On $i\alpha$ - Open Sets

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ABSTRACT

In this paper, we introduce a new class of open sets defined as follows: A subset $A$ of a topological space $(X, \tau)$ is called $i\alpha$-open set, if there exists a non-empty subset $O$ of $X$, $O \in \alpha O(X)$, such that $A \subseteq C(A \cap O)$. Also, we present the notion of $i\alpha$-continuous mapping, $i\alpha$-open mapping, $i\alpha$-irresolute mapping, $i\alpha$-totally continuous mapping, $i\alpha$-contra-continuous mapping, $i\alpha$-contra-continuous mapping and we investigate some properties of these mappings. Furthermore, we introduce some $i\alpha$-separation axioms and the mappings are related with $i\alpha$-separation axioms.

Keywords: Open totals, Topological space, Types of mappings.

1 Introduction and Preliminaries

A Generalization of the concept of open sets is now well-known important notions in topology and its applications. Levine [7] introduced semi-open set and semi-continuous function, Njastad [8] introduced $\alpha$-open set, Askander [15] introduced $i$-open set, $i$- irresolute mapping and $i$-homeomorphism, Biswas [6] introduced semi-open functions, Mashhour, Hasanein, and El-Deeb [1] introduced $\alpha$-continuous and $\alpha$-open mappings, Noiri [16] introduced totally (perfectly) continuous function, Crossley [11] introduced irresolute function, Maheshwari [14] introduced $\alpha$-irresolute mapping, Beceren [17] introduced semi $\alpha$-irresolute functions, Donchev [4] introduced contra continuous functions, Donchev and Noiri [5] introduced contra semi continuous functions, Jafari and Noiri [12] introduced Contra-$\alpha$-continuous functions, Ekici and Caldas [3] introduced clopen-$T_i$, Staum [10] introduced, ultra hausdorff, ultra normal, clopen regular and clopen normal, Ellis [9] introduced ultra regular, Maheshwari [13] introduced s-normal space, Arhangel [2] introduced $\alpha$-normal space. The main aim of this paper is to introduce and study a new class of open sets which is called $i\alpha$-open set and we present the notion of $i\alpha$-continuous mapping, $i\alpha$-totally continuity mapping and some weak separation axioms for $i\alpha$-open sets. Furthermore, we investigate some properties of these mappings. In section 2, we define $i\alpha$-open set, and we investigate the relationship with, open set, semi-open set, $\alpha$-open set and $i$-open set. In section 3, we present the notion of $i\alpha$-continuous mapping, $i\alpha$-open mapping, $i\alpha$-irresolute mapping and $i\alpha$-homeomorphism mapping, and we investigate the relationship between $i\alpha$-continuous mapping with some types of continuous mappings, the relationship between

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ia-open mapping, with some types of open mappings and the relationship between ia-
irresolute mapping with some types of irresolute mappings. Further, we compare ia-
homeomorphism with i-homeomorphism. In section 4, we introduce new class of
mappings called ia-totally continuous mapping and we introduce i-contra-continuous
mapping and ia-contra-continuous mapping. Further, we study some of their basic
properties. Finally in section 5, we introduce new weak of separation axioms for ia-
open and we conclude ia-continuous mappings related with ia-separation axioms.
Throughout this paper, we denote the topology spaces $(X, \tau)$ and $(Y, \sigma)$ simply by $X$ and
$Y$ respectively. We recall the following definitions, notations and characterizations. The
closure (resp. interior) of a subset $A$ of a topological space $X$ is denoted by $\text{Cl}(A)$ (resp.
$\text{Int}(A)$).

**Definition 1.1** A subset $A$ of a topological space $X$ is said to be

(i) semi-open set, if $\exists O \in \tau$ such that $O \subseteq A \subseteq \text{Cl}(O)$ [7]
(ii) $\alpha$-open set, if $A \subseteq \text{Int(Cl(Int(A)))}$ [8]
(iii) i-open set, if $A \subseteq \text{Cl}(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \phi$ [15]
(iv) clopen set, if $A$ is open and closed.

The family of all semi-open (resp. $\alpha$-open, i-open, clopen) sets of a topological space is
denoted by $\text{SO}(X)$ (resp. $\text{aO}(X), \text{iO}(X), \text{CO}(X)$). The complement of semi-open (resp. $\alpha$-
open, i-open) sets of a topological space $X$ is called semi-closed (resp. $\alpha$-closed, i-
closed) sets.

**Definition 1.2** Let $X$ and $Y$ be a topological spaces, a mapping $f : X \rightarrow Y$ is said to be

(i) semi-continuous [7] if the inverse image of every open subset of $Y$ is semi-open set
in $X$.
(ii) $\alpha$-continuous [1] if the inverse image of every open subset of $Y$ is an $\alpha$-open set in $X$.
(iii) i-continuous [15] if the inverse image of every open subset of $Y$ is an i-open set in $X$.
(iv) totally (perfectly) continuous [16] if the inverse image of every open subset of $Y$ is
clopen set in $X$.
(v) irresolute [11] if the inverse image of every semi-open subset of $Y$ is semi-open set
in $X$.
(vi) $\alpha$-irresolute [14] if the inverse image of every $\alpha$-open subset of $Y$ is an $\alpha$-open subset
in $X$.
(vii) semi $\alpha$-irresolute [17] if the inverse image of every $\alpha$-open subset of $Y$ is semi-
clopen subset in $X$.
(viii) i-irresolute [15] if the inverse image of every i-open subset of $Y$ is an i-open subset
in $X$.
(ix) contra-continuous [4] if the inverse image of every open subset of $Y$ is closed set in
$X$.
(x) contra semi continuous [5] if the inverse image of every open subset of $Y$ is semi-
closed set in $X$.
(xi) contra $\alpha$-continuous [12] if the inverse image of every open subset of $Y$ is an $\alpha$-
closed set in $X$.
(xii) semi-open [6] if the image of every open set in $X$ is semi-open set in $Y$.
(xiii) $\alpha$-open [1] if the image of every open set in $X$ is an $\alpha$-open set in $Y$.
(xiv) i-open [15] if the image of every open set in $X$ is an i-open set in $Y$.

**Definition 1.3** Let $X$ and $Y$ be a topology space, a bijective mapping $f : X \rightarrow Y$ is said
to be i-homeomorphism [15] if $f$ is an i-continuous and i-open.
Lemma 1.4 Every open set in a topological space is an $i\alpha$-open set [15].

Lemma 1.6 Every semi-open set in a topological space is an $i\alpha$-open set [15].

Lemma 1.8 Every $\alpha$-open set in a topological space is an $i\alpha$-open set [15].

2 Sets That are $i\alpha$-Open Sets and Some Relations With Other Important Sets

In this section, we introduce a new class of open sets which is called $i\alpha$-open set and we investigate the relationship with, open set, semi-open set, $\alpha$-open set and $i\alpha$-open set.

Definition 2.1 A subset $A$ of the topological space $X$ is said to be $i\alpha$-open set if there exists a non-empty subset $O$ of $X$, $O \in \alpha O(X)$, such that $A \subseteq Cl(A \cap O)$. The complement of the $i\alpha$-open set is called $i\alpha$-closed. We denote the family of all $i\alpha$-open sets of a topological space $X$ by $i\alpha O(X)$.

Example 2.2 Let $X=\{a,b,c\}$, $\tau=\{\emptyset,\{b\},\{c\},\{b,c\}\}$, $i\alpha O(X)=\{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$ and $\sigma=\{\emptyset,\{a\},\{b\},\{c\}\}$. Note that $i\alpha O(X) \subseteq i\alpha O(X)$.

Example 2.3 Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a,d\},\{b,c\}\}$, $i\alpha O(X)=\{\emptyset,\{a\},\{b\},\{c\},\{d\}\}$. Clearly, the identity mapping $i: X \rightarrow X$ is an $i\alpha$-continuous.

Lemma 2.5 Every $i\alpha$-open set in any topological space is an $i\alpha$-open set.

Proof. Let $X$ be any topological space and $A \subseteq X$ be any $i\alpha$-open set. Therefore, $A \subseteq Cl(A \cap O)$, where $\exists O \in \tau$ and $O \neq X, \emptyset$. Since, every open is an $\alpha$-open[8], then $\exists O \in \alpha O(X)$. We obtain $A \subseteq Cl(A \cap O)$, where $\exists O \in \alpha O(X)$ and $O \neq X, \emptyset$. Thus, $A$ is an $i\alpha$-open set. The following example shows that $i\alpha$-open set need not be $i\alpha$-open set.

Example 2.4 Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a\},\{b\},\{c\}\}$, $i\alpha O(X)=\{\emptyset,\{a\},\{b\},\{c\}\}$. Clearly, the identity mapping $i: X \rightarrow X$ is an $i\alpha$-continuous.

Proposition 3.3 Every $i\alpha$-continuous mapping is an $i\alpha$-continuous.
Proof. Let \( f : X \to Y \) be an \( i\alpha \)-continuous mapping and \( V \) be any open subset in \( Y \). Since, \( f \) is an \( i\alpha \)-continuous, then \( f^{-1}(V) \) is an \( i\alpha \)-open set in \( X \). Since, every \( i\alpha \)-open set is an \( \alpha \)-open set by lemma 2.5, then \( f^{-1}(V) \) is an \( \alpha \)-open set in \( X \). Therefore, \( f \) is an \( \alpha \)-continuous.

Remark 3.4 The following example shows that \( \alpha \)-continuous mapping need not be continuous, semi-continuous, \( \alpha \)-continuous and \( i\alpha \)-continuous mappings.

Example 3.5 Let \( X=\{a,b,c\} \) and \( Y=\{1,2,3\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( SO(X)=\alpha O(X)=iO(X)=\{\emptyset,\{b\},\{b,c\},X\} \), \( \sigma=\{\emptyset,\{2\},Y\} \). A mapping \( f : X \to Y \) is defined by \( f(a) = (2), f(b) = (1), f(c) = (3) \). Clearly, \( f \) is an \( \alpha \)-continuous, but \( f \) is not continuous, \( f \) is not semi-continuous, \( f \) is not \( \alpha \)-continuous and \( f \) is not \( i\alpha \)-continuous because for open subset \( \{2\} \),
\[ f^{-1}(\{2\}) = \{a\} \notin \tau \text{ and } f^{-1}(\{2\}) = \{a\} \notin SO(X) = \alpha O(X) = iO(X). \]

Definition 3.6 Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be \( i\alpha \)-open, if the image of every open set in \( X \) is an \( i\alpha \)-open set in \( Y \).

Example 3.7 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( \sigma=\{\emptyset,\{a\},Y\} \), and \( i\alpha O(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( i\alpha \)-open.

Proposition 3.8 Every \( i\alpha \)-open mapping is an \( i\alpha \)-open.

Proof. Let \( f : X \to Y \) be an \( i\alpha \)-open mapping and \( V \) be any open set in \( X \). Since, \( f \) is an \( i\alpha \)-open, then \( f(V) \) is an \( i\alpha \)-open set in \( Y \). Since, every \( i\alpha \)-open set is an \( \alpha \)-open set by lemma 2.5, then \( f(V) \) is an \( \alpha \)-open set in \( Y \). Therefore, \( f \) is an \( \alpha \)-open.

Remark 3.9 The following example shows that \( \alpha \)-open mapping need not be open, semi-open, \( \alpha \)-open and \( i\alpha \)-open mappings.

Example 3.10 Let \( X=Y=\{1,2,3\} \), \( \tau=\{\emptyset,\{3\},X\} \), \( \sigma=\{\emptyset,\{1\},Y\} \), \( SO(Y)=\alpha O(Y)=iO(Y)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},Y\} \). A mapping \( f : X \to Y \) is defined by \( f(1) = 2, f(2) = 1, f(3) = 3 \). Clearly, \( f \) is an \( \alpha \)-open, but \( f \) is not open, \( f \) is not semi-open, \( f \) is not \( \alpha \)-open and \( f \) is not \( i\alpha \)-open because for open subset \( \{3\} \),
\[ f^{-1}\{3\} = \{3\} \notin \sigma \text{ and } f^{-1}\{3\} = \{3\} \notin SO(Y) = \alpha O(Y) = iO(Y). \]

Definition 3.11 Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be \( i\alpha \)-irresolute, if the inverse image of every \( i\alpha \)-open subset of \( Y \) is an \( \alpha \)-open subset in \( X \).

Example 3.12 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{b\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \) and \( \alpha O(Y)=\{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( i\alpha \)-irresolute.

Proposition 3.13 Every \( i\alpha \)-irresolute mapping is an \( \alpha \)-irresolute.

Proof. Let \( f : X \to Y \) be an \( i\alpha \)-irresolute mapping and \( V \) be any \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( i\alpha \)-irresolute, then \( f^{-1}(V) \) is an \( \alpha \)-open set in \( X \). Hence, \( \alpha \)-open set in \( X \) by lemma 2.5. Therefore, \( f \) is an \( \alpha \)-irresolute.

Remark 3.14 The following example shows that \( \alpha \)-irresolute mapping need not be irresolute, semi \( \alpha \)-irresolute, \( \alpha \)-irresolute and \( i\alpha \)-irresolute mappings.

Example 3.15 Let \( X=Y=\{a,b,c\} \), \( \tau=\{\emptyset,\{a\},X\} \), \( SO(X)=\alpha O(X)=iO(X)=\{\emptyset,\{a\},\{a,b\},\{a,c\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \), \( SO(Y)=\alpha O(Y)=iO(Y)=\{\emptyset,\{c\},\{a,c\},\{b,c\},Y\} \) and \( \alpha O(Y)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,b\},\{a,c\},\{b,c\},Y\} \). Clearly,
the identity mapping \( f : X \to Y \) is an \( \alpha \)-irresolute, but \( f \) is not irresolute, \( f \) is not \( \alpha \)-irresolute, \( f \) is not semi \( \alpha \)-irresolute and \( f \) is not \( i \)-irresolute because for semi-open, \( \alpha \)-open and \( i \)-open subset \( \{ c \} \), \( f^{-1}(c) = \{ c \} \notin SO(X)=aO(X)=iO(X) \).

**Proposition 3.16** Every \( \alpha \)-irresolute mapping is an \( \alpha \)-continuous.

**Proof.** Let \( f : X \to Y \) be an \( \alpha \)-irresolute mapping and \( V \) be any open set in \( Y \). Since, every open set is an \( \alpha \)-open set. Since, \( f \) is an \( \alpha \)-irresolute, then \( f^{-1}(V) \) is an \( \alpha \)-open set in \( X \). Therefore \( f \) is an \( \alpha \)-continuous. The converse of the above proposition need not be true as shown in the following example.

**Example 4.17** Let \( X=Y=\{a,b,c\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( \alpha \)-continuous, but \( f \) is not \( \alpha \)-irresolute because for \( \alpha \)-open set \( \{ c \} \), \( f^{-1}(c) = \{ c \} \notin iO(X) \).

**Definition 3.18** Let \( X \) and \( Y \) be a topological space, a bijective mapping \( f : X \to Y \) is said to be \( \alpha \)-homeomorphism if \( f \) is an \( \alpha \)-continuous and \( \alpha \)-open.

**Theorem 3.19** If \( f : X \to Y \) is an \( i \)-homomorphism, then \( f : X \to Y \) is an \( \alpha \)-homomorphism.

**Proof.** Since, every \( i \)-continuous mapping is an \( \alpha \)-continuous by proposition 3.3. Also, since every \( i \)-open mapping is an \( \alpha \)-open 3.8. Further, since \( f \) is bijective. Therefore, \( f \) is an \( \alpha \)-homomorphism. The converse of the above theorem need not be true as shown in the following example.

**Example 3.20** Let \( X=Y=\{a,b,c\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \). Clearly, the identity mapping \( f : X \to Y \) is an \( \alpha \)-homomorphism, but it is not \( i \)-homomorphism because \( f \) is not \( i \)-continuous since for open subset \( \{ b \} \), \( f^{-1}(b) = \{ b \} \notin iO(X) \).

**4 Mappings That are \( \alpha \)-Totally Continuous and \( \alpha \)-Contra-Continuous**

In this section, we introduce new classes of mappings called \( \alpha \)-totally continuous, \( i \)-contra-continuous and \( \alpha \)-contra-continuous.

**Definition 4.1** Let \( X \) and \( Y \) be a topological space, a mapping \( f : X \to Y \) is said to be \( \alpha \)-totally continuous, if the inverse image of every \( \alpha \)-open subset of \( Y \) is clopen set in \( X \).

**Example 4.2** Let \( X=Y=\{a,b,c\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \). The mapping \( f : X \to Y \) is defined by \( f(a) = \{a\} \), \( f(b) = f(c) = b \). Clearly, \( f \) is an \( \alpha \)-totally continuous mapping.

**Theorem 4.3** Every \( \alpha \)-totally continuous mapping is totally continuous.

**Proof.** Let \( f : X \to Y \) be \( \alpha \)-totally continuous and \( V \) be any open set in \( Y \). Since, every open set is an \( \alpha \)-open set, then \( V \) is an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous mapping, then \( f^{-1}(V) \) is clopen set in \( X \). Therefore, \( f \) is totally continuous. The converse of the above theorem need not be true as shown in the following example.

**Example 4.4** Let \( X=Y=\{a,b,c\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \), \( \sigma = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\} \). Clearly, the identity mapping is totally \( f : X \to Y \)
continuous, but \( f \) is not \( \alpha \)-totally continuous because for \( \alpha \)-open set \( \{a,c\} \), \( f^{-1}\{a,c\} \neq \{a,c\} \notin CO(X) \).

**Theorem 4.5** Every \( \alpha \)-totally continuous mapping is an \( \alpha \)-irresolute.

**Proof.** Let \( f : X \to Y \) be \( \alpha \)-totally continuous and \( V \) be an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous mapping, then \( f^{-1}(V) \) is clopen set in \( X \), which implies \( f^{-1}(V) \) open, it follow \( f^{-1}(V) \) \( \alpha \)-open set in \( X \). Therefore, \( f \) is an \( \alpha \)-irresolute■ The converse of the above theorem need not be true as shown in the following example

**Example 4.6** Let \( X=\{1,2,3\}, \tau=\{\emptyset,\{2\},X\} \), \( \alpha O(X)=\{\emptyset,\{1\},\{2,\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\} \) \( \alpha \)-irresolute, but \( f \) is not \( \alpha \)-totally continuous because for \( \alpha \)-open subset \( \{1,3\} \), \( f^{-1}\{1,3\} \neq \{1,3\} \notin CO(X) \).

**Theorem 4.7** The composition of two \( \alpha \)-totally continuous mapping is also \( \alpha \)-totally continuous.

**Proof.** Let -be any \( \alpha \) \( V \)totally continuous. Let -o \( \alpha \)be any tw \( g : Y \to Z \) and \( f : X \to Y \) open in \( Z \). Since, \( g \) is an \( \alpha \)-totally continuous, then \( g^{-1}(V) \) is clopen set in \( Z \), which implies \( f^{-1}(V) \) open set, it follow \( f^{-1}(V) \) \( \alpha \)-open set. Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen in \( X \). Therefore, \( gof : X \to Z \) is an \( \alpha \)-totally continuous■

**Theorem 4.8** If \( f : X \to Y \) be an \( \alpha \)-totally continuous and-be an \( \alpha \) \( g : Y \to Z \) irresolute, then \( gof : X \to Z \) is an \( \alpha \)-totally continuous.

**Proof.** Let \( f : X \to Y \) be \( \alpha \)-totally continuous and \( g : Y \to Z \) be \( \alpha \)-irresolute . Let \( V \) be \( \alpha \)-open set in \( Z \). Since, \( g \) is an \( \alpha \)-irresolute, then \( g^{-1}(V) \) is an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen in \( X \). Therefore, \( gof : X \to Z \) is an \( \alpha \)-totally continuous■

**Theorem 4.9** If \( f : X \to Y \) is an \( \alpha \)-totally continuous and-is an \( \alpha \) \( g : Y \to Z \) continuous, then \( gof : X \to Z \) is totally continuous.

**Proof.** Let continuous . Let -is an \( \alpha \) \( g : Y \to Z \) totally continuous and -be \( \alpha \) \( f : X \to Y \) \( V \) be an open set in \( Z \). Since, \( g \) is an \( \alpha \)-continuous, then \( g^{-1}(V) \) is an \( \alpha \)-open set in \( Y \). Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is clopen set in \( X \). Therefore, \( gof : X \to Z \) is totally continuous■

**Definition 4.10** Let \( X, Y \) be a topological spaces, a mapping \( f : X \to Y \) is said to be \( \alpha \)-contra-continuous (resp. \( i \)-contra-continuous), if the inverse image of every open subset of \( Y \) is an \( \alpha \)-closed (resp. \( i \)-closed) set in \( X \).

**Example 4.11** Let \( X=\{a,b,c\}, \tau=\{\emptyset,\{a\},X\} \), \( \sigma=\{\emptyset,\{c\},Y\} \) \( \alpha \)-contra-continuous and \( \sigma \)-contra-continuous. Clearly, the identity mapping \( f : X \to Y \) is an \( i \)-contra-continuous and \( \alpha \)-contra-continuous.

**Proposition 4.12** Every contra-continuous mapping is an \( i \)-contra-continuous.

**Proof.** Let \( f : X \to Y \) be contra continuous mapping and \( V \) any open set in \( Y \). Since, \( f \) is contra continuous, then \( f^{-1}(V) \) is closed sets in \( X \). Since, every closed set is an \( i \)-closed set, then \( f^{-1}(V) \) is an \( i \)-closed set in \( X \). Therefore, \( f \) is an \( i \)-contra-continuous■ Similarly we have the following results.
Proposition 4.13 Every contra semi-continuous mapping is an i-contra-continuous.

Proof. Clear since every semi-open set is an i-open set.

Proposition 4.14 Every contra α-continuous mapping is an i-contra-continuous.

Proof. Clear since every α-open set is an i-open set. The converse of the propositions 4.12, 4.13 and 4.14 need not be true in general as shown in the following example:

Example 4.15 Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},X\}$, $iO(X)=\{\emptyset,\{a\},\{a,b\},\{a,c\},\{b,c\},X\}$ and $\sigma=\{\emptyset,\{c\},Y\}$. Clearly, the identity mapping $f:X\to Y$ is an i-contra continuous, but $f$ is not contra-continuous, $f$ is not contra semi-continuous, $f$ is not contra α-continuous because for open subset $f^{-1}\{c\}=\{c\}$ is not closed in $X$, $f^{-1}\{c\}=\{c\}$ is not semi-closed in $X$ and $f^{-1}\{c\}=\{c\}$ is not α-closed in $X$.

Proposition 4.16 Every i-contra-continuous mapping is an iα-contra-continuous.

Proof. Let $f:X\to Y$ be an i-contra continuous mapping and $V$ any open set in $Y$. Since, $f$ is an i-contra continuous, then $f^{-1}(V)$ is an i-closed sets in $X$. Since, every i-closed set is an iα-closed, then $f^{-1}(V)$ is an iα-closed set in $X$. Therefore, $f$ is an iα-contra-continuous.

Remark 4.17 The following example shows that iα-contra-continuous mapping need not be contra semi-continuous, contra semi-continuous, contra α-continuous and i-contra-continuous mappings.

Example 4.18 Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},X\}$, $iO(X)=aO(X)=iO(X)=\{\emptyset,\{a\},\{a,b\},\{a,c\},X\}$, $aO(X)=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\}$, $\sigma=\{\emptyset,\{c\},Y\}$. A mapping continuous, -contra-is an iα $f(\{a\})=a$. Clearly, $f(\{b\})=b, f(\{a\})=c$, $f$ is defined by $f:X\to Y$ but $f$ is not contra-continuous, $f$ is not contra semi continuous, $f$ is not contra α-continuous and $f$ is not i-contra-continuous because for open subset $\{c\}, f^{-1}\{c\}=\{a\}$ is not closed, $f^{-1}\{c\}=\{a\}$ is not semi-closed, $f^{-1}\{c\}=\{a\}$ is not α-closed and $f^{-1}\{c\}=\{a\}$ is not i-closed in $X$.

Theorem 4.19 Every totally continuous mapping is an iα-contra continuous.

Proof. Let $f:X\to Y$ be totally continuous and $V$ be any open set in $Y$. Since, $f$ is totally continuous mapping, then $f^{-1}(V)$ is clopen set in $X$, and hence closed, it follows iα-closed set. Therefore, $f$ is an iα-contra-continuous. The converse of the above theorem need not be true as shown in the following example:

Example 4.20 Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},X\}$, $\sigma=\{\emptyset,\{a\},Y\}$ and $iO(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$. Clearly, the identity mapping $f$ -contra-is an iα $f:X\to Y$ continuous, but $f$ is not totally continuous because for open subset $f^{-1}\{a\}=\{a\} \notin CO(X)$.

5 Separation Axioms with iα-open Set

In this section, we introduce some new weak of separation axioms with iα-open sets.

Definition 5.1 A topological space $X$ is said to be
(i) $\text{iα-}T_0$ if for each pair distinct points of $X$, there exists iα-open set containing one point but not the other.
(ii) $\text{iα-}T_1$ (resp. clopen $\cdot T_1$ [3]) if for each pair of distinct points of $X$, there exists two iα-open (resp. clopen) sets containing one point but not the other.
(iii) $\text{iα-}T_2$ (resp. ultra hausdorff (UT2)[10]) if for each pair of distinct points of $X$ can be separated by disjoint iα-open (resp. clopen) sets.
(iv) $\alpha$–regular (resp. ultra regular [9]) if for each closed set $F$ not containing a point in $X$ can be separated by disjoint $\alpha$-open (resp. clopen) sets.

(v) clopen regular [10] if for each clopen set $F$ not containing a point in $X$ can be separated by disjoint open sets.

(vi) $\alpha$–normal (resp. ultra normal[10], s-normal[13], $\alpha$-normal[2]) if for each of non-empty disjoint closed sets in $X$ can be separated by disjoint $\alpha$-open (resp. clopen, semi-open, $\alpha$-open) sets.

(vii) clopen normal [10] if for each of non-empty disjoint clopen sets in $X$ can be separated by disjoint open sets.

(viii) $\alpha$-T$_{1/2}$ if every $\alpha$-closed is $\alpha$-closed in $X$.

**Remark 5.2** The following example shows that $\alpha$-normal need not be normal, s-normal, $\alpha$-normal spaces.

**Example 5.3** Let $X$=\{1,2,3,4,5\}, $\tau$=\{\emptyset,\{1,2,3\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\},\{1,2,3,4,5\},\{1,2,3,5\},\{1,3,4,5\},\{1,2,4,5\},\{2,3,4,5\}\} and $\alpha O(X)$=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{2,3,4,5\}\}. Clearly, the space $X$ is $\alpha$-T$_1$, $\alpha$-T$_2$, $\alpha$–regular, $\alpha$–normal and $\alpha$-T$_{1/2}$, but $X$ is not normal, s-normal and $\alpha$-normal.

**Theorem 5.4** If a mapping continuous mapping and the $\alpha$–contra-is an $\alpha$ $f : X \to Y$ space $X$ is an $\alpha$-T$_{1/2}$, then $f$ is an $\alpha$-contra-continuous.

**Proof.** Let $y$ be any open set in $V$ continuous mapping and $\alpha$–ractant-$\alpha$ $f : X \to Y$ since, $f$ is an $\alpha$-contra-continuous mapping, then $f^{-1}(V)$ is an $\alpha$-closed in $X$. Since, $X$ is an $\alpha$-T$_{1/2}$, then $f^{-1}(V)$ is $\alpha$-closed in $X$. Therefore, $f$ is an $\alpha$-contra-continuous.

**Theorem 5.5** If $f : X \to Y$ is an $\alpha$-totally continuous injection mapping and $Y$ is an $\alpha$-T$_1$, then $X$ is clopen-T$_1$.

**Proof.** Let $x$ and $y$ be any two distinct points in $X$. Since, $f$ is an injective, we have $f(x)$ and $f(y)$ in $Y$ such that $f(x) \neq f(y)$. Since, $Y$ is an $\alpha$-T$_1$, there exists $\alpha$-open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \in V$ and $f(x) \not\in V$. Therefore, we have $x \in f^{-1}(U)$ and $y \not\in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of $X$ because $f$ is an $\alpha$-totally continuous. This shows that $X$ is clopen-T$_1$.

**Theorem 5.6** If $f : X \to Y$ is an $\alpha$-totally continuous injection mapping and $Y$ is an $\alpha$-T$_0$, then $X$ is ultra-Hausdorff (U$T_2$).

**Proof.** Let $a$ and $b$ be any pair of distinct points of $X$ and $f$ be an injective, then $f(a) \neq f(b)$ in $Y$. Since $Y$ is an $\alpha$-T$_0$, there exists $\alpha$-open set $U$ containing $f(a)$ but not $f(b)$, then we have $a \in f^{-1}(U)$ and $b \not\in f^{-1}(U)$. Since, $f$ is an $\alpha$-totally continuous, then $f^{-1}(U)$ is clopen in $X$. Also $a \not\in f^{-1}(U)$ and $b \in X-f^{-1}(U)$. This implies every pair of distinct points of $X$ can be separated by disjoint clopen sets in $X$. Therefore, $X$ is ultra-Hausdorff.

**Theorem 5.7** Let $f : X \to Y$ be a closed $\alpha$-continuous injection mapping. If $Y$ is an $\alpha$–normal, then $X$ is an $\alpha$–normal.

**Proof.** Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since, $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since, $Y$ is an $\alpha$–normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint $\alpha$-open sets $V_1$ and $V_2$ respectively. Therefore, we obtain, $F_1 \subset f^{-1}(V_1)$ and $F_2 \subset f^{-1}(V_2)$. Since, $f$ is an $\alpha$-continuous, then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $\alpha$-open sets in $X$. Also, $f^{-1}(V_1) \cap f^{-1}(V_2)= f^{-1}(V_1 \cap V_2)= \emptyset$. Thus, for each pair of non-
empty disjoint closed sets in \( X \) can be separated by disjoint \( \alpha \)-open sets. Therefore, \( X \) is an \( \alpha \)-normal \( \blacksquare \)

**Theorem 5.8** If \( f : X \to Y \) is an \( \alpha \)-totally continuous closed injection mapping and \( Y \) is an \( \alpha \)-normal, then \( X \) is ultra-normal.

**Proof.** Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( X \). Since, \( f \) is closed and injective, \( f(F_1) \) and \( f(F_2) \) are disjoint closed subsets of \( Y \). Since, \( Y \) is an \( \alpha \)-normal, \( f(F_1) \) and \( f(F_2) \) are separated by disjoint \( \alpha \)-open sets \( V_1 \) and \( V_2 \) respectively. Therefore, we obtain, \( F_1 \subset f^{-1}(V_1) \) and \( F_2 \subset f^{-1}(V_2) \). Since, \( f \) is an \( \alpha \)-totally continuous, then \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are clopen sets in \( X \). Also, \( f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset \). Thus, for each pair of non-empty disjoint closed sets in \( X \) can be separated by disjoint clopen sets in \( X \). Therefore, \( X \) is ultra-normal. \( \blacksquare \)

**Theorem 5.9** Let \( f : X \to Y \) be a totally continuous closed injection mapping, if \( Y \) is an \( \alpha \)-regular, then \( X \) is ultra-regular.

**Proof.** Let \( F \) be a closed set not containing \( x \). Since, \( f \) is closed, we have \( f(F) \) is a closed set in \( Y \) not containing \( f(x) \). Since, \( Y \) is an \( \alpha \)-regular, there exists disjoint \( \alpha \)-open sets \( A \) and \( B \) such that \( f(x) \in A \) and \( f(F) \subset B \). Since, \( Y \) is \( \alpha \)-regular, there exists disjoint \( \alpha \)-open sets \( f^{-1}(A) \) and \( f^{-1}(B) \) are clopen sets in \( X \) because \( f \) is totally continuous. Moreover, since \( f \) is an injective, \( f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset \). Thus, for a pair of a point and a closed set not containing a point in \( X \) can be separated by disjoint clopen sets. Therefore, \( X \) is ultra-regular. \( \blacksquare \)

**Theorem 5.10** If \( f \) is a closed set in \( Y \) and \( y \notin F \). Take \( y = f(x) \). Since, \( f \) is totally continuous, \( f^{-1}(F) \) is clopen in \( X \). Let \( G = f^{-1}(F) \), then we have \( x \notin G \). Since, \( X \) is clopen regular, there exists disjoint open sets \( U \) and \( V \) such that \( G \subset U \) and \( x \in V \). This implies \( F = f(G) \subset f(U) \) and \( y = f(x) \in V \). Further, since \( f \) is an injective and \( \alpha \)-open, we have \( f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset \). \( f(U) \) and \( f(V) \) are \( \alpha \)-open sets in \( Y \). Thus, for each closed set \( F \) in \( Y \) and each \( y \in F \), there exists disjoint \( \alpha \)-open sets \( f(U) \) and \( f(V) \) in \( Y \) such that \( f \subset f(U) \) and \( y \in f(V) \). Therefore, \( Y \) is an \( \alpha \)-regular. \( \blacksquare \)

**Theorem 5.11** If \( f : X \to Y \) is a totally continuous injective and \( \alpha \)-open mapping from clopen normal space \( X \) into a space \( Y \), then \( Y \) is an \( \alpha \)-normal.

**Proof.** Let \( F_1 \) and \( F_2 \) be any two disjoint closed sets in \( Y \). Since, \( f \) is totally continuous, \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are clopen subsets of \( X \). Take \( U = f^{-1}(F_1) \) and \( V = f^{-1}(F_2) \). Since, \( f \) is an injective, we have \( U \cap V = f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2) = f^{-1}(\emptyset) = \emptyset \). Since, \( X \) is clopen normal, there exists disjoint open sets \( A \) and \( B \) such that \( U \subset A \) and \( V \subset B \). This implies \( F_1 = f(U) \subset f(A) \) and \( F_2 = f(V) \subset f(B) \). Further, since \( f \) is an injective \( \alpha \)-open, then \( f(A) \) and \( f(B) \) are disjoint \( \alpha \)-open sets. Thus, each pair of disjoint closed sets in \( Y \) can be separated by disjoint \( \alpha \)-open sets. Therefore, \( Y \) is an \( \alpha \)-normal. \( \blacksquare \)
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