Equivalence classes in matching covered graphs

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Abstract

A connected graph \(G\), of order two or more, is matching covered if each edge lies in some perfect matching. The tight cut decomposition of a matching covered graph \(G\) yields a list of bricks and braces; as per a theorem of Lovász \cite{5}, this list is unique (up to multiple edges); \(b(G)\) denotes the number of bricks, and \(c_4(G)\) denotes the number of braces that are isomorphic to the cycle \(C_4\) (up to multiple edges).

Two edges \(e\) and \(f\) are mutually dependent if, for each perfect matching \(M\), \(e \in M\) if and only if \(f \in M\); Carvalho, Lucchesi and Murty investigated this notion in their landmark paper \cite{2}. For any matching covered graph \(G\), mutual dependence is an equivalence relation, and it partitions \(E(G)\) into equivalence classes; this equivalence class partition is denoted by \(\mathcal{E}_G\) and we refer to its parts as equivalence classes of \(G\); we use \(\varepsilon(G)\) to denote the cardinality of the largest equivalence class.

The operation of ‘splicing’ may be used to construct bigger matching covered graphs from smaller ones; see \cite{7}; ‘tight splicing’ is a stronger version of ‘splicing’. (These are converses of the notions of ‘separating cut’ and ‘tight cut’.) In this article, we answer the following basic question: if a matching covered graph \(G\) is obtained by ‘splicing’ (or by ‘tight splicing’) two smaller matching covered graphs, say \(G_1\) and \(G_2\), then how is \(\mathcal{E}_G\) related to \(\mathcal{E}_{G_1}\) and to \(\mathcal{E}_{G_2}\) (and vice versa)?

As applications of our findings: firstly, we establish tight upper bounds on \(\varepsilon(G)\) in terms of \(b(G)\) and \(c_4(G)\); secondly, we answer a recent question of He, Wei, Ye and Zhai \cite{4}, in the affirmative, by constructing graphs that have arbitrarily high \(\kappa(G)\) and \(\varepsilon(G)\) simultaneously, where \(\kappa(G)\) denotes the vertex-connectivity.

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1 Matching covered graphs

For general graph-theoretic terminology, we follow Bondy and Murty [1]. All graphs considered here are finite and loopless; however, we do allow multiple edges. We begin by reviewing some important terminology and notation.

For a subset $X$ of the vertex set $V(G)$ of a graph $G$, we use $N_G(X)$, or simply $N(X)$, to denote the set of vertices that have at least one neighbor in $X$; and we use $\text{odd}(G - X)$ to denote the number of components of odd order in the graph $G - X$. Furthermore, we use $\partial_G(X)$, or simply $\partial(X)$, to denote the set of edges of $G$ that have exactly one end in $X$; such a set $\partial(X)$ is called a cut of $G$; the sets $X$ and $\overline{X} := V(G) - X$ are referred to as the shores of $\partial(X)$. For a vertex $v$, we simplify the notation $\partial\{\{v\}\}$ to $\partial(v)$. A cut $C$ is trivial if either shore has just one vertex; otherwise $C$ is nontrivial. A cut $C$ is a $k$-cut if $|C| = k$.

A connected graph, of order two or more, is matching covered if each edge is admissible. The following fact is easily deduced from Lemma 1.1.

Proposition 1.2 For a bipartite matchable graph $H[A, B]$, of order four or more, the following statements are equivalent:

Lemma 1.1 Let $H[A, B]$ denote a bipartite matchable graph. An edge $e$ of $H$ is inadmissible if and only if there exists a nonempty proper subset $S \subset A$ such that $|N_H(S)| = |S|$ and $e$ has one end in $N_H(S)$ and its other end is not in $S$. \hfill \Box

A connected graph, of order two or more, is matching covered if each edge is admissible. The following fact is easily deduced from Lemma 1.1.

Proposition 1.2 For a bipartite matchable graph $H[A, B]$, of order four or more, the following statements are equivalent:
(i) $H$ is matching covered.

(ii) $|N_H(S)| \geq |S| + 1$ for every nonempty proper subset $S$ of $A$.

(iii) $H - a - b$ is matchable for each pair of vertices $a \in A$ and $b \in B$. □

Using Tutte’s 1-factor Theorem, one may prove that a connected matchable graph $G$ is matching covered if and only if every barrier of $G$ is stable (that is, an independent set). Kotzig proved the following fundamental theorem: the maximal barriers of a matching covered graph $G$ partition its vertex-set $V(G)$; this partition of $V(G)$ is called the canonical partition of $G$. (See Lovász and Plummer [6, page 150].)

In this article, our main focus is the ‘equivalence class partition’ — a partition of the edge-set $E(G)$ of a matching covered graph $G$ — that was formally introduced and investigated by Carvalho, Lucchesi and Murty in their landmark paper [2].

![Illustrations of various concepts in bipartite graphs](image)

(a) $e$ is inadmissible  
(b) $\partial(X)$ is a tight cut; $B \cap X$ is a barrier  
(c) $e$ depends on $f$  
(d) $e$ and $f$ are mutually dependent

Figure 1: Illustrations of various concepts in bipartite graphs
1.1 The equivalence class partition $\mathcal{E}_G$

For a cut $C := \partial(X)$ of a matching covered graph $G$, the parities of $|X|$ and $|\overline{X}|$ are the same; we say that $C$ is an odd cut if $|X|$ is odd; otherwise, $C$ is an even cut.

For a matching covered graph $G$, an edge $e$ depends on an edge $f$, denoted as $e \xrightarrow{G} f$, if each perfect matching that contains $e$ also contains $f$. Note that, for two distinct edges $e$ and $f$, $e \xrightarrow{G} f$ if and only if $e$ is inadmissible in $G - f$. The following is easily deduced from Lemma 1.1 and Proposition 1.2; see Figure 1(c).

Corollary 1.3 Let $e$ and $f$ denote distinct edges of a bipartite matching covered graph $H[A, B]$. Then $e$ depends on $f$ if and only if there exist partitions $(A_0, A_1)$ of $A$ and $(B_0, B_1)$ of $B$ such that (i) $|A_0| = |B_0|$, (ii) $e$ joins a vertex in $A_0$ to a vertex in $B_1$, and (iii) $f$ is the only edge that joins a vertex in $B_0$ to a vertex in $A_1$. $\square$

For a matching covered graph $G$, edges $e$ and $f$ are mutually dependent, denoted as $e \leftrightarrow_{G} f$, if $e \xrightarrow{G} f$ and $f \xrightarrow{G} e$. For example: one may verify that if $\{e, f\}$ is an even 2-cut of a bipartite matching covered graph $H$, then $H - e - f$ has precisely two (matchable) components and $e \leftrightarrow_{H} f$. The converse also holds, and is easily deduced from Corollary 1.3; see Figure 1(d).

Corollary 1.4 Let $e$ and $f$ denote distinct edges of a bipartite matching covered graph $H$. The following are equivalent:

(i) $e \leftrightarrow_{G} f$.

(ii) $\{e, f\}$ is an even 2-cut of $H$. $\square$

Observe that mutual dependence is an equivalence relation; whence it partitions the edge set $E(G)$ into equivalence classes. Throughout this article, we refer to this partition of $E(G)$, denoted as $\mathcal{E}_G$, as the equivalence class partition of a matching covered graph $G$, and we refer to its parts as the equivalence classes of $G$; we also use the notation $\varepsilon(G)$ to denote the cardinality of the largest member of $\mathcal{E}_G$. Corollary 1.4 yields the following consequence.

Corollary 1.5 For a bipartite matching covered graph $H$, the following are equivalent:

(i) $\varepsilon(H) = 1$.

(ii) $H$ is free of even 2-cuts. $\square$

An equivalence class $F$ is a singleton if $|F| = 1$, and it is a doubleton if $|F| = 2$. Thus, in the case of bipartite graphs, edge-connectivity three (or more) already implies that each equivalence class is a singleton. This is in stark contrast with the case of nonbipartite graphs — as we will see in Section 4.2. The reader may verify that, for each of the graphs shown in Figure 2, the sets $\{e_1, e_2\}$, $\{f_1, f_2\}$ and $\{g_1, g_2\}$ are doubleton equivalence classes; all other equivalence classes are singleton.
An edge $e$ of a matching covered graph $G$ is *removable* if the graph $G - e$ is also matching covered. Using the fact that all matching covered graphs, distinct from $K_2$, are 2-edge-connected, one may easily verify the following.

**Lemma 1.6** For every matching covered graph $G$, distinct from $K_2$, an edge $e$ is removable if and only if no other edge depends on $e$. □

Consequently, every removable edge is the member of a singleton equivalence class. It is worth noting that every multiple edge is in fact a removable edge. Removable edges play an important role in the theory of matching covered graphs, and especially in several works of Carvalho, Lucchesi and Murty [2]. We will further discuss removable edges in Section 1.3.

The following provides another (equivalent) definition of equivalence classes of a matching covered graph, and is easily verified.

**Proposition 1.7** For a matching covered graph $G$ and a subset $F \subseteq E(G)$, the following are equivalent:

(i) $F$ is a (not necessarily proper) subset of some member of $\mathcal{E}_G$.

(ii) For each perfect matching $M$ of $G$, either $F \subseteq M$ or $F \cap M = \emptyset$. □

We infer that $\varepsilon(G) \leq n$ for any matching covered graph $G$ of order $2n$. Observe that $\mathcal{E}_{C_{2n}}$ has precisely two members, each of which is a perfect matching of the even cycle $C_{2n}$. Consequently, $\varepsilon(C_{2n}) = n$. We leave the following as an easy exercise for the reader to get acquainted with the notion of equivalence classes.

**Proposition 1.8** Let $G$ denote a matching covered graph of order $2n$, where $n \geq 2$. Then $\varepsilon(G) = n$ if and only if (i) the underlying simple graph of $G$ is either $K_4$ or $C_{2n}$ and (ii) $G$ has a perfect matching $M$, each of whose member has multiplicity precisely one. □

An equivalence class $R$ of a matching covered graph $G$ is a *removable class* if the graph $G - R$ is also matching covered. Since a matching covered graph is connected (by definition), Corollary 1.4 implies that each removable class of a bipartite matching covered graph is a singleton. The following is a consequence of a Lovász and Plummer [6, Lemma 5.4.5].

**Theorem 1.9** In every matching covered graph, each removable class is either a singleton or a doubleton. □

Now, let $R$ denote a removable class of a matching covered graph $G$. Observe that $R$ is a singleton if and only if its (only) member is a removable edge. On the other hand, if $R$ is a doubleton, then $R$ is referred to as a *removable doubleton*. Removable edges and removable doubletons play a crucial role in the theory of matching covered graphs; see [2]. (It is worth noting that a matching covered graph may have singleton or doubleton equivalence classes that are not removable.)
1.2 Splicing and separating cuts

For \( i \in \{1, 2\} \), let \( G_i \) denote a graph with a specified vertex \( v_i \) so that (i) \( G_1 \) and \( G_2 \) are disjoint and (ii) degree of \( v_1 \) in \( G_1 \) is the same as degree of \( v_2 \) in \( G_2 \). Suppose that \( \pi \) is a bijection from \( \partial G_1(v_1) \) to \( \partial G_2(v_2) \). We denote by \((G_1 \circ G_2)_{v_1, v_2, \pi}\) the graph obtained from the union of \( G_1 - v_1 \) and \( G_2 - v_2 \) by joining, for each edge \( e \in \partial G_1(v_1) \), the end of \( e \) in \( G_1 - v_1 \) to the end of \( \pi(e) \) in \( G_2 - v_2 \). We refer to \((G_1 \circ G_2)_{v_1, v_2, \pi}\) as the graph obtained by splicing \( G_1 \) at \( v_1 \) with \( G_2 \) at \( v_2 \) with respect to the bijection \( \pi \), or simply as a graph obtained by splicing \( G_1 \) and \( G_2 \). Thus, \( V(G) = (V(G_1) - v_1) \cup (V(G_2) - v_2) \); the corresponding cut \( C := \partial(V(G_1) - v_1) \) is referred to as the splicing cut; the vertices \( v_1 \) and \( v_2 \) are referred to as the splicing vertices.

The following is easy to see.

**Lemma 1.10** If a graph \( G \) is obtained by splicing two simple graphs, then \( G \) is simple, and the corresponding splicing cut is a matching of \( G \). \( \square \)

The following basic, nonetheless crucial, fact is easily proved.

**Lemma 1.11** Any graph obtained by splicing two matching covered graphs is also matching covered. \( \square \)

Figure 2 shows a few examples that are obtained by splicing smaller matching covered graphs; the splicing cut is depicted using a thick line. (In all of these examples, the choice of the splicing vertices and the choice of the permutation does not matter; it is for this reason that we have simplified the notation.) However, in general, these choices do matter. For instance, splicing two copies of the wheel \( W_5 \) (at their hubs) can result in several non-isomorphic graphs (such as the Petersen graph and the pentagonal prism) depending on the choice of the permutation.

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1Hint: fix a perfect matching \( M \) that is an equivalence class (of \( G \)) using Proposition 1.7 and consider the symmetric difference of \( M \) and any other perfect matching.
Given any cut $C := \partial_c(X)$ of a graph $G$, we denote by $G/X \to x$, or simply by $G/X$, the graph obtained from $G$ by shrinking $X$ to a single vertex $x$ (and deleting any resulting loops); we refer to $x$ as the contraction vertex. The two graphs $G/X$ and $G/\overline{X}$ are called the $C$-contractions of $G$. (Observe that $G$ may be recovered by splicing the $C$-contractions, at their contraction vertices, appropriately.) The following lemma is easily proved, and it will be useful to us later.

**Lemma 1.12** Let $C := \partial(X)$ denote an odd cut of a connected graph $G$ of even order, and let $H$ denote a $C$-contraction of $G$. Then: (i) $H$ is of even order, (ii) if $D$ is a cut of $H$ then $D$ is a cut of $G$, and (iii) for any cut $D$ of the graph $H$, $D$ is an even cut of $H$ if and only if $D$ is an even cut of $G$. $\square$

A cut $C$ of a matching covered graph $G$ is a separating cut if both $C$-contractions of $G$ are also matching covered. Consequently, $G$ has a nontrivial separating cut $C$ if and only if $G$ can be obtained by splicing two smaller matching covered graphs (and the corresponding splicing cut is precisely the separating cut $C$). The following is easy to see.

**Lemma 1.13** Let $C$ denote a separating cut of a matching covered graph $G$, and let $G_1$ denote a $C$-contraction of $G$. Then each perfect matching of $G_1$ extends to a perfect matching of $G$. $\square$

Note that if $C := \partial(X)$ is a separating cut of a matching covered graph $G$ then each of the induced subgraphs $G[X]$ and $G[\overline{X}]$ is connected. The following characterization of separating cuts is easily proved; see [3, Lemma 2.19].

**Lemma 1.14** A cut $C$ of a matching covered graph $G$ is a separating cut if and only if, for each $e \in E(G)$, there exists a perfect matching $M$ such that $e \in M$ and $|M \cap C| = 1$. $\square$

Let $C := \partial(X)$ denote a separating cut of a matching covered graph $G$ such that the induced (connected) subgraph $G[X]$ is bipartite. Using Lemma 1.14 one may easily verify that one of the color classes of $G[X]$ has precisely one more vertex than the other color class; furthermore, all edges of $C$ are incident with the bigger color class; consequently, the $C$-contraction $G/\overline{X}$ is a bipartite (matching covered) graph. This proves the following.

**Lemma 1.15** For a separating cut $C := \partial(X)$ of a matching covered graph $G$, the $C$-contraction $G/\overline{X}$ is bipartite if and only if the induced subgraph $G[X]$ is bipartite. $\square$
1.3 Tight splicing, tight cuts, bricks and braces

A cut $C$ of a matching covered graph $G$ is a tight cut if $|M \cap C| = 1$ for each perfect matching $M$ of $G$. It follows from Lemma 1.14 that each tight cut is a separating cut. For instance, if $B$ is a barrier of a matching covered graph $G$, and if $K$ is a component of $G - B$, then $\partial(V(K))$ is a tight cut of $G$; such a tight cut is referred to as a barrier cut associated with the barrier $B$, or simply a barrier cut.

If a matching covered graph $G$ is obtained by splicing two smaller matching covered graphs, say $G_1$ and $G_2$, and if the corresponding splicing cut (i.e., separating cut) is a tight cut (in $G$) then we also say that $G$ is obtained by tight splicing $G_1$ and $G_2$.

In the case of bipartite graphs, one may prove using Lemmas 1.14 and 1.15 that each separating cut is indeed a tight cut; furthermore, one may infer that each tight cut is a barrier cut; see Figure 1(b). However, in general, a separating cut need not be a tight cut. For instance, the separating cuts, depicted by thick lines, in Figures 2(a) and 2(b) are not tight cuts. On the other hand, the separating cut shown in Figure 2(c) is a barrier cut, and the corresponding splicing is a tight splicing.

For tight cuts, we have the following stronger conclusion (in comparison to Lemma 1.13).

Lemma 1.16 Let $C$ denote a tight cut of a matching covered graph $G$, and let $G_1$ denote a $C$-contraction of $G$. Then each perfect matching of $G_1$ extends to a perfect matching of $G$; furthermore, the restriction of each perfect matching of $G$ to the set $E(G_1)$ is a perfect matching of $G_1$. \[\square\]

A matching covered graph $G$ that is devoid of nontrivial tight cuts is called a brick if it is nonbipartite, or a brace if it is bipartite. A brick is solid if it is devoid of nontrivial separating cuts. Thus, bricks and braces are precisely those matching covered graphs that are free of nontrivial tight cuts; whereas braces and solid bricks are precisely those matching covered graphs that are free of nontrivial separating cuts.

The smallest simple bipartite matching covered graphs are the braces $K_2$ and $C_4$, whereas the smallest nonbipartite ones are the solid brick $K_4$ and the nonsolid brick $\overline{C_6}$. Every matching covered graph of order at least four is 2-connected. It is easy to see that if a matching covered graph of order six or more has a 2-vertex-cut then it has a nontrivial tight cut. This proves the following fact.

Proposition 1.17 Every brace of order six or more, and every brick, is 3-connected. \[\square\]

Using the fact that, in a bipartite matching covered graph, every tight cut is a barrier cut (see Figure 1(b)), one may obtain the following characterizations of braces.

Proposition 1.18 For a bipartite matching covered graph $H[A, B]$, the following statements are equivalent:

(i) $H$ is a brace.
(ii) \(|N_H(S)| \geq |S| + 2\) for every nonempty subset \(S\) of \(A\) such that \(|S| < |A| - 1\).

(iii) \(H - \{a_1, a_2, b_1, b_2\}\) is matchable for any four distinct vertices \(a_1, a_2 \in A\) and \(b_1, b_2 \in B\). \(\square\)

The first statement of the next corollary follows immediately from Corollary 1.5 and Proposition 1.17, whereas the second statement follows from Propositions 1.2 and 1.18.

**Corollary 1.19** For any brace \(H\), the following statements hold:

(i) \(\varepsilon(H) \in \{1, 2\}\), and \(\varepsilon(H) = 2\) if and only if \(H\) is \(C_4\) (up to multiple edges) and \(H\) has an even 2-cut.

(ii) If \(|V(H)| \geq 6\) then each edge is removable. \(\square\)

Lovász [5] proved that if \(e\) and \(f\) are distinct mutually dependent edges of a brick \(G\), then \(H := G - e - f\) is a (connected) bipartite matchable graph, both ends of \(e\) lie in one color class of \(H\) and both ends of \(f\) lie in the other color class; consequently, \(\{e, f\}\) is the complement of a cut of \(G\). (The reader may verify this for each of the doubleton equivalence classes shown in Figure 2.) Using this fact and Proposition 1.17 one may easily deduce the following; see [2, Lemma 2.4].

**Corollary 1.20** In any brick \(G\), each equivalence class is either a singleton or a doubleton. Consequently, \(\varepsilon(G) \in \{1, 2\}\). \(\square\)

Unlike the case of braces, the existence (and distribution) of removable edges is much harder to explain in the case of bricks. It was shown by Lovász [6] that every brick, distinct from \(K_4\) and the triangular prism \(C_6\), has a removable edge; the brick shown in Figure 2(b) has a unique removable edge.

### 1.4 Tight cut decomposition, and the invariants \(b(G)\) and \(c_4(G)\)

Now, let \(G\) denote any matching covered graph. We may apply to \(G\) a recursive procedure, called the **tight cut decomposition procedure**, to output a list of bricks and braces. If \(G\) is free of nontrivial tight cuts, then this list comprises \(G\). Otherwise, we choose a nontrivial tight cut \(C\) and obtain the two \(C\)-contractions (of \(G\), say \(G_1\) and \(G_2\); these are smaller matching covered graphs; we now apply the tight cut decomposition procedure recursively to each of \(G_1\) and \(G_2\), and then combine the resulting output lists into a single list — that is the output of an application of the tight cut decomposition procedure to \(G\).

Lovász [5] proved the following remarkable property of the tight cut decomposition procedure, and used it as one of several ingredients in his characterization of the matching lattice.

**Theorem 1.21** Any two applications of the tight cut decomposition procedure to a matching covered graph yield the same list of bricks and braces (up to multiple edges).
In light of this, we may now define a couple of useful invariants of a matching covered graph \( G \). As usual, we let \( b(G) \) denote the number of bricks yielded by any tight cut decomposition of \( G \). (This invariant plays a key role in the theory of matching covered graphs; for instance, it appears in the formula for the dimension of the perfect matching polytope.) The following is an immediate consequence of Lemma 1.15.

**Proposition 1.22** Let \( G \) denote a matching covered graph. Given any tight cut \( C \) of \( G \), the graph \( G \) is bipartite if and only if both \( C \)-contractions of \( G \) are bipartite. Consequently, \( G \) is bipartite if and only if \( b(G) = 0 \). \( \square \)

Now, we prove an easy lemma that will be useful to us in Section 4.1.

**Lemma 1.23** Let \( G \) be a matching covered graph that is neither a brick nor a brace. Then:

(i) either \( G \) has a nontrivial tight cut \( C \) such that both \( C \)-contractions are nonbipartite,

(ii) or \( G \) has a nontrivial tight cut \( D \) such that one of the \( D \)-contractions is a brace.

**Proof**: Let \( G \) be a matching covered graph that is neither a brick nor a brace; whence \( G \) has a nontrivial tight cut, say \( C := \partial(X) \). Let \( G_1 := G/X \) and \( G_2 := G/X \) denote the two \( C \)-contractions of \( G \). If \( G_1 \) and \( G_2 \) are both nonbipartite then there is nothing to prove.

Now suppose that one of \( G_1 \) and \( G_2 \) is bipartite; adjust notation so that \( G_1 \) is bipartite. If \( G_1 \) is a brace then there is nothing to prove. Now suppose that \( G_1 \) is not a brace; consequently, \( G_1 \) has nontrivial tight cut(s). We may choose a shore \( Y \) of a nontrivial tight cut (in \( G_1 \)) so that (i) \( \emptyset \notin Y \) and (ii) no proper subset of \( Y \) is a shore of a nontrivial tight cut (in \( G_1 \)). Thus \( D := \partial(Y) \) is a nontrivial tight cut of \( G_1 \), and also of \( G \). By Lemma 1.15 the \( D \)-contraction \( H := G/Y \) is a bipartite matching covered graph; by our choice of \( Y \), the graph \( H \) is free of nontrivial tight cuts. Thus \( H \) is a brace. This completes the proof. \( \square \)

For reasons that will be evident later, we let \( c_4(G) \) denote the number of braces isomorphic to \( C_4 \) (up to multiple edges) yielded by any tight cut decomposition of \( G \). The following proposition will also be useful to us in Section 4.1.

**Proposition 1.24** Let \( G \) denote a matching covered graph of order four or more. Then \( G \) has an even 2-cut if and only if the following holds: there exists an application of the tight cut decomposition procedure to \( G \) that yields a brace \( J \) that is \( C_4 \) (up to multiple edges) and has an even 2-cut. Consequently, if \( G \) is free of even 2-cuts then \( c_4(G) = 0 \).

**Proof**: Let \( G \) denote a matching covered graph of order four or more.

First suppose that \( G \) has an even 2-cut, say \( \partial(X) := \{e, f\} \). Since \( G \) is 2-connected, \( e \) and \( f \) are nonadjacent; furthermore, \( e \nleftrightarrow f \). We let \( e := v\overline{v} \) so that \( v \in X \) and \( \overline{v} \in \overline{X} \). Observe that \( C := \partial(X - v) \) and \( D := \partial(X - \overline{v}) \) are laminar tight cuts of \( G \). Let \( J \) denote the graph obtained by first contracting the shore \( X - v \) (of \( C \)) and then contracting the shore \( \overline{X} - \overline{v} \) (of \( D \)); that is, \( J := (\overline{G/(X - v)})/(\overline{X} - \overline{v}) \). Observe that \( J \) is indeed the brace \( C_4 \) and that \( \{e, f\} \) is an even 2-cut of \( J \). This proves the forward implication.
Now suppose that there exists an application of the tight cut decomposition procedure to $G$ that yields a brace $J$ that is $C_4$ (up to multiple edges) and has an even 2-cut, say $F$. By repeatedly invoking Lemma 1.12 we infer that $F$ is an even 2-cut of $G$ as well. This completes the proof of Proposition 1.24.

One may conveniently define the less well-known separating cut decomposition procedure by replacing each occurrence of ‘tight cut’ by ‘separating cut’ in the first paragraph of this section (1.4). It follows from the discussion in Section 1.3 that the output of any application of the separating cut decomposition procedure (to a matching covered graph) is a list of braces and solid bricks. However, unlike the tight cut decomposition procedure, two distinct applications of the separating cut decomposition procedure to a matching covered graph need not yield the same list of braces and solid bricks; in fact, they may even yield lists of different cardinalities.

1.5 Our results

This article is inspired by the following basic question.

**Question 1.25** If a matching covered graph $G$ is obtained by splicing (or by tight splicing) two smaller matching covered graphs, say $G_1$ and $G_2$, then how is $\mathcal{E}_G$ related to $\mathcal{E}_{G_1}$ and to $\mathcal{E}_{G_2}$ (and vice versa)?

In Section 2 we answer the above question with respect to the splicing operation (or equivalently, with respect to separating cuts). In Section 3 we answer the above question with respect to the tight splicing operation (or equivalently, with respect to tight cuts). In Section 4 we provide two applications of our findings.

In Section 4.1 we establish tight upper bounds on $\varepsilon(G)$; in particular, we prove that for any matching covered graph $G$: if $G$ is bipartite then $\varepsilon(G) \leq 1 + c_4(G)$; whereas if $G$ is nonbipartite then $\varepsilon(G) \leq 2 \cdot b(G) + c_4(G)$.

In Section 4.2 we describe a procedure to construct matching covered graphs that have arbitrarily high $\kappa(G)$ and $\varepsilon(G)$ simultaneously (where $\kappa(G)$ denotes the vertex-connectivity of $G$). This affirmatively answers a recent question of He, Wei, Ye and Zhai [4].

2 Equivalence classes and separating cuts

Throughout this section, we let $C := \partial(X)$ denote a nontrivial separating cut of a matching covered graph $G$, and we let $G_1 := G/X \rightarrow x$ and $G_2 := G/X \rightarrow x$ denote its $C$-contractions. Equivalently, $G$ is obtained by splicing the two smaller matching covered graphs $G_1$ and $G_2$. The following is an immediate consequence of Lemma 1.14.

**Lemma 2.1** If $e \xrightarrow{G} f$ then $|\{e, f\} \cap C| \in \{0, 1\}$. Consequently, each equivalence class of $G$ meets $C$ in at most one edge.
By Lemma \[1.13\] for \(i \in \{1, 2\}\), each perfect matching of \(G_i\) may be extended to a perfect matching of \(G\); equivalently, each perfect matching of \(G_i\) may be viewed as a restriction of some perfect matching of \(G\) to the set \(E(G_i)\). Using this, one may infer the following.

**Lemma 2.2** For \(i \in \{1, 2\}\), for any two edges \(e, f \in E(G_i)\): if \(e \xrightarrow{G_i} f\) then \(e \xrightarrow{G} f\). \(\square\)

Lemma 2.2 implies the first part of the following proposition, whereas Lemma 2.1 implies the second part.

**Proposition 2.3** For each \(F \in \mathcal{E}_G\) and for \(i \in \{1, 2\}\), the set \(F_i := F \cap E(G_i)\) is a (not necessarily proper) subset of some member of \(\mathcal{E}_{G_i}\). Furthermore, \(|F| = |F_1| + |F_2| - |F \cap C|\) where \(|F \cap C| \in \{0, 1\}\). \(\square\)

Consequently, the function \(\varepsilon(G)\) satisfies the following subadditivity property across separating cuts.

**Corollary 2.4** Let \(C\) denote a separating cut of a matching covered graph \(G\), and let \(G_1\) and \(G_2\) denote its \(C\)-contractions. Then \(\varepsilon(G) \leq \varepsilon(G_1) + \varepsilon(G_2)\). \(\square\)

## 3 Equivalence classes and tight cuts

Throughout this section, we let \(C := \partial(X)\) denote a nontrivial tight cut of a matching covered graph \(G\), and we let \(G_1 := G/X \rightarrow \bar{x}\) and \(G_2 := G/X \rightarrow x\) denote its \(C\)-contractions. Using Lemma 1.16 we have the following stronger conclusion (in comparison to Lemma 2.2).

**Lemma 3.1** For \(i \in \{1, 2\}\), for any two edges \(e, f \in E(G_i)\), \(e \xrightarrow{G_i} f\) if and only if \(e \xrightarrow{G} f\). \(\square\)

This yields the following two consequences; the former is a strengthening of Proposition 2.3 that is applicable in the case of tight cuts; the latter may be viewed as its converse.

**Proposition 3.2** For each \(F \in \mathcal{E}_G\) and for \(i \in \{1, 2\}\), if \(F_i := F \cap E(G_i)\) is nonempty then \(F_i\) is a member of \(\mathcal{E}_{G_i}\). Furthermore, \(|F| = |F_1| + |F_2| - |F \cap C|\) where \(|F \cap C| \in \{0, 1\}\). \(\square\)

**Proposition 3.3** For \(i \in \{1, 2\}\), let \(F_i \in \mathcal{E}_{G_i}\). Then, for \(i \in \{1, 2\}\), \(F_i\) is a (not necessarily proper) subset of some member of \(\mathcal{E}_G\). Furthermore, if \(F_1 \cap F_2\) is nonempty then (i) \(F_1\) and \(F_2\) meet in precisely one edge that lies in the cut \(C\), (ii) \(F := F_1 \cup F_2\) is a member of \(\mathcal{E}_G\), and (iii) \(|F| = |F_1| + |F_2| - 1\). \(\square\)

The following provides an alternative understanding of the second part of the above proposition. Let \(e\) denote any edge of the tight cut \(C\). For \(i \in \{1, 2\}\), let \(F_i\) denote the equivalence class of \(G_i\) that contains \(e\); note that \(F_i \cap C \in \{e\}\) and that \(F_i\) may be a singleton equivalence class. Then \(F := F_1 \cup F_2\) is an equivalence class of \(G\) and \(F \cap C \in \{e\}\). In other words, the equivalence class \(F\) (of \(G\)) is obtained by “merging” an equivalence class of \(G_1\) with an equivalence class of \(G_2\). In fact, this is how each equivalence class of \(G\), that meets the cut \(C\), is formed. This raises the following question.
Question 3.4 Is it possible for an equivalence class \( F_1 \) of \( G_1 \) to merge (i.e., union) with an equivalence class \( F_2 \) of \( G_2 \) to form an equivalence class \( F \) of \( G \) that does not meet the cut \( C \)?

In what follows, we shall answer the above question in the affirmative; in fact, we will obtain a complete characterization of the circumstances under which this phenomenon (of merging) occurs. Observe that it suffices to answer the following question. Under what circumstances do two edges, one in \( E(G_1) - C \) and another in \( E(G_2) - C \), become mutually dependent in the graph \( G \)?

Henceforth, for \( i \in \{1, 2\} \), we let \( f_i \) denote any edge in \( E(G_i) - C \), and we let \( C_i \) denote the (nonempty) subset of \( C \) that comprises edges that participate in some perfect matching of \( G_i \) with \( f_i \); in other words, \( C_i := \{ e \in C : \{ e, f_i \} \text{ extends to a perfect matching of } G_i \} \). Using these definitions, the reader may now verify the following.

Lemma 3.5 The following are equivalent:

(i) \( f_1 \xrightarrow{G_1} f_2 \).

(ii) \( e \xrightarrow{G_2} f_2 \) for each \( e \in C_1 \).

Observe that if each member of \( C_1 \) depends on \( f_2 \) in the graph \( G_2 \) then \( C_1 \subseteq C_2 \). Likewise, if each member of \( C_2 \) depends on \( f_1 \) in the graph \( G_1 \) then \( C_2 \subseteq C_1 \). This leads us to our next conclusion.

Corollary 3.6 The following are equivalent:

(i) \( f_1 \xleftarrow{G_1} f_2 \).

(ii) \( C_1 = C_2 \) and each of its members is inadmissible in \( G_i - f_i \) for each \( i \in \{1, 2\} \).

We may now easily extend this to two equivalence classes \( F_1 \) and \( F_2 \) (instead of two edges) in order to answer Question 3.4.

Corollary 3.7 For \( i \in \{1, 2\} \), let \( F_i \in \mathcal{E}_{G_i} \) such that \( F_i \cap C = \emptyset \), and let \( C_i \) denote the set \( \{ e \in C : F_i \cup \{ e \} \text{ extends to a perfect matching of } G_i \} \). Then \( F := F_1 \cup F_2 \) is an equivalence class of \( G \) if and only if the following hold:

(i) \( C_1 = C_2 \) and this set has cardinality two or more.

(ii) For \( i \in \{1, 2\} \), each member of \( C_i \) is inadmissible in the graph \( G_i - F_i \).

The above corollary provides the key insight for constructing graphs with arbitrarily high \( \kappa(G) \) and \( \varepsilon(G) \) simultaneously — by means of the tight splicing operation — in Section 4.2. The following tool will come in handy in Section 4.1 — where we establish tight upper bounds on \( \varepsilon(G) \).

\[ \text{By symmetry: the analogous statement — obtained by interchanging all subscripts “1” and “2” — holds.} \]
Corollary 3.8 Assume that $G_1$ is a brace. Let $F \in \mathcal{E}_G$ such that $F \cap C = \emptyset$ and $F_i := F \cap E(G_i)$ is nonempty for each $i \in \{1, 2\}$. Then $G_1$ is $C_4$ (up to multiple edges), and $|F_1| = 1$.

Proof: For each $i \in \{1, 2\}$, $F_i \cap C = \emptyset$; by Proposition 3.2, $F_i \in \mathcal{E}_G$. We let $C_1$ denote the set $\{e \in C : F_i \cup \{e\} \text{ extends to a perfect matching of } G_1\}$. Since $F := F_1 \cup F_2$ is a member of $\mathcal{E}_G$, we may invoke Corollary 3.7. Each member of $C_1$ is inadmissible in the graph $G_1 - F_1$; consequently, each member of $F_1$ is a non-removable edge of the brace $G_1$. It follows from Corollary 1.19 that $|V(G_1)| = 4$; whence $G_1$ is $C_4$ (up to multiple edges). Since $F_1$ is an equivalence class of $G_1$ that does not meet the cut $C$, we infer that $F_1$ is a singleton equivalence class. Thus $|F_1| = 1$. $\square$

4 Applications

4.1 Upper bounding $\varepsilon(G)$

In this section, we will establish tight upper bounds on $\varepsilon(G)$ in terms of the invariants $b(G)$ and $c_4(G)$. We begin with the class of bipartite graphs (i.e., precisely those graphs for which $b(G) = 0$).

Proposition 4.1 For every bipartite matching covered graph $G$, the following inequality holds: $\varepsilon(G) \leq 1 + c_4(G)$.

Proof: Let $G$ denote any bipartite matching covered graph. We proceed by induction on the order of $G$. If $G$ is a brace then the desired inequality holds due to Corollary 1.5.

Now suppose that $G$ is not a brace. By Proposition 1.22 and Lemma 1.28, $G$ has a nontrivial tight cut $C := \partial(X)$ so that both $C$-contractions are bipartite matching covered graphs and one of them is a brace. Adjust notation so that $G_1 := G/X$ is a brace, and $G_2 := G/X$ is a bipartite matching covered graph. By the induction hypothesis, for each $i \in \{1, 2\}$, the inequality $\varepsilon(G_i) \leq 1 + c_4(G_i)$ holds. Also, $c_4(G) = c_4(G_1) + c_4(G_2)$.

We let $F \in \mathcal{E}_G$. Our goal is to deduce that $|F| \leq 1 + c_4(G)$. By Proposition 3.2 for each $i \in \{1, 2\}$, the set $F_i := F \cap E(G_i)$ is a member of $\mathcal{E}_G_i$. Thus $|F_i| \leq 1 + c_4(G_i)$. Furthermore, $|F| = |F_1| + |F_2| - |F \cap C|$ where $|F \cap C| \in \{0, 1\}$. Note that, if $F_1 = F$ or if $F_2 = F$, then the desired inequality holds immediately.

Now suppose that each of $F_1$ and $F_2$ is a proper subset of $F$. Note that, if $|F \cap C| = 1$, then $|F| = |F_1| + |F_2| - 1 \leq 1 + c_4(G_1) + 1 + c_4(G_2) - 1 = 1 + c_4(G)$; the desired inequality holds. Now suppose that $F \cap C = \emptyset$. Since $G_1$ is a brace, it follows from Corollary 3.8 that $G_1$ is $C_4$ (up to multiple edges) and $|F_1| = 1$; whence $|F| = |F_1| + |F_2| \leq 1 + 1 + c_4(G_2) = 1 + c_4(G_1) + c_4(G_2) = 1 + c_4(G)$. This completes the proof of Proposition 4.1 $\square$

The upper bound established in Proposition 4.1 is tight since $c_4(C_{2n}) = n - 1$ and $\varepsilon(C_{2n}) = n$, where $C_{2n}$ is the simple even cycle of order $2n$; one may easily construct other such examples. We now move on to the class of nonbipartite graphs.
Proposition 4.2 For every nonbipartite matching covered graph $G$, the following inequality holds: $\varepsilon(G) \leq 2 \cdot b(G) + c_4(G)$. Furthermore, if $G$ is free of even 2-cuts then $\varepsilon(G) \leq 2 \cdot b(G)$.

Proof: Let $G$ denote any nonbipartite matching covered graph. We proceed by induction on the order of $G$. If $G$ is a brick then the desired inequality holds due to Corollary 1.20. Now suppose that $G$ is not a brick; whence $G$ has nontrivial tight cut(s).

First consider the case in which $G$ has a nontrivial tight cut $C$ so that both $C$-contractions, say $G_1$ and $G_2$, are nonbipartite. By the induction hypothesis, $\varepsilon(G_i) \leq 2 \cdot b(G_i) + c_4(G_i)$ for each $i \in \{1, 2\}$. By subadditivity (Corollary 2.4): $\varepsilon(G) \leq \varepsilon(G_1) + \varepsilon(G_2) \leq 2 \cdot b(G_1) + c_4(G_1) + 2 \cdot b(G_2) + c_4(G_2) = 2 \cdot b(G) + c_4(G)$.

Now consider the case in which $G$ does not have a nontrivial tight cut $C$ such that both $C$-contractions are nonbipartite. By Proposition 1.22 and Lemma 1.23, $G$ has a nontrivial tight cut $D := \partial(X)$ so that one of the $D$-contractions is a brace, whereas the other is a nonbipartite matching covered graph. Adjust notation so that $G_1 := G/X$ is a brace, and $G_2 := G/X$ is nonbipartite. Note that $b(G) = b(G_2)$ and that $c_4(G) = c_4(G_1) + c_4(G_2)$. By the induction hypothesis, $\varepsilon(G_2) \leq 2 \cdot b(G_2) + c_4(G_2) = 2 \cdot b(G) + c_4(G_2)$. Also, since $G_1$ is bipartite, Proposition 4.1 implies that $\varepsilon(G_1) \leq 1 + c_4(G_1)$.

We let $F \in \mathcal{E}_G$. Our goal is to deduce that $|F| \leq 2 \cdot b(G) + c_4(G)$. By Proposition 3.2, for each $i \in \{1, 2\}$, the set $F_i := F \cap E(G_i)$ is a member of $\mathcal{E}_G$. Also, $|F| = |F_1| + |F_2| - |F \cap C|$ where $|F \cap C| \in \{0, 1\}$. Note that if $F = F_1$ then $|F| \leq 1 + c_4(G_1) < 2 \cdot b(G) + c_4(G)$, and if $F = F_2$ then $|F| \leq 2 \cdot b(G) + c_4(G_2) \leq 2 \cdot b(G) + c_4(G)$.

Now suppose that each of $F_1$ and $F_2$ is a proper subset of $F$. Note that, if $|F \cap C| = 1$, then $|F| = |F_1| + |F_2| - 1 \leq 1 + c_4(G_1) + 2 \cdot b(G) + c_4(G_2) - 1 = 2 \cdot b(G) + c_4(G)$; the desired inequality holds. Now suppose that $F \cap C = \emptyset$. Since $G_1$ is a brace, it follows from Corollary 3.8 that $G_1$ is $C_4$ (up to multiple edges) and $|F_1| = 1$; whence $|F| = |F_1| + |F_2| \leq 1 + 2 \cdot b(G) + c_4(G_2) = c_4(G_1) + 2 \cdot b(G) + c_4(G_2) = 2 \cdot b(G) + c_4(G)$. This completes the proof of the first part of Proposition 4.2, the second part follows by invoking Proposition 1.24. □

One may view the above proposition as a generalization of Corollary 1.20. The upper bound established in Proposition 4.2 is tight for the simple reason that there exist (infinitely many) bricks with doubleton equivalence classes. However, it would be interesting to find tight examples (perhaps infinite families) with arbitrarily high $b(G)$; the technique used in Section 4.2 might be helpful in constructing such graphs.

4.2 Building graphs with arbitrarily high $\kappa(G)$ and $\varepsilon(G)$

In this section, we prove the following that affirmatively answers a recent question of He, Wei, Ye and Zhai [1]. The reader may find it helpful to see Figure 3 for a demonstration of the construction provided in the proof.

Proposition 4.3 For any pair of positive integers $p$ and $q$, there exists a matching covered graph $G$ such that $\kappa(G) \geq p$ and $\varepsilon(G) \geq q$. 
Proof: Clearly, we may assume that \( p \geq 2 \) and \( q \geq 2 \). We begin by choosing a pair \((H[A, B], a)\) where \( H[A, B] \) is a \((p + 1)\)-connected brace\(^4\) and \( a \in A \) is a fixed vertex. Now we construct a simple bipartite \( G_0[A_0, B_0] \) as follows.

(i) We take \( q \) disjoint copies of the pair \((H[A, B], a)\): \((H_i[A_i, B_i], a_i)\), so that \( a_i \in A_i \), for each \( i \in \{1, 2, \ldots, q\} \). We let \( A_0 := \cup_{i=1}^q A_i \) and \( B_0 := \cup_{i=1}^q B_i \).

(ii) For each \( i \in \{1, 2, \ldots, q - 1\} \), we add \( p \) edges, each joining \( a_i \) with some vertex in \( B_{i+1} \), and we denote this set of edges by \( C_i \).

(iii) We add an edge \( f_0 \) joining \( a_q \) with some vertex in \( B_1 \).

We let \( C^* := C_1 \cup C_2 \cup \cdots \cup C_{q-1} \). The reader may easily verify the following.

4.3.1 The simple bipartite graph \( G_0[A_0, B_0] \) is matching covered\(^3\), each member of \( C^* \) is inadmissible in \( G_0 - f_0 \), and \( \{f_0\} \in \mathcal{E}_{G_0} \). \( \square \)

For \( i \in \{1, 2, \ldots, q - 1\} \), we construct a simple nonbipartite graph \( J_i \) as follows.

(i) We take a new copy of the pair \((H[A, B], a)\), say \((L_i[U_i, V_i], u_i)\), so that \( u_i \in U_i \).

(ii) We add \( p \) edges, each joining \( u_i \) with some vertex in \( U_i - u_i \), and we denote this set of edges by \( C_i' \).

(iii) We add an edge \( f_i \) that has both ends in \( V_i \).

The reader may easily verify the following.

4.3.2 For \( i \in \{1, 2, \ldots, q - 1\} \), the simple nonbipartite graph \( J_i \) is a brick\(^5\), each member of \( C_i' \) is inadmissible in \( J_i - f_i \), and \( \{f_i\} \in \mathcal{E}_{J_i} \). \( \square \)

Finally, for \( i \in \{1, 2, \ldots, q - 1\} \), we let \( G_i := (G_{i-1} \circ J_i)_{a_i, u_i, \pi_i} \) where \( \pi_i \) is any bijection between \( \partial G_{i-1}(a_i) \) and \( \partial J_i(u_i) \) that maps each member of \( C_i \) to some member of \( C_i' \), and we let \( D_i \) denote the corresponding splicing cut of \( G_i \).

By Lemma 1.10 and 1.11 all of the graphs \( G_1, G_2, \ldots, G_{q-1} \) are simple and matching covered; note that \( B_0 \) is a barrier in each of them. This proves the following.

4.3.3 For \( i \in \{1, 2, \ldots, q\} \), the simple graph \( G_i \) is matching covered and \( D_i \) is a barrier cut associated with the barrier \( B_0 \). \( \square \)

For each \( i \in \{0, 1, \ldots, q - 1\} \), we let \( F_i := \{f_0, f_1, \ldots, f_i\} \).

4.3.4 For \( i \in \{0, 1, 2, \ldots, q-1\} \), each member of \( C^* \) is inadmissible in \( G_i - F_i \), and \( F_i \in \mathcal{E}_{G_i} \).

\(^3\)For instance, one may choose \( H \) to be the complete bipartite graph \( K_{p+1, p+1} \).

\(^4\)Invoke Proposition 1.3.

\(^5\)Use Proposition 1.18 to show that \( J_i \) is matching covered. Adding edges cannot create “new” tight cuts.
Figure 3: Illustration for the proof of Proposition 4.3 for $p = 3$ and $q = 3$ (and $H := K_{4,4}$)

**Proof:** We proceed by induction on $i$. By 4.3.1, the desired conclusions hold when $i = 0$.

Now suppose that $1 \leq i \leq q - 1$, and that the desired conclusions hold for $i - 1$. By the induction hypothesis, each member of $C^*$ is inadmissible in $G_{i-1} - F_{i-1}$. That is, if $e \in C^*$ and $f \in F_{i-1}$, then $e \xrightarrow{G_{i-1}} f$; consequently, since $G_i$ is obtained by tight splicing $G_{i-1}$ and $J_i$, Lemma 3.1 implies that $e \xrightarrow{G_i} f$. In other words, each member of $C^*$ is inadmissible in $G_i - F_{i-1}$; whence each member of $C^*$ is inadmissible in $G_i - F_i$.

By the induction hypothesis, $F_{i-1} \in \mathcal{E}_{G_{i-1}}$ and each member of $C_i$ is inadmissible in
$G_{i-1} - F_{i-1}$. On the other hand, by \[4.3.2\] $\{f_i\} \in \mathcal{E}_{J_i}$ and each member of $C'_i$ is inadmissible in $J_i - f_i$. Recall that $G_i := (G_{i-1} \odot J_i)_{a_i, u_i, \pi}$ where $\pi$ is a bijection between $\partial G_{i-1}(a_i)$ and $\partial J_i(u_i)$ that maps each member of $C_i$ to some member of $C'_i$; also, by \[4.3.3\] the corresponding splicing cut is a tight cut; thus we invoke Corollary \[3.7\] to infer that $F_{i-1} \cup \{f_i\}$ is an equivalence class of $G_i$. In other words, $F_i \in \mathcal{E}_{G_i}$.

We let $G := G_{q-1}$. By \[4.3.4\], $F_{q-1} \in \mathcal{E}_{G}$. Thus $\varepsilon(G) \geq q$. It remains to show that $\kappa(G) \geq p$. We consider the following ordered partition of $V(G)$.

$$((V(H_1) - a_1), (V(L_1) - u_1), (V(H_2) - a_2), (V(L_2) - u_2), \ldots, (V(L_{q-1}) - u_{q-1}), V(H_q)).$$

Since the brace $H$ is $(p+1)$-connected, the subgraph (of $G$) induced by each part (of the above partition) is $p$-connected. By our construction of $G$, and by Lemma \[1.10\], there is a matching of cardinality $p$ (or more) joining any two consecutive parts of the above partition. Using Menger’s Theorem, we infer that $G$ is $p$-connected. Thus $\kappa(G) \geq p$.

This completes the proof of Proposition \[4.3\] \[\square\]

Thus we have shown that there exist highly-connected graphs with arbitrarily large equivalence classes; this is in stark contrast to removable classes as was demonstrated by Lovász and Plummer (see Theorem \[1.9\]).

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