Manifest Form of the Spin-Local Higher-Spin Vertex $\Upsilon^{\eta}_{\omega CCC}$

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Abstract

Vasiliev generating system of higher-spin equations allowing to reconstruct nonlinear vertices of field equations for higher-spin gauge fields contains a free complex parameter $\eta$. Solving the generating system order by order one obtains physical vertices proportional to various powers of $\eta$ and $\bar{\eta}$. Recently $\eta^2$ and $\bar{\eta}^2$ vertices in the zero-form sector were presented in [1] in the $Z$-dominated form implying their spin-locality by virtue of $Z$-dominance Lemma of [2]. However the vertex of [1] had the form of a sum of spin-local terms dependent on the auxiliary spinor variable $Z$ in the theory modulo so-called $Z$-dominated terms, providing a sort of existence theorem rather than explicit form of the vertex. The aim of this paper is to elaborate an approach allowing to systematically account for the effect of $Z$-dominated terms on the final $Z$-independent form of the vertex needed for any practical analysis. Namely, in this paper we obtain explicit $Z$-independent spin-local form for the vertex $\Upsilon^{\eta}_{\omega CCC}$ for its $\omega CCC$-ordered part where $\omega$ and $C$ denote gauge one-form and field strength zero-form higher-spin fields valued in an arbitrary associative algebra in which case the order of product factors in the vertex matters. The developed formalism is based on the Generalized Triangle identity derived in the paper and is applicable to all other orderings of the fields in the vertex.
Contents

1 Introduction 3

2 Higher Spin equations 5
  2.1 Generating equations ......................................................... 5
  2.2 Perturbation theory ............................................................ 6

3 Subspace $\mathcal{H}^+$ and $Z$-dominance lemma 7
  3.1 $\mathcal{H}^+$ ................................................................. 7
  3.2 Notation .......................................................................... 8
  3.3 Contribution to $\Upsilon_{\omega_{CCC}}$ modulo $\mathcal{H}^+$ .................. 8

4 Calculation scheme 10

5 Main result $\Upsilon_{\omega_{CCC}}$ 11

6 To $z$-linear pre-exponentials 13

7 Generalised Triangle identity 15

8 Uniformization 16

9 Eliminating $\delta(\rho_j)$ and $\delta(\xi_j)$. Result 18

10 Final step of calculation 19
  10.1 Degree-four pre-exponential .................................................. 19
  10.2 Degree-six pre-exponential .................................................... 20

11 Conclusion 20

Appendix A: $B_3^m$ 21

Appendix B: Details of uniformization 21
  13.1 $d_x B_2 + W_2 \ast C$ ........................................................... 22
  13.2 $(d_x B_3^m + \omega \ast B_3^m)|_{\delta(\rho_1)} + W_{1 \omega C} \ast B_2^{\eta_{\text{loc}}}$ .... 22

Appendix C: Eliminating $\delta(\rho_j)$ and $\delta(\xi_j)$ 24
  C.1 Terms proportional to $(p_1 + p_2)^{\alpha}(p_3 + p_2)_\alpha$ .................. 24
  C.2 Term proportional to $t^{\alpha}(p_1 + p_2)_{\alpha}$ ................................ 25
  C.3 Sum of $(p_1 + p_2)^{\alpha}(p_3 + p_2)_\alpha$-proportional and $t^{\alpha}(p_1 + p_2)_{\alpha}$-proportional terms . 26
  C.4 Terms proportional to $\delta(\xi_1) - \delta(\xi_2)$ .......................... 26
  C.5 Terms proportional to $\xi_1 \delta(\xi_2)$ .......................................... 27

Appendix D: Details of the final step of the calculation 27
  D.1 $\xi_1$-independent pre-exponentials ........................................... 29
  D.2 $\xi_1$-proportional pre-exponentials ............................................ 30

Appendix E: Useful formulas 31
1 Introduction

Higher-spin (HS) gauge theory describes interacting systems of massless fields of all spins (for reviews see e.g. [3, 4]). Effects of HS gauge theories are anticipated to play a role at ultra high energies of Planck scale [5]. Theories of this class play a role in various contexts from holography [6] to cosmology [7]. HS theory differs from usual local field theories because it contains infinite tower of gauge fields of all spins and the number of space-time derivatives increases with the spins of fields in the vertex [8, 9, 10, 11]. However one may ask for spin-locality [5, 12, 13, 14] which implies space-time locality in the lowest orders of perturbation theory [13]. Even though details of the precise relation between spin-locality and space-time locality in higher orders of perturbation theory have not been yet elaborated, from the form of equations it is clear that spin-locality constraint provides one of the best tools to minimize the space-time non-locality. Moreover demanding spin-locality one actually fixes functional space for possible field redefinitions that is highly important for the predictability of the theory.

A useful way of description of HS dynamics is provided by the generating Vasiliev system of HS equations [15]. The latter contains a free complex parameter $\eta$. Solving the generating system order by order one obtains vertices proportional to various powers of $\eta$ and $\bar{\eta}$. In the recent paper [1], $\eta^2$ and $\bar{\eta}^2$ vertices were obtained in the sector of equations for zero-form fields, containing, in particular, a part of the $\phi^4$ vertex for the scalar field $\phi$ in the theory. Though being seemingly $Z$-dependent, in [1] these vertices were written in the $Z$-dominated form which implies their spin-locality by virtue of $Z$-dominance Lemma of [2]. In this paper we obtain explicit $Z$-independent spin-local form for the vertex $\Upsilon_{\omega CCC}$ starting from the $Z$-dominated expression of [1]. The label $\omega CCC$ refers to the $\omega CCC$-ordered part of the vertex where $\omega$ and $C$ denote gauge one-form and field strength zero-form HS fields valued in arbitrary associative algebra in which case the order of the product factors in $\omega CCC$ matters.

There are several ways to study the issue of (non)locality in HS gauge theory. One is reconstruction the vertices from the boundary by the holographic prescription based on the Klebanov-Polyakov conjecture [3] (see also [10, 17]). Alternatively, one can analyze vertices directly in the bulk starting from the generating equations of [15]. The latter approach developed in [13, 14, 1, 2, 18] is free from any holographic duality assumptions but demands careful choice of the homotopy scheme to determine the choice of field variables compatible with spin-locality of the vertices. The issue of (non)locality of HS gauge theories was also considered in [19] and [20] with somewhat opposite conclusions.

From the holographic point of view the vertex that contains $\phi^4$ was argued to be essentially non-local [21] or at least should have non-locality of very specific form presented in [22]. On the other hand, the holomorphic, i.e., $\eta^2$ and antiholomorphic $\bar{\eta}^2$ vertices, where $\eta$ is a complex parameter in the HS equations, were recently obtained in [1] where they were shown to be spin-local by virtue of $Z$-dominance lemma of [2]. The computation was done directly in the bulk starting from the non-linear HS system of [15].

In this formalism HS fields are described by one-forms $\omega(Y; K|x)$ and zero-forms $C(Y; K|x)$ where $x$ are space-time coordinates while $Y_A = (y_\alpha, \bar{y}_\dot{\alpha})$ are auxiliary spinor variables. Both dotted and undotted indices are two-component, $\alpha, \dot{\alpha} = 1, 2$, while $K = (k, \bar{k})$ are outer Klein
operators satisfying $k * k = \bar{k} * \bar{k} = 1$,
\[
\{k, y^\alpha\}_s = \{k, z^\alpha\}_s = \{\bar{k}, \bar{y}^\alpha\}_s = \{\bar{k}, \bar{z}^\alpha\}_s = \{k, \theta^\alpha\}_s = \{\bar{k}, \bar{\theta}^\alpha\}_s = 0, \quad (1.1)
\]
\[
[k, \bar{y}^\alpha]_s = [k, \bar{z}^\alpha]_s = [\bar{k}, y^\alpha]_s = [\bar{k}, z^\alpha]_s = [k, \bar{\theta}^\alpha]_s = [\bar{k}, \theta^\alpha]_s = 0,
\]
where $\theta$ and $\bar{\theta}$ are anticommuting spinors in the theory.

Schematically, non-linear HS equations in the unfolded form read as
\[
d_x \omega + \omega * \omega = \Upsilon(\omega, \omega, C) + \Upsilon(\omega, \omega, C) + \ldots, \quad (1.2)
\]
\[
d_x C + \omega * C - C * \omega = \Upsilon(\omega, C, C) + \Upsilon(\omega, C, C) + \ldots. \quad (1.3)
\]

As recalled in Section 2, generating equations of (15) that reproduce the form of equations (1.2) and (1.3) have a simple form as a result of doubling of spinor variables, namely
\[
\omega(Y; K|x) \rightarrow W(Z; Y; K|x), \quad C(Y; K|x) \rightarrow B(Z; Y; K|x).
\]

Equations (1.2) and (1.3) result from the generating equations of (15) upon order by order reconstruction of $Z$-dependence (for more detail see Section 2). The final form of equations (1.2) and (1.3) turns out to be $Z$-independent as a consequence of consistency of the equations of (15). This fact may not be manifest however since the $r.h.s.$'s of HS equations usually have the form of the sum of $Z$-dependent terms.

HS equations have remarkable property (23) that they remain consistent with the fields $W$ and $B$ valued in any associative algebra. For instance $W$ and $B$ can belong to the matrix algebra $Mat_n$ with any $n$. Since in that case the components of $W$ and $B$ do not commute, different orderings of the fields should be considered independently. (Mathematically, HS equations with this property correspond to $A_\infty$ strong homotopy algebra introduced by Stasheff in 24,25,26.) For instance, holomorphic ($i.e., \bar{\eta}$-independent) vertices in the zero-form sector can be represented in the form
\[
\Upsilon^\eta(\omega, C, C) = \Upsilon^\eta_{\omega CC} + \Upsilon^\eta_{\omega CC} + \Upsilon^\eta_{\omega CC}, \quad \Upsilon^m(\omega, C, C, C) = \Upsilon^m_{\omega CCC} + \Upsilon^m_{\omega CCC} + \Upsilon^m_{\omega CCC} + \Upsilon^m_{\omega CCC}, \quad (1.4)
\]
where the subscripts of the vertices $\Upsilon$ refer to the ordering of the product factors.

The vertices obtained in (1) were shown to be spin-local due to the $Z$-dominance Lemma of (2) that identifies terms that must drop from the $r.h.s.$'s of HS equations together with the $Z$-dependence. Recall that spin-locality implies that the vertices are local in terms of spinor variables for any finite subset of fields of different spins (18) (for more detail on the notion of spin-locality see (18)). Analogous vertices in the one-form sector have been shown to be spin-local earlier in (14).

The main achievement of (1) consists of finding such solution of the generating system in the third order in $C$ that all spin-nonlocal terms containing infinite towers of derivatives in $y(\bar{y})$ between $C$-fields in the (anti)holomorphic in $\eta(\bar{\eta})$ sector do not contribute to $\eta^2 (\bar{\eta}^2)$ vertices by virtue of $Z$-dominance Lemma. Thus (1) gives spin-local expressions for the vertices $\Upsilon^m(\omega, C, C, C)$ which, however, have a form of a sum of a number of $Z$-dependent terms. To make spin-locality manifest one must remove the seeming $Z$-dependence from the vertex of (1). Technically, this can be done with the help of partial integration and the Schouten identity. The aim of this paper is to show how this works in practice.

Since the straightforward derivation presented in this paper is technically involved we confine ourselves to the particular vertex $\Upsilon_{\omega CCC}^m$ (1.4). Complexity of the calculations in this paper
expresses complexity of the obtained vertex having no analogues in the literature. Indeed, this is explicitly calculated spin-local vertex of the third order in the equations, corresponding to the vertices of the fourth (and, in part, fifth) order for the fields of all spins. The example described in the paper explains the formalism applicable to all other orderings of the fields in the vertex that are also computable. So, our results are most important from the general point of view highlighting a way for the computation of higher vertices in HS theory that may be important from various perspectives and, in the first place, for the analysis of HS holography. It should be stressed that the results of [1] provided a sort of existence theorem for a spin-local vertex that was difficult to extract without developing specific tools like those developed in this paper. In particular, it is illustrated how the general statements like $Z$-dominance Lemma work in practical computations. Let us stress that at the moment this is the only available approach allowing to compute explicit form of the spin-local vertices for all spins at higher orders.

The rest of the paper is organized as follows. In Section 2, the necessary background on HS equations is presented with brief recollection on the procedure of derivation of vertices from the generating system. Section 3 reviews the notion of the $H^+$ space as well as the justification for a computation modulo $H^+$. In Section 4, we present step-by-step scheme of computations performed in this paper. Section 5 contains the final manifestly spin-local expression for $\Upsilon_{\hat{\nu}\hat{\nu}}^{\eta\eta\omega\omega CCC}$ vertex. In Sections 6, 7, 8, 9 and 10 technical details of the steps sketched in Section 4 are presented. In particular, in Section 7 we introduce important Generalised Triangle identity which allows us to uniformize expressions from [1]. Conclusion section contains discussion of the obtained results. Appendices A, B, C and D contain technical detail on the steps listed in the scheme of computation. Some useful formulas are collected in Appendix E.

2 Higher Spin equations

2.1 Generating equations

Spin-$s$ HS fields are encoded in two generating functions, namely, the space-time one-form

$$\omega(y, \bar{y}, x) = \sum_{n,m} dx^\mu \omega_{\mu \alpha_1 \ldots \alpha_n \bar{\alpha}_1 \ldots \bar{\alpha}_m}(x)y^{\alpha_1} \ldots y^{\alpha_n}\bar{y}^{\bar{\alpha}_1} \ldots \bar{y}^{\bar{\alpha}_m}, \quad s = \frac{2 + m + n}{2} \quad \text{(2.1)}$$

and zero-form

$$C(y, \bar{y}, x) = \sum_{n,m} C_{\alpha_1 \ldots \alpha_n \bar{\alpha}_1 \ldots \bar{\alpha}_m}(x)y^{\alpha_1} \ldots y^{\alpha_n}\bar{y}^{\bar{\alpha}_1} \ldots \bar{y}^{\bar{\alpha}_m}, \quad s = \frac{|m - n|}{2} \quad \text{(2.2)}$$

where $\alpha = 1, 2$ and $\bar{\alpha} = 1, 2$ are two-component spinor indices. Auxiliary commuting variables $y^\alpha$ and $\bar{y}^{\bar{\alpha}}$ can be combined into an $\mathfrak{sp}(4)$ spinor $Y^A = (y^\alpha, \bar{y}^{\bar{\alpha}})$, $A = 1, \ldots, 4$.

The vertices $\Upsilon(\omega, \omega, C, C, \ldots)$ (1.2) and $\Upsilon(\omega, C, C, \ldots)$ (1.3) result from the generating system of HS

$$d_x W + W \star W = 0, \quad \text{(2.3)}$$
$$d_x S + W \star S + S \star W = 0, \quad \text{(2.4)}$$
$$d_x B + W \star B - B \star W = 0, \quad \text{(2.5)}$$
$$S \star S = i(\theta^A \theta_A + \eta B \star \gamma + \bar{\eta} B \star \bar{\gamma}), \quad \text{(2.6)}$$
\[ S \ast B - B \ast S = 0. \] (2.7)

Apart from space-time coordinates \( x \), the fields \( W(Z; Y; K|x) \), \( S(Z; Y; K|x) \) and \( B(Z; Y; K|x) \) depend on \( Y^A, Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}}) \) and Klein operators \( K = (k, \bar{k}) \). \( W \) is a space-time one-form, \( i.e., W = dx^\nu W_\nu \) while \( S \)-field is a one-form in \( Z \) spinor directions \( \theta^A = (\theta^\alpha, \bar{\theta}^{\dot{\alpha}}), \{ \theta^A, \theta^B \} = 0, \)

\( i.e., \)
\[ S(Z; Y; K) = \theta^A S_A(Z; Y; K). \] (2.8)

\( B \) is a zero-form.

Star product is defined as follows
\[ (f \ast g)(Z; Y; K) = \frac{1}{(2\pi)^4} \int d^4 U d^4 V e^{iU_A Y_A} f(Z + U, Y + U; K) g(Z - V, Y + V; K). \] (2.9)

Elements
\[ \gamma = \theta^\alpha \theta_{\dot{\alpha}} e^{iz^\alpha \bar{z}^{\dot{\alpha}}} k \quad \text{and} \quad \bar{\gamma} = \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\alpha} e^{i\bar{z}^{\dot{\alpha}} \bar{y}^{\dot{\alpha}}} \bar{k} \] (2.10)

are central because \( \theta^3 = 0 \) since \( \theta_\alpha \) is a two-component anticommuting spinor.

### 2.2 Perturbation theory

Starting with a particular solution of the form
\[ B_0(Z; Y; K) = 0, \quad S_0(Z; Y; K) = \theta^\alpha z^\alpha + \bar{\theta}^{\dot{\alpha}} \bar{z}^{\dot{\alpha}}, \quad W_0(Z; Y; K) = \omega(Y; K), \] (2.11)
which indeed solves (2.3)-(2.7) provided that \( \omega(Y; K) \) satisfies zero-curvature condition,
\[ d\omega + \omega \ast \omega = 0, \] (2.12)

one develops perturbation theory. Starting from (2.7) one finds
\[ [S_0, B_1]_\ast = 0. \] (2.13)

From (2.9) one deduces that
\[ [Z_A, f(Z; Y; K)]_\ast = -2i \frac{\partial}{\partial Z_A} f(Z; Y; K). \] (2.14)

Hence, equation (2.13) yields
\[ [S_0, B_1] = -2i \theta^A \frac{\partial}{\partial Z_A} B_1 = -2i d_Z B_1 = 0 \implies B_1(Z; Y; K) = C(Y; K). \] (2.15)

The \( Z \)-independent \( C \)-field that appears as the first-order part of \( B \) is the same that enters equations (1.2), (1.3). The perturbative procedure can be continued further leading to the equations of the form
\[ d_Z \Phi_{k+1} = J(\Phi_k, \Phi_{k-1}, \ldots), \] (2.16)

where \( \Phi_k \) is either \( W, S \) or \( B \) field of the \( k \)-th order of perturbation theory, identified with the degree of \( C \)-field in the corresponding expression, \( i.e., \)
\[ W = \omega + W_1(\omega, C) + W_2(\omega, C, C) + \ldots, \quad S = S_0 + S_1(C) + S_2(C, C) + \ldots, \]
\[ B = C + B_2(C, C) + B_3(C, C, C) + \ldots. \]
To obtain dynamical equations (1.2), (1.3) one should plug obtained solutions into equations (2.3) and (2.5). For instance, (2.5) up to the third order in C-field is
\[ \text{d}_x C + [\omega, C] = -\text{d}_x B_2 - [W_1, C] - \text{d}_x B_3 - [W_1, B_2] - [W_2, C] + \ldots \] (2.17)

Though the fields \( W_1, W_2 \) and \( B_2, B_3 \) and hence various terms that enter (2.17) are \( Z \)-dependent, equations (2.3)-(2.7) are designed in such a way that, as a consequence of their consistency, the sum of the terms on the r.h.s. of (2.17) is \( Z \)-independent. To see this it suffices to apply \( \text{d}_Z \) realized as \( i \oint [S_0, \cdot] \) to the r.h.s. of (2.17) and make sure that it gives zero by virtue of already solved equations. For more detail we refer the reader to the review [4].

3 Subspace \( \mathcal{H}^+ \) and \( Z \)-dominance lemma

3.1 \( \mathcal{H}^+ \)

In this Section the definition of the space \( \mathcal{H}^+ \) that plays a crucial role in our computation is recollected. Function \( f(z, y|\theta) \) of the form
\[ f(z, y|\theta) = \int_0^1 dT e^{iT z_0 y} \phi(T z, y|T \theta, T) \] (3.1)

belongs to the space \( \mathcal{H}^+ \) if there exists such a real \( \varepsilon > 0 \), that
\[ \lim_{T \to 0} T^{1-\varepsilon} \phi(w, u|\theta, T) = 0 . \] (3.2)

Note that this definition does not demand any specific behaviour of \( \phi \) at \( T \to 1 \) as was the case for the space \( \mathcal{H}^+_{0} \) of [18].

In the sequel we use two main types of functions that obey (3.2):
\[ \phi_1(T z, y|T \theta, T) = \frac{\phi_0(T z, y|T \theta)}{T^{\delta_1}}, \quad \phi_2(T z, y|T \theta, T) = \vartheta(T - \delta_2) \frac{\phi_2(T z, y|T \theta)}{T^{\delta_2}} \] (3.3)

with some \( \delta_{1,2} > 0 \). (Note that the second option with \( \delta_2 > 0 \) can be interpreted as the first one with arbitrary large \( \delta_1 \). Here step-function is denoted as \( \vartheta \) to distinguish it from the anticommuting variables \( \theta \).)

Space \( \mathcal{H}^+ \) can be represented as the direct sum
\[ \mathcal{H}^+ = \mathcal{H}^+_0 + \mathcal{H}^+_1 + \mathcal{H}^+_2 , \] (3.4)

where \( \phi(w, u|\theta, T) \in \mathcal{H}^+_p \) are degree-\( p \) forms in \( \theta \) satisfying (3.2).

All terms from \( \mathcal{H}^+ \) on the r.h.s. of HS field equations must vanish by \( Z \)-dominance Lemma [2]. Following [4] this can be understood as follows. All the expressions from (2.17) have the form (3.1) and the only way to obtain \( Z \)-independent non-vanishing expression is to bring the hidden \( T \) dependence in \( \phi(T z, y|T \theta, T) \) to \( \delta(T) \). If a function contains an additional factor of \( T^\varepsilon \) or is isolated from \( T = 0 \), it cannot contribute to the \( Z \)-independent answer which is the content of \( Z \)-dominance Lemma [2]. This just means that functions of the class \( \mathcal{H}^+_0 \) cannot contribute to the \( Z \)-independent equations (1.3). Application of this fact to locality is straightforward once this is shown that all terms containing infinite towers of higher derivatives in the vertices of interest belong to \( \mathcal{H}^+_0 \) and, therefore, do not contribute to HS equations. This is what was in particular shown in [4].
In this paper we use the following notation of [1]:

\[ \omega(y_\omega, \bar{y}) \approx C(y_1, \bar{y}) \approx C(y_2, \bar{y}) \approx C(y_3, \bar{y}) \]  \hspace{1cm} (3.5)

with \( \approx \) denoting star-product with respect to \( \bar{y} \). Derivatives \( \partial_\omega \) and \( \partial_j \) act on auxiliary variables as follows

\[ \partial_\omega \alpha = \frac{\partial}{\partial y_\omega}, \quad \partial_j \alpha = \frac{\partial}{\partial y_j^\alpha}. \]  \hspace{1cm} (3.6)

After all the derivatives in \( y_\omega \) and \( y_j \) are evaluated the latter are set to zero, \( i.e., \)

\[ y_\omega = y_j = 0. \]  \hspace{1cm} (3.7)

In this paper we use the following notation of [1]:

\[ t_\alpha := -i\partial_\omega \alpha, \quad p_{j\alpha} := -i\partial_j \alpha, \]

\[ \int d^n\rho_+ := \int d\rho_1 \ldots d\rho_n \vartheta(\rho_1) \ldots \vartheta(\rho_n). \]  \hspace{1cm} (3.8)

\[ \int d^n\rho_{-} := \int d\rho_1 \ldots d\rho_n \vartheta(\rho_n) \ldots \vartheta(\rho_1). \]  \hspace{1cm} (3.9)

### 3.3 Contribution to \( \Upsilon_{\omega C C C}^m \) modulo \( \mathcal{H}^+ \)

The \( \eta^2 C^3 \) vertex in the equations on the zero-forms \( C \) resulting from equations of [15] is

\[ \Upsilon_{\omega C C C}^m(\omega, C, C, C) = -(d_x B_3^m + \omega, B_3^m) + [W_1^\eta, B_2^\eta] + [W_2^\eta, C] + d_x B_2^\eta. \]  \hspace{1cm} (3.10)

Recall, that, being \( Z \)-independent, \( \Upsilon_{\omega C C C}^m \) is a sum of \( Z \)-dependent terms that makes its \( Z \)-independence implicit.

As explained in Introduction, \( \Upsilon_{\omega C C C}^m \) can be decomposed into parts with different orderings of fields \( \omega \) and \( C \). In this paper we consider

\[ \Upsilon_{\omega C C C}^m := \Upsilon_{\omega C C C}^m(\omega, C, C, C) \Big|_{\omega C C C}. \]  \hspace{1cm} (3.11)

Since the terms from \( \mathcal{H}^+ \) do not contribute to the physical vertex such terms can be discarded. Following [1] equality up to terms from \( \mathcal{H}^+ \) referred to as weak equality is denoted as \( \approx \).

We start with the following results of [1]:

\[ \tilde{\Upsilon}_{\omega C C C}^m \approx \Upsilon_{\omega C C C}^m = -(W_{1\omega C}^\eta B_2^{\eta loc} + W_{2\omega C C}^\eta C + d_x B_2^{\eta loc}) \Big|_{\omega C C C} + \omega \ast B_3^m + d_x B_3^m \Big|_{\omega C C C}, \]  \hspace{1cm} (3.12)

where

\[ W_{1\omega C}^\eta B_2^{\eta loc} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int_0^1 d\sigma \int d^3\rho_+ \delta \left( 1 - \sum_{i=1}^{3} \rho_i \right) \frac{(\zeta_0 t^\gamma y^\alpha + \zeta_0 t^\alpha)}{(\rho_1 + \rho_2)} \times \]

\[ \times \exp \left\{ i\mathcal{T} z_0 y^\alpha + (1 - \sigma) t^\alpha \rho_1 - i \frac{\rho_1 \sigma}{\rho_1 + \rho_2} t^a p_{2a} + i \frac{\rho_2 \sigma}{\rho_1 + \rho_2} t^a p_{3a} \right\} \]

\[ + \mathcal{T} z^\alpha \left( - (\rho_1 + \rho_2 + \sigma \rho_3) t_\alpha - (\rho_1 + \rho_2)p_{1a} + (\rho_3 - \rho_1)p_{2a} + (\rho_3 + \rho_2)p_{3a} \right) \]

\[ + iy^\alpha \left( \sigma t_\alpha - \frac{\rho_1}{\rho_1 + \rho_2} p_{2a} + \frac{\rho_2}{\rho_1 + \rho_2} p_{3a} \right) \omega C C C, \]  \hspace{1cm} (3.13)
The sum of \( \eta \eta \hat{\exp} \) exponentials, hence being spin-local \([1]\). Thus

\[
W_{2\omega CC} \ast C \approx -\frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int d^4 \rho_+ \delta \left( 1 - \sum_{i=1}^4 \rho_i \right) \frac{\rho_1 (z_\alpha t_\alpha)^2}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} \times \\
\times \exp \left\{ i \mathcal{T} z_\alpha y_\alpha + i \mathcal{T} z_\alpha ( (1 - \rho_2) t_\alpha - (\rho_3 + \rho_4) p_{1\alpha} + (\rho_1 + \rho_2) p_{2\alpha} + p_{3\alpha} ) + iy_\alpha t_\alpha \\
+ \frac{\rho_1 \rho_3}{(\rho_1 + \rho_2)(\rho_3 + \rho_4)} (iy_\alpha t_\alpha + it_\alpha p_{3\alpha} ) + i \left( \frac{1 - \rho_4}{\rho_1 + \rho_2} + \frac{\rho_1 \rho_4}{\rho_3 + \rho_4} \right) \right\} \omega CCC, \\
\text{(3.14)}
\]

\[
d_x B_2^{\eta loc} \mid_{\omega CCC} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int_0^1 d\xi \int d^3 \rho_+ \delta \left( 1 - \sum_{i=1}^3 \rho_i \right) (z_\alpha y_\alpha) \left[ (\mathcal{T} z_\alpha - \xi y_\alpha) t_\alpha \right] \times \\
\times \exp \left\{ i \mathcal{T} z_\alpha y_\alpha + i \mathcal{T} z_\alpha ( (1 - \rho_2) t_\alpha - \rho_3 p_{1\alpha} + \rho_1 p_{2\alpha} + (\rho_1 + \rho_2) p_{3\alpha} ) + iy_\alpha t_\alpha \\
+ i (1 - \xi) y_\alpha \left( \frac{\rho_1}{\rho_1 + \rho_2} p_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_2} p_{2\alpha} \right) + i \xi \left( \frac{\rho_1}{\rho_1 + \rho_2} p_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_2} p_{2\alpha} \right) \\
+ i \left( \frac{1 - \xi}{\rho_1 + \rho_2} t_\alpha p_{1\alpha} - i \left( \frac{1 - \xi}{\rho_1 + \rho_2} + \frac{\xi}{\rho_1 + \rho_2} \right) t_\alpha p_{2\alpha} + i \frac{\rho_1}{\rho_1 + \rho_2} p_{3\alpha} \right) \right\} \omega CCC, \\
\text{(3.15)}
\]

\[
\omega \ast B_3^{\eta loc} \approx -\frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int d^3 \rho_+ \delta \left( 1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y_\alpha + t_\alpha)^2}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \\
\times \exp \left\{ i \mathcal{T} z_\alpha y_\alpha + i \mathcal{T} z_\alpha ( (1 - \rho_3) t_\alpha - (\rho_1 + \rho_3) p_{1\alpha} + (\rho_2 - \rho_3) p_{2\alpha} + (\rho_1 + \rho_2) p_{3\alpha} ) + iy_\alpha t_\alpha \\
+ i (1 - \xi) y_\alpha \left( \frac{\rho_1}{\rho_1 + \rho_3} p_{1\alpha} - \frac{\rho_2}{\rho_1 + \rho_3} p_{2\alpha} \right) + i \xi \left( \frac{\rho_1}{\rho_1 + \rho_3} p_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3} p_{2\alpha} \right) \\
+ i \left( \frac{1 - \xi}{\rho_1 + \rho_2} t_\alpha p_{1\alpha} - i \left( \frac{1 - \xi}{\rho_1 + \rho_2} + \frac{\xi}{\rho_1 + \rho_2} \right) t_\alpha p_{2\alpha} + i \frac{\rho_1}{\rho_1 + \rho_3} p_{3\alpha} \right) \right\} \omega CCC, \\
\text{(3.16)}
\]

\[
d_x B_3^{\eta loc} \mid_{\omega CCC} \approx \frac{\eta^2}{4} \int_0^1 d\mathcal{T} \int d^3 \rho_+ \delta \left( 1 - \sum_{i=1}^3 \rho_i \right) \int_0^1 d\xi \frac{\rho_1 (z_\alpha y_\alpha)^2}{(\rho_1 + \rho_2)(\rho_1 + \rho_3)} \times \\
\times \exp \left\{ i \mathcal{T} z_\alpha y_\alpha + i \mathcal{T} z_\alpha ( (1 - \rho_3) t_\alpha - (\rho_1 + \rho_3) p_{1\alpha} + (\rho_2 - \rho_3) p_{2\alpha} + (\rho_1 + \rho_2) p_{3\alpha} ) + it_\alpha p_{1\alpha} \\
+ i (1 - \xi) y_\alpha \left( \frac{\rho_1}{\rho_1 + \rho_2} (t_\alpha + p_{1\alpha}) - \frac{\rho_2}{\rho_1 + \rho_2} p_{2\alpha} \right) + \xi y_\alpha \left( \frac{\rho_1}{\rho_1 + \rho_3} p_{3\alpha} - \frac{\rho_3}{\rho_1 + \rho_3} p_{2\alpha} \right) \right\} \omega CCC. \\
\text{(3.17)}
\]

The sum of r.h.s.'s of \((3.13)-(3.17)\) yields \(\tilde{Y}_{\omega CCC}^{\eta}(Z;Y)\).

Note, that all terms on the r.h.s.'s of \((3.13)-(3.17)\) contain no \(p_{3\alpha} p_{1\alpha}\) contractions in the exponentials, hence being spin-local \([\textbf{4}]\). Thus \(\tilde{Y}_{\omega CCC}^{\eta}(Z;Y)\) is also spin-local.

Let us emphasize that only the full expression for \(Y_{\omega CCC}^{\eta}(Y)\) \((\textbf{3.11})\) is \(Z\)-independent, while \(\tilde{Y}_{\omega CCC}^{\eta}(Z;Y)\) \((\textbf{3.12})\) with discarded terms in \(\mathcal{H}^+\) is not. This does not allow one to find manifestly \(Z\)-independent expression for \(Y_{\omega CCC}^{\eta}\) by setting for instance \(Z = 0\) in Eqs. \((3.13)-(3.17)\).
In this paper $Z$-dependence of $\hat{\Upsilon}_\omega^{\eta\eta}(Z;Y)$ is eliminated modulo terms in $\mathcal{H}^+$ by virtue of partial integration and the Schouten identity. As a result,

$$\hat{\Upsilon}_\omega^{\eta\eta}(Z;Y) \approx \hat{\Upsilon}_\omega^{\eta\eta}(Y),$$

where $\hat{\Upsilon}_\omega^{\eta\eta}(Y)$ is manifestly spin-local and $Z$-independent. Since $\mathcal{H}_0^+$-terms do not contribute to the vertex by $Z$-dominance Lemma [2]

$$\Upsilon_\omega^{\eta\eta}(Y) = \hat{\Upsilon}_\omega^{\eta\eta}(Y).$$

Our goal is to find the manifest form of $\hat{\Upsilon}_\omega^{\eta\eta}(Y)$.

4 Calculation scheme

The calculation scheme is as follows.

• I. We start from the expression Eqs. (3.13)-(3.17) for the vertex obtained in [1].

• II. To $z$-linear pre-exponentials.
Using partial integration and the Schouten identity we transform Eqs. (3.13)-(3.17) to the form with $z$-linear pre-exponentials modulo weakly $Z$-independent (cohomology) terms. These expressions are collected in Section 6, Eqs. (6.1)-(6.4). The respective cohomology terms being a part of the vertex $\Upsilon_\omega^{\eta\eta}$ are presented in Section 5.

• III. Uniformization.
We observe that the r.h.s.'s of Eqs. (6.1)-(6.4) can be re-written modulo cohomology and weakly zero terms in a form of integrals $\int d\Gamma$ over the same integration domain $I$

$$\int d\Gamma \: z_\alpha f^\alpha(y, t, p_1, p_2, p_3|T, \xi_i, \rho_i) E \omega_{CCC},$$

where the integrand contains an overall exponential function $E$

$$E = E_z E,$$

$$E_z := \exp i \left\{ T \omega_{\alpha}(y + \mathbb{P})^\alpha \right\},$$

$$E := \exp i \left\{ -\frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (y + \mathbb{P})^\alpha y_\alpha \right.$$

$$+\frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( y + \mathbb{P} \right)^\alpha \tilde{t}_\alpha$$

$$+\frac{\rho_3}{(1 - \rho_1 - \rho_4)} (p_3 + p_2)^\alpha y_\alpha - \frac{\rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \rho_1 t^\alpha y_\alpha$$

$$+\frac{\rho_1}{(1 - \rho_3)} (p_1 + p_2)^\alpha t_\alpha + p_3 o y^\alpha + p_1 o t^\alpha \right\},$$

10
\[ \dot{t} = \frac{\rho_1}{\rho_1 + \rho_4} t , \]  
\[ \mathbb{P} = \mathcal{P} + (1 - \rho_4) t ; \]  
\[ \mathcal{P} = (1 - \rho_1 - \rho_4)(p_1 + p_2) - (1 - \rho_3)(p_3 + p_2) , \]  
\[ (4.5) \]
\[ (4.6) \]
\[ (4.7) \]

the integral over \( I \) is denoted as

\[ \int d\Gamma = \int_0^1 d\mathbb{T} \int d^3 \xi_+ \delta \left( 1 - \sum_{i=1}^3 \xi_i \right) \int d^4 \rho_+ \delta \left( 1 - \sum_{j=1}^4 \rho_j \right) \]  
\[ (4.8) \]

Eqs. (4.1)-(4.4) transformed to the form (4.1) are collected in Section 8, Eqs. (8.2)-(8.5).

- IV. Elimination of \( \delta \)-functions.
  Using partial integration and the Schouten identity we eliminate all factors of \( \delta(\rho_i) \), \( \delta(\xi_1) \) and \( \delta(\xi_2) \) from Eqs. (8.2)-(8.3). The result is presented in Section 9, Eqs. (9.1)-(9.4).

- V. Final step.
  Finally, we show in Section 10 that a sum of the r.h.s.’s of Eqs. (9.2)-(9.4) is \( \mathcal{Z} \)-independent up to \( \mathcal{H}^+ \).

By collecting all resulting \( \mathcal{Z} \)-independent terms we finally obtain the manifest expression for vertex \( \Upsilon_{\omega CCC}^{\eta\eta\omega} \), being a sum of expressions (5.2)-(5.12).

5 Main result \( \Upsilon_{\omega CCC}^{\eta\eta\omega} \)

Here the final manifestly \( \mathcal{Z} \)-independent \( \omega CCC \) contribution to the equations is presented.

Vertex \( \Upsilon_{\omega CCC}^{\eta\eta\omega} \) is

\[ \Upsilon_{\omega CCC}^{\eta\eta\omega} = \sum_{j=1}^{11} J_j \]  
\[ (5.1) \]

with \( J_i \) given in Eqs. (5.2)-(5.12). Note that the integration regions may differ for different terms \( J_j \) in the vertex, depending on their genesis.

Firstly we note that \( B_3^{\eta\eta\omega} \) (A.10), that contains a \( \mathcal{Z} \)-independent part, generates cohomologies both from \( \omega \ast B_3^{\eta\eta\omega} \) and from \( d_\mathbb{Z} B_3^{\eta\eta\omega} \),

\[ J_1 = -\frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \frac{\rho_2}{(\rho_2 + \rho_1)(\rho_2 + \rho_3)} \delta(\rho_4) E \omega CCC , \]  
\[ (5.2) \]
\[ J_2 = \frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \frac{\rho_2}{(\rho_2 + \rho_4)(\rho_2 + \rho_3)} \delta(\rho_1) E \omega CCC . \]  
\[ (5.3) \]

Recall that \( E \) and \( d\Gamma \) are defined in (4.4) and (4.8), respectively. (Note, that, here and below, the integrands on the r.h.s.’s of expressions for \( J_i \) are \( \mathcal{T} \)-independent, hence the factor of \( \int_0^1 d\mathbb{T} \) in \( d\Gamma \) equals one.)

Other cohomology terms are collected from (9.2), (9.3), (9.4), (10.1), (11.2), (11.1), (11.3), (11.4) and (11.5), respectively,
\[ J_3 = \frac{-i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{(1 - \rho_3)} \bigg\{ \frac{1}{(\rho_2 + \rho_3)(1 - \rho_3)} \left\{ \rho_2 t^\alpha (p_1 + p_2) \alpha^{\rho_2 - \rho_3} \right. \bigg\} E \omega CCC, \]

\[ J_4 = \frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{1 - \rho_3} \left( -\frac{\rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\gamma \gamma - \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\gamma \gamma \left[ -\partial_{\rho_1} + \partial_{\rho_2} \right] \right) E \omega CCC, \]

\[ J_5 = -\frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{1 - \rho_3} \left\{ 1 + \xi_1 \left( \partial_{\xi_1} - \partial_{\xi_2} \right) \right\} \left\{ -\frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha \alpha^{\rho_2} \right. \right. \]

\[ J_6 = \frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \right) \left( p_1 + p_2 \right)^\gamma (t) \gamma E \omega CCC, \]

\[ J_7 = \frac{-\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( \frac{\rho_2 \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \right) \left( y + (1 - \rho_1 - \rho_4)(p_1 + p_2) + (1 - \rho_4) t \right)^\gamma (y + t) \gamma t^\alpha \alpha E \omega CCC, \]

\[ J_8 = -\frac{\eta^2}{4} \int d\Gamma \frac{\delta(\rho_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( \frac{\rho_2 \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \right) \left( \frac{\xi_1}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \right) \left( y + (1 - \rho_1 - \rho_4)(p_1 + p_2) + (1 - \rho_4) t \right)^\gamma (y + t) \gamma t^\alpha \alpha E \omega CCC, \]

\[ J_9 = \frac{-i\eta^2}{4} \int d\Gamma \frac{\delta(\rho_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \exp \left\{ -i \xi_2 (p_1 + p_2 + t - \rho_2 (p_3 + p_2)) \alpha (y) \alpha \right. \]

\[ J_{10} = -\frac{-i\eta^2}{4} \int d\Gamma \frac{\delta(\rho_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \exp \left\{ -i \xi_2 (y + p_1 + p_2 + t - \rho_2 (p_3 + p_2)) \alpha (y) \alpha \right. \]

\[ J_{11} = \frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\rho_3)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( y + p_0 + t \right)^\gamma (\xi_1 t - \xi_2 y) \gamma + (1 - \rho_2) (p_3 + p_2) \gamma y \gamma \]

\[ + p_3 \gamma t + t^\beta \gamma y \gamma E \omega CCC. \]
Let us emphasize, that neither exponential function $E$ (4.4) nor the exponentials on the r.h.s.’s of Eqs. (5.10)-(5.12) contain $\delta_{\alpha} \partial_{z}^\alpha$ terms. Hence, as anticipated, all $J_j$ are spin-local.

One can see that though having poles in pre-exponentials these expressions are well defined. For instance a potentially dangerous factor on the r.h.s. of (5.2) is dominated by 1 as follows from the inequality $\rho_2 - (\rho_1 + \rho_2)(\rho_2 + \rho_3) = -\rho_3 \rho_1 \leq 0$ that holds due to the factor of $\prod \vartheta(\rho_i) \delta(1 - \sum \rho_i) \delta(\rho_1)$. Analogous simple reasoning applies to the r.h.s. of (5.3).

The case of (5.4)-(5.8) is a bit more tricky. By partial integration one obtains from (5.4)-(5.6)

$$J_3 + J_4 + J_5 = \frac{i f^2}{4} \int d\Gamma \delta(\varepsilon_3) \frac{1}{(\rho_2 + \rho_3)(1 - \rho_3)} \left\{ -\delta(\rho_3) t^\alpha (p_1 + p_2)_\alpha \right. \right.$$

$$+ [\delta(\rho_1) - \delta(\rho_1)] \rho_2 (p_1 + p_2)_\alpha (p_3 + p_2)_\alpha + t^\alpha (p_3 + p_2)_\alpha - \delta(\rho_1) \rho_2 t^\alpha (p_3 + p_2)_\alpha$$

$$- \delta(\varepsilon_2) \left. \frac{-\rho_2}{(\rho_2 + \rho_3)(1 - \rho_3)} (p_3 + p_2)_\alpha \gamma t_\gamma \right.$$ 

$$- \frac{\rho_3}{(\rho_2 + \rho_3)(1 - \rho_3)} t^\alpha y_\alpha + \frac{1}{(p_1 + p_4)} (p_1 + p_2)_\alpha t_\alpha \left. \right\} E \omega CCC .$$

Using that, due to the factor of $\delta(1 - \sum \rho_i)$, for positive $\rho_i$ it holds

$$\frac{\rho_2}{(\rho_3 + \rho_2)(1 - \rho_3)} - 1 = -\frac{\rho_3(1 - (\rho_3 + p_2))}{(\rho_3 + \rho_2)(1 - \rho_3)} \leq 0 \tag{5.14},$$

$$\frac{1}{(\rho_2 + \rho_3)(1 - \rho_3)} \leq \frac{1}{(\rho_2 + \rho_3)(1 - \rho_3)} - \frac{1}{(\rho_3 + \rho_2)} + \frac{1}{(\rho_1 + \rho_4)} \tag{5.15},$$

one can make sure that each of the expressions with poles in the pre-exponential in Eqs. (5.7), (5.8) and (5.13) can be represented in the form of a sum of integrals with integrable pre-exponentials. For instance, the potentially dangerous factor in (5.8), by virtue of (5.14) and (5.13) satisfies

$$\frac{\rho_2 \rho_2}{(\rho_2 + \rho_3)^3(1 - \rho_3) (\rho_1 + \rho_4)} \leq \frac{1}{(1 - \rho_3)(\rho_1 + \rho_4)} + \frac{1}{(\rho_3 + \rho_2)} + \frac{1}{(\rho_1 + \rho_4)} \tag{5.16}.$$

Each of the terms on the r.h.s. of Eq. (5.16) is integrable, because integration is over a three-dimensional compact area $\sum \rho_i = 1$ in the positive quadrant. For instance consider the first term. Swopping $\rho_4 \leftrightarrow \rho_2$ one has

$$\int d^4 \rho_+ \delta(1 - \sum \rho_i) (1 - \rho_3)(\rho_1 + \rho_2) = \int d^3 \rho_+ \delta(1 - \sum \rho_i) (1 - \rho_3)(\rho_1 + \rho_2) = \int_0^1 d\rho_1 \int_0^{1-\rho_1} d\rho_2 \frac{\log(\rho_1 + \rho_2)}{(\rho_1 + \rho_2)} = \frac{1}{2} \int_0^1 d\rho_1 \log^2(\rho_1),$$

which is integrable.

Analogously other seemingly dangerous factors can be shown to be harmless as well.

### 6 To z-linear pre-exponentials

Step II of the calculation scheme of Section 4 is to transform r.h.s.’s of Eqs. (3.13)-(3.17) to $Z$-independent terms plus terms with linear in $z$ pre-exponentials (modulo $H^+$).
To this end, from (A.10) one straightforwardly obtains that

\[
\omega \ast B_3^{\eta} \approx J_1 + \frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1 - \rho_1)(1 - \rho_3)} \left[ -\rho_2(z_\alpha(y^\alpha + t^\alpha))(p_{1\beta} + p_{2\beta})(p_2^\beta + p_3^\beta) + i \left( \delta(\rho_1) + \delta(\rho_3) \right)(1 - \rho_1)(1 - \rho_3) - \delta(\xi_2) \right] z_\alpha \left( (1 - \rho_1)(p_{1\alpha} + p_{2\alpha}) - (1 - \rho_3)(p_{2\alpha} + p_{3\alpha}) \right) \\
+ i z_\alpha(p_{1\alpha} + p_{2\alpha})(1 - \rho_1) \left( \delta(\xi_2) - \delta(\xi_1) \right) \exp \left\{ i T z_\alpha(y^\alpha + t^\alpha + (1 - \rho_1)(p_{1\alpha} + p_{2\alpha}) - (1 - \rho_3)(p_{2\alpha} + p_{3\alpha})) \right\} + \frac{i(1 - \xi_1)p_2(y^\alpha + t^\alpha)(p_{1\alpha} + p_{2\alpha})}{\rho_1 + \rho_2} + \frac{i \xi_1 p_2(y^\alpha + t^\alpha)(p_{2\alpha} + p_{3\alpha}) - i(y^\alpha + t^\alpha)p_{2\alpha}}{\rho_2 + \rho_3} \right\} \omega_{CCC},
\]

(6.1)

where \( J_1 \) is the cohomology term (5.2). Analogously,

\[
d_x B_3^{\eta} \approx J_2 - \frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\delta(\rho_4)}{(1 - \rho_1)(1 - \rho_3)} \left[ -\rho_2(z_\alpha y^\alpha)(p_{1\beta} + t_\beta + p_{2\beta})(p_2^\beta + p_3^\beta) + i \left( \delta(\rho_1) + \delta(\rho_3) \right)(1 - \rho_1)(1 - \rho_3) - \delta(\xi_2) \right] z_\alpha \left( (1 - \rho_1)(p_{1\alpha} + t^\alpha + p_{2\alpha}) - (1 - \rho_3)(p_{2\alpha} + p_{3\alpha}) \right) \\
+ iz_\alpha(p_{1\alpha} + t^\alpha + p_{2\alpha})(1 - \rho_1) \left( \delta(\xi_2) - \delta(\xi_1) \right) \exp \left\{ i T z_\alpha(y^\alpha + (1 - \rho_1)(p_{1\alpha} + t^\alpha + p_{2\alpha}) - (1 - \rho_3)(p_{2\alpha} + p_{3\alpha})) \right\} \\
+ \frac{i(1 - \xi_1)p_2 y^\alpha(p_{1\alpha} + t_\alpha + p_{2\alpha})}{\rho_1 + \rho_2} + \frac{i \xi_1 p_2 y^\alpha(p_{2\alpha} + p_{3\alpha}) - iy^\alpha p_{2\alpha} + it^\alpha p_{1\beta}}{\rho_2 + \rho_3} \right\} \omega_{CCC} \quad (6.2)
\]

with \( J_2 \) (5.3).

Using the Shouten identity and partial integration one obtains from Eqs. (3.13)-(3.13), respectively,

\[
W_{1\omega C}^\eta \ast B_2^{\eta} \approx \frac{\eta^2}{4} \int_0^1 dT \int_0^1 d\tau \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \left[ i(z_\alpha t^\alpha)\delta(1 - \tau) \right] \\
+ \frac{z_\alpha(p_{2\alpha} + p_{3\alpha})}{1 - \tau} \left( i(\delta(\sigma_1) - \delta(1 - \sigma_1)) - [y^\alpha + p_{1\alpha} + p_{2\alpha} - \sigma_2(p_{2\alpha} + p_{3\alpha})] t_\alpha \right) \exp \left\{ i T z_\alpha y^\alpha \right\} \\
+ iT z_\alpha \left( \tau(p_{1\alpha} + p_{2\alpha}) - ((1 - \tau) + \sigma_2(\tau)p_{2\alpha} + p_{3\alpha}) + (\sigma_1 + \tau(1 - \sigma_1)) t_\alpha \right) + it^\alpha p_{1\alpha} \\
+ i\sigma_1 [y^\alpha + p_{1\alpha} + p_{2\alpha} - \sigma_2(p_{2\alpha} + p_{3\alpha})] t_\alpha - i \left( \sigma_2 p_{3\alpha} - (1 - \sigma_2)p_{2\alpha} \right) y_\alpha \right\} \omega_{CCC}, \quad (6.3)
\]

\[
W_{2\omega CC}^\eta \ast C \approx -\frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\delta(\rho_3)}{(1 - \rho_1 + \rho_4)} \left[ -\rho_1 \left( \delta(\rho_4) + it^\alpha(p_{1\alpha} + p_{2\alpha}) \right) + \xi_1 \delta(\xi_2) \right] x \times \exp \left\{ iT z_\alpha y^\alpha + iT z_\alpha \left( (1 - \rho_1 - \rho_4)(p_{1\alpha} + p_{2\alpha}) - (1 - \rho_3)(p_{2\alpha} + p_{3\alpha}) + (1 - \rho_4)t^\alpha \right) \right\} \\
+ iy^\alpha \left( \frac{\xi_1 p_1}{1 - \rho_2} t_\alpha + p_{3\alpha} \right) + i \left( 1 - \rho_1 - \frac{\xi_1 p_1 p_2}{1 - \rho_2} \right) t^\alpha p_{1\alpha} - i(1 - \xi_1)p_1 t^\alpha p_{2\alpha} + i \frac{\xi_1 p_1 t^\alpha p_{3\alpha}}{1 - \rho_2} \right\} \omega_{CCC},
\]

(6.4)
\[ d_x B_2 \approx \frac{i\eta^2}{4} \int d\Gamma \delta(\zeta_3)\delta(\rho_4) \left( it^\gamma (p_{1\gamma} + p_{2\gamma}) + \delta(\rho_4) - \delta(\rho_1) \right) \times \]
\[ \times \exp \left\{ iT z_\alpha y^\alpha + iT z_\alpha \left( (1 - \rho_1 - \rho_4)(p_1^\alpha + p_2^\alpha) - (1 - \rho_3)(p_2^\alpha + p_3^\alpha) + (1 - \rho_4)t^\alpha \right) \right. \]
\[ + i(1 - \rho_2)t^\beta p_1^\beta - i\rho_2 t^\beta p_2^\beta + i\zeta_2 y^\alpha \left( (\rho_1 + \rho_2)t_\alpha + \rho_2 p_1^\alpha - (1 - \rho_2)p_2^\alpha - p_3^\alpha \right) + iy^\alpha p_3^\alpha \left\} \omega_{CCC}. \]

(6.5)

7 Generalised Triangle identity

Here a useful identity playing the key role in our computations is introduced.

For any \( F(x,y) \) consider

\[ I = \int_{[0,1]} d\tau \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) \]
\[ z^\gamma \left[ (a_2 - a_1)\gamma \delta(\xi_3) + (a_3 - a_2)\gamma \delta(\xi_1) + (a_1 - a_3)\gamma \delta(\xi_2) \right] F(\tau z_\beta p_\beta, (-\xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)\alpha P^\alpha) \]

with arbitrary \( \tau, \xi \)-independent \( P \) and \( a_i \).

Let \( G(x,y) \) be a solution to differential equation

\[ \frac{\partial}{\partial x} G(x,y) = \frac{\partial}{\partial y} F(x,y). \]

(7.2)

Hence

\[ I = \int_{[0,1]} d\tau \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) \]
\[ (a_1 - a_3)\alpha (a_3 - a_2)\alpha \frac{\partial}{\partial \tau} G(\tau z_\beta p_\beta, (-\xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)\alpha P^\alpha). \]

Note that there is a factor of \((a_1 - a_3)\alpha (a_3 - a_2)\alpha \) equal to the area of triangle spanned by the vectors \( a_1, a_2, a_3 \) on the r.h.s. of (7.3).

This identity is closely related to identity (3.24) of [13], that, in turn, expresses triangle identity of [27]. Hence, (7.3) will be referred to as Generalised Triangle identity or GT identity.

Note that, for appropriate \( G \) partial integration on the r.h.s. of (7.3) in \( \tau \) gives \( z \)-independent (cohomology) term plus \( \mathcal{H}^+ \)-term. Namely,

\[ I = - \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) \]
\[ (a_1 - a_3)\alpha (a_3 - a_2)\alpha G(0, (-\xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)\alpha P^\alpha) \]
\[ + \int d^4\xi \delta(1 - \xi_1 - \xi_2 - \xi_3) \]
\[ (a_1 - a_3)\alpha (a_3 - a_2)\alpha G(z_\beta p_\beta, (-\xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)\alpha P^\alpha). \]

The second term on the r.h.s. belongs to \( \mathcal{H}^+ \) if \( G \) is of the form (6.1) satisfying (3.2).
To prove GT identity let us perform partial integration on the r.h.s. of (7.1) with respect to \( \xi \). This yields

\[
I = \int_{[0,1]} d\tau \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) \left[ z^\gamma(a_3 - a_2)\gamma P^\alpha a_{1\alpha} + z^\gamma(a_1 - a_3)\gamma P^\alpha a_{2\alpha} + z^\gamma(a_2 - a_1)\gamma P^\alpha a_{3\alpha} \right] \times \frac{\partial}{\partial y} F(\tau z_a P^\alpha, - (\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3)_\alpha P^\alpha).
\]

The Schouten identity yields

\[
\left[ z^\gamma a_{1\gamma} P^\alpha (a_3 - a_2)_\alpha + z^\gamma a_{2\gamma} P^\alpha (a_1 - a_3)_\alpha + z^\gamma a_{3\gamma} P^\alpha (a_2 - a_1)_\alpha \right] = (7.6)
\left[ z^\gamma P^\gamma \left\{ a_{1\alpha} (a_3 - a_2)_\alpha + a_{2\alpha} (a_1 - a_3)_\alpha + a_{3\alpha} (a_2 - a_1)_\alpha \right\} + z^\gamma (a_3 - a_2)\gamma P^\alpha a_{1\alpha} + z^\gamma (a_1 - a_3)\gamma P^\alpha a_{2\alpha} + z^\gamma (a_2 - a_1)\gamma P^\alpha a_{3\alpha} \right].
\]

One can observe that

\[
\left[ z^\gamma (a_3 - a_2)\gamma P^\alpha a_{1\alpha} + z^\gamma (a_1 - a_3)\gamma P^\alpha a_{2\alpha} + z^\gamma (a_2 - a_1)\gamma P^\alpha a_{3\alpha} \right] = (7.7)
- \left[ z^\gamma a_{1\gamma} P^\alpha (a_3 - a_2)_\alpha + z^\gamma a_{2\gamma} P^\alpha (a_1 - a_3)_\alpha + z^\gamma a_{3\gamma} P^\alpha (a_2 - a_1)_\alpha \right],
\]

whence it follows (7.3).

A useful particular case of GT identity is that with \( F(x, y) = f(x + y) \), namely

\[
\int_{[0,1]} d\tau \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) z^\gamma \left[ (a_2 - a_1)\gamma \delta(\xi_3) \right. \left. + (a_3 - a_2)\gamma \delta(\xi_1) + (a_1 - a_3)\gamma \delta(\xi_2) \right] f((\tau z - \xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)_\alpha P^\alpha) = - \int_{[0,1]} d\tau \int d^3\xi_+ \delta(1 - \xi_1 - \xi_2 - \xi_3) (a_1 - a_3)\alpha (a_3 - a_2)_\alpha \partial_\tau f((\tau z - \xi_1 a_1 - \xi_2 a_2 - \xi_3 a_3)_\alpha P^\alpha).
\]

### 8 Uniformization

Step III of Section 4 is to uniformize the r.h.s. of Eqs. (6.1)-(6.5) putting them into the form (4.1), where GT identity (7.1) plays an important role. Details of uniformization are given in Appendix B (p. 22).

As a result, Eq. (3.12) yields

\[
\hat{\Upsilon}^{\eta}_{\omega_{\text{ccc}}} \mid_{\text{mod. cohomology}} \approx \sum_{j=1}^{4} F_j
\]

with \( F_j \) presented in (8.2)-(8.5).
Note that different terms of $F_j$ will be considered separately in what is follows. For the future convenience the underbraced terms are re-numerated, being denoted as $F_{j,k}$, where $j$ refers to $F_j$ while $k$ refers to the respective underbraced term in the expression for $F_j$. For instance, $F_1 = F_{1,1} + F_{1,2} + F_{1,3} + F_{1,4}$, etc.

\[-\omega \ast B_3^{\eta} \big|_{\text{mod } \delta(\rho_1) \& \delta(T)} \approx F_1 := -\eta^2 4 \int d\Gamma \frac{\delta(\xi_3)\delta(\rho_1)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left[ \rho_2(z_3 P^3)(p_{1\alpha} + p_{2\alpha})(p_2^\alpha + p_3^\alpha) \right] \]

\[-d_x B_3^{\eta} \big|_{\text{mod } \delta(\rho_1) \& \delta(T)} \approx F_2 := \eta^2 4 \int d\Gamma \frac{\delta(\xi_3)\delta(\rho_1)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left[ \rho_2(z_3 P^3)(p_{1\alpha} + p_{2\alpha})(p_2^\alpha + p_3^\alpha) \right] \]

\[-d_x B_2^{\eta} - W_2^{\eta \omega CCC} \ast C \big|_{\text{mod } \delta(T)} \approx F_3 := -\eta^2 4 \int d\Gamma \delta(\rho_3) \delta(\xi_3) \left[ \frac{i\delta(\rho_1)(z_3 P^3)}{\rho_1 + \rho_4} + \frac{i(z_3 t^3) \xi_1 \delta(\xi_2)}{\rho_1 + \rho_4} \right] \]

\[-(d_x B_3^{\eta} + \omega \ast B_3^{\eta}) \bigg|_{\delta(\rho_1) \& \delta(T)} - W_1^{\eta \omega C} \ast B_2^{\eta loc} \approx F_4 := -\eta^2 4 \int d\Gamma \frac{\delta(\xi_3)\delta(\xi_2)z_3(p_{2\alpha} + p_3^\alpha)}{(\rho_2 + \rho_3)(\rho_1 + \rho_4)} \times \]

\[
\times \left( i(\delta(\rho_1) - \delta(\rho_4))E + iE_z \left( \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_4} \right) E \right) \omega CCC. \quad (8.5)
\]

Note that

\[F_{1,2} + F_{3,4} = 0, \quad (8.6)\]
$F_{2,3} + F_{3,1} = 0. \quad (8.7)$

Let us emphasise that, by virtue (3.1), each $F_j$ is of the form (1.1) as expected.

Note that during uniformizing procedure the vertices (5.9)-(5.12) are obtained in Appendix B (p. 22).

9 Eliminating $\delta(\rho_j)$ and $\delta(\xi_j)$: Result

The fourth step of Section 4 is to eliminate all $\delta(\rho_1)$, $\delta(\xi_1)$ and $\delta(\xi_2)$ from the pre-exponentials on the $r.h.s.'s$ of Eqs. (8.2)-(8.5).

More precisely, using partial integration, the Schouten identity and Generalised Triangle identity (7.3), taking into account Eqs. (4.3)-(4.4) one finds that Eq. (8.1) yields

$$\left(\tilde{\Gamma}_{\omega CCC}^\eta - G_1 - G_2 - G_3\right)_{\text{mod cohomology}} \approx 0, \quad (9.1)$$

where

$$G_1 := J_3 + \frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) z_3 \left\{ (y^\gamma + \tilde{t}^\gamma) \frac{\rho_2 \, t^\alpha(p_{1\alpha} + p_{2\alpha})}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} E \left[ \frac{\partial}{\partial \rho_2} - \frac{\partial}{\partial \rho_3} \right] E \right. \right.$$  

$$+ (y^\gamma + \tilde{t}^\gamma) \frac{\rho_2 (p_{1\alpha} + p_{2\alpha})(p_{2\alpha} + p_{3\alpha})}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} E \left[ \frac{\partial}{\partial \rho_2} - \frac{\partial}{\partial \rho_4} \right] E + (y^\gamma + \tilde{t}^\gamma) \frac{(p_{1\alpha} + p_{2\alpha})}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} E$$  

$$+ (y^\gamma + \tilde{t}^\gamma) \frac{\rho_2 t^\alpha(p_{2\alpha} + p_{3\alpha})}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)} \left\{ \frac{\partial}{\partial \rho_2} - \frac{\partial}{\partial \rho_3} \right\} \omega_{CCC}, \quad (9.2)$$

$$G_2 := J_4 + \frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)}{1 - \rho_3} z_4 \left\{ \frac{\rho_3 (y_\alpha + \tilde{t}_\alpha) t^\gamma(y_\gamma + \mathbb{P}_\gamma)}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)} E \right. \right.$$  

$$- \frac{\rho_2 \rho_4 t_\alpha(y^\gamma + \mathbb{P}_\gamma) t_\gamma}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(1 - \rho_4)} E - \frac{\rho_2 (y_\alpha + \tilde{t}_\alpha) t^\gamma(p_{1\gamma} + p_{2\gamma})}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} E$$  

$$- \frac{\rho_2 (p_{1\alpha} + p_{2\alpha})(y^\gamma + \mathbb{P}_\gamma) t_\gamma}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(1 - \rho_4)} E + E_2 \frac{\rho_2 t^\gamma(y_\gamma + \mathbb{P}_\gamma)(y_\gamma + \tilde{t}_\gamma)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_2} \right] E$$  

$$+ E_2 \frac{\rho_2 (y_\alpha + \tilde{t}_\alpha)(p_{1\gamma} + p_{2\gamma})(y_\gamma + \mathbb{P}_\gamma)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_4} \right] E \right\} \omega_{CCC}, \quad (9.3)$$
with $J_3$, $J_4$ and $J_5$ being the cohomology terms (5.4), (5.5) and (5.6), respectively. (Details of the derivation are presented in Appendix C (p.24).)

Note that schematically

\[
G_1 + G_2 + G_3 = \int d\Gamma \delta(\xi_3)z_\alpha g^\alpha(y,t,p_1,p_2,p_3|\rho,\xi)\mathcal{E}\omega CCC + J_3 + J_4 + J_5, \tag{9.5}
\]

as expected. Let us stress that $g^\alpha(y,t,p_1,p_2,p_3|\rho,\xi)$ on the r.h.s. of (9.3) is free from a distributional behaviour.

## 10 Final step of calculation

Here this is shown that the sum of the r.h.s.’s of Eqs. (9.2)-(9.4) gives a $Z$-independent cohomology term up to terms in $\mathcal{H}^+$.

More in detail, the expression $G_1 + G_2 + G_3$ of the form (9.5) consists of two types of terms with the pre-exponential of degree four and six in $z, y, t, p_1, p_2, p_3$, respectively. That with degree-four pre-exponential separately equals a $Z$-independent cohomology term up to terms in $\mathcal{H}^+$. This is considered in Section 10.1. The term with degree-six pre-exponential is considered in Section 10.2. As a result of these calculations $J_6$ (5.7) and $J_7$ (5.8) are obtained.

### 10.1 Degree-four pre-exponential

Consider the sum of expressions with $z$-dependent degree-four pre-exponential from Eqs. (9.2), (9.3) and (9.4), denoting it as $S_4$. Partial integration yields

\[
S_4 \approx J_6 + \frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left[ \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(\rho_1 + \rho_4)} t^\alpha z_\alpha(p_3 + p_2)\gamma(t - \tilde{t})_\gamma \right. \tag{10.1}
\]

\[
+ \frac{\rho_2^2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2(\rho_1 + \rho_4)^2} t^\gamma z_\gamma(y + \mathbb{P})^\alpha t_\alpha + \frac{\rho_2^2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2(\rho_1 + \rho_4)} (y + (1 - \rho_4)t)_\gamma z^\alpha t_\alpha
\]

\[
+ \frac{\rho_2^2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)(\rho_1 + \rho_4)} t^\alpha z_\alpha(p_3 + p_2)\gamma(y + \tilde{t})_\gamma
\]

\[
+ \frac{\rho_2^2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)(\rho_1 + \rho_4)} (-\mathbb{P} + \tilde{t})_\gamma(y + \tilde{t})_\gamma z^\alpha t_\alpha \mathcal{E}\omega CCC,
\]

where the cohomology term $J_6$ is given in (5.7). It is not hard to see that the integrand of the remaining term is zero by virtue of the Schouten identity.
10.2 Degree-six pre-exponential

Terms of this type either appear in (9.2), (9.3) via differentiation in $\rho_j$ or in (9.4) via differentiation in $\xi_j$. Denoting a sum of these terms as $S_6$ we obtain

\[ S_6 = + \frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left\{ E_z (y + i)^{\gamma z \gamma} \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_1 + p_2)^\alpha \left[ (\overrightarrow{\partial}_{\rho_2} - \overrightarrow{\partial}_{\rho_1}) E \right] \right\} \tag{10.2} \]

\[ + E_z \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \left[ (p_1 + p_2)^\gamma (y + (1 - \rho_4) t_{\gamma} \gamma z_{\alpha} (y + i)^\alpha) \left[ \overrightarrow{\partial}_{\rho_2} (\overrightarrow{\partial}_{\rho_1}) E \right] \right. \]

\[ + E_z \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t_{\gamma} (y + (1 - \rho_1 - \rho_4) (p_1 + p_2))_{\gamma} z_{\alpha} (y + i)^\alpha \left[ \overrightarrow{\partial}_{\rho_2} (\overrightarrow{\partial}_{\rho_1}) E \right] \]

\[ + i \xi_1 \left\{ + \left\{ \frac{\rho_2 \rho_2}{(1 - \rho_1 - \rho_4)^3 (1 - \rho_3)^3 (p_1 + p_2)} (y + (1 - \rho_1 - \rho_4) (p_1 + p_2) + (1 - \rho_4) t_{\gamma} (y + i)^\gamma z_{\alpha} t_{\alpha} \right[ \right. \]

\[ - \frac{\rho_2 \rho_2}{(1 - \rho_1 - \rho_4)^3 (1 - \rho_3)^3} (y + i)^\gamma z_{\gamma} t_{\alpha} \right\} \times (y + \overrightarrow{\partial}_{\rho_2} + i)^\alpha (y + i) \left\}\right\} \omega_{CCC} \]

Recall that the integral measure $d\Gamma (1.8) \right\} \right\} \omega_{CCC}$ contains the factor of $\delta(1 - \sum^3 \xi_i)$). Hence taking into account the factor of $\delta(\xi_3)$ on the r.h.s. of (10.2) the dependence on $\xi_2, \xi_3$ can be eliminated by the substitution $\xi_2 \rightarrow 1 - \xi_1, \xi_3 \rightarrow 0$. Then we consider separately the terms that contain and do not contain $\xi_1$ in the pre-exponentials. As shown in Appendix D, those with $\xi_1$-proportional pre-exponentials give $J_7 (3.8)$ up to $\mathcal{H}^+$, while those with $\xi_1$-independent pre-exponentials give zero up to $\mathcal{H}^+$.

11 Conclusion

In this paper starting from $Z$-dominated expression obtained in (3.1) the manifestly spin-local holomorphic vertex $\Upsilon_{\omega_{CCC}}$ in the equation (1.3) is obtained for the $\omega_{CCC}$ ordering. Besides evaluation the expression for the vertex, our analysis illustrates how $Z$-dominance implies spin-locality.

One of the main technical difficulties towards $Z$-independent expression was uniformization, that is bringing the exponential factors to the same form, for all contributions (3.13)-(3.17) with the least amount of new integration parameters possible. Practically, some part of the uniformization procedure heavily used the Generalized Triangle identity of Section 6 playing important role in our analysis.

Let us stress that spin-locality of the vertices obtained in (3.1) follows from $Z$-dominance Lemma. However the evaluation the explicit spin-local vertex $\Upsilon_{\omega_{CCC}}$ achieved in this paper is technically involved. To derive explicit form of other spin-local vertices in this and higher orders a more elegant approach to this problem is highly desirable.

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Appendix A: \(B^{\xi} \eta_3\)

\(B^{\xi} \eta_3\) modulo \(H^+\) terms from [1] is given by

\[
B^{\xi} \eta_3 \approx \frac{-\eta^2}{4} \int d\Gamma \delta(\xi_3) \delta(\rho_3) \frac{T \rho_2 (z_3 y^{\alpha})^2}{(\rho_1 + \rho_2)(\rho_2 + \rho_3)} \exp (F) CCC,
\]

where \(d\Gamma\) is defined in [1.8],

\[
F = iT z_3 (y^{\alpha} + P_0^\alpha) + \frac{i(1 - \xi_1 \rho_2)}{\rho_1 + \rho_2} y^{\alpha}(p_{1\alpha} + p_{2\alpha}) + \frac{i \xi_1 \rho_2}{\rho_2 + \rho_3} y^{\alpha}(p_{2\alpha} + p_{3\alpha}) - iy^{\alpha} p_{2\alpha},
\]

(\(A.2\))

and performing partial integration with respect to \(T\) twice we obtain

\[
B^{\xi} \eta_3 \approx \frac{-\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3) \delta(\rho_1) \rho_2}{(1 - \rho_3)(1 - \rho_1)} \left[ \delta(T) + iz_3 P_0^\alpha + iz_3 P_0^\alpha \left( 1 + i T z_3 P_0^\alpha \right) \right] \exp (F) CCC.
\]

(\(A.4\))

Noticing that

\[
\frac{\partial}{\partial \rho_1} F = -iT z_3 (p_{1\alpha} + p_{2\alpha}) - i \frac{(1 - \xi_1 \rho_2)}{(\rho_1 + \rho_2)^2} y^{\alpha}(p_{1\alpha} + p_{2\alpha}),
\]

(\(A.5\))

\[
\frac{\partial}{\partial \rho_3} F = = iT z_3 (p_{2\alpha} + p_{3\alpha}) - i \frac{\xi_1 \rho_2}{(\rho_2 + \rho_3)^2} y^{\alpha}(p_{2\alpha} + p_{3\alpha})
\]

(\(A.6\))

and performing partial integration with respect to \(\rho_1\) and \(\rho_3\) we obtain

\[
B^{\xi} \eta_3 \approx \frac{i \eta^2}{4} \int d\Gamma \frac{\delta(\xi_3) \delta(\rho_4)}{(1 - \rho_3)(1 - \rho_1)} \left[ -i \rho_2 \delta(T) + z_3 P_0^\alpha \left( (1 - \rho_3)(1 - \rho_1) (\delta(\rho_1) + \delta(\rho_3)) - 1 \right) \right.
\]

\[
- i \rho_2 z_3 P_0^\alpha \left( \frac{\xi_2 y^{\alpha}(p_{1\alpha} + p_{2\alpha})}{\rho_1 + \rho_2} + \frac{\xi_1 y^{\alpha}(p_{2\alpha} + p_{3\alpha})}{\rho_2 + \rho_3} \right) \] \]

\[
\exp (F) CCC.
\]

(\(A.7\))

Observing that

\[
\frac{\partial F}{\partial \xi_1} = \frac{i \rho_2}{\rho_2 + \rho_3} y^{\alpha}(p_{2\alpha} + p_{3\alpha}) - \frac{i \rho_2}{\rho_1 + \rho_2} y^{\alpha}(p_{1\alpha} + p_{2\alpha})
\]

(\(A.8\))

and using the Schouten identity

\[
 z_3 (p_{2\alpha} + p_{3\alpha}) y^{\beta}(p_{1\beta} + p_{2\beta}) = z_3 y^{\alpha}(p_{2\beta} + p_{3\beta})(p_{1\beta} + p_{2\beta}) + z_3 (p_{1\alpha} + p_{2\alpha}) y^{\beta}(p_{2\beta} + p_{3\beta})
\]

(\(A.9\))

after partial integration with respect to \(\xi_1\) we obtain

\[
B^{\xi} \eta_3 \approx \frac{i \eta^2}{4} \int d\Gamma \frac{\delta(\xi_3) \delta(\rho_4)}{(1 - \rho_3)(1 - \rho_1)} \left[ -i \rho_2 \delta(T) + z_3 (p_{1\alpha} + p_{2\alpha})(1 - \rho_1) \left( \delta(\xi_2) - \delta(\xi_1) \right) \right.
\]

\[
+ z_3 P_0^\alpha \left[ (1 - \rho_1)(1 - \rho_3) \left( \delta(\rho_1) + \delta(\rho_3) \right) - \delta(\xi_2) \delta(\xi_1) + i \rho_2 z_3 y^{\alpha}(p_{1\beta} + p_{2\beta}) (p_{2\beta} + p_{3\beta}) \right] \] \]

\[
\exp (F) CCC.
\]

(\(A.10\))

The \(\delta(T)\)-proportional term gives rise to \(J_1\) (5.2) and \(J_2\) (5.3).
Appendix B: Uniformization Detail

Here some details of the transformation of integrands (6.1)–(6.5) to the form (4.1) are presented.

Uniformization can be easily achieved for Eqs. (6.1) and (6.2) modulo δ(ρ1)-proportional terms. Indeed, eliminating δ(ρ1)-proportional term from the r.h.s. of (6.1), adding an integration parameter ρ4 and a factor of δ(ρ4), one obtains (8.2). Analogously, eliminating δ(ρ1)-proportional term from the r.h.s. (6.2), adding an integration parameter ρ4, swapping ρ1 ↔ ρ4 and then adding a factor of δ(ρ1) one obtains (8.3).

To transform integrands of Eqs. (6.4) and (6.5), as well as δ(ρ1)-proportional terms of the integrands of Eqs. (6.1) and (6.2), to the form (4.1) GT identity (7.1) is used in Sections B.1 and B.2.

13.1  \( d_x B_2 + W_2 * C \)

Noticing that the exponential of (6.4) coincides with \( \mathcal{E} \) at \( \xi_2 = 0 \), while the exponential of (6.3) coincides with \( \mathcal{E} \) (4.2) at \( \xi_1 = 0 \), one can easily make sure, that only the \( \delta(\xi_2) \)-proportional term of (6.4) and the \( \delta(\rho_1) \)-proportional term of (6.5) have the desired form (4.1).

Using that \( \mathcal{E} \) (4.2) does not depend on \( \xi_3 \), swapping \( \xi_3 \leftrightarrow \xi_1 \) in the remaining part of (6.3), then swapping \( \xi_3 \leftrightarrow \xi_2 \) in the remaining part of (6.4), one then can apply GT identity (7.2) to the sum of the two obtained terms. As a result, Eqs. (6.4), (6.3) yield

\[
\frac{d_x B_2}{\rho_{\text{loc}}} + W_{2\omega \mathcal{C}} * C \approx \frac{\eta^2}{4} \int d\Gamma \delta(\rho_3)\delta(\xi_3) \left[ -\frac{(z_\alpha \tau_\alpha)}{\rho_1 + \rho_4} \delta(\xi_2) - i(z_\alpha y_\alpha)\delta(\rho_1) \right] \mathcal{E}\omega \mathcal{C} \mathcal{C} \quad \text{(B.1)}
\]

\[
+ \frac{\eta^2}{4} \int d\Gamma \delta(\rho_3) \left[ i\delta(\rho_4) - \tau_\gamma(p_{1\gamma} + p_{2\gamma}) \right] \left\{ \delta(\mathcal{T})t_\alpha y_\alpha + \delta(\xi_3)(z_\alpha \tilde{\eta}_\alpha + z_\alpha y_\alpha) \right\} \mathcal{E}\omega \mathcal{C} \mathcal{C},
\]

where the terms in the second row of formula (B.1) result from applying GT -identity. Rewriting the underlined part as the result of differentiation with respect to \( \mathcal{T} \) and performing partial integration one obtains Eq. (8.4) plus the cohomology term \( J_8 \) (8.9).

13.2  \( (d_x B_3^m + \omega * B_3^m)\vert_{\delta(\rho_1)} + W_1^m * B_2^m_{\text{loc}} \)

Uniformization of the sum of \( \delta(\rho_1) \)-proportional terms on the r.h.s.’s of (6.2) and (6.1) is done with the help of GT identity (7.8) as follows. Denoting

\[
\tilde{P} = y + p_1 + p_2 + t - \rho_2(p_3 + p_2)
\]

one can see that partial integration in \( \mathcal{T} \) yields

\[
\frac{d_x B_3^m}{\delta(\rho_1)} \approx -\frac{i\eta^2}{4} \int d\Gamma \delta(\rho_4)\delta(\xi_1) \left[ i\delta(\mathcal{T}) - z_\alpha y_\alpha \right] \exp \left\{ i\mathcal{T}z_\alpha \tilde{P}_\alpha - i\xi_2 \tilde{P}_\alpha y_\alpha \right. \\
\left. + i(1 - \rho_2)(p_2^\alpha + p_3^\alpha)y_\alpha + ip_3\alpha y_\alpha + it^\beta p_{1\beta} \right\} \omega \mathcal{C} \mathcal{C},
\]

\[
\frac{\omega * B_3^m}{\delta(\rho_1)} \approx \frac{i\eta^2}{4} \int d\Gamma \delta(\rho_4)\delta(\xi_3) \left[ i\delta(\mathcal{T}) - z_\alpha (y_\alpha + t^\alpha) \right] \exp \left\{ i\mathcal{T}z_\alpha \tilde{P}_\alpha - i\xi_2 \tilde{P}_\alpha y_\alpha \\
+ i\xi_1 \tilde{P}^\alpha t_\alpha + i(1 - \rho_2)(p_2^\alpha + p_3^\alpha)y_\alpha + ip_3\alpha y_\alpha + it^\beta p_{1\beta} \right\} \omega \mathcal{C} \mathcal{C}.
\]

(8.3)
The sum of (B.3) and (B.4) gives

\[
\left(d_x B_3^{\eta n} + \omega \ast B_3^{\eta n}\right)_{\delta(\rho_1)} \approx \frac{in^2}{4} \int d\Gamma \delta(\rho_4)\delta(\rho_1) \left[z_{\gamma}(-t^\gamma - y^\alpha)\delta(\xi_3) + z_{\gamma}t^\gamma\delta(\xi_1) + z_{\gamma}t^\gamma\delta(\xi_2)\right] \times \\
\times \exp \left\{ i T z_{\alpha} \bar{P}^{\alpha} - i \xi_2 \bar{P}^{\alpha} y_\alpha + i \xi_1 \bar{P}^{\alpha} t_\alpha + \frac{1}{2} (1 - \rho_2) (p_2^{\alpha} + p_3^{\alpha}) y_\alpha + i p_3 y^\alpha + i t^\beta p_{1\beta} \right\} \omega CCC \\
- \frac{in^2}{4} \int d\Gamma \delta(\rho_4)\delta(\rho_1)(z_{\gamma}t^\gamma)\delta(\xi_2) \exp \left\{ i T z_{\alpha} \bar{P}^{\alpha} - i \xi_2 \bar{P}^{\alpha} y_\alpha + i \xi_1 \bar{P}^{\alpha} t_\alpha + \frac{1}{2} (1 - \rho_2) (p_2^{\alpha} + p_3^{\alpha}) y_\alpha \\
+ i p_3 y^\alpha + i t^\beta p_{1\beta} \right\} \omega CCC + J_9 + J_{10} \tag{B.5}
\]

with $J_9$ (B.10) and $J_{10}$ (B.11). By virtue of GT identity (7.3) the first term weakly equals $J_{11}$ (B.12). Finally, Eq. (B.5) yields

\[
\left(d_x B_3^{\eta n} + \omega \ast B_3^{\eta n}\right)_{\delta(\rho_1)} \approx -\frac{in^2}{4} \int d\Gamma \delta(\rho_4)\delta(\rho_1)(z_{\gamma}t^\gamma)\delta(\xi_2) \exp \left\{ i T z_{\alpha} \bar{P}^{\alpha} - i \xi_2 \bar{P}^{\alpha} y_\alpha \\
+ i \xi_1 \bar{P}^{\alpha} t_\alpha + \frac{1}{2} (1 - \rho_2) (p_2^{\alpha} + p_3^{\alpha}) y_\alpha + i p_3 y^\alpha + i t^\beta p_{1\beta} \right\} \omega CCC + J_9 + J_{10} + J_{11}. \tag{B.6}
\]

Consider $W_{loc}^{\eta n} * B_2^{\eta loc}$ (3.13). This is convenient to change integration variables, moving from the integration over simplex to integration over square. As a result

\[
W_{loc}^{\eta n} * B_2^{\eta loc} \approx \frac{\eta^2}{4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 d\sigma_1 d\sigma_2 (z_\alpha t^\alpha) \times \\
\left[z_\alpha y^\alpha + \sigma_1 z_\alpha t^\alpha\right] \exp \left\{ i T z_{\alpha} y^\alpha + \frac{1}{2} (1 - \sigma_2) \sigma_1 t_\alpha p_1^{\alpha} + i \sigma_1 \sigma_2 t_\alpha p_3^{\alpha} + \frac{1}{2} (1 - \sigma_1) t^\alpha p_{1\alpha} \\
+ i T z_{\alpha} \left( (\tau_1 + \tau_2) t^\alpha + \tau_1 p_1^{\alpha} - (\tau_2 - \tau_1(1 - \sigma_2)) p_2^{\alpha} - (\tau_2 + \sigma_2 t_\alpha) p_3^{\alpha} \right) + \sigma_1 y^\alpha t_\alpha \\
- i \left( \frac{1}{2} \sigma_2 y^\alpha p_2^{\alpha} + i \sigma_2 y^\alpha p_3^{\alpha} + i \sigma_2 y^\alpha p_3^{\alpha} \right) \right\} \omega CCC. \tag{B.7}
\]

Partial integration with respect to $T$ yields

\[
W_{loc}^{\eta n} * B_2^{\eta loc} \approx -\frac{\eta^2}{4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 d\sigma_1 d\sigma_2 (z_\alpha t^\alpha) \times \\
\left[T z_{\alpha} \left( \tau_1 (p_1^{\alpha} + p_2^{\alpha}) - (\tau_2 + \sigma_2 t_\alpha) (p_2^{\alpha} + p_3^{\alpha}) \right) - i T \tau_1 (1 - \sigma_1) z_\alpha t^\alpha \right] \exp(\mathbb{F}) \omega CCC, \tag{B.8}
\]

where

\[
\mathbb{F} = i T z_{\alpha} y^\alpha + i t^\beta p_{1\beta} + \sigma_1 \left( y^\alpha t_\alpha + (p_1^{\alpha} + p_2^{\alpha}) t_\alpha - \sigma_2 (p_2^{\alpha} + p_3^{\alpha}) t_\alpha \right) - i \left( \sigma_2 p_3^{\alpha} - (1 - \sigma_2) p_2^{\alpha} \right) y_\alpha \\
+ i T z_{\alpha} \left( \tau_1 (p_1^{\alpha} + p_2^{\alpha}) - (\tau_2 + \sigma_2 t_\alpha) (p_2^{\alpha} + p_3^{\alpha}) + (\sigma_1 + \tau_1(1 - \sigma_1)) t^\alpha \right). \tag{B.9}
\]

By virtue of evident formulas

\[
\tau_1 \left( \frac{\partial}{\partial \tau_1} - \frac{\partial}{\partial \tau_2} \right) \mathbb{F} = i T z_{\alpha} \left( \tau_1 (p_1^{\alpha} + p_2^{\alpha}) + \left( \tau_1 + \tau_2 \right) - (\tau_2 + \sigma_2 t_\alpha) \right) (p_2^{\alpha} + p_3^{\alpha}) + \tau_1 (1 - \sigma_1) t^\alpha, \\
\frac{\partial}{\partial \sigma_1} \mathbb{F} = i T (1 - \tau_1) z_\alpha t^\alpha + i \left( y^\alpha + p_1^{\alpha} + p_2^{\alpha} - \sigma_2 (p_2^{\alpha} + p_3^{\alpha}) \right) t_\alpha,
\]

23
Eq. (B.7) acquires the form
\[ W_{1\omega C}^n \ast B_{2}^{\eta_{loc}} \approx \frac{\eta^2}{4} \int_{0}^{1} d\tau \int d^2\tau + \delta(1 - \tau_1 - \tau_2) \int_{0}^{1} d\sigma_1 \int_{0}^{1} d\sigma_2 \left[ iz\alpha t^\alpha \tau_1 \left( \frac{\partial}{\partial \tau_1} - \frac{\partial}{\partial \tau_2} \right) \right] \exp(\mathcal{F}) \omega CCC. \tag{B.10} \]

After partial integrations in \( \tau_1, \tau_2 \) and \( \sigma_1 \), one obtains
\[ W_{1\omega C}^n \ast B_{2}^{\eta_{loc}} \approx \frac{\eta^2}{4} \int_{0}^{1} d\tau \int d^2\tau + \delta(1 - \tau_1 - \tau_2) \int_{0}^{1} d\sigma_1 \int_{0}^{1} d\sigma_2 \left[ iz\alpha t^\alpha \delta(\tau_2) \right] \exp(\mathcal{F}) \omega CCC. \tag{B.11} \]

After a simple change of integration variables the underlined term on the r.h.s. of Eq. (B.11) cancels the r.h.s. of Eq. (B.6). Performing integration with respect to \( \tau_2 \) in the remaining part of (B.11), after the following change of the integration variables
\[ \int_{0}^{1} d\sigma_1 \int_{0}^{1} d\tau_1 \int_{0}^{1} d\sigma_2 f(\sigma_1, 1 - \sigma_1, \tau_1, \tau_2) \]
\[ = \int d^4\rho + \delta \left( 1 - \sum_{j=1}^{4} \rho_j \right) \frac{1}{(\rho_2 + \rho_3)(1 - \rho_2 - \rho_3)} f \left( \frac{\rho_1}{1 - \rho_2 - \rho_3}, \frac{\rho_1}{1 - \rho_2 - \rho_3}, \frac{\rho_1}{\rho_2 + \rho_3}, \frac{\rho_2}{\rho_2 + \rho_3} \right), \]

\[ \exp(\mathcal{F}) \tag{B.3} \]
acquires the form \( E \) (4.2). As a result, the sum of Eq. (B.11) and Eq. (B.6) by virtue Eq. (E.1) yields Eq. (8.3).

**Appendix C: Eliminating \( \delta(\rho_j) \) and \( \delta(\xi_j) \)**

To eliminate \( \delta(\rho_j) \) and \( \delta(\xi_j) \) from of the r.h.s.’s of Eqs. (8.2), (8.3) this is convenient to group similar pre-exponential terms as in Sections C.1 - C.3.

**C.1 Terms proportional to \( (p_1 + p_2)^\alpha (p_3 + p_2)^\alpha \)**

Consider \( F_{1,1} + F_{2,1} \) of (8.2) and (8.3), respectively. Partial integration with respect to \( \rho_1 \) and \( \rho_4 \) yields
\[ F_{1,1} + F_{2,1} \approx -\frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (p_1^\alpha + p_2^\alpha)(p_2^\alpha + p_3^\alpha) \times \]
\[ \times (z\gamma \mathcal{P}^\gamma) \left( \frac{\partial}{\partial \rho_4} - \frac{\partial}{\partial \rho_1} \right) E \omega CCC. \tag{C.1} \]

By direct calculation, Eq. (C.1) gives
\[ F_{1,1} + F_{2,1} \approx -\frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (p_1^\alpha + p_2^\alpha)(p_2^\alpha + p_3^\alpha) \times \]
\[ \left[ E \left( \frac{\partial}{\partial \rho_4} - \frac{\partial}{\partial \rho_1} \right) (z\gamma \mathcal{P}^\gamma) E + (z\gamma \mathcal{P}^\gamma) \mathcal{C}(z\alpha t^\alpha) \right] \omega CCC. \tag{C.2} \]
By virtue of the Schouten identity
\[
z_\alpha t^\alpha (p_1 + p_2)\gamma (p_3 + p_2)\gamma = t^\alpha (p_1 + p_2)\alpha z^\gamma (p_3 + p_2)\gamma + t^\alpha (p_3 + p_2)\alpha (p_1 + p_2)\gamma z_\gamma \tag{C.3}
\]
and its consequence
\[
z_\alpha t^\alpha (p_1 + p_2)\gamma (p_3 + p_2)\gamma E = t^\alpha (p_1 + p_2)\alpha \left[ i \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \right) E_z E + iE_z \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \right) E \right] + t^\alpha (p_2 + p_3)\alpha \left[ i \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \right) E_z E + iE_z \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \right) E \right] \tag{C.4}
\]
Eq. (C.1) yields
\[
F_{1,1} + F_{2,1} \approx + \frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left\{ \frac{(z_\gamma P^\gamma)\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \times \right.
\]
\[
\times \left( (p_1 + p_2)\alpha (p_3 + p_2)\alpha E_z \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_4} \right] E + t^\alpha (p_1 + p_2)\alpha \left[ \delta(\rho_3)E - E_z \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \right) E \right] \right.
\]
\[
+ t^\alpha (p_3 + p_2)\alpha \left[ \delta(\rho_1)E - E_z \left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \right) E \right] + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \left( t^\alpha z_\alpha (p_3 + p_2)\gamma (p_1 + p_2)\gamma E \right)
\]
\[
+ \left( z_\gamma P^\gamma \right) \left( \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\alpha (p_1 + p_2)\alpha E - \frac{1 - \rho_1 - \rho_4 - \rho_2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)} t^\alpha (p_3 + p_2)\alpha E \right) \right\} \omega CCC. \tag{C.5}
\]
One can see that $\delta(\rho_1)$- and $\delta(\rho_3)$-proportional terms on the r.h.s. of (C.5) (the underlined ones) cancel terms $F_{2,4}$ (C.3) and $F_{3,3}$ (C.4), respectively.

**C.2 Term proportional to** $t^\alpha (p_1\alpha + p_2\alpha)$

Consider term $F_{3,5}$ of $F_3$ (8.4). By virtue of the following identity
\[
\frac{\rho_2}{(\rho_2 + \rho_3)(1 - \rho_3)} (\delta(\rho_3) - \delta(\rho_2)) = 1 \tag{C.6}
\]
\[
F_{3,5} \approx - \frac{\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\rho_2}{(\rho_2 + \rho_3)(1 - \rho_3)} \left( \delta(\rho_3) - \delta(\rho_2) \right)
\]
\[
\left[ (p_2\alpha + p_1\alpha) t^\alpha (z_\gamma P^\gamma ) (1 - \rho_4) - \frac{\rho_1}{(p_1 + \rho_4)} \right] E \right\} \omega CCC. \tag{C.7}
\]
Partial integrations along with the Schouten identity
\[
t^\alpha (p_1\alpha + p_2\alpha)(p_3\gamma + p_2\gamma)z_\gamma = - t^\alpha z_\alpha (p_1\gamma + p_2\gamma)(p_2\gamma + p_3\gamma) + t^\alpha (p_3\alpha + p_2\alpha)(p_1 + p_2)\gamma z_\gamma \tag{C.8}
\]

25
and realization of the underlined terms as derivative of $E_z$ along with further partial integration yields

$$F_{3,5} \approx -\frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left[ \frac{\rho_4}{(1 - \rho_3)^2} \left( (p_{2\alpha} + p_{1\alpha}) t^\alpha z_{\gamma} t^\gamma \right) E \right. \right.$$

$$\left. + \frac{\rho_2 \rho_4}{(\rho_1 + \rho_4)(1 - \rho_3)} \left( (p_{2\alpha} + p_{1\alpha}) t^\alpha z_{\gamma} t^\gamma \right) E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_3} \right] E + \frac{\rho_2 \rho_4}{(\rho_1 + \rho_4)(1 - \rho_3)} (z_\alpha t^\alpha) \times \right.$$  

$$\times \left. (1 - \rho_1 + \rho_2) \gamma (p_3 + p_2) E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_2} \right] E - \frac{\delta(\rho_1)(p_1 + p_2) \gamma (p_3 + p_2)}{4} \right) E$$

$$\left. \right) \omega CCC. \quad (C.9) \right.$$  

One can see that the sum of the underlined $\delta(\rho_1)$-proportional terms cancel $F_{2,2} + F_{2,3}$ of $(8.3)$.  

### C.3 Sum of $(p_1 + p_2)\alpha (p_3 + p_2)\alpha$-proportional and $t^\alpha (p_{1\alpha} + p_{2\alpha})$-proportional terms

Summing up $F_{1,1} + F_{2,1}$ (C.5), $F_{3,3}$ (8.4), $F_{3,5}$ (C.9) and $F_{2,2} + F_{2,3} + F_{2,4}$ (8.3), then performing partial integrations and using the following simple identities

$$\left(1 - \rho_4\right) - \frac{\rho_1}{\rho_1 + \rho_4} = \frac{\rho_4(\rho_2 + \rho_3)}{\rho_1 + \rho_4}, \quad (C.10)$$

$$\frac{\rho_2}{\rho_1 + \rho_4} + \frac{\rho_4}{\rho_1 + \rho_4} \frac{\rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} = \frac{-\rho_2 \rho_4}{(\rho_1 + \rho_4)^2(1 - \rho_1 - \rho_4)(1 - \rho_3)}, \quad (C.11)$$

one obtains by virtue of Eqs. (4.3)-(4.7)

$$F_{1,1} + F_{2,1} + F_{2,4} + F_{3,3} + F_{3,5} + F_{2,2} + F_{2,3} = G_1 \quad (C.12)$$

with $G_1$ \ref{9.2}.  

### C.4 Terms proportional to $\delta(\xi_1) - \delta(\xi_2)$

Consider a sum of $F_{1,4}$ (8.2) and $F_{2,8}$ (8.3). Performing partial integrations with respect to $\rho_1$ and $\rho_4$, then applying the Schouten identity one obtains

$$F_{1,4} + F_{2,8} \approx -\frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_4} \right] \frac{i z_\alpha (p_{1\alpha} + p_{2\alpha})}{1 - \rho_3} \left( \delta(\xi_2) - \delta(\xi_1) \right) E \omega CCC =$$

$$= -\frac{\eta^2}{4} \int d\Gamma \delta(\xi_3) \left( \delta(\xi_2) - \delta(\xi_1) \right) \left\{ \frac{i z_\alpha t^\gamma}{1 - \rho_3} \left( E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_2} \right] E + \left( \delta(\rho_1) - \delta(\rho_2) \right) E \right) \right.$$

$$\left. + \frac{i z_\alpha (p_{1\alpha} + p_{2\alpha})}{1 - \rho_3} E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_4} \right] E \right\} \omega CCC. \quad (C.13)$$
The underlined $\delta(\rho_1)$-proportional term compensates $F_{2,9}$ of (8.3). The double underlined $\delta(\rho_2)$-proportional term vanishes due to the factor of $(\delta(\xi_2) - \delta(\xi_1))$ which after partial integrations in $\xi_1$ and $\xi_2$ produces an expression proportional to $\rho_2$.

Summing up $F_{1,4} + F_{2,8}$ (C.14) and $F_{2,9}$ (8.3), performing partial integrations with respect to $\xi$ and $\mathcal{T}$ along with the Schouten identity one obtains

$$F_{1,4} + F_{2,8} + F_{2,9} \approx G_2$$  (C.14)

with $G_2$ (9.3).

C.5 Terms proportional to $\xi_1 \delta(\xi_2)$

Consider a sum of $F_{1,3}$ (8.2), $F_{2,6}$ (8.3) and $F_{4,1}$ (8.5).

$$F_{1,3} + F_{2,6} + F_{4,1} \approx \frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\delta(\xi_2)[\delta(\rho_1) - \delta(\rho_4)]}{(\rho_2 + \rho_3)} z_\alpha \left\{ \frac{\mathcal{P}^\alpha}{(1 - \rho_3)} - \frac{\xi_1 (p_2^\alpha + p_3^\alpha)}{(\rho_1 + \rho_4)} \right\} \mathcal{E} \omega CCC.$$(C.15)

Partial integration yields

$$F_{1,3} + F_{2,6} + F_{4,1} \approx \frac{i\eta^2}{4} \int d\Gamma \frac{\delta(\xi_3)\delta(\xi_2)\xi_1}{(1 - \rho_3)} \left\{ z_\alpha t^\alpha \left[ \frac{1}{\rho_1 + \rho_4} \left( E_z \left[ \frac{\partial}{\partial \rho_2} - \frac{\partial}{\partial \rho_3} \right] E + \left[ \delta(\rho_2) - \delta(\rho_3) \right] \mathcal{E} \right) + \frac{1}{1 - \rho_3} \left( E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_2} \right] E + \left[ \delta(\rho_1) - \delta(\rho_2) \right] \mathcal{E} \right) \right] + \left[ \frac{z_\alpha (p_2^\alpha + p_3^\alpha)}{\rho_1 + \rho_4} + \frac{z_\alpha (p_1^\alpha + p_2^\alpha)}{1 - \rho_3} \right] E_z \left[ \frac{\partial}{\partial \rho_1} - \frac{\partial}{\partial \rho_4} \right] E \right\} \omega CCC.$$(C.16)

One can see that the underlined $\delta(\rho_2)$-proportional terms vanish due to the factor of $(1 - \sum \rho_i) (4.8)$, while $\delta(\rho_1)$-proportional term compensates $F_{2,7}$ (8.3) and $\delta(\rho_3)$-proportional term compensates $F_{3,2}$ (8.4).

Summing up $F_{2,7}$ (8.3), $F_{3,2}$ (8.3), $F_{4,2}$ and $F_{1,3} + F_{2,6} + F_{4,1}$ (8.5), and then performing partial integration in $\mathcal{T}$ one obtains by virtue of the Schouten identity

$$F_{1,3} + F_{2,6} + F_{4,1} + F_{2,7} + F_{3,2} + F_{4,2} \approx G_3 := \frac{i\eta^2}{4} \int d\Gamma \delta(\xi_3)\delta(\xi_2) \times \left\{ \rho_2 \left[ (\tilde{\gamma}^\gamma + y^\gamma)(y_\gamma + \mathbb{P}_\gamma)(z^\alpha t_\alpha) + i\delta(\mathcal{T}) t_\gamma (\tilde{\gamma}^\gamma - \mathbb{P}^\gamma) \right] \left( \frac{(1 - \rho_1 - \rho_4)^2}{(1 - \rho_3)(\rho_1 + \rho_4)} + \frac{\rho_2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)^2(\rho_1 + \rho_4)} \right) + \rho_3 \left[ (\tilde{\gamma}^\gamma + y^\gamma)(t^\alpha y_\alpha) \right] \left( \frac{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)^2} + \frac{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)^2} \right) \right\} \mathcal{E} \omega CCC.$$(C.17)

Since the partial integration procedure $\xi_1 \delta(\xi_2) \equiv 1 + \xi_1(\partial_{\xi_1} - \partial_{\xi_2})$, (C.17) yields $G_3$ (9.4).
Appendix D: Details of the final step of the calculation

By virtue of Eqs. (D.1)–(D.3), Eq. (10.2) yields

\[ S_0 = \frac{i \eta^2}{4} \int d\Gamma \delta(\xi_3) \]

\[ \left\{ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \mathbb{P}^\alpha y_\alpha ight. \]

\[ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (y + \mathbb{P})^\alpha \tilde{t}_\alpha \\
\]

\[ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha (-\xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (p_3 + p_2)^\alpha y_\alpha \\
\]

\[ - (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (p_3 + p_2)^2 \tilde{t}_3 + \]

\[ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha (-\frac{\rho_1 + \rho_4}{(1 - \rho_3)^2} ((p_1 + p_2)^\alpha y_\alpha \\
\]

\[ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha (-\frac{\rho_1}{(1 - \rho_3)^2} t^\alpha y_\alpha \\
\]

\[ + (y + \tilde{t})^\gamma z_\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha (-\frac{\rho_1}{(1 - \rho_3)^2} (p_1 + p_2)^\alpha t_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_1 + p_2)^\gamma (y + (1 - \rho_4) t) \gamma z_\alpha (y + \tilde{t})^\alpha (-\frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha y_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_1 + p_2)^\gamma (y + (1 - \rho_4) t) \gamma z_\alpha (y + \tilde{t})^\alpha \]

\[ \times (-\xi_1 \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(\rho_1 + \rho_4)} (y^\alpha + \mathbb{P}^\alpha) t_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_1 + p_2)^\gamma (y + (1 - \rho_4) t) \gamma z_\alpha (y + \tilde{t})^\alpha \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha t_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \gamma z_\alpha (y + \tilde{t})^\alpha \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha (y + \tilde{t})^\alpha \\
\]

\[ \times \xi_1 \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(\rho_1 + \rho_4)} t^\alpha y_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \gamma z_\alpha (y + \tilde{t})^\alpha \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha \\
\]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \gamma z_\alpha (y + \tilde{t})^\alpha \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha \tilde{t}_\alpha \]

28
Terms from the r.h.s. \( \xi \) Here we consider only pre-exponentials, omitting for brevity integrals, integral measures etc of (D.1) with \( \xi \)-independent pre-exponentials are considered in Section D.1, while those with \( \xi \)-proportional pre-exponentials are considered in Section D.2.

D.1 \( \xi \)-independent pre-exponentials

Here we consider only pre-exponentials, omitting for brevity integrals, integral measures etc of (D.1). By virtue of the Schouten identity taking into account that \( \sum \rho_i = 1 \) Eq. (D.1) yields

\[
\begin{align*}
\text{Integrand}(S_0) \mod \xi &= (y + \tilde{t})^\nu z_\nu \left\{- \frac{\rho_2 (p_1 + p_4)}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2 t^\alpha (p_1 + p_2) \alpha } \right\} y_\alpha \\
- \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^3} t^\alpha (p_1 + p_2) \alpha y_\alpha \\
- \frac{\rho_2 \rho_1}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^3} t^\alpha (p_1 + p_2) \alpha (p_1 + p_2) \alpha y_\alpha \\
+ \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^3} (p_1 + p_2) \gamma (y + (1 - \rho_4 ) t) \gamma t^\alpha y_\alpha \\
- \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^3} (p_1 + p_2) \gamma (y + (1 - \rho_4) t) \gamma (p_1 + p_2) \alpha y_\alpha \\
- \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^3} (p_1 + p_2) \gamma (y + (1 - \rho_4) t) \gamma (p_1 + p_2) \gamma t_\alpha y_\alpha \\
- (p_1 + p_2) \gamma (1 - \rho_4) t_\gamma (p_1 + p_2) \gamma t_\alpha y_\alpha \\
- t^\gamma y_\gamma (p_1 + p_2) \alpha y_\alpha - t^\gamma (1 - \rho_1 - \rho_4) (p_1 + p_2) \gamma (p_1 + p_2) \alpha t_\alpha \right\} \mathcal{E} \omega CCC = \\
= (y + \tilde{t})^\nu z_\nu \left\{ \rho_1 t^\alpha (p_1 + p_2) \alpha (p_1 + p_2) \alpha t_\alpha + (p_1 + p_2) \gamma y_\gamma t^\alpha y_\alpha \\
- (p_1 + p_2) \gamma (1 - \rho_4) t_\gamma (p_1 + p_2) \gamma t_\alpha \\
- t^\gamma y_\gamma (p_1 + p_2) \alpha y_\alpha - t^\gamma (1 - \rho_1 - \rho_4) (p_1 + p_2) \gamma (p_1 + p_2) \alpha t_\alpha \right\} \mathcal{E} \omega CCC \\
= (y + \tilde{t})^\nu z_\nu \left\{ - \rho_1 t^\alpha (p_1 + p_2) \alpha (p_1 + p_2) \alpha t_\beta \right\} \mathcal{E} \omega CCC \\
= (y + \tilde{t})^\nu z_\nu \left\{ (1 - \rho_1 - \rho_4)(1 - \rho_3)^3 \right\} \left\{ - \rho_1 t^\alpha (p_1 + p_2) \alpha (p_1 \alpha + p_2 \beta) t_\beta \\
- (p_1 + p_2) \gamma (1 - \rho_4) t_\gamma (p_1 + p_2) \gamma t_\alpha \\
- t^\gamma y_\gamma (p_1 + p_2) \gamma t_\alpha - t^\gamma (1 - \rho_1 - \rho_4) (p_1 + p_2) \gamma (p_1 + p_2) \alpha t_\alpha \right\} \mathcal{E} \omega CCC \equiv 0.
\end{align*}
\]
\[ \{y + \tilde{t}\}^\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2) \rho_1 \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \mathbb{P}^\alpha y_\alpha \]

\[ + (y + \tilde{t})^\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2) \rho_1 \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (y + \mathbb{P})^\alpha \tilde{t}_\alpha \]

\[ - (y + \tilde{t})^\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2) \rho_1 \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_3^\alpha + p_2^\alpha) y_\alpha \]

\[ - (y + \tilde{t})^\gamma \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha (p_1 + p_2) \rho_1 \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} \tilde{t}^\alpha \beta (p_3 + p_2)^\beta \tilde{t}_\beta \]

\[ - \frac{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (p_1 + p_2)^\gamma (y + (1 - \rho_4)t) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1 - \rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha y_\alpha \]

\[ \times (y^\alpha + \mathbb{P}^\alpha) t_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1 - \rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha y_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1 - \rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha \tilde{t}_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1 - \rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} y_\alpha \]

\[ \times \xi_1 \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(p_1 + p_2)} y^\alpha + \mathbb{P}^\alpha t_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha \tilde{t}_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha \]

\[ \times \xi_1 \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(p_1 + p_2)} y^\alpha + \mathbb{P}^\alpha t_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha \tilde{t}_\alpha \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2)) \gamma_\alpha (y + \tilde{t})^\alpha \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha \]

\[ \times \xi_1 \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(p_1 + p_2)} y^\alpha + \mathbb{P}^\alpha t_\alpha \]

\[ + \xi_1 \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)^3(1 - \rho_3)^3(p_1 + p_2)} (y + (1 - \rho_1 - \rho_4)(p_1 + p_2) + (1 - \rho_4)t)^\gamma (y + \tilde{t}) \gamma_\alpha \]

\[ \times \left\{ t^\alpha (y + \mathbb{P}) \alpha^\sigma (y + \tilde{t}) \sigma \right\} - \frac{\rho_2 \rho_4}{(1 - \rho_1 - \rho_4)^3(1 - \rho_3)^3} (y + \tilde{t})^\gamma \gamma_\alpha t^\alpha (y + \mathbb{P})^\sigma (y + \tilde{t}) \sigma \]

\[ + \frac{\rho_2}{(1 - \rho_1 - \rho_4)^2(1 - \rho_3)^3} (y + \tilde{t})^\gamma \gamma_\alpha (y + \mathbb{P})^\sigma (y + \tilde{t}) \sigma (p_1 + p_2)^\alpha t_\alpha \right\} \mathcal{E}_\omega CCC , \]
where \( J_7 \) is the cohomology term \([5,8]\). This yields

\[
S_6 \left|_{\xi_1} \right. \approx J_7 + \frac{n^2}{4} \int d\Gamma \delta(\xi_3) \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} (y + \mathbb{P})^\alpha (y + \tilde{\iota})^\beta \]

\[
\xi_1 (y + \tilde{\iota})^\gamma z_1 \left\{ \frac{(1 - \rho_3 - \rho_2)}{(1 - \rho_3)} t^\alpha (p_1 + p_2)_\alpha (y + \mathbb{P})^\beta (y + \tilde{\iota})^\beta 

- \rho_2 \rho_2 (p_1 + p_2)_\alpha (p_3^\alpha + p_2^\beta) (y + \tilde{\iota})^\beta - \frac{\rho_2}{(1 - \rho_3)} (p_1 + p_2)^\gamma (y + (1 - \rho_4)t)^\gamma t^\alpha y_\alpha 

- \rho_2 \rho_3 (p_1 + p_2)^\gamma (y + (1 - \rho_4)t)^\gamma (y + \mathbb{P})^\alpha t_\alpha 

- \frac{\rho_3}{(1 - \rho_1 - \rho_4)^2} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2))_\gamma (p_3 + p_2)_\alpha (y + \tilde{\iota})_\alpha 

+ \frac{\rho_3^2 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(p_1 + p_4)} t^\gamma (y + (1 - \rho_1)(p_1 + p_2))_\gamma (y + \mathbb{P})^\alpha t_\alpha 

+ \frac{\rho_2}{(1 - \rho_3)(p_1 + p_2)} t^\gamma (y + (1 - \rho_1 - \rho_4)(p_1 + p_2))_\gamma (y + \mathbb{P})^\alpha t_\alpha 

- \frac{\rho_3}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(p_1 + p_4)} t^\gamma (y + \mathbb{P})^\alpha t_\alpha 

+ \frac{1}{(1 - \rho_3)} (y + \mathbb{P})^\alpha (y + \tilde{\iota})_\alpha (p_1 + p_2)_\alpha t_\alpha \right\} \mathcal{E} \omega CCC \equiv J_7
\]

since, using the Schouten identity, one can see that the pre-exponential of the integrand on the r.h.s. of \((D.3)\) equals zero.

**Appendix E: Useful formulas**

From \((4.4)\) one has

\[
\left( \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_4} \right) E = i \left\{ \xi_1 \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} t^\alpha y_\alpha + \frac{1}{(1 - \rho_3)} (y + p_1^\alpha + p_2^\alpha) t_\alpha \right\} E
\]

\[
\left( \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \right) E = i \left\{ \xi_1 \frac{1 - \rho_3 - \rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)^2} (y + \mathbb{P})^\alpha (y + \tilde{\iota})_\alpha + \frac{\rho_1 + \rho_4}{(1 - \rho_3)^2} ((p_1 + p_2)_\alpha t_\alpha \right\} E
\]
\[
\left( \frac{\partial}{\partial \rho_2} - \frac{\partial}{\partial \rho_1} \right) E = i \left\{ \xi_1 \frac{-\rho_3}{(1 - \rho_1 - \rho_4)^2} (p_3 + p_2)^\alpha (y + \tilde{t})_\alpha \right. \\
+ \xi_1 \frac{\rho_3 \rho_4}{(1 - \rho_1 - \rho_4)(1 - \rho_3)(\rho_1 + \rho_4)} t^\alpha y_\alpha + \xi_1 \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha (y + \tilde{t})_\alpha \\
- \xi_1 \frac{\rho_2}{(1 - \rho_1 - \rho_4)(1 - \rho_3)} \frac{\rho_4}{(\rho_1 + \rho_4)^2} (y + P)^\alpha t_\alpha \\
- \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha y_\alpha - \frac{1}{(1 - \rho_3)} (p_1 + p_2)^\alpha t_\alpha \left\} E .
\]

References

[1] V. Didenko, O. Gelfond, A. Korybut, and M. Vasiliev, “Spin-Locality of $\eta^2$ and $\bar{\eta}^2$ Quartic Higher-Spin Vertices”, JHEP, vol. 12, p. 184, 2020, 2009.02811.

[2] O. Gelfond and M. Vasiliev, “Homotopy Operators and Locality Theorems in Higher-Spin Equations”, Phys. Lett. B, vol. 786, pp. 180–188, 2018, 1805.11941.

[3] M. Vasiliev, “Higher spin gauge theories: Star product and AdS space”, in "The Many Faces of the Superworld" , pp. 533-610 (2000) , https://doi.org/10.1142/9789812793850_0030 [arXiv:hep-th/9910096 [hep-th]].

[4] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. Vasiliev, “Nonlinear higher spin theories in various dimensions”, in 1st Solvay Workshop on Higher Spin Gauge Theories, pp. 132–197, 2004, hep-th/0503128.

[5] M. Vasiliev, “Current Interactions and Holography from the 0-Form Sector of Nonlinear Higher-Spin Equations”, JHEP, vol. 10, p. 111, 2017, 1605.02662.

[6] I. Klebanov and A. Polyakov, “AdS dual of the critical $O(N)$ vector model”, Phys. Lett. B, vol. 550, pp. 213–219, 2002, hep-th/0210114.

[7] A. Barvinsky, “CFT driven cosmology and conformal higher spin fields”, Phys.Rev.D, 93 (2016) 10, 103530, 1511.07625 [hep-th]

[8] A. K. Bengtsson, I. Bengtsson, and L. Brink, “Cubic Interaction Terms for Arbitrary Spin”, Nucl. Phys. B, vol. 227, pp. 31–40, 1983.

[9] F. A. Berends, G. Burgers, and H. Van Dam, “On spin three selfinteractions”, Z. Phys. C, vol. 24, pp. 247–254, 1984.

[10] E. Fradkin and M. A. Vasiliev, “On the Gravitational Interaction of Massless Higher Spin Fields”, Phys. Lett. B, vol. 189, pp. 89–95, 1987.

[11] E. Fradkin and R. Metsaev, “A Cubic interaction of totally symmetric massless representations of the Lorentz group in arbitrary dimensions”, Class. Quant. Grav., vol. 8, pp. L89–L94, 1991.
[12] O. Gelfond and M. Vasiliev, “Current Interactions from the One-Form Sector of Nonlinear Higher-Spin Equations”, *Nucl. Phys. B*, vol. 931, pp. 383–417, 2018, 1706.03718.

[13] V. E. Didenko, O. A. Gelfond, A. V. Korybut, and M. A. Vasiliev, “Homotopy Properties and Lower-Order Vertices in Higher-Spin Equations”, *J. Phys.*, vol. A51, no. 46, p. 465202, 2018, 1807.00001.

[14] V. E. Didenko, O. A. Gelfond, A. V. Korybut, and M. A. Vasiliev, “Limiting Shifted Homotopy in Higher-Spin Theory and Spin-Locality”, *JHEP*, vol. 12, p. 086, 2019, 1909.04876.

[15] M. A. Vasiliev, “More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions”, *Phys. Lett. B*, vol. 285, pp. 225–234, 1992.

[16] E. Sezgin and P. Sundell, “Massless higher spins and holography”, *Nucl. Phys. B*, vol. 644, pp. 303–370, 2002, hep-th/0205131. [Erratum: Nucl.Phys.B 660, 403–403 (2003)].

[17] E. Sezgin and P. Sundell, “Holography in 4D (super) higher spin theories and a test via cubic scalar couplings”, *JHEP*, vol. 07, p. 044, 2005, hep-th/0305040.

[18] O. Gelfond and M. Vasiliev, “Spin-Locality of Higher-Spin Theories and Star-Product Functional Classes”, *JHEP*, vol. 03, p. 002, 2020, 1910.00487.

[19] A. Fotopoulos and M. Tsulaia, “On the Tensionless Limit of String theory, Off - Shell Higher Spin Interaction Vertices and BCFW Recursion Relations”, *JHEP*, vol. 11, p. 086, 2010, 1009.0727.

[20] A. David and Y. Neiman, “Higher-spin symmetry vs. boundary locality, and a rehabilitation of dS/CFT”, *JHEP*, vol. 10, p. 127, 2020, 2006.15813.

[21] C. Sleight and M. Taronna, “Higher-Spin Gauge Theories and Bulk Locality”, *Phys. Rev. Lett.*, vol. 121, no. 17, p. 171604, 2018, 1704.07859.

[22] D. Ponomarev, “A Note on (Non)-Locality in Holographic Higher Spin Theories”, *Universe*, textbf4 (2018) no.1, 2 https://doi.org/10.3390/universe4010002 [arXiv:1710.00403 [hep-th]].

[23] M. A. Vasiliev, “Consistent Equations for Interacting Massless Fields of All Spins in the First Order in Curvatures”, *Annals Phys.*, vol. 190, pp. 59–106, 1989.

[24] J. Stasheff, *H-spaces from a homotopy point of view*, pp. 1–2. Berlin, Heidelberg: Springer Berlin Heidelberg, 1970.

[25] J. D. Stasheff, “Homotopy associativity of h-spaces. i”, *Transactions of the American Mathematical Society*, vol. 108, no. 2, pp. 275–292, 1963.

[26] J. D. Stasheff, “Homotopy associativity of h-spaces. ii”, *Transactions of the American Mathematical Society*, vol. 108, no. 2, pp. 293–312, 1963.

[27] M. Vasiliev, “Triangle identity and free differential algebra of massless higher spins”, *Nuclear Physics B*, vol. 324, no. 2, pp. 503 – 522, 1989.