A Note On Fuzzy Space and Topology

B.G. Sidharth
International Institute for Applicable Mathematics & Information Sciences
Hyderabad (India) & Udine (Italy)
B.M. Birla Science Centre, Adarsh Nagar, Hyderabad - 500 063 (India)

Abstract

We consider the problem of distance between two particles in the universe, where space is taken to be Liebnizian rather than Newtonian, this being the present day approach. We then argue that with latest inputs from physics, it is possible to define such a distance in a topological sense.

1 Introduction

We would like to give a statement of the problem in terms of an actual physical system which is the focus of attention. Increasingly the Newtonian view of a smooth background space which acts as a container in which the events in the universe take place is giving way to the view of Liebnitz in which the contents of the universe themselves give rise to space [1][2]. So we consider the universe as containing a (finite) number of elementary particles denoted by \( N \). In fact \( N \) has been taken to be of the order \( 10^{80} \). The next input is the fact that these \( N \) particles are ill defined to within an extent of the Compton scale (Cf.refs.[3][2]). All this in present physics leads to a noncommutative geometry, which is again symptomatic of the non differentiable nature of spacetime in recent studies (Cf. also ref.[4] for a discussion). Indeed space now becomes fuzzy - the points are ill defined in the minimum intervals [5][6].

Now, the question is can we define a metric for such a set of particles or elements?
2 Topological Considerations

In earlier work (Cf.ref.[2]) we had argued as follows: When we talk of a metric or the distance between two "points" or "particles", a concept that is implicit is that of topological "nearness" - we require an underpinning of a suitably large number of "open" sets [7]. Let us now abandon the absolute or background spacetime and consider, for simplicity, a Universe (or set) that consists solely of two particles. The question of the distance between these particles (quite apart from the question of the observer) becomes meaningless from the point of view of physical space. Indeed, this is so for a Universe consisting of a finite number of particles. For, we could isolate any two of them, and the distance between them would have no meaning. We can intuitively appreciate that we would in fact need distances of intermediate or more generally, other points.

In earlier work[8, 9], motivated by physical considerations we had considered a series of nested sets or neighborhoods which were countable and also whose union was a complete Hausdorff space. The Urysohn Theorem was then invoked and it was shown that the space of the subsets was metrizable. Let us examine this in more detail.

Firstly we observe that in the light of the above remarks, the concepts of open sets, connectedness and the like reenter in which case such an isolation of two points would not be possible. More formally let us define a neighborhood of a particle (or point or element) \( A \) of a set of particles as a subset which contains \( A \) and at least one other distinct element. Now, given two particles (or points) or sets of points \( A \) and \( B \), let us consider a neighborhood containing both of them, \( n(A, B) \) say. We require a non empty set containing at least one of \( A \) and \( B \) and at least one other particle \( C \), such that \( n(A, C) \subseteq n(A, B) \), and so on. Strictly, this "nested" sequence should not terminate. For, if it does, then we end up with a set \( n(A, P) \) consisting of two isolated "particles" or points, and the "distance" \( d(A, P) \) is meaningless.

We now assume the following property[8, 9]: Given two distinct elements (or even subsets) \( A \) and \( B \), there is a neighborhood \( N_{A_1} \) such that \( A \) belongs to \( N_{A_1} \), \( B \) does not belong to \( N_{A_1} \), and also given any \( N_{A_1} \), there exists a neighborhood \( N_{A_2} \) such that \( A \subseteq N_{A_2} \subseteq N_{A_1} \), that is there exists an infinite topological closeness.

From here, as in the derivation of Urysohn’s Lemma[10], we could define a mapping \( f \) such that \( f(A) = 0 \) and \( f(B) = 1 \) and which takes on all...
intermediate values. We could now define a metric, \( d(A, B) = |f(A) - f(B)| \).
We could easily verify that this satisfies the properties of a metric.

With the same motivation we will next deduce a similar result, but with
different conditions. In the sequel, by a subset we will mean a proper subset,
which is also non null, unless specifically mentioned to be so. We will also
consider Borel sets, that is the set itself (and its subsets) has a countable
covering with subsets. We then follow a pattern similar to that of a Cantor
ternary set \([7, 11]\). So starting with the set \( N \) we consider a subset \( N_1 \) which
is one of the members of the covering of \( N \) and iterate this process so that
\( N_{12} \) denotes a subset belonging to the covering of \( N_1 \) and so on.

We note that each element of \( N \) would be contained in one of the series of
subsets of a sub cover. For, if we consider the case where the element \( p \)
belongs to some \( N_{12-k} \) but not to any \( N_{1,2,3,\ldots,K} \), this would be impossible
because the latter form a cover of the former. In any case as in the derivation
of the Cantor set, we can put the above countable series of sub sets of sub
covers in a one to one correspondence with suitable sub intervals of a real
interval \((a, b)\).

**Case I**

If \( N_{1,2,3,\ldots,m} \rightarrow \) an element of the set \( N \) as \( m \rightarrow \infty \), that is if the set is closed,
we would be establishing a one to one relationship with points on the interval
\((a, b)\) and hence could use the metric of this latter interval, as seen earlier.

**Case II**

It is interesting to consider the case where in the above iterative countable
process, the limit does not tend to an element of the set \( N \), that is set \( N \) is
not closed and has what we may call singular points. We could still truncate
the process at \( N_{1,2,3,\ldots,m} \) for some \( m > L \) arbitrary and establish a one to one
relationship between such truncated subsets and arbitrarily small intervals in
\( a, b \). We could still speak of a metric or distance between two such arbitrarily
small intervals.

This case which may be termed ”Fuzzy Topology”, is of interest because of
our description of elementary particles in terms of fuzzy spacetime (Cf. also
ref.\([12]\)), where we have a length of the order of the Compton wavelength as
seen in the previous section, within which spacetime as we know it breaks
down. Such cut offs lead to a non commutative geometry and what may be
called fuzzy spaces\([2, 4, 5, 6]\).

To put it another way, in view of the fact that the number of particles or
points or elements of the universe, which is the set under consideration is
finite, it may be pointed out that the sequence \( N_{1,2,3,\ldots,m} \) would terminate for
some finite $L_1$, let us say. We could still identify these sub sets, sub covers etc. with open sets, sub sets etc. of some manifold $M$ which has interminable sequences, so that a metric by the previous arguments would exist for $M$. We could then truncate the sub sets, $N_{1,2,3...m}$ for some $L_1$ as above, keeping in view the fact that each particle is not defined to within its Compton scale, and the fact that there are a finite number of particles. There would be thus a sequence $L_1, L_2, \cdots L_N$ for each of the $N$ particles of the universe. We could consider $L = \inf(L_1, L_2$ etc), $L$ representing some fundamental irreducible scale like the Compton scale. What happens within the scale is in any case ill defined characteristic of the fuzzyness of space.

We could reformulate the above problem in the following simpler fashion. Let us take a set $S$ of $N$ open sets and put the $N$ particles of the universe in a one to one correspondence with the open sets of $S$. As $N$ is finite, there are a finite number of geometric transformations in the spirit of Smale’s original transformations, which take $S$ to $S'$ such that $S'$ is a metrizable set of open sets. In this process any overlaps of the points of the images of open sets of $S$, is not important in view of the fuzzyness associated with the Compton scale of the $N$ particles. So we have a one to one correspondence between the $N$ particles of the universe with the sub sets of a metrizable set $S'$, which in fact is all that is required.

To give a simple example, let the universe consist of just two particles, $A_1$ and $A_2$. By earlier arguments the distance between them would be ill defined (in a physical sense, not an artificial mathematical sense). We associate with $A_1$ and $A_2$ two open sets $s_1$ and $s_2$ of $S$. Then we consider a transformation $S$ to $S'$ such that the images $s_1'$ and $s_2'$ form a set that satisfies standard properties such as connected so that we can define a distance or metric. This is taken over to be a ”distance” between $A_1$ and $A_2$ remembering that in any case the points within a fundamental interval around them are indeterminate. Finally, it may be remarked that if $N$ is countable (infinite) then there are elementary topologies that can be formed (Cf.ref.[13]).

3 Remarks

We would like to reiterate the following. Interestingly, we usually consider two types of infinite sets - those with cardinal number $n$ corresponding to countable infinities, and those with cardinal number $c$ corresponding to a continuum, there being nothing in between [11]. This is the well known but
unproven Continuum hypothesis.
What we have shown with the above process is that it is possible to conceive of an intermediate possibility with a cardinal number $n^p, p > 1$.
In the above considerations three properties are important: Firstly the set must be closed i.e. it must contain all its limit points. Secondly, it must be perfect i.e. in addition, each of its points must be a limit point. Finally it must be disconnected i.e. it contains no non null open intervals. Only the first was invoked in Case I.
We notice that there is the holistic feature. A metric emerges by considering large encompassing sets. Finally, we could deviate from a strict mathematical analysis and introduce an element of physics. We could say that a point or particle $B$ would be in a neighborhood of another point or particle $A$, only if $A$ and $B$ interact”. Thus the universe would consist of a network of ”interacting” particles, reminiscent of the Feynman-Wheeler perfect absorber model encountered.

References

[1] Lucas, J.R. (1984). *Space Time, And Causality* (Oxford Clarendon Press).

[2] Sidharth, B.G. (2008). *The Thermodynamic Universe* (World Scientific, Singapore).

[3] Sidharth, B.G. *The Limits of Special Relativity* to appear in *Foundations of Physics Letters*.

[4] Sidharth, B.G. (2005). *The Universe of Fluctuations* (Springer, Netherlands).

[5] Madore, J. (1995). *An Introduction to Non-Commutative Differential Geometry* (Cambridge University Press, Cambridge).

[6] Madore, J. (1992) *Class.Quantum Grav.* 9, p.69-87.

[7] Simmons, G.F. (1965). *Introduction to Topology and Modern Analysis* (McGraw Hill Book Co. Inc., New York), pp.135.

[8] Altaisky, M.V. and Sidharth, B.G. (1999). *Chaos, Solitons and Fractals* Vol. 10, No.2-3, pp.167-176.
[9] Sidharth, B.G. (2002). Chaos, Solitons and Fractals 14, pp.167–169.

[10] Munkres, J.R. (1988). Topology (Prentice Hall India, New Delhi).

[11] Gullick, D. (1997). Encounters With Chaos (McGraw Hill, New York), p.114ff.

[12] Sidharth, B.G. (2001). Chaotic Universe: From the Planck to the Hubble Scale (Nova Science, New York).

[13] Lipschutz, S. (1965). General Topology (McGraw Hill, Singapore), p.75.