PARTICLE TRANSPORT IN TANGLED MAGNETIC FIELDS AND FERMI ACCELERATION AT RELATIVISTIC SHOCKS

Martin Lemoine
GReCO/Institut d’Astrophysique de Paris, CNRS, 98 bis boulevard Arago, F-75014 Paris, France; lemoine@iap.fr

And
Guy Pelletier
Laboratoire d’Astrophysique de Grenoble, CNRS, Université Joseph Fourier, BP 53, F-38041 Grenoble, France; and Institut Universitaire de France; guy.pelletier@obs.ujf-grenoble.fr

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ABSTRACT

This Letter presents a new method of Monte Carlo simulations of test particle Fermi acceleration at relativistic shocks. The particle trajectories in tangled magnetic fields are integrated out exactly from entry to exit through the shock, and the conditional probability of return as a function of ingress and egress pitch angles is constructed by Monte Carlo iteration. These upstream and downstream probability laws are then used in conjunction with the energy gain formula at shock crossing to simulate Fermi acceleration. For pure Kolmogorov magnetic turbulence upstream and downstream, the spectral index is found to evolve smoothly from \( s = 2.09 \pm 0.02 \) for mildly relativistic shocks with Lorentz factor \( \Gamma_s = 2 \) to \( s = 2.26 \pm 0.04 \) in the ultrarelativistic limit \( \Gamma_s \gg 1 \). The energy gain is \( \sim \Gamma_s^2 \) at first shock crossing, and \( \sim 2 \) in all subsequent cycles, as anticipated by Gallant & Achterberg. The acceleration timescale is found to be as short as a fraction of Larmor time when \( \Gamma_s \gg 1 \).

Subject headings: acceleration of particles — cosmic rays — shock waves

1. INTRODUCTION

Fermi acceleration at relativistic shocks is an important topic for understanding the formation of spectra of ultrarelativistic particles and radiation in relativistic flows such as those observed in active nuclei, microquasars, \( \gamma \)-ray bursts, and pulsar wind nebulae (see Kirk & Duffy 1999 and references therein). Of particular interest is the acceleration timescale that can be as short as a Larmor time for relativistic Fermi acceleration; the smaller return probability to the shock for downstream particles, as compared to the nonrelativistic regime, is compensated by the much larger energy gain at each cycle. However, the study of Fermi acceleration in the relativistic limit is more involved than in the nonrelativistic regime owing to the increased importance of the anisotropy of the distribution function (Gallant & Achterberg 1999).

Various methods have been used to study the relativistic regime of Fermi acceleration (see Kirk & Duffy 1999 and references therein), starting with analytical estimates by Peacock (1981), followed by semianalytical methods (Kirk & Schneider 1987; Gallant & Achterberg 1999; Kirk et al. 2000; Achterberg et al. 2001) and numerical Monte Carlo techniques (Ballard & Heavens 1992; Ostrowski 1993; Bednarz & Ostrowski 1998), which in spite of their differences converge to an asymptotic spectral index \( s \approx 2.2–2.3 \) in the ultrarelativistic limit.

In the present Letter, we propose a new numerical Monte Carlo method of simulation of the acceleration of test particles at relativistic shocks. The trajectories of particles in the upstream and downstream inhomogeneous magnetic fields are integrated out exactly from the entry of each particle through the shock until its first return to the shock. The law of probability of return to the shock as a function of ingress and egress pitch angles is then constructed by Monte Carlo iteration. Finally, we combine these probability laws, one defined for upstream and the other for downstream, with the energy gain formula at shock crossing to simulate the acceleration process. This latter use of the angular probability laws is similar to the method of Gallant et al. (2000), with some differences as discussed below.

The present method appears efficient and potentially more powerful when compared to direct Monte Carlo simulations, which follow each particle through its repeated shock crossings (e.g., Ballard & Heavens 1992; Ostrowski 1993). It allows one to simulate relativistic Fermi acceleration in any magnetic configuration, albeit for test particles only. We describe the method and numerical simulations in § 2 and then present the results for a planar shock with a fully turbulent magnetic field both upstream and downstream in § 3.

2. NUMERICAL SIMULATIONS

The hydrodynamic jump conditions at an adiabatic strong shock, neglecting magnetic fields, are given in Blandford & McKee (1976), Kirk & Duffy (1999), and Gallant (2002). The shock Lorentz factor is \( \Gamma_s \) (upstream or lab frame\(^1\)), and the relative Lorentz factor \( \Gamma_b \) between upstream and downstream \( \Gamma' \equiv \Gamma, \Gamma_{s/d}(1 - \beta_b \beta_{s/d}) \). The downstream Lorentz factor \( \Gamma_{s/d} \) (and \( \Gamma_s \)) can be obtained as a function of \( \Gamma_b \) from the relations derived from the shock jump conditions given in Gallant (2002). In particular, in the ultrarelativistic limit \( \Gamma_s \rightarrow +\infty \), one finds the well-known results \( \beta_{s/d} \rightarrow \frac{1}{2} \) and \( \Gamma_s \rightarrow \Gamma_{s/d}/2 \).

We conduct our simulations in two steps. We first perform Monte Carlo simulations of particle propagation in a given magnetic field structure following Casse, Lemoine, & Pelletier (2002; where one may find more details on the numerical procedure). These simulations are carried out separately in the downstream and in the upstream rest frames. It is possible to set up any magnetic field structure including regular and tangled components, but in the following, for the sake of simplicity, we restrict ourselves to the case of pure Kolmogorov turbulence in both upstream and downstream.

\(^1\) Unless otherwise noted, all quantities are calculated in the upstream (lab) frame; wherever needed, quantities relative to a given frame but calculated in another will be marked with the vertical bar subscript; e.g., \( \beta_{s/d} \) refers to the shock velocity measured in the downstream rest frame.
downstream and upstream rest frames. The equations of motion of each particle are integrated out exactly, the magnetic field being calculated at each point of the trajectory as a sum of plane waves, using 200 modes spaced logarithmically on three decades of wavelength and whose wavevector directions are drawn at random.\footnote{Ostrowski (1993) used a similar technique to construct the magnetic field, albeit with three modes only, while Ballard & Heavens (1992) used three-dimensional fast Fourier transform methods (see Casse et al. 2002 for a comparison of these methods).}

The trajectories are integrated over 100 scattering times upstream and 1000 scattering times downstream in order not to miss possible late returns to the shocks. The laws of return probability as a function of ingress and egress pitch angles are then constructed in the following way. We draw at random a point along a simulated trajectory that defines the point of entry through the shock. The ingress pitch angle cosine to shock normal \( \mu' \) is recorded, the trajectory is scanned to find the point of exit through the shock, and the egress pitch angle cosine \( \mu'' \) is then recorded. Iteration of the above then yields the desired law of conditional return probability \( P(\mu'; \mu'') \), which gives the probability for a particle entering with a pitch angle cosine \( \mu' \) to return to the shock with a pitch angle cosine \( \mu'' \). The simulations also give a direct measurement of the return time to the shock as a function of pitch angles.

Once the upstream and downstream laws of return probability, respectively, \( P_u(\mu_u'; \mu_u'') \) and \( P_d(\mu_d'; \mu_d'') \), are known, the simulation of the acceleration process itself can be performed as follows. We denote by \( f^{2n}(\mu_u, \epsilon_u) \) the distribution function of particles that enter the shock to downstream and that have experienced \( 2n \) shock crossings, and by \( f^{2n}(\mu_d, \epsilon_d) \) similarly upstream: particles are upstream for an even number of shock crossings, and \( f^0 \) represents the injection population. These distribution functions are normalized to the total number of particles \( N \) injected such that, in the absence of the escape from the acceleration site, after \( 2n \) shock crossings \( N = \int d\mu_u d\epsilon_u f^{2n}(\mu_u, \epsilon_u) \), and after \( 2n + 1 \) shock crossings \( N = \int d\mu_d d\epsilon_d f^{2n+1}(\mu_d, \epsilon_d) \). The conservation of particle number at shock crossing \( u \to d \) and Lorentz transforms of pitch angles and energies imply

\[
f^{2n+1}(\epsilon_d, \mu_d) d\mu_d d\epsilon_d = \left[ \int_{\beta_d}^1 d\mu_u' P_u(\mu_u', \mu_u'') f^{2n}(\epsilon_u, \mu_u'') \right] d\mu_u' d\epsilon_u, \quad (1)
\]

\[
\mu_d = \frac{\mu_u' - \beta_u}{1 - \beta_u \mu_u}, \quad \epsilon_d = \Gamma(1 - \beta_u \mu_u) \epsilon_u, \quad (2)
\]

and one obtains a similar system for shock crossing \( d \to u \):

\[
f^{2n}(\epsilon_u, \mu_u') d\mu_u' d\epsilon_u = \left[ \int_{-1}^{\beta_u} d\mu_d' P_d(\mu_d', \mu_d'') f^{2n-1}(\epsilon_d, \mu_d'') \right] d\mu_d' d\epsilon_d, \quad (3)
\]

\[
\mu_u' = \frac{\mu_d' + \beta_d}{1 + \beta_d \mu_d'}, \quad \epsilon_u = \Gamma(1 + \beta_d \mu_d') \epsilon_d. \quad (4)
\]

The terms within brackets in equations (1) and (3) correspond to the distribution function upon exit from upstream and downstream, respectively. The return probability to the shock \( P_d(\mu_d') \) as a function of ingress pitch angle can be obtained as \( P_d(\mu_d') = \int d\mu_u P_u(\mu_u'; \mu_d') \). The corresponding return probability for upstream is obviously unity. After each cycle \( u \to d \to u \), a fraction \( f_{out}^{2n+1}(\epsilon) = \int d\mu_u [1 - P_u(\mu_u')] f^{2n+1}(\mu_u'; \epsilon) \) of the particle population has escaped downstream and accumulates to form the outgoing particle population \( f_{out}(\epsilon) = \sum_{n=0}^{\infty} f_{out}^{2n+1}(\epsilon) \). By following each shock crossing, and using equations (1), (2), (3), and (4), one can follow the evolution of \( f_u, f_d, \) and \( f_{out} \) starting from a monoenergetic and isotropic initial injection distribution upstream. The distribution \( f_{out}(\epsilon) \) eventually provides the accelerated particle population. A similar formal development of the acceleration process by repeated shock crossings has been proposed independently by Vietri (2003): the flux of particles crossing the shock in the stationary regime, noted \( J_{in} \) in Vietri (2003), is related to the above as

\[
J_{in} = C \sum_{n=0}^{\infty} f_{out}^{2n+1}, \quad (10)
\]

with \( C \) a normalization constant.

This assumes that the angular probability laws do not depend on rigidity. This is true in the diffusive limit, but one might expect some weak dependence to appear in the relativistic limit \( \Gamma \gg 1 \). Indeed, we have found numerically such a weak dependence of \( P_u \) and \( P_d \) on the particle rigidity. However, it remains weak, and the change in averages to a few percent when the rigidity changes by 2 orders of magnitude. In terms of energy spectral index \( s \), this dependence introduces an error of \( \delta s = \pm 0.02 \) for \( \Gamma = 2 \) up to \( \delta s = \pm 0.04 \) for \( \Gamma = 100 \). Therefore, in the following we neglect the dependence on rigidity but keep the above errors as uncertainties on our results. Note that one can in principle incorporate this dependence on rigidity in our method at the expense of having to calculate the probability laws \( P \) for a wide range of values of the rigidity.

The present technique has significant advantages when compared to standard Monte Carlo techniques, which follow the particle trajectories on both sides of the shock through the whole acceleration process. Indeed, provided one neglects the dependence on rigidity of \( P \), one can simulate the trajectories of particles of high rigidity only (near the end of the resonance range), which are must faster to integrate than the trajectories of particles of small rigidity since the ratio of scattering time to Larmor time decreases with increasing rigidity. The direct Monte Carlo methods also suffer from the problem of a small dynamic range of the magnetic fields as compared to the wide dynamic range of particle momenta. The present method also offers a significant gain in signal, as will be obvious in the following. Finally, our method differs from Gallant et al. (2000), as they use 3D differential techniques to simulate scattering downstream and analytical methods for scattering in a regular magnetic field upstream assuming \( \Gamma \gg 1 \). In contrast, the present simulations can be applied to any shock Lorentz factor and magnetic field configuration. Furthermore, they use Monte Carlo methods to simulate the acceleration process after constructing the laws of return probability, while we directly fold over repeatedly the probability distributions in conjunction with the energy gain formula.

3. RESULTS

The downstream return probability to the shock as a function of ingress pitch angle cosine is shown in Figure 1 for \( \Gamma = 2, 100 \). The return probability for \( \Gamma = 100 \) is an asymptote that is reached to within 1% as early as \( \Gamma = 5 \). In Figure 2, we show the average energy gain \( \langle g \rangle = \langle \epsilon_i \rangle / \langle \epsilon_i \rangle \) per cycle

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\]
u \rightarrow d \rightarrow u \text{ (diamonds)} \text{ and } d \rightarrow u \rightarrow d \text{ (triangles)} \text{ for } \Gamma = 100. \text{ This energy gain is defined as the ratio of the average energies at the end } \langle \epsilon_f \rangle \text{ and at the beginning of the cycle } \langle \epsilon_i \rangle, \text{ with } \langle \epsilon \rangle = \int \mu \, d\mu \, \epsilon(f(\mu, \epsilon)) / \int \mu \, d\mu \, d\epsilon(f(\mu, \epsilon)). \text{ The average energy gain in each cycle subsequent to first shock crossing is } \approx 1.93. \text{ The average value is reached immediately after the first cycle. This is a rather dramatic confirmation of the analytical expectations of Gallant \& Achterberg (1999) and Achterberg et al. (2001), which had argued that only the first cycle should yield a gain } \approx \Gamma^2 \text{ since the anisotropy of the distribution function upstream is so pronounced that the gain in subsequent cycles is reduced to } \approx 2.\text{ An example of the spectrum of accelerated particles for } \Gamma = 100 \text{ after } 20 \text{ cycles } u \rightarrow d \rightarrow u \text{ is shown in Figure 3; the thin lines in this figure show the fractions of particles } f_{\text{out}} \text{ that escape after } 2n + 1 \text{ shock crossings. One clearly sees in this figure the piling up of populations of particles of ever decreasing size (owing to finite escape probability) and ever increasing energy, which gives rise to the accelerated population } f_{\text{out}} = \sum_{n_{\text{out}}} f^{(n_{\text{out}})}. \text{ The spectral index of the escaping population for } \Gamma = 100 \text{ is here } s = 2.26 \pm 0.04 \text{ (incorporating the error due to the dependence of } P \text{ on rigidity), in excellent agreement with previous results by Bednarz \& Ostrowski (1998), Kirk et al. (2000), and Achterberg et al. (2001).} \text{ Finally, in Figure 4, we give the average return probabilities (open squares), average asymptotic energy gains (diamonds), and spectral indices (filled circles) for values of } \Gamma \text{ comprised between 2 and 100. The average return probabilities shown in this figure are the return probabilities shown in Figure 1 weighted by the corresponding asymptotic downstream ingress pitch angle distribution, i.e., } P_{\text{ret}} = \lim_{n \rightarrow \infty} \langle P_{\text{ret}}(\mu, \epsilon) \rangle = \langle P_{\text{ret}}(\mu, \epsilon) \rangle. \text{ A naive unweighted average of the return probability shown in Figure 1 for } \Gamma = 100 \text{ would give 0.33, whereas the weighted average gives 0.40: the difference is directly related to the strong anisotropy at shock crossing. The standard nonrelativistic formula for the (energy) spectral index } s = 1 - \log(P_{\text{ret}})/\log(g), \text{ with } \langle g \rangle \text{ the average energy gain, is in relatively good agreement with the slopes obtained, provided one uses the weighted average for the return probability as above (see Fig. 4). A more general formula, which includes relativistic effects, has been proposed by Vietri (2003): } \langle P_{\text{ret}} \rangle g^{s-1} = 1. \text{ One can derive this formula and variants of it using the average return probabilities and energy gains (see text).}
by using our equations (1), (2), (3), and (4). For instance, one can insert equation (3) into equation (1) and then sum over \(n\) (shock crossing number) to go to the stationary regime and consider an energy range where \(\epsilon \gg \epsilon_{0}, \epsilon_{0}\) being the maximal injection energy. There one expects that \(\Sigma_{\epsilon_{i}} f_{\epsilon_{i}} \approx \epsilon_{i}^{-\alpha} G_{\epsilon_{i}}\); i.e., the distribution factorizes out in an energy power law times a function of pitch angle, and indeed this property is verified numerically to high accuracy. Then one introduces the energy gain per cycle \(g(\mu'_{\epsilon}, \mu'_{d}) = \epsilon_{d}/\epsilon_{i} \) and integrates over \(\mu'_{d}\) both sides of the equation. Finally, dividing one side by the other yields the following relation, which is a variant of the formula of Vietri (2003):

\[
\int_{-1}^{0} d\mu'_{d} \int_{\beta_{i}}^{1} d\mu'_{i} \tilde{P}_{\epsilon}(\mu'_{i}) \tilde{P}_{\epsilon}(\mu'_{d}, \mu'_{i}) g^{-1}(\mu'_{d}, \mu'_{i}) = 1. \tag{5}
\]

In this equation, \(\tilde{P}_{\epsilon}(\mu'_{d}, \mu'_{i}) \) simply corresponds to \(\tilde{P}_{\epsilon}\) expressed in terms of downstream pitch angle cosines, and \(P_{\epsilon}(\mu'_{d}) \equiv \int d\mu'_{i} \tilde{P}_{\epsilon}(\mu'_{d}, \mu'_{i}) \). Equation (5) is indeed verified to within the numerical noise (<1%).

Our simulations also provide a direct measurement of the acceleration timescale, which can be taken as the cycle time in the upstream rest frame when \(\Gamma_{i} \gg 1\): \(t_{\text{acc}}(\epsilon) \approx t_{\text{acc}}(\epsilon) + \Gamma_{i} t_{\text{d,d}}(\epsilon/\Gamma_{i})\), where \(t_{\text{d,d}}(\epsilon)\) and \(t_{\text{d,d}}(\epsilon)\) denote the upstream and downstream return times measured in their respective rest frames for a particle of energy \(\epsilon\). For \(f_{\beta,d}\), we find to within the noise of the simulations \(t_{\text{d,d}} \approx 1.5\frac{B_{0} \Gamma}{E_{\text{m}}} t_{\text{scatt}}(\epsilon)\), with \(t_{\text{scatt}}(\epsilon)\) the scattering time downstream. The scattering time is given as a function of Larmor time in the relativistic limit.

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