Density-Matrix Approach to a Strongly Coupled Two-Component Bose-Einstein Condensate

Andal Narayanan *

Non-Accelerator Particle Physics Group
Indian Institute of Astrophysics, Koramangala, Bangalore - 560 034, INDIA

and

Hema Ramachandran
Raman Research Institute, Sadashivnagar, Bangalore - 560 080, INDIA

The time evolution equations for average values of population and relative phase of a strongly coupled two component Bose-Einstein condensate (BEC) is derived analytically. The two components are two hyper-fine states, which are coupled by an external laser that drives fast Rabi oscillations between these states. Specifically, this derivation incorporates the two-mode model proposed in [1] for the strongly coupled hyper-fine states |1, −1⟩ and |2, 1⟩ of ^87Rb. The fast Rabi cycles are averaged out and the rate equations so derived represent the slow dynamics of the system. These include the collapse and revival of Rabi oscillations and their dependence on detuning and trap displacement as reported in experiments of [1]. A procedure for stabilising vortices is also suggested.

I. INTRODUCTION

Observations of Bose-Einstein Condensation in trapped dilute alkali atoms have opened up both experimental and theoretical challenges to understand the properties of such systems. The dynamical properties of a single condensate such as its collective excitations in

*e-mail: andal@iiap.ernet.in
a trap due to a time dependent drive has been one such area of research \[2\], \[3\]. The experimental realisation of simultaneous creation and confinement of Bose- Einstein condensates (BECs) in several hyperfine states of a given species of atom \[4\], \[5\], \[6\] has led to investigations on dynamics of two or more overlapping condensates by coupling them with an externally applied laser field. In particular experimental realisation of binary mixtures of two hyperfine states namely \(|1, -1\rangle\) and \(|2, 1\rangle\) of \(^{87}\text{Rb}\) has established the following properties of this coupled condensed system \[7\], \[8\].

(a) These two states have magnetic moments which are same to the first order. However, due to other small effects such as gravity, the nuclear magnetic moment and nonlinearity in the Zeeman shifts, the location of minima for the two states in the trap can be adjusted to be slightly different or exactly coincident.

(b) Spontaneous inter-conversion from one state to the other is not seen due to the large difference in internal energies between these two states. The hyperfine energy is 6.8 GHz. This makes the two condensates distinguishable. These can be selectively imaged by choice of an appropriate laser.

(c) These condensate states possess a relative quantum phase which can be measured. This phase evolves with time the rate being proportional to the chemical potential difference between the two condensates.

(d) An external laser drive couples these two systems and helps to coherently transfer population from one state to the other.

The mixed condensates thus offer an ideal experimental apparatus to look for macroscopic realisations of dynamical effects like standard Josephson effects \[9\].

Theoretical calculations on such Josephson like oscillations in these coupled Boson Josephson Junctions (BJJ) \[10\], \[11\] have shown several interesting dynamical effects. Recently however \[12\], an experimental observation of an unexpected behaviour of these
coupled systems was reported. In the limit of sustained and large field strengths of the external coupling laser, that is when $\Omega$, the Rabi frequency, was five to ten times larger than the trap frequency in the vertical direction, along which the two condensates sit displaced \[12\], the Rabi oscillations between the hyperfine states was found to collapse and revive. This occurred on a time scale which is large compared to the Rabi period. These slow varying modulations of the fast Rabi oscillations vanish at zero trap displacement. It was also seen to vanish when $\delta = 0$, where $\delta = \omega - \omega_d$ is the detuning of the external laser frequency ($\omega_d$) from the transition frequency between the hyperfine states ($\omega$). It was shown subsequently in the same paper that this phenomenon was due to a weak coupling between the low lying motional states of the trap. In particular a simplified two-mode model was suggested. In this two-mode model the trap ground state and first excited dipole state were coupled and couplings to all higher motional states were neglected. Thus the paper \[1\], demonstrates the possibility of quantum state engineering of topological excitations, through the interplay between the internal and motional degrees of freedom of a BEC in a TOP trap. Numerical simulations by solving the Gross-Pitaeskii (GP) equations for the coupled system were carried out in \[1\] which reproduced the experimental features.

In the present paper, we derive the essential experimental features analytically, using the density matrix approach. Equations for the fractional population ($Z$) in the hyperfine states and their relative phase ($\theta$) as a function of time have been obtained. Averaging over the Rabi period, these equations represent the slow dynamics of the system. Collapse and revivals of Rabi oscillations and their dependence on detuning and trap displacement are seen to match qualitatively with the experimental results described in \[1\]. A proposal for combining this strongly coupled regime to the weakly coupled Josephson regime is presented and its role in increasing the stability of vortices \[14\], \[15\], \[16\] is also speculated.
II. A DENSITY-MATRIX METHOD

The total wave-function of the two-component condensate is denoted by $\psi(r, t)$. Initially this wave-function just represents the total population ($N_T$) in the ground state $|1, -1\rangle$. An external laser drives the transition from this state to the $|2, 1\rangle$ state coherently. So we can write,

$$\psi(r, t) = \psi_1(r, t) + \psi_2(r, t)$$

These two states are of the form

$$|\psi_1\rangle = (\alpha_1(t)c_0(t)|\phi_0\rangle + \alpha_2(t)d_1(t)|\phi_1\rangle)|1\rangle \quad (1)$$
$$|\psi_2\rangle = (\alpha_2(t)c_0(t)|\phi_0\rangle + \alpha_1^*(t)d_1(t)|\phi_1\rangle)|2\rangle \quad (2)$$

Here $|1\rangle$ and $|2\rangle$ refer to the hyperfine (internal) states and $|\phi_0\rangle$ and $|\phi_1\rangle$ refer to the motional (external) states and

$$\alpha_1(t) = \cos\left(\frac{\Omega_{eff} t}{2}\right) - i\left(\frac{\delta}{\Omega_{eff}}\right)\sin\left(\frac{\Omega_{eff} t}{2}\right)$$
$$\alpha_2(t) = -i\left(\frac{\Omega_{01}}{\Omega_{eff}}\right)\sin\left(\frac{\Omega_{eff} t}{2}\right)$$
$$\Omega_{eff} = \sqrt{\delta^2 + \Omega^2}$$
$$c_0(t) = \cos\left(\frac{\Omega_{01} t}{2}\right) - i\left(\frac{\Delta e_{01}}{\Omega_{01}}\right)\sin\left(\frac{\Omega_{01} t}{2}\right)$$
$$d_1(t) = -i\left(\frac{2\beta <z>}{\Omega_{01}}\right)\sin\left(\frac{\Omega_{01} t}{2}\right)$$
$$\Omega_{01} = \sqrt{4\beta^2 <z>^2 + \Delta e_{01}^2}$$
$$<z>_{ij} = \int \phi_i z \phi_j dz$$
$$\beta = \frac{z_0 \delta \Omega}{\Omega_{eff}^2}$$
$$\Delta e_{01} = e_1 - e_0$$

Here $\delta$ is the detuning, $\Delta e_{01}$ is the energy difference between the two trap states, namely the ground state and the first excited dipole state. This energy difference is held fixed.
through the derivation, while in actuality they will vary with time. $z_0$ is the displacement between the two condensates and $<z>_{ij}$ is the dipole matrix element which couples the ground and excited states of the trap. In this derivation $<z>_{ij}$ is held fixed at $<z>_{01}$. The higher couplings are weak and hence neglected.

Taking equations (1) and (2) as starting points, we point out that the $|\psi_1\rangle$ and $|\psi_2\rangle$ individually satisfy the following normalized coupled Gross-Pitaeskii (GP) equation in the Thomas-Fermi limit in an isotropic trap.

$$\frac{d\psi_1(t)}{dt} = \frac{1}{i}\left\{[2z_0 < z > + \lambda_1 N_1 + N_2 + \frac{\delta}{\omega z}]\psi_1(t) + \frac{\Omega}{\omega z}\psi_2(t)\right\}$$  \hspace{1cm} (3)

$$\frac{d\psi_2(t)}{dt} = \frac{1}{i}\left\{[-2z_0 < z > + \lambda_2 N_2 + N_1 - \frac{\delta}{\omega z}]\psi_2(t) + \frac{\Omega}{\omega z}\psi_1(t)\right\}$$  \hspace{1cm} (4)

$$N_T = N_1 + N_2$$

$$a_\perp = \sqrt{\frac{\hbar}{m\omega}}$$

$$a = a_{11} \sim a_{22}$$

$$\lambda_1 = \frac{a_{11}}{a_{12}}$$

$$\lambda_2 = \frac{a_{22}}{a_{12}}$$

In writing the above set of coupled equations, time is in units of trap frequency $\omega_z$ and the spatial variables are scaled by $a_\perp$. $a_{11}$ and $a_{22}$ are respectively the s wave scattering lengths for the two hyperfine species of the condensates and $a_{12}$ is the inter-species scattering length. The various energy terms are given with respect to the trap energy level $\hbar\omega_z$. Due to a near degeneracy of $a_{11}$ and $a_{22}$ scattering lengths in the case of $^{87}Rb$ the approximation that they are equal can be safely carried out. The system is characterized then by a single scattering length $a$ and a single $\lambda = \lambda_1 \sim \lambda_2$. In deriving these equations, the spatial dependence of the GP wave-functions are integrated out (Adiabatic approximation) with respect to the trap wave-functions, namely $|\phi_0(z)\rangle$ and $|\phi_1(z)\rangle$ and treated as constants. This assumes that the specific changes in the shape of the trap wave-function is not playing a major role in the
time evolution of the system. This assumption is an approximation over what is actually experimentally seen since we are only interested in the dynamics of the fractional population of the hyperfine states.

We proceed to derive the population fractions and the relative phase differences between the two hyperfine states by noting that

\[ |1\rangle = \sqrt{N_1(t)} e^{i\varphi_1(t)} \]
\[ |2\rangle = \sqrt{N_2(t)} e^{i\varphi_2(t)} \]
\[ Z = \frac{N_1 - N_2}{N_T} \]
\[ \theta = \varphi_2 - \varphi_1 \]

By taking appropriate inner products of \(|\psi_1\rangle\) and \(|\psi_2\rangle\) with \(|\phi_0\rangle\) and \(|\phi_1\rangle\) and substituting for \(|\psi_1\rangle\) and \(|\psi_2\rangle\) in the GP equations, the form given in equations (1) and (2), the following rate equations for average values of fractional population \(\langle Z \rangle\) and relative phase \(\langle \theta \rangle\) in the two hyperfine states can be derived.

\[
\langle \dot{Z} \rangle = \left( \frac{\beta^2 <z>^2}{\Omega_{01}} \sin (\Omega_{01} t) \right) Z \quad (10)
\]
\[
\langle \dot{\theta} \rangle = 4 \left( \Delta e_{01} \frac{\beta^2 <z>^2}{\Omega_{01}^2} \sin^2 (\Omega_{01} t) \right) \left[ \frac{1}{\cos^2 \left( \frac{\Omega_{01} t}{2} \right)} + \left( \frac{\Delta e_{01}}{\Omega_{01}} \right)^2 \sin^2 \left( \frac{\Omega_{01} t}{2} \right) \right] \quad (11)
\]

In deriving the above equations the orthonormal relations of trap wave-functions are assumed to be

\[ \langle \phi_i | \phi_j \rangle = \delta_{ij} \]

Equations (1) and (2) are obtained after averaging over the fast time period namely that of \(\Omega\) in the problem. So these equations do not explicitly contain \(\Omega\). Analytical expression for \(\langle Z \rangle\) can then be derived.

\[
\langle Z \rangle = Z_0 \exp \left( \frac{\beta^2 <z>^2}{\Omega_{01}^2} \left[ 1 - \cos \Omega_{01} t \right] \right) \quad (12)
\]
In the limit of small detuning $\delta$ ($\delta \ll \Omega$), this is of the form

$$\langle Z \rangle = Z_0 \left[ 1 + 2 \beta^2 \langle z \rangle^2 \Omega^2_0 \sin^2 \left( \frac{\Omega_{01} t}{2} \right) \right]$$  \hspace{1cm} (13)

Here $Z_0$ is the initial value of the population at time $t = 0$.

**III. RESULTS AND DISCUSSIONS**

The equation (13) has all the essential features which are reported by the experiments and subsequent numerical investigation in [1].

(1) The $\langle Z \rangle$, remains a constant when $\delta$ goes to zero or when $z_0$ goes to zero. That is, in the lab frame the fast Rabi oscillations remain unmodulated.

(2) Equation (13) is derived with the implicit assumption that the condensate has a well defined overall phase which can be measured relative to a reference. A decoupling of $\langle \theta \rangle$ and $\langle Z \rangle$ in the time averaged frame over the fast variable occurs as this phase (both the slow and fast varying part) averages to zero.

(3) Though the mean-field term does not explicitly enter the expression (13), we can see that the amplitude of modulation increases with decreasing $\Delta e_{01}$ - a result which is confirmed by numerical simulations in [1] which predicts a decrease in $\Delta e_{01}$ for enhanced mean-field effects.

(4) This form of equation (13) does not give rise to the chaotic behaviour with high values of $z_0$ reported in [1].

Figure 1 gives a typical curve for the parameters given.

In the paper [13], a preparation of the vortex mode is presented in this very two-component system. In this contest it is tempting to think of the following scheme for stabilising such vortices. Starting from the strong coupling regime ($\Omega > \omega_z$), the strength
of the external laser field could be gradually decreased over time. At sometime then, when \( \Omega < \omega_z \), the population in a particular state (which is a combination of motional and internal states) gets trapped in that state itself, due to Macroscopic Quantum State Trapping (MQST) effects \([1]\), applicable in this regime. More specifically, if such trapping should occur in the first excited motional state, which could be a vortex state, then there seems to be a tremendous improvement achieved in the stability of the vortex. A schematic of such a scenario is given in Figure 2.

IV. CONCLUSIONS

In this paper we have analytically derived an expression for the rate of change of fractional population of the hyperfine states \(|1, -1\rangle\) and \(|2, 1\rangle\) of \(^{87}\text{Rb}\) in the strong coupling regime using the density matrix approach. The derivation gives analytical results for population evolution after averaging out the fast dynamical variable namely the Rabi period in the problem. This derivation is based on a two-mode model for the trap states as proposed in \([1]\). The main result of our analytical approach is presented in Equation (13). This equation reproduces most of the essential features of the two-mode model presented in \([1]\). Also the possibility to increase the stability of vortex state by modulating \(\Omega\) is also discussed.

V. ACKNOWLEDGEMENTS

AN wants to acknowledge with pleasure the useful and interesting time spent at ICTP, Trieste, Italy, with Prof. S.R. Shenoy and Dr. A. Smerzi during when some of the techniques used in these calculations were learnt.

[1] J. Williams, R. Walser, J. Cooper, E.A. Cornell and M. Holland, cond-mat-9904399 (v3)

[2] E. Goldstein and P. Meystre, Phys. Rev. A, 55, 2935 (1997)
[3] D. Gordon and C.M. Savage, Phys. Rev. A, **58**, 1440 (1998) and references therein.

[4] C.J. Myatt et al., Phys. Rev. Lett. **78**, 586 (1997)

[5] H.J. Miesner et al., Phys. Rev. Lett. **82**, 2228 (1999)

[6] J.L. Martin et al., J. Phys. B: At.Mol.Opt.Phys. **32**, 3065 (1999)

[7] M. R. Matthews et al., Phys. Rev. Lett. **81**, 243 (1998)

[8] D. S. Hall et al., Phys. Rev. Lett. **81**, 1539 (1998)

[9] I. Zapata, F. Sols, and A. Legget, Phys. Rev. A **57**, R28 (1998)

[10] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997)

[11] S. Raghavan, A. Smerzi, S. Fantoni, and S. Shenoy, Phys. Rev. A **59**, 620 (1999)

[12] D.S. Hall et al., Proc. SPIE **3270**, 98 (1998)

[13] M.R. Matthews et al., Phys. Rev. Lett. **83**, 2498 (1999)

[14] F. Dalfovo and S. Stringari, Phys. Rev. A **53**, 2477 (1996)

[15] S. Sinha, Phys. Rev. A **55**, 4325 (1997)

[16] D. S. Rokshar, Phys. Rev. Lett. **79**, 2164 (1997)

| TABLE I. Parameter values used to plot Figure 1 are given here |
|------------------|------------------|------------------|
| $N_T$            | $10^5$           | $\nu_z$          | 65 Hz           |
| $a_{11}$         | 1.0 $a_{21}$     | $a_{\perp}$      | 1.3 \(\mu m\) |
| $a_{22}$         | 1.0 $a_{21}$     | $a_{21}$         | 5.5 nm          |
FIG. 1. This plot shows the change in $\langle Z \rangle$ as a function of detuning $\delta$. The parameters are (a) $\delta = 0$, (b) $\delta = 2\pi \ast 50Hz$, (c) $\delta = 2\pi \ast 100Hz$. The values of other parameters are given in Table 1.
Schematic illustrating creation of stable vortex state

Ground State

Vortex state

Excited trap state is vortex state

Figure 2.

Andal Narayanan
Hema Ramachandran