A NOTE ON DEPTH PRESERVATION

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Abstract. We show that for a wildly ramified torus, depth is not preserved in general under local Langlands correspondence for tori.

1. Introduction

Let $K$ be a non-archimedean local field and let $W_K$ denote its Weil group. Local class field theory (LCFT) tells us that there is a canonical isomorphism $K^\times \cong W_K^{ab}$ and this isomorphism respects the numbering on the filtration subgroups $\{K_r^\times\}_{r \geq 0}$ of $K^\times$ and the upper numbering on the filtration subgroups $\{W_K^r\}_{r \geq 0}$ of $W_K$. Local Langlands Correspondence (LLC) stipulates a vast generalization of the LCFT isomorphism. In LLC, irreducible representations $\text{Irr}(G(K))$ of the $K$-points of a reductive $K$-group $G$ are expected to be parametrized by arithmetic objects called Langlands parameters $\Phi(G)$ in a certain natural way. For each $\pi \in \text{Irr}(G(K))$, Moy-Prasad theory associates an invariant called depth $\text{dep}(\pi)$. Also for each $\phi \in \Phi(G)$, one defines the notion of depth $\text{dep}(\phi)$. It is the smallest number such that $\phi$ is trivial on $W_K^r$ for all $r > \text{dep}(\phi)$. If $\phi$ associates to $\pi$ under LLC, then one expects that quite fairly $\text{dep}(\pi) = \text{dep}(\phi)$. This is known in many cases (see the introduction in [2] for a survey). However, counter examples have been constructed for inner forms of $\text{SL}_n(K)$ [2] and in the case of $\text{SL}_2(K)$ when $K$ has characteristic 2 [1].

Now let $T = R_{K'/K}\mathbb{G}_m$ where $K'$ is a finite separable extension of $K$ and $R_{K'/K}$ denotes the Weil restriction and let $\lambda_T : \chi \in \text{Irr}(T(K)) \mapsto \lambda_T(\chi) \in \Phi(T)$ under LLC. In this note, we show that $\varphi_{K'/K}(e \cdot \text{dep}(\chi)) = \text{dep}(\lambda_T(\chi))$ where $\varphi_{K'/K}$ is the Hasse-Herbrand function and $e$ is the ramification index of $K'/K$. Thus for all positive depth characters $\chi$, $\text{dep}(\lambda_T(\chi)) > \text{dep}(\chi)$. When $T$ is a tamely induced wildly ramified torus (see Sec. 7.1), we show that $T(K)$ admits characters for which depth is not preserved under LLC. In Section 8 we compute Hasse-Herbrand function for a certain wildly ramified extension of a cyclotomic field to illustrate the failure of depth preservation.

The proofs in Section 7 follow closely the proofs in [8] and [4].
2. Review of ramification groups

Let \( K \) be a non-archimedean local field and let \( L \) be a finite Galois extension of \( K \). Write \( \mathcal{O}_L, p_L \) for the ring of integers of \( L \) and the maximal ideal of \( \mathcal{O}_L \). For \( i \geq -1 \), define \( G_i \) to be the set of all \( s \in G := \text{Gal}(L/K) \) such that \( s \) operates trivially on \( \mathcal{O}_L/p_L^{i+1} \). Then \( G_{-1} = G \). The groups \( G_i \) are called ramification groups. They form a decreasing filtration of normal subgroups. Extend the definition of \( G_u \) for all real numbers \( u \geq -1 \) by setting

\[
G_u = G_i \quad \text{where} \quad i \quad \text{is the least integer} \quad \geq \quad u.
\]

This numbering of ramification groups is called lower numbering. Lower numbering behaves well with respect to intersections, i.e., if \( H \) is a subgroup of \( G \), then \( G_u \cap H = H_u \).

*Upper numbering of ramification groups.* Define \( \varphi_{L/K} : [-1, \infty) \rightarrow \mathbb{R} \) to be the map \( r \mapsto \int_{0}^{r} \frac{1}{(1+t^2)} dt \) where \( (G_0 : G_u) := (G_u : G_0)^{-1} \) for \( u \in [-1, 0) \). The function \( \varphi_{L/K} \) is called the Hasse-Herbrand function. It has the basic properties [6]:

(a) \( \varphi_{L/K} \) is continuous, piecewise linear, increasing and concave.

(b) \( \varphi_{L/K}(0) = 0 \).

(c) \( \varphi_{L/K} \) is a homeomorphism of \([-1, \infty)\) onto itself.

(d) If \( H \) is a normal subgroup of \( G \), then \( \varphi_{L/K} = \varphi_{L/H/K} \circ \varphi_{L/H} \).

If an extension \( M/K \) is not Galois, define \( \varphi_{M/K} = \varphi_{M'/K} \circ \varphi_{M/M'}^{-1} \), where \( M' \) is a Galois extension of \( K \) containing \( M \). The inverse \( \varphi_{M/K}^{-1} \) is denoted \( \psi_{M/K} \).

Define an upper numbering on ramification groups by setting \( G^v = G_u \) if \( v = \varphi_{L/K}(u) \). Upper numbering behaves well with respect to quotients, i.e., if \( H \) is a normal subgroup of \( G \), then

\[
(G/H)^v = G^v H/H.
\]

For an infinite Galois extension \( \Omega \) of \( K \), define the ramification groups on \( G = \text{Gal}(\Omega/K) \) by:

\[
G^v = \lim_{\overset{F/K \text{ finite}}{F/K \text{ finite}}} \text{Gal}(F/K)^v.
\]

Now let \( L/K \) be Galois extension of local fields and let \( F \) be a finite extension of \( K \) contained in \( L \). Write \( G = \text{Gal}(L/K) \) and \( H = \text{Gal}(L/F) \).

**Lemma 1.** For all \( r \geq 0 \), \( G^r \cap H = H^{\psi_{F/K}(r)} \).
Proof. Let $E$ be a finite Galois extension of $K$ in $L$ containing $F$. Write $I = \text{Gal}(L/E)$. Then
\[
(G/I)^r \cap (H/I) = (G/I)^{\psi_{E/K}^r} \cap (H/I) = (H/I)^{\psi_{E/K}^r} = (H/I)^{\psi_{E/F}^r \psi_{E/K}^r} = (H/I)^{\psi_{F/K}^r}.
\]
The lemma now follows by taking inverse limit over $E$. □

3. Notion of depth

Let $G = \text{Gal}(L/K)$ where $L$ and $K$ are local fields and let $M$ be a $G$-module. Define the depth of $M$ to be:
\[
\text{dep}_G(M) = \inf \{ r \geq 0 \mid M^{G^r} \neq 0 \text{ for all } s > r \}.
\]

Define the depth of a co-cycle $\varphi \in H^1(G,M)$ to be:
\[
\text{dep}_G(\varphi) = \inf \{ r \geq 0 \mid G^s \subset \ker(\varphi) \text{ for all } s > r \}.
\]

4. Depth change under induction

Let $G = \text{Gal}(L/K)$ and $H = \text{Gal}(L/F)$ where $K \subseteq F \subseteq L$ and $F/K$ finite Galois. Let $N$ be an $H$-module.

**Proposition 2.** $\text{dep}_H(N) = \psi_{F/K}(\text{dep}_G(\text{Ind}_H^G N))$.

**Proof.** By Mackey theory,
\[
\text{Res}_{G^r}(\text{Ind}_H^G N) = \bigoplus_{g \in G^r \setminus G/H} \text{Ind}_{G^r \cap H}^{G^r} N^g.
\]
Here $\text{Res}$ denotes the restriction functor and $N^g$ denotes the $g$-twisted module $N$. By Lemma 1, $G^r \cap H = H^{\psi_{F/K}^r}$. Thus
\[
(\text{Ind}_H^G N)^{G^r} \neq 0 \iff (\text{Ind}_{H^{\psi_{F/K}^r}} N^g)^{G^r} \neq 0 \text{ for some } g \in G^r \setminus G/H
\]
\[
\iff (N^g)^{H^{\psi_{F/K}^r}} \neq 0
\]
\[
\iff (N)^{H^{\psi_{F/K}^r}} \neq 0.
\]

□

5. Depth change under Shapiro’s isomorphism

Again $G = \text{Gal}(L/K)$ and $H = \text{Gal}(L/F)$ where $K \subseteq F \subseteq L$ and $F/K$ is any finite extension and let $N$ be an $H$-module.
Shapiro’s lemma states that the map 
\[ \text{Sh} : H^1(G, \text{Ind}_{H}^G N) \to H^1(H, N) \]
defined by 
\[ \gamma \mapsto (h \mapsto \gamma(h)(1)) \]
is an isomorphism. We wish to relate the depth of co-cycles under this isomorphism. We first observe the following:

**Lemma 3.** Let \( A \) be a group, \( B \) and \( C \) subgroups of \( A \) with \( C \) being normal in \( A \). Let \( M \) be a \( B \)-module. Then there is a canonical isomorphism of \( A/C \)-modules:

\[ (\text{Ind}_{A}^B M)^C \cong \text{Ind}_{B \cap C}^A M_{C \cap B} \]

**Proof.** The map \( f \in (\text{Ind}_{A}^B M)^C \mapsto \tilde{f} := (gC \mapsto f(g)) \in (\text{Ind}_{B \cap C}^A M_{C \cap B}) \) is easily verified to be the required isomorphism. \( \square \)

**Lemma 4.** For \( r \geq 0 \), Shapiro’s lemma induces an isomorphism 
\[ H^1(G/G^r, (\text{Ind}_{H}^G N)^{G^r}) \cong H^1(H/H^{\psi_F/K(r)}, N^{H^{\psi_F/K(r)}}) \]

**Proof.** We have 
\[ H^1(G/G^r, (\text{Ind}_{H}^G N)^{G^r}) \cong H^1(G/G^r, \text{Ind}_{H/G^r \cap H}^{G/G^r} N^{G^r \cap H}) \cong H^1(G/G^r, \text{Ind}_{H/H^{\psi_F/K(r)}}^{G/G^r} N^{H^{\psi_F/K(r)}}) \cong H^1(H/H^{\psi_F/K(r)}, N^{H^{\psi_F/K(r)}}) \]

The first isomorphism follows from Lemma 3, second from Lemma 1 and the last from Shapiro’s lemma. \( \square \)

Write \( H^1(G, \text{Ind}_{H}^G N)^{\text{adm}} = \bigcup_{r \geq 0} H^1(G/G^r, (\text{Ind}_{H}^G N)^{G^r}) \).

**Corollary 5.** If \( \lambda \in H^1(G, \text{Ind}_{H}^G N)^{\text{adm}} \), then \( \text{dep}_G(\lambda) = \varphi_{F/K}(\text{dep}_H(\text{Sh}(\lambda))) \).

**Proof.** Let \( \text{dep}_G(\lambda) = r \). Then \( G^s \subset \ker(\lambda) \) if \( s > r \). By Lemma 4, this implies \( H^{\psi_F/K(s)} \subset \ker(\text{Sh}(\lambda)) \) if \( s > r \). Therefore \( \text{dep}_H(\text{Sh}(\lambda)) \leq \psi_{F/K}(\text{dep}_G(\lambda)) \). The argument is reversible showing that \( \text{dep}_H(\text{Sh}(\lambda)) \geq \psi_{F/K}(\text{dep}_G(\lambda)) \). Therefore \( \text{dep}_H(\text{Sh}(\lambda)) = \psi_{F/K}(\text{dep}_G(\lambda)) \). \( \square \)

6. **Langlands correspondence for tori**

We review here the statement of local Langlands correspondence for tori as stated and proved in [8].
6.1. Special case. Let \( T = R_{K'/K} \mathbb{G}_m \) where \( K' \) is a finite separable extension of \( K \) and \( R_{K'/K} \) denotes the Weil restriction. Then \( T(K) = K'^\times \) and the group of characters \( X^*(T) \) is canonically a free \( \mathbb{Z} \)-module with basis \( W_K/W_{K'} \) where \( W_K \) (resp. \( W_{K'} \)) denotes the Weil group of \( K \) (resp. \( K' \)). From this, it follows that the complex dual \( \hat{T} \) of \( T \) is canonically isomorphic to \( \text{Ind}_{W_K}^{W_{K'}} \mathbb{C}^\times \). We get,

\[
\text{Hom}(T(K), \mathbb{C}^\times) \cong \text{Hom}(K'^\times, \mathbb{C}^\times) \\
\cong \text{Hom}(W_{K'}, \mathbb{C}^\times) \\
\cong \text{H}^1(W_{K'}, \mathbb{C}^\times) \\
\cong \text{H}^1(W_K, \text{Ind}_{W_{K'}}^{W_K} \mathbb{C}^\times) \\
\cong \text{H}^1(W_K, \hat{T}). \tag{6.2}
\]

The isomorphism 6.1 follows by class field theory and the isomorphism 6.2 by Shapiro’s lemma.

6.2. The LLC for tori in general.

**Theorem.** \([3]\) There is a unique family of homomorphisms

\[ \lambda_T : \text{Hom}(T(K), \mathbb{C}^\times) \to \text{H}^1(W_K, \hat{T}) \]

with the following properties:

1. \( \lambda_T \) is additive functorial in \( T \), i.e., it is a morphism between two additive functors from the category of tori over \( K \) to the category of abelian groups;
2. For \( T = R_{K'/K} \mathbb{G}_m \), where \( K'/K \) is a finite separable extension, \( \lambda_T \) is the isomorphism described in Section 6.1.

7. Depth change for tori under LLC

We keep the notations as in Section 6. Let \( M \) be a local field. Recall that \( M^\times \) admits a filtration \( \{M_r^\times\}_{r \geq 0} \) where \( M_0^\times \) is the units of the ring of integers and for \( r > 0 \), \( M_r^\times := \{ x \in M \mid \text{ord}_M(x - 1) \geq r \} \). Here \( \text{ord}_M \) is the valuation of \( M \) normalised so that \( \text{ord}_M(M^\times) = \mathbb{Z} \). Under local class field theory isomorphism

\[ M_r^\times \cong (W_M^r)^\text{ab}. \]

We recall that \( T(K) \) carries a Moy-Prasad filtration \( \{T(K)_r\}_{r \geq 0} \). The depth \( \text{dep}_T(\chi) \) of a character \( \chi : T(K) \to \mathbb{C}^\times \) is defined to be

\[ \inf\{ r \geq 0 \mid T(K)_s \subset \ker(\chi) \text{ for } s > r \}. \]

The group \( T(K)_0 \) is called the Iwahori subgroup of \( T(K) \). It is a subgroup of finite index in the maximal compact subgroup of \( T(K) \). When \( T = R_{K'/K} \mathbb{G}_m \),
then for $r > 0$,

\begin{align}
T(K)_r &= \{ x \in T(K) = K'^x \mid \text{ord}_K(x - 1) \geq r \} \\
(7.1) &= \{ x \in K'^x \mid \text{ord}_{K'}(x - 1) \geq er \} \\
(7.2) &= K'^x.
\end{align}

Here $\text{ord}_K$ is the valuation on $K'$ normalised so that $\text{ord}_K(K'^x) = \mathbb{Z}$ and $e$ is the ramification index of $K'/K$. The equality (7.1) follows from \cite[Sec. 4.2]{7} and the equality (7.2) follows from the fact that $\text{ord}_{K'}(\alpha) = e \cdot \text{ord}_K(\alpha)$ for all $\alpha \in K^x$.

**Theorem 6.** Let $T = R_{K'/K} \mathbb{G}_m$, where $K'/K$ is a finite separable extension of local fields of ramification index $e$. Then for $r \geq 0$, the local Langlands correspondence for tori induces an isomorphism:

\[ \text{Hom}(T(K)/T(K)_r, C^\times) \cong H^1(W_K/W_{K'}^{\varphi_{K'/K}(er)}, \hat{T}W_{K'}^{\varphi_{K'/K}(er)}). \]

**Proof.** The case $r = 0$ is a special case of \cite[Theorem 7]{4}. For $r > 0$, this follows by

\begin{align}
\text{Hom}(T(K)/T(K)_r, C^\times) &\cong \text{Hom}(K'^x/K'^x_\text{er}, C^\times) \\
(7.3) &\cong \text{Hom}(W_{K'}^{\varphi_{K'/K}(er)}, C^\times) \\
&\cong H^1(W_{K'}^{\varphi_{K'/K}(er)}, C^\times) \\
(7.4) &\cong H^1(W_K/W_{{K'}_r}^{\varphi_{K'/K}(er)}, \text{Ind}_{W_{K'}}^{W_K} C^\times W_{K'}^{\varphi_{K'/K}(er)}) \\
&\cong H^1(W_K/W_{{K'}_r}^{\varphi_{K'/K}(er)}, (\hat{T}) W_{K'}^{\varphi_{K'/K}(er)}).
\end{align}

Here, the isomorphism (7.4) follows from Lemma 4. \hfill \Box

**Corollary 7.** For $T$ as in Theorem 6, \( \varphi_{K'/K}(e \cdot \text{dep}_T(\chi)) = \text{dep}_{W_K}(\lambda_T(\chi)). \)

**Proof.** This follows from an argument analogous to the argument in the proof of the Corollary 5. \hfill \Box

**Remark 8.** The slope of the map $r \mapsto \varphi_{K'/K}(er)$ at a differentiable point $r$ is \( \frac{e}{\text{log}_e(e^{r})} \geq 1. \) Thus, when $K'/K$ is a wildly ramified extension, $\varphi_{K'/K}(er) > r$ and consequently $\text{dep}_T(\chi) < \text{dep}_{W_K}(\lambda_T(\chi))$.

When $K'/K$ is a tamely ramified extension, $\varphi_{K'/K}(r) = \frac{r}{e}$. Therefore in this case, Corollary 7 simplifies to,

\[ \text{dep}_T(\chi) = \text{dep}_{W_K}(\lambda_T(\chi)). \]

This is a special case of Depth-preservation Theorem of Yu for tamely ramified tori \cite[Sec. 7.10]{8}.
Lemma 9. For $1 \leq r$ and $\varphi_{L/F}(r) = (p - 1)\varphi_{L/K}(r)$. 

7.1. Case of a tamely induced tori. Recall that a $K$-torus is called induced if it is of the form $\Pi_{i=1}^{k} R_{L_i/K} G_m$, where $L_i$ are finite separable extensions of $K$. A $K$-torus $T$ is called tamely induced if $T \otimes_K K_1$ is an induced torus for some tamely ramified extension $K_1$ of $K$. In this section, we compare depths under LLC for such tori following the proof in [8, Sec. 7.10].

Let $T$ be a tamely induced $K$-torus. Then there exists an induced torus $T' = \prod_{i=1}^{n} R_{K_i'/K} G_m$ such that $T' \to T$ and $C_0 := \ker(T' \to T)$ is connected. Further $T'(K)_r \to T(K)_r \forall r > 0$ (see proof in [7, Lemma 4.7.4]). Let $\chi \in \text{Hom}(T(K), \mathbb{C}^\times)$ and let $\chi'$ denote its lift to $T'(K)$. Then

$$\text{dep}_T(\chi) = \text{dep}_{T'}(\chi') = \sup\{\text{dep}_{T'}(\chi'_i) \mid 1 \leq i \leq n\}. \tag{7.5}$$

Here $T'_i$ denotes $R_{K_i'/K} G_m$ and $\chi'_i = \chi'|_{T'_i}$. By functoriality, $\lambda_T(\chi)$ is the image of $\lambda_T(\chi)$ under $H^1(W_K, T) \to H^1(W_K, T')$ and therefore $\text{dep}_{W_K}(\lambda_T(\chi)) = \text{dep}_{W_K}(\lambda_T(\chi'))$. But

$$\text{dep}_{W_K}(\lambda_T(\chi')) = \sup\{\text{dep}_{W_K}(\varphi_{K_i'/K_i}(e_i \cdot \text{dep}_{T'}(\chi'_i)) \mid 1 \leq i \leq n\}.$$ \hspace{1cm} (7.6)

$$= \sup\{\varphi_{K_i'/K_i}(e_i \cdot \text{dep}_{T'}(\chi'_i)) \mid 1 \leq i \leq n\}.$$ \hspace{1cm} (7.6)

$$\geq \sup\{\text{dep}_{T'}(\chi'_i) \mid 1 \leq i \leq n\}. \tag{7.6}$$

Here $e_i$ denotes the ramification index of $K_i'/K_i$. Thus

$$\text{dep}_{W_K}(\lambda_T(\chi)) \geq \text{dep}_T(\chi). \tag{7.6}$$

Now assume $T$ is wildly ramified. We will now produce a character of $T(K)$ for which the inequality (7.6) is strict. We can assume without loss of generality that $T_0 := R_{K_0'/K} G_m$ is wildly ramified. Let $\chi'_0$ be a positive depth character of $T_0(K)$ which is trivial on $C_0 \cap T_0(K)$. Extend $\chi'_0$ trivially to a character $\chi'$ of $T'(K)$. Then since $\mathbb{C}^\times$ is divisible, the character $\chi'$ lifts to a character $\chi$ of $T(K)$. By Remark [8] $\text{dep}_{T_0}(\chi'_0) < \text{dep}_{W_K}(\lambda_T_0(\chi'_0))$. Since $\text{dep}_T(\chi) = \text{dep}_{T_0}(\chi'_0)$ and $\text{dep}_{W_K}(\lambda_T(\chi)) = \text{dep}_{W_K}(\lambda_{T_0}(\chi'_0))$, it follows that the inequality (7.6) is strict for this choice of $\chi$.

8. An Example

Let $K = \mathbb{Q}_p$, $L = K(\zeta_{p^n})$, where $\zeta_{p^n}$ denotes a primitive $p^n$th root of unity, $n \geq 1$. Then $L/K$ is a totally ramified extension of degree $(p - 1)p^{n-1}$. Consider the intermediate extension $F = K(\zeta_p)$ of $K$ of degree $p - 1$ over $K$. Then $L/F$ is a wildly ramified extension. Write $G = \text{Gal}(L/K)$ and $H = \text{Gal}(L/F)$.

Lemma 9. For $1 \leq r$ and $\varphi_{L/F}(r) = (p - 1)\varphi_{L/K}(r)$. 

Proof. We first note that since we considering abelian extensions, the jumps in filtration occur at integer values. We have for \( r \geq 1 \),
\[
\varphi_{L/K}(r) = \int_0^r \frac{dt}{(G_0 : G_t)}
\]
\[
= \int_0^1 \frac{dt}{(G_0 : G_t)} + \int_1^r \frac{dt}{(G_0 : G_t)}
\]
\[
= \frac{1}{p-1} + \int_1^r \frac{dt}{(G_0 : G_t)}
\]
\[
= \frac{1}{p-1} + \int_1^r \frac{(H_0 : H_t)}{(G_0 : G_t)} \frac{dt}{(G_0 : G_t)}
\]
\[
= \frac{1}{p-1} + \int_1^r \frac{(H_0 : H_t)}{(G_0 : H_0)} \frac{dt}{(H_0 : H_t)}
\]
\[
= \frac{1}{p-1} + \frac{1}{p-1} \int_1^r \frac{dt}{(H_0 : H_t)}.
\]
The last equality holds because \( G_t = H_t \) for \( t \geq 1 \) and \( (G_0 : H_0) = p - 1 \). Thus
\[
\varphi_{L/K}(r) = \frac{1}{p-1} + \frac{1}{p-1} (\varphi_{L/F}(r) - \int_0^1 \frac{1}{(H_0 : H_t)} dt)
\]
\[
= \frac{1}{p-1} + \frac{1}{p-1} (\varphi_{L/F}(r) - 1)
\]
\[
= \frac{\varphi_{L/F}(r)}{(p-1)}.
\]
\[\square\]

Write \( m = p^n \) and let \( G(m) = (\mathbb{Z}/m\mathbb{Z})^\times \). By [6, Chap IV, Prop. 17], \( G = G(m) \).
Define
\[
G(m)^e := \{ a \in G(m) \mid a \equiv 1 \mod p^e \}.
\]
Then \( G(m)^e = \text{Gal}(L/K(\zeta_{p^e})) \). The ramification groups \( G_u \) of \( G \) are [6, Chap IV, Prop. 18]:
\[
G_0 = G
\]
if \( 1 \leq u \leq p - 1 \) \( G_u = G(m)^1 \)
if \( p \leq u \leq p^2 - 1 \) \( G_u = G(m)^2 \)
\[
\vdots
\]
if \( p^{n-1} \leq u \) \( G_u = 1 \).

We now calculate \( \varphi_{L/F} \).
Proposition 10. The Hasse-Herbrand function of the wildly ramified extension \( L/F \) is given by

\[
\varphi_{L/F}(r) = \begin{cases} 
  k(p-1) + \frac{r-p^k+1}{p^k} & \text{if } p^k - 1 < r \leq p^{k+1} - 1 \text{ with } 0 \leq k < n - 1 \\
  (n-1)(p-1) + \frac{r-p^{n-1}+1}{p^{n-1}} & \text{if } r > p^{n-1} - 1
\end{cases}
\]

Proof. We consider various cases:

- **Case** \( 0 < r \leq 1 \)
  \[
  \varphi_{L/F}(r) = \int_0^r \frac{dt}{(H_0 : H_t)} = \frac{1}{(H_0 : H_1)} \int_0^r dr = r.
  \]

- **Case** \( 1 < r \leq p - 1 \)
  \[
  \varphi_{L/K}(r) = \int_0^r \frac{dt}{(G_0 : G_t)} = \int_0^1 \frac{dt}{(G_0 : G_1)} + \int_1^r \frac{dt}{(G_0 : G_t)} = \frac{1}{p-1} + \int_1^r \frac{dt}{(G_0 : G(m)^1)} = \frac{r}{p-1}.
  \]

Therefore, \( \varphi_{L/F}(r) = r \).

- **Case** \( p^k - 1 < r \leq p^{k+1} - 1 \) with \( 1 \leq k < n - 1 \)
  \[
  \varphi_{L/K}(r) = \int_0^r \frac{dt}{(G_0 : G_t)} = \sum_{i=0}^{k-1} \int_{(p^i-1)}^{(p^{i+1}-1)} \frac{dt}{(G_0 : G_t)} + \int_{p^k-1}^r \frac{dt}{(G_0 : G_t)} = \int_0^1 \frac{dt}{(G_0 : G_1)} + \int_1^{p-1} \frac{dt}{(G_0 : G(m)^1)} + \sum_{i=1}^{k-1} \int_{p^i-1}^{p^{i+1}-1} \frac{dt}{(G_0 : G(m)^i)}
  \]
  \[
  + \int_{p^k-1}^r \frac{dt}{(G_0 : G(m)^{k+1})} = \frac{1}{p-1} + \frac{p-2}{p-1} + \sum_{i=1}^{k-1} \frac{p^{i+1} - p^i}{(p-1)p^i} + \frac{r - p^k + 1}{(p-1)p^k} = k + \frac{r - p^k + 1}{(p-1)p^k}.
  \]

Therefore, \( \varphi_{L/F}(r) = k(p-1) + \frac{r-p^k+1}{p^k} \).
• Case $r > p^n - 1$

$$
\varphi_{L/K}(r) = \int_0^r \frac{dt}{G_0 : G_t} = \int_0^{p^n - 1} \frac{dt}{G_0 : G_1} + \int_{p^n - 1}^r \frac{dt}{G_0 : G_t} = (n - 1) + \frac{r - p^n - 1 + 1}{(p - 1)p^n - 1}.
$$

Therefore, $\varphi_{L/F}(r) = (n - 1)(p - 1) + \frac{r - p^n - 1 + 1}{p^n - 1}$.

Now write $T = R_{L/F} G_m$ and let $\lambda_T$ be as denoted in Sec. 6. It then immediately follows from Prop. 10.

**Lemma 11.** $\varphi_{L/K}(p^n - 1 - r) > r \forall r > 0$. Consequently, for all positive depth $\chi \in \text{Hom}(T(K), \mathbb{C}^\times)$, $\text{dep}_T(\chi) < \text{dep}_{W_K}(\lambda_T(\chi))$.

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