Finite-difference approach of ice build-up around a cylindrical pipe

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Abstract. Melting-freezing phenomena are largely studied due to their importance in many applications such as latent heat storage, ice build-up in different settings, solidification of moulded pieces, epitaxial growth of silicon bars used to produce semiconductor waffles, etc. The paper deals with studying the ice build-up process around a cylindrical pipe through which a coolant flows, extracting heat from the surrounding water. Such a situation can be encountered in ice rinks where pipes are immersed in a shallow water basin that freezes. The approach of the paper is to determine the required time necessary for the ice thickness of the layer that builds up around the pipe to freeze under specified conditions (coolant temperature, pipe radius, and total radius of the water domain). An implicit finite difference technique with constant space step is employed to perform this task and the results are plotted and discussed.

1. Introduction

Transient conduction heat transfer involving the phase-change phenomenon is generally categorized as a moving boundary problem because of the solid-liquid interface that travels across the substance that freezes (solidifies) or melts. The mathematical approach is particularly difficult to deal with because of the obstacles arisen in finding the solutions of the partial differential equation that governs a time-dependent phenomenon. Moreover, the system is non-homogenous and the shares of the two phases vary continuously.

The first approach of the moving boundary problem applied to the ice-building process in polar seas belongs to J. Stefan who published in 1892 the paper Über die Theorie der Eisbildung, insbesondere über die Eisbildung in Polarmeeren (About the theory of ice formation, especially ice formation in polar seas) [1]. In fact, F. Neumann has proposed a solution to the moving boundary problem under the same conditions (an initially liquid semi-infinite domain with its fixed frontier maintained at the freezing temperature) as early as 1860, but his results have been published only in 1912, twenty years after Stefan’s paper. Neumann’s and Stefan’s results demonstrate how complicated the solution of the problem can be, even for the simplest geometries and boundary conditions. More complex geometries involving two- or three-dimension conduction heat transfer and more realistic boundary conditions (such as the third kind boundary condition) cannot generally lead to analytical solutions of the conduction heat transfer equation. These situations can be dealt with numerical techniques, such as the finite-difference approach.

A phase-change process is controlled by parameters such as: the initial temperature distribution within the phase-change material, geometry (size, surface area, shape), thermal properties of the substance that freezes or melts, and freezing temperature [3], [4].
The moving boundary problem related to freezing-melting phenomena has been approached by various methods [5]: isotherm migration [6], analytical [7], experimental [8], and numerical [9], [10].

The use of the finite-difference technique became more and more popular because it allows considering a very large diversity of geometries and physical properties, under different boundary conditions [10], [11], [12], [13].

The present paper considers the one-dimension water freezing around a pipe by using an implicit finite-difference method that involves a fixed space-step mesh.

2. The physical model
A single pipe submerged in water, cools it by means of a coolant flowing along the tube. The pipe is sufficiently long to assume a one-dimension (radial) conduction heat transfer. The inner radius of the cylindrical pipe wall is \( R_0 \), its outer radius is \( R_w \), the radius of the water domain is \( R \), and consequently its thickness is \( H \). Figure 1 schematizes the pipe-water system, its geometry and the fixed temperatures. The temperature \( t_c \) of the inner wall surface is constant (boundary condition of the first kind). Liquid water is initially at temperature \( t_0 \). The freezing temperature is \( t_f \). The outer boundary of the water domain is supposed adiabatic.

![Figure 1. Schematic of the pipe-water system.](image)

3. Mathematical model
In order to simplify the approach, dimensionless temperatures are generally used. In this particular case, one defines the dimensionless temperature \( \theta \) as:

\[
\theta = \frac{t - t_c}{t_0 - t_c}. \tag{1}
\]

By using cylindrical coordinates and assuming a one-dimension heat conduction, the equations involved are:

– the conduction heat transfer equation:

\[
\frac{\partial \theta}{\partial \tau} = a \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right), \tag{2}
\]

where \( a \) stands for the thermal diffusivity of the substance, and \( \tau \) for time.
the boundary conditions:

- at the inner surface of the pipe:
  \[ r = R_i : \theta_{w,0} = 0 \]  

- at the outer surface of the pipe:
  \[ \lambda_w \left( \frac{\partial \theta_w}{\partial r} \right)_{r=R_o} = \lambda \left( \frac{\partial \theta}{\partial r} \right)_{r=R_o} , \]  
  where the right hand term refers to the liquid phase as long freezing has not yet started, respectively to the solid phase during freezing.

- at the solid-liquid interface:
  \[ \lambda_s \left( \frac{\partial \theta_s}{\partial r} \right)_{r=R_s+S} = \lambda_i \left( \frac{\partial \theta_i}{\partial r} \right)_{r=R_s+S} + \rho l \frac{dR_s}{d\tau} , \]  
  where \( \rho \) stands for the density, \( l \) for the latent heat of fusion, \( R_i \) for the radius corresponding to the position of the interface, and \( \frac{dR_s}{d\tau} = \dot{R}_s(\tau) \) for the interface rate.

From equation (5), the interface rate can be expressed as:

\[ \dot{R}_s(\tau) = \frac{1}{\rho l} \left[ \lambda_s \left( \frac{\partial \theta_s}{\partial r} \right)_{r=R_s+S} - \lambda_i \left( \frac{\partial \theta_i}{\partial r} \right)_{r=R_s+S} \right] . \]  

This equation is of extreme importance, because it allows determining the position of the interface.

- at the adiabatic boundary (the outer limit of the water domain):
  \[ r = R_w + H : \left( \frac{\partial \theta_i}{\partial r} \right)_{R_o + H} = 0 . \]  

4. Finite-difference approach

A mesh of space step \( h_w \) has been attached to the pipe wall, and another one (space step: \( h \)), to the water domain. The numbers of nodes resulted by dividing the wall thickness and the water domain depth to the corresponding space step respectively: \( N_w \) for the pipe wall, and \( N \) for the water domain. The equations of the mathematical model have been re-written according to the implicit finite-difference approach.

Equation (1) written for node \( m \) at time step number \( p \) becomes:

\[ -\frac{c_m}{c_m} - \frac{1}{c_m} \theta_{m-1}^p + \left( 2 + \frac{1}{\alpha} \right) \theta_m^p - \frac{c_m + 1}{c_m} \theta_{m+1}^p = \frac{1}{\alpha} \theta_{m-1}^{p-1} \]  

where \( \alpha \) represents the stability and convergence criterion defined as:

\[ \alpha = a \Delta \tau h^{-2} \]  

\( \Delta \tau \) is the time step, and \( c_m \) is a coefficient defined as:

\[ c_m = 2 \left[ N - \frac{R_i}{R_o - R_i} + m \right] \]
where \( N \) stands for the number of nodes in the domain, and \( R_i \) and \( R_e \) represent the inner and the outer radii of the domain respectively.

In equation (8), the coefficients and the dimensionless temperatures will have a subscript specifying the domain: “w” for the pipe wall, “S” for the solid phase, or “L” for the liquid phase.

A special approach is necessary for the nodes adjacent to the interface. In this case, on both sides of the interface there are the following three nodes: the interface itself (a moving node), and nodes \( N_s - 1 \) and \( N_s \) in the solid phase domain, respectively \( N_s + 1 \) and \( N_s + 2 \) in the liquid phase domain (see figure 2).

\[
\text{Figure 2. Position of the solid-liquid interface with respect to the mesh at time step number } p.
\]

In this case, the fractions of the space step \( h \) (\( \xi \) for the solid, and \( \psi \) for the liquid respectively) imply that the three nodes necessary to write the finite-difference equation for the heat transfer equation (2) are not equally spaced, and therefore equation (8) cannot be used.

A procedure similar to that one which has been developed to derive equation (8) from equation (2) will result in the following two equations for nodes \( N_s \) and \( N_s + 1 \):

- finite-difference equation for node \( N_s \):

\[
-\theta^p_{N_s-1} + \frac{\xi^p}{2} + 1 \left( \frac{2}{\xi^p} + \frac{1}{\alpha_s} \right) \theta^p_{N_s} = \frac{\xi^p}{2} + 1 \frac{1}{\alpha_s} \theta^p_{N_s-1} + \frac{1}{\xi^p} \theta^p_{F}.
\]  

(11)

- finite-difference equation for node \( N_s + 1 \):

\[
\frac{\psi^p}{2} \left( \frac{2}{\psi^p} + \frac{1}{\alpha_l} \right) \theta^p_{N_s+1} - \psi^p \theta^p_{N_s+2} = \frac{\psi^p}{2} \left( \frac{1}{\alpha_l} \theta^p_{N_s+1} + \theta^p_{F} \right).
\]  

(12)

Written in finite-difference form, the interface rate equation (6) becomes:

\[
\dot{R}^p_S = \frac{t_0 - t_c}{2 \rho_l F} \left[ \frac{\lambda_S}{h} \left( 3 \theta^p_{F} - 4 \theta^p_{N_s} + \theta^p_{N_s+1} + \theta^p_{N_s+2} \right) + \frac{\lambda_L}{h} \left( 3 \theta^p_{F} - 4 \theta^p_{L,N_s+1} + \theta^p_{L,N_s+2} \right) \right].
\]  

(13)

Equation (13) yields, at time step number \( p \), the interface rate. This way, by using the finite-difference scheme presented above, one obtains the temperature field at each time step, the position of the solid-liquid interface, and the interface rate, which makes it possible to calculate the new position of the interface corresponding to the next time step.

5. Results and discussion

The implicit-type finite-difference approach described above is not restricted to a maximum value of the time step (as it is the case of the explicit approach), because the scheme is unconditionally stable and convergent. Thus, the computer time necessary to perform the calculations can be shortened by
using larger time steps. There is however a drawback: for each time step, a set of finite-difference equations need to be solved simultaneously and the number of equations is equal to the number of nodes, which must be large enough to provide a good accuracy of the results. The complications arisen by the algorithm for solving large sets of equations are partially eliminated by noticing that the coefficients' matrix of the set is a tri-diagonal one and consequently the Gauss elimination technique is particularly easy to apply.

In order to solve the problem we have developed computer codes that provided the results necessary to plot the variables needed to analyse the process and to interpret these results.

We have considered two dimensions for the pipe: inner radius \( R_0 \): 5 mm, respectively 10 mm; outer radius \( R_w \): 7 mm, and 12 mm respectively. The radius of the water domain was \( R_w + 50 \) mm, \( R_w + 100 \) mm, respectively \( R_w + 150 \) mm. The fixed temperatures were \( t_0 = 15^\circ \text{C} \), and \( t_C = -10^\circ \text{C} \), \( t_C = -20^\circ \text{C} \), \( t_C = -30^\circ \text{C} \).

The pipe wall material is copper (\( \lambda_{\text{copper}} = 390 \text{ W/mK}; \ a_{\text{copper}} = 113 \text{ mm}^2/\text{s} \) [14].

The water properties are as follows [14]: \( \lambda_{\text{ice}} = 2.092 \text{ W/(mK)} \); \( a_{\text{ice}} = 0.54 \text{ mm}^2/\text{s} \); \( \rho_{\text{ice}} = 917 \text{ kg/m}^3 \);
\( \lambda_{\text{water}} = 0.603 \text{ W/(mK)} \); \( a_{\text{water}} = 0.14 \text{ mm}^2/\text{s} \); \( l = 334 \text{ kJ/kg} \).

The wall mesh has 5 nodes, and the water domain mesh has 50 nodes.

The results are presented below.

Table 1 displays the necessary time in seconds to completely freeze the water for different thicknesses of the water domain and different pipe dimensions, at different coolant temperatures.

| Coolant temperature °C | Water domain thickness \( H = 50 \) mm | Water domain thickness \( H = 100 \) mm | Water domain thickness \( H = 150 \) mm |
|------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| \( t_0 = 15^\circ \text{C} \) | \( R_w = 7 \text{ mm} \) \( R_w = 12 \text{ mm} \) | \( R_w = 7 \text{ mm} \) \( R_w = 12 \text{ mm} \) | \( R_w = 7 \text{ mm} \) \( R_w = 12 \text{ mm} \) |
| \( t_C = -30^\circ \text{C} \) | 14,240 \( R_w = 7 \text{ mm} \) 12,272 \( R_w = 12 \text{ mm} \) | 69,134 \( R_w = 7 \text{ mm} \) 59,474 \( R_w = 12 \text{ mm} \) | 173,654 \( R_w = 7 \text{ mm} \) 147,606 \( R_w = 12 \text{ mm} \) |
| \( t_C = -20^\circ \text{C} \) | 20,915 \( R_w = 7 \text{ mm} \) 17,975 \( R_w = 12 \text{ mm} \) | 101,855 \( R_w = 7 \text{ mm} \) 87,417 \( R_w = 12 \text{ mm} \) | 251,995 \( R_w = 7 \text{ mm} \) 217,370 \( R_w = 12 \text{ mm} \) |
| \( t_C = -10^\circ \text{C} \) | 40,831 \( R_w = 7 \text{ mm} \) 34,983 \( R_w = 12 \text{ mm} \) | 199,590 \( R_w = 7 \text{ mm} \) 170,842 \( R_w = 12 \text{ mm} \) | 511,281 \( R_w = 7 \text{ mm} \) 419,456 \( R_w = 12 \text{ mm} \) |

**Figure 3.** Freezing times for different pipe diameters, water domain thicknesses, and coolant temperatures.
The longest freezing time (511,281 seconds, roughly 142 hours, about 6 days) corresponds to the 14 mm diameter pipe and a 150 mm water domain thickness for a coolant temperature of –10°C, whereas a 14 mm diameter pipe with a 50 mm water domain thickness and coolant at –30°C only needs 12,272 seconds (3 hours and 24 minutes) to freeze the water. One notices that the greater the pipe diameter (the other conditions remaining unchanged), the shorter the freezing time.

Data from table 1 are presented graphically in figure 3.

A special attention required the early time steps of the process, because the initial condition specifies that at $\tau = 0$, the temperature across both domains (pipe wall and water) is uniform and equal to $t_0$. Right before the first time step, the inner wall temperature drops to $t_C$, and the process starts. For a little while, the perturbation propagates across the pipe wall until it reaches its outer surface. This phase of the process, corresponding to the heat propagation across the pipe wall, needs a special treatment and therefore we have opted for an explicit finite-difference technique that is easier to apply in this case. The explicit method that we have used was the so-called Schmidt technique [12], for which the time step corresponds to the time interval necessary for the temperature perturbation to reach the next node. Figure 4 shows the temperature distribution at the end of the heat propagation across the wall thickness, for $R_w = 12$ mm, $H = 100$ mm, $t_0 = −20°C$.

![Figure 4. Temperature of the wall at the end of the heat propagation across its thickness ($R_w = 12$ mm, $H = 100$ mm, $t_0 = −20°C$).](image)

As soon as the dimensionless freezing temperature $\theta_F$ of the outer surface of the pipe has been reached (see also figure 4), freezing begins by the growth of the ice layer on this surface. In order to apply the finite-difference technique to the two-phase domain (ice + water), the initial thickness of the ice layer is necessary. To calculate it, we have used an equation derived from the Neumann model [15], [16]:

$$S_0 = \frac{2\lambda_l (t_0 - t_F) (\Delta\tau_0)^{\frac{1}{2}}}{\rho S_l} \Delta\theta^*$$

(14)

where $S_0$ is the initial ice layer thickness corresponding to the water supercooling temperature difference $\Delta\theta^*$ that is supposed to exist during a brief time step $\Delta\tau_0$ which needs to be imposed. As long as $S_0 < h$, a linear variation of the temperature across the ice layer thickness is supposed; when the interface has surpassed the first node, one can write the finite-difference equation (11) for this node.
Figures 5 and 6 illustrate the dimensionless temperature distribution at a 6 hours interval in the wall and across the water domain for $R_w = 12$ mm, $H = 100$ mm, $t_0 = -20^\circ$C. Due to the very large difference between the water and the wall temperatures, separate plots were necessary: the wall temperature is of the order of $10^{-4}$, whereas the water temperature is between 0 and 0.9. The very low wall temperatures can be explained by the very high values of the thermal conductivity and of the diffusivity of copper. In figure 6, the temperature plot after 6 hours shows that the interface is positioned approximately at node number 31 (the dimensionless fusion temperature is in this case

![Figure 5](image1.png)

**Figure 5.** Temperature of the wall plotted on a 6 hours basis ($R_w = 12$ mm, $H = 100$ mm, $t_0 = -20^\circ$C).

![Figure 6](image2.png)

**Figure 6.** Temperature of the water domain plotted on a 6 hours basis ($R_w = 12$ mm, $H = 100$ mm, $t_0 = -20^\circ$C).
\(\theta_e = 0.8\). Liquid water is still superheated (\(\theta_{water} > \theta_e\)). After another 6 hours, the interface has reached node number 41, and water is practically isothermal at \(\theta_e\). After 18 hours from the beginning, the interface is at roughly node number 49 and the process ends after 24 hours.

Figure 7 represents the temperature plot in the case of 7 mm pipe radius and a 150 mm water domain, when the coolant temperature is \(-30^\circ C\).

Figure 7. Temperature of the water domain plotted on a 6 hours basis (\(R_w = 7\) mm, \(H = 150\) mm, \(t_0 = -30^\circ C\)).

Figure 8. The interface rate versus time (\(R_w = 12\) mm, \(H = 100\) mm).
In this case, the process takes a longer time to end, because of the large value of the water domain thickness.

For the two pipe/water domain dimensions already considered as an example above ($R_w = 12$ mm and $H = 100$ mm, respectively $R_w = 7$ mm and $H = 150$ mm), the variation of the interface rate is plotted versus time in figures 8 and 9 and the solid phase percentage versus time in figures 10 and 11.

![Figure 9](image9.png)

**Figure 9.** The interface rate versus time ($R_w = 7$ mm, $H = 150$ mm).

![Figure 10](image10.png)

**Figure 10.** The solid phase percentage versus time ($R_w = 12$ mm, $H = 100$ mm).
Figure 11. The solid phase percentage versus time ($R_w = 7$ mm, $H = 150$ mm).

The utility of the latter two graphs resides in the fact that implicitly they easily allow determining the position of the solid-liquid interface at any moment during the freezing process.

6. Conclusions
An implicit-type finite-difference approach has been developed in order to solve the moving boundary problem represented by the freezing process of water around a single cylindrical pipe. The process is supposed to occur in the radial direction (one-dimension conduction heat transfer). The approach takes into consideration the influence of the pipe wall and uses a fixed mesh for the water domain, which is why special finite-difference equations needed to be derived for the nodes on both sides in the vicinity of the solid-liquid interface. A special procedure derived from the Neumann model has been set up to initiate the freezing process as soon as the wall temperature reaches the freezing point.

Different diameters have been considered for both the pipe and the water domain, under conditions of three different values of the cooling temperature.

After running the computer codes developed by the authors, retrieved data in the cases of two illustrative examples ($R_w = 12$ mm and $H = 100$ mm, respectively $R_w = 7$ mm and $H = 150$ mm) have been used to generate different plots that allowed to analyse and interpret the results, thus acquiring an important amount of information about the dynamics of freezing process.

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