Multiplicity of ground states in quantum field models: applications of asymptotic fields

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September 28, 2018

Abstract

The ground states of an abstract model in quantum field theory are investigated. By means of the asymptotic field theory, we give a necessary and sufficient condition for that the expectation value of the number operator of ground states is finite, from which we obtain a wide-usable method to estimate an upper bound of the multiplicity of ground states. Ground states of massless GSB models and the Pauli-Fierz model with spin 1/2 are investigated as examples.

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†This work is partialy supported by Grant-in-Aid for Science Research (C) 1554019 from MEXT.
1 Preliminaries

1.1 Boson Fock spaces

Let \( \mathcal{W} \) be a Hilbert space over \( \mathbb{C} \) with a conjugation \( \bar{\cdot} \). The boson Fock space \( \mathcal{F}_b \) over \( \mathcal{W} \) is defined by

\[
\mathcal{F}_b = \mathcal{F}_b(\mathcal{W}) := \bigoplus_{n=0}^{\infty} [\otimes^n_{s} \mathcal{W}]
\]

\[
= \{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes^n_{s} \mathcal{W}, \| \Psi \|_{\mathcal{F}_b}^2 := \sum_{n=0}^{\infty} \| \Psi^{(n)} \|_{\otimes^n \mathcal{W}}^2 < \infty \},
\]

where \( \otimes^n_{s} \mathcal{W} \) denotes the \( n \)-fold symmetric tensor product of \( \mathcal{W} \) with \( \otimes^0_{s} \mathcal{W} := \mathbb{C} \).

In this paper \( (f, g)_{\mathcal{K}} \) and \( \| f \|_{\mathcal{K}} \) denote the scalar product and the norm on Hilbert space \( \mathcal{K} \) over \( \mathbb{C} \), respectively, where \( (f, g)_{\mathcal{K}} \) is linear in \( g \) and antilinear in \( f \). Unless confusions arise we omit \( \mathcal{K} \) of \( (\cdot, \cdot)_{\mathcal{K}} \) and \( \| \cdot \|_{\mathcal{K}} \). \( D(T) \) denotes the domain of operator \( T \). Moreover, for a bounded operator \( S \), we denote its operator norm by \( \| S \| \).

Fock vacuum \( \Omega \in \mathcal{F}_b \) is given by

\[
\Omega = \{ 1, 0, 0, \ldots \}.
\]

The finite particle subspace of \( \mathcal{F}_b \) is defined by

\[
\mathcal{F}_{\text{fin}} := \{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F}_b | \Psi^{(m)} = 0 \text{ for all } m \geq n \text{ with some } n \}.
\]

It is known that \( \mathcal{F}_{\text{fin}} \) is dense in \( \mathcal{F}_b \). The creation operator \( a^\dagger(f) : \mathcal{F}_b \to \mathcal{F}_b \) with test function \( f \in \mathcal{W} \) is the densely defined linear operator in \( \mathcal{F}_b \) defined by

\[
(a^\dagger(f)\Psi)^{(0)} = 0,
\]

\[
(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n (f \otimes \Psi^{(n-1)}), \quad n \geq 1,
\]

where \( S_n \) is the symmetrization operator on \( \otimes^n \mathcal{W} \), i.e., \( S_n[\otimes^n \mathcal{W}] = \otimes^n \mathcal{W} \). The annihilation operator \( a(f), f \in \mathcal{W} \), is defined by

\[
a(f) = (a^\dagger(\mathcal{F}))^* \big|_{\mathcal{F}_{\text{fin}}}.
\]

Since it is seen that \( a(f) \) and \( a^\dagger(f) \) are closable operators, their closures are denoted by the same symbols, respectively. Note that \( a^\dagger(f) (a^2 = a \text{ or } a^\dagger) \) is linear in \( f \). On
$\mathcal{F}_{\text{fin}}$ the annihilation operator and the creation operator obey canonical commutation relations,

$$[a(f), a^\dagger(g)] = (\mathcal{T}, g)_W,$$
$$[a(f), a(g)] = 0,$$
$$[a^\dagger(f), a^\dagger(g)] = 0,$$

where $[A, B] := AB - BA$. Define

$$\mathcal{F}^D_{\text{fin}} := \text{the linear hull of } \{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega, \Omega | f_j \in D, j = 1, ..., n \geq 1\}.$$

Let $S$ be a self-adjoint operator acting in $\mathcal{W}$. The second quantization of $S$,

$$d\Gamma(S) : \mathcal{F}_b \to \mathcal{F}_b,$$

is defined by

$$d\Gamma(S) := \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^{n} \frac{1 \otimes \cdots \otimes \frac{S^j}{n} \cdots \otimes 1}{n} \right),$$

with

$$D(d\Gamma(S)) := \mathcal{F}^D(S).$$

Here we define

$$(d\Gamma(S)\Psi)(0) := 0.$$ 

In particular it follows that

$$d\Gamma(S)\Omega = 0. \quad (1.1)$$

Note that

$$d\Gamma(S)a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega = \sum_{j=1}^{n} a^\dagger(f_1) \cdots a^\dagger(Sf_j) \cdots a^\dagger(f_n)\Omega. \quad (1.2)$$

From (1.2) it follows that, for $f \in D(S)$,

$$[d\Gamma(S), a(f)] = -a(Sf), \quad (1.3)$$
$$[d\Gamma(S), a^\dagger(f)] = a^\dagger(Sf) \quad (1.4)$$

on $\mathcal{F}^D(S)$. It is known that $d\Gamma(S)$ is essentially self-adjoint. The self-adjoint extension of $d\Gamma(S)$ is denoted by the same symbol $d\Gamma(S)$. It can be seen that unitary operator $e^{itd\Gamma(S)}$ acts as

$$e^{itd\Gamma(S)}a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega = a^\dagger(e^{its}f_1) \cdots a^\dagger(e^{its}f_n)\Omega.$$
Thus we see that that
\[ e^{itd\Gamma(S)} a(f) e^{-itd\Gamma(S)} = a(e^{-itS} f), \]
(1.5)
\[ e^{itd\Gamma(S)} a^\dagger(f) e^{-itd\Gamma(S)} = a^\dagger(e^{itS} f) \]
(1.6)
on \mathcal{F}_b. For a self-adjoint operator \( T \), we write its spectrum (resp. essential spectrum, point spectrum) as \( \sigma(T) \) (resp. \( \sigma_{\text{ess}}(T), \sigma_p(T) \)). It is known that
\[ \sigma(d\Gamma(S)) = \{0\} \bigcup \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \right\}, \quad \sigma_p(d\Gamma(S)) = \{0\} \bigcup \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \right\}, \]
where \{\ldots\} denotes the closure of set \{\ldots\}. The second quantization of the identity operator 1 on \( \mathcal{W} \), \( d\Gamma(1) \), is referred to as the number operator, which is written as
\[ N := d\Gamma(1). \]

We note that
\[ D(N^k) = \{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} | \sum_{n=0}^{\infty} n^{2k} \| \Psi^{(n)} \|^2 < \infty \}. \]
From (1.7) and (1.8) it follows that
\[ \sigma(N) = \sigma_p(N) = \mathbb{N} \cup \{0\}. \]

1.2 Abstract interaction systems

Let \( \mathcal{H} \) be a Hilbert space. A Hilbert space for an abstract coupled system is given by
\[ \mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b, \]
and a decoupled Hamiltonian \( H_0 \) acting in \( \mathcal{F} \) is of the form
\[ H_0 = A \otimes 1 + 1 \otimes d\Gamma(S). \]
Assumptions (A1) and (A2) are as follows.

(A1) Operator \( A \) is a self-adjoint operator acting in \( \mathcal{H} \), and bounded from below.
(A2) Operator $S$ is a nonnegative self-adjoint operator acting in $\mathcal{W}$.

Total Hamiltonians under consideration are of the form

$$H = H_0 + gH_1, \quad (1.9)$$

where $g \in \mathbb{R}$ denotes a coupling constant and $H_1$ a symmetric operator. Assumption (A3) is as follows.

(A3) $H_1$ is $H_0$-bounded with

$$\|H_1\Psi\| \leq a\|H_0\Psi\| + b\|\Psi\|, \quad \Psi \in D(H_0),$$

where $a$ and $b$ are nonnegative constants.

Under (A3), by the Kato-Rellich theorem, $H$ is self-adjoint on $D(H_0)$ and bounded from below for $g$ with $|g| < 1/a$. Moreover $H$ is essentially self-adjoint on any core of $H_0$. The bottom of $\sigma(H)$ is denoted by

$$E(H) := \inf \sigma(H),$$

which is referred to as the ground state energy of $H$. If an eigenvector $\Psi$ associated with $E(H)$ exists, i.e.,

$$H\Psi = E(H)\Psi,$$

then $\Psi$ is called a ground state of $H$. Let $E_T(B)$ be the spectral projection of self-adjoint operator $T$ onto a Borel set $B \subset \mathbb{R}$. We set

$$P_T := E_T(\{E(T)\}).$$

Then $P_H$ denotes the projection onto the subspace spanned by ground states of $H$. The dimension of $P_H\mathcal{F}$ is called the multiplicity of ground states of $H$, and it is denoted by

$$m(H) := \dim P_H\mathcal{F}.$$

If $m(H) = 1$, then we call that the ground state of $H$ is unique.
1.3 Expectation values of the number operator

For Hamiltonians like as (1.9), the existence of a ground state $\varphi_g$ such that

$$\varphi_g \in D(1 \otimes N^{1/2})$$  \hspace{1cm} (1.10)

has been shown by many authors, e.g., [3, 8, 9, 14, 15, 23, 37]. Conversely, if $\varphi_g$ exists, little attention, however, has been given to investigate whether (1.10) holds or not. Then the first task in this paper is to give a necessary and sufficient condition for

$$P_H \mathcal{F} \subset D(1 \otimes N^{1/2}).$$  \hspace{1cm} (1.11)

As we will see later, to show (1.11) is also the primary problem in estimating an upper bound of $m(H)$.

1.4 Massive and massless cases

Typical examples of Hilbert space $\mathcal{W}$ and nonnegative self-adjoint operator $S$ are

$$\mathcal{W} = L^2(\mathbb{R}^d),$$  \hspace{1cm} (1.12)

$$S = \text{ the multiplication operator by } \omega_\nu(k) := \sqrt{|k|^2 + \nu^2}.$$  \hspace{1cm} (1.13)

In the case of $\nu > 0$ (resp. $\nu = 0$), a model is referred to as a massive (resp. massless) model. Note that under (A1) and (A3),

$$D(H) = D(H_0) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_\nu)).$$  \hspace{1cm} (1.14)

In a massive case, one can see that (1.11) is always satisfied. Actually in a massive case, we have $D(d\Gamma(\omega_\nu)) \subset D(N)$ and

$$\frac{1}{\nu} \|d\Gamma(\omega_\nu) \Psi\| \geq \|N\Psi\|, \quad \Psi \in D(d\Gamma(\omega_\nu)).$$

Together with (1.14) we obtain that

$$P_H \mathcal{F} \subset D(H) \subset D(1 \otimes d\Gamma(\omega_\nu)) \subset D(1 \otimes N) \subset D(1 \otimes N^{1/2}).$$

Hence (1.11) follows. Kernel $a(k)$ of $a(f)$, $f \in L^2(\mathbb{R}^d)$, is defined for each $k \in \mathbb{R}^d$ as

$$(a(k) \Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1} \Psi^{(n+1)}(k, k_1, \ldots, k_n)$$
and

$$(a(f)\Psi)^{(n)} = \int f(k)(a(k)\Psi)^{(n)}dk$$

for $\Psi \in \mathcal{F}_{\text{fin}}^{C_{0}^{\infty}(\mathbb{R}^{d})}$, and it is directly seen that

$$\int_{\mathbb{R}^{d}} \|a(k)\Psi\|^{2}dk = \|N^{1/2}\Psi\|^{2}, \quad \Psi \in \mathcal{F}_{\text{fin}}^{C_{0}^{\infty}(\mathbb{R}^{d})}. \quad \text{(1.15)}$$

From (1.15), $a(\cdot)\Psi$ for $\Psi \in D(N^{1/2})$ can be defined as an $\mathcal{F}_{b}$-valued $L^{2}$ function on $\mathbb{R}^{d}$ by

$$a(\cdot)\Psi := s - \lim_{m \to \infty} a(\cdot)\Psi_{m} \quad \text{in} \quad L^{2}(\mathbb{R}^{d}; \mathcal{F}_{b}),$$

where $s - \lim_{m \to \infty}$ denotes the strong limit in $L^{2}(\mathbb{R}^{d}; \mathcal{F}_{b})$ and sequence $\Psi_{m} \in \mathcal{F}_{\text{fin}}^{C_{0}^{\infty}(\mathbb{R}^{d})}$ is such that $\Psi_{m} \to \Psi$ and $N^{1/2}\Psi_{m} \to N^{1/2}\Psi$ strongly as $m \to \infty$. By an informal calculation, it can be derived pointwise that

$$(1 \otimes a(k))\varphi_{g} = g(H - E(H) + \omega(k))^{-1}[H_{1}, 1 \otimes a(k)]\varphi_{g}.$$

Note that at least we have to assume $\varphi_{g} \in D(1 \otimes N^{1/2})$ for (1.16) to make a sense, and the right-hand side of (1.16) is also delicate. See e.g., [36, Lemma 2.6] and [12, p.170, Conclusion] for this point. For massive cases, $(1 \otimes a(\cdot))\varphi_{g}$ is well defined as an $\mathcal{F}$-valued $L^{2}$ function on $\mathbb{R}^{d}$, since $\varphi_{g} \in D(1 \otimes N^{1/2})$, but of course it does not make sense pointwise. From (1.16) and (1.15) it follows that

$$\|(1 \otimes N^{1/2})\varphi_{g}\|^{2} = g^{2} \int_{\mathbb{R}^{d}} \|(H - E(H) + \omega(k))^{-1}[H_{1}, 1 \otimes a(k)]\varphi_{g}\|^{2}dk. \quad \text{(1.17)}$$

We may say under some conditions that

$$\varphi_{g} \in D(1 \otimes N^{1/2}) \quad \text{and} \quad \int_{\mathbb{R}^{d}} \|(H - E(H) + \omega(k))^{-1}[H_{1}, 1 \otimes a(k)]\varphi_{g}\|^{2}dk < \infty$$

$$\implies \|(1 \otimes N^{1/2})\varphi_{g}\|^{2} = g^{2} \int_{\mathbb{R}^{d}} \|(H - E(H) + \omega(k))^{-1}[H_{1}, 1 \otimes a(k)]\varphi_{g}\|^{2}dk.$$
Because of the tedious argument involved in establishing (1.16) pointwise, a quite different method is taken to show (1.17) in this paper. We will show under some conditions that
\[ \varphi_g \in D(1 \otimes N^{1/2}) \iff \int_{\mathbb{R}^d} \| (H - E(H) + \omega(k))^{-1} [H_1, 1 \otimes a(k)] \varphi_g \|^2 dk < \infty, \quad (1.18) \]
and (1.17) follows when the right or left-hand side of (1.18) holds. The method is an application of the fact that asymptotic annihilation operators vanish arbitrary ground states. See (1.23). As a result, (1.17) and (1.18) can be valid rigorously for both massive and massless cases without using (1.16). As far as we know, this method is new, cf., see [5, 6, 19, 20]. By means of (1.18) we can find a condition for \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \).

### 1.5 Multiplicity

Generally, in the case where \( E(H) \) is discrete, the min-max principle [35] is available to estimate the multiplicity of ground states. Actually the ground state energy of a massive generalized-spin-boson (GSB) model with a sufficiently weak coupling is discrete. Hence the min-max principle can be applied for this model [3]. However, for some typical models, e.g., massless GSB models, the Pauli-Fierz model, and the Nelson model [31], etc., their ground state energy is the edge of the essential spectrum, namely it is not discrete. See also [2, 22]. Then the min-max principle does not work at all.

Instead of the min-max principle, we can apply an infinite dimensional version of the Perron-Frobenius theorem [16, 17] to show the uniqueness of its ground state. I.e., in a Schrödinger representation,
\[ (\Psi, e^{-tH} \Phi) > 0, \quad \Psi \geq 0 \ (\neq 0), \quad \Phi \geq 0 \ (\neq 0), \quad (1.19) \]
implies \( m(H) = 1 \). Property (1.19) is called that \( e^{-tH} \) is positivity-improving. The Perron-Frobenius theorem has been applied for some models, e.g., the Nelson model in [8], and the spinless Pauli-Fierz model in [24]. It is, however, for e.g., the Pauli-Fierz model with spin 1/2, \( H_{PF} \), we can not apply the Perron-Frobenius theorem, since, as far as we know, a suitable representation for \( e^{-tH_{PF}} \) to be positivity-improving can not be constructed.

In this paper, applying the fact \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \), we establish a wide-usable method to estimate an upper bound of the multiplicity of ground states under some conditions.
1.6 Main results and strategies

The main results are (m1) and (m2).

(m1) We give a necessary and sufficient condition for $P_H \mathcal{F} \subset D(1 \otimes N^{1/2})$.

(m2) We prove $m(H) \leq m(A)$ under some conditions.

Strategies are as follows. It is proven that

$$\varphi_g \in D(1 \otimes N^{1/2}) \iff \sum_{m=1}^{\infty} \left\| (1 \otimes a(e_m))\varphi_g \right\|^2 < \infty,$$

(1.20)

where $\{e_m\}_{m=1}^{\infty}$ is an arbitrary complete orthonormal system of $\mathcal{W}$. When the left or right-hand side of (1.20) holds, it follows that

$$\sum_{m=1}^{\infty} \left\| (1 \otimes a(e_m))\varphi_g \right\|^2 = \left\| (1 \otimes N^{1/2})\varphi_g \right\|^2.$$

(1.21)

Let us define an asymptotic annihilation operator by

$$a_+(f)\Psi := s - \lim_{t \to \infty} e^{-itH}e^{itH_0}(1 \otimes a(f))e^{-itH_0}e^{itH}\Psi.$$  

(1.22)

Of course some conditions on $\Psi$ and $f$ are required to show the existence of $a_+(f)\Psi$. It is well known [1, 18], however, that (1.22) exists for an arbitrary ground state of $H$, $\Psi = \varphi_g$, and $a_+(f)$ vanishes $\varphi_g$, i.e.,

$$a_+(f)\varphi_g = 0$$

(1.23)

for some $f$. (1.23) is applied for (m1). We decompose $a_+(f)\Psi$ as

$$a_+(f)\Psi = (1 \otimes a(f))\Psi - gG(f)\Psi$$

with some operator $G(f) : \mathcal{F} \to \mathcal{F}$. From (1.23) it follows that

$$(1 \otimes a(f))\varphi_g = gG(f)\varphi_g.$$  

(1.24)

We define the bounded operator $T_{\varphi_g} : \mathcal{W} \to \mathcal{F}$ by

$$T_{\varphi_g}f := G(f)\varphi_g, \quad f \in \mathcal{W}.$$  

(1.25)

I.e.,

$$(1 \otimes a(f))\varphi_g = gT_{\varphi_g}f.$$  

(1.26)
It is seen that \( T_{\varphi} \) is an \( \mathcal{F} \)-valued integral operator such that
\[
T_{\varphi} f = \int_{\mathbb{R}^d} f(k) \kappa_{\varphi}(k) dk
\]
with some kernel \( \kappa_{\varphi}(k) \in \mathcal{F} \). See (2.29) for details. Note that
\[
\sum_{m=1}^{\infty} \|T_{\varphi} e_m\|^2 = \text{Tr}(T_{\varphi}^* T_{\varphi}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi}(k)\|^2 dk. \tag{1.27}
\]
Using (1.20), (1.26) and (1.27), we see that
\[
\varphi \in D(1 \otimes N^{1/2}) \iff g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi}(k)\|^2 dk < \infty,
\]
and by (1.21),
\[
\|(1 \otimes N^{1/2})\varphi\|^2 = g^2 \int_{\mathbb{R}^d} \|\kappa_{\varphi}(k)\|^2 dk. \tag{1.28}
\]
Thus we can obtain that
\[
P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \iff \int \|\kappa_{\varphi}(k)\|^2 dk < \infty \text{ for all } \varphi \in P_H \mathcal{F}.
\]
To show (m2) we apply the method in [27], by which we can prove that
\[
\dim(P_H \mathcal{F} \cap D(1 \otimes N^{1/2})) \leq \frac{1}{1 - \delta(g)} m(A),
\]
where
\[
\delta(g) = o(g) + \sup_{\varphi \in P_H \mathcal{F} \cap D(1 \otimes N^{1/2})} \frac{\|(1 \otimes N^{1/2})\varphi\|^2}{\|\varphi\|^2}.
\]
By (1.28) and the fact
\[
\lim_{g \to 0} \sup_{\varphi \in P_H \mathcal{F}} \frac{\int_{\mathbb{R}^d} \|\kappa_{\varphi}(k)\|^2 dk}{\|\varphi\|^2} < \infty,
\]
we see that \( \lim_{g \to 0} \delta(g) = 0 \). Hence for sufficiently small \( g \),
\[
\dim(P_H \mathcal{F} \cap D(1 \otimes N^{1/2})) \leq m(A)
\]
is proven. Together with the fact \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \) under some conditions, we get
\[
\dim(H) = \dim(P_H \mathcal{F}) \leq m(A).
\]
We organize this paper as follows.

Section 2 is devoted to show \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \). In Section 3, we estimate the multiplicity of ground states. In Sections 4, we give examples including massless GSB models and the Pauli-Fierz model. Finally in Section 5 we give appendixes.
2 Equivalent conditions to $P_H \mathcal{F} \subset D(1 \otimes N^{1/2})$

2.1 The number operator

Let $\{e_m\}_{m=1}^{\infty}$ be a complete orthonormal system of $\mathcal{W}$. We define

$$A_M, \quad M = 1, 2, ..., $$

by

$$A_M := (N + 1)^{-1/2} \left( \sum_{m=1}^{M} a^\dagger(e_m)a(e_m) \right) (N + 1)^{-1/2}. $$

Lemma 2.1 We have

1. $A_M$ has a unique bounded operator extension $\overline{A_M}$,
2. $\overline{A_M}$ is uniformly bounded in $M$ as $\| \overline{A_M} \| \leq 1$,
3. $s - \lim_{M \to \infty} A_M = N(N + 1)^{-1}$.

Proof: Let us define

$$\mathcal{F}_\omega := \left[ \bigoplus_{n=0}^{\infty} \left\{ \sum_{i_1 \leq \cdots \leq i_n}^{\text{finite}} \alpha_{i_1, \ldots, i_n} a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}) \Omega \left| \alpha_{i_1, \ldots, i_n} \in \mathbb{C} \right. \right\} \right] \cap \mathcal{F}_{\text{fin}}.$$

Note that $\mathcal{F}_\omega$ is dense in $\mathcal{F}_b$. Let

$$\phi = a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}) \Omega, \quad i_1 \leq \cdots \leq i_n.$$ 

Then

$$A_M \phi = \frac{1}{n + 1} \sum_{j=1}^{n} a^\dagger(e_{i_1}) \cdots a^\dagger(\sum_{m=1}^{M} (e_m, e_{i_j})e_m) \cdots a^\dagger(e_{i_n}) \Omega$$

$$= \beta_{i_1, \ldots, i_n}(M) \phi, \quad (2.1)$$

where

$$\beta_{i_1, \ldots, i_n}(M) := \begin{cases} \frac{n}{n+1}, & i_n \leq M, \\ \frac{n-1}{n+1}, & i_{n-1} \leq M < i_n, \\ \vdots & \vdots \\ \frac{1}{n+1}, & i_1 \leq M < i_2, \\ 0, & M < i_1. \end{cases}$$
Let $\Psi \in \mathcal{F}_\omega$ be such that
\[ \Psi = \sum_{i_1 \leq \cdots \leq i_n} \alpha_{i_1, \ldots, i_n} a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}) \Omega. \] (2.2)
We see that
\[ \|\Psi\|^2 = \sum_{i_1 \leq \cdots \leq i_n} |\alpha_{i_1, \ldots, i_n}|^2. \]
From (2.1) it follows that
\[ A_M \Psi = \sum_{i_1 \leq \cdots \leq i_n} \alpha_{i_1, \ldots, i_n} \beta_{i_1, \ldots, i_n}(M) a^\dagger(e_{i_1}) \cdots a^\dagger(e_{i_n}). \]
Then
\[ 0 \leq \|A_M \Psi\|^2 = \sum_{i_1 \leq \cdots \leq i_n} |\alpha_{i_1, \ldots, i_n}|^2 \beta_{i_1, \ldots, i_n}(M)^2 \]
\[ \leq \sum_{i_1 \leq \cdots \leq i_n} |\alpha_{i_1, \ldots, i_n}|^2 \left( \frac{n}{n+1} \right)^2 = \left( \frac{n}{n+1} \right)^2 \|\Psi\|^2. \]
Note that $A_M$ leaves $\otimes_n \mathcal{W}$ invariant. Hence for an arbitrary $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_\omega$, we have
\[ \|A_M \Psi\|^2 = \sum_{n=0}^\infty \|A_M \Psi^{(n)}\|^2 \leq \sum_{n=0}^\infty \left( \frac{n}{n+1} \right)^2 \|\Psi^{(n)}\|^2 \leq \|\Psi\|^2. \]
Since $\mathcal{F}_\omega$ is dense in $\mathcal{F}_b$, (1) and (2) follow. Let $\Psi$ be as (2.2). We see that
\[ s - \lim_{M \to \infty} A_M \Psi = \frac{n}{n+1} \Psi. \]
Hence for an arbitrary $\Phi \in \mathcal{F}_\omega$,
\[ s - \lim_{M \to \infty} A_M \Phi = N(N+1)^{-1} \Phi. \]
For an arbitrary $\Phi \in \mathcal{F}_b$ and an arbitrary $\epsilon > 0$, we can choose $\Phi_\epsilon \in \mathcal{F}_\omega$ such that
\[ \|\Phi - \Phi_\epsilon\| < \epsilon. \]
Since $\|A_M\| \leq 1$, we obtain that
\[ \|A_M \Phi - N(N+1)^{-1} \Phi\|
\leq \|A_M \Phi - A_M \Phi_\epsilon\| + \|A_M \Phi_\epsilon - N(N+1)^{-1} \Phi_\epsilon\| + \|N(N+1)^{-1}(\Phi_\epsilon - \Phi)\|
\leq 2\epsilon + \|A_M \Phi_\epsilon - N(N+1)^{-1} \Phi_\epsilon\|. \]
Then
\[ \lim_{M \to \infty} \|A_M \Phi - N(N + 1)^{-1} \Phi\| < 2 \epsilon \]
for an arbitrary \(\epsilon\). Thus (3) follows. \(\square\)

**Lemma 2.2** Let \(\{e_m\}_{m=1}^\infty\) be an arbitrary complete orthonormal system in \(\mathcal{W}\). Then (1) and (2) are equivalent.

(1) \(\Psi \in D(N^{1/2})\).

(2) \(\Psi \in \bigcap_{m=1}^\infty D(a(e_m))\) and
\[ \sum_{m=1}^\infty \|a(e_m)\Phi\|^2 < \infty. \] (2.3)

Moreover when (1) or (2) holds, it follows that
\[ \|N^{1/2} \Psi\|^2 = \sum_{m=1}^\infty \|a(e_m)\Phi\|^2. \]

**Proof:** (1) \(\Rightarrow\) (2)

We see that \(\mathcal{F}_\omega \subset D(N^{1/2})\) and
\[ e^{-tN^{1/2}} \mathcal{F}_\omega \subset \mathcal{F}_\omega, \]
which implies that \(\mathcal{F}_\omega\) is a core of \(N^{1/2}\) by [33, X.49]. Then, for \(\Psi \in D(N^{1/2})\), there exists a sequence \(\Psi_\epsilon \in \mathcal{F}_\omega\) such that \(s - \lim_{\epsilon \to 0} \Psi_\epsilon = \Psi\) and \(s - \lim_{\epsilon \to 0} N^{1/2} \Psi_\epsilon = N^{1/2} \Psi\). It is well known that
\[ \|a(f)\Phi\| \leq \|f\| \|N^{1/2} \Phi\|, \quad \Phi \in D(N^{1/2}). \]

Hence from the fact \(\Psi \in D(N^{1/2})\), it follows that \(\Psi \in D(a(e_m))\). We have
\[ \sum_{m=1}^M \|a(e_m)\Psi_\epsilon\|^2 = \langle (N + 1)^{1/2} \Psi_\epsilon, A_M (N + 1)^{1/2} \Psi_\epsilon \rangle \]
\[ \leq \| (N + 1)^{1/2} \Psi_\epsilon \|^2 \]
\[ \leq \|N^{1/2} \Psi_\epsilon\|^2 + \| \Psi_\epsilon\|^2. \] (2.4)

From this it follows that \(a(e_m)\Psi_\epsilon\) is a Cauchy sequence in \(\epsilon\). Since \(a(e_m)\) is a closed operator, \(\lim_{\epsilon \to 0} a(e_m)\Psi_\epsilon = a(e_m) \Psi\) follows. Hence we obtain that, as \(\epsilon \to 0\) on the both sides of (2.4),
\[ \sum_{m=1}^M \|a(e_m)\Psi\|^2 \leq \|N^{1/2} \Psi\|^2 + \| \Psi\|^2. \]
Taking $M \to \infty$ on the both sides above, we have
\[
\sum_{m=1}^{\infty} \|a(e_m^r)\Psi\|^2 \leq \|N^{1/2}\Psi\|^2 + \|\Psi\|^2.
\]
Thus the desired results follow.

(2) $\Rightarrow$ (1) We see that
\[
\inf_{N} \sum_{m=1}^{\infty} \|a(e_m^r)\Psi\|^2 = \lim_{M \to \infty} \sum_{m=1}^{M} \sum_{n=0}^{\infty} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)})
\]
\[
= \lim_{M \to \infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)}).
\]
Since $\sum_{m=1}^{M} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)})$ is monotonously increasing as $M \uparrow \infty$ and by (2.3),
\[
\lim_{M \to \infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)}) < \infty,
\]
we have by the Lebesgue monotone convergence theorem and (3) of Lemma 2.1,
\[
\lim_{M \to \infty} \sum_{n=0}^{\infty} \sum_{m=1}^{M} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)})
\]
\[
= \sum_{n=0}^{\infty} \lim_{M \to \infty} \sum_{m=1}^{M} (a(e_m^r)\Psi^{(n)}, a(e_m^r)\Psi^{(n)})
\]
\[
= \sum_{n=0}^{\infty} \lim_{M \to \infty} ((N+1)^{1/2}\Psi^{(n)}, A_{M}(N+1)^{1/2}\Psi^{(n)})
\]
\[
= \sum_{n=0}^{\infty} n(\Psi^{(n)}, \Psi^{(n)})
\]
\[
= \sum_{n=0}^{\infty} n\|\Psi^{(n)}\|^2 < \infty.
\]
This yields that $\Psi \in D(N^{1/2})$. \hfill $\Box$

### 2.2 Weak commutators

In subsections 2.2-2.4, we consider the case where
\[
\mathcal{W} = \oplus^{D} L^{2}(\mathbb{R}^{d}) \cong L^{2}(\mathbb{R}^{d} \times \{1, \ldots, D\})
\]  \hfill (2.5)

and
\[
S = [\omega]
\]  \hfill (2.6)
such that \([\omega] : \oplus^D L^2(\mathbb{R}^d) \to \oplus^D L^2(\mathbb{R}^d)\) is the multiplication operator defined by
\[
[\omega](\oplus_{j=1}^D f_j) = \oplus_{j=1}^D \omega f_j,
\]
with
\[
\omega(\cdot) : \mathbb{R}^d \to [0, \infty), \quad (\omega f)(k) = \omega(k) f(k).
\]
The creation operator and the annihilation operator of \(F_b(W)\) are denoted by
\[
a^\sharp(f, j) := a(0 \oplus \cdots \oplus \delta^j \oplus \cdots \oplus 0), \quad f \in L^2(\mathbb{R}^d), \quad j = 1, \ldots, D,
\]
which satisfy on \(F_{\text{fin}}\),
\[
[a(f, j), a^\dagger(g, j')] = (\delta_{jj'} f, g),
\]
\[
[a^\dagger(f, j), a^\dagger(g, j')] = 0,
\]
\[
[a(f, j), a(g, j')] = 0.
\]
By (1.5)-(1.4),
\[
e^{it(1 \otimes d\Gamma(\omega))}(1 \otimes a(f, j)) e^{-it(1 \otimes d\Gamma(\omega))} = 1 \otimes a(e^{-it\omega f, j}), \quad (2.8)
\]
\[
e^{it(1 \otimes d\Gamma(\omega))}(1 \otimes a^\dagger(f, j)) e^{-it(1 \otimes d\Gamma(\omega))} = 1 \otimes a^\dagger(e^{it\omega f, j}) \quad (2.9)
\]
follow on \(H \otimes F_{\text{fin}}\), and
\[
[1 \otimes d\Gamma(\omega), 1 \otimes a(f, j)] = -1 \otimes a(\omega f, j), \quad (2.10)
\]
\[
[1 \otimes d\Gamma(\omega), 1 \otimes a^\dagger(f, j)] = 1 \otimes a^\dagger(\omega f, j) \quad (2.11)
\]
follow for \(f \in D(\omega)\) on \(H \otimes F_{\text{fin}}^{D(\omega)}\). Let \(S\) and \(T\) be operators acting in a Hilbert space \(K\). We define a quadratic form \([S, T]_W^D\) with a form domain
\[
D \subset D(S^*) \cap D(S) \cap D(T^*) \cap D(T)
\]
by
\[
[S, T]_W^D(\Psi, \Phi) := (S^*\Psi, T\Phi) - (T^*\Psi, S\Phi), \quad \Psi, \Phi \in D.
\]

**Proposition 2.3** (1) Let \(f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)\). Then (2.8) and (2.9) can be extended on \(D(1 \otimes d\Gamma([\omega]))\). (2) Let \(\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)\). Then
\[
[1 \otimes d\Gamma(\omega), 1 \otimes a(f, j)]_W^{D(1 \otimes d\Gamma([\omega]))}(\Psi, \Phi) = (\Psi, -(1 \otimes a(\omega f, j))\Phi), \quad (2.12)
\]
\[
[1 \otimes d\Gamma(\omega), 1 \otimes a^\dagger(f, j)]_W^{D(1 \otimes d\Gamma([\omega]))}(\Psi, \Phi) = (\Psi, (1 \otimes a^\dagger(\omega f, j))\Phi). \quad (2.13)
\]
Proof: Let $\Psi \in D(1 \otimes d\Gamma([\omega]))$. Since $\mathcal{H} \otimes \mathcal{F}_{\text{fin}}$ is a core of $1 \otimes d\Gamma([\omega])$, there exists a sequence $\Psi_\epsilon$ such that $\Psi_\epsilon \to \Psi$ and $(1 \otimes d\Gamma([\omega]))\Psi_\epsilon \to (1 \otimes d\Gamma([\omega]))\Psi$ strongly as $\epsilon \to 0$. It follows that

$$
eq e^{it(1 \otimes a(f, j))}(1 \otimes a(f, j))e^{-it(1 \otimes d\Gamma([\omega]))}\Psi_\epsilon = 1 \otimes a(e^{-it\omega f, j})\Psi_\epsilon, \quad (2.14)$$

$$
eq e^{it(1 \otimes d\Gamma([\omega]))}(1 \otimes a(f, j))e^{-it(1 \otimes d\Gamma([\omega]))}\Psi_\epsilon = 1 \otimes a(e^{it\omega f, j})\Psi_\epsilon. \quad (2.15)$$

Using well known inequalities

$$
norm{(1 \otimes a(f, j))\Psi} \leq \norm{f/\sqrt{\omega}}\norm{(1 \otimes d\Gamma([\omega]))^{1/2}\Psi},$$
$$
norm{(1 \otimes a(f, j))\Psi} \leq \norm{f/\sqrt{\omega}}\norm{(1 \otimes d\Gamma([\omega]))^{1/2}\Psi} + \norm{f}\norm{\Psi},$$

we see that the both hand sides of (2.14) and (2.15) converge strongly as $\epsilon \to 0$. Since $1 \otimes a^2(f, j)$ is closed, (1) follows.

We shall prove (2). Let $\Psi, \Phi \in \mathcal{H} \otimes \mathcal{F}_{\text{fin}}^{D([\omega])}$. Then

$$(1 \otimes d\Gamma([\omega]))(1 \otimes a(f, j))\Phi) - ((1 \otimes a(f, j))\Psi, (1 \otimes d\Gamma([\omega]))\Phi) = (\Psi, -(1 \otimes a(f, j))\Phi).$$

Since $\mathcal{H} \otimes \mathcal{F}_{\text{fin}}^{D([\omega])}$ is a core of $1 \otimes d\Gamma([\omega])$, there exists a sequence $\Psi_\epsilon$ such that $\Psi_\epsilon \to \Psi$ and $(1 \otimes d\Gamma([\omega]))\Psi_\epsilon \to (1 \otimes d\Gamma([\omega]))\Psi$ strongly as $\epsilon \to 0$. By using the closedness of $1 \otimes a(f, j)$ and a similar limiting argument as that of (1), we obtain (2.12). (2.13) can be similarly proven.  

\[\square\]

2.3 Asymptotic fields

Define on $D(H)$,

$$a_t(f, j) := e^{-itH}e^{itH_0}(1 \otimes a(f, j))e^{-itH_0}e^{itH} = e^{-itH}(1 \otimes a(e^{-it\omega f, j}))e^{itH}.$$ 

Note that

$$H_0 = A \otimes 1 + 1 \otimes d\Gamma([\omega]),$$

Assumption (B1) is as follows.

(B1) $\omega$ satisfies (1) and (2).

(1) The Lebesgue measure of $K_\omega := \{k \in \mathbb{R}^d | \omega(k) = 0\}$ is zero.
(2) There exists a subset $K \subset \mathbb{R}^d$ with Lebesgue measure zero such that

$$\omega \in C^3(\mathbb{R}^d \setminus K)$$

and

$$\frac{\partial \omega}{\partial k_n}(k) \neq 0, \quad n = 1, \ldots, d, \quad k = (k_1, \ldots, k_d) \in \mathbb{R}^d \setminus K.$$  

**Example 2.4** A typical example of $\omega$ is $\omega(k) = |k|^p$ with $p > 0$. In this case

$$K_\omega = \{0\}$$

and

$$K = \bigcup_{n=1}^{d} \{(k_1, \ldots, k_d) \in \mathbb{R}^d | k_n = 0\}.$$  

**Lemma 2.5** Suppose (2) of (B1). Then for $f \in C^2_0(\mathbb{R}^d \setminus K)$,

$$\left| \int_{\mathbb{R}^d} e^{i\omega(k)} f(k) dk \right| \leq \frac{c}{s^2}$$

with some constant $c$.

**Proof:** We have, for $1 \leq m, n \leq d$,

$$e^{i\omega} = -\frac{1}{s^2} \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial e^{i\omega}}{\partial k_m} \right)$$

on $\mathbb{R}^d \setminus K$. Hence it follows that by integration by parts,

$$\int_{\mathbb{R}^d} e^{i\omega(k)} f(k) dk = -\frac{1}{s^2} \int_{\mathbb{R}^d} e^{i\omega(k)} \frac{\partial}{\partial k_m} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} f(k) \right) \right) dk.$$  

Thus we have

$$\left| \int_{\mathbb{R}^d} e^{i\omega(k)} f(k) dk \right| \leq \frac{1}{s^2} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial k_m} \left( \left( \frac{\partial \omega}{\partial k_m} \right)^{-1} \frac{\partial}{\partial k_n} \left( \left( \frac{\partial \omega}{\partial k_n} \right)^{-1} f(k) \right) \right) \right| dk.$$  

Since the integrand of the right-hand side above is integrable, the lemma follows. \(\square\)

**Proposition 2.6** Suppose (B1). Let $f \in C^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Then

$$s - \lim_{t \to \infty} \alpha_t(f, j) \varphi_g = 0, \quad j = 1, \ldots, D. \quad (2.16)$$
Proof: Note that it follows that
\[ \|a_t(f, j)\Psi\| \leq \|f/\sqrt{\omega}\|\| (1 \otimes d\Gamma([\omega])^{1/2})e^{itH}\Psi\|, \quad j = 1, \ldots, D, \]
and by the closed graph theorem,
\[ \|H_0\Psi\| \leq c_1\|H\Psi\| + c_2\|\Psi\|, \quad \Psi \in D(H), \]
with some constants \( c_1 \) and \( c_2 \), and
\[ \| (1 \otimes d\Gamma([\omega])^{1/2})\Psi\| \leq c_3\|(H_0 + 1)\Psi\| \]
with some constant \( c_3 \). Thus it follows that
\[ \|a_t(f, j)\Psi\| \leq c_4\|f/\sqrt{\omega}\|\|(H + 1)\Psi\| \] (2.17)
with some constant \( c_4 \). Let \( \mathcal{D} \) be a core of \( A \) and
\[ \Psi = G \otimes a^\dagger(f_1, j_1) \cdots a^\dagger(f_n, j_n)\Omega, \]
(2.18)
where \( G \in \mathcal{D} \) and \( f_l \in C^\infty_0(\mathbb{R}^d \setminus K) \), \( l = 1, \ldots, n \). We see that for an arbitrary \( \delta \in \mathbb{R} \),
\[ a(e^{-it(\omega-\delta)} f, j)\Psi = \sum_{l=1}^n (e^{it(\omega-\delta)} \bar{f}, f_l)G \otimes a^\dagger(f_1, j_1) \cdots a^\dagger(f_l, j_l) \cdots a^\dagger(f_n, j_n)\Omega, \]
where \( \hat{X} \) means neglecting \( X \). Since \( ff_l \in C^2_0(\mathbb{R}^d \setminus K) \), by Lemma 2.5 we see that
\[ |(e^{it(\omega-\delta)} \bar{f}, f_l)| \leq \frac{c_5}{|t|^2} \]
with some constant \( c_5 \). Hence
\[ s - \lim_{t \to \infty} a(e^{it(\omega-\delta)} f, j)\Psi = 0 \]
follows. Let \( \mathcal{E} \) be the set of the linear hull of vectors such as (2.18), which is a core of \( H_0 \). Thus there exists \( \Psi_\epsilon \in \mathcal{E} \) such that
\[ \Psi_\epsilon \to \varphi_g, \quad H_0\Psi_\epsilon \to H_0\varphi_g \]
strongly as \( \epsilon \to 0 \), which yields that
\[ \lim_{\epsilon \to 0} \|(H_0 + 1)^{1/2}(\Psi_\epsilon - \varphi_g)\| = 0. \]
Let \( \|(H_0 + 1)^{1/2}(\Psi - \varphi_g)\| < \epsilon \). We obtain that
\[
\|a_t(f, j)\varphi_g\|
= \|(1 \otimes a(e^{-it(\omega - E(H))}f, j))\varphi_g\|
\leq \|(1 \otimes a(e^{-it(\omega - E(H))}f, j))\Psi\| + \|(1 \otimes a(e^{-it(\omega - E(H))}f, j))(\Psi - \varphi_g)\|
\leq \|(1 \otimes a(e^{-it(\omega - E(H))}f, j))\Psi\| + C\| (H_0 + 1)^{1/2}(\Psi - \varphi_g)\|
\leq \|(1 \otimes a(e^{-it(\omega - E(H))}f, j))\Psi\| + \epsilon.
\]
Then
\[
\lim_{t \to \infty} \|a_t(f, j)\varphi_g\| < C\epsilon
\]
for an arbitrary \( \epsilon \). Then the proposition follows. \( \square \)

In addition to (B1), we introduce assumptions (B2)-(B4).

(B2) There exists an operator
\[
T_j(k) : \mathcal{F} \to \mathcal{F}, \quad k \in \mathbb{R}^d, \quad j = 1, \ldots, D,
\]
such that
\[
D(T_j(k)) \supset D(H), \quad \text{a. e. } k \in \mathbb{R}^d,
\]
and
\[
[1 \otimes a(f, j), H]_{W^D(H)}(\Psi, \Phi) = \int_{\mathbb{R}^d} f(k)(\Psi, T_j(k)\Phi)dk.
\]

(B3) Let \( \Psi \in D(H) \) and \( f \in C_0^2(\mathbb{R}^d \setminus \widetilde{K}) \) with some measurable set \( \widetilde{K} \subset \mathbb{R}^d \) such that \( K \subset \widetilde{K} \) and its Lebesgue measure is zero. Then
\[
\left| \int_{\mathbb{R}^d} dk f(k)(\Psi, e^{-is(H - E(H) + \omega(k)))}T_j(k)\varphi_g) \right| \in L^1([0, \infty), ds).
\]

(B4) \( \|T_j(\cdot)\varphi_g\| \in L^2(\mathbb{R}^d) \).

Lemma 2.7 Suppose (B1)-(B4). Let \( f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d) \). Then it follows that
\[
\int_{\mathbb{R}^d} \|f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g\|dk < \infty \quad (2.19)
\]
and
\[
(1 \otimes a(f, j))\varphi_g = g \int_{\mathbb{R}^d} f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g dk. \quad (2.20)
\]
**Proof:** Noting that

\[
\|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g\| \leq \|T_j(k)\varphi_g\|/\omega(k), \quad k \notin K_\omega,
\]

we see that

\[
\int_{\mathbb{R}^d} \|f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g\|\,dk
\leq \left( \int_{|k|<1} \frac{|f(k)|^2}{\omega(k)}\,dk \right)^{1/2} \left( \int_{|k|<1} \omega(k)\|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g\|^2\,dk \right)^{1/2}
+ \left( \int_{|k|\geq 1} |f(k)|^2\,dk \right)^{1/2} \left( \int_{|k|\geq 1} \|(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g\|^2\,dk \right)^{1/2}
\leq (\|f/\sqrt{\omega}\| + \|f\|)\|T_j(\cdot)\varphi_g\| < \infty. \tag{2.21}
\]

Then (2.19) follows. We divide a proof of (2.20) into three steps.

**(Step 1)** Let \( f \in C^2_0(\mathbb{R}^d \setminus \bar{K}), \ f/\sqrt{\omega} \in L^2(\mathbb{R}^d), \) and \( \Psi, \Phi \in D(H) \). Then

\[
(\Psi, (1 \otimes a(f, j))\varphi_g) = -ig \int_0^\infty \left( \int_{\mathbb{R}^d} (\Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g)\,dk \right)\,ds. \tag{2.22}
\]

**Proof of Step 1**

Let \( \Psi, \Phi \in \mathcal{D} := C^\infty(\mathbb{R}^d) \otimes D(d\Gamma(\omega)) \). Note that \( \mathcal{D} \) is a core of \( H \). We see that by (2.12) of Proposition 2.3 and (B2),

\[
\begin{align*}
\frac{d}{dt}(\Psi, a_t(f, j)\Phi)
&= -i(He^{itH}\Psi, (1 \otimes a(e^{-it\omega f}))e^{itH}\Phi) - i(e^{itH}\Psi, (1 \otimes a(\omega e^{-it\omega f}))e^{itH}\Phi) \\
&\quad + i((1 \otimes a^1(e^{it\omega j}))e^{itH}\Psi, He^{itH}\Phi) \\
&= -ig(H_1e^{itH}\Psi, (1 \otimes a(e^{-it\omega f}))e^{itH}\Phi) + ig((1 \otimes a^1(e^{it\omega j}))e^{itH}\Psi, H_1e^{itH}\Phi) \\
&= ig[1 \otimes a(e^{-it\omega f}), H_1^{D,H}]_{W^((H)}(e^{itH}\Psi, e^{itH}\Phi) \\
&= ig \int_{\mathbb{R}^d} f(k)e^{-it\omega l(k)}(\Psi, e^{-itH}T_j(k)e^{itH}\Phi)\,dk.
\end{align*}
\]

Then we obtain that for \( \Psi, \Phi \in \mathcal{D}, \)

\[
(\Psi, a_t(f, j)\Phi)
= (\Psi, (1 \otimes a(f, j))\Phi) + ig \int_0^t \left( \int_{\mathbb{R}^d} f(k)e^{-is\omega l(k)}(\Psi, e^{-isH}T_j(k)e^{isH}\Phi)\,dk \right)\,ds. \tag{2.23}
\]

Let \( \Psi, \Phi \in D(H) \). There exist sequences \( \Psi_m, \Phi_n \in \mathcal{D} \) such that \( \lim_{m \to \infty} \Psi_m = \Psi \) and \( \lim_{n \to \infty} \Phi_n = \Phi \) strongly. (2.23) holds true for \( \Psi, \Phi \) replaced by \( \Psi_m, \Phi_n, \) respectively.
By a simple limiting argument as \( m \to \infty \) and then \( n \to \infty \), we get (2.23) for \( \Psi, \Phi \in D(H) \). By Proposition 2.6 and (2.23) we have

\[
0 = \lim_{t \to \infty} \langle \Psi, a_t(f, j) \varphi_g \rangle \\
= \langle \Psi, (1 \otimes a(f, j)) \varphi_g \rangle + ig \int_0^\infty \left( \int_{\mathbb{R}^d} \langle \Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g \rangle dk \right) ds.
\]

Thus (2.22) follows.

(Step 2) (2.20) holds true for \( f \) such that \( f \in C_0^2(\mathbb{R}^d \setminus \tilde{K}) \) and \( f/\sqrt{\omega} \in L^2(\mathbb{R}^d) \).

Proof of Step 2

By (B3) and the Lebesgue dominated convergence theorem, we have

\[
-ig \int_0^\infty \left( \int_{\mathbb{R}^d} \langle \Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g \rangle dk \right) ds \\
= -ig \lim_{\epsilon \to 0} \int_0^\infty ds e^{-\epsilon s} \left( \int_{\mathbb{R}^d} \langle \Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g \rangle dk \right).
\]

By (B4),

\[
\int_{\mathbb{R}^d} dk \int_0^\infty \left| e^{-s t}(\Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g) \right| ds \\
\leq \|\Psi\| \left( \int_{\mathbb{R}^d} |f(k)||T_j(k)\varphi_g| dk \right) \int_0^\infty e^{-\epsilon s} ds < \infty.
\]

Hence Fubini’s theorem yields that \( f dk \) and \( f ds \) can be exchanged, i.e.,

\[
-ig \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon s} \left( \int_{\mathbb{R}^d} \langle \Psi, f(k)e^{-is(H-E(H)+\omega(k))}T_j(k)\varphi_g \rangle dk \right) ds \\
= -ig \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \left( \int_0^\infty \langle \Psi, f(k)e^{-is(H-E(H)+\omega(k)-i\epsilon)}T_j(k)\varphi_g \rangle ds \right) dk \\
= g \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \langle \Psi, f(k)(H-E(H)+\omega(k)-i\epsilon)^{-1}T_j(k)\varphi_g \rangle dk.
\]

We can check that, for \( k \notin K_\omega \),

\[
\left| \langle \Psi, f(k)(H-E(H)+\omega(k)-i\epsilon)^{-1}T_j(k)\varphi_g \rangle \right| \\
\leq \|\Psi\| |f(k)||T_j(k)\varphi_g|, \tag{2.24}
\]

\[
\int_{\mathbb{R}^d} |f(k)||T_j(k)\varphi_g| dk \\
\leq (\|f/\sqrt{\omega}\| + \|f\|)|T_j(\cdot)\varphi_g| < \infty, \tag{2.25}
\]

\[
s - \lim_{\epsilon \to 0} \langle H-E(H)+\omega(k)-i\epsilon\rangle^{-1}\varphi_g = (H-E(H)+\omega(k))^{-1}\varphi_g. \tag{2.26}
\]
(2.24), (2.25) and (2.26) imply that by the Lebesgue dominated convergence theorem,
\[ g \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} (\Psi, f(k)(H - E(H) + \omega(k) - i\epsilon)^{-1}T_j(k)\varphi_g) dk = g \int_{\mathbb{R}^d} (\Psi, f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g) dk. \]

Since, by (2.25) we have
\[ (\Psi, a(f, j)\varphi_g) = g \int_{\mathbb{R}^d} (\Psi, f(k)(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g) dk \]
we obtain (2.20).

\( \square \)

(Step 3) (2.20) holds true for \( f \) such that \( f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d) \).

Proof of Step 3

Set
\[ g(k) := \begin{cases} f(k)/\sqrt{\omega(k)}, & |k| < 1, \\ f(k), & |k| \geq 1. \end{cases} \]

Since \( g \in L^2(\mathbb{R}^d) \), there exists a sequence \( g_\epsilon \in C_0^\infty(\mathbb{R}^d \setminus \tilde{K}) \) such that \( g_\epsilon \to g \) strongly as \( \epsilon \to 0 \). Define
\[ f_\epsilon(k) := \begin{cases} \sqrt{\omega(k)}g_\epsilon(k), & |k| < 1, \\ g_\epsilon(k), & |k| \geq 1. \end{cases} \]

Hence \( f_\epsilon \in C_0^\infty(\mathbb{R}^d \setminus \tilde{K}) \) by (2) of (B1), and it follows that
\[ \int_{\mathbb{R}^d} |f(k) - f_\epsilon(k)|^2 / \omega(k) dk \to 0, \tag{2.27} \]
\[ \int_{|k| > 1} |f(k) - f_\epsilon(k)|^2 dk \to 0, \tag{2.28} \]
as \( \epsilon \to 0 \). We see that, by (2.27) and (2.28),
\[ \|(1 \otimes a(f))\varphi_g - (1 \otimes a(f_\epsilon))\varphi_g\| \leq \|(f - f_\epsilon)/\sqrt{\omega}\|\|(1 \otimes d\Gamma([\omega])^{1/2})\varphi_g\| \to 0 \]
and
\[ \left\| \int_{\mathbb{R}^d} (f(k) - f_\epsilon(k))(H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g dk \right\| \leq \left\{ \left( \int_{|k| < 1} \frac{|f(k) - f_\epsilon(k)|^2}{\omega(k)} dk \right)^{1/2} + \left( \int_{|k| \geq 1} |f(k) - f_\epsilon(k)|^2 dk \right)^{1/2} \right\} \|T_j(\cdot)\varphi_g\| \to 0 \]
as \( \epsilon \to 0 \). Then we can extend (2.20) to \( f \) such that \( f, f/\sqrt{\omega} \in L^2(\mathbb{R}^d) \). \( \square \)
2.4 Main theorem I

Set
\[ \kappa_{\varphi_k^j}(k) := (H - E(H) + \omega(k))^{-1}T_j(k)\varphi_g, \quad k \notin K. \]

We define
\[ T_{\varphi_k^j} : L^2(\mathbb{R}^d) \to \mathcal{F}, \quad j = 1, \ldots, D, \]
by
\[ T_{\varphi_k^j}f := \int_{\mathbb{R}^d} f(k)\kappa_{\varphi_k^j}(k)dk, \]
where the integral is taken in the strong sense in \( \mathcal{F} \). I.e., we have
\[ (1 \otimes a(f, j))\varphi_g = gT_{\varphi_k^j}f. \]

Proposition 2.8

(1) \( T_{\varphi_k^j} \) is a Hilbert-Schmidt operator if and only if
\[ \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk < \infty. \]

(2) Suppose that \( T_{\varphi_k^j} \) is a Hilbert-Schmidt operator, Then
\[ \sum_{m=1}^{\infty} \|T_{\varphi_k^j}e_m\|^2 = \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk \]
for an arbitrary complete orthonormal system \( \{e_m\}_{m=1}^{\infty} \) in \( L^2(\mathbb{R}^d) \).

Proof: The adjoint of \( T_{\varphi_k^j} \),
\[ T^*_{\varphi_k^j} : \mathcal{F} \to L^2(\mathbb{R}^d), \quad j = 1, \ldots, D, \]
is referred to as a Carleman operator (see e.g., [38, p.141]) with kernel \( \kappa_{\varphi_k^j} \), i.e.,
\[ T^*_{\varphi_k^j}\Phi(\cdot) := (\kappa_{\varphi_k^j}(\cdot), \Phi). \]

It is known [38, Theorem 6.12] that \( T^*_{\varphi_k^j} \) is a Hilbert-Schmidt operator if and only if
\[ \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk < \infty. \]
Moreover suppose that \( T^*_{\varphi_k^j} \) is a Hilbert-Schmidt operator. Then
\[ \text{Tr}(T_{\varphi_k^j} T^*_{\varphi_k^j}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk \]
is also known, which implies that \( T_{\varphi_k^j} \) is a Hilbert-Schmidt operator if and only if
\[ \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk < \infty, \]
and
\[ \text{Tr}(T^*_{\varphi_k^j} T_{\varphi_k^j}) = \text{Tr}(T_{\varphi_k^j} T^*_{\varphi_k^j}) = \int_{\mathbb{R}^d} \|\kappa_{\varphi_k^j}(k)\|^2 dk. \]
Thus the proposition follows.

The main theorem in this section is as follows.
Theorem 2.9 Suppose (B1)-(B4). Then (1), (2) and (3) are equivalent.

(1) \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \),

(2) \( T_{\varphi_k} \) is a Hilbert-Schmidt operator for all \( j = 1, \ldots, D \) and all \( \varphi_k \in P_H \mathcal{F} \),

(3) \( \int_{\mathbb{R}^d} \| (H - E(H) + \omega(k))^{-1} T_j(k) \varphi_k \|^2 dk < \infty \) for all \( j = 1, \ldots, D \) and all \( \varphi_k \in P_H \mathcal{F} \).

Suppose that one of (1), (2) and (3) holds, it follows that for an arbitrary ground state \( \varphi_g \),

\[
\|(1 \otimes N^{1/2}) \varphi_g \|^2 = g^2 \sum_{j=1}^D \int_{\mathbb{R}^d} \|(H - E(H) + \omega(k))^{-1} T_j(k) \varphi_k \|^2 dk. \tag{2.30}
\]

Proof: Let \( \{ e_m \}_{m=1}^\infty \) be a complete orthonormal system of \( L^2(\mathbb{R}^d) \) such that \( e_m / \sqrt{\omega} \in L^2(\mathbb{R}^d) \). It is proven in Lemma 2.2 that \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \) if and only if

\[
\sum_{j=1}^D \sum_{m=1}^\infty \| (1 \otimes a(e_m, j)) \varphi_g \|^2 < \infty \tag{2.31}
\]

for an arbitrary \( \varphi_g \in P_H \mathcal{F} \). By (2.29),

\[
(1 \otimes a(e_m, j)) \varphi_g = g T_{\varphi_k} e_m.
\]

Hence \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \) if and only if

\[
g^2 \sum_{j=1}^D \sum_{m=1}^\infty \| T_{\varphi_k} e_m \|^2 < \infty, \quad \varphi_g \in P_H \mathcal{F}.
\]

That is to say, \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \) if and only if \( T_{\varphi_k} \) is a Hilbert-Schmidt operator for all \( j = 1, \ldots, D \), and all \( \varphi_k \in P_H \mathcal{F} \), i.e., by Proposition 2.8, \( P_H \mathcal{F} \subset D(1 \otimes N^{1/2}) \) if and only if

\[
g^2 \sum_{j=1}^D \int_{\mathbb{R}^d} \| \kappa_{\varphi_k}(k) \|^2 dk < \infty, \quad \varphi_g \in P_H \mathcal{F}.
\]

Then the first half of the theorem is proven. Moreover by Lemma 2.2, when \( \varphi_g \in D(1 \otimes N^{1/2}) \),

\[
\|(1 \otimes N^{1/2}) \varphi_g \|^2 = \sum_{j=1}^D \sum_{m=1}^\infty \| (1 \otimes a(e_m, j)) \varphi_g \|^2,
\]

for all \( \varphi_g \in P_H \mathcal{F} \).
which yields that
\[ \| (1 \otimes N^{1/2}) \varphi_g \|^2 = g^2 \sum_{j=1}^D \text{Tr}(T_{\varphi_g}^* T_{\varphi_g}) = g^2 \sum_{j=1}^D \int_{\mathbb{R}} \| \kappa_{\varphi_g}(k) \|^2 dk. \]

Thus the proof is complete. \(\square\)

**Remark 2.10** In [6] a more general formula than (2.30) is obtained.

### 3 Proof of \( m(H) \leq m(A) \)

#### 3.1 Quadratic forms

We revive \( H = H_0 + gH_1 \), where \( H_0 = A \otimes 1 + 1 \otimes d\Gamma(S) \), and (A.1) – (A.3) are assumed. Set
\[ \overline{H}_0 := H_0 - E(H_0). \]

Actually
\[ E(H_0) = E(A). \]

The quadratic form \( \beta_0 \) associated with \( \overline{H}_0 \) is defined by
\[ \beta_0(\Psi, \Phi) := (\overline{H}_0^{1/2} \Psi, \overline{H}_0^{1/2} \Phi), \quad \Psi, \Phi \in D(\overline{H}_0^{1/2}). \]

Define a symmetric form by
\[ \beta_{H_1}(\Psi, \Phi) := (\Psi, H_1 \Phi), \quad \Psi, \Phi \in D(\overline{H}_0). \]

Since \( \| H_1 \Psi \| \leq a \| H_0 \Psi \| + b \| \Psi \| \), it follows that
\[ \| H_1 \Psi \| \leq a \| \overline{H}_0 \Psi \| + b' \| \Psi \|, \]
where \( b' = b + a |E(H_0)| \). Then \( H_1(\overline{H}_0 + \mu)^{-1} \) and \( (\overline{H}_0 + \mu)^{-1} H_1, \mu > 0 \), are bounded operators with
\[ \| H_1(\overline{H}_0 + \mu)^{-1} \| \leq a + b'/\mu, \quad \| (\overline{H}_0 + \mu)^{-1} H_1 \| \leq a + b'/\mu. \]

By an interpolation argument [33, Section IX], \( (\overline{H}_0 + \mu)^{-1/2} H_1(\overline{H}_0 + \mu)^{-1/2} \) is also a bounded operator with
\[ \| (\overline{H}_0 + \mu)^{-1/2} H_1(\overline{H}_0 + \mu)^{-1/2} \| \leq a + b'/\mu. \]
Then

\[ |\beta_H(\Psi, \Phi)| \leq (a + b'/\mu)\beta_0(\Psi, \Psi) + (a + b'/\mu)\|\Psi\|^2, \quad \Psi \in D(H_0), \quad (3.1) \]

for an arbitrary \( \mu > 0 \). By (3.1), a polarization identity [32] and a limiting argument, \( \beta_{H_1}(\Psi, \Phi) \) can be extended to \( \Psi, \Phi \in D(H_1/2) \). The extension of \( \beta_{H_1} \) is denoted by \( \tilde{\beta}_{H_1} \), and which satisfies

\[ |\tilde{\beta}_{H_1}(\Psi, \Psi)| \leq (a + b'/\mu)\beta_0(\Psi, \Psi) + (a + b'/\mu)\|\Psi\|^2, \quad \Psi \in D(H_1/2). \quad (3.2) \]

Thus we see that, for a sufficiently small \( g \),

\[ \beta_H := \beta_0 + g\tilde{\beta}_{H_1} \]

is a semibounded closed quadratic form on \( D(H_1/2) \times D(H_1/2) \). Then by the representation theorem for forms [28, p.322, Theorem 2.1], there exists a unique self-adjoint operator \( H' \) such that \( D(H') \subset D(H_1/2) \) and

\[ \beta_H(\Psi, \Phi) = (\Psi, H'\Phi), \quad \Psi \in D(H_1/2), \quad \Phi \in D(H'). \]

On the other hand, we can see directly that \( D(H) \subset D(H_0/2) \) and

\[ \beta_H(\Psi, \Phi) = (\Psi, H\Phi), \quad \Psi \in D(H_0/2), \quad \Phi \in D(H_0). \]

which yields that

\[ H' = H. \]

I.e., \( H \) is a unique self-adjoint operator associated with the quadratic form \( \beta_H \). We generalize this fact in the next subsection.

### 3.2 Abstract results

As was seen in the previous subsection, self-adjoint operator \( H = H_0 + gH_1 \) is defined through the quadratic form \( \beta_H \). In this subsection, as a mathematical generalization, we define a total Hamiltonian \( H_q \) through an abstract quadratic form, and estimate an upper bound of \( \dim \left\{ P_{H_q}F \cap D(1 \otimes N^{1/2}) \right\} \).

**Remark 3.1** Hamiltonians of the Nelson model without ultraviolet cutoffs are defined as the self-adjoint operator associated with a semibounded quadratic form. See [21, 31]. As far as we know, it can not be represented as the form \( H_0 + gH_1 \).
Let $\beta_{\text{int}}$ be a symmetric quadratic form with form domain $D(H^1_0/2)$ such that

$$|\beta_{\text{int}}(\Psi, \Psi)| \leq a\beta_0(\Psi, \Psi) + b(\Psi, \Psi), \quad \Psi \in D(H^1_0/2), \quad (3.3)$$

with some nonnegative constants $a$ and $b$. Define the quadratic form $\beta$ on $D(H^1_0/2)$ by

$$\beta := \beta_0 + g\beta_{\text{int}}.$$

**Proposition 3.2** Let $|g| < 1/a$. Then there exists a unique self-adjoint operator $H_q$ associated with $\beta$ such that its form domain is $D(H^1_0/2)$,

$$\beta(\Psi, \Phi) = (\Psi, H_q \Phi), \quad \Psi \in D(H^1_0/2), \Phi \in D(H_q),$$

and

$$\beta(\Psi, \Phi) = (H_{q+}^{1/2}\Psi, H_{q+}^{1/2}\Phi) - (H_{q-}^{1/2}\Psi, H_{q-}^{1/2}\Phi), \quad \Psi, \Phi \in D(H^1_0/2),$$

where

$$H_{q+} := H_q E_{H_q}((0, \infty)), \quad H_{q-} := -H_q E_{H_q}((-\infty, 0]).$$

**Proof:** From (3.3) it follows that

$$|g\beta_{\text{int}}(\Psi, \Psi)| \leq |g|a\beta_0(\Psi, \Psi) + |g|b(\Psi, \Psi).$$

Hence by the KLMN theorem [33, Theorem X.17], the proposition follows. \(\square\)

Assumptions (Gap) and (N) are as follows.

**(Gap)** $\inf \sigma_{\text{ess}}(A) - E(A) > 0.$

**(N)** $\lim_{g\to 0} \sup_{\Psi \in (P_{H_q} F) \cap D(1 \otimes N^{1/2})} \frac{\| (1 \otimes N^{1/2}) \Psi \|}{\| \Psi \|} = 0.$

Suppose that $\sigma_p(S) \not\supseteq 0$. Then by the facts that

$$\inf \sigma(d\Gamma(S)[\otimes_{n=1}^{\infty} W]) \geq 0,$$

$$\sigma_p(d\Gamma(S)[\otimes_{n=1}^{\infty} W]) \not\supseteq 0,$$

$$\sigma(d\Gamma(S)[\otimes W]) = \sigma_p(d\Gamma(S)[\otimes W]) = \{0\},$$

$d\Gamma(S)$ is nonnegative self-adjoint operator, and has a unique ground state $\Omega$ with eigenvalue 0. We have a lemma.
Lemma 3.3 Assume (A1), (A2), (Gap), (N) and \( \sigma_p(S) \neq 0 \). Then there exists \( \delta(g) > 0 \) such that
\[
\lim_{g \to 0} \delta(g) = 0
\]
and, for \( g \) with \( \delta(g) < 1 \),
\[
\dim \left\{(P_{H_q}F) \cap D(1 \otimes N^{1/2})\right\} \leq \frac{1}{1 - \delta(g)} m(A).
\]

Proof: Let \( \epsilon > 0 \) be such that
\[
[E(A), E(A) + \epsilon) \cap \sigma(A) = \{E(A)\}
\]
and we set
\[
P_\epsilon := E_A([E(A), E(A) + \epsilon)), \quad P_e := 1 - P_\epsilon.
\]
Furthermore let
\[
P_\Omega := E_{d\Gamma(S)}(\{0\}).
\]
We fix a \( \varphi_g \in (P_{H_q}F) \cap D(1 \otimes N^{1/2}) \). Using the inequality
\[
1 \otimes 1 \leq 1 \otimes N + 1 \otimes P_\Omega
\]
in the sense of form, we have
\[
(\varphi_g, \varphi_g) \leq ((1 \otimes N^{1/2}) \varphi_g, (1 \otimes N^{1/2}) \varphi_g) + (\varphi_g, (1 \otimes P_\Omega) \varphi_g)
\leq \| (1 \otimes N^{1/2}) \varphi_g \|^2 + \| (P_\epsilon \otimes P_\Omega) \varphi_g \|^2 + \| (P_e \otimes P_\Omega) \varphi_g \|^2. \tag{3.4}
\]
Let \( Q := P_\epsilon \otimes P_\Omega \). In Proposition 5.1 we shall show that
\[
\varphi_g \in D(H_{0}^{1/2}), \quad Q \varphi_g \in D(H_{0}^{1/2}) \tag{3.5}
\]
and
\[
H_{0}^{1/2} Q \varphi_g = Q H_{0}^{1/2} \varphi_g. \tag{3.6}
\]
Hence we have
\[
0 = (Q \varphi_g, (H_q - E(H_q)) \varphi_g)
= \beta(Q \varphi_g, \varphi_g) - E(H_q)(Q \varphi_g, \varphi_g)
= \beta_0(Q \varphi_g, \varphi_g) + g \beta_{\text{int}}(Q \varphi_g, \varphi_g) - E(H_q)(Q \varphi_g, \varphi_g).
\]
From this we have

\[-g\beta_{\text{int}}(Q\varphi_g, \varphi_g) = (\overline{\mathcal{H}}_0^{1/2}Q\varphi_g, \overline{\mathcal{H}}_0^{1/2}\varphi_g) - E(H_q)(Q\varphi_g, \varphi_g).\]  (3.7)

Since we have by (3.6)

\[
(\overline{\mathcal{H}}_0^{1/2}Q\varphi_g, \overline{\mathcal{H}}_0^{1/2}\varphi_g) = (\overline{\mathcal{H}}_0^{1/2}Q\varphi_g, \overline{\mathcal{H}}_0^{1/2}Q\varphi_g) = \int_{[E(A), \infty) \times [0, \infty)} (\lambda + \mu - E(A)) d\|(E_A(\lambda) \otimes E_d\Gamma(S)(\mu))Q\varphi_g\|^2
\]

\[
= \int_{[E(A)+\epsilon, \infty) \times \{0\}} (\lambda + \mu - E(A)) d\|(E_A(\lambda) \otimes E_d\Gamma(S)(\mu))Q\varphi_g\|^2
\]

\[
\geq \epsilon(Q\varphi_g, Q\varphi_g),
\]

then (3.7) implies that

\[-g\beta_{\text{int}}(Q\varphi_g, \varphi_g) \geq (\epsilon - E(H_q))(Q\varphi_g, \varphi_g).\]  (3.8)

We shall estimate \(|\beta_{\text{int}}(Q\varphi_g, \varphi_g)|\).

\[
\beta_0(\varphi_g, \varphi_g) = (\varphi_g, H_q\varphi_g) - g\beta_{\text{int}}(\varphi_g, \varphi_g)
\]

\[
\leq E(H_q)\|\varphi_g\|^2 + |g| (a\beta_0(\varphi_g, \varphi_g) + b(\varphi_g, \varphi_g)),
\]

which yields that, since \(|g| < 1/a,\)

\[
\beta_0(\varphi_g, \varphi_g) \leq \frac{E(H_q) + |g|b}{1 - a|g|}(\varphi_g, \varphi_g).
\]

Then we have

\[
|\beta_{\text{int}}(\varphi_g, \varphi_g)| \leq (a\beta_0(\varphi_g, \varphi_g) + b(\varphi_g, \varphi_g)) \leq c_{\text{int}}(\varphi_g, \varphi_g),
\]

where

\[
c_{\text{int}} := \frac{a(E(H_q) + |g|b)}{1 - a|g|} + b.
\]

From the polarization identity

\[
\beta_{\text{int}}(Q\varphi_g, \varphi_g) = \frac{1}{4} \{(\beta_{\text{int}}((1 + Q)\varphi_g, (1 + Q)\varphi_g) - \beta_{\text{int}}((1 - Q)\varphi_g, (1 - Q)\varphi_g))
\]

\[
- i (\beta_{\text{int}}((1 + iQ)\varphi_g, (1 + iQ)\varphi_g) - \beta_{\text{int}}((1 - iQ)\varphi_g, (1 - iQ)\varphi_g))\},
\]

it follows that

\[
|\beta_{\text{int}}(Q\varphi_g, \varphi_g)| \leq 2c_{\text{int}}(\varphi_g, \varphi_g).\]  (3.9)
Note that
\[
|\beta(\Psi, \Psi) - \beta_0(\Psi, \Psi)| = |g||\beta_{\text{int}}(\Psi, \Psi)| \leq |g|(a + b)\|(\mathcal{P}_0 + 1)^{1/2}\Psi\|^2.
\]

Then
\[
\lim_{g \to 0} \sup_{\Psi \in D(\mathcal{P}_0^{1/2})} \frac{|\beta(\Psi, \Psi) - \beta_0(\Psi, \Psi)|}{\|(\mathcal{P}_0 + 1)^{1/2}\Psi\|^2} \leq \lim_{g \to 0} |g|(a + b) = 0,
\]

which implies that for \( z \in \mathbb{C} \) with \( \Im z \neq 0 \),
\[
\lim_{g \to 0} \|(H_q - z)^{-1} - (\mathcal{P}_0 - z)^{-1}\| = 0. \tag{3.10}
\]

See Proposition 5.2 for a proof of (3.10). Thus it follows that
\[
\lim_{g \to 0} E(H_q) = E(\mathcal{P}_0) = 0. \tag{3.11}
\]

Then there exists a constant \( c > 0 \) such that for all \( g \) with \( |g| < c \), it obeys that
\[
\epsilon - E(H_q) > 0.
\]

Then by (3.8) and (3.9), for \( g \) with \( |g| < c \),
\[
\|Q\varphi_g\|^2 \leq |g| \frac{|\beta_{\text{int}}(Q\varphi_g, \varphi_g)|}{\epsilon - E(H_q)} \leq 2|g| \frac{c_{\text{int}}}{\epsilon - E(H_q)}\|\varphi_g\|^2.
\]

Let
\[
c(g) := \sup_{\Psi \in (P_{H_q}^N \cap D(1 \otimes N^{1/2}))} \frac{\|(1 \otimes N^{1/2})\Psi\|}{\|\Psi\|}.
\]

Together with (3.4) we have
\[
(\varphi_g, \varphi_g) \leq c(g)^2\|\varphi_g\|^2 + 2|g| \frac{c_{\text{int}}}{\epsilon - E(H_q)}\|\varphi_g\|^2 + \|(P_e \otimes P_\Omega)\varphi_g\|^2. \tag{3.12}
\]

Setting
\[
\delta(g) := c(g)^2 + 2|g| \frac{c_{\text{int}}}{\epsilon - E(H_q)},
\]

we see that by (3.11) and (N),
\[
\lim_{g \to 0} \delta(g) = 0.
\]

Then by (3.12) there exists \( g_* \leq c \) such that for \( g \) with \( |g| < g_* \),
\[
(\varphi_g, \varphi_g) \leq (1 - \delta(g))^{-1} (\varphi_g, (P_e \otimes P_\Omega)\varphi_g). \tag{3.13}
\]
Let \( \{ \varphi^j_{gj} \}_{j=1}^M \), \( M \leq \infty \), be a complete orthonormal system of \( (P_{Hq}\mathcal{F}) \cap D(1 \otimes N^{1/2}) \). Then by (3.13),
\[
(\varphi^j_{gj}, \varphi^j_{gj}) \leq (1 - \delta(g))^{-1} (\varphi^j_{gj}, (P_{\epsilon} \otimes P_{\Omega})\varphi^j_{gj}).
\] (3.14)
Summing up from \( j = 1 \) to \( M \), we have
\[
\dim \{ (P_{Hq}\mathcal{F}) \cap D(1 \otimes N^{1/2}) \} \leq (1 - \delta(g))^{-1} \sum_{j=1}^M (\varphi^j_{gj}, (P_{\epsilon} \otimes P_{\Omega})\varphi^j_{gj}).
\]
Since
\[
\sum_{j=1}^M (\varphi^j_{gj}, (P_{\epsilon} \otimes P_{\Omega})\varphi^j_{gj}) = \sum_{j=1}^M (\varphi^j_{gj}, (P_A \otimes P_{\Omega})\varphi^j_{gj}) \leq \text{Tr}(P_A \otimes P_{\Omega}) = \text{Tr}P_A \times \text{Tr}P_{\Omega} = m(A),
\]
we obtain that
\[
\dim \{ (P_{Hq}\mathcal{F}) \cap D(1 \otimes N^{1/2}) \} \leq (1 - \delta(g))^{-1} m(A).
\]
Thus the lemma is proven. \( \square \)

From Lemma 3.3, corollaries immediately follow.

**Corollary 3.4** We assume the same assumptions as in Lemma 3.3. Suppose that
\[
P_{Hq}\mathcal{F} \subset D(1 \otimes N^{1/2}).
\] (3.15)
Then
\[
m(H_q) \leq (1 - \delta(g))^{-1} m(A).
\]

In addition, suppose that \( g \) is such that \( \delta(g) < 1/2 \) and \( m(A) = 1 \). Then \( m(H) = 1 \).

**Proof:** Since \( P_{Hq}\mathcal{F} \cap D(1 \otimes N^{1/2}) = P_{Hq}\mathcal{F} \), the corollary follows from Lemma 3.3. \( \square \)

**Corollary 3.5 [Overlap]** We assume the same assumptions as in Lemma 3.3 and (3.15). Let \( g \) be such that \( \delta(g) < 1 \). Then for an arbitrary ground state \( \varphi_g \), it follows that
\[
(\varphi_g, (P_A \otimes P_{\Omega})\varphi_g) \neq 0.
\]

**Proof:** By (3.13) it is seen that
\[
0 < \|\varphi_g\|^2 \leq (1 - \delta(g))^{-1} (\varphi_g, (P_{\epsilon} \otimes P_{\Omega})\varphi_g) = (1 - \delta(g))^{-1} (\varphi_g, (P_A \otimes P_{\Omega})\varphi_g).
\]
Hence the corollary follows. \( \square \)
3.3 Main theorem II

We assume (2.5) and (2.6), i.e., \( H = H_0 + gH_1 \) and \( H_0 = A \otimes 1 + 1 \otimes d\Gamma([\omega]) \). Now we are in the position to state the main theorem in this section.

**Theorem 3.6** Suppose that (B1)-(B4), (A1), (A3), (Gap). We assume that

\[
\int_{\mathbb{R}^d} \| (H - E(H) + \omega(k))^{-1} T_j(k) \varphi_g \|^2 dk < \infty,
\]

and

\[
\sup_{\varphi_g \in P_H} \sum_{j=1}^D \int_{\mathbb{R}^d} \frac{\| (H - E(H) + \omega(k))^{-1} T_j(k) \varphi_g \|^2}{\| \varphi_g \|^2} dk < \infty.
\] (3.16)

Then there exists a constant \( g_* \) such that for \( g \) with \( |g| < g_* \), it follows that

\[ m(H) \leq m(A). \]

**Proof:** By Theorem 2.9, it follows that \( P_HF \subset D(1 \otimes N^{1/2}) \) and

\[
\| (1 \otimes N^{1/2}) \varphi_g \|^2 = g^2 \sum_{j=1}^D \int_{\mathbb{R}^d} \| (H - E(H) + \omega(k))^{-1} T_j(k) \varphi_g \|^2 dk.
\]

By (3.16) we have

\[
\lim_{g \to 0} \sup_{\varphi_g \in P_H} \frac{\| (1 \otimes N^{1/2}) \varphi_g \|}{\| \varphi_g \|} = \lim_{g \to 0} |g| \sup_{\varphi_g \in P_H} \left( \frac{\sum_{j=1}^D \int_{\mathbb{R}^d} \| (H - E(H) + \omega(k))^{-1} T_j(k) \varphi_g \|^2 dk}{\| \varphi_g \|^2} \right)^{1/2} = 0.
\]

From this and Corollary 3.4, the theorem follows. \( \square \)

4 Examples

4.1 GSB models

GSB models are a generalization of the spin-boson model, which was introduced and investigated in [3]. Examples of GSB models are e.g., \( N \)-level systems coupled to a Bose field, lattice spin systems, the Pauli-Fierz model with the dipole approximation neglected \( A^2 \) term, a Fermi field coupled to a Bose field, etc. See [3, p. 457].

The Hilbert space on which GSB Hamiltonians act is

\[
\mathcal{F}_{GSB} := \mathcal{H} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)),
\]
where $\mathcal{H}$ is a Hilbert space. Let $a(f)$ and $a^\dagger(f), f \in L^2(\mathbb{R}^d)$, be the annihilation operator and the creation operator on $\mathcal{F}_b(L^2(\mathbb{R}^d))$, respectively. We use the same notations $a(f)$ and $a^\dagger(f)$ as those of Subsection 1.1. We set
\[
\phi(\lambda) := \frac{1}{\sqrt{2}} (a^\dagger(\bar{\lambda}) + a(\lambda)), \quad \lambda \in L^2(\mathbb{R}^d).
\]

GSB Hamiltonians are defined by
\[
H_{\text{GSB}} := H_{\text{GSB},0} + \alpha H_{\text{GSB},1}.
\]

Here $\alpha \in \mathbb{R}$ is a coupling constant, and
\[
H_{\text{GSB},0} := A \otimes 1 + 1 \otimes d\Gamma(\omega_{\text{GSB}}),
\]
\[
H_{\text{GSB},1} := \sum_{j=1}^J B_j \otimes \phi(\lambda_j),
\]

where $\omega_{\text{GSB}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a multiplication operator by $\omega_{\text{GSB}}(k)$ such that
\[
\omega_{\text{GSB}}(\cdot) : \mathbb{R}^d \to [0, \infty)
\]

and $\overline{X}$ denotes the closure of $X$. Assumption (GSB1)-(GSB5) are as follows.

(GSB1) Operator $A$ satisfies (A1). Set $\overline{A} := A - E(A)$.

(GSB2) $\lambda_j, \lambda_j/\sqrt{\omega_{\text{GSB}}}, j = 1, \ldots, J \in L^2(\mathbb{R}^d)$.

(GSB3) $B_j, j = 1, \ldots, J$, is a symmetric operator, $D(\overline{A}^{1/2}) \subset \cap_{j=1}^J D(B_j)$ and there exist constants $a_j$ and $b_j$ such that
\[
\|B_j f\| \leq a_j \|\overline{A}^{1/2} f\| + b_j \|f\|, \quad f \in D(\overline{A}^{1/2}).
\]

Moreover
\[
|\alpha| < \left(\sum_{j=1}^J a_j \|\lambda_j/\sqrt{\omega_{\text{GSB}}}\|\right)^{-1}.
\]

(GSB4) $\omega_{\text{GSB}}$ satisfies that

(1) $\omega_{\text{GSB}}(\cdot)$ is continuous,

(2) $\lim_{|k| \to \infty} \omega_{\text{GSB}}(k) = \infty$, 

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there exist constants $C > 0$ and $\gamma > 0$ such that
\[ |\omega_{\text{GSB}}(k) - \omega_{\text{GSB}}(k')| \leq C|k - k'|\gamma(1 + \omega_{\text{GSB}}(k) + \omega_{\text{GSB}}(k')) . \]

\[(\text{GSB}5) \] \( \lambda_j, j = 1, \ldots, J, \) is continuous.

**Proposition 4.1** Assume \((\text{GSB}1)-(\text{GSB}3)\). Then \(H_{\text{GSB}}\) is self-adjoint on
\[ D(H_{\text{GSB},0}) = D(A \otimes 1) \cap D(1 \otimes d\Gamma(\omega_{\text{GSB}})) \]
and bounded from below. Moreover it is essentially self-adjoint on any core of \(H_{\text{GSB},0}\).

**Proof:** We can show that
\[ \|H_{\text{GSB},1}\Psi\| \leq \left( \sum_{j=1}^{J} a_j \|\lambda_j/\sqrt{\omega_{\text{GSB}}}\| \right) \|H_{\text{GSB},0}\Psi\| + b\|\Psi\| \tag{4.1} \]
for \(\Psi \in D(H_{\text{GSB},0})\) with some constant \(b\). Then by the Kato-Rellich theorem, the proposition follows. for details. \(\square\)

We introduce assumptions.

\[(\text{IR}) \] \( \lambda_j/\omega_{\text{GSB}} \in L^2(\mathbb{R}^d), j = 1, \ldots, J. \)

\[(\text{GSB6}) \] \( \omega_{\text{GSB}} \) satisfies \((B1)\) with \(\omega\) replaced by \(\omega_{\text{GSB}}\).

\[(\text{GSB7}) \] \( \lambda_j \in C^2(\mathbb{R}^d), j = 1, \ldots, J. \)

**Proposition 4.2** We assume \((\text{GSB1})-(\text{GSB5}), \ (\text{IR}) \) and \((\text{Gap})\). Then there exists a constant \(\alpha_* > 0\) such that for \(\alpha\) with \(|\alpha| < \alpha_*\), \(H_{\text{GSB}}\) has a ground state \(\varphi_g\) such that \(\|(1 \otimes N^{1/2})\varphi_g\| < \infty.\)

**Proof:** See [3, Theorem 1.3]. \(\square\)

**Remark 4.3** In [3, Theorem 1.3], it is actually supposed that self-adjoint operator \(A\) has a compact resolvent, i.e., \(\sigma(A) = \sigma_p(A)\). However it can be extended to \(A\) satisfying \((\text{Gap})\). See [7, Appendix].
Let \( f \in C^2_0(\mathbb{R}^d \setminus K) \) and \( \Psi, \Phi \in D(H_{\text{GSB}}) \). We have

\[
[a(f), H_{\text{GSB}}, I]_{\text{W}} \mathcal{D}(H_{\text{GSB}}) = (\Psi, \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1)\Phi) = \int_{\mathbb{R}^d} f(k)(\Psi, \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1)\Phi) dk
\]

where

\[
T_{\text{GSB}}(k) := \sum_{j=1}^{J} \lambda_j(k)(B_j \otimes 1).
\]

Our main theorem in this subsection is as follows.

**Theorem 4.4** Suppose (GSB1)-(GSB3), (IR), (GSB 6) and (GSB 7). Then it follows that

\[
P_{H_{\text{GSB}}} F_{\text{GSB}} \subset D(1 \otimes N^{1/2}),
\]

and

\[
\|(1 \otimes N^{1/2}) \varphi_g\|^2 = \alpha^2 \int_{\mathbb{R}^d} \|(H_{\text{GSB}} - E(H_{\text{GSB}}) + \omega_{\text{GSB}}(k))^{-1}T_{\text{GSB}}(k)\varphi_g\|^2 dk.
\]

In addition, suppose (Gap). Then there exists \( \alpha_{**} \) such that for \( \alpha \) with \( |\alpha| < \alpha_{**} \), it follows that

\[
m(H_{\text{GSB}}) \leq m(A).
\]

**Proof:** We shall check assumptions (B1)-(B4) and (3) of Theorem 2.9 with the following identifications.

\[\mathcal{F} = \mathcal{F}_{\text{GSB}}, \quad H_0 = H_{\text{GSB},0}, \quad H_1 = H_{\text{GSB},1}, \quad \omega = \omega_{\text{GSB}}, \quad D = 1, \quad T_{j=1}(k) = T_{\text{GSB}}(k).\]

(B1) and (B2) have been already checked. We have

\[
\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{\text{GSB}} - E(H_{\text{GSB}}) + \omega_{\text{GSB}}(k))}T_{\text{GSB}}(k)\varphi_g) dk
\]

\[
= \int_{\mathbb{R}^d} f(k) \sum_{j=1}^{J} \lambda_j(k)e^{-is\omega_{\text{GSB}}(k)}(\Psi, e^{-is(H_{\text{GSB}} - E(H_{\text{GSB}}))}(B_j \otimes 1)\varphi_g) dk
\]

\[
= \sum_{j=1}^{J}(\Psi, e^{-is(H_{\text{GSB}} - E(H_{\text{GSB}}))(B_j \otimes 1)\varphi_g}) \int_{\mathbb{R}^d} f(k)\lambda_j(k)e^{-is\omega_{\text{GSB}}(k)} dk.
\]
Since $f \lambda_j \in C_0^2(\mathbb{R}^d \setminus K)$, we see that by Lemma 2.5,

$$\left| \int_{\mathbb{R}^d} f(k) \lambda_j(k) e^{-i\omega_{\text{GSB}}(k)} dk \right| \in L^1([0, \infty), ds),$$

which implies, together with (4.5), that (B3) follows. We have

$$\int_{\mathbb{R}^d} \| T_{\text{GSB}}(k) \varphi_\omega \|^2 dk \leq J \sum_{j=1}^J \left( \int_{\mathbb{R}^d} |\lambda_j(k)|^2 dk \right) \| (B_j \otimes 1) \varphi_\omega \|^2 < \infty,$$

and

$$\int_{\mathbb{R}^d} \| (H_{\text{GSB}} - E(H_{\text{GSB}}) + \omega_{\text{GSB}}(k))^{-1} T_{\text{GSB}}(k) \varphi_\omega \|^2 dk$$

$$\leq \int_{\mathbb{R}^d} \frac{1}{\omega_{\text{GSB}}(k)^2} \| T_{\text{GSB}}(k) \varphi_\omega \|^2 dk$$

$$\leq J \sum_{j=1}^J \left( \int_{\mathbb{R}^d} |\lambda_j(k)|^2 \omega_{\text{GSB}}(k)^2 \right) \| (B_j \otimes 1) \varphi_\omega \|^2 < \infty.$$

Thus (B4) and (3) of Theorem 2.9 follow. Hence (4.2) and (4.3) are proven. We check (3.16) in Theorem 3.6 to show (4.4). Note that

$$\|(B_j \otimes 1) \varphi_\omega \| \leq a_j \|(A_1^{1/2} \otimes 1) \varphi_\omega \| + b_j \| \varphi_\omega \|$$

(4.6)

and

$$\|(A_1^{1/2} \otimes 1) \varphi_\omega \| \leq \| H_{\text{GSB},0}^{1/2} \varphi_\omega \|.$$ 

Since

$$\|H_{\text{GSB},0}^{1/2} \varphi_\omega \|^2 = (\varphi_\omega, H_{\text{GSB},0}^{1/2} \varphi_\omega) \leq (c_1 E(H_{\text{GSB}}) + c_2) \| \varphi_\omega \|^2$$

with some constants $c_1$ and $c_2$, we have

$$\|(B_j \otimes 1) \varphi_\omega \| \leq (a_j (c_1 E(H_{\text{GSB}}) + c_2)^{1/2} + b_j) \| \varphi_\omega \|.$$

Thus by (4.6),

$$\lim_{g \to 0} \sup_{\| \varphi_\omega \| \leq 1} g^2 \int_{\mathbb{R}^d} \| (H_{\text{GSB}} - E(H_{\text{GSB}}) + \omega_{\text{GSB}}(k))^{-1} T_{\text{GSB}}(k) \varphi_\omega \|^2 dk$$

$$\leq \lim_{g \to 0} g^2 J \sum_{j=1}^J (a_j (c_1 E(H_{\text{GSB}}) + c_2)^{1/2} + b_j)^2 \| \lambda_j / \omega_{\text{GSB}} \|^2 = 0.$$

Then (4.4) follows from Theorem 3.6. □
Corollary 4.5 Assume (GSB1)-(GSB4), (GSB6), (GSB7), (IR) and (Gap). Then there exists $\alpha^+\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!

Proof: It follows from Proposition 4.2 and Theorem 4.4.

4.2 The Pauli-Fierz model

The Pauli-Fierz model describes a minimal interaction between electrons with spin 1/2 and a quantized radiation field quantized in the Coulomb gauge. The asymptotic field for $H_{\text{PF}}$ is studied in e.g., [13, 22]. The Hilbert space for state vectors of the Pauli-Fierz Hamiltonian is given by

$$\mathcal{F}_{\text{PF}} := L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})) \otimes F_{\text{PF}}.$$ 

Formally the annihilation operator and the creation operator of $\mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\}))$ is denoted by

$$a^\sharp(f, j) = \int f(k) a^\sharp(k, j) dk.$$ 

The Pauli-Fierz Hamiltonian with ultraviolet cutoff $\hat{\varphi}$ is defined by

$$H_{\text{PF}} := \frac{1}{2m} (p \otimes 1 - eA_{\hat{\varphi}})^2 + V \otimes 1 + 1 \otimes H_t - \frac{e}{2m} (\sigma \otimes 1) \cdot B_{\hat{\varphi}},$$

where $m > 0$ and $e \in \mathbb{R}$ denote the mass of an electron and the charge of an electron, respectively. We regard $e$ as a coupling constant. $p$ denotes the momentum operator of an electron, i.e.,

$$p = (p_1, p_2, p_3) = (-i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3}),$$

and $V$ is an external potential. We identify $\mathcal{F}_{\text{PF}}$ as

$$\mathcal{F}_{\text{PF}} \cong \mathbb{C}^2 \otimes \int_{\mathbb{R}^3} \mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\}))dx,$$

where $\int_{\mathbb{R}^3} \cdots dx$ denotes a constant fiber direct integral [35]. $A_{\hat{\varphi}}$ and $B_{\hat{\varphi}}$ denote a quantized radiation field and a quantized magnetic field with ultraviolet cutoff $\hat{\varphi}$, respectively, which are given by, under identification (4.7),

$$A_{\hat{\varphi}} := 1 \otimes \int_{\mathbb{R}^3} A_{\hat{\varphi}}(x)dx, \quad B_{\hat{\varphi}} := 1 \otimes \int_{\mathbb{R}^3} B_{\hat{\varphi}}(x)dx.$$
with

\[ A_\hat{\phi}(x) := \sum_{j=1,2} \int \frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} e(k, j) \left\{ e^{-ik_x} a_j^\dagger(k, j) + e^{ik_x} a(k, j) \right\} dk \]

and

\[ B_\hat{\phi}(x) := \text{rot}_x A_\hat{\phi}(x) \]

\[ = \sum_{j=1,2} \int \frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} (-ik \times e(k, j)) \left\{ e^{-ik_x} a_j^\dagger(k, j) - e^{ik_x} a(k, j) \right\} dk. \]

Here

\[ \omega_{PF}(k) := |k| \]

and \( \hat{\phi} \) denotes an ultraviolet cutoff function.

\[ H_f := d\Gamma([\omega_{PF}]) \]

is the second quantization of the multiplication operator

\[ [\omega_{PF}] : L^2(\mathbb{R}^3 \times \{1, 2\}) \to L^2(\mathbb{R}^3 \times \{1, 2\}) \]

such that

\[ ([\omega_{PF}] f)(k, j) = \omega_{PF}(k) f(k, j). \]

Vector

\[ e(k, j) = (e_1(k, j), e_2(k, j), e_3(k, j)) \in \mathbb{R}^3, \quad j = 1, 2, \]

denotes a polarization vector satisfying

\[ e(k, 1) \times e(k, 2) = k/|k|, \quad |e(k, j)| = 1, \quad j = 1, 2. \]

Note that

\[ e(-k, 1) = -e(k, 1), \quad e(-k, 2) = e(k, 2). \]

Finally \( \sigma := (\sigma_1, \sigma_2, \sigma_3) \) denotes \( 2 \times 2 \) Pauli matrices satisfying the anticommutation relations,

\[ \{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad i, j = 1, 2, 3, \]

where \( \{A, B\} := AB + BA \). Assumptions (PF1)- (PF3) are as follows.

(PF1)

\[ (1) \sqrt{\omega_{PF}} \hat{\phi}, \hat{\phi}/\sqrt{\omega_{PF}}, \hat{\phi}/\omega_{PF} \in L^2(\mathbb{R}^3) \text{ and } \hat{\phi}(k) = \hat{\phi}(-k) = \overline{\hat{\phi}(k)}. \]
(2) \( V \) is \( \Delta \)-bounded with a relative bound strictly less than one.

(PF2)

(1) \( \hat{\varphi} \in C^\infty(\mathbb{R}^3) \).

(2) \( e(\cdot, j) \in C^\infty(\mathbb{R}^3 \setminus Q) \), \( j = 1, 2 \), with some measurable set \( Q \) with its Lebesgue measure zero.

(PF3) The ground state energy of self-adjoint operator

\[
h_p := -\frac{1}{2m} \Delta + V
\]

acting in \( L^2(\mathbb{R}^3) \) is discrete.

Let \( H_{PF,0} \) be \( H_{PF} \) with \( \epsilon = 0 \), i.e.,

\[
H_{PF,0} := H_p \otimes 1 + 1 \otimes H_t,
\]

where

\[
H_p := \begin{pmatrix} h_p & 0 \\ 0 & h_p \end{pmatrix}
\]

acting in \( L^2(\mathbb{R}^3; \mathbb{C}^2) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \).

In what follows, simply we write \( T \otimes 1 \) for \( \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \otimes 1 \) unless confusions arise.

We note that \( H_{PF,0} \) is self-adjoint on

\[
D(H_{PF,0}) = D(\Delta \otimes 1) \cap D(1 \otimes H_t).
\]

Note that

\[
(p \otimes 1) \cdot A_{\hat{\varphi}} = A_{\hat{\varphi}} \cdot (p \otimes 1)
\]

on \( D(H_{PF,0}) \). We set

\[
H_{PF} = H_{PF,0} + \epsilon H_{PF,1},
\]

where, by (4.8),

\[
H_{PF,1} := -\frac{1}{m} (p \otimes 1) \cdot A_{\hat{\varphi}} + \frac{\epsilon}{2m} A_{\hat{\varphi}} \cdot A_{\hat{\varphi}} - \frac{1}{2m} (\sigma \otimes 1) \cdot B_{\hat{\varphi}}.
\]

**Proposition 4.6** Assume (PF1), (PF2). Then \( H_{PF} \) is self-adjoint on \( D(H_{PF,0}) \) and bounded from below. Moreover it is essentially self-adjoint on any core of \( H_{PF,0} \).

**Proof:** See [25, 26]. \( \square \)
Proposition 4.7 Suppose (PF1) and (PF3). Then there exists a constant $e_* \leq \infty$ such that for $e$ with $|e| \leq e_*$, $H_{PF}$ has a ground state such that $\varphi_g \in D(1 \otimes N^{1/2})$.

Proof: See e.g., [9, 10, 15, 29, 30, 23]. \hfill \Box

Remark 4.8 For some $V$, we can take $e_* = \infty$. See [15, 29].

Remark 4.9 Spinless Pauli-Fierz Hamiltonians are defined by

$$H_{PF}^{sl} := \frac{1}{2m}(p \otimes 1 - eA \hat{\varphi})^2 + V \otimes 1 + 1 \otimes H_f,$$

which acts in $\mathcal{F} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_0(L^2(\mathbb{R}^3 \times \{1, 2\}))$. It can be proven that $H_{PF}^{sl}$ has a ground state $\varphi_g$ such that $\varphi_g \in D(1 \otimes N^{1/2})$, and it is unique [24]. Then it follows that $P_{H_{PF}^{sl}} \mathcal{F} \subset D(1 \otimes N^{1/2})$.

We have

$$[1 \otimes a(f, j), H_{PF}, I]_{W}^{D(H_{PF})}(\Psi, \Phi) = \langle \Psi, \{-\frac{1}{m} K^{(1)}(x, j) \cdot (p \otimes 1 - eA \hat{\varphi}) - \frac{1}{2m} K^{(2)}(x, j) \cdot (\sigma \otimes 1)\} \Phi\rangle,$$

where

$$K^{(1)}(x, j) := (\mathcal{F}, \lambda_j^{(1)}(x)), \quad K^{(2)}(x, j) := (\mathcal{F}, \lambda_j^{(2)}(x)),$$

and

$$\lambda_j^{(1)}(x, k) := \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx} e(k, j),$$

$$\lambda_j^{(2)}(x, k) := \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx} (-ik \times e(k, j)).$$

Hence we have

$$[1 \otimes a(f, j), H_{PF}, I]_{W}^{D(H_{PF})}(\Psi, \Phi) = \int f(k) (\Psi, T_{PF_j}(k)\Psi) dk,$$

where

$$T_{PF_j}(k) := T_{PF_j}^{(1)}(k) + T_{PF_j}^{(2)}(k),$$

$$T_{PF_j}^{(1)}(k) := -\frac{1}{m} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx} e(k, j) \cdot (p \otimes 1 - eA \hat{\varphi}),$$

$$T_{PF_j}^{(2)}(k) := -\frac{1}{2m} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx} (-ik \times e(k, j)) \cdot (\sigma \otimes 1).$$
Let $H_{PF}^0$ be $H_{PF}$ with $V = 0$. Then the binding energy is defined by

$$E_{\text{bin}} := E(H_{PF}^0) - E(H_{PF}).$$

**Proposition 4.10** Assume that $h_p$ has a ground state in $L^2(\mathbb{R}^3)$. Then

$$E_{\text{bin}} \geq -E(h_p).$$

**Proof:** See Appendix 5.3. \qed

Assumption (V) is as follows.

(V) Potential $V = V_+ - V_-$ ($V_+(x) = \max\{0, V(x)\}$, $V_-(x) = \min\{0, V(x)\}$) satisfies that (1) $\lim_{|x| \to \infty} V_-(x) = V_\infty < \infty$, (2) $|x|^2 V_- \in L^\infty_{\text{loc}}(\mathbb{R}^3)$, (3) $E_{\text{bin}} > V_\infty$.

A typical example of $V$ in (V) is Coulomb potential $-e^2/|x|$.

**Lemma 4.11** Suppose (V). Then it follows that

$$\sup_{\varphi_g \in H_{PF}} \frac{\|(|x| \otimes 1) \varphi_g\|}{\|\varphi_g\|} < c_{\text{exp}}$$

with some constant $c_{\text{exp}}$.

**Proof:** See Appendix 5.4. \qed

The main theorem in this subsection is as follows.

**Theorem 4.12** Assume (PF1), (PF2), and (V). Then it follows that

$$P_{H_{PF}^0} \mathcal{F}_{PF} \subset D(1 \otimes N^{1/2}),$$

and

$$\|(1 \otimes N^{1/2}) \varphi_g\|^2 = e^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF j}(k) \varphi_g\|^2 dk.$$  \hspace{1cm} (4.10)

In addition, assume (PF3), then there exists a constant $e_{**}$ such that for $e$ with $|e| < e_{**}$,

$$m(H_{PF}) \leq m(H_p).$$  \hspace{1cm} (4.11)

**Remark 4.13** Suppose that e.g., $V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $V_-$ is infinitesimally small with respect to $\Delta$. Then, by a Feynman-Kac formula, it is shown that $e^{-t(h_p - E(h_p))}$ is positivity improving in $L^2(\mathbb{R}^3)$. Hence $h_p$ has a unique ground state in $L^2(\mathbb{R}^3)$. Then $m(H_p) = 2.$
To prove Theorem 4.12 it is sufficient to check (B1)-(B4) and (3) of Theorem 2.9 with the following identifications.

\[ \mathcal{F} = \mathcal{F}_{PF}, \quad H_0 = H_{PF,0}, \quad H_1 = H_{PF,1}, \quad \omega = \omega_{PF}, \quad D = 2, \quad T_j(k) = T_{PF,j}(k). \]

Let

\[ K := \bigcup_{n=1}^{3} \{(k_1, k_2, k_3) \in \mathbb{R}^3 | k_n = 0\} \]

and

\[ \tilde{K} := K \cup Q \cup \{0\}. \]

**Lemma 4.14** Assume \((PF1)\) and \((PF2)\). Then for \(f \in C^0_0(\mathbb{R}^3 \setminus \tilde{K})\) and \(\Psi \in D(H_{PF}), \)

\[ \left| \int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{PF}-E(H_{PF})+\omega_{PF}(k)))}T_{PF,j}^{(l)}(k)\varphi_{\tilde{g}})dk \right| \in L^1([0, \infty), ds), \quad l = 1, 2. \]

**Proof:** Note that

\[ \sum_{\mu=1,2,3} [K^{(1)}_{\mu}(x, j), (p \otimes 1 - eA_{\tilde{g}})_{\mu}] = 0 \]

on \(D(H_{PF}).\) We see that

\[ \int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{PF}-E(H_{PF})+\omega_{PF}(k)))}T_{PF,j}^{(l)}(k)\varphi_{\tilde{g}})dk \]

\[ = -\frac{1}{m} \sum_{\mu=1,2,3} ( (p \otimes 1 - eA_{\tilde{g}})_{\mu}e^{-is(H_{PF}-E(H_{PF}))}\Psi, K^{(1)}_{\mu}(s, x, j)\varphi_{\tilde{g}}), \]

where

\[ K^{(1)}(s, x, j) := (e^{is\omega_{PF}} f, \lambda^{(1)}_j(x)). \]

Since

\[ e^{is\omega_{PF}} = \frac{\omega_{PF}(k)}{k_{\mu}} \frac{1}{is} \frac{\partial}{\partial k_{\mu}} e^{is\omega_{PF}}, \quad \mu = 1, 2, 3, \]

we have

\[ K^{(1)}(s, x, j) = \frac{1}{s} (K^{(1)}_1(s, x, j) + x_{\mu}K^{(1)}_2(s, x, j)), \]

where

\[ K^{(1)}_1(s, x, j) := -i \int_{\mathbb{R}^3} e^{-i(s\omega_{PF}(k)+kx)} \frac{\partial}{\partial k_{\mu}} \left( \frac{\omega_{PF}(k)}{k_{\mu}} \sqrt{2\omega_{PF}(k)} \hat{\varphi}(k) f(k)e(k, j) \right) dk, \]

\[ K^{(1)}_2(s, x, j) := \int_{\mathbb{R}^3} e^{-i(s\omega_{PF}(k)+kx)} \frac{\partial}{\partial k_{\mu}} \left( \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)e(k, j) \right) dk. \]
From the fact that $\hat{\varphi} \in C^\infty(\mathbb{R}^3)$ and $f \in C^2_0(\mathbb{R}^3 \setminus \tilde{K})$, it follows that for $\nu = 1, 2, 3$,

$$
\frac{\partial}{\partial k_\mu} \left( \frac{\omega_{PF}(k)}{k_\mu} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)e_\nu(k, j) \right) \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}),
$$

$$
\frac{\partial}{\partial k_\mu} \left( \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)e_\nu(k, j) \right) \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}).
$$

Thus by [34, Theorem XI.19 (c)] there exist constants $c_1$ and $c_2$ such that

$$
\sup_x |K_{(1)_{\mu}}(s, x, j)| \leq \frac{c_l}{1 + s}, \quad l = 1, 2, \quad \mu = 1, 2, 3.
$$

(4.12)

By this we have

$$
\|K_{(1)_{\mu}}(s, x, j)\varphi_g\| \leq \frac{1}{s(s + 1)}(c_1\|\varphi_g\| + c_2\|(|x| \otimes 1)\varphi_g\|).
$$

(4.13)

Since

$$
\|(p \otimes 1 - eA\varphi)\mu\Psi\| \leq c'_1\|(H_{PF} - E(H_{PF}))\Psi\| + c'_2\|\Psi\|
$$

(4.14)

with some constants $c'_1$ and $c'_2$, we conclude that

$$
-\frac{1}{m} \sum_{\mu=1,2,3} ((p \otimes 1 - eA\varphi)\mu e^{-is(H_{PF} - E(H_{PF}))}\Psi, K_{(1)_{\mu}}(s, x, j)\varphi_g)
$$

$$
\leq \frac{3}{m}(c'_1\|(H_{PF} - E(H_{PF}))\Psi\| + c'_2\|\Psi\|)(c_1\|\varphi_g\| + c_2\|(|x| \otimes 1)\varphi_g\|) \frac{1}{s(1 + s)}.
$$

(4.15)

From this it follows that

$$
-\frac{1}{m} \sum_{\mu=1,2,3} ((p \otimes 1 - eA\varphi)\mu e^{-is(H_{PF} - E(H_{PF}))}\Psi, K_{(1)_{\mu}}(s, x, j)\varphi_g) \in L^1([0, \infty), ds).
$$

Similarly we can estimate

$$
\int_{\mathbb{R}^d} f(k)(\Psi, e^{-is(H_{PF} - E(H_{PF}) + \omega_{PF}(k))}TPF_{(2)}(k)\varphi_g)dk
$$

$$
= -\frac{1}{2m} \sum_{\mu=1,2,3} ((\sigma_\mu \otimes 1)e^{-is(H_{PF} - E(H_{PF}))}\Psi, K_{(2)_{\mu}}(s, x, j)\varphi_g),
$$

where

$$
K_{(2)}(s, x, j) := (e^{is\omega_{PF}} f, \lambda_{(2)}(x)).
$$

We have

$$
K_{(2)}(s, x, j) = \frac{1}{s}(K_{(1)}^{(2)}(s, x, j) + x\mu K_{(2)}(s, x, j)),
$$
where
\[
K_1^{(2)}(s, x, j) := -i \int_{\mathbb{R}^3} e^{-i(s\omega_{PF}(k) + kx)} \frac{\partial}{\partial k} \left( \frac{\omega_{PF}(k)}{k} \frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)(-ik \times e(k, j)) \right) dk,
\]
\[
K_2^{(2)}(s, x, j) := \int_{\mathbb{R}^3} e^{-i(s\omega_{PF}(k) + kx)} \frac{\partial}{\partial k} \left( -\frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)(-ik \times e(k, j)) \right) dk.
\]

Since for \( \mu, \nu = 1, 2, 3 \),
\[
\frac{\partial}{\partial k} \left( \frac{\omega_{PF}(k)}{k} \frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)(-ik \times e(k, j)) \right) \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),
\]
\[
\frac{\partial}{\partial k} \left( -\frac{\hat{\phi}(k)}{\sqrt{2\omega_{PF}(k)}} f(k)(-ik \times e(k, j)) \right) \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),
\]
we can see that there exist constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) such that
\[
\sup_x |K_{l\mu}^{(2)}(s, x, j)| \leq \frac{\tilde{c}_l}{1 + s}, \quad l = 1, 2, \quad \mu = 1, 2, 3. \tag{4.16}
\]

Then we conclude that
\[
\left| -\frac{1}{2m} \sum_{\mu=1,2,3} ((\sigma_\mu \otimes 1)e^{-is(H_{PF} - E(H_{PF}))}\Psi, K_{l\mu}^{(2)}(s, x, j)\varphi_g) \right| \\
\leq \frac{3}{2m} \|\Psi\| (\tilde{c}_1\|\varphi_g\| + \tilde{c}_2 (|x| \otimes 1)\|\varphi_g\|) \frac{1}{s(1 + s)}.
\]

Thus
\[
-\frac{1}{2m} \sum_{\mu=1,2,3} ((\sigma_\mu \otimes 1)e^{-is(H_{PF} - E(H_{PF}))}\Psi, K_{l\mu}^{(2)}(s, x, j)\varphi_g) \in L^1([0, \infty), ds).
\]

Hence the lemma is proven.

\[\square\]

**Lemma 4.15** We have
\[
\|T_{PF_{l}}^{(l)}(\cdot)\varphi_g\| \in L^2(\mathbb{R}^3), \quad l = 1, 2.
\]
Proof: It follows that
\[
\|T_{PF_j}^{(1)}(k)\varphi_g\| \leq \frac{1}{m} \sum_{\mu=1,2,3} \frac{1}{\sqrt{2\omega_{PF}(k)}} |\hat{\varphi}(k)\| (p \otimes 1 - eA_{\varphi})_{\mu} \|e(k,j)_{\mu}\| \|p \otimes 1 - eA_{\varphi}\| \mu \|\varphi_g\|
\]
and
\[
\|T_{PF_j}^{(2)}(k)\varphi_g\| \leq \frac{1}{2m} \sum_{\mu=1,2,3} \frac{1}{\sqrt{2\omega_{PF}(k)}} |\hat{\varphi}(k)\| (k \times e(k,j))_{\mu} \|\sigma_{\mu} \otimes 1\| \|\varphi_g\|
\]
Since $\sqrt{\omega_{PF}}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega_{PF}} \in L^2(\mathbb{R}^3)$, the lemma follows.

Lemma 4.16 Suppose that $\Psi \in D(H_{PF,0}) \cap D(|x| \otimes 1)$ and $H_{PF}\Psi \in D(|x| \otimes 1)$. Then for $\mu = 1, 2, 3$,

(1) $(x_{\mu} \otimes 1)\Psi \in D(H_{PF})$,

(2) $[x_{\mu} \otimes 1, H_{PF}]\Psi = \frac{i}{m} (p \otimes 1 - eA_{\varphi})_{\mu} \Psi$.

In particular, it follows that $(x_{\mu} \otimes 1)\varphi_g \in H_{PF}$ with
\[
\frac{i}{m} (p \otimes 1 - eA_{\varphi})_{\mu} \varphi_g = [x_{\mu} \otimes 1, H_{PF}]\varphi_g = (H_{PF} - E(H_{PF}))(x_{\mu} \otimes 1)\varphi_g.
\]
Proof: See Appendix 5.5.

Lemma 4.17 Assume (PF1). Then
\[
\int_{\mathbb{R}^3} \|(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}T_{PF_j}^{(l)}(k)\varphi_g\|^2 dk < \infty, \quad l = 1, 2.
\]
Proof: Note that by Lemma 4.16
\[
T_{PF_j}^{(l)}(k)\varphi_g
= -\frac{1}{m} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx}e(k,j) \cdot (p \otimes 1 - eA_{\varphi})_{\mu} \varphi_g
= -\frac{1}{m} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx}e(k,j) \cdot (-im)[x \otimes 1, H_{PF}]\varphi_g
= -\frac{1}{m} \frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e^{-ikx}e(k,j) \cdot (-im)(H_{PF} - E(H_{PF}))(x \otimes 1)\varphi_g.
\]

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Hence we have

\[
(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}T_{PF}^{(1)}(k)\varphi_g \\
= i\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e(k, j)(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} e^{-ikx}(H_{PF} - E(H_{PF}))(x \otimes 1)\varphi_g \\
= i\frac{\hat{\varphi}(k)}{\sqrt{2\omega_{PF}(k)}} e(k, j)(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} \\
\times (H_{PF}(k) - E(H_{PF})) e^{-ikx}(x \otimes 1)\varphi_g,
\]

where we used that \( e^{ikx} \) maps \( D(H_{PF,0}) \) onto itself (see Appendix 5.5) and on \( D(H_{PF}) \),

\[
H_{PF}(k) := e^{-ikx}H_{PF}e^{ikx} \\
= \frac{1}{2m}(p \otimes 1 + k - eA_\phi)^2 + V \otimes 1 + 1 \otimes H_t - \frac{1}{2m} (\sigma \otimes 1) \cdot B_\phi \\
= H_{PF} + \frac{1}{m}(p \otimes 1 - eA_\phi) \cdot k + \frac{1}{2m} |k|^2.
\]

Thus we have

\[
\| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} (H_{PF}(k) - E(H_{PF})) e^{-ikx}(x_\mu \otimes 1)\varphi_g \| \\
\leq \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}(H_{PF} - E(H_{PF})) e^{-ikx}(x_\mu \otimes 1)\varphi_g \| \quad (4.17) \\
+ \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} \frac{1}{m}(p \otimes 1 - eA_\phi) \cdot k e^{-ikx}(x_\mu \otimes 1)\varphi_g \| \quad (4.18) \\
+ \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} \frac{1}{2m} |k|^2 e^{-ikx}(x_\mu \otimes 1)\varphi_g \|. \quad (4.19)
\]

We have

\[
| (4.17) | \leq \| (|x| \otimes 1)\varphi_g \| \quad (4.20)
\]

and

\[
| (4.19) | \leq \frac{1}{2m} \frac{|k|^2}{\omega_{PF}(k)} \| (|x| \otimes 1)\varphi_g \| = \frac{1}{2m} |k| \| (|x| \otimes 1)\varphi_g \|. \quad (4.21)
\]

Note that by (4.14),

\[
\| (p \otimes 1 - eA_\phi)_{\mu}(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}\varphi_g \| \\
\leq c'_1 \| (H_{PF} - E(H_{PF}))(H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}\varphi_g \| \\
+ c'_2 \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1}\varphi_g \| \\
\leq c'_1 \| \varphi_g \| + \frac{c'_2}{\omega_{PF}(k)} \| \varphi_g \|.
\]

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Then
\[ |(4.18)| \leq \frac{3}{m} |k| (c_1' + \frac{c_2'}{\omega_{PF}(k)}) \|(|x| \otimes 1) \varphi_g\| = \frac{3}{m} (c_1'|k| + c_2') \|(|x| \otimes 1) \varphi_g\|. \tag{4.22} \]
Together with (4.20), (4.21), (4.22), we have
\[
\| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF_j}^{(1)}(k) \varphi_g \|
\leq \frac{3}{2m} \frac{|\hat{\psi}(k)|}{\sqrt{2 \omega_{PF}(k)}} \left( \frac{1}{|k|} + \frac{3}{2m} (c_1'|k| + c_2') \right) \|(|x| \otimes 1) \varphi_g\|. \tag{4.23} \]
Since \( \sqrt{\omega_{PF}} \hat{\psi}, \hat{\psi} / \sqrt{\omega_{PF}} \in L^2(\mathbb{R}^3) \),
\[
\int_{\mathbb{R}^3} \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF_j}^{(1)}(k) \varphi_g \|^2 dk < \infty \tag{4.24} \]
follows. Moreover we have
\[
\| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF_j}^{(2)}(k) \varphi_g \|
\leq \frac{3}{2m} \frac{|\hat{\psi}(k)|}{\sqrt{2 \omega_{PF}(k)}} \|\varphi_g\|. \tag{4.25} \]
Hence
\[
\int_{\mathbb{R}^3} \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF_j}^{(2)}(k) \varphi_g \|^2 dk < \infty \tag{4.26} \]
follows. Thus by (4.24) and (4.26), we get the desired results. \( \square \)

Proof of Theorem 4.12
Lemmas 4.14, 4.15 and 4.17 correspond to assumptions (B3), (B4) and (3) of Theorem 2.9, respectively. Then (4.9) and (4.10) follow from Theorem 2.9. By (4.23) and (4.25), we have
\[
\frac{\| (1 \otimes N^{1/2}) \varphi_g \|^2}{\| \varphi_g \|^2}
= e^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PF_j}^{(j)}(k) \varphi_g \|^2 dk
\leq 6e^2 \int \left\{ \frac{3 |\hat{\psi}(k)|}{\sqrt{2 \omega_{PF}(k)}} \left( \frac{1}{|k|} + \frac{3}{2m} (c_1'|k| + c_2') \right) \right\}^2 dk \frac{\|(|x| \otimes 1) \varphi_g\|^2}{\| \varphi_g \|^2}
+ 6e^2 \int \left\{ \frac{3}{2m} \frac{|\hat{\psi}(k)|}{\sqrt{2 \omega_{PF}(k)}} \right\}^2 dk.
\]
Since, by Lemma 5.3, 
\[ \frac{\|(|x| \otimes 1) \varphi_g\|^2}{\|\varphi_g\|^2} \leq c_{\text{exp}}, \]
we obtain
\[ \lim_{\varepsilon \to 0} \sup_{\varphi_g \in P_{HF} F_{PF}} \varepsilon^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PFj}(k) \varphi_g \|^2 dk = 0. \]
Thus (4.11) follows from Theorem 3.6.

\[ \square \]

**Remark 4.18** Although, in [27], formula (4.10):
\[ \|(1 \otimes N^{1/2}) \varphi_g\|^2 = e^2 \sum_{j=1,2} \int_{\mathbb{R}^3} \| (H_{PF} - E(H_{PF}) + \omega_{PF}(k))^{-1} T_{PFj}(k) \varphi_g \|^2 dk \]
has been used to show \( m(H_p) \leq 2 \), there is no exact proof to derive this formula in it. See Subsection 1.3.

In [27] it has been also proven that \( 2 \leq m(H_{PF}) \) under some conditions on \( V \). We state a theorem.

**Theorem 4.19** In addition to (PF1) – (PF3) and (V), we assume \( m(H_p) = 2 \) and \( V(x) = V(-x) \). Then there exists a constant \( e_{***} \) such that for \( e \) with \( |e| < e_{***} \), \( m(H_{PF}) = 2 \).

**Proof:** \( m(H_{PF}) \leq 2 \) follows from Theorem 4.12 and \( 2 \leq m(H_{PF}) \) from [27]. We omit details.

\[ \square \]

### 4.3 Concluding remarks

We can apply the method stated in this paper to a wide class of interaction Hamiltonians in quantum field models. Hamiltonian \( H_{CD} \) of the Coulomb-Dirac system is defined as an operator acting in
\[ \mathcal{F}_{CD} = \mathcal{F}_f(\oplus^4 L^2(\mathbb{R}^3)) \otimes \mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})), \]
where \( \mathcal{F}_f(\oplus^4 L^2(\mathbb{R}^3)) \) denotes a fermion Fock space over \( \oplus^4 L^2(\mathbb{R}^3) \). The Coulomb-Dirac system describes an interaction of positrons and relativistic electrons through photons in the Coulomb gauge. Operator \( H_{CD} \) is of the form
\[ H_{CD} = H_f \otimes 1 + 1 \otimes d\Gamma(\omega_{CD}) + e H_{\text{rad}} + e^2 H_{\text{Coulomb}}, \]
where \( H_t \) denotes a free Hamiltonian of \( \mathcal{F}_1(\bigoplus^4 L^2(\mathbb{R}^3)) \), \( \omega_{CD} \) the multiplication operator by \( \omega_{CD}(k) = |k| \), and \( H_{\text{rad}}, H_{\text{Coulomb}} \) interaction terms. \( H_{\text{CD}} \) has been investigated in [11], where the self-adjointness and the existence of a ground state are proven under some conditions. It is known that \( m(H_t) = 1 \). Then using the method in this paper we can also show

\[
m(H_{\text{CD}}) \leq m(H_t) = 1,
\]

i.e., the ground state of \( H_{\text{CD}} \) is unique for a sufficiently small \( e \). We omit details.

## 5 Appendix

### 5.1 Proofs of (3.5) and (3.6)

**Proposition 5.1 ((3.5) and (3.6))** We have \( \varphi_g \in D(\mathcal{H}_0^{1/2}) \), \( Q\varphi_g \in D(\mathcal{H}_0^{1/2}) \) and

\[
\mathcal{H}_0^{1/2} Q\varphi_g = Q\mathcal{H}_0^{1/2} \varphi_g.
\]

Proof: From the fact that the form domain \( Q(H) \) of \( H \) satisfies

\[ D(H) \subset Q(H) = D(\mathcal{H}_0^{1/2}), \]

it follows that \( \varphi_g \in D(\mathcal{H}_0^{1/2}) \). Since \([e^{-it\mathcal{H}_0^{1/2}}, Q] = 0\), we have

\[
\frac{e^{-it\mathcal{H}_0^{1/2}} - 1}{t} Q\varphi_g = Q\frac{e^{-it\mathcal{H}_0^{1/2}} - 1}{t} \varphi_g, \quad t > 0.
\]

Since \( \varphi_g \in D(\mathcal{H}_0^{1/2}) \), we see that

\[
s - \lim_{t \to 0} Q\frac{e^{-it\mathcal{H}_0^{1/2}} - 1}{t} \varphi_g = -Q\mathcal{H}_0^{1/2} \varphi_g.
\]

Hence the left-hand side of (5.2) converges, which implies that \( Q\varphi_g \in D(\mathcal{H}_0^{1/2}) \) and (5.1) follows. 

### 5.2 Proof of (3.10)

**Proposition 5.2 ((3.10))** Assume that

\[
\lim_{\varphi \to 0} \sup_{\Psi \in D(\mathcal{H}_0^{1/2})} \frac{\beta_0(\Psi, \Psi) - \beta(\Psi, \Psi)}{((\mathcal{H}_0 + 1)^{1/2} \Psi)^2} = 0.
\]

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Then for $z \in \mathbb{C}$ with $\Im z \neq 0$,

$$\lim_{g \to 0} \| (H_q - z)^{-1} - (H_0 - z)^{-1} \| = 0.$$  

Proof: Set

$$K_0 := (\mathcal{P}_0 + 1)^{-1/2}.$$  

We have

$$\begin{align*}
(\Psi, K_0 H_q K_0 \Psi) &= (\Psi, K_0 (H_{q+}^{1/2} H_{q+}^{1/2} - H_{q-}^{1/2} H_{q-}^{1/2}) K_0 \Psi) \\
&= (\Psi, K_0 (H_{q+}^{1/2} H_{q+}^{1/2}) K_0 \Psi) - (\Psi, K_0 (H_{q-}^{1/2} H_{q-}^{1/2}) K_0 \Psi) \\
&= (H_{q+}^{1/2} K_0 \Psi, H_{q+}^{1/2} K_0 \Psi) - (H_{q-}^{1/2} K_0 \Psi, H_{q-}^{1/2} K_0 \Psi) \\
&= \beta(K_0 \Psi, K_0 \Psi) \quad (5.4) \\
&\leq (1 + |g| a) \beta_0(K_0 \Psi, K_0 \Psi) + |g| b(K_0 \Psi, K_0 \Psi) \\
&\leq C(g) \| \Psi \|^2,
\end{align*}$$

where

$$C(g) := 1 + |g|(a + b).$$

Then $K_0 H_q K_0$ is a bounded operator and

$$\| K_0 H_q K_0 \| < C(g).$$

Since the range of $K_0$ equals to $D(\mathcal{P}_0^{1/2})$, we have from (5.3)

$$\lim_{g \to 0} \sup_{\Psi \in \mathcal{F}} \frac{|\beta(K_0 \Psi, K_0 \Psi) - \beta_0(K_0 \Psi, K_0 \Psi)|}{\| \Psi \|^2} = 0. \quad (5.5)$$

By (5.4), i.e.,

$$\beta(K_0 \Psi, K_0 \Psi) = (\Psi, K_0 H_q K_0 \Psi)$$

and

$$\beta_0(K_0 \Psi, K_0 \Psi) = (\Psi, K_0 \mathcal{P}_0 K_0 \Psi),$$

(5.5) implies that

$$\lim_{g \to 0} \sup_{\Psi \in \mathcal{F}} \frac{|(\Psi, K_0 (H_q - \mathcal{P}_0) K_0 \Psi)|}{\| \Psi \|^2} = 0.$$  

Hence we obtain that

$$\lim_{g \to 0} \| K_0 (H_q - \mathcal{P}_0) K_0 \| = 0. \quad (5.6)$$
Moreover, for $z \in \mathbb{C}$ with $\Im z \neq 0$,

$$|\beta_0((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi)| \leq |\beta((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi)| + |g\beta_0((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi)| \leq |((H_q - z)^{-1}\Psi, H_q(H_q - z)^{-1}\Psi)| + a|g|\beta_0((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi) + b|g|((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi).$$

Then

$$\|P^{1/2}_0(H_q - z)^{-1}\Psi\|^2 = \beta_0((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi) \leq \frac{1}{1 - a|g|} \left\{((H_q - z)^{-1}\Psi, H_q(H_q - z)^{-1}\Psi) + b|g|((H_q - z)^{-1}\Psi, (H_q - z)^{-1}\Psi)\right\} = \int_{\inf \sigma(H_q)}^{\infty} \left(\frac{1}{1 - |g|a(\lambda - z)^2}\right) d\lambda \|\mathcal{E}_{H_q}(\lambda)\|\Psi\|^2 \leq d(g)\|\Psi\|^2,$$

where

$$d(g) := \sup_{\lambda \in \mathbb{R}} \left|\frac{1}{1 - |g|a(\lambda - z)^2}\right|.$$

Thus we see that

$$\|K_0^{-1}(H_q - z)^{-1}\| < \|P^{1/2}_0(H_q - z)^{-1}\| + \|(H_q - z)^{-1}\| \leq \sqrt{d(g)} + \frac{1}{|\Im z|} := D(g).$$

Directly we have

$$\|(H_q - z)^{-1} - (\overline{P}_0 - z)^{-1}\| \leq \|(H_q - z)^{-1}K_0^{-1}\|\|K_0(H_q - \overline{P}_0)K_0\|\|K_0^{-1}(\overline{P}_0 - z)^{-1}\| \leq D(g)D(0)\|K_0(H_q - \overline{P}_0)K_0\|.$$

By (5.6) and the fact

$$\lim_{g \to 0} D(g) < \infty,$$

we obtain that

$$\lim_{g \to 0} \|(H_q - z)^{-1} - (\overline{P}_0 - z)^{-1}\| = \lim_{g \to 0} D(g)D(0)\|K_0(H_q - \overline{P}_0)K_0\| = 0.$$

Thus the proposition is proven. □
5.3 Proof of Proposition 4.10

Proof of Proposition 4.10

We see an outline of a proof. See [15, 21] for details. It is enough to show that for an arbitrary \( \epsilon > 0 \), there exists a vector \( \Psi_\epsilon \in D(H_{PF}) \) such that

\[
(\Psi_\epsilon, (H_{PF} - (E(H_{PF}^0) + \epsilon + E(h_p)))\Psi_\epsilon) < 0.
\]  

(5.7)

We identify \( \mathcal{F}_{PF} \) with \( L^2(\mathbb{R}^3; \oplus^2 \mathcal{F}_b) \), i.e., \( \oplus^2 \mathcal{F}_b \)-valued \( L^2 \)-function over \( \mathbb{R}^3 \). Let \( f \) be a ground state of \( h_p \). Assume that \( f \) is real and \( \|f\| = 1 \). For \( \epsilon > 0 \), let

\[
\Phi_\epsilon = \sum_{m}^{\infty} f_m^\epsilon \otimes \Phi_m^\epsilon
\]  

(5.8)

be such that \( f_m^\epsilon \in C_0^\infty(\mathbb{R}^3) \), \( \Phi_m^\epsilon \in \oplus^2 \mathcal{F}_{\text{fin}} \) and

\[
(\Phi_\epsilon, H_{PF}^0 \Phi_\epsilon) \leq E(H_{PF}^0) + \epsilon.
\]  

(5.9)

Actually, since the linear hull of vectors such as (5.8) is a core of \( H_{PF} \), there exists \( \Phi_\epsilon \) such as (5.8) satisfying (5.9). Note that \( f\Phi_\epsilon \in \mathcal{F} \). Let

\[
U_y := e^{iy \cdot p} \otimes 1, \quad y \in \mathbb{R}^3.
\]

We can see that \( fU_y \Phi_\epsilon \in D(H_{PF}) \) and

\[
\Omega_y := (fU_y \Phi_\epsilon, (H_{PF} - (E(H_{PF}^0) + \epsilon + E(h_p)))fU_y \Phi_\epsilon)_{\mathcal{F}_{PF}}
\]

\[
= \int_{\mathbb{R}^3} \{(\Phi_\epsilon(x), (H_{PF}^0 \Phi_\epsilon)(x))_{\oplus^2 \mathcal{F}_b} - (E(H_{PF}^0) + \epsilon)\|\Phi_\epsilon(x)\|_{\oplus^2 \mathcal{F}_b}^2\} f(x-y)^2 dx
\]

\[
+ \int_{\mathbb{R}^3} (\Phi_\epsilon(x), ((p \otimes 1) \Phi_\epsilon)(x))_{\oplus^2 \mathcal{F}_b} f(x-y) (pf)(x-y) dx.
\]

Note that \( fU_y \Phi_\epsilon \) and \( H_{PF,0} fU_y \Phi_\epsilon \) are strongly continuous in \( y \). Then \( \Omega_y \) is continuous in \( y \). We have

\[
\int_{\mathbb{R}^3} \Omega_y dy
\]

\[
= \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dx \{(\Phi_\epsilon(x), (H_{PF}^0 \Phi_\epsilon)(x))_{\oplus^2 \mathcal{F}_b} - (E(H_{PF}^0) + \epsilon)\|\Phi_\epsilon(x)\|_{\oplus^2 \mathcal{F}_b}^2\} f(x-y)^2
\]

\[
= \int_{\mathbb{R}^3} dx \{(\Phi_\epsilon(x), (H_{PF}^0 \Phi_\epsilon)(x))_{\oplus^2 \mathcal{F}_b} - (E(H_{PF}^0) + \epsilon)\|\Phi_\epsilon(x)\|_{\oplus^2 \mathcal{F}_b}^2\}
\]

\[
= (\Phi_\epsilon, (H_{PF}^0 - (E(H_{PF}^0) + \epsilon)\Phi_\epsilon) < 0.
\]

Here we used that \( \|f\| = 1 \), and by the fact that \( f \) is real,

\[
\int_{\mathbb{R}^3} f(x-y)(p_\mu f)(x-y) dy = (f, p_\mu f) = 0, \quad \mu = 1, 2, 3.
\]

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Thus we conclude that there exists \( y_0 \in \mathbb{R}^3 \) such that \( \Omega_{y_0} < 0 \), which yields that

\[
(f U_{y_0} \Phi_\epsilon, (H_{PF} - (E(H_{PF}^0) + \epsilon + E(h_p))f U_{y_0} \Phi_\epsilon) < 0.
\]

The proof is complete. \( \square \)

### 5.4 Proof of Lemma 4.11

We show a more general lemma than Lemma 4.11.

**Lemma 5.3** Assume the following facts.

1. \( \lim_{|x| \to \infty} V_-(x) = V_\infty < \infty \).
2. A positive function \( G \) satisfies that
   - \( (i) \ G \in C^1(\mathbb{R}^3 \setminus \mathcal{N}) \cap C(\mathbb{R}^3) \) with some compact set \( \mathcal{N} \),
   - \( (ii) \ \sup_{x \in \mathbb{R}^3 \setminus \mathcal{N}} |\nabla G(x)| < \infty \),
   - \( (iii) \ G^2V_- \in L^\infty_{\text{loc}}(\mathbb{R}^3) \),
   - \( (iv) \ \lim_{|x| \to \infty} |G(x)||x|^{-l} = d < \infty \) with some \( l \geq 0 \) and \( d > 0 \).
3. \( E_{\text{bin}} > V_\infty \).

Then there exists a constant \( c_{\text{exp}} \) such that

\[
\sup_{\varphi_g \in \mathcal{P}_{H_{PF}}} \frac{\| (G \otimes 1) \varphi_g \|}{\| \varphi_g \|} \leq c_{\text{exp}}. \quad (5.10)
\]

**Proof:** This proof is a generalization of [15]. Let \( \chi_{|x|>R} \in C^\infty(\mathbb{R}^3) \) be a function such that \( \chi_{|x|>R}(x) = 0 \) for \( |x| < R \), and \( \chi_{|x|>R}(x) = 1 \) for \( |x| > R + 1 \). Since \( \mathcal{N} \) is compact, \( \chi_{|x|>R}G \in C^1(\mathbb{R}^3) \) for a sufficiently large \( R \). Since \( (1 - \chi_{|x|>R})G \) is a bounded operator, it is enough to prove (5.10) with \( G \) replaced by \( \chi_{|x|>R}G \). We reset \( \chi_{|x|>R}G \) as \( G \).

1. Suppose that \( G \in C_0^\infty(\mathbb{R}^3) \). Then we have

\[
(G \varphi_g, (H_{PF} - E(H_{PF}))G \varphi_g) = \frac{1}{2m} (\varphi_g, |\nabla G|^2 \varphi_g).
\]

The left-hand side above is

\[
(G \varphi_g, (H_{PF} - E(H_{PF}))G \varphi_g) = (G \varphi_g, (H_{PF}^0 - E(H_{PF}^0) + V) G \varphi_g) \geq E_{\text{bin}} \|G \varphi_g\|^2 + (G \varphi_g, V G \varphi_g) \geq E_{\text{bin}} \|G \varphi_g\|^2 + (G \varphi_g, -V_- G \varphi_g).
\]
Hence
\[ E_{\text{bin}} \| G\varphi_g \|^2 \leq \frac{1}{2m} (\varphi_g, |\nabla G|^2 \varphi_g) - (G\varphi_g, VG\varphi_g). \] (5.11)

Let \( \epsilon \) satisfy
\[ E_{\text{bin}} - V_\infty - \epsilon > 0. \]

There exists \( R \) such that
\[ |V_-(x) - V_\infty| \leq \epsilon, \quad |x| > R. \]

Let
\[ a := \sup_{|x| < R} G^2(x)V_-(x), \quad b := \frac{1}{2m} \sup_{x \in \mathbb{R}^3} |\nabla G(x)|^2. \]

Then
\[ E_{\text{bin}} \| G\varphi_g \|^2 \leq b \| \varphi_g \|^2 + a \| \varphi_g \|^2 + (V_\infty + \epsilon) \| G\varphi_g \|^2, \]
and
\[ \| G\varphi_g \|^2 \leq \frac{a + b}{E_{\text{bin}} - V_\infty - \epsilon} \| \varphi_g \|^2. \] (5.12)

(2) Suppose that \( G \in C^\infty(\mathbb{R}^3) \). Let \( \chi \in C^\infty(\mathbb{R}^3) \) be
\[ \chi(x) := \begin{cases} 
1, & |x| \leq 1, \\
\theta(x), & 1 < |x| < 2, \\
0, & |x| \geq 2,
\end{cases} \]
such that \( \partial \theta(x)/\partial x_\mu \leq 0, \mu = 1, 2, 3, \sup_{x \in \mathbb{R}^3} |\theta(x)| \leq 1 \) and \( \sup_{x \in \mathbb{R}^3} |\nabla \chi(x)| \leq c \) with some constant \( c \). Define \( \chi_n(x) := \chi(x/n) \) and set
\[ G_n := \chi_n G. \]

Note that
\[ |\nabla \chi_n(x)| = \begin{cases} 
0, & |x| < n', \\
\leq \frac{c}{n-1}, & n' \leq |x| \leq 2n', \\
0, & |x| > 2n'.
\end{cases} \] (5.13)

It follows that by (5.11),
\[ E_{\text{bin}} \| G_n\varphi_g \|^2 \leq \frac{1}{2m} (\varphi_g, |\nabla G_n|^2 \varphi_g) + (G_n\varphi_g, V G_n\varphi_g). \]

Since
\[ |\nabla G_n|^2 = |\nabla \chi_n \cdot G_n|^2 + |\chi_n \cdot \nabla G_n|^2 + 2(\nabla \chi_n \cdot G_n) \cdot (\chi_n \cdot \nabla G_n), \]

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we have by (5.13) and assumption (iv) of (2), for sufficiently large \( n \),
\[
\frac{1}{2m} (\varphi_g, |\nabla G_n|^2 \varphi_g) \\
\leq \frac{1}{2m} \frac{c^2}{n^2} \int_{|x| \leq n^l \leq 2n^l} |G(x)|^2 \|\varphi_g(x)\|_{F_b}^2 \mathrm{d}x \\
+ \frac{1}{m} \left( \frac{c^2}{n^2} \int_{|x| \leq n^l \leq 2n^l} |G(x)|^2 \|\varphi_g(x)\|_{F_b}^2 \mathrm{d}x \right)^{1/2} (\varphi_g, |\nabla G|^2 \varphi_g)^{1/2} \\
\leq \left( (2c^2 d/m) + b + \sqrt{8c^2 db/m} \right) \|\varphi_g\|^2. \tag{5.14}
\]
Moreover
\[
\sup_{|x| < R} G_n^2(x) |V(x)| \leq \sup_{|x| < R} G^2(x) |V(x)| = a.
\]
Thus we obtain that
\[
(E_{\text{bin}} - V_{\infty} - \epsilon) \|G_n \varphi_g\|^2 \leq \left( b + a + (2c^2 d/m) + \sqrt{8c^2 db/m} \right) \|\varphi_g\|^2. \tag{5.15}
\]
Since \( G_n \) is monotonously increasing in \( n \) and \( \lim_{n \to \infty} G_n(x) = G(x) \), by (5.15) and the Lebesgue monotone convergence theorem, we see that \( G \|\varphi_g(\cdot)\|_{F_b} \in L^2(\mathbb{R}^3) \) and
\[
\lim_{n \to \infty} \|G_n \varphi_g\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} G_n(x)^2 \|\varphi_g(x)\|_{F_b}^2 \mathrm{d}x = \|G \varphi_g\|^2.
\]
In particular by (5.14),
\[
\lim_{n \to \infty} \frac{1}{2m} (\varphi_g, |\nabla G_n|^2 \varphi_g) \leq b \|\varphi_g\|^2.
\]
Thus (5.12) follows for \( G \in C^\infty \).

(3) Suppose that \( G \in C^1(\mathbb{R}^3) \). Let \( \rho \in C_0^\infty(\mathbb{R}^3) \) and \( \rho > 0 \) such that \( \int_{\mathbb{R}^3} \rho(x) \mathrm{d}x = 1 \). Set \( \rho_\epsilon = \rho(\cdot/\epsilon) \epsilon^{-3} \). Define
\[
G_\epsilon := \rho_\epsilon * G.
\]
Since \( G_\epsilon \in C^\infty(\mathbb{R}^3) \), we have
\[
\|G_\epsilon \varphi_g\|^2 \leq \frac{a_\epsilon + b_\epsilon}{E_{\text{bin}} - V_{\infty} - \epsilon} \|\varphi_g\|^2,
\]
where \( a_\epsilon \) and \( b_\epsilon \) are \( a \) and \( b \) with \( G \) replaced by \( G_\epsilon \), respectively. Note that
\[
\sup_{|x| < R} G_\epsilon(x) = \sup_{|x| < R} \int_{\mathbb{R}^3} \rho(z) G(x - \epsilon z) \mathrm{d}z \leq \sup_{|x| < R} G(x),
\]
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with some \( R' \), and
\[
\sup_{x \in \mathbb{R}^3} |\nabla G(x)| = \sup_{x \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \rho(z) \nabla G(x - \varepsilon z) dz \right| \leq \sup_{x \in \mathbb{R}^3} |\nabla G(x)|,
\]
which yields that
\[
a_{\varepsilon} \leq \sup_{|x| < R'} G^2(x)|V(x)| := a', \quad b_{\varepsilon} \leq b.
\]
Hence we obtain that
\[
\|G_{\varepsilon}\varphi_g\|^2 \leq \left( \frac{a' + b}{E_{\text{bin}} - V_\infty - \varepsilon} \right) \|\varphi_g\|^2.
\]
From this,
\[
\liminf_{\varepsilon \to 0} \|G_{\varepsilon}\varphi_g\|^2 < \infty
\]
follows. Since
\[
\liminf_{\varepsilon \to 0} G^2_{\varepsilon}(x)\|\varphi_g(x)\|^2_{F_{\beta}} = G^2(x)\|\varphi_g(x)\|^2_{F_{\beta}},
\]
Fatou’s lemma yields that \( G\|\varphi_g(\cdot)\|_{F_{\beta}} \in L^2(\mathbb{R}^3) \) and
\[
\|G\varphi_g\|^2 \leq \liminf_{\varepsilon \to 0} \|G_{\varepsilon}\varphi_g\|^2 \leq \left( \frac{a' + b}{E_{\text{bin}} - V_\infty - \varepsilon} \right) \|\varphi_g\|^2.
\]
Then
\[
\frac{\|G\varphi_g\|^2}{\|\varphi_g\|^2} \leq \frac{a' + b}{E_{\text{bin}} - V_\infty - \varepsilon}
\]
follows. The proof is complete. \( \square \)

**Remark 5.4** It can be also proven that \( \varphi_g \in D(e^{\beta|x|} \otimes 1) \) for some \( \beta \) under some conditions on \( V \), e.g., \( V(x) = -e^2/|x| \). See [15, 21].

### 5.5 Proof of Lemma 4.16

**Proof of Lemma 4.16**

We prove the lemma for \( \mu = 1 \). Proofs for \( \mu = 2, 3 \) are the same as that of \( \mu = 1 \).
Let \( \varepsilon = (a, 0, 0) \in \mathbb{R}^3 \). We see that \( e^{i\varepsilon x} \) maps \( D(H_{PF,0}) \) onto itself. Thus
\[
e^{i\varepsilon x} H_{PF}\Psi = H_{PF}e^{i\varepsilon x}\Psi + (H_{PF}(\varepsilon) - H_{PF})e^{i\varepsilon x}\Psi, \quad \Psi \in D(H_{PF,0}),
\]
where
\[
H_{PF}(\varepsilon) := \frac{1}{2m}(p \otimes 1 + \varepsilon - eA_{\hat{\phi}}) + V \otimes 1 + 1 \otimes H_I - \frac{1}{2m}(\sigma \otimes 1) \cdot B_{\hat{\phi}}
\]
\[
= H_{PF} + \frac{1}{m}(p \otimes 1 - eA_{\hat{\phi}}) \cdot \varepsilon + \frac{1}{2m}|\varepsilon|^2.
\]
We have
\[ \left( \frac{e^{ix} - 1}{a} \right) H_{PF} \Psi = H_{PF} \left( \frac{e^{ix} - 1}{a} \right) \Psi + \frac{1}{a} (H_{PF}(\epsilon) - H_{PF}) e^{ix} \Psi. \]

By the assumption we see that
\[ s - \lim_{a \to 0} \left( \frac{e^{ix} - 1}{a} \right) H_{PF} \Psi = i(x_1 \otimes 1) H_{PF} \Psi. \tag{5.16} \]

Directly we see that
\[ \frac{1}{a} (H_{PF}(\epsilon) - H_{PF}) e^{ix} \Psi = \left( \frac{1}{m} (p \otimes 1 - eA_{\hat{\varphi}}) + \frac{1}{2m} a \right) e^{ix} \Psi \]
\[ = e^{ix} \left( \frac{1}{m} (p \otimes 1 - \epsilon - eA_{\hat{\varphi}}) + \frac{1}{2m} a \right) \Psi \]
\[ \to \frac{1}{m} (p \otimes 1 - eA_{\hat{\varphi}}) \Psi \tag{5.17} \]
strongly as \( a \to 0 \). From (5.16) and (5.17) it follows that
\[ s - \lim_{\epsilon \to 0} H_{PF} \left( \frac{e^{ix} - 1}{a} \right) \Psi = \frac{1}{m} (p \otimes 1 - eA_{\hat{\varphi}}) \Psi \tag{5.17} \]
for \( \Psi \in D(H_{PF}) \). Thus (1) of Lemma 4.16 is proven. Let
\[ \Psi \in S := C^\infty_0(\mathbb{R}^3; \mathbb{C}^2) \otimes \bigoplus_{n=0}^\infty s_n C^\infty_0(\mathbb{R}^3). \]

Then we have
\[ [(x_1 \otimes 1), H_{PF}] \Psi = \frac{i}{m} (p \otimes 1 - eA_{\hat{\varphi}}) \Psi. \tag{5.18} \]

Hence
\[ \|[x_1 \otimes 1, H_{PF}] \Psi\| \leq c \|(H_0 + 1)^{1/2} \Psi\| \]
follows with some constant \( c \). Then the closure \([[(x_1 \otimes 1), H_{PF}]_S\) is well defined on \( D(H_0^{1/2}) \) and satisfies
\[ [(x_1 \otimes 1), H_{PF}] \Psi = \frac{i}{m} (p \otimes 1 - eA_{\hat{\varphi}}) \Psi \]
for \( \Psi \in D(H_0^{1/2}) \). In particular,
\[ [(x_1 \otimes 1), H_{PF}] \varphi_g = \frac{i}{m} (p \otimes 1 - eA_{\hat{\varphi}}) \varphi_g \]
follows, since \( \varphi_g \in D((x_1 \otimes 1) H_{PF}) \cap D(H_{PF}(x_1 \otimes 1)) \).

Acknowledgements I am very grateful to A. Arai and M. Hirokawa for useful comments and discussions.
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