Complex nonlinear Fourier transform and its inverse

Pavle Saksida\textsuperscript{1,2}
\textsuperscript{1}Faculty of Mathematics and Physics, University of Ljubljana
\textsuperscript{2}Institute of Mathematics, Physics and Mechanics, Ljubljana
E-mail: pavle.saksida@fmf.uni-lj.si

Abstract. We study the nonlinear Fourier transform associated to the integrable systems of AKNS-ZS type. Two versions of this transform appear in connection with the AKNS-ZS systems. These two versions can be considered as two real forms of a single complex transform \( F^c \). We construct an explicit algorithm for the calculation of the inverse transform \( (F^c)^{-1}(h) \) for an arbitrary argument \( h \). The result is given in the form of a convergent series of functions in the domain space and the terms of this series can be computed explicitly by means of finitely many integrations.

1. Introduction
A partial differential equation in temporal and one spatial dimensions is integrable, if it can be rewritten in the form of the so called zero-curvature condition. Let

\[ E(q(x,t)) = 0 \]  

be a partial differential equation for the unknown function \( q(x,t) \). If the equation \( E \) is integrable, then there exists a pair of matrix-valued functions

\[ L_q(x,t; z), A_q(x,t; z) : [a,b] \times \mathbb{R}_+ \times \mathbb{C} \rightarrow \text{Mat}^{n \times n} \]

such that

\[ (L_q)_t - (A_q)_x + [L_q, A_q] = E(q) \cdot M(z), \]

where \( M(z) \) is a matrix which is constant or it depends only on the parameter \( z \), called the spectral parameter. This means that for every \( z \) the connection \( d + L_q dt + A_q dz \) on the trivial \( \mathbb{C}^n \)-bundle over \([a,b] \times \mathbb{R}_+\) is flat if and only if \( q(x,t) \) is a solution of equation (1).

Fix \( t_0 \) and let \( \Phi(x,t_0;z) \) be the solution of the linear initial value problem

\[ \Phi_x = L_q(x,t_0; z) \cdot \Phi, \quad \Phi(a,t_0;z) = Id. \]  

Denote

\[ \text{Hol}_{q(x,t_0)}(z) = \Phi(b,t_0;z). \]

The fact that holonomies of flat connections along contractible loops are trivial implies that

\[ \text{Hol}_{q(x,t)}(z) = P(b,t;z) \cdot \text{Hol}_{q(x,0)}(z) \cdot P(a,t;z)^{-1}, \]  

where \( P(a,t;z) \) is the parallel transport along the path from \( a \) to \( t \) and \( P(b,t;z) \) is the parallel transport along the path from \( b \) to \( t \).
where $P(x_0, t; z)$ is the solution of

$$P_t = A_q(x_0, t; z) \cdot P, \quad P(x_0, 0; z) = Id.$$ 

If $[a, b] = \mathbb{R} = (\infty, \infty)$, and if $q(x, 0)$ has some suitable asymptotic behaviour at both infinities, then often the matrix $P(-\infty, t; z) = P(\infty, t; z)$ is independent of $q(x, t)$ and one can explicitly compute $\text{Hol}_{q(x,t)}(z)$ from $\text{Hol}_{q(x,0)}$ by means of equation (3).

Let $f(x)$ be a function of one (spacial) variable, belonging to some suitable function space. Solution of the linear initial value problem (2) whose coefficient matrix is $L_f$ induces the map

$$\mathcal{F} : f(x) \mapsto \text{Hol}_{f(z)}$$

from our function space into a suitable space of matrix-valued functions. Suppose we are able to compute the inverse map $\mathcal{F}^{-1}$. Then the above discussion can be summarized into the following scheme:

$$q(x, 0) \xrightarrow{\mathcal{F}} \text{Hol}_{q(x,0)}(z) \xrightarrow{\text{time evolution}} \text{Hol}_{q(x,t)}(z) \xrightarrow{\mathcal{F}^{-1}} q(x, t).$$

If we were able to compute all the above arrows, this scheme would yield the solution of the Cauchy problem for our equation (1). This procedure is called the inverse scattering transform method. It is reminiscent of the algorithm for solving the Cauchy problems for certain linear partial differential equations by means of the Fourier analysis. This can be summarized in the diagram

$$q(x, 0) \xrightarrow{\mathcal{F}} F(q(x, 0))(z) \xrightarrow{\text{time evolution}} F(q(x, t_0)) \xrightarrow{\mathcal{F}^{-1}} q(x, t_0).$$

This similarity was first exposed by Ablowitz, Kaup, Newell and Segur in [1] and [2]. The operator $\mathcal{F}$ is called the inverse scattering transform or, often, the nonlinear Fourier transform. Important results on the connection of the nonlinear Fourier transform and integrable systems were presented e.g. in [6], [7], [8]. Deep analytical issues concerning the nonlinear Fourier transform were studied in [3], [4], [5], [9], [10], [11]. In certain cases the derivative $\mathcal{F}$ is (essentially) the linear Fourier transform $F$. This fact was exploited in [12], [13] to shed some light on certain Poisson geometric aspects of nonlinear integrable systems and their linearizations. The integrable systems are only one of the fields in which the nonlinear Fourier transform plays an important role. Other such fields are orthogonal polynomials, random matrices and others. An introduction and overview to these topics is given in [15].

One of the essential differences between the usual, linear Fourier transform $F$ and the nonlinear transform $\mathcal{F}$ lies in the fact that while the calculation of $F$ and $F^{-1}$ are problems of similar difficulty, the calculation of $\mathcal{F}^{-1}$ is considerably more difficult than the calculation of $\mathcal{F}$. Finding $\mathcal{F}^{-1}$ is a difficult inverse problem. It can be rephrased in the form of a suitable Riemann-Hilbert problem. But this Riemann-Hilbert problem is almost never explicitly soluble. Therefore, it yields an explicit expression of

$$\mathcal{F}^{-1}[\text{time evolution}(\mathcal{F}(q(x, 0))))] = q(x, t_0)$$

only for some exceptional initial conditions $q(x, 0)$. In this paper, our goal will be to find the preimage $\mathcal{F}^{-1}(h(z))$ of an arbitrary element $h(z)$ for the AKNS-ZS class of integrable equations. Since there is no hope of finding a general closed analytic expression, we will give the solution in the form of a series which will be convergent in a suitable topology. This convergent series will provide a tool by means of which one can calculate the approximations of $\mathcal{F}^{-1}(h(z))$ to any desired accuracy.

The AKNS-ZS systems are a relatively large and a very important class of integrable partial differential equations. Some of the most salient integrable equations belong to this class. Notable examples are
(i) Nonlinear Schrödinger equations: \( iq_t = -q_{xx} + 2\kappa |q|^2 q, \quad \kappa \pm 1. \)
(ii) modified KdV equation: \( q_t = 6q^2 q_x - q_{xxx}. \)
(iii) sine-Gordon equation: \( q_{tx} = -\sin q. \)

The common feature of the AKNS-ZS systems is the form of the \( L_q \)-matrices associated to them. These \( L_q \)-matrices are of the form

\[
L_q = \begin{pmatrix}
\frac{i\kappa}{2} q(x, t) & q(x, t) \\
\kappa q(x, t) & -\frac{i\kappa}{2}
\end{pmatrix},
\]

where \( \kappa \) is equal to +1 or to -1. We shall construct a complexified version \( F_c \) of the nonlinear Fourier transform associated to the AKNS-ZS systems. The nonlinear transforms \( F_1 \) and \( F_2 \) for the cases \( \kappa = +1 \) and \( \kappa = -1 \) will turn out to be two different real forms of \( F_c \). In addition to that, \( F_c \) is important in its own right. Namely, it is the transform associated to the vector nonlinear Schrödinger system

\[
i q_t = q_{xx} - 2r q^2, \quad i r_t = r_{xx} - 2q r^2.
\]

We will construct the iterative scheme by means of which one can calculate arbitrarily good approximations of \( (F_c)^{-1} \) for an arbitrary choice of the argument. From this scheme the procedures for calculation of the real forms \( F_1 \) and \( F_2 \) can be derived immediately.

In section 2 we provide the precise definition of the complex nonlinear Fourier transform \( F_c \) and of its real forms. We present our main result in theorem 3 of section 3.

2. The complex nonlinear Fourier transform and its real forms

We begin by giving a precise definition of the complex nonlinear transform \( F_c \). The \( L \)-matrix of the Lax pair for the vector nonlinear Schrödinger system (4) is

\[
L_{(q,r)} = \begin{pmatrix}
\frac{i\kappa}{2} q(x) & q(x) \\
r(x) & -\frac{i\kappa}{2}
\end{pmatrix},
\]

therefore, \( F_c \) will be given by the holonomy of a matrix of the above form. It turns out that the expressions become more manageable if one changes the gauge by the gauge transform

\[
G(x; z) = \begin{pmatrix}
e^{-\frac{i\kappa x}{2}} & 0 \\
0 & e^{\frac{i\kappa x}{2}}\end{pmatrix}.
\]

The transformed matrix

\[
L^g_{(q,r)} = G x G^{-1} + G \cdot L_{(q,r)} \cdot G^{-1}
\]

is

\[
L^g_{(q,r)}(x; z) = \begin{pmatrix}
0 & e^{-\frac{i\kappa x}{2}} q(x) \\
e^{\frac{i\kappa x}{2}} r(x) & 0
\end{pmatrix}.
\]

In this paper, we shall confine ourselves to the case where the operator \( F_c \) will be defined on a suitable \( L^2 \)-space over a finite interval. Let the space \( L^2_c[0, 2\pi] \) be defined by

\[
L^2_c[0, 2\pi] = \{ (f(x), g(x)) : f(x), g(x) \in L^2[0, 2\pi] \}.
\]

This space is equipped with the obvious \( L^2 \)-norm and with the complex structure \( J \), given by

\[
J(f, g) = (-g, f).
\]

(6)
The holonomy of $L^g_{q,r}$ can be defined by the following Dyson’s series

$$\text{Hol}_{(q,r)}(z) = I + \sum_{k=1}^{\infty} \int_{\Delta_k} L^g_{q,r}(x_1; z) L^g_{q,r}(x_2; z) \cdots L^g_{q,r}(x_k; z) \, d\vec{x} = \begin{pmatrix} h^d(z) & h(z) \\ k(z) & k^d(z) \end{pmatrix}, \quad (7)$$

where

$$\Delta_k = \{(x_1, x_2, \ldots, x_k); 2\pi \geq x_1 \geq x_2 \geq \ldots \geq x_k \geq 0\}$$

is the so-called ordered simplex of dimension $k$. Whenever the functions $q(x), r(x)$ are continuous, the above holonomy can be obtained by solving the initial value problem

$$\Phi_x(x; z) = L^g_{(q,r)} \cdot \Phi(x; z), \quad \Phi(0; z) = Id$$

and then evaluating the solution at $x = 2\pi$. By contrast, the expression (7) is defined for every $(q, r) \in L^2_c[0, 2\pi]$.

Finally, we define the target space of the operator $\mathcal{F}^c$. Let $l^2_{c,\mathbb{Z}}$ denote the space of square-integrable bi-infinite complex sequences. Its complexification is the space

$$l^2_{c,\mathbb{C}} = \left\{ \left\{ a(z) \right\}_{z \in \mathbb{Z}}, \left\{ b(z) \right\}_{z \in \mathbb{Z}} \right\}, \left\{ a(z) \right\}_{z \in \mathbb{Z}}, \left\{ b(z) \right\}_{z \in \mathbb{Z}} \in l^2_{c,\mathbb{C}} \right\}.

This space is also equipped with the natural complex structure

$$K\left( \left\{ a(z) \right\}_{z \in \mathbb{Z}}, \left\{ b(z) \right\}_{z \in \mathbb{Z}} \right) = \left( -\left\{ b(z) \right\}_{z \in \mathbb{Z}}, \left\{ a(z) \right\}_{z \in \mathbb{Z}} \right). \quad (8)$$

**Definition 1** The complex nonlinear Fourier transform, associated to the AKNS-ZS systems is the map

$$\mathcal{F}^c: L^2_c[0, 2\pi] \longrightarrow l^2_{c,\mathbb{C}},$$

given by the formula

$$\mathcal{F}^c(q, r) = \left\{ \left\{ h(z) \right\}_{z \in \mathbb{Z}}, \left\{ k(z) \right\}_{z \in \mathbb{Z}} \right\},$$

where the values $(h(z), k(z))$ are defined by (7). More explicitly, (7) gives

$$(\mathcal{F}^c(q, r))(z) = \frac{\sum_{k=1}^{\infty} \int_{\Delta_{2k-1}} e^{-iz(x_1-x_2+\ldots+x_{2k-1})} q(x_1)r(x_2) \cdots q(x_{2k-1}) \, d\vec{x}}{\sum_{k=1}^{\infty} \int_{\Delta_{2k-1}} e^{iz(x_1-x_2+\ldots+x_{2k-1})} r(x_1)q(x_2) \cdots r(x_{2k-1}) \, d\vec{x}}. \quad (9)$$

The fact that the target space of $\mathcal{F}^c$ is indeed the space of pairs of square-integrable sequences $l^2_{c,\mathbb{C}}$ is proved in [14]. We shall see below that for $(q, r)$ close enough to the origin the entire holonomy $\text{Hol}_{(q,r)}$ is indeed completely determined by its off-diagonal part.

The essential theorem about $\mathcal{F}^c$, which is also proved in [14], is the following.

**Theorem 1** The map

$$\mathcal{F}^c: L^2_c[0, 2\pi] \longrightarrow l^2_{c,\mathbb{C}}$$

is holomorphic with respect to the complex structures $J$ and $K$, given by (6) and (8).

It should be mentioned that analyticity of $\mathcal{F}^c$ or of any of its real forms should not be taken for granted. It is true that $\mathcal{F}^c$ remains holomorphic if we replace the spaces $L^2_c[0, 2\pi]$ and $l^2_{c,\mathbb{C}}$ by the appropriate Schwartz spaces. But it is not analytic if we replace them by $L^2$-spaces on the entire real line. In this case $\mathcal{F}^c$ is not even $C^3$. This follows from a result, obtained by Muscalu, Tao and Thiele in [11].
Another essential property of $\mathcal{F}^c$ is the fact that its derivative at the origin is given by the usual linear Fourier transform. Indeed, let $(t(x), u(x)) \in L^2_c[0, 2\pi]$ be an arbitrary element. Then

$$D_0 \mathcal{F}^c(t, u) = \frac{d}{ds}|_{s=0} \mathcal{F}^c(st, su).$$

From (7) we immediately get

$$(D_0 \mathcal{F}^c(t, u))(z) = \left( \int_0^{2\pi} t(x)e^{-ixz} \, dx, \int_0^{2\pi} e^{ixz} u(x) \, dx \right) = \left( F(t)(z), F(u)(-z) \right),$$

where $F$ denotes the linear Fourier transform. The above calculation gives the weak derivative, but since $\mathcal{F}^c$ is holomorphic, this is also the Fréchet derivative.

The complex spaces $L^2_c[0, 2\pi]$ and $\mathcal{C}_{2,\mathbb{Z}}$ can be equipped with two real structures which yield important real forms of $\mathcal{F}^c$. The first is given by

$$\tau_1(q, r) = (-\overline{r}, -\overline{q}) \text{ on } L^2_c\mathbb{H}$$

and the second by

$$\tau_2(q, r) = (\overline{r}, \overline{q}) \text{ on } L^2_c\mathbb{H}$$

These real structures have natural lifting on $L$-matrices and their holonomies. Note that matrices $L_{(q,r)}$ and $L_q^g$ take values in the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$. The two natural real forms of $\mathfrak{sl}(2; \mathbb{C})$ are the real Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$. The corresponding real structures are given by

$$\tilde{\tau}_1(\alpha) = -\alpha^*, \quad \tilde{\tau}_2(\alpha) = -J \cdot \alpha^* \cdot J,$$

where $J = \text{diag}(1, -1)$ and $\alpha^*$ is the conjugate transpose of the matrix $\alpha \in \mathfrak{su}(2; \mathbb{C})$. These two real structures induce the real structures on the corresponding Lie group $SL(2, \mathbb{C})$, given by

$$\tilde{\sigma}_1(g) = (g^{-1})^*, \quad \tilde{\sigma}_2(g) = J \cdot (g^{-1})^* \cdot J$$

and the associated real forms are of course the groups $SU(2)$ and $SU(1, 1)$.

Let us consider the restriction of $\mathcal{F}^c$ on the real forms $R_1 = \{(q(x), -\overline{q(x)})\}$ and $R_2 = \{(q(x), \overline{q(x)})\}$ of the real structures $\tau_1$ and $\tau_2$, respectively. The corresponding $L$-matrices are $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$ valued functions

$$L_q^{1,1} = \begin{pmatrix} 0 & \frac{e^{-ixz}q(x)}{q(x)} \\ -\frac{e^{ixz}q(x)}{q(x)} & 0 \end{pmatrix} \quad \text{and} \quad L_q^{2,2} = \begin{pmatrix} 0 & \frac{e^{-ixz}q(x)}{q(x)} \\ e^{ixz}q(x) & 0 \end{pmatrix},$$

and their respective holonomies are of the form

$$\text{Hol}_{q}^{1}(z) = \begin{pmatrix} h_1^{1}(z) & h_1^{1}(z) \\ -\overline{h_1^{1}(z)} & \overline{h_1^{1}(z)} \end{pmatrix}, \quad \text{and} \quad \text{Hol}_{q}^{2}(z) = \begin{pmatrix} h_2^{1}(z) & h_2^{1}(z) \\ h_2^{-1}(z) & h_2^{-1}(z) \end{pmatrix}.$$
3. Inverse of the complex nonlinear Fourier transform

In the previous section we have seen that $\mathcal{F}^c$ is a holomorphic map and that its derivative at the origin is given by the linear Fourier transform. The norms on $L^2_\mathbb{C}[0, 2\pi]$ and $l^2_{\mathbb{C}, Z}$ are obtained by means of Pythagoras’ theorem. Thus for every $(t, u) \in L^2_\mathbb{C}[0, 2\pi]$, we have

$$\|D_0 \mathcal{F}^c(t, u)\| = \|\{F(t)(z)\}_{z \in \mathbb{Z}}\|^2 + \|\{F(u)(z)\}_{z \in \mathbb{Z}}\|^2.$$  

By Plancharel theorem, this gives

$$\|D_0 \mathcal{F}^c(t, u)\|^2 = \|t\|^2 + \|u\|^2 = \|(t, u)\|^2.$$

In other words, $D_0 \mathcal{F}^c$ is an isometry. Theorem 1 and the analytic version of the inverse mapping theorem for Hilbert spaces provide the proof of the following theorem.

**Theorem 2** There exist neighbourhoods $V \subset l^2_{\mathbb{C}, Z}$ and $U \subset L^2_\mathbb{C}[0, 2\pi]$ of the respective origins and the map

$$\mathcal{G}^c : V \longrightarrow U$$

which is the inverse of the complex nonlinear Fourier transform $\mathcal{F}^c$. The map $\mathcal{G}^c$ is holomorphic with respect to the complex structures $K$ and $J$.

Our next task will be to construct an iterative procedure by means of which one can compute $\mathcal{G}^c(h(z), k(z)) = (\mathcal{F}^c)^{-1}(h(z), k(z))$ for an arbitrary element $(h(z), k(z)) \in V$ to any desired degree of accuracy.

Let $w \mapsto (q(x; w), r(x; w))$ be a holomorphic curve in $L^2_\mathbb{C}[0, 2\pi]$ defined on a disc $\Delta_\alpha \subset \mathbb{C}$ whose center is the origin, and let $(q(x; 0), r(x; 0)) = (0, 0)$. Suppose we can expand our curve into a power series around 0 on the entire $\Delta_\alpha$. We have

$$(q(x; w), r(x; w)) = \sum_{n=1}^{\infty} \frac{w^n}{n!} (q_n(x), r_n(x)). \tag{12}$$

Since $\mathcal{F}^c$ is holomorphic, so is the composition $w \mapsto \mathcal{F}^c(q(x; w), r(x; w))$. This holomorphic curve in $l^2_{\mathbb{C}, Z}$ can be represented in the form of the convergent power series

$$\mathcal{F}^c(q(x; w), r(x; w)) = \sum_{n=1}^{\infty} \frac{w^n}{n!} (h_n(z), k_n(z)).$$

We shall give the formulae which connect the terms $(q_n, r_n)$ and $(h_n, k_n)$ of the above expansions; Let

$$L^N_{(q(w), r(w))}(x; z) = \begin{pmatrix} 0 & L_1 & L_2 & \cdots & \frac{1}{N!} L_N \\ 0 & 0 & L_1 & \cdots & \frac{1}{(N-1)!} L_{N-1} \\ 0 & 0 & 0 & \cdots & \frac{1}{(N-2)!} L_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L_1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where

$$L_k = \begin{pmatrix} 0 & e^{i\pi x} q_k(x) & 0 \\ e^{i\pi x} r_k(x) & 0 & 0 \end{pmatrix},$$

and $(q_k, r_k)$ are the terms of the expansion (12). Because the matrix $L^N$ is strictly upper triangular, any product of more than $N$ matrices of this form is equal to zero. In particular

$$L^N_{(q, r)}(x_1; z) \cdot L^N_{(q, r)}(x_2, z) \cdots L^N_{(q, r)}(x_m; z) = 0, \quad \text{if} \; m \geq N + 1.$$
Therefore, the Dyson series
\[
\Hol_{(q(w),r(w))}^N(z) = I + \sum_{k=1}^{\infty} \int_{\Delta_k} L_{(q(w),r(w))}^N(x_1; z) \cdot L_{(q(w),r(w))}^N(x_2; z) \cdots L_{(q(w),r(w))}^N(x); z) \, d\vec{x}
\]  
(13)

for the holonomy of \( L_{(q(w),r(w))}^N \) is actually a finite sum and is therefore explicitly computable in the form of a finite sum of integrals. This holonomy is again an upper-triangular block-Toeplitz matrix of the form
\[
\Hol_{(q(w),r(w))}^N(z) = \begin{pmatrix}
I & H_1(z) & H_2(z) & \cdots & \frac{1}{N!} H_N(z)
0 & I & H_1(z) & \cdots & \frac{1}{(N-1)!} H_{N-1}(z)
0 & 0 & I & \cdots & \frac{1}{(N-2)!} H_{N-2}(z)
\vdots & \vdots & \vdots & \ddots & \vdots
0 & 0 & 0 & \cdots & H_1(z)
0 & 0 & 0 & \cdots & I
\end{pmatrix}.
\]

Let us now consider the usual \( 2 \times 2 \) holonomy \( \Hol_{(q(w),r(w))}^N(z) \) as a function of \( w \in \Delta_\alpha \subset \mathbb{C} \). The composition \( w \mapsto \mathcal{F}^c(q(w), r(w)) \) is a holomorphic curve in \( L^2_{\mathbb{Z}} \), which means that the anti-diagonal of \( \Hol_{(q(w),r(w))}^N \) is holomorphic. In a completely analogous way one can prove that also the diagonal is holomorphic, so \( w \mapsto \Hol_{(q(w),r(w))}^N \) is a matrix-valued holomorphic curve, defined on the disc \( \Delta_\alpha \). (For the details, see [14].) Therefore, we can expand it into the Taylor series around the origin in \( \Delta_\alpha \),
\[
\Hol_{(q(w),r(w))}^N(z) = I + \sum_{n=1}^{\infty} \frac{w^n}{n!} \hat{H}_n(z) = I + \sum_{n=1}^{\infty} \frac{w^n}{n!} \begin{pmatrix} h_n^x(z) & h_n^y(z) \\ k_n^x(z) & k_n^y(z) \end{pmatrix}.
\]

It is easy to see that the blocks of \( \Hol_{(q(w),r(w))}^N \) and the terms of the above expansion are equal,
\[
\hat{H}_n(z) = H_n(z), \quad \text{for every } n \in \mathbb{N}.
\]

(14)

On the one hand, we have \( \Hol_{(q(w),r(w))}^N(z) = \Psi^N(2\pi; z) \), where \( \Psi^N(x; z) \) is the solution of the initial value problem
\[
\Psi^N_x(x; z) = L_{(q(w),r(w))}^N \cdot \Psi^N(x; z), \quad \Psi^N(0; z) = I d
\]

and this is equivalent to the system
\[
(\psi_n)_x = \sum_{k=1}^{n} \binom{n}{k} L_k \cdot \psi_{n-k}, \quad \psi_n(0; z) = 0 \text{ for } n > 0 \text{ and } \psi_0(x; z) = I d,
\]

where \( \psi_n \) are the \( 2 \times 2 \) blocks of \( \Psi^N \). On the other hand, expanding the \( 2 \times 2 \) initial value problem
\[
\Phi_x(x; z; w) = L_{(q(w),r(w))} \cdot \Phi(x; z; w), \quad \Phi(0; z; w) = I d
\]

with respect to the parameter \( w \) yields precisely the same system for the terms \( \phi_n(x; z) \) of the expansion \( \Phi(x; z; w) = I + \sum_{n=1}^{\infty} (w^n/n!) \phi_n(x; z) \). The uniqueness part of the fundamental existence theorem for the ordinary differential equations gives \( \phi_n(x; z) = \psi_n(x; z) \) and upon evaluation at \( x = 2\pi \) this yields (14).
Equation (14) enables us to use the finite expression (13) for the calculation of the components \( h_n(z) \) and \( k_n(z) \) of the complex nonlinear Fourier transform

\[
\mathcal{F}^r(q(w), r(w)) = \sum_{n=1}^{\infty} \frac{w^n}{n!} (h_n(z), k_n(z))
\]

of the holomorphic curve \((q(w), r(w))\). A straightforward calculation gives

\[
h_n(z) = \sum_{k_1=1}^{n} \sum_{|J|=n} \alpha^J \int_{\Delta_{2k-1}} e^{-iz(x_1-x_2+...+x_{2k-1})} q_{j_1}(x_1) r_{j_2}(x_2) \cdots r_{j_{2k-1}}(x_{2k-1}) \, d\vec{x}
\]

\[
k_n(z) = \sum_{k_1=1}^{n} \sum_{|J|=n} \alpha^J \int_{\Delta_{2k-1}} e^{iz(x_1-x_2+...+x_{2k-1})} r_{j_1}(x_1) q_{j_2}(x_2) \cdots r_{j_{2k-1}}(x_{2k-1}) \, d\vec{x},
\]

where \( J = (j_1, j_2, \ldots, j_{2k-1}) \) are multi-indices with \( j_r \geq 1 \), and

\[
\alpha^J = \left( \begin{array}{c} n \\ j_1, j_2, \ldots, j_{2k-1} \end{array} \right)
\]

are the multinomial coefficients. The summation limit \( \hat{n} \) is equal to \( n/2 \) when \( n \) is even and to \((n + 1)/2 \) when \( n \) is odd. Rewriting formulae (15) and (16) in a more compact way yields the following proposition.

**Proposition 1** For every positive integer \( n \), we have

\[
h_n(z) = \int_{0}^{2\pi} q_n(x) e^{-izx} \, dx + \mathcal{H}_n(q_1, \ldots, q_{n-2}, r_1, \ldots, r_{n-2})
\]

\[
k_n(z) = \int_{0}^{2\pi} r(x) e^{izx} \, dx + \mathcal{K}_n(q_1, \ldots, q_{n-2}, r_1, \ldots, r_{n-2}),
\]

where for \( n > 1 \),

\[
\mathcal{H}_n(z) = \sum_{k=2}^{\hat{n}} \sum_{|J|=n} \alpha^J \int_{\Delta_{2k-1}} e^{-iz(x_1-x_2+...+x_{2k-1})} q_{j_1}(x_1) r_{j_2}(x_2) \cdots r_{j_{2k-1}}(x_{2k-1}) \, d\vec{x}
\]

\[
\mathcal{K}_n(z) = \sum_{k=2}^{\hat{n}} \sum_{|J|=n} \alpha^J \int_{\Delta_{2k-1}} e^{iz(x_1-x_2+...+x_{2k-1})} r_{j_1}(x_1) q_{j_2}(x_2) \cdots r_{j_{2k-1}}(x_{2k-1}) \, d\vec{x},
\]

and \( \mathcal{H}_1(z) = \mathcal{K}_1(z) = 0 \).

**Proof:** All that remains to be checked is the fact that the expressions \( \mathcal{H}_n \) and \( \mathcal{K}_n \) depend only on \( q_1, \ldots, q_{n-2}, r_1, \ldots, r_{n-2} \). Recall that all components \( j_r \) of \( J \) are strictly positive and that \( |J| = n \). Therefore the terms of \( \mathcal{H}_n \) which contain the functions \( q_j \) and \( r_k \) with the highest values of indices \( j \) and \( k \) are integrals over \( \Delta_3 \) of the products of the form \( q_{\sigma(1)}(x_1) r_{\sigma(2)}(x_2) q_{\sigma(n-2)}(x_3) \), where \( \sigma \) is an arbitrary permutation of the symbols \( \{1, 1, n-2\} \). Such terms of \( \mathcal{K}_n \) are the integrals over \( \Delta_3 \) of all the products \( r_{\sigma(1)}(x_1) q_{\sigma(2)}(x_2) r_{\sigma(n-2)}(x_3) \), which proves our claim. \( \Box \)

The first summands on the right-hand sides of formulae (17) and (18) are the linear Fourier transforms of the functions \( q_n \) and \( r_n \). Therefore, equations (17) and (18) can be solved for \( q_n \) and \( r_n \). In our case, the inverse linear Fourier transform \( G \) is given by

\[
G(l(z))(x) = \sum_{z=-\infty}^{\infty} l(z) e^{izx}.
\]
Proposition 2 Let \( w \mapsto (h(z; w), k(z; w)) \) be a holomorphic curve in \( L^2_{c,Z} \) which starts at the origin and lies entirely in the neighbourhood \( V \subset L^2_{c,Z} \), given in theorem 2. Let its Taylor expansion be

\[
(h(z; w), k(z; w)) = \sum_{n=1}^{\infty} \frac{w^n}{n!} (h_n(z), k_n(z)).
\]

Then the inverse image of \((\mathcal{F}^c)^{-1}(h(z; w), k(z; w))\) is the convergent series

\[
(h(z; w), k(z; w)) = \sum_{n=1}^{\infty} \frac{w^n}{n!} (q_n(x), r_n(x))
\]

of elements in \( L^2_{c}[0, 2\pi] \) whose components are given by

\[
\begin{align*}
q_n(x) &= G(h_n)(x) - G(H_n(q_1, \ldots, q_{n-2}, 1, \ldots, n-2))(x) \quad (21) \\
r_n(x) &= G(k_n)(-x) - G(K_n(q_1, \ldots, q_{n-2}, 1, \ldots, n-2))(-x). \quad (22)
\end{align*}
\]

The terms \( H_n \) and \( K_n \) are given by (19) and (20).

Let now \((h(z), k(z))\) be an arbitrary element in \( V \subset L^2_{c,Z} \). If we assume \( V \) to be convex, the line \( w \mapsto (wh(z), wk(z)) \), defined for \( |w| \leq 1 \), is a holomorphic curve in \( V \). If we apply the above proposition to such a line and evaluate at \( w = 1 \), we get the following theorem.

Theorem 3 Let \((h(z), k(z))\) be an arbitrary element of \( V \subset L^2_{c,Z} \). The inverse image \((\mathcal{F}^c)^{-1}(h(z), k(z))\) is given by a convergent series

\[
(\mathcal{F}^c)^{-1}(h(z), k(z)) = \sum_{n=1}^{\infty} \frac{1}{n!} (q_n(x), r_n(x))
\]

whose terms are given by the formulae

\[
\begin{align*}
q_1(x) &= G(h_1)(x) \\
q_{2n}(x) &= 0 \\
q_{2n+1}(x) &= -G(H_{2n+1}(q_1, \ldots, q_{2n-1}, 1, \ldots, n-1))(x)
\end{align*}
\]

and

\[
\begin{align*}
r_1(x) &= G(k_1)(-x) \\
r_{2n}(x) &= 0 \\
r_{2n+1}(x) &= -G(K_{2n+1}(q_1, \ldots, q_{2n-1}, 1, \ldots, n-1))(-x),
\end{align*}
\]

where

\[
\begin{align*}
H_{2n+1} &= \sum_{k=2}^{n+1} \sum_{|J| = 2n+1} \alpha^J \int \Delta_{2k-1} e^{-iz(x_1 - x_2^2 + \ldots + x_{2k-1})} q_{j_1}(x_1)q_{j_2}(x_2) \cdots q_{j_{2k-1}}(x_{2k-1}) \\
K_{2n+1} &= \sum_{k=2}^{n+1} \sum_{|J| = 2n+1} \alpha^J \int \Delta_{2k-1} e^{iz(x_1 - x_2^2 + \ldots + x_{2k-1})} r_{j_1}(x_1)q_{j_2}(x_2) \cdots r_{j_{2k-1}}(x_{2k-1}).
\end{align*}
\]

All the components \( j_\alpha \) of the multi-indices \( J = (j_1, j_2, \ldots, j_k) \) are odd positive integers and \( \alpha^J \) is the multinomial coefficient.
Proof: The expressions have been derived above and the convergence is a consequence of the holomorphy of $\mathcal{F}^c$, proved in [14]. It only remains to prove that all the even terms $(q_{2n} \cdot r_{2n})$ are equal to zero. This is equivalent to the claim that all the components of the multi-indices $J$ are odd. This can be proved by induction. Suppose that for some $n$ the term $q_{2n}$ is non-zero. Then by formula (19), there should exist a multi-index $J = (j_1, \ldots, j_{2k-1})$ such that $j_1 + \ldots + j_{2k-1} = 2n$. But by the induction hypothesis this would mean that the sum of an odd number of odd summands would have to be even, which is impossible. In the same way we can show that $r_{2n} = 0$, for every $n \in \mathbb{N}$.

From the above result and from formulae (10), (11) of section 2, we immediately get the expressions for the inverses $\mathcal{F}_i^{-1}$ of the real forms $\mathcal{F}_i$, $i = 1, 2$. For $i = 1, 2$, we have

$$\mathcal{F}_i^{-1}(h(z))(x) = \sum_{n=1}^{\infty} \frac{1}{n!} q_{n,i}(x),$$

where

$$q_{1,i}(x) = G(h)(x)$$

$$q_{2n,i}(x) = 0$$

$$q_{2n+1,i}(x) = -G(\mathcal{H}_{2n+1,i}(q_1, \ldots, q_{2n-1}))(x)$$

and

$$\mathcal{H}_{2n+1,1}(z) = \sum_{k=2}^{\infty} (-1)^{k-1} \sum_{|J|=2n+1} \alpha^J \int_{\Delta_{2k-1}} e^{-iz(x_1-x_2+\ldots+x_{2k-1})} q_{j_1}(x_1) q_{j_2}(x_2) \cdots q_{j_{2k-1}}(x_{2k-1}),$$

$$\mathcal{H}_{2n+1,2}(z) = \sum_{k=2}^{\infty} \sum_{|J|=2n+1} \alpha^J \int_{\Delta_{2k-1}} e^{-iz(x_1-x_2+\ldots+x_{2k-1})} q_{j_1}(x_1) q_{j_2}(x_2) \cdots q_{j_{2k-1}}(x_{2k-1}).$$

References

[1] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 Nonlinear-evolution equations of physical significance Phys. Rev. Lett. 31 125–27

[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform - Fourier analysis for nonlinear problems Stud. Appl. Math. 53 249-315

[3] Christ M and Kiselev A 1998 Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results J. Amer. Math. Soc. 11 771–97

[4] Christ M and Kiselev A 2001 Maximal functions associated to filtrations J. Funct. Anal. 179 499–25

[5] Frayer C, Hryniv R O, Mykytyuk Ya V and Perry P A 2009 Inverse scattering for Schrödinger operators with Miura potentials: I. Unique Riccati representatives and ZS-AKNS systems Inverse Problems 25 11500–32

[6] Fokas A S 2002 Integrable nonlinear evolution equations on the half-line Commun. Math. Phys. 230 1–39

[7] Fokas A S and Gelfand I M 1994 Integrability of linear and nonlinear evolution equations and the associated nonlinear Fourier transform Lett. Math. Phys. 32 189–210

[8] Fokas A S and Sung L Y 2005 Generalized Fourier transforms, their nonlinearization and the imaging of the Brain Notices Amer. Math. Soc. 52 1178–92

[9] Hryniv R O, Mykytyuk Ya V and Perry P A 2010 Inverse scattering for Schrödinger operators with Miura potentials: II. Different Riccati representatives Comm. Partial Differential Equations 36 1587–623

[10] Muscalu C, Tao T and Thiele C 2003 A Carleson type theorem for a Cantor group model of the scattering transform Nonlinearity 16 219-46

[11] Muscalu C, Tao T and Thiele C 2003 A counterexample to a multilinear end-point question by Christ and Kiselev Math. Res. Lett. 10 237–46

[12] Saksida P 2011 On zero-curvature condition and Fourier analysis J. Phys. A: Math. Gen. 44 85203–85222

[13] Saksida P 2012 Lax pairs and Fourier analysis: The case of sine-Gordon and Klein-Gordon equations J. Phys.: Conf. Ser. 343 012109 (10pp)

[14] Saksida P 2013 On the nonlinear Fourier transform associated with periodic AKNS-ZS system and its inverse J. Phys. A: Math. Theor. 46 465204 (22pp)

[15] Tao T and Thiele C 2012 Nonlinear Fourier analysis (arXiv:1201.5129)