CONVERGENCE OF DIRECT RECURSIVE ALGORITHM FOR IDENTIFICATION OF PREISACH HYSTERESIS MODEL WITH STOCHASTIC INPUT

D. RACHINSKI\(^*\) AND M. RUDERMAN\(^\dagger\)

Abstract. We consider a recursive iterative algorithm for identification of parameters of the Preisach model, one of the most commonly used models of hysteric input-output relationships. The classical identification algorithm due to Mayergoyz defines explicitly a series of test inputs that allow one to find parameters of the Preisach model with any desired precision provided that (a) such input time series can be implemented and applied; and, (b) the corresponding output data can be accurately measured and recorded. Recursive iterative identification schemes suitable for a number of engineering applications have been recently proposed as an alternative to the classical algorithm. These recursive schemes do not use any input design but rather rely on an input-output data stream resulting from random fluctuations of the input variable. Furthermore, only recent values of the input-output data streams are available for the scheme at any time instant. In this work, we prove exponential convergence of such algorithms, estimate explicitly the convergence rate, and explore which properties of the stochastic input and the algorithm affect the guaranteed convergence rate.

Key words. Identification problem, recursive algorithm, exponential convergence rate, model of hysteresis, input-output operator, stochastic input.

AMS subject classifications. 93E12, 47J40, 74N30

1. Introduction. Preisach model [22], with its roots originated in magnetism [18,19], is well-suitable for describing various rate-independent hysteresis phenomena in a closed input-output analytic form. The Preisach hysteresis operator \(y(k) = (H[x])(k)\) models an input-output relationship with an erasable memory of previous input states \(x(j)\), where \(j < k\). The input-output time series \((x, y)(k)\) constitute monotonic piecewise continuous hysteresis curves, see Figure 1 (right).

\(\rho\)

Preisach density function

memory interface

\(\alpha\)

\(\beta\)

hysteresis map

\(x\)

\(y\)

Fig. 1. Preisach density function \(\rho(\alpha, \beta)\) over the plane \((\alpha, \beta)\) (left) and input-output curves \(y = (H[x])(t)\) of the Preisach operator (right).

\(\ast\)Department of Mathematical Sciences, The University of Texas at Dallas, USA (dmitry.rachinskiy@utdallas.edu). The author acknowledges the support of NSF through grant DMS-1413223. Questions, comments, or corrections to this document may be directed to that email address.

\(\dagger\)Department of Computer Science and Engineering, Nagoya Institute of Technology, Japan (ruderman.michael@nitech.ac.jp).
According to the representation theorem of Mayergoyz, the Preisach hysteresis model can be fitted to any set of input-output data that possesses two properties: the so-called wiping-out of local input extremum values and congruency of any two hysteresis loops created by the same two consecutive input extrema; see [18] for more details of the formalism and properties of the Preisach hysteresis model.

The use of models of hysteresis in magnetism, as well as in other natural, engineering, and social sciences, confronts with problems of identification of parameters, which requires to obtain a sufficient number of observations of hysteresis system states [2,8]. The phenomenological nature of the Preisach model, which is based on superposition of multiple elementary hysteresis operators called non-ideal relays (switches), and the representation theorem of Mayergoyz allow to apply the model to various hysteretic systems using a phenomenological argument or the black box approach without complex analysis of such systems based on first principles [3, 7, 10, 11, 15–17, 20]. Importantly, the representation theorem provides the Preisach model with straightforward means for identification of parameters from dedicated input-output data. Here it should be recalled that the Preisach model is entirely parameterized by the so-called Preisach density function \( \rho = \rho(\alpha, \beta) \) defined over the \((\alpha, \beta)\) plane, as schematically shown in Figure 1 (left). The \( \alpha \) and \( \beta \) coordinates, with \( \beta \geq \alpha \), are the ‘on’ and ‘off’ switching thresholds of a single relay operator, and the \( \rho \)-function represents the weighted \((\alpha, \beta)\)-distribution of these relays over the possible threshold values. Each relay switches from state 0 (‘off’) to state 1 (‘on’) and backwards in response to variations of the same input time series called the input \( x(k) \) of the Preisach model; the output time series \( y(k) \) of the model is defined as the sum of states of all the relays. The identification algorithm of Mayergoyz requires to discretize the Preisach model and assumes a sufficient number of measured first-order descending (FOD) input-output curves [18]. The higher the required accuracy of hysteresis modeling is, the smaller the discrete step on the \((\alpha, \beta)\) plane should be (cf. with Figure 1, left), and the more accurate FOD curve measurements are required. The volume of data, which should be first recorded before the direct Mayergoyz identification method can be applied for calculation of the density function \( \rho \), rapidly increases with increasing target accuracy.

From this point of view, the motivation for this work is to explore the possibilities of applying recursive/iterative identification methods for which only the previous (or, a few recent) values of the input-output data stream should be available at any instant. Recursive (also known as on-line) identification methods can be required when dealing with time-varying processes, i.e., where the hysteresis behavior changes (usually slowly) with time depending on certain non-controllable operational or environmental conditions, such as varying ambient temperature, force fields, temporal relaxation, etc. The applicability of the direct Mayergoyz identification method can also be limited by a strong uncontrollable noise capable of corrupting the designed test input signal. Furthermore, in various engineering applications which use hysteresis models, e.g., for process monitoring or control, a simplified identification with no special (expert) knowledge as for design of dedicated input processes (such as a process creating the FOD curves) can be required. Here an algorithm which is capable of using data generated by a stochastic input processes (with certain well-defined properties) may be advantageous.

Surprisingly, little number of recursive methods suitable for identification of parameters of the Preisach hysteresis model has been analyzed theoretically or experimentally. A gradient-based recursive identification scheme has been proposed in [28]
and experimentally evaluated on a magnetostrictive actuator system which exhibits hysteresis between the relative displacement and the actuator current. The error convergence for this scheme has been estimated for relatively rough discretization of the \((\alpha, \beta)\) plane (up to 25 steps in the input domain). Another direct recursive identification method has been proposed in [24]. This method updates the values of the Preisach density function in the so-called switching region of the \((\alpha, \beta)\) plane, which defines an output increment in response to a random input increment at each iteration step. This method has been tested and evaluated numerically in the context of modeling the hysteretic constitutive relationship between the magnetic field and magnetization in a ferromagnetic material. However no rigorous analysis of the convergence and its properties, such as the convergence rate, has been done insofar.

In this paper, we prove the exponential convergence of the recursive identification algorithm proposed in [24]. We first show that the error of the algorithm monotonically decreases (Section 2) and establish an estimate for the rate of convergence for any input time series (Section 3). Our main result (Section 4) establishes that the error \(E_k\) almost surely converges to zero exponentially, \(\|E_k\| \leq Ce^{-k\lambda}\|E_0\|\), with the rate satisfying an explicit estimate \(\lambda \geq \mu(N_0, \langle T \rangle, \langle 1/T \rangle) > 0\) if the input time series generates a regular Markov chain in the space of states of the Preisach model (the latter condition is generic and is satisfied for a wide class of input Markov processes). Here \(N_0\) is a function of the number of nodes in the domain of the density function on the \((\alpha, \beta)\) plane, \(T\) is a certain stopping time associated with the input process, and \(\langle T \rangle, \langle 1/T \rangle\) are the mean values of \(T\) and its inverse. Essentially, \(T\) is the length of the time interval during which fluctuations of the input randomly create all the segments of all the FOD input-output curves prescribed by the classical Mayergoz identification algorithm. If all this information was recorded, a complete identification of the density function was achievable from the input-output data collected within \(T\) time steps. The recursive algorithm considered here is not capable of recovering the density function exactly as it updates the approximation to the density function at each time step without recording past input-output data. However, we guarantee that the error decreases by a certain percentage over \(T\) time steps, \(\|E_{k+T}\| \leq e^{-c/(N_0T)}\|E_k\|\).

It is worth noting that the random time \(T\) is a characteristic of the input process. Overall, our estimates suggest that the smaller the mean value of \(T\) is and the less nodes the grid has (smaller \(N_0\)), the faster is the guaranteed exponential convergence of the recursive algorithm. We test these conclusions by considering two examples of random input processes and comparing their rates of convergence in Section 5.

2. Monotonic decrease of the error. Let \(\rho = \rho(p)\) be a density function of the Preisach operator where \(p = (\alpha, \beta)\) denotes a point in the half-plane \(\Pi = \{ p = (\alpha, \beta) : \alpha \leq \beta \}\). Each point \(p\) represents a non-ideal relay with the threshold values \(\alpha\) and \(\beta\). Denote by \(S_k = S(t_k)\) the subset of the half-plane \(\Pi\) where relays are ‘on’ (in state 1) at a moment \(t_k = k \geq 0\) (we normalize the time step to 1); the other relays are ‘off’ (in state 0). Then the output \(y_k = y(t_k)\) of the Preisach model at this moment equals

\[
y_k = \int_{S_k} \rho(p) \, dp
\]

where \(dp = d\alpha d\beta\). The set \(S_k\) is called the state of the Preisach model. The state \(S_k\) and the output \(y_k\) change in response to variations of the input \(x_k = x(t_k)\) of the model as the non-ideal relays switch between states 0 and 1. For discrete time input
series, which is the case considered here,

$$S_{k+1} = \begin{cases} 
S_k \cup \{(\alpha, \beta) \in \Pi : \beta \leq x_{k+1}\} & \text{if } x_{k+1} > x_k \\
S_k \setminus \{(\alpha, \beta) \in \Pi : \alpha \geq x_{k+1}\} & \text{if } x_{k+1} < x_k \\
S_k & \text{if } x_{k+1} = x_k 
\end{cases}$$

This rule defines the time series of the state and output, $S_k$ and $y_k$, for any given input sequence $x_k$, $k \geq 0$ and a given initial state $S_0$ (see, for example, [4–6,9,12–14,21,23,29] for more details).

In the following identification algorithm we use an estimate $\hat{\rho}_k = \hat{\rho}_k(p)$, $p \in \Pi$ of the density function, which is updated at each moment $t_k$. The estimate $\hat{y}_k = \hat{y}(t_k)$ of the output and the error $e_k = y_k - \hat{y}_k$ between the observed and modeled output values equal, respectively,

$$\hat{y}_k = \int_{S_k} \hat{\rho}_k(p) dp; \quad e_k = \int_{S_k} (\rho(p) - \hat{\rho}_k(p)) dp.$$

Comparing states $S_k$ and $S_{k+1}$, denote $\Omega_k^+ = S_{k+1} \setminus S_k$. This is the set of those relays that switch on during the time step from $t_k$ to $t_k + 1$. Similarly, consider the set $\Omega_k^- = S_k \setminus S_{k+1}$ of relays that switch off during the same time step. Then, over this time step, the error between the observed output and the output modeled with the density function $\hat{\rho}_k$ receives the increment

$$\Delta e_k = \int_{\Omega_k^+} (\rho(p) - \hat{\rho}_k(p)) dp - \int_{\Omega_k^-} (\rho(p) - \hat{\rho}_k(p)) dp.$$

Following [24], we update the estimate of the true density function $\rho$ according to the formula

$$(2.1) \quad \hat{\rho}_{k+1}(p) = \hat{\rho}_k(p) + r_k(p) \quad \text{with} \quad r_k(p) = \begin{cases} 
0 & \text{if } p \notin \Omega_k = \Omega_k^+ \cup \Omega_k^- \\
\frac{\Delta e_k}{A_{\Omega_k}} & \text{if } p \in \Omega_k^+ \\
-\frac{\Delta e_k}{A_{\Omega_k}} & \text{if } p \in \Omega_k^- 
\end{cases}$$

where $A_{\Omega_k}$ denotes the area of the switching region $\Omega_k$.

Let us first show that the mean square distance between the exact and estimated density is monotonically non-increasing with iterations.

**Lemma 2.1.** For any initial estimate $\hat{\rho}_0$, the mean square distance

$$||\hat{\rho}_k - \rho|| = \left( \int_{\Pi} (\hat{\rho}_k(p) - \rho(p))^2 dp \right)^{1/2}$$

between $\hat{\rho}_k$ and $\rho$ is a non-increasing function of $k$.

**Proof.** According to the update rule,

$$||\hat{\rho}_{k+1} - \rho||^2 = \int_{\Pi} (\hat{\rho}_{k+1}(p) + r_k(p) - \rho(p))^2 dp$$

$$= \int_{\Pi} (\hat{\rho}_k(p) - \rho(p))^2 dp + \int_{\Omega_k} r_k^2(p) dp + 2 \int_{\Omega_k} r_k(\hat{\rho}_k(p) - \rho(p)) dp$$
in the update rule (2.1) with \(N\) written as (2.2) 

\[ \|\hat{\rho}_k - \rho\|^2 + \frac{(\Delta e_k)^2}{A_{\Omega_k}} \left( \int_{\Omega_k^+} (\hat{\rho}_k(p) - \rho(p)) \, dp - \int_{\Omega_k^-} (\hat{\rho}_k(p) - \rho(p)) \, dp \right) \]

\[= \|\hat{\rho}_k - \rho\|^2 + \frac{(\Delta e_k)^2}{A_{\Omega_k}} - \frac{2(\Delta e_k)^2}{A_{\Omega_k}} = \|\hat{\rho}_k - \rho\|^2 - \frac{(\Delta e_k)^2}{A_{\Omega_k}}. \]

We see that

\[
\|\hat{\rho}_{k+1} - \rho\|^2 = \|\hat{\rho}_k - \rho\|^2 - \frac{(\Delta e_k)^2}{A_{\Omega_k}},
\]

hence the mean square distance between the exact and estimated Preisach density functions not only never increases, but actually strictly decreases each time the output increment error \(\Delta e\) is non-zero. The aim of the following argument is to show that, for reasonable inputs, the error \(\Delta e\) is non-zero “often enough” (as long as \(\hat{\rho}_k \neq \rho\) to ensure the exponential convergence \(\hat{\rho}_k \to \rho\).

3. Estimation of the rate of convergence. From now on, we will consider the discretized version of the Preisach model. We will assume that the input is a discrete Markov process \(X_k\) with \(N + 1\) states \(x_i = i\varepsilon\), where \(\varepsilon = 1/N\) and \(i = 0, \ldots, N\). In this discrete setting, the set \(S\) of all possible states \(S\) of the model is also finite. Furthermore, we can assume without loss of generality that the density \(\rho\) (as well as its approximations \(\hat{\rho}_k\)) is concentrated at the finite set of nodes

\[
\{ (\alpha, \beta) = ((\ell - 1/2)\varepsilon, (m - 1/2)\varepsilon) : 1 \leq \ell \leq m \leq N \}
\]

that form a uniform rectangular mesh within the right triangle

\[
\Lambda_{supp} = \{ (\alpha, \beta) : 0 \leq \alpha \leq \beta \leq 1 \}
\]

of the half-plane \(\Pi\); these nodes will be denoted \(p_i = (\alpha_i, \beta_i)\), where \(i \in I\) with the total number of nodes \(#(I) = M = N(N + 1)/2\). In other words, we consider the density functions of the form \(\rho(p) = \varepsilon^2 \sum_{i=1}^{M} \rho(p_i) \delta(p - p_i)\). It is convenient to rescale the mesh step to unity by setting

\[
y_k = \sum_{i \in I} \rho(p_i), \quad \hat{y}_k = \sum_{i \in I} \hat{\rho}_k(p_i), \quad \Delta e_k = \sum_{i \in I} (\rho(p_i) - \hat{\rho}_k(p_i)) - \sum_{i \in I} (\rho(p_i) - \hat{\rho}_k(p_i))
\]

and, simultaneously, setting

\[
A_{\Omega_k} = N_k := \#(i \in I : p_i \in \Omega_k)
\]

in the update rule (2.1) with \(N_k\) denoting the number of the mesh nodes in the switching region \(\Omega_k\). We note that for this discrete model, at each time step either \(\Omega_k^+ = \emptyset\), \(\Omega_k = \Omega_k^+\), or \(\Omega_k = \emptyset\), \(\Omega_k = \Omega_k^+\). With this notation, formula (2.2) can be written as

\[
\|E_{k+1}\|^2 = \|E_k\|^2 - N_k(E_{k+1}(p) - E_k(p))^2, \quad \forall p \in \Omega_k,
\]

where \(E_k\) is the error

\[
E_k(p) = \hat{\rho}_k(p) - \rho(p)
\]
and \( \|E_k\| = \left( \sum_{i \in I} E_k(p_i)^2 \right)^{1/2} \). Relationship (3.2) can be also written as
\[
(3.3) \quad \|E_{k+1}\|^2 = \|E_k\|^2 - \|E_k - E_{k+1}\|^2.
\]

We define the rate of convergence of the iterative scheme (2.1) as
\[
(3.4) \quad \lambda = -\limsup_{k \to \infty} \frac{1}{k} \ln \frac{\|E_k\|}{\|E_0\|} = -\limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \ln \frac{\|E_i\|}{\|E_{i-1}\|}.
\]

In order to estimate the rate of convergence of the identification algorithm, we will use a certain stopping time \( T \) associated with the input process. With a given realization of the input process \( X_k \) we associate successive time intervals

\[
1 \leq k \leq T_1, \quad T_1 + 1 \leq k \leq T_1 + T_2, \quad T_1 + T_2 + 1 \leq k \leq T_1 + T_2 + T_3, \ldots
\]

During each time interval we require the process to create a set of Preisach states satisfying the following properties. We require that for every node \( p_i = (\alpha_i, \beta_i) \) there would be two states \( S(t_{k'}) \) and \( S(t_{k'} + 1) \) achieved at two successive moments\(^1\) within the time interval \([T_1 + \cdots + T_{k-1} + 1, T_1 + \cdots + T_k]\) such that
\[
(S(t_{k'}) \setminus S(t_{k'} + 1)) \cup (S(t_{k'} + 1) \setminus S(t_{k'})) = \Lambda_i
\]

where
\[
(3.5) \quad \Lambda_i = \{ (\alpha, \beta) : \alpha_i - \epsilon/2 \leq \alpha \leq \beta \leq \beta_i + \epsilon/2 \}.
\]

As soon as this requirement is satisfied for all the nodes \( (\alpha_i, \beta_i), i \in I, \) of the mesh, the new time interval \([T_1 + \cdots + T_k + 1, T_1 + \cdots + T_{k+1}]\) starts. The set (3.5) is a right triangle with the vertex \((\alpha_i - \epsilon/2, \beta_i + \epsilon/2)\) and the hypothenuse on the line \( \alpha = \beta \).

Therefore, we will simply say that the input creates all the possible triangles on the half-plane \( \Pi \) within each time interval \([T_1 + \cdots + T_{k-1} + 1, T_1 + \cdots + T_k]\).

For example, this requirement will be satisfied if for every \((\alpha_i, \beta_i)\) there will be three successive time moments \( t_k = k, t_{k+1} = k + 1 \) and \( t_{k+2} = k + 2 \) during the time interval \([T_1 + \cdots + T_k + 1, T_1 + \cdots + T_{k+1}]\) such that either
\[
(3.6) \quad x(t_k) = \beta_j + \epsilon/2, \quad x(t_{k+1}) = \alpha_i - \epsilon/2, \quad x(t_{k+2}) = \beta_i + \epsilon/2
\]

for some \( \beta_j \geq \beta_i \), or
\[
(3.7) \quad x(t_k) = \alpha_j - \epsilon/2, \quad x(t_{k+1}) = \beta_i + \epsilon/2, \quad x(t_{k+2}) = \alpha_i - \epsilon/2
\]

for some \( \alpha_j \leq \alpha_i \).

A less restrictive requirement is that the same holds but the moments \( t_k, t_{k+1} \) are not necessarily successive and the input stays between \( \alpha_i - \epsilon/2 \) and \( \beta_j + \epsilon/2 \) for \( t_k \leq t \leq t_{k+1} \) in the first case (between \( \alpha_j - \epsilon/2 \) and \( \beta_i + \epsilon/2 \) for \( t_k \leq t \leq t_{k+1} \) in the second case).

**Theorem 3.1.** The rate of convergence (3.4) of the iterative scheme (2.1) satisfies the estimate
\[
(3.8) \quad \lambda \geq \frac{1}{32N_0} \liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{m(k)} \frac{1}{1 + T_i}
\]

\(^1\)It is easy to extend the algorithm and results to the case when these moments are not necessarily successive. We consider successive moments for simplicity.
where \( m = m(k) \) denotes the largest integer such that \( k \geq T_1 + \cdots + T_m \) and

\[
N_0 = \frac{N(N+1)^2}{4} + \frac{N^2(N^2-1)}{24}.
\]

In the next section, we will show that the limit in (3.8) is positive almost surely and can be found from the stationary distribution of the input process under natural assumptions.

**Proof of Theorem 3.1.** Define \( \tau_0 = 0, \tau_m = T_1 + \cdots + T_m \) and assume that the input creates all the possible triangles during each time interval \([\tau_{k-1}, \tau_k]\). Set

\[
\sigma = \|E_{\tau_{m-1}}\|^2 - \|E_{\tau_m}\|^2.
\]

According to formula (3.3),

\[
\sigma = M \sum_{i=1}^{\tau_m-1} \|E_{k+1} - E_k\|^2.
\]

We split this sum into \( M \) sums, where \( M \) is the total number of nodes:

\[
\sigma = M \sum_{i=1}^{\tau_m-1} \theta_i \sum_{k=\tau_m-1}^{\tau_m} \|E_{k+1} - E_k\|^2
\]

where the coefficients \( \theta_i \) satisfy

\[
\theta_1 + \cdots + \theta_M = 1.
\]

Now, we replace the upper limit \( \tau_m - 1 \) in the internal sum by the moment \( t_i \in [\tau_{m-1}, \tau_m - 1] \) when the switching area during the time step from \( t_i \) to \( t_i + 1 \) is the triangle \( \Lambda_i = \{(\alpha, \beta) : \alpha_i - \varepsilon/2 \leq \alpha \leq \beta \leq \beta_i + \varepsilon/2\} \). In this way, we obtain the estimate

\[
\sigma \geq M \sum_{i=1}^{\tau_m-1} \theta_i \sum_{k=\tau_m-1}^{t_i} \|E_{k+1} - E_k\|^2.
\]

At the time step from \( t_i \) to \( t_i + 1 \) we have

\[
\|E_{t_i+1} - E_{t_i}\|^2 = \frac{(A_{t_i}^i)^2}{N_i}
\]

where \( A_{t_i}^i = \sum_{p \in \Lambda_i} E_i(p) \) denotes the sum of \( E_i(p) \) over the triangle \( \Lambda_i \) at the moment \( t \) and \( N_i \) is the number of nodes in the triangle. Hence,

\[
\sigma \geq M \sum_{i=1}^{\tau_m-1} \theta_i \frac{(A_{t_i}^i)^2}{N_i} + M \sum_{i=1}^{\tau_m-1} \theta_i \sum_{k=\tau_m-1}^{t_i-1} \|E_{k+1} - E_k\|^2.
\]

Using the inequality

\[
\sum_{k=\tau_m-1}^{t_i-1} \|E_{k+1} - E_k\|^2 \geq \frac{1}{t_i - \tau_m-1} \left( \sum_{k=\tau_m-1}^{t_i-1} (E_{k+1} - E_k) \right)^2,
\]

we have

\[
\sigma \geq M \sum_{i=1}^{\tau_m-1} \theta_i \frac{(A_{t_i}^i)^2}{N_i} + M \sum_{i=1}^{\tau_m-1} \theta_i \frac{1}{t_i - \tau_m-1} \left( \sum_{k=\tau_m-1}^{t_i-1} (E_{k+1} - E_k) \right)^2.
\]
we obtain
\[ \sum_{k=m-1}^{t_i-1} \| E_{k+1} - E_k \|^2 \geq \sum_{k=m}^{t_i-1} \| E_{k+1} - E_{k-1} \|^2 \]
and therefore
\[ \sigma \geq \sum_{i=1}^{M} \theta_i \left( \frac{(A_{\xi i})^2}{N_i} + \frac{\| E_{t_i} - E_{m-1} \|^2}{T_m} \right). \]

Let us consider one term in this sum:
\[ (3.10) \]
\[ \frac{(A_{\xi i})^2}{N_i} + \frac{\| E_{t_i} - E_{m-1} \|^2}{T_m} \geq \frac{1}{N_i} \left( \sum_{p \in \Lambda_i} E_{t_i}(p) \right)^2 + \frac{1}{T_m} \sum_{p \in \Lambda_i} (E_{t_i}(p) - E_{m-1}(p))^2. \]
The right hand side of this inequality can be rewritten as
\[ \frac{1}{N_i} \left( \sum_{p \in \Lambda_i} (E_{m-1}(p) + \xi_p) \right)^2 + \frac{1}{T_m} \sum_{p \in \Lambda_i} \xi_p^2 \]
where \( \xi_p = E_{t_i}(p) - E_{m-1}(p) \). Considering the minimization problem
\[ V(\xi_1, \ldots, \xi_{N_i}) = \frac{1}{N_i} \left( \sum_{p \in \Lambda_i} (E_{m-1}(p) + \xi_p) \right)^2 + \frac{1}{T_m} \sum_{p \in \Lambda_i} \xi_p^2 \rightarrow \min \]
with respect to the variables \( \xi_p \), we obtain the equations
\[ \frac{1}{N_i} \left( \sum_{p \in \Lambda_i} (E_{m-1}(p) + \xi_p) \right) + \frac{\xi_p}{T_m} = 0, \quad p \in \Lambda_i, \]
for the extremum, hence the minimum is achieved at the point
\[ \xi_1 = \cdots = \xi_{N_i} = -\frac{T_m}{N_i (1 + T_m)} \sum_{p \in \Lambda_i} E_{m-1}(p) \]
and the global minimum value equals
\[ V_{\text{min}} = \frac{1}{N_i (1 + T_m)} \left( \sum_{p \in \Lambda_i} E_{m-1}(p) \right)^2 = \frac{(A_{\tau m-1})^2}{N_i (1 + T_m)}. \]
Thus, this expression is the lower estimate for the the right hand side of inequality (3.10) and we conclude that
\[ \sigma \geq \frac{1}{1 + T_m} \sum_{i=1}^{M} \theta_i (A_{\xi i})^2 \]
We rearrange the sum in the right hand side as follows:

$$\sigma \geq \frac{1}{4(1 + T_m)} \sum_{i=1}^{M} \left( \frac{\theta_i(A^{m-1}_i)^2}{N_i} + \frac{\theta_j(A^{m-1}_j)^2}{N_j} + \frac{\theta_k(A^{m-1}_k)^2}{N_k} + \frac{\theta_\ell(A^{m-1}_\ell)^2}{N_\ell} \right)$$

where $j = j(i), k = k(i), \ell = \ell(i)$ are defined so that $\Lambda_j = \{(\alpha, \beta) : \alpha_i + \varepsilon/2 \leq \alpha \leq \beta \leq \beta_i + \varepsilon/2\}, \Lambda_k = \{(\alpha, \beta) : \alpha_i - \varepsilon/2 \leq \alpha \leq \beta \leq \beta_i - \varepsilon/2\}, \Lambda_\ell = \{(\alpha, \beta) : \alpha_i + \varepsilon/2 \leq \alpha \leq \beta \leq \beta_i - \varepsilon/2\}$ (for the nodes $(\alpha_i, \beta_i)$ on the diagonal $\beta = \alpha$ we set $A_j = A_k = A_\ell = 0$). We choose

$$\theta_i = \frac{N_i}{N_0}$$

with

$$N_0 = \sum_{i=1}^{M} N_i = \frac{1}{2} \sum_{k=1}^{N} k(k + 1)(N - k + 1) = \frac{1}{2} \sum_{k=1}^{N} (N + 1)k + Nk^2 - k^3$$

$$= \frac{N(N + 1)^2}{4} + \frac{N^2(N + 1)(2N + 1)}{12} - \frac{N^2(N + 1)^2}{8} = \frac{N(N + 1)^2}{4} + \frac{N^2(N - 1)}{24}$$

where the total number of nodes is $M = N(N + 1)/2$. With this choice of $\theta_i$,

$$\sigma \geq \frac{1}{4(1 + T_m)N_0} \sum_{i=1}^{M} (A^{m-1}_i)^2 + (A^{m-1}_j)^2 + (A^{m-1}_k)^2 + (A^{m-1}_\ell)^2.$$ 

We now note that for each node $i$

$$A^{m-1}_i - A^{m-1}_j - A^{m-1}_k + A^{m-1}_\ell = E_{m-1}(i).$$

Minimizing the function $W(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2$ under the constraint $\xi_1 + \xi_2 + \xi_3 + \xi_4 = E_{m-1}(i)$, we see that $\min W = (E_{m-1}(i))^2/4$, hence

$$\sigma = \|E_{m-1}\|^2 - \|E_m\|^2 \geq \frac{1}{4(1 + T_m)N_0} \sum_{i=1}^{M} (E_{m-1}(i))^2 \geq \frac{\|E_{m-1}\|^2}{16(1 + T_m)N_0}.$$ 

That is,

$$\|E_{m}\|^2 \leq \left(1 - \frac{1}{16(1 + T_m)N_0}\right) \|E_{m-1}\|^2.$$ 

As $\ln(1 - x) \leq -x$ for $0 < x < 1$, this implies

$$-\ln \|E_{m}\| + \ln \|E_{m-1}\| \geq \frac{1}{32(1 + T_m)N_0}.$$ 

Summing, we obtain

$$-\ln \|E_k\| + \ln \|E_0\| \geq \frac{1}{32N_0} \sum_{i=1}^{m} \frac{1}{1 + T_i}$$

where we use the largest $m = m(k)$ such that $k \geq T_1 + \cdots + T_m$. Therefore, the rate of convergence (3.4) of the iterative scheme satisfies (3.8).
4. Main result. As the input increment $X_{k+1} - X_k$ and the state $S_k$ of the Preisach model at a moment $k$ define the state $S_{k+1}$ at the next moment, the input Markov chain $X_k$ defines the Markov chain $S_k$ in the state space $S$ of the Preisach model describing the random evolution of the state.

Now, let us consider two more auxiliary Markov chains defined by the input process. One is $Y_k = S_{T_1 + \cdots + T_k}$. That is, any realization of the Markov chain $S_k$ generates a realization of the auxiliary process $Y_k = S_{\tau_k}$ where all the triangles (3.5) are created between any two moments $\tau_{k-1} = T_1 + \cdots + T_{k-1}$ and $\tau_k = T_1 + \cdots + T_k$\footnote{According to the definition of the times $\tau_k$, each moment $\tau_k$ is minimal in the sense that at least one triangle is not created between the moments $\tau_{k-1}$ and $\tau_{k-1}$.}. The finite state space of the Markov chain $Y_k$ is the space $S$ of states of the Preisach model. Here and henceforth we assume that the Markov chain $S_k$ almost surely creates all the triangles (3.5) in finite time, hence the Markov chain $Y_k = S_{\tau_k}$ is well defined.

We will assume that properties of the input Markov chain $X_k$ ensure that the Markov chain $Y_k = S_{\tau_k}$ is regular (ergodic), that is some power of the transition probability matrix of the process $Y_k$ is strictly positive.

Secondly, we consider an auxiliary Markov chain $Z_k = (S_{T_1 + \cdots + T_k}, T_k)$ with a countable number of states. A pair $(\hat{S}, \hat{T})$ belongs to the state space $Z$ of this Markov chain if and only if there is a Preisach state $\hat{S}$ such that $P(S_{T_1} = \hat{S}, T_1 = \hat{T}, S_0 = \hat{S}) > 0$. Let us observe that the transition probability $P(Z_{k+1} = Z'|Z_k = Z)$ from a state $Z = (S, \tau) \in Z$ to a state $Z' = (S', \tau') \in Z$ of the Markov chain $Z_k$ is, by its definition, independent of the component $\tau$ of the initial state:

$$P(S_{T_1 + \cdots + T_{k+1}} = S', T_{k+1} = \tau' \mid S_{T_1 + \cdots + T_k} = S, T_k = \tau) =: p_{S,S',\tau'}.$$  

As $Y_k = S_{\tau_k}$ is a regular Markov chain, for every Preisach state $S \in S$ there is at least one positive integer $T$ such that $(S, T) \in Z$.

Lemma 4.1. Assume that the Markov chain $Y_k = S_{T_1 + \cdots + T_k}$ is regular. Then, the Markov chain $Z_k = (S_{T_1 + \cdots + T_k}, T_k)$ is (a) irreducible; (b) positive recurrent; and, (c) aperiodic.

Proof. Irreducibility. Consider any pair of states $\tilde{Z} = (\tilde{S}, \tilde{T}), \hat{Z} = (\hat{S}, \hat{T}) \in Z$. By definition of the state space $Z$, there is at least one state $\hat{Z} = (\hat{S}, \hat{T}) \in Z$ such that $P(S_{T_1} = \hat{S}, T_1 = \hat{T}, S_0 = \hat{S}) > 0$. On the other hand, as the process $Y_k = S_{\tau_k}$ is regular, there is a positive integer $k_0$ such that $P(S_{T_1 + \cdots + T_{k_0}} = \tilde{S} | S_0 = \hat{S}) > 0$. Combining these two estimates and using the time invariance of the Markov chain $S_k$, we obtain $P(S_{T_1 + \cdots + T_{k_0+1}} = \tilde{S}, T_{k_0+1} = \tilde{T} | S_0 = \hat{S}) > 0$. But this is the probability that the process $Z_k$ reaches the state $\hat{Z}$ from the state $\tilde{Z}$ in $k_0 + 1$ steps. As the probability is positive, we conclude that $Z_k$ is irreducible.

Positive recurrence. For any state $\tilde{Z} = (\tilde{S}, \tilde{T}) \in Z$ consider the state $\hat{Z} = (\hat{S}, \hat{T}) \in Z$ such that $\hat{p} := P(S_{T_1} = \hat{S}, T_1 = \hat{T}, S_0 = \hat{S}) > 0$. Consider again a sufficiently large integer $k_0$ such that the regularity of the process $Y_k = S_{\tau_k}$ implies $p_m := \min\{P(S_{T_1 + \cdots + T_{k_0}} = \hat{S} | S_0 = \hat{S}) : \hat{S} \in S\} > 0$. We see that the probability that the process $Z_k$ starting from the state $\tilde{Z}$ returns to this state after $k_0 + 1$ time steps, $P(S_{T_1 + \cdots + T_{k_0+1}} = \tilde{S}, T_{k_0+1} = \tilde{T} | S_0 = \hat{S})$, is greater or equal than $p_m \hat{p} > 0$, hence the Markov chain $Z_k$ is recurrent. Furthermore, $P(S_{T_1 + \cdots + T_{k_0+1}} = \hat{S}, T_{k_0+1} = \tilde{T} | S_0 = \hat{S}) \geq p_m \hat{p}$ for any $\hat{S} \in S$. Therefore, the probability that the first return time
to the state $\hat{Z}$ exceeds $n(k_0 + 1)$ is less or equal than $(1 - p_m\hat{p})^n$ for any integer $n$. Hence, the expected value of this first return time does not exceed the value

$$\sum_{k=1}^{k_0+1} k + (1 - p_m\hat{p}) \sum_{k=k_0+2}^{2k_0+2} k + (1 - p_m\hat{p})^2 \sum_{k=2k_0+3}^{3k_0+3} k + \cdots$$

As this sum is finite, the expected value of the first return time is finite, hence the process $Z_k$ is positive recurrent.

**Aperiodicity.** Using the same notation as above, $\hat{p} = P(S_{T_k} = \hat{S}, T_1 = T|S_0 = \hat{S}) > 0$. From the regularity of the process $Y_k = S_{T_k}$ it follows that $P(k) := P(S_{T_1} + \cdots + T_k = \hat{S}|S_0 = \hat{S}) > 0$ for any $k \geq k_0$. Therefore, $P(S_{T_1} + \cdots + T_{k+1} = \hat{S}, T_1 = T|S_0 = \hat{S}) > 0$, that is the process $Z_k$ returns to the state $\hat{Z} = (\hat{S}, \hat{T})$ with positive probability after $k$ time steps for any $k \geq k_0 + 1$. Hence the state $\hat{Z}$ is aperiodic. Since this is true for any $\hat{Z} \in Z$, the Markov chain $Z_k$ is aperiodic.

Lemma 4.1 ensures that the finite Markov chain $Y_k$ has a strictly positive and unique stationary distribution $\pi_Y$. Furthermore, the Markov chain $Z_k$ also has a unique stationary distribution $\pi_Z$ and the ergodicity property

$$\frac{1}{k} \sum_{i=1}^{k} f(Z_k) \xrightarrow{a.s.} E_{\pi_Z}[f(Z)]$$

holds for any initial state $Z_1$ and any function $f$ with a finite expected value

$$E_{\pi_Z}[f(Z)] < \infty$$

with respect to the distribution $\pi_Z$ (see, for example, [27]). In particular, using $f(Z) = T$ where $Z = (S, T)$, we obtain

$$\frac{1}{k} \sum_{i=1}^{k} T_k \xrightarrow{a.s.} E_{\pi_Z}[T_k] =: T_*$$

(4.2) (the condition $T_* < \infty$ will be established in the proof of Theorem 4.2 below). That is, the time average of the time intervals $T_k$ converges to the mean $T_*$ of $T_k$ with respect to the stationary distribution of the chain $Z_k$. Similarly,

$$\frac{1}{k} \sum_{i=1}^{k} \frac{1}{1 + T_k} \xrightarrow{a.s.} E_{\pi_Z} \left[ \frac{1}{1 + T_k} \right] =: T_*^{**}$$

(4.3)

Let us remark that the mean value $T_*$ is defined by

$$T_* = \sum_{(S', \tau') \in Z} \tau' \cdot (\pi_Z)_{S', \tau'}$$

where $(\pi_Z)_{S', \tau'}$ is the probability to find the process $Z_k$ in the state $(S_k, T_k) = (S', \tau')$ according to the stationary distribution of this process. According to (4.1), for the stationary distribution,

$$\sum_{(S, \tau) \in Z} (\pi_Z)_{S, \tau} \cdot p_{SS', \tau'} = \sum_{S \in S} \Pi_S \cdot p_{SS', \tau'}$$

(4.4)
where $\Pi_S = \sum_s (\pi_Z)_{S,s}$ and we sum over all the pairs $(S, \tau) \in \mathcal{Z}$ with the fixed $S$. Summing (4.4) with respect to $\tau'$ over all pairs $(S', \tau') \in \mathcal{Z}$ with the fixed $S'$, we obtain
\[
\Pi_{S'} = \sum_{S \in \mathcal{S}} P_{S,S'} \Pi_S
\]
where $P_{S,S'} = \sum_{\tau'} p_{S,S',\tau'}$. As the stationary distribution $\pi_Y$ of the Markov chain $Y_k = X_{T_1 + \cdots + T_k}$ satisfies the same equation with the same transition probabilities $P_{S,S'}$ and the same normalization condition, the uniqueness implies $\Pi = \pi_Y$. Hence, (4.4) is equivalent to
\[
(\pi_Z)_{S',\tau'} = \sum_{S \in \mathcal{S}} (\pi_Y)_S \cdot p_{S,S',\tau'}
\]
and, furthermore, the mean $T_* = E_{\pi_Z}[T_k]$ of the time $T_k$ satisfies
\[
E_{\pi_Z}[T_k] = \sum_{S \in \mathcal{S}} (\pi_Y)_S \sum_{(S',\tau') \in \mathcal{Z}} \tau' \cdot p_{S,S',\tau'}.
\]
That is, we average the time required to create all the triangles (3.5) (that is, the transition time from $Y_k$ to $Y_{k+1}$) over the stationary distribution $\pi_Y$ of the initial state $Y_k = S$ (regardless of the destination state $Y_{k+1} = S'$). Similarly,
\[
E_{\pi_Z} \left[ \frac{1}{1 + T_k} \right] = \sum_{S \in \mathcal{S}} (\pi_Y)_S \sum_{(S',\tau') \in \mathcal{Z}} \frac{p_{S,S',\tau'}}{1 + \tau'}.
\]
Now, we are ready to formulate the main result.

**Theorem 4.2.** Assume that the Markov chain $Y_k = S_{T_1 + \cdots + T_k}$ induced by the input process $X_k$ in the Preisach state space $\mathcal{S}$ is regular. Then the error of the iterative scheme (2.1) exponentially decreases and the rate of the exponential convergence (3.4) satisfies the estimate
\[
\lambda \geq \frac{E_{\pi_Z} \left[ \frac{1}{1 + T_k} \right]}{32 N_0 \cdot E_{\pi_Z}[T_k]} > 0
\]
where $N_0$ is given by (3.9) and $E_{\pi_Z}[T_k]$, $E_{\pi_Z} \left[ \frac{1}{1 + T_k} \right]$ are defined by (4.5), (4.6).

According to (4.7), the lower bound for the rate of the exponential convergence is controlled by the average time that the input process takes to create all the triangles on the half-plane $\Pi$. One can assume that the shorter is this time, the faster is the convergence. We will test this conjecture with numerical examples in the next section. Also, the lower bound decreases as $O(N^{-4})$ with the increasing density of the mesh on the half-plane $\Pi$.

**Proof of Theorem 4.2.** From $T_k \geq 1$ it follows that $E_{\pi_Z} \left[ \frac{1}{1 + T_k} \right] < \infty$, hence Lemma 4.1 implies relation (4.3) [27].

Let us show that also $E_{\pi_Z}[T_k] < \infty$. Indeed, we assumed that the Markov chain $S_k$ starting from any initial state $S_0$ almost surely creates all the triangles (3.5) in finite time. Therefore, there is an integer $\tau_*$ such that the process $S_k$ creates all the triangles (3.5) within the first $\tau_*$ time steps with a positive probability $p_{S_0} \geq p_* > 0$. 

for any \( S_0 \in \mathcal{S} \). Hence, the probability (4.1) satisfies \( p_{S,S',\tau'} \leq (1 - p_\ast)^n \) for every \( \tau' \geq n\tau_\ast \) and every \( S,S' \in \mathcal{S} \) and consequently formula (4.5) implies \( E_{\pi_Z}[T_k] < \infty \).

As the mean \( E_{\pi_Z}[T_k] \) of the times \( T_k \) is finite, using Lemma 4.1 we conclude that (4.2) is valid.

Finally, let us see that (4.8) holds for
\[
\frac{1}{k} \sum_{i=1}^{m(k)} T_i \overset{a.s.}{\rightarrow} 1
\]

where \( m = m(k) \) denotes the largest integer such that \( k \geq T_1 + \cdots + T_m \). For any \( \varepsilon > 0 \), the estimate
\[
\liminf_{k \to \infty} \frac{1}{k} \sum_{i=1}^{m(k)} T_i < 1 - \varepsilon
\]

implies the existence of a subsequence \( T_{i_n} \) satisfying
\[
T_1 + \cdots + T_{i_n} \leq \frac{T_{i_n + 1}}{\varepsilon}.
\]

Since \( p_{S,S',\tau'} \leq (1 - p_\ast)^{\tau' / \tau_\ast - 1} \) for every positive integer \( \tau' \) and every \( S,S' \in \mathcal{S} \), the probability that a sequence \( T_i \) contains at least one element \( T_i = \tau' \) and, simultaneously, satisfies \( \varepsilon(T_1 + \cdots + T_m) \leq T_{i+1} = \tau' \) can be estimated as follows:
\[
P(T_{i+1} = \tau', T_1 + \cdots + T_i \leq T_{i+1} / \varepsilon \text{ for at least one } i) \leq \frac{\tau'}{\varepsilon} (1 - p_\ast)^{\tau' / \tau_\ast - 1}.
\]

Therefore,
\[
P(T_{i+1} \geq \tau', T_1 + \cdots + T_i \leq T_{i+1} / \varepsilon \text{ for at least one } i) \leq \sum_{j=\tau'}^{\infty} \frac{j}{\varepsilon} (1 - p_\ast)^{j / \tau_\ast - 1}.
\]

As the right hand side of this expression tends to zero with \( \tau' \to \infty \) and \( T_{i_n} \to \infty \) in (4.10), we conclude that relationship (4.9) holds with zero probability for each \( \varepsilon > 0 \).

This proves (4.8).

Now, estimate (3.8) can be rewritten as
\[
\lambda \geq \frac{1}{32N_0} \liminf_{k \to \infty} \left( \frac{1}{k} \sum_{i=1}^{m(k)} T_i \right) \left( \frac{1}{m(k)} \sum_{i=1}^{m(k)} T_i \right)^{-1} \left( \frac{1}{m(k)} \sum_{i=1}^{m(k)} \frac{1}{1 + T_i} \right).
\]

Combining this formula with relations (4.2), (4.3) and (4.8), and taking into account that \( m(k) \to \infty \) as \( k \to \infty \) almost surely (because the Markov chain \( S_{T_1 + \cdots + T_k} \) is, by assumption, well defined), we obtain (4.7).

5. Numerical examples. In this section, we consider two examples of the input process and evaluate the convergence rate of the norm \( \| E_k \| \) of the error for the iterative scheme (2.1), and that depending on the quantities \( T_\ast \) and \( N_0 \) as in (3.9) and (4.2), respectively. These are the two factors that affect the convergence rate guaranteed by Theorem 4.2. The following numerical computations were performed using the discrete dynamic Preisach (DDP) modeling algorithm which is suitable for
realization of the iterative scheme (2.1). The DDP algorithm has been introduced in [25] and applied for inverse hysteresis control in [26].

As the density function $\rho(\alpha, \beta)$ of the Preisach model, we used the $\delta$-function $\rho = \delta(p - p_0)$ located at the point $p_0 = (\alpha, \beta) = (N\varepsilon/2, N\varepsilon/2)$. The convergence in this case is relatively slow, because the algorithm makes a nonzero correction to the approximating density $\hat{\rho}$ only when the switching region contains the point $p_0$. Moreover, the correction is each time spread over multiple (at least $(N - 1)/2$) nodes. The initial approximation of the scheme was the uniform density $\hat{\rho}_0(\alpha, \beta) = a$ with a small positive $a$.

Two meshes with $N^I = 31$ and $N^{II} = 101$ were considered (the odd number is conditioned by the numerical implementation of DDP algorithm). The number of nodes for these meshes, $M^I = 496$ and $M^{II} = 5151$, differs by one order of magnitude.

5.1. Stochastic inputs. We will consider two types of the input Markov chain $X_k$. The first input denoted by $X^A_k$ is the Bernoulli (memoryless) process:

$$P(X^A_k = i\varepsilon) = \frac{1}{N + 1}, \quad i = 0, \ldots, N.$$ 

The second input process denoted by $X^B_k$ is a modification of the Bernoulli process where the sign of the input increment alternates at each time step:

$$P(X^B_{k+1} = j\varepsilon|X^B_k = i\varepsilon) = \begin{cases} 0, & j < i \\ \frac{j}{N + 1}, & j \geq i \end{cases} \quad \text{for odd } k;$$

$$P(X^B_{k+1} = j\varepsilon|X^B_k = i\varepsilon) = \begin{cases} \frac{j}{N + 1}, & j \leq i \\ 0, & j > i \end{cases} \quad \text{for even } k,$$

where $i, j = 0, \ldots, N$. Thus, a specific feature of this process is that it produces continuously oscillating input sequences. Note that both processes $A$ and $B$ have no knowledge of dynamics in the state space $S$ of the Preisach model.

We now show that the above two stochastic input processes satisfy the conditions of Theorem 4.2.

**Lemma 5.1.** The Markov chain $Y_k = S_{T_1 + \ldots + T_k}$ induced by each of the processes $A$ and $B$ in the Preisach state space $S$ is regular.

**Proof.** A state $S$ of the Preisach model will be identified with a polyline connecting the point $(\alpha, \beta) = (0, N\varepsilon)$ with the line $\alpha = \beta$ on the half-plane $\Pi$. This polyline consists of vertical and horizontal links where the length of each link is a multiple of $\varepsilon$. The relays are in state 1 below the polyline and in state 0 above the polyline [18].

The following finite sequence of input values will be called a *standard sequence*:

$$0, \; x_1, \; 0, \; x_2, \; x_1, \; 0, \; x_2, \; 0, \; x_3, \; 0, \; x_4, \; x_1, \; 0, \; x_4, \; x_2, \; x_3, \; 0, \; x_4, \; \ldots,$$

$$0, \; x_N, \; x^1, \; 0, \; x_N, \; x_2, \; 0, \; x_N, \; x_3, \; \ldots; \; 0, \; x^N, \; x^{N-1}, \; 0, \; x^N, \; 0,$$

where $x_i = i\varepsilon$. This input sequence produces all the possible triangles (3.5). By definition, each of the processes $A$ and $B$ has a positive transition probability from
any state $x_i$ to any state $x_j$. Therefore, for any given initial state, the input makes the standard sequence of steps and creates all the triangles with a positive probability. Hence, the process $Y_k = S_{T_1} + \cdots + T_k$ in the Preisach state space is well defined (that is the times $T_i$ are almost surely all finite).

Now, it suffices to show that, given any pair of states $S', S'' \in S$, the transition probability of the Markov chain $Y_k = S_{T_1} + \cdots + T_k$ from $S'$ to $S''$ is positive. Denote by $P_0', \ldots, P_K'$, the corners of the polyline $S'$ and by $P_0'', \ldots, P_K''$, the corners of the polyline $S''$, where $P_0'' = P_0'' = (0, \varepsilon N)$, $\alpha_i''' = \beta_i'''$, $\alpha_i''' = \beta_i'''$, and we use the notation $P_i' = (\alpha_i', \beta_i')$, $P_i'' = (\alpha_i'', \beta_i'')$. For any state $S'' = P_0'' P_1'' \cdots P_K''$, $\in S$ (except for the state $S'' = \{(\alpha, \beta) : \alpha = 0, \beta \in [0, N\varepsilon]\}$ consisting of one vertical link), we define an input sequence associated with $S''$; this sequence is given by:

- $\beta_2', \alpha_3', \beta_4', \alpha_5', \ldots, \alpha_{K''-1}', \beta_{K''}'$ if the link $P_0'' P_1''$ is vertical and the link $P_{K''-1}'' P_{K''}'$ is horizontal;
- $\beta_2', \alpha_3', \beta_4', \alpha_5', \ldots, \alpha_{K''-1}', \alpha_{K''}'$ if both the links $P_0'' P_1''$ and $P_{K''-1}'' P_{K''}'$ are vertical;
- $\beta_2', \alpha_3', \beta_4', \alpha_5', \ldots, \beta_{K''-1}', \beta_{K''}'$ if both the links $P_0'' P_1''$ and $P_{K''-1}'' P_{K''}'$ are horizontal;
- $\beta_2', \alpha_3', \beta_4', \alpha_5', \ldots, \beta_{K''-1}', \alpha_{K''}'$ if the link $P_0'' P_1''$ is horizontal and the link $P_{K''-1}'' P_{K''}'$, $\in S''$ induces the transition of the process $S_k$ from the state consisting of one vertical link to the state $S''$.

Depending on a few features of the states $S', S''$, we will define explicitly a finite sequence of input values, which induces the transition of the Markov chain $Y_k$ from the state $Y_k = S'$ to the state $Y_{k+1} = S''$; the probability of such transition is positive, because the input sequence is finite. We call such an input sequence a transition input sequence.

First, consider any state $S' = P_0' P_1' \cdots P_K'$, which has a vertical link $P_{K'-1}' P_K'$. Consider the modified standard sequence (5.1) where we replace every instance of 0, except for the first and the last one, with the pair of entries 0, $x_1$:

(5.2)

\[
\begin{align*}
0, & \quad x_1, \\
0, & \quad x_2, \quad x_1, \quad 0, \quad x_1, \quad x_2, \\
0, & \quad x_3, \quad x_1, \quad 0, \quad x_1, \quad x_3, \quad x_2, \quad 0, \quad x_1, \quad x_3, \\
& \vdots \\
0, & \quad x_N, \quad x_1, \quad 0, \quad x_1, \quad x_N, \quad x_2, \quad 0, \quad x_1, \quad x_N, \quad x_3, \quad \cdots, \quad 0, \quad x_1, \quad x_N, \\
& \quad 0.
\end{align*}
\]

- The input sequence (5.2) induces the transition of the Markov chain $Y_k = S_{T_1} + \cdots + T_k$ from the state $S'$ to the state $S'' = \{(\alpha, \beta) : \alpha = 0, \beta \in [0, N\varepsilon]\}$ consisting of one vertical link.
- If the polyline $S'' = P_0'' P_1'' \cdots P_K''$, has more than one link and its link $P_{K''-1}'' P_{K''}'$ is vertical, then a transition input sequence from $S'$ to $S''$ is the sequence (5.2), where we omit the successive four entries $x_1, \beta_{K''-1}'', \beta_{K''}'', 0$, which is followed by the input sequence associated with the state $S''$.
- If the polyline $S'' = P_0'' P_1'' \cdots P_K''$, has more than two links and its link $P_{K''-1}'' P_{K''}'$ is horizontal, then a transition input sequence from $S'$ to $S''$ is the sequence (5.2), where we omit the successive four entries $x_1, \alpha_{K''}', \alpha_{K''-1}'', 0$, which is followed by the input sequence associated with the state $S''$.
- If $S'' = P_0'' P_1'' = \{(\alpha, \beta) : \alpha \in [0, N\varepsilon], \beta = N\varepsilon\}$ (i.e., $S''$ consists of one
horizontal link), or if \( S'' = P''_0 P''_1 P''_2 \) has two links and the link \( P''_1 P''_2 \) is horizontal and has the length of at least \( 2\varepsilon \), then a transition input sequence from \( S' \) to \( S'' \) is the sequence (5.2), where we omit the successive three entries \( x_1, \alpha''_{K''}, 0 \), which is followed by the input sequence associated with the state \( S'' \).

- Finally, if \( S'' = P''_0 P''_1 P''_2 \) has two links and the link \( P''_1 P''_2 \) has the length \( \varepsilon \) (i.e., \( P''_1 P''_2 = \{(\alpha, \beta) : \alpha \in [0, \varepsilon], \beta = \varepsilon\} \)), then a transition input sequence from \( S' \) to \( S'' \) is the standard sequence (5.1), where we omit the first two entries, 0, \( x_1 \), and add an extra entry \( x_1 \) after the last 0.

These cases define transitions with positive probability to all the destination states \( S'' \in S \) from all the initial states \( S' = P'_0 P'_1 \cdots P'_K \) with the horizontal link \( P'_{K' - 1} P'_{K'} \). In each case, the transition input sequence creates all the triangles (3.5); the last triangle is created at the last step after which the destination state \( S'' \) is achieved.

The transition sequences from \( S' \) to \( S'' \) are similar in the complementary case when the link \( P'_{K' - 1} P'_{K'} \) of the initial state \( S' \) is horizontal. In this case, a transition sequence consists of the values \( x_N, a, 0 \) followed by the transition sequence defined above for the transition to the state \( S'' \) (that is, we just add three entries at the beginning of the transition sequence). Here \( a \) is any input value satisfying \( a \neq 0, a \neq x_N, a \neq \alpha''_{K''} \).

5.2. Comparison of convergence rates. First we compare the convergence rates of \( \|E_k\| \) for the processes \( X^A \) and \( X^B \). Figure 2 presents simulation results shown by thick lines and exponential fits of the form \( \|E_k\| = c \exp(-k\delta) \) (thin lines). For both input processes, the scheme (2.1) clearly demonstrates exponential convergence. The free parameters \( c \) and \( \delta \) of the fit are determined by using a standard nonlinear least-squares algorithm. Note that the exponential rate \( \delta \) is an empirical measure of the convergence rate \( \lambda \) estimated from below in Theorems 3.1 and 4.2. According to Theorem 4.2, \( \lambda \geq T_{**}/(32N_0T_*) \). We have estimated the time average \( T_\star \) of the times \( T_k \) and the time average \( T_{**} \) of the inverse times \( 1/(1 + T_k) \) (cf. (4.2), (4.3)) for the simulations shown in Figure 2. We found that \( T_{**} \approx 1/T_\star \), hence \( \lambda \geq \mu := 1/(32N_0T_*^2) \). In particular, the lower estimate \( \mu \) of the convergence rate increases with the decrease of the average time \( T_\star \) required by an input process to
create all the triangles (3.5). Figure 2 shows that, in line with this observation, the actual empirical convergence rate $\delta$ is also higher for the process $A$, which has smaller characteristic time $T_*$ than the process $B$. Table 1 compares the ratio $\delta_A/\delta_B$ of the empirical convergence rates for processes $A$ and $B$ with the ratio of the theoretical lower estimates $\mu_A/\mu_B \approx (T_B^*)^2/(T_A^*)^2$ of these rates guaranteed by Theorem 4.2.

| Input     | $N$ | $T_A^*$ | $T_B^*$ | $\delta_A/\delta_B$ | $(T_B^*)^2/(T_A^*)^2$ |
|-----------|-----|---------|---------|----------------------|------------------------|
| $X^A$, $X^B$ | 31  | 15778   | 25934   | 1.64                 | 2.70                   |

The convergence of the error of iterative scheme (2.1) for processes $X^A$ and $X^B$ is further compared with that obtained using a deterministic input, denoted as $X^C$, which constructs successively all the triangles (3.5). This is the same input sequence as used by the classical deterministic identification algorithm of Mayersgoyz, but applied repeatedly as our algorithm is a recursive scheme. The input sequence is a concatenation of inputs generating a mesh of first-order descending (FOD) curves [18, 19]. The convergence rate $\delta^C$ for this input is higher than for stochastic inputs $X^A, B$ (as expected, because the deterministic input has the shortest time $T_*$ (which is also deterministic in this case, $T_k = T_*^C$ for all $k$).

Next, we compare the convergence rate for two different discretization meshes with $N^I = 31$ and $N^{II} = 101$ input states, respectively, and the corresponding meshes (3.1) on the Preisach operator half-plane $\Pi$. Figure 3 presents simulation results obtained for the Bernoulli input process (type $A$). For each of the two meshes we observe the exponential convergence of the error. The convergence is much faster for the mesh with a smaller number of nodes, $N^I$. This agrees with the fact that the theoretical lower bound of the convergence rate $\mu = T_*/(32N_0T_*)$ is inverse proportional to $N_0 = O(N^4)$ (cf. (3.9)). Table 2 compares the ratio $\delta^I/\delta^{II}$ of the empirical convergence rates for the two meshes with the ratio of the theoretical lower estimates $\mu^I/\mu^{II} \approx (N^{II})^4/(N^I)^4$ of these rates guaranteed by Theorem 4.2.
Table 2
Evaluated features of convergence rates for two meshes $N^I$ and $N^{II}$.

| Input | $N^I$ | $N^{II}$ | $\delta^I/\delta^{II}$ | $(N^{II})^4/(N^I)^4$ |
|-------|-------|----------|------------------------|-----------------------|
| $X^A$ | 31    | 101      | 16.40                  | 112.68                |

6. Conclusions. The classical deterministic identification algorithm of Mayergoz recovers the density function of the Preisach model from input-output data that are obtained by testing the system with a series of deterministic test inputs which create a sufficiently dense mesh of first-order descending hysteresis curves. In this work, we have analyzed a recursive iterative identification algorithm that uses an input-output data stream generated by a random input process. This algorithm updates the values of the density function in the switching region at each time step. We have shown that the error converges to zero exponentially and obtained an explicit estimate of the convergence rate for a wide class of stochastic input processes. The rate of convergence guaranteed by this estimate depends on the target accuracy of the algorithm and on a certain stopping time $T$, which is a characteristic of the input process. Essentially, during the time interval of length $T$ the input creates an input-output data set that contains sufficient information for recovering the density function. We have tested the recursive algorithm with two examples of stochastic input processes and the deterministic input sequence prescribed by the Mayergoz algorithm and evaluated the rate of convergence numerically. We have found that the convergence rate decreases with increasing mean value of $T$ and the number of nodes in the domain of the density function, which is in agreement with our theoretical estimate of the convergence rate.

REFERENCES
[1] P. Andrei, L. Oniciuc, A. Stancu, and L. Stoleriu, Identification techniques for phenomenological models of hysteresis based on the conjugate gradient method, J. Magn. Magn. Mater., 316 (2007), pp. 330–333.
[2] B. Appelbe, D. Flynn, H. McNamara, P. O’Kane, A. Pimenov, A. Pokrovskii, D. Rachinskii, and A. Zhezherun, Rate-independent hysteresis in terrestrial hydrology, IEEE Control Syst. Mag., 29 (2009), pp. 44–69.
[3] B. Appelbe, D. Rachinskii, and A. Zhezherun, Hopf bifurcation in a van der Pol type oscillator with magnetic hysteresis, Phys. B, 403 (2008), pp. 301–304.
[4] M. Brokate, A. Pokrovskii, and D. Rachinskii, Asymptotic stability of continuum sets of periodic solutions to systems with hysteresis, J. Differential Equations, 319 (2006), pp. 94–109.
[5] M. Brokate, A. Pokrovskii, D. Rachinskii, and O. Rasskazov, Differential equations with hysteresis via a canonical example, in The Science of Hysteresis, I. Mayergoz and G. Bertotti, eds., vol. I, chap. II, Elsevier, Academic Press, 2005, pp. 125–291.
[6] M. Brokate and J. Sprekels, Hysteresis and Phase Transitions, Springer, Berlin, 1996.
[7] R. Cross, H. McNamara, A. Pokrovskii, and D. Rachinskii, A new paradigm for modelling hysteresis in macroeconomic flows, Phys. B, 403 (2008), pp. 231–236.
[8] D. Davino, P. Krejčí, and C. Visoné, Fully coupled modeling of magneto-mechanical hysteresis through thermodynamic compatibility, Smart Mater. Struct., 22 (2013), 095009.
[9] P. Diamond, D. Rachinskii, and M. Yumagulov, Stability of large cycles in a nonsmooth problem with Hopf bifurcation at infinity, Nonlinear Anal., 42 (2000), pp. 1017–1031.
[10] M. Dimian and P. Andrei, Noise-Driven Phenomena in Hysteretic Systems (Signals and Communication Technology), Springer, 2013.
[11] R. Iyer and X. Tan, Control of hysteretic systems through inverse compensation, IEEE Control Syst. Mag., 29 (2009), pp. 83–99.
[12] M. A. Krasnosel’skii and A. V. Pokrovskii, Systems with Hysteresis, Springer, New York,
REFERENCES

[10] A. Krasnosel’skii and D. Rachinskii, Bifurcation of forced periodic oscillations for equations with Preisach hysteresis, J. Phys.: Conf. Ser., 22 (2005), pp. 93–102.

[11] A. Krasnosel’skii and D. Rachinskii, On a bifurcation governed by hysteresis nonlinearity, NoDEA Nonlinear Differential Equations Appl., 9 (2002), pp. 93–115.

[12] P. Krejčí, J. P. O’Kane, A. Pokrovskii, and D. Rachinskii, Stability results for a soil model with singular hysteretic hydrology, J. Phys.: Conf. Ser., 268 (2011), 012016.

[13] P. Krejčí, J. P. O’Kane, A. Pokrovskii, and D. Rachinskii, Properties of solutions to a class of differential models incorporating Preisach hysteresis operator, Phys. D, 241 (2012), pp. 2010–2028.

[14] P. Krejčí and K. Kuhnen, Compensation of complex hysteresis and creep effects in piezoelectrical actuated systems - a new Preisach modeling approach, IEEE Trans. Autom. Control, 54 (2009), pp. 537–550.

[15] I. D. Mayergoyz, Mathematical Models of Hysteresis and their Application, Elsevier, Academic Press, 2003.

[16] The Science of Hysteresis, I. Mayergoyz and G. Bertotti, eds., Elsevier, Academic Press, 2005.

[17] A. Pimenov, T. C. Kelly, A. Korobeinikov, M. J. O’Callaghan, A. V. Pokrovskii, and D. Rachinskii, Memory effects in population dynamics: spread of infectious disease as a case study, Math. Model. Nat. Phenom., 7 (2012), pp. 204–226.

[18] A. Pimenov and D. Rachinskii, Linear stability analysis of systems with Preisach memory, Discrete Contin. Dyn. Syst. Ser. B, 11 (2009), pp. 997–1018.

[19] F. Preisach, Über die magnetische Nachwirkung (in German), Z. Physik, 94 (1935), pp. 277–302.

[20] D. Rachinskii, Asymptotic stability of large-amplitude oscillations in systems with hysteresis, NoDEA Nonlinear Differential Equations Appl., 6 (1999), pp. 267–288.

[21] M. Ruderman, Direct recursive identification of the Preisach hysteresis density function, J. Magn. Magn. Mater., 348 (2013), pp. 22–26.

[22] M. Ruderman and T. Bertram, Discrete Dynamic Preisach Model for Robust Inverse Control of Hysteresis Systems, 49th IEEE Conference on Decision and Control, 2010, pp. 3463-3468.

[23] M. Ruderman and T. Bertram, Control of magnetic shape memory actuators using observer-based inverse hysteresis approach, IEEE Trans. Control Syst. Technol., 22 (2014), pp. 1181-1189.

[24] R. Serfozo, Basics of Applied Stochastic Processes, Springer, 2009.

[25] X. Tan and J. Baras, Adaptive identification and control of hysteresis in smart materials, IEEE Trans. Autom. Control, 50 (2005), pp. 827-839.

[26] A. Visintin, Differential Models of Hysteresis, Springer, Berlin, 1994.