The Topology of Conjugate Varieties

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Abstract

Serre [Se 64] and Abelson [Ab 74] have produced examples of conjugate algebraic varieties which are not homeomorphic. We show that if the field of definition of a polarized projective variety coincides with its field of moduli then all of its conjugates have the same topological type. This immediately extends the class of varieties known to possess invariant topological type to all canonically embedded varieties. We also show that (normal) complete intersections in projective space and, more generally in homogeneous varieties, satisfy the condition.

1 Introduction

If \( V \) is an algebraic variety defined over \( k \), a finitely generated extension of \( \mathbb{Q} \), for each embedding \( \sigma : k \to \mathbb{C} \) we can extend scalars to form the complex algebraic variety \( V_\sigma \) defined by the following cartesian square (base change or extension of scalars)

\[
V_\sigma := V \times_{\mathbb{Q}} \mathbb{C} \to \quad V
\]

\[
\begin{array}{c}
\downarrow \\
\text{Spec } \mathbb{C} \\
\eta \to \quad \text{Spec } k
\end{array}
\]

The complex points of this variety \( V_\sigma(\mathbb{C}) \) form a topological space and the topological type of two such spaces, \( V_\sigma(\mathbb{C}) \) and \( V_\tau(\mathbb{C}) \) for two different embeddings \( k \to \mathbb{C} \) can be compared. We will refer to varieties \( V_\sigma \) and \( V_\tau \) obtained in this manner as conjugate varieties.

Serre [Se 64] and Abelson [Ab 74] have produced examples of conjugate varieties whose complex points constitute non-homeomorphic topological spaces. Since the publication of these papers there has been little published work in this area. The principal result of the research reported upon here is a sufficient condition for the topological spaces of complex points of conjugate varieties to be homeomorphic. We begin by recalling definitions due to Matsusaka-Shimura-Koizumi [Ko 72].
(1.0.1) Definition. For a divisor $X$ on a projective variety $V$ define the class $\mathcal{P}(X)$ to be the class of all divisors $X'$ on $V$ such that there are integers $m, n$ with $mX \equiv nX'$ (algebraic equivalence). If $\mathcal{P}(X)$ contains an ample divisor then it is called a polarization (this term is also applied to the ample divisor in the class). An isomorphism of projective varieties $f : V \to W$ is said to be a isomorphism of polarized projective varieties if there are polarizations $\mathcal{P}, \mathcal{P}'$ on $V$, and $W$ respectively such that the map on divisors induced by $f$ takes $\mathcal{P}$ to $\mathcal{P}'$.

(1.0.2) Definition. The field of moduli for polarized projective variety $(V, \mathcal{P})$ is the field $k$ such that for $\sigma \in \text{Aut}(\mathbb{C})$, $\sigma$ is in fact in $\text{Aut}(\mathbb{C}/k)$ if and only if $V^\sigma \simeq V$ as polarized varieties.

Compare this field to the field of definition of the variety which is the field $K$ such that $\sigma \in \text{Aut}(\mathbb{C})$ is in fact $\text{Aut}(\mathbb{C}/k)$ iff and only if $V^\sigma = V$.

A discussion of when fields of moduli exist in general for varieties can be found in [Ko 72]. An example of a variety (a hyperelliptic curve) whose field of moduli differs from its field of definition can be found in [Sh 72].

(1.0.3) Theorem. If $V$ is a polarized projective variety defined over $k$, a finitely generated extension of $\mathbb{Q}$, and if the field of moduli for $V$ coincides with $k$, then the topological type of $V_\sigma(\mathbb{C})$ is independent of $\sigma$.

The proof of this Theorem relies on a strengthened form of Thom’s stratified Isotopy Theorem which is given in §2 below. The Theorem itself is proved in §3.

It was previously known that certain types of varieties whose topology is rather easily described, such as non-singular curves, abelian varieties, $K - 3$ surfaces and simply connected non-singular surfaces of general type, have topological types which do not vary under conjugation of their fields of definition. The above Theorem extends this list to include all canonically embedded varieties.

In §4, the criterion given above will be used to show that (normal) complete intersections in projective space, and, more generally, in homogeneous varieties, also belong on this list by showing that they satisfy the condition of the Theorem as well. In §5 we point out a few of the many questions that remain open in this area.

2 Stratified Isotopy Theorems and the Topology of Conjugate Varieties

2.1 Preliminaries from Algebraic Geometry

Our intention is to review the proof of the “stratified isotopy Theorem” as it applies to algebraic varieties with a view to establishing that the stratification described by the Theorem can be defined without extension of the base fields of the varieties involved. This material is essentially contained in [Ve 76].

In the course of the analysis we will frequently rely on two sets of well-known and indeed basic facts from algebraic geometry which are stated here with an emphasis on
the relevant fields of definition. The first set of facts deals with the singular locus of a
variety defined over a field $k$ of characteristic 0 (here we do not need any restriction
on the nature of the extension $k/Q$).

Let $X$ be a variety of dimension $n$ over a characteristic zero field $k$, then for any
point $x \in X$ the following are equivalent:

1. $\Omega^1_{X,x}$ (the module of differentials at $x$) is a free module of rank $n$ over the local
   ring $\mathcal{O}_{X,x}$ of $X$ at $x$;
2. $\mathcal{O}_{X,x}$ is a regular local ring.

and, if either of these conditions obtains at $x$ we say $X$ is smooth at $x$.

(2.1.1) Fact. There is an everywhere dense Zariski open set $U \subset X$ which is smooth
and the Zariski closed set $X - U$ is defined over $k$.

Remark: For a discussion of fields of definition for arbitrary subsets of schemes see [EGA IV, §4.8].

The second set of facts is just Hironaka’s well-known resolution of singularities. Once again our only restriction is that we work over characteristic 0 fields.

Let $X$ be variety defined over $k$ and $\mathcal{J}$ a coherent sheaf of ideals defining a closed
sub-scheme $D$ then we make the usual:

(2.1.2) Definition. A blow-up of $X$ at $D$, otherwise known as a monoidal transformation of $X$ with
center $D$, is a pair $(P, f)$ consisting of a variety $P$ and a morphism $f : P \to X$
such that $f^{-1}(\mathcal{J})$ is an invertible sheaf on $P$ and, for any other pair $(P', f')$ with
$f' : P' \to X$ and $f'^{-1}(\mathcal{J})$ an invertible sheaf on $P'$, there is a unique morphism
$g : P' \to P$ such that

commutes.

A general procedure for constructing $P$ is to define

$$P := \text{Proj}(\bigoplus_{d=0}^{\infty} \mathcal{J}^d)$$

where we set $\mathcal{J}^0 = \mathcal{O}_X$. There is a natural map $P \to X$ (given by $\mathcal{O}_X \to \bigoplus \mathcal{J}^d$) and
the universal property is proved in [H 77] Ch II, §7. In particular the field of definition
for $P$ is just the field of definition of $D$ or, equivalently, of $\mathcal{J}$.

Hironaka has shown,

(2.1.3) Theorem. Let $X$ be a variety defined over $k$, characteristic 0, then there is
a closed subscheme $D$ of $X$ such that:
1. the set of closed points of $D$ is the singular locus of $X$; and

2. if $f : \tilde{X} \to X$ is the monoidal transform of $X$ at $D$ then $\tilde{X}$ is smooth.

Proof. [Hi 64] (Main Theorem 1) □

For our purposes we note in particular that since $D$ is defined over $k$ by Fact (2.1.1) above we have that $\tilde{X}$ and $f$ are defined over $k$ as well.

(2.1.4) Definition. A divisor with normal crossings $D$ in a smooth variety $X$ is a divisor such that for any $x \in D \subset X$ with local ring $\mathcal{O}_{X,x}$ and maximal ideal $m_{X,x} = (z_1, \ldots, z_n)$, each component of $D$ passing through $x$ is described by precisely one ideal $(z_i)$.

(2.1.5) Theorem. Let $X$ be a smooth variety defined over $k$, $W$ a nowhere dense sub-scheme of $X$, then there exists a finite set of monoidal transforms

$$f_i : X_{i+1} \to X_i$$

with smooth centers $D_i$, for $0 \leq i < r$ and $X_0 = X$ such that

1. $X_r$ is smooth;

2. if $\tilde{f}_i$ is the composition of the $f_j$ for $0 \leq j < i$, then $D_i \subset \tilde{f}_i(W)$ for all $i$; and

3. $\tilde{f}_r^{-1}(W)$ is an invertible sheaf whose support is a divisor with normal crossings.

Proof. [Hi 64] (Cor 3, to Main Theorem II). □

Since we know that the singular locus of a variety $X$ over $k$ is nowhere dense and hence its inverse image under the monoidal transform $f$ from Theorem (2.1.3) is nowhere dense, we can summarize the above by saying that Hironaka’s resolution of singularities starts with an arbitrary variety $\tilde{X}$, defined over $k$ characteristic 0, and produces a smooth variety $X'$ and a morphism $f : X' \to X$ such that the inverse image of the singular locus of $X$ becomes a divisor with normal crossings in $X'$ and $X'$, $D$ and $f : X \to X$ are defined over $k$ as well.

2.2 Stratifications and stratified isotopy in the Real Analytic Category

The most natural setting for the study of stratifications of singular spaces is the category of real analytic subspaces of smooth (real analytic) manifolds and proper maps between them. A brief sketch of the aspects of the theory used below is given here. The application to complex algebraic varieties follows. The best current reference is [G-M 88].

Let $M$ be a real analytic manifold, $Z \subset M$ a closed subset and

$$Z = \bigcup_{i \in S} S_i$$
(\(S\) a partially ordered set) a decomposition of \(Z\) as a union of a locally finite collection of disjoint locally closed “pieces” or “strata” satisfying the boundary condition

\[ S_i \cap \bar{S}_j \neq \emptyset \Leftrightarrow S_i \subset \bar{S}_j \Leftrightarrow i = j \text{ or } i < j \]

(in the last case we also write \(S_i < S_j\)). Such a decomposition is called a Whitney Stratification if and only if it also satisfies:

1. each \(S_i\) is smooth (not necessarily connected), and
2. each pair \((S_i, S_j)\) satisfies the \(a\) and \(b\) conditions, namely, if we have a collection of points \(\{x_i\} \subset S_i\) such that \(\{x_i\} \to y \in S_j\) and another set of points \(\{y_i\} \subset S_j\) with \(\{y_i\} \to y\) such that the secant lines \(\overline{x_iy_i} \to l\) and the tangent planes \(T_{x_i}S_i \to \tau\) then we have
   - \(a\): \(T_yS_j \subset \tau\); and
   - \(b\): \(l \subset \tau\)

These conditions ensure that the pieces \(S_i\) “fit together” well at an infinitesimal level (see [B-C-R 87] for examples). The conditions are local and can be tested by taking local coordinates in \(M\) about \(y\). The validity of the conditions is independent of the choice of coordinate system. It is a theorem (Hironaka-Hardt) that any subanalytic manifold admits such a stratification.

If a map behaves well with respect to stratifications we say it is a stratified map. Specifically, let \(Y_1 \subset M_1\), \(Y_2 \subset M_2\) be Whitney stratified subsets of manifolds \(M_1\), \(M_2\), and let \(f : M_1 \to M_2\) be a real analytic map such that \(f| Y_1\) is proper and \(f(Y_1) \subset Y_2\), then \(f\) is stratified if for each stratum \(A \subset Y_2\) we have \(f^{-1}(A)\) a union of connected components of strata of \(Y_1\), say \(f^{-1}(A) = \cup S_i\) and \(f\) takes each \(S_i\) submersively to \(A\) (surjection on tangent spaces). There are two key results on stratified maps which are often referred to as the 1st and 2nd (stratified) Isotopy Theorems.

**Theorem.** For \(Z \subset M\) a Whitney stratified subset of a real analytic manifold, \(f : Z \to \mathbb{R}^n\) proper and such that the restriction to each stratum \(f| A : A \to \mathbb{R}^n\) is a submersion, then there is a stratum preserving homeomorphism \(h : Z \to \mathbb{R}^n \times (f^{-1}(0) \cap Z)\) such that

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & \mathbb{R}^n \times (f^{-1}(0) \cap Z) \\
\downarrow f & & \downarrow \text{pr}_1 \\
\mathbb{R}^n & \xrightarrow{id} & \mathbb{R}^n
\end{array}
\]

commutes. In particular, the fibers of \(f| Z\) are homeomorphic by a stratum preserving homeomorphism.
(2.2.7) Theorem. \( A \subset M, \ B \subset N \) subanalytic subsets of real analytic manifolds, \( F : A \to B \) a proper subanalytic map. Then there exist stratifications \( S, \ T \) of \( A, \ B \) into smooth subanalytic manifolds such that \( f \) is stratified with respect to \( S \) and \( T \). Furthermore, given any locally finite collection of subanalytic subsets \( C \) of \( A \) (resp. \( D \) of \( B \) we can choose \( S \) (resp. \( T \)) such that each elements of \( C \) (resp. \( D \)) is a union of strata of \( S \) (resp. \( T \)).

By the 1st isotopy Theorem one obtains local topological triviality of the \( f \) along connected components of strata of \( B \).

For further discussion and guidance to the literature see [G-M 88] Part I, Chapter 1, pp. 36 - 44.

We wish to employ this theory in the context of complex algebraic varieties and to take our stratifications to be constructible sets whose fields of definition we can control. Following a suggestion of Bernstein, Beilinson, Deligne [BBD 81] Chapter 6, we find that such a version of stratified isotopy theory has been given by Verdier.

2.3 Whitney Stratifications à la Verdier

We now define a notion of Whitney stratification which is adapted to algebraic varieties. The properties \( a \) and \( b \) above will be replaced by a single property \( w \) which is also local in nature. Hence we will always assume that smooth complex algebraic varieties have been equipped with coordinate charts given by their underlying real analytic manifold structure. Nothing will depend on the choice of coordinates (see comments below).

We use the following notion of distance between sub-vector spaces in a finite dimensional Euclidean space \( E, \delta(F,G) \) defined by

\[
\delta(F,G) := \sup_{\|x\| = 1} \text{dist}(x,G)
\]

In particular, \( \delta(F,G) = 0 \Rightarrow F \subset G \).

(2.3.8) Definition.

A Verdier-Whitney stratification of (the complex points of) a complex algebraic variety \( V \) where \( V \) is a \( k \)-variety of finite type, \( k \) a finitely generated extension of \( Q \), is a finite disjoint partition of \( V \) by smooth constructible sets \( A_i \)

\[
V = \bigcup_{i=1}^{n} A_i
\]

such that

1. the “boundary property” holds, namely \( A_i \cap A_j \neq \emptyset \) implies \( A_i \supset A_j \) and

2. if \( A_i \supset A_j \) with \( i \neq j \) then the pair \((A_i, A_j)\) satisfies the following property “\( w \)” at every point \( y \in A_i \):

Consider \( A_\alpha \) and \( A_\beta \) as real analytic manifolds and take coordinate patches around \( y \) to some Euclidean space \( E \), then there exists a neighborhood \( U \subset E \)
of (the image of) \( y \) and a positive real number \( C \) such that \( \forall x \in U \cap A_i \) and \( y' \in U \cap A_j \) (here \( A_i \) and \( A_j \) are taken to mean the images of some small open subsets around \( y \) in \( E \)) we have

\[
\delta(T_{y'}A_j, T_xA_i) \leq C \| x - y' \|
\]

where \( T_xA_\alpha \) is the tangent plane to \( A_\alpha \) at \( x \) and \( \delta \) is as defined above.

A number of remarks are called for here.

As we are restricting ourselves to algebraic varieties, it is sufficient to consider stratifications with finite collections of subsets. The analytic cases require infinite collections of subsets. This permits a certain amount of simplification in the definition and the subsequent arguments. It also permits us to speak of the (common) field of definition of the stratification as being the smallest field containing the fields of definition of the \( A_\alpha \).

On the other hand the condition \( w \) (so-called by Verdier) replaces the more familiar conditions ‘\( a \)’ and ‘\( b \)’ above. Condition \( w \) implies condition \( a \) simply because it is a uniform version of it but \( w \) does not imply \( b \) in general (consider the logarithmic spiral at 0). The key fact however is that this implication does hold when \( A_\alpha \) is a smooth subanalytic subspace of a real analytic space and \( A_\beta \) is a smooth analytic subspace of \( \overline{A_\alpha} \) (Kuo).

Verification of property \( w \) does not depend on the choice of coordinates (for this it is important that the strata \( A_i \) are required to be smooth). The next key fact is that using resolution of singularities, we can stratify arbitrary complex algebraic varieties in much the same way as real analytic manifolds.

The first Theorem we will require is,

\((2.3.9)\) Theorem. \( \text{If } V \text{ is a complex algebraic variety as above and } V_\beta \text{ a finite family of constructible subsets of } V \text{ then there is a Verdier-Whitney stratification of } V \text{ such that each } V_\beta \text{ is obtained as the union of strata. The stratification is defined over the (common) field of definition of } V \text{ and the } V_\beta. \)

The proof of this is based on

\((2.3.10)\) Theorem. \( V \text{ as above, } M, M' \text{ smooth, connected, locally closed subsets such that } M \cap M' = \emptyset, M' \subset \overline{M} \text{ and } \overline{M} - M' \text{ is closed (all for the Zariski topology), then there is a Zariski open } Y \subset M' \text{ containing all of the points } y \in M' \text{ such that } (M, M') \text{ has the property } w \text{ at } y \text{ and } M' - V \text{ is Zariski closed. } Y \text{ is defined over the (common) field of definition of } V, M, \text{ and } M'. \)

\[\text{Proof of (2.3.10). When } M, M' \text{ are locally closed smooth subanalytic subspaces of a second countable real analytic space } X \text{ a corresponding Theorem is proved by} \]

1. defining a subset \( V \subset M' \) by removing “bad points” to arrive at an open subanalytic subset of \( X \) which is dense in \( M' \), and then

2. showing that \( (M, M') \) has property \( w \) at all points of \( V \) by taking coordinate charts in which \( M' \) is an open subset of an affine space \( F \). There is an affine
space $G$ such that locally $F \oplus G = X$, there is a “blow-up” $W$ of $X$ in which $M'$ is described as a divisor with normal crossings $\prod q_i z_i^{n_i}$ and there is a map $\pi : W \to X = F \oplus G$. Additional work involving an analysis of the matrix representation of $d\pi$ then gives the desired result.

In what follows we show how $V$ is defined in the complex algebraic case, show that $V$ is a Zariski open and describe its field of definition. We do not review the proof that $(M, M')$ has the property at all points of $V$ since we are principally interested in showing that the stratifications can be taken to be algebraic and have the right fields of definition. The reference for all omitted parts of proofs is [Ve 76].

We use resolution of singularities to find a smooth complex algebraic variety $W$ and a proper morphism $\pi : W \to V$ with $\pi(W) = \overline{M}$ and $\pi^{-1}(M') = D \subset W$ is a divisor whose singularities are at worst normal crossings. By (2.1.3) and (2.1.5) resolution of singularities takes place without extension of the field of definition so that both $W$ and $D$ are defined over the same field as $V$. The next step is to produce the desired open set by removing “bad points”, in this case, the normal crossings singularities.

Consider the subset $D_q \subset D$ of points of $D$ where at least $q$ irreducible local components of $D$ meet (this set is empty for $q \gg 0$) let $\tilde{D}_q$ be the normalization of $D_q$ (separating the points lying on the various components) and let $i : \tilde{D}_q \to D_q$ be the normalization map. $\tilde{D}_q$ is a smooth algebraic variety defined over the same field as $D$ so we can consider

$$d(\pi \circ i_q) : \Omega^1_{\tilde{D}_q} \to (\pi \circ i_q)^* \Omega^1_M$$

where $\Omega^1$ is the sheaf of differentials. We now have one more correction to make, namely we must consider the points where this map is not surjective so that we do not have a submersion. These form a Zariski closed subset $R_q \subset D_q$ [EGA IV, § 17.15.13] defined by a Jacobian condition and hence this singular locus is defined over the same field as $D_q$.

$i_q(R_q)$ is closed since $i_q$ is finite and hence a closed map (the “going-down” Theorem) and the collection of $i_q(R_q)$ is finite so that $S := \pi(\bigcup i_q(R_q)) \subset \overline{M}$ is Zariski closed. We thus have that

$$Y := M' \cap (M - S) \subset \overline{M}$$

is a Zariski open subset of $M$ defined without extension of the base field such that $Y$ is dense in $M'$ and it is smooth by construction. The proof of (2.3.10) now proceeds by considering the local analysis of the smooth real analytic varieties underlying $M$, $M'$ and $Y$ as very briefly described above. $\square$

Proof of Theorem (2.3.9). We need two Lemmas.

(2.3.11) Lemma. Let $X$ be an algebraic variety, $Y_\beta$ a finite family of constructible subsets of $X$, then there is another finite family of subsets of $X$, $B_\alpha$ satisfying:
(a) for all $\alpha$, $\overline{B}_\alpha$ and $B_\alpha - B_\alpha$ are Zariski closed and the $B_\alpha$ are connected and smooth;
(b) the $B_\alpha$ partition $X$ and each $Y_\beta$ is the union of a collection of $B_\alpha$;
(c) $\overline{B}_\alpha \cap B_\beta \neq \emptyset \Rightarrow B_\beta \subset \overline{B}_\alpha$

and the $B_\alpha$ can be defined without extending the (common) field of definition of the $Y_\beta$.

Proof. Since the $Y_\beta$ are locally closed we have that the sets $Y_\beta$ and $Y_\beta - Y_\beta$ are Zariski closed and we replace the family $Y_\beta$ with the family of Zariski closed sets $\{X, \overline{Y}_\beta, \overline{Y}_\beta - Y_\beta\}$ which we continue to refer to as $Y_\beta$.

Let $\mathcal{F}$ be the largest collection of Zariski closed subsets of $S$ which is such that:

- for all $\beta$, $Y_\beta \in \mathcal{F}$;
- $\mathcal{F}$ is closed under intersection;
- for all $Z \in \mathcal{F}$ the irreducible components of $Z$ are in $\mathcal{F}$; and
- $Z \in \mathcal{F}$ implies that the set of singular points of $Z$, $\text{sing } Z \in \mathcal{F}$.

We can get at least one such collection with these properties by taking the family of sets consisting of the $Y_\beta$ and their singular loci $(Y_\beta)_{\text{sing}}$ and then closing this collection under taking of irreducible components (defined over the field of definition of the $Y_\beta$) and intersections. In particular $\mathcal{F}$ can be constructed without extending the field of definition of the $Y_\beta$.

Then we define

$$B_\alpha = Z_\alpha - \bigcup_{Z_\beta \subset Z_\alpha, Z_\beta \neq Z_\alpha} Z_\beta$$

where the $Z_\gamma$ are the irreducible sets in $\mathcal{F}$. It is clear that the $B_\alpha$ have the properties set out in the Lemma. $\square$

(2.3.12) Lemma. Let $V \subset X$ be a constructible set which is connected and smooth and let $Z \subset \overline{V} - V$ be Zariski closed, then there is a (finite) partition $B_\alpha$ of $Z$ such that such that for all $\alpha$, $\overline{B}_\alpha$, $\overline{B}_\alpha - B_\alpha$ are Zariski closed in $X$, the $B_\alpha$ are smooth and connected and the pairs $(V, B_\alpha)$ have property $w$. The $B_\alpha$ are defined over the (common) field of definition of $X$, $V$ and $Z$.

Proof. By induction. The statement is true for $Z = \emptyset$. Assume $Z \neq \emptyset$ so we apply Lemma (2.3.11) to get a finite collection of smooth connected subsets $U_\alpha \subset Z$, $U_\alpha \cap U_\beta = \emptyset$ defined over the field of definition of $Z$ such that the $U_\alpha$, $\overline{U}_\alpha - U_\alpha$, $Z - U_\alpha$ are all Zariski closed and $Z - \bigcup \alpha U_\alpha$ is a Zariski closed set of lower dimension. By Theorem (2.3.10) there is an open dense $W_\alpha \subset U_\alpha$ such that the $\overline{U}_\alpha - W_\alpha$ are Zariski closed and the pairs $(V, W_\alpha)$ have the property $w$.

Now $Z_1 := Z - \bigcup W_\alpha$ is Zariski closed with dimension lower than $Z$ so we can apply the induction hypothesis to it. Combine the $B_\alpha$ thus obtained with the
$W_\alpha$ to get a new collection of $B_\alpha$. Since the $W_\alpha$ provided by Theorem (2.3.10) are constructed without extending fields of definition we are done. \(\Box\)

Returning to the Proof of Theorem (2.3.9) we start by restricting to the case of $V$ irreducible and proceed once again by induction.

For $V = \emptyset$ the Theorem is trivially true so suppose $V \neq \emptyset$ and replace the $V_\beta$ in the statement of the Theorem with the family $\{V, V - V_\beta, \overline{V_\beta} - V_\beta\}$ as in the proof of Lemma (2.3.11) so that this new family (which we still call $Y_\beta$) is made up of Zariski closed sets.

Since $V$ is irreducible there is a $\beta_0$ such that $Y_{\beta_0}$ is open and dense in $V$. Hence $V_1 := V - Y_{\beta_0}$ is closed and is of lower dimension than $V$. We also clearly have that $\beta \neq \beta_0 \Rightarrow Y_\beta \subset X_1$.

Let $B_\alpha$ be a partition of $V_1$ coming from Lemma (2.3.12) and apply Lemma (2.3.11) to the $Y_\beta$ and $B_\alpha$ together. This produces a common refinement $\{C_\gamma\}$ which still has property $w$ because, in general, if $M, M'$ are locally closed subsets of an algebraic variety $X$ which are smooth with $M' \subset M$ and $M' \cap M = \emptyset$, then if there is a locally closed and smooth $M'' \subset M'$, the pair $(M, M'')$ will have the property $w$ if the pair $(M, M')$ does.

Now assume for the moment the following

Claim: If $V = V_\alpha$ (finite union) and the $V_\alpha$ are Zariski closed, then if Theorem (2.3.9) is true for the $V_\alpha$ it is true for $V$.

$V_1$ is a finite union of irreducibles of dimension lower than $V$ so apply the induction hypothesis to $V_1$ and the $C_\gamma$ to get a Whitney stratification of $V_1$ and add $Y_{\beta_0}$.

If we can now prove the Claim we have just made we will both complete the proof for the irreducible case and for the general case as well. So let $V = Y \cup Z$ be a union of irreducibles. Apply the result in the irreducible case to $Z$, $Y_{\beta} \cap Z$, $Y \cap Z$ to get a Whitney stratification of $Z$ such that $Y_{\beta} \cap Z$ and $Y \cap Z$ are unions of strata. Now apply the result again to $Y$, $Y_{\beta} \cap Y$ and the $A_\alpha$ such that $A_\alpha \subset Y \cap Z$ to get a Whitney stratification of $Y$ such that the $A_\alpha$ and $Y_{\beta} \cap Y$ are unions of strata. Take $B_\beta$ and those $A_\alpha$ such that $A_\alpha \subset Z - (Y \cap Z)$. Note that property $w$ still holds by the remark made above. Continue by induction. Nothing in any of these procedures requires an extension of fields of definition. \(\Box\)

### 2.4 Stratified Morphisms

The next ingredient is the demonstration that quite general algebraic-geometric morphisms behave well with respect to stratifications.

Recall that a morphism of stratified spaces $f : X \rightarrow Y$ is called a *stratified* morphism if it is proper and if the inverse image of a stratum of $Y$ under $f$ is a union of strata of $X$ and each component of these strata is mapped smoothly (as a real analytic manifold) to $Y$.

We now have a version of the 2nd isotopy theorem for complex algebraic varieties.
(2.4.13) Theorem. (Verdier) If \( f : X \to Y \) is a morphism of complex algebraic varieties and \( f \) is proper then there are Verdier-Whitney stratifications \( S \) and \( T \) of \( X \) and \( Y \), defined over the (common) field of definition of \( X \), \( Y \) and \( f \), such that \( f \) is transverse to \( S \) and \( T \).

**Proof.**

**Step 1** We first show that if \( X \to Y \) is proper and \( S \) a stratification of \( X \) such as given in Theorem (2.3.9), then there is a Zariski open \( U \subset f(X) \subset Y \) which is smooth in \( Y \) and (Zariski) dense in \( f(x) \) such that \( f_{mid}f^{-1}(U) \to U \) takes the connected components of \( S \cap f^{-1}(U) \) submersively to \( U \).

Set
\[
X_q = \bigcup_{\dim S_\alpha = q} S_\alpha
\]
this is closed, smooth, constructible subset of \( X \) and we write \( f_q \) for \( f \mid X_q : X_q \to Y \).

Once again, let \( R_q \) be the set of points in \( X_q \) where
\[
df_q : \Omega^1_{X_q} \to f_q^*(\Omega^1_Y)
\]
is not onto and set
\[
U_q = Y - Y_{reg} \cap \left( \bigcup_{q' \leq q} f_q'(R_{q'}) \right)
\]
where \( Y_{reg} \) is the set of regular points of \( Y \) (as noted above this set is defined without extension of the field of definition in characteristic 0). The proof now proceeds by induction on \( q \) to show that
\[
f_q \mid U_q : f_q^{-1}(U_q) \to U_q
\]
maps the connected components of \( S \cap f_q^{-1}(U_q) \) to \( f_q(U_q) \) where we always have \( U_q \) Zariski dense in \( Y \). Note that \( Y - U_q \) is automatically a Zariski closed subset since the \( f_q'(R_{q'}) \) are.

So let \( S_\alpha \subset X_q \) be a stratum. If \( \dim S_\alpha < q \) then there is a \( q' < q \) such that \( f_{q'} \) maps the connected components of \( S \cap f_{q'}^{-1}(U_{q'}) \) to \( U_{q'} \). Thus we can assume \( \dim S_\alpha = q \) so \( S_\alpha \subset X_q \) is Zariski open and smooth and is a connected component of \( W = X_q - X_{q-1} \) which is also open and smooth.

Now \( R_q \cap f^{-1}(U_{q-1}) \subset W \) since \( \forall q' < q \)
\[
coker (df_{q'}) \to coker (df_q)
\]
is surjective over points of \( X_q \) as \( f_q \) agrees with \( f_{q'} \) on \( X_{q'} \). \( f_q(R_q) \cap U_{q-1} \) is a Zariski closed subset and \( U_{q-1} - (f_q(R_q) \cap U_{q-1}) \) is dense in \( U_{q-1} \) and \( f_q \mid U_q \) takes \( S \cap f_{q'}^{-1}(U_{q}) \) submersively to \( f_q(U_q) \).

But \( f : X \to Y \) is proper so \( f_q(R_q) \) is empty for \( q \gg 0 \) and we can form \( U := \bigcap_q U_q \). This is dense in \( Y \) with \( Y - U \) an algebraic Zariski closed set and
by construction \( f \) takes the connected components of \( S \cap f^{-1}(U) \) submersively to \( U \).

**Step 2** Now take \( f : X \to Y \) with stratifications \( S \) and \( T \) respectively as guaranteed by Theorem (2.3.9). By step 1 we find a Zariski open \( U \subset f(X) \) smooth in \( Y \) and dense in \( f(X) \) with \( f | f^{-1}(U) \) submersive on connected components. We now re-stratify \( Y \) per Theorem (2.3.9) with \( U \) a union of strata. Take \( f^{-1} \) of these strata and restratify \( X \). \( f \) now behaves as we want on \( f^{-1}(U) \). Consider \( f | X - f^{-1}(U) \), find a \( U' \subset f(X - f^{-1}(U)) = f(X) - f(U) \subset Y \) which is Zariski open, smooth in \( Y \) and dense in \( f(X) - f(U) \) and repeat the above restratification process. Since \( \dim U' < \dim U \) the procedure terminates after a finite number of steps. Using Theorem (2.3.9) we will always have that the \( U, U', \ldots \) will be unions of strata and similarly for the \( f^{-1}(U), f^{-1}(U'), \ldots \) and \( f \) will clearly take each connected component of the stratifications of the \( f^{-1}(U), f^{-1}(U'), \ldots \) to \( f(U), f(U'), \ldots \)

\[ \square \]

### 2.5 Stratified Isotopy

Lastly we state a version of the 1st isotopy Theorem:

**(2.5.14) Theorem.** (Thom) \( X, Y \) are real analytic spaces, \( S \) and \( T \) stratifications, \( f : X \to Y \) proper and submersive on the connected components of the strata of \( X \). Set \( y_0 \in Y \). Write \( X_0 = f^{-1}(y_0) \), \( S_0 = X_0 \cap S \). Then there is an open neighborhood (in the complex topology) \( y_0 \in V \subset Y \) and a homeomorphism \( \phi : (f^{-1}(V), S \cap f^{-1}(V)) \to (X_0 \times V, S_0 \times V) \) preserving the stratifications and compatible with projections to \( V \).

**Proof.** Classically this is proved using techniques from differential topology and is quite difficult. Using the condition \( w \) in place of the more standard \( a \) and \( b \) conditions Verdier is able to give a fairly self-contained proof in a few pages [Ve 76]. Nonetheless we will pass this over in silence since our objective is not to see how the result is proved but rather to show how it can be applied to give useful transversality properties for the algebraically defined stratifications described above. See [Ve 76] for the missing details. \( \square \)

When combined with Verdier’s results this gives:

**(2.5.15) Corollary.** Let \( X \to Y \) be a proper morphism of algebraic varieties, then the topological type of the fibers of \( f \) over a connected component of a stratum of \( T \) is constant.

**Proof.** By Theorem (2.4.13) there are stratifications \( S \) and \( T \) of \( X \) and \( Y \) respectively, defined over the common fields of definition of \( X, Y \) and \( f \), such that the inverse image of any stratum of \( T \) is a union of strata of \( S \).
∪S_d and each connected component of a stratum of S is mapped submersively onto a stratum of T. Thus only the last statement requires discussion. Consider a connected component of a stratum of T, and call it W. Partition W into subsets such that the topological type of the fibers of f are constant on each member of the partition. The sets partitioning W are then open by Theorem (2.5.14) and they are disjoint by construction. Since W is connected only one of the sets in the partition is non-empty. □

3 Principal Results

3.1 A Sufficient Condition for Topological Stability Under Conjugation

The results described above can be applied to give a sufficient criterion for an algebraic variety and its conjugates to have the same topological type.

(3.1.1) Theorem. Let V be a k-variety, k a finitely generated extension of Q and suppose there exists a family f : Y → B, that is, a proper morphism of complex algebraic varieties such that:

(a) all of the conjugate complex algebraic varieties V_σ are isomorphic as k-varieties to fibers of f (in other words for each σ : k → C there is a point b_σ ∈ B and k-isomorphism V_σ ≃ f^{-1}(b_σ)); and

(b) f : Y → B arises by base extension from Q, that is, there are Q varieties Y/Q and B/Q and a morphism f/Q : Y/Q → B/Q such that

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & Y/Q \\
\downarrow \beta_Y & & \downarrow f/Q \\
B & \xrightarrow{\beta_B} & B/Q
\end{array}
$$

commutes,

then the topological type of V_σ(C) is independent of σ.

Proof. By Corollary (2.5.15) we can stratify Y and B with stratifications S and T defined over Q so that the map f is topologically locally trivial over each connected component of the strata. We need only show therefore that the points b_σ, b_τ corresponding to conjugate varieties V_σ, V_τ must lie in a single connected component of a stratum of S.

Since v_σ := f^{-1}(b_σ), v_τ := f^{-1}(b_τ) are k-isomorphic to varieties which differ only by conjugation of their field of definition these fibers of f are mapped to
the same subscheme $v_\alpha / Q$ of $Y_\alpha / Q$ by $\beta_\alpha$. This is in turn mapped to a subscheme of $B / Q$; call it $b_\alpha / Q$. Now we claim $v_\alpha / Q$ and hence $b_\alpha / Q$ are irreducible. If not, the $v_\alpha$ divide up into subsets which are interchanged by some $\phi \in \text{Aut}(C)$ (each $v_\alpha$ is irreducible since it is isomorphic to a variety $V_\alpha$). Thus the largest irreducible closed subschemes of $Y_\alpha / Q$ containing the images of these subsets are not equal to $Y_\alpha$. But $Y_\alpha$ is irreducible since $Y$ is, so we have a contradiction. Thus $b_\alpha / Q$ is defined by a sheaf of prime ideals $P$ with local ring $O_P$ and residue field $k_P$.

Take the inverse image of $b_\alpha / Q$ under $\beta_B$ in $B \simeq B / Q \times C$. Call this $b_\alpha / C$. The points $b_\alpha$ and $b_\beta$ lie in $b_\alpha / C$ by commutativity of the diagram in Theorem (3.1.1) and call the residue fields of these points $k_\alpha$ and $k_\beta$. Now $\beta_B$ is an open map so that we have $k_\alpha \simeq k_P \otimes C$ and similarly for $k_\beta$.

The stratification $T$ of $B$ is defined over $Q$ and we claim that points of a subscheme of $B$, defined over $Q$ and with residue fields isomorphic to $k_P$ over $C$ must lie in a single irreducible component of a stratum of $T$.

Assume not. By Verdier’s results $b_\alpha / C(C)$, the set of complex points of $b_\alpha / C$ is a union of strata $\sqcup T_i$ and each $T_i$ is defined over $Q$. Assume that we have $b_\alpha \in T_1$ and $b_\beta \in T_2$. We may assume $T_1 \sqcup T_2$ is all of $b_\alpha / C(C)$. By the definition of a stratification, if $T_1 \cap T_2 \neq \emptyset$ then $T_1 \supset T_2$. This implies that one of the points $b_\alpha$ or $b_\beta$ lies in a zariski closed subset of $b_\alpha / C$ and hence does not have residue field isomorphic to $k_P$ over $C$. So we must have $T_1 \cap T_2 = \emptyset$ and similarly $T_2 \cap T_1 = \emptyset$. But then $b_\alpha / Q$ is not irreducible. Thus the points $b_\alpha$ and $b_\beta$ lie in a single irreducible component of the stratification of $B$. Finally, over $C$, irreducibility in the zariski topology implies connectedness in the complex topology so $b_\alpha$ and $b_\beta$ lie in a single connected component of the stratification and hence the topological types of $f^{-1}(b_\alpha)$ and $f^{-1}(b_\beta)$ are the same. □

Remark: Shimura has used the existence of non-homeomorphic conjugate varieties to show that the irreducible components of Chow varieties are not necessarily defined over $Q$ [Sh 68]. I am grateful to Professor J-P. Serre for providing this reference.

### 3.2 Corollaries

Following, for example [BBD 81], it is easy to see that any complex projective variety $V$ (say defined over a field $k$) can be embedded in a family defined over $Q$. One merely considers the coefficients $c_{\alpha \beta}$ of the homogeneous ideal defining $V$ as indeterminates. This produces a family $f : Y \to S$ over an affine base $S$ with the original variety $V$ isomorphic to the fiber of $f$ over the point of $S$ corresponding to the $c_{\alpha \beta}$. This family clearly contains all of the conjugates of $V$, since these are obtained by conjugating the coefficients in its homogeneous ideal. Furthermore, since $S$ is affine this family is actually “arises via base extension from $Q$” in the sense of Theorem (3.1.1).

Fortunately we cannot use this technique to show that all varieties have invariant topological type under conjugation due to the fact that the families we obtain in
this way may not be irreducible. Consider as an example the hyperelliptic curve $C$ constructed by Shimura [Sh 72]

$$y^2 = a_0 x^m + \sum_{r=1}^{m} (a_r x^{m+r} + (-1)^r \bar{a}_r x^{m-r})$$

If we treat the $a_r$ and $\bar{a}_r$ as independent indeterminates $a_r$, $b_r$ we get a family of curves $F$ and $C$ lies over a point $p_0$ in the locus where $a_r = b_r$. The family has two components which are interchanged by complex conjugation. We cannot use it in Theorem (3.1.1) therefore to show that the conjugates of $C$ are homeomorphic (although, of course, this can be shown in other ways).

If the field of moduli of $V$ coincides with its field of definition however, we can use the BBD type family in Theorem (3.1.1) by virtue of,

(3.2.2) Proposition. Let $V$ be a projective variety whose field of definition $k$, a finitely generated extension of $\mathbb{Q}$, coincides with its field of definition. Define the family $f: Y \to S$ as above, by letting the coefficients $\{c_{\alpha\beta}\}$ of the homogeneous ideal of $V$ vary. Then all of the conjugates of $V$ are isomorphic to fibers $f^{-1}(b_i)$ where the $b_i$ lie in a single connected component of a Verdier-Whitney stratification of $S$ and hence have the same topological type.

Proof. Examining the proof of Theorem (3.1.1) we see that we did not need there the full assumption that $Y$ is irreducible, but merely that the conjugates of $V$ do not lie in separate irreducible components of $Y$.

Consider the action of $\sigma \in \text{Aut}(C)$ on $f: Y \to S$. Since the field of moduli of $V$ coincides with its field of definition $k$ we have that $Y \to S$ is stable under $\sigma$ if and only if $\sigma \in \text{Aut}(C/k)$ (that is, the field of definition of $f: Y \to S$ is $k$). Let $Y = \bigcup Y_i$ be a decomposition of $Y$ into irreducible components with fields of definition $k_i$. If $Y_i \neq Y$ then $k \subset k_i$ and $k \neq k_i$. Let the fibers of $f$ corresponding to the conjugates $V_\sigma$ be $v_\sigma$ and suppose, for example, that $v_\sigma$ and $v_\tau$ were in $Y_i$ and $Y_j$ respectively. Then there is some $\phi \in \text{Aut}(C/k)$ taking $Y_i$ to $Y_j$. But $\phi$ fixes $v_\sigma$ and $v_\tau$ so both must be in $Y_i \cap Y_j$. This argument applies to all of the $v_\sigma$ and all components $Y_i$ hence all of the conjugates of $V$ can be identified with fibers of $f$ lying in a single component $Y_i$ (we may pick any component). We now replace $Y$ with $Y_i$, which is irreducible, and apply the theorem.

☐

(3.2.3) Corollary. The topological type of a canonically embedded variety is invariant under conjugation.

Proof. The canonical embedding is given over the field of definition and hence the field of definition coincides with the field of moduli. ☐
This enlarges the class of varieties previously known to have conjugate invariant
topological type. Further examples are given in §4 where we show that complete
intersections in homogeneous varieties have fields of moduli which coincide with
the fields of definition.

For a proof that Serre’s original examples of non-homeomorphic conjugate vari-
eties do not have fields of definition which coincide with their fields of moduli see

\[\text{Re} 94\]

4 Complete Intersection Type Varieties

4.1 Generalities

We can use Theorem \((3.1.1)\) to exhibit further classes of varieties which have
topological type invariant under conjugation. These classes of varieties, which
include (normal) complete intersection varieties in projective space, may be
parametrized by vector spaces of sections of vector bundles and this linear struc-
ture provides a natural method of descending from \(\mathbb{C}\) to \(\mathbb{Q}\).

The Deformation Theory of these varieties has been studied by a number of
authors including [K-S 58], [S 75], [B 83] and [W 84] in a series of papers with
results extending from smooth hypersurfaces through to the more general cases.
The most general result along these lines is:

\[(4.1.1)\] Theorem. (Wehler) Let \(Z = G/H\) be a non-singular homogeneous
complex variety, quotient of a simple, simply connected Lie group \(G\) by a parabolic
subgroup \(H\), \(E = \bigoplus_{j=1}^{r} \mathcal{O}_Z(d_j)\) a vector bundle, \(s \in H^0(Z, E)\) a section and \(X\)
a complex variety described by the zero locus of \(s\) such that \(\text{codim} X = r\) and \(X\) is not a \(K\)-3 surface. Then the vector space \(H^0(Z, E)\) parametrizes a complete
set of small deformations of \(X\) and these deformations are given by the family

\[Y := \{(z, s) \mid z \in Z, s \in H^0(Z, E), s(z) = 0\} \rightarrow H^0(Z, E)\]

If we are to apply the Theorem to this case we must show that the family \(Y \rightarrow H^0(Z, E)\) satisfies the conditions of Theorem \((3.1.1)\). To do this we reprove the
theorem using algebraic geometric techniques to obtain a family over \(\mathbb{Q}\) which
will have the required properties. The proofs given here therefore are modelled on
Wehler’s but adapted to algebraic rather than analytic geometry. The key in both
the analytic and algebraic approach is to establish a close connection between the
deformation theory of the objects and their Hilbert schemes as will be explained
shortly. For another approach to establishing the algebraic deformation theory
of complete intersections see [Ma 68].
4.2 Comparison of Hilbert Scheme and Deformation Functors

Let $Z$ be an arbitrary non-singular projective variety over a field $k$ and $E \to Z$ an algebraic $k$ vector bundle over $Z$. Consider the scheme $X \subset Z$ defined by a global section $s_0 \in H^0(Z,E)$. We construct this scheme as follows: the section $s$ defines a map of sheaves $\mathcal{O}_Z \to E$ sending the section 1 of $\mathcal{O}_Z$ to the section $s$ (we are abusing notation here by not distinguishing between the vector bundle and its associated locally free sheaf). There is a dual map $\check{s} : \check{E} \to \mathcal{O}_Z$ and $X$ is said to be defined by $s$ if $\mathcal{O}_X$ fits into an exact sequence of sheaves

$$\check{E} \xrightarrow{\check{s}} \mathcal{O}_Z \to \mathcal{O}_X \to 0$$

$X$ is sometimes referred to as the zero scheme of $s$.

(4.2.2) Definition. The Hilbert functor of $X$ (in $Z$), is the functor $\text{Hilb}$ from $\mathcal{C}$, the category of local artin rings with residue field $k$ to the category $\text{Sets}$ which assigns to each object $A$ in $\mathcal{C}$ the set of schemes $Y$ which fit into the following diagram

\[
\begin{array}{cccc}
Z \supset X \simeq Y \times_k A & \longrightarrow & Y \subset Z \times \text{Spec} A \\
\downarrow & & \downarrow \\
\text{Spec} k & \longrightarrow & \text{Spec} A
\end{array}
\]

with $Y$ flat over $\text{Spec} A$.

(4.2.3) Definition. The (affine) projective cone (hereinafter simply “the cone”) $C_X$ of (or on) a projective variety $X \subset \mathbb{P}^n$ is given by

$$C_X := \text{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(1) \oplus \mathcal{O}_X(2) \oplus \ldots)$$

The vertex $p$ of $C_X$ is the (closed) subscheme defined by the augmentation ideal, $\ker \epsilon$, where

$$\epsilon : (\mathcal{O}_X \oplus \mathcal{O}_X(1) \oplus \mathcal{O}_X(2) \oplus \ldots) \to \mathcal{O}_X$$

$\text{Spec}(\epsilon)$ thus defines a map $X \to C_X$ whose image is $p$.

For future use we recall that the cone with vertex removed (épointé)

$$C_X - p \simeq \mathbb{V}(\mathcal{O}_X(-1)) \simeq \text{Spec}(\oplus \mathcal{O}_X(n))$$

where $\mathbb{V}$ denotes the operation of taking the vector bundle associated to a locally free sheaf. There is an action of $\mathbb{G}_m$ on $C_X$ (and on $C_X - p$) with integral weights.
coming from the action on each tensor power $\mathcal{O}(n)$. For details see [EGA II §8.4 - 8.6].

Next we define a deformation functor $\text{Def}_{C_X}$ as the functor which assigns to each object $A$ in $\mathcal{C}$ the set of deformations $\mathcal{O}_{C_X,p}(A)$ of the $k$-algebra $\mathcal{O}_{C_X,p}$ which is the local ring of $C_X$ at $p$.

There is a natural morphism $h$ from the Hilbert functor to this deformation functor obtained by assigning to a scheme $Y$ as above the local ring of the vertex of the projective cone on $Y$. This is naturally a deformation of the local ring of the vertex of the projective cone on $X$. The morphism thus consists of “forgetting the embedding in $\mathbb{P}^n$ or, in terms of the underlying rings, forgetting the gradings.

Comparison Theorems between Hilbert functors and Deformation functors go back (at least) to Schlessinger [Sch 71] and can be found in [P 74], [K 79] and [Wa 92]. We will use a version due to Kleppe which employs André-Quillen cohomology to avoid unnecessary smoothness conditions.

Since $X$ is given as the zero-scheme of a section of a vector bundle we want to describe its Hilbert scheme in the same way. So define functors $F_{s_0}$ from the category $\mathcal{C}$ to $\text{Sets}$ which, for any given section $s_0 \in H^0(Z, E)$, assign to an object $A$ of $\mathcal{C}$ the set of zero schemes in $Z \times \text{Spec } A$ of sections $s_A \in H^0(Z \times \text{Spec } A, E \times \text{Spec } A)$ which reduce to $s_0$ over the closed point $k$ under the map $\text{Spec } k \to \text{Spec } A$.

If we further assume that $\text{codim } X = \text{rank } E$ we have that $X$ is (at least) a local complete intersection. Moreover, such zero schemes $s_A$ are flat over $\text{Spec } A$. By [EGA IV §19.3.8] the zero scheme defined by any of the $s_A$ is a local complete intersection as well. Thus we obtain elements of $\text{Hilb}_X(A)$ as sections of vector bundles $E \times \text{Spec } A$ and a morphism of functors $F_{s_0} \xrightarrow{f} \text{Hilb}_X$.

We now study this morphism of functors.

**4.2.4 Proposition.** Let $Z$, $E$ and $A$ be as above and let $X$ be the zero set of a section $s_0$ with codim $X = \text{rank } E$, write $\mathcal{I}_X \subset \mathcal{O}_Z$ for the ideal sheaf of $X$ and suppose that

$$H^1(Z, E \otimes \mathcal{I}_X) = 0$$

then the above morphism of functors $f : F \to \text{Hilb}_X$ is surjective on tangent spaces.

**Proof.** The tangent space to $H$ at $h_0$, is $\text{Hilb}_{s_0}(k[\epsilon])$ and it is standard that

$$T(H, h_0) = \text{Hom}_{\mathcal{O}_Z}(\mathcal{I}_X, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$$

(for the first isomorphism see [G 61], for the second see [H 77] Ch. II, §8). We also have

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X) \simeq H^0(X, N_{X/Z}) \simeq H^0(X, E|_X)$$
where $N$ is isomorphic to the tangent space to $\text{Hilb}$. But the tangent space to $F := F(k[\epsilon])$ is the vector space of sections of $E \times \text{Spec}(k[\epsilon]) \to Z \times \text{Spec}(k[\epsilon])$ which reduce to $s_0$ over $k$ and this is just $H^0(Z, E)$. Consider the standard short exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_Z \to \mathcal{O}_X \to 0$$

tensor it with $E$ and take cohomology to get

$$0 \to H^0(Z, E \otimes \mathcal{I}_X) \to H^0(Z, E) \to H^0(X, E|_X)$$

$$\to H^1(Z, E \otimes \mathcal{I}_X) \to \ldots$$

Hence if $H^1(Z, E \otimes \mathcal{I}_X) = 0$ we have the desired surjectivity. $\square$

So far we have developed portions of a triangle of functors

To fill in the morphism marked with "?" note that the local ring of the vertex of the cone of the zero scheme of $s_A$ is naturally a deformation of the local ring of the vertex of the cone of the zero scheme of $s_0$. It is clear that the triangle commutes because $g$ is just the composition of $f$ and $h$. We now invoke the comparison Theorem relating the Hilbert functor and the deformation functor

(4.2.5) Theorem. Suppose that $V$ is projectively normal and that $T^1(V)$, the tangent space to $\text{Def}_{C_V}$, is negatively graded (in a sense to be made precise below), then the natural morphism of functors

$$h : \text{Hilb}_V \to \text{Def}_{C_V}$$

is smooth.

Proof. See [K 79] $\square$

As we will see in the case of the zero schemes of sections of certain vector bundles, this result together with Proposition (4.2.4) will enable us to establish the existence of families of the desired type. The codim $X = \text{rank } E$ condition that we have been placing on our zero schemes of sections of $E$ ensures projective normality (assuming that the varieties are normal in the first place) by a straightforward adaptation of the proof for complete intersections in projective space. It is the grading condition in the comparison theorem which is the more difficult of the two to verify and we will reduce it to a condition on vanishing of cohomology.

Recall that $\text{Def}_{C_X}$ is the deformation functor of the vertex on the projective cone $C_X$ over $X$ and that there is an action by $\mathbb{G}_m(k)$ on $C_X$ with weights
ranging over the integers. \( T^1(X) := \text{Def}_{C_X}(k[\epsilon]) \) is a vector space and the action of \( \mathbb{G}_m \) on on \( C_X \) becomes an action on \( T^1(X) \), so that we get a decomposition \( T^1(X) = \bigoplus_{\nu=-\infty}^\infty T^1(\nu) \) and \( T^1(X) \) becomes a graded vector space (as \( p \) is an isolated singularity in \( C_X \), in fact \( T^1(\nu) = 0 \) for \( \nu \gg 0 \) and for \( \nu \ll 0 \) but we will not need this). We say that \( T^1(X) \) is negatively graded if \( T^1(X)(\nu) = 0 \) for \( \nu > 0 \). To demonstrate that this condition obtains we show that \( \text{Hilb}(k[\epsilon]) \) too has a grading, which is negative in this sense and that the map \( Th \) is surjective and respects both gradings.

\[ \text{(4.2.6) Proposition.} \quad \text{Let } X \text{ be the zero scheme of the section } s_0 \text{ of } E \to Z \text{ and assume that } X \text{ projectively normal (e.g. codim } X \text{ = rank } E). \text{ Suppose the conclusion of Proposition } (1.2.4) \text{ holds, that is, the morphism of functors } F_{s_0} \to \text{Hilb}_X \text{ is surjective on tangent spaces. Let } g : F_{s_0} \to \text{Def}_{C_X} \text{ be the natural morphism taking an element of } F_{s_0}(A) \text{ to an element of } \text{Def}_{C_x}(A). \text{ Denote the map on tangent spaces by } H^0(Z, E) \xrightarrow{Tg} T^1(X). \text{ If this map is surjective then } T^1(X) \text{ is negatively graded and the morphism of functors } h \text{ is smooth.} \]

\[ \text{Proof.} \quad \text{We have a triangle of tangent maps} \]

\[
\begin{array}{ccc}
H^0(Z, E) & \xrightarrow{Tf} & \text{Hilb}(k[\epsilon]) \\
\downarrow & & \downarrow Tg \\
\text{Hilb}(k[\epsilon]) & \xrightarrow{Th} & \text{Def}_{C_X}(k[\epsilon])
\end{array}
\]

which commutes because the triangle of underlying maps does and we are assuming that \( Tf \) is onto. If we now further assume that \( H^0(Z, E) \to T^1(X) \) is onto then \( \text{Hilb}(k[\epsilon]) \to T^1(X) \) must also be onto as well.

\( \text{Hilb}(k[\epsilon]) \simeq H^0(X, N_{X/Z}) \) as noted above and \( H^0(X, N_{X/Z}) \simeq H^0(C_X, N_{C_X}) \) where \( C_X \) is the projective cone over \( X \). By projective normality the singularity at the vertex \( p \) of \( C_X \) has depth=2 so global sections of \( C_X - p \) extend to global sections of \( C_X \). Since \( X \) is a local complete intersection, this is true of all coherent sheaves on \( C_X \) by local duality, and so, in particular

\[
H^0(C_X, N_{C_X}) \simeq H^0(C_X - p, N_{C_X}) \simeq H^0(C_X - p, N_{C_X-p})
\]

But

\[
C_X - p \simeq \text{V}(\mathcal{O}_X(-1)) - \text{zerosection}
\]

so there is a natural affine map \( \pi : C_X - p \to X \) and \( N_{C_X-p} \simeq \pi^*N_X \) and we can compute

\[
H^0(C_X-p, N_{C_X-p}) \simeq H^0(C_X-p, \pi^*N_X) \simeq H^0(X, \pi_*\pi^*N_X) \simeq \bigoplus_{\nu=-\infty}^0 H^0(X, N_X(\nu))
\]

where \( N_X(\nu) := N_X \otimes \mathcal{O}(\nu) \) as usual. The second isomorphism comes from the fact that \( \pi \) is affine and the conclusion that \( H^0(X, N_X(\nu)) = 0 \) for \( \nu > 0 \) comes
from the fact that the sections must extend over all of $C_X$ and hence cannot have any poles at $p$.

The grading thus produced on $\text{Hilb}(k[\epsilon])$ arises from the action of $G_m$ on $C_X$ and the morphism $\text{Hilb} \to \text{Def}$ is contracted precisely by passing to the projective cone to go from the varieties in projective space to abstract varieties. Hence the action of $G_m$ on $T^1(X)$ and $\text{Hilb}(k[\epsilon])$ is compatible with this morphism. The result is that $T^1(X)$ must also be negatively graded and hence we can apply the comparison Theorem to get that this morphism is smooth. □

Finally it is not difficult to reduce the requirement “$H^0(Z,E) \to T^1(X)$ surjective” to a statement of vanishing of cohomology.

The tangent sheaf to a variety $V$ is defined by $\Theta_V := \text{Hom}_{O_V}(\Omega_V, O_V)$, where $\Omega_V$ is the sheaf of differentials of $V$. Since $C_X \subset A^{n+1}$ and $C_X - p$ is smooth we have that

$$0 \to \Theta_{C_X}{|_{C_X-p}} \to \Theta_{A^{n+1}}{|_{C_X-p}} \to \mathcal{N}_{C_X}{|_{C_X-p}} \to 0$$

is exact. Note that $C - p$ is not affine so there is a longer exact cohomology sequence

$$0 \to H^0(C_X - p, \Theta_{C_X}{|_{C_X-p}}) \to H^0(C_X - p, \Theta_{A^{n+1}}{|_{C_X-p}}) \to$$

$$H^0(C_X - p, \mathcal{N}_{C_X}{|_{C_X-p}}) \to H^1(C_X - p, \Theta_{C_X}{|_{C_X-p}}) \to H^1(C_X - p, \Theta_{A^{n+1}}{|_{C_X-p}}) \to$$

The theory of the “cotangent complex” \cite{Li-Sch 67} gives us the following exact sequence

$$0 \to H^0(C, \Theta_{C_X}) \to H^0(C, \Theta_{A^{n+1}}{|_{C_X}}) \to H^0(C_X, \mathcal{N}_{C_X})$$

$$\to T^1(X) \to H^1(X, \Theta_Z{|_X}) \to \ldots$$

Once again since $X$ is projectively normal, sections over $C_X - p$ extend to $C_X$ so that

$$H^0(C_X, \Theta_{C_X}) \simeq H^0(C_X - p, \Theta_{C_X}{|_{C_X-p}})$$

$$H^0(C_X, \Theta_{A^{n+1}}{|_{C_X}}) \simeq H^0(C_X - p, \Theta_{A^{n+1}}{|_{C_X-p}})$$

and

$$H^0(C_X, \mathcal{N}_{C_X}) \simeq H^0(C_X - p, \mathcal{N}_{C_X-p})$$

Putting all of this together we see that if

$$H^1(C_x - p, \Theta_{A^{n+1}}{|_{C_X-p}}) \simeq H^1(X, \Theta_Z{|_X}) = 0$$

then

$$T^1(X) \hookrightarrow H^1(C_X - p, \Theta_{C_X}{|_{C_X-p}}) = 0$$
To summarize,

**Proposition.** For $X$ the zero scheme of a section $S$ bundle $E$ over a non-singular projective variety $Z$ with codim $X = \text{rank } E$ where

$$H^1(Z, E \otimes \mathcal{I}_X) = H^1(X, \Theta_Z|_X) = 0$$

and $X$ is normal we have the commutative triangle

\[
\begin{array}{c}
\mathcal{F}_{s_0} \\
\downarrow \quad \downarrow \\
\text{Hilb}_X & \quad & \text{Def}_{C_X}
\end{array}
\]

where all of the arrows are smooth.

**Proof.** We only need to show that $\mathcal{F}_{s_0} \to \text{Hilb}_X$ is smooth since then the third side of the triangle will be smooth by [Sch 67]. We know that this arrow is surjective on tangent spaces hence we only need to show that $\mathcal{F}_{s_0}$ is less obstructed than $\text{Hilb}_X$. In other words, if $B \xrightarrow{\phi} A$ is a surjection in $\mathcal{C}$ with $\text{ker}(\phi)^2 = 0$ and $(\mathcal{m}_B)\ker(\phi) = 0$ (so $\ker(\phi)$ is a $k$-vector space) and if for $\xi_0 \in \text{Hilb}_X(A)$ which is in the image of $f$ so, $\xi_0 = f(\zeta_0)$, $\zeta_0 \in \mathcal{F}_{s_0}(A)$, there is a $\xi \in \text{Hilb}_X(B)$ such that $\text{Hilb}_X(\phi)(\xi) = \xi_0$ then there is a $\zeta \in \mathcal{F}_{s_0}(B)$ such that $\mathcal{F}_{s_0}(\phi)(\zeta) = \zeta_0$. The obstruction to the existence of such $\zeta$ lies in $\text{Ext}^2(L_{X/Z}, \mathcal{O}_X)$ where $L_{X/Z}$ is the cotangent complex of $X$. This is a two term complex

$$0 \to \mathcal{I}_X/(\mathcal{I}_X)^2 \to i^*(\Omega^1_Z) \to 0$$

since $X$ is a local complete intersection [I 71] Ch. III, §3.2 and the $\text{Ext}^2$ terms vanish since $H^1(X, \Theta_Z|_X) = 0$. □

Since we know that the formal scheme representing $\text{Hilb}_X$ is algebraizable [G 61], we now have the same conclusion for $\text{Def}_{C_X}$. All infinitesimal deformations of $X$ come from small deformations and all small deformations lie in $Z$. Indeed, by versality $H^0(Z, E)$ is a complete deformation space and the family

$$Y := \{(z, s) \mid z \in Z, s \in H^0(Z, E), s(z) = 0\} \subset Z \times H^0(Z, E) \to H^0(Z, E)$$

is a universal family both in the sense of the deformation functor and the Hilbert functor. These results are valid for $X$ defined over any field.
4.3 Application of the theory

We now apply this theory to our problem. If a normal $k$-variety $V$ ($k$ a finitely generated extension of $\mathbb{Q}$) is given as the zero scheme of a section $s \in H^0(Z, E)$ (everything defined over $k$) and if we have

- $\text{codim} V = \text{rank} E$
- $H^1(Z, E \otimes \mathcal{I}_V) = H^1(V, \Theta_V) = 0$

then we know that the $k$-vector space $H^0(Z, E)$ parametrizes a complete family of deformations of $V$. There is a map

$$H^0(Z, E) \to H^0(Z, E) / Q \otimes k$$

giving a $Q$ structure to $H^0(Z, E)$. This induces a morphism of functors $F^k_s \to F^Q_s$.

We observe that this morphism is smooth since first, if $A' \to A$ is a surjection in $\mathcal{C}$ then

$$F^k(A') \to F^k(A) \times_{F^Q(A)} F^Q(A')$$

must also be onto. To see this note that the arrow $\beta$ in

$$F^k(A) \times_{F^Q(A)} F^Q(A') \to F^Q(A')$$

is onto and hence $\beta'$ is also and that $F^k(A') \to F^Q(A')$ is onto as well. Secondly the morphism also induces a bijection on tangent spaces since $H^0(Z, E)_k$ and $H^0(Z, E)_Q$ are vector spaces of the same dimension over fields of the same cardinality.

Thus the projection $H^0(Z, E) \to H^0(Z, E)_Q$ induces a map of deformation spaces between the functors $\text{Def}^k$ and $\text{Def}^Q$ and $\text{Hilb}^k_X$ and $\text{Hilb}^Q_X$.

We have a universal family $Y/Q \to H^0(Z, E)_Q$ and the above ensures that we recover the corresponding universal family over $k$ by extension of scalars

$$Y_k \to Y / Q$$

$$H^0(Z, E)_k \to H^0(Z, E)_Q$$

But there is a unique extension of scalars from $Q$ to $C$ and this gives a diagram of the sort described at the beginning of this §
Since we can now consider $V$, $E$, $Z$, and $s$ as objects defined over the complex numbers it is clear that all of the conjugate varieties $V_{\sigma}$ can be obtained by conjugating the section $s$, that is, $V_{\sigma}$ is the zero scheme of $s_{\sigma} := \sigma(s)$. Hence all of the conjugates $V$ are in the family $Y$ if $V$ is. Thus the conditions of Theorem (3.1.1) are satisfied and the independence of the topological type from variation with $\sigma$ is ensured.

It only remains to spell out some types of varieties which are defined as the zero sections of vector bundles satisfying our conditions.

If we assume that $Z$ is homogeneous, that is

$$Z \simeq G/H$$

where $G$ is a simple, simply connected, split algebraic group over $\mathbb{Q}$, and $H$ a parabolic subgroup, then by algebraic versions of Bott’s vanishing Theorems [Dm 76] we get that $H^1(Z, \Theta_Z) = 0$ for all $i > 0$. If we further suppose that $E \simeq \oplus_{j=1}^r \mathcal{O}_Z(d_j)$ with $d_j > 0$ and $X$ defined by a section $s$ such that

- $\text{codim } X = r$
- $X$ is not a $K$-3 surface
- $\dim Z \geq 3$

then a reworking of calculations of Borcea using an algebraic version of the Kodaira-Nakano-Akizuki vanishing theorem shows that $H^1(Z, E \otimes \mathcal{I}_X) = 0$. For details see [Re 94].

If we now assume that $X$ is normal and defined over $k$ that is, the section $s$ satisfies

$$s \in H^0(Z, E)/\mathbb{Q} \otimes k$$

then we have a class of $k$ varieties whose topology is independent of the embedding of $k$. This class includes complete intersections in projective space. A somewhat different proof is available for the special case of complete intersections in projective space which does not require normality [Re 94].

We note finally that while the case of $K$-3 surfaces must be treated separately, the result is the same since all $K$-3’s are homeomorphic (indeed diffeomorphic) and conjugates of $K$-3 varieties are $K$-3.

5 Comments and Open Questions

The above discussion can be used to shed a bit of light on the nature of the Serre-Abelson examples. Both authors construct their varieties as quotients using finite group actions. In both cases the action is varied under conjugation. The difference lies in the part of the homotopy type which is affected by conjugation.

In Serre, the variety acted upon is a product of a diagonal hypersurface by an abelian variety and the group action on the abelian variety makes the $\pi_1$ of
the variety into a module in demonstrably different ways under conjugation. In Abelson’s example the group acts on complete intersection which is constructed via a representation of the group and all of this varies under conjugation. Special choices of the group allow one to demonstrate variation in the Postnikov tower.

It is not difficult to see, in both cases that the field of moduli of the varieties thus constructed is smaller than their field of definition. The group action creates some “symmetries” which produce this result.

This suggests the following vague question:

A) Is it possible to produce examples of non-homeomorphic conjugate varieties without using (finite) group actions?

This question can be made more specific in a number of ways.

Because of the use of group actions the examples of Serre and Abelson are rather rigid. A small deformation of one of their examples no longer maintains the structure required to compare it with its conjugates. One approach might therefore be to ask,

A1) Is it possible to construct examples of non-homeomorphic conjugate varieties which are stable under small deformations? This seems unlikely.

Another way to cut out finite group actions is to ask for simply-connected examples,

A2) Are there examples of simply connected non-homeomorphic conjugate varieties?

A useful source of new examples may be provided by Shimura varieties.

Note: It may be useful to employ Kollar’s notions of "essentially large" fundamental groups here instead.

Finally, along these lines one has the fundamental question,

A4) Are the simply connected covering spaces of conjugate algebraic varieties analytically isomorphic?

The criterion developed in this paper does not provide an indication of the minimum "necessary" conditions under which the topology of varieties remains stable under conjugation. Neither does it give any indication of the “part” (if any) of the topological type which are conjugations invariant (over and above the étale homotopy type which is clearly invariant). A Theorem of Deligne [?] shows that the nilpotent completion of the fundamental group of an algebraic variety is algebraically determined. One is led to ask,

B) Is the entire rational homotopy type a conjugation invariant?

One might also pose the following question which seems to lie somewhere between A) and B),

C) Is simple connectivity a conjugation invariant?
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