ACCURATE APPROXIMATIONS OF SOME EXPRESSIONS INVOLVING TRIGONOMETRIC FUNCTIONS

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Abstract. The aim of this paper is to apply an original computation method due to Malešević and Makragić \cite{5} to the problem of approximating some trigonometric functions. Inequalities of Wilker-Cusa-Huygens are discussed, but the method can be successfully applied to a wide class of problems. In particular, we improve the estimates recently obtained by Mortici \cite{1} and moreover we show that they hold true also on some extended intervals.

Keyword. Wilker inequality, trigonometric approximation

1. Introduction

In the reference \cite{2} J. B. Wilker presented the inequality

\[ 2 \left( \frac{\sin x}{x} \right)^2 + \tan \frac{x}{x} < 2, \quad (1) \]

for \( x \in (0, \pi/2) \) and he asked for largest constant \( c \) in

\[ 2 + cx^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \tan \frac{x}{x} \quad \text{for} \quad c > 0, \quad (2) \]

and for \( x \in (0, \pi/2) \). Recently, Wilker inequality is a lot studied in different paper works. In the paper \cite{3}, J.S.Sumner, A.A.Jagers, M. Vowe, J. Anglesio proved the following double inequality

\[ 2 + \frac{16}{\pi^2} x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \tan \frac{x}{x} < 2 + \frac{8}{45} x^3 \tan x, \quad (3) \]

for \( x \in (0, \pi/2) \). In the paper \cite{1}, C. Mortici has proved the following two statements:

\textbf{Theorem 1.1.} For every \( x \in (0, 1) \) we have:

\[ 2 + \left( \frac{8}{45} - a(x) \right) x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \tan \frac{x}{x} < 2 + \left( \frac{8}{45} - b(x) \right) x^3 \tan x, \quad (4) \]

where \( a(x) = \frac{8}{945} x^2, \quad b(x) = \frac{8}{945} x^2 - \frac{16}{14175} x^4. \)

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Theorem 1.2. For every \( x \in \left( \frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} \right) \) in the left-hand side and for every \( x \in \left( \frac{\pi}{3} - \frac{1}{2}, \frac{\pi}{2} \right) \) in the right-hand side the following inequalities are true:

\[
2 + \left( \frac{16}{\pi^4} + c(x) \right) x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} < 2 + \left( \frac{16}{\pi^4} + d(x) \right) x^3 \tan x, \tag{5}
\]

where

\[
c(x) = \left( \frac{160}{\pi^5} - \frac{16}{\pi^4} \right) \left( \frac{\pi}{2} - x \right), \quad d(x) = \left( \frac{160}{\pi^5} - \frac{16}{\pi^4} \right) \left( \frac{\pi}{2} - x \right) + \left( \frac{960}{\pi^6} - \frac{96}{\pi^5} \right) \left( \frac{\pi}{2} - x \right)^2.
\]

Theorem 1.1. and Theorem 1.2. describe a subtly analysis of Wilker inequality by C. Mortici. The method of proving inequalities in this paper was given in the paper [4] and it is based on use of appropriate approximations of some trigonometric polynomials:

In [5] is considered a method of proving trigonometric inequalities for mixed trigonometric polynomials with finite Taylor series. The method presents by C. Mortici. Theorem 1.1. and Theorem 1.2. describe a subtly analysis of Wilker inequality by C. Mortici. The method of proving inequalities in this paper was given in the paper [4]. More precisely, we extend the domains \((0, 1)\) and \((\frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2})\) from the previous theorems to \((0, \frac{\pi}{2})\). We give the next two statements.

Theorem 2.1. For every \( x \in \left( 0, \frac{\pi}{2} \right) \) the following inequalities are true:

\[
2 + \left( \frac{8}{45} - a(x) \right) x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} < 2 + \left( \frac{8}{45} - b_1(x) \right) x^3 \tan x, \tag{6}
\]

where \(a(x) = \frac{8}{945} x^2, b_1(x) = \frac{8}{945} x^2 - \frac{\alpha}{14175} x^4\) and \(\alpha = \frac{480\pi^6}{\pi^6} - \frac{46220\pi^4}{\pi^6} + \frac{3628800}{\pi^6} = 17.15041\ldots\)

Theorem 2.2. For every \( x \in \left( 0, \frac{\pi}{2} \right) \) the following inequalities are true:

\[
2 + \left( \frac{16}{\pi^4} + c(x) \right) x^3 \tan x < \left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} < 2 + \left( \frac{16}{\pi^4} + d(x) \right) x^3 \tan x, \tag{7}
\]

where

\[
c(x) = \left( \frac{160}{\pi^5} - \frac{16}{\pi^4} \right) \left( \frac{\pi}{2} - x \right), \quad d(x) = \left( \frac{160}{\pi^5} - \frac{16}{\pi^4} \right) \left( \frac{\pi}{2} - x \right) + \left( \frac{960}{\pi^6} - \frac{96}{\pi^5} \right) \left( \frac{\pi}{2} - x \right)^2.
\]

In [3] is considered a method of proving trigonometric inequalities for mixed trigonometric polynomials:

\[
f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x > 0, \tag{8}
\]

for \( x \in (\delta_2, \delta_1), \delta_2 < \delta_1, \) where \(\alpha_i \in \mathbb{R} \setminus \{0\}, p_i, q_i, r_i \in \mathbb{N}_0\) and \(n \in \mathbb{N}.\) One method of proving inequalities in form [3] is based on transformation, using the sum of sine and cosine of multiple angles.
Let us mention some facts from [5]. Let \( \varphi : [a, b] \rightarrow \mathbb{R} \) be a function which is differentiable on a segment \([a, b]\) and differentiable arbitrary number of times on a right neighbourhood of the point \( x = a \) and denote by \( T_m^{\varphi, a}(x) \) the Taylor polynomial of the function \( \varphi(x) \) in the point \( x = a \) of the order \( m \). If there is some \( \eta > 0 \) such that holds: \( T_m^{\varphi, a}(x) \geq \varphi(x) \), for \( x \in (a, a + \eta) \subset [a, b] \); then let us define \( T_m^{\varphi, a}(x) = T_m^{\varphi, a}(x) \) and \( T_m^{\varphi, a}(x) \) present an upward approximation of the function \( \varphi(x) \) on right neighbourhood \((a, a + \eta)\) of the point \( a \) of the order \( m \). Analogously, if there is some \( \eta > 0 \) such that holds: \( T_m^{\varphi, a}(x) \leq \varphi(x) \), for \( x \in (a, a + \eta) \subset [a, b] \); then let us define \( T_m^{\varphi, a}(x) = T_m^{\varphi, a}(x) \) and \( T_m^{\varphi, a}(x) \) present a downward approximation of the function \( \varphi(x) \) on right neighbourhood \((a, a + \eta)\) of the point \( a \) of the order \( m \). Let us note that it is possible to analogously define upward and downward approximations on some left neighbourhood of a point. According to the paper [5] following Lemmas are true:

**Lemma 2.3.** (i) For the polynomial \( T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i2i+1}{(2i+1)!}, \) where \( n = 4k+1, k \in \mathbb{N}_0 \), it is valid:

\[
\left( \forall t \in [0, \sqrt{(n+3)(n+4)}] \right) T_n(t) \leq T_{n+4}(t) \leq \sin t, \tag{9}
\]

\[
\left( \forall t \in [-\sqrt{(n+3)(n+4)}, 0] \right) T_n(t) \leq T_{n+4}(t) \leq \sin t. \tag{10}
\]

For the value \( t = 0 \) the inequalities in (9) and (10) turn into equalities. For the values \( t = \pm \sqrt{(n+3)(n+4)} \) the equalities \( T_n(t) = T_{n+4}(t) \) and \( T_n(t) = T_{n+4}(t) \) are true, respectively.

(ii) For the polynomial \( T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i2i+1}{(2i+1)!}, \) where \( n = 4k+3, k \in \mathbb{N}_0 \), it is valid:

\[
\left( \forall t \in [0, \sqrt{(n+3)(n+4)}] \right) T_n(t) \leq T_{n+4}(t) \leq \sin t, \tag{11}
\]

\[
\left( \forall t \in [-\sqrt{(n+3)(n+4)}, 0] \right) T_n(t) \geq T_{n+4}(t) \geq \sin t. \tag{12}
\]

For the value \( t = 0 \) the inequalities in (11) and (12) turn into equalities. For the values \( t = \pm \sqrt{(n+3)(n+4)} \) the equalities \( T_n(t) = T_{n+4}(t) \) and \( T_n(t) = T_{n+4}(t) \) are true, respectively.

Let us notice that for the function \( \sin x \) we have following order:

\[
T_{13}^{\sin, 0}(x) \leq T_9^{\sin, 0}(x) \leq T_5^{\sin, 0}(x) \leq \ldots \leq \sin x \leq \ldots \leq T_{13}^{\sin, 0}(x) \leq T_9^{\sin, 0}(x) \leq T_5^{\sin, 0}(x) \quad \text{for} \quad x \in [0, \sqrt{20}]. \tag{13}
\]

**Lemma 2.4.** (i) For the polynomial \( T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i2i}{(2i)!}, \) where \( n = 4k, k \in \mathbb{N}_0 \), it is valid:

\[
\left( \forall t \in [-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}] \right) T_n(t) \geq T_{n+4}(t) \geq \cos t. \tag{14}
\]
For the value $t = 0$ the inequality in (13) turns into equality. For the values $t = \pm \sqrt{(n+3)(n+4)}$ the equality $T_n(t) = T_{n+4}(t)$ is true.

(ii) For the polynomial $T_n(t) = \sum_{i=0}^{n/2} \frac{(-1)^i t^{2i}}{(2i)!}$, where $n = 4k + 2$, $k \in \mathbb{N}_0$, it is valid:

$$\left( \forall t \in [-\sqrt{(n+3)(n+4)}, \sqrt{(n+3)(n+4)}] \right) T_n(t) \leq T_{n+4}(t) \leq \cos t. \quad (15)$$

For the value $t = 0$ the inequality in (15) turns into equality. For the values $t = \pm \sqrt{(n+3)(n+4)}$ the equality $T_n(t) = T_{n+4}(t)$ is true.

Let us notice that for the function $\cos x$ we have following order:

$$T_{12}^{\cos,0}(x) \leq T_{8}^{\cos,0}(x) \leq T_{4}^{\cos,0}(x) \leq \ldots \leq \cos x \leq \ldots \leq T_{12}^{\cos,0}(x) \leq T_{8}^{\cos,0}(x) \leq T_{4}^{\cos,0}(x) \text{ for } x \in \left[0, \sqrt{12}\right].$$

Proofs of previous Lemmas given above are presented in the paper [7].

3. Proofs

In order to prove Theorem 2.1. and Theorem 2.2. we will separately observe left and right sides of inequalities.

The proof of Theorem 2.1.

Transforming inequality (10) we have following considerations.

(A) Proving the left side of inequality

$$2 + \left(\frac{8}{45} - a(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}, \quad (17)$$

for $x \in \left(0, \frac{\pi}{2}\right)$. The inequality (17) is equivalent to the mixed trigonometric inequality

$$f(x) = 1 - 8x^2 + h_1(x) \cos 4x + h_2(x) \cos 2x + h_3(x) \sin 2x$$

$$= 1 - 8x^2 - \cos 4x - 8x^2 \cos 2x + 4x - 4 \left(\frac{8}{45} - a(x)\right)x^5 \sin 2x > 0, \quad (18)$$

for $x \in \left(0, \frac{\pi}{2}\right)$, and $h_1(x) = -1 < 0$, $h_2(x) = -8x^2 < 0$, $h_3(x) = 4x - 4 \left(\frac{8}{45} - a(x)\right)x^5$.

Now let us consider two cases:

(A/I) $x \in (0, 1.57]$ Let us determine sign of the polynomial $h_3(x)$. As we see, that polynomial is the polynomial of $7^{th}$ degree

$$h_3(x) = P_7(x) = 4x - 4 \left(\frac{8}{45} - a(x)\right)x^5 = 32 \frac{945}{945} x^7 - 32 \frac{45}{45} x^5 + 4x. \quad (19)$$

Using the factorization of the polynomial $P_7(x)$ we have

$$P_7(x) = \frac{4}{945} x (8x^6 - 168x^4 + 945) = \frac{4}{945} xP_6(x), \quad (20)$$

where

$$P_6(x) = 8x^6 - 168x^4 + 945, \quad (21)$$
for $x \in (0, 1.57]$. Introducing the substitution $s = x^2$ we can notice that the polynomial $P_6(x)$ can be transformed into polynomial of 3$^{rd}$ degree

$$P_3(s) = 8s^3 - 168s^2 + 945, \quad (22)$$

for $s \in (0, 2.4649]$. Using MATLAB software we can determine the real numerical factorization of the polynomial

$$P_3(s) = \alpha(s - s_1)(s - s_2)(s - s_3), \quad (23)$$

where $\alpha = 8$ and where

$$s_1 = \frac{1}{8} \left(18172 + 84I\sqrt{21495}\right)^{1/3} - \frac{98}{3 \left(18172 + 84I\sqrt{21495}\right)^{1/3} + 7}$$

and

$$s_2 = \frac{3}{4} I\sqrt{3} \left(\frac{1}{6} \left(18172 + 84I\sqrt{21495}\right)^{1/3} - \frac{392}{3 \left(18172 + 84I\sqrt{21495}\right)^{1/3} + 7}\right) = -2.253 \ldots,$$

$$s_3 = \frac{1}{4} \left(18172 + 84I\sqrt{21495}\right)^{1/3} + \frac{196}{\left(18172 + 84I\sqrt{21495}\right)^{1/3} + 7} = 2.528 \ldots,$$

$$s_3 = 20.724 \ldots;$$

for $I = \sqrt{-1}$ (imaginary unit). The polynomial $P_6(x)$ has exactly three simple real roots with a symbolic radical representation and corresponding numerical values $s_1$, $s_2$, $s_3$ given at $(23)$. Since $P_6(0) > 0$ it follows that $P_3(s) > 0$ for $s \in (s_1, s_2)$, so we have following conclusions:

$$P_3(s) > 0 \quad \text{for} \quad s \in (0, 2.4649] \subset (s_1, s_2) \Rightarrow P_6(x) > 0 \quad \text{for} \quad x \in (0, 1.57] \subset (0, \sqrt{s_2})$$

$$\Rightarrow P_7(x) > 0 \quad \text{for} \quad x \in (0, 1.57] \subset (0, \sqrt{s_2}).$$

where $\sqrt{s_2} = 1.589 \ldots > 1.57$.

According to the Lemmas 2.3. and 2.4. and description of the method based on $(13)$ and $(10)$, the following inequalities: $\cos y < T_{k}^{\cos,0}(y) \quad (k = 20)$, $\cos y < T_{k}^{\cos,0}(y) \quad (k = 16)$, $\sin y > T_{k}^{\sin,0}(y) \quad (k = 11)$ are true, for $y \in \left(0, \sqrt{(k+3)(k+4)}\right)$. For $x \in (0, 1.57]$ it is valid:

$$f(x) > Q_{20}(x) = 1 - 8x^2 - T_{20}^{\cos,0}(4x) - 8xT_{16}^{\cos,0}(2x) + P_7(x)T_{11}^{\sin,0}(2x), \quad (26)$$

where $Q_{20}(x)$ is the polynomial

$$Q_{20}(x) = \frac{16}{9280784638125}x^{10} \left(262144x^{10} - 5203625x^8 + 69322260x^6 - 665557650x^4 + 3412527300x^2 - 5237832600\right)$$

$$-665557650x^4 + 3412527300x^2 - 5237832600$$

$$= \frac{16}{9280784638125}x^{10}Q_{10}(x), \quad (27)$$

for $x \in (0, 1.57]$. Then, we have to determine sign of the polynomial

$$Q_{10}(x) = 262144x^{10} - 5203625x^8 + 69322260x^6 - 665557650x^4 + 3412527300x^2 - 5237832600, \quad (28)$$
for \( x \in (0, 1.57] \), which is the polynomial of 10th degree. By substitution \( t = x^2 \) we can transform the polynomial \( Q_{10}(x) \) into polynomial

\[
Q_5(t) = 262144t^5 - 5203625t^4 + 69322260t^3 - 665557650t^2 + 3412527300t - 5237832600,
\]

for \( t \in (0, 2.4649) \). The first derivative of the polynomial \( Q_5(t) \) is the polynomial of 4th degree

\[
Q_5'(t) = 1312070t^4 - 20814500t^3 + 207966780t^2 - 1331115300t + 3412527300.
\]

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
Q_5(t) = \alpha(t^2 + p_1t + q_1)(t^2 + p_2t + q_2),
\]

where \( \alpha = 1310720 \), \( p_1 = -11.655 \ldots \), \( q_1 = 34.966 \ldots \), \( p_2 = -4.224 \ldots \), \( q_2 = 74.457 \ldots \). Also, holds that inequalities \( p_1^2 - 4q_1 < 0 \) and \( p_2^2 - 4q_2 < 0 \) are true. The polynomial \( Q_5(t) \) has no real roots. Let us remark that roots and constants \( p_1, q_1, p_2, q_2 \) can be represented in symbolic form. The polynomial \( Q_5(t) \) is positive function for \( t \in R \) therefore the polynomial \( Q_5(t) \) is monotonically increasing function for \( t \in R \). Further, the function \( Q_5(t) \) has real root in \( a_1 = 2.464993 \ldots > 2.4649 \) and \( Q_5(0) < 0 \), so we have that the function \( Q_5(t) < 0 \) for \( t \in (0, a_1) \) which follows that the function \( Q_{10}(x) < 0 \) for \( x \in (0, 1.57] \).

After all we can conclude following:

\[
\begin{align*}
Q_{10}(x) < 0 & \quad \text{for } x \in (0, 1.57] \\
\implies Q_{20}(x) > 0 & \quad \text{for } x \in (0, 1.57] \\
\implies f(x) > 0 & \quad \text{for } x \in (0, 1.57].
\end{align*}
\]

Let us remark that we can easily calculate the real root \( a_1 \) of the polynomial \( Q_5(t) \), and with arbitrary accuracy because \( Q_5(t) \) is a strictly increasing polynomial function. This also determines \( x^* = \sqrt{a_1} = 1.570029 \ldots (> 1.57) \) as the first positive root of the polynomial \( Q_{20}(x) \) defined at \( [24] \).

(A/II) \( x \in \left(1.57, \frac{\pi}{2}\right) \) Let us define the function:

\[
\begin{align*}
\hat{f}(x) &= f\left(\frac{\pi}{2} - x\right) = 1 - 8\left(\frac{\pi}{2} - x\right)^2 + \hat{h}_1(x)\cos 4x + \hat{h}_2(x)\cos 2x + \hat{h}_3(x)\sin 2x \\
&= 1 - 8\left(\frac{\pi}{2} - x\right)^2 - \cos 4x + 8\left(\frac{\pi}{2} - x\right)^2 \cos 2x \\
&\quad + \left(4\left(\frac{\pi}{2} - x\right) - 4\left(\frac{8}{45} - a\left(\frac{\pi}{2} - x\right)\right)\right)\sin 2x,
\end{align*}
\]

where \( x \in (0, c_1) \) for \( c_1 = \frac{\pi}{2} - 1.57 = \frac{\pi}{2} - \frac{457}{100} \approx 0.00079 \ldots \) and \( \hat{h}_1(x) = -1 < 0, \hat{h}_2(x) = 8\left(\frac{\pi}{2} - x\right)^2 > 0, \hat{h}_3(x) = 4\left(\frac{\pi}{2} - x\right) - 4\left(\frac{8}{45} - a\left(\frac{\pi}{2} - x\right)\right)\left(\frac{\pi}{2} - x\right)^5 \).

We are proving that the function \( \hat{f}(x) > 0 \).

Again, it is important to find sign of the polynomial \( \hat{h}_3(x) \). As we see, that polynomial is the polynomial of 7th degree or

\[
\begin{align*}
\hat{h}_3(x) &= \hat{P}_7(x) = -3245\pi^7 x^7 + 16135\pi^6 x^6 + 3245\pi^5 x^5 - 16\pi^4 + 4\pi^3 x^4 + \left(16\pi^2 \pi^2 + \frac{2}{27}\pi^3\right) x^3 \\
&\quad + \left(\frac{16}{9}\pi^2 - \frac{2}{27}\pi^4\right)x^2 + \left(-\frac{8}{9}\pi^3 + \frac{1}{45}\pi^5\right)x \\
&\quad + \left(-4 + \frac{2}{9}\pi - \frac{1}{270}\pi^6\right)x + 2\pi - \frac{1}{45}\pi^5 + \frac{1}{3780}\pi^7.
\end{align*}
\]
Using the factorization of the polynomial \( \hat{P}_7(x) \) we have

\[
\hat{P}_7(x) = \frac{1}{3780} (-2x + \pi) \left( 64x^6 - 192\pi x^5 + (240\pi^2 - 1344) x^4 \\
+ (-160\pi^3 + 2688\pi) x^3 + (60\pi^4 - 2016\pi^2) x^2 + (-12\pi^5 + 672\pi^3) x \\
+ \pi^6 - 84\pi^4 + 7560 \right) = \frac{1}{3780} (-2x + \pi) \hat{P}_6(x),
\]

where

\[
\hat{P}_6(x) = 64x^6 - 192\pi x^5 + (240\pi^2 - 1344) x^4 + (-160\pi^3 + 2688\pi) x^3 \\
+ (60\pi^4 - 2016\pi^2) x^2 + (-12\pi^5 + 672\pi^3) x + \pi^6 - 84\pi^4 + 7560,
\]

for \( x \in (0, c_1) \). The second derivate of the polynomial \( \hat{P}_6(x) \) is the polynomial of 4th degree

\[
\hat{P}_6''(x) = 1920x^4 - 3840\pi x^3 + (2880\pi^2 - 16128) x^2 \\
+ (-960\pi^3 + 16128\pi) x + 120\pi^4 - 4032\pi^2.
\]

Factorization of \( \hat{P}_6''(x) \) is given by

\[
\hat{P}_6''(x) = 24 (20x^2 - 20\pi x + (5\pi^2 - 168))(\pi - 2x)^2 = 24 (\pi - 2x)^2 \hat{P}_2(x),
\]

where

\[
\hat{P}_2(x) = 20x^2 - 20\pi x + (5\pi^2 - 168)
\]

is quadratic polynomial with two simple real roots:

\[
\hat{P}_2(x) = \alpha(x - x_1)(x - x_2),
\]

with values \( \alpha = 20, x_1 = -1.327 \ldots, x_2 = 4.469 \ldots \). It holds that next inequalities are true

\[
\hat{P}_2(x) < 0 \quad \text{for} \quad x \in (0, c_1) \subset (x_1, x_2)
\]

\[
\implies \hat{P}_6''(x) < 0 \quad \text{for} \quad x \in (0, c_1) \subset (x_1, x_2).
\]

Therefore, for chosen interval \( x \in (0, c_1) \) the polynomial \( \hat{P}_6''(x) \) has no roots. Since \( \hat{P}_6''(0) < 0 \), the polynomial \( \hat{P}_6''(x) \) is negative function for \( x \in (0, c_1) \) and \( \hat{P}_6(x) \) is monotonically decreasing function for \( x \in (0, c_1) \).

Furthermore, as the polynomial \( \hat{P}_6(c_1) > 0 \) it follows that the polynomial \( \hat{P}_6(x) \) is positive function for \( x \in (0, c_1) \), and the polynomial \( \hat{P}_6(x) \) is monotonically increasing function for \( x \in (0, c_1) \). Because of \( \hat{P}_6(0) > 0 \) we conclude following:

\[
\hat{P}_6(x) > 0 \quad \text{for} \quad x \in (0, c_1)
\]

\[
\implies \hat{P}_7(x) > 0 \quad \text{for} \quad x \in (0, c_1).
\]

According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: \( \cos y < T^{\cos, 0}_k(y)(k = 0) \), \( \cos y > T^{\cos, 0}_k(y)(k = 2) \),
\( \sin y > T^{\sin, 0}_k(y)(k = 3) \) are true, for \( y \in \left( 0, \sqrt{(k+3)(k+4)} \right) \). For \( x \in (0, c_1) \) it is valid:

\[
\hat{f}(x) = f \left( \frac{\pi}{2} - x \right) > \hat{Q}_{10}(x) = 1 - 8 \left( \frac{\pi}{2} - x \right)^2 - T^{\cos, 0}_0(4x)
\]

\[
+ 8 \left( \frac{\pi}{2} - x \right)^2 T^{\cos, 0}_2(2x) + \hat{P}_7(x)T^{\sin, 0}_3(2x),
\]
where $\hat{Q}_{10}(x)$ is the polynomial

$$
\hat{Q}_{10}(x) = \frac{128}{2835}x^{10} - \frac{64}{405}x^9 + \left(\frac{32}{135}\right)x^8 - \frac{64}{63}x^7 + \left(-\frac{16}{81}\right)x^6 + \left(+\frac{325}{135}\right)x^5
$$

$$
+ \left(\frac{8}{81}\pi^4 - \frac{368}{135}\pi^2 + \frac{440}{45}\right)x^4 + \left(-\frac{4}{135}\pi^6 + \frac{40}{27}\pi^4 - \frac{32}{9}\pi^2\right)x^3
$$

$$
+ \left(\frac{2}{405}\pi^6 - \frac{1}{9}\pi^4 + \frac{32}{9}\pi^2 - \frac{32}{3}\right)x^2 + \left(-\frac{1}{2835}\pi^7 + \frac{22680}{27}\pi^5 - \frac{160}{3}\pi^3 + \frac{40}{3}\pi\right)x
$$

$$
+ \left(-\frac{1}{135}\pi^6 + \frac{4}{9}\pi^4 - 4\pi^2 - 8\pi\right)x^2 + \left(\frac{1}{1890}\pi^7 - \frac{2}{45}\pi^5 + 4\pi\right)x
$$

$$
= -\frac{1}{5670}x(-2x + \pi)Q_8(x).
$$

Then, we have to determine sign of the polynomial

$$
\hat{Q}_8(x) = 128x^8 - 384\pi x^7 + (480\pi^2 - 2880) x^6 + (-320\pi^3 + 5952\pi) x^5
$$

$$
+ (120\pi^4 - 4752\pi^2 + 4032) x^4 + (-24\pi^5 + 1824\pi^3 - 8064\pi) x^3
$$

$$
+ (2\pi^6 - 348\pi^4 + 6048\pi^2 - 30240) x^2 + (36\pi^5 - 2016\pi^3 + 22680\pi) x
$$

$$
- 3\pi^6 + 252\pi^4 - 22680,
$$

for $x \in (0, c_1)$. The fourth derivate of the polynomial $\hat{Q}_8(x)$ is the polynomial of 4th degree

$$
\hat{Q}_8^{(iv)}(x) = 215040x^4 - 322560\pi x^3 + (172800\pi^2 - 1036800)x^2
$$

$$
+ (-38400\pi^3 + 714240\pi)x + 2880\pi^4 - 114048\pi^2 + 96768.
$$

Using MATLAB software we can determine the real numerical factorization of the polynomial

$$
\hat{Q}_8^{(iv)}(x) = \alpha(x - x_1)(x - x_2)(x - x_3)(x - x_4),
$$

with values $\alpha = 2.15 \ldots 10^5$, $x_1 = -0.976 \ldots$, $x_2 = 0.674 \ldots$, $x_3 = 1.505 \ldots$, $x_4 = 3.509 \ldots$. The polynomial $\hat{Q}_8^{(iv)}(x)$ has exactly four simple real roots with a symbolic radical representation and the corresponding numerical values: $x_1, x_2, x_3, x_4$. Therefore, the polynomial $\hat{Q}_8^{(iv)}(x)$ has no roots for $x \in (0, c_1)$. Since $\hat{Q}_8^{(iv)}(0) < 0$, we can conclude that the polynomial $\hat{Q}_8^{(iv)}(x)$ is negative function for $x \in (0, c_1)$, which follows that the function $\hat{Q}_8(x)$ is monotonically decreasing for $x \in (0, c_1)$. Doing the same procedure for all derivates up to $\hat{Q}_8(x)$ we have the following:

$$
\hat{Q}_8'(c_1) > 0 : \hat{Q}_8''(x) > 0 \text{ for } x \in (0, c_1) \implies \hat{Q}_8''(x) \nearrow \text{ for } x \in (0, c_1),
$$

$$
\hat{Q}_8''(c_1) < 0 : \hat{Q}_8''(x) < 0 \text{ for } x \in (0, c_1) \implies \hat{Q}_8''(x) \searrow \text{ for } x \in (0, c_1),
$$

$$
\hat{Q}_8'(c_1) > 0 : \hat{Q}_8'(x) > 0 \text{ for } x \in (0, c_1) \implies \hat{Q}_8'(x) \nearrow \text{ for } x \in (0, c_1).
$$

After all, we conclude following:

$$
\hat{Q}_8(x) < 0 \text{ for } x \in (0, c_1)
$$

$$
\implies \hat{Q}_{10}(x) > 0 \text{ for } x \in (0, c_1)
$$

$$
\implies f(x) = f\left(\frac{\pi}{2} - x\right) > 0 \text{ for } x \in (0, c_1)
$$

$$
\implies f(x) > 0 \text{ for } x \in \left(0, \frac{\pi}{2}\right).
$$

Hence we proved that the function $f(x)$ is positive on interval $x \in (0, 1.57)$ we conclude that the function $f(x)$ is positive on whole interval $x \in \left(0, \frac{\pi}{2}\right)$.
(B) Let us now prove the right side of inequality. If we write inequality in the following form
\[
\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{8}{45} - b_1(x)\right) x^3 \tan x \quad \text{for} \quad x \in \left(0, \frac{\pi}{2}\right),
\]  
where
\[
b_1(x) = \frac{8x^2}{945} - \frac{ax^4}{14175}, \quad a = \frac{480\pi^6 - 40320\pi^4 + 3628800}{\pi^8} = 17.15041 \ldots
\]  
The inequality is equivalent to the mixed trigonometric inequality
\[
f(x) = 8x^2 - 1 + h_1(x) \cos 4x + h_2(x) \cos 2x + h_3(x) \sin 2x = 8x^2 - 1 + \cos 4x + 8x^2 \cos 2x + (4 \left(\frac{\pi}{45} - b_1(x)\right) x^3 - 4x) \sin 2x > 0,
\]  
for \(x \in (0, \frac{\pi}{2})\), and \(h_1(x) = 1 > 0, h_2(x) = 8x^2 > 0, h_3(x) = 4 \left(\frac{\pi}{45} - b_1(x)\right) x^3 - 4x\).

Now let us consider two cases:

(B/I) \(x \in (0, 1.53]\) Let us determine sign of the polynomial \(h_3(x)\). As we see, that polynomial is the polynomial of 9th degree
\[
h_3(x) = P_9(x) = 4 \left(\frac{8}{45} - b_1(x)\right) x^5 - 4x = 4 \left(\frac{a}{14175}\right) x^9 - \frac{32}{45} \pi x^7 + \frac{32}{45} \pi x^5 - 4x.
\]  
Using factorization of the polynomial \(P_9(x)\) we have
\[
P_9(x) = (8\pi^6 - 672\pi^4 + 60480)x^6 + (-168\pi^6 + 15120\pi^2)x^4 + 3780\pi^4 x^2 + 945\pi^6)
\]  
for \(x \in (0, 1.53]\). Introducing the substitution \(s = x^2\) we can notice that the polynomial \(P_9(x)\) can be transformed into polynomial of 3rd degree
\[
P_3(s) = (8\pi^6 - 672\pi^4 + 60480)s^3 + (-168\pi^6 + 15120\pi^2)s^2 + 3780\pi^4 s + 945\pi^6,
\]  
\(s \in (0, 2.3409]\). Using MATLAB software we can determine the real numerical factorization of the polynomial
\[
P_3(s) = \alpha(s - s_1)(s^2 + ps + q),
\]  
where \(\alpha = 2712.294\ldots, s_1 = -2.221\ldots, p = -6.751\ldots, q = 150.759\ldots\) whereby the inequality \(p^2 - 4q < 0\) is true. The polynomial \(P_3(s)\) has exactly one real root with a symbolic radical representation and corresponding numerical value \(s_1\). Since \(P_3(0) > 0\) it follows that \(P_3(s) > 0\) for \(s \in (s_1, \infty)\), so we have following conclusions:
\[
\begin{align*}
P_3(s) > 0 & \quad \text{for} \quad s \in (0, 2.3409]\subset (s_1, \infty) \\
\implies P_9(x) > 0 & \quad \text{for} \quad x \in (0, 1.53]\subset (0, \frac{\pi}{2}) \\
\implies P_3(s) < 0 & \quad \text{for} \quad x \in (0, 1.53]\subset (0, \frac{\pi}{2}).
\end{align*}
\]  
According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: \(\cos y > \sum_{k=0}^{\infty} \cos^k(y)(k = 22), \cos y > \sum_{k=0}^{\infty} \cos^k(y)(k = 22)\).
14), \sin y < T_{6}^{\sin,0}(y)(k = 13) are true, for \( y \in \left(0, \sqrt{(k+3)(k+4)}\right) \). For \( x \in (0, 1.53] \) it is valid:

\[
f(x) > Q_{22}(x) = 8x^2 - 1 + 8x^2 P_{14}^{\cos,0}(2x) + P_{22}^{\cos,0}(4x) + P_{6}(x) T_{13}^{\sin,0}(2x),
\]

where \( Q_{22}(x) \) is the polynomial

\[
Q_{22}(x) = \left(\frac{-33554432}{2143861254109875} + \frac{1024}{140032} + \frac{4096}{273648375\pi^2} + \frac{8192}{6081075\pi^8}\right) x^{22}
+ \left(\frac{343732764375}{787456} + \frac{147349125\pi^2}{512} + \frac{7016625\pi^4}{4096} + \frac{155925\pi^8}{8192}\right) x^{20}
+ \left(\frac{97692469875}{4672} + \frac{267907\pi^2}{512} - \frac{127575\pi^4}{4096} + \frac{2835\pi^8}{8192}\right) x^{18}
+ \left(\frac{39092625}{588} + \frac{291767\pi^2}{512} + \frac{14175\pi^4}{4096} + \frac{15\pi^8}{3\pi}\right) x^{16}
+ \left(\frac{5108103}{2752} + \frac{14175\pi^2}{512} + \frac{675\pi^4}{4096} + \frac{15\pi^8}{3\pi}\right) x^{14}
+ \left(\frac{467775}{128} + \frac{2835\pi^2}{256} + \frac{135\pi^4}{4\pi} + \frac{3\pi^8}{\pi}\right) x^{12}
+ \left(\frac{1}{64} + \frac{945\pi^2}{45\pi} + \frac{\pi^4}{\pi}\right) x^{10}
= \frac{1}{2143861254109875} \pi^{10} Q_{12}(x).
\]

Then, we have to determine sign of the polynomial

\[
Q_{12}(x) = (524288\pi^8 - 5969040\pi^6 + 501399360\pi^4 - 45125942400) x^{12}
+ (-13646556\pi^8 + 232792560\pi^6 - 19554575040\pi^4 + 1759911753600) x^{10}
+ (270011280\pi^8 - 64017954000\pi^6 - 85791433936000\pi^4 + 8713557066968000) x^8
+ (-914714339360000\pi^8 + 6049696653000\pi^6 - 508174518852000\pi^4 + 4573570669668000) x^6
+ (-90745449795000\pi^8 + 762261778278000\pi^6 - 68603560045020000),
\]

for \( x \in (0, 1.53) \), which is the polynomial of 12\textsuperscript{th} degree. Introducing the substitution \( s = x^2 \) we can notice that the polynomial \( Q_{12}(x) \) can be transformed into polynomial of 6\textsuperscript{th} degree

\[
Q_{6}(s) = (524288\pi^8 - 5969040\pi^6 + 501399360\pi^4 - 45125942400) s^6
+ (-13646556\pi^8 + 232792560\pi^6 - 19554575040\pi^4 + 1759911753600) s^5
+ (270011280\pi^8 - 64017954000\pi^6 - 85791433936000\pi^4 + 8713557066968000) s^4
+ (-914714339360000\pi^8 + 6049696653000\pi^6 - 508174518852000\pi^4 + 4573570669668000) s^3
+ (90745449795000\pi^8 - 762261778278000\pi^6 - 68603560045020000) s^2
+ (-90745449795000\pi^8 + 762261778278000\pi^6 - 68603560045020000) s
+ 302484382650\pi^8 - 9074544979500\pi^6 + 762261778278000\pi^4
- 68603560045020000,
\]
for $s \in (0, 2.3409]$. The second derivate of the polynomial $Q_b(x)$ is the polynomial of 4th degree

\[
Q_b''(s) = 30(524288\pi^8 - 5969040\pi^6 + 501399360\pi^4 - 45125942400)s^4 + 20(-13646556\pi^8 + 232792560\pi^6 - 19554575040\pi^4 + 1759911753600)s^3 + 12(270011280\pi^8 - 6401795400\pi^6 + 537750813600\pi^4 - 4839757322400)s^2 + 6(-4003360515\pi^8 + 115232317200\pi^6 - 9679514644800\pi^4 + 87115631803200)s + 772244550000\pi^8 - 2419878661200\pi^6 + 203269807540800\pi^4 - 18294282678672000.
\]

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
Q_b''(s) = \alpha(s - s_1)(s - s_2)(s^2 + ps + q),
\]

with values $\alpha = 8.853 \ldots 10^{10}$, $s_1 = -3.45 \ldots$, $s_2 = 5.381 \ldots$, $p = -9.49 \ldots$, $q = 53.32 \ldots$. Also, holds that inequality $p^2 - 4q < 0$ is true. The polynomial $Q_b''(s)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values: $s_1, s_2$. Since we have that $Q_b''(0) < 0$ that follows $Q_b''(s) < 0$ for $s \in (0, 2.3409] \cup (s_1, s_2)$.

Further, the function $Q_b''(s)$ is monotonically decreasing function for $s \in (0, 2.3409]$, $Q_b(1.53) > 0$ and has the first positive root for $s = 2.472 \ldots$ which follows $Q_b(s) > 0$ for $s \in (0, 2.3409]$. The function $Q_b(s)$ is monotonically increasing for $s \in (0, 2.3409]$, has the first positive root $b = 2.358 \ldots$ and holds $Q_b(1.53) < 0$, which follows:

\[
Q_b(s) < 0 \quad \text{for} \quad s \in (0, 2.3409] \subset (0, b) \quad \Rightarrow \quad Q_{12}(x) > 0 \quad \text{for} \quad x \in (0, 1.53]
\]

\[
Q_{22}(x) > 0 \quad \text{for} \quad x \in (0, 1.53] \quad \Rightarrow \quad f(x) > 0 \quad \text{for} \quad x \in (0, 1.53].
\]

We can easily calculate the real root $b$ of the polynomial $Q_b(s)$, and with arbitrary accuracy because of the monotonic increasing of the polynomial function. This also applies to $x^* = \sqrt{b} = 1.53579 \ldots > 1.53$ (and $x^* < \frac{\pi}{2}$) which is the first positive root of the polynomial $Q_{22}(x)$ defined at (63).

(B/II) $x \in (1.53, \frac{\pi}{2})$ Let us define the function

\[
f(x) = f\left(\frac{\pi}{2} - x\right) = 8\left(\frac{\pi}{2} - x\right)^2 - 1 + h_1(x) \cos 4x + \hat{h}_2(x) \cos 2x + \hat{h}_3(x) \sin 2x
\]

\[
= 8\left(\frac{\pi}{2} - x\right)^2 - 1 + \cos 4x - 8\left(\frac{\pi}{2} - x\right)^2 \cos 2x + 4\left(\frac{\pi}{2} - x\right)^5 \sin 2x + 4\left(\frac{\pi}{2} - x\right)^5 \sin 2x,
\]

where $x \in (0, c_2)$ for $c_2 = \frac{\pi}{2} - 1.53 = \frac{\pi}{2} - \frac{153}{100} = 0.04079 \ldots$, and $\hat{h}_1(x) = 1 > 0$, $\hat{h}_2(x) = 8\left(\frac{\pi}{2} - x\right)^2 > 0$, $\hat{h}_3(x) = 4\left(\frac{8}{45} - b_1\left(\frac{\pi}{2} - x\right)\right)\left(\frac{\pi}{2} - x\right)^5 - 4\left(\frac{\pi}{2} - x\right)$.

We are proving that the function $f(x) > 0$. 

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Again, it is important to find sign of the polynomial $\hat{h}_3(x)$. As we see, that polynomial is the polynomial of $9^{th}$ degree

$$\hat{h}_3(x) = \hat{P}_3(x) = 4 \left( \frac{8}{45} - b_1 \left( \frac{\pi}{2} - x \right) \right) \left( \frac{\pi}{2} - x \right)^5 - 4 \left( \frac{\pi}{2} - x \right).$$

(67)

Let us determine the sign of the polynomial

$$\hat{P}_3(x) = \frac{1}{945\pi^8} \left( x(\pi - x)(\pi - 2x)(\pi^{12} - 12\pi^{11}x + 60\pi^{10}x^2 - 160\pi^9x^3 + 240\pi^8x^4 - 192\pi^7x^5 + 64\pi^6x^6 - 168\pi^5x^7 + 7056\pi^4x^8 - 192\pi^3x^9 + 483840\pi^2x^{10} + 1935360\pi x^{11} - 160\pi^9x^{12} + 16128\pi^8x^{13} - 1451520\pi^7x^{14} + 30240\pi^6x^{15} - 168\pi^5x^{16} + 111440\pi^4x^{17} + 665280\pi^3x^{18} + 1935360\pi^2x^{19} - 1451520\pi x^{20} + 483840\pi^2x^{21}) \right).$$

for $x \in (0, c_2)$ where

$$\hat{P}_3(x) = (64\pi^6 - 5376\pi^4 + 483840)x^6 + (-192\pi^7 + 16128\pi^5 - 1451520\pi)x^5 + (240\pi^8 - 2150\pi^6 + 1935360\pi^2)x^4 + (-160\pi^9 + 16128\pi^7 - 1451520\pi^3)x^3 + (60\pi^{10} - 7056\pi^8 + 665280\pi^4)x^2 + (-12\pi^{11} - 168\pi^9 + 14112\pi^{10} + 30240\pi^6)x + 120\pi^{10} - 14112\pi^8 + 1330560\pi^4.$$

(69)

The second derivate of the polynomial $\hat{P}_3(x)$ is the polynomial of $4^{th}$ degree

$$\hat{P}_6^{(2)}(x) = 30 (64\pi^6 - 5376\pi^4 + 483840)x^4 + 20 (-192\pi^7 + 16128\pi^5 - 1451520\pi)x^3 + 12 (240\pi^8 - 2150\pi^6 + 1935360\pi^2)x^2 + 6(-160\pi^9 + 16128\pi^7 - 1451520\pi^3)x + 120\pi^{10} - 14112\pi^8 + 1330560\pi^4.$$

(70)

The polynomial $\hat{P}_6^{(2)}(x)$ has no real numerical roots for interval $x \in (0, c_2)$ whereby the function $\hat{P}_6^{(2)}(x)$ is positive function for $x \in (0, c_2)$. That further means that the function $\hat{P}_6^{(2)}(x)$ is monotonically increasing function for $x \in (0, c_2)$. The function $\hat{P}_6^{(2)}(x)$ has root for $x = \frac{\pi}{2}$, also holds that $\hat{P}_6^{(2)}(c_2) < 0$, so we can conclude that $\hat{P}_6^{(2)}(x) < 0$ for $x \in (0, c_2)$ and the function $\hat{P}_6^{(2)}(x)$ is monotonically decreasing for $x \in (0, c_2)$. The function $\hat{P}_6^{(2)}(x)$ has no roots for $x \in (0, c_2)$ and $\hat{P}_6^{(2)}(c_2) > 0$ so we have the following:

$$\hat{P}_6^{(2)}(x) > 0 \quad \text{for} \quad x \in (0, c_2) \quad \implies \quad \hat{P}_6^{(2)}(x) < 0 \quad \text{for} \quad x \in (0, c_2).$$

(71)

According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: $\cos y > T_{k}^{\cos, 0}(y)(k = 2), \cos y < T_{k}^{\cos, 0}(y)(k = 4), \sin y < T_{k}^{\sin, 0}(y)(k = 1)$ are true, for $y \in \left(0, \sqrt{(k + 3)(k + 4)}\right)$. For $x \in (0, c_2)$ it is valid:

$$\hat{f}(x) = f \left( \frac{\pi}{2} - x \right) > \hat{Q}_{10}(x) = 8 \left( \frac{\pi}{2} - x \right)^2 - 1 + T_{4}^{\cos, 0}(4x) - 8 \left( \frac{\pi}{2} - x \right)^2 T_{4}^{\cos, 0}(2x) + \hat{P}_3(x) T_{4}^{\sin, 0}(2x),$$

(72)
where $Q_{10}(x)$ is the polynomial

$$Q_{10}(x) = \left( \frac{2048}{\pi^4} \frac{256}{945 \pi^2} + \frac{1024}{45 \pi^4} \right) x^{10} + \left( \frac{128}{165 \pi} - \frac{512}{5 \pi} \right) x^9 + \left( \frac{128}{165 \pi} - \frac{512}{5 \pi} \right) x^9 + \left( \frac{128}{165 \pi} - \frac{512}{5 \pi} \right) x^9$$

Then, we have to determine sign of the polynomial

Then, we have to determine sign of the polynomial

$$Q_8(x) = (128 \pi^6 - 10752 \pi^4 + 976780) x^8 + (576 \pi^7 + 48384 \pi^5$$

$$-4354560 \pi x^7 + (1120 \pi^8 - 96768 \pi^6 + 870912 \pi^2) x^6$$

$$+(-1232 \pi^9 + 112896 \pi^7 - 1016064 \pi^5) x^5 + (840 \pi^{10}$$

$$-81480 \pi^8 + 746408 \pi^4) x^4 + (-364 \pi^{11} + 38136 \pi^9$$

$$-3810240 \pi^7) x^2 + (98 \pi^{12} - 11802 \pi^{10} - 7560 \pi^8 + 1270080 \pi^6) x^2$$

$$+(-15 \pi^{13} + 2184 \pi^{11} + 7560 \pi^9 - 272160 \pi^7) x + \pi^{14} - 1685 \pi^{12}$$

$$-1890 \pi^{10} + 34020 \pi^8,$$

for $x \in (0, c_2)$. The fourth derivate of the polynomial $Q_8(x)$ is the polynomial of 4th degree

$$Q^{(4)}_8(x) = 1680(128 \pi^6 - 10750 \pi^4 + 976780) x^8 + 840(576 \pi^7 + 48384 \pi^5$$

$$-4354560 \pi x^7 + 360(1120 \pi^8 - 96768 \pi^6 + 870912 \pi^2) x^6$$

$$+120(-1232 \pi^9 + 112896 \pi^7 - 1016064 \pi^5) x + 20160 \pi^{10}$$

$$-1955520 \pi^8 + 182891520 \pi^4.$$

Using MATLAB software we can determine the real numerical factorization of the polynomial

$$Q^{(4)}_8(x) = \alpha (x^2 + p_1 x + q_1) (x^2 + p_2 x + q_2),$$

with values $\alpha = 7.29 \ldots 10^7$, $p_1 = -0.798 \ldots$, $q_1 = 1.417 \ldots$, $p_2 = -6.27 \ldots$, $q_2 = 11.111 \ldots$. Also, holds that inequalities $p_1^2 - 4q_1 < 0$ and $p_2^2 - 4q_2 < 0$ are true. The polynomial $Q^{(4)}_8(x)$ has no simple roots but has two pairs of complex conjugate. Roots and constants $p_1, q_1, p_2, q_2$ can be represented in symbolic form. The polynomial $Q^{(4)}_8(x)$ has no simple roots for $x \in \left(0, \frac{\pi}{2}\right)$ and $Q^{(4)}_8(0) > 0$. That means that $Q^{(4)}_8(x) > 0$ for $x \in (0, c_2) \subset \left(0, \frac{\pi}{2}\right)$ and the function $Q^{(4)}_8(x)$ is monotonically increasing for $x \in (0, c_2)$. Further, $Q^{(4)}_8(c_2) < 0$ and the function $Q^{(4)}_8(x)$ has the first positive root $x = 1.00733 \ldots$ which follows that $Q^{(4)}_8(x) < 0$ for $x \in (0, c_2) \subset (0, 1.00733 \ldots)$ and the function $Q^{(4)}_8(x)$ is monotonically decreasing function for $x \in (0, c_2)$. $Q^{(4)}_8(c_2) > 0$ and the function $Q^{(4)}_8(x)$ has the first positive root $x = 0.45455 \ldots$ which follows that $Q^{(4)}_8(x) > 0$ for $x \in (0, c_2) \subset (0, 0.45455 \ldots)$ and the function $Q^{(4)}_8(x)$ is monotonically increasing function for $x \in (0, c_2)$. $Q^{(4)}_8(0) > 0$ and the function $Q^{(4)}_8(x)$ has the first positive root $x = 1.16834 \ldots$ which follows that $Q^{(4)}_8(x) > 0$ for $x \in (0, c_2) \subset (0, 1.16834 \ldots)$ and the function $Q^{(4)}_8(x)$ is monotonically increasing function.
for \( x \in (0, c_2) \). Since we have that \( \hat{Q}_8(c_2) < 0 \) and the function \( \hat{Q}_8(x) \) has the first positive root \( x = 0.04383 \ldots \) we can conclude following:

\[
\hat{Q}_8(x) < 0 \quad \text{for} \quad x \in (0, c_2) \subset (0, 0.04383 \ldots)
\]

\[
\Rightarrow \quad \hat{Q}_{10}(x) > 0 \quad \text{for} \quad x \in (0, c_2)
\]

\[
\Rightarrow \quad \hat{f}(x) = f \left( \frac{\pi}{2} - x \right) > 0 \quad \text{for} \quad x \in (0, c_2)
\]

\[
\Rightarrow \quad f(x) > 0 \quad \text{for} \quad x \in \left(1.53, \frac{\pi}{2}\right).
\]

Hence we proved that the function \( f(x) \) is positive on interval \( x \in (0, 1.53) \) we conclude that the function \( f(x) \) is positive on whole interval \( x \in \left(0, \frac{\pi}{2}\right) \).

**The proof of Theorem 2.2.**

Transforming inequality \( \text{(7)} \) we have the following considerations.

*(C)* Let us prove the left side of the inequality

\[
2 + \left(\frac{16}{\pi^4} + c(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x},
\]

for \( x \in \left(0, \frac{\pi}{2}\right) \). The inequality \( \text{(78)} \) is equivalent to the mixed trigonometric inequality

\[
f(x) = \frac{1}{3} - 8x^2 + h_1(x) \cos 4x + h_2(x) \cos 2x + h_3(x) \sin 2x
\]

\[
= 1 - 8x^2 - \cos 4x - 8x^2 \cos 2x + \left(4x - 4 \left(\frac{16}{\pi^4} + c(x)\right)x^3\right) \sin 2x > 0,
\]

for \( x \in \left(0, \frac{\pi}{2}\right) \), and \( h_1(x) = -1 < 0, h_2(x) = -8x^2 < 0, h_3(x) = 4x - 4 \left(\frac{16}{\pi^4} + c(x)\right)x^3 \).

Now let us consider two cases:

*(C/I) \( x \in (0, 0.98) \)* Let us determine sign of the polynomial \( h_3(x) \). As we see, that polynomial is the polynomial of 6th degree

\[
h_3(x) = P_6(x) = 4x - 4 \left(\frac{16}{\pi^4} + c(x)\right)x^5 = \left(\frac{64}{\pi^4} - \frac{64}{\pi^4}\right)x^5 + \left(-\frac{384}{\pi^4} + \frac{32}{\pi^2}\right)x^5 + 4x.
\]

Using factorization of the polynomial \( P_6(x) \) we have

\[
P_6(x) = \frac{4(-2x + \pi)(8x^2 - 80)x^4 + 8\pi x^3 + 4\pi^2 x^3 + 2\pi^3 x + \pi^4) x}{\pi^5}
\]

\[
= \frac{4(-2x + \pi)x}{\pi^5} P_4(x),
\]

where

\[
P_4(x) = \left(8\pi^2 - 80\right)x^4 + 8\pi x^3 + 4\pi^2 x^2 + 2\pi^3 x + \pi^4,
\]

for \( x \in (0, 0.98) \). Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
P_4(x) = \alpha(x - x_1)(x - x_2)(x^2 + px + q),
\]

with values \( \alpha = -1.043 \ldots, x_1 = -1.524 \ldots, x_2 = 25.663 \ldots, p = 0.046 \ldots, q = 2.387 \ldots \). Also, holds that inequality \( \alpha^2 - 4q < 0 \) is true. The polynomial \( P_4(x) \) has exactly two simple real roots with a symbolic radical representation and corresponding numerical values \( x_1, x_2 \). Since \( P_4(0) > 0 \) it follows that \( P_4(x) > 0 \) for \( x \in (x_1, x_2) \), so we have following conclusion:

\[
P_4(x) > 0 \quad \text{for} \quad x \in (0, 0.98) \subset (x_1, x_2)
\]

\[
\Rightarrow \quad P_6(x) > 0 \quad \text{for} \quad x \in (0, 0.98).
\]
According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: \( \cos y < T_k^{\cos,0}(y)(k = 12) \), \( \cos y < T_h^{\cos,0}(y)(k = 8) \), \( \sin y > T_k^{\sin,0}(y)(k = 7) \) are true, for \( y \in \left(0, \sqrt{(k + 3)(k + 4)}\right) \). For \( x \in (0, 0.98) \) it is valid:

\[
f(x) > Q_{13}(x) = 1 - 8x^2 - T_4^{\cos,0}(4x) - 8x^2T_8^{\cos,0}(2x) + P_b(x)T_7^{\sin,0}(2x),
\]  

where \( Q_{13}(x) \) is the polynomial

\[
Q_{13}(x) = \left(-\frac{1024}{63\pi^4} + \frac{512}{315\pi^3}\right)x^{13} + \left(\frac{1024}{105\pi^4} - \frac{256}{315\pi^3} - \frac{16384}{467775}\right)x^{12}
+ \left(\frac{512}{3\pi^5} - \frac{256}{15\pi^4}\right)x^{11} + \left(-\frac{512}{5\pi^4} + \frac{128}{15\pi^3} + \frac{3376}{14175}\right)x^{10}
+ \left(-\frac{2560}{3\pi^5} + \frac{256}{3\pi^4}\right)x^9 + \left(\frac{512}{\pi^5} - \frac{128}{3\pi^4} + \frac{64}{3\pi^3}\right)x^8
+ \left(\frac{1280}{\pi^5} - \frac{128}{\pi^4}\right)x^7 + \left(-\frac{768}{\pi^4} + \frac{64}{\pi^3} + \frac{64}{45}\right)x^6
= \frac{-16}{467775\pi^5}x^6Q_7(x),
\]

for \( x \in (0, 0.98) \). Then, we have to determine sign of the polynomial

\[
Q_7(x) = (-47520\pi^2 + 475200)x^7 + (1024\pi^5 + 23760\pi^3 - 285120\pi)x^6
+ (498960\pi^2 - 4989600)x^5 + (-6963\pi^5 - 249480\pi^3
+ 2993760\pi)x^4 + (-2494800\pi^2 + 24948000)x^3 + (297000\pi^5
+ 1247400\pi^3 - 14968800\pi)x^2 + (3742200\pi^2 - 3742200)x
- 41580\pi^5 - 1871100\pi^3 + 22453200\pi,
\]

for \( x \in (0, 0.98) \), which is the polynomial of \( 7^{th} \) degree. The third derivate of the polynomial \( Q_7(x) \) is the polynomial of \( 4^{th} \) degree

\[
Q''_7(x) = 210(-47520\pi^2 + 475200)x^4 + 120(1024\pi^5 + 23760\pi^3
- 285120\pi)x^3 + 60(498960\pi^2 - 4989600)x^2 + 24(-6963\pi^5
- 2494800\pi^3 + 2993760\pi)x - 14968800\pi^2 + 149688000\pi.
\]

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
Q''_7(x) = \alpha(x - x_1)(x - x_2)(x - x_3)(x - x_4),
\]

with values \( \alpha = 1.301 \ldots 10^6, x_1 = -14.400 \ldots, x_2 = -0.776 \ldots, x_3 = 0.174 \ldots, x_4 = 0.768 \ldots \). The polynomial \( Q''_7(x) \) has exactly four simple real roots with a symbolic radical representation and the corresponding numerical values \( x_1, x_2, x_3, x_4 \).

The polynomial \( Q''_7(x) \) has two simple real roots on \( x \in \left(0, \frac{\pi}{2}\right) \) for \( x = x_3 \) and \( x = x_4 \). Also holds that \( Q''_7(0) > 0 \). That means that \( Q''_7(x) > 0 \) for \( x \in (0, x_3) \cup (x_4, \infty) \) and \( Q''_7(x) < 0 \) for \( x \in (x_3, x_4) \) so the function \( Q''_7(x) \) is monotonically increasing for \( x \in (0, x_3) \cup (x_4, \infty) \) and monotonically decreasing for \( x \in (x_3, x_4) \). \( Q''_7(0) > 0 \), and \( Q''_7(0.98) > 0 \) and the function \( Q''_7(x) \) has no real roots on \( x \in \left(0, \frac{\pi}{2}\right) \). That means that \( Q''_7(x) > 0 \) for \( x \in \left(0, \frac{\pi}{2}\right) \) so the function \( Q''_7(x) \) is monotonically increasing for \( x \in \left(0, \frac{\pi}{2}\right) \). \( Q''_7(0) < 0, Q''_7(0.98) > 0 \) the function \( Q''_7(x) \) has real root for \( x = 0.30395 \ldots \). That means that \( Q''_7(x) < 0 \) for \( x \in (0, 0.30395 \ldots) \) and \( Q''_7(x) > 0 \) for \( x \in (0.30395 \ldots, \infty) \) so the function \( Q''_7(x) \) is monotonically decreasing for
Let us note that $x^* = 0.98609\ldots$ is also the first positive root of the approximation of the function $f(x)$, i.e. of the polynomial $Q_{13}(x)$, defined at $x = 0.98$.

(C/II) $x \in (0.98, \frac{\pi}{2})$ Let us define the function

\[
\hat{f}(x) = f(x) = 1 - 8 \left( \frac{\pi}{2} - x \right)^2 + \hat{h}_1(x) \cos 4x + \hat{h}_2(x) \cos 2x + \hat{h}_3(x) \sin 2x \\
= 1 - 8 \left( \frac{\pi}{2} - x \right)^2 - \cos 4x + 8 \left( \frac{\pi}{2} - x \right)^2 \cos 2x \\
+ \left( 4 \left( \frac{\pi}{2} - x \right) - 4 \left( \frac{16}{\pi^2} + c \left( \frac{\pi}{2} - x \right) \right) \left( \frac{\pi}{2} - x \right)^5 \right) \sin 2x,
\]

where $x \in (0, c_3)$ for $c_3 = \frac{\pi}{2} - 0.98 = \frac{\pi}{2} - \frac{49}{50} (\approx 0.59079\ldots)$ and $\hat{h}_1(x) = -1 < 0$, $\hat{h}_2(x) = 8 \left( \frac{\pi}{2} - x \right)^2 > 0$, $\hat{h}_3(x) = 4 \left( \frac{\pi}{2} - x \right) - 4 \left( \frac{16}{\pi^2} + c \left( \frac{\pi}{2} - x \right) \right) \left( \frac{\pi}{2} - x \right)^5$.

We are proving that the function $\hat{f}(x) > 0$.

It is important to find sign of the polynomial $\hat{h}_3(x)$. As we see, that polynomial is the polynomial of $6^{th}$ degree

\[
\hat{h}_3(x) = \hat{P}_6(x) = 4 \left( \frac{\pi}{2} - x \right) - 4 \left( \frac{16}{\pi^2} + c \left( \frac{\pi}{2} - x \right) \right) \left( \frac{\pi}{2} - x \right)^5
\]

\[
= \left( \frac{640}{\pi^6} - \frac{64}{\pi^4} \right) x^6 + \left( -\frac{1536}{\pi^5} + \frac{160}{\pi^3} \right) x^5 + \left( \frac{1440}{\pi^4} - \frac{160}{\pi} \right) x^4 + \left( -\frac{640}{\pi^2} + 80 \right) x^3 + \left( \frac{120}{\pi} - 20 \pi \right) x^2 + (2\pi^2 - 4) x.
\]

Using factorization of the polynomial $\hat{P}_6(x)$ we have

\[
\hat{P}_6(x) = \frac{1}{\pi^6} (2x(\pi - 2x)(\pi^6 - 8\pi^5x + 24\pi^4x^2 - 32\pi^3x^3 + 16\pi^2x^4 - 2\pi^3x^5 + 56\pi^3x^2 - 208\pi^2x^2 + 304\pi x^3 - 160x^4)) = \frac{2x(\pi - 2x)}{\pi^5} \hat{P}_5(x),
\]

where

\[
\hat{P}_5(x) = (16\pi^2 - 160) x^4 + (304\pi - 32\pi^2) x^3 + (24\pi^4 - 208\pi^2) x^2 \\
+ (56\pi^3 - 8\pi^5) x + \pi^6 - 2\pi^4,
\]

for $x \in (0, c_3)$. Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
\hat{P}_4(x) = \alpha(x - x_1)(x - x_2)(x^2 + px + q),
\]

where $\alpha = -2.086\ldots$, $x_1 = -24.992\ldots$, $x_2 = 3.094\ldots$, $p = -3.188\ldots$, $q = 4.927\ldots$ whereby the inequality $p^2 - 4q < 0$ is true. The polynomial $\hat{P}_4(x)$ has exactly two simple real roots with a symbolic radical representation and the corresponding numerical values $x_1, x_2$. Since we have that $\hat{P}_5(0) > 0$ and knowing roots of the polynomial $\hat{P}_4(x)$ we have the following:

\[
\hat{P}_5(x) > 0 \quad \text{for} \quad x \in (0, c_3) \subset (x_1, x_2) \\
\Rightarrow \hat{P}_6(x) > 0 \quad \text{for} \quad x \in (0, c_3).
\]
According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: \( \cos y < \mathcal{T}_{\text{cos}}^\text{inv}(y)(k = 8) \), \( \cos y > \mathcal{T}_{\text{cos}}^\text{inv}(y)(k = 6) \), \( \sin y < \mathcal{T}_{\text{sin}}^\text{inv}(y)(k = 7) \) are true, for \( y \in \left(0, \sqrt{(k+3)(k+4)} \right) \). For \( x \in (0, c_3) \) it is valid:

\[
\hat{f}(x) = f\left(\frac{\pi}{2} - x\right) > \hat{Q}_{13}(x) = 1 - 8\left(\frac{\pi}{2} - x\right)^2 - \mathcal{T}_{\text{cos}}(4x) + 8\left(\frac{\pi}{2} - x\right)^2 \mathcal{T}_{\text{cos}}(2x) + \mathcal{P}_0(x) \mathcal{T}_{\text{sin}}(2x),
\]

where \( \hat{Q}_{13}(x) \) is the polynomial

\[
\hat{Q}_{13}(x) = \left(\frac{1024}{63\pi^4} + \frac{512}{315\pi^4}\right)x^{13} + \left(\frac{4096}{105\pi^4} - \frac{256}{63\pi^4}\right)x^{12} + \left(\frac{512}{3\pi^2} - \frac{5632}{105\pi^4}\right)x^{11} + \left(\frac{2560}{3\pi^4} + \frac{32}{63}\pi - \frac{320}{7}\pi + \frac{1408}{3\pi^3}\right)x^9 + \left(\frac{736}{3\pi^4} + \frac{2048}{\pi^3} + \frac{1280}{3\pi^4} + \frac{208}{45}\pi\right)x^7 + \left(\frac{480}{\pi} + \frac{2880}{3\pi^3} + \frac{64}{3}\pi\right)x^5 + \left(\frac{240}{\pi} - 150\pi\right)x^3
\]

Then we have to determine the sign of the polynomial

\[
\hat{Q}_{10}(x) = \left(-128\pi^2 - 1280 \right) x^{10} + \left(320\pi^3 - 3072\pi\right)x^8 + \left(-320\pi^4 + 4224\pi^2 - 13440\right)x^6 + \left(160\pi^5 - 4640\pi^3 + 32256\pi\right)x^4 + \left(-40\pi^6 + 22160\pi^4 - 167200\right)x^2 + \left(4\pi^7 - 1504\pi^5 + 23040\pi^3 - 161280\pi\right)x + \left(64\pi^6 - 19320\pi^4 + 161280\pi^2 - 100800\pi + 1980\pi^5 - 226800\pi^3 + 10080\pi^3\right)x
\]

for \( x \in (0, c_3) \) which is the polynomial of 10\textsuperscript{th} degree. The sixth derivate of the polynomial \( \hat{Q}_{10}(x) \) is the polynomial of 4\textsuperscript{th} degree

\[
\hat{Q}_{10}^{(vi)}(x) = \left(151200(-128\pi^2 - 1280)x^4 + 60480(320\pi^3 - 3072\pi)x^3 + \right.\left.20160(-320\pi^4 + 4224\pi^2 - 13440)x^2 + 5040(160\pi^5 - 4640\pi^3 + 32256\pi)x + 122880\pi^2 - 259200\pi^4 + 2661120\pi^2 + 48384000\right)
\]

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
\hat{Q}_{10}^{(vi)}(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4),
\]

with values \( \alpha = 2.523 \cdots 10^6 \), \( x_1 = -9.183 \ldots \), \( x_2 = -0.226 \ldots \), \( x_3 = 1.117 \ldots \), \( x_4 = 1.796 \ldots \). The polynomial \( \hat{Q}_{10}^{(vi)}(x) \) has exactly four simple real roots with a symbolic radical representation and the corresponding numerical values: \( x_1, x_2, x_3, x_4 \).

Since polynomial \( \hat{Q}_{10}^{(vi)}(x) \) has root for \( x = x_3 \) whereby the \( \hat{Q}_{10}^{(vi)}(0) > 0 \) we have the following \( \hat{Q}_{10}^{(vi)}(x) > 0 \) for \( x \in (0, c_3) \) and also the polynomial \( \hat{Q}_{10}^{(vi)}(x) \) is monotonically increasing function for \( x \in (0, c_3) \).

Further, \( \hat{Q}_{10}^{(vi)}(x) \) has the first positive root for \( x = 0.16300 \ldots \) and \( \hat{Q}_{10}^{(vi)}(c_3) > 0 \) which gives us that \( \hat{Q}_{10}^{(vi)}(x) < 0 \) for \( x \in (0, 0.16300 \ldots) \) and \( \hat{Q}_{10}^{(vi)}(x) > 0 \) for \( x \in (0, c_3) \).
\(Q\), \(Q\), \(Q\) respectively, \(Q\) is monotonically increasing function for \(x \in (0, 0.16300 \ldots c_1)\), also \(\hat{Q}^{(iv)}(x)\) is monotonically decreasing function for \(x \in (0, 0.16300 \ldots c_3)\). \(\hat{Q}^{(iv)}(x)\) has the first positive root for \(x = 0.55589 \ldots x\) and \(\hat{Q}^{(iv)}(0) < 0 \) and \(\hat{Q}^{(iv)}(c_3) > 0\) which gives us that \(\hat{Q}^{(iv)}(0) < 0\) for \(x \in (0, 0.55589 \ldots)\) and \(\hat{Q}^{(iv)}(x) > 0\) for \(x \in (0.55589 \ldots c_3)\), also \(\hat{Q}^{(iv)}(0) < 0\) for \(x \in (0, 0.55589 \ldots)\) and monotonically increasing function for \(x \in (0.55589 \ldots c_3)\). \(\hat{Q}^{(iv)}(x)\) has no root for \(x \in (0, c_3)\) and \(\hat{Q}^{(iv)}(0) > 0\) and \(\hat{Q}^{(iv)}(0) > 0\) which gives us that \(\hat{Q}^{(iv)}(x) > 0\) for \(x \in (0, c_3)\), also \(\hat{Q}^{(iv)}(x)\) is monotonically increasing function for \(x \in (0, 0.16300 \ldots c_3)\).

\(\hat{Q}^{(iv)}(x)\) has the first positive root for \(x = 0.64192 \ldots x\) and \(\hat{Q}^{(iv)}(c_3) < 0\) which gives us that \(\hat{Q}^{(iv)}(x) < 0\) for \(x \in (0, c_3) \subset (0, 0.64192 \ldots)\). \(\hat{Q}^{(iv)}(x)\) is monotonically decreasing function for \(x \in (0, c_3)\).

\(\hat{Q}^{(iv)}(x)\) has no real root for \(x \in (0, c_3)\) and \(\hat{Q}^{(iv)}(c_3) > 0\) which gives us that \(\hat{Q}^{(iv)}(x) > 0\) for \(x \in (0, c_3)\), also \(\hat{Q}^{(iv)}(x)\) is monotonically increasing function for \(x \in (0, c_3)\). \(\hat{Q}^{(iv)}(x)\) has real root \(x = 0.66825 \ldots\) and \(\hat{Q}^{(iv)}(c_3) < 0\) which gives us following

\[
\begin{align*}
\phi_{10}(x) &< 0 \quad \text{for } x \in (0, c_3) \subset (0, 0.66825 \ldots) \\
\Rightarrow \quad Q_{10}(x) &> 0 \quad \text{for } x \in (0, c_3) \\
\Rightarrow \quad \hat{\varphi}(x) = f\left(\frac{\pi}{2} - x\right) &> 0 \quad \text{for } x \in (0, c_3) \\
\Rightarrow \quad f(x) &> 0 \quad \text{for } x \in \left(0.98, \frac{\pi}{2}\right). 
\end{align*}
\]

Hence we proved that the function \(f(x)\) is positive for \(x \in (0, 0.98)\), we conclude that the function \(f(x)\) is positive for whole interval \(x \in \left(0, \frac{\pi}{2}\right)\).

\(D\) Let us now prove the right side of the inequality

\[
\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{16}{\pi^2} + d(x)\right) x^3 \tan x \quad \text{for } x \in \left(0, \frac{\pi}{2}\right). 
\]

The inequality \(103\) is equivalent to the mixed trigonometric inequality

\[
f(x) = 8x^2 - 1 + h_1(x) \cos 4x + h_2(x) \cos 2x + h_3(x) \sin 2x
\]
\[
= 8x^2 - 1 + \cos 4x + 8x^2 \cos 2x + \left(4 \left(\frac{16}{\pi^2} + d(x)\right) x^3 \sin 2x\right) > 0, 
\]

for \(x \in (0, \frac{\pi}{2})\), and \(h_1(x) = 1 > 0\), \(h_2(x) = 8x^2 > 0\), \(h_3(x) = 4 \left(\frac{16}{\pi^2} + d(x)\right) x^5 - 4x\).

Now, let us consider two cases:

\(D/D\) \(x \in (0, 1.43)\) Let us determine sign of the polynomial \(h_3(x)\). As we see, that polynomial is the polynomial of 7th degree

\[
h_3(x) = P_7(x) = 4 \left(\frac{16}{\pi^2} + d\right) x^5 - 4x
\]
\[
= 4 \left(\frac{16}{\pi^2} + \frac{160}{\pi^3} - \frac{16}{\pi^3}\right) \left(\frac{\pi}{2} - x\right) + \left(\frac{960}{\pi^5} - \frac{96}{\pi^4}\right) \left(\frac{\pi}{2} - x\right)^3 x^5 - 4x 
\]
\[
= \frac{3840}{\pi^6} - \frac{384}{\pi^2} x^2 + \left(-\frac{480}{\pi^5} + \frac{448}{\pi^4}\right) x^6 + \left(\frac{1344}{\pi^4} - \frac{128}{\pi^2}\right) x^5 - 4x. 
\]

Using factorization of the polynomial \(P_7(x)\) we have

\[
P_7(x) = -4x(-2x + \pi)
\]
\[
\frac{(32\pi^2 x^4 - 48 \pi^2 x^3 + \pi^3 + 2 \pi^4 x + 4 \pi^3 x^2 + 8 \pi^2 x^3 - 320 \pi x^4 + 480 \pi^5)}{\pi^6}
\]
\[
= -4x(-2x + \pi)P_5(x) 
\]

\[
= -4x(-2x + \pi)x^5, 
\]
where

\[ P_5(x) = (480 - 48\pi^2)x^5 + (32\pi^3 - 320\pi)x^4 + 8\pi^2 x^3 + 4\pi^3 x^2 + 2\pi^4 x + \pi^5, \]  
(107)

for \( x \in (0, 1.43) \). The first derivate of the polynomial \( P_5(x) \) is the polynomial of 4th degree

\[ P_5'(x) = 5(480 - 48\pi^2)x^4 + 4(32\pi^3 - 320\pi)x^3 + 24\pi^2 x^2 + 8\pi^3 x + 2\pi^4. \]  
(108)

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[ P_5'(x) = \alpha(x^2 + p_1 x + q_1)(x^2 + p_2 x + q_2), \]  
(109)

where \( \alpha = 31.294 \ldots, p_1 = 1.004 \ldots, q_1 = 0.647 \ldots, p_2 = -2.68 \ldots, q_2 = 9.614 \ldots \)

whereby the inequalities \( p_1^2 - 4q_1 < 0 \) and \( p_2^2 - 4q_2 < 0 \) are true.

The polynomial \( P_5'(x) \) has no real roots for interval \( x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).

Further, the polynomial \( P_5(x) \) also has no real roots for \( x \in \left( 0, \frac{\pi}{2} \right) \).

Since the function \( P_5(x) \) has real roots at \( x = 0 \) and \( x = \frac{\pi}{2} \), we have the following conclusion

\[
\begin{align*}
P_5(x) &> 0 \quad \text{for} \quad x \in (0, 1.43] \\
\implies P_5'(x) &< 0 \quad \text{for} \quad x \in (0, 1.43].
\end{align*}
\]  
(110)

According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: \( \cos y > \sum k=0 (y)(k = 10), \sin y < T_k^{\sin, 0}(y)(k = 1) \)

are true, for \( y \in \left( 0, \sqrt{(k + 3)}(k + 4) \right) \). For \( x \in (0, 1.43] \) it is valid:

\[
f(x) > Q_{12}(x) = 8x^2 - 1 + x^{5/2} \sum_{k=0}^{10} (4k) + 8x^2 \sum_{k=0}^{10} (2k) + P_5(x)T_{1}^{\sin, 0}(2x),
\]  
(111)

where \( Q_{12}(x) \) is the polynomial

\[
Q_{12}(x) = -\frac{32}{14175} x^6 + \left( \frac{512}{\pi^6} + \frac{512}{\pi^4} - \frac{3376}{14175} \right) x^{10} + \left( \frac{17920}{3\pi^6} - \frac{1792}{3\pi^4} \right) x^9 + \left( \frac{7680}{\pi^6} - \frac{2560}{\pi^4} + \frac{512}{3\pi^2} + \frac{32}{35} \right) x^8 + \left( \frac{860}{\pi^6} + \frac{860}{\pi^4} \right) x^7 + \left( \frac{2688}{\pi^6} - \frac{256}{\pi^4} - \frac{16}{45} \right) x^6
\]  
(112)

Then, we have to determine sign of the polynomial

\[
Q_6(x) = 2\pi^6 x^6 + (211\pi^6 - 453600\pi^2 + 4536000)x^4 + (529200\pi^3 - 529200\pi)x^3 + (-810\pi^6 - 151200\pi^4 + 226800\pi^2 - 6804000)x^2
\]  
(113)

and

\[
+ (-793800\pi^3 + 793800\pi)x + 315\pi^6 + 226800\pi^4 - 2381400\pi^2,
\]

for \( x \in (0, 1.43] \). The second derivate of the polynomial \( Q_6(x) \) is the polynomial of 4th degree

\[
Q_6''(x) = 60\pi^6 x^4 + 12(211\pi^6 - 453600\pi^2 + 4536000)x^2 + 6(529200\pi^3 - 529200\pi)x - 1620\pi^6 - 302400\pi^4 + 4536000\pi^2 - 13608000,
\]  
(114)

Using MATLAB software we can determine the real numerical factorization of the polynomial
Further, it is important to find sign of the polynomial $\hat{h}(x)$ using factorization of the polynomial $Q_6(x)$. Since the function $Q_6(x)$ has no real roots for interval $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $Q_6(0) > 0$ which gives that $Q_6''(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, and it means that the function $Q_6(x)$ is monotonically increasing function for $x \in \left(0, \frac{\pi}{2}\right)$.

Further, the polynomial $Q_6(x)$ also has no real roots for $x \in \left(0, \frac{\pi}{2}\right)$, $Q_6(0) > 0$, which gives that $Q_6(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$.

Since the function $Q_6(x)$ has real roots at $x = 1.436 \ldots$ and $Q_6(0) = -1.108 \ldots 10^6 < 0$ we have the following:

$$Q_6(x) > 0 \text{ for } x \in (0, 1.43], \quad Q_6(x) < 0 \text{ for } x \in (0, 1.43], \quad Q_{12}(x) > 0 \text{ for } x \in (0, 1.43], \quad f(x) > 0 \text{ for } x \in (0, 1.43].$$

Let us notice that $x^* = 1.43649 \ldots$ is also the first positive root of the approximation of the function $f(x)$, i.e. of the polynomial $Q_{12}(x)$, defined at $\textbf{112}$.

**(D/II) $x \in \left(1.43, \frac{\pi}{2}\right)$** Let us define the function

$$\hat{f}(x) = f'(\frac{\pi}{2} - x) = 8 \left(\frac{\pi}{2} - x\right)^2 - 1 + \hat{h}_1(x) \cos 4x + \hat{h}_2(x) \cos 2x + \hat{h}_3(x) \sin 2x$$

$$= 8 \left(\frac{\pi}{2} - x\right)^2 - 1 + \cos 4x - 8 \left(\frac{\pi}{2} - x\right)^2 \cos 2x$$

$$+ \left(4 \left(\frac{16}{\pi^4} + d \left(\frac{\pi}{2} - x\right)\right)\left(\hat{h}_3(x) = 4 \left(\frac{16}{\pi^4} + d \left(\frac\pi2 - x\right)\right)\left(\frac\pi2 - x\right)^5 - 4 \left(\frac\pi2 - x\right)\right)\right) \sin 2x,$$

where $x \in (0, c_4)$ for $c_4 = \frac{\pi}{2} - 1.43 = \frac{\pi}{2} - \frac{143}{100} = 0.14 \ldots$ and $\hat{h}_1(x) = 1 > 0$, $\hat{h}_2(x) = 8 \left(\frac{\pi}{2} - x\right)^2 - 1 > 0$, $\hat{h}_3(x) = 4 \left(\frac{16}{\pi^4} + d \left(\frac\pi2 - x\right)\right)\left(\frac\pi2 - x\right)^5 - 4 \left(\frac\pi2 - x\right)$.

We are proving that the function $\hat{f}(x) > 0$.

Further, it is important to find sign of the polynomial $\hat{h}_3(x)$. As we see, that polynomial is the polynomial of $\pi^{th}$ degree

$$\hat{h}_3(x) = \hat{P}_7(x) = 4 \left(\frac{16}{\pi^4} + d \left(\frac\pi2 - x\right)\right)\left(\frac\pi2 - x\right)^5 - 4 \left(\frac\pi2 - x\right)$$

$$= 4 \left(\frac{16}{\pi^4} + \frac{160}{\pi^4} - \frac{16}{\pi^4}\right) x + \left(960 \frac{\pi^6}{\pi^4} - 96 \frac{\pi^6}{\pi^4}\right) x^2 \left(\frac\pi2 - x\right)^5 - 2\pi + 4x$$

$$= \left(\frac{3840}{\pi^8} + \frac{384}{\pi^4}\right) x^5 + \left(\frac{8960}{\pi^8} - \frac{896}{\pi^4}\right) x^3 \left(\frac\pi2 - x\right)^5 + \left(-\frac{8064}{\pi^8} + \frac{800}{\pi^4}\right) x^5$$

$$+ \left(\frac{3360}{\pi^8} - \frac{320}{\pi^4}\right) x^4 + \left(40 - \frac{560}{\pi^2}\right) x^3 + 8\pi x^2 + (-2\pi^2 + 4 x).$$

Using factorization of the polynomial $\hat{P}_7(x)$ we have:

$$\hat{P}_7(x) = -\frac{1}{\pi^6} \left(2x(\pi - 2x)(\pi^7 - 2\pi^5 x - 24\pi^7 x^2 + 112\pi^4 x^3 - 176\pi^3 x^4 + 96\pi^2 x^5 - 2\pi^5 - 3\pi^4 x + 272\pi^3 x^2 - 1136\pi^2 x^3 + 1760\pi x^4 - 960\pi^5)\right)$$

$$= -\frac{2x(\pi - 2x)}{\pi^6} \hat{P}_3(x),$$
\begin{align*}
\dot{P}_0(x) &= (96\pi^2 - 960)x^5 + (1760\pi - 176\pi^3)x^4 + (112\pi^4 - 1136\pi^2)x^3 \\
&\quad + (272\pi^3 - 24\pi^5)x^2 - (4\pi^4 + 2\pi^6)x + \pi^7 - 2\pi^5, \\
\end{align*}

(120)

for $x \in (0, c_4)$. The first derivate of the polynomial $\dot{P}_0(x)$ is the polynomial of 4th degree

\begin{align*}
\dot{P}_0'(x) &= 5(96\pi^2 - 960)x^4 + 4(1760\pi - 176\pi^3)x^3 + 3(112\pi^4 - 1136\pi^2)x^2 \\
&\quad + 2(272\pi^3 - 24\pi^5)x - (4\pi^4 + 2\pi^6),
\end{align*}

(121)

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[ \dot{P}_0(x) = \alpha(x^2 + p_1 x + q_1)(x^2 + p_2 x + q_2), \]

(122)

where $\alpha = -62.589 \ldots$, $p_1 = -0.461 \ldots$, $q_1 = 7.871 \ldots$, $p_2 = -4.146 \ldots$, $q_2 = 4.693 \ldots$ whereby the inequalities $p_1^2 - 4q_1 < 0$ and $p_2^2 - 4q_2 < 0$ are true. 

The polynomial $\dot{P}_0(x)$ has no real roots for interval $x \in (0, c_4)$, $\dot{P}_0(0) < 0$ which gives that $\dot{P}_0'(x) < 0$ for $x \in (0, c_4)$, and it means that the function $\dot{P}_0(x)$ is monotonically increasing function for $x \in (0, c_4)$. Further, the polynomial $\dot{P}_0(x)$ also has no real roots for $x \in (0, c_4)$, $\dot{P}_0(0) > 0$, which gives that $\dot{P}_0'(x) > 0$ for $x \in (0, c_4)$.

Since the function $\dot{P}_0(x)$ has first positive root at $x = \frac{\pi}{2}$ and $\dot{P}_0(0) = 0$ we have the following:

\[ \dot{P}_0(x) > 0 \quad \text{for} \quad x \in (0, c_4) \]

(123)

According to the Lemmas 2.3. and 2.4. and description of the method based on (14) and (17), the following inequalities: $\cos y > T_{k,0}^{\cos,0}(y)(k = 6)$, $\cos y < T_{k,0}^{\cos,0}(y)(k = 4)$, $\sin y < T_{k,0}^{\sin,0}(y)(k = 5)$ are true, for $y \in (0, \sqrt{(k + 3)(k + 4)})$. For $x \in (0, c_4)$ it is valid:

\begin{align*}
\hat{f}(x) &= f\left(\frac{\pi}{2} - x\right) > \dot{Q}_{12}(x) = 8 \left(\frac{\pi}{2} - x\right)^2 - 1 + \sum_{k=1}^{\infty} T_{k,0}^{\cos,0}(4x) \\
&\quad - 8 \left(\frac{\pi}{2} - x\right)^2 T_{4}^{\cos,0}(2x) + \dot{P}_0(x)\dot{T}_{4}^{\sin,0}(2x),
\end{align*}

(124)

where $\dot{Q}_{12}(x)$ is the polynomial

\begin{align*}
\dot{Q}_{12}(x) &= -\frac{1024}{\pi^6} + \frac{512}{\pi^4} + \frac{7168}{3\pi^2} + \frac{3584}{15\pi^2} \left(\frac{\pi^6}{\pi^6}\right) x^{11} \\
&\quad + \frac{5120}{\pi^6} + \frac{13312}{3\pi^4} + \frac{640}{\pi^2} \left(\frac{\pi^6}{\pi^6}\right) x^{10} + \left(\frac{3584}{3\pi^6} + \frac{6272}{\pi^4} + \frac{256}{3\pi^4} \right) x^9 \\
&\quad + \left(\frac{7680}{\pi^6} + \frac{11520}{\pi^4} + \frac{1216}{\pi^2} + \frac{32}{3} \right) x^8 + \left(\frac{17920}{\pi^6} + \frac{1280}{\pi^4} + \frac{32}{15} \right) x^7 \\
&\quad + \left(\frac{16128}{\pi^6} + \frac{7040}{3\pi^2} + \frac{8}{15} \right) x^6 + \left(\frac{16}{15} + \frac{6720}{\pi^6} \right) x^5 \\
&\quad + \left(\frac{1120}{\pi^2} + \frac{4}{3} \right) x^4 + \frac{304}{3} x^3 \\
&\quad = -\frac{4 x^4}{45 \pi^2} \dot{Q}_6(x).
\end{align*}

(125)

Then, we have to determine sign of the polynomial

\begin{align*}
\dot{Q}_6(x) &= (-1152\pi^2 + 11520)x^6 + (2688\pi^3 - 26880\pi)x^7 + (-2400\pi^4 \\
&\quad + 29952\pi^2 - 57600)x^6 + (960\pi^5 - 23520\pi^3 + 134400\pi)x^5 \\
&\quad + (-120\pi^6 + 13680\pi^3 - 129600\pi^2 + 86400)x^4 + (-24\pi^7 \\
&\quad - 4800\pi^5 + 70560\pi^3 - 201600\pi)x^3 + (6\pi^6 + 712\pi^6 - 26400\pi^4 \\
&\quad + 171440\pi^2)x^2 + (60\pi^7 + 720\pi^5 - 75600\pi^3)x - 15\pi^8 \\
&\quad - 1140\pi^6 + 12600\pi^4,
\end{align*}

(126)
for \( x \in (0, c_4) \). The fourth derivative of the polynomial \( \hat{Q}_8(x) \) is the polynomial of 4\(^{th}\) degree

\[
\hat{Q}_8^{(iv)}(x) = 1680(-1152\pi^2 + 11520) x^4 + 840(2688\pi^3 - 26880\pi)x^3 \\
+ 360(-2400\pi^4 + 29952\pi^2 - 57600)x^2 + 120(960\pi^5 - 23520\pi^3) \\
+ 134400\pi)x - 2880\pi^6 + 328320\pi^4 - 3110400\pi^2 + 2073600.
\]

Using MATLAB software we can determine the real numerical factorization of the polynomial

\[
\hat{Q}_8^{(iv)}(x) = \alpha(x - x_1)(x - x_2)(x^2 + px + q),
\]

where \( \alpha = 2.523\ldots 10^5 \), \( x_1 = 0.627\ldots \), \( x_2 = 1.89\ldots \), \( p = -1.146\ldots \), \( q = 1.963\ldots \) whereby the inequality \( p^2 - 4q < 0 \) is true.

The polynomial \( \hat{Q}_8^{(iv)}(x) \) has no real roots for interval \( x \in (0, c_4) \), \( \hat{Q}_8^{(iv)}(0) > 0 \) which gives that \( \hat{Q}_8^{(iv)}(x) > 0 \) for \( x \in (0, c_4) \), and it means that the function \( \hat{Q}_8^{(iv)}(x) \) is monotonically increasing function for \( x \in (0, c_4) \).

Further, the polynomial \( \hat{Q}_8''(x) \) also has no real roots for \( x \in (0, c_4) \), \( \hat{Q}_8''(0) > 0 \), which gives that \( \hat{Q}_8''(x) > 0 \) for \( x \in (0, c_4) \), and means that polynomial \( \hat{Q}_8''(x) \) is monotonically increasing function for \( x \in (0, c_4) \). The polynomial \( \hat{Q}_8''(x) \) also has no real roots for \( x \in (0, c_4) \), \( \hat{Q}_8''(c_4) < 0 \), which gives that \( \hat{Q}_8''(x) < 0 \) for \( x \in (0, c_4) \), and means that polynomial \( \hat{Q}_8''(x) \) is monotonically decreasing function for \( x \in (0, c_4) \). The polynomial \( \hat{Q}_8''(x) \) also has no real roots for \( x \in (0, c_4) \), \( \hat{Q}_8''(c_4) > 0 \), which gives that \( \hat{Q}_8''(x) > 0 \) for \( x \in (0, c_4) \), and means that polynomial \( \hat{Q}_8''(x) \) is monotonically increasing function for \( x \in (0, c_4) \). The polynomial \( \hat{Q}_8(x) \) has first positive real root at \( x = 0.38641\ldots > c_4 \), \( \hat{Q}_8(c_4) < 0 \), which gives the following:

\[
\begin{align*}
\hat{Q}_8(x) &< 0 \quad \text{for } x \in (0, c_4) \\
\implies \hat{Q}_{12}(x) &> 0 \quad \text{for } x \in (0, c_4) \\
\implies \hat{f}(x) &= f\left(\frac{\pi}{2} - x\right) > 0 \quad \text{for } x \in (0, c_4) \\
\implies f(x) &> 0 \quad \text{for } x \in \left(1.43, \frac{\pi}{2}\right).
\end{align*}
\]

Hence we proved that the function \( f(x) \) is positive for \( x \in (0, 1.43] \), we conclude that the function \( f(x) \) is positive for whole interval \( x \in \left(0, \frac{\pi}{2}\right) \).

4. Conclusion

With proving Theorem 2.1. and Theorem 2.2. is proved that is possible to extend interval defined for inequalities given in Theorem 1.1. by [4] and Theorem 1.2. by [5]. The subject of future paper work is to determine the maximum interval for which the inequalities given in previous theorems are true.

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