On Atkin-Lehner correspondences on Siegel spaces

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Abstract

We introduce a higher dimensional Atkin-Lehner theory for Siegel-Parahoric congruence subgroups of $GSp(2g)$. Old Siegel forms are induced by geometric correspondences on Siegel moduli spaces which commute with almost all local Hecke algebras. We also introduce an algorithm to get equations for moduli spaces of Siegel-Parahoric level structures, once we have equations for prime levels and square prime levels over the level one Siegel space. This way we give equations for an infinite tower of Siegel spaces after N. Elkies who did the genus one case.

Introduction

Classical Atkin-Lehner theory for $GL(2)$ has two conceptual ingredients: The first one is Casselman’s theory of local new-forms for $GL(2)$ [Cas], generalized to $GSp(4)$-case by R. Schmidt for square-free level [Sch]. But, at the moment there is no general local theory of new-forms available for $GSp(2g)$. The second ingredient of Atkin-Lehner theory is strong multiplicity one for cuspidal automorphic representations of $GL(2)$. But, multiplicity one fails to hold in higher $GSp(2g)$'s. Therefore, we regard all Siegel forms with the same Hecke eigenvalues as old Siegel-forms to be old forms again. Despite this loss, Atkin-Lehner geometric theory can be generalized to Siegel-parahoric congruence subgroups of $GSp(2g)$.

We use elements of the Weyl group of $GSp(2g)$ to construct new congruence subgroups sandwiched between the Siegel-parahoric congruence group $\Gamma^P(n)$ and what we call the diagonal congruence group $\Gamma^D(n)$. These groups are all defined in terms of the mod-$n$ reduction of elements in $Sp(2g, \mathbb{Z})$. We shall use these congruence subgroups to introduce geometric correspondences on Siegel moduli spaces with explicit moduli interpretations. These correspondences induce an injection of a number of copies of Siegel modular forms with Siegel-parahoric $n$-level structure inside the space of Siegel modular forms of Siegel-parahoric level $pn$ for $(p,n) = 1$. Since our correspondences commute with all $q$-Hecke correspondences for $(q, pn) = 1$, any satisfactory local definition of $p$-old Siegel forms will imply that our correspondences introduce a higher dimensional Atkin-Lehner theory for $p$-old Siegel forms of Siegel parahoric level $pn$. By a satisfactory local theory of new-forms, we mean that any new eigenform of almost all local Hecke algebras should be an eigenform of all local
Hecke algebras prime to the level. Also, eigenforms with eigenvalues repeated in lower levels should be considered as old-forms.

Next, we give algebraic equations for moduli spaces of Siegel-Parahoric level structures, by taking fiber products of the same moduli spaces with prime power level structure, which in turn can be constructed via algebraic equations for Siegel moduli spaces of Siegel-Parahoric prime level and square prime level structures. This way we can give equations for an infinite tower of level structures after N. Elkies who did the genus one case [Elk].

1 Atkin-Lehner theory

1.1 Siegel-parahoric subgroup of $GSp(2g)$

The Chevalley group scheme $GSp(2g)$ is defined as the set of matrices $P \in GL(2g)$ with $\lambda(P)J$ where $\lambda(P) \in GL(1)$ and $J = antidiag(I_g, -I_g)$ where $I_g$ denotes the $g \times g$ identity matrix. The representation $\lambda$ is called the multiplier representation. Similar to above, we also use two by two matrices whose entries are $g \times g$ matrices to represent elements of $GSp(2g)$. The symplectic group scheme $Sp(2g)$ is defined to be the kernel of the multiplier representation, which is the space of transformations on the symplectic space $\mathbb{Z}^{2g}$ with its standard alternating form:

$\langle, \rangle: \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \rightarrow \mathbb{Z}$

$\langle (u, v), (z, w) \rangle \mapsto u^t w - v^t z$

Let $T \cong G^{g+1}_m$ denote the maximal torus in $GSp(2g)$. An element of $T$ is a diagonal matrix $diag(a_1, ..., a_g, d_1, ..., d_g)$ with $a_i, d_i$ equal to the multiplier. Let $M$ denote the subgroup of $GSp(2g)$ which respects the standard decomposition $\mathbb{Z}^{2g} \cong \mathbb{Z}^g \oplus \mathbb{Z}^g$. The subgroup $M$ consists of diagonal elements in $GSp(2g)$ in their two by two representation. These elements are of the form $diag(A, D)$ with $A^t D = \lambda I_g$. The subgroup of $Sp(2g, \mathbb{Z})$ which fixes only the first direct summand $\mathbb{Z}^g \subset \mathbb{Z}^{2g}$ is denoted by $U$ which is the space of all $\mathbb{Z}$-valued bilinear symmetric forms. Elements of $U$ are upper-triangular with $I_g$ on diagonal entries and a symmetric matrix $B$ on the upper right corner. The subgroup $P = M \rtimes U$ is a maximal parabolic subgroup whose elements are zero in the lower left $g \times g$ corner. This parabolic subgroup is also called the Siegel-parahoric subgroup. Fix the Borel subgroup contained in $P$ consisting of the matrices with $A, B, 0, 0$ entries with $A$ upper-triangular and $D$ lower-triangular. Weyl groups of $GSp(2g)$ and $P$ with respect to the maximal torus $T$ are denoted by $W_G$ and $W_P$ respectively. $W_G \cong S_g \ltimes (\pm 1)^g$ and $W_P \cong S_g$ act on diagonal matrices $diag(a_1, ..., a_g, d_1, ..., d_g)$ by permutation or exchange of the $a_i$'s and the $d_i$'s.

1.2 Congruence subgroups of $Sp(2g, \mathbb{Z})$

A discrete subgroup $\Gamma \subset Sp(2g, \mathbb{Q})$ is called a congruence group, if it contains $\Gamma(n) = \Gamma^{id}(n)$ for some positive integer $n$, where $\Gamma(n)$ is the kernel of reduction map
1.2 Congruence subgroups of $Sp(2g, \mathbb{Z})$

modulo $n$ on $Sp(2g, \mathbb{Z})$:

$$\Gamma(n) = \{ \gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv I_{2g} \in Sp(2g, \mathbb{Z}/n\mathbb{Z}) \pmod{n} \}.$$ 

The Siegel-parahoric congruence group is defined by

$$\Gamma^P(n) = \{ \gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \pmod{n} \}$$

and the diagonal congruence group by

$$\Gamma^D(n) = \{ \gamma \in Sp(2g, \mathbb{Z}) | \gamma \equiv \text{diag}(\ast, \ldots, \ast), \pmod{n} \}.$$ 

$\Gamma^P(n)$ and $\Gamma^D(n)$ are examples of congruence groups. The significance of congruence groups is that they carry arithmetic information. In this paper, we are only interested in congruence subgroups of $Sp(2g, \mathbb{Z})$.

By the theory of Tits systems [Hum], every parabolic subgroup of $GSp(2g)$ is conjugate to a "standard" parabolic subgroup which contains $B$. Each standard parabolic subgroup $P_I$ corresponds to one of the $2^{q+1}$ subsets $I \subseteq S$ where $S$ is a minimal generating set for the Weyl group $W_G$ consisting of involutions $\rho_i$ which are elements of order 2. $S$ can be taken as the set of simple reflections corresponding to the base of the root system determined by $B$. In fact, $P_I = BW_I B$ where $W_I \subseteq W_G$ is the subgroup generated by elements in $I$. In particular, $P_0 = B$ and $P_S = GSp(2g)$.

By Bruhat decomposition theorem, $GSp(2g) = \cup B\sigma B$ is a decomposition to disjoint subsets where $\sigma$ runs in the Weyl group $W_G$. Two such double cosets coincide if and only if the middle Weyl elements coincide. One can assume that axioms of Tits systems are satisfied. Namely,

(T1) For $\rho \in S$ and $\sigma \in W_G$ we have $\rho B\sigma \subset B\sigma B \cup B\rho B\sigma B$.

(T2) For $\rho \neq \rho$ we have $\rho B\rho \neq B$.

An expression $\sigma = \rho_1 \ldots \rho_k$ with $\rho_i \in S$ is called reduced if $k$ is as small as possible. The minimal length $k$ of a reduced expression is denoted by $\ell(\sigma)$. By convention, $\ell(\sigma) = 0$ if and only if $\sigma = id$ and $\ell(\sigma) = 1$ if and only if $\sigma \in S$. Tits axioms imply that for $\rho \in S$ we have $\ell(\rho \sigma) = \ell(\sigma) \pm 1$. This implies that for the reduced form $\sigma = \rho_1 \ldots \rho_k$ and $I = \{\rho_1, \ldots, \rho_k\}$, the parabolic subgroup $P_I$ is generated by $B$ and $\sigma B\sigma^{-1}$ or by $B$ and $\sigma$. Given a choice of a Borel subgroup $S$ is precisely the set of elements in $W_G$ such that $B \cup B\sigma B$ is a group. $P_I$ is conjugate to $P_J$ implies that $P_I = P_J$ and $P_I \subset P_J$ implies that $I \subset J$.

We define a parabolic congruence subgroup $\Gamma^{P_I}(n) \subset Sp(2g, \mathbb{Z})$ to be the set of elements which reduce to $P_I$ modulo $n$:

$$\Gamma^{P_I}(n) = \{ \gamma \in Sp(2g, \mathbb{Z}) | \gamma \in P_I \subset GSp(2g, \mathbb{Z}/n\mathbb{Z}) \pmod{n} \}.$$ 

Fix a generating set $S$ for $W_G$ consisting of $g - 1$ pairs in $S_g \subset W_g$ and a nonzero element in $(\pm 1)^g$. Now, there exists a generating set $w_1, \ldots, w_g$ for the Weyl group such that $w_i$ have increasing length if represented in reduced form in term of involutions in $S$. 

Assume that \( w_{k+1} = \rho_k w_k \) for all \( k \) where \( \rho_k \) is an involution. Also assume \( w_1, ..., w_g \) generate \( S_g \). We have \( \Gamma^P \equiv \Gamma^P(n) \) where \( P \) is the maximal parabolic we fixed in notations. The fact that \( B \) together with \( I_k = \{ w_1, ..., w_k \} \) can not generate any of \( w_{k+1}, ..., w_g \) implies that each \( P_{I_k} \) has exactly \( \frac{q^k}{(k+1)}2^g \) conjugates of the form \( \sigma P_{I_k} \sigma^{-1} \) for \( \sigma \) in \( W_G \). To these we associate \( \frac{q^k}{(k+1)}2^g \) parabolic congruence subgroups of the form \( \sigma \Gamma^P \equiv \Gamma^P(n) \sigma^{-1} \) for \( \sigma \) in \( W_G \). In particular, \( \Gamma^P(n) \) has \( g!2^g \) conjugates in \( Sp(2g, \mathbb{Z}) \) and each \( \Gamma^P(n) \) has \( k + 2 \) conjugates in \( \Gamma^P(n+1) \).

We have made a nested family of parabolic congruence groups contained in the maximal parabolic congruence group \( \Gamma^P \) consisting of levels 0 to \( g-1 \). The \( k \)-th level is formed by \( \frac{q^k}{(k+1)}2^g \) congruence groups. Each group in level \( k \) contains \( k+1 \) groups in level \( k-1 \) for \( k = 1 \) to \( g-1 \). There are \( 2^g \) congruence groups in level \( g-1 \) and the congruence groups in level 0 are the \( g!2^g \) conjugates by elements in \( W_g \) of \( \Gamma^P(p) \) which are lying inside \( Sp(2g, \mathbb{Z}) \).

### 1.3 Siegel moduli spaces

A general reference for the arithmetic of Siegel moduli spaces is [Fal-Cha]. Let \( A_g \) denote the moduli stack of principally polarized abelian schemes of relative dimension \( g \). By a symplectic principal level-\( n \) structure, we mean a symplectic isomorphism \( \alpha : A[n] \to (\mathbb{Z}/n\mathbb{Z})^{2g} \), where \( (\mathbb{Z}/n\mathbb{Z})^{2g} \) is equipped with the standard non-degenerate skew-symmetric pairing.

Let \( \zeta_n \) denote an \( n \)-th root of unity for \( n \geq 3 \). The moduli scheme classifying the principally polarized abelian schemes over \( Spec(\mathbb{Z}[\zeta_n, 1/n]) \) together with a symplectic principal level-\( n \) structure is a scheme over \( Spec(\mathbb{Z}[\zeta_n, 1/n]) \) and will be denoted by \( A_g(n) \). The symplectic group \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) acts on \( A_g(n) \) as a group of symmetries by acting on level structures. We will recognize these moduli spaces and their etale quotients under the action of subgroups of \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) as Siegel spaces.

A \( \Gamma(n) \)-level structure of type I on \( (A, \lambda) \) is choice of a subgroup \( H \subset A[n] \) of order \( n^g \) which is totally isotropic with respect to the Weil pairing induced by \( \lambda \). A \( \Gamma(n) \)-level structure of type II on \( (A, \lambda) \) is choice of a principally polarized isogeny \( (A_1, \lambda_1) \to (A_2, \lambda_2) \) of degree \( n^g \). By a principally polarized isogeny, we mean an isogeny \( \sigma : A_1 \to A_2 \) such that \( \sigma \circ \lambda_2 \circ \sigma^{-1} \) is multiplication by an integer. For \( n \geq 3 \) type I and type II \( \Gamma(n) \)-level structures induce isomorphic moduli schemes over \( Spec(\mathbb{Z}[1/n]) \) [DJo]. We denote this moduli scheme by \( A_g^P(n) \). There exists a natural involution

\[
w_n^P : A_g^P(n) \to A_g^P(n)
\]

taking \( (\sigma : (A_1, \lambda_1) \to (A_2, \lambda_2)) \) to \( (\sigma^{-1} : (A_2, (\lambda_2)^{-1}) \to (A_1, (\lambda_1)^{-1})) \) which we call the Atkin-Lehner involution.

A \( \Gamma(n) \)-level structure of type I on \( (A, \lambda) \) is choice of \( g \) subgroups \( H_i \subset A[n] \) of order \( n^i \) with \( H_1 \subset ... \subset H_g \) where \( H_g \) is totally isotropic. A \( \Gamma(n) \)-level structure of type II on \( (A, \lambda) \) is choice of a chain of \( g \) isogenies \( (A_0, \lambda_0) \to ... \to (A_g, \lambda_g) \) each of degree \( n \) which satisfy \( n.id_{A_i} = \alpha^i \circ \lambda_0^{-1} \circ (\alpha^i)^g \circ \lambda_g \circ \alpha^{-i} \) for all \( i = 1, ..., g. \)
1.4 The $\Gamma^P_l(p)$-level structure

In case $n \geq 3$ type I and type II $\Gamma^B(n)$-level structures induce isomorphic moduli schemes over $\text{Spec}(\mathbb{Z}[1/n])$ [DJo]. We denote this moduli scheme by $A^B_g(n)$. There also exists a natural involution

$$w^B_n : A^B_g(n) \to A^B_g(n)$$

taking $((A_0, \lambda_0) \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} (A_g, \lambda_g))$ to $((A_g, (\lambda_g)^{-1}) \xrightarrow{\alpha^t} \ldots \xrightarrow{\alpha^t} (A_0, (\lambda_0)^{-1}))$ which commutes with the Atkin-Lehner involution under the natural projection between the Siegel spaces.

$$A^B_g(n) \xrightarrow{w^B_n} A^B_g(n) \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow
A^P_g(n) \xrightarrow{w^P_n} A^P_g(n)$$

A $\Gamma^T(n)$-level structure on $(A, \lambda)$ is choice of $2g$ subgroups $H_i \subset A[n]$ for $i = 1$ to $2g$, each isomorphic to $(\mathbb{Z}/n\mathbb{Z})$ such that $H_1 \oplus \ldots \oplus H_g$ and $H_{g+1} \oplus \ldots \oplus H_{2g}$ are totally isotropic subgroups of order $n^g$ which do not intersect with $H_i \oplus H_{g+i}$ hyperbolic for $i = 1$ to $g$. For $A$ and $A'$ abelian schemes over the schemes $S$ and $S'$ respectively, we define a morphism from $(S, A, \lambda, H_1, \ldots, H_{2g})$ to $(S', A', \lambda', H'_1, \ldots, H'_{2g})$ to be a pair of morphisms $(f, g)$ where $f : S \to S'$ and $g : A \to A'$ satisfy $g^*(\lambda') = \lambda$ and $g(H_i) = H'_i$ for all $1 \leq i \leq 2g$. Also we want the pair $(f, g)$ to induce an isomorphism $A \simeq S \times S' A'$. Having these morphisms defined, we have formed a category $A^T_g(n)$. The functor $\pi : A^T_g(n) \to \text{Sch}$ defined by $\pi(S, A, \lambda, H_1, \ldots, H_{2g}) = S$ makes $A^T_g(n)$ into a stack in groupoids over $S$. The 1-morphism of stacks $\pi' : A^T_g(n) \to A_g$ defined by $\pi(S, A, \lambda, H_1, \ldots, H_{2g}) = (S, A, \lambda)$ is representable and is a proper surjective morphism. For $n \geq 3$ we get a separated scheme of finite type $A^T_g(n)$ which is smooth over $\text{Spec}(\mathbb{Z}[1/n])$.

Let $\mathbb{H}_g$ denote the Siegel upper half-space, which consists of the set of complex symmetric $g \times g$ matrices $\Omega$ with $\Im(\Omega)$ positive definite. As a complex analytic stack $A_g/\mathbb{C}$ is the quotient of Siegel upper half-space $\mathbb{H}_g$ by the action of $Sp(2g, \mathbb{Z})$ via Möbius transformations. The family of principally polarized abelian varieties over $\mathbb{H}_g$ is given by $A(\Omega) = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega.\mathbb{Z}^g)$. To any congruence subgroup $\Gamma \subset Sp(2g, \mathbb{Z})$ one can associate the quotient $\Gamma/\mathbb{H}_g$ which is a Siegel moduli-space with some extra level structure. The corresponding level structure can be made explicit. Indeed, $\Gamma(n) \backslash \mathbb{H}_g$ corresponds to $A_g(n)/\mathbb{C}$ whose quotient under the action of the symplectic group $Sp(2g, \mathbb{Z}/n\mathbb{Z})$ is $A_g/\mathbb{C}$. Any congruence subgroup $\Gamma(n) \subset \Gamma \subset Sp(2g, \mathbb{Z})$ corresponds to the quotient of $A_g(n)/\mathbb{C}$ under the action of the finite group $\Gamma/\Gamma(n)$. This helps to associate explicit level structures to the space $\Gamma/\mathbb{H}_g$ which makes it a moduli space.

1.4 The $\Gamma^P_l(p)$-level structure

The Atkin-Lehner involution can be easily generalized from $GL(2)$ [Atk-Leh] to the higher dimensional case $GSp(2g)$. This generalization involves Siegel-parahoric and
Borel congruence groups. On the other hand, local considerations show that $p$-old Siegel modular forms with respect to the Siegel-parahoric or Borel congruence groups of level $pn$ contain several copies of Siegel forms of level $n$. This implies that a single Atkin-Lehner involution would not do the job of geometrically generating the $p$-old part. In this section, we intend to introduce geometric correspondences which complement the role of Atkin-Lehner involution.

Let $p$ be a prime not dividing the positive integer $n$ and let $\mathcal{A}^T_g(p)/\mathcal{C}$ and $\mathcal{A}^{T,P}_g(p,n)/\mathcal{C}$ denote the Siegel spaces associated to congruence groups $\Gamma^T(p)$ and $\Gamma^{T,P}(p,n) = \Gamma^T(p) \cap \Gamma^P(n)$, respectively. The group $\Gamma^T(p)$ remains invariant under conjugation by elements in $\mathcal{W}_g$. So $(\pm 1)^g$ acts on $\mathcal{A}^T_g(p)/\mathcal{C}$ and $\mathcal{A}^{T,P}_g(p,n)/\mathcal{C}$ by $2^g$ involutions. Let $\mathcal{A}^{P_{ik}}_g(p)/\mathcal{C}$ and $\mathcal{A}^{P_{ik},P}_g(p,n)/\mathcal{C}$ denote the Siegel spaces associated to the congruence groups $\Gamma^{P_{ik}}(p)$ and $\Gamma^{P_{ik},P}(p,n) = \Gamma^{P_{ik}}(p) \cap \Gamma^P(n)$ respectively. We have a chain of etale maps

$$\mathcal{A}^{B,P}_g(p,n) \rightarrow ... \rightarrow \mathcal{A}^{P_{ik},P}_g(p,n) \rightarrow \mathcal{A}^{P_{ik+1},P}_g(p,n) \rightarrow ... \rightarrow \mathcal{A}^P_g(pn).$$

Since each congruence group $\Gamma^{P_{ik+1}}(p)$ on the $(k+1)$-th level contains $k+2$ conjugates (by Weyl elements) of $\Gamma^{P_{ik}}(p)$ on the $k$-th level, we expect that for each $k$ we get $k + 2$ copies of forms on $\mathcal{A}^{P_{ik+1},P}_g(p,n)$ injecting in forms on $\mathcal{A}^{P_{ik},P}_g(p,n)$. We will use the geometry of $\mathcal{A}^{B,P}_g(p,n)$ to give a geometric construction of these $k+2$ copies.

In order to simplify the notations, let us forget the $\Gamma^P(n)$-level structure which is auxiliary. We get a chain of etale maps

$$\mathcal{A}^B_g(p) \rightarrow ... \rightarrow \mathcal{A}^{P_{ik}}_g(p) \rightarrow \mathcal{A}^{P_{ik+1}}_g(p) \rightarrow ... \rightarrow \mathcal{A}^P_g(p)$$

which corresponds to a chain of congruence groups

$$\Gamma^B(p) \leftrightarrow ... \leftrightarrow \Gamma^{P_{ik}}(p) \leftrightarrow \Gamma^{P_{ik+1}}(p) \leftrightarrow ... \leftrightarrow \Gamma^P(p).$$

Each $\Gamma^{P_{ik}}(p)$ maps to $\Gamma^{P_{ik+1}}(p)$ by $k+1$ maps: natural inclusion and conjugation by representatives $\sigma_{k+1} \in W_k \backslash W_{k+1}$ followed by inclusion, where $W_k$ is the subgroup of the Weyl group generated by $w_1, ..., w_k$. Inclusion induces the natural projection map $\pi_{P_{ik},P_{i+1}} : \mathcal{A}^{P_{ik}}_g(p) \rightarrow \mathcal{A}^{P_{ik+1}}_g(p)$. Conjugation by $\sigma_{k+1}$ induces an inclusion

$$\sigma_{k+1} \Gamma^{P_{ik}}(p) \sigma_{k+1}^{-1} = \Gamma^{P_{ik+1}}(p) \subseteq \Gamma^{P_{ik+1}}(p).$$

This inclusion corresponds to another projection from a different moduli-space

$$\mathcal{A}^{\sigma_{k+1} P_{ik} \sigma_{k+1}^{-1}}_g(p) \rightarrow \mathcal{A}^{P_{ik+1}}_g(p).$$

Conjugation by $\sigma_{k+1}$ identifies $\mathcal{A}^{P_{ik}}_g(p)$ with $\mathcal{A}^{\sigma_{k+1} P_{ik} \sigma_{k+1}^{-1}}_g(p)$. The moduli-space $\mathcal{T}_g(p)$ is the appropriate moduli space to geometrically realize all the endomorphisms

$$\nu_p : \mathcal{T}_g(p) \rightarrow \mathcal{T}_g(p).$$

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induced by conjugation via elements $\sigma$ in the Weyl group $W_G$. In fact, the following diagrams are commutative

\[
\begin{array}{ccc}
A_g^T(p) & \xrightarrow{\sigma} & A_g^T(p) \\
\downarrow & & \downarrow \\
A_g^{P_1}(p) & \xrightarrow{\sigma} & A_g^{P_1P\sigma^{-1}}(p) \\
\downarrow & & \downarrow \\
A_g^{P_1}(p) & \xrightarrow{\sigma} & A_g^{P_1P\sigma^{-1}}(p)
\end{array}
\]

1.5 The geometry of Siegel spaces $A_g^B(p)$ and $A_g^{P_1,P}(p)$

In this section we try to geometrically characterize fibers of natural maps between moduli spaces, we have already introduced.

**Definition** We say that two points $x$ and $y$ on $A_g^{P_1,P}(p,n)$ are $\sigma$-connected, for an element $\sigma$ in the Weyl group, if there exists a chain of points $x = x_1, \ldots, x_t = y$ on $A_g^{P_1,P}(p,n)$ such that for each $i$ there are points $x'_i$ and $x'_{i+1}$ on $A_g^{T,P}(p,n)$ mapping to $x_i$ and $x_{i+1}$ respectively, with $x_{i+1} = v_p^\sigma(x_i)$ where

\[
v_p^\sigma : A_g^{T,P}(p,n) \to A_g^{T,P}(p,n)
\]

is the endomorphism induced by the action of $\sigma$ on $p$-level structure.

**Proposition 1.1** Every fiber of the map $\pi_k : A_g^{P_{1k}}(p,n) \to A_g^{P_{1k+1}}(p,n)$ is an equivalence class of $\rho_{k+1}$-connected points.

**Proof.** The $n$-level structure is auxiliary. Let $x$ be a point on the Siegel upper half-plane $\mathbb{H}_g$ and let $[x]^T$ denote the equivalence class containing $x$ defined by left quotient of the Siegel upper space by $\Gamma^T(p)$. We have $\sigma.[y]^T = [\sigma y]^T$. The group $\Gamma^{P_{1k+1}}(p)$ is generated $\Gamma^{P_{1k}}(p)$ and $\rho_{k+1}$. So every element in $\Gamma^{P_{1k+1}}(p)$ can be written as a product of elements of the form $\gamma_i\rho_{k+1}$ with $\gamma_i \in \Gamma^{P_{1k}}(p)$. Define the equivalence class $[x]^{P_{1k}}$ similarly. The classes $[x]^{P_{1k}}$ and $[\gamma_i\rho_{k+1},x]^{P_{1k}}$ are $\rho_{k+1}$-connected. So the equivalence class $[x]^{P_{1k+1}}$ is obtained by joining the $\rho_{k+1}$-connected points. $\Box$

**Definition** For a subset $W \subset W_g$, we say that two points $x$ and $y$ on $A_g^{P_{1,P}}(p,n)$ are $W$-connected, if there exists a chain of points $x = x_1, \ldots, x_t = y$ on $A_g^{P_{1,P}}(p,n)$, such that for each $i$, $x_i$ and $x_{i+1}$ are $\sigma$-connected for some $\sigma \in W$.

**Proposition 1.2** Every fiber of the map $A_g^{B,P}(p,n) \to A_g^{P_{1k+1}}(p,n)$ is an equivalence class of $W_k$-connected points.

**Proposition 1.3** Every fiber of the map $A_g^P(pn) \to A_g^P(n)$ is an equivalence class of $\sigma$-connected points for some nonzero representative $\sigma$ of $W_P \backslash W_G$. 

Proof. The Bruhat-Tits decomposition modulo $p$ implies that the congruence groups $\Gamma^P(p)$ and $\sigma \Gamma^P(p)\sigma^{-1}$ generate $Sp(2g,\mathbb{Z})$. This is a consequence of the fact that any two conjugates of a maximal parabolic subgroup over $\mathbb{F}_p$ generate the whole algebraic group $GSp(2g,\mathbb{F}_p)$. □

1.6 $p$-old Siegel modular forms on $A^P_g(p)$

Let $B^P_g(n)$ denote the universal abelian variety over $A^P_g(n)$. The Hodge bundle $\omega$ is defined to be the pull back via the zero section $i_0 : A^P_g(n) \to B^P_g(n)$ of the line bundle $\wedge^{top}\Omega^1_{B_g(n)/A_g(n)}$. The Hodge bundle is an ample invertible sheaf on $A^P_g(n)$. Let $R$ be a $\mathbb{Z}[1/n]$-module. By a $\Gamma^P(n)$-Siegel modular form of weight $k$ with coefficients in $R$, we mean an element in $H^0(A^P_g(n)/R, \omega^\otimes k/R)$. The same notation is used for other congruence subgroups, but in this paper we focus on Siegel modular forms with respect to Siegel parahoric congruence subgroup $P$ and Borel congruence subgroup $B$.

If we pull back the Hodge bundle $\omega$ to the Siegel upper half-space, the pull back is canonically isomorphic to $O_{\mathbb{H}_g} \otimes_\mathbb{C} \wedge^q(\mathbb{C}^g)$. A complex modular form of weight $k$ becomes an expression of the form $f(\Omega), (dz_1 \wedge ... \wedge dz_g)^\otimes k$ where $f$ is an $\Gamma^P(n)$-invariant complex holomorphic function on $\mathbb{H}_g$ which is holomorphic at $\infty$. For genus $\geq 2$ the condition, holomorphic at infinity, is automatically satisfied by Koecher principle. Trivializing $\omega$ on $\mathbb{H}_g$, complex modular forms of weight $k$ are identified with holomorphic functions $f(\Omega)$ on $\mathbb{H}_g$, satisfying the transformation rule $f[\gamma]_k = f$ for all $\gamma \in \Gamma^P(n)$ where

$$f[\gamma]_k(\Omega) = \eta(\gamma)^{pk/2} \det(C.\Omega + D)^{-k} f(\gamma.\Omega).$$

Let $l$ be a prime not dividing $pn$. To any Siegel modular form $f$ of weight $k$ and level $n$, one associates an irreducible admissible representation $\pi = \bigotimes \pi_\sigma$ of $GSp(2g, \mathbb{A}_f)$ over $\mathbb{Q}_l$ [Asg-Sch]. This association is not unique, but we use it as a motivation to understand the notion of $p$-old form. Let $U$ denote the open subgroup of $GSp(2g, \mathbb{A}_f)$ associated to the congruence group $\Gamma^P(n)$. If $(p,n) = 1$ and $\pi^U \neq 0$, then $\pi_p$ is spherical, and it is the unique irreducible subquotient of some unramified principal series representation $\pi_\chi$ with respect to Borel subgroup $B(\mathbb{Q}_p)$. One can show that $\pi_{GSp(2g, \mathbb{Z}_p)}$ is one-dimensional. So the number of copies of modular forms with respect to $\Gamma^P(n)$ inside modular forms with respect to $\Gamma^P(pn)$ is equal to the dimension of $\pi_{GSp(2g, \mathbb{Z}_p)}$. The mod-$p$ Bruhat-Tits decomposition implies that, we have the following decomposition

$$GSp(2g, \mathbb{Q}_p) = \bigsqcup_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g \subset W_g} B(\mathbb{Q}_p)\sigma\Gamma^P(p).$$

So to specify $f \in \pi_{\Gamma^P(p)}$ it is enough to specify it on elements $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$. Because $W_P \simeq S_g$ and the subgroup $(\mathbb{Z}/2\mathbb{Z})^g \subset W_G$ is a complete set of representatives for $W_P/W_G$. Therefore, the space of $p$-old forms on $A^P_g(pn)$ consists of $2^g$ copies of
1.7 Atkin-Lehner correspondences

forms on $A_g^P(n)$. The vector space $\pi_1^{P}(p)$ has a basis consisting of functions $f_1, \ldots, f_{2g}$ where $f_i$ is supported on $B(\mathbb{Q}_p)\sigma_i \Gamma^P(p)$ and $f_i(\sigma_i) = 1$. The group of involutions $(\mathbb{Z}/2\mathbb{Z})^g \subset W_G$ acts on the space of $p$-old forms by $f(z) \mapsto f^\sigma(z) := f(\sigma.z)$ for $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$.

Similar considerations show that we expect $g!2^g$ copies of forms on $A_g^P(n)$ inside the space of $p$-old forms on $A_g^{B,P}(p,n)$. Following Atkin-Lehner theory, we need a geometric characterization of the space of $p$-old forms.

1.7 Atkin-Lehner correspondences

Let $\pi_{T,P} : A_g^T(p) \to A_g^P(n)$ and $\pi_{P_1,P_2} : A_g^{P_1}(p) \to A_g^{P_2}(p)$ denote the natural projection maps induced by inclusions of the corresponding congruence groups. Pulling back forms from level $n$ to level $np$ using the natural projection map $\pi_n : A_g^{B,P}(p,n) \to A_g^n(p)$ induces the first copy of $p$-old forms in $H^0(A_g^{B,P}(p,n), \omega^{\otimes k})$. For simplicity, let us forget the auxiliary level structure and consider the projection $\pi_1 : A_g^B(p) \to A_g$ and $p$-old forms in $H^0(A_g^B(p), \omega^{\otimes k})$. At first glance, it seems that geometric correspondences of the form $D^\sigma_B(p) = \pi_T^{*} v_p^\sigma \pi_T^{*}$ should induce more copies of $p$-old Siegel forms on $A_g^B(p)$ out of the pull-back copy.

But $\pi_T \circ \pi_1$ commutes with $v_p^\sigma$ for all $\sigma \in W_G$. Therefore, correspondences of above type generate the same copy as the pull back copy. To disturb the symmetry of the picture, we use Atkin-Lehner involution. $\pi_T \circ w_p^B \circ \pi_1$ no longer commutes with $v_p^\sigma$ and we can hope that correspondences of the form $C^\sigma_B(p) = \pi_T^{*} v_p^\sigma \pi_T^{*} w_p^B$ could generate more copies of $p$-old forms.

To generate $p$-old forms in $H^0(A_g^B(p), \omega^{\otimes k})$ we should use $g!2^g$ correspondences $C^\sigma_B(p)$ for $\sigma \in W_G$. 

**Main Theorem 1.4** The linear subspaces of $H^0(A^P_{g}(p), \omega \otimes k)$ generated by Atkin-Lehner correspondences $C_B(p)\pi^*_n$ where $C_B(p)$ is defined by $\pi_{T,B}^*\pi^*_T\pi^*_B w_p^*$

$$A^T_{g}(p, n) \xrightarrow{v_p^*} A^T_{g}(p, n)$$

$$A^B_{g}(p, n) \xrightarrow{w_p^B} A^B_{g}(p, n)$$

for $\sigma$ varying in $W_G$ give $g!2^g$ linearly independent copies of $H^0(A^P_{g}(n), \omega \otimes k)$ inside $p$-old forms of level $pn$ living on $A^B_{g}(p, n)$.

The corresponding theorem for $H^0(A^P_{g}(p), \omega \otimes k)$ can also be proved. However, in order to generate $p$-old forms in $H^0(A^P_{g}(p), \omega \otimes k)$ we should divide the space of $p$-old forms on $A^B_{g}(p)$ by the action of $(\mathbb{Z}/2\mathbb{Z})^{2g} \subset W_G$.

Main theorem is proved in a few stages. In the first stage, we prove that Atkin-Lehner involution induces a second copy of level-$n$ forms inside $p$-old part of level-$np$ forms which has trivial intersection with the pull-back copy.

**Proposition 1.5** $\pi^*_{B,P} H^0(A^P_{g}(p, n), \omega \otimes k)$ and $w^B_{p} \pi^*_B P H^0(A^P_{g}(p, n), \omega \otimes k)$ as subspaces of $H^0(A^B_{g}(p, n), \omega \otimes k)$ have trivial intersection.

**Proof.** Let $f$ and $g$ be nonzero Siegel modular forms in $\pi^*_{B,P} H^0(A^P_{g}(p, n), \omega \otimes k)$ with $w^B_{p} f = g$. Since $w^B_{p}$ is an involution, $f \pm g$ are eigenforms of $w^B_{p}$ and the proposition follows from the following

**Lemma 1.6** Any Siegel form which is eigenform of $w^B_{p}$ on $A^B_{g}(p, n)$ and also pull back of a Siegel form on $A^P_{g}(p, n)$ vanishes if $P \subsetneq B$.

**Proof.** The zero locus of such an eigenform is pull back of the zero locus of a form living on $A^P_{g}(p, n)$ and also $w^B_{p}$-invariant. This contradicts density of Hecke orbit [Cha]. □

In the second stage, we show that Atkin-Lehner correspondences induce $g!2^g$ non-intersecting copies of level-$n$ forms inside $p$-old part of level-$np$ forms.

**Lemma 1.7** Let $D_B^p(p) = \pi_{T,B}^* v^*_{p} \pi^*_T \pi^*_B$ where $\pi_{T,B} : A^T_{g}(p, n) \rightarrow A^B_{g}(p, n)$ is the natural projection. Then, for $\sigma, \sigma' \in W_G$ the correspondences $D_B^p(p) D_B^p(p)$ and $D_B^p(p)$ acting on any linear subspace of $H^0(A^B_{g}(p, n), \omega \otimes k)$ generate the same image subspaces.

**Proof.** For simplicity, let us forget the auxiliary $n$-level structure. Then, lemma follows from commutativity of the following diagram

$$A^T_{g}(p) \xrightarrow{v_p^*} A^T_{g}(p) \xrightarrow{v_p'^*} A^T_{g}(p)$$

$$\downarrow \quad \downarrow$$

$$A^\sigma_{g}^B \sigma' \sigma^{-1} \sigma^{-1} (p) \xrightarrow{\sigma} A^\sigma'_{g} \sigma' \sigma^{-1} (p) \xrightarrow{\sigma'} A^\sigma'_{g} B (p)$$

and, $v^*_{p} \circ v^*_{p}' = v^*_{p} \sigma'$ and $\sigma \circ \sigma' = \sigma \sigma' \sigma^{-1}$ hold for $\sigma, \sigma' \in W_G$. □
**Proposition 1.10** Every two linear subspaces of \( H^0(\mathcal{A}_g^{B,P}(p,n),\omega^{\otimes k}) \) generated by correspondences \( C_B^\sigma(p)\pi_n^\ast \) acting on \( H^0(\mathcal{A}_g^P(n),\omega^{\otimes k}) \) for \( \sigma \in \mathcal{W}_G \) have trivial intersection.

**Proof.** By the previous lemma, it is enough to show that the linear subspaces generated by correspondences \( \pi_{T,B}v_p^{\sigma}\pi_p^{T,B}w_p^{B*}\pi_1^* \) and \( w_p^{B*}\pi_1^* \) have such intersection. Suppose \( \pi_{T,B}v_p^{\sigma}\pi_p^{T,B}f = g \) for nonzero Siegel modular forms on \( \mathcal{A}_g^B(p) \) which are elements of \( w_p^{B*}\pi_n^*H^0(\mathcal{A}_g^P(n),\omega^{\otimes k}) \). Let \( d = \text{deg}(\pi_{T,B}) \). Then \( df \pm g \) are eigenforms of \( \pi_{T,B}v_p^{\sigma}\pi_p^{T,B} \) with eigenvalue \( \pm d \). Now, the truth of proposition is a consequence of the first stage and the following

**Lemma 1.9** Any nonzero Siegel modular form on \( \mathcal{A}_g^{B,P}(p,n) \) which is an eigenform of \( \pi_{T,B}v_p^{\sigma}\pi_p^{T,B} \) for a non-zero \( \sigma \) in \( \mathcal{W}_G \) is pull back of a Siegel form on \( \mathcal{A}_g^{P',P}(p,n) \) where \( P' \) is the parabolic subgroup generated by \( B \) and \( \sigma B \sigma^{-1} \).

**Proof.** This is a consequence of propositions 1.1. and 1.2. □

In the final stage we show that the above-mentioned \( g!2^g \) copies of level-\( n \) Siegel forms inside the space of Siegel forms of level \( pn \) are indeed \( p \)-old forms of level-\( pn \) and they are linearly independent.

**Proposition 1.10** The Atkin-Lehner correspondences \( C_B^\sigma(p)\pi_n^\ast \) commute with all Hecke correspondences which generate the local Hecke algebras \( H_q \) for \( q \) relatively prime to \( pn \).

**Proof.** The action of Atkin-Lehner correspondences can be interpreted in terms of the \( p \)-torsion of abelian varieties representing points of the moduli-space. By geometric base-change one can see that such an action commutes with those interpreted in terms of the \( q \)-torsion points. □

**Proof of the main theorem.** The \( g!2^g \) copies of level-\( n \) Siegel forms induced by Atkin-Lehner correspondences are contained in the space of \( p \)-old forms by previous proposition. Recall that in this paper Siegel modular forms with eigenvalues repeated from lower levels are considered to be old. The space of Siegel forms on \( \mathcal{A}_g^{B,P}(p,n) \) is finite-dimensional and has a basis of all prime to \( pn \) Hecke eigenforms. So is the case for any of the \( g!2^g \) Atkin-Lehner copies, by previous proposition. Fix a basis for the space of Siegel forms of level \( pn \) whose elements are eigenforms of all prime to \( pn \) local Hecke algebras. Suppose the \( g!2^g \) Atkin-Lehner copies are linearly dependent. It means that a non-zero eigenform \( f \) of all prime to \( pn \) local Hecke algebras is generated by some basis elements of the Atkin-Lehner copies which have the same eigenvalues. Consider the vector space \( V \) of all Siegel forms with the same Hecke eigenvalues \( f \) and consider the \( g!2^g \) Atkin-Lehner copies in it. \( V \) is invariant under the action of all the \( g!2^g \) correspondences \( \mathcal{D}_B^\sigma(p) = \pi_{T,B}v_p^{\sigma}\pi_p^{T,B} \) for \( \sigma \in \mathcal{W}_G \), because these correspondences commute with the Atkin-Lehner copies. By this symmetry, \( df + \pi_{T,P}v_p^{\sigma}\pi_p^{T,P}f \) is also a Hecke eigenform in \( V \) with the same Hecke eigenvalues as \( f \). There exists a \( \sigma \in \mathcal{W}_G \) which gives a nonzero Siegel form \( df + \pi_{T,B}v_p^{\sigma}\pi_p^{T,B}f \) generated by the Atkin-Lehner copies. But such a vector is a
Siegel form which can be pulled back from a lower Siegel space by proposition 1.2. This contradicts linear dependency. □

Having constructed the Atkin-Lehner copies of $p$-old forms on $A_g^{B,P}(p,n)$ one can get the $2^g$ copies of $p$-old forms on $A_g^P(pn)$ by pushing forward all the $g!2^g$ $p$-old copies down to $A_g^P(pn)$.

**Theorem 1.11** The linear subspaces of $H^0(A_g^P(pn), \omega^{\otimes k})$ generated by Atkin-Lehner correspondences $C_p^\sigma(p)\pi^*_\sigma$ for $\sigma$ varying in $W_G$ where $C_p^\sigma(p)$ is defined by

$$\sum \pi_{T,B}^* u_p^{\sigma*} \pi_{T,B}^* w_p^{B*}$$

where the sum ranges over $\eta \in S_g \subset W_G$ give $2^g$ linearly independent copies of $H^0(A_g^P(n), \omega^{\otimes k})$ inside $p$-old forms of level $pn$ living on $A_g^P(pn)$.

**Proof.** This is a direct consequence of the main theorem and proposition 1.5. □

## 2 Algebraic equations for towers of Siegel spaces

### 2.1 Construction of $A_g^P(p^k)$ from $A_g^P(p)$'s and $A_g^P(p^2)$'s

In this section, we follow Elkies, who did the genus one case [Elk]. Fix a prime $p > 1$. For positive $k$, the Siegel moduli space $A_g^P(p^k)$ parametrizes principally polarized abelian varieties with a cyclic $p^k$-isogeny, or equivalently sequences of $p$-isogenies

$$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k$$

such that the composite isogeny $A_{j-1} \rightarrow A_{j+1}$ of degree $p^{2g}$ is cyclic for each $j$ with $0 < j < k$. Thus for each $m = 0, 1, \ldots, k$ there are $k + 1 - m$ maps $\pi_j : A_g^P(p^k) \rightarrow A_g^P(p^{m})$ obtained by extracting for some $j = 0, 1, \ldots, k - m$ the cyclic $p^{m}$-isogeny $A_j \rightarrow A_{j+m}$ from the above sequence. In particular we have a tower of maps

$$A_g^P(p^k) \rightarrow A_g^P(p^{k-1}) \rightarrow A_g^P(p^{k-2}) \rightarrow \cdots \rightarrow A_g^P(p^2) \rightarrow A_g^P(p)$$

each map being of degree $p^g$. Each $A_g^P(p^k)$ also has an Atkin-Lehner involution $w_p = w_p^{(k)}$, taking a cyclic $p^k$-isogeny to its dual isogeny, and the above sequence to the sequence

$$A_k \rightarrow \cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0$$

of dual isogenies. We thus have

$$w_p^{(m)} \circ \pi_j = \pi_{k-m-j} \circ w_p^{(k)},$$

where $\pi_j, \pi_{k-m-j}$ are our $j$th and $(k-m-j)$th maps from $A_g^P(p^k)$ to $A_g^P(p^n)$.

Now the explicit formulas for $A_g^P(p), A_g^P(p^2)$, together with the involutions $w_p^{(1)}, w_p^{(2)}$ of these moduli spaces and the map $\pi_0 : A_g^P(p^2) \rightarrow A_g^P(p)$ between them, suffice to exhibit the entire tower explicitly:
2.2 Construction of $\mathcal{A}_g^P(n)$ from $\mathcal{A}_g^P(p^k)$'s

\begin{proposition}
For $k \geq 2$ the product map
\[ \pi = \pi_0 \times \pi_1 \times \pi_2 \times \cdots \times \pi_{k-2} : \mathcal{A}_g^P(p^k) \to (\mathcal{A}_g^P(p^2))^{k-1} \]
is a 1:1 map from $\mathcal{A}_g^P(p^k)$ to the set of $(P_1, P_2, \ldots, P_{k-1}) \in (\mathcal{A}_g^P(p^2))^{k-1}$ such that
\[ \pi_0(w_p^{(2)}(P_j)) = w_p^{(1)}(\pi_0(P_{j+1})) \]
for each $j = 1, 2, \ldots, k-2$.
\end{proposition}

**Proof.** Informally speaking, we get from $\mathcal{A}_g^P(p^2)$ up to $\mathcal{A}_g^P(p^k)$ by iterating $k-2$ times the involution $w_p^{(2)}$ composed with the "$p$-valued involution" $\pi_0^{-1}w_p^{(1)}\pi_0$. Of course the maps $\pi_j : \mathcal{A}_g^P(p^k) \to \mathcal{A}_g^P(p^m)$ (for $m \geq 2$) are then simply
\[ (P_1, \ldots, P_{k-1}) \mapsto (P_{j+1}, \ldots, P_{j+m-1}), \]
and the involution $w_p^{(k)}$ is
\[ (P_1, P_2, \ldots, P_{k-2}, P_{k-1}) \leftrightarrow (w_p^{(2)}P_{k-1}, w_p^{(2)}P_{k-2}, \ldots, w_p^{(2)}P_2, w_p^{(2)}P_1), \]
i.e. reversing the order of $P_1, \ldots, P_{k-1}$ and applying $w_p^{(2)}$ to each coordinate.

\begin{proof}
It is clear that the map is 1:1 to its image, because a sequence of $p$-isogenies is determined by the $p^2$-isogenies $A_{j-1} \to A_{j+1}$ parametrized by the $j$th coordinate of $\pi$ ($0 < j < k$). Now $(P_1, \ldots, P_{k-1})$ is in the image of $\pi$ if and only if the $p^2$-isogenies parametrized by $P_1, \ldots, P_{k-1}$, regarded as sequences $A_0^i \to A_1^i \to A_2^i$ of $p$-isogenies, fit together to form a sequence with $A_i^1 = A_{i+j}$, i.e. if and only if the isogenies $A_j^1 \to A_2^j$ and $A_0^{j+1} \to A_1^{j+1}$ coincide for each $j = 1, 2, \ldots, k-2$. But these isogenies are represented by the points $\pi_1(P_j)$ and $\pi_0(P_{j+1})$ on $\mathcal{A}_g^P(p)$. Thus the necessary and sufficient condition is that
\[ \pi_1(P_j) = \pi_0(P_{j+1}) \]
for each $j = 1, 2, \ldots, k-2$; applying $w_p^{(1)}$ to both sides, and then commutativity of the Atkin-Lehner involutions to $w_p^{(1)}(\pi_1(P_j))$, then yields the equivalent form of what we seek. \end{proof}

\section{Construction of $\mathcal{A}_g^P(n)$ from $\mathcal{A}_g^P(p^k)$'s}

\begin{proposition}
Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime decomposition of an integer $n \geq 2$ then if $\pi_j : \mathcal{A}_g^P(p_j^{\alpha_j}) \to \mathcal{A}_g^P(1)$ denotes the natural projection forgetting the level structure for $j = 1, \ldots, k$, then $\mathcal{A}_g^P(n)$ is nothing but the fiber product of $\mathcal{A}_g^P(p_j^{\alpha_j})$'s over $\mathcal{A}_g^P(1)$ via $\pi_j$.
\end{proposition}

In fact, the following is true:
Proposition 2.2 Let \( n \) and \( m \) relatively prime natural numbers. Let \( \pi' : A_g^P(n) \to A_g(1) \) and \( \pi'' : A_g^P(m) \to A_g(1) \) denote the natural projections forgetting the level structures, then
\[
A_g^P(nm) = A_g^P(n) \times_{A_g(1)} A_g^P(m)
\]
where \( \times \) is the fiber product over \( A_g(1) \) via \( \pi' \) and \( \pi'' \).

**Proof.** Considering the moduli interpretation of the above mentioned moduli spaces, the fact that \( n \)-level structures are independent of \( m \)-level structures for \( (n, m) = 1 \) and fixing these two we can get an \( mn \)-level structure gives a one-to-one map from the right hand side to the left. \( \square \)

By the previous section, once we have algebraic equations for \( A_g^P(p) \) and \( A_g^P(p^2) \) and morphisms over \( A_g(1) \) we get algebraic equations for \( A_g^P(p^k) \) over \( A_g(1) \) for all primes \( p \) and this will suffice to get algebraic equations for all \( A_g^P(n) \) as we desire.

### 2.3 Compactification of Siegel moduli spaces

The space of Siegel modular forms can also be formulated in the language of schemes. Let \( S \) be a base scheme. A modular form of weight \( k \) is a rule which assigns to each principally polarized abelian variety \( (A/S, \lambda) \) a section \( f(A/S, \lambda) \) of \( \omega^k_{A/S} \) over \( S \) depending only on the isomorphism class of \( (A/S, \lambda) \) commuting with arbitrary base change. Here \( \omega_{A/S} \) is the top wedge of tangent bundle at origin of \( A \) over \( S \).

To define Siegel modular forms of higher level, one should equip principally polarized abelian varieties with level structures. Let \( \zeta_n \) denote an \( n \)-th root of unity where \( n \geq 3 \). On a principally polarized abelian scheme \( (A, \lambda) \) over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \) of relative dimension \( g \) we define a symplectic principal level-\( n \) structure to be a symplectic isomorphism \( \alpha : A[n] \to (\mathbb{Z}/n\mathbb{Z})^{2g} \) where \( (\mathbb{Z}/n\mathbb{Z})^{2g} \) is equipped with the standard non-degenerate skew-symmetric pairing
\[
\langle , \rangle : (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to \mathbb{Z}/n\mathbb{Z}
\]
\[
\langle (u, v), (z, w) \rangle \mapsto u.w^t - v.z^t
\]

Let \( S \) be a scheme over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \). The moduli scheme classifying the principally polarized abelian schemes over \( S \) together with a symplectic principal level-\( n \) structure is a scheme over \( S \) and will be denoted by \( A_g(n) \). The moduli scheme \( A_g(n) \) over \( S \) can be constructed from \( A_g(n) \) over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \) by base change.

\( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) acts as a group of symmetries on \( A_g(n) \) by acting on level structures. We will recognize these moduli spaces and their equivariant quotients under the action of subgroups of \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) as Siegel spaces. We restrict our attention to Siegel spaces over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \). \( A_g(n) \) is connected and smooth over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \). The condition \( n \geq 3 \) is to guarantee that we get a moduli scheme, instead of getting only a moduli stack. The natural morphism \( A_g(n) \to A_g(m) \) where \( m, n \) are positive integers \( \geq 3 \) and \( m|n \) is a finite and etale morphism over \( \text{Spec}(\mathbb{Z}[\zeta_n, 1/n]) \).
2.4 Algebraic equations for \( A_2^2(2) \)

Let \( B_g(n) \) denote the universal abelian variety over \( A_g(n) \). The Hodge bundle \( \omega \) is defined to be the pull back via the zero section \( i_0 : A_g(n) \rightarrow B_g(n) \) of the line bundle \( \wedge^{tor}\Omega_{B_g(n)/A_g(n)} \). The Hodge bundle is an ample invertible sheaf on \( A_g(n) \) and can be naturally extended to a bundle \( \omega \) on \( A_g^*(n) \). We could define the minimal compactification \( A_g^*(n) \) by the formula

\[
A_g^*(n) = \text{proj}(\oplus_{k \geq 0} H^0(A_g(n), \omega^{\otimes k})).
\]

The graded ring above is regarded as a \( \mathbb{Z}[\zeta, 1/n] \)-algebra. The scheme \( A_g^*(n) \) is equipped with a stratification by locally closed subschemes which are geometrically normal and flat over \( \text{Spec} \mathbb{Z}[\zeta, 1/n] \). Each of these strata is canonically isomorphic to a moduli space \( A_i(n) \) for some \( i \) between 0 and \( g \). The map \( A_g(n) \rightarrow A_g(m) \) can be extended uniquely to \( A_g^*(n) \rightarrow A_g^*(m) \) for \( m|n \). These maps when restricted to strata, induce the corresponding natural maps between lower genera Siegel spaces \( A_i(n) \rightarrow A_i(m) \). The action \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) on \( A_g(n) \) naturally extends to an action on the compactified Siegel space \( A_g^*(n) \). This action is compatible with the maps \( A_g^*(n) \rightarrow A_g^*(m) \) for \( m|n \).

Let \( K(n) \) denote the subgroup of \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) fixing the \( g \) first \( (\mathbb{Z}/n\mathbb{Z}) \)-basis elements of \( (\mathbb{Z}/n\mathbb{Z})^{\oplus 2g} \) on which \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \) acts. Since \( A_g^*(n) \) is a projective scheme, we can define the quotient projective schemes \( A_g^{P*}(n) \) to be the geometric quotient of \( A_g^*(n) \) by \( K(n) \). This quotient provides us with a compactification of \( A_g^{P*}(n) \) which is the moduli scheme of principally polarized abelian schemes \( (A, \lambda) \) over \( \text{Spec} \mathbb{Z}[\zeta, 1/n] \), together with \( g \) elements in \( A[n] \) generating a symplectic subgroup isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^g \). Again we have natural maps \( A_g^{P*}(n) \rightarrow A_g^{P*}(m) \) for \( m|n \).

We define the Hodge bundle \( \omega \) on \( A_g^{P*}(n) \) to be the quotient of the Hodge bundle \( \omega \) on \( A_g^*(n) \) under the action of the corresponding subgroup \( K(n) \) of \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \). This is possible because the line bundle \( \omega \) on the space \( A_g^*(n) \) is \( Sp(2g, \mathbb{Z}/n\mathbb{Z}) \)-linearizable. A Siegel modular form of weight \( k \) and full level \( n \) is a global section of \( \omega^k \) on \( A_g^*(n) \). Over the complex numbers, this corresponds to a Siegel modular form of weight \( k \) with respect to \( \Gamma(n) \). In this paper, by a Siegel modular form of weight \( k \) and level \( n \) we mean a global section of \( \omega^k \) on \( A_g^{P*}(n) \). This corresponds to the congruence subgroup \( \Gamma_0(n) \).

2.4 Algebraic equations for \( A_2^2(2) \)

In this section, we follow Lee and Weintraub [Lee-Wei1-5]. For construction of compactifications of \( A_2^2(1) \) look at [Lee-Wei3]. The compactification we have built in the previous section is called the Satake compactification which is a projective algebraic variety with severe singularities. It would be also handy to introduce algebraic equations for smooth compactifications as was constructed by Igusa for \( g = 2 \) and generalized by Mumford and his collaborators to general genus and extended to schemes by Falting and Chai [Fal-Chai] which is called the toroidal compactification.
The Siegel moduli space $A^{\ast}_{2}(2)$ is related another variety which appeared in the work of Deligne and Mostow [Del-Mos], constructed by means of Mumfords geometric invariant theory.

Let $S$ denote the set $\{1, 2, 3, 4, 5, 6\}$, and let $\mathbb{P}^{5}$ denote the space of functions of $S$ to $\mathbb{P}^{1} = \mathbb{P}^{1}(\mathbb{C})$. There is a natural action of $PGL_{2}(\mathbb{C})$ on the space $\mathbb{P}^{1}$ induced by the linear fractional transformation of $PGL_{2}(\mathbb{C})$ on $\mathbb{P}^{1}$. The subspace of injective functions can be identified with $(\mathbb{P}^{1})^{6} - \Delta$, and its quotient with the moduli space $\mathcal{M}$ of nonsingular curves with level 2 structure. By a stable point (resp. semi-stable point) of $\mathbb{P}^{5}$, we mean a point with the property that no more than two (resp. three) elements in $S$ have the same image. The group $PGL_{2}$ operates freely on the subspace of stable points, and its quotient space $\mathcal{Q}_{st}$, is a quasi-projective variety. To compactify $\mathcal{Q}_{st}$, we consider the space of semi-stable points. Define an equivalence relation for which two stable points are equivalent if and only if they have the same $PGL_{2}$-orbit and if two points are semi-stable but not stable they are equivalent if they induce the same partition of $S$ into two sets of three elements $S_{1}$ and $S_{2}$ such that each function separates them and on constant on one of them. The quotient space $\mathcal{Q}_{sst}$ of semi-stable points module this relation is a projective variety, and contains $\mathcal{Q}_{st}$ as a Zariski open set. In fact, $\mathcal{Q}_{cusp} = \mathcal{Q}_{sst} - \mathcal{Q}_{st}$, consists of ten isolated singular points. To desingularize this variety, we blow up these points and obtain a nonsingular variety $\mathcal{Q}_{sst}^{\sim}$.

**Proposition 2.3** $\mathcal{Q}_{sst}^{\sim}$ is isomorphic to the Igusa compactification $A^{\sim}_{2}(2)$

The Igusa compactification $A^{\sim}_{2}(2)$ of $A_{2}(2)$ may be constructed by desingularizing, or blowing up, the Satake compactification of $A^{\ast}_{2}(2)$. Lee and Weintraub construct a birational transformation $f : A^{\ast}_{2}(2) \rightarrow \mathcal{Q}_{sst}$. To begin, they identify $\mathcal{Q}_{sst}$, with a classical object, Segres cubic threefold. From this it follows that $\mathcal{Q}_{sst}$ is isomorphic to the threefold in $\mathbb{P}^{5}$ defined by the homogeneous equations

$$\sum_{i=1}^{6} x_{i} = 0$$

$$\sum_{i=1}^{6} x_{i}^{3} = 0$$

known as Segres cubic threefold. Since Segres time it has been known that considering the dual hypersurface to the symmetric quartic threefold defined by the equations

$$\sum_{i=1}^{6} x_{i} = 0$$

$$\left(\sum_{i=1}^{6} x_{i}^{4}\right) - \left(\sum_{i=1}^{6} x_{i}^{2}\right)^{2} = 0$$
yields Segres cubic threefold. van der Geer [vdG] has shown that the quartic threefold defined above can be identified with the Satake compactification $A^*_2(2)$, so one obtains a birational transformation

$$\tilde{f} : A^*_2(2) \to Q_{sst}$$

Alternately, we may consider the Igusa compactification $A\sim_2(2)$. Note that the Satake compactification $A^*_2(2)$ is a hypersurface in $\mathbb{P}^5$ defined by a single function $F(x_1, ..., x_5) = 0$. The derivatives $\partial F/\partial x_i$, on the one hand define the coordinate functions of the projective dual, and on the other hand generate the ideal $I = (\partial F/\partial x_1, ..., \partial F/\partial x_5)$ that defines the boundary components $\partial$. Since $A\sim_2(2)$ is defined by blowing up $A^*_2(2)$ along $\partial$, it follows that $\tilde{f}$ lifts to a morphism from $A\sim_2(1)$ to the projective dual of $Q_{sst}$. one can blow down the ten components of the Humbert surface in $A\sim_2(1)$ to points to get a complex analytic space $A^*_2(2)$. Then, from the definitions, we have the mapping $\tilde{f}$ in the following diagram:

$$\tilde{f} : A\sim_2(2) \to Q_{sst}$$

$$\tilde{f} : A^*_2(2) \to Q^*_sst$$

It can be shown that $\tilde{f}$ and $f^\sim$ are isomorphisms.

By division by the two-level structure on $Q_{sst}$, we can find the ring of invariants of $A^*_2(1)$ and the relative morphism of $A^*_2(2)$ over $A^*_2(1)$.

### 2.5 Algebraic equations for $A^*_2(3)$

In this section, we follow Hoffman and Weintraub [Hof-Wei]. Here, one uses a variety $B$ defined over $\mathbb{Q}(\sqrt{-3})$ such that $B$ over $\mathbb{C}$ is isomorphic with $A^*_2(3)$. Felix Klein initiated the study of the moduli spaces of genus 2 curves and the coverings defined by "Stufe". Two of his students, H. Burkhardt and H. Maschke, took up the case where Stufe = 3. Burkhardt managed to write down an explicit equation for this moduli space. The general idea is this: Consider the 9 thetanullwerte

$$X_{\alpha,\beta} = \theta \left( \begin{array}{cc} 0 & 0 \\ \alpha & \beta \end{array} \right) (\tau, 0), \quad \alpha \in 1/3\mathbb{Z}/\mathbb{Z} \quad \beta \in 1/3\mathbb{Z}/\mathbb{Z}$$

These 9 values have the property that as $\tau \mapsto \gamma \tau$ with $\gamma \in \Gamma_2(1)$ they undergo a linear transformation, which is the identity up to scalar multiples for $\gamma \in \Gamma_2(3)$. In other words, we have a projective representation of the finite simple group of order 25920, $G = \Gamma_2(1)/\Gamma_2(3) = Sp(4;F_3)$. This representation splits into two invariant subspaces of dimensions 4 and 5 respectively, the spaces of

$$Z_{\alpha,\beta} = (X_{\alpha,\beta} - X_{-\alpha,-\beta})/2 \quad Y_{\alpha,\beta} = (X_{\alpha,\beta} + X_{-\alpha,-\beta})/2$$
Maschke studied the action of $G$ on the $Z$’s, Burkhardt studied the action on the $Y$’s, and both managed to find the ring of $G$-invariant forms in their respective cases. Let

$$Y_0 = Y_{0,0}, \quad 2Y_1 = Y_{1/3,0}, \quad 2Y_2 = Y_{0,1/3}, \quad 2Y_3 = Y_{1/3,1/3}, \quad 2Y_4 = Y_{1/3,2/3}.$$ 

Burkhardt found the invariant form of degree 4:

$$J_4 = Y_0^4 - Y_0(Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3) + 3Y_1Y_2Y_3Y_4.$$

**Proposition 2.4** Let $\mathcal{B}_0 \subset \mathbb{P}^4$ be the quartic hypersurface defined by $J_4 = 0$. There is an isomorphism between a Zariski open subset of $\mathcal{A}_2(3)$ and a Zariski open subset of $\mathcal{B}_0$. Let $\mathcal{B}$ be the variety obtained by resolving the 45 nodes on $\mathcal{B}_0$. The map above extends to an isomorphism with the Igusa compactification: $\mathcal{B}_0 \simeq \mathcal{A}_2^*(3)$. 

In this form the proposition was first proved by van der Geer [vdG2], who asserted some thing stronger, namely that these results were true for the corresponding schemes over $\mathbb{Z}[1/3; \epsilon]$, where $\epsilon$ is a primitive cube root of unity (the existence of a model of $\mathcal{A}_2^*(3)$ over that ring being a consequence of Faltings’ theory [Cha-Fal]).

Since we have the group action explicitly, we can find the ring of invariants of $\mathcal{A}_2^*(1)$ and the relative morphism of $\mathcal{A}_2^*(3)$ over $\mathcal{A}_2^*(1)$.

### 2.6 Algebraic equations for $\mathcal{A}_2^*(4)$

In this section, we follow van Geemen and Nygaard [vGe-Nyg] and its exposition by Okazaki and Yamauchi [Oka-Yam]. Let $\mathcal{A}_2(2,4,8)$ be the moduli space of abelian surfaces with some level structure which has been studied by van Geemen and Nygaard. It is the quotient space of the Siegel upper half plane of degree 2 by the arithmetic subgroup $\Gamma(2,4,8)$ of the symplectic group $Sp_4(\mathbb{Z})$. This congruence subgroup $\Gamma(2,4,8)$ is contained in the principal congruence subgroup $\Gamma(4) := \{ \gamma \in Sp_4(\mathbb{Z}) \mid \gamma \equiv 1_4 \mod 4 \}$. $\mathcal{A}(2,4,8)$ is a quasi-projective smooth threefold. By [?] we have the projective model $\mathcal{A}_2^*(2,4,8)$ the Satake compactification of $\mathcal{A}_2(2,4,8)$ which is defined over $\mathbb{Q}$ in $\mathbb{P}^{13}$ as follows:

\[
\begin{align*}
Y_0^2 &= Q_0(X_0, X_1, X_2, X_3) := X_0^2 + X_1^2 + X_2^2 + X_3^2 \\
Y_1^2 &= Q_1(X_0, X_1, X_2, X_3) := X_0^2 - X_1^2 + X_2^2 - X_3^2 \\
Y_2^2 &= Q_2(X_0, X_1, X_2, X_3) := X_0^2 + X_1^2 - X_2^2 - X_3^2 \\
Y_3^2 &= Q_3(X_0, X_1, X_2, X_3) := X_0^2 - X_1^2 - X_2^2 + X_3^2 \\
Y_4^2 &= Q_4(X_0, X_1, X_2, X_3) := 2(X_0X_1 + X_2X_3) \\
Y_5^2 &= Q_5(X_0, X_1, X_2, X_3) := 2(X_0X_2 + X_1X_3) \\
Y_6^2 &= Q_6(X_0, X_1, X_2, X_3) := 2(X_0X_3 + X_1X_2) \\
Y_7^2 &= Q_7(X_0, X_1, X_2, X_3) := 2(X_0X_1 - X_2X_3) \\
Y_8^2 &= Q_8(X_0, X_1, X_2, X_3) := 2(X_0X_2 - X_1X_3) \\
Y_9^2 &= Q_9(X_0, X_1, X_2, X_3) := 2(X_0X_3 - X_1X_2).
\end{align*}
\]
Since we have the group action of $Sp_4(\mathbb{Z})$ explicitly, we can find the ring of invariants of $A^*_2(4)$ and $A^*_2(1)$ and the relative morphism of $A^P_2(4)$ over $A^*_2(1)$.

### 2.7 Algebraic equations for $A^P_2(2^k)$ and $A^P_2(3.2^k)$

By section 2.1, we can get algebraic equations for $A^P_2(2^k)$ using algebraic equations for $A^P_2(4)$ and $A^P_2(2)$ over $A^*_2(1)$. By section 2.2, using algebraic equations for $A^P_2(2^k)$ and $A^P_2(3)$ over $A^P_2(1)$ one can get algebraic equations for $A^P_2(3.2^k)$.

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