Abstract. The primary goal of this paper is to abstract notions, results and constructions from the theory of categories to the broader setting of plots. Loosely speaking, a plot can be thought of as a non-associative non-unital category with a “relaxed” composition law: Besides categories, this includes as a special case graphs and neocategories in the sense of Ehresmann, Gabriel’s quivers, Mitchell’s semicategories, and composition graphs, precategories and semicategories in the sense of Schröder. Among other things, we formulate an “identity-free” definition of isomorphisms, equivalences, and limits, for which we introduce regular representations, punctors, $M$-connections, and $M$-factorizations. Part of the material will be used in subsequent work to lay the foundation for an abstract theory of “normed structures” serving as a unifying framework for the development of fundamental aspects of the theory of normed spaces, normed groups, etc., on the one hand, and measure spaces, perhaps surprisingly, on the other.

Contents

1. Introduction ................................................. 2
2. Background and philosophy .................................. 8
3. Preliminaries ................................................. 12
4. Plots and parenthesizations .................................. 14
5. Some appealing examples .................................... 31
6. Special classes of arrows and related notions .................. 38
7. Plot homomorphisms ......................................... 52
8. Closing remarks .............................................. 72
Acknowledgements ............................................ 72
1. INTRODUCTION

I believe that one of the greatest legacies of A. Grothendieck to posterity, largely anticipated by C. Ehresmann’s view on mathematics [19], is neither a theorem nor a theory, but instead a meta-principle, which may be stated as “Think general to get closer to the essence of things.”

With this in mind, the primary goal of the present paper is to carry over a few fundamental notions, results and constructions from the theory of categories to the broader setting of plots, which can be informally understood as non-associative non-unital categories with a “relaxed” composition law (see Definitions 2 and 3 for details).

Let us stress in this respect that, while our use of “non-associative” means that associativity is not assumed, it does not mean that associativity is disallowed. In other words, “non-associative” stands for “not necessarily associative”, in the same way as “non-commutative” means “not necessarily commutative” (and idem for “non-unital”).

Besides categories, plots do essentially include as a special case graphs and neo-categories in the sense of Ehresmann [5] [18], quivers (or multidigraphs) in the sense of P. Gabriel [22], semicategories in the sense of B. Mitchell [50], and composition graphs, precategories and semicategories in the sense of L. Schröder [58]-[59], so providing in the first place a unifying framework for coping with fundamental aspects of all of these structures, and not only, by a top-to-bottom approach (see Remark 1).

Plots represent another step up the ladder of abstraction, namely the step on which one’s foot lands in the attempt to generalize the heck out of the notion of (partial) magma (see Example 1). They may perhaps look similar to categories, but are in fact profoundly different from them, and quite more general, and much more problematic. So it won’t surprise that people with conflicting feelings about categories have much more of them about plots and similar structures. It is however startling to learn from private conversations and discussions on public forums that even those working in category theory and related areas may feel the same way about the subject.

There are, for instance, people dismissing any visible interest in (Mitchell’s) semicategories based on the fact that these can be turned into categories, “with no loss of information” (this seems to be quite a common assessment), by adjoining an identity for each object (a process here referred to as forced unitization), to the extent of claiming that semicategories are not really more general than categories. The case
Salvatore Tringali

deserves attention, for the same considerations would also apply, if correct, to the non-associative setting (see Section 7.3), with the result that studying non-unital plots would look like a needless complication. But what should it mean, in formal terms, that unitizing a non-unital structure implies “no loss of information”? In the “small case”, the forced unitization of a (Mitchell’s) semicategory boils down to the existence of a left adjoint $L$ to the obvious forgetful functor from the ordinary category of small categories and functors [3, Section 1.4, Example 6] to the ordinary category of small semicategories and semifunctors [50, Section 4, p. 328], to the effect that a number of properties of the latter can be actually recovered from properties of the former (see also Example 40 and Proposition 36). Nonetheless, while faithful and injective on objects, $L$ is neither full nor dense (and thus not an equivalence of categories). So?

E.g., it is known [54, Section 4.7.1] that the only finite monoids with no non-trivial homomorphic images are the finite simple groups and the two-elements monoids (one of these actually being a simple group), as it follows from the classification of congruence-free finite semigroups. Nevertheless, there exists a whole infinite family of finite semigroups with no non-trivial homomorphisms not included on this list. Doesn’t this count as a significant loss of information due to the structural rigidity induced by the presence of identities? The question is quite pertinent, for a semigroup can be canonically identified with a one-object semicategory, in the same way as a monoid can with a one-object category (cf. Example 1). Moreover, every semigroup can be unitized to become a monoid, and even in more than one way (see [54, Section 1.2.3] for remarks on this). Should we perhaps conclude that the theory of semigroups is not really more general than the theory of monoids?

Maybe the truth is just that certain oversimplifications are pointless and exceedingly rough: To indulge with the obvious, the degree to which it is useful to have a way, whether functorial or not, of unitizing a (non-unital) structure does definitely depend on the particular situation at hand. It is, therefore, desirable to develop as much as possible of the “theory of plots” with no regard to the presence of identities. Yet, we will see that certain parts assume a more “natural” and better motivated form when identities are made available somehow: This is notably the case, e.g., with natural transformations, $M$-limits and related notions, for which we do not, however, rely on the process of forced unitization mentioned above, but instead on the alternative approach of adjoining a local identity only if none is already present (see Sections 7.3 and 7.4 for details).

As for the motivations of the paper, my initial interest in plots was mostly out of curiosity, for the desire of finding a raison d’être for the following:

- Many popular categories don’t have all of their binary products or exponentials, so that the corresponding bifunctors are only partially defined. And
even when binary products or exponentials may all exist, the corresponding bifunctors are not, in general, associative, or even weakly associative (that is, associative up to an isomorphism, either canonical or not).

- Relevant notions of category theory, such as limits, colimits, (binary) factorizations, etc., do not need associativity to be defined (and even developed, at least to a certain extent), for they involve the commutativity of triangles or squares, that is the composition of just one or two morphisms. Of course, some form of associativity is still necessary in a great number of situations, and trying to understand to which extent this is really the case looks like an interesting challenge and may improve our comprehension of certain problems.

- There is a certain degree of redundancy in the common definition of categorical limits by diagrams. The observation is nothing new, and an alternative approach, favored by some authors, is in fact based on the use of quivers (in place of diagrams). But, though the two approaches are perfectly equivalent, the latter looks “less natural” than the former as far as we are talking of categories. However, this is no longer the case if we pass from functors, i.e. homomorphisms of categories, to punctors, i.e. homomorphisms of plots, and we let a diagram be a punctor going from a quiver (which, as mentioned, is just a particular plot) to an arbitrary plot.

These speculations led straight to the idea of abstracting (partial) magmas in the same way as monoids are abstracted by categories and semigroups by Mitchell’s semicategories. But curiosity is not always enough of a motivation for the pursuit of a research program, so everything laid in a drawer for several months.

Things changed when the reading of the Elephant [35] prompted the idea of using plots to lay the foundation for an alternative approach to the categorification of (fragments of) first-order logic based on what we refer to as a “semantic domain”. This is, in the simplest case, a 5-tuple $\mathcal{D} = (\mathcal{S}, \otimes, \mathcal{O}, \mathcal{P}, \xi)$ consisting of

- a category $\mathcal{S} = (C_0, C_1, s, t, \diamond)$, termed the (semantic) category of $\mathcal{D}$.
- a bifunctor $\otimes$ from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$, called the (semantic) tensor of $\mathcal{D}$.
- a distinguished object $\mathcal{O} \in \mathcal{S}$, called the (semantic) origin of $\mathcal{D}$.
- a “system of points” $\mathcal{P}$, i.e. a collection of $\mathcal{S}$-morphisms, referred to as the points of $\mathcal{D}$, that on the one hand behaves as a right ideal, in the sense that $f \diamond g \in \mathcal{P}$ for any pair $(f, g)$ of composable $\mathcal{S}$-morphisms such that $g \in \mathcal{P}$, and on the other is closed under tensorization, namely $f \otimes g \in \mathcal{P}$ for all $f, g \in \mathcal{P}$ (here and later, unless differently stated, we use the diagrammatic notation for the composition of arrows). In particular, we may assume that the source of each point of $\mathcal{D}$ is a tensorization of the origin $\mathcal{O}$, i.e. an object
of the form $(O \otimes \cdots \otimes O)_{\wp[S]}$ for some fundamental parenthesesization $\wp$; see Section 4.2 for the notations and terminology used here.

- a function $\xi : \text{Prs} \to \mathcal{C}_1$, called the (semantic) pullback of $\mathcal{D}$, taking a fundamental parenthesesization $\wp$ of length $n$ to a $S$-morphism $O \to (O \otimes \cdots \otimes O)_{\wp[S]}$.

More in general, $S$ may be an arbitrary plot (not necessarily a category) and $\otimes$ a partial bipunctor (see Section 7), but the case above is somewhat paradigmatic. In particular, there is a canonical way to turn a monoidal category [9, Definition 6.1.1], say $\mathcal{M}$, into a semantic domain: The semantic tensor is just the tensor product of $\mathcal{M}$, the semantic origin is the identity, say $I$, of $\mathcal{M}$, the points are the so-called "elements" of $\mathcal{M}$, that is all and the only arrows whose source is isomorphic to $I$ (or even better, all and the only elements of $\mathcal{M}$ whose source is a tensorization of $I$), and the semantic pullback is defined in the obvious way through the left (respectively, right) unitor.

Semantic domains are not discussed in further details here (in particular, we will not explain how to use them to define a categorical semantic, which would take us too far away from the main point). They will be developed in a separate paper, and later applied, in the first place, to the construction of a unifying theory of "normed structures", subsuming fundamental aspects of the theories of normed (vector) spaces, normed algebras, normed groups, etc., but also, and perhaps surprisingly, of the theory of measure spaces. This is made possible by the introduction, on the one hand, of premorphisms, a "natural" generalization of (algebraic) homomorphisms for which equality is replaced with preorders, and on the other hand, of complemented prehilbertian lattices, an "analogue" of normed spaces (nothing to do with normed vector lattices, or normed Riesz spaces, as they are sometimes called; see [47] and [65]) where (semi)vector spaces are replaced by lattices.

Basically, the story is as follows: A homomorphism of algebraic structures of the same type, as intended in the context of universal algebra, is a function with the property of "preserving the operations". Then, one notes that, with a little effort of imagination, a seminorm $\| \cdot \|$ (say, on a real vector space exhibits almost the same behaviour:

- Its target is a certain "reference (pre)ordered structure" (in our toy example, this is the set of non-negative real numbers with their standard structure of totally ordered semifield, regarded as a totally ordered semivector space over itself).
- It preserves the additive identity, which is loosely equivalent to saying that the norm "maps" a nullary operation to a nullary operation.
• It relates a sum (of vectors) to a sum (of scalars) by means of an inequality, disregarding for a moment the fact that the one addition and the other are not exactly the same, subjected as they are to “slightly” different axioms.

• It equates the product of a scalar by a vector to a product of two scalars, which is informally the same as saying that it preserves the products.

The next step is to emphasize something obvious in itself, i.e., that equalities and inequalities share the property of being orders, either total or not. And it is just by using orders (or better, preorders) and relaxing equalities to inequalities that we are allowed to relate structures of possibly different types (e.g., a vector space and a semivector space) and claim that a measure is, in the essence, a norm.

Nothing of the above will be, however, developed here: It was mentioned to stimulate, if possible, the readers’ interest, and to convince them that, if nothing else, there is a program at the shoulders of this paper, which might otherwise look like a mathematical extravagance and be branded as an end in itself.

Now returning to plots, the origin of their name is mainly of a literary nature, and an implicit tribute to Ehresmann’s work (see the excerpt at the end of Section 2). It is probably not a best choice, but we have some reasonable motivation for avoiding alternative terms, and especially “graph”: The latter is, in particular, used in so many different contexts and senses (a graph can be oriented or not, with or without isolated nodes, multiple arrows, loops, etc.) that it would have led, for our purposes at least, to certain ambiguities.

There is another remark that I feel as necessary, before proceeding: Even if containing some results, the paper is not mainly concerned with proving theorems. Its aim is primarily to introduce a language for dealing with structures which turn out to appear quite “naturally” in applications, and to suggest that further levels of abstractions going beyond the scope of category theory are possibly more interesting than one would suspect.

As was mentioned before, the material serves mainly as a reference for subsequent work. But it may perhaps be interesting in its own right, or useful for other researchers, all the more that, in spite of contributions by prominent scholars to the pursuit of possible generalizations of category theory following up the original ideas of Ehresmann’s school on neocategories and Mitchell’s paper on semicategories, there doesn’t exist any survey or book on the subject (see Section 2 for a detailed bibliography).

The reason for the gap seems primarily due to the fact that the “loss of identities”, let alone when combined with non-associativity, is widely perceived as being too drastic a reduction to allow for an appropriate extension of fundamental aspects
of categories - a reasonable precondition, somebody will argue, for hoping in “significant applications”. Of course, this feeling is not ungrounded, but based on concrete difficulties, which may not, however, prove anything but the limits of a certain view.

E.g., while functors go through from categories to plots with no substantial modification (see Section 7), nobody has worked out, as far as I am aware, an answer to the following:

Is there any “sensible” way to extend isomorphisms, limits, equivalences, natural transformations, adjoints, etc. to “non-unital (respectively, non-associative) categories”? And what about Yoneda's lemma and other fundamentals?

And this may certainly have been one of the brakes to a systematic development of the subject. For what I can say, the only previous attempt in this direction is due to Schröder [57], and related to “associative structures” that he calls semicategories: These however are kind of “unital structures” (though not in the usual sense of categories), with the result that isomorphisms, equivalences, adjoints, etc. can be still defined in essentially the same manner as for categories [57, Section 2].

As a matter of fact, the paper looks into the above questions and provides some (partial) answers to them, trying to advocate for the “soundness” of the proposed approach. This is done step by step by generalizing a lot of accessory categorical notions, results, and constructions. While most of these are carried over with none or minor adjustments, some others are subtler: we incorporate all of them in our presentation for future reference and completeness (but we omit proofs if somewhat trivial). Readers are advised to look carefully at definitions, comments, and remarks, even if they are familiar with categories, as notations and terminology may be unconventional.

In particular, the material includes an “identity-free” definition of isomorphisms, equivalences, and limits, based on the introduction of regular representations, punctors, $\mathcal{M}$-connections, and $\mathcal{M}$-factorizations, and curved out from the following elementary facts, where we let $f: A \to B$ be a morphism in a certain category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, \diamond)$:

- $f$ is an isomorphism if and only it is monic and split epi, or epic and split mono, or split mono and split epi; see [3, Definition 2.7 and Exercise 2.8.5].
- The property of $f$ of being epic/split mono (respectively, monic/split epi) can be equivalently formulated in terms of the injectivity/surjectivity of the left (respectively, right) regular representation of $f$, namely the map $\mathcal{T}_A \to \mathcal{T}_B : g \mapsto g \diamond f$ (respectively, $\mathcal{S}_B \to \mathcal{S}_A : g \mapsto f \diamond g$), where for $X \in \mathcal{C}_0$ we let $\mathcal{T}_X$ (respectively, $\mathcal{S}_X$) be the class of all arrows in $\mathcal{C}$ whose target (respectively, source) is $X$; cf. Section 4.3.
The above “naturally” leads to the definition of an isomorphism in a plot as an arrow that is concurrently cancellative (i.e., both of its regular representations are injective) and split (i.e., both of its regular representations are surjective). Such a definition returns the standard notion of isomorphism on the level of categories, but does not depend at all on the presence of local identities (i.e., we say, is “identity-free”); see Section 6 for details.

The first corollary of a similar change of view is trivially that isomorphisms need no longer be invertible, which may sound all the more striking (or awkward, according to one’s sensitivity) when considering the fact that, even in the case of a unital plot (namely, in presence of local identities), an isomorphism (in the sense of our definitions) is not necessarily invertible, and viceversa an invertible arrow is not necessarily an isomorphism, unless some form of associativity is given. Of course, this has annoying consequences, but is somehow in the “natural order of things”, and we will try to convince the readers that, in spite of a number of difficulties, the insights implied by the new view are possibly more interesting than one would be inclined to believe at first.

1.1. Organization. The plan of the paper is as follows. Section 2 is a survey of the literature on Ehresmann’s graphs and their generalizations (save for philosophical musings). In Section 3, we fix general notation and terminology. Section 4 includes the definition of fundamental notions such as plots, parenthesizations, and regular representation. In Section 5, we give examples (others are disseminated all through the article) and introduce \( M \)-connections et similia (Example 9). In Section 6, we define isomorphisms and other special sets of arrows, and consider a few related notions such as \( M \)-orphic and \( M \)-equivalent objects. Finally, Section 7 deals with punctors, categories of plots, basic constructions, and \( M \)-limits and the like.

2. Background and philosophy

This section is primarily an overview of the existing literature on “horizontal” generalizations of categories, but it also includes some remarks that I feel as necessary as a consequence of (sometimes heated) exchanges with colleagues and friends.

Though I have tried my best to provide a complete list of what I believe are the most relevant references on the topic, there are undoubtedly many sins of omission for which I beg to apologize in advance. The readers will note that the survey does not intentionally include higher-dimensional categories, for I tend to think of them as a “vertical” generalization of categories, and hence a “horizontal” specialization of higher-dimensional analogues of “horizontal” generalizations of categories.

To best of my knowledge, the first attempt to abstract aspects of category theory to more primitive settings, possibly better suited for particular applications, tracks back to Ehresmann, by the introduction of neocategories (and, as a special
case, Ehresmann’s graphs) in his 1965 monograph on categories and structures [18],
motivated by early work on sketches [5] (neocategories are also called multiplicative
graphs in Ehresmann’s work and later literature). But the subject has not
received much attention as a topic per se for many years, except for contributions
by Ehresmann’s school (e.g., see [16], [13]-[15], and [43]) and few others (e.g., [29]).

However, it has been taken up again in the last decade in connection to recent
trends in the theory of $C^*$-algebras (e.g., see [20] and the bibliography therein),
leading to the study of associative structures, somewhat similar to Ehresmann’s
neocategories, which practitioners in the area call semigroupoids (sometimes written
as semi-groupoids). This has provided new insights in the study of the so-called
Cuntz-Krieger algebras, combinatorial objects which can be attached, for instance,
to infinite 0–1 matrices to investigate some of their properties.

Also, and more specifically, Ehresmann’s original ideas have been further general-
ized by L. Schröder and coauthors through the introduction of composition graphs,
precategories and semicategories (see [57]-[59] and references therein), with appli-
cations, e.g., to the theory of extensions of categories: In particular, composition
graphs, in the sense of Schröder, differ from Ehresmann’s multiplicative graphs for
the fact that objects do not need be unital (see Definition 2 for the terminology).
For the record, the possibility of a similar weakening of axioms had been previously
alluded to, but not pursued in [15].

Even Mitchell’s semicategories (i.e., non-unital categories), which differ to some
significant extent from Schröder’s semicategories (see Remark 1 for details), have not
had a really different fate, though at some point they have attracted the attention
of various leading authors. As possibly suggested by the name, they were first
considered by Mitchell (who is also recognized as the one who coined the term) in
relation to J.M. Howie and J.R. Isbell’s work on epimorphisms and dominions (see
[50] and references therein). They are mentioned by B. Tilson in [64, Appendix B,
pp. 193–196], in respect to monoid decomposition theory. They appear in [52]
(referred to as semigroupoids), in connection with semidirect decompositions of
finite ordered semigroups, and in [23, Section 2], related to the study of sheaves
on quantaloids. They are used by A. Joyal in his notes on quasi-categories [37,
Section 16.13] to prove that the functor sending a small category to its Karoubi
envelope has the structure of a monad with a left adjoint comonad, by R. Street
and coauthors in [6] to talk about weak Hopf algebras, and by J. Kock in [40]
(while written as semi-categories) to work out the basic theory of weak arrows in an
attempt to find a satisfactory answer to the comparison problem [45] (the interested
reader may want to refer to [4] for recent developments in the subject). In this way,
Mitchell’s semicategories appear to be “naturally” linked to Simpson’s conjectures
(see [36] and references therein). Additionally, they play a certain role in a series
of papers by I. Stubble on categorical structures enriched in quantaloids (see [62]
and references therein), partly based on his 2003 PhD dissertation [61] and previous work by F. Borceux and coauthors [51]. Research in these lines has eventually led to revisit some aspects concerned with sheaves on locales by placing them in the context of the theory of quantale modules (see [53] and references therein). Finally, semicategories are central in a series of three papers by P.W. Michor in which the author develops part of the theory of categories enriched over the usual category of Banach spaces [49].

On a related note, semifunctors (or alternatively, semi-functors) between categories, which can be thought of as functors that need not preserve identities, have long been investigated in (theoretical) computer science, especially in regard to the modelization of untyped $\lambda$-calculus, and they are mentioned by A.P. Freyd and A. Shedrov in [21, Section 1.2(84), p. 16] under the name of pre-functors. The notion was first introduced by S. Hayashi in [28] (but a similar idea, that of a neofunctor, is due to Ehresmann; see [5, Section I.2] and references therein), and is the object of a 1995 paper by R. Hoofman and I. Moerdijk [31], establishing the existence of “an equivalence between, on the one hand, the theory of categories, semifunctors and natural transformations between such semifunctors, and, on the other hand, the theory of Karoubi-complete categories, ordinary functors and ordinary natural transformations”, from which it follows that standard properties and constructions of functors extend automatically to semifunctors (between categories), at least as far as the ones and the others can be regarded as 1-arrows between small 2-categories.

On the other hand, Mitchell’s semicategories have been only mildly popular in computer science, where they are mostly called semigroupoids. Sort of remarkable exceptions are represented, for instance, by recent work due to W. Kahl [38], who has shown how to profit from these structures to define a generalisation of relational categories that serves as a versatile framework for the manipulation of finite binary relations in functional programming languages such as Haskell, and to M. Hyland and coauthors [34], who provide a category-theoretic formulation of Engeler-style models for the untyped $\lambda$-calculus.

While certainly far from completeness, this overview aims to suggest that interest in possible generalizations of categories and their theory, particularly based on the idea of relaxing the axioms concerned with local identities or associativity, has been rather limited and occasional (if not at all erratic), with the result that the topic has never really “taken off”. Of course, this is true only if we leave aside the vast literature on Gabriel’s quivers [22] and their applications (e.g., see [7], [26], [44] and the references therein), where the basic object of study is actually a generalization of Ehresmann’s graphs (the only difference is that the source and the target maps in the latter are required to be surjective). But Gabriel’s quivers are a somewhat extreme example, and it may be interesting to have them fit into a bigger picture.
Now, to quote an excerpt from [50, Section 4], “[…] if one grants an interest in one object categories without identities - i.e., semigroups - then one should grant an interest in several object categories without identities”, and the paper proceeds with Mitchell’s definition of a semicategory, even though “[…] at the risk of being censured by the mathematical community.” We are not in the position to give any guarantee for the authentic interpretation of Mitchell’s real intentions: Some will discern sparks of wit and humor between the lines of his words, others will be urged to reflect on the fierce resistance opposed, either passively or actively, by prominent personalities and fringes of the mathematical community to novelty. After all, M. Barr reports in an old thread from the category theory mailing list that early comments by P.A. Smith on S. Eilenberg and S. Mac Lane’s pioneer work on categories [17] were all but positive: “P.A. Smith said he never read a more trivial paper in his life.” And S. Lang has always had a sort of “idiosyncrasy” for categories, only partially tempered by time. Not to mention the expression “general abstract nonsense”, mostly used by practitioners as an indication of deep mathematical perspective, but often intended by others in a sarcastic (if not derogatory) sense, still today that we have already got huge evidence that looking at questions from the point of view of categories is capable to spread one’s horizons beyond any apparent imagination and to open new paths towards unknown universes just waiting for exploration (which may sound perhaps too “lyrical”, but looks much more than sufficient to deserve some interest and efforts in the area).

Category theory is a useful way to think of mathematics and about mathematics in its wholeness, with a special emphasis on structures and foundations: Indeed, not only categories provide a powerful language to expose inherent structural similarities, they also allow intricate and subtle mathematical results from several, apparently unrelated fields to be stated, and often proved, in full generality in a much simpler way than without.

So the question is: Why couldn’t it be the same with “looser” structures than categories? Mathematics exhibits an inherent tendency towards increasing levels of generality. Yet, generality is often met with diffidence, to such an extent that we have invented catch phrases like “generality for the sake of generality” bandied about in erudite debates about abstract thinking to support everything and its opposite. Cui prodest?

Interesting structures arising in numerous applications, from functional analysis to topology passing through discrete mathematics and the theory itself of categories, can be “naturally” regarded as plots, but apparently not as categories. Even ignoring any other implication, shouldn’t this be enough of a motivation to dig into the subject? A possible answer comes from Ehresmann [19]:
“The same development which leads to a new literature where novels do not need to have a plot, to an abstract music, sometimes written by a computer, to abstract sculpture and painting, which do not intend to give an ordinary representation of real objects, this same development toward abstraction leads to a kind of Mathematics much less motivated by possible applications than by a profound desire to find in each problem the very essence of it, the general structure on which it depends. [...] For the Platonists among the mathematicians, the motivation of their work lies in this search for the true structure in a given situation and in the study of such an abstract structure for itself. For the more pragmatic mathematician, the purpose of his efforts is to solve a preassigned problem [...] with any means at his disposal, avoiding as much as possible the introduction of new general concepts. But all mathematicians agree that the value of a work in mathematics is best proved if it stimulates new research, and the main range of applications of Mathematics is Mathematics itself.”

So it should be finally clear where the name of the kind of structures considered in this paper comes from. And after that we are ready to get to the heart of the matter.

3. Preliminaries

For the sake of exposition, and especially to avoid misunderstandings, let us first clarify some general points, mostly related to notation and terminology that may differ slightly from author to author or are specific for this manuscript.

We use as a foundation for our work the Tarski-Grothendieck (TG) axiomatic set theory; other choices would be equally possible, but this issue exceeds the scope of the paper, and we can just pass over it without worrying too much. In particular, we fix one and for all an uncountable universe \( \Omega \), and then we refer to the elements of \( \Omega \) as \( \Omega \)-sets (or simply sets) and to an arbitrary set in the ontology of TG as a class, a family, or a collection (a class which is not an \( \Omega \)-set will be called proper).

We follow [1], [3] and [8] for fundamental aspects and results concerning the theory of categories, while we refer to [11], [10] and [41], respectively, for notation and terminology from set theory, algebra and functional analysis used but not defined here.

We write \( \mathbb{Z} \) for the integers, \( \mathbb{N}^+ \) for the positive integers, \( \mathbb{Q} \) for the rationals, \( \mathbb{R} \) for the real numbers, and \( \mathbb{R}^+ \) for the positive real numbers. Then, we let \( \mathbb{N} := \{0\} \cup \mathbb{N}^+ \) and \( \mathbb{R}_0^+ := \{0\} \cup \mathbb{R}^+ \). Each of these sets is regarded as a subset of \( \mathbb{R} \) and endowed with its usual addition \( + \), multiplication \( \cdot \), absolute value \( | \cdot |_\infty \) and order \( \leq \) (as customary, we write \( \geq \) for the dual of \( \leq \), and \( < \) and \( > \), respectively, for the strict orders induced by \( \leq \) and \( \geq \)). Also, we use \# \( X \) or \( |X| \) for the cardinality of a set \( X \), unless a statement to the contrary is made (see Section 4.4).
Given a totally ordered set \( \mathbb{I} = (I, \preceq) \) and a family \((X_i)_{i \in \mathbb{I}}\) of classes indexed by \( \mathbb{I} \), we use \( \prod_{i \in \mathbb{I}} X_i \) and \( \bigsqcup_{i \in \mathbb{I}} X_i \), respectively, for the Cartesian product and coproduct of \((X_i)_{i \in \mathbb{I}}\), which we write as \( X_1 \times \cdots \times X_n \) and \( X_1 \oplus \cdots \oplus X_n \), respectively, if \( \mathbb{I} \) is finite and \( \alpha := |\mathbb{I}| \neq 0 \). Also, we may use \( X^{\times \alpha} \) and \( \times \in \alpha \) in place of \( \prod_{i \in \mathbb{I}} X_i \) and \( \bigsqcup_{i \in \mathbb{I}} X_i \), respectively, if \( X_i = X_j =: X \) for all \( i, j \). Lastly, we denote an element of \( \prod_{i \in \mathbb{I}} X_i \) by \((x_i : X_i)_{i \in \mathbb{I}}\), or simply by \((x_i)_{i \in \mathbb{I}}\), as is customary, if the \( X_i \) are understood from the context.

If \( X \) and \( Y \) are classes, we define a (binary) relation on the pair \((X, Y)\) as a triple \((X, Y, \varphi)\) where \( R \subseteq X \times Y \) (we often identify the triple with \( R \) if \( X \) and \( Y \) are implied by the context), and we let a (partial) function (or map, or mapping) from \( X \) to \( Y \) be a relation \((X, Y, \varphi)\) such that, on the one hand, \( \varphi \subseteq D \times Y \) for some (possibly empty) subclass \( D \) of \( X \), and on the other hand, for every \( x \in D \) there exists a unique \( y \in Y \), denoted by \( \varphi(x) \) or \( \varphi_x \), and termed the image of \( x \) under \( \varphi \), such that \((x, y) \in \varphi \) (this is also expressed by the notation \( x \mapsto y \), or by saying that \( \varphi \) sends, or maps, \( x \) to \( y \)). If the two conditions above are satisfied, then \( D \) is univocally determined by \( \varphi \), and we call \( D \), \( X \) and \( Y \), respectively, the domain, source and target of \( \varphi \).

We generally write a function \((X, Y, \varphi)\) as \( \varphi : X \to Y \), and we denote its domain by \( \text{dom}(\varphi) \). We may say that \( \varphi \) is a function from \( X \) to \( Y \) (or some variation of this), or refer to \( \varphi \) as the function \( X \to Y : x \mapsto \varphi_x \) if we are given an “explicit formula” to describe the correspondence between an element \( x \in \text{dom}(\varphi) \) and its image \( \varphi_x \) under \( \varphi \). In particular, \( \varphi \) is termed a total function (from \( X \) to \( Y \)) if \( \text{dom}(\varphi) = X \), in which case we write \( \varphi : X \to Y \) for \( \varphi : X \to Y \). Then, we use \( 1_X \) for the identity map of \( X \), i.e. the map \( X \to X : x \mapsto x \).

For a function \( \varphi : X \to Y \) we take the (direct) image, or range, of \( \varphi \) to be the class \( \{\varphi(x) : x \in \text{dom}(\varphi)\} \), here denoted by \( \text{im}(\varphi) \) or \( \varphi(X) \). Also, for \( S \subseteq Y \) we write \( \varphi^{-1}(S) \) for the inverse image of \( S \) under \( \varphi \), viz the class \( \{x \in \text{dom}(\varphi) : \varphi(x) \in S\} \). Then, if \( \varphi \) is total and bijective, we let \( \varphi^{-1} \) represent the function \( Y \to X : y \mapsto \varphi^{-1}(\{y\}) \), namely the (functional) inverse of \( \varphi \).

If \( X, Y, Z, W \) are classes and \( \varphi : X \to Z \), \( \psi : Y \to W \) are (partial) mappings, we use \( \psi \circ \varphi \) for the (functional) composition of \( \varphi \) with \( \psi \), that is the (partial) function \( X \to W : x \mapsto \psi(\varphi(x)) \) whose domain is the inverse image of \( \text{dom}(\psi) \) under \( \varphi \). Then, in particular, we refer to \( \psi \circ 1_X \) as the restriction of \( \psi \) to \( X \), denoted by \( \psi|_{1_X} \), and we call \( 1_Y \circ \varphi \) the corestriction of \( \varphi \) to \( Y \), denoted by \( \varphi|_{1_Y} \); we say that \( \varphi \) is a restriction of \( \psi \), or equivalently that \( \psi \) is an extension of \( \varphi \), if \( \varphi = 1_Z \circ \psi \circ 1_X \). In addition to this, if \( U \) and \( V \) are classes, we let \( U \times_{\varphi, \psi} V \) be the (canonical) pullback of \((\varphi|_U, \psi|_V)\), i.e. the class \( \{(x, y) \in U \times V : x \in \text{dom}(\varphi), y \in \text{dom}(\psi) \text{ and } \varphi(x) = \psi(y)\} \).

Given two collections of classes \((X_i)_{i \in \mathbb{I}}\) and \((Y_i)_{i \in \mathbb{I}}\) indexed by a totally ordered set \( \mathbb{I} \), along with a family of functions \((\varphi_i : X_i \to Y_i)_{i \in \mathbb{I}}\), we let \((\varphi_i)_{i \in \mathbb{I}}\) indicate the map

---

Salvatore Tringali

13
\(\prod_{i \in \mathbb{I}} X_i \rightarrow \prod_{i \in \mathbb{I}} Y_i\) whose domain is the class \(\prod_{i \in \mathbb{I}} \text{dom}(\varphi_i)\) and which sends \((x_i)_{i \in \mathbb{I}}\) to \((\varphi_i(x_i))_{i \in \mathbb{I}}\); we may denote \(\langle \varphi_i \rangle_{i \in \mathbb{I}}\) by \(\langle \varphi_1, \ldots, \varphi_\alpha \rangle\) if \(\mathbb{I}\) is finite and \(\alpha := |\mathbb{I}| \neq 0\).

We set \([n] := \{1, \ldots, n\}\) for \(n \in \mathbb{N}^+\) and \([0] := \emptyset\). Lastly, for a class \(X\) and an equivalence relation \(R\) on \(X\), we use \(x\) mod \(R\) or \([x]_R\) for the equivalence class of an element \(x \in X\) in the quotient of \(X\) by \(R\), which we denote by \(X/R\).

4. Plots and parenthesizations

In this section, we formally introduce plots and related notions (in particular, parenthesizations and regular representations), and we derive a few basic results. As a matter of fact, we give two slightly different, but equivalent definitions of a plot, the second of which is the ‘official’ one used throughout.

4.1. Fundamental definitions. What is formally a plot? Which relation is there between plots, on the one hand, and categories, Mitchell’s semicategories, Gabriel’s quivers, etc., on the other? Answers to these questions are given below.

Definition 1. Given a class \(X\), we let a (partial) (binary) operation, or composition law, on \(X\) be any map \(\varphi : X \times X \rightarrow X\), and then we define the dual (operation) of \(\varphi\), here denoted by \(\varphi^{\text{op}}\), as the function \(X \times X \rightarrow X\) sending all and the only pairs \((x,y) \in X \times X\) such that \((y,x) \in \text{dom}(\varphi)\) to \(\varphi(y,x)\). We say that \(\varphi\) is:

1. left pre-associative if for all \(x,y,z \in X\) such that \((x,y),(y,z) \in \text{dom}(\varphi)\) it holds that \((\varphi(x,y),z) \in \text{dom}(\varphi)\) implies \((x,\varphi(y,z)) \in \text{dom}(\varphi)\), and whenever this is the case then \(\varphi(\varphi(x,y),z) = \varphi(x,\varphi(y,z))\).
2. right pre-associative if \(\varphi^{\text{op}}\) is left pre-associative.
3. pre-associative if it is both left and right pre-associative.
4. strongly associative if it is pre-associative and \((\varphi(x,y),z) \in \text{dom}(\varphi)\) for all \(x,y,z \in X\) such that \((x,y),(y,z) \in \text{dom}(\varphi)\).
5. associative if for all \(x,y,z \in X\) with \((x,y),(y,z) \in \text{dom}(\varphi)\) it holds \(\varphi(\varphi(x,y),z) = \varphi(x,\varphi(y,z))\) whenever \((\varphi(x,y),z),(x,\varphi(y,z)) \in \text{dom}(\varphi)\).
6. left dissociative if for all \(x,y,z \in X\) the following holds: If \((x,y) \in \text{dom}(\varphi)\) then \((\varphi(x,y),z) \in \text{dom}(\varphi)\) implies \((y,z) \in \text{dom}(\varphi)\), and then subsequently \((x,\varphi(y,z)) \in \text{dom}(\varphi)\) and \(\varphi(x,\varphi(y,z)) = \varphi(x,\varphi(y,z))\).
7. right dissociative if \(\varphi^{\text{op}}\) is left dissociative.
8. dissociative if it is both left and right dissociative.

Lastly, \(\varphi\) is possibly called a total (binary) operation on \(X\) if \(\varphi\) is a total function.

The interested reader is exhorted to familiarize with the list of properties above, for they will be often referred to in the sequel, and in fact used to single out sufficient conditions ensuring that some relevant facts from category theory carry over to plots under very mild (though possibly tricky) assumptions.
Definition 2. A plot $P$ is any 6-tuple $(C_0, C_1, s, t, c, i)$, where:

1. $C_0$ and $C_1$ are classes, whose elements are respectively referred to as objects (or vertices, or nodes) and morphisms (or arrows, or edges, or arcs);
2. $s$ and $t$ are functions $C_1 \to C_0$ which assign, to every arrow $f$, an object $A$, called its source, and an object $B$, called its target, respectively;
3. $c$ is a (partial) binary operation on $C_1$, whose domain is a (possibly empty) subclass of the (canonical) pullback $C_1 \times_{t,s} C_1$;
4. $i$ is a (partial) function $C_0 \to C_1$, referred to as the identity of $P$: The elements in $\text{dom}(i)$ are called (the) unital objects of $P$, those in $\text{im}(i)$ (the) local identities;

and the following axioms are satisfied:

- $(p.1)$ $s(c(f, g)) = s(f)$ and $t(c(f, g)) = t(g)$ for every $(f, g) \in \text{dom}(c)$;
- $(p.2)$ $s(i(A)) = t(i(A)) = A$ for all $A \in \text{dom}(i)$;
- $(p.3)$ $(f, i(A)), (i(A), g) \in \text{dom}(c)$ for every $A \in \text{dom}(i)$ and all $f, g \in C_1$ such that $t(f) = s(g) = A$, and also $c(f, i(A)) = f$ and $c(i(A), g) = g$.
- $(p.4)$ $i$ is maximal, in the sense that if $i : C_0 \to C_1$ is another function for which Axioms $(p.2)$-$(p.3)$ are satisfied then $\text{dom}(i) \subseteq \text{dom}(i)$.

We refer to the elements of $\text{dom}(c)$ as the composable pairs (or pairs of composable arrows) of $P$, and to $c$ as the composition (or composition law) of $P$.

We say that $P$ is: a quiver (or multidigraph) if $\text{dom}(c) = \emptyset$; monic (or posetal) if for all $A, B \in C_0$ there exists at most one arrow $f \in C_1$ such that $s(f) = A$ and $t(f) = B$; epic if for each $A \in C_0$ there exists $f \in C_1$ with $s(f) = A$ or $t(f) = A$; unital if $i$ is total; saturated if $\text{dom}(c) = C_1 \times_{t,s} C_1$; and associative/strongly associative/pre-associative/dissociative/left or right pre-associative/left or right dissociative if $c$ is.

Then, we let a magmoid be a saturated plot, a semigroupoid be a pre-associative plot, a semicategory (or semicat) a saturated semigroupoid (or, equivalently, a pre-associative magmoid), and a category (or cat) a unital semicategory.

Remark 1. For the sake of comparison, Gabriel’s quivers [22], Ehresmann’s neo-categories [19, Section 1], Mitchell’s semicategories [50, Section 4] and (ordinary) categories can be respectively identified with quivers, unital plots, semicategories and categories in the sense of Definition 2, by curtailing, if necessary, the composition law or the identity as appropriate (see Remark 2 for more discussion on this point). On another hand, composition graphs (respectively, precategories) in the sense of Schröder [58] are a special kind of plots (respectively, of pre-associative plots), where every object $A$ is assigned a morphism $u_A$ whose source and target are equal to $A$ (this arrow is still referred to as an identity in [58], though it needs not behave as such). The same is true also for Schröder’s semicategories [57], which are strongly associative plots of the form $(C_0, C_1, s, t, c, i)$, where every object $A$ has one associated arrow $u_A$ with $s(u_A) = t(u_A) = A$ such that $c(f, u_A) = f$ for all $f \in C_1$.
with \( t(f) = A \) and \((f, u_A) \in \text{dom}(c)\), and \( c(u_A, g) = g \) for all \( g \in C_1 \) with \( s(g) = A \) and \((u_A, g) \in \text{dom}(c)\).

**Remark 2.** From a formal point of view, the notions of quiver, multiplicative graph and semicategory from Definition 2 slightly differ from the ones available in the literature; cf., e.g., [22], [19, Section 1] and [19, Section 1], respectively. The main difference comes from the introduction of unital objects, which are even implicit to the theory of categories, but overshadowed by the global character, in that setting, of the identity (which is, in fact, “restricted” in the case of plots).

Unital objects make the transition from categories to Mitchell’s semicategories and weaker structures look “smoother” than it would otherwise, and yield the existence of an intermediate level of abstraction between the classical notions of functor and semifunctor (see Section 7). Furthermore, the idea of unital objects allows us to literally regard categories as a special kind of plots (a category in the sense of Definition 2 is precisely a category in the sense of [3, Definition 1.1]), to the effect that terminology and statements relating to plots are immediately applied to categories.

**Remark 3.** For all practical purposes and intents, we will occasionally refer to the objects of a plot \( P \) as \( P \)-objects (or \( P \)-nodes, or \( P \)-vertices), to the morphisms of \( P \) as \( P \)-morphisms (or \( P \)-arrows, \( P \)-arcs, or \( P \)-edges), and to the property of \( P \) of being epic, monic, pre-associative, strongly associative, or dissociative as epicity, monicity, pre-associativity, strong associativity, or dissociativity, respectively. Moreover, we introduce the adjective “plotal” and use it in the relation to plots in the very same way as “categorical” is used in relation to categories. Finally, we say that a collection of arrows in a plot are parallel if all of them have the same source and target.

**Remark 4.** While strongly associative functions are associative and pre-associative, and every left/right dissociative mapping is left/right pre-associative, there is no general implication between associativity, either strong or not, and dissociativity. However, the composition law of a saturated plot (e.g., a category) is strongly associative if and only if it is left/right dissociative if and only if it is left/right pre-associative, to the effect that the distinction among associativity, dissociativity, and pre-associativity is effective only in the “non-saturated case”.

Given a plot \( P = (C_0, C_1, s, t, c, i) \), it is immediate that every \( A \in C_0 \) has at most one local identity: For if \( A \in C_0 \) and \( i_1, i_2 \) are local identities on \( A \) then that \((i_1, i_2)\) is a composable pair of \( P \) by Axiom (p.2) and \( i_1 = c(i_1, i_2) = i_2 \). It follows, in view of Axiom (p.4), that \( i \) is entirely (and uniquely) determined by the datum of the 5-tuple \((C_0, C_1, s, t, c)\), which suggests the opportunity to redefine \( P \) by curtailing the 6-tuple \((C_0, C_1, s, t, c, i)\) to the 5-tuple \((C_0, C_1, s, t, c)\), while keeping the same terminology as in Definition 2. Thus, we are led to the following:

**Definition 3.** A plot \( P \) is any 5-tuple \((C_0, C_1, s, t, c)\), where:
1. \( C_0 \) and \( C_1 \) are classes, whose elements are respectively referred to as objects (or vertices, or nodes) and morphisms (or arrows, or edges, or arcs);

2. \( s \) and \( t \) are functions \( C_1 \to C_0 \) which assign, to every arrow \( f \), an object \( A \), called its source, and an object \( B \), called its target, respectively;

3. \( c \) is a (partial) binary operation on \( C_1 \), whose domain is a (possibly empty) subclass of the (canonical) pullback \( C_1 \times_{t,s} C_1 \);

and the following axiom is satisfied:

\[(q.1) \quad s(c(f, g)) = s(f) \quad \text{and} \quad t(c(f, g)) = t(g) \quad \text{for every} \quad (f, g) \in \text{dom}(c).\]

The identity of \( P \) is then the unique (partial) function \( i : C_0 \to C_1 \) for which:

\[(q.2) \quad s(i(A)) = t(i(A)) = A \quad \text{for all} \quad A \in \text{dom}(i);\]

\[(q.3) \quad (f, i(A)), (i(A), g) \in \text{dom}(c) \quad \text{for every} \quad A \in \text{dom}(i) \quad \text{and all} \quad f, g \in C_1 \quad \text{such that} \quad t(f) = s(g) = A, \quad \text{and furthermore} \quad c(f, i(A)) = f \quad \text{and} \quad c(i(A), g) = g.\]

\[(q.4) \quad i \text{ is maximal, in the sense that if} \quad \hat{i} : C_0 \to C_1 \text{ is another function for which} \quad \text{Axioms} \ (q.2)-(q.3) \text{ are satisfied then} \quad \text{dom}(i) \subseteq \text{dom}(i).\]

As for the rest, the other terminology introduced by Definition 2 is adapted in the obvious way (we may omit the details).

The reader is advised that, for the remainder of the paper, plots and related notions such as quivers, semicategories and categories will be intended, unless differently stated, only in the sense of Definition 3.

**Remark 5.** For the sake of simplicity, we adopt the convenient convention that, when a statement is made about a freshly introduced element involving the composition of two arrows \( f \) and \( g \) in a plot \( P = (C_0, C_1, s, t, c) \), then the statement is implicitly supposed to include the requirement that the composition is allowed, namely \( (f, g) \in \text{dom}(c) \). More in general, the same principle applies to any partial operation, as in the following example.

**Example 1.** Basic examples of plots are borrowed from classical algebra, for which we refer to [10, Sections I.1-I.4], the main difference being that we allow for partial operations. Other, and more interesting, examples will be given later (see, e.g., Section 5).

Specifically, a partial magma is here any pair \( A \) consisting of a (possibly empty) class \( A \), called the carrier (or the underlying class), and a (partial) binary operation \( \star \) on it; if \( A \) is a set, then \( A \) is termed small. The theory of partial magmas is distinct from the theory of total algebras (groups, semigroups, etc.) and the theory of their transformations; a systematic exposition of the subject can be found, e.g., in [48], where many relevant notions are however defined in a slightly different way than we do below (and magmas are called groupoids).

Given a partial magma \( A = (A, \circ) \), we abuse notation on a systematic basis and write \( x \in A \) to mean that \( x \) is an element of \( A \) in contexts or statements involving,
along with \( x \), the algebraic structure of \( A \) (the principle applies not only to elements, but also to subsets). E.g., we say that \( a \in A \) is left (respectively, right) cancellative if \( x \circ y = x \circ z \) (respectively, \( y \circ x = z \circ x \)) for \( y, z \in A \) implies \( y = z \). We set that \( A \) is trivial if \( A \) is a singleton, a partial semigroup if \( \circ \) is pre-associative (there is no general agreement on the notion of partial semigroup, but this is not so important here), and unital if there exists an element \( e \in A \), which is provably unique and termed the identity of \( A \), such that \( x \circ e = e \circ x = x \) for all \( x \in A \) (recall Remark 5). A unital partial semigroup is then called a partial monoid.

If \( A \) is unital and \( e \) is its identity, an element \( x \in A \) is then referred to as a unit if there exists \( \bar{x} \in A \) such that \( x \circ \bar{x} = \bar{x} \circ x = e \); if \( A \) is associative, then \( \bar{x} \) is uniquely determined and called the inverse of \( x \) (in \( A \)). We write \( A^\times \) for the (class of) units of \( A \) (the empty set, if \( A \) is not unital), and we say that \( A \) is a partial group if it is a partial monoid and \( A = A^\times \). On a similar note, \( A \) is called a partial quasigroup if both of the equations \( x \circ \xi_1 = z \) and \( \xi_2 \circ y = z \) have unique solutions \( \xi_1, \xi_2 \in A \) for all \( x, y, z \in A \) (every partial group is clearly a partial quasigroup); cf. [60, Ch. 1].

Then, we let (total) magmas, semigroups, etc. be, respectively, partial magmas, partial semigroups, etc. whose operation is a total function, and we give some examples of magmas and partial magmas which are, in general, neither unital nor associative:

(a) If \( A \) is an arbitrary class, then \( \circ \) can be the map \( (x, y) \mapsto x \), or the map \( (x, y) \mapsto y \); In the former case, \( A \) is called the left-zero magma of \( A \), while in the latter it is termed the right-zero magma of \( A \); cf. [33, Section 1.1, p. 3].

(b) If \( m, n \in \mathbb{N}^+ \) and \( A \) is the carrier of a semiring \( S \) with addition \( + \) and multiplication \( \cdot \), then \( \circ \) can be the map \( (x, y) \mapsto x^m + y^n \), in which case \( A \) is named the Fermat magma of \( S \). Here, a partial semiring is a triple \( S = (S, +, \cdot) \) such that \( (S, +) \) and \( (S, \cdot) \) are partial semigroups and \( \cdot \) distributes over \( + \) (cf. [25, p. 7]), in the sense that for all \( x, y, z \in S \) the following holds:

1. \( x \cdot (y + z) \) is defined if and only if \( xy + xz \) is, and then \( x \cdot (y + z) = xy + xz \);
2. \( (x + y) \cdot z \) is defined if and only if \( xz + yz \) is, and then \( (x + y) \cdot z = xz + yz \).

If \( S \) is a set, then \( S \) is said to be small, while it is called total, or simply a semiring, if + and \( \cdot \) are total functions. Moreover, we let a rng, that is “ring” without the ‘i’ of “identity”, be a semiring \( S = (S, +, \cdot) \) whose additive semigroup is a commutative group with identity \( 0_S \) and \( 0_S \cdot x = x \cdot 0_S = 0_S \) for all \( x \in S \). A ring is then a rng whose multiplicative semigroup is a monoid, and a field is a rng \( \mathbb{K} = (K, +, \cdot) \) such that \( (K \setminus \{0_K\}, \cdot) \) is a commutative group, with \( 0_K \) being the identity of \( (K, +) \).

(c) If \( A = \mathbb{R} \), then \( \circ \) can be the map \( (x, y) \mapsto x^y \) with domain \( \mathbb{R}^+ \times \mathbb{R} \), in which case \( A \) is named the exponential magma of \( \mathbb{R} \).
(d) If $A$ is the carrier of a partial group $\mathbb{G} = (G, \cdot)$, then $\circ$ can be the map sending $(x, y)$ to $x \cdot \hat{y}$, where $\hat{y}$ is the inverse of $b$ in $\mathbb{G}$. It is seen that $\bigcirc$ is a partial quasigroup: We call it the difference quasigroup of $\mathbb{G}$ if $\mathbb{G}$ is written additively, or the division quasigroup of $\mathbb{G}$ if it is written multiplicatively.

(e) If $A$ is the class of all sets, then $\circ$ can be either the map $(X, Y) \to X \setminus Y$, where $X \setminus Y$ is the relative complement of $Y$ in $X$, or the function $(X, Y) \to Y^X$, where $Y^X$ is the set of all mappings $X \to Y$.

(f) If $\mathcal{P} = (C_0, C_1, s, t, c)$ is a plot, then $\bigcirc$ can be the pair $(C_1, c)$, which we refer to as the compositive magma of $\mathcal{P}$ (a partial semigroup if $\mathcal{P}$ is a semicat).

Every partial magma $\mathbb{A} = (A, \circ)$ can be made into a plot $\mathcal{P} = (C_0, A, s, t, \circ)$ by assuming that $C_0 := \{\emptyset\}$, while $s$ and $t$ are the only possible mappings $C_1 \to C_0$. We refer to $\mathcal{P}$ as the canonical plot of $\mathbb{A}$, here denoted by $\text{Plt}(\mathbb{A})$. Trivially, $\text{Plt}(\mathbb{A})$ is saturated if and only if $\circ$ is total, while it is unital, associative, pre-associative, and so on if and only if $\mathbb{A}$ is, which ultimately shows that the above is a generalization of quite similar constructions relating to monoids, in the familiar setting of categories [3, Example 13, Section 1.4, p. 11], and semigroups, in the context of Mitchell’s semicategories [23, Example 2.1]. Notably, the process is also reversible: If $\mathcal{P} = (C_0, C_1, s, t, c)$ is the canonical plot of a magma $\mathbb{A}$ then $\mathbb{A}$ is precisely the compositive magma of $\mathcal{P}$.

Given a plot $\mathcal{P}$, we denote (the class of) the objects of $\mathcal{P}$ by $\text{Ob}(\mathcal{P})$ and (the class of) the arrows of $\mathcal{P}$ by $\text{hom}(\mathcal{P})$; we occasionally refer to $\text{Ob}(\mathcal{P})$ as the object-class of $\mathcal{P}$ and to $\text{hom}(\mathcal{P})$ as the hom-class of $\mathcal{P}$. As is usual in category theory, we write $A \in \mathcal{P}$ in place of $A \in \text{Ob}(\mathcal{P})$ and $f \in \mathcal{P}$ for $f \in \text{hom}(\mathcal{P})$ if it is clear that $A$ is an object and $f$ a morphism from the “ambient plot” $\mathcal{P}$.

**Example 2.** We let the empty quiver be the 5-uple $(\emptyset, 0, 1_\emptyset, 1_\emptyset, 1_\emptyset)$, and we say that a plot is non-empty if it is not the empty quiver, and discrete if its hom-class is empty. A discrete category is, instead, a unital plot whose arrows are only the local identities (which agrees with the usual definition; cf. [8, Example 1.2.6.c]).

We denote by $\text{sr}_{\mathcal{P}}$ and $\text{tr}_{\mathcal{P}}$ the maps taking a $\mathcal{P}$-arrow to its source and target, respectively. Then, given objects $A, B \in \mathcal{P}$ we write $(f : A \to B)_{\mathcal{P}}, (A \xrightarrow{f} B)_{\mathcal{P}}, (f : B \leftarrow A)_{\mathcal{P}},$ or $(B \xleftarrow{f} A)_{\mathcal{P}}$, like with categories, to mean that $f$ is a $\mathcal{P}$-morphism with $\text{sr}_{\mathcal{P}}(f) = A$ and $\text{tr}_{\mathcal{P}}(f) = B$, which is expressed simply by $f : A \to B$, $A \xrightarrow{f} B$, $f : B \leftarrow A$, or $B \xleftarrow{f} A$ if $\mathcal{P}$ is implied from the context; in such a case, we possibly say that $f$ is a $\mathcal{P}$-arrow from $A$ to $B$, or write that $f$ is a $\mathcal{P}$-morphism $A \to B$. Also, we let $\text{hom}_{\mathcal{P}}(A, B)$ represent the class of all arrows $(f : A \to B)_{\mathcal{P}},$ and use $\text{hom}_{\mathcal{P}}(A)$ in place of $\text{hom}_{\mathcal{P}}(A, B)$ when $A = B$. 

We use \( \diamond P \) for the binary operation of \( P \), and we adopt the diagrammatic notation for the composition of morphisms, unless otherwise specified (as in the case of the set-theoretic composition of functions; see Section 3). Namely, taking Remark 5 in mind, we write \( f \diamond P g \) for the image of \( (f, g) \) under \( \diamond P \), and we refer to \( f \diamond P g \) as the composition of \( f \) with \( g \) (in \( P \)), or the product of \( f \) by \( g \) (in \( P \)). Lastly, we denote the identity map of \( P \) by \( \text{id}_P \). In most situations, the subscript ‘\( P \)’ is removed from everywhere, and we may write \( fg \) in place of \( f \diamond P g \) for easiness, especially if this doesn’t lead to confusion.

4.2. Playing around with parentheses. In this section, we let \( P \) be an arbitrary plot, unless differently specified, and we denote by \( \diamond \) its composition. If \( n \) is a positive integer and \( P \) is a semicategory, then for any \( n \)-tuple \( (f_1, \ldots, f_n) \) of \( P \)-morphisms with \( t(f_i) = s(f_{i+1}) \) for each \( i \in [n-1] \) the arrow \( f_1 \cdots f_n \) is always well-defined: There is no need for parentheses to indicate a specific order of evaluation for \( n \geq 3 \), because associativity makes everything be unambiguous (see Remark 4), and there is also no need for further conditions on the composability of an arrow of the form \( f_1 \cdots f_j \) with another of the form \( f_{j+1} \cdots f_k \), for \( i, j, k \in \mathbb{N}^+ \) and \( i \leq j < k \leq n \), because all of these compositions are then made possible by the very fact that \( P \) is saturated. But what about the case when \( P \) is not strongly associative or not full? The pursuit of an answer to this question leads to the notion of parenthesization; cf. [10, Section I.2, Definition 4].

**Definition 4.** Following [10, Section I.7.1], denote by \( \emptyset = (O, \ast) \) the free magma on the set \( \{ \emptyset \} \). Then, let \( \text{Prs}_1 := \{1_O\} \) and, for an integer \( n \geq 2 \), define recursively \( \text{Prs}_n \) as the set of all functions \( \varphi : O^* \rightarrow O \) for which there exist \( k \in [n-1] \) and \( \varphi_1 \in \text{Prs}_k \), \( \varphi_2 \in \text{Prs}_{n-k} \) such that \( \varphi = \varphi_1 \ast \varphi_2 \), i.e.

\[
\varphi(x_1, \ldots, x_n) = \varphi_1(x_1, \ldots, x_k) \ast \varphi_2(x_{k+1}, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n \in O \). We call \((\varphi_1, \varphi_2)\) a (binary) splitting of \( \varphi \), or say that \( \varphi \) splits into the product of \( \varphi_1 \) by \( \varphi_2 \). The members of the set \( \text{Prs} := \bigcup_{n=1}^{\infty} \text{Prs}_n \) are then termed fundamental parenthesizations, and given \( n \in \mathbb{N}^+ \) we say that \( \varphi \in \text{Prs}_n \) is a fundamental \( n \)-parenthesization, or a fundamental parenthesization of length \( n \), if \( \varphi \in \text{Prs}_n \).

By the basic properties of free magmas, every fundamental parenthesization \( \varphi \) of length \( n \geq 2 \) has a unique splitting \((\varphi_1, \varphi_2)\), hence called the (binary) splitting of \( \varphi \). Thus we are allowed for the next:

**Definition 5.** Given an integer \( n \geq 1 \) and a fundamental \( n \)-parenthesization \( \varphi \), we define inductively the function \( \varphi[P] : \text{hom}(P)^{\times n} \rightarrow \text{hom}(P) \) as follows: If \( n = 1 \), then \( \varphi[P] \) is the identity map of \( \text{hom}(P) \); otherwise, \( \varphi[P] := \varphi_1[P] \circ \varphi_2[P] \), where
$(\varphi_1, \varphi_2)$ is the binary splitting of $\varphi$, which is just a shorthand notation to say that
\[
\varphi[\mathcal{P}](f_1, \ldots, f_n) := \varphi_1[\mathcal{P}](f_1, \ldots, f_k) \circ \varphi_2[\mathcal{P}](f_{k+1}, \ldots, f_n)
\]
for all $n$-tuples $(f_1, \ldots, f_n) \in \text{dom}(\varphi[\mathcal{P}])$, with $\text{dom}(\varphi[\mathcal{P}])$ being the class of all and the only $n$-tuples $(f_1, \ldots, f_n)$ of $\mathcal{P}$-morphisms such that
1. $(f_1, \ldots, f_k) \in \text{dom}(\varphi_1[\mathcal{P}])$ and $(f_{k+1}, \ldots, f_n) \in \text{dom}(\varphi_2[\mathcal{P}])$;
2. $(\varphi_1[\mathcal{P}](f_1, \ldots, f_k), \varphi_2[\mathcal{P}](f_{k+1}, \ldots, f_n)) \in \text{dom}(\circ)$.

We call $\varphi[\mathcal{P}]$ an $n$-parenthesization of $\mathcal{P}$, or a parenthesization of $\mathcal{P}$ of length $n$.

We use parenthesizations in the case of a non-associative plot to establish precedence rules in the order of compositions, when more than two morphisms are in play and we want them to be composed all together in one expression.

**Remark 6.** For the record, let us mention that “Prs” comes from the verb “to parse”, and not from the noun “parenthesis”.

Let $\varphi$ be a fundamental $n$-parenthesization ($n \in \mathbb{N}^+$). For a tuple $(f_1, \ldots, f_n) \in \text{dom}(\varphi[\mathcal{P}])$ we usually write $\varphi[\mathcal{P}](f_1, \ldots, f_n)$ as $(f_1 \odot \cdots \odot f_n)_{\varphi[\mathcal{P}]}$, which is then further simplified, if there is no danger of confusion, according to the conventions introduced at the end of Section 4.1. On another hand, we use $f_1 \odot \cdots \odot f_n$ for $(f_1 \circ \cdots \circ f_n)_{\varphi[\mathcal{P}]}$ if $n \leq 2$ or $\mathcal{P}$ is associative, since then the latter expression does not actually depend on $\varphi$, and we write $(f_1 \circ \cdots \circ f_n)_{\varphi[\mathcal{P}]}$ as $(f^n)_{\varphi[\mathcal{P}]}$ if $f_1 = \cdots = f_n =: f$. Moreover, given $S_1, \ldots, S_n \subseteq \text{hom}(\mathcal{P})$ we write $(S_1 \odot \cdots \odot S_n)_{\varphi[\mathcal{P}]}$ for
\[
\{(f_1 \circ \cdots \circ f_n)_{\varphi[\mathcal{P}]} : (f_1, \ldots, f_n) \in \text{dom}(\varphi[\mathcal{P}]) \cap (S_1 \times \cdots \times S_n)\}.
\]
In particular, we use $(S^n)_{\varphi[\mathcal{P}]}$ for $(S_1 \circ \cdots \circ S_n)_{\varphi[\mathcal{P}]}$ when $S_1 = \cdots = S_n =: S$ and, like with the case of morphisms, we write $S_1 \circ \cdots \circ S_n$ in place of $(S_1 \circ \cdots \circ S_n)_{\varphi[\mathcal{P}]}$ if $\mathcal{P}$ is associative or $n \leq 2$. Additionally, if $S_i = \{f_i\}$ for some $i$, then we may abuse notation and replace $S_i$ with $f_i$ in the above expressions.

Based on this notation, we can state and prove the following proposition (which describes the behaviour of parenthesizations with respect to composition).

**Proposition 1.** Let $(O, \odot)$ denote the free magma on the set $\{\emptyset\}$, and let $\varphi$ be a fundamental parenthesization of length $n$ ($n \in \mathbb{N}^+$). Also, for each $i \in [n]$ let $\omega_i$ be a fundamental parenthesization of length $\ell_i$ ($\ell_i \in \mathbb{N}^+$), and set $L := L_0 + \ell_n$. Then the function
\[
\tilde{\varphi} := \varphi \circ \langle \omega_1 \circ p_1, \ldots, \omega_n \circ p_n \rangle,
\]
where $p_i$ is, for each $i \in [n]$, the projection
\[
O^\times L \to O^\times L_i : (x_1, \ldots, x_L) \mapsto (x_{L_i+1}, \ldots, x_{L_i+\ell_i}),
\]
is a fundamental parenthesization of length $L$. Moreover, it holds that

$$
\tilde{\varphi}[P] = \varphi[P] \circ \langle \omega_1[P] \circ q_1, \ldots, \omega_n[P] \circ q_n \rangle,
$$

where $q_i$ is, for each $i \in [n]$, the projection

$$
\text{hom}(P)^{\times L} \to \text{hom}(P)^{\times L_i} : (f_1, \ldots, f_L) \mapsto (f_{L_{i+1}}, \ldots, f_{L_{i+L_i}}).
$$

**Proof.** We proceed by strong induction on $n$. The claim is trivial for the base case $n = 1$. If, on the other hand, $n \geq 2$ then $\varphi = \varphi_1 \circ \varphi_2$, where $\varphi_1$ and $\varphi_2$ are, respectively, fundamental parenthesizations of lengths $k$ and $n - k$ for some $k \in [n - 1]$. This implies that $\tilde{\varphi} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$, where

$$
\tilde{\varphi}_1 := \varphi_1 \circ \langle \omega_i \circ p_i \rangle_{i \in [k]}, \quad \tilde{\varphi}_2 := \varphi_2 \circ \langle \omega_{k+i} \circ p_{k+i} \rangle_{i \in [n-k]}.
$$

Thus $\tilde{\varphi}$ is a fundamental parenthesization, since $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are such by the inductive hypothesis. Furthermore, $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ is the (binary) splitting of $\tilde{\varphi}$, to the effect that

$$
\tilde{\varphi}[P] := \tilde{\varphi}_1[P] \circ \tilde{\varphi}_2[P],
$$

where (again by the inductive hypothesis)

$$
\tilde{\varphi}_1[P] = \varphi_1[P] \circ \langle \omega_i[P] \circ q_i \rangle_{i \in [k]}, \quad \tilde{\varphi}_2[P] = \varphi_2[P] \circ \langle \omega_{k+i}[P] \circ q_{k+i} \rangle_{i \in [n-k]}.
$$

Considering that $\varphi[P] := \varphi_1[P] \circ \varphi_2[P]$, this completes the proof. \hfill \Box

We conclude with the notion of compositive subclass (which is, in fact, an abstraction of the notion of multiplicative subset, say, from the theory of magmas).

**Definition 6.** We say that a subclass $\mathcal{M}$ of $\text{hom}(P)$ is closed under the composition law of $P$, or simply that $\mathcal{M}$ is a compositive subclass of $P$, if $f \circ g \in \mathcal{M}$ whenever $f, g \in \mathcal{M}$ and $(f, g)$ is a composable pair of $P$.

The following property is straightforward by induction (we omit the details):

**Proposition 2.** Let $\mathcal{M}$ be a compositive subclass of $P$, and let $n$ be a positive integer. Then, for every fundamental parenthesization $\varphi$ of length $n$ it holds that $(f_1 \circ \cdots \circ f_n)\varphi[P] \in \mathcal{M}$ whenever $(f_1, \ldots, f_n) \in \text{dom}(\varphi[P])$.

Parenthesizations are an essential technical passage for dealing with non-associative structures like plots.

4.3. **Duality and regular representations.** Given a plot $P = (C_0, C_1, s, t, c)$ the 5-uple $(C_0, C_1, t, s, c^{\text{op}})$ is itself a plot, which we call the dual (plot) of $P$, and we write as $P^{\text{op}}$; cf. [8, Section 1.10]. Note that $(P^{\text{op}})^{\text{op}} = P$, which informally means that “dualization is involutive”, and $\text{id}_P = \text{id}_{P^{\text{op}}}$; cf. [3, Section 1.6(2)].

**Remark 7.** The dual of a left/right pre-associative plot is right/left pre-associative, and the same applies to any other form of associativity specified by Definition 1. In a similar vein, the dual of a unital/saturated plot is unital/saturated. It follows, in particular, that the dual of a semicategory/category is itself a semicategory/category.
Remark 8. Even if a plot and its dual have the same elements as arrows, it is sometimes useful to distinguish between a morphism \( f \in P \) and its “clone” in \( P^{op} \), by denoting the latter by \( f^{op} \).

Thus the duality principle of category theory carries over to plots immediately: It is sufficient to take [8, Metatheorem 1.10.2] and replace each occurrence of the word “category” with the term “plot”. Like with categories, this is quite useful, in the first place because it allows to cut by half the number of definitions, theorems, and other mathematical statements, as in the case of the following:

Definition 7. If \( A, B \) are objects in a plot \( P \), we define

\[
\text{hom}_P(-, B) := \{ f \in \text{hom}(P) : \text{tr}_P(f) = B \}
\]

and \( \text{hom}_P(A, -) := \text{hom}_{P^{op}}(-, A) \). Moreover, if \( f : A \to B \) is an arrow in \( P \), we let \( \text{hom}_P(-, A_{off}) \) be the class of all arrows \( g \in \text{hom}_P(-, A) \) such that \((g, f)\) is a composable pair of \( P \), and we set \( \text{hom}_P(B_{off}, -) := \text{hom}_{P^{op}}(-, B_{off}) \).

We denote by \( \varrho_P(f) \) the right regular representation of \( f \) (in \( P \)), i.e. the map

\[
\text{hom}_P(-, A_{off}) \to \text{hom}_P(-, B) : g \mapsto g \circ_P f,
\]

and we let the left regular representation of \( f \) (in \( P \)) be the right regular representation of \( f \) as an arrow of \( P^{op} \), which is written here as \( \lambda_P(f) \), and is explicitly given by the function \( \text{hom}_P(B_{off}, -) \to \text{hom}_P(A, -) : g \mapsto f \circ_P g \).

Regular representations are a common motive in mathematics, and in the special case of plots they are somewhat reminiscent of the Yoneda lemma. In fact, there is more than a vague analogy in this respect, but we will not discuss the question here; for the moment, let us instead concentrate on the following lemma.

Lemma 1. Let \( (f : A \to B, g : B \to C) \) be a composable pair of \( P \).

(i) If \( P \) is right dissociative, then \( \text{dom}(\lambda_P(f) \circ \lambda_P(g)) = \text{hom}_P(C_{off}, -) \subseteq \text{hom}_P(C_{off}, f) \) and accordingly \( \varrho_P(f \circ g)(h) = (\varrho_P(g) \circ \varrho_P(f))(h) \) for \( h \in \text{hom}_P(-, A_{off}) \).

(ii) If \( P \) is right pre-associative, it holds that

\[
\text{dom}(\lambda_P(f) \circ \lambda_P(g)) = \text{hom}_P(C_{off}, -) \subseteq \text{hom}_P(C_{off}, f)
\]

and then \( \lambda_P(f \circ g)(h) = (\lambda_P(f) \circ \lambda_P(g))(h) \) for \( h \in \text{dom}(\lambda_P(f) \circ \lambda_P(g)) \).

(iii) If \( P \) is left dissociative, then \( \text{hom}_P(C_{off}, -) \subseteq \text{hom}_P(C_{off}, (-, \lambda_P(g)) \) and moreover \( \lambda_P(f \circ g)(h) = (\lambda_P(f) \circ \lambda_P(g))(h) \) for \( h \in \text{hom}_P(C_{off}, -) \).

(iv) If \( P \) is left pre-associative, it happens that

\[
\text{dom}(\varrho_P(g) \circ \varrho_P(f)) = \text{hom}_P(-, A_{off}) \subseteq \text{hom}_P(-, A_{off}),
\]

and hence \( \varrho_P(f \circ g)(h) = (\varrho_P(g) \circ \varrho_P(f))(h) \) for \( h \in \text{dom}(\varrho_P(g) \circ \varrho_P(f)) \).
Proof. (i) For brevity’s sake, set \( r_{fg} := \varrho_P(f \circ g) \), \( r_f := \varrho_P(f) \), and \( r_g := \varrho_P(g) \), and pick \( h \in \text{hom}_P(-, A \downarrow f \circ g) \), so that \((h, f \circ g)\) is a composable pair of \( P \). Then, by right dissociativity, the same is true for \((h, f)\), and then also for \((h \circ f, g)\), and in addition to this \((h \circ f) \circ g = h \circ (f \circ g)\). In other words, we have that \( h \in \text{dom}(r_g \circ r_f) \) and \((r_g \circ r_f)(h) = r_f(h) \circ g = r_{fg}(h)\), which ultimately proves the first part of the claim.

(ii) Let \( \ell_{fg} := \lambda_P(f \circ g) \), \( \ell_f := \lambda_P(f) \), and \( \ell_g := \lambda_P(g) \), and pick a morphism \( h \) in \( \text{dom}(\lambda_P(f) \circ \lambda_P(g)) \), in such a way that \( g \circ h \) is well-defined and \((f, g \circ h)\) is a pair of composable \( P \)-arrows. This yields that \( \text{dom}(\lambda_P(f) \circ \lambda_P(g)) = \text{hom}_P(C_{fg}, -) \). Furthermore, since \( P \) is right pre-associative, \((f \circ g, h)\) is composable too, namely \( h \in \text{hom}_P(C_{fg}, -) \), and \( \ell_{fg}(h) = (f \circ g) \circ h = f \circ (g \circ h) = f \circ \ell_g(h) = (\ell_f \circ \ell_g)(h) \).

(iii) and (iv) follow at once from the above (by duality and Remark 7).

The above will be greatly useful in Section 6. However, we will profit from (the language of) regular representations already in Section 4.4, in regard to the problem of sizing a plot (which is why we introduced them here). In this respect, we have the following:

**Proposition 3.** Let \( P \) be a plot. Then

\[
\text{hom}_P(A, -) \cap \text{hom}_P(B, -) = \text{hom}_P(-, A) \cap \text{hom}_P(-, B) = \emptyset
\]

for all \( A, B \in \text{Ob}(P) \) with \( A \neq B \), and

\[
\text{hom}(P) = \bigcup_{A \in \text{Ob}(P)} \text{hom}(A, -) = \bigcup_{B \in \text{Ob}(P)} \text{hom}(-, B).
\]

**Proof.** The proof is straightforward, and we may omit the details.  

4.4. **Sizing a plot.** Like with categories (see, e.g., [3, Section 1.8]), one concern about plots is related to their size. In the categorical case, this is completely determined by the arrows solely, which is however no longer true for plots, due to the possible presence of isolated objects, namely objects which are neither the source nor the target of any morphism (a more formal definition is given below). Through this section, we let \( P \) be an arbitrary plot.

**Definition 8.** We say that \( P \) is \( \Omega \)-small (or simply small) if \( \text{Ob}(P) \cup \text{hom}(P) \) is a set, and \( \Omega \)-large (or simply large) otherwise. Moreover, \( P \) is called locally \( \Omega \)-small (or simply locally small) if \( \text{hom}_P(A, B) \) is a set for all \( A, B \in \text{Ob}(P) \), while it is termed a one-object, two-object, etc. (respectively, one-arrow, two-arrow, etc.) plot if it has one, two, etc. objects (respectively, arrows). Lastly, \( P \) is termed finite if both \( \text{Ob}(P) \) and \( \text{hom}(P) \) are finite sets, and infinite otherwise.

We look for sufficient conditions ensuring that a plot is small, an obvious remark in this respect being that a plot is small/locally small if and only if the same is true
for its dual. For our purposes, we need first to adapt the notion of degree from the theory of finite multidigraphs, as shown in a while.

To start with, let $\text{Card}(\Omega)$ be the class of all cardinal numbers relative to elements of the universe $\Omega$, equipped with the ordinary addition $+$, multiplication $\cdot$ and total order $\leq$. We extend $+$, $\cdot$ and $\leq$ to $\text{Card}_\infty(\Omega) := \text{Card}(\Omega) \cup \{\Omega\}$ by taking

1. $\Omega + \kappa := \kappa + \Omega := \Omega$ and $\kappa \leq \Omega$ for all $\kappa \in \text{Card}_\infty(\Omega)$;
2. $\Omega \cdot 0 := 0 \cdot \Omega := 0$ and $\Omega \cdot \kappa := \kappa \cdot \Omega := \Omega$ for every $\kappa \in \text{Card}_\infty(\Omega) \setminus \{0\}$.

Here, as is usual, we abuse notation by writing the extended operations and ordering with the same symbols as those used for $\text{Card}(\Omega)$. Then, for a class $X$ we use $\|X\|_\Omega$, or simply $\|X\|$, for the relative cardinality of $X$ over $\Omega$, which is set equal to $|X|$ if $X \in \Omega$, and to $\Omega$ otherwise. We refer to an element of $\text{Card}_\infty(\Omega)$ as an extended cardinal (over $\Omega$).

**Remark 9.** In the sequel, unless otherwise specified, every statement involving extended cardinals and sums, products and orderings of extended cardinals is implicitly referred to the structure of $\text{Card}_\infty(\Omega)$, as given above. This will be especially the case with minima, maxima and related notions (e.g., suprema).

**Remark 10.** The notation that we are using here for the relative cardinality of a set is reminiscent of the typical one for norms. This is not by chance, absolutely not, for we will show in a forthcoming paper, as mentioned before in the introduction, that any measure can be understood as a particular norm, which especially applies, of course, to the counting measure of a set.

**Definition 9.** Given an object $A \in \mathcal{P}$, we set

$$\deg_{\mathcal{P}}(A) := \| \text{hom}_{\mathcal{P}}(-, A) \|,$$

where $\deg_{\mathcal{P}}(A)$ and $\deg_{\mathcal{P}}(A)$, respectively, the indegree and the outdegree of $A$ in $\mathcal{P}$. Then, we define

$$\deg_{\mathcal{P}}(A) := \deg_{\mathcal{P}}(A) + \deg_{\mathcal{P}}(A),$$

and refer to $\deg_{\mathcal{P}}(A)$ as the degree of $A$ in $\mathcal{P}$. In particular, we say that $A$ is

1. isolated (in $\mathcal{P}$), or an isolated object (of $\mathcal{P}$), if $\deg_{\mathcal{P}}(A) = 0$;
2. an object of small degree (in $\mathcal{P}$) if $\deg_{\mathcal{P}}(A) < \Omega$.

Lastly, $\mathcal{P}$ is said to have small degree if all of its objects have small degree.

Suppose $X$ is a class. For a map $f : X \to \text{Card}_\infty(\Omega)$, we want to give a formal meaning to the “sum” $\sum_{x \in X} f(x)$. For assume first that $f(x) \neq 0$ for every $x \in X$.

1. If $X$ is empty, then $\sum_{x \in X} f(x) := 0$.
2. If $X$ is finite and non-empty, then $\sum_{x \in X} f(x) := f(x_1) + \cdots + f(x_n)$, where $n := |X|$ and $(x_i)_{i \in [n]}$ is any numbering of $X$. 
3. If $X$ is infinite, then $\sum_{x \in X} f(x) := \max(\sup_{x \in X} f(x), \|X\|)$; this definition is well-posed, for $\text{Card}_\infty(\Omega)$ is well-ordered with respect to $\leq$ (see, e.g., [30]). Finally, if $f^{-1}(0)$ is non-empty, then we let $\sum_{x \in X} f(x) := \sum_{x \in X \setminus f^{-1}(0)} f(x)$. With this notation in hand, we have the degree sum formula below (cf. [7, Section I.1, p. 4]), in whose proof we need the following:

Proposition 4. Let $\text{Ob}_{\text{in}}(P)$ and $\text{Ob}_{\text{out}}(P)$ denote the subclasses of $\text{Ob}(P)$ whose elements are all and the only objects $A$ for which $\deg_{\text{P}, \text{in}}(A) \neq 0$ and $\deg_{\text{P}, \text{out}}(A) \neq 0$, respectively. In addition to this, define

$$\sigma_{\text{in}} := \sup_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{in}}(A), \quad \sigma_{\text{out}} := \sup_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{out}}(A),$$

and let $\sigma_{\text{min}} := \min(\sigma_{\text{in}}, \sigma_{\text{out}})$ and $\sigma_{\text{max}} := \max(\sigma_{\text{in}}, \sigma_{\text{out}})$. Then

$$\max(\sigma_{\text{max}}, \|\text{Ob}_{\text{in}}(P)\|, \|\text{Ob}_{\text{out}}(P)\|) \leq \|\text{hom}(P)\| \leq \sigma_{\text{min}} \cdot \|\text{Ob}(P)\|.$$

(N.B.: The supremum of the empty set is assumed equal to the cardinal zero.)

Proof. It is straightforward by Proposition 3, using the axiom of choice and the fact that a class $X$ is proper if and only if $\alpha \leq \|X\|$ for every $\alpha \in \text{Card}(\Omega)$. \hfill $\square$

Theorem 1 (Degree sum formula). It holds that

$$\sum_{A \in \text{Ob}(P)} \deg_{P}(A) = \sum_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{in}}(A) + \sum_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{out}}(A) = 2 \cdot \|\text{hom}(P)\|.$$

Proof. We freely use the notation of Proposition 4, and assume without loss of generality that $\deg_{P}(A) \neq 0$ for all $A \in \text{Ob}(P)$, which implies that $\sigma_{\text{in}} \cdot \sigma_{\text{out}} \neq 0$, and accordingly $\sigma_{\text{min}} \neq 0$. We proceed by a case-by-case analysis.

If $\text{Ob}(P)$ is a finite set, then the claim follows at once from Proposition 3 and the basic properties (namely, associativity and commutativity) of the operations of $\text{Card}_\infty(\Omega)$. Thus we restrict to the case when $\text{Ob}(P)$ is infinite. Then

$$\|\text{Ob}_{\text{in}}(P)\| = \|\text{Ob}(P)\| \quad \text{or} \quad \|\text{Ob}_{\text{out}}(P)\| = \|\text{Ob}(P)\|,$$

since $P$ has no isolated objects, and hence $\text{Ob}(P) = \text{Ob}_{\text{in}}(P) \cup \text{Ob}_{\text{out}}(P)$. So we get from Proposition 4 that $\text{hom}(P)$ cannot be a finite set, and indeed

$$\max(\sigma_{\text{max}}, \|\text{Ob}(P)\|) \leq \|\text{hom}(P)\| = 2 \cdot \|\text{hom}(P)\|. \quad (2)$$

Now, let us define $\sigma := \sup_{A \in \text{Ob}(P)} \deg_{P}(A)$, and also

$$\Xi := \sum_{A \in \text{Ob}(P)} \deg_{P}(A), \quad \Xi_{\text{in}} := \sum_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{in}}(A), \quad \Xi_{\text{out}} := \sum_{A \in \text{Ob}(P)} \deg_{\text{P}, \text{out}}(A).$$

If $\sigma \leq \|\text{Ob}(P)\|$, then (1), (2) and Proposition 4 yield that $\Xi = \Xi_{\text{in}} + \Xi_{\text{out}} = \|\text{Ob}(P)\| = 2 \cdot \|\text{hom}(P)\|$, and we are done. Otherwise, $\sigma = \sigma_{\text{max}} = \sigma_{\text{in}} + \sigma_{\text{out}},$
for \( \sigma_{\text{max}} \leq \sigma \leq \sigma_{\text{in}} + \sigma_{\text{out}} \) and \( \sigma \) is either \( \Omega \) or an infinite cardinal. Therefore, we see by (2) and another application of Proposition 4 that \( \Xi = \Xi_{\text{in}} + \Xi_{\text{out}} = \sigma_{\text{max}} = 2 \cdot \| \hom(P) \| \).

**Corollary 1.** Let \( P \) be a plot. If \( P \) is epic and \( \hom(P) \) is a set, or \( P \) has small degree and \( \Ob(P) \) is a set, then \( P \) is small.

**Proof.** Suppose first that \( P \) is epic and \( \hom(P) \) is a set. If \( \Ob(P) \) is also a set, then we have nothing to prove; otherwise, using that \( P \) is epic, \( \deg_P(A) \neq 0 \) for every \( A \in \Ob(P) \), with the result that \( \| \Ob(P) \| \leq 2 \cdot \| \hom(P) \| \) by the degree sum formula, and \( P \) is small.

Now assume that \( \Ob(P) \) is a set and \( P \) has small degree, so that \( \deg_P(A) < \Omega \) for every \( A \in \Ob(P) \). Since non-empty subsets of cardinal numbers are bounded in \( \text{Card}(\Omega) \), there then exists a cardinal \( \alpha \) such that \( \deg_P(A) \leq \alpha \) for every \( A \in \Ob(P) \). It follows, again from the degree sum formula, that \( 2 \cdot \| \hom(P) \| < \Omega \). Namely, \( \hom(P) \) is a set, and our proof is complete. \( \square \)

In particular, Corollary 1 gives that a category is small if and only if its hom-class is a set (a category is a unital plot, and all unital plots are clearly epic). On another hand, we observe that the first part of the same corollary (namely, the one relative to epicity) is totally trivial in the categorical case, where the availability of a local identity for each object saves the use of the axiom of choice.

This said, let us show how plots can be used to abstract (basic) facts from the “additive” theory of groups, as in the following (cf. [63, Lemma 2.1]):

**Proposition 5.** Let \( X, Y, Z_1, Z_2 \) be classes with \( Z_1 \subseteq Z_2 \). Then:

(i) \( X \circ_P Y = Y \circ_{P^{\op}} X \), and hence \( |X \circ_P Y| = |Y \circ_{P^{\op}} X| \).

(ii) \( Y \circ_P Z_1 \subseteq Y \circ_P Z_2 \), and hence \( |Y \circ_P Z_1| \leq |Y \circ_P Z_2| \).

Moreover, if \( X_1, \ldots, X_n \) are classes and \( \varphi \) is a fundamental parentheses of length \( n \) (\( n \in \mathbb{N}^+ \)), then \( |(X_1 \circ_P \cdots \circ_P X_n)_{\varphi[P]}| \leq \prod_{i=1}^n |X_i| \).

**Proof.** Points (i) and (ii) are straightforward, so we only prove the last assertion. For set \( X := (X_1 \circ_P \cdots \circ_P X_n)_{\varphi[P]} \). For \( n = 1 \) the claim is obvious, and for \( n = 2 \) it boils down to the fact that \( |X_1 \circ_P X_2| \leq |X_1 \times X_2| \). So assume \( n \geq 3 \) and let the statement hold for \( k \in \llbracket n - 1 \rrbracket \). If \((\varphi_1, \varphi_2)\) is the binary splitting of \( \varphi \) and \( \varphi_1 \) is of length \( k \), then

\[
X = (X_1 \circ_P \cdots \circ_P X_k)_{\varphi_1[P]} \circ_P (X_{k+1} \circ_P \cdots \circ_P X_n)_{\varphi_2[P]}.
\]

It follows from the induction basis that

\[
|X| \leq |(X_1 \circ_P \cdots \circ_P X_k)_{\varphi_1[P]}| \cdot |(X_{k+1} \circ_P \cdots \circ_P X_n)_{\varphi_2[P]}|,
\]

and then the inductive hypothesis gives that \( |X| \leq \prod_{k=1}^n |X_i| \). \( \square \)
Some other “combinatorial” properties of plots will be discussed later, with the introduction of monos and epis in Section 6.

4.5. **Identities and neutral arrows.** Through this section, $\mathbf{P}$ is a fixed plot, whose composition we denote by $\circ$. For an object $A \in \mathbf{P}$ it may well happen that there exists an arrow $\alpha : A \to A$ such that $\alpha \circ f = f$ for every $f \in \text{hom}_\mathbf{P}(A_\downarrow \alpha, -)$, but $\alpha$ is not a local identity, because it is not true that $g \circ \alpha = g$ for all $g \in \text{hom}_\mathbf{P}(-, A_\downarrow \alpha)$, or because $\text{hom}_\mathbf{P}(A_\downarrow \alpha, -) \nsubseteq \text{hom}_\mathbf{P}(A, -)$; it is the case, e.g., with the dual of the canonical plot of a non-trivial left-zero magma, where any arrow is a left identity in the sense of the following:

**Definition 10.** We say that an arrow $\alpha : A \to A$ of $\mathbf{P}$ is

1. left neutral if $\alpha \circ f = f$ for every $f \in \text{hom}_\mathbf{P}(A_\downarrow \alpha, -)$.
2. right neutral if it is left neutral in $\mathbf{P}^{\text{op}}$.
3. neutral if it is both left and right neutral.
4. a left identity if $\alpha$ is left neutral and $\text{hom}_\mathbf{P}(A_\downarrow \alpha, -) = \text{hom}_\mathbf{P}(A, -)$.
5. a right identity if it is a left identity of $\mathbf{P}^{\text{op}}$.

Then, an object $A \in \mathbf{P}$ is called a left/right unital object if there exists at least one left/right identity of $\mathbf{P}$ from $A$ to $A$.

All of these notions trivialize on the level of categories, as implied by the next proposition (cf. also the comments to Definition 3):

**Proposition 6.** The following condition hold:

(i) Every left/right identity of $\mathbf{P}$ is left/right neutral, and the converse of this implication is also true when $\mathbf{P}$ is saturated.

(ii) If $A$ is a left and right unital object of $\mathbf{P}$, and $\ell_A$ and $r_A$ are, respectively, a left and a right identity of $A$, then $A$ is unital and $\ell_A = r_A = \text{id}_\mathbf{P}(A)$. Viceversa, every unital object of $\mathbf{P}$ is left and right unital.

(iii) Let $\alpha : A \to A$, $\beta : B \to B$ be left/right neutral arrows of $\mathbf{P}$, and suppose $(\alpha, \beta)$ is a composable pair of $\mathbf{P}$. Then also $\alpha \circ \beta$ is left/right neutral.

*Proof.* Points (i) and (ii) are immediate from our definitions. As for Point (iii), assume that $\alpha$ and $\beta$ are left neutral (the other case being analogous). Since $(\alpha, \beta)$ is a composable pair, necessarily $A = B$, and then $\gamma := \alpha \circ \beta = \beta$, using that $\alpha$ is left neutral. It follows that $\text{hom}_\mathbf{P}(-, A_\downarrow \gamma) = \text{hom}_\mathbf{P}(-, A_\downarrow \beta)$, with the result that $\gamma \circ f = \beta \circ f = f$ for all $f \in \text{hom}_\mathbf{P}(-, A_\downarrow \gamma)$, because $\beta$ is left neutral too. \[\square\]

We will return on left and right identities in Section 7. For the moment, we have nothing to add to what has been already said, and we move on to other subjects.
4.6. Substructures. Plots come with a “natural” notion of substructure, to compare with the one of subcategory from [8, Definition 1.5.3].

Definition 11. Let $P = (C_0, C_1, s, t, c)$ and $Q = (D_0, D_1, \sigma, \tau, d)$ be plots. We say that $Q$ is a subplot of $P$ if the following conditions hold:

1. $D_0 \subseteq C_0$, $D_1 \subseteq C_1$ and $\text{dom}(d) \subseteq \text{dom}(c)$.
2. $\sigma$ and $\tau$ are, respectively, the corestrictions of $s_{|D_1}$ and $t_{|D_1}$ to $D_0$.
3. $d$ is the corestriction of $c_{|D_1 \times D_1}$ to $D_1$.

This is compactly expressed by writing $Q \leq P$, or $Q \leq P$ if $D_0 \cup D_1 \subset C_0 \cup C_1$ (in the latter case, $Q$ is said a proper subplot of $P$).

Remark 11. The roles of restrictions and corestrictions in Definition 11 are interchangeable: Given classes $X, X', Y, Y'$ and a function $f : X \to Y$, the restriction to $X'$ of the corestriction of $f$ to $Y'$ is equal to the corestriction to $Y'$ of the restriction of $f$ to $X'$.

Let $P$ be a plot and $Q$ a subplot of $P$. We say that $Q$ is a unital subplot of $P$ if $Q$ is a unital plot in itself, and the same principle applies to other properties such as associativity, epicity, etc. In particular, $Q$ is called a subquiver/subsemigroupoid/subsemicat of $P$ if $Q$ is a quiver/semigroupoid/semicategory in itself.

Remark 12. A subplot of a strongly associative plot needs not even be pre-associative (see Example 6), which may come as a relief to some readers. However, it is easily seen that saturated subplots of pre-associative plots are strongly associative.

Remark 13. Every subplot of a small/locally small plot is itself small/locally small.

We refer to $Q$ as a wide subplot of $P$ if $\text{Ob}(Q) = \text{Ob}(P)$, and as a full subplot (of $P$) if $\text{hom}_Q(A, B) = \text{hom}_P(A, B)$ for all $A, B \in \text{Ob}(Q)$; then, $Q$ is named a fully wide subplot if it is both full and wide (as a subplot of $P$). In addition, $Q$ is termed an identitive subplot of $P$ if $Q \leq P$ and $\text{id}_P(A) \in \text{hom}(Q)$ whenever $A$ is an object of $Q$ and a unital object of $P$; in particular, every full subplot of $P$ is also an identitive subplot. This is compactly written as $Q \leq_1 P$, and as $Q \leq_1 P$ if $Q \leq P$ also holds; of course, every identitive subplot of a unital plot is a unital subplot. Thus, we say that $Q$ is a subcategory of $P$ if $Q$ is a unital identitive subsemicategory of $P$; this returns the usual notion of subcategory in the case when $P$ is a category.

Remark 14. We notice that $Q$ can well be a category in itself even if $P$ is neither saturated, nor associative, nor unital; however, $Q$ is a subcategory of $P$ only if $\text{id}_Q(A) = \text{id}_P(A)$ for every object $A \in Q$ which is unital in $P$.

Example 3. We can associate any plot $P = (C_0, C_1, s, t, c)$ with a canonical quiver: This is the 5-tuple $(C_0, C_1, s, t, c_{|\emptyset})$, which is obviously a fully wide subplot of $P$, and as such a subcategory of $P$. The subobject $\sigma$ of $s$ and $\tau$ of $t$ are determined by $\sigma = s_{|D_1}$ and $\tau = t_{|D_1}$, respectively. The corestriction $c_{|D_1 \times D_1}$ is determined by $c$ on $D_1 \times D_1$. The graph of $d$ is determined by $d$ on $D_1$. This is clearly a positive example of a subsemicategory (or subsemigroupoid) of $P$; it is also a subquiver because it is unital.
here denoted by \( \text{QUIV}(P) \) and termed the (underlying) quiver of \( P \). Also, \( P \) can be canonically “projected” onto an (undirected) (multi)graph, whose nodes are precisely the objects of \( P \) and whose arrows are all and the only pairs \( (f, \{A, B\}) \) for which \( f \in \text{hom}_P(A, B) \cup \text{hom}_P(B, A) \): We call it the (underlying) (multi)graph of \( P \), and write it as \( \text{GRAPH}(P) \). Note that \( P \) and \( \text{QUIV}(P) \) have then the same underlying graph.

The next proposition is straightforward, but worth to mention, for it proves that “being a subplot” is a partial order on the class of small plots.

**Proposition 7.** Let \( P, Q \) and \( R \) be plots. The following holds:

(i) \( P \leq_1 P \), and \( P \leq Q \) if \( P \leq_1 Q \).

(ii) If \( P \leq Q \) and \( Q \leq P \), then \( P = Q \).

(iii) If \( P \leq Q \) and \( Q \leq R \), then \( P \leq R \).

**Proof.** It follows at once from our definitions (we omit the details). \( \square \)

The following example, on another hand, generalizes standard constructions from the algebraic theory of magmas and semigroups (see Remark 15).

**Example 4.** Pick a plot \( P = (C_0, C_1, s, t, c) \) and classes \( C'_0 \) and \( C'_1 \). Let \( D_1 \) be the intersection of all subclasses of \( C_1 \) such that \( C_1 \cap C'_1 \subseteq D_1 \) and \( c(f, g) \in D_1 \) for \( (f, g) \in (D_1 \times D_1) \cap \text{dom}(c) \), and set \( D_0 := (C_0 \cap C'_0) \cup \text{im}(s_{D_1}) \cup \text{im}(t_{D_1}) \). Then denote by \( \sigma \) and \( \tau \), respectively, the corestrictions of \( s_{D_1} \) and \( t_{D_1} \) to \( D_0 \), and by \( d \) the corestriction of \( c_{D_1} \times D_1 \) to \( D_1 \).

It is immediate that \( Q := (D_0, D_1, \sigma, \tau, d) \) is a subplot of \( P \) for which \( C_0 \cap C'_0 \subseteq D_0 \) and \( C_1 \cap C'_1 \subseteq D_1 \). Moreover, it is not difficult to prove (we may omit the details) that \( Q \leq R \) for any other subplot \( R = (E_0, E_1, u, v, e) \) of \( P \) for which \( C_0 \cap C'_0 \subseteq E_0 \), \( C_1 \cap C'_1 \subseteq E_1 \) and \( (f, g) \in \text{dom}(e) \) if \( (f, g) \in (E_1 \times E_1) \cap \text{dom}(c) \) and \( c(f, g) \in E_1 \). This is why we call \( Q \) the smallest subplot of \( P \) generated by \( (C'_0, C'_1) \), which is written here as \( \text{PLT}_1(C'_0, C'_1)_P \).

It is seen that \( \text{PLT}_1(C'_0, C'_1)_P \) is saturated/left pre-associative/left dissociative/associative/strongly associative whenever \( P \) is, and the same is true by replacing “left” with “right” in this statement (again, we may omit the details).

That said, we let \( \text{PLT}_1(C'_0, C'_1)_P \) denote the smallest subplot of \( P \) generated by the pair \( (C'_0, \text{id}_P(s(C_1 \cap C'_1) \cup \text{id}_P(t(C_1 \cap C'_1) \cup C'_1)) \). It is clear that

\[
\text{PLT}_1(C_0, C_1)_P \leq \text{PLT}_1(C_0, C_1)_P \leq_1 P,
\]

and \( \text{PLT}_1(C_0, C_1)_P \leq_1 Q \) for any identitive subplot of \( P \) such that \( \text{PLT}_1(C_0, C_1)_P \leq Q \), which is why we refer to \( \text{PLT}_1(C_0, C_1)_P \) as the smallest identitive subplot of \( P \).

Given a plot \( P \) and a class \( C \), we let the hom-subplot of \( P \) generated by \( C \) be the smallest subplot of \( P \) generated by the pair \( (\text{sr}_P(C) \cup \text{tr}_P(C), C) \), while we let
the obj-subplot of $P$ generated by $C$ be the smallest subplot of $P$ generated by $(C, \bigcup_{A,B \in C} \text{hom}_P(A, B))$. Both of these notions are now adapted in the obvious way to define the identitive hom-subplot and identitive obj-subplot generated by $C$.

On a related note, we define the wide (respectively, full) subplot/identitive subplot of $P$ generated by $C$ as the subplot/identitive subplot of $P$ generated by the pair $(\text{Ob}(P), C)$ (respectively, by $(C, \text{hom}(P))$).

**Remark 15.** It follows from Example 1 that, for a small magma $A$ and a (possibly empty) subset $S$ of $A$, the submagma of $A$ generated by $S$ is the uniquely determined submagma $B$ of $A$ such that $\text{Plt}(B)$ is equal to the wide subplot of $\text{Plt}(A)$ generated by $S$, which is in fact a semigroup, by Proposition 4, if $A$ is associative; see [10, Section I.1.4], if necessary, for terms used here without definition.

The next example is somewhat complementary to Example 4:

**Example 5.** Given a plot $P = (C_0, C_1, s, t, c)$ and a pair of classes $(C'_0, C'_1)$, let us set $D_1 := C_1 \cap C'_1$ and $D_0 := (C_0 \cap C'_0) \cup \text{im}(s_{|D_1}) \cup \text{im}(t_{|D_1})$, and then denote by $\sigma$ and $\tau$, respectively, the corestrictions of $s_{|D_1}$ and $t_{|D_1}$ to $D_0$, and by $d$ the corestriction of $c_{|D_1 \times D_1}$ to $D_1$. The tuple $Q := (D_0, D_1, \sigma, \tau, d)$ is a subplot of $P$, which is written here as $\text{Plt}^\uparrow(C'_0, C'_1)_P$ and called the relative subplot of $P$ generated by $(C'_0, C'_1)$. The name comes from the observation that, if $R = (E_0, E_1, u, v, e)$ is another subplot of $P$ with the property that $E_1 \subseteq C_1 \cap C'_1$ and $E_0 \subseteq (C_0 \cap C'_0) \cup s(E_1) \cup t(E_1)$, then $R \leq \text{Plt}^\uparrow(C'_0, C'_1)_P$. It is straightforward to see that $\text{Plt}^\uparrow(C'_0, C'_1)_P \leq \text{Plt}^\downarrow(C'_0, C'_1)_P$, and the equality holds if and only if $D_1$ is a compositive subclass of $P$ (see Definition 6).

We will return on subplots in Section 7. However, the notion is occasionally used in Section 5, and this is the motivation for having introduced it here.

5. SOME APPEALING EXAMPLES

Below are described a few plots that appear “naturally” in applications. The objective is primarily to motivate interest in the subject and convince skeptical readers that a further level of abstraction, crossing the limits of category theory, can be worthwhile and may lead, over time, to interesting developments. Since every category is, in our approach, a true and lawful plot, and specifically a saturated associative unital plot, we will look, as a rule of thumb, for examples of plots which fail to be unital, associative, or saturated. More will be given later, e.g. with the introduction of deunitization (Example 30), unitization (Section 7.3), products (Section 7.5.1), coproducts (Section 7.5.2), and augmentations (Section 7.5.3).

As far as we are aware, most of the entries of this section are original material, and this is in fact meant to be the case if no reference is provided. We remark for all practical purposes that, in dealing with topological notions relating to $R$ and
its subsets, the standard Euclidean topology will be always implied (and thus not explicitly mentioned).

**Example 6.** Let $\mathbf{P} = (\mathcal{C}_0, \mathcal{C}_1, s, t, c)$ be a plot and $\mathcal{R} = (\mathcal{C}_1, \mathcal{C}_1, R)$ a binary relation on $\mathcal{C}_1$. Take $\mathcal{D}_1$ to be the subclass of $\mathcal{C}_1$ consisting of those $f$ such that $(f, g) \in R$ or $(g, f) \in R$ for some $g \in \mathcal{C}_1$. If $\mathcal{D}_0 := s(\mathcal{D}_1) \cup t(\mathcal{D}_1)$, then $\text{Plt}^\uparrow(\mathcal{D}_0, \mathcal{D}_1)_\mathbf{P}$ is an epic subplot of $\mathbf{P}$ (see Example 5), here denoted by $\mathbf{P}_{/\mathcal{R}}$ and called the restriction of $\mathbf{P}$ to $\mathcal{R}$. No matter which properties $\mathbf{P}$ has, $\mathbf{P}_{/\mathcal{R}}$ can be very “wild” for an arbitrary $\mathcal{R}$.

**Example 7.** Some significant examples of pre-associative unital plots are presented by Schröder in [58, Section 1] (see also the references therein): These notably include the wide subplot of $\text{Cat}$, the usual category of small categories [3, Section 1.4, Example 6], generated by the topologically algebraic functors [1, Section VI.25], and for any given category $\mathbf{C}$ the wide subplot of $\mathbf{C}$ generated by its regular epimorphisms [8, Definition 4.3.1]. In both cases, we end up with pre-associative idempotent subplots of saturated associative unital plots, which however fail, at least in general, to be saturated or associative in their own right (cf. Remark 12).

**Example 8.** Let $X$ be a class and $\mathcal{R} = (X, X, R)$ a binary relation on $X$. One says that $\mathcal{R}$ is: reflexive if $(x, x) \in R$ for all $x \in X$; transitive if $(x, z) \in R$ whenever $(x, y), (y, z) \in R$; a preorder if it is reflexive and transitive.

Consider now the 5-tuple $\mathbf{P} = (\mathcal{C}_0, \mathcal{C}_1, s, t, c)$, where $\mathcal{C}_0 := X$ and $\mathcal{C}_1 := R$, $s$ and $t$ are the functions $R \to X$ sending $(x_1, x_2)$ to $x_1$ and $x_2$, respectively, and $c$ is the map $R \times R \to R$ defined as follows: If $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are in $R$ then $(p_1, p_2) \in \text{dom}(c)$ if and only if $y_1 = x_2$ and $p := (x_1, y_2) \in R$, in which case $c(p_1, p_2) := p$.

It is seen that $\mathbf{P}$ is a pre-associative plot, here denoted by $\text{Plt}(\mathcal{R})$ and referred to as the canonical plot of $\mathcal{R}$. Note that $\text{Plt}(\mathcal{R})$ has as many unital objects as the elements $x \in X$ such that $(x, x) \in \mathcal{R}$, but it is not unital unless $\mathcal{R}$ is reflexive. Also, if $\mathcal{R}$ is transitive then $\text{Plt}(\mathcal{R})$ is a semicategory. It follows that $\text{Plt}(\mathcal{R})$ is a category if $\mathcal{R}$ is a preorder, so the above is ultimately a generalization of [8, Example 1.2.6.b].

The next example will turn out to be useful in regard to the definition of $\mathcal{M}$-morphic (and, in particular, isomorphic) objects, in Section 6.4, and $\mathcal{M}$-limits and like, in Section 7.6.

**Example 9.** Let $\mathbf{P}$ be a plot and $\mathcal{M}$ a (possibly empty) subclass of $\text{hom}(\mathbf{P})$; in particular, we don’t assume that $\mathbf{P}$ is compositive. E.g., $\mathbf{P}$ may be a semigroup (regarded, of course, as a semicategory, as in Example 1) and $\mathcal{M}$ the class of its irreducible elements, or $\mathbf{P}$ a non-commutative monoid and $\mathcal{M}$ the class of its units,
or P the “underlying category” of a model category and M the class of its weak equivalences \cite{32}, or P arbitrary and M any of the special classes of P-arrows introduced later in Section 6, and so on.

Given arrows f, g ∈ P, we say that f is left contiguous to g (in P), and write \((f \bullet g)_P\), if \(s\pi(f) = s\pi(g)\) or \(t\pi(f) = t\pi(g)\). Then, f is said right contiguous to g (in P) if \(g^\text{op}\) is left contiguous to \(f^\text{op}\) in \(P^\text{op}\), which is written as \((f \bullet g)_P\).

Finally, f is defined to be contiguous to g (in P) if f is left or right contiguous to g, which is denoted by \((f \bullet g)_P\). In particular, we use \(f \bullet g\) in place of \((f \bullet g)_P\).

\[
\text{If } \text{P is implied from the context, and we do something similar with } (f \bullet g)_P \text{ and } (f \rightarrow g)_P. \]

Of course, f \(\bullet\) g if and only if g \(\bullet\) f, but also if and only if \((f \bullet g)_P\) in \(P^\text{op}\).

An \(M\)-connection (of P) is now any n-tuple \(\gamma = (f_1, \ldots, f_n)\) of P-morphisms \((n \in \mathbb{N}^+)\), referred to as the edges of \(\gamma\), such that \(f_i \in \mathcal{M}\) and either \(n = 1\) or there exists a permutation \(\sigma\) of \([n - 1]\) for which \((f_{\sigma(1)}, \ldots, f_{\sigma(n-1)})\) is itself an \(\mathcal{M}\)-connection (of P) with \(s\pi(f_1) = s\pi(f_{\sigma(1)})\) and \(f_{\sigma(n-1)} \bullet \cdots \bullet f_n\); we refer to the source of \(f_1\), the target of \(f_n\) and the integer n as the start vertex, the end vertex and the length of \(\gamma\), respectively, and call \(\gamma\), in particular, an \(\mathcal{M}\)-path, or a polarized \(\mathcal{M}\)-connection, (of P) if \(t\pi(f_1) = s\pi(f_{i+1})\) for each \(i \in [n - 1]\), in which case \(\gamma\) is also referred to as an \(\mathcal{M}\)-path from \(s\pi(f_1)\) to \(t\pi(f_n)\). Then, we let an \(\mathcal{M}\)-copath (of P) be an \(\mathcal{M}\)-path of \(P^\text{op}\).

With this in hand, we define a 5-tuple \((\mathcal{P}_0, \mathcal{P}_1, s, t, c)\) as follows: \(\mathcal{P}_0\) is just the object class of P, while \(\mathcal{P}_1\) is the class of all \(\mathcal{M}\)-connections of P; s and t are the functions \(\mathcal{P}_1 \rightarrow \mathcal{P}_0\) taking an \(\mathcal{M}\)-path to its start and end vertex, respectively; and c is the mapping \(\mathcal{P}_1 \times \mathcal{P}_1 \rightarrow \mathcal{P}_1\) sending a pair \((\gamma, \sigma)\) of \(\mathcal{M}\)-paths, say \(\gamma = (f_1, \ldots, f_m)\) and \(\sigma = (g_1, \ldots, g_n)\), to the \((m + n)\)-tuple \((f_1, \ldots, f_m, g_1, \ldots, g_n)\), and this if and only if \(t\pi(f_m) = s\pi(g_1)\).

In fact, \((\mathcal{P}_0, \mathcal{P}_1, s, t, c)\) is a semicategory, here written as \(\text{Net}_\mathcal{M}(P)\) and termed, as is expected, the semicategory of \(\mathcal{M}\)-connections of P, and the \(\mathcal{M}\)-paths of P generate a wide subsemicategory of it, which we denote by \(\text{Path}_\mathcal{M}(P)\) and call the semicategory of \(\mathcal{M}\)-paths of P. Furthermore, we set \(\text{Copath}_\mathcal{M}(P) := \text{Path}_\mathcal{M}(P^\text{op})\), and call \(\text{Copath}_\mathcal{M}(P)\) the semicategory of \(\mathcal{M}\)-copaths of P (recall Remark 7).

Note that neither of these can ever be a category; however, both of them can be made into a category, though it looks somewhat “unnatural”, by allowing “empty paths” around each vertex, which is the same as adjoining identities (as formalized in Section 7.3), and also a way to construct, for \(\mathcal{M} = \text{hom}(P)\), the free category on the underlying quiver of P.

As mentioned before, \(\mathcal{M}\)-paths enter in the definition of \(\mathcal{M}\)-limits (and \(\mathcal{M}\)-colimits), which is done by the notion of \(\mathcal{M}\)-factorization (and \(\mathcal{M}\)-cofactorization). For we let an \(\mathcal{M}\)-factorization (of P) be any pair \((\gamma, \phi)\) consisting of an \(\mathcal{M}\)-path \(\gamma\) of length \(n\) and a fundamental \(n\)-parenthesization \(\phi\) such that \(\gamma \in \text{dom}(\phi[\mathcal{P}])\).
Then, we define a new 5-tuple \((F_0, F_1, u, v, d)\) as follows: \(F_0\) is again the object class of \(P\), whereas \(F_1\) is the class of all \(M\)-factorizations of \(P\); \(u\) and \(v\) are the maps \(F_1 \to F_0\) that send an \(M\)-factorization \((\gamma, \varphi)\) to the start and end vertex of \(\gamma\), respectively; and \(d\) is the function \(F_1 \times F_1 \to F_1\) taking a pair \((\Phi_1, \Phi_2)\) of \(M\)-factorizations, say \(\Phi_i = (\gamma_i, \varphi_i)\), to \((\Phi, \varphi)\) where \(\Phi\) is the composition of \(\gamma_1\) with \(\gamma_2\) in \(\text{Path}_M(P)\) and \(\varphi\) is the unique fundamental factorization splitting into the product of \(\varphi_1\) by \(\varphi_2\), and this if and only if \((\varphi_1[p]\gamma_1), \varphi_2[p]\gamma_2) \in \text{dom}(\varphi[p]).\) It is apparent that \((F_0, F_1, u, v, d)\) is a plot: We call it the \(M\)-factorial plot of \(P\) and denote it by \(\text{Fact}_M(P)\). Lastly, we set \(\text{Cofact}_M(P) := \text{Fact}_M(P^\text{op})\) and refer to \(\text{Cofact}_M(P)\) as the plot of \(M\)-cofactorizations of \(P\).

Any explicit reference to \(M\) is dropped from the above terminology and notations in the limit case when \(M\) is just the whole hom-class of \(P\), for which, e.g., we speak of connections in place of \(M\)-connections, and write \(\text{Fact}(P)\) instead of \(\text{Fact}_M(P)\).

Remark 16. Incidentally, it might be interesting to use \(M\)-factorizations to lay the foundation for a general theory of factorization subsuming key aspects of various types of factorizations that have been studied to date, e.g. by D.D. Anderson and A.M. Frazier [2] in the context of integral domains, by A. Geroldinger and F. Halter-Koch [24] in the setting of commutative (unital) semigroups, or by R. Exel [20] in the kind of associative structures referred to in his work as semigroupoids. Seen this way, divisibility, primality, atomicity, and so forth would become properties of morphisms, which could perhaps give rise to interesting insights. However, I have only vague ideas on this point at present, so the above should not be taken too seriously by the readers.

Functional analysis is another factory of “natural” examples.

**Example 10.** Let \(\text{Met}\) be the usual category of metric spaces and Lipschitz functions (we rely on [55, Section 1.2, p. 4] for notation and terminology used here without definition): The objects are metric spaces, i.e. pairs \((X, d)\) for which \(X\) is a (possibly empty) set and \(d : X \times X \to \mathbb{R}_0^+\) a metric, while the arrows are triples \((X_1, X_2, f)\), where \(X_i = (X_i, d_i)\) is a metric space and \(f\) is a Lipschitz function from \(X_1\) to \(X_2\), that is a function \(f : X_1 \to X_2\) such that, for some \(\kappa \in \mathbb{R}^+\),

\[d_2(f(x), f(y)) \leq \kappa \cdot d_1(x, y) \quad \text{for all } x, y \in X_1, \quad (3)\]

the infimum over all constant \(\kappa \in \mathbb{R}^+\) for which (3) is satisfied being denoted by \(\text{Lip}(f)\) and called the Lipschitz constant of \(f\) (the source and target maps as well the composition law are defined in the expected way).

If \(C\) is now the class of all (metric) contractions of \(\text{Met}\), i.e. morphisms \(f \in \text{Met}\) with \(\text{Lip}(f) \leq 1\), we let \(\text{Met}_{\leq 1}\) denote the hom-subplot of \(\text{Met}\) generated by \(C\): One proves that \(\text{Met}_{\leq 1}\) is an epic wide subsemicategory of \(\text{Met}\) (since \(C\) is a compositive subclass of \(\text{Met}\)), though not a unital one (essentially because the local identities
of \( \text{Met} \) have a Lipschitz constant equal to one): We call it the semicategory of metric spaces and contractions, and write it as \( \text{Met}_{\leq 1} \). This has in turn a rather interesting subplot, i.e. the full subsemicategory, \( \text{ChMet}_{\leq 1} \), of (Cauchy-)complete metric spaces, whose relevance descends from the fact that each arrow \( f : \mathcal{X} \to \mathcal{X} \) in \( \text{ChMet}_{\leq 1} \) for which \( \mathcal{X} \) is non-empty has a unique fixed point (by the Banach contraction theorem).

**Remark 17.** Much of what has been said above can be repeated for semimetric spaces [55, Section 1.33] (all the more that doing so would imply no significant complication), but for concreteness we have focused here on the metric case.

**Example 11.** This is sort of a variant on Example 10. For simplicity, we shall restrict on the case of real or complex normed (vector) spaces. However, most of the subsequent considerations could be repeated in a much greater generality (net of technical complications) by replacing normed spaces with more abstract “normed structures” or norms with seminorms. For terminology used here but not defined the reader is referred to [41, Sections 1-4, 7-8, 9]. Also, vector and normed spaces considered in the sequel are implicitly assumed to be defined on sets, and not on proper classes.

To start with, let \( K \) be a small field and \( |\cdot| \) an absolute value on \( K \); we refer to the pair \( \mathcal{K} := (K, |\cdot|) \) as a normed field. Then, we write \( \text{Nor}(\mathcal{K}) \) for the category of normed spaces over \( \mathcal{K} \) (or normed \( \mathcal{K} \)-spaces) and bounded (\( K \)-linear) operators: The objects are normed \( \mathcal{K} \)-spaces whose carrier is a set and the arrows are triples \((\mathcal{V}_1, \mathcal{V}_2, f)\), where \( \mathcal{V}_i = (\mathcal{V}_i, \|\cdot\|_i) \) is a \( \mathcal{K} \)-normed space and \( f \) a bounded operator from \( \mathcal{V}_1 \) to \( \mathcal{V}_2 \), viz a homomorphism \( f : \mathcal{V}_1 \to \mathcal{V}_2 \) of \( K \)-vector spaces such that

\[
\|f(v)\|_2 \leq \kappa \quad \text{for all } v \in \mathcal{V}_1 \text{ with } \|v\|_1 = 1,
\]

for some constant \( \kappa \in \mathbb{R}^+ \), the infimum over such constants being denoted by \( \|f\|_* \) and termed the operator norm of \( f \) (the composition as well as the source and target maps are defined in the expected way, and we may omit the details).

If \( \mathcal{C} \) is now the class of all contractions of \( \text{Nor}(\mathcal{K}) \), that is morphisms \( f \in \text{Nor}(\mathcal{K}) \) with \( \|f\|_* \leq 1 \), we denote by \( \text{Nor}_{\leq 1}(\mathcal{K}) \) the hom-subplot of \( \text{Nor}(\mathcal{K}) \) generated by \( \mathcal{C} \). It is easy to see that \( \mathcal{C} \) is a compositional subclass of \( \text{Nor}(\mathcal{K}) \), and then \( \text{Nor}_{\leq 1}(\mathcal{K}) \) is an epic wide subsemicategory of \( \text{Nor}(\mathcal{K}) \), here called the semicat of normed \( \mathcal{K} \)-spaces and contractions. Clearly, \( \text{Nor}_{\leq 1}(\mathcal{K}) \) is not a category, essentially because the local identities of \( \text{Nor}(\mathcal{K}) \) are only weakly contractive (their operator norm is equal to one). But \( \text{Nor}_{\leq 1}(\mathcal{K}) \) has a remarkable subsemicategory, that is the full subsemicategory, \( \text{Ban}_{\leq 1}(\mathcal{K}) \), of Banach \( \mathcal{K} \)-spaces, which makes it interesting, like with the metric case of Example 10, in view of the Banach contraction theorem (e.g., in the real or complex case).
**Example 12.** This is another example taken from functional analysis, for which we shall use the same notation and terminology of Example 11.

To start with, let $V_i = (V_i, ||\cdot||_i)$ be a normed space over $K$ (with $i = 1, 2$). A homomorphism $f : V_1 \to V_2$ of $K$-linear spaces is termed a compact operator from $V_1$ to $V_2$ if the image under $f$ of any bounded subset $X$ of $V_1$ is relatively compact in $V_2$, i.e. if the topological closure of $f(X)$ is a compact subset of $V_2$, where all topological properties of $V_i$ and its subsets refer to the metric topology induced by $||\cdot||_i$ on the underlying set; cf. [55, Section 1.2] and [42, Section 4.1, p. 200].

Every compact operator from a normed $K$-space to another is bounded in the sense of (4); cf. [42, Section 4.1, Theorem (2)]. Moreover, given morphisms $f : V_1 \to V_2$ and $g : V_2 \to V_3$ of $\text{Nor}(K)$ such that either of $f$ or $g$ is compact, the composition of $f$ with $g$ is a compact operator from $V_1$ to $V_3$; cf. [42, Section 4.1, Theorem (1)]. It follows that the subclass $C$ of all compact operators of $\text{Nor}(K)$ is compositive, and thus the hom-subplot of $\text{Nor}(K)$ generated by $C$ is an epic wide subsemicategory: We call it the semicategory of normed spaces and compact (linear) operators, and denote it by $\text{CmpNor}(K)$; this is not a category, for the local identity of an object $V \in \text{Nor}(K)$ is a compact operator if and only if $V$ is finite dimensional (as implied, e.g., by [55, Theorems 1.21, 1.22]). As is well-known, the full subsemicategory of $\text{CmpNor}(K)$ of Banach $K$-spaces, here written as $\text{CmpBan}(K)$, plays a central role in the (classical) spectral theory of bounded operators (say, in the real or complex case).

**Remark 18.** For the record, we mention that the above example can be easily generalized to other operator ideals; see [55, Ch. 11].

The next example is borrowed from point-set topology.

**Example 13.** Let $\text{Top}_*$ be the standard category of pointed topological spaces and based maps (cf. [8, Example 3.1.6.i]). I.e., objects are triples $(X, O_X, x_0)$, indeed called pointed topological spaces, consisting of a set $X$, a topology $O_X$ on $X$ and a distinguished element $x_0 \in X$ (called a base-point); arrows are triples of the form $(\mathcal{X}, \mathcal{Y}, f)$, where $\mathcal{X} = (X, O_X, x_0)$ and $\mathcal{Y} = (Y, O_Y, y_0)$ are pointed topological spaces and $f$ is a based map from $\mathcal{X}$ to $\mathcal{Y}$, to wit a continuous function from $(X, O_X)$ to $(Y, O_Y)$ such that $f(x_0) = y_0$; the source and target maps as well as the composition law are defined in the expected way.

Given a pointed topological space $\mathcal{X} = (X, O_X, x_0)$ and a subset $S$ of $X$, let $\text{cl}_X(S)$ be the (topological) closure of $S$ in $(X, O_X)$. Then, denote by $C$ the class of all compactly supported functions of $\text{Top}_*$, i.e. arrows $f : (X, O_X, x_0) \to (Y, O_Y, y_0)$ of $\text{Top}_*$ such that the set $\text{supp}_c(f) := \text{cl}_X(X \setminus f^{-1}(y_0))$, here called the closed support of $f$, is compact in $(X, O_X)$. The interest in this kind of functions is related, e.g., to the classical theory of distributions [55, Ch. 6] and Fourier transforms [55, Ch. 7].
Lemma 2. If \( f : (X, \mathcal{O}_X, x_0) \to (Y, \mathcal{O}_Y, y_0) \) and \( g : (Y, \mathcal{O}_Y, y_0) \to (Z, \mathcal{O}_Z, z_0) \) are arrows of \( \text{Top}_* \), and if \( f \) is compactly supported, then the composition \( fg \) of \( f \) with \( g \) in \( \text{Top}_* \) is compactly supported.

Proof. Since \( y_0 \in g^{-1}(z_0) \), then \( f^{-1}(y_0) \subseteq f^{-1}(g^{-1}(z_0)) \), so that \( \text{cl}_X(X \setminus (fg)^{-1}(z_0)) \subseteq \text{cl}_X(X \setminus f^{-1}(y_0)) \). Thus, \( \text{cl}_X(X \setminus (fg)^{-1}(z_0)) \) is a compact subset of \( (X, \mathcal{O}_X) \), that is \( fg \) is a compactly supported function from \( (X, \mathcal{O}_X, x_0) \) to \( (Z, \mathcal{O}_Z, z_0) \), as \( f \) is compactly supported, \( \text{cl}_X(X \setminus (fg)^{-1}(z_0)) \) is a closed set in \( (X, \mathcal{O}_X) \), and closed subsets of compact sets are compact [41, Section 3.2, Theorem (2)].

It follows from Lemma 2 that \( \mathcal{C} \) is a compositive subclass of \( \text{Top}_* \), and then the hom-subplot of \( \text{Top}_* \) generated by \( \mathcal{C} \) is an epic wide subsemicategory, here denoted by \( \text{CST}_* \), and referred to as the semicategory of pointed topological spaces and compactly supported functions. It is found that \( \text{CST}_* \) is not a category, basically because the local identity of a pointed topological space \( (X, \mathcal{O}_X, x_0) \) in \( \text{Top}_* \) is compactly supported only if \( X \setminus \{x_0\} \) is compact in \( (X, \mathcal{O}_X) \), which is not the case, e.g., when \( X \) is infinite and \( \mathcal{O}_X \) is discrete.

The last example of this section is motivated by control theory.

Example 14. This example is partially based on Examples 10 and 11 for notation and terminology. Let a pointed metric space be a triple \( (X, d_X, x_0) \), where \( (X, d) \) is a non-empty metric space and \( x_0 \) is a distinguished point of \( X \). Then, we define an asymptotically stable transformation to be any triple \( (\mathcal{X}, \mathcal{Y}, f) \) such that \( \mathcal{X} = (X, d_X, x_0) \) and \( \mathcal{Y} = (Y, d_Y, y_0) \) are pointed metric spaces and \( f \) is a function from \( X \) to \( Y \) for which both of the following conditions hold:

(i) \( f(x_0) = y_0 \) and \( d_Y(f(x), y_0) \to 0 \) as \( d_X(x, x_0) \to 0 \).

(ii) \( d_Y(f(x), y_0) \to 0 \) as \( d_X(x, x_0) \to \infty \).

That is, \( f \) is continuous in \( x_0 \), where it takes value \( y_0 \), and tends to \( y_0 \) far away from \( x_0 \) (with respect to the metric topologies induced on \( X \) and \( Y \), respectively, by \( d_X \) and \( d_Y \); see [55, Section 1.2]). More generally, similar notions may be given for semimetric spaces, but again, we choose to focus on the metric case for concreteness.

Roughly speaking, the idea is that an asymptotically stable transformation describes a “system” \( \mathcal{G} \) evolving from an “initial state”, in a neighborhood of which \( \mathcal{G} \) is “well-behaved”, to an “asymptotic equilibrium state”, in such a way that the original and final “configurations” are the same, regardless of the “strain” affecting the system in the intermediate stages of its “history”, that is independently of its “transient behaviour”.

Given two asymptotically stable transformations \( (\mathcal{X}, \mathcal{Y}, f) \) and \( (\mathcal{Y}, \mathcal{Z}, g) \), where \( \mathcal{X} = (X, d_X, x_0) \), \( \mathcal{Y} = (Y, d_Y, y_0) \) and \( \mathcal{Z} = (Z, d_Z, z_0) \), it is not difficult to check that the (set-theoretic) composition of \( f \) with \( g \), here written as \( fg \), yields an asymptotically stable transformation from \( \mathcal{X} \) to \( \mathcal{Z} \): For \( f(x_0) = y_0 \) and \( g(y_0) = z_0 \) imply
\(fg(x_0) = z_0\), while \(g(y) \to z_0\) as \(y \to y_0\) gives \(f(x) \to z_0\) when either \(d_X(x, x_0) \to 0\) or \(d_X(x, x_0) \to \infty\), since in both cases it holds \(f(x) \to y_0\).

It follows that pointed metric spaces and asymptotically stable transformations are, respectively, the objects and the arrows of an epic semicategory, here denoted by \(\text{AsyMet}_*\), where the source and target maps as well as the composition law are defined in the most intuitive way: This is not a category, essentially because, given a pointed metric space \(X = (X, d_X, x_0)\), the identity map of \(X\) does not, in general, tend to \(x_0\) as \(d(x, x_0) \to \infty\).

The careful readers will have observed that, with the only exception of Example 1.5 in [59] (which is implicitly referred to in our Example 7) and Example 6, all plots considered in this section are pre-associative (and a number of them are semicategories). Other examples will be considered in the subsequent sections, and some of them will not even be pre-associative. For the moment, we may however be content with those above.

6. Special classes of arrows and related notions

Whenever a class \(X\) is equipped with a binary operation \(*\), special attention is always paid to those elements in \(X\) that are “cancellable” or whatever else with respect to \(*\) (see Example 1). This is all the more true in the case of plots, for which we are going to show that, maybe a little bit surprisingly, there is a “natural” notion of isomorphism, effectively generalizing the one proper of category theory along with a number of basic properties. This is made possible by a simple change of perspective, boiling down to what we call the “point of view of regular representations”.

6.1. Monomorphisms and epimorphisms. We start with monic and epic arrows; cf. [8, Sections 1.7 and 1.8]. Throughout, unless differently specified, we let \(P\) be a fixed plot, whose composition law we write here as \(\diamond\), and we use the notation of Section 4.3 for left and right regular representations.

**Definition 12.** We say that a morphism \(f \in P\) is

1. monic (in \(P\)), or a monomorphism (or a mono) (of \(P\)), if \(\varrho_P(f)\) is injective.
2. epic (in \(P\)), or an epimorphism (or an epi) (of \(P\)), if \(f^\text{op}\) is monic in \(P^\text{op}\).
3. cancellative (in \(P\)) if it is both monic and epic.

We denote by \(\text{mono}(P)\) and \(\text{epi}(P)\), respectively, the classes of all monic and epic arrows of \(P\), and by \(\text{canc}(P)\) the one of cancellative \(P\)-morphisms.

**Example 15.** Every left/right neutral arrow is obviously epic/monic. In particular, every local identity is cancellative; cf. [8, Propositions 1.7.2(1) and 1.8.2(1)].
Occasionally, we may refer to monomorphisms and epimorphisms, respectively, as right and left cancellative arrows. This is motivated by the next proposition, whose proof is obvious in the light of Definition 14 (we may omit the details):

**Proposition 8.** Let \( f : A \to B \) be a morphism of \( P \). Then:

(i) \( f \) is epic if and only if \( \lambda_P(f) \) is injective.

(ii) \( f \) is monic if and only if \( g_1 \circ f = g_2 \circ f \) for \( g_1, g_2 \in \text{hom}_P(-, A_{\downarrow f}) \) implies \( g_1 = g_2 \).

(iii) \( f \) is epic if and only if \( f \circ g_1 = f \circ g_2 \) for \( g_1, g_2 \in \text{hom}_P(B_{\downarrow f}, -) \) implies \( g_1 = g_2 \).

In other words, monomorphisms and epimorphisms behave like, and in fact are, respectively, fundamental parenthesizations of lengths \( 2 \) are, respectively, fundamental parenthesizations of lengths \( n \) if \( X \subseteq M \subseteq \text{canc}(P) \).

**Proof.** Set \( X := (X_1 \circ \cdots \circ X_n)_{\varphi[P]} \). The assertion is straightforward if \( n = 1 \), and for \( n = 2 \) it follows from considering that if \( X_2 \subseteq M \subseteq \text{mono}(P) \) then the right regular representation of any morphism \( g \in X_2 \) is an injection \( X_1 \to \text{hom}(P) \), while if \( X_1 \subseteq M \subseteq \text{epi}(P) \) then the left regular representation of any arrow \( f \in X_1 \) is an injection \( X_2 \to \text{hom}(P) \). So let \( n \geq 3 \) and suppose that the claim is true for every \( k \in \{n-1\} \). Moreover, denote by \((\varphi_1, \varphi_2)\) the (binary) splitting of \( \varphi \), where \( \varphi_1 \) and \( \varphi_2 \) are, respectively, fundamental parenthesizations of lengths \( k \) and \( n-k \) for some \( k \in \{n-1\} \). Then, Definition 5 implies that

\[
|X| = |(X_1 \circ \cdots \circ X_k)_{\varphi_1[P]} \circ (X_{k+1} \circ \cdots \circ X_n)_{\varphi_2[P]}|,
\]

to the effect that, by the induction basis, \(|(X_1 \circ \cdots \circ X_k)_{\varphi_1[P]}| \leq |X| \) if the arrows of \( M \) are right cancellative and \( X_2, \ldots, X_n \subseteq M \), \(|(X_{k+1} \circ \cdots \circ X_n)_{\varphi_2[P]}| \leq |X| \) if the arrows of \( M \) are left cancellative and \( X_1, \ldots, X_{n-1} \subseteq M \), and finally

\[
\max(|(X_1 \circ \cdots \circ X_k)_{\varphi_1[P]}|, |(X_{k+1} \circ \cdots \circ X_n)_{\varphi_2[P]}|) \leq |X|
\]

if \( X_1, \ldots, X_n \subseteq M \subseteq \text{canc}(P) \). This concludes the proof (by strong induction). \( \square \)
Unlike with categories, neither of \( \text{mono}(P) \), \( \text{epi}(P) \), or \( \text{canc}(P) \) needs be closed under the composition law of \( P \), as shown by the next example:

**Example 16.** Let \( A \) be a set of three elements, here denoted by 0, 1 and 2, and define a binary operation \( \star \) on \( A \) according to the following Cayley table:

\[
\begin{array}{c|ccc}
\star & 0 & 1 & 2 \\
\hline
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 2 \\
2 & 1 & 2 & 1
\end{array}
\]

Then 1 is a cancellative element in the magma \( A = (A, \star) \), but \( 1 \star 1 = 0 \), and 0 is neither left nor right cancellative (by the way, note that \( 1 = 0 \star 0 \)). On another hand, 2 is right, but not left cancellative (in \( A \)). Since the left cancellative/right cancellative/cancellative elements of a magma are, respectively, the epic/monic/cancellative arrows of its canonical plot (Example 1), this leads to the desired conclusion.

However, sufficient conditions for \( \text{mono}(P) \), \( \text{epi}(P) \), or \( \text{canc}(P) \) to be compositive subclasses of \( P \) are provided by the next proposition:

**Proposition 10.** If \( P \) is left/right dissociative and \( (f, g) \) is a composable pair consisting of epic/monic arrows, then \( f \circ g \) is epic/monic too.

*Proof.* We may assume by duality that \( P \) is right dissociative. Pick arrows \( h_1, h_2 \in P \) such that \( (h_1, f \circ g), (h_2, f \circ g) \) are composable pairs, and suppose that \( h_1 \circ (f \circ g) = h_2 \circ (f \circ g) \). So the assumptions on \( P \) imply that \( (h_i, f) \) is itself a composable pair, and then it goes the same with \( (h_i \circ f, g) \); furthermore, \( (h_i \circ f) \circ g = h_i \circ (f \circ g) \). Therefore \( (h_1 \circ f) \circ g = (h_2 \circ f) \circ g \), with the result that \( h_1 \circ f = h_2 \circ f \), since \( g \) is monic, and so \( h_1 = h_2 \), for \( f \) is monic too (by Proposition 8). \( \square \)

The following result is a sort of converse of Proposition 10.

**Proposition 11.** Assume \( P \) is strongly associative and let \( (f, g) \) be a composable pair of \( P \) such that \( f \circ g \) is monic/epic. Then \( f \) is monic/\( g \) is epic.

*Proof.* By duality, we may focus on the case when \( f \circ g \) is monic. So pick \( h_1, h_2 \in \text{hom}(P) \) such that \( (h_i, f) \) is a composable pair. Then, taking Remark 5 in mind and using that \( P \) is strongly associative, we get that \( (h_i \circ f) \circ g = h_i \circ (f \circ g) \). Since \( f \circ g \) is monic, it follows that if \( h_1 \circ f = h_2 \circ f \) then \( h_1 = h_2 \) (Proposition 8), and this in turn implies that \( f \) is monic too. \( \square \)

We notice for all practical purposes that Proposition 10, Proposition 11 and Remark 15 together provide, in view of Remark 4, a comprehensive generalization of Propositions 1.7.2 and 1.8.2 from [8].

Monomorphisms and epimorphisms supply other examples of “natural” structures encountered in the everyday practice that are not categories.
Example 17. We let \( \text{MONO}(P) \), \( \text{EPI}(P) \) and \( \text{CANC}(P) \) be, respectively, the restrictions of \( P \) to \( \text{mono}(P) \times \text{mono}(P) \), \( \text{epi}(P) \times \text{epi}(P) \) and \( \text{canc}(P) \times \text{canc}(P) \), these being regarded as binary relations on the hom-class of \( P \) (see Example 6). What said above implies that \( \text{MONO}(P) \), \( \text{EPI}(P) \) and \( \text{CANC}(P) \) need not be saturated subplots of \( P \), even if \( P \) is saturated, since the composition of two monos/epis may not be a mono/epi; however, Proposition 10 and Remark 4 give that all of them are semicats/categories whenever this is the case for \( P \).

Example 18. We say that a \( P \)-morphism is opaque if it is neither monic nor epic, and transparent otherwise; in view of Example 1, a null magma, that is a magma \( \mathbb{M} = (M, \star) \) with a distinguished element \( 0_M \) such \( a \star b = 0_M \) for all \( a, b \in \mathbb{M} \), furnishes an extreme instance of a plot where every arrow is opaque (save that the magma is trivial), and the case of a group is exactly complementary. If \( \mathcal{O} \) is the class of the opaque arrows of \( P \), then we let \( \text{OPA}(P) \) be the restriction of \( P \) to the binary relation \( \mathcal{O} \times \mathcal{O} \) on \( \text{hom}(P) \). By Remark 15, this is a plot with no left/right neutral arrow, but not much more than this, at least in general, as is implied by Example 16. However, if \( P \) is strongly associative, it follows from Proposition 10 that \( \mathcal{O} \) is a compositive subclass of \( P \), to the effect that \( \text{OPA}(P) \) is the hom-subplot of \( P \) generated by \( \mathcal{O} \), and hence a semicat/category whenever \( P \) is.

By way of example, we now characterize monomorphisms and epimorphisms of the canonical plot of a binary relation and some special magmas.

Example 19. With the notation of Example 8, any arrow of \( \text{PLOT}(\mathcal{R}) \) is cancellative, no matter if \( \text{PLOT}(\mathcal{R}) \) is, or is not, unital, associative, etc.

Example 20. Following Example 1(a), let \( P \) be first the canonical plot of the left-zero magma on a certain class \( A \): It is rather evident that \( \text{mono}(P) = A \), while \( \text{epi}(P) = \emptyset \) unless \( A \) is a singleton, in which case \( \text{canc}(P) = A \). Following Example 1(c), on the other hand, let \( P \) now be the canonical plot of the exponential magma of \( \mathbb{R} \): Then \( \text{mono}(P) = \mathbb{R} \setminus \{0\} \), while \( \text{epi}(P) = \mathbb{R}^+ \setminus \{1\} \). Finally, following Example 1(e), let \( P \) be the canonical plot of the magma \((A, \star)\), where \( A \) is the class of all sets and \( \star \) is the binary operation \( A \times A \to A \) sending a pair \((X, Y)\) to the relative complement of \( Y \) in \( X \). Then \( \text{mono}(P) = \{\emptyset\} \), while \( \text{epi}(P) = \emptyset \).

We conclude the section by mentioning constant and coconstant arrows, the former being a generalization of the notion of constant function.

Definition 13. A morphism \( f : A \to B \) in \( P \) is constant if the right regular representation of \( f \) (in \( P \)) is a constant map, and coconstant if \( f \) is constant in \( P^{\text{op}} \).

Namely, a morphism \((f : A \to B)_P\) is constant if and only if \( g_1 \circ f = g_2 \circ f \) for all \( g_1, g_2 \in \text{hom}_P(-, A_{/f}) \), while it is coconstant if and only if \( f \circ h_1 = f \circ h_2 \) for all \( h_1, h_2 \in \text{hom}_P(B_{/f}, -) \). Loosely speaking, a constant/coconstant arrow is as
far as possible from the condition of being monic/epic. In particular, all isolated arrows are constant and coconstant, while a right identity \( i : A \to A \) is constant if and only if \( \text{hom}_P(-, A) \) is a singleton (by duality, a similar condition holds for left identities).

6.2. Sections and retractions. While mono splits and right invertible arrows, along with their duals (namely, epi splits and left invertible arrows), are undistinguished on the level of categories [3, Section 2.1.1], this is no longer the case for plots, where it is quite “natural” to differentiate among a greater number of situations, according to the properties of right and left regular representations (see also Section 6.3). N.B.: We use here the same notation of the previous section.

**Definition 14.** We say that a \( P \)-morphism \( A \xrightarrow{f} B \) is

1. right split (in \( P \)), or a retraction or a right split (of \( P \)), if \( \bar{\rho}_P(f) \) is surjective.
2. left split (in \( P \)), or a section or a left split (of \( P \)), if \( f^{\text{op}} \) is right split in \( P^{\text{op}} \).
3. split (in \( P \)), or a split (of \( P \)), if it is both epi and mono split.

We write \( l\text{-spl}(P) \) and \( r\text{-spl}(P) \), respectively, for the classes of all left and right splits of \( P \), and we let \( \text{spl}(P) := l\text{-spl}(P) \cap r\text{-spl}(P) \).

**Example 21.** A right neutral arrow \( \alpha : A \to A \) of \( P \) is right split if and only if it is a right identity, because the image of \( \text{hom}_P(-, A) \downarrow \alpha \) under \( \bar{\rho}_P(\alpha) \) is \( \text{hom}_P(-, A) \downarrow \alpha \), as well as a left neutral one is left split if and only if it is a left identity (by duality); cf. Remark 15.

The next result is immediate from the above definition (we omit the details):

**Proposition 12.** Let \( f : A \to B \) be a morphism of \( P \). Then:

(i) \( f \) is left split if and only if \( \lambda_P(f) \) is surjective.
(ii) \( f \) is right split if and only if for every \( h \in \text{hom}_P(\lnot, B) \) there exists at least one arrow \( g_h \in \text{hom}_P(-, A_{\downarrow f}) \) such that \( g_h \circ f = h \).
(iii) \( f \) is left split if and only if for every \( h \in \text{hom}_P(A, \lnot) \) there exists at least one arrow \( g_h \in \text{hom}_P(B_{\downarrow f}, \lnot) \) such that \( f \circ g_h = h \).

Instead, the following result generalizes Propositions 1.7.4 and 1.8.3 from [8], and gives a link between splits and cancellative morphisms:

**Proposition 13.** Let \( P \) be strongly associative and pick an arrow \( (f : A \to B)_{P} \).

(i) If \( f \) is right split and there is a right neutral \( \beta \in \text{hom}_P(B) \), then \( f \) is epic.
(ii) If \( f \) is left split and there is a left neutral \( \alpha \in \text{hom}_P(A) \), then \( f \) is monic.

**Proof.** By duality, it is enough to prove point (i). For suppose there exists a right neutral arrow \( \beta : B \to B \) and let \( f \) be a right split. Then, we can find a morphism \( g \in \text{hom}_P(-, A_{\downarrow f}) \) such that \( g \circ f = \beta \), and now the claim follows from Remark 15 and Proposition 11.

\( \square \)
Proposition 13 is the reason why right and left splits are called, on the level of categories, epi and mono splits. This terminology is, however, misleading for plots, where nothing similar holds (not in general), so it will not be used here.

One question is now worth asking: Are splits closed under composition? Similarly to the case of epimorphisms, monomorphisms and cancellative arrows (Example 16), this is answered in the negative by the following:

**Example 22.** With the notation of Example 16, it is immediate to see from the Cayley table (5) that 1 is split in Plt(A), but 1 ⋆ 1 = 0, and 0 is neither left nor right split. On another hand, 2 is left, but not right split.

However, pre-associativity comes again to one’s assistance.

**Proposition 14.** Let \((f : A \to B, g : B \to C)\) be a composable pair of \(P\). If \(P\) is left/right pre-associative and \(f, g\) are right/left split, then \(f \circ g\) is right/left split.

**Proof.** By duality, it is enough to focus on the case when \(P\) is left pre-associative and both of \(f\) and \(g\) are right split. For pick \(h \in \text{hom}_P(\cdot, C)\). Since \(g\) is right split, there exists \(\ell_h \in \text{hom}_P(\cdot, B)\) for which \(\ell_h \circ g = h\). But \(\ell_h \in \text{hom}_P(\cdot, B)\) and \(f\) is right split too, so we can find a morphism \(k_h \in \text{hom}_P(\cdot, A)\) such that \(k_h \circ f = \ell_h\), and hence \((k_h \circ f) \circ g = h\). Now use point (iv) of Lemma 1. □

The next result is kind of a converse of Proposition 14.

**Proposition 15.** Assume \(P\) is left/right dissociative and let \((f, g)\) be a composable pair of \(P\) such that \(f \circ g\) is left/right split. Then \(f\) is left split/g is right split.

**Proof.** We may suppose, by duality, that \(P\) is left dissociative and \(f \circ g\) is left split. For each \(h \in \text{hom}_P(A, \cdot)\) there then exists \(k_h \in \text{hom}_P(C_{f \circ g}, \cdot)\) such that \((f \circ g) \circ k_h = h\), to the effect that, by left dissociativity, \((f, g \circ k_h)\) is a (well-defined and) composable pair of \(P\) and \(h = f \circ (g \circ k_h)\). The claim follows. □

Like with monic and epic arrows, left and right splits furnish further examples of plots “naturally” arising from practice that are not, in general, categories.

**Example 23.** We use \(l-\text{Spl}(P), r-\text{Spl}(P)\) and \(\text{Spl}(P)\), respectively, for the restrictions of \(P\) to the binary relations \(l-\text{spl}(P) \times l-\text{spl}(P), r-\text{spl}(P) \times r-\text{spl}(P)\) and \(\text{spl}(P) \times \text{spl}(P)\) on \(\text{hom}(P)\). By Example 22, these may not be saturated subplots of \(P\), no matter if \(P\) is or not, but all of them are semicategories/categories when this is the case for \(P\), as is implied by Proposition 14 and Remark 4.

**Example 24.** We say that a \(P\)-morphism \(f\) is singular if it is neither left nor right split, otherwise \(f\) is called regular; e.g., a non-trivial null magma is a plot where all arrows are atomic (see Example 18), while a group is at the opposite extreme. If \(S\) is the class of the atomic arrows of \(P\), then we let \(\text{Sng}(P)\) be the restriction of \(P\) to
the binary relation $S \times S$ on $\text{hom}(P)$. This is never a unital plot (by Example 21), nor it is saturated, at least in general (as it follows from Example 16). However, if $P$ is dissociative, Proposition 14 gives that $S$ is a compositive subclass of $P$, with the result that $\text{SNG}(P)$ is the hom-subplot of $P$ generated by $S$, and hence a semicat/category whenever $P$ is.

We conclude with another couple of simple, but instructive examples.

**Example 25.** In the light of Example 1(e), let $A$ be the class of all sets, $\star$ the binary operation $A \times A \to A$ mapping a pair $(X,Y)$ to the relative complement of $Y$ in $X$, and $P$ the canonical plot of $(A,\star)$. It is an easy exercise that $\text{l-spl}(P) = \emptyset$ and $\text{r-spl}(P) = \{\emptyset\}$ (cf. Example 20); let us note, in this respect, that $\emptyset$ is a right identity of $P$, but not an identity (recall Example 21).

**Example 26.** Pick $m, n \in \mathbb{N}^+$ and let $\star$ be the map $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+: (x,y) \mapsto x^m y^n$. Then every element of $\mathbb{R}^+$ is cancellative and split when viewed as a morphism of the canonical plot of the magma $(\mathbb{R}^+, \star)$, which is in turn neither unital nor associative unless $m = n = 1$.

6.3. **Invertible arrows and isomorphisms.** One benefit of taking the “point of view of regular representations” is the possibility of defining isomorphisms even in absence of local identities. Not only this gives a generalization of the corresponding categorical notion, but also a number of basic properties are saved.

A consequence of the approach is the “natural” distinction between isomorphisms and invertible arrows, though the two concepts collapse onto each other in presence of identities. As mentioned before in the introduction, this has problematic consequences, e.g. with respect to the binary relation “being isomorphic” on the object-class, for which appropriate solutions are however possible. N.B.: We keep on using the notation from the previous section.

**Definition 15.** Taking Remark 5 in mind, we say that a $P$-morphism $A \xrightarrow{f} B$ is

1. right invertible (in $P$) if $A \in \text{dom}(\text{id}_P)$ and there exists a $P$-morphism $f^\times : B \to A$, called a right inverse of $f$, such that $f \circ f^\times = \text{id}_P(A)$.

2. left invertible (in $P$) if $f^{\text{op}}$ is right invertible as an arrow of $P^{\text{op}}$, the arrow $f^\times : B \to A$ implicit to this definition being named a left inverse of $f$.

3. invertible (in $P$) if it is both left and right invertible.

We write $\text{l-inv}(P)$ and $\text{r-inv}(P)$, respectively, for the classes of all left and right invertible morphisms of $P$, and we let $\text{inv}(P) := \text{l-inv}(P) \cap \text{r-inv}(P)$.

The next propositions gives a link between invertible and split/cancellative arrows.

**Proposition 16.** Let $f : A \to B$ be an arrow of $P$. If $f$ is right (respectively, left) split and $B$ (respectively, $A$) is unital, then $f$ is left (respectively, right) invertible.
Proof. Obvious (we may omit the details).

**Proposition 17.** Let $f : A \to B$ be an arrow of $\mathcal{P}$. If $f$ is right/left invertible and $\mathcal{P}$ is left/right dissociative, then $f$ is left/right split.

**Proof.** Assume $f$ is left invertible and $\mathcal{P}$ is right dissociative (the other case follows by duality). Then $B$ is a unital object of $\mathcal{P}$ and there exists a morphism $f_\times : B \to A$ such that $f_\times \circ f = \text{id}_B$. Pick $g \in \text{hom}_\mathcal{P}(\_, B)$; the pair $(g, \text{id}_B)$ is composable (by the definition itself of identities), and by right dissociativity $g = g \circ (f_\times \circ f) = (g \circ f_\times) \circ f$, using the convention of Remark 5. This completes the proof.

**Proposition 18.** Let $f : A \to B$ be an arrow of $\mathcal{P}$. If $f$ is right/left invertible and $\mathcal{P}$ is left/right pre-associative, then $f$ is monic/epic (cf. Proposition 11).

**Proof.** Suppose that $f$ is left invertible and $\mathcal{P}$ is right pre-associative (the other case follows by duality). Then $B$ is a unital object of $\mathcal{P}$ and there exists $(f_\times : B \to A)_{\mathcal{P}}$ such that $f_\times \circ f = \text{id}_B$. Pick $g_1, g_2 \in \text{hom}_\mathcal{P}(\_, B)$ such that $f \circ g_1 = f \circ g_2$; the pair $(g_i, \text{id}_B)$ is composable, and by right pre-associativity $g_i = (f_\times \circ f) \circ g_i = f_\times \circ (f \circ g_i)$, taking in mind the convention of Remark 5. The claim follows.

As for unicity, we have the following result (whose proof we omit):

**Proposition 19.** Let $f : A \to B$ be an invertible arrow (of $\mathcal{P}$), and denote by $f_\times$ and $f^\times$, respectively, a left and a right inverse of $f$. If $\mathcal{P}$ is associative, then $f_\times$ and $f^\times$ are unique and $f_\times = f^\times$ (cf. Example 1).

**Remark 19.** If an arrow $(f : A \to B)_{\mathcal{P}}$ has a unique left inverse $f_\times$ and a unique right inverse $f^\times$, and additionally $f_\times = f^\times$ (as in the case of Proposition 19), then we say that $f$ is strongly invertible, and refer to $f_\times$ (or, equivalently, to $f^\times$) as the inverse of $f$ (in $\mathcal{P}$), which we denote by $(f^{-1} : B \to A)_{\mathcal{P}}$, or simply by $f^{-1} : B \to A$ or $f^{-1}$ provided that $\mathcal{P}$ is implied from the context.

**Example 27.** Every local identity is invertible, and is in fact the inverse of itself.

The next proposition, in its turn, gives some sufficient conditions for the invertibility of the composition of invertible arrows:

**Proposition 20.** Let $(f : A \to B, g : B \to C)$ be a composable pair (of $\mathcal{P}$) and assume that $\mathcal{P}$ is pre-associative. Then the following holds:

(i) If $f$ and $g$ are right invertible, and $f^\times$ and $g^\times$ are right inverses of $f$ and $g$, respectively, then $g^\times \circ f^\times$ is (well-defined and) a right inverse of $f \circ g$.

(ii) If $f$ and $g$ are left invertible, and $f_\times$ and $g_\times$ are left inverses of $f$ and $g$, respectively, then $g_\times \circ f_\times$ is (well-defined and) a left inverse of $f \circ g$. 
Proof. Let $f^\times$ and $g^\times$ be right inverses of $f$ and $g$, respectively (the other case follows by duality). Then $(f, g \diamond g^\times)$ is a composable pair, from which it is found (by right pre-associativity) that the same is true for $(f \diamond g, g^\times)$, and also $f = f \diamond (g \circ g^\times) = (f \circ g) \circ g^\times$. This in turn implies that $((f \circ g) \circ g^\times, f^\times)$ is composable too, and then by left pre-associativity $g^\times \circ f^\times$ is well-defined, $(g \circ f, g^\times \circ f^\times)$ is composable in its own turn, and $\text{id}_P(A) = f \circ f^\times = (f \circ g) \circ (g^\times \circ f^\times)$.

Sort of a converse of Proposition 20 is also possible:

**Proposition 21.** Let $(f : A \to B, g : B \to C)$ be a composable pair (of $P$) and assume that $P$ is left/right dissociative and $f \circ g$ is right/left invertible. Then $f$ is right invertible/g is left invertible.

**Proof.** By duality, we may suppose that $P$ is left dissociative and $f \circ g$ is right invertible. There then exists a morphism $u \in \text{hom}_P(C \downarrow f \circ g, -)$ such that $(f \circ g) \circ u = \text{id}_P(A)$. By left dissociativity, this implies that both $g \circ u$ and $f \circ (g \circ u)$ are well-defined, and moreover $(f \circ g) \circ u = f \circ (g \circ u)$, viz $f$ is right invertible.

So, at long last, we have come to the notion of isomorphism.

**Definition 16.** We say that a $P$-morphism $A \xrightarrow{f} B$ is

1. a right isomorphism (of $P$) if it is monic and left split.
2. a left isomorphism (of $P$) if $f^{op}$ is a right isomorphism of $P^{op}$.
3. an isomorphism, or a iso, (of $P$) if it is both a left and a right invertible.
4. an automorphism (of $P$) if $A = B$ and $f$ is an isomorphism.

We denote by l-iso($P$) and r-iso($P$), respectively, the classes of all left and right invertible $P$-morphisms, and we set iso($P$) := l-iso($P$) $\cap$ r-iso($P$).

**Remark 20.** The reader may want to compare the above with point (3) of Proposition 1.9.2 in [8]: What is a derived property of isomorphisms in the categorical case, is assumed here to hold by definition.

**Example 28.** By Examples 15 and 21, every local identity is an isomorphism; cf. point (1) of Proposition 1.9.2 in [8].

In absence of identities, it is hard to think of an alternative definition of isomorphisms which may be more “natural” than the one above, as far as one embraces, as we are doing, the “point of view of regular representations”: If injectivity is to “being cancellative” as surjectivity is to “being split”, then it seems quite reasonable to associate bijectivity to “being an isomorphism”. All the more that this leads to the abstraction of properties and constructions related to the “ordinary” (that is, categorical) notion of isomorphism, as in the case of the next proposition, which is a generalization of [8, Proposition 1.9.3].
Proposition 22. Let \( f : A \to B \) be a \( \mathbf{P} \)-morphism, and suppose that \( A \) and \( B \) are unital and \( \mathbf{P} \) is dissociative. If \( f \) is a right/left isomorphism, then \( f \) is a strongly invertible isomorphism, and the same is true for its inverse.

Proof. By duality, it is enough to consider the case when \( f \) is a right isomorphism (i.e., right split and monic). Then, \( f \) is left invertible (by Proposition 16), and hence epic (by Remark 4 and Proposition 18). Let \( f_x \) be a left inverse of \( f \). Since \( B \) is unital, \((f, f_x \circ f)\) is composable, and then it goes the same for the pairs \((f, f_x)\) and \((f \circ f_x, f)\), where we use that \( \mathbf{P} \) is right dissociative. Furthermore, it holds
\[
f = f \circ (f_x \circ f) = (f \circ f_x) \circ f = \text{id}_{\mathbf{P}}(A) \circ f,
\]
so that \( f \circ f_x = \text{id}_{\mathbf{P}}(A) \), because \( f \) is monic. To wit, \( f \) is invertible, and \( f_x \) is the inverse of \( f \) (by Proposition 17 and, again, Remark 4). It follows from Proposition 17 that \( f \) is also left split (we use here that \( \mathbf{P} \) is left dissociative), which in turn implies that \( f \) an isomorphism. The rest is obvious (we may omit the details). \( \square \)

The following is sort of an inverse of Proposition 22:

Proposition 23. Let \( f : A \to B \) be a \( \mathbf{P} \)-morphism, and suppose that \( f \) is right/left invertible and \( \mathbf{P} \) is left/right dissociative. Then \( f \) is a right/left isomorphism.

Proof. It is immediate by Propositions 17 and 18, Remark 4 and Definition 16. \( \square \)

Apart from giving an “identity-free” generalization of the notion itself of isomorphism to more abstract scenarios than categories (in view of Proposition 22), another positive aspect of this view is that the change of perspective has the benefit of making pieces of the complex mosaic of category theory (related to fundamental properties of morphisms) look, in some sense, more “homogeneous” than in the “classical” approach, in that all of them are now put onto one common ground - that of regular representations -, which is, if nothing else, at least conceptually attractive.

Now, an obvious question is: Are we really enlarging the range of category-like structures for which the above notion of isomorphism is “non-vacuous” beyond the scope of unital semigroupoids? And the answer is positive, as shown by the next examples:

Example 29. The one-object plot of Example 26 is neither unital nor associative, but nonetheless all of its arrows are automorphisms.

Example 30. Define the tuple \( (\mathcal{C}_0, \mathcal{C}_1, s, t, c) \) as follows: \( \mathcal{C}_0 \) is just the object class of \( \mathbf{P} \), while \( \mathcal{C}_h \) is the same as the hom-class of \( \mathbf{P} \) save for the fact that we remove any local identity; \( s \) and \( t \) are, each in turn, the restrictions of the source and target maps of \( \mathbf{P} \) to \( \mathcal{C}_h \); and \( c \) is the corestriction to \( \mathcal{C}_h \) of the restriction to \( \mathcal{C}_h \times \mathcal{C}_h \) of the composition law of \( \mathbf{P} \). It is easy to see that \( (\mathcal{C}_0, \mathcal{C}_1, s, t, c) \) is a plot: We call it the deunitization of \( \mathbf{P} \), and denote it by \( \mathcal{P} \). The deunitization of a saturated plot does
not need be saturated, although it can be such (see Section 7.3), but \( \mathcal{P} \) is associative whenever \( \mathcal{P} \) is. In particular, the deunitization of a category \( \mathcal{C} \) is an associative plot, and every \( \mathcal{C} \)-isomorphism which is not a local identity, continues being an isomorphism in \( \mathcal{C} \), but is no longer an invertible arrow. And the deunitization of a groupoid is a somewhat extreme example of an associative isoid where no arrow is either left or right invertible.

The above motivates the following:

**Definition 17.** We say that a plot is a left/right isoid if each of its arrows is a left/right isomorphism. An isoid is then a plot that is both a left and a right isoid, and a groupoid is an isoid that is also a category.

However, it is not all a bed of roses, and somewhat awkward situations may arise: For instance, with the notation of Examples 16 and 22, it is immediate to see that 1 is an isomorphism, but \( 1 \circ 1 \) is none of monic, epic, right split or left split. This shows that, as before with the case of cancellative and split arrows, isomorphisms do not form, at least in general, a compositive subclass. But we have the following positive result, which generalizes point (2) of [8, Proposition 1.9.2]:

**Proposition 24.** Let \( (f : A \to B, g : B \to C) \) be a composable pair of \( \mathcal{P} \).

(i) If \( f \) and \( g \) are left/right isomorphisms and \( \mathcal{P} \) is left/right dissociative and right/left pre-associative, then \( f \circ g \) is a left/right isomorphism too.

(ii) If \( \mathcal{P} \) is dissociative and \( f, g \) are isomorphisms, then \( f \circ g \) is an iso too.

*Proof.* It is straightforward from Remark 4 and Propositions 10 and 14. \( \square \)

Like with cancellative, split and invertible morphisms, we can now use isomorphisms to build new plots of general interest.

**Example 31.** We write \( \text{l-ISO}(\mathcal{P}) \), \( \text{r-ISO}(\mathcal{P}) \) and \( \text{ISO}(\mathcal{P}) \), respectively, for the restrictions of \( \mathcal{P} \) to the binary relations \( \text{l-iso}(\mathcal{P}) \times \text{l-iso}(\mathcal{P}) \), \( \text{r-iso}(\mathcal{P}) \times \text{r-iso}(\mathcal{P}) \) and \( \text{iso}(\mathcal{P}) \times \text{iso}(\mathcal{P}) \) on \( \text{hom}(\mathcal{P}) \). None of these needs be a saturated subplot of \( \mathcal{P} \), no matter if \( \mathcal{P} \) is or not (see Example 22), but all of them are semicategories/categories if this is true for \( \mathcal{P} \), by Proposition 24 and Remark 4, and in the case of categories we have also \( \text{l-ISO}(\mathcal{P}) = \text{r-ISO}(\mathcal{P}) = \text{ISO}(\mathcal{P}) = \text{Inv}(\mathcal{P}) \), by Proposition 22.

We are thus left with one question: What should it mean, in the abstract setting of plots, that a pair \((A, B)\) of \( \mathcal{P} \)-objects are “isomorphic”? Asking for the existence of an isomorphism \( f : A \to B \) is no longer satisfactory, as far as “being an isomorphism” must be an equivalence relation on the object-class of \( \mathcal{P} \), which seems essential, for clustering objects into “isomorphism classes”, whatever these may be on the level of plots, is likely to be a fundamental step for going deeper into the theory and obtaining some non-trivial extension of more categorical results.
and constructions. So the new question is: Can we find a natural extension of the categorical point of view even in this respect? Again, the answer is in the positive, as shown in the forthcoming (short) subsection.

6.4. $\mathcal{M}$-morphic objects and the like. Let $P$ be a plot and $\mathcal{M}$ a (possibly empty) subclass of $\text{hom}(P)$. Given objects $A, B \in P$, we say that $A$ is $\mathcal{M}$-connected to $B$ (in $P$), and we write $(A \overset{\mathcal{M}}{\sim} B)_P$, or just $A \overset{\mathcal{M}}{\sim} B$ if no confusion is possible, if there exists an $\mathcal{M}$-connection $\gamma$ (of $P$) from $A$ to $B$; in particular, $A$ is said to be connected to $B$ if $\gamma$ is a connection (see Example 9), in which case we use $(A \overset{\equiv}{\sim} B)_P$, or simply $A \overset{\equiv}{\sim} B$, in place of $(A \overset{\mathcal{M}}{\sim} B)_P$.

Note that “being $\mathcal{M}$-connected” is a symmetric and transitive relation on the object class of $P$, but not necessarily reflexive (and hence, not an equivalence), unless $\text{hom}_P(A)$ is non-empty for each $A \in \text{Ob}(P)$, which is the case when $P$ is unital. Thus we say that $A$ is $\mathcal{M}$-equivalent to $B$ (in $P$), here written as $(A \overset{\mathcal{M}}{\simeq} B)_P$, or simply as $A \overset{\mathcal{M}}{\simeq} B$ when $P$ is implied from the context, if the pair $(A, B)$ is in the equivalence generated by the binary relation “being $\mathcal{M}$-connected” on $\text{Ob}(P)$, that is $A = B$ or $A$ is $\mathcal{M}$-connected to $B$ (the reason for this notation and terminology will be clear in a while).

More specifically, $A$ is called isoequivalent to $B$ (in $P$) if $A \overset{\mathcal{M}}{\simeq} B$ for $\mathcal{M} = \text{Iso}(P)$. Monoequivalent and epiequivalent objects are defined in a similar way, by assuming, respectively, $\mathcal{M} = \text{Mono}(P)$ and $\mathcal{M} = \text{Epi}(P)$ in the above definitions.

**Remark 21.** Let us observe that $A$ is $\emptyset$-equivalent to $B$ if and only if $A = B$, while $A$ is $\text{hom}(P)$-equivalent to $B$ if and only if $A$ and $B$ are in the same connected component of $P$, where we define an $\mathcal{M}$-connected component of $P$ as any full subplot $Q$ of $P$ such that $\text{Ob}(Q)$ is an equivalence class in the quotient of $\text{Ob}(P)$ by the binary relation $\mathcal{M}$, and refer to this as a connected component of $P$ in the special case when $\mathcal{M} = \text{hom}(P)$.

The next lemma shows that “being $\mathcal{M}$-equivalent objects” is an abstraction of the categorical notion of connectedness, and it is used below to prove a characterization of $\mathcal{M}$-equivalences in terms of $\mathcal{M}$-paths.

**Lemma 3.** Suppose that $\gamma = (f_1, \ldots, f_n)$ is an $\mathcal{M}$-connection from $A$ to $B$. There then exist $k \in \mathbb{N}^+$, objects $X_0, \ldots, X_k \in P$, and indices $i_1, \ldots, i_k \in \llbracket n \rrbracket$ with $i_1 < \cdots < i_k$ such that $(f_{i_1}, \ldots, f_{i_k})$ is an $\mathcal{M}$-connection from $A$ to $B$ and $f_{i_t} \in \text{hom}_P(X_{i_t-1}, X_{i_t})$ for each $t = 1, \ldots, k$. Furthermore, if $\mathcal{M} \cap \text{hom}_P(C)$ is non-empty for any $C \in \text{Ob}(P)$, then $A$ is $\mathcal{M}$-connected to $B$ if and only if there exist an integer $n \geq 1$, objects $X_0, \ldots, X_{2n} \in P$ with $X_0 = A$ and $X_{2n} = B$, and arrows $f_1, \ldots, f_{2n} \in P$ such that $f_{2i-1} \in \text{hom}(X_{2i-1}, X_{2i-2})$ and $f_{2i} \in \text{hom}_P(X_{2i-1}, X_{2i})$ for each $i \in \llbracket 2n \rrbracket$. In particular, if $P$ is unital then $A$ is
connected to \( B \) (in the sense of our definitions) if and only if \( A \) is connected to \( B \) in the sense of [8, Example 2.6.7.e].

**Proof.** The first part is a routine induction on the length of \( \gamma \): The induction basis is obvious, and the inductive step follows at once from the definition of an \( \mathcal{M} \)-connection. As for the second part, it is enough to observe that, by the standing assumptions, a sequence of \( \mathbf{P} \)-arrows of the form \((f : X \to Y, g : Y \to Z)\) can always be turned into a longer sequence of the form \((e_X : X \leftarrow X, f : X \to Y, e_Y : Y \leftarrow Y, g : Y \to Z)\) by insertion of arbitrary morphisms \(e_X \in \mathcal{M} \cap \text{hom}_P(X)\) and \(e_Y \in \mathcal{M} \cap \text{hom}_P(Y)\), and similarly for a sequence of the form \((h : X \leftarrow Y, \ell : Y \leftarrow Z)\).

\(\square\)

**Proposition 25.** It holds that \( A \) is \( \mathcal{M} \)-equivalent to \( B \) if and only if \((A, B)\) is in the equivalence generated by the binary relation “There exists an \( \mathcal{M} \)-path (respectively, an \( \mathcal{M} \)-copath) from \( X \) to \( Y \)” on the object class of \( \mathbf{P} \).

**Proof.** In the light of Lemma 3, it suffices to consider that \((A, B)\) is in the equivalence relation described by the statement if and only if \( A = B \) or there can be found an integer \( n \geq 1 \) and objects \( X_0, X_1, \ldots, X_n \in \mathbf{P} \) such that \( X_0 = A, X_n = B \) and for each \( i \in [n] \) there exists an \( \mathcal{M} \)-path (respectively, an \( \mathcal{M} \)-copath) from \( X_{i-1} \) to \( X_i \) or from \( X_i \) to \( X_{i-1} \).

\(\square\)

On another hand, we let \( A \) be \( \mathcal{M} \)-morphic to \( B \) (in \( \mathbf{P} \)) if there exists \((\gamma, \varphi) \in \text{Fact}_{\mathcal{M}}(\mathbf{P})\) such that \( \gamma \) is an \( \mathcal{M} \)-path from \( A \) to \( B \), which is denoted by writing \((A \overset{\mathcal{M}}{\longrightarrow} B)_{\mathbf{P}}\), or simply \( A \overset{\mathcal{M}}{\longrightarrow} B \) if there is no likelihood of confusion.

**Proposition 26.** If \( \mathcal{M} \) is a compositive subclass of \( \mathbf{P} \), then \( A \overset{\mathcal{M}}{\longrightarrow} B \) if and only if there exists a \( \mathbf{P} \)-morphism \( f : A \to B \) in \( \mathcal{M} \). Also, \( A \overset{\text{Inv}(\mathbf{P})}{\simeq} B \) if and only if there exists an \( \text{Inv}(\mathbf{P}) \)-path from \( A \) to \( B \), and \( A \overset{\text{Iso}(\mathbf{P})}{\simeq} B \) when \( \mathbf{P} \) is unital only if \( A \overset{\text{Iso}(\mathbf{P})}{\simeq} B \).

**Proof.** It is obvious from our definitions (we may omit the details).

We say that \( A \) is homomorphic to \( B \) (in \( \mathbf{P} \)) if \( A \) is \( \mathcal{M} \)-morphic to \( B \) for \( \mathcal{M} = \text{hom}(\mathbf{P}) \). This sounds suggestive, for then “being homomorphic” boils down exactly, in the light of Proposition 26, to the expected notion of homomorphism for the case when \( A \) and \( B \) are “structures” (groups, rings, lattices, posets, etc.). In addition, \( A \) is said

- to be isomorphic to \( B \) (in \( \mathbf{P} \)) if \( A \overset{\mathcal{M}}{\longrightarrow} B \) for \( \mathcal{M} = \text{Iso}(\mathbf{P}) \).
- to embed, or to be embedable, into \( B \) if \( A \overset{\mathcal{M}}{\longrightarrow} B \) for \( \mathcal{M} = \text{Mono}(\mathbf{P}) \).
- to split through \( B \) if \( A \overset{\mathcal{M}}{\longrightarrow} B \) for \( \mathcal{M} = \text{Spl}(\mathbf{P}) \).
Similar considerations like those expressed above for homomorphisms then extend to “being isomorphic” and “being embedable” (regarded as binary relations on the objects of $P$), and something else is true for the former, as is implied by the following:

**Proposition 27.** If $P$ is dissociative then $A \xrightarrow{\text{Inv}(P)} B$ if and only if $A \xrightarrow{\text{Iso}(P)} B$.

**Proof.** Immediate by Propositions 22 and 26 (again, we may omit the details). □

So we see that, in a perfect accordance with the critical separation between isomorphisms and invertible arrows observed before in Section 6.3, an effective generalization of the categorical notion of “isomorphic objects” to the setting of plots is not “being isomorphic”, as far as the goal is to get an equivalence relation out of the heck of a similar concept, but instead “being isoequivalent”. Therefore, we are “naturally” led to the conclusion that, from the essential perspective of plots, $A \cong \text{Iso}(P)B$ has not much to do with the existence of an isomorphism from $A$ to $B$ or vice versa (let alone an invertible arrow), although this is “accidentally” the case when $P$ is a unital semigroupoid, and hence, in particular, a category (by Propositions 24, 26 and 27), as a consequence of the “inherent rigidity” of certain structures.

### 6.5. Endomorphisms and orders.

The material of this section is somewhat subsidiary to the core of the paper. However, as it deals with basic elements of the language of plots, we find appropriate to mention it here, all the more that part of this will be used later in reference to constant functors (see Proposition 35).

Let $P$ be a plot. An arrow $f \in P$ is called an endomorphism (of $P$) if $f \in \text{hom}_P(A)$ for some object $A \in P$; in particular, every automorphism is an endomorphism. We use $\text{end}(P)$ for the class of all the endomorphisms of $P$, and $\text{End}(P)$ for the restriction of $P$ to the binary relation $\text{end}(P) \times \text{end}(P)$ on $\text{hom}(P)$.

Now, pick an endomorphism $f \in P$. We say that $f$ is a periodic arrow (of $P$) if there exist $n, p \in \mathbb{N}^+$ and fundamental parenthesizations $\varphi_n$ and $\varphi_{n+p}$ of lengths $n$ and $n + p$, respectively, such that the $n$-tuple $(f, \ldots, f)$ is in the domain of $\varphi_n[P]$, the $p$-tuple $(f, \ldots, f)$ is in the domain of $\varphi_p[P]$, and the following holds:

$$(f^n)_{\varphi_n[P]} \circ (f^p)_{\varphi_p[P]} = (f^n)_{\varphi_n[P]},$$

in which case it is an easy exercise to prove, by a routine induction, that

$$(f^n)_{\varphi_n[P]} = (f^n)_{\varphi_n[P]} \circ (f^p)_{\varphi_p[P]} = ((f^n)_{\varphi_n[P]} \circ (f^p)_{\varphi_p[P]}) \circ (f^p)_{\varphi_p[P]} =$$

$$= (((f^n)_{\varphi_n[P]} \circ (f^p)_{\varphi_p[P]}) \circ (f^p)_{\varphi_p[P]}) \circ (f^p)_{\varphi_p[P]} = \cdots,$$

where all the relevant compositions are seen to exist. We refer to the smallest $n$ for which (6) holds as the index of $f$, here written as $\text{ind}_P(f)$, and to the smallest $p$
relative to this \( n \) as the period of \( f \), here denoted by \( \text{prd}_P(f) \); both of these are set equal to \( \Omega \) if \( f \) is not periodic. Accordingly, we let

\[
\text{ord}_P(f) := \text{ind}_P(f) + \text{prd}_P(f) - 1,
\]

and we call \( \text{ord}_P(f) \) the order of \( f \) (as is usual, when there is no danger of confusion, we omit the subscript “\( P \)” from the notation). Note how this generalizes the notions of period, index and order from the theory of semigroups; cf. \[33, \text{p. 10}\]. In particular, \( f \) is said to be simply periodic (in \( P \)) if it is periodic with \( \text{ind}_P(f) = 1 \), and idempotent (in \( P \)) if it is simply periodic with \( \text{prd}_P(f) = 1 \). Of course, every local identity is idempotent.

**Question 1.** Is there any “natural” way to define a “non-trivial” notion of order for arrows that are not necessarily endomorphisms?

At present, we have neither ideas on this nor a formal definition of what should be meant by the words “natural” and “non-trivial” in the statement of the question (in fact, this is part of the question itself), but we hope that somebody else may find the challenge stimulating and end up with an answer, which is why we have mentioned it here.

## 7. Plot homomorphisms

In this section, we present punctors, which are a generalization of both functors [3, Definition 1.2] and semifunctors (in the sense of Mitchell) [50, Section 4], and we prove some of their basic properties. Again, the main goal is to show that fundamental features of the theory of categories can be “naturally” extended to plots, for which punctors give a satisfactory notion of “structure-preserving map”.

### 7.1. Punctors and categories of plots.

The idea of punctors is not new (see \[58, \text{Section 1}\]), though we “smooth” it here by the introduction of unital punctors, which are “naturally” motivated by the notion of unital object, and represent a better surrogate of ordinary functors.

**Definition 18.** A punctor is a triple \((P, Q, F)\), where \( P \) and \( Q \) are plots and \( F \) is a pair \((F_o : \text{Ob}(P) \to \text{Ob}(Q), F_h : \text{hom}(P) \to \text{hom}(Q))\) of maps such that:

1. If \( A, B \in \text{Ob}(P) \) and \( f \in \text{hom}_P(A, B) \), then \( F_h(f) \in \text{hom}_Q(F_o(A), F_o(B)) \).
2. If \((f, g)\) is a composable pair of \( P \), then \((F_h(f), F_h(g))\) is a composable pair of \( Q \) and \( F_h(f \circ_P g) = F_h(f) \circ_Q F_h(g) \).

The triple \((P, Q, F)\) is then more usually written as \( F : P \to Q \), and one refers to \( F \) as a punctor from \( P \) to \( Q \), and to \( F_o \) and \( F_h \), respectively, as the object and the arrow component of \( F \), while \( P \) and \( Q \) are named the source and target (plot) of \( F \) (we say that the members of a family of punctors are parallel if all of them have
the same source and target). In addition to this, \( F \) is called unital if \( F_h(i) \) is a local identity of \( Q \) whenever \( i \) is a local identity of \( P \).

**Remark 22.** If \( F : P \to Q \) is a punctor, we abuse notation on a systematic basis and don’t distinguish between \( F \) and its components \( F_o : \text{Ob}(P) \to \text{Ob}(Q) \) and \( F_h : \text{hom}(P) \to \text{hom}(Q) \), to the extent of writing \( F(A) \) instead of \( F_o(A) \) for \( A \in \text{Ob}(P) \) and \( F(f) \) in place of \( F_h(f) \) for \( f \in \text{hom}(P) \).

**Remark 23.** In our approach, an ordinary functor from a category \( C \) to a category \( D \) is literally a unital punctor from \( C \) to \( D \), which seems enough to justify the use of the term “functor”, in the sequel, as a regular systematic alternative to “unital punctor”. On another hand, a semifunctor in the sense of Mitchell [50, Section 4] is but a punctor between saturated associative plots.

**Remark 24.** A one-object plot \( P \) is essentially the same thing as a partial magma (see Example 1). Punctors between one-object saturated plots thus correspond to homomorphisms of partial magmas, here understood as triples \( (A_1, A_2, f) \) such that \( A_i = (A_i, \star_i) \) is a partial magma and \( f \) is a function \( A_1 \to A_2 \) with the property that \( f(a \star_1 b) = f(a) \star_2 f(b) \) for all \( a, b \in A_1 \), where the composition on the right-hand side is required to exist whenever \( a \star_1 b \) is well-defined; cf. [10, Definition I.1.3].

**Remark 25.** A punctor \( F : P \to Q \) maps connections, factorizations, and cofactorizations of \( P \), respectively, to connections, factorizations, and cofactorizations of \( Q \).

**Example 32.** For any plot \( P = (C_0, C_1, s, t, c) \) the pair \((1_{C_0}, 1_{C_1})\) is clearly a unital punctor, called the identity punctor of \( P \). On the other hand, the unique punctor from the empty quiver to \( P \), which is unital too, is referred to as the empty punctor of \( P \).

**Example 33.** If \( F : P \to Q \) is a punctor, the pair consisting of the functions \( \text{Ob}(P^{\text{op}}) \to \text{Ob}(Q^{\text{op}}) : A \mapsto F(A) \) and \( \text{hom}(P^{\text{op}}) \to \text{hom}(Q^{\text{op}}) : f \mapsto F(f) \) is a punctor \( P^{\text{op}} \to Q^{\text{op}} \): This is denoted by \( F^{\text{op}} \) and termed the dual of \( F \). Note that \( (F^{\text{op}})^{\text{op}} = F \), which can be expressed by saying that “dualization of punctors is involutive”; cf. Definition 3.41 and Remark 3.42 in [1].

The next lemma accounts for the effect of punctors on parenthesized “products”, and will be used later in this section to define the evaluation punctor of a plot.

**Lemma 4.** Let \( F : P \to Q \) be a punctor, and for a fixed integer \( n \geq 1 \) let \( \varnothing \) be a fundamental parenthesization of length \( n \) and \( X_1, \ldots, X_n \) subclasses of \( \text{hom}(P) \) such that \( X_1 \times \cdots \times X_n \subseteq \text{dom}(\varnothing[P]) \). Then

\[
F(\langle X_1 \diamond_P \cdots \diamond_P X_n \rangle_{\varnothing[P]}) = (F(X_1) \diamond_Q \cdots \diamond_Q F(X_n))_{\varnothing[Q]}.
\]
Proof. It is straightforward by induction on \( n \) (we may omit the details).

It is convenient to extend to punctors the notions of restriction and corestriction proper of functions (a class can be viewed as a partial magma whose binary operations have an empty domain, to the effect that a partial function can be thought of as a homomorphism of partial magmas). Incidentally, this provides a way to define new plots from old ones by using punctors, as in the following:

**Example 34.** Let \( F : \mathcal{P} \rightarrow \mathcal{Q} \) be a punctor, and let \( F_o : \text{Ob}(\mathcal{P}) \rightarrow \text{Ob}(\mathcal{Q}) \) and \( F_h : \text{hom}(\mathcal{P}) \rightarrow \text{hom}(\mathcal{Q}) \) be its components.

If \( \mathcal{P}' = (\mathcal{C}_0, \mathcal{C}_1, s, t, c) \) is a subplot of \( \mathcal{P} \), we refer to the relative subplot of \( \mathcal{Q} \) generated by \( (F_o(\mathcal{C}_0), F_h(\mathcal{C}_1)) \) as the (direct) image of \( \mathcal{P}' \) under \( F \), and we denote it by \( F(\mathcal{P}') \); in particular, \( F(\mathcal{P}) \) is simply named the (direct) image of \( F \).

On the other hand, if \( \mathcal{Q}' = (\mathcal{D}_0, \mathcal{D}_1, u, v, d) \) is a subplot of \( \mathcal{Q} \), the relative subplot of \( \mathcal{P} \) generated by \( (F_o^{-1}(\mathcal{D}_0), F_h^{-1}(\mathcal{D}_1)) \) is called the inverse image of \( \mathcal{Q}' \) under \( F \), and is denoted by \( F^{-1}(\mathcal{Q}') \); specifically, \( F^{-1}(\mathcal{Q}) \) is termed the inverse image of \( F \).

It is seen that \( F(\mathcal{P}') \) is a subcategory of \( \mathcal{Q} \) whenever \( \mathcal{P} \) and \( \mathcal{Q} \) are categories, \( \mathcal{P}' \) is a subcategory of \( \mathcal{P} \) and \( F \) is a functor. Moreover, \( F(F^{-1}(\mathcal{Q}')) \leq \mathcal{Q}' \) and \( \mathcal{P}' \leq F^{-1}(F(\mathcal{P}')) \), which generalizes the set-theoretic property according to which, given a function \( f : X \rightarrow Y \) and subsets \( S \subseteq X \) and \( T \subseteq Y \), it holds \( S \subseteq f^{-1}(f(S)) \) and \( f(f^{-1}(T)) \subseteq T \).

In addition, it is easy to check that the pair \((J_o, J_h)\), where \( J_o \) is the restriction of \( F_o \) to \( \text{Ob}(\mathcal{P}') \) and \( J_h \) is the restriction of \( F_h \) to \( \text{hom}(\mathcal{P}') \), is a punctor from \( \mathcal{P}' \) to \( \mathcal{Q} \), which we call the restriction of \( F \) to \( \mathcal{P}' \), and write as \( F_{\mathcal{P}} \) (note that \( F_{\mathcal{P}} = F \)).

Similarly, if we let \( L = (L_o, L_h) \) be the restriction of \( F \) to \( F^{-1}(\mathcal{Q}') \), and then we take \( K_o \) to be the corestriction of \( L_o \) to \( \text{Ob}(\mathcal{Q}') \) and \( K_h \) the corestriction of \( L_h \) to \( \text{hom}(\mathcal{Q}') \), then the pair \((K_o, K_h)\) is a punctor from \( F^{-1}(\mathcal{Q}') \) to \( \mathcal{Q}' \), here referred to as the corestriction of \( F \) to \( \mathcal{Q}' \), and denoted by \( F_{\mathcal{Q}} \) (observe that \( F_{\mathcal{Q}}(\mathcal{P}) = F(\mathcal{P}) \)).

The last remark leads to the following:

**Definition 19.** We say that a punctor \( G : \mathcal{P} \rightarrow \mathcal{Q} \) is an extension of a punctor \( F : \mathcal{P}' \rightarrow \mathcal{Q}' \) if \( \mathcal{P}' \leq \mathcal{P} \), \( \mathcal{Q}' \leq \mathcal{Q} \) and the restriction to \( \mathcal{P}' \) of the corestriction of \( G \) to \( \mathcal{Q}' \) is precisely \( F \).

Punctors are classified in the very same way as functors in the restricted sense of category theory: Though some definitions are almost identical in the two cases (save for the notion of “fullness”, which is subtler), we include all of them here for completeness and future reference; cf. [8, Definition 1.5.1]. The main novelty is represented here by the notions of \( M \)-density and \( M \)-equivalence.

**Definition 20.** Let \( F : \mathcal{P} \rightarrow \mathcal{Q} \) be a punctor and, for \( X, Y \in \text{Ob}(\mathcal{P}) \), denote by \( F_{X,Y} \) the map \( \text{hom}_\mathcal{P}(X,Y) \rightarrow \text{hom}_\mathcal{Q}(F(X), F(Y)) : f \mapsto F(f) \), here called, as expected, the component of \( F \) relative to the pair \((X,Y)\). We say that \( F \) is:
1. an endopunctor (of $P$) if $P = Q$, and an endofunctor if $F$ is unital.
2. faithful if $F_{X,Y}$ is injective for all $X, Y \in \text{Ob}(P)$.
3. full if $F_{X,Y}$ is surjective for all $X, Y \in \text{Ob}(P)$.
4. fully faithful if $F$ is both full and faithful.
5. injective on objects if the object component, say $F_o$, of $F$ is injective.
6. an embedding (of plots) if it is faithful and injective on objects.
7. an isomorphism (of plots) if it is fully faithful and $F_o$ is bijective.
8. an automorphism (of $P$) if $F$ is an isomorphism and $P = Q$.

Suppose now that $M$ is a certain subclass of $Q$. We say that $F$ is:
9. $M$-dense if for each $B \in \text{Ob}(Q)$ there exists $A \in \text{Ob}(P)$ such that $(F(A) \simeq_M B)_{\mathcal{Q}}$.
10. an $M$-equivalence (of plots) if it is fully faithful and $M$-dense.

In particular, we refer to $F$ as a dense, or essentially surjective, punctor if $F$ is $M$-dense for $M = \text{hom}(Q)$. A fully faithful and essentially surjective punctor is then simply called an equivalence (of plots).

Many of the constructions that we have encountered so far are, either implicitly or explicitly, related to various types of punctors from the above list.

**Example 35.** The corestriction of a punctor $F : P \to Q$ to $F(P)$ is full, which generalizes the fact that the corestriction of a function $f : X \to Y$ to $f(X)$ is onto.

**Example 36.** Suppose $P = (C_0, C_1, s, t, c)$ and $Q = (D_0, D_1, u, v, d)$ are plots, with $Q \leq P$. Then clearly, the pair $I := (I_o : D_0 \to C_0 : A \mapsto A, I_h : D_1 \to C_1 : f \mapsto f)$ is an embedding of plots, here called the inclusion (punctor) of $Q$ into $P$. Observe that $I$ is a functor if $Q \leq 1_P$ (the condition is not necessary), which is the case if $P$ is a category and $Q$ is a subcategory of $P$; cf. [8, Definition 1.5.3].

Now, given a subclass $M$ of $\text{hom}(P)$, we say that $Q$ is an $M$-dense subplot of $P$ if the inclusion of $Q$ into $P$ is an $M$-dense punctor, and we let an $M$-skeleton of $P$ be any full and $M$-dense subplot $S$ of $P$ in which no two distinct objects are $M$-equivalent. Moreover, we say that $P$ is $M$-skeletal if $P$ is an $M$-skeleton of itself. This yields a generalization of the categorical notion of skeleton (see [1, Definition 4.12]), which is recovered in the special case when $P$ is a category and $M = \text{Iso}(P)$.

Any $M$-skeleton $S$ of $P$ is $M$-equivalent to $P$, for the inclusion of $S$ into $P$ is clearly an $M$-equivalence. Also, $P$ has always an $M$-skeleton: If $O_M$ is the quotient of $\text{Ob}(P)$ by $\simeq_M$ and $\chi$ a choice function $O_M \to \text{Ob}(P)$ sending an equivalence class to a distinguished representative of itself, then the full subplot of $P$ generated by the image of $\chi$ is an $M$-skeleton of $P$. The whole is, in turn, a generalization of points (1) and (3) of [1, Proposition 4.14] to the setting of plots.

**Example 37.** The identity functor of a plot $P$ is a unital automorphism of $P$. 
Example 38. A plot $P$ is called self-dual if $P$ is equivalent to $P^{op}$. In particular, every quiver is self-dual, and in fact isomorphic to its dual.

The next example will be used later in Section 7.6, to define $M$-limits and the like.

Example 39. With the same notation as in Example 9, we let $Ev_M[P]$ denote the pair $(E_o, E_h)$ where $E_o$ is the identity of $\text{Ob}(P)$ and $E_h$ the map from the hom-class of $\text{Fact}_M(P)$ to the hom-class of $P$ taking an $M$-factorization $(\gamma, \varphi)$ to $\varphi[P](\gamma)$. It follows from Lemma 4 that $Ev_M[P]$ is a functor $\text{Fact}_M(P) \to P$. In particular, we use $Ev_P$ for $Ev_M[P]$ in the case when $M = \text{hom}(P)$, and refer to $Ev_P$ as the evaluation functor of $P$. In fact, $Ev_P$ is full, but $Ev_M[P]$ is in general neither full nor faithful.

If $P$ and $Q$ are plots, we write $\text{Punct}(P, Q)$ for the class of all functors from $P$ to $Q$, and $\text{Funct}(P, Q)$ for the subclass of $\text{Punct}(P, Q)$ consisting of functors; note that $\text{Punct}(P, Q) = \text{Funct}(P, Q)$ if $\text{dom}(\text{id}_P) = \emptyset$, while $\text{Funct}(P, Q) = \emptyset$ if $P$ has at least one unital object, but $Q$ has none.

Example 40. If $F : P \to Q$ and $G : P \to Q$ are functors, and we let $F = (F_o, F_h)$ and $G = (G_o, G_h)$, it is then routine to check that the same is also true for the pair $(G_o \circ F_o, G_h \circ F_h)$, which we denote by $G \circ F$ or $GF$, and call the (functional) composition of $F$ with $G$.

If $P_0$ is the class of all small (i.e., $\Omega$-small) plots, $P_1$ is the union of all the sets of the form $\text{Punct}(P, Q)$ with $P, Q \in P_0$, and $s$ and $t$ are the functions $P_1 \to P_0$ sending a functor $F : P \to Q$ to $P$ and $Q$, respectively, and $c$ is the the function $P_1 \times P_1 \to P_1$ whose domain $D$ is the canonical pullback of $(t, s)$ and which maps a pair of functors $(F, G) \in D$ to $G \circ F$, it now follows from the above that $(P_0, P_1, s, t, d)$ is a (large) category, where the local identity of an object $P$ is the identity functor of $P$. This category is denoted here by $\text{Plot}(\Omega)$, or more simply by $\text{Plot}$ (if $\Omega$ is clear from the context), and called the category of small plots and functors.

In fact, $\text{Plot}(\Omega)$ has a number of rather interesting substructures, which includes, among the many others: the full subcat, $\text{uPlot}_1(\Omega)$, of (that is, generated by) unital small plots and functors; the full subcat, $\text{Quiv}(\Omega)$, of small quivers; the full subcat, $\text{Mgd}(\Omega)$, of small magmoids; the full subcat, $\text{Sgrpd}(\Omega)$, of small semigroupoids; the full subcat, $\text{Semicat}(\Omega)$, of small semicategories; the full subcat, $\text{Cat}(\Omega)$, of small categories and functors; the wide subcat, $\text{Cat}_1(\Omega)$, of $\text{Cat}(\Omega)$ of unital functors; and last but not least, the full subcat, $\text{Grpd}(\Omega)$, of $\text{Cat}_1(\Omega)$ of small groupoids (again, any reference to $\Omega$ is omitted in the notation if there is no danger of ambiguity).
The categories listed in Example 40, and a few others, will be the main subject of the second part of this work, where we prove the existence of some remarkable adjunctions between a few of them.

7.2. Basic properties of functors. In this section, we collect a few basic properties of functors (and functors), and compare them, when possible, with similar properties of their “categorical cousins”. In particular, our first proposition describes a “combinatorial” property of embeddings, which generalizes an analogous property of injective functions.

Lemma 5. If $F : P \to Q$ is an embedding (of plots), then both the object and the arrow components of $F$ are injective.

Proof. If $f_1 : A_1 \to B_1$ $(i = 1, 2)$ is a $P$-arrow and $F(f_i) = g$, then $F(A_1) = F(A_2)$ and $F(B_1) = F(B_2)$. Thus $A_1 = A_2 := A$ and $B_1 = B_2 := B$, since $F$ is injective on objects. It follows that $f_1 = f_2$, for $F$ is also faithful. □

Proposition 28. Let $F : P \to Q$ be an embedding (of plots), and for a fixed integer $n \geq 1$ let $\varphi$ be a fundamental parenthesisization of length $n$ and $X_1, \ldots, X_n$ subclasses of $\text{hom}(P)$ such that $X_1 \times \cdots \times X_n \subseteq \text{dom}(\varphi[P])$. Then

$$|(X_1 \diamond_P \cdots \diamond_P X_n)_{\varphi[P]}| = |(F(X_1) \diamond_Q \cdots \diamond_Q F(X_n))_{\varphi[Q]}|.$$

Proof. It is an immediate consequence of Lemmas 4 and 5. □

Although elementary, Proposition 28 is often useful, for instance, when we have to estimate the size, say, of a sumset naturally defined in an additive semigroup and it turns out to be advantageous to embed everything in a monoid (see Section 7.3).

The next result gives sufficient conditions for a functor to be a functor.

Proposition 29. Let $F : P \to Q$ be a functor and pick $B \in \text{Ob}(P)$.

(i) If $F$ is injective on objects and $B$ is unital in $P$, then $F(B)$ is unital in $F(P)$. In particular, $F(P)$ is a unital plot if $P$ is too.

(ii) Assume that $F$ is an embedding and $F(B)$ is unital in $F(P)$, and let $j_B$ be the local identity of $F(B)$ in $F(P)$. Then $F^{-1}(j_B)$ is a neutral morphism in $P$ (recall Lemma 5), and $B$ is unital if $P$ is saturated.

Proof. (i) Let $B$ be a unital object of $P$ and set $i_B := \text{id}_P(B)$. We claim that $F(i_B)$ is a local identity of $F(B)$ in $F(P)$. Since $F$ is injective on object if and only if $F^{\text{op}}$ is too, it is enough, by duality, to prove that $F(i_B)$ is a left identity in $F(P)$. For let $g : F(B) \to D$ be an arrow in $F(P)$. There then exists a morphism $f : A \to C$ in $P$ for which $g = F(f)$, so that $F(A) = F(B)$ and $F(C) = D$. Since $F$ is injective on objects, it follows that $A = B$. Therefore, $(i_A, f)$ is a composable pair of $P$ and $i_B \diamond_P f = f$, to the effect that $(F(i_B), g)$ is composable in $Q$ and

$$g = F(f) = F(i_B \diamond_P f) = F(i_B) \diamond_Q F(f) = F(i_B) \diamond_Q g.$$
To wit, \( i_B \) is a left identity in \( F(P) \), and the first part of the claim is proved.

(ii) In view of Lemma 5, there exists a unique arrow \( i_B : B \to B \) in \( P \) such that \( F(i_B) = j_B \), where \( j_B \) stands for the local identity of \( F(B) \) in \( F(P) \). We have to show that \( i_B \) is neutral \( P \). Since \( F \) is an embedding if and only if \( F^{op} \) is too, it will be enough, by duality, to prove that \( i_B \) is just left neutral. For pick \( f \in \text{hom}_P(B_{\downarrow f}, -) \).

Then \( F(i_B \cdot_P f) = F(i_B) \cdot_P F(f) = j_B \cdot_Q F(f) = F(f) \), which implies \( i_B \cdot_P f = f \), again by Lemma 5. This ultimately means that \( i_B \) is left neutral in \( P \), but also that \( i_B \) is a left identity if \( \text{hom}_P(B_{\downarrow f}, -) = \text{hom}_P(B, -) \), which is particularly the case when \( P \) is saturated. □

The following is a “natural” extension of Definition 1.7.5 in [8].

**Definition 21.** Let \( F : P \to Q \) be a punctor, and pick a subclass \( M \) of \( \text{hom}(P) \) and a subclass \( N \) of \( \text{hom}(Q) \). We say that \( F \) is

1. \((M, N)\)-reflecting if \( f \in M \) whenever \( F(f) \in N \) for some \( f \in \text{hom}(P) \).
2. \((M, N)\)-preserving if, conversely, \( F(f) \in N \) whenever \( f \in M \).

More specifically, we say that \( F \) reflects/preserves monos (respectively, epis, retractions, sections, or isomorphisms) if \( F \) is \((M, N)\)-reflecting/preserving for \( M \) and \( N \) being, each in turn, the monomorphisms (respectively, epimorphisms, right splits, left splits, or isomorphisms) of \( P \) and \( Q \).

These notions are used in the next proposition, which generalizes Propositions 1.7.4 and 1.8.4 from [8], and describes the effect of certain kinds of punctors on monomorphisms and epimorphisms.

**Proposition 30.** A faithful punctor reflects both epis and monos.

*Proof.* Let \( F : P \to Q \) be a punctor and \( f : A \to B \) a morphism of \( P \) for which \( F(f) \) is monic in \( Q \). Pick parallel arrows \( g_1, g_2 \in \text{hom}_P(A_{\downarrow f}, -) \) such that \( g_1 \cdot_P f = g_2 \cdot_P f \). Then \( F(g_1) \cdot_Q F(f) = F(g_2) \cdot_Q F(f) \), to the effect that \( F(g_1) = F(g_2) \) for \( F(f) \) is monic in \( Q \). But \( F \) is faithful, so \( g_1 = g_2 \). This completes the proof since \( F \) reflects monos/epis if and only if \( F^{op} \) reflects epis/monos, and \( F \) is faithful if and only if \( F^{op} \) is too. □

An analogous result holds for right or left splits.

**Proposition 31.** Let \( F : P \to Q \) be a fully faithful punctor and suppose that \( P \) is saturated. Then \( F \) reflects both left and right splits.

*Proof.* Pick a morphism \( (f : B \to C)_P \) such that \( F(f) \) is a right split of \( Q \) (the other case follows by duality, using that \( F \) is an embedding if and only if \( F^{op} \) is too). We want to prove that \( f \) is a right split of \( P \). For let \( p \in \text{hom}_P(-, B) \) denote by
A the source of \( p \). Then, we get from the standing assumptions on \( F(f) \) that there exists an arrow \( h \in \text{hom}_Q(-, F(B)) \) such that \( h \circ F(f) = F(p) \). Thus, \( h \) is a \( Q \)-morphism \( F(A) \to F(B) \), which in turn implies, since \( F \) is full, that there exists at least one arrow \( g : A \to B \) such that \( F(g) = h \). Now, \( (g, f) \) is a composable pair, for \( P \) is saturated, and \( F(g \circ_P f) = F(g) \circ_Q F(f) = F(p) \), from the above. But \( F \) is faithful, so we have \( g \circ_P f = h \), and the proof is complete. □

Nothing similar to Proposition 31 is available for categories, since in that setting “being right/left split” is a local property equivalent to left/right invertibility (see Section 6.3), and local identities are preserved by functors, to the effect that the next property in that case is absolutely trivial (and hence overlooked); cf. also [3, Section 2.1.1].

**Proposition 32.** Every functor preserves both left and right invertible arrows, that is sends left/right invertible arrows to left/right invertible arrows.

**Proof.** Obvious (cf. Proposition 1.9.4 in [8]). □

As an application of these simple, but useful results, we include here a couple of examples serving as a complement to the material of Section 6.

**Example 41.** Following notation and terminology of Example 12, suppose that \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are \( \mathcal{K} \)-normed spaces, and let \( f : \mathcal{V}_1 \to \mathcal{V}_2 \) be a compact operator. For the sake of simplicity, we assume here that \( \mathcal{K} \) is either the real or the complex field.

**Lemma 6.** \( f \) is monic in \( \text{CmpNor}(\mathcal{K}) \) if and only if it is injective.

**Proof.** Since there exists an obvious faithful functor from \( \text{CmpNor}(\mathcal{K}) \) to the usual category of sets and total functions, the “if” part follows at once from Proposition 30 and [8, Example 1.7.7.a]. So let us assume that \( f \) is not injective. We claim that there then exist arrows \( g_1, g_2 : \mathcal{V} \to \mathcal{V}_1 \) in \( \text{CmpNor}(\mathcal{K}) \) such that \( f \circ g_1 = f \circ g_2 \), but \( g_1 \neq g_2 \) (compact operators are essentially functions, so we use the same notation of functions for their composition).

For pick a non-zero \( v \in \mathcal{V}_1 \) such that \( f(v) = 0_{\mathcal{V}_2} \), where \( 0_{\mathcal{V}_i} \) is the zero vector of \( \mathcal{V}_i \), and let \( \mathcal{V} = \mathcal{K} \) (thinking of \( \mathcal{K} \) as a \( \mathcal{K} \)-normed space). Then define \( g_1 \) and \( g_2 \) to be the \( \text{Nor}(\mathcal{K}) \)-arrows \( \mathcal{K} \to \mathcal{V}_1 : k \mapsto 0_{\mathcal{V}_1} \) and \( \mathcal{K} \to \mathcal{V}_1 : k \mapsto kv \), respectively. Both of these are compact operators, for their ranges are finite dimensional, but nonetheless \( f \circ g_1 = f \circ g_2 \). Namely, \( f \) is not monic in \( \text{CmpNor}(\mathcal{K}) \). □

**Lemma 7.** \( f \) is epic in \( \text{CmpNor}(\mathcal{K}) \) if and only if \( f(\mathcal{V}_1) \) is dense in \( \mathcal{V}_2 \).

**Proof.** Similarly to the proof of Lemma 6, the “if” part is straightforward, because there exists an obvious faithful functor from \( \text{CmpNor}(\mathcal{K}) \) to the usual category of Hausdorff topological spaces and continuous functions, so that we can profit from Proposition 30 and [8, Example 1.8.5.c]. Let us assume, on the other hand,
that the range of \( f \) is not dense in \( V_2 \). We claim that there then exist morphisms \( g_1, g_2 : V_2 \to V \) in \( \text{CmpNor}(K) \) such that \( g_1 \circ f = g_2 \circ f \), but \( g_1 \neq g_2 \).

For pick a vector \( v_0 \) not in the (topological) closure of \( f(V_1) \) in \( V_2 \). In the light of a well-known corollary to the Hahn-Banach theorem [55, Theorem 3.5, p. 77], there then exists a continuous linear functional \( g_1 : V_2 \to K \) such that \( g_1(v) = 0_K \) for \( v \in f(V_1) \) and \( g_1(v_0) = 1 \) (where \( 0_K \) is, of course, the zero of \( K \)). Thus, if we let \( g_2 \) be the functional \( V_2 \to K : v \to 0_K \), then \( g_1 \) and \( g_2 \) are obviously compact operators (for their range is a finite-dimensional \( K \)-normed space), and \( g_1 \circ f = g_2 \circ f \). However \( g_1 \neq g_2 \), so \( f \) is not epic in \( \text{CmpNor}(K) \). □

The question of giving a non-trivial characterization of the left and right split arrows of \( \text{CmpNor}(K) \) looks more elusive. A related question is as follows:

**Question 2.** Is it possible to get rid of the Hahn-Banach theorem in the proof of Lemma 7, particularly in view of the extension of the above characterizations to categories of more abstract “normed structures”?

Our second example is concerned with the category of small plots (cf. Example 40).

**Example 42.** Let \( F : P \to Q \) be a punctor between small plots. It is then easily seen that \( F \) is monic in \( \text{Plot}(\Omega) \) if and only if it is an embedding, where we profit from Lemma 5 and the fact that injective functions are right cancellative in the usual category, \( \text{Set}(\Omega) \), of small sets; see [8, Example 1.7.7.a]. In a similar way, \( F \) is an epimorphism of \( \text{Plot}(\Omega) \) if and only if \( F \) is full and surjective on objects, using that surjective functions are precisely the epic arrows of \( \text{Set}(\Omega) \); see [8, Example 1.8.5.a]. Finally, since the isomorphisms of a category are split, invertible and cancellative, it is rather straightforward to conclude that \( F \) is an isomorphism of \( \text{Plot}(\Omega) \) if and only if it is an isomorphism in the sense of Definition 20; cf. [8, Example 1.9.6.g]. That is, \( \text{Plot}(\Omega) \) is a balanced category (a plot is said to be balanced if an arrow is iso if and only if it is cancellative).

We say that a plot \( P \) is isomorphic to a plot \( Q \) if there exists an isomorphism of plots from \( P \) to \( Q \). By virtue of Example 42, this is perfectly consistent with the abstract notion of “isomorphic objects” introduced before, for if \( P \) and \( Q \) are small plots then \( P \) is isomorphic to \( Q \), in the sense of the above definition, if and only if \( P \) is isomorphic to \( Q \) in the sense of Section 6.4 (cf. the comments following Proposition 27).

Things are more delicate with the notion of “equivalent plots”, which is somewhat meaningless in view of our approach. In order to clarify what we mean, let \( T \) be a certain subplot of \( \text{Plot}(\Omega) \), e.g. any of the subcategories introduced by Example 40, and for each plot \( P \in T \) let \( \mathcal{H}_P \) be a collection of subfamilies of \( \text{hom}(P) \). Afterwards take \( \mathcal{H} \) to be a subclass of \( \text{hom}(T) \) consisting of punctors \( F : P \to Q \) such that
If (we may assume with no loss of generality that \( \Omega \) is sufficiently large that \( \mathcal{H} \) needs not be a compositive class of \( \mathcal{T} \), and in particular, the composition of an \( \mathcal{M} \)-equivalence \( F \) with an \( \mathcal{N} \)-equivalence \( G \), where \( F,G \in \mathcal{H} \), is not necessarily an \( \mathcal{N} \)-equivalence. Furthermore, if \( F : P \to Q \) is an \( \mathcal{M} \)-equivalence in \( \mathcal{H} \), there is no way, at least in general, to “turn over” \( F \) into an \( \mathcal{N} \)-equivalence \( G : Q \to P \) for some \( \mathcal{N} \in \mathcal{F} \), not even assuming that the class \( \mathcal{F} := \bigcup_{P \in \mathcal{T}(\Omega)} \mathcal{F}_P \) is “closed under inverse images”, in the sense that \( H^{-1}(\mathcal{M}) \in \mathcal{F} \) whenever \( H \in \mathcal{H} \) and \( \mathcal{M} \in \mathcal{F} \).

Partial results in this direction are provided by Proposition 34 below, which refines Example 3.35(1) and point (2) of Proposition 3.36 in [1], and for which we need the following straightforward generalization of [1, Proposition 3.30].

**Proposition 33.** Let \( F : P \to Q \) and \( G : Q \to R \) be punctors, and set \( H := G \circ F \).

(i) If \( F \) and \( G \) are both iso morphisms (respectively, embeddings), then so is \( H \).

(ii) If \( F \) and \( G \) are both faithful (respectively, full), then so is \( H \).

(iii) If \( H \) is an embedding (respectively, faithful), then so is \( F \).

(iv) If \( F \) is surjective on objects and \( H \) is full, then \( G \) is full.

*Proof.* (i) It is immediate by Example 42 and Proposition 24 (we may assume with no loss of generality that \( \Omega \) is sufficiently large that \( P, Q \) and \( R \) are \( \Omega \)-small plots).

(ii) Pick \( A,B \in \text{Ob}(P) \), and suppose that \( F \) and \( G \) are faithful (respectively, full). Then \( F_{A,B} \) and \( G_{F(A),F(B)} \) are injective (respectively, surjective) maps, so their composition is such too. But \( G_{F(A),F(B)} \circ F_{A,B} = H_{A,B} \), and we are done.

(iii) If \( H \) is faithful, then the same is true with \( F \), by Example 42 and Proposition 11. If, on the other hand, \( H \) is faithful, then the map \( H_{XY} \) is injective for all \( X,Y \in \text{Ob}(P) \), and since \( H_{XY} = G_{F(X),F(Y)} \circ F_{XY} \), we get from Proposition 11, together with the fact that injective functions are monomorphisms in \( \text{Set}(\Omega) \), that \( F_{XY} \) is itself injective, which ultimately implies that \( F \) is faithful.

(iv) Pick \( C,D \in \text{Ob}(Q) \). Since \( F \) is surjective on objects, there exist \( A,B \in \text{Ob}(P) \) such that \( F(A) = C \) and \( F(B) = D \), to the effect that \( H_{A,B} = G_{C,D} \circ F_{A,B} \). But now \( H_{A,B} \) is surjective, and since surjections are epimorphisms in \( \text{Set}(\Omega) \), it follows from Proposition 11 that \( G_{C,D} \) is onto, and hence \( G \) is full. \( \square \)

**Proposition 34.** The following conditions hold:

(i) Given a plot \( P \) and a subclass \( \mathcal{M} \) of \( \text{hom}(P) \), the functor \( 1_P \) is an \( \mathcal{M} \)-equivalence.
(ii) If $F : P \to Q$ is an $\mathcal{M}$-equivalence and $G : Q \to R$ an $\mathcal{N}$-equivalence, where $\mathcal{M}$ and $\mathcal{N}$ are, respectively, subclasses of $Q$ and $R$ with the property that $G(\mathcal{M}) \subseteq \mathcal{N}$, then $G \circ F$ is an $\mathcal{N}$-equivalence.

*Proof.* (i) is obvious, so let us focus on point (ii). It suffices to show, in view of Proposition 33, that $H := G \circ F$ is $\mathcal{N}$-dense. For pick $C \in \text{Ob}(R)$. Since $G$ is $\mathcal{N}$-dense, there exists $B \in \text{Ob}(Q)$ such that $G(B) \simeq_{\mathcal{N}} C$ (in $R$). On the other hand, $F$ is $\mathcal{M}$-dense, so there exists as well an object $A \in P$ such that $F(A) \simeq_{\mathcal{M}} B$ (in $Q$). We have to prove that $H(X) \simeq_{\mathcal{N}} C$ for some $X \in \text{Ob}(P)$.

If $F(A) = B$ then $H(A) = G(B) \simeq_{\mathcal{N}} C$, and we are done. Otherwise, there exists an $\mathcal{M}$-connection $(f_1, \ldots, f_n)$ of $Q$ from $F(A)$ to $B$, to the effect that $(G(f_1), \ldots, G(f_n))$ is, by Remark 25 and the assumption that $G(\mathcal{M}) \subseteq \mathcal{N}$, an $\mathcal{N}$-connection of $R$ from $H(A)$ to $G(B)$. To wit, $H(A) \simeq_{\mathcal{N}} G(B)$, and hence $H(A) \simeq_{\mathcal{N}} C$ by the transitivity of $\simeq_{\mathcal{N}}$. ∎

We conclude the section with a few words about constant functors. In the setting of categories, a functor $F : C \to D$ is said to be constant if it maps every object of $C$ to a fixed object $A \in D$ and every morphism in $C$ to the local identity of $A$ in $D$. However, this can be recovered as a consequence of a more general definition, which applies to plots and doesn’t need the presence of identities.

**Definition 22.** A functor $F : P \to Q$ is called constant (respectively, coconstant) if there don’t exist parallel functors $G_1$ and $G_2$ to $P$ (respectively, from $Q$) such that $F \circ G_1 \neq F \circ G_2$ (respectively, $G_1 \circ F \neq G_2 \circ F$).

For what it is worth, note that a functor between small plots is constant (respectively, coconstant) if and only if it is constant (respectively, coconstant) as an arrow of $\text{Plot}(\Omega)$, which is consistent with Definition 13. The next proposition is now straightforward (we may omit the details):

**Proposition 35.** Let $F : P \to Q$ be a functor and assume $\text{hom}(Q)$ is non-empty. Then $F$ is constant if and only if there exist $B \in \text{Ob}(Q)$ and $i_B \in \text{hom}_Q(B)$ such that $F(A) = B$ for every $A \in \text{Ob}(P)$ and $F(f) = i_B$ for all $f \in \text{hom}(P)$. In particular, $i_B$ is an idempotent arrow (of $Q$) if the domain of the composition law of $P$ is non-empty, and a local identity (of $Q$) if $F$ is a functor and $\text{dom}(\text{id}_P) \neq \emptyset$.

It is thus apparent that a constant functor in the sense of categories is just a constant functor between unital non-empty plots.

### 7.3. Unitizations

At this point, we briefly address the problem of making a generic plot into a unital plot. For suppose $P = (C_0, C_1, s, t, c)$ is a plot, and for an object $A \in P$ let $A^P$ denote the pair $(C_1, A)$; note that $A^P$ is not in $C_1$. Then set $D_0 := C_0$ and $D_1 := C_1 \cup \{A^P : A \in C_0\}$, and define the maps $u, v : D_1 \to D_0$
by \( u(A^\sharp P) := v(A^\sharp P) := A \) for \( A \in \mathcal{C}_0 \) and \( u_{\downarrow \mathcal{C}_1} := s, v_{\downarrow \mathcal{C}_1} := t \). Finally, let \( \mathcal{D} \) be the class

\[
\mathcal{D} := \bigcup_{A \in \mathcal{C}_0} \left( \{(A^\sharp P) \times \text{hom}_P(A, -)) \cup (\text{hom}_P(-, A) \times \{A^\sharp P\}) \cup \{(A^\sharp P, A^\sharp P)\} \right),
\]

and take \( d \) to be the mapping \( \mathcal{D}_1 \times \mathcal{D}_1 \to \mathcal{D}_1 \) whose domain is \( \text{dom}(c) \cup \mathcal{D} \), and which sends a pair \((f, g)\) to \( c(f, g) \) if \( f, g \in \mathcal{C}_1 \), to \( f \) if \( g \notin \mathcal{C}_1 \), and to \( g \) if \( f \notin \mathcal{C}_1 \).

The tuple \( (\mathcal{D}_0, \mathcal{D}_1, u, v, d) \) is clearly a unital plot, here written as \( P^\sharp \) and termed the \textit{forced} unitization of \( P \), to stress that we are unitizing \( P \) by “adjoining a local identity even where a local identity is already present”. Then we denote by \( P^\flat \) the wide subplot of \( P^\sharp \) generated by the class \( \mathcal{C}_1 \cup \{A^\sharp P : A \notin \text{dom}(\text{id}_P)\} \), and refer to \( P^\flat \) as the \textit{conditional} unitization of \( P \), to remark that we are “adjoining a local identity if and only if a local identity is not already present”. Furthermore, we may occasionally use the expression “forced (respectively, conditional) unitization” to address the process itself of turning a plot into its forced (respectively, conditional) unitization.

\textbf{Remark 26.} By Remark 13 and the fact that \(|\text{hom}(P^\sharp)| \leq |\text{Ob}(P)| + |\text{hom}(P)|\), it is immediate that \( P \) is small/locally small if and only if this is the case with \( P^\sharp \), or equivalently with \( P^\flat \). Also, \( P \) is saturated if and only if \( P^\sharp \) is, and each form of associativity described by Definition 1 applies to \( P \) if and only if it applies to \( P \) too.

Different plots can have the same conditional unitization (i.e., the “underlying process” is not “reversible”), which is not the case with forced unitization, since obviously \( \sharp(P^\sharp) = P \). Moreover, neither monomorphisms, nor epimorphisms, nor isomorphisms are preserved under unitizations. These are, however, minor drawbacks when contrasted with the fact that \( P \preceq_1 P^\flat \preceq P^\sharp \) and \( P = P^\flat \) if and only if \( P \) is unital. In particular, the latter observation suggests that we can profit from conditional unitization to carry over natural transformations, limits and other fundamental notions from categories to the broader setting of plots, and what is more remarkable, we can do it in a “conservative way”, in the sense that the new approach recovers the classical definitions in the special case of unital plots. However, before giving the details for this, we want to prove that the inclusion \( I : \mathbf{uPlot}_1(\Omega) \to \mathbf{Plot}(\Omega) \) has a left adjoint.

For let \( L \) denote the pair \((L_o, L_h)\), where \( L_o \) and \( L_h \) are, respectively, the functions taking a small plot \( P \) to \( P^\sharp \) and a punctor \( F : P \to Q \) between small plots to the unique extension of \( F \) to a functor \( P^\sharp \to Q^\sharp \). Note that \( L \) is a punctor, and in fact a unital punctor, since it takes the identity functor of a small plot \( P \) to the identity functor of \( P^\sharp \).

\textbf{Proposition 36.} \( L \) is left adjoint to \( I \).
Proof. Let $\varepsilon$ and $\eta$ be, each in turn, the functions sending a unital small plot $P$ to the unique extension of $1_P$ to a functor $P^\sharp \to P$ and an arbitrary small plot $P$ to the inclusion of $P$ into $P^\sharp$. In fact, $\varepsilon$ and $\eta$ are natural transformations $L \circ I \Rightarrow 1_{\text{Plot}(\Omega)}$ and $1_{\text{Plot}(\Omega)} \Rightarrow I \circ L$, respectively, and it is routine to check that for every unital plot $P$ and every plot $Q$ the pair $(\varepsilon, \eta)$ satisfies the following zig-zag equations:

$$1_L(P) = \varepsilon_L(P) \circ L(\eta_P), \quad 1_I(Q) = I(\varepsilon_Q) \circ \eta_I(Q).$$

In view of [8, Theorem 3.1.5], this proves that $L$ is left adjoint to $I$. \hfill $\square$

The previous proposition is essentially adapted from a similar result for semicategories, but contrarily to the case of semicategories, we don’t know at present if $I$ has also a right adjoint; cf. [37, Section 16.13]. The question will be reconsidered in the second part of this work, in relation to the Karoubi envelope of a saturated plot.

7.4. Natural transformations. Following Eilenberg and Mac Lane’s original program [17], the next notion on our agenda is that of a “natural transformation” between functors. As already evidenced by Schröder in the case of his precategories (cf. Remark 1 and the comments to Definition 1.7 in [58]), this point is rather delicate, for different approaches are possible, at least in principle, and whether or not theorems can be salvaged from category theory does really depend on which definitions are chosen. In particular, our idea is based on conditional unitization. This is not intended as a definite solution, but it looks like an appropriate compromise for unitization, while failing to remedy to the difficulties coming with the lack of associativity, does at least fix issues related to the absence of identities.

**Definition 23.** We let a natural transformation be any triple of the form $(F, G, \varepsilon)$ consisting of parallel functors $F, G : P \to Q$ and a map $\varepsilon : \text{Ob}(P) \to \text{hom}(Q^\flat)$ such that:

1. $\varepsilon_A \in \text{hom}_{Q^\flat}(F(A), G(A))$ for every $P$-object $A$.
2. For each $P$-morphism $f : A \to B$ it holds that $(F(f), \varepsilon_B) \text{ and } (\varepsilon_A, G(f))$ are composable pairs of $Q^\flat$, and also $F(f) \circ Q^\flat \varepsilon_B = \varepsilon_A \circ Q^\flat G(f)$.

We usually write $(F, G, \varepsilon)$ as $\varepsilon : F \Rightarrow G : P \to Q$, or simply as $\varepsilon : F \Rightarrow G$, and we identify the triple with $\varepsilon$ if there is no likelihood of confusion.

A natural transformation $\varepsilon : F \Rightarrow G : P \to Q$ is called a natural monomorphism if $\varepsilon_A$ is a local identity of $Q^\flat$ or a monic arrow in $Q$ for every $A \in \text{Ob}(P)$; natural epimorphisms and natural isomorphisms are then defined in a similar way. Moreover, we allow for expressions like “$\varepsilon$ is a natural transformation from $F$ to $G$”, “$\varepsilon$ is a natural isomorphism $F \Rightarrow G$”, or variations of these, whose meaning is self-explanatory.
Example 43. For a punctor $F : P \to Q$ the mapping $\text{Ob}(P) \to \text{hom}(Q^\flat) : A \mapsto \text{id}_{Q^\flat}(A)$ is a natural isomorphism $F \Rightarrow F$, called the identity natural isomorphism on $F$.

Given plots $P$ and $Q$, we denote by $\text{Nat}(F,G)$ the class of all natural transformations $\varepsilon : F \Rightarrow G : P \to Q$, which is written as $\text{Nat}(F)$ when $F = G$. Afterwards we form a tuple $(C_0, C_1, s, t, c)$ as follows: First, we set $C_0 := \text{Punct}(P, Q)$ and $C_1 := \bigcup_{F,G \in C_0} \text{Nat}(F,G)$; then, we let $s$ and $t$ be, respectively, the mappings $C_1 \to C_0$ taking a natural transformation $\varepsilon : F \Rightarrow G : P \to Q$ to $F$ and $G$; lastly, we define $c$ as the map $C_1 \times C_1 \to C_1$ sending a pair $(\varepsilon : F \Rightarrow G, \eta : G \Rightarrow H)$ of natural transformations between punctors $P \to Q$ to the function $\tau : \text{Ob}(P) \to \text{hom}(Q^\flat) : A \mapsto \varepsilon_A \circ Q^\flat \eta_A$, and this if and only if $(\varepsilon_A, \eta_A)$ is a composable pair of $Q^\flat$ for every object $C \in C_0$ and $\tau$ is itself a natural transformation $F \Rightarrow H$ (cf. [57, Remark 1.8]).

In the light of Example 43, it is found that $(C_0, C_1, s, t, c)$ is a unital plot, here called the punctor plot of the pair $(P, Q)$ and denoted by $\text{Punct}(P, Q)$, or simply by $\text{Punct}(P)$ if $P = Q$. Furthermore, we use $\text{Funct}(P, Q)$ for the full subplot of $\text{Punct}(P, Q)$ whose objects are functors, i.e. unital punctors, and we refer to it as the punctor plot of $(P, Q)$.

Now we want to show that, under suitable assumptions, our definition of a punctor plot can be somewhat simplified, making it apparent how the above notions generalize analogous ones related to natural transformations on the level of categories.

Proposition 37. Let $F$, $G$ and $H$ be parallel punctors from $P$ to $Q$, and let $\varepsilon$ and $\eta$ be natural transformations $F \Rightarrow G$ and $G \Rightarrow H$, respectively. Furthermore, assume that $Q$ is strongly associative. Then $(\varepsilon, \eta)$ is a composable pair of $\text{Punct}(P, Q)$ if and only if $(\varepsilon_C, \eta_C)$ is a composable pair of $Q^\flat$ for each $C \in \text{Ob}(P)$.

Proof. Suppose $(\varepsilon_C, \eta_C)$ is composable in $Q^\flat$ for every $C \in \text{Ob}(P)$, the other direction being obvious, and pick a $P$-arrow $f : A \to B$. Considering that $\varepsilon$ is a natural transformation $F \Rightarrow G$, also $(F(f), \varepsilon_B)$ and $(\varepsilon_A, G(f))$ are then composable pairs of $Q^\flat$, and $F(f) \circ Q^\flat \varepsilon_B = \varepsilon_A \circ Q^\flat G(f)$, hence Remark 26 and the strong associativity of $Q$ give

$$F(f) \circ Q^\flat (\varepsilon_B \circ Q^\flat \eta_B) = (F(f) \circ Q^\flat \varepsilon_B) \circ Q^\flat \eta_B = (\varepsilon_A \circ Q^\flat G(f)) \circ Q^\flat \eta_B \quad (7)$$

(recall Remark 5, too). But $\eta$ is a natural transformation $G \Rightarrow H$, so again $(G(f), \eta_B)$ and $(\eta_A, H(f))$ are composable pairs of $Q^\flat$ for which $G(f) \circ Q^\flat \eta_B = \eta_A \circ Q^\flat H(f)$, and

$$(\varepsilon_A \circ Q^\flat G(f)) \circ Q^\flat \eta_B = \varepsilon_A \circ Q^\flat (G(f) \circ Q^\flat \eta_B) = \varepsilon_A \circ Q^\flat (\eta_A \circ Q^\flat H(f)). \quad (8)$$
Then, putting (7) and (8) together gives, once more based on the strong associativity of $Q'$, that $F(f) \circ_{Q'} (\varepsilon_B \circ_{Q'} \eta_B) = (\varepsilon_A \circ_{Q'} \eta_A) \circ_{Q'} H(f)$, which implies, by the arbitrariness of $f$, that $\text{Ob}(P) \to \text{hom}(Q') : C \mapsto \varepsilon_C \circ_{Q'} \eta_C$ is a natural transformation $F \Rightarrow H$.

The next result proves that relevant properties of $\text{Punct}(P, Q)$ depend on $Q$, but not on $P$, which looks interesting for it suggests that we can try to get information about $P$, no matter how “meagre” it may be, by making $Q$ range over a suitable class of “nice test categories” (and looking at punctors $P \to Q$ and natural transformations thereof).

**Corollary 2.** If $Q$ is strongly associative, then also $\text{Punct}(P, Q)$ is. In particular, if $Q$ is a semicategory, then the same is true for $\text{Punct}(P, Q)$.

**Proof.** Let $\varepsilon : F \Rightarrow G$, $\eta : G \Rightarrow H$, and $\sigma : H \Rightarrow L$ be arrows in $\text{Punct}(P, Q)$ such that $(\varepsilon, \eta)$ and $(\eta, \sigma)$ are composable, in such a way that $(\varepsilon_C, \eta_C)$ and $(\eta_C, \sigma_C)$ are composable in $Q'$ for each $C \in \text{Ob}(Q')$. Since $Q$ is strongly associative, this implies, in view of Remark 26, that $(\varepsilon_C \circ \eta_C) \circ \sigma_C$ and $\varepsilon_C \circ (\eta_C \circ \sigma_C)$ are defined and equal. Thus, $(\varepsilon \circ \eta, \sigma)$ and $(\varepsilon, \eta \circ \sigma)$ are composable in $\text{Punct}(P, Q)$ by Proposition 37, and $(\varepsilon \circ \eta) \circ \sigma = \varepsilon \circ (\eta \circ \sigma)$. To wit, $\text{Punct}(P, Q)$ is a strongly associative plot. The rest is obvious.

In particular, it follows from Corollary 2 and the above comments that the punctor plot of a semicategory is, in fact, a category, and especially $\text{Punct}(P, Q)$ belongs to $\text{Cat}(\Omega)$ if both of $P$ and $Q$ are small, as implied by the following:

**Proposition 38.** Let $P$ and $Q$ be plots. If $P$ is small and $Q$ is small/locally small, then also $\text{Punct}(P, Q)$ is small/locally small.

**Proof.** Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be, respectively, the classes of all functions $\text{Ob}(P) \to \text{Ob}(Q)$ and $\text{hom}(P) \to \text{hom}(Q)$, and $\mathcal{N}$ the class of all functions $\text{Ob}(P) \to \text{hom}(Q')$. The object class of $\text{Punct}(P, Q)$ is, by definition, a subclass of $\mathcal{H}_0 \times \mathcal{H}_1$, while $|\text{Nat}(F, G)| \leq |\mathcal{N}|$ for every pair $(F, G)$ of punctors from $P$ to $Q$.

Now, if both of $P$ and $Q$ are small, then $\mathcal{H}_0 \times \mathcal{H}_1$ and $\mathcal{N}$ are sets (see Remark 26), to the effect that $\text{Punct}(P, Q)$ has small degree and $\text{Punct}(P, Q)$ is itself a set, which is enough to conclude, by Corollary 1, that $\text{Punct}(P, Q)$ is small too.

If, on the other hand, $P$ is small and $Q$ is locally small, then given punctors $F, G : P \to Q$ it is straightforward to see that $\text{Nat}(F, G)$ is a set, and hence $\text{Punct}(P, Q)$ is locally small: In fact, $\text{Nat}(F, G)$ is not larger than $\text{Ob}(P) \times \bigcup_{A \in \text{Ob}(P)} \text{hom}_{Q'}(F(A), G(A))$, and the latter is a set for $P$ is small and $\text{hom}_{Q'}(F(A), G(A))$ is a set for every $A \in \text{Ob}(P)$.
Many other things could be said about natural transformations, but they are somewhat subsidiary to the primary goal of the paper, and we hope to discuss them in a later note.

7.5. **Constructions on plots.** We have already seen how it is possible to construct plots from magmas (Example 1), relations (Example 8) and other plots (Proposition 4), possibly by the aid of punctors (Examples 34). Now we consider another couple of fundamental constructions that produce new plots from old ones, namely products and coproducts. Both of them can be almost verbatim carried over from categories to plots; however, we include relevant details here for future reference and the sake of exposition.

7.5.1. **Products of plots.** We start with products. For let \((P_i)_{i \in I}\) be a family of plots indexed by an ordered class \(I\). We define a tuple \(P = (P_0, P_1, s, t, c)\) as follows:

1. \(P_0 := \prod_{i \in I} \text{Ob}(P_i)\) and \(P_1 := \prod_{i \in I} \text{hom}(P_i)\);
2. \(s\) is the function \(P_1 \to P_0 : (f_i)_{i \in I} \mapsto (sr_i(f_i))_{i \in I}\);
3. \(t\) is the function \(P_1 \to P_0 : (f_i)_{i \in I} \mapsto (tr_i(f_i))_{i \in I}\);
4. \(c\) is the map \(P_1 \times P_1 \to P_1\) whose domain is the class \(\prod_{i \in I} \text{dom}(\phi_{P_i})\), and which sends an element \(((f_i, g_i))_{i \in I}\) in that class to \((f_i \circ_P g_i)_{i \in I}\).

It is easy to check that \(P\) is a plot, here called the (canonical) product (plot) of the class \((P_i)_{i \in I}\) and written as \(\text{PLOT} : \prod_{i \in I} P_i\), or simply as \(\prod_{i \in I} P_i\) if there is no likelihood of confusion. The notation is further simplified to \(P_1 \times \cdots \times P_\alpha\) if \(I\) is finite and \(\alpha := |I| \neq 0\), or to \(P^{\times \alpha}\) if \(P_1 = \cdots = P_\alpha =: P\). In particular, \(\prod_{i \in I} P_i\) is the empty quiver when \(\alpha = 0\).

**Proposition 39.** The product plot \(P\) satisfies any of the properties specified by Definition 1 if and only if each of its components does. Similarly, \(P\) is saturated/unital if and only if \(P_i\) is saturated/unital for every \(i \in \alpha\). Furthermore, a \(P\)-morphism \((f_i)_{i \in I}\) is monic/epic/iso if and only if \(f_i\) is monic/epic/iso in \(P_i\) for each \(i \in I\).

**Proof.** It is straightforward by our definitions (we may omit the details). \(\square\)

For each \(j \in I\) the pair \((\pi_j, \sigma_j)\), with \(\pi_j, \sigma_j\) the map \(P_0 \to \text{Ob}(P_j) : (A_i)_{i \in I} \mapsto A_j\) and \(\pi_j, \sigma_j\) the map \(P_1 \to \text{hom}(P_j) : (f_i)_{i \in I} \mapsto f_j\), is now a functor, here termed the canonical projection of \(P\) onto \(P_j\); accordingly, we refer to \(P_j\) as the \(j\)-th component of \(P\). It is easily seen that \(A \in \text{Ob}(P)\) is unital if and only if \(\pi_j(A)\) is unital in \(P_j\) for each \(j \in I\), to the effect that \(P\) is a category if and only if the same is true for every member of the family \((P_i)_{i \in I}\). This means that the above construction is just a straightforward generalization of the analogous one for products of categories; cf. [3, Section 1.6.1]. Moreover, the following holds (cf. Proposition 1.6.6 in [8]):

**Proposition 40.** If \(Q\) is a plot and \((F_i : Q \to P_i)_{i \in I}\) is a family of punctors, there exists a unique punctor \(F : Q \to P\) such that \(\pi_i \circ F = F_i\) for each \(i\).
Proof. If $F_o$ is the map $\text{Ob}(Q) \to \text{Ob}(P) : A \mapsto (F_i(A))_{i \in I}$ and $F_h$ is the map $\text{hom}(Q) \to \text{hom}(P) : f \mapsto (F_i(f))_{i \in I}$, the pair $F := (F_o, F_h)$ is clearly a functor $Q \to P$ for which $\pi_i \circ F = F_i$ for each $i$. The rest is routine. \qed

In particular, since $P$ is clearly small whenever $\alpha$ is a small ordinal and $P_i$ is a small plot for each $i \in I$, it follows from Propositions 39 and 40 that each of the categories of plots described in Example 40 is a category with (small) products, which is another point in common with the theory of categories; cf. [8, Example 2.1.7.b].

Definition 24. For an integer $n \geq 2$, we let an $n$-ary punchor/functor be a functor/functor of the form $F : Q_1 \times \cdots \times Q_n \to Q$, i.e. whose source is the product of a family of $n$ plots. In particular, this is called a bipunctor if $n = 2$.

Bipunctors will be essential for the definition of semantic domains, which we mentioned in the introduction and will be presented in a separate work.

7.5.2. Coproducts of plots. Now it is the turn of coproducts, for we define a second tuple $K = (C_0, C_1, \sigma, \tau, d)$ as follows (we continue with the same notation as in Section 7.5.1):

1. $C_0 := \coprod_{i \in I} \text{Ob}(P_i)$ and $C_1 := \coprod_{i \in I} \text{hom}(P_i)$;
2. $\sigma$ is the function $C_1 \to C_0 : (f_i, i) \mapsto (\text{sr}_{P_i}(f_i), i)$;
3. $\tau$ is the function $C_1 \to C_0 : (f_i, i) \mapsto (\text{tr}_{P_i}(f_i), i)$;
4. $d$ is the map $C_1 \times C_1 \to C_1$ whose domain consists of all and only pairs $((f_i, i), (g_i, i))$ such that $(f_i, g_i) \in \text{dom}(P_i)$, and which maps a pair like this to $(f_i \circ P_i g_i, i)$.

It is easily seen that $K$ is a plot, here termed the (canonical) coproduct (plot) of the class $(P_i)_{i \in I}$ and denoted by $\text{PLOT} \coprod_{i \in I} P_i$, or simply by $\coprod_{i \in I} P_i$ if there is no danger of ambiguity. The notation is further simplified to $P_1 \oplus \cdots \oplus P_\alpha$ if $\alpha$ is finite and $\neq 0$, or to $P^{\oplus \alpha}$ if $P_1 = \cdots = P_\alpha =: P$. Note that $\coprod_{i \in I} P_i$ is the empty quiver when $\alpha = 0$.

Proposition 41. The coproduct plot $K$ satisfies any of the properties specified by Definition 1 if and only if each of its components does. Similarly, $K$ is saturated/unital if and only if $P_i$ is saturated/unital for every $i \in [\alpha]$. Furthermore, a $K$-morphism $(f, i)$ is monic/epic/iso if and only if $f$ is monic/epic/iso in $P_i$.

Proof. It is just about unravelling our definitions (we may omit the details). \qed

For each $j \in I$ the pair $(\kappa_{j,o}, \kappa_{j,h})$, where $\kappa_{j,o}$ is the function $\text{Ob}(P_j) \to C_0 : A \mapsto (A, j)$ and $\kappa_{j,h}$ the function $\text{hom}(P_j) \to C_1 : f \mapsto (f, j)$, is again a functor, here referred to as the canonical injection of $P_j$ into $K$; we call $P_j$ as the $j$-th component of $K$. It is easily seen that $A \in \text{Ob}(P)$ is unital if and only if $\kappa_j(A)$ is unital in
$P_j$ for each $j \in I$, to the effect that $P$ is a category if and only if the same is true for every member of the family $(P_i)_{i \in I}$. It follows that the above generalizes the usual construction for the coproduct of a family of categories; cf. [3, Section 1.6.1]. Furthermore, we have the following (cf. Proposition 1.6.6 in [8]):

**Proposition 42.** If $Q$ is a plot and $(F_i : P_i \to Q)_{i \in I}$ is a family of functors, there exists a unique functor $F : K \to Q$ such that $F \circ \kappa_i = F_i$ for each $i$.

**Proof.** Let $F_o$ be the map $\text{Ob}(K) \to \text{Ob}(Q) : (A,i) \mapsto F_i(A)$ and $F_h$ the map $\text{hom}(K) \to \text{hom}(Q) : (f,i) \mapsto F_i(f)$. Then, the pair $F := (F_o,F_h)$ is a functor $K \to Q$, and it holds $F \circ \kappa_i = F_i$ for each $i$. The rest is trivial. □

As an application of the above, suppose that $P$ is a plot and let $M$ be any given subclass of $\text{hom}(P)$. Then $P$ is isomorphic to the disjoint union of its own $M$-connected components (see Remark 21).

### 7.5.3. Augmentations of plots.

Let $P$ be a plot and $\zeta$ a binary operation $I \times I \to I$. Define the 5-tuple $(C_0, C_1, s, t, c)$ as follows: $C_0$ is just the object class of $P$ and $C_1$ the quotient of $\text{hom}(P) \times I$ by the equivalence relation $\mathcal{R}$ which amalgamates two pairs $(f,i)$ and $(g,j)$ if and only if $f = g = \text{id}_P(A)$ for some object $A \in P$; $s$ and $t$ are the maps $C_1 \to C_0$ taking an equivalence class $(f,i) \mod \mathcal{R}$ to $sr_P(f)$ and $tr_P(f)$, respectively; and $c$ is the function $C_1 \times C_1 \to C_1$ mapping a pair $((f,i) \mod \mathcal{R}, (g,j) \mod \mathcal{R})$ to $(f \circ_P g, \zeta(i,j)) \mod \mathcal{R}$, and this if and only if $(f,g)$ is composable in $P$ and $(i,j) \in \text{dom}(\zeta)$.

It is clear that $(C_0, C_1, s, t, c)$ is a plot, which we denote by $\zeta \ast P$ and call the augmentation of $P$ by $\zeta$. The reason for the name is due to the fact that, when $\zeta$ is a total function and $I$ is non-empty, $\zeta \ast P$ makes $|I|$ copies of each non-identity arrow of $P$, blowing up, in particular, the number of isomorphisms when $I$ is sufficiently large. Since $\zeta \ast P$ is a category whenever $\zeta$ is, this provides a good toy example for illustrating the “working principle” of $\mathcal{M}$-limits, which are the subject of the next section.

### 7.6. $\mathcal{M}$-limits and the like.

There are relevant situations where the categorical notion of limit (and its dual) turns out to be too “rigid”, and limits fail to exist. But, instead of accepting this as a fact of life, mathematicians, who are well-known for being obstinate people, have sharpened their wits and introduced, among the others, weak limits and sublimits. The former, in particular, have found remarkable applications in generalized homotopy theory [12], and paved the ground for impressive developments and applications in and out of the category theory. But while limits are possibly too “rigid”, it is as well true that weak limits and sublimits are often too “feeble” to make their use effective, simply for the fact, e.g., that there may be too many of them. This leads to the idea of “interpolating” these views
by requiring that the mediating morphisms implied by the categorical definition of limits and the like have a prescribed “shape”, i.e. belong to a given class $\mathcal{M}$. This is, in fact, the basic insight for the introduction of $\mathcal{M}$-limits, weak $\mathcal{M}$-limits, and $\mathcal{M}$-sublimits (along with their duals), where we use $\mathcal{M}$-factorizations as a “natural remedy” to the fact that, in absence of some form of associativity, certain classes of morphisms, particularly relevant for applications, are not compositive, as observed in Section 6. An extensive discussion of the subject would take us too far from our initial goals, so here we just present the topic and hope to develop it in the near future.

In this section, $\mathbb{P}$ is a plot and $\mathcal{M}$ a fixed class of $\mathbb{P}^\flat$-morphisms, which we allow to be absolutely arbitrary, unless a statement to the contrary is made. We start with the notion of a diagram, which is just adapted from the analogous one for categories.

**Definition 25.** A diagram of type $\mathbf{J}$ in $\mathbb{P}$ is any punctor $D : \mathbf{J} \to \mathbb{P}$ for which $\mathbf{J}$ is a quiver. The diagram is called finite, countable, or small whenever $\mathbf{J}$ is. More generally, we say that $D$ is $\kappa$ for a cardinal $\kappa$ if $|\text{Ob}(\mathbf{J}) \cup \text{hom}(\mathbf{J})| \leq \kappa$.

The punctor $D$ in the above definition can be thought of as indexing a collection of objects and morphisms in $\mathbb{P}$ patterned on $\mathbf{J}$; this is why $D$ is also called a $\mathbf{J}$-shaped diagram, with $\mathbf{J}$ being termed the index (or pattern) of $D$. Although there is no formal difference between a diagram and a punctor, the change in terminology reflects a change in perspective, in that one usually fixes $\mathbf{J}$, and lets $D$ (and secondarily $\mathbb{P}$) be variable.

Like with categories, the actual objects and morphisms in a pattern $\mathbf{J}$ are largely irrelevant, only the way in which they are interrelated matters. However, contrarily to the case of categories, where limits can be equivalently defined by using either of (categorical) diagrams (i.e., functors) or multidigraphs (recall what we said in the introduction), the distinction between the two approaches doesn’t exist on the level of plots, and in fact doesn’t even make sense any longer. Somehow, this does justice to the common observation that, in the categorical definition of a limit as a universal cone for a certain $\mathbf{J}$-shaped diagram, we never use the identities and the composition law of $\mathbf{J}$, and hence neither the associativity of $\mathbf{J}$, which advocates, from an essentialist point of view, for another level of abstraction to remove what is, in some sense, a needless redundancy.

There is always some ambiguity in the term “finite limit” because you can consider the source of a “diagram” to be either a graph or a category. This matters not for the notion of limit because of the free/forgetful adjunction between graphs and categories, but the free category on a finite graph may not be finite; it is however $\mathcal{L}$-finite, as you would expect. This is another situation where it seems more natural,
even from within the theory of categories, to view diagrams as graph morphisms rather than functors.

**Definition 26.** Given a diagram $D$ of type $J$ in $P$, a cone to $D$ is any pair $(L, \phi)$ consisting of an object $L \in P$ and a function $\phi$ from $\text{Ob}(J) \to \text{hom}(P^\flat)$ such that

1. $\phi_A \in \text{hom}_P(L, D(A))$ for every object $A \in J$.
2. $\phi_A \circ D(f) = \phi_B$ for each $J$-morphism $f : A \to B$.

We say that a cone $(L, \phi)$ to $D$ is a weak $\mathcal{M}$-limit (respectively, an $\mathcal{M}$-sublimit) for $D$ if for any other cone $(N, \psi)$ to $D$ there exists at least (respectively, at most) one factorization $\Phi = (\gamma, \wp)$ of minimal length such that $\gamma$ is an $\mathcal{M}$-path from $N$ to $L$ (in $P$), called a mediating $\mathcal{M}$-factorization, and for any $J$-arrow $f : X \to Y$ it holds

$$\psi_X = \text{Ev}_\mathcal{M}[P](\Phi) \circ P \phi_X$$

(i.e., $\psi_X = \wp[P](\gamma) \circ P \phi_X$), $\forall X \in \text{Ob}(J)$;

in particular, $(L, \phi)$ is called a (strong) $\mathcal{M}$-limit if there exists a unique such $\Phi$, in which case $(L, \phi)$ is called a universal cone to $D$ and $\Phi$ a universal $\mathcal{M}$-factorization.

A limit for a diagram $D : J \to P$ is just an $\mathcal{M}$-limit with $\mathcal{M} = \text{hom}(P)$; weak limits and sublimits are defined similarly. On another hand, a weak $\mathcal{M}$-colimit for $D$ is just a weak $\mathcal{M}$-limit for $D^{\text{op}}$, the “dual diagram” of $D$ (see Definition 33); $\mathcal{M}$-colimits and $\mathcal{M}$-sublimits, as well as colimits and subcolimits, are now defined in the expected way (we may omit the details). Every $\mathcal{M}$-limit (respectively, $\mathcal{M}$-colimit) is both a weak $\mathcal{M}$-limit (respectively, weak $\mathcal{M}$-colimits) and an $\mathcal{M}$-sublimit (respectively, an $\mathcal{M}$-subcolimit).

The definition of weak $\mathcal{M}$-limits, $\mathcal{M}$-limits, and $\mathcal{M}$-sublimits are general enough to subsume several constructions useful in practice. The next proposition generalizes a well-known property of categorical limits:

**Proposition 43.** If $P$ is associative and $(L, \phi)$ is an $\mathcal{M}$-limit for a diagram $D : J \to P$, then either $(L, \phi)$ is the unique universal cone to $D$, or $L$ is a unital object of $P$ and, for any other universal cone $(N, \psi)$ to $D$ such that $(N, \psi) \neq (L, \phi)$, the mediating morphism $m : N \to L$ is a strongly invertible $P$-arrow. In addition, the same conclusions hold by replacing “limit” with “colimit” and “cone” with “cocone”.

**Proof.** It is enough to consider the case of limits (by duality). For suppose that $(L, \phi)$ and $(N, \psi)$ are different universal cones to the same diagram $D : J \to P$, and denote by $u$ and $v$, respectively, the mediating $\mathcal{M}$-morphisms $L \to N$ and $N \to L$ implied by Definition 26. Then $u \circ v = \text{id}_P(L)$ and $v \circ u = \text{id}_P(N)$, which is possible, by the definition itself of $P^\flat$, only if $N$ and $L$ are unital objects of $P$, and $u$ and $v$ are inverse to each other. □

In particular, note that two limits for the same diagram need not, at least in general, be isomorphic, but this is actually the case if $P$ is, e.g., dissociative, as
implied by Proposition 23. Further properties of $\mathcal{M}$-limits and their variations will be investigated in future work. For the moment, we conclude our exposition with the following:

**Example 44.** Let $C$ be a category with binary products and $\zeta$ an associative total binary operation on $\text{hom}(G)$. At least in general, the augmentation of $G$ by $\zeta$ will fail to have binary products.

### 8. Closing remarks

This paper is just the first step of a march. We have introduced plots and shown how they result into a unifying setting for a number of fundamental structures, most notably including categories, Gabriel’s quivers and Ehresmann’s multiplicative graphs. We have proved that it is even possible to generalize to plots the notion itself of isomorphism and make it “identity-free”, and we have shown, as a “natural” consequence of our approach, how isomorphisms have not really much to do with invertible arrows. Furthermore, we have introduced punctors, as an abstraction of (categorical) functors, and proved that the former retain interesting (though possibly “elementary”) properties of the latter. Lastly, we have defined $\mathcal{M}$-connections and used them, on the one hand, to generalize the notion of “isomorphic objects” (by means of $\mathcal{M}$-equivalences), and on the other hand, to define $\mathcal{M}$-limits and the like (owing to $\mathcal{M}$-factorizations). But a number of basic problems, let alone deeper questions, stand wide open, and saying that the author feels like having just added a comma in a book of billions and billions of words, is not merely rhetoric.

E.g., is there any “appropriate” way to carry over the geometric representation of commutative diagrams to the non-associative setting of plots? Nobody can disagree with the extreme usefulness of commutative diagrams in working with categories. And how far can be pushed the fact that the plotal approach to the definition of monomorphisms, left splits, isomorphisms, and their duals does depend on the regular representations? Regular representations are clearly reminiscent of Yoneda’s lemma, and having Yoneda’s lemma extended to plots could lead to new exciting developments.

### Acknowledgements

The author is indebted to Giacomo Canevari (LJLL, UPMC - Paris, FR), Andrea Gagna (Universiteit Leiden - Leiden, NL), and Fosco Loregian (SISSA - Trieste, IT) for useful conversations. He is also thankful to André Ehresmann for an unforgivable afternoon spent discussing aesthetics, Platonism and mathematics in a typical Parisian café, in place I. Stravinski.
References

[1] J. Adámek, H. Herrlich, and G. Strecker (2006), Abstract and concrete categories: The joy of cats, Reprints in Theory and Applications of Categories, No. 17.
[2] D.D. Anderson and A.M. Frazier (2011), On a general theory of factorization in integral domains, Rocky Mountain J. Math. 41, No. 3, 663–705.
[3] S. Awodey (2010), Category Theory, Oxford Univ. Press (2nd edition).
[4] C. Barwick and C. Schommer-Pries (2012), On the Unicity of the Homotopy Theory of Higher Categories, preprint, arXiv:math.AT/1112.0040.
[5] A. Bastiani and C. Ehresmann (1972), Categories of sketched structures, Cahiers Topologie Géom. Différ. Catégories 13, No. 2, 104–214.
[6] G. Böhm, S. Lack and R. Street, Idempotent splittings, colimit completion, and weak aspects of the theory of monads, J. Pure and Appl. Algebra (to appear).
[7] B. Bollobás (1998), Modern Graph Theory, Springer.
[8] F. Borceux (1994), Handbook of Categorical Algebra 1 - Basic Category Theory, Vol. 50 of the Encyclopedia of Mathematics and its Applications, Cambridge University Press.
[9] F. Borceux (1994), Handbook of Categorical Algebra 2 - Categories and Structures, Vol. 51 of the Encyclopedia of Mathematics and its Applications, Cambridge University Press.
[10] N. Bourbaki (1998), Elements of Mathematics - Algebra I: Ch. 1-3, reprint ed., Springer.
[11] N. Bourbaki (2004), Theory of Sets, Springer (2nd revised edition).
[12] E. Brown (1965), Abstract homotopy theory, Trans. AMS 119, No. 1, 79–85.
[13] L. Coppey (1980), Quelques problèmes typiques concernant les graphes multiplicatifs, Diagrammes 3, C1–C46.
[14] L. Coppey (1982), Sur quelques structures de base pour définir les structures, Diagrammes 7, C1–C23.
[15] L. Coppey (1990), Graphes structuraux, Diagrammes 24, 33–76.
[16] F. Cury, Systèmes de générateurs et relations pour les catégories enrichies, Diagrammes 1, C1–C21.
[17] S. Eilenberg and S. Mac Lane, Relations between homology and homotopy groups of spaces, Annals of Mathematics 46, 480–509.
[18] C. Ehresmann (1965), Catégories et structures, Dunod.
[19] C. Ehresmann (1966), Trends toward unity in mathematics, Cahiers Topologie Géom. Différ. Catégories 8, No. 1, 1–7.
[20] R. Exel (2011), Semigroupoid C*-algebras, J. Math. Anal. Appl. 377, No. 1, 303–318.
[21] P.A. Freyd and A. Scedrov, Categories, Allegories, North-Holland, Amsterdam.
[22] P. Gabriel (1972), Unzerlegbare Darstellungen I, manuscipta Math. 6, No. 1, 71–103.
[23] W.D. Garraway (2005), Sheaves for an involutive quantaloid, Cahiers Topologie Géom. Différ. Catégories 46, No. 4, pp. 243–274.
[24] A. Geroldinger and F. Halter-Koch (2006), Non-Unique Factorizations: Algebraic, Combinatorial and Analytic Theory, Vol. 278 of the Pure and Applied Mathematics Series, Chapman & Hall/CRC.
[25] K. Glazek (2002), A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences (with Complete Bibliography), Kluwer Acad. Publ. (1st edition).
[26] J.L. Gross and J. Yellen (eds.) (2003), Handbook of Graph Theory (Discrete Mathematics and Its Applications), 1st ed., CRC Press.
[27] M. Gross and R. Pandharipande, *Quivers, curves, and the tropical vertex*, preprint, arxiv:math.AG/0909.5153.

[28] S. Hayashi (1985), *Adjunction of Semifunctors: Categorical Structures in Nonextensional λ-Calculus*, Theoretical Computer Science 41, 95–104.

[29] C. Henry (1986), *Sur quelques problèmes de plongement en algèbre : II. Extensions de Kan et prolongements de foncteurs à des graphes multiplicatifs*, Diagrammes 15, H1–H13.

[30] C.S. Hönig (1954), *Proof of the well-ordering of cardinal numbers*, Proc. Amer. Math. Soc. 5, 312.

[31] R. Hoofman and I. Moerdijk (1995), *A Remark on the Theory of Semi-Functors*, Mathematical Structures in Computer Science 5, 1–8.

[32] M. Hovey (1999), *Model Categories*, Vol. 63 in the Mathematical Surveys and Monographs Series, American Math. Soc..

[33] J.M. Howie (2003), *Fundamentals of semigroup theory*, Clarendon Press (reprint).

[34] M. Hyland, M. Nagayama, J. Power, and G. Rosolini (2006), *A Category Theoretic Formulation for Engeler-style Models of the Untyped λ-Calculus*, Electronic Notes in Theoretical Computer Science 161, 43–57.

[35] P.T. Johnstone (2002), *Sketches of an Elephant: A Topos Theory Compendium*, Clarendon Press.

[36] A. Joyal and J. Kock (2007), “Weak units and homotopy 3-types.” In *Categories in Algebra, Geometry and Mathematical Physics: Conference and Workshop in Honor of Ross Street’s 60th Birthday*, Contemporary Math. 431, 257–276.

[37] A. Joyal (2008), *Notes on quasi-categories*, http://www.math.uchicago.edu/~may/IMA/Joyal.pdf (version: 2008-06-22).

[38] W. Kahl (2006), “Semigroupoid interfaces for relation-algebraic programming in Haskell.” In *Relations and Kleene Algebra in Computer Science* (ed. by R.A. Schmidt), Lecture Notes in Computer Science, Springer, 4136, 235–250.

[39] B. Keller (2011), *On cluster theory and quantum dilogarithm identities*, arxiv:math.RT/1102.4148.

[40] J. Kock (2006), *Weak identity arrows in higher categories*, Internat. Math. Res. Papers, 1–54.

[41] G. Köthe (1969), *Topological Vector Spaces I*, Vol. 159 of the Grundlehren der mathematischen Wissenschaften Series, New York: Springer-Verlag (2nd edition, translated from German by D.J.H. Garling).

[42] G. Köthe (1979), *Topological Vector Spaces II*, Vol. 237 of the Grundlehren der mathematischen Wissenschaften Series, New-York: Springer-Verlag.

[43] C. Lair (1987), *Trames et sémantiques catégoriques des systèmes de trames*, Diagrammes 18, CL1–CL47.

[44] L. Le Bruyn (2007), *Noncommutative Geometry and Cayley-smooth Orders*, Vol. 290 of the Pure and Applied Mathematics Series, Chapman and Hall/CRC Press.

[45] T. Leinster (2002), *A Survey of Definitions of n-Category*, Theory Appl. Categ. 10, 1–70.

[46] T. Leinster (2004), *Higher Operads, Higher Categories*, Vol. 298 of the London Mathematical Society Lecture Note Series, Cambridge University Press.

[47] W.A.J. Luxemburg and A.C. Zaanen (1971), *Riesz spaces. I*, North-Holland.

[48] E.S. Lyapin and A.E. Evseev (1997), *The theory of partial algebraic operations*, Transl. from the Russian by J. M. Cole, Vol. 414 of the Mathematics and its Applications Series, Kluwer Acad. Publ.
[49] P.W. Michor (1976), *Banach-Semikategorien*, Sitzungsberichte Österreichische Akademie Wiss., Abt II, 185, 181–204 (part I), 205–219 (part II), 221–238 (part III).

[50] B. Mitchell (1972), *The dominion of Isbell*, Trans. Amer. Math. Soc. 167, 319–331.

[51] M.-A. Moens, U. Berni-Canani and F. Borceux (2002), *On regular presheaves and regular semicategories*, Cahiers Topologie Géom. Différ. Catégories 43, No. 2, 163–190.

[52] J.-E. Pin, A. Pinguet and P. Weil (1999), *Ordered Categories and Ordered Semigroups*, Comm. Algebra 30, 5651–5675.

[53] P. Resende and E. Rodrigues (2010), *Sheaves as Modules*, Appl. Categor. Struct. 18, 199–217.

[54] J. Rhodes and B. Steinberg (2009), *The q-theory of Finite Semigroups*, Springer.

[55] W. Rudin (1991), *Functional Analysis*, McGraw-Hill (2nd edition).

[56] C. Simpson (1998), *Homotopy types of strict 3-groupoids*, arXiv:math.CT/9810059v1.

[57] L. Schröder (2000), *Isomorphisms and Splittings of Idempotents in Semicategories*, Cahiers Topologie Géom. Différ. Catégories 41, No. 2, 143–153.

[58] L. Schröder (2002), *Monads on composition graphs*, Appl. Categ. Structures 10, 221–236.

[59] L. Schröder and P. Mateus (2002), *Universal Aspects of Probabilistic Automata*, Math. Struct. Comput. Sci. 12, No. 4, 481–512.

[60] J.D.H. Smith (2006), *An Introduction to Quasigroups and their Representations*, Studies in Advanced Mathematics, Chapman & Hall/CRC.

[61] I. Stubble (2003), *Categorical structures enriched in a quantaloid: categories and semicategories*, PhD thesis, Université Catholique de Louvain-la-Neuve.

[62] I. Stubble (2007), *Q-modules are Q-suplattices*, Theory Appl. Categ. 19, No. 4, 50–60.

[63] T. Tao and V. Vu (2006), *Additive Combinatorics*, Vol. 105 of the Cambridge Studies in Advanced Mathematics Series, Cambridge University Press.

[64] B. Tilson (1987), *Categories as algebra: An essential ingredient in the theory of monoids*, Journal of Pure and Applied Algebra 48, pp. 83–198.

[65] A.C. Zaanen (1983), *Riesz spaces. II*, North-Holland.

Institut Camille Jordan (ICJ), Université Jean-Monnet, 23 rue Paul Michelon, 42023 Saint-Étienne cedex 2, France

E-mail address: tringali@ann.jussieu.fr