Fractional Vortices and Lumps

Minoru Eto$^{1,2}$, Toshiaki Fujimori$^3$, Sven Bjarke Gudnason$^{1,2}$, Kenichi Konishi$^{1,2}$, Takayuki Nagashima$^3$, Muneto Nitta$^4$, Keisuke Ohashi$^5$ and Walter Vinci$^{1,2,5}$

1 Department of Physics, University of Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
2 INFN, Sezione di Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
3 Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
4 Department of Physics, Keio University, Hiyoshi, Yokohama, Kanagawa 223-8521, Japan
5 Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, CB3 0WA, UK

Abstract

We study what might be called fractional vortices, vortex configurations with the minimum winding from the viewpoint of their topological stability, but which are characterized by various notable substructures in the transverse energy distribution. The fractional vortices occur in diverse Abelian or non-Abelian generalizations of the Higgs model. The global and local features characterizing these are studied, and we identify the two crucial ingredients for their occurrence—the vacuum degeneracy leading to non-trivial vacuum moduli $M$, and the BPS nature of the vortices. Fractional vortices are further classified into two kinds. The first type of such vortices appear when $M$ has orbifold $\mathbb{Z}_n$ singularities; the second type occurs in systems in which the vacuum moduli space $M$ possesses either a deformed geometry or some singularity. These general features are illustrated with several concrete models.

* Now at Theoretical Physics Laboratory at RIKEN since April 2009.
† Now at the department of physics in Kyoto University since April 2009.
1 Introduction

Vortices appear in many different areas of physics, from fluid and plasma dynamics, solid-state physics, particle physics to cosmology. Usually certain topological properties lie behind their stability in time and in space. Typically, the energy distribution in the plane perpendicular to the vortex axis is peaked around the vortex axis, with a well-defined finite width in the vortex profile. This is certainly the case for the single (i.e. minimum-vorticity) type II Abrikosov-Nielsen-Olesen (ANO below) vortex [1, 2], where the origin of the vortex stability lies in the first homotopy group, $\pi_1(U(1)) = \mathbb{Z}$. As it turns out, however, when the gauge group and/or the matter content of the system are of more general kind than the standard Abelian-Higgs model (Landau-Ginzburg model)—$U(1)$ gauge group and one charged scalar field—, a variety of interesting generalized vortex solutions appear.

The present paper is concerned with a class of vortex-like solutions which might be called “fractional vortices”. They are characterized by the minimal quantized vorticity (winding or magnetic flux) from the point of view of topological stability; nevertheless, their transverse profiles exhibit various non-trivial substructures as if they were made of smaller vortices, a little like a multi-vortex solution in the standard type II superconductor. But in contrast to the latter case the tension of each of the sub-peak is not quantized, and their relative weights, distances and shapes depend on the details of the system, such as the coupling constants, the scalar VEVs (vacuum expectation values) and the symmetry breaking pattern, etc. In all cases, a sub-peak cannot be removed by sending its position to infinity while keeping others in a finite region.\footnote{An analogous phenomenon occurs in a simple extension of the Abelian Higgs model—the Landau–Ginzburg model—with two coupled scalar fields. This can be realized in a certain unconventional superconductor [3]. On a broader prospect, our fractional vortices share also some features with the fractional instantons—the torons [4] or the calorons [5] which exist when the base space has a period such as a torus. Fractional vortices in a torus or a cylinder have also been studied [6, 7], see also [8].}

The aim of this work is to show that such a phenomenon occurs very generally, and to make a preliminary study of these solutions, trying to find what characterizes the occurrence and substructures of these fractional vortices. As the arena of our study, we consider various Abelian and non-Abelian extensions of the Higgs model. The degrees of freedom will be a set of complex scalar fields with various charges and gauge fields. For definiteness and for simplicity, we take the models whose Lagrangians have the form of the bosonic sector of $\mathcal{N} = 2$ supersymmetric gauge theories. They are natural generalizations of the Abelian-Higgs model in the Bogomol’nyi-Prasad-Sommerfield (BPS) limit. The constant which characterizes the VEVs and which forces the system into the Higgs phase, arises as the Fayet-Iliopoulos (FI) term in the supersymmetric
setting. Although most of our results are independent of supersymmetry, we shall mention also results more specific to the supersymmetric versions of the models, when appropriate.

It will be shown that the fractional vortices can be further classified into two different classes. The first type exists when the vacuum moduli $\mathcal{M}$ has an orbifold $\mathbb{Z}_n$ singularity. The second type occurs when the vacuum moduli $\mathcal{M}$ has a 2-cycle with a deformed geometry.

This paper is organized as follows. In Sec. 2 we review the semi-local vortices in the extended Abelian-Higgs model (EAH). In Sec. 3 general properties of vortices in degenerate vacua are discussed. In Sec. 4 we present various concrete models admitting fractional vortices based on a $\mathbb{CP}^1$ target space. The first two models, in particular, provides the simplest examples of the fractional vortices of the first and second types, respectively. In Sec. 5 we discuss an $SO(N) \times U(1)$ gauge theory, which exhibits fractional vortices of the second type. Some useful technical details are collected in the Appendices.

2 Semi-local vortex in the Extended Abelian-Higgs model

Something quite non-trivial occurs already in the Abelian-Higgs model, if the number of charged fields is greater than one [10, 11, 12]. The model is

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu q (\mathcal{D}^\mu q)^\dagger - \frac{\lambda}{2} (qq^\dagger - \xi)^2 , \quad (2.1)$$

where $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ is the standard covariant derivative, $q = (q_1, q_2, \ldots, q_{N_f})$ represents a set of complex scalar matter fields of the equal charge. This model is sometimes called the semi-local model since not all global symmetries, i.e. $G = U(N_f)$ here, are gauged. Even if we restrict to the minimum vorticity, the vortex profile turns out to depend on the particular solution considered.

In order to have a finite-energy (the energy per unit length—the tension) configuration, the complex scalar fields must asymptotically approach a vacuum configuration,

$$\mathcal{M} \equiv \{q_i\}, \quad \sum_{i=1}^{N_f} |q_i|^2 = \xi , \quad (2.2)$$

far from the vortex center. By the $SU(N_F)$ global and $U(1)$ local symmetry they can be chosen to be

$$\langle q \rangle = (q_0, 0, \ldots, 0), \quad q_0 = \sqrt{\xi} , \quad (2.3)$$

breaking the global symmetry to $SU(N_f - 1) \times U(1)$ In other words, the vacuum moduli space is

$$\mathcal{M} = \mathbb{CP}^{N_f-1} = SU(N_f)/[SU(N_f - 1) \times U(1)] . \quad (2.4)$$
The vacuum configurations $M$ represent a non-trivial $U(1)$ fibration over the vacuum moduli space $\mathbb{C}P^{N_f-1}$. As the $U(1) \subset U(N_f)$ part is gauged, its breaking does not lead to any further vacuum degeneracy. Since the first homotopy group of the vacuum configurations $M$ is trivial, vortex solutions may not necessarily be stable.

Indeed, for $\beta \equiv \lambda/e^2 > 1$ (i.e. type II superconductors), an ANO vortex solution embedded in the first flavor is found to be unstable against fluctuations of the extra fields ($i = 2, 3, \ldots, N_f$) which increase its size; the vortex flux spreads out all over the transverse space [11, 12].

For $\beta < 1$ (i.e. type I superconductors), instead, an ANO vortex [1, 2] embedded in one of the flavors is found to be stable. The origin of the stability of such a vortex can be traced to the fact that the asymptotic scalar field must be actually the vacuum configuration modulo gauge transformations. The vortex winds a non-trivial $U(1)$ fiber over the vacuum moduli $\mathbb{C}P^{N_f-1}$: the relevant homotopy is

$$\pi_1(U(1)) = \mathbb{Z},$$

just as in the case of the ANO vortex (which indeed it is).

In the interesting special (BPS) case, $\beta = 1$, we find a family of degenerate vortex solutions with the same tension, $T = 2\pi \xi$. Except for the special point of the vortex moduli space (i.e. the space of solutions), which represents the ANO vortex (sometimes called a “local vortex”), the vortex has a power-like tail in the profile function, and the width of the vortex (thickness of the string) can be of an arbitrary size.

Far from the vortex center, the vortex configuration essentially reduces to the $\mathbb{C}P^{N_f-1}$ sigma-model lump (or two-dimensional Skyrmion), characterized by

$$\pi_2(\mathbb{C}P^{N_f-1}) = \mathbb{Z}.$$  

In terms of an effective potential as a function of the vortex radius, the $k = 1$ (minimum-winding) sector of the system has the minimum at the origin for $\beta < 1$; at infinity for $\beta > 1$ (a “run-away vacuum” behavior); and has no potential—a flat direction—in the BPS case.

3 Vortices in degenerate vacua

There are two crucial ingredients which lead to the interesting varieties of degenerate vortex solutions in systems such as the extended Abelian-Higgs model (with $\beta = 1$) just considered:

---

\footnote{This type of vortex solutions has been termed “semi-local vortices”. Again although this is not an entirely adequate terminology we shall stick to it as it is commonly used in the literature.}

3
the vacuum degeneracy and the BPS saturated nature of the vortices. The first means that, in contrast to the cases in which the vacuum moduli is trivial (a point), we must consider in general all vortex solutions in all possible points of the vacuum moduli simultaneously. See Fig. 1. Even if we restrict ourselves to the minimally winding vortex solutions only—we shall do so in this article for definiteness—the vortices represent non-trivial fiber bundles over the vacuum moduli \( \mathcal{M} \). The BPS saturated nature of the vortices implies that the vortex equations linearize, and in turn half of the equations (the matter equations of motion) being solved by the moduli-matrix Ansatz, as is well known \[20, 21, 22\], see Eqs. (A.5), (A.6). The other equations (the gauge field equations) reduce, in the strong coupling limit or anyway sufficiently far from the vortex center, to the vacuum equations for the scalar fields. In other words, the vortex solutions tend to sigma model lumps.

\[\text{3This is the case for the standard Abelian-Higgs model, of course, but so it is in the case of the } U(N) \text{ gauge theory with } N_f = N \text{ squarks in the fundamental representation. The latter model and its generalizations, after the discovery of the non-Abelian vortices in the color-flavor locked vacuum } \[13][14], \text{ has attracted considerable attention } \[15-36].\]


3.1 Structures of the vacuum moduli

Let us first consider what we regard as the global aspect of our vortices. More precisely, our first concern is the vacuum moduli \( \mathcal{M} \) on each point of which the vortex solutions are defined. Let the symmetry group of the underlying system be

\[
K = L \otimes G_F ,
\]

where \( L \) is the local gauge group, while \( G_F \) is the global symmetry group. Let \( M \) be the manifold of the minima of the scalar potential, the vacuum configuration \( M = \{ q_i : q_i^\dagger T_i^j q_j = \xi^j \} \). The vacuum moduli \( \mathcal{M} \) is given by the points

\[
p \in \mathcal{M} = M / F ,
\]

where the fiber \( F \) is the sum of the gauge orbits of a point in \( M \)

\[
f \in F = \{ q^g : q^g = g q \} , \quad g \in L / L_0 ,
\]

where we have taken into account the possibility that a given vacuum configuration might leave a subgroup \( L_0 \subset L \) unbroken:

\[
L_0^{(q)} = \{ \ell_0 \in L : \ell_0 q = q \} .
\]

In other words the vacuum moduli are made of the points of \( M \) in which gauge-equivalent points are identified.

A subgroup of the global group

\[
\tilde{G}^{(q)} \subset G_F , \quad \text{such that } \tilde{G}^{(q)} = \{ g_f \in G_F : g_f q = \ell q \} , \quad \ell \in L ,
\]

represents the unbroken global symmetry group of the system. A vortex solution is defined on each point of \( \mathcal{M} \), in the sense that the scalar configuration along a sufficiently large circle (\( S^1 \)) surrounding it traces a non-trivial orbit in \( F \) (hence a point in \( \mathcal{M} \)). The existence of a vortex solution at a point \( f \in F \) requires that

\[
\pi_1 (F, f) \neq 1 ;
\]

a vortex corresponds to a non-trivial element of \( \pi_1 (F, f) \). The field configuration on a disk \( D^2 \) encircled by \( S^1 \) traces \( \mathcal{M} \), apart from points at finite radius where it goes off \( M \) (hence from \( \mathcal{M} \)).

\[\footnote{In other words, \( \tilde{G}^{(q)} \) is the subgroup of \( G_F \) i.e. transformations which can be “undone” by—or equivalent to—a local gauge transformation.}\]
Far from the vortex center the fields trace, along a circle, the gauge orbits \((f)\) regarded as a single point of \(\mathcal{M}\). Inside the circle, the fields provide a map from \(R^2(S^2)\) to \(\mathcal{M}\). In other words it represents an element of \(\pi_2(\mathcal{M}, p)\), where \(p\) is the gauge orbit containing \(f\), or

\[ p = \pi(f) \quad (3.7) \]

\(\pi\) is the projection of the fiber onto a point of the basis \(\mathcal{M}\). The exact sequence of homotopy groups for the fiber bundle reads

\[ \cdots \to \pi_2(M, f) \to \pi_2(\mathcal{M}, p) \to \pi_1(F, f) \to \pi_1(M, f) \to \pi_1(\mathcal{M}, p) \to \cdots (3.8) \]

where \(\pi_2(M/F, f) \sim \pi_2(\mathcal{M}, p)\). Note that in our application of such a sequence to the physical, vortex solutions, the reference point \(f\) or \(p\) appearing in the definition of the homotopy groups, corresponds to the field configurations along the large circle \(S^1\) encircling the given solution, see Fig. 2.

Given the points \(f, p\) and the space \(\mathcal{M}\), the vortex solution is still not unique. Any exact symmetry of the system (internal symmetry \(\tilde{G}^{(q)}\) as well as spacetime symmetries such as Poincaré invariance) broken by an individual vortex solution gives rise to vortex zero modes (moduli), \(V\). The vortex-center position moduli \(V \sim \mathbb{C}\), for instance, arise as a result of the breaking of the translation invariance in \(\mathbb{R}^2\). The breaking of the internal symmetry \(\tilde{G}^{(q)}\) (Eq. (3.5)) by the individual vortex solution gives rise to orientational zero-modes in the \(U(N)\) models extensively studied in last several years. See [37, 38, 39] for more recent results on this issue.

\[ \text{This notion can be made more precise by considering the strong-coupling limit of our systems where the gauge field equations reduce to the vacuum condition, see Eq. (3.9) below. In the presence of an ANO like sub-peaks, however, this correspondence leads to a singular lump, as will be seen in several examples below.} \]
Our main interest here, however, is the vortex moduli which arises from the non-trivial vacuum moduli \( \mathcal{M} \) itself. Due to the BPS nature of our vortices, the gauge field equation (see Eq. (A.3))

\[
F_{I12}^I = g_I^2 (q^I T^I q - \xi^I),
\]

reduces, in the strong-coupling limit (or in any case, sufficiently far from the vortex center), to the vacuum equation defining \( M \). This means that a vortex configuration can be approximately seen as a non-linear \( \sigma \)-model (NL\( \sigma \)M) lump with target space \( \mathcal{M} \) (for non-trivial element of \( \pi_2(\mathcal{M}) \)). Various distinct maps

\[
S^2 \mapsto \mathcal{M},
\]  

(3.10)
of the same homotopy class correspond to physically inequivalent solutions; each of these corresponds to a vortex with the equal tension

\[
T_{\text{min}} = -\xi^I \int d^2x F_{I12}^I > 0,
\]

(3.11)
because of their BPS nature. They thus represent non-trivial vortex moduli.

The semi-local vortices of the extended-Abelian Higgs (EAH) model reviewed in the previous section arise precisely this way. In the EAH model with \( N \) flavors of (scalar) electrons,

\[
M = S^{2N-1}, \quad F = S^1, \quad \mathcal{M} = S^{2N-1}/S = \mathbb{C}P^{N-1},
\]

(3.12)
and the homotopy sequence reads

\[
\cdots \rightarrow \pi_2 \left( S^{2N-1} \right) \rightarrow \pi_2 \left( \mathbb{C}P^{N-1} \right) \rightarrow \pi_1 \left( S^1 \right) \rightarrow \pi_1 \left( S^{2N-1} \right) \rightarrow \cdots
\]

(3.13)

\[
\begin{array}{c|c|c|c|}
\pi_2 \left( S^{2N-1} \right) & \pi_2 \left( \mathbb{C}P^{N-1} \right) & \pi_1 \left( S^1 \right) & \pi_1 \left( S^{2N-1} \right) \\
1 & \mathbb{Z} & \mathbb{Z} & 1 \\
\end{array}
\]

The usual argument tells us then that \( \pi_2(\mathbb{C}P^{N-1}) \) and \( \pi_1(S^1) \) are isomorphic: each (i.e. minimum) vortex solution corresponds to a minimal \( \sigma \)-model lump solution. As in this model the vacuum moduli \( \mathcal{M} \) is a (smooth) manifold, the above relations do not depend on the reference point \( f \) (or \( p \)).

In most cases discussed below, however, the base space \( \mathcal{M} \) will be various kinds of singular manifolds: a manifold with singularities. The nature of the singularity depends on the system and on the particular point(s) of \( \mathcal{M} \). Some of them are simple conic (orbifold) singularities, due to the fact that some discrete (e.g. \( \mathbb{Z}_N \)) symmetry is restored at that point. The fiber (the gauge
orbits) is smaller by some discrete quotient, with respect to $F$ at neighbouring points. Other
singularities at isolated points, or along some submanifold, reflect an even more drastic change of
$F$ such as a different unbroken gauge group at those points, as compared to that in surrounding
regular (or less singular) points of $\mathcal{M}$. The fiber itself goes through a discrete change in its
dimension and in the type, at or along the singularity(ies).

3.2 Classification of fractional vortices

There are basically two distinct causes or mechanisms leading to the appearance of multiple peaks
in the energy density even if the vortex under consideration has a minimum vorticity required by
the regularity and the topological stability. The first type is related to the presence of orbifold
singularities in $\mathcal{M}$. For example, let us consider a $\mathbb{Z}_2$ point $p_0$ such as the one appearing in a
simple $U(1)$ model with two scalars (Section 4.1). At this singularity, both elements of $\pi_2 (\mathcal{M}, p)$
and $\pi_1 (F, f)$ make a discontinuous change. The minimum element of $\pi_1 (F_0, f_0)$ is half of that of
$\pi_1 (F, f)$ defined off the singularity, and similarly for $\pi_2 (\mathcal{M}, p_0)$ with respect to $\pi_2 (\mathcal{M}, p)$, $p \neq p_0$.
Even though the exact sequence such as Eq. (3.8) continues to hold on and off the orbifold point,
the vortex defined near such a point will look like a doubly-wound vortex, with two centers (if
the vortex moduli parameters are chosen appropriately).

Another cause for the appearance of fractional peaks, which we call the second type, is best
understood by considering the strong coupling limit where the vortex reduces to a sigma-model
lump, as already noted. Even if the base point $p$ is a perfectly generic, regular point of $\mathcal{M}$, not
close to any singularity, the field configurations in the transverse plane ($S^2$) trace a 2-cycle in
the vacuum moduli space $\mathcal{M}$. The energy distribution reflects the nontrivial structure of $\mathcal{M}$ as
the volume of the target space is mapped into the transverse plane, $\mathbb{C}

\[ E = 2 \int_\mathbb{C} \frac{\partial^2 K}{\partial \phi^i \partial \phi^\dagger_j} \partial \phi^i \partial \phi^\dagger_j = 2 \int_\mathbb{C} \partial \partial K. \quad (3.14) \]

Let us consider the case in which a 2-cycle in the vacuum moduli space $\mathcal{M}$ (the target space
of the non-linear sigma model) is endowed with a deformed geometry, with regions of relatively
larger scalar curvature, and possibly with some singularities. When the field configuration sweeps
such regions, the energy density will show sub-peaks as illustrated in Fig. 3.

\footnote{The existence of the directions in the target space, which are not related to any isometry, is necessary for the
fractional lumps of the second type. Such directions are parametrized by so-called quasi-Nambu-Goldstone modes
in the context of supersymmetric theories \cite{40} while the directions of isometries correspond to Nambu-Goldstone
modes.}
The field configuration may also simply hit one of the singularities (conic or not), which could represent a sick point in the non-linear sigma model limit. Even at finite coupling, the vortex tension density will exhibit a similar substructure. The existence of the singularity is, however, not essential for the occurrence of fractional vortices of the second type, in contrast to the first type.

4 Models based on $\mathbb{C}P^1$

Several concrete models will be studied below. The fractional vortices appearing in these systems are caused by one or the other of the above mechanisms, or by collaboration of the two. The actual manifestation of these singularities could sometimes look quite complicated. We first discuss in this section models where the base space (vacuum moduli) is a $\mathbb{C}P^1$ with one or two singularities, or a smooth but deformed $\mathbb{C}P^1$.

4.1 Abelian Higgs model with two fields of different charges

– fractional vortex of the first type –

The first model is a simple extension of the Abelian-Higgs model with $N_f = 2$ flavors $H = (A, B)$ but with unequal charges. We assign the $U(1)$ charges ($\{m, n\}$) to the fields $A$ and $B$, respectively. The gauge transformations take the form,

$$H = (A, B) \rightarrow (e^{im(x)}A, e^{in(x)}B). \quad (4.1)$$
For simplicity, we assume that the charges are relatively prime, i.e., \( \text{g.c.d}\{m, n\} = 1 \). The vacuum manifold (\( D \)-flatness condition) is topologically equivalent to \( S^3 \) and the vacuum moduli are topologically the same as \( \mathbb{C}P^1 \) but with some conical singularities

\[
M = \{A, B \mid m|A|^2 + n|B|^2 = \xi\} ,
\]

\[
\mathcal{M} = M/U(1) \simeq W\mathbb{C}P_{(m,n)}^1 \simeq \mathbb{C}P^1/(\mathbb{Z}_m \times \mathbb{Z}_n) .
\]

The vacuum moduli can be also described by the following quotient

\[
(A, B) \sim (\lambda^m A, \lambda^n B) , \quad \lambda \in \mathbb{C}^* .
\]

Clearly, \( A = 0 \) is a \( \mathbb{Z}_n \) fixed point and \( B = 0 \) is a \( \mathbb{Z}_m \) fixed point. The \( U(1) \) gauge symmetry is broken at every point of the vacuum moduli, thus topologically stable vortices can appear.

Such a vortex solution is characterized by the broken \( U(1) \)-winding number \( \nu \) given in Eq. (4.4). The BPS energy density and mass are

\[
\mathcal{E} = -\xi F_{12} + \partial_i^2 J , \quad J = \frac{1}{2} HH^\dagger ,
\]

\[
T = \int dx^2 \mathcal{E} = 2\pi \xi \nu .
\]

By using the moduli matrix method in Appendix, it can be expressed as

\[
\nu = -\frac{1}{2\pi} \int dx^2 F_{12} = \frac{1}{\pi} \int dx^2 \partial \bar{\partial} \log |s|^2 ,
\]

\[
\bar{W} = -i \bar{\partial} \log s ,
\]

\[
H = (A, B) = (s^{-m} A_0(z) , \ s^{-n} B_0(z)) ,
\]

where \( \nu \) is a positive number, \( s \) is an everywhere non-zero function and \textit{the moduli matrices} \( A_0(z) \) and \( B_0(z) \) are polynomial functions of \( z \). The first equation determines the asymptotic behavior of \( s \) as

\[
|s|^2 \to |z|^{2\nu} \quad \text{as} \quad |z| \to \infty .
\]

We choose the boundary condition

\[
(A, B) \to (A_{\text{vev}} e^{im\nu \theta}, B_{\text{vev}} e^{in\nu \theta}) \in M \quad \text{as} \quad |z| \to \infty .
\]

The BPS equations (see Appendix) lead to the master equation

\[
\bar{\partial}\partial \log \omega = -\frac{\epsilon^2}{4} \left[ m \omega^{-m} |A_0|^2 + n \omega^{-n} |B_0|^2 - \xi \right] ,
\]

where \( \omega \equiv ss^\dagger \).
Before going into details, let us make a comment. If we fix \( A \equiv 0 \) (\( B \equiv 0 \)) everywhere, we can think of the system as just the Abelian-Higgs model with one complex scalar field \( B \) (\( A \)) whose \( U(1) \) charge is \( n \) (\( m \)). The vortices there are the normal ANO solutions, though the \( k \)-vortex solutions will have the \( U(1) \)-winding number \( k/n \) (\( k/m \)) with tension \( T_B = 2\pi \xi k/n \) (\( T_A = 2\pi \xi k/m \)). Indeed, when only one field is active while the other is inert, the \( U(1) \) gauge coupling constant and the FI term can be rescaled such that the system looks exactly as the standard Abelian-Higgs model with unit \( U(1) \) charge. What we are trying to study in this section is an intermediate situation between two kinds of vortices where both fields contribute non-trivially. Such intermediate states should have the energy \( T \equiv mT_A = nT_B \), and we shall see configurations which have \( m \) peaks in one limit and \( n \) peaks in another limit.

First we choose a generic point such as \( A_{\text{vev}} \neq 0 \) and \( B_{\text{vev}} \neq 0 \). The moduli matrices behave asymptotically as follows

\[
A_0(z) = s^m A \rightarrow |z|^{m\nu} e^{im\nu \theta} A_{\text{vev}} \quad , \quad B_0(z) = s^n B \rightarrow |z|^{n\nu} e^{in\nu \theta} B_{\text{vev}} \quad , \quad \text{as} \quad |z| \rightarrow \infty . \tag{4.13}
\]

Holomorphy of \( A_0, B_0 \) requires \( m\nu \in \mathbb{Z}_+ \) and \( n\nu \in \mathbb{Z}_+ \). As we have chosen \( m \) and \( n \) to be relatively prime, this is satisfied by \( \nu \equiv k \in \mathbb{Z}_+ \). Thus we have obtained the non-trivial condition for \( A_0, B_0 \)

\[
\nu = k : \quad A_0(z) = A_{\text{vev}} z^{mk} + \mathcal{O}(z^{mk-1}) \quad , \quad B_0(z) = B_{\text{vev}} z^{nk} + \mathcal{O}(z^{nk-1}) . \tag{4.14}
\]

Note that \( k \) vortices have \((m+n)k\) moduli parameters with the boundary vacuum modulus. They may correspond to positions and sizes of the fractional vortices.

When we choose the special point \( A_{\text{vev}} = 0 \) (\( \mathbb{Z}_m \) fixed point) or \( B_{\text{vev}} = 0 \) (\( \mathbb{Z}_n \) fixed point) as a boundary condition, the conditions for the moduli matrix drastically change. Say \( |A_{\text{vev}}| = \sqrt{\xi/m} \) and \( B_{\text{vev}} = 0 \). Immediately we get \( \nu = k/m \) and the conditions

\[
\nu = \frac{k}{m} : \quad A_0 = \sqrt{\frac{\xi}{m}} z^k + \cdots \quad , \quad B_0 = bz^\beta + \cdots , \tag{4.15}
\]

where \( \beta \) is a semi-positive definite integer less than \( n\nu = \frac{n}{m}k \). If we set \( B_0 = 0 \), the solution is identical to the ANO vortex as we mentioned before. When \( B_0 \) is not zero, the solutions significantly differ from the ANO solution and also from the semi-local vortices in EAH model. Similarly, if we choose \( |B_{\text{vev}}| = \sqrt{\xi/n} \) and \( A_{\text{vev}} = 0 \), the \( U(1) \) winding number becomes \( \nu = k/n \) and the conditions change as

\[
\nu = \frac{k}{n} : \quad A_0 = az^\alpha + \cdots \quad , \quad B_0 = \sqrt{\frac{\xi}{n}} z^k + \cdots , \tag{4.16}
\]
where $\alpha$ is a semi-positive definite integer less than $m\nu = \frac{m}{n}k$. Note that the $U(1)$ charge $\nu$ is fractionally quantized at the conical singularities. The present model thus nicely illustrates the first mechanism for the fractional vortices discussed in the previous section.

From this point on, we shall concentrate on the special concrete case $m = 2$ and $n = 1$, in order to illustrate in detail the properties of a fractional vortex. The vacuum moduli $\mathcal{M} = W\mathbb{C}P^1_{(2,1)} \simeq \mathbb{C}P^1/\mathbb{Z}_2$ has a $\mathbb{Z}_2$ conical singularity at $B_{\text{vev}} = 0$ (north pole), see Fig. 4. We consider the minimal-energy vortex configuration ($k = 1$). When we choose a generic point ($B_{\text{vev}} \neq 0$) as the boundary condition, the minimal configuration has $U(1)$ winding $\nu = 1$ whose energy is

$$T_{\text{min}} = 2\pi \xi , \quad (\nu = 1) .$$

The corresponding moduli matrix is given by

$$A_0(z) = A_{\text{vev}} z^2 + a_1 z + a_2 , \quad B_0(z) = B_{\text{vev}} z + b_1 , \quad a_1, a_2, b_1 \in \mathbb{C}^3 ,$$

with $2|A_{\text{vev}}|^2 + |B_{\text{vev}}|^2 = \xi$. Although this is the minimal-energy configuration, we have three complex moduli parameters $a_1, a_2, b_1$. Remember that $A$ ($B$) is zero at a point where $A_0$ ($B_0$) is zero. Note that $A_0$ has two zeros and $B_0$ has one zero because $A$ winds twice and $B$ winds once when we go around the boundary, $S^1$ at spatial infinity. An important observation is that the $U(1)$ gauge symmetry is not generally recovered at the zeros. Only when $A$ and $B$ vanish simultaneously, the $U(1)$ gauge symmetry is recovered (this would happen if some of the zeros of $A_0$ and $B_0$ are coincident).
Consider now the vortex at the special point of the vacuum moduli, $B_{vev} = 0$. The minimal configuration $k = 1$ corresponds to $\nu = 1/2$ and has a tension

$$T_{\text{min}}^{\text{special}} = \pi \xi, \quad (\nu = 1/2).$$

(4.19)

The moduli matrix takes the form (for $k = 1$)

$$A_0 = A_{vev} z + a, \quad B_0 = b, \quad A_{vev} = \sqrt{\xi/2},$$

(4.20)

where $a, b$ are the moduli parameters. Comparing this with Eq. (4.18) with $B_{vev} = 0$, one immediately sees that the latter is not a minimal-energy solution.

The vortex (energy, and magnetic flux) profiles can be approximately determined from the strong-coupling limit consideration. The gauge theory reduces to the non-linear sigma model whose target space is the vacuum moduli $\mathcal{M}$ in Eq. (4.3). The Kähler potential is given, in the supersymmetric version of our model, by

$$K = |A|^2 e^{-2V} + |B|^2 e^{-V} + \xi V.$$  (4.21)

Integrating out the $U(1)$ vector multiplet $V$, we get the following Kähler potential in terms of an inhomogeneous coordinate: $\varphi = 2\sqrt{\xi} A/B^2$

$$K = \xi \log f(\varphi, \bar{\varphi}) + \xi f^{-1}(\varphi, \bar{\varphi}), \quad f(\varphi, \bar{\varphi}) \equiv 1 + \sqrt{1 + 2|\varphi|^2}. \quad (4.22)$$

Note that the first term is due to the magnetic flux $F_{12}$ and the second term corresponds to the surface term $\partial_i J$ in Eq. (4.5). All the regular BPS solutions are analytically solved by

$$2\xi |s|^2 = |B_0|^2 + \sqrt{|B_0|^4 + 8\xi |A_0|^2} = |B_0|^2 f(\varphi, \bar{\varphi}), \quad \varphi = \varphi(z) = \frac{2\sqrt{\xi} A_0(z)}{B_0(z)^2}. \quad (4.23)$$

Only the solutions which have points where $A$ and $B$ simultaneously vanish cannot be seen in this limit, because the $U(1)$ gauge symmetry would remain unbroken there. Such solutions contain small lump singularities and we should go back to the original gauge theory in order to observe the configurations correctly. A numerical result is shown in Fig. 5.

As we move in the vacuum moduli space $\mathcal{M}$ by varying the VEVs $A_{vev}, B_{vev}$ (or $\varphi_{vev} \equiv 2\sqrt{\xi} A_{vev}/B_{vev}^2$) and change the vortex moduli parameters the tension density profile shows varying substructures. Since the zeros of the fields do not imply necessarily the restoration of a $U(1)$ gauge symmetry, the positions of the peaks do not always coincide with the zeros of $A, B$. Although it is very complicated to specify the positions of peaks analytically, it is easy to visualize it numerically. In Fig. 5 we have shown the zeros of $A, B$ and the peaks. We observe that there
Fig. 5: The energy (the left-most and the 2nd left panels) and the magnetic flux (the 2nd right panels) density are shown, together with the boundary values $(A, B)$ (the right-most panels) for the minimal lump of the first type in the strong gauge coupling limit. The moduli parameters are fixed as $a_1 = 0, a_2 = 1, b_1 = −1$ in Eq. (4.18). The red dots are zeros of $A$ and the black one is the zero of $B$. $\xi = 1$. The last figures illustrates the minimum lump defined at exactly the orbifold point (see Eq. (4.20)) with $A_{\text{vev}} = 1/\sqrt{2}$, and with $b = 0.8$. 
are no direct relations between the zeros of fields and the positions of the peaks, except at the two poles, \( A_{\text{vev}} = 0 \) (south pole) and \( B_{\text{vev}} = 0 \) (north pole), of the space \( \mathcal{M} \).

An axially symmetric peak appears at the zero \( z = z^S \) of \( B_0(z) \) in the limit \( A_{\text{vev}} \to 0 \); as \( A_{\text{vev}} \) departs from 0, it decomposes into two sub-peaks. We cannot remove one of the two sub-peaks pushing its position to infinity. This feature can be easily observed for large \( |\varphi_{\text{vev}}| \gg 1 \) where positions of the two peaks are naturally approximated by the zeros \( z = z_i^S \) (\( i = 1, 2 \)) of \( A_0(z) \).

\( A_{\text{vev}}(z_1^S + z_2^S) = -a_1, A_{\text{vev}}z_1^S z_2^S = a_2, B_{\text{vev}}z^N = -b_1. \) The energy density \( E = 2\partial \bar{\partial} K \) at those points is given by

\[
E|_{z = z_i^S} = 2\xi |\varphi_{\text{vev}}|^2 \frac{|z_1^S - z_2^S|^2}{|z_i^S - z^N|^4}. \tag{4.24}
\]

For instance, if the three zeros get separated by large distances, then we see that the sub-peaks are diluted. If, instead, only one of the zeros, \( z = z_2^N \) is pushed toward infinity, that is \( |z_2^N - z^S|, |z_1^N - z_2^N| \gg |z_1^N - z^S| \), the peaks at \( z = z_1^N \) becomes singular. In either case, the isolated one peak is not allowed as a vortex(lump) solution. This solution consisting the two sub-peaks is one of typical examples of fractional vortices. Only when \( B_{\text{vev}} = 0 \), they become independent.

Such a limiting configuration is no longer a minimal energy configuration, however. The minimal configuration at exactly \( B_{\text{vev}} = 0 \) (with (4.20)) has only one peak. Its tension is half of the minimal configuration for \( B_{\text{vev}} \neq 0 \).

The reason why the minimum vortex at \( B_{\text{vev}} \neq 0 \) must have twice the energy with respect to the minimal object at \( B_{\text{vev}} = 0 \) is as follows. Our vacuum moduli has a \( \mathbb{Z}_2 \) singularity at \( |B_{\text{vev}}| = 0 \). If the vacuum is chosen at \( B_{\text{vev}} \neq 0 \) the solution touches the singularity at a finite point in the \( z \)-plane and would get singular there. To remove such a singularity, the solution must wrap twice around the vacuum moduli. On the other hand, if one is at exactly the \( \mathbb{Z}_2 \) point the solution never touches it and a regular solution can be constructed with just a single winding.

As discussed in Section 3, these characteristics of the vortex-energy profile are thus deeply rooted in the property of the vacuum moduli \( \mathcal{M} \) itself and to its singularity structure. In understanding the qualitative features of the vortex tension distribution and their dependence on \( p = \varphi_{\text{vev}} = 2\sqrt{\xi} A_{\text{vev}}/B_{\text{vev}}^2 \) just described, the crucial fact is that the elements of the homotopy groups corresponding to regular configurations make a discontinuous jump at \( p = \varphi = \infty \) (\( B_{\text{vev}} = 0 \)). In fact

\[
\frac{\pi_2(\mathcal{M}, p)}{\pi_2(\mathcal{M}, \infty)} = \mathbb{Z}_2, \quad \frac{\pi_1(F, f)}{\pi_1(F, f_0)} = \mathbb{Z}_2, \quad p \neq \infty. \tag{4.25}
\]
The $S^1$ fiber itself reduces to half at the orbifold singularity

\[
f = \pi^{-1}(p) = S^1, \quad f_0 = \pi^{-1}(\infty) = S^1/\mathbb{Z}_2. \tag{4.26}
\]

Thus even though $\pi_2(M) = 1$ and $\pi_1(M) = 1$, just as in the case of the EAH model Eq. (3.13), each minimum element of $\pi_2(\mathcal{M}, p)$ is a double cover of the minimum element of $\pi_2(\mathcal{M}, \infty)$, just as the fiber at the generic $p$ ($S^1$: $\alpha = 0 \to 2\pi$ in Eq. (4.1)) is a double cover of the fiber at $B = 0$ ($\alpha = 0 \to \pi$ in Eq. (4.1)). This is the (global) reason for the double peaks observed in Fig. 5.

The argument here can be easily extended to more general cases with the multiple flavors $H = (A, B, C, D, \ldots)$ with generic $U(1)$ charges $Q = (m, n, o, q, \ldots)$, which are all relatively prime. The moduli manifold is then $\mathcal{M} = \mathbb{C}P^{N_f-1}_{(n, m, \cdots)} \simeq \mathbb{C}P^{N_f-1}/(\mathbb{Z}_m \times \mathbb{Z}_n \times \cdots)$. Near a $\mathbb{Z}_m$ singular point, $(|A_{\text{vev}}|, |B_{\text{vev}}|, |C_{\text{vev}}|, \cdots) = (\sqrt{\xi/m}, 0, 0, \cdots, m)$ peaks appear in the energy distribution.

### 4.2 An Abelian $U(1) \times U(1)$ Higgs model

**fractional vortex of the second type**

The next system we consider, which has the same target space as in the previous model, is a $U(1)_1 \times U(1)_2$ gauge theory with three flavors of scalar electrons $H = (A, B, C)$ with charges $Q_1 = (2, 1, 1)$ for $U(1)_1$ and $Q_2 = (0, 1, -1)$ for $U(1)_2$. The gauge transformations act as

\[
(A, B, C) \to (e^{i2\alpha(x)} A, e^{i\alpha(x)+i\beta(x)} B, e^{i\alpha(x)-i\beta(x)} C). \tag{4.27}
\]

Note that the transformation $(\alpha, \beta) = (\pi, \pm\pi)$ leaves the fields invariant: the true gauge group is

\[
[U(1)_1 \times U(1)_2]/\mathbb{Z}_2.
\]

Some details about this model are given in the Appendix. The vacuum manifold and vacuum moduli space are

\[
\mathcal{M} = \{A, B, C \mid 2|A|^2 + |B|^2 + |C|^2 = \xi_1, |B|^2 - |C|^2 = \xi_2\}, \tag{4.28}
\]

\[
\mathcal{M} = \mathcal{M}/[U(1)_1 \times U(1)_2]/\mathbb{Z}_2. \tag{4.29}
\]

Here $\xi_1$ is the FI term for the first $U(1)_1$ and $\xi_2$ is that for the second $U(1)_2$. Note that the existence of the supersymmetric vacuum requires $\xi_1 \geq |\xi_2|$. Below we shall mainly be interested in the case of $\xi_2 = 0$ (in the Appendix the case with a non-vanishing $\xi_2$ is also discussed).

When $\xi_2 = 0$, the vacuum manifold is the same as one in Section 4.1. In fact, the Kähler potential of the vacuum manifold is the same as Eq. (4.22) with replacements $\xi \to \xi_1$ and
\[ \varphi = 2\sqrt{2} \xi A/B^2 \rightarrow 2\sqrt{2} \xi A/(BC). \] Although the vacuum moduli manifolds are the same, there is an important difference. The singular point in the previous section was a \( \mathbb{Z}_2 \) conical (orbifold) singularity whereas the singular point \( |B| = |C| = 0 \) here represents a theory with a restored \( U(1)_2 \) gauge symmetry, i.e., in a Coulomb phase. Since the \( \mathbb{Z}_2 \) action has been modded out from the beginning, the singular point is not a \( \mathbb{Z}_2 \) fixed point and therefore can be smeared by the introduction of non-vanishing \( \xi_2 \) as illustrated in Fig. 6. (See however the discussions below.)

Fig. 6: A sketch of the vacuum moduli space.

BPS vortex solutions can be treated by the moduli matrix formalism. The tension is determined by the \( U(1)_1 \) winding number \( \nu_1 \) (and the \( U(1)_2 \) winding number \( \nu_2 \) if \( \xi_2 \neq 0 \)) as

\[
E = -\xi_1 F^{(1)}_{12} - \xi_2 F^{(2)}_{12} + \partial_i^2 J, \quad J = \frac{1}{2} HH^\dagger, \tag{4.30}
\]

\[
T = \int dx^2 E = 2\pi \xi_1 \nu_1 + 2\pi \xi_2 \nu_2, \tag{4.31}
\]

In the moduli-matrix formalism, the winding number can be expressed as

\[
\nu_I = -\frac{1}{2\pi} \int dx^2 F^{(I)}_{12} = \frac{1}{\pi} \int dx^2 \bar{\partial} \log |s_I|^2, \quad \bar{W}_I = -i\bar{\partial} \log s_I, \tag{4.32}
\]

\[
H = (A, B, C) = (s_1^{-2} A_0(z), \ s_1^{-1} s_2^{-1} B_0(z), \ s_1^{-1} s_2 C_0(z)), \tag{4.33}
\]

where \( s_1 \) and \( s_2 \) are everywhere non-zero functions and the holomorphic functions \( A_0, B_0, C_0 \) are the elements of the moduli matrix. The BPS equation for the gauge fields reduces to the master equations

\[
\bar{\partial} \partial \log \omega_1 = -\frac{e^2}{4} [\omega_1^{-1} (2\omega_1^{-1}|A_0|^2 + \omega_2^{-1}|B_0|^2 + \omega_2|C_0|^2) - \xi_1], \tag{4.34}
\]

\[
\bar{\partial} \partial \log \omega_2 = -\frac{g^2}{4} [\omega_2^{-1} (\omega_2^{-1}|B_0|^2 - \omega_2|C_0|^2) - \xi_2], \tag{4.35}
\]

where \( \omega_i \equiv s_i s_i^\dagger, \ i = 1, 2 \).
To specify a solution, we need to choose the winding numbers $\nu_1$ and $\nu_2$. The condition for $\nu_1$ and $\nu_2$ depends on the boundary condition $(A, B, C) \to (A_{\text{vev}}, B_{\text{vev}}, C_{\text{vev}}) \in M$ as $|z| \to \infty$. So we first need to determine $(A_{\text{vev}}, B_{\text{vev}}, C_{\text{vev}})$, (see the Appendix). It should be noted that, when $\xi_2 = 0$, although the magnetic flux $F_{12}^{(2)}$ does not contribute to the tension it contributes to the tension density through the surface term, $J$ of Eq. (4.30). In other words $\nu_2$ as well as $\nu_1$ is needed to determine a solution. $F_{12}^{(2)}$ is non-trivial even when $\xi_2 = 0$.

Let us concentrate on the minimal energy configuration viz. $\xi_2 = 0$ in the following. We first choose a generic point $(|B_{\text{vev}}| = |C_{\text{vev}}| \neq 0)$ as the boundary condition. The minimal configuration is given by two different choices of the winding numbers (the generic tension formula is given in Eq. (A.19))

$$T_{\text{min}} = \pi \xi_1, \quad \text{with} \quad (\nu_1, \nu_2) = (1/2, \pm 1/2). \quad (4.36)$$

The corresponding moduli matrix is given by

$$A_0(z) = A_{\text{vev}} z + a, \quad B_0(z) = B_{\text{vev}} z + b, \quad C_0(z) = C_{\text{vev}}, \quad \text{for} \quad (\nu_1, \nu_2) = (1/2, 1/2), \quad (4.37)$$

$$A_0(z) = A_{\text{vev}} z + a, \quad B_0(z) = B_{\text{vev}}, \quad C_0(z) = C_{\text{vev}} z + c, \quad \text{for} \quad (\nu_1, \nu_2) = (1/2, -1/2).$$

The two complex parameters $(a, b)$ represent the moduli parameters.

At exactly $B_{\text{vev}} = C_{\text{vev}} = 0$ the $U(1)_2$ gauge symmetry is restored at infinity: the system is in a Coulomb phase. We shall not discuss this case: it is beyond the reach of the moduli-matrix formalism. On the contrary, at an orbifold singularity such as those considered in the previous subsection the system remains in a Higgs phase even though with a different property from the surrounding vacua.

Numerical solutions with various choices of $A_{\text{vev}}, B_{\text{vev}}$ and $C_{\text{vev}}$ are shown in Fig. 7. When a point $|B_{\text{vev}}| = |C_{\text{vev}}| \simeq 0$ is chosen as the boundary condition, one peak is observed near $z_a = -a/A_{\text{vev}}$ where $A(z_a) = 0$. On the other hand, another single peak appears near $z_b = -b/B_{\text{vev}}$ where $B(z_b) = 0$, when we choose $|A_{\text{vev}}| = 0$ as the boundary condition. These two peaks are smeared in the intermediate values of $(A_{\text{vev}}, B_{\text{vev}}, C_{\text{vev}})$. Although the vacuum manifold is the same as in Section 4.1, the vortex solutions are significantly different. The difference can be clearly seen in the winding numbers and tensions in Eqs. (4.17) and (4.36). In the model of Section 4.1 the $\mathbb{Z}_2$ quotient requires us to choose $\nu = 1$ except for at the $\mathbb{Z}_2$ fixed point. However, exactly in that point there are no such discrete symmetries, so there exist solutions with $\nu_1 = 1/2$. On the contrary, in the model of this section, we cannot choose the singular point (in the Coulomb phase) as the boundary condition.

In order to understand better the characteristics of the vortex energy profile in this model,
we study the underlying sigma model, of which the Kähler potential with non-vanishing $\xi_2$ is given by

$$K = \xi_1 f_\lambda(\varphi, \bar{\varphi})^{-1} + \xi_1 \log f_\lambda(\varphi, \bar{\varphi}) - \frac{\xi_1 - \xi_2}{2} \log(1 - \lambda f_\lambda(\varphi, \bar{\varphi})),$$

where $f_\lambda(\varphi, \bar{\varphi})$ is a function of an inhomogeneous coordinate $\varphi = \sqrt{\xi_1 A}$. By setting $\xi_2 = 0$ ($\lambda = 0$), we find the same Kähler potential as in Eq. (4.22). Inside the disk of an arbitrary large radius the vortices reduce to sigma-model lumps,

$$\varphi = \varphi(z) \equiv \frac{\sqrt{\xi_1 A_0(z)}}{B_0(z)C_0(z)},$$

characterized by $\pi_2(\mathcal{M})$. There the minimal lump solution has a tension $T_{\text{lump}} = \pi(\xi_1 - |\xi_2|)$. In the strong coupling limit giving rise to the above non-linear sigma model, solutions for $s_1$ and $s_2$ are explicitly given by

$$|s_1|^2 = \frac{f_\lambda(\varphi, \bar{\varphi}) |B_0C_0|}{\sqrt{1 - \lambda f_\lambda(\varphi, \bar{\varphi})}} \frac{1}{\xi_1}, \quad |s_2|^2 = \sqrt{1 - \lambda f_\lambda(\varphi, \bar{\varphi})} \frac{|B_0|}{|C_0|}.$$
Since the vortex-lump solutions in the sigma-model limit, suffer from the Coulomb singularity, we introduce a small $\xi_2$ to regularize such a singularity. For instance, the analytic solution $|s_2|^2 = |B_0|/|C_0|$ in the sigma model limit ($\xi_2 = 0$) leads to the singular magnetic flux for $U(1)_2$ since $F^{(2)}_{12} = -2\partial\bar{\partial}\log|s_2|^2 = -\partial\bar{\partial}\log|z|^2 = -\pi\delta^{(2)}(z)$. The price we have to pay is that the two degenerate configurations (Eq. (4.37)) are now split. Suppose $-\xi_1 < \xi_2 < 0$, then the energy changes as

\[
T_{(1/2,1/2)} = \pi(\xi_1 + \xi_2) = T_{\text{lump}} < T_{(1/2,-1/2)} = \pi(\xi_1 - \xi_2),
\]

although the lump solution cannot distinguish the two. The difference $2\pi|\xi_2|$ between the two is caused by the existence of an ANO-vortex-like singular peak attached on a tip of the lump solution for $(\nu_1, \nu_2) = (1/2, -1/2)$. Actually, the solution for $|s_2|^2$ with $\xi_2 < 0$ is still singular at the zero of $C_0(z)$ since no configuration with $C = 0$ can satisfy the $D$ term condition for $\xi_2 < 0$. Therefore, we consider only the vortex solution with $(\nu_1, \nu_2) = (1/2, 1/2)$ in Eq. (4.37) as a smooth lump solution. The vortex configurations in a model with $\xi_2$ in Fig. 8 (strong gauge coupling) are quite similar to the ones in the model with $\xi_2 = 0$, Fig. 7 (finite gauge coupling). With $\varphi_{\text{vev}} \neq 0, \infty$, we again observe two peaks. The positions of the two are estimated by zeros of $A_0(z)$ and $B_0(z)$, $z_a = -a/A_{\text{vev}}$, $z_b = -b/B_{\text{vev}}$, which are mapped onto the two different poles of the vacuum moduli space $\mathcal{M}$. Note that the two peaks for the fractional vortices in the previous subsection are mapped onto the same pole (the south pole $\varphi_{\text{vev}} = 0$). The energy density $E = 2\partial\bar{\partial}K$ at those zeros is

\[
E|_{z=z_a} = \frac{\xi_1(1-\lambda^2)}{2} \frac{|\varphi_{\text{vev}}|^2}{|z_a-z_b|^2}, \quad E|_{z=z_b} = \frac{1 - |\lambda|}{|\lambda||\varphi_{\text{vev}}|^2} \frac{1}{|z_a-z_b|^2}.
\]

(4.42)

From this observation we see that the energy density for the fractional vortices gets diluted as $|z_a-z_b|$ is increased.

In spite of the similar structure of $\mathcal{M}$, the difference in the gauge group, matter content and in the fiber, manifest themselves in distinct ways that the vacuum manifold is covered by the vortex-sigma model lump solution here, as compared to the previous model. A comparison of Eqs. (4.17) and (4.36) shows that the minimum lump solution in a generic point of $\mathcal{M}$ here clearly is seen to be half of the corresponding lump element $\pi_2(\mathcal{M})$ in Section 4.1 (on the $\mathbb{Z}_2$ singularity of the latter the solutions are similar). In the present model there is no jump in the homotopy-group elements at the singularity of $\mathcal{M}$. A sub-peak appears simply because the target space $\mathcal{M}$ which is a distorted sphere is warped. If the target space were the standard $\mathbb{C}P^1$ sigma model (a perfect sphere) only one peak would have appeared.

Just as the model considered in Section 4.1 showed a good example of the first mechanism for
Fig. 8: The energy density (left-most) and the magnetic flux density $F_{12}^{(1)}$ (2nd from the left), $F_{12}^{(2)}$ (2nd from the right) and the boundary condition (right-most) for the lump of the second type in the strong gauge coupling limit with $a = -1$ and $b = 1$ in Eq. (4.37), $\xi_1 = 2$ and $\xi_2 = -0.1$.

the vortex substructures discussed in Section 3.2, the present model nicely illustrates the second mechanism for the fractional vortex.

### 4.3 A model with $U(1) \times SU(2)$

The third and last example of a model with the target space of the droplet type is a gauge theory with a $U(1) \times SU(2)$ gauge group with Higgs fields $H = (A, B)$

|     | $U(1)$ | $SU(2)$ |
|-----|--------|---------|
| $A$ | 2      | 1       |
| $B$ | 1      | 1       |
namely, a complex scalar filed $A$ of $U(1)$ charge 2, and two complex scalars $B = (\vec{B}_1, \vec{B}_2)$ in the fundamental representation of the $SU(2)$ group, and with the Abelian charge 1, while $A$ is a singlet. The latter is conveniently denoted by a color-flavor mixed $2 \times 2$ matrix, $B$. There is furthermore a global symmetry $SU(2)_f$. The gauge group acts on the fields as

$$(A, B) \to (e^{2i\alpha} A, e^{i\alpha} g B), \quad e^{i\alpha} \in U(1), \quad g \in SU(2).$$

(4.43)

Note that the transformation $(\alpha, g) = (\pm \pi, -\frac{1}{2})$ is a symmetry, thus the gauge group is really $[U(1) \times SU(2)]/\mathbb{Z}_2 \simeq U(2)$.

The vacuum manifold is given by the $D$-flatness condition

$$M = \{(A, B) \mid 2|A|^2 1_2 + 2BB^\dagger = \xi 1_2\}. \quad (4.44)$$

By using $SU(2)$ gauge and flavor symmetries, the vacuum configuration for $B$ fields can be taken in the form $B = B(1)_2$, showing that the vacuum manifold $M$ is an $S^3$ defined by $|A|^2 + |B|^2 = \xi/2$. The vacuum moduli space $\mathcal{M}$ is a $U(1)$ quotient of this $S^3$ with the weighted charges given above, topologically the same as $\mathbb{C}P^1$. At a generic point ($B \neq 0$) of the vacuum moduli, the flavor symmetry $SU(2)_f$ is broken but the color-flavor diagonal symmetry $SU(2)_{c+f}$ is preserved. At the special point $B = 0$, neither the $SU(2)$ gauge nor the flavor symmetry is broken. To be precise, we can compute the Kähler metric on the vacuum moduli space by eliminating all the gauge multiplets from the Kähler potential

$$K = |A|^2 e^{-2V} + \text{Tr} \left( BB^\dagger e^{-V-V'} \right) + \xi V, \quad (4.45)$$

with $V$ being a $U(1)$ vector multiplet and $V'$ being an $SU(2)$ vector multiplet. Once we eliminate $V'$ from the Lagrangian, we obtain

$$K = |A|^2 e^{-2V} + \left| (2 \det B)^\frac{1}{2} \right|^2 e^{-V} + \xi V. \quad (4.46)$$

Comparing this with Eq. (4.21), it is easy to see that the two spaces have the same metric, if we replace the inhomogeneous coordinate $\varphi = 2A/B^2 \to A/\det B$. Thus the vacuum moduli space is of the droplet form as in Fig. 3 but there is no conical singularity. Instead, the $SU(2)$ gauge symmetry is unbroken at the tip of the droplet ($B = 0$).

Construction of the BPS vortex solutions in this model is summarized in the Appendix in detail. The energy formulae are the same as Eqs. (4.5) and (4.6) with $J = |A|^2/2 + \text{Tr} (BB^\dagger)/2$ and the solutions are determined by $U(1)$ winding number $\nu$

$$\nu = -\frac{1}{2\pi} \int dx^2 F_{12} = \frac{1}{\pi} \int dx^2 \partial \bar{\partial} \log |s|^2, \quad (4.47)$$

$$W_{U(1)} = -i\bar{\partial} \log s, \quad W_{SU(2)} = -iS^{-1} \bar{\partial} S', \quad (4.48)$$

$$H = (A, B) = \left(s^{-2} A_0(z), s^{-1} S^{-1} B_0(z)\right), \quad (4.49)$$
where \( s \in \mathbb{C}^* \) and \( S' \in SL(2, \mathbb{C}) \) and \( A_0 \) and all the elements of \( B_0(z) \) must be holomorphic functions of \( z \). The BPS equations for the gauge the gauge fields lead to the master equations

\[
\ddbar \log \omega = -\frac{e^2}{8} \left( \frac{1}{\omega} \text{Tr} \left( B_0 B_0^\dagger \Omega'^{-1} \right) + \frac{2}{\omega^2} |A_0|^2 - \xi \right),
\]

\[
\ddbar \left( \Omega' \partial \Omega'^{-1} \right) = \frac{g^2}{4\omega} \left( B_0 B_0^\dagger \Omega'^{-1} - \frac{1}{2} \text{Tr} \left( B_0 B_0^\dagger \Omega'^{-1} \right) \right),
\]

(4.50)

where \( \omega \equiv ss^\dagger \) and \( \Omega' \equiv S'S'^\dagger \).

The first equation determines the asymptotic behavior of \(|s|^2 \sim |z|^{2\nu} \) as \(|z| \to \infty \). The choice of \( \nu \) must be consistent with the given boundary condition \((A, B) \to (A_{\text{vev}}, B_{\text{vev}} 1_2)\) satisfying \(|A_{\text{vev}}|^2 + |B_{\text{vev}}|^2 = \xi/2\). To find consistent solutions, it is useful to consider the following \( SU(2) \) gauge invariant \( I = \det B = s^{-2} \det B_0(z) \). When a generic point \((B_{\text{vev}} \neq 0)\) is chosen, holomorphy of \( A_0(z) \) and \( \det B_0(z) \) requires \( 2\nu \in \mathbb{Z}_+ \) because of the asymptotic behavior

\[
A_0 \sim A_{\text{vev}} z^{2\nu} + \cdots, \quad \det B_0 \sim B_{\text{vev}}^2 z^{2\nu} + \cdots, \quad \text{as} \quad |z| \to \infty.
\]

(4.51)

In this way the vortices in this system are characterized by the half quantized \( U(1) \) winding number \( \nu \in \mathbb{Z}_+/2 \).

The minimal configuration with tension \( T = \pi \xi \) is described by the moduli matrices

\[
A_0 = A_{\text{vev}} z + a, \quad \det B_0 = B_{\text{vev}}^2 (z + b).
\]

(4.52)

Note that the matrix \( B_0(z) \) is not uniquely determined by these conditions. The simplest one is \( B_0 = B_{\text{vev}} \text{diag}(z + b, 1) \). This matrix breaks the color-flavor symmetry \( SU(2)_{c+f} \) of the vacuum into \( U(1)_{c+f} \). Henceforth, the generic configurations are generated by \( SU(2)_{c+f} \), so that the vortex has an internal orientation \( \mathbb{C}P^1 \simeq SU(2)_{c+f}/U(1)_{c+f} \) (Nambu-Goldstone mode). The moduli space of the single vortex is

\[
\mathcal{V} = \mathbb{C} \times \mathcal{V}^{U(2)}_{k=1},
\]

\[
\mathcal{V}^{U(2)}_{k=1} = \mathbb{C} \times \mathbb{C}P^1,
\]

(4.53)

which is a product of the orientational zero-modes \((b)\) and a center of mass and a “size” parameter.

When we choose \( A_{\text{vev}} = 0 \) as the boundary condition, the vortex is a semi-local extension of the so-called \( U(2) \) local vortex. In fact, if we turn off the size moduli, viz. \( a = 0 \), it is precisely the \( U(2) \) local vortex. The transverse width of the vortex becomes large as \(|a|\) increases while the orientational moduli are still localized near the vortex core. This is completely different from the so-called non-Abelian semi-local vortex in \( U(N_c) \) gauge theories with \( N_f > N_c \) Higgs fields in the fundamental representation.
We see that the global features of the vortices of this model are similar to that of the $U(1) \times U(1)$ model (except, of course, for the internal, orientational modes which are present here). The key fact is that the non-Abelian sector of this model (with the matter fields $B$) is actually a $U(2) \sim SU(2) \times U(1)/\mathbb{Z}_2$ theory. Taking into account the $U(1)$ charge of $A$, we see that the fiber is generated by $\pi_1(S^1/\mathbb{Z}_2)$, on and off the orbifold singularities ($B = 0$).

The system at $B_{\text{vev}} = 0$ is in a (non-Abelian) Coulomb phase classically and no vortex solutions exist. In the $U(1)^2$ model of Section 4.2 we avoided the problem by turning on the FI parameter for the second $U(1)$ factor. The same cannot be done here, as no FI-like term exists for a non-Abelian gauge group. Therefore there exist neither a smooth non-linear sigma-model limit nor any regular lump solutions. Such solutions would necessarily be singular.

### 4.4 An alternative $U(1)^2$ model: the lemon space

The model of Section 4.2 was chosen to have a minimal fiber $F$ with the fewest possible fields. However, the energy density (in the lump limit) and the Kähler metric turned out to be rather elaborate. In this subsection, we analyze a related model, with a slightly different charge assignment, $Q_1 = (1, 1, 1)$ for $U(1)_1$ and $Q_2 = (0, 1, -1)$ for $U(1)_2$ as

$$(A, B, C) \rightarrow (e^{i\alpha(x)}A, e^{i\alpha(x)+i\beta(x)}B, e^{i\alpha(x)-i\beta(x)}C) .$$

An important difference from Section 4.2 is that the gauge symmetry is now $U(1)_1 \times U(1)_2$, without a $\mathbb{Z}_2$ division. The points $(A, B, C)$ and $(-A, B, C)$ related by $(\alpha, \beta) = (\pi, \pm \pi) \in \mathbb{Z}_2$, are distinct points. The vacuum manifold and vacuum moduli space are given by

$$M = \{A, B, C \mid |A|^2 + |B|^2 + |C|^2 = \xi_1, |B|^2 - |C|^2 = \xi_2\} ,$$

$$\mathcal{M} = M / (U(1)_1 \times U(1)_2) .$$

We see that $A = 0$ is a $\mathbb{Z}_2$ orbifold point, whereas the point $B = C = 0$ represents a system in Coulomb phase (which can be Higgsed and regularized by $\xi_2 \neq 0$). See Fig. 9. Clearly this model shares aspects both of the simple $U(1)$ model of Section 4.1 and of the $U(1) \times U(1)$ model of Section 4.2. In the following we shall consider mainly the case of $\xi_2 = 0$, except when we consider the sigma-model limit, which is well defined only for a non-vanishing $\xi_2$.

The vortex Ansatz is

$$(A, B, C) = \left(s_1^{-1}A_0(z), s_1^{-1}s_2^{-1}B_0(z), s_1^{-1}s_2C_0(z)\right) ,$$

(4.57)
with the gauge field equations

\[ \bar{\partial} \partial \log \omega_1 = -\frac{e^2}{4} \left[ \omega_1^{-1} (|A_0|^2 + \omega_2^{-1}|B_0|^2 + \omega_2|C_0|^2) - \xi_1 \right], \tag{4.58} \]

\[ \bar{\partial} \partial \log \omega_2 = -\frac{g^2}{4} \left[ \omega_1^{-1} (\omega_2^{-1}|B_0|^2 - \omega_2|C_0|^2) - \xi_2 \right], \tag{4.59} \]

where \( \omega_i \equiv s_i s_i^\dagger \), for \( i = 1, 2 \). In order to avoid repetition, all the details are summarized in the Appendix. As in the case of Section 4.2, the winding numbers are \( \nu_1 \) and \( \nu_2 \) for \( U(1)_1 \) and \( U(1)_2 \), respectively. The tension depends only on \( \nu_1 \) for \( \xi_2 = 0 \):

\[ T = 2\pi \xi_1 \nu_1 , \quad \nu_1 \in \mathbb{Z}_+ . \tag{4.60} \]

The minimal-energy solutions with the generic boundary condition \( (0 < |A_{\text{vev}}|^2 < \xi_1) \) have \( T = 2\pi \xi_1 \) and are obtained by the following three different moduli matrices

\[ A_0 = A_{\text{vev}} z + a , \quad B_0 = B_{\text{vev}} , \quad C_0 = C_{\text{vev}} z^2 + c_1 z + c_2 , \quad (\nu_1, \nu_2) = (1, -1) , \tag{4.61} \]

\[ A_0 = A_{\text{vev}} z + a , \quad B_0 = B_{\text{vev}} z + b , \quad C_0 = C_{\text{vev}} z + c , \quad (\nu_1, \nu_2) = (1, 0) , \tag{4.62} \]

\[ A_0 = A_{\text{vev}} z + a , \quad B_0 = B_{\text{vev}} z^2 + b_1 z + b_2 , \quad C_0 = C_{\text{vev}} , \quad (\nu_1, \nu_2) = (1, 1) . \tag{4.63} \]

As they obey different boundary condition for \( \nu_2 \), they belong to different topological sectors. Each configuration has three moduli parameters.

Near the \( \mathbb{Z}_2 \) orbifold point we observe two peaks. Although the energy density always looks the same, the magnetic fluxes, especially of the second \( U(1)_2 \), depends on the value of \( \nu_2 \). In Fig. 10 we show several numerical solutions for Eq. (4.63). We also show a couple of solutions for Eq. (4.62) in Fig. 11. In almost all regions, the configuration consists of one peak or two peaks but sometimes we observe three peaks simultaneously.
Fig. 10: The energy density (left-most) and the magnetic flux density $F_{12}^{(1)}$ (2nd from the left), $F_{12}^{(2)}$ (2nd from the right) and the boundary condition (right-most) for Eq. (4.63) with $\xi_1 = 1$ and $\xi_2 = 0$ and $e_1 = 1, e_2 = 2.$
Fig. 11: The energy density (left-most) and the magnetic flux density $F^{(1)}_{12}$ (2nd from the left), $F^{(2)}_{12}$ (2nd from the right) and the boundary condition (right-most) for Eq. (4.62) with $\xi_1 = 1$ and $\xi_2 = 0$ and $e_1 = 1, e_2 = 2$.

On the other hand, at exactly a singular vacuum $A_{vev} = 0$ (the singular point on $\mathcal{M}$), the minimal vortex with tension $T = \pi \xi_1$ is given by

$$A_0 = a, \quad B_0 = B_{vev} z + b, \quad C_0 = C_{vev}, \quad (\nu_1, \nu_2) = (1/2, 1/2), \quad (4.64)$$

$$A_0 = a, \quad B_0 = B_{vev}, \quad C_0 = C_{vev} z + c, \quad (\nu_1, \nu_2) = (1/2, -1/2). \quad (4.65)$$

At $A = 0$ ($\varphi = 0$) a $\mathbb{Z}_2$ symmetry remains unbroken which is a typical orbifold singularity. As a result, the $U(1)_1$ fiber $F$ is the half ($\alpha = 0 \rightarrow \pi$) at the orbifold point as compared to that in
other points of the vacuum moduli, where $\alpha = 0 \rightarrow 2\pi$. The global structure of the vortex-sigma model lumps in this model is thus somewhat similar to the model of Section 4.1. At the $\mathbb{Z}_2$ orbifold singularity $\pi_1(F)$ and $\pi_2(M)$ make a jump, and this explains the appearance of the double peaks.

As in the model in Section 4.2 we cannot take $B_{\text{vev}} = C_{\text{vev}} = 0$ as a boundary condition since the second $U(1)_2$ is unbroken at infinity.

These aspects can be made more explicit in the strong gauge coupling limit $e_1, e_2 \rightarrow \infty$, where the Kähler potential is simpler (than in the model of Section 4.2) by construction and the solutions can be analytically solved. Since the Coulomb phase leads to singular solutions, we here turn on the another FI parameter $\xi_2$ ($|\xi_2| < \xi_1$) for $U(1)_2$. Working in a supersymmetric context, elimination of the gauge superfields $V_1$ and $V_2$ from

$$K = |A|^2 e^{-V_1} + |B|^2 e^{-V_1-V_2} + |C|^2 e^{-V_1+V_2} + \xi_1 V_1 + \xi_2 V_2 ,$$

yields the Kähler potential

$$K = \xi_1 \log \left( 1 + \sqrt{\lambda^2 + (1-\lambda^2)}|\tilde{\varphi}|^2 \right) - |\xi_2| \log \left( |\lambda| + \sqrt{\lambda^2 + (1-\lambda^2)}|\tilde{\varphi}|^2 \right) ,$$

where the inhomogeneous coordinate $\tilde{\varphi}$ and $\lambda$ are defined by $\tilde{\varphi} \equiv \frac{2BC}{A}$ and $\lambda \equiv \frac{\xi_2}{\xi_1}$. The BPS solutions are given by the holomorphic functions

$$\tilde{\varphi}(z) = \frac{2B_0(z)C_0(z)}{A_0^2(z)},$$

and characterized by the quantized tension

$$T = 2\pi \sum_i \xi_i \nu_i , \quad \nu_i \equiv \frac{1}{\pi} \int dx^2 \partial \bar{\partial} \log |s_i|^2 ,$$

where $|s_i|^2$ is given by

$$|s_1|^2 = \frac{|A_0|^2}{1-\lambda^2} \left( 1 + \sqrt{\lambda^2 + (1-\lambda^2)}|\tilde{\varphi}|^2 \right) ,$$

$$|s_2|^2 = \frac{|A_0|^2 - \lambda + \sqrt{\lambda^2 + (1-\lambda^2)}|\tilde{\varphi}|^2}{|C_0|^2} \frac{2(1-\lambda)}{2(1+\lambda)} \frac{|B_0|^2}{|A_0|^2 \lambda + \sqrt{\lambda^2 + (1-\lambda^2)}|\tilde{\varphi}|^2} .$$

### 4.5 An alternative $U(1) \times SU(2)$ model

The next example with the same base space is the $U(1) \times SU(2)$ theory with the same Lagrangian as in Section 4.3 except for a different $U(1)$ charge assignment for the $A$ field:
The gauge group action on the fields is

\[(A, B) \rightarrow (e^{i\alpha} A, e^{i\alpha} g B), \quad e^{i\alpha} \in U(1), \quad g \in SU(2).\] (4.72)

Note that \((\alpha, g) = (\pm \pi, -\frac{1}{2})\) is not an identity operator, so that the gauge group is truly \(U(1) \times SU(2)\). The vacuum manifold is given by

\[M = \{(A, B) \mid |A|^2 + 2BB^\dagger = \xi_1\}.\] (4.73)

By using \(SU(2)\), we can bring \(B = B_1\). Then we see that the vacuum manifold is isomorphic to \(S^3\) and the vacuum moduli space is isomorphic to \(S^3/U(1) \simeq \mathbb{C}P^1\). The vacuum manifold is the lemon space as drawn in Fig. 9. There are two singularities: one (at \(A = 0\)) is a \(\mathbb{Z}_2\) conical singularity and the other is a Coulomb singularity where \(SU(2)\) gauge symmetry is restored (\(B = 0\)).

The construction of the BPS vortex in this model is the same as the one in Section 4.3. However a difference appears in Eq. (4.49) as

\[H = \left( A, B \right) = \left( s^{-1}A_0(z), s^{-1}S^{-1}B_0(z) \right).\] (4.74)

This makes a difference for the choice of the \(U(1)\) winding number \(\nu\) which should be chosen to be consistent with a given boundary condition \((A, B) \rightarrow (A_{\text{vev}}, B_{\text{vev}} 1_2)\) satisfying \(|A_{\text{vev}}|^2 + 2|B_{\text{vev}}|^2 = \xi\). To see this, we again make use of the holomorphic \(SU(2)\) gauge invariant \(I = \det B = s^{-2}\det B_0(z)\). For a generic point \(A_{\text{vev}} \neq 0\) or \(B_{\text{vev}} \neq 0\), the asymptotic behavior is of the form

\[A_0 \rightarrow A_{\text{vev}} z^\nu + \cdots, \quad \det B_0 \rightarrow B_{\text{vev}}^2 z^{2\nu} + \cdots.\] (4.75)

Holomorphy requires \(\nu\) to be semi-positive integer. Thus the minimal configuration with \(\nu = 1\) has the mass \(T = 2\pi \xi\) and it is generated by the moduli matrix

\[A_0 = A_{\text{vev}} z + a, \quad \det B_0 = b_0^2 (z^2 + b_1 z + b_2).\] (4.76)

These conditions do not uniquely determine the matrix \(B_0(z)\). The moduli matrix is the same as one for \(U(2)\) local vortex, for instance

\[B_0(z) = B_{\text{vev}} \begin{pmatrix} 1 & c_1 z + c_2 \\ 0 & z^2 + b_1 z + b_2 \end{pmatrix}.\] (4.77)
Thus the moduli space for the single vortex is

$$\mathcal{V}_{\text{gen}} = \mathbb{C} \times \mathcal{V}^{U(2)}_{k=2},$$

(4.78)

where $\mathcal{V}^{U(2)}_{k=2}$ stands for the moduli space of $k = 2$ $U(2)$ local vortices and the first factor $\mathbb{C}$ corresponds to the complex parameter $a$ in $A_0$. Note that the minimal configuration includes two non-Abelian vortices since the vacuum moduli space has a $\mathbb{Z}_2$ singularity. Once we choose a generic point as the boundary condition, we cannot remove one of two non-Abelian vortices from the configuration. If we do that, the configuration meets a singularity.

Only when we choose the special point $A_{\text{vev}} = 0$, we can avoid the $\mathbb{Z}_2$ conical singularity. The $U(1)$ winding number $\nu$ can be a half integer and the minimal configuration is obtained by $\nu = 1/2$

$$A_0 = a, \quad \det B_0 = B^2_{\text{vev}}(z + b). \quad (4.79)$$

This reflects the fact that at the orbifold point $A = 0$ of $\mathcal{M}$, the $U(1)$ fiber makes a jump (becomes a half of what is at regular points). The vortex moduli space is the same as that of $k = 1$ $U(2)$ local vortex [13, 14] and is given by

$$\mathcal{V}_{\text{sp}} = \mathcal{V}^{U(2)}_{k=1}, \quad \mathcal{V}^{U(2)}_{k=1} = \mathbb{C} \times \mathbb{C} P^1. \quad (4.80)$$

As in Section 4.3 we cannot take the singular point $B_{\text{vev}} = 0$ as a boundary condition for constructing vortex solutions. The lumps in the strong gauge coupling limit always hit the singularity.

5 $U(1) \times SO(N)$ model

We now consider the fractional vortices occurring in a model with gauge group $U(1) \times SO(N)$. Vortices with orientational modes (non-Abelian vortices) in these models, in a maximally color-flavor locked vacuum, have recently been constructed and studied [37, 38, 39].

For our purposes here, we shall consider only the even-dimensional orthogonal groups, i.e. $N = 2M$. The matter content is $N_f = N$ flavors of squarks in the fundamental (vector) representation of the $SO(N)$ group, all with the same unit charge with respect to the $U(1)$ group:

|         | $U(1)$ | $SO(N)$ |
|---------|--------|---------|
| $H$     | 1      | $\Box$  |
As the $\mathbb{Z}_2$ element (i.e. $-1$) of the $SO(N)$ group is also an element of $U(1)$, the gauge group is really $U(1) \times SO(N)/\mathbb{Z}_2$.

The vacuum moduli have been studied in Ref. [41] and it turns out that it has a rather rich structure. By color and flavor transformation, the scalar VEV can be put in the canonical form

$$\langle H \rangle = \text{diag} (v_1, v_2, \cdots, v_{2M}) , \quad \sum_{i=1}^{2M} v_i^2 = \xi , \quad v_i \in \mathbb{R}.$$  \hspace{1cm} (5.1)

Note that, in contrast to the $U(N)$ models with $N_f = N$ flavors, where vacuum conditions force the VEV of $H$ to be proportional to an $N \times N$ unit matrix, the weaker condition here leaves the possibility of having arbitrary values $v_i$ subject to the constraint, $\sum_{i=1}^{2M} v_i^2 = \xi$. A large vacuum degeneracy is present here.

At a generic point in $\mathcal{M}$, where $v_i \neq 0$, $\forall i$, and all distinct, the gauge and flavor groups

$$L = \frac{U(1) \times SO(2M)}{\mathbb{Z}_2} , \quad G_F = SU(2M) ,$$  \hspace{1cm} (5.2)

are completely broken. The fiber $F$ is given by the $L$ orbits of the points $H = \text{diag}(v_1, v_2, \ldots, v_{2M})$.

On the points where some (at least two) of the $v_i$’s vanish, the unbroken gauge group $L_0$ is strictly smaller than $L$. The gauge orbit $F$ is now generated by $L/L_0$ and has a smaller dimension than in the case of a vortex constructed on a generic point of $\mathcal{M}$. Thus even though in all cases

$$\pi_1(F) = \mathbb{Z} ,$$  \hspace{1cm} (5.3)

its actual (e.g.) minimal element goes through discontinuous changes whenever we hit a singularity (or a singularity curve) on $\mathcal{M}$. Also, in such a point, the global symmetry group $G_F$ is different from that at surrounding points, and the consequent internal vortex moduli also undergoes a discontinuous change. As a singular surface (e.g. with a given number of vanishing $v_i$’s) contains a smaller subspace of singular points (some of the remaining $v_i$’s vanishing there), etc., one ends up with a rather rich structure of a (stratified) singular manifold $\mathcal{M}$, and of the vortices and related sigma-model lumps as the fiber defined over it. We shall leave the study of these varieties of phenomena for a separate investigation: here we will take all vacuum moduli to be non-vanishing.

Our fractional vortex solution is closely related to the “fractional lump” which was found by some of us recently [41]. We choose in the following the scalar VEV to be of color-flavor diagonal form, and moreover proportional to the unit matrix form,

$$\langle H \rangle = \sqrt{\xi} 1_{2M}$$  \hspace{1cm} (5.4)
leaving a residual global color-flavor symmetry $SO(N)_{c+f}$ unbroken. The standard moduli-matrix Ansatz is

$$H = s^{-1}(z, \bar{z}) S'^{-1}(z, \bar{z}) H_0(z) ,$$

(5.5)

where $s \in U(1)^C, S' \in SO(N)^C$. The gauge field BPS equations lead to

$$\partial \bar{\partial} \log \omega = -\frac{e^2}{4N} \left( \frac{1}{\omega} \Tr (\Omega_0 \Omega'^{-1}) - v^2 \right) ,$$

(5.6)

$$\bar{\partial} (\Omega' \partial \Omega'^{-1}) = \frac{g^2}{8\omega} \left( \Omega_0 \Omega'^{-1} - J^\dagger \left( \Omega_0 \Omega'^{-1} \right)^T J \right) ,$$

(5.7)

where $\omega = ss^\dagger, \Omega' = S'S'^\dagger, \Omega_0 = H_0 H_0^\dagger$. $J$ is the invariant tensor of $SO(N = 2M)$

$$J = \left( \begin{array}{cc} 0 & 1_M \\ 1_M & 0 \end{array} \right) .$$

(5.8)

The tension of the vortex remains

$$T = 2v^2 \int_C d^2x \bar{\partial} \partial \log \omega = \pi v^2 k .$$

(5.9)

Following the construction of [38], we have the constraint

$$H_0^T J H_0 = z^k J + O(z^{k-1}) ,$$

(5.10)

for vortex solution of winding number $k$.

In order to study the minimal winding vortex configuration more concretely, we choose

$$H_0 = \left( \begin{array}{cc} z 1_M - Z & C \\ 0 & 1_M \end{array} \right) , \quad Z = \text{diag}(z_1, z_2, \ldots, z_M) , \quad C = \text{diag}(c_1, c_2, \ldots, c_M) .$$

(5.11)

To solve the master equations (5.6) and (5.7), we set

$$\Omega' = \text{diag} \left( e^{x_1'}, \ldots, e^{x_M'}, e^{-x_1'}, \ldots, e^{-x_M'} \right) ,$$

(5.12)

where the determinant one is manifest. Taking $\omega = e^\psi$, we obtain

$$\bar{\partial} \partial \psi = -\frac{e^2}{8M} \left[ \sum_{i=1}^M \left( |z - z_i|^2 + |c_i|^2 \right) e^{-(\psi + x_i')} - e^{-(\psi - x_i')} \right] - v^2 ,$$

(5.13)

$$\bar{\partial} \partial x'_i = -\frac{g^2}{8} \left[ (|z - z_i|^2 + |c_i|^2) e^{-(\psi + x_i')} - e^{-(\psi - x_i')} \right] , \quad \forall i \in [1, M] .$$

(5.14)

If we now take the infinite gauge coupling limit $e \to \infty, g \to \infty$, we obtain the following lump solution

$$e^{x_i'} = \sqrt{|z - z_i|^2 + |c_i|^2} ,$$

(5.15)

$$e^\psi = \frac{2}{v^2} \sum_{i=1}^M e^{x_i'} = \frac{2}{v^2} \sum_{i=1}^M \sqrt{|z - z_i|^2 + |c_i|^2} ,$$

(5.16)
which has the energy density

$$\mathcal{E} = 2\xi \bar{\partial} \partial \log \left\{ \sum_{i=1}^{M} \sqrt{|z - z_i|^2 + |c_i|^2} \right\}. \quad (5.17)$$

This is the fractional lump solution found in Ref. [41].

The vortex energy profile in the strong-coupling approximation for the $U(1) \times SO(6)$ model is shown in Figure 12. Three fractional peaks are clearly seen. The positions of the peaks can be understood as follows. If $c_i = 0$ one of the $\hat{U}(1) \subset U(1) \times SO(2M)$, constructed as the diagonal combination of $U(1)$ and one of the $U(1)$ Cartan subalgebra of $SO(2M)$, is restored at the points $z = z_i$ ($i = 1, 2, \ldots M$). The sharp peak in the right panel of Fig. 12 can be thought locally (in $z$) to be an ANO vortex. If $c_i \neq 0$ the situation around a fractional peak at $z = z_i$ is more similar to the power-behaved semi-local vortex of the EAH model. The number of peaks reflects obviously the rank of the group considered (here rank$\{SO(6)\} = 3$), but the number of the possible fractional peaks depends on the point of the vacuum moduli (a particular VEV) considered. For instance, if two of $v_i$ are taken to be zero, the maximum number of the fractional peaks would be two, and so on.

In the supersymmetric version of the models based on the $U(1) \times SO(N)$ gauge groups, the Kähler potential in terms of a meson $M$ has been determined in Ref. [41],

$$K = \xi \log \text{Tr} \sqrt{MM^\dagger}. \quad (5.18)$$

Fig. 12: The energy density of three fractional vortices (lumps) in the $U(1) \times SO(6)$ model in the strong coupling approximation. The positions are $z_1 = -\sqrt{2} + i\sqrt{2}, z_2 = -\sqrt{2} - i\sqrt{2}, z_3 = 2$. Left panel: the size parameters are chosen as $c_1 = c_2 = c_3 = 1/2$. Right panel: the size parameters are chosen as $c_1 = 0, c_2 = 0.1, c_3 = 0.3$. Notice that one peak is singular ($z_1$) and the other two are regularized by the finite (non-zero) parameters $c_{2,3}$.
If we relax the vacuum moduli to be equal \( v_i \), thus having the possibility of distinct \( \{v_i\} \)'s in Eq. (5.1), it will prove convenient to work directly with the mesons of \( SO(2M) \)

\[
M = \begin{pmatrix}
  e^u(z - a) & \pm ia \\
  \pm ia & e^{-u}(z + a)
\end{pmatrix}
\]  

(5.19)

with \( a, u \in \mathbb{R} \). The meson VEV will be \( \text{diag}(v_1^2, v_2^2) = \text{diag}(e^u, e^{-u}) \). Using the Kähler potential (5.18) we readily obtain the energy density

\[
\mathcal{E} = \xi \bar{\partial} \partial \log \left( |z - a \tanh(u)|^2 + \frac{a^2}{\cosh^2(u)} \right) .
\]  

(5.20)

Furthermore, we can construct a typical example of fractional vortices, in a \( U(1) \times SO(2N) \) model in the lump limit as follows

\[
\mathcal{E} = 2\xi \bar{\partial} \partial \log \left( \sum_{i=1}^{N} m_i \sqrt{|z - a_i \tanh(u_i)|^2 + \frac{a_i^2}{\cosh^2(u_i)}} \right),
\]  

(5.21)

with \( v_{2i-1}^2 = m_i e^{u_i}, v_{2i}^2 = m_i e^{-u_i} \). For each \( SO(2) \) subgroup, we have in this construction a possibility for amplification \( m_i \), and \( a_i, u_i \) which serve as a position- and an effective size-parameters. One can observe that \( m_i \) controls the relative weight of the energy distributed to the \( i \)-th fractional vortex.

6 Conclusion

In this paper we have given a simple account of the fractional vortices which have minimally quantized magnetic flux (winding) but with non-trivial substructures in the energy distribution in the transverse plane. They could often appear in various generalizations of the Abelian Higgs model. The common characteristic features these models share are a non-trivial vacuum degeneracy and the BPS saturated nature of the vortex solutions. We have generalized the moduli matrix formalism \([20, 21, 22]\) to clarify all possible moduli parameters of the minimally quantized fractional vortices.

The vacuum moduli \( \mathcal{M} \) in these models turns out, in general, to be a singular manifold, i.e., a manifold with singularities. Vortex solutions approach the vacuum configuration far from the center, and trace various closed gauge orbits \( F \).

We have classified fractional vortices into two types; the first type appears when \( \mathcal{M} \) has a \( \mathbb{Z}_n \) singularity where the gauge symmetry is not restored while the second type occurs when \( \mathcal{M} \) has a 2-cycle with a deformed geometry. The existence of a singularity is not essential for
the fractional vortices of the second type. Indeed, we have observed that smooth fractional lump solutions become singular as the smooth manifold $\mathcal{M}$ is deformed into a singular manifold (e.g. when some FI parameters are turned off). Even when $\mathcal{M}$ has such singularities, we have found smooth fractional vortex solutions. The vortices share the same properties as those of the corresponding lumps wrapping on $\mathcal{M}$ smoothened.

An interesting aspect of our analysis, especially relevant to the systems with the first type of fractional vortices and lumps, is the fact that the latter often represent *a generalized fiber bundle over the singular manifold* $\mathcal{M}$. At a singularity (or on a singular surface) the fiber space $F$ undergoes a discontinuous change either in its dimension or in its nature, or both. The vortex moduli also make a jump at such a point (points). These observations seem to point towards interesting physical applications as well as some novel kind of mathematical structures.

**Note added**

After completing this paper, a new paper by D. Tong and B. Collie which discusses precisely the second type of our fractional vortices – lumps, though in a different context, was posted on the ArXiv [9].

**Acknowledgements**

One of us (K.K.) thanks Daniele Dorigoni, Sergio Spagnolo, and Mario Salvetti for discussions. W.V. is supported by Della Riccia grant which supports Italian young researchers for working abroad. W.V. thanks the Department of Applied Mathematics and Theoretical Physics (DAMTP) of Cambridge for the nice hospitality. W.V, S.B.G. and M.E. thank David Tong for the nice and useful discussions. M.E. and K.O. are supported by the Research Fellowships of the Japan Society for the Promotion of Science for Research Abroad. The work of M.N. is supported in part by Grant-in-Aid for Scientific Research (No. 20740141) from the Ministry of Education, Culture, Sports, Science and Technology-Japan. One of us (K.K.) thanks the organizers and the participants of the Workshop “Crossing Boundaries”, Minneapolis, 14 -17 May 2009, in honor of the 60th birthday of M. Shifman, for providing him with an opportunity to discuss this work, and especially David Tong for a discussion on the unpublished versions of our (and their) work.
A General BPS vortex equations

A.1 General arguments

Let $W^I_\mu$ denote the gauge fields for $G = U(1)^n \times G'$ gauge group, where $I = -(n-1), \cdots, -1, 0$ are for the Abelian gauge fields $U(1)^n$, the rest referring to the arbitrary simple group $G'$. We write the complex scalar (Higgs) fields as

$$H^\alpha, \quad \alpha = (r_1, A_1), (r_2, A_2), \ldots,$$

where $r_i$ and $A_i$ stand for the color and flavor indices of the $i$-th matter in the representation $R_i$. A gauge transformation $G$ induces

$$\delta H^\alpha = i (\Lambda H)^\alpha = i \sum_I \Lambda^I (T^I)^{\alpha}_\beta H^\beta = i \sum_{I \leq 0} \Lambda^I (T^I)^{\alpha}_\beta H^\beta + i \sum_{I > 0} \Lambda^I (T^I)^{\alpha}_\beta H^\beta,$$

Here $T^I$ for $I \leq 0$ is the charge matrix for $U(1)_I$, i.e, $Q^I_i$ are the $U(1)_I$ charges of the matter fields present

$$(T^I)^{\alpha}_\beta = Q^I_i \delta_{\alpha \beta}, \quad \text{for} \quad I \leq 0.$$

$T^I (I \geq 1)$ denotes the generators of the gauge group $G'$ in the (in general, reducible) representation

$$T^I = t^{(1)}_I \otimes 1_{N^{(1)}} + t^{(2)}_I \otimes 1_{N^{(2)}} + \cdots.$$

Accordingly, the covariant derivative is given by

$$(\mathcal{D}_\mu H)^\alpha = [(\partial_\mu - iW^\mu) H]^\alpha = (\delta^\alpha_\beta \partial_\mu - iW^\mu (T^I)^{\alpha}_\beta) H^\beta;$$

in our models the energy (tension) has the following form and is semi-positive definite:

$$T = \int d^2x \left[ \frac{1}{2g_f^2} (F^I_{12})^2 + \sum_\alpha \left( |(\mathcal{D}_1 H)^\alpha|^2 + |(\mathcal{D}_2 H)^\alpha|^2 \right) + \sum_I g_f^2 (H^\dagger T^I H - \xi_I)^2 \right],$$

where $\xi_I = 0$, ($\forall I > 0$) and $\xi_I$ for $I \leq 0$ are assumed such that a supersymmetric vacuum exists and the theory is in Higgs phase.

The vacuum manifold is the zero locus of the superpotential

$$M_v = \{ H \mid D^I = H^\dagger T^I H - \xi_I = 0, \forall I \} \quad \text{(A.1)}$$

and the vacuum moduli is

$$\mathcal{M}_v = M/G \simeq H/\mathbb{C}.$$
These quotients may be ill-defined where the vacuum moduli space is singular or when some gauge symmetry is restored.

When we choose a point in the vacuum manifold where some of $U(1)$ gauge symmetries are spontaneously broken as $G = U(1)^n \times G' \to U(1)^{n-r} \times g'$ ($g' \in G'$), the topologically stable vortices appear with support from the non-trivial first homotopy group

\[ \pi_1(U(1)^r) = \bigoplus_{i=1}^r \mathbb{Z}. \]

This is related to $\pi_2(M)$ by the homotopy sequence (3.8). We now introduce the complex coordinates to write down the BPS equations

\[ z = x + iy, \quad \bar{z} = x - iy, \quad \partial \equiv \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \equiv \frac{1}{2}(\partial_x + i\partial_y), \]

\[ \mathcal{D} \equiv \frac{1}{2}(\mathcal{D}_1 + i\mathcal{D}_2), \quad \mathcal{D} \equiv \frac{1}{2}(\mathcal{D}_1 - i\mathcal{D}_2), \quad W = \frac{1}{2}(W_1^I - iW_2^I)T^I, \quad \bar{W} = \frac{1}{2}(W_1^I + iW_2^I)T^I. \]

The Bogomol’nyi completion reads

\[ T = \int d^2x \left[ \frac{1}{2g_2^2}F_{12}^I + g_1^2(H^\dagger T^I H - \xi_I)^2 + \sum_\alpha |(D_1 H \pm iD_2 H)^\alpha|^2 \mp \xi_I F_{12}^I \right]. \]

This shows that the minimum of the tension is given by either the “negative chirality” (or left-winding) solutions satisfying the Bogomol’nyi equations

\[ F_{12}^I = g_1^2(H^\dagger T^I H - \xi_I), \quad \mathcal{D} H = 0, \quad T_{\text{min}} = -\xi_I \int d^2x F_{12}^I > 0; \quad (A.3) \]

or by the “positive chirality” (right-winding) solutions such that

\[ F_{12}^I = -g_1^2(H^\dagger T^I H - \xi_I), \quad \mathcal{D} H = 0, \quad T_{\text{min}} = \xi_I \int d^2x F_{12}^I > 0; \]

Therefore, the tension is expressed by

\[ T_{\text{min}} = \sum_I 2\pi \nu_I \xi_I, \quad \nu_I = \mp \frac{1}{2\pi} \int dx^2 F_{12}^I, \quad (A.4) \]

where $-$ is for the negative chirality and $+$ is for the positive chirality. The rational number $\nu_I$ (for $I \leq 0$) stands for how many times the corresponding solution winds $U(1)_I \sim S^1$ when we go around once the boundary circle $S^1$ on the $z$-plane.

From now on, we concentrate on the BPS states with the negative chirality which are the solutions of Eq. (A.3). The matter part of the above vortex equation can be solved by the moduli matrix $H_0(z)$ which is a color-flavor mixed matrix whose elements are holomorphic (polynomials) in $z$, for the left-handed vortex solution Eq. (A.3) \[ H = S^{-1}H_0(z), \quad \bar{W} = -iS^{-1}\bar{\partial}S, \quad (A.5) \]
where \( S \) is an element of the complex extension of the gauge group
\[
S(z, \bar{z}) = e^{i \sum_i \Lambda^I(z, \bar{z}) T^I} \in G^C, \quad \Lambda^I(z, \bar{z}) \in \mathbb{C}. \tag{A.6}
\]
For a given \( H_0 \), \( S \) is determined by the first equation in Eq. (A.3). We assume existence and uniqueness of the solutions. Thus all the complex constants (coefficients of the polynomial functions) appearing in \( H_0(z) \) represent the vortex moduli parameters. Finally, \( S \) and \( H_0(z) \) are defined up to a \( V(z) \) transformation which does not change (A.5) and keeps \( S \) inside \( G^C \)
\[
H_0(z) \to V(z) H_0(z), \quad S \to V(z) S,
\]
where \( V(z) \) is a holomorphic matrix belonging to the complexified gauge group \( G^C \). When the vortex moduli space is seen as a complex manifold (whose local coordinates are the moduli parameters appearing in \( H_0(z) \)), the \( V(z) \) transformations act as the transition functions in two overlapping patches. Thus the (vortex) moduli space of the 1/2 BPS vortices is formally expressed by
\[
\mathcal{M}_{\text{vor}} = \{H_0 \mid H_0 \sim V H_0, \ \bar{\partial} H_0 = 0, \ \bar{\partial} V = 0, \ V \in G^C\}. \nonumber
\]

Generally speaking, one should choose a boundary condition when one wishes to solve some partial differential equations. Our strategy of solving the BPS equations (A.3) is somehow upside-down to such an ordinary way. In fact, we have not fixed any boundary conditions yet. Of course, we are talking about the topological solitons which are characterized by the boundary conditions. So the remaining task is to figure out the condition for the moduli matrix which yields solutions satisfying the correct boundary condition. Note that the condition for \( H_0(z) \) depends on the boundary condition which is nothing but VEV \( \langle H \rangle \), a point on the vacuum manifold. We have to be careful to specify the moduli matrix especially when we choose a singular point as the boundary condition. Furthermore, the configuration (energy distribution) may change as varying the VEV even if the moduli matrix is fixed.

The concrete conditions for the moduli matrix \( H_0(z) \) depend on details of the models, such as gauge groups, representations of the matter fields and the \( U(1) \) charges. The case of \( G = U(1) \times G' \) with \( G' \) being an arbitrary simple group have been studied in Ref. [38]. In what follow, we will explain two typical cases. i) \( G = U(1) \) gauge theory with matter fields whose \( U(1) \)-charges are distinct. ii) there are more than one Abelian group such as \( G = U(1)_1 \times U(1)_2 \). These models, and especially, their moduli spaces, have not been studied so far.
A.2  \( G = U(1)_1 \times U(1)_2 \)

Let us consider a \( G = U(1)_1 \times U(1)_2 \) gauge theory with three Higgs fields \( H = (A, B, C) \) with the following \( U(1) \)-charges: \( Q_1 = (m, 1, 1) \) under the first \( U(1)_1 \) and \( Q_2 = (0, 1, -1) \) under \( U(1)_2 \). The vacuum manifold and the vacuum moduli are

\[
M = \{ A, B, C \mid m|A|^2 + |B|^2 + |C|^2 = \xi_1, \ |B|^2 - |C|^2 = \xi_2 \} \simeq S^3 , \quad \text{(A.7)}
\]

\[
\mathcal{M} = M/(U(1)_1 \times U(1)_2) \simeq \mathbb{CP}^1/\mathbb{Z}_m . \quad \text{(A.8)}
\]

Here \( \xi_{1,2} \) are the FI-terms of \( U(1)_{1,2} \). We choose them in the region \(-\xi_1 \leq \xi_2 \leq \xi_1 (\xi_1 \geq 0)\) to get the system in a Higgs vacuum. In the main text of this paper we set \( \xi_2 = 0 \) but here we consider a more general situation. At generic point of \( M \), both \( U(1)_1 \) and \( U(1)_2 \) are broken, hence we have topologically stable vortex solutions with topological characters \( \nu_I \)

\[
\nu_I = \frac{1}{\pi} \int dx^2 \partial \bar{\partial} \log |s_I|^2 , \quad \bar{W}^I = -i \partial \bar{\partial} \log s_I , \quad (I = 1, 2) , \quad \text{(A.9)}
\]

\[
H = (A, B, C) = (s_1^{-m}A_0(z), s_1^{-1}s_2^{-1}B_0(z), s_1^{-1}s_2C_0(z)) , \quad \text{(A.10)}
\]

The solutions are determined by \( U(1)_1 \times U(1)_2 \) winding numbers \( (\nu_1, \nu_2) \). Eq. (A.9) determines the asymptotic behavior of \( s_I \)

\[
|s_I|^2 \to |z|^{2\nu_I} \quad \text{as} \quad |z| \to \infty . \quad \text{(A.11)}
\]

Then the Higgs fields asymptotically behaves as

\[
(A_0, B_0, C_0) \simeq (|z|^{\nu_1}A_{vvev}, |z|^{\nu_1+\nu_2}B_{vvev}, |z|^{\nu_1-\nu_2}C_{vvev}) , \quad \text{as} \quad |z| \to \infty , \quad \text{(A.12)}
\]

\((A_{vvev}, B_{vvev}, C_{vvev})\) being a point in the vacuum manifold.

A.2.1 \( m = 1 \)

Let us first consider a generic point, i.e., \( (A_{vvev}, B_{vvev}, C_{vvev}) \neq (0, 0, 0) \). Holomorphy forces us to choose \( \nu_1 \in \mathbb{Z}_+, \nu_1 + \nu_2 \in \mathbb{Z}_+ \) and \( \nu_1 - \nu_2 \in \mathbb{Z}_+ \). Let us rewrite

\[
\nu_1 \equiv k_1 \in \mathbb{Z}_+ , \quad \nu_1 + \nu_2 \equiv k_2 \in \mathbb{Z}_+ , \quad 2k_1 \geq k_2 \geq 0 . \quad \text{(A.13)}
\]

The tension can be expressed as

\[
\frac{T}{2\pi} = \nu_1 \xi_1 + \nu_2 \xi_2 = (\xi_1 - \xi_2)k_1 + \xi_2 k_2 \geq \frac{\xi_1 + \xi_2}{2} k_2 \geq 0 . \quad \text{(A.14)}
\]

Thus the moduli matrix for \((k_1, k_2)\) configuration is

\[
A_0(z) = A_{vvev}z^{k_1} + \cdots , \quad B_0(z) = B_{vvev}z^{k_2} + \cdots , \quad C_0(z) = C_{vvev}z^{2k_1-k_2} + \cdots . \quad \text{(A.15)}
\]
The number of complex moduli parameters is \( \dim \mathcal{M}(k_1, k_2) = 3k_1 \). The minimum configuration depends on \( \xi_2 \). When \( 0 < \xi_2 < \xi_1 \), the minimum configuration is \( T_{(1,0)} = 2\pi(\xi_1 - \xi_2) \). When \( -\xi_1 < \xi_2 < 0 \), \( T_{(1,2)} = 2\pi(\xi_1 + \xi_2) \) is minimum. The next lightest configuration has always \( T_{(1,1)} = 2\pi \xi_1 \). If \( \xi_2 = 0 \), all these three are degenerate, namely \( T_{(1,0)} = T_{(1,1)} = T_{(1,2)} \).

Let us next choose the case of \( A_{\text{vev}} = 0 \) with \( B_{\text{vev}}, C_{\text{vev}} \neq 0 \). In this case, the condition changes as \( \nu_1 + \nu_2 \equiv k_1 \in \mathbb{Z}_+ \) and \( \nu_1 - \nu_2 \equiv k_2 \in \mathbb{Z}_+ \). In this case, the \( U(1) \)-charges \( \nu_{1,2} \) are half quantized and the energy is

\[
\frac{T}{2\pi} = \frac{\xi_1 + \xi_2}{2} k_1 + \frac{\xi_1 - \xi_2}{2} k_2 \geq 0 .
\]

(A.16)

The moduli matrix for \( (k_1, k_2) \) configuration should be chosen as

\[
A_0(z) = az^\alpha + \cdots , \quad B_0(z) = B_{\text{vev}} z^{k_1} + \cdots , \quad C_0(z) = C_{\text{vev}} z^{k_2} + \cdots ,
\]

where \( \alpha \) is an arbitrary positive integer less than \( \nu_1 = (k_1 + k_2)/2 \). The number of complex moduli parameters is \( \dim \mathcal{M}(k_1,k_2) = k_1 + k_2 + \alpha + 1 \). The minimum configuration is \( T_{(1,0)} = \pi(\xi_1 + \xi_2) \) for \( -\xi_1 < \xi_2 < 0 \) while \( T_{(0,1)} = \pi(\xi_1 - \xi_2) \) for \( 0 < \xi_2 < \xi_1 \). \( T_{(1,1)} = 2\pi \xi_1 \) is the third lightest configuration. Note that these tensions are one half of that of the vortices for \( A_{\text{vev}} \neq 0 \). If \( \xi_2 = 0 \), we again observe degeneracy \( T_{(1,0)} = T_{(0,1)} < T_{(1,1)} \).

The vacuum \( B_{\text{vev}} = C_{\text{vev}} = 0 \) is possible when \( \xi_2 = 0 \). We shall not consider this case since the second \( U(1) \) factor is now in a Coulomb phase.

\[A.2.2\]

\( m = 2 \)

Let us repeat the analysis for case \( m = 2 \). First we consider \( (A_{\text{vev}}, B_{\text{vev}}, C_{\text{vev}}) \neq (0,0,0) \). We find the condition \( 2\nu_1 \in \mathbb{Z}_+, \nu_1 + \nu_2 \in \mathbb{Z}_+ \) and \( \nu_1 - \nu_2 \in \mathbb{Z}_+ \). Rewrite

\[
\nu_1 + \nu_2 \equiv k_1 \in \mathbb{Z}_+ \; , \quad \nu_1 - \nu_2 \equiv k_2 \in \mathbb{Z}_+ .
\]

(A.18)

Note that \( 2\nu_1 = k_1 + k_2 \in \mathbb{Z}_+ \) is automatically satisfied. The energy is given by

\[
\frac{T}{2\pi} = \frac{\xi_1 + \xi_2}{2} k_1 + \frac{\xi_1 - \xi_2}{2} k_2 \geq 0 .
\]

(A.19)

Since the \( U(1) \) charges are half quantized, this tension formula is slightly different from Eq. \[A.14\] with \( m = 1 \). The moduli matrix for \( (k_1, k_2) \) configuration is

\[
A_0(z) = A_{\text{vev}} z^{k_1+k_2} + \cdots , \quad B_0(z) = B_{\text{vev}} z^{k_1} + \cdots , \quad C_0(z) = C_{\text{vev}} z^{k_2} + \cdots .
\]

(A.20)

Dimension of the moduli space is \( \dim \mathcal{M}(k_1,k_2) = 2k_1 + 2k_2 \). The minimal energy configuration is \( T_{(1,0)} = \pi(\xi_1 - \xi_2) \) for \( \xi_2 > 0 \) or \( T_{(1,1)} = \pi(\xi_1 + \xi_2) \) for \( \xi_2 < 0 \). If \( \xi_2 = 0 \), these two are degenerate.
Configurations with $A_{\text{vev}} = 0$, or $B_{\text{vev}} = 0$ (or $C_{\text{vev}} = 0$) as the boundary condition are not special but belong to the above category. $B_{\text{vev}} = C_{\text{vev}} = 0$ is only possible when $\xi_2 = 0$. Again, in this case the $U(1)_2$ gauge symmetry is restored (Coulomb phase).

### A.2.3 $m \geq 3$

As we have seen, the difference coming from the choice of $m$ appears only when $(A_{\text{vev}}, B_{\text{vev}}, C_{\text{vev}}) \neq (0, 0, 0)$:

$$m \nu_1 \equiv k_1, \quad \nu_1 + \nu_2 = k_2, \quad \Rightarrow \quad \nu_1 - \nu_2 = 2\nu_1 - k_2 = \frac{2k_1}{m} - k_2 \in \mathbb{Z}_+.$$  \hspace{1cm} (A.21)

When $m$ is odd, we should choose $k_1 = m_{\text{odd}} k_1'$ with $k_1' \in \mathbb{Z}_+$. Then $k_2$ can be taken in the region $2k_1' \geq k_2 \geq 0$. The corresponding tension is

$$\frac{T_{\text{odd}}}{2\pi} = \frac{\xi_1 - \xi_2}{m_{\text{odd}}} k_1 + \xi_2 k_2 = (\xi_1 - \xi_2) k_1' + \xi_2 k_2.$$ \hspace{1cm} (A.22)

The moduli matrix is

$$A_0(z) = A_{\text{vev}} z^{m_{\text{odd}} k_1'} + \cdots, \quad B_0(z) = B_{\text{vev}} z^{k_2} + \cdots, \quad C_0(z) = C_{\text{vev}} z^{2k_1' - k_2} + \cdots.$$ \hspace{1cm} (A.23)

The dimension of the moduli space is $\dim_{\mathbb{C}} \mathcal{M}^{\text{odd}}_{(k_1, k_2)} = (m_{\text{odd}} + 2) k_1'$.

When $m$ is even, we must choose $k_1 = \frac{m_{\text{even}}}{2} k_1'$ with $k_1' \in \mathbb{Z}_+$. Thus the $U(1)$ charges are half quantized. Holomorphy forces $k_1' \geq k_2 \geq 0$. The tension is given by

$$\frac{T_{\text{even}}}{2\pi} = \frac{\xi_1 - \xi_2}{m_{\text{even}}} k_1 + \xi_2 k_2 = \frac{\xi_1 - \xi_2}{2} k_1' + \xi_2 k_2.$$ \hspace{1cm} (A.24)

The moduli matrix is given by

$$A_0(z) = A_{\text{vev}} z^{\frac{m_{\text{even}}}{2} k_1'} + \cdots, \quad B_0(z) = B_{\text{vev}} z^{k_2} + \cdots, \quad C_0(z) = C_{\text{vev}} z^{k_1' - k_2} + \cdots.$$ \hspace{1cm} (A.25)

The dimension of the moduli space is $\dim_{\mathbb{C}} \mathcal{M}^{\text{even}}_{(k_1, k_2)} = (\frac{m_{\text{even}}}{2} + 1) k_1'$.

### A.3 $G = U(1) \times SU(N)$

Let us consider $G = U(1) \times G'$ and $G' = SU(N)$ with Higgs fields $H = (A, B)$ where $A$ is a singlet scalar field and $B$ is a collection of $N$ fields in the fundamental representation $\mathbf{N}$ of $SU(N)$, written as an $N \times N$ matrix. We assign the $U(1)$ charges $(m, 1)$ to the fields $(A, B)$, respectively. The vacuum manifold is given by the $D$-flatness conditions and the vacuum moduli is obtained by dividing it by $G$ as

$$M = \left\{ (A, B) \mid m |A|^2 \mathbf{1}_N + N \mathbf{B} \mathbf{B}^\dagger = \xi \mathbf{1}_N \right\}, \quad M = M/G \simeq \mathbb{C}P^1/\mathbb{Z}_m.$$ \hspace{1cm} (A.26)
The vacuum condition is simplified by using $B = B_{\text{vev}} 1_N$ as
\begin{equation}
m A_{\text{vev}}^2 + N B_{\text{vev}}^2 = \xi .
\end{equation}

The moduli matrix formalism is summarized as
\begin{equation}
(A, B) = s^{-1} (A_0(z), S^{-1} B_0(z)) , \quad W_U(1) = -i \partial \bar{\partial} \log s , \quad W_{SU(N)} = -i S^{-1} \bar{\partial} S , \tag{A.28}
\end{equation}
with $s \in \mathbb{C}^*$ and $S \in SL(N, \mathbb{C})$. $U(1)$ winding number is given by
\begin{equation}
\nu = \frac{1}{\pi} \int dx^2 \bar{\partial} \partial \log |s|^2 , \quad \Rightarrow \quad |s|^2 \sim |z|^{2\nu} \quad \text{as} \quad |z| \to \infty . \tag{A.29}
\end{equation}

When we have some non-Abelian gauge group $G'$, we can consider the holomorphic $G'$ invariant
\begin{equation}
I \equiv \det B = s^{-N} \det B_0(z) \to |z|^{-N\nu} \det B_0(z) \equiv B_{\text{vev}}^N \quad \text{as} \quad |z| \to \infty . \tag{A.30}
\end{equation}

Since all the elements in $B_0(z)$ is holomorphic in $z$, $\det B_0(z)$ is also a polynomial function of $z$. Combining this and the behavior of $A$
\begin{equation}
A = s^{-1} A_0(z) \to |z|^{-m\nu} A_0(z) \equiv A_{\text{vev}} , \quad \text{as} \quad |z| \to \infty , \tag{A.31}
\end{equation}
we can consistently determine $\nu$.

When we choose $(A_{\text{vev}}, B_{\text{vev}}) \neq (0, 0)$, the moduli matrix asymptotically behave as
\begin{equation}
A_0(z) \to |z|^{m\nu} A_{\text{vev}} , \quad \det B_0(z) \to |z|^{\nu N} B_{\text{vev}}^N , \quad \text{as} \quad |z| \to \infty . \tag{A.32}
\end{equation}

Thus we should choose the $U(1)$ winding number $\nu$ in such a way that
\begin{equation}
m\nu \equiv k \in \mathbb{Z}_+ , \quad N\nu = \frac{N}{m} k = \frac{N'}{m'} k \in \mathbb{Z}_+ , \tag{A.33}
\end{equation}
where we have assumed $N = n_0 N'$ and $m = n_0 m'$ with $n_0 \in \mathbb{Z}_+$ ($n_0 = \text{g.c.d}(N, m)$). Thus $k$ must be $k = m' k'$ with $k' \in \mathbb{Z}_+$, so that the $U(1)$ winding number is fractionally quantized $\nu = \frac{k'}{n_0}$ if $n_0 \geq 2$. The corresponding tension is $T = 2\pi \xi \frac{k'}{n_0}$. The moduli matrices for the configuration with $k'$ may be chosen as
\begin{equation}
A_0(z) = A_{\text{vev}} z^{\frac{m}{n_0} k'} + \cdots , \quad \det B_0(z) = B_{\text{vev}}^N z^{\frac{N}{n_0} k'} + \cdots . \tag{A.34}
\end{equation}

Note that we have specified only $\det B_0(z)$, so there are many possibilities for $B_0(z)$ per se, it is nothing but the orientational moduli. The moduli space for $k' = 1$ is $M_{k'=1} \simeq C^\frac{N+m}{n_0} \times CP^{N-1}$.

When we choose $A_{\text{vev}} = 0$ and $B_{\text{vev}} \neq 0$ as the boundary condition, we should satisfy
\begin{equation}
A \to |z|^{-m\nu} A_0(z) = 0 , \quad I \to |z|^{-N\nu} \det B_0(z) = B_{\text{vev}}^N \quad \text{as} \quad |z| \to \infty . \tag{A.35}
\end{equation}

Thus we can choose $N\nu \equiv k \in \mathbb{Z}_+$, such that the $U(1)$ winding is $1/N$ quantized and corresponding tension is $T = 2\pi \xi \frac{k}{N}$. The moduli matrices are
\begin{equation}
A_0(z) = a z^\alpha + \cdots , \quad \det B_0 = B_{\text{vev}}^N z^k + \cdots , \tag{A.36}
\end{equation}
where $\alpha$ is a semi-positive definite integer $\alpha < m\nu = m\frac{k}{N}$. 

42
References

[1] A. A. Abrikosov, Sov. Phys. JETP 5 (1957) 1174 [Zh. Eksp. Teor. Fiz. 32 (1957) 1442].

[2] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61 (1973) 45.

[3] E. Babaev, Phys. Rev. Lett. 89, 067001 (2002) [arXiv:cond-mat/0111192].

[4] G. ’t Hooft, Commun. Math. Phys. 81 (1981), 267.

[5] B. J. Harrington and H. K. Shepard, Phys. Rev. D17 (1978) 2122; Phys. Rev. D18 (1978) 2990; D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. 53 (1981) 43.

[6] D. Tong, Phys. Rev. D 66, 025013 (2002) [arXiv:hep-th/0202012].

[7] M. Eto, Y.Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 72, 025011 (2005) [arXiv:hep-th/0412048]; M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, Phys. Rev. D 73, 085008 (2006) [arXiv:hep-th/0601181]; M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, Nucl. Phys. B 788, 120 (2008) [arXiv:hep-th/0703197].

[8] F. Bruckmann, Phys. Rev. Lett. 100, 051602 (2008) [arXiv:0707.0775 [hep-th]]; D. Harland, [arXiv:0902.2303 [hep-th]]; W. Brendel, F. Bruckmann, L. Janssen, A. Wipf and C. Wozar, [arXiv:0902.2328 [hep-th]].

[9] B. Collie and D. Tong, [arXiv:0905.2267 [hep-th]].

[10] T. Vachaspati and A. Achucarro, Phys. Rev. D 44, 3067 (1991); A. Achucarro and T. Vachaspati, Phys. Rept. 327 (2000) 347 [arXiv:hep-ph/9904229].

[11] A. Achucarro, K. Kuijken, L. Perivolaropoulos and T. Vachaspati, Nucl. Phys. B 388, 435 (1992).

[12] M. Hindmarsh, Nucl.Phys. B392 (1993) 461. e-Print: hep-ph/9206229. M. Hindmarsh, R. Holman, T. W. Kephart and T. Vachaspati, Nucl.Phys. B404 (1993) 794. e-Print: hep-th/9209088

[13] A. Hanany and D. Tong, JHEP 0307 (2003) 037 [arXiv:hep-th/0306150].

[14] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673 (2003) 187 [arXiv:hep-th/0307287].

[15] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, Nucl. Phys. B 686 (2004) 119 [arXiv:hep-th/0312233].

[16] D. Tong, Phys. Rev. D 69 (2004) 065003 [arXiv:hep-th/0307302].

[17] M. Shifman and A. Yung, Phys. Rev. D 70 (2004) 045004 [arXiv:hep-th/0403149].
[18] A. Hanany and D. Tong, JHEP **0404** (2004) 066 [arXiv:hep-th/0403158].

[19] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71** (2005) 045010 [arXiv:hep-th/0412082].

[20] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129]; Phys. Rev. Lett. **93**, 161601 (2004) [arXiv:hep-th/0404198]; Phys. Rev. D **70**, 125014 (2004) [arXiv:hep-th/0405194].

[21] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. **96**, 161601 (2006) [arXiv:hep-th/0511088].

[22] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39** (2006) R315 [arXiv:hep-th/0602170]; “Solitons in supersymmetric gauge theories: Moduli matrix approach,” [arXiv:hep-th/0607225].

[23] M. Eto, L. Ferretti, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Nucl.Phys. **B780** 161-187, 2007 [arXiv: hep-th/0611313].

[24] M. Shifman and A. Yung, Rev. Mod. Phys. **79** (2007) 1139 [arXiv:hep-th/0703267].

[25] D. Tong, “TASI lectures on solitons,” [arXiv:hep-th/0509216].

[26] D. Tong, “Quantum Vortex Strings: A Review,” [arXiv:0809.5060 [hep-th]].

[27] K. Hashimoto and D. Tong, JCAP **0509** (2005) 004 [arXiv:hep-th/0506022].

[28] R. Auzzi, M. Shifman and A. Yung, Phys. Rev. D **73** (2006) 105012 [Erratum-ibid. D **76** (2007) 109901] [arXiv:hep-th/0511150].

[29] M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, Phys. Rev. D **74** (2006) 065021 [arXiv:hep-th/0607070].

[30] M. Shifman and A. Yung, Phys. Rev. D **73** (2006) 125012 [arXiv:hep-th/0603134].

[31] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, N. Yokoi, Phys. Rev. D **76** (2007) 105002 [arXiv:0704.2218 [hep-th]].

[32] R. Auzzi, M. Eto and W. Vinci, JHEP **0711** (2007) 090 [arXiv:0709.1910 [hep-th]].

[33] R. Auzzi, M. Eto and W. Vinci, JHEP **0802** (2008) 100 [arXiv:0711.0116 [hep-th]]. M. Eto, [arXiv:0810.4895 [hep-th]].

[34] L. Ferretti and K. Konishi, “Duality and confinement in SO(N) gauge theories,” in “"Sense of Beauty in Physics: a volume in honour of Adriano Di Giacomo”, Ed. by M. D’Elia, et. al., Edizioni PLUS (Univ. Pisa. Press), 2000, [arXiv:hep-th/0602252].

[35] M. Edalati and D. Tong, JHEP **0705** (2007) 005 [arXiv:hep-th/0703045]; D. Tong, JHEP **0709** (2007) 022 [arXiv:hep-th/0703235]; M. Shifman and A. Yung, Phys. Rev. D **77** (2008)
125016 [arXiv:0803.0158 [hep-th]]; M. Shifman and A. Yung, Phys. Rev. D 77 (2008) 125017
[arXiv:0803.0698 [hep-th]].

[36] D. Dorigoni, K. Konishi and K. Ohashi, Phys. Rev. D 79 (2009) 045011.

[37] L. Ferretti, S. B. Gudnason and K. Konishi, Nucl. Phys. B 789, 84 (2008) [arXiv:0706.3854 [hep-th]].

[38] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, Phys. Lett. B 669, 98 (2008) [arXiv:0802.1020 [hep-th]].

[39] M. Eto et al., JHEP (in press) [arXiv:0903.4471 [hep-th]].

[40] K. Higashijima, M. Nitta, Prog. Theor. Phys. 103 (2000) 635. e-Print: hep-th/9911139;
    Prog. Theor. Phys. 103 (2000) 833. e-Print: hep-th/9911225

[41] M. Eto, T. Fujimori, S. B. Gudnason, M. Nitta and K. Ohashi, Nucl. Phys. B 815, 495 (2009) [arXiv:0809.2014 [hep-th]].