Probing noncommutative phase-space effects through thermal diffusion of Gaussian states

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Noncommutative phase-space and its effects have been studied in different settings in physics, in order to unveil a better understanding of phase-space structures. Here, we use the thermal diffusion approach to study how noncommutative effects can influence the time evolution of a one-mode Gaussian state when in contact with a thermal environment obeying the Markov approximation. Employing the cooling process and considering the system of interest as a one-mode Gaussian state, we show that the fidelity comparing the Gaussian state of the system in different times and the asymptotic thermal state is useful to sign noncommutative effects. Besides, by using the monotonicity behavior of the fidelity, we discuss some aspects of non-Markovianity during the dynamics.

1. INTRODUCTION

Quantum dissipative systems is a research field broadly investigated in theoretical and experimental physics because they are one of the best examples of open quantum system applications [1, 2]. There is a fairly deep interest in dissipative acting on quantum systems, ranging from quantum optics to condensed matter physics as, for instance, entangled states in optical cavities [3], dissipative optomechanical systems [4], in the Rabi model [5], and in systems interacting with Gaussian dissipative reservoir [6]. Besides, there exist a practical appeal to control dissipation in quantum technologies, such as quantum information processing [7] and quantum computation [8]. Then, in general, one can think that dissipation comes from the interaction of quantum systems to some external agent, such as a thermal environment.

On the other hand, from a theoretical point of view, exciting questions arise when new quantum features can affect quantum thermodynamic processes, in particular, that features underlying the context of phase-space noncommutativity (NC) extension of quantum mechanics. The noncommutativity in the configuration space has been firstly suggested by Snyder [9] as a propose to avoid divergences in the quantum field theory. Additionally, there is a fairly deep consensus that in the Planck scale (∼ 10^{-32} cm), the notion of space-time has to drastically rectified in a consistent formulation of quantum mechanics and gravity [10, 11], such that noncommutativity must be assumed at high energy scales. There is a large number of studies concerning the implications of what has been conventionally called by noncommutative quantum mechanics (NCQM), for instance, in the context of 2D-harmonic oscillator [12, 13], the gravitational quantum well [14, 15], and in relativist dispersion relations [16].

It can be observed that for systems described by Hamiltonians at most quadratic in their coordinates, the noncommutative effects can be effectively mapped to the standard quantum mechanics as an external magnetic field acting on the system [13, 15, 20]. This fact allows to obtain a convenient correspondence between the NCQM and the standard quantum mechanics (SQM) which is suitable to investigate how NC effects could impact some particular dynamics. The aim of this work is to use this property to study how NC effects can influence a thermalization process. We considered the so called Gaussian states as our system evolving under the dynamics of the NC harmonic oscillator. Then, we put the system in a contact with a thermal environment in order to probe how the time evolution of the system state is influenced by NC effects. We assume the cooling process, i. e., the system and the thermal environment are associated to the mean numbers of photons \( \bar{n} \) and \( \bar{m} \), respectively, such that \( \bar{n} > \bar{m} \). To quantify our study we use the quantum fidelity which has a well known form when both states are Gaussian.

This work is organized as follows. In section 2 we introduce the necessary information on noncommutative quantum mechanics and the Seiberg-Witten map. Besides, we provide the theoretical framework to treat thermal diffusion of Gaussian states. In particular, the section 3 is devoted to show how to map NC effects as effective external fields acting on the harmonic oscillator Hamiltonian and its equations of motion. Section 3 is dedicated to show how NC effects can affect the cooling dynamics of a Gaussian state when in contact with a thermal environment. Finally, in section 5 we draw our conclusions and final remarks.

2. THEORETICAL FRAMEWORK

In this section we provide the theoretical framework to treat noncommutativity in phase-space and the thermal diffusion equation for Gaussian states.
2.1. Noncommutative quantum mechanics in phase-space

Noncommutative quantum mechanics is based on the deformed Heisenberg-Weyl algebra [18,14,21,22] and it is represented by the commutation relations given by,

\[ [\hat{q}_i, \hat{q}_j] = i\theta_{ij}, \quad [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad i, j = 1, ..., d, \]

where \( \theta_{ij} \) and \( \epsilon_{ij} \) are invertible antisymmetric real constant \((d \times d)\) matrices, and one can define the matrix \( \Sigma_{ij} = \delta_{ij} + \theta_{ik} \epsilon_{kj}/\hbar^2 \), which is also invertible if \( \theta_{ik} \epsilon_{kj} = -\hbar^2 \delta_{ij} \). Writing \( \theta_{ij} = \epsilon_{ij} + \zeta_{ij} = \zeta_{ij} \), with \( \epsilon_{ii} = 0, \epsilon_{ij} = -\epsilon_{ji} \), one can interpret \( \eta \) and \( \zeta \) as being new constants in the quantum theory, which have been extensively studied recently [13, 20, 23–26]. Furthermore, there is a way of connecting the Hilbert space of the NCQM to that of the SQM, which is represented by the following relations,

\[ [\hat{Q}_i, \hat{P}_j] = 0, \quad [\hat{Q}_i, \hat{Q}_j] = i\hbar \delta_{ij}, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad i, j = 1, ..., d. \]

This is implemented through the Seiberg-Witten (SW) map, given by \( \hat{q}_i = \nu \hat{Q}_i - (\theta/2\nu \hbar) \epsilon_{ij} \hat{P}_j \) and \( \hat{p}_i = \mu \hat{P}_i + (\zeta/2\mu \hbar) \epsilon_{ij} \hat{Q}_j \), where \( \nu \) and \( \mu \) are arbitrary parameters fulfilling the condition \( \theta \zeta = 4\hbar^2 \mu \nu (1 - \mu \nu) \).

Once the SW map is applied to the Hamiltonian of a system, to describe the state of such a system, one can use the density matrix \( \hat{\rho} = \mid \Psi \rangle \langle \Psi \mid \), which can be used to define the associated Wigner function through the Weyl transform [27, 28],

\[
W(Q_i, P_i) = h^{-1} \rho^W = h^{-1} \int d\eta \exp[iP\eta/h]\Psi(Q_i - y/2)\Psi(Q_i + y/2),
\]

which can be naturally generalized to a statistical mixture, with “W” standing for the Weyl transform. The marginal integration of the Wigner function results in,

\[
\psi^*(Q_i)\psi(Q_i) = \int dP_i W(Q_i, P_i) \quad (3)
\]

\[
\psi^*(P_i)\psi(P_i) = \int dQ_i W(Q_i, P_i), \quad (4)
\]

i.e., the distribution probability for position and momentum, respectively. The Wigner function can be used to obtain the expectation value of an observable \( \mathcal{O} \) as,

\[
\langle \mathcal{O} \rangle = \int \int dQ_i dP_i W(Q_i, P_i) \mathcal{O}^W(Q_i, P_i). \quad (5)
\]

where \( \mathcal{O}^W(Q_i, P_i) \) is the Weyl transform of the operator \( \mathcal{O} \).

2.2. Gaussian States and Thermal Diffusion

One important class of states with extensive theoretical and experimental applications are the so called Gaussian states (GS) which has been largely used in quantum information and quantum communication [29,30], quantum metrology [31], quantum optics [32] etc. Gaussian states are a subset of the more general class used to treat continuous variables (CV) systems [29,30]. An important aspect of the GS is that they are completely characterized by their first and second moments. Introducing a vector \( \vec{R}(Q_1, P_1, Q_2, P_2) \) to collect the coordinates of a two-dimensional system, the first moments can be rearranged in the vector \( \vec{d} = (\langle Q_1 \rangle, \langle P_1 \rangle, \langle Q_2 \rangle, \langle P_2 \rangle) \). The set of all second moments are collected in the so called covariance matrix (CM), given by \( \sigma = \sigma_{ii} \oplus \sigma_{ij} \) for a two-mode Gaussian state, where,

\[
\sigma_{ii} = \left( \begin{array}{cc} \sigma_{Q_iQ_i} & \sigma_{P_iQ_i} \\ \sigma_{Q_iP_i} & \sigma_{P_iP_i} \end{array} \right), \quad (6)
\]

with \( \sigma_{AB} = (AB + BA) - 2(A)\langle B \rangle \). It can be shown that for a bona-fide two-mode Gaussian state the CM satisfies the relation \( \sigma + i\Omega \geq 0 \), where,

\[
\Omega = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)^{\otimes 2}, \quad (7)
\]

such that \( [\vec{R}, \vec{R}] = i\Omega \). In particular, for GS the Wigner function assumes a gentle form given by [29],

\[
W_G(\vec{R}) = \exp\left[-(1/2)(\vec{R} - \vec{d})\sigma^{-1}(\vec{R} - \vec{d})\right]/(2\pi)^{2n}\sqrt{\text{Det}(|\sigma|)}, \quad (8)
\]

where \( n \) is the number of modes of the system. An especial type of GS is the thermal state. A one-mode thermal state is represented by [29],

\[
\rho^n(\bar{m}) = \sum_{m=0}^{\infty} \frac{\bar{m}^m}{(\bar{m} + 1)^{m+1}} |m\rangle\langle m|, \quad (9)
\]

where \( \bar{m} = \text{exp}[\hbar\omega/k_BT - 1] \) is the mean number of photons in the bosonic mode, \( k_B \) is the Boltzmann constant, \( T \) is the associated temperature, and \( \{|m\} \) is the Fock basis, with first moments and CM given by \( \bar{R} = (0, 0) \) and \( \sigma^{th} = (2\bar{m} + 1)\mathbb{I}_{2 \times 2} \), respectively, where \( \mathbb{I}_{2 \times 2} \) represents a two-by-two identity matrix.

The propagation of a general Gaussian state in a noisy and dissipative channel where each mode is coupled with a different and uncorrelated Markovian environment modeled by a stationary continuum of oscillators can be described, in the interaction picture, by the following master equation,

\[
\frac{d}{dt} \rho = -i[H, \rho] + \mathcal{L}(\rho), \quad (10)
\]

where the first and second term on the right-hand side represent the unitary and dissipative part of the dynamics. It can be shown that when the system of interest is
where the dynamics is easily evolved in terms of the first moments and covariance matrix [36, 37],

\[
\dot{\sigma} = \Gamma \sigma + \Gamma (2\bar{m} + 1)I_{2x2},
\]

\[
\dot{\sigma} = - (\Gamma / 2) \sigma,
\]

with solutions,

\[
\sigma(t) = e^{-\Gamma t}\sigma(0) + (1 - e^{-\Gamma t})(2\bar{m} + 1)I_{2x2},
\]

\[
\bar{\sigma}(t) = e^{-\Gamma t/2}\bar{\sigma}(0),
\]

where \( \Gamma \) is the decay rate and \( \bar{m} \) is the mean number of photons of the thermal environment, and \( \sigma(0) \) and \( \bar{\sigma}(0) \) are the initial covariance matrix and first moments of the system. Assuming a mean number of photons to the system as being \( \bar{n} \), we remark that for the cooling process, \( \bar{n} > \bar{m} \), whereas for the heating process, \( \bar{n} < \bar{m} \). Here, we adopt the cooling process to investigate the NC effects. To conclude this section, we are interested in the time evolution of the following two-dimensional Gaussian Wigner function,

\[
W_G(\vec{R}) = \frac{1}{\pi^2} \exp \left[ - \left( (Q_1(t) - x_0)^2 - (Q_2(t) - y_0)^2 \right) \right]
\]

\[
\times \exp \left[ - \left( (P_1(t) - p_{x_0})^2 - (P_2(t) - p_{y_0})^2 \right) \right],
\]

where \( x_0, y_0, p_{x_0} \) and \( p_{y_0} \) are arbitrary initial parameters.

3. MAPPING NONCOMMUTATIVE EFFECTS AS EXTERNAL FIELDS

Let us consider the Hamiltonian of a noncommutative harmonic oscillator,

\[
\hat{H}^{NC}(\hat{q}_i, \hat{p}_i) = \frac{\hat{p}_i^2}{2m} + m\omega^2\frac{\hat{q}_i^2}{2},
\]

where \( m \) and \( \omega \) are the mass and frequency of the system, respectively. Applying the SW map to obtain the standard version of the eq. (14) and then performing the Weyl transform, one gets [12, 13],

\[
H(Q_i, P_i) = \alpha^2 Q_i^2 + \beta^2 \gamma^2 P_i^2 + \gamma \sum_{i,j=1}^2 \epsilon_{ij} P_i Q_j,
\]

where we defined,

\[
\alpha^2 = \frac{\nu^2 m \omega^2}{2} + \frac{\zeta^2}{9m^2 \nu^2 \hbar^2}, \quad \beta^2 = \frac{\mu^2}{2m} + \frac{m \omega^2 \theta^2}{8 \nu^2 \hbar^2}, \quad \gamma = \frac{\theta}{2\hbar} \frac{m \omega^2}{2m \nu^2 \hbar^2} + \frac{\zeta}{2m \nu^2 \hbar^2}.
\]

Following ref. [13], we use the Heisenberg equation to obtain a set of uncoupled four equations of motions,

\[
Q_1(t) = x_0 \cos(\Omega t) \cos(\gamma t) + y_0 \cos(\Omega t) \sin(\gamma t)
\]

\[
+ \frac{\beta}{\alpha} [p_{x_0} \sin(\Omega t) \sin(\gamma t) + p_{y_0} \cos(\Omega t) \cos(\gamma t)]
\]

\[
Q_2(t) = y_0 \cos(\Omega t) \cos(\gamma t) - x_0 \cos(\Omega t) \sin(\gamma t)
\]

\[
- \frac{\beta}{\alpha} [p_{x_0} \sin(\Omega t) \sin(\gamma t) - p_{y_0} \cos(\Omega t) \cos(\gamma t)]
\]

\[
P_1(t) = p_{x_0} \cos(\Omega t) \cos(\gamma t) + p_{y_0} \cos(\Omega t) \sin(\gamma t)
\]

\[
- \frac{\beta}{\alpha} [y_0 \sin(\Omega t) \sin(\gamma t) + x_0 \sin(\Omega t) \cos(\gamma t)]
\]

\[
P_2(t) = p_{y_0} \cos(\Omega t) \cos(\gamma t) - p_{x_0} \cos(\Omega t) \sin(\gamma t)
\]

\[
+ \frac{\alpha}{\beta} [x_0 \sin(\Omega t) \sin(\gamma t) - y_0 \sin(\Omega t) \cos(\gamma t)]
\]

where \( \Omega = 2\alpha \beta = \omega \sqrt{(2\mu \nu - 1)^2 + \xi^2} \) and \( \xi = (1/2\hbar) \left[ m \omega \theta + \zeta/(m \omega) \right] \).

From eq. (15) we can note that the influence of the noncommutative parameters \( \theta \) and \( \zeta \) on the dynamics of the system is effectively a magnetic field-like term in the orthogonal direction to the plane of the oscillator. More explicitly, from the Hamiltonian of a harmonic oscillator in an external magnetic field [33], one has,

\[
B_0 \sim \frac{m \omega^2 \theta}{q \hbar} + \frac{\zeta}{q \hbar},
\]

where \( q \) is the effective charge associated to the harmonic oscillator.

4. THERMAL DIFFUSION WITH NC EFFECTS

In order to see how NC effects can influence the thermal diffusion process, we consider a Gaussian Wigner function as in eq. (13) evolving onto the dynamics dictated by eqs. (17) [20]. Assuming we are interested in the phase-space \( (Q_1, P_1) \) as our system, the respective Wigner function is obtained tracing out the coordinates \( (Q_2, P_2) \), i. e.,

\[
W_{sys}(Q_1, P_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dQ_2 dP_2 W_G(\vec{R}) = W_1
\]

It is important to stress that \( W_{sys}(Q_1, P_1) \) will be Gaussian during all the time evolution, such that we can assume that the first moments and the covariance matrix are sufficient to characterize the state of the system and they are given by \( \langle \vec{d}_{sys} \rangle \) and \( \sigma_{sys} \), i. e.,

\[
\langle \vec{d}_{sys} \rangle = \langle (Q_1)_{w_1}, (P_1)_{w_1} \rangle, \quad \sigma_{sys} = \left( \begin{array}{c} \sigma_{Q_1 Q_1} w_1 \\ \sigma_{Q_1 P_1} w_1 \\ \sigma_{P_1 Q_1} w_1 \\ \sigma_{P_1 P_1} w_1 \end{array} \right)
\]

where \( \langle \sigma_{AB} \rangle_{w_1} = \langle AB + BA \rangle_{w_1} - 2 \langle A \rangle_{w_1} \langle B \rangle_{w_1} \), and the mean values are obtained over the Wigner function \( W_1 \).

After that, the system is placed in thermal contact with an environment represented by a mean number of photons \( \bar{m} \), such that the first moments are null and \( \sigma^{th}(\bar{m}) = (2\bar{m} + 1)I_{2x2} \). For an environment fulfilling...
the Born-Markovian approximation, we assume a decay rate $\Gamma$. Besides, we consider $\langle \hat{B}_i^\dagger(0)\hat{B}_i(\nu) \rangle = \bar{n}\delta(\nu)$ and $\langle \hat{B}_i(0)\hat{B}_i(\nu) \rangle = M\delta(\nu)$, where $\hat{B}$ is the bosonic operator associated to the environment and the second expression relates the squeezing feature of the environment. Here, we assume that $M = 0$, i. e., the environment is just a thermal one. Furthermore, we consider that the thermal environment interacts only with the sub-space $(Q_1, P_1)$ or, in other words, the interaction between the sub-space $(Q_2, P_2)$ and the thermal environment is sufficiently weak such that it does not cause any effects on the evolution of the state $W_{\text{sys}}(Q_1, P_1)$. During the time evolution of the system in contact with the thermal environment, we assume the cooling process, i. e., we indexed to the system a mean number of photons $\bar{n}$, with $\bar{n} \gg \bar{m}$ and, for a Markovian evolution, $\bar{n} \rightarrow \bar{m}$ for a time sufficiently large. The time evolution of the first moments and CM during the thermal diffusion reads,

$$\sigma_{\text{sys}}(t) = \bar{n}e^{-t\Gamma}\sigma_{\text{sys}}(0) + (1 - e^{-t\Gamma})(2\bar{m} + 1)\mathbb{I}_{2\times 2}, \quad (24)$$

$$\hat{d}_{\text{sys}}(t) = e^{-t\Gamma/2}\hat{d}_{\text{sys}}(0). \quad (25)$$

In order to quantify how NC effects mapped effectively as external fields influence the cooling process, we use the quantum fidelity which, for two Gaussian states, has the well known expression $[34,35]$,

$$F(\sigma_1, \hat{d}_1; \sigma_2, \hat{d}_2) = \frac{2}{\sqrt{\Delta + \delta - \delta}} e^{-\frac{1}{2}d^{\dagger}d^{-1}d^{\dagger}}, \quad (26)$$

where $\Delta \equiv \text{Det}[\sigma_1 + \sigma_2]$, $\delta = (\text{Det}[\sigma_1] - 1)(\text{Det}[\sigma_2] - 1)$, $\hat{d} \equiv d^{\dagger} - d_2$, and $\sigma_+ = \sigma_1 + \sigma_2$. The fidelity is bounded by $0 \leq F \leq 1$, with $F = 0$ for two completely different states and $F = 1$ for two identical states. Here we consider $(\sigma_1, \hat{d}_1)$ and $(\sigma_2, \hat{d}_2)$ as being our system of interest during the time evolution and the asymptotic thermal state for complete cooling, respectively.

In Fig. 1 we have plotted the quantum fidelity as a function of time for the Gaussian state of the system $(Q_1, P_1)$ and the asymptotic thermal state for complete cooling, assuming $(\bar{n}, \bar{m}) = (4, 2)$ in order to have the cooling process of the system. We consider three different cases, i. e., the absence of NC effects, $B_0 = 0$ (black, solid line) and two cases with NC effects represented by $B_0 = 0.5aU$ (red, dotted line) and $B_0 = 1aU$ (blue, dashed line) where $aU$ means arbitrary unity. Moreover, two initially located system states were assumed: $(x_0, p_{x_0}) = (1, 1)$ (figure (a)) and $(x_0, p_{x_0}) = (0, 0)$ (figure (b)). The black curves corresponds exactly to the process of cooling of a Gaussian state when in contact with a thermal environment, i. e., the fidelity increases monotonically up to unity. The only difference is the initial value of the fidelity, which is easily explained by noting that in the second case the system starts the dynamics with $(\hat{d}_{\text{sys}}) = (0, 0)$, i. e., it is more closely to the asymptotic state for complete cooling process. For $B_0 \neq 0$, we observe that the monotonicity of the fidelity is not fulfilled, though as the time increases the state of the system goes to the asymptotic state. It can be also noted an difference of phase relative to the two values of $B_0$ when we change the initial position of the system in the phase-space.

4.1. Non-Markovian-like Effect

As we can note from the time evolution of the quantum fidelity in Fig. 1 for $B_0 \neq 0$ the fidelity does not increase monotonically up to unity. The presence of oscillations in the fidelity during a thermalization process has been largely studied in the literature as been associated to an information backflow from the thermal environment to the system $[39,40]$. Non-Markovian effects is a current field of research with several applications in quantum information $[11,12]$ and quantum thermodynamics $[43,44]$, both in theoretical and experimental areas $[43]$. As we have mentioned above, quantum fidelity obeys a significant property, i. e., monotonicity $[46]$,

$$F(\Lambda \rho_1, \Lambda \rho_2) \geq F(\rho, \rho_2), \quad (27)$$

where $\Lambda$ denotes a completely positive map, which serves as a characteristic feature of Markovian dynamics (see Fig. 1). Noticing that Markovian evolution guarantees...
a completely positive trace preserving dynamical map \( \Lambda(t) \), i.e., \( \rho(0) \rightarrow \rho(t) = \Lambda(t)\rho(0) \), which also forms a one-parameter semigroup obeying the composition law \[ \Lambda(t_1)\Lambda(t_2) = \Lambda(t_1 + t_2), \] with \( t_1, t_2 \geq 0 \). Therefore, any violation of the inequality in Eq. (27) is a clear signature of non-Markovian dynamics, indicating that the associated dynamical map does not obey the composition law. It is worth to mention that deviation from (27) is sufficient, though not necessary, reflection of non-Markovianity. Following the standard procedure, we used the time derivative of the quantum fidelity to indicate what we call non-Markovian-like effect, i.e., an effective non-Markovian behavior due exclusively to \( B_0 \neq 0 \). Therefore, for non-Markovian-like effect we have

\[
\frac{d}{dt} F(\sigma_1, \vec{d}_1; \sigma_2, \vec{d}_2) < 0. \tag{28}
\]

In order to visualize this type of effect in our system, we depicted in Fig. 2 the time evolution of the time derivative presented in Fig. 1 using the same parameters. We note clearly the existence of non-Markovian-like effects during the cooling process. Obviously, the same effect would be present if the heating process was considered, i.e., \( \bar{n} < \bar{m} \). From the Fig. 2 one can note that the intensity of the effect is greater as the intensity of the field increases when the system starts out of the origin, and the intensity of the effect is lower as the intensity of the field decreases when the system starts at the origin.

5. CONCLUSIONS

In this work we used the possibility of mapping non-commutative effects as external fields to study how these effects can influence a cooling process of a system described by a Gaussian state. Using the well known thermal diffusion approach, we investigated the time evolution of the state of the system in two situations, first out of the origin and second at the origin of the phase-space, remembering that the asymptotic thermal state for complete cooling remains at the origin during the process. It can be observed that the evolution to the asymptotic state depends on the the value of \( B_0 \) and on the initial localization of the system in the phase-space. Besides, for \( B_0 \neq 0 \) the system takes a longer time to reach the asymptotic state.

We also discuss the conceptual meaning of a non-Markovian-like effect during the time evolution of the system from an initial state to the asymptotic thermal state. It must be stressed that the thermal environment introduces a Markovian dynamics and that the decay rate \( \Gamma \) does not depend on time. Therefore, the negativity of the time derivative of the quantum fidelity is exclusively due to \( B_0 \neq 0 \). We believe that this work can contribute to unveil important features of noncommutative effects mapped as external fields on the dynamics of Gaussian states. Finally, in describing these effects through Gaussian states and thermal diffusion, studies concerning the experimental simulation of NC effects can emerge in the future.

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[1] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2002).
[2] Á. Rivas and S. F. Huega, Open Quantum Systems. An Introduction (Springer, New York, 2012).
[3] M. J. Kastoryano, F. Reiter, and A. S. Sørensen, Dissipative Preparation of Entanglement in Optical Cavities, Phys. Rev. Lett. 106, 090502 (2011).
[4] O. Kyriienko, T. C. H. Liew, and I. A. Shelykh, Optomechanics with Cavity Polaritons: Dissipative Coupling and Unconventional Bistability, Phys. Rev. Lett. 112, 076402 (2014).
[5] A. Le Boite, M.-J. Hwang, and M. B. Plenio, Metastability in the driven-dissipative Rabi model, Phys. Rev. A
[95.1] M. Jarzyna and M. Zwierz, Parameter estimation in the presence of the most general Gaussian dissipative reservoir, Phys. Rev. A 95, 012109 (2017).

[7] A. Bermudez, T. Schaeetz, and M. B. Plenio, Dissipation-Assisted Quantum Information Processing with Trapped Ions, Phys. Rev. Lett. 110, 110502 (2013).

[8] W. H. Zurek, Reversibility and Stability of Information Processing Systems, Phys. Rev. Lett. 53, 391 (1984).

[9] H. S. Snyder, Phys. Rev. 71, 38 (1946).

[10] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. 172, 187 (1995).

[11] N. Seiberg, “Emergent Spacetime”, arXiv:hep-th/0601234.

[12] O. Bertolami and J. G. Rosa, C. M. L. de Aragão, P. Casalino, and D. Zappalà, Noncommutative Gravitational Quantum Well, Phys. Rev. D 73, 105005 (2011).

[13] A. E. Bernardini and O. Bertolami, Probing phase-space noncommutativity through quantum beating, missing information and the thermodynamic limit, Phys. Rev. A 88, 012101 (2013).

[14] O. Bertolami, J. G. Rosa, C. M. L. de Aragão, P. Casalino, and D. Zappalà, Noncommutative Gravitational Quantum Well, Phys. Rev. D 72, 025010 (2005).

[15] R. Banerjee, B. D. Roy, and S. Samanta, Remarks on the Noncommutative Gravitational Quantum Well, Phys. Rev. D74, 045015 (2006).

[16] Kh. P. Gnatenko and V. M. Tkachuk, Upper bound on the momentum scale in noncommutative phase space of canonical type arXiv:1905.03245 (2019).

[17] P. Leal and O. Bertolami, Relativistic dispersion relation and putative metric structure in noncommutative phase-space, Phys. Lett. B 793, 240 (2019).

[18] J. F. G. Santos, A. E. Bernardini, and C. Bastos, Probing phase-space noncommutativity through quantum mechanics and thermodynamics of free particles and quantum rotors, Physica A 438, 340 (2015).

[19] J. F. G. Santos and A. E. Bernardini, Gaussian fidelity distorted by external fields, Physica A 445, 75 (2016).

[20] J. F. G. Santos and A. E. Bernardini, Quantum engines and the range of the second law of thermodynamics in the noncommutative phase-space, Eur. Phys. J. Plus 132, 260 (2017).

[21] C. Bastos and O. Bertolami, Berry phase in the gravitational quantum well and the Seiberg–Witten map, Phys. Lett. A 372, 34 (2008).

[22] J. Gamboa, M. Loewe, and J. C. Rojas, Phys. Rev. D 64, 067901 (2001).

[23] A. Saha, S. Gangopadhyay, and S. Saha, Noncommutative quantum mechanics of a harmonic oscillator under linearized gravitational waves, Phys. Rev. D 83, 025004 (2011).

[24] C. Bastos, N.C. Dias, and J.N. Prata, Wigner Measures in Noncommutative Quantum Mechanics, Commun. Math. Phys. 299, 3 (2010).

[25] C. Bertolami, and N. Dias, J. Prata, Noncommutative Graphene, Int. J. Mod. Phys. A 28, 16 (2013).

[26] J. B. Geloun, F. G. Scholtz, Coherent states in noncommutative quantum mechanics, J. Math. Phys. 50, 043505 (2009).

[27] W. B. Case, Wigner functions and Weyl transforms for pedestrians, Am. J. Phys. 76, 937 (2008).