Quaternionic Quantum Mechanics and Noncommutative Dynamics

Stephen L. Adler

Institute for Advanced Study

Princeton, NJ 08540

In this talk I shall first make some brief remarks on quaternionic quantum mechanics, and then describe recent work with A.C. Millard in which we show that standard complex quantum field theory can arise as the statistical mechanics of an underlying noncommutative dynamics.

In quaternionic quantum mechanics, the Dirac transition amplitudes $\langle \psi | \phi \rangle$ are quaternion valued, that is, they have the form $r_0 + r_1 i + r_2 j + r_3 k$, where $r_{0,1,2,3}$ are real numbers and $i,j,k$ are quaternion imaginary units obeying $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The Schrödinger equation takes the form

$$\frac{\partial |\psi\rangle}{\partial t} = -\tilde{H} |\psi\rangle,$$

$$\tilde{H} = -\tilde{H}^\dagger.$$ 

A systematic study of quaternionic quantum mechanics is given in my recent book [1] on “Quaternionic Quantum Mechanics and Quantum Fields.” For information on how to obtain the book and updates on more recent work, see my web home page (http://www.sns.ias.edu/~adler/Html/quaternionic.html).
In the book, I discuss many aspects of the generalization of standard complex quantum mechanics to quaternionic Hilbert space. Let me here focus on the one obvious question, can quaternionic quantum mechanics be relevant to physics? A key result in the book relating to this is that in quaternionic quantum mechanics, the $S$-matrix is always a complex matrix with no dependence on the quaternionic units $j, k$, for appropriate ray representative choices for the states in Hilbert space. Hence the asymptotic dynamics for quaternionic quantum mechanics is always an effective complex theory, and this means that a quaternionic Hilbert space dynamics may play a role as an underlying dynamics for the standard model.

The reason for my long-standing interest in quaternionic quantum mechanics is that it offers the possibility of elegant substructure models for quarks and leptons along lines outlined long ago by Harari and Shupe. Recently [2], I gave a simple set of rules for constructing composite quarks and leptons as triply occupied quaternionic quasiparticles. The mixed symmetry states obtained this way correspond precisely to the three spin $1/2$ quark-lepton families of the standard model, plus one additional family of (possibly massive) spin $3/2$ quarks. The fact that the spin $1/2$ state multiplicities come out right could of course be fortuitous; the next step in the program is to try to give a systematic derivation of the rules from an underlying quaternionic Hilbert space or other noncommutative dynamics.

Let me turn now to the major topic of my talk, which is to sketch how standard complex quantum field theory can arise as the statistical mechanics of an underlying noncommutative dynamics. This investigation grew out of my analysis [1, 3] of the problem of taking the step from quaternionic quantum mechanics to quaternionic quantum field theory, which requires a generalization beyond the canonical quantum mechanical formalism. To do this, I suggested a formalism that I termed “generalized quantum dynamics,” which is a noncommutative
generalization of classical Lagrangian and Hamiltonian dynamics. Let me describe briefly how it works, focusing for simplicity on the bosonic case (all results generalize properly when fermionic grading is included).

Let \( \{q_r\} \) be a set of noncommuting coordinates, which act as linear operators on an underlying real, complex, or quaternionic Hilbert space, and let \( \{\dot{q}_r\} \) be their time derivatives. From these quantities we form a polynomial operator Lagrangian \( L[\{q_r\}, \{\dot{q}_r\}] \), in which the order of factors is significant. Because the coordinates \( \{q_r\} \) do not commute with one another, we cannot define a coordinate derivative \( \delta L/\delta q_r \). To deal with this problem, we use the cyclic invariance property of the trace: let us define the total trace Lagrangian

\[
L \equiv \text{Tr} L \equiv \text{Re} \text{Tr} L ,
\]

where \( \text{Tr} \) denotes the trace over the underlying Hilbert space and where \( \text{Re} \) denotes the real part. If we vary all of the \( q_r \) and \( \dot{q}_r \), in the corresponding variation of the total trace Lagrangian we can use cyclic invariance of the trace to reorder factors so that all of the \( \delta q_r \) and \( \delta \dot{q}_r \) stand on the right, allowing us to define derivatives of \( L \) with respect to the \( q_r \) and \( \dot{q}_r \) by

\[
\delta L = \sum_r \text{Tr} \left( \frac{\delta L}{\delta q_r} \delta q_r + \frac{\delta L}{\delta \dot{q}_r} \delta \dot{q}_r \right) .
\]

Using this definition, it is now easy to demonstrate the following properties:

1. Introducing the total trace action defined by \( S = \int_{-\infty}^{\infty} dt L \) and requiring stationarity as expressed by \( 0 = \delta S \), implies the operator Euler-Lagrange equations

\[
\frac{\delta L}{\delta q_r} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_r} = 0 .
\]
(2) If we now define the operator momenta \( \{ p_r \} \) and total trace Hamiltonian \( H \) by

\[
p_r = \frac{\delta L}{\delta q_r} ,
\]

\[
H = \text{Tr} \sum_r p_r \dot{q}_r - L ,
\]

then the operator Euler-Lagrange equations can be rewritten as operator Hamilton equations

\[
\frac{\delta H}{\delta q_r} = -\dot{p}_r , \quad \frac{\delta H}{\delta p_r} = \dot{q}_r .
\]

(4a)

(3) For any total trace functional \( A[q_r, p_r, t] \), the time evolution is given by

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{ A, H \} ,
\]

where the final term is a generalized Poisson bracket of its arguments, defined for any two total trace functionals \( A \) and \( B \) by

\[
\{ A, B \} = \text{Tr} \sum_r \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} - \frac{\delta B}{\delta q_r} \frac{\delta A}{\delta p_r} \right) .
\]

(5a)

Applying this to the total trace Hamiltonian \( H \), and assuming that the dynamics is time translation invariant, so that \( \partial H / \partial t = 0 \), we have

\[
\frac{dH}{dt} = \{ H, H \} = 0 ,
\]

(6)

telling us that the total trace Hamiltonian \( H \) is a constant of the motion. Note that there is no corresponding conserved *operator* Hamiltonian.

(4) The generalized Poisson bracket obeys the Jacobi identity \[4\]

\[
\{ A, \{ B, C \} \} + \{ C, \{ A, B \} \} + \{ B, \{ C, A \} \} = 0 .
\]

(7)

The derivation of the Jacobi identity uses only associativity of the multiplication in the underlying Hilbert space, which can be real, complex, or quaternionic (but not octonionic).
As a consequence of the Jacobi identity [5], the symplectic structure of generalized quantum dynamics is closely analogous to that of classical mechanics.

(5) Let us define the anti-self-adjoint operator $\tilde{C}$ by

$$\tilde{C} = \sum_r [q_r, p_r].$$

(8) Then when $H$ involves no noncommutative constants, a simple argument based on cyclic invariance of the trace [6] shows that $d\tilde{C}/dt = 0$, so that $\tilde{C}$ is a conserved operator.

(6) In analogy with classical mechanics, let us define a canonical transformation in generalized quantum dynamics by

$$\delta p_r = -\frac{\delta G}{q_r}, \quad \delta q_r = \frac{\delta G}{p_r},$$

(9) where $G$ is any total trace functional; in particular, when $G = H dt$, the canonical transformation corresponds to an infinitesimal time step. Let us now define a phase space measure in operator phase space by

$$d\mu = \prod_{r, m, n, A} d\langle m|q_r|n\rangle^A d\langle m|p_r|n\rangle^A,$$

(10) where $A$ indexes the real components of the indicated matrix elements (i.e., $A = 0$ in real Hilbert space, $A = 0, 1$ in complex Hilbert space, and $A = 0, 1, 2, 3$ in quaternionic Hilbert space). Then one can show [6] that $d\mu$ is invariant under the general canonical transformation of Eq. (9), and in particular is conserved under time evolution.

The existence of a conserved phase space measure implies that generalized quantum dynamics obeys a generalized Liouville’s theorem, and this in turn means that the methods of statistical mechanics are applicable. The canonical ensemble [6] has the form

$$\rho = Z^{-1} e^{-\gamma H - \text{Tr} \lambda \tilde{C}},$$

$$\int d\mu \rho = 1 \Rightarrow Z = \int d\mu \rho e^{-\gamma H - \text{Tr} \lambda \tilde{C}}.$$
Here $\tau$ and $\tilde{\lambda}$ are ensemble parameters which are determined by requiring that the ensemble averages $\langle H \rangle_{AV}$ and $\langle \tilde{C} \rangle_{AV}$ have specified values. For the corresponding form of the microcanonical ensemble, and its relation to the canonical ensemble, see [7]. Introducing the standard canonical form for an anti-self-adjoint operator, we can write

$$\langle \tilde{C} \rangle_{AV} = i_{eff} D ,$$  \hspace{1cm} (12a)

with $D$ real diagonal and with

$$i^2_{eff} = -1 , \quad i_{eff}^4 = -i_{eff} , \quad [i_{eff} , D] = 0 .$$  \hspace{1cm} (12b)

Henceforth we shall assume that the ensemble does not favor any state in the underlying Hilbert space, which implies that $D$ is proportional to the unit operator with a proportionality constant that we denote by $\hbar$, so that the canonical form of Eq. (12a) reads

$$\langle \tilde{C} \rangle_{AV} = i_{eff} \hbar .$$  \hspace{1cm} (12c)

(7) Let us now look at the Ward identities for averages of the dynamical variables over $\rho$, that are analogous to the generalized equipartition theorem in classical statistical mechanics (which we recall states that for $x_r$ a coordinate or momentum, we have $\langle x_s \partial H / \partial x_r \rangle_{AV} = \beta^{-1}\delta_{rs}$, with $\beta$ the inverse temperature in units of Boltzmann’s constant). A detailed calculation [6] shows that equipartition of the conserved operator $\tilde{C}$ in generalized quantum dynamics gives the canonical commutation relations of complex quantum field theory, with $i_{eff}$ playing the role of the imaginary unit and with the averages over $\rho$ playing the role of the Wightman functions. In other words, canonical, complex quantum field theory can arise as an emergent property of a deterministic underlying noncommutative operator dynamics!
To conclude, both our discussion of the $S$-matrix in quaternionic quantum mechanics, and of the statistical properties of generalized quantum dynamics, show that noncommutative dynamics hides itself beneath effective complex quantum mechanical structures. This means that noncommutative generalizations of standard quantum mechanics and quantum field theory could be the vehicle for an elegant solution to the problem of unifying the forces—one is not restricted to searching for a solution solely within the framework of quantum mechanics in complex Hilbert space.

This work was supported in part by the Department of Energy under Grant #DE-FG02-90ER40542. I wish to acknowledge the hospitality of the Aspen Center for Physics where part of this work was done.

References

[1] S. L. Adler, “Quaternionic Quantum Mechanics and Quantum Fields”, Oxford, New York, 1995.

[2] S. L. Adler, Phys. Letters B332, 358 (1994).

[3] S. L. Adler, Nucl. Phys. B415, 195 (1994).

[4] S. L. Adler, G. V. Bhanot, and J. D. Weckel, J. Math. Phys. 35, 531 (1994).

[5] S. L. Adler and Y.-S. Wu, Phys. Rev. D 49, 6705 (1994).

[6] S. L. Adler and A. C. Millard, “Generalized Quantum Dynamics as Pre-Quantum Mechanics”, Nucl. Phys. B (in press).

[7] S. L. Adler and L. P. Horwitz, “Microcanonical Ensemble and Algebra of Conserved Generators for Generalized Quantum Dynamics”, hep-th/9606023, submitted to J. Math. Phys.