{p - 1} \int_0^t \left( \int_0^{\sigma^{-1}(\theta)} \frac{X_{\theta} - Y_{\theta}}{\zeta(\theta)} \, d\theta \right) (r) \, d\tilde{W}_r \right]
\]

\[
- \frac{1}{2(p - 1)^2} \int_0^t \left[ \int_0^{\sigma^{-1}(\theta)} \frac{X_{\theta} - Y_{\theta}}{\zeta(\theta)} \, d\theta \right] (r) \, d\tilde{W}_r
\]

\[
+ \frac{1}{2(p - 1)^2} \int_0^t \left[ \int_0^{\sigma^{-1}(\theta)} \frac{X_{\theta} - Y_{\theta}}{\zeta(\theta)} \, d\theta \right] (r) \, d\tilde{W}_r
\]

\[
\leq \exp \left[ \frac{p}{(p - 1)^2} \left( C + C'T + C''T^{2\alpha_0} \right) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2\pi}{4}Kt})^{3}} \frac{|x - y|^2}{|x - y|^2} \right].
\]

This, together with the Fatou lemma and (3.18), leads to the desired Harnack inequality. \( \square \)
Remark 3.3 By a Lamperti transform and Theorem 3.1, we can derive Harnack type inequalities for one-dimensional SDE by multiplicative noise with $H > 1/2$:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB^H_t, \quad X_0 = x \in \mathbb{R}, \quad t \in [0, T].$$

As a direct application of the Harnack type inequalities derived above, by [22, Proposition 4.1] we get the strong Feller property on $P_T$.

Corollary 3.4 Assume (H1). Then $P_T$ is strong Feller, i.e. the relation

$$\lim_{|y-x| \to 0} P_T f(y) = P_T f(x).$$

holds for each $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

To present some more applications of Theorem 3.1 let us introduce some notations and another assumption.

Observe that the solution $X$ of equation (2.5) is not a Markov process. As a consequence, $(P_T)_{T \geq 0}$ does not consist of a Markov semigroup. Thus, we construct the following semigroup in discrete time, i.e. for any Borel set $A$ in $\mathbb{R}^d$,

$$P_T(x, A) := P_T I_A(x), \quad P^n_T(x, A) := \int_{\mathbb{R}^d} P_{n-1}^T(x, dy) P_T(y, A), \quad n \geq 2.$$

In general, $(P^n_T f)(x) = \int_{\mathbb{R}^d} f(y) P^n_T(x, dy)$, $x \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Next, we shall focus on the existence and uniqueness of invariant probability measure for the discrete semigroup $(P^n_T)_{n \geq 1}$, and if so, discuss its entropy-cost inequality. To this end, we assume moreover (H2)

$$\langle x, b(t, x) \rangle \leq L|x|^2, \quad \forall t \in [0, T], x \in \mathbb{R}^d,$$

where $L \in \mathbb{R}$ satisfies $Me^{2LT} < 1$, $M$ is a positive constant given in Lemma 3.6 below.

Theorem 3.5 Assume (H1) and (H2). Then the semigroup $(P^n_T)_{n \geq 1}$ has a unique invariant measure $\mu$.

In order to verify this theorem, a preliminary estimate is necessary.

Lemma 3.6 Assume (H1) and (H2). Then there exists a positive constant $M$ (independence of $L$) such that

$$\mathbb{E}|X^x_T|^2 \leq Me^{2LT}(1 + |x|^2).$$

Proof. According to the change-of-variables formula [35 Theorem 4.3.1] and (H2), we get

$$|X^x_T|^2 = |x|^2 + 2 \int_0^T \langle X^x_t, b(t, X^x_t) \rangle dt + 2 \int_0^T \langle \sigma^*(t) X^x_t, dB^H_t \rangle$$

$$\leq |x|^2 + 2L \int_0^T |X^x_t|^2 dt + 2 \int_0^T \langle \sigma^*(t) X^x_t, dB^H_t \rangle. \quad (3.19)$$
Next, we are to estimate the term $\int_0^T \langle \sigma^x(t) X^x_t, dB_t^H \rangle$. Due to the fractional integration by parts formula (2.1), the above Riemann-Stieltjes integral can then be expressed as

$$\int_0^T \langle \sigma^x(t) X^x_t, dB_t^H \rangle = (-1)^{\alpha} \int_0^T D^\alpha_{0+} (\sigma^x(\cdot) X^x)(r) D^{1-\alpha}_{T-} B^H_{T-}(r) dr,$$

(3.20)

where $1 - H < \alpha < \alpha_0$, $D^\alpha_{0+}$ and $D^{1-\alpha}_{T-}$ are given by, respectively,

$$D^\alpha_{0+} (\sigma^x(\cdot) X^x)(r) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{\sigma^x(r) X^x_r}{r^\alpha} + \alpha \int_0^r \frac{\sigma^x(r) X^x_r - \sigma^x(s) X^x_s}{(r-s)^{1+\alpha}} ds \right),$$

(3.21)

and

$$D^{1-\alpha}_{T-} B^H_{T-}(r) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{B^H_r - B^H_T}{(T-r)^{1-\alpha}} + (1-\alpha) \int_r^T \frac{B^H_r - B^H_s}{(s-r)^{2-\alpha}} ds \right).$$

(3.22)

By (3.22), we get, for $H > \lambda > 1 - \alpha$,

$$|D^{1-\alpha}_{T-} B^H_{T-}(r)| \leq c \|B^H\|_\lambda (T-r)^{\lambda+\alpha-1},$$

(3.23)

where and in what follows, $c$ denotes a generic constant.

On the other hand, by (3.21) and noting the fact that $\sigma$ is also Hölder continuous of order $\alpha_0$, we arrive at

$$|D^\alpha_{0+} (\sigma^x(\cdot) X^x)(r)| \leq c \left( \|X^x\|_\infty r^{-\alpha} + \|X^x\|_\infty r^{\alpha_0-\alpha} + \int_0^r \frac{|X^x_r - X^x_s|}{(r-s)^{1+\alpha}} ds \right).$$

(3.24)

Observe that by the fractional integration by parts formula (2.1), (i) of (H1) and the Gronwall lemma, we conclude that

$$\|X^x\|_\infty \leq c (1 + |x| + \|B^H\|_\lambda),$$

$$|X^x_r - X^x_s| \leq c \left[ (1 + |x|)(|r-s| + \|B^H\|_\lambda (|r-s| + |r-s|^{\lambda} + |r-s|^{\lambda+\alpha_0})) \right].$$

Substituting the two previous estimates into (3.24) yields

$$|D^\alpha_{0+} (\sigma^x(\cdot) X^x)(r)|$$

$$\leq c \left[ (1 + |x|)(r^{-\alpha} + r^{1-\alpha} + r^{\alpha_0-\alpha}) + \|B^H\|_\lambda \left( r^{-\alpha} + r^{1-\alpha} + r^{\alpha_0-\alpha} + r^{\lambda-\alpha} + r^{\lambda+\alpha_0-\alpha} \right) \right].$$

(3.25)

Combining (3.20), (3.23) with (3.25), we obtain

$$\left| \int_0^T \langle \sigma^x(t) X^x_t, dB_t^H \rangle \right| \leq c (1 + |x|^2 + \|B^H\|^2_\lambda).$$

This, together with (3.19) and the Gronwall lemma, confirms the assertion. \hfill \Box

Now, we proceed with the proof of Theorem 3.5.

**Proof of Theorem 3.5** \textbf{Existence:} We will make use of Krylov-Bogoliubov’s method. Let $x_0 \in \mathbb{R}^d$ and define

$$\mu_n := \frac{\sum_{k=1}^n \delta_{x_0} D^k T}{n}, \quad n \geq 1,$$
i.e. for each $f \in \mathcal{B}_b(\mathbb{R}^d)$, $\mu_n(f) = \sum_{k=1}^{n} P_T^k f(x_0)$.

Next, we will prove the tightness of $\{\mu_n\}_{n \geq 1}$.

Firstly, based on induction argument we shall show that $\{\int_{\mathbb{R}^d} |x|^2 P_T^n(x, dx)\}_{n \geq 1}$ is bounded. When $n = 1$, it follows directly from Lemma 3.6 that

$$\int_{\mathbb{R}^d} |x|^2 P_T(x_0, dx) \leq Me^{2LT} (1 + |x_0|^2) =: a(1 + |x_0|^2).$$

Suppose that

$$\int_{\mathbb{R}^d} |x|^2 P_T^{n-1}(x_0, dx) \leq a + a^2 + \cdots + a^{n-1} + a^{n-1} |x_0|^2$$

holds, then by Lemma 3.6 again we obtain

$$\int_{\mathbb{R}^d} |x|^2 P_T^n(x_0, dx) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x|^2 P_T^{n-1}(x_0, dy) P_T(y, dx) = \int_{\mathbb{R}^d} \mathbb{E}|X_T^n|^2 P_T^{n-1}(x_0, dy)$$

$$\leq a + a \int_{\mathbb{R}^d} |y|^2 P_T^{n-1}(x_0, dy)$$

$$\leq a + a (a + a^2 + \cdots + a^{n-1} + a^{n-1} |x_0|^2)$$

$$= a + a^2 + \cdots + a^n + a^n |x_0|^2.$$

Hence, we have, for any $n \geq 1$,

$$\int_{\mathbb{R}^d} |x|^2 P_T^n(x_0, dx) \leq \frac{a}{1-a} + |x_0|^2.$$

Consequently, there holds

$$\int_{\mathbb{R}^d} |x|^2 \mu_n(dx) = \frac{1}{n} \sum_{k=1}^{n} \int_{\mathbb{R}^d} |x|^2 P_T^k(x_0, dx) \leq \frac{a}{1-a} + |x_0|^2.$$

Using the Chebyshev inequality, we have

$$\sup_{n} \mu_n(\{|x|^2 > r\}) \leq \frac{1}{r} \left( \frac{a}{1-a} + |x_0|^2 \right) \to 0, \ r \to \infty,$$

which shows the tightness of $\{\mu_n\}_{n \geq 1}$.

So, from the Prohorov theorem, there exists a probability $\mu$ and a subsequence $\mu_{n_k}$ such that $\mu_{n_k} \to \mu$ weakly as $n_k \to \infty$. To simplify notation, we will denote $\mu_n \to \mu$ weakly as $n \to \infty$.

Now, we will prove that $\mu$ is a invariant probability measure for $(P_T^n)_{n \geq 1}$.

Denote $\mathcal{C}_b(\mathbb{R}^d)$ by the set of all bounded continuous functions on $\mathbb{R}^d$.

For any $f \in \mathcal{C}_b(\mathbb{R}^d)$, it follows from Corollary 3.4 that $P_T^m f \in \mathcal{C}_b(\mathbb{R}^d)$, $\forall n \geq 1$. Furthermore, we conclude that, for all $l \in \mathbb{N}$,

$$\mu(P_T^l f) = \lim_{n \to \infty} \mu_n(P_T^l f)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_T^{k+l} f(x_0)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P_T^m f(x_0) + \lim_{n \to \infty} \frac{1}{n} \sum_{m=n+1}^{n+l} P_T^m f(x_0) - \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{l} P_T^m f(x_0)$$

13
\[ \mu(f), \]
i.e. \( \mu \) is invariant for \( (P^m_n)_{n \geq 1} \).

Uniqueness: By Theorem 3.1 and [32, Theorem 1.4.1], the proof is completed. \( \square \)

To conclude this section, we present below the entropy-cost inequality for \( \mu \).

**Corollary 3.7** Assume (H1) and (H2). Then for the above invariant measure \( \mu \), the entropy-cost inequality

\[ \mu(P^*_T f \log P^*_T f) \leq (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3} W^2_2(\mu, f_\mu) \]

holds for non-negative \( f \in \mathcal{B}(\mathbb{R}^d) \) with \( \mu(f) = 1 \), where \( P^*_T \) is the adjoint operator of \( P_T \) in \( L^2(\mu) \) and \( W^2_2 \) is the \( L^2 \)-Wasserstein distance induced by the Euclidian metric, i.e. for any two probability measures \( \mu_1, \mu_2 \) on \( \mathbb{R}^d \),

\[ W^2_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu, \mu_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy), \]

where \( \mathcal{C}(\mu_1, \mu_2) \) is the set of all couplings of \( \mu_1 \) and \( \mu_2 \).

**Proof.** Applying Theorem 3.1 to \( P^*_T f \) in place of \( f \), we have

\[ P_T(\log P^*_T f)(y) \leq \log P_T(P^*_T f)(x) + (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3} |x - y|^2, \ x, y \in \mathbb{R}^d. \tag{3.26} \]

Integrating both sides of (3.26) with respect to \( \pi \in \mathcal{C}(\mu, f_\mu) \) yields

\[ \mu(P^*_T f \log P^*_T f) \leq \mu(\log P_T(P^*_T f)) + (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy). \tag{3.27} \]

Observe that from the Jensen inequality and invariance of \( \mu \), we have

\[ \mu(\log P_T(P^*_T f)) = \log \mu(P^*_T f) = \log \mu(f^T) = 0. \]

Therefore, (3.27) becomes

\[ \mu(P^*_T f \log P^*_T f) \leq (C + C'T + C''T^{2\alpha_0}) \frac{T^{2(1-H)}}{(1 - e^{-\frac{2}{3}KT})^3} \inf_{\pi \in \mathcal{C}(\mu, f_\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy). \]

This completes the proof. \( \square \)

**Acknowledgement** The author would like to thank Professor Feng-Yu Wang for his encouragement and comments that have led to improvements of the manuscript.

**References**

[1] S. Aida and H. Kawabi, *Short time asymptotics of a certain infinite dimensional diffusion process*, Stochastics Analysis and Related Topics 48(2001), 77–124.
[2] S. Aida and T. Zhang, *On the small time asymptotics of diffusion processes on path groups*, Potential Anal. 16(2002), 67–78.

[3] E. Alòs, O. Mazet and D. Nualart, *Stochastic calculus with respect to Gaussian processes*, Ann. Probab. 29(2001), 766–801.

[4] F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer-Verlag, London, 2008.

[5] S. Bobkov, I. Gentil and M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. 80(2001), 669–696.

[6] L. Coutin and Z. Qian, *Stochastic analysis, rough path analysis and fractional Brownian motions*, Probab. Theory Related Fields 122(2002), 108–140.

[7] L. Decreusefond and A. S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Anal. 10(1998), 177–214.

[8] X. L. Fan, *Derivative formula, integration by parts formula and applications for SDEs driven by fractional Brownian motion*, [arXiv:1206.0961](https://arxiv.org/abs/1206.0961).

[9] X. L. Fan, *Harnack inequality and derivative formula for SDE driven by fractional Brownian motion*, Science in China-Mathematics 561(2013), 515–524.

[10] X. L. Fan, *Derivative formulae and Harnack inequalities for SDEs with fractional noises*, [arXiv:1308.5309](https://arxiv.org/abs/1308.5309).

[11] F. Gong and F. Y. Wang, *Heat kernel estimates with application to compactness of manifolds*, Q. J. Math. 52(2001), 171–180.

[12] A. Guillin and F. Y. Wang, *Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality*, J. Differential Equations 253(2012), 20–40.

[13] H. Kawabi, *The parabolic Harnack inequality for the time dependent Ginzburg-Landau type SPDE and its application*, Potential Anal. 22(2005), 61–84.

[14] W. Liu, *Harnack inequality and applications for stochastic evolution equations with monotone drifts*, J. Evol. Equ. 9(2009), 747–770.

[15] T. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana 14(1998), 215–310.

[16] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Boston, 1988.

[17] D. Nualart, *The Malliavin Calculus and Related Topics, Second edition*, Springer-Verlag, Berlin, 2006.

[18] D. Nualart and Y. Ouknine, *Regularization of differential equations by fractional noise*, Stochastic Process. Appl. 102(2002), 103–116.

[19] D. Nualart and A. Răscaţanu, *Differential equations driven by fractional Brownian motion*, Collect. Math. 53(2002), 55–81.
[20] S. X. Ouyang, *Harnack inequalities and applications for stochastic equations*, Ph.D. Thesis, Bielefeld University, 2009.

[21] S. X. Ouyang, M. Röckner and F. Y. Wang, *Harnack inequalities and applications for Ornstein-Uhlenbeck semigroups with jump*, Potential Anal. 36(2012), 301–315.

[22] G. Da Prato, M. Röckner and F.Y. Wang, *Singular stochastic equations on Hilberts space: Harnack inequalities for their transition semigroups*, J. Funct. Anal. 257(2009), 992–1017.

[23] M. Röckner and F. Y. Wang, *Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds*, Forum Math. 15(2003), 893–921.

[24] M. Röckner and F. Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. 203(2003), 237–261.

[25] M. Röckner and F. Y. Wang, *Log-Harnack inequality for stochastic differential equations in Hilbert spaces and its consequences*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13(2010), 27–37.

[26] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publishers, Yvendon, 1993.

[27] F. Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields 109(1997), 417–424.

[28] F. Y. Wang, *Harnack inequalities for log-Sobolev functions and estimates of log-Sobolev constants*, Ann. Probab. 27(1999), 653–663.

[29] F. Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. 35(2007), 1333–1350.

[30] F. Y. Wang, *Harnack inequalities on manifolds with boundary and applications*, J. Math. Pures Appl. 94(2010), 304–321.

[31] F.Y. Wang, *Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds*, Ann. Probab. 39(2011), 1449–1467.

[32] F. Y. Wang, *Harnack Inequalities for Stochastic Partial Differential Equations*, Springer, 2013.

[33] F. Y. Wang and C. Yuan, *Harnack inequalities for functional SDEs with multiplicative noise and applications*, Stochastic Process. Appl. 121(2011), 2692–2710.

[34] L. C. Young, *An inequality of the Hölder type connected with Stieltjes integration*, Acta Math. 67(1936), 251–282.

[35] M. Zähle, *Integration with respect to fractal functions and stochastic calculus I*, Probab. Theory Related Fields 111(1998), 333–374.

[36] X. C. Zhang, *Derivative formulas and gradient estimates for SDEs driven by α-stable processes*, Stochastic Process. Appl. 123(2013), 1213–1228.