Nonergodic metallic and insulating phases of Josephson junction chains

Manuel Pino*, Lev B. Ioffe*a,b, and Boris L. Altshuler*a,c

*Department of Physics and Astronomy, Rutgers, The State University of New Jersey, New Brunswick, NJ 08901; aLaboratoire de Physique Théorique et Hautes Énergies, CNRS, UMR 7589, 75252 Paris Cedex 05, France; and cPhysics Department, Columbia University, New York, NY 10027

Contributed by Boris L. Altshuler, November 20, 2015 (sent for review August 24, 2015; reviewed by Eugene Bogomolny and Leonid I. Glazman)

Nonergodic metallic and insulating phases of Josephson junction chains

The remarkable feature of the closed quantum systems is the appearance of many-body localization (MBL) (1). Under certain conditions the states of a many-body system are localized in the Hilbert space resembling the celebrated Anderson localization (2) of single particle states in a random potential. MBL implies that macroscopic states of an isolated system depend on the initial conditions (i.e., the time averaging does not result in equipartition distribution and the entropy never reaches its thermodynamic value). Variation of macroscopic parameters (e.g., temperature) can delocalize the many-body state. However, the delocalization does not imply the recovery of the equipartition. Such a nonergodic behavior in isolated physical systems is the subject of this paper.

We argue that regular Josephson junction arrays (JJAs) under the conditions that are feasible to implement and control experimentally demonstrate both MBL and nonergodic behavior. A great advantage of the Josephson circuits is the possibility to disentangle them from the environment, as was demonstrated by the quantum information devices (3). At low temperatures the conductivity σ of JJA is finite (below we call such behavior metallic), whereas as T → 0 JJA becomes either a superconductor (σ → ∞) or an insulator (σ → 0) (4, 5). We predict that at a critical temperature Tc JJA undergoes a true phase transition into an MBL insulator (σ = 0 for T > Tc). Remarkably, already in the metallic state JJA becomes nonergodic and cannot be properly described by conventional statistical mechanics.

In the bad metal regime the dynamical evolution starting from a particular initial state does not lead to the thermodynamic equilibrium, so that even extensive quantities such as entropy differ from their thermodynamic values. Starting with the seminal paper of Fermi et al. (6) the question of nonergodicity in nonintegrable systems was extensively studied (7). However, the difference between the long time asymptotics of the extensive quantities and their thermodynamic values was not analyzed to the best of our knowledge. We believe that the behavior of JJA that we describe here can be observed in a variety of nonlinear systems.

The ground state of the model Eq. 1 is determined by the ratio of the Josephson and charging energies, EJ/Ec, that controls the strength of quantum fluctuations: JJA is an insulator at EJ/Ec < η and a superconductor at EJ/Ec > η (4, 5) with η ≈ 0.63 (Supporting Information, section 1 and Fig. S1). The quantum transition at EJ/Ec = η belongs to the Berezinsky–Kosterlitz–Thouless universality class (8). Away from the ground state in addition to EJ/Ec there appears dimensionless parameter U/Ec, where U is the energy per superconducting island (U = kBT in the thermodynamic equilibrium at T ≪ EJ).

The main qualitative finding of this paper is the appearance of a nonergodic and highly resistive “bad metal” phase at high temperatures, T/EJ > 1, which at T ≥ Tc ≈ EJ/Ec undergoes the transition to the MBL insulator. In contrast to the T = 0 behavior, these results are robust (e.g., they are insensitive to the presence of static random charges). The full phase diagram in the variables EJ/Ec, T/EJ is shown in Fig. 1. We confirmed numerically that the bad metal persists in the classic (EJ ≫ Ec) regime and survives in the presence of static random charges.

Significance

Conventional equilibrium statistical physics that aims to describe dynamical systems with many degrees of freedom relies crucially on the equipartition postulate: After evolving for a sufficiently long time, the probabilities to find the system in states with the same energy are equal. Time averaging is thus assumed to be equivalent to the averaging over the energy shell—the famous ergodic hypothesis. In this study we show that this hypothesis is not correct for a large class of quantum many-body models that can be implemented in the laboratory. These models are predicted to show a novel type of behavior that we name bad metal, which is neither a many-body insulator nor a conventional conductor.

Author contributions: M.P., L.B.I., and B.L.A. performed research, analyzed data, and wrote the paper.

Reviews: E.B., Université Paris-Sud, CNRS, Laboratoire de Physique Théorique et Modèles; and L.I.G., Yale University.

The authors declare no conflict of interest.

1To whom correspondence should be addressed. Email: bla@phys.columbia.edu.

This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1520033113/-DCSupplemental.
limit although $T_c \to \infty$; it is characterized by the exponential growth of the resistance with $T$ and violation of thermodynamic identities. We support these findings by semiquantitative theoretical arguments. Finally, we present the results of numerical diagonalization and TDMRG [time density matrix renormalization group (9)] of quantum systems that demonstrate both the nonergodic bad metal and the MBL insulator.

It is natural to compare the nonergodic state of JJA with a conventional glass characterized by infinitely many metastable states. The glass entropy does not vanish at $T=0$, that is, when heated from $T=0$ to the melting temperature the glass releases less entropy than the crystal [Kauzmann paradox (10)]. Similarly to glasses JJA demonstrates nonergodic behavior in both quantum and classical regimes. However, the ergodicity violation emerges as high rather than low temperatures transforming the Kauzmann paradox that arises at low temperatures into an apparent temperature divergence at high temperatures (discussed below).

**Qualitative Arguments for MBL Transition**

In a highly excited state $U \gg E_J$ the charging energy dominates: $E_c q^2 \sim U \gg E_J$. Accordingly, the value of the charge, $|q_i|$, and charge difference on neighboring sites, $\delta_i = |q_i - q_{i+1}|$, are of the order of $q - \delta_i \sim \sqrt{U/E_c}$. The energy cost of a unit charge transfer between two sites $\delta E \sim \sqrt{U/E_c}$ exceeds the matrix element of the charge transfer, $E_J/2$, as long as $U \gg U_{MBL} = E_J^2/E_c$. According to refs. 1 and 2 the system is a nonergodic MBL insulator under this condition. Thus, we expect the transition to MBL phase at $T_c/E_J \propto E_J/E_c$ with the numerical prefactor close to unity, as follows from the computation presented in Supporting Information, section 2 and Fig. S2. This computation also shows that resistivity is indeed infinite in this phase, contrary to the statement of work (11) that true MBL transition is impossible in translationally invariant systems.

If $E_J/E_C \gg 1$ and $U \ll E_J$ the conductivity limited by thermally activated phase slips is exponentially large, $\sigma \sim \exp(E_J/T)$ (12, 13). As we show below, at $T \gg E_J$ in the metallic phase conductivity is exponentially small, $\sigma \sim \exp(-T/E_J)$, even far from the transition, $E_J \ll T \ll E^2/E_C$. The resistance in this state can exceed $R_0 = h/(2e)^2$ dramatically and still display the “metallic” temperature behavior $(dR/dT > 0)$ (see Fig. 3).

Classical regime is realized at $E_C \to 0$ for fixed $E_J$ and $T$. One can express the charge of an island $q$ through the dimensionless time $\tau = \sqrt{E_c/E_J} t$ as $q = \sqrt{E_c/E_J} \exp(d q/d\tau)$. Because $q \sim \sqrt{T/E_J} \gg 1$, one can neglect the charge quantization and use the equations of motion

$$\frac{d^2}{dt^2} \phi_i = \sin(\phi_{i+1} - \phi_i) + \sin(\phi_{i-1} - \phi_i).$$  \[2\]

Here $i = 1, \ldots, L$ and the boundary conditions are $\phi_0 = \phi_{L+1} = 0$. The ergodicity of this classic problem was discussed in ref. 14; however, the available time scales were too short to make convincing conclusions.

We solve Eq. 2 for the various initial conditions corresponding to a given total energy and compute the energy $U_2$ contained in a part of the whole chain of the length $1 \ll l \ll L$ as a function of $\tau$.

The ergodicity implies familiar thermodynamic identities; for example

$$\left( \langle U_2^2 \rangle - \langle U_2 \rangle^2 \right)/T^2 = d(dU_2)/dT.$$  \[3\]

This relation between the average energy of the subsystem, $\langle U_2 \rangle$, and its second moment $\langle U_2^2 \rangle$ turns out to be invalid for a bad metal. To demonstrate this we evaluated the average energy per site in this subsystem, $u = \langle U_2 \rangle/E_J$, and the temporal fluctuations of this energy, $w_2 = \sqrt{\langle U_2^2 \rangle - \langle U_2 \rangle^2}/E_J$ (Fig. 3). Here $\langle \ldots \rangle$, and the bar denote correspondingly averaging over the time and over the ensemble of the initial conditions. (Given the evolution time $\tau_0$, averaging means over the time interval $\tau < \tau_0$ after initial evolution for time $\tau_0$.)

From Eq. 1 it follows that $u = T/(2E_J)$ at $T \gg E_J$, $u(T)$-function is evaluated for arbitrary $T/E_J$ in Supporting Information, section 3. One can thus rewrite Eq. 3 as

$$w = T^2 du/(EdT) \approx 2u^2.$$  \[4\]

Results of the numerical solution of Eq. 2 are compared with Eq. 3 in Fig. 2. For any given evolution time, $\tau_0$, the computed $w_2(u)$-dependence saturates instead of increasing as $u^2$ (Eq. 4). At a fixed $u$, $w_2(u)$ increases with time extremely slowly. Below we argue that $w_2(u)$ does not reach its thermodynamic value even at $\tau \to \infty$. Violation of the thermodynamic identity implies that temperature is ill-defined, so the average energy $u$ rather than $T$ is the proper control parameter. The effective temperature, defined as $T_e(u) = E_J/\int du/w(u)$, is shown in Fig. 2. Insert.

**Qualitative Interpretation**

Large dispersion of charges on adjacent islands, $i,i+1$ at $u \gg 1$, implies quick change of the phase differences, $\phi_i - \phi_{i+1}$, with time. Typical current between the two neighbors $E_J \sin(\phi_i - \phi_{i+1})$ time-averages almost to zero. However, accidentally the frequencies, $\omega_0 = d\phi_i/d\tau_0$ get close. In the classical limit the difference $\omega_0 - \omega_{i+1}$ can be arbitrary small. Such pair of islands is characterized by one periodic in time phase difference. Contrarily, three consecutive islands with close frequencies $|\omega_0 - \omega_{i+1} - \omega_0 - \omega_{i-1}| \leq 1$ experience chaotic dynamics that contains arbitrary small frequencies, similarly to work (15). For uncorrelated frequencies with variances $u \gg 1$, a triad of
islands \((i-1,i,i+1)\) is chaotic with the probability \(1/u \ll 1\), that is, such triads are separated by large quiet regions of a typical size \(r_i \sim u \gg 1\). The low-frequency noise generated by a chaotic island decreases exponentially deep inside a quiet region. Provided that \(u < u_{\text{rev}}(m = 1)\) the superconducting order parameter \(z_i(\omega) = \int \frac{d\tau}{2\pi} \exp(i\phi + i\omega \tau)\) at site \(i\) satisfies the recursion relation

\[
z_{i+1}(\omega) = z_i(\omega) \frac{1}{2\omega i u},
\]

which implies the log-normal distribution for the resistances \(R_{i+1}\) of quiet regions (Supporting Information, section 4):

\[
\left\langle \ln^2(R/R_i) \right\rangle = \ln R_i
\]

\[
\ln R_i = \ln(R) = u \ln(u),
\]

where \(R_i\) is the typical resistance of a quiet region. The resistance of the whole array is the sum of the resistances of the quiet regions. The mean number of these regions in the chain equals to \(N = L/r_i \gg 1\), its fluctuations being negligible. For the log-normal distribution the average resistance of a quiet region, \(\langle R \rangle\), is given by \(\langle R \rangle \approx R_i^{1/2}\). For the resistivity, \(\rho_i\), we thus have \(\rho = \sqrt{\langle R \rangle}/L = L^{1/2}/r_i\). According to Eq. 7

\[
\ln(\rho_i) \approx \frac{3}{2}(\gamma + \ln u) r_i \approx u(\ln u + \gamma),
\]

where \(\gamma = 0.577\) is Euler constant.

To determine the current caused by voltage \(V\) across the chain, we solved Eq. 2 with modified boundary conditions, \(\phi_0 = 0, \phi_{L+1} = V\). The results confirm the prediction Eq. 8 (Fig. 3). The range of the resistances set by realistic computation time is too small to detect the logarithmic factor in Eqs. 7 and 8; however, a relatively large slope, \(d\ln \rho_i/d\ln u \approx 3.0\), at \(u \approx 3.5\) is consistent with Eq. 8 that gives \(d\ln \rho_i/d\ln u \approx 2.5 + 1.5 \ln u - 1/u\).

The qualitative picture of triads separated by log-normally distributed resistances of silent regions allows one to understand the long time relaxation of \(w(t)\) in the subsystem of length \(L\) (Fig. 2). Each resistance can be viewed as a barrier with a tunneling rate \(\sim 1/R\). For a given time \(t\) the barriers with \(r \ll R\) can be considered impenetrable, whereas the barriers with \(r \gg R\) can be neglected. As a result, the barriers with \(R \geq r\) break the system into essentially independent quasiequilibrium regions (QER) of the typical size

\[
l_i \sim \frac{r_i}{\sqrt{2\pi} \ln R_i} \exp\left[\frac{\ln^2(r_i/R_i)}{2\ln R_i}\right].
\]

If \(l \gg r_i\) and \(r \lesssim \exp[\sqrt{\ln(1/\alpha)}]\ln R_i\), the subsystem contains \(l_i/l \gg 1\) QER, so that \(w \alpha l_i/l\). At longer times, \(l_i \gg l\), the subsystem is in equilibrium with a particular QER and \(w \propto l_i/l\). The full dependence on time can be interpolated as

\[
w_i = \frac{w_{\infty}}{1 + \beta \exp(-\alpha \ln(\tau/\tau_0))},
\]

where \(\alpha = (2 \ln R_i)^{-1}\), \(\beta = \sqrt{\pi/\alpha}(l/R_i)\) and \(\tau_0 = R_i l\).

The numerical simulations confirm that the energy variance \(w\) relaxes in agreement with Eq. 10 as shown in Fig. 4. The best fit to Eq. 10 yields parameters close to the expected, \(\ln R_i = \ln \rho(u)/\rho_i\), \(r_i \approx u\). Extrapolation to infinite times and sizes gives \(w_{\infty}(\infty) \approx 10.0\) of \(w_{\infty}(l)\) to large \(l\), which is significantly smaller than thermodynamic value \(w_{\infty} = 10.0\) at \(u = 3.5\).

Another test of the ergodicity follows from the fluctuation-dissipation theorem (FDT) that relates conductivity and current fluctuations. In the low frequency limit the noise power spectrum is \(S(\omega) = 2T_{\text{eff}}/R\), where \(T_{\text{eff}}\) is the effective temperature (Fig. 5), which we extracted from the numerical data. We found that \(T_{\text{eff}} > T_{\text{Th}}\) for \(u \ll 1\), where \(T_{\text{Th}}\) is the thermodynamic temperature. In particular, \(T_{\text{eff}} \approx 1.6 T_{\text{Th}}\) for \(u = 3.5\), which is close to \(T_{\text{Th}}(u)\) shown in Fig. 2. Inset. Note that both the energy and current fluctuations are less than expected in equilibrium.

**Quantum Behavior**

In contrast to a classical limit \(E_C \rightarrow 0\) in the quantum regime \(E_C > 0\) we expect at \(T = T_{\text{Th}}\) a MBL phase transition between two nonergodic states: the insulator (\(\rho = \infty\)) and a bad metal (\(\rho < \infty\)). For \(E_J > E_C\) the bad metal can be described classically at \(T < T_{\text{Th}} \sim E_J^2/E_C\). Our previous discussion suggests that the bad metal is nonergodic in a broad range of the parameters, \(T/E_J\) and \(E_J/E_C\). To verify this conjecture numerically we reduced the
Hilbert space of the model Eq. 1 to a finite number of charging states at each site, q_i = 0, ±1, ±2 (RHS model). We analyzed the time evolution of entanglement entropy, S{Ψ}, of the left half of the system. The entropy was averaged over the initial states from the ensemble of product states in the charge basis, S_i(L) = ⟨S{Ψ}⟩_{Ψ_i}, that correspond to zero total charge. As a result, we obtained the Gibbs entropy at T = ∞ [all states have the same weight exp(−H/J)]

Fig. 6, Inset shows the time dependence of the entropy at E_j/E_C = 0.3 that corresponds to the bad metal regime (discussed below). A slow saturation of the entropy follows its quick initial increase. It is crucial that the saturation constant, S_α(L), is significantly less than its maximal value, S_τ(L) = L ln5 expected at T = ∞ equilibrium. Furthermore, dS_α(L)/dL < ln5, indicating that S_α ≈ S_τ is extensive and the system is essentially nonergodic.

Fig. 6 presents S_α as a function of E_j/E_C. Note that S_α is measurably less than S_τ for E_j/E_C < 0.6 - 1. For E_j/E_C = 0, the entropy saturation is quick, and the accuracy of the simulations does not allow us to distinguish S_α from S_τ (see Figs. S3 and S4). We thus are unable to conclude whether the system is truly ergodic or weakly nonergodic at E_j/E_C ≫ 1. The former behavior would imply a genuine phase transition between bad and good metals, whereas the latter corresponds to a cross-over.

Deep in the insulator the time dependence of the entropy is extremely slow, roughly linear in ln t in a wide time interval (Supporting Information, section 5). This resembles the results of the works (16, 17) for the conventional disordered insulators. The extremely long relaxation times can be attributed to rare pairs of almost degenerate states localized within different halves of the system. The exponential decay with distance of the tunneling amplitude that entangles them leads to the exponentially slow relaxation.

To locate the MBL transition we analyzed the time dependence of the charge fluctuations. In a metal the charge fluctuations relaxation rate depends weakly (as a power law) on the sample size, in contrast to the exponential dependence in the insulator. Comparing the dependencies of the rates on the system size for different E_j/E_C (Fig. 7) we see that the transition happens in the interval 0.05 < E_j/E_C < 0.3.

The variances of the charge in the RHS model at T = ∞ and in the problem Eq. 1 at finite T coincide at T = 2E_C. Thus, we expect that the results of the quantum simulations describe the behavior of the model Eq. 1 at T/E_j ∼ E_C/E_j yielding the hyperbola shown in Fig. 1. The MBL transition at E_j/E_C ∼ 0.2 in T = ∞ RHS model corresponds to the transition temperature T_c ∼ 10E_j in model Eq. 1. The transition line shown in Fig. 1 is a natural connection of this point with T_c ∼ E_j/E_C asymptotic at E_j/E_C ≫ 1, discussed above. The maximum of the transition temperature is natural. Indeed, at E_j < E_C the charge fluctuates weakly at low T, in the opposite limit E_j < E_C the phase is well-defined, whereas at E_j ∼ E_C the quantum fluctuations are largest.

Possible Experimental Realization

MBL and the violation of the ergodicity can be observed only at sufficiently low temperatures when one can neglect the effects of thermally excited quasiparticles that form the environment to model Eq. 1. This limits temperatures to T < 0.1Δ, where Δ is the superconducting gap. To explore the phase diagram one has to vary both T/E_j and E_j/E_C in the intervals 1 < T/E_j < 5 and 0.1 < E_j/E_C < 5. The former condition can be satisfied if each junction is implemented as a superconducting quantum interference device loop with individual Josephson energy E_j(0) ∼ T_{max} = 0.1Δ so that E_j = 2 cos(π/φ)/πE_j, where Φ is flux through the loop. The latter condition requires enhancing ground capacitance of each island, which should exceed the capacitance of the junctions in order for the model Eq. 1 to be relevant. Realistic measurements of such array include
1174. 4,6,8 was computed using exact Ulam Problem and current noise. Here we predict a fast Phys Rev B through the τ = T ÷ averaging out temporal fluctuations requires exponentially long times. Moreover, at u ÷ 1 the resistance increases with u factorially, leading to a strong heating in the computation of resistance unless the measurement current is factorially small. Observation of a small current against the background of a low-frequency noise requires increasingly long times. Accordingly, for the realistic evolution times τ ≈ 10⁶ the resistance can be computed only for u < 4.0.

Simulation of the Quantum Problem. The time dependence of the entropy and the charge fluctuations for system of sizes L = 4,6,8 was computed using exact diagonalization in a symmetric subspace under charge conjugation. The tDMRG method was used for larger sizes but accuracy limits the range of times that we could study. In all simulations we impose the particle number conservation and open boundary conditions.

Acknowledgments. We thank I. L. Aleiner, M. Feigelman, S. Flach, V. E. Kravtsov, and A. M. Polyakov for useful discussions. This work was supported in part by Templeton Foundation Grant 40381, Army Research Office Grant W911NF-13-1-0431, and Agence Nationale de la Recherche QuDec.

Methods

Simulation of the JJA in the Classical Regime. At large u averaging out temporal fluctuations requires exponentially long times. Moreover, at u ÷ 1 the resistance increases with u factorially, leading to a strong heating in the computation of resistance unless the measurement current is factorially small. Observation of a small current against the background of a low-frequency noise requires increasingly long times. Accordingly, for the realistic evolution times τ ≈ 10⁶ the resistance can be computed only for u < 4.0.

Simulation of the Quantum Problem. The time dependence of the entropy and the charge fluctuations for system of sizes L = 4,6,8 was computed using exact diagonalization in a symmetric subspace under charge conjugation. The tDMRG method was used for larger sizes but accuracy limits the range of times that we could study. In all simulations we impose the particle number conservation and open boundary conditions.

Acknowledgments. We thank I. L. Aleiner, M. Feigelman, S. Flach, V. E. Kravtsov, and A. M. Polyakov for useful discussions. This work was supported in part by Templeton Foundation Grant 40381, Army Research Office Grant W911NF-13-1-0431, and Agence Nationale de la Recherche QuDec.
26. Rasmussen KO, Cretegny T, Kevrekidis PG, Gronbech-Jensen N (2000) Statistical mechanics of a discrete nonlinear system. Phys Rev Lett 84(17):3740–3743.
27. Flach S, Gorbach AV (2008) Discrete breathers – Advances in theory and applications. Phys Rep 467(1–3):1–116.
28. Feigel’man MV, Ioffe LB, Mezard M (2010) Superconductor-insulator transition and energy localization. Phys Rev B 82(18):184534.
29. Cuevas E, Feigel’man M, Ioffe L, Mezard M (2012) Level statistics of disordered spin-1/2 systems and materials with localized Cooper pairs. Nat Commun 3:1128.
30. Buonsante P, Vezzani A (2007) Ground-state fidelity and bipartite entanglement in the Bose-Hubbard model. Phys Rev Lett 98(11):110601.
31. Pino M, Prior J, Somoza AM, Jaksch D, Clark SR (2012) Reentrance and entanglement in the one-dimensional Bose-Hubbard model. Phys Rev A 86:023631.