EQUIVARIANT HOPF GALOIS EXTENSIONS AND HOPF CYCLIC COHOMOLOGY

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Abstract
We define the notion of equivariant $\times$-Hopf Galois extension and apply it as a functor between the categories of SAYD modules of the $\times$-Hopf algebras involving in the extension. This generalizes the result of Jara-Stefan and Böhm-Stefan on associating a SAYD modules to any ordinary Hopf Galois extension.

Introduction
Hopf cyclic cohomology, which is now a well known cohomology theory in noncommutative geometry, was invented by A. Connes and H. Moscovici in [CM98] as a computational tool for calculating the local index formula of spectral triples associated to hypoelliptic operators on manifolds. Since then, the theory has been evolving to cover many other cases including algebras and coalgebras endowed with (co)symmetry from Hopf algebras, bialgebras, and Hopf algebras with several objects such as para-Hopf algebras and $\times$-Hopf algebras [CM00, CM01, BS, Cra, Gor, HKRS1, HKRS2, JS, Kay05, Kay06, KP, KR02, KR05, KR04, KR03, Ra]. The main application of Hopf cyclic cohomology is to produce special cyclic cocycles on (co)algebras endowed with a (co)symmetry by a Hopf algebra. Generally speaking, Hopf cyclic cohomology of a Hopf algebra is more computable than cyclic cohomology of (co)algebras upon which the Hopf algebra (co)acts. This accessibility is the main motivation and application of Hopf cyclic cohomology. The Hopf cyclic cohomology of Hopf (co)module (co)algebras with coefficients in SAYD modules was first defined in [HKRS2, HKRS1], where it was

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shown that the modular pair in involution defined by Connes and Moscovici in [CM00], is in fact, a one dimensional SAYD module. Later on, Jara and Stefan in [JS] associated a canonical SAYD module to any Hopf-Galois extension such that the relative cyclic complex of the extension is isomorphic with the cyclic complex defined on the Hopf algebra with coefficients in the associated SAYD module. Soon after, Khalkhali and the second author showed in [KR05] that the cyclic module obtained by Jara-Stefan is of the form of cyclic module defined in [KR02] which is dual (in the sense of Connes' cyclic category) of Connes-Moscovici cocyclic module.

The first step for defining Hopf cyclic cohomology of Hopf algebras with several objects was taken in [CM01], where it was shown, among other things, that the symmetry on the crossed product algebra of diffeomorphism group and algebra of smooth functions on the frame bundle of a not necessarily flat manifold is defined by a Hopf algebroid generalizing Connes-Moscovici Hopf algebra in the case the manifold in question is flat. The cohomology defined in [CM01] was abstractly generalized for a class of Hopf algebroids called Para Hopf algebras in [KR04]. The Hopf cyclic cohomology of a coring with symmetry from a Para Hopf algebra defined in [Ra]. Finally Hopf cyclic cohomology of \(\times\)-Hopf algebras with coefficients in the SAYD modules was defined by Böhm and Stefan in [BS], where it was also shown that the result of Jara-Stefan is valid for \(\times\)-Hopf Galois extensions.

Let us recall very briefly here the result of Jara-Stefan from [JS]. Let \(\nabla : A \rightarrow A \otimes H\) define a comodule algebra and \(B\) be the subalgebra of coinvariants for this coaction. Then the extension \(B \subseteq A\) is called Galois if the canonical map \(\text{can} : A \otimes_B A \rightarrow A \otimes H\) is bijective. One uses the map \(\text{can}\) iteratively to transfer the cyclic structure on the relative cyclic complex of \(B \subseteq A\) to get a cyclic module on the other side depending only on the Hopf algebra \(H\) and the SAYD module \(A_B := A/[A,B]\).

In this paper we start from the fact that relative cyclic homology of an extension \(B \subseteq A\) is in fact Hopf cyclic homology of the ring \(A\) over \(B\) with coefficients in \(B\) as a SAYD module over the \(\times_B\)-Hopf algebra \(B^e := B \otimes B^{op}\). It is also easy to observe that \(A_B = B \otimes_{B^e} A\). On the other hand, if \(M\) is a SAYD module over \(B^e\), in the sense of [BS], then the cyclic homology of \(B\)-ring \(A\) with coefficients in \(M\) under symmetry of \(B^e\) is well-defined. So in case the extension is \(H\)-Galois iterative application of the map \(\text{can}\) takes us to a cyclic complex which is again the dual of the Connes-Moscovici cocyclic module associated to the Hopf algebra \(H\) with coefficients in \(M \otimes_{B^e} A\). The mentioned transfer of cyclic structures works due to the fact that the map \(\text{can}\) happens to be \(B^e\)-equivariant. This means that, if one assumes the existence of a \(\times\) Hopf algebra \(K\) acting on \(A\) such that the
map can is equivariant then to any SAYD module $M$ over $K$ one associates a SAYD module $\tilde{M} := M \otimes_K A$ over $H$. One then shows that the Hopf cyclic homology of the ring $A$ under symmetry of $K$ with coefficients in $M$ is isomorphic with the dual cyclic module of Hopf cocyclic module of $H$ with coefficients in $\tilde{M}$. The next step is to upgrade everything to the level of a $\times$-Hopf Galois extension, which is stated as Theorem 3.5.

The plan of the paper is as follows. In Section 1 we recall the basics of co(cyclic) modules and duality in the cyclic category. In Section 2 we review the concepts of left and right $\times$-Hopf algebras and basics of Hopf cyclic (co)homologies together with some examples. In Section 3 we define the equivariant $\times$-Hopf Galois extensions and prove the main result of the paper which is summarized in Theorem 3.5 and finally we bring some non trivial examples of this result.

Throughout the paper we assume all objects are $\mathbb{C}$-vector spaces although everything works for $k$-modules, where $k$ is a commutative ring. We use the Sweedler summations: for comultiplication of coalgebras or corings i.e., $\Delta(e) = e^{(1)} \otimes e^{(2)}$, for coactions i.e, $\nabla(m) = m_{<0>} \otimes m_{<1>}$; for translation map i.e, $\text{can}^{-1}(b) = b_{<->} \otimes b_{<+>}$; and finally for the “antipode” of $\times$-Hopf algebras, i.e, $\nu^{-1}(1 \otimes b) = b^{-} \otimes b^{+}$.

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3
1 Basics of (co)cyclic modules

In this section we recall the definitions of cyclic and cocyclic modules from C-Book (see also Lo). Recall that a cosimplicial module is given by datum $(C_n, \delta_i, \sigma_i)$ where $C_n$, $n \geq 0$ is a $\mathbb{C}$-module. The maps $\delta_i : C^n \to C^{n+1}$ are called cofaces, and $\sigma_i : C^n \to C^{n-1}$ called codegeneracies. These are $\mathbb{C}$-module maps satisfying the following cosimplicial relations:

$$
\delta_j \delta_i = \delta_i \delta_{j-1}, \quad \text{if } i < j,
$$

$$
\sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad \text{if } i \leq j,
$$

$$
\sigma_j \delta_i = \begin{cases} 
\delta_i \sigma_j, & \text{if } i > j + 1, \\
\delta_{i-1} \sigma_j, & \text{if } i = j + 1, \\
\text{Id}, & \text{if } i = j \text{ or } i = j + 1,
\end{cases}
$$

(1.1)

A cocyclic module is a cosimplicial module equipped with extra morphisms, $\tau : C^n \to C^n$, called cocyclic maps such that the following extra relations hold.

$$
\tau \delta_i = \delta_{i-1} \tau, \quad 1 \leq i \leq n + 1,
$$

$$
\tau \sigma_i = \sigma_{i-1} \tau, \quad 1 \leq i \leq n,
$$

$$
\tau \sigma_0 = \sigma_n \tau^2, \quad \tau^{n+1} = \text{Id}.
$$

(1.2)

In a dual manner, one can define a cyclic module as a simplicial module with extra cyclic maps. More precisely: A cyclic module is given by data, $C = (C_n, \delta_i, \sigma_i, \tau_n)$, where $C_n$, $n \geq 0$ is a $\mathbb{C}$-module and $\delta_i : C_n \to C_{n-1}$, $\sigma_i : C_n \to C_{n+1}$, $0 \leq i \leq n$, and $\tau : C_n \to C_n$, are called faces, degeneracies and cyclic maps, respectively, are $\mathbb{C}$-module maps, satisfying the following relations:

$$
\delta_i \delta_j = \delta_{j-1} \delta_i, \quad \text{if } i < j,
$$

$$
\sigma_i \sigma_j = \sigma_{j+1} \sigma_i, \quad \text{if } i \leq j,
$$

$$
\delta_i \sigma_j = \begin{cases} 
\sigma_j \delta_i, & \text{if } i < j, \\
\text{Id}, & \text{if } i = j \text{ or } i = j + 1,
\end{cases}
$$

$$
\text{Id}, & \text{if } i = j + 1,
\end{cases}
$$

and

$$
\delta_i \tau = \tau \delta_{i-1}, \quad 1 \leq i \leq n,
$$

$$
\delta_0 \tau = \delta_n, \quad \sigma_i \tau = \tau \sigma_{i-1}, \quad 1 \leq i \leq n,
$$

$$
\sigma_0 \tau = \tau^2 \sigma_n, \quad \tau^{n+1} = \text{Id}.
$$

(1.3)
Remark 1.1. [Duality] Let $C = (C^n, d_i, s_i, t)$ be a cocyclic module. We denote its cyclic dual by $\tilde{C}$. It is shown in [C-Book] that with the following operators $\tilde{C}^n := C^n$ is a cyclic module.

\begin{align*}
\delta_i &:= s_i : \tilde{C}_n \to \tilde{C}_{n-1}, \quad 0 \leq i \leq n - 1, \quad \delta_n := \delta_0 \tau_n, \\
\sigma_i &:= d_{i+1} : \tilde{C}_n \to \tilde{C}_{n+1}, \quad 0 \leq i \leq n - 1, \\
\tau &:= t^{-1}.
\end{align*}

(1.5)

Conversely, one obtains a cocyclic module from a cyclic module as follows. Let $C = (C_n, \delta_i, \sigma_i, \tau)$ be a cyclic module. Its cyclic dual is denoted by $\hat{C}$ and defined by $\hat{C}^n = C_n$ with the following cofaces, codegeneracies and cyclic operator.

\begin{align*}
d_0 &:= \tau_n \sigma_{n-1}, \quad d_i := \sigma_{i-1} : \hat{C}^n \to \hat{C}^{n+1}, \quad 1 \leq i \leq n, \\
s_i &:= \delta_i : \hat{C}^n \to \hat{C}^{n-1}, \quad 0 \leq i \leq n - 1, \\
t &:= \tau^{-1}.
\end{align*}

(1.6)

2 Hopf cyclic homology of $\times_R$-Hopf algebras

We refer the reader to [B, BW, Sch, Sch98] for details about bialgebroids and $\times$-Hopf algebras, however in Subsection 2.1 we recall the basics of $\times_R$-Hopf algebras. We review the stable anti Yetter-Drinfeld modules for $\times_R$-Hopf algebras in Subsection 2.2 from [BS]. Finally in Subsection 2.3 we bring the cyclic modules of module algebras and module corings over $\times_R$-Hopf algebras with coefficients in stable anti Yetter-Drinfeld modules.

2.1 Preliminaries of $\times_R$-Hopf algebras

Let $R$ and $\mathcal{K}$ be algebras over $\mathbb{C}$. In addition let the source and the target maps be the $\mathbb{C}$-algebra homomorphisms $s : R \to \mathcal{K}$ and $t : R^{op} \to \mathcal{K}$, such that their ranges commute with one another. We equip $\mathcal{K}$ with an $R$-bimodule structure by $r_1 k r_2 = s(r_1) t(r_2) k$ for $r_1, r_2 \in R$ and $k \in \mathcal{K}$. Similarly, $\mathcal{K} \otimes_R \mathcal{K}$ is endowed with the natural $R$-bimodule structure, i.e., $r_1 (k_1 \otimes_R k_2) r_2 := s(r_1) k_1 \otimes_R t(r_2) k_2$. We assume that there are $R$-bimodule maps called the coproduct $\Delta : \mathcal{K} \to \mathcal{K} \otimes_R \mathcal{K}$ and the counit $\varepsilon : \mathcal{K} \to R$ via which $\mathcal{K}$ is an $R$-coring. The data $(\mathcal{K}, s, t, \Delta, \varepsilon)$ is called a left $R$-bialgebroid if for $k_1, k_2 \in \mathcal{K}$ and $r \in R$ the following identities hold

\begin{enumerate}
  \item $k^{(1)} \otimes_R k^{(2)} = k^{(1)} \otimes_R k^{(2)} s(r),$
\end{enumerate}
ii) $\Delta(1_K) = 1_K \otimes_R 1_K$, and $\Delta(k_1 k_2) = k_1^{(1)} k_2^{(1)} \otimes_R k_1^{(2)} k_2^{(2)}$.

iii) $\varepsilon(1_K) = 1_R$ and $\varepsilon(k_1 k_2) = \varepsilon(k_1 \varepsilon(k_2))$.

For a left bialgebroid $K$ over an algebra $R$, the following identities hold for $r, r_1, r_2, r_3, r_4 \in R$ and $k, k_1, k_2 \in K$ [BW 4.31].

a) $\varepsilon(s(r)) = \varepsilon(t(r)) = r$.

b) $s(\varepsilon(k^{(1)})) k^{(2)} = t(\varepsilon(k^{(2)})) k^{(1)} = k$.

c) $\varepsilon(k_1 k_2) = \varepsilon(k_1 \varepsilon(k_2))$.

d) $\Delta(s(r)) = s(r) \otimes_R 1_K$ and $\Delta(t(r)) = 1_K \otimes_R t(r)$.

e) $\Delta(s(r_1) t(r_2) k s(r_3) t(r_4)) = s(r_1) k^{(1)} s(r_3) \otimes_R t(r_4) k^{(2)} t(r_4)$.

Similarly a right $R$-bialgebroid $B$ is an $R$-coring $(B, s, t, \Delta, \varepsilon)$, where $s, t : R \to B$ are algebra and anti-algebra homomorphisms with commuting ranges, and $B$ is an $R$-bimodule with the right multiplications by $s$ and $t$, such that the following conditions are satisfied for $b, b_1, b_2 \in B$ and $r \in R$.

i) $b^{(1)} \otimes_R t(r) b^{(2)} = s(r) b^{(1)} \otimes_R b^{(2)}$.

ii) $\Delta(1_B) = 1_B \otimes_R 1_B$ and $\Delta(b_1 b_2) = b_1^{(1)} b_2^{(1)} \otimes_R b_1^{(2)} b_2^{(2)}$.

iii) $\varepsilon(1_B) = 1_R$ and $\varepsilon(b_1 b_2) = \varepsilon(s(\varepsilon(b_1)) b_2)$.

For a right bialgebrid $B$ over an algebra $R$, the following identities hold for $r, r_1, r_2, r_3, r_4 \in R$ and $b, b_1, b_2 \in B$.

a) $\varepsilon(s(r)) = \varepsilon(t(r)) = r$.

b) $b = b^{(1)} s(\varepsilon(b^{(2)})) = b^{(2)} t(\varepsilon(b^{(1)}))$.

c) $\varepsilon(b_1 b_2) = \varepsilon(t(\varepsilon(b_1)) b_2)$.

d) $\Delta(s(r)) = 1_B \otimes_R s(r)$ and $\Delta(t(r)) = t(r) \otimes_R 1_B$.

e) $\Delta(s(r_1) t(r_2) b s(r_3) t(r_4)) = t(r_2) b^{(1)} t(r_4) \otimes_R s(r_1) b^{(2)} s(r_3)$.
A left $R$-bialgebroid $(\mathcal{K}, s, t, \Delta, \varepsilon)$ is said to be a left $\times_R$-Hopf algebra provided that the map

$$\nu : \mathcal{K} \otimes_{R^{op}} \mathcal{K} \rightarrow \mathcal{K} \otimes_R \mathcal{K}, \quad k \otimes_{R^{op}} k' \mapsto k^{(1)} \otimes_R k^{(2)} k', \quad (2.1)$$

is bijective. Here in the left hand side of (2.1) the $R^{op}$-module structures are given by right and left multiplication by $t(r)$ for $r \in R$, however in the right hand side the $R$-module structures are given by the original actions of $R$ on $\mathcal{K}$. One notes that $\nu$ and $\nu^{-1}$ are right $\mathcal{K}$-linear. The role of antipode in $\times_R$-Hopf algebras is played by the following map.

$$\mathcal{K} \rightarrow \mathcal{K} \otimes_{R^{op}} \mathcal{K}, \quad k \mapsto \nu^{-1}(k \otimes_R 1) \quad (2.2)$$

For our convenience we use $k^- \otimes_{R^{op}} k^+$ for $\nu^{-1}(k \otimes_R 1)$.

**Example 2.1.** Let $R$ be an algebra over $\mathbb{C}$. The simplest left $\times_R$-Hopf algebra is $\mathcal{K} = R \otimes R^{op}$ with the source and target maps defined by

$$s : R \rightarrow \mathcal{K}, \quad r \mapsto r \otimes 1; \quad t : R^{op} \rightarrow \mathcal{K}, \quad r \mapsto 1 \otimes r,$$

comultiplication defined by

$$\Delta : \mathcal{K} \rightarrow \mathcal{K} \otimes_R \mathcal{K}, \quad r_1 \otimes r_2 \mapsto (r_1 \otimes 1) \otimes_R (1 \otimes r_2),$$

counit given by

$$\varepsilon : \mathcal{K} \rightarrow R, \quad \varepsilon(r_1 \otimes r_2) = r_1 r_2,$$

and

$$\nu((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 \otimes 1 \otimes r_3 \otimes r_4 r_2,$$

$$\nu^{-1}((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 \otimes 1 \otimes r_2 r_3 \otimes r_4,$$

where $r, r_1, r_2, r_3, r_4 \in R$.

The following properties are a symmetrical version of [Sch Proposition 3.7]. We refer the reader to [BS Lemma 2.14] for more properties of a right $\times$-Hopf algebras.

**Proposition 2.2.** Let $R$ be an algebra over $\mathbb{C}$ and $\mathcal{K}$ a left $\times_R$-Hopf algebra. The following identities hold for $k, k' \in \mathcal{K}$ and $r \in R$.

i) $k^{-(1)} \otimes_R k^{-(2)} k^+ = k \otimes_R 1_{\mathcal{K}}$.

ii) $k^{(1)-} \otimes_{R^{op}} k^{(2)} = k \otimes_{R^{op}} 1_{\mathcal{K}}.$
iii) \((kk')^- \otimes \mathcal{R}^\text{op} (kk')^+ = k^- k'^- \otimes \mathcal{R}^\text{op} k'^+ k^+\).

iv) \(1\_\mathcal{K} \otimes \mathcal{R} \mathcal{K}^+ = 1\_\mathcal{K} \otimes \mathcal{R} \mathcal{K}^+\).

v) \(k^\text{−−}(1) \otimes_R k^\text{+−}(2) \otimes \mathcal{R} k^+(2) = k^- k^\text{−−} \otimes \mathcal{R}^\text{op} k^+(2)\).

vi) \(k^- \otimes \mathcal{R}^\text{op} k^+ = (1) \otimes \mathcal{R} k^- = k^+(1) \otimes \mathcal{R}^\text{op} k^+(2)\).

vii) \(k = k^− t(\varepsilon(k^+))\).

viii) \(s(\varepsilon(k)) = k^- k^++\).

ix) \(k^- \otimes \mathcal{R}^\text{op} k^+ t(r) = t(r) k^- \otimes \mathcal{R}^\text{op} k^+\).

Let \(R\) be an algebra over \(\mathbb{C}\). A right \(R\)-bialgebroid \(B = (B, s, t, \Delta, \varepsilon)\), is said to be a right \(\times R\)-Hopf algebra provided that the map

\[
\nu : B \otimes \mathcal{R} B \longrightarrow B \otimes_R B, \quad b \otimes_R b' \mapsto bb'(1) \otimes_R b'(2),
\]

is bijective. In the domain of the map \((2.3)\), \(\mathcal{R}^\text{op}\)-right and left module structures of \(B\) are given by right and left multiplication by \(t(r)\) respectively, for \(r \in R\). In the codomain of the map \((2.3)\), \(R\)-module structures are given by right multiplication by \(s(r)\) and \(t(r)\).

One notes that \(\nu\) and \(\nu^{-1}\) are left \(B\)-linear maps. We denote the image of \(\nu\) by the following index notation,

\[
\nu^{-1}(1 \otimes \mathcal{R} b) = b^− \otimes_R b^+.
\]

**Example 2.3.** Let \(R\) be an algebra over \(\mathbb{C}\). The \(B = R \otimes \mathcal{R}^\text{op}\) is a right \(\times R\)-Hopf algebra where the source and target maps are defined by

\[
s : R \longrightarrow B, \quad r \longmapsto r \otimes 1; \quad t : \mathcal{R}^\text{op} \longrightarrow B, \quad r \longmapsto 1 \otimes r,
\]

cumultiplication by

\[
\Delta : B \longrightarrow B \otimes_B B, \quad r_1 \otimes r_2 \longmapsto (1 \otimes r_2) \otimes_B (r_1 \otimes 1),
\]

counit by

\[
\varepsilon : B^\text{c} \longrightarrow B, \quad \varepsilon(r_1 \otimes r_2) = r_2 r_1,
\]

and

\[
\nu((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 \otimes r_4 r_2 \otimes r_3 \otimes 1,
\]

\[
\nu^{-1}((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 r_4 \otimes r_2 \otimes r_3 \otimes 1.
\]
2.2 Stable anti Yetter-Drinfeld-modules

In this subsection for the reader’s convenience we briefly recall the definitions of module, comodule and stable anti Yetter-Drinfeld-module for $\times_R$-Hopf algebras. Also we present an example of a stable anti Yetter-Drinfeld (SAYD) module for the $\times_R$-Hopf algebra $R \otimes R^{op}$ which plays an important role in the sequel section.

A right module over a $\times_R$-Hopf algebra $K$ is a right $K$-module $M$, which is naturally an $R$-bimodule by $mr = m \triangleleft s(r)$, and $mr = m \triangleleft s(r)$. Similarly, by a left module over a right $\times_R$-Hopf algebra $B$ we mean a left $B$-module $M$, which is naturally an $R$-bimodule by $rm = s(r)\triangleright m$, and $mr = t(r)\triangleright m$.

A left (right) comodule over a $\times_R$-Hopf algebra $C$ is defined by a left(right) $R$-module $M$, together with a left(right) $R$-module coaction map, $M \rightarrow C \otimes_R M$, $m \mapsto m_{<1>} \otimes_R m_{<0>},$ (2.4)

$( M \rightarrow M \otimes_R C, m \mapsto m_{<0>} \otimes_R m_{<1>})$ satisfying coassociativity and counitality axioms.

By a comodule over a $\times$-Hopf algebra we mean a comodule over the underlying coring. One notes that a left comodule $M$ of a left bialgebroid $K$ can be equipped with an $R$-bimodule structure by introducing a right $R$-action

$$mr := \varepsilon(m_{<1>} s(r)) \triangleright m_{<0>},$$ (2.5)

for $r \in R$ and $m \in M$. It is checked in [BS] that the left $K$-coaction on left comodule $M$ is an $R$-bimodule map. That is for $r,r' \in R$ and $m \in M$,

$$(rmr')_{<1>} \otimes_R (rmr')_{<0>} = s(r)m_{<1>} s(r') \otimes_R m_{<0>}.$$. (2.6)

Furthermore, for any $m \in M$ and $r \in R$,

$$m_{<1>} \otimes_R m_{<0>} r = m_{<1>} t(r) \otimes_R m_{<0>}.$$ (2.7)

Here we recall the definition of stable anti-Yetter-Drinfeld modules over $\times$-Hopf algebras from [BS]. Let $\mathcal{K}$ be a left $\times_R$-Hopf algebra and $M$ be a right $\mathcal{K}$-module and a left $\mathcal{K}$-comodule. We say $M$ is a right-left anti Yetter-Drinfeld module provided that the following conditions hold.

i) The $R$-bimodule structures on $M$, underlying its module and comodule structures, coincide. That is, for $m \in M$ and $r \in R$,

$$mr = m \triangleleft s(r), \quad \text{and} \quad rm = m \triangleleft t(r),$$

where $rm$ denotes the left $R$-action on the left $\mathcal{K}$-comodule $M$ and $mr$ is the canonical right action defined in (2.5), i.e. $mr = \varepsilon(m_{<1>} s(r))m_{<0>}$. 9
ii) For $k \in K$ and $m \in M$,

\[(m \triangleleft k)_{<1>} \otimes (m \triangleright k)_{<0>} = k^{(2)+}_{<1>} \otimes_R m_{<0>} \triangleleft k^{(2)-}. \quad (2.8)\]

The anti Yetter-Drinfeld module $M$ is said to be stable if in addition, for any $m \in M$, $m_{<0>}, m_{<1>} = m$. Similarly one defines a left-right AYD module over a right $\times_R$-Hopf algebra $B$ by a left $B$-module and right $B$-comodule $M$ satisfying

i) $mr = t(r) \triangleright m$, and $rm = s(r) \triangleright m$, where $mr$ denotes the right $R$-action on the right $B$-comodule $M$ and $rm$ is the canonical left $R$-action $rm = m_{<0>} \varepsilon(s(r)m_{<1>}).

ii) For $b \in B$ and $m \in M$,

\[(b \triangleright m)_{<0>} \otimes (b \triangleright m)_{<1>} = b^{(1)+} \triangleright m_{<0>} \otimes_R b^{(2)} m_{<1>} b^{(1)-}. \quad (2.9)\]

The anti Yetter-Drinfeld module $M$ is said to be stable if in addition, for any $m \in M$, we have $m_{<1>}, m_{<0>} = m$.

**Remark 2.4.** Left or right $\times_R$-Hopf algebras extend the notion of Hopf algebras. In fact if $K$ is a bialgebra over a commutative ring $R$, with co-product $k \mapsto k^{(1)} \otimes_R k^{(2)}$, then the bijectivity of the map $\nu$ is equivalent to the fact that $K$ is a Hopf algebra. In this case the inverse of the map $\nu$ can be defined as

\[\nu^{-1}(k \otimes 1) = k^- \otimes_R \varepsilon k^+ := k^{(1)} \otimes_R \varepsilon S(k^{(2)}),\]

where $S$ denotes the antipode of the Hopf algebra $K$. Therefore the condition \[(2.8)\] for a left $\times_R$-Hopf algebra is equivalent to the following relation

\[(mk)_{<1>} \otimes (mk)_{<0>} = S(k^{(3)}) m_{<1>} k^{(1)} \otimes m_{<0>} k^{(2)}.\]

Here we recall the definition of a character for $\times_R$-Hopf algebras. A map $\delta$ is called a right character [B, Lemma 2.5], for the $\times_R$-Hopf algebra $K$ if it satisfies the following conditions:

\[\delta(k s(r)) = \delta(k) r, \quad \text{for} \quad k \in K \quad \text{and} \quad r \in R, \quad (2.10)\]

\[\delta(k_1 k_2) = \delta(s(\delta(k_1)) k_2), \quad \text{for} \quad k_1, k_2 \in K, \quad (2.11)\]

\[\delta(1_K) = 1_R. \quad (2.12)\]

As an example, for any right $\times_R$-Hopf algebra the counit $\varepsilon$ is a right character. The following example is similar to [BS, Example 2.18] for right $\times_R$-Hopf algebras.
Example 2.5. Let \( \mathcal{K} \) be a left \( \times_R \)-Hopf algebra, \( \sigma \in \mathcal{K} \) a group-like element and the map \( \delta : \mathcal{K} \to R \) a right character. The following action and coaction,

\[
    r \triangleleft k = \delta(s(r)k), \quad \text{and} \quad r \mapsto s(r)\sigma \otimes 1
\]

(2.13)
define a right \( \mathcal{K} \)-module and left \( \mathcal{K} \)-comodule structure on \( R \), respectively. These action and coaction amount to a right-left anti Yetter-Drinfeld module on \( R \) if and only if, for all \( r \in R \) and \( k \in \mathcal{K} \),

\[
    s(\delta(k))\sigma = t(\delta(k^{(2)})k^{(2)+}\sigma k^{(1)}), \quad \text{and} \quad \varepsilon(\sigma s(r)) = \delta(s(r)).
\]

(2.14)

The anti Yetter-Drinfeld module \( R \) is stable if in addition \( \delta(s \otimes r) = r \), for all \( r \in R \). We specialize this example to the left \( \times_R \)-Hopf algebra \( \mathcal{K} = R \otimes R^\text{op} \) explained in the Example 2.1.

To this end, we like to know all group-like elements and right characters of \( \mathcal{K} \). One can easily characterize all homogenous group-like elements of \( \mathcal{K} \).

An element \( x \otimes y \in \mathcal{K} \) is a group-like element if and only if, \( xy = yx = 1 \), because

\[
    \Delta(x \otimes y) = (x \otimes y) \otimes_R (x \otimes y) = (x \otimes y) \otimes_R (x \otimes 1)(1 \otimes y) = (1 \otimes x)(x \otimes y) \otimes_R (1 \otimes y) = (x \otimes yx) \otimes_R (1 \otimes y).
\]

On the other hand, one sees \( \Delta(x \otimes y) = (x \otimes 1) \otimes_R (1 \otimes y) \). Therefore \( yx = 1 \). Since \( x \otimes y \) is a group like element, we have \( 1 = \varepsilon(x \otimes y) = xy \). Conversely one easily sees that if \( xy = yx = 1 \) then \( x \otimes y \) is a group-like element for \( \mathcal{K} \).

We claim that a map \( \delta : \mathcal{K} \to R \) is a right character if and only if \( \delta(s \otimes r) = \theta(s)\theta(r) \) for some algebra map \( \theta : R \to R \). Indeed, let \( \delta \) be a right character on \( \mathcal{K} \), then

\[
    \delta(s \otimes r) = \delta((1 \otimes r)(s \otimes 1)) = \delta((1 \otimes r)s(s)) = \delta((1 \otimes r)s).
\]

We define \( \theta \) by \( \theta(r) := \delta(1 \otimes r) \). It is obvious that \( \theta \) is unital. We show that \( \theta \) is an algebra map,

\[
    \theta(rr') = \delta(1 \otimes rr') = \delta((1 \otimes r')(1 \otimes r)) = \delta(s(\delta((1 \otimes r'))(1 \otimes r)))
    = \delta((1 \otimes r)s(\delta((1 \otimes r')))) = \delta(\delta((1 \otimes r'))(1 \otimes r) \delta(r' \otimes 1) = \theta(r)\theta(r').
\]

Conversely, let \( \theta \) be an algebra endomorphism of \( R \), we consider the map \( \delta : \mathcal{K} \to R, \quad s \otimes r \mapsto \theta(r)s \). We show that \( \delta \) satisfies (2.10). Indeed,
\[ \delta((s \otimes r)s(r')) = \delta(sr' \otimes r) = \theta(r)sr' = \delta(s \otimes r)r'. \] The satisfaction of (2.11) by \( \delta \) is as follows.

\[ \delta((s \otimes r)(s' \otimes r')) = \delta(ss' \otimes r)r' = \theta(r')\theta(r)ss' = \delta(\theta(r)ss' \otimes r'). \]

Also since \( \delta(1R \otimes 1R) = \theta(1R)1R = 1R \), the map \( \delta \) satisfies (2.12). Therefore \( \delta \) is a right character.

**Proposition 2.6.** Let \( \mathcal{K} \) be the left \( \times \)-Hopf algebra \( R \otimes R^{op} \), \( x \otimes x^{-1} \in \mathcal{K} \) a homogenous group like element and \( \delta \) a right character on \( \mathcal{K} \). Then the action and coaction defined in (2.13) amount to a right-left SAYD module on \( R \) if and only if \( x \) belongs the center of the algebra \( R \) and \( \theta = \text{Id} \).

**Proof.** Let \( k \otimes t \in \mathcal{K} \) and \( r \in R \). By the characterization of all right characters on \( \mathcal{K} \) the action and coaction defined in (2.13) reduce to

\[ r \triangleright (k \otimes t) = \theta(t)rk, \quad \text{and} \quad r \longmapsto (rx \otimes x^{-1}) \otimes 1. \]

Now we show \( x \) is in the center of the algebra \( R \). For this we use the stability condition,

\[ r = \delta(s(r)(x \otimes x^{-1})) = \delta((r \otimes 1)(x \otimes x^{-1})) = \delta(rx \otimes x^{-1}), \quad \forall r \in R. \]

In the previous equality let \( r = x^{-1} \). We have

\[ y = \delta(1 \otimes x^{-1}) = \theta(x^{-1}). \]

On the other hand,

\[ r = \delta(s(r)(x \otimes x^{-1})) = \delta(rx \otimes x^{-1}) = \theta(x^{-1})rx. \]

By multiplying the both sides by \( x^{-1} \), we obtain \( \theta(x^{-1})r = rx^{-1} \). Therefore \( x^{-1}r = rx^{-1} \) for all \( r \in R \). Thus \( x \) is also in the center of the algebra \( R \).

We shall show the AYD condition implies that the map \( \theta \) is identity. The AYD condition is equivalent to

\[ s(\delta(k))\sigma = t(\delta(k^{(2)}))k^{(2)}\sigma k^{(1)}. \]

Let \( k = g \otimes h \) and \( \sigma = x \otimes x^{-1} \). We have

\[ \theta(h)gx \otimes x^{-1} = (h \otimes 1)(x \otimes x^{-1})(g \otimes 1) = hxg \otimes x^{-1}, \]
and therefore $\theta(h)gx = hxg$. Since $x$ is in the center of $R$, $\theta(h)gx = hgx$. By multiplying both sides by $x^{-1}$ from right, we obtain $\theta(h)g = hg$ and for $g = 1$ we have $\theta(h) = h$ for $\forall h \in R$. Thus we obtain the claimed action and coaction. Conversely, it is easy to check that if $x$ belongs the center of $R$ and $\theta = Id$, then the following $R \otimes R^{op}$-action and coaction define a right-left SAYD module $R$ over the left $\times_R$-Hopf algebra $R \otimes R^{op}$. 

$$d_0(m \otimes_{K} \tilde{t}) = m \otimes_{K} 1 \otimes_{S} t_0 \otimes_{S} \cdots \otimes_{S} t_n,$$

$$d_i(m \otimes_{K} \tilde{t}) = m \otimes_{K} t_0 \otimes_{S} \cdots \otimes_{S} t_i \otimes_{S} t_{i+1} \otimes_{S} \cdots \otimes_{S} t_n,$$  

$$1 \leq i \leq n,$$

$$d_n(m \otimes_{K} \tilde{t}) = m_{<0>} \otimes_{K} t_0 \otimes_{S} \cdots \otimes_{S} t_n \otimes_{S} m_{<1>} \otimes_{K} 1_T,$$

$$s_i(m \otimes_{K} \tilde{t}) = m \otimes_{K} t_0 \otimes_{S} \cdots \otimes_{S} t_i t_{i+1} \otimes_{S} \cdots t_n, \quad 0 \leq i \leq n - 1,$$

$$t_n(m \otimes_{K} \tilde{t}) = m_{<0>} \otimes_{K} t_1 \otimes_{S} \cdots \otimes_{S} t_n \otimes_{S} m_{<1>} \otimes_{K} 1_T.$$  

Here $\tilde{t}$ stands for $t_0 \otimes_{S} \cdots \otimes_{S} t_n$. 

### 2.3 Cyclic homology of module algebras and module corings

In this subsection we recall from [BS] the cyclic homology of module algebras and module corings with coefficients in SAYD modules under the symmetry of $\times_R$-Hopf algebras. In the case of module algebras, first we introduce a cocyclic module for left $\times_R$-Hopf algebras and then we dualize it, as explained in the Section 2.2 to find a cyclic module.

Let $R$ be an algebra over $C$ and $K$ be a left $\times_R$-Hopf algebra. A left $K$-module algebra $A$ [BS, Definition 2.3] is a $C$-algebra and a left $K$-module such that for all $k \in K$, $a, a' \in A$ and $r \in R$, the following identities hold,

$$k \triangleright 1_A = s(\varepsilon(k)) \triangleright 1_A,$$

$$k \triangleright (aa') = (k^{(1)} \triangleright a)(k^{(2)} \triangleright a'),$$

$$t(r) \triangleright a = a(s(r) \triangleright a'),$$  

(multiplication is $R$-balanced.)

**Proposition 2.7.** Let $S$ be an algebra over $C$, $K$ a left $\times_S$-Hopf algebra, $T$ a left $K$-module algebra, and $M$ a right-left SAYD module over $K$. Let

$$K^C_n(T, M) = M \otimes_{K} T \otimes_{S} (n+1).$$

Then the following cofaces, codegeneracies and cyclic map define a cocyclic module structure on $K^C_n(T, M)$.
Proof. Since $T$ is an $\mathcal{K}$-module algebra, by $S$-balanced property of $T$, we have $k \triangleright 1_T = s(\varepsilon(k)) 1_T = t(\varepsilon(k)) 1_T$. Therefore, cofaces are well-defined. Here we consider $T^{\otimes S(n+1)}$ as a $\mathcal{K}$-module with diagonal action. By (2.17) the codegeneracies are well-defined. We show that the cocyclic map is well-defined:

$$
  t_n(m \triangleleft k \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n)
  = (m \triangleleft k)_{<0>} \otimes_\mathcal{K} t_1 \otimes S \cdots \otimes S t_n \otimes S ((m \triangleleft k)_{<-1>}) \triangleright t_0
  = m_{<0>} \triangleleft k^{(2)}_{<2>} \otimes_\mathcal{K} t_1 \otimes S \cdots \otimes S t_n \otimes S k^{(2)}_{<1>} \triangleright m_{<0>}^{(1)} \triangleright t_0
  = m_{<0>} \triangleleft k^{-2} \otimes_\mathcal{K} t_1 \otimes S \cdots \otimes S t_n \otimes S k^+ m_{<-1>}^+ k^{-1} \triangleright t_0
  = m_{<0>} \otimes_\mathcal{K} k_{<2>}^{(2)} \triangleright (t_1 \otimes S \cdots \otimes S t_n \otimes S k^+ m_{<-1>}^+ k_{<-1>}^{-1} \triangleright t_0)
  = m_{<0>} \otimes_\mathcal{K} k_{<-2>}^{(2)} \triangleright t_1 \otimes S \cdots \otimes S k_{<-2>}^{(2)} \otimes S(k_{<-1>}^+ m_{<-1>}^+ k_{<-1>}^{-1} \triangleright t_0)
  = m_{<0>} \otimes_\mathcal{K} k_{<-2>}^{(2)} \triangleright t_1 \otimes S \cdots \otimes S m_{<-1>}^+ k_{<-1>}^{-1} \triangleright t_0
  = t_n(m \otimes_\mathcal{K} k_{<-1>}^+ \triangleright t_0 \otimes S \cdots \otimes S m_{<-1>}^+ m_{<-1>}^+ k_{<-1>}^{-1} \triangleright t_0)
  = t_n(m \otimes_\mathcal{K} k_{<-1>}^+ \triangleright (t_0 \otimes S \cdots \otimes S t_n)).
$$

In the above, we use the SAYD condition (2.8) in the second equality, Proposition 2.2(v) in the third equality, the diagonal action of $\mathcal{K}$ on $T^{\otimes S(n+1)}$ in the fifth equality and Proposition 2.2(i) in the seventh equality. Now we check some of the cocyclicity conditions. To see $t_n^{n+1} = Id$, we have

$$
  t_n^{n+1}(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_{n+1})
  = m_{<0>} \otimes_\mathcal{K} m_{<-n-1>}^+ t_0 \otimes S \cdots \otimes S m_{<-1>}^+ t_n
  = m_{<0>} \otimes_\mathcal{K} m_{<-1>}^+ \triangleright (t_0 \otimes S \cdots \otimes S t_n)
  = m_{<0>} m_{<-1>} \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n
  = m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n.
$$

(2.20)

In the above, we use the left diagonal action of $\mathcal{K}$ on $T^{\otimes S(n+1)}$ in the second equality and the stability condition in the last equality.
Now we show $t_n s_0 = s_n t_{n+1}^2$.

\[
s_n t_{n+1}^2 (m \otimes \mathcal{K} t_0 \otimes S \cdots \otimes_S t_{n+1}) = s_n t_{n+1} (m_{<n>} \otimes \mathcal{K} t_1 \otimes S \cdots \otimes_S t_{n+1} \otimes_S m_{<-1>} \triangleright t_0) = s_n (m_{<n>} \otimes \mathcal{K} t_2 \otimes S \cdots \otimes_S t_{n+1} \otimes_S m_{<-1>} \triangleright t_0 \otimes_S m_{<n> \cdot (-1)} \triangleright t_1) = s_n (m_{<n>} \otimes \mathcal{K} t_2 \otimes S \cdots \otimes_S t_{n+1} \otimes_S m_{<-1>} \triangleright t_0 \otimes_S m_{<-1> \cdot (2)} \triangleright t_1) = m_{<n>} \otimes \mathcal{K} t_2 \otimes S \cdots \otimes_S t_{n+1} \otimes_S m_{<-1>} \triangleright (t_0 t_1) = t_n (m \otimes \mathcal{K} t_0 t_1 \otimes S t_2 \otimes S \cdots \otimes_S t_{n+1}) = t_n s_0 (m \otimes \mathcal{K} t_0 \otimes S \cdots \otimes_S t_{n+1}).
\]

(2.21)

In the above, we use the left $\mathcal{K}$-coaction property of $M$ in the third equality and left $\mathcal{K}$-module algebra property of $T$ in the fifth equality. The relation $t_n d_0 = d_n$ is obvious. The other cocyclicity conditions can be easily verified.

The cyclic cohomology of the preceding cocyclic module is denoted by $\mathcal{K} HC^*(T, M)$, and is called Hopf cyclic cohomology of $T$ with coefficients in $M$ under the symmetry of $\mathcal{K}$.

To get a cyclic module we apply the duality procedure to the previous cocyclic module as follows. Let $S$ be an algebra over $\mathbb{C}$, $\mathcal{K}$ a left $\times_S$-Hopf algebra, $T$ a left $\mathcal{K}$-module algebra and $M$ a right-left SAYD module over $\mathcal{K}$. Let

\[
\mathcal{K}C_n(T, M) = M \otimes \mathcal{K} T^{\otimes (n+1)}.
\]

(2.22)

The following faces, degeneracies and cyclic map define a cyclic module structure on $\mathcal{K} C_*(T, M)$.

\[
\begin{align*}
\delta_i (m \otimes \mathcal{K} \tilde{t}) &= m \otimes \mathcal{K} t_0 \otimes S \cdots \otimes_S t_i \otimes_S m_{<-1> \cdot (i+1)} \triangleright t_n, \quad 0 \leq i \leq n - 1, \\
\delta_n (m \otimes \mathcal{K} \tilde{t}) &= m_{<n> \cdot (-1)} \otimes \mathcal{K} t_n (m_{<-n> \cdot (-1)} \triangleright t_0) \otimes_S m_{<-n+1> \cdot (-1)} \triangleright t_1 \otimes_S \cdots \otimes_S m_{<-1> \cdot (-1)} \triangleright t_{n-1}, \\
\sigma_i (m \otimes \mathcal{K} \tilde{t}) &= m \otimes \mathcal{K} t_0 \otimes_S \cdots \otimes_S t_i \otimes_S 1 \otimes_S t_{i+1} \otimes_S \cdots \otimes_S t_n, \quad 0 \leq i \leq n, \\
\tau_n (m \otimes \mathcal{K} \tilde{t}) &= m_{<n>} \otimes \mathcal{K} t_n \otimes S m_{<-n>} \triangleright t_0 \otimes_S \cdots \otimes_S m_{<-1>} \triangleright t_{n-1}.
\end{align*}
\]

(2.23)

Now we recall the cyclic homology of a module coring with coefficients in a SAYD module under the symmetry of a $\times_R$-Hopf algebra. Let $\mathcal{B}$ be a right $\times_R$-Hopf algebra. A right $\mathcal{B}$-module coring $C$ is an $R$-coring and right $\mathcal{B}$-module such that $R$-bimodule structure of $C$ coincides with the one induced
by $\mathcal{B}$. In addition, it is assumed that the counit $\varepsilon$ and comultiplication $\Delta$ are right $\mathcal{B}$ linear. That is we consider the right $\mathcal{B}$-module structure of $R$ by $r \triangleright b := \varepsilon(s(r)b)$ and right $\mathcal{B}$ module structure of $C \otimes_R C$ is by the diagonal action. This means:

$$\varepsilon_C(c \triangleright b) = \varepsilon_C(c) \triangleright b = \varepsilon_{\mathcal{B}}(s(\varepsilon_C(c))b), \quad (2.24)$$

$$\Delta_C(c \triangleright b) = \Delta(c) \triangleright b = c^{(1)} \triangleright b^{(1)} \otimes_R c^{(2)} \triangleright b^{(2)}. \quad (2.25)$$

We define

$$\tilde{C}_{\mathcal{B},n}(C, M) = C^{\otimes R(n+1)} \otimes_{\mathcal{B}} M, \quad (2.26)$$

and following operators on $\tilde{C}_{\mathcal{B},n}(C, M)$.

$$\delta_i(\tilde{c} \otimes_{\mathcal{B}} m) = c_0 \otimes_R \cdots \otimes_R \varepsilon_C(c_i) \otimes_R \cdots \otimes_R c_n \otimes_{\mathcal{B}} m, \quad 0 \leq i \leq n,$$

$$\sigma_i(\tilde{c} \otimes_{\mathcal{B}} m) = c_0 \otimes_R \cdots \otimes_R \Delta_C(c_i) \otimes_R \cdots \otimes_R c_n \otimes_{\mathcal{B}} m, \quad 0 \leq i \leq n, \quad (2.27)$$

$$\tau_n(\tilde{c} \otimes_{\mathcal{B}} m) = c_n \triangleleft m_{<1>} \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \otimes_{\mathcal{B}} m_{<0>}.$$

Here $\tilde{c} = c_0 \otimes_R \cdots \otimes_R c_n$ and $C^{\otimes R(n+1)}$ is a right $\mathcal{B}$-module by diagonal action. After some identifications the following theorem coincides with Proposition 2.19 in [BS]. For reader convenience we just check that the cyclic operator is well defined.

**Proposition 2.8.** Let $\mathcal{B}$ be a right $\times_{R^*}$-Hopf algebra, $M$ a left-right SAYD module over $\mathcal{B}$ and $C$ a right $\mathcal{B}$-module coring. Then the operators defined in [2.27] define a cyclic module structure on $\tilde{C}_{\mathcal{B},n}(C, M)$.

**Proof.** Since $C$ is a right $\mathcal{B}$-module coring we have, $(c\lhd b^{(1)}) (r \triangleright b^{(2)}) = (cr) \triangleright b$, for $c \in C$ and $r \in R$. Therefore $(c_1 \triangleright b^{(1)}) (\varepsilon(c_2) \triangleright b^{(2)}) = (c_1 \varepsilon(c_2)) \triangleright b$ for $c_1, c_2 \in R$, shows that faces are well-defined. Also since comultiplication is a right $\mathcal{B}$-module map, using [2.25] degeneracies are well-defined.
The following calculation shows that the cyclic map is well-defined:
\[
\tau_n(c_0 \otimes_R \cdots \otimes_R c_n \otimes_B b \triangleright m) = \\
c_n \triangleleft (b \triangleright m)_{<1>} \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \otimes_B (b \triangleright m)_{<0>} = \\
c_n \triangleleft b^{(2)} m_{<1>} b^{(1)} \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \otimes_B b^{(1)} \triangleright m_{<0>} = \\
((c_n \triangleleft b^{(2)} m_{<1>} b^{-}) \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \triangleleft b^{(1)} \otimes_B m_{<0>} = \\
((c_n \triangleleft b^{(n+2)} m_{<1>} b^{-}) \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \triangleleft b^{(n+1)} \otimes_B m_{<0>} = \\
(c_n \triangleleft b^{(n+1)} m_{<1>}) \otimes_R c_0 \otimes_R \cdots \otimes_R c_{n-1} \otimes_B b^{(n)} \triangleright m_{<0>} = \\
\tau_n(c_0 \otimes_R \cdots \otimes_R c_n \otimes_B b \triangleright m) = \\
\tau_n((c_0 \otimes_R \cdots \otimes_R c_n) \triangleleft b \otimes_B m).
\]

We use the AYD condition \cite{BS} in the second equality, \cite{BS} Lemma 2.14.(v) in the third equality, the diagonal action in the fourth equality and \cite{BS}, Lemma 2.14.(i) in the sixth equality. We leave to the reader to check that \(\delta_i, \sigma_j\) and \(\tau\) satisfy all conditions of a cyclic module.

The cyclic homology of this cyclic module is denoted by \(\widetilde{HC}_{B,*}(C, M)\).

In a special case, \(B\) is a right \(B\)-module coring by multiplication map \(B \otimes_R B \to B\). To synchronize with the Hopf cyclic complex of Hopf algebras we identify
\[
\widetilde{C}_{B,n}(B, M) \xrightarrow{\varphi_n} B^\otimes R^m \otimes_{R^\text{op}} M,
\]
where \(\varphi\) defined by
\[
\varphi_n : B^\otimes R^{(n+1)} \otimes_B M \to B^\otimes R^m \otimes_{R^\text{op}} M,
\]
\[
\varphi_n(b_0 \otimes_R \cdots \otimes_R b_n \otimes_B m) = (b_0 \otimes_R \cdots \otimes_R b_{n-1}) \triangleleft b_n^- \otimes_{R^\text{op}} b_n^+ m.
\]

**Proposition 2.9.** The map \(\varphi_n\) defined in (2.29) is a well-defined isomorphism of vector spaces.

**Proof.** By \cite{BS} Lemma 2.14. (iii)(ii), the map \(\varphi\) is well-defined. We define the inverse of map \(\varphi\) by
\[
\varphi_n^{-1}(b_1 \otimes_R \cdots \otimes_R b_n \otimes_{R^\text{op}} m) = b_1 \otimes_R \cdots \otimes_R b_n \otimes_R 1_B \otimes_B m.
\]
This map is well-defined because for any \(r^o \in R^\text{op}\) we have \(r^o m = t(r) \triangleright m\) and \((b_1 \otimes_R \cdots \otimes_R b_n) \triangleleft r^o = b_1 t(r) \otimes_R \cdots \otimes_R b_n\). One can easily check that
\[ \varphi \varphi^{-1} = Id \text{ by [BS] Lemma 2.14. (iv), and } \varphi^{-1} \varphi = Id \text{ by [BS] Lemma 2.14.(i)}. \]

Therefore we transfer the cyclic structure of \( \widetilde{C}_{B^*}(B, M) \) to \( B^{\otimes R^p} \otimes _{R^p} M \). The resulting operators are recorded below.

\[
\begin{align*}
\delta_i (\tilde{b} \otimes R^p m) &= b_1 \otimes R \cdots \otimes R \varepsilon(b_{i+1}) \otimes R \cdots \otimes R b_n \otimes R^p m, & 0 \leq i \leq n-1 \\
\delta_n (\tilde{b} \otimes R^p m) &= (b_1 \otimes R \cdots \otimes R b_{n-1}) \triangleleft b_{n}^- \otimes R^p b_{n}^+ m, \\
\sigma_i (\tilde{b} \otimes R^p m) &= b_1 \otimes R \cdots \otimes R \Delta(b_{i+1}) \otimes R \cdots \otimes R b_n \otimes R^p m, & 0 \leq i \leq n-1, \\
\sigma_n (\tilde{b} \otimes R^p m) &= b_1 \otimes R \cdots \otimes R b_n \otimes R 1_B \otimes R^p m, \\
\tau_n (\tilde{b} \otimes R^p m) &= (m_{<1>} \otimes R b_1 \otimes R \cdots \otimes R b_{n-1}) \triangleleft b_{n}^- \otimes R^p b_{n}^+ m_{<0>},
\end{align*}
\]

where \( \tilde{b} = b_1 \otimes R \cdots \otimes R b_n \).

### 3 Equivariant Hopf Galois extensions

In this section we first define equivariant Hopf Galois extensions. An equivariant Hopf Galois extension is a quadruple \((K, B, T, S)\) satisfying certain properties as stated in Definition 3.2. In the Subsection 3.1 we show that any equivariant Hopf Galois extension \((K, B, T, S)\) defines a functor from the category of SAYD modules over \( K \) to the category of SAYD modules over \( B \) such that their Hopf cyclic complexes with corresponding coefficients are isomorphic. In the Subsection 3.2 we introduce an example of equivariant Hopf Galois extension and explicitly illustrate the functor between the categories of SAYD modules.

Let \( B \) be a right \( \times _R \)-Hopf algebra. We say the algebra \( T \) is a right \( B \)-comodule algebra via \( \nabla : T \to T \otimes R B \) if \( T \) is \( R \)-bimodule and right \( B \)-comodule satisfying the following conditions

i) \((t_1 r)t_2 = t_1 (rt_2)\), (multiplication in \( T \) is \( R \)-balanced).

ii) \(1_{T_{<0>}} \otimes _R 1_{T_{<1>}} = 1_T \otimes _R 1_B\),

iii) \((tt')_{<0>} \otimes _R (tt')_{<1>} = t_{<0>} t'_{<0>} \otimes _R t_{<1>} t'_{<1>}\).

The coinvariants subalgebra \( S \subseteq T \) is defined by

\[
S := T^B = \{ s \in T | \quad \text{where } \nabla(s) = s \otimes R 1_B \}. \quad (3.1)
\]
Definition 3.1. Let $R$ be an algebra over $\mathbb{C}$, $\mathcal{B}$ a right $\times_R$-Hopf algebra, $T$ a right $\mathcal{B}$-comodule algebra, $S = T^\mathcal{B}$, $\mathcal{K}$ a left $\times_S$-Hopf algebra, and $T$ a left $\mathcal{K}$-module algebra. $T$ is called a $\mathcal{K}$-equivariant $\mathcal{B}$-Galois extension of $S$, if the canonical map

$$\text{can} : T \otimes_S T \rightarrow T \otimes_R \mathcal{B}, \quad t' \otimes_S t \mapsto t' t_{<0>} \otimes_R t_{<1>}$$

is bijective and the right coaction of $\mathcal{B}$ over $T$ is $\mathcal{K}$-equivariant, i.e.,

$$(kt)_{<0>} \otimes_R (kt)_{<1>} = kt_{<0>} \otimes_R t_{<1>}, \quad k \in \mathcal{K}, \ t \in T. \tag{3.3}$$

We denote a $\mathcal{K}$-equivariant $\mathcal{B}$-Galois extension described in the definition by $\mathcal{K}T(S)^\mathcal{B}$. One observes that by $\mathcal{K} := B \otimes B^{op}$ in the above definition we recover the ordinary Hopf Galois extension since for any $\mathcal{B}$-comodule algebra $T$ the Galois map (3.2) is always $B \otimes B^{op}$-equivariant.

One defines left $\mathcal{K}$-module structures on $T \otimes_S T$ and $T \otimes_R \mathcal{B}$, respectively, as follows.

$$k \triangleright (t_1 \otimes_S t_2) = k^{(1)} t_1 \otimes_S k^{(2)} t_2, \quad \text{and} \quad k \triangleright (t \otimes_R b) = kt \otimes_R b. \tag{3.4}$$

The $\mathcal{K}$-equivariant condition of $\mathcal{K}T(S)^\mathcal{B}$ implies that

$$\text{can}(k \triangleright (t' \otimes_S t)) = k \triangleright \text{can}(t' \otimes_S t), \quad k \in \mathcal{K}, \ t, t' \in T. \tag{3.5}$$

We denote the inverse of the Galois map (3.2) by the following index notation

$$\text{can}^{-1}(1 \otimes_R b) := b_{<->} \otimes_S b_{<>}. \tag{3.6}$$

One has the following properties for the maps $\text{can}$ and $\text{can}^{-1}$.

Lemma 3.2. Let $\mathcal{K}T(S)^\mathcal{B}$ be a $\mathcal{K}$-equivariant $\mathcal{B}$-Galois extension. Then the following properties hold.

\begin{enumerate}
  \item $\text{can}(k \triangleright t) b_{<->} \otimes_S b_{<>} = (k^{(1)} t)(k^{(2)} b_{<->}) \otimes_S k^{(3)} b_{<>}.$
  \item $(bb')_{<->} \otimes_S (bb')_{<>} = b'_{<->} b_{<>} \otimes_S b_{<>} b'_{<>}.$
  \item $b_{<->} \otimes_R b_{<>} b_{<>} \otimes_S b_{<>} = b^+_{<->} \otimes_R b^- \otimes_S b^+_{<>}.$
  \item $b_{<->} \otimes_S b_{<>} b_{<>} \otimes_R b_{<>} = b^{(1)}_{<->} \otimes_S b^{(1)}_{<>} \otimes_R b^{(2)}.$
  \item $b_{<->} b_{<>} b_{<>} \otimes_R b_{<>} = 1 \otimes_R b.$
\end{enumerate}
vi) $t_{<0>} t_{<1>} <_-> \otimes S t_{<1>} <_+> = 1 \otimes_S t$.

vii) $1_B <-> \otimes_S 1_B <_+> = 1_T \otimes_S 1_T$.

eviii) $t(r) <-> \otimes_S t(r) <_+> = r > 1_T \otimes_S 1_T$.

ix) $s(r) <-> \otimes_S s(r) <_+> = 1_T \otimes_S 1_T \triangleleft r$.

Proof. One easily sees that (i) is equivalent to $\mathcal{K}$-equivariant property of the map $can^{-1}$. One can find a proof of the rest in [KS, pages 268-270].

3.1 Equivariant Hopf Galois extension as a functor

Let $\mathcal{K} T(S)^B$ be a $\mathcal{K}$-equivariant $B$-Galois extension, and $M$ be a right-left SAYD module over $\mathcal{K}$. We let $B$ act on $\tilde{M} := M \otimes_K T$ on the left by

$$b \triangleright (m \otimes_K t) = m_{<0>} \otimes_K b_{<+>} (m_{<-1>} \triangleright (tb_{<->})). \quad (3.7)$$

We also let $B$ coact on $\tilde{M}$ from the right by

$$(m \otimes_K t) \mapsto m \otimes_K t_{<0>} \otimes_R t_{<1>}.$$

$$(3.8)$$

Theorem 3.3. Let $R$ be an algebra over $\mathbb{C}$, $\mathcal{B}$ a right $\times_R$-Hopf algebra, $T$ a right $\mathcal{B}$-comodule algebra, $S = T^B$, $\mathcal{K}$ a left $\times_S$-Hopf algebra, $T$ a left $\mathcal{K}$-module algebra and $M$ be a right-left SAYD module over $\mathcal{K}$. If $\mathcal{K} T(S)^B$ is a $\mathcal{K}$-equivariant $B$-Galois extension, then $\tilde{M} := M \otimes_K T$ is a left-right SAYD module over $B$ by the action and coaction defined in $(3.7)$ and $(3.8)$.

Proof. The above coaction is obviously well-defined by the $\mathcal{K}$-equivariant property of the coaction of $\mathcal{B}$ over $T$. We shall show that the action $(3.7)$ is well-defined. We show that

$$b \triangleright (m \triangleleft k \otimes_K t) = b \triangleright (m \otimes_K k \triangleright t), \quad b \in \mathcal{B}, \quad m \in M, \quad k \in \mathcal{K}, \quad t \in T.$$
Indeed,

\[ b \triangleright (m \triangleleft k \otimes K t) = (mk)_{<0>} \otimes_K b_{<+>} ((mk)_{<-1>} \triangleright (tb_{<-})) \]
\[ = m_{<0>} (k(2)^- \otimes_K b_{<+>} (k(2)^+ m_{<-1>} k(1) \triangleright (tb_{<-}))) \]
\[ = m_{<0>} \otimes_K k(2)^- \triangleright (b_{<+>} (k(2)^+ m_{<-1>} k(1) \triangleright (tb_{<-}))) \]
\[ = m_{<0>} \otimes_K (k(2)^- b_{<+>})(k(2)^- k(2)^+ m_{<-1>} k(1) \triangleright (tb_{<-}))) \]
\[ = m_{<0>} \otimes_K (k(2)^- b_{<+>})(m_{<-1>} k(1) \triangleright (tb_{<-}))) \]
\[ = m_{<0>} \otimes_K b_{<+>} ((k \triangleright t) b_{<-})) \]
\[ = b \triangleright (m \otimes_K k \triangleright t). \]

We use the SAYD condition \[ (2.8) \] for \( M \) over \( K \) in the second equality, the \( K \)-module algebra property of \( T \) in the fourth equality, the Lemma \[ (2.2)(i) \] in the fifth equality, the \( K \)-module algebra property of \( T \) in the sixth equality and the Lemma \[ (3.2)(i) \] in the seventh equality. Next we show that the action \[ (3.7) \] is \( S \)-balanced both in \( b_{<+>} \otimes_S b_{<+>} \) and \( m_{<-1>} \otimes_S m_{<0>} \). That is for any \( q, p \in S, m \in M, k \in K \) and \( t_1, t_2, t_3 \in T \), we should show

\[ pm \otimes_K t_1 (k \triangleright t_2 t_3) = m \otimes_K t_1 (t(p) k \triangleright t_2 t_3), \tag{3.9} \]

which is obvious because

\[ pm \otimes_K t_1 (k \triangleright t_2 t_3) = m \triangleleft (t(p) \otimes_K t_1 (k \triangleright t_2 t_3) \]
\[ = m \otimes_K t_1 (t(p) \triangleright t_1 (k \triangleright t_2 t_3)) = m \otimes_K t_1 (t(p) k \triangleright t_2 t_3), \]

where \( t \) is the target map of the left bialgebroid \( K \). In the above we use \( pm = m \triangleleft t(p) \) and \( \Delta(t(p)) = 1_K \otimes_S t(p) \). This allows then to substitute \( k \otimes_S m = n_{<-1>} \otimes_S n_{<0>} \) for any \( n \in M \). To finish the argument that the action \[ (3.7) \] is \( S \)-balanced we should also show

\[ m_{<0>} \otimes_K t_1 (m_{<-1>} \triangleright t_2 t_3 q) = m_{<0>} \otimes_K qt_1 (m_{<-1>} \triangleright t_2 t_3), \tag{3.10} \]

which is clear because

\[ m_{<0>} \otimes_K t_1 (m_{<-1>} \triangleright t_2 t_3 q) = m_{<0>} \otimes_K t_1 (m_{<-1>} t(q) \triangleright t_2 t_3) \]
\[ = m_{<0>} q \otimes_K t_1 (m_{<-1>} \triangleright t_2 t_3) = m_{<0>} \triangleleft s(q) \otimes_K t_1 (m_{<-1>} \triangleright t_2 t_3) \]
\[ = m_{<0>} \otimes_K s(q) \triangleright t_1 (m_{<-1>} \triangleright t_2 t_3) = m_{<0>} \otimes_K qt_1 (m_{<-1>} \triangleright t_2 t_3), \]

where \( s \) is the source map of the left bialgebroid \( K \). In the above we use \[ (2.7) \] in the second equality. Thus it makes sense to substitute \( t_3 \otimes_S t_1 =
Next we show that (3.7) is associative. For this, let \( g, h \in \mathcal{B} \) and \( m \otimes \mathcal{K} t \in \widetilde{M} \).

\[
g \triangleright (h \triangleright (m \otimes \mathcal{K} t)) = g \triangleright (m_{<0>} \otimes \mathcal{K} h_{<+>}(m_{<1>} \triangleright (th_{<+>}))) = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<0>} \rightarrow (th_{<+>}))
\]

\[
eq m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>}\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]

\[
= m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\} = m_{<0>} \otimes \mathcal{K} g_{<+>\{m_{<1>} \rightarrow (th_{<+>}))g_{<+>}\}
\]
We use the $\mathcal{K}$-equivariant property of the coaction of $B$ on $T$ and the $B$-comodule algebra property of $T$ in the second equality, the Lemma 3.2(iv) in the third equality and the Lemma 3.2(iii) in the fourth equality.

The stability condition of $\tilde{M}$ over $B$ follows from
\[
\begin{align*}
(m \otimes_K t)_{<1>} (m \otimes_K t)_{<0>} &= t_{<1>} (m \otimes_K t)_{<0>} \\
&= m_{<0>} \otimes_K (t_{<1>} \cdot_{<+>} (m_{<1>} \cdot_{<>} (t_{<0>} \cdot_{<1>} <_{<+>}))) \\
&= m_{<0>} \otimes_K t(m_{<1>} \cdot_{>1} 1_T) = m_{<0>} \otimes_K t(t(\varepsilon(m_{<-1}))) \cdot_{1_T} 1_T \\
&= m_{<0>} \otimes_K t(\varepsilon(m_{<-1})) \cdot_{>} t = m_{<0>} \cdot_{<} t(\varepsilon(m_{<-1})) \otimes_K t \\
&= m \otimes_K t,
\end{align*}
\]

Lemma 3.2 (vi) is applied in the third equality, the module algebra property in fourth equality and the counitality of the coaction on $M$ is used in the last equality.

Let us denote the category of SAYD modules over a left $\times$-Hopf algebra $K$ by $K\text{SAYD}_K$. Its objects are all right-left SAYD modules over $K$ and its morphisms are all $K$-linear-colinear maps. Similarly one denotes by $B\text{SAYD}_B$ the category of left-right SAYD modules of a right $\times$-Hopf algebra $B$. We see that Theorem 3.3 amounts to the object map of a functor $F$ from $K\text{SAYD}_K$ to $B\text{SAYD}_B$. Let $\phi : M \rightarrow N$ be a morphism in $K\text{SAYD}_K$ we define $F(\phi)$ to be $\phi \otimes_K 1_{T}$.

**Proposition 3.4.** The assignment $F : K\text{SAYD}_K \rightarrow B\text{SAYD}_B$ defines a covariant functor.

**Proof.** Using Theorem 3.3 we see that $F$ is an object map. We now prove that it is also a morphism map. Let $\phi : M \rightarrow N$ be a morphism in $K\text{SAYD}_K$. The colinearity of $F(\phi)$ is obvious. The following shows that $F(\phi)$ is also linear,
\[
(\phi \otimes Id)(b \cdot (m \otimes_K t)) = (\phi \otimes Id)(m_{<0>} \otimes_K b_{<+>} (m_{<1>} \cdot_{<+>} (tb_{<-,>}))) = \\
\phi(m_{<0>} \otimes_K b_{<+>} (m_{<-1>} \cdot_{<-,>} (tb_{<-1>}))) = \phi(m)_{<0>} \otimes_K b_{<+>} (\phi(m)_{<-1>} \cdot_{<-,>} (tb_{<-1>})) = \\
b \cdot (\phi(m) \otimes t) = b \cdot ((\phi \otimes Id)(m \otimes_K t)).
\]

We use the comodule map property of $\phi$ in the third equality. The covariant property of the functor is obvious.

Let us recall that $K C_n(T,M) = M \otimes_K T^{\otimes n+1}$ and $C_{B,n}(B,\tilde{M}) := B^{\otimes n} \otimes_{R^{op}} \tilde{M}$ are the cyclic modules defined in (2.23) and (2.31) respectively. Now we define the following maps.
and the

$$\omega_n : M \otimes_K T^{\otimes s(n+1)} \rightarrow \mathcal{B}^{\otimes R^n} \otimes_{R^{\text{op}}} \tilde{M}, \quad (3.11)$$
given by

$$\omega_n(m \otimes_K t_0 \otimes S \cdots \otimes S t_n) = t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R t_{2<2>} \cdots t_{n<2>} \otimes_R \cdots \otimes_R t_{n<n>} \otimes_{R^{\text{op}}} (m \otimes_K t_{0<1>} \cdots t_{n<0>}),$$

and

$$\omega_n^{-1}(b_1 \otimes_R \cdots \otimes_R b_{n+1} \otimes_{R^{\text{op}}} m \otimes_K t) = m \otimes_K t b_{1<->} \otimes_S b_{1<->} b_{2<->} \otimes_S \cdots \otimes_S b_{n-1<->} b_{n<->}.$$  

**Theorem 3.5.** Let $K T(S)^B$ be a $K$-equivariant $B$-Galois extension, and $M$ be a right-left SAYD module over $K$. Then $\omega_n$ defines an isomorphism of cyclic modules between $K C_s(T, M)$ and $\tilde{C}_{B,s}(\mathcal{B}, M)$, which are defined in (2.23) and (2.24) respectively. Here $\tilde{M} := M \otimes_K T$ is the left-right SAYD module over $\mathcal{B}$ introduced in Theorem 3.3.

**Proof.** The map $\omega_n$ is the composition of the following maps

$$M \otimes_K T^{\otimes s(n+1)} \xrightarrow{\alpha_n} \tilde{M} \otimes_R \mathcal{B}^{\otimes R^n} \xrightarrow{\psi_n} \mathcal{B}^{\otimes R^n} \otimes_{R^{\text{op}}} \tilde{M}. \quad (3.12)$$

Here $\alpha_n$ and $\psi_n$ are given by

$$\alpha_n(m \otimes_K t_0 \otimes S \cdots \otimes S t_n) = m \otimes_K t_0 t_{1<0>} \cdots t_{n<0>} \otimes_R t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R t_{n-1<n-1>} \otimes_R t_{n<n>},$$

$$\alpha_n^{-1}(m \otimes_K t \otimes_R b_1 \otimes_R \cdots \otimes_R b_n) = m \otimes_K t b_{1<->} \otimes_S b_{1<->} b_{2<->} \otimes_S \cdots \otimes_S b_{n-1<->} b_{n<->} \otimes_S b_{n<->}.$$ 

$$\psi_n(m \otimes_K t \otimes_R b_1 \otimes_R \cdots \otimes_R b_n) = b_1 \otimes_R \cdots \otimes_R b_n \otimes_{R^{\text{op}}} m \otimes_K t,$$

$$\psi_n^{-1}(b_1 \otimes_R \cdots \otimes_R b_n \otimes_{R^{\text{op}}} m \otimes_K t) = m \otimes_K t \otimes_R b_1 \otimes_R \cdots \otimes_R b_n.$$ 

One can easily use the $K$-equivariant property of the coaction of $\mathcal{B}$ over $T$ and the $K$-module algebra property of $T$ to see that the map $\alpha_n$ is well-defined. Also using $\Delta(s(r)) = 1_B \otimes_R s(r)$ and $r \triangleright (b_1 \otimes_R \cdots \otimes_R b_n) = b_1 s(r) \otimes_R \cdots \otimes_R b_n$, the morphism $\psi_n$ is obviously well-defined. We show that $\omega$ is a map of cyclic modules. By multiplicity of the coproduct of $\mathcal{B}$, right $\mathcal{B}$-comodule algebra property of $T$ and the right $\times_R$-Hopf algebra property.
Next we show that the maps $\delta_i, 0 \leq i \leq n - 1$ commute with $\omega$ as follows.

$$b^{(2)}(t(\varepsilon(t(r)b^{(1)}))) = t(r)b,$$
the maps $\delta_i, 0 \leq i \leq n - 1$ commute with $\omega$ as follows.

$$\delta_i(\omega_n(m \otimes t_0 \otimes S \cdots \otimes t_n)) = \delta_i(t_{1<1} \otimes t_{2<1} \cdots t_{n<1} \otimes_R t_{2<2} \cdots t_{n<2}) \otimes_R t_{n<0} \otimes_R t_{n<0} \otimes_R t_{n<0})$$

$$= t_{1<1} \otimes t_{2<1} \cdots t_{n<1} \otimes_R \varepsilon(t_{i+1<i+1} \cdots t_{n<i+1}) \otimes_R$$

$$= t_{1<1} \otimes t_{2<1} \cdots t_{n<1} \otimes_R \cdots \otimes_R t_{1<1} \cdots t_{n<1} \otimes_R t_{i+2<i+1} \cdots t_{n<i+1} \otimes_R$$

$$= t_{1<1} \otimes t_{2<1} \cdots t_{n<1} \otimes_R \cdots \otimes_R t_{i+1<i+1} \otimes_R t_{i+2<i+1} \cdots t_{n<i+1} \otimes_R$$

$$= \omega_{n-1}(m \otimes t_0 \otimes S \cdots \otimes t_i \cdots \otimes S t_n)$$

Next we show that $\omega$ commutes with the last face morphism.

$$\delta_n(\omega_n(m \otimes t_0 \otimes S \cdots \otimes t_n)) = \delta_n(t_{1<1} t_{2<1} \cdots t_{n<1} \otimes_R$$

$$t_{2<2} \cdot \cdot \cdot t_{n<2} \otimes_R \cdots \otimes_R t_{n<0} \otimes_R \cdots \otimes_R t_{n<0})$$

$$= (t_{1<1} \cdots t_{n<1} \otimes_R \cdots \otimes_R t_{n-1<n-1} \otimes_R t_{n<n-1}) \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$= t_{1<1} \cdots t_{n-1<n-1} \otimes_R \cdots \otimes_R t_{n-1<n-1} \otimes_R \cdots \otimes_R t_{n-1<n-1} \otimes_R$$

$$= t_{1<1} \cdots t_{n-1<n-1} \otimes_R \cdots \otimes_R t_{n-1<n-1} \otimes_R \cdots \otimes_R t_{n-1<n-1} \otimes_R$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$\cdots \otimes_R t_{n<n-1} \otimes_R t_{n<n-1}$$

$$= \omega_{n-1}(m \otimes t_0 \otimes S \cdots \otimes S t_n).$$

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We apply the right diagonal action of $B$ over $B^{\otimes R(n-1)}$ in the third equality, the lemma [BS 2.14 .(ii)] in the fourth equality, the action (3.7) in the fifth equality, the Lemma (3.2)(vi) in the sixth equality, the $\mathcal{K}$-module algebra property of $T$ in the seventh equality and $\mathcal{K}$-equivariant property of the coaction of $B$ over $T$ in the eighth equality. By multiplicity of coproduct of $B$ the commutativity of $\omega$ and $\sigma_i$, $0 \leq i \leq n - 1$, can be verified as follows.

$$
\sigma_i(\omega_n(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n)) \\
= \sigma_i(t_{1<1}> t_{2<1>} \cdots t_{n<1>} \otimes_R t_{2<2>} \cdots t_{n<2>} \otimes_R \\
\cdots \otimes_R t_{n<n>} \otimes_R (m \otimes_\mathcal{K} t_0 t_{1<0>} \cdots t_{n<0})))) \\
= t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R (t_{i<i+1}> \cdots t_{n<i+1>})(1) \otimes_R (t_{i+1<i+1>} \cdots t_{n<i+1>})(2) \otimes_R \cdots \otimes_R \\
t_{n<n>} \otimes_R (m \otimes_\mathcal{K} t_0 t_{1<0>} \cdots t_{n<0})))) \\
= t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R t_{i+1<i+1>} \cdots t_{n<i+1>} \otimes_R t_{i+1<i+2>} \cdots t_{n<i+2>} \cdots \otimes_R \\
t_{n<n+1>} \otimes_R (m \otimes_\mathcal{K} t_0 t_{1<0>} \cdots t_{n<0})))) \\
= \omega_{n+1}(m \otimes_\mathcal{K} t_0 \otimes S \cdots t_i \otimes S \cdots t_n) \\
= \omega_{n+1}(\sigma_i(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n)).
$$

Using Lemma [BS 2.14.(iv)], the commutativity of $\omega$ with the last degeneracy follows from

$$
\sigma_n(\omega_n(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n)) \\
= \sigma_n(t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R t_{2<2>} \cdots t_{n<2>} \otimes_R \\
\cdots \otimes_R t_{n<n>} \otimes_R (m \otimes_\mathcal{K} t_0 t_{1<0>} \cdots t_{n<0})))) \\
= t_{1<1>} t_{2<1>} \cdots t_{n<1>} \otimes_R t_{2<2>} \cdots t_{n<2>} \otimes_R \cdots \otimes_R \\
t_{n<n>} \otimes_R (1_T \otimes R (m \otimes_\mathcal{K} t_0 t_{1<0>} \cdots t_{n<0})))) \\
= \omega_{n+1}(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n) \\
= \omega_{n+1}(\sigma_n(m \otimes_\mathcal{K} t_0 \otimes S \cdots \otimes S t_n)).
$$
Finally, we show that $\omega$ commutes with the cyclic maps.

\[
\begin{align*}
\tau_n(\omega_n(m \otimes K t_0 \otimes S \cdots \otimes S t_n)) &= \tau_n(t_{1<1}t_{2<1} \cdots t_{n<1} \otimes R t_{2<2} \cdots t_{n<2} \otimes R \\
&\quad \cdots \otimes R t_{n<\langle n \rangle} \otimes_{R^\text{op}} (m \otimes K t_{0<0} \cdots t_{n<0})) \\
&= (t_{0<1}t_{1<0} \cdots t_{n<\langle n \rangle} \otimes R t_{1<1} \cdots t_{n<2} \otimes R \\
&\quad \cdots \otimes R t_{n-1\langle n \rangle-1} t_{n-1\langle n \rangle} \otimes_{R^\text{op}} t_{n<\langle n \rangle}^\text{op} \\
t_{n<\langle n \rangle} + (m \otimes K t_{0<0} \otimes t_{1<0} \cdots t_{n<0}) \\
&= (t_{0<1}t_{1<1} \cdots t_{n-1\langle n \rangle} \otimes R t_{1<2} \cdots t_{n<2} \otimes R \\
&\quad \cdots \otimes R t_{n-1\langle n \rangle-1} t_{n-n<\langle n \rangle} \otimes_{R^\text{op}} t_{n<\langle n \rangle}^\text{op} \\
t_{n<\langle n \rangle} + (m \otimes K t_{0<0} \otimes t_{1<0} \cdots t_{n<0}) \\
&= t_{0<1}t_{1<1} \cdots t_{n-1\langle n \rangle} \otimes R t_{1<2} \cdots t_{n-1\langle n \rangle} \otimes R \\
&\quad \cdots \otimes R t_{n-1\langle n \rangle-1} t_{n-1\langle n \rangle} \otimes_{R^\text{op}} t_{n<\langle n \rangle}^\text{op} \\
t_{n<\langle n \rangle} + (m \otimes K t_{0<0} \otimes t_{1<0} \cdots t_{n<0}) \\
&= t_{0<1}t_{1<1} \cdots t_{n-1\langle n \rangle} \otimes R t_{1<2} \cdots t_{n-1\langle n \rangle} \otimes R \\
&\quad \cdots \otimes R t_{n-1\langle n \rangle-1} t_{n-1\langle n \rangle} \otimes_{R^\text{op}} t_{n<\langle n \rangle}^\text{op} \\
m_{<\langle n \rangle} \otimes K t_{n<\langle n \rangle} (m_{<\langle n \rangle} \otimes K t_{0<0} \cdot \cdots t_{n<0}, t_{n<\langle n \rangle}^{\text{op}}) \\
&= t_{0<1}t_{1<1} \cdots t_{n-1\langle n \rangle} \otimes R t_{1<2} \cdots t_{n-1\langle n \rangle} \otimes R \\
&\quad \cdots \otimes R t_{n-1\langle n \rangle-1} t_{n-1\langle n \rangle} \otimes_{R^\text{op}} t_{n<\langle n \rangle}^\text{op} \\
m_{<\langle n \rangle} \otimes K t_{n<\langle n \rangle} (m_{<\langle n \rangle} \otimes K t_{0<0} \cdot \cdots t_{n<0}, t_{n<\langle n \rangle}^{\text{op}}) \\
&= \omega_n(m_{<\langle n \rangle} \otimes K t_{n} \otimes S m_{<\langle n \rangle} t_{0} \otimes S \cdots \otimes S m_{<\langle n \rangle} t_{n-1}) \\
&= \omega_n \tau_n(m \otimes K t_{0} \otimes S \cdots \otimes S t_{n}).
\end{align*}
\]

We use the coaction property of $T$ over $B$ in the second equality, the right diagonal action of $B$ over $B^{\otimes R \langle n \rangle}$ in the third equality, Lemma [BS, 2.14.(ii)] in the fourth equality, the action $[5.7]$ in the fifth equality, Lemma [3.2](vi) in the sixth equality, the $K$-module algebra property of $T$ in the seventh equality and $K$-equivariant property of the coaction of $B$ over $T$ in the eighth equality.

\[\square\]

### 3.2 Examples

As the first example, one notes that by using Proposition [2.6] and Theorem [3.3] for $K = S \otimes S^{\text{op}}$ and $M = S$, we see that Theorem [3.5] generalizes Theorem 2.6 at [BS].

We like to illustrate the SAYD module constructed in Theorem 3.3 in an explicit and nontrivial example. Let $\mathcal{H}$ and $\mathcal{F}$ denote two Hopf algebras such
that $\mathcal{H}$ acts on $\mathcal{F}$ and makes it a left module algebra. We let $A := \mathcal{F} \rtimes \mathcal{H}$ be the usual crossed product algebra, that is $\mathcal{F} \otimes \mathcal{H}$ as a vector space with multiplication $(f \triangleright h)(g \triangleright v) = f(h(1) \triangleright g) \triangleright h(2)v$, and $1 \triangleright v$ as its unit. One knows that $\nabla : A \to A \otimes \mathcal{H}$ defined by $\nabla(f \triangleright h) = f \triangleright h(1) \otimes h(2)$ defines a comodule algebra over the Hopf algebra $\mathcal{H}$. One also easily verifies that $B := A^\mathcal{H} = \mathcal{F} \otimes \mathbb{C} \simeq \mathcal{F}$, and $A(B)^\mathcal{H}$ is a Hopf Galois extension. We let $\mathcal{K} := B \otimes \mathcal{F} \otimes B^a$ be the usual left $\times_B$-Hopf algebra, i.e., as an algebra its $\times_B$ tensor product of algebras and its coring structure is given by,

$$s(b) = b \otimes 1 \otimes 1, \quad t(b) = 1 \otimes 1 \otimes b,$$

$$\Delta(b_1 \otimes f \otimes b_2) = b_1 \otimes f(1) \otimes 1 \otimes_B 1 \otimes f(2) \otimes b_2, \quad (3.13)$$

$$\varepsilon(b_1 \otimes f \otimes b_2) = \varepsilon(f)b_1b_2,$$

One observes that

$$\nu^{-1}(b_1 \otimes f \otimes b_2) - \otimes_B (b_1 \otimes f \otimes b_2)^+ = b_1 \otimes f(1) \otimes 1 \otimes_B b_2 \otimes S(f(2))g \otimes b_4. \quad (3.14)$$

So we see that

$$(b_1 \otimes f \otimes b_2)^- \otimes_B (b_1 \otimes f \otimes b_2)^+ = b_1 \otimes f(1) \otimes 1 \otimes_B b_2 \otimes S(f(2)) \otimes 1. \quad (3.15)$$

Now let $N$ be a right-left SAYD module over the Hopf algebra $\mathcal{F}$ and let also $x \otimes x^{-1} \in B \otimes B^a$ be a group-like element with $x \in Z(B)$, the center of $B$. We define the following action and coaction on $M := x \otimes x^{-1}B \otimes N = B \otimes N$.

$$\nabla(b \otimes n) = bx \otimes n_{<1>} \otimes x^{-1} \otimes_B 1 \otimes n_{<0>}, \quad (3.16)$$

$$(b \otimes n) \cdot (b_1 \otimes f \otimes b_2) = b_2bb_1 \otimes n \cdot f \quad (3.17)$$

**Lemma 3.6.** Via the above action and coaction $x \otimes x^{-1}B \otimes N$ is a right-left SAYD module over $\mathcal{K}$.

**Proof.** It is obvious that the action and coaction are well-defined. In the following we check the AYD condition,

$$\nabla((b \otimes n) \cdot (b_1 \otimes f \otimes b_2)) = \nabla(b_2bb_1 \otimes n \cdot f) =$$

$$b_2bb_1x \otimes (nf)_{<1>} \otimes x^{-1} \otimes_B 1 \otimes (n \cdot f)_{<0>} =$$

$$b_2bb_1x \otimes S(f(3))n_{<1>}f(1) \otimes x^{-1} \otimes_B 1 \otimes n_{<0>} \cdot f(2) =$$

$$(b_2 \otimes S(f(3)) \otimes 1)(bx \otimes n_{<1>} \otimes x^{-1})(b_1 \otimes f(1) \otimes 1) \otimes_B (1 \otimes n_{<0>}) \cdot (1 \otimes f(2) \otimes 1) =$$

$$k_{(2)}^+n_{<1>}k_{(1)} \otimes_B n_{<0>} \cdot k_{(2)}^-.$$ 

The stability condition follows from the stability of $N$. \qed
One then defines a left action of $K$ on $A$ by
\[
(b_1 \otimes f \otimes b_2) \triangleright (g \triangleright h) = b_1 f_1 (g (h_1 \triangleright (S(f_2)b_2))) \triangleright h_2. \tag{3.18}
\]

**Lemma 3.7.** Let $F$ be commutative. Then the above action makes $A$ a $K$-module algebra.

**Proof.** Let us first prove that the equation (3.18) defines an action. Indeed for $k_1 := b_1 \otimes e \otimes b_2$ and $k_2 := b_3 \otimes f \otimes b_4$ by using the facts that $F$ is commutative and $F$ is $H$-module algebra we have
\[
k_1 \triangleright (k_2 \triangleright (g \triangleright h)) = k_1 \triangleright (b_3 f_1 (g (h_1 \triangleright (S(f_2)b_4)))) \triangleright h_2) =
(b_1 b_3 e_1 f_1 (g (h_1 \triangleright (S(f_2)b_4))) \triangleright h_2) \triangleright h_2) = k_1 k_2 \triangleright (g \triangleright h).
\]

Obviously the action defined in (3.18) is unital. We use $h \triangleright 1_F = \varepsilon(h)1_F$ to see that
\[
s(b) \triangleright (f \triangleright h) = (b \otimes 1 \otimes 1) \triangleright (f \triangleright h) = bf \triangleright h. \nonumber
\]
\[
t(b) \triangleright (g \triangleright h) = (1 \otimes 1 \otimes b) \triangleright (g \triangleright v) = g(h_1 \triangleright b) \triangleright h_2. \nonumber
\]

This tells us that the multiplication of $K$ is $B$-balanced. Now we show that $k \triangleright (a_1 a_2) = (k_1 \triangleright a_1)(k_2 \triangleright a_2)$. Let $k = (b_1 \otimes f \otimes b_2)$, $a_1 = g \triangleright h$, and $a_2 = l \triangleright v$, we have
\[
k \triangleright (a_1 a_2) = k \triangleright (gh_1 \triangleright l \triangleright h_2 \triangleright v) = b_1 f_1 (g (h_1 \triangleright l)(h_2 \triangleright v_1)(S(f_2)b_2)) \triangleright h_3 \triangleright v_2). \nonumber
\]

On the other hand by using the fact that $F$ is $H$-module algebra we see
\[
(k_1 \triangleright a_1)(k_2 \triangleright a_2) = ((b_1 \otimes f_1 \otimes 1) \triangleright (g \triangleright h))((1 \otimes f_2 \otimes b_2) \triangleright (l \triangleright v)) = (b_1 f_1 (g (h_1 \triangleright S(f_2))) \triangleright h_2)(f_3 l(v_1) \triangleright (S(f_4)b_2)) \triangleright v_2) = b_1 f_1 (g (h_1 \triangleright S(f_2)))(h_2 \triangleright (f_3 l(v_1) \triangleright (S(f_4)b_2))) \triangleright v_2) = b_1 f_1 (g (h_1 \triangleright S(f_2)))(h_2 \triangleright f_3)(h_3 \triangleright l)(h_4 v_1) \triangleright (S(f_4)b_2)) \triangleright h_3 \triangleright v_2) = b_1 f_1 (g (h_1 \triangleright l)(h_2 v_1) \triangleright (S(f_2)b_2)) \triangleright h_3 \triangleright v_2). \nonumber
\]

Finally we check the condition (2.16). For $k := b_1 \otimes f \otimes b_2$ we have,
\[
k \triangleright 1_A = (b_1 \otimes f \otimes b_2) \triangleright (1 \triangleright 1) = b_1 f_1 (S(f_2)b_2) \triangleright 1 =
\varepsilon(f)b_2 \triangleright 1 = (\varepsilon(f)b_1 b_2 \otimes 1 \otimes 1) \triangleright (1 \triangleright 1) = s(\varepsilon(k)) \triangleright 1_A. \tag{2.16}
\]

\[
\Box
\]
One easily sees that the coaction $\nabla : A \to A \otimes H$ is $K$ equivariant.

We now let the group of all unit elements of $B$, which is denoted by $B^\times$, define a category whose objects and morphisms are both elements of $B^\times$. More precisely, for $b_1, b_2 \in B^\times$, $\text{Mor}(b_1, b_2)$ consists of the unique element denoted by $\hat{c}$, where $c = b_2 b_1^{-1}$. We let $B^\times \times F \text{SAYD}_F$ be the product of categories. Here $F \text{SAYD}_F$ is the category whose objects are left-right SAYD modules over $F$ and morphisms are $F$-linear and $F$-colinear maps. We define the following functor

$$\Phi : B^\times \times F \text{SAYD}_F \to K \text{SAYD}(H)^K,$$

$$\Phi((x, N)) = x \otimes x^{-1} B \otimes N,$$

$$\Phi(\hat{y}, \phi) : x \otimes x^{-1} B \otimes N_1 \to (xy \otimes (xy)^{-1}) B \otimes N_2,$$

$$\Phi(\hat{y}, \phi)(b \otimes n) = by^{-1} \otimes \phi(n).$$

Here $y \in B^\times$ and $\phi \in F \text{Hom}_F(N_1, N_2)$.

**Proposition 3.8.** The above assignment $\Phi$ defines a covariant functor.

**Proof.** We need to show that $\Phi$ is a morphism map. Indeed, first we see that $\Phi(\hat{y}, \phi)$ is a $K$-colinear map. Using the facts that $\phi$ is a $F$-colinear map, that $B$ is commutative, and also the bialgebroid structure of $K = B \otimes F \otimes B$, we see

$$\nabla(\Phi(\hat{y}, \phi)(b \otimes n)) = \nabla(by^{-1}, \phi(n)) =$$

$$by^{-1} xy \otimes \phi(n)_{<0>} \otimes (xy)^{-1} \otimes B 1 \otimes \phi(n)_{<0>} =$$

$$bx \otimes \phi(n_{<0>}) \otimes x^{-1} y^{-1} \otimes B 1 \otimes \phi(n_{<0>}).$$

Now we prove that $\Phi(\hat{y}, \phi)$ is a $K$-linear map.

$$\Phi(\hat{y}, \phi)((b \otimes n) \cdot (b_1 \otimes f \otimes b_2)) = \Phi(\hat{y}, \phi)(b_2 bb_1 \otimes n \cdot f) =$$

$$(b_2 bb_1 y^{-1} \otimes \phi(n \cdot f)) = (b_2 bb_1 y^{-1} \otimes \phi(n) \cdot f) =$$

$$(b_2 bb_1 y^{-1} \otimes \phi(n \cdot f)) = (\Phi(\hat{y}, \phi))(b_2 bb_1 \otimes n \cdot f).$$

Finally one uses again the commutativity of $B$ to see that $\Phi((\hat{y}_1, \phi_1) \circ (\hat{y}_2, \phi_2)) = \Phi(\hat{y}_1 y_2, \phi_1 \phi_2) = \Phi(\hat{y}_1, \phi_1) \circ \Phi(\hat{y}_2, \phi_2).$
As a result one composes the functors Φ and F defined in Propositions 3.4 and 3.8 respectively to get the following functor

\[ F \circ \Phi \colon B^\times \times \mathcal{F}SAYD \rightarrow \mathcal{H}SAYD^H. \]  

(3.20)

One notes that in the simplest possible case of this example, i.e., when \( \mathcal{F} := B = C, \ N := C, \ x = 1_C \) the resulting SAYD module is \( \mathcal{H} \) with the standard action and coaction, i.e, the action and coaction are defined by the adjoint action and the comultiplication of \( \mathcal{H} \) respectively.

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