The No Cloning Theorem versus the Second Law of Thermodynamics

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Asher Peres’ proof that a violation of the No Cloning Theorem would imply a violation of the Second Law of Thermodynamics is shown not to take into account the algorithmic-information’s contribution to the Thermodynamical Entropy of the semi-permeable membranes of Peres’ engine.

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I. INTRODUCTION

Two results have changed, in the last two decades, our way of looking at the Foundation of Quantum Mechanics: the No-Cloning Theorem (by Dieks, Wooters and Zurek [1], [2]) and the comprehension of the algorithmic-information’s contribution to the thermodynamical entropy in presence of Maxwell’s demons (by Landauer, Bennett and Zurek [3], [4], [5], [6], [7], [8]).

The No-cloning theorem, stating the impossibility of building a quantum gate able to clone two non-orthogonal states, would seem to have no connection with Quantum Thermodynamics; that this is not the case, anyway, is implied by its equivalence with the Theorem of Indistinguishability for nonorthogonal states lying at the heart of the irreducibility of Quantum Information Theory to the classical one.

In [1] (as well as in the 9th chapter of his wonderful book [1]) Asher Peres claims that the Theorem of Indistinguishability for nonorthogonal states is necessary in order of preserving the Second Law of Thermodynamics; his proof of this statement is based on the analysis of a cyclic thermodynamical engine in which some “magic” semi-permeable membranes, assumed ad absurdum to be able to distinguish nonorthogonal states, are used in a suitable way in order of lowering the Universe’s entropy.

As we will show, anyway, such a proof is not correct, since it doesn’t take into account the Landauer-Bennett-Zurek’s results on Maxwell’s demon, that imply that also the algorithmic-information of Peres’ semi-permeable membranes contribute to the thermodynamical entropy, preventing the Second Principle to be violated.

This consideration, already presented in the remark 7.3.10 of my PHD-thesis [10], is here extensively analyzed.

II. NO-CLONING THEOREM AND INDISTINGUISHABILITY OF NONORTHOGONAL STATES

Let us us consider a quantum gate \( \hat{U} \) with two input edges and two outputs edges such that there exist a normalized start state \( |s > \) and two distinct vectors \( |\psi_1 > \) and \( |\psi_2 > \) such that:

\[
\hat{U} |\psi_1 > |s > = |\psi_1 > |\psi_1 > 
\]

and:

\[
\hat{U} |\psi_2 > |s > = |\psi_2 > |\psi_2 > 
\]

Taking the inner product of eq (2.1) and eq (2.2) one obtains the equation:

\[
< \psi_1 |\psi_2 >^2 = < \psi_1 |\psi_2 > 
\]

from which it follows that

\[
< \psi_1 |\psi_2 > = 0 
\]

The No-Cloning Theorem, stating the impossibility of a device able of cloning two nonorthogonal states, is then proved.

Let us now consider a different situation in which Alice codifies her answer to Bob’s marriage proposal sending him one of two possible states \( |\psi_1 > \) and \( |\psi_2 > \) (with the previously concorded rule that \( |\psi_1 > \) means yes while \( |\psi_2 > \) means no). To know Alice’s answer, Bob makes on the received state the measurement described by the positive-operator-valued-measure \( \{ M_j \}^2_{j=1} \) with outcome \( j \). Depending on the outcome on the measurement Bob tries to guess what the index \( i \) was using some rule \( i = f(j) \), where \( f(\cdot) \) represents the rule he uses to make the guess.

We will now prove the Theorem of Indistinguishability of Nonorthogonal States stating that if \( |\psi_1 > \) and \( |\psi_2 > \) are nonorthogonal it follows that Bob cannot infer if Alice has accepted his marriage proposal.

Introduced the operators:

\[
\hat{E}_i := \sum_{j; f(j) = i} \hat{M}_j \hat{M}_j^\dagger 
\]

the condition that Bob can infer Alice’s answer may be formalized by the constraint:

\[
< \psi_i |\hat{E}_i |\psi_i > = 1 \quad i = 1, 2 
\]
Since $\sum_1 \hat{E}_i = 1$ it follows that $\sum_1 <\psi_1|\hat{E}_i|\psi_1> = 1$; assuming the distinguishability condition of eq. (2.4) it follows that $<\psi_1|\hat{E}_2|\psi_1> = 0$ and thus:

$$\sqrt{\hat{E}_2}\psi_1 = 0 \quad (2.7)$$

Let us now suppose ad absurdum that $<\psi_1|\psi_2> \neq 0$.

It follows that there exist a state $|\psi_\perp>$ and two complex numbers $\alpha$ and $\beta$ such that:

$$|\psi_2> = \alpha |\psi_1> + \beta |\psi_\perp> \quad (2.8)$$

$$|\alpha|^2 + |\beta|^2 = 1 \quad (2.9)$$

$$|\beta| < 1 \quad (2.10)$$

$$<\psi_1|\psi_\perp> = 0 \quad (2.11)$$

Consequentlly:

$$\sqrt{\hat{E}_2}|\psi_2> = \beta \sqrt{\hat{E}_2}|\psi_\perp> \quad (2.12)$$

and hence:

$$<\psi_2|\hat{E}_2|\psi_2> = |\beta|^2 <\psi_\perp|\hat{E}_i|\psi_\perp> \leq |\beta|^2 \sum_i <\psi_\perp|\hat{E}_i|\psi_\perp> = |\beta|^2 < 1 \quad (2.13)$$

which contradicts the absurdum hypothesis.

Beside their apparent diversity, the No-Cloning Theorem and the Theorem of Indistinguishability of Nonorthogonal States may be easily proved to be equivalent:

if Bob was able to distinguish the two nonorthogonal states $|\psi_1>$, $|\psi_2>$ Alice used to answer his marriage proposal, he could clone them by making the measurement distinguishing them and making at will multiple copies of the state Alice had given him.

Contrary, if the non-orthogonal states $|\psi_1>$, $|\psi_2>$ were clonable, Bob could easily distinguish them by cloning them repeatedly in order of obtaining the states $|\psi_1>^{\otimes n}$, $|\psi_2>^{\otimes n}$ whose inner product tends to zero when $n \to \infty$ and are, consequentially, asymptotically distinguishable by projective measurements.

### III. BENNETT'S THEOREM ON MAXWELL'S DEMONS IN CLASSICAL THERMODYNAMICS

Almost all the greatest physicists of the last two centuries has, at some point, fought against one of the deepest problems of Thermodynamics: Maxwell’s demon.

Let us introduce it with Maxwell’s own words:

"One of the best established facts in thermodynamics is that it is impossible in a system enclosed in an envelope which permits neither change of volume nor passage of heat, and in which both the temperature and the pressure are everywhere the same, to produce any inequality of temperature or of pressure without the expenditure of work. This is the second law of thermodynamics, and it is undoubtedly true as long as we can deal with bodies only in mass, and have no power of perceiving or handling the separate molecules of which they are made up. But if we conceive a being whose faculties are so sharpened that he can follow every molecule in its course, such a being, whose attributes are still as essentially finite as our own, would be able to do what is at present impossible to us. For we have seen that the molecules in a vessel full of air at uniform temperature are moving with velocities by no means uniform, though the mean velocity of any great number of them, arbitrary selected, is almost exactly uniform. Now let us suppose that such a vessel is divided in two portions, A and B, by a division in which there is a small hall, and that a being, who can see the individual molecules, opens and closes this hole so as to allow only the lower ones to pass from B to A. He will see, thus, without expenditure of work, raise the temperature of B and lower that of A, in contradiction with the second law of thermodynamics"; cited from the last but one section "Limitation of the Second Law of Thermodynamics" of the 22th chapter of [11].

In the 220 years after the publication of Maxwell’s book an enormous literature tried to exorcize Maxwell’s demon in different ways; an historical review may be found in the first chapter "Overview" as well as in the "Chronological Bibliography with Annotations and Selected Quotations" of the wonderful books edited by Harvey S. Leff and Andrew F. Rex [12], [13].

All these exorcisms were based on the idea that, to accomplish his task, Maxwell’s demon necessarily causes a thermodynamical-entropy’s raising causing the Second Law to be preserved:

they anyway strongly differed in identifying the element of the demon’s dynamical evolution which is necessarily thermodynamically-irreversible:

coming to recent times, most of the Scientific Community (not only of Physics: cfr. e.g. the third chapter "Maxwell’s Demons" of [14]) strongly believed in Leon Brillouin’s exorcism [17], identifying such an element in the demon’s information-acquisition’s process.

When anyone thought that the "The-end" script had at least appeared to conclude "The Exorcist" movie, Charles H. Bennett showed in 1982 [1], [5], [14], basing on the previous work by Rolf Landauer on the Thermodynamics of Computation [8], that:

1. Maxwell’s Demon was still alive owing to the nullity of Brillouin’s exorcism: the demon’s acquisition process may be done in a completely thermodynamically-reversible way

2. the necessarily-thermodinamically-irreversible element is instead demon’s information-erasure’s process

The corner-stone of the Thermodynamics of Computation is Landauer’s Principle:
in this framework an arbitrary function is called logically-reversible if it is injective while it is called thermodynamically-reversible if there exist a physical device computing it in a thermodynamically-reversible way; Landauer’s Principle states the equivalence of logical-reversibility and thermodynamical-reversibility.

An immediate consequence of Landauer’s Principle is that the erasure of information is thermodynamically-irreversible:

to prove it, it is sufficient to observe that to any logically-irreversible function one may associate a logically-reversible function different from the original one in that the output is augmented by some of the input’s information (usually called garbage); assuming ad absurdum that garbage’s erasure is thermodynamically-reversible, it would then follow that the original function would be thermodynamically-reversible too, contradicting the hypothesis.

We can at last introduce Bennett’s exorcism of Maxwell’s demon: conceptually Maxwell’s demon may be formalized as a computer that:

1. gets the input \((s, v)\) from a device measuring both the side \(s\) from which the molecule arrives and its velocity

2. computes a certain semaphore-function \((s, v) \mapsto (s, v)\) giving as output a \(1\) if the molecule must be left to pass while gives as output a \(0\) if the molecule must be stopped:

   specifically, the semaphore-function may be defined through the following Mathematica expression \([17]\):

   \[
   p[s_\cdot, v_\cdot] := If[s = Left, \begin{cases} If[v \leq v_T, 0, 1], \\ If[v > v_T, 0, 1] \end{cases} \] (3.1)

   where \(v_T\) is a fixed threshold velocity

3. gives the output \(p[(s,v)]\) to a suitable device that operates on the molecule in the specified way

Both the first and the third phases of this process, taking into account also the involved devices, may be made in a thermodynamically-reversible way.

As to the second step, anyway, let us observe that the semaphore-function \(p\) is logically-irreversible and hence, by Landauer’s Principle, also thermodynamically-irreversible.

As above specified, such a thermodynamically-irreversibility may be avoided conserving the garbage; let us, precisely, suppose, that the demon-computer computes the thermodynamically-reversibly-computable function \(\tilde{p}\):

\[
\tilde{p}[s_\cdot, v_\cdot] := \begin{cases} If[s = Left, \begin{cases} If[v \leq v_T, (s, v), 0], \\ If[v > v_T, (s, v), 1] \end{cases} \end{cases} \] (3.2)

Let us suppose to make the demon-computer operate \(n\) times on \(n\) different molecules.

When \(n\) grows the demon, with no expenditure of work, raises the temperature of \(B\) and lowers that of \(A\).

But let us now analyze more carefully Clausius’s formulation of the Second Principle: it states that no thermodynamical transformation is possible that has as its only result the passage of heat from a body at lower temperature to a body at higher temperature.

In the above process the passage of heat from \(A\) to \(B\) is not the only result: another result is the storage in the demon-computer’s memory of the \(n\)-ple of inputs \((\{s_1, v_1\}, \ldots , \{s_n, v_n\})\).

To make the passage of heat from \(A\) to \(B\) to become the only result of the process we could think that the demon, at the end, erases his memory; but this, as we have seen, cannot be done in a thermodynamically-reversible way: such an erasure causes an increase of entropy that may be proved to be greater than or equal to the entropy-decrease produced by the passage of heat from \(A\) to \(B\).

Bennett’s exorcism of Maxwell’s demon, has, anyway, a far reaching consequence; supposed that the gas is described by the thermodynamical ensemble \((X, P)\), let us introduce the Bennett’s entropy of \(P\):

\[
S_{\text{Bennett}}(P) := H(P) + I(P) \quad (3.3)
\]

where:

\[
H(P) := - \log_2 P > (3.4)
\]

is Shannon’s entropy of the distribution \(P\) (i.e. its Gibbs’ entropy in thermodynamical language), while:

\[
I(P) := \begin{cases} \min\{|x| : U(x) = P\} & \text{if } \exists x : U(x) = P, \\ +\infty & \text{otherwise} \end{cases} \quad (3.5)
\]

is its prefix-algorithmic-information (denoted simply as algorithmic information form here and beyond), i.e. the length of the shortest program computing it on the fixed Chaitin universal computer \(U\) (demanding for details we recall that a Chaitin universal computer is a universal computer with prefix-free halting set and the property that, up to an input-independent additive constant, it describes algorithmically any output in a way more concise that any other computer).

Bennett’s exorcism of Maxwell’s Demon implies Bennett’s Theorem stating that the thermodynamical entropy of the ensemble \((X, P)\) is equal to to its Bennett’s entropy:

\[
S_{\text{therm}}(P) = S_{\text{Bennett}}(P) \neq H(P) \quad (3.6)
\]

To understand why Bennett’s exorcism implies equation (3.6) let us consider some example:

let us suppose, for simplicity, that the initial equilibrium probability distribution is such that the molecules have one of only two possible velocities \(v_L\) and \(v_H\), respectively lower and higher than the threshold velocity
$v_T$

$$v_L < v_T < v_H \quad (3.7)$$

Let us start from the case in which:

$$P(v = v_L) = P(v = v_T) = \frac{1}{2} \quad (3.8)$$

Supposing that the demon memorizes in the cbit $x_n$ the value of the semaphore function of the $n^{th}$ molecule that he observes, we have that, at the beginning, the string $\vec{x}_n := x_1 \cdots x_n$ seems to increase its length in an algorithmically-random way, i.e. in a way such that, informally speaking $^1$:

$$I(\vec{x}_n) \approx n \quad (3.9)$$

As the distribution of the molecules becomes more and more disuniform, with the slow molecules accumulating on the left side and the speed molecules accumulating on the right side (i.e. when the temperature’s difference among the two sides arises), the probability distribution of $x_n$ becomes more and more unfair preferring for $x_n = 0$, so that the string $\vec{x}_n$ increases its deviation from Borel-normality.

Such an increasing regularity of $x_n$ corresponds to the fact that its algorithmic-information becomes to increase more and more slowly.

Reasoning in terms of a finite number $N$ of molecules $^2$, after a certain number $n_{ord}$ of measurements made by the demon, the system reaches the state in which all the slow molecules are on the left side of the vessel, while all the speed molecules are on the right side; from that point further the demon stops every molecule so that:

$$x_n = 0 \; \forall n > n_{ord} \quad (3.10)$$

At this point, in which the demon has completed its task of lowering the probabilistic information of the gas so that such a probabilistic information ceases to decrease, the algorithmic information of the string $\vec{x}_n$ ceases to increase:

$$I(\vec{x}_n) = I(\vec{x}_{n_{ord}}) \; \forall n > n_{ord} \quad (3.11)$$

The whole process may, consequentially, be seen as a transfer of information from the gas to demon’s memory in which an amount of gas’ probabilistic information is transferred to the demon as algorithmic information.

Let us now consider the case in which the initial distribution of molecules’ velocities is unfair:

$$P(v = v_L) = 1 - \alpha \quad (3.12)$$
$$P(v = v_H) = \alpha \quad (3.13)$$

where $\alpha \neq \frac{1}{2}$.

The qualitative behaviour of the process is analogous to the previously discussed one although the greater is the difference $|\alpha - \frac{1}{2}|$ the littler is the amount of gas’ probabilistic information converted into algorithmic information.

Now, as we have already stressed, the involved thermodynamical process doesn’t violate the Second Principle of Thermodynamics since the passage of heat from the low-temperatures-source A to the high-temperatures-source B is not the only result: an other result is the memorization in demon’s memory of the sequence $\{x_n\}$.

Such a memorization, that as we have seen is a transfer of information from the gas to the demon as well as a transfer of a portion of the overall information of the Universe from probabilistic to algorithmic form, corresponds to an accumulation in algorithmic form of useful-energy (i.e. of energy that may be transformed in work), i.e. in an accumulation of thermodynamical-entropy in algorithmic-form that has to be counted in the Universe’s overall thermodynamical balance preventing, indeed, the Second Principle to be violated.

IV. THERMODYNAMICAL ENTROPY, STATISTICAL MECHANICS AND THE KOLMOGOROVIAN FOUNDATION OF INFORMATION THEORY

Despite Richard Feynman’s strongly authoritative acclamation of the Landauer-Bennett’s results on Maxwell’s Demons (cfr. the section 5.1.1 ”Maxwell’s Demon and the Thermodynamics of Measurement” of [14] and its appreciation by Nobel prize awarded theoretical physicists such as Murray Gell-Mann (cfr. e.g. the 15th chapter ”Time’s arrows” of [15]), these, and in particular Bennett’s Theorem, are far from having been accepted by the Theoretical Physics’ community.

The objections (implicitly or explicitly) moved to Bennett’s Theorem are essentially the following:

1. the Maxwell-demon’s issue simply shows that the Second Law has a statistical validity

2. the action of Maxwell’s demon moves the system out of thermodynamical equilibrium: consequentially the thermodynamical entropy ceases to be defined

3. the interdisciplinary attitude of Algorithmic Physics is not necessary to understand Thermodynamics

$^1$ the sign $\approx$ of ”rough-equality” may be formalized more precisely by the condition that the l.r.h. grows at least as quicker than the r.h.s. so that eq. (3.11) may be rigorously considered as a different way of stating Chaitin’s condition $\lim_{n \to \infty} I(\vec{x}_n) - n = +\infty$

$^2$ This must be considered only as an artifice to clarify the argument, since one has to remember that the thermodynamical limit $N \to \infty$ has to be taken at the end; the qualitative behaviour here described becomes, in this limit, an asymptotic one
The first objection, i.e. the claim that the Second Law has only a statistical validity, is the kovari as far as the Mathematical-Physics’ literature is concerned.

Such a claim, anyway, false: though Statistical Mechanics (historically pioneered by Maxwell, Thomson and Boltzmann: cfr. e.g. the chap.3-7 of [20]) allows to obtain the Equilibrium Thermodynamics of a macroscopic thermodynamical system deriv- ing it from a probabilistic description of its underlying microscopic dynamics, Thermodynamics is a perfectly self-consistent physical theory predicting the value and dynamical evolution of all the thermodynamical observables of thermodynamical systems, (generally not in ther- modynamical equilibrium), with certainty. This occurs, in particular, as to the Second Law of Thermodynamics stating that in any thermodynamical cycle of any iso- lated thermodynamical system (generally not in ther- modynamical equilibrium) one has with certainty that:

$$\int \frac{\delta Q}{T} \leq 0$$

(4.1)

where $\delta Q$ is heat’s amount absorbed by the system while $T$ is its temperature.

The source of the erroneous claim that Maxwell- demon’s issue simply shows the statistical validity of the Second Law may be understood in terms of the following words by Joel Lebowitz:

"The various ensembles commonly used in statistical mechanics are to be thought of as nothing more than mathematical tools for describing behaviour which is practically the same for "almost all" individual macroscopic systems in the ensemble. While these tools can be very useful and some theorems that are proven about them are very beautiful they must not be confused with the real thing going in a single system. To do that is to commit the scientific equivalent of idolatry, i.e. substituting representative images for reality"; cited from the Introduction of [27].

Such an idolatric attitude for which a thermodynamical system is confused with its modelling through Statistical Mechanics is, indeed, a typical mental attitude of some Mathematical Physicists that had often induced even authoritative scientists to assert trivially erroneous statements of Thermodynamics; this is the case, for example, of Giovanni Gallavotti’s analysis of brownian mo- tors that, misunderstanding the celebrated analysis by Richard Feynman of a "ratchet and pawl heat engine":

"Let us try to invent a device which will violate the Second Law of Thermodynamics, that is a gadget which will generate work from a heat reservoir with everything at the same temperature. Let us say we have a box of gas at a certain temperature, and inside there is a an axes with vanes in it, etc. Because of the bombardments of gas molecules on the vane, the vane oscillates and jiggles. All we have to do is to hook into the other end of the axle a wheel which can turn only one way- the ratchet and pawl. Then when the shaft tries to jiggle one way, it will not turn, and when it jiggles the other, it will turn. Then the wheel will slowly turn, and perhaps we might even tie a flea onto a string hanging from a drum on the shaft, and lift the flea! Now let us ask if this is possible. According to Carnot’s hypothesis, it is impossible. But is we just look at it we see, prima facie, that it seems quite possible. So we must look more closely"; cited from the chapter 46 of [24], streights Jeann Perrin’s restatement of the idolatric claim that the Second Law has only a statistical validity: "But is must be remembered ... that the brownian movement, which is a fact beyond dispute, provides an experimental proof (deduced from the molecular agitation hypothesis) by which of means Maxwell, Gibbs and Boltzmann robbed Carnot’s Principle of its claim to rank as an absolute truth and reduced it to the mere expression of a very high probability"; cited from the 51th section "The brownian movement and Carnot’s Principle" of [21], claiming that:

"It is important to keep in mind that here we are somewhat stretching the validity of thermodynamics laws: the above machines are very idealized objects, like the daemon. They cannot be realized in any practical way; one can arrange them to perform one cycle, ..., but what one needs to violate the second law is the possibility of performing as many energy producing cycles as required.Otherwise their existence "only" proves that the second law has only a statistical validity, a fact that had been well establishes with the work of Boltzmann. In fact an accurate analysis of the actual possibility of building walls semipermeable to colloids and of exhibiting macroscopic violation of the second principle runs into grave difficulties: it is not possible to realize a perpetual motion of the second kind by using the properties of Brownian motion. It is possible to obtain a single violation of Carnot’s law (or of a few of them) of the type described by Perrin, but as time elapses and the machine is left running, isolated and subject to physical laws with no daemon or other extraterrestrial being intervening (or performing work accounted for), the violations (i.e. the energy produced per cycle) vanish because the cycle will be necessarily performed as many times in one direction (apparently violating Carnot’s principle and producing work) as in the opposite direction (using it). This is explained in an analysis on Feynman, see [22] chapter 46, where the semi-permeable wall is replaced by a wheel with an anchor mechanism, a "ratchet and a pawl", allowing it to rotate only in one direction under the impulses communicated by the colloidal particle collisions with the valves of a second wheel rigidly bound to the same axis. Feynman’s analysis is really beautful, and remarkable as an example of how one can still say something interesting on perpetual motion. It also brings important
insight into the related so-called "reversibility paradox" (that microscopic dynamics generates an irreversible macroscopic world); cited from the section 8.1 "Brownian motion and Einstein's Theory" of [26].

i.e. that a brownian motor can violate Carnot's Law for a few cycles, a thing that, if it was true (that unfortunately this is not the case is shown, for example, in [25]), would have allowed Gallavotti to definitively resolve the energetic problem of the World saving it from the slavery of oil.

The second objection (implicitly or explicitly) moved to Bennett's Theorem, namely that since the action of Maxwell's demon moves the system out of thermodynamical equilibrium the thermodynamical entropy ceases to be defined, is based again on the idolatric attitude of making confusion between a thermodynamical system and its modelization through Statistical Mechanics denounced by Lebowitz:

the fact that there doesn't exist a universally accepted notion of entropy in Nonequilibrium Statistical Mechanics is consequently seen a synonymous of the false statement that the notion of thermodynamical entropy of a thermodynamical system not in equilibrium is not defined.

Such a confusion appears, for example, in the following passage by Gallavotti:

"One of the key notions in equilibrium statistical mechanics is that of entropy; its extension is surprisingly difficult, assuming that it really can be extended. In fact we expect that, in a system that reaches under forcing a stationary state, entropy is produced at a constant rate, so that there is no way of defining an entropy value for the system, except perhaps by saying that its entropy is \(-\infty\). Although one should keep in mind that there is no universally accepted notion of entropy in systems out of equilibrium, even when in a stationary state, we shall take the attitude that in a stationary state only the entropy creation rate is defined: the system entropy decreases indefinitely, but at a constant rate. Defining "entropy" and "entropy production" should be considered an open problem"; cited from the section 9.7 "Entropy Generation. Time Reversibility and Fluctuation Theorem. Experimental Tests of the Chaotic Hypothesis" of [26].

or in the following passage by Olivier Penrose:

"Even in thermodynamics, where entropy is defined only for equilibrium states, the definition of entropy can depend on what problem we are interested in and on what experimental techniques are available"; cited from [24].

that is implicitly a kind of self-criticism as to the following analysis of Maxwell's demon:

"The large number of distinct observational states that the Maxwell demon must have in order to make significant entropy reductions possible may be thought of as a large memory capacity in which the demon stores the information about the system which he acquires as he works reducing the entropy. As soon as the demon's memory is completely filled, however, \(\cdots\) he can achieve no further reduction of the Boltzmann entropy. He gains nothing for example, by deliberately forgetting or erasing some of his stored information in order to make more memory capacity available; for the erasure being a setting process, itself increases the entropy by an amount of at least as great as the entropy decrease made possible by the newly available memory capacity"; cited from [27].

in which, as it has been observed by Harvey S. Leff and Andrew F. Rex in the 1st chapter "Overview" of their wonderful book [12], Olivier Penrose arrived very near to the right Bennett's exorcism, though lacking to make the final intellectual step to understand that erasure is the fundamental act that saves Maxwell's demon.

Let us now explicitly show how Lebowitz's remark against idolatry allows to confute the first objection to Bennett's Theorem, namely that the Second Law of Thermodynamics has only statistical validity: let us analyze, at this purpose, the following pass in which Maxwell himself explains what he wanted to show through the introduction of his demon:

"This is only one of the instances in which conclusions which we have drawn from our experience of bodies consisting of an immense number of molecules may be found not to be applicable to the more delicate observations and experiments which we may suppose made by one who can perceive and handle the individual molecules which we deal with only in large masses. In dealing with masses of matter, while we do not perceive the individual molecules, we are compelled to adopt what I have described as the statistical method of calculation, and to abandon the strict dynamical method, in which we follow every motion by the calculus. It would be interesting to enquire how far those ideas about the nature and method of science which have been derived from examples of scientific investigation in which the dynamical method is followed are applicable to our actual knowledge of concrete things, which, as we have seen, is of an essentially statistical nature, because no one has yet discovered any practical method of tracing the path of a molecule, or of identifying it at different times."; cited from the last but one section "Limitation of the Second Law of Thermodynamics" of the 22th chapter of [11].

and the following pass by Thomson (later Lord Kelvin):

"'Dissipation of Energy' follows in nature from the fortuitous concourse of atoms. The lost motivity is essentially not restorable otherwise than by an agency dealing with individual atoms"; cited from [24].

They don't say that the Second Principle of Thermodynamics have only a statistical validity, but a differ-
ent thing: that the usual link existing between such a principle (that, not falling in the idolatry denounced by Lebowitz, one have to remember to have an its own validity in Thermodynamics) and Statistical Mechanics have to be modified as soon as entities able to handle individual molecules are involved.

Having followed Lebowitz’s advise of not falling into the idolatric attitude of making confusion among a physical thermodynamical system and its modelisation through Statistical Mechanics and, hence, preserving us from the error of confusing the difficulties involved in the definition of entropy in Nonequilibrium Statistical Mechanics from the difficulties involved in defining entropy in Nonequilibrium Thermodynamics, let us briefly recall these latter:

given a thermodynamical system made of N different species, the thermodynamical entropy of a thermodynamical state X is defined as:

\[ S_{\text{therm}}(X) := \int_{\text{REV}}^{X} \frac{\delta Q}{T} \] (4.2)

where the integral is over a thermodynamically-reversible transformation starting in a fixed reference thermodynamical state O (to be ultimatively fixed by the Third Law of Thermodynamics requiring that \( \lim_{T \to 0} S_{\text{therm}}(X) = 0 \)) and ending in the state X.

If X is a state of thermodynamical equilibrium the thermodynamical entropy may be expressed as a function of the internal energy \( U \), of the volume \( V \) and of the number of moles of each contributing specie \( N_k \):

**X equilibrium state ⇒**

\[ S_{\text{therm}}(X) = S_{\text{therm}}[U(X), V(X), N_k(X)] \] (4.3)

If X is not a state of thermodynamical equilibrium, anyway, its thermodynamical entropy cannot be expressed anymore as a function of the internal energy \( U \), of the volume \( V \) and of the number of moles of each contributing specie \( N_k \):

**X nonequilibrium state ⇒**

\[ S_{\text{therm}}(X) \neq S_{\text{therm}}[U(X), V(X), N_k(X)] \] (4.4)

This fact is often erroneously expressed as the claim that thermodynamical entropy is not defined out of equilibrium: this is simply false, since the the operational definition of \( S_{\text{therm}}(X) \) through eq.4.2 continues to hold.

Simply one has, denoting with lower case letters the (intensive) densities of (extensive) quantities, that the equation eq.4.2 of Classical Thermodynamics must be generalized by its expression in Generalized Thermodynamics, having the form 29:

\[ s_{\text{therm}}(X, t) = s_{\text{therm}}[u(X, t), v(X, t), n_k(X, t), \nabla u(X, t), \nabla v(X, t), \nabla n_k(X, t), \nabla^2 u(X, t), \nabla^2 v(X, t), \nabla^2 n_k(X, t), \ldots] \] (4.5)

and reducing to eq.13 in the equilibrium case.

Under conditions explicitely formalizable, furthermore, the Local Equilibrium Condition, stating that the local and instananeous relations between the thermal and mechanical properties of a physical system are the same as for a uniform system at equilibrium, holds. In this case eq.14 reduces to:

\[ s_{\text{therm}}(X, t) = s_{\text{therm}}[u(X, t), v(X, t), n_k(X, t)] \] (4.6)

i.e.:

\[ ds = (\frac{\partial s}{\partial u})_{v_n,k} + (\frac{\partial s}{\partial v})_{u_n,k} + \sum_{k=1}^{N} (\frac{\partial s}{\partial n_k})_{u,v,n_k'} \ (\text{for } k' \neq k) \] (4.7)

Consequentially, if the one-parameter family of nonequilibrium thermodynamical states \( X_t \) satisfy the Local-Equilibrium Condition, one has that the temperature in the point \( \vec{x} \) of the system at time t may be simply expressed as:

\[ T(X_t, \vec{x}, t) = [(\frac{\partial s}{\partial u})_{v_n,k}]^{-1} \] (4.8)

Returning at last to our Maxwell’s demon, let us observe that its way of taking the whole thermodynamical system out of the thermodynamical equilibrium satisfies the conditions under which the Local-Equilibrium Condition holds.

The third objection moved (implicitly or explicitely) to Bennett’s Theorem, namely that the intedisciplinary attitude of Algorithmic Physics is not necessary to understand Thermodynamics, is certainly the subtler one.

To analyze it, let us observe that the involved thermodynamical system is the compound system Gas + Demon.

As a mechanical system such a compound system is a classical dynamical system \( (X_{\text{compound}}, H_{\text{compound}}) \) with phase space:

\[ X_{\text{compound}} := X_{\text{Gas}} \cup X_{\text{Demon}} \] (4.9)

and hamiltonian:

\[ H_{\text{compound}} := H_{\text{Gas}} + H_{\text{Demon}} + H_{\text{interaction}} \] (4.10)

where \( H_{\text{Gas}} \in C^\infty(X_{\text{Gas}}) \), \( H_{\text{Demon}} \in C^\infty(X_{\text{Demon}}) \), and \( H_{\text{interaction}} \in C^\infty(X_{\text{compound}}) \).

The mechanical description of the whole process is defined by the Hamiltonian flow \( T_t : X_{\text{compound}} \to X_{\text{compound}} \) induced by Hamilton’s equation:

\[ \frac{dx}{dt} = \{H, x\} \] (4.11)

associating to any initial state \( x_{\text{compound}}^{(IN)} = (x_{\text{Gas}}^{(IN)}, x_{\text{Demon}}^{(IN)}) \in X_{\text{compound}} \) the final state \( x_{\text{OUT}} := \lim_{t \to \infty} T_t x_{\text{IN}} \).
The strategy of Statistical Mechanics would consist in deriving the macroscopic thermodynamical variables of the thermodynamical system Gas+Demon as properly-defined functions of a suitable statistical ensemble \((X_{\text{compound}}, P_{\text{compound}})\).

The ensemble \((X, P)\) involved in the formulation of Bennett’s Theorem, instead, doesn’t take into account the demon: as we saw in the last section, it is the equilibrium statistical ensemble \((X_{\text{gas}}, P_{\text{eq}})\) that Statistical Mechanics would associate to the dynamical system \((X_{\text{gas}}, H_{\text{gas}})\), the underlying reason for that deriving substantially from the Algorithmic Physics’ attitude, as we will now explain.

Algorithmic Physics is, by definition, that discipline analyzing physical processes looking at them as computational processes according to the following correspondence’s table:

| PHYSICAL PROCESS | COMPUTATIONAL PROCESS |
|------------------|-----------------------|
| initial state    | input                 |
| dynamical evolution | computation         |
| final state      | output                |

and consquentially applying the conceptual instruments of Computation’s Theory.

Essentially owing to the overwhelming ”new age” folklore by which it has been popularized in the divulgative literature, the interdisciplinary nature of Algorithmic Physics is looked by many theoretical and mathematical physicists with great mistrust; as a consequence, also the beautiful and serious insight it has produced, such as the investigations concerning the foundations of Computational Physics (i.e. of the discipline studying the computer-simulation of physical systems) such as Stephen Wolfram’s notion of computational irreducibility (i.e. the situation in which the faster way of predicting the final state of a dynamical system of known laws of motion is to simulate its whole dynamical evolution and to see what happens at the end) or his analyses concerning the rule of Undecidability in Physics 3 concretized by the work of Chris Moore and many others 35, 36, are ignored.

The approach underlying Bennett’s Theorem is a partial application of the Algorithmic Physics’ approach in which not the whole hamiltonian flow \(T_t: X_{\text{compound}} \mapsto X_{\text{compound}}\) is seen as a computational process, but only its restriction as to the Demon \(T_t|_{X_{\text{Demon}}}: X_{\text{Demon}} \mapsto X_{\text{Demon}}\).

The dynamical evolution of the gas \(T_t|_{X_{\text{gas}}}: X_{\text{gas}} \mapsto X_{\text{gas}}\) continues to be described through Mechanics, i.e. owing to the enormous number of involved degrees of freedom, through Statistical Mechanics.

The third objection to Bennett’s theorem is based on the observation that such an (hybrid) recourse to Algorithmic Physics is, at last, completely avoidable:

why, for particular compound systems, should one to give up the usual, traditional approach of Statistical Mechanics to derive the thermodynamical entropy in the usual way from the partition function of a Gibbs’s ensemble for the dynamical system \((X_{\text{compound}}, H_{\text{compound}})\)?

And which should be exactly these particular compound systems?

A minimal answer to the last question is immediate: those particular compound systems in which \(X_{\text{Demon}}\) and \(H_{\text{Demon}}\) are such to result in the scattering pattern that, looking at the demon with the eyes of Algorithmic Physics, corresponds to the computational-process of computing the semaphore function \(p\) and making to pass or to reflect the molecule correspondingly as described in the last section; as we will see in the next section, anyway, such a class of particular compound systems may be considerably enlarged through a suitable characterization of the notion of an intelligent system.

As to the former question, namely why for these compound systems one should indeed to give up the usual Statistical Mechanics’ approach, the answer is: simply because it is simpler.

The third objection to Bennett’s Theorem is, with this regard, correct: there is no necessity of adopting the hybrid Algorithmic-Physics’ approach on which Bennett’s Theorem is based on:

simply, given an arbitrary many-body physical system like \(X_{\text{gas}}\), whenever its interaction with another physical systems gives rise to a scattering-cross-section \(d\sigma/d\Omega\) of the particular kind specified above, the usual Statistical Mechanics’ approach, though still perfectly applicable, is not the simpler approach since it doesn’t catch the particular structural peculiarity of the analyzed system, structural peculiarity that allows an alternative, more concise, explication that, according to Occam’s Razor, have consquentially to be preferred.

Such a passage from a purely probabilistic approach to a hybrid mix of two approaches, the probabilistic and the algorithmic one, reflects itself in the the link between Thermodynamics and Information Theory: in terms of the three different approaches to the definition of information introduced by A.N. Kolmogorov in his fundamental papers on the Foundation of Information Theory 37, such a passage is exactly a passage from an interpretation of thermodynamical entropy in terms...
of the probabilistic approach alone to an interpretation
of thermodynamical entropy as a hybrid mix of the prob-
abilistic and the algorithmic approaches.

V. ZUREK’S THEOREM ON MAXWELL’S
DEMONS IN QUANTUM THERMODYNAMICS

Bennett’s work on Maxwell’s demon has been gen-
eralized by Wojciech Hubert Zurek in many respects [8, 9, 32, 33].

The first point analyzed by Zurek concerns the charac-
terization of the particular structural peculiarity of the
dynamical system \((X_{\text{compound}}, H_{\text{compound}})\) under which the
mix probabilistic + algorithmic approach underlying
Bennett’s Theorem may be applied:

up to this point we have assumed that Maxwell’s de-
mon acts on a particular molecule in a very particular
way: if the molecule arrives from the left side the demon
makes it to pass unaltered if and only if its velocity is
less or equal to a given threshold-velocity, acting in the
opposite way if the molecule arrives from the right side.

Such a behaviour of the demon, as it was first observed
by Leo Szilard in his basic 1929’s paper [39], appears as
a kind of intelligence.

While, taken too literally, Szilard’s paper was unfor-
nately also the source of many confusionary specu-
lations concerning the contribution of the Subject (or
the Cartesian Cogito in more philosophically palatable
terms) to the Object’s thermodynamical entropy, some-
times appealing to the wrong claim that Subject’s mea-
surements are necessary thermodynamically-irreversible
processes (whose falsity, as we have seen, may be di-
rectly derived by Landauer’s Principle; in the quantum
case we are going to introduce, anyway, it was time be-
fore directly derived by Yakir Aharonov, Peter Bergmann
and Joel Lebowitz [40]), it had the great merit of intu-
itively suggesting the essential structural peculiarity of
Maxwell’s demon, allowing Zurek to generalize Bennett’s
Theorem starting from the following questions:

1. which is exactly the kind of intelligence showed by
Maxwell’s demon?

2. can Bennett’s Theorem to be generalized to sys-
tems having the same kind of intelligence?

Observing that, according to the way we characterized
it in the previous section, the structural peculiarity of
\(X_{\text{demon}}\) and \(H_{\text{demon}}\) is to give rise to a scattering cross-
section \(d\sigma\) that, from the point of view of Algorithmic
Physics, corresponds to the prescribed algorithm of leav-
ing to pass or reflecting elastically a molecule accord-
ing to the value of a computed semaphore-binary predicate
\(p(s,v)\), one could think that Bennett’s Theorem might
be generalized to any situation in which \(X_{\text{demon}}\) and
\(H_{\text{demon}}\) give rise to a scattering cross-section \(d\sigma\) that,
from the point of view of Algorithmic Physics, corre-
sponds to an algorithm leaving to pass or reflecting elas-
tically a molecule according to a computed arbitrary bi-
nary predicate.

But one immediately realizes that such a generalization
is wrong in that not every chosen semaphore-function

Indeed, Szilard tells us, the particular ”intelligence” of
the semaphore-predicate \(p\) derives from the fact that the
resulting algorithm performed by the demon acts on each
molecule in order of lowering the probabilistic informa-
tion of the gas: we are tempted to say that it acts in a
clever way exactly since his behaviour seems to be tele-
ological, finalized to the objective of taking the gas in a
more ordered state.

Let us observe, finally, that such an ordering-process
made by the demon acts necessarily out of thermodynam-
ical equilibrium, since its ”intelligence” is accomplished
precisely by creating a ”clever” disuniformity in the spati-
al distribution of a thermodynamical variable (in this
case the temperature).

We are consequentially led to the following generaliza-
ation of Bennett’s theorem:

the thermodynamical entropy of a classical many-body
system \((X_{m.b}, H_{m.b})\):

- preliminary prepared in a state of thermodynam-
ical equilibrium described, in Classical Statistical
Mechanics, by the statistical ensemble \((X_{m.b}, P_{eq})\)

- in a second time made to interact with an other
physical system \((X_{int}, H_{int})\) that is intelligent:

may be expressed as:

\[
S_{\text{therm}}(P_{eq}) = I_{\text{prob}}(P_{eq}) + I_{\text{alg}}(P_{eq})
\]

where \(I_{\text{prob}}(P_{eq})\) and \(I_{\text{alg}}(P_{eq})\) are, respectively, the
probabilistic information and the algorithmic informa-
tion of the ensemble \((X_{m.b}, P_{eq})\), where the intelligence
condition of \((X_{int}, H_{int})\) is defined in the following way:

1. the scattering cross-section \(d\sigma\), seen from the point
of view of Algorithmic Physics, corresponds to a deter-
nministic algorithm \(f\) acting on a single molecule

2. in a way that takes the many-body system out of
thermodynamical equilibrium

3. reducing its probabilistic information

Such a definition of an intelligent system is, indeed,
a strengthening of Gell-Mann’s notion of information

gathering and using system (IGUS) defined as a complex
adaptive system able to make observations

A part from having generalized it to a strongly larger
class of intelligent systems, Zurek’s main extension of
Bennett’s work concerns its extension to the quantum
domain:
Zurek’s theorem 4, the quantum analogue of Bennett’s theorem, states that the thermodynamical entropy of a quantum many-body system \((\mathcal{H}_{m,b}, \hat{H}_{m,b})\):

- preliminary prepared in a state of thermodynamical equilibrium described, in Quantum Statistical Mechanics, by the quantum statistical ensemble \((\mathcal{H}_{m,b}, \rho_{eq})\)
- in a second time made to interact with an other physical system \((\mathcal{H}_{int}, \hat{H}_{int})\)

may be expressed as:

\[
S_{\text{therm}}(\rho_{eq}) = \int_{\text{prob}} I_{\text{prob}}(\rho_{eq}) + I_{\text{alg}}(\rho_{eq})
\]  

(5.2)

where \(I_{\text{prob}}(\rho_{eq})\) is the quantum probabilistic information of the density operator \(\rho_{eq}\), namely its Von Neumann’s entropy:

\[
I_{\text{prob}}(\rho_{eq}) := -\text{Tr}\rho_{eq}\log\rho_{eq}
\]  

(5.3)

while \(I_{\text{alg}}(\rho_{eq})\) is the quantum algorithmic information of the ensemble \((\mathcal{H}_{m,b}, \rho_{eq})\) 5, namely:

\[
I(\rho_{eq}) := \begin{cases} 
\min \{ |x| : U(x) = \rho_{eq} \} & \text{if } \exists x : U(x) = \rho_{eq}, \\
+\infty & \text{otherwise.}
\end{cases}
\]  

(5.4)

i.e. the length of the shortest program computing it on the fixed Chaitin quantum universal computer \(U\), where the intelligence-condition of the quantum system \((\mathcal{H}_{int}, \hat{H}_{int})\) is defined exactly as in the classical case through the following conditions:

1. the scattering cross-section \(d\sigma/dE\) seen from the point of view of Algorithmic Physics, corresponds to an algorithm \(f\) acting on a single molecule
2. in a way that takes the many-body system out of thermodynamical equilibrium

3. reducing its probabilistic information

It is important to remark, at this point, that such a definition of intelligence, applied in the quantum domain, is indeed subtler, owing to the entanglement’s phenomenon between the quantum many-body system \((\mathcal{H}_{m,b}, \hat{H}_{m,b})\) and \((\mathcal{H}_{int}, \hat{H}_{int})\) having no classical analogue, that is itself used by \((\mathcal{H}_{int}, \hat{H}_{int})\) as to realize its teleological action.

VI. THE THERMODYNAMICAL COST OF ERASING THE MEMBRANES’ MEMORY OF PERES’ ENGINE

In the previous sections we have introduced all the ingredients required to present the contribution of this paper, namely the confutation of the claim, presented by Asher Peres in \(\text{[4]}\) as well as in the \(9^{th}\) chapter of his wonderful book \(\text{[1]}\), that the Theorem of Indistinguishability for nonorthogonal states is necessary in order of preserving the Second Law of Thermodynamics.

Peres’s argument is based on the assumption that the thermodynamical-entropy of a quantum system is described by Von Neumann’s entropy, assumption that he deeply analyzes explicitly reporting the celebrated original calculus by which Von Neumann, in the section5.2 of \(\text{[1]}\), computed the thermodynamical entropy of a quantum mixture \(\{p_i, |\phi_i><\phi_i|\}_{i=1}^n\) as if each \(|\phi_i><\phi_i|\) was a specie of ideal gas enclosed in a large impenetrable box, and inferring that the thermodynamical mixing entropy of the different species is \(I_{\text{prob}}(\sum_i p_i|\phi_i><\phi_i|)\).

Peres reviews Von Neumann’s procedure in the following way:

"It also assumes the existence of semipermeable membranes which can be used to perform quantum tests. These membranes separate orthogonal states with perfect efficiency. The fundamental problem here is whether it is legitimate to treat quantum states in the same way as varieties of classical ideal gases. This issue was clarified by Einstein in the early days of the "old" quantum theory as follows: consider an ensemble of quantum systems, each one enclosed in a large impenetrable box, so as to prevent any interaction between them. These boxes are enclosed in an even larger container, where they behave as an ideal gas, because each box is so massive that classical mechanics is valid for its motion (\(\cdots\)). The container itself has ideal walls and pistons which may be, according to our needs, perfectly conducting, or perfectly insulating, or with properties equivalent to those of semipermeable membranes. The latter are endowed with automatic devices able to peak inside the boxes and to test the state of the quantum system enclosed therein." (from the section9.3 of \(\text{[1]}\))

There is a point, anyway, of this review in which, deliberately, Peres moves away from Von Neumann’s original

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4 I would like to advise the reader that my presentation of Zurek’s theorem differs slightly from Zurek’s own ideas for which we strongly demand to the previously cited original Zurek’s papers
5 Zurek claims that the assumption of the Church-Turing’s Thesis eliminates any dependence from the particular universal computer \(U\) adopted by (or better constituting) the demon. He, in particular, claims that, by the Church-Turing’s Thesis, it doesn’t matter if \(U\) is a classical computer or a quantum computer; so he substantially claims that Quantum Algorithmic Information Theory collapses to Classical Algorithmic Information Theory, an arbitrary assumption for whose discussion I demand once more to \(\text{[1]}\).
treatment:

he doesn’t assume that the membranes separate nonorthogonal states with perfect efficiency as, instead, Von Neumann does:

"Each system \( s_1, \ldots, s_n \) is confined in a box \( K_1, \ldots, K_n \) whose walls are impenetrable to all transmission effects – which is possible for this system because of the lack of interaction" (from the section 5.2 of [45])

The reason why Peres, contrary to Von Neumann, doesn’t make such an assumption is that, according to him, this would imply a violation of the Second Law of Thermodynamics; his argument is the following: if semi-permeable membranes which unambiguously distinguish non-orthogonal states were possible, one could use them to realize the following cyclic thermodynamical transformation for a mixture of two species of 1-qubit’s states, the \( |0<0>|\)-specie and the \( \frac{1}{2}(|0>+|1>|(<0>+<1|))\)-specie, both with the same concentration \( \frac{1}{2} \):

- in the initial state the two species occupy two chambers with equal volumes, with the \( |0<0>|\)-specie occupying the right-half of the left-half of the vessel and the \( \frac{1}{2}(|0>+|1>|(<0>+<1|))\)-specie occupying the left-half of the right-half of the vessel

- the first step of the process is an isothermal expansion by which the \( |0<0>|\)-specie occupies all the left-half of the vessel while the \( \frac{1}{2}(|0>+|1>|(<0>+<1|))\)-specie occupies all the right-half of the vessel; this expansion supplies an amount of work:

\[
\Delta L_1 = +nT\ln 2
\]

\( T \) being the temperature of the reservoir.

- at this stage the impenetrable partitions separating the two species are replaced by the "magic"-semi-permeable membranes having the ability of distinguishing non-orthogonal states; precisely one of them is transparent to the \( |0<0>|\)-specie and reflect the \( \frac{1}{2}(|0>+|1>|(<0>+<1|))\)-specie while the other membrane has the opposite properties; then, by a double frictionless piston, it is possible to bring the engine, without expenditure of work or heat transfer, to a state in which all the two species occupy with the same concentration only the left-hand of the vessel, the right-hand of the vessel remaining empty; we can represent mathematically the state of affairs of the system by the following decomposition:

\[
\mathcal{E}_1 := \{\left(\frac{1}{2}|0<0|\right), \left(\frac{1}{2}\frac{1}{2}(|0>+|1>|(<0>+<1|))\}\}
\]

\[
\rho := \begin{pmatrix}
\frac{3}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\]

- since the state of the mixture-of-species is completely determined by \( \rho \), and not by a particular its decomposition, to represent the actual state of affairs by \( \mathcal{E} \) or by the Schatten’s decomposition of \( \rho \):

\[
\mathcal{E}_1 := \{(\rho_-,|e_-><e_-|), (\rho_+|e_+><e_+|)\}
\]

\[
\rho_\pm := \frac{1}{4}(2 \pm \sqrt{2})
\]

\[
|e_\pm := (1 \pm \sqrt{2})(0 > +|1>)
\]

is absolutely equivalent

- let us now replace the two "magic" membranes with ordinary membranes able to distinguish only orthogonal species; since the \( |e_-><e_-|\)-specie and the \( |e_+><e_-|\)-specie are orthogonal, the reversible diffusion of the two species separate them, with the \( |e_+><e_-|\)-specie occupying the left-half of the vessel and the \( |e_-><e_-|\)-specie occupying the right-half of the vessel.

- finally an isothermal compression takes the system in a situation in which the volume and the pressure are the same of the initial state; such a compression requires an expenditure of work of:

\[
\Delta L_2 = -nT[\rho_1 \log \rho_1 + \rho_2 \log \rho_2]
\]

- finally a suitable unitary evolution takes the system again in the initial state.

The net work made by the engine during the cycle is:

\[
\Delta L = \Delta L_1 + \Delta L_2 > 0
\]

so that the whole thermodynamical cycle converts the heat extracted by the reservoir in a positive amount of work of \( \Delta L \).

This, according to Peres, violates the Second Principle, proving that the "magic" membranes able to separate nonorthogonal states with perfect efficiency cannot exist.

Such a proof, anyway, in not correct, owing to Zurek’s theorem; the key point touches the conceptual deepness underlying eq 5.2 whose complete comprehension requires to explicitly analyze the bug in Von Neumann’s proof that \( S_{therm}(\rho) = I_{prob}(\rho) \).

The key point is based in the own definition of the semi-permeables membranes of Einstein’s method: as correctly observed by Peres the semipermeable membranes are endowed with automatic devices able to peak inside the boxes and to test the state.

What Peres seems unfortunately not to catch is that a semi-permeable membrane is then an intelligent system operating in the following way:

1. gets the input \( (s,i) \) from a device measuring both the side \( s \) from which the \( |\phi_i><\phi_i|\)-specie arrives and its kind, i.e. the classical information codified by its label \( i \).
2. computes a certain semaphore-function p such that 
\( (s,i) \xrightarrow{P} p[(s,i)] \) giving as output a 0 if the \( |\phi_i| >> \phi_i\)-specie must be left to pass while gives as output a one if the \( |\phi_i| >> \phi_i\)-specie must be stopped
3. gives the output \( p[(s,i)] \) to a suitable device that operates on the \( |\phi_i| >> \phi_i\)-specie in the specified way.

The argument of Bennett’s exorcism concerning the necessity of taking into account the algorithmic-information of the sequences of successive recorded \( (s,i) \)'s in the membrane’s memory thus apply.

But this must be done, in particular, in the cases of Peres’-engine:

- taking into account also the algorithmic-information of the semi-permeable’s membranes, one sees that it is greater than or equal to the universe’s entropy decrease corresponding to the work made by the engine, so that, by eq. \( 5.2 \)

\[
\Delta S_{\text{therm}} \geq 0 \quad (6.9)
\]

and Peres’ arguments falls down.

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