Minimally doubled chiral fermions with C, P and T symmetry on the staggered lattice

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Recently, the interest in local lattice actions for chiral fermions has revived, with the proposition of new local actions in which only the minimal number of doublers appear. The trigger role of graphene having a minimally doubled, chirally invariant, Dirac-like excitation spectrum can not be neglected. The challenge is to construct an action which preserves enough symmetries to be useful in lattice gauge calculations. We present a new approach to obtain local lattice actions for fermions using a reinterpretation of the staggered lattice approach of Kogut and Susskind. This interpretation is based on the similarity with the staggered lattice approach in FDTD simulations of acoustics and electromagnetism. It allows us to construct a local action for chiral fermions which has all discrete symmetries and the minimal number of fermion flavors, but which is non-Hermitian in real space. However, we argue that this will not pose a threat to the usability of the theory.

I. INTRODUCTION

The notorious fermion doubling problem has always complicated calculations of lattice field theories in which strict chiral symmetry of the fermions is required. The no-go theorem of Nielsen and Ninomiya ensures that every local, unitary and chirally symmetric lattice theory will have a degenerate excitation spectrum, with at least two independent fermion flavors \[ N_f \]. Eliminating these fermion doublers requires explicit breaking of chiral symmetry on the lattice, as proposed by Wilson \[ 4 \]. Although chiral symmetry is restored in the continuum limit, this situation is often disadvantageous. Other solutions to circumvent the fermion doubling problem include overlap fermions, which are highly nonlocal, or domain wall fermions, which require an extra dimension. Both approaches are thus computationally expensive.

A possible solution of a different kind is the use of the rooting procedure. If the action is quadratic in the fermion fields, as is the case in lattice gauge theory, they can be integrated out. The resulting determinant, describing \( N_f \) fermion flavors, can be exponentiated to a power \( 1/N_f \). This reduces the number of fermions to one in all orders of perturbation theory. Although lattice gauge theory constitutes the majority of all lattice calculations, this is a restricted approach, which is also criticized of introducing non-perturbative artefacts \[ 5, 6, 7 \].

The rooting procedure is virtually always used in combination with the staggered lattice formulation of Kogut and Susskind \[ 8, 9, 10 \]. This is a simple construction to reduce the number of fermion flavours in the lattice theory. However, on a four dimensional Euclidean lattice, not two but four fermion flavours survive the reduction. The observation of a minimally doubled Dirac-like excitation spectrum in graphene has triggered the search for new lattice actions for local, chiral fermions with the bare minimum of different flavours \[ 11, 12, 13, 14, 15, 16 \]. However, these all violate at least some of the discrete symmetries of the continuum theory, and interactions will generate unwanted relevant operators in the effective continuum theory \[ 17 \]. This paper tries to offer a workaround for this problem.

Since the presence of fermion doublers is a property of the equations of motion, it can be investigated at the classical level. The fermion doubling problem is often attributed to the first order derivatives appearing in the Dirac Hamiltonian or action. However, first order partial differential equations also occur in acoustics and electromagnetics. Although a naive discretization also results in an analogue of the doubling problem, this problem can be completely circumvented. The solution is, not incidentally, called the staggered grid approach and is commonly used in numerical simulations using the FDTD (finite-difference time-domain) method. We apply this machinery to the equations of motion of the Dirac fermion. Removing all doublers with this approach is still forbidden by the Nielsen-Ninomiya theorem. The reason for this is the spinor nature of the fields, which have no natural position on the lattice, in contrast with the scalar and vector fields in acoustics and electromagnetics.

In addition, formulating an action on a completely discretized spacetime lattice without introducing new doublers is a non-trivial procedure. We illustrate a possible approach, which results in an action with no time doublers and all required discrete symmetries. This action is non-Hermitian in real space, but will not introduce unphysical effects in the calculation of expectation values. We start with \((1 + 1)\)-dimensional fermions, where the doubling problem only appears when both space and time are discretized. The Kogut-Susskind construction can completely eliminate the fermion doubling problem on the Hamiltonian lattice, so we try to discretize time without increasing the number of doublers. We pay special attention to the conservation of the discrete symmetries P, C and T, and briefly comment on how to incorporate gauge fields. The second section considers the more interesting case of fermions in \((3 + 1)\) dimensions.
II. STAGGERED LATTICE IN \((1 + 1)\) DIMENSIONS

At first, the original Kogut-Susskind construction on the Hamiltonian lattice in \((1 + 1)\) dimensions is reviewed. This is the only simple approach which escapes the Nielsen-Ninomiya theorem, since only one dimension is discretized. By reinterpreting this approach from a point of view borrowed from the field of FDTD simulations, we illustrate how to construct a lattice action which has all the required symmetries without introducing extra time doublers. The way to circumvent the Nielsen-Ninomiya theorem this time, is by giving up Hermiticity.

A. The Kogut-Susskind construction on the Hamiltonian Lattice

The staggered fermion formulation on the Hamiltonian lattice in \((1 + 1)\) dimensions starts by introducing an auxiliary lattice with twice the number of sites as the physical lattice. On each site \(k\), a one-component fermionic variable \(\phi(k)\) is introduced, which is related to the continuum fermion field by

\[
\psi_1(ka_x) = \frac{\phi(2k - 1)}{\sqrt{a_x}}, \quad \psi_2(ka_x) = \frac{\phi(2k)}{\sqrt{a_x}}.
\]

The constant \(a_x\) will always denote the lattice spacing of the physical lattice. The Dirac Hamiltonian

\[
H = \int dx \psi^\dagger(x) \left[ -i \frac{d}{dx} + m\beta \right] \psi(x),
\]

with \(\alpha = \sigma_x\) and \(\beta = \sigma_z\), is discretized using an asymmetric definition for the derivatives

\[
\begin{align}
\frac{d\psi_1}{dx}(ka_x) &= \frac{\psi_1((k + 1)a_x) - \psi_1(ka_x)}{a_x}, \\
\frac{d\psi_2}{dx}(ka_x) &= \frac{\psi_2(ka_x) - \psi_2((k - 1)a_x)}{a_x}.
\end{align}
\]

This results in a pair of equations of motion which have non-trivial solutions if, in Fourier space,

\[
\omega^2 = E(\kappa)^2 \text{ with } E(\kappa) = \sqrt{m^2 + \frac{4}{a_x^2} \sin^2(\kappa/2)}.
\]

There is only one low energy sector and thus only one fermion flavour. A chiral transformation can be implemented as

\[
\gamma^5 : \psi_1(ka_x) \rightarrow \psi_2(ka_x), \quad \psi_2(ka_x) \rightarrow \psi_1((k + 1)a_x),
\]

which is equivalent to the continuum chiral transformation up to order \(O(a_x)\). More details about this construction can be found in [3].

B. Reinterpretation of the finite differences

The auxiliary lattice and the asymmetric definition of the finite differences in [3], appear to be rather random. Furthermore, this approximations for the derivatives are only accurate up to order \(O(a_x)\), while a symmetric definition would be accurate up to order \(O(a_x^2)\). This deficiency can be cured by a simple reinterpretation of the staggered fermion construction.

In FDTD simulations of acoustics or electromagnetics, which are also governed by first order partial differential equations, the concept of a staggered lattice designates that different fields are discretized at different positions. FIG. 1 illustrates the standard unit cell for both areas. Essential to this approach is the fact that the equation of motion of one field is not dependent on spatial derivatives of that same field.

Applying the same technique to the Dirac equation, we immediately arrive to a lattice scheme where the \(\psi_1\) field is discretized half in between the discretization points of the \(\psi_2\) field. Thus, the Dirac Hamiltonian (2) can be discretized as

\[
H = a_x \sum_n \psi_1^\dagger(ka_x) \left[ -i \frac{d}{dx}((k + 1)a_x) + m\psi_1(ka_x) \right] - \psi_2^\dagger((k + 1)a_x) \left[ -i \frac{d}{dx}((k + 1)a_x) \right]
\]

\[
+ \psi_2((k + 1)a_x) \left[ -i \frac{d}{dx}((k + 1)a_x) \right],
\]

with derivatives which are approximated by

\[
\begin{align}
\frac{d\psi_1}{dx}((k + 1/2)a_x) &= \frac{\psi_1((k + 1)a_x) - \psi_1(ka_x)}{a_x}, \\
\frac{d\psi_2}{dx}(ka_x) &= \frac{\psi_2((k + 1)a_x) - \psi_2((k - 1)a_x)}{a_x}.
\end{align}
\]

These are accurate up to order \(O(a_x^2)\). The derivatives are now symmetric and the arbitrariness is shifted to the allocation of the two field components to one unit cell, which has no physical content and is only necessary if we want to label the lattice fields by integers. With the
assignment
\[
\psi_1(k a_x) = \frac{\chi_1(k)}{\sqrt{a_x}}, \quad \psi_2((k + 1/2) a_x) = \frac{\chi_2(k)}{\sqrt{a_x}}, \quad (8)
\]
we arrive at the same theory as the Kogut-Susskind formulation on the mathematical level. The discrete-space Hamiltonian of the \(\chi_i\) fields is given by
\[
H = \sum_k \chi_1^\dagger(k) \left[ -i \frac{\chi_2(k) - \chi_2(k - 1)}{a_x} + m \chi_1(k) \right]
+ \chi_2^\dagger(k) \left[ -i \frac{\chi_1(k + 1) - \chi_1(k)}{a_x} - m \chi_2(k) \right]. \quad (9)
\]

The difference in physical interpretation between (1) and (8) only comes in to play when adding extra fields such as gauge fields. The best solution is then to discretize the gauge field twice as fine as the fermion field. There will be two gauge units cells in each fermion unit cell. This is equivalent to adding a gauge variable to each link in the auxiliary lattice of Kogut and Susskind. Wilczek pointed out why a finer discretization of the gauge field is an appropriate concept \([19]\). We will comment in more detail on the incorporation of gauge fields for the case of the spacetime lattice.

Before continuing this discretization to the dimension of time, we elaborate on the symmetries of the discrete-space formulation \([9]\). We focus on the discrete symmetries, which are, besides chiral invariance, \(P\) (invariance under parity transformations), \(C\) (invariance under charge conjugation) and \(T\) (invariance under time inversions). The chiral transformation on the lattice was already defined in \([5]\). With this new interpretation, we see that it is not possible to simply swap the spinor components \(\psi_1\) and \(\psi_2\) in the lattice model, since they are defined at different points in space. As a consequence, this swapping should be accompanied with a shift of the lattice over half a lattice spacing. This operation transforms \(\psi_1\) to \(\psi_2\) in the same unit cell, but shifts \(\psi_2\) to \(\psi_1\) in the next unit cell (or vice versa, depending on the direction of the shift and the definition of the unit cell). It is obvious that the kinetic term in \([9]\) is invariant under this transformation, while the mass term changes sign. The new definition given in \([8]\) thus renders the idea of a chiral transformation which deviates from the continuum prescription more acceptable.

The parity transformation is defined in the continuum theory as \(\psi(x) \rightarrow \beta \psi(-x)\) and is nicely mapped to the lattice variables as
\[
P : \chi_1(k) \rightarrow \chi_1(-k), \quad \chi_2(k) \rightarrow -\chi_2(-k - 1), \quad (10)
\]
which leaves \([9]\) invariant. Charge conjugation in \((1 + 1)\) dimensions is defined as \(\psi \rightarrow i\psi^*\), and thus also depends on swapping the fermion components. Using the same prescription as for chiral invariance, the Hamiltonian \([9]\) is invariant under
\[
C : \chi_1(k) \rightarrow \chi_1^\dagger(k), \quad \chi_2(k) \rightarrow \chi_2^\dagger(k + 1). \quad (11)
\]
Finally there is invariance under time reversal, which is an antilinear operation. In the \((1 + 1)\) dimensional continuum time reversal is established as \(\psi(x) \rightarrow \beta \psi(x)\), while the antilinearity results in a complex conjugation of the prefactors. The Hamiltonian \([9]\) is indeed invariant under
\[
T : \chi_1(k) \rightarrow \chi_1(k), \quad \chi_2(k) \rightarrow -\chi_2(k) + \text{complex conjugation of prefactors.} \quad (12)
\]

In the next paragraph, we try to discretize time, by going to the action formalism, as is used in Monte Carlo simulations of lattice field theory. The discrete symmetries \(P, C\) and \(T\) are difficult to preserve in an action formulation with discrete time, without introducing new doublers. These symmetries are however of key importance to ensure a correct continuum limit when interactions or gauge fields are added. Absence of these symmetries would allow the appearance of new relevant operators in the effective continuum theory \([17]\).

C. One-fermion chiral action on the spacetime lattice

In FDTD simulations, time is discretized as well. The time evolution of the initial configuration of the fields is approximated using a leap frog algorithm, which implies that the different fields are also staggered in the direction of time. With the correspondence
\[
\psi_1(k a_x, l a_t) = \frac{\chi_1(k, l)}{\sqrt{a_x}}, \quad \psi_2((k + 1/2) a_x, (l + 1/2) a_t) = \frac{\chi_2(k, l)}{\sqrt{a_x}}, \quad (13)
\]
the FDTD equations for the free Dirac field would be
\[
\begin{align*}
\frac{1}{a_t} \left[ \chi_1(k, l) - \chi_1(k, l - 1) \right] &= -\frac{1}{a_x} \left[ \chi_2(k, l - 1) - \chi_2(k - 1, l - 1) \right] + \frac{m}{a_x} \left[ \chi_1(k, l) + \chi_1(k, l - 1) \right], \\
\frac{1}{a_t} \left[ \chi_2(k, l) - \chi_2(k, l - 1) \right] &= -\frac{1}{a_x} \left[ \chi_1(k + 1, l) - \chi_1(k, l) \right] - \frac{m}{a_x} \left[ \chi_2(k, l) + \chi_2(k, l - 1) \right].
\end{align*} \quad (14)
\]

These equations of motion result in a unique fermion flavour. The objective is thus to construct a discrete spacetime action which returns these equations as extremization conditions. But firstly, we determine the dispersion relation of
these equations. After Fourier transforming \( \chi_i(k, l) = \exp(ik \cdot \xi - i\omega l)\chi_i(\omega, \kappa) \), non-trivial solutions only exist if
\[
\frac{4}{a_{\kappa}^2} \sin^2(\omega/2) - m^2 \cos^2(\omega/2) - \frac{4}{a_{\xi}^2} \sin^2(\kappa/2) = 0.
\] (15)

Although the \( \omega \)-dependent factor in the term proportional to \( m^2 \) looks strange, the corresponding solution
\[
\omega = \pm 2 \arcsin \left[ \frac{m^2 a_{\kappa}^2 + 4 \sin^2(\kappa/2) a_{\xi}^2}{4 m^2 a_{\kappa}^2 + 4 \sin^2(\kappa/2) a_{\xi}^2} \right] \] (16)
is, for \( ma_{\kappa} \) sufficiently smaller than 1, closer to the continuum solution \( \omega/\kappa = \pm (m^2 + \kappa^2/a_{\xi}^2)^{1/2} \) than the solution of the expected dispersion relation
\[
\frac{4}{a_{\kappa}^2} \sin^2(\omega/2) - m^2 - \frac{4}{a_{\xi}^2} \sin^2(\kappa/2) = 0,
\] (17)
as is illustrated in FIG. 2. In particular, the dispersion relation (15) yields real solutions of \( \omega \) for all values of \( \kappa \) in the domain \( [-\pi, +\pi] \).

In order to find a discrete-time action, we can try a naive discretization of the continuous-time action corresponding to the discrete-space Hamiltonian (9) on a staggered spacetime grid. We temporarily restrict our attention to the massless Dirac fermion field for brevity of the expressions, which results in the following Minkowski action
\[
S = a_t \sum_{k, l} \frac{\chi_i^\dagger(k, l) + \chi_i^\dagger(k, l - 1)}{2} \left[ \frac{i \chi_1(k, l) - \chi_1(k, l - 1)}{a_t} + \frac{i \chi_2(k, l - 1) - \chi_2(k - 1, l - 1)}{a_x} \right] + \frac{\chi_i^\dagger(k, l) + \chi_i^\dagger(k, l - 1)}{2} \left[ \frac{i \chi_2(k, l) - \chi_2(k, l - 1)}{a_t} + \frac{i \chi_1(k + 1, l) - \chi_1(k, l)}{a_x} \right].
\] (18)

This action has all discrete symmetries of the continuum theory. However, due to the averaging of the \( \chi_i^\dagger \) fields, there will be a line of poles for \( \omega = \pi \) in the fermion propagator, which is even worse than an extra fermion flavour and is thus highly unphysical. Correspondingly, the extremalization conditions \( \partial S/\partial \chi_i^\dagger = 0 \) do not reduce to the optimal leap frog equations (14). To circumvent this problem, averaging should be avoided. An arbitrary choice for one of the two positions \( \chi_i^\dagger(k, l - 1) \) or \( \chi_i^\dagger(k, l) \) results in
\[
S = a_t \sum_{k, l} \frac{\chi_i^\dagger(k, l - 1)}{2} \left[ \frac{i \chi_1(k, l) - \chi_1(k, l - 1)}{a_t} + \frac{i \chi_2(k, l - 1) - \chi_2(k - 1, l - 1)}{a_x} \right] + \chi_i^\dagger(k, l - 1) \left[ \frac{i \chi_2(k, l) - \chi_2(k, l - 1)}{a_t} + \frac{i \chi_1(k + 1, l) - \chi_1(k, l)}{a_x} \right].
\] (19)
This time the extremalization conditions $\partial S/\partial \chi_i = 0$ are given by \((14)\), as wanted. The price we have to pay is the loss of Hermiticity at order $\mathcal{O}(a_i)$. However, the extremalization conditions $\partial S/\partial \chi_i^\dagger = 0$ and $\partial S/\partial \chi_i = 0$ do not conflict, since

$$\frac{\partial S}{\partial \chi_i(k,t)} = \left( \frac{\partial S}{\partial \chi_i^\dagger(k,t-1)} \right)^\dagger. \quad (20)$$

As a consequence, both extremalization conditions result in compatible equations of motion for $\chi$ and $\chi_i^\dagger$, but shifted over one time step. The inverse fermion propagator in momentum space is given by

$$D^{-1} = \begin{bmatrix} 2a e^{i\omega/2} \sin(\omega/2) & 2a e^{-i\kappa/2} \sin(\kappa/2) \\ 2a e^{i\omega+ix/2} / \sqrt{2} & 2a e^{i\omega/2} \sin(\omega/2) \end{bmatrix}, \quad (21)$$

which is Hermitian if not for the global factor $\exp(i\omega/2)$.

The action \((19)\) does not only sacrifice Hermiticity, it requires a special treatment concerning symmetries in addition. A chiral transformation is now accompanied by a shift over half the lattice spacing, both in the direction of space and in the direction of time

$$\gamma^5 : \begin{cases} \chi_1(k,l) \to \chi_2(k,l), \\
\chi_2(k,l) \to \chi_1(k+1,l+1). \end{cases} \quad (22)$$

A parity transformation is still given by

$$P : \begin{cases} \chi_1(k,l) \to \chi_1(-k,l), \\
\chi_2(k,l) \to -\chi_2(-k-1,l). \end{cases} \quad (23)$$

The action \((19)\) is invariant under both transformations. It is however not invariant under charge conjugation

$$\chi_1(k,l) \to \chi_2^\dagger(k,l), \quad \chi_2(k,l) \to \chi_1^\dagger(k+1,l+1),$$

or time inversion

$$\chi_1(k,l) \to \chi_1(k,-l), \quad \chi_2(k,l) \to -\chi_2(k,-l-1),$$

+ complex conjugation of prefactors.

Only a combined CT transformation maps the action \((19)\) back to itself

$$\chi_1(k,l) \to -\chi_2^\dagger(k,-l-1),$$

$$\chi_2(k,l) \to \chi_1^\dagger(k+1,-l-1),$$

+ complex conjugation of prefactors.

The lack of time reversal invariance can easily be seen from the structure on the action, which is depicted in FIG. 3 The time derivative of $\psi_i$ is defined at the position half in between the points were the two fields $\psi_1$ appearing in the derivative are defined. We have no $\psi_i^\dagger$ at that position to complete the discrete analogue of $\psi_i^\dagger \frac{\partial}{\partial t} \psi_i$ in the action. Averaging leads to a line of poles in the inverse propagator, while adding a single nearby

$\psi_i^\dagger$ breaks time reversal symmetry. Under the suggested transformation laws for charge conjugation or time reversal, the expression for $S$ with front factor $\chi_i^\dagger(k,l-1)$ is transformed to the equivalent expression with front factor $\chi_i^\dagger(k,l)$. Another possible solution consists of adding extra discretization points for the fields at the intermediate position. This results in time doublers and is exactly the approach we are trying to avoid.

However, there is a way to restore the symmetries by allowing more deviations of order $\mathcal{O}(a)$ between the lattice and the continuum theory. In all the proposed transformation laws, the transformation of the fields $\chi_i^\dagger$ is implicitly assumed to be the complex conjugate of the transformation of the fields $\chi_i$. By dropping this restriction, it is possible to define a time inversion and charge conjugation transformation which independently map the action \((19)\) to itself. This is allowed, since this lattice action is only meaningful in the definition of a partition function, in which $\chi_i^\dagger$ and $\chi_i^\dagger$ are independent integration variables. It makes no sense to define $\chi_i^\dagger$ as the conjugate momentum of $\chi_i$ and to define a (anti)commutation relation on both fields. Canonical quantization is only possible in a Hamiltonian framework, in which the time dimension is continuous. Only for $a_i \to 0$ will the transformation laws match and the Hermiticity of the action be restored. The correct symmetry transformations can easily be deduced by further implementing the idea of staggered lattices. If we discretize $\psi_i$ in between two $\psi_i$'s, the perfect discrete analogue of $\psi_i^\dagger \frac{\partial}{\partial t} \psi_i$ can be constructed. In order to retain action \((19)\) in this new interpretation, we define

$$\begin{cases} \psi_1(ka_x,la_t) = \frac{\chi_i(k,l)}{\sqrt{a_t}}, \\
\psi_1^\dagger(ka_x,(l+1/2)a_t) = \frac{\chi_i^\dagger(k,l)}{\sqrt{a_t}}, \\
\psi_2((k+1/2)ka_x,(l+1/2)a_t) = \frac{\chi_i^\dagger(k,l)}{\sqrt{a_t}}, \\
\psi_2^\dagger((k+1/2)ka_x,(l+1)a_t) = \frac{\chi_i^\dagger(k,l)}{\sqrt{a_t}}. \end{cases} \quad (24)$$

Alternatively, we could have used $\psi_2^\dagger((k+1/2)ka_x,la_t)$ in the last line, but this would require a small modification in the second term of action \((19)\), as well as a separate transformation law for $\chi_i^\dagger$ in case of a chiral transformation. Again, this renewed interpretation does not change the mathematical content of the model. However, at the physical level, it explicitly destroys any hope
for a Hermitian action, since the complex conjugate of the fields can not be defined on the same lattice. In other words, at the physical level, the complex conjugate \( \chi_i(k, l) \) only equals \( \chi_i^\dagger(k, l) \) up to corrections of order \( \mathcal{O}(a) \). Furthermore, this point of view can result in actual differences when adding interactions and it changes the way in which correlation functions should be interpreted: \( \langle \chi_i^\dagger(k, l) \chi_i(k', l) \rangle \) is not an equal-time correlation function, except for \( a_t \to 0 \). And last but not least, it facilitates the search for symmetry transformations. Using [24], we define a time reversal transformation as

\[
T : \begin{cases}
\chi_1(k, l) \to \chi_1^\dagger(k, -l), \\
\chi_1^\dagger(k, l) \to \chi_1^\dagger(k, -l - 1), \\
\chi_2(k, l) \to -\chi_2(k, -l - 1), \\
\chi_2^\dagger(k, l) \to -\chi_2^\dagger(k, -l - 2), \\
+ \text{ complex conjugation of prefactors.}
\end{cases}
(25)
\]

This is indeed an invariant transformation for the action in [19]. To define a charge conjugation transformation, which maps \( \psi_1 \) to \( \psi_2^\dagger \) and vice versa in the continuum limit, we have to shift the lattice over half a lattice spacing in the space direction. This implies

\[
C : \begin{cases}
\chi_1(k, l) \to \chi_2^\dagger(k, l - 1), \\
\chi_1^\dagger(k, l) \to \chi_2^\dagger(k, l), \\
\chi_2(k, l) \to \chi_1^\dagger(k + 1, l), \\
\chi_2^\dagger(k, l) \to \chi_1(k + 1, l + 1),
\end{cases}
(26)
\]

which maps [19] to itself as well.

As a conclusion, we point out that the problematic term in the fermion action is \( \psi^\dagger \frac{\partial}{\partial \tau} \psi \). It does not change shape under unitary transformations of the Dirac spinor \( \psi \) and is thus always present. Treating this term on a discretized spacetime lattice with local approximations for the time derivative, will require to choose between a number of possibilities which all have disadvantages. Averaging is the worst solution, leading to a line of poles in the fermion propagator. Spreading the derivative over two lattice sites introduces time doublers. The only way to circumvent this problem is by giving up on Hermiticity of the action and a strict relation between the fields and their complex conjugate on the lattice. This is allowed, since the complex conjugate fields are independent variables in a partition function.

And finally, it is possible to show that the non-Hermiticity of the action does not cause any unphysical effects. By performing independent Fourier transformations on the fields \( \chi_i \), and \( \chi_i^\dagger \), given by

\[
\begin{align*}
\chi_1(k, l) &= \sum_{\kappa, \omega} \chi_1(\kappa, \omega)e^{i(k\kappa - l\omega t)}, \\
\chi_1^\dagger(k, l) &= \sum_{\kappa, \omega} \chi_1^\dagger(\kappa, \omega)e^{-i(k\kappa - l(\omega + t/2))}, \\
\chi_2(k, l) &= \sum_{\kappa, \omega} \chi_2(\kappa, \omega)e^{i(k/2 + \omega t) - \omega(l + t/2)}, \\
\chi_2^\dagger(k, l) &= \sum_{\kappa, \omega} \chi_2^\dagger(\kappa, \omega)e^{-i(k/2 - \omega l - \omega(t + 1))},
\end{align*}
(27)
\]

the original action can be made Hermitian in the new fields

\[
S = a_t \sum_{\kappa, \omega} \left[ \chi_1(\kappa, \omega) \right]^\dagger \left[ \begin{array}{cc}
-\frac{2}{a_t} \sin(\omega/2) & -\frac{2}{a_t} \sin(\kappa/2) \\
-\frac{2}{a_t} \sin(\kappa/2) & -\frac{2}{a_t} \sin(\omega/2)
\end{array} \right] \chi_2(\kappa, \omega).
(28)
\]

Furthermore, it is now appropriate to consider the fields \( \chi_i(k, \omega) \) and \( \chi_i^\dagger(k, \omega) \) as Hermitian conjugates at the physical level. This way, the non-Hermiticity of the action in real space can be regarded as a consequence of performing non-matching Fourier transformations on the fields in the physical (Hermitian) action defined in momentum space. The incompatibility of these transformations implies a change in the integration element of the partition function, which is given by

\[
\prod_{m,n} \mathcal{D} \chi_1(k, l) \mathcal{D} \chi_2(k, l) \mathcal{D} \chi_2^\dagger(k, l) \mathcal{D} \chi_1^\dagger(k, l)
= \prod_{\omega, \kappa} [\exp(i\omega/2)]^2
\times \prod_{\omega, \kappa} \mathcal{D} \chi_1(\kappa, \omega) \mathcal{D} \chi_2(\kappa, \omega) \mathcal{D} \chi_2^\dagger(\kappa, \omega) \mathcal{D} \chi_1^\dagger(\kappa, \omega).
(29)
\]

The integration element thus acquires an overall phase factor, which will drop out in the calculation of expectation values. This proves that expectation values calculated with the real-space action will be well defined and result in physical values. We attribute this to the fact that all steps in our derivation have a clear physical interpretation and motivation, and are not mere mathematical tricks.

D. Adding gauge fields on the spacetime lattice

We now continue this line of reasoning to formulate an action for chiral fermions interacting with gauge fields. Just as in the case of the Hamiltonian lattice, the way to go is by making the lattice for the gauge fields twice as fine as the lattice for the fermions. There will be four gauge field unit cells in one fermion unit cell. We label the parallel transporters living on the links as

\[
U_x(k, l) = \mathcal{P} \exp \left( i \int_{k a_x}^{(k+l/2)a_x} A_x(x, l a_t) \, dx \right),
(30a)
\]

\[
U_t(k, l) = \mathcal{P} \exp \left( i \int_{l a_t}^{(l+1/2)a_t} A_t(k a_x, t) \, dt \right),
(30b)
\]

with \( \mathcal{P} \) being the path ordering operator, and this for \( k \) and \( l \) being either integers or half integers.
The correct way to add these transporters to the fermion action [19] is completely specified by definition (24). The result is given by

\[
S = a_t \sum_{k,l} \left[ \frac{1}{a_t} \left( U^{-1}_t(k, l - \frac{1}{2})\chi_1(k, l) - U_t(k, l - 1)\chi_1(k, l - 1) \right) + \frac{1}{a_x} \left( U^{-1}_x(k, l - \frac{1}{2})\chi_2(k, l) - U_x(k - \frac{1}{2}, l - \frac{1}{2})\chi_2(k - \frac{1}{2}, l) \right) \right]
\]

(31)

The action in (31) still has the discrete symmetries P, C, T and chiral symmetry, when accompanied by the appropriate transformation laws for the parallel transporters. In these transformation laws, it is important to include the possible shifts over half a distance \(a_x\) and/or \(a_t\), as is required in the definitions of chiral transformations and charge conjugations. These are again effects of order \(O(a)\), which do not exist in the continuum limit. In the continuum theory for example, a chiral transformation is only defined on the fermion fields and has no effect on the gauge fields. However, this discrepancy does not change the content of these transformations. Not only is the required shift of order \(O(a)\), but since it is a pure shift it has no influence on the the continuum theory which is required to have translational invariance. The only implication of these transformation laws is that chiral transformations and charge conjugations on the lattice can not be defined without performing such a shift.

III. MINIMALLY DOUBLED, CHIRAL FERMIONS ON THE STAGGERED LATTICE IN \((3+1)\) DIMENSIONS

We are ready to apply the analogous approach to the chiral fermion action in \((3+1)\) dimensions. We start again with the Kogut-Susskind construction on the Hamiltonian lattice. However, fermion doublers will now already exist without discretizing time. We will briefly elaborate on the reason why we can not circumvent those by staggering the spinor components. Nextly, we will proceed to the action formalism and define a unit cell in discrete space time which does not introduce extra time doublers. As such, we will have defined a non-Hermitian action which describes two chiral fermions.

Since gauge fields can be added in a completely parallel way, we will not repeat this argument and focus completely on the free, massless fermion field.

A. Staggered fermions on the Hamiltonian lattice

We consider the three dimensional discretized space lattice. Keeping our approach to staggered lattices in mind, if the evolution equation for one field depends on the spatial derivative of another field, the second field’s discretization position should be shifted over half a lattice distance in that spatial direction. The Hermiticity of the Dirac operator ensures that the evolution equation of the second field will depend on the same spatial derivative of the first field, which is required for the consistency of this approach. We choose to work with the standard representation of the \(\gamma\)-matrices, although nearly any representation is possible. Exceptions are those representations that result in the evolution equation of a field component being dependent on the spatial derivative of that same component. This is the case if one of the matrices \(\alpha^i = \gamma^0\gamma^i\) for \(i = 1, 2, 3\) has non-zero diagonal entries, which is the case i.e. the chiral representation or the Majorana representation.

In the standard representation, we start by placing \(\psi_1\) in the center of the unit cell. Since \(\partial\psi_1/\partial t\) depends on \(\partial\psi_3/\partial z\), \(\psi_3\) should be discretized half a lattice distance \(a_z\) further on in the \(z\) direction. The evolution equation of \(\psi_1\) also depends on the \(x\) derivative as well as the \(y\) derivative of \(\psi_3\). Continuing this line of reasoning results in the unit cell of FIG. 4 where all spinor components appear two times in the unit cell. We have the following rules for the discretization points:

\[
\psi_1: \quad (ia_x, ja_y, ka_z)
\]

\[
\psi_2: \quad (i + \frac{1}{2})a_x, (j + \frac{1}{2})a_y, (k + \frac{1}{2})a_z
\]

\[
\psi_3: \quad (ia_x, j + \frac{1}{2})a_y, (k + \frac{1}{2})a_z
\]

\[
\psi_4: \quad (i + \frac{1}{2})a_x, j + \frac{1}{2})a_y, (k + \frac{1}{2})a_z
\]

\[
(ia_x, (j + \frac{1}{2})a_y, ka_z)
\]

Exactly the same construction was found by Susskind
indeand interchange the fields over half a lattice spacing. A chiral transformation is obtained by shifting the lattice will not repeat his argument. Let us only point out that independent fermion flavours could be reobtained. We fermion system in detail [20] and illustrated how the two different spin orientations. The other factor two points generate. A factor two of this degeneracy is due to the tum space ($\kappa$). While there is only one low-energy region in the momen-
tum space ($\kappa_x, \kappa_y, \kappa_z \in [-\pi, +\pi]^{\mathbb{Z}^3}$, it is fourfold de-
generate. A factor two of this degeneracy is due to the different spin orientations. The other factor two points at two different fermion flavours, which is a logical con-
sequence of having twice the number of fields in the unit cell. Suskind worked out the symmetries of this two fermion system in detail [20] and illustrated how the two independent fermion flavours could be reobtained. We will not repeat his argument. Let us only point out that a chiral transformation is obtained by shifting the lattice over half a lattice spacing $a_z$ in the $z$ direction. This does indeed interchange the fields $\psi_1 \leftrightarrow \psi_3$ and $\psi_2 \leftrightarrow \psi_4$.

\[
\begin{align*}
\omega^2 - m^2 &= \frac{4}{a_x^2} \sin^2(\kappa_x/2) - \frac{4}{a_y^2} \sin^2(\kappa_y/2) \\
& \quad - \frac{4}{a_z^2} \sin^2(\kappa_z/2) = 0.
\end{align*}
\] (32)

While there is only one low-energy region in the momentum space ($\kappa_x, \kappa_y, \kappa_z \in [-\pi, +\pi]^{\mathbb{Z}^3}$, it is fourfold de-
generate. A factor two of this degeneracy is due to the different spin orientations. The other factor two points at two different fermion flavours, which is a logical con-
sequence of having twice the number of fields in the unit cell. Suskind worked out the symmetries of this two fermion system in detail [20] and illustrated how the two independent fermion flavours could be reobtained. We will not repeat his argument. Let us only point out that a chiral transformation is obtained by shifting the lattice over half a lattice spacing $a_z$ in the $z$ direction. This does indeed interchange the fields $\psi_1 \leftrightarrow \psi_3$ and $\psi_2 \leftrightarrow \psi_4$.

Another way to look at the fermion doubling in this unit cell is by constructing the Voronoi cell or Wigner-Seitz cell, which is in fact the physical unit cell in which every field appears only once, and which is given in FIG. 5. For this smaller cell, the physical domain in momentum space is enlarged and a second low-energy region is discovered around the corners of the Brillouin zone, corresponding to the second fermion flavor. Both pictures are of course completely equivalent.

Having this structure on the Hamiltonian lattice, we can proceed to the action on a discrete spacetime grid by performing the completely analogous approach as in $(1+1)$ dimensions. But let us first elaborate on the fact that the fermion flavor pair can not be reduced on this Hamiltonian lattice. The Dirac spinor in $(3+1)$ di-
mensons has 4 components, just as the 4-tuple $(p, v_x, v_y, v_z)$ in acoustics. However, the placement of the acoustics variables as given in FIG. 4 is not simply a trick to make the math work. This construction follows naturally from the physical interpretation of a scalar density $p$ and a vector field $\vec{v}$, and gives as such a clear representation of the conservation equation $\partial p/\partial t + \nabla \cdot \vec{v} = 0$ in a discrete setting. Such an interpretation is not possible for the four spinor components $\psi_i$, since spinors have no obvious classical, geometrical representation.

We can try to find a representation of the $\gamma$-matrices for which a unit cell similar to that of the quadruple $(p, v_x, v_y, v_z)$ is useful. This would be the case if the time derivative of $\psi_1$ depends on the $x$-derivative of $\psi_2$, the $y$-derivative of $\psi_3$ and the $z$-derivative of $\psi_4$. Although it is possible to construct three $\alpha$-matrices with the required properties, such as $\alpha^1 = 1 \otimes \sigma_x$, $\alpha^2 = \sigma_z \otimes \sigma_z$ and $\alpha^3 = \sigma_x \otimes \sigma_y$, it is now no longer possible to construct a matrix $\beta$ which anticommutes with all $\alpha$’s. Also, trying to calculate $\gamma^5 \sim \alpha^1 \alpha^2 \alpha^3$ yields an answer which is proportional to the unit matrix. These conclusions clearly indicate that the fermion doubling problem should be completely attributed to the spinorial character of the field components and not, as is often thought, to the fact that the Dirac equation consists of a set of first order partial differential equations. First order partial differential equations introduce no problems whatsoever for acoustic or electromagnetic problems on space(time) lattices.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Unit cell for the discretization of the fermion field components on the three-dimensional Hamiltonian lattice.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Unit cell (dashed lines) and Voronoi or Wigner-Seitz cell (full lines) for the discretization of the fermion field components on the three dimensional Hamiltonian lattice, in upper sight, (a) and the corresponding Brillouin zones in momentum space (b).}
\end{figure}

\subsection{B. Local lattice action for a minimally doubled pair of chiral fermions}

To construct a local action on a spacetime lattice, we should first look at a way to discretize time in the classical equations of motion. Since the time derivatives of $\psi_1$ and $\psi_2$ depend only on (space derivatives of) $\psi_3$ and $\psi_4$, and vice versa, we should shift the discretization points of $\psi_3$ and $\psi_4$ in the time direction. In addition, the $(1+1)$-dimensional case has taught that the discretization points of the complex conjugate fields $\psi_1^\dagger$ should be shifted over half a lattice spacing $a_t$ in the time direction. The resulting unit cell is depicted in FIG. 6.
The unit cell in FIG. [a] corresponds to the definitions

\[
\frac{\chi_{1a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_1(i, j, k, l) \quad \frac{\chi_{1a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_1(i, j, k, l + 1/2)
\]

\[
\frac{\chi_{1b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_1(i + 1/2, j + 1/2, k, l) \quad \frac{\chi_{1b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_1(i + 1/2, j + 1/2, k, l + 1/2)
\]

\[
\frac{\chi_{2a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_2(i + 1/2, j, k + 1/2, l) \quad \frac{\chi_{2a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_2(i + 1/2, j, k + 1/2, l + 1/2)
\]

\[
\frac{\chi_{2b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_2(i, j + 1/2, k + 1/2, l) \quad \frac{\chi_{2b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_2(i, j + 1/2, k + 1/2, l + 1/2)
\]

\[
\frac{\chi_{3a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_3(i, j, k + 1/2, l + 1/2) \quad \frac{\chi_{3a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_3(i, j, k + 1/2, l + 1)
\]

\[
\frac{\chi_{3b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_3(i + 1/2, j + 1/2, k + 1/2, l + 1) \quad \frac{\chi_{3b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_3(i + 1/2, j + 1/2, k + 1/2, l + 1)
\]

\[
\frac{\chi_{4a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_4(i + 1/2, j, k, l + 1/2) \quad \frac{\chi_{4a}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_4(i + 1/2, j, k, l + 1)
\]

\[
\frac{\chi_{4b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_4(i, j + 1/2, k, l + 1/2) \quad \frac{\chi_{4b}(i, j, k, l)}{\sqrt{a_x a_y a_z}} = \psi_4(i, j + 1/2, k, l + 1)
\]

(33)

in which the factor \(a_x, a_y, a_z\) and \(a_t\) in the arguments of the fields \(\psi_i\) were omitted for brevity. With these fields, we write down the discrete action

\[
S = i a_t \sum_{ijkl} \chi_{1a}(i, j, k, l) \left[ \frac{\chi_{1a}(i, j, k, l + 1) - \chi_{1a}(i, j, k, l)}{a_t} + \frac{\chi_{3a}(i, j, k, l) - \chi_{3a}(i, j, k, l - 1)}{a_z} \right]
\]

\[
+ \chi_{2a}(i, j, k, l) \left[ \frac{\chi_{2a}(i, j, k, l + 1) - \chi_{2a}(i, j, k, l)}{a_t} + \frac{\chi_{3a}(i + 1/2, j, k, l) - \chi_{3a}(i, j, k, l)}{a_z} \right]
\]

\[
+ \chi_{3a}(i, j, k, l - 1) \left[ \frac{\chi_{3a}(i, j, k, l) - \chi_{3a}(i, j, k, l - 1)}{a_t} + \frac{\chi_{1a}(i, j, k, l + 1) - \chi_{1a}(i, j, k, l)}{a_z} \right]
\]

\[
+ \chi_{4a}(i, j, k, l - 1) \left[ \frac{\chi_{4a}(i, j, k, l) - \chi_{4a}(i, j, k, l - 1)}{a_t} + \frac{\chi_{2a}(i, j, k, l) - \chi_{2a}(i, j, k, l)}{a_z} \right]
\]

(34)

\[
+ \text{analogous expressions for the } b\text{-fields.}
\]

Symmetry transformation which leave this action invariant can easily be deduced from the continuum prescription and the definitions (33), along with a possible shift over half a lattice distance if the mapping of the fields would mismatch. A chiral transformation is accomplished by shifting the lattice over half a distance \(a_z\) in the \(z\) direction and half a distance \(a_t\) in the \(t\) direction. All of this is a straightforward generalization of the \((1+1)\)-dimensional case and will not be written down explicitly. It is furthermore obvious that no new doublers were introduced by this approach to time discretization. We thus propose the action in (34) as a non-Hermitian action for chiral fermions with ultralocal approximations for the derivatives, which has the discrete symmetries \(P\), \(C\) and \(T\), and which describes only two fermion flavours.
IV. CONCLUSION

We share the opinion of the authors of [17] that a Hermitian lattice action for minimally doubled, chiral fermions can not have all the required discrete symmetries $P$, $C$ and $T$. However, we argue that we can solve this problem by losing the Hermiticity condition at order $O(a)$ for the discrete action, without this having to yield a physically ill-defined theory. We have illustrated a possible approach by reinterpreting the staggered lattice approach in a language borrowed from the FDTD community. In addition, we believe that these results imply a more stringent version of the Nielsen-Ninomiya theorem. A Hermitian lattice action for local, chiral fermions which has $P$, $C$ and $T$ symmetry will result in at least four fermion flavours. If the action is non-Hermitian, the Nielsen-Ninomiya theorem is no longer applicable, but, from our point of view, not all doublers can be removed without introducing non-physical effects or breaking symmetries. The reason for this is that physicality requires the theory to be Hermitian for $a_t \to 0$, and two different fermions will survive on the Hamiltonian lattice. This pair of fermions cannot be further reduced, due to the spinorial character of the field components.

What we have given are not proofs of these statements, but some easy to understand arguments obtained by reinterpreting this problem using the staggered lattice machinery as it is applied in FDTD simulations. It remains to be seen wether these arguments are generally valid, and if these statements can be proven on a more rigorous level.

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