On some weighted Hölder spaces as a possible functional framework for the thin film equation and other parabolic equations with a degeneration at the boundary of a domain.

S.P. Degtyarev

Institute for applied mathematics and mechanics of Ukrainian National academy of sciences, Donetsk, Ukraine,
State Institute for applied mathematics and mechanics, Donetsk, Ukraine,
R. Luxemburg str., 74, Donetsk, Ukraine, 83114
E-mail: degtyar@i.ua

Abstract

The present paper is devoted to studying of some weighted Hölder spaces. These spaces are designed in the way to serve as a framework for studying different statements for the thin film equations in weighted classes of smooth functions in the multidimensional setting. These spaces can serve also for considering of other equations with the degeneration on the boundary of the domain of definition.

Key words: weighted Hölder spaces, interpolation inequalities, degenerate parabolic equations
MSC: 26B35, 26D20, 35K65

1 Introduction.

The present paper is devoted to studying of some weighted Hölder spaces $C^{m+\gamma, \frac{m+\gamma}{m}}_{n, \omega\gamma}$. These spaces are designed in the way to serve as a framework for consideration of different statements for the thin film equations in weighted classes of smooth functions in the multidimensional setting. These spaces can serve also for considering of other equations with the degeneration on the boundary of the domain of definition, for example, in the spirit of [1].

The literature on the subject of the thin film equations is very numerous but almost all results with sufficient regularity are devoted to the case of one spatial
variable. As a possible target for an application of the spaces $C_{n,\omega \gamma}^{m+\gamma, \frac{m+\gamma}{m}}$ we only mention the papers [2]-[16].

The spaces $C_{n,\omega \gamma}^{m+\gamma, \frac{m+\gamma}{m}}$ arise at the considering linearised version of the thin film equations. Let us explain this on the example for the thin film equation in the case of partial wetting (see, for example, [2] for the accurate statement). Consider the thin film equation of fourth order for an unknown function $h(x, t)$ (compare [17])

$$\frac{\partial h}{\partial t} + \nabla (h^n \nabla h - \beta \nabla h) = f(x, t) \text{ in } \Omega,$$  \hspace{1cm} (1.1)

where $n > 0$ is fixed, $\Omega$ is a half space $\Omega = \{(x, t) : x = (x', x_N) \in R^N, x_N > 0, t > 0\}$. Consider also partial wetting conditions at $x_N = 0$

$$h(x', 0, t) = 0, \quad \frac{\partial h}{\partial x_N}(x', 0, t) = 1$$ \hspace{1cm} (1.2)

and an initial condition

$$h(x, 0) = w(x).$$ \hspace{1cm} (1.3)

From (1.2) it follows that we must have for $w(x)$

$$w(x', 0) = 0, \quad \frac{\partial w}{\partial x_N}(x', 0) = 1.$$ \hspace{1cm} (1.4)

Consequently, we have

$$w(x) \sim x_N, \quad x_N \rightarrow 0.$$ \hspace{1cm} (1.5)

The linearization of equation (1.1) at the initial datum $w(x)$ means that we denote in (1.1) $h = w + u$ and extract linear with respect to $u$ part (we also drop lower order terms). Formally, one can just replace $h^n$ by $w^n$ in (1.1) and replace $h$ by $u$ in other places of this equation. Taking into account (1.5) and replacing $w$ by just $x_N$, we arrive at

$$\frac{\partial u}{\partial t} + \nabla (x_N^n \nabla u - \beta \nabla u) = f(x, t) \text{ in } \Omega.$$ \hspace{1cm} (1.6)

For second order equations this procedure is described in details in, for example, [18], [19], [1], and for fourth order see [16], [2], [3] formula (13), [4] formula (7).

If we are going to consider equations (1.6) (and correspondingly (1.1)) in classes of Hölder functions we have to consider $f(x, t)$ in (1.6) from some (may be weighted) Hölder class. This leads to the consideration of $\nabla (x_N^n \nabla u)$ from the same weighted Hölder class. In our definition below this will be the class $C_{n,\omega \gamma}^{m+\gamma, \frac{m+\gamma}{m}}$. In the case of second order equations such classes were used in fact in [20]-[22], [1], where the papers [20]-[22] are based on the Carnot-Carathéodory metric and the paper [1] is based on classes $C_{n,\omega \gamma}^{m+\gamma, \frac{m+\gamma}{m}}$. Note that we consider
the framework of classes $C^{m+\gamma}_{n,\omega,\gamma}\frac{m+\gamma}{m}$ as an alternative for considering the Carnot-Carathéodory metric for studying degenerate equations in classes of smooth functions [20], [22], [16].

Note that in the case of elliptic equations more simple weighted Hölder classes with unweighted Hölder constants can be used [23], [24]. The reason is that in the elliptic case no agreement between smoothness in $x$-variables and $t$-variable is needed.

Let us turn now to exact definitions and to the main results.

Denote $H = \{x = (x', x_N) \in R^N : x_N > 0\}$, $Q = \{(x, t) : x \in H, -\infty < t < \infty\}$. And we note at once that all the reasoning and statement below are valid in evident way also for $Q^+ = \{(x, t) : x \in H, t \geq 0\}$ instead of $Q$. Let $m$ be a positive integer and let $n$ be a positive number, $n < m$. Denote

$$\omega = n/m < 1.$$ 

Let $C^{\gamma}_{\omega,\gamma}(\overline{H})$, $\gamma \in (0, 1)$, be the weighted Hölder space of continuous functions $u(x)$ with the finite norm

$$|u|^{(\gamma)}_{\omega,\gamma,\overline{H}} \equiv \|u\|_{C^{\gamma}_{\omega,\gamma}(\overline{H})} \equiv |u|^{(0)}_{\overline{H}} + \langle u \rangle^{(\gamma)}_{\omega,\gamma,\overline{H}},$$ (1.7)

where

$$|u|^{(0)}_{\overline{H}} = \max_{x \in \overline{H}}|u(x)|, \langle u \rangle^{(\gamma)}_{\omega,\gamma,\overline{H}} = \sup_{x, \overline{x} \in \overline{H}} \left( x_N^\omega \right)^{\omega / \gamma} \frac{|u(x) - u(\overline{x})|}{|x - \overline{x}|^{\gamma}}, x_N^* = \max\{x_N, \overline{x}_N\}.$$ (1.8)

Thus $\langle u \rangle^{(\gamma)}_{\omega,\gamma,\overline{H}}$ represents a weighted Hölder constant of the function $u(x)$. We suppose that

$$n < m, \quad \text{if } n \text{ is a noninteger } (1 - \omega)\gamma = \gamma \left( 1 - \frac{n}{m} \right) < \min\{n, 1 - \{n\}\},$$ (1.9)

where for a real number $a$, $\{a\}$ is the fractional part of $a$, $[a]$ is the integer part of $a$. This assumption is technical and it allows us, for example, to consider the functions $x_N^{n-j}$ as elements of $C^{\gamma}_{\omega,\gamma}(\overline{H})$ for all integer $j < n$.

**Remark 1** Note that in terms of the Carnot-Carathéodory metric seminorm (1.8) is equivalent to

$$\langle u \rangle^{(\gamma)}_{\omega,\gamma,\overline{H}} \approx \sup_{x, \overline{x} \in \overline{H}} \frac{|u(x) - u(\overline{x})|}{s(x, \overline{x})^{\gamma}},$$

where the Carnot-Carathéodory distance is defined as

$$s(x, \overline{x}) = \frac{|x - \overline{x}|}{|x - \overline{x}|^{\omega} + x_N^{\omega} + \overline{x}_N^{\omega}}.$$
In the case of $m = 2$, $n \in (0,1)$ this was proved in [1] and the general case is quite similar but one should also take into account Proposition 4 below.

In the similar way we define the Hölder seminorms with respect to each variable separately

\[ \langle u \rangle_{\alpha, \gamma, x_i, \overline{H}}^{(\gamma)} = \sup_{x, \overline{H}} (x^\gamma) \frac{|u(x) - u(\overline{x})|}{h^\gamma}, \quad x_N = \max\{x_N, \overline{x}_N\}, \quad i = 1, N, \] (1.10)

where $x = (x_1, x_2, \ldots, x_N)$, $\overline{x} = (x_1, \ldots, x_i + h, \ldots, x_N)$, $h > 0$.

In the standard way we denote by $\langle u \rangle_{x_i, \overline{H}}^{(\gamma)}$, $\langle u \rangle_{x', \overline{H}}^{(\gamma)}$ and $\langle u \rangle_{x, \overline{H}}^{(\gamma)}$ usual unweighted Hölder seminorms with respect to each variable separately, with respect to $x' = (x_1, \ldots, x_{N-1})$ or with respect to all $x$-variables.

Define a weighted Hölder space $C_{n, \omega^\gamma}(\overline{H})$ as the space of continuous functions $u(x)$ with the finite norm

\[ ||u||_{C_{n, \omega^\gamma}(\overline{H})} = \sup_{x, \overline{H}} (x^\gamma) \frac{|u(x) - u(\overline{x})|}{h^\gamma}. \] (1.11)

Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multiindex, $|\alpha| = \alpha_1 + \ldots + \alpha_N$, $D^\alpha u = D^{\alpha_1}_{x_1} \ldots D^{\alpha_N}_{x_N} u$.

Note that we do not include in the definition of the norm the term $|D^{m-n}_{x_N} u|_{\omega^\gamma, \overline{H}}^{(\gamma)}$ in the case of an integer $n$. The reason is that this term is finite only in the case of the special behaviour of $x^\gamma_N D^m_{x_N} u \to 0$ at $x_\overline{N} \to 0$. This issue will be explained below. For the spaces with the finite term $|D^{m-n}_{x_N} u|_{\omega^\gamma, \overline{H}}^{(\gamma)}$ in the case of an integer $n$ we use the notation with cap. That is the space $C_{n, \omega^\gamma}(\overline{H})$ is the space with the finite norm

\[ ||u||_{C_{n, \omega^\gamma}(\overline{H})} = \sup_{x, \overline{H}} (x^\gamma) \frac{|u(x) - u(\overline{x})|}{h^\gamma}. \] (1.12)

We will show below that the norm (1.11) is equivalent to the norm

\[ ||u||_{C_{n, \omega^\gamma}(\overline{H})} = ||u||_{C_{n, \omega^\gamma}(\overline{H})} + \sum_{i=1}^{N} \langle x^\gamma_N D^m_{x_i} u \rangle_{\omega^\gamma, x_i, \overline{H}}^{(\gamma)}. \] (1.13)

and the norm (1.12) in the case of an integer $n$ is equivalent to the norm

\[ ||u||_{C_{n, \omega^\gamma}(\overline{H})} = ||u||_{C_{n, \omega^\gamma}(\overline{H})} + \sum_{i=1}^{N} \langle x^\gamma_N D^m_{x_i} u \rangle_{\omega^\gamma, x_i, \overline{H}}^{(\gamma)}. \] (1.14)
We also consider a space \( C^{\gamma,m}_{\omega,\gamma}(Q) \) of functions \( u(x, t) \) with the finite norm

\[
|u|^{(\gamma,m)}_{\omega,\gamma} \equiv \|u\|_{C^{\gamma,m}_{\omega,\gamma}(Q)} \equiv |u|^{(0)}_{\omega,\gamma} + \langle u \rangle^{(\gamma,m)}_{\omega,\gamma},
\]

where

\[
\langle u \rangle^{(\gamma,m)}_{\omega,\gamma,x} \equiv \langle u \rangle^{(\gamma)}_{\omega,\gamma,x} + \langle u \rangle^{(m)}_{t},
\]

\[
\langle u \rangle^{(\gamma)}_{\omega,\gamma,x} \equiv \sup_{x, x' \in Q} (x^*_N)^{\omega,\gamma} |u(x, t) - u(x', t)|, \quad x^*_N = \max\{x_N, x_N^x\},
\]

and \( \langle u \rangle^{(m)}_{t} \) is the usual Hölder constant of \( u \) over \( Q \) with respect to \( t \) with the exponent \( \gamma/m \). Analogously to (1.11), (1.12) we consider the space \( \hat{C}^{m+\gamma,m}_{\omega,\gamma}(Q) \) with the finite norm

\[
|\hat{u}|^{(m+\gamma)}_{n,\omega,\gamma} \equiv \|u\|_{\hat{C}^{m+\gamma,m}_{\omega,\gamma}(Q)} =
\]

\[
= |u|^{(0)}_{Q} + \sum_{0<|\alpha|<m-n} |D^\alpha_x u|^{(\gamma)}_{Q} + \sum_{j=0}^{N} \sum_{0<|\alpha|<m-j, \alpha_N \neq m-n} |D^\alpha_x u|^{(\gamma)}_{\omega,\gamma,Q} + |D_t u|^{(\gamma)}_{\omega,\gamma,Q},
\]

and the space \( \hat{\hat{C}}^{m+\gamma,m+\gamma}_{\omega,\gamma}(Q) \) with the finite norm

\[
|\hat{\hat{u}}|^{(m+\gamma)}_{n,\omega,\gamma} \equiv \|u\|_{\hat{\hat{C}}^{m+\gamma,m+\gamma}_{\omega,\gamma}(Q)} =
\]

\[
= |u|^{(0)}_{Q} + \sum_{0<|\alpha|<m-n} |D^\alpha_x u|^{(\gamma)}_{Q} + \sum_{j=0}^{N} \sum_{0<|\alpha|<m-j} |D^\alpha_x u|^{(\gamma)}_{\omega,\gamma,Q} + |D_t u|^{(\gamma)}_{\omega,\gamma,Q},
\]

And again we will show that the norm (1.17) is equivalent to the norm

\[
|\hat{u}|^{(m+\gamma)}_{n,\omega,\gamma} = |u|^{(0)}_{Q} + \sum_{i=1}^{N} \langle x^*_N D^\alpha_x u \rangle^{(\gamma)}_{\omega,\gamma,Q} + \langle D_t u \rangle^{(\gamma/m)}_{Q},
\]

and the norm (1.18) in the case of an integer \( n \) is equivalent to the norm

\[
|\hat{\hat{u}}|^{(m+\gamma)}_{n,\omega,\gamma} = |u|^{(0)}_{Q} + \sum_{i=1}^{N} \langle x^*_N D^\alpha_x u \rangle^{(\gamma)}_{\omega,\gamma,Q} + \langle D_t u \rangle^{(\gamma/m)}_{Q}.
\]
Namely, we have the following estimate which is one of the main results of the present paper.

Recall that \( \langle D_t u \rangle_{t, \overline{Q}}^{(\gamma/m)} \) is the usual Hölder constant of \( D_t u \) over \( \overline{Q} \) with respect only to \( t \) with the exponent \( \gamma/m \) and \( \langle x_N^\alpha D_x^\alpha u \rangle_{x, \overline{Q}}^{(\gamma)} \) is the weighted Hölder constants of the "pure" derivatives \( x_N^\alpha D_x^\alpha u \) with respect only to the corresponding variables \( x_i \) with the same index \( i, i = 1, N \). That is

\[
\langle x_N^\alpha D_x^\alpha u \rangle_{x, \overline{Q}}^{(\gamma)} = \sup_{(x,t),(\overline{x},\overline{t}) \in \overline{Q}} (x_N^\alpha)^{\omega_{x,t}} \left| \frac{u(x,t) - u(\overline{x},\overline{t})}{|x - \overline{x}|^\gamma} \right|, \quad x_N^\ast = \max\{x_N, \overline{x}_N\},
\]

where \( \sup \) is taken over \( x = (x_1, \ldots, x_i, \ldots, x_N) \), \( \overline{x} = (\overline{x}_1, \ldots, \overline{x}_i, \ldots, \overline{x}_N) \).

**Theorem 2** Let \( u(x,t) \) be continuous in \( \overline{Q} \) and the right hand side in (1.21) below is finite. Then for some \( C = C(N, \gamma, m, n) \)

\[
\langle u \rangle_{n, \omega_{x,t}, \overline{Q}}^{(m+\gamma/m)} \equiv \sum_{j=0}^{j \leq n} \sum_{|\alpha| = m-j} \langle x_N^\alpha D_x^\alpha u \rangle_{x, \overline{Q}}^{(\gamma)} + \sum_{j=0}^{j \leq n} \sum_{|\alpha| = m-j} \langle x_N^\alpha D_x^\alpha u \rangle_{x, \overline{Q}}^{(\gamma)} +
\]

\[
+ \langle D_t u \rangle_{x, \overline{Q}}^{(\gamma/m)} + \sum_{j=0}^{j \leq m-n} \sum_{|\alpha| = |m-n+(1-\omega)\gamma|-j} \langle D_x^\alpha D_x^j u \rangle_{x, \overline{Q}}^{(\{m-n+(1-\omega)\gamma\})} +
\]

\[
+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha| = |m-n+\gamma|-j} \langle D_x^\alpha D_x^j u \rangle_{x, \overline{Q}}^{(\{m-n+\gamma\})} +
\]

\[
+ \sum_{j=1}^{j \leq m-n} \sum_{|\alpha| = j} \langle D_x^\alpha u \rangle_{x, \overline{Q}}^{(1-\frac{m-n+\gamma}{m})} \leq C \left( \sum_{i=1}^{N} \langle x_N^\alpha D_x^\alpha u \rangle_{x, \overline{Q}}^{(\gamma)} + \langle D_t u \rangle_{x, \overline{Q}}^{(\gamma/m)} \right), \quad (1.21)
\]

where, \( [a] \) and \( \{a\} \) are the integer and the fractional parts of a real number \( a \) correspondingly and in the left hand side of (1.21) included only those terms that are finite.

Moreover,

\[
x_N^{n-j} D_x^\alpha u(x,t) \to 0, \quad x_N \to 0, \quad 0 \leq j < n, \quad \alpha = (\alpha_1, \ldots, \alpha_N), \quad |\alpha| = m-j, \quad \alpha_N < m-j.
\]

If \( u(x) \) is continuous in \( \overline{H} \) and the right hand side in (1.23) below is finite then for some \( C = C(N, \gamma, m, n) \)
\[ \langle u \rangle_{n, \omega \gamma, \Pi}^{(m+\gamma)} \equiv \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-j} \langle x_{N}^{n-j} D_{x}^{\alpha} u \rangle_{\omega \gamma, \Pi}^{(\gamma)} + \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-n+\gamma-j} \langle D_{x}^{\alpha} D_{x}^{j} u \rangle_{\omega \gamma, x_{N}^{\gamma}, \Pi}^{(m-n+\gamma)} + \]

\[ + \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=m-n+(1-\omega)\gamma-j} \langle D_{x}^{\alpha} D_{x}^{j} u \rangle_{\omega \gamma, x_{N}^{\gamma}, \Pi}^{(m-n+(1-\omega)\gamma)} \leq C \sum_{i=1}^{N} \langle x_{N}^{n} D_{x}^{m} u \rangle_{\omega \gamma, x_{N}^{\gamma}, \Pi}. \]

(1.23)

and in the left hand side of (1.23) included only those terms that are finite. Moreover,

\[ x_{N}^{n-j} D_{x}^{\alpha} u(x) \to 0, x_{N} \to 0, \quad 0 \leq j < n, \alpha = (\alpha_{1}, \ldots, \alpha_{N}), |\alpha| = m-j, \alpha_{N} < m-j. \]

(1.24)

Note that Theorem 2 is an analog for weighted Hölder spaces of well known properties of standard Hölder spaces. We are going to use these known properties so we formulate them in the next section.

Let us stress that the assumption that the terms in the left hand side of (1.21), (1.23) are finite is essential. Consider in \{(x_{1}, x_{2}) : x_{2} \geq 0\} for \(m = 2\) the function \(u(x) = x_{1}^{2} x_{2}^{2-n}\), where \(n \in [0, 1)\). For this function the right hand side of (1.23) is zero but the Hölder seminorms of the mixed derivative \(x_{N}^{n-j} D_{x}^{2} u\) in the left hand side are infinite.

The further content of the paper is as follows. In section 2 we formulate some known results about classical Hölder spaces and prove some useful statements about weighted Hölder spaces for further using. Section 3 is devoted to the proof of Theorem 2. In section 4 we consider properties of mixed and lower order derivatives of functions from the space \(C_{n, \omega \gamma, m+\gamma}^{m+\gamma}(\Omega)\). In section 5 we study traces of functions from \(C_{n, \omega \gamma, m+\gamma}^{m+\gamma}(\Omega)\) at \(x_{N} = 0\). Section 6 contains some interpolations inequalities for functions from \(C_{n, \omega \gamma, m+\gamma}^{m+\gamma}(\Omega), C_{n, \omega \gamma}^{m+\gamma}(\Pi)\). In section 7 we consider the spaces \(C_{n, \omega \gamma, m+\gamma}^{m+\gamma}(\Omega_{T}), C_{n, \omega \gamma}^{m+\gamma}(\Omega)\) in the case of arbitrary smooth domain. At last, section 8 devoted to some properties of functions from \(C_{n, \omega \gamma, 0, m+\gamma}^{m+\gamma}(\Omega_{T}), \) where the last is the closed subspace of \(C_{n, \omega \gamma}^{m+\gamma, m+\gamma}(\Omega_{T})\) consisting of functions \(u(x, t)\) with the property \(u(x, 0) \equiv u_{t}(x, 0) \equiv 0\) in \(\Omega_{T}\).

2 Auxiliary assertions.

Let \(M\) be a positive integer. In the space \(R^{M}\) we use standard Hölder spaces
$C^\overline{l}(R^M)$, where $\overline{l} = (l_1, l_2, ..., l_M)$, $l_i$ are arbitrary positive non-integers. The norm in such spaces is defined by

$$\|u\|_{C^\overline{l}(R^M)} = |u|_{R^M} + \sum_{i=1}^{M} \langle u \rangle_{x_i,R^M}^{(l_i)},$$

(2.1)

where $[l_i]$ is the integer part of the number $l_i$, $D_{x_i}^{[l_i]}u$ is the derivative of order $[l_i]$ with respect to the variable $x_i$ of a function $u$.

**Proposition 3** Seminorm (2.2) can be equivalently defined by ([25], [26], [27])

$$\langle u \rangle_{x_i,R^M}^{(l_i)} \simeq \sup_{x \in R^M, h > 0} \frac{|\Delta_h^{k} u(x)|}{h^{l_i}}, \quad k > l_i,$$

(2.3)

where $\Delta_h^{k} u(x) = u(x_1, ..., x_i + h, ..., x_N) - u(x)$ is the difference from a function $u(x)$ with respect to the variable $x_i$ with a step $h$, $\Delta_h^{k} u(x) = \Delta_h^{k-1} u(x)$, $k > l_i$.

The same is also valid not only for the whole space $R^M$ but also for it’s subsets of the form $R^M \cap \{x_{i_j}, x_{i_2}, ..., x_{i_K} \geq 0\}$ with $K \leq M$. Note that below we prove an analogous statement for weighted spaces.

It is known that functions from the space $C^\overline{l}(R^M)$ have also mixed derivatives up to definite orders and all derivatives are Hölder continuous with respect to all variables with some exponents in accordance with ratios between the exponents $l_i$. Namely, if $\overline{k} = (k_1, ..., k_M)$ with nonnegative integers $k_i$, $k_i \leq [l_i]$, and

$$\omega = 1 - \sum_{i=1}^{N} \frac{k_i}{l_i} > 0,$$

(2.4)

then (see for example [26])

$$D_{x}^{\overline{l}}u(x) \in C^{\overline{l}}(R^M), \quad \|D_{x}^{\overline{l}}u\|_{C^{\overline{l}}(R^M)} \leq C \|u\|_{C^{\overline{l}}(R^M)},$$

(2.5)

where

$$\overline{l} = (d_1, ..., d_M), \quad d_i = \omega l_i.$$

Moreover, relation (2.5) is valid not only for $R^M$ but for any domain $\Omega \subset R^M$ with sufficiently smooth boundary and we have

$$\|D_{x}^{\overline{l}}u\|_{C^{\overline{l}}(\Omega)} \leq C \|u\|_{C^{\overline{l}}(\Omega)}.$$
For special domains of the form $\Omega_+ = R^M \cap \{x_{i_1}, x_{i_2}, ..., x_{i_K} \geq 0\}$ we have even more strong inequality just for seminorms
\[
\sum_{\vec{k}} \sum_{i=1}^M \langle D_{x_{i_1}}^k u \rangle_{x_{i_1}, \Omega_+}^{(d_i)} \leq C \sum_{i=1}^M \langle u \rangle_{x_{i_1}, \Omega_+}^{(l_i)}. \tag{2.8}
\]
Here the sum is taken over all $\vec{k}$ with the property (2.4) and $d_i$ are defined in (2.6).

The analog of this estimate for an arbitrary smooth domain $\Omega$ (including bounded domains) is
\[
\sum_{\vec{k}} \sum_{i=1}^M \langle D_{x_{i_1}}^k u \rangle_{x_{i_1}, \Omega}^{(\hat{d}_i)} \leq C \left( \sum_{i=1}^M \langle u \rangle_{x_{i_1}, \Omega}^{(d_i)} + |u|_{H^\infty} \right), \tag{2.9}
\]
with arbitrary $\hat{d}_i \leq d_i$. Note that inequalities (1.21) and (1.23) are in fact a particular cases of (2.8) for weighted spaces.

It turns out that the weighted space $C^\gamma_{\omega\gamma}(H)$ is embedded into the usual space $C^{\gamma-\omega\gamma}(\Omega)$. Namely, we have the following assertion.

**Proposition 4** Let a function $u(x) \in C^\gamma_{\omega\gamma}(H)$. Then $u(x)$ is continuous in $\Omega$ and
\[
\langle u \rangle_{x, \Omega}^{(\gamma-\omega\gamma)} \leq C \langle u \rangle_{\omega\gamma, x, \Omega}^{(\gamma)} \tag{2.10}
\]

**Proof.**

We consider the Hölder property with the exponent $\gamma-\omega\gamma$ of the function $u(x)$ with respect to the variable $x_N$ and with respect to the variables $x'$ separately.

Consider the ratio with $h > 0$
\[
A_h \equiv \frac{|u(x', x_N + h) - u(x', x_N)|}{h^{\gamma-\omega\gamma}} = h^{\omega\gamma} \frac{|u(x', x_N + h) - u(x', x_N)|}{h^{\gamma}} \leq (x_N + h)^{\omega\gamma} \frac{|u(x', x_N + h) - u(x', x_N)|}{h^{\gamma}} \leq \langle u \rangle_{\omega\gamma, x, \Omega}^{(\gamma)}.
\]
Thus it is proved that at least on open set $H$
\[
\langle u \rangle_{x_N, H}^{(\gamma-\omega\gamma)} \leq \langle u \rangle_{\omega\gamma, x, \Omega}^{(\gamma)}. \tag{2.11}
\]
Let now $\vec{h} = (h_1, ..., h_{N-1})$. Consider the expression
\[
A_h \equiv \frac{|u(x' + \vec{h}, x_N) - u(x', x_N)|}{|\vec{h}|^{\gamma-\omega\gamma}}.
\]

9
If $|\overline{h}| \leq x_N/2$ we can write

$$A_h = |\overline{h}|^{-\gamma} \left| u(x' + \overline{h}, x_N) - u(x', x_N) \right| \leq C x_N^{\omega^\gamma} \left| u(x' + \overline{h}, x_N) - u(x', x_N) \right| \leq C \langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)}.$$  \hfill (2.12)

If now $|\overline{h}| > x_N/2$, then we estimate $A_h$ as

$$A_h \leq \frac{|u(x' + \overline{h}, x_N) - u(x', x_N + 2|\overline{h}|)|}{|\overline{h}|^{\gamma}} + \frac{|u(x' + \overline{h}, x_N + 2|\overline{h}|) - u(x', x_N + 2|\overline{h}|)|}{|\overline{h}|^{\gamma}} + \frac{|u(x', x_N + 2|\overline{h}|) - u(x', x_N)|}{|\overline{h}|^{\gamma}} \equiv \sum_{i=1}^{3} I_i.$$  

The estimates for $I_1$ and $I_3$ follow from (2.11) and the estimate for $I_2$ follows from (2.12) because in this case $|\overline{h}| \leq (x_N + 2|\overline{h}|)/2$. Thus in this case

$$A_h \leq C (\langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)} + \langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)}) \leq C \langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)}.$$  

Consequently, it is proved that on open set $H$

$$\langle u \rangle_{x', H}^{(\gamma-\omega)} \leq C \langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)}.$$  \hfill (2.13)

From (2.11) and (2.13) it follows that

$$\langle u \rangle_{x', H}^{(\gamma-\omega)} \leq C \langle u \rangle_{\omega^{\gamma}, x', \overline{H}}^{(\gamma)}.$$  

This means that $u(x)$ has a finite limit as $x_N \to 0$ and consequently can be defined at $x_N = 0$ as a continuous function with (2.10). Thus the proposition follows. $\blacksquare$

We need also the analog of relation (1.7) for weighted seminorm.

**Proposition 5** Let $l = m + \gamma > 0$ be noninteger, $m = [l]$, $\gamma \in (0, 1)$, and let a function $u(y) \in C_{\omega^{\gamma}}([0, \infty))$, $\omega \in (0, 1)$, in the sense that

$$\langle D_y^m u \rangle_{\omega^{\gamma}, y}^{(\gamma)} = \sup_{y,h>0} (y + h)^{\omega^\gamma} \frac{D_y^m u(y + h) - D_y^m u(y)}{h^\gamma} < \infty.$$  \hfill (2.14)

Then for any integer $k > l$

$$\langle D_y^m u \rangle_{\omega^{\gamma}, y}^{(\gamma)} \leq C_k \sup_{y,h>0} y^{\omega^\gamma} \frac{\Delta_h^k u(y)}{h^l} \equiv C_k \langle \langle u \rangle_{\omega^{\gamma}, y}^{(l)} \rangle^{(k)}.$$  \hfill (2.15)

where $\Delta_h^k u(y)$ is the $k$-th difference with the step $h$, $\Delta_h^1 u(y) = \Delta_h u(y) = u(y + h) - u(y)$, $\Delta_h^2 u(y) = \Delta_h (\Delta_h^1 u(y))$. Note that the inverse inequality to (2.15) is evident because of the mean value theorem.
The idea of the proof is taken from [28] and demonstrates also the main idea of the proof of Theorem 2. Let \( \varepsilon \in (0, 1) \) be fixed and will be chosen later. To prove (2.15) we represent
\[
\langle D^m_y u \rangle_{\omega, y} \lesssim \sup_{y, h \geq \varepsilon y} (y + h)^\omega |D^m_y u(y + h) - D^m_y u(y)| \frac{h^\gamma}{h^\gamma} + \]
\[
\sup_{y, 0 < h < \varepsilon y} (y + h)^\omega |D^m_y u(y + h) - D^m_y u(y)| \frac{h^\gamma}{h^\gamma} = \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} + \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon -)} .
\]

We are going to consider the two cases for the relation between \( \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} \) and \( \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon -)} \).

Suppose first that
\[
\langle D^m_y u \rangle_{\omega, y}^{(\varepsilon -)} \leq \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} , \tag{2.17}
\]
and consequently
\[
\langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} \leq \langle D^m_y u \rangle_{\omega, y} \leq 2 \langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} . \tag{2.18}
\]

We prove that in this case
\[
\langle D^m_y u \rangle_{\omega, y}^{(\varepsilon +)} \leq C_{\varepsilon, k} \sup_{y, h > 0} y^{\omega \gamma} |\Delta^k_y u(y)| \frac{h^l}{h^l} . \tag{2.19}
\]

The proof is by contradiction. Suppose that (2.19) is not valid. Then for any positive integer \( p \) there exists a function \( u_p(y) \in C^k_{\omega, y}([0, \infty)) \) with
\[
\sup_{y, h \geq \varepsilon y} (y + h)^\omega |D^m_y u_p(y + h) - D^m_y u_p(y)| \frac{h^\gamma}{h^\gamma} \geq p \sup_{y, h > 0} y^{\omega \gamma} |\Delta^k_y u_p(y)| . \tag{2.20}
\]

Consider the functions
\[
w_p(y) = \frac{u_p(y)}{\langle D^m_y u_p \rangle_{\omega, y}^{(\varepsilon +)}} . \tag{2.21}
\]

For such functions we have by the definition and by (2.20), (2.18)
\[
\langle D^m_y w_p \rangle_{\omega, y}^{(\varepsilon +)} = 1, \quad \sup_{y, h \geq \varepsilon y} (y + h)^\omega |D^m_y w_p(y + h) - D^m_y w_p(y)| \frac{h^\gamma}{h^\gamma} \geq \frac{1}{2} , \tag{2.22}
\]
\[
\sup_{y, h > 0} y^{\omega \gamma} |\Delta^k_y w_p(y)| \frac{h^l}{h^l} \leq \frac{1}{p} . \tag{2.23}
\]
It follows from the second relation in (2.22) that there exist sequences \( \{y_p\} \subset [0, \infty) \) and \( \{h_p\} \subset (0, \infty) \) with
\[
(y_p + h_p)^{\omega \gamma} \frac{|D^m w_p(y_p + h_p) - D^m w_p(y_p)|}{h_p} \geq \frac{1}{4},
\]
(2.24)

Now we apply the scaling arguments. Define the sequence of scaled functions
\[
\{v_p(z)\}, \quad z \in [0, \infty),
\]
(2.25)

It follows from this definition and from (2.22) - (2.24) that
\[
\langle D^m z v_p \rangle_{\omega \gamma, z} = 1, \quad \sup_{z, h > 0} z^{\omega \gamma} \frac{|\Delta^k h v_p(z)|}{h^l} \leq \frac{1}{p},
\]
(2.26)
\[
(z_n + 1)^{\omega \gamma} |D^m z v_p(z_p + 1) - D^m v_p(z_p)| \geq \frac{1}{4},
\]
(2.27)

where \( z_p = y_p/h_p \). Let now \( P_m^{(p)}(z) \) be the Taylor polynomial of the degree \( m \) for the function \( v_p(z) \) at the point, for example, \( z = 1 \). Since \( D^m P_m^{(p)}(z) = const \) and \( k > m \) in (2.26), we have for the functions \( r_p(z) = v_p(z) - P_m^{(p)}(z) \)
\[
\langle D^m z r_p \rangle_{\omega \gamma, z} = 1, \quad \sup_{z, h > 0} z^{\omega \gamma} \frac{|\Delta^k h r_p(z)|}{h^l} \leq \frac{1}{p},
\]
(2.28)
\[
(z_p + 1)^{\omega \gamma} |D^m z r_p(z_p + 1) - D^m r_p(z_p)| \geq \frac{1}{4},
\]
(2.29)

From Proposition 4, the first relation in (2.26), and from the fact that \( D^i r_p(1) = 0, \ i = 0, m \) it follows that
\[
\|r_p\|_{C^{m+(1-\omega)\gamma}(K)} \leq C(K) = CR^m,
\]
(2.30)
where \( K \) is a compact set in \([0, \infty)\), \( K \subseteq [0, R] \), \( R > 0 \). From this and the Arzela theorem we conclude that (at least for a subsequence) \( D^i r_p(z), \ i = 0, m \), uniformly converge on compact sets \( K \) to some function \( r(z) \) and it’s derivatives
\[
D^i r_p(z) \to_K D^i r(z), \quad i = 0, m.
\]
(2.31)

This, together with the first relation in (2.28), in particular, gives
\[
\langle D^m z r \rangle_{\omega \gamma, z} + \langle D^m r \rangle_{z}^{(1-\omega)\gamma} \leq 1.
\]
(2.32)

Let now \( z, h > 0 \) be fixed. From (2.28) it follows that
and letting \( p \to \infty \) we obtain \( \Delta^k_r(z) = 0 \). As \( z \) and \( h \) are arbitrary we conclude that

\[
\Delta^k_r(z) \equiv 0, \quad z, h > 0,
\]

and consequently \( r(z) \) is a polynomial of degree not greater than \( k - 1 \). Moreover \( D^m_r(z) \) is not a constant because of (2.29). Indeed, consider the sequence \( \{ z_p \} \).

Since we are considering \( A_1(\varepsilon) \) with the condition \( h \geq \varepsilon y \), we have \( 0 \leq z_p = y_p/h_p \leq 1/\varepsilon \). Therefore for a subsequence \( z_p \to z_0, n \to \infty \). Then it follows from (2.29) and (2.31) that

\[
(z_0 + 1)^{\omega \gamma} |D^m_r(z_0 + 1) - D^m_r(z_0)| \geq \frac{1}{4}
\]

that is \( D^m_r(z) \) is not a constant polynomial. But this fact contradicts to (2.32) since a non constant polynomial can not have finite seminorms as those in (2.32).

This contradiction shows that (2.19) is valid with some constant \( C_{\varepsilon,k} \) and in this case we have also (2.15) with such \( C_{\varepsilon,k} \) by virtue of (2.18).

Suppose now that \( \langle D^m_y u \rangle^{(\varepsilon+)}_{\omega \gamma,y} \leq \langle D^m_y u \rangle^{(\varepsilon-)}_{\omega \gamma,y} \). In this case we have instead of (2.18)

\[
\langle D^m_y u \rangle^{(\varepsilon-)}_{\omega \gamma,y} \leq 2 \langle D^m_y u \rangle^{(\varepsilon-)}_{\omega \gamma,y}.
\] (2.33)

We prove in this case the estimate

\[
\langle D^m_y u \rangle^{(\varepsilon-)}_{\omega \gamma,y} \leq C_{\varepsilon,k} \sup_{y,h>0} y^{\omega \gamma} \frac{|\Delta^k_h u(y)|}{h^{k}} + C_{k} \varepsilon^{1-\gamma} \langle D^m_y u \rangle^{(\gamma)}_{\omega \gamma,y},
\] (2.34)

where \( C_k \) does not depend on \( \varepsilon \in (0,1/(8k)) \). We apply some local considerations around arbitrary point in \( [0,\infty) \) . Let \( y_0 > 0 \) and \( 0 < h < \varepsilon y_0 \) be fixed. Let \( B = [y_0/4,7y_0/4] \) be a ball with center in \( y_0 \) and of radius \( 3y_0/4 \). Denote by \( \eta(y) \in C^\infty([0,\infty)) \) a smooth function with the properties

\[
\eta(y) \equiv 1, |y - y_0| \leq \frac{1}{4} y_0, \quad \eta(y) \equiv 0, |y - y_0| \geq \frac{1}{2} y_0, \quad |D^s_y \eta(y)| \leq C_s y_0^{-s}. \] (2.35)

Without loss of generality we can assume that

\[
D^i_y u(y_0) = 0, \quad i = 0,1, ..., m.
\] (2.36)

If it is not the case we can consider \( u(y) = u(y) - P^{(m)}_{y_0}(y) \) instead of \( u(y) \), where \( P^{(m)}_{y_0}(y) \) is the Taylor polynomial of \( u(y) \) of power \( m \) at the point \( y_0 \). It
is possible because $\Delta_h D^m_y u(y) \equiv \Delta_h D^m_y \overline{u}(y)$ and $\Delta^k_h u(y) \equiv \Delta^k_h \overline{u}(y)$. Denote also $v(y) = u(y)\eta(y)$. Keeping in mind the definition of $\langle D^m_y u \rangle^{(\gamma)}(\varepsilon-)$, we have by virtue of the properties of $\eta$ in (2.35) and $h < \varepsilon y_0 < y_0/4$

$$A_{\gamma}(y_0, h)(\varepsilon) \equiv (y_0 + h)\omega_{\gamma} \frac{|D^m_y u(y_0 + h) - D^m_y u(y_0)|}{h^\gamma} =$$

$$= (y_0 + h)^{\omega_{\gamma}} \frac{|D^m_y v(y_0 + h) - D^m_y v(y_0)|}{h^\gamma} \equiv (y_0 + h)^{\omega_{\gamma}} A. \quad (2.37)$$

Note that the truncated function $v(y) = u(y)\eta(y) \in C^{m+\gamma}([0, \infty))$, that is to the usual space without a weight. Thus by (2.3) we have ($l = m + \gamma$)

$$A \leq C \sup_{y, h > 0} \frac{\Delta^k_y v(y)}{h^l}. \quad (2.38)$$

The ratio in the right hand side of this inequality has the form

$$\frac{\Delta^k_y v(y)}{h^l} = \frac{\Delta^k_h (u(y)\eta(y))}{h^l} = \sum_{i=0}^{k} C_i \frac{\Delta^i_h u(y_i^{(u)}) \Delta^{k-i}_h \eta(y_i^{(\eta)})}{h^l} =$$

$$= \frac{\Delta^k_h u(y)}{h^l} \eta(y_k) + \sum_{i=0}^{k-1} C_i \frac{\Delta^i_h u(y_i^{(u)}) \Delta^{k-i}_h \eta(y_i^{(\eta)})}{h^l} = I_k + \sum_{i=0}^{k-1} I_i, \quad (2.39)$$

where $y_i^{(u)} = y + n_i h$, $y_i^{(\eta)} = y + m_i h$, and $n_i \leq k$, $m_i \leq k$, $C_i \leq C(k)$ are some integers. Evidently, by virtue of (2.35)

$$|I_k| \leq \sup_{y \in B, h > 0} \frac{|\Delta^k_h u(y)|}{h^l}. \quad (2.40)$$

Let us estimate expressions $I_i$ in (2.39). First, it follows from (2.35) and the mean value theorem that

$$|\Delta^{k-i}_h \eta(y_i^{(\eta)})| \leq C_k h^{k-i} y_0^{-(k-i)}. \quad (2.41)$$

Besides, as it follows from (2.36),

$$|D^i_y u(y)| \leq C|y - y_0|^{m+\gamma-i} \langle D^m_y u \rangle^{(\gamma)}_B, \quad y \in B, i = 0, m.$$ Since $\varepsilon < 1/(8k)$ is sufficiently small and $h < \varepsilon y_0$, it follows from the last inequality and the mean value theorem that

$$|\Delta^i_y u(y_i^{(u)})| \leq C_k \left\{ \begin{array}{ll}
 h^i y_0^{m+\gamma-i} \langle D^m_y u \rangle^{(\gamma)}_B, & i \leq m,
 h^{m+\gamma} \langle D^m_y u \rangle^{(\gamma)}_B, & m < i \leq k - 1.
\end{array} \right. \quad (2.42)$$
From (2.41) and (2.42) we have \((h < \varepsilon y_0)\)
\[
|I_i| \leq C_k h^{-1} h^{-(k-l)} y_0^{m+\gamma-i} \langle D_y^m u \rangle_B^{(\gamma)} =
\]
\[
= C_k h^{-(k-l)} y_0^{-(k-l)} \langle D_y^m u \rangle_B^{(\gamma)} \leq C_k \varepsilon^{(k-l)} \langle D_y^m u \rangle_B^{(\gamma)}, \quad i \leq m,
\]
and
\[
|I_i| \leq C_k h^{-1} h^{-(k-l)} y_0^{m+\gamma} \langle D_y^m u \rangle_B^{(\gamma)} =
\]
\[
= C_k h^{-1} h^{-(k-l)} y_0^{-(k-l)} \langle D_y^m u \rangle_B^{(\gamma)} \leq C_k \varepsilon^{(k-l)} \langle D_y^m u \rangle_B^{(\gamma)}, \quad m < i \leq k - 1.
\]
From (2.37) - (2.40), (2.43), and (2.44) it follows that the expression \(A_2^{(y_0,h)}(\varepsilon)\) in (2.37) is estimated as follows
\[
A_2^{(y_0,h)}(\varepsilon) \leq C_k (y_0 + h)^{\omega \gamma} \sup_{y \in B, h > 0} \left| \frac{\Delta_h^k u(y)}{h^l} \right| + C_k \varepsilon^{1-\gamma} (y_0 + h)^{\omega \gamma} \langle D_y^m u \rangle_B^{(\gamma)}.
\]
Since \(h \leq \varepsilon y_0\) and on the ball \(B\) we have \(y_0/4 \leq y \leq 7y_0/4\), we infer
\[
A_2^{(y_0,h)}(\varepsilon) \leq C_k \sup_{y \in B, h > 0} y^{\omega \gamma} \left| \frac{\Delta_h^k u(y)}{h^l} \right| + C_k \varepsilon^{1-\gamma} \langle D_y^m u \rangle_{\omega^{\gamma},y,B}^{(\gamma)}.
\]
As the point \(y_0\) is arbitrary, we obtain (2.34).

Combining now estimates (2.41) and (2.34), we obtain with \(\varepsilon < 1/8k\)
\[
\langle D_y^m u \rangle_{\omega^{\gamma},y} \leq C_{\varepsilon,k} \sup_{y > 0} y^{\omega \gamma} \left| \frac{\Delta_h^k u(y)}{h^l} \right| + C_k \varepsilon^{1-\gamma} \langle D_y^m u \rangle_{\omega^{\gamma},y}^{(\gamma)}.
\]
Choosing now \(\varepsilon\) in the last term sufficiently small and absorbing this term in the left hand side, we arrive at the assertion of the proposition.

As a corollary we have the following assertion.

**Proposition 6** Let a function \(u(x)\) be defined in \(\overline{H}\) and
\[
\langle u \rangle_{\omega^{\gamma},x,\overline{H}}^{(\gamma)} \equiv \sup_{x, x_N \in \overline{H}} (x_N + h_N)^{\omega \gamma} \left| \frac{u(x + \overline{h}) - u(x)}{h}\right| < \infty, \quad \gamma \in (0, 1).
\]
Then for any integer \(k \geq 1\) there is a constant \(C_k^{(i)} = C^{(i)}(k, N, \gamma, \omega), \quad i = 1, 2\) with
\[
\langle u \rangle_{\omega^{\gamma},x,\overline{H}}^{(\gamma)} \leq C_k^{(i)} \sup_{x, x_N \in \overline{H}} x_N^{\omega \gamma} \left| \frac{\Delta_h^k u(x)}{h^l} \right|^{\gamma} \equiv C_k^{(i)} \langle \langle u \rangle \rangle_{\omega^{\gamma},x,\overline{H}}^{(\gamma)(k)}
\]
and
\[ \langle u \rangle^{(\gamma)(k)}_{\omega, x, \overline{h}} \leq C_k^{(2)} \langle u \rangle^{(\gamma)}_{\omega, x, \overline{h}}. \] (2.46)

**Proof.**
We prove only (2.45) because (2.46) can be checked directly.

It is enough to verify the weighted Hölder property with respect to \( x' = (x_1, \ldots, x_{N-1}) \) and \( x_N \) separately. Let first \( x' \) be fixed and \( \overline{h} = (0, \ldots, 0, h) \), \( h > 0 \).

Then
\[ (x_N + h_N)^{\omega^\gamma} \frac{|u(x + \overline{h}) - u(x)|}{|h|^\gamma} = \]
\[ = (x_N + h_N)^{\omega^\gamma} \frac{|u(x', x_N + h) - u(x', x_N)|}{h^\gamma} \leq C_k^{(1)} \langle \langle u \rangle \rangle^{(\gamma)(k)}_{\omega, x, \overline{h}} \] (2.47)

by Proposition 5. Let now \( x_N \) be fixed and \( \overline{h} = (h_1, \ldots, h_{N-1}, 0) = (h', 0) \) with \( h_N = 0 \). Then

\[ (x_N + h_N)^{\omega^\gamma} \frac{|u(x + \overline{h}) - u(x)|}{|h|^\gamma} = x_N^{\omega^\gamma} \left( \frac{|u(x' + h', x_N) - u(x', x_N)|}{|h'|^\gamma} \right) \leq \]
\[ \leq x_N^{\omega^\gamma} \sup_{x', h'} \frac{\Delta_{h'}^k u(x', x_N)}{|h'|^\gamma} = \sup_{x', h'} x_N^{\omega^\gamma} \frac{\Delta_{h'}^k u(x', x_N)}{|h'|^\gamma} \leq C_k^{(1)} \langle \langle u \rangle \rangle^{(\gamma)(k)}_{\omega, x, \overline{h}} \] (2.48)

by (2.3). The assertion of the proposition follows now from (2.47) and (2.48).

**Corollary 7** *The seminorms*

\[ \langle u \rangle^{(\gamma)}_{\omega, \overline{h}} = \sup_{x, \overline{x} \in \overline{h}} (x^*_N)^{\omega^\gamma} \frac{|u(x) - u(\overline{x})|}{|x - \overline{x}|^\gamma}, \quad x^*_N = \max \{ x_N, \overline{x}_N \} \]

with \( x^*_N = \max \{ x_N, \overline{x}_N \} \) and

\[ \langle \widehat{u} \rangle^{(\gamma)}_{\omega, \overline{h}} = \sup_{x, \overline{x} \in \overline{h}} (\widehat{x}_N)^{\omega^\gamma} \frac{|u(x) - u(\overline{x})|}{|x - \overline{x}|^\gamma}, \quad \widehat{x}_N = \min \{ x_N, \overline{x}_N \} \]

with \( \widehat{x}_N = \min \{ x_N, \overline{x}_N \} \) are equivalent.
For the proof of this corollary it is enough to choose $k = 1$ in (2.45).

Let now we are given a function $u(x, t) \in C_{n, \omega^\gamma, m+\gamma}^{m, \frac{m+\gamma}{m}} (\mathcal{Q})$. We are going to construct an analog of Taylor polynomial for such a function at the point $O = (0, 0)$. Under this we mean some "power-like" function $Q_u(x, t)$ with the same asymptotic at $(x, t) \to (0, 0)$ as that of $u(x, t)$. The simplest situation for constructing of such a function is when $u(x, t)$ is smooth with respect to the tangent space variables $x'$ and its derivatives with respect to these variables need not weights near $\{x_N = 0\}$. We are going to achieve this situation by the smoothing process

$$u_\varepsilon(x, t) \equiv \int_{R^{N-1} - \infty}^\infty \int u(y', x_N, \tau) \omega_\varepsilon(x' - y', t - \tau) dy' d\tau,$$

where $\omega_\varepsilon(x', t) = \varepsilon^{-N} \omega(x'/\varepsilon, t/\varepsilon), \varepsilon > 0, \omega(x', t) \in C^\infty$ is a mollifier with compact support and with unit total integral. For such more smooth function we have

$$x_{N}^{n-j} D^\alpha u_\varepsilon(x, t) \to 0, x_N \to 0, \quad 0 \leq j < n, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = m-j, \alpha_N < m-j.$$  \hspace{1cm} (2.50)

This means that for the mixed derivatives $D^\alpha u_\varepsilon(x, t)$ of order $|\alpha| = m - j > \alpha_N$ (that is when a derivative $D^\alpha u_\varepsilon(x, t)$ does not coincide with the pure derivative $D^{m-j}_{x_N} u_\varepsilon(x, t)$) we have the property as in (2.39). It turns out that (2.50) is valid in fact for the function $u(x, t)$ itself without smoothing. But this will be proved later on.

**Lemma 8.** Let $u(x, t) \in C_{n, \omega^\gamma, m+\gamma}^{m, \frac{m+\gamma}{m}} (\mathcal{Q})$. Then (2.50) is valid.

**Proof.**

Show first that for a function $u(x, t) \in C_{n, \omega^\gamma, m+\gamma}^{m, \frac{m+\gamma}{m}} (\mathcal{Q})$ for a positive integer $j \leq m$

$$|D^{m-j}_{x_N} u(x, t)| \leq F(j; x_N) \equiv \left\{ \begin{array}{ll}
C x_N^{-(n-j)}, & 0 \leq j < n, \\
C (1 + |\ln x_N|), & j = n, \\
C, & n < j \leq m,
\end{array} \right.$$  \hspace{1cm} (2.51)

Really, for $j = 0$ this estimate follows directly from the definition of the space $C_{n, \omega^\gamma}^{m, m} (\mathcal{Q})$. Since the functions from $C_{n, \omega^\gamma}^{m+\gamma, \frac{m+\gamma}{m}} (\mathcal{Q})$ for $x_N > 0$, for $j = 1$ we have for $x_N \leq 2$ (if $1 < n$)

$$D^{m-1}_{x_N} u(x, t) = -\int_{x_N}^{1} \xi^{-n} [\xi^n D^m_\xi u(x', \xi, t)] d\xi + D^{m-1}_{x_N} u(x', 1, t).$$

Consequently,
\[
|D_{x_N}^{m-1}u(x,t)| \leq C_1 \int_{x_N}^{1} \xi^{-n}d\xi + C_2 \leq \begin{cases} 
Cx_N^{-(n-1)}, & n > 1, \\
C(1 + |\ln x_N|), & n = 1, = F(1;x_N), \quad x_N \leq 2, \\
C, & n < 1
\end{cases}
\]

that is (2.54) is proved for \( j = 1 \). Now (2.51) for \( j = 2 \) follows from (2.52) and so on by induction for \( j \leq m \).

Let now \( \varepsilon > 0, j < n, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = m - j, \alpha_N < m - j \). Denoting \( \alpha' = (\alpha_1, ..., \alpha_{N-1}) \), we have

\[
|D_{x_N}^{n-j}D_{x}^{\alpha}u(\varepsilon,x,t)| = \left| \int_{R^{n-1}-\infty}^{\infty} \left( D_{x_N}^{n-j}u(y',x_N,\tau) \right) D_{x'}^{\alpha}D_{x'}^{\alpha'}(x' - y', t - \tau)dy'd\tau \right| 
\leq Cx_N^{n-j}F(m - \alpha_N; x_N) = C \begin{cases} 
\int_{x_N}^{n-j}x_{n-j}^{-\gamma}(m - \alpha_N < n, \\
\int_{x_N}^{n-j}(1 + |\ln x_N|), m - \alpha_N = n, = \\
\int_{x_N}^{n-j}, m - \alpha_N > n
\end{cases}
\]

This proves the lemma. \( \blacksquare \)

**Lemma 9** Let a function \( u(x,t) \in C^{m+\gamma, m+n}_{n,\omega\gamma}(Q) \) satisfy (2.50) without smoothing that is

\[
x_N^{n-j}D_{x}^{\alpha}u(x,t) \to 0, x_N \to 0, \quad 0 \leq j < n, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = m-j, \alpha_N < m - j.
\]

Denote

\[
a = \lim_{(x,t) \to (0,0)} x_N D_{x_N}^{m}u(x,t).
\]

and denote

\[
\tilde{Q}_u(x_N) = \begin{cases} 
\frac{bax_N^{m-n}}{n}x_N, & n \text{ is a noninteger}, \\
\frac{ba\ln^{(m-n)}}{n}x_N, & n \text{ is an integer}.
\end{cases}
\]

Here

\[
\ln^{(k)}x_N \equiv \int_{0}^{x_N} d\xi_k \int_{0}^{\xi_{k-1}} d\xi_{k-1} \cdots \int_{0}^{\xi_2} \ln \xi_1 d\xi_1,
\]
is a constant which is chosen from the condition $a = x_N^m D_{x_N}^m (\tilde{Q}_u(x_N))$. Then

$$
\lim_{(x,t) \to (0,0)} x_N^{n-j} D_x^n [u(x,t) - \tilde{Q}_u(x_N)] = 0, \quad \alpha = m-j, 0 \leq j < n, \quad (2.57)
$$

$$
x_N^{n-j} D_x^n \tilde{Q}_u(x_N) \equiv \text{const}, \quad \alpha = m-j, 0 \leq j \leq n, \alpha_N < m-n. \quad (2.58)
$$

If $n$ is an integer and $\langle D_{x_N}^{m-n} u \rangle^{(\gamma, \gamma/m)}_{\omega, Q} < \infty$ is finite, then

$$
D_{x_N}^{m-n} \tilde{Q}_u(x_N) \equiv \tilde{Q}_u(x_N) \equiv 0. \quad (2.59)
$$

**Proof.**

Note first that since $m-j$ is an integer and $m-j > m-n > 0$, we have $m-j \geq 1$. Now if the derivative $D_x^n$ contains at least one differentiation in $x'$ then $D_x^n [u(x,t) - \tilde{Q}_u(x_N)] = D_x^n u(x,t)$ and we have (2.57) by (2.53). Let now $D_x^m = D_{x_N}^{m-j}$. Then by the construction

$$
D_{x_N}^m u(x,t) = \frac{a}{x_N^m} + o(x_N^{-m}) = D_{x_N}^m \tilde{Q}_u(x_N) + o(x_N^{-m}), \quad (x,t) \to (0,0). \quad (2.60)
$$

Integrating this relation with respect to $x_N \to \xi$ on the interval $[x_N, 1]$ for example, we find for $j < n$

$$
D_{x_N}^{m-j} u(x,t) = \frac{b_j a}{x_N^{m-j}} + o(x_N^{-(m-j)}) = D_{x_N}^{m-j} \tilde{Q}_u(x_N) + o(x_N^{-(m-j)}), \quad (x,t) \to (0,0), \quad (2.61)
$$

where $b_j$ are some definite constants and these constants agree with the condition $a = x_N^m D_{x_N}^m (\tilde{Q}_u(x_N))$. In particular, if $n$ is an integer

$$
D_{x_N}^{m-n} u(x,t) = b_n a \ln x_N + o(\ln x_N), \quad (x,t) \to (0,0). \quad (2.62)
$$

From (2.61) we obtain (2.57). Relations (2.58) follows directly from the definition of $\tilde{Q}_u(x_N)$ by the construction taking into account that $\tilde{Q}_u(x_N)$ depends on $x_N$ only. If now for an integer $n$ we have $\langle D_{x_N}^{m-n} u \rangle^{(\gamma, \gamma/m)}_{\omega, Q} < \infty$ then it follows from (2.62) that we must have $a = 0$ in this relation. But in this case $\tilde{Q}_u(x_N) \equiv 0$. This proves (2.59). \qed

**Lemma 10** Denote

$$
Q_u(x,t) = \tilde{Q}_u(x_N) + \sum_{|\alpha| \leq m-n} \frac{a_\alpha}{\alpha!} (x - \bar{c})^\alpha + d^{(1)} t, \quad (2.63)
$$

19
where $\tilde{Q}_u(x_N)$ is defined in (2.55), $\alpha = (\alpha_1, ..., \alpha_N)$, $\alpha! = \alpha_1!...\alpha_N!$, $\varepsilon = (0, ..., 1) \in R^N$, $(x - \varepsilon)^\alpha = x_1^{\alpha_1}...x_{N-1}^{\alpha_{N-1}}(x_N - 1)^{\alpha_N}$.

$$ a_\alpha = D^\alpha_x(u - \tilde{Q}_u(x_N))|_{x=\varepsilon,t=0}, \quad a^{(1)} = D_t(u - \tilde{Q}_u(x_N))|_{x=\varepsilon,t=0}. $$

Then the function $Q_u(x,t)$ has the following properties

$$ x_n^{-j}D^\alpha_x[u(x,t) - Q_u(x,t)]|_{(x,t)=(0,0)} = 0, \quad j < n, \quad |\alpha| = m - j, \quad (2.64) $$

$$ D^\alpha_x[u(x,t) - Q_u(x,t)]|_{(x,t)=(\varepsilon,0)} = 0, \quad |\alpha| \leq m - n, \quad D_t[u(x,t) - Q_u(x,t)]|_{(x,t)=(\varepsilon,0)} = 0. \quad (2.65) $$

$$ x_n^{-j}D^\alpha_xQ_u(x,t) \equiv \text{const}, \quad |\alpha| = m - j, 0 \leq j \leq n, \quad \alpha_N < m - n, \quad D_tQ_u(x,t) \equiv \text{const}. \quad (2.66) $$

If $n$ is an integer and $(D^{m-n}u)_{\omega \gamma Q}^{(\gamma, \gamma/m)} < \infty$ then

$$ D^{m-n}_{x_N}Q_u(x,t) \equiv \text{const}. \quad (2.67) $$

At last for $j \leq n$ and $|\alpha| = [m - n + (1 - \omega)\gamma] - j$

$$ D^\beta_xD^j_{x_N}Q_u(x,t) \quad \text{does not depend on } x' \text{ and } t. \quad (2.68) $$

The proof of this lemma follows from Lemma 9 directly by the construction of $Q_u(x,t)$ with the taking into account that for $j < n$ we have $(\beta = (\beta_1,...,\beta_N))$

$$ D^\beta_x \left( \sum_{|\alpha| \leq m - n} \frac{a_\alpha}{\alpha!} (x - \varepsilon)^\alpha + a^{(1)}t \right) \equiv 0, \quad |\beta| = m - j. $$

We prove in addition two useful lemmas about Hölder spaces. First we prove some lemma that makes the verification of the Hölder condition for functions on domains with boundaries more simple. This is done by restricting the general position of two different points of a domain to the situation when the two points are away from a boundary in some sense.

**Lemma 11** Let $\gamma \in (0,1)$ and $\omega \in [0,1)$ (the case $\omega = 0$ corresponds to the nonweighted case). Let a function $u(y)$, $y \in R^+ \equiv (0,\infty)$ satisfy the condition

$$ \sup_{0 < h \leq \varepsilon y} y^{\omega\gamma} \frac{|u(y+h) - u(y)|}{h^{\gamma}} \equiv (u)_{\omega \gamma, R^+}^{(\gamma)(\varepsilon -)} < \infty. \quad (2.69) $$

Then $u(y)$ is continuous on $[0,\infty)$ and

$$ (u)_{\omega \gamma, R^+}^{(\gamma)(\varepsilon -)} \leq C \varepsilon^{-1} (u)_{\omega \gamma, R^+}^{(\gamma)(\varepsilon -)}. \quad (2.70) $$
Proof.
Due to Corollary 7 and (2.69) it is enough to verify that
\[
\sup_{\varepsilon y \leq h} y^{\omega\gamma} \frac{|u(y + h) - u(y)|}{h^{\gamma}} \leq C_\gamma \varepsilon^{-1} \langle u \rangle_{\omega, R}^{(\gamma)(\varepsilon^{-})}.
\] (2.71)

Let \( y, h \geq \varepsilon y > 0 \) be arbitrary, \( \Delta_h u(y) \equiv u(y + h) - u(y) \). Denote ([a] is the integer part of a)
\[
M = \begin{cases} 
\left[ \log_{(1+\varepsilon)} \left( \frac{y + h}{y} \right) \right], & \text{if } \log_{(1+\varepsilon)} \left( \frac{y + h}{y} \right) \text{ is a noninteger}, \\
\log_{(1+\varepsilon)} \left( \frac{y + h}{y} \right) - 1, & \text{if } \log_{(1+\varepsilon)} \left( \frac{y + h}{y} \right) \text{ is an integer}.
\end{cases}
\]

Consider the difference
\[
u(y + h) - u(y) = \Delta_h u(y) = \sum_{i=1}^{M} (u(y_{i+1}) - u(y_i)),
\]
where
\[
y_1 = y, \quad y_i = y_{i-1} + \varepsilon y_{i-1} = (1 + \varepsilon)y_{i-1} = (1 + \varepsilon)^{i-1}y, \quad i \leq M, \quad y_{M+1} = y + h,
\]
so that \( y_{i+1} - y_i \equiv \varepsilon y_i \). We have
\[
y^{\omega\gamma} \frac{|u(y + h) - u(y)|}{h^{\gamma}} \leq \sum_{i=1}^{M} y_i^{\omega\gamma} \frac{|u(y_{i+1}) - u(y_i)|}{y_{i+1} - y_i^{\gamma}} \left( \frac{|y_{i+1} - y_i|^{\gamma}}{h^{\gamma}} \right) \leq \langle u \rangle_{\omega, R}^{(\gamma)(\varepsilon^{-})} \sum_{i=1}^{M} \left( \frac{|y_{i+1} - y_i|}{h} \right)^{\gamma} \equiv \langle u \rangle_{\omega, R}^{(\gamma)(\varepsilon^{-})} S.
\]

On the other hand
\[
S \equiv \sum_{i=1}^{M} \left( \frac{|y_{i+1} - y_i|}{h} \right)^{\gamma} \leq \sum_{i=1}^{M} \left( \frac{\varepsilon(1 + \varepsilon)^{i-1}y}{h} \right)^{\gamma} = \varepsilon^{\gamma} \left( \frac{y}{h} \right)^{\gamma} \sum_{i=1}^{M} (1 + \varepsilon)^{(i-1)} = \varepsilon^{\gamma} \left( \frac{y}{h} \right)^{\gamma} \frac{(1 + \varepsilon)^{\gamma M} - 1}{(1 + \varepsilon)^{\gamma} - 1}.
\]

But according to the definition of the number \( M \)
\[
(1 + \varepsilon)^{\gamma M} \leq \left( \frac{y + h}{y} \right)^{\gamma},
\]
so that
\[ S = \left( \frac{\varepsilon^\gamma}{(1 + \varepsilon)^\gamma - 1} \right) \left[ \left( \frac{y}{h} \right)^\gamma \left( 1 + \varepsilon \right)^{\gamma M} - 1 \right] \leq C\gamma \varepsilon^{-1+\gamma} \left( \frac{y + h}{h} \right)^\gamma \leq C\gamma \varepsilon^{-1}. \]

From this (2.71) follows for \( y > 0 \). That is on the open set \((0, \infty)\)

\[ \langle u \rangle_{(0, \infty)}^{(\gamma)} \leq C\gamma \varepsilon^{-1} \langle u \rangle_{(0, \infty)}^{(\gamma)} \langle u \rangle_{(0, \infty)}^{(\gamma)} \cdot \]

Then from Corollary 7 and the proof of Proposition 4 it follows that

\[ \langle u \rangle_{(0, \infty)}^{(\gamma - \omega^\gamma)} \leq C\gamma \varepsilon^{-1} \langle u \rangle_{(0, \infty)}^{(\gamma)} \langle u \rangle_{(0, \infty)}^{(\gamma)} \cdot \]

This means that \( u(y) \) has a finite limit as \( y \to 0 \) and consequently can be defined at \( y = 0 \) as a continuous function with (2.70). Thus the lemma follows. \( \square \)

**Corollary 12** Let \( \gamma \in (0, 1) \) and \( \omega \in [0, 1) \) (the case \( \omega = 0 \) corresponds to the nonweighted case). Let a function \( u(x), x \in H \) satisfy the condition

\[ \sup_{\overline{h} \in H, |\overline{h}| \leq \varepsilon x_N} x_N^{\omega^\gamma} \frac{|u(x + \overline{h}) - u(x)|}{|\overline{h}|^\gamma} \equiv \langle u \rangle_{\omega^\gamma, \overline{H}}^{(\gamma)} < \infty. \] (2.72)

Then \( u(x) \) is continuous on \( H \) and

\[ \langle u \rangle_{\omega^\gamma, \overline{H}}^{(\gamma)} \leq C\gamma \varepsilon^{-1-\gamma} \langle u \rangle_{\omega^\gamma, H}^{(\gamma - \omega^\gamma)}. \] (2.73)

**Proof.**

In view of Lemma 11 for arbitrary fixed \( x' \) we have

\[ \langle u(x', \cdot) \rangle_{(0, \infty)}^{(\gamma)} \leq C\gamma \varepsilon^{-1} \langle u \rangle_{\omega^\gamma, H}^{(\gamma - \omega^\gamma)}. \] (2.74)

Therefore it is enough to consider the Hölder property of \( u(x) \) with respect to the tangent variables \( x' \) only under the condition \( |\overline{h}| \geq \varepsilon x_N \). That is for a given \( \overline{h'} = (h_1, \ldots, h_{N-1}) \), and for \( x_N > 0 \) with \( |\overline{h'}| \geq \varepsilon x_N \) we must estimate the expression

\[ A(x, \overline{h'}) \equiv x_N^{\omega^\gamma} \frac{|u(x' + \overline{h'}, x_N) - u(x', x_N)|}{|\overline{h'}|^\gamma}. \]

We estimate \( A(x, \overline{h'}) \) as follows

\[ A(x, \overline{h'}) \leq x_N^{\omega^\gamma} \frac{|u(x' + \overline{h'}, x_N + \varepsilon^{-1} |\overline{h'}|) - u(x', x_N + \varepsilon^{-1} |\overline{h'}|)|}{|\overline{h'}|^\gamma} + x_N^{\omega^\gamma} \frac{|u(x' + \overline{h'}, x_N + \varepsilon^{-1} |\overline{h'}|) - u(x', x_N)|}{|\overline{h'}|^\gamma} + x_N^{\omega^\gamma} |u(x', x_N + \varepsilon^{-1} |\overline{h'}|) - u(x', x_N)|. \]
In view of (2.72) and (2.74) the proof of the corollary is completed now exactly as in Lemma 11. □

Now we prove a simple lemma about compactness and convergence in weighted Hölder spaces. This assertion is "almost well known". But the experience of the author shows that the following simple fact is not generally known: after convergence of a sequence from a Hölder space in a weaker Hölder space the limit belongs to the original space. This fact is a very useful tool in applications because smooth functions are not dense in Hölder spaces. The precise statement is as follows.

**Proposition 13** Let $\gamma \in (0, 1)$, $\omega \in [0, 1)$. Let $K \subset \overline{H}$ be a compact domain in $\overline{H}$ with smooth boundary. Let $U \subset C^{\gamma}_{\omega\gamma}(K)$ be a bounded subset in $C^{\gamma}_{\omega\gamma}(K)$ that is

$$u(x) \in U \Rightarrow \|u\|_{C^{\gamma}_{\omega\gamma}(K)} \leq M \quad (2.75)$$

for some constant $M > 0$.

Then there exists a sequence $\{u_n(x)\} \subset U$ and a function $u_0(x) \in C^{\gamma}_{\omega\gamma}(K)$ from the same space $C^{\gamma}_{\omega\gamma}(K)$ such that for any $\gamma' \in (0, \gamma)$

$$\|u_n - u_0\|_{C^{\gamma'}_{\omega\gamma}(K)} + \|u_n - u_0\|_{C^{(1-\omega)\gamma'}(K)} \to_{n \to \infty} 0, \quad \|u_0\|_{C^{\gamma}_{\omega\gamma}(K)} \leq M. \quad (2.76)$$

**Proof.**

From Proposition 4 and from (2.75) it follows that

$$u(x) \in U \Rightarrow \|u\|_{C^{(1-\omega)\gamma}(K)} \leq CM.$$  

Thus, as it is well known, there exists a sequence $\{u_n(x)\} \subset U$ and a function $u_0(x) \in \bigcap_{\gamma' \in (0, \gamma)} C^{(1-\omega)\gamma'}(K)$ with

$$\|u_n - u_0\|_{C^{(1-\omega)\gamma'}(K)} \to_{n \to \infty} 0, \quad \gamma' \in (0, \gamma). \quad (2.77)$$

Let us show that $u_0(x)$ belongs to the original space $C^{\gamma}_{\omega\gamma}(K)$ and the estimate in (2.76) is valid. Let $x \in K$ and $\overline{h} \neq 0 \in H$ be fixed and such that $x + \overline{h} \in K$. Consider the expression

$$A_n(x, \overline{h}) = x^{\omega\gamma} \frac{|u_n(x + \overline{h}) - u_n(x)|}{|\overline{h}|^\gamma} \leq M \quad (2.78)$$

and suppose that $x_N > 0$ because in the case $x_N = 0$ we have $A_n(x, \overline{h}) = 0$. From (2.77) it follows that $u_n(x) \to u_0(x)$ uniformly on $K$. Therefore letting $n \to \infty$ in (2.78), we obtain

$$A_0(x, \overline{h}) = x^{\omega\gamma} \frac{|u_0(x + \overline{h}) - u_0(x)|}{|\overline{h}|^\gamma} \leq M.$$
Since $x$ and $\overline{h}$ are arbitrary we infer from the last inequality and from (2.77)

\[ u_0(x) \in C^\gamma_{\omega\gamma}(K), \quad \|u_0\|_{C^\gamma_{\omega\gamma}(K)} \leq M. \]

Let us show now that

\[ \|u_n - u_0\|_{C^\gamma_{\omega\gamma}(K)} \to_{n\to\infty} 0, \quad \gamma' \in (0, \gamma). \tag{2.79} \]

Let $x \in \overline{K}$ and $\overline{h} \neq 0 \in H$ be such that $x + \overline{h} \in \overline{K}$ and let $\gamma' \in (0, \gamma)$. Denote $v_n(x) = u_n(x) - u_0(x)$. and consider the expression

\[ A_n(x, \overline{h}) = x_N^{\omega\gamma'} \frac{|v_n(x + \overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma'}}. \]

Let we are given an $\varepsilon > 0$. If $x_N = 0$ then $A_n(x, \overline{h}) = 0$ therefore we suppose that $x_N > 0$. Denote $R_K = \inf \{ R > 0 : K \subset \{ 0 \leq x_N \leq R \} \}$ and consider two cases. If $|\overline{h}| \leq \varepsilon x_N$ then we have

\[ A_n(x, \overline{h}) = x_N^{\omega\gamma'} \frac{|v_n(x + \overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma'}} \leq \varepsilon^{\gamma - \gamma'} R_K^{(1 - \omega)(\gamma - \gamma')} \left( x_N^{\omega\gamma'} \frac{|v_n(x + \overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma'}} \right) \leq \varepsilon^{\gamma - \gamma'} R_K^{(1 - \omega)(\gamma - \gamma')} \left( \langle u_n \rangle_{\omega\gamma', K} + \langle u_0 \rangle_{\omega\gamma', K} \right) \leq \varepsilon^{\gamma - \gamma'} 2MR_K^{(1 - \omega)(\gamma - \gamma')} \tag{2.80} \]

If now $|\overline{h}| > \varepsilon x_N$ then

\[ A_n(x, \overline{h}) = \left( \frac{x_N}{|\overline{h}|} \right)^{\omega\gamma'} \frac{|v_n(x + \overline{h}) - v_n(x)|}{|\overline{h}|^{(1 - \omega)\gamma'}} \leq \varepsilon^{\omega\gamma'} \langle u_n - u_0 \rangle_{K}^{(1 - \omega)\gamma'}. \tag{2.81} \]

Since $x$ and $\overline{h}$ are arbitrary, from (2.80) and (2.81) it follows that

\[ \langle u_n - u_0 \rangle_{\omega\gamma', K} \leq \varepsilon^{\gamma - \gamma'} C(M, K) + \varepsilon^{\omega\gamma'} \langle u_n - u_0 \rangle_{K}^{(1 - \omega)\gamma'}. \]

Taking into account (2.77), we have

\[ \|u_n - u_0\|_{C^\gamma_{\omega\gamma}(K)} \leq \varepsilon^{\gamma - \gamma'} C(M, K) + \varepsilon^{\omega\gamma'} \|u_n - u_0\|_{C^{(1 - \omega)\gamma'}(K)}. \]

From this we see that the left hand side can be made arbitrary small for large $n$ by choosing first $\varepsilon$ sufficiently small and then $n \geq N(\varepsilon)$ sufficiently large.

This completes the proof of the proposition.
3 Proof of Theorem 2

We prove only (1.21) since (1.23) is a consequence of (1.21) for functions without dependence on $t$. We use the idea of scaling arguments from [28] and the reasoning by contradiction exactly as in the proof of Proposition 5.

On the base of Proposition 6 we can turn to prof of the estimate

$$
\langle \langle u \rangle \rangle_{n, \omega \gamma, Q}^{(m+\gamma)/(2s)} = \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-j} \langle \langle x_n^{-j} D_x^\alpha u \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma)/(2s)} + \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-j} \langle \langle x_n^{-j} D_x^\alpha u \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma/m)/(2s)} +
$$

$$
+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=|m-n+(1-\omega)\gamma|-j} \langle \langle D_x^\alpha D_{x'}^j u \rangle \rangle_{\omega \gamma, x, Q}^{(m-n+(1-\omega)\gamma)/(2s)} + \langle \langle D_t u \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma)/(4)} + \langle \langle D_{t'} u \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma/m)/(4)} +
$$

$$
+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=|m-n+\gamma|-j} \langle \langle D_x^\alpha D_{x'}^j u \rangle \rangle_{\omega \gamma, x, Q}^{(m-n+\gamma)/(2s)} +
$$

$$
+ \sum_{j=1}^{j \leq m-n} \sum_{|\alpha|=j} \langle \langle D_x^\alpha u \rangle \rangle_{\omega \gamma, x, Q}^{(1-\omega) \alpha},
$$

where $s = m + 1$ and for a function $v(x, t)$ we denote $(\varepsilon, \beta \in (0, 1))$

$$
\langle \langle v \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma)(k)} = \sup_{(x, t) \in Q, \alpha \in \mathcal{P}} \frac{|\Delta^k_{h, x} v(x, t)|}{|h|^\gamma} \leq \sup_{(x, t) \in Q, \alpha \in \mathcal{P}, |\alpha| \geq \varepsilon N} \frac{|\Delta^k_{h, x} v(x, t)|}{|h|^\gamma} +
$$

$$
+ \sup_{(x, t) \in Q, \alpha \in \mathcal{P}, |\alpha| \leq \varepsilon N} \frac{|\Delta^k_{h, x} v(x, t)|}{|h|^\gamma} \equiv \langle \langle v \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma)(k)(\varepsilon, \beta)} + \langle \langle v \rangle \rangle_{\omega \gamma, x, Q}^{(\gamma)(k)(\beta)} ,
$$

$$
\langle \langle v \rangle \rangle_{x', Q}^{(\gamma)(k)} = \sup_{(x', t') \in Q, \alpha' \in \mathcal{P}, |\alpha'| \geq \varepsilon N} \frac{|\Delta^k_{h, x'} v(x', t')|}{|h|^\gamma} \leq \sup_{(x', t') \in Q, \alpha' \in \mathcal{P}, |\alpha'| \geq \varepsilon N} \frac{|\Delta^k_{h, x'} v(x', t')|}{|h|^\gamma} +
$$

$$
+ \sup_{(x', t') \in Q, \alpha' \in \mathcal{P}, |\alpha'| \leq \varepsilon N} \frac{|\Delta^k_{h, x'} v(x', t')|}{|h|^\gamma} \equiv \langle \langle v \rangle \rangle_{x', Q}^{(\gamma)(k)(\varepsilon, \beta)} + \langle \langle v \rangle \rangle_{x', Q}^{(\gamma)(k)(\beta)} ,
$$

25
\[
\langle \langle v \rangle \rangle_{t, Q}^{(\alpha) (\kappa)}(\kappa) \equiv \sup_{(x,t) \in Q, h > 0} \frac{|\Delta_{h,x}^{k} v(x,t)|}{h^\beta} \leq \sup_{(x,t) \in Q, h \leq e_N} \frac{|\Delta_{h,x}^{k} v(x,t)|}{h^\beta} + \\
+ \sup_{(x,t) \in Q, h \leq e_N} \frac{|\Delta_{h,t}^{k} v(x,t)|}{h^\beta} \equiv \langle \langle v \rangle \rangle_{t, Q}^{(\alpha) (\kappa) (+)} + \langle \langle v \rangle \rangle_{t, Q}^{(\alpha) (\kappa) (-)}, \tag{3.4}
\]

\[
\Delta_{h,x}^{k} v(x,t) = \Delta_{h,x}^{k} v(x,t) = v(x + h) - v(x), \Delta_{h,x}^{k} v(x,t) = \Delta_{h,x}^{k} v(x,t),
\]

\[
\Delta_{h,t}^{k} v(x,t) = \Delta_{h,t}^{k} v(x,t) = v(x, t + h) - v(x, t), \Delta_{h,t}^{k} v(x,t) = \Delta_{h,t}^{k} v(x,t).
\]

We first prove (3.1) under the additional restriction (2.53) on functions \( u(x,t) \), that is we suppose that

\[
x_N^{-j} D_x^a u(x,t) \rightarrow 0, x_N \rightarrow 0, 0 \leq j < n, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = m-j, \alpha_N < m-j. \tag{3.5}
\]

According to the definitions in (3.2), (3.4) we represent left hand side of (3.1) as

\[
\langle \langle u \rangle \rangle_{n, \omega_N, Q}^{(m+\gamma)(2s)} \leq \langle \langle u \rangle \rangle_{n, \omega_N, Q}^{(m+\gamma)(2s)(\pm)} + \langle \langle u \rangle \rangle_{n, \omega_N, Q}^{(m+\gamma)(2s)(\mp)}, \tag{3.6}
\]

where correspondingly

\[
\langle \langle u \rangle \rangle_{n, \omega_N, Q}^{(m+\gamma)(2s)(\pm)} (u) \equiv \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle x_N^{-j} D_x^a u \rangle \rangle_{\omega_N, x, Q}^{(\gamma/m)(2s)(\pm)} + \tag{3.7}
\]

\[
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle x_N^{-j} D_x^a u \rangle \rangle_{l, Q}^{(\gamma/m)(2s)(\pm)} + \\
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{l, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \sum_{j=1}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{l, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{\omega_N, x, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{\omega_N, x, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{\omega_N, x, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \sum_{j=0}^{j=n} \sum_{|\alpha| = m-j} \langle \langle D_x^a D_x^j \rangle \rangle_{\omega_N, x, Q}^{(m-n+(1-\omega)\gamma)(2s)(\pm)} + \\
+ \langle \langle D_t u \rangle \rangle_{\omega_N, x, Q}^{(\gamma)(4)(\pm)} + \langle \langle D_t u \rangle \rangle_{l, Q}^{(\gamma/m)(4)(\pm)}.
\]
Suppose first that
\[ \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)}, \] (3.8)
and consequently
\[ \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq 2 \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)}. \] (3.9)

Let us show that on the class of functions \( u \) with this condition
\[ \langle \langle u \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq C_\varepsilon \left( \sum_{i=1}^{N} \langle x_N^m D_{x_i}^m u \rangle_{\omega,\gamma, x_i, Q}^{(\gamma)} + \langle D_t u \rangle_{t, Q}^{(\gamma/m)} \right). \] (3.10)

The proof is by contradiction. Suppose that (3.10) is not valid. Then there exists a sequence \( \{ u_p(x, t) \} \subset C_{n,\omega,\gamma, Q}, \ p = 1, 2, \ldots, \) with the property (3.5) and with
\[ \langle \langle u_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \geq p \left( \sum_{i=1}^{N} \langle x_N^m D_{x_i}^m u_p \rangle_{\omega,\gamma, x_i, Q}^{(\gamma)} + \langle D_t u_p \rangle_{t, Q}^{(\gamma/m)} \right) \] (3.11)
and
\[ \langle \langle u_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq \langle \langle u_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq 2 \langle \langle u_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)}. \] (3.12)

Denote \( v_p(x, t) \equiv u_p(x, t)/\langle \langle u_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)}. \) For the functions \( \{ v_p \} \) we have from (3.11)
\[ 1 = \langle \langle v_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \geq p \left( \sum_{i=1}^{N} \langle x_N^m D_{x_i}^m v_p \rangle_{\omega,\gamma, x_i, Q}^{(\gamma)} + \langle D_t v_p \rangle_{t, Q}^{(\gamma/m)} \right). \]

And from the last inequality and from (3.11) we infer that
\[ \sum_{i=1}^{N} \langle x_N^m D_{x_i}^m v_p \rangle_{\omega,\gamma, x_i, Q}^{(\gamma)} + \langle D_t v_p \rangle_{t, Q}^{(\gamma/m)} \leq \frac{1}{p}, \]
\[ 1 \leq \langle \langle v_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq 2 \langle \langle v_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \leq 2. \] (3.13)

It follows from the second inequality in (3.13) that there is a term in the definition of \( \langle \langle v_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} \),
\[ \langle \langle v_p \rangle \rangle_{n,\omega,\gamma, Q}^{(m+\gamma)(2\varepsilon)} = \sum_{j=0}^{n} \sum_{|j|=m-j}^{\mathbb{Z}} \langle \langle x_N^n D_{x}^n v_p \rangle \rangle_{\omega,\gamma, x, Q}^{(2\varepsilon)} + \] (3.14)
\[
+ \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-j} \langle \langle x_N^{n-j} D_x^\alpha v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma/m)(2s)(\epsilon^+)} + \\
+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+(1-\omega)\gamma]-j} \langle \langle D_x^\alpha D_{xN}^j v_p \rangle \rangle_{x, \overline{Q}}^{(m-n+(1-\omega)\gamma)(2s)(\epsilon^+)} + \\
+ \sum_{j=1}^{j \leq m-n} \sum_{|\alpha|=j} \langle \langle D_x^\alpha v_p \rangle \rangle_{x, \overline{Q}}^{(1-\frac{1}{m-n} + \frac{2}{m})(2s)(\epsilon^+)} + \\
+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+\gamma]-j} \langle \langle D_x^\alpha D_{xN}^j v_p \rangle \rangle_{x', \overline{Q}}^{(m-n+\gamma)(2s)(\epsilon^+)} + \\
+ \sum_{j=0}^{j \leq n} \sum_{|\alpha|=m-j} \langle \langle x_N^{n-j} D_x^\alpha v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma/2s)(\epsilon^+)} + \langle \langle D_t v_p \rangle \rangle_{\omega, x', \overline{Q}}^{(\gamma)(4)(\epsilon^+)} + \langle \langle D_t v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma/m)(4)(\epsilon^+)} \geq \frac{1}{2},
\]

which is not less than some absolute constant \( \nu = \nu(m, n, N) > 0 \). This is valid at least for a subsequence of indexes \( \{p\} \). It can not be the sequence of terms \( \langle \langle D_t v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma/m)(4)(\epsilon^+)} \) because of \( (3.13) \). We suppose, for example, that for some multiindex \( \bar{\alpha}, |\bar{\alpha}| = m, \)

\[
\langle \langle x_N^m D_x^\bar{\alpha} v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma)(2s)(\epsilon^+)} \geq \nu > 0, \quad p = 1, 2, \ldots \tag{3.15}
\]

The all reasonings below are completely the same for all other terms in \( (3.14) \).

From \( (3.15) \) and from the definition of \( \langle \langle x_N^m D_x^\bar{\alpha} v_p \rangle \rangle_{\omega, x, \overline{Q}}^{(\gamma)(2s)(\epsilon^+)} \) in \( (3.12) \) it follows that there exist sequences of points \( \{(x^{(p)}, t^{(p)}) \in \overline{Q}\} \) and vectors \( \{\bar{h}^{(p)} \in \overline{I}\} \) with

\[
h_p \equiv |\bar{h}^{(p)}| \geq \varepsilon x_N^{(p)}, \quad p = 1, 2, \ldots \tag{3.16}
\]

and with

\[
\left( x_N^{(p)} \right)^{\omega, \gamma} \frac{|\Delta_{x}^{m} \left[ (x_N^{(p)})^n D_x^\bar{\alpha} v_p (x^{(p)}, t^{(p)}) \right]|}{h_p^{r}} \geq \frac{\nu}{2} > 0. \tag{3.17}
\]

We make in the functions \( \{v_p\} \) the change of the independent variables \((x, t) \rightarrow (y, \tau)\)

\[
x_i = x_i^{(p)} + y_i h_p, \quad i = 1, N-1, \quad x_N = y_N h_p; \quad t = t^{(p)} + h_p^{m-n} \tau \tag{3.18}
\]

and denote
Thus from the second inequality in (3.13) and Proposition 6 it follows that the rescaled functions $w^{(p)}(y, \tau)$ satisfy

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)} &= \sum_{j=0}^{j=m-n-\gamma} \sum_{m-j}^{j} \langle \langle y_{N}^{n-j} D_{y}^{\alpha} w_{p} \rangle \rangle_{\omega, \gamma, \tau}^{(\gamma)} + \\
&+ \sum_{j=0}^{j=m-n-\gamma} \sum_{m-j}^{j} \langle \langle y_{N}^{n-j} D_{y}^{\alpha} w_{p} \rangle \rangle_{\gamma}^{(\gamma/m)} + \\
&+ \sum_{j=1}^{j=m-n} \sum_{m-j}^{j} \langle \langle D_{y}^{\alpha} w_{p} \rangle \rangle_{\gamma}^{(1+\gamma/m)} + \\
&+ \sum_{j=0}^{j=m-n} \sum_{m-j}^{j} \langle \langle D_{y}^{\alpha} w_{p} \rangle \rangle_{\gamma}^{(1+\gamma/m)} + \\
&+ \sum_{j=0}^{j=m-n} \sum_{m-j}^{j} \langle \langle y_{N}^{n-j} D_{y}^{\alpha} w_{p} \rangle \rangle_{\tau}^{(\gamma/\alpha)} + \\
&+ \langle \langle D_{\tau} w_{p} \rangle \rangle_{\gamma}^{(4)} + \langle \langle D_{\tau} w_{p} \rangle \rangle_{\tau}^{(4)} = \langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)}.
\end{align*}

And also (see (1.21))

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} &= \langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)/\gamma}.
\end{align*}

Thus from the second inequality in (3.13) and Proposition 6 it follows that

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} &\leq C \langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} \leq 2C = C.
\end{align*}

From (3.13) and (3.21) we have

\begin{align*}
\sum_{i=1}^{N} \langle \langle y_{N}^{n} D_{y}^{\alpha} w_{p} \rangle \rangle_{\omega, \gamma, \tau}^{(\gamma)} + \langle \langle D_{\tau} w_{p} \rangle \rangle_{\tau}^{(\gamma/m)} &\leq \frac{1}{p}.
\end{align*}

And from 2.17 we obtain

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} \leq 2C = C.
\end{align*}

Thus from the second inequality in (3.13) and Proposition 6 it follows that

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} &\leq C \langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} \leq 2C = C.
\end{align*}

From (3.13) and (3.21) we have

\begin{align*}
\sum_{i=1}^{N} \langle \langle y_{N}^{n} D_{y}^{\alpha} w_{p} \rangle \rangle_{\omega, \gamma, \tau}^{(\gamma)} + \langle \langle D_{\tau} w_{p} \rangle \rangle_{\tau}^{(\gamma/m)} &\leq \frac{1}{p}.
\end{align*}

And from 2.17 we obtain

\begin{align*}
\langle \langle w_{p} \rangle \rangle_{n, \omega, \gamma, \tau}^{(m+\gamma)\gamma} \leq 2C = C.
\end{align*}
where
\[ y_N^{(p)} \equiv x_N^{(p)}/h_p, \quad \overline{y}^{(p)} \equiv \overline{x}^{(p)}/h_p, \quad |\overline{y}^{(p)}| = 1, \quad P^{(p)} \equiv (0', y_N^{(p)}). \quad (3.25) \]

Note also that the functions \( w_p(y, \tau) \) inherit property (3.5)

\[ y_n^{-j}D_x^\alpha w_p(y, \tau) \to 0, \quad y_N \to 0, \quad 0 \leq j < n, \quad \alpha = (\alpha_1, ..., \alpha_N), \quad |\alpha| = m-j, \quad \alpha_N < m-j. \quad (3.26) \]

Denote by \( Q_p(y, \tau) \equiv Q_{w_p}(y, \tau) \) the "Taylor" function \( Q_{w_p}(y, \tau) \) for the function \( w_p(y, \tau) \), which was constructed in Lemma 10 and denote \( r_p(y, \tau) \equiv w_p(y, \tau) - Q_p(y, \tau) \). From Lemma 10 it follows that

\[ y_n^{-j}D_y^\alpha r_p(y, \tau)|_{(y, \tau)=(0,0)} = 0, \quad j < n, \quad |\alpha| = m-j, \quad (3.27) \]
\[ D_y^\alpha r_p(y, \tau)|_{(y, \tau)=(\overline{y},0)} = 0, \quad |\alpha| \leq m-n, \quad D_\tau r_p(y, \tau)|_{(y, \tau)=(\overline{y},0)} = 0. \quad (3.28) \]

Recall that

\[ y_n^{-j}D_y^\alpha Q_p(y, \tau) \equiv const, \quad |\alpha| = m-j, \quad j \leq n, \quad \alpha_N < m-n, \quad D_\tau Q_p(y, \tau) \equiv const, \quad (3.29) \]

and also

\[ D_{y_N}^{m-n}Q_p(y, \tau) \equiv const \quad \text{if the seminorm} \quad \langle \langle D_{x_N}^{m-n}u \rangle \rangle_{\gamma, x_Q} < \infty \quad \text{is finite} \quad (3.30) \]

and it is included in the left hand side of (3.31) and the seminorm \( \langle \langle D_{y_N}^{m-n}w_p \rangle \rangle_{\gamma, y_Q} \) is included in (3.20). Consequently, from (3.29), (3.30) and from the definition of Hölder classes in view of (3.22) it follows that

\[ \langle r_p \rangle^{(m+\gamma, \frac{m+\gamma}{m})}_{n, \omega, \gamma, Q} = \langle w_p - Q_p(y, \tau) \rangle^{(m+\gamma, \frac{m+\gamma}{m})}_{n, \omega, \gamma, Q} \leq C. \quad (3.31) \]

For the same reason we have from (3.23)

\[ \sum_{i=1}^N \langle y_{N i}^{(p)} D_{y_{N i}}^{m, \gamma} r_p \rangle_{\omega, \gamma, i, Q} + \langle D_\tau r_p \rangle_{\gamma, \tau, Q} \leq \frac{1}{p} \quad (3.32) \]

and from (3.24)

\[ \left( y_{N}^{(p)} \right)^{\omega, \gamma} |\Delta_{y_{N}}^{2s} (y_{N}^{(p)}) \rangle^{\gamma} D_{y}^{\alpha} r_{p}(P^{(p)}, 0) | \geq \nu. \quad (3.33) \]
From (3.26), (3.27), (3.28), and (3.31) it follows that the sequence of functions \( \{r_p(y, \tau)\} \) is bounded in \( C^{m+\gamma', \frac{m+\gamma'}{m}}(K_\delta) \) for any compact set \( K_\delta \subset Q \cap \{\delta \leq x_N \leq \delta^{-1}\} \), \( \delta \in (0, 1) \). Therefore there exists a function \( r(y, \tau) \in C^{m+\gamma', \frac{m+\gamma'}{m}}(Q \cap \{x_N > 0\}) \) with (at least for a subsequence)

\[
r_p \to r \text{ in } C^{m+\gamma', \frac{m+\gamma'}{m}}(K_\delta), \quad p \to \infty, \quad \forall K_\delta \subset Q \cap \{\delta \leq y_N \leq \delta^{-1}\}, \quad \gamma' < \gamma. \tag{3.34}
\]

At the same time, since the sequences \( \{y_N^{(p)}\} \), \( \{\tau^{(p)}\} \), and \( \{P^{(p)}\} \) are bounded (recall that \( y_N^{(p)} = x_N^{(p)}/h_p \leq \varepsilon^{-1} \) since \( h_p \geq \varepsilon x_N^{(p)} \))

\[
y_N^{(p)} \to y_N^{(0)}, \quad \tau^{(p)} \to \tau^{(0)}, \quad P^{(p)} \to P^{(0)}, \quad p \to \infty, \tag{3.35}
\]

where \( y_N^{(0)} \) is a nonnegative number, \( \tau^{(0)} \in \mathbf{H} \) is a unit vector, \( P^{(0)} = (0', y_N^{(0)}) \in \mathbf{H} \). From (3.27) and (3.31) (together with (2.10) and the Arzela theorem) it follows that the functions \( y_N^n D_y^\alpha r_p(y, \tau) \) are uniformly convergent (for a subsequence) on any compact set \( K_R \subset Q \cap \{0 \leq y_N \leq R\} \), \( R > 0 \),

\[
y_N^n D_y^\alpha r_p(y, \tau) \Rightarrow y_N^n D_y^\alpha r(y, \tau), \quad p \to \infty.
\]

Thus we can choose a compact set \( K_R \) and take the limit of relation (3.33) on this set. This gives

\[
|\Delta_{\tau^{(0)}}^{2\nu}(y_N^{(0)})^n D_y^\alpha r(P^{(0)}, 0)| \geq \nu > 0. \tag{3.36}
\]

Moreover, from (2.10) and (3.31) it follows that uniformly in \( p \)

\[
\langle y_N^n D_y^\alpha r_P^{(\gamma'-\omega)} \rangle_{y, \tau, Q}^{(\gamma')} + \langle y_N^n D_y^\alpha r_P^{(\gamma/m)} \rangle_{\tau, Q}^{(\gamma/m)} \leq C. \tag{3.37}
\]

Together with (3.27) this means that the sequence \( \{y_N^n D_y^\alpha r_P\} \) is bounded in the space \( C^{\gamma'-\omega' \frac{m}{\gamma}'}(K_R) \) for any compact set \( K_R \). Therefore for any \( \gamma' < \gamma \) the sequence \( \{y_N^n D_y^\alpha r_P\} \) converges to \( y_N^n D_y^\alpha r \) in the space \( C^{\gamma'-\omega', \frac{m}{\gamma}'}(K_R) \) and for the limit \( y_N^n D_y^\alpha r \) we have with the same exponent \( \gamma \)

\[
\langle y_N^n D_y^\alpha r \rangle_{y, \tau, Q}^{(\gamma'-\omega)} + \langle y_N^n D_y^\alpha r \rangle_{\tau, Q}^{(\gamma/m)} \leq C. \tag{3.38}
\]

Further, from (3.32) it follows that

\[
y_N^n D_y^m r(y, \tau) \text{ does not depend on } y_N, \quad D_y^m r(y, \tau) \text{ does not depend on } y_i, \quad i = 1, N-1, \quad D_{r^i} r(y, \tau) \text{ does not depend on } \tau.
\]

Really, let us prove the first assertion. Let \( y = (y', y_N), y_N > 0, \tau, \) and \( h > 0 \) be fixed. Then we have directly from the definition and from (3.32)
where terms in (3.40), we see with integer \( k \).

Integrating this equality in \( y_N \), we find

\[
y^\omega \gamma \left| (y_N + h)^n D^m_{y_N} r_p(y', y_N + h, \tau) - y_N^n D^m_{y_N} r_p(y', y_N, \tau) \right| \leq \left< y_N^n D^m_{y_N} r_p \right>_{\omega, y, \gamma, y} \leq \frac{1}{p}.
\]

Making use of (3.34) and taking limit in this inequality as \( p \to \infty \) we obtain

\[
(y_N + h)^n D^m_{y_N} r_p(y', y_N + h, \tau) = y_N^n D^m_{y_N} r_p(y', y_N, \tau).
\]

Since \( y, \tau, \) and \( h \) are arbitrary this proves the first assertion in (3.39). Other assertions are completely analogous.

Now from the first assertion in (3.39) we have with some functions \( a(y', \tau) \)

\[
D^m_{y_N} r(y, \tau) = \frac{a(y', \tau)}{y_N^n}.
\]

Integrating this equality in \( y_N \), we find

\[
r(y, \tau) = \begin{cases} 
  b_0(y', \tau) y_N^{m-n} + \sum_{i=1}^m b_i(y', \tau) y_N^{-i}, & n \text{ is a noninteger,} \\
  b_0(y', \tau) \ln^{(m-n)} y_N + \sum_{i=1}^m b_i(y', \tau) y_N^{-i}, & n \text{ is an integer,}
\end{cases}
\]

(3.40)

where \( b_0(y', \tau) \) and \( b_i(y', \tau) \) are some functions and \( \ln^{(m-n)} y_N \) is defined in (2.56).

Making use again of (3.39) and taking into account the independence of all terms in (3.40), we see

\[
D^m_{y_N} \ldots D^1_{y_N} b_i(y', \tau) \equiv 0 \text{ in } Q, \quad i = 0, m -
\]

at least in the sense of distributions. This means, as it is well known, that the functions \( b_i(y', \tau) \) are polynomials in \( y' \) of degree not greater than \( m \) and in \( \tau \) of degree not greater than 2. Consequently, the function \( y_N^n D^m_{y_N} r_p(y, \tau) \) has the form

\[
y_N^n D^m_{y_N} D^\alpha r(y, \tau) = P_0(y', \tau) y_N^n \ln^{(k)} y_N + \sum_{j=1}^{m+1} P_j(y', \tau) y_N^{d_j},
\]

(3.41)

where \( P_0(y', \tau) \) and \( P_j(y', \tau) \) are some polynomials, \( k \) is an integer, and for each \( j = 1, m + 1 \) the number \( d_j \) is either an integer or a number of the form \( k_j + n \) with integer \( k_j \). Now relation (3.36) means that the function \( y_N^n D^\alpha_{y_N} r_p(y, \tau) \) in (3.31) is not a constant identically. At last, as it can be checked directly, a nonconstant function of the form (3.41) can not have a finite values of \( \left< y_N^n D^m_{y_N} D^\alpha_{y_N} r_p \right>_{\gamma, y, \gamma, y} \) and \( \left< y_N^n D^m_{y_N} D^\alpha_{y_N} r_p \right>_{\gamma, y, \gamma, y} \) under our assumption \( \gamma - \omega \gamma < n \) over unbounded halfspace \( Q \) (it is enough to consider the term in (3.41) with the maximal growth at infinity). This contradict to (3.38).
This contradiction proves estimate \((3.10)\) on the class of functions \(u(x, t)\) with \((3.8)\). Note again that all the above reasonings for the term \(\langle x_N^n D_x^\alpha u \rangle_{\omega_\gamma, x, Q}^{(\gamma)(2s) (+)}\) from \((3.14)\) with \((3.15)\) are completely the same for other terms in \((3.14)\). For any other term in \((3.14)\) we obtain an analog of relations \((3.36)\) and \((3.41)\) with the same contradiction.

We now turn to the estimate of the value of \(\langle u \rangle_{n, \omega_\gamma, x, Q}^{(m+\gamma)(2s)(\varepsilon)}\) in \((3.7)\). Our goal is to obtain the estimate (compare \((2.34)\))

\[
\langle u \rangle_{n, \omega_\gamma, x, Q}^{(m+\gamma)(2s)(\varepsilon)} \leq C\varepsilon \left( \sum_{i=1}^{N} \langle x_N^i D_x^m u \rangle_{\omega_\gamma, x, Q}^{(\gamma)} + \langle D_t u \rangle_{x, Q}^{(\gamma/m)} \right) + C\varepsilon^\gamma \langle u \rangle_{n, \omega_\gamma, x, Q}^{(m+\gamma)(2s)}.
\]

(3.42)

All terms in the definition of \(\langle u \rangle_{n, \omega_\gamma, x, Q}^{(m+\gamma)(2s)(\varepsilon)}\) in \((3.7)\) are estimated completely similarly. We estimate the most complex term with a degenerate factor

\[
\langle x_N^{n-j} D_x^\alpha u \rangle_{\omega_\gamma, x, Q}^{(\gamma)(2s)(\varepsilon)} = \sup_{(x,t)\in Q, x, t, |x| \leq \varepsilon x_N} |x_N^{\omega_\gamma} \frac{\Delta_{h,x}^s \langle x_N^{n-j} D_x^\alpha u(x, t) \rangle_{x, t}}{|h|^\gamma}|, \quad |\alpha| = m - j, j \leq n.
\]

(3.43)

It can be checked directly that it is enough to consider the Hölder property of \(x_N^{n-j} D_x^\alpha u\) with respect to the tangent variables \(x'\) and with respect to the variable \(x_N\) separately. This corresponds to the obtaining separately the estimates for two cases of step \(h : h = (h, 0) = (h_1, ..., h_{N-1}, 0)\) and \(h = (0, ..., 0, h)\), where \(h > 0\). We first obtain the estimate \((3.42)\) with respect to the tangent variables, that is we estimate the expression

\[
\langle x_N^{n-j} D_x^\alpha u \rangle_{\omega_\gamma, x, Q}^{(\gamma)(2s)(\varepsilon)} \equiv \sup_{(x,t)\in Q, h \in R^{N-1}, |h'| \leq \varepsilon x_N} x_N^{\omega_\gamma} \frac{\Delta_{h'}^{s} \langle x_N^{n-j} D_x^\alpha u(x, t) \rangle_{x, t}}{|h'|^\gamma},
\]

(3.44)

where \(v(x, t) = \Delta_{h', x}^s u(x, t)\). Let a point \((x, t) = (x_0, t_0) = (x_0', x_0^0, t_0)\) be fixed and fix also a vector \(h' \in R^{N-1}, |h'| \leq \varepsilon x_N^0\). Suppose that \(\varepsilon \in (0, 1/32m)\). Consider the expression

\[
A \equiv \langle x_N^0 \rangle^{\omega_\gamma} \frac{|x_N^0|^{n-j} \Delta_{h', x}^s D_x^\alpha v(x_0, t_0)|}{|h'|^\gamma}, \quad |\alpha| = m - j, j \leq n.
\]

(3.45)
Make in the functions \(u(x, t)\) and \(v(x, t)\) the change of variables \((x, t) \rightarrow (y, \tau)\), \(v(x, t) \rightarrow v(y, \tau)\)

\[
x' = x_0' + (x_N^0) y', \quad x_N = (x_N^0) y_N, \quad t = t_0 + (x_N^0)^{m-n} \tau
\]

(3.46)

and denote \(\overline{d} = \overline{h}/x_N^0, \quad |\overline{d}| \leq \varepsilon < 1/32m, \quad P_1 \equiv (y_0, \tau_0) \equiv (0', 1, 0), \) that is \((x_0, t_0) \rightarrow (y_0, \tau_0)\). In the new variables the expression \(A\) takes the form

\[
A = (x_N^0)^{\omega+\gamma-n-m-\gamma} \frac{|\Delta_{d,y}^\alpha D_y^\alpha v(0', 1, 0)|}{|\overline{d}|^\gamma}.
\]

(3.47)

Denote for \(\rho < 1\)

\[
Q_{\rho} \equiv \{(y, \tau) \in Q : |y'| \leq \rho, |y_N - 1| \leq \rho, |\tau| \leq (\rho)^{m-n}\}
\]

and consider the function \(v(y, \tau)\) on this cylinder. Note first, that since \(y_N \geq 1/4\) on \(Q_{3/4}\), the function \(v(y, \tau)\) belongs to the usual smooth class \(C^{m+\gamma, 1+\gamma/m}(Q_{3/4})\).

Considering this function on \(Q_{1/4} \subset Q_{3/4}\) and applying (2.9), we obtain

\[
\frac{|\Delta_{d,y}^\alpha D_y^\alpha v(0', 1, 0)|}{|\overline{d}|^\gamma} \leq C \langle D_y^\alpha v(y, \tau) \rangle_{y, Q_{1/4}}^{(\gamma)} \leq C \left( \sum_{i=1}^{N} \langle D_{y_i}^m v \rangle_{y_i, Q_{1/4}}^{(\gamma)} + |v|_{Q_{1/4}}^{(0)} \right) \leq C \left( \sum_{i=1}^{N-1} \langle D_{y_i}^m u \rangle_{y_i, Q_{3/4}}^{(\gamma)} + \langle D_{y_N}^m v \rangle_{y_N, Q_{3/4}}^{(\gamma)} + |v|_{Q_{1/4}}^{(0)} \right). \tag{3.48}
\]

Note that we drop the term \(\langle D_{\tau} v \rangle_{Q_{1/4}}^{(\gamma/m)}\) in the right hand side of (3.48) because this estimate can be obtained at a fixed \(\tau\) with respect to the variables \(y\) only. Now we go back to the variables \((x, t)\) in the last estimate and obtain (\(|\alpha| = m - j\))

\[
(x_N^0)^{\gamma+m-j} \frac{|\Delta_{h,x}^\alpha D_{x}^\alpha v(x_0, t_0)|}{|h|^\gamma} \leq C \left( (x_N^0)^{m+\gamma} \sum_{i=1}^{N-1} \langle D_{x_i}^m u \rangle_{x_i, Q_{(3/4)}^{(\gamma)}}^{(\gamma)} + (x_N^0)^{m+\gamma} \langle D_{x_N}^m v \rangle_{x_N, Q_{(1/4)}^{(\gamma)}}^{(\gamma)} + |v|_{Q_{(1/4)}^{(\gamma)}}^{(0)} \right), \tag{3.49}
\]

where for \(\rho \in (0, 1)\),

\[
Q_{\rho x_N^0} \equiv \{(x, t) \in Q : \rho^{|x'|} x_N^0, \rho x_N^0, \rho x_N^0 - x_N^0, \rho x_N^0, |t_0| \leq \rho x_N^0 \} = \rho x_N^0 \}
\]

34
Before proceeding further with the estimate of the expression $A$ in (3.45), note that since $x_N \sim x_N^0$ on the set $Q_{(1/4)x_N^0}$ we have just from the definition of the Hölder constants

$$
(x_N^0)^n (D^m_{x_N} v)^{(\gamma)}_{x_N, Q_{(1/4)x_N^0}} \leq C \left( (x_N^n D^m_{x_N} v)^{(\gamma)}_{x_N, Q_{(1/4)x_N^0}} + (x_N^0)^{n-\gamma} |D^m_{x_N} v|^{(0)}_{Q_{(1/4)x_N^0}} \right).
$$

(3.50)

Substituting this estimate in (3.49), dividing both parts of obtained inequality by $(x_N^0)^{\gamma + m - n - \omega \gamma}$, and taking into account that $v(x, t) = \Delta_{h', x'}^s u(x, t)$, we obtain

$$
A \leq C \left( (x_N^0)^{n+\omega \gamma} \sum_{i=1}^{N-1} (D^m_{x_i} u)^{(\gamma)}_{x_i, Q_{(1/4)x_N^0}} + (x_N^0)^{\omega \gamma} (x_N^n D^m_{x_N} v)^{(\gamma)}_{x_N, Q_{(1/4)x_N^0}} \right) +
$$

$$
+ C \left( (x_N^0)^{\omega \gamma - \gamma} |\Delta_{h', x'}^s (x_N^n D^m_{x_N} u)^{(0)}_{Q_{(1/4)x_N^0}} + (x_N^0)^{-m + m + \omega \gamma} |\Delta_{h', x'}^s u(x, t)|^{(0)}_{Q_{(1/4)x_N^0}} \right) \leq
$$

$$
\leq C \sum_{i=1}^{N} (x_N^n D^m_{x_N} u)^{(\gamma)}_{x_i, Q_{(1/4)x_N^0}} +
$$

$$
+ C \left( (x_N^0)^{-\gamma} |x_N^{\omega \gamma} \Delta_{h', x'}^s (x_N^n D^m_{x_N} u)^{(0)}_{Q_{(1/4)x_N^0}} +
$$

$$
+ (x_N^0)^{-(\gamma + m)} (x_N^{\omega \gamma} |\Delta_{h', x'}^s (x_N^n u(x, t))|^{(0)}_{Q_{(1/4)x_N^0}} \right).
$$

(3.51)

At the same time for the last two terms in the right hand side of (3.51) we have

$$
(x_N^0)^{-\gamma} |x_N^{\omega \gamma} \Delta_{h', x'}^s (x_N^n D^m_{x_N} u)^{(0)}_{Q_{(1/4)x_N^0}} =
$$

$$
\left( \frac{|h'|}{x_N^0} \right)^\gamma \left( x_N^{\omega \gamma} |\Delta_{h', x'}^s (x_N^n D^m_{x_N} u)^{(0)}_{Q_{(1/4)x_N^0}} \right) \leq
$$

$$
\leq C \varepsilon^{\gamma} \langle x_N^n D^m_{x_N} u \rangle^{(\gamma)}_{\omega \gamma, x_n, Q_{(1/4)x_N^0}} \leq C \varepsilon^{\gamma} \langle \langle u \rangle \rangle^{(m + \gamma)(2\gamma)}_{n \omega \gamma, Q_{(1/4)x_N^0}},
$$

$$
(x_N^0)^{-(\gamma + m)} |x_N^{\omega \gamma} \Delta_{h', x'}^s (x_N^n u(x, t))|^{(0)}_{Q_{(1/4)x_N^0}} =
$$

$$
= \left( \frac{|h'|}{x_N^0} \right)^{m+\gamma} \left( x_N^{\omega \gamma} |\Delta_{h', x'}^s (x_N^n u(x, t))|^{(0)}_{Q_{(1/4)x_N^0}} \right) \leq
$$

35
\[
\leq C\varepsilon^m + \gamma \left( \sum_{|\beta|=m} \langle x_N^\beta D_x^\gamma u \rangle_{\omega, \gamma, x, x', \overline{Q}} \right) \leq C\varepsilon^\gamma \langle \langle u \rangle \rangle_{m, \omega, \gamma, \overline{Q}},
\]
where we made use of the mean value theorem and of Proposition 6. Substituting these two inequalities in (3.51) and taking into account the definition of the expression A in (3.45), we get
\[
\langle \langle x_N^{n-j} D_x^\alpha u \rangle \rangle_{\omega, \gamma, \overline{Q}} \leq C\varepsilon^\gamma \langle \langle u \rangle \rangle_{m, \omega, \gamma, \overline{Q}}.
\]
We turn now to the obtaining the same estimate for \(\langle \langle x_N^{n-j} D_x^\alpha u \rangle \rangle_{\omega, \gamma, \overline{Q}}\) with respect to the variable \(x_N, |\alpha| = m - j\)
\[
\langle \langle x_N^{n-j} D_x^\alpha u \rangle \rangle_{\omega, \gamma, \overline{Q}} \leq
\]
\[
\leq C\varepsilon \left( \sum_{i=1}^N \langle x_N^{n} D_x^{m} u \rangle_{\omega, \gamma, x, \overline{Q}} + \langle D_t u \rangle_{\gamma, \overline{Q}} \right) + C\varepsilon^\gamma \langle \langle u \rangle \rangle_{m, \omega, \gamma, \overline{Q}}. \quad (3.52)
\]
We consider only the case \(j < n\) because for an integer \(n\) in the case \(j = n\) the function \(x_N^{n-j} D_x^\alpha u = D_x^\alpha u\) has no a degeneration and all the estimates below are completely the same and become simpler. The schema of the reasonings is quite similar to the proof of (3.52) above.

Let \(Q_u(x, t)\) be the polynomial from (2.63) with the properties (2.66), (2.67) for the function \(u(x, t)\) under the consideration. If we consider the function \(v(x, t) = u(x, t) - Q_u(x, t)\) instead of the function \(u(x, t)\) itself, we see that all terms in both sides of (3.53) remain unchanged because of (2.66)–(2.68). Therefore it is enough to prove that
\[
\langle \langle x_N^{n-j} D_x^\alpha v \rangle \rangle_{\omega, \gamma, x, \overline{Q}} \leq C\varepsilon \left( \sum_{i=1}^N \langle x_N^{n} D_x^{m} v \rangle_{\omega, \gamma, x, \overline{Q}} + \langle D_t v \rangle_{\gamma, \overline{Q}} \right) + C\varepsilon^\gamma \langle \langle v \rangle \rangle_{m, \omega, \gamma, \overline{Q}}. \quad (3.53)
\]
It is important for us that the function \(v(x, t)\) possess the property
\[
x_N^{\omega_{\gamma}} x_N^{n-j-\gamma} |D_x^\gamma v(x, t)| \leq \langle x_N^{n-j} D_x^\alpha v \rangle_{\omega, \gamma, x, \overline{Q}}, \quad (x, t) \in \overline{Q}. \quad (3.55)
\]
Really, from (2.64) it follows that \(x_N^{n-j} D_x^\alpha v(x, t)|_{x_N=0} = 0\) and we obtain
\[ x_N^{\omega_2} x_N^{n-j-\gamma} |D_x^\alpha v(x, t)| = \]
\[ = x_N^{\omega_2} \frac{|x_N^{n-j} D_x^\alpha v(x, t) - [x_N^{n-j} D_x^\alpha v(x, t)]|_{x_N=0}}{x_N^{\gamma}} \leq \langle x_N^{n-j} D_x^\alpha v \rangle_{\omega_2, x_N; \mathbb{Q}}. \]

As above, let a point \((x_0, t_0) = (x_0, x_N, t_0)\) and \(0 < h < \varepsilon x_N\) be fixed, \(0 < \varepsilon < 1/(32m)\). Consider the expression

\[ A = (x_N^0)^{\omega_2} |\Delta_{h,x_N}^{2s} (x_N^0)^{n-j} D_x^\alpha v(x_0, t_0)| = \frac{(x_N^0)^{\omega_2}}{h^\gamma} B. \tag{3.56} \]

We have

\[ B = \Delta_{h,x_N}^{2s} (x_N^0)^{n-j} D_x^\alpha v(x_0, t_0) = \]
\[ = \sum_{i=1}^{2s} C_i \Delta_{h,x_N}^{i} (x_N^0 + \theta)^{n-j} \Delta_{h,x_N}^{2s-i} D_x^\alpha v(x_0', x_N^0 + \theta, t_0) + \]
\[ + (x_N^0 + \theta)^{n-j} \Delta_{h,x_N}^{2s} D_x^\alpha v(x_0, x_N^0, t_0) = \sum_{i=1}^{2s} B_i + B_0, \tag{3.57} \]

where \(C_i\) are some constants, and by \(h_\theta\) here and below we denote all possible expressions of the form \(h_\theta = C \cdot h\) with \(0 \leq C \leq C(m)\). Consider \(B_i\) with \(i \geq 1\). Making use of the mean value theorem to estimate \(\Delta_{h,x_N}^{i} [(x_N^0 + \theta)^{n-j}]\) and keeping in mind the assumption \(h \leq \varepsilon x_N\), we have

\[ B_i \leq \sum_{h_\theta} C (x_N^0 + h_\theta)^{n-j-i} |D_x^\alpha v(x_0', x_N^0 + h_\theta, t_0)| \leq \]
\[ \leq \varepsilon^i \sum_{h_\theta} C (x_N^0 + h_\theta)^{n-j-\gamma} |D_x^\alpha v(x_0', x_N^0 + h_\theta, t_0)| h^\gamma. \]

Therefore, in view of the definition of the expression \(A\) in (3.56),

\[ A_i = (x_N^0)^{\omega_2} \frac{B_i}{h^\gamma} \leq \]
\[ \leq \varepsilon^i \sum_{h_\theta} C (x_N^0 + h_\theta)^{n-j-\gamma+\omega_2} |D_x^\alpha v(x_0', x_N^0 + h_\theta, t_0)| \leq C \varepsilon^i \langle x_N^{n-j} D_x^\alpha v \rangle_{\omega_2, x_N; \mathbb{Q}}. \tag{3.58} \]

Consider now the expression \(B_0\) in (3.57). The considerations in this case are similar to the previous case of the variables \(x'\). Denote
\[ w(x, t) \equiv \Delta_{h,x_N}^x v(x', x_N, t) \] so that
\[ \Delta_{h,x_N}^{2\alpha} D_x^\alpha v(x_0^0, x_N^0, t_0) = \Delta_{h,x_N}^x D_x^\alpha w(x_0^0, x_N^0, t_0) \]
and consider the expression
\[ A_0 \equiv \frac{(x_N^0)^{\alpha \gamma}}{h^\gamma} B_0. \]

As above, make in the functions \( v(x, t) \) and \( w(x, t) \) the change of variables \((3.46)\) and denote \( d = h/x_N^0 \), \( d \leq \varepsilon < 1/32m \), \( d_0 = h_0/x_N^0 \leq C(m)\varepsilon \), \( P_1 \equiv (y_0, \tau_0) \equiv (0', 1, 0) \), that is \((x_0, t_0) \to (y_0, \tau_0)\). In the new variables the expression \( A_0 \) takes the form
\[
A_0 = (x_N^0)^{\alpha \gamma + n - m - \gamma} (1 + d_0)^{n-j} \frac{|\Delta_{d,y_N}^x D_y^\alpha w(0', 1, 0)|}{d^\gamma} \leq C (x_N^0)^{\alpha \gamma + n - m - \gamma} \frac{|\Delta_{d,y_N}^x D_y^\alpha w(0', 1, 0)|}{d^\gamma}. \tag{3.59}
\]

Denote for \( \rho < 1 \)
\[ Q_\rho \equiv \{(y, \tau) \in Q : |y| \leq \rho, |y_N - 1| \leq \rho, |\tau| \leq (\rho)^{m-n}\} \]
and consider the function \( w(y, \tau) \) on this cylinder. As above, since \( y_N \geq 1/4 \) on \( Q_{3/4} \), the function \( w(y, \tau) \) belongs to the usual smooth class \( C^{m+\gamma, 1+\gamma/m}(\overline{Q_{3/4}}) \).

Considering, as above, this function on \( \overline{Q_{1/4}} \subset \overline{Q_{3/4}} \) and applying \((2.9)\), we obtain
\[
\frac{|\Delta_{d,y_N}^x D_y^\alpha w(0', 1, 0)|}{d^\gamma} \leq C \langle D_y^\alpha w(y, \tau) \rangle_{y_N, \overline{Q_{1/4}}}^{(\gamma)} \leq C \left( \sum_{i=1}^{N-1} \langle D_{y_i}^m w(y, \tau) \rangle_{y_N, \overline{Q_{1/4}}}^{(\gamma)} + |w|_{y_N, \overline{Q_{1/4}}}^{(0)} \right). \tag{3.60}
\]

Note that we again drop the term \( \langle D_\tau v \rangle_{\tau, \overline{Q_{1/4}}}^{(\gamma/m)} \) in the right hand side of \((3.60)\) because this estimate can be obtained at a fixed \( \tau \) with respect to the variables \( y \) only. Now we go back to the variables \((x, t)\) in the last estimate and obtain \((|\alpha| = m-j)\)
\[
\frac{|\Delta_{d,y_N}^x D_y^\alpha w(0', 1, 0)|}{d^\gamma} \leq (x_N^0)^{\alpha \gamma + n - m - \gamma} \frac{|\Delta_{h,x_N}^x D_x^\alpha w(x_0^0, t_0)|}{h^\gamma} \leq C \left( (x_N^0)^{m + \gamma} \sum_{i=1}^{N-1} \langle D_{x_i}^m v \rangle_{x_i, \overline{Q_{(3/4)}^{1/4}}}^{(\gamma)} + \right). \tag{3.61}
\]
\[ + \left( x_N^0 \right)^{m+\gamma} \left\langle D^{m}_{x_N} v \right\rangle_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} + \left| w\left|_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \right. \right. \],

where again for \( \rho \in (0, 1) \)

\[ Q_{\rho x_N^0} = \{(x, t) \in Q \mid |x| \leq \rho x_N^0, |x_N - x_N^0| \leq \rho x_N^0, |t - t_0| \leq (\rho x_N^0)^{m-n}\}. \]

Substituting this estimate in (3.59), we obtain

\[ A_0 \leq C \left( \left( x_N^0 \right)^{\omega^\gamma + n} \sum_{i=1}^{N-1} \left\langle D^{m}_{x_i} v \right\rangle_{x_i, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} + \right. \]

\[ + \left( x_N^0 \right)^{\omega^\gamma + n} \left\langle D^{m}_{x_N} v \right\rangle_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} + \left( x_N^0 \right)^{\omega^\gamma + n - m - \gamma} \left| w\left|_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \right. \right. \].

Again, since \( x_N \sim x_N^0 \) on the set \( \mathbb{Q}_{(3/4), 0}^N \), for \( i = 1, N - 1 \) in the first term of (3.62)

\[ \left( x_N^0 \right)^{\omega^\gamma + n} \left\langle D^{m}_{x_i} v \right\rangle_{x_i, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} \leq C \left( x_N^0 \right)^{\omega^\gamma} \left( x_N^n D^{m}_{x_i} v \right)_{x_i, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} \leq C \left( x_N^n D^{m}_{x_i} v \right)_{\omega^\gamma, x_i, \mathbb{Q}}^{(\gamma)}. \]

Making further use of (3.50) and then (3.55) we have on \( \mathbb{Q}_{(3/4), 0}^N \) for the second term

\[ \left( x_N^0 \right)^{\omega^\gamma + n} \left\langle D^{m}_{x_N} v \right\rangle_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} \leq C \left( x_N^0 \right)^{\omega^\gamma} \left( x_N^n D^{m}_{x_N} v \right)_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} \leq \]

\[ C \left( x_N^0 \right)^{\omega^\gamma} \left( \left| x_N^n D^{m}_{x_N} v \right|_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(0)} + \left. \left| x_N^n D^{m}_{x_N} v \right|_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \right. \right. \]

\[ \leq C \left( x_N^0 \right)^{\omega^\gamma} \left( x_N^n D^{m}_{x_N} v \right)_{x_N, \mathbb{Q}_{(3/4), 0}^N}^{(\gamma)} \leq C \left( x_N^n D^{m}_{x_N} v \right)_{\omega^\gamma, x_N, \mathbb{Q}}^{(\gamma)}. \]

The third term in (3.62) we estimate as follows \( (s = m + 1) \)

\[ \left( x_N^0 \right)^{\omega^\gamma + n - m - \gamma} \left| w\left|_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \right. \right. = \left( \frac{h}{x_N^0} \right)^{m+\gamma} \left( x_N^0 \right)^{\omega^\gamma + n} \left( \frac{\Delta h_{x_N} v(x', x_N, t)}{h^m+\gamma} \right)_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \leq \]

\[ \leq C \varepsilon^{m+\gamma} \left( x_N^0 \right)^{\omega^\gamma + n} \left( \frac{\Delta h_{x_N} D^{m}_{x_N} v(x', x_N + h\theta, t)}{h^\gamma} \right)_{\mathbb{Q}_{(1/4), 0}^N}^{(0)} \].
where the mean value theorem was used. Making again use of (3.50) and (3.55) we have on $Q_{(3/4)_{2}^{0}}$ as above

$$
\left( x_{N}^{0} \right)^{\omega_{\gamma}+n} \frac{\Delta_{h,x_{N}} D_{x_{N}}^{m} v(x', x_{N} + h_{0}, t)}{h^{\gamma}} \leq \left( x_{N}^{0} \right)^{\omega_{\gamma}+n} \left( D_{x_{N}}^{m} v \right)^{(\gamma)}_{x_{N}, Q_{(3/4)_{2}^{0}}} \leq C \left( x_{N}^{0} \right)^{\omega_{\gamma}} \left( D_{x_{N}}^{m} v \right)^{(\gamma)}_{x_{N}, Q_{(3/4)_{2}^{0}}}.
$$

(3.66)

From (3.63) - (3.66) it follows that

$$
A_{0} \leq C \sum_{i=1}^{N} \langle x_{N}^{0} D_{x_{i}}^{m} v \rangle^{(\gamma)}_{\omega_{\gamma}, x_{i}, Q}
$$

and from this and from (3.58), (3.56) it follows that

$$
\left( x_{N}^{0} \right)^{\omega_{\gamma}} \frac{\left| \Delta_{h,x_{N}}^{2s} \left[ \left( x_{N}^{0} \right)^{n-j} D_{x}^{m} v(x_{0}, t_{0}) \right] \right|}{h^{\gamma}} \leq C \sum_{i=1}^{N} \langle x_{N}^{0} D_{x_{i}}^{m} v \rangle^{(\gamma)}_{\omega_{\gamma}, x_{i}, Q} + C \varepsilon \langle \left( x_{N}^{0} D_{x}^{m} v \right)^{(\gamma)}_{\omega_{\gamma}, x_{N}, Q} \rangle.
$$

(3.67)

Since $(x_{0}, t_{0})$ and $h \leq \varepsilon x_{N}$ are arbitrary, this means (3.54) and therefore (3.53).

Other terms in the definition of $\langle \langle u \rangle \rangle^{(m+\gamma)(2s)}_{n, \omega_{\gamma}, Q}$ in (3.7) are estimated completely similarly. The smoothness with respect to the $t$ - variable is estimated identically to the estimates of the smoothness with respect to $x'$ with the taking into account the relation between the dimensions of $x$ and $t : x \sim x_{N}^{0}, t \sim x_{N}^{m-n}$ or $x \sim h, t \sim h^{m-n}$. Note that all terms in the definition of $\langle \langle u \rangle \rangle^{(m+\gamma)(2s)}_{n, \omega_{\gamma}, Q}$ have the same total dimension with respect to this relation of the dimensions. Now from the alternative (3.9) and from (3.10) with (3.12) it follows that for arbitrary function $u(x, t) \in C_{n, \omega_{\gamma}}^{m+\gamma, \frac{m+\gamma}{m}}(Q)$ we have

$$
\langle \langle u \rangle \rangle^{(m+\gamma)(2s)}_{n, \omega_{\gamma}, Q} \leq C_{\varepsilon} \left( \sum_{i=1}^{N} \langle x_{N}^{n} D_{x_{i}}^{m} u \rangle^{(\gamma)}_{\omega_{\gamma}, x_{i}, Q} + \langle D_{t} u \rangle^{(\gamma/m)}_{t, Q} \right) + C \varepsilon \langle \langle u \rangle \rangle^{(m+\gamma)(2s)}_{n, \omega_{\gamma}, Q}.
$$

Finally, choosing $\varepsilon$ in this estimate sufficiently small and absorbing the last term in the left hand side, we arrive at (3.1). This completes the proof of Theorem 2 under assumption (3.5).

We now remove this assumption. Let $u(x, t) \in C_{n, \omega_{\gamma}}^{m+\gamma, \frac{m+\gamma}{m}}(Q)$ and let $u_{\varepsilon}(x, t)$ be defined by (2.49) so it satisfies (2.50), (3.5). By what was proved above
\[
\langle u_\varepsilon \rangle_{n, \omega, \gamma, Q}^{(m+\gamma)} \leq C \left( \sum_{i=1}^{N} \langle x_N^n D_x^m u_{\varepsilon} \rangle_{\omega, \gamma, x, Q}^{(\gamma)} + \langle D_t u_{\varepsilon} \rangle_{t, Q}^{(\gamma/m)} \right),
\]
where the constant \( C \) does not depend on \( \varepsilon \). It can be directly verified that
\[
\langle x_N^n D_x^m u_{\varepsilon} \rangle_{\omega, \gamma, x, Q}^{(\gamma)} \leq C \langle x_N^n D_x^m u \rangle_{\omega, \gamma, x, Q}^{(\gamma)}, \quad \langle D_t u_{\varepsilon} \rangle_{t, Q}^{(\gamma/m)} \leq \langle D_t u \rangle_{t, Q}^{(\gamma/m)}
\]
and therefore uniformly in \( \varepsilon \)
\[
\langle u_\varepsilon \rangle_{n, \omega, \gamma, Q}^{(m+\gamma)} \leq C \left( \sum_{i=1}^{N} \langle x_N^n D_x^m u \rangle_{\omega, \gamma, x, Q}^{(\gamma)} + \langle D_t u \rangle_{t, Q}^{(\gamma/m)} \right). \tag{3.68}
\]
Let \( \delta \in (0, 1) \) and let \( Q_\delta = \{(x, t) : \delta \leq x_N \leq \delta^{-1}, |x'| \leq \delta^{-1}, |t| \leq \delta^{-1}\} \). It is well known that for each \( \delta \in (0, 1) \) the sequence \( \{u_\varepsilon \} \) is bounded in the standard space \( C^{m+\gamma, \frac{m+\gamma}{m}}(Q_\delta) \). Therefore for a subsequence
\[
\lim_{\varepsilon \to 0} u_\varepsilon = u \text{ on each } Q_\delta \text{ in the space } C^{m+\gamma, \frac{m+\gamma}{m}}(Q_\delta) \tag{3.69}
\]
with \( \gamma' < \gamma \). Besides, all weighted terms \( x_N^{n-j} D_x^a u_{\varepsilon} \) and \( x_N^{n-j} D_x^a u \) in the definition of \( \langle u_\varepsilon \rangle_{n, \omega, \gamma, Q}^{(m+\gamma)} \), \( j \leq n \), \( |\alpha| = m-j \), converge to \( x_N^{n-j} D_x^a u \) and \( x_N^{n-j} D_x^a u \) in the space \( C^{(1-\omega)\gamma', \frac{\gamma'}{m}}(K_\delta) \) for each \( K_\delta = \{(x, t) : 0 \leq x_N \leq \delta^{-1}, |x'| \leq \delta^{-1}, |t| \leq \delta^{-1}\} \)
\[
x_N^{n-j} D_x^a u_{\varepsilon} \to C^{(1-\omega)\gamma', \frac{\gamma'}{m}}(K_\delta) x_N^{n-j} D_x^a u, \quad x_N^{n-j} D_x^a u_{\varepsilon} \to C^{(1-\omega)\gamma', \frac{\gamma'}{m}}(K_\delta) x_N^{n-j} D_x^a u. \tag{3.70}
\]
And the same is valid for the term \( D_t u_{\varepsilon} \). Exactly the same reasonings as in the proof of Proposition \([3]\) show that
\[
\sum_{j=0}^{n} \sum_{|\alpha| = m-j} \langle x_N^{n-j} D_x^a u \rangle_{\omega, \gamma, Q}^{(\gamma/m)} + \sum_{j=0}^{n} \sum_{|\alpha| = m-j} \langle x_N^{n-j} D_x^a u \rangle_{t, Q}^{(\gamma/m)} + \langle D_t u \rangle_{\omega, \gamma, Q}^{(\gamma/m)} \leq
\]
\[
\leq C \limsup_{\varepsilon \to 0} \left( \sum_{j=0}^{n} \sum_{|\alpha| = m-j} \langle x_N^{n-j} D_x^a u_{\varepsilon} \rangle_{\omega, \gamma, Q}^{(\gamma/m)} + \langle D_t u_{\varepsilon} \rangle_{t, Q}^{(\gamma/m)} \right) +
\]
\[
+ \sum_{j=0}^{n} \sum_{|\alpha| = m-j} \langle x_N^{n-j} D_x^a u_{\varepsilon} \rangle_{t, Q}^{(\gamma/m)} + \langle D_t u_{\varepsilon} \rangle_{t, Q}^{(\gamma/m)} \right) \leq
\]
\[
\leq C \left( \sum_{i=1}^{N} \langle x_N^n D_x^m u \rangle_{\omega, \gamma, x, Q}^{(\gamma)} + \langle D_t u \rangle_{t, Q}^{(\gamma/m)} \right),
\]

41
Note that this is valid for the pure derivative $D^{m-n}_x u$ in the case of an integer $n$ only if this derivative and its seminorms are included in the definition of the space. Besides, since the functions $u_\varepsilon$ satisfy (2.50), (3.5), it follows from (3.70) that the function $u(x,t)$ itself satisfies (1.22). The same reasoning holds true also for the term \[ \sum_{j=1}^{j \leq m-n} \sum_{|\alpha|=j} \langle D_x^n u_\varepsilon \rangle_{t, \Omega}^{(1-n/m + \varepsilon)} \] in the definition of \( \langle u_\varepsilon \rangle_{n, \omega, \Omega}^{(m+\gamma)} \) and therefore

\[ \sum_{j=1}^{j \leq m-n} \sum_{|\alpha|=j} \langle D_x^n u_\varepsilon \rangle_{t, \Omega}^{(1-n/m + \varepsilon)} \leq C \limsup_{\varepsilon \to 0} \sum_{j=1}^{j \leq m-n} \sum_{|\alpha|=j} \langle D_x^n u_\varepsilon \rangle_{t, \Omega}^{(1-n/m + \varepsilon)} \leq C \left( \sum_{i=1}^{N} \langle x_N^m u \rangle_{\omega, x, \Omega}^{(\gamma)} + \langle D_t u \rangle_{\Omega}^{(\gamma/m)} \right). \]

Note again that this is valid for the pure derivative $D^{m-n}_x u$ in the case of an integer $n$ only if this derivative and its seminorms are included in the definition of the space.

At last, since $u_\varepsilon(x,t)$ is smooth with respect to $x'$ and $t$ for any $x_N > 0$, we have in the open domain $Q = \Omega \cap \{x_N > 0\}$ by the same reasonings as above uniformly in $x_N$

\[ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+(1-\omega)\gamma]-j} \langle D_x^n D_{x_N}^j u \rangle_{x', \Omega}^{\{m-n+(1-\omega)\gamma\}} + \]

\[ + \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+\gamma]-j} \langle D_x^n D_{x_N}^j u \rangle_{\omega, x', \Omega}^{\{m-n+\gamma\}} \leq C \limsup_{\varepsilon \to 0} \left( \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+(1-\omega)\gamma]-j} \langle D_x^n D_{x_N}^j u_\varepsilon \rangle_{x', \Omega}^{\{m-n+(1-\omega)\gamma\}} + \right. \]

\[ + \sum_{j=0}^{j \leq m-n} \sum_{|\alpha|=[m-n+\gamma]-j} \langle D_x^n D_{x_N}^j u_\varepsilon \rangle_{\omega, x', \Omega}^{\{m-n+\gamma\}} \right) \leq C \left( \sum_{i=1}^{N} \langle x_N^m u \rangle_{\omega, x, \Omega}^{(\gamma)} + \langle D_t u \rangle_{\Omega}^{(\gamma/m)} \right). \quad (3.71) \]

But from estimates (2.51) it follows that the functions $D^{j}_{x_N} u$, $j < m - n$, are continuous at $x_N \to 0$. Then from the above estimate it follows that at a
fixed \( t_0 > 0 \) the sequence of the functions \( \{ D^{j}_{x_N} u(\cdot, x_N, t_0) \} \) with \( x_N \) as a parameter is bounded in \( C^{m-n+(1-\omega)\gamma-j}(R^{N-1}) \). Therefore, in view of Proposition \[13\] this sequence converges (at least for a subsequence) on compact sets \( K \) in \( R^{N-1} \) as \( x_N \rightarrow 0 \) in the space \( C^{m-n+(1-\omega)\gamma-j}(K) \), \( \gamma' < \gamma \), to some function \( v(x', t_0) \in C^{m-n+(1-\omega)\gamma-j}(R^{N-1}) \) with the same estimate of the highest semi-norm \( \langle D^{\alpha}_{x'} v \rangle_{x', R^{N-1}}^{(m-n+(1-\omega)\gamma)} \). This means that at a fixed \( t_0 > 0 \) the functions \( D^{j}_{x_N} u, j \leq m-n \), have traces at \( x_N = 0 \) from the space \( C^{m-n+(1-\omega)\gamma-j}(R^{N-1}) \) and estimate \[3.71\] is valid also at \( x_N = 0 \) that is \( D^{\alpha}_{x} D^{j}_{x_N} u, j < m-n, |\alpha| = m-n+(1-\omega)\gamma - j \), exist in the usual classical sense at \( x_N = 0 \) and\[
\begin{align*}
&\sum_{j=0}^{j \leq m-n} \sum_{|\alpha| = |m-n+(1-\omega)\gamma|-j} \langle D^{\alpha}_{x} D^{j}_{x_N} u \rangle_{x, Q}^{(m-n+(1-\omega)\gamma)} + \\
&+ \sum_{j=0}^{j \leq m-n} \sum_{|\alpha| = |m-n+\gamma|-j} \langle D^{\alpha}_{x} D^{j}_{x_N} u \rangle_{\omega, x', Q}^{(m-n+\gamma)} \leq \\
&\leq C \left( \sum_{i=1}^{N} \langle x^{n}_N D^{m}_{x_i} u \rangle_{\omega_N, x_i, Q}^{(\gamma)} + \langle D_{t} u \rangle_{(\gamma/m)}^{(\gamma/m)} \right).
\end{align*}
\]

This completes the proof of \[1.21\] and of the Theorem \[2\].

4 Mixed and lower order derivatives of functions from \( C^{m+\gamma, m+\gamma}_{n, m} (Q) \).

In this section we consider the mixed derivatives \( x^{n-(m-|\alpha|)} D^{\alpha}_{x} u \) of order \( m-n < |\alpha| \leq m \) and also the lower order derivatives \( D^{\alpha}_{x} u \) of order \( |\alpha| \leq m-n \). The last lower order derivatives do not require a weight in general but the situation in the case of an integer \( n \) differs from that in the case of a noninteger \( n \). Therefore we consider these two cases separately. Besides, we concentrate on the local behaviour of functions near the singular boundary \( \{ x_N = 0 \} \) and assume in this section that all functions under consideration have compact support in \( \overline{Q} \) or in \( \mathcal{P} \). This permits us to avoid the consideration of possible behaviour of functions at infinity. Particularly, we will show that for functions with compact support of fixed dimensions norms \[1.11\], \[1.17\] and \[1.13\], \[1.20\] are equivalent.

Below we need the following lemma which is valid for both cases of an integer and a noninteger \( n \).

**Lemma 14** Let \( n \in (0, m) \) be arbitrary and a function \( u(x, t) \in C^{m+\gamma, m+\gamma}_{n, m} (Q) \) in the sense of \[1.19\] has compact support. Then...
\[
\sum_{j=0}^{j=n} \langle x_N^{n-j} D_{x_N}^{m-j} u \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)} \leq C \langle x_N^n D_{x_N}^m u \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)}. \tag{4.1}
\]

**Proof.** The proof is by induction. For \( j = 0 \) there is nothing to prove. Let us prove that for \( j < n \)
\[
\langle x_N^{n-j} D_{x_N}^{m-j} u \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)} \leq C \langle x_N^{n-j+1} D_{x_N}^{m-j+1} u \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)}. \tag{4.2}
\]
Since the function \( u(x, t) \) has a compact support, we have
\[
x_N^{n-j} D_{x_N}^{m-j} u(x, t) = -x_N^{n-j} \int_{x_N}^{x_N} T_{n-j+1} \left[ T_{n-j+1} u(x', \xi_N, t) \right] d\xi_N. \tag{4.3}
\]
Denote for brevity
\[
f(x, t) \equiv x_N^{n-j} D_{x_N}^{m-j} u(x, t), \quad \omega \equiv \left[ T_{n-j+1} D_{x_N}^{m-j+1} u(x', \xi_N, t) \right] \equiv a(x', \xi_N, t)
\]
and transform the representation for \( f(x, t) \) making the change of the variable \( \xi_N = \eta x_N \) in the corresponding integral
\[
f(x, t) = \int_{1}^{\infty} \eta^{-n+j-1} a(x', \eta x_N, t) d\eta.
\]
Let \( x_N, \eta_N \) be fixed, \( 0 < x_N < \eta_N \). We have
\[
|x_N^{\omega} [f(x', \eta_N, t) - f(x', x_N, t)]| =
\]
\[
= \int_{1}^{\infty} \eta^{-n+j-1-\omega} (\eta x_N)^{\omega} \left[ a(x', \eta x_N, t) - a(x', \eta x_N, t) \right] d\eta \leq
\]
\[
\leq \langle a \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)} (\eta_N - x_N)^{\gamma} \int_{1}^{\infty} \eta^{-n+j-1+(1-\omega)\gamma} d\eta \leq C \langle a \rangle_{\omega \gamma, x_N, \mathcal{Q}}^{(\gamma)} (\eta_N - x_N)^{\gamma},
\]
where \(-n + j - 1 + (1 - \omega)\gamma < -1\) in view of (1.9) and the assumption \( j < n \). This proves (4.2) and the lemma follows by induction. 

\[\blacksquare\]
4.1 The case of an integer \( n \).

We suppose in this subsection that \( n \) is an integer, \( 0 < n < m \). Consider first the derivatives \( x_N^{-j}D_x^n u(x,t) \) of order \( |\alpha| = m - j \), \( j \leq n \). In this case we have two different situations for the derivatives \( D_x^n u(x,t) \) of the particular order \( |\alpha| = m - n \) with \( \alpha_N < m - n \) and for the derivative \( D_x^{m-n} u(x,t) \).

**Proposition 15** Let \( n \in (0,m) \) be an integer. Let a function \( u(x,t) \in C_{n,\omega,\gamma/\omega}^{m+\gamma/\omega, m+\gamma/\omega}(Q) \) in the sense of (1.19) has compact support. Then all seminorm of the function \( u(x,t) \) in the left hand side of (1.21) except for, may be, seminorms of \( D_x^{m-n} u(x,t) \) are finite and

\[
\langle u \rangle_{n,\omega,\gamma/\omega}^{m+\gamma/\omega, m+\gamma/\omega} = \sum_{j=0}^{\min(n,m)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle x_N^{-j}D_x^\alpha u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle x_N^{-j}D_x^\alpha u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle x_N^{-j}D_x^\alpha u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle x_N^{-j}D_x^\alpha u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)}
\]

\[+ \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)}\]

\[+ \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)}\]

\[+ \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)}\]

\[+ \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} + \sum_{j=0}^{\min(m-n,n)} \sum_{\alpha \neq (0,\ldots,m-n)} \langle D_x^\alpha D_x^j x_N^n u \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)}\]

\[(4.4)\]

**Proof.** The proof essentially follows from the proof of Theorem 2. Consider the smoothed function \( u_\varepsilon(x,t) \) as in (2.49) and in the end part of the proof of Theorem 2

\[
u_\varepsilon(x,t) \equiv \int_{\mathbb{R}^{k-1}} \int_{-\infty}^{\infty} u(y',x_N,\tau)\omega_\varepsilon(x' - y',t - \tau)dy'd\tau.
\]

From Lemma 14 from (2.51) with \( j = n \), and from the way of the construction of \( u_\varepsilon(x,t) \) it follows that for this function all seminorms in the left hand side of (4.4) are finite (including the seminorm \( \langle x_N^{-n}D_x^{m-n}u_\varepsilon \rangle_{\omega,\gamma/\omega}^{(\gamma/\omega)} \) and thus this function satisfies (4.4). The rest of the proof coincides with the end part of the proof of Theorem 2.

\[\square\]

Consider now the derivative \( D_x^{m-n} u(x,t) \).
Lemma 16 Let \( n \in (0, m) \) be an integer and a function \( u(x, t) \in C^{m+n,\gamma}_{n,\omega\gamma} (\bar{Q}) \) in the sense of (1.19) has compact support. Then \( D_{x_N}^{m-n} u(x, t) \) is bounded and

\[
\langle D_{x_N}^{m-n} u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \leq C \langle x_N^n D_{x_N}^m u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}}
\]  

(4.5)

if and only if

\[
[x_N^n D_{x_N}^m u(x, t)]|_{x_N=0} = 0.
\]  

(4.6)

**Proof.** From (2.60) and (2.61) it follows that condition (4.6) is equivalent to the condition

\[
[x_N D_{x_N}^{m-n+1} u(x, t)]|_{x_N=0} = 0.
\]

Denote \( a(x, t) \equiv x_N D_{x_N}^{m-n+1} u(x, t) \) and note that by Lemma 14 \( \langle a(x, t) \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \leq C \langle x_N^n D_{x_N}^m u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \). We have the following representation

\[
D_{x_N}^{m-n} u(x, t) = - \int_{x_N}^{\infty} D_{x_N}^{m-n+1} u(x', \xi_N, t) d\xi_N = - \int_{x_N}^{\infty} \frac{1}{\xi_N} a(x', \xi_N, t) d\xi_N.
\]  

(4.7)

From this it directly follows that \( D_{x_N}^{m-n} u(x, t) \) is bounded at \( x_N = 0 \) if and only if \( a(x', 0, t) \equiv 0 \) that is if and only if (4.6) is valid. Further, suppose that \( a(x', 0, t) \equiv 0 \). Let \( x_N, \bar{Q}_N \) be fixed, \( 0 < x_N < \bar{Q}_N \). We have

\[
[x_N^{\omega\gamma} \left[ D_{x_N}^{m-n} (x', \bar{Q}_N, t) - D_{x_N}^{m-n} (x', x_N, t) \right]] =
\]

\[
= \int_{\xi_N}^{\infty} \frac{1}{\xi_N} [x_N^{\omega\gamma} a(x', \xi_N, t)] d\xi_N \leq \int_{x_N}^{\bar{Q}_N} \frac{1}{\xi_N} [\xi_N^{\omega\gamma} [a(x', \xi_N, t) - a(x', 0, t)]] d\xi_N \leq
\]

\[
\leq \langle a(x, t) \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \int_{x_N}^{\bar{Q}_N} \frac{\xi_N^{\omega\gamma}}{\xi_N} d\xi_N \leq C \langle a(x, t) \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} (\bar{Q}_N - x_N)^{\gamma}.
\]

In view of the definition of \( a(x, t) \) this means that

\[
\langle D_{x_N}^{m-n} u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \leq C \langle x_N^n D_{x_N}^{m-n+1} u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}} \leq C \langle x_N^n D_{x_N}^m u \rangle^{(\gamma)}_{\omega\gamma, x_N, \bar{Q}}
\]

hence, the lemma. ■
Corollary 17 Let $n \in (0, m)$ be an integer. Let a function $u(x, t) \in C_{n, \omega_{\gamma}}^{m+\gamma} (\mathbb{Q})$ in the sense of (1.19) has compact support and condition (4.6) is fulfilled. Then all seminorm of the function $u(x, t)$ in the left hand side of (4.21) are finite and

$$
\langle u \rangle^{(m+\gamma, m+\gamma)}_{n, \omega_{\gamma}, \mathbb{Q}} = \sum_{j=0}^{j\leq n} \sum_{|\alpha|=m-j} \langle x_{N}^{\gamma} D_{x}^{\alpha} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma, m)} + \sum_{j=0}^{j\leq n} \sum_{|\alpha|=m-j} \langle x_{N}^{\gamma} D_{t}^{\alpha} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma, m)} +
$$

$$
+ \sum_{j=0}^{j\leq m-n} \sum_{|\alpha|=m-n-j} \langle D_{x, N}^{\alpha} D_{x,N}^{\gamma} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(1, \omega)} + \sum_{j=0}^{j\leq m-n} \sum_{|\alpha|=m-n-j} \langle D_{x, N}^{\alpha} D_{x,N}^{\gamma} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma)}
$$

$$
+ \sum_{j=0}^{j\leq m-n} \sum_{|\alpha|=m-n-j} \langle D_{t, N}^{\alpha} D_{t,N}^{\gamma} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(1, \omega)} + \sum_{j=0}^{j\leq m-n} \sum_{|\alpha|=m-n-j} \langle D_{t, N}^{\alpha} D_{t,N}^{\gamma} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma)}
$$

The corollary follows from Proposition [15] and Lemma [16].

Consider now derivatives $D_{x}^{\alpha} u$ of order $|\alpha| < m - n$. If condition (4.6) is fulfilled then all such derivatives are characterized by the following statement.

Corollary 18 Let $n \in (0, m)$ be an integer. Let a function $u(x, t) \in C_{n, \omega_{\gamma}}^{m+\gamma} (\mathbb{Q})$ in the sense of (1.19) has compact support and condition (4.6) is fulfilled. For multiindex $\alpha$ with $|\alpha| = j < m - n$ denote $v_{\alpha} = D_{x}^{\alpha} u$. The following estimate is valid

$$
\sum_{i=1}^{N} \langle D_{x,i} v_{\alpha} \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma, m)} + \langle v_{\alpha} \rangle_{t, \mathbb{Q}}^{(1, \gamma - m + \gamma)} \leq C \left( \sum_{i=1}^{N} \langle x_{N}^{\gamma} D_{x,i}^{m} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma)} + \langle D_{t}^{\gamma/m} u \rangle_{t, \mathbb{Q}} \right).
$$

(4.9)

This corollary directly follows from (4.8).

In general case we have a weaker statement about smoothness of the derivatives $D_{x}^{\alpha} u$ of order $|\alpha| < m - n$. In particular, the derivative $D_{x,N}^{m-n-1} u$ belongs only to a kind of Zygmund space $Z^{1}$ - see (4.12) below.

Proposition 19 Let $n \in (0, m)$ be an integer. Let a function $u(x, t) \in C_{n, \omega_{\gamma}}^{m+\gamma} (\mathbb{Q})$ has compact support. For $v_{\alpha} = D_{x}^{\alpha} u$ with $|\alpha| = j < m - n - 1$ we have

$$
\sum_{i=1}^{N} \langle D_{x,i} v_{\alpha} \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma, m)} + \langle v_{\alpha} \rangle_{t, \mathbb{Q}}^{(1, \gamma - m + \gamma)} \leq C \left( \sum_{i=1}^{N} \langle x_{N}^{\gamma} D_{x,i}^{m} u \rangle_{\omega_{\gamma}, \mathbb{Q}}^{(\gamma)} + \langle D_{t}^{\gamma/m} u \rangle_{t, \mathbb{Q}} \right).
$$

(4.10)
For \( v_\alpha = D_\alpha u \) with \( |\alpha| = m - n - 1 > 0 \) we have

\[
\sum_{xN<n-n-1} \sum_{|\alpha|=m-n-1} \langle D_x v_\alpha \rangle_{\omega_N,\overline{Q}}^{(\gamma/m)} + \sum_{i=1}^{N-1} \langle D_x D_{xN}^{m-n-1} u \rangle_{\omega_N,\overline{Q}}^{(\gamma/m)} + \sum_{|\alpha|=m-n-1} \langle v_\alpha \rangle_{t,\overline{Q}}^{(1-m-n-1+\gamma/m)} \leq C \left( \sum_{i=1}^{N} \langle xN^{m} D_x u \rangle_{\omega_N,\overline{Q}}^{(\gamma)} + \langle v_N \rangle_{t,\overline{Q}}^{(\gamma/m)} \right) \quad (4.11)
\]

and also

\[
[D_{xN}^{m-n-1} u]^{(1-(\gamma/m))}_{xN,x'} + [D_{xN}^{m-n-1} u]^{(1-(\gamma/m))}_{xN,t} \leq C \left( \sum_{i=1}^{N} \langle xN^{m} D_x u \rangle_{\omega_N,\overline{Q}}^{(\gamma)} + \langle v_N \rangle_{t,\overline{Q}}^{(\gamma/m)} \right). \quad (4.12)
\]

where

\[
[v]^{(1-(\gamma/m))}_{xN,x',Q} = \sup_{\theta>0,\tau>0, (x,t) \in \overline{Q}} \frac{\| \Delta_{xN}^{\gamma} \Delta_{\tau}^{\gamma} v(x,t) \|}{\theta |h|^{(1-(\gamma/m))}}, \quad (4.13)
\]

\[
[v]^{(1-(\gamma/m))}_{xN,x',t} = \sup_{\theta>0,\tau>0, (x,t) \in \overline{Q}} \frac{\| \Delta_{xN}^{\gamma} \Delta_{\tau}^{\gamma} v(x,t) \|}{\theta \tau^{(1-(\gamma/m))}}. \quad (4.14)
\]

**Proof.** The proof of (4.10) and (4.11) is completely similar to the proof of Proposition 15 in view of (2.51). Let us prove (4.12). Since estimates for both terms in (4.12) are completely similar, we estimate only the term \( [D_{xN}^{m-n-1} u]^{(1-(\gamma/m))}_{xN,x'} \).

Denote \( a(x,t) = xN D_{xN}^{m-n} u \). We have from (4.8) and from Proposition 4

\[
\langle a(x,t) \rangle_{\gamma/m}^{(\gamma/m)} + \langle a(x,t) \rangle_{\gamma}^{(\gamma)} xN,Q \leq C \left( \sum_{i=1}^{N} \langle xN^{m} D_x u \rangle_{\omega_N,\overline{Q}}^{(\gamma)} + \langle v_N \rangle_{t,\overline{Q}}^{(\gamma/m)} \right). \quad (4.15)
\]

From the definition of \( a(x,t) \) we have

\[
D_{xN}^{m-n-1} u = \int_{xN}^{\infty} d\xi \int_{xN}^{\infty} D_{xN}^{m-n+1} u(x',\eta,t) d\eta = \int_{xN}^{\infty} d\xi \int_{xN}^{\infty} \frac{1}{\eta} a(x',\eta,t) d\eta. \quad (4.16)
\]
Let $\overline{h} \in R^{N-1}$ and $\theta > 0$ be fixed. Denote $b(x', \eta, t) = a(x', \eta + \overline{h}, t) - a(x', \eta, t)$. Then by simple direct calculations

$$\left| \Delta_{\overline{h},x,N}^2 \Delta_{\overline{h},x} D_{x,N}^m u(x, t) \right| = \left| \Delta_{\overline{h},x,N}^2 \int_{x_N}^{\infty} \int_{\xi}^{\eta} \frac{1}{\eta} b(x', \eta, t) d\eta \right| =$$

$$= \int_{x_N}^{x_N+\theta} d\xi \int_{\xi}^{\xi+\theta} \frac{1}{\eta} b(x', \eta, t) d\eta \leq |b^{(0)}|_{\overline{Q}^2} \int_{x_N}^{x_N+\theta} d\xi \int_{\xi}^{\xi+\theta} \frac{1}{\eta} d\eta =$$

$$= |b^{(0)}|_{\overline{Q}^2} \int_{x_N}^{x_N+\theta} (\ln(\xi + \theta) - \ln \xi) d\xi \leq C |b^{(0)}|_{\overline{Q}^2} \leq C \left( x_N D_{x,N}^{m-n+1} u \right)^{(\gamma)} \left( 1 - \omega \right)^{\gamma} \overline{h}.$$ 

Since $\overline{h}$ and $\theta$ are arbitrary, this proves estimate (1.12) for the first term. The estimate of the second term is completely analogous. This completes the proof of the proposition.

$$\blacksquare$$

### 4.2 The case of a noninteger $n$.

In this case we have the following propositions analogous to Propositions 15, 19.

**Proposition 20** Let $n \in (0, m)$ be a noninteger. Let a function $u(x, t) \in C^{m+\gamma, m+\gamma}_{n, \omega \gamma} (\overline{Q})$ in the sense of (1.19) has compact support.

Then

$$\langle u \rangle^{(m+\gamma, m+\gamma)}_{n, \omega \gamma, \overline{Q}} \equiv \sum_{j=0}^{j<n} \sum_{|\alpha|=m-j} \langle x_N^{m-j} D_x^\alpha u \rangle^{(\gamma, \gamma/m)}_{\omega \gamma, \overline{Q}} +$$

$$+ \sum_{j=0}^{j<n} \sum_{|\alpha|=m-j} \langle x_N^{m-j} D_x^\alpha u \rangle^{(\gamma, \gamma/m)}_{\omega \gamma, \overline{Q}} + \langle D_t u \rangle^{(\gamma, \gamma/m)}_{\omega \gamma, \overline{Q}} +$$

$$+ \sum_{j=0}^{j<m-n} \sum_{|\alpha|=m-n-(1-\omega)\gamma-j} \langle D_x^\alpha D_{x,N}^j u \rangle^{(m-n+(1-\omega)\gamma)}_{\omega \gamma, \overline{Q}} +$$

$$+ \sum_{j=0}^{j<m-n} \sum_{|\alpha|=m-n+\gamma-j} \langle D_x^\alpha D_{x,N}^j u \rangle^{(m-n+\gamma)}_{\omega \gamma, \overline{Q}} +$$

$$+ \sum_{j=1}^{j<m-n} \sum_{|\alpha|=j} \langle D_x^\alpha u \rangle^{(1-\omega \gamma/m)}_{\omega \gamma, \overline{Q}} \leq C \left( \sum_{i=1}^{N} \langle x_N^m D_{x_i}^{m} u \rangle^{(\gamma)}_{\omega \gamma, \overline{Q}, i} + \langle D_t u \rangle^{(\gamma/m)}_{\omega \gamma, \overline{Q}} \right). \quad (4.17)$$
The proof of this proposition is completely analogous to the proof of Proposition 10.

In addition we have the following proposition.

**Proposition 21** Let \( n \in (0, m) \) be a noninteger. Let a function \( u(x, t) \in C^{m+\gamma, m+\gamma}_{n, \omega \gamma} (\bar{Q}) \) in the sense of \((1.19)\) has compact support in a set \( \{ x_N \leq R \} \), \( R > 0 \). Then

\[
\langle D_{x_N}^{[m-n]} u \rangle_{x_N, \bar{Q}}^{(1-\{n\})} \leq C |x_N^{[n]} D_{x_N}^{[m-n]+1} u|_{0, \bar{Q}}^{(0)} \leq C(R) \langle x_N^{[n]} D_{x_N}^{[m-n]+1} u \rangle_{\omega, \bar{Q}}^{(\gamma)} \leq C(R) \langle x_N^{n} D_{x_N}^{m} u \rangle_{\omega, \bar{Q}}^{(\gamma)}.
\]

(4.18)

The proof of this proposition directly follows from the Newton-Leibnitz formula and from \((1.17)\).

5 Traces of functions from \( C^{m+\gamma, m+\gamma}_{n, \omega \gamma} (\bar{Q}) \) at \( \{ x_N = 0 \} \).

As it was proved in Theorem 2 for \( u(x, t) \in C^{m+\gamma, m+\gamma}_{n, \omega \gamma} (\bar{Q}) \) we have

\[
\sum_{j=0}^{j \leq m-n} \sum_{|\alpha| = |m-n+(1-\omega)\gamma| - j} \langle D_{x_N}^{\alpha} D_{x_N}^{j} u \rangle_{x_N, \bar{Q}}^{\{m-n+(1-\omega)\gamma\}} + \sum_{j=1}^{j \leq m-n} \sum_{|\alpha| = j} \langle D_{x_N}^{\alpha} u \rangle_{x_N, \bar{Q}}^{\{1-\frac{m-n}{m} + \frac{\gamma}{m}\}} \\
\leq C \left( \sum_{i=1}^{N} \langle x_N^{m} D_{x_N}^{m} u \rangle_{\omega, x_i, \bar{Q}}^{(\gamma)} + \langle D_{t} u \rangle_{x_i, \bar{Q}}^{(\gamma/m)} \right),
\]

(5.1)

where the terms for \( j = m - n \) are included if \( n \) is an integer and if \( D_{x_N}^{m-n} u \) is bounded. From this estimate it follows that for \( j \leq m - n \) and for a fixed \( x_N > 0 \) the function \( D_{x_N}^{j} u(x', x_N, t) \) belongs to the space \( C^{m-n+(1-\omega)\gamma-j, 1-\frac{m-n}{m} + \frac{\gamma}{m}}_{x', t} (R^{N-1} \times R^1) \). This means that we have the following statement.

**Proposition 22** A function \( u(x, t) \in C^{m+\gamma, m+\gamma}_{n, \omega \gamma} (\bar{Q}) \) with compact support and its derivatives \( D_{x_N}^{j} u, j \leq m - n \), have traces at \( x_N = 0 \) from the spaces \( D_{x_N}^{j} u(x', 0, t) \in C^{m-n+(1-\omega)\gamma-j, 1-\frac{m-n}{m} + \frac{\gamma}{m}}_{x', t} (R^{N-1} \times R^1) \) and

\[
\| D_{x_N}^{j} u(x', 0, t) \|_{C^{m-n+(1-\omega)\gamma-j, 1-\frac{m-n}{m} + \frac{\gamma}{m}}_{x', t} (R^{N-1} \times R^1)} \leq C \left( \sum_{i=1}^{N} \langle x_N^{m} D_{x_N}^{m} u \rangle_{\omega, x_i, \bar{Q}}^{(\gamma)} + \langle D_{t} u \rangle_{x_i, \bar{Q}}^{(\gamma/m)} \right).
\]

(5.2)
Now we consider the question of the extension of functions \( v(x', t) \) from the class \( C^{n-\alpha} \) to the region \( Q \).

**Theorem 23** There exists an operator \( E : C^{n-\alpha}(\Omega) \rightarrow C^{n-\alpha}(\Omega) \) defined on functions with compact supports in \( B_R = \{ |x'| \leq R, |t| \leq R \} \) with the property:

for a given function \( v(x', t) \in C^{n-\alpha}(\Omega) \) with compact support in \( B_R \) the function \( w(x, t) = Ev \in C^{n-\alpha}(\Omega) \) has compact support and satisfies

\[
w(x', 0, t) = v(x', t), \quad \|w\|_{C^{n-\alpha}(\Omega)} \leq C \|v\|_{C^{n-\alpha}(\Omega)}, \quad (5.3)
\]

where the constant \( C \) does not depend on \( v \).

**Proof.**

The proof is similar to the proof of corresponding Lemma 2.4 from [1]. Let we are given a function \( v(x', t) \in C^{n-\alpha}(\Omega) \) with compact support. Consider the following boundary problem with \( t \) as a parameter for an unknown function \( u(x, t) \)

\[
\Delta u(x, t) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} = 0, \quad x \in H, t \in \mathbb{R}^1,
\]

\[
u(x', 0, t) = v(x', t), \quad x' \in \mathbb{R}^N, t \in \mathbb{R}.
\]

It is well known (see [29], [30], for example) that for a fixed \( t > 0 \) problem (5.4), (5.5) has the unique bounded solution \( u(x, t) \) with

\[
\|u(\cdot, t)\|_{C^{n-\alpha}(\Omega)} \leq C \|v(\cdot, t)\|_{C^{n-\alpha}(\Omega)}.
\]

In Lemma 2.4 from [1] also proved that

\[
\left\langle \frac{\partial u(x, t)}{\partial t} \right\rangle_{t, \Omega} \leq C \left\langle \frac{\partial v(x', t)}{\partial t} \right\rangle_{t, \Omega}.
\]

Therefore it is enough to consider the properties of \( u(x, t) \) with respect to the variables \( x \). For this we will use the following inequality (see [31], Chapter 5.4)

\[
|D^\alpha_x u(x, t)| \leq C_{\alpha} x_N^{-|\alpha|+l} \|v(\cdot, t)\|_{C^{l}(\mathbb{R}^N)}, \quad |\alpha| \geq l,
\]

We now prove the estimate
\[
\sum_{i=1}^{N} \langle x_N^m D^m_{x_i} u(\cdot, t) \rangle_{\omega, x_i, \Omega}^{(\gamma)} \leq C \| v(\cdot, t) \|_{C_0^{m-n+(1-\omega)\gamma}(R^{N-1})}. \tag{5.9}
\]

Since it is important to prove (5.9) for \( x_N < 1 \) only (for \( x_N > 1 \), such the estimate follows from the local estimates and is well-known), we consider only the case \( x_N < 1 \). We also use the well-known interpolation inequality
\[
\langle v(x) \rangle_{x, \Omega}^{(\gamma)} \leq C \left( \| v \|^{(0)}_{\Omega} \right)^{1-\gamma} \left( \| v \|^{(1)}_{\Omega} \right)^{\gamma}, \tag{5.10}
\]

which is valid for functions \( v(x) \in C^1(\Omega) \), where \( \Omega \) is a domain (possibly unbounded) with sufficiently smooth boundary (see, for example, [32], Ch.1). It is important that the constant \( C \) does not depend on the size of the domain \( \Omega \) under scaling. Consider first the tangent variables \( x_i, i = 1, N-1 \).

Let \( x_N \) be fixed. Then by (5.9) and (5.10),
\[
x_N^{\omega_i^{(\gamma)}} \langle x_N^m D^m_{x_i} u(\cdot, x_N) \rangle_{x, R^{N-1}}^{(\gamma)} \leq C \left( \| x_N^m D^m_{x_i} u(\cdot, x_N) \|^{(0)}_{R^{N-1}} \right)^{1-\gamma} \left( \| x_N^m D^m_{x_i} u(\cdot, x_N) \|^{(1)}_{R^{N-1}} \right)^{\gamma} \leq C \| u \|_{C_0^{m-n+(1-\omega)\gamma}(R_{x_i}^{N-1})} \left( x_N^m x_{x_N}^{m-1+(m-n+(1-\omega)\gamma)} \right)^{1-\gamma} \left( x_N x_{x_N}^{-1} \right)^{\gamma} \leq C \| u \|_{C_0^{m-n+(1-\omega)\gamma}(R_{x_N}^{N-1})}. \tag{5.11}
\]

By the definition, this means that
\[
\langle x_N^m D^m_{x_i} u(\cdot, t) \rangle_{\omega, x_i, \Omega}^{(\gamma)} \leq C \| v(\cdot, t) \|_{C_0^{m-n+(1-\omega)\gamma}(R^{N-1})}, \quad i = 1, N-1.
\]

We consider now \( \langle x_N^m D^m_{x_N} u(\cdot, t) \rangle_{\omega, x_N, \Omega}^{(\gamma)} \). Let \( x_N \) and \( \bar{x}_N \) be fixed. We fix some \( \varepsilon_0 \in (0, 1/16) \) and consider the two cases, assuming without loss of generality that \( \varepsilon_0 \leq x_N \). Let first
\[
|x_N - \bar{x}_N| = (x_N - \bar{x}_N) \geq \varepsilon_0 x_N.
\]

Then
\[
x_N^{\omega_i^{(\gamma)}} \frac{|x_N^m D^m_{x_N} u(x, t) - \bar{x}_N^m D^m_{x_N} u(\bar{x}, t)|}{|x_N - \bar{x}_N|^{\gamma}} \leq C \left( |x_N^{-1} \gamma \bar{D}_{x_N}^m u(x, t)| + |\bar{x}_N^{-1} \gamma D_{x_N}^m u(\bar{x}, t)| \right). \tag{5.12}
\]

In this case, as above

52
\[ |x_N^{n-(1-\omega)\gamma} D_{x_N}^m u(x, t)| \leq C \|v\|_{C_x^{m-n+(1-\omega)\gamma}(R_T^{N-1})} x_N^{n-(1-\omega)\gamma} x_N^{-m+(1-n+(1-\omega)\gamma)} = \]

\[ = C |u|_{C_x^{m-n+(1-\omega)\gamma}(R_T^{N-1})} \leq C \|v\|_{C_x^{m-n+(1-\omega)\gamma}(R_T^{N-1})}, \]

and similarly for \[|x_N^{n-(1-\omega)\gamma} D_{x_N}^m u(\xi, t)|.\]

Let now

\[ 0 < (x_N - \xi_N) \leq \varepsilon_0 x_N, \quad (5.14) \]

and let also

\[ \Pi(x_N) = \{ y \in R_T^N : x_N - 2\varepsilon_0 x_N \leq y_N \leq x_N + 2\varepsilon_0 x_N \} \quad (5.15) \]

Then, taking into account that on \( \Pi(x_N) \) we have \( y_N \sim x_N \), as in the previous case

\[ x_N^{\omega\gamma} |x_N^{n} D_{x_N}^m u(x, t) - \bar{x}_N^{n} D_{x_N}^m u(\bar{x}, t)| \leq x_N^{\omega\gamma} \langle y_N^{n} D_{y_N}^m u(y, t) \rangle_{y, \Pi(x_N)}^{(\gamma)} \leq \]

\[ \leq C \left( x_N^{\omega\gamma} |y_N^{n-\gamma} D^2 u|^{(0)}_{\Pi(x_N)} + x_N^{\omega\gamma} x_N^{n} \langle D_{y_N}^m u(y, t) \rangle_{y, \Pi(x_N)}^{(\gamma)} \right) \equiv A_1 + A_2. \quad (5.16) \]

Here \( A_1 \) is estimated in the same way as in \( (5.13) \), and \( A_2 \) - in similar way after the estimate

\[ \langle D_{y_N}^m u(y, t) \rangle_{y, \Pi(x_N)}^{(\gamma)} \leq C \left( |D_{y_N}^m u(y, t)|^{(0)}_{\Pi} \right)^{1-\gamma} \left( |D_{y_N}^{m+1} u(y, t)|^{(1)}_{\Pi} \right)^{\gamma}. \]

This gives

\[ \langle x_N^{n} D_{x_N}^m u(\cdot, t) \rangle_{\omega\gamma, x_N, \Pi}^{(\gamma)} \leq C \|v(\cdot, t)\|_{C^{m-n+(1-\omega)\gamma}(R_T^{N-1})} \]

Now fix \( \eta(x, t) \in C^\infty(\bar{Q}) \) with compact support and with \( \eta(x', 0, t) \equiv 1 \) on \( B_R \). Then we can define \( w(x, t) \equiv E\eta(x, t) \equiv u(x, t)\eta(x, t) \). This completes the proof of the theorem.

\[ \blacksquare \]
6 Some interpolations inequalities for functions from $C_{n,ωγ}^{m+γ} \frac{m+γ}{m} (Q)$, $C_{n,ωγ}^{m+γ} (H)$.

In this section we prove some interpolation inequalities for functions from the spaces $C_{n,ωγ}^{m+γ} \frac{m+γ}{m} (Q)$, $C_{n,ωγ}^{m+γ} (H)$. These inequalities are consequences of (1.21), (1.23) and they are useful in applications.

**Theorem 24** Let a function $u(x) \in C_{n,ωγ}^{m+γ} (H)$ and $α = (α_1, ..., α_N)$, $|α| = m$, be a multiindex, $k \in \{1, 2, ..., N\}$. Then for any $ε > 0$ ($ε$ may be chosen big or small)

$$\langle x^n D^α_x u \rangle_{α,ωγ,ωγ} \leq C \varepsilon^{-α_k - γ} \sum_{i=1, i \neq k}^N \langle x^n D^m_x u \rangle_{m,ωγ,ωγ} + C \varepsilon^{-α_k} \langle x^n D^m_x u \rangle_{m,ωγ,ωγ}, \quad k < N; \quad (6.1)$$

$$\langle x^n D^α_x u \rangle_{α,ωγ,ωγ} \leq C \varepsilon^{-α_k - (1-ω)γ} \sum_{i=1}^{N-1} \langle x^n D^m_x u \rangle_{m,ωγ,ωγ} + C \varepsilon^{-α_k} \langle x^n D^m_x u \rangle_{m,ωγ,ωγ}, \quad k = N; \quad (6.2)$$

where the constants $C$ does not depend on $ε$, $u$.

**Proof.**

Let $ε > 0$ be fixed. Consider the function $v_ε(y) = u(y_1, y_2, ..., εy_k, ...y_{N-1}, y_N) \in C_{n,ωγ}^{m+γ} (H)$. Then from (1.23) we have

$$\langle y^n D^α_y v_ε \rangle_{α,ωγ,ωγ} \leq C \sum_{i=1}^N \langle y^n D^m_y v_ε \rangle_{m,ωγ,ωγ}, \quad (6.3)$$

where the constant $C$ does not depend on $ε$. Now make in (6.3) the change of the variables

$$y = e(x): \quad y_i = x_i, i \neq k, \quad y_k = ε^{-1} x_k$$

and take into account that $v_ε(y) \circ e(x) = u(x)$. This gives (6.1), (6.2) and completes the proof.

**Theorem 25** Let a function $u(x) \in C_{n,ωγ}^{m+γ} \frac{m+γ}{m} (Q)$ and $α = (α_1, ..., α_N)$, $|α| = m$, be a multiindex, $k \in \{1, 2, ..., N\}$. Then for any $ε > 0$
Theorem 26 Let a function $u(x, t) \in C_{n, \omega t}^{m+\gamma+m} (Q)$ has compact support. Let the support of $u(x, t)$ is included in $Q_R = \{ |x| \leq R, |t| \leq R \}$. Then for an integer $0 \leq j < n$ and for an arbitrary $h > 0$

$$
\sum_{|\alpha|=m-j} |x^{n-j} D_x^\alpha u(x, t)|_{\omega t, x, Q}^{(0)} \leq C h^{(1-\omega)\gamma} \sum_{|\alpha|=m-j} \langle x^{n-j} D_x^\alpha u(x, t) \rangle_{\omega t, x, Q} +
$$

$$
+ \frac{(1 + R)}{h} \sum_{|\alpha|=m-j-1} |x^{n-j-1} D_x^\alpha u(x, t)|_{\omega t, x, Q}^{(0)}, \quad n - j \geq 1,
$$

where the constants $C$ does not depend on $\varepsilon, u$.

The proof of this theorem is identical to that of the previous theorem.
\[
\sum_{|\alpha|=m-j} |x_n^{n-j} D_x^\alpha u(x, t)|^{(0)}_{Q} \leq C \left( h^{(1-\omega)\gamma} \sum_{|\alpha|=m-j} \langle x_N D_x^\alpha u(x, t) \rangle_{\omega \gamma, x, Q} \right) + (6.10)
\]

\[
+ \frac{(1 + R^{n-j})}{h} \sum_{|\alpha|=m-j-1} |D_x^\alpha u(x, t)|^{(0)}_{Q}, \quad n - j < 1.
\]

**Proof.**

Let \( n-1 > 0 \). Consider first the estimate of a derivative \( D_x^\alpha u \), \(|\alpha| = m-1\) with respect to \( x_N \). Let \( h > 0 \) and \( \varepsilon \in (0, 1) \) be fixed. We have

\[
x_N^n D_x^n D_x^\alpha u(x, t) = \left( x_N^n D_x^n D_x^\alpha u(x, t) - x_N^n \frac{\Delta_{h,x_N} D_x^\alpha u(x, t)}{h} \right) +
\]

\[
+ x_N^n \frac{\Delta_{h,x_N} D_x^\alpha u(x, t)}{h} =
\]

\[
= -x_N^n \int_0^1 \left[ D_x^n D_x^\alpha u(x', x_N + \theta h, t) - D_x^n D_x^\alpha u(x, t) \right] d\theta + x_N^n \frac{\Delta_{h,x_N} D_x^\alpha u(x, t)}{h} = A_1 + A_2
\]

and evidently

\[
|A_2| \leq CR \frac{|x_N^{n-1} D_x^\alpha u(x, t)|^{(0)}_{Q}}{h}.
\]

The expression \( A_1 \) we represent as

\[
A_1 = -\int_0^1 \Delta_{h,x_N} \left[ x_N^n D_x^n D_x^\alpha u(x, t) \right] d\theta +
\]

\[
+ \int_0^1 \left[ (x_N + \theta h)^n - x_N^n \right] D_x^n D_x^\alpha u(x', x_N + \theta h) d\theta = A_{11} + A_{12}.
\]

The estimate of \( A_{11} \) is

\[
|A_{11}| \leq \left( \int_0^1 \frac{|\Delta_{h,x_N} \left[ x_N^n D_x^n D_x^\alpha u(x, t) \right]|}{(\theta h)^{(1-\omega)\gamma}} d\theta \right)^{\gamma (1-\omega)} \leq
\]

56
To estimate $A_{12}$ we apply the integration by parts. This gives ($y = \theta h$)

$$A_{12} = \frac{1}{h} \left[ (x_N + h)^n - x_N^n \right] \frac{D^a_x u(x', x_N + h)}{h} - \frac{n}{h} \int_0^h (x_N + y)^{n-1} D^a_x u(x', x_N + y) dy$$

and we obtain

$$|A_{12}| \leq C (1 + R) \frac{|x_N^{n-1} D^a_x u(x, t)|^{(0)}_{\mathcal{Q}}}{h}$$.  \hspace{1cm} (6.14)

Note that in the case $n < 1$ the derivative $D^a_x u(x, t)$ and we have

$$\left| \int_0^h (x_N + y)^{n-1} D^a_x u(x', x_N + y) dy \right| \leq |D^a_x u(x, t)|^{(0)}_{\mathcal{Q}} h \int_0^h (x_N + y)^{n-1} dy \leq CR^n |D^a_x u(x, t)|^{(0)}_{\mathcal{Q}}$$

From (6.11)-(6.14) it follows that

$$|x_N^n D_{x_N} D^a_x u(x, t)|^{(0)}_{\mathcal{Q}} \leq C h^{(1-\omega)\gamma} \langle x_N^n D_{x_N} D^a_x u(x, t) \rangle^{(\gamma)}_{\omega\gamma, x_N, \mathcal{Q}} +$$

$$+ \left\{ \begin{array}{l}
(1 + R) \frac{|x_N^{n-1} D^a_x u(x, t)|^{(0)}_{\mathcal{Q}}}{h}, n \geq 1, \\
(1 + R^n) \frac{|D^a_x u(x, t)|^{(0)}_{\mathcal{Q}}}{h}, n < 1.
\end{array} \right. $$

The estimates of other derivatives $x_N^n D_{x_N} D^a_x u(x, t)$ are completely similar and give for an arbitrary $h > 0$

$$\sum_{|\alpha| = m} |x_N^n D_{x_N} D^a_x u(x, t)|^{(0)}_{\mathcal{Q}} \leq C \left( h^{(1-\omega)\gamma} \sum_{|\alpha| = m} \langle x_N^n D^a_x u(x, t) \rangle^{(\gamma)}_{\omega\gamma, x, \mathcal{Q}} + \right.$$

$$+ \frac{(1 + R)}{h} \sum_{|\alpha| = m-1} |x_N^{n-1} D^a_x u(x, t)|^{(0)}_{\mathcal{Q}}, \hspace{1cm} n \geq 1.$$
\[ \frac{1}{h} \sum_{|\alpha|=m-1} |D^\alpha_x u(x,t)|^{(0)}_{|x|} + (1 + R^n) \], \quad n < 1. \]

The estimates of derivatives \( D^\alpha_x u(x,t) \) of order \(|\alpha| = m - j, j < n\), are obtained in the same way.

\[ \text{Corollary 27} \quad \text{The norms (1.17) and (1.19) and also the norms (1.18) and (1.20) are equivalent.} \]

This assertion follows from the previous theorem, from section 4 and from Theorem 2.

\section{The spaces \( C^{m+\gamma}_{m} (\Omega) \), \( C^{m+\gamma}_{n,\omega\gamma} (\overline{\Omega}) \) in the case of an arbitrary smooth domain.}

Let \( \Omega \) be a domain in \( R^N \) (bounded or unbounded) with boundary \( \partial \Omega \) of the class \( C^{m+\gamma} \). Let \( d(x) \) be a function of the class \( C^{1+\gamma}(\overline{\Omega}) \) with the property

\[ \nu \cdot \text{dist}(x, \partial \Omega) \leq d(x) \leq \nu^{-1} \cdot \text{dist}(x, \partial \Omega), \quad \text{dist}(x, \partial \Omega) \leq 1, \quad \nu > 0. \quad (7.1) \]

As such a function can serve, for example, the bounded solution of the problem

\[ \Delta d(x) = -1, \quad x \in \Omega, \quad d(x)|_{\partial \Omega} = 0. \]

For \( x, \overline{x} \in \Omega \) we denote \( d(x, \overline{x}) = \max\{d(x), d(\overline{x})\} \) and for a function \( v(x) \) denote

\[ \langle v \rangle_{\omega\gamma}^{(\gamma)} = \sup_{x, \overline{x} \in \Omega} d(x, \overline{x})^{\omega\gamma} \frac{|u(\overline{x}) - u(x)|}{|x - \overline{x}|^\gamma}. \]

Define the space \( C^{m+\gamma}_{\omega\gamma}(\overline{\Omega}) \) as the space of functions \( u(x) \) with the finite norm

\[ \|u\|_{C^{m+\gamma}_{\omega\gamma}(\overline{\Omega})} \equiv \|u\|_{C^{m+\gamma}_{\omega\gamma}(\overline{\Omega})}^{(0)} + \langle u \rangle_{\omega\gamma}^{(\gamma)}. \quad (7.2) \]

And define the space \( C^{m+\gamma}_{n,\omega\gamma}(\overline{\Omega}) \) as the space of continuous in \( \overline{\Omega} \) functions \( u(x) \) with the finite norm

\[ \|u\|_{C^{m+\gamma}_{n,\omega\gamma}(\overline{\Omega})} \equiv \|u\|_{C^{m+\gamma}_{n,\omega\gamma}(\overline{\Omega})}^{(0)} + \sum_{|\alpha|=m} \langle d(x)^m D^\alpha_x u(x) \rangle_{\omega\gamma}^{(\gamma)}. \quad (7.3) \]

For \( T > 0 \) denote \( \Omega_T = \{(x, t) : x \in \Omega, t \in (0, T)\} \) and define the space \( C^{m+\gamma}_{n,\omega\gamma} (\overline{\Omega}_T) \) as the space of continuous in \( \overline{\Omega}_T \) functions \( u(x, t) \) with the finite norm.
\[ \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0}} \equiv \|u\|_{C^{m,0}_{n,\omega,\gamma,0}} \equiv \|u(0)\| + \sum_{|\alpha|=m} \langle d(x)^n D_x^\alpha u(x,t) \rangle^{(\gamma)}_{\omega,\gamma,\Omega_T} + \langle D_t u \rangle^{(\gamma/m)}_{\Omega_T}. \] (7.4)

Theorem 28 Let \( \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0}} \leq \infty \). Then

\[ \sum_{j=0}^{j=n} \sum_{|\alpha|=m-j} \langle d(x)^{n-j} D_x^\alpha u \rangle^{(\gamma/m)}_{\omega,\Omega_T} + \sum_{j=0}^{j=n} \sum_{|\alpha|=m-j} \langle d(x)^{n-j} D_x^\alpha u \rangle^{(\gamma/m)}_{\omega,\Omega_T} + \sum_{j=1}^{j=m-n} \sum_{|\alpha|=m-j} |D_x^\alpha u|^{(\gamma/m)}_{\omega,\Omega_T} \leq C \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0}}. \] (7.5)

If \( n \) is an integer and \( d(x)^n D_x^\alpha u|_{\partial \Omega} \equiv 0 \) for \( |\alpha| = m \) then also

\[ \sum_{|\alpha|=m-n} |D_x^\alpha u|^{(\gamma/m)}_{\omega,\Omega_T} \leq C \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0}}. \] (7.6)

The proof of this theorem follows from the results of sections 1, 4 and 6 by localisation and considering the functions \( u(x,t)\eta(x) \), where \( \eta(x) \in C^\infty(\Omega) \) and has sufficiently small support near \( \partial \Omega \). After corresponding change of the variables \( v(x,t) = u(x,t)\eta(x) \) can be considered in a half-space \( Q \). The proof is pretty standard with the making use of the interpolation inequalities and therefore we omit it.

8 Spaces \( C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T) \).

We denote by \( C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T) \) the closed subspace of \( C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T) \) consisting of functions \( u(x,t) \) with the property \( u(x,0) \equiv u_t(x,0) \equiv 0 \) in \( \overline{\Omega} \).

Proposition 29 Let \( u(x,t) \in C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T) \), \( T \leq 1 \). Then for \( 1 \leq j \leq n \), \( |\alpha| = m-j \) and with some \( \delta > 0 \)

\[ |d(x)^n D_x^\alpha u|^{(\gamma/m)}_{\Omega_T} \leq C T^\delta \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T)}. \] (8.1)

And for \( |\alpha| < m-n \)

\[ |D_x^\alpha u|^{(\gamma/m)}_{\Omega_T} \leq C T^\delta \|u\|_{C^{m+\gamma, m+\gamma}_{n,\omega,\gamma,0} (\Omega_T)}. \] (8.2)
Proof.
First of all, since $D_x^0 u(x, 0) \equiv 0$,

$$|d(x)^n D_x^0 u(x, 0)|_{\overline{\Omega}_T} \leq C \langle d(x)^{n-j} D_x^0 u \rangle_{t, \overline{\Omega}_T}^{(\gamma/m)} T^{\gamma/m} \leq C T^{\gamma/m} \|u\|_{C_{n, \omega, 0}^{m+\gamma, m+\gamma}(\overline{\Omega}_T)}. \tag{8.3}$$

Further, let $t, \overline{t} \in [0, T]$. Then

$$\frac{|d(x)^n D_x^0 u(x, \overline{t}) - d(x)^n D_x^0 u(x, t)|}{|\overline{t} - t|^{\gamma/m}} =$$

$$= d^{j\omega}(x) \frac{|d(x)^{n-j} D_x^0 u(x, \overline{t}) - d(x)^{n-j} D_x^0 u(x, t)|}{|\overline{t} - t|^{j/m}} \leq$$

$$\leq C T^{j/m} \langle d(x)^{n-j} D_x^0 u \rangle_{t, \overline{\Omega}_T}^{(j/m)} \leq C T^{j/m} \|u\|_{C_{n, \omega, 0}^{m+\gamma, m+\gamma}(\overline{\Omega}_T)}.$$

This means

$$\langle d(x)^n D_x^0 u \rangle_{t, \overline{\Omega}_T}^{(\gamma/m)} \leq C T^{j/m} \|u\|_{C_{n, \omega, 0}^{m+\gamma, m+\gamma}(\overline{\Omega}_T)} \tag{8.4}.$$

Consider now the smoothness with respect to $x$ - variables. Note that the function $d(x)^n D_x^0 u(x, t)$ has bounded gradient in $x$ - variables (since $|\alpha| = m - j < m$)

$$\frac{\partial}{\partial x_i} d(x)^n D_x^0 u(x, t) = n \frac{\partial d(x)}{\partial x_i} \left[ d(x)^{n-1} D_x^0 u(x, t) \right] + d(x)^n D_x D_x^0 u(x, t),$$

where both terms are bounded in $\overline{\Omega}_T$ by $C \|u\|_{C_{n, \omega, 0}^{m+\gamma, m+\gamma}(\overline{\Omega}_T)}$.

Let now $x, \overline{x} \in \overline{\Omega}$. Consider the difference

$$A \equiv d(x, \overline{x})^{\omega \gamma} \frac{|d(\overline{x})^n D_x^0 u(\overline{x}, t) - d(x)^n D_x^0 u(x, t)|}{|\overline{x} - x|^{\gamma}}.$$

If $|\overline{x} - x| \geq t^{1/m}$ then

$$A \leq C \frac{|d(\overline{x})^{n-j} D_x^0 u(\overline{x}, t)| + |d(x)^{n-j} D_x^0 u(x, t)|}{t^{j/m}} \leq$$

$$\leq C \langle d(x)^{n-j} D_x^0 u \rangle_{t, \overline{\Omega}_T}^{(j/m)} T^{j/m} \leq C T^{j/m} \|u\|_{C_{n, \omega, 0}^{m+\gamma, m+\gamma}(\overline{\Omega}_T)}.$$

Let now $|\overline{x} - x| < t^{1/m}$. Then

$$A = \left( \frac{|d(\overline{x})^n D_x^0 u(\overline{x}, t) - d(x)^n D_x^0 u(x, t)|}{|\overline{x} - x|} \right) |\overline{x} - x|^{1-\gamma} \leq$$

$$60.$$
\[ \leq C |\nabla_x [d(x)^n D^\alpha_x u(x, t)]|^{(0)}_{\Omega_T} T^{\frac{m-n}{m}} \leq C T^{\frac{m-n}{m}} \|u\|_{C^{m+n, \gamma}_{n, \omega, \gamma, 0}}^{m+n, \gamma}_{n, \omega, \gamma, 0} (\Omega_T). \]

Thus, we have proved that

\[ \langle d(x)^n D^\alpha x u \rangle^{(\gamma)}_{\omega, \gamma, \Omega_T} \leq C T^{\delta} \|u\|_{C^{m+n, \gamma}_{n, \omega, \gamma, 0}}^{m+n, \gamma}_{n, \omega, \gamma, 0} (\Omega_T), \quad (8.5) \]

where \( \delta = (1 - \gamma)/m \). The estimate \((8.1)\) follows from \((8.3)-(8.5)\).

Let now \(|\alpha| < m - n\). If \(n\) is an integer, then \(D^\alpha_x u\) either has bounded derivatives in \(x\) of order not greater than \(m - n\) or such derivatives have the Hölder property with arbitrary exponent \(\gamma' \in (\gamma, 0)\) (if \(D^{m-n}_x u\) is not bounded). In both cases the proof of \((8.2)\) is completely analogous to the proof of \((8.1)\).

\[ \square \]

Note that the above property for unweighted space is well known - [33], [34].

References

[1] Degtyarev, S.P.: Classical solvability of multidimensional two-phase Stefan problem for degenerate parabolic equations and Schauder’s estimates for a degenerate parabolic problem with dynamic boundary conditions, Nonlinear Differential Equations and Applications (NoDEA). 22(2), 185-237 (2015).

[2] Knüpfer, H.: Well-posedness for the Navier slip thin-film equation in the case of partial wetting. Comm. Pure Appl. Math. 64(9), 1263–1296 (2011).

[3] Giacomelli, L., Knüpfer H., Otto, F.: Smooth zero-contact-angle solutions to a thin-film equation around the steady state. J. Differential Equations 245(6), 1454–1506 (2008).

[4] Giacomelli, L., Knüpfer, H.: A free boundary problem of fourth order: classical solutions in weighted Hölder spaces. Commun. Partial Differ. Equations, 35(10-12), 2059-2091 (2010).

[5] Giacomelli, L., Gnann, M.V., Knüpfer, H., Otto, F.: Well-posedness for the Navier-slip thin-film equation in the case of complete wetting. J. Differ. Equations 257(1), 15-81 (2014).

[6] Giacomelli, L., Gnann, M.V., Otto, F.: Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3. Eur. J. Appl. Math. 24(5), 735-760 (2013).

[7] Boutat, M., Hilout, S., Rakotoson, J.-E., Rakotoson, J.-M.: A generalized thin-film equation in multidimensional space. Nonlinear Anal. 69(4), 1268–1286 (2008).
[8] Bertsch, M., Giacomelli, L., Karali, G.: Thin-film equations with "partial wetting" energy: existence of weak solutions. Phys. D. 209(1-4), 17–27 (2005).

[9] Dal Passo, R., Garcke, H., Grün, G.: On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. SIAM J. Math. Anal. (2), 321–342 (1998).

[10] Dal Passo, R., Giacomelli, L., Shishkov, A.: The thin film equation with nonlinear diffusion. Commun. Partial Differ. Equations. 26 (9-10) 1509-1557 (2001).

[11] Giacomelli, L., Shishkov, A.: Propagation of support in one-dimensional convected thin-film flow. Indiana Univ. Math. J. 54 (4), 1181-1215 (2005).

[12] Novick-Cohen, A., Shishkov, A.: The thin film equation with backwards second order diffusion. Interfaces Free Bound. 12 (4), 463-496 (2010).

[13] Liang, B.: Mathematical analysis to a nonlinear fourth-order partial differential equation. Nonlinear Anal. textbf{74}(11), 3815–3828 (2011).

[14] Liu, C., Tian, Y.: Weak solutions for a sixth-order thin film equation. Rocky Mt. J. Math. 41 (5), 1547-1565 (2011).

[15] Liu, C.: Qualitative properties for a sixth-order thin film equation. Math. Model. Anal. 15 (4), 457-471 (2010).

[16] Dominik John: On Uniqueness of Weak Solutions for the Thin-Film Equation. ArXiv: http://arxiv.org/abs/1310.6222 (2013).

[17] Degtyarev S.P.: Liouville Property for Solutions of the Linearized Degenerate Thin Film Equation of Fourth Order in a Halfspace. Results in Mathematics. DOI 10.1007/s00025-015-0467-x, (2015).

[18] Bazalii, B.V., Degtyarev, S.P.: On classical solvability of the multidimensional Stefan problem for convective motion of a viscous incompressible fluid. Math. USSR Sb. 60(1), 1–17 (1988).

[19] Bizhanova, G.I., Solonnikov, V.A.: On problems with free boundaries for second-order parabolic equations. St. Petersburg Math. J. 12(6), 949–981 (2001).

[20] Daskalopoulos, P., Hamilton : Regularity of the free boundary for the porous medium equation. J.Amer.Math.Soc. 11 (4), 899–965 (1998).

[21] Sunghoon Kim, Ki-Ahm Lee : Smooth solution for the porous medium equation in a bounded domain. J.Differ.Equations. 247 (4), 1064–1095 (2009).
[22] Bazalij, B.V., Krasnoshchek, N.V.: Regularity of solution to a multidimensional free boundary problem for an equation of a porous medium. Siberian Advances in Mathematics. 13 (3), 1–53 (2003).

[23] Goulaouic, C. Shimakura, N.: Regularite holderienne de certains problemes aux limites elliptiques degeneres. Annali della Scuola Normale Superiore de Pisa, X (1), 79-108 (1983).

[24] Bazaliy, B.V., Degtyarev, S.P.: A boundary value problem for elliptic equations that degenerate on the boundary in weighted Hölder spaces. Sb. Math. 204 (7-8), 958–978 (2013).

[25] Triebel, H.: Theory of function spaces II. Reprint of the 1992 edition. Modern Birkhauser Classics. Basel: Birkhauser (2010).

[26] Solonnikov, V.A.: Estimates for solutions of a non-stationary linearized system of Navier–Stokes equations. In: Boundary value problems of mathematical physics. Part 1, Collection of articles, Trudy Mat. Inst. Steklov. 70, 213–317. Nauka, Moscow–Leningrad (1964).

[27] Golovkin, K.K.: On equivalent normalizations of fractional spaces. In: Automatic programming, numerical methods and functional analysis, Trudy Mat. Inst. Steklov., 66, 364–383. Acad. Sci. USSR, Moscow–Leningrad (1962)(Russian); English transl. Amer. Math. Soc. Transl. 81, 257–280 (1969).

[28] Simon, L.: Schauder estimates by scaling. Calc. Var. Partial Differ. Equ. 5 (5), 391-407 (1997).

[29] Ladyzhenskaya, O.A., Uraltseva, N.N.: Linear and Quasilinear Equations ofElliptic Type. Second edn. Nauka, Moscow, (1973).

[30] Solonnikov, V.A.: General boundary value problems for systems elliptic in thesense of A. Douglis and L. Nirenberg. II. (Russian). Trudy Mat. Inst. Steklov(Proc. Steklov Inst. Math.) 92, 233–297 (1966).

[31] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, No. 30, pp. xiv+290. Princeton University Press,Princeton (1970).

[32] Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems.Progress in Nonlinear Differential Equations and their Applications, vol. 16.Birkhauser (1995)
[33] Ladyzhenskaja, O.A., Solonnikov, V.A., Ural’tseva, N.N.: Linear and quasi-linear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968, xi+648 pp.

[34] Bizhanova, G.I.: Investigation of solvability of the multidimensional two-phase Stefan and the nonstationary filtration Florin problems for second order parabolic equations in weighted Hölder spaces of functions. Journal of Mathematical Sciences. 84 (1), 823–844 (1997).