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Realizations of $SU_q(2)$ Algebra in Terms of $q$-Deformed Multimode Boson Oscillators

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Abstract. We consider some version of $q$-deformed multimode boson oscillators. The realization of $SU_q(2)$ algebra in terms of $q$-deformed multimode boson oscillators which involves $q$-interference between oscillators of different modes and the realization of $SU_q(2)$ algebra in terms of $q$-deformed multimode boson oscillators in which each oscillator mode has its own deformation parameter are constructed.

1. Introduction
Quantum groups and quantum algebras have been shown to arise in many problems of current physical and mathematical interest. Much effort is now being devoted to the construction of their representations and recently many realizations have been usefully devised using $q$-deformations of boson and fermion operators [1, 2, 3].
On the other hand, quantum groups are a subject of great activity at present and although their direct physical interpretation is still lacking, it is of particular importance to study the possible physical implications of these deformations. These structures which first emerged in connection with the quantum inverse scattering theory [4], the quantum Yang – Baxter equation .... The algebras may be described as a deformation, depending on one or more parameters of the ordinary Lie algebras [5].
In this paper, we introduction some versions of $q$-deformed multimode boson oscillators. The realization of $SU_q(2)$ algebra in terms of $q$-deformed multimode boson oscillators which involves $q$-interference between oscillators of different modes and the realization of $SU_q(2)$ algebra in terms of $q$-deformed multimode boson oscillators in which each oscillator mode has its own deformation parameter are constructed.

2. The Realizations of $SU_q(2)$ Algebra in Terms of $q$-Deformed Multimode Boson Oscillators which Involves $q$-Interference between Oscillators of Different Modes
In the classical case, the $SU(2)$ generators satisfy the commutation relations

\[
[J_0, J_-] = -J_-, \\
[J_0, J_+] = J_+, \\
[J_+, J_-] = 2J_0. 
\]

(1)

Realization of $SU(2)$ algebra in terms of the (ordinary) boson operators as follows
\[ J_+ = a_j^\dagger a_2, \]
\[ J_- = a_j a_1^\dagger, \]
\[ J_0 = \frac{1}{2} (N_1 - N_2). \]

In this work, we consider a version of deformation which involves some \( q \)-interference between oscillators of different modes. The creation and annihilation oscillator operators obeying bosonic commutation relations

\[ a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij} q^N, \]

\[ [a_i, a_j] = 0, (i \neq j) \]

where the deformation parameter \( q \) being real, and \( N \) is the total oscillator number operators,

\[ N = \sum_{i=1}^{k} N_i; \]
\[ [N_i, a_j] = -\delta_{ij} a_j, \]
\[ [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger. \]

Now the oscillators of different mode enter the theory not quite independently but with some \( q \) interference, which results from the presence of the factor \( q^N \) in the r. h. s. of 3. Equation 3 gives

\[ a_i (a_i^\dagger)^m = q^m (a_i^\dagger)^m a_i + q^{m-1} m (a_i^\dagger)^{m-1} q^N. \]

The \( q \)-oscillator algebra 3 can be realised in the Fock space spanned by the orthonormalised eigenstates of the oscillator number operators

\[ |n_1, n_2, \ldots, n_k\rangle = \frac{q^{-n_1-1}}{\sqrt{n_1! n_2! \ldots n_k!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \ldots (a_k^\dagger)^{n_k} |0\rangle \]

and in this space the following relations hold:

\[ a_i^\dagger a_i = q^{N-1} N_i, \]
\[ a_i a_i^\dagger = q^{N} (N_i + 1), \]

and hence

\[ \sum_{i=1}^{k} a_i^\dagger a_i = q^{N-1} N, \]
\[ \sum_{i=1}^{k} a_i a_i^\dagger = q^{N} (N + k), \]

where \( k \) is the number of modes.

The realisation of Lie algebra in terms of (ordinary) bosons are useful not only as a convenient mathematical tool, but also because of their applications in physics. In the case of quantum algebras it turns out that boson realizations are possible in terms the \( q \)-deformed boson operators already introduced above. In the case of \( SU_q(2) \) algebra for two mode oscillator eqs. 3 and 4 read:

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\[ a_i a_i^\dagger - q a_i^\dagger a_i = q_i^N, \quad (i = 1, 2) \]
\[ a_1 a_2 = q a_2 a_1, \]
\[ a_1 a_2 = a_2 a_1. \]

The realisation of \( SU_q(2) \) algebra based on the \( q \)-oscillator algebra 10 can be performed in the Fock space spanned by the orthonormalized of \( N_i \) defined as
\[ |j, m\rangle = \frac{q^{j-(j-\frac{1}{2})}}{\sqrt{(j+m)!(j-m)!}} (a_1^\dagger)^j (a_2^\dagger)^m |0\rangle, \]
and the generators can be mapped onto \( q \) deformed bosons as follows
\[ J_+ = q^{1-N} a_1^\dagger a_2, \]
\[ J_- = q^{1-N} a_2^\dagger a_1, \]
\[ J_0 = \frac{1}{2} (N_1 - N_2). \]

In fact, using 10, we can check that these generators satisfy the algebras 1.

3. The Realizations of \( SU_q(2) \) Algebra in Terms of \( q \)-Deformed Multimode Boson Oscillators in which each Oscillator Mode Has Its Own Deformation Parameter

Now, we consider a version of \( q \)-deformed multimode boson oscillators in which each oscillator mode has its own deformation parameter of form:
\[ a_i a_j^\dagger - q a_j a_i^\dagger = \delta_{ij} q_i^{2N}, \]
\[ q_i^{-1} a_i a_j - q_i^{-1} a_j a_i = 0, \]
\[ [N, a_i] = -a_i, \]
where \( N \) is the total oscillator number operators,
\[ N = \sum_{i=1}^{k} N_i, \]
\[ [N_i, a_j] = -\delta_{ij} a_j, \]
\[ [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger. \]

The basic of the Fock space is defined by repeated action of the creation operators on the vacuum state:
\[ |n_1, n_2, \ldots, n_k\rangle = \frac{1}{\sqrt{n_1!n_2!\ldots n_k!}} \prod_{i=1}^{k} q_i^{-n_i} \sum_{j>i}^{n_i} \frac{n_i(n_i-1)}{2} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \ldots (a_k^\dagger)^{n_k} |0\rangle, \]
where \( k \) is the number of the modes. In this space the following relations hold:
\[ a_i a_i^\dagger = q_i^{2(N-1)} N_i, \]
\[ a_i a_i^\dagger = q_i^{2N} (N_i + 1). \]

In particular, for two mode oscillator equations 13 and 14 read:
\[ a_i a_i^\dagger - q_i^{2N} a_i^\dagger a_i = q_i^{2N}, \quad (i = 1, 2) \]
The realisation of SU_q(2) algebra based on the q-deformed boson oscillator can be performed in the Fock space spanned by the orthonormalized eigenstates of N defined as

$$\langle jm \rangle = \frac{1}{\sqrt{(j+m)(j-m)+[j+m][j+m-1]}} q_1^{-(j-m)(j-m-1)} q_2^{-(j+m)(j+m-1)} \left( a_1^\dagger \right)^{j+m} \left( a_2^\dagger \right)^{j-m} \{|0\rangle\}
$$

with the identification

$$J_+ = q_1^{1-N} q_2^{1-N} a_1^\dagger a_2^\dagger$$
$$J_- = q_1^{1-N} q_2^{1-N} a_2^\dagger a_1^\dagger$$
$$J_0 = \frac{1}{2} \left[ q_1^{2(1-N)} a_1^\dagger a_1 - q_2^{2(1-N)} a_2^\dagger a_2 \right]
$$

Using algebra 20, we can check that these generators satisfy the algebras 13.

4. Conclusions
In this paper we have constructed generators of of SU_q(2) algebra in terms of q-deformed multimode boson oscillators which involves q-interference between oscillators of different modes (2.11) and in terms of q-deformed multimode boson oscillators in which each oscillator mode has its own deformation parameter (3.9). These realisations of SU_q(2) algebra in terms of q-deformed multimode boson oscillators have subsequently found applications in the SU_q(2) rotator model, as for example, rotational spectra of diatomic molecules.

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References
[1] R. J. Finkelstein, Lett. in Math. Phys. 34, (1996), 169.
[2] R. N. Mohapatra, Phys. Lett. B 242 (1990), 407.
[3] Y. Yang, and Z. Yu, Mod. Phys. Lett. A 8 (1993), 3025.
[4] O.W. Greenberg, Phys. Rev. Lett. 64, 705.
[5] D. Bonatsos and A. Klein, Nucl. Phys. A 469 (1987), 253.