L_∞-ALGEBRAS, CARTAN HOMOTOPIES AND PERIOD MAPS

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Abstract. We prove that, for every compact Kähler manifold, the period map of its Kuranishi family is induced by a natural $L_∞$-morphism. This implies, by standard facts about $L_∞$-algebras, that the period map is a “morphism of deformation theories” and then commutes with all deformation theoretic constructions (e.g. obstructions).

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**Introduction**

The homotopy Lie algebra approach to deformation theory over a field $\mathbb{K}$ of characteristic 0 is based on two creeds.

First creed: let $\mathcal{M}$ be a moduli space (for some classification problem) and $x$ a point of $\mathcal{M}$; then the geometry of the formal neighbourhood of $x$ in $\mathcal{M}$ is encoded into an (homotopy class of) $L_\infty$-algebra(s). More precisely, there exists an $L_\infty$-algebra $g$, defined up to quasiisomorphism, such that, for any local Artinian $\mathbb{K}$-algebras $B$ with maximal ideal $m_B$ and residue field $\mathbb{K}$ one has

$$\{\phi: \text{Spec}(B) \to \mathcal{M} \mid \phi(\text{Spec}(\mathbb{K})) = x\} = \text{Def}_g(B),$$

where

$$\text{Def}_g(B) = \frac{\{\text{Solutions of the Maurer-Cartan equation in } g \otimes_{\mathbb{K}} m_B\}}{\text{homotopy equivalence}}.$$  

Second creed: every "natural" morphism between formal pointed moduli spaces is induced by an $L_\infty$-morphism between the associated $L_\infty$-algebras.

In this paper we shall explicate the first creed for Grassmannians using a general construction that also applies to other moduli spaces (dg-Grassmannians, Quot and Hilbert schemes, Brill-Noether loci etc.) and the second creed for the universal period map, intended as a natural morphism of moduli spaces: from deformations of a compact Kähler manifold to deformations of the Hodge filtration of its De Rham cohomology.

More concretely, let’s denote by

$$\mathcal{X} \to (S, 0), \quad \mathcal{X} = \bigcup_{t \in S} X_t,$$

the Kuranishi family of a compact Kähler manifold $X = X_0$; let $p \geq 0$ be a fixed integer and consider the period map [33, 10.1.2]

$$\mathcal{P}^p: (S, 0) \to \text{Grass}(H^*(X, \mathbb{C})) = \prod_i \text{Grass}(H^i(X, \mathbb{C})),

\text{where } t \mapsto F^p H^*(X_t, \mathbb{C}) = \prod_i F^p H^i(X_t, \mathbb{C}).$$

Griffiths proved [11] that $\mathcal{P}^p$ is a holomorphic map and its differential $d\mathcal{P}^p$ is the same of the contraction map

$$i: H^1(X, T_X) \to \bigoplus_i \text{Hom} \left(F^p H^i(X, \mathbb{C}), \frac{H^i(X, \mathbb{C})}{F^p H^i(X, \mathbb{C})}\right), \quad i_\xi(\omega) = \xi \lrcorner \omega.$$  

It is also known [3, 21, 26] that obstructions to deformations of $X$ are contained in the kernel of

$$i: H^2(X, T_X) \to \bigoplus_i \text{Hom} \left(F^p H^i(X, \mathbb{C}), \frac{H^{i+1}(X, \mathbb{C})}{F^p H^{i+1}(X, \mathbb{C})}\right), \quad i_\xi(\omega) = \xi \lrcorner \omega$$

(this fact is known as Kodaira’s Principle ambient cohomology annihilates obstruction). However the proofs of [3, 21, 26] are not completely satisfying because the period map plays only a marginal role in them, while the most natural way of proving Kodaira’s Principle would be to show that the period map is a “morphism of deformation theories”, i.e. that $\mathcal{P}^p$ commutes with every deformation theoretic construction: for instance obstruction theories.

It is well known that the deformations of a compact complex manifold $X$ are governed
by the $L_\infty$-algebra underlying the Kodaira-Spencer algebra $K_X := \oplus_i A^{0,i}_X(T_X)$.

The main results of this paper are:

1. We explicitly describe an $L_\infty$-algebra $C^p$ such that
   \[ \{ \phi: \text{Spec}(B) \to \text{Grass}(H^*(X, \mathbb{C})) \mid \phi(\text{Spec}(\mathbb{C})) = F^p H^*(X, \mathbb{C}) \} = \text{Def}_{C^p}(B). \]

2. We explicitly describe a linear $L_\infty$-morphism $K_X \to C^p$ inducing $P^p$ at the level of deformation functors.

These results give us an algebraic description of the period map and imply, by general theory of $L_\infty$-algebras, that $P^p$ is a morphism of deformation theories.

The paper is divided in three parts: roughly speaking, in the first we make a functorial construction of an $L_\infty$ structure on the mapping cone of a morphism of differential graded Lie algebras and we introduce the notion of Cartan homotopy. A slightly expanded version of this part is available as [5].

In the second part we describe an $L_\infty$-algebra governing deformations of subcomplexes and then giving a local description of Grassmannians.

Finally in the third part we exhibit an $L_\infty$-morphism inducing the universal period map of a compact Kähler manifolds.

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Keywords and general notation.

We assume that the reader is familiar with the notion and main properties of differential graded Lie algebras and $L_\infty$-algebras (we refer to [6, 9, 14, 17, 18, 22] as introduction of such structures); however the basic definitions are recalled in this paper in order to fix notation and terminology.

For the whole paper, $K$ is a field of characteristic 0; every vector space is intended over $K$.

$\textbf{Art}$ is the category of local Artinian $K$-algebras with residue field $K$. For $A \in \textbf{Art}$ we denote by $m_A$ the maximal ideal of $A$.

Part 1. $L_\infty$ structures on mapping cones

Let $\chi: L \to M$ be a morphism of differential graded Lie algebras over a field $K$ of characteristic 0. In the paper [23] one of the authors has introduced, having in mind the example of embedded deformations, the notion of Maurer-Cartan equation and gauge action for the triple $(L, M, \chi)$; these notions reduce to the standard Maurer-Cartan equation and gauge action of $L$ when $M = 0$. More precisely there are defined two functors of Artin rings $MC_\chi, \text{Def}_\chi: \textbf{Art} \to \textbf{Set}$, where $\textbf{Art}$ is the category of local Artinian $K$-algebras with residue field $K$, in the following way:

\[ MC_\chi(A) = \left\{ (x, e^a) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid dx + \frac{1}{2} [x, x] = 0, \ e^a \ast \chi(x) = 0 \right\}, \]

where for $a \in M^0 \otimes m_A$ we denote by $ad_a: M \otimes m_A \to M \otimes m_A$ the operator $ad_a(y) = [a, y]$ and

\[ e^a \ast y = y + \sum_{n=0}^{+\infty} \frac{ad_a^n}{(n+1)!} ([a, y] - da) = y + \frac{e^{ad_a} - 1}{ad_a} ([a, y] - da), \quad y \in M^1 \otimes m_A. \]
Then one defines
\[ \text{Def}_\chi(A) = \frac{\text{MC}_\chi(A)}{\text{gauge equivalence}}, \]
where two solutions of the Maurer-Cartan equation are gauge equivalent if they belong to the same orbit of the gauge action
\[ (\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)) \times \text{MC}_\chi(A) \overset{\sim}{\longrightarrow} \text{MC}_\chi(A) \]
given by the formula
\[ (e^l, e^{dm}) \mapsto (e^l \ast x, e^{dm} e^{-\chi(l)}) = (e^l \ast x, e^{dm \ast (-\chi(l))}). \]
The \( \ast \) in the rightmost term in the above formula is the Baker-Campbell-Hausdorff multiplication; namely \( e^x e^y = e^{x+y}. \)
Several examples in [23] illustrate the utility of this construction in deformation theory. In the same paper it is also proved that \( \text{Def}_\chi \) is the truncation of an extended deformation functor [20, Def. 2.1] \( F \) such that \( T^i F = H(C_\chi) \), where \( C_\chi \) is the differential graded vector space \( C_\chi = (\oplus_i C^i_\chi, \delta), \)
\[ C^i_\chi = L^i \oplus M^{i-1}, \quad \delta(l, m) = (dl, \chi(l) - dm), \quad l \in L, m \in M. \]
By a general result [20, Thm. 7.1] there exists an \( L_\infty \) structure on \( C_\chi \), defined up to homotopy, whose associated deformation functor is isomorphic to \( \text{Def}_\chi \).

The main result of this part is to describe explicitly a canonical (and hence functorial) \( L_\infty \) structure on \( C_\chi \) with the above property; this is done in an elementary way, without using the theory of extended deformation functors, so that the result of this part can be used to reprove the main results of [23] without using [20].

1. Conventions on graded vector spaces

In this paper we will work with \( \mathbb{Z} \)-graded vector spaces; we write a graded vector space as \( V = \bigoplus_{n \in \mathbb{Z}} V^n \), and call \( V^n \) the degree \( n \) component of \( V \); an element \( v \) of \( V^n \) is called a degree \( n \) homogeneous element of \( V \).

We adopt the convention according to which degrees are ‘shifted on the left’. By this we mean that, for every integer \( n \), \( V[n] \simeq \mathbb{K}[n] \otimes V \) where \( \mathbb{K}[n] \) denotes the graded vector space consisting in the field \( \mathbb{K} \) concentrated in degree \( -n \). Note that, with this convention the canonical isomorphism \( V \otimes \mathbb{K}[n] \simeq V[n] \) is \( v \otimes 1_{[n]} \mapsto (-1)^{n \deg(v)} v_{[n]} \) and we have the following isomorphism, usually called decalage
\[ V_1[1] \otimes \cdots \otimes V_n[1] \overset{\sim}{\rightarrow} (V_1 \otimes \cdots \otimes V_n)[n], \]
\[ v_1[1] \otimes \cdots \otimes v_n[1] \mapsto (-1)^{\sum_{i=1}^n (n-i) \deg(v_i)} (v_1 \otimes \cdots \otimes v_n)[n]. \]

Denote by \( \bigotimes^n V, \bigodot^n V \) and \( \bigwedge^n V \) the \( n \)-th tensor, symmetric and exterior powers of \( V \) respectively. As with ordinary vector spaces, one can identify \( \bigodot^n V \) and \( \bigwedge^n V \) with suitable subspaces of \( \bigotimes^n V \), called the subspace of symmetric and antisymmetric tensors respectively. The decalage induces a canonical isomorphism
\[ \bigodot^n(V[1]) \overset{\sim}{\rightarrow} \left( \bigwedge^n V \right)[n]. \]

Remark 1.1. Using the natural isomorphisms
\[ \text{Hom}^i(V,W[l]) \simeq \text{Hom}^{i+l}(V,W) \]
and the decalage isomorphism, we obtain natural identifications
\[ \text{dec}: \text{Hom}^i \left( \bigotimes^k V, W \right) \xrightarrow{\sim} \text{Hom}^{i+k-l} \left( \bigotimes^k (V[1]), W[l] \right), \]
where
\[ \text{dec}(f)(v_1[1] \otimes \cdots \otimes v_k[1]) = (-1)^{ki + \sum_{j=1}^{k} (k-j) \text{deg}(v_j)} f(v_1 \otimes \cdots \otimes v_k)[l]. \]

By the above considerations
\[ \text{dec}: \text{Hom}^i \left( \bigwedge^k V, V \right) \xrightarrow{\sim} \text{Hom}^{i+k-1} \left( \bigwedge^k (V[1]), V[1] \right). \]

2. DIFFERENTIAL GRADED LIE ALGEBRAS AND $L_\infty$-ALGEBRAS

A differential graded Lie algebra (DGLA for short) is a Lie algebra in the category of graded vector spaces, endowed with a compatible degree 1 differential. More explicitly, it is the data \((V,d,[,])\), where \(V\) is a graded vector space, the Lie bracket \([,] : V \wedge V \to V\) satisfies the graded Jacobi identity:
\[ [v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{\text{deg}(v_1) \text{deg}(v_2)} [v_2, [v_1, v_3]], \]
and \(d : V \to V\) is a degree 1 differential, i.e.,
\[ d^2 = 0, \quad d[v_1, v_2] = [dv_1, v_2] + (-1)^{\text{deg}(v_1)} [v_1, dv_2]. \]

Via the decalage isomorphisms one can look at the Lie bracket of a DGLA \(V\) as a morphism
\[ q_2 \in \text{Hom}^1(V[1] \odot V[1], V[1]), \quad q_2(v_1[1] \odot w[1]) = (-1)^{\text{deg}(v)} [v, w][1], \]
Similarly, the suspended differential \(q_1 = d[1] = \text{id}_{V[1]} \otimes d\) is a morphism of degree 1
\[ q_1 : V[1] \to V[1], \quad q_1(v_1[1]) = - (dv_1)[1]. \]
Up to the canonical bijective linear map \(V \to V[1], v \mapsto v[1]\), the suspended differential \(q_1\) and the bilinear operation \(q_2\) are written simply as
\[ q_1(v) = -d v; \quad q_2(v \odot w) = (-1)^{\text{deg}(v)} [v, w]. \]

Define morphisms \(q_k \in \text{Hom}^1(\odot^k(V[1]), V[1])\) by setting \(q_k \equiv 0\), for \(k \geq 3\). The map
\[ \sum_{n \geq 1} q_n : \bigoplus_{n \geq 1} \bigotimes^n V[1] \to V[1] \]
extends to a coderivation of degree 1
\[ Q : \bigoplus_{n \geq 1} \bigotimes^n V[1] \to \left( \bigoplus_{n \geq 1} \bigotimes^n V[1] \right) \]
on the reduced symmetric coalgebra cogenerated by \(V[1]\), by the formula
\[ Q(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k, n-k)} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}, \]
where \( S(k, n-k) \) is the set of unshuffles and \( \varepsilon(\sigma) = \pm 1 \) is the Koszul sign, determined by the relation in \( \bigodot^n V[1] \)

\[
v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma)v_1 \odot \cdots \odot v_n.
\]
The axioms of differential graded Lie algebra are then equivalent to \( Q \) being a codifferential, i.e., \( Q^2 = 0 \). This description of differential graded Lie algebras in terms of the codifferential \( Q \) is called the Quillen construction \([25]\). By dropping the requirement that \( q_k \equiv 0 \) for \( k \geq 3 \) one obtains the notion of \( L_\infty \)-algebra (or strong homotopy Lie algebra), see e.g. \([17, 18, 14]\); namely, an \( L_\infty \) structure on a graded vector space \( V \) is a sequence of linear maps of degree 1

\[
q_k: \bigodot^k V[1] \to V[1], 
\]
such that the induced coderivation \( Q \) on the reduced symmetric coalgebra cogenerated by \( V[1] \), given by the Formula 2.1 is a codifferential, i.e., \( Q^2 = 0 \). This condition in particular implies \( q_1^2 = 0 \), i.e., an \( L_\infty \)-algebra is in particular a differential complex. Note that, by the above discussion, every DGLA can be naturally seen as an \( L_\infty \)-algebra; namely, a DGLA is an \( L_\infty \)-algebra with vanishing higher multiplications \( q_k \), \( k \geq 3 \). Note that, via the decalage isomorphisms of Remark 1.1, every component \( q_k \) of an \( L_\infty \) structure on \( V \) can be seen as morphism

\[
\mu_k \in \text{Hom}^{2-k}(\wedge^k V, V).
\]

A morphism \( f_\infty \) between two \( L_\infty \)-algebras \( (V, q_1, q_2, q_3, \ldots) \) and \( (W, p_1, p_2, p_3, \ldots) \) is a sequence of linear maps of degree 0

\[
f_n: \bigodot^n V[1] \to W[1], 
\]
such that the morphism of coalgebras

\[
F: \bigoplus_{n \geq 1} \bigodot^n V[1] \to \bigoplus_{n \geq 1} \bigodot^n W[1]
\]
induced by \( \sum_n f_n: \bigoplus_{n \geq 1} \bigodot^n V[1] \to W[1] \) commutes with the codifferentials induced by the two \( L_\infty \) structures on \( V \) and \( W \) \([6, 14, 17, 18, 22]\). An \( L_\infty \)-morphism \( f_\infty \) is called linear (sometimes strict) if \( f_n = 0 \) for every \( n \geq 2 \). We note that a linear map \( f_1: V[1] \to W[1] \) is a linear \( L_\infty \)-morphism if and only if

\[
p_n(f_1(v_1) \odot \cdots \odot f_1(v_n)) = f_1(q_n(v_1 \odot \cdots \odot v_n)), \quad \forall n \geq 1, v_1, \ldots, v_n \in V[1].
\]
For instance, morphisms between DGLAs are linear morphisms between the corresponding \( L_\infty \)-algebras.

If \( f_\infty \) is an \( L_\infty \)-morphism between \( (V, q_1, q_2, q_3, \ldots) \) and \( (W, p_1, p_2, p_3, \ldots) \), then its linear part

\[
f_1: V[1] \to W[1]
\]
satisfies the equation \( f_1 \circ q_1 = p_1 \circ f_1 \), i.e., \( f_1 \) is a map of differential complexes \( (V[1], q_1) \to (W[1], p_1) \). An \( L_\infty \)-morphism \( f_\infty \) is called a quasiisomorphism of \( L_\infty \)-algebras if its linear part \( f_1 \) is a quasiisomorphism of differential complexes. A major result in the theory of \( L_\infty \)-algebra is the following homotopical transfer of structure theorem (see \([6, 16]\) for a proof).
**Theorem 2.2.** Let \((V, q_1, q_2, q_3, \ldots)\) be an \(L_\infty\)-algebra and \((C, \delta)\) be a differential complex. If there exist two morphisms of differential complexes

\[ \iota: (C[1], \delta[1]) \to (V[1], q_1) \quad \text{and} \quad \pi: (V[1], q_1) \to (C[1], \delta[1]) \]

which are homotopy inverses, then there exist an \(L_\infty\)-algebra structure \((C, \langle \rangle_1, \langle \rangle_2, \ldots)\) on \(C\) extending its differential complex structure, and making \((V, q_1, q_2, \ldots)\) and \((C, \langle \rangle_1, \langle \rangle_2, \ldots)\) be quasiisomorphic \(L_\infty\)-algebras via an \(L_\infty\)-quasiisomorphism \(\iota\Gamma\) extending \(\iota\).

In case \(\pi \iota = \Id_{C[1]}\), explicit formulas for such a transfer are described in [6, 16] and [32] in terms of summation over rooted trees [15, Definition 6]

\[ \langle \rangle_n = \sum_{\Gamma \in T_n} \varepsilon_\Gamma Z_\Gamma(t, \pi, K, q_i), \]

where \(K \in \Hom^{-1}(V[1], V[1])\) is an homotopy between \(\iota \pi\) and \(\Id_{V[1]}\), \(T_n\) is the set of rooted trees with \(n\) tails, \(\varepsilon_\Gamma = \pm 1\) is a sign depending on the combinatorics of the tree \(\Gamma\) and \(\text{Aut} \Gamma\) is the group of automorphisms of \(\Gamma\). Each tail edge of a tree is decorated by the operator \(\iota\), each internal edge is decorated by the suspended operator \(K\) and the root edge is decorated by the suspended operator \(\pi\); every internal vertex \(v\) carries the operation \(q_r\), where \(r\) is the number of edges having \(v\) as endpoint. Then \(Z_\Gamma\) is the evaluation of such a decorated graph according to the usual operadic rules; see [6, Thm. 2.3.1] for an explicit recursive formula.

3. The suspended mapping cone of \(\chi: L \to M\).

The suspended mapping cone of the DGLA morphism \(\chi: L \to M\) is the graded vector space

\[ C_\chi = \text{Cone}(\chi)[-1], \]

where \(\text{Cone}(\chi) = L[1] \oplus M\) is the mapping cone of \(\chi\). More explicitly,

\[ C_\chi = \bigoplus_i C_i, \quad C_i = L \oplus M^{i-1}. \]

The suspended mapping cone has a natural differential \(\delta \in \Hom^1(C_\chi, C_\chi)\) given by

\[ \delta(l, m) = (dl, \chi(l) - dm), \quad l \in L, m \in M. \]

Denote by \(\langle \langle \rangle \rangle_1 \in \Hom^1(C_\chi[1], C_\chi[1])\) the suspended differential, namely

\[ \langle \langle (l, m) \rangle \rangle_1 = (-dl, -\chi(l) + dm), \quad l \in L, m \in M. \]

**Remark 3.1.** If \(\chi\) is injective, then the projection on the second factor induces a quasiisomorphism of differential complexes \(\pi_2: C_\chi \to (M/\text{Im}(\chi))[-1]\). In particular, it induces isomorphisms \(H^i(C_\chi) \cong H^{i-1}(\text{Coker}(\chi))\), for every \(i\).

Setting \(M[t, dt] = \mathbb{K}[t, dt] \otimes M\), then

\[ H_\chi = \{(l, m(t, dt)) \in L \times M[t, dt] \mid m(0, 0) = 0, m(1, 0) = \chi(l)\} \]

is a differential graded Lie algebra. The differential on \(H_\chi\) is \((l, m(t, dt)) \mapsto (dl, dm(t, dt))\); since the differential on \(H_\chi\) has degree 1, the suspended differential \(q_1: H_\chi[1] \to H_\chi[1]\) is the opposite differential:

\[ q_1(l, m(t, dt)) = -(dl, dm(t, dt)). \]
The integral operator \( \int_a^b : \mathbb{K}[t, dt] \to \mathbb{K} \) extends naturally to a linear map of degree \(-1\)
\[
\int_a^b : M[t, dt] \to M, \quad \int_a^b (\sum_i t^i m_i + t^i dt \cdot n_i) = \sum_i \left( \int_a^b t^i dt \right) n_i.
\]

**Lemma 3.2.** The complexes \( C_\chi[1] \) and \( H_\chi[1] \) are homotopic; more precisely, if one denotes by
\[
\iota \in \text{Hom}^0(C_\chi[1], H_\chi[1]), \quad \pi \in \text{Hom}^0(H_\chi[1], C_\chi[1]), \quad K \in \text{Hom}^{-1}(H_\chi[1], H_\chi[1])
\]
the linear maps defined as
\[
\iota(l, m) = (l, t \chi(l) + dt \cdot m), \quad \pi(l, m(t, dt)) = \left( l, \int_0^1 m(t, dt) \right)
\]
\[
K(l, m) = \left( 0, t \int_0^1 m - \int_0^t m \right),
\]
then \( \iota \) and \( \pi \) are morphisms of complexes and
\[
\pi \iota = \text{Id}_{C_\chi[1]}, \quad \text{Id}_{H_\chi[1]} - \iota \pi = K q_1 + q_1 K.
\]

**Proof.** Straightforward. \(\square\)

4. **The \( L_\infty \) structure on \( C_\chi \)**

By Quillen construction [25], the differential graded Lie algebra \( H_\chi \) carries an \( L_\infty \) structure
\[
q_k : \bigotimes_{n=0}^k (H_\chi[1]) \to H_\chi[1],
\]
where \( q_k = 0 \) for every \( k \geq 3 \),
\[
q_1(l, m(t, dt)) = (-dl, -dm(t, dt))
\]
and
\[
q_2((l_1, m_1(t, dt)) \odot (l_2, m_2(t, dt))) = (-1)^{\deg_{H_\chi}(l_1, m_1(t, dt))} ([l_1, l_2], [m_1(t, dt), m_2(t, dt)]).
\]

Results of Section 3 tell us that we can apply the homotopy transfer of structure theorem, to induce on \( C_\chi \) an \( L_\infty \)-algebra structure making \( C_\chi \) and \( H_\chi \) be quasiisomorphic \( L_\infty \)-algebras. Moreover, the linear maps of degree 1
\[
\langle \rangle_n : \bigotimes_{n=0}^n C_\chi[1] \to C_\chi[1], \quad n \geq 1,
\]
defining the induced \( L_\infty \)-algebra structure on \( C_\chi \) are explicitly described in terms of summation over rooted trees. In our case, the properties
\[
\pi q_1 K = K q_1 = 0,
\]
\[
q_2(\text{Im } K \otimes \text{Im } K) \subseteq \ker \pi \cap \ker K, \quad q_k = 0 \quad \forall \ k \geq 3,
\]
imply that, fixing the number of tails, there exists at most one isomorphism class of trees giving a nontrivial contribution.

- One tail: the only tree is
\[
\xymatrix{ \bullet \ar@{-}[r] & \bullet \ar@{-}[r] & \bullet \ar@{--}[r] & l \ar@/_/[r] & q_1 \ar@{-}[r] & \pi
\}
\]
giving by operadic evaluation the formula
\[
\langle (l, m) \rangle_1 = \pi q_1 \iota(l, m) = (-dl, -\chi(l) + dm).
\]
Two tails:

Again by operadic evaluation, this graph gives
\[ \langle \gamma_1 \circ \gamma_2 \rangle_2 = \pi q_2 \circ (t(\gamma_1) \circ t(\gamma_2)) \].

\[ \langle \gamma_1 \circ \cdots \circ \gamma_n \rangle_n = \frac{(-1)^{n-2}}{2} \sum_{\sigma \in S_n} \varepsilon(\sigma) \pi q_2(t(\gamma_{\sigma(1)}) \circ K q_2(t(\gamma_{\sigma(2)}) \circ \cdots \circ K q_2(t(\gamma_{\sigma(n-1)}) \circ t(\gamma_{\sigma(n)})) \cdots)) \].

The factor 1/2 in the above formula accounts for the cardinality of the automorphism group of the graph involved.

**Remark 4.1.** The above construction of the \( L_\infty \) structure on \( C_\chi \) commutes with tensor products of differential graded commutative algebras. This means that if \( R \) is a DGCA, then the \( L_\infty \)-algebra structure on the suspended mapping cone of \( \chi \otimes \text{id}_R : L \otimes R \rightarrow M \otimes R \) is naturally isomorphic to the \( L_\infty \)-algebra \( C_\chi \otimes R \).

A more refined description involving the original brackets in the differential graded Lie algebras \( L \) and \( M \) is obtained decomposing the symmetric powers of \( C_\chi [1] \) into types:

\[ n \odot (C_\chi [1]) = n \odot \text{Cone}(\chi) = \bigoplus_{\lambda + \mu = n} \left( \begin{array}{c} \mu \\ \lambda + \mu = n \end{array} \right) \left( \odot M \right) \otimes \left( \odot L[1] \right) \].

The operation \( \langle \rangle_2 \) decomposes into
\[ l_1 \otimes l_2 \mapsto (-1)^{\text{deg}_L(l_1)} l_1, l_2 \in L; \quad m_1 \otimes m_2 \mapsto 0; \]
\[ m \otimes l \mapsto \frac{(-1)^{\text{deg}_M(m)+1}}{2} [m, \chi(l)] \in M. \]

For later use, we point out that, via decalage isomorphisms, the maps \( \langle - \rangle_1 \) and \( \langle - \rangle_2 \) corresponds to
\[ \mu_1 \in \text{Hom}^1(C_\chi, C_\chi), \quad \mu_2 \in \text{Hom}^0(\bigwedge^2 C_\chi, C_\chi), \]
\[ \mu_1(l, m) = (dl, \chi(l) - dm), \]
\[ \mu_2((l_1,m_1) \wedge (l_2,m_2)) = \left([l_1,l_2], \frac{1}{2}[m_1,\chi(l_2)] + \frac{(-1)^{\deg L(l_1)}}{2}[\chi(l_1),m_2]\right). \]

For every \( n \geq 2 \) it is easy to see that \( \langle \gamma_1 \circ \cdots \circ \gamma_{n+1} \rangle \) can be nonzero only if the multivector \( \gamma_1 \circ \cdots \circ \gamma_n \) belongs to \( \bigotimes^n M \otimes L[1] \). For \( n \geq 2 \), \( m_1, \ldots, m_n \in M \) and \( l \in L[1] \) the formula for \( \langle - \rangle_{n+1} \) described above becomes

\[
\langle m_1 \circ \cdots \circ m_n \circ l \rangle_{n+1} = (-1)^{n-1} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ K q_2((dt)m_{\sigma(2)} \circ \cdots \circ K q_2((dt)m_{\sigma(n)} \circ t\chi(l)) \cdots))).
\]

Define recursively a sequence of polynomials \( \phi_t(1) \in \mathbb{Q}[t] \subseteq \mathbb{K}[t] \) and rational numbers \( I_n \) by the rule

\[
\phi_1(t) = t, \quad I_n = \int_0^1 \phi_n(t)dt, \quad \phi_{n+1}(t) = \int_0^t \phi_n(s)ds - tI_n.
\]

By the definition of the homotopy operator \( K \) we have, for every \( m \in M \)

\[
K((\phi_n(t)dt)m) = -\phi_{n+1}(t)m.
\]

Therefore, for every \( m_1, m_2 \in M \) we have

\[
K q_2((dt \cdot m_1) \circ \phi_n(t)m_2) = (-1)^{\deg_M(m_1)}\phi_{n+1}(t)[m_1, m_2].
\]

Therefore, we find:

\[
\langle m_1 \circ \cdots \circ m_n \circ l \rangle_{n+1} = (-1)^{n-1} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ K q_2((dt)m_{\sigma(2)} \circ \cdots \circ K q_2((dt)m_{\sigma(n)} \circ t\chi(l)) \cdots))).
\]

\[
= (-1)^{n-1+\deg_M(m_\sigma(n))} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ K q_2((dt)m_{\sigma(2)} \circ \cdots \circ K q_2((dt)m_{\sigma(n)} \circ t\chi(l)) \cdots))).
\]

\[
= (-1)^{n-1+\sum_{i=1}^{n} \deg_M(m_{\sigma(i)})} \sum_{\sigma \in S_n} \varepsilon(\sigma)\pi q_2((dt)m_{\sigma(1)} \circ \phi_n(t)[m_{\sigma(2)}, \ldots, m_{\sigma(n)}, \chi(l)] \cdots))
\]

\[
= (-1)^{n+\sum_{i=1}^{n} \deg_M(m_{\sigma(i)})} I_n \sum_{\sigma \in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots, m_{\sigma(n)}, \chi(l)] \cdots] \in M
\]

**Theorem 4.2.** For every \( n \geq 2 \) we have

\[
\langle m_1 \circ \cdots \circ m_n \circ l \rangle_{n+1} = -(-1)^{\sum_{i=1}^{n} \deg_M(m_{\sigma(i)})} \frac{B_n}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma)[m_{\sigma(1)}, [m_{\sigma(2)}, \ldots, m_{\sigma(n)}, \chi(l)] \cdots],
\]

where the \( B_n \) are the Bernoulli numbers, defined by the series expansion identity

\[
\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots
\]

**Proof.** Since \( B_{2k+1} = 0 \) for every \( k > 0 \), it is sufficient to prove that \( B_n = -n!I_n \) for every \( n \geq 1 \). Consider the polynomials \( \psi_0(t) = 1 \) and \( \psi_n(t) = n!(\phi_n(t) - I_n) \) for \( n \geq 1 \).

Then, for any \( n \geq 1 \),

\[
\frac{d}{dt} \psi_n(t) = n \cdot \psi_{n-1}(t), \quad \int_0^1 \psi_n(t)dt = 0.
\]
Therefore the $\psi_n(t)$ satisfy the recursive relations (see e.g. [28]) of the Bernoulli polynomials $B_n(t)$, defined by the series expansion identity

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{xe^{tx}}{e^x - 1}.$$ 

In particular $B_n = B_n(0) = \psi_n(0) = -n! I_n$ for every $n \geq 1$. \hfill \Box

Remark 4.3. Recently, the relevance of Bernoulli numbers in deformation theory has been also remarked by Ziv Ran in [27]. In particular, Ran’s “JacoBer” complex seems to be closely related to the coderivation $Q$ defining the $L_\infty$ structure on $C_\chi$.

5. The functors $MC_\chi$ and $Def_\chi$ revisited

Having introduced an $L_\infty$ structure on $C_\chi$ in Section 4, we have a corresponding Maurer-Cartan functor \cite{6, 14} $MC_{C_\chi} : \mathbf{Art} \to \mathbf{Set}$, defined as

$$MC_{C_\chi}(A) = \left\{ \gamma \in C_\chi[1] \otimes m_A \left| \sum_{n \geq 1} \frac{\langle \gamma \odot_n \rangle n}{n!} = 0 \right. \right\}, \quad A \in \mathbf{Art}.$$ 

Writing $\gamma = (l, m)$, with $l \in L^1 \otimes m_A$ and $m \in M^0 \otimes m_A$, the Maurer-Cartan equation becomes

$$0 = \sum_{n=1}^{\infty} \frac{\langle (l, m) \odot_n \rangle n}{n!} = \langle (l, m) \rangle_1 + \frac{1}{2} \langle (l \odot 2) \rangle_2 + \langle m \otimes l \rangle_2 + \frac{1}{2} \langle m \odot 2 \rangle_2 + \sum_{n \geq 2} \frac{n + 1}{(n + 1)!} \langle m \odot_n \otimes l \rangle_{n+1}$$

$$= \left( -dl - \frac{1}{2}[l, l] - \chi(l) + dm - \frac{1}{2}[m, \chi(l)] + \sum_{n \geq 2} \frac{1}{n!} \langle m \odot_n \otimes l \rangle_{n+1} \right) \in (L^2 \oplus M^1) \otimes m_A.$$ 

According to Theorem 4.2, since $\deg_M(m) = \deg_{C_\chi[1]}(m) = 0$, we have

$$\langle m \odot_n \otimes l \rangle_{n+1} = \frac{B_n}{n!} \sum_{\sigma \in S_n} [m, [m, \cdots, [m, \chi(l)] \cdots]] = -B_n \text{ad}_m^n(\chi(l)).$$ 

The Maurer-Cartan equation on $C_\chi$ is therefore equivalent to

$$\left\{ \begin{array}{l}
 dl + \frac{1}{2}[l, l] = 0 \\
 \chi(l) - dm + \frac{1}{2}[m, \chi(l)] + \sum_{n=2}^{\infty} \frac{B_n}{n!} \text{ad}_m^n(\chi(l)) = 0.
\end{array} \right.$$ 

Since $B_0 = 1$ and $B_1 = -\frac{1}{2}$, we can write the second equation as

$$[m, \chi(l)] - dm + \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}_m^n(\chi(l)) = [m, \chi(l)] - dm + \frac{\text{ad}_m}{e^{\text{ad}_m} - 1}(\chi(l)) = 0.$$
Applying the invertible operator $\frac{e^{ad_m} - 1}{ad_m}$ we get

$$0 = \chi(l) + \frac{e^{ad_m} - 1}{ad_m}([m, \chi(l)] - dm) = e^m * \chi(l).$$

Therefore, the Maurer-Cartan equation for the $L_\infty$-algebra structure on $C_\chi$ is equivalent to

$$\begin{cases}
  dl + \frac{1}{2}[l, l] = 0 \\
e^m * \chi(l) = 0
\end{cases}$$

and the Maurer-Cartan functor $MC_\chi$ described in the introduction is precisely the Maurer-Cartan functor corresponding to the $L_\infty$ structure on $C_\chi$.

Recall that the deformation functor associated to an $L_\infty$-algebra $\mathfrak{g}$ is $\text{Def}_\mathfrak{g} = MC_\mathfrak{g} / \sim$, where $\sim$ denotes homotopy equivalence of solutions of the Maurer-Cartan equation: two elements $\gamma_0$ and $\gamma_1$ of $MC_\mathfrak{g}(A)$ are called homotopy equivalent if there exists an element $\gamma(t, dt) \in MC_{[t, dt]}(A)$ with $\gamma(0) = \gamma_0$ and $\gamma(1) = \gamma_1$.

We have already identified the functor $MC_{C_\chi}$ with the functor $MC_\chi$. Now we want to show that, under this identification, the homotopy equivalence on $MC_{C_\chi}$ is the same thing of gauge equivalence on $MC_\chi$ described in the introduction, so that

$$\text{Def}_\chi \simeq \text{Def}_{C_\chi}.$$ 

We will need the following lemma (see the Appendix A for a proof).

**Lemma 5.1.** Let $\mathfrak{g}$ be a differential graded Lie algebra and let $A \in \mathsf{Art}$. Then, for any $x$ in $MC_\mathfrak{g}(A)$ and any $g(t) \in \mathfrak{g}^0[t] \otimes m_A$, with $g(0) = 0$, the element $e^{g(t)} * x$ is an element of $MC_{[t, dt]}(A)$. Moreover all the elements of $MC_{[t, dt]}(A)$ are obtained in this way.

We first show that homotopy implies gauge. Let $(l_0, m_0)$ and $(l_1, m_1)$ be homotopy equivalent elements of $MC_\chi(A)$. Then there exists an element $(\tilde{l}, \tilde{m})$ of $MC_{C_\chi(s, ds)}(A)$ with $(\tilde{l}(0), \tilde{m}(0)) = (l_0, m_0)$ and $(\tilde{l}(1), \tilde{m}(1)) = (l_1, m_1)$. According to Remark 4.1, the Maurer-Cartan equation for $(\tilde{l}, \tilde{m})$ is

$$\begin{cases}
  d\tilde{l} + \frac{1}{2}[\tilde{l}, \tilde{l}] = 0 \\
e^{\tilde{m}} * \chi(\tilde{l}) = 0
\end{cases}$$

The first of the two equations above tells us that $\tilde{l}$ is a solution of the Maurer-Cartan equation for $L[s, ds]$. So, by Lemma 5.1, there exists a degree zero element $\lambda(s)$ in $L[s] \otimes m_A$ with $\lambda(0) = 0$ such that $\tilde{l} = e^{\lambda} * l_0$. Evaluating at $s = 1$ we find $l_1 = e^{\lambda_1} * l_0$. As a consequence of $\tilde{l} = e^{\lambda} * l_0$, we also have $\chi(\tilde{l}) = e^{\chi(\lambda)} * \chi(l_0)$. Set $\tilde{m} = \tilde{m} * m_0$, so that $l = \tilde{m} * m_0 * (-\chi(\lambda))$ and the second Maurer-Cartan equation is reduced to $e^{\tilde{m}} * (e^{m_0} * \chi(l_0)) = 0$, i.e., to $e^{\tilde{m}} = 0 = 0$, where we have used the fact that $(l_0, m_0)$ is a solution of the Maurer-Cartan equation in $C_\chi$. This last equation is equivalent to the equation $d\tilde{m} = 0$ in $(C_\chi[s, ds])^0 \otimes m_A$. If we write $\tilde{m}(s, ds) = \mu^0(s) + ds \mu^{-1}(s)$, then the equation $d\tilde{m} = 0$ becomes

$$\begin{cases}
  \mu^0 - d_M \mu^{-1} = 0 \\
d_M \mu^0 = 0,
\end{cases}$$

where $d_M$ is the differential in the DGLA $M$. The solution is, for any fixed $\mu^{-1}$,

$$\mu^0(s) = \int_0^s d\sigma d_M \mu^{-1}(\sigma) = -d_M \int_0^s d\sigma \mu^{-1}(\sigma)$$
Set $\nu = -\int_0^1 ds \mu^{-1}(s)$. Then $m_1 = \tilde{m}(1) = (d_M \nu) \cdot m_0 \cdot (-\chi(\lambda_1))$. Summing up, if $(l_0, m_0)$ and $(m_1, l_1)$ are homotopy equivalent, then there exists $(d\nu, \lambda_1) \in (dM^{-1} \otimes m_A) \times (L^0 \otimes m_A)$ such that

$$
\begin{align*}
  l_1 &= e^{s\lambda_1} \ast l_0 \\
  m_1 &= d\nu \cdot m_0 \cdot (-\chi(\lambda_1)),
\end{align*}
$$

i.e., $(l_0, m_0)$ and $(m_1, l_1)$ are gauge equivalent.

We now show that gauge implies homotopy. Assume $(l_0, m_0)$ and $(m_1, l_1)$ are gauge equivalent. Then then there exist $(d\nu, \lambda_1) \in (dM^{-1} \otimes m) \times (L^0 \otimes m)$ such that

$$
\begin{align*}
  l_1 &= e^{s\lambda_1} \ast l_0 \\
  m_1 &= d\nu \cdot m_0 \cdot (-\chi(\lambda_1)).
\end{align*}
$$

Set $\tilde{l}(s, ds) = e^{s\lambda_1} \ast l_0$. By Lemma 5.1, $\tilde{l}$ satisfies the equation $d\tilde{l} + \frac{1}{2} [\tilde{l}, \tilde{l}] = 0$. Set $\tilde{m} = (d(s\nu)) \cdot m_0 \cdot (-\chi(s\lambda_1))$. Reasoning as above, we find

$$
e^{\tilde{m} \ast \chi(\tilde{l})} = e^{d(s\nu) \ast 0} = 0.
$$

Therefore, $(\tilde{l}, \tilde{m})$ is a solution of the Maurer-Cartan equation in $C_\chi[s, ds]$. Moreover $\tilde{l}(0) = l_0$, $\tilde{l}(1) = l_1$, $\tilde{m}(0) = m_0$ and $\tilde{m}(1) = d\nu \cdot m_0 \cdot (-\chi(\lambda_1)) = m_1$, i.e. $(l_0, m_0)$ and $(m_1, l_1)$ are homotopy equivalent.

Summing up, we have

$$\text{Def}_{C_\chi} = \frac{\text{MC}_{C_\chi}}{\text{homotopy}} \simeq \frac{\text{MC}_{C_\chi}}{\text{gauge}} = \text{Def}_\chi.
$$

6. Functoriality

In the above section we have shown how to a morphism of differential graded Lie algebras $\chi: L \to M$ is associated a canonical $L_\infty$ structure on $C_\chi$. We will now discuss the functorial aspects of this construction. Denote by $L_\infty$ the category of $L_\infty$-algebras and by $M$ the category of morphisms of differential graded Lie algebras; objects in $M$ are DGLA morphisms $\chi: L \to M$; morphisms in $M$ are commutative squares

$$
L_1 \xrightarrow{\chi_1} L_2 \\
M_1 \xrightarrow{\chi_2} M_2
$$

of DGLA morphisms.

It is immediate to observe that the above commutative square induces a linear $L_\infty$-morphism $(f_L, f_M): C_\chi_2 \to C_\chi_2$ and then $C: M \to L_\infty$ is a functor. Moreover both MC and Def are functors from the category $L_\infty$ to the category of functors of Artin rings [14, 20, 22]. The functor MC acts on the morphisms of $L_\infty$ in the following way: let $f_\infty: V \to W$ be an $L_\infty$-morphism, then

$$
\text{MC}_{f_\infty}: \text{MC}_V \to \text{MC}_W
$$

is the natural transformation given, for $A \in \text{Art}$ and $v \in \text{MC}_V(A) \subseteq V[1]^0 \otimes m_A$, by

$$
\text{MC}_{f_\infty}(v) = \sum_{n=1}^{\infty} \frac{1}{n!} f_n(v \otimes n) \in W[1]^0 \otimes m_A.
$$
The natural transformation $MC_{f_\infty}$ preserves the homotopy equivalence and then induces a natural transformation
\[ \text{Def}_{f_\infty} : \text{Def}_V \to \text{Def}_W. \]

Recall from Section 2 that an $L_\infty$-morphism $f_\infty$ is called a quasiisomorphism of $L_\infty$-algebras if its linear part $f_1$ is a quasiisomorphism of differential complexes. Using the fact that every quasiisomorphism of $L_\infty$-algebras induces an isomorphism of the associated deformation functors [14], the next theorem becomes evident.

**Theorem 6.1** ([23]). Consider a commutative diagram of morphisms of differential graded Lie algebras
\[
\begin{array}{ccc}
L_1 & \xrightarrow{f_L} & L_2 \\
\downarrow{\chi_1} & & \downarrow{\chi_2} \\
M_1 & \xrightarrow{f_M} & M_2
\end{array}
\]
and assume that $(f_L, f_M) : C_{\chi_1} \to C_{\chi_2}$ is a quasiisomorphism of complexes (e.g. if both $f_L$ and $f_M$ are quasiisomorphisms). Then the natural transformation $\text{Def}_{\chi_1} \to \text{Def}_{\chi_2}$ is an isomorphism.

7. **Cartan homotopies**

In this section we formalize, under the notion of *Cartan homotopy*, a set of standard identities that often arise in algebra and geometry [3, Appendix B].

**Definition 7.1.** Let $(L, d, [\cdot, \cdot])$ and $(M, d, [\cdot, \cdot])$ be two differential graded Lie algebras and denote by $\delta$ the standard differential on $\text{Hom}^*(L, M)$. A linear map $i \in \text{Hom}^{-1}(L, M)$ is called a *Cartan homotopy* if for every $a, b \in L$ we have:
\[ i([a, b]) = [i(a), \delta i(b)], \quad [i(a), i(b)] = 0. \]

Notice that, according to the definition of $\delta$, for every $a \in L$ we have
\[ \delta i(a) = d(i(a)) + i(da). \]

For later use we point out that $[i(a), [i(b), \delta i(c)]] = [i(a), i([b, c])] = 0$ for every $a, b, c$. It is moreover easy to verify that $\delta i$ is a morphism of differential graded Lie algebras and
\[ i([a, b]) = (-1)^{\deg(a)}[\delta i(a), i(b)] = \frac{1}{2}[i(a), \delta i(b)] + \frac{(-1)^{\deg(a)}}{2}[\delta i(a), i(b)]. \]

**Example 7.2.** The name Cartan homotopy has a clear origin in differential geometry. Namely, let $M$ be a differential manifold, $\mathcal{X}(M)$ be the Lie algebra of vector fields on $M$, and $\mathcal{E}n^*(\Omega^*(M))$ be the Lie algebra of endomorphisms of the de Rham algebra of $M$. The Lie algebra $\mathcal{X}(M)$ can be seen as a DGLA concentrated in degree zero, and the graded Lie algebra $\mathcal{E}n^*(\Omega^*(M))$ has a degree one differential given by $[d, -]$, where $d$ is the de Rham differential. Then the contraction
\[ i : \mathcal{X}(M) \to \mathcal{E}n^*(\Omega^*(M))[-1] \]
is a Cartan homotopy and its differential is the Lie derivative
\[ \delta i = \mathcal{L} : \mathcal{X}(M) \to \mathcal{E}n^*(\Omega^*(M)). \]

In fact, by classical Cartan’s homotopy formulas [1, Section 2.4], for any two vector fields $X$ and $Y$ on $M$, we have
\( L = \mathcal{L} \)  
\( \mathcal{L}_X = dX + i_X d = [d, i_X]; \)

(2) \( i[X,Y] = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X = [\mathcal{L}_X, i_Y] = [i_X, \mathcal{L}_Y]; \)

(3) \( [i_X, i_Y] = 0. \)

Note that the first Cartan formula above actually states that \( \delta i = L. \) Indeed \( \mathcal{X}(M) \) is concentrated in degree zero and then its differential is trivial.

**Example 7.3.** The composition of a Cartan homotopy with a morphism of DGLAs is a Cartan homotopy. If \( i: L \to M[-1] \) is a Cartan homotopy and \( \Omega \) is a differential graded-commutative algebra, then its natural extension

\[
i \otimes \text{Id}: L \otimes \Omega \to (M \otimes \Omega)[-1], \quad a \otimes \omega \mapsto i(a) \otimes \omega,
\]

is a Cartan homotopy.

**Proposition 7.4.** Let \( i: L \to M[-1] \) be a Cartan homotopy and \( \chi = \delta i: L \to M. \) Then the linear map

\[
\tilde{i}: L \to C_\chi, \quad \tilde{i}(a) = (a, i(a))
\]

is a linear \( L_\infty \)-morphism.

**Proof.** By decalage isomorphism, the \( L_\infty \) structure on \( C_\chi \) is given by the higher brackets

\[
\mu_1(l, m) = (dl, \chi(l) - dm),
\]

\[
\mu_2(l, m) \wedge (h, k) = \left( [l, h], \frac{1}{2}[m, \chi(h)] + \frac{(-1)^{\deg(l)}}{2} [\chi(l), k] \right)
\]

and for \( n \geq 3 \)

\[
\mu_n(l_1, m_1) \wedge \cdots \wedge (l_n, m_n) = \frac{B_{n-1}}{(n-1)!} \sum_{\sigma \in S_n} \pm [m_{\sigma(1)}, \cdots, [m_{\sigma(n-1)}, \chi(l_{\sigma(n)})] \cdots]
\]

It is straightforward to check that \( \tilde{i} \) commutes with every bracket, i.e.

\[
\tilde{i}(dx) = \mu_1(\tilde{i}(x)), \quad \tilde{i}([x, y]) = \mu_2(\tilde{i}(x) \wedge \tilde{i}(y)),
\]

and for \( n \geq 3 \)

\[
\mu_n(\tilde{i}(x_1) \wedge \cdots \wedge \tilde{i}(x_n)) = 0.
\]

Therefore \( \tilde{i} \) is a linear \( L_\infty \)-morphism. \( \square \)

**Appendix A: Gauge vs. Homotopy**

In this Appendix we briefly discuss the relation between homotopy and gauge equivalence for solutions of the Maurer-Cartan equation for a given differential graded Lie algebra \( L \). We also give a proof of Lemma 5.1, which is here presented as Corollary A.1.

**Proposition A.1.** Let \( (L, d, [\ , \ ]) \) be a differential graded Lie algebra such that:

(1) \( L = M \oplus C \oplus D \) as graded vector spaces.

(2) \( M \) is a differential graded subalgebra of \( L \).

(3) \( d: C \to D[1] \) is an isomorphism of graded vector spaces.

Then, for every \( A \in \text{Art} \) there exists a bijection

\[
\alpha: MC_M(A) \times (C^0 \otimes m_A) \to MC_L(A), \quad (x, c) \mapsto e^c \ast x.
\]
Proof. This is essentially proved in [31, Section 5] using induction on the length of $A$ and the Baker-Campbell-Hausdorff formula. Here we sketch a simpler proof based on formal theory of deformation functors [30, 29, 4, 19].

The map $\alpha$ is a natural transformation of homogeneous functors, so it is sufficient to show that $\alpha$ is bijective on tangent spaces and injective on obstruction spaces. Recall that the tangent space of $MC_L$ is $Z^1(L)$, while its obstruction space is contained in $H^2(L)$. The functor $A \mapsto C^0 \otimes m_A$ is smooth with tangent space $C^0$ and therefore tangent and obstruction spaces of the functor

$$A \mapsto MC_M(A) \times (C^0 \otimes m_A)$$

are respectively $Z^1(M) \oplus C^0$ and $H^2(M)$. The tangent map is

$$Z^1(M) \oplus C^0 \ni (x, c) \mapsto e^c \ast x = x - dc \in Z^1(M) \oplus d(C^0) = Z^1(M) \oplus D^1 = Z^1(L)$$

and it is an isomorphism. The inclusion $M \hookrightarrow L$ is a quasi-isomorphism, therefore the obstruction to lifting $x$ in $M$ is equal to the obstruction to lifting $x = e^0 \ast x$ in $L$. We conclude the proof by observing that, according to [4, Prop. 7.5], [19, Lemma 2.21], the obstruction maps of Maurer-Cartan functor are invariant under the gauge action. 

**Corollary A.1.** Let $M$ be a differential graded Lie algebra, $L = M[t,dt]$ and $C \subseteq M[t]$ the subspace consisting of polynomials $g(t)$ with $g(0) = 0$. Then for every $A \in \text{Art}$ the map $(x,g[t]) \mapsto e^{g(t)} \ast x$ induces an isomorphism

$$MC_M(A) \times (C^0 \otimes m_A) \cong MC_L(A).$$

**Proof.** The data $M$, $C$ and $D = d(C)$ satisfy the condition of Proposition A.1. 

**Corollary A.2.** Let $M$ be a differential graded Lie algebra. Two elements $x_0, x_1 \in MC_M(A)$ are gauge equivalent if and only if they are homotopy equivalent.

**Proof.** If $x_0$ and $x_1$ are gauge equivalent, then there exists $g \in M^0 \otimes m_A$ such that $e^g \ast x_0 = x_1$. Then, by Corollary A.1. $x(t) = e^tg \ast x_0$ is an element of $MC_M(t,dt)(A)$ with $x(0) = x_0$ and $x(1) = x_1$, i.e., $x_0$ and $x_1$ are homotopy equivalent.

Vice versa, if $x_0$ and $x_1$ are homotopy equivalent, there exists $x(t) \in MC_M(t,dt)(A)$ such that $x(0) = x_0$ and $x(1) = x_1$. By Corollary A.1., there exists $g(t) \in M^0[t] \otimes m_A$ with $g(0) = 0$ such that $x(t) = e^{g(t)} \ast x_0$. Then $x_1 = e^{g(1)} \ast x_0$, i.e., $x_0$ and $x_1$ are gauge equivalent. 

### Part 2. dg-Grassmann functors

Let $W$ be vector space over $\mathbb{K}$. The total Grassmannian of $W$ is

$$\text{Grass}(W) = \{ \text{linear subspaces of } W \}.$$ 

The group $\text{Aut}(W)$ of linear automorphisms of $W$ acts on $\text{Grass}(W)$. Denoting by $\text{Grass}(V,W) \subseteq \text{Grass}(W)$ the orbit of a subspace $V \subseteq W$ we have

$$\text{Grass}(V,W) = \frac{\text{Aut}(W)}{\text{Aut}(V,W)}, \quad \text{where} \quad \text{Aut}(V,W) = \{ g \in \text{Aut}(W) \mid g(V) = V \}.$$
The infinitesimal neighborhood of $V$ in $\text{Grass}(W)$ is the formal moduli space for the functor $\text{Grass}_{V,W}: \text{Art} \to \text{Set}$:

$$\text{Grass}_{V,W}(A) = \{ \phi: \text{Spec}(A) \to \text{Grass}(W) \mid \phi(\text{Spec}(\mathbb{K})) = V \}$$

$$= \{ \text{free } A \text{ submodules } V_A \subseteq W \otimes A \mid V_A \otimes A \mathbb{K} = V \}$$

$$= \{ f(V \otimes A) \subseteq W \otimes A \mid f \in \text{Aut}(W \otimes A), f|_V = \text{Id} \}.$$

Since $\{ f \in \text{Aut}(W \otimes A) \mid f|_V = \text{Id} \} = \exp(\mathfrak{gl}(W) \otimes m_A)$ we can write

$$\text{Grass}_{V,W}(A) = \frac{\exp(\mathfrak{gl}(W) \otimes m_A)}{\exp(L^0_{V,W} \otimes m_A)}$$

where

$$L^0_{V,W} = \{ g \in \mathfrak{gl}(W) \mid g(V) \subseteq V \}$$

and the action is given by

$$\exp(L^0_{V,W} \otimes m_A) \times \exp(\mathfrak{gl}(W) \otimes m_A) \to \exp(\mathfrak{gl}(W) \otimes m_A), \quad (e^a, e^m) \mapsto e^m e^{-a}.$$ 

In conclusion, the functor $\text{Grass}_{V,W}$ coincides with the deformation functor $\text{Def}_{\chi}$, where $\chi: L^0_{V,W} \to \mathfrak{gl}(W)$ is the inclusion. Indeed the differential graded Lie algebras $L^0_{V,W}$ and $\mathfrak{gl}(W)$ are concentrated in degree 0, $\text{MC}_{\chi}(A) = \exp(\mathfrak{gl}(W) \otimes m_A)$ and the gauge action is given by

$$\exp(L^0_{V,W} \otimes m_A) \times \exp(\mathfrak{gl}(W) \otimes m_A) \to \exp(\mathfrak{gl}(W) \otimes m_A), \quad (e^a, e^m) \mapsto e^m e^{-\chi(a)}.$$

8. THE COARSE DG-GRASSMANNIAN

The considerations of the above section suggest the following generalization from vector spaces to differential complexes. Let $(W, d)$ be a dg-vector space and denote by:

1. $\text{Aut}(W)$ the group of automorphisms of the graded vector space $W$;
2. $\text{Aut}(W, d)$ the group of automorphisms of the differential graded vector space $(W, d)$, i.e. the subgroup of $\text{Aut}(W, d)$ consisting of linear automorphisms which commute with the differential $d$;
3. $\text{Aut}^0(W, d)$ the subgroup of $\text{Aut}(W, d)$ of automorphisms inducing the identity in cohomology.

Define the coarse Grassmannian $\text{Grass}(W)$ as the quotient

$$\text{Grass}(W) = \frac{G(W)}{\text{Aut}^0(W, d)}, \quad \text{where } G(W) = \{ \text{subcomplexes of } (W, d) \}.$$ 

Note that $\text{Aut}(W, d)$ acts on $G(W)$ and then the quotient group $\text{Aut}(W, d)/\text{Aut}^0(W, d)$ acts on $\text{Grass}(W)$.

If $d = 0$ then $\text{Aut}^0(W, d) = \{ \text{Id} \}$ and then $\text{Grass}(W)$ is the standard Grassmannian. Notice that the cohomology functor gives a map

$$h: \text{Grass}(W) \to \text{Grass}(H^*(W)), \quad h(V) = \text{Im}(H^*(V) \to H^*(W)).$$ 

Denoting by $\text{Grass}(W)^s \subseteq \text{Grass}(W)$ the “open” subset consisting of subcomplexes $V \subseteq W$ such that $H^*(V) \to H^*(W)$ is injective, we shall prove later (Theorems 9.3 and 10.6) that, is some sense, the map $h: \text{Grass}(W)^s \to \text{Grass}(H^*(W))$ is a local isomorphism; this fact justifies the quotient of $G(W)$ by the action of $\text{Aut}^0(W, d)$. 

Given a subcomplex $V \subseteq W$ we denote $\text{Grass}(V,W) = G(V,W)/\text{Aut}^0(W)$, where $G(V,W)$ the set of subcomplexes of $W$ that are isomorphic to $V$ as graded vector spaces; equivalently $U \in G(V,W)$ if and only if $U \in G(W)$ and there exists $f \in \text{Aut}(W)$ such that $f(V) = U$ and then we have a natural identification

$$\left\{ f \in \text{Aut}(W) \mid df(V) \subseteq f(V) \right\} \simeq G(V,W), \quad f \mapsto f(V),$$

where $\text{Aut}(V,W)$ is the subgroup of $\text{Aut}(W)$ consisting of the automorphisms $g$ such that $g(V) = V$.

**Remark 8.1.** To put some structure on $\text{Grass}(W)$ the natural choice is to take $\text{Grass}(V,W)$ as its components; then try to define $\text{Grass}(V,W)$ as categorical quotient (in a suitable category: varieties, schemes, stacks, ...) using the identification

$$\text{Grass}(V,W) = \left\{ f \in \text{Aut}(W) \mid df(V) \subseteq f(V) \right\} / \text{Aut}^0(W,d) \times \text{Aut}(V,W).$$

Note that $(\varphi,\psi) \in \text{Aut}^0(W,d) \times \text{Aut}(V,W)$ acts of $\left\{ f \in \text{Aut}(W) \mid df(V) \subseteq f(V) \right\}$ by the formula $(\varphi,\psi) \cdot f = \varphi f \psi^{-1}$.

Unfortunately we may not expect that $\text{Grass}(V,W)$ is separated in general. Consider $W^0 = W^1 = \mathbb{K} \oplus \mathbb{K}$, $W^i = 0$ for $i \neq 0,1$, and $d: W^0 \to W^1$ of rank 1.

If $V^0 = V^1 = \mathbb{K}$, then $V$ is a subcomplex of $W$ if and only if $V^0 = \ker(d)$ or $V^1 = \text{Im}(d)$. Therefore $G(V,W) \subseteq \mathbb{P}(W^0) \times \mathbb{P}(W^1) = \mathbb{P}^1 \times \mathbb{P}^1$ is the union of two intersecting lines. In particular $G(V,W)$ is singular at the point $V^0 = \ker(d)$, $V^1 = \text{Im}(d)$.

The group $\text{Aut}^0(W,d)$ acts transitively on $\{ V \mid V_0 \neq \ker(d) \}$, $\{ V \mid V_1 \neq \text{Im}(d) \}$ and then $\text{Grass}(V,W)$ contains three point and it is not separated (same type of $\{ xy = 0 \}/\mathbb{K}^*$).

9. **Infinitesimal Study**

Let $(W,d)$ be a complex of vector spaces and $V \subseteq W$ a subcomplex; denote by $L_W$ and $L_{V,W}$ the differential graded Lie algebras

$$L_W = \text{Hom}^*(W,W), \quad L_{V,W} = \{ g \in \text{Hom}^*(W,W) \mid g(V) \subseteq V \}.$$

An Artinian algebra can be seen as a differential complex with trivial differential; for every $A \in \text{Art}$ we still denote by $d$ the differential on $W \otimes A$. Since the differential on $A$ is trivial, we have

$$d: W^i \otimes A \to W^{i+1} \otimes A, \quad d(v \otimes a) = d(v) \otimes a.$$

Let $V \subseteq W$ be a subcomplex and consider the functors

$$\text{Aut}_W, \text{Aut}_{W,d}, \text{Aut}^0_{W,d}, \text{Aut}_{V,W}: \text{Art} \to \text{Groups}$$

defined as

$$\text{Aut}_W(A) = \{ f \in \text{Hom}_d^*(W \otimes A, W \otimes A) \mid f \equiv \text{Id} \pmod{m_A} \}$$

$$\text{Aut}_{W,d}(A) = \{ f \in \text{Aut}_W(A) \mid fd = df \}$$

$$\text{Aut}_{V,W}(A) = \{ f \in \text{Aut}_W(A) \mid f(V \otimes A) = V \otimes A \}$$

$$\text{Aut}^0_{W,d}(A) = \{ f \in \text{Aut}_{W,d}(A) \mid H^*(f) \text{ is the identity on } H^*(W \otimes A, d) \}$$
The above functors are smooth and homogeneous \[29, 19\]. Their tangent spaces are
\[
T^1 \text{Aut}_W = \text{Hom}^0(W,W) = L^0_W
\]
\[
T^1 \text{Aut}_{W,d} = \{ f \in \text{Hom}^0(W,W) \mid df = df \} = Z^0(\text{Hom}^*(W,W))
\]
\[
T^1 \text{Aut}_{W,V} = \{ f \in \text{Hom}^0(W,W) \mid f(V) \subseteq V \} = L^0_{V,W}
\]
\[
T^1 \text{Aut}_{W,d}^0 = \{ f \in \text{Hom}^0(W,W) \mid df = df, f(\ker(d)) \subseteq \text{Im}(d) \}
\]

**Lemma 9.1.** For every \( A \in \text{Art} \), the exponential map gives isomorphisms
\[
\exp: \text{Hom}^0(W,W) \otimes m_A \to \text{Aut}_W(A)
\]
\[
\exp: Z^0(\text{Hom}^*(W,W)) \otimes m_A \to \text{Aut}_{W,d}(A)
\]
\[
\exp: B^0(\text{Hom}^*(W,W)) \otimes m_A \to \text{Aut}_{W,d}^0(A)
\]
\[
\exp: L^0_{V,W} \otimes m_A \to \text{Aut}_{V,W}(A)
\]

**Proof.** First we note that For every \( i \) there exists a natural exact sequence
\[
0 \to B^i(\text{Hom}^*(W,W)) \to Z^i(\text{Hom}^*(W,W)) \to \text{Hom}^i(H^*(W), H^*(W)) \to 0
\]
and therefore
\[
T^1 \text{Aut}_{W,d}^0 = B^0(\text{Hom}^*(W,W)) = \{ df + fd \mid f \in \text{Hom}^{-1}(W,W) \}.
\]
Since all the functors are smooth and homogeneous, it is sufficient to prove that the exponential induces isomorphisms on the tangent spaces. \( \square \)

The considerations made in Section 8 lead us to define the functors
\[
M_{V,W}, \text{Grass}_{V,W} : \text{Art} \to \text{Sets}
\]
\[
M_{V,W}(A) = \{ f \in \text{Aut}_W(A) \mid df(V \otimes A) \subseteq f(V \otimes A) \},
\]
\[
\text{Grass}_{V,W} = \frac{M_{V,W}}{\text{Aut}_{V,W}^0 \times \text{Aut}_{V,W}^0}.
\]

**Proposition 9.2.** In the notation above, let \( \chi : L_{V,W} \to L_W \) the inclusion. Then there exist natural isomorphisms of functors
\[
\text{MC}_\chi \cong M_{V,W}, \quad \text{Def}_\chi \cong \text{Grass}_{V,W}.
\]

**Proof.** Note that, since \( V \) is a subcomplex of \((W,d)\), we have \( d(V \otimes A) \subseteq V \otimes A \), and so
\[
M_{V,W}(A) = \{ f \in \text{Aut}_W(A) \mid (f^{-1}df - d)(V \otimes A) \subseteq V \otimes A \} = \{ f \in \text{Aut}_W(A) \mid f^{-1}df - d \in L^1_{V,W} \otimes m_A \},
\]
and then, by Lemma 9.1 and by the identity \( e^{-a} \ast 0 = e^{-a}de^a - d \),
\[
M_{V,W}(A) = \{ e^a \in \exp(L^0_W \otimes m_A) \mid e^{-a} \ast 0 \in L^1_{V,W} \otimes m_A \}.
\]
Recall that
\[
\text{MC}_\chi(A) = \left\{ (x,e^a) \in (L^1_{V,W} \otimes m_A) \times \exp(L^0_W \otimes m_A) \mid dx + \frac{1}{2}[x,x] = 0, \ e^{-a} \ast 0 = \chi(x) \right\}.
\]
Since \( \chi \) is injective and the set of solutions of the Maurer-Cartan equation in \( L^1_W \otimes m_A \) is preserved by the gauge action \( l \mapsto e^{-a} \ast l \), we have
\[
\text{MC}_\chi(A) = \{ e^a \in \exp(L^0_W \otimes m_A) \mid e^{-a} \ast 0 \in L^1_{V,W} \otimes m_A \}.
\]
and then the isomorphism $\text{MC}_\chi \simeq M_{V,W}$.

The gauge action

$$(\exp(L^0_{V,W} \otimes m_A) \times \exp(B^0(L_W \otimes m_A))) \times \text{MC}_\chi(A) \to \text{MC}_\chi(A)$$

becomes

$$(e^l, e^{dm}) \circ e^a = e^{dm} e^a e^{-\chi(l)} = e^{dm} e^a e^{-l}.$$  

According to Lemma 9.1 we have

$$\exp(L^0_{V,W} \otimes m_A) \times \exp(B^0(L_W \otimes m_A)) = \text{Aut}_{V,W}(A) \times \text{Aut}_{W,d}^0(A)$$

and it is immediate to observe that the gauge action is identified with the natural group action

$$\text{Aut}_{V,W}(A) \times \text{Aut}_{W,d}^0(A) \times M_{V,W}(A) \to M_{V,W}(A)$$

and then we have the isomorphism of quotient functors

$$\text{Grass}_{V,W} = \text{Def}_\chi.$$  

Note that the isomorphism $\text{Def}_\chi \to \text{Grass}_{V,W}$ has a natural explicit description: it is the map

$$e^a \mapsto (e^a(V \otimes A), d) \subseteq (W \otimes A, d).$$

**Theorem 9.3.** Let $V$ be a subcomplex of $(W, d)$ such that the inclusion $V \hookrightarrow W$ induces an injective morphism in cohomology $H^*(V) \hookrightarrow H^*(W)$. Identifying $H^*(V)$ with its image in $H^*(W)$, then the cohomology functor $H^*$ gives a natural transformation of functors

$$H^* : \text{Grass}_{V,W} \to \text{Grass}_{H^*(V), H^*(W)} \simeq \prod_i \text{Grass}_{H^*(V), H^*(W)}.$$  

**Proof.** Recall that

$$\text{Grass}_{H^*(V), H^*(W)}(A) = \{\text{free } A \text{ submodules } F_A \subseteq H^*(W) \otimes A \mid F_A \otimes_A \mathbb{K} = H^*(V)\}.$$  

Given $e^a \in \text{MC}_\chi(A)$ denote by $d_a = e^{-a} de^a : W \otimes A \to W \otimes A$; then $(V \otimes A, d_a)$ is a subcomplex of $(W \otimes A, d_a)$ and $e^a$ is a morphism of complexes $(W \otimes A, d_a) \to (W \otimes A, d)$. By local flatness criteria, the cohomology of $(V \otimes A, d_a)$ is a free $A$-module and the map

$$H^*(e^a(V \otimes A), d) \to H^*(W \otimes A, d) \simeq H^*(W) \otimes A$$

is injective since it factors as

$$H^*(e^a(V \otimes A), d) \simeq H^*(V \otimes A, d_a) \xrightarrow{H^*(e^a)} H^*(W \otimes A, d)$$

where the isomorphism on the left is $H^*(e^{-a})$. Therefore the map

$$h : \text{MC}_\chi(A) \to \text{Grass}_{H^*(V), H^*(W)}(A),$$

$$h(e^a) = \text{image of the natural map } H^*(e^a(V \otimes A)) \to H^*(W) \otimes A$$

is well defined and factors to a natural transformation of functors

$$H^* : \text{Grass}_{V,W} \longrightarrow \text{Grass}_{H^*(V), H^*(W)}.$$  

$\square$
10. Homotopy Invariance

Lemma 10.1. If $V \hookrightarrow W$ is a quasiisomorphism, then the inclusion

$$L_{V,W} \hookrightarrow L_W.$$ 

and the projection

$$L_{V,W} \rightarrow L_V, \quad f \mapsto f|_V$$

are quasiisomorphisms of DGLA.

Proof. We have two short exact sequences

$$0 \rightarrow L_{V,W} \rightarrow L_W \rightarrow \text{Hom}^*(V,W/V) \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}^*(W/V,W) \rightarrow L_{V,W} \rightarrow L_V \rightarrow 0$$

Since the complex $W/V$ is acyclic and the bifunctor $\text{Hom}^*$ commutes with cohomology, the complexes $\text{Hom}^*(V,W/V)$ and $\text{Hom}^*(V,W/V)$ are acyclic. □

Lemma 10.2. Assume we have an exact diagram of differential graded vector spaces

$$
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & 0 \\
\downarrow & & \downarrow{\alpha} & & \downarrow{\psi} & & \downarrow{\beta} & & \downarrow{E} & \\
0 & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & 0
\end{array}
$$

If $\psi\alpha$ is a quasiisomorphism, then also $\beta\varphi$ is a quasiisomorphism.

Proof. For every $i$ in $\mathbb{Z}$ the map $H^i(\psi\alpha): H^i(A) \rightarrow H^i(D)$ is an isomorphism. Since $H^i(\psi\alpha) = H^i(\psi)\circ H^i(\alpha)$, the map $H^i(\alpha): H^i(A) \rightarrow H^i(C)$ is injective and $H^i(\psi): H^i(C) \rightarrow H^i(D)$ is surjective. Therefore, the long exact sequences

$$\cdots \rightarrow H^{i-1}(E) \xrightarrow{\delta^{i-1}} H^i(A) \xrightarrow{H^i(\alpha)} H^i(C) \xrightarrow{H^i(\beta)} H^i(E) \xrightarrow{\delta^i} H^{i+1}(A) \rightarrow \cdots$$

and

$$\cdots \rightarrow H^{i-1}(D) \xrightarrow{\delta^{i-1}} H^i(B) \xrightarrow{H^i(\varphi)} H^i(C) \xrightarrow{H^i(\psi)} H^i(D) \xrightarrow{\delta^i} H^{i+1}(B) \rightarrow \cdots$$

can be refined as

$$H^{i-1}(E) \xrightarrow{\delta^{i-1}} 0 \rightarrow H^i(A) \xrightarrow{H^i(\alpha)} H^i(C) \xrightarrow{H^i(\beta)} H^i(E) \xrightarrow{\delta^i} 0$$

and

$$H^{i-1}(D) \xrightarrow{\delta^{i-1}} 0 \rightarrow H^i(B) \xrightarrow{H^i(\varphi)} H^i(C) \xrightarrow{H^i(\psi)} H^i(D) \xrightarrow{\delta^i} 0$$
and we have an exact diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^i(B) & \xrightarrow{H^i(\phi)} & H^i(C) & \xrightarrow{H^i(\psi)} & H^i(D) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H^i(A) & \xrightarrow{H^i(\alpha)} & H^i(E) & \xrightarrow{H^i(\beta)} & H^i(E) & \rightarrow & 0 \\
\end{array}
\]

Now it is an easy exercise on linear algebra to show that \( H^i(\beta\phi) = H^i(\beta) \circ H^i(\phi) \) is an isomorphism, i.e., \( \beta\phi \) is a quasiisomorphism. Note that, by symmetry, if \( \beta\phi \) is a quasiisomorphism, then also \( \psi\alpha \) is a quasiisomorphism. □

**Lemma 10.3.** Consider a subcomplex \( U \) of \( V \subseteq W \) and denote

\[
L_{U,V,W} = L_{U,W} \cap L_{V,W} = \{ f \in \text{Hom}^*(W,W) \mid f(U) \subseteq U, f(V) \subseteq V \}.
\]

1. If \( U \hookrightarrow V \) is a quasiisomorphism, then the two inclusions

\[
L_{U,V,W} \hookrightarrow L_{V,W}, \quad L_{U,V,W} \hookrightarrow L_{U,W}
\]

are quasiisomorphisms of DGLA.

2. If \( V \hookrightarrow W \) is a quasiisomorphism, then the inclusions \( L_{U,V,W} \hookrightarrow L_{U,W} \) and the projection

\[
L_{U,V,W} \twoheadrightarrow L_{U,V}, \quad f \mapsto f|_V
\]

are quasiisomorphisms of DGLA.

**Proof.** Assume that \( U \hookrightarrow V \) is a quasiisomorphism; consider the exact sequences

\[
0 \rightarrow L_{U,V,W} \xrightarrow{(\text{id}|_V,\text{id})} L_{U,W} \oplus L_{V,W} \xrightarrow{\tau} L_V \rightarrow 0, \quad \tau(f,g) = f - g|_V
\]

and

\[
0 \rightarrow L_{U,V} \xrightarrow{\text{id},0} L_{U,W} \oplus L_{V,W} \xrightarrow{\pi_2} L_{V,W} \rightarrow 0.
\]

The composition \( \tau \circ (\text{id},0) : L_{U,V} \rightarrow L_V \) coincides with the inclusion \( L_{U,V} \hookrightarrow L_V \). By Lemma 10.1, the inclusion \( L_{U,V} \hookrightarrow L_V \) is a quasiisomorphism. Therefore, by Lemma 10.2 the map \( \pi_2 \circ (\text{id}|_V,\text{id}) : L_{U,V,W} \rightarrow L_{V,W} \) is a quasiisomorphism, i.e., \( L_{U,V,W} \hookrightarrow L_{V,W} \) is a quasiisomorphism.

The acyclicity of \( V/U \), together with the exact sequence

\[
0 \rightarrow L_{U,V,W} \rightarrow L_{U,W} \xrightarrow{\sigma} \text{Hom}^* \left( \frac{V}{U}, \frac{W}{V} \right) \rightarrow 0, \quad \sigma(g) = g|_V,
\]

imply that \( L_{U,V,W} \hookrightarrow L_{U,W} \) is a quasiisomorphism.

The proof of the second statement is very similar. Assume that \( V \hookrightarrow W \) is a quasiisomorphism and consider the exact sequences

\[
0 \rightarrow L_{U,V,W} \xrightarrow{(\text{id}|_V,\text{id})} L_{U,V} \oplus L_{V,W} \xrightarrow{\tau} L_V \rightarrow 0, \quad \tau(f,g) = g|_V - f
\]

and

\[
0 \rightarrow L_{V,W} \xrightarrow{(0,\text{id})} L_{U,V} \oplus L_{V,W} \xrightarrow{\pi_1} L_{U,V} \rightarrow 0.
\]
The projection $L_{V,W} \to L_V$ is a quasiisomorphism by Lemma 10.1. This projection can be written as the composition $\tau \circ (0, \text{id}) : L_{V,W} \to L_V$, hence by Lemma 10.2 the map $\pi_1 \circ (\text{id}|_V, \text{id}) : L_{U,V,W} \to L_{U,V}$ is a quasiisomorphism, i.e., the projection $L_{U,V,W} \to L_{U,V}$ is a quasiisomorphism. Moreover $W/V$ is acyclic and then the inclusion $L_{U,V,W} \hookrightarrow L_{U,W}$ is a quasiisomorphism. □

Lemma 10.4. Assume we have a commutative diagram of inclusions of differential graded vector spaces

$$
\begin{array}{ccc}
U & \to & V \\
\downarrow & & \downarrow \\
Z & \to & W
\end{array}
$$

If the horizontal arrows are quasiisomorphisms then $\text{Grass}_{U,Z} \simeq \text{Grass}_{U,W} \simeq \text{Grass}_{V,W}$.

Proof. In the notation of Lemma 10.3, there exists a commutative diagram of DGLA, where every arrow is either the natural inclusion or the natural projection:

$$
\begin{array}{ccc}
L_{U,Z} & \leftarrow & L_{U,Z,W} \to L_{U,W} \leftarrow L_{U,V,W} \to L_{V,W} \\
\downarrow \eta & & \downarrow \rho & & \downarrow \chi \\
L_Z & \leftarrow & L_{Z,W} \to L_W \leftarrow L_W \to L_W
\end{array}
$$

By Lemma 10.3 the horizontal arrows are quasiisomorphism and then, Theorem 6.1 gives an isomorphism of deformation functors

$$
\text{Grass}_{U,Z} \simeq \text{Def}_\eta \simeq \text{Def}_\rho \simeq \text{Grass}_{U,W} \simeq \text{Def}_\chi \simeq \text{Grass}_{V,W}.
$$

□

Remark 10.5. In the same hypothesis of Lemma 10.4, if in addition $U = V \cap Z$, then the isomorphism $\text{Grass}_{U,Z} \simeq \text{Grass}_{V,W}$ can be done in a more explicit and easy way. In fact the two inclusion

$$
\alpha : L_{Z,W} \cap L_{V,W} \hookrightarrow L_{U,V,W}, \quad \beta : L_{Z,W} \cap L_{V,W} \hookrightarrow L_{U,Z,W}
$$

are quasiisomorphisms since their cokernels are the acyclic complexes

$$
\text{Coker}(\alpha) = \text{Hom}^* \left( \frac{Z}{U}, \frac{W}{Z} \right), \quad \text{Coker}(\beta) = \text{Hom}^* \left( \frac{V}{U}, \frac{W}{V} \right).
$$

Then, according to Lemma 10.4 every horizontal arrow of

$$
\begin{array}{ccc}
L_{U,Z} & \leftarrow & L_{Z,W} \cap L_{V,W} \hookrightarrow L_{V,W} \leftarrow L_{L_W} \to L_{L_W} \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \chi \\
L_Z & \leftarrow & L_{Z,W} \hookrightarrow L_W \to L_W
\end{array}
$$

is a quasiisomorphism. The isomorphism $\text{Grass}_{V,W} \simeq \text{Grass}_{U,Z}$ can then be explicitly described as follows: given $A \in \text{Art}$ and an element $e^\alpha(V \otimes A)$ in $\text{Grass}_{V,W}(A)$, there exists (and it is unique up to gauge) an element $e^\alpha \in \exp(\text{Hom}_0^0(W, W) \otimes \mathfrak{m}_A)$ such that $e^\alpha(V \otimes A) = e^\alpha(V \otimes A)$ and $e^\alpha(Z \otimes A) \subseteq Z \otimes A$. Then the isomorphism $\text{Grass}_{V,W} \simeq \text{Grass}_{U,Z}$ is given by the map

$$
e^\alpha(V \otimes A) \mapsto e^\alpha(U \otimes A) \subseteq Z \otimes A.
$$

Theorem 10.6. Let $V$ be a subcomplex of $(W,d)$ such that the inclusion $V \hookrightarrow W$ induces an injective morphism in cohomology $H^*(V) \hookrightarrow H^*(W)$. Identifying $H^*(V)$
with its image in $H^*(W)$, then the cohomology functor $H^*$ gives an isomorphism of functors
\[ H^* : \text{Grass}_{V, W} \rightarrow \text{Grass}_{H^*(V), H^*(W)} \cong \prod_i \text{Grass}_{H^i(V), H^i(W)}. \]

Proof. It is possible to find “harmonic representatives” $H_V \subseteq V$ and $H_W \subseteq W$ with $H_V = H_W \cap V$. Indeed, the injectivity of $H^*(V) \rightarrow H^*(W)$ implies the two equalities
\[ Z^*(V) = V \cap Z^*(W), \quad B^*(V) = V \cap B^*(W). \]
Therefore it is possible to find a subspace $H_W \subseteq Z^*(W)$ such that $H_W \oplus B^*(W) = Z^*(W)$ and $(H_W \cap V) \oplus B^*(V) = Z^*(V)$. Set $H_V = H_W \cap V$. Then the diagram
\[
\begin{array}{ccc}
H_V & \rightarrow & V \\
\downarrow & & \downarrow \\
H_W & \rightarrow & W
\end{array}
\]
satisfies the hypothesis of Lemma 10.4 and Remark 10.5 and we have an isomorphism $\text{Grass}_{V, W} \cong \text{Grass}_{H_V, H_W}$ given by
\[ e^\alpha(V \otimes A) \mapsto e^\alpha(H_V \otimes A), \]
where $e^\alpha \in \exp(\text{Hom}^0(W, W) \otimes \mathfrak{m}_A)$ is such that $e^\alpha(V \otimes A) = e^\alpha(V \otimes A)$ and $e^\alpha(H_W \otimes A) \subseteq H_W \otimes A$.

Now, according to Theorem 9.3 it is sufficient to note that the natural map
\[ e^\alpha(H_V \otimes A) \rightarrow H^*(e^\alpha(V \otimes A), d) = H^*(e^\alpha(V \otimes A), d) \]
is an isomorphism. \qed

Part 3. The universal period map

In this part we work over the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Unless otherwise specified, the symbol $\otimes$ denotes tensor product over $\mathbb{C}$.

11. The Kodaira-Spencer DGLA

We will follow the same general notation of [33]; in particular, for a differentiable manifold $M$ we denote by $\mathcal{A}^p_M$ the sheaf of differentiable $p$-forms with complex coefficients and by $d : \mathcal{A}^p_M \rightarrow \mathcal{A}^{p+1}_M$ the De Rham differential.

We think an almost complex structure on $M$ as a subsheaf $\mathcal{V} \subseteq \mathcal{A}^1_M$ of locally free $\mathcal{A}^0_M$-modules such that $\mathcal{V} \oplus \overline{\mathcal{V}} = \mathcal{A}^1_M$.

For a complex manifold $X$ we denote by:
\begin{itemize}
  \item $T_{X, \mathbb{C}} = T^{1,0}_X \oplus T^{0,1}_X$ the complexified differential tangent bundle.
  \item $T_X \cong T^{1,0}_X$ the holomorphic tangent bundle.
  \item $\mathcal{A}^{p,q}_X$ the sheaf of differentiable $(p, q)$-forms and by $\mathcal{A}^{p,q}_X(T_X)$ the sheaf of $(p, q)$-forms with values in $T_X$.
  \item $A^{p,q}_X$ and $A^{p,q}_X(T_X)$ the vector spaces of global sections of $\mathcal{A}^{p,q}_X$ and $\mathcal{A}^{p,q}_X(T_X)$ respectively.
\end{itemize}
Recall that an almost complex structure \( \mathcal{V} \subseteq A^1_X \) is called integrable if there exists a structure of complex manifold on \( X \) such that \( \mathcal{V} = A^{1,0}_X \).

The direct sum

\[
A_X = \bigoplus_i A^i_X, \quad \text{where} \quad A^i_X = \bigoplus_{p+q=i} A^{p,q}_X,
\]

endowed with the wedge product \( \wedge \), is a sheaf of graded algebras; we denote by \( \text{Der}^{a,b}(A_X) \) the sheaf of its \( \mathbb{C} \)-linear derivations of bidegree \( (a,b) \). Notice that \( \partial \) and \( \overline{\partial} \) are global sections of \( \text{Der}^{1,0}(A_X) \) and \( \text{Der}^{0,1}(A_X) \) respectively.

The direct sum \( \text{Der}^*(A_X) = \bigoplus_k \oplus_{a+b=k} \text{Der}^{a,b}(A_X) \) is a sheaf of differential graded Lie algebras, with its natural bracket

\[
[f, g] = fg - (-1)^{\deg(f) \deg(g)} gf
\]

and differential \([d, -] = [\partial + \overline{\partial}, -]\).

Similarly \( A^{0,*}_X(T_X) \) is a sheaf of DGLA, where the bracket \([ , , ]\) and the differential \( D \) are defined in local coordinates by the formulas:

\[
D \left( \phi \frac{\partial}{\partial z_i} \right) = -\overline{\partial}(\phi) \frac{\partial}{\partial z_i}, \quad \phi \in A^{0,*}_X,
\]

\[
\left[ f dz^l \frac{\partial}{\partial z_i}, g dz^j \frac{\partial}{\partial z_j} \right] = \overline{\partial} f \wedge \overline{\partial} g \left( f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i} \right).
\]

The contraction of differential forms with vector fields is used to define two injective morphisms of sheaves: the contraction map

\[
i: A^{0,*}_X(T_X) \to \text{Der}^*(A_X)[-1], \quad a \mapsto i_a, \quad i_a(\omega) = a \cdot \omega,
\]

and the holomorphic Lie derivative

\[
l: A^{0,*}_X(T_X) \to \text{Der}^*(A_X), \quad a \mapsto l_a = [\partial, i_a], \quad l_a(\omega) = \partial(a \cdot \omega) + (-1)^{\deg(a)} a \cdot \partial \omega.
\]

**Lemma 11.1.** In the notation above, for every \( a, b \in A^{0,*}_X(T_X) \) we have

\[
i_{Da} = -[\overline{\partial}, i_a], \quad i_{[a,b]} = [i_a, [\partial, i_b]], \quad [i_a, i_b] = 0.
\]

In particular, since \([\partial, i_b] = [d, i_b] + i_{Db}\), the contraction map \( i \) is a Cartan homotopy and the holomorphic Lie derivative is a morphism of sheaves of differential graded Lie algebras.

**Proof.** Since locally \( A_X \) is generated as \( \mathbb{C} \)-algebra by \( A^{0,0}_X \oplus A^{1,0}_X \oplus A^{0,1}_X \), if \( \min(a,b) \leq -2 \) then \( \text{Der}^{a,b}(A_X) = 0 \), while if \( \min(a,b) \leq -1 \) then every \( h \in \text{Der}^{a,b}(A_X) \) is \( C^\infty \)-linear.

The elements of the third formula belong to \( \text{Der}^{-2,*}(A_X) \) and therefore vanish. Every term of first two formulas belongs to \( \text{Der}^{-1,*}(A_X) \) and then it is sufficient to check equalities of such derivations when applied to \( dz_i \), where \( z_1, \ldots, z_n \) are local holomorphic coordinates. This is straightforward and it is left to the reader: see also Lemma 7 of [21]. \( \square \)

The contraction map gives a natural isomorphism of vector spaces

\[
i: A^{0,1}_X(T_X) \to \text{Hom}_{A^0}(A^{1,0}_X, A^{0,1}_X).
\]

For every sufficiently small \( \xi \in A^{0,1}_X(T_X) \), the graph of \( i_\xi \in \text{Hom}_{A^0}(A^{1,0}_X, A^{0,1}_X) \) determines a variation of the almost complex structure of \( X \) given by

\[
A^{1,0}_\xi = \{ \omega \in A^1_X \mid \overline{\pi}(\omega) = i_\xi \pi(\omega) \} = \{ \omega \in A^1_X \mid \pi(\omega) = \xi \cdot \pi(\omega) \},
\]
where \( \pi : A^1_X \to A^{1,0}_X \) and \( \overline{\pi} : A^1_X \to A^{0,1}_X \) are the projections. Then denote
\[
A^{0,1}_\xi \quad \text{and} \quad A^{p,q}_\xi = \bigwedge^p A^{1,0}_\xi \otimes \bigwedge^q A^{0,1}_\xi.
\]
The sheaf of \( \xi \)-holomorphic functions is by definition
\[
\mathcal{O}_\xi = \{ f \in A^0_X \mid df \in A^{1,0}_\xi \}
\]
and then, according to the definition of \( A^{1,0}_\xi \), we have
\[
\mathcal{O}_\xi = \{ f \in A^0_X \mid \overline{\partial} f = \xi \partial f \} = \{ f \in A^0_X \mid (\partial + i\xi) f = 0 \}.
\]

Since \( i\xi \) is a nilpotent derivation of degree 0 of the sheaf of graded algebras \( A_X \), its exponential
\[
e^{i\xi} = \exp(\xi \partial) : A_X \to A_X
\]
is an isomorphism of graded algebras.

**Lemma 11.2.** In the above notation \( e^{i\xi}(A^{1,0}_X) = A^{1,0}_\xi \).

*Proof.* For every \( \omega \in A^1_X \) we have \( e^{i\xi}(\omega) = \omega + \xi \partial \pi(\omega) \), therefore
\[
\pi e^{i\xi}(\omega) = \pi(\omega), \quad \overline{\pi} e^{i\xi}(\omega) = \overline{\pi}(\omega) + \xi \partial \pi(\omega)
\]
and then \( e^{i\xi}(\omega) \in A^{1,0}_\xi \) if and only if \( \overline{\pi}(\omega) = 0 \). \( \square \)

The Newlander-Nirenberg theorem [24], [13, Thm. 5.5] implies (see e.g. [2, Lecture 1]) that the following four conditions are equivalent:

1. The almost complex structure \( A^{1,0}_\xi \) is integrable.
2. \( \xi \) is a solution of the Maurer-Cartan equation [13, Equation 5.86], [2, Equation 2.5]
\[
D\xi + \frac{1}{2}[\xi,\xi] = 0.
\]
3. For every \( x \in X \) there exist \( f_1, \ldots, f_n \in \mathcal{O}_{\xi,x}, n = \dim X \), such that \( df_1, \ldots, df_n \) are a basis of the \( A^{0,1}_{X,x} \)-module \( A^{1,0}_{\xi,x} \).
4. \( \mathcal{F}^1_\xi \subseteq \mathcal{F}^1_\xi \), where \( \mathcal{F}^1_\xi = \bigoplus_{p \geq 1} A^{p,q}_\xi = (A^{1,0}_\xi) \subseteq A_X \) is the graded ideal sheaf generated by \( A^{1,0}_\xi \).

**Definition 11.3.** The Kodaira-Spencer algebra of a complex manifold \( X \) is the differential graded Lie algebra \( K_X = \bigoplus_i A^{0,i}_X(T_X) \) of global sections of \( \bigoplus_i A^{0,i}_X(T_X) \).

Denoting by \( \text{Def}_X : \text{Art} \to \text{Set} \) the functor of infinitesimal deformations of \( X \):
\[
\text{Def}_X(B) = \frac{\text{deformations of } X \text{ over } \text{Spec}(B)}{\sim}.
\]
A deformation of \( X \) over \( \text{Spec}(B) \) may be interpreted as a morphism \( \mathcal{O}_B \to \mathcal{O}_X \) of sheaves of \( B \)-algebras such that \( \mathcal{O}_B \) is flat over \( B \) and the induced map \( \mathcal{O}_B \otimes_B \mathbb{C} \to \mathcal{O}_X \) is an isomorphism.

The following result is well known [3], [8], [14, Ex. 3.4.1]: a detailed proof will also appear in [12].
Theorem 11.4. There exists an isomorphism of functors

\[ \mathcal{O} : \text{Def}_{K_X} \rightarrow \text{Def}_X \]

defined in the following way: given a local Artinian \( \mathbb{C} \)-algebra \( B \) and a solution of the Maurer-Cartan equation \( \xi \in A^{0,1}_X(T_X) \otimes m_B \) we set

\[ \mathcal{O}_\xi = \ker(A^{0,0}_X \otimes B \xrightarrow{\partial + i\xi} A^{0,1}_X \otimes B) = \{ f \in A^{0,0}_X \otimes B \mid \partial f = \xi \cdot \partial f \} \]

and the map \( \mathcal{O}_\xi \rightarrow \mathcal{O}_X \) is induced by the projection \( A^{0,0}_X \otimes B \rightarrow A^{0,0}_X \otimes \mathbb{C} = A^{0,0}_X \).

12. The period map

Let \( X \) be a fixed complex manifold; we shall denote by \( A^*_X = F^0 \supseteq F^1 \supseteq \cdots \) the Hodge filtration of differential forms on \( X \), i.e. for every \( p \geq 0 \)

\[ F^p = \bigoplus_{i \geq 0} A^i_{X,j}. \]

Theorem 12.1. Let \( p \) be a fixed nonnegative integer and consider the inclusion of differential graded Lie algebras

\[ L_{F^p,A_X} = \{ f \in \text{Hom}^*(A_X, A_X) \mid f(F^p) \subseteq F^p \} \xrightarrow{\chi} \text{Hom}^*(A_X, A_X) = L_{A_X}. \]

Then the linear map

\[ p^p : K_X \rightarrow C_X, \quad p^p(\xi) = (l_\xi, i_\xi) = ([\partial, i_\xi], i_\xi) \]

is a linear \( L_\infty \)-morphism.

In particular \( p^p \) induces a natural transformation of functors:

\[ p^p : \text{Def}_X \rightarrow \text{Def}_X = \text{Grass}_{F^p,A_X}. \]

Proof. According to Lemma 11.1 and Proposition 7.4, the Lie derivative

\[ K_X = A^{0,*}_X(T_X) \xrightarrow{l} \text{Hom}^*(A_X, A_X) \]

is a morphism of differential graded Lie algebras and the map

\[ \tilde{i} : K_X \rightarrow C_l = K_X \oplus \text{Hom}^*(A_X, A_X)[-1], \quad \tilde{i}(\xi) = (\xi, i_\xi) \]

is a linear \( L_\infty \)-morphism.

The morphism \( p^p \) is the composition of \( \tilde{i} \) and the linear \( L_\infty \)-morphism \( C_l \rightarrow C_X \) induced by the horizontal arrows of the following commutative diagram of differential graded Lie algebras

\[
\begin{array}{ccc}
A^{0,*}_X(T_X) & \xrightarrow{l} & L_{F^p,A_X} \\
\downarrow t & & \downarrow \chi \\
\text{Hom}^*(A_X, A_X) & \xrightarrow{\sim} & L_{A_X}
\end{array}
\]

\( \square \)

Corollary 12.2. If \( \xi \in A^{0,1}_X(T_X) \) is a solution of the Maurer-Cartan equation, then in the associative algebra \( \text{Hom}^*(A_X, A_X) \) we have the equality

\[ e^{-i_\xi}d + i_\xi = d + e^{-i_\xi} \ast 0 = d + l_\xi = d + [\partial, i_\xi]. \]
Proof. The first equality follows from the explicit description of the gauge action in the DGLA Hom\(^*\)(\(A_X, A_X\)). Taking \(p = 0\) in Theorem 12.1, the pair \(p^0(\xi) = (l_\xi, i_\xi)\) satisfies the Maurer-Cartan equation in \(C_X\) and then \(e^{i_\xi} \ast l_\xi = 0\). Obviously the above equality can be also proved directly as a consequence of Cartan homotopy formulas (Lemma 11.1).

**Theorem 12.3.** In the same notation of Theorem 12.1, assume that \(X\) is a compact Kähler manifold and denote by

\[
P^p : \text{Def}_X \rightarrow \text{Grass}_{H^* (F^p), H^* (A_X)}
\]

the composition of the natural transformation \(p^p : \text{Def}_X \rightarrow \text{Grass}_{F^p, A_X}\) and the cohomology isomorphism \(H^* : \text{Grass}_{F^p, A_X} \rightarrow \text{Grass}_{H^* (F^p), H^* (A_X)}\) (see Theorem 10.6). Then \(P^p\) is the universal period map.

**Proof.** Let \(B \in \text{Art}\) and \(\xi \in \text{MC}_{K_X}(B)\); by definition

\[
P^p(\xi) = H^*(e^{i_\xi}(F^p \otimes B)) \subseteq H^*(A_X \otimes B) = H^*(A_X) \otimes B.
\]

On the other hand, the period of the infinitesimal deformation \(O_\xi = \ker(\partial + l_\xi)\) is the \(B\)-submodule \(H^*(F^p_\xi) \subseteq H^*(A_X \otimes B)\), where \(F^p_\xi\) is the complex of global sections of the differential ideal sheaf \(F^p_\xi \subseteq A_X \otimes B\) generated by \((dO_\xi)^p\).

It is sufficient to prove that \(e^{i_\xi}(F^p \otimes B) = F^p_\xi\); since \(e^{i_\xi} : A_X \otimes B \rightarrow A_X \otimes B\) is a morphism of sheaves of \(B\)-algebras, it is sufficient to prove that \(e^{-i_\xi}(dO_\xi) \subseteq A^{1,0}_X \otimes B\).

This equality, together rank considerations, will imply that

\[
e^{-i_\xi}(F^p_\xi) = \bigoplus_{i \geq p, \ j} A^{i,j}_X \otimes B.
\]

Since \(e^{i_\xi}\) is the identity on \(A^{0,0}_X \otimes B\), by Corollary 12.2 we can write

\[
e^{-i_\xi}(dO_\xi) = e^{-i_\xi} d e^{i_\xi} O_\xi = (\partial + \bar{\partial} + l_\xi) O_\xi = \partial O_\xi \subseteq \partial A^{0,0}_X \otimes B \subseteq A^{1,0}_X \otimes B.
\]

**Remark 12.4.** In the statement of Theorem 12.3 the Kähler assumption is used in cohomological sense; more precisely we only require that the cohomology of the complex \(F^p\) injects into the De Rham cohomology of \(X\).

**Corollary 12.5** (Griffiths). The differential of the universal period map is

\[
dP^p = i : H^1(X, T_X) \rightarrow \bigoplus_i \text{Hom} \left( F^p H^i(X, \mathbb{C}), \frac{H^i(X, \mathbb{C})}{F^p H^i(X, \mathbb{C})} \right).
\]

**Proof.** Recall from Remark 3.1 that the projection on the second factor induces an identification

\[
H^j(C_X) \xrightarrow{\sim} H^{j-1}(\text{Hom}^*(F^p, A_X/F^p)) = \bigoplus_i \text{Hom}(H^i(F^p), H^{i+j-1}(A_X/F^p)).
\]

Via this identification, the \(L_\infty\)-morphism \(p^p : K_X \rightarrow C_X\), induces in cohomology the map

\[
H^j(p^p) = i : H^j(K_X) \rightarrow \bigoplus_i \text{Hom}(H^i(F^p), H^{i+j-1}(A_X/F^p)).
\]

The differential of \(P^p : \text{Def}_X \rightarrow \text{Grass}_{F^p, A_X} \simeq \text{Grass}_{H^* (F^p), H^* (X, \mathbb{C})}\) is therefore

\[
dP^p = i : H^1(K_X) \rightarrow \bigoplus_i \text{Hom}(H^i(F^p), H^i(A_X/F^p)).
\]

\[\square\]
Corollary 12.6 (Kodaira’s Principle [3, 21, 26]). The obstructions to deformations of $X$ are contained in the kernel of
\[ i: H^2(X, T_X) \to \bigoplus_i \text{Hom} \left( F^p H^i(X, \mathbb{C}), \frac{H^{i+1}(X, \mathbb{C})}{F^p H^{i+1}(X, \mathbb{C})} \right), \]
for every $p \geq 0$.

Proof. We use the same general argument of [21, Section 5]: since the period map $\mathcal{P}^p: \text{Def}_X \to \text{Grass} H^*(F^p, H^*(X, \mathbb{C}))$ is induced by the $L_\infty$-morphism $p^p: K_X \to C_X$, the linear map
\[ H^2(\mathcal{P}^p): H^2(X, T_X) \to H^2(C_X) \]
is a morphism of obstruction theories, i.e., it commutes with the natural obstruction maps for $\text{Def}_X$ and $\text{Grass} H^*(F^p, H^*(X, \mathbb{C}))$ [23]. In particular, obstructions to deforming the complex structure of the Kähler manifold $X$ are mapped to obstructions of the functor $\text{Grass} H^*(F^p, H^*(X, \mathbb{C}))$. Since the latter is unobstructed, the obstructions to deforming $X$ are annihilated by $H^2(p^p)$. By the proof of Corollary 12.5, $H^2(p^p) = i$. $\square$

13. TRASVERSALITY

Consider a fixed compact Kähler manifold $X$ and a differential graded commutative $\mathbb{C}$-algebra $(\Omega, d_\Omega)$. Let
\[ \text{Id} \otimes d_\Omega: A_X \otimes \Omega \to A_X \otimes \Omega, \quad (\text{Id} \otimes d_\Omega)(a \otimes \omega) = (-1)^{\deg(a)}a \otimes d_\Omega(\omega) \]
the trivial extension of $d_\Omega$. Then $\text{Id} \otimes d_\Omega$ is a differential of the graded algebra $A_X \otimes \Omega$ inducing a flat connection
\[ \text{Id} \otimes d_\Omega: H^*(A_X) \otimes \Omega^i \to H^*(A_X) \otimes \Omega^{i+1}. \]
Assume now that $B$ is a commutative unital $\mathbb{C}$-algebra and let $\phi: B \to \Omega^0$ be a morphism of graded unital $\mathbb{C}$-algebras.

Via the natural isomorphisms
\[ A_X \otimes \Omega = (A_X \otimes B) \otimes_B \Omega, \quad H^*(A_X) \otimes \Omega = H^*(A_X \otimes B) \otimes_B \Omega \]
the operator $\text{Id} \otimes d_\Omega$ induces the differential
\[ \nabla: (A_X \otimes B) \otimes_B \Omega \to (A_X \otimes B) \otimes_B \Omega, \]
\[ \nabla((a \otimes b) \otimes_B \omega) = (-1)^{\deg(a)}(a \otimes 1) \otimes_B d_\Omega(\phi(b)\omega) \]
and the flat connection
\[ \nabla: H^*(A_X \otimes B) \otimes_B \Omega^i \to H^*(A_X \otimes B) \otimes_B \Omega^{i+1}, \]
that, by analogy with the case of $B$ a power series ring and $\Omega = \bigwedge_B^* \Omega_B$, we shall call Gauss-Manin connection.

Assume now that $B \in \text{Art}$ and consider a deformation of $X$ over $\text{Spec}(B)$ determined by the Kuranishi data $\xi \in \text{MC}_{K_X}(B) \subseteq A_X^{0,1}(T_X) \otimes m_B$.

The classical Griffiths’ trasversality theorem [10], [33, Prop. 10.18] generalizes to the following result:

Proposition 13.1 (Trasversality). Let $F^p_\xi$ be the complex of global sections of the differential ideal sheaf $F^p_\xi \subseteq A_X \otimes B$ generated by $(d\mathcal{O}_\xi)^p$. Then
\[ \nabla(H^*(F^p_\xi) \otimes_B \Omega^i) \subseteq H^*(F^{p-1}_\xi) \otimes_B \Omega^{i+1}. \]
Proof. It is not restrictive to assume \( B \subseteq \Omega^0 \subseteq \Omega \) and \( \phi \) the inclusion. For notational simplicity denote by \( \delta : A^0_X(T_X) \otimes \Omega \to A^0_X(T_X) \otimes \Omega \) the trivial extension of \( d_\Omega \), i.e. \( \delta(\xi \otimes b) = (-1)^{\deg(\xi)} \xi \otimes d_\Omega(b) \).

It is sufficient to prove that \( \nabla(F^p_{\xi} \otimes_B \Omega^i) \subseteq F^{p-1}_{\xi} \otimes_B \Omega^{i+1} \).

The contraction map \( i \) extends naturally to a Cartan homotopy (Example 7.3)

\[
i : A^0_X(T_X) \otimes \Omega \to \text{Hom}^*(A_X, A_X) \otimes \Omega \subseteq \text{Hom}^*((A_X \otimes B) \otimes_B \Omega, (A_X \otimes B) \otimes_B \Omega)
\]

such that \( i_{\delta x} = -[\nabla, i_x] ; \) in particular \( i_x([i_x, \nabla]) = -[i_x, i_{\delta x}] = 0 \) for every \( x \). For every nilpotent \( a \in \text{Hom}^0((A_X \otimes B) \otimes_B \Omega, (A_X \otimes B) \otimes_B \Omega) \) we have the equality

\[
\nabla, e^a = \nabla e^a - e^a \nabla = e^a(e^{-a} \nabla e^a - \nabla) = e^a([\nabla, a] - e^a \sum_{n \geq 2} \frac{(-ad_a)^n}{n!} \nabla).
\]

Since \( \xi \in MC_{K_X}(B) \), for \( a \in A_X \otimes B \) and \( \omega \in \Omega \) we have \( i_\xi(a \otimes_B \omega) = i_\xi(a) \otimes_B \omega \); therefore

\[
\nabla, e^{i_\xi} = e^{i_\xi} \left(-\frac{1}{2}[i_\xi, i_\delta] + \frac{1}{6}[i_\xi, [i_\xi, i_\delta]] + \cdots \right) = -e^{i_\xi} i_{\delta \xi}.
\]

Now it is easy to conclude the proof: we have seen in the proof of Theorem 12.3 that \( F^p_{\xi} = e^{i_\xi}(F^p \otimes B) \) and then every \( v \in F^p_{\xi} \otimes_B \Omega \) can be written as \( v = e^{i_\xi}(u) \), with \( u \in (F^p \otimes B) \otimes_B \Omega \). Therefore

\[
\nabla(v) = \nabla(e^{i_\xi}(u)) = [\nabla, e^{i_\xi}](u) + e^{i_\xi}([\nabla, u]) = e^{i_\xi}(-i_{\delta \xi}(u) + \nabla(u)) \in e^{i_\xi}((F^{p-1} \otimes B) \otimes_B \Omega) = F^{p-1}_{\xi} \otimes_B \Omega.
\]

\( \square \)

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