MAXIMIZERS OF THE HAWKING MASS IN ASYMPTOTICALLY FLAT MANIFOLDS

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Abstract. We study the Hawking mass in an end of an asymptotically flat manifold which is $C^3$-close to an exterior region of one half of the three dimensional Schwarzschild manifold. Such an end is foliated by spheres of Willmore type by a result of Lamm, Metzger and Schulze (see [LMS11]). We show that, amongst all spherical surfaces, the leaves of this foliation maximize the Hawking mass if the area is prescribed. In fact, the leaves are the only surfaces of Willmore type which have non-negative Hawking mass and are not null-homologous. If the end is conformally flat, we show that this holds even without any assumption on the topology. The main ingredients in the proof are a careful application of the first variational formula for the area functional to estimate the center of gravity of a sphere with positive Hawking mass as well as estimates of integral type to derive geometric properties of spheres of Willmore type.

1. Introduction

Let $(M, g)$ be an asymptotically flat Riemannian three manifold with non-negative scalar curvature. Under suitable decay conditions on the metric, such a manifold possesses a global non-negative invariant called the ADM mass and denoted by $m_{\text{ADM}}$ (see [ADM61, SY79, Bar86]). On the other hand, finding the right notion of quasi-local mass corresponding to this global invariant remains an interesting open problem (see [Pen82]). A promising candidate is the so-called Hawking mass $m_H$ defined by

$$m_H(\Sigma) := \frac{|\Sigma|}{(16\pi)^{\frac{3}{2}}} \left( 16\pi - \int_{\Sigma} H^2 d\mu \right),$$

where $\Sigma$ is a compact surface bounding a region $\Omega$ whose mass is to be determined. With the help of the Hawking mass, the ADM mass can be quantified in terms of the local geometry: in a celebrated work, Huisken and Ilmanen used a weak version of the inverse mean curvature flow to prove the Riemannian Penrose inequality which states that the ADM mass of an asymptotically flat manifold is bounded from below by the Hawking mass of any connected outward minimizing surface (see [HI01]). A different version of the Penrose inequality, where the comparison surface is required to be minimal but not necessarily connected, was later on shown by Bray using a quasistatic flow (see [Bra01]). More recently, Huisken introduced a concept of isoperimetric mass which only relies on $C^0$-data of the metric and provides a notion of quasi-local as well as global mass. The global mass can be shown to agree with the ADM mass in case the latter is well-defined. It turns out that the isoperimetric mass can be characterized in terms of the Hawking mass of outward minimizing surfaces (see [Hui06] or [JL16] for a more detailed discussion).

While the Hawking mass enjoys such desirable connections to the global geometry, there are unfortunately many surfaces with negative Hawking mass. This is a contrast to some other concepts of quasi-local mass such as the Brown-York mass (see [ST02]). It was therefore a crucial insight by Christodoulou and Yau that the Hawking mass of a closed stable constant mean curvature surface is non-negative (see [CY88]). This suggested that such surfaces are suitable to test the gravitational field of an asymptotically flat manifold and motivated the study of the isoperimetric problem in such spaces. As some of the following results require stronger decay conditions on the metric than asymptotical flatness, we make the following definition: the metric $g$ is said to be $C^k$-close to

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Schwarzschild with decay coefficient $\eta$ and ADM mass $m$ if in the chart at infinity there holds $g = g_S + h$, where $h$ a symmetric two tensor satisfying
\[
|\partial_j h| \leq \eta r^{-2-j},
\]for any $0 \leq j \leq k$. Here, $g_S$ is the Schwarzschild metric with mass $m$, $\partial$ the Euclidean derivative and $r$ the radial parameter in the chart at infinity. The Schwarzschild space models a static, single black hole and a space which is $C^k-$close can be understood to be a small perturbation. The first breakthrough in the study of the isoperimetric problem was accomplished by Huisken and Yau, who used a volume-preserving version of the mean curvature flow to show that an exterior region of an asymptotically flat manifold which is $C^4$-close to Schwarzschild and has non-negative scalar curvature is foliated by embedded stable constant mean curvature spheres. Such a foliation induces a natural coordinate system and also gives rise to a geometric center of mass. This result was later refined by Qing and Tian who showed uniqueness of this foliation (see [QT07]). Using an ingenious argument, Bray showed in his PhD-thesis that the centered spheres are the unique non-null-homologous isoperimetric surfaces in the Schwarzschild manifold (see [Bra09]). This provided evidence that the leaves of the foliation in [HY96] might actually be isoperimetric. In another breakthrough, Eichmair and Metzger extended the idea of Bray and showed in [EM13] that a foliation as in [HY96] exists even if the manifold is only $C^2$-close to Schwarzschild. Furthermore, they proved that in the exterior region the leaves are in fact the unique isoperimetric surfaces enclosing a sufficiently large volume. It was later shown by Chodosh, Eichmair, Shi and Yu that a unique minimizer of the isoperimetric problem exists even if the manifold is only asymptotically flat and satisfies a certain decay condition on the scalar curvature (see [CESY16]). On the other hand, studying the uniqueness of stable constant mean curvature spheres which are not necessarily isoperimetric turned out to be a more difficult problem. As a first step in this direction, Brendle showed a Heintze-Karcher type inequality and used a conformal flow in an elegant way to show that the centered spheres are the only constant mean curvature surfaces contained in one half of the Schwarzschild manifold (see [Bre13]). Finally, in a series of crucial results, Chodosh and Eichmair obtained the unconditional characterization of stable constant mean curvature surfaces in asymptotically flat manifolds which are $C^0-$close to Schwarzschild and whose scalar curvature is non-negative and satisfies a certain decay condition. By comparing certain mass flux integrals ([CE17a]) and using a Lyapunov-Schmidt analysis to study null-homologous surfaces ([CE17b]), they showed that the leaves of the foliation are the only stable compact constant mean curvature surfaces without any assumption on their homology class. In the proof, the result [CEE16] by Carlotto, Chodosh and Eichmair played an important part where they showed among other things that any asymptotically flat manifold with non-negative scalar curvature admitting an unbounded area minimizing minimal surface must be isometric to the flat Euclidean space. The results in [CE17a] seem to be optimal in some sense (see also [BE14]). Moreover, it should be noted that they stand in stark contrast to the situation in the Euclidean space. The presence of positive mass seems to rule out all but one isoperimetric surface.

For any concept of quasi-local mass, it is natural to look for regions which contain a maximal amount of mass. Usually, one can only hope to find such regions if one fixes a certain geometric quantity such as the volume of the region $\Omega$ or the area of its boundary $\Sigma$. In the case of the Hawking mass, the latter seems to be the more obvious quantity. While isoperimetric surfaces enjoy non-negative Hawking mass, they in fact maximize Huisken’s quasi-local isoperimetric mass when fixing the volume of $\Omega$. Hence, when studying the Hawking mass, it might be a more natural problem to directly look for maximizers of the Hawking mass when fixing the area. This approach is equivalent to finding area-constrained minimizers of the Willmore functional $\mathcal{W}$ which is defined to be
\[
\mathcal{W}(\Sigma) := \frac{1}{2} \int_{\Sigma} H^2 d\mu.
\]
While the isoperimetric problem can be formulated solely in terms of $C^0$-data, the Hawking mass depends on higher order quantities and therefore seems to be more complicated to investigate. In
fact, the Euler-Lagrange equation for the Willmore functional is a fourth-order elliptic equation and cannot be studied with the same techniques as the second order problem of finding constant mean curvature surfaces. Nevertheless, using a continuity method and integral curvature estimates, Lamm, Metzger and Schulze showed the following result (see [LMS11] and section 2 for a more precise statement).

**Theorem 1.1.** Let \((M, g)\) be an asymptotically flat manifold which is \(C^3\)—close to Schwarzschild, with mass \(m > 0\) and decay coefficient \(0 < \eta < \eta_0\) for some constant \(\eta_0\) depending only on \(m\), and satisfies \(0 \leq S_c \leq \eta^{-5}\). Then there exists a constant \(\lambda_0 > 0\) and a compact set \(K\) depending only on \(m\) and \(\eta_0\) such that \(M \setminus K\) is foliated by surfaces of Willmore type \(\Sigma_\lambda\) where \(\lambda \in (0, \lambda_0)\). Moreover, every sufficiently centered, strictly mean-convex sphere \(\Sigma \subset M \setminus K\) which is of Willmore type belongs to the foliation.

Here, a surface of Willmore type is a critical point of the area prescribed Willmore energy. More precisely, every \(\Sigma_\lambda\) satisfies the equation

\[
\Delta H + H(Rc(\nu, \nu) + |\dot{A}|^2 + \lambda) = 0.
\]

As for the isoperimetric problem, the positivity of the ADM mass is related to uniqueness which is evidently violated in the Euclidean space. The leaves of the foliation enjoy various desirable properties: the Hawking mass is positive and non-decreasing along the foliation and approaches the ADM mass as \(\lambda \to 0\). Given the results obtained for the isoperimetric problem, one is tempted to believe that in an exterior region, the leaves \(\Sigma_\lambda\) are the global maximizers of the Hawking mass and perhaps the only surfaces of Willmore type with non-negative Hawking mass and a sufficiently large area. Up to now, this has not even been known in Schwarzschild. In fact, a result comparable to the one obtained in [Bra01] cannot be expected as one can easily construct spheres which are close and homologous to the event horizon, but have arbitrarily large Hawking mass (see Remark 5.4). Still, we can show the following result:

**Theorem 1.2.** Let \((M, g)\) be an asymptotically flat manifold which is \(C^3\)-close to Schwarzschild, with mass \(m > 0\) and decay coefficient \(0 < \eta \leq \eta_0\) for some \(\eta_0(m) > 0\), and satisfies \(0 \leq S_c \leq \eta^{-5}\). Given \(m, \eta\) and any positive number \(\Lambda\), there exists a compact set \(K\) which is foliated by surfaces of Willmore type \(\Sigma_\lambda\), \(\lambda \in (0, \lambda_0)\), such that the following holds: Any immersed closed surface \(\Sigma \subset (M \setminus K)\) with genus(\(\Sigma\)) \(\leq \Lambda\) satisfies the inequality

\[
m \geq m_H(\Sigma)
\]

with equality if and only if the non-compact component of \((M \setminus \Sigma, g)\) is isometric to an exterior region of the Schwarzschild manifold and \(\Sigma\) is a centered sphere. Moreover, if \(\lambda \in (0, \lambda_0)\) is such that \(|\Sigma| = |\Sigma_\lambda|\), then there holds

\[
m_H(\Sigma_\lambda) \geq m_H(\Sigma)
\]

with equality if and only if \(\Sigma = \Sigma_\lambda\). On the other hand, if \(|\Sigma| < |\Sigma_\lambda|\) for every \(\lambda \in (0, \lambda_0)\) then \(\Sigma\) is null-homologous and satisfies \(m_H(\Sigma) < m_H(\Sigma_\lambda)\) for any \(\lambda \in (0, \lambda_0)\).

The assumption on the topology of \(\Sigma\) is technical (it is used to compare the actual Willmore energy with the Euclidean Willmore energy in the asymptotic chart) and we do not expect it to be necessary. In fact, if \((M, g)\) is conformally flat near infinity we can already remove this condition and obtain an analog of [Bra09] as a special case:

**Theorem 1.3.** In the situation of the previous theorem, assume that there is a compact set \(\tilde{K}\) such that \((M \setminus \tilde{K}, g)\) is conformally flat. Then one may take \(\Lambda = \infty\). In particular, outside of a large centered coordinate ball, the centered spheres are the global area constrained maximizers of the Hawking mass in the Schwarzschild space without any assumption on the topology.

A drawback of Theorem 1.1 is that it requires \((M, g)\) not only to be \(C^3\)-close to Schwarzschild, but also \(\eta\) in (1) to be small and an additional decay condition on the scalar curvature. This is for technical reasons as the continuity method used in [LMS11] relies on the invertibility of the
linearization of the Willmore operator. The proof of Theorem 1.2 continues to work assuming only $C^3$-closeness with no condition on $\eta$ or the decay of the scalar curvature, provided the same holds true for Theorem 1.1. Consequently, it would be interesting to know if the conditions in Theorem 1.1 can be weakened. However, this would probably require a completely different idea. As has been stated above, the assumption that a compact set $K$ needs to be removed is necessary even if $(M, g)$ is one half of the Schwarzschild manifold. This is in stark contrast to the situation regarding the isoperimetric problem. It would be interesting to determine the optimal size of $K$, at least in Schwarzschild.

It is of course also of interest to study other area-prescribed critical points of the Hawking mass. Assuming non-negative Hawking mass, that is, $\mathcal{W} \leq 4\pi$, we obtain the following theorem as a by-product of Theorem 1.2:

**Theorem 1.4.** Let $(M, g), \Lambda$ and $K$ be as in Theorem 1.2 and let $\Sigma \subset M \setminus K$ be a closed surface of Willmore type which is not null-homologous and satisfies $m_H(\Sigma) \geq 0$ as well as $\text{genus}(\Sigma) \leq \Lambda$. Then $\Sigma = \Sigma_\lambda$ for some $\lambda \in (0, \lambda_0)$. If $(M \setminus K, g)$ is conformally flat, one may take $\Lambda = \infty$.

Again, we expect that the conditions on $(M, g)$ can be weakened and that the assumption on the topology is redundant. Moreover, we expect that the assumption that $\Sigma$ is null-homologous can be replaced by a lower bound on the area. This condition is certainly necessary, as a neighborhood of a non-degenerate critical point of the scalar curvature is foliated by small spheres of Willmore type (see [LMS18]). However, so far, we cannot rule out that very outlying surfaces with large area might be of Willmore type, too.

We now describe the proofs of Theorem 1.2 and Theorem 1.4. Many of the difficulties arise because the Euler-Lagrange equation of the Willmore functional is a fourth order equation, while the constant mean curvature equation is of second order. Consequently, many useful tools in the theory of second order elliptic equations such as maximum and comparison principles are not available and thus, different techniques have to be developed. The central approach is to prove the existence of an area-prescribed maximizer of the Hawking mass and then to show that the maximizer is part of the foliation $\{\Sigma_\lambda\}$. Under the assumptions of Theorem 1.2, it is rather standard to see that a maximizing sequence $\Sigma_k$ must eventually consist of embedded spheres with positive Hawking mass. The main difficulty is then to exclude that the sequence approaches the obstacle $K$ (which would be problematic to proving regularity) or becomes outlying and potentially escapes to infinity (which would be detrimental to proving compactness). In either of these scenarios, the ratio $|\Sigma_k|^2/r_{\min}(\Sigma_k)$ must become large. The crucial observation is then that by integrating a suitable vector field with a singularity at the minimal point of $\Sigma_k$ and carefully analyzing the resulting equation, one finds that the ratio can only become large if the Hawking mass becomes negative, a contradiction. This argument is based on the first variational formula for the area functional and is inspired by a similar idea used by Simon in [Sim93]. In the choice of the singular vector field, the conformal structure of the Schwarzschild manifold is explicitly exploited. Moreover, the estimates for surfaces with small traceless part of the second fundamental by Müller and de Lellis (see [DLM+05, DLM06]) play an important part. Once the existence of a maximizer has been established, it remains to show that the uniqueness statement in Theorem 1.1 can be applied. To this end, we use integral curvature estimates in the spirit of Kuwert and Schätzle (see [KS01]) to show that the maximizing surface is strictly mean convex and sufficiently round. We then use a hidden divergence structure of the Einstein tensor, the so-called Pohozaev identity, to deduce that the Hawking mass of the maximizer must be close to the ADM-mass $m$. This identity has been used in [LMS11] in a similar context. Then, we can refine the estimate obtained from the first variational formula for the area functional to deduce that the maximizer must actually be centered enough for the uniqueness statement of Theorem 1.1 to hold. This completes the proof of Theorem 1.2 and we will obtain Theorem 1.4 as an easy by-product.

The rest of this paper is organized as follows. In section 2, we fix some notation and collect many useful properties of asymptotically flat manifolds which are $C^3$-close to Schwarzschild. In section 3, we show that closed surfaces of non-negative Hawking mass which avoid a large compact
set must be embedded and spherical. We then prove an estimate for the barycenter to show that a maximizing sequence of the Hawking mass must be homologous to a centered sphere and stay away from the obstacle \( K \). In section 4, we prove integral curvature estimates for surfaces of Willmore type which are sufficiently centered and satisfy a smallness assumption on the traceless part of the second fundamental form. In section 5, we turn these estimates into \( L^\infty \)-estimates of certain geometric quantities and show that spheres of Willmore type with non-negative Hawking mass must be mean convex, star shaped and as round as one could expect. We then proceed to prove Theorem 1.2 as well as Theorem 1.4.

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2. Preliminaries

Let \((M, g)\) be an an asymptotically flat Riemannian manifold which is \( C^3 \)-close to Schwarzschild with mass \( m > 0 \) and decay coefficient \( \eta > 0 \). More precisely, we assume that there is a compact set \( K \) such that \( M \setminus K \) is diffeomorphic to \( \mathbb{R}^3 \setminus B_\sigma(0) \) for some \( \sigma > m/2 \) and that the following estimate holds on all of \( M \):

\[
r^2 |g - gs| + r^3 |\nabla g - \nabla_S| + r^4 |Rc_g - Rc_S| + r^5 |\nabla_S Rc_g - \nabla_S Rc_S| \leq \eta.
\]  

(2)

Here, \( r \) denotes the radial function on the asymptotic chart \( \mathbb{R}^3 \setminus B_\sigma(0) \), \( \nabla \) the gradient of the ambient space and \( Rm_g \) and \( Rc_g \) the Riemann curvature tensor and the Ricci curvature of the ambient space, respectively. \( gs \) denotes the Schwarzschild metric with mass \( m \), which is defined by

\[
gs := \left(1 + \frac{m}{2r}\right)^4 ge = \phi^4 ge.
\]

The subscripts \( S \) and \( e \) indicate that the geometric quantity is computed with respect to the Schwarzschild and the Euclidean metric, respectively. On the other hand, we will usually omit the subscript \( g \). It can be seen that \( m \) and \( \eta \) are geometric invariants, that is, they do not depend on the chosen chart at infinity. Moreover, the definition of being \( C^3 \)-close to Schwarzschild is equivalent to the definition made in the introduction. Finally, we assume that \((M, g)\) has non-negative scalar curvature and that

\[
r^5 Sc_g \leq \eta.
\]

(3)

Such manifolds were called asymptotically Schwarzschild in [LMS11] and we will sometimes adopt this terminology. The Schwarzschild manifold is the model space and has vanishing scalar curvature while the Ricci curvature is given by

\[
Rc_S(\cdot, \cdot) = mr^{-3} \phi^{-2} (ge(\cdot, \cdot) - 3 ge(\partial_r, \cdot) ge(\partial_r, \cdot)).
\]

(4)

We consider an immersed closed surface \( \Sigma \subset M \) and denote its first fundamental form by \( \gamma \), its connection by \( \nabla \), its second fundamental form by \( A \), the traceless part by \( \bar{A} \), the mean curvature by \( h \) and the area element by \( d\mu \). Moreover, we denote the induced curvature by \( Rc^\Sigma \) and \( Rm^\Sigma \), respectively. \( \Sigma \) can also be regarded as an embedded surface in \((\mathbb{R}^3 \setminus B_\sigma(0), ge)\) or \((\mathbb{R}^3 \setminus B_\sigma(0), gs)\). We indicate the corresponding geometric quantities by the subscripts \( e \) and \( S \). If we want to emphasize the correspondence to \( g \), we sometimes use the subscript \( g \). We use the letter \( c \) for any constant that only depends on \( m, \eta \) in a non-decreasing way. The meaning of such a constant will be different in most of the following inequalities. If a constant has a geometric dependency, we will explicitly state it. We fix a chart at infinity and extend the radial parameter \( r \) in a smooth way to all of \( M \). We define \( r_{\min} \) and \( r_{\max} \) to be the minimal and maximal value of the radial function on \( \Sigma \), respectively. As all of our results concern surfaces which are contained in the asymptotic region, we will always assume that \( r_{\min} \geq R_0 \) for some positive constant \( R_0(\eta, m) \) which is to be
Theorem 2.1. summarizes their main results (Theorem 0.1 and Theorem 0.2 in [LMS11]), the uniqueness statement asymptotically Schwarzschild manifold is foliated by such surfaces. The following theorem summarizes their main results (Theorem 0.1 and Theorem 0.2 in [LMS11]), the uniqueness statement will be essential in the proof of Theorem 1.2.

**Theorem 2.1.** Let \((M,g)\) be an asymptotically Schwarzschild manifold with mass \(m\) and decay coefficient \(\eta < \eta_0\) for some \(\eta_0 > 0\) depending only on \(m\). Then there exists a constant \(\lambda_0 > 0\) and a compact set \(K\) depending only on \(m, \eta_0\) such that \(M \setminus K\) is foliated by embedded spheres of Willmore type \(\Sigma_\lambda\) where \(\lambda \in (0, \lambda_0)\). Moreover, there are constants \(\chi, \tau > 0\) which only depend on \(m, \eta_0\) such that any strictly mean convex sphere \(\Sigma \subset M \setminus K\) of Willmore type satisfying \(\tau_e \leq \hat{\tau}\) and \(R_e \leq \chi r^2_{\text{min}}\) belongs to the foliation.

The spheres \(\Sigma_\lambda\) from the previous theorem solve (6) with the same number \(\lambda\). These surfaces will appear at various points in the rest of this paper. We now collect some useful lemmas to study the Hawking mass in asymptotically Schwarzschild manifolds. Before we start, we need an estimate for integrals of negative powers of the radial function in the asymptotic chart. The following lemma is an easy consequence of the divergence theorem and dates back to [HY96].

**Lemma 2.2.** Let \(\Sigma\) be an orientable surface satisfying \(r_{\text{min}} \geq R_0\) for some \(R_0(\eta,m)\) sufficiently large. There holds

\[
\int_\Sigma r^{-3}d\mu \leq cr^{-1}_{\text{min}} |H|_{L^2(\Sigma)}.
\]

**Proof.** See for example Lemma 1.4 in [LMS11].

In the next lemma, various geometric quantities are compared when computed with respect to the different background metrics \(g, g_S\) and \(g_e\). To this end, we denote the Einstein tensor by \(G = \text{Rc} - \frac{1}{2} \text{Sc}\).

**Lemma 2.3.** Let \(\Sigma\) be an orientable surface satisfying \(r_{\text{min}} \geq R_0\) for some \(R_0(m, \eta)\) sufficiently large. There is a universal constant \(c\) such that

\[
\nu_S = \phi^{-2} \nu_e, \quad |\nu_S - \nu| \leq c m r^{-2},
\]

\[
\nabla_S = \nabla_e + \phi^{-1} \nabla_e \phi, \quad |\nabla_S - \nabla| \leq c m r^{-3},
\]

\[
H_S = \phi^{-2} H_e - 2 m r^{-2} \phi^{-3} \nu_e, \quad |H_S - H| \leq c m r^{-3} + c m r^{-2} |A|,
\]

\[
\dot{A}_S = \phi^2 \dot{A}_e, \quad |\dot{A}_S - \dot{A}| \leq c m r^{-3} + c m r^{-2} |A|,
\]

\[
|A_S - A_e| \leq c m r^{-2} + c m r^{-1} A_S, \quad |A_S - A| \leq c m r^{-3} + c m r^{-2} |A|,
\]
such that Lemma 2.2 is valid, this term can be estimated via

\[ \int H^2 d\mu = 16\pi (1 - \text{genus}(\Sigma)) + 4 \int \nabla G(\nu, \nu) d\mu + 2 \int |\hat{A}|^2 d\mu, \tag{7} \]

where \( G \) is the Einstein tensor. Comparing the Euclidean and Schwarzschild version of this equality, we find that the only differing term is the integral of the Einstein tensor. Choosing \( R_0 \) large enough such that Lemma 2.2 is valid, this term can be estimated via

\[ \left| \int G_S(\nu_S, \nu_S) d\mu_S \right| \leq cm \int r^{-3} d\mu_S \leq cm r^{-1} \int H^2 d\mu_S. \tag{8} \]

This implies the second estimate in the second set. The first estimate in the third set has been proven in [LMS11, Lemma 1.6]. Moreover, there holds

\[ \left| \int G(\nu, \nu) d\mu - \int G_S(\nu_S, \nu_S) d\mu_S \right| \leq cm \int r^{-4} d\mu \leq cm r^{-2} \int |H|^2 d\mu, \tag{9} \]

where we used (2) in the first step and Lemma 2.2 in the second step. Now, (7) and (9) imply

\[ \left| |H|^2 - |H_S|^2 \right| \leq cm r^{-2} (|H|^2 + |\hat{A}|^2 - |\hat{A}_S|^2) \leq cm r^{-2} (|H|^2 + |\hat{A}|^2) \leq cm r^{-2} (|H|^2 + cm \text{genus}(\Sigma)), \]

where we used (7) again to estimate \(|\hat{A}|^2_S\) in the last inequality. \(\square\)

**Remark.** It is easy to see that the second set holds for any conformally flat metric which is asymptotically Schwarzschild.

We will also need the following straightforward consequence of the previous lemma.

**Lemma 2.4.** Let \( \Sigma \) be an immersed surface satisfying \( r_{\text{min}} \geq R_0 \) for some constant \( R_0(m, \eta) > 0 \). Then the following estimates hold

\[ (1 - cr^{-1}_{\text{min}})|\Sigma| \leq |\Sigma| \leq (1 + cr^{-1}_{\text{min}})|\Sigma|, \]

\[ (1 - cr^{-1}_{\text{min}})|\Sigma|^\frac{1}{2} \leq 2\sqrt{\pi} R_e \leq (1 + cr^{-1}_{\text{min}})|\Sigma|^\frac{1}{2}. \]

In the next section, it will become clear that we primarily need to be concerned with topological spheres. Müller and de Lellis showed (see [DLM*05]) that in the Euclidean space, a surface \( \Sigma \) with small traceless part of the second fundamental form is \( W^{2,2}\)-close to a round sphere \( S \) and that there is a conformal parametrization mapping \( S \) onto \( \Sigma \). Using the conformal parametrization,
geometric quantities on $S$ and $\Sigma$ can be related. The corresponding quantities on $S$ will be indicated by a tilde. The exact statement of the result in [DLM+05, DLM06] is as follows.

**Lemma 2.5.** Let $\Sigma \subset \mathbb{R}^3$ be a surface satisfying $|\tilde{A}_e|_{L^2(\Sigma)} < 8\pi$. Then $\Sigma$ is a sphere and there exists a universal constant $c$ independent of $\Sigma$ and a conformal parametrization $\psi : S := S^2_{\eta,m}(R_e) \to \mathbb{R}^3$ such that

$$|\psi - \text{id}|_{L^2(S)} \leq cR_e^2 |\tilde{A}_e|_{L^2(\Sigma)} ,$$
$$|\nabla_e \psi - \nabla_e \text{id}|_{L^2(S)} \leq cR_e |\tilde{A}_e|_{L^2(\Sigma)} ,$$
$$|\nabla_e^2 \psi - \nabla_e^2 \text{id}|_{L^2(S)} \leq c|\tilde{A}_e|_{L^2(\Sigma)} ,$$
$$|\nu_e \circ \psi - \tilde{\nu}_e \circ \text{id}|_{L^2(S)} \leq cR_e |\tilde{A}_e|_{L^2(\Sigma)} ,$$
$$|\nabla_e \nu_e \circ \psi - \nabla_e \tilde{\nu}_e \circ \text{id}|_{L^2(S)} \leq c|\tilde{A}_e|_{L^2(\Sigma)} ,$$
$$|h^2 - 1|_{L^\infty(S)} \leq c|\tilde{A}_e|_{L^2(\Sigma)} ,$$

where $\text{id} : S \to \mathbb{R}^3$ is the identity, $h$ the conformal factor of $\psi$ and $\nu_e, \tilde{\nu}_e$ the outward normals of $\psi$ and $\text{id}$, respectively. Moreover, the Sobolev embedding theorem implies

$$|\psi - \text{id}|_{L^\infty(S)} \leq cR_e |\tilde{A}_e|_{L^2(\Sigma)}$$

and one easily obtains

$$|\tilde{A}_e - A_e|_{L^2(S)} \leq c |\tilde{A}_e|_{L^2(\Sigma)} .$$

Using the previous lemma, we can relate $R_e, r_{\min}$ and $r_{\max}$ under certain circumstances.

**Lemma 2.6.** Let $|\tilde{A}_e|_{L^2} < 8\pi$. There holds

$$(1 - \tau_e - c|\tilde{A}_e|_{L^2})R_e \leq r_{\min} \leq r_{\max} \leq (1 + \tau_e + c|\tilde{A}_e|_{L^2})R_e .$$

For any constant $0 < \tau_0 < 1$, there exist constants $M, \epsilon$ depending only on $\tau_0$ such that

$$M^{-1} R_e \leq r_{\min} \leq r_{\max} \leq MR_e ,$$

provided $|\tilde{A}_e|_{L^2} < \epsilon$ and $\tau_e \leq \tau_0$. Moreover, $M, \epsilon^{-1}$ are increasing in $\tau_0$.

**Proof.** All claims follow easily from the previous lemma and the definition of $\tau_e$. 

Finally, we need the Michael-Simon-Sobolev inequality which can, for instance, be found as Proposition 5.4 in [HY96].

**Lemma 2.7.** If $r_{\min} \geq R_0$ for some $R_0(\eta, m)$ sufficiently large, then any smooth function $f$ satisfies

$$\int_{\Sigma} |f|^2 d\mu \leq c \left( \int_{\Sigma} (|\nabla f| + |f H|) d\mu \right)^2 .$$

3. **Surfaces of non-negative Hawking mass**

The aim of this section is to prove several useful properties of surfaces which enjoy non-negative Hawking mass. Moreover, we will study the existence of an area constrained maximizer of the Hawking mass. If the exterior region is conformally flat, the existence of such a surface can be shown without further assumptions. Otherwise, we can only show that a maximizer exists amongst all surfaces with a certain bound on the genus, a technical condition which we expect to be removable. In any case, it turns out that the maximzer is an embedded sphere which is not null-homologous. We first show that a surface of non-negative Hawking mass is embedded and in particular orientable.

**Lemma 3.1.** Let $\Sigma$ be an immersed surface such that $r_{\min} \geq R_0$ for some $R_0 = R_0(\eta, m)$. If $\Sigma$ is not embedded, then $m_H(\Sigma) < 0$. 
Proof. This follows from a small modification of a classical inequality by Li and Yau, see [LY82]. For the convenience of the reader, we briefly describe the argument. If $\Sigma$ is not embedded, then its image in $M$ has a double point $p$. Let $x$ be the position vector field in the asymptotic chart. For any $\sigma > 0$, we consider the continuous vector field

$$X_\sigma := \frac{x - p}{g(x - p, x - p)_\sigma},$$

where $g(x - p, x - p)_\sigma = \max\{g(x - p, x - p), \sigma\}$. If $g(x - p, x - p) < \sigma$, a straightforward computation gives

$$\text{div } X_\sigma = \sigma^{-2}(2 + \mathcal{O}(|x - p|r^{-2})). \tag{10}$$

Likewise, if $g(x - p, x - p) > \sigma$, then

$$\text{div } X_\sigma = g(x - p, \nu)^2 g(x - p, x - p)^{-2} + \mathcal{O}(r^{-2}|x - p|^{-1}). \tag{11}$$

Let $f := g(x - p, \nu)g(x - p, x - p)^{-1}$ and $\Sigma_\sigma := \Sigma \cap \{g(x - p, x - p) > \sigma\}$. Then, from the divergence theorem, (10) and (11) it follows that

$$\int_{\Sigma} H g(X_\sigma, \nu) = \int_{\Sigma} \text{div } X_\sigma d\mu = \sigma^{-2} \int_{\Sigma_\sigma} (2 + \mathcal{O}(|x - p|r^{-2})) d\mu + \int_{\Sigma \setminus \Sigma_\sigma} f^2 + \mathcal{O}(r^{-2}|x - p|^{-1}) d\mu.$$

Letting $\sigma$ to zero, noting that $g(X_\sigma, \nu) = f$ for $g(x - p, x - p) > \sigma$ and applying Young's inequality $Hf \leq \frac{1}{4}H^2 + f^2$ we infer

$$W(\Sigma) \geq 8\pi + \int_{\Sigma} \mathcal{O}(r^{-2}|x - p|^{-1}) d\mu. \tag{12}$$

Here, we used that $p$ is a double point. Similarly as before, one can apply the divergence theorem to

$$Y = \frac{x - p}{g(x - p, x - p)^2} r^{-2}$$

and find that

$$\int_{\Sigma} |x - p|^{-1}r^{-2} d\mu \leq c\min r^{-1} \int_{\Sigma} H^2 d\mu + c \int_{\Sigma} r^{-3} d\mu,$$

provided $R_0$ is sufficiently large. Estimating the third term with Lemma 2.2 and inserting into (12) we find that $W(\Sigma) > 4\pi$, provided $R_0$ is sufficiently large. In view of (5), this completes the proof. 

We proceed to show that surfaces of non-negative Hawking mass must be spheres or have a large genus, if $(M, g)$ is not conformally flat in the asymptotic region. We expect that the latter case does not occur.

**Lemma 3.2.** Let $\Sigma$ be an immersed surface with $m_H(\Sigma) \geq 0$ and $r_{\min} \geq R_0$. Moreover, let genus$(\Sigma) \leq \theta R_0^2$ for some constant $\theta(\eta, m)$ if the asymptotic region $M \setminus \{r \leq R_0\}$ is not conformally flat. Then $\Sigma$ is an embedded sphere which satisfies $|A|_{L^2(\Sigma)}^2 \leq c_r^{\min} \leq cR_0^{-1}.$

**Proof.** First, let $(M, g)$ be conformally flat. As $m_H(\Sigma) \geq 0$ there holds $W(\Sigma) \leq 4\pi$. According to Lemma 2.3 and the remark below we then infer $W_\nu(\Sigma) \leq 5\pi$ provided $R_0$ is sufficiently large. Here, $W_\nu$ denotes the Euclidean Willmore energy. If $\Sigma$ is not a sphere, then the proof of the Willmore conjecture by Marquez and Neves, see [MN14], implies that $W_\nu(\Sigma) \geq 2\pi^2$. Hence, $\Sigma$ must be an embedded sphere. If $(M, g)$ is not conformally flat we first note that the previous lemma implies that $\Sigma$ is orientable. If $\Sigma$ satisfies genus$(\Sigma) \leq \theta R_0^2$, we use the last inequality in Lemma 2.3 to deduce that $W_\Sigma(\Sigma) \leq 4\pi + c\eta R_0^{-1} + c\eta \theta \leq 5\pi$, provided $R_0$ is sufficiently large and $\theta = \theta(m, \eta)$ is sufficiently small. As before, this implies $W_\nu(\Sigma) \leq 6\pi$ provided $R_0$ is sufficiently large and thus, $\Sigma$ must be an embedded sphere. Since there also holds $W(\Sigma) \leq 4\pi$, the integrated Gauss equation (7) and Lemma 2.2 imply

$$\int_{\Sigma} |A|^2 d\mu \leq 2(4W(\Sigma) - 16\pi) + c_{\min}^{-1} \leq c_{\min}^{-1} \leq cR_0^{-1}.$$
Before we proceed, we calculate the integral of the Einstein tensor in the integrated Gauss equation (7). This will turn out to be useful at various points. The main idea is that the Einstein tensor has a hidden divergence structure and that the position vector field is a conformal killing vector field in Schwarzschild. The divergence structure implies the so-called Pohozaev identity which was used in a similar context in [LMS11].

**Lemma 3.3.** Let $\Sigma$ be a sphere satisfying $|\hat{\mathcal{A}}_x|^{2}_{L^2(\Sigma)} < 8\pi$ and the constraints of Lemma 2.4. If $\Sigma$ is homologous to a centered sphere, there holds

$$
\int_{\Sigma} G(\nu, \nu) d\mu = -16\pi^2 m |\Sigma|^{1/4} + \mathcal{O}(|\hat{\mathcal{A}}_x|^{4}_{L^2(\Sigma)} r_{\min}^{-1} + r_{\min}^{-2} + \eta r_{\min}^{-2}).
$$

If $\Sigma$ is null-homologous one has

$$
\int_{\Sigma} G(\nu, \nu) d\mu = \mathcal{O}(|\hat{\mathcal{A}}_x|^{4}_{L^2(\Sigma)} r_{\min}^{-1} r_{\min}^{-2}).
$$

**Proof.** We assume that $\eta = 0$, the case $\eta \neq 0$ then easily follows from Lemma 2.3, Lemma 2.2 and the fact that the assumptions imply a bound on $|H|^2_{L^2(\Sigma)}$. The Pohozaev identity states that for any smooth vector field $X$ and any bounded domain $\Omega$ with smooth boundary one has

$$
\frac{1}{2} \int_{\Omega} G \cdot DX d\mu - \frac{1}{6} \int_{\Omega} \text{Sc} \text{div} X = \int_{\partial \Omega} G(X, \nu) d\mu,
$$

see [LMS11, 5.12]. Here, $DX = L_X g - 1/3 \text{tr}(LX)g$ is the conformal Killing operator, where $L$ the Lie derivative. The position vector $x$ is a conformal killing field in the Euclidean and Schwarzschild space. Moreover, using (4) and Lemma 2.3 one easily verifies that

$$
\int_{S_{\mu}(0)} G_S(x, \nu_S) d\mu_S = -8\pi m.
$$

If $\Sigma$ is homologous to a centered sphere we then take $\Omega$ to be the domain bounded by $\Sigma$ and a large coordinate sphere and deduce

$$
\int_{\Sigma} G_S(x, \nu_S) d\mu_S = -8\pi m.
$$

(13)

On the other hand, one has

$$
\int_{\Sigma} G_S(x, \nu_S) d\mu_S = 0
$$

(14)

if $\Sigma$ is null-homologous. If $X = a_e$ we deduce in any case that

$$
\int_{\Sigma} G_S(a_e, \nu_S) d\mu_S = 0.
$$

(15)

This is clear if $\Sigma$ is null-homologous, otherwise we take $\Omega$ to be a domain which is bounded by $\Sigma$ and a centered coordinate sphere with radius tending to infinity. Using Lemma 2.5, $|\Sigma|^{1/2} \geq c R_e$, see Lemma 2.4, and $\tilde{x} = R_e \tilde{\nu}_e + a_e$ (which holds for any round sphere) we obtain

$$
|\Sigma|^{-\frac{1}{2}} \int_{\Sigma} G_S(x, \nu_S) d\mu_S = |\Sigma|^{-\frac{1}{2}} \int_{\Sigma} G_S(\tilde{x}, \nu_S) d\mu_S + \mathcal{O}(|\hat{\mathcal{A}}_e|^{4}_{L^2(\Sigma)} r_{\min}^{-1})
$$

$$
= |\Sigma|^{-\frac{1}{2}} \int_{\Sigma} G_S(R_e \tilde{\nu}_e + a_e, \nu_S) d\mu_S + \mathcal{O}(|\hat{\mathcal{A}}_e|^{4}_{L^2(\Sigma)} r_{\min}^{-1})
$$

$$
= R_e |\Sigma|^{-\frac{1}{2}} \int_{\Sigma} G_S(\tilde{\nu}_e, \nu_S) d\mu_S + \mathcal{O}(|\hat{\mathcal{A}}_e|^{4}_{L^2(\Sigma)} r_{\min}^{-1})
$$

$$
= R_e |\Sigma|^{-\frac{1}{2}} \int_{\Sigma} G_S(\nu_S \phi^2, \nu_S) d\mu_S + \mathcal{O}(|\hat{\mathcal{A}}_e|^{4}_{L^2(\Sigma)} r_{\min}^{-1}).
$$

(16)
In the first inequality, we replaced $x$ by $\tilde{x}$ and estimated the error by
\[
|\Sigma|^{\frac{1}{2}} \int_{\Sigma} |G_S||x - \tilde{x}|d\mu_S \leq |\tilde{A}_e|_{L^2(\Sigma)} |\Sigma|^{\frac{1}{2}} R_e \int_{\Sigma} r^{-3}d\mu_S \leq c|\tilde{A}_e|_{L^2(\Sigma)} r_{\min}^{-1}
\]
using Lemma 2.5, Lemma 2.2 and the fact that the assumptions imply a bound on $|H|^{2}_{L^2(\Sigma)}$. In the third inequality we used (15) and in the fourth inequality we used $\nu_S = \phi^2 \nu_e$. If $\Sigma$ is null-homologous we proceed to estimate $|\phi^2 - 1| \leq cr^{-1}$ as well as $|R_e|\Sigma|^{-\frac{1}{2}} - (2\sqrt{\pi})^{-1} \leq cr_{\min}^{-1}$, see Lemma 2.4. Combining this with Lemma 2.2 gives
\[
\frac{2\sqrt{\pi} R_e}{|\Sigma|^{\frac{1}{2}}} \int_{\Sigma} G_S(\nu_S \phi^2, \nu_S)d\mu_S - \int_{\Sigma} G_S(\nu_S, \nu_S)d\mu_S \leq c^{-1}_{\min} \int_{\Sigma} r^{-3}d\mu_S + \int_{\Sigma} r^{-4}d\mu_S \leq cr_{\min}^{-2}.
\]
Then the claim follows from (14) and (16). If $\Sigma$ is homologous to a centered sphere there holds $\tau_e < 1$ and with Lemma 2.6 we can estimate
\[
|\phi^2(r) - \phi^2(R_e)| \leq \frac{c(R_e - r)}{R_e r} \leq \frac{c r_e + c|\tilde{A}_e|_{L^2(\Sigma)}}{r}.
\]
Similarly, one can show that
\[
||\Sigma||^{\frac{1}{2}} R_e \phi^2(R_e) - (2\sqrt{\pi})^{-1} \leq cr_{\min}^{-1}(\tau_e + |\tilde{A}_e|_{L^2(\Sigma)}).
\]
As before, combining this with Lemma 2.2 to estimate the error terms the claim then follows from (13) and (16) (recall that $|\tilde{A}_e|_{L^2(\Sigma)} = |\tilde{A}_S|_{L^2(\Sigma)}$ by conformal invariance). \qed

In the next lemma, we prove an a-priori estimate for the centering parameter $\tau_e$ of a spherical surface of non-negative Hawking mass which is homologous to one of the leaves of the foliation from Theorem 2.1. This will be a central ingredient in the proofs of Theorem 1.2 and Theorem 1.4. The argument uses the first variational formula of the area functional and is motivated by the first section in [Sim93] and the proof of Theorem 1.4 in [Sch18].

**Lemma 3.4.** Let $\Sigma$ be an embedded sphere homologous to $\Sigma_{\lambda_0}$ with $m_H(\Sigma) \geq 0$ and $r_{\min} \geq R_0$ where $R_0(m, \eta)$ is sufficiently large. Then $\Sigma$ satisfies $\tau_e \leq 1 - \iota$ for some $\iota > 0$ which only depends on $m, \eta$.

**Proof.** We first assume that $\eta = 0$. We suppose for contradiction that $1 > \tau_e \geq 1 - \iota$ for some $\iota > 0$ to be chosen. By the previous lemma, there holds
\[
|\tilde{A}_e|_{L^2(\Sigma)} = |\tilde{A}_S|_{L^2(\Sigma)} < cr_{\min}^{-1}
\]
and we may assume that $R_0 \geq R_0^{\frac{1}{\iota}} \geq 1/\iota$. For the approximate sphere, there holds $r_{\min} = R_e(1 - \tau_e) \leq \iota R_e$. Using Lemma 2.5 and (17), we infer
\[
r_{\min} \leq r_{\min}^0 + c R_e |\tilde{A}_S|_{L^2(\Sigma)} \leq \iota R_e + c R_e r_{\min}^{-\frac{1}{2}} \leq c R_e.
\]
Here, we used that $r_{\min}^{0} \geq R_0^{\frac{1}{\iota}} \geq 1/\iota$. We thus may assume that $R_e^{-1} \leq c r_{\min}^{-1}$. Next, we consider a point $p \in \Sigma$, fix a positive number $\sigma$ and define the vector field
\[
X_{\sigma} := \phi^{\sigma}(r) \frac{x - p}{|x - p|^2},
\]
where $|x - p|_{\sigma} := \max\{|x - p|, \sigma\}$. Let $e_i$ be an orthonormal base of $\Sigma$ with respect to the Euclidean metric and choose Euclidean coordinates $\partial_k$ with $\partial_i = e_i$ for $i \in \{1, 2\}$. Now,
\[
ge_{e}(\nabla_{e_i}(x - p), e_i) = 2 - \Gamma^{e}_{ij}(x^l - p^l) \partial_k \cdot e_i = 2 - 2\phi^{-1} \partial_j \phi \delta^e_i(x^l - p^l) = 2 - 4\phi^{-1} \partial_j \phi \delta^e_i \cdot (x - p),
\]
where $\Gamma^e_{ij}$ denote the Christoffel symbols of $g_S$. We also calculate
\[
\partial_i \phi^{A}(x - p) \cdot e_i = 4\phi^{A} \partial_i \phi \cdot (x - p) + \partial_r.
\]
Hence, if $|x - p| < \sigma$, there holds
\[
\text{div}_{\Sigma} X_{\sigma} = \sigma^{-2} (2\phi^{A} - 4\phi^{A} \partial_r) \cdot (x - p) + \sigma^{-1}.
\]
On the other hand, if $|x - p| > \sigma$, then
\[
\partial_i |x - p|^{-2} (x - p) \cdot e_i = -2|x - p|^2 |x - p|^{-4} = -2|x - p|^{-2} + 2(x - p)^2 |x - p|^{-4}.
\]
Denoting $Z := (x - p)/|x - p|^2$ this gives
\[
\text{div}_\Sigma X_\sigma = 2\phi^4 Z_\perp^2 - 4\phi^3 \partial_r \phi \partial_r \cdot Z_\perp.
\] (20)
As in the proof of Lemma 3.1 the divergence theorem, (19), (20) and letting $\sigma \to 0$ yields
\[
2\pi \phi^8(p) + \int_\Sigma (2\phi^4 |Z_\perp|^2 - 4\phi^3 \partial_r \phi \partial_r \cdot Z_\perp) d\mu_S = -\int_\Sigma \phi^6 g_e(Z_\perp, \nu_e) H_S d\mu_S,
\] (21)
where we used that $\Sigma$ is embedded by Lemma 3.1. Next,
\[
-\int_\Sigma \phi^6 Z_\perp \cdot \nu_e H_S d\mu_S = \frac{1}{8} \int_\Sigma \phi^8 H_S^2 d\mu_S + 2 \int_\Sigma \phi^4 |Z_\perp|^2 d\mu_S - 2 \int_\Sigma \left( \frac{\phi^4 H_S}{4} + (Z_\perp \cap \nu_e \phi^2) \right)^2 d\mu_S.
\] (22)
In order to proceed, we note that $\phi^4 H_S/4 = \phi^2 H_e/4 + \phi \partial_r \phi \partial_r \cdot \nu_e$. Using this identity and discarding the negative terms we obtain
\[
-2 \int_\Sigma \left( \frac{\phi^4 H_S}{4} + (Z_\perp \cap \nu_e \phi^2) \right)^2 d\mu_S \leq \int_\Sigma \phi^3 H_e \partial_r \phi \partial_r \cdot \nu_e d\mu_S - 4 \int_\Sigma \phi^3 \partial_r \phi Z_\perp \cdot \partial_r d\mu_S.
\] (23)
Combining (21), (22) and (23) we arrive at the estimate
\[
2\pi \phi^8(p) + \int_\Sigma \phi^7 H_e \partial_r \phi \partial_r \cdot \nu_e d\mu_S \leq \frac{1}{8} \int_\Sigma \phi^8 H_S^2 d\mu_S.
\] (24)
Now, we would like to estimate the second term on the left hand side. Since $|\partial_r \phi| \leq cr^{-2}$ and $|\phi^7 - 1| \leq cr^{-1}$ we can replace $\phi^7$ by 1 and use Hölder’s inequality as well as Lemma 2.2 to estimate the resulting error term by $|H_e|_{L^2(\Sigma)} |r^{-3}|_{L^2(\Sigma)} \leq cr^{-2}_{\min}$. Likewise, we use Hölder’s inequality and Lemma 2.5 to replace $H_e$ by $\tilde{H}_e = 2R_e^{-1}$, the resulting error term can then be estimated by $|\tilde{H}_e|_{L^2(\Sigma)} r^{-1}_{\min} \leq cr^{-3}_{\min}$. Here, we also used (17). As $\partial_r \phi = mr^{-2}$, we are then left to compute
\[
-mr_e^{-1} \int_\Sigma r^{-2} \partial_r \cdot \nu_e d\mu_e = -mr_e^{-1} \int_{S_R(0)} r^{-2} \partial_r \cdot \nu_e = -4\pi mr_e^{-1},
\]
where $R$ is chosen such that $S_R(0) \cap \Sigma = \emptyset$. Here, we used the divergence theorem and $\text{div}_e r^{-2} \partial_r = 0$. Inserting these three estimates into (24) we obtain
\[
16\pi - cr^{-2}_{\min} - cr_e^{-1} \leq \int_\Sigma H_S^2 \frac{\phi^8(q)}{\phi^8(p)} d\mu_S(q).
\] (25)
Now we chose $p \in \Sigma$ such that $r(p) = r_{\min}$. Let $\tilde{x}$ be the embedding of the approximating sphere and $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 := \{ x | x \in \Sigma, |x| \leq R_e \}$ and $\Sigma_2 := \{ x | x \in \Sigma, |x| > R_e \}$. Clearly, Lemma 2.5 and the estimate (17) imply $|x| \geq R_e/2$ for $x \in \Sigma_2$ if $R_0$ is sufficiently large. Using $R_e^{-1} \leq cr_{\min}^{-1}$, it is easy to see that for such $x$ we have $\phi(x)/\phi(p) \leq 1 - m(4r_{\min})^{-1}$, provided $R_0$ is sufficiently large and $\iota$ is chosen to be small. Hence, we obtain that $\phi^\theta(x)/\phi^\theta(p) \leq 1 - mr_{\min}^{-1}$ for $x \in \Sigma_2$, provided $R_0$ is sufficiently large. This yields
\[
16\pi - cr^{-2}_{\min} - cr_e^{-1} \leq \int_{\Sigma_2} H_S^2 d\mu_S - mr_{\min}^{-1} \int_{\Sigma_2} H_S^2 d\mu_S.
\] (26)
Finally, using Lemma 2.3 to replace $|H_S|_{L^2(\Sigma)}$ by $|H_e|_{L^2(\Sigma)}$ and Lemma 2.5 as well as (17) to express the integral in terms of the approximating sphere we obtain
\[
r_{\min}^{-1} \int_{\Sigma_2} H_S^2 d\mu_S \geq r_{\min}^{-1} \int_{\Sigma_2} \tilde{H}_e^2 d\mu_e - cr_{\min}^{-2} \geq \kappa(\iota) r_{\min}^{-1} - cr_{\min}^{-2},
\]
in the last step we used the fact that if $R_e \geq r_{\min}$, then the area of the part of the approximating sphere lying outside of the ball $B_{R_e}$ is bounded from below by $\kappa(\iota) R_e^2$ for some positive constant.
\( \kappa(\iota) \) which only depends on \( \iota \) and also decreases in \( \iota \). Combining this with (26) and using the estimates 

\[-R^{-1}_e \geq -\tau^{-1}_{\min}, \quad \text{as well as} \quad -r^{-1}_{\min} \geq -R^{-1}_0 \tau^{-1}_{\min} \geq -\iota r^{-1}_{\min}, \]

we obtain the estimate

\[
16\pi - \int_\Sigma H^2_S d\mu_S < -(m\kappa(\iota) - c)r^{-1}_{\min}. \tag{27}
\]

As \( m\kappa(\iota) - c < 0 \) for \( \iota > 0 \) sufficiently small, this completes the proof if \( \eta = 0 \). If \( \eta > 0 \), then the last inequality in Lemma 2.3 implies \(|H|_{L^2(\Sigma)} - |H_S|_{L^2(\Sigma)}| \leq cr^{-2}_{\min}\) as \( \text{genus}(\Sigma) = 0 \). Hence, the claim follows from (27) if \( R_0 \) is sufficiently large. \( \Box \)

**Remark.** Revisiting the proof of the previous lemma, it is easy to see that for the conclusion to hold, it suffices that \( \Sigma \) is a spherical surface which satisfies the estimates (17) as well as (18).

Before we can show the main result of this section, we need another lemma to ensure that spheres which are null-homologous do not maximize the Hawking mass. We conjecture that all null-homologous surfaces of sufficiently large area have negative Hawking mass. Unfortunately, this is not implied by the proof of the next lemma if the surface is sufficiently outlying.

**Lemma 3.5.** Let \( \Sigma \) be an orientable surface satisfying \( r_{\min} \geq R_0 \). If \( \Sigma \) is null-homologous, there holds \( m_H(\Sigma) \leq m/4 \), provided \( R_0(m, \eta) \) is sufficiently large.

**Proof.** Let \( \Sigma \) be a null-homologous surface with positive Hawking mass. From (7) and Lemma 2.2 it is easy to see that \( \Sigma \) must satisfy (17). From the remark below Lemma 3.4 it then follows that there is a small number \( \kappa > 0 \) and some \( R_0 > 0 \) such that \( r_{\min} \geq R_0 \) implies that \( \Sigma \) must satisfy \( r_{\min} \geq \kappa R_e \) for \( \Sigma \) to have positive Hawking mass. But then (7), Lemma 3.3 as well as (17) imply that

\[ m_H(\Sigma) \leq c|\Sigma| \tau^{-\frac{3}{2}}r^{-\frac{5}{2}}_{\min} \leq cr^{-1}_{\min} \frac{R^2_e}{r^2_{\min}} \leq cr^{-1}_{\min} \kappa^{-\frac{1}{2}} \leq m/4, \]

provided \( R_0 \) is chosen sufficiently large. This completes the proof. \( \Box \)

The next lemma is the main result of this section. We show that given a surface avoiding a large obstacle there is a surface of Willmore type with the same area and bigger Hawking mass avoiding a smaller obstacle. The main difficulty is to show that the minimizing sequence avoids the smaller obstacle so that one can conclude that the limit is a weak solution of (6). We will see in section 5 that the surface of Willmore type in fact avoids the bigger obstacle as well. This is because the obstacle acts as a barrier.

**Lemma 3.6.** Let \( \bar{K} \supset \{ r \geq R_0 \} \) be a compact set such that \( M \setminus \bar{K} \) is foliated by surfaces of Willmore type. Then there exists a compact set \( \bar{K} \supset K \) such that \( M \setminus K \) is also foliated by surfaces of Willmore type and the following property holds: Let \( \Sigma \) be a surface in \( M \setminus K \) which is not null-homologous and satisfies \( \text{genus} \leq \theta R^2_0 \) if \( (M, g) \) is not conformally flat, where \( \theta \) is the constant from Lemma 3.2. Then there exists an embedded sphere \( \Sigma^* \subset M \setminus K \) of Willmore type satisfying \( |\Sigma| = |\Sigma^*| \) as well as \( m_H(\Sigma) \leq m_H(\Sigma^*) \).

**Proof.** There holds \( \hat{K} \subset \{ r \leq \hat{\Lambda} R_0 \} \) for some \( \hat{\Lambda} \geq 1 \) and we chose \( \hat{K} \supset \bar{K} \) to be a compact set which is foliated by surfaces of Willmore type and contains \( \{ r \leq \Lambda R_0 \} \) for some \( \Lambda > 1 \) to be determined. Let \( \Sigma \subset M \setminus K \) be a surface which is not null-homologous and satisfies the genus bound if \( M \) is not asymptotically flat. Since the round spheres are isoperimetric in the Euclidean sense, there holds \( R_e \geq \Lambda R_0 \). It is easy to see that if \( \Lambda \) is chosen sufficiently large, there is a centered sphere \( S \) contained in \( M \setminus \hat{K} \) such that \( |S| = |\Sigma| \). Furthermore, if \( R_0 \) is sufficiently large we may assume that \( m_H(S) \geq m/2 \) (this is an easy consequence of Lemma 2.3 and the fact that the centered spheres in Schwarzschild have Hawking mass equal to \( m \)). Hence, we may assume that \( m_H(\Sigma) \geq m/2 \). Now, let \( \Sigma_k \subset M \setminus \hat{K} \) be a minimizing sequence for the Willmore energy of surfaces satisfying the genus bound, \( |\Sigma_k| = |\Sigma| \) and \( m_H(\Sigma_k) \geq m/2 \). By the previous lemma, all \( \Sigma_k \) are homologous to \( \Sigma \) and by Lemma 3.1 and Lemma 3.2 every \( \Sigma_k \) is an embedded sphere which satisfies \( |\hat{\Lambda}|_{L^2(\Sigma_k)} \leq c R^{-1}_0 \). Moreover, by Lemma 3.4 every \( \Sigma_k \) satisfies \( \tau_e < 1 - \iota \) for some \( \iota > 0 \).
Finally, since $|\Sigma_k| = |\Sigma|$ we may assume that $R_e(\Sigma_k) \geq \Lambda / 2R_0$. It then follows from Lemma 2.6 that $r_{\min}(\Sigma_k) \geq 2\Lambda R_0$ for all $k \in \mathbb{N}$, provided $\Lambda$ is chosen sufficiently large. We have already seen that the $\Sigma_k$ satisfy uniform estimates on $\tau_e(\Sigma_k), |\hat{\mathbf{A}}|_{L^2(\Sigma_k)}$ and the area bound implies a uniform estimate on $R_e(\Sigma_k)$. Then, Lemma 2.5 implies uniform $W^{2,2}$-estimates and exactly in the same way as in [LM13, section 4] the $\Sigma_k$ converge weakly in $W^{2,2}$ to a $W^{2,2}$-sphere $\Sigma^*$ with $|\Sigma^*| = |\Sigma|$. By lower semi-continuity of the Willmore energy (see Proposition 3.1 in [LM13]), there holds $m_H(\Sigma) \geq m_H(\Sigma_k)$ for all $k \in \mathbb{N}$. As $W^{2,2}$-convergence implies $C^0$-convergence, there also holds $r_{\min}(\Sigma^*) \geq 2\Lambda R_0$, so $\Sigma^*$ avoids the obstacle $\tilde{K}$ and consequently minimizes the Willmore energy amongst all area-preserving variations. Now, regularity of $\Sigma^*$ can be proven almost completely verbatim to [LM13, section 4] which is a variation of the arguments in [KMS14] and [Sch12]. \qed

4. Integral Curvature Estimates

The aim of this section is to prove integral curvature estimates for surfaces of Willmore type which are sufficiently round. Throughout this section, we only assume that $\Sigma$ is spherical, that $R_e, r_{\min}, r_{\max}$ are comparable, that $r_{\min} \geq R_0$ and that $\int_{\Sigma} |A|^2 d\mu$ is uniformly bounded (this is automatic for spheres with non-negative Hawking mass). We will then state all estimates in terms of $R_e$. As before, we denote the connection of $\Sigma$ by $\nabla$ and the connection of the ambient space by $\nabla$.

In order to obtain estimates for the different components of the second fundamental form, we follow the approach of [KS01, section 2]. However, we need to take the effect of the ambient curvature into account. We need the following lemma which follows from a straight-forward computation:

**Lemma 4.1.** Let $\psi \in \Omega^k(\Sigma), p \in \Sigma$ and $e_1, e_2$ be an orthonormal frame of $\Sigma$ at $p$. For vector fields $X_i \in \{e_1, e_2\}$ we have

$$ \langle \nabla \nabla^* \psi - \nabla^* \nabla \psi \rangle (X_1, \ldots, X_k) = \psi(R^\Sigma(X_1, e_i, e_j), X_2, \ldots, X_k) $$

$$ + \sum_{j=2}^k \psi(e_i, \ldots, X_j, \ldots, X_k) - \nabla^* T_\psi(X_1, \ldots, X_k), $$

where $\nabla^* = - \text{div}$ is the adjoint of $\nabla$ and

$$ T_\psi(X_0, \ldots, X_k) = \nabla_{X_0} \psi(X_1, \ldots, X_k) - \nabla_{X_i} \psi(X_0, \ldots, X_k). $$

We would now like to use the previous lemma to express certain geometric quantities in terms of the Willmore operator $W$ defined by $W = \Delta H + H \text{Rc}(\nu, \nu) + H|\hat{\mathbf{A}}|^2$. To this end, we first need the following lemma.

**Lemma 4.2.** The following identities hold:

$$ \Delta \hat{\mathbf{A}} = (\nabla^2 H - \frac{1}{2} \gamma \Delta H) + \frac{1}{2} H^2 \hat{\mathbf{A}} + \hat{\mathbf{A}} \ast \hat{\mathbf{A}} \ast \hat{\mathbf{A}} + \text{Rm} \ast \hat{\mathbf{A}} + \nabla \text{Rm} \ast 1, \tag{28} $$

$$ \nabla^* \nabla^2 H = - \nabla(\Delta H) + \text{Rm} \ast \nabla H + \hat{\mathbf{A}} \ast \hat{\mathbf{A}} \ast \nabla \hat{\mathbf{A}} + \text{Rm} \ast \hat{\mathbf{A}} \ast \hat{\mathbf{A}} - \frac{1}{4} H^2 \nabla H, \tag{29} $$

$$ \nabla^* \nabla^2 H = - \nabla(\Delta H) + \text{Rm} \ast \nabla H + \hat{\mathbf{A}} \ast \hat{\mathbf{A}} \ast \nabla H - \frac{1}{4} H^2 \nabla H, \tag{30} $$

$$ \nabla^* (\nabla^2 \hat{\mathbf{A}}) = \nabla(\nabla^* \hat{\mathbf{A}}) + \nabla \hat{\mathbf{A}} \ast \hat{\mathbf{A}} \ast \hat{\mathbf{A}} + \text{Rm} \ast \hat{\mathbf{A}} + \text{Rm} \ast \hat{\mathbf{A}} \ast \hat{\mathbf{A}} \ast \nabla \hat{\mathbf{A}}. \tag{31} $$

**Proof.** With the convention

$$ A_{ij} = A(X_i, X_j) = -g(\nabla_{X_i} X_j, \nu) $$

the orthonormal frame satisfies

$$ \nabla X_i \cdot X_j = \nabla X_i \cdot X_j + A_{ij} \nu = A_{ij} \nu. $$

Chosing $\psi = A$ in Lemma 4.1 we obtain

$$ T_A(X_1, X_2, X_3) = X_1(g(\nabla_{X_1} X_3, \nu)) - X_2(g(\nabla_{X_3} X_3, \nu)) = \text{Rm}(X_1, X_2, X_3, \nu) $$
as $\nabla \nu$ is tangential. We clearly have $\nabla^* T(X_1, X_2) = X_i (\text{Rm}(X_1, X_1, X_2, \nu))$ which implies
\[ \nabla^* T_A = \nabla \text{Rm} * 1 + \text{Rm} * A. \]

A straightforward calculation gives
\[ \nabla^* A = -\nabla H + \text{Rm} * 1. \]

Evaluated at $(X_1, X_2)$, this yields (with slight abuse of notation)
\[ \Delta A = \nabla^2 H + A(\text{Rm} \Sigma(X_1, e_i, e_i), X_2) + A(e_i, \text{Rm} \Sigma(X_1, e_i, X_2)) + \nabla \text{Rm} * 1 + \text{Rm} * A. \quad (32) \]

If we furthermore assume that the $e_i$ are principal directions, we obtain
\[ A(\text{Rm} \Sigma(X_1, e_i, e_i), X_2) + A(e_i, \text{Rm} \Sigma(X_1, e_i, X_2)) = \text{Rm} \Sigma(X_1, e_i, e_i) A(X_2, X_2) + \text{Rm} \Sigma(X_1, e_i, 2X_2) A(e_i, e_i). \]

If $X_1, X_2$ are distinct, both terms vanish. Otherwise we can assume that $X_1 = X_2 = e_1$. Then, using the Gauss equation
\[ \text{Rm} \Sigma(a, b, c, d) = \text{Rm}(a, b, c, d) + A(a, d) A(b, c) - A(a, c) A(b, d) \]
the right hand side becomes (again with abuse of notation)
\[ \text{Rm} \Sigma(e_1, e_2, e_1, e_1)(A(e_1, e_1) - A(e_2, e_2)) = \text{Rm} * A + A_{11} A_{22}(A_{11} - A_{22}). \]
\[ = \text{Rm} * A + (\hat{A}_{11} + \frac{1}{2} H)(\hat{A}_{22} + \frac{1}{2} H)(\hat{A}_{11} - \hat{A}_{22}) \]
\[ = \text{Rm} * A - 2(\hat{A}_{11} + \frac{1}{2} H)(\hat{A}_{11} - \frac{1}{2} H) \hat{A}_{11}, \]
where we used that $\hat{A}$ is trace free. Combining this with (32) clearly implies
\[ \Delta A = \nabla^2 H + \frac{1}{2} H^2 \hat{A} + \hat{A} * \hat{A} * \hat{A} + \text{Rm} * A + \nabla \text{Rm} * 1. \]

As $\nabla \gamma = 0$, there holds $\Delta (\gamma H) = \gamma \Delta H$ and we obtain the first claim. Choosing $\psi = \nabla H$ we find $T_{\nabla H} = 0$ by the symmetry of second derivatives and evaluating at $X_1 = e_1$ we obtain (again, with abuse of notation)
\[ \nabla^* \nabla^2 H = \nabla \nabla^* \nabla H - \text{Rm} \Sigma(X_1, e_i, e_i, H) \nabla H(e_j) \]
\[ = \nabla \nabla^* \nabla H - \text{Rm} \Sigma(e_1, e_2, e_1) e_1(H) \]
\[ = -\nabla (\Delta H) + \text{Rm} * \nabla H + (\hat{A}_{11}^2 - \frac{1}{4} H^2) e_1(H) \]
\[ = -\nabla (\Delta H) + \text{Rm} * \nabla H + \hat{A} * \hat{A} * \nabla \hat{A} + \text{Rm} * \hat{A} * \hat{A} - \frac{1}{4} H^2 \nabla H, \]
as $\nabla^* \hat{A} = -\frac{1}{2} \nabla H + \text{Rm} * 1$. This implies the second and third claim. Finally, if $\psi = \nabla \hat{A}$, then at $(X_1, X_2, X_3, X_4)$ we have
\[ T_{\nabla \hat{A}} = \hat{A} (\text{Rm} \Sigma(X_1, X_2, X_3, X_4)) + \hat{A} (\text{Rm} \Sigma(X_1, X_2, X_4)) = \text{Rm} * \hat{A} + \hat{A} * A * A, \]
hence
\[ \nabla^* T_{\nabla \hat{A}} = \nabla \text{Rm} * \hat{A} + \text{Rm} * \hat{A} * A + \text{Rm} * \nabla \hat{A} + \nabla \hat{A} * A * A. \]
This gives the very rough identity
\[ \nabla^* (\nabla^2 \hat{A}) = \nabla (\nabla^* \nabla \hat{A}) + \nabla \hat{A} * A * A + \nabla \text{Rm} * \hat{A} + \text{Rm} * \hat{A} * A + \text{Rm} * \nabla \hat{A}. \]
Remark. In fact, one can actually show the more precise identity
\[
\hat{A} \cdot \nabla^2 \hat{A} = \hat{A} \cdot \nabla^2 H + \frac{1}{2} H^2 |\hat{A}|^2 - |\hat{A}|^4 + \hat{A} \cdot \hat{A} \cdot Rm + 2\hat{A} \cdot \nabla \text{Re}(\nu, \nu)^T, \tag{33}
\]
see (1.7) in [LMS11]. This makes use of the more refined identity \(\nabla^* \hat{A} = -\frac{1}{2} \nabla H + \text{Re}^T(\cdot, \nu)\).

For the following proof we remark that \(|\text{Re}| \leq cR^{-4}_e + cmR^{-3}_e \leq cR^{-3}_e\) and \(|\nabla \text{Re}| \leq c\eta R^{-5}_e + cmR^{-4}_e \leq cR^{-4}_e\). Moreover, since \(|H|_{L^2(\Sigma)}^2 \geq 16\pi\), there evidently holds \(|A|_{L^2} \geq 1\), provided \(r_{\text{min}} \geq R_0\) (c.f. Lemma 2.3).

Lemma 4.3. Let \(\Sigma\) be a spherical surface satisfying \(r_{\text{min}} \geq R_0\), the assumptions of Lemma 2.6 and a uniform bound on \(|A|_{L^2(\Sigma)}\). Then, there holds
\[
\int_{\Sigma} |\nabla A|^2 d\mu \leq -c \int_{\Sigma} HWd\mu + \frac{c}{R^2_e} \int_{\Sigma} |A|^2 d\mu + c \int_{\Sigma} |\hat{A}|^4 d\mu,
\]
provided \(R_0\) is chosen to be sufficiently large. In this estimate, we may replace \(\nabla A\) by \(\nabla \hat{A}\) or \(\nabla H\).

Proof. Multiplying (28) by \(\hat{A}\), integrating and using that \(\hat{A}\) is trace free we obtain
\[
\int_{\Sigma} |\nabla \hat{A}|^2 d\mu + \frac{1}{2} \int_{\Sigma} H^2 |\hat{A}|^2 d\mu \leq - \int_{\Sigma} \nabla H \cdot \nabla |\hat{A}|^2 d\mu + c \int_{\Sigma} |\hat{A}|^4 d\mu + \frac{c}{R^2_e} \int_{\Sigma} |\hat{A}|^2 d\mu + \frac{c}{R^2_e} \int_{\Sigma} |\hat{A}|^2 d\mu
\]
\[
\leq \frac{1}{2} \int_{\Sigma} |\nabla H|^2 d\mu + \frac{c}{R^2_e} \int_{\Sigma} |\nabla H| d\mu + c \int_{\Sigma} |\hat{A}|^4 d\mu + \frac{c}{R^2_e} \int_{\Sigma} |A|^2 d\mu,
\]
where we used \(\nabla H = -\nabla^* \hat{A} + \text{Re}^3\). \(|\hat{A}| |\hat{A}| \leq R^{-1}_e |A|^2 + R_e |\hat{A}|^2\) and estimated
\[
\int_{\Sigma} |\hat{A}| d\mu \leq c \int_{\Sigma} \text{Re}^3 |\hat{A}|^2 d\mu + cR^{-3}_e \int_{\Sigma} 1 d\mu \leq cR^3_e \int_{\Sigma} |\hat{A}|^4 d\mu + cR^3_e \int_{\Sigma} |\hat{A}|^4 d\mu + cR^3_e \int_{\Sigma} |A|^2 d\mu.
\]
Integrating by parts and using the definition of \(W\) we obtain
\[
\frac{1}{2} \int_{\Sigma} |\nabla H|^2 d\mu = \frac{1}{2} \int_{\Sigma} (-HW + H^2 \text{Re}(\nu, \nu) + H^2 |\hat{A}|^2) d\mu
\]
\[
\leq \frac{1}{2} \int_{\Sigma} (-HW + H^2 |\hat{A}|^2) d\mu + cR^3_e \int_{\Sigma} |A|^2 d\mu.
\]
Next, \(|\nabla H| \leq c |\nabla \hat{A}| + cR^{-3}_e\) and hence
\[
\int_{\Sigma} |\nabla H| R^{-3}_e d\mu \leq cR^{-1}_e \int_{\Sigma} |\nabla \hat{A}|^2 d\mu + cR^{-3}_e \int_{\Sigma} |A|^2 d\mu.
\]
The claimed inequality now follows from combining these estimates, absorbing the \(|\nabla \hat{A}|^2\) term and using \(|\nabla A|^2 \leq c |\nabla \hat{A}|^2 + cR^{-6}_e\). Finally, we clearly have \(|\nabla H| \leq c |\nabla A|\). \(\square\)

Lemma 4.4. Under the assumptions of the previous lemma, there holds
\[
\int_{\Sigma} |\nabla^2 H|^2 d\mu + \int_{\Sigma} |A|^2 |\nabla A|^2 d\mu + \int_{\Sigma} |A|^4 |\hat{A}|^2 d\mu \leq c \int_{\Sigma} |W|^2 d\mu + cR^{-3}_e \int_{\Sigma} WH d\mu + c \left( |\hat{A}|^2 |\nabla \hat{A}|^2 + |\hat{A}|^6 \right) d\mu + cR^6_e \int_{\Sigma} |A|^2 d\mu.
\]

Proof. Multiplying (29) with \(\nabla H\) and integrating we obtain
\[
\int_{\Sigma} |\nabla^2 H|^2 d\mu + \frac{1}{4} \int_{\Sigma} H^2 |\nabla H|^2 d\mu \leq \int_{\Sigma} |\Delta H|^2 d\mu + cR^{-3}_e \int_{\Sigma} |\nabla H|^2 d\mu + c \int_{\Sigma} |\hat{A}|^2 |\nabla \hat{A}| |\nabla H| d\mu
\]
\[
+ cR^{-3}_e \int_{\Sigma} |\hat{A}|^2 |\nabla H| d\mu.
\]
The last term can be estimated by
\[ c(\kappa) R^{-6}_\varepsilon \int |A|^2 d\mu + \kappa/2 \int |\nabla A|^2 |A|^2 d\mu. \]

Next, we have
\[ \int |\Delta H|^2 d\mu \leq c \int |W|^2 d\mu + c R^{-6}_\varepsilon \int |A|^2 d\mu + c \int |\hat{A}|^4 |A|^2 d\mu \]
and \[ \int |\hat{A}|^4 |A|^2 d\mu \leq c(\kappa) \int |\hat{A}|^6 d\mu + \kappa \int |A|^4 |\hat{A}|^2 d\mu. \] There also holds
\[ \int |\hat{A}|^2 \nabla \hat{A} \nabla H d\mu \leq \kappa/2 \int |A|^2 |\nabla A|^2 d\mu + c(\kappa) \int |\nabla \hat{A}|^2 |\hat{A}|^2 d\mu \]
and the term \[ R^{-3}_\varepsilon \int |\nabla H|^2 d\mu \] can be estimated with the previous lemma and the trivial estimate \[ \int |\hat{A}|^4 d\mu \leq 2 R^3 \int |\hat{A}|^6 d\mu + 2 R^{-3}_\varepsilon \int |A|^2 d\mu. \] Hence, so far we have shown that
\[ \int |\nabla^2 H|^2 d\mu + \frac{1}{4} \int H^2 |\nabla H|^2 d\mu \leq c \int |W|^2 d\mu + c(\kappa) R^{-6}_\varepsilon \int |A|^2 d\mu + \kappa \int |\hat{A}|^2 |A|^4 d\mu \quad (34) \]
\[ + \kappa \int |A|^2 |\nabla A|^2 d\mu + c(\kappa) \int |\nabla \hat{A}|^2 |\hat{A}|^2 d\mu - c R^{-3}_\varepsilon \int WH d\mu + c(\kappa) \int |\hat{A}|^6 d\mu. \]
From the estimate \[ |\nabla A|^2 \leq c |\nabla \hat{A}|^2 + c R^{-6}_\varepsilon \] and (28) it follows that
\[ \frac{1}{c} \int H^2 |\nabla A|^2 d\mu - c R^{-6}_\varepsilon \int |A|^2 d\mu \leq \int H^2 |\nabla \hat{A}|^2 d\mu \]
\[ = - \int H^2 \hat{A} \cdot \nabla^2 H d\mu - \frac{1}{2} \int H^4 |\hat{A}|^2 d\mu \]
\[ + \int (H^2 \hat{A} \ast \hat{A} \ast \hat{A} + H^2 \hat{A} \ast (Rm \ast A + \nabla Rm) d\mu \]
\[ + \int H \ast \nabla H \ast \hat{A} \ast \nabla \hat{A} d\mu. \]
The second and third row can be estimated by
\[ \kappa \int |A|^4 |\hat{A}|^2 d\mu + c(\kappa) \int |\hat{A}|^6 d\mu + c(\kappa) R^{-6}_\varepsilon \int |A|^2 d\mu + \kappa \int |A|^2 |\nabla A|^2 d\mu + \int c(\kappa) |\hat{A}|^2 |\nabla \hat{A}|^2 d\mu. \]
Using partial integration and the identity \[ -\nabla \ast \hat{A} = \frac{1}{2} \nabla H + Rm \ast 1, \] the first term in the first row can be computed to be
\[ - \int H^2 \hat{A} \nabla^2 H d\mu = \frac{1}{2} \int H^2 \nabla H^2 d\mu + \int H^2 \ast \nabla H \ast Rmd\mu + \int \nabla H \ast \nabla H \ast \hat{A} \ast H d\mu. \]
In the last equation, the third term can be estimated as before and the second term can be estimated by
\[ \frac{c(\kappa)}{R^6_\varepsilon} \int |A|^2 d\mu + \kappa \int |A|^2 |\nabla A|^2 d\mu. \]
Finally, we note that
\[ \int |A|^4 |\hat{A}|^2 d\mu \leq c \int |H^2 |\nabla A|^2 d\mu + c \int |\hat{A}|^2 |\nabla \hat{A}|^2 d\mu + \frac{c}{R^6_\varepsilon} \int |A|^2 d\mu. \]
Combining all these inequalities, choosing \[ \kappa > 0 \] sufficiently small, absorbing all terms when possible and noting that
\[ \int |A|^4 |\hat{A}|^2 d\mu \leq \int |H^4| |\hat{A}|^2 d\mu + \int |\hat{A}|^6 d\mu \]
implies the statement. \qed
Lemma 4.5. Under the assumptions of Lemma 4.3 there holds
\[\int_{\Sigma} |\nabla^2 \hat{A}|^2 d\mu + \int_{\Sigma} |A|^4 |\hat{A}|^2 d\mu + \int_{\Sigma} |\nabla A|^2 |A|^2 d\mu \]
\[\leq c \int_{\Sigma} |W|^2 d\mu + c \int_{\Sigma} (|\nabla \hat{A}|^2 |A|^2 + |\hat{A}|^6) d\mu + cR_e^{-6} \int_{\Sigma} |A|^2 d\mu,\]
provided \(R_0\) is sufficiently large.

Proof. Multiplying (31) by \(\nabla \hat{A}\) and applying (28) we obtain
\[\int_{\Sigma} |\nabla^2 \hat{A}|^2 d\mu \leq \int_{\Sigma} |\Delta \hat{A}|^2 + c \int_{\Sigma} |\nabla \hat{A}|^2 |A|^2 d\mu + cR_e^{-4} \int_{\Sigma} |\hat{A}| |\nabla \hat{A}| d\mu \]
\[+ cR_e^{-3} \int_{\Sigma} |\nabla \hat{A}| |\nabla A| d\mu + cR_e^{-3} \int_{\Sigma} |\nabla^2 \hat{A}|^2 d\mu \]
\[\leq c \int_{\Sigma} |\nabla^2 H|^2 d\mu + c \int_{\Sigma} |A|^4 |\hat{A}|^2 d\mu + c \int_{\Sigma} |\hat{A}|^6 d\mu + c \int_{\Sigma} |\nabla A|^2 |A|^2 d\mu \]
\[+ \frac{c}{R_e^6} \int_{\Sigma} |A|^2 d\mu + \frac{c}{R_e^6} \int_{\Sigma} |\nabla A|^2 d\mu,\]
where we used \(|\hat{A}| |\nabla \hat{A}| \leq c|A||\nabla A|\). The claim now follows from the two previous lemmas and from estimating
\[R_e^{-3} \int_{\Sigma} HW|d\mu \leq 2R_e^{-6} \int_{\Sigma} |A|^2 d\mu + 2 \int_{\Sigma} |W|^2 d\mu.\]

We now need the following Sobolev-type interpolation inequality.

Lemma 4.6. Let \(\Sigma\) be a compact surface satisfying \(r_{\min} \geq R_0\). If \(R_0\) is chosen sufficiently large, then any smooth \(k\)-form \(\psi\) satisfies
\[|\psi|^4_{L^\infty(\Sigma)} \leq c|\psi|^2_{L^2(\Sigma)} \left( \int_{\Sigma} (|\nabla^2 \psi|^2 + |\psi|^2 H^4) d\mu \right)\]

Proof. The proof of Theorem 5.6. in [KS02] carries over to our setting as \(\Sigma\) is compact and since the Michael-Simon-Sobolev inequality holds in an exterior region of an asymptotically Schwarzschild manifold, see Lemma 2.7. One then easily adapts the proof of Lemma 2.8 in [KS01].

We also need the following multiplicative Sobolev-inequality.

Lemma 4.7. Under the assumptions of the previous lemma we have
\[\int_{\Sigma}(|\hat{A}|^4 |\nabla \hat{A}|^2 + |\hat{A}|^6) d\mu \leq c \int_{\Sigma} |\hat{A}|^2 d\mu \int_{\Sigma} (|\nabla^2 \hat{A}|^2 + |A|^2 |\nabla A|^2 + |A|^4 |\hat{A}|^2) d\mu.\]

Proof. This is a straightforward adaption of Lemma 2.5 in [KS01]. Again, the proof only relies on the Michael-Simon-Sobolev inequality, Young’s inequality and Hölder’s inequality.

At this point, a small curvature assumption allows us to absorb the term on the left hand side of the previous equation in Lemma 4.5. This finally yields an \(L^\infty\)-estimate for \(\hat{A}\).

Lemma 4.8. Under the assumptions of Lemma 4.3 there exists a constant \(\epsilon(m, \eta) > 0\) such that if
\[\int_{\Sigma} |\hat{A}|^2 d\mu \leq \epsilon,\]
then
\[\int_{\Sigma} |\nabla^2 \hat{A}|^2 d\mu + \int_{\Sigma} |A|^2 |\nabla A|^2 d\mu + \int_{\Sigma} |A|^4 |\hat{A}|^2 d\mu \leq c \int_{\Sigma} |W|^2 d\mu + cR_e^{-6} \int_{\Sigma} |A|^2 d\mu.\]
In particular, we have
\[ |\hat{A}|^4_{L^\infty(\Sigma)} \leq c|\hat{A}|^2_{L^2(\Sigma)} \left( \int_\Sigma |\mathcal{W}|^2 d\mu + R_c^{-6} \int_\Sigma |\hat{A}|^2 d\mu \right). \]

Proof. This is a direct consequence of the previous three lemmas.

It turns out that the integral curvature estimates also imply an improved estimate for \( |\hat{A}|_{L^2(\Sigma)} \).

**Lemma 4.9.** Under the assumptions of Lemma 4.3 there exists a constant \( \epsilon(m, \eta) > 0 \) such that if
\[ \int_\Sigma |\hat{A}|^2 d\mu \leq \epsilon, \]
then
\[ \int_\Sigma |\hat{A}|^2 d\mu \leq cR_c^2 \int_\Sigma (|\nabla \hat{A}|^2 + |\text{Rc}(\nu, \cdot)^T|^2) d\mu. \]

Proof. Integrating (33) and using integration by parts as well as the identity \( \nabla^* \hat{A} = -\frac{1}{2} \nabla H + \text{Rc}(\nu, \cdot)^T \) yields
\[ \int_\Sigma (|\nabla \hat{A}|^2 + H^2|\hat{A}|^2) d\mu \leq c \int_\Sigma (|\nabla \hat{A}|^2 + |\hat{A}|^4 + |\text{Rc}(\nu, \cdot)^T|^2 + R_c^{-3}|\hat{A}|^2) d\mu. \]
Next, using Lemma 2.7 and Hölder’s inequality, we obtain
\[ \int_\Sigma |\hat{A}|^4 d\mu \leq |\hat{A}|^2_{L^2(\Sigma)} \left( \int_\Sigma (|\nabla \hat{A}|^2 + H^2|\hat{A}|^2) d\mu \right). \]
Hence, this term can be absorbed. Again with the Michael-Simons-Sobolev inequality and Hölder’s inequality we get
\[ \int_\Sigma |\hat{A}|^2 d\mu \leq c|\Sigma| \int_\Sigma (|\nabla \hat{A}|^2 + H^2|\hat{A}|^2) d\mu. \]
From this the claim follows. \( \square \)

5. A-Priori Curvature Estimates and the Proof of the Main Results

We now use the results from the previous section to derive a-priori estimates for spherical surfaces \( \Sigma \) of Willmore type with non-negative Hawking mass. The main goal is to prove an improved estimate for \( |\hat{A}|_{L^2(\Sigma)} \) and to show that \( \Sigma \) is strictly mean convex. The a-priori estimates will be enough to conclude that the minimizers constructed in Lemma 3.6 are part of the foliation \( \{ \Sigma_\lambda \} \). We start with the following lemma.

**Lemma 5.1.** Let \( \Sigma \) be a spherical surface of Willmore type with non-negative Hawking mass and \( \tau_{\min} \geq R_0 \). If \( R_0(\eta, m) \) is sufficiently large, then the following estimates hold:
\[ |\hat{A}|_{L^\infty(\Sigma)} \leq cR_c^{-2}, \]
\[ |\hat{A}|_{L^2(\Sigma)} \leq cR_c^{-2}, \]
\[ |\nabla H|_{L^2(\Sigma)} \leq cR_c^{-4}, \]
\[ |\Delta H|_{L^2(\Sigma)} \leq cR_c^{-6}. \]

Proof. By Lemma 3.2 and Lemma 3.4 there holds \( |\hat{A}|^2_{L^2(\Sigma)} \leq c r_{\min}^{-1} \leq cR_0^{-1} \) and \( \tau_c < 1 - \iota \). Hence, the constraints of Lemma 2.6 are satisfied, provided \( R_0 \) is sufficiently large. Next, since the Hawking mass is non-negative and since \( \Sigma \) is a sphere, we infer that \( |\hat{A}|^2_{L^2(\Sigma)} \) is bounded. Consequently, we can use the results from the previous section. Choosing \( R_0 \) sufficiently large such that \( |\hat{A}|^2_{L^2(\Sigma)} \leq cR_0^{-1} < \epsilon \), we can therefore apply Lemma 4.9 as well as Lemma 4.8 to obtain
\[ |\hat{A}|^4_{L^\infty(\Sigma)} \leq c|\hat{A}|^2_{L^2(\Sigma)}(|\mathcal{W}|^2_{L^2(\Sigma)} + R_c^{-6}) \] (36)
as well as
\[ |\tilde{A}|^2_{L^2(\Sigma)} \leq cR_e^{-2} + cR_e^2|\nabla H|^2_{L^2(\Sigma)}. \] (37)
In order to estimate \(|\nabla H|^2_{L^2(\Sigma)}\), we first observe that by Lemma 2.3 and Lemma 2.5 there holds
\[ |H - \tilde{H}|_{L^2(\Sigma)} \leq cR_e^{-1} + c|\tilde{A}|_{L^2(\Sigma)}. \] (38)
Hence, by the divergence theorem we infer
\[ \int_{\Sigma} |\nabla H|^2 d\mu = -\int_{\Sigma} \Delta H(H - \tilde{H}) d\mu \leq c(R_e^{-1} + |\tilde{A}|_{L^2(\Sigma)})(\Delta H)_{L^2(\Sigma)}. \] (39)
Now, as \(\Sigma\) is of Willmore type there holds \(W + \lambda H = 0\) and consequently
\[ |W|^2_{L^2(\Sigma)} = \lambda^2 |H|^2_{L^2(\Sigma)} \leq c|\nabla H|^4_{L^2(\Sigma)} + c|\tilde{A}|^4_{L^\infty(\Sigma)} + cR_e^{-6}. \] (40)
Here we used that by multiplying (6) with \(H\) and integrating one finds
\[ \lambda |H|^2_{L^2(\Sigma)} = \int_{\Sigma} |\nabla H|^2 d\mu - \int_{\Sigma} |\tilde{A}|^2 |H|^2 d\mu - \int_{\Sigma} H^2 \text{Rc}(\nu, \nu) d\mu. \]
On the other hand, by the definition of \(W\) one finds
\[ |\Delta H|^2_{L^2(\Sigma)} \leq |W|^2_{L^2(\Sigma)} + c|\tilde{A}|^4_{L^\infty(\Sigma)} + cR_e^{-6}. \] (41)
Combining (36,39,40,41) gives
\[ |W|^2_{L^2(\Sigma)} \leq c(|\tilde{A}|_{L^2(\Sigma)} + R_e^{-1})|\nabla H|^2_{L^2(\Sigma)} + CR_e^{-6}. \]
Using \(|\tilde{A}|^2_{L^2(\Sigma)} \leq cR_0^{-1}\) we can absorb if \(R_0\) is sufficiently large and obtain \(|W|^2_{L^2(\Sigma)} \leq cR_e^{-6}. \)
Combining this with (36), (41) gives \(|\Delta H|^2_{L^2(\Sigma)} \leq cR_e^{-6}. \) But then (39) implies \(|\nabla H|^2_{L^2(\Sigma)} \leq cR_e^{-4} + c|\tilde{A}|_{L^2(\Sigma)}R_e^{-3}. \) Plugging this into (37) gives
\[ |\tilde{A}|^2_{L^2(\Sigma)} \leq cR_e^{-2} + cR_e^{-1}|\tilde{A}|_{L^2(\Sigma)} \leq cR_e^{-2} + \frac{1}{2}|\tilde{A}|^2_{L^2(\Sigma)}, \]
which implies the desired estimate for \(|\tilde{A}|^2_{L^2(\Sigma)}\). The remaining estimates now follow from (36) and (39). \(\square\)

In order to prove \(L^\infty\) estimates for \(H\), we need the following Bochner-type inequality. This is very similar to Lemma 4.8 in [LMS11]

**Lemma 5.2.** Under the assumptions of the previous lemma there holds
\[ \int_{\Sigma} |\nabla^2 H|^2 d\mu + \frac{1}{4} \int_{\Sigma} H^2 |\nabla H|^2 d\mu \leq cR_e^{-6}. \]

**Proof.** We multiply (30) by \(\nabla H\) to obtain the Bochner type identity
\[ \int_{\Sigma} |\nabla^2 H|^2 d\mu + \frac{1}{4} \int_{\Sigma} H^2 |\nabla H|^2 d\mu \leq \int_{\Sigma} |\Delta H|^2 d\mu + \int_{\Sigma} |\tilde{A}|^2 |\nabla H|^2 d\mu \leq |\Delta H|^2_{L^2(\Sigma)} + cR_e^{-3} |\nabla H|^2_{L^2(\Sigma)} + |\tilde{A}|^2_{L^\infty(\Sigma)} |\nabla H|^2_{L^2(\Sigma)} \leq cR_e^{-6}. \]
We used the previous lemma in the last inequality. \(\square\)

The next lemma establishes the strict mean convexity of spheres of Willmore type with positive Hawking mass.

**Lemma 5.3.** Under the assumptions of Lemma 5.1 we have
\[ |H - \tilde{H}|_{L^\infty(\Sigma)} \leq cR_e^{-2}. \]
In particular, \(\Sigma\) is strictly mean convex. Moreover, \(\Sigma\) is convex in the Euclidean sense.
Lemma 3.3 and (7) implies that $m \mid \Sigma$ by continuity. On the other hand, Lemma 5.1 implies $\operatorname{Minimizing Hull Property 1.4}$ in [HI01]. Hence, the proof of Lemma 3.4 we find that $M$ constant weak inverse mean curvature flow agree, which can only happen if we arrive at $F \text{rom (38) and Lemma 5.1 we obtain } |H - \hat{H}_e|_{L^2(\Sigma)} \leq c R_e^{-1}$. Now, Lemma 4.6 and Lemma 5.2 imply

$$|H - \hat{H}_e|_{L^2(\Sigma)}^2 \leq \int \nabla^2 H^2 d\mu + \int H^4 |H - \hat{H}_e|^2 d\mu \leq c R_e^{-2} (R_e^{-6} + R_e^{-4} \int |H - \hat{H}_e|^2 d\mu + |H - \hat{H}_e|^4 \int |H - \hat{H}_e|^2 d\mu) \leq c R_e^{-8} + c R_e^{-4} |H - \hat{H}_e|_{L^\infty(\Sigma)}^4.$$ 

Here, we estimated $H^4 \leq c |H - \hat{H}_e|^4 + c |\hat{H}_e|^4$ and used $\hat{H}_e \leq c R_e^{-1}$. After absorbing the second term on the right hand side, the first claim follows. To show that $\Sigma$ is convex in the Euclidean sense, we consider the Gauss equation $K = 8 \pi H^2 - 2 |\lambda_e|^2$. From the first claim and Lemma 2.3, it is easy to see that $H_e \geq 2 R_e^{-1} - c R_e^{-2}$, while Lemma 5.1 and Lemma 2.3 imply that $\hat{A}_e \leq c R_e^{-2}$. Hence, $K_e > 0$ if $R_0$ is sufficiently large.

We are now in the position to prove the main result.

**Proof of Theorem 1.2 and Theorem 1.3.** We start with the second inequality. Let $K, \hat{K}$ be compact regions as in Lemma 3.6 which are foliated by surfaces of Willmore type $\Sigma_\lambda$ where $\lambda \in (0, \lambda_0)$ and $\tilde{\lambda} \in (0, \lambda_0)$ respectively. Obviously, $\lambda_0 \geq \lambda_0$. Let $\Sigma \subset M \setminus K$ be a surface in the exterior region which satisfies the genus bound from Lemma 3.6 if $(M, g)$ is not conformally flat. It is well-known that $m_H(\Sigma_\lambda)$ approaches $m > 0$ as $\lambda \to 0$, see [SWW09]. Consequently, after possibly enlarging $K$, we may assume by Lemma 3.5 that $\Sigma$ is not null-homologous and has non-negative Hawking mass. By Lemma 3.6, there exists a spherical surface of Willmore type $\Sigma^* \subset M \setminus \hat{K}$ with $m_H(\Sigma^*) \geq m_H(\Sigma)$ as well as $|\Sigma^*| = |\Sigma|$ and by Lemma 5.3 we know that $\Sigma^*$ is strictly mean convex. Now, we fix a number $\lambda \in (0, \lambda_0)$ such that the leave $\Sigma_\lambda$ satisfies $|\Sigma^*| = |\Sigma_\lambda|$. In order to see that such a number exists, we argue as follows: According to the previous lemma, $\Sigma_\lambda$ is mean convex and convex in the Euclidean sense, in particular it is star-shaped. Then, by a result by Li and Wei, see [LW17], the smooth inverse mean curvature flow exists for all times and foliates the non-compact component of $M \setminus \Sigma_\lambda$. But then Lemma 2.3 in [HI01] implies that the smooth and weak inverse mean curvature flow agree, which can only happen if $\Sigma_\lambda$ is outward minimizing, see Minimizing Hull Property 1.4 in [HI01]. Hence, $|\Sigma^*| = |\Sigma| \geq |\Sigma_\lambda|$ and the existence of $\lambda$ follows by continuity. On the other hand, Lemma 5.1 implies $|\hat{A}_e|_{L^2(\Sigma^*)}^2 \leq c R_e^{-2}$ which in combination with Lemma 3.3 and (7) implies that $m_H(\Sigma^*) \geq m - c R_e^{-1}$. Let $p \in \Sigma^*$ such that $r(p) = r_{\text{min}}$. As in the proof of Lemma 3.4 we find that

$$2 \pi \phi^8(p) + \int \phi^3 H_e \partial_r \phi \partial_r \nu_e d\mu_S \leq \frac{1}{8} \int \phi^8 H_e^2 d\mu_S.$$ 

In the first integral, we replace $\phi$ by 1 and $H_e$ by $\hat{H}_e$ using the previous lemma and Lemma 2.3. In the second integral, we replace $H_S$ by $H$ and $d\mu_S$ by $d\mu$ using Lemma 2.3. Thus we find

$$2 \pi \phi^8(p) - 4 \pi m R_e^{-1} = 2 \pi \phi^8(p) - m R_e^{-1} \int r^{-2} \partial_r \nu_e d\mu_e \leq \frac{1}{8} \int \phi^8 H^2 d\mu + c R_e^{-2}.$$ 

In the first equation, we used the divergence theorem and that $\text{div}_e(r^{-2} \partial_r) = 0$. Rearranging we arrive at

$$m - c R_e^{-1} \leq m_H(\Sigma^*) \leq \frac{|\Sigma^*|^2}{(16 \pi)^2} \left( \int \phi^8(q) - \phi^8(p) H^2 d\mu(q) + 32 \pi m R_e^{-1} + c R_e^{-2} \right).$$

According to Lemma 2.4 there holds $|\Sigma^*|^2 - 2 \sqrt{c} |\hat{A}_e| \leq c$ and it follows that there is some positive constant $M$ which we do not need to determine such that

$$- c R_e^{-1} \leq M |\Sigma|^\frac{1}{2} \int (r^{-1} - r_{\text{min}}^{-1}) H^2 d\mu.$$
By Lemma 5.1 and Lemma 2.3, there holds $|\hat{A}_e|_{L^2(\Sigma')} \leq cR_{e}^{-1}$. Then, by Lemma 2.5 we infer that
\[-cR_{e}^{-1} \leq M e \int_{\Sigma'} (\hat{r}^{-1} - \hat{r}_{\min}^{-1}) \hat{H}^2 d\mu.\]
If we now assume that $\tau_e \geq \hat{\tau}$, where $\hat{\tau}$ is the constant from Theorem 2.1, then a definite share of the approximating sphere satisfies $\hat{\tau} \geq (1 + \kappa)\hat{r}_{\min}$, for some $\kappa > 0$ only depending on $\hat{\tau}$. As in the proof of Lemma 3.4 it then follows that the right hand side is uniformly bounded from above by a negative constant. However, this is impossible if $R_0$, that is, $\tilde{K}$, is sufficiently large so we may assume that $\tau_e \leq \tilde{\tau}$. But then by Theorem 2.1, we infer $\Sigma^* = \Sigma_\lambda$ as $\Sigma^*$ is strictly mean convex, in particular, $\Sigma^* \subset M \setminus K$. This implies that $m_H(\Sigma) \leq m_H(\Sigma^*) = m_H(\Sigma_\lambda)$ with equality if and only if $\Sigma = \Sigma^* = \Sigma_\lambda$ and proves the second inequality. On the other hand, we note that by Theorem 3.2 in [LMS11], the Hawking mass is non-decreasing along the foliation and we have seen before that $m_H(\Sigma_\lambda)$ approaches $m$ as $\lambda \to 0$. This proves the first inequality. If equality holds for $\Sigma = \Sigma_\lambda$, then it is easy to see from Lemma 3.1 in [LMS11] that all leaves $\Sigma_{\lambda'}$ for $\lambda' \leq \lambda$ are umbilical constant mean curvature spheres. By a standard argument (see for example [HI01]), it follows that the leaves foliate an exterior region of the Schwarzschild manifold.

**Remark 5.4.** The assumption that a compact set $K$ needs to be removed cannot be dropped, even if $(M, g)$ is one half of the Schwarzschild manifold. In order to see this, one can connect two centered spheres close to the horizon $\{r = m/2\}$ by a small minimal tube and smoothen the resulting surface. Such a surface has about twice the area of the horizon and Willmore energy close to zero, hence the Hawking mass is about twice the Hawking mass of the horizon. Iterating this construction produces embedded spheres (which are even homologous to the horizon) of arbitrarily large Hawking mass. It is easy to see that this construction will not increase the Hawking mass if $r_{\min} \geq (1 + \sqrt{3}/2)m$. Hence, we conjecture that $K = \{r \leq (1 + \sqrt{3}/2)m\}$ is optimal in Theorem 1.3 if $\eta = 0$.

**Proof of theorem 1.4.** Let $\Sigma$ be a surface of Willmore type with non-negative Hawking mass which is not null-homologous. By the proof of the previous theorem, there holds $\tau_e \leq \tilde{\tau}$ provided the compact set $K$ is sufficiently large. As $\Sigma$ must be strictly mean convex, we infer $\Sigma = \Sigma_\lambda$ for some $\lambda \in (0, \lambda_0)$.

**Remark.** There certainly needs to be some sort of assumption on the area if one considers outlying surfaces, as for small areas the behavior of the Willmore energy is governed by the scalar curvature rather than the ADM mass, see [LMS18]. It would be interesting to know if Theorem 1.4 can be improved to include all surfaces of non-negative Hawking mass and sufficiently large area.

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