Non-commutativity on another Minkowski space-time: Vierbein formalism and Higgs approach

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Abstract: We extend the non-commutative coordinates relationship into other than the Minkowski space-time. We clarify the non-commutativity dependency to the geometrical structure. As well as, we find an inverse map between Riemann’s normal and local coordinates. Furthermore, we show that behavior of the corresponding coordinates non-commutativity like as a tensor. All results summarized for the Schwarzschild metric. And they summarized for the space-time in the presence of weak gravitational waves.

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INTRODUCTION

Space-time at local is flat and the asymptotic conception of a more general space-time. It also has a correlation characteristic. When a pseudo-Riemann’s manifold gets the notion of correlation, then a system of measurements is available. One can always span each neighborhood of an event \( 'P' \) in space-time by local coordinates if it is possible to exist a unique geodesic joining between any points of the neighborhood and \( 'P'' \). Therefore, any Riemannian manifold equipped with normal coordinates \( \mathbb{R}^n \). Minkowski space-time has an asymptotic conception for any solutions to Einstein’s equation which is descriptive of a free-falling frame in this notion. Minkowski space-time always is available if the measurements are in the inertial frame. Vierbein field is a tool for generalizing a theorem from the tangential space to its main one. In general relativity, a vierbein field is a set of four orthonormal vector fields. Express of all tensorial quantities on a manifold is by using the frame field and its dual co-frame field. The set of the vierbein field contains \( "d"-vector \) fields. Their numbers are proportional to the dimension of space-time. One of them is time-like while rests are space-like. They are on a Lorentz manifold and interpreted as a model of space-time. It is important to recognize that frame fields are geometric quantities. They made based on the metric tensor of a general space-time. Vierbein fields make sense independent of a choice of a coordinate chart. They do the notions of orthogonality and length. Therefore, vierbein formalism is an official approach. By these, Minkowski version of some physical quantities generalizes into a general space-time. Vierbein field \( a^\alpha_\mu \) has two kinds of indices. \( \mu \) indicates the coordinates of a general space-time and \( \alpha \) is only a counter. Changes in \( \mu \) occur by the metric tensor while the number of vierbein fields enumerated by \( \alpha \). The formalism is invariance under the Lorentz transformation \( \alpha \) as well as \( \mu \). Vierbein counters are rising by \( \eta_{\alpha\beta} \), while the space-time indices with the metric \( g_{\mu\nu} \). Lorentz indices run from zero to \( "d" \), dimension of space-time \( \mathbb{R}^d \). Vierbein is a square root of the metric tensor \( g_{\mu\nu} \).

\[
g_{\mu\nu}(x) = a^\alpha_\mu(y)\eta_{\alpha\beta}a^\beta_\nu(y). \quad (1)
\]

Clearly, in the whole of a manifold, the vierbein field given by \( a^\mu_\alpha(y) = \delta^\mu_\alpha \) when gravity is absent. There is another way to find the non-commutativity dependency to the geometrical characteristics. This way named Higgs approach. For a small neighborhood of \( 'P'' \) in \( M \), global and local coordinates related to each other. Also, the local metric of this region achieves by employing the Higgs approach.

We deal with the non-commutative coordinates when we attempt to make of change in physics for short distances \( \theta \).

We can introduce the non-commutative framework with a replacement of Hermitian operators \( \hat{y}^\mu \) instead of local coordinates \( y^\mu \). So that obeys the following relation

\[
[y^\mu, y^\nu] = i\theta_{\mu\nu}(\hat{y}), \quad (2)
\]

which for Minkowski space-time, Eq. (2) gets the following form

\[
[y^\mu, y^\nu] = i\theta_{\mu\nu} \quad (3)
\]

\( \theta_{\mu\nu} \) is a constant and real parameter \( \theta \). In the limit of \( T_\mu M \), the results of the non-commutative version of physics depend on the choice of an event in the space-time. For, we determine \( \theta(y) \) up to the first order of the Riemann curvature tensor. Since many properties of tangential spaces induced from the base manifolds direct. So, the non-commutative relationship on \( T_\mu M \) comes from \( M \). Obviously that why we are studying for the main non-commutativity of coordinates on the \( M \).

The purpose of this paper is to recognize the dependence of \( \theta \) on coordinates. We also find another mapping other than vierbein formalism. So that, it helps us to get...
back the Minkowskian coordinates non-commutativity into the original manifold. Then, we compare the results of the two proposed methods.

**PRESENTATION OF THE THEORY**

We make two sets of vierbein fields based on the given metrics, the metric of weak gravitational waves and Schwarzschild universe. We assume an event of space-time "\(P\)" (with the coordinate \(Y_p\)). Suppose that its relevant geometry quantities are slowly varying functions. Local and global coordinates \((y\) and \(x)\) span the pseudo-Riemann manifolds in the partial and whole states. For a point \(Y_p\) of space-time, there is a fundamental relationship linking them to one another.

\[
x^{\mu} = Y_{\mu} + y^{\mu} + A^{\mu}_{\alpha}(Y_{\mu})y_{\alpha}y^{\gamma} + B^{\mu}_{\alpha\beta}(Y_{\mu})y_{\alpha}y_{\beta}y_{\gamma} + \cdots
\]

\[(4)\]

\(A^{\mu}_{\alpha\beta}\) and \(B^{\mu}_{\alpha\beta\gamma}\) are the Christopher coefficients and their derivatives. The amount of them calculated at the event point. The set of coordinates \(Y_p\) indicates a global framework. Also the validity of Eq. (4) restricted to a small neighborhood of \(T_pM\). Obviously that, the variation of the general coordinate \(y^{\mu}\) is small compared to the local coordinate \(y'\). Nevertheless, metric expansion becomes

\[
g_{\alpha\beta}(y) = g_{\alpha\beta}(Y_{\mu}) - \frac{1}{3}R_{\alpha\mu\beta\nu}(Y_{\mu})y^{\mu}y^{\nu} + \cdots
\]

\[(5)\]

of course, \(g_{\alpha\beta}(Y_{\mu}) \approx \eta_{\alpha\beta}\), and

\[
\Gamma_{\alpha\beta\gamma}(y) = \Gamma_{\alpha\beta\gamma}(Y_{\mu}) + \partial_{\gamma}g_{\alpha\beta}(Y_{\mu})y^{\mu} + \cdots
\]

\[(6)\]

Hence, at the origin of a framework, Riemann’s curvature tensors will be

\[
R_{\beta\mu\nu}(Y_{\mu}) = \left(\partial_{\nu}\Gamma^{\alpha}_{\beta\mu} - \partial_{\mu}\Gamma^{\alpha}_{\beta\nu}\right)|_{\text{calculated at } Y_{\mu}}
\]

\[(7)\]

**Static case of \(T_pM\)- Schwarzschild universe** is a static solution to Einstein’s equation \[4, 13\]. We assume that \(T_pM\) is a falling space along a geodesic of Schwarzschild space-time. Hyper-surface of space-time with constant time is perpendicular to the geodesic. So, its affine connections will be time-independent \[4, 15\]. Based on Eq. (6), we provide a set of vierbein fields which specialized for Schwarzschild universe. In matrix representation, vierbein fields found to be: \[14, 15\]

\[
\quad
\]

which simplify to the following expression

\[
a^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} - \xi^{\mu}_{\alpha}R^{\alpha}_{\mu k l}y^k y^l
\]

\[(9)\]

where \(\xi^{\mu}_{\alpha} = 1\) and \(\xi^{\mu}_{\alpha} = 3\). To get the coordinates non-commutative relationship on the \(M\) from the version of \(T_pM\), we can write

\[
\bar{\theta}_{\mu\nu} = a^{\alpha}_{\beta}b_{\alpha\beta}\nu.
\]

\[(10)\]

Now, by substitution \(y^{\mu}\) with \(\bar{y}^{\mu}\) and Eq. (9) in Eq. (10), we get coordinate dependency of \(\bar{\theta}(\bar{y})\)

\[
\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} - \xi^{\alpha}_{\mu}R^{\alpha}_{\mu k l}y^k y^l.
\]

\[(11)\]

When \(\theta^{\alpha\beta\gamma} = 0\), the special case of Eq. (11) becomes

\[
\bar{\theta}_{ij} = \theta_{ij} - \frac{1}{6}\theta_{ik}R^{k}_{ljm}y^{m} - \frac{1}{6}\theta_{jk}R^{k}_{lij}y^{m}.
\]

\[(12)\]

**Higgs mapping**- In this section, we set a local framework on a time-independent small subset of \(M\) at an event point \(Y_p\) of space-time. Then, global coordinate \(x^\mu\) belonging to \(M\) can be written in term of the coordinate \(Y_p\) also local coordinate \(y^\mu\). The coordinates \(y^\mu\) can span the small neighborhood around of \(Y_p\). Therefore,

\[
x^{\mu} = y^{\mu} + A^{\mu}_{\alpha}(Y_{\mu})y^{\alpha}y^{\gamma} + B^{\mu}_{\alpha\beta}(Y_{\mu})y^{\alpha}y^{\beta}y^{\gamma} + \cdots
\]

\[(13)\]

Where \(Y_p\) is a general coordinate of \(Y_p\). Since, Eq. (13) derived from the definition of Eq. (13), so \(A^{\mu}_{\alpha}(Y_{\mu})\) is zero. By excluding the upper-order terms, Eq. (13) becomes:

\[
x^{\mu} = y^{\mu} + y^{\mu} + B^{\mu}_{\alpha\beta}(Y_{\mu})y^{\alpha}y^{\beta}y^{\gamma}.
\]

\[(14)\]

\(B^{\mu}_{\alpha\beta}\) is a combination of Christopher coefficients, \(B^{\mu}_{\alpha\beta\gamma} = \lambda^{\mu}\Gamma^{\mu}_{\alpha\beta\gamma}\), in which \(\lambda^{\mu}\) calculated for
Schwarzschild universe by Higgs approach [16]. By this choice, Christopher coefficients independent from time. Therefore, for Schwarzschild universe,

$$x^\mu = Y^\mu(P) + y^\mu + \lambda^\mu \Gamma_{\alpha\beta\nu}(P)y^\nu y^\beta.$$  \hspace{1cm} (15)

The coefficients $\lambda^0$ and $\lambda^i$’s get the values by comparing $dx^\mu \eta_{\mu\nu}dx^\nu = \{-1 - 4\lambda^\mu \Gamma_{\alpha\beta\nu}(P)y^\nu y^\beta\}dy^\mu dy^\nu + \{1 + 2\lambda^\mu \Gamma_{\alpha\beta\nu}(P)y^\nu y^\beta\}dy^\mu dy^\nu + \cdots$. Our calculations show that, $\lambda^{\text{temporal}} = -\frac{1}{2}$ and $\lambda^{\text{spatial}} = \frac{1}{2}$. With the condition: $\theta^0_i = 0$, we have

$$\tilde{\theta}^i_j = \theta^i_j - \frac{1}{6} \theta^{kj} R^i_{\ mkn} y^m y^n - \frac{1}{6} \theta^{jk} R^i_{\ mkn} y^m y^n.$$  \hspace{1cm} (16)

which specialized for Schwarzschild universe. One can see that $\tilde{\theta}^i_j$ in Eq. (16) is an anti-symmetric tensor, because they transform by the tetrad approach.

**Dynamic case of $T_p M$:** Due to general relativity, the curvature of space-time affected by massive bodies. Also, the gravitational waves are one of the solutions to Einstein’s equation, which occurs due to the change in the distance of the two massive bodies. So, gravitational waves are ripples in the curvature of space-time which generated by gravitational interactions. These waves are coming from the depths of space and outward from their source at the speed of light. They have various amplitudes and frequencies. We especially assume a state that the space-time filled with weak gravitational effects. These effects are static waves and depend on time only. Gravitational waves explained by the assumption of small fluctuation in the metric tensor [17–20]. They propagate in the z-direction and is at least time and no direction dependent. In this case, we decompose the metric tensor as:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + h_{ij}(t)dx^i dx^j.$$  \hspace{1cm} (17)

Latin indices run from 1 to 3 while Greek take on the values 0,1,2,3. Indeed, the first order perturbation correction $h_{ij}$ is a function of coordinates which we assume to be only time dependent. The simple form of gravitational background is as follows:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_{12} & 0 \\ 0 & h_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (18)

with $\mathcal{D}^\mu h_{\mu\nu} = 0$, where $\mathcal{D}^\mu$ denoting the covariant derivative. By setting $h_{12} = h_{21} = \frac{-a}{x}$, and $\frac{\partial}{\partial x}$ polarization are familiar [21]. For the explaining of the frame field of $T_p M$, it is necessary to use a set of localized coordinates. We mention that $T_p M$ contains a first order of Riemann’s curvature tensor. The non zero components of the Christopher connections are $\Gamma^1_{02} = \Gamma^2_{01} = \frac{a}{x^2}$. Also, only non zero independent component of Riemann’s curvature tensor is: $R^3_{020} = \frac{a}{x^2}$. It is possible to find vierbein field based on the modified version of the relation found in Ref. [14–16]. In Ref. [14–16] has offered a relation to make the vierbein field. This validity is only for the static state of $T_p M$. So that, for the case of gravitational waves, we have to change it. Our finding vierbein field satisfies the corresponding motion equation: $\mathcal{D}^\mu \alpha^\mu_{\nu} = \partial_\mu \alpha^\mu_{\nu} - \alpha^\mu_{\sigma} \Gamma^\nu_{\mu\sigma} = 0$. Also, we know that the non-zero elements of Christopher connections are $\Gamma^i_{0j} = \Gamma^j_{0i} = \frac{a}{x}$. So, solving the equation of motion of time-dependent only vierbein fields are $\alpha^\mu_{\nu} = \int dt a^\mu_{\nu} \Gamma^\nu_{\mu\sigma}$ which reduced to $a^1_{12} = \frac{a}{x} \Gamma^2_{01} = \frac{a}{x} \frac{a}{x}$ and $a^1_{11} = a^{11} \Gamma^1_{11} = \frac{a}{x} \frac{a}{x}$. In the absence of gravity, we find coordinates such that $a^\mu_{\nu}(y) = \delta^\mu_{\nu}$. So that, we can set $a^\alpha_{\mu} = \delta^\alpha_{\mu} + \Gamma^\alpha_{\mu\nu}$. In the language of Riemann’s curvature tensors, we present components of the vierbein field:

$$\alpha^\mu_{\nu} = \delta^\mu_{\nu} - \int^t \int^i \dd t \dd \tau R^\alpha_{\mu0}(\tau)$$

whose form is as follows:

$$\alpha^\mu_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{a}{x} & 0 \\ 0 & \frac{a}{x} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (19)

In this way, we can replicate Eq. (10): $\tilde{\theta}_{\mu\nu} = a^\mu_{\alpha} \theta_{\alpha\beta} a^\beta_{\nu}$. With direct substitution and in the matrix representation, we have

$$\begin{pmatrix} 0 & \tilde{\theta}_{01} & \tilde{\theta}_{02} & \tilde{\theta}_{03} \\ \tilde{\theta}_{10} & 0 & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{20} & \tilde{\theta}_{21} & 0 & \tilde{\theta}_{23} \\ \tilde{\theta}_{30} & \tilde{\theta}_{31} & \tilde{\theta}_{32} & 0 \end{pmatrix} = \begin{pmatrix} \theta_{10} & 0 & \theta_{01} - \frac{a}{x} \theta_{02} & \theta_{03} \\ -\frac{a}{x} \theta_{10} + \theta_{20} & \theta_{21} & 0 & -\frac{a}{x} \theta_{23} \\ 0 & \theta_{31} - \frac{a}{x} \theta_{32} & \theta_{30} \\ -\frac{a}{x} \theta_{31} + \theta_{32} & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (20)
Therefore, in the presence of gravitational waves, the non-commutativity dependency on coordinates is achievable. Here, we should use the extension of the metric interrupt. In this manner, the obtained result simplified to

\[ \tilde{\theta}_{\mu \nu} = \theta_{\mu \nu} - \theta_{\mu \beta} \int^i d\hat{t} R^\beta_{0 \alpha}(\hat{t}) - \theta_{\beta \nu} \int^i d\hat{t} R^\beta_{0 \alpha}(\hat{t}), \]  

(21)

in which, \( \int^i d\hat{t} d\hat{t} R^\beta_{0 \alpha}(\hat{t}) = \frac{h_{\mu \alpha}(\hat{t})}{2} \).

CONCLUSION

This work is in the first order of Riemann’s curvature tensors and at the level of tangential space. In these limits, by employing vierbein field, we found coordinates dependency of generalized non-commutative relationship. We made local fields which satisfied the complement relation of Eq.(1). So, we derived the coordinates dependency for other than the Minkowski space-time. We specialized the generalized non-commutativity coordinates relationship for Schwarzschild universe as well as for space-time in the presence of weak gravitational waves. Also, we showed that behavior of the coordinates non-commutativity like as a tensor. Based on Higgs approach, we found a different way to find the coordinates non-commutativity dependence on local coordinates. Obtained results showed that these methods are comparable.

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