Finite volume analysis of scattering theory in the scaling Potts model

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Abstract

We perform a detailed investigation of the scaling Potts field theory using the truncated conformal space approach. The aim is to check the scattering theory obtained from the S matrix bootstrap, both in the paramagnetic and ferromagnetic phases. It turns out that the bootstrap S matrices explain the finite size spectrum perfectly well. This finding is relevant in the view of a recent controversy surrounding the scaling limit of the lattice Potts model. In the conclusions we discuss possible scenarios to resolve the conflict between the perturbed CFT picture and lattice results, but the whole issue remains enigmatic.

1 Introduction

The three-state Potts model is one of the most important models in statistical mechanics. It is a generalization of the Ising model where the spin can take three values. At criticality, it is known to be integrable [1] and the corresponding conformal field theory is also known [2, 3].

Away from criticality, integrability on the lattice is lost. However in the vicinity of the critical point one can pass to a description by a continuum field theory, the scaling Potts model, which is uniquely specified by the operator content of the conformal field theory at the critical point [3]. The scaling Potts model is known to be integrable [1] and its exact S matrix has been constructed using the bootstrap [5, 6, 7, 8], and also from an integrable lattice regularization or an integrable lattice regularization [9]. The bootstrap S matrix is diagonal i.e. there is only transmission without reflection.

Recent investigations show that the scattering theory of the lattice model is not diagonal and it is suggested that it stays so even in the scaling limit [10]. A detailed numerical verification of this claim has also been carried out, and it turns out that the lattice spectrum does not agree with the predictions of the bootstrap S matrices [11] (although there is partial agreement for a subset of the states). This is really surprising, since the identification of the scaling limit proceeds via well-known arguments that use the notion of universality of critical behavior.

However, the bootstrap S matrices themselves have never been compared to the scaling Potts theory in detail. It is well-known that bootstrap S matrices are usually conjectures, and their correspondence to the microscopic (e.g. Lagrangian) description must be established.

The thermodynamic Bethe Ansatz for the Potts model was constructed [12], but while it shows that the bootstrap scattering theory agrees in the ultraviolet with the conformal field theory

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1 A detailed analysis of the scaling q-state Potts model scattering theory for general $q \leq 4$, and the corresponding TBA system can be found in [13, 14].
predictions, it does not eventually test whether the $S$ matrices really describe the scattering of multi-particle states. A simple way to see this is that the TBA equation is eventually the same for the Potts model and the tetra-critical Ising model, which is another perturbed conformal field theory with the same ultraviolet central charge, but the scattering in these two models show subtle differences, and they describe two different universality classes in terms of critical phenomena, with a different set of critical exponents.

In view of this situation it is worthwhile to examine whether the bootstrap $S$ matrices really describe the scattering theory of the scaling Potts model defined as a perturbed conformal field theory. This is the aim of this paper, accomplished using the truncated conformal space approach (TCSA) [15] to construct the finite volume spectrum of the scaling Potts field theory and comparing to the predictions made on the basis of the bootstrap $S$ matrices. While the TCSA has been applied to the Potts model previously in [16], that work mainly focused on the evolution of the particle spectrum in the non-integrable case when both the thermal and the magnetic perturbations are added. Here we stay with the purely thermal perturbation, but explore it much deeper in order to determine whether the conjectured bootstrap $S$ matrices really describe the spectrum.

The outline of the paper is as follows. In section 2 we collect some necessary facts about the scaling Potts model. Section 3 outlines the TCSA method, and in section 4 we discuss the predictions of the bootstrap $S$ matrices for the finite volume spectrum. The detailed numerical comparison is made in section 5, while section 6 is reserved for the conclusions.

2 Scaling Potts model as perturbed conformal field theory

The three-state Potts quantum chain has three degrees of freedom per site. The Hilbert space is of the form

$$\mathcal{H}_{\text{chain}} = \prod_i (\mathbb{C}^3)_i$$

where $i$ labels the sites of the chain, and the quantum space $\mathbb{C}^3$ at site $i$ has the basis $|\alpha\rangle$, $\alpha = 0, 1, 2$. The dynamics of the Potts chain is defined by the Hamiltonian

$$H_{\text{chain}} = -J \sum_i \sum_{\alpha=0}^2 P^\alpha_i P^\alpha_{i+1} - Jg \sum_i \tilde{P}_i$$

(2.1)

where

$$P^\alpha = |\alpha\rangle\langle\alpha| - \frac{1}{3}$$

$$\tilde{P} = \frac{1}{3} \sum_{\alpha, \alpha'=0}^2 (1 - \delta_{\alpha\alpha'}) |\alpha\rangle\langle\alpha'|$$

The critical point is at $g = 1$: for $g > 1$, the cyclic symmetry $\mathbb{Z}_3$ acting as

$$|\alpha\rangle \to |\alpha + 1 \text{ mod } 3\rangle$$

is unbroken and the model is in a paramagnetic phase, while for $g < 1$ it is broken and the model is in a ferromagnetic phase. In the neutral sector of the model there is a duality under

$$g \leftrightarrow 1/g$$

(2.2)

which is the analogue of the Kramers-Vannier duality in the Ising model. The critical point is self-dual and there the lattice model can be solved exactly [1].

The scaling (continuum) limit of the lattice Potts model is obtained by taking

$$g \to 1, \quad a \to 0$$

such that

$$(g - 1)a^{-6/5} = \text{fixed}$$

(2.3)
where \( g \) is the lattice Potts coupling and \( a \) is the lattice spacing. Since the gap on the lattice scales as

\[
\Delta(g) \propto (g - 1)^{5/6}
\]

this means a finite mass gap in the continuum limit

\[
a^{-1} \Delta(g) \to m = \text{finite}
\]

On the other hand, if one sets the critical value

\[
g = 1
\]

and takes the continuum limit \( a \to 0 \) only afterward, then one arrives to a minimal conformal field theory with central charge

\[
c = \frac{4}{5}
\]

\[2, 3\]. The Kac table of conformal weights is

\[
\{ h_{r,s} \} = \begin{pmatrix}
0 & 1 & 2 & 13 & 3 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{5}{4} & \frac{3}{4}
\end{pmatrix}
\]

\( r = 1, \ldots, 4 \) \( s = 1, \ldots, 5 \)

The \( D_4 \) conformal field theory is invariant under the permutation group \( S_3 \) generated by two elements \( Z \) and \( C \) with the relations

\[
Z^3 = 1 \quad C^2 = 1 \quad CZC = Z^{-1}
\]

which have the signatures

\[
\text{sign } Z = +1 \quad \text{sign } C = -1
\]

The sectors of the Hilbert space are products of the irreducible representations of the left and right moving Virasoro algebras which can be specified by giving their left and right conformal weights as

\[\mathcal{S}_{h,h} = \mathcal{V}_h \otimes \mathcal{V}_h\]

The complete Hilbert space of the model is

\[
\mathcal{H} = \mathcal{S}_{0,0} \oplus \mathcal{S}_{\frac{1}{2},\frac{1}{2}} \oplus \mathcal{S}_{\frac{3}{2},\frac{3}{2}} \oplus \mathcal{S}_{3,3}
\]

\[
\oplus \mathcal{S}^+_{\frac{1}{2},\frac{1}{2}} \oplus \mathcal{S}^-_{\frac{1}{2},\frac{1}{2}} \oplus \mathcal{S}^+_{\frac{3}{2},\frac{3}{2}} \oplus \mathcal{S}^-_{\frac{3}{2},\frac{3}{2}}
\]

\[
\oplus \mathcal{S}^+_{\frac{5}{2},\frac{5}{2}} \oplus \mathcal{S}^-_{\frac{5}{2},\frac{5}{2}} \oplus \mathcal{S}_{0,3} \oplus \mathcal{S}_{3,0}
\]

(2.4)

which corresponds to a modular invariant partition function on the torus \[17\]. Note that not all of the possible representations occur in the Hilbert space; there is another modular invariant partition function called \( A_4 \) which includes all sectors of diagonal form \( \mathcal{S}_{h,h} \) allowed by the Kac table exactly once:

\[
\mathcal{H} = \bigoplus_h \mathcal{V}_h \otimes \mathcal{V}_h
\]

It corresponds to the scaling limit of a higher multicritical Ising class fixed point (with symmetry \( \mathbb{Z}_2 \)).

The sectors on the first line of (2.4) are invariant under the action of the permutation group \( S_3 \), the two pairs on the second line each form the two-dimensional irreducible representation, which is characterized by the following action of the generators:

\[
C |\pm\rangle = |\mp\rangle
\]

\[
Z |\pm\rangle = e^{\pm \frac{2\pi i}{3}} |\pm\rangle
\]

3
while the ones on the third line form the one-dimensional signature representation where each element is represented by its signature. These sectors are in one-to-one correspondence with the families of conformal fields, and the primary field (the one with the lowest conformal weight) in the family corresponding to \( S_{h,\bar{h}} \) has left and right conformal weights \( h \) and \( \bar{h} \); they are denoted \( \Phi_{h,\bar{h}} \) with an optional upper ± index for fields forming a doublet of \( S_3 \). In a family all other fields have conformal weights that differ from those of the primary by natural numbers. The conformal spin \( s = h - \bar{h} \) gives the behavior under spacial translations; translational invariant fields must be spinless i.e. \( h = \bar{h} \).

Relevant fields are exactly those for which \( h + \bar{h} < 2 \). It is then obvious that the only \( S_3 \)-invariant spinless relevant field is

\[
\Phi_{2,\bar{2}}
\]

which means that the Hamiltonian of the scaling limit of the off-critical Potts model is uniquely determined

\[
H = H_* + \tau \int dx \Phi_{2,\bar{2}}
\]  

(2.5)

The coupling constant \( \tau \) is related to the lattice coupling in the scaling limit as

\[
\tau \propto (g - g_c) a^{-6/5}
\]

and \( \tau > 0 \) in the paramagnetic, while \( \tau < 0 \) in the ferromagnetic phase. The magnetization operator is the doublet field

\[
\Phi_{\pm}
\]

and there is also another relevant field doublet

\[
\Phi_{\mp}
\]

which is the subleading magnetization operator.

The scaling Potts model (2.5) is integrable [4], and its spectrum and the scattering matrix can be determined exactly [5, 4, 6, 7, 8] using the exact \( S \) matrix bootstrap developed in [18] or an integrable lattice regularization [9]. In the paramagnetic phase, the vacuum is non-degenerate and the spectrum consists of a pair of particles \( A \) and \( \bar{A} \) of mass \( m \) which form a doublet under \( S_3 \):

\[
C|A(\theta)\rangle = |\bar{A}(\theta)\rangle \\
Z|A(\theta)\rangle = e^{2\pi i \theta} |A(\theta)\rangle
\]

\[
C|\bar{A}(\theta)\rangle = |A(\theta)\rangle \\
Z|\bar{A}(\theta)\rangle = e^{-2\pi i \theta} |\bar{A}(\theta)\rangle
\]  

(2.6)

The generator \( C \) is identical to charge conjugation, the charge of \( A \) is \( Q = 1 \) while that of \( \bar{A} \) is \( Q = -1 \) (where \( Q \) is defined mod 3). Due to two-dimensional Lorentz invariance, the energy and momentum of the particles is parametrized by the rapidity \( \theta \):

\[
E = m \cosh \theta \quad p = m \sinh \theta
\]

The two-particle scattering amplitudes are

\[
S_{AA}(\theta_{12}) = S_{A\bar{A}}(\theta_{12}) = \frac{\sinh \left( \frac{\theta_{12}}{2} + \frac{\pi i}{3} \right)}{\sinh \left( \frac{\theta_{12}}{2} - \frac{\pi i}{3} \right)}
\]

\[
S_{A\bar{A}}(\theta_{12}) = S_{\bar{A}A}(\theta_{12}) = -\frac{\sinh \left( \frac{\theta_{12}}{2} + \frac{\pi i}{3} \right)}{\sinh \left( \frac{\theta_{12}}{2} - \frac{\pi i}{3} \right)}
\]  

(2.7)

where \( \theta_{12} = \theta_1 - \theta_2 \) is the rapidity difference of the incoming particles. This \( S \) matrix was confirmed by the field theoretic thermodynamic Bethe Ansatz [12]. It can also be obtained by Bethe Ansatz from an integrable \( Z_3 \) invariant lattice spin chain [9], thought to be in the same universality class as the usual Potts chain defined by (2.1).
In the ferromagnetic phase the vacuum is three-fold degenerate

\[ |0\rangle_a \quad a = -1, 0, 1 \]

where the action of \( S_3 \)

\[ Z|0\rangle_a = |0\rangle_{a+1 \mod 3} \quad C|0\rangle_a = |0\rangle_{-a} \]

and the excitations are kinks of mass \( m \) interpolating between adjacent vacua \([7]\). The kink of rapidity \( \theta \), interpolating from \( a \) to \( b \) is denoted by

\[ K_{ab}(\theta) \quad a - b = \pm 1 \mod 3 \]

The scattering processes of the kinks are of the form

\[ K_{ab}(\theta_1) + K_{bc}(\theta_2) \rightarrow K_{ad}(\theta_1) + K_{dc}(\theta_2) \]

with the scattering amplitudes equal to

\[ S \left( \begin{array}{ccc} d \\ a \\ b \\ c \end{array} \right) (\theta_{12}) = \begin{cases} S_{AA}(\theta_{12}) & \text{if } b = d \\ S_{A\bar{A}}(\theta_{12}) & \text{if } a = c \end{cases} \] (2.8)

The exact relation between the mass gap and the coupling \( \tau \) is also known \([19]\):

\[ |\tau| = \kappa m^{6/5} \]

\[ \kappa = \frac{\Gamma \left( \frac{3}{10} \right) \Gamma \left( \frac{2}{5} \right) \Gamma \left( \frac{3}{5} \right)^{6/5}}{4 \times 2^{1/5} \pi^{8/5} \Gamma \left( \frac{1}{10} \right)} \sqrt{\frac{\Gamma \left( -\frac{1}{2} \right) \Gamma \left( \frac{7}{10} \right)}{\Gamma \left( -\frac{1}{2} \right) \Gamma \left( \frac{7}{5} \right)}} = 0.164303 \ldots \]

3 TCSA for the scaling Potts model

To test whether the \( S \) matrices in the previous subsection describe the scattering in the field theory correctly, we can turn to the truncated conformal space approach \([15]\). This consists of putting the system on a cylinder with compactified spatial dimension of circumference \( L \) and evaluating the excitation spectrum numerically. The matrix elements of the Hamiltonian between the \( \tau = 0 \) eigenstates can be evaluated exactly using conformal field theory techniques. Continuing to Euclidean time \( t_E = -it \) and mapping the cylinder to the complex plane using

\[ z = e^{2\pi \left( t_E + ix \right)} , \quad z = e^{2\pi \left( t_E - ix \right)} \]

the explicit Hamiltonian is

\[ H_{\text{TCSA}} = \frac{2\pi}{L} \left( H_0 + \tau \frac{L^{6/5}}{(2\pi)^{1/5}} B \right) \]

where \( H_0 \) is the matrix

\[ (H_0)_{ij} = \left( h_i + \frac{\tau}{12} \right) \delta_{ij} \] (3.1)

and \( B \) is given by the matrix elements of the perturbing field on the complex plane

\[ B_{ij} = \langle i| \Phi_{\frac{1}{6}} \Phi_{\frac{1}{6}} (z = 1) \rangle_{s_i, s_j} \]

where \( s_i \) and \( s_j \) are the conformal spins of the states \( |i\rangle \) and \( |j\rangle \) (the selection rule comes from translational invariance). The Hamiltonian can be made dimensionless in units of the mass gap \( m \):

\[ h_{\text{TCSA}}^\pm = \frac{1}{m} H_{\text{TCSA}} = \frac{2\pi}{l} \left( H_0 \pm \tau \frac{L^{6/5}}{(2\pi)^{1/5}} B \right) \]

\[ l = mL \]
where ± corresponds to the sign of \( \tau \), i.e. to the para/ferromagnetic phases. To calculate the matrix elements \( B \) it is necessary to know the conformal three-point couplings which can be obtained from the literature \([20, 21, 22]\); for the present calculations the construction in \([23]\) was used.

It turns out that the perturbing operator acts separately in the following four sectors:

\[
H_0 = S_{0,0} \oplus S_{\frac{3}{2}, \frac{3}{2}} \oplus S_{3,3} \\
H_\pm = S_{\frac{1}{2}, \frac{1}{2}} \oplus S_{\frac{3}{2}, \frac{3}{2}} \\
H_1 = S_{\frac{5}{2}, \frac{5}{2}} \oplus S_{3,3} \oplus S_{0,0} \oplus S_{3,0}
\]

so the Hamiltonian can be diagonalized separately in each of them. It is also the case that the Hamiltonian is exactly identical in the sectors \( H_+ \) and \( H_- \). The reason for this is that the charge conjugation symmetry \( C \) acts on the sectors \( H_+ \) and \( H_- \) by swapping them.

The spectrum is invariant under \( \tau \to -\tau \) in sectors \( H_0 \) and \( H_1 \). The latter fact is the consequence of a \( \mathbb{Z}_2 \) symmetry in these sectors under which the parities in \( H_0 \) are even:

\[
even : S_{0,0} \quad S_{\frac{3}{2}, \frac{3}{2}}
\]

and odd:

\[
odd : S_{\frac{3}{2}, \frac{3}{2}} \quad S_{3,3}
\]

so the perturbing operator \( \Phi_{\frac{3}{2}, \frac{3}{2}} \) is odd. In \( H_1 \) this \( \mathbb{Z}_2 \) acts by swapping

\[
S_{0,0} \leftrightarrow S_{3,0} \quad S_{\frac{3}{2}, \frac{3}{2}} \leftrightarrow S_{\frac{3}{2}, \frac{3}{2}}
\]

This symmetry leaves the fixed point Hamiltonian \( H_* \) and the conformal fusion rules in these sectors invariant\(^2\) away from the critical point, it is the continuum counterpart of the Kramers-Vannier duality \([22]\) under the scaling limit \([23]\).

4 Predictions for the finite volume spectrum

4.1 Vacuum states in the ferromagnetic phase and vacuum tunneling

For \( \tau > 0 \) one expects a single vacuum state, while for \( \tau < 0 \) there should be three of them, corresponding to spontaneous breaking of the \( \mathbb{Z}_3 \) subgroup of \( S_3 \). In a finite volume \( L \), the degeneracies between these states must be lifted by tunneling. Using the formalism developed in \([24]\), tunneling in the three dimensional vacuum subspace can be described by the following effective Hamiltonian

\[
H = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
m \\
\sqrt{2\pi m L} e^{-mL} + O(e^{-2mL})
\end{pmatrix}
\]

where the matrix part is just the connectivity structure between the vacua introduced by the kinks, and the value of the matrix element is obtained by borrowing the dilute instanton gas result derived in \([24]\). The eigenvectors of this matrix are

\[
|\vartheta\rangle_L = \sum_{a=0,\pm1} e^{ia\vartheta} |0\rangle_a \quad \vartheta = 0, \pm \frac{2\pi}{3}
\]

with eigenvalues

\[
E_{\text{tunnel}}(\vartheta) = 2 \cos \vartheta \frac{m}{\sqrt{2\pi m L}} e^{-mL} + O(e^{-2mL})
\]

Two of the vacua remain degenerate

\[
E_{\text{tunnel}}(2\pi/3) = E_{\text{tunnel}}(-2\pi/3)
\]

\(^2\)The conformal fusion rules do not allow the extension of this symmetry to the \( H_\pm \).
which is due to the presence of the charge conjugation $\mathbb{Z}_2$ subgroup of $S_3$ which acts as
\[
C| \pm 2\pi/3 \rangle_L = | \mp 2\pi/3 \rangle_L \\
C|0 \rangle_L = |0 \rangle_L
\]
therefore the degeneracy holds exactly (i.e. beyond the dilute instanton gas approximation). The splitting between these and the lowest lying state is given by
\[
\Delta E_{\text{vac}} = E_{\text{tunnel}}(\pm 2\pi/3) - E_{\text{tunnel}}(0) = 3\frac{m}{\sqrt{2\pi m L}} \cosh \theta + O(e^{-m L}) \tag{4.1}
\]
The vacuum states can also be described by thermodynamic Bethe Ansatz (TBA): the state $|0 \rangle$ can be described by the TBA developed by Zamolodchikov\cite{25,12}, while the vacua $| \pm 2\pi/3 \rangle$ can be obtained by a twist\cite{26}. The results from TBA also agree with the asymptotics\eqref{4.1}.

In connection with the TBA, note that the detailed structure of the $A_4$ theory differs from that of $D_4$ also by including sectors absent from the latter. Although it has only one copy of the vacuum built upon
\[
\Phi \frac{\mp 2\pi}{3}
\]
it also contains a third vacuum state which in the ultraviolet limit corresponds to the field
\[
\Phi \frac{\mp \pi}{3}
\]
which is absent from the $D_4$ spectrum. Furthermore, while in the paramagnetic phase its scattering theory is identical to that of the $D_4$, in the ferromagnetic phase its kink structure is markedly different\cite{0} as shown in figure 4.1.

As a result of the above considerations, while TBA provides a very strong argument in favour of the correctness of the bootstrap $S$-matrices, it cannot confirm their details since it cannot even distinguish between the $A_4$ and $D_4$ systems. Detailed confirmation is only possible by considering excited states, which is the motivation for the analysis that follows in the sequel.

### 4.2 Multi-particle states in the paramagnetic phase

The finite volume dependence of two-particle energy levels can be described using the standard Bethe-Yang equations, which are a specialization of Lüscher’s formalism for finite size corrections\cite{27,28} to two-dimensional integrable field theories. The spectrum of multi-particle states
\[
|A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)\rangle
\]
can be obtained by solving the Bethe-Yang equations which describe the periodicity of the wave function:
\[
e^{imL \sinh \theta} \prod_{j \neq k} S_{a_k a_j}(\theta_k - \theta_j) = 1 \quad k = 1, \ldots n \tag{4.2}
\]
The energy relative the vacuum state can be evaluated as
\[
E(L) - E_0(L) = m \sum_{k=1}^n \cosh \theta_k + O(e^{-m L})
\]

![Figure 4.1: Vacuum adjacency diagrams for kinks in the ferromagnetic phase of the $D_4$ and $A_4$ models](image)
which is exact to all orders in $1/L$ i.e. up to corrections that vanish exponentially with the volume, as indicated.

On the basis of $Z$ quantum numbers, the states contained in sectors $\mathcal{H}_0$ and $\mathcal{H}_1$ are the $Q = 0$ states

$$A \bar{A}, AAA, \bar{A}A, AAA$$

almost all of them occur in degenerate pairs under $\mathcal{C}$; an exception is the state containing two stationary particles, $A$ and $\bar{A}$, as this is necessarily $\mathcal{C}$-even.

In finite volume, the $\mathcal{C}$-even combinations are in $\mathcal{H}_0$, while the $\mathcal{C}$-odd combinations in $\mathcal{H}_1$ and the degeneracy, which is exact in the Bethe-Yang approximation, is lifted by corrections decaying exponentially with increasing volume. In addition, $\mathcal{H}_0$ is expected to contain the vacuum, which is $\mathcal{C}$-even. On the other hand, the states in $\mathcal{H}_\pm$ are expected to be the $Q = \pm 1$ states $H^\pm$: $\bar{A}, AA, A \bar{A}, \bar{A} \bar{A}, AAA$ and the two sectors are related to each other by $\mathcal{C}$, with exactly identical spectra.

### 4.3 Multi-particle states in the ferromagnetic phase

In this case the kink structure necessitates the diagonalization of the multi-particle transfer matrix $[29, 30]$.

The two-particle states allowed by periodic boundary condition are of the form $K_{ab}^{\theta_1, \theta_2}$ which is six dimensional corresponding to the allowed sequences

$$aba = 010, 020, 101, 202, 121, 101, 212$$

and the states in the sector are overall neutral in terms of topological charge. The transfer matrices acting on this space, which describe the monodromy of the wave function corresponding to the kinks of rapidity $\theta_1$ and $\theta_2$ are

$$T_1(\theta_1, \theta_2)_{ab}^{\theta_1, \theta_2} = \delta^a_c S \left( \begin{array}{cc} d & b \\ c & d \end{array} \right) (\theta_1 - \theta_2) = \delta^a_c S_{A\bar{A}}(\theta_1 - \theta_2)$$

$$T_2(\theta_1, \theta_2)_{ab}^{\theta_1, \theta_2} = \delta^a_c S \left( \begin{array}{cc} b & a \\ d & c \end{array} \right) (\theta_2 - \theta_1) = \delta^a_c S_{A\bar{A}}(\theta_2 - \theta_1)$$

$T_1$ and $T_2$ commute, and so they can be diagonalized simultaneously. The eigenvalues come in doubly degenerate pairs:

$$\ell^{(q)}_1(\theta_1, \theta_2) = e^{\theta_1 - \theta_2} S_{A\bar{A}}(\theta_1 - \theta_2)$$

$$\ell^{(q)}_2(\theta_1, \theta_2) = e^{-\theta_1 - \theta_2} S_{A\bar{A}}(\theta_2 - \theta_1)$$

where $q = +1, 0, -1$ i.e. whenever $T_1$ has eigenvalue $\ell^{(q)}_1$, $T_2$ has eigenvalue $\ell^{(q)}_2$ with the same eigenvector, and each of the three distinct eigenvalue pairs occurs twice corresponding to degeneracy under charge conjugation $\mathcal{C}$. The quantization relations therefore read

$$e^{imL \sinh \theta_1} e^{\theta_2 - \theta_1} S_{A\bar{A}}(\theta_1 - \theta_2) = 1$$

$$e^{imL \sinh \theta_2} e^{-\theta_2 + \theta_1} S_{A\bar{A}}(\theta_2 - \theta_1) = 1$$

This can be interpreted in the following way: there is a kink-particle correspondence

$$K_{01}, K_{12}, K_{20} \leftrightarrow A$$

$$K_{02}, K_{10}, K_{21} \leftrightarrow \bar{A}$$
and there is a twist operator on the circle, which can be either the identity, $Z$ or $Z^{-1}$. When calculating the monodromy of the wave-function, any kink taken around the circle gains the phase factor from the momentum and the scattering amplitudes with the other kinks which can be evaluated using the kink-particle correspondence. In addition there is a phase factor which is the eigenvalue of the corresponding particle under the twist operator, which accounts for the cases $q = +1, 0, -1$, respectively.

This picture generalizes to higher particle states: the transfer matrix for the kink of rapidity $\theta_1$ can be written as

$$\mathcal{T}_1(\theta_1, \theta_2, \ldots, \theta_n)_{a_1, a_2, \ldots, a_n} = \delta_{a_1}^{b_1} S(b_2^{\pm} a_3) (\theta_1 - \theta_2) S(b_3^{\pm} a_4) (\theta_1 - \theta_3) \times \ldots S(b_n^{\pm} a_1) (\theta_1 - \theta_n)$$

and due to the periodic boundary conditions the transfer matrices $\mathcal{T}_k$ for the other particles $k = 2, \ldots, n$ can be obtained by cyclically rotating the indices $1, \ldots, n$ in this formula. All these transfer matrices commute and can be diagonalized simultaneously. An explicit computation shows that the three-particle states are described by

$$e^{i m L \sinh \theta_1} e^{\frac{i}{2} \theta_1^3} S_{AA}(\theta_1 - \theta_2) S_{AA}(\theta_1 - \theta_3) = 1$$

and similar equations can be written for four-particle states.

Due to periodic boundary conditions, all the states are of total charge zero. On the basis of their $Z$ quantum numbers, the $q = 0$ states are expected to be in $\mathcal{H}_0$ and $\mathcal{H}_1$, coming in pairs under $C$ which are degenerate in the Bethe-Yang approximation, so their degeneracy is only lifted by corrections exponentially decreasing with the volume. This is identical to the spectrum expected in the paramagnetic phase, in perfect agreement with the fact that the spectrum in $\mathcal{H}_0$ and $\mathcal{H}_1$ is invariant under the Kramers-Vannier duality $\tau \rightarrow - \tau$.

The $q = \pm 1$ states are expected to be in the sectors $\mathcal{H}_{\pm}$, which again have exactly identical spectra due to the fact that $C$ swaps them with other; these sectors also contain the vacua $|\pm 2\pi/3\rangle$. As stated above, there are no states for which the total kink charge is nonzero; in particular, there are no one-kink states. Therefore these states are genuinely different from the paramagnetic case when sectors $\mathcal{H}_{\pm}$ contained states with nonzero total charge. However, these sectors are characterized by the presence of the twist $Z$ or $Z^{-1}$, which in essence carries a twist quantum number $q$ similar to the charge $Q$, distinguishing them from each other and the untwisted states.

## 5 Analysis of the TCSA spectrum

Here we show that the finite size spectrum derived from TCSA numerics is fully consistent with the theoretical picture presented in subsection 4 and so the detailed correspondence between the scattering matrices ($2.7, 2.8$) and the scaling field theory ($2.5$) hold, both in the paramagnetic and the ferromagnetic regime.

We diagonalized the Hamiltonian using the TCSA program developed for the calculations in [31]. It can be done separately in the four sectors. We cut off the Hilbert space by imposing an upper limit on the eigenvalues of the conformal Hamiltonian (3.1) by considering only states with

$$h_i + \tilde{h}_i < e_{\text{cut}}$$

and restricted to the zero-momentum sector in which

$$h_i - \tilde{h}_i = 0$$

The dimensions of the Hilbert spaces for the values of the cutoff used in the calculation are summarized in the following table:
The gap in the ferromagnetic and paramagnetic phases. The dots are the TCSA data, while the continuous line shows the prediction (4.1) for the vacuum split and (5.1) for the volume dependent particle mass, respectively.

### Figure 5.1

| $\mathcal{H}_0$ | $\mathcal{H}_1$ | $\mathcal{H}_\pm$ |
|---------------|---------------|----------------|
| 1695          | 1384          | 900            |
| 1744          | 1422          | 1225           |
| 1438          | 2574          | 2262           |
| 2591          | 4682          | 1287           |
| 4708          | 7652          | N/A            |
| 8174          | 8144          |                 |

One technical issue is that the TCSA converges slowly; however, it is ultraviolet finite due to the fact that the conformal weight of the perturbing operator is less than $\frac{1}{2}$. Convergence can be improved using the TCSA renormalization group [32, 33, 34] which predicts that the cutoff dependence of energy levels computed from TCSA is

$$E_i(L, e_{\text{cut}}) = E_i(L, \infty) + b_i(L) \frac{1}{e_{\text{cut}}} + \text{subleading corrections}$$

Using this result, the computed spectra can be extrapolated to infinite cutoff, which results in a substantial improvement of accuracy.

### 5.1 The gap

First we analyze the gap in the spectrum, which is the difference between the ground state energy in sectors $\mathcal{H}_\pm$ and $\mathcal{H}_0$. As shown in figure 5.1, the gap vanishes exponentially with $L$ in the ferromagnetic phase in accordance with (4.1), which corresponds to a triply degenerate vacuum related to spontaneously broken $\mathbb{Z}_3$ symmetry. In the paramagnetic phase the gap approaches unity, i.e. in proper units $m$ as $L$ grows, corresponding to the one-particle states $A$ and $\bar{A}$, which are expected to be the lowest lying states in sectors $\mathcal{H}_\pm$. For a more precise comparison, one can also use Lüscher’s formalism for finite volume mass corrections [35] (see also [36]) which gives the following theoretical prediction for the finite volume mass gap in the paramagnetic phase:

$$\Delta E(L) = m - \mu g^2 e^{-\mu L}$$

$$\times m \cosh \theta e^{-m L \cosh \theta} + O(e^{-2\mu L})$$

$$g = - \text{Res}_{\theta=2\pi/3} S_{AA}(\theta) = \sqrt{3} \quad \mu = \sqrt{m_A^2 - m_A^2} = \frac{\sqrt{3}}{2} m$$

The numerical accuracy of the agreement with the expected gap in the range $2.5 < mL < 10$ is of order $10^{-3}$ or better. This gives a good test of the $S$ matrices, but again, it is not really sensitive to details such as whether one deals with the $D_4$ or the $A_4$ model. It also does not give a test of the individual two-particle $S$-matrices separately.
5.2 Excited states in $\mathcal{H}_0$ and $\mathcal{H}_1$

To test the $S$ matrices, we first examine the sector $\mathcal{H}_0$. Figure 5.2 shows the agreement between the two-particle line predicted by the Bethe-Yang equations from the bootstrap $S$ matrices and the TCSA data. The continuous lines are two-particle states, but to explain all the states shown it is also necessary to consider three-particle $AAA$ levels (dashed lines) and four-particle $AAAA$ levels (dotted lines), which could be done by a simple extension of the formalism formulated in subsection 4. The lowest lying two-particle level contains stationary $A$ and $\bar{A}$ particles; the Bethe-Yang prediction for this level is constant:

$$\Delta E(L) = 2m$$

but it has exponential corrections similar to those for the mass gap in (5.1), which clearly show up on the plot. It only occurs in the $C$-even sector $\mathcal{H}_0$.

The agreement is spectacular, but we must keep in mind that this sector is independent of the phase, and even of the choice of the universality class $A_4$ or $D_4$.

The same agreement is observed in sector $\mathcal{H}_1$ shown in figure 5.3 but this is more interesting, given that this sector occurs only in the $D_4$ theory. Comparing the numerical spectra it can also be seen that states with the same particle contents are almost degenerate between $\mathcal{H}_0$ and $\mathcal{H}_1$, split only by corrections decaying exponentially with $L$.

5.3 $\mathcal{H}_\pm$ for $\tau > 0$

In the paramagnetic phase, the two-particle and three-particle levels can be interpreted in terms of $AA$ and $AAA$ states in the $\mathcal{H}_+$ and their charge conjugates in the $\mathcal{H}_-$ sector, as expected from the description given in subsection 4. This is demonstrated in figure 5.4 which shows one of the sectors, as the spectrum in the other one is identical.

5.4 $\mathcal{H}_\pm$ for $\tau < 0$

In the ferromagnetic phase, just as expected, this sector turns out to contain the two-kink states described by the $q = \pm 1$ case of the Bethe-Yang equations (4.3). This is shown in figure 5.5 where the two-particle lines are shown by continuous lines. The other states in the plot can
Figure 5.3: The spectrum in the $\mathcal{H}_1$ sector

Figure 5.4: The spectrum in the $\mathcal{H}_\pm$ sectors for $T > T_c$. The lowest lying (one-particle) level has been omitted from the plot, since it was already analyzed in detail in 5.1.
be explained using the three-kink equation (4.4) and the four-kink generalization of the transfer matrix formalism, and are shown by dashed and dotted lines, respectively.

5.5 Explicit determination of the phase-shift functions

Although the agreement between the Bethe-Yang predictions and the TCSA spectrum seems quite convincing, it is not obvious whether this is true for the phase-shifts, since the dependence on the phase-shift is a subleading term in the finite size corrections. However, we can extract the phase-shifts directly from two-particle states. For $A \bar{A}$ we can consider the states in the zero-momentum subspaces of $H_0$ and $H_1$ (identical in both regimes) which are described by

$$mL \sinh \theta + \delta_{A \bar{A}}(2\theta) = 2\pi I \quad I \in \mathbb{N}$$

$$\delta_{A \bar{A}}(\theta) = -i \log S_{A \bar{A}}(\theta)$$

that follows from taking the logarithm of the appropriate Bethe-Yang equation. The phase-shift function can be extracted using

$$\theta(L) = \cosh^{-1}(\Delta E(L)/2m)$$
$$\delta_{A \bar{A}}(2\theta(L)) = 2\pi I - mL \sinh \theta(L)$$

and plotted as a parametric curve in the variable $l = mL$. The two sectors give exponentially close results, which are visually indistinguishable, so only one of them is shown in figure 5.6 (a). It is also possible to extract the same phase shift function from the paramagnetic $H_{\pm}$ sectors, which are described by the same equations, but with

$$I \in \mathbb{N} \pm \frac{1}{3}$$

The numerical results from this extraction are shown in figure 5.6 (b).

For the case of $AA$, only the paramagnetic phase $H_{\pm}$ sectors have such two-particle states because these are charged. The phase-shifts determined from the first two such levels are shown in figure 5.7. For large values of $\theta$ (corresponding to small $mL$) the observed deviations can be
The simplest scenario is that the lattice calculations are simply too far from the critical point, and so not in the scaling regime yet. The results in [11] however seem to indicate that the scaling regime of the lattice model was reached at least for the low-energy degrees of freedom; the

attributed to finite-size effects decaying exponentially with the volume (which are neglected in the Bethe-Yang description), while for small values of $\theta$ (large $mL$) they reflect the truncation error inherent in the TCSA method.

Note that these phase-shifts enter in the determination of many other levels, such as three and four-particle states, for which the agreement between the theoretical prediction and TCSA data is as good for the two-particle levels. This further strengthens the agreement, and also confirms that the $AA$ phase-shift correctly describes the scattering of like-charge kinks in the ferromagnetic phase, although a direct determination from two-particle states (due to their absence from the $T < T_c$ spectrum) is not possible.

6 Conclusions and open questions

The main conclusion is very simple: the $S$ matrices [27] and [28] do describe the scattering theory of the $\tau > 0$ and $\tau < 0$ phase of the scaling Potts model, defined as the perturbed conformal field theory [26]. Everyone familiar with the field of integrability would say that this was to be expected. However, this leaves us with an enigma: why is it that these $S$ matrices are incompatible with the lattice results as shown in [10] [11]?
lattice calculation also reproduces the correct $S$-matrix $S_{AA}$ for the $C$-odd neutral states (sector $H_1$). The lattice data however show no trace of the exponential degeneracy between the $C$-even multi-particle states in sector $H_0$ and their $C$-odd counterparts in sector $H_1$, and yield a scattering length for the $C$-even states that is inconsistent with $S_{AA}$. In this scenario this could be due to the contribution of some irrelevant operator with an unusually large coefficient, and when approaching the critical point on the lattice the behaviour would be described by the scaling Potts model for a range of energy values which excludes very small rapidities (and also large ones, due to the presence of the cut-off), but extends longer and longer as $g \rightarrow 1$ such that the scattering amplitude determined from the lattice tends to $S_{AA}$ everywhere (except at $\theta = 0$, where it always remains $-1$, whereas $S_{AA}(0) = +1$).

The above scenario is, however, not really supported by the lattice data in [11] (although it is not ruled out conclusively, either). Therefore there is also another, more exotic scenario: it is possible that the low energy behaviour of the off-critical lattice Potts model is not the integrable scaling Potts model, but some other (probably non-integrable) field theory $X$. We know that the critical Potts model in the scaling limit is described by the $c = 4/5$ CFT, which has a single $S_3$-invariant relevant operator, so $X$ cannot have the $c = 4/5$ $D_4$ CFT as its ultraviolet limit. If this scenario makes any sense, the open question is: what is theory $X$?

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