ABSTRACT BIVARIANT CUNTZ SEMIGROUPS II
RAMON ANTOINE, FRANCESC PERERA, AND HANNES THIEL

Abstract. We previously showed that abstract Cuntz semigroups form a closed symmetric monoidal category. This automatically provides additional structure in the category, such as a composition and an external tensor product, for which we give concrete constructions in order to be used in applications.

We further analyse the structure of not necessarily commutative Cu-semirings and we obtain, under mild conditions, a new characterization of solid Cu-semirings $R$ by the condition that $R \cong [R, R]$.

1. Introduction

The Cuntz semigroup is a geometric refinement of $K$-theory that was introduced by Cuntz [Cun78] in his groundbreaking studies of simple $C^*$-algebras. It is a partially ordered semigroup that is constructed from positive elements in the stabilization of the algebra, in a similar way to how the Murray-von Neumann semigroup in $K$-theory is built from projections.

There has been an intensive use of Cuntz semigroups in $C^*$-algebra theory, particularly related to classification results. One of the most prominent instances of this appeared in Toms’ example [Tom08] of two simple $C^*$-algebras that are indistinguishable by $K$-theoretic and tracial data, yet their Cuntz semigroups are not order-isomorphic. For the largest agreeable class of simple, separable, nuclear, finite and $Z$-stable $C^*$-algebras that can be classified using $K$-theoretic and tracial information, the Cuntz semigroup features prominently since, suitably paired with $K_1$, it is functorially equivalent to the Elliott invariant; see [BPT08] and [ADPS14].

Further, for $C^*$-algebras of stable rank one, the Cuntz semigroup has additional fine structure that was key in solving a number of open problems, such as the Blackadar Handelman conjecture and the Global Glimm Halving Problem; see [APRT18].

The formal framework to study Cuntz semigroups of $C^*$-algebras is provided by the category $Cu$ of abstract Cuntz semigroups, which was introduced by Coward, Elliott and Ivanescu in [CEI08], and which was studied in detail in [APT18] and later in [APT18b]. The objects of this category, called Cu-semigroups, are partially ordered semigroups that satisfy the order-theoretic analogues of being a complete topological space without isolated points. The Cu-morphisms are natural models for $^*$-homomorphisms between $C^*$-algebras. The following result was established in [APT18] and [APT18b].

Theorem. The category $Cu$ of abstract Cuntz semigroups is a symmetric monoidal closed category.

This means that there are bifunctors

$$- \otimes - : Cu \times Cu \to Cu, \text{ and } [\cdot, \cdot] : Cu \times Cu \to Cu,$$
such that \( \otimes \) is associative, symmetric, has as unit object the semigroup \( \mathbb{N} = \{0, 1, \ldots, \infty\} \) and, for any Cu-semigroup \( T \), the functors \( \cdot \otimes T \) and \( [T, \cdot] \) are an adjoint pair. For Cu-semigroups \( S \) and \( T \), the construction of the internal-hom \([S, T]\) is based on paths of the so-called generalized Cu-morphisms, which are natural models for completely positive contractive order-zero maps between \( C^* \)-algebras; see Section 2 for more details. We refer to the semigroups \([S, T]\) as bivariant Cu-semigroups.

Further, we show in [APT18c] that Cu is complete and cocomplete, and that the functor that assigns to each \( C^* \)-algebra its Cuntz semigroup is compatible with products and ultraproducts.

The fact that Cu is a closed category automatically adds additional features well known to category theory (see, for instance, [Kel05]). For example, one obtains a composition product given in the form of a Cu-morphism:

\[
\circ: [T, P] \otimes [S, T] \to [S, P],
\]

which is the generalization of the composition of morphisms in a category to a notion of composition between internal-hom objects in a closed category; see the comments after [APT18b, Proposition 5.11].

Although the said features can be derived from general principles, in our setting they become concrete, and this is very useful in applications. In this direction, and bearing in mind that \([S, T]\) is a semigroup built out of paths of morphisms from \( S \) to \( T \), the composition product can be realized as the composition of paths. Another important example is the evaluation map which, for Cu-semigroups \( S \) and \( T \) is a Cu-morphism \( e_{S,T}: [S, T] \otimes S \to T \) such that \( e_{S,T}(x \otimes a) \) can be interpreted as the evaluation of \( x \in [S, T] \) at \( a \in S \). We therefore also write \( x(a) := e_{S,T}(x \otimes a) \). The evaluation map can be used to concretize the adjunction between the internal-hom bifunctor and the tensor product; see Proposition 3.22.

Likewise, the tensor product of generalized Cu-morphisms induces an external tensor product

\[
\boxtimes: [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2],
\]

which is associative and, like in KK-Theory, compatible with the composition product; see Proposition 3.21. This means that, for elements \( x_k \in [S_k, T_k] \) and \( y_k \in [T_k, P_k] \) (for \( k = 1, 2 \)), we have

\[
(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).
\]

In Section 4, we study the ideal structure of bivariant Cu-semigroups. Given an ideal \( J \) in \( S \), and an ideal \( K \) in \( T \), we show that there is a natural identification of \([S/J, K]\) with an ideal in \([S, T]\); see Propositions 4.2 and 4.3. However, in general, not every ideal of \([S, T]\) arises this way. Indeed, in Example 4.7, we construct a simple Cu-semigroup \( S \) such that \([S, S]\) is not simple.

In Section 5, we deepen our study of Cu-semirings and their semimodules, which was started in [APT18] Chapters 7 and 8. Given a Cu-semigroup \( S \), the composition product turns \([S, S]\) into a Cu-semiring; see Proposition 5.1. Further, the evaluation map defines natural left \([S, S]\)-action on \( S \); see Proposition 5.3. Finally, \([S, T]\) has both a natural left \([T, T]\)-action and a compatible right \([S, S]\)-action; see Proposition 5.6.

For any Cu-semiring \( R \), the internal-hom construction makes it possible to define a left regular representation-like map \( \pi_R: R \to [R, R] \), which is always a multiplicative order-embedding (and unital in case the unit of \( R \) is a compact element); see Definition 5.9, Theorem 5.12 and Proposition 5.16. We study when \( \pi_R: R \to [R, R] \) is an isomorphism; see Theorem 5.23. This is closely related to the property of being solid, which means that the multiplication defines an isomorphism between \( R \otimes R \) and \( R \); see [APT18, Definition 7.1.6].
Acknowledgements

This work was initiated during a research in pairs (RiP) stay at the Oberwolfach Research Institute for Mathematics (MFO) in March 2015. The authors would like to thank the MFO for financial support and for providing inspiring working conditions.

Part of this research was conducted while the third named author was visiting the Universitat Autònoma de Barcelona (UAB) in September 2015 and June 2016, and while the first and second named authors visited Münster Universität in June 2015 and 2016. Part of the work was also completed while the second and third named authors were attending the Mittag-Leffler institute during the 2016 program on Classification of Operator Algebras: Complexity, Rigidity, and Dynamics. They would like to thank all the involved institutions for their kind hospitality.

The two first named authors were partially supported by MINECO (grants No. MTM2014-53644-P and No. MTM2017-83487-P), and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The third named author was partially supported by the Deutsche Forschungsgemeinschaft (SFB 878 Groups, Geometry & Actions).

2. Preliminaries

Let $S$ be a positively ordered semigroup. Recall (cf. [GHK+03, Definition I-1.11, p.57]) that an additive auxiliary relation on $S$ is a binary relation $\prec$ on $S$ satisfying the following conditions for all $a, a', b, b' \in S$:

1. If $a \prec b$ then $a \leq b$.
2. If $a' \leq a \prec b \leq b'$ then $a' \prec b'$.
3. We have $0 \prec a$.
4. The relation $\prec$ is compatible with addition.

An important example of an auxiliary relation that we will use in the sequel is the way-below relation, originally coming from Domain Theory (see [GHK+03]):

Let $S$ be a positively ordered semigroup, and let $a, b \in S$. Recall that $a$ is way-below $b$ (we also say that $a$ is compactly contained in $b$), denoted $a \ll b$, if whenever $(c_n)_n$ is an increasing sequence in $S$ for which the supremum exists and that satisfies $b \leq \sup_n c_n$, then there exists $k \in \mathbb{N}$ with $a \leq c_k$.

We say that $a$ is compact if $a \ll a$, and we let $S_c$ denote the submonoid of compact elements in $S$.

Definition 2.1 ([CE10]; see also [APT18, Definition 3.1.2]). A Cu-semigroup, also called abstract Cuntz semigroup, is a positively ordered semigroup $S$ that satisfies the following axioms (O1)-(O4):

1. Every increasing sequence $(a_n)_n$ in $S$ has a supremum $\sup_n a_n$ in $S$.
2. For every element $a \in S$ there exists a sequence $(a_n)_n$ in $S$ with $a_n \ll a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_n a_n$.
3. If $a' \ll a$ and $b' \ll b$ for $a', b', a, b \in S$, then $a' + b' \ll a + b$.
4. If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in $S$, then $\sup_n (a_n + b_n) = \sup_n a_n + \sup_n b_n$.

A Cu-morphism between Cu-semigroups $S$ and $T$ is an additive map $f : S \to T$ that preserves order, the zero element, the way-below relation and suprema of increasing sequences. In case $f$ is not required to preserve the way-below relation, then we say it is a generalized Cu-morphism. The set of Cu-morphisms (respectively, generalized Cu-morphisms) is denoted by $\text{Cu}(S, T)$ (respectively, by $\text{Cu}[S, T]$).

We let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.
A notion central to the construction of tensor products is that of bimorphisms, which we now recall.

**Definition 2.2** ([APT18 Definition 6.3.1]). Let $S, T$ and $P$ be Cu-semigroups. A map $\varphi : S \times T \to P$ is a Cu-bimorphism if it satisfies the following conditions:

1. $\varphi$ is a positively ordered monoid morphism in each variable.
2. We have that $\sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k)$, for every increasing sequences $(a_k)_k$ in $S$ and $(b_k)_k$ in $T$.
3. If $a', a \in S$ and $b', b \in T$ satisfy $a' \ll a$ and $b' \ll b$, then $\varphi(a', b') \ll \varphi(a, b)$.

The set of Cu-bimorphisms is denoted by $\text{BiCu}(S \times T, P)$. Equipped with pointwise order and addition, this set is a positively ordered monoid. Similarly, the set of Cu-morphisms between two Cu-semigroups is also a positively ordered monoid.

**Theorem 2.3** ([APT18 Theorem 6.3.3]). Let $S$ and $T$ be Cu-semigroups. Then there exists a Cu-semigroup $S \otimes T$ and a Cu-bimorphism $\omega : S \times T \to S \otimes T$ such that for every Cu-semigroup $P$ the following universal property holds:

1. For every Cu-bimorphism $\varphi : S \times T \to P$ there exists a (unique) Cu-morphism $\tilde{\varphi} : S \otimes T \to P$ such that $\varphi = \tilde{\varphi} \circ \omega$.
2. If $\alpha_1, \alpha_2 : S \otimes T \to P$ are Cu-morphisms, then $\alpha_1 \leq \alpha_2$ if and only if $\alpha_1 \circ \omega \leq \alpha_2 \circ \omega$.

Thus, for every Cu-semigroup $P$, the assignment

$$(\alpha : S \otimes T \to P) \mapsto (\alpha \circ \omega : S \times T \to P)$$

defines a natural isomorphism of positively ordered monoids

$$\text{Cu}(S \otimes T, P) \cong \text{BiCu}(S \times T, P).$$

The existence of a natural tensor product turns Cu into a symmetric monoidal category; see [APT18 6.3.7]. As mentioned above, the tensor product functor $- \otimes T$ has a right adjoint $[T, -]$, and thus Cu is also a closed category. We recall some details; see [APT18 Section 3] for a full account.

Let $(S, \prec)$ be an ordered semigroup equipped with an additive auxiliary relation $\preceq$, and let $I_Q = \mathbb{Q} \cap (0, 1)$. A path on $S$ is a map $f : I_Q \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda' < \lambda$. The set of paths on $S$ is denoted by $P(S)$, which becomes a semigroup under pointwise addition. It is often the case that we write $f_\lambda = f(\lambda)$, and refer to a path as $f = (f_\lambda)_{\lambda \in I_Q}$.

Given paths $f, g$ in $P(S)$, write $f \preceq g$ if for every $\lambda \in I_Q$, there is $\mu \in I_Q$ such that $f(\lambda) \preceq g(\mu)$. Set $f \sim g$ provided $f \preceq g$ and $g \preceq f$, and let $\tau(S, \prec) := P(S)/\sim$.

Let $[f]$ denote the equivalence class of a path $f$. Then $\tau(S, \prec)$ becomes an ordered semigroup by setting $[f] + [g] = [f + g]$ and $[f] \leq [g]$ provided $f \preceq g$. It was proved in [APT18] Theorem 3.15 that, for $S$ as above, the semigroup $\tau(S)$ is a Cu-semigroup.

**Remark 2.4.** The above construction is also referred to as the $\tau$-construction in [APT18]. It defines a functor $\tau : \mathbb{Q} \rightarrow \text{Cu}$, where $\mathbb{Q}$ is the category of positively ordered semigroups $S$ with an additive auxiliary relation $\prec$ that additionally satisfy axioms (O1) and (O4). In this way, $\tau$ is a coreflector of the inclusion functor $\iota : \text{Cu} \rightarrow \mathbb{Q}$; see [APT18 Theorem 4.12]. Objects (respectively, morphisms) in the category $\mathbb{Q}$ are termed $\mathbb{Q}$-semigroups (respectively, $\mathbb{Q}$-morphisms).

If now $S$ and $T$ are Cu-semigroups, it is clear that $\text{Cu}[S, T]$ is also an ordered semigroup (with pointwise order and addition), and satisfies axioms (O1) and (O4) by taking pointwise suprema of increasing sequences. Given $\varphi, \psi \in \text{Cu}[S, T]$, we define $\varphi \prec \psi$ provided $\varphi(a') \ll \psi(a)$ whenever $a' \ll a$. This is easily seen to be an additive auxiliary relation on $\text{Cu}[S, T]$. Thus, $\text{Cu}[S, T]$ is a $\mathbb{Q}$-semigroup in the sense of [Remark 2.4].
Definition 2.5 ([APT18b] Definition 5.3). Let $S$ and $T$ be $\Cu$-semigroups. The internal hom from $S$ to $T$ is the $\Cu$-semigroup
\[[S, T] := \tau(\Cu[S, T], \prec)\].
We also call $[S, T]$ the bivariant $\Cu$-semigroup, or the abstract bivariant Cuntz semigroup of $S$ and $T$.

The proof that $\Cu$ is a closed category requires the use of the so-called endpoint map, which is made precise below.

Definition 2.6 ([APT18b] Definition 5.5). Let $S$ and $T$ be $\Cu$-semigroups. We let $\sigma_{S, T} : [S, T] \to \Cu[S, T]$ be defined by
\[\sigma_{S, T}(f)(a) = \sup_{\lambda \in I_0} f_\lambda(a),\]
for a function $f = (f_\lambda)_{\lambda} \in \Cu[S, T]$ and $a \in S$. We refer to $\sigma_{S, T}$ as the endpoint map.

Theorem 2.7 ([APT18b] Theorem 5.9). Let $S, T$ and $P$ be $\Cu$-semigroups. Then there are natural positively ordered monoid isomorphisms
\[\Cu(S, [T, P]) \cong \BiCu(S \times T, P) \cong \Cu(S \otimes T, P).\]
The first isomorphism is given by
\[(\alpha : S \to [T, P]) \mapsto (\tilde{\alpha} : S \times T \to P),\]
where $\tilde{\alpha}(a, b) = \sigma_{T, P}(\alpha(a))\beta(a, b)$, for $(a, b) \in S \times T$. The second is given by
\[(\beta : S \otimes T \to P) \mapsto ((a, b) \mapsto \beta(a \otimes b)), \text{ for } (a, b) \in S \times T.\]

3. Concretization of categorical constructions for $\Cu$

In this section, we give concrete pictures of general constructions in closed, symmetric, monoidal categories for the category $\Cu$. This will be used in the next section, and we start below with the analysis of the unit and counit maps.

Definition 3.1. Given $\Cu$-semigroups $S$ and $T$, the unit map is the $\Cu$-morphism $d_{S, T} : S \to [T, S \otimes T]$ that under the identification
\[\Cu(S, [T, S \otimes T]) \cong \Cu(S \otimes T, S \otimes T)\]
corresponds to the identity map on $S \otimes T$.

In the result below we shall use that, if $S$ is a $\Cu$-semigroup, and $a \in S$, then there is $(a_\lambda)_{\lambda \in I_0}$ such that $a' \prec a_\lambda$ whenever $\lambda' \prec \lambda$, $a_\lambda = \sup_{\lambda' \prec \lambda} a_{\lambda'}$, and $\sup \lambda a_\lambda = s$; see [APT18b] Proposition 2.8.

Proposition 3.2. Let $S$ and $T$ be $\Cu$-semigroups, and let $a \in S$. Let $(a_\lambda)_{\lambda \in I_0}$ be a path in $(S, \ll)$ with endpoint $a$. Then for each $\lambda \in I_0$, the map $a_\lambda \otimes : T \to S \otimes T$, sending $b \in T$ to $a_\lambda \otimes b$, is a generalized $\Cu$-morphism. Moreover, $(a_\lambda \otimes : T \to S \otimes T)$ is a path in $(\Cu[T, S \otimes T], \prec)$, and we have $d_{S, T}(a) = [(a_\lambda \otimes : T \to S \otimes T)]$.

Proof. The map $\omega : S \times T \to \Cu[T, S \otimes T]$, given by $\omega(s, t) = s \otimes t$, is a $\Cu$-bimorphism. This implies that $s \otimes : T \to S \otimes T$ is a generalized $\Cu$-morphism for each $s \in S$. Moreover, using that $\omega$ preserves the joint way-below relation, we obtain that $s' \prec s \otimes : T \to S \otimes T$ is a generalized $\Cu$-morphism for each $s' \prec s$. In particular, if $(s_\lambda)_{\lambda \in I_0}$ is a path in $S$, then $(s_\lambda \otimes : T \to S \otimes T)$ is a path in $(\Cu[T, S \otimes T], \prec)$. We define $\alpha : S \to [T, S \otimes T]$ by sending $s \in S$ to $[(s_\lambda \otimes : T \to S \otimes T)]$ for some choice of path $(s_\lambda)_{\lambda}$ in $S$ with endpoint $a$. It is straightforward to check that $\alpha$ is a well-defined $\Cu$-morphism.

Let us show that $\alpha = d_{S, T}$. Consider the bijections
\[\Cu(S, [T, S \otimes T]) \cong \BiCu(S \times T, S \otimes T) \cong \Cu(S \otimes T, S \otimes T)\]
from Theorem 2.7 Under the first bijection, \( \alpha \) corresponds to the Cu-bimorphism \( \bar{\alpha} \) given by
\[
\bar{\alpha}(s, t) = \sigma_{T, S \otimes T}(\alpha(s))(t),
\]
for \( s \in S \) and \( t \in T \), where \( \sigma_{T, S \otimes T} \) is the endpoint map. We compute
\[
\bar{\alpha}(s, t) = \sigma_{T, S \otimes T}(\alpha(s))(t) = \sup_{\lambda \in I_q}(s_\lambda \otimes \lambda t) = s \otimes t,
\]
for every path \((s_\lambda)_\lambda \) with endpoint \( s \in S \), and every \( t \in T \). It follows that \( \bar{\alpha} \) corresponds to \( \text{id}_{S, T} \) under the second bijection. By definition of \( d_{S, T} \), this shows that \( \alpha = d_{S, T} \), as desired.

**Notation 3.3.** Given Cu-semigroups \( S, T \) and \( P \), and a Cu-bimorphism \( \alpha : S \times T \to P \), we shall often use the notation \( \bar{\alpha} : S \to \llbracket T, P \rrbracket \) to refer to the Cu-morphism that corresponds to \( \alpha \) under the identification in Theorem 2.7.

**Corollary 3.4.** Let \( S \) and \( T \) be Cu-semigroups. Then the composition
\[
\sigma_{T, S \otimes T} \circ d_{S, T} : S \xrightarrow{d_{S, T}} \llbracket T, S \otimes T \rrbracket \xrightarrow{\sigma_{T, S \otimes T}} \text{Cu}(T, S \otimes T).
\]
satisfies \( (\sigma_{T, S \otimes T} \circ d_{S, T})(a) = a \otimes \cdot \), for every \( a \in S \). In particular
\[
(\sigma_{T, S \otimes T} \circ d_{S, T})(a)(b) = a \otimes b,
\]
for \( a \in S \) and \( b \in T \).

**Proof.** Let \( a \in S \) and \( b \in T \). Choose a path \((a_\lambda)_\lambda \) in \( S \) with endpoint \( a \). Then \( d_{S, T}(a) = [(a_\lambda \otimes \cdot)_\lambda] \) by Proposition 3.3. The supremum of the maps \( a_\lambda \otimes \cdot \) in \( \text{Cu}(S, T \otimes S) \) is the map \( a \otimes \cdot \). Thus, \( (\sigma_{T, S \otimes T} \circ d_{S, T})(a) = a \otimes \cdot \), as desired. \( \square \)

**Definition 3.5.** Given Cu-semigroups \( S \) and \( T \), the counit map (also called evaluation map) is the Cu-morphism \( e_{S, T} : [S, T] \otimes S \to T \) that under the identification
\[
\text{Cu}([S, T] \otimes S, T) \cong \text{Cu}([S, T], [S, T])
\]
corresponds to the identity map on \([S, T] \). Given \( x \in [S, T] \) and \( a \in S \), we also use \( x(a) \) to denote \( e_{S, T}(x \otimes a) \), but note that \( x(a) = x'(a) \) for all \( a \in S \) does not imply \( x = x' \).

**Proposition 3.6.** Let \( S \) and \( T \) be Cu-semigroups, let \( x \in [S, T] \), and let \( a \in S \). Then \( e_{S, T}(x \otimes a) = \sigma_{S, T}(x)(a) \). Thus, if \( \mathbf{f} = (f_\lambda)_\lambda \) is a path in \( \text{Cu}(S, T) \), then
\[
[f](s) = e_{S, T}([f] \otimes a) = \sup_{\lambda < 1} f_\lambda(a).
\]

**Proof.** Consider the bijections
\[
\text{Cu}([S, T], [S, T]) \cong \text{BiCu}([S, T] \times S, T) \cong \text{Cu}([S, T] \otimes S, T)
\]
from Theorem 2.7. To simplify notation, we denote the identity map on \([S, T] \) by \( \text{id} \). Under the first bijection, \( \text{id} \) corresponds to the Cu-bimorphism \( \text{id} \) satisfying
\[
\text{id}(y, s) = \sigma_{S, T}(\text{id})(y)(s),
\]
for all \( y \in [S, T] \) and \( s \in S \). We obtain that
\[
e_{S, T}(x \otimes a) = \text{id}(x, a) = \sigma_{S, T}(\text{id})(x)(a) = \sigma_{S, T}(x)(a). \quad \square
\]

**Remark 3.7.** Let \( \varphi : S \to T \) be a Cu-morphism, and let \( a \in S \). Considering \( \varphi \) as an element of \([S, T] \), the notation \( \varphi(a) \) for \( e_{S, T}(\varphi \otimes a) \) is consistent with the usual notation of \( \varphi(a) \) for the evaluation of \( \varphi \) at \( a \).
Lemma 3.8. Let $S$ be a Cu-semigroup. Let $ev_1: \Cu[N,S] \to S$ be given by $ev_1(f) = f(1)$ for $f \in \Cu[N,S]$. Then $ev_1$ is an isomorphism of $\Q$-semigroups. That is, $ev_1$ is an additive order-isomorphism and we have $f \prec g$ if and only if $ev_1(f) \prec ev_1(g)$, for $f, g \in \Cu[N,S]$.

It follows that $(\Cu[N,S], \prec)$ is a Cu-semigroup (naturally isomorphic to $S$ via $ev_1$). Moreover, the endpoint map $\sigma_{N,S}: [N,S] \to \Cu[N,S]$ from [Definition 2.6] is an isomorphism.

Proof. It is straightforward to prove that $ev_1$ is an isomorphism of $\Q$-semigroups. By [APT18b Proposition 4.10], the endpoint map of a Cu-semigroup is an isomorphism. Thus, the endpoint maps $\varphi_S$ and $\varphi_{\Cu[N,S]}$ are isomorphisms. By definition, $\sigma_{N,S} = \varphi_{\Cu[N,S]}$. Since $ev_1$ is an isomorphism, so is $\tau(ev_1)$. \hfill $\Box$

Definition 3.9. Given a Cu-semigroup $S$, we let $i_S: S \to [N,S]$ be the Cu-morphism that under the identification

$$\Cu(S,[N,S]) \cong Cu(S \otimes N,S)$$

corresponds to the natural isomorphism $r_S: S \otimes N \to S$.

We leave the proof of the following result to the reader.

Proposition 3.10. Let $S$ be a Cu-semigroup. Then $i_S: S \to [N,S]$ is an isomorphism. The inverse of $i_S$ is $ev_1 \circ \sigma_{N,S}$, where $ev_1$ is evaluation at 1 as in [Lemma 3.8] and where $\sigma_{N,S}: [N,S] \to \Cu[N,S]$ denotes the endpoint map from [Definition 2.6].

We now introduce and study the external tensor product map. To this end, let first $S_k$ and $T_k$ be Cu-semigroups, and let $\varphi_k: S_k \to T_k$ be (generalized) Cu-morphisms, for $k = 1, 2$. Recall from the comments after [APT18b Theorem 2.10] that the map $\varphi_1 \times \varphi_2: S_1 \times S_2 \to T_1 \otimes T_2$, defined by

$$(\varphi_1 \times \varphi_2)(a_1, a_2) := f_1(a_1) \otimes f_2(a_2),$$

for $a_1 \in S_1$ and $a_2 \in S_2$, is a (generalized) Cu-bimorphism. We denote the induced (generalized) Cu-morphism by $\varphi_1 \otimes \varphi_2: S_1 \otimes S_2 \to T_1 \otimes T_2$, and we call it the tensor product of $\varphi_1$ and $\varphi_2$.

Next, we generalize this construction and define an external tensor product between elements of internal-homs.

Definition 3.11. Given Cu-semigroups $S_1, S_2, T_1$ and $T_2$, we define the external tensor product map $\boxtimes: [S_1,T_1] \otimes [S_2,T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$ as the Cu-morphism that under the identification

$$\Cu([S_1,T_1] \otimes [S_2,T_2], [S_1 \otimes S_2, T_1 \otimes T_2])$$

$$\cong \Cu([S_1,T_1] \otimes [S_2,T_2] \otimes S_1 \otimes S_2, T_1 \otimes T_2),$$

corresponds to the composition

$$(e_{S_1,T_1} \otimes e_{S_2,T_2}) \circ (id_{S_1,T_1} \otimes \sigma \otimes id_{S_2}),$$

where $\sigma: [S_2,T_2] \otimes S_1 \to S_1 \otimes [S_2,T_2]$ denotes the flip isomorphism.

Given $x_1 \in [S_1, T_1]$ and $x_2 \in [S_2, T_2]$, we denote the image of $x_1 \otimes x_2$ under this map by $x_1 \boxtimes x_2$, and we call it the external tensor product of $x_1$ and $x_2$.

Remark 3.12. Let $\varphi_1: S_1 \to T_1$ and $\varphi_2: S_2 \to T_2$ be Cu-morphisms. Using [APT18b Proposition 5.11], we identify $\varphi_1$ with a compact element in $[S_1, T_1]$, and similarly for $\varphi_2$. It is easy to see that the element $\varphi_1 \boxtimes \varphi_2$ from [Definition 3.11] agrees with the compact element in $[S_1 \otimes S_2, T_1 \otimes T_2]$ that is identified with the tensor product map $\varphi_1 \otimes \varphi_2: S_1 \otimes S_2 \to T_1 \otimes T_2$ from the comments before the above definition.
Notice that there is a certain ambiguity with the notation $\varphi_1 \otimes \varphi_2$, in that it may refer to a Cu-morphism (identified with a compact element in $[S_1 \otimes S_2, T_1 \otimes T_2]$), and also to an element in $[S_1, T_1] \otimes [S_2, T_2]$. However, the precise meaning will be clear from the context.

**Theorem 3.13.** Let $S_1, S_2, T_1$ and $T_2$ be Cu-semigroups, and let $f = (f_\lambda)_\lambda$ and $g = (g_\lambda)_\lambda$ be paths in $\text{Cu}([S_1, T_1])$ and $\text{Cu}([S_2, T_2])$, respectively. For each $\lambda$, consider the generalized Cu-morphism $f_\lambda \otimes g_\lambda : S_1 \otimes S_2 \to T_1 \otimes T_2$. Then $(f_\lambda \otimes g_\lambda)_\lambda$ is a path in $\text{Cu}([S_1 \otimes S_2, T_1 \otimes T_2])$ and we have

$$[f] \otimes [g] = [(f_\lambda \otimes g_\lambda)_\lambda].$$

**Proof.** To show that $(f_\lambda \otimes g_\lambda)_\lambda$ is a path, let $\lambda', \lambda \in I_Q$ satisfy $\lambda' < \lambda$. To show that $f_{\lambda'} \otimes g_{\lambda'} \prec f_\lambda \otimes g_\lambda$, let $t', t \in S_1 \otimes S_2$ satisfy $t' \preceq t$. By properties of the tensor product in Cu, we can choose $n \in \mathbb{N}$, elements $a'_k, a_k \in S_1$ and $b'_k, b_k \in S_2$ satisfying $a'_k \ll a_k$ and $b'_k \ll b_k$ for $k = 1, \ldots, n$, and such that

$$t' \leq \sum_{k=1}^n a'_k \otimes b'_k, \quad \text{and} \quad \sum_{k=1}^n a_k \otimes b_k \leq t.$$ 

We have $f_{\lambda'} \prec f_\lambda$ and $g_{\lambda'} \prec g_\lambda$, and therefore $f_{\lambda'}(a'_k) \ll f_\lambda(a_k)$ and $g_{\lambda'}(b'_k) \ll g_\lambda(b_k)$ for $k = 1, \ldots, n$. Using this at the third step we deduce that

$$(f_{\lambda'} \otimes g_{\lambda'})(t') \leq (f_\lambda \otimes g_\lambda)
\left(\sum_{k=1}^n a'_k \otimes b'_k\right)
= \sum_{k=1}^n f_{\lambda'}(a'_k) \otimes g_{\lambda'}(b'_k)
\ll \sum_{k=1}^n f_\lambda(a_k) \otimes g_\lambda(b_k)
= (f_\lambda \otimes g_\lambda)
\left(\sum_{k=1}^n a_k \otimes b_k\right) \leq (f_\lambda \otimes g_\lambda)(t).$$

Thus, given paths $p = (p_\lambda)_\lambda$ and $q = (q_\lambda)_\lambda$ in $\text{Cu}([S_1, T_1])$ and $\text{Cu}([S_2, T_2])$, respectively, then $(p_\lambda \otimes q_\lambda)_\lambda$ is a path in $\text{Cu}([S_1 \otimes S_2, T_1 \otimes T_2])$. Moreover, it is tedious but straightforward to check that the map $[S_1, T_1] \times [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$ that sends a pair $([p] \otimes [q])$ to $([p_\lambda \otimes q_\lambda])$ is a well-defined Cu-bimorphism. We let $\alpha : [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$ be the induced Cu-morphism.

To show that $[f] \otimes [g] = [(f_\lambda \otimes g_\lambda)_\lambda$, we will prove that the external tensor product $\Box$ and the map $\alpha$ correspond to the same Cu-morphism under the bijection

$$\text{Cu}([S_1, T_1] \otimes [S_2, T_2], [S_1 \otimes S_2, T_1 \otimes T_2]) \cong \text{Cu}([S_1, T_1] \otimes [S_2, T_2], [S_1 \otimes S_2, T_1 \otimes T_2]).$$

Thus, given $p = (p_\lambda)_\lambda$ and $q = (q_\lambda)_\lambda$ in $\text{Cu}([S_1, T_1])$, we will prove that the external tensor product $\Box$ and the map $\alpha$ correspond to the same Cu-morphism under the bijection

$$(p_\lambda \otimes q_\lambda)_\lambda = [p_\lambda](s_1) \otimes [q_\lambda](s_2) = p_1(s_1) \otimes q_1(s_2).$$

Using Theorem 2.7 at the first step, we obtain that

$$\alpha([p_\lambda \otimes q_\lambda](s_1 \otimes s_2)) = [\alpha([p] \otimes [q])](s_1 \otimes s_2) = [\sigma_{S_1 \otimes S_2, T_1 \otimes T_2}([p] \otimes [q])](s_1 \otimes s_2) = (p_1 \otimes q_1)(s_1 \otimes s_2).$$
It follows that $\Box = \alpha$ and therefore

$$[f] \Box [g] = \Box ([f] \otimes [g]) = \alpha([f] \otimes [g]) = [(f \lambda \otimes g \lambda)\lambda].$$

The following result shows that the external tensor product is associative.

**Proposition 3.14.** Let $S_1, S_2, T_1, T_2, P_1$ and $P_2$ be Cu-semigroups, let $x \in [S_1, S_2]$, $y \in [T_1, T_2]$, and let $z \in [P_1, P_2]$. For $k = 1, 2$, we identify $(S_k \otimes T_k) \otimes P_k$ with $S_k \otimes (T_k \otimes P_k)$ using the natural isomorphism from the monoidal structure of $\text{Cu}$ (see comments after [AP][TS] Theorem 2.10). Then

$$(x \Box y) \Box z = x \Box (y \Box z).$$

**Proof.** Given $f \in \text{Cu}[S_1, S_2]$, $g \in \text{Cu}[T_1, T_2]$ and $h \in \text{Cu}[P_1, P_2]$, it is straightforward to check that

$$(f \otimes g) \otimes h = f \otimes (g \otimes h),$$

as generalized Cu-morphisms $S_1 \otimes T_1 \otimes P_1 \to S_2 \otimes T_2 \otimes P_2$. The result follows by applying [Theorem 3.13].

**Problem 3.15.** Study the order-theoretic properties of the external tensor product map $\Box : [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$. In particular, when is this map an order-embedding, when is it surjective?

We recall below the definition of the composition product and analyse its relation with the external tensor product.

**Definition 3.16.** Given Cu-semigroups $S$, $T$ and $P$, we define the composition product

$$\circ : [T, P] \otimes [S, T] \to [S, P]$$

as the Cu-morphism that under the identification

$$\text{Cu}([T, P] \otimes [S, T], [S, P]) \cong \text{Cu}([T, P] \otimes [S, T] \otimes [S, P])$$

corresponds to the composition $\varepsilon_{T,P} \circ (\text{id}_{[T, P]} \otimes \varepsilon_{S,T})$. Given $x \in [S, T]$ and $y \in [T, P]$, we denote the image of $y \otimes x$ under the composition product by $y \circ x$.

Given $x \in [S, T]$, we let $x^* : [T, P] \to [S, P]$ be given by $x^*(y) := y \circ x$ for $y \in [T, P]$. Analogously, given $y \in [T, P]$, we let $y_* : [S, T] \to [S, P]$ be given by $y_*(x) := y \circ x$ for $x \in [S, T]$.

The composition product for the internal-hom satisfies the axioms for the hom sets in an enriched category. In particular, the product is associative and the identity element $\text{id}_S \in \text{Cu}(S, S) \subseteq [S, S]$ acts as a unit for the composition product (see [Kec05] Section 1.6). We recall these facts in the following proposition:

**Proposition 3.17.** Let $S, T, P$ and $Q$ be Cu-semigroups, let $x \in [S, T]$, $y \in [T, P]$, and let $z \in [P, Q]$. Then

$$(z \circ y) \circ x = z \circ (y \circ x).$$

Further, for the identity Cu-morphisms $\text{id}_S \in \text{Cu}(S, S)$ and $\text{id}_T \in \text{Cu}(T, T)$, we have

$$\text{id}_T \circ x = x = x \circ \text{id}_S.$$

It follows that $[S, S]$ and $[T, T]$ are (not necessarily commutative) Cu-semirings and that $[S, T]$ has a natural left $[S, S]$- and right $[T, T]$-semimodule structure; see Propositions 5.1 and 5.6 in the next section. In the following proposition we give an explicit description of the composition product.

**Proposition 3.18.** Let $S, T, P$ be Cu-semigroups, and let $f = (f_\lambda)_\lambda$ and $g = (g_\lambda)_\lambda$ be paths in $\text{Cu}[S, T]$ and $\text{Cu}[T, P]$, respectively. For each $\lambda$, consider the generalized Cu-morphism $g_\lambda \circ f_\lambda : S \to P$. Then $(g_\lambda \circ f_\lambda)_\lambda$ is a path in $\text{Cu}[S, P]$ and

$$[g] \circ [f] = [(g_\lambda \circ f_\lambda)_\lambda].$$
Proof. It is easy to check that \((g_\lambda \circ f_\lambda)_\lambda\) is a path. Moreover, it is tedious but straightforward to check that the map \([T, P] \times [S, T] \to [S, P]\) that sends a pair \(([p], [q])\) to \([[q_\lambda \circ p_\lambda]]_\lambda\) is a well-defined \(\Cu\)-bimorphism. We let \(\alpha : [T, P] \otimes [S, T] \to [S, P]\) be the induced \(\Cu\)-morphism.

To show that \([g] \circ [f] = [[g_\lambda \circ f_\lambda]]_\lambda\), we will prove that the composition product \(\circ\) and the map \(\alpha\) correspond to the same \(\Cu\)-morphism under the bijection

\[
\Cu([T, P] \otimes [S, T], [S, P]) \cong \Cu([T, P] \otimes [S, T] \otimes S, P)
\]

from Theorem 2.7.

Let \(p = (p_\lambda)_\lambda\) and \(q = (q_\lambda)_\lambda\) be paths in \(\Cu[S, T]\) and \(\Cu[T, P]\), respectively, and let \(s \in S\). Set \(p_1 := \sup_{\lambda < 1} p_\lambda\) and \(q_1 := \sup_{\lambda < 1} q_\lambda\). By definition, we have

\[
\tilde{\alpha}([q] \otimes [p] \otimes s) = e_{T, P} \circ (\id_{[T, P]} \otimes e_{S, T})([q] \otimes [p] \otimes s) = e_{T, P}([q] \otimes e_{S, T}([p] \otimes s))
\]

On the other hand, using Theorem 2.7 at the first step, we obtain that

\[
\tilde{\alpha}([q] \otimes [p] \otimes s) = \sigma_{S, P}(\alpha([q] \otimes [p]))(s) = \sigma_{S, P}([[q_\lambda \circ p_\lambda]]_\lambda)(s)
\]

It follows that \(\circ = \alpha\) and therefore

\[
[g] \circ [f] = \alpha([g] \otimes [f]) = \alpha([g_\lambda \circ f_\lambda]).
\]

Note that, in Proposition 3.18, the composition product of two \(\Cu\)-morphisms, viewed as compact elements in the internal-hom set, is the usual composition of morphisms as maps.

Next we show that the composition product is compatible with the evaluation map in the expected way. It will follow later that the evaluation map \(e_{S, S} : [S, S] \otimes S \to S\) defines a natural left \([S, S]\)-semimodule structure on \(S\); see Proposition 5.3.

**Lemma 3.19.** Let \(S, T\) and \(P\) be \(\Cu\)-semigroups, let \(x \in [S, T]\), and let \(y \in [T, P]\). Then

\[
\sigma_{S, P}(y \circ x) = \sigma_{T, P}(y) \circ \sigma_{S, T}(x).
\]

**Proof.** Let \(f = (f_\lambda)_\lambda\) be a path in \(\Cu[S, T]\) representing \(x\), and let \(g = (g_\lambda)_\lambda\) be a path in \(\Cu[T, P]\) representing \(y\). Let \(a \in S\). By Proposition 3.18 we have \(y \circ x = [(g_\lambda \circ f_\lambda)]_\lambda\). Using this at the first step, we obtain that

\[
\sigma_{S, P}(y \circ x)(a) = \sup_{\lambda \in [0, 1]} (g_\lambda \circ f_\lambda)(a) = \sup_{\mu \in [0, 1]} \left(\sup_{\lambda \in [0, 1]} f_\lambda(a)\right) = \sigma_{T, P}(y)(\sigma_{S, T}(x)(a)),
\]

as desired.

By combining Lemma 3.19 with Proposition 3.6, we obtain:

**Proposition 3.20.** Let \(S, T\) and \(P\) be \(\Cu\)-semigroups, let \(x \in [S, T]\), let \(y \in [T, P]\), and let \(a \in S\). Then

\[
(y \circ x)(a) = y(x(a)).
\]

Moreover, for the identity \(\Cu\)-morphism \(\id_S \in \Cu(S, S)\), we have \(\id_S(a) = a\).

The following result shows that the external tensor product and the composition product commute.

**Proposition 3.21.** Let \(S_1, S_2, T_1\) and \(T_2\) be \(\Cu\)-semigroups. Given \(x_k \in [S_k, T_k]\) and \(y_k \in [T_k, P_k]\) for \(k = 1, 2\), we have

\[
(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).
\]
Proof. Let \( f^{(k)} = (f^{(k)}_\lambda)_\lambda \) be a path in \( \text{Cu}[S_k, T_k] \) representing \( x_k \), for \( k = 1, 2 \), and let \( g^{(k)} = (g^{(k)}_\lambda)_\lambda \) be a path in \( \text{Cu}[T_k, P_k] \) representing \( y_k \), for \( k = 1, 2 \). Given \( \lambda \), it is straightforward to check that
\[
(g^{(2)}_\lambda \otimes g^{(1)}_\lambda) \circ (f^{(2)}_\lambda \otimes f^{(1)}_\lambda) = (g^{(2)}_\lambda \circ f^{(2)}_\lambda) \otimes (g^{(1)}_\lambda \circ f^{(1)}_\lambda).
\]
Using this at the second step, and using Theorem 3.13 and Proposition 3.18 at the first and last step, we obtain that
\[
(g_2 \otimes g_1) \circ (x_2 \otimes x_1) = \left[ (g^{(2)}_\lambda \otimes g^{(1)}_\lambda) \circ (f^{(2)}_\lambda \otimes f^{(1)}_\lambda) \right] \lambda
= \left[ (g^{(2)}_\lambda \circ f^{(2)}_\lambda) \otimes (g^{(1)}_\lambda \circ f^{(1)}_\lambda) \right] \lambda
= (g_2 \circ x_2) \otimes (g_1 \circ x_1).
\]

In the last part of this section, we revisit the unit and counit maps, their functorial properties, and how they can be used to implement the adjunction between the tensor product and the internal-hom functors.

**Proposition 3.22.** Let \( S, T \) and \( P \) be \( \text{Cu} \)-semigroups. Then the bijection
\[
\text{Cu}(S, [T, P]) \cong \text{Cu}(S \otimes T, P)
\]
from [Theorem 2.7] identifies a \( \text{Cu} \)-morphism \( f: S \rightarrow [T, P] \) with
\[
ev_{T, P} \circ (f \otimes \text{id}_T): S \otimes T \xrightarrow{f \otimes \text{id}_T} [T, P] \otimes T \xrightarrow{\text{id}_P \otimes T} P.
\]
Conversely, a \( \text{Cu} \)-morphism \( g: S \otimes T \rightarrow P \) is identified with
\[
g_* \circ d_{S,T}: S \xrightarrow{d_{S,T}} [T, S \otimes T] \xrightarrow{g_*} [T, P].
\]
In particular, we have
\[
f = (ev_{T, P} \circ (f \otimes \text{id}_T))_\star \circ d_{S,T}, \quad \text{and} \quad g = ev_{T, P} \circ (g_* \circ d_{S,T} \otimes \text{id}_T).
\]

**Proof.** Let \( f: S \rightarrow [T, P] \) be a \( \text{Cu} \)-morphism. Under the natural bijection from [Theorem 2.7] \( f \) corresponds to the \( \text{Cu} \)-morphism \( \bar{f}: S \otimes T \rightarrow P \) with
\[
\bar{f}(s \otimes t) = \sigma_{T, P}(f(s))(t),
\]
for a simple tensor \( s \otimes t \in S \otimes T \). On the other hand, we have
\[
(ev_{T, P} \circ (f \otimes \text{id}_T))(s \otimes t) = ev_{T, P}(f(s) \otimes t) = \sigma_{T, P}(f(s))(t),
\]
for a simple tensor \( s \otimes t \in S \otimes T \). Thus \( \bar{f} \) and \( ev_{T, P} \circ (f \otimes \text{id}_T) \) agree on simple tensors, and consequently \( \bar{f} = ev_{T, P} \circ (f \otimes \text{id}_T) \), as desired.

Let \( g: S \otimes T \rightarrow P \) be a \( \text{Cu} \)-morphism. Set \( \alpha := g_* \circ d_{S,T} \). Under the natural bijection from [Theorem 2.7] \( \alpha \) corresponds to the \( \text{Cu} \)-morphism \( \bar{\alpha}: S \otimes T \rightarrow P \) with
\[
\bar{\alpha}(s \otimes t) = \sigma_{T, P}(\alpha(s))(t),
\]
for a simple tensor \( s \otimes t \in S \otimes T \). It is straightforward to verify that \( \sigma_{T, P} \circ g_* = g_* \circ \sigma_{T, S \otimes T} \). Using this at the third step, and using [Corollary 3.3] at the fourth step, we deduce that
\[
\bar{\alpha}(s \otimes t) = \sigma_{T, P}(\alpha(s))(t) = (\sigma_{T, P} \circ g_*) \circ d_{S,T}(s)(t)
= (g_* \circ \sigma_{T, S \otimes T} \circ d_{S,T})(s)(t) = g(s \otimes t),
\]
for every simple tensor \( s \otimes t \in S \otimes T \). Thus, \( \bar{\alpha} = g \), as desired. \( \square \)

Applying the previous result to the identity morphisms, we obtain:

**Corollary 3.23.** Let \( S \) and \( T \) be \( \text{Cu} \)-semigroups. Then
\[
id_{[S, T]} = (e_{S,T})_\star \circ d_{[S, T], S} \quad \text{and} \quad id_{S \otimes T} = ev_{S \otimes T} \circ (d_{S, T} \otimes \text{id}_T).
\]
Given \( \text{Cu} \)-semi-groups \( S \) and \( T \), we consider the unit map \( d_{S,T}: S \to [T,S \otimes T] \) from Definition 3.24. Next, we introduce a more general form of the unit map.

**Definition 3.24.** Let \( S, T \) and \( T' \) be \( \text{Cu} \)-semi-groups. We define the **general left unit map** \( S \otimes [T',T] \to [T',S \otimes T] \) as the \( \text{Cu} \)-morphism that under the identification

\[
\text{Cu}(S \otimes [T',T], [T',S \otimes T]) \cong \text{Cu}(S \otimes [T',T] \otimes T', S \otimes T)
\]

corresponds to the map \( \text{id}_S \otimes e_{T,T'} \). Given \( a \in S \) and \( x \in [T',T] \), we denote the image of \( a \otimes x \) under this map by \(.ax \).

Analogously, we define the **general right unit map** \( [T',T] \otimes S \to [T',T \otimes S] \) as the \( \text{Cu} \)-morphism that under the identification

\[
\text{Cu}([T',T] \otimes S, [T',T \otimes S]) \cong \text{Cu}([T',T] \otimes T', S \otimes T)
\]

corresponds to the map \((e_{T,T'} \otimes \text{id}_S) \circ (\text{id}_{[T',T]} \otimes \sigma)\), where \( \sigma \) denotes the flip isomorphism. Given \( a \in S \) and \( x \in [T',T] \), we denote the image of \( x \otimes a \) under this map by \( xa \).

We leave the proof of the following result to the reader.

**Proposition 3.25.** Let \( S, T \) and \( T' \) be \( \text{Cu} \)-semi-groups, let \( a \) be an element in \( S \), and let \( x \) be an element in \( [T',T] \). Let \( i_S: S \to [\mathbb{N},S] \) be the isomorphism from Definition 3.9 and let \( l_{T'}: \mathbb{N} \otimes T' \to T' \) and \( r_{T'}: T' \otimes \mathbb{N} \to T' \) be the natural \( \text{Cu} \)-isomorphism. Then

\[
ax = (i_S(a) \otimes x) \circ l_{T'}^{-1} = d_{S,T}(a) \circ x = (\text{id}_S \otimes x) \circ d_{S,T}(a).
\]

and analogously \( xa = (x \otimes i_S(a)) \circ r_{T'}^{-1} \). Further, for the unit map \( d_{S,T}: S \to [T,S \otimes T] \), we have \( d_{S,T}(a) = a(\text{id}_T) \) for every \( a \in S \).

Finally, similar to KK-theory for \( C^* \)-algebras, we have a general form of the product that simultaneously generalizes the composition product and the external tensor product; see [Bla98, Section 18.9, p.180f].

Let \( P, S_1, S_2, T_1 \) and \( T_2 \) be \( \text{Cu} \)-semi-groups. We let

\[
\Xi_P: [S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2],
\]

be the \( \text{Cu} \)-morphism that under the identification

\[
\text{Cu}([S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2], [S_1 \otimes S_2, T_1 \otimes T_2])
\]

\[
\cong \text{Cu}([S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2] \otimes S_1 \otimes T_2)
\]

corresponds to the composition

\[
(e_{S_1 \otimes P, T_1} \otimes \text{id}_{S_2}) \circ (\text{id}_{[S_1 \otimes P, T_1]} \otimes e_{S_2, P \otimes T_2}) \circ (\text{id}_{[S_1 \otimes P, T_1]} \otimes \sigma_{[S_2, P \otimes T_2], S_1 \otimes T_2}),
\]

where \( \sigma_{[S_2, P \otimes T_2], S_1} \) denotes the flip isomorphism.

Given \( x \in [S_1 \otimes P, T_1] \) and \( y \in [S_2, P \otimes T_2] \), we have

\[
x \Xi_P y = (x \Xi_P y) \circ (\text{id}_{S_1} \Xi_P y).
\]

Specializing to the case \( P = \mathbb{N} \), we obtain the external tensor product, after applying the usual isomorphisms \( S_1 \otimes \mathbb{N} \cong S_1 \) and \( \mathbb{N} \otimes T_2 \cong T_2 \).

Specializing to the case \( T_2 = S_1 = \mathbb{N} \), we obtain the composition product, after applying the usual isomorphisms \( \mathbb{N} \otimes P \cong P \cong P \otimes \mathbb{N} \), \( \mathbb{N} \otimes S_2 \cong S_2 \), and \( T_1 \otimes \mathbb{N} \cong T_1 \).
4. Bivariant Cuntz semigroups of ideals and quotients

A sub-Cu-semigroup of a Cu-semigroup $T$ is a submonoid $S \subseteq T$ that is a Cu-semigroup for the partial order inherited from $T$ and such that the inclusion $S \rightarrow T$ is a Cu-morphism. It is easy to see that $S$ is a sub-Cu-semigroup of $T$ if and only if $S$ is closed under passing to suprema of increasing sequences and if the way-below relation in $S$ and $T$ agree.

**Lemma 4.1.** Let $S$ and $T$ be Cu-semigroups, and let $T' \subseteq T$ be a sub-Cu-semigroup. Then the inclusion map $ι: T' \rightarrow T$ induces an order-embedding $ι_*: [S,T'] \rightarrow [S,T]$.

**Proof.** Let $f = (f_λ)_λ$ be a path in $Cu[S,T']$. Then $\overline{f} := (ι \circ f_λ)_λ$ is a path in $Cu[S,T]$ and we have $ι_*(\overline{f}) = [\overline{f}]$; see the comments after Remark 5.4 in [APT18].

To show that $ι_*$ is an order-embedding, let $x,y \in [S,T']$ with $ι_*(x) ≤ ι_*(y)$. Choose paths $f$ and $g$ in $Cu[S,T']$ representing $x$ and $y$, respectively. We have $(ι \circ f_λ)_λ ≼ (ι \circ g_λ)_λ$. Thus, for every $λ \in I_0$, there is $μ \in I_0$ such that $ι \circ f_λ ≼ ι \circ g_μ$. Using that $T' \subseteq T$ is a sub-Cu-semigroup, for such $λ$ and $μ$ we deduce that $f_λ ≼ g_μ$. (We use that for $a',a \in T'$ we have $a' ≼ a$ in $T'$ if and only if $ι(a') ≼ ι(a)$ in $T$.) It follows that $f ≼ g$, and hence $x ≤ y$, as desired. □

Recall that an ideal of a Cu-semigroup $S$ is a submonoid $J \subseteq S$ that is closed under passing to suprema of increasing sequences and that is downward-hereditary. Every ideal is in particular a sub-Cu-semigroup. (See [APT18 Section 5.1] for an account on ideals and quotients.)

**Proposition 4.2.** Let $S$ and $T$ be Cu-semigroups, and let $J$ be an ideal of $T$. Let $ι: J \rightarrow T$ denote the inclusion map. Then the induced Cu-morphism $ι_*: [S,J] \rightarrow [S,T]$ is an order-embedding that identifies $[S,J]$ with an ideal of $[S,T]$. Moreover, $x \in [S,T]$ belongs to $[S,J]$ if and only if for some (equivalently, for every) path $(f_λ)_λ$ representing $x$, each $f_λ$ takes image in $J$.

**Proof.** By Lemma 4.1 $ι_*$ is an order-embedding. Hence, $ι_*$ identifies $[S,J]$ with a submonoid of $[S,T]$ that is closed under passing to suprema of increasing sequences.

Let $x \in [S,T]$ be represented by a path $f = (f_λ)_λ$ in $Cu[S,T]$. If each $f_λ$ takes values in $J$, then we can consider $f$ as a path in $Cu[S,J]$ whose class is an element $x' \in [S,J]$ satisfying $ι_*(x') = x$. Conversely, assume that $x$ belongs to $[S,J]$. Then there is a path $g = (g_μ)_μ$ in $Cu[S,J]$ with $ι_*(\overline{g}) = x$. Let $λ \in I_0$. Since $f ≼ g$, we can choose $μ \in I_0$ with $f_λ ≼ g_μ$. Since $g_μ$ takes values in $J$, and since $J$ is downward-hereditary, it follows that $f_λ$ takes values in $J$, as desired.

A similar argument shows that $[S,J]$ is downward-hereditary in $[S,T]$. □

**4.3.** Given $S$, let us study whether the functor $[S, -]: Cu \rightarrow Cu$ is exact. More precisely, let $J \triangleleft T$ be an ideal, with inclusion map $ι: J \rightarrow T$ and with quotient map $π: T \rightarrow T/J$. This induces the following Cu-morphisms:

$[S,J] \xrightarrow{ι_*} [S,T] \xrightarrow{π_*} [S,T/J]$.

By Proposition 4.2 $ι_*$ identifies $[S,J]$ with an ideal in $[S,T]$. Since $π \circ ι$ is the zero map, so is $π_* \circ ι_*$. Thus, $π_*$ vanishes on the ideal $[S,J] \triangleleft [S,T]$. It follows that $π_*$ induces a Cu-morphism

$\overline{π}_*: [S,T]/[S,J] \rightarrow [S,T/J]$.

**Problem 4.4.** Study the order-theoretic properties of the Cu-morphism $\overline{π}_*$ from Paragraph 4.3 in particular, when is $\overline{π}_*$ an order-embedding, when is it surjective?

We are currently not aware of any example for $S$ and $J \triangleleft T$ such that the map $\overline{π}_*: [S,T]/[S,J] \rightarrow [S,T/J]$ is not an isomorphism.
The following result and its proof are analogous to Proposition 4.2.

**Proposition 4.5.** Let $S$ and $T$ be Cu-semigroups, let $J \triangleleft S$, and let $\pi: S \to S/J$ denote the quotient map. Then the induced Cu-morphism $\pi^*: [S/J, T] \to [S, T]$ is an order-embedding that identifies $[S/J, T]$ with an ideal in $[S, T]$. Moreover, $x \in [S, T]$ belongs to $[S/J, T]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing $x$, each $f_\lambda$ vanishes on $J$.

4.6. By Propositions 4.2 and 4.5, ideals in $S$ and $T$ naturally induce ideals in $[S, T]$. More precisely, if $J \triangleleft S$ and $K \triangleleft T$, then we can identify $[S/J, K]$ with an ideal in $[S, T]$. Let $\text{Lat}(P)$ denote the ideal lattice of a Cu-semigroup $P$. We obtain a natural map
\[
\text{Lat}(S)^{op} \times \text{Lat}(T) \to \text{Lat}([S, T]).
\]

However, this map need not be injective. For example, consider $S = Z$ and $T = \mathbb{N} \oplus Z$ with the ideal $J = 0 \oplus Z$. Note that every generalized Cu-morphism $Z \to \mathbb{N} \oplus Z$ necessarily takes values in the ideal $0 \oplus Z$. It follows that in this case $[S, J] = [S, T]$.

The following example shows that the above map is also not surjective in general. In fact, the example shows that there exists a simple Cu-semigroup $S$ such that $[S, S]$ is not simple.

**Example 4.7.** Let $S := \{0, 1\} \cup \{\infty\}$, considered with order and addition as a subset of $\mathbb{P}$, with the convention that $a + b = \infty$ whenever $a + b > 1$ in $\mathbb{P}$. It is easy to check that $S$ is a simple Cu-semigroup.

Given $t \in \{0\} \cup [1, \infty]$, let $\varphi_t: S \to S$ be the map given by $\varphi_t(a) := ta$, where $ta$ is given by the usual multiplication in $\mathbb{P}$ applying the above convention that an element is $\infty$ as soon as it is larger than $1$. Then $\varphi_t$ is a generalized Cu-morphism. One can show that every generalized Cu-morphism $S \to S$ is of this form. Hence, $\text{Cu}(S, S)$ is isomorphic to $\{0\} \cup [1, \infty]$, identifying $<$ with $\leq$. It follows that
\[
[S, S] = \tau(\text{Cu}(S, S), <) \cong \tau(\{0\} \cup [1, \infty], \leq) \cong \{0\} \cup [1, \infty] \cup (1, \infty],
\]
which is a disjoint union of compact elements corresponding to $\{0\} \cup [1, \infty]$ and nonzero soft elements corresponding to $(1, \infty]$. (Similar to the decomposition of $Z$ and $R_n$.) In particular, $[S, S]$ contains a compact infinite element $\infty$, and a noncompact infinite element $\infty'$. The set $J := \{x : x \leq \infty'\}$ is an ideal in $[S, S]$. We have $\infty \notin J$, which shows that $[S, S]$ is not simple.

**Problem 4.8.** Characterize when $[S, T]$ is simple. In particular, given simple Cu-semigroups $S$ and $T$, give necessary and sufficient criteria for $[S, T]$ to be simple.

## 5. Cu-semirings and Cu-semimodules

A (unital) Cu-semiring is a Cu-semigroup $R$ together with a Cu-bimorphism $R \times R \to R$, denoted by $(r_1, r_2) \mapsto r_1r_2$, and a distinguished element $1 \in R$, called the unit of $R$, such that $r_1(r_2r_3) = (r_1r_2)r_3$ and $1 = r = 1r$ for all $r, r_1, r_2, r_3 \in R$. This concept was introduced and studied in [APT18 Chapter 7], where it is further assumed that the product is commutative. We will not make this assumption here.

We let $\mu_R: R \otimes R \to R$ denote the Cu-morphism induced by multiplication in $R$.

The following result follows from the general properties of the composition product for the internal-hom (see [Kei05 Section 1.6]).

**Proposition 5.1.** Let $S$ be a Cu-semigroup. Then $[S, S]$ is a Cu-semiring with product given by the composition product $\circ: [S, S] \otimes [S, S] \to [S, S]$, and with unit element given by the identity map $\text{id}_S \in [S, S]$. 


Remark 5.2. Let $S$ be a Cu-semigroup. The identity map $id_S : S \to S$ is a Cu-morphism. Therefore, the unit of the Cu-semiring $[S, S]$ is compact.

In Example 5.3, we will see that $[S, S]$ is noncommutative in general.

Given a Cu-semiring $R$, a left Cu-semimodule over $R$ is a Cu-semigroup $S$ together with a Cu-bimorphism $R \times S \to S$, denoted by $(r, a) \mapsto ra$, such that for all $r_1, r_2 \in R$ and $a \in S$, we have $(r_1 r_2)a = r_1(r_2a)$ and $1a = a$. We also say that $S$ has a left action of $R$ if $S$ is a left Cu-semimodule over $R$. Right Cu-semimodules are defined analogously. If $R_1$ and $R_2$ are Cu-semirings, we say that a Cu-semigroup $S$ is a $(R_1, R_2)$-Cu-semimodule if it has a left $R_1$-action and a right $R_2$-action that satisfy $r_1( ar_2) = (r_1a)r_2$ for all $r_1 \in R_1$, $r_2 \in R_2$ and $a \in S$.

We refer the reader to [APT18, Chapter 7] for a discussion on commutative Cu-semirings and their Cu-semimodules.

Proposition 5.3. Let $S$ be a Cu-semigroup. Then $e_{S,S} : [S, S] \otimes S \to S$ defines a left action of $[S, S]$ on $S$.

**Proof.** It follows directly from Proposition 3.20 that the action of $[S, S]$ on $S$ is associative and that $id_S$ acts as a unit. □

5.4. Let $R$ be a Cu-semiring, let $S$ be a Cu-semigroup, and let $T$ be a Cu-semigroup with a left $R$-action $\alpha : R \otimes T \to T$. Consider the general left unit map $R[ [S, T] \to [S, R \otimes T]$ from Definition 3.23. Postcomposing with $\alpha_* : [S, R \otimes T] \to [S, T]$ we obtain a Cu-morphism $\alpha_S : R \otimes [S, T] \to [S, R \otimes T]$.

Let $r \in R$ and $x \in [S, T]$. We denote $\alpha_S(r \otimes x)$ by $rx$. Choose a path $f = (f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}$ representing $x$, and pick a path $(r_\lambda)_{\lambda \in R}$ with endpoint $r$. For each $\lambda$, let $r_\lambda f : S \to T$ be given by $s \mapsto r_\lambda f(s)$. Then $(r_\lambda f_\lambda)$ is a path in $Cu[S, T]$ and

$$rx = [(r_\lambda f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}].$$

Proposition 5.5. Let $R$ be a Cu-semiring with compact unit, let $S$ be a Cu-semigroup, and let $T$ be a Cu-semigroup with a left $R$-action $\alpha : R \otimes T \to T$. Then the map $\alpha_S : R \otimes [S, T] \to [S, R \otimes T]$ from Paragraph 5.4 defines a left $R$-action on $[S, T]$.

**Proof.** Let $r, r' \in R$ and $x \in [S, T]$. Choose a path $f = (f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}$ representing $x$. Choose paths $(r_\lambda)_{\lambda \in R}$ and $(r'_\lambda)_{\lambda \in R}$ with endpoints $r$ and $r'$, respectively. Then $(r_\lambda r'_\lambda)$ is a path in $R$ with endpoint $rr'$. Using the description of the $R$-action on $[S, T]$ from the end of Paragraph 5.4 we deduce that

$$(rr')x = [(r_\lambda r'_\lambda f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}] = [(r_\lambda (r'_\lambda f_\lambda))_{\lambda \in \mathbb{Cu}[S, T]}] = r[(r'_\lambda f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}] = r(rr')x.$$ 

Let 1 denote the unit element of $R$. For every $f \in Cu[S, T]$, we have $1f = f$. Since 1 is compact, the constant function with value 1 is a path in $R$ with endpoint 1. It follows that

$$1x = [(1f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}] = [(f_\lambda)_{\lambda \in \mathbb{Cu}[S, T]}] = x.$$ □

Proposition 5.6. Let $S$ and $T$ be Cu-semigroups. Then the composition product $\circ : [T, T] \otimes [S, T] \to [S, T]$ defines a left action of the Cu-semiring $[T, T]$ on $[S, T]$. Analogously, we obtain a right action of $[S, S]$ on $[S, T]$. These actions are compatible and thus $[S, T]$ is a $(T, [T, T], [S, S])$-Cu-semimodule.

**Proof.** This follows directly from the associativity of the composition product; see Proposition 3.17. □

Remark 5.7. Let $S$ and $T$ be Cu-semigroups. By Proposition 5.3, the evaluation map $ev_T : [T, T] \otimes T \to T$ from Definition 3.5 defines a left action of $[T, T]$ on $T$. By Proposition 5.5, this induces a left action of $[T, T]$ on $[S, T]$. This action agrees with that from Proposition 5.6.
Remark 5.10. The Cu-morphism $\mathbf{id}$ is a generalized Cu-morphism given by left multiplication with $\mu$. We have $
abla(\mu)$.

Let $k, l \in \mathbb{N}$. We claim that $[\mathbb{N}^k, \mathbb{N}^l]$ can be identified with $M_{l,k}(\mathbb{N})$, the $l \times k$-matrices with entries in $\mathbb{N}$, with order and addition defined entrywise. Thus, as a Cu-semigroup, $[\mathbb{N}^k, \mathbb{N}^l]$ is isomorphic to $\mathbb{N}^{kl}$. However, the presentation as matrices allows to expatiate the composition product.

First, let $\varphi: \mathbb{N}^k \to \mathbb{N}^l$ be a generalized Cu-morphism. For each $x \in \mathbb{N}$, consider the vector $\varphi(x)$ in $\mathbb{N}^k$ and let $x_{1,j}, \ldots, x_{l,j}$ denote its coefficients. This defines a matrix $x = (x_{i,j})_{i,j}$ with $l \times k$ entries in $\mathbb{N}$. It is readily verified that the coefficients of $\varphi(v)$ are obtained by multiplication of the matrix $x$ with the vector of coefficients of $v$. We identify $\varphi$ with the associated matrix $x$ in $M_{l,k}(\mathbb{N})$.

Let $\varphi, \psi: \mathbb{N}^k \to \mathbb{N}^l$ be generalized Cu-morphisms with associated matrices $x$ and $y$ in $M_{l,k}(\mathbb{N})$. It is straightforward to check that $\varphi \prec \psi$ if and only if $x_{i,j} \leq y_{i,j}$ for each $i, j$, and thus the claim follows.

Given $k, l, m \in \mathbb{N}$, consider the composition product

$$[\mathbb{N}^k, \mathbb{N}^m] \otimes [\mathbb{N}^k, \mathbb{N}^l] \to [\mathbb{N}^{kl}, \mathbb{N}^m].$$

After identifying $[\mathbb{N}^k, \mathbb{N}^m]$ with $M_{l,k}(\mathbb{N})$, identifying $[\mathbb{N}^k, \mathbb{N}^m]$ with $M_{m,l}(\mathbb{N})$, and identifying $[\mathbb{N}^k, \mathbb{N}^m]$ with $M_{m,k}(\mathbb{N})$, the composition product is given as a map

$$M_{m,l}(\mathbb{N}) \otimes M_{l,k}(\mathbb{N}) \to M_{m,k}(\mathbb{N}).$$

It is straightforward to check that this map is induced by matrix multiplication. In particular, the Cu-semiring $[\mathbb{N}^k, \mathbb{N}^l]$ can be identified with the matrix ring $M_{k,k}(\mathbb{N})$. Thus, for $k \geq 2$, the Cu-semiring $[\mathbb{N}^k, \mathbb{N}^l]$ is not commutative.

Given a Cu-semiring $R$, recall that $\mu_R: R \otimes R \to R$ denotes the Cu-morphism induced by multiplication.

Definition 5.9. Given a Cu-semiring $R$, we let $\pi_R: R \rightarrow [R, R]$ be the Cu-morphism that corresponds to $\mu_R$ under the identification $\text{Cu}(R, [R, R]) \cong \text{Cu}(R \otimes R, R)$.

Remark 5.10. The Cu-morphism $\pi_R$ can be regarded as a kind of left regular representation of $R$.

Lemma 5.11. We have $\pi_R = (\mu_R) \circ d_{R,R}$ and $e_{R,R} \circ (\pi_R \otimes \mathbf{id}_R) = \mu_R$.

Proof. The first equality follows from Proposition 3.22. It is straightforward to show that $e_{R,R} \circ (\mu_R) \circ d_{R,R} = \mu_R \circ e_{R,R} \circ d_{R,R}$. Further, we have $e_{R,R} \circ (d_{R,R} \otimes \mathbf{id}_R) = \mathbf{id}_{R,R}$ by Corollary 3.23. Using these equations, we deduce that

$$e_{R,R} \circ (\pi_R \otimes \mathbf{id}_R) = e_{R,R} \circ ((\mu_R) \circ d_{R,R} \otimes \mathbf{id}_R)$$

$$= e_{R,R} \circ (\mu_R) \circ (d_{R,R} \otimes \mathbf{id}_R)$$

$$= \mu_R \circ e_{R,R} \circ (d_{R,R} \otimes \mathbf{id}_R) = \mu_R.$$

\[\square\]

Theorem 5.12. Let $R$ be a Cu-semiring. Then $\pi_R: R \rightarrow [R, R]$ is multiplicative. If the unit element of $R$ is compact, then $\pi_R$ is unital.

Proof. Let $M: [R, R] \otimes [R, R] \rightarrow [R, R]$ denote the composition map. We need to show that $M \circ (\pi_R \otimes \pi_R) = \pi_R \circ \mu_R$.

Given $r, s \in R$, choose paths $r = (r_\lambda)$ and $s = (s_\lambda)$ in $(R, \ll)$ with endpoints $r$ and $s$, respectively. For each $\lambda$, let $f_\lambda: R \rightarrow R$ and $g_\lambda: R \rightarrow R$ be the generalized Cu-morphism given by left multiplication with $r_\lambda$ and $s_\lambda$, respectively. By
Proposition 3.2 we have $d_{R,R}(r) = [(r_\lambda \otimes \cdot)_{\lambda}]$, where $r_\lambda \otimes \cdot : R \to R \otimes R$ is the map sending $t \in R$ to $r_\lambda \otimes t$. We also have $\mu_R \circ (r_\lambda \otimes \cdot) = f_\lambda$. Since $\pi_R = (\mu_R) \circ d_{R,R}$ by Lemma 5.11 it follows that $\pi_R(r) = [(f_\lambda)_{\lambda}]$. Likewise, we deduce $\pi_R(s) = [(g_\lambda)_{\lambda}]$.

By Proposition 3.18 we obtain $M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_{\lambda}]$.

As the product in $R$ is associative, the composition $f_\lambda \circ g_\lambda$ is the generalized Cu-morphism $h_\lambda$ defined by left multiplication with $r_\lambda s_\lambda$. Notice that $(r_\lambda s_\lambda)_{\lambda}$ is a path in $(R, \leq)$ with endpoint $rs$. Therefore, $\pi_R(rs) = [(h_\lambda)_{\lambda}]$. Altogether, we get the desired equality

$$M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_{\lambda}] = [(h_\lambda)_{\lambda}] = \pi_R(\mu_R(r \otimes s)).$$

To show the second statement, let us assume that the unit $1_R$ is a path in $(R, \leq)$ with endpoint $1_R$. It follows easily as in the first part of the proof that $\pi_R(1_R) = [(id_R)_{\lambda}] = id_R$. □

Definition 5.13. Let $R$ be a Cu-semiring with unit $1_R$. Then $\varepsilon_R : \llbracket R, R \rrbracket \to R$ is the generalized Cu-morphism given by

$$\varepsilon_R([f]) = \sup_{\lambda} f_\lambda(1_R),$$

for a path $f = (f_\lambda)_{\lambda}$ in Cu$[R,R]$.

Remark 5.14. Let $\sigma_{R,R} : \llbracket R, R \rrbracket \to \text{Cu}[R,R]$ denote the endpoint map as in Definition 2.6. Then $\varepsilon_R(x) = \sigma_{R,R}(x)(1_R)$ for every $x \in \llbracket R, R \rrbracket$.

Lemma 5.15. We have $\varepsilon_R \circ \pi_R = id_{R,R}$.

Proof. Given $r \in R$, choose a path $(r_\lambda)_{\lambda}$ in $(R, \leq)$ with endpoint $r$, and for each $\lambda$ let $f_\lambda : R \to R$ be given by left multiplication with $r_\lambda$. As in the proof of Theorem 5.12 we obtain $\pi_R(r) = [(f_\lambda)_{\lambda}]$, whence

$$\varepsilon_R(\pi_R(r)) = \varepsilon_R([(f_\lambda)_{\lambda}]) = \sup_{\lambda} f_\lambda(1_R) = \sup_{\lambda} (r_\lambda 1_R) = r.$$ □

Proposition 5.16. Let $R$ be a Cu-semiring. Then $\pi_R : R \to \llbracket R, R \rrbracket$ is a multiplicative order-embedding. Thus, in a natural way, $R$ is a sub-semiring of $\llbracket R, R \rrbracket$. If the unit of $R$ is compact, then $R$ is even a unital sub-semiring of $\llbracket R, R \rrbracket$.

Proof. By Lemma 5.15 we have $\varepsilon_R \circ \pi_R = id_{R,R}$, which implies that $\pi_R$ is an order-embedding. By Theorem 5.12 $\pi_R$ is a (unital) multiplicative Cu-morphism. □

An element $a$ in a Cu-semigroup $S$ is soft if for every $a' \in S$ with $a' \ll a$ there exists $k \in \mathbb{N}$ with $(k + 1)a' \leq ka$; see [APT18] Definition 5.3.1. The following result will be used below.

Lemma 5.17. Let $S$ and $T$ be Cu-semigroups, let $\varphi : S \to T$ be a generalized Cu-morphism, and let $a \in S$ be soft. Then $\varphi(a)$ is soft.

Proof. To verify that $\varphi(a)$ is soft, let $x \in T$ satisfy $x \ll \varphi(a)$. Using that $\varphi$ preserves suprema of increasing sequences, we can choose $a' \in S$ with $a' \ll a$ and $x \leq \varphi(a')$. (Indeed, applying (O2) in $S$, choose a $\ll$-increasing sequence $(a_n)_n$ in $S$ with supremum $a$. Then $\varphi(a) = \sup_n \varphi(a_n)$, whence there is $n$ with $x \leq \varphi(a_n)$. Since $a$ is soft, we can choose $k \in \mathbb{N}$ such that $(k + 1)a' \leq ka$. Then

$$(k + 1)x \leq (k + 1)\varphi(a') = \varphi((k + 1)a') \leq \varphi(ka) = k\varphi(a).$$ □

Let $\mathbb{P} = [0, \infty]$, with natural order and addition. Recall that $\mathbb{P}$ is isomorphic to the Cuntz semigroup of the Jacelon-Razak algebra (see Jac13 and Roh13). The usual multiplication of real numbers extends to $\mathbb{P}$. This gives $\mathbb{P}$ the structure of a commutative Cu-semiring.
Example 5.18. Let \( M_1 = [0, \infty) \sqcup (0, \infty) \), a disjoint union of compact elements \([0, \infty)\) and nonzero soft elements \((0, \infty)\). Recall that \( M_1 \) denotes the Cuntz semigroup of a \(_3\)-factor; see \cite{APT18} Example 4.14 and \cite{APT18} Proposition 4.15. We identify \( [0, \infty] \) with the sub-Cu-semigroup of soft elements in \( M_1 \), and we define the Cu-morphism \( \varrho: M_1 \to [0, \infty] \subseteq M_1 \) by fixing all soft elements and by sending a compact to the soft element of the same value.

We define a product on \( M_1 \) as follows: We equip the compact part \([0, \infty)\) with the usual multiplication of real numbers, and similarly for the product in \((0, \infty)\). The product of any element with 0 is 0. Given a nonzero compact element \( a \) and a nonzero soft element \( b \), their product is defined as the soft element \( ab := \varrho(a)b \).

This gives \( M_1 \) the structure of a commutative Cu-semiring. Moreover, we may identify \( [0, \infty] \) with the (nonunital) sub-Cu-semiring of soft elements in \( M_1 \). The map \( \varrho: M_1 \to [0, \infty] \) is multiplicative. One can show that the map \( \pi_{M_1}: M_1 \to \llbracket M_1, M_1 \rrbracket \) is an isomorphism.

Example 5.19. We have \([0, \infty], [0, \infty] \cong M_1 \). The map \( \pi: [0, \infty, [0, \infty] \to \llbracket [0, \infty], [0, \infty] \rrbracket \) embeds \([0, \infty] \) as the sub-Cu-semiring of soft elements in \( M_1 \). In particular, \( \pi \) is not unital.

Proof. We have \([0, \infty], [0, \infty] \cong M_1 \) by \cite{APT18} Proposition 5.13. By Proposition 5.16 \( \pi \) is multiplicative order-embedding. Note that every element of \([0, \infty] \) is soft. By Lemma 5.17 a generalized Cu-morphism maps soft elements to soft elements. Thus, the image of \( \pi \) is contained in the soft elements of \( M_1 \). It easily follows that \( \pi \) identifies \([0, \infty] \) with the soft elements in \( M_1 \). Since the unit of \( M_1 \) is compact, it also follows that \( \pi \) is not unital. \( \qed \)

We finally turn our attention to solid semirings. Recall from \cite{APT18} Definition 7.1.5 that a Cu-semiring \( R \) is said to be solid if \( \mu_R: R \otimes R \to R \) is an isomorphism. In \cite{APT18}, all Cu-semirings were required to be commutative, and thus a solid Cu-semiring was assumed to be commutative. Next, we show that this assumption is not necessary since a Cu-semiring is automatically commutative as soon as \( \mu_R \) is injective.

Lemma 5.20. Let \( R \) be a Cu-semiring such that \( \mu_R: R \otimes R \to R \) is injective. Then \( R \) is commutative and \( \mu_R \) is an isomorphism (and consequently \( R \) is solid.)

Proof. To show that \( R \) is commutative, let \( a, b \in R \). We have

\[
\mu_R(1 \otimes a) = a = \mu_R(a \otimes 1),
\]

and thus \( 1 \otimes a = a \otimes 1 \) in \( R \otimes R \). Consider the shuffle Cu-morphism \( \alpha: R \otimes R \otimes R \to R \otimes R \otimes R \) that satisfies \( \alpha(x \otimes y \otimes z) = y \otimes x \otimes z \) for every \( x, y, z \in R \). Then

\[
1 \otimes b \otimes a = \alpha(b \otimes 1 \otimes a) = \alpha(b \otimes a \otimes 1) = a \otimes b \otimes 1
\]

in \( R \otimes R \otimes R \). By the associativity of the product in \( R \), we get \( ba = ab \), as desired.

Thus, \( R \) is commutative and \( 1 \otimes a = a \otimes 1 \) in \( R \otimes R \), for every \( a \in R \). Using \cite{APT18} Proposition 7.1.6], this implies that \( R \) is solid. \( \qed \)

Let \( R \) be a solid Cu-semiring, and let \( S \) be a Cu-semigroup. It was shown in \cite{APT18} Corollary 7.1.8 that any two \( R \)-actions on \( S \) agree. (Since \( R \) is commutative, we need not distinguish between left and right \( R \)-actions.) Thus, \( S \) either has a (unique) \( R \)-action, or it does not admit any \( R \)-action, which means that having an \( R \)-action is a property rather than an additional structure for \( S \), which justifies the following definition.

Definition 5.21. Let \( R \) be a solid Cu-semiring, and let \( S \) be a Cu-semigroup. We say that \( S \) is \( R \)-stable if \( S \) has an \( R \)-action.
Remark 5.22. In [APT18], we said that $S$ ‘has $R$-multiplication’ if it has an $R$-action. Given a solid ring $R$, it was shown [APT18, Theorem 7.1.12] that $S$ is $R$-stable if and only if $S \cong R \otimes S$.

Recall that a $C^*$-algebra $A$ is said to be $Z$-stable if $A \cong Z \otimes A$, and similarly one defines being UHF-stable and $\mathbb{C}_\infty$-stable. Thus, the terminology of being ‘$R$-stable’ for $Cu$-semigroups is analogous to the terminology used for $C^*$-algebras.

Theorem 5.23. Given a $Cu$-semiring $R$, consider the following statements:

1. $R$ is solid, that is, $\mu: R \otimes R \rightarrow R$ is an isomorphism.
2. The map $e_{R,R}: [R, R] \otimes R \rightarrow R$ is an isomorphism.
3. The map $\pi_R \otimes id_R: R \otimes R \rightarrow [R, R] \otimes R$ is an isomorphism.
4. The map $\pi_R: R \rightarrow [R, R]$ is an isomorphism.
5. The map $\varepsilon: [R, R] \rightarrow R$ is an isomorphism.

Then the following implications hold:

$(1) \iff (2) \implies (3) \iff (4) \iff (5)$.

Further, if $R$ satisfies (1) and (3), then it satisfies (2). The $Cu$-semiring $\mathbb{F}$ satisfies (1), (2) and (3), but not (4); see Example 5.17. The $Cu$-semiring $M_1$ satisfies (3) and (4) but neither (1) nor (2); see Example 5.18.

Proof. By Lemma 5.15, we have $\varepsilon \circ \pi_R = id_R$. It follows that $\varepsilon$ is an isomorphism if and only if $\pi_R$ is, which shows the equivalence of (4) and (5). It is obvious that (4) implies (3). To show that (2) implies (1), assume that $e_{R,R}$ is an isomorphism. We have

$$(\varepsilon \otimes id_R) \circ (\pi_R \otimes id_R) = id_R \otimes id_R,$$

which shows that $\pi_R \otimes id_R$ is an order-embedding. Hence, $e_{R,R} \circ (\pi_R \otimes id_R)$ is an order-embedding. By Lemma 5.11, we have $e_{R,R} \circ (\pi_R \otimes id_R) = \mu_R$, whence $\mu_R$ is an order-embedding. By Lemma 5.20, this implies that $R$ is solid.

Using again that $e_{R,R} \circ (\pi_R \otimes id_R) = \mu_R$, if any two of the three maps $e_{R,R}$, $\pi_R \otimes id_R$ and $\mu_R$ are isomorphisms, then so is the third. This shows that (2) implies (3), and that the combination of (1) and (3) implies (2).

Question 5.24. Given a solid $Cu$-semiring $R$, is the evaluation map $e_{R,R}: [R, R] \otimes R \rightarrow R$ an isomorphism?

Remark 5.25. Let $R$ be a solid $Cu$-semiring. The answer to Question 5.24 is ‘yes’ in the following cases:

1. If the unit of $R$ is compact; see Remark 5.30 below.
2. If $R$ satisfies (O5) and (O6). This follows from the classification of solid $Cu$-semirings with (O5) obtained in [APT18, Theorem 8.3.13] which shows that each such $Cu$-semiring is either isomorphic to $\mathbb{F}$ or has a compact unit. In either case, Question 5.24 has a positive answer.

In particular, a $Cu$-semiring $R$ with compact unit is solid if and only if the evaluation map $e_{R,R}: [R, R] \otimes R \rightarrow R$ is an isomorphism.

Theorem 5.26. Let $R$ be a solid $Cu$-semiring with compact unit, and let $S$ and $T$ be $Cu$-semigroups. Assume that $T$ is $R$-stable. Then $[S, T]$ is $R$-stable, and hence $[S, T] \cong R \otimes [S, T]$.

Proof. Since the unit of $R$ is compact, it follows from Proposition 5.5 that $[S, T]$ has a left $R$-action. Since $R$ is solid, this implies that $[S, T]$ is $R$-stable.

Lemma 5.27. Let $R$ be a solid $Cu$-semiring, let $S$ and $T$ be $Cu$-semigroups, and let $f, g: R \otimes S \rightarrow T$ be a generalized $Cu$-morphisms. Assume that $T$ is $R$-stable. Then $f \leq g$ if and only if $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$. □
If the unit of $R$ is compact, then $f \prec g$ if and only if $f(1 \otimes a') \ll g(1 \otimes a)$ for all $a', a \in S$ with $a' \ll a$.

Proof. The forward implications are obvious. To show the converse of the first statement, assume that $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$. To verify $f \leq g$, it is enough to show that $f(r \otimes a) \leq g(r \otimes a)$ for all $r \in R$ and $a \in S$. Note that $R \otimes S$ and $T$ are $R$-stable. Since $R$ is solid, every generalized Cu-morphism between $R$-stable Cu-semigroups is automatically $R$-linear; see [APT18, Proposition 7.1.6]. Thus, given $r \in R$ and $a \in S$, we obtain

$$f(r \otimes a) = f(r(1 \otimes a)) = rf(1 \otimes a) \leq rg(1 \otimes a) = g(r \otimes a).$$

To show the converse of the second statement, assume that $f(1 \otimes a') \ll g(1 \otimes a)$ for all $a', a \in S$ with $a' \ll a$. To verify $f \prec g$, it is enough to show that $f(r' \otimes a') \ll g(r \otimes a)$ for all $r', r \in R$ and $a', a \in S$ with $r' \ll r$ and $a' \ll a$. Given such $r', r, a', a$, we use at the second step that multiplication in $R$ preserves the joint way-below relation, to deduce

$$f(r' \otimes a') = r'f(1 \otimes a') \ll rg(1 \otimes a) = g(r \otimes a).$$

\[\Box\]

Proposition 5.28. Let $R$ be a solid Cu-semiring with compact unit, and let $S$ and $T$ be Cu-semigroups. Assume that $T$ is $R$-stable. Let $\alpha: S \to R \otimes S$ be the Cu-morphism given by $\alpha(a) = 1 \otimes a$, for $a \in S$. Then the induced map $\alpha^*: [R \otimes S, T] \to [S, T]$ is an isomorphism.

Proof. Consider $\alpha_Q^*: Cu[R \otimes S, T] \to Cu[S, T]$ given by sending a generalized Cu-morphism $f: R \otimes S \to T$ to the generalized Cu-morphism $\alpha_Q(f)$ given by

$$\alpha_Q(f)(a) = f(1 \otimes a),$$

for $a \in S$. It follows from [Lemma 5.27] that $\alpha_Q^*$ is an isomorphism of $\mathcal{Q}$-semigroups. Since $\alpha^*$ is obtained by applying the functor $\tau$ to $\alpha_Q^*$ (see the comments after [APT18] Remark 5.4), it follows that $\alpha^*$ is an isomorphism, as desired. \[\Box\]

Corollary 5.29. Let $R$ be a solid Cu-semiring with compact unit, and let $T$ be an $R$-stable Cu-semigroup. Then there is a natural isomorphism $[R, T] \cong T$.

Proof. Applying Proposition 5.28 for $S := \mathbb{N}$, we obtain $[R, T] \cong [\mathbb{N}, T]$. By Proposition 3.10 we have a natural isomorphism $[\mathbb{N}, T] \cong T$. \[\Box\]

Remark 5.30. Let $R$ be a solid Cu-semiring with compact unit. Since $R$ is $R$-stable itself, it follows from Corollary 5.29 that $[R, R] \cong R$. It follows that the evaluation map $e_{R,R}: [R, R] \otimes R \to R$ is an isomorphism.

For the solid Cu-semiring $\mathcal{P}$, we have seen in [APT18] Proposition 5.13 that $[\mathcal{P}, \mathcal{P}] \cong M_1 \not\cong \mathcal{P}$. This shows that Proposition 5.28 and Corollary 5.29 cannot be generalized to solid Cu-semirings without compact unit.

References

[ADPS14] R. Antoine, M. Dadarlat, F. Perera, and L. Santiago, Recovering the Elliott invariant from the Cuntz semigroup, Trans. Amer. Math. Soc. 366 (2014), 2907–2922. [MR 3180735] [Zbl 06303118].

[APRT18a] R. Antoine, F. Perera, L. Robert, and H. Thiel, $\mathcal{C}^*$-algebras of stable rank one and their Cuntz semigroups, preprint (arXiv:1809.03894 [math.OA]), 2018.

[APRT18] R. Antoine, F. Perera, and H. Thiel, Tensor products and regularity properties of Cuntz semigroups, Mem. Amer. Math. Soc. 251 (2018), viii+191. [MR 3796007].

[APT18b] R. Antoine, F. Perera, and H. Thiel, Abstract bivariant Cuntz semigroups, Int. Math. Res. Not., rny143 (2018), http://dx.doi.org/10.1093/imrn/rny143.

[APT18c] R. Antoine, F. Perera, and H. Thiel, Cuntz semigroups of ultraproduct $\mathcal{C}^*$-algebras, in preparation, 2018.
ABSTRACT BIVARIANT CUNTZ SEMIGROUPS II

[Bla98] B. Blackadar, \textit{K-theory for operator algebras}, second ed., Mathematical Sciences Research Institute Publications 5, Cambridge University Press, Cambridge, 1998. MR 1656031 Zbl 0959.46072.

[BPT08] N. P. Brown, F. Perera, and A. S. Toms, The Cuntz semigroup, the Elliott conjecture, and dimension functions on C*-algebras, \textit{J. Reine Angew. Math.} 621 (2008), 191–211. MR 2431254 Zbl 1158.46040.

[CEI08] K. T. Coward, G. A. Elliott, and C. Ivanescu, The Cuntz semigroup as an invariant for C*-algebras, \textit{J. Reine Angew. Math.} 623 (2008), 161–193. MR 2458043 Zbl 1161.46029.

[Cun78] J. Cuntz, Dimension functions on simple C*-algebras, \textit{Math. Ann.} 233 (1978), 145–153. MR 0467332 Zbl 0354.46043.

[GHK+03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, \textit{Continuous lattices and domains}, Encyclopedia of Mathematics and its Applications 93, Cambridge University Press, Cambridge, 2003. MR 1975381 Zbl 1088.06001.

[Jac13] B. Jacelon, A simple, monotracial, stably projectionless C*-algebra, \textit{J. Lond. Math. Soc. (2)} 87 (2013), 365–383. MR 3046276 Zbl 1275.46047.

[Kel05] G. M. Kelly, Basic concepts of enriched category theory, \textit{Repr. Theory Appl. Categ.} (2005), vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301 Zbl 1086.18001.

[Rob12] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, \textit{Adv. Math.} 231 (2012), 2802–2836. MR 2970466 Zbl 1286.46043.

[Rob13] L. Robert, The cone of functionals on the Cuntz semigroup, \textit{Math. Scand.} 113 (2013), 161–186. MR 3145179 Zbl 1286.46061.

[Tom08] A. S. Toms, On the classification problem for nuclear C*-algebras, \textit{Ann. of Math. (2)} 167 (2008), 1029–1044. MR 2415391 Zbl 1181.46037.

[WZ09] W. Winter and J. Zacharias, Completely positive maps of order zero, \textit{Münster J. Math.} 2 (2009), 311–324. MR 2554517 Zbl 1190.46042.

Ramon Antoine, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
Email address: ramon@mat.uab.cat

Francesc Perera, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
Email address: perera@mat.uab.cat

Hannes Thiel, Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
Email address: hannes.thiel@uni-muenster.de