Boson sampling cannot be faithfully simulated by only the lower-order multi-boson interferences

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To simulate noisy boson sampling approximating it by only the lower-order multi-boson interferences (e.g., by a smaller number of interfering bosons and classical particles) is very popular idea. I show that the output data from any such classical simulations can be efficiently distinguished from that of the quantum device they try to simulate, even with finite noise in the latter. The distinguishing datasets can be the experimental estimates of some large probabilities, a wide class of such is presented. This is a sequel of Quantum 5, 423 (2021), where I present more accessible account of the main result enhanced by additional insight on the contribution from the higher-order multi-boson interferences in presence of noise.

1 Introduction

Presence of weak noise is accounted in the boson sampling idea [1]. For \( N \) interfering bosons the number of classical computations required for the noiseless boson sampling is estimated to be \( O(N^2N) \) [2]. However, a polynomial in \( N \) classical simulation could become possible for some strong enough noise. In the experiments [3, 4, 5] there is some amount of noise. Is it weak noise or strong noise?

Due to an exponentially large in \( N \) space of outcomes (exponentially small probabilities) the experimentalists check only some lower-order correlations. The universal feature of noise effect, be it photon distinguishability [6], photon losses [7], or unstable noisy network [8, 9, 10], is that the higher orders of multi-boson interference (precise definition is given below) affected stronger by presence of noise (see also the discussion [11]). Efficient classical approximations exploit this effect of noise [12, 13, 14, 15, 16, 17, 18, 19]. When amplitudes of noise in boson sampling are not scaling down with its size, one can call such noise finite noise. It is known that for such noise the correlation between the output distribution of noiseless boson sampling and that of the noisy one tend to zero [9] and the total variation distance between the two distributions cannot be small [18]. Efficient classical approximation of boson sampling can be constructed for finite noise amplitudes by imposing a cutoff at a fixed order \( K = O(1) \) (as \( N \) scales up) of multi-boson interferences [9, 13, 17, 18, 19] (see also explanation in Aaronson’s blog Ref. [20]). Such lower-order approximations are the focus of this work.

I show that classical simulations accounting for only the lower-order multi-boson interferences can be efficiently distinguished from the quantum device with finite noise they try to simulate. Despite the fact that the sensitivity to noise is proportional to the order of quantum correlations, the higher-order correlations (higher-order multi-boson interferences) still make up the difference. Thus experimentalists should find ways to check the higher-order correlations. One way is presented below. This is accessible exposition of the main result in the recent paper [21] enhanced by additional insight on the contribution from the higher orders of multi-boson interferences in the presence of noise.

The text is structured as follows. I present and discuss the main result of Ref. [21] (section 2), then give an additional insight on the higher-order correlations in the presence of noise (section 3), and then discuss the relation to classical simulations and the recent experiment on boson sampling (section 4). Conclusion (section 5) contains the main message to be taken from this text.
2 How to distinguish noisy boson sampling from classical approximations

The computational complexity of boson sampling is related to the fact that the number of possible “quantum paths” in the quantum transition of $N$ bosons through a unitary linear interferometer is exponential in $N$, due to bosons being identical particles. Different quantum paths of $N$ bosons are given by different permutations of bosons by the symmetric group of $N$ objects, each particular path of bosons is composed of multi-boson interferences. Let us define what is meant by the “multi-boson interferences of order $\ell$”. To this goal, we partition the full multi-boson interference into some disjoint classes. As is known, a permutation can be decomposed into a product of disjoint cycles. A permutation $\pi \in S_N$ of $X \equiv \{1, \ldots, N\}$ is a cycle of length $\ell$ if it cyclically permutes $\ell$ elements, e.g., $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_\ell \rightarrow i_1$ ($i_\alpha \in X$, $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$). The key property of the disjoint cycles is that the cycles of length $\ell$ map to the multi-boson interferences of exactly $\ell$ bosons [22]. For example, cycles of length $\ell = 1$, i.e., the fixed points, correspond to “lone”, or classical, bosons, not interfering with other bosons, and cycles of length $\ell = 2$ correspond to two-boson interferences, responsible for the second-order correlations. Moreover, the quantum multi-boson correlation function of order $K$, which describes the joint detection of only $K$ bosons out of $N$ (averaging on $N - K$ bosons in the output probability), depend only on the cycles (the multi-boson interferences) of orders $\ell \leq K$ [22].

Bosons are indistinguishable, i.e., can interfere on a unitary interferometer, only to the degree given by the overlap of their states in the internal degrees of freedom, called the internal states (for photons there are infinitely many of such, due to the continuous spectral shape). Assuming a constant overlap $0 < \xi \leq 1$ of the internal states of two bosons, one can show that multi-boson interferences of orders $\ell \geq 2$ (given by the cycles of length $\ell$) acquire a weight equal to $\xi^\ell$ [21] (for $\ell = 1$, i.e., for the lone bosons, the weight is equal to 1; the output probability formula is reproduced in appendix A).

Let us define “the lower-order interferences” by combining all the orders $\ell$ of multi-boson interferences satisfying $\ell \leq K$, with some fixed $K = O(1)$ for the total number of bosons $N$ scaling to infinity. Due to the fact that the higher-order interferences have much stronger noise sensitivity then the lower-order ones, it seems reasonable enough to assume that the former do not matter in the presence of finite noise and, therefore, one could faithfully approximate such noisy boson sampling device by accounting only for the latter. Will this simple idea work? Below I argue that no, it will not: One can efficiently distinguish the output data set coming from such classical simulators and that from the boson sampling they try to simulate.

The above negative answer applies not only to the noise due to partial distinguishability of bosons, but also to other sources of noise and to their combined effect as well. Here we consider also imperfect transmission (losses) of bosons through the device, accounted for by a uniform transmission coefficient $0 < \eta \leq 1$. The probability of a single boson being transmitted by the interferometer is $\eta^2$. The so-called dark counts of detectors (not related to particle detection), which follow the usual Poisson distribution $\pi(n) = \frac{\nu^n}{n!} e^{-\nu}$, are also accounted with some uniform rate $\nu$ for all detectors. Moreover, there are equivalence relations between action of various sources of noise on boson sampling (the proof can be found in Ref. [18]). For instance, noise in interferometer [9] is equivalent to a combined action of boson losses exactly compensated by dark counts of detectors (a special case of the shuffled bosons model of Ref. [7]). We consider a linear interferometer with a unitary matrix $U_{kl}$, where there are $M$ input and output ports ($1 \leq k, l \leq M$), and impose no relation between $M$ and $N$, except that $M \geq N$ (as is common in recent experiments with large numbers of bosons [3, 4, 5]). Introduce the density of bosons parameter $\rho = N/M$.

Our goal is to bound from below the total variational distance between the probability distributions of a noisy boson sampling $p_{\text{bm}}$ with some noise parameters $\xi, \eta, \nu$ (see an example for $\nu = \eta = 1$ and arbitrary $\xi$ in appendix A) and a classical simulation accounting for the lower-order multi-boson interferences $p_{\text{bm}}(K)$, where such a simulation is obtained by imposing a cutoff at an order $K = O(1)$ of the multi-boson correlations. One can impose the cutoff, for example, by allowing only $K$ (random) bosons to interfere
supplemented by $N - K$ "lone" bosons passing through an interferometer one by one [13, 17, 18]. The general way is to limit the disjoint cycle lengths by $K$ in the symmetric group $S_N$ describing the multi-boson interferences [21]. The total variational distance reads

$$D(p, p^{(K)}) = \frac{1}{2} \sum_{m} |p_m - p_m^{(K)}|, \quad (1)$$

where the sum runs over all possible configurations $m = (m_1, \ldots, m_M)$ of bosons in the output ports, with $m_l$ being the number of bosons in output port $l$. The difference in probability of any subset $\Omega$ of the output configurations $m$ bounds the total variation distance from below,

$$D(p, p^{(K)}) \geq |P_{\Omega} - P_{\Omega}^{(K)}|, \quad P_{\Omega} \equiv \sum_{m \in \Omega} p_m. \quad (2)$$

Observe that the equality is necessarily achieved for a certain subset $\Omega$, depending on $U$ and other parameters of the setup.

Consider first one specific choice of $\Omega$: the probability to detect zero bosons at a single output port, say port $l = 1$, and arbitrary numbers of bosons in other output ports, $m_1 = 0$, $m_2 + \ldots + m_M = N$. Denote this probability by $P_1 (P_1^{(K)})$ and the difference $\Delta P_1 \equiv P_1 - P_1^{(K)}$. Assuming that $K \ll \sqrt{N}$ we obtain [21]:

$$D(p, p^{(K)}) \geq |\langle \Delta P_1 \rangle| \approx \frac{(\xi \eta \rho)^{K+1}}{1 + \xi \eta \rho} e^{-1 - \nu - \eta \rho} \equiv W_1, \quad (3)$$

$$\frac{(\langle \Delta P_1 \rangle)^2 - (\Delta P_1)^2}{W_1^2} \approx \frac{(1 - \rho)(K + 1)^2}{N},$$

where the averaging is performed over the Haar-random interferometers $U^1$. Observe that, in contrast, single output probability, i.e., $p_m$ and $p_m^{(K)}$, vanishes exponentially with $N$.

The vanishing of the relative variance as $N$ scales up, given that $K \ll \sqrt{N}$, implies (by the standard Chebyshev’s inequality) that the lower bound in Eq. (3) applies almost surely over the Haar-random interferometers (only some subset of the interferometers having vanishing Haar measure does not satisfy the bound).

Furthermore, for a wide class of interferometers having one balanced output port $|U_{k,1}| = \frac{1}{\sqrt{M}}$ the difference in probability of no boson counts in the balanced output port satisfies [21]

$$D(p, p^{(K)}) \geq |\Delta P_1| \geq W_1 \left(1 - O\left(\frac{K^2}{N}\right)\right), \quad (4)$$

where the minus sign indicates a negative correction. Such interferometers contain a wide class: $U = F(1 + V)$ with the Fourier interferometer $F_{kl} = \frac{1}{\sqrt{M}} e^{2\pi i kl/M}$ and an arbitrary $(M - 1)$-dimensional unitary interferometer $V$.

If the parameters $\xi, \eta, \nu$ remain fixed when the total number of bosons $N$ scales up, then such boson sampling has finite noise. In this case the correlation between the output distribution of noiseless boson sampling and that of the noisy one tend to zero [9]. Moreover, the total variation distance between the two distributions cannot be small [18]. Finite noise is experimentally relevant, since most of the experimental noise amplitudes remain finite with scaling up the size of a quantum device (except for the transmission $\eta$ [15]). Eqs. (3)-(4) point that with only a polynomial in $N$ number of runs of the quantum device with finite noise, one would have accumulated a data set sufficient to distinguish the output probability distribution from that produced by any classical simulation accounting for only the lower-order multi-boson interferences. Consider the no-collision boson sampling with $M \sim N^2$, i.e. with the density of bosons $\rho \sim 1/N$. In this case the lower bound $W_1$ in Eq. (3) scales as $O(\rho^{K+1}) = O(N^{-K-1})$ in the total number of bosons. To tell apart the two distributions one has to only estimate the probability $P_1$ using $T \gg N^{K+2}$ output datasets, to reduce the statistical error $R \sim \frac{1}{\sqrt{T}}$ in them well below the lower bound in Eq. (3).

In the strong collision regime $M \sim N$ (a finite density of bosons $\rho \sim 1$) – which is the regime of current boson sampling experiments [3, 24, 5] – we get $|\langle \Delta P_1 \rangle| = O(1)$, i.e., the lower bound is independent of the total number of bosons $N$. In this regime of boson sampling the approximation by the lower-order multi-boson interferences would be exposed after a fixed number of runs $T$ of the quantum device with arbitrarily large number of bosons $N$, dependent only on the (finite) noise amplitudes and $K$.
\[ T = T(\xi, \eta, \nu, \rho, K). \] This fact indicates that the strong collision regime is even worst approximated by taking onto account only the lower-order multi-boson interferences.

More generally, numerical simulations show that a similar lower bound as in Eq. (3) can be expected for the probability difference to detect no bosons in \( L < M \) output ports \((|\Omega| = L)\) [21]. Such a probability can serve to rule out the classical simulation based on the lower-order interferences. The probabilities to detect no bosons in \( L < M \) output ports can be used, therefore, as witnesses of boson sampling, they are given by the matrix permanents of some positive-semidefinite Hermitian matrices, built from the interferometer matrix, presented in appendix D. Such matrix permanents can be efficiently estimated by the algorithm of Ref. [23].

3  The set of lower-order interferences is an exponentially small fraction of all multi-boson interferences

Additional insight is provided by the asymptotic estimate on the fraction of the lower-order multi-boson interferences in the presence of a finite-amplitude noise, counting the \( \ell \)th-order interferences with the weight function \( \xi^\ell \). Surprisingly, even for a finite noise, \( \xi^{-1} = O(1) \), the relative contribution from the lower-order interferences is vanishing exponentially fast in the total number of bosons \( N \).

Let us first consider the noiseless case. For the total number \( Z_N^{(K)} \) of permutations in \( S_N \) decomposable into the disjoint cycles of length \( \ell \leq K = O(1) \) we get as \( N \to \infty \) (see details in appendix B)

\[
\frac{\mathcal{F}_N^{(K)}}{Z_N^{(K)}!} = \frac{Z_N^{(K)}}{N!} < \frac{1 + o(1)}{\sqrt{2\pi N}} \exp \left\{ -N \left[ \ln N - 1 \frac{1}{K} - \frac{e - 1}{N^{1/\pi}} - \ln \xi^{-1} \right] \right\}. \tag{5}
\]

In the presence of noise, where noise is due to partial distinguishability of bosons with a uniform overlap \( \xi \), we estimate the ratio of the number \( Z_N^{(K)}(\xi) \) of permutations in \( S_N \) decomposable into the disjoint cycles of lengths \( \ell \leq K \), weighted by \( \xi \) as above, to all weighted permutations \( Z_N(\xi) \) (see details in appendix C):

\[
\frac{Z_N^{(K)}(\xi)}{Z_N(\xi)} < \frac{\mathcal{F}_N^{(K)}}{\xi^N} \frac{1 + o(1)}{\sqrt{2\pi N}} \exp \left\{ -N \left[ \ln N - 1 \frac{1}{K} - \frac{e - 1}{N^{1/\pi}} - \ln \xi^{-1} \right] \right\}. \tag{6}
\]

Therefore, as in the noiseless case (\( \xi = 1 \)), the fraction of permutations with the disjoint cycles of lengths not exceeding \( K \) is exponentially vanishing in \( N \) for \( K = O(1) \) and any finite noise \( \xi^{-1} = O(1) \).

4  Implication for experimental verifications of boson sampling

The above results imply that to validate boson sampling against classical simulations one must go beyond the lower-order correlations (involving only the lower-order interferences). Otherwise an efficient classical approximation could be found for the data sets containing only the (standard in the field) lower-order correlations. And indeed, recently an efficient classical simulation was found [24] for such experimental data of Ref. [5], namely the marginal probabilities, depending only the lower-order interferences (though the Gaussian variant of boson sampling was considered, the main idea and the conclusions are expected to hold). The above discussion, however, predicts that such a classical simulator can be efficiently distinguished from the boson sampling by looking at an output probability of detecting no bosons in a (fixed) subset of output ports, such as in Eq. (3).

5  Conclusion

We have considered the classical approximations of finite-noise boson sampling by the lower-order multi-boson interferences, which includes the only known to date efficient approximations for a finite noise, e.g., by a smaller number interfering bosons padded by classical particles. It is argued that the set of sampling data coming from such approximations can be efficiently distinguished from the boson sampling data they simulate. The output probabilities counting no bosons in fixed subsets of output ports can serve as the efficient distinguishers between the output distribution coming from quantum device
and classical simulations. Surprisingly, for boson sampling with a large number of bosons at input on a smaller-sized interferometer, than is required for the no-collision regime, the number of runs of the quantum device sufficient for distinguishing it from the classical simulations does not depend on the number of bosons, but solely on the amplitudes of noise. The results leave an open problem for the future: Can we classically simulate, both efficiently and faithfully, the dataset from the boson sampling with finite amplitudes of noise, i.e., as in the current experiments on boson sampling?

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A Output probability of boson sampling with partially distinguishable bosons

Consider $N$ single bosons in internal states $|\psi_1\rangle,\ldots,|\psi_N\rangle$ at inputs $k = 1,\ldots,N$ of a unitary interferometer $U_{kl}$ of size $M$. Assuming that $\langle\psi_j|\psi_k\rangle = \xi$, for $j \neq k$, the probability $p(\xi)$ to count $m_1,\ldots,m_M$ bosons in the output ports has the following form (see Ref. [21])

$$p_m(\xi) = \frac{1}{m!} \sum_\sigma \sum_\pi \xi^{N-C_1(\pi^{-1})} \prod_{i=1}^N U_{\sigma(i),l_i} U^*_{\sigma(i),l_i},$$

where $m! = m_1 \ldots m_M!$, $1 \leq l_1 \leq \ldots \leq l_N \leq M$ is the multi-set of output ports corresponding to occupations $m = (m_1,\ldots,m_M)$, $C_1$ is the number of fixed points in permutation, and in the second expression we use the relative permutation $\pi \equiv \sigma^{-1}$. The weight $\xi^{N-C_1(\pi)}$ is due to multi-boson interferences (the cycles of length $\ell \geq 2$).

The expression in Eq. (7) can be understood without any derivation as follows. We have $N$ input bosons (in an order given by permutation $\sigma$) distributed over at most $N$ output ports $1 \leq l_1 \leq \ldots \leq l_N \leq M$ (i.e., some port indices could coincide). The probability is a sum over all possible products of the quantum amplitude, $\prod_{i=1}^N U_{\sigma(i),l_i}$, of the transition $\sigma(i) \rightarrow l_i$, $i = 1,\ldots,N$, and the conjugate amplitude $\prod_{i=1}^N U^*_{\sigma(i),l_i}$ with arbitrary $\pi$-permuted transition (since bosons are identical, our labeling them by $\sigma$ has no physical meaning), weighted by the overlap of the internal states of bosons $\xi^{N-C_1(\pi)}$ and divided by the number $m!$ of identical terms in the sum over the permutations $\sigma$ and $\pi$.

B The fraction of permutations with only lower-order cycles

Let us estimate the fraction of permutations, in the group $S_N$ of permutations of $N$ objects, that have no disjoint cycle of length greater than $K$. To this goal we will use the generating function method for the cycle sum $Z_N$ [25]. We set $Z_0 = 1$ and for $N \geq 1$

$$Z_N(t_1,\ldots,t_N) = \sum_{\sigma} \prod_{k=1}^N t_k^{\xi_k(\sigma)} = \left( \frac{d}{dx} \right)^N \exp \left\{ \sum_{k=1}^{\infty} t_k \frac{x^k}{k} \right\} \bigg|_{x=0},$$

where $(C_1,\ldots,C_N)$ is the cycle type of permutation, with $C_k$ being the number of cycles of length $k$, and $t_k$ being the corresponding control parameter (in the exponent only the terms up to $x^N$ contribute). For example, by setting $t_k = 1$ for all $k$ we get $Z_N(1,\ldots,1) = N!$, i.e., the number of permutations in $S_N$. To count all the permutations with the cycle type $(C_1,\ldots,C_K,0,\ldots,0)$ we must use $t_k = 1$ for $1 \leq k \leq K$ and zero otherwise. The fraction of such permutations can be estimated for $N \gg 1$ by employing the following asymptotic formula [26]

$$F_N^{(K)}(x) \equiv \frac{Z_N(1,\ldots,1,0,\ldots,0)}{N!} = \frac{1}{N!} \left( \frac{d}{dx} \right)^N \exp \left\{ \frac{K^x}{x} \right\} \bigg|_{x=0} = \frac{1 + o(1)}{(N!)^{K/2} (2\pi x)^{K/2}} \sqrt{\frac{\pi}{K}}.$$

with (for $K \geq 2$)

$$R_{N,K} = \sum_{s=1}^{K-1} \left( \frac{K}{s} - 1 \right) \frac{K}{s} \frac{s-1}{s!} \frac{N^{K-s}}{s! (K-s)!} - \frac{1}{K} \sum_{s=2}^{K} \frac{1}{s},$$

(10)
Let us estimate $R_{N,K}$ from above and from below. The first sum in Eq. (10) dominates (since $K = O(1)$ as $N$ scales up). For $1 \leq s \leq K-1$ one can see that $(\frac{s}{K} + 1) \ldots (\frac{s}{K} + s - 1) < s!$, hence

$$R_{N,K} < \sum_{s=1}^{K-1} \frac{(\frac{s}{K} + 1) \ldots (\frac{s}{K} + s - 1)}{s!(K-s)!} N^{\frac{K-s}{K}} < \sum_{s=1}^{K-1} \frac{N^{\frac{K-s}{K}}}{(K-s)!} < (e-1)N^{\frac{K-1}{K}}.$$  

(11)

Using the Stirling approximation [27]

$$N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^N e^{r_N}, \quad r_N > \frac{1}{12N+1},$$

we obtain for the fraction of permutations with cycles of length not exceeding $K$

$$F_{N}^{(K)} < \frac{1 + o(1)}{\sqrt{2\pi N}} \exp \left\{ -N \left[ \ln \frac{N-1}{K} - \frac{e-1}{N^2 \pi} \right] \right\},$$

(12)

For $K = O(1)$, as $N$ scales up the fraction of permutations with the disjoint cycles of lengths not exceeding $K$ is vanishing exponentially in $N$.

Let us also work out explicitly a simple example. Consider the subset of permutations having at least $N - K$ fixed points (i.e., $C_1 \geq N - K$). Since the maximal length of a cycle is equal to $K$, this example gives the subset of permutations having no cycles of length exceeding $K$ (precisely only such permutations are taken into account by the efficient classical approximations in Refs. [13, 17, 18]). The total number of such permutations can be decomposed into disjoint subsets with $N - K + s$ fixed points with $0 \leq s \leq K$:

$$D_K(N) = \sum_{s=0}^{K} \binom{N}{s} d_s,$$

(13)

where $d_s$ is the number of permutations of $s$ objects without fixed points, i.e., the derangements (see Ref. [25]), given as follows

$$d_s = s! \sum_{j=0}^{s} \frac{(-1)^j}{j!}.$$

(14)

Using that $N!/(N-s)! = N^s \left(1 + O\left(\frac{s^2}{N}\right)\right)$, for $K = O(1)$ we get the following estimate on the total number of permutations with at least $N-K$ fixed points

$$D_K(N) = O \left( N^K \right).$$

(15)

As predicted by Eq. (12) the fraction $D_K(N)/N!$ is exponentially vanishing with $N$.

In conclusion, the fraction of permutations in $S_N$ that have no disjoint cycle of length greater than $K = O(1)$ is exponentially vanishing as $N$ scales up.

C The fraction of weighted permutations with only lower-order cycles

Let us first make an observation on the physical significance of the relative permutation $(\pi)$ in the output probability formula in Eq. (7). As distinct from the relative permutation $\pi$, the other permutation $(\sigma)$ gives a spurious order of identical bosons and appears also in the classical limit $\xi \to 0$ of completely distinguishable bosons (or classical particles, classically indistinguishable). For distinguishable bosons (classical particles) $C_1(\pi) = N$ (\pi is trivial permutation) and the probability becomes proportional to the matrix permanent of a positive matrix

$$p_m(0) = \frac{1}{m!} \sigma \prod_{i=1}^{N} |U_{\sigma(i),l_i}|^2.$$

(16)

Hence, only one (of the two) permutations in the expression for the output probability in Eq. (7) is mapped to the the multi-boson interferences, the other one permutes classical particles (see the full theory in Ref. [22]). This simple fact also applies to the probability in the ideal case of completely indistinguishable bosons

$$p(1) = \frac{1}{m!} \left( \sum_{\sigma} \prod_{i=1}^{N} |U_{\sigma(i),l_i}|^2 \right)^2,$$

(17)

where, due to bosons being identical particles (no labels), from the two permutations (in the amplitude and the complex conjugate amplitude) only one could correspond to different multi-boson interferences. The other, quite similarly, can be accounted for by the classical particles (classically indistinguishable).

Cycles of length $\ell$ in the permutation $\pi$ are weighted by $\xi^{\ell}$ in Eq. (7), since each such cycle involves exactly $\ell$ overlaps of different internal states, e.g., cycle $k_1 \to k_2 \to \ldots \to k_\ell \to k_1$ corresponds to the product of the overlaps

$$\prod_{i=1}^{\ell} \langle \psi_{k_{i+1}} | \psi_{k_i} \rangle = \xi^{\ell}, \quad (\ell + 1 \to 1).$$

(18)
For example, a cutoff in the summation $\xi^K$ (as in Refs. [13, 17, 18]) is equivalent to retaining only the subset of permutations $\pi$ with at least $C_1 = N - K$ fixed points. There are $D_K(N) = O(N^K)$ of such permutations $\pi$ for $K = O(1)$, see Eq. (15) of appendix B. This is also a subset of permutations $\pi$ having no cycles of length greater than $K$.

Consider the group of permutations $\pi \in S_N$, where each permutation is weighted by $\xi^{N-C_1(\pi)}$, i.e., each cycle is weighted as in Eq. (18) by the two-boson overlap $\xi$. If we retain only the permutations with cycles of lengths $\ell \leq K$, the discarded permutations exponentially dominate by their number, as shown in appendix B, but with individual contributions weighted down by higher powers of the overlap $\xi$. Our goal is to estimate the relative fraction of weighted permutations $\pi$ having no cycles of length greater than $K$, similar as in Appendix B for the unweighted permutations.

We start with estimating the total sum of the weighted permutations in $S_N$. This can be done using the same generating function method, used in Appendix B. Using the expression for $Z_N$ of Eq. (8) with $t_1 = 1/\xi$ and $t_k = 1$ for all $k \geq 2$, we obtain

$$Z_N(\xi) \equiv \sum_{\pi} \xi^{N-C_1(\pi)} = \xi^N Z_N(1/\xi, 1, \ldots, 1) = \xi^N \left. \left( \frac{d}{dx} \right)^N \frac{x^{1/\xi - 1} x}{1 - x} \right|_{x = 0} = \xi^N N! \sum_{n=0}^{N} \frac{(1/\xi - 1)^n}{n!},$$

where we have used Leibniz’s rule for the $N$-order derivative of a product.

Let us now estimate the contribution $Z_N^{(K)}(\xi)$ to $Z_N(\xi)$ of Eq. (19) coming from the permutations $\pi$ with the disjoint cycles of lengths not exceeding $K$. The simplest bound follows from setting $\xi = 1$ (completely indistinguishable bosons):

$$Z_N^{(K)}(\xi) \leq Z_N^{(K)}(1),$$

where $Z_N^{(K)}(1)$ is the total number of permutations with the disjoint cycles of lengths $\ell \leq K$, considered in appendix B. Hence, utilizing the asymptotic bound on $F_N^{(K)}$ from Eqs. (12) and (20) we obtain (observing that $1/\xi - 1 > 0$)

$$\frac{Z_N^{(K)}(\xi)}{Z_N(\xi)} < \frac{F_N^{(K)}}{\xi^N} < \frac{1 + o(1)}{\sqrt{2\pi N}} \exp \left\{ -N \left[ \frac{\ln N - 1}{K} - e - 1 \right] - \xi^{-1} \right\}. \tag{21}$$

Therefore, as in the noiseless case ($\xi = 1$), the fraction of permutations with the disjoint cycles of lengths not exceeding $K$ is exponentially vanishing in $N$ for $K = O(1)$ and any constant noise $\xi^{-1} = O(1)$.

Thus, for constant noise (parameter $\xi$ bounded from below) the permutations with the lower-order disjoint cycles, i.e., with lengths bounded by $K = O(1)$, even if unweighted by the overlap $\xi$, correspond to an exponentially vanishing fraction of all weighted permutations.

D Output probabilities with contribution from the higher-order interferences

Here we look for the output probabilities which reveal the contribution from higher-order cycles (higher multi-boson interferences). To this goal, consider a probability $P_{\Omega}$ that all the bosons are detected in some subset $\Omega$ of $M$ output ports. Such probability is obtained by summation in Eq. (7) over the output configurations $m$, i.e., the occupations of output ports $l_i \in \Omega$, $i = 1, \ldots, N$, or equivalently, by summation of $m! \sqrt{\pi m}$ over independent output port indices $l_i \in \Omega$, $i = 1, \ldots, N$.

Let us introduce a positive semi-definite Hermitian matrix

$$A_{kj} \equiv \sum_{l \in \Omega} U_{kl} U_{jl}^*. \tag{22}$$

Then, performing the summation as above indicated, we get the probability $P_{\Omega}$ as follows

$$P_{\Omega} = \frac{1}{N!} \sum_{\pi} \xi^{N-C_1(\pi)} \prod_{i=1}^{N} A_{\sigma(i), \pi\sigma(i)} = \sum_{\pi} \xi^{N-C_1(\pi)} \prod_{i=1}^{N} A_{i, \pi(i)}, \tag{23}$$

where now there is only one relative permutation $\pi$, since $\sigma$ has no effect (as seen by reordering the terms in the product; this is a consequence of the different physical meaning of the two permutations, discussed in appendix C). We can simplify the result even more, by observing that the
\( N - C_1(\pi) \) in the \( \xi \)-factor counts the number of off-diagonal elements (i.e., \( \pi(i) \neq i \)) in the matrix \( A \). Hence, by introducing the rescaled matrix as follows

\[
A_{kj} = \begin{cases} 
A_{kk}, & j = k \\
\xi A_{kj}, & j \neq k 
\end{cases}
\]

we obtain the probability in the form of a matrix permanent

\[
P_\Omega = \text{per}_A. \tag{25}
\]

Such probabilities can be approximated by polynomial classical computations \([23]\).

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