THE STRONG SLOPE CONJECTURE FOR TWISTED GENERALIZED WHITEHEAD DOUBLES

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ABSTRACT. The Slope Conjecture proposed by Garoufalidis asserts that the degree of the colored Jones polynomial determines a boundary slope, and its refinement, the Strong Slope Conjecture proposed by Kalfagianni and Tran asserts that the linear term in the degree determines the topology of an essential surface that satisfies the Slope Conjecture. Under certain hypotheses, we show that twisted, generalized Whitehead doubles of a knot satisfies the Slope Conjecture and the Strong Slope Conjecture if the original knot does. Additionally, we provide a proof that there are Whitehead doubles which are not adequate.

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2010 Mathematics Subject Classification. Primary 57M25, 57M27
1. Introduction

Let $K$ be a knot in the 3–sphere $S^3$. The Slope Conjecture of Garoufalidis [9] and the Strong Slope Conjecture of Kalfagianni and Tran [20] propose relationships between a quantum knot invariant, the degrees of the colored Jones function of $K$, and a classical invariant, the boundary slope and the topology of essential surfaces in the exterior of $K$.

The colored Jones function of $K$ is a sequence of Laurent polynomials $J_{K,n}(q) \in \mathbb{Z}[q^\pm 1]$ for $n \in \mathbb{N}$, where $J_{\bigcirc,n}(q) = \frac{a^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ for the unknot $\bigcirc$ and $\frac{J_K(q)}{J_{\bigcirc,n}(q)}$ is the ordinary Jones polynomial of $K$. Since the colored Jones function is $q$–holonomic [11, Theorem 1], the degrees of its terms are given by quadratic quasi-polynomials for suitably large $n$ [10, Theorem 1.1 & Remark 1.1]. For the maximum and minimum degrees $d_+[J_{K,n}(q)]$ and $d_- [J_{K,n}(q)]$, we set these quadratic quasi-polynomials to be

$$\delta_K(n) = a(n)n^2 + b(n)n + c(n) \quad \text{and} \quad \delta_K(n) = a^*(n)n^2 + b^*(n)n + c^*(n)$$

for rational valued periodic functions $a(n), b(n), c(n)$ and $a^*(n), b^*(n), c^*(n)$ with integral period. Now define the sets of Jones slopes of $K$:

$$js(K) = \{4a(n) \mid n \in \mathbb{N}\} \quad \text{and} \quad js(K) = \{4a^*(n) \mid n \in \mathbb{N}\}.$$ 

Allowing surfaces to be disconnected, we say a properly embedded surface in a 3–manifold is essential if each component is orientable, incompressible, and boundary-incompressible. A number $p/q \in \mathbb{Q} \cup \{\infty\}$ is a boundary slope of a knot $K$ if there exists an essential surface in the knot exterior $E(K) = S^3 - \text{int} \mathcal{N}(K)$ with a boundary component representing $p[\mu] + q[\lambda] \in H_1(\partial E(K))$ with respect to the standard meridian $\mu$ and longitude $\lambda$. Now define the set of boundary slopes of $K$:

$$bs(K) = \{r \in \mathbb{Q} \cup \{\infty\} \mid r \text{ is a boundary slope of } K\}.$$ 

Since a Seifert surface of minimal genus is an essential surface, $0 \in bs(K)$ for any knot. Let us also remark that $bs(K)$ is always a finite set [13, Corollary].

Garoufalidis conjectures that Jones slopes are boundary slopes.

**Conjecture 1.1 (Slope Conjecture [9]).** For any knot $K$ in $S^3$, every Jones slope is a boundary slope. That is $js(K) \cup js^*(K) \subset bs(K)$.

Garoufalidis’ Slope Conjecture concerns only the quadratic terms of $\delta_K(n)$ and $\delta_K^*(n)$. Recently Kalfagianni and Tran have proposed the Strong Slope Conjecture which asserts that the topology of the surfaces whose boundary slopes are Jones slopes may be predicted by the linear terms of $\delta_K(n)$ and $\delta_K^*(n)$. Define

$$jx_K = \{2b(n) \mid n \in \mathbb{N}\} \quad \text{and} \quad jx_K^* = \{2b^*(n) \mid n \in \mathbb{N}\}.$$ 

**Conjecture 1.2 (Strong Slope Conjecture [20]).** For a slope $p/q \in js(K)$ with $p, q$ coprime and $q > 0$, there exists an essential surface $F$ with boundary slope $p/q$ such that $\frac{\chi(F)}{|\partial F| q} \in jx(K)$.

Similarly, for a slope $p^*/q^* \in js(K)$ with $p^*, q^*$ coprime and $q^* > 0$, there exists an essential surface $F^*$ with boundary slope $p/q$ such that $-\frac{\chi(F^*)}{|\partial F^*| q^*} \in jx^*(K)$. 

Let \( \overline{K} \) denote the mirror image of a knot \( K \), and let \( -X = \{ -x_1, \ldots, -x_m \} \) for a set \( X = \{ x_1, \ldots, x_m \} \). Then, as noted in [9, §1.4], \( d_+[J_{K,n}(q)] = -d_+[\overline{J}_{K,n}(q)] \). Hence \( \delta_K(n) = -\delta_{\overline{K}}(n) \). It is known to the experts (and a straightforward exercise) that a knot satisfies the Slope Conjecture and the Strong Slope Conjecture if and only if its mirror does; see Proposition 1.6.

1.1. The Yoked Strong Slope Conjecture. In [4] we introduce the following properties and conjecture to refine the Strong Slope Conjecture when a quadratic term \( a(n) \) or \( a^*(n) \) may have period \( \geq 2 \).

**Property 1.3 (Properties YSS\((n)\) and YSS\(^*(n)\)).** For a given integer \( n \in \mathbb{N} \), we say a knot \( K \subset S^3 \) has Property YSS\((n)\) if \( \delta_K(n) = a(n)n^2 + b(n)n + c(n) \) and there is an essential surface \( F_n \) in the exterior \( F_n \) of \( K \) such that

- 4a(n) is the boundary slope of \( F_n \), and
- writing 4a(n) = \( p/q \) for coprime integers \( p,q \) with \( q > 0 \), \( 2b(n) = \chi(F_n)/|\partial F_n|q \).

Property YSS\(^*(n)\) is defined for \( K \) using \( \delta_K^*(n) \) in a similar manner.

**Conjecture 1.4 (The Yoked Strong Slope Conjecture).** Assume \( a(n) \) and \( a^*(n) \) are the quadratic terms of \( \delta_K(n) \) and \( \delta_K^*(n) \) respectively.

- For each Jones slope \( p/q \in js(K) \) there is an integer \( n \in \mathbb{N} \) such that \( 4a(n) = p/q \) and \( K \) has Property YSS\((n)\).
- For each Jones slope \( p/q \in js^*(K) \) there is an integer \( n \in \mathbb{N} \) such that \( 4a^*(n) = p/q \) and \( K \) has Property YSS\(^*(n)\).

Let us point the reader to [4] and [3] for more complete discussions about these properties and conjecture. However the following remark will be useful here.

**Remark 1.5.** The Yoked Strong Slope Conjecture implies the Strong Slope Conjecture. If the periods of \( a(n) \) and \( a^*(n) \) are 1 for a knot \( K \), then the Yoked Strong Slope Conjecture is equivalent to the Strong Slope Conjecture for \( K \). Presently, no knots are known for which the periods of \( a(n) \) and \( a^*(n) \) are not 1.

**Proposition 1.6.** If \( K \) satisfies the Slope Conjecture and the Yoked Strong Slope Conjecture, then so does its mirror \( \overline{K} \). Similarly, if \( K \) satisfies the Slope Conjecture and the Strong Slope Conjecture, then so does its mirror \( \overline{K} \).

**Proof.** We only show the first statement. The latter statement follows very similarly and is known to the experts.

Let \( \delta_K(n) = a(n)n^2 + b(n)n + c(n) \) and \( \delta_K^*(n) = a^*(n)n^2 + b^*(n)n + c^*(n) \). Since \( \delta_K(n) = -\delta_{\overline{K}}(n) \), \( \delta_K^*(n) = -a^*(n)n^2 - b^*(n)n - c^*(n) \) and \( \delta_{\overline{K}}(n) = -a(n)n^2 - b(n)n - c(n) \). It follows that \( js(\overline{K}) = -js^*(K) \) and \( js^*(\overline{K}) = -js(K) \).

Assume that \( p/q \in js(\overline{K}) = -js^*(K) \). Then \( \frac{p}{q} = -p^*/q^* \in -js^*(K) \) (where \( q = q^* > 0 \)). By the assumption that \( K \) satisfies the Slope Conjecture and the Yoked Strong Slope Conjecture, there is an integer \( n \in \mathbb{N} \) such that \( 4a^*(n) = p^*/q^* \) and there exists an essential surface \( F^* \subset E(K) \)
with boundary slope \( p^*/q^* \) so that \( 2b^*(n) = -\frac{\chi(F^*)}{|\partial F^*|q^*} \). Take the mirror image \( F^* \subset E(K) \) of \( F^* \subset E(K) \). Then \( F^* \) has the boundary slope \( 4(-a^*(n)) = -4a^*(n) = -p^*/q^* = \bar{p}/\bar{q} \), and it satisfies \( \chi(F^*) = \chi(F^*) = -2b^*(n) = 2(-b^*(n)) \), so \( K \) has Property YSS(n).

Similarly if \( \bar{p}^*/\bar{q}^* \in js^*(\bar{K}) \), we find an integer \( n \in \mathbb{N} \) so that there is an essential surface \( F \subset E(\bar{K}) \) whose boundary slope is \( 4(-a(n)) = \bar{p}^*/\bar{q}^* \) and satisfies \( 2(-b(n)) = -\frac{\chi(F)}{|\partial F|q^*} \).

**Example 1.7.** Let \( K \) be a knot which appears in the following list with \( \delta_K(n) = a(n)n^2 + b(n)n + c(n) \), \( \delta_K(n) = a^*(n)n^2 + b^*(n)n + c^*(n) \). Then \( a(n), a^*(n), b(n) \) and \( b^*(n) \) are constant, and \( c(n), c^*(n) \) have period at most two. Moreover, \( K \) satisfies the Slope Conjecture, Properties YSS(1) and YSS*(1), and hence also the Yoked Strong Slope Conjecture. Note also that if \( K \) is nontrivial, then \( b(n) = b \leq 0 \) and \( b^*(n) = b^* \geq 0 \).

1. Torus knots [9, 20, Theorem 3.9].
2. Adequate knots [8], [20, Lemma 3.6, 3.8], and hence alternating knots.
3. Non-alternating knots with up to 9 crossings except for 820, 943, 944 [9], [20, 18]. (820, 943, 944 satisfy the Yoked Strong Slope Conjecture, but for these knots the coefficient \( b(n) \) has period 3.)
4. Iterated cables of knots in (1), (2), and (3) above [20, Proposition 3.2, Theorem 3.9 and Corollary 5.5].
5. Graph knots [27, 4].

**1.2. Main Results.** In this article we give further supporting evidence for the Strong Slope Conjecture and the Yoked Strong Slope Conjecture by examining them for the Whitehead doubles of a knot \( K \), and more generally for its twisted generalized Whitehead doubles \( W^\omega_\tau(K) \) defined below.

Let \( V \) be a standardly embedded solid torus in \( S^3 \) with a preferred meridian-longitude \((\mu_V, \lambda_V)\), and take a pattern \((V, k^\omega_V)\) where \( k^\omega_V \) is a knot in the interior of \( V \) illustrated by Figure 1.1. We always assume \( \omega \neq 0 \), for otherwise, \( k^\omega_V \) is the unknot contained in a 3-ball in \( V \). Given a knot \( K \) in \( S^3 \) with a preferred meridian-longitude \((\mu_K, \lambda_K)\), let \( f: V \to S^3 \) an orientation preserving embedding which sends the core of \( V \) to the knot \( K \subset S^3 \) such that \( f(\mu_V) = \mu_K \) and \( f(\lambda_V) = \lambda_K \). Then the image \( f(k^\omega_V) \) is called a \( \tau \)-twisted, \( \omega \)-generalized Whitehead double of \( K \) and is denoted by \( W^\omega_\tau(K) \). When \( \omega = 1, \tau = 0 \), \( W^0_1 \) is the (untwisted) negative Whitehead double of \( K \). Note that the mirror image \( W^\omega_\tau(K) \) of \( W^\omega_\tau(K) \) is \( W^\omega_\tau(K) \).

For notational simplicity, in what follows, we use the following notation.

**Convention 1.8.** For a given knot \( K \), let \( N_K \) be the smallest nonnegative integer such that \( d_+[J_{K,n}(q)] \) is a quadratic quasi-polynomial \( \delta_K(n) = a(n)n^2 + b(n)n + c(n) \) for \( n \geq 2N_K + 1 \), and \( N^*_K \) the smallest nonnegative integer such that \( d_-[J_{K,n}(q)] \) is a quadratic quasi-polynomial \( \delta^*_K(n) = a^*(n)n^2 + b^*(n)n + c^*(n) \) for \( n \geq 2N^*_K + 1 \). We put \( a_1 := a(2N_K + 1), b_1 := b(2N_K + 1), a^*_1 := a^*(2N^*_K + 1), \) and \( b^*_1 := b^*(2N^*_K + 1) \).

The aim of this paper is to establish the Slope Conjecture and the Strong Slope Conjecture for twisted generalized Whitehead doubles in the following form.
**Theorem 1.9.** Let $K$ be a knot. We assume that the period of $\delta_K(n)$ and $\delta^*_K(n)$ are less than or equal to 2 and that $b_1 \leq 0$ and $b^*_1 \geq 0$. Assume further that if $b_1 = 0$, then $a_1 \neq \frac{2}{5}$ and that if $b^*_1 = 0$, then $a^*_1 \neq \frac{2}{5} - \frac{1}{8}$.

1. If $K$ satisfies the Slope Conjecture, then all of its twisted generalized Whitehead doubles also satisfy the Slope Conjecture.
2. If $K$ has Property YSS(1) and YSS*(1), then all of its twisted generalized Whitehead doubles satisfy the Yoked Strong Slope Conjecture.

**Remark 1.10.**

1. The hypotheses that $b_1 \leq 0$ and $b^*_1 \geq 0$ are actually implied by $K$ being a non-trivial knot that satisfies the Strong Slope Conjecture since the only essential surface with boundary and positive Euler characteristic is the disk. See also [20, Conjecture 5.1].
2. The hypothesis that the quasi-polynomials have period $\leq 2$ allows simplifications in the proofs that lead to Theorem 1.9. This is in part due to the pattern for a twisted generalized Whitehead double having wrapping number 2 and its effect upon the colored Jones polynomial for the satellite, see Proposition 2.7. Indeed, allowing periods $> 2$ significantly complicates Propositions 2.12 and 2.13.
3. As mentioned in Remark 1.5, the coefficients $a(n)$ and $a^*(n)$ are constant for all known examples. Hence the hypothesis in Theorem 1.9(2) that $K$ has Property YSS(1) and YSS*(1) holds for all known knots for which the Strong Slope Conjecture holds.

The extra condition that $a_1 \neq \frac{2}{5}$ (if $b_1 = 0$) and $a^*_1 \neq \frac{2}{5} - \frac{1}{8}$ (if $b^*_1 = 0$) in Theorem 1.9 excludes at most two values of $\tau$. If we take $\tau = 0$, we have the following which is proven in Section 7.

**Corollary 1.11.** Any knot obtained by a finite sequence of cabling, untwisted $\omega$–generalized Whitehead doublings with $\omega \neq 0$ and connected sums of adequate knots or torus knots satisfies the Slope Conjecture and the Yoked Strong Slope Conjecture.

**Convention 1.12.** Observe that the conditions on the companion knot $K$ in Theorem 1.9 are symmetric with respect to mirroring. Furthermore, a knot satisfies the Slope Conjecture and the (Yoked) Strong Slope Conjecture if and only if its mirror does by Proposition 1.6. Hence we restrict
our attention to \(\tau\)-twisted \(\omega\)-generalized Whitehead doubles for which \(\omega > 0\) for the remainder of this article.

2. Colored Jones polynomials of generalized Whitehead doubles and their degrees

The goal of this section is to prove Propositions 2.1 and 2.2 below which give their maximum degree and minimum degree of the colored Jones function of a \(\tau\)-twisted, \(\omega\)-generalized Whitehead double of a knot \(K\) under the hypotheses of Theorem 1.9.

**Proposition 2.1 (maximum-degree).** Let \(K\) be a knot in \(S^3\) and \(N_K^\tau\) the smallest nonnegative integer such that \(d_+ [J_{K,n}(q)]\) is a quadratic quasi-polynomial \(\delta_K(n) = a(n)n^2 + b(n)n + c(n)\) for \(n \geq 2N_K^\tau + 1\). We put \(a_1 := a(2N_K^\tau + 1)\), \(b_1 := b(2N_K^\tau + 1)\), and \(c_1 := c(2N_K^\tau + 1)\). We assume that the period of \(\delta_K(n)\) is less than or equal to 2 and that \(b_1 \leq 0\). Assume further that if \(b_1 = 0\), then \(a_1 \neq \frac{1}{4}\). Then the maximum degree of the colored Jones polynomial of its \(\tau\)-twisted \(\omega\)-generalized Whitehead double with \(\omega > 0\) is given by the quadratic polynomial

\[
\delta_{W^\omega_\tau(K)}(n) = \begin{cases} 
(4a_1 - \tau)n^2 + (-4a_1 + 2b_1 + \tau - \frac{1}{2})n + a_1 - b_1 + c_1 + \frac{1}{2} & (a_1 > \frac{1}{4}), \\
-\frac{n}{2} + C_+(K,\tau) + \frac{1}{2} & (a_1 < \frac{1}{4}), \\
-\frac{n}{2} + C'_-(K,\tau) + \frac{1}{2} & (a_1 = \frac{1}{4}, b_1 \neq 0),
\end{cases}
\]

where \(C_+(K,\tau)\) and \(C'_-(K,\tau)\) are numbers that only depend on the knot \(K\) and the number \(\tau\).

**Proposition 2.2 (minimum-degree).** Let \(K\) be a knot in \(S^3\) and \(N_K^\tau\) the smallest nonnegative integer such that \(d_- [J_{K,n}(q)]\) is a quadratic quasi-polynomial \(\delta_K^\tau(n) = a^\ast(n)n^2 + b^\ast(n)n + c^\ast(n)\) for \(n \geq 2N_K^\tau + 1\). We put \(a_1^\ast := a^\ast(2N_K^\tau + 1)\), \(b_1^\ast := b^\ast(2N_K^\tau + 1)\), and \(c_1^\ast := c^\ast(2N_K^\tau + 1)\). We assume that the period of \(\delta_K^\tau(n)\) is less than or equal to 2, \(b_1^\ast \geq 0\). Assume further that if \(b_1^\ast = 0\), then \(a_1^\ast \neq \frac{1}{4} - \frac{1}{4}\). Then the minimum degree of the colored Jones polynomial of its \(\tau\)-twisted \(\omega\)-generalized Whitehead double with \(\omega > 0\) is given by the quadratic polynomial

\[
\delta_{W^\omega_\tau(K)}(n) = \begin{cases} 
(4a_1^\ast - \frac{2a_1 - 1}{2})n^2 + (-4a_1^\ast + 2b_1^\ast + \omega - 1 + \tau)n + a_1^\ast - b_1^\ast + c_1^\ast + \frac{1}{2} & (a_1^\ast < \frac{1}{4} - \frac{1}{4}), \\
-\omega n^2 + \frac{2\omega - 1}{2}n + C_-(K,\tau) + \frac{1}{2} & (a_1^\ast > \frac{1}{4} - \frac{1}{4}), \\
-\omega n^2 + \frac{2\omega - 1}{2}n + C'_-(K,\tau) + \frac{1}{2} & (a_1^\ast = \frac{1}{4} - \frac{1}{4}, b_1^\ast \neq 0),
\end{cases}
\]

where \(C_-(K,\tau)\) and \(C'_-(K,\tau)\) are numbers that only depend on the knot \(K\) and the number \(\tau\).

**Remark 2.3.** Propositions 2.1 and 2.2 say that even when \(\delta_K(n)\) and \(\delta_K^\tau(n)\) have period 2, \(\delta_{W^\omega_\tau(K)}(n)\) and \(\delta_{W^\omega_\tau(K)}^\tau(n)\) are usual polynomials rather than quasi-polynomials.

**Proof of Propositions 2.1 and 2.2.** In Subsection 2.1 we first derive a formula of the colored Jones function of \(W^\omega_\tau(K)\) using a slightly different normalization that simplifies calculation. For knot \(K\) and a nonnegative integer \(n\), set

\[
J_{K,n}(q) := \frac{J_{K,n+1}(q)}{J_{\text{rot},n+1}(q)}
\]
so that $J_{\bigcirc,n}(q) = 1$ for the unknot $\bigcirc$ and $J_{K,1}(q)$ is the ordinary Jones polynomial of a knot $K$. Let $\delta'_K(n)$ and $\delta''_K(n)$ be the maximum and minimum degrees of this normalized colored Jones function. With respect to this normalization we first establish Propositions 2.12 and 2.13 in Subsection 2.2, which derive Propositions 2.1 and 2.2 by using the transformation described below.

To derive Propositions 2.1 and 2.2 from Propositions 2.12 and 2.13, we apply the transformation

\[
< n > J'_{K,n}(q) = < n > J_{K,n+1}(q) = < n > \frac{J_{K,n+1}(q)}{(-1)^n} = (-1)^n J_{K,n+1}(q).
\]

This implies:

\[
d_{\pm} \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}} d_{\pm}[J_{K,n}(q)] = d_{\pm}[J_{K,n+1}(q)].
\]

Since $d_{\pm} \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}} = \pm \frac{n}{2}$, we have

\[
\delta'_K(n) = \delta_K(n + 1) - \frac{1}{2} n,
\]

\[
\delta''_K(n) = \delta''_K(n + 1) + \frac{1}{2} n
\]

and

\[
\delta_{W^\omega(K)}(n) = \delta_{W^\omega(K)}(n - 1) + \frac{1}{2} n - \frac{1}{2},
\]

\[
\delta''_{W^\omega(K)}(n) = \delta''_{W^\omega(K)}(n - 1) - \frac{1}{2} n + \frac{1}{2}.
\]

Note also that

\[
\alpha_0 = a_1, \beta_0 = 2a_1 + b_1 - \frac{1}{2}, \gamma_0 = a_1 + b_1 + c_1,
\]

\[
\alpha_0^* = a_1^*, \beta_0^* = 2a_1^* + b_1^* + \frac{1}{2}, \gamma_0^* = a_1^* + b_1^* + c_1^*.
\]

\[\square\]

2.1. Computations of colored Jones polynomials of $W^\omega(K)$. In this subsection we will compute the normalized colored Jones polynomial $J'_{W^\omega(K),n}(q)$ instead of $J_{W^\omega(K),n}(q)$. As we mentioned at the end of Section 1, for any knot $K$ we may restrict attention to $\tau$-twisted, $\omega$-generalized Whitehead doubles of $W^\omega(K)$ with $\omega > 0$. We begin by recalling the following functions with respect to $q$ for non-negative integers $s, t, u$. See [25].

\[
< s > := (-1)^s [s + 1], \quad [s] = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad [s]! = \prod_{t=1}^{s} [t]
\]

\[
< s, t, u > := (-1)^{i+j+k} \frac{[i + j + k + 1][i][j][k]}{[s][t][u]!},
\]

where $i = \frac{i + u - s}{2}$, $j = \frac{u + s - i}{2}$, and $k = \frac{s + t - u}{2}$,

\[
\delta(u; s, t) := (-1)^{\frac{u + s}{2}} q^{-\frac{1}{2}(u^2 - s^2 - t^2 + 2u - 2s - 2t)},
\]

and
\[
\left\langle \begin{array}{ccc}
A & B & E \\
D & C & F
\end{array} \right\rangle = \prod_{i=1}^{3} \prod_{j=1}^{4} \left[ b_i - a_j \right]! \sum_{\max\{a_i\} \leq s \leq \min\{b_j\}} \frac{(-1)^s [s+1]!}{\prod_{i=1}^{3} [b_i - s]! \prod_{j=1}^{4} [s - a_j]!}.
\]

where \(a_1 = \frac{A+B+E}{2}, a_2 = \frac{B+D+E}{2}, a_3 = \frac{C+D+E}{2}, a_4 = \frac{A+C+E}{2}, \Sigma = A + B + C + D + E + F, b_1 = \frac{\Sigma - A - D}{2}, b_2 = \frac{\Sigma - E - F}{2}, \) and \(b_3 = \frac{\Sigma - B - C}{2}.\)

We will also use the following equalities introduced by Masbaum and Vogel [25].

\[
\sum_{u} = \sum_{\mathclap{\begin{array}{c} < u > \\
< s,t,u > \end{array}}} = \frac{s}{t} = \frac{s}{t} = \frac{s}{t}.
\]

Here the sum is over those colors \(u\) such that the triple \((s, t, u)\) satisfies \(s \equiv t \equiv u \equiv 0 \pmod{2}\) and \(|s - t| \leq u \leq s + t.\)

\[
\frac{s}{t} = \delta(u; s, t)^{-1}, \quad \frac{s}{t} = \delta(u; s, t)\]

\[
\frac{s}{t} = \delta(0; s, s)^{-1}, \quad \frac{s}{t} = \delta(0; s, s)\]

\[
\frac{s}{t} = \frac{s}{t} = \frac{s}{t} = \frac{s}{t}.
\]

\[
\frac{B E}{A D C} = \left\langle \begin{array}{ccc}
A & B & E \\
D & C & F
\end{array} \right\rangle = \left\langle \begin{array}{ccc}
A & B & E \\
D & C & F
\end{array} \right\rangle = \left\langle \begin{array}{ccc}
A & B & E \\
D & C & F
\end{array} \right\rangle = \left\langle \begin{array}{ccc}
A & B & E \\
D & C & F
\end{array} \right\rangle.
\]

In the following we will use the following symbols.

\begin{align*}
2 & = \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\end{tikzpicture} \\
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\draw (0,-1.5) -- (0,-0.5);
\draw (0,0.5) -- (0,1.5);
\end{tikzpicture} & = \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\end{tikzpicture} \\
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\draw (0,-1.5) -- (0,-0.5);
\draw (0,0.5) -- (0,1.5);
\end{tikzpicture} & = \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\end{tikzpicture}
\end{align*}

Figure 2.1.
Lemma 2.4.

\begin{align*}
\frac{\alpha}{n} &= \frac{\alpha}{n} \quad \text{for each handedness:}
\end{align*}

\begin{align*}
\frac{\alpha}{n} &= \frac{\alpha}{n} \\
\frac{\alpha}{n} &= \frac{\alpha}{n}
\end{align*}

Proof. The first equality is due to the framed isotopy shown here for each handedness:

\begin{align*}
\frac{\alpha}{n} &= \frac{\alpha}{n} \\
\frac{\alpha}{n} &= \frac{\alpha}{n}
\end{align*}

Next we exploit the formula (2.8) and then formula (2.10). \square

Lemma 2.5.

\begin{align*}
\frac{2}{n} \sum_{i=0}^{n} \frac{\langle 2i \rangle}{\langle n, n, 2i \rangle} q^{-\alpha(i+1)}
\end{align*}

Proof. Apply formula (2.12) and then (2.11). \square

Lemma 2.6. For a 0 framed diagram of any knot $K$:

\begin{align*}
\langle n \rangle_{K,n}(q) = \langle n \rangle
\end{align*}

Proof. It follows from [24, Section 5] that the right hand side describes $(-1)^n J_{K,n+1}(q)$. On the other hand, 2.3 shows that

\begin{align*}
\langle n \rangle_{K,n}(q) = (-1)^n J_{K,n+1}(q).
\end{align*}

Thus we obtain the desired equality. \square

Proposition 2.7.

\begin{align*}
J'_{W_{\omega}(K),n}(q) = \frac{1}{\langle n \rangle} \sum_{j,k=0}^{n} \frac{\langle 2j \rangle}{\langle n, n, 2j \rangle} \frac{\langle 2k \rangle}{\langle n, n, 2k \rangle} q^{-\omega(j+1) - \tau(k+1)} J'_{K,2k}(q).
\end{align*}

Proof. We will compute $J'_{W_{\omega}(K),n}(q)$ using the graphical calculus of Masbaum and Vogel [25, 24] following the method of Tanaka [33]. To this end we need a diagram of $W_{\omega}(K)$ whose blackboard framing is 0. Note that since two strands in the $\omega$–twist region and the $\tau$–twist region run in opposite directions, to obtain a correct 0–framing of $W_{\omega}(K)$ we need to add some curls indicated
in Figure 2.2. In Figure 2.3, we use \( D(K) \) to mean a double of \( K \), i.e. two parallel copies of \( K \) whose blackboard framing is 0.

![Diagram](image)

**Figure 2.2.** Addition of curls to get a 0–framing to twist regions

![Diagram](image)

**Figure 2.3.** A diagram of \( W^\tau(K) \) with trivial writhe

Using the above formulas, we compute \( J'_{W^\tau(K), n}(q) \) graphically in the manner of [24] and [33]. As shown below, we begin by expressing \( < n > J'_{W^\tau(K), n}(q) \) diagrammatically with Lemma 2.6. Then we apply Lemma 2.4 twice, once for each of the \( \tau \) and \( \omega \) twist regions. Next we apply Lemma 2.5. Finally, we again apply Lemma 2.6 for the diagrammatic expression of \( < 2k > J'_{K, 2k}(q) \).
\[ < n > J'_{WZ(K),n}(q) = \]

\[
= \sum_{j,k=0}^{n} \frac{< 2j > < 2k >}{< n, n, 2j > < n, n, 2k >} q^{-\omega j(j+1) - \tau k(k+1)} \langle n \ n \ 2j \ n \ n \ 2k \rangle J'_{K,2k}(q) \]

Therefore we have:

\[ J'_{WZ(K),n}(q) = \frac{1}{< n >} \sum_{j,k=0}^{n} \frac{< 2j > < 2k >}{< n, n, 2j > < n, n, 2k >} \langle n \ n \ 2j \ n \ n \ 2k \rangle q^{-\omega j(j+1) - \tau k(k+1)} J'_{K,2k}(q). \]

\[ \square \]
Remark 2.8. Other formulas of the colored Jones polynomial of the twisted Whitehead double of a knot $K$, are given by Tanaka [33] and Zheng [35].

2.2. Computations of the maximum and minimum degrees. For a rational function $f(q) = \frac{f_1(q)}{f_2(q)}$ with $f_1(q), f_2(q) \in \mathbb{Q}[q^\pm]$ and $f_2(q) \neq 0$, we extend the maximum (resp. minimum) degree of $f(q)$ as $d_+[f_1(q)] - d_+[f_2(q)]$ (resp. $d_-[f_1(q)] - d_-[f_2(q)]$).

Let $\delta'_K(n)$ (resp. $\delta'^*_{K}(n)$) be the maximum degree (resp. the minimum degree) of $J'_{K,n}(q) \in \mathbb{Z}[q^\pm]$. Propositions 2.12 and 2.13 give the maximum degree and the minimum degree of $J'_{K,n}(q) \in \mathbb{Z}[q^\pm]$. We note that $\delta'^*_{K}(n) = -\delta'_K(n)$ [9, §1.4]. For convenience we recall the maximum and minimum degrees of the functions which appear in the expression of $J'_{W^\tau(\mathcal{K}),n}(q)$ in Proposition 2.7.

Lemma 2.9. $d_+[< n >] = \pm \frac{1}{2}n$.

Lemma 2.10 ([12]). The maximum and minimum degrees of $< s, t, u >$ are given by

$$d_+ [< s, t, u >] = \pm \frac{s + t + u}{4}.$$ 

Lemma 2.11 ([12]). The maximum and minimum degrees of $\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle$ are given by

$$d_+ \left[ \left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle \right]$$

$$= \pm \frac{1}{2} \left[ -\Sigma^2 - \frac{1}{2}(A^2 + B^2 + C^2 + D^2 + E^2 + F^2 - \Sigma) + \sum_{i=1}^{3} b_i(b_i - 1) + \sum_{j=1}^{4} a_j(a_j + 1) 
- 3M^2 + M(1 + 2\Sigma) \right],$$

where $\Sigma, a_j, b_i$ are as in (2.7) and $M = \min b_i$.

To ease our computations, we define

$$f(j, k; q) = \frac{< 2j >}{< n, n, 2j >} \frac{< 2k >}{< n, n, 2k >} \left\langle \begin{array}{ccc} n & n & 2j \\ n & n & 2k \end{array} \right\rangle q^{-\omega_j(j+1) - \tau_k(k+1)} J'_{K, 2k}(q)$$

so that by Proposition 2.7 we have

$$< n > J'_{W^\tau(\mathcal{K}), n}(q) = \sum_{j,k=0}^{n} f(j, k; q).$$

Hence our computations of $d_+_J'_{W^\tau(\mathcal{K}), n}(q)$ reduce to an understanding the extrema of $d_+ [f(j, k; q)]$ for $0 \leq j, k \leq n$.

To that end we first record the following computations of degrees. From Lemma 2.10, we have that

$$d_+ \left[ \frac{< 2j >}{< n, n, 2j >} \right] = \pm \left( \frac{n}{2} + \frac{j}{2} \right).$$
From Lemma 2.11, we have that
\[ d_\pm \left( \begin{array}{ccc} n & n & 2j \\ 2 & n & 2k \end{array} \right) = \pm \left( \begin{array}{c} \frac{1}{2}(j + k + n) \\ \frac{1}{2}(-j^2 - 2jk - k - 2n + 2jn + 2kn - n^2) \end{array} \right) (j + k \leq n), \]
\[ d_\pm \left( \begin{array}{ccc} n & n & 2j \\ 2 & n & 2k \end{array} \right) = \pm \left( \begin{array}{c} \frac{1}{2}(j + k + n) \\ \frac{1}{2}(-j^2 - 2jk - k - 2n + 2jn + 2kn - n^2) \end{array} \right) (j + k \geq n). \]

**Proposition 2.12** (Normalized maximum degree). Let \( K \) be a knot in \( S^3 \) and \( N'_k \) the smallest nonnegative integer such that \( d_+ [J'_{K,n}(q)] \) is a quadratic quasi-polynomial \( \delta'_K(n) = \alpha(n)n^2 + \beta(n)n + \gamma(n) \) for \( n \geq 2N'_K \). We put \( \alpha_0 := \alpha(2N'_K), \beta_0 := \beta(2N'_K), \) and \( \gamma_0 := \gamma(2N'_K) \). We assume that the period of \( \delta'_K(n) \) is less than or equal to 2 and that \(-2\alpha_0 + \beta_0 + \frac{1}{2} \leq 0 \). Assume further that if \(-2\alpha_0 + \beta_0 + \frac{1}{2} = 0 \), then \( \tau \neq 4\alpha_0 \). Then, for suitably large \( n \), the maximum degree of the colored Jones polynomial of its \( \tau \)-twisted \( \omega \)-generalized negative Whitehead double is given by
\[ \delta_{W_\tau(K)}(n) = \begin{cases} (4\alpha_0 - \tau)n^2 + (2\beta_0 - \tau)n + \gamma_0 & (\alpha_0 > \frac{\tau}{4}) \\ -n + C_+(K, \tau) & (\alpha_0 < \frac{\tau}{4}) \\ -n + C'_+(K, \tau) & (\alpha_0 = \frac{\tau}{4}, -2\alpha_0 + \beta_0 + \frac{1}{2} \neq 0) \end{cases} \]
where \( C_+(K, \tau) \) and \( C'_+(K, \tau) \) are numbers that only depend on the knot \( K \) and the number \( \tau \).

**Proof.** In light of (2.14), to determine the maximum degree of \( J_{W_\tau(K),n}(q) \), we need to understand the maximum degrees of the functions \( f(j, k; q) \) for \( 0 \leq j, k \leq n \).

Since the period of \( \delta'_K(n) \) is less than or equal to 2, it follows that \( \alpha(2k) = \alpha_0, \beta(2k) = \beta_0, \) and \( \gamma(2k) = \gamma_0 \) and so
\[ d_+ [J'_{K,2k}(q)] = \alpha(2k)(2k)^2 + \beta(2k)2k + \gamma(2k) = 4\alpha_0 k^2 + 2\beta_0 k + \gamma_0 \]
for \( k \geq N'_K \).

Due to (2.16) the argument splits into two cases.

**Case 1.** \( j + k \leq n \).

From the equalities (2.15) and (2.16), one obtains
\[
d_+ [f(j, k; q)] = (-\frac{n}{2} + \frac{j}{2}) + (\frac{n}{2} + \frac{k}{2}) + \frac{1}{2}(j + k + n)
- \omega j(j + 1) - \tau k(k + 1) + d_+ [J'_{K,2k}(q)]
= -\omega j^2 - (\omega - 1)j - \tau k^2 - (\tau - 1)k - \frac{n}{2} + d_+ [J'_{K,2k}(q)]
= -\omega(j + \frac{\omega - 1}{2\omega})^2 + \omega(\frac{\omega - 1}{2\omega})^2 - \tau k^2 - (\tau - 1)k - \frac{n}{2} + d_+ [J'_{K,2k}(q)].
\]

Since \( \omega > 0 \) so that \( \omega - 1 \geq 0 \), this is maximized uniquely at \( j = 0 \) for a fixed \( k \). Thus,
\[
\max_{0 \leq j \leq n-k} d_+ [f(j, k; q)] = d_+ [f(0, k; q)]
= -\tau k^2 - (\tau - 1)k - \frac{n}{2} + d_+ [J'_{K,2k}(q)].
\]

For \( k \geq N'_K \), by (2.18) this becomes
\[
d_+ [f(0, k; q)] = -\tau k^2 - (\tau - 1)k - \frac{n}{2} + 4\alpha_0 k^2 + 2\beta_0 k + \gamma_0
= (4\alpha_0 - \tau)k^2 + (2\beta_0 - \tau + 1)k - \frac{n}{2} + \gamma_0.
\]
Assume first that $\alpha_0 \neq \frac{\tau}{4}$, we may write

\[
d_+ (f(0, k; q)) = (4\alpha_0 - \tau) \left( k + \frac{2\beta_0 - \tau + \frac{1}{2}}{2(4\alpha_0 - \tau)} \right) - \frac{(2\beta_0 - \tau + 1)^2}{4(4\alpha_0 - \tau)} - \frac{n}{2} + \gamma_0
\]

for $n \geq k \geq N'_K$.

First suppose $\alpha_0 > \frac{\tau}{4}$. Note that $-\frac{2\beta_0 - \tau + 1}{2(4\alpha_0 - \tau)} < \frac{\tau}{2}$ for sufficiently large $n \geq N'_K$. Then Since $N'_K \geq 0$, for $N'_K \leq k \leq n$, $d_+[f(0, k; q)]$ is uniquely maximized at $k = n$. Hence, we get that

\[
\max_{N'_K \leq k \leq n} d_+ [f(0, k; q)] = d_+ [f(0, n; q)],
\]

for sufficiently large $n$.

Moreover, this maximum tends to $\infty$ as $n$ goes to $\infty$. Thus obviously,

\[
\max_{0 \leq k \leq N'_K - 1} d_+ [f(0, k; q)] < d_+ [f(0, n; q)],
\]

for sufficiently large $n$, and hence we have:

\[
\max_{0 \leq k \leq n} d_+ [f(0, k; q)] = d_+ [f(0, n; q)] = (4\alpha_0 - \tau)n^2 + (2\beta_0 - \tau + \frac{1}{2})n + \gamma_0.
\]

Assume now $\alpha_0 < \frac{\tau}{4}$. Then we proceed to show that $\max_{N'_K \leq k \leq n} d_+ [f(0, k; q)]$ is realized at $k = N'_K$.

Since $0 \geq -2\alpha_0 + \beta_0 + \frac{1}{2} = -2(\alpha_0 - \frac{\tau}{4}) + \beta_0 - \frac{\tau}{4} + \frac{1}{2}$ by assumption, then we have $2\beta_0 - \tau + 1 < 0$.

Therefore, if $\alpha_0 < \frac{\tau}{4}$, so that $\frac{2\beta_0 - \tau + 1}{2(4\alpha_0 - \tau)} > 0$, (2.19) shows that $d_+[f(0, k; q)]$ is monotonically decreasing with respect to $k$ for $k \geq N'_K \geq 0$. Thus $\max_{N'_K \leq k \leq n} d_+ [f(0, k; q)]$ is uniquely realized at $k = N'_K$. In the sum $\sum_{k=0}^{n} f(0, k; q)$, there may exist some cancellations, but there may exist at most $N'_K$ cancellations. Therefore, there exists $0 \leq k_0 \leq n$ such that $d_+[f(0, k_0; q)]$ gives the maximum degree of the sum $\sum_{k=0}^{n} f(0, k; q)$. Hence

\[
d_+ \left[ \sum_{j,k=0}^{n} f(j, k; q) \right] = d_+ \left[ \sum_{k=0}^{n} f(0, k; q) \right] = d_+[f(0, k_0; q)] = -\tau k_0^2 - (\tau - 1)k_0 - \frac{n}{2} + d_+[J'_{K,2k_0}(q)] = C_+(K, \tau) - \frac{n}{2},
\]

where we put

\[
C_+(K, \tau) = -\tau k_0^2 - (\tau - 1)k_0 + d_+[J'_{K,2k_0}(q)],
\]

which only depends on the knot $K$ and the number $\tau$.

Let us assume $\alpha_0 = \frac{\tau}{4}$. Then by the assumption $-2\alpha_0 + \beta_0 + \frac{1}{2} \neq 0$, and hence $2\beta_0 - \tau + 1 \neq 0$. Furthermore, $\beta_0 - \frac{\tau}{4} + \frac{1}{2} = -2(\alpha_0 - \frac{\tau}{4}) + \beta_0 - \frac{\tau}{4} + \frac{1}{2} = -2\alpha_0 + \beta_0 + \frac{1}{2} \leq 0$ by the assumption in Proposition 2.12. Hence we have $2\beta_0 - \tau + 1 < 0$. Thus

\[
d_+[f(0, k; q)] = (2\beta_0 - \tau + 1)k - \frac{n}{2} + \gamma_0
\]
is monotonically decreasing with respect to $k$ for $N'_K \leq k \leq n$. Hence $\max_{N'_K \leq k \leq n} d_+ [f(0, k; q)]$ is uniquely realized at $k = N'_K$. In the sum $\sum_{k=0}^{n} f(0, k; q)$, there may exist some cancellations, but there may exist at most $N'_K$ cancellations. Therefore, there exists $0 \leq k'_0 \leq n$ such that $d_+ [f(0, k_0; q)]$ gives the maximum degree of the sum $\sum_{k=0}^{n} f(0, k; q)$. Hence

\begin{equation}
(2.22) \quad d_+ \left[ \sum_{j, k=0}^{n} f(j, k; q) \right] = d_+ \left[ \sum_{k=0}^{n} f(0, k; q) \right] = d_+ [f(0, k'_0; q)] = -\tau k'_0^2 - (\tau - 1)k'_0 + \frac{n}{2} + d_+[J_{K,2k'_0}(q)] = C'_+(K, \tau) - \frac{n}{2},
\end{equation}

where we put

\[ C'_+(K, \tau) = -\tau k'_0^2 - (\tau - 1)k'_0 + d_+[J_{K,2k'_0}(q)], \]

which only depends on the knot $K$ and the number $\tau$.

**Case 2.** $j + k \geq n$.

From the equalities (2.15) and (2.16), we see that

\[
d_+ [f(j, k; q)] = (-\frac{n}{2} + \frac{j}{2}) + (-\frac{n}{2} + \frac{k}{2}) + \frac{1}{2}(-j^2 - 2jk - k^2 + 2n + 2jn + 2kn - n^2) - \omega j(j+1) - \tau k(k+1) + d_+[J_{K,2k}(q)]
\]

\[
= -\frac{2\omega}{2} + \frac{1}{2}j^2 + (n-k-\omega + \frac{1}{2})j + (-\tau - \frac{1}{2}j)k + (\frac{1}{2} + n - \tau)k - \frac{n^2}{2} + d_+[J_{K,2k}(q)].
\]

Since $j \geq n - k > \frac{1}{2\omega+1} (n-k-\omega + \frac{1}{2})$, this is maximized at $j = n - k$ for a fixed $k$. Therefore this case is contained in the case $j + k \leq n$.

Finally we determine the maximum degree of $J_{W^{\omega}(K),n}(q) = \frac{1}{<n>} \sum_{j, k=0}^{n} f(j, k; q)$; see (2.14).

If $\alpha_0 > \frac{1}{\tau}$, then by (2.20) we have

\begin{equation}
(2.23) \quad \delta'_{W^{\omega}(K)}(n) = d_+ [f(0, n; q)] - \frac{n}{2} = (4\alpha_0 - \tau)n^2 + (2\beta_0 - \tau)n + \gamma_0.
\end{equation}

If $\alpha_0 < \frac{1}{\tau}$, then by (2.21) we have

\begin{equation}
(2.24) \quad \delta'_{W^{\omega}(K)}(n) = (C_+(K, \tau) - \frac{n}{2}) - \frac{n}{2} = -n + C_+(K, \tau).
\end{equation}

If $\alpha_0 = \frac{1}{\tau}$ and $-2\alpha_0 + \beta_0 + \frac{1}{2} \neq 0$, then by (2.22) we have

\begin{equation}
(2.25) \quad \delta'_{W^{\omega}(K)}(n) = (C'_+(K, \tau) - \frac{n}{2}) - \frac{n}{2} = -n + C'_+(K, \tau).
\end{equation}
Proposition 2.13 (Normalized minimum degree). Let $K$ be a knot in $S^3$ and $N^*_{K} \geq 0$ the smallest nonnegative integer such that $d_-[J_{K,n}(q)]$ is a quadratic quasi-polynomial $\delta^*_K(n) = \alpha^*(n)n^2 + \beta^*(n)n + \gamma^*(n)$ for $n \geq 2N^*_{K}$. We put $\alpha^*_0 := \alpha^*(2N^*_{K})$, $\beta^*_0 := \beta^*(2N^*_{K})$, and $\gamma^*_0 := \gamma^*(2N^*_{K})$. We assume that the period of $\delta^*_K(n)$ is less than or equal to 2, $-2\alpha^*_0 + \beta^*_0 - \frac{1}{2} \geq 0$. Assume further that if $-2\alpha^*_0 + \beta^*_0 - \frac{1}{2} = 0$, then $\tau \neq 4\alpha^*_0 + \frac{1}{2}$. Then, for suitably large $n$, the minimum degree of the colored Jones polynomial of its $\tau$-twisted $\omega$-generalized negative Whitehead double is given by

$$\delta^*_{W^*\omega(K)}(n) = \begin{cases} 
(4\alpha^*_0 - \frac{2\omega - 1}{2} - \tau)n^2 + (2\beta^*_0 - \frac{2\omega + 1}{2} - \tau)n + \gamma^*_0 & (\alpha^*_0 < \frac{\omega}{4} - \frac{1}{8}) \\
-\omega n^2 - \omega n + C_-(K, \tau) & (\alpha^*_0 > \frac{\omega}{4} - \frac{1}{8}) \\
-\omega n^2 - \omega n + C'_-(K, \tau) & (\alpha^*_0 = \frac{\omega}{4} - \frac{1}{8}, -2\alpha^*_0 + \beta^*_0 - \frac{1}{2} \neq 0) 
\end{cases}$$

where $C_-(K, \tau)$ and $C'_-(K, \tau)$ are numbers that only depend on the knot $K$ and the number $\tau$.

Proof. In light of (2.14), to determine the minimum degree of $J_{W^*\omega(K),n}(q)$, we need to understand the minimum degrees of the functions $f(j,k;q)$ for $0 \leq j,k \leq n$.

Since the period of $\delta^*_K(n)$ is less than or equal to 2, it follows that $\alpha^*(2k) = \alpha^*_0$, $\beta^*(2k) = \beta^*_0$, and $\gamma^*(2k) = \gamma^*_0$ and so

$$d_-[J_{K,2k}(q)] = \alpha^*(2k)(2k)^2 + \beta^*(2k)2k + \gamma^*(2k) = 4\alpha^*_0k^2 + 2\beta^*_0k + \gamma^*_0$$

for $k \geq N^*_{K}$.

Again, due to (2.16) the argument splits into two cases.

Case 1*. $j + k \leq n$.

From the equalities (2.15) and (2.16), one obtains

$$d_-[f(j,k;q)] = -\omega j^2 - (\omega + 1)j - \tau k^2 - (\tau + 1)k + \frac{n}{2} + d_-[f(k,2k)]$$

$$= -\omega \left(j + \frac{\omega + 1}{2\omega}\right)^2 + \omega \left(\frac{\omega + 1}{2\omega}\right)^2 - \tau k^2 - (\tau + 1)k + \frac{n}{2} + d_-[f(K,2k)].$$

Since $\frac{\omega + 1}{2\omega} > 0$ and $0 \leq j \leq n - k$, this is minimized at $j = n - k$ for a fixed $k$. Therefore this case is included in the next case.

Case 2*. $j + k \geq n$.

From the equalities (2.15) and (2.16), one obtains

$$d_-[f(j,k;q)] = -(\omega - \frac{1}{2})j^2 - (n - k + \omega + \frac{1}{2})j - (\tau - \frac{1}{2})k^2 - (n + \tau + \frac{1}{2})k + \frac{n^2}{2} + d_-[J_{K,2k}(q)].$$

Since $\omega - \frac{1}{2} > 0$ and $n - k + \omega + \frac{1}{2} > 0$, this is minimized uniquely at $j = n$ for a fixed $k$. Thus,

$$\min_{n - k \leq j \leq n} d_-[f(j,k;q)] = d_-[f(n,k;q)]$$

$$= -\omega n^2 - (\omega + \frac{1}{2})n - (\tau - \frac{1}{2})k^2 - (\tau + \frac{1}{2})k + d_-[J_{K,2k}(q)].$$

For $k \geq N^*_{K}$, by (2.27) this becomes
for sufficiently large $n$ increasing for $K$ which only depends on the knot degree of the sum. Therefore, there exists $0 \leq \tau \leq \min \tau$. This implies
\[
\min_{N'_K \leq k \leq n} d_\star[f(n, k; q)] = d_\star[f(n, N'_K; q)].
\]

In the sum $\sum_{k=0}^n f(n, k; q)$, there may exist some cancellations, but there may exist at most $N'_K$ cancellations. Therefore, there exists $0 \leq m_0 \leq n$ such that $d_\star[f(n, m_0; q)]$ gives the minimum degree of the sum $\sum_{k=0}^n f(n, k; q)$. Hence
\[
\text{(2.29)} \quad d_\star[\sum_{j,k=0}^n f(j, k; q)] = d_\star[\sum_{k=0}^n f(n, k; q)] = d_\star[f(n, m_0; q)]
\]
\[
= -\omega n^2 - (\omega + \frac{1}{2})n - (\tau - \frac{1}{2})m_0^2 - (\tau + \frac{1}{2})m_0 + d_\star[J'_K,2m_0(q)]
\]
\[
= -\omega n^2 - (\omega + \frac{1}{2})n + C_\star(K, \tau),
\]
where we put
\[
C_\star(K, \tau) = -(\tau - \frac{1}{2})m_0^2 - (\tau + \frac{1}{2})m_0 + d_\star[J'_K,2m_0(q)],
\]
which only depends on the knot $K$ and the number $\tau$.

If instead $\alpha_0^* < \frac{\tau}{4} - \frac{1}{8}$, then $4\alpha_0^* - \frac{\tau}{4} < 0$ and $-\frac{2\beta_0^* - \frac{\tau}{4} - \frac{1}{2}}{4(4\alpha_0^* + \frac{\tau}{4} - \frac{1}{2})} < \frac{\tau}{4}$ for sufficiently large $n \geq N'_K$. Then since $N'_K \geq 0$, for $N'_K \leq k \leq n$, $d_\star[f(n, k; q)]$ is uniquely minimized at $k = n$. Hence, we get that
\[
\min_{N'_K \leq k \leq n} d_\star[f(n, k; q)] = d_\star[f(n, n; q)],
\]
for sufficiently large $n$.

Moreover, this minimum tends to $-\infty$ as $n$ goes to $\infty$. Thus obviously,
\[
\min_{0 \leq k \leq N'_K - 1} d_\star[f(n, k; q)] > d_\star[f(n, n; q)],
\]
for sufficiently large $n$, and hence we have:
\[
\text{(2.30)} \quad \min_{0 \leq k \leq n} d_\star[f(n, k; q)] = d_\star[f(n, n; q)] = (4\alpha_0^* - \tau - \omega + \frac{1}{2})m_0^2 + (2\beta_0^* - \tau - \omega - 1)n + \gamma_0^*.
\]

Let us assume $\alpha_0^* = \frac{\tau}{4} - \frac{1}{8}$. Then by the assumption $-2\alpha_0^* + \beta_0^* - \frac{1}{2} \neq 0$, and hence $2\beta_0^* - \tau - \frac{1}{2} \neq 0$. Furthermore $\beta_0^* - \frac{\tau}{4} - \frac{1}{4} = \beta_0^* - \frac{\tau}{4} - \frac{1}{4} - 2(\alpha_0^* - \frac{\tau}{4} + \frac{1}{4}) = -2\alpha_0^* + \beta_0^* - \frac{1}{2} \geq 0$ by the assumption.
in Proposition 2.13. Hence we have $2\beta_0^* - \tau - \frac{1}{2} > 0$. Thus

$$d_\tau[f(n, k; q)] = (2\beta_0^* - \tau - \frac{1}{2})k - \omega n^2 - \frac{2\omega + 1}{2}n + \gamma_0^*$$

is monotonically increasing with respect to $k$ for $N^*_K \leq k \leq n$.

This implies

$$\min_{N^*_K \leq k \leq n} d_\tau[f(n, k; q)] = d_\tau[f(n, N^*_K; q)].$$

In the sum $\sum_{k=0}^n f(n, k; q)$, there may exist some cancellations, but there may exist at most $N^*_K$ cancellations. Therefore, there exists $0 \leq m'_0 \leq n$ such that $d_\tau[f(n, m'_0; q)]$ gives the minimum degree of the sum $\sum_{k=0}^n f(n, k; q)$. Hence

$$(2.31) \quad d_\tau[\sum_{j,k=0}^n f(j, k; q)] = d_\tau[\sum_{k=0}^n f(n, k; q)]$$

$$= d_\tau[f(n, m'_0; q)]$$

$$= -\omega n^2 - (\omega + \frac{1}{2})n - (\tau - \frac{1}{2})m'_0^2 - (\tau + \frac{1}{2})m'_0 + d_\tau[J'_K, 2m'_0(q)]$$

$$= -\omega n^2 - (\omega + \frac{1}{2})n + C'_\tau(K, \tau),$$

where we put

$$C'_\tau(K, \tau) = -(\tau - \frac{1}{2})m'_0^2 - (\tau + \frac{1}{2})m'_0 + d_\tau[J'_K, 2m'_0(q)],$$

which only depends on the knot $K$ and the number $\tau$.

Finally we determine the minimum degree of $J_{W^{(1)}_Z(K), n}(q) = \frac{1}{n} \sum_{j,k=0}^n f(j, k; q)$; see (2.14).

If $\alpha_0^* > \frac{7}{4} - \frac{1}{8}$, then

$$(2.32) \quad \delta^*_W(K)(n) = -\omega n^2 - (\omega + \frac{1}{2})n + C_\tau(K, \tau) + \frac{n}{2} = -\omega n^2 - \omega n + C_\tau(K, \tau).$$

If $\alpha_0^* < \frac{7}{4} - \frac{1}{8}$, then

$$(2.33) \quad \delta^*_W(K)(n) = d_\tau[f(n, n; q)] + \frac{n}{2} = (4\alpha_0^* - \tau - \omega + \frac{1}{2})n^2 + (2\beta_0^* - \tau - \omega - \frac{1}{2})n + \gamma_0^*.$$  

If $\alpha_0^* = \frac{7}{4} - \frac{1}{8}$ and $-2\alpha_0^* + \beta^* - \frac{1}{2} \neq 0$, then we have

$$(2.34) \quad \delta^*_W(K)(n) = -\omega n^2 - (\omega + \frac{1}{2})n + C_\tau(K, \tau) + \frac{n}{2} = -\omega n^2 - \omega n + C_\tau(K, \tau).$$

\[\Box\]

**Remark 2.14.** Let $K$ be a knot which has a $B$-adequate diagram $D_B(K)$ whose blackboard framing is 0. It follows that a $B$-adequate diagram $D_B(W^1_B(K))$ of $W^1_B(K)$ can be obtained from $D_B(K)$. Let $c_+(D_B(K))$ be the number of positive crossings of $D_B(K)$. From [8], we have that $\alpha_0 = c_+(D_B(K))$. Moreover, one can see that $c_+(D_B(W^1_B(K))) = 4c_+(D_B(K))$. So, the formula for the case $a_1 > \frac{7}{4}$ in (2.17) agrees with Lemma 6 (b) in [8].
Remark 2.15. In Proposition 2.1, for technical reason, if $b_1 = 0$, then we assume that $a_1 \neq \frac{7}{4}$. When $b_1 = 0$, it is conjectured that $K$ is cabled [20, Conjecture 5.1]. For example, if $K$ is an $(a, b)$-torus knot $T_{a, b}$, then we may have $\delta_{T_{a, b}}(n)$ with $4a_1 = ab$ and $b_1 = 0$ [9].

We close this section by computing $\delta_{W_{a,b}^\omega(T_{a,b})}(n)$.

Proposition 2.16. Let $a$ and $b$ be integers with $a > b > 1$. Then the maximum degree of the colored Jones polynomial $J^\omega_{W_{a,b}^\omega(K)}(n)$ of ab-twisted $\omega$-generalized Whitehead double of $K = T_{a,b}$ is given by $-\frac{a}{4}n + \frac{b}{2}$.

Proof. The colored Jones polynomial of $K = T_{a,b}$ is explicitly computed in [26]:

\[ J_{K,n}(q) = \frac{q^{\frac{1}{4}ab(n+2)}}{q^\frac{a+1}{2} - q^\frac{a+1}{2}} \sum_{k=-\frac{a}{4}}^{\frac{1}{2}} (q^{-abk^2 + (a-b)k + \frac{1}{2}} - q^{-(a+b)k - \frac{1}{2}}) \]

We note that if $n$ is even, then $k$ is an integer in the summand. We define the functions $f_\pm(\ell)$ on $\mathbb{Z}$ by

\[ f_\pm(\ell) := -ab\ell^2 + (a \mp b)\ell \pm \frac{1}{2} \]

Since

\[ f_\pm(\ell) = -ab(\ell - \frac{a \mp b}{2ab})^2 + \frac{(a \mp b)^2}{4ab} \pm \frac{1}{2} \]

and $0 < \frac{a \mp b}{2ab} < \frac{1}{2}$, $f_\pm(\ell)$ is maximized at $\ell = 0$ and $f_-(0) < f_+(0) = \frac{1}{2}$. Hence the maximum degree of $J'_{K,n}(q)$ for even $n$ is calculated by

\[ f_\pm(\ell) = \frac{1}{4}ab(n+2) - \frac{n+1}{2} + \frac{1}{2} = \frac{ab}{4}n^2 + \frac{ab-1}{2}n, \]

and the term of the maximum degree is $q^{\frac{ab}{4}n^2 + \frac{ab-1}{2}n}$. Therefore, the term of the maximum degree of $J'_{K,2k}(q)$ is given by

\[ q^{abk^2 + (ab-1)k}. \]

Recall from (2.13) that

\[ f(j, k; q) = \begin{cases} 2j > & < n, 2j > < n, k > \begin{pmatrix} n & n & 2j \\ n & n & 2k \end{pmatrix} q^{-\omega j(j+1) - ab(k+1)} J'_{K,2k}(q), \\ \end{cases} \]

and then by Proposition 2.7 we have

\[ < n > J_{W_{a,b}^\omega(K),n}(q) = \sum_{j,k=0}^{n} f(j, k; q). \]

From the proof of Proposition 2.12, we have that $d_+[f(j, k; q)]$ is maximized uniquely at $j = 0$ for a fixed $k$. Moreover, Since we have that $< n, n, 0 > = < n >$ and $\begin{pmatrix} n & n & 0 \\ n & n & 2k \end{pmatrix} = < n, n, 2k >$, we calculate

\[ f(0, k; q) = \begin{cases} 1 < n > < 2k > < n, n, 2k > \begin{pmatrix} n & n & 0 \\ n & n & 2k \end{pmatrix} q^{-ab(k+1)} J'_{K,2k}(q), \\ \end{cases} \]

\[ = \begin{cases} 1 < n > < 2k > q^{-ab(k+1)} J'_{K,2k}(q). \end{cases} \]
From (2.36), the term of the maximum degree of \( f(0, k; q) \) is calculated as

\[
(-1)^n q^{-\frac{2}{q}} k^q - abk^{q-1} q^{a+bk^2 + (ab-1)k} = (-1)^n q^{-\frac{2}{q}}.
\]

Hence the term of the maximum degree of \( J'_{W^{ab}(K), n}(q) \) is given by

\[
(-1)^n q^{-\frac{2}{q}} \sum_{k=0}^{n} (-1)^n q^{-\frac{2}{q}} = (n+1)q^{-n}.
\]

Hence \( \delta'_{W^{ab}(K)}(n) = -n \). Apply the transformation given in the proof of Proposition 2.1, we have

\[
\delta_{W^{ab}(K)}(n) = \delta'_{W^{ab}(K)}(n-1) + \frac{1}{2}n - \frac{1}{2} = -\frac{1}{2}n + \frac{1}{2}.
\]

□

3. Computations of slopes and Euler characteristics for generalized Whitehead doubles

3.1. Exteriors of twisted, generalized Whitehead doubles and those of two-bridge links.

We start with a 2–bridge link \( k_1 \cup k_2 \), which is expressed as \([2, 2\omega, -2]\) with \( \omega \geq 1 \) depicted in Figure 3.1 below. Then \( k_2 \) lies in an unknotted solid torus \( V = S^3 - \text{int}N(k_1) \). Let us perform \( \tau \) twist along \( k_1 \) to obtain a knot \( k^\tau_2 \) \((\tau \in \mathbb{Z})\), which is embedded in \( V \). Note that \( k_1 \cup k^\tau_2 \) does not form a 2–bridge link in general, but its exterior is orientation preservingly homeomorphic to the exterior of the 2–bridge link \( k_1 \cup k_2 \). If \((\omega, \tau) = (1, 0)\), then \( k_1 \cup k^0_2 \) is the negative Whitehead link.

Let us take preferred meridian-longitude pairs \((\mu_1, \lambda_1), (\mu, \lambda)\) of \( k_1, k^\tau_2 \), respectively. Then take an orientation preserving embedding \( f : V \to S^3 \) which sends the core of \( V \) to a knot \( K \) and \( f(\mu_1) = \lambda_K \) and \( f(\lambda_1) = \mu_K \), where \((\mu_K, \lambda_K)\) is a preferred meridian-longitude pair of \( K \). The image \( f(k^\tau_2) \) is \( W^\tau(K) \), a \( \tau \)–twisted, \( \omega \)–generalized Whitehead double of \( K \).

![Figure 3.1](image)

**Figure 3.1.** \( k_1 \cup k_2 \) is a two bridge link \([2, 2\omega, -2] = L_{\frac{2\omega+1}{2\omega}}\).

This observation shows that the exterior of \( W^\tau(K) \) is the union of the exterior \( E(K) \) and \( V - \text{int}N(k_2) \); the latter is the exterior of the two-bridge link \( k_1 \cup k_2 \), which is expressed as \([2, 2\omega, -2]\).

Since \( k_2 = k^\tau_2 \) has winding number 0 in \( V \), and is therefore null-homologous in \( V \), we have:
Lemma 3.1. Let $F$ be an essential surface in $V - \text{int}N(k_2) = V - \text{int}N_0(k_2)$ such that $F \cap \partial V$ has slope $r_V$ and $F \cap \partial N(k_2)$ has slope $r$. Apply a twist to obtain $k_2$ and an essential surface $F_\tau$, the image of $F$, in $V - \text{int}N(k_2)$. Then $F_\tau \cap \partial V$ has slope $r_V + \tau$ and $F \cap \partial N(k_2)$ has slope $r$.

In the following subsections 3.2–3.7, we will investigate essential surfaces in the exterior $E(k_1 \cup k_2)$ of a two-bridge link $[2, \omega, -2]$ in details.

3.2. Essential surfaces in two-bridge link exteriors. Here we extend the work in [17, Section 5] to catalogue all the properly embedded essential surfaces in the exterior of the two-bridge link $L_{(4\omega - 1)/8\omega}$ for integers $\omega \geq 1$.

Hatcher-Thurston show how a certain collection of “minimal edge paths” in the Farey diagram from 1/0 to $p/q$ are in correspondence with the properly embedded incompressible and $\partial$–incompressible surfaces with boundary in the exterior of the two-bridge knot $L_{p/q}$ [15]. Floyd-Hatcher extend this to two-bridge links of two components [6] from which Hoste-Shanahan discern the boundary slopes of such surfaces [17], building upon work of Lash [21].

Here, for use with satellite constructions, we use the works of Floyd-Hatcher [6] and Hoste-Shanahan [17] to catalog all the properly embedded essential surfaces in the exterior of the generalized Whitehead link $L_{(4\omega - 1)/8\omega}$, their Euler characteristics, their boundary slopes, and number of boundary components.

Remark 3.2. While [6] uses the continued fraction convention $[x_1, x_2, \ldots, x_n] = 1/(x_1 + 1/(x_2 + \ldots + 1/x_n))$, [17] appears to use the convention $[x_0, x_1, x_2, \ldots, x_n] = x_0 + 1/(x_1 + 1/(x_2 + \ldots + 1/x_n))$. To remain consistent with this notation and the depiction of $L_{3/8}$ in [17, Figure 1], the link $L_{(4k - 1)/8k}$ is actually obtained by $-1/k$ surgery on the middle circle of [17, Figure 9] which produces $2k$ right-handed crossings.

We refer the reader to both the original paper [6] and Hoste-Shanahan’s recounting of it [17, Section 2] for details on the Floyd-Hatcher algorithm. Here we briefly recall the algorithm and quickly work through the application of it for the Whitehead link $L_{3/8}$ based on the more general treatment for the links $L_{(4\omega - 1)/8\omega}$ given in [17, Section 5].

3.3. The Algorithm. Figure 3.2 shows three diagrams. The diagram $D_1$ is the common Farey diagram. Pair adjacent triangles into quadrilaterals containing a diagonal so that a vertex is an endpoint of either all or none of the diagonals of the incident quadrilaterals. The diagram $D_0$ is obtained by switching the diagonal in each of the quadrilaterals. The diagram $D_t$ is obtained by replacing these diagonals with inscribed quadrilaterals. Actually, $D_t$ represents a parameterized family of diagrams for $t \in [0, \infty)$: with appropriate parameterizations of the edges of the quadrilaterals by $[0, 1]$ the vertices of the inscribed quadrilaterals in $D_t$ are located at either $t$ or $1/t$.

The diagrams $D_0 = D_\infty$ and $D_1$ arise as limits where the inscribed quadrilaterals degenerate to diagonals. The edges of $D_1$ are labeled $A$ and $C$, the edges of $D_0 = D_\infty$ are labeled $B$ and $D$, and these induce labels on $D_t$. Orientations are chosen on a basic set of edges in $D_t$ and passed to the rest of the edges of $D_t$ by the action of the Möbius transformations in which the ideal triangle with vertices $\{1/0, 0/1, 1/1\}$ is a fundamental domain. We omit the orientations in Figure 3.2; see [17] for details.
For a two bridge link $L_{p/q}$ (where $q$ is even), Floyd-Hatcher show that a properly embedded essential surface in the exterior of the link is carried by one of finitely many branched surfaces associated to “minimal edge paths” in $D_t$ from $1/0$ to $p/q$. A minimal edge path in $D_t$ is a consecutive sequence of edges of $D_t$ (ignoring their orientations) such that the boundary of any face of $D_t$ contains at most one edge of the path. Then for each minimal edge path, a branched surface is assembled from the sequence of edges by stacking four blocks of basic branched surface $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_D$ corresponding to the labels $A, B, C, D$ that are positioned according to the endpoints and orientation of its edge and whether $t < 1$ or $t > 1$. These blocks of basic branched surfaces are illustrated in Figure 3.3 for $t > 1$ (cf. [17, Figure 2] and [6, Figure 3.1]) and are weighted in terms of the parameters $\alpha > \beta > 0$ where $t = \alpha/\beta$ and the extra integral parameter $n$ between 0 and $\beta$ for $\Sigma_A$ or between 0 and $\alpha - \beta$ for $\Sigma_D$. (This extra parameter $n$ allows for the construction of homeomorphic but non-isotopic surfaces with the same boundary slopes, see [17, 6].) For $t < 1$, the blocks are rotated 180° corresponding to an exchange of the components of $L_{p/q}$ and the parameters $\alpha$ and $\beta$ are swapped in the figure.

In this manner, every minimal edge path in $D_t$ for $t \in (0,1) \cup (1,\infty)$ produces a weighted branched surface, with weights in terms of the parameters $\alpha$ and $\beta$ such that $t = \alpha/\beta$ (along with auxiliary parameters for instances of the blocks $\Sigma_A$ and $\Sigma_D$). These minimal edge paths $\gamma$ in $D_t$ with their parameters $\alpha, \beta$ describe specific surfaces $F_{\gamma,\alpha,\beta}$ which may have multiple components and may be non-orientable. If it is non-orientable, then we may replace $F_{\gamma,\alpha,\beta}$ by the boundary of a tubular neighborhood (a twisted I–bundle over $F_{\gamma,\alpha,\beta}$), which is orientable and associated with parameters $2\alpha, 2\beta$; so the resulting orientable essential surface is associated with $F_{\gamma,2\alpha,2\beta}$. In the following we omit parameters $\alpha, \beta$ and assume that $F_{\gamma}$ is orientable, but it may have multiple components.

Taking the limits $t \to 0$ or $t \to \infty$ so that $\alpha = 0$ or $\beta = 0$ produces surfaces associated to minimal edges paths in $D_0 = D_\infty$. Taking the limits $t \to 1$ so that $\alpha = \beta$ also produces surfaces associated to minimal edge paths in $D_1$. However, since $\alpha - \beta = 0$ in this case, the basic surface $\Sigma_A$ with its extra parameter $n$ may be used in place of $\Sigma_D$ to produce more surfaces.

Floyd and Hatcher [6] establish the following classification of essential surfaces in the exterior of two-bridge links.
Figure 3.3. The four basic weighted branched surfaces (reproduced from [17, Figure 2], see also [6, Figure 3.1]) along with the corresponding number of saddles for the carried surface, when $\alpha \geq \beta$. When $\alpha < \beta$, rotate the images $180^\circ$ and swap $\alpha$ and $\beta$.

Theorem 3.3 ([6]). Let $L_{p/q}$ be a two-bridge link (with $q$ even). The orientable incompressible and meridionally incompressible surfaces in $S^3 - N(L_{p/q})$ without peripheral components are (up to isotopy) exactly the orientable surfaces carried by the collection of branched surfaces associated to minimal edge paths in $D_t$ from $1/0$ to $p/q$ for $t \in [0, \infty]$.

Remark 3.4. Let $S$ be a properly embedded surface in the exterior of a link $L$ in $S^3$. Then $S$ is meridionally incompressible if for any embedded disk $D$ in $S^3$ with $D \cap S = \partial D$ such that $L$
intersects $D$ transversally in a single interior point, there is an annulus embedded in $S$ whose boundary is $\partial D$ and a component of $\partial S$ that is a meridian of $L$. A component of $S$ is peripheral if it is isotopic through the exterior of $L$ into $\partial N(L)$. If $S$ has a $\partial$–compressing disk, then either $L$ is a split link or the $\partial$–compressible component of $S$ is either compressible or peripheral. Hence the surfaces in Theorem 3.3 are also $\partial$–incompressible.

3.4. Euler characteristics of carried surfaces. The Euler characteristic of a surface carried by one of these weighted branched surfaces associated to an edge path in $D_t$ may be calculated from the branch pattern associated to the edge path and the weights $\alpha$ and $\beta$.

Lemma 3.5. Let $S$ be the surface carried by the weighted branched surface associated to an edge path $\gamma$ in $D_t$ where $t = \alpha/\beta$. If $\alpha \geq \beta$, then

$$\chi(S) = (\alpha + \beta) - \sum s_i(\alpha, \beta)$$

where $s_i(\alpha, \beta)$ is the number of saddles of the surface carried by the basic branched surface associated to the label of the $i$th edge of $\gamma$ and weighted by $\alpha$ and $\beta$ as shown in Figure 3.3. If $\alpha < \beta$, exchange $\alpha$ and $\beta$.

Proof. As shown in Figure 3.3, when $\alpha \geq \beta$, each basic weighted branched surface of type $\Sigma_A, \Sigma_B, \Sigma_C, \Sigma_D$ carries $\beta, \frac{\alpha - \beta}{2}, \beta, \alpha - \beta$ saddles respectively and a number of vertical disks that together meet each of the upper and lower levels in a total of $\alpha + \beta$ arcs. Since the weighted branched surfaces are assembled from a stack of copies of these basic weighted branched surfaces, the height function induces a Morse function on a carried surface $S$ whose singularities correspond to the saddles on the interior of the surface and the $\alpha + \beta$ half-maxima and $\alpha + \beta$ half-minima on the boundary of the surface at the extrema of the link. Hence $\chi(S)$ is calculated as $\alpha + \beta$ minus the total number of saddles. When $\alpha < \beta$, the blocks are rotated and $\alpha$ and $\beta$ are swapped. \(\square\)

3.5. Boundaries slopes and count of boundary components. Note that surfaces carried by these branched surfaces are given by non-negative integral weights $\alpha$ and $\beta$ (with the auxiliary integral parameters $n$ as needed), and these weights indicate the algebraic (and geometric) intersection numbers of the surface with the meridians $\mu_1, \mu_2$ of the two components of $\mathcal{L}_{p/q}$.

Hoste and Shanahan use a certain blackboard framing $\lambda_1, \lambda_2$ of the two components of $\mathcal{L}_{p/q}$ to further keep track of how the branched surfaces associated to minimal edge paths in $D_t$ intersect this framing. They then determine how to correct this framing to the canonical framings $\lambda_1^0, \lambda_2^0$ of the individual unknot components of the two-bridge link. From this, one then obtains the boundary slopes of the carried surfaces in terms of the canonical framings of the components.

Furthermore, by a calculation in the homology of a torus, the greatest common divisor (gcd) of the algebraic intersection numbers of the boundary of a surface with the meridian and longitudinal framing of a component of $\mathcal{L}_{p/q}$ produces the number of boundary components of the surface meeting that component of $\mathcal{L}_{p/q}$.

3.6. Applying the Algorithm to the Whitehead Link– 2–bridge link $[2, 2, -2]$. As a warm-up example, in this subsection, we apply the algorithm to the Whitehead Link, which is the 2–bridge link $[2, 2, -2]$. Figure 3.4 shows the portions of the diagrams $D_0 = D_{\infty}$, $D_t$, and $D_1$ that carry the
minimal edge paths from $1/0$ to $3/8$. Table 3.1 lists these minimal edge paths with their names as given in each [17] and [6], the branch pattern of the induced branched surface (i.e. the sequence of edge labels), and the Euler characteristic of the carried surface corresponding to weights $\alpha \geq \beta$. Table 3.2 lists for each of these paths the boundary slopes of the carried surfaces relative to the canonical meridian-longitude framings of the two unknotted components of the two-bridge links and the count of the number of boundary components on each link component. These are also calculated from the given preliminary data of algebraic intersections of the boundary components with the meridians and blackboard framed longitudes and the boundary slopes in terms of the blackboard framing; refer to [17] for details. Note that for each of the paths $\gamma_i$, $i \in \{1, 2, 3, 5, 6\}$, when $\beta = 0$ so that $\alpha/\beta = \infty$ the associated essential surface is disjoint from the 2nd link component. Table 3.2 summarizes the relevant data. When $\alpha < \beta$ we may continue to use the two tables, but with $\alpha$ and $\beta$ swapped and with the two link components swapped.

3.7. Applying the Algorithm to a generalized Whitehead Link -- 2–bridge link $[2, 2\omega, -2]$. Let us apply the algorithm to a generalized Whitehead Link, which is 2–bridge link $[2, 2\omega, -2]$. Figure 3.5 shows the portions of the diagrams $D_0 = D_\infty$, $D_t$, and $D_1$ that carry the minimal edge paths from $1/0$ to $(4\omega - 1)/8\omega$. We obtain Tables 3.4, 3.5 and 3.6 corresponding to Tables 3.1, 3.2 and 3.3. Note that for each of the paths $\gamma_i$, $i \in \{1, 2, 3, 4, 5, 6\}$, when $\beta = 0$ so that $\alpha/\beta = \infty$ the associated surface is disjoint from the second link component. Indeed, for $i \in \{1, 2, 3, 4, 5\}$, when $\alpha = 1$ and $\beta = 0$, the associated surface is a once-punctured torus Seifert surface for the first link component. For $i = 6$, when $\alpha = 2$ and $\beta = 0$, the associated surface is a twice punctured torus disjoint from the second component.

Figure 3.5. The diagrams $D_1$, $D_t$, and $D_0 = D_\infty$ that carry the minimal paths from $1/0$ to $(4\omega - 1)/8\omega$. 
Table 3.1. The minimal edge paths in \( D_t \) from 0/1 to 3/8, the branch patterns of their supporting branched surfaces, and the Euler characteristics for the surfaces when \( t = \alpha/\beta \geq 1 \) are shown. The HS path name is established in [17, Table 2]. The FH path name is established in [6, Section 5(III) and Figure 5.3] using \( l = m = n = 1 \); see also the top left of [6, Figure 5.4].

| HS path | FH path | path picture | branch pattern | \( \chi \) |
|---------|---------|--------------|----------------|---------|
| \( \gamma_1 \) | 1-6 | \( ADAADA \) | \(-\alpha - \beta\) |
| \( \gamma_2 \) | 3-6 | \( ADAADA \) | \(-\alpha - \beta\) |
| \( \gamma_3 \) | 2-6 | \( ADAADA \) | \(-\alpha - \beta\) |
| \( \gamma_5 \) | 5-6 | \( ADCDA \) | \(-\alpha\) |
| \( \gamma_6 \) | 5-10 | \( ABBCBBA \) | \(-\alpha\) |
| \( \gamma_5' \) | 5 | \( ACA \) | \(-\alpha\) |

Table 3.2. The two boundary slopes and number of boundary components of each surface carried by a branched surface associated to a minimal edge path from 0/1 to 3/8 for the first and second link component are presented as a pair. The table shows the case \( \alpha \geq \beta > 0 \) or (with an even integer \( 0 \leq X \leq 2\beta \) for \( \gamma_5' \) where \( \alpha = \beta > 0 \)) so that \( t = \alpha/\beta \in [1, \infty) \). For \( t = \infty \) when \( \alpha > \beta = 0 \), the surface is disjoint from the second component so the second coordinate in the last three columns are \( \emptyset, \emptyset, 0 \) respectively. For \( t \in [0, 1] \), apply the homeomorphism of the two-bridge link that swaps its two components, i.e. exchange coordinates and swap \( \alpha \) and \( \beta \).

| HS alg. int. with blackboard framing | slopes with canonical framing | number of boundary components |
|-------------------------------------|-------------------------------|------------------------------|
| \( \gamma_1 \) \( (\alpha + 2\beta, \alpha, 2\alpha + \beta, \beta) \) | \( (1 + 2\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta} + 1) \) | \( (\gcd(2\beta, \alpha), \gcd(2\alpha, \beta)) \) |
| \( \gamma_2 \) \( (\alpha, \alpha, \beta, \beta) \) | \( (1, 1) \) | \( (\alpha, \beta) \) |
| \( \gamma_3 \) \( (\alpha, \alpha, \beta, \beta) \) | \( (1, 1) \) | \( (\alpha, \beta) \) |
| \( \gamma_5 \) \( (\alpha - 2\beta, \alpha, -2\alpha - \beta, \beta) \) | \( (1 - 2\frac{\alpha}{\beta}, -2\frac{\alpha}{\beta} - 1) \) | \( (\gcd(2\beta, \alpha), \gcd(2\alpha, \beta)) \) |
| \( \gamma_6 \) \( (-3\alpha, \alpha, -\beta, \beta) \) | \( (-3, -1) \) | \( (\alpha, \beta) \) |
| \( \gamma_5' \) \( (-3\beta + X, \beta, -\beta - X, \beta) \) | \( (-3 + \frac{\alpha}{\beta}, -1 - \frac{X}{\beta}) \) | \( (\gcd(X, \beta), \gcd(X, \beta)) \) |
Table 3.3. Summary of data in terms of canonical framing.

| HS branch pattern | $\chi$ | boundary slopes | number of boundary components |
|-------------------|--------|----------------|-----------------------------|
| $\gamma_1$ ADAADA | $-\alpha - \beta$ | $(\frac{2\beta}{\alpha}, \frac{2\beta}{\alpha})$ | $(\text{gcd}(2\beta, \alpha), \text{gcd}(2\alpha, \beta))$ |
| $\gamma_2$ ADAADA | $-\alpha - \beta$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_3$ ADAADA | $-\alpha - \beta$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_4$ ADCDA | $-\alpha$ | $(-2\frac{\alpha}{\beta}, -2\frac{\alpha}{\beta} - 2)$ | $(\text{gcd}(2\beta, \alpha), \text{gcd}(2\alpha, \beta))$ |
| $\gamma_5$ ABBCBBA | $-\alpha$ | $(-4, -2)$ | $(\alpha, \beta)$ |
| $\gamma_5'$ ACA | $-\alpha$ | $(-4 + \frac{\alpha}{\beta}, -2 - \frac{\alpha}{\beta})$ | $(\text{gcd}(X, \alpha), \text{gcd}(X, \beta))$ |

Table 3.4. The minimal edge paths in $D_t$ from $0/1$ to $(4k - 1)/8k$ for positive integers $k$, the branch patterns of their supporting branched surfaces, and the Euler characteristics for the surfaces when $t = \alpha/\beta \geq 1$ are shown. The HS path name is established in [17, Table 2]. The FH path name is established in [6, Section 5(III) and Figure 5.3] using $l = n = 1$ and $m = k$; see also the top left of [6, Figure 5.4]. (Compare with Table 3.1 for $L_{3/8}$.)

| HS path | FH path | path picture | branch pattern | $\chi$ |
|---------|---------|--------------|----------------|-------|
| $\gamma_1$ | 1-6 | | ADAADA | $-\alpha - \beta$ |
| $\gamma_2$ | 3-6 | | ADAADA | $-\alpha - \beta$ |
| $\gamma_3$ | 2-6 | | ADAADA | $-\alpha - \beta$ |
| $\gamma_4$ | 4-6 | | ADAADA | $-\alpha - \beta$ |
| $\gamma_5$ | 5-6 | | ADC$^{2k-1}$DA | $-\alpha + 2(1 - k)\beta$ |
| $\gamma_6$ | 5-10 | | AB(BCB)$^{2k-1}$BA | $(1 - 2k)\alpha$ |
| $\gamma_5'$ | 5 | | AC$^{2k-1}$A | $(1 - 2k)\beta = (1 - 2k)\alpha$ |

4. Slope conjecture for twisted generalized Whitehead doubles

In this section we prove Theorem 1.9(1).

For a given knot $K$, recall that $N_K$ the smallest nonnegative integer such that $d_+[J_{K,n}(q)]$ is a quadratic quasi-polynomial $\delta_K(n) = a(n)n^2 + b(n)n + c(n)$ for $n \geq 2N_K + 1$, and $N^*_K$ the smallest nonnegative integer such that $d_+[J_{K,n}(q)]$ is a quadratic quasi-polynomial $\delta^*_K(n) = a^*(n)n^2 +$
Table 3.5. The two boundary slopes and number of boundary components of each surface carried by a branched surface associated to a minimal edge path from 0/1 to (4k − 1)/8k for the first and second link component are presented as a pair. The table shows the case $\alpha \geq \beta > 0$ or (with $X = 2(n_1 - n_2 + n_3 - \cdots + n_{2k-1})$ for integers $0 \leq n_i \leq 1$) for $\gamma_0^*$ where $\alpha = \beta > 0$ so that $t = \alpha/\beta \in [1, \infty)$. For $t = \infty$ when $\alpha > \beta = 0$, the surface is disjoint from the second component so the second coordinate in the last two columns are $\emptyset$ and 0 respectively. For $t \in [0, 1]$, apply the homeomorphism of the two-bridge link that swaps its two components, i.e. exchange coordinates and swap $\alpha$ and $\beta$. (Compare with Table 3.2 for $L_{3/8}$.)

| HS path $(\lambda_1, \mu_1, \lambda_2, \mu_2)$ | slopes with canonical framing $\gamma_0^*$ | number of boundary components $\alpha, \beta$ |
|-----------------------------------------------|---------------------------------------------|-----------------------------------------------|
| $\gamma_1$ $(\alpha, 2\beta, \alpha, 2\alpha + \beta, \beta)$ | $(2\beta, \alpha, \alpha)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$ |
| $\gamma_2$ $(\alpha, \alpha, \beta, \beta)$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_3$ $(\alpha, \alpha, \beta, \beta)$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_4$ $(\alpha - \beta, \alpha, -2\alpha + \beta, \beta)$ | $(-2\beta, -2\alpha)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$ |
| $\gamma_5$ $(\alpha - \beta, \alpha, -2\alpha + (3 - 4k)\beta, \beta)$ | $(-2\beta, -2\alpha + 2 - 4k)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$ |
| $\gamma_6$ $(1 - 4k)\alpha, \alpha, -\beta, \beta)$ | $(-4k, -2)$ | $(\alpha, \beta)$ |
| $\gamma_0^*$ $(-1 + 2k)\beta + X, \beta, (1 - 2k)\beta - X, \beta)$ | $(-2 - 2k + \frac{\beta}{2}, -2k - \frac{\beta}{2})$ | $(\gcd(X, \beta), \gcd(X, \beta))$ |

Table 3.6. Summary of data in terms of canonical framing. (Compare with Table 3.3 for $L_{3/8}$.)

| HS path $(\lambda_1, \mu_1, \lambda_2, \mu_2)$ | boundary slopes $\chi$ | number of boundary components $\beta > 0$ | $\beta = 0$ |
|-----------------------------------------------|--------------------------|-----------------------------------------------|-----------------------------------------------|
| $\gamma_1$ $-\alpha - \beta$ | $(2\beta, \alpha)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$, $(\gcd(2\beta, \alpha), 0)$ |
| $\gamma_2$ $-\alpha - \beta$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_3$ $-\alpha - \beta$ | $(0, 0)$ | $(\alpha, \beta)$ |
| $\gamma_4$ $-\alpha - \beta$ | $(-2\beta, -2\alpha)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$, $(\gcd(2\beta, \alpha), 0)$ |
| $\gamma_5$ $-\alpha + 2(1 - k)\beta$ | $(-2\beta, -2\alpha + 2 - 4k)$ | $(\gcd(2\beta, \alpha), \gcd(2\alpha, \beta))$, $(\gcd(2\beta, \alpha), 0)$ |
| $\gamma_6$ $(1 - 2k)\alpha$ | $(-4k, -2)$ | $(\alpha, \beta)$ |
| $\gamma_0^*$ $(1 - 2k)\alpha$ | $(-2 - 2k + \frac{\beta}{2}, -2k - \frac{\beta}{2})$ | $(\gcd(X, \beta), \gcd(X, \beta))$ |

Then we put $a_1 := a(2N_K^* + 1)$, $b_1 := b(2N_K^* + 1)$, and $b_1^* := b^*(2N_K^* + 1)$. For convenience we state Theorem 1.9(1) again here.

**Theorem 1.9(1).** Let $K$ be a knot. We assume that the period of $\delta_K(n)$ and $\delta_K^*(n)$ are less than or equal to 2 and that $b_1 \leq 0$ and $b_1^* \geq 0$. Assume further that that if $b_1 = 0$, then $a_1 = \frac{\beta}{\alpha}$ and that if $b_1^* = 0$, then $a_1^* = \frac{\beta}{\alpha} - \frac{\gamma}{\delta}$. If $K$ satisfies the Slope Conjecture, then all of its twisted generalized Whitehead doubles also satisfy the Slope Conjecture.

**Proof.** It follows from Proposition 2.1 that if $r_W \in jsW^*_K$, then there exists $r_K \in js_K$ such that $r_W = 4r_K - 4r$ if $a_1 > \frac{\beta}{\alpha}$, or $r_W = 0$ if $a_1 \leq \frac{\beta}{\alpha}$. Similarly, Proposition 2.2 shows that if $r_W \in js_K^*$, then there exists $r_K \in js_K^*$ such that $r_W = 4r_K - 4\omega + 2 - 4\tau$ if $a_1^* < \frac{\beta}{\alpha}$, or
\[ r_W = -4\omega \text{ if } a_1^+ \geq -\frac{2}{3} - \frac{1}{8} \text{.} \] This gives
\[ js_{W^\tau(K)} \subset \{4js(K) - 4\tau, 0\} \quad \text{and} \quad js_{\partial W^\tau(K)} \subset \{4js(K) - 4\omega + 2 - 4\tau, -4\omega\} \text{.} \]

Let us find essential surfaces in \( E(W^\tau(K)) \) whose boundary slopes are these Jones slopes.

Recall that \( k_1 \cup k_2 \) is a 2–bridge link expressed as \([2, 2\omega, -2] \; (\omega \geq 1) \) depicted in Figure 3.1; \((\mu_1, \lambda_1)\) denotes a preferred meridian-longitude pair of \( k_1 \). As in Figure 3.1 take a solid torus \( V = S^3 - \text{int}N(k_1) \) which contains \( k_2 \) in its interior; let \((\mu_V, \lambda_V)\) be the standard meridian-longitude pair of \( V \subset S^3 \). Performing \(-1/\tau–surgery\) on \( k_1 \), equivalently \( \tau–twisting\) along \( \mu_V \), we obtain \( k^\tau_\omega \) which is the image of \( k_2 \) (Figure 1.1). Let \( f \) be an orientation preserving embedding \( f : V \to S^3 \) which sends \( V \) to \( N(K) \) and \( f(\mu_V) = \mu_K \) and \( f(\lambda_V) = \lambda_K \), where \((\mu_K, \lambda_K)\) is a preferred meridian-longitude pair of \( K \). Then \( W^\tau(K) = f(k^\tau_\omega) \) is the \( \tau–twisted\) generalized Whitehead double of \( K \). Thus the exterior \( E(W^\tau(K)) \) is the union of \( E(K) \) and \( f(V - \text{int}N(k^\tau_\omega)) \). The boundary of \( f(V - \text{int}N(k^\tau_\omega)) \) consists of two tori \( T_W = \partial N(W^\tau(K)) \) and \( T_K = f(\partial V) = \partial E(K) \). Then \((f(\mu_2), f(\lambda_2))\) is a preferred meridian-longitude pair \((\mu_W, \lambda_W)\) of \( W^\tau(K) \).

1. **Realization of the Jones slopes arising from the maximum degree.**

We divide into two cases depending upon \( a_1 > \frac{\gamma}{4} \) or \( a_1 \leq \frac{\gamma}{4} \); see Proposition 2.1.

**Case 1.** \( a_1 > \frac{\gamma}{4} \). Since \( K \) satisfies the Slope Conjecture, the Jones slope \( 4a_1 \) is realized by a boundary slope of an essential surface \( S_K \subset E(K) \).

**Claim 4.1.** There exists an essential surface \( F'_\omega \) in \( V - \text{int}N(k^\tau_\omega) \) such that each component of \( F'_\omega \cap \partial V \) has slope \( 4a_1 \) and each component of \( F'_\omega \cap \partial N(k^\tau_\omega) \) has \( 16a_1 - 4\tau \).

**Proof.** Let us take an essential surface \( F_{\gamma_1} \) in \( S^3 - \text{int}N(k_1 \cup k_2) = V - \text{int}N(k_2) \) associated to the minimal edge path \( \gamma_1 \) described in Section 3. Then it has a pair of boundary slopes \((2a, 2b)\) on \( k_1, k_2 \). Then \( F_{\gamma_1} \) has boundary slopes \( \frac{2a}{\alpha} \) on \( \partial N(k_1) \) and \( \frac{2b}{\beta} \) on \( \partial N(k_2) \). Using the preferred meridian-longitude \((\mu_V, \lambda_V)\) of \( V \) instead of \((\mu_1, \lambda_1)\) of \( k_1 \), \( F_{\gamma_1} \cap \partial V \) has slope \( \frac{\alpha}{\beta} \). Choose \( \alpha, \beta \) so that \( \frac{\alpha}{\beta} = 4a_1 - \tau > 0 \), i.e. \( \frac{\alpha}{\beta} = 8a_1 - 2\tau > 0 \). Then \( F_{\gamma_1} \subset V - \text{int}N(k_2) \) has boundary slope \( 16a_1 - 4\tau \) on \( \partial N(k_2) \) and \( 4a_1 - \tau \) on \( \partial V \). Now we apply \( \tau–twisting\) along \( \mu_V \) which changes \( V - \text{int}N(k_2) \) to \( V - \text{int}N(k^\tau_\omega) \); we denote the image of \( F_{\gamma_1} \) by \( F'_\omega \). By Lemma 3.1 each component of \( F'_\omega \cap \partial V \) has slope \( 4a_1 = (4a_1 - \tau + \tau) \), and each component of \( F'_\omega \cap \partial N(k^\tau_\omega) \) has slope \( 16a_1 - 4\tau \) as desired. \( \square \)

Let us take the image \( f(F'_\omega) \) in \( f(V - \text{int}N(k^\tau_\omega)) \), and denote it by \( S'_\omega \). Write \( T_K = \partial E(K) = f(\partial V) \) and \( T_W = \partial N(W^\tau(K)) = f(\partial N(k^\tau_\omega)) \). By construction \( S'_\omega \) is essential in \( f(V - \text{int}N(k^\tau_\omega)) \) and each component of \( S'_\omega \cap T_K \) has slope \( 4a_1 \) and each component of \( S'_\omega \cap T_W \) has slope \( 16a_1 - 4\tau \).

To build a required essential surface \( S \subset E(W^\tau(K)) \) we take \( m \) parallel copies \( mS'_\omega \) of the essential surface \( S'_\omega \) and \( n \) parallel copies \( nS_K \) of the essential surface \( S_K \), and then glue them along their boundaries to obtain a connected surface \( S = mS'_\omega \cup nS_K \subset E(W^\tau(K)) \). Even when both \( S'_\omega \) and \( S_K \) are orientable, \( S \) may not be orientable. If \( S \) is non-orientable, then consider a regular neighborhood of \( S \) in \( E(W^\tau(K)) \), which is a twisted \( f–bundle \) of \( S \) whose \( \partial I–subbundle \) is an orientable double cover of \( S \). We use the same symbol \( S \) to denote this \( \partial I–subbundle \). Note that \( S_K \) and \( S'_\omega \) are orientable, so \( S \cap E(K) \) consists of parallel copies of \( S_K \) and similarly \( S \cap f(V - \text{int}N(k^\tau_\omega)) \) consists of parallel copies of \( S'_\omega \). Since \( \partial E(K) \) is incompressible in
2. Realization of the Jones slopes arising from the minimum degree. We divide into two cases depending upon \( a_1^* < \frac{7}{4} - \frac{1}{8} \) or \( a_1^* \geq \frac{7}{4} - \frac{1}{8} \).

**Case 1.** \( a_1^* < \frac{7}{4} - \frac{1}{8} \). By the assumption, \( K \) satisfies the Slope Conjecture, hence the Jones slope \( 4a_1^* \) of \( K \) is realized by a boundary slope of an essential surface \( S_K^* \subset E(K) \).

**Claim 4.2.** There exists an essential surface \( F_{\gamma_5}^* \) in \( V - \text{int}N(k_2^*) \) such that each component of \( F_{\gamma_5}^* \cap \partial V \) has slope \( 4a_1^* \) and each component of \( F_{\gamma_5}^* \cap \partial N(k_2^*) \) has slope \( 16a_1^* - 4\tau + 2 - 4\omega \).

**Proof.** Let us take an essential surface \( F_{\gamma_5} \) with \( k = \omega \) in \( S^3 - \text{int}N(k_1 \cup k_2) = V - \text{int}N(k_2) \) associated to the minimal edge path \( \gamma_5 \) described in Section 3. Then it has a pair of boundary slopes \( (-2\frac{\beta}{\alpha}, -2\frac{\alpha}{\beta} + 2 - 4\omega) \) on \( k_1, k_2 \). \( F_{\gamma_5} \) has boundary slopes \( -\frac{2\beta}{\alpha} \) on \( \partial N(k_1) \) and \(-2\frac{\alpha}{\beta} + 2 - 4\omega \) on \( \partial N(k_2) \). Using \((\mu_V, \lambda_V)\) instead of \((\mu_1, \lambda_1)\), \( F_{\gamma_5} \cap \partial V \) has slope \(-\frac{2\beta}{\alpha}\). Choose \( \alpha, \beta \) so that \(-\frac{2\beta}{\alpha} = 4a_1^* - \tau \), i.e. \(-\frac{\beta}{\alpha} = -8a_1^* + 2\tau > 1 \). Then \( F_{\gamma_5} \subset V - \text{int}N(k_2) \) has boundary slope \( 16a_1^* - 4\tau + 2 - 4\omega \) on \( \partial N(k_2) \) and \( 4a_1^* - \tau \) on \( \partial V \). Now we apply \( \tau \)-twisting along \( \mu_V \) which changes \( V - \text{int}N(k_2) \) to \( V - \text{int}N(k_2^*) \); we denote the image of \( F_{\gamma_5} \) by \( F_{\gamma_5}^* \). By Lemma 3.1 each component of \( F_{\gamma_5}^* \cap \partial V \) has slope \( 4a_1^* (= 4a_1^* - \tau + \tau) \), and each component of \( F_{\gamma_5}^* \cap \partial N(k_2^*) \) has slope \( 16a_1^* - 4\tau + 2 - 4\omega \) as desired. 

Let us take \( S^*_W = f(F_{\gamma_5}^*) \subset f(V - \text{int}N(k_2^*)) \). Then it is essential in \( f(V - \text{int}N(k_2^*)) \) and each component of \( S^*_W \cap T_K \) has slope \( 4a_1^* \) and each component of \( S^*_W \cap T_V \) has slope \( 16a_1^* - 4\tau + 2 - 4\omega \). Take \( m \) parallel copies \( mS^*_W \) of \( S^*_W \) and \( n \) parallel copies \( nS_K^* \) of \( S_K^* \), and then glue them along their boundaries to obtain a connected surface \( S^* = mS^*_W \cup nS_K^* \) in \( E(W^*_W(K)) \). If \( S^* \) is non-orientable, then we re-take \( S^* \) as the \( \partial I \)–subbundle of the regular neighborhood of \( S^* \) in \( E(W^*_W(K)) \), which is an orientable double cover of \( S^* \). Note that \( S_K^* \) and \( S^*_W \) are orientable, so \( S^* \cap E(K) \) consists of parallel copies of \( S_K^* \) and similarly \( S^* \cap (V - \text{int}N(k_2^*)) \) consists of parallel copies of \( S^*_W \). Since \( \partial E(K) \) is incompressible in \( E(W^*_W(K)) \) and \( S^*_W \), \( S^*_W \) is essential in \( f(V - \text{int}N(k_2^*)) \) and \( E(K) \) respectively, \( S^* \) is incompressible in \( E(W^*_W(K)) \). Since it cannot be an annulus, \( S^* \) is the desired essential surface.

**Case 2.** \( a_1^* \geq \frac{7}{4} - \frac{1}{8} \). In this case Proposition 2.2 shows that the Jones slope is \(-4\omega \). Take an (orientable) essential surface \( F_{\gamma_6} \) with \( k = \omega , \alpha = 2, \beta = 0 \). (At the end of the argument in this case, we explain why we need to choose \( \alpha = 2, \beta = 0 \) rather than \( \alpha = 1, \beta = 0 \).) A symmetry of \( k_1 \) and \( k_2 \) induces an orientation preserving homeomorphism \( \varphi \) of \( S^3 - \text{int}N(k_1 \cup k_2) = V - \text{int}N(k_2) \) which exchanges the components \( \partial N(k_1) = \partial V \) and \( \partial N(k_2) \). Let us set \( F = \varphi(F_{\gamma_6}) \subset V - \text{int}N(k_2) \). Then
it follows from Table 3.6 that $F$ has two boundary components with boundary slopes $(0, -4\omega)$. We apply $\tau$-twisting along $\mu_V$ which changes $V - \text{int}N(k_2)$ to $V - \text{int}N(k_1^*)$, and we denote the image of $F$ by $F_{\omega*}^\tau$. By Lemma 3.1, $F_{\omega*}^\tau$ has the boundary slope $-4\omega$ on $\partial N(k_1^*)$. Hence, $S^* = f(F_{\omega*}^\tau) \subset f(V - \text{int}N(k_1^*))$ is an essential surface such that $S^* \cap T_K = \emptyset$ and each component of $S^* \cap T_W$ has slope $-4\omega$. Thus the Jones slope $-4\omega$ is a boundary slope of $W_{\omega*}^\tau(K)$. Finally we explain why we choose $\alpha = 2$, $\beta = 0$. If we choose $\alpha = 1$, $\beta = 0$ in the above, then $F_{\gamma_b}$ has a single boundary component on $\partial N(k_1)$. Then $S^* = f(F_{\omega*}^\tau)$ has a single boundary component on $T_W$. If $F_{\gamma_b}$ (with $\alpha = 1$, $\beta = 0$), hence $S^*$, is orientable, then its boundary slope would be $0$. However $S^* \cap T_W$ has slope $-4\omega$, a contradiction. Hence $F_{\gamma_b}$ with $\alpha = 1$, $\beta = 0$ is non-orientable. The surface corresponding to $\alpha = 2$, $\beta = 0$ is an orientable double cover of the surface corresponding to $\alpha = 1$, $\beta = 0$.

This completes the proof of Theorem 1.9(1). \qed

5. Strong slope conjecture for twisted generalized Whitehead doubles

This section is devoted to a proof of Theorem 1.9(2), which we state again below.

**Theorem 1.9(2).** Let $K$ be a knot. We assume that the period of $\delta_K(n)$ and $\delta_K^*(n)$ are less than or equal to 2 and that $b_1 \leq 0$ and $b_1^* \geq 0$. Assume further that that if $b_1 = 0$, then $a_1 \neq \frac{1}{2}$ and that if $b_1^* = 0$, then $a_1^* \neq \frac{1}{2} - \frac{1}{b_2}$. If $K$ has Property YSS(1) and YSS$^*(1)$, then all of its twisted generalized Whitehead doubles satisfy the Yoked Strong Slope Property.

**Remark 5.1.** Even when $\delta_K(n)$ and $\delta_K^*(n)$ have period 2, $\delta_{W^*_\omega(K)}(n)$ and $\delta_{W^*_\omega(K)}^*(n)$ are usual polynomials rather than quasi-polynomials (Remark 2.3). So the Yoked Strong Slope Conjecture is equivalent to the Strong Slope Conjecture for $W^*_\omega(K)$ (Remark 1.5).

**Proof.** Write $\delta_{W^*_\omega(K)}(n) = a_W(n)n^2 + b_W(n)n + c_W(n)$ and $\delta_{W^*_\omega(K)}^*(n) = a_W^*(n)n^2 + b_W^*(n)n + c_W^*(n)$. It follows from Remark 2.3 that coefficients of $\delta_{W^*_\omega(K)}(n)$ and $\delta_{W^*_\omega(K)}^*(n)$ are constants and so we may write $a_W(n) = a_W$, $b_W(n) = b_W$, $c_W(n) = c_W$, $a_W^*(n) = a_W^*$, $b_W^*(n) = b_W^*$, $c_W^*(n) = c_W^*$. Then we show that essential surfaces $S$ and $S^*$ in $E(W^*_\omega(K))$ given in the proof of Theorem 1.9(1) satisfy the condition of the Strong Slope Conjecture:

$$S \text{ has boundary slope } p/q = 4a_W \quad \text{and} \quad \frac{\chi(S)}{|\partial S|q} = 2b_W,$$

and

$$S^* \text{ has boundary slope } p^*/q^* = 4a_W^* \quad \text{and} \quad -\frac{\chi(S^*)}{|\partial S^*|q^*} = 2b_W^*.$$

It is convenient to note the following.

**Lemma 5.2.** Let $F$ be a properly embedded surface in a knot exterior $E$ such that a component of $\partial F$ has slope $p/q$. Let $\overline{F}$ be the frontier of a tubular neighborhood $N(F)$ in $E$, i.e., $\overline{F}$ is the $\partial I$-subbundle of an $I$-bundle over $F$. Then $\partial \overline{F}$ has slope $p/q$ and

$$\frac{\chi(\overline{F})}{|\partial \overline{F}|q} = \frac{\chi(F)}{|\partial F|q}. $$
Proof. If $F$ is orientable, then $N(F) = F \times I$, whose frontier $\tilde{F}$ consists of two copies of $F$. So each component of $\partial F$ has slope $p/q$ and $\frac{\chi(\tilde{F})}{|\partial F|q} = 2\frac{\chi(F)}{|\partial F|q} = \frac{\chi(F)}{|\partial F|q}$. Assume now that $F$ is non-orientable. Then $\tilde{F}$ is the orientable double cover of $F$, and $\partial \tilde{F}$ consists of two parallel loops with slope $p/q$. Hence $\frac{\chi(\tilde{F})}{|\partial F|q} = 2\frac{\chi(F)}{|\partial F|q} = \frac{\chi(F)}{|\partial F|q}$.

1. Jones surfaces arising from the maximum degree.

Case 1-1. $a_1 > \frac{a}{4}$. Write $a_1 = r/s$ where $r$ and $s$ are coprime integers and $s > 0$. Then, as a ratio of coprime integers, the denominator of $4a_1$ is $s / \gcd(4, s)$. Since $K$ has Property YSS(1), there is a properly embedded essential surface $S_K$ in the exterior of $K$ whose boundary slope is $4a_1$ and

$$\frac{\chi(S_K)}{|\partial S_K| \cdot \frac{s}{\gcd(4, s)}} = 2b_1.$$ 

When addressing the Slope Conjecture for $W_\omega^\tau(K)$ in this case, we constructed a properly embedded essential surface $S = mS_K \cup nS^\tau_\omega$ in the exterior of $W_\omega^\tau(K)$ by joining $m$ copies of $S_K$ in $E(K)$ to $n$ copies of the surface $S^\tau_\omega$ in $V - N(k_\omega)$. This requires that $m|\partial S_K| = n|\partial S^\tau_\omega \cap T_K|$.

The surface $S^\tau_\omega$ is identified with a surface of type $F_{\gamma_1}$ in the exterior of the $[2, 2\omega, -2]$ two-bridge link, where $\frac{a}{\omega} = 8a_1 - 2\tau = \frac{8r - 2r_s}{s} > 0$ so that $S^\tau_\omega$ has boundary slope $4a_1$ on $\partial V$. We choose $\beta = 2s$, $\alpha = 2(8r - 2r_s)$ so that $F_{\gamma_1} = F_{\gamma_1, \alpha, \beta}$ is orientable; see Subsection 3.3.

Then, using Table 3.6, we calculate the following:

- $\chi(S^\tau_\omega) = -\alpha - \beta = -2(8r - (2\tau - 1)s)$,
- slope of $\partial S^\tau_\omega$ on $T_W$ is $2\frac{a}{\omega} = 2(\frac{8r - 2r_s}{s}) = \frac{16r - 4r_s}{s}$,
- $|\partial S^\tau_\omega \cap T_K| = \gcd(2\beta, \alpha) = \gcd(4s, 2(8r - 2r_s)) = 4 \gcd(4, s)$, and
- $|\partial S^\tau_\omega \cap T_W| = \gcd(2\alpha, \beta) = \gcd(4(8r - 2r_s), 2s) = 2 \gcd(16, s)$.

The boundary of $S$ consists of $n$ copies of the boundary of $S^\tau_\omega$ on $T_W$, so we have

- $|\partial S| = n|\partial S^\tau_\omega \cap T_W| = 2n \gcd(16, s)$.

Moreover, the boundary slope of $S$ is the slope of $\partial S^\tau_\omega$ on $T_W$, and so this has denominator $\frac{s}{\gcd(16r - 4r_s, s)} = \frac{s}{\gcd(16, s)}$. We may now calculate

$$\frac{\chi(S)}{|\partial S| \cdot \frac{s}{\gcd(16, s)}} = \frac{m\chi(S_K) + n\chi(S^\tau_\omega)}{2n \gcd(16, s) \cdot \frac{s}{\gcd(16, s)}} = \frac{2b_1 m|\partial S_K| \cdot \frac{s}{\gcd(4, s)} - 2n(8r - (2\tau - 1)s)}{2ns} = \frac{8b_1 n \gcd(4, s) \cdot \frac{s}{\gcd(4, s)} - 2n(8r - (2\tau - 1)s)}{2ns} = \frac{8b_1 ns - 2n(8r - (2\tau - 1)s)}{2ns} = \frac{4b_1 - 8r/s + (2\tau - 1)}{2} = 2(-4a_1 + 2b_1 + \tau - \frac{1}{2}) = 2b_W$$

as desired.
If the glued surface \( S = mS_K \cup nS_w^\tau \) is non-orientable, then as in the proof of Theorem 1.9(1), we replace \( S \) by the frontier \( \tilde{S} \) of the tubular neighborhood of \( S \), but Lemma 5.2 shows that \( \tilde{S} \) and \( S \) has the same boundary slope and \( \frac{\chi(\tilde{S})}{|\partial \tilde{S}|} \cdot \frac{\gcd(16, s)}{\gcd(16, s)} = \frac{\chi(S)}{|\partial S|} \cdot \frac{\gcd(16, s)}{\gcd(16, s)} \). Thus the essential surface \( S \) or \( \tilde{S} \) (when \( S \) is non-orientable) is the desired essential surface.

**Case 1-2.** \( a_1 \leq \frac{r}{4} \). In this situation, \( S \) is a minimal genus Seifert surface of \( W_\omega^\tau(K) \), which is a once punctured torus. Hence

\[
\frac{\chi(S)}{|\partial S|} = \frac{\chi(S)}{1} = -1 = 2\left(-\frac{1}{2}\right) \in jx_{W_\omega^\tau(K)}.
\]

2. Jones surfaces arising from the minimum degree.

**Case 2-1.** \( a_1^\tau < \frac{r}{4} - \frac{1}{2} \). We follow the same argument in Case 1-1. Write \( a_1^\tau = r/s \) for some coprime \( r \) and \( s > 0 \). Then, as a ratio of coprime integers, the denominator of \( 4a_1^\tau \) is \( s/\gcd(4, s) \).

Since \( K \) has Property \( YSS^*(1) \), there is a properly embedded essential surface \( S_K^* \) in the exterior of \( K \) whose boundary slope is \( 4a_1^\tau \) and

\[
\frac{\chi(S_K^*)}{|\partial S_K^*|} = \frac{\chi(S_K^*)}{\gcd(4, s)} = -2b^*_s.
\]

When addressing the Slope Conjecture for \( W_\omega^\tau(K) \) in this case, we constructed a properly embedded essential surface \( S^* = mS_K^* \cup nS_w^{\tau*} \) in the exterior of \( W_\omega^\tau(K) \) by joining \( m \) copies of \( S_K^* \) in \( E(K) \) to \( n \) copies of the surface \( S_w^{\tau*} \) in \( V \sim N(k_\omega^\tau) \). This requires that

\[
m|\partial S_K^*| = n|\partial S_w^{\tau*} \cap T_K|.
\]

The surface \( S_w^{\tau*} \) is identified with a surface of type \( F_{\gamma_5} \) (with \( k = \omega \)) in the exterior of the \((2, 2\omega, -2)\) two-bridge link where \( \frac{a}{b} = -8a_1^\tau + 2\tau = \frac{8r-2r\tau}{s} > 1 \) so that \( S_w^{\tau*} \) has boundary slope \( 4a_1^\tau \) on \( \partial V \). We choose \( \beta = 2s \) and \( \alpha = -2(8r - 2\tau s) \), so that \( F_{\gamma_5} = F_{\gamma_5, \alpha, \beta} \) is orientable; see Subsection 3.3. Then, using Table 3.6, we calculate the following:

- \( \chi(S_w^{\tau*}) = -\alpha + 2(1 - \omega)\beta = 16r - 4(\tau + \omega - 1)s \),
- slope of \( \partial S_w^{\tau*} \) on \( T_W \) is \( -2\frac{b}{s} + 2 - 4\omega = -2(\frac{8r-2r\tau s}{s}) + 2 - 4\omega = \frac{16r - 4r\tau s + 2s - 4\omega s}{s} \),
- \( |\partial S_w^{\tau*} \cap T_K| = \gcd(2\beta, \alpha) = \gcd(4s, -16r + 4\tau s) = 4 \gcd(4, s) \), and
- \( |\partial S_w^{\tau*} \cap T_W| = \gcd(2\alpha, \beta) = \gcd(-32r + 8\tau s, 2s) = 2 \gcd(16, s) \).

The boundary of \( S^* \) consists of \( n \) copies of the boundary of \( S_w^{\tau*} \) on \( T_W \), so we have

- \( |\partial S^*| = n|\partial S_w^{\tau*} \cap T_W| = 2n \gcd(16, s) \).
Moreover, the boundary slope of $S^*$ is the slope of $\partial S^*_{\tau}$ on $T_W$, and so this has denominator \[
\frac{s}{\gcd(16r-4r\tau+2s-4\omega, s)} = \frac{s}{\gcd(16, s)} \]
We may now calculate
\[
\frac{\chi(S^*)}{|\partial S^*|} \cdot \frac{s}{\gcd(16, s)} = \frac{m\chi(S^*_K) + n\chi(S^*_\tau)}{2n \gcd(16, s) \cdot s} + n(16r - 4(\tau + \omega - 1)s)
\]
\[
= -2b_1^* m|\partial S^*_K| \cdot \frac{s}{\gcd(16, s)} + n(16r - 4(\tau + \omega - 1)s)
\]
\[
= -8b_1^* n \gcd(4, s) \cdot \frac{s}{\gcd(16, s)} + n(16r - 4(\tau + \omega - 1)s)
\]
\[
= -8b_1^* n s + n(16r - 4(\tau + \omega - 1)s)
\]
\[
= -4b_1^* + 8r/s - 2(\tau + \omega - 1)
\]
\[
= 8a_1^* - 4b_1^* - 2(\tau + \omega - 1)
\]
\[
= -2(-4a_1^* + 2b_1^* + \tau + \omega - 1)
\]
\[
= -2b_1^* W
\]
as desired.

If the glued surface $S^* = mS^*_K \cup nS^*_\tau$ is non-orientable, then we replace $S$ by the frontier $\widetilde{S}^*$ of the tubular neighborhood of $S^*$. By Lemma 5.2 $\widetilde{S}^*$ and $S^*$ has the same boundary slope and
\[
\frac{\chi(S^*)}{|\partial S^*|} \cdot \frac{s}{\gcd(16, s)} = \frac{\chi(S^*)}{|\partial S^*|} \cdot \frac{s}{\gcd(16, s)}.
\]
Thus the essential surface $S^*$ or $\widetilde{S}^*$ (when $S^*$ is non-orientable) is the desired essential surface.

**Case 2-2.** $a_1^* \geq \frac{\tau}{4} - \frac{1}{8}$. In this case, the argument in Case 2 in the proof of Theorem 1.9(1) and Table 3.6 show that $|\partial S^*|$ = 2 and $\chi(S^*) = 2 - 4\omega$. Note that $\partial S^* \cap T_W$ consists of two simple closed curves each of which has slope $-4\omega$ ($\partial S^* \cap T_K = \emptyset$). Hence
\[
\frac{\chi(S^*)}{|\partial S^*|} = \frac{\chi(S^*)}{|\partial S^*|} = \frac{-4\omega + 2}{2} = -2\omega + 1 = -2(\frac{2\omega - 1}{2}) \in jx_{W^*}(k).
\]
Thus $S^*$ satisfies the required condition.

This completes the proof of Theorem 1.9(2).

6. **Examples – twisted generalized Whitehead doubles of torus knots and connected sums of torus knots**

In this section we take a closer look at some concrete examples.

6.1. **Twisting number $\tau$ and Jones surfaces.** The maximum degree and the minimum degree of the colored Jones function of a torus knot $K = T_{p,q}$ with relatively prime integers $p, q > 0$ are explicitly computed by [9].

\[
d_+[J_{K,n}(q)] = d_+[J_{K,n}(q)] = \frac{pq}{4} n^2 - \frac{pq}{4} - (1 + (-1)^n) \frac{(p-2)(q-2)}{8}
\]
and
\[
d_-[J_{K,n}(q)] = d_-[J_{K,n}(q)] = \frac{pq - p - q}{2} n - \frac{pq - p - q}{2}.
\]
Since $d_+[J_{K,n}(q)]$ and $d_-[J_{K,n}(q)]$ are quadratic quasi-polynomials for all integers $n$, in the proof of Proposition 2.12 and 2.13, $N'_K = 0$ and $N'^*_K = 0$, respectively. Note also that $d_+[J_{K,0}(q)] = d_+[J_{K,1}(q)] = 0$ and $d_-[J_{K,0}(q)] = d_-[J_{K,1}(q)] = 0$ from the above formulas. Hence $C_+(K, \tau) = 0$ and $C_-(K, \tau) = 0$. Therefore, it follows from Propositions 2.1, 2.2 and Proposition 2.16 (for $\tau = pq$) that for $\omega > 0$ we have:

$$\delta_{W^*_\tau(K)}(n) = \begin{cases} (pq - \tau)n^2 + (-pq + \tau - \frac{1}{2})n + \frac{1}{2} & (\tau < pq), \\ -\frac{n}{2} + \frac{1}{2} & (\tau \geq pq) \end{cases}$$

and

$$\delta_{W^*_\tau(K)}(n) = \begin{cases} (-\omega + \frac{1}{2} - \tau)n^2 + (pq - p - q + \omega - 1 + \tau)n - pq + p + q + \frac{1}{2} & (\tau \geq \frac{1}{2}), \\ -\omega n^2 + (\omega - \frac{1}{2})n + \frac{1}{2} & (\tau < \frac{1}{2}) \end{cases}.$$

This shows that a twisted generalized Whitehead double of $K$ has two Jones surfaces, one comes from the maximum degree and the other comes from the minimum degree. Those Jones surface are of a different nature depending upon the twisting number $\tau$. For instance, if $\frac{1}{2} < \tau < pq$, then both Jones surfaces intersect the essential companion torus $\partial E(T_{p,q})$. Otherwise, one of Jones surfaces is contained in companion solid torus and the other intersects the companion torus $\partial E(T_{p,q})$.

6.2. Twisted generalized Whitehead doubles of inadequate knots. As we mentioned in Section 1 adequate knots satisfy the assumption in Theorem 1.9. We say that a knot is inadequate if it is neither $A$-adequate nor $B$-adequate [32]. Let us give examples of inadequate knots which still satisfy the assumption in Theorem 1.9.

Assume $p, q > 0$ are relatively prime integers. Since $T_{p,-q} = \overline{T_{p,q}}$,

$$\delta_{T_{p,-q}}(n) = -\delta_{T_{p,q}}(n) = -\frac{pq - p - q}{2}n + \frac{pq - p - q}{2},$$

$$\delta_{T_{p,-q}}^*(n) = -\delta_{T_{p,q}}^*(n) = -\frac{pq}{4}n^2 + \frac{pq}{4} + (1 + (-1)^n)\frac{(p - 2)(q - 2)}{8}.$$ Putting $K = T_{p,q}$ and $K' = T_{p',q'}$, we see:

$$\delta_{K \sharp K'}(n) = \frac{pq}{4}n^2 - \frac{(p' - 1)(q' - 1)}{2}n - \frac{pq}{4} - (1 + (-1)^n)\frac{(p - 2)(q - 2)}{8} + \frac{(p' - 1)(q' - 1)}{2},$$

$$\delta_{K \sharp K'}^*(n) = -\frac{pq}{4}n^2 + \frac{(p - 1)(q - 1)}{2}n + \frac{pq}{4} + (1 + (-1)^n)\frac{(p - 2)(q - 2)}{8} - \frac{(p - 1)(q - 1)}{2}.$$

Combining these formulas with [4, Theorem 2.1], the assumption in Theorem 1.9 holds for $K \sharp K'$. So, the twisted generalized Whitehead double of $K \sharp K'$ satisfies the Yokota Strong Slope Conjecture. On the other hand, when $p$ and $q$ are odd, $\frac{pq}{4} \not\in \mathbb{Z}$ and so, $K \sharp K'$ is inadequate [8, Lemma 6].

**Question 6.1.** Is a twisted generalized Whitehead double of an inadequate knot also inadequate?

7. Proof of Corollary 1.11

The aim of this section is to prove Corollary 1.11.
Corollary 1.11. Any knot obtained by a finite sequence of cabling, untwisted $\omega$–generalized Whitehead doublings with $\omega \neq 0$ and connected sums of adequate knots or torus knots satisfies the Strong Slope Conjecture (Yoked Strong Slope Conjecture).

This result follows from a more general proposition below (Proposition 7.4), for which we introduce the following technical condition.

**Definition 7.1.** We say that $K$ satisfies **Condition $\delta$** if

1. $\delta_K(n) = an^2 + bn + c(n)$ and $\delta^*_K(n) = a^*n^2 + b^*n + c^*(n)$ have period at most 2,
2. $b \leq 0$ and $b^* \geq 0$,
3. $4a, 4a^* \in \mathbb{Z}$, and
4. both $b = 0 \implies a \neq 0$ and $b^* = 0 \implies a^* \neq 0$.

**Remark 7.2.** This Condition $\delta$ is slightly stronger than the Condition $\delta$ in [4]. They are the same except for the addition of item (4).

**Lemma 7.3.** A knot satisfies Condition $\delta$ if and only if its mirror satisfies Condition $\delta$.

**Proof.** Let $\overline{K}$ denote the mirror of $K$. Then the symmetry $d_+[J_K, n(q)] = -d_+[J_{\overline{K}}, n(q)]$ implies that $\delta_{\overline{K}} = -\delta_K$ and $\delta^*_K = -\delta_K$ and the result follows immediately. $\square$

**Proposition 7.4.** Let $\mathcal{K}$ be the maximal set of knots of which each satisfies the Strong Slope Conjecture and Condition $\delta$. Then $\mathcal{K}$ is closed under connected sum, cabling and untwisted $\omega$–generalized Whitehead doubling with $\omega \neq 0$.

To prove this proposition, we prepare some lemmas.

**Lemma 7.5.** Assume that $K_i \in \mathcal{K}$. Then $K_1 \sharp K_2 \in \mathcal{K}$.

**Proof.** By Claim 4.4 in [4] we only need to see that (4) in Condition $\delta$ holds. Write $\delta_{K_1}(n) = a_i n^2 + b_i n + c_i(n)$ and $\delta^*_{K_1}(n) = a^*_i n^2 + b^*_i n + c^*_i(n)$. Then

$$\delta_{K_1 \sharp K_2}(n) = \delta_{K_1}(n) + \delta_{K_2}(n) - \frac{1}{2} n + \frac{1}{2}$$

$$= (a_1 + a_2)n^2 + (b_1 + b_2 - \frac{1}{2})n + (c_1(n) + c_2(n) + \frac{1}{2}).$$

Since $b_i \leq 0$, we have $b_1 + b_2 - \frac{1}{2} < 0$.

Due to the symmetry $\delta^*_K = -\delta_K$ where $\overline{K}$ is the mirror of $K$, we have

$$\delta^*_{K_1 \sharp K_2}(n) = -\delta_{K_1 \sharp K_2}$$

$$= -(\delta_{K_1} + \delta_{K_2} - n + \frac{1}{2})$$

$$= \delta^*_{K_1} + \delta^*_{K_2} + n - \frac{1}{2}$$

$$= (a^*_1 + a^*_2)n^2 + (b^*_1 + b^*_2 + \frac{1}{2})n + (c^*_1(n) + c^*_2(n) - \frac{1}{2}).$$

Since $b^*_i \geq 0$, we have $b^*_1 + b^*_2 + 1/2 > 0$.

Hence (4) obviously holds for $K_1 \sharp K_2$. $\square$
Lemma 7.6. Assume that $K \in \mathcal{K}$. Then its $(p,q)$–cable $K_{p,q}$ $(q > 1)$ belongs to $\mathcal{K}$.

Proof. By Claim 4.5 in [4] we only need to see that (4) in Condition $\delta$ holds. If $\delta_K(n) = an^2 + bn + c(n)$, then we have [4, Proposition 3.1]:

$$
\delta_{K_{p,q}}(n) = \begin{cases} 
q^2an^2 + \left(qb + \frac{(q-1)(p-4pq)}{2}\right)n \\
\quad + (a(q-1)^2 - (b + \frac{q}{2})(q-1) + c(i)) & \text{for } \frac{q}{q} < 4a, \\
\frac{pq(n^2-1)}{4} + C_j(K_{p,q}) & \text{for } \frac{q}{q} \geq 4a.
\end{cases}
$$

where $i \equiv (2) q(n-1) + 1$, $j \equiv (2) n$, and $C_j(K_{p,q})$ is a number that only depends on the knot $K$ and the numbers $p$ and $q$.

Assume first that $\frac{q}{q} < 4a$. If $q^2a = 0$, then $a = 0$. Thus $p/q < 0$ and $p < 0$ (because $q > 1$). Then $qb + \frac{(q-1)(p-4pq)}{2} = qb + \frac{q(q-1)}{2} < 0$ (because $b \leq 0$). Hence, if $qb + \frac{(q-1)(p-4pq)}{2} = 0$, then $q^2a \neq 0$. In the case where $\frac{q}{q} \geq 4a$, the linear term is 0, but the quadratic term $\frac{pq}{4}$ is not 0.

For $\delta_{K_{p,q}}^*(n)$, we have the following from [4, Proposition 3.2].

$$
\delta_{K_{p,q}}^*(n) = \begin{cases} 
q^2a^*n^2 + \left(qb^* + \frac{(q-1)(p-4pq^*)}{2}\right)n \\
\quad + (a^*(q-1)^2 - (b^* + \frac{q}{2})(q-1) + c^*(i)) & \text{for } \frac{q}{q} > 4a^*, \\
\frac{pq(n^2-1)}{4} + C_j^*(K_{p,q}) & \text{for } \frac{q}{q} \leq 4a^*,
\end{cases}
$$

where $i \equiv (2) q(n-1) + 1$, $j \equiv (2) n$, and $C_j^*(K_{p,q})$ is a number that only depends on the knot $K$ and the numbers $p$ and $q$. Then the analogous result for the coefficients of $\delta_{K_{p,q}}^*(n)$ follows in a similar straightforward manner.

Therefore $K_{p,q}$ also satisfies Condition $\delta$. \qed

Lemma 7.7. Assume that $K \in \mathcal{K}$. Then its untwisted $\omega$–generalized Whitehead double $W_\omega^0(K)$ $(\omega \neq 0)$ also belongs to $\mathcal{K}$.

Proof. Write $\delta_K(n) = an^2 + bn + c(n)$ and $\delta_K^*(n) = a^*n^2 + b^*n + c^*(n)$. Since $K$ satisfies Condition $\delta$, $\delta_K(n)$ and $\delta_K^*(n)$ have period at most 2 and we have $b \leq 0$, $b^* \geq 0$, $4a, 4a^* \in \mathbb{Z}$, and both $a \neq 0$ if $b = 0$ and $a^* \neq 0$ if $b^* = 0$.

Case 1: $\omega > 0$. Then $\omega \geq 1$, since $\omega \in \mathbb{Z}$. Since $\tau = 0$, Propositions 2.1 and 2.2 show that

$$
\delta_{W_\omega^0(K)}(n) = \begin{cases} 
4an^2 + (-4a + 2b - \frac{1}{2})n + a - b + c_1 + \frac{1}{2} & (a > 0), \\
-\frac{n}{2} + C_+(K,0) + \frac{1}{2} & (a < 0), \\
-\frac{n}{2} + C_+(K,0) + \frac{1}{2} & (a = 0, b \neq 0),
\end{cases}
$$

and

$$
\delta_{W_\omega^0}(n) = \begin{cases} 
(4a^* - \frac{2\omega - 1}{2})n^2 + (-4a^* + 2b^* + \omega - 1)n + a^* - b^* + c_1^* + \frac{1}{2} & (a^* < -\frac{1}{8}), \\
-\omega n^2 + \frac{2\omega - 1}{2}n + C_- (K,0) + \frac{1}{2} & (a^* > -\frac{1}{8}), \\
-\omega n^2 + \frac{2\omega - 1}{2}n + C'_-(K,0) + \frac{1}{2} & (a^* = -\frac{1}{8}, b^* \neq 0).
\end{cases}
$$

We now check Condition $\delta$ for $W_\omega^0(K)$. Obviously (1) in Condition $\delta$ holds; actually $\delta_{W_\omega^0(K)}(n)$ and $\delta_{W_\omega^0}(n)$ are usual polynomials rather than quasi-polynomials. Since $4(4a), 0, 4(4a^*-(2\omega-1)/2)$ and $-4\omega$ are integers, we have (3) in Condition $\delta$. 

\begin{thebibliography}{99}
\end{thebibliography}
To address (2) and (4) in Condition \( \delta \) we first examine the coefficients of \( \delta_{W_0^*}(K)(n) \) and then the coefficients of \( \delta_{W_0^*}(K)(n) \). When \( a > 0 \), since \( b \leq 0 \) (by Condition \( \delta \) for \( K \)), we have \(-4a + 2b - \frac{1}{2} < 0\). Thus the coefficient of linear term of \( \delta_{W_0^*}(K)(n) \) is negative; in particular it is not 0. When \( a \leq 0 \), the coefficient of linear term of \( \delta_{W_0^*}(K)(n) \) is \(-\frac{1}{2} < 0\). When \( a^* < -\frac{1}{8} \), since \( b^* \geq 0 \) (by Condition \( \delta \) for \( K \)) and \( \omega \geq 1 \) (by the assumption in Case 1), \(-4a^* + 2b^* + \omega - 1 > 0\). Thus the coefficient of the linear term of \( \delta_{W_0^*}(K)(n) \) is positive; in particular it is not 0. When \( a^* \geq -\frac{1}{8} \), then the coefficient of the linear term of \( \delta_{W_0^*}(K)(n) \) is \( \frac{2a^* - 1}{2} > 0 \). Hence (2) and (4) in Condition \( \delta \) hold.

It remains to show that \( W_0^* (K) \) enjoys the Strong Slope Conjecture. Note that since \( K \) satisfies the Strong Slope Conjecture and Condition \( \delta \), it also satisfies the properties \( YSS(1) \) and \( YSS^*(1) \); see Remark 1.5. Since \( \tau = 0 \), to apply Theorem 1.9(2) we need to check extra conditions: if \( b = 0 \), then \( a \neq \frac{7}{4} = 0 \), and if \( b^* = 0 \), then \( a^* \neq \frac{7}{4} - \frac{1}{8} = -\frac{1}{8} \).

If \( b = 0 \), then by Condition \( \delta \) for \( K \), we have \( a \neq 0 \). Furthermore, since \( 4a^* \in \mathbb{Z} \) by Condition \( \delta \) for \( K \), even if \( b^* = 0 \), \( a^* \neq -\frac{1}{8} \) and the above condition is satisfied. Hence, Theorem 1.9(2) shows that \( W_0^* (K) \) satisfies the Yoked Strong Slope Conjecture, and thus the Strong Slope Conjecture.

**Case 2:** \( \omega < 0 \). Then \( W_0^* (K) \) is the mirror of \( W_0^* (\overline{K}) \). Since \( -\omega > 0 \), we have seen above that \( W_0^* (\overline{K}) \) satisfies Condition \( \delta \) and the Strong Slope Conjecture. Then by Lemma 7.3 and Proposition 1.6, \( W_0^* (K) \) also satisfies Condition \( \delta \) and the Strong Slope Conjecture.

**Proof of Proposition 7.4.** The proof follows from Lemmas 7.5, 7.6 and 7.7. 

**Proof of Corollary 1.11.** For the trivial knot, \( b = \frac{1}{2} > 0 \), \( b^* = -\frac{1}{2} < 0 \), and hence the trivial knot does not belong to \( K \). Note that an untwisted \( \omega \)-generalized Whitehead double of the trivial knot is also the trivial knot. However the trivial knot satisfies the Strong Slope Conjecture (and the Yoked Strong Slope Conjecture). Note also that a cable of the trivial knot is a (possibly trivial) torus knot and satisfies the Strong Slope Conjecture (and the Yoked Strong Slope Conjecture) and Condition \( \delta \) as mentioned below.

Therefore by Proposition 7.4, it is sufficient to see that nontrivial torus knots and nontrivial adequate knots belong to \( K \). We first recall:

- Any nontrivial torus knot satisfies the Strong Slope Conjecture and Condition \( \delta \). More precisely, if \( K \) is a positive torus knot, then \( 0 < 4a \in \mathbb{Z} \), \( b = 0 \), and \( a^* = 0 \), \( b^* > 0 \). If \( K \) is a negative torus knot, then \( a = 0 \), \( b < 0 \), and \( 0 > 4a^* \in \mathbb{Z} \), \( b^* > 0 \).
- Any nontrivial adequate knot satisfies the Strong Slope Conjecture and Condition \( \delta \). More precisely, \( a \geq 0 \), \( b \leq 0 \), and \( a^* \leq 0 \), \( b^* \geq 0 \). See [20, Lemma 3.6]. Furthermore, if \( b = 0 \), then \( K \) is a torus knot and \( a > 0 \) ([20, Lemma 3.8]). Similarly, if \( b^* = 0 \), then \( K \) is a torus knot and \( a^* < 0 \).

Then the proof follows from Proposition 7.4.

**8. Non-adequate Whitehead doubles**

As shown by Kalfagianni and Tran [20], adequate knots and their iterated cables satisfy the Strong Slope Conjecture. We may expect that most Whitehead doubles are not adequate. However, to the best our knowledge, there are no explicit such examples. So for completeness we give explicit family of Whitehead doubles which are not adequate. Recall that \( W_0^1(K) \) is the (untwisted)
negative Whitehead double of \( K \). In the following, for notational simplicity, we denote \( W_0^0(K) \) by \( W^0_- (K) \). We also denote the (untwisted) positive Whitehead double of \( K \) by \( W^0_+ (K) \), which may be written as \( W_2^0(K) \).

**Theorem 8.1.** Let \( K \) be the torus knot \( T_{2,-(2m+1)} \) for \( m \geq 2 \). Then \( W_- (K) \) is not adequate.

To prove Theorem 8.1, we prepare two lemmas below. Let us denote the Turaev genus of \( K \) by \( g_T(K) \), which is defined in [5], and denote the minimal crossing number of \( K \) by \( c(K) \).

**Lemma 8.2.** If \( W_-(T_{p,-q}) \) is adequate for \( p, q > 0 \), then \( g_T(W_-(T_{p,-q})) = 1 \).

**Proof.** Since \( T_{p,-q} = T_{p,q} \),

\[
\delta_{T_{p,-q}}(n) = -\delta_{T_{p,q}}(n) = -\frac{pq-p-q}{2} n + \frac{pq+p-q}{2},
\]

\[
\delta^*_{T_{p,-q}}(n) = -\delta^*_{T_{p,q}}(n) = -\frac{pq}{4} n^2 + \frac{pq}{4} + (1 + (-1)^n) \frac{(p-2)(q-2)}{8},
\]

and then by Propositions 2.1 and 2.2, we obtain:

\[
\delta_{W_-(T_{p,-q})}(n) = -\frac{1}{2} n + C'_+(K,0) + \frac{1}{2},
\]

\[
\delta^*_{W_-(T_{p,-q})}(n) = -(pq + \frac{1}{2}) n^2 + pqn + \frac{1}{2},
\]

Then, it follows that

\[
(8.1) \quad \delta_{W_-(T_{p,-q})}(n) - \delta^*_{W_-(T_{p,-q})}(n) = (pq + \frac{1}{2}) n^2 - (pq + \frac{1}{2}) n + C'_+(K,0).
\]

Then [19, Theorem 1.1] asserts:

\[
(8.2) \quad c(W_-(T_{p,-q})) = 2(pq + \frac{1}{2}) = 2pq + 1, \quad \text{and}
\]

\[
(8.3) \quad -(pq + \frac{1}{2}) = 1 - g_T(W_-(T_{p,-q})) - c(W_-(T_{p,-q}))/2 = 1 - g_T(W_-(T_{p,-q})) - (pq + \frac{1}{2}).
\]

Hence, we have \( g_T(W_-(T_{p,-q})) = 1 \) as desired. \( \square \)

**Lemma 8.3.** For \( m \geq 1 \), we have \( g_T(W_-(T_{2,-(2m+1)})) \geq 2m - 1 \).

**Proof.** Let \( \widehat{HF}(K) \) be the knot Floer homology of a knot \( K \) in \( S^3 \) [29, 31]. The homological width \( w_{HF}(K) \) of \( K \) is defined by [2] as:

\[
w_{HF}(L) = 1 + \max \{ \delta \mid \widehat{HF}_\delta(K) \neq 0 \} - \min \{ \delta \mid \widehat{HF}_\delta(K) \neq 0 \},
\]

where

\[
\widehat{HF}_\delta(K) = \bigoplus_{s-t = \delta} \widehat{HF}_{s,t}(K, s).
\]

Since \( w_{HF}(K) \leq g_T(K) + 1 \) for a knot [22, Theorem 1.1] and

\[
g_T(W_-(T_{2,-(2m+1)})) = g_T(W_+(T_{2,-(2m+1)})) = g_T(W_+(T_{2,2m+1})),
\]

Claim 8.4 below shows

\[
(8.4) \quad 2m = w_{HF}(W_+(T_{2,2m+1})) \leq g_T(W_+(T_{2,2m+1})) + 1 = g_T(W_-(T_{2,-(2m+1)})) + 1.
\]

Thus \( g_T(W_-(T_{2,-(2m+1)})) \geq 2m - 1 \). \( \square \)
Claim 8.4. \( w_{HF}(W_+(T_{2,2m+1})) = 2m. \)

Proof. Following [16, Theorem 2.7], we have

\[
\widehat{HFK}(W_+(T_{2,2m+1}), 1) \cong \mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(2)} \oplus \mathbb{F}^2_{(4)} \cdots \oplus \mathbb{F}^2_{(2m-2)},
\]

where \( \mathbb{F}^k_t \) denotes a \( \mathbb{Z}/2\mathbb{Z} \) vector space of dimension \( k \) supported in homological grading \( t \).

Recall that for a torus knot \( T_{2,2m+1} \), we have \( \tau(T_{2,2m+1}) = g(T_{2,2m+1}) = m \) [30], where \( \tau \) is the Ozsváth and Szabó concordance invariant [28] and \( g \) is the Seifert genus.

Apply [14, Theorem 1.2] to obtain

\[
\widehat{HFK}(W_+(T_{2,2m+1}), 1) \cong \mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-3)} \cdots \oplus \mathbb{F}^2_{(-2m+1)},
\]

where \( V \) is isomorphic to a finite direct sum of \( F = \mathbb{Z}/2\mathbb{Z} \). Note that \( \mathbb{F}^2_{-1} \) means the quotient of the remaining group by a subgroup of dimension 2, supported in homological grading 1.

Comparing (7.5) and (7.6), we see that

\[
V \cong \mathbb{F}^2_{(1)} \oplus \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-3)} \cdots \oplus \mathbb{F}^2_{(-2m+1)}.
\]

It follows that we have:

\[
\widehat{HFK}(W_+(T_{2,2m+1}), 1) \cong \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-3)} \cdots \oplus \mathbb{F}^2_{(-2m+1)}.
\]

Moreover, since [23, Section 2.2] gives a symmetry

\[
\widehat{HFK}_t(K, 1) \cong \widehat{HFK}_{t-2}(K, -1),
\]

we obtain

\[
\widehat{HFK}(W_+(T_{2,2m+1}), -1) \cong \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-3)} \cdots \oplus \mathbb{F}^2_{(-2m+1)}.
\]

Let us compute \( \widehat{HFK}(W_+(T_{2,2m+1}), 0) \). Since \( g(W_+(T_{2,2m+1})) = 1 \), \( \tau(W_+(T_{2,2m+1})) = 1 \) by [14, Theorem 1.5], and \( d_2 = 0 \) by [14, Theorem 1.2], we may apply [14, Proposition 6.2] to obtain:

\[
\widehat{HFK}_*(W_+(T_{2,2m+1}), 0) \cong \widehat{HFK}_{*+1}(W_+(T_{2,2m+1}), 1) \oplus \widehat{HFK}_{*-1}(W_+(T_{2,2m+1}), -1)/\mathbb{F}(0).
\]

Hence, we get:

\[
\widehat{HFK}(W_+(T_{2,2m+1}), 0) \cong \mathbb{F}^4_{(-1)} \oplus \mathbb{F}^4_{(-2)} \oplus \mathbb{F}^4_{(-4)} \cdots \oplus \mathbb{F}^4_{(-2m)}.
\]

Now following (6.7), (6.8) and (6.9), we observe:

\[
\max\{\delta \mid \widehat{HF}_\delta(K) \neq 0\} = 2m, \quad \min\{\delta \mid \widehat{HF}_\delta(K) \neq 0\} = 1.
\]

Therefore we have the desired equality

\[
w_{HF}(W_+(T_{2,2m+1})) = 1 + 2m - 1 = 2m.
\]

\( \square \)

Proof of Theorem 8.1. Assume for a contradiction that \( W_-(T_{2,2m+1}) \) is adequate. Following Lemma 8.2 \( gr(W_-(T_{2,2m+1})) = 1 \). On the contrary, Lemma 8.3 shows that \( gr(W_-(T_{2,2m+1})) \geq 2m - 1 \). Since \( m \geq 2 \), \( gr(W_-(T_{2,2m+1})) \geq 3 \), a contradiction. \( \square \)
Theorem 8.1 says that $W_-(K)$ is not adequate, and we can see that a modification of the diagram of $W_-(K)$ in Figure 2.3 is $A$-adequate. So it is not $B$-adequate.

REFERENCES

[1] T. Abe, The Turaev genus of an adequate knot, Topology. Appl. 156 (2009), 2704–2712.
[2] J.A. Baldwin and A.S. Levine, A combinatorial spanning tree model for knot Floer homology, Adv. Math. 231 (2012), 1886-1939.
[3] K.L. Baker, C.R.S. Lee, K. Motegi and T. Takata, Notes on the Strong Slope Conjecture, in preparation.
[4] K.L. Baker, K.Motegi and T.Takata, The strong slope conjecture for graph knots, arXiv:1809.01039.
[5] O.T. Daubach, D. Futer, E. Kalfagianni, X.S. Lin, and N. Stoltzfus, The Jones polynomial and graphs on surfaces, J. Combin. Theory Ser. B 98 (2008), 384–399.
[6] W. Floyd and A. Hatcher; The space of incompressible surfaces in a 2–bridge link complement, Trans. Amer. Math. Soc. 305 (1988), 575–599.
[7] W. Floyd and U. Oertel; Incompressible surfaces via branched surfaces, Topology 23 (1984), 117–125.
[8] D. Futer, E. Kalfagianni and J. Purcell; Slopes and colored Jones polynomials of adequate knots, Proc. Amer. Math. Soc., 139 (2011), 1889–1896.
[9] S. Garoufalidis; The Jones slopes of a knot, Quantum Topology 2 (2011), 43–69.
[10] S. Garoufalidis; The degree of a $q$-holonomic sequence is a quadratic quasi-polynomial, Electron. J. Combin. 18(2) (2011), 23 pp.
[11] S. Garoufalidis and T.T. Le; The colored Jones function is $q$-holonomic, Geom. Topol. 9 (2005), 1253–1293.
[12] S. Garoufalidis and R. van der Veen; Quadratic integer programming and the slope conjecture, New York J. Math. 22 (2016), 907–932.
[13] A.E. Hatcher; On the boundary curves of incompressible surfaces, Pacific J. Math. 99 (1982), 373–377.
[14] M. Hedden; Knot Floer homology and Whitehead doubles, Geom. Topol. 11 (2007), 101–163.
[15] A. Hatcher and W. Thurston; Incompressible surfaces in 2–bridge knot complements, Invent. Math. 79 (1985), 225–246.
[16] M. Hedden and P. Ording; The Ozsváth–Szabó and Rasmussen concordance invariants are not equal, Amer. J. Math. 130 (2008), 441–453.
[17] J. Hoste and P. Shanahan; Computing boundary slopes of 2–bridge links, Math. Comp. 76 (2007), 1521–1545.
[18] J. Howie: Coiled surfaces and slope conjectures, in preparation.
[19] E. Kalfagianni; A Jones slopes characterization of $A$ adequate knots, Indiana Univ. Math. J. 67(1) (2018) 205–219.
[20] E. Kalfagianni and A.T. Tran; Knot cabling and the degree of the colored Jones polynomial, New York J. Math. 21 (2015), 905–941.
[21] A.E. Lash; Boundary curve space of the Whitehead link complement, Ph.D. thesis, University of California, Santa Barbara, 1993.
[22] A.M. Lowrance, On knot Floer width and Turaev genus, Algebr. Geom. Topol. 8 (2008) 1141–1162.
[23] C. Manolescu, An introduction to knot Floer homology, arXiv:math/14017107.
[24] G. Masbaum; Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556.
[25] G. Masbaum and P. Vogel; 3–valent graphs and the Kauffman bracket, Pacific J. Math. 164 (1994), 361–381.
[26] H. R. Morton; The coloured Jones function and Alexander polynomial for torus knot, Math. Proc. Cambridge Philos. Soc. 117 (1995), 129–135.
[27] K. Motegi and T. Takata; The slope conjecture for graph knots, Math. Proc. Camb. Philos. Soc. 162 (2017), 383–392.
[28] P. Ozsváth and Z. Szabó, Knot Floer homology and the four-ball genus. Geom. Topol., 7 (2003), 615–639.
[29] P. Ozsváth and Z. Szabó, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), 58–116.
[30] J.A. Rasmussen, Floer homology and knot complements, Ph.D. thesis, Harvard University, 2003, arXiv:math/0509499.
[31] A. Stoimenow, Coefficients and non-triviality of the Jones polynomial, J. Reine Angew. Math. 657 (2011), 1–55.
[32] T. Tanaka; The colored Jones polynomials of doubles of knots, J. Knot Theory 17 No.8 (2008), 925–937.
[33] V. G. Turaev; A simple proof of the Murasugi and Kauffman theorems on alternating links, Enseign. Math. 33 (1987), 203–225.
[34] H. Zheng; Proof of the volume conjecture for Whitehead doubles of a Family of torus knots, Chin. Ann. Math., 26 B(4) (2007), 375–388.
(i) once punctured torus

(ii) non-orientable surface of

\[ \omega = 2 \]