Flat Galactic Rotation Curves from Geometry in Weyl Gravity

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We searched for a resolution of the flat galactic rotation curve problem from geometry instead of assuming the existence of dark matter. We observed that the scale independence of the rotational velocity in the outer region of galaxies could point out to a possible existence of local scale symmetry and therefore the gravitational phenomena inside such regions should be described by the unique local scale symmetric theory, namely Weyl’s theory of gravity. We solved field equations of Weyl gravity and determined the special geometry in the outer region of galaxies. In order to understand the scale dependent description of gravitational phenomena, we compared individual terms of so called Einstein–Weyl theory and concluded that while the outer region of galaxies are described by the Weyl term, the inner region of galaxies are described by the Einstein–Hilbert term.

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I. INTRODUCTION

According to current cosmological paradigm, there is a “dark side of the universe,” which consists of Dark Energy and Dark Matter. Former is the proposed explanation to the observation of cosmic speed–up [1, 2], and the latter is the proposed explanation to observation of flat rotation curves of galaxies [3, 4], gravitational lensing observations [5, 6], and many other astrophysical and cosmological observations [7–13]. Other than the existence of these dark components of the universe, contemporary ΛCDM cosmology (Λ for Dark Energy and CDM for Cold Dark Matter) also assumes that Einstein’s theory of gravity, i.e. general relativity, is valid at all length scales.

Although ΛCDM cosmology successfully explains almost all the cosmological observations it has several important theoretical drawbacks. Dark Energy (see e.g. reviews [14–19]) is smooth and persistent as it produces negative pressure and thus causes repulsive gravitational effect on expansion. Even though diverse phenomenology is proposed to explain the nature of dark energy, to date there is no satisfactory model everyone agrees on. In contrast, Dark Matter [20–22] needs to be nonrelativistic matter, because it should have a mass to produce the required phenomenology in the Newtonian limit, and needs to be dark, because it should not interact with baryonic matter electromagnetically, otherwise we would have seen it. Ambitious experiments designed to observe elusive dark matter particles gave negative results so far [23, 24]. Another related important problem is that the ratio of dark energy and dark matter density parameters is of the order one, so called the coincidence problem.

These theoretical problems led many physicists to seek alternative explanations to cosmological observations by challenging assumption of ΛCDM cosmology on the theory of gravity: that the general relativity may not be valid in all length scales. Successful tests of predictions of general relativity in the Solar system (see e.g. [26]) and the results of table–top experiments [27–29] on the range of validity of the Newtonian potential, which is the non–relativistic, weak field limit of general relativity, confirms that the general relativity is valid at least from a few microns to the size of the Solar system. However, we have no compelling evidence for the validity of the general relativity larger than the scale of the Solar system. Gravity could be behaving fundamentally different in large scales, such as galactic or cosmic scales. This idea motivated the study of so called modified or alternative theories of gravity.

There are many diverse ways to modify Einstein’s general relativity. Simplest of which is to have an arbitrary function of the scalar curvature as the Lagrangian density. Such theories are simply called the $f(R)$ theories of gravity [30–34]. There are several works attacking the flat galactic rotational velocity curve problem in $f(R)$ gravity [35–37]. In [35] and [36] the gravitational potential in the weak field limit is modified and this modification has either logarithmic [35] or power law [36] behavior. Whereas [37] accepts the gravitational potential has the Newtonian form, but the “mass” required for the flat rotation curve phenomenology is declared to be “geometric mass” coming from a special solution for the metric in the region where rotation velocity of the stars is almost the same irrespective of their distance to the galactic center.

In this article we will adopt a similar strategy as Böhmer et al. [54] in conformally invariant Weyl gravity [38, 39], except the fact that we will not interpret the effect of geometry as some kind of “geometric matter.” The Weyl gravity has as its fundamental symmetry invariance under the local scale transformation, so called Weyl transformation, of the metric tensor. So it is not a modification of the general relativity, but it is truly an alternative theory written a few years after Einstein and Hilbert formulated general relativity. Weyl gravity do not have the correct Newtonian limit [40], and this pre-
vented for almost a century for this theory to be taken seriously to describe the gravitational phenomena. Then, Mannheim and Kazanas in the 1990's considered the phenomenology of a Schwarzschild-like solution to attack both of the dark problems. Their approach is, however, claimed to be flawed. Thus the Weyl gravity approach to flat galactic rotation curve problem is far from complete. Taking into account Yoon's critique one needs to find new solutions of Weyl gravity which are exempt from problems mentioned in [43]. This is the reason that we embarked on the present project and this aim determines also the scope of this paper.

Given the surface matter density of a regular spiral galaxy as \( \sigma(r) = \sigma_0 e^{-r/\rho_0} \) with \( \rho_0 \) being a characteristic length specific to the galaxy, the rotational velocities of stars located at a distance of nearly \( 2.2 \rho_0 \) up to the edge of the galaxy (which could be at about \( 6 \rho_0 - 20 \rho_0 \) depending the galaxy), have almost the same value [42]. We will call this region of flat velocity curve the outer region and the region of raising velocity curve (from \( r = 0 \) to \( r = 2.2 \rho_0 \)) the inner region following [42]. Newtonian gravity produced by the luminous matter can be seen solely responsible of the rising velocity curve from the center up to the peak value at \( r = 2.2 \rho_0 \). After the peak point, however, the effect of Newtonian potential diminishes and related to that it is expected that the rotational velocities of stars would diminish depending on how much a particular star is away from the centre of the galaxy. Hence almost constancy of the rotational velocities beyond the peak value at \( r = 2.2 \rho_0 \) points to the fact that something is different in the outer region of the galaxy. The current paradigm asserts the existence of non-luminous dark matter in the outer region of the galaxy. In this paper we would like to follow the idea that in fact gravity is fundamentally different in the outer region of the galaxy and perhaps beyond.

The scale independence of the rotational velocity in the outer region of galaxies could point out to a possible existence of local scale symmetry in such regions. Therefore the gravitational phenomena in the outer region of galaxies could be described by the unique local scale symmetric metric theory, namely Weyl's theory of gravity. Then we have to determine the special geometry in which all the objects rotating around a distant galactic center have the same rotational velocities. Here we solve the field equations of Weyl gravity and by finding non-trivial exact solutions, we determine special geometry of the outer region of galaxies in several cases of the geometry of the solid angle. Even though the simplest of our solutions is the same as given in [37] in the context of \( f(R) \) gravity, we manage to find further solutions with the required properties. At this point we would like to emphasize again that we are not proposing “existence of geometric matter.” We are only claiming that the scale invariance of rotation curves in the outer region of galaxies could be due to the special form of the metric in that region, and this metric is an exact solution of the Weyl gravity. Thus we propose that the fundamental theory of gravity beyond the inner region of galaxies could be the Weyl gravity. We will attempt to reconcile this proposal with the fact that general relativity is very successful inside the Solar system and perhaps in the inner galactic region in the last section.

We remark that the scope of the present paper is set at the level of [35, 37] and [11]. This means that what we are looking in this paper is (are) some solution(s) to Weyl gravity that describes the outer regions of galaxies where the almost constant rotational velocities of the stars are observed irrespective to their distance from the galactic center. We do not attempt to do any fitting to actual data or in that respect we are not concerned with the problem of “almost” flatness of the rotation curves. These problems will be commented upon in the last section of this paper and planned to be thoroughly investigated as part of the future research.

The outline of this paper is as follows. In the next section we are going to review the basic structure of Weyl gravity and its field equations. Then in section (III) the tangential velocity of matter particles on circular orbits will be derived in terms of one of metric functions à la Böhmer et al. [37]. Afterwards in section (IV) four distinct solutions of Weyl gravity in the region of flat galactic rotation curves will be presented. Section (V) will contain our discussion of the results and comments on the scale dependent modelling of gravitational interactions.

II. WEYL GRAVITY AND THE FIELD EQUATIONS

Action of Weyl gravity is given in terms of contraction of two Weyl tensors as

\[
S = \alpha \int d^4x \sqrt{-g} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \tag{1}
\]

where \( C_{\mu\nu\rho\sigma} \) is the Weyl tensor, which is defined as the anti-symmetric part of the Riemann tensor, and \( \alpha \) is a dimensionless parameter. As described in the introduction, this action is invariant under Weyl transformation, which is the local scale transformation of the metric tensor, given by

\[
g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{\Omega(x)} g_{\mu\nu}. \tag{2}
\]

The invariance of (1) under this transformation is due to the fact that Weyl tensor does not transform under it,

\[
C_{\mu\nu\rho\sigma} \rightarrow \tilde{C}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}. \tag{3}
\]

It can be shown [42] that (1) is the unique metric action invariant under the Weyl transformation (2).

Since the Weyl tensor is the anti-symmetric part of the Riemann tensor, its self-contraction can be written in a special combination of self contractions of Riemann and
Ricci tensors, and the scalar curvature. This expression is given by

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2. \tag{4}$$

In four dimensions there is another special combination of these terms, so called Gauss–Bonnet term,

$$G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \tag{5}$$

which is topological. This means that it does not contribute to the field equations and therefore only the part of $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ modulo Gauss–Bonnet term,

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = 2(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) \pmod{G} \tag{6}$$

contributes to the field equations.

Thus the action of Weyl gravity that we are going to use is

$$S = 2\alpha \int d^4x \sqrt{-g} (-\frac{1}{3} R^2 + R_{\mu\nu} R^{\mu\nu}). \tag{7}$$

We are going to analyze the vacuum solutions of field equations of this theory in the constant rotation velocity galactic region.

We present the vacuum field equations of this theory as a combination of two terms. Contribution of the first term in the action \(7\), that is \(R^2\), to the field equations is given by

$$H_{\mu\nu} = 2R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) + 2 (g_{\mu\nu} \nabla \nabla - \nabla \mu \nabla \nu) R, \tag{8}$$

and contribution of the second term in the action \(7\), that is \(R_{\mu\nu} R^{\mu\nu}\), to the field equations is

$$K_{\mu\nu} = \Box \left( R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) - \nabla^\lambda \nabla_\mu R^\lambda_\nu - \nabla^\lambda \nabla_\nu R^\lambda_\mu + 2R_{\mu\lambda \nu \rho} R^\lambda_{\rho \sigma} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta}. \tag{9}$$

Then the field equations of the Weyl gravity \(7\) are given by

$$B_{\mu\nu} = -\frac{1}{3} H_{\mu\nu} + K_{\mu\nu} = 0, \tag{10}$$

where \(B_{\mu\nu}\) is called the Bach tensor \([44]\). Since it is conformally invariant, trace of the Bach tensor vanishes.

III. FLAT ROTATION CURVE GALACTIC REGION

Since our aim is to determine the form of the metric in the constant rotation velocity galactic region we consider a spherically symmetric form for the metric:

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2_k, \tag{11}$$

where \(d\Omega^2_k \equiv \frac{1}{1-kx^2} dx^2 + (1-kx^2) dy^2 + dz^2\) corresponds to two dimensional hyperbola, torus and sphere geometries, with \(k\) taking values -1, 0, 1, respectively. In the sphere geometry we may use angular coordinates \(\theta\) and \(\phi\) to write \(d\Omega^2_0 = d\theta^2 + \sin^2 \theta d\phi^2\). Since a spiral galaxy has a disc geometry, stars move on geodesics in the equatorial plane, and therefore in the sphere geometry we might have \(\theta = \pi/2\).

The kinematical condition that the tangential velocity of a circular orbit is constant gives us a constraint to determine one of these metric functions completely. Since we are analyzing timelike geodesics that stars follow we have \(ds^2 < 0\), which allows us to write

$$1 = Ai^2 - \frac{r^2}{B(r)} - r^2 \dot{\phi}^2, \tag{12}$$

where dotted quantities represent derivative with respect to affine parameter \(s\). For the case of circular timelike geodesics we further have \(\dot{r} = 0\), we then have

$$1 = Ai^2 - r^2 \dot{\phi}^2. \tag{13}$$

Derivative of this expression with respect to \(r\) gives us further that

$$\frac{1}{2} A i^2 = r \dot{\phi}^2. \tag{14}$$

Tangential velocity can be obtained from the expression given in \([12]\) as

$$v_\phi = \frac{dl}{ds} = \frac{r d\phi}{\sqrt{-g_{\phi\phi}} dl}, \tag{15}$$

where \(l\) is the length measured on the geodesic. Using \([14]\), we obtain the tangential velocity as

$$v_\phi^2 = \frac{r A'}{2 A}, \tag{16}$$

which was also obtained previously in \([35]\) and \([37]\).

Since we are interested constant \(v_\phi\) and since \([10]\) is a first order differential equation in \(A(r)\), it can be easily integrated. Setting \(v_\phi = w = \text{constant}\) we obtain the form of the metric as

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2_k, \tag{17}$$

where \(w = v_\phi^2 \approx 10^{-6}\) if we work with natural units in which \(c = 1\), and \(r_c\) is an integration constant.

IV. SOLUTIONS IN WEYL GRAVITY

We have only one metric component left to determine. For that we need to solve only one field equation, and make sure that the other field equations are trivially satisfied by this solution. Due to the Bianchi identities and trace relations, all field equations can be written in terms
of the “rr” component. So it’s enough to solve just “rr” component of the field equations. This equation can also be obtained by varying the action with respect to $B(r)$ after substituting the metric into the Lagrangian. Thus we determine the “rr” field equation,

$$BB'' r^2 + wBB' r - \frac{1}{4} B'^2 r^2 - B^2 (w - 1)^2 + \frac{k^2}{(w - 1)^2} = 0,$$

(18)

where $B'(r)$ is the derivative of $B(r)$ with respect to $r$.

For the sake of the completeness we also write down the other components of the field equations. Due to the spherical symmetry “$\theta$θ” and “$\phi$φ” field equations are the same and given by

$$B[B'' r^3 + 3wB'' r^2 + (3w - 2)B' r] + \frac{1}{2} B'' B' r^3$$

$$+ \frac{w}{2} B'^2 r^2 - 2(w - 1)^3 B^2 + \frac{2k^2}{w - 1} = 0,$$

(19)

whereas the “tt” field equation is

$$B[B'' r^3 + (2w + 1)B'' r^2 - (w^2 - 4w + 2)B' r]$$

$$+ \frac{1}{2} B'' B' r^3 + \frac{3w - 1}{4} B'^2 r^2$$

$$- (w - 1)^3 B^2 + \frac{k^2}{w - 1} = 0.$$

(20)

These equations are satisfied by all the solutions of equation (18) presented in this paper.

Simplest solution of equation (18) is the obvious constant solution, $B(r) = C$, with

$$C = \frac{k}{(w - 1)^2}$$

(21)

This solution is exactly equivalent to the “constant $\lambda$” solution described in [37], except the fact that it is obtained as a solution of the Weyl gravity instead of the $f(R)$ gravity. In fact, this was the only solution that could be obtained exactly in [37]. In the present paper we are going to show that there are exact solutions of Weyl gravity in which $B(r)$ (or $\lambda$ in the language of [37]) is not a constant. These solutions are our main contribution to the idea that the galactic flat rotation curves can be explained via geometry instead of dark matter. We now present these solutions.

A. Solution specific to torus $T^2$ geometry

We first find an analytical solution for the 2-torus $T^2$ ($k = 0$) case. We perform a transformation $B(r) = [b(r)]^n$ after which the equation (18) we want to solve becomes

$$b[b'' r^2 + wb' r - \frac{1}{n}(1 - w)^2 b] + (\frac{3n}{4} - 1)(b')^2 r^2 = 0,$$

(22)

which has a simple form. After a choice $n = \frac{4}{3}$, we get a second order linear differential equation which can be solved easily,

$$b'' r^2 + wb' r - \frac{3}{4}(1 - w)^2 b = 0.$$

(23)

To solve this equation we make a solution ansatz $b(r) = r^\alpha$. Then the characteristic equation

$$\alpha^2 - (1 - w) \alpha - \frac{3}{4}(1 - w)^2 = 0,$$

(24)

has to have real solutions since the discriminant is positive. The general solution for $b(r)$ is given by

$$b(r) = C_1 r^{\alpha_1} + C_2 r^{\alpha_2},$$

(25)

where $C_{1,2}$ are the integration constants, $\alpha_1 = \frac{2}{3}(1 - w)$ and $\alpha_2 = -\frac{1}{3}(1 - w)$ are the roots of (24). Thus from the “rr” field equation we find the solution for $B(r)$ as

$$B(r) = (C_1 r^{\alpha_1} + C_2 r^{\alpha_2})^\frac{2}{3}.$$

(26)

After inserting the expressions for $\alpha_1$ and $\alpha_2$ into this solution one gets

$$B(r) = r^{2(1-w)} (C_1 + C_2 r^{-2(1-w)})^\frac{2}{3},$$

(27)

which is independent of $n$. Therefore the metric that satisfies the field equations is

$$ds^2 = - \left( \frac{r}{r_c} \right)^{2w} dt^2 + \frac{r^{2(w-1)} dr^2}{(C_1 + C_2 r^{2(w-1)})^\frac{2}{3}} + r^2 (dx^2 + dy^2).$$

(28)

B. Solution valid for sphere $S^2$ and hyperbolic plane $H^2$ geometries

In order to simplify the equation (18) we perform the following transformation:

$$B(r) = \frac{2k}{(1 - w)^2 r^{2(1-w)}} F(r),$$

(29)

which yields

$$F(r) \left[ r^2 F''(r) + (3 - 2w) r F'(r) \right] - \frac{r^2}{4} F'(r)^2$$

$$+ \frac{(1 - w)^2}{4r^{4(1-w)}} = 0.$$

(30)

After a change of variable, $r = z^{1/(1-w)}$, we get

$$F(z) \left[ z^2 F''(z) + 3z F'(z) \right] - \frac{z^2}{4} F'(z)^2 + \frac{1}{4z^4} = 0.$$

(31)

Now we multiply this expression with $2z^4 F' F^{-3/2}$ and obtain

$$2z^6 F' F'' F^{-1/2} + 6z^6 F' F^{-1/2} = \frac{1}{2} F' F^{-3/2}$$

$$+ \frac{1}{2} F' F^{-3/2} = 0.$$

(32)
Then the all solutions of (36) are given by

$$\frac{d}{dz} \left( \sqrt[3]{z^6 F^2 F^{-1/2} - F^{-1/2}} \right) = 0,$$

from which we can easily solve for the first derivative of 

$$F'(z) = \left(1 + \frac{1}{16C_1/3 \sqrt{F(z)}} \right)^{1/2} \frac{1}{z^3},$$

(34)

where we have chosen the integration constant as 

$$\sqrt{16C_1/3}$$

for later convenience. To integrate this equation we make another useful definition,

$$v(z) = 1 + \sqrt[3]{16C_1/3 \sqrt{F(z)}}$$

(35)

with

$$dF = \frac{3}{8C_1} (v - 1)dv.$$

Then the integration of

$$\frac{dF}{\left(1 + \frac{1}{16C_1/3 \sqrt{F}}\right)^{1/2}} = \frac{dz}{z^3}$$

becomes a cubic algebraic equation of 

$$\sqrt[3]{\nu}$$

given by

$$\left(\sqrt[3]{\nu}\right)^3 - 3\sqrt[3]{\nu} = 2q(z),$$

(36)

where 

$$q(z) = -\frac{C_1}{2} + C_2.$$ 

Using Cardano’s method for this type of cubic equations we further write

$$\sqrt[3]{\nu} = a + b,$$

and then substituting this into (35) yields

$$a^3 + b^3 + 3(ab - 1)(a + b) - 2q = 0.$$

If we now take 

$$ab = 1,$$

then

$$a^3 + b^3 - 2q = 0.$$

This way we reach a second order equation

$$x^2 - 2qx + 1 = 0,$$

where 

$$x$$

stands for both 

$$a$$

and 

$$b.$$ 

This equation is described by the relations,

$$a^3 + b^3 = 2q, a^3b^3 = 1.$$

Then the all solutions of (36) are given by

$$v_1(z) = (h + h^{-1})^2$$

$$v_2(z) = (e^{i4\pi/3}h + e^{i2\pi/3}h^{-1})^2$$

$$v_3(z) = (e^{i2\pi/3}h + e^{i4\pi/3}h^{-1})^2$$

where

$$h \equiv \left[q(z) + \sqrt{q(z)^2 - 1}\right]^{1/3}.$$

As seen above, there are two kinds of solutions for 

$$v(z).$$ 

If 

$$h$$

is real, there are one real and two complex solutions, and if 

$$h$$

is complex, then all solutions are real.

Using the relations (29) and (35), and then transforming the coordinate back to 

$$r$$

from 

$$z,$$

all metric function solutions are found to be

$$B_i(r) = \frac{3k^2(1 - w)}{8(1 - w)^2 C_1} (1 + u_i(r))^2 \quad (i = 1, 2, 3),$$

(37)

with

$$u_1(r) = h(r)^2 + h(r)^{-2}$$

(38)

$$u_2(r) = \frac{1}{2} (1 + i\sqrt{3})h(r)^2 - (1 + i\sqrt{3})h(r)^{-2}$$

(39)

$$u_3(r) = \frac{1}{2} (1 + i\sqrt{3})h(r)^2 + (1 + i\sqrt{3})h(r)^{-2}$$

(40)

where

$$h(r) = \left( - \frac{C_1}{r^2(1 - w)} + C_2 + \sqrt{\left( - \frac{C_1}{r^2(1 - w)} + C_2 \right)^2 - 1} \right).$$

These solutions describe the spacetimes with two dimensional hyperbola and sphere geometries for 

$$k = -1$$

and 

$$k = 1,$$

respectively.

This way we determined general geometries as solutions of Weyl gravity, which allow stars moving in any of these geometries to have the same rotational velocity no matter their individual distance from the galactic center. According to the observations, such a region starts from the surface of the central bulge at about 

$$r \approx 2.2v_0$$

to the edge of the galaxy [42]. Therefore our solutions only describe this outer region of the galaxy. The geometry of the central bulge, i.e. the inner region of the galaxy, is described by the Einstein’s theory of gravity. How the two theories could be reconciled at the border of inner and outer regions is commented upon in the next section.

V. DISCUSSION OF THE RESULTS AND CONCLUSIONS

In this article we searched a resolution of flat galactic rotation curve problem from geometry instead of assuming the existence of dark matter. The first observation we made is that the scale independence of the rotational velocity in the outer region of galaxies could point out to a possible existence of local scale symmetry inside such regions. Therefore the gravitational phenomena in the outer region of galaxies should be described by the unique local scale symmetric metric theory, namely Weyl’s theory of gravity. Then we had to determine the special geometry in which all the objects rotating around a distant galactic center have the same rotational velocity. We solved field equations of Weyl gravity and determined the special geometry of the outer region of galaxies in several cases of the geometry of the solid angle.
The fact that the phenomenology of raising rotation velocity curves in the central bulge of galaxies is explained very successfully by general relativity brings us to the natural question of how these two theories can be reconciled inside the galaxy or in that respect in any other system strongly dependent on gravitational physics. To resolve this issue in our model, we propose that the true theory of gravity is the combination of general relativity and the Weyl gravity with an action given by

$$S = \frac{M_P}{2} \int d^4x \sqrt{-g} \left[ R + 2\tilde{\alpha} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \right], \quad (41)$$

where $M_P$ is the Planck mass. Since the self–contraction of Weyl tensor has already mass dimension 4, the coefficient $\tilde{\alpha}$ here has dimension of $[\text{length}]^2$ and related to $\alpha$ of (1) by $M_P^2 \tilde{\alpha} = \alpha$.

The relative values of the Einstein–Hilbert term $R$ and the Weyl term $\tilde{\alpha} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ would determine whether one is dominant in a specific setting or both of them should be considered in equal footing [46, 47]. We consider a galaxy to be made of pressureless matter (stars) with a mass density decreasing exponentially from the center of the galaxy to its edge. However, the mass enclosed by a disk (or sphere) of radius $r$ increases from the center of the galaxy to its edge as more and more mass is taken into account. The relative values of the Einstein–Hilbert and the Weyl terms could be calculated in the framework of general relativity via their expressions in terms of hydrodynamical quantities given by

$$R(r) = \kappa (\rho(r)c^2 - 3\beta(r)), \quad C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}(r) = \frac{4}{3} \left( \frac{6Gm(r)}{c^4\rho^3} - \kappa \rho(r)c^2 \right)^2, \quad (42)$$

where $\kappa = \frac{8\pi\sigma}{c^4}$ is the so called Einstein’s constant [48].

Given the surface matter density of a regular spiral galaxy as $\sigma(r) = \sigma_0 e^{-r/r_0}$ with $r_0$ being a characteristic length specific to the galaxy, it is not hard to determine the values of $R$ and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ by using the above relations (42). As explained in (42), in the maximum disk prescription the rotation velocity data is well explained up to the border of the central bulge by the Newtonian theory and the border of the central bulge is normalized to be at $r = 2.2 r_0$. Therefore if our proposal to make sense, the Weyl term, $\tilde{\alpha} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, should dominate over the Einstein–Hilbert term, $R$, starting at $r = 2.2 r_0$. Since $R$ is proportional to the mass density $\rho$ when pressure $p$ vanishes, it decreases exponentially starting from a central value at the center of the galaxy. In contrast, the Weyl term is proportional to the mass enclosed inside the radius $r$ away from the center. Thus it is an increasing function of $r$ up to the edge of the galaxy. We interpret this to be a scale dependent turning on/off of two parts of the Einstein–Weyl theory.

It is pretty clear that the relative value of Einstein–Hilbert and Weyl terms depends also on the value of $\tilde{\alpha}$. This coefficient and the Weyl term it multiplies has different meanings depending on the context the theory is utilized. In the ultraviolet, the Weyl term is needed as a counterterm to make theory of general relativity renormalizable [49, 51]. Then $\tilde{\alpha}$ is a dimensional coupling that depends on scale in [51, 52] according to a $\beta$–function which turns out to be negative. In the infrared, however, the same Weyl term could be considered as a high–derivative correction to general relativity, very similar in concept as the chiral Lagrangian of QCD [51], but now the coefficient of Weyl term has different meaning than ultraviolet. Astrophysical scales are obviously in the infrared and cosmological scales are in the far infrared. In such scales, we again interpret $\tilde{\alpha}$ as scale dependent coefficient which could have different values depending on the characteristics of the system at that scale.

The requirement of dominance can be quantified by requiring Weyl term to be two orders of magnitude larger than Einstein–Hilbert term at $r = 2.2 r_0$, which allowed us to determine the required value of $\tilde{\alpha}$. Using fairly average values for $r_0$ ($\approx 1 \, \text{kpc}$) and for the total mass of a galaxy ($M \approx 10^{12} \, M_\odot$), an easy algebra gave us that $\sqrt{\tilde{\alpha}} \approx 10 \, \text{Mpc}$. This value is much larger than radius of any galaxy. In fact it is on the order of radius of typical galaxy cluster. This means that starting from edge of central bulge of a galaxy up to the radius of the cluster, of which it is a member of, the Weyl term dominates the Einstein–Hilbert term and therefore the geometry should be described by the Weyl gravity. Thus, a further check of astrophysical relevance of our approach will be theoretical explanation of the gravitational lensing data of galaxy clusters and elliptical galaxies (see [11] for many references). In this respect, study of gravitational lensing in the Weyl gravity is very important and warrants a separate publication, which is currently under preparation [53].

We also plan to investigate possible effect of Einstein–Hilbert term on the breaking of the scale symmetry in the outer regions of the galaxies. The rotation curves on the outer regions of galaxies are not exactly flat and this points out slight violation of the scale symmetry on which the Weyl theory of gravity is based. This is another indication that the resolution of flat galactic rotation curve problem cannot be found in pure Weyl gravity and supports our proposal that the full theory should be the Einstein–Weyl gravity.

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