On local antimagic total labeling of amalgamation graphs

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Abstract

Let $G = (V, E)$ be a connected simple graph of order $p$ and size $q$. A graph $G$ is called local antimagic (total) if $G$ admits a local antimagic (total) labeling. A bijection $g : E \rightarrow \{1, 2, \ldots, q\}$ is called a local antimagic labeling of $G$ if for any two adjacent vertices $u$ and $v$, we have $g^+(u) \neq g^+(v)$, where $g^+(u) = \sum_{e \in E(u)} g(e)$, and $E(u)$ is the set of edges incident to $u$. Similarly, a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ is called a local antimagic total labeling of $G$ if for any two adjacent vertices $u$ and $v$, we have $w_f(u) \neq w_f(v)$, where $w_f(u) = f(u) + \sum_{e \in E(u)} f(e)$. Thus, any local antimagic (total) labeling induces a proper vertex coloring of $G$ if vertex $v$ is assigned the color $g^+(v)$ (respectively, $w_f(u)$). The local antimagic (total) chromatic number, denoted $\chi_{la}(G)$ (respectively $\chi_{lat}(G)$), is the minimum number of induced colors taken over local antimagic (total) labeling of $G$. In this paper, we determined $\chi_{lat}(G)$ where $G$ is the amalgamation of complete graphs.

Keywords: Local antimagic (total) chromatic number, Amalgamation, Complete graphs

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1 Introduction

Consider a $(p, q)$-graph $G = (V, E)$ of order $p$ and size $q$. In this paper, all graphs are simple. For positive integers $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$. Let $g : E(G) \rightarrow [1, q]$ be a bijective edge labeling that induces a vertex labeling $g^+ : V(G) \rightarrow \mathbb{N}$ such that $g^+(v) = \sum_{uv \in E(G)} g(uv)$. We say $g$ is a local antimagic labeling of $G$ if $g^+(u) \neq g^+(v)$ for each $uv \in E(G)$ \cite{1}. The number of distinct colors induced by $g$ is called the color number of $g$ and is denoted by $c(g)$. The number

$$\chi_{la}(G) = \min\{c(g) \mid g \text{ is a local antimagic labeling of } G\}$$

is called the local antimagic chromatic number of $G$ \cite{1}. Clearly, $\chi_{la}(G) \geq \chi(G)$. Let $f : V(G) \cup E(G) \rightarrow [1, p+q]$ be a bijective total labeling that induces a vertex labeling $w_f : V(G) \rightarrow \mathbb{N}$, where

$$w_f(u) = f(u) + \sum_{uv \in E(G)} f(uv)$$

and is called the weight of $u$ for each vertex $u \in V(G)$. We say $f$ is a local antimagic total labeling of $G$ (and $G$ is local antimagic total) if $w_f(u) \neq w_f(v)$ for each $uv \in E(G)$. Clearly, $w_f$ corresponds to a proper vertex coloring of $G$ if each vertex $v$ is assigned the color $w_f(v)$. If no ambiguity, we shall drop the subscript $f$. Let $w(f)$ be the number of distinct vertex weights induced by $f$. The number

$$\min\{w(f) \mid f \text{ is a local antimagic total labeling of } G\}$$

is called the local antimagic total chromatic number of $G$, denoted $\chi_{lat}(G)$. Clearly, $\chi_{lat}(G) \geq \chi(G)$. It is well known that determining the chromatic number of a graph $G$ is NP-hard \cite{13}. Thus, in general, it is also very difficult to determine $\chi_{la}(G)$ and $\chi_{lat}(G)$.

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For a graph $G$, the graph $H = G \lor K_1$ is obtained from $G$ by joining a new vertex to every vertex of $G$. We refer to [3] for notation not defined in this paper. In [3], the authors proved that every graph is local antimagic. In [8], the authors proved that every graph is local antimagic total. We shall need the following theorem in [8].

**Theorem 1.1:** Let $G$ be a graph of order $p \geq 2$ and size $q$ with $V(G) = \{v_i | 1 \leq i \leq p\}$.

(a) $\chi(G) \leq \chi_{lat}(G) \leq \chi_{la}(G \lor K_1) - 1$.

(b) Suppose $\chi_{lat}(G) = \chi(G \lor K_1) - 1$ with a corresponding local antimagic total labeling $f$ of $G$. If $\sum_{i=1}^{p} f(v_i) \neq w_f(v_j)$, $1 \leq j \leq p$, then $\chi_{la}(G \lor K_1) = \chi(G \lor K_1)$.

For $m \geq 2$ and $1 \leq i \leq m$, let $G_i$ be a simple graph with an induced subgraph $H$. An **amalgamation** of $G_1, \ldots, G_m$ over $H$ is the simple graph obtained by identifying the vertices of $H$ of each $G_i$ so that the new obtained graph contains a subgraph $H$ induced by the identified vertices. Suppose $G$ is a graph with a proper subgraph $K_r$, $r \geq 1$. Let $A(mG, K_r)$ be the amalgamation of $m \geq 2$ copies of $G$ over $K_r$. Note that there may be many non-isomorphic $A(mG, K_r)$ graphs. For example, $A(2P_3, K_2)$ may be either $K_{1,3}$ or $P_3$. When $r = 1$, the graph is also known as one-point union of graphs. Note that $A(mK_2, K_1) \cong K_{1,m}$ and $A(mK_3, K_1)$ is the friendship graph $f_m$, $m \geq 2$. In [9] Theorem 2.4, the authors completely determined $\chi_{la}(A(mC_n, K_1))$ for $m \geq 2$, $n \geq 3$ where $2 \leq \chi(A(mC_n, K_1)) \leq 3$. Motivated by this, in this paper, we determine the $\chi_{lat}(A(mK_n, K_r))$ and $\chi_{la}(A(mK_n, K_r) \lor K_1)$ where $\chi(A(mK_n, K_r)) = n$ for $m \geq 1$, $n \geq 2, r \geq 0$. Sharp upper bounds on $\chi_{lat}(mK_n)$ are also obtained for odd $n \geq 3$.

### 2 Amalgamations of Complete Graphs

Let $f$ be a total labeling of a simple $(p, q)$-graph $G$. Let $V(G) = \{u_1, \ldots, u_p\}$. We define a total labeling matrix which is similar to the labeling matrix of an edge labeling introduced in [13].

Suppose $f : V(G) \cup E(G) \to S$ is a mapping, where $S$ is a set of labels. A **total labeling matrix** $M$ of $f$ for $G$ is a $p \times p$ symmetric matrix in which the $(i, i)$-entry of $M$ is $f(u_i)$; the $(i, j)$-entry of $M$ is $f(u_i u_j)$ if $u_i u_j \in E$ and is * otherwise. If $f$ is a local antimagic total labeling of $G$, then a total labeling matrix of $f$ is called a **local antimagic total labeling matrix** of $G$. Clearly the $i$-th row sum (and $i$-th column sum) is $w_f(u_i)$, where *'s are treated as zero. Thus the condition of a total labeling matrix being a local antimagic total labeling matrix is the $i$-th row sum different from the $j$-th row sum when $u_i u_j \in E$.

For $m \geq 2$ and $n \geq r \geq 1$, let $V(A(mK_n, K_r)) = \{v_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(A(mK_n, K_r)) = \{(v_{i,j}, v_{i,k} | 1 \leq i \leq m, 1 \leq j < k \leq n\}$, where $v_{i,j} = \cdots = v_{m,j}$ for each $n - r + 1 \leq j \leq n$. For convenience, let $u_j = v_{n,j}$ for $n - r + 1 \leq j \leq n$. Note that $A(mK_n, K_r) \equiv mK_{n-r} \lor K_r$. We first list the vertices of the $m$ copies of $K_{n-r}$ in lexicographic order followed by $u_{n-r+1}, \ldots, u_n$. Now let us show the structure of a total labeling matrix $M$ of the graph $A(mK_n, K_r)$ under this list of vertices as a block matrix. Namely,

$$
M = \begin{pmatrix}
L_1 & \star & \star & \cdots & \star & B_1 \\
\star & L_2 & \star & \cdots & \star & B_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\star & \cdots & \star & L_i & \star & B_i \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\star & \star & \cdots & \star & L_m & B_m \\
B_1^T & B_2^T & \cdots & B_i^T & \cdots & B_m^T & A
\end{pmatrix},
$$

where $L_i$ is an $(n-r) \times (n-r)$ symmetric matrix, $B_i$ is an $(n-r) \times r$ matrix, $1 \leq i \leq m$, and $A$ is an $r \times r$ symmetric matrix. Here $\star$ denotes an $(n-r) \times (n-r)$ matrix whose entries are *'s. Thus, the
corresponding total labeling matrix of the $i$-th $K_n$ is
\[ M_i = \begin{pmatrix} L_i & B_i \\ B_i^T & A \end{pmatrix}. \tag{2.2} \]

Now, a local antimagic total labeling for the graph $A(mK_n, K_r)$ is obtained if we use the integers in $[1, N]$ for all entries of the upper triangular part of all $L_i$’s and $A$, and all entries of all $B_i$’s such that the row sums of each matrix $M_i$ are distinct, where $N = \frac{m(n+1)}{2} - \frac{(m-1)r(r+1)}{2}$. We may extend the case to $r = 0$. We let $A(mK_n, K_0) = mK_n$ by convention. For this case, all of $B_i$’s and $A$ in (2.1) and (2.2) do not exist.

For a given matrix $B$, we shall use $R_i(B)$ and $C_j(B)$ to denote the $i$-th row sum and the $j$-th column sum of $B$, respectively. Also we shall use $(B)$ to denote the sum of the main diagonal of $B$ if $B$ is a square matrix. Suppose $S$ is a finite subset of $\mathbb{Z}$. Let $S^−$ and $S^+$ be a decreasing sequence and an increasing sequence of $S$, respectively.

We shall keep the notation defined above in this section.

**Lemma 2.1:** Suppose $m \leq n$. Let $M = (m_{i,j})$ be an $m \times n$ matrix with the following properties:

(a) $m_{k,j} = m_{j,k}$ for all $j, k$, where $1 \leq j < k \leq m$;

(b) $m_{j,j} < m_{k,k}$ if $1 \leq j < k \leq m$;

(c) for $j_1 < k_1$ and $j_2 < k_2$, $(j_1, k_1) < (j_2, k_2)$ in lexicographic order implies that $m_{j_1, k_1} < m_{j_2, k_2}$.

Then $R_j(M)$ is a strictly increasing function of $j$.

**Proof.**
\[
R_{j+1}(M) - R_j(M) = \sum_{k=1}^{n} (m_{j+1,k} - m_{j,k}) \\
= \sum_{k=1}^{j} (m_{j+1,k} - m_{j,k}) + (m_{j+1,j} - m_{j,j} + m_{j+1,j+1} - m_{j,j+1}) + \sum_{k=j+2}^{n} (m_{j+1,k} - m_{j,k}) \\
= \sum_{k=1}^{j} (m_{k,j+1} - m_{k,j}) + (m_{j+1,j+1} - m_{j,j}) + \sum_{k=j+2}^{n} (m_{j+1,k} - m_{j,k}) > 0.
\]

Note that the empty sum is treated as $0$. This completes the proof. \[ \square \]

**Lemma 2.2:** For positive integers $t$ and $m$, let $S(a) = [m(a-1) + 1, ma]$, $1 \leq a \leq t$. We have:

(i) $\{S(a) \mid 1 \leq a \leq t\}$ is a partition of $[1, nt]$.

(ii) If $a < b$, then every term of $S(a)$ is less than every term of $S(b)$.

(iii) For any $1 \leq a, b \leq t$, the sum of the $i$-th term of $S^+(a)$ and that of $S^−(b)$ is independent of the choice of $i$, $1 \leq i \leq m$.

(iv) For any $1 \leq a_i, b_i \leq t$, $\sum_{i=1}^{k} (i$-th term of $S^+(a_i)) + \sum_{i=1}^{k} (i$-th term of $S^−(b_i))$ is independent of the choice of $i$, $1 \leq i \leq m$.

**Proof.** The first two parts are obvious. For (iii), the $i$-th terms of $S^+(a)$ and $S^−(b)$ are $m(a-1) + i$ and $m(b-1) + (m+1-i)$, respectively. So the sum is $m(a+b-1) + 1$ which is independent of $i$. The last part follows from (iii). \[ \square \]
Before providing results about $\chi_{lat}(A(mK_n, K_r))$ for some $m, n, r$, we define a ‘sign matrix’ $S_n$ for even $n$.

Let $S_2 = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$ be a $2 \times 2$ matrix and $S_4 = \begin{pmatrix} S_2 & S_2 \\ S_2 & -S_2 \end{pmatrix}$ be a $4 \times 4$ matrix. Let $S_{4k}$ be a $(4k) \times (4k)$ matrix given by the following block matrix, where $k \geq 2$:

$$S_{4k} = \begin{pmatrix} S_4 & \cdots & S_4 \\ \vdots & \ddots & \vdots \\ S_4 & \cdots & S_4 \end{pmatrix}. $$

Let $S_{4k+2}$ be a $(4k + 2) \times (4k + 2)$ matrix as the following block matrix, where $k \geq 1$:

$$S_{4k+2} = \begin{pmatrix} S_{4k} & S_2 \\ \vdots & \vdots \\ S_2 & S_2 \end{pmatrix}. $$

We shall keep these notation in this section.

**Remark 2.1**: It is easy to see that each row and column sum of $S_n$ are zero. Moreover, the diagonal sum of $S_{4k}$ is zero.

**Theorem 2.3**: For $m \geq 2$, $n$ even and $n > r \geq 0$, $\chi_{lat}(A(mK_n, K_r)) = n$.

**Proof.** Let $S$ be the $(n - r) \times n$ matrix obtained from $S_n$ by removing the last $r$ rows of $S_n$. First, define an $(n - r) \times n$ matrix $M'$ in which $(M')_{j,k} = (M)_{i,j}$ for $1 \leq j < k \leq n - r$. Assign the increasing sequence $[1, (n - r)(n + r - 1)/2]$ in lexicographic order to the upper part of the off-diagonal entries of $M'$, denoted $(j,k)$ if in row $j$ and column $k$ for $1 \leq j < k \leq n$. Next, assign $[(n - r)(n + r - 1)/2 + 1, (n - r)(n + r - 1)/2 + (n - r)]$ to the entries of the main diagonal of $M'$ in natural order.

Now, define an $(n - r) \times n$ ‘guide matrix’ $M$ whose $(j,k)$-th entry is $(S)_{j,k}(M')_{j,k}$, $1 \leq j \leq n - r$ and $1 \leq k \leq n$.

**Stage 1**: We shall assign labels to the upper triangular entries of $L_i$’s and all the entries of $B_i$’s. Note that if $r = 0$, all of $B_i$’s and $A$ do not exist. There are $N_1 = (n - r)(n + r - 1)/2 + (n - r) = \frac{(n + r + 1)(n - r)}{2}$ entries needed to be filled for each $i$.

Now we shall use labels in $[1, mN_1]$ to fill in the $m$ submatrices $(L_i, B_i)$, $1 \leq i \leq m$. We use the sequences $S(a)$ defined in Lemma 2.2 where $t = N_1$. The $(j,k)$-entry of $M_i$ is the $i$-th term of $S^+(a)$ or $S^-(a)$ if the corresponding $(j,k)$-entry of $M$ is $+a$ or $-a$ respectively, where $1 \leq j \leq n - r$.

By Lemma 2.2 (iv), $R_j(M_i)$ are the same for all $i$, $1 \leq i \leq m$. In other words, $w(v_{i,j})$ is a constant function for a fixed $j$, $1 \leq j \leq n - r$.

**Stage 2**: Note that when $r = 0$, the total labeling matrix $M$ does not have the last row and column of block matrices so that we only need to perform Stage 1 above. Thus, we now assume $r \neq 0$. Also note that all integers in $[1, mN_1]$ are used up in Stage 1. Use the increasing sequence $[mN_1 + 1, mN_1 + r(r - 1)/2]$ in lexicographic order for the off-diagonal entries of $A$. Lastly, use $[mN_1 + r(r - 1)/2 + 1, N]$ in natural order for the diagonals of $A$. The lower triangular part duplicates the upper triangular part.

Consider the matrix $M_m$. Clearly it satisfies the conditions of Lemma 2.1. Hence $R_j(M_m)$ is a strictly increasing function of $j$, $1 \leq j \leq n$. Thus $w(v_{i,j}) = w(v_{m,j})$ is a strictly increasing function of $j$, $1 \leq j \leq n - r$, for $1 \leq i \leq m$. 


By the structure of $B_i$ and $A$, and by Lemma \ref{lem:local}(i), we have

$$w(u_{n-r+j}) = R_j(A) + \sum_{i=1}^{m} R_j(B_i^T) = R_j(A) + \sum_{i=1}^{m} C_j(B_i)$$

$$< R_j(A) + \sum_{i=1}^{m} C_j(B_i) = w(u_{n-r+j+1}),$$

for $1 \leq j \leq r - 1$.

Thus $\chi_{lat}(A(mK_n, K_r)) = n$ since $\chi(A(mK_n, K_r)) = n$. \hfill \qed

\textbf{Remark 2.2:} Suppose $f$ is a local antimagic total labeling of a graph $G$ and $M$ is the corresponding total labeling matrix. From the proof of Theorem \ref{thm:local}(b) we can see that, if $(M)$ does not equal to every row (also column) sum of $M$, then $f$ induces a local antimagic labeling of $G \lor K_1$. Moreover, $\chi(G \lor K_1) \leq \chi_{lat}(G) \leq \chi_{lat}(G) + 1$.

\textbf{Corollary 2.4:} For $m \geq 2$, $n$ even and $n > r \geq 0$, $\chi_{la}(A(mK_{n+1}, K_{r+1})) = \chi_{la}(A(mK_n, K_r) \lor K_1) = n + 1$.

\textbf{Proof.} Note that $A(mK_{n+1}, K_{r+1}) \cong A(mK_n, K_r) \lor K_1$. Since $\chi_{la}(A(mK_n, K_r) \lor K_1) \geq \chi(A(mK_n, K_r) \lor K_1) = n + 1$, we only need to show $\chi_{la}(A(mK_n, K_r) \lor K_1) \leq n + 1$. Keep the total labeling matrix $M$ of $A(mK_n, K_r)$ in the proof of Theorem \ref{thm:local}.

Since each diagonal of $M$ is the largest entry in the corresponding column, $(M)$ is larger than each row sum of $M$. Thus, $(M)$ is greater than all vertex weights of $A(mK_n, K_r)$. By Remark \ref{rem:local} we get that $\chi_{la}(A(mK_n, K_r) \lor K_1) \leq n + 1$. \hfill \qed

\textbf{Example 2.1:} We take $m = 3$, $n = 6$ and $r = 0$.

The guide matrix is

$$M = \begin{pmatrix}
+16 & -1 & +2 & -3 & +4 & -5 \\
-1 & +17 & -6 & +7 & -8 & +9 \\
+2 & -6 & -18 & +10 & +11 & -12 \\
-3 & +7 & +10 & -19 & -13 & +14 \\
+4 & -8 & +11 & -13 & +20 & -15 \\
-5 & +9 & -12 & +14 & -15 & +21
\end{pmatrix}$$

The above matrices give $\chi_{lat}(3K_6) = 6$. Let $v$ be the vertex of $K_1$. If the main diagonal labels are the edge labels of $3K_6 \lor K_1$ incident with $v$, then the induced label of $v$ is 981. Thus, the matrices give $\chi_{la}(3K_6 \lor K_1) = 7$.

Deleting edge $v_{2,6}$ of label 63 from $3K_6 \lor K_1$, we get a local antimagic labeling of $(3K_6 \lor K_1) - v_{2,6}$. Thus, by symmetry, $\chi_{la}(3K_6 \lor K_1 - e) = 7$ for $e$ not belonging to any $K_6$.

Delete the edge $v_{3,1}, v_{3,2}$ that has label 1 and reduce all other labels by 1. We get $\chi_{lat}(3K_6 - e) = 6$ by symmetry for $e$ that belongs to any $K_6$. Now, if the main diagonal labels are the edge labels of $(3K_6 - e) \lor K_1$ incident with $v$, then we have $\chi_{la}(3K_6 \lor K_1 - e) = 7$ for $e$ that belongs to any $K_6$. \hfill \qed

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
& $v_{1,1}$ & $v_{1,2}$ & $v_{1,3}$ & $v_{1,4}$ & $v_{1,5}$ & $v_{1,6}$ & sum \\
\hline
$v_{1,1}$ & 46 & 3 & 4 & 9 & 10 & 13 & 87 \\
$v_{1,2}$ & 3 & 49 & 18 & 49 & 25 & 13 & 138 \\
$v_{1,3}$ & 4 & 15 & 54 & 25 & 30 & 19 & 171 \\
$v_{1,4}$ & 9 & 19 & 28 & 57 & 39 & 40 & 192 \\
$v_{1,5}$ & 10 & 24 & 31 & 39 & 48 & 207 & 220 \\
$v_{1,6}$ & 16 & 25 & 36 & 40 & 45 & 61 & 222 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
& $v_{2,1}$ & $v_{2,2}$ & $v_{2,3}$ & $v_{2,4}$ & $v_{2,5}$ & $v_{2,6}$ & sum \\
\hline
$v_{2,1}$ & 47 & 2 & 5 & 8 & 11 & 14 & 87 \\
$v_{2,2}$ & 2 & 50 & 17 & 20 & 23 & 26 & 138 \\
$v_{2,3}$ & 5 & 17 & 53 & 29 & 32 & 35 & 171 \\
$v_{2,4}$ & 8 & 20 & 29 & 56 & 38 & 41 & 192 \\
$v_{2,5}$ & 11 & 23 & 32 & 38 & 59 & 44 & 207 \\
$v_{2,6}$ & 14 & 26 & 38 & 41 & 44 & 62 & 222 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
& $v_{3,1}$ & $v_{3,2}$ & $v_{3,3}$ & $v_{3,4}$ & $v_{3,5}$ & $v_{3,6}$ & sum \\
\hline
$v_{3,1}$ & 48 & 1 & 6 & 9 & 13 & 87 \\
$v_{3,2}$ & 1 & 51 & 16 & 21 & 22 & 138 \\
$v_{3,3}$ & 6 & 16 & 52 & 30 & 33 & 171 \\
$v_{3,4}$ & 7 & 21 & 30 & 55 & 37 & 102 \\
$v_{3,5}$ & 12 & 22 & 33 & 31 & 60 & 207 \\
$v_{3,6}$ & 13 & 27 & 34 & 42 & 43 & 222 \\
\hline
\end{tabular}
\end{table}
Example 2.2: We take $m = 3$, $n = 6$ and $r = 1$. The guide matrix is obtained from the guide matrix of Example 2.1 by deleting the last row. So we have

$$M_1 = \begin{pmatrix} 46 & 3 & 4 & 9 & 10 & 15 & 87 \\ 3 & 49 & 18 & 19 & 24 & 25 & 138 \\ 4 & 18 & 54 & 28 & 31 & 36 & 171 \\ 9 & 19 & 28 & 57 & 39 & 40 & 192 \\ 10 & 24 & 31 & 39 & 58 & 45 & 207 \\ 15 & 25 & 36 & 40 & 45 & 61 & 222 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 47 & 2 & 5 & 8 & 11 & 14 & 87 \\ 2 & 50 & 17 & 20 & 23 & 26 & 138 \\ 5 & 17 & 53 & 29 & 32 & 35 & 171 \\ 8 & 20 & 29 & 56 & 38 & 41 & 192 \\ 11 & 23 & 32 & 38 & 59 & 44 & 207 \\ 14 & 26 & 38 & 41 & 44 & 61 & 221 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 48 & 1 & 6 & 7 & 12 & 13 & 87 \\ 1 & 51 & 16 & 21 & 22 & 27 & 138 \\ 6 & 16 & 52 & 30 & 33 & 34 & 171 \\ 7 & 21 & 30 & 55 & 37 & 42 & 192 \\ 12 & 22 & 33 & 37 & 60 & 43 & 207 \\ 13 & 27 & 34 & 42 & 43 & 61 & 220 \end{pmatrix}.$$ 

The last column of each matrix is the corresponding row sum. Now $w(u_6) = 222 + 221 + 220 - 2 \times 61 = 541$. Hence $\chi_{lat}(A(3K_6, K_1)) = 6$. Since $\chi(M) = 856$, $\chi_{la}(A(3K_6, K_1) \lor K_1) = 7$. ■

Corollary 2.5: For $m \geq 2$, $\chi_{la}(mK_{4k+1}) = 4k + 1$.

Proof. Consider the total labeling matrix of $mK_{4k}$ defined in the proof of Theorem 2.3. For each matrix $M_i = L_i$, $1 \leq i \leq m$, we add the $(n+1)$-st extra column at the right of $M_i$ with entry $\ast$. For each row of this matrix, swap the diagonal entry with the entry of the $(n+1)$-st column. Add the $(n+1)$-st extra row to this matrix and let the $(n+1, n+1)$-entry be $\ast$ and then make the resulting matrix $Q_i$ to be symmetric. Then $Q_i$ is a labeling matrix of the $i$-th copy of $K_{4k+1}$.

By Remark 2.1 and Lemma 2.2 (iv), all the diagonal sums of $M_i$’s are the same, $1 \leq i \leq n$. Thus the $j$-th row sum of $Q_i$ is independent of $i$, $1 \leq j \leq n+1$. Hence we have $\chi_{la}(mK_{4k+1}) \leq 4k + 1$. Since $\chi(mK_{4k+1}) = 4k + 1$, $\chi_{la}(mK_{4k+1}) = 4k + 1$. ■

Example 2.3: We take $m = 3$, $n = 4$. So

$$\mathcal{M} = \begin{pmatrix} +7 & -1 & +2 & -3 \\ -1 & +8 & -4 & +5 \\ +2 & -4 & -9 & +6 \\ -3 & +5 & +6 & -10 \end{pmatrix},$$

and

$$M_1 = \begin{pmatrix} 19 & 3 & 4 & 9 & 35 \\ 3 & 22 & 12 & 13 & 50 \\ 4 & 12 & 27 & 16 & 59 \\ 9 & 13 & 16 & 30 & 68 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 20 & 2 & 5 & 8 & 35 \\ 2 & 23 & 11 & 14 & 50 \\ 5 & 11 & 26 & 17 & 59 \\ 8 & 14 & 17 & 29 & 68 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 21 & 1 & 6 & 7 & 35 \\ 1 & 24 & 10 & 15 & 50 \\ 6 & 10 & 25 & 18 & 59 \\ 7 & 15 & 18 & 28 & 68 \end{pmatrix}.$$ 

Thus $\chi_{lat}(3K_4) = 4$. 

6
Let

\[
Q_1 = \begin{pmatrix}
* & 3 & 4 & 9 & 19 & 35 \\
3 & * & 12 & 13 & 22 & 50 \\
4 & 12 & * & 16 & 27 & 59 \\
9 & 13 & 16 & * & 30 & 68 \\
19 & 22 & 27 & 30 & * & 98 \\
\end{pmatrix},
Q_2 = \begin{pmatrix}
* & 2 & 5 & 8 & 20 & 35 \\
2 & * & 11 & 14 & 23 & 50 \\
5 & 11 & * & 17 & 26 & 59 \\
8 & 14 & 17 & * & 29 & 68 \\
20 & 23 & 26 & 29 & * & 98 \\
\end{pmatrix},
\]

\[
Q_3 = \begin{pmatrix}
* & 1 & 6 & 7 & 21 & 35 \\
1 & * & 10 & 15 & 24 & 50 \\
6 & 10 & * & 18 & 25 & 59 \\
7 & 15 & 18 & * & 28 & 68 \\
21 & 24 & 25 & 28 & * & 98 \\
\end{pmatrix}.
\]

Thus \(\chi_{lat}(3K_5) = 5\). \(\blacksquare\)

**Theorem 2.6:** For \(m \geq 2\), odd \(n \geq 3\) and \(n > r \geq 3\), \(\chi_{lat}(A(mK_n, K_r)) = n\).

**Proof.** Suppose \(r\) is odd so that \(n - r\) is even.

**Stage 1:** Using the same approach of the proof of Theorem 2.3 we construct an \((n-r) \times (n-r)\) guide matrix \(M\).

Similar to the proof of Theorem 2.3, we use the guide matrix \(M\) for all entries of \(L_i, 1 \leq i \leq m\), using labels in \([1, mN_2]\), where \(N_2 = (n-r)(n-r+1)/2\). Thus, \(R_j(L_i)\) is a function only depending on \(j\) and is strictly increasing, for \(1 \leq i \leq m\) and \(1 \leq j \leq n-r\).

**Stage 2:** Use \([mN_2 + 1, mN_2 + (m(n-r)+1)r]\) to form an \((m(n-r)+1) \times r\) magic rectangle \(\Omega\). Note that the existence of this magic rectangle is referred to in [4]. Let

\[
\begin{pmatrix}
B_1 \\
B_2 \\
\vdots \\
B_m \\
\alpha
\end{pmatrix} = \Omega,
\]

where \(\alpha = (A_{1,1}, A_{2,2}, \ldots, A_{r,r})\). Now, for a fixed \(i\), \(w(v_{i,j}) = R_j(L_i) + R_j(B_i)\). So \(w(v_{i,j})\) is a function only depending on \(j\) and is strictly increasing, for \(1 \leq i \leq m\) and \(1 \leq j \leq n-r\).

**Stage 3:** Use the increasing sequence \([mN_2 + (m(n-r)+1)r + 1, N]\) in lexicographic order for the remaining entries of the upper triangular part of \(A\). For \(1 \leq j \leq r\),

\[
w(u_{n-r+j}) = \sum_{i=1}^{m} R_j(B_i^T) + R_j(A) = \sum_{i=1}^{m} C_j(B_i) + A_{j,j} + \sum_{l=1}^{r} A_{j,l}
\]

\[
= C_j(\Omega) + \sum_{l=1}^{j-1} A_{j,l} + A_{j,j+1} + \sum_{l=j+2}^{r} A_{j,l}
\]

\[
< C_j(\Omega) + \sum_{l=1}^{j-1} A_{j+1,l} + A_{j+1,j} + \sum_{l=j+2}^{r} A_{j+1,l}
\]

(since \(r \geq 3\), there is at least one non-empty sum)

\[
= C_{j+1}(\Omega) + \sum_{l=1}^{r} A_{j+1,l} = w(u_{n-r+j+1}).
\]

So \(w(u_{n-r+j})\) is a strictly increasing function of \(j\) for \(1 \leq j \leq r\).
Thus, we have \( \chi \) 

\[ \chi = \mathcal{R}_n \left( L_m \right) + \mathcal{R}_n \left( B_m \right) = C_n \left( L_m \right) + \mathcal{R}_1 \left( \alpha \right) \quad \text{(since } \Omega \text{ is a magic rectangle)} \]

\[ \mathcal{R}_1 \left( \beta \right) < \mathcal{R}_1 \left( \alpha \right) < \mathcal{R}_1 \left( \beta^T \right) + \sum_{k=1}^{n} A_{1,k} < w(u_{n-r+1}). \]

Thus, we have \( \chi \) 

Suppose \( r \) is even so that \( n-r+1 \) is even.

(a) Suppose \( m \) is odd so that \( m(n-r) \) and \( r-1 \) are odd and at least 3.

**Stage 1(a):** We use a modification of the proof of Theorem 2.3. Firstly we use the \((n-r) \times (n-r+1)\) sign matrix \( S \). Next, we define an \((n-r) \times (n-r+1)\) matrix \( M \) by assigning the increasing sequence \([1, (n-r)(n-r+1)/2]\) in lexicographic order to the off-diagonal entries of the upper triangular part of \( M \). Now, assign \([(n-r)(n-r+1)/2 - (n-r-1), (n-r)(n-r+1)/2]\) to the entries of the main diagonal of \( M \) in natural order. The \((n-r) \times (n-r+1)\) guide matrix \( M \) is defined the same way as in the proof of Theorem 2.3.

Write \( B_i = (\beta_i \ X_i) \), where \( \beta_i \) and \( X_i \) are \((n-r) \times 1\) and \((n-r) \times (r-1)\) matrices, respectively. Similar to Stage 1 of the odd case, for \((L_i \ \beta_i)\) use the labels in \([1, mZ_1] \cup [mZ_2 - (n-r)] + 1, mZ_2]\), \(1 \leq i \leq m\), where \( Z_1 = (n-r)(n-r+1)/2\) and \( Z_2 = Z_1 + (n-r)\).

Thus, \( \mathcal{R}_j \left( L_i \ \beta_i \right) \) is a function only depending on \( j \) and is a strictly increasing function of \( j \), \( 1 \leq i \leq m \) and \( 1 \leq j \leq n-r \).

**Stage 2(a):** Use \([mZ_1 + 1, mZ_1 + (n-r)(r-1)]\) to form an \((n-r) \times (r-1)\) magic rectangle \( \Omega \). Let

\[ \left( \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \right) = \Omega. \]

Now, for a fixed \( i \), \( w(v_{i,j}) = \mathcal{R}_j \left( L_i \ \beta_i \right) + \mathcal{R}_j \left( X_i \right) \). So \( w(v_{i,j}) \) is a function only depending on \( j \) and is a strictly increasing function of \( j \) for \( 1 \leq j \leq n-r \) and \( 1 \leq i \leq m \).

**Stage 3(a):** Use the increasing sequence \([mZ_2 + 1, mZ_2 + (r-1)r/2]\) in lexicographic order for the off-diagonal entries of the upper triangular part of \( A \) and then use \([mZ_2 + (r-1)r/2 + 1, N]\) in natural order for the diagonals of \( A \). By Lemma 2.3, \( \mathcal{R}_j (A) \) is a strictly increasing function of \( j \) for \( 1 \leq j \leq r \).

Now, for \( 2 \leq j \leq r \),

\[ w(u_{n-r+j}) = \sum_{i=1}^{m} \sum_{k=1}^{n-r} (B_i^T)_{j,k} + \sum_{i=1}^{m} \sum_{k=1}^{n-r} A_{j,t} = \sum_{i=1}^{m} \sum_{k=1}^{n-r} (B_i)_{k,j} + \sum_{l=1}^{r} A_{j,t} = \mathcal{C}_{j-1} (\Omega) + \mathcal{R}_j (A). \]

So \( w(u_{n-r+j}) \) is a strictly increasing function of \( j \) for \( 2 \leq j \leq r \).

Next

\[ w(u_{n-r+1}) = \sum_{i=1}^{m} \sum_{k=1}^{n-r} (B_i^T)_{1,k} + \sum_{i=1}^{m} \sum_{k=1}^{n-r} A_{1,t} = \sum_{i=1}^{m} \sum_{k=1}^{n-r} (B_i)_{k,1} + \mathcal{R}_1 (A) \]

\[ < \sum_{i=1}^{m} \sum_{k=1}^{n-r} (B_i)_{k,2} + \mathcal{R}_2 (A) = w(u_{n-r+2}). \]
Now
\[
w(v_{m,n-r}) = R_{n-r}(L_m) + R_{n-r}(B_m) = C_{n-r}(L_m) + R_{n-r}(B_m)
\]
\[
< C_1(B_m) + \sum_{k=1}^{r} A_{1,k} = R_1(B_m^T) + \sum_{k=1}^{r} A_{1,k} < w(u_{n-r+1}).
\]

Thus we have \(\chi_{lat}(A(mK_n, K_r)) = n\).

(b) Suppose \(m\) is even so that \(m(n-r)\) is even.

**Stage 1(b):** Similar to Stage 1 of the odd \(r\) case we define an \((n-r) \times (n-r+1)\) guide matrix \(M\).

Write \(B_i = (\beta_i \ X_i)\), where \(\beta_i\) and \(X_i\) are \((n-r) \times 1\) and \((n-r) \times (r-1)\) matrices, respectively. Similar to Stage 1 of the odd \(r\) case, for \((L_i \ \beta_i)\) use labels in \([1, mN_3]\), \(1 \leq i \leq m\), where \(N_3 = (n-r+3)(n-r)/2\). Thus, \(R_j(L_i \ \beta_i)\) is a function only depending on \(j\) and is a strictly increasing function of \(j\), \(1 \leq i \leq m\) and \(1 \leq j \leq n-r\).

**Stage 2(b):** \([mN_3 + 2, mN_3 + (m(n-r) + 1)(r-1) + 1]\) to form an \((m(n-r) + 1) \times (r-1)\) magic rectangle \(\Omega\). We will assign \(mN_3 + 1\) to \(A_{1,1}\) in the next stage.

Let
\[
\left(\begin{array}{c}
X_1 \\
X_2 \\
\vdots \\
X_m \\
\alpha
\end{array}\right) = \Omega,
\]

where \(\alpha\) is a \(1 \times (r-1)\) matrix.

Same as Stage 2(a), we have \(w(v_{i,j})\) is a function only depending on \(j\) and is a strictly increasing function for \(1 \leq j \leq n-r\) and \(1 \leq i \leq m\).

**Stage 3(b):** Let \(A_{1,1} = mN_3 + 1\). Use the increasing sequence \([mN_3 + (m(n-r) + 1)(r-1) + 2, N]\) in lexicographic order for the remaining entries of the upper triangular part of \(A\).

By the same proof of Stage 3 of the odd \(r\) case, we have \(w(u_{n-r+j})\) is a strictly increasing function of \(j\), \(2 \leq j \leq r\). By a similar proof of Stage 3(a) we have \(w(u_{n-r+1}) < w(u_{n-r+2})\).

Now
\[
w(v_{m,n-r}) = R_{n-r}(L_m) + R_{n-r}(B_m) = \sum_{k=1}^{n-r} (L_m)_{n-r,k} + (\beta_m)_{n-r,1} + \sum_{k=1}^{r-1} (X_m)_{n-r,k}
\]
\[
= \sum_{k=1}^{n-r-1} (L_m)_{k,n-r} + (L_m)_{n-r,n-r} + (\beta_m)_{n-r,1} + \sum_{k=1}^{r-1} (X_m)_{n-r,k}
\]
\[
< \sum_{k=1}^{n-r-1} (\beta_m)_{k,1} + A_{1,1} + (\beta_m)_{n-r,1} + \sum_{k=2}^{r} A_{1,k} < w(u_{n-r+1}).
\]

Thus we have \(\chi_{lat}(A(mK_n, K_r)) = n\).

\(\square\)

**Corollary 2.7:** For \(m \geq 2\), \(n\) odd and \(n > r \geq 3\), \(\chi_{lat}(A(mK_{n+1}, K_{r+1}) = \chi_{lat}(A(mK_n, K_r) \vee K_1) = n + 1\).
Proof. Similar to Corollary 2.3, we only need to show that $\chi_{la}(A(mK_n, K_r) \vee K_1) \leq n + 1$. Keep the construction in the proof of Theorem 2.6.

(A) Suppose $r$ is odd so that $n - r \geq 2$ is even.

$$w(v_{m,n-r}) = \sum_{i=1}^{n-r} (L_m)_{n-r,i} + \sum_{i=1}^{r} (B_m)_{n-r,i}$$

$$= \sum_{i=1}^{n-r} (L_m)_{i,n-r} + \sum_{i=1}^{r} A_{i,i}$$

(since $\Omega$ is a magic rectangle)

$$< \sum_{i=1}^{n-r} (L_m)_{i,i} + \sum_{i=1}^{r} A_{i,i} \leq (M).$$

Next

$$(M) = \sum_{i=1}^{m} \sum_{j=1}^{n-r} (L_i)_{j,j} + A_{1,1} + \sum_{k=2}^{r} A_{k,k} < \sum_{i=1}^{m} \sum_{j=1}^{n-r} (B_i)_{j,1} + A_{1,1} + \sum_{k=2}^{r} A_{k,k}$$

$$< \sum_{i=1}^{m} \sum_{j=1}^{n-r} (B_i)_{j,1} + A_{1,1} + \sum_{k=2}^{r} A_{1,k}$$

$$= w(u_{n-r+1}).$$

Thus we have $w(v_{m,n-r}) < (M) < w(u_{n-r+1}).$

(B) Suppose $r$ is even and $m$ is odd. Since each diagonal of $M$ is the largest entry in the corresponding column, $(M)$ is larger than all the vertex weights.

(C) Suppose $r$ is even and $m$ is even.

$$(M) = \sum_{i=1}^{m} \sum_{j=1}^{n-r} (L_i)_{j,j} + \sum_{k=1}^{r} A_{k,k} < \sum_{i=1}^{m} \sum_{j=1}^{n-r} (X_i)_{j,1} + \sum_{k=1}^{r} A_{2,k}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n-r} (X_i^T)_{1,j} + \sum_{k=1}^{r} A_{2,k} = w(u_{n-r+2}).$$

$$w(u_{n-r+1}) = \sum_{i=1}^{m} \sum_{k=1}^{n-r} (\beta_i)_{k,1} + A_{1,1} + \sum_{k=2}^{r} A_{1,k}$$

$$< \sum_{i=1}^{m} \sum_{k=1}^{n-r} (L_i)_{k,k} + A_{1,1} + \sum_{k=2}^{r} A_{1,k}$$

$$< \sum_{i=1}^{m} \sum_{k=1}^{n-r} (L_i)_{k,k} + A_{1,1} + \sum_{k=2}^{r} A_{k,k} = (M)$$

From Remark 2.2, we have $\chi_{la}(A(mK_n, K_r) \vee K_1) \leq n + 1$. 

Example 2.4: We take $m = 2$, $n = 7$ and $r = 3$. The guide matrix is

$$M = \begin{pmatrix} +7 & -1 & +2 & -3 \\ -1 & +8 & -4 & +5 \\ +2 & -4 & -9 & +6 \\ -3 & +9 & +6 & -10 \end{pmatrix}; \quad L_1 = \begin{pmatrix} 13 & 2 & 3 & 6 \\ 2 & 15 & 8 & 9 \\ 3 & 8 & 18 & 11 \\ 6 & 9 & 11 & 20 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 14 & 1 & 4 & 5 \\ 1 & 16 & 7 & 10 \\ 4 & 7 & 17 & 12 \\ 5 & 10 & 12 & 19 \end{pmatrix}. $$

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Thus $\chi_{\text{tot}}(A(2K_7, K_3)) = 7$. Since $(M) = 234$, $\chi_{\text{tot}}(A(2K_7, K_3) \vee K_1) = 8$. 

**Example 2.5:** We take $m = 3$, $n = 7$ and $r = 4$. The guide matrix is

$$M = \begin{pmatrix}
13 & 2 & 3 & 6 & 6 & 9 & 20 \\
2 & 15 & 8 & 9 & 11 & 20 & 41 \\
3 & 8 & 18 & 11 & 2 & 5 & 14 \\
6 & 9 & 11 & 20 & * & * & 148 \\
\end{pmatrix} \quad \Omega = \begin{pmatrix}
25 & 30 & 47 & 126 \\
34 & 39 & 29 & 136 \\
36 & 44 & 22 & 142 \\
41 & 28 & 33 & 148 \\
\end{pmatrix} \quad A = \begin{pmatrix}
32 & 48 & 49 \\
48 & 46 & 50 \\
49 & 50 & 24 \\
\end{pmatrix}.$$ 

$L_1|\beta_1 = \begin{pmatrix} 46 & 3 & 4 & 9 \\ 3 & 49 & 12 & 13 \\ 4 & 12 & 54 & 16 \end{pmatrix}$, 
$L_2|\beta_2 = \begin{pmatrix} 47 & 2 & 5 & 8 \\ 2 & 50 & 11 & 14 \\ 5 & 11 & 53 & 17 \end{pmatrix}$, 
$L_3|\beta_3 = \begin{pmatrix} 48 & 1 & 6 & 7 \\ 1 & 51 & 10 & 15 \\ 6 & 10 & 52 & 18 \end{pmatrix}$. 

$$\Omega = \begin{pmatrix}
27 & 32 & 37 & 61 \\
20 & 34 & 42 & 55 \\
31 & 39 & 26 & 57 \\
\end{pmatrix} \quad A = \begin{pmatrix}
9 & 27 & 32 & 37 \\
13 & 20 & 34 & 42 \\
16 & 31 & 39 & 26 \\
8 & 36 & 41 & 19 \\
14 & 29 & 43 & 24 \\
17 & 40 & 21 & 35 \\
48 & 1 & 6 & 7 \\
1 & 51 & 10 & 15 \\
6 & 10 & 52 & 18 \\
\end{pmatrix}.$$ 

$$M = \begin{pmatrix}
46 & 3 & 4 & 9 \\
3 & 49 & 12 & 13 \\
4 & 12 & 54 & 16 \\
\end{pmatrix} \quad \begin{pmatrix}
27 & 32 & 37 & 61 \\
20 & 34 & 42 & 55 \\
31 & 39 & 26 & 57 \\
\end{pmatrix} \quad \begin{pmatrix}
48 & 1 & 6 & 7 \\
1 & 51 & 10 & 15 \\
6 & 10 & 52 & 18 \\
\end{pmatrix}.$$
Thus $\chi_{lat}(A(3K_7, K_4)) = 7$. Since $\binom{m}{2} = 700$, $\chi_{lat}(A(3K_7, K_4) \lor K_4) = 8$. \qed

**Example 2.6:** We take $m = 2$, $n = 7$ and $r = 4$. The guide matrix is

$$
M = \begin{pmatrix}
+7 & -1 & +2 & -3 \\
-1 & +8 & -4 & +5 \\
+2 & -4 & -9 & +6
\end{pmatrix}, \quad (L_1|\beta_1) = \begin{pmatrix}
13 & 2 & 3 \\
2 & 15 & 8 \\
3 & 8 & 18
\end{pmatrix} \begin{pmatrix} 6 \end{pmatrix}, \quad (L_2|\beta_2) = \begin{pmatrix}
14 & 1 & 4 \\
1 & 16 & 7 \\
4 & 7 & 17
\end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}.
$$

$$
\Omega = \begin{pmatrix}
X_1 \\
X_2 \\
\alpha
\end{pmatrix} = \begin{pmatrix}
27 & 25 & 38 \\
24 & 31 & 35 \\
21 & 32 & 37 \\
39 & 22 & 29 \\
36 & 26 & 28 \\
33 & 34 & 23 \\
30 & 40 & 20
\end{pmatrix}, \quad A = \begin{pmatrix}
19 & 30 & 40 & 20 \\
30 & 41 & 42 & 43 \\
40 & 42 & 44 & 45 \\
20 & 43 & 45 & 46
\end{pmatrix}.
$$

$$
M = \begin{pmatrix}
13 & 2 & 3 & * & * & * & 6 & 27 & 25 & 38 & 114 \\
2 & 15 & 8 & * & * & * & 9 & 24 & 31 & 35 & 124 \\
3 & 8 & 18 & * & * & * & 11 & 21 & 32 & 37 & 130 \\
* & * & * & 14 & 1 & 4 & 5 & 39 & 22 & 29 & 114 \\
* & * & * & 1 & 16 & 7 & 10 & 36 & 26 & 28 & 124 \\
* & * & * & 4 & 7 & 17 & 12 & 33 & 34 & 23 & 130 \\
6 & 9 & 11 & 5 & 10 & 12 & 19 & 30 & 40 & 20 & 162 \\
27 & 24 & 21 & 39 & 36 & 33 & 30 & 41 & 42 & 43 & 336 \\
25 & 31 & 32 & 22 & 26 & 34 & 40 & 42 & 44 & 45 & 341 \\
38 & 35 & 37 & 29 & 28 & 23 & 20 & 43 & 45 & 46 & 344
\end{pmatrix}.
$$

Thus $\chi_{lat}(A(2K_7, K_4)) = 7$. Since $D(M) = 243$, $\chi_{lat}(A(2K_7, K_4) \lor K_4) = 8$. \qed

**Theorem 2.8:** For $m \geq 2$ and $n = 4k + 3 \geq 3$, $\chi_{lat}(mK_n) \leq n + 1$.

**Proof.** Consider $S_{4k+2}$. Change each diagonal entry of $S_{4k+2}$ from $\pm 1$ to $\pm 2$. Let this new sign matrix be $S'$. Now define $S_{4k+3}$ by extending $S'$ to a $4k + 3$ square matrix by adding the last column and row. Use $+1$, $-1$ or $+2$ for each entry of this new column and row such that the row sum and the column sums of $S_{4k+3}$ are zero.

Define a symmetric matrix $M'$ of order $4k + 3$ by using the increasing sequence $[1,(2k+1)(4k+3)]$ in lexicographic order for the upper triangular entries of $M'$, and use $*$ for all diagonal entries.

Now let $M$ be the guide matrix whose $(j,l)$-th entry is $*$ if $j = l$; and $(S_{4k+3})_{j,l}(M')_{j,l}$, $1 \leq j,l \leq 4k + 3$ if $j \neq l$.

**Stage 1:** Using the same procedure as Stage 1 in the proof of Theorem 2.3 fill the off-diagonals of the $m$ submatrices $L_i$, $1 \leq i \leq m$, with labels in $[1,mN_4]$, where $N_4 = \frac{n(n+1)}{2}$.

**Stage 2:** Let $T(2a-1) = \{mN_4+2l-1+2m(a-1) \mid 1 \leq l \leq m \}$ and $T(2a) = \{mN_4+2l+2m(a-1) \mid 1 \leq l \leq m \}$, $1 \leq a \leq 2k+1$. For $1 \leq j \leq 4k + 2$, the $(j,j)$-entry of $L_i$ is the $i$-th term of $T^-(j)$ or $T^+(j)$, if the corresponding $(j,j)$-entry of $M$ is $-2$ or $+2$, respectively.

**Stage 3:** Let $U(1) = \{mN_4+(4k+2)m+2l-1 \mid 1 \leq l \leq \lceil m/2 \rceil \}$ and $U(2) = \{mN_4+(4k+2)m+2l \mid 1 \leq l \leq \lfloor m/2 \rfloor \}$. Let $U$ be the compound sequence $U^+(1)U^+(2)$, i.e., list the terms of $U^+(1)$ first and then follow with the terms of $U^+(2)$. The $(4k+3,4k+3)$-entry of $L_i$ is the $i$-th term of $U$.

For a fixed $j$, $1 \leq j \leq 4k + 2$, according to the structures of $S(a)$'s and $T(a)$'s, $(L_{i+1})_{j,l} - (L_i)_{j,l}$ is $(S_{4k+3})_{j,l}$ for $1 \leq i \leq m - 1, 1 \leq l \leq 4k + 3$. Thus, $m(e_{i,j})$ is a constant for a fixed $j$.

Similarly, $(L_{i+1})_{4k+3,l} - (L_i)_{4k+3,l}$ is $(S_{4k+3})_{4k+3,l}$ for $1 \leq i \leq m - 1, 1 \leq l \leq 4k + 2$. Finally, for $1 \leq i \leq m - 1$,

$$(L_{i+1})_{4k+3,k} = (L_i)_{4k+3,k} = 0$$

if $i \neq \lfloor m/2 \rfloor$. \qed
Thus, \( \{ w(v_i, 4k+3) \mid 1 \leq i \leq m \} \) consists of two different values.

Since each \( L_i \) satisfies the condition of Lemma 2.8, \( R_j(L_i) \) is a strictly increasing function of \( j \). Hence \( \chi_{lat}(mK_n) \leq n + 1 \).

**Example 2.7:** Take \( n = 7 \) and \( m = 2 \). So

\[
S_0 = \begin{pmatrix}
+1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 \\
-1 & +1 & -1 & +1 & -1 & +1
\end{pmatrix} \rightarrow S_7 = \begin{pmatrix}
+2 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +2 & -1 & +1 & -1 & +1 & -1 & +1 \\
+1 & -1 & -2 & +1 & -1 & +1 & -1 & +1 \\
-1 & +1 & -2 & +1 & -1 & +1 & -1 & +1 \\
-1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & +1 & -1 & +1 & -1 & +1 & -1
\end{pmatrix}
\]

So \( \chi_{lat}(2K_7) \leq 8 \).

**Theorem 2.9:** For \( m \geq 2 \) and \( n = 4k + 1 \geq 5 \), \( \chi_{lat}(mK_n) \leq \min\{n + 3, n - 1 + m\} \).

**Proof.** Similar to the proof of Theorem 2.8, we will define a sign matrix \( S \) and a guide matrix \( M \) of order \( 4k + 1 \). We define a \( (4k) \times (4k) \) matrix \( S_{4k} \) first.

Let \( S'_1 = \begin{pmatrix}
+1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1
\end{pmatrix} \) and \( S'_{4k} = \begin{pmatrix}
S'_{4k-4} \\
\vdots \\
S'_4 \\
S'_4 \\
S'_4
\end{pmatrix} \) when \( k \geq 2 \).

Now, we define a symmetric sign matrix \( S_{4k+1} \) of order \( 4k + 1 \) using the same method as in the proof of Theorem 2.8. Note that the \((4k + 1, 4k + 1)\)-entry of \( S_{4k+1} \) is +4. The definition of a guide matrix \( M \) and Stage 1 are similar to the proof of Theorem 2.8.

**Stage 2:** Using a similar procedure to Stage 2 in the proof of Theorem 2.8, fill all diagonals of each \( L_i \) except \((L_i)_{4k+1, 4k+1} \) by using \( T(j) \) defined in the proof of Theorem 2.8 \( 1 \leq j \leq 4k \).

Now we have to fill \([mN_4 + 4km + 1, mN_4 + 4km + m]\) in the \((L_i)_{4k+1, 4k+1} \), here \( N_4 = \frac{n(n-1)}{2} \). Suppose \( m = 4s + m_0 \), for some \( s \geq 0 \) and \( 0 \leq m_0 < 4 \).

Suppose \( s \geq 1 \). Let \( U(a) = \{mN_4 + 4km + a + d \mid 0 \leq l \leq s \} \) for \( 1 \leq a \leq m_0 \); and \( U(a) = \{mN_4 + 4km + a + d \mid 0 \leq l \leq s - 1 \} \) for \( m_0 < a \leq 4 \). Let \( U \) be the compound sequence \( U^+(1)U^+(2)U^+(3)U^+(4) \). Let the \((4k + 1, 4k + 1)\)-entry of \( L_i \) be the \( i \)-th term of \( U \).

Using a proof similar to Theorem 2.8, we have \( \chi_{lat}(mK_n) \leq n + 3 \).
Suppose \( s = 0 \). That means \( 2 \leq m \leq 4 \). Then fill \([mN_4 + 4km + 1, mN_4 + 4km + m]\) to the \((4k + 1, 4k + 1)\)-entry of \(L_i\), respectively.

Using a proof similar to that above, \(R_j(L_i)\) is a strictly increasing function of \(j\) and hence we have \(\chi_{lat}(mK_n) \leq n - 1 + m\).

Combining these two cases, we conclude that \(\chi_{lat}(mK_n) \leq \min\{n + 3, n - 1 + m\} \).

\[\square\]

**Corollary 2.10:** Let \(m \geq 2\) and odd \(n \geq 3\), \(\chi_{lat}(A(mK_n, K_1)) = n\).

**Proof.** From the proofs of Theorems 2.8 and 2.9, we see that the \((n, n)\)-th entries of all \(L_i\)'s are the largest \(m\) labels. If we change \((L_i)_{n,n}\) to \(N\) and denote this new matrix by \(M_i\), then the matrix \(M\) becomes a total labeling matrix of \(A(mK_n, K_1)\) with \(n\) difference row sums. Note that, \((L_i)_{n,n}\)'s are identified to \(A_{1,1}\). Thus \(\chi_{lat}(A(mK_n, K_1)) = n\) .

\[\square\]

**Example 2.8:** Let \(n = 9, m = 2\). Then

\[
S'_8 = \begin{pmatrix}
+1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 \\
+1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 \\
-1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
\rightarrow
S_8 = \begin{pmatrix}
+2 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 \\
-1 & +2 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +2 & -1 & +1 & -1 & +1 & -1 & +1 \\
+1 & -1 & -1 & +2 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & +2 & -1 & +1 & -1 & +1 \\
-1 & +1 & -1 & -1 & -1 & +2 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +2 & -1 & +1 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & +2 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +2 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}
\]

Hence

\[
M = \begin{pmatrix}
* & -1 & +2 & -3 & +4 & -5 & +6 & -7 & -8 \\
-1 & * & -9 & +10 & -11 & +12 & -13 & +14 & -15 \\
+2 & -9 & * & -16 & +17 & -18 & +19 & -20 & -21 \\
-3 & +10 & -16 & * & -22 & +23 & -24 & +25 & -26 \\
+4 & -11 & +17 & -22 & * & -27 & +28 & -29 & -30 \\
-5 & +12 & -18 & +23 & -27 & * & -31 & +32 & -33 \\
+6 & -13 & +19 & -24 & +28 & -31 & * & +34 & -35 \\
-7 & +14 & -20 & +25 & -29 & +32 & +34 & * & +36 \\
-8 & -15 & -21 & -26 & -30 & -33 & +35 & +36 & *
\end{pmatrix}
\]

We have

\[
L_1 = \begin{pmatrix}
73 & 2 & 3 & 6 & 7 & 10 & 11 & 14 & 16 & 142 \\
2 & 74 & 18 & 19 & 22 & 23 & 26 & 27 & 30 & 241 \\
3 & 18 & 77 & 32 & 33 & 36 & 37 & 40 & 42 & 318 \\
6 & 19 & 32 & 78 & 44 & 45 & 48 & 49 & 52 & 373 \\
7 & 22 & 33 & 44 & 81 & 54 & 55 & 58 & 60 & 414 \\
10 & 23 & 36 & 45 & 54 & 82 & 62 & 63 & 66 & 441 \\
11 & 26 & 37 & 48 & 55 & 62 & 87 & 67 & 69 & 462 \\
14 & 27 & 40 & 49 & 58 & 63 & 67 & 88 & 71 & 477 \\
16 & 30 & 42 & 52 & 60 & 66 & 69 & 71 & 89 & 495 \\
75 & 1 & 4 & 5 & 8 & 9 & 12 & 13 & 15 & 142 \\
1 & 76 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 241 \\
4 & 17 & 79 & 31 & 34 & 35 & 38 & 39 & 41 & 318 \\
5 & 20 & 31 & 80 & 43 & 46 & 47 & 50 & 51 & 373 \\
8 & 21 & 34 & 43 & 83 & 53 & 56 & 57 & 59 & 414 \\
9 & 24 & 35 & 46 & 53 & 84 & 61 & 64 & 65 & 441 \\
12 & 25 & 38 & 47 & 56 & 61 & 85 & 68 & 70 & 462 \\
13 & 28 & 39 & 50 & 57 & 64 & 68 & 86 & 72 & 477 \\
15 & 29 & 41 & 51 & 59 & 65 & 70 & 72 & 90 & 492
\end{pmatrix}
\]

\[
L_2 = \begin{pmatrix}
8 & 21 & 34 & 43 & 83 & 53 & 56 & 57 & 59 & 414 \\
9 & 24 & 35 & 46 & 53 & 84 & 61 & 64 & 65 & 441 \\
12 & 25 & 38 & 47 & 56 & 61 & 85 & 68 & 70 & 462 \\
13 & 28 & 39 & 50 & 57 & 64 & 68 & 86 & 72 & 477 \\
15 & 29 & 41 & 51 & 59 & 65 & 70 & 72 & 90 & 492
\end{pmatrix}
\]
Thus we have

$$\chi_{\text{lat}}(2K_n) \leq 10.$$ Clearly $(M)$ is larger than all vertex weights, so $\chi_{\text{lat}}((2K_n) \lor K_1) \leq 11.$

If we change $(L_2)_{9,9}$ to 89 and identify the vertices $v_{1,9}$ with $v_{2,9},$ then we have a local antimagic total labeling for $A(2K_9, K_1)$ and hence $\chi_{\text{lat}}(A(2K_9, K_1)) = 9.$ Clearly $(M)$ is larger than all vertex weights, so $\chi_{\text{lat}}(A(2K_9, K_1) \lor K_1) = 10.$

**Corollary 2.11:** Let $m \geq 2$ and odd $n \geq 3,$

$$\chi_{\text{lat}}(mK_{n+1}) = \chi_{\text{lat}}((mK_n) \lor K_1) \leq \begin{cases} \min\{n + 4, n + m\} & \text{if } n \equiv 1 \pmod{3}, \\ n + 2 & \text{if } n \equiv 3 \pmod{3}. \end{cases}$$

**Proof.** From the proofs of Theorems 2.8 and 2.9, we see that the diagonals of $M$ are the largest $mn$ labels. Thus, $D(M) > w(v_{m,n}).$ So we have the corollary.

By Corollary 2.10 and the same argument above, we have

**Corollary 2.12:** Let $m \geq 2$ and odd $n \geq 3,$ $\chi_{\text{lat}}(A(mK_{n+1}, K_2)) = \chi_{\text{lat}}(A(mK_n, K_1) \lor K_1) = n + 1.$

We have an ad hoc result for $A(mK_n, K_2)$ for $n$ odd as follows.

**Theorem 2.13:** For $2 \leq m \leq 3$ and odd $n \geq 3,$ $\chi_{\text{lat}}(A(mK_n, K_2)) = n$ and $\chi_{\text{lat}}(A(mK_n, K_2) \lor K_1) = n + 1.$

**Proof.** The guide matrix is obtained from the guide matrix $M$ of order $n$ defined in the proof of Theorem 2.8 or Theorem 2.9 by deleting the last two rows. Apply the same procedure as Stage 1 in the proof of Theorem 2.8 or Theorem 2.9. It is equivalent to defining the matrix $(L_i, B_i)$ of order $(n - 2) \times n$ by deleting the last two rows of $L_i,$ defined in the proof of Theorem 2.8 or Theorem 2.9. Note that $(M)_{n-1,n} = -\frac{n(n-1)}{2}$ is not used in this stage. So the labels used are $[1, mN],$ where $N_2 = \frac{(n-2)(n+1)}{2}.$

**Stage 2:** Similar to Stage 2 of the proof of Theorem 2.8 or Theorem 2.9, we define the set $T(j)$ by labels $[mN_2 + 1, mN_2 + m(n - 2) + m]$ for $1 \leq j \leq n - 1$ and fill the diagonals of $L_i$'s. Note that, $mN_2 + m(n - 2) + m \leq N$ and $T(n - 1)$ is not used in this stage.

By Lemma 2.1 each $j$-th row sum of each $(L_i, B_i)$ is a constant, and each $j$-th row sum of a fixed matrix $(L_i, B_i)$ is a strictly increasing function of $j,$ $1 \leq j \leq n - 2.$

**Stage 3:** Use the remaining 3 labels to fill in the matrix $A_{1,2}, A_{1,1}, A_{2,2}$ in natural order. It is easy to see that the last two row sums of $M$ are distinct and larger than the other row sums of $M.$

It is easy to see that the diagonal entries of $M$ are the largest entries in the corresponding columns, and $(M)$ is greater than all weights of vertices.

Thus we have $\chi_{\text{lat}}(A(mK_n, K_2)) = n$ and $\chi_{\text{lat}}(A(mK_n, K_2) \lor K_1) = n + 1.$

**Example 2.9:** Take $n = 7$ and $m = 2.$

$$w(u_6) = 230 + 229 - (50 + 52) = 357$$ and $w(u_7) = 241 + 240 - (50 + 53) = 378.$ So $\chi_{\text{lat}}(A(2K_7, K_2)) = 7.$

| $$(L_1, B_1) = \begin{pmatrix} 41 & 2 & 3 & 6 & 7 & 10 & 12 & 82 \\ 2 & 14 & 15 & 18 & 19 & 22 & 132 \\ 3 & 4 & 17 & 23 & 25 & 28 & 29 & 169 \\ 6 & 15 & 23 & 48 & 32 & 33 & 35 & 192 \\ 7 & 18 & 25 & 32 & 49 & 38 & 40 & 209 \\ 10 & 19 & 28 & 33 & 38 & 52 & 50 & 230 \\ 12 & 22 & 29 & 35 & 40 & 50 & 53 & 241 \end{pmatrix},$$ (L_2, B_2) = \begin{pmatrix} 43 & 1 & 4 & 5 & 8 & 9 & 11 & 81 \\ 1 & 44 & 13 & 16 & 17 & 20 & 21 & 132 \\ 4 & 13 & 45 & 24 & 26 & 27 & 30 & 169 \\ 5 & 16 & 24 & 46 & 31 & 34 & 36 & 192 \\ 8 & 17 & 26 & 31 & 51 & 37 & 39 & 209 \\ 9 & 20 & 27 & 34 & 37 & 52 & 50 & 229 \\ 11 & 21 & 30 & 36 & 39 & 50 & 53 & 240 \end{pmatrix}.$$}
3 Conclusion and Open Problems

In this paper, \( \chi_{lat}(A(mK_n, K_1)) \) and \( \chi_{lat}(A(mK_n, K_1) \lor K_1) \) are determined except the following remaining case.

**Problem 3.1:** For \( m \geq 2 \) odd, \( n \geq 3, n > r, r = 0, 2 \), determine \( \chi_{lat}(A(mK_n, K_r)) \).

From the motivation, we end this paper with the following problem.

**Problem 3.2:** Determine \( \chi_{lat}(A(mC_n, K_2)) \) for \( m \geq 2, n \geq 3 \).

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