Searching, Sorting, and Cake Cutting in Rounds

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Abstract

We study searching and sorting in rounds motivated by a fair division question: given a cake cutting problem with $n$ players, compute a fair allocation in at most $k$ rounds of interaction with the players. Rounds interpolate between the simultaneous and the fully adaptive settings, also capturing parallel complexity.

We find that proportional cake cutting in rounds is equivalent to sorting with rank queries in rounds. We design a protocol for proportional cake cutting in rounds, while lower bounds for sorting in rounds with rank queries were given by Alon and Azar. Inspired by the rank query model, we then consider two basic search problems: ordered and unordered search.

In unordered search, we get an array $x = (x_1, \ldots, x_n)$ and an element $z$ promised to be in $x$. We have access to an oracle that receives queries of the form “Is $z$ at location $i$?” and answers “Yes” or “No”. The goal is to find the location of $z$ with success probability at least $p$ in at most $k$ rounds of interaction with the oracle. We show that the expected query complexity of

- randomized algorithms on a worst case input is $np\left(\frac{k+1}{2k}\right) + O(1)$;
- deterministic algorithms on a worst case input distribution is $np(1 - \frac{k-1}{2k}p) + O(1)$.

These bounds apply even to fully adaptive unordered search, where the gap, i.e. the ratio between the two complexities, converges to $2 - p$ as the size of the array grows.

In ordered search, we get sorted array $x = (x_1, \ldots, x_n)$ and element $z$ promised to be in $x$. We have access to an oracle that gets comparison queries. Here we find that the expected query complexity of randomized algorithms on a worst case input and deterministic algorithms on a worst case input distribution is essentially the same: $pk \cdot n^\frac{1}{k} + O(1 + pk)$.

1 Introduction

We study search in rounds motivated by the following fair division question: given a cake cutting instance with $n$ players, compute a fair allocation using at most $k$ rounds of interaction with the players. Rounds interpolate between the simultaneous setting, where all the communication between the mediator and the players has to take place in one round, and the fully adaptive setting, where the mediator can ask one query at a time. Rounds also capture the parallel complexity [Val75] and emerge naturally when considering the complexity of approximate fairness with few cuts [BN19].

We find that proportional cake cutting among $n$ players in $k$ rounds is equivalent to sorting with rank queries in $k$ rounds. In the rank query model, there is an array $x$ that can be accessed

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via queries of the form “Is rank\((x_i) \leq j\)?”. We give a protocol for proportional cake cutting in \(k\) rounds (which can thus be run in parallel), while a lower bound for sorting with rank queries in \(k\) rounds was given by [AA88a]. The connection between cake cutting and sorting was first made in [WS07]. Inspired by the rank query model, we consider two basic search problems: ordered and unordered search.

**Unordered search.** In the unordered search problem, we are given an array \(x = (x_1, \ldots, x_n)\) and an element \(z\) promised to be in the array. The size \(n\) of the array is known, but the elements of the array or the element \(z\) are not known. Instead, we have access to an oracle that answers queries of the form: “Is \(z\) at location \(i\)?”, where the answers can be “Yes” or “No”. The goal is to find the location of \(z\) with success probability at least \(p \in [0, 1]\) in at most \(k\) rounds of interaction with the oracle. We show that the expected query complexity of

- randomized algorithms on a worst case input is \(n p (\frac{k+1}{2k}) + O(1)\);
- deterministic algorithms on a worst case input distribution is \(n p (1 - \frac{k-1}{2k} p) + O(1)\).

The uniform distribution is the worst case for unordered search.

These bounds apply even to the fully adaptive unordered search problem, where the gap, i.e. the ratio between the two complexities, converges to \(2 - p\) as the size of the array grows. In particular, the gap is strictly larger than 1 even for success probability \(p > 1/2\).

**Ordered search.** In the ordered search problem, we are given a sorted array \(x = (x_1, \ldots, x_n)\) and an element \(z\) promised to be in the array. We have access to an oracle that answers comparison queries of the form: “How is \(z\) compared to the element at location \(i\)?”, where the answers can be \(<, =, \text{ and } >\). Here we find that the expected query complexity of randomized algorithms on a worst case input and deterministic algorithms on a worst case input distribution is essentially the same, namely \(p k \cdot n^{\frac{1}{k}} + O(1 + pk)\). The uniform distribution is the worst case.

**Applications of search in rounds and connections to other problems.** The problem of searching in rounds is important in distributed settings, where large data sets (e.g. with genomic data) are analyzed using cloud computing. In such scenarios, each query takes resources to complete and is executed by a separate processor. Thus an algorithm that runs in \(k\) rounds can be viewed as follows: a central machine issues in each round \(j\) a set of queries, one to each processor, then waits for the answers before issuing the next set of parallel queries in round \(j + 1\). The question then is how many processors are needed to achieve a parallel search time of \(k\), or equivalently, what is the query complexity in \(k\) rounds. For example, when \(k = 1\), the algorithm must issue all the queries it will ever ask at once. Finding a solution often requires more queries when \(k = 1\) compared to the fully adaptive case of \(k = \infty\), since with one round the algorithm has only one chance to ask for information and must be prepared in advance for every possible configuration of answers.

The unordered search problem with success probability \(p < 1/2\) makes sense in settings such as blockchain [Nak08], where the success probability of an individual miner is small and a lower bound quantifies how much work is needed to achieve even such a small probability of success. Another example is in security applications where potential attackers have some number of attempts available to guess the password to a server before getting locked out.

Parallel complexity is a fundamental concept with a long history in areas such as sorting and optimization (see, e.g. the work of Nemirovski [Nem94] on the parallel complexity of optimization.
and the more recent results on submodular optimization [BS18]). An overview on parallel sorting algorithms is given in the book by Akl [Akl14] and many works on sorting and selection in rounds can be found in [Val75, Pip87, Bol88, AAV86b, WZ99, GGK03], aiming to understand the tradeoffs between the number of rounds of interaction and the query complexity. Another setting of interest is active learning, where there is an “active” learner that can submit queries—taking the form of unlabeled instances—to be annotated by an oracle (e.g., a human) [Set12]. However each round of interaction with the human annotator has a cost, which can be captured through a budget on the number of rounds.

1.1 Our results

In this section we summarize our results for unordered search, ordered search, and cake cutting in rounds as well as the connections to sorting.

1.2 Unordered Search

Next we include the definition of the unordered search problem.

Definition 1 (Unordered search). Let \( k, n \in \mathbb{N} \) and \( p \in [0, 1] \), with \( k, n \geq 1 \). We are given an array \( x = (x_1, \ldots, x_n) \) and an element \( z \) promised to be in the array. The size \( n \) is known, but the elements of \( x \) or the element \( z \) are not. Instead, an algorithm has access to an oracle that takes queries of the form “Is \( z \) at location \( i \)” and answers “Yes” or “No”.

The task is to find the location of \( z \), i.e. the index \( \ell \) for which \( z = x_\ell \), with success probability at least \( p \) in at most \( k \) rounds of interaction with the oracle. An index must be queried before getting returned as the solution.

The requirement that an algorithm queries an index before returning it as the solution only makes a difference of \( \pm 1 \) in the bounds.

Theorem 1 (Unordered search, randomized algorithms). Let \( k, n \in \mathbb{N} \) with \( k, n \geq 1 \) and \( p \in [0, 1] \). For unordered search on an array \( x = (x_1, \ldots, x_n) \) in \( k \) rounds:

- there is a randomized \( k \)-round algorithm that succeeds with probability \( p \) and asks at most \( np \cdot \frac{k+1}{2k} + p + \frac{p}{n} \) queries in expectation on each input.
- if a randomized \( k \)-round algorithm succeeds with probability at least \( p \) on every instance, then it asks at least \( np \cdot \frac{k+1}{2k} \) in expectation on a worst case input.

For \( k = 1 \) rounds, the expected number of queries is exactly \( np \).

We also look at the performance of deterministic algorithms searching for an element \( z \) that is chosen at random from the array \( x \) according to some distribution \( \Psi = (\Psi_1, \ldots, \Psi_n) \). That is, \( z = x_i \) with probability \( \Psi_i \). The uniform distribution will in fact turn out to be the worst case for unordered search.

Theorem 2 (Unordered search, deterministic algorithms on random input). Let \( k, n \in \mathbb{N} \) with \( k, n \geq 1 \) and \( p \in [0, 1] \). For unordered search on an array \( x = (x_1, \ldots, x_n) \) in \( k \) rounds, where the search element \( z \) is drawn from \( x \) according to a distribution \( \Psi = (\Psi_1, \ldots, \Psi_n) \):
• there is a deterministic k-round algorithm that succeeds with probability at least $p$ and asks at most $np\left(1 - \frac{k-1}{2k}p\right) + p + \frac{2}{n} + 1$ queries in expectation.

• if a deterministic k-round algorithm succeeds with probability at least $p$, then it asks at least $np(1 - \frac{k-1}{2k}p)$ queries in expectation when $\Psi$ is the uniform distribution.

For $k = 1$ rounds, the expected number of queries is at least $np$ and at most $\lceil np \rceil$.

Figure 1: The query complexity of fully adaptive unordered search for $n = 2^{10}$ elements, with success probability $p$ ranging from 0 to 1. The bounds (both upper and lower) for randomized algorithms are shown in red and for deterministic algorithms when the input is drawn from the uniform distribution in blue. The difference between each upper bound and the corresponding lower bound is very small so the two blue lines appear merged into one (and similarly for the red lines).

1.2.1 Fully adaptive unordered search

By taking $k = n$, the bounds in Theorem 1 and 2 characterize the query complexity of fully adaptive unordered search algorithms.

Corollary 1 (Fully adaptive unordered search). Consider the fully adaptive unordered search problem, where we have to find an element $z$ in an unsorted array $\mathbf{x} = (x_1, \ldots, x_n)$. For each $p \in [0, 1]$:

- There is a randomized algorithm that on each instance, succeeds with probability $p$ and asks at most $np \cdot \frac{n+1}{2n} + p + \frac{p}{n}$ queries in expectation. Moreover, every randomized algorithm that succeeds with probability at least $p$ on each instance asks at least $np \cdot \frac{n+1}{2n}$ queries in expectation in the worst case.

- For each distribution $\Psi$, if $z$ is drawn from $\mathbf{x}$ according to $\Psi$, then there is a deterministic algorithm that succeeds with probability at least $p$ and asks at most

$$np \left(1 - \frac{n-1}{2n}p\right) + p + \frac{2}{n} + 1$$

queries in expectation. Moreover, if a deterministic algorithm succeeds with probability at least $p$, then it must ask at least $np(1 - \frac{n-1}{2n}p)$ queries in expectation when $\Psi$ is the uniform distribution.
Thus for unordered search with success probability $p$, the expected query complexity of fully adaptive randomized algorithms is of the order $p\left(\frac{n+1}{2}\right)$ and that of fully adaptive deterministic algorithms searching for a uniform random element is of the order $p\left(n - \frac{n-1}{2}p\right)$. Observe the complexity of deterministic algorithms searching for a uniform random element is quadratic in $p$, in contrast to that of randomized algorithms on worst case input, which is linear in $p$.

Let $gap(n, p)$ denote the gap for fully adaptive unordered search, i.e. the ratio of the expected number of queries asked by an optimal deterministic algorithms with success probability at least $p$ on a worst case input distribution to the expected number of queries asked by an optimal randomized algorithms with success probability at least $p$ on a worst case input.

**Corollary 2** (Gap for fully adaptive unordered search). For fully adaptive unordered search with success probability at least $p \in (0,1]$, the ratio between the expected query complexity of randomized algorithms on worst case input and deterministic algorithms on worst case input distribution, denoted $gap(n, p)$, converges to

$$\lim_{n \to \infty} gap(n, p) = 2 - p.$$  \hspace{1cm} (1)

**Proof.** Corollary 1 shows that

$$\frac{np \left(1 - \frac{n-1}{2n}p\right)}{np \cdot \left(\frac{n+1}{2n}\right) + p + \frac{p}{n}} \leq gap(n, p) \leq \frac{np \left(1 - \frac{n-1}{2n}p\right) + p + \frac{2}{n} + 1}{np \left(\frac{n+1}{2n}\right)}.$$  \hspace{1cm} (2)

Taking the limit as $n$ goes to $\infty$ in (2), we obtain that the gap for fully adaptive unordered search is $2 - p$ as the array size grows: $\lim_{n \to \infty} gap(n, p) = 2 - p$. \hfill \Box

### 1.3 Ordered Search

The ordered search problem we consider is defined next.

**Definition 2** (Ordered search). Let $k, n \in \mathbb{N}$ and $p \in [0,1]$, with $k, n \geq 1$. We are given a sorted array $x = (x_1, \ldots, x_n)$ and an element $z$ promised to be in the array. The size $n$ is known, but the array elements or $z$ are not and they cannot be accessed directly. Instead, an algorithm has access to an oracle that takes comparison queries of the form “How is $z$ compared to the element at location $i$?” and answers $<$, $=$, or $\geq$.

The task is to find the location of $z$, i.e. the index $\ell$ for which $z = x_\ell$, with success probability at least $p$ in at most $k$ rounds of interaction with the oracle. An index must be queried before getting returned as the solution.

For ordered search the number of rounds need not be larger than $\lceil \log_2 n \rceil$, since binary search is an optimal fully adaptive algorithm with success probability 1. We have the following bounds.

**Theorem 3** (Ordered search, deterministic and randomized algorithms). Let $k, n \geq 1$ and $p \in [0,1]$, where $k, n \in \mathbb{N}$. For ordered search in $k$ rounds on an array with $n$ elements, the expected query complexity of the best

- randomized algorithm that succeeds with probability $p$ on a worst case input is at most $pk\lceil n^{\frac{1}{k}} \rceil$
  and at least $pkn^{\frac{1}{k}} - 2pk$.  

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• deterministic algorithm that succeeds with probability $p$ on a worst case input distribution is at most $k[\lceil pn^{1/k} \rceil] + 2$ and at least $pkn^{1/k} - 2pk$.

The uniform distribution is the worst case for deterministic algorithms. When $k = 1$, the lower bound is $np$ and the upper bound is $\lceil np \rceil$.

Thus for ordered search there is essentially no gap between deterministic and randomized algorithms. In this case, the deterministic algorithm that gets an input drawn from an arbitrary distribution is able to extract enough randomness from the answers to queries from round 1 while also partitioning a relevant portion of the array via the round 1 queries so that by the beginning of round 2 it has caught up with an optimal randomized algorithm.

1.4 Cake Cutting in Rounds and Sorting with Rank Queries

Our starting point was the cake cutting challenge of finding a proportional allocation with contiguous pieces in $k$ rounds. The cake is the interval $[0, 1]$ and the goal is to divide it among $n$ players with private additive valuations. A proportional allocation, where each player gets a piece worth $1/n$ of the total cake according to the player’s own valuation, always exists and can be computed in the standard (RW) query model for cake cutting. (See Section 5 for details)

We establish a connection between proportional cake cutting with contiguous pieces and sorting in rounds in the rank query model. In the latter, we have oracle access to a list $x$ of $n$ elements that we cannot inspect directly. The oracle accepts rank queries of the form “How is rank($x_i$) compared to $j$?”, where the answer is $<, =, \text{or } >$.

The reduction from sorting to cake cutting appears in the work of Woeginger and Sgall [WS07], which gave a lower bound for proportional cake cutting in the fully adaptive case.

**Theorem 4. (Informal).** For any number of rounds $k \geq 1$, the following problems are equivalent:

- computing a proportional cake allocation with contiguous pieces for $n$ agents in the standard (RW) query model
- sorting a list of $n$ items with rank queries.

The query complexity of both problems is $\Theta(k \cdot n^{1+\frac{1}{k}})$.

The model of sorting with rank queries was studied by Alon and Azar [AA88a] in the context of sorting lower bounds; they gave a lower bound of $\Omega(kn^{1+\frac{1}{k}})$ for randomized algorithms.

2 Related work

Valiant [Val75] initiated the study of parallelism using the number of comparisons as a complexity measure and showed that $p$ processor parallelism can offer speedups of at least $O\left(\frac{p}{\log \log p} \right)$ for problems such as sorting and finding the maximum of a list of $n > p$ elements. The connection to the problem of sorting in rounds is straightforward since one parallel step of the $p$ processors (e.g. $p$ comparisons performed in parallel) can be viewed as one round of computations. Häggkvist and Hell [HH81] showed that $O\left( n^{1+\frac{k-1}{2^k-1}} \log n \right)$ comparisons suffice to sort an array in $k$ rounds.

\footnote{Equivalently, the queries are “Is rank($x_i$) $\leq j$?”, where the answer is yes or no}
Bollobás and Thomason [BT83] improved the case of 2 rounds by showing that $O(n^{3/2} \log n)$ comparisons suffice. Pippenger [Pip87] obtained a connection between expander graphs and sorting and proved that $O\left(n^{1+\frac{1}{k}} (\log n)^{2-\frac{2}{k}}\right)$ comparisons are enough. This was improved to $O\left(n^{3/2} \frac{\log n}{\sqrt{\log \log n}}\right)$ by Alon and Azar [AA88b] who also showed that $\Omega(n^{1+1/k}(\log n)^{1/k})$ comparisons are necessary.

Bollobás [Bol88] generalized the latter upper bound to $O\left(n^{1+\frac{1}{k}} (\log n)^{2-\frac{2}{k}} (\log \log n)^{1-\frac{1}{k}}\right)$ for $k$ rounds. The best upper bound known to us is due to Wigderson and Zuckerman [WZ99] who obtained a $k$-rounds algorithm that performs $O(n^{1+\frac{1}{k+o(1)}})$ comparisons. For randomized algorithms, Alon, Azar, and Vishkin [AAV86a] obtained an algorithm that runs in $k$ rounds and issues $O(n^{1+1/k})$ queries, thus demonstrating that randomization helps in the comparison model. Local search in rounds was considered in [BL22].

In fault detection problems, the goal is to identify all the defective items from a finite set items via a minimum number of tests. More formally, there is a universe of $U$ of $n$ items, $d$ of which are defective. Each test is executed on a subset $S \subseteq U$ and says whether $S$ is contaminated (i.e. has at least one defective item) or pure (i.e. none of the items in $S$ are defective). Important questions in group testing include how many tests are needed to identify all the defective items and how many stages are needed, where the tests performed in round $k+1$ can depend on the outcome of the tests in round $k$. An example of group testing is to identify which people from a set are infected with a virus, given access to any combination of individual blood samples; combining their samples allows detection using a smaller number of tests compared to checking each sample individually.

The group testing problem was first posed by Dorfman [Dor43] and a lower bound of $\Omega\left(d^2 \log \frac{n}{d}\right)$ for the number of tests required in the one round setting was given in [DRR89]. One round group testing algorithms with an upper bound of $O(d^2 \log n)$ on the number of tests were designed in [AMS06,PR08,INR10,NPR11]. Two round testing algorithms were studied in [DBGV05,EGH07]. The setting where the number of rounds is allowed is given by some parameter $r$ and the number of defective items is not known in advance was studied for example in [DP94,CDZ15,Dam19,GV20]; see the book [SAJ19] for a survey. The cake cutting model was introduced by Steinhaus [Ste48] to study the allocation of a heterogeneous resource among agents with complex preferences. Cake cutting was studied in mathematics, political science, economics [RW98,BT96,Mou03], and computer science [Pro13,BCE+16], where multiple protocols were implemented on the Spliddit platform [GP14]. There is a hierarchy of fairness notions that includes proportionality, where each player gets a piece it values at least $1/n$, and envy-freeness (where no player prefers the piece of another player), equitability, and necklace splitting [Alo87], with special cases such as consensus halving and perfect partitions. See [Bra15] and [Pro16] for recent surveys.

Cake cutting protocols are often studied in the Robertson-Webb [WS07] query model, where a mediator asks the players queries until it has enough information to output a fair division. Even and Paz [EP84] devise an algorithm for computing a proportional allocation with connected pieces that asks $O(n \log n)$ queries, with matching lower bounds due to Woeginger and Sgall [WS07] and Even and Paz [EP06].

For the query complexity of exact envy-free cake cutting (possibly with disconnected pieces), a lower bound of $\Omega(n^2)$ was given by Procaccia [Pro09] and an upper bound of $O\left(n^{n+n\frac{n}{n}}\right)$ by Aziz and Mackenzie [AM16]. A simpler algorithm for 4 agents with lower query complexity was given in [ACF+18]. An upper bound on the query complexity of equitability was given by Cechlarova, Dobos, and Pillarova [CDP13] and a lower bound by Procaccia and Wang [PW17]. The query
complexity of envy-freeness, perfect, and equitable partitions with minimum number of cuts was studied by Branzei and Nisan [BN22].

The issue of rounds in cake cutting was studied in [BN19], where the goal is to bound the communication complexity of protocols depending on the fairness notion. The query complexity of proportional cake cutting with different entitlements was studied by [Seg18], which showed a lower bound of $2n - 2$ cut queries and an upper bound of $O(n \log n)$. The query complexity of consensus halving was studied in [DFH20], which studied monotone valuations and introduced a generalization of the Robertson-Webb query model for this class. The query complexity of cake cutting in one round, i.e. in the simultaneous setting, was studied in [BBKP14].

Many other works analyzed the complexity of fair division in models such as cake cutting, multiple divisible goods, and indivisible goods. The complexity of cake cutting (e.g. envy-free division, consensus halving, necklace splitting) was studied, e.g., in [DFH21, GHS20, Che20, GHI+20, FRHSZ20, FRHHH22, AG20, PR19, BCF+19, DEG+22]. Indivisible goods were studied, e.g., in [OPS21] for their query complexity and in [MS21, CKMS21] for algorithms. Cake cutting with separation was studied in [ESS21], fair division of a graph or graphical cake cutting in [BCE+17, BS21], multi-layered cakes in [IM21], fair cutting in practice in [KOS22], and cake cutting where some parts are good and others bad in [SH18] and when the whole cake is a “bad” in [FH18]. Branch-choice protocols were developed and analyzed in [GI21], which have the same expressive power as the GCC protocols from prior work [BCKP16] but are a simpler extensive-form game model. A body of work analyzed truthful cake cutting both in the standard (Robertson-Webb) query model [MT10, BM15] and in the direct revelation model [CLPP13, BST23, BLS22, Tao22].

3 Ordered search

In this section we focus on ordered search and prove Theorem 3, which quantifies the expected query complexity of randomized algorithms in the worst case and deterministic algorithms on a worst case input distribution. The omitted proofs of this section can be found in Appendix A. Since we arrived at ordered search by considering the locate task in the rank query model, in the appendix we also show that Locate \footnote{Recall that rank queries have the form “How is rank($x_i$) compared to $i$?”}, with rank queries is equivalent to ordered search with comparison queries.

Thus from now on, we focus on ordered search with comparison queries. At a high level, we first design a deterministic algorithm $D$ for ordered search that succeeds on every instance and asks at most $k \cdot \lceil n \frac{k}{k} \rceil$ queries in the worst case. Using $D$, we then design a randomized algorithm $R$ that succeeds with probability at least $p$ and asks at most $pk \lceil n \frac{p}{k} \rceil$ queries in expectation. The randomized algorithm $R$ has an all-or-nothing structure: with probability $1 - p$, do nothing; and with probability $p$, run $D$.

Then, using $D$ and $R$, we show how when the search element $z$ is drawn from the array $x$ according to an arbitrary distribution $\Psi = (\Psi_1, \ldots, \Psi_n)$ \footnote{The element $z$ being drawn from $\Psi$ means that $z = x_i$ with probability $\Psi_i$.}, one can design a deterministic algorithm $D_\Psi$ that asks at most $k \lceil p n \frac{1}{k} \rceil + 2$ queries in expectation and succeeds with probability at least $p$.

The distribution-dependent deterministic algorithm $D_\Psi$ simulates the execution of $R$ by identifying in the first round a sub-array $y_\Psi$ of the array $x$ (contiguous or consisting of at most two
contiguous sub-parts) that has length roughly $np$ and probability mass roughly $p$. In the first round, $D_\Psi$ queries the endpoints of $y_\Psi$ while simultaneously querying locations in the sub-array $y_\Psi$ that partition it into blocks of roughly equal size. However, if the first round queries reveal that $z$ is outside $y_\Psi$, then $D_\Psi$ gives up. Otherwise, it runs $D$ on $y_\Psi$ in the remaining $k - 1$ rounds.

Afterwards, we prove a lower bound for randomized algorithms, which has the same leading term as the upper bound achieved by $D_\Psi$, thus showing that the randomized query complexity on a worst case input has the same order as the deterministic query complexity on a worst case input distribution. The uniform distribution will turn out to represent the worst case.

### 3.1 Randomized ordered search algorithms

The maximum number of rounds for ordered search is $\lceil \log n \rceil$, since by asking one query per round the element can always be found if it exists using the classical binary search algorithm.

**Proposition 1.** For each $k \in \{1, \ldots, \lceil \log n \rceil \}$, there is a deterministic $k$-round algorithm for ordered search that succeeds on every input and asks at most $k \cdot \lceil n^{\frac{1}{k}} \rceil$ queries in the worst case.

**Proof.** We design a $k$-round algorithm recursively, using induction on $k$.

**Base case:** $k = 1$. Let $A_1$ be the following algorithm:

- Query all the elements of the array simultaneously. Return the correct location based on the results of the queries.

Then $A_1$ runs in one round, succeeds on every input, and the number of queries is at most $n$.

**Induction hypothesis.** For $k \geq 2$, assume there is a $(k - 1)$-round algorithm $A_{k-1}$ that always succeeds and asks at most $(k - 1) \cdot \lceil n^{\frac{1}{k-1}} \rceil$ queries on each array of length $n$.

**Induction step.** Using the induction hypothesis, we will design a $k$-round algorithm $A_k$ with the required properties. For each $s \in [n]$, write $n = s \cdot u_s + v_s$, for $u_s = \lfloor \frac{n}{s} \rfloor$ and $v_s = n \pmod{s}$. Let $A_k(s)$ be the following algorithm:

1. In round 1, query locations $i_1, \ldots, i_s \in [n]$ with the property that $1 < i_1 < \ldots < i_s = n$. Let $i_0 = 0$. Then these queries create $s$ contiguous blocks $B_1, \ldots, B_s$, such that $B_j = [i_{j-1} + 1, i_j]$ for $j \in [s]$.

   For each $j \in [s]$, set the size of each block $B_j$ to $\lfloor \frac{n}{s} \rfloor$ if $j \leq s - v_s$ and to $\lceil \frac{n}{s} \rceil$ if $j > s - v_s$. This uniquely determines indices $i_1, \ldots, i_s$.

   If the element searched for is found at one of these $s$ locations, then return that location and halt. Otherwise, identify the index $\ell \in [s]$ for which the block $B_\ell$ contains the answer.

2. Given index $\ell$ from step (i) such that block $B_\ell = [i_{\ell-1} + 1, i_\ell]$ contains the answer, we observe that position $i_\ell$ is the only one from block $B_\ell$ that has been queried so far. If $i_{\ell-1} + 1 \geq i_\ell$, let $B_\ell = [i_{\ell-1} + 1, i_\ell - 1]$ and run algorithm $A_{k-1}$ on block $B_\ell$. Else, halt.

We first show algorithm $A_k(s)$ is correct for every choice of $s$, and then obtain $A_k$ by optimizing $s$.

Algorithm $A_k(s)$ is correct if the choice of indices $i_1, \ldots, i_s$ is valid. This is the case if the sizes of the blocks $B_1, \ldots, B_s$ sum up to $n$. We have $\sum_{j=1}^s |B_j| = \lfloor n/s \rfloor \cdot (s - v_s) + \lceil n/s \rceil \cdot v_s$. 


(a) If \( v_s = 0 \) then \([n/s] = [n/s] = u_s\), so the sum of block sizes is \( \sum_{j=1}^{s} |B_j| = u_s \cdot (s - v_s) + u_s \cdot v_s = n\).

(b) If \( v_s > 0 \) then \([n/s] = u_s + 1\), so \( \sum_{j=1}^{s} |B_j| = u_s \cdot (s - v_s) + (u_s + 1) \cdot v_s = u_s \cdot s + v_s = n\).

Combining (a) and (b), we get that the block sizes are valid. Thus \( A_k(s) \) does not skip any indices, so it always finds the element.

Next we argue that there is a choice of \( s \) such that by setting \( A_k = A_k(s) \), we obtain a \( k \)-round algorithm that issues at most \( k \cdot [n^{1/k}] \) queries.

For a fixed \( s \in [n] \), the array size at the beginning of round 2 is at most \( m(s) = \max_{j \in [s]} |B_j| - 1 \), since the rightmost element of each block \( B_j \) has been queried in round 1 while the rest of block \( B_j \) has not been queried. Then \( m(s) = \max \{ \left\lfloor \frac{n}{s} \right\rfloor - 1, \left\lceil \frac{n}{s} \right\rceil - 1 \} = \left\lfloor \frac{n}{s} \right\rfloor - 1 \).

The total number of queries of algorithm \( A_k(s) \) is at most

\[
f(s) = s + (k - 1) \cdot \left\lfloor m(s)^{1/k} \right\rfloor = s + (k - 1) \cdot \left\lfloor \left( \left\lfloor \frac{n}{s} \right\rfloor - 1 \right)^{1/k} \right\rfloor. \tag{3}
\]

Taking \( s = \left\lceil n^{1/k} \right\rceil \) in (3), we get

\[
f\left( \left\lceil n^{1/k} \right\rceil \right) = \left\lceil n^{1/k} \right\rceil + (k - 1) \cdot \left\lfloor \left( \left\lfloor \frac{n}{\left\lceil n^{1/k} \right\rceil} \right\rfloor - 1 \right)^{1/k} \right\rfloor \leq \left\lceil n^{1/k} \right\rceil + (k - 1) \cdot \left\lfloor \left( \frac{n}{\left\lceil n^{1/k} \right\rceil} \right)^{1/k} \right\rfloor \leq \left\lceil n^{1/k} \right\rceil + (k - 1) \cdot \left\lfloor \left( \frac{n}{\left\lceil n^{1/k} \right\rceil} \right)^{1/k} \right\rfloor = k \cdot \left\lceil n^{1/k} \right\rceil.
\]

Setting \( A_k = A_k(\left\lceil n^{1/k} \right\rceil) \), we obtain a correct \( k \)-round algorithm that issues at most \( k \cdot \left\lceil n^{1/k} \right\rceil \) queries on every array with \( n \) elements. This completes the induction step and the proof.

**Corollary 3.** For each \( p \in [0, 1] \), there is a randomized algorithm for ordered search that runs in \( k \) rounds, succeeds with probability \( p \), and asks at most \( k \cdot p \cdot \left\lceil n^{1/k} \right\rceil \) queries in expectation in the worst case.

**Proof.** Consider the following randomized algorithm:

- With probability \( p \), run the deterministic algorithm \( A_k \) from Proposition 1.
- With probability \( 1 - p \), do nothing.

On every input, this algorithm succeeds with probability \( p \) and issues at most \( k \cdot p \cdot \left\lceil n^{1/k} \right\rceil \) queries in expectation. \( \square \)
3.2 Deterministic ordered search algorithms on random input

Let \( x = (x_1, \ldots, x_n) \) be an array with \( n \) elements and \( z \) the element being searched for the ordered search problem. Given \( \Psi = (\Psi_1, \ldots, \Psi_n) \) with \( \Psi_i \geq 0 \) \( \forall i \in [n] \) and \( \sum_{i=1}^n \Psi_i = 1 \), we say that the input distribution is \( \Psi \) if \( z = x_i \) with probability \( \Psi_i \).

**Proposition 2.** Let \( p \in (0,1) \), \( n \geq 1 \). For each input distribution \( \Psi = (\Psi_1, \ldots, \Psi_n) \), there is a one-round deterministic algorithm for both ordered and unordered search that succeeds with probability at least \( p \) and asks at most \( \lceil np \rceil \) queries.

**Proof.** Sort the elements of \( x \) in decreasing order by \( \Psi \) and let \( \pi \) be the permutation obtained, that is, \( \Psi_{\pi_1} \geq \ldots \geq \Psi_{\pi_n} \). Let \( \ell \) be the smallest index for which \( \sum_{i=1}^{\ell} \Psi_{\pi_i} \geq p \). Let \( q = \sum_{i=1}^\ell \Psi_{\pi_i} \geq p \).

Consider the following algorithm \( A \):

- Query elements \( x_{\pi_1}, \ldots, x_{\pi_\ell}, \) i.e. compare each of them with \( z \). If there is \( i \in [\ell] \) such that \( z = x_{\pi_i} \), then return \( \pi_i \).

By choice of \( \ell \), the success probability of this algorithm is \( q \geq p \). The number of queries is \( \ell \). Let \( m = \lceil np \rceil \). Then \((m - 1)/n < p \leq m/n \). By Lemma 23, \( \Psi_{\pi_1} + \ldots + \Psi_{\pi_m} \geq m/n \). Since \( \ell \) is the smallest index with \( \Psi_{\pi_1} + \ldots + \Psi_{\pi_{\ell}} \geq p \), it follows that \( \ell \leq m = \lceil np \rceil \). \( \square \)

**Proposition 3.** Let \( p \in (0,1) \) and \( k, n \in \mathbb{N} \) with \( n \geq 1 \), \( k \geq 2 \). For each input distribution \( \Psi = (\Psi_1, \ldots, \Psi_n) \), there is a \( k \)-round deterministic algorithm for ordered search that succeeds with probability at least \( p \) and asks at most \( k\lceil pn^{\frac{1}{k}} \rceil + 2 \) queries.

We first explain the high level idea of the proof. Given input distribution \( \Psi \), we find a sub-array \( y \) that has probability mass roughly \( p \) and also roughly \( np \) elements. In round 1, the algorithm submits \( pn^{\frac{1}{k}} \) queries as equally spaced as possible in the sub-array \( y \). If the answers from round 1 reveal the element is not in \( y \), the algorithm gives up. Otherwise, it continues by calling the \((k - 1)\)-round deterministic algorithm from Proposition 1 on the relevant block of \( y \) identified to contain the element. When the block has length \( T \), the deterministic algorithm from Proposition 1 always succeeds and issues \((k - 1)T \frac{1}{n^{\frac{1}{k}}} \) queries.

Since the block identified at the beginning of round 2 has length roughly \( np/(pn^{\frac{1}{k}}) = n^{\frac{k-1}{k}} \), the total expected number of queries is approximately

\[
\frac{pn^{\frac{1}{k}}} + p \cdot (k - 1) \left( n^{\frac{k-1}{k}} \right)^{\frac{1}{k-1}} = pkn^{\frac{1}{k}}.
\]

The proof finds such a sub-array \( y \) by converting the input distribution \( \Psi \) into a probability density \( v \) on \([0,1]\), such that \( v(x) = n \cdot \Psi_i \) for each \( i \in [n] \) and \( x \in [(i-1)/n, i/n] \). An application of the intermediate value theorem (Lemma 25) ensures the existence of an interval of length \( p \) with probability mass \( p \) on the circle obtained by bending the interval \([0,1]\) so that 0 coincides with 1. This interval is used to obtain the sub-array \( y \) that the algorithm searches in.

**Proof of Proposition 3.** First, we show there exists an interval \([i,j]\) on the array viewed on the circle (i.e. where index \( n + 1 \) is the same as index 1) that has probability mass roughly \( pn \) and length roughly \( pn \) as well. Second, we use the interval identified to simulate the randomized algorithm from Proposition 1.
Formally, given $\Psi$, we define a probability density function $v : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ by

$$v(x) = n \cdot \Psi_i \quad \forall i \in [n] \text{ and } x \in [(i - 1)/n, i/n].$$

Then $\int_0^1 v(x) \, dx = \sum_{i=1}^n \frac{1}{n} \cdot n \Psi_i = \sum_{i=1}^n \Psi_i = 1$. By Lemma 25, there exists a point $c \in [0, 1]$ such that one of the following holds:

(a) $\int_c^{c+p} v(x) \, dx = p$, where $0 \leq c \leq 1 - p$;

(b) $\int_0^c v(x) \, dx + \int_{c+1-p}^1 v(x) \, dx = p$, where $1 - p < c < 1$.

**Case (a).** In this case there exists $c \in [0, 1-p]$ such that $\int_c^{c+p} v(x) \, dx = p$.

We first make a few observations and then define the protocol. Let $i, j \in [n]$ be such that

$$\frac{i-1}{n} \leq c < \frac{i}{n} \quad \text{and} \quad \frac{j-1}{n} \leq c + p < \frac{j}{n}.$$

Let $T = j - i + 1$. Then $np \leq T \leq np + 2$. Since each interval $[(\ell - 1)/n, \ell/n]$ corresponds to element $x_{i+\ell}$ of the array, we have $\sum_{\ell=i}^j \Psi_i \geq p$. By choice of $i$ and $j$, we have:

- the sub-array $y = [x_i, \ldots, x_j]$ has length $T \leq np + 2$ and probability mass $\sum_{\ell=i}^j \Psi_i \geq p$.
- if $T \geq 2$, the sub-array $\bar{y} = [x_{i+1}, \ldots, x_{j-1}]$ has length $T - 2 \leq np$ and probability mass $\sum_{\ell=i+1}^j \Psi_i \leq p$.

![Figure 2](image.png)

Figure 2: Given distribution $\Psi = (\Psi_1, \ldots, \Psi_n)$, define probability density $v : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ by $v(x) = n \cdot \Psi_\ell$ for all $\ell \in [n]$ and $x \in [(\ell - 1)/n, \ell/n]$. The left figure shows an interval $[c, c+p]$ of length $p$ and probability mass $\int_c^{c+p} v(x) \, dx = p$. The right figure shows the queried sub-array $y = [x_i, \ldots, x_j]$, which has length $T = j - i + 1 \leq np + 2$ and probability mass $\sum_{\ell=i}^j \Psi_i \geq p$. When $T \geq 2$, the sub-array $\bar{y} = [x_{i+1}, \ldots, x_{j-1}]$ has length $T - 2 \leq np$ and probability mass $\sum_{\ell=i+1}^j \Psi_i \leq p$.

Let $A$ be the following $k$-round protocol:

**Step a.(i)** If $T \leq 2$: query locations $i$ and $j$ in round 1. If the element is found, return it and halt.

**Else:** since the element is guaranteed to be in the array $x$, it must be the case that $T \geq 3$.

Let $r = [p \cdot n^{\frac{k}{2}}]$. Query in round 1 locations $i$ and $j$, together with additional locations $t_1, \ldots, t_r$ set as equally spaced as possible.

More precisely, require $i + 1 \leq t_1 \leq \ldots \leq t_r = j - 1$, with $t_0 = i$. For each $\ell \in [r]$, let

$$B_\ell = [x_{t_{\ell-1}+1}, \ldots, x_{t_\ell}]$$

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be the \( \ell \)-th block created by the queries \( t_1, \ldots, t_r \). Define indices \( t_1, \ldots, t_r \) so that each block \( B_\ell \) has size at most \( \left\lceil \frac{T-2}{r} \right\rceil \), which is possible since the sub-array \( \tilde{y} \) has length \( T-2 \) and there are \( r \) blocks.

If the element is found at one of the indices \( i, j, t_1, \ldots, t_r \) queried in round 1, then return it and halt. Otherwise, continue to step a.(ii).

**Step a.(ii)** If the answers to round 1 queries show the element is not at one of the indices \( [i, \ldots, j] \), then halt. Else, let \( B_\ell = [x(t_{\ell-1}+1), \ldots, x(t_\ell)] \) be the block identified to contain the element, where location \( t_\ell \) has been queried. Run the \((k-1)\)-round deterministic protocol from Proposition 1 on the sub-array \( \tilde{y} = [x(t_{\ell-1}+1), \ldots, x(t_{\ell-1})] \), which always succeeds and asks at most \((k-1) \cdot (\text{len}(\tilde{y}))^{\frac{1}{k-1}} \) queries.

We now analyze the success probability and expected number of queries of algorithm \( \mathcal{A} \) described in steps a.(i-ii).

**Success probability.** The algorithm is guaranteed to find the element precisely when it is located in the sub-array \([x_i, \ldots, x_j]\). Since \( \sum_{\ell=i}^j \Psi_\ell \geq p \), the success probability of the algorithm is at least \( p \).

**Expected number of queries.** We count separately the expected queries for round 1 and the remainder. The number of queries issued in round 1 is at most

\[
2 + r = 2 + \left\lceil p \cdot \frac{n^\frac{1}{k}}{1} \right\rceil.
\]

The algorithm continues beyond round 1 when the element is in the sub-array \( \tilde{y} = [x_{i+1}, \ldots, x_{j-1}] \), which has length \( T-2 \leq np \) and probability mass \( \sum_{t=i+1}^{j-1} \Psi_t \leq p \).

Thus with probability at least \( 1-p \), the algorithm halts at the end of round 1. With probability at most \( p \), it continues beyond round 1 by running step a.(ii). The number of queries in step a.(ii) is bounded by

\[
(k-1)\left(\left\lceil \frac{T-2}{r} \right\rceil - 1 \right)^{\frac{1}{k-1}}
\]

by Proposition 1 since \( \text{len}(\tilde{y}) \leq \left\lceil \frac{T-2}{r} \right\rceil - 1 \). Since \( T-2 \leq np \) and \( r = \left\lceil p \cdot n^\frac{1}{k} \right\rceil \), the expected number of queries from step a.(ii) can be bounded by

\[
p \cdot (k-1) \left(\left\lceil \frac{T-2}{r} \right\rceil - 1 \right)^{\frac{1}{k-1}} + (1-p) \cdot 0 = p \cdot (k-1) \left(\left\lceil \frac{T-2}{p \cdot n^\frac{1}{k}} \right\rceil - 1 \right)^{\frac{1}{k-1}}.
\]

\[
\leq p \cdot (k-1) \left(\left\lceil \frac{np}{p \cdot n^\frac{1}{k}} \right\rceil - 1 \right)^{\frac{1}{k-1}} \quad \text{(Since } T-2 \leq np \text{)}
\]

\[
\leq p \cdot (k-1) \cdot \left(\left\lceil \frac{np}{p \cdot n^\frac{1}{k}} \right\rceil - 1 \right)^{\frac{1}{k-1}} \quad \text{(Since } \frac{np}{p \cdot n^\frac{1}{k}} \leq \frac{np}{p \cdot n^\frac{1}{k}} \text{)}
\]

\[
\leq p \cdot (k-1) \cdot \left(\frac{np}{p \cdot n^\frac{1}{k}} \right)^{\frac{1}{k-1}} = p \cdot (k-1) \cdot n^\frac{1}{k}.
\]

Combining (4) and (5), the expected number of queries of algorithm \( \mathcal{A} \) is at most

\[
2 + \left\lceil p \cdot n^\frac{1}{k} \right\rceil + p \cdot (k-1) \cdot n^\frac{1}{k} \leq k\left\lceil p n^\frac{1}{k} \right\rceil + 2.
\]
Case (b). In this case, there exists $c \in (1-p, 1)$ such that $\int_0^c v(x) \, dx + \int_{c+1-p}^1 v(x) \, dx = p$. Let $i, j [n]$ be such that $(i-1)/n \leq c \leq i/n$ and $(j-1)/n \leq c + p - 1 \leq j/n$. By choice of $i$ and $j$, we have $np \leq T \leq np + 2$. Then

- the sub-array $y = [x_1, \ldots, x_i, x_j, \ldots, x_n]$ has length $T = n + i - j + 1 \leq np + 2$ and probability mass $\sum_{\ell=1}^T \Psi_\ell + \sum_{\ell=j}^n \Psi_\ell \geq p$.
- the sub-array $\tilde{y} = [x_1, \ldots, x_{i-1}, x_j+1, \ldots, x_n]$ has length $T - 2 \leq np$ and probability mass $\sum_{\ell=1}^{i-1} \Psi_\ell + \sum_{\ell=j+1}^n \Psi_\ell \leq p$.

![Figure 3: Given distribution $\Psi = (\Psi_1, \ldots, \Psi_n)$, define $v : [0,1] \to \mathbb{R}_{\geq 0}$ by $v(x) = n \cdot \Psi_\ell$ for all $\ell \in [n]$ and $x \in [(\ell-1)/n, \ell/n]$. The left figure shows point $c$ with probability mass $\int_0^c v(x) \, dx + \int_{c+1-p}^1 v(x) \, dx = p$. The right figure shows the queried sub-array consisting of two parts: $y = [x_1, \ldots, x_i, x_j, \ldots, x_n]$, of length $T = n + i - j + 1 \leq np + 2$ and probability mass $\sum_{\ell=1}^T \Psi_\ell + \sum_{\ell=j}^n \Psi_\ell \geq p$. When $T \geq 2$, the sub-array $\tilde{y} = [x_1, \ldots, x_{i-1}, x_j+1, \ldots, x_n]$ has length $T - 2 \leq np$ and probability mass $\sum_{\ell=1}^{i-1} \Psi_\ell + \sum_{\ell=j+1}^n \Psi_\ell \leq p$.](image)

Let $A$ be the same $k$-round protocol as in case (a), but where the array $y$ is treated as if it were contiguous when making queries:

**Step b.(i)** If $T \leq 2$: query locations $i$ and $j$ in round 1. If the element is found, return it.

**Else,** $T \geq 3$. Let $r = \lceil p \cdot n^\frac{1}{k} \rceil$. Query in round 1 locations $i$ and $j$, together with additional locations $t_1, \ldots, t_r \in \{1, \ldots, i-1, j+1, \ldots, n\}$, set as equally spaced as possible so that for each $\ell \in [r]$, the size of each block $B_\ell = [x_{(t_{\ell-1}+1)}, \ldots, x_{t_\ell}]$ is at most $\lceil \frac{T - 2}{r} \rceil$. At most one of the blocks may skip over the the indices in $\{i, \ldots, j\}$. If the element is found at one of the queried locations then return it and halt. Else, go to step b.(ii).

**Step b.(ii)** If round 1 indicates that the element is not at one of the indices $\{1, \ldots, i, j, \ldots, n\}$, then halt. Otherwise, let $B_\ell = [x_{(t_{\ell-1}+1)}, \ldots, x_{t_\ell}]$ be the block identified to contain the element, where location $t_\ell$ has been queried. Run the $(k-1)$-round deterministic protocol from Proposition 1 on the sub-array $\tilde{y} = [x_{(t_{\ell-1}+1)}, \ldots, x_{(t_{\ell-1})}]$, which always succeeds and asks at most $(k-1) \cdot (\text{len}(\tilde{y}))^\frac{1}{k-1}$ queries.

Next we bound the success probability and expected number of queries when the algorithm executes steps b.(i) and b.(ii).

**Success probability.** The algorithm finds the element when its location is one of $[1, \ldots, i, j, \ldots, n]$. Since $\sum_{\ell=1}^i \Psi_\ell + \sum_{\ell=j}^n \Psi_\ell \geq p$, the success probability is at least $p$.

**Expected number of queries.** The expected number of queries in round 1 is at most $2 + r = 2 + \lceil pn^\frac{1}{k} \rceil$, while the number of queries after round 2 is at most

$$p \cdot (k-1) \left( \left\lceil \frac{T - 2}{r} \right\rceil - 1 \right)^\frac{1}{k-1} \leq p \cdot (k-1) \cdot n^\frac{1}{k}.$$
Thus the total expected number of queries is at most $2 + \lceil pn^{\frac{k}{k}} \rceil + p \cdot (k - 1) \cdot n^{\frac{1}{k}} \leq 2 + k\lceil pn^{\frac{k}{k}} \rceil$, which completes the proof.

### 3.3 Ordered search lower bounds

**Proposition 4.** Every $k$-round randomized algorithm for unordered search that succeeds with probability at least $p$ on every instance asks in expectation at least $pk \cdot n^{\frac{1}{k}} - 2pk$ queries when $k \geq 2$; and at least $pn$ queries when $k = 1$ on a worst case input, where $n$ is the size of the input vector.

**Proof.** For proving the required lower bound, it will suffice to assume the input is drawn from the uniform distribution, meaning the algorithm is given an ordered array with $n$ elements together with a uniform random element $z$. The reason is that if a lower bound holds for a randomized algorithm when the input is uniformly distributed, then by an average argument the same lower bound also holds for a worst case input.

Let $A_k$ be an optimal $k$-round randomized algorithm that succeeds with probability $p$ when facing the uniform distribution as input. Let $q_k(n, p)$ be the expected number of queries of algorithm $A_k$ as a function of $n$ and $p$.

In round 1, the algorithm has some probability $\delta_m$ of asking $m$ queries, for each $m \in \{0, \ldots, n\}$. Moreover, for each such $m$, there are different (but finitely many) choices for the positions of the $m$ queries of round 1. However, since the algorithm is optimal, it suffices to restrict attention to the best way of positioning the queries in round 1, breaking ties arbitrarily if there are multiple equally good options.

For each $m \in \{0, \ldots, n\}$, we define the following variables:

- $\delta_m$ is the probability that the algorithm asks $m$ queries in round one.
- $b_{m,i}$ is the size of the $i$-th block demarcated by the indices queried in round 1, for $i \in \{0, 1, 2, 3\}$.

![Block Diagram](image)

Figure 4: Array with $n = 15$ elements. The $m = 3$ locations issued in round 1 are illustrated in gray. The resulting blocks demarcated by these queries are marked, such that the $i$-th block has length $b_{m,i}$, for $i \in \{0, 1, 2, 3\}$.

- $b_{m,i}$ is the size of the $i$-th block demarcated by the indices queried in round 1, excluding those indices, counting from left to right, for all $i \in \{0, \ldots, m\}$. Thus $\sum_{i=0}^{m} b_{m,i} = n - m$. An illustration with an array and the blocks formed by the queries issued in round 1 can be found in Figure 1.

- $\alpha_{m,i}$ the success probability of finding the element in the $i$-th block (as demarcated by the indices queried in round 1), given that the element is in this block.
The expected number of queries of the randomized algorithm is

\[ q_k(n, p) = \sum_{m=0}^{n} \delta_m \left[ m + \left( \frac{n - m}{n} \right) \sum_{i=0}^{m} \left( \frac{b_{m,i}}{n - m} \right) \cdot q_{k-1}(b_{m,i}, \alpha_{m,i}) \right] \]

\[ = \sum_{m=0}^{n} \delta_m \left[ m + \frac{1}{n} \sum_{i=0}^{m} b_{m,i} \cdot q_{k-1}(b_{m,i}, \alpha_{m,i}) \right], \quad (7) \]

where the variables are related by the following constraints:

\[ \sum_{i=0}^{m} b_{m,i} = n - m, \quad \forall m \in \{0, \ldots, n\} \quad (8) \]

\[ \sum_{m=0}^{n} \delta_m = 1 \quad (9) \]

\[ p_m = \frac{m}{n} + \frac{n - m}{n} \cdot \sum_{i=0}^{m} \frac{b_{m,i}}{n - m} \cdot \alpha_{m,i} = \frac{m}{n} + \frac{1}{n} \cdot \sum_{i=0}^{m} b_{m,i} \cdot \alpha_{m,i}, \quad \forall m \in \{0, \ldots, n\} \quad (10) \]

\[ p = \sum_{m=0}^{n} \delta_m \cdot p_m \quad (11) \]

\[ b_{m,i} \geq 0, \quad \forall m \in \{0, \ldots, n\}, i \in \{0, \ldots, m\} \quad (12) \]

\[ 0 \leq \alpha_{m,i} \leq 1, \quad \forall m \in \{0, \ldots, n\}, i \in \{0, \ldots, m\} \quad (13) \]

\[ \delta_m \geq 0, \quad \forall m \in \{0, \ldots, n\} \quad (14) \]

Let \( \{\gamma_\ell\}_{\ell=1}^\infty \) be the sequence given by \( \gamma_1 = 0 \) and \( \gamma_\ell = 2\ell \) for \( \ell \geq 2 \). We will prove by induction on \( k \) that

\[ q_k(n, p) \geq p(k \cdot n^{\frac{1}{2}} - \gamma_k) \quad \forall n, k \geq 1 \text{ and } p \in [0, 1]. \quad (15) \]

**Base case.** Proposition 5 gives \( q_1(n, p) \geq np \), so (15) holds with \( \gamma_1 = 0 \) for all \( n \geq 1 \) and \( p \in [0, 1] \).

**Induction hypothesis.** Suppose \( q_\ell(m, s) \geq s(\ell \cdot m^{\frac{1}{2}} - \gamma_\ell) \) for all \( \ell \in [k-1], m \geq 1, s \in [0, 1] \).

**Induction step.** We will show that (15) holds for \( k \geq 2 \), where \( n \geq 1 \) and \( p \in [0, 1] \). The bound clearly holds when \( p = 0 \), so we will focus on the scenario \( p > 0 \). For each \( m \in \{0, \ldots, n\} \), define

\[ r_k(n, m, p) = m + \frac{1}{n} \cdot \sum_{i=0}^{m} b_{m,i} \cdot q_{k-1}(b_{m,i}, \alpha_{m,i}) \quad (16) \]

By definition of \( q_k(n, p) \),

\[ q_k(n, p) = \sum_{m=0}^{n} \delta_m \cdot r_k(n, m, p). \quad (17) \]
The induction hypothesis implies \( q_{k-1}(b_{m,i}, \alpha_{m,i}) \geq \alpha_{m,i} \cdot \left( (k-1) \left( b_{m,i} \right)^{\frac{1}{k-1}} - \gamma_{k-1} \right) \), which substituted in (16) gives

\[
 r_k(n, m, p) \geq m + \left( \frac{k-1}{n} \right) \sum_{i=0}^{m} \alpha_{m,i} \cdot (b_{m,i})^{\frac{k}{k-1}} - \left( \frac{\gamma_{k-1}}{n} \right) \sum_{i=0}^{m} \alpha_{m,i} \cdot b_{m,i}. \tag{18}
\]

Given a choice of \( \alpha_{m,i}, b_{m,i} \) for all \( m \in \{0, \ldots, n\} \) and \( i \in \{0, \ldots, m\} \), let \( i_0, \ldots, i_m \in \{0, \ldots, m\} \) be such that \( 0 \leq \alpha_{m,i_0} \leq \ldots \leq \alpha_{m,i_m} \leq 1 \). Then we can decompose \( p_m \) using a telescoping sum:

\[
p_m = \frac{m}{n} + \frac{1}{n} \sum_{i=0}^{m} b_{m,i} \cdot \alpha_{m,i} \tag{By (10)}
\]

\[
= \alpha_{m,i_0} \left[ \frac{m}{n} + \frac{1}{n} \cdot \sum_{\ell=0}^{m} b_{m,i_\ell} \right] + \left\{ \sum_{j=1}^{m} \left( \alpha_{m,i_j} - \alpha_{m,i_{j-1}} \right) \cdot \left[ \frac{m}{n} + \frac{1}{n} \sum_{\ell=j}^{m} b_{m,i_\ell} \right] \right\} + \left( 1 - \alpha_{m,i_m} \right) \cdot \frac{m}{n}. \tag{19}
\]

We can similarly decompose the right hand side of inequality (18), obtaining:

\[
r_k(n, m, p) \geq m + \frac{k-1}{n} \cdot \sum_{i=0}^{m} \alpha_{m,i} \cdot (b_{m,i})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{i=0}^{m} \alpha_{m,i} \cdot b_{m,i} \tag{By (18)}
\]

\[
= \alpha_{m,i_0} \left[ m + \frac{k-1}{n} \cdot \sum_{\ell=0}^{m} (b_{m,i_\ell})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=0}^{m} b_{m,i_\ell} \right]
\]

\[
+ \sum_{j=1}^{m} \left( \alpha_{m,i_j} - \alpha_{m,i_{j-1}} \right) \cdot \left[ m + \frac{k-1}{n} \cdot \sum_{\ell=j}^{m} (b_{m,i_\ell})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=j}^{m} b_{m,i_\ell} \right] + \left( 1 - \alpha_{m,i_m} \right) \cdot m. \tag{20}
\]

Let \( w_{m,0} = \alpha_{m,i_0}, w_{m,j} = \alpha_{m,i_j} - \alpha_{m,i_{j-1}} \) for all \( j \in \{1, \ldots, m\} \), and \( w_{m,m+1} = 1 - \alpha_{m,i_m} \). Then we can rewrite (19) and (20) as follows:

\[
r_k(n, m, p) \geq \sum_{j=0}^{m} w_{m,j} \cdot \left[ m + \frac{k-1}{n} \cdot \sum_{\ell=j}^{m} (b_{m,i_\ell})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=j}^{m} b_{m,i_\ell} \right] + \left( w_{m,m+1} \cdot m \right) \tag{21}
\]

\[
p_m = \sum_{j=0}^{m} w_{m,j} \cdot \left[ \frac{m}{n} + \frac{1}{n} \cdot \sum_{\ell=j}^{m} b_{m,i_\ell} \right] + \left( w_{m,m+1} \cdot \frac{m}{n} \right). \tag{22}
\]

For each \( m \in \{0, \ldots, n\} \) and \( j \in \{0, \ldots, m+1\} \), define

\[
p_{m,j} = \begin{cases} 
\frac{m}{n} + \frac{1}{n} \cdot \sum_{\ell=j}^{m} b_{m,i_\ell} & \text{if } j \in \{0, \ldots, m\}, \\
\frac{m}{n} & \text{if } j = m + 1.
\end{cases} \tag{23}
\]

\[
r_k^j(n, m, p) = \begin{cases} 
\frac{m}{n} + \frac{k-1}{n} \cdot \sum_{\ell=j}^{m} (b_{m,i_\ell})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=j}^{m} b_{m,i_\ell} & \text{if } j \in \{0, \ldots, m\}, \\
m & \text{if } j = m + 1.
\end{cases} \tag{24}
\]
Substituting the definition of $r_k^j(n, m, p)$ in (21) and that of $p_{m,j}$ in (22) yields
\[
r_k(n, m, p) \geq \sum_{j=0}^{m+1} w_{m,j} \cdot r_k^j(n, m, p) \quad \text{and} \quad p_m = \sum_{j=0}^{m+1} w_{m,j} \cdot p_{m,j} \cdot (25)
\]

Combining (11), (17), and (25), we obtain
\[
q_k(n, p) = \sum_{m=0}^{n} \delta_m \cdot r_k(n, m, p) \geq \sum_{m=0}^{n} \delta_m \cdot \left( \sum_{j=0}^{m+1} w_{m,j} \cdot r_k^j(n, m, p) \right).
\]
\[
p = \sum_{m=0}^{n} p_m = \sum_{m=0}^{n} \left( \sum_{j=0}^{m+1} w_{m,j} \cdot p_{m,j} \right) \cdot (26)
\]

Let $S_{m,j} = \sum_{\ell=j}^{m} b_{m,\ell}$, for all $m \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. Then for $j \in \{0, \ldots, m\}$, we have $n \cdot p_{m,j} = m + S_{m,j}$, so $S_{m,j} = n \cdot p_{m,j} - m$. Since $S_{m,j} \geq 0$, we have $n \cdot p_{m,j} \geq m$. In summary,
\[
\sum_{\ell=j}^{m} b_{m,\ell} = n \cdot p_{m,j} - m, \quad \forall j \in \{0, \ldots, m\} \quad (27)
\]
\[
n \cdot p_{m,j} \geq m, \quad \forall j \in \{0, \ldots, m\} \quad (28)
\]

Next we will lower bound $r_k^j(n, m, p)$ and consider two cases, for $m \geq 1$ and $m = 0$.

Case $m \geq 1$. If $j = m + 1$, we are in the scenario where the algorithm asks $m$ queries in round 1 and no queries in the later rounds. Formally, since $p_{m,m+1} = m/n$, we have $m = n \cdot p_{m,m+1}$. Using the identity for $r_k^{m+1}(n, m, p)$ in (24), we obtain
\[
r_k^{m+1}(n, m, p) = m = n \cdot p_{m,m+1} \\
\geq p_{m,m+1} \cdot kn^{1/2} - \gamma_k \cdot p_{m,m+1}. \quad \text{(By Corollary 6.)}
\]

Thus from now on we can assume $j \in \{0, \ldots, m\}$. Observe that by definition of $p_{m,0}$ in (23),
\[
 p_{m,0} = m/n + \sum_{\ell=0}^{m} b_{m,\ell}/n = m/n + (m - n)/n = 1.
\]
For all $j \in \{0, \ldots, m\}$, using (24) and Jensen’s inequality, we obtain
\[
r_k^j(n, m, p) = m + \frac{k-1}{n} \cdot \sum_{\ell=j}^{m} (b_{m,\ell})^{k-1} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=j}^{m} b_{m,\ell} \\
\geq m + \frac{(k-1)(m-j+1)}{n} \left( \sum_{\ell=j}^{m} b_{m,\ell} \right)^{k-1} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=j}^{m} b_{m,\ell}. \quad (29)
\]
\[
\sum_{\ell=0}^{m} b_{m,\ell} = n \cdot p_{m,j} - m \quad \text{by (27), the inequality in (29) can be rewritten as}
\]
\[
r_k^j(n, m, p) \geq m \left(1 + \frac{\gamma_{k-1}}{n}\right) + \frac{(k-1)(n \cdot p_{m,j} - m)^{k-1}}{n \cdot (m-j+1)^{k-1}} - \gamma_{k-1} \cdot p_{m,j}. \quad (30)
\]
When \( j = 0 \), substituting \( \sum_{\ell=0}^{m} b_{m,i_{\ell}} = n - m \) in (30), we obtain

\[
\begin{align*}
r_{k}^{0}(n, m, p) & \geq m \left( 1 + \frac{\gamma_{k-1}}{n} \right) + \frac{(k-1)(n-m)^{\frac{k}{k-1}}}{n \cdot (m+1)^{\frac{1}{k-1}}} - \gamma_{k-1} \\
& \geq kn^{\frac{1}{k}} - \gamma_{k} \\
& = p_{m,0} \cdot kn^{\frac{1}{k}} - p_{m,0} \cdot \gamma_{k}.
\end{align*}
\]

(By Lemma 2.)

Thus from now on we can assume \( j \in \{1, \ldots, m\} \). Using \( j \geq 1 \) in (30), we further get

\[
r_{k}^{j}(n, m, p) \geq m \left( 1 + \frac{\gamma_{k-1}}{n} \right) + \frac{(k-1)(n \cdot p_{m,j} - m)^{\frac{k}{k-1}}}{n \cdot m^{\frac{1}{k-1}}} - \gamma_{k-1} \cdot p_{m,j}.
\]

(By (31))

In this range of \( m \) and \( j \), we have \( m/n \leq p_{m,j} \leq 1 \) and \( 1/2 < m \leq n \cdot p_{m,j} \) by inequality (28). Applying Lemma 13 with \( c = p_{m,j} \) in (31), we obtain:

\[
r_{k}^{j}(n, m, p) \geq m \left( 1 + \frac{\gamma_{k-1}}{n} \right) + \frac{(k-1)(n \cdot p_{m,j} - m)^{\frac{k}{k-1}}}{n \cdot m^{\frac{1}{k-1}}} - \gamma_{k-1} \cdot p_{m,j}.
\]

(By (31))

\[
\geq p_{m,j} \cdot kn^{\frac{1}{k}} - \gamma_{k} \cdot p_{m,j}.
\]

(By Lemma 13)

**Case** \( m = 0 \). This corresponds to the scenario where the algorithm asks zero queries in round 1. Since \( j \in \{0, \ldots, m + 1\} \), it follows that \( j = 0 \) or \( j = 1 \).

If \( j = 0 \), then by definition of \( p_{m,j} \) we have \( p_{0,0} = 0/n + (1/n) \cdot \sum_{\ell=0}^{0} b_{0,i_{\ell}} = 0 + b_{0,i_{0}}/n \). Since there is only one block, \( b_{0,i_{0}} = n \). Thus \( p_{0,0} = 1 \). We get

\[
r_{k}^{0}(n, 0, p) = 0 + \frac{k-1}{n} \cdot \sum_{\ell=0}^{0} (b_{0,i_{\ell}})^{\frac{k}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot \sum_{\ell=0}^{0} b_{0,i_{\ell}}
\]

\[
= (k-1) \cdot n^{\frac{1}{k-1}} - \frac{\gamma_{k-1}}{n} \cdot n
\]

\[
\geq kn^{\frac{1}{k}} - \gamma_{k}
\]

\[
= p_{0,0} \cdot kn^{\frac{1}{k}} - p_{0,0} \cdot \gamma_{k}.
\]

(By Corollary 5)

\[
\geq kn^{\frac{1}{k}} - \gamma_{k}
\]

\[
= p_{0,0} \cdot kn^{\frac{1}{k}} - p_{0,0} \cdot \gamma_{k}.
\]

(By (31))

If \( j = 1 \), then since \( m = 0 \) we are in the case \( j = m + 1 \). Since \( p_{m,m+1} = m/n \), we have \( p_{0,1} = 0/n = 0 \). Informally, this corresponds to the scenario where the algorithm asks \( m = 0 \) queries in round 1 and no queries in the later rounds either. Formally,

\[
r_{k}^{1}(n, 0, p) = 0 = p_{0,1} \cdot kn^{\frac{1}{k}} - p_{0,1} \cdot \gamma_{k}.
\]

(32)

**Combining cases** \( m \geq 1 \) and \( m = 0 \). We obtain

\[
r_{k}^{j}(n, m, p) \geq p_{m,j} \cdot kn^{\frac{1}{k}} - p_{m,j} \cdot \gamma_{k} \quad \forall m \in \{0, \ldots, n\}, \forall j \in \{0, \ldots, m + 1\}
\]

(33)

Summing inequality (33) over all \( m \in \{0, \ldots, n\} \) and \( j \in \{0, \ldots, m + 1\} \) and using identity (26) that expresses the total expected number of queries \( q_{k}(n, p) \) as a weighted sum of the \( r_{k}^{j}(n, m, p) \)
terms, we obtain
\[
q_k(n, p) = \sum_{m=0}^{n} \delta_m \cdot \left( \sum_{j=0}^{m+1} w_{m,j} \cdot r^j_k(n, m, p) \right)
\]
\[
\geq \sum_{m=0}^{n} \delta_m \cdot \left( \sum_{j=0}^{m+1} w_{m,j} \cdot \left( p_{m,j} \cdot kn^\frac{1}{k} - p_{m,j} \cdot \gamma_k \right) \right) 
\text{(By inequality (33))}
\]
\[
= \sum_{m=0}^{n} \delta_m \cdot p_m \cdot \left( kn^\frac{1}{k} - \gamma_k \right) 
\text{(Since } p_m = \sum_{j=0}^{m+1} w_{m,j} \cdot p_{m,j} \text{ by (25))}
\]
\[
= p \cdot \left( kn^\frac{1}{k} - \gamma_k \right). 
\text{(Since } p = \sum_{m=0}^{n} \delta_m \cdot p_m \text{ by (11))}
\]
This completes the induction step and the proof.

\[\square\]

**Proposition 5.** *The randomized query complexity of both ordered and unordered search with success probability p in k = 1 rounds is at least pn, where n is the size of the input vector.*

*Proof.* Let \(A_1\) be an optimal 1-round algorithm that succeeds with probability \(p\) when given an input drawn from the uniform distribution. Let \(q_1(n, p)\) be the expected number of queries of \(A_1\) as a function of the input size \(n\) and the success probability \(p\). The expected number of queries is
\[
q_1(n, p) = \sum_{m=0}^{n} \delta_m \cdot m,
\]
where
\[
\sum_{m=0}^{n} \delta_m = 1 \quad (34)
\]
\[
p = \sum_{m=0}^{n} \delta_m \cdot \left( \frac{m}{n} \right) \quad (35)
\]
\[
\delta_m \geq 0 \quad \forall m \in \{0, \ldots, n\}. \quad (36)
\]
Thus we have \(q_1(n, p) = \sum_{m=0}^{n} \delta_m \cdot m = np\).

\[\square\]

### 4 Unordered search

In this section we analyze the unordered search problem and prove Theorems 1 and 2. These quantify the expected query complexity of randomized algorithms in the worst case and deterministic algorithms on worst case input distribution. The omitted proofs of this section can be found in Appendix B.

At a high level, an optimal randomized algorithm for unordered search also has an all-or-nothing structure:

(i) with probability \(1 - p\), do nothing; and
(ii) with probability $p$, select a uniform random permutation $\pi$ over $[n]$. For all $j \in [k]$, define $S_j = \{\pi_1, \ldots, \pi_{m_j}\}$, where $m_j = \lceil npj/k \rceil$. Then in each round $j$, query the locations of $S_j$ that have not been queried in the previous $j - 1$ rounds. Once the element is found, return it and halt.

On the other hand, since the unordered search problem has less structure, a deterministic algorithm receiving an element drawn from some distribution $\Psi$ will no longer be able to extract enough randomness from the answers to the first round queries to simulate the optimal randomized algorithm. Instead, the deterministic algorithm will establish in advance a fixed set of $np$ locations and query those in the same manner as step (ii) of the optimal randomized algorithm.

However, since the search space becomes smaller as an algorithm checks more locations, the fact that the deterministic algorithm is forced to stop after at most $np$ queries, regardless of whether it found the element or not, is a source of inefficiency. This is the main reason for which a deterministic algorithm receiving a random input cannot do as well as the optimal randomized algorithm that either did nothing or searched all the way until finding the solution.

4.1 Unordered search upper bounds

A simple observation is that the maximum number of rounds for unordered search is $n$. However, since with each location queried the only information an algorithm receives is whether the element is at that location or not, a deterministic algorithm cannot do better than exhaustive search in the worst case.

**Observation 1.** For each $k \in \{1, \ldots, n\}$, there is a deterministic $k$-round algorithm for ordered search that always succeeds and asks at most $n$ queries in the worst case:

- In each round $j \in [k]$, issue $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ at locations not previously queried. When the item is found, return it and halt.

However, randomized algorithms can do better by querying locations uniformly at random.

**Proposition 6.** Let $p \in (0, 1]$ and $k, n \in \mathbb{N}$. For each array of length $n$ and input distribution $\Psi = (\Psi_1, \ldots, \Psi_n)$, there is a $k$-round deterministic algorithm for unordered search that succeeds with probability at least $p$ when the input is drawn from $\Psi$ and asks at most

$$np \left(1 - \frac{p(k-1)}{2k}\right) + p + \frac{2}{n} + \left(\lceil np \rceil - np\right)$$

queries in expectation.

**Proof.** Let $\pi$ be a permutation of $[n]$ such that $\Psi_{\pi_1} \geq \ldots \geq \Psi_{\pi_n}$. For each $j \in [k]$, let $S_j \subseteq [n]$ be the top $\lceil np \cdot \frac{j}{k} \rceil$ array positions in the ordering given by $\pi$, that is:

$$S_j = \{\pi_1, \ldots, \pi_{m_j}\}, \text{ where } m_j = \left\lceil np \cdot \frac{j}{k} \right\rceil.$$

Consider the following algorithm.

In each round $j \in [k]$: Query the locations in $S_j$ that have not been queried in the previous $j - 1$ rounds. Once the element is found, return its location and halt immediately.
Success probability. To bound the success probability of the algorithm, observe that the subsets $S_j$ are nested, that is: $S_j \subseteq \ldots \subseteq S_k$. By the end of round $k$, the algorithm has only queried locations from $S_k$ and either found the element or exhausted $S_k$.

For all $j \in [k]$, denote the probability that the sought element is in $S_j$ by

$$\phi_j = \sum_{\ell \in S_j} \Psi_\ell.$$  

Lemma 23 gives $\phi_j \geq |S_j|/n$. Then the success probability is $p = \sum_{\ell \in S_k} \Psi_\ell \geq \frac{|S_k|}{n} = \frac{\lceil n p \ell \rceil}{n} \geq p$.

Expected number of queries. Next we bound the expected number of queries. For each $j \in [k]$, let $A_j$ be the event that the algorithm halts exactly at the end of round $j$. On event $A_j$, the algorithm issues a total of $|S_j|$ queries. Moreover, the probability of event $A_j$ is

$$\Pr(A_j) = \begin{cases} 
\phi_j - \phi_{j-1} & \text{if } 1 \leq j \leq k-1, \text{ where } \phi_0 = 0.
\end{cases}$$ (37)

Let $S_0 = \emptyset$. For each $j \in \{0, \ldots, k\}$, define

$$\eta_j = \phi_j - \frac{|S_j|}{n}.$$ (38)

We have $\eta_j \geq 0$ since $\phi_j \geq |S_j|/n$.

Then the expected number $q_k$ of queries issued on input distribution $\Psi$ can be bounded by:

$$q_k = \sum_{j=1}^{k} \Pr(A_j) \cdot |S_j| = (1 - \phi_{k-1}) \cdot |S_k| + \sum_{j=1}^{k-1} (\phi_j - \phi_{j-1}) \cdot |S_j|$$ (By (37))

$$= 
\left(1 - \frac{|S_{k-1}|}{n} - \eta_{k-1}\right) \cdot |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j|}{n} + \eta_j - \frac{|S_{j-1}|}{n} - \eta_{j-1}\right) \cdot |S_j| \quad \text{(By definition of } \eta_j)$$

$$= \left(1 - \frac{|S_{k-1}|}{n}\right) \cdot |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j| - |S_{j-1}|}{n}\right) \cdot |S_j| - \eta_{k-1} \cdot |S_k| + \sum_{j=1}^{k-1} (\eta_j - \eta_{j-1}) \cdot |S_j|. \quad (39)$$

We have $0 = |S_0| \leq |S_1| \leq \ldots \leq |S_k|$, and so $\sum_{j=1}^{k-1} (\eta_j - \eta_{j-1}) \cdot |S_j| \leq \eta_{k-1} \cdot |S_{k-1}|$. Thus

$$-\eta_{k-1} \cdot |S_k| + \sum_{j=1}^{k-1} (\eta_j - \eta_{j-1}) \cdot |S_j| \leq -\eta_{k-1} \cdot |S_k| + \eta_{k-1} \cdot |S_{k-1}| \leq 0. \quad (40)$$

Using (40) in (39) gives

$$q_k \leq \left(1 - \frac{|S_{k-1}|}{n}\right) \cdot |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j| - |S_{j-1}|}{n}\right) \cdot |S_j|. \quad (41)$$
We observe that
\[
\left(1 - \frac{|S_{k-1}|}{n}\right) \cdot |S_k| - \sum_{j=1}^{k-1} \frac{|S_{j-1}|}{n} \cdot |S_j| = |S_k| - \sum_{j=1}^{k-1} \frac{|S_{j+1}|}{n} \cdot |S_j|.
\] (42)

Adding \(\sum_{j=1}^{k-1} |S_j|^2/n\) to both sides of (42), we obtain
\[
\left(1 - \frac{|S_{k-1}|}{n}\right) \cdot |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j| - |S_{j-1}|}{n}\right) \cdot |S_j| = |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j| - |S_{j+1}|}{n}\right) \cdot |S_j|.
\] (43)

Substituting (43) in (41) and using the identity \(|S_j| = \lceil np \cdot j/k \rceil\) gives
\[
q_k \leq |S_k| + \sum_{j=1}^{k-1} \left(\frac{|S_j| - |S_{j+1}|}{n}\right) \cdot |S_j| = [np] + \sum_{j=1}^{k-1} \left(\frac{[np \cdot j/k] - [np \cdot j+1/k]}{n}\right) \cdot \left[\frac{np \cdot j}{k}\right].
\] (44)

Applying Lemma 16 with \(x = np\) gives
\[
\sum_{j=1}^{k-1} \left(\frac{[np \cdot j/k] - [np \cdot j+1/k]}{n}\right) \cdot \left[\frac{np \cdot j}{k}\right] \leq -\frac{(np)^2(k-1)}{2k} + [np] + 1.
\] (45)

Combining (44) and (45) gives:
\[
q_k \leq [np] + \frac{1}{n} \left( -\frac{(np)^2(k-1)}{2k} + [np] + 1 \right) = np \left(1 - \frac{p(k-1)}{2k}\right) + \frac{[np]}{n} + \frac{1}{n} + [np] - np
\]
\[
\leq np \left(1 - \frac{p(k-1)}{2k}\right) + p + 2 + [np] - np.
\] (46)

This completes the proof.

Proposition 7. There is a randomized \(k\)-round algorithm for unordered search that succeeds with probability \(p\) and issues at most \(np \cdot \frac{k+1}{2k} + p + \frac{p}{n}\) queries in expectation on each input.

Proof. Consider the following algorithm, which has an all-or-nothing structure.

\(\diamond\) With probability \(1 - p\): do nothing\(^4\).

\(\diamond\) With probability \(p\): run the following protocol:

- Choose a uniform random permutation \(\pi = (\pi_1, \ldots, \pi_n)\) of \([n]\). For each \(j \in [k]\), define \(m_j = \lceil n \cdot j/k \rceil\) and \(S_j = \{\pi_1, \ldots, \pi_{m_j}\}\).

- In each round \(j \in [k]\): query all the locations in \(S_j\) that have not been queried yet. Whenever the element is found, return its location and halt immediately.

\(^4\)That is, halt without any queries or return.
Success probability. If the algorithm finishes execution in exactly \( j \geq 1 \) rounds, then the number of queries issued is \(|S_j| = \lceil nj/k \rceil\). By the end of the \( k \)-th round, the number of queries issued would be \( \lceil nk/k \rceil = n \). Thus if the algorithm enters round 1 then it doesn’t stop until finding where the element is, so the success probability is exactly \( p \).

Expected number of queries. Let \( A_j \) be the event that the algorithm halts exactly at the end of round \( j \). On event \( A_j \), the algorithm issues \( \lceil nj/k \rceil \) queries. The probability of event \( A_j \) is

\[
\Pr(A_j) = p \cdot \frac{\lceil n \cdot \frac{j}{k} \rceil - \lceil n \cdot \frac{j-1}{k} \rceil}{n}.
\]

Then the expected number of queries issued by the algorithm is

\[
q_k(n, p) = \sum_{j=1}^{k} \Pr(A_j) \cdot \lceil nj/k \rceil.
\]

Using (47), we can rewrite this as

\[
q_k(n, p) = np + \frac{p}{n} \sum_{j=1}^{k} \left( \frac{n \cdot \frac{j}{k} - n \cdot \frac{j+1}{k}}{k} \right) \cdot \frac{n \cdot j}{k}.
\]

Applying Lemma 16 with \( x = n \) to bound the expression in (48) yields

\[
q_k(n, p) \leq np + \frac{p}{n} \left( -\frac{n^2(k-1)}{2k} + [n] + 1 \right) = np \cdot \frac{k+1}{2k} + p + \frac{p}{n}.
\]

This completes the proof.

4.2 Unordered search lower bounds

Proposition 8. Let \( p \in (0, 1] \) and \( k, n \in \mathbb{N} \). Every \( k \)-round deterministic algorithm for unordered search on an array of size \( n \) with success probability \( p \) when the input is drawn from the uniform distribution asks at least \( np \left( 1 - \frac{k-1}{2k} p \right) \) queries in expectation.

Proof. For each \( \ell \in \mathbb{N} \), let \( A_{\ell} \) be an optimal \( \ell \)-round randomized algorithm that succeeds with probability \( p \) when facing the uniform distribution as input. Let \( q_\ell(n, p) \) be the expected number of queries of algorithm \( A_\ell \) when given an array of length \( n \).

Since \( A_k \) is deterministic, it asks a fixed number \( m \) of queries in round 1. Moreover, since the input is drawn from the uniform distribution, each location is equally likely to contain the answer, and so the actual locations do not matter, but rather only their number. Thus the probability of finding the answer in round 1 is \( m/n \). Let \( \alpha \) be the probability that the algorithm finds the element in one of the later rounds in \( \{2, \ldots, k\} \), given that the element was not found in the first round.

Given these observations, the expected number of queries of the deterministic algorithm can be written as

\[
q_k(n, p) = m + \left( \frac{n-m}{n} \right) \cdot q_{k-1}(n-m, \alpha),
\]
where the variables are related by the following constraints:

\[
\begin{align*}
  p &= m + \left( \frac{n-m}{n} \right) \cdot \alpha \\
  0 &\leq \alpha \leq 1.
\end{align*}
\]  

(51)

We prove by induction on \( k \) that that

\[
q_k(n, p) \geq np \left( 1 - \frac{k-1}{2k} \cdot p \right).
\]  

(52)

**Base case.** Proposition 5 shows that \( q_1(n, p) \geq np \).

**Induction hypothesis.** Suppose \( q_\ell(v, s) \geq vs \left( 1 - \frac{\ell-1}{2\ell} \cdot s \right) \) for all \( \ell \in [k-1], v \in \mathbb{N}, \) and \( s \in [0, 1] \).

**Induction step.** We prove (52) holds for \( k \) and all \( n \in \mathbb{N}, p \in [0, 1] \). The induction hypothesis gives

\[
q_{k-1}(n-m, \alpha) \geq (n-m)\alpha \left( 1 - \frac{k-2}{2k-2} \cdot \alpha \right),
\]  

(53)

which substituted in (50) yields

\[
q_k(n, p) = m + \left( \frac{n-m}{n} \right) \cdot q_{k-1}(n-m, \alpha) \geq m + \frac{\alpha(n-m)^2}{n} \left( 1 - \frac{k-2}{2k-2} \cdot \alpha \right).
\]  

(54)

Since \( \alpha = (np-m)/(n-m) \) by (51), we obtain

\[
q_k(n, p) \geq m + \frac{(np-m)(n-m)}{n} \left( 1 - \frac{k-2}{2k-2} \cdot \left( \frac{np-m}{n-m} \right) \right)
\]

\[
\geq np \left( 1 - \frac{k-1}{2k} \cdot p \right) \quad \text{(By Lemma 17)}
\]

This completes the induction step and the proof.

**Proposition 9.** Let \( p \in (0, 1] \) and \( k, n \in \mathbb{N} \). Every \( k \)-round randomized algorithm for unordered search on an array of size \( n \) with success probability \( p \) asks at least \( np \cdot \frac{k+1}{2k} \) queries in expectation in the worst case.

**Proof.** For proving the required lower bound, it will suffice to assume the input is drawn from the uniform distribution, meaning the algorithm is given an ordered array with \( n \) elements together with a uniform random element \( z \). By an average argument, such a lower lower bound will also hold for a worst case input.

Let \( A_k \) be a \( k \)-round randomized algorithm that succeeds with probability \( p \) when facing the uniform distribution as input and denote by \( q_k(n, p) \) the expected number of queries asked by \( A_k \) on the uniform distribution.

In round 1, the algorithm has some probability \( \delta_m \) of asking \( m \) queries, for each \( m \in \{0, \ldots, n\} \). Moreover, for each such \( m \), there are different (but finitely many) choices for the positions of the \( m \) queries of round 1. However, since the goal is to minimize the number of queries, it suffices to
restrict attention to the best way of positioning the queries in round 1, breaking ties arbitrarily between different equally good options. For unordered search, each queried location is equivalent to any other since a query only reveals whether the element is there or not.

For each \( m \in \{0, \ldots, n\} \), we define the following variables:

- \( \delta_m \) is the probability that the algorithm asks \( m \) queries in the first round.
- \( \alpha_m \) is the probability that the algorithm finds the element in one of the rounds in \( \{2, \ldots, k\} \), given that it didn’t find it in the first round.

The probability of finding the element in the first round is \( m/n \), so the probability that the algorithm may need to continue to one of the rounds in \( \{2, \ldots, k\} \) is \((n - m)/n\). The expected number of queries of \( A_k \) on the uniform distribution is

\[
q_k(n, p) = \sum_{m=0}^{n} \delta_m \left( m + \left( \frac{n-m}{n} \right) \cdot q_{k-1}(n-m, \alpha_m) \right),
\]

where the variables are related by the following constraints:

\[
\sum_{m=0}^{n} \delta_m = 1 \tag{56}
\]

\[
p_m = \frac{m}{n} + \left( \frac{n-m}{n} \right) \cdot \alpha_m, \quad \forall m \in \{0, \ldots, n\} \tag{57}
\]

\[
p = \sum_{m=0}^{n} \delta_m \cdot p_m \tag{58}
\]

\[
0 \leq \alpha_m \leq 1, \quad \forall m \in \{0, \ldots, n\} \tag{59}
\]

\[
\delta_m \geq 0, \quad \forall m \in \{0, \ldots, n\}. \tag{60}
\]

**Base case.** Proposition 5 gives \( q_1(n, p) \geq np \), as required.

**Induction hypothesis.** Suppose \( q_\ell(v, s) \geq vs \cdot \frac{\ell+1}{2\ell} \) for all \( \ell \in [k-1] \), \( v \in \mathbb{N} \), and \( s \in [0, 1] \).

**Induction step.** Using the induction hypothesis in (55) gives

\[
q_k(n, p) = \sum_{m=0}^{n} \delta_m \left( m + \left( \frac{n-m}{n} \right) \cdot q_{k-1}(n-m, \alpha_m) \right)
\]

\[
\geq \sum_{m=0}^{n} \delta_m \left( m + \left( \frac{n-m}{n} \right)^2 \cdot \alpha_m \cdot \frac{k}{2k-2} \right). \tag{61}
\]

Substituting \( \alpha_m = (n \cdot p_m - m)/(n - m) \) from (57) in (61) gives

\[
q_k(n, p) \geq \sum_{m=0}^{n} \delta_m \left( m + \frac{(n-m)(n \cdot p_m - m)}{n} \cdot \frac{k}{2k-2} \right). \tag{62}
\]
Lemma 18 gives
\[ m + \frac{(n-m)(n \cdot p_m - m)}{n} \cdot \frac{k}{2k-2} \geq n \cdot p_m \left( \frac{k+1}{2k} \right). \] (63)

Using (63) in (62) gives
\[ q_k(n, p) \geq \sum_{m=0}^{n} \delta_m \left( n \cdot p_m \cdot \frac{k+1}{2k} \right) = n \cdot \left( \frac{k+1}{2k} \right) \cdot \sum_{m=0}^{n} \delta_m \cdot p_m \]
\[ = np \cdot \left( \frac{k+1}{2k} \right). \]

Since \( p = \sum_{m=0}^{n} \delta_m \cdot p_m \) by (58)

\( \Box \)

5 Proportional cake cutting and sorting

In this section we study cake cutting in rounds and discuss the connection between sorting with rank queries and proportional cake cutting. We first introduce the cake cutting model.

Cake cutting model. The resource (cake) is represented as the interval \([0, 1]\). There is a set of players \( N = \{1, \ldots, n\} \), such that each player \( i \in N \) is endowed with a private valuation function \( V_i \) that assigns a value to every subinterval of \([0, 1]\). These values are induced by a non-negative integrable value density function \( v_i \), so that for an interval \( I \), \( V_i(I) = \int_{x \in I} v_i(x) \, dx \). The valuations are additive, so \( V_i\left( \bigcup_{j=1}^{m} I_j \right) = \sum_{j=1}^{m} V_i(I_j) \) for any disjoint intervals \( I_1, \ldots, I_m \subseteq [0, 1] \). The value densities are non-atomic, and sets of measure zero are worth zero to a player. W.l.o.g., the valuations are normalized to \( V_i([0, 1]) = 1 \), for all \( i = 1 \ldots n \).

A piece of cake is a finite union of disjoint intervals. A piece is connected (or contiguous) if it consists of a single interval. An allocation \( A = (A_1, \ldots, A_n) \) is a partition of the cake among the players, such that each player \( i \) receives the piece \( A_i \), the pieces are disjoint, and \( \bigcup_{i \in N} A_i = [0, 1] \). An allocation \( A \) is said to be proportional if \( V_i(A_i) \geq 1/n \) for all \( i \in N \).

Query complexity of cake cutting. All the discrete cake cutting protocols operate in a query model known as the Robertson-Webb model (see, e.g., the book of [RW98]), which was explicitly stated by [WS07]. In this model, the protocol communicates with the players using the following types of queries:

- **Cut\(_i\)(\( \alpha \))**: Player \( i \) cuts the cake at a point \( y \) where \( V_i([0, y]) = \alpha \), where \( \alpha \in [0, 1] \) is chosen arbitrarily by the center\(^5\). The point \( y \) becomes a cut point.

- **Eval\(_i\)(\( y \))**: Player \( i \) returns \( V_i([0, y]) \), where \( y \) is a previously made cut point.

An RW protocol asks the players a sequence of cut and evaluate queries, at the end of which it outputs an allocation demarcated by cut points from its execution (i.e. cuts discovered through queries). Note that the value of a piece \([x, y]\) can be determined with two Eval queries, \( Eval_i(x) \) and \( Eval_i(y) \).

\(^5\)Ties are resolved deterministically, using for example the leftmost point with this property.
When a protocol runs in $k$ rounds, then multiple RW queries (to the same or different agents) can be issued at once in each round. Note the choice of queries submitted in round $j$ cannot depend on the results of queries from the same or later rounds (i.e. $j, j+1, \ldots, k$).

5.1 Upper bound

We will devise a protocol that finds a proportional allocation of the cake in $k$ rounds of interaction, which will also give a protocol for sorting with rank queries. For the special case of one round, a proportional protocol was studied in [BBKP14, MO12]. Our high level approach is to iteratively divide the cake into subcakes and assign agents to each subcake. The algorithm can be found in Appendix C, together with the omitted proofs of this section.

**Proposition 1.** There is an algorithm that runs in $k$ rounds and computes a proportional allocation with a total of $O(\frac{kn^{1+1/k}}{k})$ RW queries.

A key step in connecting cake cutting with sorting will be the following reduction, which reduces sorting a vector of $n$ elements with rank queries to proportional (contiguous) cake cutting with $n$ agents. Rank queries have the form “How is rank($x_j$) compared to $k$?”, where the answer can be “$<$”, “$=$”, or “$>$”.

**Proposition 2.** There exists a polynomial time reduction from sorting $n$ elements with rank queries to proportional cake cutting with $n$ agents. The reduction holds for any number of rounds.

The reduction from sorting to cake cutting was shown in Woeginger and Sgall [WS07]. We formalize the connection to rank queries and note the reduction is round-preserving.

**Proposition 3.** There is a deterministic sorting algorithm in the rank query model that runs in $k$ rounds and asks a total of $O(\frac{kn^{1+1/k}}{k})$ queries.

**Proof.** By the reduction from sorting to cake cutting in Proposition 2, the upper bound follows from Proposition 1. In short, the sorting algorithm is as follows. In the first round issue $n^{1/k} - 1$ evenly spaced queries to each element, dividing the array into $n^{1/k}$ blocks. Each element either has its exact rank revealed, or is found to belong to a particular block. Then recursively call the sorting algorithm in each block. \hfill $\Box$

5.2 Lower bound

In this section we first show a lower bound for sorting in the rank query model; for deterministic algorithms this bound improves upon the bound in [AA88a] by a constant factor and the proof is simpler (see Appendix D). Deterministic algorithms are relevant specifically for fair division, since some studies find that it is preferable to avoid randomness in the allocations if possible when dealing with human agents.

**Proposition 10.** Let $c(k, n)$ be the minimum total number of queries required to sort $n$ elements in the rank query model by the best deterministic algorithm in $k$ rounds. Then $c(k, n) \geq \frac{k}{2e} n^{1+1/k} - kn$.

Alon and Azar [AA88a] show a lower bound of $\Omega(\frac{kn^{1+1/k}}{k})$ for randomized sorting with rank queries, which together with the reduction in Proposition 2 implies the next corollary.

**Corollary 4.** Let $A$ be an algorithm that runs in $k$ rounds for solving proportional cake cutting with contiguous pieces for $n$ agents. If $A$ succeeds with constant probability, then it issues $\Omega(\frac{kn^{1+1/k}}{k})$ queries in expectation.
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A Ordered Search Appendix

In this section we include the omitted proofs for ordered search.

First we show that Locate with rank queries is equivalent to ordered search with comparison queries. Recall that rank queries have the form “How is \( \text{rank}(x_j) \) compared to \( i \)?”, where the answer can be “<”, “=”, or “>”. Locate is the following problem: given a vector \( x = (x_1, \ldots, x_n) \) and an element \( x_i \), find its rank via rank queries. We show that locate and ordered search are equivalent.

Lemma 1. Ordered search with comparison queries is equivalent to Locate with rank queries.

Proof. We show the two problems are equivalent by mapping the locations in the locate problem to the elements in the ordered search problem.

More precisely, given any vector \( x = (x_1, \ldots, x_n) \) that can be accessed via rank queries, construct a vector \( y = (y_1, \ldots, y_n) \) that can be accessed via comparison queries, with \( y_i = i \) for each \( i \in [n] \). For any \( i \), let \( r_i = \text{rank}(x_i) \) be the rank of \( x_i \) in the vector \( x \). Observe that \( r_i \in y \). Then we have a bijection between each rank query “How is \( \text{rank}(x_i) \) compared to \( \ell \)” and the corresponding comparison query: “How is \( r_i \) compared to \( y_\ell \)?”.

For the other direction, suppose we are given an ordered search instance consisting of a vector \( y = (y_1, \ldots, y_n) \) and an element \( u \) promised to be in \( y \). We have access to an oracle that answers comparison queries. Then we construct a locate instance as follows: define \( x = (x_1, \ldots, x_n) \), where \( x_i = i \) for each \( i \in [n] \), and set \( z = \text{rank}(u) \); that is, \( z \) is the rank of \( u \) in the vector \( y \). Again we have a bijection between each comparison query “How is \( y_i \) compared to \( y_\ell \)” and the corresponding rank query: “How is \( \text{rank}(x_i) \) compared to \( \ell \)?”, where \( \ell = \text{rank}(x_\ell) \).

The next lemmas are used in the proofs of the ordered search upper and lower bounds.

Lemma 2. Let \( k \geq 2, n \geq 1 \), and the sequence \( \{\gamma_\ell\}_{\ell=1}^\infty \) with \( \gamma_1 = 0 \) and \( \gamma_\ell = 2\ell \) for all \( \ell \geq 2 \). Then

\[
x \left(1 + \frac{\gamma_{k-1}}{n}\right) + (k-1) \cdot \frac{(n-x)^{k-1}}{n \cdot (x+1)^{k-1}} - \gamma_{k-1} \geq kn^\frac{1}{k} - \gamma_k \quad \forall x \in (1/2, n].
\] (64)

Proof. Let \( t = \left(\frac{n-x}{x+1}\right)^{\frac{1}{k-1}} \). Then \( t \) is decreasing in \( x \). Since \( x \in (1/2, n] \), we have \( 0 \leq t < \left(\frac{2n-1}{3}\right)^{\frac{1}{k-1}} \).

Expressing \( x \) in terms of \( t \) we get

\[
x = \frac{n - tk^{-1}}{tk^{-1} + 1}.
\] (65)

Substituting (65) in (64), we get that (64) is equivalent to

\[
t^k \cdot (k-1)(n+1) - t^{k-1} \cdot \left(kn^{\frac{k+1}{k}} + n + \gamma_{k-1}(n+1) - n\gamma_k\right) + \left(n^2 + n\gamma_k - kn^{\frac{k+1}{k}}\right) \geq 0
\]

\[
\forall 0 \leq t < \left(\frac{2n-1}{3}\right)^{\frac{1}{k-1}}.
\] (66)

We consider two cases, for \( k = 2 \) and \( k \geq 3 \).
Case $k = 2$. Since $\gamma_1 = 0$ and $\gamma_2 = 4$, inequality (66) is equivalent to

$$t^2 \cdot (n + 1) - t \cdot (2n\sqrt{n} - 3n) + n^2 - 2n\sqrt{n} + 4n \geq 0 \quad \forall 0 \leq t < \frac{2n - 1}{3}. \quad (67)$$

Inequality (67) holds by Lemma 3.

Case $k \geq 3$. Since $\gamma_1 = 0$ and $\gamma_\ell = 2\ell$ for $\ell \geq 2$, inequality (66) can be simplified to

$$t^k \cdot (k - 1)(n + 1) - t^{k-1} \cdot \left( k \cdot n^{\frac{k+1}{k}} - n + 2k - 2 \right) + n^2 + 2kn - kn^{1+\frac{1}{k}} \geq 0$$

$$\forall 0 \leq t < \left( \frac{2n - 1}{3} \right)^{\frac{1}{k-1}}. \quad (68)$$

Inequality (68) holds by Lemma 4 for all $k \geq 3$. This completes the proof. \hfill \square

Lemma 3. Let $n \geq 1$. Then for all $t \in [0, (2n - 1)/3)$, we have

$$t^2 \cdot (n + 1) - t \cdot (2n\sqrt{n} - 3n) + n^2 - 2n\sqrt{n} + 4n \geq 0. \quad (69)$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = t^2 \cdot (n + 1) - t \cdot (2n\sqrt{n} - 3n) + n^2 - 2n\sqrt{n} + 4n$. Then

$$f'(t) = 2t(n + 1) - (2n\sqrt{n} - 3n) \quad \text{and} \quad f''(t) = 2(n + 1). \quad (70)$$

Thus $f$ is convex and the global minimum is at $t^*$ for which $f'(t^*) = 0$, that is, $t^* = \frac{2n\sqrt{n} - 3n}{2n + 2}$. Evaluating $f(t^*)$ gives

$$f(t^*) = \left( \frac{2n\sqrt{n} - 3n}{2n + 2} \right)^2 \cdot (n + 1) - \left( \frac{2n\sqrt{n} - 3n}{2n + 2} \right) \cdot (2n\sqrt{n} - 3n) + n^2 - 2n\sqrt{n} + 4n$$

$$= \frac{11n^2 - 8n\sqrt{n} + 16n + 4n^2\sqrt{n}}{4n + 4}$$

$$> 0. \quad \text{(Since } 11n^2 > 8n\sqrt{n} \text{ for } n \geq 1)$$

Thus $f(t) \geq f(t^*) > 0$ for all $t \in \mathbb{R}$, which implies the inequality required by the lemma. \hfill \square

Lemma 4. Let $n \geq 1$ and $k \geq 3$. Then

$$t^k \cdot (k - 1)(n + 1) - t^{k-1} \cdot \left( k \cdot n^{\frac{k+1}{k}} - n + 2k - 2 \right) + n^2 + 2kn - kn^{1+\frac{1}{k}} \geq 0, \quad \forall t \in \left[ 0, n^{\frac{1}{k-1}} \right). \quad (71)$$

Proof. Dividing both sides of (71) by $n^2$, we get that (71) holds if and only if

$$\left( \frac{t^k}{n^{\frac{1}{k-1}}} \right)^k \cdot (k - 1) \left( 1 + \frac{1}{n} \right) \cdot n^{\frac{1}{k-1}} - \left( \frac{t^{k-1}}{n^{\frac{k}{k-1}}} \right)^{k-1} \cdot \left( k \cdot n^{\frac{1}{k}} - 1 + \frac{2k - 2}{n} \right) + 1 + \frac{2k}{n} - \frac{k}{n^{\frac{k}{k-1}}} \geq 0$$

$$\forall t \in \left[ 0, n^{\frac{1}{k-1}} \right). \quad (72)$$

If $t = 0$ then (72) is equivalent to $n + 2k - kn^{\frac{1}{k}} \geq 0$, which holds by Corollary 6.
For \( t > 0 \), let \( x = n^{\frac{1}{k-1}}/t \). Since \( 0 < t < n^{\frac{1}{k-1}} \), we have \( x > 0 \). Substituting \( t \) by \( x \) we obtain that (72) is equivalent to

\[
x^k \cdot \left( 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k-1}}} \right) - x \cdot \left( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} \right) + (k-1) \left( 1 + \frac{1}{n} \right) \cdot n^{\frac{1}{k-1}} \geq 0, \quad \forall x > 1.
\]  

(73)

Define the function \( f : (0, \infty) \to \mathbb{R} \), where \( f(x) \) is given by the left hand side of (73). Then

\[
f'(x) = k \left( 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k-1}}} \right) x^{k-1} - \left( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} \right) x^{k-2}.
\]

(74)

By Corollary 6, we have \( 1 + 2k/n - k/n^{k-1} > 0 \) for all \( n \geq 1 \) and \( k \geq 3 \). Thus \( f''(x) > 0 \) for all \( x > 0 \), so \( f \) is convex on \((0, \infty)\). Observe that \( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} > 0 \) for all \( n \geq 1 \) and \( k \geq 3 \). Then there is a global minimum of \( f \) at a point \( x \in (0, \infty) \) with \( f'(x) = 0 \), or equivalently,

\[
\bar{x} = \left( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} \right)^{\frac{1}{k-1}} \left( 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k-1}}} \right)^{-\frac{1}{k-1}}.
\]  

(75)

Evaluating \( f \) at \( \bar{x} \) and rearranging terms gives

\[
f(\bar{x}) = \bar{x} \left[ \left( 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k-1}}} \right) \bar{x}^{k-1} - \left( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} \right) \right] + (k-1) \left( 1 + \frac{1}{n} \right) \cdot n^{\frac{1}{k-1}}
\]

\[
= \bar{x} \left( 1 + \frac{1}{k} \right) \left( k \cdot n^{\frac{1}{k-1}} - 1 + \frac{2k-2}{n} \right) + (k-1) \left( 1 + \frac{1}{n} \right) \cdot n^{\frac{1}{k-1}}
\]

\[
= (k-1) \left( 1 + \frac{1}{n} \right) \cdot n^{\frac{1}{k-1}} - (k-1) \left( n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{k} - \frac{2}{kn} \right)^{\frac{1}{k-1}} \left( \frac{n}{n+2k-kn^{\frac{1}{k}}} \right)^{\frac{1}{k-1}}.
\]  

(76)

Thus \( f(x) > 0 \) \( \forall x > 1 \) whenever the next two properties are met

1. \( f(\bar{x}) > 0 \) \( \text{when } \bar{x} > 1 \). This is equivalent to

\[
(n+2k-kn^{\frac{1}{k}}) \left( 1 + \frac{1}{n} \right)^{k-1} > \left( n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} \right)^{k}
\]

\[\text{whenever } n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} > 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k}}} \]  

(77)

Lemma 6 implies that inequality (77) holds under condition (78).

2. \( f(1) \geq 0 \) when \( \bar{x} < 1 \). To show this, observe that for all \( n \geq 1 \) and \( k \geq 3 \), we have

\[
f(1) = \left( 1 + \frac{1}{n} \right) \left( 2 - k \cdot n^{\frac{1}{k}} + (k-1) \cdot n^{\frac{1}{k-1}} \right)
\]

\[
\geq 0. \quad \text{(By Lemma 5)}
\]

Thus \( f(1) \geq 0 \) for all \( n \geq 1 \) and \( k \geq 3 \), which completes property 2.
Since both properties 1 and 2 hold, we have that \( f(x) > 0 \) for all \( x > 1 \), so (73) holds. Equivalently, (71) holds as required by the lemma.

**Lemma 5.** Let \( n \geq 1 \) and \( k \geq 3 \), where \( k, n \in \mathbb{N} \). Then \( 2 - k \cdot n^{\frac{1}{k}} + (k - 1) \cdot n^{\frac{1}{k-1}} \geq 0 \).

**Proof.** Consider the function \( f : [2, \infty) \to \mathbb{R} \) given by \( f(x) = x n^{\frac{1}{x}} \). We first show an upper bound on \( f' \) and then use it to upper bound \( f(k) - f(k-1) \). We have

\[
f'(x) = n^{\frac{1}{x}} \left( 1 - \frac{\ln(n)}{x} \right).
\]

Let \( y = n^{\frac{1}{x}} \). Then \( \ln(y) = \frac{1}{x} \ln(n) \). Since \( x \geq 2 \), we have \( y \in (1, \sqrt{n}] \). Then

\[
n^{\frac{1}{y}} \left( 1 - \frac{\ln(n)}{x} \right) = y \left( 1 - \ln(y) \right).
\]

The function \( g : (1, \infty) \to \mathbb{R} \) given by \( g(y) = y (1 - \ln(y)) \) has \( g'(y) = -\ln(y) < 0 \). Therefore \( g(y) < g(1) = 1 \) for all \( y > 1 \). Using the identity in (79), we get that \( f'(x) < 1 \) for all \( x \geq 2 \).

Then for all \( k \geq 3 \),

\[
k \cdot n^{\frac{1}{x}} = f(k) \leq f(k-1) + \max_{x \in [2, \infty)} f'(x) < f(k-1) + 1 = (k-1) \cdot n^{\frac{1}{k-1}} + 1.
\]

Inequality (80) implies the lemma statement.

**Corollary 5.** Let \( n \geq 1 \) and \( k \geq 2 \). Suppose \( \{\gamma_{\ell}\}_{\ell=1}^{\infty} \) is the sequence given by \( \gamma_1 = 0 \) and \( \gamma_\ell = 2\ell \) for \( \ell \geq 2 \). Then \( (k - 1) \cdot n^{\frac{1}{k-1}} - \gamma_{k-1} \geq kn^{\frac{1}{k}} - \gamma_k \).

**Proof.** If \( k = 2 \), the required inequality is \( n \geq 2\sqrt{n} - 4 \), or \( (\sqrt{n} - 1)^2 + 3 \geq 0 \). The latter holds for all \( n \geq 1 \). If \( k \geq 3 \), the required inequality is \( (k - 1) \cdot n^{\frac{1}{k-1}} + 2 \geq kn^{\frac{1}{k}} \), which holds by Lemma 5.

**Lemma 6.** Let \( n \geq 1 \) and \( k \geq 3 \), where \( k, n \in \mathbb{N} \). Then

\[
(n + 2k - kn^{\frac{1}{k}}) \left( 1 + \frac{1}{n} \right)^{k-1} > \left( n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} \right)^k
\]

whenever \( n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} > 1 + \frac{2k}{n} - \frac{k}{n^{\frac{1}{k}}} \).

**Proof.** If \( n = 1 \) then the condition in (82) is equivalent to \( 1 - 1/k + 2 - 2/k > 1 + 2k - k \), which holds if and only if \( 2 - 3/k > k \) (†). Since \( k \geq 3 \), inequality (†) does not hold so condition (82) is not met either.

Thus from now on we can assume \( n \geq 2 \). By Lemma 10, condition (82) implies \( k < n \). We show (81) holds when \( n \geq 2 \) and \( k < n \) by considering separately a few ranges of \( k \).
Case I: $n/2 < k < n$ and $k \geq 3$. Then $k < n < 2k$. When $n = 2k - 1$ inequality (81) holds by Lemma 9.

Thus from now on we can assume $k < n \leq 2k - 2$. To show inequality (81), we will first bound separately several of the terms in the inequality and then combine the bounds.

For $k \geq 3$, we have $k < 2k - 2$. Moreover, since $n < 2k$, we have $n^{\frac{1}{k}} < (2k)^{\frac{1}{k}} \leq 2$, and so $2k > kn^{\frac{1}{k}}$. Thus $n + 2k - kn^{\frac{1}{k}} > n$, which implies

$$
\left(n + 2k - kn^{\frac{1}{k}}\right) \left(1 + \frac{1}{n}\right)^{k-1} > n \left(1 + \frac{1}{n}\right)^{k-1}.
$$

Moreover, since $n \geq 2$, we have $2k \leq kn \cdot n^{\frac{1}{k}}$, and so $2k - 2 - n < kn \cdot n^{\frac{1}{k}}$. Since $n \leq 2k - 2$, we also have $2k - 2 - n \geq 0$, and so

$$
0 \leq \frac{2k - 2 - n}{kn \cdot n^{\frac{1}{k}}} < 1.
$$

Let $r = (2k - 2 - n)/(kn \cdot n^{\frac{1}{k}})$. Inequality (84) gives $0 \leq r < 1$. We consider two sub-cases:

- If $n = 2k - 2$ then $r = 0$. We have

$$
\left(1 + \frac{2k - 2 - n}{kn \cdot n^{\frac{1}{k}}}\right)^k = (1 + r)^k = 1 = e^0 = e^{\left(\frac{2k-2-n}{kn \cdot n^{\frac{1}{k}}}\right)}.
$$

- Else $k < n < 2k - 2$. Then $0 < r < 1$. We have

$$
\left(1 + \frac{2k - 2 - n}{kn \cdot n^{\frac{1}{k}}}\right)^k = (1 + r)^k
$$

(By definition of $r$)

$$
= \left[(1 + r)^{\frac{1}{r}}\right]^{kr}
\leq e^{kr}
\quad \text{(Since } (1 + r)^{\frac{1}{r}} \leq e \text{ for } r \in (0, 1).)

= e^{\left(\frac{2k-2-n}{kn \cdot n^{\frac{1}{k}}}\right)}.
$$

Combining inequalities (85) and (86) from the two sub-cases, we obtain

$$
\left(1 + \frac{2k - 2 - n}{kn \cdot n^{\frac{1}{k}}}\right)^k \leq e^{\left(\frac{2k-2-n}{kn \cdot n^{\frac{1}{k}}}\right)} \quad \forall n \in \mathbb{N} \text{ with } k < n \leq 2k - 2.
$$

Using (87), we can upper bound the right hand side of inequality (81) as follows:

$$
\left(n^{\frac{1}{k}} - k + \frac{2}{n} - \frac{2}{kn}\right)^k = \left(n^{\frac{1}{k}} + \frac{2k - 2 - n}{kn}\right)^k
= n \left(1 + \frac{2k - 2 - n}{kn \cdot n^{\frac{1}{k}}}\right)^k
\leq ne^{\left(\frac{2k-2-n}{kn \cdot n^{\frac{1}{k}}}\right)}.
$$
By Lemma 11, we have
\[ e^{\frac{2k - 2 - n}{n^2}} \leq \left( 1 + \frac{1}{n} \right)^{k-1}. \]  
(89)

Combining (83), (88), and (89), gives:
\[
\left( n^{\frac{1}{n^2}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} \right)^k \leq ne^{\frac{2k - 2 - n}{n^2}} \quad \text{(By (88))}
\]
\[
\leq n \left( 1 + \frac{1}{n} \right)^{k-1} \quad \text{(By (89))}
\]
\[
< \left( n + 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{1}{n} \right)^{k-1}. \quad \text{(By (83))}
\]

In summary,
\[
\left( n^{\frac{1}{n^2}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} \right)^k \leq \left( n + 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{1}{n} \right)^{k-1} \forall n \in \mathbb{N} \text{ with } k < n \leq 2k - 2.
\]

This is the required inequality (81), which completes case I.

**Case II:** \(3 \leq k \leq n/2\). Then \(n \geq 2k\) and \(k \geq 3\). Then \(t^k \geq 2k\). For \(k = 3\), the required inequality (81) is equivalent to
\[
\left( x + 6 - 3x^{\frac{1}{3}} \right) \left( 1 + \frac{1}{x} \right)^2 - \left( x^{\frac{1}{3}} - \frac{1}{3} + \frac{2}{x} - \frac{2}{3x} \right)^3 \quad \forall x \geq 6^{\frac{1}{3}},
\]
which holds (see, e.g., [wol]). Thus from now on we can assume \(k \geq 4\) with \(k \in \mathbb{N}\).

Let \(f : (0, \infty) \to \mathbb{R}\) be
\[
f(x) = \left( x + 2k - kx^{\frac{1}{k}} \right) \left( 1 + \frac{1}{x} \right)^{k-1} - \left( x^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{x} - \frac{2}{kx} \right)^k.
\]

Using Bernoulli’s inequality gives
\[
\left( 1 + \frac{1}{n} \right)^{k-1} \geq 1 + \frac{k - 1}{n}. \quad \text{(90)}
\]

Since \(n \geq 2k\), we have \(2/n \leq 1/k\). Thus
\[
-\frac{1}{k} + \frac{2k - 2}{kn} \leq -\frac{1}{k} + \frac{k - 1}{k^2} = -\frac{1}{k^2}. \quad \text{(91)}
\]

Using (90) and (91), we can lower bound \(f(n)\) as follows:
\[
f(n) = \left( n + 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{1}{n} \right)^{k-1} - \left( n^{\frac{1}{k}} - \frac{1}{k} + \frac{2k - 2}{kn} \right)^k
\]
\[
\geq \left( n + 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{k - 1}{n} \right) - \left( n^{\frac{1}{k}} - \frac{1}{k} \right)^k \quad \text{(By (90) and (91))}
\]
\[
\geq \left( 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{k - 1}{n} \right) + n - \left( n^{\frac{1}{k}} - \frac{1}{k} \right)^k. \quad \text{(92)}
\]
If $n^{\frac{1}{k}} \leq 2$, then $2k - kn^{\frac{1}{k}} \geq 0$, which together with (92) yields $f(n) \geq n - \left( \frac{n^{\frac{1}{k}}}{\frac{1}{k^2}} \right)^k \geq 0$, as required.

Thus from now on we will assume $n^{\frac{1}{k}} > 2$, that is, $n > 2^k$. Then $2k - kn^{\frac{1}{k}} < 0$. Together with $n \geq 2k$, this implies
\[ \left( 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{k - 1}{n} \right) > \left( 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{k - 1}{2k} \right). \] (93)

Inequality (93) together with $(k - 1)/2k < 1/2$ yields
\[ \left( 2k - kn^{\frac{1}{k}} \right) \left( 1 + \frac{k - 1}{n} \right) > 1.5 \cdot \left( 2k - kn^{\frac{1}{k}} \right). \] (94)

Combining (92) and (94) gives
\[ f(n) \geq 1.5 \cdot \left( 2k - kn^{\frac{1}{k}} \right) + n - \left( \frac{n^{\frac{1}{k}}}{\frac{1}{k^2}} \right)^k. \] (95)

Next we expand and truncate $\left( n^{\frac{1}{k}} - \frac{1}{k^2} \right)^k$ via Lemma 7, yielding
\[ - \left( n^{\frac{1}{k}} - \frac{1}{k^2} \right)^k \geq -n + \frac{n^{1 - \frac{1}{k}}}{k} - \frac{(k - 1)n^{1 - \frac{2}{k}}}{2k^3}. \] (96)

Using (96), we can further bound $f(n)$ by
\[ f(n) \geq 1.5 \cdot \left( 2k - kn^{\frac{1}{k}} \right) + \frac{n^{1 - \frac{1}{k}}}{k} - \frac{(k - 1)n^{1 - \frac{2}{k}}}{2k^3}. \] (Combining (95) and (96))
\[ \geq 0. \] (By Lemma 8)

Thus $f(n) > 0$, as required. This completes the analysis for the range $n > 2^k$ and case II.

**Wrapping up.** We obtain that inequality (81) holds under condition (82), as required.

**Lemma 7.** Let $k, n \in \mathbb{N}$ with $n \geq 1$ and $k \geq 3$. Then
\[ \left( n^{\frac{1}{k}} - \frac{1}{k^2} \right)^k \leq n - \frac{n^{1 - \frac{1}{k}}}{k} + \frac{(k - 1)n^{1 - \frac{2}{k}}}{2k^3}. \] (97)

**Proof.** Let $t = n^{\frac{1}{k}}$. Then $t \geq 1$. The required inequality (97) is equivalent to
\[ \left( t - \frac{1}{k^2} \right)^k \leq t^k - \frac{t^{k-1}}{k} + \frac{(k - 1)t^{k-2}}{2k^3}. \] (98)

For $i \in [k + 1]$ let $c_i$ be the $i$-th term in the binomial expansion of $(t - 1/k^2)^k$. In particular,
\[ c_1 = t^k; \quad c_2 = -\frac{t^{k-1}}{k}; \quad c_3 = \frac{(k - 1)t^{k-2}}{2k^3}; \quad c_{k+1} = (-1)^k \frac{1}{k^{2k}}. \] (99)
Let us bound the ratio \( \frac{c_i}{c_{i+1}} \) for \( i \in [k] \):

\[
\left| \frac{c_i}{c_{i+1}} \right| = \frac{k^2 \cdot t_i}{k - i + 1} \geq tk > 1 .
\]  

(100)

Since \( c_{2i} < 0 \) and \( c_{2i+1} > 0 \) for all \( i \), inequality (100) implies

\[
c_i + c_{i+1} \leq 0 \quad \forall i \in [k] \text{ with } i \in 2\mathbb{N}.
\]  

(101)

We bound the term \( \left( t - \frac{1}{k^2} \right)^k \) by considering two cases. If \( k \) is even, then

\[
\begin{align*}
\left( t - \frac{1}{k^2} \right)^k &= c_1 + c_2 + c_3 + \sum_{i=2}^{k/2} (c_{2i} + c_{2i+1}) \\
&\leq c_1 + c_2 + c_3 .
\end{align*}
\]

(By definition of \( c_i \).

Since \( c_{2i} + c_{2i+1} < 0 \) by (101))

If \( k \) is odd, then

\[
\begin{align*}
\left( t - \frac{1}{k^2} \right)^k &= \sum_{i=1}^{k+1} c_i \\
&< \sum_{i=1}^{k} c_i = c_1 + c_2 + c_3 + \sum_{i=2}^{(k-1)/2} (c_{2i} + c_{2i+1}) \\
&\leq c_1 + c_2 + c_3 .
\end{align*}
\]

(By definition of \( c_i \).

Since \( c_{k+1} < 0 \).

Since \( c_{2i} + c_{2i+1} < 0 \) by (101))

Thus for both odd and even \( k \), we have

\[
\left( t - \frac{1}{k^2} \right)^k \leq c_1 + c_2 + c_3 = t^k - \frac{t^{k-1}}{k} + \frac{(k-1)t^{k-2}}{2k^3} .
\]

(102)

Thus in both cases (98) holds, as required.

\[ \square \]

**Lemma 8.** Let \( k, n \in \mathbb{N} \) with \( n \geq 2 \) and \( k \geq 4 \). Then

\[
1.5 \left( 2k - kn^\frac{1}{k} \right) + \frac{n^{1-\frac{1}{k}}}{k} - \frac{(k-1) \cdot n^{1-\frac{2}{k}}}{2k^3} \geq 0 .
\]

(103)

**Proof.** Let \( t = n^\frac{1}{k} \). Then \( t > 1 \) since \( n \geq 2 \). For \( k = 4 \), inequality (103) with \( n \) substituted by \( t^4 \) is equivalent to \( 1.5(8 - 4t) + t^2/4 - 3t^2/128 \geq 0 \), which holds for all \( t > 1 \).

Thus from now on we can assume \( k \geq 5 \). The left hand side of (103), where \( n \) is substituted by \( t^k \), can be bounded as follows:

\[
\begin{align*}
1.5 (2k - kt) + \frac{t^{k-1}}{k} - \frac{(k-1) \cdot t^{k-2}}{2k^3} &\geq 1.5(2k - kt) + \frac{t^{k-2}}{2k^2} (2kt - 1) \quad \text{(Since } k - 1 < k) \\
&\geq 1.5 \left( 2k - kt + \frac{t^{k-1}}{2k} \right) . \quad \text{(Since } t > 1 \text{ and } k \geq 4)
\end{align*}
\]

(104)
Let $g : (0, \infty) \to \mathbb{R}$ be $g(t) = 2k - kt + \frac{t^{k-1}}{2k}$. We will show that $g(t) \geq 0$ for all $t > 1$. We have

$$g'(t) = -k + \frac{k-1}{2k} \cdot t^{k-2} \quad \text{and} \quad g''(t) = \frac{(k-1)(k-2)}{2k} \cdot t^{k-3}.$$  \hspace{1cm} (105)

Thus $g$ is convex on $(0, \infty)$. The global minimum is $t^*$ with $g'(t^*) = 0$, so $t^* = \left(\frac{2k^2}{k-1}\right)^{\frac{1}{k-2}}$. Then

$$g(t) \geq g(t^*) = 2k - k \cdot t^* + t^* \cdot \left(\frac{t}{k-2} \cdot \frac{(k-2)}{2k} \cdot \left(\frac{k-2}{k-1}\right) = 2k - \left(\frac{2k^2}{k-1}\right)^{\frac{1}{k-2}} \cdot \left(\frac{k}{k-1}\right) \right.$$  \hspace{1cm} (106)

Since $2^{k-2} > \left(\frac{2k^2}{k-1}\right)$ for $k \geq 5$, we get

$$2 > \left(\frac{2k^2}{k-1}\right)^{\frac{1}{k-2}} > \left(\frac{2k^2}{k-1}\right)^{\frac{1}{k-2}} \cdot \left(\frac{k}{k-1}\right) \quad \forall k \geq 5.$$  \hspace{1cm} (107)

Using (107) in (106), we obtain

$$g(t) \geq k \left(2 - \left(\frac{2k^2}{k-1}\right)^{\frac{1}{k-2}} \cdot \left(\frac{k}{k-1}\right)\right) > 0 \quad \forall t > 1, k \geq 5. \hspace{1cm} (108)$$

Combining (104) and (108), we obtain $1.5(2k - kt) + t^{k-1}/k - (k-1) \cdot t^{k-2}/(2k^3) \geq 0$ for all $t > 1$ and $k \geq 5$. This completes the proof. \hfill \Box

**Lemma 9.** Let $k \in \mathbb{N}$ with $k \geq 3$. Then

$$\left(4k - 1 - k \cdot (2k - 1)^{\frac{1}{k}}\right) \left(1 + \frac{1}{2k-1}\right)^{k-1} > \left(2k - 1\right)^{\frac{k}{k-1}} - \frac{1}{k \cdot (2k - 1)} \right)^k.$$  \hspace{1cm} (109)

**Proof.** Since $k \geq 3$, we have $2k - 1 \leq 2^k$, so $(2k - 1)^{\frac{1}{k}} \leq 2$. Then

$$4k - 1 - k \cdot (2k - 1)^{\frac{1}{k}} \geq 2k - 1. \hspace{1cm} (110)$$

Meanwhile,

$$\left(2k - 1\right)^{\frac{k}{k-1}} - \frac{1}{k \cdot (2k - 1)} \right)^k < 2k - 1. \hspace{1cm} (111)$$

Combining (110) and (111), we obtain

$$\left(4k - 1 - k \cdot (2k - 1)^{\frac{1}{k}}\right) \left(1 + \frac{1}{2k-1}\right)^{k-1} > \left(4k - 1 - k \cdot (2k - 1)^{\frac{1}{k}}\right)$$

(Since $1 + 1/(2k - 1) > 1$)

$$\geq 2k - 1 \hspace{1cm} (By \hspace{1cm} (110))$$

$$> \left(2k - 1\right)^{\frac{k}{k-1}} - \frac{1}{k \cdot (2k - 1)} \right)^k. \hspace{1cm} (By \hspace{1cm} (111))$$

Thus the required inequality (110) holds, which completes the proof. \hfill \Box
Lemma 10. Let $n \geq 2$ and $k \geq 3$, where $k, n \in \mathbb{N}$. Suppose 
\[ \frac{1}{n^k} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} > 1 + \frac{2k}{n} - \frac{k}{n^{k-1}}. \] (112) 
Then $k < n$.

Proof. We will show the constraint in (112) is incompatible with the range $k \geq n$. Let $t = n^{\frac{1}{k}}$. Then $n = t^k$. Since $k \geq n$, we have $t = n^{\frac{1}{k}} \leq k^{\frac{1}{k}}$. We have 
\[ n^{\frac{1}{k}} - \frac{1}{k} + \frac{2}{n} - \frac{2}{kn} > 1 + \frac{2k}{n} - \frac{k}{n^{k-1}}, \forall k \geq n \iff \] (113) 
\[ kt^{k+1} - t^k(k + 1) + k^2t - 2k^2 + 2k - 2 > 0, \forall t \in (0, k^{\frac{1}{k}}], \] (114) 
where (114) is obtained from (113) by multiplying both sides by $kn$, substituting $n = t^k$, and rearranging.

In order to upper bound the left hand side of (114), we define a function $f : [0, \infty) \to \mathbb{R}$ by 
\[ f(x) = x^k \left( k^{1 + \frac{1}{k}} - k - 1 \right) + k^2x - 2k^2 + 2k - 2. \]
For $0 \leq t \leq k^{\frac{1}{k}}$, we have $kt^{k+1} \leq kt^k k^{\frac{1}{k}}$, so the left hand side of (114) can be upper bounded as follows: 
\[ kt^{k+1} - t^k(k + 1) + k^2t - 2k^2 + 2k - 2 \leq t^k \left( k^{1 + \frac{1}{k}} - k - 1 \right) + k^2t - 2k^2 + 2k - 2 \]
= $f(t)$. (115) 
Observe $f'(x) = kx^{k-1} \left( k^{1 + \frac{1}{k}} - k - 1 \right) + k^2$ and $f''(x) = k(k - 1)x^{k-2} \left( k^{1 + \frac{1}{k}} - k - 1 \right)$. By Lemma 24 the function $f$ is convex for all $k \geq 3$. Thus $f$ has a global maximum on the interval $[0, k^{\frac{1}{k}}]$ which is attained at one of the endpoints. We check the value of the function is negative at both endpoints of $[0, k^{\frac{1}{k}}]$:
\[ \bullet \quad f(0) = -2k^2 + 2k - 2 = -k^2 - (k - 1)^2 - 1 < 0. \]
\[ \bullet \quad f(k^{\frac{1}{k}}) = 2k^2k^{\frac{1}{k}} - 3k^2 + k - 2 < 0 \text{ by Lemma 12}. \]
By convexity, it follows that $f(t) < 0$ for all $t \in [0, k^{\frac{1}{k}}]$. Combining this fact with (115), we get 
\[ kt^{k+1} - t^k(k + 1) + k^2t - 2k^2 + 2k - 2 \leq f(t) < 0 \quad \forall t \in [0, k^{\frac{1}{k}}], \] (116) 
which implies (114) cannot hold when $k \geq n$. Thus condition (112) in the lemma statement rules out the range $k \geq n$. This completes the proof. \qed

Lemma 11. Let $k, n \in \mathbb{N}$ such that $k \geq 3$ and $k < n \leq 2k - 2$. Then 
\[ \left(1 + \frac{1}{n}\right)^{\frac{k-1}{n}} \geq e^{\frac{2k-2-n}{2k-n^2}}. \] (117)
Proof. Taking log on both sides of (117), the required inequality is equivalent to
\[
\ln \left(1 + \frac{1}{n}\right) \geq \frac{2}{n^{1 + \frac{1}{k}}} - \frac{1}{(k - 1) \cdot n^\frac{1}{k}}. \tag{118}
\]
We first show several independent inequalities and then combine them to obtain the inequality required by the lemma. Recall that \(\ln(1 + x) \geq \frac{x}{1 + x}\) for all \(x > -1\) (see, e.g., [ber]). Taking \(x = 1/n\) yields
\[
\ln \left(1 + \frac{1}{n}\right) \geq \frac{1}{n + 1} = \frac{1}{n + 1}. \tag{119}
\]
Since \(n > k \geq 3\), we get \(n \geq 3\). By Lemma 24, we obtain \(n^{\left(1 + \frac{1}{n}\right)} \geq n + 1\). Since \(k < n\), we get
\[
n^{\left(1 + \frac{1}{k}\right)} \geq n^{\left(1 + \frac{1}{n}\right)} \geq n + 1. \tag{120}
\]
Next we will show that
\[
\frac{1}{n + 1} \geq \frac{2}{n^{1 + \frac{1}{k}}} - \frac{1}{(k - 1) \cdot n^\frac{1}{k}}, \tag{121}
\]
which is equivalent to
\[
(k - 1) \cdot n^{1 + \frac{1}{k}} \geq 2(k - 1)(n + 1) - n(n + 1). \tag{122}
\]
By (120) we have
\[
(k - 1)n^{\left(1 + \frac{1}{n}\right)} \geq (k - 1)(n + 1). \tag{123}
\]
Since \(n > k - 1\), we have \(k - 1 > 2(k - 1) - n\), which multiplied by \(n + 1\) on both sides gives
\[
(k - 1)(n + 1) \geq 2(k - 1)(n + 1) - n(n + 1). \tag{124}
\]
Combining (123) and (124) yields
\[
(k - 1)n^{\left(1 + \frac{1}{n}\right)} \geq (k - 1)(n + 1) \geq 2(k - 1)(n + 1) - n(n + 1). \tag{By (123) and (124)}
\]
Thus (122) holds, so (121) holds as well. Combining (119) and (121) yields
\[
\ln \left(1 + \frac{1}{n}\right) \geq \frac{1}{n + 1} \geq \frac{2}{n^{1 + \frac{1}{k}}} - \frac{1}{(k - 1) \cdot n^\frac{1}{k}}. \tag{By (119) and (121)}
\]
Thus (118) holds, which is equivalent to the required inequality (117). This completes the proof. \(\square\)

Lemma 12. Let \(k \in \mathbb{N}\) such that \(k \geq 3\). Then \(2k^2 \cdot k^\frac{1}{k} - 3k^2 + k - 2 < 0\).

Proof. Let \(f : (0, \infty) \to \mathbb{R}\) be \(f(x) = 2x^2 \cdot x^\frac{1}{x} - 3x^2 + x - 2\). We check separately for \(k \in \{3, 4, 5, 6\}\):
• $f(3) = 18 \cdot 3^{\frac{1}{2}} - 26 < -0.01 < 0$ and $f(4) = 32 \cdot 4^{\frac{1}{4}} - 46 < -0.7 < 0$.
• $f(5) = 50 \cdot 5^{\frac{1}{2}} - 72 < -3 < 0$ and $f(6) = 72 \cdot 6^{\frac{1}{3}} - 104 < -6 < 0$.

Thus it remains to show the required inequality when $k \geq 7$. The function $x^{\frac{1}{k}}$ has a global maximum at $e^{\frac{1}{k}}$ (see, e.g., Wolfram Alpha [wol]). Then

$$f(x) = 2x^2 \cdot x^{\frac{1}{k}} - 3x^2 + x - 2 \leq 2x^2 \cdot e^{\frac{1}{k}} - 3x^2 + x - 2$$
$$< -0.11x^2 + x - 2$$
$$< 0 \quad \forall x \geq 7.$$ (125)

Thus $f(k) < 0$ for all $k \geq 3, k \in \mathbb{N}$, as required.

**Lemma 13.** Let $k \geq 2, n \geq 1, c \in [1/n, 1]$, and the sequence $\{\gamma_\ell\}_{\ell=1}^\infty$ with $\gamma_1 = 0$ and $\gamma_\ell = 2\ell$ for $\ell \geq 2$. Then

$$x \left(1 + \frac{\gamma_{k-1}}{n}\right) + (k - 1) \cdot \left(\frac{nc - x}{n \cdot x^{\frac{1}{k-1}}} - \gamma_{k-1} \cdot c \geq c \cdot kn^{\frac{1}{k}} - \gamma_k \cdot c, \quad \forall x \in (1/2, nc].ight. (126)$$

**Proof.** When $x = nc$, inequality (126) is equivalent to

$$nc \left(1 + \frac{\gamma_{k-1}}{n}\right) - \gamma_{k-1} \cdot c \geq c \cdot kn^{\frac{1}{k}} - \gamma_k \cdot c \iff n - kn^{\frac{1}{k}} \geq -\gamma_k.$$ (127)

(Dividing both sides by $c$ and re-arranging terms.)

Since $n - kn^{\frac{1}{k}} \geq 1 - k$ for all $n \geq 1$, it follows that (127) holds if $\gamma_k \geq k - 1$, which is the case since $\gamma_k = 2k$. Thus (126) holds when $x = nc$.

From now on we can assume $x \in (1/2, nc)$. Let $t = (nc/x - 1)^{\frac{1}{k-1}}$. Then $0 < t < (2nc - 1)^{\frac{1}{k-1}}$.

Equivalently, $x = \frac{nc}{1 + t^{k-1}}$, which substituted in (126) gives

$$x \left(1 + \frac{\gamma_{k-1}}{n}\right) + (k - 1) \cdot \left(\frac{nc - x}{n \cdot x^{\frac{1}{k-1}}} - \gamma_{k-1} \cdot c \geq ckn^{\frac{1}{k}} - \gamma_k \cdot c \iff \right.$$ $$\left(\frac{nc}{1 + t^{k-1}}\right) \left(1 + \frac{\gamma_{k-1}}{n}\right) + (k - 1) \cdot \left(\frac{nc - \frac{nc}{1 + t^{k-1}}}{n \cdot \left(\frac{nc}{1 + t^{k-1}}\right)^{\frac{1}{k-1}}} - \gamma_{k-1} \cdot c \geq ckn^{\frac{1}{k}} - \gamma_k \cdot c. \right. (128)$$

Multiplying both sides of (128) by $(1 + t^{k-1}) / c$ and simplifying, we see (128) is equivalent to

$$(k - 1) \cdot t^k - t^{k-1} \cdot \left(kn^{\frac{1}{k}} - \gamma_k + \gamma_{k-1}\right) + \left(n - kn^{\frac{1}{k}} + \gamma_k\right) \geq 0.$$ (129)

We will show that (129) holds, which will imply inequality (126) for all $x \in (1/2, nc)$. We consider two cases, depending on whether $k = 2$ or $k \geq 3$.

**Case** $k = 2$. Since $\gamma_1 = 0$ and $\gamma_2 = 4$, inequality (129) is equivalent to

$$t^2 - t \left(2\sqrt{n} - 4 + 0\right) + \left(n - 2\sqrt{n} + 4\right) \geq 0 \iff (t - (\sqrt{n} - 2))^2 + 2\sqrt{n} \geq 0,$$ (130)

where (131) was obtained from (130) by re-arranging terms. Inequality (131) clearly holds, which implies (129) and completes the analysis for $k = 2$. 

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Case $k \geq 3$. We define a function $h : (0, \infty) \to \mathbb{R}$ to capture the left hand side of (129). Then we will show $h$ is non-negative on the entire domain, which will imply (129). Let

$$h(t) = (k-1) \cdot t^k - t^{k-1} \cdot \left( kn^{\frac{1}{k}} - \gamma_k + \gamma_{k-1} \right) + \left( n - kn^{\frac{1}{k}} + \gamma_{k} \right) .$$

(132)

The first and second derivatives of $h$ are

$$h'(t) = t^{k-1} \cdot k(k-1) - t^{k-2} \cdot (k-1) \left( kn^{\frac{1}{k}} - \gamma_k + \gamma_{k-1} \right)$$

$$h''(t) = t^{k-2} \cdot k(k-1)^2 - t^{k-3} \cdot (k-1)(k-2) \left( kn^{\frac{1}{k}} - \gamma_k + \gamma_{k-1} \right) .$$

(133)

On $(0, \infty)$ we have:

- the function $h'$ has a unique root at $t_1 = n^{\frac{1}{k}} + \frac{\gamma_{k-1} - \gamma_k}{k}$;
- the function $h''$ has a unique root at $t_2 = \left( \frac{k-2}{k-1} \right) \left( \frac{n^{\frac{1}{k}} + \frac{\gamma_{k-1} - \gamma_k}{k}}{\frac{k}{k-1}} \right) t_1$.

Clearly $t_2 < t_1$. Since $n \geq 1$ and $\gamma_k - \gamma_{k-1} = 2$ when $k \geq 3$, we have

$$n^{\frac{1}{k}} \geq 1 > \frac{2}{k} = \frac{\gamma_k - \gamma_{k-1}}{k} \geq 0 .$$

Thus $n^{\frac{1}{k}} + (\gamma_{k-1} - \gamma_k)/k > 0$, so $t_2 > 0$. We obtain $0 < t_2 < t_1$. Moreover, $h'(t) < 0$ for $t < t_1$ and $h'(t) > 0$ for $t > t_1$; similarly $h''(t) < 0$ for $t < t_2$ and $h''(t) > 0$ for $t > t_2$. Thus $h$ is

- concave and decreasing on $(0, t_2)$;
- convex and decreasing on $(t_2, t_1)$;
- convex and increasing on $(t_1, \infty)$.

Thus $h$ has a unique global minimum at $t_1$, so the required inequality (129) holds if $h(t_1) \geq 0$. We have

$$h(t_1) = (k-1) \cdot t_1^k - t_1^{k-1} \cdot \left( kn^{\frac{1}{k}} - \gamma_k + \gamma_{k-1} \right) + \left( n - kn^{\frac{1}{k}} + \gamma_{k} \right)$$

$$= n - kn^{\frac{1}{k}} + \gamma_k - \left( n^{\frac{1}{k}} + \frac{\gamma_{k-1} - \gamma_k}{k} \right)^k .$$

(134)

Since $\gamma_k = 2k$ and $\gamma_{k-1} = 2(k-1)$ for $k \geq 3$, we have

$$n^{\frac{1}{k}} + \frac{\gamma_{k-1} - \gamma_k}{k} = n^{\frac{1}{k}} - \frac{2}{k} \geq 1 - \frac{2}{k} > 0 .$$

(135)

Using (135) in (134) gives

$$h(t_1) = n - kn^{\frac{1}{k}} + 2k - \left( n^{\frac{1}{k}} - \frac{2}{k} \right)^k$$

$$> 0 .$$

(By Lemma 14.)

Thus $h(t_1) \geq 0$, and so inequality (129) also holds in the case $k \geq 3$. 

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Combining the cases. In both cases $k = 2$ and $k \geq 3$, inequality (126) holds, which implies (127) for all $x \in (1/2, 1]$. This completes the proof. \hfill \square

Lemma 14. Let $n \geq 1$ and $k \geq 3$, where $k, n \in \mathbb{N}$. Then $n - kn^\frac{1}{k} + 2(k - 1) - \left(n \frac{1}{k} - \frac{2}{k}\right)^k \geq 0$.

Proof. Define $f : [1 - \frac{2}{k}, \infty) \to \mathbb{R}$ as

$$f(x) = \left(x + \frac{2}{k}\right)^k - k \left(x + \frac{2}{k}\right) + 2(k - 1) - x^k = \left(x + \frac{2}{k}\right)^k - kx - x^k + 2k - 4.$$  \hspace{1cm} (136)

The lemma statement requires showing $f \left(n \frac{1}{k} - \frac{2}{k}\right) \geq 0$. We will show that $f(x) \geq 0$ for all $x \geq 1 - 2/k$, which will imply the required inequality. We divide the range of $x$ in two parts and analyze each separately.

Case $x \in [1 - \frac{2}{k}, 1]$. We consider a few sub-cases depending on the value of $k$:

- If $k = 3$, then $f(x) = \left(x + \frac{2}{3}\right)^3 - 3x - x^3 + 2 \cdot 3 - 4 = \frac{1}{27} (54x^2 - 45x + 62).$ Then $\Delta < 0$, so $f(x) > 0$ for all $x \in \mathbb{R}$.

- If $k \geq 4$, then using the inequalities $1 - 2/k \leq x \leq 1$ in the definition of $f$ from (136) gives

$$f(x) \geq 1^k - k \cdot 1 - 1^k + 2k - 4 = k - 4 \geq 0.$$ \hspace{1cm} (137)

Case $x > 1$. Then

$$f(x) \geq x^k + \binom{k}{1} \cdot x^{k - 1} \cdot \frac{2}{k} + \binom{k}{2} \cdot x^{k - 2} \cdot \left(\frac{2}{k}\right)^2 - kx - x^k + 2k - 4$$

$$= 2x^{k - 1} + \frac{2(k - 1)}{k} \cdot x^{k - 2} - kx + 2k - 4.$$ \hspace{1cm} (138)

When $k = 3$, using inequality (138), we obtain $f(x) \geq 2x^2 - 5x/3 + 2 \geq 0$ $\forall x \in [1, \infty)$. Thus from now on we can assume $k \geq 4$. Using $x > 1$ and $k \geq 4$, we obtain

$$f(x) \geq 2x^{k - 1} + \frac{2(k - 1)}{k} \cdot x^{k - 2} - kx + 2k - 4 \hspace{1cm} \text{(By (138))}$$

$$> 2x^{k - 2} - kx + k,$$ \hspace{1cm} (139)

Let $f_1 : (0, \infty) \to \mathbb{R}$ be $f_1(x) = 2x^{k - 2} - kx + k$. The derivatives are $f_1'(x) = 2(k - 2)x^{k - 3} - k$ and $f_1''(x) = 2(k - 2)(k - 3)x^{k - 4}$. Since we are in the case $k > 3$, we have $f_1''(x) > 0$ for $x > 0$. Thus the function $f_1$ is convex and has a unique global minimum at the point $x^*$ for which $f_1'(x^*) = 0$, that is, at

$$x^* = \left(\frac{k}{2(k - 2)}\right)^{\frac{1}{k - 2}}.$$ \hspace{1cm} (140)

Since $k \geq 4$, we have $\frac{k}{2(k - 2)} \leq 1$, and so

$$f_1(x^*) = 2 \left(\frac{k}{2(k - 2)}\right)^{\frac{k - 2}{k - 3}} - k \left(\frac{k}{2(k - 2)}\right)^{\frac{1}{k - 3}} + k \geq 2 \left(\frac{k}{2(k - 2)}\right)^{\frac{k - 2}{k - 3}} - k \cdot 1 + k > 0.$$ \hspace{1cm} (141)
Combining (139) and (141) gives \( f(x) \geq f_1(x) \geq f_1(x^*) > 0 \ \forall x > 0 \). In particular, the required inequality holds for all \( x > 1 \), which completes the case.

Combining the cases, we obtain \( f(x) \geq 0 \) for all \( x \geq 1 - 2/k \), and so \( f \left( n^k - 2/k \right) \geq 0 \). This completes the proof of the lemma.

**Corollary 6.** For each \( n \geq 1 \) and \( k \geq 3 \), we have: \( n + 2k > kn^k + 2 \).

**Proof.** Lemma 14 yields \( n + 2k \geq kn^k + 2 + \left( n^k - 2/k \right)^k \). Since \( k \geq 3 \), we also have \( n^k \geq 1 > 2/k \), so \( \left( n^k - 2/k \right)^k > 0 \). Thus \( n + 2k > kn^k + 2 \), as required.

### B Unordered Search Appendix

In this section we include the omitted proofs for unordered search.

Recall that rank queries have the form “How is rank \( (x_j) \) compared to \( i \)?”, where the answer can be “<”, “=”, or “>”. The select task with rank queries is: “Given \( q \), find index \( i \in [n] \) such that \( x_i \) has rank \( q \)”.

**Lemma 15.** (restated): Unordered search with comparison queries is equivalent to select with rank queries.

**Proof.** The only useful type of comparison in unordered search is comparing the searched element \( z \) to the item at index \( i \) of the list \( x = (x_1, \ldots, x_n) \), for some \( i \in [n] \). To see this, consider the graph of items with edges where comparisons are made. Any component disconnected from the given \( z \) gives no information on the location of \( z \); it is dominated by making no comparisons. Any component connected to the given \( z \) is dominated by instead comparing every node in the component to \( z \) directly.

If \( x_i = z \), this gives the answer; otherwise it only eliminates \( i \) without providing further information on the other indices. Similarly, in Select, every query either is for the correct item (which reveals the answer entirely), or eliminates that item without providing further information on the other items. Thus unordered search is equivalent to Select.

**B.1 Unordered search upper bounds**

**Lemma 16.** Let \( x \in \mathbb{R} \) and \( k \in \mathbb{N} \), where \( x, k > 0 \). Then

\[
\sum_{j=1}^{k-1} \left( \left\lfloor x \cdot \frac{j}{k} \right\rfloor - \left\lfloor x \cdot \frac{j+1}{k} \right\rfloor \right) \left( x \cdot \frac{j}{k} \right) \leq -\frac{x^2(k-1)}{2k} + \lceil x \rceil + 1.
\]

(142)

**Proof.** For every \( j \in [k] \), let \( b_j = \lfloor xj/k \rfloor - xj/k \). The left hand side of (142) can be rewritten as

\[
\sum_{j=1}^{k-1} \left( \left\lfloor x \cdot \frac{j}{k} \right\rfloor - \left\lfloor x \cdot \frac{j+1}{k} \right\rfloor \right) \left( x \cdot \frac{j}{k} \right) = \sum_{j=1}^{k-1} \left( x \cdot \frac{j}{k} + b_j - x \cdot \frac{j+1}{k} - b_{j+1} \right) \left( x \cdot \frac{j}{k} + b_j \right)
\]

(143)

\[
\sum_{j=1}^{k-1} \left( -\frac{x^2j}{k^2} + b_j (b_j - b_{j+1}) + \frac{x}{k} (j(b_j - b_{j+1}) - b_j) \right).
\]

(144)
Combining (144) with (145), we get
\[
\sum_{j=1}^{k-1} \frac{x}{k} \cdot \left( j (b_j - b_{j+1}) - b_j \right) = -b_k \cdot \frac{x(k-1)}{k} \leq 0 .
\] (145)

Combining (144) with (145), we get
\[
\sum_{j=1}^{k-1} \left( \left\lfloor \frac{x \cdot j}{k} \right\rfloor - \left\lfloor \frac{x \cdot (j+1)}{k} \right\rfloor \right) \leq \sum_{j=1}^{k-1} \left( -\frac{x^2 j}{k^2} + b_j (b_j - b_{j+1}) \right) = -\frac{x^2 (k-1)}{2k} + \sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) .
\] (146)

Next we bound the summation term in (146). If \( b_j \geq b_{j+1} \geq b_{j+2} \) for some \( j \in [k-2] \), then
\[
b_j (b_j - b_{j+1}) + b_{j+1} (b_{j+1} - b_{j+2}) \leq b_j (b_j - b_{j+2}) .
\] (147)

Thus if there is a (weakly) decreasing sequence \( b_j \geq b_{j+1} \geq \ldots \geq b_{j+t} \) for some \( t \geq 2 \) and \( j \in [k-t] \), then applying inequality (147) iteratively gives
\[
\sum_{i=j}^{j+t-1} b_i (b_i - b_{i+1}) \leq b_j (b_j - b_{j+t}) .
\] (148)

We will use inequality (147) to collapse some of the terms in the sum \( \sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) \).

Towards this end, let \( G = ([k], E) \) be a line graph where the vertices are \( \{1, \ldots, k\} \) and the edges \( E = \{(j, j+1) \mid j \in [k-1]\} \). For each \( j \in [k-1] \), if \( b_j \geq b_{j+1} \) then edge \((j, j+1)\) is colored with black and depicted as oriented down, and otherwise it is colored with yellow and oriented up.

We also give each vertex \( j \in [k] \) a color \( c_j \in \{R, B\} \), such that \( c_1 = c_k = R \). Furthermore, for each \( j \in [k-1] \), if \( b_j < b_{j+1} \) then both endpoints of the edge are colored red: \( c_j = c_{j+1} = R \). All other vertices are colored blue (B). See Figure 5 for an illustration.

Let \( \ell_1 = 1 < \ldots < \ell_m = k \) be the red vertices in \( G \) and \( L = \{\ell_1, \ldots, \ell_m\} \). For all \( i \in [m-1] \):

- if the path from \( \ell_i \) to \( \ell_{i+1} \) has black edges, then \( b_{\ell_i} \geq \ldots \geq b_{\ell_{i+1}} \) and so inequality (147) gives
  \[
  \sum_{j=\ell_i}^{\ell_{i+1}-1} b_j (b_j - b_{j+1}) \leq b_{\ell_i} \left( b_{\ell_i} - b_{\ell_{i+1}} \right) .
  \] (149)

- else, the path from \( \ell_i \) to \( \ell_{i+1} \) has no black edges. Then \( \ell_{i+1} = \ell_i + 1 \), and so the next inequality trivially holds:
  \[
  b_{\ell_i} \left( b_{\ell_i} - b_{\ell_{i+1}} \right) \leq b_{\ell_i} \left( b_{\ell_i} - b_{\ell_{i+1}} \right) .
  \] (150)

Combining (149) and (150), we can bound the sum of all \( b_j \)'s as follows:
\[
\sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) \leq \sum_{i=1}^{m-1} b_{\ell_i} \left( b_{\ell_i} - b_{\ell_{i+1}} \right) .
\] (151)

Since \( b_j \in [0,1) \) for all \( j \), we have \( b_{\ell_i} \left( b_{\ell_i} - b_{\ell_{i+1}} \right) \leq (1 - b_{\ell_{i+1}}) \) and \( b_{\ell_{i+1}} \left( b_{\ell_{i+1}} - b_{\ell_{i+2}} \right) \leq b_{\ell_{i+1}} \). Thus adjacent terms in (151) sum to at most 1. Then
Figure 5: Given $k \geq 2$ and numbers $b_1, \ldots, b_k \in [0, 1)$, we construct a graph with edges $(j, j+1)$ for each $j \in [k-1]$. For each $j \in [k]$, if $b_j \geq b_{j+1}$, the edge from $j$ to $j+1$ is oriented downwards and is colored with black. If $b_j < b_{j+1}$, the edge from $j$ to $j+1$ is oriented upwards and is colored with yellow. The endpoints of all the yellow edges are added to the set $L$, together with special vertices 1 and $k$. All the vertices in $L$ are colored red and the vertices in $[k] \setminus L$ are colored blue. For the graph in the picture we have $k = 14$ and $L = \{1, 4, 5, 6, 7, 8, 13, 14\}$. Each element $\ell_j$ of $L$ is marked in red near the corresponding node.

• If $m - 1$ is even, then
  \[ \sum_{i=1}^{m-1} b_{\ell_i} (b_{\ell_i} - b_{\ell_{i+1}}) \leq \frac{m - 1}{2} < \frac{m}{2}. \]  

• If $m - 1$ is odd, then
  \[ \sum_{i=1}^{m-1} b_{\ell_i} (b_{\ell_i} - b_{\ell_{i+1}}) \leq \left\lfloor \frac{m - 1}{2} \right\rfloor + b_{\ell_i} (b_{\ell_i} - b_{\ell_{i+1}}) \leq \left\lfloor \frac{m - 1}{2} \right\rfloor + 1 = \frac{m}{2}. \]  

Combining (152) and (153), we obtain
  \[ \sum_{i=1}^{m-1} b_{\ell_i} (b_{\ell_i} - b_{\ell_{i+1}}) \leq \frac{m}{2}. \]  

Combining (151) with (154) while summing over all $j \in [k-1]$ gives
  \[ \sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) \leq \frac{m}{2}. \]  

Let $D = \{j \in [k-1] \mid b_j < b_{j+1}\}$ and $\Delta = |D|$. Since $b_j = \lfloor x_j/k \rfloor - x_j/k$, we have $b_j \leq b_{j+1}$ if and only if $\lfloor x_j/k \rfloor - x_j/k \leq \lfloor x(j+1)/k \rfloor - x(j+1)/k$ (†). Since $x/k > 0$, inequality (†) implies
  \[ \lfloor x_j/k \rfloor + 1 \leq \lfloor x(j+1)/k \rfloor \quad \forall j \in D. \]  

Consider the elements of $D$ in sorted order: $d_1 < \ldots < d_\Delta$. We will show by induction that
  \[ \lfloor x \cdot d_i/k \rfloor \geq i \quad \text{for all } i \in [\Delta]. \]  

The base case is \( i = 1 \). Indeed \( \lceil x \cdot d_1/k \rceil \geq 1 \) since \( x \cdot d_1/k > 0 \). We assume inequality (157) holds for \( i \) and show this implies the inequality for \( i + 1 \). We have
\[
\left\lceil \frac{x \cdot d_{i+1}}{k} \right\rceil \geq \left\lceil \frac{x \cdot d_i}{k} \right\rceil + 1 \quad \text{(Since } d_{i+1} > d_i \text{ and } d_i, d_{i+1} \in \mathbb{N})\]
\[
\geq i + 1 \quad \text{(By (156)).}
\]
This completes the induction, so (157) holds. Now we can bound the size of \( D \). Since \( d_{\Delta} \in [k-1] \), we have \( \lceil x \cdot k/k \rceil \geq \lceil x \cdot d_{\Delta}/k \rceil \). By (157), we have \( \lceil x \cdot d_{\Delta}/k \rceil \geq \Delta \). Thus
\[
\left\lfloor x \right\rfloor = \left\lceil x \cdot \frac{k}{k} \right\rceil \geq \left\lceil x \cdot \frac{d_{\Delta}}{k} \right\rceil \geq \Delta. \tag{158}
\]

Observe that \( \Delta \) is equal to the number of yellow edges in the graph \( G \), since \( j \in D \) if and only if the edge \((j, j+1)\) is yellow. Thus the number of endpoints of yellow edges in \( G \) is at most \( 2\Delta \).

Since \( |L| = m \) and \( L \) consists precisely of all the endpoints of yellow edges together with vertices 1 and \( k \), we have \( m = |L| \leq |\{1, k\}| + 2\Delta = 2 + 2\Delta \). Since \( \Delta \leq \lceil x \rceil \) by (158), we obtain
\[
m \leq 2 + 2\Delta \leq 2 + 2\lceil x \rceil. \tag{159}
\]
Combining (159) with (155), we get
\[
\sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) \leq \left\lceil x \right\rceil + 1. \tag{160}
\]
Combining (160) with (146) gives
\[
\sum_{j=1}^{k-1} \left( \left\lfloor x \cdot \frac{j}{k} \right\rfloor - \left\lfloor x \cdot \frac{j+1}{k} \right\rfloor \right) \left\lfloor x \cdot \frac{j}{k} \right\rfloor \leq \frac{x^2(k-1)}{2k} + \sum_{j=1}^{k-1} b_j (b_j - b_{j+1}) \leq \frac{x^2(k-1)}{2k} + \left\lceil x \right\rceil + 1.
\]
This completes the proof.

\[\Box\]

**B.2 Unordered search lower bounds**

In this section we include the omitted proofs for unordered search.

**Lemma 17.** Let \( k, n \in \mathbb{N} \) and \( p \in [0, 1] \). Suppose \( k \geq 2 \). Then
\[
x + \frac{(np-x)(n-x)}{n} \left( 1 - \frac{k-2}{2k-2} \cdot \frac{np-x}{n-x} \right) \geq np \left( 1 - \frac{k-1}{2k} \cdot p \right). \tag{161}
\]

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be
\[
f(x) = x + \frac{(np-x)(n-x)}{n} \left( 1 - \frac{k-2}{2k-2} \cdot \frac{np-x}{n-x} \right) - np \left( 1 - \frac{k-1}{2k} \cdot p \right). \tag{162}
\]
Then the required inequality (161) is equivalent to showing \( f(x) \geq 0 \) for all \( x \in [0, n] \).
Expanding the terms in the expression for \( f(x) \), we get

\[
f(x) \geq 0 \iff x + np - x - \frac{x(np - x)}{n} - \left( \frac{k - 2}{2k - 2} \right) \frac{(np - x)^2}{n} - \left( \frac{k - 2}{2k - 2} \right) \frac{(np - x)^2}{n} - np + np^2 \left( \frac{k - 1}{2k} \right) \geq 0,
\]

which after simplification is equivalent to

\[
x^2k^2 - x \cdot 2knp + n^2p^2 \geq 0. \tag{163}
\]

The quadratic equation in (164) has a unique global minimum at \( x^* = np/k \), with \( f(x^*) = 0 \). Thus (164) holds, so (163) holds and so \( f(x) \geq f(x^*) = 0 \) \( \forall x \in [0, n] \) as required.

**Lemma 18.** Let \( k, n, m \in \mathbb{N} \), where \( k \geq 2 \), \( n \geq 1 \), and \( m \in \{0, \ldots, n\} \). Let \( \gamma \in [m/n, 1] \). Then

\[
m + \frac{(n - m)(n \cdot \gamma - m)}{n} \cdot \frac{k}{2k - 2} - n \cdot \gamma \left( \frac{k + 1}{2k} \right) \geq 0. \tag{165}
\]

**Proof.** Inequality (165) is equivalent to

\[
m + n\gamma \cdot \frac{k}{2k - 2} - m \cdot \frac{k}{2k - 2} - m\gamma \cdot \frac{k}{2k - 2} + \frac{m^2}{n} \cdot \frac{k}{2k - 2} - n \cdot \gamma \cdot \frac{k + 1}{2k} \geq 0. \tag{166}
\]

Multiplying both sides of (166) by \( 2nk(k - 1) \) and rearranging, we get that (165) is equivalent to

\[
\gamma n(n - mk^2) + mn(k^2 - 2k) + m^2k^2 \geq 0. \tag{167}
\]

If \( n \geq mk^2 \) then (167) clearly holds. Else, assume \( n < mk^2 \). Since \( m/n \leq \gamma \leq 1 \), we have

\[
\gamma n(n - mk^2) \geq n(n - mk^2). \tag{168}
\]

Using (168), we can bound the left hand side of (167) as follows:

\[
\gamma n(n - mk^2) + mn(k^2 - 2k) + m^2k^2 \geq n(n - mk^2) + mn(k^2 - 2k) + m^2k^2 = (n - mk)^2 \geq 0.
\]

Thus (167) holds when \( n < mk^2 \) as well, which implies (165) holds in all cases, as required.

## C Proportional cake cutting and sorting in rounds

In this section we give the algorithm for proportional cake cutting and discuss the reduction from sorting with rank queries to cake cutting.

### C.1 Cake cutting algorithm

For the algorithm, we will maintain two values, \( a_i \) and \( b_i \) for each agent \( i \), such that \( \text{Cut}_i(a_i) \) and \( \text{Cut}_i(b_i) \) lie inside her subcake and \( b_i - a_i = m_i/n \), where \( m_i \) is the number of agents in her subcake.
Algorithm 1. Initialize $a_i = 0$ and $b_i = 1$ for all agents $i \in [n]$. In the first round, we divide the cake into $z = \lceil n/k \rceil$ subcakes to be specified later, and then assign $m_1, m_2, \ldots, m_z$ agents to each subcake respectively, with $m_j = \lceil \frac{n}{z} \rceil$ for $j \leq n - z\lfloor \frac{n}{z} \rfloor$ and $m_j = \lceil \frac{n}{z} \rceil$ for $j > n - z\lfloor \frac{n}{z} \rfloor$. Note that $\sum_{j=1}^z m_j = n$.

Formally, for each agent $i$, ask the following queries

- $Cut_i \left( \frac{1}{n} \cdot m_1 \right), Cut_i \left( \frac{1}{n} \cdot m_1 + m_2 \right), \ldots, Cut_i \left( \frac{1}{n} \cdot \sum_{j=1}^{z-1} m_j \right)$.

Let $S_j$ be the set of agents we assign to the $j$th subcake. Let $c_j$ for $j = 0, 1, 2, \ldots, z$ be the points demarking subcakes, with $c_0 = 0$ and $c_z = 1$; i.e. the $j$th subcake is $[c_{j-1}, c_j)$. For $k = 1, 2, \ldots, z-1$, we choose $c_k$ to be the $m_k$-th smallest among $Cut_i \left( \frac{1}{n} \cdot \sum_{j=1}^k m_j \right)$ for $i \in [n] \setminus (\bigcup_{j=1}^{k-1} S_j)$.

Again for $k = 1, 2, \ldots, z-1$, we choose $S_k$ to be the $m_k$ agents $i \in [n] \setminus (\bigcup_{j=1}^{k-1} S_j)$ such that $Cut_i \left( \frac{1}{n} \cdot \sum_{j=1}^k m_j \right) \leq c_k$. Then $S_2$ is the remaining agents, $[n] \setminus (\bigcup_{j=1}^{z-1} S_j)$.

For each agent $i \in S_k$, set $a_i = \frac{1}{n} \cdot \sum_{j=1}^{k-1} m_j$ and $b_i = \frac{1}{n} \cdot \sum_{j=1}^{k} m_j$. Note that

$$c_{k-1} \leq Cut_i(a_i) < Cut_i(b_i) \leq c_k \quad \text{and} \quad b_i - a_i = m_k/n.$$

In successive rounds we recurse on each subcake, with one caveat: we have no way to ask an agent $i \in S_k$ to cut fractions of $[c_{k-1}, c_k)$ since we don’t necessarily know $Eval_i(c_{k-1})$ and $Eval_i(c_k)$.

To get around this, we instead ask the agents to cut fractions of $[Cut_i(a_i), Cut_i(b_i))$, since we do know the values $i$ has on those endpoints ($a_i$ and $b_i$ respectively).

Allocation: After the final round, each subcake contains only a single agent. We then allocate each subcake to its corresponding agent.

Example 1. Let $n = 1000$ and $k = 3$. Algorithm 1 works as follows in each round:

1. Round 1: everyone is asked to mark their $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}$ points. These are used to separate the agents into 10 subcakes, each containing 100 agents.

2. Round 2: Everyone is asked to mark their $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}$ points within their respective value interval $[a_i, b_i]$. For example, for the second subcake each agent marks their $\frac{11}{100}, \frac{12}{100}, \ldots, \frac{19}{100}$ points. Again these are used to separate each subcake further into 10 subcakes, each containing 10 agents.

3. Round 3: Everyone is asked to mark their $1/10, \ldots, 9/10$ points within their respective value interval. This time when assigning agents to subcakes, the algorithm assigns only 1 to each, so we’re done.

Proposition 1 (restated): There is an algorithm that runs in $k$ rounds and computes a proportional allocation with a total of $O(kn^{1+1/k})$ RW queries.

Proof. We claim that after $j$ rounds each subcake contains at most $n^{1-j/k}$ agents. In the base case, after 0 rounds, the sole subcake contains all $n$ agents. In the inductive case, we assume that after $j$ rounds each subcake contains at most $n^{1-j/k}$ agents. Consider an arbitrary subcake containing $m$ agents. The subcakes carved from it will each contain at most $m^{1-1/(k-j)}$ agents. This is at most $(n^{1-j/k})^{1-1/(k-j)} = n^{1-\frac{j}{k}}$ agents. This concludes the induction. Then after $k$ rounds each subcake contains at most $n^{1-k/k} = 1$ agents. Thus the algorithm generates an allocation in $k$
rounds. Furthermore, for every agent $i$, because after the final round both $\text{Cut}(a_i)$ and $\text{Cut}(b_i)$ lie inside agent $i$’s subcake and that subcake contains only one agent, agent $i$ receives at least $1/n$ value.

To argue the bound on the number of queries, we proceed by induction on $k$. For $k = 1$, the bound is $n^2$, which is satisfied since we issue $n - 1$ queries for each of $n$ agents. In the inductive case, in the first round we issue $\lceil n^{1/k} \rceil - 1$ queries for every agent, for a total of $n^{1+1/k}$. By the inductive assumption, the remaining number of queries is

$$\sum_{j=1}^{\lceil n^{1/k} \rceil} (k-1)m_j^{1+1/(k-1)} \leq (k-1)^{\lceil n^{1/k} \rceil} \leq (k-1)n^{1+1/k}$$

Combining, we get at most $kn^{1+1/k}$ queries in total. 

\[\square\]

### C.2 Sorting to cake cutting reduction

Here we prove the reduction of sorting to proportional cake cutting where the sorting is not with comparisons, but rather with queries that, given an item $p$ and index $i$ return whether the rank of $p$ is less than, equal to, or greater than $i$. The bulk of the work has already been done by Woeginger and Sgall [WS07] through the introduction of a set of cake valuations and an adversary protocol. We present again their valuations and adversary protocol without proving the relevant lemmas; we would refer the reader to their paper for the proofs. Then we perform the last few steps to prove the reduction.

**Definition 3.** [WS07] Let the $\alpha$-point of an agent $p$ be the infimum of all numbers $x$ such that $\mu_p([0,x]) = \alpha$. In other words, $\text{Cut}_p(\alpha) = x$.

We fix $0 < \epsilon < 1/n^4$. The choice is not important.

**Definition 4.** [WS07] For $i = 1, \ldots, n$ let $X_i \subset [0,1]$ be the set consisting of the $n$ points $i/(n+1) + k\epsilon$ with integer $1 \leq k \leq n$. Further let $X = \bigcup_{1 \leq i \leq n} X_i$.

By definition every agent’s 0-point is at 0. The positions of the $i/n$-points with $1 \leq i \leq n$ are fixed by the adversary during the execution of the protocol. In particular, the $i/n$-points of all agents are distinct elements of $X_i$. Note that this implies that all $i/n$-points are left of all $(i+1)/n$ points.

**Definition 5.** [WS07] Let $I_{p,i}$ be a tiny interval of length $\epsilon$ centered around the $i/n$-point of agent $p$.

We place all the value of each agent $p$ in her $I_{p,i}$ for $i = 1, \ldots, n$. More precisely, for $i = 0, \ldots, n$ she has a sharp peak of value $i/(n^2 + n)$ immediately to the left of her $i/n$ point and a sharp peak of value $(n - i)/(n^2 + n)$ immediately to the right of her $i/n$ point. Note that the measure between the $i/n$ and $(i+1)/n$ points is indeed $1/n$. Further note that the value $\mu_p(I_{p,i}) = 1/(n+1)$. Also note that the $I_{p,i}$ are all disjoint except for the $I_{p,0}$, which are identical. Finally note that every $\alpha$-point of an agent $p$ lies one one of that agent’s $I_{p,i}$s.

**Definition 6.** [WS07] If $x \in I_{p,i}$, then let $c_p(x)$ be the corresponding $i/n$-point of agent $p$. 

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Lemma 19. [WS07] For every protocol \( P \) there exists a primitive protocol \( P' \) such that for every cake cutting instance of the restricted form described above,

1. \( P \) and \( P' \) make the same number of cuts.
2. if \( P \) assigns to agent \( p \) a piece \( J \) of measure \( \mu_p(J) \geq 1/n \), then also \( P' \) assigns to agent \( p \) a piece \( J' \) of measure \( \mu_p(J') \geq 1/n \).

It is also true that given \( P \), protocol \( P' \) can be quickly constructed. This follows directly from Woeginger and Sgall’s constructive proof of the above lemma. This implies that we, the adversary, may assume w.l.o.g. that the protocol is primitive. We can now define the adversary’s strategy. Fix a permutation \( \pi \) on \([n]\). Suppose at some point the protocol asks \( Cut_p(i/n) \). With multiple queries in the same round, answer the queries in an arbitrary order.

1. If \( \pi(p) < i \), then the adversary assigns the \( i/n \) point of agent \( p \) to the smallest point in the set \( X_i \) that has not been used before.
2. If \( \pi(p) > i \), then the adversary assigns the \( i/n \) point of agent \( p \) to the largest point in the set \( X_i \) that has not been used before.
3. If \( \pi(p) = i \), then the adversary assigns the \( i/n \) point of agent \( p \) to the \( i \)th smallest point in the set \( X_i \).

This strategy immediately precipitates the following lemma.

Lemma 20. [WS07] If \( \pi(p) \leq i \leq \pi(q) \) and \( p \neq q \), then the \( i/n \) point of agent \( p \) strictly precedes the \( i/n \) point of agent \( q \).

At the end, the protocol must assign intervals to agents. Let \( y_0, y_1, \ldots, y_n \) be the boundaries of these slices; i.e. \( y_0 = 0, y_n = 1 \), and all other \( y_j \) are cuts performed. Then there is a permutation \( \phi \) of \([n]\) such that for \( i = 1, \ldots, n \) the interval \([y_{i-1}, y_i]\) goes to agent \( \phi(i) \).

Lemma 21. [WS07] If the primitive protocol \( P' \) is fair, then \( y_i \in X_i \) for \( 1 \leq i \leq n-1 \) and the interval \([y_{i-1}, y_i]\) contains the \((i-1)/n\)-point and the \(i/n\)-point of agent \( \phi(i) \).

Lemma 22. [WS07] For any permutation \( \sigma \neq id \) of \([n]\), there exists some \( i \) with

\[ \sigma(i + 1) \leq i \leq \sigma(i). \]

We can now claim that \( \phi = \pi^{-1} \). To prove this, suppose for sake of contradiction \( \phi \neq \pi^{-1} \); then \( \pi \circ \phi \neq id \) and by Lemma 4, there exists an \( i \) such that

\[ \pi(\phi(i + 1)) \leq i \leq \pi(\phi(i)) \tag{169} \]

Then let \( p = \phi(i + 1) \) and \( q = \phi(i) \). Further let \( z_p \) be the \( i/n \) point of agent \( p \) and \( z_q \) be the \( i/n \) point of agent \( q \). By Lemma 2, we have \( z_p < z_q \). By Lemma 3, we have \( z_p \in [y_{i+1}, y_{i+1}] \) and \( z_q \in [y_{i-1}, y_i] \). But this implies \( z_p \geq y_i \geq z_q \), in contradiction with \( z_p < z_q \). Therefore \( \phi = \pi^{-1} \).

With this preliminary work out of the way, we are finally ready to state and prove the reduction.
Figure 6: A potential value distribution for four agents. Each agent receives a spike in value in each of $X_0, X_1, X_2, X_3, X_4$ ($X_0$ is not shown). Each spike has total value $1/5$, so to get the required $1/4$ value an agent’s slice must include parts of multiple $X_i$. Note that Agent 1 receives the first slot in $X_1$, Agent 2 receives the second slot in $X_2$, etc. Further note that slot 1 is allocated to Agent 1 in $X_2$ and slot 4 is allocated to Agent 4 in $X_3$. This implies that the slices must be allocated to agents 1, 2, 3, 4 in order.

Proposition 4. There exists a polynomial time reduction from sorting an $n$ element with rank queries to proportional cake cutting with $n$ agents. The reduction holds for any number of rounds.

Proof. After an evaluation query $\text{Eval}(p, x)$, where $x = \text{Cut}(p', i/n)$ and $p \neq p'$, there are only two possible answers: $i/(n+1)$ and $(i+1)/(n+1)$. This reveals whether the $i/n$ point of $p$ is left or right of that of $p'$. This only reveals new information if $\pi(p') = i$. In this case, the information is whether $\pi(p) < i$ or $\pi(p) > i$. After a cut query $\text{Cut}_p(i/n)$, there are only three answers. These correspond exactly to $\pi(p) < i$, $\pi(p) = i$, and $\pi(p) > i$. Thus w.l.o.g., all queries are cut queries. Then given a sorting problem with rank queries, we can construct a proportional cake cutting instance such that any solution assigns slices according to the inverse permutation of the unsorted elements of the original sorting problem. The sorting problem can then be solved without any additional queries. Furthermore, each query in the cake cutting instance can be answered using at most one query in the sorting instance. This completes the reduction. Because of the one-to-one correspondence between queries, it immediately follows that the reduction holds for any number of rounds.

D Sorting lower bound

Our approach for the lower bound builds on the work in [AAV86a]. To show that this sorting problem is hard, we find a division between two regions of the array such that one must be solved in future rounds while the other still needs to be solved in the current round.

Proposition 10 (restated): Let $c(k, n)$ be the minimum total number of queries required to sort $n$ elements in $k$ rounds in the rank query model. Then $c(k, n) \geq \frac{k}{2e}n^{1+1/k} - kn$.

Proof. We proceed by induction on $k$. For $k = 1$, note that if any two items $p_j, p_k$ have no query for indices $i, i + 1$ then the adversary can assign those positions to those items and the solver will be unable to determine their true order. Thus for $i = 2, 4, \ldots n$ at least $n - 1$ queries are necessary, for a total of $\lceil n/2 \rceil (n - 1)$. Then

$$\lceil n/2 \rceil (n - 1) \geq (n/2 - 1/2)(n - 1) = n^2/2 - n + 1/2 > n^2/(2e) - n.$$
For $k > 1$, assume the claim holds for all pairs $(k', n')$ where either $(k' < k)$ or $(k' = k$ and $n' < n$).

If $n^{1/k} \leq 2e$, then

$$\frac{n^{1/k}}{2e} - 1 \leq 0 \iff \frac{k}{2e} n^{1+1/k} - kn \leq 0$$

so the bound is non-positive, and is thus trivially satisfied. Thus we may assume $n^{1/k} > 2e$.

If there are no queries in the first round, then we have

$$c(k, n) \geq c(k-1, n) \geq \frac{(k-1)}{2e} n^{1+\frac{1}{k-1}} - (k-1)n = \frac{k}{2e} n^{1+1/k} \left[ \left( 1 - \frac{1}{k} \right) n^{\frac{1}{k-1}} + \frac{2e}{kn^{1/k}} \right] - kn$$

From here it suffices to show $\left(1 - \frac{1}{k}\right) n^{1/(k^2-k)} + \frac{2e}{kn^{1/k}} \geq 1$.

Recall the AM–GM inequality $\alpha a + \beta b \geq a^\alpha b^\beta$ with $a, b, \alpha, \beta > 0$ and $\alpha + \beta = 1$. Taking $\alpha = 1 - 1/k$, $\beta = 1/k$, $a = n^{1/k^2-1/k}$, and $b = 2e/n^{1/k}$, we get

$$\left(1 - \frac{1}{k}\right) n^{1/(k^2-k)} + \frac{2e}{kn^{1/k}} \geq (2e)^{1/k} \geq 1$$

so we may assume there is at least one query in the first round.

Take any $k$-round algorithm for sorting a set $V$ of $n$ elements using rank queries. Let $x$ be the maximum integer such that there exist $x$ items with no queries in $[1, x]$ but there do not exist $x + 1$ items with no queries in $[1, x+1]$. Note that since there is at least one query, it follows that $x < n$. Let $S$ be one such set of $x$ items. Then at least $n - x$ items have a query in $[1, x+1]$. At this point the adversary announces that every element of $S$ precedes every element of $V - S$. The adversary also announces the item at position $x + 1$. We call this item $p_{\text{mid}}$. None of the $n - x$ queries help to sort the items in $S$ since they are either at $x + 1$ or for an item not in $S$, so we also need $c(k-1, x)$ queries to sort $S$. Additionally, none of the $n - x$ queries help to sort the items in $V - S - \{p_{\text{mid}}\}$, so we also need an additional $c(k, n-x-1)$ queries to sort $V - S - \{p_{\text{mid}}\}$. This implies the following inequality.

$$c(k, n) \geq c(k, n-x-1) + (n-x) + c(k-1, x)$$

We consider two cases.

**Case** $x \geq k/\ln 2$. By the inductive assumption,

$$c(k, n) \geq \frac{k}{2e} (n-x-1)^{1+1/k} - k(n-x-1) + (n-x) + \frac{k-1}{2e} x^{1+1/(k-1)} - (k-1)x$$

$$= \frac{k}{2e} n^{1+1/k} \left[ \left( 1 - \frac{x+1}{n} \right)^{1+1/k} + \left( 1 - \frac{1}{k} \right) \frac{x^{1+1/(k-1)}}{n^{1+1/k}} + \frac{2e}{kn^{1/k}} \right] - kn + k$$

In the AM-GM inequality $\alpha a + \beta b \geq a^\alpha b^\beta$, taking

$$\alpha = 1 - 1/k, \beta = 1/k, a = \frac{x^{1+1/(k-1)}}{n^{1+1/k}}, \text{ and } b = \frac{2e}{kn^{1/k}},$$

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we get
\[ c(k, n) \geq \frac{k}{2e} n^{1+1/k} \left[ \left( 1 - \frac{x + 1}{n} \right)^{1+1/k} + \frac{x}{n^{1-1/k}} \cdot \left( \frac{2e^{1/k}}{n^{1/k}} \right) \right] - kn + k \quad (174) \]

Now, since \((1 + \frac{1}{k})^k \nearrow e\), we have \(e^{1/k} > 1 + 1/k\). This yields
\[ c(k, n) \geq \frac{k}{2e} n^{1+1/k} \left[ \left( 1 - \frac{x + 1}{n} \right)^{1+1/k} + 2^{1/k} \frac{x}{n} \left( 1 + \frac{1}{k} \right) \right] - kn + k \quad (175) \]

Then recall Bernoulli’s Inequality: \((1 - a)^t \geq 1 - at\) if \(t \geq 1\) and \(a \leq 1\). This yields
\[ c(k, n) \geq \left[ \frac{k}{2e} n^{1+1/k} - kn \right] + \frac{k + 1}{2e} n^{1/k} \left( \frac{2^{1/k} - 1}{x} - 1 \right) + k \quad (176) \]

Then since by L'Hôpital’s rule \(k(2^{1/k} - 1) \to \ln 2\) from above, we have
\[ c(k, n) \geq \left[ \frac{k}{2e} n^{1+1/k} - kn \right] + \frac{k + 1}{2e} n^{1/k} \left( \frac{x \ln 2 \ln(2)}{k} - 1 \right) + k \quad (177) \]

Then by invoking our case assumption that \(x \geq k/\ln 2\), we get
\[ c(k, n) \geq \left[ \frac{k}{2e} n^{1+1/k} - kn \right] + \frac{k}{2e} n^{1+1/k} - kn, \]

as required.

**Case** \(x < k/\ln 2\). From inequality \((171)\), we get
\[ c(k, n) \geq c(k, n - x - 1) + (n - x) + c(k - 1, x) \geq c(k, n - x - 1) + n - \frac{k}{\ln 2}. \]

By the inductive hypothesis,
\[ c(k, n) \geq \frac{k}{2e} (n - x - 1)^{1+1/k} - nk + n - \frac{k}{\ln 2} = \frac{k}{2e} n^{1+1/k} \left( 1 - \frac{x + 1}{n} \right)^{1+1/k} - nk + n - \frac{k}{\ln 2} \]

Using the Bernoulli inequality \((1 - a)^t \geq 1 - at\) with \(t \geq 1\) and \(a \leq 1\), we get
\[ c(k, n) \geq \frac{k}{2e} n^{1+1/k} \left[ 1 - \frac{x + 1}{n} \left( 1 + \frac{1}{k} \right) \right] - nk + n - \frac{k}{\ln 2} \]
\[ = \left[ \frac{k}{2e} n^{1+1/k} - nk \right] - \frac{k}{2e} n^{1/k}(x + 1) \left( 1 + \frac{1}{k} \right) + n - \frac{k}{\ln 2} \]
\[ \geq \left[ \frac{k}{2e} n^{1+1/k} - nk \right] - \frac{k + 1}{2e} n^{1/k} \left( \frac{k}{\ln 2} + 1 \right) + n - \frac{k}{\ln 2} \quad (178) \]
At this point we want to show \( n > \frac{k+1}{2e} n^{1/k} (1 + k/\ln 2) + k/\ln 2 \). It suffices to show both of the following inequalities

\[
(i) \quad \frac{3n}{4} > \frac{k+1}{2e} n^{1/k} \left( \frac{k}{\ln 2} + 1 \right) \quad \text{and} \quad (ii) \quad \frac{n}{4} > \frac{k}{\ln 2}
\]

(179)

Inequality \((i)\) holds if and only if

\[ n^{1-1/k} > \frac{2(k+1)}{3e} \left( \frac{k}{\ln 2} + 1 \right) \]

Since \( n > (2e)^k \), we get that \( n^{1-1/k} > (2e)^{k-1} \). For \( k \geq 2 \), we obtain \( (2e)^{k-1} > \frac{2(k+1)}{3e} \left( \frac{k}{\ln 2} + 1 \right) \), which concludes \((i)\).

To show \((ii)\), recall that \( n > (2e)^k \). Then for \( k \geq 2 \) we get \( (2e)^k > 4k/\ln(2) \), which implies \((ii)\). This concludes the second case and the proof of the theorem.

\[ \square \]

### E Motivation for Rank Queries

Rank queries make sense in situations such as unscrambling a list, searching genetic information, or where the elements of the list are not directly comparable to each other. We provide some high level examples for these three scenarios.

- **List unscrambling.** Suppose we have two lists, the first with student names and corresponding student IDs (sorted alphabetically by name), and a second one with student IDs and grades (sorted by grade). If one wants to find out whether Alice’s grade is in the top 10 marks, then this corresponds to a rank query: “Is \( \text{rank}(Alice) < 10? \)”.

- **Searching genetic information.** Assume we have an algorithm that analyzes an individual’s DNA and assigns it to one of \( k \) likelihood groups according to how likely they are to develop a specific disease, where \( i < j \) means that people in group \( i \) are more likely to develop the disease than people in group \( j \). This algorithm is very expensive computationally, because it needs to exhaustively look for bad combinations of genes and mutations. However, obtaining a crude estimation of the likelihood is much easier (e.g. because a particular mutation increases the likelihood of developing the disease). Thus, by using such a heuristic, we can quickly answer questions of the form “Does DNA sequence \( X \) belong to one of the first \( p \) groups?” for most DNA sequences. This scenario motivates the question of whether it is possible to solve problems such as ordering the individuals in decreasing risk using queries for this form.

- **Ranking incomparable items.** Consider a talent show, where there are \( n \) participants, each with different skills (e.g. singing, dancing, martial arts). The goal is to rank them from 1 to \( n \), but the participants are not directly comparable to each other because their skills are different. Thus the jury will give each participant a grade by comparing them with performers of different levels of ability (tiers) from the general population, where the tiers range from 1 to \( n \). For example, if a participant named Bob is a singer, a question can be whether Bob’s performance is comparable to a top or second best performance (i.e. “Is \( \text{rank}(Bob) \leq 2? \)”).
Note in this case two participants can receive the same grade, however ties can be broken lexicographically to obtain a distinct rank for each participant. However the bounds hold if there are ties in the rank as well.

F Folklore lemmas

Here we include a few folklore lemmas that we use, together with their proofs for completeness.

Lemma 23. Let \( y = (y_1, \ldots, y_n) \) with \( y_1 \geq \ldots \geq y_n \geq 0 \) and \( \sum_{i=1}^{n} y_i = 1 \). Then \( \sum_{j=1}^{i} y_j \geq i/n \) \( \forall i \in [n] \).

Proof. Let \( i \in [n] \). Since \( y \) is decreasing, we have \( (\sum_{j=1}^{i} y_j)/i \geq (\sum_{j=i+1}^{n} y_j)/(n-i) \) (†).

Assume by contradiction that \( y_1 + \ldots + y_i < \frac{i}{n} \) (‡). Adding \( y_{i+1} + \ldots + y_n \) to both sides of (‡), we get

\[
1 = y_1 + \ldots + y_n \leq \frac{i}{n} + y_{i+1} + \ldots + y_n \leq \frac{i}{n} + \frac{n-i}{i} \cdot \left( \sum_{j=1}^{i} y_j \right) \quad \text{(By (†))}
\]

\[
< \frac{i}{n} + \frac{n-i}{i} \cdot \left( \frac{i}{n} \right) \quad \text{(By (‡))}
\]

\[
= 1. \quad \text{(180)}
\]

We obtained \( 1 < 1 \), thus the assumption in (‡) must have been false and the lemma holds.

Lemma 24. Let \( x \in \mathbb{R}_{\geq 3} \). Then \( x^{1+\frac{1}{x}} > x + 1 \).

Proof. Raising both sides to the power \( 1/(x+1) \), the inequality is equivalent to \( x^{\frac{1}{x}} > (x+1)^{\frac{1}{x+1}} \), or \( (1/x)\ln (x) > (1/(x+1))\ln (x+1) \) (†).

Define \( g(x) = (\ln x)/x \). Its derivative is \( g'(x) = (1 - \ln x)/x^2 \). Thus \( g \) is increasing on \([1, e)\) and decreasing on \([e, +\infty)\). It follows that (†) holds for \( x \geq 3 \) and the lemma follows.

The next lemma shows that if \( v \) is an integrable function defined on \([0, 1]\), then there is an interval \( I \) of length \( p \) on the circle where the interval \([0, 1]\) is bent such that the point 0 coincides with 1, with the property that \( \int_{I} v(x) dx = p \).

Lemma 25. Let \( v : [0, 1] \to \mathbb{R}_{\geq 0} \) be an integrable function with \( \int_{0}^{1} v(x) dx = 1 \). Then there exists \( a \in [0, 1] \) such that one of the following holds:

- \( \int_{a}^{a+p} v(x) dx = p \), where \( 0 \leq a \leq 1 - p \);
- \( \int_{0}^{a} v(x) dx + \int_{a+1-p}^{1} v(x) dx = p \), where \( 1 - p < a < 1 \).

Proof. We define a new function \( g : [0, 1] \to \mathbb{R}_{\geq 0} \), such that

\[
g(x) = \begin{cases} 
\int_{x}^{x+p} v(y) dy & \text{if } 0 \leq x \leq 1 - p, \\
\int_{0}^{1} v(y) dy + \int_{0}^{x+p-1} v(y) dy & \text{if } 1 - p < x \leq 1.
\end{cases}
\]
To prove the lemma it suffices to show that there exists $c \in [0, 1]$ such that $g(c) = p$. Indeed, the function $g$ is continuous and so integrable. Let $F : [0, 1] \to \mathbb{R}_{\geq 0}$ be $F(x) = \int_0^x v(y) \, dy$. Using this notation, we get:

\[
\int_0^1 g(x) \, dx = \left[ \int_0^{1-p} \int_x^{x+p} v(y) \, dy \, dx \right] + \left[ \int_1^{1-p} \left( \int_x^1 v(y) \, dy \right) + \left( \int_0^{x+p-1} v(y) \, dy \right) \, dx \right] \\
= \int_0^{1-p} \left( F(x+p) - F(x) \right) \, dx + \int_1^{1-p} \left( (F(1) - F(x)) + (F(x+p-1) - F(0)) \right) \, dx \\
= \int_0^{1-p} F(x+p) \, dx - \int_0^{1-p} F(x) \, dx + \int_1^{1-p} 1 \, dx - \int_1^{1-p} F(x) \, dx + \int_1^{1-p} F(x+p-1) \, dx \\
\quad \text{(Since } F(1) = 1 \text{ and } F(0) = 0. ) \\
= \int_0^{1-p} F(x+p) \, dx - \int_0^{1-p} F(x) \, dx + p - \int_1^{1-p} F(x) \, dx + \int_1^{1-p} F(x+p-1) \, dx. 
\] (181)

We have

\[
\int_1^{1-p} F(x+p-1) \, dx = \int_0^p F(y) \, dy \quad \text{and} \quad \int_0^{1-p} F(x+p) \, dx = \int_0^1 F(z) \, dz. 
\] (182)

Using (182) in (181) yields

\[
\int_0^1 g(x) \, dx = \int_0^1 F(z) \, dz - \int_0^{1-p} F(x) \, dx + p - \int_1^{1-p} F(x) \, dx + \int_0^p F(y) \, dy. 
\] (183)

Notice that \( \int_p^1 F(z) \, dz + \int_0^p F(y) \, dy = \int_0^1 F(x) \, dx \) and \(- \int_0^{1-p} F(x) \, dx - \int_1^{1-p} F(x) \, dx = - \int_0^1 F(x) \, dx. \)

Therefore, the four integrals in (183) cancel each other and we get \( \int_0^1 g(x) \, dx = p. \) \( \dagger \)

Applying the first mean value theorem for definite integrals in \( \dagger \), there exists $c \in [0, 1]$ with the property that $g(c) = \frac{1}{1-p} \int_0^1 g(x) \, dx = p$, which concludes the proof. \( \square \)