Three generation magnetized orbifold models

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Abstract

We study three generation models in the four-dimensional spacetime, which can be derived from the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory on the orbifold background with a non-vanishing magnetic flux. We classify the flavor structures and show possible patterns of Yukawa matrices. Some examples of numerical studies are also shown.

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1 Introduction

Extra dimensional field theories, in particular string-derived extra dimensional field theories, play important roles in particle physics, e.g. as an origin of the flavor structure including the hierarchy of quark/lepton masses and mixing angles. How to derive chiral theory is a key issue when our starting point is extra dimensional theory. Introduction of a magnetic flux in extra dimensional space is one of interesting ways to obtain chiral theory. Indeed, several studies have been carried out on models with magnetic fluxes in field theories and superstring theories [1, 2, 3, 4, 5, 6, 7, 8]. Furthermore, magnetized D-brane models are T-duals of intersecting D-brane models and within the latter framework several interesting models have been constructed [4, 5, 6, 9, 10, 11].

Zero-modes are quasi-localized on the torus with the magnetic flux. The number of zero-modes, which corresponds to the generation number, is determined by the value of the magnetic flux in the same way as that the generation number is determined by the intersecting number in intersecting D-brane models. Yukawa couplings among zero-modes in four-dimensional effective theory are obtained by overlap integral of zero-mode profiles in extra dimensions. Large or suppressed Yukawa couplings can be derived depending on the size of overlap integral. That is, when zero-modes are quasi-localized far away from each other, their couplings in 4D effective field theory are suppressed. On the other hand, when their localized points are close to each other, 4D effective Yukawa couplings would be of $\mathcal{O}(1)$. Thus, magnetized torus models are quite interesting to derive realistic models, in particular a realistic flavor structure. However, it is still a challenging issue to derive realistic mass matrices of quarks and leptons.

Orbifolding the extra dimensions is another way to derive chiral theory [13]. In Ref. [14], magnetized orbifold models have been studied. Phenomenological aspects in magnetized orbifold models are different from those in magnetized torus models. Some of zero-modes are projected out by the orbifold projection. However, odd modes as well as even modes could correspond to zero-modes, although odd modes correspond to only massive modes on orbifolds without the magnetic flux. Then, the generation number is smaller than the number of the magnetic flux, i.e. one in magnetized torus models with the same magnetic flux. Thus, a new type of flavor structure can appear in magnetized orbifold models. Hence, it is quite important to study in detail phenomenological aspects of magnetized orbifold models. That is our purpose in this paper. We classify three generation models and study predicted patterns of Yukawa matrices.

The paper is organized as follows. In section 2, we give a review on magnetized orbifold models. In section 3, we classify three generation models on the orbifold with magnetic fluxes. In section 4, we study Yukawa couplings in three generation models and we show explicitly an example of numerical studies on our models. Section 5 is devoted to conclusion and discussion. In Appendix, we show explicitly all of possible Yukawa matrices in our three generation models.

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1 See for a review [12] and references therein.
2 Other geometrical backgrounds with a magnetic flux have also been studied [15, 16].
2 Magnetized extra dimensions

Here, we give a review on extra dimensional models with a magnetic flux on torus and orbifold backgrounds [7, 14].

2.1 $U(N)$ gauge theory on $(T^2)^3$

We start with $\mathcal{N} = 1$ ten-dimensional $U(N)$ super Yang-Mills theory. We consider the background $R^{3,1} \times (T^2)^3$, whose coordinates are denoted by $x_\mu$ ($\mu = 0, \cdots, 3$) for the uncompact space $R^{3,1}$ and $y_m$ ($m = 4, \cdots, 9$) for the compact space $(T^2)^3$. At the first stage, we use orthogonal coordinates of the compact space and choose the torus metric such that $y_m$ is identified by $y_m + n_m$ with $n_m = \text{integer}$, i.e. $y_m \sim y_m + 1$. At the end of this subsection, we will extend it by introducing the complex structure. Also, we can extend the following discussions to $\mathcal{N} = 1$ super Yang-Mills theory on $R^{3,1} \times (T^2)^n$.

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4g^2} \text{Tr} \left( F^{MN} F_{MN} \right) + \frac{i}{2g^2} \text{Tr} \left( \bar{\lambda} \Gamma^M D_M \lambda \right),$$

where $M,N = 0, \cdots, 9$. Here, $\lambda$ denotes gaugino fields, $\Gamma^M$ is the gamma matrix for ten-dimensions and the covariant derivative $D_M$ is given as

$$D_M \lambda = \partial_M \lambda - i[A_M, \lambda], \quad (1)$$

where $A_M$ is the vector field. Furthermore, the field strength $F_{MN}$ is given by

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]. \quad (2)$$

The gaugino fields $\lambda$ and the vector fields $A_m$ corresponding to the compact directions are decomposed as

$$\lambda(x,y) = \sum_n \chi_n(x) \otimes \psi_n(y),$$
$$A_m(x,y) = \sum_n \varphi_{n,m}(x) \otimes \phi_{n,m}(y).$$

Here, we concentrate on zero-modes, $\psi_0(y)$ and we denote them as $\psi(y)$ by omitting the subscript “0”. Furthermore, the internal part $\psi(y)$ is decomposed as a product of the $i$-th $T^2$ parts, i.e. $\psi(i)(y_{2i+2}, y_{2i+3})$. Each of $\psi(i)(y_{2i+2}, y_{2i+3})$ is two-component spinor, $\psi(i) = (\psi(i)_+, \psi(i)_-)^T$, and their chirality for the $i$-th $T^2$ part is denoted by $s_i$. We use the gamma matrix $\tilde{\Gamma}^m$ corresponding to the $i$-th $T^2$ as

$$\tilde{\Gamma}^{2i+2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Gamma}^{2i+3} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3)$$

and the total gamma matrices are obtained as their direct products with the four-dimensional part.
Here, we introduce the magnetic flux in the background as $F_{45}$, $F_{67}$ and $F_{89}$, which are given by

$$
F_{45} = 2\pi \begin{pmatrix}
M_1^{(1)} 1_{N_1 \times N_1} & 0 \\
0 & M_n^{(1)} 1_{N_n \times N_n}
\end{pmatrix},
$$

$$
F_{67} = 2\pi \begin{pmatrix}
M_1^{(2)} 1_{N_1 \times N_1} & 0 \\
0 & M_n^{(2)} 1_{N_n \times N_n}
\end{pmatrix},
$$

$$
F_{89} = 2\pi \begin{pmatrix}
M_1^{(3)} 1_{N_1 \times N_1} & 0 \\
0 & M_n^{(3)} 1_{N_n \times N_n}
\end{pmatrix}.
$$

This background breaks the gauge group $U(N)$ as $U(N) \rightarrow \prod_{a=1}^n U(N_a)$ with $N = \sum_a N_a$. We concentrate on an Abelian flux, although in general non-Abelian magnetic fluxes, which reduce ranks of gauge groups, are possible [17] [18] [19].

Here we focus on the $U(N_a) \times U(N_b)$ part, which has the magnetic flux,

$$
F_{2i+2,2i+3} = 2\pi \begin{pmatrix}
M_a^{(i)} 1_{N_a \times N_a} & 0 \\
0 & M_b^{(i)} 1_{N_b \times N_b}
\end{pmatrix},
$$

for $i = 1, 2, 3$. We use the following gauge,

$$
A_{2i+2} = 0, \quad A_{2i+3} = F_{2i+2,2i+3} y_{2i+2}.
$$

Similarly, the gaugino fields $\lambda$ and their $i$-th torus parts are decomposed as

$$
\lambda(x, y) = \begin{pmatrix}
\lambda_{aa}(x, y) \\
\lambda_{ab}(x, y)
\end{pmatrix}, \quad \psi(i)(y) = \begin{pmatrix}
\psi_{(i)+}^{aa}(y) \\
\psi_{(i)+}^{ab}(y)
\end{pmatrix}.
$$

The fields $\lambda_{aa}$ and $\lambda_{bb}$ correspond to the gaugino fields under the unbroken gauge group $U(N_a) \times U(N_b)$. On the other hand, $\lambda_{ab}$ and $\lambda_{ba}$ correspond to bi-fundamental matter fields, $(N_a, \tilde{N}_b)$ and $(\tilde{N}_a, N_b)$, under the unbroken gauge group $U(N_a) \times U(N_b)$. The Dirac equations for these gaugino fields corresponding to zero-modes are obtained as

$$
\begin{pmatrix}
\bar{\partial}_{(i)+} \psi_{(i)+}^{aa} \\
\bar{\partial}_{(i)+} \psi_{(i)+}^{bb}
\end{pmatrix} = 0,
$$

$$
\begin{pmatrix}
\bar{\partial}_{(i)+} \psi_{(i)+}^{ab} \\
\bar{\partial}_{(i)+} \psi_{(i)+}^{ba}
\end{pmatrix} = 0,
$$

$$
\begin{pmatrix}
\partial_{(i)-} \psi_{(i)-}^{aa} \\
\partial_{(i)-} \psi_{(i)-}^{bb}
\end{pmatrix} = 0,
$$

$$
\begin{pmatrix}
\partial_{(i)-} \psi_{(i)-}^{ab} \\
\partial_{(i)-} \psi_{(i)-}^{ba}
\end{pmatrix} = 0.
$$
where $\tilde{\partial}_i = \partial_{2i+2} + i \partial_{2i+3}$ and $\partial_i = \partial_{2i+2} - i \partial_{2i+3}$. The gaugino fields, $\psi^{\alpha a}$ and $\psi^{\beta b}$, for the unbroken gauge symmetry have no effect from the magnetic flux in their Dirac equations. Hence, they have the same zero-modes as those on $(T^2)^3$ without the magnetic flux. On the other hand, the magnetic flux appears in the zero-mode equations of $\psi^{ab}$ and $\psi^{ba}$ corresponding to bi-fundamental matter fields, $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_b)$. Furthermore, they satisfy the following boundary conditions,

$$
\begin{align*}
\psi^{ab}_{s_i} (y_{2i+2} + 1, y_{2i+3}) & = e^{2\pi i s_i (M_a^{(i)} - M_b^{(i)})} \psi^{ab}_{s_i} (y_{2i+2}, y_{2i+3}), \\
\psi^{ba}_{s_i} (y_{2i+2} + 1, y_{2i+3}) & = e^{2\pi i s_i (M_a^{(i)} - M_b^{(i)})} \psi^{ba}_{s_i} (y_{2i+2}, y_{2i+3}), \\
\psi^{ab}_{s_i} (y_{2i+2}, y_{2i+3} + 1) & = \psi^{ab}_{s_i} (y_{2i+2}, y_{2i+3}), \\
\psi^{ba}_{s_i} (y_{2i+2}, y_{2i+3} + 1) & = \psi^{ba}_{s_i} (y_{2i+2}, y_{2i+3}),
\end{align*}
$$

because of Eq. (5).

For the $i$-th $T^2$ with $M_a^{(i)} - M_b^{(i)} > 0$, the fields $\psi^{ab}_{(i)+}$ and $\psi^{ba}_{(i)+}$ have $|M_a^{(i)} - M_b^{(i)}|$ normalizable zero-modes, while $\psi^{ab}_{(i)-}$ and $\psi^{ba}_{(i)-}$ have no normalizable zero-modes. Thus, we can derive chiral theory. When $M_a^{(i)} - M_b^{(i)} < 0$, $\psi^{ab}_{(i)-}$ and $\psi^{ba}_{(i)-}$ have $|M_a^{(i)} - M_b^{(i)}|$ normalizable zero-modes. The normalizable wavefunction for the $j$-th zero mode is obtained as

$$
\Theta^j (y_{2i+2}, y_{2i+3}) = N_j e^{-M\pi y_{2i+2}} \vartheta \left[ \begin{array}{c} j/M \\ 0 \end{array} \right] (M(y_{2i+2} + iy_{2i+3}), Mi),
$$

for $M = |M_a^{(i)} - M_b^{(i)}|$ and $j = 0, 1, \cdots, N - 1$, where $N_j$ is a normalization constant and

$$
\vartheta \left[ \begin{array}{c} j/M \\ 0 \end{array} \right] (M(y_{2i+2} + i y_{2i+3}), Mi) = \sum_n e^{-M\pi (n+j/M)^2 + 2\pi i (n+j/M)y_{2i+2} + iy_{2i+3}},
$$

that is, the Jacobi theta-function. Furthermore, we can introduce the complex structure modulus $\tau$ by replacing the above Jacobi theta-function as

$$
\vartheta \left[ \begin{array}{c} j/M \\ 0 \end{array} \right] (M(y_{2i+2} + i y_{2i+3}), Mi) \rightarrow \vartheta \left[ \begin{array}{c} j/M \\ 0 \end{array} \right] (M(y_{2i+2} + \tau y_{2i+3}), M\tau).
$$

The total number of bi-fundamental zero-modes is given by $\prod_{i=1}^{3} |M_a^{(i)} - M_b^{(i)}|$ and all of them have the same six-dimensional chirality sign $\prod_{i=1}^{3} (M_a^{(i)} - M_b^{(i)})$. Since the ten-dimensional chirality of gaugino fields is fixed, bi-fundamental zero-modes for either $(N_a, \bar{N}_b)$ or $(\bar{N}_a, N_b)$ appear for a fixed four-dimensional chirality. That is, the total number of bi-fundamental zero-modes for $(N_a, \bar{N}_b)$ is equal to

$$
I_{ab} = \prod_{i=1}^{3} (M_a^{(i)} - M_b^{(i)}).
$$

When $I_{ab} < 0$, this means that there appear $|I_{ab}|$ independent zero modes for $(N_a, N_b)$. It is also convenient to introduce the notation, $I_{ab}' \equiv M_a^{(i)} - M_b^{(i)}$. Zero-mode wavefunctions are given by a product of two-dimensional parts, i.e.

$$
\Theta^{i_1, i_2, i_3} (y) = \Theta^{i_1} (y_4, y_5) \Theta^{i_2} (y_6, y_7) \Theta^{i_3} (y_8, y_9),
$$

where $\Theta^{i_1, i_2, i_3} (y)$ is the three-dimensional Jacobi theta-function.
for \( i_1 = 0, \ldots, (|M_a^{(1)}| - M_b^{(1)}| - 1) \), \( i_2 = 0, \ldots, (|M_a^{(2)}| - M_b^{(2)}| - 1) \) and \( i_3 = 0, \ldots, (|M_a^{(3)}| - M_b^{(3)}| - 1) \).

### 2.2 \( U(N) \) gauge theory on magnetized orbifolds \( T^6/(Z_2 \times Z_2') \)

Here we review on the \( T^6/(Z_2 \times Z_2') \) orbifold with a magnetic flux \([13]\).

#### 2.2.1 \( T^2/Z_2 \) orbifold

First, let us study the \( U(N) \) gauge theory on the orbifold \( T^2/Z_2 \) with the coordinates \((y_4, y_5)\), which transform as

\[
y_4 \to -y_4, \quad y_5 \to -y_5,
\]

under the \( Z_2 \) orbifold twist. Here, we associate the \( Z_2 \) twist with the \( Z_2 \) action in the gauge space as

\[
A_\mu(x, -y) = PA_\mu(x, y)P^{-1}, \quad A_m(x, y) = -PA_m(x, y)P^{-1},
\]

and the \( Z_2 \) boundary conditions for gaugino fields,

\[
\lambda_\pm(x, -y) = \pm P\lambda_\pm(x, y)P^{-1},
\]

where the \( Z_2 \) projection \( P \) must satisfy \( P^2 = 1 \).

We focus on the \( U(N_a) \times U(N_b) \) block \([4], [6]\) and consider the spinor fields, \( \lambda_{ab}^a, \lambda_{ab}^b, \lambda_{ba}^a \) and \( \lambda_{ba}^b \), in particular bi-fundamental fields \( \lambda_{ab}^{\pm a} \) and \( \lambda_{ba}^{\pm a} \), where \( \pm \) denotes the chirality \( s_i \) in the extra dimension. Without the \( Z_2 \) projection, there are \( |M_a - M_b| \) zero modes for \( \lambda_{ab}^{a} \) and \( \lambda_{ba}^{b} \). For example, when \( M_a - M_b > 0 \), \( \lambda_{ab}^{a} \) as well as \( \lambda_{ba}^{b} \) has \( (M_a - M_b) \) zero modes with the wavefunctions \( \Theta^j \) for \( j = 0, \ldots, (M_a - M_b - 1) \). When we consider the \( Z_2 \) projection, either even or odd modes of them remain. Here note that

\[
\Theta^j(y_4, -y_5) = \Theta^{M-j}(y_4, y_5),
\]

where \( \Theta^M(y_4, y_5) = \Theta^0(y_4, y_5) \). That is, even and odd functions are given by

\[
\begin{align*}
\Theta^j_{\text{even}} &= \frac{1}{\sqrt{2}}(\Theta^j + \Theta^{M-j}), \\
\Theta^j_{\text{odd}} &= \frac{1}{\sqrt{2}}(\Theta^j - \Theta^{M-j}),
\end{align*}
\]

respectively. For example, when we consider the projection \( P \) such that \( \lambda_{ab}^a(x, -y) = \lambda_{ab}^a(x, y) \), only zero-modes corresponding to \( \Theta^j_{\text{even}} \) remain and the number of zero-modes is equal to \( (M_a - M_b)/2 + 1 \) for \( (M_a - M_b) = \text{even} \) and \( (M_a - M_b + 1)/2 \) for \( (M_a - M_b) = \text{odd} \). On the other hand, when we consider the projection \( P \) such that \( \lambda_{ab}^a(x, -y) = -\lambda_{ab}^a(x, y) \), only zero-modes corresponding to \( \Theta^j_{\text{odd}} \) remain and the number of zero-modes is equal to \( (M_a - M_b)/2 - 1 \) for \( (M_a - M_b) = \text{even} \) and \( (M_a - M_b - 1)/2 \) for \( (M_a - M_b) = \text{odd} \). The same holds true for \( \lambda_{ba}^b \). Table 1 shows the numbers of zero-modes with even and odd wavefunctions for \( M \leq 10 \).
Table 1: The numbers of zero-modes for even and odd wavefunctions.

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|---|----|----|----|----|----|----|----|----|----|----|----|
| even | 1  | 1  | 2  | 2  | 3  | 3  | 4  | 4  | 5  | 5  | 6  |
| odd  | 0  | 0  | 0  | 1  | 1  | 2  | 2  | 3  | 3  | 4  | 4  |

2.2.2 $T^6/(Z_2 \times Z_2')$

Here, we can extend the previous analysis on the two-dimensional orbifold $T^2/Z_2$ to the $U(N)$ gauge theory on the six-dimensional orbifold $T^6/(Z_2 \times Z_2')$. We consider two independent twists, $Z_2$ and $Z_2'$. The $Z_2$ twist acts on the six-dimensional coordinates $y_m$ ($m = 4, \cdots, 9$) as

$$y_m \rightarrow -y_m \quad \text{(for } m = 4, 5, 6, 7),$$

and the $Z_2'$ twist acts as

$$y_m \rightarrow -y_m \quad \text{(for } m = 4, 5, 6, 9),$$

$$y_n \rightarrow y_n \quad \text{(for } n = 8, 9),$$

If the magnetic flux is vanishing, we realize four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories for the orbifold $T^6/(Z_2 \times Z_2')$. The bi-fundamental matter fields $\lambda^{ab}_{s_1,s_2,s_3}$, $\lambda^{ba}_{s_1,s_2,s_3}$ with the chirality $s_i$ corresponding to the $i$-th $T^2$ are also introduced. Their $Z_2$ boundary conditions are given by

$$\lambda_{s_1,s_2,s_3}(x, -y_m, y_n) = s_1 s_2 P \lambda_{s_1,s_2,s_3}(x, y_m, y_n) P^{-1},$$

with $m = 4, 5, 6, 7$ and $n = 8, 9$ for $\lambda^{aa}_{s_1,s_2,s_3}$, $\lambda^{ab}_{s_1,s_2,s_3}$, $\lambda^{ba}_{s_1,s_2,s_3}$ and $\lambda^{bb}_{s_1,s_2,s_3}$. Similarly, the $Z_2'$ boundary conditions are given by

$$\lambda_{s_1,s_2,s_3}(x, -y_m, y_n) = s_1 s_3 P' \lambda_{s_1,s_2,s_3}(x, y_m, y_n) P'^{-1},$$

with $m = 4, 5, 8, 9$ and $n = 6, 7$. Then, depending on the projections $P$ and $P'$, even or odd modes for the $i$-th torus remain such as $\Theta_{\text{even}}^i$ or $\Theta_{\text{odd}}^i$. Their products such as $\prod_{i=3}^3 \Theta_{\text{even,odd}}^{i,M}(y_{2i+2}, y_{2i+3})$ provide with zero-modes on the $T^6/(Z_2 \times Z_2')$.

3 Three generation magnetized orbifold models

In this section, we consider the $U(N_a) \times U(N_b) \times U(N_c)$ models, which lead to three families of bi-fundamental matter fields, $(N_a, N_b)$ and $(N_a, N_c)$. Such a gauge group is derived by starting with the $U(N)$ group and introducing the following form of the magnetic flux,

$$F_{45} = 2\pi \begin{pmatrix} M^{(1)}_a 1_{N_a \times N_a} & 0 \\ 0 & M^{(1)}_b 1_{N_b \times N_b} \\ & M^{(1)}_c 1_{N_c \times N_c} \end{pmatrix}.$$
where $N = N_a + N_b + N_c$. For $N_a = 4$, $N_b = 2$ and $N_c = 2$, we can realize the Pati-Salam gauge group up to $U(1)$ factors, some of which may be anomalous and become massive by the Green-Schwarz mechanism. Then, the bi-fundamental matter fields, $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_c)$ correspond to left-handed and right-handed matter fields. In addition, the bi-fundamental matter fields $(N_b, \bar{N}_c)$ correspond to higgsino fields. We assume that supersymmetry is preserved at least locally at the $a-b$ sector, $b-c$ sector and $c-a$ sector. Then, the number of Higgs scalar fields are the same as the number of higgsino fields. There are no tachyonic modes at the tree level. Indeed, in intersecting D-brane models it would be one of convenient ways towards realistic models to derive the Pati-Salam model at some stage and to break the gauge group to $SU(3) \times SU(2)_L \times U(1)$. (See e.g. Ref. [11, 20] and references therein.) At the end of this section, we give a comment on breaking of $SU(4) \times SU(2)_L \times SU(2)_R$ to $SU(3) \times SU(2)_L \times U(1)$.

In both cases with and without orbifolding, the total number of chiral matter fields is a product of the numbers of zero-modes corresponding to the $i$-th $T^2$ for $i = 1, 2, 3$. That is, the three generations are realized in the models, where the $i$-th $T^2$ has three zero-modes while each of the other tori has a single zero mode. Thus, there are two types of flavor structures. That is, in one type the three zero-modes corresponding to both left-handed matter fields $(N_a, \bar{N}_b)$ and right-handed matter fields $(\bar{N}_a, N_c)$ appear in the same $i$-th $T^2$, while each of the other tori has a single zero mode for $(N_a, \bar{N}_b)$ as well as $(\bar{N}_a, N_c)$. In the other type, three zero-modes of $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_c)$ are originated from different tori. The Yukawa coupling for 4D effective field theory is evaluated by the following overlap integral of zero-mode wavefunctions

$$Y_{ij} = \int d^6y \psi_L^i(y) \psi_R^j(y) \phi_H(y),$$

where $\psi_L(y)$, $\psi_R(y)$ and $\phi_H(y)$ denote zero-mode wave-functions of the left-handed, right-handed matter fields and Higgs field, respectively. Note that the integral corresponding to each torus is factorized in the Yukawa coupling. In the second type of flavor structure, one obtains the following form of Yukawa matrices,

$$Y_{ij} = a_i b_j,$$

at the tree-level, because the flavor structure of left-handed and right-handed matter fields are originated from different tori. This matrix, $Y_{ij}$, has rank one and that is not phenomenologically

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3 See for the supersymmetric conditions e.g. Ref. [7, 8].

4 See for the Pati-Salam model in heterotic orbifold models e.g. Ref. [21], where $SU(4) \times SU(2)_L \times SU(2)_R$ is broken to the standard gauge group by vacuum expectation values of scalar fields, $(4, 1, 2)$ and $(\bar{4}, 1, 2)$, while in the intersecting D-brane models $SU(4) \times SU(2)_L \times SU(2)_R$ is broken by splitting D-branes, that is, vacuum expectation values of adjoint scalar fields.
Table 2: Possible patterns of wavefunctions with non-vanishing Yukawa couplings for the first torus.

|     | $\lambda^{ab}$ | $\lambda^{ca}$ | $\lambda^{bc}$ |
|-----|----------------|----------------|----------------|
| I   | even           | even           | even           |
| II  | even           | odd            | odd            |
| II' | odd            | even           | odd            |
| III | odd            | odd            | even           |

interesting, unless certain corrections appear. Hence, we concentrate on the first type of the
flavor structure. In the first type, the flavor structure is originated from the single torus, where
both three zero-modes of $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_c)$ appear. We assign this torus with the first torus.
On the other hand, the other tori, the second and third tori, do not lead to flavor-dependent
aspects. That is, Yukawa matrices are obtained as the following form,

$$Y_{ij} = a^{(2)} a^{(3)} a^{(1)}_{ij},$$

where the structure of $a^{(1)}_{ij}$ is determined by only the first torus corresponding to three zero-
modes $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_c)$ while the other tori contribute to overall factors $a^{(2)}$ and $a^{(3)}$. Thus, we concentrate on the single torus, where both of three zero-modes $(N_a, \bar{N}_b)$ and $(\bar{N}_a, N_c)$ appear, i.e. the first torus.

Zero-mode wavefunctions are classified into even and odd modes under the $Z_2$ twist. Only
even or odd modes remain through the orbifold projection. Furthermore, the 4D Yukawa
 couplings are non-vanishing for combinations among (even, even, even) wavefunctions and
(even, odd, odd) wavefunctions, while Yukawa couplings vanish for combinations among (even,
even, odd) wavefunctions and (odd, odd, odd) wavefunctions. Thus, we study only the former
case with non-vanishing Yukawa couplings, that is, the combinations among (even, even, even)
wavefunctions and (even, odd, odd) wavefunctions. Hence, we are interested in four types
of combinations of wavefunctions for the first torus, as shown in Table 2. The II’ type of
combinations is obtained by exchanging the left and right-handed matter fields in the II type.
Thus, we study explicitly the three types, I, II and III.

We can realize three even zero-modes when $|I^{(1)}_{ab}| = 4, 5$, as shown in Table 1. On the
other hand, three odd zero-modes can appear when $|I^{(1)}_{ab}| = 7, 8$. Furthermore, the consistency
condition on magnetic fluxes requires

$$|I^{(1)}_{bc}| = |I^{(1)}_{ab}| \pm |I^{(1)}_{ca}|.$$

Thus, the number of Higgs and higgsino fields are constrained. Table 3 shows all of possible
magnetic fluxes for the three types, I, II and III. The fourth and fifth columns of the table show
possible sizes of magnetic fluxes for $|I^{(1)}_{bc}|$ and the number of zero-modes corresponding to the
Higgs fields. As a result, flavor structures of our models with Yukawa couplings are classified
into 20 classes. However, the model with $\left(|I^{(1)}_{ab}|, |I^{(1)}_{ca}|, |I^{(1)}_{bc}|\right) = (5, 7, 2)$ has no zero-modes for the Higgs fields. Thus, we do not consider this case, but we will study the other 19 classes
the numbers of Higgs zero modes

|    | $I_{ab}^{(1)}$ | $I_{ca}^{(1)}$ | $I_{bc}^{(1)}$ |
|----|---------------|---------------|---------------|
| I  | 4             | 4             | 8             | 5             |
|    | 4             | 4             | 0             | 1             |
|    | 4             | 5             | 9             | 5             |
|    | 4             | 5             | 1             | 5             |
|    | 5             | 5             | 10            | 6             |
|    | 5             | 5             | 0             | 1             |
| II | 4             | 7             | 11            | 5             |
|    | 4             | 7             | 3             | 1             |
|    | 4             | 8             | 12            | 5             |
|    | 4             | 8             | 4             | 1             |
|    | 5             | 7             | 12            | 5             |
|    | 5             | 7             | 2             | 0             |
|    | 5             | 8             | 13            | 6             |
|    | 5             | 8             | 3             | 1             |
| III| 7             | 7             | 14            | 8             |
|    | 7             | 7             | 0             | 1             |
|    | 7             | 8             | 15            | 8             |
|    | 7             | 8             | 1             | 1             |
|    | 8             | 8             | 16            | 9             |
|    | 8             | 8             | 0             | 1             |

Table 3: The number of Higgs fields of $(T^2)^1$ with non-vanishing Yukawa couplings.

in Table 3. Therefore, we study possible flavor structures explicitly by deriving the coupling selection rule and evaluating values of Yukawa couplings in these 19 classes. That is the purpose of the next section.

Before explicit study on flavor structures of 19 classes in the next section, we give a comment on breaking of $SU(4) \times SU(2)_L \times SU(2)_R$. At any rate, we need the $SU(3) \times SU(2)_L \times U(1)$ gauge group at low energy. When the magnetic flux and orbifold projections lead to the $SU(4) \times SU(2)_L \times SU(2)_R$ gauge group from $U(8)$ as we have discussed so far, we need further breaking of $SU(4) \times SU(2)_L \times SU(2)_R$ to $SU(3) \times SU(2)_L \times U(1)$. Such breaking can be realized by assuming non-vanishing vacuum expectation values (VEVs) of Higgs fields like adjoint scalar fields for $SU(4)$ and $SU(2)_R$ and/or bi-fundamental scalar fields like $(4, 1, 2)$ and $(\bar{4}, 1, 2)$ on fixed points. Note that our models have degree of freedom to add any modes at the fixed points from the viewpoint of point particle field theory. The above breaking may affect the structure of Yukawa matrices as higher dimensional operators. However, we will show results on Yukawa matrices without such corrections.

Alternatively, magnetic fluxes and/or orbifold projections break $U(8)$ into $U(3) \times U(1)_1 \times U(2)_L \times U(1)_2 \times U(1)_3$. The gauge group $U(3) \times U(1)_1$ would correspond to $U(4)$ and $U(1)_2 \times U(1)_3$ would correspond to $U(2)_R$. We assume that all the bi-fundamental matter fields under
U(3) × U(1), i.e. extra colored modes, are projected out. The bi-fundamental matter fields for U(3) × U(1) and U(3) × U(1) correspond to up and down sectors of right-handed quarks, respectively. Similarly, up and down sectors of Higgs fields and right-handed charged leptons and neutrinos are obtained. In this case, the classification of this section and patterns of Yukawa matrices, which will be studied in the next section and Appendix, are available for up-sector and down-sector quarks as well as the lepton sector. However, the up sector and down sector can correspond to different classes of Table 3. On the other hand, the up sector and down sector correspond to the same class in Table 3, when the SU(4) × SU(2)L × SU(2)R is broken by VEVs of Higgs fields on fixed points as discussed above.

4 Yukawa couplings in three generation models

4.1 Yukawa interactions

Following [7, 23], first we show computation of Yukawa interactions on the torus with the magnetic flux. Omitting the gauge structure and spinor structure, the Yukawa coupling among left, right-handed matter fields and Higgs field corresponding to three zero-mode wavefunctions, Θi,M1(z), Θj,M2(z) and (Θk,M3(z))*, is written by

\[ Y_{ijk} = c \int dz d\bar{z} \Theta^{i,M_1}(z) \Theta^{j,M_2}(z)(\Theta^{k,M_3}(z))^*, \] (8)

where z = x4 + τy5, M1 ≡ I_{a1}^{(1)}, M2 ≡ I_{a2}^{(1)}, M3 ≡ I_{c1}^{(1)} and c is a flavor-independent contribution due to the other tori. Note that M1 + M2 = M3. Because of the gauge invariance, not the wavefunction Θk,M3(z), but (Θk,M3(z))* appears in the Yukawa coupling [7].

By using the formula of the \( \vartheta \) function,

\[ \vartheta \left[ \frac{r/N_1}{s/N_2} \right] (z_1, N_1 \tau) \times \vartheta \left[ \frac{s/N_2}{0} \right] (z_2, N_2 \tau) \]

\[ = \sum_{m \in \mathbb{Z}} \vartheta \left[ \frac{r+s+N_1m}{N_1+N_2} \right] (z_1 + z_2, \tau(N_1 + N_2)) \]

\[ \times \vartheta \left[ \frac{N_2r-N_1s+N_1N_2m}{N_1N_2(N_1+N_2)} \right] (z_1 N_2 - z_2 N_1, \tau N_1 N_2 (N_1 + N_2)), \]

we can decompose \( \Theta^{i,M_1}(z) \Theta^{j,M_2}(z) \) as

\[ \Theta^{i,M_1}(z) \Theta^{j,M_2}(z) = \sum_{m \in \mathbb{Z}} \Theta^{i+j+M_1m,M_3}(z) \times \vartheta \left[ \frac{M_2j-M_1i+M_1M_2m}{M_1M_2M_3} \right] (0, \tau M_1 M_2 M_3). \]

Wavefunctions satisfy the orthogonal condition

\[ \int dz d\bar{z} \Theta^{i,M}(\Theta^{j,M})^* = \delta_{ij}. \]
Then, the integral of three wavefunctions is represented by

\[ Y_{ijk} = c \int dz d\bar{z} \Theta^{i,M_1} \Theta^{j,M_2} (\Theta^{k,M_3})^* = c |M_3|^{-1} \sum_{m=0}^{M_3-1} \vartheta \left[ \frac{M_{2i-M_1j+M_1m}}{M_{1M_2M_3}} 0 \right] (0, \tau M_1 M_2 M_3) \times \delta_{i+j+M_1m,k+M_3\ell}, \]

where \( \ell = \text{integer} \). Thus, we have the selection rule for allowed Yukawa couplings as

\[ i + j = k, \]

where \( i, j \) and \( k \) are defined up to mod \( M_1, M_2 \) and \( M_3 \), respectively. In addition, the Yukawa coupling \( Y_{ijk} \), in particular its flavor-dependent part, is written by the \( \vartheta \) function. When \( g.c.d.(M_1, M_3) = 1 \), a single \( \vartheta \) function appears in \( Y_{ijk} \). When \( g.c.d.(M_1, M_3) = g \neq 1 \), \( g \) terms appear in \( Y_{ijk} \) as

\[ Y_{ijk} = c \sum_{n=1}^{g} \vartheta \left[ \frac{M_{2k-M_3j+M_2M_3\ell_0}}{M_{1M_2M_3}} + \frac{n}{g} 0 \right] (0, \tau M_1 M_2 M_3), \]

where \( \ell_0 \) is an integer corresponding to a particular solution of \( M_3\ell_0 = M_1m_0 + i + j - k \) with integer \( m_0 \).

Zero-mode wavefunctions on the orbifold with the magnetic flux are obtained as even or odd linear combinations of wavefunctions on the torus with the magnetic flux \( (7) \). Thus, it is straightforward to extend the above computations of Yukawa couplings on the torus to Yukawa couplings on the orbifold. As a result, Yukawa couplings on the orbifold are obtained as proper linear combinations of Yukawa couplings on the torus, i.e. linear combinations of \( \vartheta \) functions.

Here we introduce the following short notation for the Yukawa coupling,

\[ \eta_N = \vartheta \left[ \frac{N}{M} 0 \right] (0, \tau M), \]

where

\[ M = M_1 M_2 M_3. \]

Since the value of \( M \) is unique in one model, we omit the value of \( M \) as well as \( \tau \) for a compact presentation of long equations.

Four models in Table 3 has \( |I_{bc}^{(1)}| = 0 \), where the Higgs zero-mode corresponds to the even function, that is, the constant profile. We can repeat the above calculation for this case, that is, the case where, one of wavefunctions in \( (8) \), e.g. \( \Theta^{i,M_1}(z) \) is constant. As a result, the Yukawa matrix is proportional to the \((3 \times 3)\) unit matrix, \( Y_{jk} = c' \delta_{jk} \). That is not realistic. Thus, we will not consider such models.

At any rate, we can apply the above selection rule and \( \eta_N \) for 20 classes of models, which have been classified in section 3, in order to analyze explicitly all of possible patterns of Yukawa matrices. In the next subsection, we show one example of Yukawa matrix among 20 classes of models. In Appendix, we show all of possible Yukawa matrices for 15 classes of models in Table 3 except models with \( I_{bc}^{(1)} = 0 \) and the model without zero-modes for the Higgs fields.

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5 See for the selection rule in intersecting D-brane models, e.g. Ref. [24, 25].
fields and Higgs fields correspond to odd, odd and even wavefunctions, respectively. Their

Let us study the model with $(I_{ab}^{(1)}, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (7, 7, 14)$. Following Table 3, we consider the combination of zero-mode wavefunctions, where zero-modes of left and right-handed matter fields and Higgs fields correspond to odd, odd and even wavefunctions, respectively. Their wavefunctions are shown in Table 4. Hereafter, for concreteness, we denote left and right-handed matter fields and Higgs fields by $L_i$, $R_j$ and $H_k$, respectively. This model has eight zero-modes for Higgs fields.

Then, their Yukawa couplings $Y_{ijk} L_i R_j H_k$ are written by

$$Y_{ijk} H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4 + y_{ij}^5 H_5 + y_{ij}^6 H_6 + y_{ij}^7 H_7,$$

where

$$y_{ij}^0 = \begin{pmatrix} -y_c & 0 & 0 \\ 0 & -y_e & 0 \\ 0 & 0 & -y_g \end{pmatrix}, \quad y_{ij}^1 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} y_d & 0 \\ -\frac{1}{\sqrt{2}} y_d & 0 & -\frac{1}{\sqrt{2}} y_f \\ 0 & -\frac{1}{\sqrt{2}} y_f & 0 \end{pmatrix},$$

$$y_{ij}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} y_a & 0 & -\frac{1}{\sqrt{2}} y_c \\ 0 & 0 & \frac{1}{\sqrt{2}} y_g \\ -\frac{1}{\sqrt{2}} y_e & \frac{1}{\sqrt{2}} y_g & 0 \end{pmatrix}, \quad y_{ij}^3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} y_b & \frac{1}{\sqrt{2}} y_f \\ \frac{1}{\sqrt{2}} y_b & \frac{1}{\sqrt{2}} y_h & 0 \\ \frac{1}{\sqrt{2}} y_f & 0 & 0 \end{pmatrix},$$

$$y_{ij}^4 = \begin{pmatrix} \frac{1}{\sqrt{2}} y_g & 0 & \frac{1}{\sqrt{2}} y_c \\ 0 & \frac{1}{\sqrt{2}} y_a & 0 \\ \frac{1}{\sqrt{2}} y_c & 0 & 0 \end{pmatrix}, \quad y_{ij}^5 = \begin{pmatrix} \frac{1}{\sqrt{2}} y_h & 0 & -\frac{1}{\sqrt{2}} y_d \\ 0 & 0 & \frac{1}{\sqrt{2}} y_b \\ -\frac{1}{\sqrt{2}} y_d & \frac{1}{\sqrt{2}} y_b & 0 \end{pmatrix},$$

$$y_{ij}^6 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} y_e & 0 \\ -\frac{1}{\sqrt{2}} y_e & 0 & -\frac{1}{\sqrt{2}} y_c \\ 0 & -\frac{1}{\sqrt{2}} y_c & \frac{1}{\sqrt{2}} y_a \end{pmatrix}, \quad y_{ij}^7 = \begin{pmatrix} -y_f & 0 & 0 \\ 0 & -y_d & 0 \\ 0 & 0 & -y_b \end{pmatrix}, \quad (10)$$

and

$$y_a = \eta_0 + 2\eta_{39} + 2\eta_{96} + 2\eta_{294},$$

Table 4: Zero-mode wavefunctions in the 7-7-14 model.

4.2 An illustrating example: 7-7-14 model
Figure 1: The $N$-dependence of $\log \eta_N$ in the 7-7-14 model ($M = 686$), where $\lambda = 0.22$ is chosen to the Cabibbo angle. The solid, dashed and dotted curves correspond to $\tau = i, 1.5i$ and $0.5i$, respectively. Note that $\eta_N$ has a periodicity $\eta_{N+nM} = \eta_N$ with an integer $n$.

Here we have used the short notation $\eta_N$ defined in Eq. (9) with the omitted value $M = M_1M_2M_3 = 686$.

### 4.3 Numerical examples in 7-7-14 model

Here, we give examples of numerical studies by using the 7-7-14 model, which is discussed in the previous subsection. For such studies, the numerical values of $\eta_N$ defined in Eq. (9) are useful. The $N$-dependence of $\eta_N$ is shown in Fig. 1.

We assume that both the up-sector and the down-sector of quarks as well as their Higgs fields have the Yukawa matrix, which is led in the 7-7-14 model. Such situation is realized in the case that we start with the $U(8)$ gauge group and break it to $U(4) \times U(2)_L \times U(2)_R$ by the magnetic flux, and then the Pati-Salam gauge group is broken to the Standard gauge group by assuming VEVs of Higgs fields on fixed points. Alternatively, we break the $U(8)$ gauge group to $U(3) \times U(1)_1 \times U(2)_L \times U(1)_2 \times U(1)_3$ by magnetic fluxes and orbifold projections as discussed in section 3. Then, both the up-sector and down-sector of quarks can correspond to the Yukawa matrix led in the 7-7-14 model, although the up-sector and down-sector can generically correspond to different patterns of Yukawa matrices. In both cases, VEVs of the up-sector and down-sector Higgs fields are independent.

First, we consider the case that VEVs of $H^0_d$, $H^7_d$ and $H^0_u$ are non-vanishing and the other...
VEVs vanish. In this case, the relevant Yukawa couplings are

\[
Y_{ijk}^u H_k = \begin{pmatrix} -y_c & -y_e & -y_g \\ -y_f H_7^d & -\frac{1}{\sqrt{2}} y_e H_6^d & 0 \\ -\frac{1}{\sqrt{2}} y_c H_6^d & -\frac{1}{\sqrt{2}} y_e H_7^d & -\frac{1}{\sqrt{2}} y_c H_7^d \end{pmatrix} H_0^u, \\
Y_{ijk}^d H_k = \begin{pmatrix} -\frac{1}{\sqrt{2}} y_c H_6^d & 0 & -\frac{1}{\sqrt{2}} y_e H_7^d \\ -\frac{1}{\sqrt{2}} y_e H_6^d & -\frac{1}{\sqrt{2}} y_c H_7^d & 0 \\ -\frac{1}{\sqrt{2}} y_e H_7^d & -\frac{1}{\sqrt{2}} y_c H_7^d & -\frac{1}{\sqrt{2}} y_e H_6^d \end{pmatrix}.
\]

Let us assume \( \langle H_6^d \rangle = -\langle H_7^d \rangle \) for their VEVs. Then, quark mass ratios are obtained from these matrices as

\[
\left( \frac{m_u}{m_t}, \frac{m_c}{m_t}, \frac{m_t}{m_t} \right) \sim (7.6 \times 10^{-4}, 6.8 \times 10^{-2}, 1.0), \\
\left( \frac{m_d}{m_b}, \frac{m_s}{m_b}, \frac{m_b}{m_b} \right) \sim (7.5 \times 10^{-4}, 5.1 \times 10^{-2}, 1.0),
\]

for \( \tau = i \). Furthermore, the mixing angles are obtained as

\[
|V_{CKM}| \sim \begin{pmatrix} 0.97 & 0.24 & 0.0025 \\ 0.24 & 0.95 & 0.20 \\ 0.046 & 0.19 & 0.98 \end{pmatrix}.
\]

Similarly, for \( \tau = 1.5i \), quark mass ratios are obtained as

\[
\left( \frac{m_u}{m_t}, \frac{m_c}{m_t}, \frac{m_t}{m_t} \right) \sim (2.1 \times 10^{-5}, 1.8 \times 10^{-2}, 1.0), \\
\left( \frac{m_d}{m_b}, \frac{m_s}{m_b}, \frac{m_b}{m_b} \right) \sim (1.4 \times 10^{-4}, 1.7 \times 10^{-2}, 1.0),
\]

and the mixing angles are obtained as

\[
|V_{CKM}| \sim \begin{pmatrix} 0.99 & 0.13 & 0.00029 \\ 0.13 & 0.98 & 0.13 \\ 0.017 & 0.13 & 0.99 \end{pmatrix}.
\]

Let us consider another type of VEVs. We assume that VEVs of \( H_0^u, H_2^u, H_1^d \) and \( H_7^d \) are non-vanishing and the other VEVs vanish. Furthermore, we consider the case with \( \langle H_0^u \rangle = -\langle H_2^u \rangle \) and \( \langle H_1^d \rangle = \langle H_7^d \rangle /3 \). In this case, the mass ratios are given by

\[
\left( \frac{m_u}{m_t}, \frac{m_c}{m_t}, \frac{m_t}{m_t} \right) \sim (2.9 \times 10^{-5}, 2.5 \times 10^{-2}, 1.0), \\
\left( \frac{m_d}{m_b}, \frac{m_s}{m_b}, \frac{m_b}{m_b} \right) \sim (4.4 \times 10^{-3}, 0.18, 1.0),
\]

for \( \tau = i \), and the mixing angles are given by

\[
|V_{CKM}| \sim \begin{pmatrix} 0.98 & 0.22 & 0.018 \\ 0.22 & 0.98 & 0.0014 \\ 0.017 & 0.0052 & 1.0 \end{pmatrix}.
\]
Similarly, for $\tau = 1.5i$ the mass ratios and the mixing angles are given by

\[
\begin{align*}
(m_u, m_c, m_t)/m_t & \sim (5.6 \times 10^{-6}, 4.7 \times 10^{-3}, 1.0), \\
(m_d, m_s, m_b)/m_b & \sim (3.3 \times 10^{-3}, 7.1 \times 10^{-2}, 1.0), \\
|V_{CKM}| & \sim \begin{pmatrix}
0.98 & 0.22 & 0.0034 \\
0.22 & 0.98 & 0.000081 \\
0.0033 & 0.00081 & 1.0
\end{pmatrix}.
\end{align*}
\]

Thus, these values can realize experimental values of quark masses and mixing angles at a certain level by using a few parameters, i.e. $\tau$ and a couple of VEVs of Higgs fields. If we consider more non-vanishing VEVs of Higgs fields, we could obtain more realistic values. For example, we assume that VEVs of $H^0_u, H^1_u, H^2_u, H^3_d$ and $H^4_d$ are non-vanishing and they satisfy $-\langle H^0_u \rangle = \langle H^1_u \rangle = \langle H^2_u \rangle$ and $\langle H^1_d \rangle = -\langle H^4_d \rangle/2$ while the other VEVs vanish. For $\tau = 1.5i$, we realize the mass ratios, $m_u/m_t \sim 2.7 \times 10^{-5}$, $m_c/m_t \sim 3.5 \times 10^{-3}$, $m_d/m_b \sim 7.3 \times 10^{-3}$ and $m_s/m_b \sim 7.5 \times 10^{-2}$, and mixing angles, $V_{us} \sim 0.2$, $V_{cb} \sim 0.03$ and $V_{ub} \sim 0.006$. When we consider more non-vanishing VEVs of Higgs fields, it is possible to derive completely realistic values. Similarly, we can study other classes of models and they have a rich flavor structure.

5 Conclusion

We have studied three generation magnetized orbifold models. We have classified their flavor structures and studied explicitly possible patterns of Yukawa matrices. Our models have a rich flavor structure, especially compared with the corresponding models without orbifolding. Realistic quark masses and mixing angles can be derived within the framework of magnetized orbifold models. We can extend our numerical studies including the lepton sector.

Here, we have studied the models, where all of three generations are originated from bulk modes. However, we have degree of freedom to put some of three generations of quarks and leptons on certain orbifold fixed points. In addition, we can assume that some Higgs fields are localized on certain orbifold fixed points. In such cases, we would have more variety of flavor structure. Furthermore, it is possible to consider localized magnetic fluxes on orbifold fixed points, which are independent of the bulk magnetic flux. Since such localized magnetic fluxes would affect profiles of zero-modes, that is one of interesting extensions of our models.

We have restricted ourselves to Abelian fluxes, but we can also extend our analysis to models with non-Abelian fluxes, which can reduce ranks of gauge groups. Moreover, although we have concentrated on the factorizable torus, $(T^2)^3$, it would be interesting to study possibilities for extensions to non-factorizable orbifolds[27].

Acknowledgement

H. A. is supported by the Grant-in-Aid for the Global COE Program “Weaving Science Web beyond Particle-matter Hierarchy” from the Ministry of Education, Culture, Sports, Science

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6 See e.g. [26].
and Technology of Japan. K.-S. C. and T. K. are supported in part by the Grant-in-Aid for Scientific Research No. 20-08326 and No. 20540266 from the Ministry of Education, Culture, Sports, Science and Technology of Japan. T. K. is also supported in part by the Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

A Possible patterns of Yukawa matrices

In this appendix, we show explicitly all of possible Yukawa matrices for 15 classes of models in Table Ex except the models with $I_{bc}^{(1)} = 0$ and the model without zero-modes for the Higgs fields.

A.1 (Even-Even-Even) wavefunctions

Here, we study the patterns of Yukawa matrices in the models, where zero-modes of left, right-handed matter fields and Higgs fields correspond to even, even and even functions, respectively.

A.1.1 4-4-8 model

Let us study the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (4, 4, 8)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

|   | $L_i(\lambda^{ab})$ | $R_j(\lambda^{ca})$ | $H_k(\lambda^{bc})$ |
|---|------------------|------------------|------------------|
| 0 | $\Theta^{0,4}$  | $\Theta^{0,4}$  | $\Theta^{0,8}$  |
| 1 | $\frac{1}{\sqrt{2}} (\Theta^{1,4} + \Theta^{3,4})$ | $\Theta^{2,4}$  | $\frac{1}{\sqrt{2}} (\Theta^{1,8} + \Theta^{7,8})$ |
| 2 | $\Theta^{2,4}$  | $\Theta^{2,4}$  | $\frac{1}{\sqrt{2}} (\Theta^{2,8} + \Theta^{6,8})$ |
| 3 | -                | -                | $\Theta^{3,8} + \Theta^{5,8}$ |
| 4 | -                | -                | $\Theta^{4,8}$  |

This model has five zero-modes for the Higgs fields. Yukawa couplings $Y_{ijk}L_iR_jH_k$ are given by

$$Y_{ijk}H_k = \begin{pmatrix} y_a H_0 + y_c H_4 \\ y_4 H_3 + y_b H_1 \\ y_c H_2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} y_a H_0 + y_c H_4 \\ y_4 H_3 + y_b H_1 \\ y_c H_2 \end{pmatrix} + \begin{pmatrix} y_4 H_0 + y_b H_4 \\ y_b H_3 + y_d H_1 \\ y_e H_0 + y_a H_4 \end{pmatrix},$$

where

$$y_a = \eta_0 + 2\eta_32 + \eta_64, \quad y_b = \eta_4 + \eta_28 + \eta_36 + \eta_60,$$
$$y_c = \eta_8 + \eta_24 + \eta_40 + \eta_56, \quad y_d = \eta_12 + \eta_20 + \eta_44 + \eta_52,$$
$$y_e = 2\eta_16 + 2\eta_48,$$

in the short notation $\eta_N$ defined in Eq. (9) with $M = M_1M_2M_3 = 128$. 

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A.1.2 4-5-9 model

Here we show the model with \(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}| \) = \((4, 5, 9)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| \(L_i(\lambda^{ab})\) | \(R_j(\lambda^{ca})\) | \(H_k(\lambda^{bc})\) |
|-----------------|-----------------|-----------------|
| 0 \(\Theta^{0,4}\) | 0 \(\Theta^{0,3}\) | 1 \(\Theta^{0,9}\) |
| 1 \(\frac{1}{\sqrt{2}}(\Theta^{1,4} + \Theta^{3,4})\) | \(\frac{1}{\sqrt{2}}(\Theta^{1,5} + \Theta^{4,5})\) | \(\frac{1}{\sqrt{2}}(\Theta^{1,9} + \Theta^{8,9})\) |
| 2 \(\Theta^{2,4}\) | \(\frac{1}{\sqrt{2}}(\Theta^{2,5} + \Theta^{3,5})\) | \(\frac{1}{\sqrt{2}}(\Theta^{2,9} + \Theta^{7,9})\) |
| 3 - | - | \(\frac{1}{\sqrt{2}}(\Theta^{3,9} + \Theta^{6,9})\) |
| 4 - | - | \(\frac{1}{\sqrt{2}}(\Theta^{4,9} + \Theta^{5,9})\) |

This model has five zero-modes for Higgs fields. Yukawa couplings \(Y_{ijk}L_iR_jH_k\) are given by

\[
Y_{ijk}H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4,
\]

where

\[
y_{ij}^0 = \begin{pmatrix} 
\eta_0 & \sqrt{2}\eta_{36} & \sqrt{2}\eta_{72} \\
\sqrt{2}\eta_{45} & \eta_9 + \eta_{81} & \eta_{27} + \eta_{63} \\
\eta_{90} & \sqrt{2}\eta_{54} & \sqrt{2}\eta_{18} 
\end{pmatrix},
\]

\[
y_{ij}^1 = \begin{pmatrix} 
\frac{1}{\sqrt{2}}(\eta_{20} + \eta_{40}) & \eta_4 + \eta_{76} & \eta_{32} + \eta_{68} \\
\eta_5 + \eta_{65} & \frac{1}{\sqrt{2}}(\eta_{31} + \eta_{41} + \eta_{49} + \eta_{59}) & \frac{1}{\sqrt{2}}(\eta_{13} + \eta_{23} + \eta_{67} + \eta_{77}) \\
\sqrt{2}\eta_{50} & \eta_{44} + \eta_{64} & \eta_{22} + \eta_{58} 
\end{pmatrix},
\]

\[
y_{ij}^2 = \begin{pmatrix} 
\frac{1}{\sqrt{2}}(\eta_{20} + \eta_{40}) & \eta_{14} + \eta_{64} & \eta_{8} + \eta_{28} \\
\eta_{35} + \eta_{55} & \frac{1}{\sqrt{2}}(\eta_{11} + \eta_{19} + \eta_{71} + \eta_{89}) & \frac{1}{\sqrt{2}}(\eta_{17} + \eta_{37} + \eta_{53} + \eta_{73}) \\
\sqrt{2}\eta_{10} & \eta_{26} + \eta_{46} & \eta_{62} + \eta_{82} 
\end{pmatrix},
\]

\[
y_{ij}^3 = \begin{pmatrix} 
\frac{1}{\sqrt{2}}(\eta_{60} + \eta_{80}) & \eta_{24} + \eta_{84} & \eta_{12} + \eta_{48} \\
\eta_{15} + \eta_{75} & \frac{1}{\sqrt{2}}(\eta_{21} + \eta_{39} + \eta_{51} + \eta_{69}) & \frac{1}{\sqrt{2}}(\eta_{3} + \eta_{33} + \eta_{57} + \eta_{87}) \\
\sqrt{2}\eta_{30} & \eta_{6} + \eta_{26} & \eta_{42} + \eta_{78} 
\end{pmatrix},
\]

\[
y_{ij}^4 = \begin{pmatrix} 
\frac{1}{\sqrt{2}}(\eta_{60} + \eta_{80}) & \eta_{16} + \eta_{56} & \eta_{52} + \eta_{88} \\
\eta_{25} + \eta_{65} & \frac{1}{\sqrt{2}}(\eta_{11} + \eta_{29} + \eta_{61} + \eta_{79}) & \frac{1}{\sqrt{2}}(\eta_{7} + \eta_{43} + \eta_{47} + \eta_{83}) \\
\sqrt{2}\eta_{70} & \eta_{34} + \eta_{74} & \eta_{2} + \eta_{38} 
\end{pmatrix},
\]

in the short notation \(\eta_N\) defined in Eq. (9) with \(M = M_1M_2M_3 = 180\).

A.1.3 4-5-1 model

Here we show the model with \(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}| \) = \((4, 5, 1)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs field.

| \(L_i(\lambda^{ab})\) | \(R_j(\lambda^{ca})\) | \(H_k(\lambda^{bc})\) |
|-----------------|-----------------|-----------------|
| 0 \(\Theta^{0,4}\) | 0 \(\Theta^{0,3}\) | 1 \(\Theta^{0,1}\) |
| 1 \(\frac{1}{\sqrt{2}}(\Theta^{1,4} + \Theta^{3,4})\) | \(\frac{1}{\sqrt{2}}(\Theta^{1,5} + \Theta^{4,5})\) | \(\frac{1}{\sqrt{2}}(\Theta^{1,9} + \Theta^{8,9})\) |
| 2 \(\Theta^{2,4}\) | \(\frac{1}{\sqrt{2}}(\Theta^{2,5} + \Theta^{3,5})\) | \(\frac{1}{\sqrt{2}}(\Theta^{2,9} + \Theta^{7,9})\) |
This model has a single zero-modes for the Higgs field. Yukawa couplings $Y_{ijk} L_i R_j H_k$ are given

$$Y_{ijk} H_k = \begin{pmatrix} y_0 & \sqrt{2}\eta_5 & \sqrt{2}\eta_8 \\ \eta_0 & \sqrt{2}\eta_6 & \sqrt{2}\eta_2 \end{pmatrix} \eta_1 H_{10}.$$

Here we have used the short notation $\eta_N$ defined in Eq. (8) with the omitted value $M = M_1 M_2 M_3 = 20$.

### A.1.4 5-5-10 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (5, 5, 10)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| $L_i(\lambda^{ab})$ | $R_j(\lambda^{ca})$ | $H_k(\lambda^{bc})$ |
|---------------------|---------------------|---------------------|
| $\Theta^{0,5}$      | $\Theta^{0,5}$      | $\Theta^{0,10}$     |
| $\frac{1}{\sqrt{2}} (\Theta^{1,5} + \Theta^{4,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{1,5} + \Theta^{4,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{1,5} + \Theta^{4,5})$ |
| $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ |
| $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ |
| $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ | $\frac{1}{\sqrt{2}} (\Theta^{2,5} + \Theta^{3,5})$ |

This model has six zero-modes for Higgs fields. Yukawa couplings $Y_{ijk} L_i R_j H_k$ are obtained as

$$Y_{ijk} H_k = \begin{pmatrix} y_a H_0 + y_c H_5 \\ y_b H_1 + y_e H_4 \\ y_e H_1 + y_c H_4 \\ y_b H_0 + y_d H_3 \\ y_e H_2 + y_d H_3 \\ \frac{1}{\sqrt{2}} (y_d H_1 + y_e H_2 + y_h H_3 + y_e H_4) \end{pmatrix}.$$  

$$y_a = \eta_0 + 2\eta_{50} + 2\eta_{100}, \quad y_b = \eta_5 + \eta_{45} + \eta_{55} + \eta_{95} + \eta_{105}, \quad y_c = \eta_{10} + \eta_{40} + \eta_{60} + \eta_{90} + \eta_{110}, \quad y_d = \eta_{15} + \eta_{35} + \eta_{65} + \eta_{85} + \eta_{115}, \quad y_e = \eta_{20} + \eta_{30} + \eta_{70} + \eta_{80} + \eta_{120}, \quad y_f = 2\eta_{25} + 2\eta_{75} + \eta_{125},$$

in the short notation $\eta_N$ defined in Eq. (8) with $M = M_1 M_2 M_3 = 250$.

### A.2 (Even-Odd-Odd) wavefunctions

Here, we study the patterns of Yukawa matrices in the models, where zero-modes of left, right-handed matter fields and Higgs fields correspond to even, odd and odd functions, respectively.

#### A.2.1 4-7-11 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (4, 7, 11)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.
This model has five zero-modes for the Higgs fields. Yukawa couplings $Y_{ijk} L_i R_j H_k$ are given by

$$Y_{ij}^k H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4,$$

where

$$
\begin{align*}
y_{ij}^0 & = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}(\eta_4 - \eta_{36}) & \sqrt{2}(\eta_{92} - \eta_{48}) & \sqrt{2}(\eta_{128} - \eta_{40}) \\
\eta_{81} - \eta_{59} + \eta_{95} + \eta_{73} & \eta_{139} - \eta_{29} + \eta_{125} + \eta_{15} & \eta_{51} - \eta_{17} - \eta_{37} + \eta_{103} \\
\sqrt{2}(\eta_{150} - \eta_{18}) & \sqrt{2}(\eta_{62} - \eta_{16}) & \sqrt{2}(\eta_{26} - \eta_{14})
\end{pmatrix}, \\
y_{ij}^1 & = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}(\eta_{80} - \eta_{52}) & \sqrt{2}(\eta_{8} - \eta_{36}) & \sqrt{2}(\eta_{96} - \eta_{124}) \\
\eta_{3} - \eta_{25} - \eta_{129} + \eta_{151} & \eta_{85} - \eta_{113} - \eta_{41} + \eta_{69} & \eta_{135} - \eta_{107} - \eta_{47} - \eta_{19} \\
\sqrt{2}(\eta_{74} - \eta_{102}) & \sqrt{2}(\eta_{146} - \eta_{118}) & \sqrt{2}(\eta_{58} - \eta_{30})
\end{pmatrix}, \\
y_{ij}^2 & = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}(\eta_{144} - \eta_{32}) & \sqrt{2}(\eta_{76} - \eta_{120}) & \sqrt{2}(\eta_{12} - \eta_{100}) \\
\eta_{87} - \eta_{109} - \eta_{45} + \eta_{67} & \eta_{1} - \eta_{11} - \eta_{13} + \eta_{153} & \eta_{89} - \eta_{23} - \eta_{131} + \eta_{65} \\
\sqrt{2}(\eta_{10} - \eta_{122}) & \sqrt{2}(\eta_{78} - \eta_{34}) & \sqrt{2}(\eta_{142} - \eta_{4})
\end{pmatrix}, \\
y_{ij}^3 & = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}(\eta_{148} - \eta_{104}) & \sqrt{2}(\eta_{148} - \eta_{104}) & \sqrt{2}(\eta_{72} - \eta_{16}) \\
\eta_{171} - \eta_{115} - \eta_{39} + \eta_{17} & \eta_{83} - \eta_{27} - \eta_{27} + \eta_{17} & \eta_{5} - \eta_{61} - \eta_{13} + \eta_{49} \\
\sqrt{2}(\eta_{94} - \eta_{38}) & \sqrt{2}(\eta_{6} - \eta_{50}) & \sqrt{2}(\eta_{82} - \eta_{138})
\end{pmatrix}, \\
y_{ij}^4 & = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}(\eta_{24} - \eta_{108}) & \sqrt{2}(\eta_{64} - \eta_{20}) & \sqrt{2}(\eta_{152} - \eta_{68}) \\
\eta_{53} - \eta_{31} - \eta_{123} + \eta_{101} & \eta_{141} - \eta_{57} - \eta_{7} + \eta_{13} & \eta_{79} - \eta_{145} - \eta_{9} + \eta_{75} \\
\sqrt{2}(\eta_{30} - \eta_{46}) & \sqrt{2}(\eta_{90} - \eta_{134}) & \sqrt{2}(\eta_{2} - \eta_{86})
\end{pmatrix},
\end{align*}
$$

in the short notation $\eta_N$ defined in Eq. (1) with $M = M_1 M_2 M_3 = 308$.

A.2.2 4-7-3 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (4, 7, 3)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

$$
\begin{array}{c|c|c|c}
& L_i(\lambda^{ab}) & R_j(\lambda^{ca}) & H_k(\lambda^{bc}) \\
\hline
0 & \Theta^{0,4} & \sqrt{2}(\Theta^{1,4} - \Theta^{0,7}) & \sqrt{2}(\Theta^{1,11} - \Theta^{10,11}) \\
1 & \frac{1}{\sqrt{2}} (\Theta^{1,4} + \Theta^{3,4}) & \sqrt{2}(\Theta^{2,5} - \Theta^{5,7}) & \frac{1}{\sqrt{2}} (\Theta^{2,11} - \Theta^{9,11}) \\
2 & \Theta^{2,4} & \sqrt{2}(\Theta^{3,7} - \Theta^{4,7}) & \frac{1}{\sqrt{2}} (\Theta^{3,11} - \Theta^{8,11}) \\
\end{array}
$$

This model has a single zero-modes for Higgs fields. Yukawa couplings $Y_{ijk} L_i R_j H_k$ are obtained as

$$Y_{ij}^k H_k = \frac{1}{\sqrt{2}} H_0 \begin{pmatrix}
\sqrt{2}(\eta_4 - \eta_{32}) & \sqrt{2}(\eta_{20} - \eta_{8}) & \sqrt{2}(\eta_{40} - \eta_{16}) \\
\eta_{17} + \eta_{25} + \eta_{11} - \eta_{31} & \eta_1 + \eta_{41} + \eta_{13} - \eta_{29} & \eta_{19} + \eta_{23} - \eta_{5} - \eta_{37} \\
\sqrt{2}(\eta_{38} - \eta_{10}) & \sqrt{2}(\eta_{22} - \eta_{34}) & \sqrt{2}(\eta_{2} - \eta_{26})
\end{pmatrix},$$
in the short notation $\eta_N$ defined in Eq. (9) with $M = M_1M_2M_3 = 84$.

### A.2.3 4-8-12 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (4, 8, 12)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| $L_i(\lambda_{cb})$ | $R_j(\lambda_{ca})$ | $H_k(\lambda_{bc})$ |
|--------------------|--------------------|--------------------|
| 0                  | $\Theta^{0,4}$     | $\frac{1}{\sqrt{2}}(\Theta^{1,8} - \Theta^{7,8})$ |
| 1                  | $\frac{1}{\sqrt{2}}(\Theta^{1,4} + \Theta^{3,4})$ | $\frac{1}{\sqrt{2}}(\Theta^{2,8} - \Theta^{6,8})$ |
| 2                  | $\Theta^{2,4}$     | $\frac{1}{\sqrt{2}}(\Theta^{3,8} - \Theta^{5,8})$ |
| 3                  | -                  | -                  |
| 4                  | -                  | -                  |

This model has five zero-modes for the Higgs fields. Yukawa couplings $Y_{ijk}L_iR_jH_k$ are given by

$$Y_{ijk}H_k = y_{ij}^0H_0 + y_{ij}^1H_1 + y_{ij}^2H_2 + y_{ij}^3H_3 + y_{ij}^4H_4,$$

where

$$y_{ij}^0 = \begin{pmatrix} y_b & 0 & -y_l \\ y_k & 0 & y_h \\ -y_f & 0 & y_d \end{pmatrix}, \quad y_{ij}^1 = \begin{pmatrix} 0 & y_c - y_k & 0 \\ \frac{1}{\sqrt{2}}(y_b - y_h) & 0 & \frac{1}{\sqrt{2}}(y_f - y_l) \\ 0 & 0 & 0 \end{pmatrix},$$

$$y_{ij}^2 = \begin{pmatrix} -y_j & 0 & y_d \\ 0 & \frac{1}{\sqrt{2}}(y_a - y_m) & 0 \\ y_d & 0 & -y_j \end{pmatrix}, \quad y_{ij}^3 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}(y_f - y_l) & 0 & \frac{1}{\sqrt{2}}(y_b - y_h) \\ 0 & y_c - y_k & 0 \end{pmatrix},$$

$$y_{ij}^4 = \begin{pmatrix} y_h & 0 & -y_f \\ 0 & \frac{1}{\sqrt{2}}(y_e - y_i) & 0 \\ -y_l & 0 & y_b \end{pmatrix},$$

and

$$y_a = \eta_9 + \eta_{96} + \eta_{192} + \eta_{96}, \quad y_b = \eta_4 + \eta_{100} + \eta_{188} + \eta_{92},$$

$$y_c = \eta_8 + \eta_{104} + \eta_{184} + \eta_{88}, \quad y_d = \eta_{12} + \eta_{108} + \eta_{180} + \eta_{84},$$

$$y_e = \eta_{16} + \eta_{112} + \eta_{176} + \eta_{80}, \quad y_f = \eta_{20} + \eta_{116} + \eta_{172} + \eta_{76},$$

$$y_g = \eta_{24} + \eta_{120} + \eta_{168} + \eta_{72}, \quad y_h = \eta_{28} + \eta_{124} + \eta_{164} + \eta_{68},$$

$$y_i = \eta_{32} + \eta_{128} + \eta_{160} + \eta_{64}, \quad y_j = \eta_{36} + \eta_{132} + \eta_{156} + \eta_{60},$$

$$y_k = \eta_{40} + \eta_{136} + \eta_{152} + \eta_{56}, \quad y_l = \eta_{44} + \eta_{140} + \eta_{148} + \eta_{52},$$

$$y_m = \eta_{48} + \eta_{144} + \eta_{144} + \eta_{48},$$

in the short notation $\eta_N$ defined in Eq. (9) with $M = M_1M_2M_3 = 384$.

### A.2.4 4-8-4 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (4, 8, 4)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.
This model has a single zero-modes for Higgs fields. Yukawa couplings $Y_{ijk} L_i R_j H_k$ are obtained as

$$Y_{ij}^k H_k = H_0 \begin{pmatrix} y_b & 0 & -y_c \\ 0 & \frac{1}{\sqrt{2}}(y_a - y_d) & 0 \\ -y_c & 0 & y_b \end{pmatrix},$$

where

$$y_a = \eta_0 + 2\eta_{32} + \eta_{64}, \quad y_b = \eta_4 + \eta_{28} + \eta_{36} + \eta_{60},$$

$$y_c = \eta_{12} + \eta_{20} + \eta_{44} + \eta_{52}, \quad y_d = 2\eta_{16} + 2\eta_{48},$$

in the short notation $\eta_N$ defined in Eq. with $M = M_1 M_2 M_3 = 128$.

### A.2.5 5-7-12 model

Here we show the model with $(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (5, 7, 12)$. The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

|      | $L_i(\lambda^{ab})$ | $R_j(\lambda^{ca})$ | $H_k(\lambda^{bc})$ |
|------|----------------|-----------------|------------------|
| 0    | $\Theta^{0.5}$ | $\frac{1}{\sqrt{2}}(\Theta^{1.7} - \Theta^{6.7})$ | $\frac{1}{\sqrt{2}}(\Theta^{11.12} - \Theta^{11.12})$ |
| 1    | $\frac{1}{\sqrt{2}}(\Theta^{1.5} + \Theta^{4.5})$ | $\frac{1}{\sqrt{2}}(\Theta^{2.7} - \Theta^{5.7})$ | $\frac{1}{\sqrt{2}}(\Theta^{10.12} - \Theta^{10.12})$ |
| 2    | $\frac{1}{\sqrt{2}}(\Theta^{2.5} + \Theta^{3.5})$ | $\frac{1}{\sqrt{2}}(\Theta^{3.7} - \Theta^{4.7})$ | $\frac{1}{\sqrt{2}}(\Theta^{8.12} - \Theta^{9.12})$ |
| 3    | -               | -               | $\frac{1}{\sqrt{2}}(\Theta^{5.12} - \Theta^{7.12})$ |
| 4    | -               | -               | -                |

This model has five zero-modes for the Higgs fields. Yukawa coupling $Y_{ijk} L_i R_j H_k$ are given by

$$Y_{ij} H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4,$$

where

$$y_{ij}^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_5 - \eta_{65}) \\ \eta_{173} - \eta_{103} + \eta_{187} + \eta_{163} \\ \eta_{67} - \eta_{137} - \eta_{153} + \eta_{17} \\ \eta_{113} - \eta_{143} - \eta_{27} + \eta_{197} \end{pmatrix},$$

$$y_{ij}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{170} - \eta_{110}) \\ \eta_{72} - \eta_{142} - \eta_{58} + \eta_{82} \\ \eta_{178} - \eta_{18} - \eta_{122} + \eta_{158} \\ \eta_{62} - \eta_{202} - \eta_{118} + \eta_{22} \end{pmatrix},$$

$$y_{ij}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{175} - \eta_{135}) \\ \eta_{117} - \eta_{13} - \eta_{117} + \eta_{93} \\ \eta_{3} - \eta_{207} - \eta_{123} + \eta_{87} \\ \eta_{83} - \eta_{27} - \eta_{57} + \eta_{153} \end{pmatrix},$$

$$y_{ij}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{165} - \eta_{45}) \\ \eta_{117} - \eta_{39} - \eta_{129} + \eta_{81} \\ \eta_{69} - \eta_{141} - \eta_{111} + \eta_{99} \end{pmatrix},$$

$y_{ij}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{15} - \eta_{195}) \\ \eta_{177} - \eta_{33} - \eta_{117} + \eta_{93} \\ \eta_{3} - \eta_{207} - \eta_{123} + \eta_{87} \\ \eta_{83} - \eta_{27} - \eta_{57} + \eta_{153} \end{pmatrix}$,
\[ y_{ij}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{100} - \eta_{140}) & \sqrt{2}(\eta_{80} - \eta_{200}) & \sqrt{2}(\eta_{160} - \eta_{20}) \\ \eta_{88} - \eta_{208} + \eta_{28} + \eta_{52} + \eta_{88} & \eta_{72} - \eta_{32} - \eta_{52} + \eta_{88} & \eta_{8} - \eta_{148} - \eta_{188} + \eta_{92} \\ \eta_{184} - \eta_{144} + \eta_{124} + \eta_{164} & \eta_{4} - \eta_{36} - \eta_{116} + \eta_{164} & \eta_{176} - \eta_{104} - \eta_{64} + \eta_{76} \end{pmatrix}, \]

\[ y_{ij}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}(\eta_{145} - \eta_{205}) & \sqrt{2}(\eta_{95} - \eta_{225}) & \sqrt{2}(\eta_{105} - \eta_{155}) \\ \eta_{107} - \eta_{37} - \eta_{47} + \eta_{23} & \eta_{73} - \eta_{143} - \eta_{193} + \eta_{157} & \eta_{167} - \eta_{97} - \eta_{13} + \eta_{83} \\ \eta_{61} - \eta_{31} - \eta_{121} + \eta_{11} & \eta_{179} - \eta_{109} - \eta_{59} + \eta_{11} & \eta_{1} - \eta_{71} - \eta_{181} + \eta_{169} \end{pmatrix}, \]

in the short notation \( \eta_N \) defined in Eq. (9) with \( M = M_1 M_2 M_3 = 420 \).

### A.2.6 5-8-13 model

Here we show the model with \((|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (5, 8, 13)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| \( L_i(\lambda^{ab}) \) | \( R_j(\lambda^{ca}) \) | \( H_k(\lambda^{bc}) \) |
|-----------------|-----------------|-----------------|
| 0 \( \Theta^{0,0} \) | \( \frac{1}{\sqrt{2}}(\Theta^{1,8} - \Theta^{7,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{1,13} - \Theta^{12,13}) \) |
| 1 \( \frac{1}{\sqrt{2}}(\Theta^{1,5} + \Theta^{4,5}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{2,8} - \Theta^{6,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{2,13} - \Theta^{11,13}) \) |
| 2 \( \frac{1}{\sqrt{2}}(\Theta^{2,5} + \Theta^{3,5}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{3,8} - \Theta^{5,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{3,13} - \Theta^{10,13}) \) |
| 3 \( \frac{1}{\sqrt{2}} \) | \( \frac{1}{\sqrt{2}}(\Theta^{4,13} - \Theta^{9,13}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{5,13} - \Theta^{8,13}) \) |
| 4 \( \frac{1}{\sqrt{2}} \) | \( \frac{1}{\sqrt{2}} \) | \( \frac{1}{\sqrt{2}}(\Theta^{6,13} - \Theta^{7,13}) \) |
| 5 \( \frac{1}{\sqrt{2}} \) | \( \frac{1}{\sqrt{2}} \) | \( \frac{1}{\sqrt{2}} \) |

This model has six zero-modes for the Higgs fields. Yukawa couplings \( Y_{ijk} L_i R_j H_k \) are given by

\[
Y_{ijk}^{k} H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4 + y_{ij}^5 H_5,
\]

where

\[
y_{ij}^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} (\eta_5 - \eta_{125}) & \sqrt{2} (\eta_{190} - \eta_{70}) & \sqrt{2} (\eta_{135} - \eta_{255}) \\ \eta_{203} + \eta_{213} - \eta_{63} - \eta_{187} & \eta_{122} - \eta_{138} + \eta_{18} - \eta_{242} & \eta_{73} - \eta_{57} + \eta_{177} - \eta_{47} \\ \eta_{109} - \eta_{21} + \eta_{99} - \eta_{229} & \eta_{86} - \eta_{174} + \eta_{226} - \eta_{34} & \eta_{239} - \eta_{151} + \eta_{31} - \eta_{61} \end{pmatrix},
\]

\[
y_{ij}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} (\eta_{115} - \eta_{245}) & \sqrt{2} (\eta_{210} - \eta_{50}) & \sqrt{2} (\eta_{15} - \eta_{145}) \\ \eta_{197} + \eta_{137} - \eta_{67} - \eta_{93} & \eta_{2} - \eta_{258} + \eta_{102} - \eta_{158} & \eta_{193} - \eta_{63} + \eta_{223} - \eta_{167} \\ \eta_{11} - \eta_{141} + \eta_{219} - \eta_{171} & \eta_{206} - \eta_{54} + \eta_{106} - \eta_{154} & \eta_{119} - \eta_{249} + \eta_{89} - \eta_{41} \end{pmatrix},
\]

\[
y_{ij}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} (\eta_{85} - \eta_{45}) & \sqrt{2} (\eta_{110} - \eta_{150}) & \sqrt{2} (\eta_{135} - \eta_{255}) \\ \eta_{23} + \eta_{163} - \eta_{53} - \eta_{227} & \eta_{202} - \eta_{58} - \eta_{98} - \eta_{162} & \eta_{7} - \eta_{137} + \eta_{57} - \eta_{33} \\ \eta_{189} - \eta_{59} + \eta_{19} - \eta_{149} & \eta_{6} - \eta_{254} + \eta_{214} - \eta_{46} & \eta_{201} - \eta_{71} + \eta_{111} - \eta_{241} \end{pmatrix},
\]

\[
y_{ij}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} (\eta_{235} - \eta_{155}) & \sqrt{2} (\eta_{90} - \eta_{170}) & \sqrt{2} (\eta_{105} - \eta_{255}) \\ \eta_{77} + \eta_{157} - \eta_{53} - \eta_{27} & \eta_{118} - \eta_{142} + \eta_{222} - \eta_{38} & \eta_{207} - \eta_{183} + \eta_{103} - \eta_{233} \\ \eta_{131} - \eta_{259} + \eta_{181} - \eta_{51} & \eta_{194} - \eta_{66} + \eta_{14} - \eta_{246} & \eta_{1} - \eta_{129} + \eta_{209} - \eta_{79} \end{pmatrix},
\]

\[
y_{ij}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} (\eta_{135} - \eta_{255}) & \sqrt{2} (\eta_{170} - \eta_{150}) & \sqrt{2} (\eta_{255} - \eta_{155}) \\ \eta_{73} - \eta_{57} + \eta_{177} - \eta_{47} & \eta_{239} - \eta_{151} + \eta_{31} - \eta_{61} & \eta_{201} - \eta_{71} + \eta_{111} - \eta_{241} \\ \eta_{119} - \eta_{249} + \eta_{89} - \eta_{41} & \eta_{7} - \eta_{137} + \eta_{57} - \eta_{33} & \eta_{207} - \eta_{183} + \eta_{103} - \eta_{233} \end{pmatrix},
\]
Here we show the model with (A.3.1 7-7-14 model) handed matter fields and Higgs fields correspond to odd, odd and even functions, respectively.

A.3.1 (Odd-Odd-Even) wavefunctions

η in the short notation η_N defined in Eq. (9) with M = M_M2M_3 = 520.

A.2.7 5-8-3 model

Here we show the model with (A.2.7 5-8-3 model) Yukawa couplings obtained as

\[ Y_{ij} H_k = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} \sqrt{2}(\eta_5 - \eta_{35}) & \sqrt{2}(\eta_{50} - \eta_{10}) & \sqrt{2}(\eta_{25} - \eta_{55}) \\ \eta_{43} - \eta_{37} - \eta_{13} + \eta_{53} & \eta_{2} - \eta_{38} - \eta_{58} + \eta_{22} & \eta_{47} - \eta_{7} - \eta_{17} + \eta_{23} \\ \eta_{29} - \eta_{11} - \eta_{59} + \eta_{19} & \eta_{46} - \eta_{34} - \eta_{14} + \eta_{26} & \eta_{1} - \eta_{41} - \eta_{31} + \eta_{49} \end{array} \right), \]

in the short notation η_N defined in Eq. (9) with M = M_M2M_3 = 120.

A.3 (Odd-Odd-Even) wavefunctions

Here, we study the patterns of Yukawa matrices in the models, where zero-modes of left, right-handed matter fields and Higgs fields correspond to odd, odd and even functions, respectively.

A.3.1 7-7-14 model

Here we show the model with (A.3.1 7-7-14 model) subsections 4.2 and 4.3 in detail. The zero-mode wavefunctions of left, right-handed matter fields and Higgs fields are shown in Table 4.

This model has eight zero-modes for the Higgs field. Yukawa couplings Y_{ijk}L_jR_jH_k are obtained as

\[ Y_{ijk} H_k = y^0_{ij} H_0 + y^1_{ij} H_1 + y^2_{ij} H_2 + y^3_{ij} H_3 + y^4_{ij} H_4 + y^5_{ij} H_5 + y^6_{ij} H_6 + y^7_{ij} H_7, \]

where y^k_{ij} is shown in Eq. (10) with M = M_M2M_3 = 686.
### A.3.2 7-8-15 model

Here we show the model with \(|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}| = (7, 8, 15)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

|   | \(L_{ij}(\lambda^{ab})\) | \(R_{ij}(\lambda^{ca})\) | \(H_{ik}(\lambda^{bc})\) |
|---|------------------|------------------|------------------|
| 0 | \(\frac{1}{\sqrt{2}} (\Theta^{1,4} - \Theta^{0,4})\) | \(\frac{1}{\sqrt{2}} (\Theta^{1,8} - \Theta^{7,8})\) | \(\frac{1}{\sqrt{2}} (\Theta^{0,15} + \Theta^{14,15})\) |
| 1 | \(\frac{1}{\sqrt{2}} (\Theta^{2,7} - \Theta^{3,7})\) | \(\frac{1}{\sqrt{2}} (\Theta^{2,8} - \Theta^{6,8})\) | \(\frac{1}{\sqrt{2}} (\Theta^{2,15} + \Theta^{13,15})\) |
| 2 | \(-\frac{1}{\sqrt{2}} (\Theta^{3,7} - \Theta^{4,7})\) | \(\frac{1}{\sqrt{2}} (\Theta^{3,7} - \Theta^{5,8})\) | \(\frac{1}{\sqrt{2}} (\Theta^{3,15} + \Theta^{12,15})\) |
| 3 | \(-\frac{1}{\sqrt{2}} (\Theta^{4,7} - \Theta^{5,7})\) | \(-\frac{1}{\sqrt{2}} (\Theta^{4,15} + \Theta^{11,15})\) | \(-\frac{1}{\sqrt{2}} (\Theta^{5,15} + \Theta^{10,15})\) |
| 4 | \(-\frac{1}{\sqrt{2}} (\Theta^{5,7} - \Theta^{6,7})\) | \(-\frac{1}{\sqrt{2}} (\Theta^{6,15} + \Theta^{9,15})\) | \(-\frac{1}{\sqrt{2}} (\Theta^{7,15} + \Theta^{8,15})\) |

This model has eight zero-modes for the Higgs fields. Yukawa couplings \(Y_{ijk} L_{ij} R_{jk} H_k\) are given by

\[
Y_{ijk}^k H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4 + y_{ij}^5 H_5 + y_{ij}^6 H_6 + y_{ij}^7 H_7,
\]

where

\[
y_{ij}^0 = \begin{pmatrix}
\eta_{225} - \eta_{15}, & \eta_{330} - \eta_{90}, & \eta_{405} - \eta_{195}, \\
\eta_{345} - \eta_{135}, & \eta_{390} - \eta_{30}, & \eta_{285} - \eta_{75}, \\
\eta_{375} - \eta_{255}, & \eta_{270} - \eta_{150}, & \eta_{165} - \eta_{45}
\end{pmatrix},
\]

\[
y_{ij}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{113} - \eta_{97} - \eta_{127} + \eta_{337}, & \eta_{218} - \eta_{202} - \eta_{22} + \eta_{398}, & \eta_{233} - \eta_{307} - \eta_{83} + \eta_{293}, \\
\eta_{233} - \eta_{23} - \eta_{247} + \eta_{383}, & \eta_{338} - \eta_{82} - \eta_{142} + \eta_{278}, & \eta_{397} - \eta_{187} - \eta_{37} + \eta_{173}, \\
\eta_{353} - \eta_{143} - \eta_{367} + \eta_{263}, & \eta_{382} - \eta_{38} - \eta_{262} + \eta_{158}, & \eta_{277} - \eta_{67} - \eta_{57} + \eta_{53}
\end{pmatrix},
\]

\[
y_{ij}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{1} - \eta_{299} - \eta_{339} + \eta_{391}, & \eta_{106} - \eta_{314} - \eta_{34} + \eta_{286}, & \eta_{211} - \eta_{119} - \eta_{29} + \eta_{181}, \\
\eta_{121} - \eta_{89} - \eta_{359} + \eta_{271}, & \eta_{226} - \eta_{194} - \eta_{354} + \eta_{166}, & \eta_{331} - \eta_{299} - \eta_{149} + \eta_{61}, \\
\eta_{241} - \eta_{301} - \eta_{361} + \eta_{151}, & \eta_{346} - \eta_{74} - \eta_{374} + \eta_{46}, & \eta_{389} - \eta_{179} - \eta_{269} + \eta_{59}
\end{pmatrix},
\]

\[
y_{ij}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{111} - \eta_{321} - \eta_{351} + \eta_{279}, & \eta_{6} - \eta_{414} - \eta_{246} + \eta_{174}, & \eta_{99} - \eta_{309} - \eta_{41} + \eta_{69}, \\
\eta_{9} - \eta_{201} - \eta_{369} + \eta_{159}, & \eta_{114} - \eta_{306} - \eta_{366} + \eta_{54}, & \eta_{219} - \eta_{111} - \eta_{261} + \eta_{51}, \\
\eta_{120} - \eta_{81} - \eta_{249} + \eta_{39}, & \eta_{234} - \eta_{186} - \eta_{354} + \eta_{66}, & \eta_{339} - \eta_{291} - \eta_{381} + \eta_{171}
\end{pmatrix},
\]

\[
y_{ij}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{223} - \eta_{407} - \eta_{377} + \eta_{167}, & \eta_{118} - \eta_{302} - \eta_{358} + \eta_{62}, & \eta_{13} - \eta_{197} - \eta_{253} + \eta_{43}, \\
\eta_{103} - \eta_{313} - \eta_{257} + \eta_{47}, & \eta_{2} - \eta_{418} - \eta_{362} + \eta_{58}, & \eta_{107} - \eta_{317} - \eta_{373} + \eta_{163}, \\
\eta_{17} - \eta_{193} - \eta_{137} + \eta_{73}, & \eta_{122} - \eta_{298} - \eta_{342} + \eta_{178}, & \eta_{227} - \eta_{403} - \eta_{347} + \eta_{283}
\end{pmatrix},
\]

\[
y_{ij}^5 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{335} - \eta_{295} - \eta_{265} + \eta_{55}, & \eta_{230} - \eta_{190} - \eta_{370} + \eta_{50}, & \eta_{125} - \eta_{85} - \eta_{365} + \eta_{155}, \\
\eta_{215} - \eta_{415} - \eta_{145} + \eta_{65}, & \eta_{110} - \eta_{310} - \eta_{250} + \eta_{170}, & \eta_{5} - \eta_{925} - \eta_{355} + \eta_{275}, \\
\eta_{215} - \eta_{305} - \eta_{25} + \eta_{185}, & \eta_{10} - \eta_{410} - \eta_{130} + \eta_{290}, & \eta_{115} - \eta_{325} - \eta_{235} + \eta_{95}
\end{pmatrix},
\]

\[
y_{ij}^6 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\eta_{393} - \eta_{183} - \eta_{153} + \eta_{57}, & \eta_{342} - \eta_{78} - \eta_{258} + \eta_{162}, & \eta_{237} - \eta_{27} - \eta_{363} + \eta_{267}, \\
\eta_{327} - \eta_{303} - \eta_{33} + \eta_{177}, & \eta_{222} - \eta_{198} - \eta_{138} + \eta_{282}, & \eta_{117} - \eta_{93} - \eta_{243} + \eta_{387}, \\
\eta_{207} - \eta_{417} - \eta_{87} + \eta_{297}, & \eta_{102} - \eta_{318} - \eta_{18} + \eta_{402}, & \eta_{3} - \eta_{213} - \eta_{123} + \eta_{333}
\end{pmatrix},
\]
\[ y_{ij}^7 = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\eta_{281} - \eta_{71} - \eta_{41} + \eta_{169} \\
\eta_{386} - \eta_{34} - \eta_{146} + \eta_{274} \\
\eta_{349} - \eta_{139} - \eta_{251} + \eta_{379} \\
\eta_{401} - \eta_{191} - \eta_{79} + \eta_{289} \\
\eta_{334} - \eta_{86} - \eta_{26} + \eta_{394} \\
\eta_{229} - \eta_{19} - \eta_{31} + \eta_{341} \\
\eta_{319} - \eta_{311} - \eta_{199} + \eta_{409} \\
\eta_{214} - \eta_{206} - \eta_{94} + \eta_{326} \\
\eta_{109} - \eta_{101} - \eta_{11} + \eta_{221}
\end{array} \right), \]

in the short notation \( \eta_N \) defined in Eq. (3) with \( M = M_1M_2M_3 = 840 \).

### A.3.3 7-8-1 model

Here we show the model with \((|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (7, 8, 1)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| \( L_i(\lambda^{ab}) \) | \( R_j(\lambda^{ca}) \) | \( H_k(\lambda^{bc}) \) |
|-----------------|-----------------|-----------------|
| 0 \( \frac{1}{\sqrt{2}}(\Theta^{1,4} - \Theta^{6,7}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{1,8} - \Theta^{4,8}) \) | \( \Theta^{7,4} \) |
| 1 \( \frac{1}{\sqrt{2}}(\Theta^{2,7} - \Theta^{5,7}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{2,8} - \Theta^{6,8}) \) | - |
| 2 \( \frac{1}{\sqrt{2}}(\Theta^{3,7} - \Theta^{4,7}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{3,8} - \Theta^{5,8}) \) | - |

This model has a single zero-mode for the Higgs field. Yukawa couplings \( Y_{ijk}L_iR_jH_k \) are given by

\[ Y_{ijk}H_k = \frac{1}{\sqrt{2}} H_0 \left( \begin{array}{c}
\sqrt{2}(\eta_5 - \eta_{35}) \\
\sqrt{2}(\eta_{50} - \eta_{10}) \\
\sqrt{2}(\eta_{25} - \eta_{55}) \\
\eta_{44} - \eta_{73} - \eta_{13} + \eta_{53} \\
\eta_2 - \eta_{38} - \eta_{58} + \eta_{22} \\
\eta_7 - \eta_{77} - \eta_{17} + \eta_{33} \\
\eta_{29} - \eta_{11} - \eta_{59} + \eta_{19} \\
\eta_46 - \eta_{34} - \eta_{14} + \eta_{26} \\
\eta_1 - \eta_{41} - \eta_{31} + \eta_{41}
\end{array} \right), \]

in the short notation \( \eta_N \) defined in Eq. (3) with \( M = M_1M_2M_3 = 56 \).

### A.3.4 8-8-16 model

Here we show the model with \((|I_{ab}^{(1)}|, |I_{ca}^{(1)}|, |I_{bc}^{(1)}|) = (8, 8, 16)\). The following table shows zero-mode wavefunctions of left, right-handed matter fields and Higgs fields.

| \( L_i(\lambda^{ab}) \) | \( R_j(\lambda^{ca}) \) | \( H_k(\lambda^{bc}) \) |
|-----------------|-----------------|-----------------|
| 0 \( \frac{1}{\sqrt{2}}(\Theta^{1,8} - \Theta^{4,7}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{1,8} - \Theta^{4,7}) \) | \( \Theta^{0,16} \) |
| 1 \( \frac{1}{\sqrt{2}}(\Theta^{2,7} - \Theta^{5,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{2,8} - \Theta^{6,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{1,16} + \Theta^{15,16}) \) |
| 2 \( \frac{1}{\sqrt{2}}(\Theta^{3,7} - \Theta^{4,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{3,8} - \Theta^{5,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{2,16} + \Theta^{14,16}) \) |
| 3 \( \frac{1}{\sqrt{2}}(\Theta^{4,7} - \Theta^{5,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{4,8} - \Theta^{6,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{3,16} + \Theta^{13,16}) \) |
| 4 \( \frac{1}{\sqrt{2}}(\Theta^{5,7} - \Theta^{6,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{5,8} - \Theta^{7,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{6,16} + \Theta^{10,16}) \) |
| 5 \( \frac{1}{\sqrt{2}}(\Theta^{6,7} - \Theta^{7,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{6,8} - \Theta^{8,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{7,16} + \Theta^{9,16}) \) |
| 6 \( \frac{1}{\sqrt{2}}(\Theta^{7,7} - \Theta^{8,8}) \) | \( \frac{1}{\sqrt{2}}(\Theta^{7,8} - \Theta^{9,8}) \) | \( \Theta^{8,16} \) |

This model has eight zero-modes for the Higgs fields. Yukawa couplings \( Y_{ijk}L_iR_jH_k \) are obtained as

\[ Y_{ijk}H_k = y_{ij}^0 H_0 + y_{ij}^1 H_1 + y_{ij}^2 H_2 + y_{ij}^3 H_3 + y_{ij}^4 H_4 + y_{ij}^5 H_5 + y_{ij}^6 H_6 + y_{ij}^7 H_7 + y_{ij}^8 H_8, \]

25
where

\[
y_{ij}^0 = \begin{pmatrix} -y_g & 0 & 0 \\
0 & -y_e & 0 \\
0 & 0 & -y_g \end{pmatrix}, \quad y_{ij}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -y_d & 0 \\
-y_d & 0 & -y_f \\
0 & -y_f & 0 \end{pmatrix},
\]

\[
y_{ij}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} y_a & 0 & -y_e \\
0 & 0 & 0 \\
-y_e & 0 & y_i \end{pmatrix}, \quad y_{ij}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & y_b & 0 \\
y_b & 0 & y_h \\
0 & y_h & 0 \end{pmatrix},
\]

\[
y_{ij}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & y_c + y_g & 0 \\
y_c + y_g & 0 & y_i \\
0 & y_i & 0 \end{pmatrix}, \quad y_{ij}^5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & y_h & 0 \\
y_h & 0 & y_b \\
0 & y_b & 0 \end{pmatrix},
\]

\[
y_{ij}^6 = \frac{1}{\sqrt{2}} \begin{pmatrix} y_i & 0 & -y_e \\
0 & 0 & y_c + y_g \\
-y_e & 0 & y_a \end{pmatrix}, \quad y_{ij}^7 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -y_f & 0 \\
y_f & 0 & -y_d \\
0 & -y_d & 0 \end{pmatrix},
\]

\[
y_{ij}^8 = \begin{pmatrix} -y_c & 0 & 0 \\
0 & -y_e & 0 \\
0 & 0 & -y_c \end{pmatrix},
\]

and

\[
y_e = \eta_0 + 2(\eta_{128} + 2\eta_{256} + 2\eta_{384}) + \eta_{512},
\]

\[
y_b = \eta_8 + \eta_{120} + \eta_{136} + \eta_{248} + \eta_{264} + \eta_{376} + \eta_{392} + \eta_{504},
\]

\[
y_c = \eta_{16} + \eta_{112} + \eta_{144} + \eta_{240} + \eta_{272} + \eta_{368} + \eta_{400} + \eta_{496},
\]

\[
y_d = \eta_{24} + \eta_{104} + \eta_{156} + \eta_{232} + \eta_{280} + \eta_{360} + \eta_{408} + \eta_{488},
\]

\[
y_e = \eta_{32} + \eta_{96} + \eta_{164} + \eta_{224} + \eta_{288} + \eta_{352} + \eta_{416} + \eta_{480},
\]

\[
y_f = \eta_{40} + \eta_{88} + \eta_{172} + \eta_{216} + \eta_{296} + \eta_{344} + \eta_{424} + \eta_{472},
\]

\[
y_g = \eta_{48} + \eta_{80} + \eta_{180} + \eta_{208} + \eta_{304} + \eta_{336} + \eta_{432} + \eta_{464},
\]

\[
y_h = \eta_{56} + \eta_{72} + \eta_{188} + \eta_{200} + \eta_{312} + \eta_{328} + \eta_{440} + \eta_{456},
\]

\[
y_i = 2(\eta_{64} + \eta_{192} + \eta_{320} + \eta_{448}),
\]

in the short notation \(\eta_N\) defined in Eq. (4) with \(M = M_1 M_2 M_3 = 1024\).

References

[1] N. S. Manton, Nucl. Phys. B 193, 502 (1981); G. Chapline and R. Slansky, Nucl. Phys. B 209, 461 (1982); S. Randjbar-Daemi, A. Salam and J. A. Strathdee, Nucl. Phys. B 214, 491 (1983); C. Wetterich, Nucl. Phys. B 222, 20 (1983); P. H. Frampton and K. Yamamoto, Phys. Rev. Lett. 52, 2016 (1984); P. H. Frampton and T. W. Kephart, Phys. Rev. Lett. 53, 867 (1984); K. Pilch and A. N. Schellekens, Nucl. Phys. B 256, 109 (1985);

[2] E. Witten, Phys. Lett. B 149, 351 (1984).
[3] C. Bachas, arXiv:hep-th/9503030.

[4] M. Berkooz, M. R. Douglas and R. G. Leigh, Nucl. Phys. B 480, 265 (1996) arXiv:hep-th/9606139.

[5] R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, JHEP 0010, 006 (2000) arXiv:hep-th/0007024.

[6] C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, Phys. Lett. B 489, 223 (2000) arXiv:hep-th/0007090.

[7] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0405, 079 (2004) arXiv:hep-th/0404229.

[8] J. Troost, Nucl. Phys. B 568, 180 (2000) arXiv:hep-th/9909187.

[9] G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan and A. M. Uranga, J. Math. Phys. 42, 3103 (2001) arXiv:hep-th/0011073; JHEP 0102, 047 (2001) arXiv:hep-ph/0011132.

[10] R. Blumenhagen, B. Kors and D. Lust, JHEP 0102, 030 (2001) arXiv:hep-th/0012156.

[11] M. Cvetic, G. Shiu and A. M. Uranga, Phys. Rev. Lett. 87, 201801 (2001) arXiv:hep-th/0107143; Nucl. Phys. B 615, 3 (2001) arXiv:hep-th/0107166.

[12] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, Ann. Rev. Nucl. Part. Sci. 55, 71 (2005) arXiv:hep-th/0502005; R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rept. 445, 1 (2007) arXiv:hep-th/0610327.

[13] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B 261, 678 (1985); Nucl. Phys. B 274, 285 (1986).

[14] H. Abe, T. Kobayashi and H. Ohki, JHEP 0809, 043 (2008) arXiv:0806.4748 [hep-th].

[15] J. P. Conlon, A. Maharana and F. Quevedo, JHEP 0809, 104 (2008) arXiv:0807.0789 [hep-th].

[16] F. Marchesano, P. McGuirk and G. Shiu, arXiv:0812.2247 [hep-th].

[17] G. ’t Hooft, Nucl. Phys. B 153, 141 (1979).

[18] J. Alfaro, A. Broncano, M. B. Gavela, S. Rigolin and M. Salvatori, JHEP 0701, 005 (2007) arXiv:hep-ph/0606070; D. Hernandez, S. Rigolin and M. Salvatori, arXiv:0712.1980 [hep-ph].

[19] G. von Gersdorff, Nucl. Phys. B 793, 192 (2008) arXiv:0705.2410 [hep-th].

[20] R. Blumenhagen, L. Gorlich and T. Ott, JHEP 0301, 021 (2003) arXiv:hep-th/0211059; M. Cvetic and I. Papadimitriou, Phys. Rev. D 67, 126006 (2003) arXiv:hep-th/0303197; M. Cvetic, T. Li and T. Liu, Nucl. Phys. B 698, 163 (2004) arXiv:hep-th/0403061.
[21] T. Kobayashi, S. Raby and R. J. Zhang, Phys. Lett. B 593, 262 (2004) arXiv:hep-ph/0403065; Nucl. Phys. B 704, 3 (2005) arXiv:hep-ph/0409098.

[22] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology,” Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[23] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, arXiv:0810.5509 [hep-th].

[24] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0307, 038 (2003) arXiv:hep-th/0302105.

[25] T. Higaki, N. Kitazawa, T. Kobayashi and K. j. Takahashi, Phys. Rev. D 72, 086003 (2005) arXiv:hep-th/0504019.

[26] H. M. Lee, H. P. Nilles and M. Zucker, Nucl. Phys. B 680, 177 (2004) arXiv:hep-th/0309195.

[27] S. Forste, T. Kobayashi, H. Ohki and K. j. Takahashi, JHEP 0703, 011 (2007) arXiv:hep-th/0612044.