PoSAT: Proof-of-Work Availability and Unpredictability, without the Work

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Abstract. An important feature of Proof-of-Work (PoW) blockchains is full dynamic availability, allowing miners to go online and offline while requiring only 50% of the online miners to be honest. Existing Proof-of-stake (PoS), Proof-of-Space and related protocols are able to achieve this property only partially, either putting the additional assumption that adversary nodes to be online from the beginning and no new adversary nodes come online afterwards, or use additional trust assumptions for newly joining nodes. We propose a new PoS protocol PoSAT which can provably achieve dynamic availability fully without any additional assumptions. The protocol is based on the longest chain and uses a Verifiable Delay Function for the block proposal lottery to provide an arrow of time. The security analysis of the protocol draws on the recently proposed technique of Nakamoto blocks as well as the theory of branching random walks. An additional feature of PoSAT is the complete unpredictability of who will get to propose a block next, even by the winner itself. This unpredictability is at the same level of PoW protocols, and is stronger than that of existing PoS protocols using Verifiable Random Functions.

1 Introduction

1.1 Dynamic Availability

Nakamoto’s invention of Bitcoin [23] in 2008 brought in the novel concept of a permissionless Proof-of-Work (PoW) consensus protocol. Following the longest chain protocol, a block can be proposed and appended to the tip of the blockchain if the miner is successful in solving the hash puzzle. The Bitcoin protocol has several interesting features as a consensus protocol. An important one is dynamic availability. Bitcoin can handle an uncertain and dynamic varying level of consensus participation in terms of mining power. Miners can join and leave as desired without any registration requirement. This is in contrast to most classical Byzantine Fault Tolerant (BFT) consensus protocols, which assumes a fixed and known number of consensus nodes. Indeed, Bitcoin has been continuously available since the beginning, a period over which the hashrate has varied over a range of 14 orders of magnitude. Bitcoin has been proven to be secure as long
as the attacker has less than 50% of the online hash power (the static power case is considered in \[15,23,24\] and variable hashing power case is considered in \[16,17\]).

Recently proof-of-stake (PoS) protocols have emerged as an energy-efficient alternative to PoW. Instead of solving a difficult hash puzzle, nodes participate in a lottery to win the right to append a block to the blockchain, with the probability of winning proportional to a node’s stake in the total pool. This replaces the resource intense mining process of PoW, while ensuring fair chances to contribute and claim rewards.

There are broadly two classes of PoS protocols: those derived from classical BFT protocols and those inspired by Nakamoto’s longest chain protocol. Attempts at blockchain design via the BFT approach include Algorand \([8,18]\), Tendermint \([6]\) and Hotstuff \([32]\). Motivated and inspired by Nakamoto longest chain protocol are the PoS designs of Snow White \([3]\) and the Ouroboros family of protocols \([1,10,19]\). One feature that distinguish the PoS longest chain protocols from the BFT protocols is that they inherit the dynamic availability of Bitcoin: the chain always grows regardless of the number of nodes online. But do these PoS longest chain protocols provide the same level of security guarantee as PoW Bitcoin in the dynamic setting?

1.2 Static vs Dynamic Adversary

Two particular papers focus on the problem of dynamic availability in PoS protocols: the sleepy model of consensus \([25]\) and Ouroboros Genesis \([1]\). In both papers, it was proved that their protocols are secure if less than 50% of the online nodes are adversary. This condition is the same as the security guarantee in PoW Bitcoin, but there is an additional assumption: all adversary nodes are always online starting from genesis and no new adversary nodes can join. While this static adversary assumption seems reasonable (why would an adversary go to sleep?), in reality this can be a very restrictive condition. In the context of Bitcoin, this assumption would be analogous to the statement that the hash power of the adversary is fixed in the past decade (while the total hashing power increased 14 orders of magnitude!) More generally, in public blockchains, PoW or PoS, no node is likely to be adversarial during the launch of a new blockchain token - adversaries only begin to emerge later during the lifecycle.

The static adversary assumption underlying these PoS protocols is not superfluous but is in fact necessary for their security. Newly joined adversary nodes can use their stake not only to participate in the lottery for winning a block near the current tip of the blockchain, but can use the same stake to participate in all past lotteries to win blocks all the way back to the genesis and then grow a chain *instantaneously* from the genesis to surpass the current longest chain (Figure 1(a)). Thus, due to this “costless simulation”, newly joined adversary nodes not only increase the current online adversary stake, but effectively increase past online adversary stake as well. In contrast, PoW does not suffer from the same issue because it would take a long time to grow such a chain from the past and that chain will always be behind the current longest chain. Thus, PoW provides
an *arrow of time*, meaning nodes cannot “go back in time” to mine blocks for the times at which they were not online. This property is key in endowing PoW protocols with the ability to tolerate fully dynamic adversaries (Figure 1(b)).

We point out that some protocols including Ouroboros Praos [10] and Snowhite [3] require that nodes discard chains that fork off too much from the present chain. This feature was introduced to handle nodes with expired stake (or nodes that can perform key grinding) taking over the longest chain. While they did not specifically consider the dynamic adversary issue we highlighted, relying on previous checkpoints can potentially solve the aforementioned security threat. However, as was eloquently argued in Ouroboros Genesis [1], these checkpoints are unavailable to offline clients and newly joining nodes require advice from a trusted party (or a group inside which a majority is trusted). This trust assumption is too onerous to satisfy in practice and is not required in PoW. Ouroboros Genesis was designed to require no trusted joining assumption while being secure to long-range and key-grinding attacks. However, they are not secure against dynamic participation by the adversary: they are vulnerable to the aforementioned attack.

This opens the following question:

*Is there a dynamically available PoS protocol which has full PoW security guarantee, without additional trust assumptions?*

1.3 PoSAT achieves PoW dynamic availability

We answer the aforementioned question in the affirmative. Given that arrow-of-time is a central property of PoW protocols, we design a new PoS protocol,
PoS with Arrow-of-Time (PoSAT), also having this property using randomness generated from Verifiable Delay Functions (VDF). VDFs are built on top of iteratively sequential functions, i.e., functions that are only computable sequentially: 
\[ f^\ell(x) = f \circ f \circ \ldots \circ f(x), \]
along with the ability to provide a short and easily verifiable proof that the computed output is correct. Examples of such functions include (repeated) squaring in a finite group of unknown order \[7,28\], i.e., 
\[ f(x) = 2^x \]
and (repeated) application of secure hash function (SHA-256) \[21\], i.e., 
\[ f(x) = H(x). \]
While VDFs have been designed as a way for proving the passage of a certain amount of time (assuming a bounded CPU speed), it has been recently shown that these functions can also be used to generate an unpredictable randomness beacon \[13\]. Thus, running the iteration till the random time \(L\) when 
\[ \text{RandVDF}(x) = f^L(x) < \tau \]
is within a certain threshold will result in \(L\) being a geometric random variable. We will incorporate this randomized VDF functionality to create an arrow-of-time in our protocol.

The basic idea of our protocol is to mimic the PoW lottery closely: instead of using the solution of a Hash puzzle based on the parent block’s hash as proof of work, we instead use the randomized VDF computed based on the parent block randomness and the coin’s public key as the proof of stake lottery. In a PoW system, we are required to find a string called "nonce" such that 
\[ H(\text{block}, \text{nonce}) < \tau, \]
a hash-threshold. Instead in our PoS system, we require 
\[ \text{RandVDF}(\text{prevRand}, pk) < \tau, \]
where \(\text{prevRand}\) is the randomness from the parent block and \(pk\) is the public key associated with the mining coin. There are three differences, the first two are common in existing PoS systems: (1) we use "prevRand" instead of "block" in order to prevent grinding attacks on the content in the PoS system, (2) we use the public-key "pk" of staking coin instead of PoW "nonce" to simulate a PoS lottery, (3) instead of using a hash, we use the RandVDF, which requires sequential function evaluation thus creating an "arrow of time."

The first two aspects are common to many PoS protocols and is most similar to an earlier PoS protocol \[14\], however, crucially we use the RandVDF function instead of a Verifiable random function (VRF) and a time parameter inside the argument used in that protocol. This change allows for full dynamic availability: if adversaries join late, they cannot produce a costless simulation of the time that they were not online and build a chain from genesis instantaneously. It will take the adversary time to grow this chain (due to the sequential nature of the RandVDF), by which time, the honest chain would have grown and the adversary will be unable to catch up. Thus, PoSAT behaves more like PoW (Figure 1(b)) rather than existing PoS based on VRF’s (Figure 1(a)). We show that this protocol achieves full dynamic availability: if \(\lambda_h(t), \lambda_a(t)\) denote the online honest and adversarial stake at time \(t\) respectively, it is secure as long as 
\[ \lambda_h(t) > e\lambda_a(t) \quad \text{for all } t, \]
where \(e\) is Euler’s number 2.7182....

We observe that the security of this protocol requires a stronger condition than PoW protocols which only require \(\lambda_h(t) > \lambda_a(t)\). The reason for this is
that an adversary can potentially do parallel evaluation of VDF on all possible blocks. Since the randomness in each of the blocks is independent from each other, the adversary has many random chances to increase the chain growth rate to out-compete the honest tree. This is a consequence of the nothing-at-stake phenomenon: the same stake can be used to grind on the many blocks. The factor $e$ is the resulting amplification factor for the adversary growth rate. This is avoided in PoW protocols due to the conservation of work inherent in PoW which requires the adversary to split its total computational power among such blocks.

We solve this problem in PoSAT by reducing the rate at which the block randomness is updated and hence reducing the block randomness grinding opportunities of the adversary. Instead of updating the block randomness at every depth of the blocktree, we only update it once every $c$ levels (called an epoch). The larger the value of the parameter $c$, the slower the block randomness is updated. The common source of randomness used to run the VDF lottery remains the same for $c$ blocks starting from the genesis and is updated only when (a) the current block to be generated is at a depth that is a multiple of $c$, or (b) the coin used for the lottery is successful within the epoch of size $c$. The latter condition is necessary to create further independent winning opportunities for the node within the period $c$ once a slot is obtained with that coin. This is illustrated in Figure 2. For $c = 1$, this corresponds to the protocol discussed earlier. Finally, since there is no explicit value of time-stamp in the block mining protocol (unlike other longest chain PoS protocols), we do not require that blocks have increasing time-stamps.

The following security theorem is proved about PoSAT for general $c$, giving a condition for security (liveness and persistency) under all possible attacks.

**Theorem 1.** PoSAT with parameter $c$ is secure as long as

$$\frac{\lambda_h(t)}{1 + \phi_c \lambda_a(t)} > \phi_c \lambda_a(t) \quad \text{for all } t, \quad (1)$$

where $\Delta$ is the network delay between honest nodes, $\lambda_{\text{max}}$ is a constant such that $\lambda_h(t) \leq \lambda_{\text{max}}$ for all $t > 0$, $\phi_c$ is a constant, dependent on $c$, given in $\phi_1 = e$ and $\phi_c \to 1$ as $c \to \infty$. 

We remark that in our PoS protocol, we have a known upper bound on the rate of mining blocks (by assuming that the entire stake is online). We can use this information to set $1 + \lambda_{\text{max}}\Delta$ as close to 1 as desired by simply setting the mining threshold appropriately. Furthermore, by setting $c$ large, $\phi_c \approx 1$ and thus PoSAT can achieve the same security threshold as PoW under full dynamic availability. We note that using a large $c$ does not degrade the dynamic availability of the protocol. This is because, in our protocol, even nodes that join in between the interval of length $c$ can start mining blocks at the tip of the chain. Furthermore, the protocol still remains fully unpredictable even with $c$ large. Thus, the only effect is that the latency to confirm a transaction increases linearly in $c$ (see Section 3). We note that this effect requires us to keep $c$ finite (unlike Ouroboros where $c$ can be chosen arbitrarily large). The constant $\phi_c$ is the amplification of the adversarial chain growth rate due to nothing-at-stake, which we calculate using the theory of branching random walks [29]. The right hand side of (1) can therefore be interpreted as the growth rate of a private adversary tree with the adversary mining on every block. Hence, condition (1) can be interpreted as the condition that the private Nakamoto attack [23] does not succeed. However, Theorem 1 is a security theorem, i.e. it gives a condition under which the protocol is secure under all possible attacks. Hence what Theorem 1 says is therefore that among all possible attacks on PoSAT, the private attack is the worst attack. We prove this by using the technique of blocktree partitioning and Nakamoto blocks, introduced in [11], which reduce all attacks to a union of private attacks.

Assuming $\lambda_{\text{max}}\Delta$ to be small and $c$ large, the comparison of PoSAT with other protocols is shown in Table 1.3. Here we use $A_a$ to be the largest adversary fraction of the total stake online at any time during the execution ($A_a = \sup_t \lambda_a(t)$). Protocols whose security guarantee assumes all adversary nodes are online all the time effectively assumes that $\lambda_{h}(t) > A_a$. Thus existing protocols have limited dynamic availability (or compromise on the potential to join late without any trusted setup).

|                     | Sleepy / Snow White / Algorand | Genesis | PoSAT                          |
|---------------------|---------------------------------|---------|--------------------------------|
| Dynamic Availability| $\lambda_{h}(t) > A_a$          | $\lambda_{h}(t) > A_a$ | $\lambda_{h}(t) > A_a$ | $\lambda_{h}(t) > \phi_c\lambda_a(t)$ |
| Trusted-set for Late-joining | Yes | Yes | No | No | No |
| Predictability      | Global | Local | Local | Local | None |

### 1.4 PoSAT has PoW Unpredictability

Another key property of PoW protocols is their ability to be unpredictable: no node (including itself) can know when a given node will be allowed to propose a block ahead of the proposal slot. This is because any other node that
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wishes to predict the leadership slot of another node also needs to perform the RandVDF sequentially and will only know its leadership status at the particular time at which the node is eligible to propose a block. We point out that PoSAT with any parameter $c$ remains unpredictable due to the unpredictability of the RandVDF till the threshold is actually reached. We refer the reader to Fig. 2(a) where if the randomness source is at the beginning of the epoch it is clear that the unpredictability of the randomized VDF implies unpredictability in our protocol. However, in case the miner has already created a block within the epoch (Fig. 2(b)), the randomness source is now her previous block. This can be thought of as a continuation of the iterative sequential function from the beginning of the epoch and hence it is also unpredictable as to when the function value will fall below a threshold. Thus PoSAT achieves true unpredictability, matching the PoW gold standard, where even an all-knowing adversary has no additional predictive power.

The first wave of PoS protocols such as Sleepy model of consensus [25] and Ouroboros [19] are fully predictable as they rely on mechanisms for proposer election that provide global knowledge of all proposers in an epoch ahead of time. The concept of Verifiable Random Functions (VRF), developed in [12,22], was pioneered in the blockchain context in Algorand [8,18], as well as applied in Ouroboros Praos [10] and Snow White [3]. The use of a private leader election using VRF enables no one else other than the proposer to know of the slots when it is allowed to propose blocks. However, unlike Bitcoin, the proposer itself can predict. Thus, these protocols still allow local predictability. The following vulnerability is caused by local predictability: a rational node may then willingly sell out his slot to an adversary. In Ouroboros Praos, such an all-knowing adversary needs to corrupt only 1 user at a time (the proposer) adaptively in order to do a double-spend attack. He will first let the chain build for some time to confirm a transaction, and then get the bribed proposers one at a time to build a competing chain. Algorand is more resilient, but even there, in each step of the BFT algorithm, a different committee of nodes is selected using a VRF based sortition algorithm. These nodes are locally predictable as soon as the previous block is confirmed by the BFT - and thus an all-knowing adversary only needs to corrupt a third of a committee. Assuming each committee is comprised of $K$ nodes ($K$ being a constant), the adversary only needs to corrupt $\frac{K}{3}$ of the nodes.

We summarize the predictability of various protocols in Table 1.3.

1.5 Related Work

Our design is based on frequent updates of randomness to run the VDF lottery. PoS protocols that update randomness at each iteration have been utilized in practice as well as theoretically proposed [14] - they do not use VDF and have neither dynamic availability nor unpredictability. Furthermore, they still face nothing-at-stake attacks. In fact, the amplification factor of $e$ we discussed earlier has been first observed in a Nakamoto private attack analysis in [14]. This analysis was subsequently extended to a full security analysis against all attacks.
where it was shown that the private attack is actually the worst attack. In \cite{30}, the idea of $c$-correlation was introduced to reduce the rate of randomness update and to reduce the severity of the nothing-at-stake attack; we borrowed this idea from them in the design of our VDF-based protocol, \textit{PoSAT}.

There have been attempts to integrate VDF into the proof-of-space paradigm \cite{9} as well as into the proof-of-stake paradigm \cite{20} both using a VRF concatenated with a VDF. But the VDF runs for a fixed duration depending on the input and hence is predictable, and furthermore these protocols do not have security proofs for dynamic availability. We note that recent work \cite{5} formalized that a broad class of PoS protocols suffer from either of the two vulnerabilities: (a) use recent randomness, thus being subject to nothing-at-stake attacks or (b) use old randomness, thus being subject to prediction based attacks (even when only locally predictable). We note that \textit{PoSAT} with large $c$ completely circumvents both vulnerabilities using the additional VDF primitive since it is able to use old randomness while still being fully unpredictable.

We want to point out that dynamic availability is distinct and complementary to dynamic stake, which implies that the set of participants and their identities in the mining is changing based on the state of the blockchain. We note that there has been much existing work addressing issues on the dynamic stake setting - for example, the $s$-longest chain rule in \cite{1}, whose adaptation to our setting we leave for future work. We emphasize that the dynamic availability problem is well posed even in the static stake setting (the total set of stakeholders is fixed at genesis).

1.6 Outline

The rest of the paper is structured as follows. Section 2 presents the VDF primitive we are using and the overall protocol. Section 3 presents the model. Section 4 presents a sketch of the analysis, the details of which are in Appendix 4.

2 Protocol

2.1 Primitives

In this section, we give an overview of VDFs and refer the reader to detailed definitions in Appendix A.

\textbf{Definition 1 (from} \cite{4}). A VDF $V = (\text{Setup}, \text{Eval}, \text{Verify})$ is a triple of algorithms as follows:

\begin{itemize}
  \item \text{Setup}(\lambda, \tau) \rightarrow \text{pp} = (ek, vk) is a randomized algorithm that produces an evaluation key $ek$ and a verification key $vk$.
  \item \text{Eval}(ek, \text{input}, \tau) \rightarrow (O, \text{proof}) takes an input $\in X$, an evaluation key $ek$, number of steps $\tau$ and produces an output $O \in Y$ and a (possibly empty) proof.
  \item \text{Verify}(vk, \text{input}, O, \text{proof}, \tau) \rightarrow \text{Yes}, \text{No} is a deterministic algorithm takes an input, output, proof, $\tau$ and outputs $\text{Yes}$ or $\text{No}$.
\end{itemize}
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VDF.Eval is usually comprised of sequential evaluation: $f^\beta(x) = f \circ f \circ \ldots \circ f(x)$ along with the ability to provide a short and easily verifiable proof. In particular, there are three separate functions VDF.Start, VDF.Iterate and VDF.Prove (the first function is used to initialize, the second one operates for the number of steps and the third one furnishes a proof). This is illustrated in Figure 3a on the left.

While VDFs have been designed as a way for proving the passage of a certain amount of time, it has been recently shown that these functions can also be used to generate an unpredictable randomness beacon [13]. Thus, running the iteration till the random time $L$ when $\text{RandVDF}(x) = f^L(x) < \tau$. This is our core transformation to get a randomized VDF. This is shown in Figure 3b on the right. Instead of running for a fixed number of iterations, we run the VDF iterations till it reaches a certain threshold. Our transformation is relatively general purpose and most VDFs can be used with our construction. For example, a VDF (which is based on squaring in a group of unknown order) is an ideal example for our construction [26,31]. In the recent paper [13], for that sequential function, a new method for obtaining a short proof whose complexity does not depend (significantly) on the number of rounds is introduced - our protocol can utilize that VDF as well. They show furthermore that they obtain a continuous VDF property which implies that partial VDF computation can be continued by a different party - we do not require this additional power in our protocol.

Normally, a VDF will satisfy correctness and soundness. And we require RandVDF to also satisfy correctness and soundness as defined below.

**Definition 2 (Correctness).** A RandVDF is correct if for all $\lambda, \tau$, parameters $(ek, vk) \leftarrow \text{Setup}(\lambda)$, and all $input \in X$, if $(O, proof) \leftarrow \text{Eval}(ek, input, \tau)$ then $\text{Verify}(vk, input, O, proof) = \text{Yes}$.

**Definition 3 (Soundness).** A RandVDF is sound if for all algorithms $A$ that run in time $O(\text{poly}(t, \lambda))$

$$\Pr\left[\begin{array}{c}
\text{Verify}(vk, input, O, proof) = \text{Yes} \\
O \neq \text{Eval}(ek, input, \tau)
\end{array}\right] \leq \text{negl}(\lambda)$$

2.2 Protocol description

The pseudocode for the PoSAT is given in Algorithm 1.

**Proposer selection.** To be elected one of the leaders, each coin first decides on where to append the next block, in its local view of the blocktree. An honest coin $n$, operating according to PoSAT, ideally would select the tip of the blockchain as its parent block $\text{parentBlk}_n$. RandVDF is used to compute an unpredictable randomness beacon based on the source of randomness $\text{RandSource}_n$, the coin’s evaluation key $\text{RandVDF}.ek_n$ and the difficulty of this computation $s_n$. The difficulty parameter $s_n$ is proportional to the current stake of the coin $n$. At a granular level, the $\text{RandVDF.Eval}(input_n, \text{RandVDF.ek}_n, s_n)$ is an iterative function that is constructed using an amalgamation of three distinct functions:
Fig. 3: VDF.Eval(input, ek, τ) requires the number of iterations that VDF.Iterate() should run, as one of its argument. Here intState denotes the internal state maintained by the VDF. On the other hand, RandVDF.Eval(input, ek, s) requires the expected number of number of iterations RandVDF.Iterate() (denoted by s) should run. At the end of each iteration, Hash(Output) is compared against the function threshold(s).

- **RandVDF.Start(input, RandVDF.ek, IntState)** initializes the iteration by setting some initial value of Output. In PoSAT, the source of randomness RandSource is taken as input. Note that IntState is just some internal state of the RandVDF.Eval().
- **RandVDF.Iterate(output, RandVDF.ek, IntState)** is the iterator function that updates output in each iteration using RandVDF.ek and IntState. At the end of each iteration, it is checked whether Hash(output) is less than Threshold(s). If No, current output is taken as input to RandVDF.Prove() described below. Note that the number of iterations, rand_iter, that would be required to pass this threshold is unpredictable, which lends to randomness in RandVDF.Eval().
- **RandVDF.Prove(output, RandVDF.ek, IntState)** operates on output using RandVDF.ek and IntState to generate proof that certifies the iterative computation done in the previous step.

The source of randomness RandSource can be updated in two ways:

- a block, mined by some other coin, is received such that its depth is multiple of $c$ (Algorithm 1 line 27) or
– if coin \( n \) wins a leader election and mines its own block (Algorithm 1 line 15).

When the coin \( n \) wins leader election, it updates the \( \text{RandSource}_n \) to output \( n \) obtained from computation of \( \text{RandVDF.Eval()} \) and then use this \( \text{RandSource}_n \) as the \( \text{input}_n \) in the next computation of \( \text{RandVDF.Eval()} \). This leads to randomness concatenation until the coin \( n \) updates \( \text{RandSource}_n \) because of the first way described above.

**Content of the block.** Once a coin is elected as a leader, all unconfirmed transactions in its buffer are added to the content. Along with the transactions, the content of the block also includes the identity of the coin that won the leader election, \( \text{input}_n, \text{RandSource}_n, \text{proof}_n, \text{rand_iter}_n \) from \( \text{RandVDF.Eval()} \). This allows other nodes to verify the work done by the coin using \( \text{RandVDF.Verify()} \). The \textit{state} variable in the content contains the hash of parent block, which ensures that the content of the parent block cannot be altered. Finally, the leader and the content is signed with the secure signature \( \text{SIGN}.sk_n \). Note that the leader election is independent of the content of the block and content of previous blocks, in order to avoid any grinding on the content. However, this allows the adversary to create multiple blocks with the same header but different content. In particular, after one leader election, the adversary can create multiple blocks appending to different parent blocks, as long as those parent blocks share the same common source of randomness. Such copies of a block with the same header but different contents are known as a “forkable string” in [19]. We show in the section 4 that the PoSAT is secure against all such variations of attacks.

### 3 Model

We will adopt a continuous-time model. Like the \( \Delta \)-synchronous model in [24], we assume there is a bounded communication delay \( \Delta \) seconds between the \( n \) honest nodes (the particular value of latency of any transmission inside this bound is chosen by the adversary).

The blockchain is run on a network of \( n \) honest nodes and a set of adversary nodes. The fraction of online honest stake, and online adversary stake at time \( t \) is referred to as \( \lambda_h(t), \lambda_a(t) \). We assume these functions are fixed \textit{a priori} deterministically, and they satisfy

\[
\lambda_a(t) \leq (1 - \eta_a)\lambda_h(t) \quad \forall \quad t \geq 0.
\]

(2)

where \( 0 < \eta_a < 1 \). Also, we assume there exists constants \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) such that

\[
\lambda_{\text{min}} \leq \lambda_h(t) \leq \lambda_{\text{max}} \quad \forall \quad t \geq 0.
\]

(3)

The existence of \( \lambda_{\text{max}} \) is obvious since we are in a proof-of-stake system, and \( \lambda_{\text{max}} \) denotes the rate of mining if every single stakeholder is online. We need
Algorithm 1 PoSAT(λ)

1: procedure Initialize( )
2:   BlkTree ← genesis  ▷ Blocktree
3:   parentBlk ← genesis  ▷ Block to mine on
4:   unCnfTx ← φ  ▷ Blk content: Pool of unconfirmed txs
5:   RandSource ← RandGenesis  ▷ Randomness embedded into genesis

6: procedure PosMining(coin)
7:   time ← SystemTime
8:   (RANDVDF.ek, RANDVDF.vk, (SIGN.vk, SIGN.sk) ← coin.Keys()
9:   stakeBk ← SearchChainUp(parentBlk)  ▷ Update the stake
10:  stake ← coin.Stake(stakeBk)
11:  s ← UpdateThreshold(parentBlk)
12:  input ← RandSource
13:  // Calling RANDVDF.EVAL()
14:  (input, output, proof, rand_iter) ← RANDVDF.EVAL(input, RANDVDF.ek, s)
15:  RandSource ← output  ▷ Update source of randomness
16:  state ← Hash(parentBlk)
17:  content ← ⟨unCnfTx, coin, input, RandSource, proof, rand_iter, state⟩
18:  return ⟨header, content, Sign(content, SIGN.sk)⟩

19: // Function to listen messages and update the blocktree
20: procedure ReceiveMessage(X)
21:   if X is a valid tx then
22:      unCnfTx ← unCnfTx ∪ {X}
23:   else if IsValidBlock(X) then
24:      if parentBlk.Height() < X.Height() then
25:         ChangeMainChain(X)  ▷ If the new chain is longer
26:      if parentBlk.Height() % c == c − 1 then
27:         RandSource ← X.content.RandSource
28:      else if YourBlockIsInPresentEpoch() then
29:         RandSource ← YourLatestBlockInEpoch().RandSource
30:      else
31:         RandSource ← RandSource
32:   procedure IsValidBlock(X)  ▷ returns true if a block is valid
33:      if not IsUnspent(X.content.coin) then return False
34:      s ← UpdateThreshold(GetParentBlk(X))
35:      if Hash(X.content.RandSource) > Threshold(s) then return False
36:      then
37:         return True
38:      else
39:         return False
40: procedure Main( )
41:   Initialize()
42:   StartThread(ReceiveMessage)
43:   while True do
44:      block ← PosMining(coin)
45:      SendMessage(block)  ▷ Broadcast to the whole network
to assume a minimum mining rate in order to guarantee that within a bounded time, confirmation can be guaranteed.

The evolution of the blockchain can be modeled as a process \( \{(T(t), C(t), T^{(p)}(t), C^{(p)}(t)) : t \geq 0, 1 \leq p \leq n\} \), \( n \) being the number of honest miners, where:

- \( T(t) \) is a tree, and is interpreted as the mother tree consisting of all the blocks that are mined by both the honest and the adversary nodes up until time \( t \) (including private blocks at the adversary).
- \( T^{(p)}(t) \) is an induced (public) sub-tree of the mother tree \( T(t) \) in the view of the \( p \)-th honest node at time \( t \).
- \( C^{(p)}(t) \) is a longest chain in the tree \( T^{(p)}(t) \), and is interpreted as the longest chain in the local view of the \( p \)-th honest node.
- \( C(t) \) is the common prefix of all the local chains \( C^{(p)}(t) \) for \( 1 \leq p \leq n \).

The process evolution is as follows.

- \( M0: T(0) = T^{(p)}(0) = C^{(p)}(0), 1 \leq p \leq n \) is a single root block (genesis).
- \( M1: \) The adversary blocks are mined according to multiple independent Poisson processes of rate \( \lambda_h(t) \) at time \( t \), one for every block mined at levels \( c, 2c, \ldots, nc, \ldots \). The adversary appends a copy of the mined block from a block at level \( nc \) to any of its descendents in the next \( c-1 \) levels \( nc, nc+1, \ldots nc+c-1 \) (including blocks that may be mined in the future). We note that the same block cannot be appended in the chain twice. We refer the reader to Figure 4 for a visual representation.
- \( M2: \) Honest blocks are mined at a total rate of \( \lambda_h(t) \) at time \( t \) across all the honest nodes at the tip of the chain held by the mining node \( p \), i.e., \( C^{(p)}(t) \).
- \( M3: \) The adversary can replace \( T^{(p)}(t^-) \) by another sub-tree \( T^{(p)}(t) \) from \( T(t) \) as long as the new sub-tree \( T^{(p)}(t) \) is an induced sub-tree of the new tree \( T^{(p)}(t) \), and can update \( C^{(p)}(t^-) \) to a longest chain in \( T^{(p)}(t) \). \(^1\)

We highlight the capabilities of the adversary in this model:

- \( A1: \) Can choose to mine on multiple blocks of the tree \( T(t) \) at any time.
- \( A2: \) Can delay the communication of blocks between the honest nodes, but no more than \( \Delta \) time.
- \( A3: \) Can broadcast privately mined blocks at times of its own choosing: when private blocks are made public at time \( t \) to node \( p \), then these nodes are added to \( T^{(p)}(t^-) \) to obtain \( T^{(p)}(t) \). Note that by property \( M3(i) \), when private blocks appear in the view of some honest node \( p \), they will also appear in the view of all other honest nodes by time \( t + \Delta \).
- \( A4: \) Can switch the \( p \)-th honest node’s mining from one longest chain to another of equal length at any time, even when its view of the tree does not change. In this case, \( T^{(p)}(t) = T^{(p)}(t^-) \) but \( C^{(p)}(t) \neq C^{(p)}(t^-) \).

Proving the security (persistence and liveness) of the protocol boils down to providing a guarantee that the chain \( C(t) \) converges fast as \( t \to \infty \) and that honest blocks enter regularly into \( C(t) \) regardless of the adversary’s strategy.

\(^1\) All jump processes are assumed to be right-continuous with left limits, so that \( C(t), T(t) \) etc. include the new arrival if there is a new arrival at time \( t \).
Fig. 4: The mining process. There is a separate randomness generated for every block in the modulo $c$ position. Blocks generated from that randomness can attach to any block inside the next $c - 1$ blocks (including blocks mined in the future).

4 Security Analysis

Our goal is to generate a transaction ledger that satisfies persistence and liveness as defined in [15]. Together, persistence and liveness guarantee robust transaction ledger; honest transactions will be adopted to the ledger and be immutable.

Definition 4 (from [15]). A protocol $\Pi$ maintains a robust public transaction ledger if it organizes the ledger as a blockchain of transactions and it satisfies the following two properties:

- (Persistence) Parameterized by $\tau \in \mathbb{R}$, if at a certain time a transaction $tx$ appears in a block which is mined more than $\tau$ time away from the mining time of the tip of the main chain of an honest node (such transaction will be called confirmed), then $tx$ will be confirmed by all honest nodes in the same position in the ledger.

- (Liveness) Parameterized by $u \in \mathbb{R}$, if a transaction $tx$ is received by all honest nodes for more than time $u$, then all honest nodes will contain $tx$ in the same place in the ledger forever.

The theorem below shows that the the private attack threshold yields the true security threshold:

Theorem 2. If

$$\phi_c \lambda_a(t) < \frac{\lambda_h(t)}{1 + \lambda_{\max} \Delta},$$

then the PoSAT generate transaction ledgers satisfying persistence (parameterized by $\tau = \sigma$) and liveness (parameterized by $u = \sigma$) in Definition 4 with probability at least $1 - e^{-\Omega(\sigma^{1-\epsilon})}$, for any $\epsilon > 0$. 
4.1 Approach

In order to prove Theorem 2, we utilize the concept of blocktree partitioning and Nakamoto blocks that were introduced in [11]. We provide a brief overview of these concepts here.

Let $\tau^h_i$ and $\tau^a_i$ be the mining time of the $i$-th honest and adversary blocks respectively; $\tau^h_0 = 0$ is the mining time of the genesis block, which we consider as the 0-th honest block.

**Definition 1.** **Blocktree partitioning** Given the mother tree $T(t)$, define for the $i$-th honest block $b_i$, the adversary tree $T_{orig}^{i}(t)$ to be the sub-tree of the mother tree $T(t)$ rooted at $b_i$ and consists of all the adversary blocks that can be reached from $b_i$ without going through another honest block. The mother tree $T(t)$ is partitioned into sub-trees $T_{orig}^{0}(t), T_{orig}^{1}(t), \ldots T_{orig}^{j}(t)$, where the $j$-th honest block is the last honest block that was mined before time $t$.

See Figure 2 in [11] for an example.

The sub-tree $T_{orig}^{i}(t)$ is born at time $\tau^h_i$ as a single block $b_i$ and then grows each time an adversary block is appended to a chain of adversary blocks from $b_i$. Let $D^{dyn}_i(t)$ denote the depth of $T_{orig}^{i}(t)$; $D^{dyn}_i(\tau^h_i) = 0$.

**Definition 2.** [27] The $j$-th honest block mined at time $\tau^h_j$ is called a **loner** if there are no other honest blocks mined in the time interval $[\tau^h_j - \Delta, \tau^h_j + \Delta]$.

**Definition 3.** Given honest block mining times $\tau^h_i$'s, define a honest fictitious tree $T_h(t)$ as a tree which evolves as follows:

1. $T_h(0)$ is the genesis block.
2. The first mined honest block and all honest blocks within $\Delta$ are all appended to the genesis block at their respective mining times to form the first level.
3. The next honest block mined and all honest blocks mined within time $\Delta$ of that are added to form the second level (which first level blocks are parents to which new blocks is immaterial).
4. The process repeats.

Let $D^{dyn}_h(t)$ be the depth of $T_h(t)$.

**Definition 4.** **(Nakamoto block)** Let us define:

$$E_{ij} = \text{event that } D^{dyn}_i(t) < D^{dyn}_h(t - \Delta) - D^{dyn}_h(\tau^h_i + \Delta) \text{ for all } t > \tau^h_j + \Delta.$$  \hspace{1cm} (4)

The $j$-th honest block is called a **Nakamoto block** if it is a loner and

$$F_j = \bigcap_{i=0}^{j-1} E_{ij}$$  \hspace{1cm} (5)

occurs.

The concepts of blocktree partitioning and Nakamoto blocks is illustrated in Figure 5.
Fig. 5: The blocktree is partitioned into a fictitious honest chain, consisting of only honest blocks, and a set of adversarial trees, each emanating from a honest block. The security of the system is viewed as a race between the adversary trees and the fictitious honest chain. While there may be multiple adversary trees simultaneously racing with the honest chain, the growth rate of each tree is bounded by the growth rate of the adversary chain in the private attack. An honest block is a Nakamoto block when all the previous adversary trees never catch up with the honest chain past that block. A Nakamoto block is guaranteed to stay in the longest chain forever. If Nakamoto blocks exist, then the ledger is secure.
Lemma 1. (Theorem 3.2 in [11]) (Nakamoto blocks stabilize) If the $j$-th honest block is a Nakamoto block, then it will be in the longest chain $C(t)$ for all $t > \tau_j^h + \Delta$.

Lemma 1 states that Nakamoto blocks remain in the longest chain forever. The question is whether they exist and appear frequently regardless of the adversary strategy. If they do, then the protocol has liveness and persistency: honest transactions can enter the ledger frequently through the Nakamoto blocks, and once they enter, they remain at a fixed location in the ledger. More formally, we have the following result.

Lemma 2. (Lemma 4.4 in [11]) Define $B_{s,s+t}$ as the event that there is no Nakamoto blocks in the time interval $[s,s+t]$. If

$$P(B_{s,s+t}) < q_t < 1$$

for some $q_t$ independent of $s$ and the adversary strategy, then the protocol generates transaction ledgers satisfying persistence (parameterized by $\tau = \sigma$) and liveness (parameterized by $u = \sigma$) in Definition 4 with probability at least $1 - q_\sigma$.

Lemma 2 reduces the problem of PoSAT guaranteeing persistence and liveness to that of bounding the probability that there are no Nakamoto blocks in a long duration. To prove Lemma 2, we follow a similar style of reasoning as in [11]:

1. Show that the probability that the $j$-th honest block is a Nakamoto block is lower bounded by some $p > 0$ for all $j$ and for all adversary strategy, in the parameter regime when the private attack growth rate is less than the honest chain growth rate.

2. Bootstrap from (1) to bound the probability of the event $B_{s,s+t}$, an event of no occurrence of Nakamoto blocks for a long time.

Intuitively, if (1) holds, then one would expect that the chance that Nakamoto blocks do not occur over a long time is low, provided that a block being Nakamoto is close to independent of another block being Nakamoto if the mining times of the two blocks are far apart. We perform the bootstrapping in (2) by exploiting this fact.

Without loss of generality, we assume that the adversarial power is boosted such that $\lambda_a(t) = (1 - \eta_a)\lambda_h(t)$. In order to prove Lemma 2 we first perform the following transformations:

– We show that the existence of Nakamoto blocks in the dynamic available system with time-varying number of online nodes is implied by the existence of Nakamoto blocks in a static system with constant number of nodes at all times. This is accomplished by a time-warping argument which is explained in detail in section B.1 of the appendix. At a high level, the system with dynamic availability is shown to be equivalent to a system with static availability but where time runs at a different rate $\alpha(t)$ (the warped time). Also, let $B_{s,s+t}^{\text{static}}$ be the event of no occurrence of Nakamoto blocks in the interval $[\alpha(s), \alpha(s+t)]$ in the static system. Then, we have the following result.
Lemma 3. For any time interval \([s, s+t]\) in the time horizon of dynamic available system, we have

\[ B_{s,s+t} \subseteq B_{\alpha(s),\alpha(s+t)}^{\text{static}}. \]

The proof for this lemma is given in section \[ B.2 \].

For the static system when epoch length \( c = 1 \), the existence of Nakamoto blocks has already been analyzed in [11] (for Model (c) in Figure 4). We extend this analysis to the case \( c > 1 \). Observe that the root \( b_i \) (i-th honest block) of an adversarial tree \( T_{\text{orig}}^i(t) \) can lie at any level within an epoch. For convenience of our analysis, we consider a more powerful adversary where it is able to diversify the common source of randomness immediately after the root. This powerful adversary then updates its source of randomness at every level that differs from level of \( b_i \) by a multiple of \( c \). Now, the growth rate of this adversarial tree (mined by the powerful adversary) can be obtained by transforming the tree into a tree of superblocks where each level corresponds to an epoch of \( c \) levels in the original tree (Figure 6). The growth rate for \( c > 1 \) is slower than for \( c = 1 \). See Figure 7.

![Figure 6](image)

Fig. 6: An example of an adversarial tree with \( c = 4 \) under our assumption of the more powerful adversary. Blocks with same color share the same common source of randomness. The adversary is able to diversify the common randomness immediately at the first level. This tree can be transformed into a tree of superblocks; one level in this transformed tree corresponds to an epoch of \( c \) levels in the original tree.

Let \( B_{\alpha(s),\alpha(s+t)}^{\text{superblock}} \) be the event of no occurrence of Nakamoto blocks in the interval \([\alpha(s), \alpha(s+t)]\) in the static system with adversarial tree being composed of only superblocks under the aforementioned powerful adversary. Then, we have the following result.
Lemma 4. For any time interval \([s, s + t]\) in the time horizon of dynamic available system, we have

\[
B_{\alpha(s), \alpha(s+t)}^{\text{static}} \subseteq B_{\alpha(s), \alpha(s+t)}^{\text{superblock}}.
\]

The proof for this lemma is given in section \(C\).

\[
\begin{align*}
\text{LEGEND} \\
\text{PRIVATE} \\
\text{ADVERSARIAL} \\
\text{BLOCKS}
\end{align*}
\]

Fig. 7: Adversarial tree for \(c = 1\) vs \(c > 1\).

With Lemma 5, we show below that in the regime \(\phi_{c, \lambda}(t) < \frac{\lambda_h(t)}{1 + \lambda_{\max} t}\), Nakamoto blocks has a non-zero probability of occurrence.

Lemma 5. If

\[
\phi_{c, \lambda}(t) < \frac{\lambda_h(t)}{1 + \lambda_{\max} t},
\]

then there is a \(p > 0\) such that the probability the \(j\)th honest block is a Nakamoto block is greater than \(p\) for all \(j\).

The proof of this result can be found in section \(F.1\) of the appendix.

Having established the fact that Nakamoto blocks occurs with non-zero frequency, we can bootstrap on Lemma 5 to get a bound on the probability that in a time interval \([s, s + t]\) in the static system, there are no Nakamoto blocks in the interval \([\alpha(s), \alpha(s + t)]\) in the static system, i.e. a bound on \(P(B_{\alpha(s), \alpha(s+t)}^{\text{superblock}})\).

Lemma 6. If

\[
\phi_{c, \lambda}(t) < \frac{\lambda_h(t)}{1 + \lambda_{\max} t},
\]

then for any \(\epsilon > 0\) there exist constants \(\bar{a}_\epsilon, \bar{A}_\epsilon\) so that for all \(s, t \geq 0\),

\[
P(B_{\alpha(s), \alpha(s+t)}^{\text{superblock}}) \leq \bar{A}_\epsilon \exp(-\bar{a}_\epsilon t^{1-\epsilon})
\]

(7)

where \(\bar{a}_\epsilon\) is a function of \(\lambda_{\min}\).

The proof of this result can be found in section \(F.2\) of the appendix. Then, combining Lemma 6 with Lemma 3, Lemma 4 and Lemma 2 implies Theorem 1.
5 Discussion

In this section, we discuss some of the practical considerations in adopting PoSAT.

We note that unlike other longest-chain based PoS protocols, we do not require nodes to accept / reject blocks based on time-stamps, thus we eliminate the requirement of clock synchronization. Thus PoSAT bears closer resemblance to a proof-of-work protocol, which do not require clock synchronization, than other longest-chain proof-of-stake protocols.

In PoSAT, a separate RandVDF needs to be run for each public-key. In a purely decentralized implementation, all nodes may not have the same rate of computing VDF. This may disadvantage nodes whose rate of doing sequential computation is slower. One approach to solve this problem is to build open-source hardware for VDF - this is already under way through the VDF Alliance. Even under such a circumstance, it is to be expected that nodes that can operate their hardware in idealized circumstances (for example, using specialized cooling equipment) can gain an advantage. A desirable feature of our protocol is that gains obtained by a slight advantage in the VDF computation rate are bounded. For PoSAT, a combination of the VDF computation rate and the stake together yields the net power wielded by a node, and as long as a majority of such power is controlled by honest nodes, we can expect the protocol to be safe.

In our PoSAT specification, the mining threshold was assumed to be fixed. This threshold was chosen based on the entire stake being online - this was to ensure that forking even when all nodes are present remains small, i.e., $\lambda_{\max} \Delta$ remained small. In periods when far fewer nodes are online, this leads to a slowdown in confirmation latency. A natural way to mitigate this problem is to use a variable mining threshold based on past history, similar to the adaptation inherent in Bitcoin. A formal analysis of Bitcoin with variable difficulty was carried out in \cite{16,17}, we leave a similar analysis of our protocol for future work.

In our protocol statement, we have used the RandVDF directly on the randomness prevRand and the public key. The RandVDF ensures that any other node can only predict a given node’s leadership slot at the instant that it actually wins the VDF lottery. However, this still enables an adversary to predict the leadership slots of nodes that are offline and can potentially bribe them to come online to favor the adversary. In order to eliminate this exposure, we can replace the hash in the mining condition by using a verifiable random function \cite{12,22} (which is calculated using the node’s secret key but can be checked using the public key). This ensures that an adversary which is aware of all the public state as well as private state of all online nodes (including their VRF outputs) still cannot predict the leadership slot of any node ahead of the time at which they can mine the block. This is because, such an adversary does not have access to the VRF output of the offline nodes.

Finally, while we specified PoSAT in the context of proof-of-stake, the ideas can apply to other mining modalities - the most natural example is proof-of-space. We note that existing proof-of-space protocols like Chia \cite{9}, use a VDF for a fixed time, thus making the proof-of-space challenge predictable. In proof-
of-space, if the predictability window is large, it is possible to use slow-storage mechanisms such as magnetic disks (which are asymmetrically available with large corporations) to answer the proof-of-space challenges. Our solution of using a RandVDF can be naturally adapted to this setting, yielding unpredictability as well as full dynamic availability.

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Appendix

A Verifiable Delay Function (VDF)

In this section, we give a brief description of VDFs, starting with its definition.

**Definition 5 (from [4]).** A VDF $V = (\text{Setup}, \text{Eval}, \text{Verify})$ is a triple of algorithms as follows:

- $\text{Setup}(\lambda, \tau) \rightarrow \text{pp} = (ek, vk)$ is a randomized algorithm that takes a security parameter $\lambda$ and a desired puzzle difficulty $\tau$ and produces public parameters $\text{pp}$ that consists of an evaluation key $ek$ and a verification key $vk$. We require $\text{Setup}$ to be polynomial-time in $\lambda$. By convention, the public parameters specify an input space $X$ and an output space $Y$. We assume that $X$ is efficiently sampleable. $\text{Setup}$ might need secret randomness, leading to a scheme requiring a trusted setup. For meaningful security, the puzzle difficulty $\tau$ is restricted to be sub-exponentially sized in $\lambda$.

- $\text{Eval}(ek, \text{input}, \tau) \rightarrow (O, \text{proof})$ takes an input $\in X$ and produces an output $O \in Y$ and a (possibly empty) proof. $\text{Eval}$ may use random bits to generate the proof but not to compute $O$. For all $\text{pp}$ generated by $\text{Setup}(\lambda, \tau)$ and all input $\in X$, algorithm $\text{Eval}(ek, \text{input}, \tau)$ must run in parallel time $\tau$ with $\text{poly}(\log(\tau), \lambda)$ processors.

- $\text{Verify}(vk, \text{input}, O, \text{proof}) \rightarrow \text{Yes, No}$ is a deterministic algorithm takes an input, output and proof and outputs Yes or No. Algorithm $\text{Verify}$ must run in total time polynomial in $\log \tau$ and $\lambda$. Notice that $\text{Verify}$ is much faster than $\text{Eval}$.

B Dynamic available system to static system

B.1 Time Warping

One can imagine another frame of reference, say $\mathcal{F}^{\text{static}}$, where there is another local clock such that the mining rate for honest users will appear to be constant (say $\lambda_h$) but the clock is skewed when viewed from the original frame of reference $\mathcal{F}^{\text{dyn}}$. That is, if the honest mining rate in frame of reference $\mathcal{F}^{\text{dyn}}$ at time $t$ is $\lambda_h(t) = 10\lambda_h$, then, the local clock in the frame of reference $\mathcal{F}^{\text{static}}$ is skewed $10 \times$ faster and the honest mining rate in $\mathcal{F}^{\text{static}}$ is $\lambda_h$. Clearly, this skew, will be varying with respect to the local clock at the original frame of reference $\mathcal{F}^{\text{dyn}}$. Let $\alpha(t)$ be the local clock time in the frame of reference $\mathcal{F}^{\text{static}}$ for some time $t$ in the frame of reference $\mathcal{F}^{\text{dyn}}$. This is illustrated in Fig 8.
Thus,

\[ \lambda_h(u)du = \lambda_h d\alpha \implies \alpha(t) = \int_0^t \frac{\lambda_h(u)}{\lambda_h} du, \]  

and

\[ \lambda_h(u)du = \lambda_h d\alpha \implies \int_{t_1}^{t_2} \lambda_h(t)dt = \lambda_h [\alpha(t_2) - \alpha(t_1)] \]  

Note that \( \alpha(t) \) is an increasing function in \( t \). Next, we will construct the relationship between the ordering of same set of events in both the frames of references \( F_{dyn} \) and \( F_{static} \).

**Lemma 7.** The ordering of events in the frame of reference \( F_{dyn} \) is same as in the frame of reference \( F_{static} \).

**Proof.** Suppose there are two events \( E_1 \) and \( E_2 \) that happen in the original frame of reference \( F_{dyn} \) such that \( t_{E_1} < t_{E_2} \), that is, \( E_1 \) happen before \( E_2 \) in \( F_{dyn} \). By contradiction, assume that \( E_2 \) happen before \( E_1 \) in the frame of reference \( F_{static} \). By equation 8, that implies, \( \alpha(t_{E_2}) < \alpha(t_{E_1}) \). However, this contradicts the fact that \( \alpha(t) \) is an increasing function in \( t \).

Lemma 7 guarantees that if there is any Nakamoto block occurring in the time interval \([s, s + t]\) in the frame of reference \( F_{dyn} \), then that Nakamoto block is also occurring in the time interval \([\alpha(s), \alpha(s + t)]\) in the frame of reference \( F_{static} \).

Recall that in the frame of reference \( F_{static} \), the honest mining rate is constant. Now, we will discuss about the adversarial mining rate in this frame of reference \( F_{static} \). For adversarial mining rate in the reference frame \( F_{static} \), we have

\[
\int_{t_1}^{t_2} \lambda_a(t)dt = (1 - \eta_a) \int_{t_1}^{t_2} \lambda_h(t)dt = (1 - \eta_a) \int_{\alpha(t_1)}^{\alpha(t_2)} \lambda_h d\alpha \\
= (1 - \eta_a)\lambda_h [\alpha(t_2) - \alpha(t_1)]
\]  

(10)
Equation 10 says that the number of adversarial blocks mined (at a rate $\lambda_a(t)$) in the frame of reference $F^{\text{dyn}}$ in an interval $[t_1, t_2]$ is equal to the number of adversarial blocks mined in the reference frame $F^{\text{static}}$ with a mining rate of $(1 - \eta_h)\lambda_h$ in the time interval $[\alpha(t_1), \alpha(t_2)]$. That implies, in terms of the number of adversarial blocks that could be mined, the adversary with mining rate $\lambda_a(t)$ in $F^{\text{dyn}}$ is identical to the adversary with constant mining rate $(1 - \eta_h)\lambda_h$ in $F^{\text{static}}$. We define the following events for the reference frames $F^{\text{dyn}}$ (with honest and adversarial mining rates being $\lambda_h(t)$ and $\lambda_a(t)$) and $F^{\text{static}}$ (with honest and adversarial mining rates being $\lambda_h$ and $(1 - \eta_h)\lambda_h$, respectively):

- $B_{s,s+t}$ - event that there is no Nakamoto block in the time interval $[s, s + t]$ in the frame of reference $F^{\text{dyn}}$,
- $B^*_{\alpha(s),\alpha(s+t)}$ - event that there is no Nakamoto block in the time interval $[\alpha(s), \alpha(s + t)]$ in the frame of reference $F^{\text{const}}$ where $s, t$ are defined over the time horizon of the frame of reference $F^{\text{dyn}}$. Note that the blocktree partitioning is done over the frame of reference $F^{\text{dyn}}$.

Let $D^*_i(.)$ and $D_h(.)$ be the corresponding depth functions of the adversarial tree rooted in $i$-th honest block and fictitious honest tree, respectively, defined in the frame of reference $F^{\text{static}}$. That is, the domain of both $D^*_i(.)$ and $D_h(.)$ is the time horizon in in the frame of reference $F^{\text{static}}$. Note that the domain of both $D^*_i(.)$ and $D^{\text{dyn}}_i(.)$ is in the time horizon of the frame of reference $F^{\text{dyn}}$. Thus, $D_i^{\text{dyn}}(t) = D^*_i(\alpha(t))$ and $D_h^{\text{dyn}}(t) = D_h(\alpha(t))$ for all $t > 0$ in the time horizon of $F^{\text{dyn}}$.

**B.2 Proof of Lemma 3**

Suppose we have an adversary $A$ operating in $F^{\text{dyn}}$ with a mining rate $\lambda_a(t)$. Consider the time interval $[s, s + t]$ in $F^{\text{dyn}}$ and its equivalent time interval $[\alpha(s), \alpha(s + t)]$ in $F^{\text{static}}$. Suppose that $B_{s,s+t}$ happens in $F^{\text{dyn}}$. This implies that, in $F^{\text{dyn}}$, for every honest block $b_j$ mined at $\tau^h_j \in [s, s + t]$, there exists some minimum time $t' > \tau^h_j + \Delta$ and some honest block $b_i$ mined at $\tau^h_i$ such that

$$D_i^{\text{dyn}}(t') \geq D_h^{\text{dyn}}(t' - \Delta) - D_h^{\text{dyn}}(\tau^h_i + \Delta).$$

Due to Lemma 7, the sample path $\omega$ of the evolution of the blockchain in $F^{\text{dyn}}$ in the interval $[\tau^h_i, t']$ happens in the same order in $F^{\text{static}}$ in the time interval $[\alpha(\tau^h_i), \alpha(t')]$. Then, we have

$$D^*_i(\alpha(t')) \geq D_h(\alpha(t' - \Delta)) - D_h(\alpha(\tau^h_i + \Delta))$$

Note that this catch-up event is happening due to the actions of the adversary $A$ whose mining rate is $\lambda_a(t)$ in the frame of reference $F^{\text{dyn}}$. If, in the frame of reference $F^{\text{static}}$, adversary $A$ had the boosted mining rate $(1 - \eta_h)\lambda_h$ and followed the same sample path as in $\omega$, then for some $u \leq \alpha(t')$, we have

$$D^*_i(\alpha(u)) \geq D_h(\alpha(u - \Delta)) - D_h(\alpha(\tau^h_i + \Delta))$$
That implies $b_j$ is not a Nakamoto block in $F^{\text{static}}$. Since, $b_j$ is any arbitrary honest block with $\tau^h_j \in [s, s+t]$, therefore this is true for all honest blocks $j'$ with $\tau^h_j \in [s, s+t]$. Hence, $B_{s,s+t} \subseteq B_{\alpha(s),\alpha(s+t)}^{\text{static}}$. This concludes our lemma.

Therefore, it is sufficient to analyse the security of the $c$–correlation in the frame of reference $F^{\text{static}}$ with honest and adversarial mining rates being $\lambda_h$ and $(1 - \eta_a)\lambda_h$, respectively. In the rest of the section, we will be analysing the security of the system in the reference frame $F^{\text{static}}$ and take $\lambda_a = (1 - \eta_a)\lambda_h$.

Owing to the transformation from dynamic available system to static system, we modify our definition of Nakamoto block. From herein, we will consider the time horizon to be over dynamic available system (reference frame $F^{\text{dyn}}$) and thus blocktree partitioning will be done over dynamic available system. However, in order to leverage the constant honest and adversarial mining rate in static system (reference frame $F^{\text{static}}$), we will define the depths of both the fictitious honest tree and the adversarial tree over the time horizon of the static system.

**Definition 5. (Nakamoto block)** Let us define:

\[ E_{ij} = \text{event that } D^i_s(\alpha(t)) < D^i_h(\alpha(t - \Delta)) - D^i_h(\alpha(\tau^h_i + \Delta)) \]  \hspace{1cm} (11)

for all $t > \tau^h_j + \Delta$ in frame of reference $F^{\text{dyn}}$.

The $j$-th honest block is called a *Nakamoto block* if it is a loner and

\[ F_j = \bigcap_{i=0}^{j-1} E_{ij} \]  \hspace{1cm} (12)

occurs.

**C Proof of Lemma 4**

For our convenience, let $A$ be the original adversary in $F^{\text{static}}$ (defined in section B) and $\hat{A}$ be the more powerful adversary that is able to diversify its source of randomness immediately in the first level after the root. We can state the following result on the dominant strategy for the adversary $A$.

**Lemma 8.** The optimal adversarial strategy to grow the tree fast is to only fork at the parents of blocks such that level of that parent block is $(c - 1) \mod c$.

**Proof.** Proof is similar to that of Lemma 13 in [2].

This optimal adversarial strategy also holds for $\hat{A}$. Going forward, we will assume that both adversaries operate under this optimal strategy.

Observe that the tree obtained by removing all the blocks until the first update of source of randomness (after root) in the adversarial tree mined by $A$ can be an adversarial tree mined by $\hat{A}$. The number of levels that would be removed by this operation from the adversarial tree mined by $A$ is at most $c - 1$. Referring to Fig 6, let $D_1(.)$ be the depth function of the adversarial tree composed of superblocks, with honest block $b_i$ being its root. Then, transforming
the adversarial tree mined by $\hat{A}$ into an adversarial tree of superblocks, for any $t > 0$ in the frame of reference $\mathcal{F}^{dyn}$, we have

$$D_s^i(\alpha(t)) < cD_i(\alpha(t)), \quad (13)$$

where $\alpha(t)$ is the corresponding time in the frame of reference $\mathcal{F}^{static}$ and $D_s^i(\alpha(t))$ is the depth function of the adversarial tree mined by $A$ (defined in section $B$).

We modify our definition of Nakamoto blocks from section $B$ to incorporate the transformation to the adversarial tree composed of superblocks.

**Definition 6. (Nakamoto block)** Let us define:

$$E_{ij} = \text{event that } cD_i(\alpha(t)) < D_h(\alpha(t - \Delta)) - D_h(\alpha(\tau_{ij}^h + \Delta)) \quad (14)$$

for all $t > \tau_{ij}^h + \Delta$ in frame of reference $\mathcal{F}^{dyn}$.

The $j$-th honest block is called a *Nakamoto block* if it is a loner and

$$F_j = \bigcap_{i=0}^{j-1} E_{ij} \quad (15)$$

occurs.

Using equation $13$ and this updated definition of Nakamoto block, we have our lemma.

**D Definitions and Preliminary Lemmas**

Let $U_j$ be the event that the $j$-th honest block $b_j$ is a loner, i.e.,

$$V_j = \{\tau_{j-1}^h < \tau_j^h - \Delta\} \cap \{\tau_j^h + \Delta > \tau_{j+1}^h + \Delta\}$$

Let $\hat{F}_j = V_j \cap F_j$ be the event that $b_j$ is a Nakamoto block. We define the following “potential” catch up event:

$$\hat{A}_{ik} = \{cD_i(\alpha(\tau_k^h + \Delta)) \geq D_h(\alpha(\tau_{k-1}^h)) - D_h(\alpha(\tau_i^h + \Delta))\}, \quad (16)$$

which is the event that the adversary launches a private attack starting from honest block $b_i$ and catches up the fictitious honest chain right before honest block $b_k$ is mined. The following lemma shows that event $\hat{F}_j$ can be represented with $\hat{A}_{ik}$’s.

**Lemma 9.** For each $j$,

$$\hat{F}_j^c = F_j^c \cup V_j^c = \left( \bigcup_{(i,k):0 \leq i < j < k} \hat{A}_{ik} \right) \cup V_j^c. \quad (17)$$
Proof.

\[ V_j \cap E_{ij} = V_j \cap \{ cD_i(\alpha(t)) < D_h(\alpha(t - \Delta)) - D_h(\alpha(\tau^h_i + \Delta)) \text{ for all } t > \tau^h_i + \Delta \} \]

\[ = V_j \cap \{ cD_i(\alpha(t + \Delta)) < D_h(\alpha(t)) - D_h(\alpha(\tau^h_i + \Delta)) \text{ for all } t > \tau^h_i \} \]

\[ = V_j \cap \{ cD_i(\alpha(\tau^h_k - \Delta)) < D_h(\alpha(\tau^h_k - \Delta))) - D_h(\alpha(\tau^h_i + \Delta)) \text{ for all } k > j \} \]

\[ = V_j \cap \{ cD_i(\alpha(\tau^h_k + \Delta)) < D_h(\alpha(\tau^h_k + \Delta)) - D_h(\alpha(\tau^h_i + \Delta)) \text{ for all } k > j \} \]

Since \( \hat{F}_j = F_j \cap V_j = \bigcap_{0 \leq i < j} E_{ij} \cap V_j \), by the definition of \( A_{ik} \) we have \( \hat{F}_j = \left( \bigcap_{(i,k):0 \leq i < j \leq k} A_{ik}^c \right) \cap V_j \). Taking complement on both side, we can conclude the proof.

Next, define the following events

\[ U_j = \{ \alpha(\tau^h_{i-1}) < \alpha(\tau^h_j) - \frac{\lambda_{\max}}{\lambda_h} \Delta \} \bigcap \{ \alpha(\tau^h_{j+1}) > \alpha(\tau^h_j) + \frac{\lambda_{\max}}{\lambda_h} \Delta \} \]

(18)

\[ \hat{B}_{ik} = \{ cD_i(\alpha(\tau^h_k) + \frac{\lambda_{\max}}{\lambda_h}) \geq D_h(\alpha(\tau^h_k - 1)) - D_h(\alpha(\tau^h_i) + \frac{\lambda_{\max}}{\lambda_h}) \} \]

(19)

Lemma 10. For any pair of \( i, k \),

\[ A_{ik} \subseteq \hat{B}_{ik}. \]

Proof. Using equation [8] we have

\[ \alpha(\tau^h_k + \Delta) = \int_0^{\tau^h_k + \Delta} \frac{\lambda_h(u)}{\lambda_h} du = \int_0^{\tau^h_k} \frac{\lambda_h(u)}{\lambda_h} du + \int_{\tau^h_k}^{\tau^h_k + \Delta} \frac{\lambda_h(u)}{\lambda_h} du \]

\[ \leq \alpha(\tau^h_k) + \frac{\lambda_{\max}}{\lambda_h} \Delta \]

Similarly, \( \alpha(\tau^h_k + \Delta) \leq \alpha(\tau^h_k) + \frac{\lambda_{\max}}{\lambda_h} \Delta \). Because \( D_h(.) \) and \( D_i(.) \) are increasing functions over their domain, we have

\[ D_i(\alpha(\tau^h_k + \Delta)) \leq D_i(\alpha(\tau^h_k) + \frac{\lambda_{\max}}{\lambda_h} \Delta) \text{ and } \]

\[ D_h(\alpha(\tau^h_k + \Delta)) \leq D_h(\alpha(\tau^h_k) + \frac{\lambda_{\max}}{\lambda_h} \Delta) \]

\[ \square \]

Lemma 11. For all \( j \),

\[ U_j \subseteq V_j. \]

Proof. This can be proved using the fact that \( \int_{\tau^h_{j-1} + \Delta}^{\tau^h_j + \Delta} \frac{\lambda_h(u)}{\lambda_h} du \leq \frac{\lambda_{\max}}{\lambda_h} \Delta \) and

\[ \int_{\tau^h_{j-1}}^{\tau^h_j + \Delta} \frac{\lambda_h(u)}{\lambda_h} du \leq \frac{\lambda_{\max}}{\lambda_h} \Delta. \]

\[ \square \]
Let \( R_m = \alpha(\tau^h_m) - \alpha(\tau^h_{m+1}) \) and \( \Delta' = \frac{\Delta_{\text{max}}}{\lambda_h} \Delta \). Then, \( U_j \) and \( \hat{B}_{ik} \) can be re-written as:

\[
U_j = \{ \Delta' < R_{j-1} \} \bigcap \{ R_j > \Delta' \} \\
\hat{B}_{ik} = \left\{ cD_i(\alpha(\tau^h_j) + \sum_{m=1}^{k-1} R_m + \Delta') \geq D_h(\alpha(\tau^h_{k-1})) - D_h(\alpha(\tau^h_k) + \Delta') \right\}
\]

(20)

**Remark 1.** By time-warping, \( R_m \) is an IID exponential random variable with rate \( \lambda_h \).

Define \( X_d, d > 0 \), as the time it takes in the static system (that is in the frame of reference \( \mathcal{F}^{\text{static}} \), see section II) for \( D_h \) to reach depth \( d \) after reaching depth \( d - 1 \). In other words, \( X_d \) is the difference between the times \( \alpha(t_1) > \alpha(t_2) \), where \( t_1 \) is the minimum time \( t \) in frame of reference \( \mathcal{F}^{\text{dyn}} \) such that \( D_h(\alpha(t)) = d \), and, \( t_2 \) is the minimum time \( t \) in \( \mathcal{F}^{\text{dyn}} \) such that \( D_h(\alpha(t)) = d - 1 \).

Let \( \zeta_i^h = \alpha(\tau^h_j) \), that is, \( \zeta_i^h \) is the time of mining of \( j \)-th honest block in the frame of reference \( \mathcal{F}^{\text{static}} \). Similarly, we define \( \zeta_j^a = \alpha(\tau^a_j) \) for the \( j \)-th adversarial block. Then, we can rewrite the event \( \hat{B}_{ik} \) as:

\[
\hat{B}_{ik} = \left\{ cD_i(\zeta_i^h + \sum_{m=1}^{k-1} R_m + \Delta') \geq D_h(\zeta_{k-1}^h) - D_h(\zeta_k^h + \Delta') \right\}.
\]

Also, let \( \delta_i^h = \zeta_i^h - \zeta_{i-1}^h \) and \( \delta_j^a = \zeta_j^a - \zeta_{j-1}^a \) denote the time intervals for subsequent honest and adversary arrival events in the frame of reference \( \mathcal{F}^{\text{static}} \).

### E Growth rate of adversarial tree of superblocks

We first give a description of the (dual of the) adversarial tree consisting of superblocks in terms of a Branching Random Walk (BRW). Recall that \( D_h(\alpha(t)) \) is the depth of the adversarial tree of superblocks in the time horizon of static system, that is, in the frame of reference \( \mathcal{F}^{\text{static}} \) and \( t \) is in the frame of reference \( \mathcal{F}^{\text{dyn}} \). For notational convenience, suppose \( \alpha(t) = t' \). Thus, for any time \( t \) in \( \mathcal{F}^{\text{dyn}} \), \( t' \) is the corresponding time in \( \mathcal{F}^{\text{static}} \). From herein, the analysis will be done in the time horizon of reference \( \mathcal{F}^{\text{static}} \).

Each vertices at generation \( k \geq 2 \) can be labelled as a \( k \)-tuple of positive integers \((i_1, \ldots, i_k)\) with \( i_j \geq c \) for \( 2 \leq j \leq k \); the vertex \( v = (i_1, \ldots, i_k) \in \mathcal{I}_k \) is the \((i_k - c + 1)\)-th child of vertex \((i_1, \ldots, i_{k-1})\) at level \( k - 1 \). At \( k = 1 \) generation, we have \( i_1 \geq 1 \). Let \( \mathcal{I}_k = \{(i_1, \ldots, i_k) : i_j \geq 1 \text{ for } i_j \geq c \text{ for } 2 \leq j \leq k \} \), and set \( \mathcal{I} = \cup_{k>0} \mathcal{I}_k \). For such \( v \) we also let \( v^j = (i_1, \ldots, i_j), j = 1, \ldots, k, \) denote the ancestor of \( v \) at level \( j \), with \( v^k = v \). For notation convenience, we set \( v^0 = 0 \) as the root of the tree.

Next, let \( \{\mathcal{E}_v\}_{v \in \mathcal{I}} \) be an i.i.d. family of exponential random variables of parameter \( \lambda \). For \( v = (i_1, \ldots, i_k) \in \mathcal{I}_k \), let \( W_v = \sum_{j \leq k} \mathcal{E}_{v(i_1, \ldots, i_{k-1}, j)} \) and let
\( S_v = \sum_{j \leq k} W_{v^j} \). This creates a labelled tree, with the following interpretation:
for \( v = (i_1, \ldots, i_j) \), the \( W_{v^j} \) are the waiting time for \( v^j \) to appear, measured from the appearance of \( v^{j-1} \), and \( S_v \) is the appearance time of \( v \). Observe that starting from any \( v \in \mathcal{I}_1 \), we obtain a standard BRW. For any \( v = (i_1, \cdots, i_k) \in \mathcal{I}_k \), we can write \( S_v = S^1_v + S^2_v \) where \( S^1_v \) is the appearance time for the ancestor of \( v \) at level 1 while \( S^2_v = S_v - S^1_v \).

Let \( S^*_k = \min_{v \in \mathcal{I}_k} S_v \). Note that \( S^*_k \) is the time of appearance of a block at level \( k \) and therefore we have
\[
\{ D_0(t') \geq k \} = \{ S^*_k \leq t' \}.
\] (21)

Fixing \( i_1 \in \mathcal{I}_1 \), let \( S^*_k = \min_{v \in \mathcal{I}_k} S^2_v \). Observe that \( S^*_k \) is the minimum of all standard BRW. Introduce, for \( \theta_c < 0 \), the moment generating function
\[
A_c(\theta_c) = \log \sum_{v \in \mathcal{I}_2} E(e^{\theta_c S^2_v}) = \log \sum_{v^i = i_1} E(e^{\sum_{i=1}^{\infty} \theta_c E_i})
= \log \sum_{j=1}^{\infty} (E(e^{\theta_c E_j}))^j = \log \frac{E(e^{\theta_c E_1})}{1 - E(e^{\theta_c E_1})}.
\]

Due to the exponential law of \( E_1 \), \( E(e^{\theta_c E_1}) = \frac{\lambda_a}{\lambda_a - \theta_c} \) and therefore \( A_c(\theta_c) = \log(\frac{\lambda_a}{\lambda_a - \theta_c}) \).

An important role is played by \( \theta_c^* \), which is the negative solution to the equation \( A_c(\theta_c) = \theta_c A_c(\theta_c) \) and let \( \eta_c \) satisfy that
\[
\sup_{\theta_c < 0} \left( \frac{A_c(\theta_c)}{\theta_c} \right) = \frac{A_c(\theta_c^*)}{\theta_c^*} = \frac{1}{\lambda_a \eta_c}.
\]

Indeed, we have the following.

**Proposition 1.**
\[
\lim_{k \to \infty} \frac{S^*_k}{k} = \lim_{k \to \infty} \frac{S^*_k}{k} = \sup_{\theta_c < 0} \left( \frac{A_c(\theta_c)}{\theta_c} \right) = \frac{1}{\lambda_a \eta_c}, \ a.s.
\]

In fact, much more is known.

**Proposition 2.** There exist explicit constants \( c_1 > c_2 > 0 \) so that the sequence \( S^*_k - k/\lambda_a \eta_c - c_1 \log k \) is tight, and
\[
\liminf_{k \to \infty} S^*_k - k/\lambda_a \eta_c - c_2 \log k = \infty, \ a.s.
\]

Note that Propositions 2 and (21) imply in particular that \( D_i(t') \leq c_\eta_a t' \) for all large \( t' \), a.s., and also that
\[
\text{if } c_\eta_a > \lambda_a \text{ then } D_i(t') > t' \text{ for all large } t', \ a.s. \quad (22)
\]

Let us define \( \phi_c := c \eta_c \), then \( \phi_c \lambda_a \) is the growth rate of private \( c \)-correlated NaS tree.
Theorem 3. For \( x > 0 \) so that \( \eta_c \lambda_a t' + x \) is an integer,
\[
P(D_0(t') \geq \eta_c \lambda_a t' + x) \leq e^{-\theta_c t'} e^{(m-1) \Lambda_c(\theta_c^*)} g(t').
\] (23)

where \( m = \eta_c \lambda_a t' + x \), \( g(t') = \sum_{i_1 \geq 1} \int_0^{t'} \frac{\lambda_{i_1}^{i_1-1} e^{-\lambda_a u}}{\Lambda(i_1)} e^{\theta_c u} du \) and \( \theta_c^* \) is the solution for the equation \( \Lambda_c(\theta) = \theta \Lambda_c(\theta) \).

Proof. Consider \( m = \eta_c \lambda_a t' + x \). Note that by (21),
\[
P(D_0(t') \geq m) = P(S_m^c \leq t') \leq \sum_{v \in I_m} P(S_v \leq t') = \sum_{v \in I_m} P(S_1^c + S_2^c \leq t')
\]
\[
= \sum_{v \in I_m} \int_0^{t'} P(S_v^c \leq t' - u) du
\]
\[
= \sum_{i_1 \geq 1 \atop i_2 \geq c, \ldots, i_m \geq c} \int_0^{t'} \lambda_{i_1}^{i_1-1} e^{-\lambda_a u} e^{\theta_c u} du
\] (24)

For \( v = (i_1, \ldots, i_k) \), set \( |v_{-1}| = i_2 + \cdots + i_k \). Then, we have that \( S_v^c \) has the same law as \( \sum_{v_{-1}} E_j \). Thus, by Chebycheff’s inequality, for \( v \in I_m \),
\[
P(S_v^c \leq t' - u) \leq E e^{\theta_c^* S_v^c e^{-\theta_c^*(t' - u)}} = \left( \frac{\lambda_a}{\lambda_a - \theta_c^*} \right)^{|v_{-1}|} e^{-\theta_c^*(t' - u)}.
\] (25)

And
\[
\sum_{i_2 \geq c, \ldots, i_m \geq c} \left( \frac{\lambda_a}{\lambda_a - \theta_c^*} \right)^{|v_{-1}|} = \sum_{i_2 \geq c, \ldots, i_m \geq c} \left( \frac{\lambda_a}{\lambda_a - \theta_c^*} \right)^{i_1-1} \sum_{j=2}^{m} \left( \frac{\lambda_a}{\lambda_a - \theta_c^*} \right)^{i_j} = e^{(m-1) \Lambda_c(\theta_c^*)}.
\] (26)

Combining (23), (26) and (24) yields
\[
P(D_0(t') \geq m) \leq e^{-\theta_c t'} e^{(m-1) \Lambda_c(\theta_c^*)} \sum_{i_1 \geq 1} \int_0^{t'} \frac{\lambda_{i_1}^{i_1-1} e^{-\lambda_a u}}{\Lambda(i_1)} e^{\theta_c u} du
\]
\[
= e^{-\theta_c t'} e^{(m-1) \Lambda_c(\theta_c^*)} \sum_{i_1 \geq 1} \int_0^{t'} \frac{\lambda_{i_1}^{i_1-1} e^{-\lambda_a u}}{\Lambda(i_1)} e^{\theta_c u} du
\]
\[
= e^{-\theta_c t'} e^{(m-1) \Lambda_c(\theta_c^*)} g(t').
\] (27)

where \( g(t') = \sum_{i_1 \geq 1} \int_0^{t'} \frac{\lambda_{i_1}^{i_1-1} e^{-\lambda_a u}}{\Lambda(i_1)} e^{\theta_c u} du \).

From proposition 1 we have
\[
\phi_c = \frac{e^{\theta_c^*}}{\lambda_a} \left( \frac{1}{\log \left( \frac{\theta_c^*}{\theta_c^* \Lambda_c(\theta_c^*)} \right)} \right),
\] (28)
where $\theta^*_c$ is the unique negative solution of

$$A_c(\theta_c) = \theta_c \dot{A}_c(\theta_c)$$

(29)

## F Proofs

**Proposition 3.** Let $Y_d$, $d \geq 1$, be i.i.d random variables, exponentially distributed with rate $\lambda_h$. Then, each random variable $X_d$ is less than $\Delta' + Y_d$, where $\Delta' = \frac{\lambda_{\max}}{\lambda_h} \Delta$.

**Proof.** Let $h_i$ be the first block that comes to some depth $d-1$ within $T_h$. Then, in the frame of reference $F_{\text{static}}$, every honest block that arrives within interval $[\alpha(\tau^h_i), \alpha(\tau^h_i + \Delta)]$ will be mapped to the same depth as $h_i$, i.e., $d-1$. Hence, $T_h$ will reach depth $d$ only when an honest block arrives after time $\alpha(\tau^h_i + \Delta)$ in the frame of reference $F_{\text{static}}$. Now, due to time warping, in $F_{\text{static}}$, we know that the difference between $\alpha(\tau^h_i + \Delta)$ and the arrival time of the first block after $\alpha(\tau^h_i + \Delta)$ is exponentially distributed with rate $\lambda_h$ due to the memoryless property of the exponential distribution. This implies that for each depth $d$, $X_d = \alpha(\tau^h_i + \Delta) - \alpha(\tau^h_i) + Y_d \leq \int_{\tau^h_i + \Delta}^{\tau^h_i + \Delta} \frac{\lambda_h(u)}{\lambda_h} du + Y_d = \Delta' + Y_d$ for some random variable $Y_d$ such that $Y_d, d \geq 1$, are IID and exponentially distributed with rate $\lambda_h$. \qed

**Proposition 4.** For any constant $a$,

$$P\left(\sum_{d=a}^{n+a} X_d > n(\Delta' + \frac{1}{\lambda_h})(1 + \delta)\right) \leq e^{-n\Omega(\delta^2(1 + \Delta' \lambda_h)^2)}$$

Proposition 4 is proved using chernoff bound and Proposition 3.

**Proposition 5.** Probability that there are less than

$$\frac{n \lambda_a(1 - \delta)}{\lambda_h}$$

adversarial superblock arrival events in the modified adversarial tree from time $\zeta^h_a$ to $\zeta^h_{n+1}$ is upper bounded by

$$e^{-n\Omega(\delta^2 \frac{\lambda_a}{\lambda_h})}$$

**Proof.** Let $N_a(\zeta^h_a, \zeta^h_{n+1})$ be the number of adversarial superblock arrivals from time $\zeta^h_a$ to $\zeta^h_{n+1}$ in the frame of reference $F_{\text{static}}$. Observe that most adversarial superblocks would be made if all the superblocks are mined in the first level of the adversarial tree. Assuming this, we have

$$P(N_a(t') = k) = \frac{e^{-\lambda_a t'} (\lambda_a t')^k}{k!}.$$
Then, using Chebychev inequality
\[
P\left( N_a(ζ_h^0, c^h_{n+1}) < n \frac{λ_a(1 - δ)}{λ_h} \right) = \int_{c^h_0}^{∞} p_{c^h_0, n+1}(t)P\left( N_a(0, t') < n \frac{λ_a(1 - δ)}{λ_h} \right) dt'
= \int_{c^h_0}^{∞} p_{c^h_0, n+1}(t')\mathbb{E}[e^{-θN_a(0,t')}]e^{θn\frac{λ_a(1 - δ)}{λ_h}} dt' \tag{30}
\]

Optimizing over θ, we have
\[
\frac{d}{dθ} \left[ (e^{-θ} - 1)λ_a t' + θn \frac{λ_a(1 - δ)}{λ_h} \right] = 0
\]
\[
e^{-θ^*} = \frac{n(1 - δ)}{λ_h t'}
\]
\[
θ^* = -\ln \left( \frac{n(1 - δ)}{λ_h t'} \right) \tag{31}
\]

From equations 30 and 31 and using the fact that \( h(x) = 2^{-x + (1 + x)\ln(1 + x)} \) is a decreasing function and \( h(0) = 1 \) yield
\[
P\left( N_a(ζ_h^0, c^h_{n+1}) < n \frac{λ_a(1 - δ)}{λ_h} \right) < e^{-nλ(\frac{λ_a(1 - δ)}{λ_h})^2}
\]

**Proposition 6.** Define \( B_n \) as the event that there are at least \( n \) adversarial block arrivals while \( D_h \) grows from depth 0 to \( n \):
\[
B_n = \left\{ \sum_{i=1}^{cn} X_i \geq n \sum_{i=0}^{n} δ_i^a \right\}
\]

If
\[
λ_a < \frac{λ_h}{c(1 + λ_hΔ')},
\]
then,
\[
P(B_n) \leq e^{-A_0 n},
\]
where,
\[
A_0 = -cwΔ' + c \ln \frac{λ_h - w}{λ_h} + \ln \frac{λ_a + w}{λ_a} > 0
\]
and,
\[
w = \frac{λ_h - λ_a}{2} + \frac{c + 1 - (c + 1)^2 + c^2Δ'^2[λ_a + λ_h]^2 + 2cΔ'(c - 1)[λ_a + λ_h]}{2cΔ'}
\]
Proof. Using Chebychev inequality and proposition for any $t > 0$, we have

$$P(B_n) \leq E \left[ \prod_{j=0}^{n} e^{-w\delta_j} \right] E \left[ \prod_{j=1}^{cn} e^{wX_j} \right]$$

$$= \left[ \frac{\lambda_a}{\lambda_a + w} \right]^n \left[ \frac{e^{w\Delta' \lambda_h}}{\lambda_h - w} \right]^{cn}$$

$$= e^{-n \left[ -cw\Delta' + c \ln \frac{\lambda_h - w}{\lambda_h} + \ln \frac{\lambda_h + w}{\lambda_a} \right]}$$

Optimizing over $w$, we have

$$\frac{d}{dw} \left[ -cw\Delta' + c \ln \frac{\lambda_h - w}{\lambda_h} + \ln \frac{\lambda_h + w}{\lambda_a} \right] = 0$$

$$= -c\Delta' - \frac{c}{\lambda_h - w} + \frac{1}{\lambda_a + w} = 0$$

$$= -c\Delta' \frac{\lambda_a + w}{(\lambda_h - w)\lambda_a} = c\Delta'$$

$$= c\Delta'w^2 + [c\Delta'(\lambda_a - \lambda_h) - c - 1]w + [\lambda_h - c\lambda_a - c\Delta'\lambda_a \lambda_h] = 0$$

$$w = \frac{\lambda_h - \lambda_a}{2} + \frac{c + 1 - \sqrt{(c+1)^2 + c^2\Delta'^2\lambda_a + \lambda_h^2 + 2c\Delta'(c-1)\lambda_a \lambda_h}}{2c\Delta'}$$

$$w = \frac{\lambda_h - \lambda_a}{2} + \frac{c + 1 - \sqrt{(c+1)^2 + c^2\Delta'^2\lambda_a + \lambda_h^2}}{2c\Delta'}$$

Lemma 12. There exists a constant $\gamma > 0$ such that

$$P(\tilde{B}_{ik}) \leq e^{-\gamma(k - i - 1)}$$

Proof. Let $N_a(\xi^h_i, \xi^h_k + \Delta')$ be the number of adversarial superblock arrivals in the interval $[\xi^h_i, \xi^h_k + \Delta']$ in the frame of reference $\mathcal{F}_{\text{static}}$. Define

$$\tilde{C}_{ik} = \text{event that } cN_a(\xi^h_i, \xi^h_k + \Delta') \geq D_h(\xi^h_i) - D_h(\xi^h_k + \Delta')$$

Then, we have

$$\tilde{B}_{ik} \subseteq \tilde{C}_{ik}.$$
\[ P(\tilde{B}_{ik}) \leq P \left( N_a(\xi^b_i, \xi^h_k + \Delta') < (1 - \delta)(k - i) \frac{\lambda_a}{\lambda_h} \right) + P \left( \tilde{C}_{ik} \mid N_a(\xi^b_i, \xi^h_k + \Delta') \geq (1 - \delta)(k - i) \frac{\lambda_a}{\lambda_h} \right) \]

\[ \leq e^{-\Omega((k-i)\delta \lambda_a/\lambda_h)} + P \left( \tilde{C}_{ik} \mid N_a(\xi^b_i, \xi^h_k + \Delta') \geq (1 - \delta)(k - i) \frac{\lambda_a}{\lambda_h} \right) \]

\[ \leq e^{-\Omega((k-i)\delta \lambda_a/\lambda_h)} + \sum_{x=\lceil 1 - \delta(k-i) \frac{\lambda_a}{\lambda_h} \rceil}^{\infty} e^{-A_0 x} \sum_{x=\lceil 1 - \delta(k-i) \frac{\lambda_a}{\lambda_h} \rceil}^{\infty} \sum_{x=\lceil 1 - \delta(k-i) \frac{\lambda_a}{\lambda_h} \rceil}^{\infty} P \left( \sum_{i=1}^{c x} X_i \geq \sum_{i=0}^{j} \delta^a_i \right) \]

where \((a)\) is due to proposition 5 which says that in frame of reference \( F^{static} \) there are more than \((1 - \delta)(k - i) \lambda_a/\lambda_h \) adversarial superblock arrival events in the time period \([\xi^b_i, \xi^h_k + \Delta']\) except with probability \( e^{-\Omega((k-i)\delta \lambda_a/\lambda_h)} \). \((b)\) is by union bound, \((c)\) is because although we are given number of adversarial superblock arrivals, \( N_a(\xi^b_i, \xi^h_k + \Delta') = x \), the number of normal adversarial block arrivals can range from \( x \) to \( cx \), \((d)\) is because for \( j \in \{x, x+1, \cdots, cx\} \), \( \sum_{i=1}^{c x} X_i \geq \sum_{i=0}^{j} \delta^a_i \) and \((e)\) is by proposition 6 which states that

\[ P(\sum_{i=1}^{c x} X_i \geq \sum_{i=0}^{n} \delta^a_i) \leq e^{-A_0 n} \]

for large \( n \). Additionally, \((f)\) is because \( \sum_{n=a}^{\infty} ne^{-bn} = e^b e^{-ab} \left[ \frac{a(e^b-1)+1}{(e^b-1)^2} \right] \).
Hence,
\[
P(\bar{B}_{ik}) < e^{-O((k-i)\delta^2 \lambda_a / \lambda_h)} + \frac{e^{-A_0(1-\delta)(k-i) \Delta h \lambda_a} }{1 - e^{-A_0}} \]
\[
+ (c-1)e^{-A_0(1-\delta)(k-i) \Delta h \lambda_a} e^{A_0} \left[ \frac{(1-\delta)(k-i) \Delta h (e^{A_0} - 1) + 1}{(e^{A_0} - 1)^2} \right]
\leq C_2 e^{-C_5(k-i)} + C_4 (k-i) e^{-C_5(k-i)} \quad (32)
\]
for large enough \(k-i\) and appropriately chosen constants \(C_2, C_3, C_4, C_5 > 0\) as functions of the fixed \(\delta\). Finally, since \(P(\bar{B}_{ik})\) decreases as \(k-i\) grows and is smaller than 1 for all \(k > i+1\), we obtain the desired inequality for a sufficiently small \(\gamma \leq C_3\).

\[
\square
\]

F.1 Proof of Lemma 5

Observe that
\[
\phi_c \lambda_a(t) \leq \frac{\lambda_h(t)}{1 + \lambda_{\text{max}} \Delta}
\Rightarrow \phi_c \lambda_a \leq \frac{\lambda_h}{1 + \lambda_{\text{max}} \Delta}
\]

In this proof, let \(r_h := \frac{\lambda_h}{1 + \lambda_{\text{max}} \Delta}\). Also, in this proof, we consider the time horizon to be defined over the dynamic available system. For notational convenience, we represent the adversarial tree composed of superblocks as as \(T_i(t)\), where \(t\) is defined over the time horizon of the frame of reference \(F_{\text{dyn}}\).

As remarked at the end of the section B of the appendix, we define our time over the dynamic available system, that is, the frame of reference \(F_{\text{dyn}}\). The random processes of interest start from time 0. To look at the system in stationarity, let us extend them to \(-\infty < t < \infty\). More specifically, define \(\tau_{h-1}^b, \tau_{h2}^b, \ldots\) such that together with \(\tau_0^b, \tau_1^b, \ldots\) in the dynamic available system, we have a double-sided infinite random process. Also, for each \(i < 0\), we define an independent copy of a random adversary tree \(T_i\) with the same distribution as \(T_0\). And we extend the definition of \(T_h(t)\) and \(D_h^{\text{dyn}}(t)\) in \(F_{\text{dyn}}\) to \(t < 0\): the last honest block mined at \(\tau_{h-1}^b < 0\) and all honest blocks mined within \((\tau_{h-1}^b - \Delta, \tau_{h}^b)\) appear in \(T_h(t)\) at their respective mining times to form the level \(-1\), and the process repeats for level less than \(-1\); let \(D_h^{\text{dyn}}(t)\) be the level of the last honest arrival before \(t\) in \(T_h(t)\), i.e., \(D_h^{\text{dyn}}(t) = \ell\) if \(\tau_{hi}^b \leq t < \tau_{i+1}^b\) and the \(i\)-th honest block appears at level \(\ell\) of \(T_h(t)\). We also extend the definition of \(D_h(\alpha(t))\) in the frame of reference \(F_{\text{static}}\): \(D_h(\alpha(t)) = D_h^{\text{dyn}}(t)\) for all \(t\) defined in \(F_{\text{dyn}}\).

These extensions allow us to extend the definition of \(E_{ij}\) to all \(i, j, -\infty < i < j < \infty\), and define \(E_j\) and \(\bar{E}_j\) to be:

\[
E_j = \bigcap_{i<j} E_{ij}
\]
\[ \hat{E}_j = E_j \cap V_j. \]

Note that \( \hat{E}_j \subset \hat{F}_j \), so to prove that \( \hat{F}_j \) has a probability bounded away from 0 for all \( j \), all we need is to prove that \( \hat{E}_j \) has a non-zero probability.

Recall that we have defined the events \( U_j \) and \( \hat{B}_{ik} \) in section D of the appendix as:

\[
U_j = \{ \Delta' < R_{j-1} \} \bigcap \{ R_j > \Delta' \}
\]
\[
\hat{B}_{ik} = \left\{ cD_i(\zeta^h_i) + \sum_{m=i}^{k-1} R_m + \Delta' \geq D_h(\zeta^h_{k-1}) - D_h(\zeta^h_i + \Delta') \right\}
\]

where \( R_m \) are i.i.d exponential random variable with mean \( \frac{1}{\lambda h} \) and \( \Delta' = (1 + \eta_{\max}) \Delta \).

Following the idea in Lemma 9 and using Lemma 10 and Lemma 11, we have

\[
\hat{E}_j = E_j \cap V_j = \bigcap_{i<j} E_{ij} \cap V_j = \left( \bigcap_{i<j<k} \hat{A}_{ik} \right) \cap V_j \supset \left( \bigcap_{i<j<k} \hat{B}_{ik} \right) \cap U_j.
\]

Let \( E'_j = \bigcap_{i<j<k} \hat{B}_{ik} \) and \( \hat{E}'_j = E'_j \cap U_j \). Then, \( P(\hat{E}_j) \geq P(\hat{E}'_j) \) and so we just need to prove that \( \hat{E}'_j \) has a non-zero probability. Observe that \( \hat{E}'_j \) has a time-invariant dependence on \( \{Z_i\} \), which means that \( p = P(\hat{E}'_j) \) does not depend on \( j \). Then we can just focus on \( P(\hat{E}'_0) \). This is the last step to prove.

\[
P(\hat{E}'_0) = P(E'_0 | U_0)P(U_0)
= P(E'_0 | U_0)P(R_0 > \Delta')P(R_{-1} > \Delta')
= e^{-2\lambda h \Delta'} P(E'_0 | U_0).
\]

where we used Remark 1 in the last step. It remains to show that \( P(E'_0 | U_0) > 0 \).

We have

\[
E'_0 = \text{event that } cD_i \left( \sum_{m=i}^{k-1} R_m + \Delta' + \zeta^h_i \right) < D_h(\zeta^h_{k-1}) - D_h(\zeta^h_i + \Delta')
\]

for all \( k > 0 \) and \( i < 0 \), then

\[
(\bar{E}'_0) = \bigcup_{k>0,i<0} \hat{B}_{ik}.
\]

Let us fix a particular \( n > 2\lambda h \Delta' > 0 \), and define:

\[
G_n = \text{event that } D_m(3n/\lambda h + \zeta^h_m) = 0
\]

for \( m = -n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n \).
Then
\[ P(E'_0|U_0) \geq P(E'_0|U_0, G_n)P(G_n|U_0) \]
\[ = \left(1 - P(\cup_{k>0,i<0} \hat{B}_{ik}|U_0, G_n)\right)P(G_n|U_0) \]
\[ \geq \left(1 - \sum_{k>0,i<0} P(\hat{B}_{ik}|U_0, G_n)\right)P(G_n|U_0) \]
\[ \geq (1 - a_n - b_n)P(G_n|U_0) \quad (34) \]

where
\[ a_n := \sum_{(i,k): -n \leq i < 0 < k \leq n} P(\hat{B}_{ik}|U_0, G_n) \quad (35) \]
\[ b_n := \sum_{(i,k): i < -n \text{ or } k > n} P(\hat{B}_{ik}|U_0, G_n). \quad (36) \]

Consider two cases:

**Case 1:** $-n \leq i < 0 < k \leq n$:

\[ P(\hat{B}_{ik}|U_0, G_n) = P(\hat{B}_{ik}|U_0, G_n, \sum_{m=i}^{k-1} R_m + \Delta' \leq 3n/\lambda_h) \]
\[ + P(\sum_{m=i}^{k-1} R_m + \Delta' > 3n/\lambda_h|U_0, G_n) \]
\[ \leq P(\sum_{m=i}^{k-1} R_m + \Delta' > 3n/\lambda_h|U_0, G_n) \]
\[ \leq P(\sum_{m=i}^{k-1} R_m > 5n/(2\lambda_h)|U_0) \]
\[ \leq P(\sum_{m=i}^{k-1} R_m > 5n/(2\lambda_h))/P(U_0) \]
\[ \leq A_1 e^{-\gamma_1 n} \]

for some positive constants $A_1, \gamma_1$ independent of $n,k,i$. The last inequality follows from the fact that $R_i$'s are iid exponential random variables of mean $1/\lambda_h$. Summing these terms, we have:

\[ a_n = \sum_{(i,k): -n \leq i < 0 < k \leq n} P(\hat{B}_{ik}|U_0, G_n) \]
\[ \leq \sum_{(i,k): -n \leq i < 0 < k \leq n} A_1 e^{-\alpha_1 n} := \bar{a}_n, \]

which is bounded and moreover $\bar{a}_n \to 0$ as $n \to \infty$. 

Case 2: $k > n$ or $i < -n$:

For $0 < \varepsilon < 1$, let us define event $W_{ik}^\varepsilon$ to be:

$$W_{ik}^\varepsilon = \text{event that } D_h(\zeta_k^h) - D_h(\zeta_i^h + \Delta') \geq (1 - \varepsilon) \frac{r_k}{\lambda_h} (k - i - 1).$$  \hspace{1cm} (37)

Then we have

$$P(\hat{B}_{ik}|U_0, G_n) \leq P(\hat{B}_{ik}|U_0, G_n, W_{ik}) + P(W_{ik}^\varepsilon|U_0, G_n).$$

We first bound $P(W_{ik}^\varepsilon|U_0, G_n)$:

$$P(W_{ik}^\varepsilon|U_0, G_n) \leq P(W_{ik}^\varepsilon|U_0, G_n) \leq P(W_{ik}^\varepsilon|U_0, G_n) \leq e^{-\Omega(\varepsilon^2(k-i-1))}$$

for some positive constants $A_2, \gamma_2$ independent of $n, k, i$, where the second inequality follows from the Erlang tail bound (as $\zeta_k^h - \zeta_i^h$ is sum of IID exponentials due to time-warping) and the third inequality follows from Proposition [1].

Meanwhile, we have

$$P(\hat{B}_{ik}|U_0, G_n, W_{ik})$$

$$\leq P(cD_1(\sum_{m=i}^{k-1} R_m + \Delta' + \zeta_k^h) \geq (1 - \varepsilon) \frac{r_k}{\lambda_h} (k - i - 1)|U_0, G_n, W_{ik}^\varepsilon)$$

$$\leq P(cD_1(\sum_{m=i}^{k-1} R_m + \Delta' + \zeta_i^h) \geq (1 - \varepsilon) \frac{r_k}{\lambda_h} (k - i - 1)$$

$$|U_0, G_n, W_{ik}^\varepsilon, \sum_{m=i}^{k-1} R_m + \Delta' \leq (k - i - 1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h}$$

$$+ P(\sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} |U_0, G_n, W_{ik}^\varepsilon)$$

$$\leq P(\sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} |U_0, G_n, W_{ik}^\varepsilon)$$

$$+ e^{-\theta_a (k-i-1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} + (k-i-1) \Lambda_c(\theta_a) g((k-i-1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h})}$$

$$\leq P(\sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} |U_0, G_n, W_{ik}^\varepsilon)$$

$$+ e^{-\theta_a (k-i-1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} + (k-i-1) \Lambda_c(\theta_a) g((k-i-1) \frac{r_k + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h})}$$
where (a) follows from Theorem 3. (b) follows from $\frac{A_v(\theta^*)}{\nu_v^2} = \frac{1}{\lambda_v \theta_v} = \frac{\alpha}{\phi_v \lambda_v}$. The second term can be bounded as:

$$P\left( \sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} | U_0, G_n, W_{ik}^\ast \right)$$

$$= P\left( \sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} | U_0, W_{ik}^\ast \right)$$

$$\leq P\left( \sum_{m=i}^{k-1} R_m + \Delta' > (k - i - 1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} \right) / P(U_0, W_{ik}^\ast)$$

$$\leq A_3 \epsilon^{-\gamma_3(k-i-1)}$$

for some positive constants $A_3, \gamma_3$ independent of $n, k, i$. The last inequality follows from the fact that $(r_h + \phi_c \lambda_a)/(2 \phi_c \lambda_a) > 1$ and the $R_i$’s have mean $1/\lambda_h$, while $P(U_0, W_{ik}^\ast)$ is a event with high probability as we showed in (38). Then we have

$$P(\tilde{B}_{ik}|U_0, G_n)$$

$$\leq A_2 e^{-\alpha_2(k-i-1)}$$

$$+ e^{-\alpha_2(k-i-1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} - 1} e^{-A_v(\theta^*)} g((k - i - 1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h})$$

$$+ A_3 \epsilon^{-\gamma_3(k-i-1)}. \tag{39}$$

Summing these terms, we have:

$$b_n = \sum_{(i,k): i < -n \text{ or } k > n} P(\tilde{B}_{ik}|U_0, G_n)$$

$$\leq A_2 e^{-\alpha_2(k-i-1)}$$

$$+ e^{-\alpha_2(k-i-1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} - 1} e^{-A_v(\theta^*)} g((k - i - 1) \frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h})$$

$$+ A_3 \epsilon^{-\gamma_3(k-i-1)}$$

$$:= \bar{b}_n$$

which is bounded and moreover $\bar{b}_n \to 0$ as $n \to \infty$ when we set $\epsilon$ to be small enough such that $\frac{r_h + \phi_c \lambda_a}{2 \phi_c \lambda_a} \frac{1}{\lambda_h} < 1$.

Substituting these bounds in (34) we finally get:

$$P(E_n'|U_0) > [1 - (\bar{a}_n + \bar{b}_n)] P(G_n|U_0) \tag{40}$$

By setting $n$ sufficiently large such that $\bar{a}_n$ and $\bar{b}_n$ are sufficiently small, we conclude that $P(E_n') > 0$. 
F.2 Proof of Lemma 6

We divide the proof in to two steps. In the first step, we prove for $\varepsilon = 1/2$. Recall that we have defined event $\hat{B}_{ik}$ as:

$$\hat{B}_{ik} = \text{event that } cD_i(\sum_{m=i}^{k-1} R_m + \Delta' + \zeta^h) \geq D_h(\zeta^h_{k-1}) - D_h(\zeta^h_{i} + \Delta').$$

Similar to inequality (39), we have

$$P(\hat{B}_{ik}) \leq e^{-c_1(k-i-1)}$$

for some positive constants $c_1$.

And by Lemma 9, 10, 11, we have

$$\hat{F}_{c_j} = F_{c_j} \cup V_{c_j} \subseteq \left( \bigcup_{(i,k):i<j<k} \hat{B}_{ik} \right) \cup U_{c_j}.$$

Let us define $s' = \alpha(s)$ and $s' + t' = \alpha(s + t)$. Thus, for time $s$ in $F^{dyn}$, $s'$ is the corresponding time in $F^{static}$. Also, $t'$ is the corresponding difference between the two times in $F^{static}$. Due to Lemma 3, the ordering of mining of honest blocks happen in same order in both frame of reference $F^{dyn}$ and $F^{static}$. For details, see section B.

Divide $[s', s' + t']$ into $\sqrt{t'}$ sub-intervals of length $\sqrt{t'}$, so that the $r$ th sub-interval is:

$$\mathcal{J}_r := [s' + (r-1)\sqrt{t'}, s' + r\sqrt{t'}].$$

Now look at the first, fourth, seventh, etc sub-intervals, i.e. all the $r = 1 \mod 3$ sub-intervals. Introduce the event that in the $\ell$-th $1 \mod 3$th sub-interval, an adversary tree that is rooted at a honest block arriving in that sub-interval or in the previous ($0 \mod 3$) sub-interval catches up with a honest block in that sub-interval or in the next ($2 \mod 3$) sub-interval. Formally,

$$C_{\ell} = \bigcap_{j: \zeta^h_j \in \mathcal{J}_{3\ell+1}} U_{c_j} \cup \left( \bigcup_{(i,k):\zeta^h_k - \sqrt{t'} < \zeta^h_i < \zeta^h_j < \zeta^h_k + \Delta' < \zeta^h_k + \sqrt{t'}} \hat{B}_{ik} \right).$$

Note that for distinct $\ell$, the events $C_{\ell}$’s are independent. Also, we have

$$P(C_{\ell}) \leq P(\text{no arrival in } \mathcal{J}_{3\ell+1}) + 1 - p < 1$$

for large enough $t$, where $p$ is a uniform lower bound such that $P(\hat{F}_{j}) \geq p$ for all $j$.

Introduce the atypical events:

$$B = \bigcup_{(i,k):\zeta^h_k \in [s', s' + t']} \hat{B}_{ik},$$

and

$$\hat{B} = \bigcup_{(i,k):\zeta^h_k < s', s' + t' < \zeta^h_k + \Delta'} \hat{B}_{ik}.$$
The events \( B \) and \( \hat{B} \) are superset of the events that an adversary tree catches up with an honest block far ahead. Then we have

\[
P(B^{\text{superblock}}) \leq P\left( \bigcap_{j: \xi_j^h \in [s', s' + t']} U_j^c \right) + P(B) + P(\hat{B}) + P\left( \bigcap_{t=0}^{\sqrt{t'/3}} C_t \right)
\]

\[
= P\left( \bigcap_{j: \xi_j^h \in [s', s' + t']} U_j^c \right) + P(B) + P(\hat{B}) + (P(C_t))^{\sqrt{t'/3}}
\]

\[
\leq e^{-c_2 t'} + P(B) + P(\hat{B}) + (P(C_t))^{\sqrt{t'/3}} \quad (44)
\]

for some positive constant \( c_2 \) when \( t' \) is large, where the equality is due to independence. Next we will bound the atypical events \( B \) and \( \hat{B} \). Consider the following events

\[
D_1 = \{ \#\{i: \xi_i^h \in (s' - \sqrt{t'} - \Delta', s' + t' + \sqrt{t'} + \Delta)\} > 2\lambda_h t' \}
\]

\[
D_2 = \{ \exists i, k: \xi_i^h \in (s', s' + t'), (k - i) < \frac{\sqrt{t'}}{2\lambda_h}, (\xi_i^h - \xi_i^h + \Delta') > \sqrt{t'} \}
\]

\[
D_3 = \{ \exists i, k: \xi_k^h + \Delta \in (s', s' + t'), (k - i) < \frac{\sqrt{t'}}{2\lambda_h}, (\xi_k^h - \xi_i^h + \Delta') > \sqrt{t} \}
\]

In words, \( D_1 \) is the event of atypically many honest arrivals in \((s' - \sqrt{t'} - \Delta', s' + t' + \sqrt{t'} + \Delta)\) while \( D_2 \) and \( D_3 \) are the events that there exists an interval of length \( \sqrt{t'} \) with at least one endpoint inside \((s', s' + t')\) with atypically small number of arrivals. Since, by time-warping, the number of honest arrivals in \((s', s' + t')\) (in frame of reference \( F^{\text{static}} \)) is Poisson with parameter \( \lambda_h t' \), we have from the memoryless property of the Poisson process that \( P(D_1) \leq e^{-c_0 t'} \) for some constant \( c_0 = c_0(\lambda_a, \lambda_h) > 0 \) when \( t' \) is large. On the other hand, using the memoryless property and a union bound, and decreasing \( c_0 \) if needed, we have that \( P(D_2) \leq e^{-c_0 \sqrt{t'}} \). Similarly, using time reversal, \( P(D_3) \leq e^{-c_0 \sqrt{t'}} \).

Therefore, again using the memoryless property of the Poisson process,

\[
P(B) \leq P(D_1 \cup D_2 \cup D_3) + P(B \cap D_1^c \cap D_2^c \cap D_3^c)
\]

\[
\leq e^{-c_0 t'} + 2e^{-c_0 \sqrt{t'}} + \sum_{i=1}^{2\lambda_h t'} \sum_{k: k - i > \sqrt{t'} / 2\lambda_h} P(\hat{B}_{ik}) \quad (45)
\]

\[
\leq e^{-c_3 \sqrt{t'}}, \quad (46)
\]

for large \( t' \), where \( c_3 > 0 \) are constants that may depend on \( \lambda_a, \lambda_h \) and the last inequality is due to \( (41) \). We next claim that there exists a constant \( \alpha > 0 \) such that, for all \( t' \) large,

\[
P(\hat{B}) \leq e^{-\alpha t'}. \quad (47)
\]
Indeed, we have that

$$P(\tilde{B}) = \sum_{i<k} \int_0^s P(\zeta_i^h \in d\theta) P(\hat{B}_{ik}, \zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta)$$

$$\leq \sum_{i} \int_0^s P(\zeta_i^h \in d\theta) \sum_{k > i} P(\hat{B}_{ik})^{1/2} P(\zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta)^{1/2}. \tag{48}$$

The tails of the Poisson distribution yield the existence of constants $c, c' > 0$ so that

$$P(\zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta) \leq \sum_{i} \int_0^s P(\zeta_i^h \in d\theta) \sum_{k > i} P(\hat{B}_{ik})^{1/2} \frac{P(\zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta)^{1/2}}{2}. \tag{49}$$

and

$$P(\zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta) \leq \frac{1}{2} e^{-\gamma (s' + t' - \theta - \Delta')}. \tag{50}$$

(41) and (49) yield that there exists a constant $\gamma > 0$ so that

$$\sum_{k > i} P(\hat{B}_{ik})^{1/2} P(\zeta_k^h - \zeta_i^h + \Delta' > s' + t' - \theta - \Delta')^{1/2} \leq e^{-2\gamma (s' + t' - \theta - \Delta')}. \tag{51}$$

Substituting this bound in (48) and using that $\sum_i P(\zeta_i^h \in d\theta) = d\theta$ gives

$$P(\tilde{B}) \leq \sum_i \int_0^{s'} P(\zeta_i^h \in d\theta) e^{-2\gamma (s' + t' - \theta - \Delta')}$$

$$\leq \int_0^{s'} e^{-2\gamma (s' + t' - \theta - \Delta')} d\theta \leq \frac{1}{2\gamma} e^{-2\gamma (t' - \Delta')} \leq e^{-\gamma t'}, \tag{52}$$

for $t'$ large, proving (47).

Combining (46), (52) and (44) concludes the proof of step 1.

In step two, we prove for any $\varepsilon > 0$ by recursively applying the bootstrapping procedure in step 1. Assume the following statement is true: for any $\theta \geq m$ there exist constants $\bar{b}_\theta, \bar{A}_\theta$ so that for all $s', t' \geq 0$,

$$\hat{q}[s', s' + t'] \leq \bar{A}_\theta \exp(-\bar{b}_\theta t'^{1/\theta}). \tag{53}$$

By step 1, it holds for $m = 2$.

Divide $[s', s' + t']$ into $t' \frac{m}{m-1}$ sub-intervals of length $t' \frac{m}{m-1}$, so that the $r$ th sub-interval is:

$$J_r := [s' + (r - 1)t' \frac{m}{m-1}, s' + rt' \frac{m}{m-1}].$$

Now look at the first, fourth, seventh, etc sub-intervals, i.e. all the $r = 1 \mod 3$ sub-intervals. Introduce the event that in the $\ell$-th $1 \mod 3$ sub-interval, an adversary tree that is rooted at a honest block arriving in that sub-interval...
or in the previous \((0 \mod 3)\) sub-interval catches up with a honest block in that sub-interval or in the next \((2 \mod 3)\) sub-interval. Formally,

\[
C_\ell = \bigcap_{j: j^e \in J_{m+1}} U_j^c \cup \bigcup_{(i, k): \zeta^h_i - t' \frac{m}{m-1} < \zeta^h_i < \zeta^h_i + \Delta'} \hat{B}_{i, k}.
\]

Note that for distinct \(\ell\), the events \(C_\ell\)'s are independent. Also by \((53)\), we have

\[
P(C_\ell) \leq A_m \exp(-\bar{a}_m t' \frac{1}{m-1}).
\]

Introduce the atypical events:

\[
B = \bigcup_{(i, k): \zeta^h_i \in [s', s' + t']} \hat{B}_{i, k},
\]

and

\[
\hat{B} = \bigcup_{(i, k): \zeta^h_i < s', s' + t' < \zeta^h_i + \Delta'} \hat{B}_{i, k}.
\]

The events \(B\) and \(\hat{B}\) are the events that an adversary tree catches up with an honest block far ahead. Following the calculations in step 1, we have

\[
P(B) \leq e^{-c_1 t' \frac{m}{m-1}}
\]

\[
P(\hat{B}) \leq e^{-\gamma t'},
\]

for large \(t'\), where \(c_1\) and \(\gamma\) are some positive constant.

Then we have

\[
\bar{q}[s', s' + t'] \leq P\left( \bigcap_{j: j^e \in [s', s' + t']} U_j^c \right) + P(B) + P(\hat{B}) + P\left( \bigcup_{\ell=0}^{t' \frac{m-1}{m-1}/3} C_\ell \right)
\]

\[
= P\left( \bigcap_{j: j^e \in [s, s+t]} U_j^c \right) + P(B) + P(\hat{B}) + (P(C_\ell))\left( t' \frac{m-1}{m-1}/3 \right)
\]

\[
\leq e^{-c_1 t'} + e^{-c_1 t' \frac{m}{m-1}} + e^{-\gamma t'}
\]

\[
+ \left( A_m \exp(-\bar{a}_m t'^{1/(2m-1)}) \right) t' \frac{m-1}{m-1}/3
\]

\[
\leq \bar{A}' \exp(-\bar{b}_m t' \frac{m}{m-1})
\]

for large \(t'\), where \(\bar{A}'\) and \(\bar{b}_m\) are some positive constant.

So we know the statement in \((53)\) holds for all \(\theta \geq \frac{2m-1}{m}\). Start with \(m_1 = 2\), we have a recursion equation \(m_k = \frac{m_{k-1} - 1}{m_{k-1}}\) and we know \((53)\) holds for all \(\theta \geq m_k\). It is not hard to see that \(m_k = \frac{k+1}{k}\) and thus \(\lim_{k \to \infty} m_k = 1\).

Recall that \(t' = \alpha(s + t) - \alpha(t) \geq \frac{\lambda_m}{\lambda_m} t\). So, for some constant \(a_0\) which is a function of \(\lambda_{\min}\), we can rewrite \((53)\) as

\[
\bar{q}[\alpha(s), \alpha(s + t)] \leq \bar{A}' \exp(-\bar{a}_0 t^{1/\theta})
\]

which concludes the lemma.