On the Distribution of Partially-Symmetric Codes for Automorphism Ensemble Decoding

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Abstract—Automorphism Ensemble (AE) decoding has recently drawn attention as a possible alternative to list decoding of polar codes. In this letter, we investigate the distribution of Partially-Symmetric Reed-Muller (PS-RM) codes, a family of polar codes yielding good performances under AE decoding. We prove the existence of these codes for almost all code dimensions for code lengths $N \leq 256$. Moreover, we analyze the absorption group of this family of codes under SC decoding, proving that valuable permutations in AE decoding always exist. Finally, we experimentally show that PS-RM codes can outperform state-of-the-art polar-code-construction algorithms in terms of error-correction performance for short code lengths, while reducing decoding latency.

I. INTRODUCTION

Polar codes, introduced in 2009 [1], have been adopted in the 5th generation of mobile networks (5G) [2] in late 2016. During the standardization process, it was agreed that CRC-aided Successive Cancellation List (CA-SCL) [3] would be adopted as the standard decoding algorithm [4]. CA-SCL is hard output, limiting its use in iterative receivers, moreover its sequential schedule coupled with comparisons of up to $2L$ candidate codewords at each estimated information bit increase its decoding latency.

Recently, ensemble decoding was proposed for polar codes [5]. Ensemble decoding offers various decoding possibilities such as both soft or hard output, and parallel or serial implementation. In ensemble decoding, $M$ instances of the same polar decoder are run independently for different permuted versions of the received signal. That decoder could be, for example, the hard-output Successive-Cancellation (SC) decoder [5] or the soft-output decoders Belief Propagation (BP) [6] or Soft Cancellation (SCAN) [7]. Preliminary hardware-implementation results showed that ensemble decoding with BP can outperform other variants of BP decoding both in terms of throughput and area efficiency [8].

Very recently, polar code design for ensemble decoding has been gaining momentum, e.g., [5], [9]–[14]. The use of affine automorphisms effectively improves decoding performance of ensemble decoding. The resulting decoder is labeled as an Automorphism Ensemble (AE) decoder in [9]. However, AE decoding does not seem to be as effective for long codes. Authors in [11] introduce Partially-Symmetric (PS) codes to prove that $M$ should grow exponentially with the code length to get good error-correction performance, while reliability-based design only provides a limited affine automorphism group when the code length grows [15]. On the other hand, PS codes, known to have an advantageous affine automorphism group, exhibit good performance under AE decoding for practical decoding parameters [10], [12]. The output of AE-SC decoding permits to cluster the automorphisms into groups providing the same codeword candidate [14]. The number of these groups, also called equivalence classes, gives an upper bound on $M$.

To yield good performance under AE decoding, two criteria are crucial in the design of short polar codes. First, the minimum distance of the code should be maximized to avoid poor performance under Maximum-Likelihood (ML) decoding [6]. Second, the code should be partially or fully symmetric, exhibiting a meaningful affine automorphism group for AE decoding [10]–[12]. In this paper, the feasibility of these high-performance codes, termed Partially-Symmetric Reed-Muller (PS-RM) codes in the remainder, is investigated for code lengths $N \leq 256$. We prove that the absorption group of a PS-RM code has a promising structure for AE decoding, allowing better error-correction performance with reduced decoding latency with respect to short 5G polar codes under CA-SCL decoding.

II. PRELIMINARIES

A. Polar Codes

A $(N = 2^k, K)$ polar code of length $N$ and dimension $K$ is a binary block code defined by a kernel $T_2 \triangleq [1_1]$, the transformation matrix $T_N = T_2^{2^n}$, the information set $\mathcal{I} \subseteq [N]$ with $[N] = \{0, 1, \ldots, N-1\}$ and the frozen set $\mathcal{F} = [N] \setminus \mathcal{I}$. The encoding is performed as $\mathbf{x} = \mathbf{u} \cdot T_N$ with the input vector $\mathbf{u} = (u_0, \ldots, u_{N-1})$ generated by assigning $u_i = 0$ if $i \in \mathcal{F}$ and storing information in the $K$ bit-channels stated in $\mathcal{I}$. In polar coding, $\mathcal{I}$ includes the $K$ most reliable bit channels resulting from channel polarization [1]. If the $K$ most reliable channels are chosen among the ones composing a larger Reed-Muller (RM) code, the resulting code distance is maximized and the code is referred to as a $RM$-polar [16]. RM-polar codes have a lower ML bound and show enhanced performance under ensemble decoding [6].

The structure of $T_N$ permits to define a universal partial order (UPO) $j \preceq i$ between two virtual channels $i, j \in [N]$ that is valid for every communication channel [17]. A polar code defined by $\mathcal{I}$ is compliant with the UPO if $\forall i, j \in [N]$ with $i \geq j$, $j \in \mathcal{I}$ implies that $i \in \mathcal{I}$. A polar code fulfilling
the UPO can be described through minimum information set \( T_{\text{min}}^N \subseteq \{N\} [10] \) as
\[
\mathcal{I} = \bigcup_{j \in \mathcal{I}^N_{\text{min}}} \{ i \in [N], j \preceq i \}.
\] (1)

In the following, a code that maximizes the code distance and that is compliant with the UPO will also be referred to as a RM-polar by a slight abuse of notation.

B. Monomial Codes

Let us define \( M^{[n]} \) as the set of monomials in \( n \) variables over \( \mathbb{F}_2 \). A monomial \( m \in M^{[n]} \) is defined as a product of the variables \( \{V_0, \ldots, V_{n-1}\} \) and its degree \( \deg(m) \) corresponds to the number of variables in its product. Similarly, a negative monomial \( \overline{m} \) is defined as a product of the variables \( \{\overline{V}_0, \ldots, \overline{V}_{n-1}\} \) with \( \overline{V}_i = 1 \oplus V_i \). A monomial \( \overline{m}_i \in M^{[n]} \) is connected to the integer \( l \in [N] \) as \( \overline{m}_i = \prod_{i \leq S} \overline{V}_i \), with \( l = \sum_{i \in S} 2^i, S \subseteq [n] \). We have \( \deg(\overline{m}_i) = n - |S| \) and there are \( \binom{n}{l} \) degree-k monomials. A boolean function \( f : \mathbb{F}_2^n \to \mathbb{F}_2 \) can be seen as a map associating a bit to each integer in \([N]\). The evaluation function \( \text{eval} : \mathbb{F}_2^n \to \mathbb{F}_2^n \) checks the output of \( f \) for all elements of \( \mathbb{F}_2^n \) in increasing order, namely:
\[
\text{eval}(f) = (f(b_0), f(b_1), \ldots, f(b_{N-1})),
\] (2)

with \( b_k \in \mathbb{F}_2^n \) being the binary representation of \( k \in [N] \). Monomial codes of length \( N = 2^n \) are a family of codes that can be obtained as evaluations of monomials in \( n \) binary variables. A monomial code \( C(N,K,G) \) is generated by \( K \) monomials forming the generating monomial set \( G \) of the code. Polar codes can be described through this formalism [18] since \( \forall l \in [N], \text{eval}(\overline{m}_l) \) correspond to the \( l^{th} \) row of \( T_N \) [18]. The notion of partial order also exists among the monomials, namely:

**Definition 1** (Partial ordering). Given two monomials \( \overline{m}_{t_1}, \overline{m}_{t_2} \) of degrees \( s_1 \) and \( s_2 \), where \( \overline{m}_{t_1} = \overline{V}_{i_0} \cdots \overline{V}_{i_{s_1-1}}, \) with \( i_0 < \cdots < i_{s_1-1} \) and \( \overline{m}_{t_2} = \overline{V}_{j_0} \cdots \overline{V}_{j_{s_2-1}} \), with \( j_0 < \cdots < j_{s_2-1} \), we say that \( \overline{m}_{t_1} \preceq \overline{m}_{t_2} \) in two cases: if \( s_1 = s_2 = s \), when \( i_l \leq j_l \) for all \( l = 0, \ldots, s-1 \), while if \( s_1 < s_2 \), when there exists a divisor \( \overline{m}'_{t_2} \) of \( \overline{m}_{t_2} \) with \( \deg(\overline{m}_{t_1}) = \deg(\overline{m}'_{t_2}) \) and \( \overline{m}_{t_1} \preceq \overline{m}'_{t_2} \).

A monomial code is called decreasing if \( \forall \overline{m}_{t_1}, \overline{m}_{t_2} \in M^{[n]} \) such that \( \overline{m}_{t_1} \preceq \overline{m}_{t_2} \), if \( \overline{m}_{t_1} \in G \), then \( \overline{m}_{t_1} \in \hat{G} \). In this case, \( \hat{G} \) can be also described with the minimal monomial set \( \mathcal{G}_{\text{min}}^N \) containing the few monomials necessary to generate \( \hat{G} \)
\[
\hat{G} = \bigcup_{\overline{m}_i \in \mathcal{G}_{\text{min}}^N} \{ \overline{m}_i \in M^{[N]}, \overline{m}_i \preceq \overline{m}_j \}.
\] (3)

A polar code compliant with the UPO framework is a provably decreasing monomial [18]. Thus, \( T_{\text{min}}^N \) and \( \hat{G}_{\text{min}}^N \) generate the same code. Next, we use the notation \( T_{\text{min}}^N \) for both sets.

C. Automorphism Group of Polar Codes

Let \( \Pi(N) \) be the set of all permutations on \( \{0, \ldots, N-1\} \). An automorphism \( \pi \in \Pi(N) \) of a code \( C \) is a permutation of \( N \) elements mapping every codeword \( x \in C \) into another codeword \( \pi(x) \in C \). The automorphism group \( \text{Aut}(C) \) of a code \( C \) is the set containing all automorphisms of the code. For monomial codes, the affine automorphism group \( A \subseteq \text{Aut}(C) \), formed by the automorphism that can be written as affine transformations of \( n \) variables, is of particular interest. An affine transformation of \( n \) variables is described by
\[
z \mapsto z' = Az + b,
\] (4)

where \( z, z' \in \mathbb{F}_2^n \), the matrix \( A \in \mathbb{F}_2^{n \times n} \) is invertible and \( b \in \mathbb{F}_2^n \). The variables in (4) are the binary representations of code bit indices, and thus affine transformations represent code bit permutations. The automorphism group of any RM code with order \( r \) and variable \( n \), denoted \( R(r,n) \), is known to be the complete affine group \( GA(n) \) [19]. The affine automorphism group of codes compliant with the UPO is the block-lower-triangular affine (BLTA) group [10], [20], namely the group of affine transformations having a BLT transformation matrix. The BLTA group is recovered by the sizes of the blocks alongside the diagonal, defining the profile \( S = (s_1, \ldots, s_t) \), with \( s_1 + \cdots + s_t = n \).

D. AE Decoder, Partial Derivatives and Code Symmetry

Given a decoder \( \text{dec} \) for a code \( C \), the corresponding automorphism decoder \( \text{adec} \) is given by
\[
\text{adec}(y, \pi) = \pi^{-1}(\text{dec}(\pi(y))),
\] (5)

where \( y \) is the received signal and \( \pi, \pi^{-1} \in \text{Aut}(C) \). The AE decoder [9] consists of \( M \) \( \text{adec} \) instances running in parallel with \( \pi \in A \). The codeword candidate of AE is selected using a least-squares metric. The absorption group of \( \text{dec} \), denoted \( [1], \) gathers all permutations such that \( \forall y \in \mathbb{R}^N, \text{adec}(y, \pi) = \text{dec}(y) \) [14]. The absorption group of \( \text{SC}([1]) \subseteq A \) is also a BLTA group [21]. It is possible to use \([1]\) to cluster \( A \) into \([A]/[1] \) equivalence classes, namely subsets of automorphisms always providing same results under \( \text{SC} \) [14].

Given a monomial code \( C(N,K,G) \), its partial derivative \( \frac{\partial}{\partial \overline{V}_i} \) is the monomial code \( C_i \left( N, K, \overline{G}_i \right) \). \( \overline{G}_i \) is composed of all monomials in \( G \) including the variable \( \overline{V}_i \). By definition, there exist \( n \) partial derivatives \( C_0, \ldots, C_{n-1} \), of dimensions \( K_0, \ldots, K_{n-1} \). As an example, the partial derivatives of the code \( C \) \((8,4, G = \{1, \overline{V}_0, \overline{V}_1, \overline{V}_0 \overline{V}_1\}) \) are calculated as:
\[
\frac{\partial}{\partial \overline{V}_0} = \{ \frac{\partial}{\partial \overline{V}_0} = 0, \frac{\partial}{\partial \overline{V}_0} = 1, \frac{\partial}{\partial \overline{V}_1} = 0, \frac{\partial}{\partial \overline{V}_0 \overline{V}_1} = \overline{V}_1 \} = \{1, \overline{V}_1\},
\]
\[
\frac{\partial}{\partial \overline{V}_1} = \{ \frac{\partial}{\partial \overline{V}_1} = 0, \frac{\partial}{\partial \overline{V}_0} = 0, \frac{\partial}{\partial \overline{V}_1} = 1, \frac{\partial}{\partial \overline{V}_0 \overline{V}_1} = \overline{V}_0 \} = \{1, \overline{V}_0\},
\]
\[
\frac{\partial}{\partial \overline{V}_2} = \{ \frac{\partial}{\partial \overline{V}_2} = 0, \frac{\partial}{\partial \overline{V}_2} = 0, \frac{\partial}{\partial \overline{V}_1} = 0, \frac{\partial}{\partial \overline{V}_0 \overline{V}_1} = 0 \} = 0.
\]
The code symmetry \( t \) is defined as the number of partial derivatives having the lowest dimension [11] and coincides
with the last block of size $s_1 \times s_t$ of $\mathcal{A} = \text{BLTA}(s_1, \ldots, s_t = t)$ [10], [14]. Having $t \neq 1$ for this block is crucial for good performance under AE decoding [10], [12]. RM codes are fully symmetric codes, i.e., $t = n$, and are thus affine invariant while polar codes are usually non-symmetric, i.e., $t = 1$ [12]. Partially-symmetric codes correspond to codes having a symmetry $2 \leq t \leq n - 1$ [11].

III. PARTIALLY-SYMMETRIC REED-MULLER CODES

Codes known to have good performance under AE decoding have peculiar properties. However, it is not clear if such codes actually exist for every code dimension $K$. In this section, the feasibility of these codes is studied for lengths $32 \leq N \leq 256$.

A. Definition

The definition of PS-RM codes is given.

Definition 2 (Partially-Symmetric Reed-Muller code). A $C(N, K, G)$ PS-RM code is a RM-polar code and is $t$-symmetric with $t > 1$.

A PS-RM code is compliant with the UPO. By being $t$-symmetric, a PS-RM code exhibits an advantageous affine automorphism group $A = \text{BLTA}(s_1, \ldots, s_t = t > 1)$. The code distance is maximised at dimension $K$ since it is a RM-polar code. Hence, a PS-RM code combines the most important properties required for good performance under AE decoding [6], [10], [12]. Similar codes were introduced in [11]. Some of the existing polar codes constructions for AE decoding [10], [13], [14] can be described through our formalism. However, this is the first time that such a construction is formally defined, and its major characteristics highlighted. The dimensions of the partial derivatives of a PS-RM code satisfy

$$K_0 \geq \cdots \geq K_{n-1}. \quad (6)$$

If (6) is not verified, given $j > i$ and $K_j > K_i$, the monomial set $G$ would be composed of more monomials including the variable $V_j$ than monomials including $V_i$. This leads to a contradiction with the decreasing property of $G$.

Since the code symmetry $t$ is defined as the number of partial derivatives having the lowest dimension [11], a $t$-symmetric code of length $N$ verifies:

$$K_{n-t-1} > K_{n-t} = \cdots = K_{n-1}. \quad (7)$$

B. Proposed Generator of PS-RM Codes

In the following, the PS-RM code generator with length $N$ is based on the minimal information set $\mathcal{I}_{\min}^N$ defined in (1). If the same generators, according to the decimal numeric system, are used to generate a code of length $N$ and $2N$, Theorem 1 shows that the code of length $2N$ exhibits useful affine automorphism group properties based on the code of length $N$.

Theorem 1 (Affine automorphism group of generated PS-RM codes). By using the same decimal generators for $\mathcal{I}_{\min}^N$ and $\mathcal{I}_{\min}^{2N}$, if $\mathcal{I}_{\min}^N$ generates a $C(N, K, \mathcal{I})$ RM-polar code with $A(C) = \text{BLTA}(s_1, s_2, \ldots, s_t)$, then $\mathcal{I}_{\min}^{2N}$ generates a $C(2N, K', \mathcal{I}')$ PS-RM code with $A(C') = \text{BLTA}(s_1, s_2, \ldots, s_t + 1)$.

Proof. Despite having the same generators in the decimal numeric system, within the polynomial formalism, monomial generators of $C'$ are all appended by the variable $\overline{V}_n$, hence $\partial C'/\partial \overline{V}_n = C$ and $A \left( \frac{\partial C'}{\partial \overline{V}_n} \right) = \text{BLTA}(s_1, s_2, \ldots, s_t)$. As mentioned in [15], $A(C) = A \left( \frac{\partial C}{\partial \overline{V}_n} \right)$ is a partition of the affine automorphism group $A(C')$ on variables $\{\overline{V}_0, \ldots, \overline{V}_{n-1}\}$ ($\overline{V}_n$ excluded). Thus, we have $A(C') = \text{BLTA}(s_1, s_2, \ldots, s_t + 1)$ or $A(C') = \text{BLTA}(s_1, s_2, \ldots, s_t, 1)$. By definition, the dimensions of the partial derivatives are not increasing, namely $K_n \leq K_{n-1}$ (6). However, by appending the variable $\overline{V}_n$ to every generator in $\mathcal{I}_{\min}$, monomials including $\overline{V}_n$ are as generated as monomials including $\overline{V}_{n-1}$ leading to $K_n = K_{n-1}$. Hence, the possibility that $A(C') = \text{BLTA}(s_1, s_2, \ldots, s_t, 1)$ is discarded. As a consequence, the generated code is $s_t + 1$-symmetric, i.e., $A(C') = \text{BLTA}(s_1, s_2, \ldots, s_t + 1)$.

Theorem 1 permits to easily generate a PS-RM code with large symmetry by using known generators generating a PS-RM code of length $N$. Moreover, if $\mathcal{I}_{\min}^N$ generates a non-symmetric ($t = 1$) RM-polar code of length $N$, by using the same generators to construct $\mathcal{I}_{\min}^{2N}$, $\mathcal{I}_{\min}^{2N}$ is generating a PS-RM code of length $2N$ of symmetry $t = 2$. Hence, more PS-RM codes are expected for longer code lengths; for $N = \{32, 64, 128, 256\}$, 22, 83, 515, and 4275 PS-RM codes have been found using a restricted number of generators, i.e., here with $|\mathcal{I}_{\min}^N| \leq 3$. Limiting the number of generators to 3 is motivated by [10, Fig. 3], since a larger $|\mathcal{I}_{\min}^N|$ would likely produce non-symmetric codes. For $N = 256$ and $|\mathcal{I}_{\min}^{256}| = 3$, 81.5%, 17.2%, and 1.3% of the codes respectively have a symmetry of 2, 3, and 4. If $|\mathcal{I}_{\min}^{256}| = 1$, 56.5% of the codes are at least $t = 4$ symmetric.

Proposition 1 (Code rate of generated PS-RM codes). If the same generators according to the decimal system are used to generate the codes $C(N, K, \mathcal{I}_{\min}^N)$ and $C(2N, K', \mathcal{I}_{\min}^{2N})$, then $K'_{2N} \geq K_{2N}$.

Proof. Let us denote $\mathcal{I}$ and $\mathcal{I}'$ as the information sets of $C$ and $C'$, respectively. Since $\frac{\partial C'}{\partial \overline{V}_n} = C$, $\mathcal{I}'$ is decomposed as $\mathcal{I}$ in the upper half (between 0 and $N - 1$) and $K''$ indices are implied by the partial order in the lower half (between $N$ and $2N - 1$). The partial order implies a code with more information bits in the second part leading to $K'' \geq K$ extra information bits (with $K'' = K$ if $K = N$). Thus, we have $K'_{2N} = K''_{2N} + K_{2N} \geq K_{2N}$.

Proposition 1 states that using the same set of generators for a longer code produces a code with a higher code rate. Hence, only PS-RM codes are expected for moderate to high code rates using Theorem 1. As an example, for $N = \{32, 64, 128, 256\}$, the codes generated by $\mathcal{I}_{\min}^N = \{27\}$ have rates $4/32 < 19/64 < 60/128 < 158/256$ and symmetry $2 < 3 < 4 < 5$. 

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with the desired
all located between
code construction, but requires
the generator proposed in [14] is used. This method is based on
dimensions are not achievable. For the remaining dimensions,
23
is explained with Theorem 1 and Proposition 1.

Next, we investigate the distribution of PS-RM codes, i.e.,
the maximum symmetry value found for PS-RM codes of
lengths \(N = \{32, 64, 128, 256\}\).

Fig. 1 depicts the distribution of PS-RM codes of lengths
\(N = \{32, 64, 128, 256\}\). For all code lengths, two dimensions
do not possess a single PS-RM code, always located second-
to-last and fourth-to-last before the dimension of \(R(2, n)\). The
existence of at least one PS-RM code for all moderate to high
code rates is verified. Fig. 2 shows the percentage of codes
with a symmetry greater or equal to a value. It can be seen
that, for a fixed symmetry, this percentage grows with \(N\). For
example, for \(N = 256\), 52.3% of the dimensions support a
PS-RM code with \(t \geq 4\) whereas this percentage drops to
37.1%, 25.5% and 23.8% for \(N = \{128, 64, 32\}\). As depicted
in Fig. 1, a high code rate facilitates large symmetry, which
is explained with Theorem 1 and Proposition 1.

The set \(I_{\min}^N\) composed of generators having values be-
tween 0 and \(N/2 - 1\) is used to generate PS-RM codes, as
suggested in Section III-B. As expected by Proposition 1, the
low code rates are difficult to obtain with this method. As
an example, 23 dimensions are not achievable for \(N = 256\),
all located between \(K = 10\) and \(K = 44\). For \(N = 128\), 8
dimensions are not achievable. For the remaining dimensions,
the generator proposed in [14] is used. This method is based on
code construction, but requires \(A(C)\) to be performed. Since
\(A(C)\) has many different patterns, an extensive search of codes
with the desired \(A(C)\) is more involved than our method.

D. Absorption Group of PS-RM Codes

Permutations in AE-SC must be carefully chosen. Some
permutations are absorbed by SC decoding resulting in no
gain [9], [21]. These permutations form the absorption group
\([1] = BLTA(S_1)\) of SC [21], where \(S_1\) denotes the profile
absorbed by SC. [1] depends on \(T\) [14], [21]. The following
proposition shows a result on the SC absorption group for
most of the PS-RM codes.

Proposition 2 (SC absorption group of PS-RM codes). A
\(C(N, K, G)\) PS-RM code with \(n \leq K \leq N - n - 1\), has a
SC absorption group

\[ [1] = BLTA(S_1) = BLTA(s_1', \ldots, s_t' = 1). \]  \(\text{(8)}\)

Proof. We have \(s_i' > 1 \iff F \subseteq [N/2]\) or all information
bits are in the last \(N/2\) bits [21, Alg. 1, lines 12–23]. Let us
prove that it is impossible for PS-RM codes. We divide the
proof in two cases, either the monomial set \(G\) includes at least
one monomial including the variable \(V_{n-1}\) or it does not.

In the first case, for dimensions slightly greater than \(n\), i.e.,
close to the dimension of \(R(1, n)\), at least one monomial in \(G\)
belongs to the upper half part. Indeed, \(\overline{m}_{N/2-1} = V_{n-1} \in G\),
thus at least one information bit is in the upper part leading to
\(s_i' = 1\). For dimensions slightly smaller than \(N - n - 1\), i.e.,
close to the dimension of \(R(n - 2, n)\), the RM-polar property
ensures that \(\overline{m}_{N/2} = \prod_{i=0}^{n-2} V_i \notin G\), hence \(F \nsubseteq [N/2]\) leading to
\(s_i' = 1\). For intermediate rate, the previous statement remains,
lower and upper half part are mandatory composed of frozen
and information bits, resulting in an absorption group with
\(s_i' = 1\).

If no monomial in \(G\) includes the variable \(V_{n-1}\), [12, Theorem 2]
states that the code \(C\) is non-symmetric, i.e.,
\(A(C) = BLTA(s_1, \ldots, s_t = 1)\). Hence, since \([1]\) is smaller

![Image](image_url)
of the investigated PS-RM codes of length $M$ classes, reaches its ML bound for $L = 0$ symmetric codes under AE-SC, denoting AE-SC with $M$ bounds are approximated using the truncated union bound.

the impact of PS-RM codes under AE-SC decoding. ML adec instances, transmitted with BPSK modulation over the

in the lower right corner of transformation matrix $A$ (4) enhance AE decoding performance. Thus, all PS-RM codes exhibit valuable permutations as some of these permutations are not absorbed according to Proposition 2.

If $K < n$ or $K > N - n - 1$, $S_2$ may have $s_i' > 1$ for the reasons invoked in the proof of Proposition 2, leading to automorphisms that are not beneficial [21]. Thus, the symmetries of these codes are not depicted in Fig. 1.

Table I recapitulates the statistics on the absorption groups $[i]$ of codes from Fig. 1. Overall, the absorption groups are mostly composed of a single block on the top left of the BLTA structure, permuting variables of low indices together. Such absorption groups represent $66.6\%$, $71.1\%$, $78.5\%$ and $84.1\%$ of the investigated PS-RM codes of length 32, 64, 128 and 256.

IV. SIMULATION RESULTS

In this section, we present simulation results of partially-symmetric codes under AE-M-SC, denoting AE-SC with $M$ adec instances, transmitted with BPSK modulation over the AWGN channel. Short and long codes are shown to illustrate the impact of PS-RM codes under AE-SC decoding. ML bounds are approximated using the truncated union bound.

In Fig. 3, the $(128, 60, I_{128} = \{27\})$ PS-RM code, decoded under AE-8-SC, outperforms by $0.25$ dB the 5G polar code under CA-SCL decoding for the same list size $M = L = 8$. The PS-RM code, having 2205 equivalence classes, reaches its ML bound for $M = 32$. The 5G polar code, that is neither RM-polar nor partially symmetric, is not suitable for AE-SC having only 3 equivalence classes.

For $N = 1024$ and $K = 512$, Fig. 4 shows the performance of $C_1$ defined by $T_{1024} = \{63, 121\}$ and $C_2$ defined by $T_{min2}^{1024} = \{183, 207, 241, 391, 928\}$. $C_3$ is retrieved with Theorem 1 and is PS-RM with symmetry $t = 7$, while $C_2$, retrieved with Algorithm 1 in [14], has symmetry $t = 3$ but is not a PS-RM code. Thus, $C_2$ has a small minimum code distance with respect to $C_1$, thus has a worse ML bound. However, this effect is countered by the fact that $C_2$ has only $192$ minimum distance codewords, leading to an acceptable ML bound with respect to $C_1$. The large symmetry of $C_1$, as well as its length $N = 1024$, dramatically decreases its performance under SC [11]. However, its large symmetry value allows for more diversity under AE-SC. $C_1$ beats $C_2$ with an unpractical number of automorphisms $M = 2048$. Thus, PS-RM codes may not be the best solution if AE is applied for longer codes. We also note the relatively poor performance of AE-SC for $N = 1024$ with respect to CA-SCL, corroborating our restricted scope of code lengths studied for PS-RM codes.

V. CONCLUSIONS

In this paper, the distribution of codes yielding good performance under AE decoding was given. These codes are RM-polar and partially symmetric. The distribution showed that almost any dimension exhibits at least one code for $N \leq 256$. Moreover, the percentage of higher symmetric codes grows with code length. We also proved that all PS-RM codes exhibit valuable equivalence classes under SC, confirming the existence of good codes under AE-SC for most dimensions.

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