TWO NEW ANALYTICAL SOLUTIONS AND TWO NEW GEOMETRICAL SOLUTIONS FOR THE WEIGHTED FERMAT-TORRICELLI PROBLEM IN THE EUCLIDEAN PLANE

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Abstract. We obtain two analytic solutions for the weighted Fermat-Torricelli problem in the Euclidean Plane which states that: Given three points in the Euclidean plane and a positive real number (weight) which correspond to each point, find the point such that the sum of the weighted distances to these three points is minimized. Furthermore, we give two new geometrical solutions for the the weighted Fermat-Torricelli problem (weighted Fermat-Torricelli point), by using the floating equilibrium condition of the weighted Fermat-Torricelli problem (first geometric solution) and a generalization of Hofmann’s rotation proof under the condition of equality of two given weights (second geometric solution).

1. Introduction

We state the weighted Fermat-Torricelli problem in $\mathbb{R}^2$:

**Problem 1.** Given a triangle $\triangle A_1 A_2 A_3$ with vertices $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, $A_3 = (x_3, y_3)$, find a fourth point $A_F = (x_F, y_F)$ which minimizes the objective function

$$f(x, y) = \sum_{i=1}^{3} B_i \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

(1.1)

where $B_i$ is a positive real number (weight) which corresponds to $A_i$.

By replacing $B_1 = B_2 = B_3$ in (1.1), we obtain the (unweighted) Fermat-Torricelli problem which was first stated by Pierre de Fermat (1643).

The solution of the weighted Fermat-Torricelli problem (Problem 1) is called the weighted Fermat-Torricelli point $A_F$.

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The existence and uniqueness of the weighted Fermat-Torricelli point and a complete characterization of the "floating case" and "absorbed case" has been established by Y. S Kupitz and H. Martini (see [9], theorem 1.1, reformulation 1.2 page 58, theorem 8.5 page 76, 77). A particular case of this result for three non-collinear points in $\mathbb{R}^2$, is given by the following theorem:

**Theorem 1.** [1], [9] Let there be given a triangle $\triangle A_1A_2A_3$, $A_1, A_2, A_3 \in \mathbb{R}^2$ with corresponding positive weights $B_1, B_2, B_3$.

(a) The weighted Fermat-Torricelli point $A_F$ exists and is unique.

(b) If for each point $A_i \in \{A_1, A_2, A_3\}$

$$\| \sum_{j=1, i \neq j}^3 B_j \vec{u}(A_i, A_j) \| > B_i,$$

(b1) the weighted Fermat-Torricelli point $A_F$ (weighted floating equilibrium point) does not belong to $\{A_1, A_2, A_3\}$ and

(b2)

$$\sum_{i=1}^3 B_i \vec{u}(A_F, A_i) = \vec{0},$$

where $\vec{u}(A_k, A_l)$ is the unit vector from $A_k$ to $A_l$, for $k, l \in \{0, 1, 2, 3\}$ (Weighted Floating Case).

(c) If there is a point $A_i \in \{A_1, A_2, A_3\}$ satisfying

$$\| \sum_{j=1, i \neq j}^3 B_j \vec{u}(A_i, A_j) \| \leq B_i,$$

then the weighted Fermat-Torricelli point $A_F$ (weighted absorbed point) coincides with the point $A_i$ (Weighted Absorbed Case).

A direct consequence of the weighted floating case and the weighted absorbed case of theorem [1] gives Corollary [1] (Torricelli’s theorem) and Corollary [1] (Cavalieri’s alternative) (see in [1], p 236).

**Corollary 1.** If $B_1 = B_2 = B_3$ and all three angles of the triangle $\triangle A_1A_2A_3$ are less than $120^\circ$, then $A_F$ is the isogonal point (interior point) of $\triangle A_1A_2A_3$ whose sight angle to every side of $A_1A_2A_3$ is $120^\circ$ (Torricelli’s theorem).

**Corollary 2.** If $B_1 = B_2 = B_3$ and one of the angles of the triangle $\triangle A_1A_2A_3$ is equal or greater than $120^\circ$, then $A_F$ is the vertex of the obtuse angle of $\triangle A_1A_2A_3$ (Cavalieri’s alternative).
Concerning the solution of the weighted Fermat-Torricelli problem with the use of analytic geometry and trigonometry, we mention the works of [3], [8], [10], [5], [4], [14] and [18].

Recently, an analytic solution, which express explicitly the coordinates of the weighted Fermat-Torricelli point with respect to the coordinates of the three points \(A_i\) and the three weights \(B_1, B_2, B_3\) for the weighted Fermat-Torricelli problem with respect to the weighted floating case of Theorem 1 has been derived in [13] and for the case \(B_1 = B_2 = B_3 = 1\) has been derived in [11].

In this paper, we present two new analytical solutions for the weighted Fermat-Torricelli problem in \(\mathbb{R}^2\) in the weighted floating case of Theorem 1. The first analytical solution gives the coordinates of the weighted Fermat-Torricelli point as a function of the coordinates of the three non collinear points and the three given weights (real positive numbers) in a different way from [13] and [11] (Theorem 2, Section 2).

The second analytical solution gives the location of the weighted Fermat-Torricelli point as a function of two inscribed angles of the circumscribed circle which passes form the three non collinear points and the three given weights by applying a coordinate independent approach given in [18] (Theorem 3, Section 3).

The first geometrical solution of the weighted Fermat-Torricelli point with ruler and compass focuses on constructing the intersection of two Simpson lines (weighted case) by applying the duality of the weighted Fermat-Torricelli problem which was introduced in [13] (Problem 2, Section 4).

Finally, the second geometric solution of the weighted Fermat-Torricelli point focuses on finding the angle of rotation of the three non collinear points about one of them and generalizes Hofmann’s rotation proof ([1], [6], [12]) regarding the equality of the given weights (Problem 3, Corollary 3, Section 4).

2. Analytical solution of the weighted Fermat-Torricelli problem

We obtain an analytic solution for the floating case of Theorem 1, i.e. the weighted Fermat-Torricelli point \(A_F\) is an interior point of \(\triangle A_1A_2A_3\), such that the coordinates \(x_F\) and \(y_F\) of \(A_F\) are expressed explicitly as a function of \(x_i, y_i\) and \(B_i\), for \(i = 1, 2, 3\), by using analytic geometry in \(\mathbb{R}^2\).

We denote by \(a_{ij}\) the length of the linear segment \(A_iA_j\) and \(\alpha_{ikj}\) the angle \(\angle A_iA_kA_j\) for \(i, j, k = 0, 1, 2, 2', 3, 3', i \neq j \neq k\) (See fig. 1).
Without loss of generality, we set \( A_1 = (0,0), A_2 = (a_{12},0), A_3 = (x_3,y_3). \)

We need the following two lemmata:

**Lemma 1.** \([1], [14]\) Under the condition (1.2) and the weighted floating equilibrium condition (1.3) the following equation is satisfied:

\[
\frac{B_i}{\sin \alpha_{203}} = \frac{B_2}{\sin \alpha_{103}} = \frac{B_3}{\sin \alpha_{102}} = C,
\]

where \( C = \sqrt{(B_1+B_2+B_3)(B_2+B_3-B_1)(B_1+B_3-B_2)(B_1+B_2-B_3)} \)

**Lemma 2.** \([1], [5], [11], [14]\) Under the condition (1.2) and the weighted floating equilibrium condition (1.3) the angle \( \alpha_{0ij} \) is expressed as a function of \( B_1, B_2, \) and \( B_3 \):

\[
\alpha_{0ij} = \arccos \left( \frac{B^2_k - B^2_i - B^2_j}{2B_iB_j} \right)
\]

for \( i,j,k = 1,2,3, \) and \( i \neq j \neq k. \)

**Theorem 2.** Under the condition (1.2) of the weighted floating case, the coordinates of the weighted Fermat-Torricelli point \( A_F (x_F,y_F) \) are given by the following relations:

\[
x_F = -\frac{(a_{12} - x'_2)(x'_3y_3 - x_3y'_3) + d_3(x_3 - x'_3)}{(a_{12} - x'_2)(y'_3 - y_3) - (y_3 - x'_3)y'_2}, \tag{2.3}
\]

\[
y_F = \frac{y'_2(a_{12}y_3 - x'_2y_3 - a_{12}y'_3 + x'_3y'_3)}{x_3y'_2 - x'_3y_2 + a_{12}y_3 - x'_2y_3 - a_{12}y'_3 + x'_3y'_3}. \tag{2.4}
\]

where

\[
x'_2 = -\frac{B_3}{B_2} \left( x_3 \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} + y_3 \sqrt{1 - \left( \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} \right)^2} \right), \tag{2.5}
\]

\[
y'_2 = \frac{B_3}{B_2} \left( x_3 \sqrt{1 - \left( \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} \right)^2} - y_3 \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} \right), \tag{2.6}
\]

\[
x'_3 = -a_{12} \frac{B_3}{B_2} \left( \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} \right) \tag{2.7}
\]

and

\[
y'_3 = -a_{12} \frac{B_3}{B_2} \left( \sqrt{1 - \left( \frac{B^2_1 - B^2_2 - B^2_3}{2B_2B_3} \right)^2} \right), \tag{2.8}
\]
**Proof of Theorem** We apply the weighted Torricelli configuration which is similar to the configuration used in and we construct two similar triangles $\triangle A_1A_2A_3'$ and $\triangle A_1A_3A_2'$, such that:

\[
\begin{align*}
\alpha_{13'}2 &= \alpha_{132} = \pi - \alpha_{102}, \quad (2.9) \\
\alpha_{213'} &= \alpha_{312'} = \pi - \alpha_{203}, \quad (2.10) \\
\end{align*}
\]

and

\[
\alpha_{123'} = \alpha_{123} = \pi - \alpha_{103}. \quad (2.11)
\]

From (2.11), (2.11) and (2.11), we derive that the point of intersection of the two circles which pass from $A_1, A_3', A_2$ and $A_1, A_2', A_3$, respectively, is the weighted Fermat-Torricelli point $A_F$ (fig. 1). Therefore, we obtain that $A_F$ is the intersection point of the lines (weighted Simpson lines) defined by $A_2A_2'$ and $A_3A_3'$.

Thus, we have:

\[
\begin{align*}
x_2' &= a_{12'} \cos(\alpha_{213} + \pi - \alpha_{203}), \quad (2.12) \\
y_2' &= a_{12'} \sin(\alpha_{213} + \pi - \alpha_{203}), \quad (2.13) \\
x_3' &= a_{13'} \cos(\pi - \alpha_{203}), \quad (2.14)
\end{align*}
\]
By applying the sine law in \( \triangle A_1 A_2 A_3 \) and \( \triangle A_1 A_3 A_2' \), we get, respectively:

\[ a_{13'} = a_{12} \frac{B_2}{B_3}, \tag{2.16} \]

and

\[ a_{12'} = a_{13} \frac{B_3}{B_2}. \tag{2.17} \]

By replacing (2.16), (2.17) and (2.2) from lemma 2 in (2.12), (2.13), (2.14) and (2.15), we obtain (2.1), (2.6), (2.7) and (2.8).

The equations of the lines defined by \( A_2 A_2' \) and \( A_3 A_3' \), respectively, are as follows:

\[ \frac{y_2'}{x_2' - a_{12}} = \frac{y}{x - a_{12}} \tag{2.18} \]

and

\[ \frac{y_3 - y_3'}{x_3 - x_3'} = \frac{y_3 - y}{x_3 - x} \tag{2.19} \]

Solving (2.18) and (2.19) with respect to \((x, y)\) we derive the point of intersection \( A_F = (x_F, y_F) \), and the coordinates \( x_F \) and \( y_F \) are given by (2.3) and (2.4), respectively.

\[ \square \]

3. AN EXPLICIT ANGULAR SOLUTION OF THE WEIGHTED FERMAT-TORRICELLI PROBLEM

It is well known that the barycenter \( A_m \) of \( \triangle A_1 A_2 A_3 \) is constructed by the relation

\[ a_{im} = \frac{2}{3} a_{i,j,k} = \frac{1}{3} \sqrt{2a_{ij}^2 + 2a_{ik}^2 - a_{jk}^2}, \]

where \( a_{i,j,k} \) is the length of the midline that connects the vertex \( A_i \) with the midpoint of the line segment \( A_j A_k \) for \( i, j, k = 1, 2, 3 \), \( i \neq j \neq k \) and the median minimizes the objective function

\[ a_{m1}^2 + a_{m2}^2 + a_{m3}^2. \]

A natural question to ask is if the location of the weighted Fermat-Torricelli problem could be given with respect to the lengths of the sides of \( \triangle A_1 A_2 A_3 \) and the constant positive weights \( B_1, B_2, B_3 \).

A positive answer to this question is given by the following lemma ([18, Corollary 2]):

\[ \frac{y_3 - y_3'}{x_3 - x_3'} = \frac{y_3 - y}{x_3 - x} \tag{2.19} \]
Lemma 3. ([18, Corollary 2]) The explicit solution of the weighted Fermat-Torricelli problem in $\mathbb{R}^2$, under the condition (1.2) (weighted floating case) is given by:

\[ \alpha_{013} = \arccot \left( \frac{\sin(\alpha_{213}) - \cos(\alpha_{213}) \cot(\arccos \frac{B_2^2 - B_3^2 - B_1^2}{2B_1B_3})}{a_{13} - \cos(\alpha_{213}) + \sin(\alpha_{213}) \cot(\arccos \frac{B_2^2 - B_3^2 - B_1^2}{2B_1B_3})} \right) \]

(3.1)

and

\[ a_{10} = \frac{\sin \left( \alpha_{013} + \arccos \frac{B_2^2 - B_3^2 - B_1^2}{2B_1B_3} \right) a_{13}}{\sin \left( \arccos \frac{B_2^2 - B_3^2 - B_1^2}{2B_1B_3} \right)} \]

(3.2)

where

\[ \alpha_{213} = \arccos \left( \frac{a_{12}^2 + a_{13}^2 - a_{23}^2}{2a_{12}a_{13}} \right) \]

(3.3)

and $\alpha_{013}$ and $a_{10}$ depend on $B_1, B_2, B_3, a_{13}, a_{12}$ and $a_{23}$.

Proof of Lemma 3: We assume that the weighted floating case occurs (see theorem 1, Case b), in order to locate it in the interior of the $\triangle A_1A_2A_3$.

From the cosine law in $\triangle A_1A_0A_2$, and $\triangle A_1A_0A_3$ we get, respectively:

\[ a_{02}^2 = a_{01}^2 + a_{12}^2 - 2a_{01}a_{12} \cos(\alpha_{213} - \alpha_{013}) \]

(3.4)

and

\[ a_{03}^2 = a_{01}^2 + a_{13}^2 - 2a_{01}a_{13} \cos(\alpha_{013}) \]

(3.5)

From (3.4) and (3.5), $a_{02}$ and $a_{03}$ are expressed with respect to the two variables $a_{01}$ and $\alpha_{013}$:

\[ a_{0i} = a_{0i}(a_1, \alpha_{013}), \]

for $i = 2, 3$. By differentiating (1.1) with respect to $a_{01}$ and $\alpha_{013}$, respectively, we get:

\[ B_1 + B_2 \frac{\partial a_{02}}{\partial a_{01}} + B_3 \frac{\partial a_{03}}{\partial a_{01}} = 0, \]

(3.6)

and

\[ B_2 \frac{\partial a_{02}}{\partial \alpha_{013}} + B_3 \frac{\partial a_{03}}{\partial \alpha_{013}} = 0. \]

(3.7)

From Appendix A, by replacing (A.1) and (A.2) in (3.6), we obtain:

\[ B_2 \cos(\alpha_{012}) + B_3 \cos(\alpha_{013}) = -B_1 \]

(3.8)

By replacing (A.5) and (A.6) in (3.7), we obtain:

\[ -B_2 \sin(\alpha_{012}) + B_3 \sin(\alpha_{013}) = 0 \]

(3.9)
By squaring both parts of (3.8) and (3.9) and by adding the two derived equations, we get:

$$\cos(\alpha_{203}) = \frac{B_1^2 - B_2^2 - B_3^2}{2B_2B_3}.$$  \hspace{1cm} (3.10)

Similarly by expressing the objective function with respect to the two variables $a_2$, $\alpha_{023}$, and with respect to the two variables $a_3$, $\alpha_{031}$, we derive, respectively:

$$\cos(\alpha_{103}) = \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3}.$$ \hspace{1cm} (3.11)

and

$$\cos(\alpha_{102}) = \frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2}.$$ \hspace{1cm} (3.12)

From the sine law in $\triangle A_1A_0A_2$, $\triangle A_1A_0A_3$, we get, respectively:

$$\frac{a_{12}}{\sin(\alpha_{102})} = \frac{a_{01}}{\sin(\alpha_{213} - \alpha_{013} + \alpha_{102})}$$ \hspace{1cm} (3.13)

and

$$\frac{a_{13}}{\sin(\alpha_{103})} = \frac{a_{01}}{\sin(\alpha_{013} + \alpha_{103})}.$$  \hspace{1cm} (3.14)

By eliminating $a_{01}$ from (3.13) and (3.14), we obtain:

$$\alpha_{013} = \arccot\left(\frac{\sin(\alpha_{213}) - \cos(\alpha_{213}) \cot(\alpha_{102}) - \frac{a_{13}}{a_{12}} \cot(\alpha_{103})}{- \cos(\alpha_{213}) - \sin(\alpha_{213}) \cot(\alpha_{102}) + \frac{a_{13}}{a_{12}}}ight)$$ \hspace{1cm} (3.15)

By replacing (3.12) and (3.11) in (3.15), we obtain (3.1). From the sine law in $\triangle A_1A_0A_3$, we derive (3.2).

The values of $a_{01}$ and $\alpha_{013}$ give the location of the weighted Fermat-Torricelli point $A_F$. \hfill \Box

**Remark 1.** The explicit solution of the weighted Fermat-Torricelli problem is similar with the definition of a complex number in a polar form:

$$z = r \exp(i(\alpha_{213} - \alpha_{013}))$$

where the absolute value of $z$ is $r = a_1$ and the argument of $z$ is $\arg z = \alpha_{213} - \alpha_{013}$.

Let $C(Q, R)$ be the inscribed circle with center $Q$ and radius $R$ which passes from the vertex $A_i$, for $i = 1, 2, 3$.

Each of the three central angles is given by the relation:

$$c_{iQj} = 2\alpha_{imj},$$ \hspace{1cm} (3.16)

such that:

$$c_{1Q2} + c_{2Q3} + c_{1Q3} = 2\pi$$
or
\[ c_{1Q2} = 2\pi - c_{1Q3} - c_{2Q3}, \]  
(3.17)
for \( i \neq m \neq j, i, m, j = 1,2,3 \). From the sine law in \( \triangle A_1A_2A_3 \) and taking into account (3.16), we get:
\[ \frac{a_{13}}{\sin(\frac{c_{1Q3}}{2})} = \frac{a_{12}}{\sin(\frac{c_{1Q2}}{2})} = 2R. \]  
(3.18)
By replacing (3.16), (3.17) (3.18) in (3.1) and (3.2) of lemma 3, we derive the following result:

**Theorem 3.** An explicit angular solution of the weighted Fermat-Torricelli problem in \( \mathbb{R}^2 \), under the condition (1.2) is given by:

\[ \cot \alpha_{013} = \frac{\sin(\frac{c_{1Q2}}{2}) - \cos(\frac{c_{2Q3}}{2}) \cot(\arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_2})}{\sin(\frac{c_{1Q3}}{2})} \]  
(3.19)
and
\[ a_{10} = 2R \frac{\sin(\alpha_{013} + \arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3}) \sin(\frac{c_{3Q2}}{2})}{\sin(\arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3})}, \]  
(3.20)
where \( \alpha_{013} \) and \( a_{10} \) depend on \( B_1, B_2, B_3, c_{1Q3}, c_{2Q3}, \) and \( R \).

**Remark 2.** We conclude that by setting \( R = 1 \) (unit radius circumscribed circle) in (3.20), the explicit solution depends only on five given elements: \( B_1, B_2, B_3, c_{1Q3}, c_{2Q3} \). This unique result holds only if the inequalities (1.2) of the weighted floating case of Theorem 1, Case (b) are satisfied.

**Remark 3.** We note that lemma 3 and theorem 3 provide an analytic solution for the weighted Fermat-Torricelli problem in \( \mathbb{R}^2 \) without using the coordinates of the points \( A_i \), for \( i = 1, 2, 3 \) (Coordinate independent approach) taking into account only given Euclidean elements (lengths and angles).

4. Two new geometrical solutions of the weighted Fermat-Torricelli point in the Euclidean plane

We present two new geometrical solutions to find the weighted Fermat-Torricelli point in the weighted floating case.

The first solution deals with the position of \( A_1' \) and \( A_3' \) which shall give the position of the two Simpson lines defined by \( A_2 \) and \( A_2' \) and \( A_3 \), \( A_3' \) (fig. 2).

The second solution deals with the generalization of Hofmann’s rotation proof ([12], [6]) for \( B_1 = B_2 \), such that \( B_1 \)
Problem 2. Construct the solution of Problem 1 (Weighted Fermat-Torricelli problem) using ruler and compass, under the condition (1.2).

Solution of Problem 2. We need to construct the vertices $A_3'$ and $A_2'$ (see fig. 2).

First, we construct the vertex $A_3'$. We select a point $K$ which belongs to the linear segment $A_1A_2$, such that $\|A_1K\| = B_3$ and we construct a triangle $\triangle A_1KL$ with the other two sides $\|A_1L\| = B_2$ and $\|KL\| = B_3$.

Thus, lemma 1 yields $\angle A_2A_1A_3' = \alpha_{213}' = \pi - \alpha_{203}$.

We need to calculate $a_{13}'$, in order to find the location of $A_3'$.

Take a point $G$ to the line defined by $A_1A_2$, such that $\|A_1A_2\| = a_{12}$, $\|A_2G\| = B_3$ where $\|A_1G\| = a_{12} + B_3$ and construct with a ruler and compass the perpendicular linear segment at the point $G$ to the line defined by $A_1A_2$ and take a point $H$ such that $|GH| = B_2$. We denote by $I$ the point of intersection of the line defined by $A_2H$ and the perpendicular line at the point $A_1$ with respect to the line defined by $A_1A_2$ (fig. 3). Taking into account the similar triangles $\triangle A_1A_2I$ and $\triangle A_2GH$, we get:

$$a_{13}' = a_{12} \frac{B_2}{B_3} \quad (4.1)$$

Similarly, we construct the vertex $A_2'$. We select a point $M$ which belongs to the linear segment $A_1A_3$, such that $\|A_1M\| = B_2$ and we
construct a triangle \( \triangle A_1MN \) with the other two sides \( \|A_1N\| = B_3 \) and \( \|MN\| = B_1 \).

Thus, lemma \( \square \) yields \( \angle A_3A_1A_2' = \alpha_{312} = \pi - \alpha_{203} \).

We need to calculate \( a_{12}' \), in order to find the location of \( A_2' \).

Take a point \( Q \) to the line defined by \( A_1A_3 \), such that \( \|A_1A_3\| = a_{13} \), \( \|A_2G\| = B_3 \) where \( \|A_1Q\| = a_{13} + B_3 \) and construct with a ruler and compass the perpendicular linear segment at the point \( Q \) to the line defined by \( A_1A_3 \) and take a point \( R \) such that \( |RQ| = B_3 \). We denote by \( P \) the point of intersection of the line defined by \( A_3R \) and the perpendicular line at the point \( A_1 \) with respect to the line defined by \( A_1A_3 \) (fig. 4). Taking into account the similar triangle \( \triangle A_1A_3P \) and \( \triangle A_3QR \), we get:

\[
a_{12}' = a_{13} \frac{B_3}{B_2}.
\]  

(4.2)

\( \square \)

**Problem 3.** Solve Problem 1 (Weighted Fermat-Torricelli problem) by generalizing Hofmann’s rotation, under the condition (1.2) and \( B_1 = B_2 \), such that \( B_1 + B_2 + B_3 = 1 \), \( B_1 > \frac{1}{4} \) and \( \alpha_{132} > \pi - \arccos \left( -1 + \frac{(1-2B_1)^2}{2B_1^2} \right) \).

**Solution of Problem** We consider a weight \( B_i \) which corresponds to the vertex \( A_i \) in \( \mathbb{R}^2 \), for \( i = 1, 2, 3 \).

By replacing \( B_1 = B_2 \) in (2.2) of lemma \( \square \) we derive that:
Figure 4.

Figure 5.
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\[ \alpha_{203} = \alpha_{103} = \arccos \left( 1 - \frac{1}{2B_1} \right) \]  

(4.3)

and

\[ \alpha_{102} = \arccos \left( -1 + \frac{(1 - 2B_1)^2}{2B_1^2} \right) \]  

(4.4)

which yields \( B_1 > \frac{1}{4} \).

Taking into account (4.3) and (4.4), we rotate the triangle \( \triangle A_1A_2A_3 \) about \( A_3 \) through \( \pi - \alpha_{102} = 2\alpha_{103} - \pi \text{rad} \) and we obtain the triangle \( \triangle A_3A_1'A_2' \). Let \( A_F' \) be the corresponding weighted Fermat-Torricelli point of \( \triangle A_1'A_2'A_3 \), for \( B_1' = B_1 \) and \( B_2 = B_2' \) (fig. 5). Thus, the points \( A_2, A_F, A_F' \) and \( A_3 \) are collinear (fig. 5), because \( \triangle A_FA_F'A_3 \) is an isosceles triangle and

\[ \angle A_FA_F'A_3 = \angle A_F'A_FA_3 = \pi - \alpha_{103}. \]

\[ \square \]

Corollary 3. [6, 12, 1] For \( B_1 = B_2 = B_3 \) the solution of Problem 3 is given by rotating the \( \triangle A_1A_2A_3 \) about \( A_3 \) through 60°.

Proof. By replacing \( B_1 = B_2 = B_3 = 1 \) in the solution of Problem 3, we deduce that the rotation about \( A_3 \) need to be \( \pi - 120° = 60° < \alpha_{132} \) (Hofmann’s rotation).

\[ \square \]

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APPENDIX A.

We mention two methods of the length of a linear segment with respect to (I) a variable length and (II) a variable angle, which have been used, in order to find the weighted Fermat-Torricelli point.

I. A method of differentiating the length of a linear segment with respect to the length of a variable linear segment is given first in [14, Proposition 2.6, (b), Remark 2.4, Corollary 3.3], [15, formula (4), p. 413] and has been explained in detail in [17, 16, Corollary 2]. Specifically, by differentiating (3.4) with respect to \( a_1 \), and by replacing in the derived equation \( \cos(\alpha_{213} - \alpha_{013}) \) taken from (3.4), we obtain:

\[ \frac{\partial a_2}{\partial a_1} = \cos(\alpha_{102}). \]  

(A.1)
Similarly, by differentiating (3.5) with respect to $a_1$, and by replacing in the derived equation $\cos(\alpha_{013})$ taken from (3.5), we obtain:

$$\frac{\partial a_3}{\partial a_1} = \cos(\alpha_{103}).$$

(A.2)

We mention a method of differentiating the length of a linear segment with respect to a variable angle, which have been used, in order to find the weighted Fermat-Torricelli point ([14 Proposition 2.6 (b)]) in $\mathbb{R}^2$. By mentioning this technique of differentiation, we correct some typographical errors which appear in [14]. Specifically, by differentiating (3.4) with respect to $\alpha_{013}$, we get:

$$\frac{\partial a_{02}}{\partial \alpha_{013}} = -a_{01} \frac{a_{12}}{a_{02}} \sin(\alpha_{213} - \alpha_{013})$$

(A.3)

From the sine law in $\triangle A_1A_0A_2$, we get:

$$\frac{a_{12}}{\sin(\alpha_{102})} = \frac{a_{02}}{\sin(\alpha_{213} - \alpha_{013})}$$

(A.4)

By replacing (A.4) in (A.3), we obtain:

$$\frac{\partial a_{02}}{\partial \alpha_{013}} = -a_{01} \sin(\alpha_{102}).$$

(A.5)

Similarly, by differentiating (3.5) with respect to $\alpha_{013}$, we obtain:

$$\frac{\partial a_{03}}{\partial \alpha_{013}} = a_{01} \sin(\alpha_{103}).$$

(A.6)

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