Abstract

We describe a regression-based method, generally referred to as the Least Squares Monte Carlo (LSMC) method, to speed up exposure calculations of a portfolio. We assume that the portfolio contains several exotic derivatives that are priced using Monte-Carlo on each real world scenario and time step. Such a setting is often referred to as a Monte Carlo over a Monte Carlo or a Nested Monte Carlo method.

Keywords

American Monte-Carlo · Least-Squares Monte-Carlo · AMC · LSMC · Nested Simulation

1 Introduction

Least Squares Monte-Carlo (LSMC) is a technique based on least squares regression, which we describe in this paper. We think of LSMC as a special case of a larger class of methods that are referred to as American Monte-Carlo in the literature [1], [2]. The term AMC has its origins in the work of Longstaff and Schwartz [3]. Therein, the authors describe a method to price American Options that relies on building a conditional expectation function using a least squares regression technique over a set of explanatory variables. In the simplest case, the set of explanatory variables would include the current state of the underlying risk factors. In its original form, the method uses two sets of Monte-Carlo simulations. One simulation is used to build the conditional expectation function by regressing over the stock price and the indicator that the option is in the money using backward propagation of state variables. Once the backward pass is done, a different set of Monte-Carlo paths are used to move forward to price the instrument. During the forward pass, the already constructed conditional expectation functions are used at every observation date to determine whether it is optimal to exercise or not.

In its modern usage, in the context of risk management, the term AMC (or LSMC) is used to handle the Nested Monte-Carlo (NMC). As an illustration, consider Potential Future Exposure (PFE) or Expected Exposure (EE) for a portfolio consisting of exotic instruments.² To estimate an EE and PFE profiles, we value the portfolio on a set of future market, or “outer”, scenarios generated across time. Suppose we use 5,000 such scenarios, and use 5,000 risk-neutral, “inner”, paths to obtain a single price estimate on each outer scenario. Under this set-up, the computational cost for each exotic instrument will be proportional to \(5,000 \times 5,000 = 25,000,000\) on each time step. Clearly, we need to apply some clever techniques to reduce the computation cost.

One such solution has been proposed by Barrie and Hibbert [5]. Berrie and Hibbert’s method reduces the overall computational cost by decreasing the number of inner paths. Their method aims at reducing Monte-Carlo errors by regressing the estimated prices against a set of explanatory variables generated in the

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² For detailed example of PFE and EE see [4].
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outer loop. We refer to price estimates obtained using this method as LSMC price estimates. To illustrate the LSMC method, consider a vanilla call option under Geometric Brownian Motion with constant drift and volatility. We use a GBM model in the outer loop to generate a set of underlying stock prices and use GBM model within the inner loop to price the option. Assume that the option matures in one year (time step 360), and we are interested in computing prices at time steps 15, 30, \ldots, 360. On each time step, in a typical NMC setting, we use 5,000 inner paths to obtain MC price estimates. Under the LSMC setting, we use 30 inner paths, for example, per outer scenario to obtain a set of MC prices on a given time step. When these prices are plotted against 5,000 underlying (outer) spot values on a given time step, we expect to see some relationship. We assume that this relationship can be explained by

\[ y_{mc,i} = \beta_0 + \beta_1 S_i + \beta_2 S_i^2 + \beta_3 S_i^3 + \xi_i \]  

where \( y_{mc,i} \) is the MC price, \( S_i \) is the spot price of the equity, and \( \xi_i \) is the error on the \( i \)th scenario.

Once the preliminary \( 5,000 \times 30 \) simulation is done, we have 5,000 MC price estimates and 5,000 spot values that we use to build regression model (1.1) to obtain coefficient estimates \( \{ \hat{\beta}_i \}_{i=0} \). These coefficient estimates are then used to obtain LSMC price estimates given by

\[ \hat{y}_{mc,i} = \hat{\beta}_0 + \hat{\beta}_1 S_i + \hat{\beta}_2 S_i^2 + \hat{\beta}_3 S_i^3. \]

Figure 1 compares the LSMC price estimates to the Black-Scholes analytic prices. The graphs are provided for time steps 15 and 345. In order to obtain the “hockey stick” shape near maturity of the option, we incorporate a dummy variable

\[ d_i = \mathbb{I}\{ S_i \geq K \}, \]  

where \( K \) is the strike price

into equation (1.1) to yield

\[ y_{mc,i} = \beta_0 + \beta_1 S_i + \beta_2 S_i^2 + \beta_3 S_i^3 + d_i + \beta_4 d_i S_i + \beta_5 d_i S_i^2 + \beta_6 d_i S_i^3 + \xi_i. \]

Figure 1 reveals that the LSMC price estimates are much closer to the true analytic prices than the original MC prices. Interestingly enough, Barrie and Hibbert’s method can be viewed as a variance reduction technique, though it was not presented as such in their original report. In this paper we look at the method more closely and present a variance formulation that could be used to evaluate the accuracy of price estimates. We also look at the PFE and EE profiles for a set of exotic instruments to test the accuracy of this method.

2 Least Squares Monte-Carlo

We assume, at outset, that \( n \) Monte-Carlo scenarios have been generated over \( k \) time points in the set \( T := \{ t_1, t_2, \ldots, t_k \} \), where \( t_i < t_j \), for \( i < j \). These time points may represent important dates such as payment dates, fixing dates, etc. At any time \( t \in T \), we have \( n \) vectors \( x_{i,t} \), for \( i = 1, 2 \ldots, n \), representing
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the cross sectional information obtained from the scenarios at time \( t \) and absorbing all the information up to time \( t \). These vectors form the explanatory variables for the regression model for time \( t \in T \).

Suppose that we are interested in obtaining the time \( t \) price of an exotic instrument maturing at time \( T \). If \( f(x_{t,i}) \) denotes the discounted payoff of this instrument, then the time \( t \) price is obtained under the conditional expectation

\[
y_i := E^Q \left[ f(x_{t,i}) \mid \mathcal{F}_t \right].
\] (2.1)

One can estimate the integral on the right-hand-size of equation (2.1) using a MC method. Doing so, will generate a sequence of MC estimates \( y_{mc,i}, i = 1, 2, \ldots, n \) at each time \( t \in T \). These prices form the corresponding set of response variables for our regression model. We can write the relationship as

\[
y_{mc,i} = \frac{1}{p} \sum_{k=1}^{p} f_k(x_{t,i}),
\] (2.3)

where \( x_{t,i} \) is generated under risk neutral measure, and \( f_k(\cdot) \) is the value of the discounted payoff function on the \( k^{\text{th}} \) path. The Monte-Carlo price can represented as the sum of the true value \( y_i \) and a Monte-Carlo error \( \xi_{mc,i} \), or

\[
y_{mc,i} = y_i + \xi_{mc,i}.
\] (2.4)

Now, suppose we let \( p \to \infty \) in equation (2.2), then

\[
\lim_{p \to \infty} y_{mc,i} = \lim_{p \to \infty} \sum_{j=0}^{m} \beta_j b_j(x_{t,i}) + \xi_i
\]

\[
= \sum_{j=0}^{m} \beta_j b_j(x_{t,i}) + \lim_{p \to \infty} \xi_i.
\] (2.5)

As the number of paths tends to infinity, \( \xi_i \) will approach \( \xi_{d,i} \). \(^3\) In case of a perfect model, \( \xi_{d,i} \) equals zero. Generally, \( \xi_{d,i} \) is \( \mathcal{F}_t \) measurable (deterministic) and is often a function of \( x_{t,i} \). From equation (2.5), we conclude that

\[
y_i = \sum_{j=0}^{m} \beta_j b_j(x_{t,i}) + \xi_{d,i}.
\] (2.6)

Substituting the right-hand-side of equation (2.6) into equation (2.4) yields the final model

\[
y_{mc,i} = \sum_{j=0}^{m} \beta_j b_j(x_{t,i}) + \xi_{d,i} + \xi_{mc,i}.
\]

The total error \( \xi_i \) from equation (2.2) can be viewed as the sum of deterministic part \( \xi_{d,j} \) and a random part \( \xi_{mc,i} \). In practice, we do not observe \( \xi_{d,i} \) directly nor can accurately measure \( \xi_{mc,i} \) unless we use high number of paths. However, we can estimate the variance of \( \xi_{mc,i} \) using the discounted payoff.

For sufficiently large number of paths, \( y_{mc} \) follows normal distribution with mean \( 0 \) and \( \Sigma \). The non-biased estimator for \( \Sigma_{mc} \) then is given by

\[
[\hat{\Sigma}_{mc}]_{ij} = \frac{1}{p(p-1)} \sum_{k=1}^{p} (f_k(X_{T,i}) - y_{mc,i})(f_k(X_{T,j}) - y_{mc,j}).
\] (2.7)

\(^3\) \( \lim_{p \to \infty} y_{mc,i} = y_i \) by the Strong Law of Large Numbers.
In case $\Sigma_{mc} \equiv \sigma_{mc}^2 I$, the MC variance estimator becomes

$$\hat{\sigma}_{mc}^2 = \frac{1}{n} \sum_{j=1}^{n} [\Sigma_{mc}]_{jj}.$$ 

In a matrix notation, we can write the regression model (2.2) as

$$y_{mc} = X\beta + \xi.$$

then, the coefficient estimator becomes

$$\hat{\beta} = (X^T X)^{-1} X^T y_{mc},$$

and

$$\hat{y}_{mc} = H y_{mc}$$

with

$$H = X(X^T X)^{-1} X^T.$$ 

H is also known as an orthogonal projection. Consequently, the total variance of $\hat{y}_{mc}$ is lower than the total variance of $y_{mc}$. This is given by the following theorem and is proven in appendix A.

**Theorem 2.1.** Let $Y : \Omega \to \mathbb{R}^m$ be a random vector having a finite variance, and let $H$ be an orthogonal projection onto a linear subspace of $\mathbb{R}^m$. Then,

$$\text{tr}(\text{Cov}(HY)) \leq \text{tr}(\text{Cov}(Y)),$$

where $\text{tr}(\cdot)$ denotes the trace operator.

By Theorem 2.1, we know that the total variance reduction is expected under an orthogonal projection

$$\text{Cov}(\hat{Y}_{mc}) = \text{Cov}(HY_{mc}) = H \text{Cov}(Y_{mc}) H \quad (2.9)$$

where $\text{Cov}(Y_{mc}) = \Sigma_{mc}$ represents the covariance matrix of $\xi_{mc}$ that is estimated using the discounted payoffs generated in the inner loop. Furthermore, when the covariance matrix is equal to $\Sigma = \sigma I$ for some constant $\sigma$ the ratio of total LSMC variance and total MC variance is given by

$$\frac{\text{tr}(\text{Cov}(y_{mc}))}{\text{tr}(\text{Cov}(y_{mc}))} = \frac{\text{rank}(X)}{n} = \frac{m}{n},$$

which follows from the properties of $H$.

### 3 Illustrative Results

In this section, we test the accuracy of the LSMC methodology. An Arithmetic Asian Option is used to check the fit of various degree polynomials. After we obtain a good fit, we look at the EE and PFE profiles of a Barrier Option, Target Accrual Redemption Note, Accumulating Forward Contract and Asian Option. For the reader’s convenience, we describe the payoffs of these instruments in appendix B.

#### 3.1 LSMC Analysis

Consider an Arithmetic Asian Put Option that matures at time $T$ with $s$ time fixings $\{t_i\}_{i=1}^s$ and weights $\{w_i\}_{i=1}^s$. The payoff at maturity takes the form

$$\text{payoff} = (K - \sum_{i=1}^{s} w_i S_{t_i})^+, \quad 0 \leq t_i \leq T \quad \forall i,$$

where $S_{t_i}$ is the price of the underlying equity at time $t_i$. For simplicity, assume that $S$ follows a GBM with constant drift and volatility in the outer and inner loops.\(^4\) Let the observation time be $t_k$ for $1 \leq k \leq s$. We consider orthogonal Forsythe polynomials [6] for basis functions using $B_{t_k} = \sum_{i=1}^{k} w_i S_{t_i}$ as the explanatory variable in the regression model. We perform the following steps to obtain LSMC estimates:

\(^4\)The drift in the outer loop is set at 0.1, while the drift in the inner loop is set at 0.05.
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- Transform the explanatory variables to take values between $-1$ and $1$ using
  \[ B^*_t,j := \frac{2B_{t,k} - \max(B_{t,k}) - \min(B_{t,k})}{\max(B_{t,k}) - \min(B_{t,k})} \]

- Obtain regression matrix $X$ using Forsythe polynomial expansion
- Estimate the coefficients $\beta$ in model (2.8)
- Use the coefficient estimates to obtain LSMC prices

We set a numerical experiment with 1, 10, 30, 50, 100 and 10,000 inner paths to compute and regress $Y_{mc}$ against $X$ to get $\hat{Y}_{mc}$. To study the accuracy of the fit and variance reduction, we compare LSMC estimates to MC prices obtained using 131,072 Sobol paths. The results are summarized in Figures 2, 3 and 4.

Figure 2: MC vs. LSMC Price:
First time step fit using polynomial of degree five

![Figure 2: MC vs. LSMC Price](image)

Note that fully estimated covariance matrix was used to obtain the ratio of total of MC variance and total of LSMC variance. As one can see, the ratio of total of MC to LSMC variance is close to $\frac{\text{rank}(X)}{n} = \frac{6}{5000} = 0.0012$. That is, using regression, we are able to reduce the variance by about 99.9%. From Figure 2, one may infer that we can obtain very accurate price estimates using 10 paths per outer scenario. Then, assuming that a typical Nested Monte-Carlo requires 5,000 inner paths to get a reasonable PFE and EE profile, we can speed up the computation by a factor of 500 using LSMC. Furthermore, one can argue that a higher polynomial degree could provide a better fit. Though this may be true when one is dealing with a closed-form functional approximation, high-degree polynomials tend to overfit data with noise. These results are most evident in Figure 5, where we plot the sum of squared deterministic residuals (SSE) $\hat{\xi}_d$.

Let $\bar{Y}$, $\hat{Y}$, $Y_{mc}$, $\hat{Y}_{mc}$ be the Actual value, the Actual fit, MC price and LSMC price, respectively. Figure 5 compares SSE values computed with

$\hat{\xi}_d = \bar{Y} - \hat{Y}$, and $\bar{\xi}_d = Y_{mc} - \hat{Y}$. 

Figure 3: Monte-Carlo Variance Density:
First time step

Figure 4: LSMC Variance Density:
First time step fit using polynomial of degree five
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Figure 5: Actual vs. LSMC Sum of Squared Errors:
First time step fit with varying degree polynomials

- LSMC
- Actual

1 MC Paths
2 4 6 8 10 12 14 16 18 20
0 200 400 600 800
Degree
SSE

10 MC Paths
2 4 6 8 10 12 14 16 18 20
0 20 40 60 80
Degree
SSE

50 MC Paths
2 4 6 8 10 12 14 16 18 20
0 10 20 30 40
Degree
SSE

1000 MC Paths
2 4 6 8 10 12 14 16 18 20
0 1 2 3 4 5
Degree
SSE

10000 MC Paths
2 4 6 8 10 12 14 16 18 20
0 1 2 3 4 5
Degree
SSE
Table 1: Risk-neutral models used for pricing of instruments

| Instrument | Model                        | Underlying         |
|------------|------------------------------|--------------------|
| Asian      | Hull-White-Black-Scholes     | Foreign Exchange   |
| Forward    | Geometric Brownian Motion    | Equity             |
| Barrier    | Hull-White-Black-Scholes     | Foreign Exchange   |
| TARN       | Libor Market Model           | Libor Rate         |

Figure 6: Actual vs. LSMC PFE Profile at 95th Percentile

We define Actual fit as the least squares fit of price estimates computed using a large number (131,072) of Sobol paths. The SSE of Actual fit that has been computed using $\tilde{\xi}_{mc}$ has a very small (quasi) MC error and serves as a measure of accuracy. From Figure 5, we observe monotonic decrease of SSE when we select higher degree polynomials for the Actual fit. This does not happen when we fit data with noise. In general, the less accurate are the price estimates, the sooner SSE starts to increase. That is, higher degree polynomials tend to overfit data with noise. With that in mind, we use the third degree polynomial for PFE and EE estimation: it appears to be the highest degree at which SSE does not increase in the graphs presented.

3.2 PFE and EE Results

For PFE and EE computations, we use a joint Hull-White two-factor model for interest rates and Heston stochastic volatility model for FX and Equity to generate outer, real scenarios. We use different models for pricing, which vary depending on the instrument. Table 1 summarizes the risk-neutral models used in the inner loop for each instrument. For computational efficiency, with no loss of generality, all instruments mature in one year and have fixing dates set at 15 day intervals. The profiles obtained for PFE and EE per instrument are summarized in Figures 6 and 7, respectively.

The PFE and EE profiles for the Actual data series in Figures 6 and 7 are obtained using 4096 Sobol paths in the inner loop. For the LSMC PFE and EE profiles, we used varying numbers of paths that were determined by numerical experiments. We found that smooth instruments such as Asian, TARN and Accumulative Forward require smaller numbers of paths. In most cases, we were able to obtain fairly accurate price estimates using anywhere between 10 and 30 inner paths. To get accurate LSMC Barrier price estimates, we used anywhere between 30 and 64 inner paths. On average, we were able to speed up the computation of PFE and EE profiles by a factor of 60.
4 Conclusion

In this paper, we were able to show that the least squares Monte-Carlo approach proposed by Barrie and Hibbert, can capture the tails of the distribution with a high degree of accuracy. This method, however, does call for numerical experiments for model calibration where an appropriate number of inner paths and basis functions need to be selected. Though we were able to capture all instrument prices with third degree polynomial basis functions, one should exercise care and diligence when selecting a higher degree. As mentioned previously, high-degree polynomials may overfit data with noise, which would result in worse price estimates.

In this paper, we have also proved that the total variance of the least squares estimates is no greater than the total variance of the original MC estimates. Indeed, this result should be anticipated since the LSMC method can be viewed as a smoothing method. Our current results suggest that LSMC should perform better in aggregate risk metrics such as CVA and/or CVA sensitivity, which is a topic for a future study.

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A Least Squares Linear Regression

In this section we review the linear regression model and a reduction of variance theorem. Linear regression modelling and estimation techniques can be found in many linear regression textbooks such as [7]. We start with a linear regression model of the form

\[ Y = X\beta + \xi \]  \hspace{1cm} (A.1)

with a response variable \( Y \), regression variables \( X \) and error terms \( \xi \). The least squares estimator \( \hat{\beta} \) is the solution to

\[ \min_{\beta \in \mathbb{R}^n} ||Y - X\beta||^2, \]  \hspace{1cm} (A.2)

where \( ||u|| \) is \( L_2 \)-norm of vector \( u \). The solution to problem (A.2) is given by

\[ \hat{\beta} = (X^TX)^{-1}X^TY, \]

and the estimator of \( Y \) is

\[ \hat{Y} = X(X^TX)^{-1}X^TY \equiv HY, \]  \hspace{1cm} (A.3)

provided that \( X^TX \) is invertible.

\( H \), commonly known as the hat matrix, maps the response variables \( Y \) onto \( \hat{Y} = X\hat{\beta} \) by minimizing the sum of squared errors \( \xi^T\xi \). \( H \) is also the orthogonal projection onto the column space of \( X \). Clearly, \( H \) is symmetric and idempotent.

We can also look at the problem (A.1) from a statistical point of view and compute the variance of our mean estimator \( \hat{Y} \). Suppose \( \xi \) is a random vector with finite variance, then the covariance of the mean response \( \hat{Y} \) is given by

\[ \text{Cov}(\hat{Y}) = \text{Cov}(HY) \]

\[ = H\text{Cov}(Y)H \]

\[ = H\Sigma H, \]  \hspace{1cm} (A.4)

where \( \Sigma \) is the covariance matrix of \( \xi \). Furthermore, we can conclude that the total variance of the mean response estimator \( \hat{Y} \) is less than or equal to the total variance of the original vector \( Y \) by the following theorem.

**Theorem.** Let \( Y : \Omega \to \mathbb{R}^m \) be a random vector having a finite variance, and let \( H \) be an orthogonal projection onto a linear subspace of \( \mathbb{R}^m \). Then,

\[ \text{tr}(\text{Cov}(HY)) \leq \text{tr}(\text{Cov}(Y)), \]

where \( \text{tr}(\cdot) \) denotes the trace operator.

**Proof.** Expressing \( Y = HY + (I - H)Y \), then

\[ \text{tr} \left( \text{Cov}(Y) \right) = \text{tr} \left( \text{Cov}(HY + (I - H)Y) \right) \]

\[ = \text{tr} \left( H\text{Cov}(Y)H + (I - H)\text{Cov}(Y)(I - H) + H\text{Cov}(Y)(I - H) + (I - H)\text{Cov}(Y)H \right) \]

\[ = \text{tr} \left( H\text{Cov}(Y)H \right) + \text{tr} \left( (I - H)\text{Cov}(Y)(I - H) \right) \]

\[ + \text{tr} \left( H\text{Cov}(Y)(I - H) \right) + \text{tr} \left( (I - H)\text{Cov}(Y)H \right) \]

\[ = \text{tr} \left( \text{Cov}(HY) \right) + \text{tr} \left( \text{Cov}((I - H)Y) \right) + 2\text{tr} \left( H\text{Cov}(Y)(I - H) \right) \]

\[ = \text{tr} \left( \text{Cov}(HY) \right) + \text{tr} \left( \text{Cov}((I - H)Y) \right) + 2\text{tr} \left( \text{Cov}(Y)(I - H)H \right) \]

\[ = \text{tr} \left( \text{Cov}(HY) \right) + \text{tr} \left( \text{Cov}((I - H)Y) \right) + 0 \]

\[ \geq \text{tr} \left( \text{Cov}(HY) \right) \]

by properties of \( H \) and \( \text{tr}(\cdot) \) operator. \( \square \)
B Definition of Instruments

B.1 Notation

- $S_t$ denotes the price of the underlying at time $t$.
- $\tau$ denotes a stopping time.
- $\tau \wedge t := \min(\tau, t)$.
- $T$ is maturity of the instrument unless otherwise stated.
- $T_i < T_j \leq T \ \forall i < j$.
- $T_0 = 0$.

B.2 Payoff Functions

- K-Strike Arithmetic Asian Option with weights $\{w_i\}_{i=1}^n$:
  
  \begin{align*}
  \text{put} &= (K - \sum_{i=1}^n w_i S_{T_i})^+ \\
  \text{call} &= \left(\sum_{i=1}^n w_i S_{T_i} - K\right)^+
  \end{align*}

- K-Strike Up-and-Out Barrier Option with barrier $B > S_0$ and rebate $R$:
  
  \begin{align*}
  \text{put} &= (K - S_T)^+ I\{\tau > T\} + R I\{\tau \leq T\} \\
  \text{call} &= (S_T - K)^+ I\{\tau > T\} + R I\{\tau \leq T\}
  \end{align*}

  where $\tau = \min\{t : S_t \geq B\}$. The rebate is paid at time $\tau$ and $(S_T - K)^+$ is paid at maturity.

- Accumulating Forward Contract with $n$-payments and barrier $B$:

  \begin{align*}
  \text{short} &= \sum_{i=1}^n (\alpha |U_i| + \beta |D_i|)(K - S_{T_i \wedge \tau}) \\
  \text{long} &= \sum_{i=1}^n (\alpha |U_i| + \beta |D_i|)(S_{T_i \wedge \tau} - K)
  \end{align*}

  where

  \begin{align*}
  U_i &:= \{S_{T \wedge t} > K : T_{i-1} < (\tau \wedge t) < T_i\} \\
  D_i &:= \{S_{T \wedge t} \leq K : T_{i-1} < (\tau \wedge t) < T_i\} \\
  |A| &\in \mathbb{R} \\
  \alpha, \beta &\in \mathbb{R} \\
  |A| &\text{ denotes the cardinality of set } A
  \end{align*}

- Target Accrual Redemption Note with barrier $B$ on accrual payments, LIBOR rate $S$ and fixed rates $K_1$ and $K_2$:

  \begin{align*}
  \text{receiver} &= \sum_{i=1}^n (K_1 - S_{T_i} I\{\tau > T_i\} - K_2 I\{\tau \leq T_i\}) \\
  \text{payer} &= \sum_{i=1}^n (S_{T_i} I\{\tau > T_i\} + K_2 I\{\tau \leq T_i\} - K_1)
  \end{align*}

  where $\tau = \min\{T_i : \sum_{i=1}^n S_{T_i} \geq B\}$. 
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