Spectral analysis of a difference operator with a growing potential

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Abstract. In this paper, we study the spectral properties of a second order difference operator with a growing potential. The operator acts in the complex Hilbert space $\ell_2(Z)$ of square summable complex sequences indexed by the integers. This operator is a discrete analogue of a second order differential operator with a growing complex potential. The study is based on a method of similar operators developed by A. G. Baskakov and his collaborators. This method allows us to reduce the study of the operator to one with a block-diagonal matrix. Asymptotic estimates of eigenvalues, eigenvectors, and spectral projections of a difference operator are obtained.

1. Introduction

In this paper, we explore the spectral properties of difference operators and the corresponding discrete Sturm-Liouville operators [1]. To this end, we employ the method of similar operators. The method has its origins in various similarity and perturbation techniques, such as the classical perturbation methods of celestial mechanics, Ljapunov’s kinematic similarity method, Friedrichs’ method of similar operators used in quantum mechanics, and Turner’s method of similar operators (see [2–4]). The method has been extensively developed and used in the works of A. G. Baskakov and his collaborators (see [2–7] and references therein). It provides a foundation for finding estimates of eigenvalues and eigenvectors of difference and differential operators with a growing potential.

Let $\ell_2(Z)$ be the complex Hilbert space of square summable sequences indexed by the integers. The inner product and the norm in $\ell_2(Z)$ are defined by

$$ (x, y) = \sum_{n \in Z} x(n)y(n) $$

and

$$ ||x|| = \left( \sum_{n \in Z} |x(n)|^2 \right)^{1/2}, \quad x, y \in \ell_2(Z), $$

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respectively.

We consider the closed linear difference operator $E : D(E) \subset \ell_2(Z) \to \ell_2(Z)$, generated by the expression

$$(Ex)(n) = a(n)x(n) + 2x(n) - x(n - 1) - x(n + 1),$$

with the domain $D(E) \subset \ell_2(Z)$ defined by

$$D(E) = \{ x \in \ell_2(Z), \sum_{n \in Z} |x(n)|^2 |a(n)|^2 < \infty \}.$$ 

In the standard basis of the space $\ell_2(Z)$, the matrix of the operator $E$ is given by

$$
\begin{bmatrix}
-1 & a(-2) + 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & -1 & a(-1) + 2 & 1 & 0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & 0 & -1 & a(1) + 2 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & 0 & -1 & a(2) + 2 & 1 & 0 \\
\end{bmatrix}.
$$

The sequence $a : Z \to C$ is such that $\lim_{|n| \to \infty} |a(n)| = \infty$. We also assume that this sequence satisfies one of the following two groups of conditions:

1) $a(i) \neq a(j)$ for $i \neq j$ and $d_j = \inf_{i \neq j} |a(i) - a(j)| \to \infty$ as $|j| \to \infty$, $i, j \in Z$;
2) $a(i) = a(-i)$ for $i \in Z$ and $d_j \to \infty$ as $j \to \infty$, $i, j \in Z_+ = \{0\} \cup N$.

The first group of conditions corresponds to a growing potential in a general form, while the second one corresponds to an even growing potential.

In [1], a finite-dimensional analogue of the operator $E$ was considered and its eigenvalue estimates were obtained by a variation method. In this paper, we provide estimates of eigenvalues and spectral projections of the operator $E$ using the method of similar operators as in [2–4]. It is important to note that the method is commonly used to study the spectral properties for differential operators [5–7]. It has, however, also been used for difference operators (see [8–11]).

We write $E = A - B$, where $A : D(A) = D(E) \subset \ell_2(Z) \to \ell_2(Z)$ is defined by

$$(Ax)(n) = a(n)x(n) + 2x(n), n \in Z,$$

and $B$ is a bounded operator given by

$$(Bx)(n) = x(n + 1) + x(n - 1), n \in Z.$$

We shall treat the operator $E$ as the perturbation of $A$ by $B$.

For a complex Hilbert space $\ell_2(Z)$ we denote by $\text{End} \ell_2$ the Banach algebra of all bounded linear operators in $\ell_2(Z)$. The operator $B$ belongs to $\text{End} \ell_2$ and $A$ is a normal closed linear operator. In the standard scheme of the method of similar operators, the unperturbed operator is a differential one and the perturbation is a multiplication operator by a potential function [2–7]. In this paper, the multiplication operator is chosen as the unperturbed one. This choice works because the matrix of the multiplication operator in the standard basis of $\ell_2(Z)$ is diagonal and, therefore, the spectral properties of this operator are easy to obtain.

The spectrum $\sigma(A)$ of the operator $A$ can be written as

$$\sigma(A) = \bigcup_{i \in J} \sigma_i = \bigcup_{i \in J} \{ \lambda_i \},$$
where $J \in \{Z, Z_+\}$, $Z_+ = N \cup \{0\}$. In the case of the first group of conditions, $\lambda_l = a(l) + 2$, $l \in Z$, are simple eigenvalues. The corresponding eigenvectors are $e_l$, $l \in Z$, where $e_l(k) = \delta_{lk}$, $k \in Z$, and $\delta_{lk}$ is the standard Kronecker delta. The spectral projections $P_l = P(\sigma_l, A)$ are given by $P_l x = (x, e_l) e_l = x(l) e_l$, $l \in Z$, $x \in \ell_2(Z)$.

In the other case, $\lambda_l = a(l) + 2$, $l \in N$, are eigenvalues of multiplicity two, and $\lambda_0 = a(0) + 2$ is a simple eigenvalue. The corresponding eigenvectors are $e_l$, $l \in Z$. The spectral projections $P_l = P(\sigma_l, A)$ are given by $P_l x = (x, e_l) e_l + (x, e_{-l}) e_{-l}$, $l \in N$, and $P_0 x = (x, e_0) e_0$, $x \in \ell_2(Z)$.

We will use the notation $H_k = \text{Im} P_k$, $k \in J$, $P_{(m)} = \sum_{|l| \leq m} P_l$, and $H_{(m)} = \text{Im} P_m$.

2. Materials and methods

Let $\mathcal{H}$ denote an abstract complex Hilbert space. We begin with the following definition.

**Definition 1.** Two linear operators $A_i : D(A_i) \subset \mathcal{H} \rightarrow \mathcal{H}$, $i = 1, 2$, are called *similar*, if there exists a continuously invertible operator $U \in \text{End}\mathcal{H}$ such that

$$A_1 U x = U A_2 x, \ x \in D(A_2), \ \text{and} \ \text{Im} P_k = \sum_{|l| \leq m} P_l, \ \text{and} \ H_{(m)} = \text{Im} P_m.$$ 

The operator $U$ is called the *similarity transform* of $A_1$ into $A_2$.

Directly from the definition (1), we have the following result about the spectral properties of similar operators.

**Lemma 1.** Let $A_i : D(A_i) \subset \mathcal{H} \rightarrow \mathcal{H}$, $i = 1, 2$, be two similar operators with the similarity transform $U$. Then the following properties hold.

1) We have $\sigma(A_1) = \sigma(A_2)$, $\sigma_p(A_1) = \sigma_p(A_2)$, and $\sigma_c(A_1) = \sigma_c(A_2)$, where $\sigma_p$ denotes the point spectrum and $\sigma_c$ denotes the continuous spectrum;

2) Assume that the operator $A_2$ admits a decomposition $A_2 = A_{21} \oplus A_{22}$ with respect to a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A_{21} = A_2|\mathcal{H}_1$ and $A_{22} = A_2|\mathcal{H}_2$ are the restrictions of $A_2$ to the respective subspaces. Then the operator $A_1$ admits a decomposition $A_1 = A_{11} + A_{12}$ with respect to a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A_{11} = A_1|\mathcal{H}_1$ and $A_{12} = A_1|\mathcal{H}_2$ are the restrictions of $A_1$ to the respective invariant subspaces. Moreover, if $P$ is the projection onto $\mathcal{H}_1$ parallel to $\mathcal{H}_2$, then $P = U P U^{-1}$ is the projection onto $\mathcal{H}_1$ parallel to $\mathcal{H}_2$.

3) If $\lambda_0$ is an eigenvalue of the operator $A_2$ and $x$ is a corresponding eigenvector, then $y = U x$ is an eigenvector of the operator $A_1$ corresponding to the same eigenvalue $\lambda_0$.

We shall need to extend Property (2) in the Lemma 1 to the case of countable direct sums. To this end, we assume that the abstract Hilbert space $\mathcal{H}$ can be written as

$$\mathcal{H} = \bigoplus_{l \in J} \mathcal{H}_l,$$

where each $\mathcal{H}_l, l \in J$, is a closed nonzero subspace of $\mathcal{H}$, $\mathcal{H}_l$ is orthogonal to $\mathcal{H}_l$ for $l \neq j \in J$, and each $x \in \mathcal{H}$ satisfies $x = \sum_{l \in J} x_l$, where $x_l \in \mathcal{H}_l$ and $\|x\|^2 = \sum_{l \in J} |x_l|^2$. In other words, we have a disjunctive resolution of the identity

$$P = \{P_l, l \in J \},$$

that is a system of idempotents with the following properties

1) \( P^*_l = P_l, l \in J \);
2) \( P_j P_l = \delta_{jl} P_l, j, l \in J \);  
3) The series $\sum_{l \in J} P_l x$ converges unconditionally to $x \in \mathcal{H}$ and $\|x\|^2 = \sum_{l \in J} \|P_l x\|^2$;
4) Equalities $P_l x = 0$, $l \in J$, imply $x = 0 \in \mathcal{H}$;
5) $\mathcal{H}_l = \text{Im} P_l, x_l = P_l x, l \in J$. 

3
**Definition 2.** We say that a closed linear operator \( A : D(A) \subseteq \mathcal{H} \to \mathcal{H} \) is represented as an orthogonal direct sum of bounded operators \( A_l \in \text{End} \mathcal{H}_l, l \in J \), that is

\[
A = \bigoplus_{l \in J} A_l,
\]

if the following three properties hold.

1. \( D(A) = \{ x \in \mathcal{H} : \sum_{l \in J} \| A_l x_l \|^2 < \infty, x_l = P_l x, l \in J \} \) and \( \mathcal{H}_l \subseteq D(A) \) for all \( l \in J \).
2. For each \( l \in J \), the subspace \( \mathcal{H}_l \) is an invariant subspace of the operator \( A_l \) and \( A_l \) is the restriction of \( A \) to \( \mathcal{H}_l \). The operators \( A_l, l \in J \), are called the parts of the operator \( A \).
3. \( Ax = \sum_{l \in J} A_l x_l, x \in D(A) \), where \( x_l = P_l x, l \in J \), and the series converges unconditionally in \( \mathcal{H} \).

**Definition 3.** Given a continuously invertible operator \( U \in \text{End} \mathcal{H} \) and an orthogonal decomposition of \( \mathcal{H} \), a \( U \)-orthogonal decomposition of \( \mathcal{H} \) is the orthogonal direct sum

\[
\mathcal{H} = \bigoplus_{l \in J} U \mathcal{H}_l.
\]

**Definition 4.** Given a continuously invertible operator \( U \in \text{End} \mathcal{H} \), we say that a closed linear operator \( A : D(A) \subseteq \mathcal{H} \to \mathcal{H} \) is a \( U \)-orthogonal direct sum of bounded linear operators \( \tilde{A}_l, l \in J \), if \( \tilde{A}_l = UA_lU^{-1}, l \in J \), and

\[
A = \bigoplus_{l \in J} \tilde{A}_l.
\]

We remark that \( U \)-orthogonal decompositions and direct sums can be viewed as orthogonal with respect to the inner product

\[
\langle x, y \rangle_U = \langle Ux, Uy \rangle, x, y \in \mathcal{H}.
\]

An example of direct sums of operators is provided by the operator \( A \) from the introduction. In particular, \( A \) is an orthogonal direct sum of operators \( A_l = A| \mathcal{H}_l = (a(l) + 2)I_l \), where \( I_l \) is the identity operator on \( \mathcal{H}_l = \text{Im} P_l, l \in J \). In other words, \( A = \bigoplus_{l \in \mathbb{Z}} (a(l) + 2)I_l \) in the first case, and

\[
A = \bigoplus_{l \in \mathbb{Z}_+} (a(l) + 2)I_l
\]

in the second case. This representation is with respect to the orthogonal decomposition of \( \ell_2(\mathbb{Z}) \) given by \( \ell_2(\mathbb{Z}) = \bigoplus_{l \in J} \mathcal{H}_l \).

Consider a new resolution of the identity

\[
\mathcal{P}^{(m)} = \{ P_{l(m)} \} \cup \{ P_l, |l| > m, l \in J \}.
\]

Then the operator \( A \) may also be represented as an orthogonal direct sum

\[
A = A_{(m)} \oplus \left( \bigoplus_{|l| > m, l \in J} A_l \right) = A_{(m)} \oplus \left( \bigoplus_{|l| > m, l \in J} (a(l) + 2)I_l \right), \tag{3}
\]

where \( A_{(m)} \) is the restriction of \( A \) to \( \mathcal{H}_{(m)} = \text{Im} P_{(m)} \). The representation (3) is with respect to the orthogonal decomposition \( \ell_2(\mathbb{Z}) = H_{(m)} \oplus \bigoplus_{|l| > m, l \in J} \mathcal{H}_l \). Observe that

\[
A_{(m)} = \bigoplus_{|l| > m, l \in J} (a(l) + 2)I_l
\]

with respect to the decomposition \( H_{(m)} = \bigoplus_{|l| < m, l \in J} H_j \).

The method of similar operators constructs a similarity transform for an operator \( A - B : D(A) \subseteq \mathcal{H} \to \mathcal{H} \), where the spectrum of the operator \( A \) is known and has certain properties, and the operator \( B \) is \( A \)-bounded.
Definition 5 [2]. Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator. A linear operator $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ is $A$-bounded if $D(B) \supset D(A)$ and $\|B\|_A = \inf\{c > 0 : \|Bx\| \leq c(\|x\| + \|Ax\|), x \in D(A)\} < \infty$.

The space $\mathcal{L}_A(\mathcal{H})$ of all $A$-bounded linear operators is a Banach space with respect to the norm $\|\cdot\|_A$. Moreover, given $\lambda_0 \in \rho(A)$, where $\rho(A) = \mathcal{C} \setminus \sigma(A)$ is the resolvent set of $A$, we have $B \in \mathcal{L}_A(\mathcal{H})$ if and only if $B(\lambda_0 I - A)^{-1} \in \text{End}\mathcal{H}$ and $\|B\|_A = \|B(\lambda_0 I - A)^{-1}\|_{\text{End}\mathcal{H}}$ defines an equivalent norm in $\mathcal{L}_A(\mathcal{H})$.

The method of similar operators uses the commutator transform $ad_A : D(ad_A) \subset \text{End}\mathcal{H} \to \text{End}\mathcal{H}$ defined by

$$ad_A X = AX - XA, \quad X \in D(ad_A),$$

where the domain $D(ad_A)$ contains all $X \in \text{End}\mathcal{H}$ such that the following two properties hold:

1. $XD(A) \subset D(A)$;
2. The operator $ad_A X : D(A) \to \mathcal{H}$ (uniquely) extends to a bounded operator $Y \in \text{End}\mathcal{H}$; we then let $ad_A X = Y$.

The key notion of the method of similar operators is that of an admissible triplet.

Definition 6 [2]. Let $\mathcal{M}$ be a linear subspace of $\mathcal{L}_A(\mathcal{H})$, $J : \mathcal{M} \to \mathcal{M}$, and $G : \mathcal{M} \to \text{End}\mathcal{H}$.

The collection $(\mathcal{M}, J, G)$ is called an admissible triplet for the operator $A$, and the space $\mathcal{M}$ is the space of admissible perturbations, if the following six properties hold.

1. $\mathcal{M}$ is a Banach space that is continuously embedded in $\mathcal{L}_A(\mathcal{H})$, i.e., $\mathcal{M}$ has a norm $\|\cdot\|_*$ such that there is a constant $C > 0$ that yields $\|X\|_* = C\|X\|_*$ for any $X \in \mathcal{M}$.
2. $J$ and $G$ are bounded linear operators; moreover, $J$ is an idempotent.
3. $(GX)D(A) \subset D(A)$ and

$$(ad_A GX)x = (X - JX)x, \quad x \in D(A), X \in \mathcal{M};$$

moreover $GX \in \text{End}\mathcal{H}$ is the unique solution of the equation

$$ad_A Y = AY - YA = X - JX,$$

that satisfies $JY = 0$.
4. $XGY, (GX)Y \in \mathcal{M}$ for all $X, Y \in \mathcal{M}$, and there is a constant $\gamma > 0$ such that

$$\|G\| \leq \gamma, \max\{\|XGY\|_*, \|GXY\|_*\} \leq \gamma \|X\|_*\|Y\|_*.$$
5. $J((GX)Y) = 0$ for all $X, Y \in \mathcal{M}$.
6. For every $X \in \mathcal{M}$ and $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A)$, such that $\|X(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.

To formulate the main theorem of the method of similar operators [2] for an operator $A - B$, we use the function $\Phi : \mathcal{M} \to \mathcal{M}$ given by

$$\Phi(X) = BGX - (GX)(JB) - (GX)J(BGX) + B. \quad (4)$$

Theorem 1 [2]. Assume that $(\mathcal{M}, J, G)$ is an admissible triplet for an operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and $B \in \mathcal{M}$. Assume also that

$$4\gamma\|J\|\|B\|_* < 1,$$

where $\gamma$ comes from the Property 4 of Definition 6. Then the operator $A - B$ is similar to the operator $A - JX_*$, where $X_* \in \mathcal{M}$ is the (unique) fixed point of the function $\Phi$ given by equation (4), and the similarity transform of $A - B$ into $A - JX_*$ is given by $I + GJX_* \in \text{End}\mathcal{H}$. Moreover, the map $\Phi : \mathcal{M} \to \mathcal{M}$ is a contraction in the ball $\{X \in \mathcal{M} : \|X - B\|_* \leq 3\|B\|_*\}$, and the fixed point $X_*$ can be found as a limit of simple iterations: $X_0 = 0, X_1 = \Phi(X_0) = B$, etc.
3. Results and discussions.

**Theorem 2.** There is a number \( k \in \mathbb{Z}_+ \) and a continuously invertible operator \( U \in \text{End} \ell_2 \) such that operator \( \mathcal{E} = A - B \) is similar to the operator \( A - X_0 \), where \( X_0 \in \text{End} \ell_2 \), \( \mathcal{E} U = U (A - X_0) \) and the subspaces \( \mathcal{H}(k) = \text{Im} P(k), \mathcal{H}_l = \text{Im} P_l, |l| > k, l \in J \), are invariant for \( X_0 \). The operator \( A - X_0 \) is the orthogonal direct sum

\[
A - X_0 = A - \left( X_{0(k)} \oplus \left( \bigoplus_{|l| > k, l \in J} X_{0(l)} \right) \right)
\]

with respect to orthogonal decomposition \( \mathcal{H} = \mathcal{H}(k) \oplus \left( \bigoplus_{|l| > k, l \in J} \mathcal{H}_l \right) \) and the dimension of \( \mathcal{H}(m) \) is \( 2m + 1 \). Moreover, the operator \( \mathcal{E} \) is the \( U \)-orthogonal direct sum

\[
\mathcal{E} = U \left( A - \left( X_{0(k)} \oplus \left( \bigoplus_{|l| > k, l \in J} X_{0(l)} \right) \right) \right) U^{-1}
\]

with respect to the \( U \)-orthogonal decomposition \( \mathcal{H} = U \mathcal{H}(k) \oplus \left( \bigoplus_{|l| > k, l \in J} U \mathcal{H}_l \right) \).

**Proof.** Let \( \mathcal{M} = \text{End} \ell_2 \subset \mathcal{L}_A(\ell_2) \) and

\[
||X|| = ||XA^{-1} Ax|| \leq ||XA^{-1}||_\infty ||Ax|| < ||XA^{-1}||_\infty (||Ax|| + ||x||),
\]

\[
||X||_A \leq ||XA^{-1}||_\infty.
\]

We deduce Property 1 of the Definition 6.

Given a resolution of identity \( \mathcal{P} \) as in (2), it is often convenient to represent an operator \( X \in \text{End} \ell_2 \) in terms of its matrix. We write such matrices as \( \tilde{X} = (X_{jl}) \), where \( X_{jl} = P_j X P_l, j, l \in J \). In the case when the matrix of a linear operator is diagonal, this operator is the orthogonal direct sum (see definition 2).

Any operator \( Z \in \text{End} \ell_2 \) can be expressed (see [8]) as

\[
Z = \lim_{n \to \infty} \sum_{|p| \leq n} \left( 1 - \frac{|p|}{n} \right) Z_p, Z_0 = \sum_{i-j=p} Z_{ij}.
\]

The operator \( Z_p, p \in \mathbb{Z} \), is the \( p \)-th diagonal of the operator matrix of the operator \( Z \).

We define a family of transformers \( J_k X \) as follows:

\[
J_k X = P(k) X P(k) + \sum_{|i| > k, i \in J} P_i X P_i, k \geq 0,
\]

moreover,

\[
JX = J_0 X = \sum_{i \in J} P_i X P_i,
\]

where the series \( \sum_{i \in J} P_i X P_i \) is convergent, because the operator \( X \) belongs to \( \text{End} \ell_2 \). Obviously, \( ||J_k|| = 1 \), because

\[
||J_k X|| \leq ||P(k) X P(k)||_2^2 + \sum_i ||P_i X P_i||_2^2 \leq ||X||_2^2.
\]

If \( X = \sum_i P_i X P_i \) (i.e., the matrix of the operator \( X \) is diagonal and \( X \) is an orthogonal direct sum of operators \( X_{ii} \)), then \( J_k X = X \) and \( ||J_k X|| = ||X|| \).
Let us proceed to the construction of the operator $G_0X : End\ell_2 \rightarrow End\ell_2$. First, we define it on the operator blocks $X_{ij} = P_iXP_j$, where $X \in End\ell_2$. For each $X_{ij}, i \neq j$, set $G_0X_{ij} = Y_{ij}$, where $Y_{ij}$ is the solution of the equation

$$AY_{ij} - Y_{ij}A = X_{ij}, \quad i \neq j, \quad i, j \in J,$$

and $Y_{ii} = 0$ for each $i \in J$. Note that the last equation can be represented in the form

$$A_i Y_{ij} - Y_{ij}A_j = X_{ij},$$

(5)

where $A_i = A|\mathcal{H}_i$, and $\mathcal{H}_i = \text{Ran}P_i$. Since $\sigma(A_i) \cap \sigma(A_j) = \emptyset, i, j \in J$, it follows that equation (5) is solvable and the following inequality holds:

$$||Y_{ij}|| \leq \frac{c||X_{ij}||}{\text{dist}(\sigma_i, \sigma_j)} \leq \frac{c||X_{ij}||}{|a(i) - a(j)|} \leq c(\min_{i \neq j}|a(i) - a(j)|)^{-1}.$$

In addition, set $Y_{ii} = 0$ for each $i \in J$.

Now we form the operator $G_0X$ from the operator blocks $Y_{ij} = (G_0X)_{ij}$ as follows:

$G_0X = \sum_{i,j}(GX)_{ij}$ (see[2], [3]).

The transformer $G_0X$ belongs to the space $End\ell_2$ and $||G_0|| \leq d_0^{-1}$.

In addition, we define a family of transformers $G_kX$ by setting

$$G_kX = G_0X - G_0(Q_kXQ_k) = G_0X - Q_k(G_0X)Q_k, \quad k \geq 0.$$

It follows from the last relation that

$$||G_kX|| \leq cd_k^{-1}||X||, \quad \gamma = \gamma_k = cd_k^{-1}.$$

If $X, Y \in End\ell_2$, then $X(G_kY), (G_kX)Y$ also belong to $End\ell_2$ and $||X(G_kY)|| \leq \gamma_k||X||||Y||, \quad ||(G_kX)Y|| \leq \gamma_k||X||||Y||$.

It follows from the preceding argument that condition 4 in Definition 6 holds with $\gamma = \gamma_k$ of the order of $d_k^{-1}$.

Operator $J_0$ is idempotent, because

$$J_0^2X = J_0(J_0X) = J_0(\sum_{i \in J}P_iXP) = \sum_{j \in J}P_j(\sum_{i \in J}P_iXP_i)P_j = \sum_{j \in J}P_jXP_j = J_0X.$$

It implies that condition 2 of Definition 6 holds true.

Let us check condition 3 of Definition 6. Let $Q_n = \sum_{||x|| < n, i \in J}P_i$. We consider operator $AQ_nGX_0A^{-1}$ and represent it as

$$AQ_nGX_0A^{-1}x = Q_nG_0Xx + Q_n(X - J_0X)A^{-1}x, \quad x \in \ell_2(Z).$$

Since $Q_nG_0X \rightarrow G_0Xx, Q_n(X - J_0X)A^{-1} \rightarrow (X - J_0X)A^{-1}$ as $n \rightarrow \infty$, then $AQ_nGX_0A^{-1}x \rightarrow y_0 \in \ell_2(Z)$. Let $Q_nG_0XA^{-1}x \rightarrow x_0 = G_0XA^{-1}x$, then due to the closedness of operator $A$ we have $x_0 \in D(A)$ and $Ax_0 = y_0$, where $y_0 = \lim_{n \rightarrow \infty} y_n$.

Condition 6 is obvious since

$$||X(A - \lambda_2I)^{-1}|| \leq ||X||||A^{-1}(A - \lambda_2I)^{-1}||$$

at that, the first factor is finite and the other can be chosen arbitrarily small.

Thus, we have proven

Lemma 2. For each $k \geq 0$ the triple $(End\ell_2, J_k, G_k)$ is admissible for operator $A$.

Lemma 2, Theorem 1 and estimate (3) imply Theorem 2.
For the class of difference operators under consideration, one may also use as the admissible space of perturbations the space \( \text{End}_1 \ell_2 \subset \text{End}_2 \ell_2 \), which is defined as follows. An operator \( Y \in \text{End}_1 \ell_2 \) belongs to the space \( \text{End}_2 \ell_2 \) if the diagonals of the matrix of \( Y \) in the standard basis of \( \ell_2 \) are summable. We note that the perturbation \( B \) belongs to \( \text{End}_1 \ell_2 \) because the matrix of the operator \( B \) is tri-diagonal. The space \( \text{End}_1 \ell_2 \) was chosen as the admissible space of perturbations in [10–11]. In this case, an analog of Theorem 2 may be proved along the same lines, but the operator \( X \) in the theorem must also belong to \( \text{End}_1 \ell_2 \). If the sequence \( d_i, i \in J \), is such that \( \sum_{i \in J} d_i^2 < \infty \) then the admissible space of perturbations may be chosen as the ideal \( \mathcal{S}_2(\ell_2) \) of Hilbert-Schmidt operators. The operator \( X_0 \) in Theorem 2 will then also belong to the ideal \( \mathcal{S}_2(\ell_2) \). In general, however, we do not have \( B \in \mathcal{S}_2(\ell_2) \). This is why we have to start with a preliminary similarity transformation of the operator \( A - B \) into the operator \( A - \tilde{B} \), where \( \tilde{B} \in \mathcal{S}_2(\ell_2) \) [7].

**Lemma 3.** Under the assumptions of Theorem 2 we have

\[
||P_i(X - B)P_i|| \leq c_1 d_i^{-1}, \ |i| > k, i \in J,
\]

\[
||P_i(X - B - BG_k)P_i|| \leq c_2 d_i^{-2}, \ |i| > k, i \in J.
\]

**Proof.** We have

\[
X - B = BG_kX - G_kXJ_kB - G_kXJ_k(BG_kX); \ J_k(X - B) = J_k(BG_kX)
\]

and

\[
||P_i(X - B)P_i|| = ||P_i(BG_kX)P_i|| \leq d_i^{-1}||B|| \cdot ||X|| \leq 3d_i^{-1} \cdot ||B||^2 \leq c_1 d_i^{-1}, \ |i| > k, i \in J, c_1 > 0.
\]

Analogically,

\[
||P_i(X - B - BG_k)P_i|| \leq ||P_i(BG_k(X - B))P_i|| \leq c_2 d_i^{-2}.
\]

Assume that Theorem 2 holds true. Then the similarity of operators \( A - B \) and \( A - J_kX \) yields \( \sigma(A - B) = \sigma(A - J_kX) = \sigma(A(k)) \cup (\bigcup_{|i| > k} A_i) \), where \( A(k) = (A - Q_kX_0Q_k)/\mathcal{H}(k) \), \( \mathcal{H}(l) = \text{Ran}Q_l \) and \( A_i = (P_iA - P_iX_0)/\mathcal{H}_i \). Since operator \( X_0 \) is unknown and we know only the first and the second approximation, then \( A_i = (P_iA - P_iB - P_i(X_0 - B))/\mathcal{H}_i \) and the first two operators are known, while for the third operator we know only estimate from Lemma 3. Analogy, \( A_i = (P_iA - P_iB - P_i(BG_k(X_0 - B)))/\mathcal{H}_i \), and first three operators are known, while for the forth operator we know only estimate from Lemma 3.

The following theorem describes the spectral properties of the operator \( \mathcal{E} \).

**Theorem 3.** Let the sequence \( a : Z \to C \) satisfy conditions 1). There is a number \( k \in Z_+ \) such that the spectrum \( \sigma(\mathcal{E}) \) of the operator \( \mathcal{E} \) satisfies

\[
\sigma(\mathcal{E}) = \sigma_{(k)} \cup \left( \bigcup_{|l| > k} \sigma_l \right),
\]

where \( \sigma_{(k)} \) consists of no more than \( 2k + 1 \) eigenvalues. The sets \( \sigma_l, |l| > k \), are one-point sets \( \sigma_l = \{ \mu_l \} \) and

\[
\mu_l = a(l) + 2 + O(d_l^{-1}),
\]

\[
\mu_l = a(l) + 2 - \frac{a(l + 1) - 2a(l) + a(l - 1)}{(a(l + 1) - a(l))(a(l - 1) - a(l))} + O(d_l^{-3}), \quad l \in Z.
\]
The corresponding eigenvectors $\tilde{e}_i, |i| > k$, satisfy the asymptotic estimates

$$||\tilde{e}_i - \tilde{y}_i|| = O(d_i^{-2}), |i| > k,$$

where $\tilde{y}_i \in \ell_2(Z)$ and

$$\tilde{y}_i(k) = \begin{cases} 1, & i = k; \\ (a(i \pm 1) - a(i))^{-1}, & k = i \pm 1; \\ 0, & \text{in other cases.} \end{cases}$$

The eigenvectors $\tilde{e}_i, i \in Z$, constitute the Riesz basis in the space $\ell_2(Z)$.

**Theorem 4.** Let the sequence $a : Z \rightarrow C$ satisfy conditions 2). There is a number $k \in Z_+$ such that the spectrum $\sigma(\mathcal{E})$ satisfies

$$\sigma(\mathcal{E}) = \sigma_{(k)} \cup \bigcup_{l > k} \sigma_l,$$

where $\sigma_{(k)}$ consists of no more than $2k + 1$ eigenvalues. The sets $\sigma_l, l > k$, satisfy $\sigma_l = \{\mu_l, \mu_{-l}\}$ and

$$\mu_{\pm l} = a(l) + 2 - \frac{a(l + 1) - 2a(l) + a(l + 1)}{(a(l + 1) - a(l))(a(l - 1) - a(l))} + O(d_i^{-3}), l > k.$$

Let $\tilde{P}_n, n \in J$, be the spectral projections $\tilde{P}_n = P(\sigma_n, \mathcal{E})$ corresponding to the sets $\sigma_n, n \in J$, described in Theorem 3 or 4.

**Theorem 5.** We have

$$\left\| \tilde{P}_n - P_n \right\| = O(d_n^{-1}),$$

$$\left\| \sum_{n \geq m} \tilde{P}_n - \sum_{n \geq m} P_n \right\| = O(d_n^{-1}),$$

where $m > k, N > k, m, N \in J$.

The spectral projections $\tilde{P}_n$ also satisfy the following estimates of uniform unconditional equiconvergence of spectral expansions:

$$\left\| P(\sigma_{(k)}, \mathcal{E}) + \sum_{|i| > k} \tilde{P}_i - \sum_{|i| < l} P_i \right\| = O(d_i^{-1}), \text{ for } J = Z,$$

and

$$\left\| P(\sigma_{(k)}, \mathcal{E}) + \sum_{i > k} \tilde{P}_i - \sum_{i = 0}^{l} P_i \right\| = O(d_i^{-1}), \text{ for } J = Z_+,$$

where $l \in Z_+, l > k$.

We need the following definition in the monograph [12].

**Definition 7.** Let $\mathcal{C} : D(\mathcal{C}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator in the space $\mathcal{H}$ whose spectrum can be represented as the union

$$\sigma(\mathcal{C}) = \bigcup_{k \in J} \sigma_k,$$
of pairwise disjoint sets $\sigma_k, k \in \mathbb{Z}$, and let $P_k$ be the Riesz projection corresponding to the spectral set $\sigma_k$. An operator $C$ is said to be *spectral with respect to the expansion* (6) (or *generalized-spectral*) if the series $\sum_k P_k x$ is convergent for any vector $x \in \mathcal{H}$.

If $\sigma_k = \{\lambda_k\}, k \in J, -$ are one-point sets and $CP_k = \lambda_k P_k$ for all $k$, except for finitely many, then spectral with respect to the expansion (6) operator $C$ is spectral operator. Operator $A$ is the spectral operator of scalar type if $AP_k = \lambda_k P_k$ for all $k \in J$.

The following assertion is a straightforward consequence of Theorem 5.

**Theorem 6.** The operator $L$ is spectral operator.

4. Conclusion

In this paper we studied the spectral properties of a second order difference operator with a growing potential. The operator acts in the complex Hilbert space $\ell_2(\mathbb{Z})$ of square summable complex sequences indexed by the integers. This operator is a discrete analogue of a second order differential operator with a growing complex potential. Asymptotic estimates of eigenvalues, eigenvectors and the spectral projections were obtained. The main method of this exploration was the method of similar operators, which allowed us to reduce the study of the operator to one with a block-diagonal matrix. This method was thoroughly described in the paper and its basic definitions, such as admissible triplet and commutator transform, were given. Main results of the paper were formulated in five theorems and two lemmas, proofs of one theorem and two lemmas were also provided. The article is ten pages long and list of references consists of twelve issues.

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