Research Article

Majorization for Certain Classes of Analytic Functions Defined by Fournier–Ruscheweyh Integral Operator

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1 Introduction and Definitions

For the two functions \( u \) and \( v \) which are analytic in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), we can define the majorization for these two functions as follows (see [1]):

\[ u(z) \leq v(z) (z \in D). \] (1)

If there exists a function \( \psi(z) \) that is analytic in \( D \), then

\[ |\psi(z)| \leq 1 \text{ and } u(z) = \psi(z) v(z) (z \in D). \] (2)

For the two functions \( u \) and \( v \), if the function \( u \) is subordinate to the function \( v \) defined as \( u(z) < v(z) \), if there is a Schwarz function \( w \), that is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), \( z \in D \), such that \( u(z) = v(w(z)) \), \( z \in D \).

Now, on combining subordination and majorization, we define quasi-subordination as follows. For two functions \( u \) and \( v \), we say that \( u \) is quasi-subordinate to \( v \) (see [2]) and it is defined as

\[ u(z) \leq_q v(z) (z \in D), \] (3)

If there are two functions \( \psi(z) \) and \( \omega(z) \) that are analytic in \( D \), then \((u(z)/\psi(z)) \) is analytic in \( D \) and

\[ |\psi(z)| \leq 1 \text{ and } \omega(0) = 0, |\omega(z)| \leq |z| \leq 1 (z \in D), \] (4)

satisfying

\[ u(z) = \psi(z) \omega(w(z)) (z \in D). \] (5)

**Remark 1**

(i) If we put \( \psi(z) = 1 \) in (5), we have the usual definition of subordination

(ii) If we put \( \omega(z) = z \) in (6), we have the usual definition of majorization

Let \( A \) denote the class of all functions of the form

\[ f(z) = z + \sum_{n=0}^{\infty} a_n z^n \text{, } (z \in D), \] (6)

which are analytic in open unit disk \( D \).

The function class \( \Phi \) has been introduced and studied by Li and Srivastava [3] and is defined as

\[ \Phi = \left\{ k(t): k(t) \geq 0, (0 \leq t \leq 1), \int_0^1 k(t) dt = 1 \right\}. \] (7)
Fournier and Ruscheweyh [3, 4] considered an integral operator with a nonnegative function:

\[ k_a: [0, 1] \rightarrow \mathbb{R} \text{ such that } \int_0^1 k_a(t)dt = 1. \]  

(8)

By substituting suitable values of parameter \( a \), there are lots of special cases of function \( k_a(t) \). We therefore consider the Fournier–Ruscheweyh integral operator to be in the following modified form [3] (see [5]):

\[ \mathcal{J}_k^a f(z) = \int_0^1 k_a(t) \frac{f(tz)}{t} dt, \quad (f \in A). \]  

(9)

where the real-valued functions \( k_a \) and \( k_{a-1} \) fulfill the requirements:

(1) For an acceptable parameter \( a \),

\[ k_{a-1}(t) \in \Phi, k_a(t) \in \Phi \text{ and } k_a(1) = 0. \]  

(10)

(2) There exists a constant \( \lambda (-1 < \lambda \leq 2) \) such that

\[ \lambda k_a(t) - tk_a'(t) = (\lambda + 1)k_{a-1}(t), \]  

(11)

where \( t \in (0, 1) \) and \(-1 < \lambda \leq 2\).

For \( \mathcal{J}_k^a \) operator, we have

\[ z(\mathcal{J}_k^a u(z))' = -\lambda(\mathcal{J}_k^a u(z)) + (\lambda + 1)\mathcal{J}_k^{a-1} u(z). \]  

(12)

Remark 2

(i) If we take

\[ k_a(t) = \left( \frac{2^a}{\mu(a)} \right) \left( \log \frac{1}{t} \right)^{a-1} = k_1(a > 0), \]  

(13)

in (9), we get the integral operator \( \mathcal{J}_k^a \) as

\[ \mathcal{J}_k^a = \frac{2^a}{z\mu(a)} \int_0^z \left( \log \frac{z}{t} \right)^{a-1} f(t) dt, \quad (f \in A, a > 0). \]  

(14)

The integral operator \( \mathcal{J}_k^a \) is exactly the same as the transformation \( I_k^1 \) given by Flett [6] and studied subsequently by Li [7], Li and Srivastava [8], and many others. In the case when \( a > 1 \), then we have \( \lambda = 1 \).

(ii) If we take

\[ k_a(t) = \left( \frac{a+b}{a} \right) a(1-t)^{a-1} t^b = k_2, \quad (a > 0, b > -1), \]  

(15)

in (9), we get the Jung–Kim–Srivastava integral operator \( Q_k^a \) [9] (see [10–12]) as

\[ Q_k^a f(z) = \left( \frac{a+b}{a} \right) \frac{a}{z^a} \int_0^z \left( 1 - \frac{t}{z} \right)^{a-1} t^{b-1} f(t) dt. \]  

(16)

\[ (f \in A, a > 0, b > -1), \]  

where

\[ \left( \begin{array}{c} a \\ b \end{array} \right) = \frac{\mu(a+1)}{\mu(b+1)\mu(a-b+1)} \left( \begin{array}{c} a \\ b \end{array} \right), \quad (a, b \in \mathbb{C}). \]  

(17)

In terms of known Gamma functions, the integral operator \( Q_k^a \) is analogous to the convolution operator \( L(a, b) \) by Carlson and Shaffer [13]. In the case when \( a > 1, b > -1, \) and \( 0 < a + b \leq 3 \), we have \( \lambda = a + b - 1 \).

Now, we describe the following classes of analytical functions using integral operator (9).

Definition 1. The function \( f \in A \) is said to be in the class \( S_k^a [M, N; \mu] \) if and only if

\[ 1 + \frac{1}{\mu} \left( \frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - 1 \right) \leq \frac{1 + Mz}{1 + Nz}, \]  

(18)

with \(-1 < N < M < 1, k, \mu \in \mathbb{C}, \) and \( C = \mathbb{C} \setminus \{0\} \).

If we take the value of \( k \) as defined in (13) and (15), this class becomes \( S_k^{a1} [M, N; \mu] \) and \( S_k^{a2} [M, N; \mu] \), respectively.

Definition 2. The function \( f \in A \) is said to be in the class \( R_k^a (\mu) \) if and only if

\[ \left[ \frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - \mu \right] \geq \left( \frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} - 1 \right) < e^c, \quad (z \in D), \]  

(19)

where \( a \geq 0, k \in \Phi, \) and \( \mu \geq 0 \).

If we take the value of \( k \) as defined in (13) and (15), this class becomes \( R_k^{a1} (\mu) \) and \( R_k^{a2} (\mu) \), respectively.

Definition 3. The function \( f \in A \) is said to be in the class \( T_k^a (\theta) \) if and only if

\[ e^{\theta i} \left( \frac{z(\mathcal{J}_k^a f(z))'}{\mathcal{J}_k^a f(z)} \right) < e^{\cos \theta + i \sin \theta}, \quad (z \in D), \]  

(20)

where \( a \geq 0, k \in \Phi, \) and \(-\Pi/2 < \theta < \Pi/2\).

If we take the value of \( k \) as defined in (13) and (15), this class becomes \( T_k^{a1} (\theta) \) and \( T_k^{a2} (\theta) \), respectively.

A majorization problem for the normalized class of starlike functions has been investigated by MacGregor [1] and further studied by Altıntas et al. [14]. Recently, a number of researchers have studied several majorization problems for univalent and multivalent functions or meromorphic and multivalent meromorphic functions involving different operators and different classes [14–20, 22–24]. By motivating the above work, the majorization problems of the classes \( S_k^{a1} [M, N, \mu], R_k^{a1} (\mu), \) and \( T_k^{a1} (\theta) \) are investigated as follows.

2. Problem of Majorization for the Classes \( S_k^{a1} [M, N, \mu], R_k^{a1} (\mu), \text{ and } T_k^{a1} (\theta) \)

Theorem 1. Let the function \( f \in A \) and assume that \( g \in S_k^{a1} [M, N, \mu] \). If \( \mathcal{J}_k^a f(z) \) is majorized by \( \mathcal{J}_k^a g(z) \) in \( D \), then

\[ |\mathcal{J}_k^a f(z)| \leq |\mathcal{J}_k^a g(z)| \text{ for } |z| \leq \rho_0, \]  

(21)
where \( \rho_0 \) is the smallest positive root of the equation
\[
\| (M - N) + (1 + \lambda)N \| p^3 - (2|N| + \lambda + 1)p^2 \\
- (2 + \| (M - N) + (1 + 1)N \|) \rho + (\lambda + 1) = 0,
\]
where \(-1 \leq N < M \leq 1, k \in \Phi, \mu \in C^*, -1 < \lambda \leq 2, \) and \((\lambda + 1) \leq \| (M - N) + (1 + 1)N \|).

**Proof.** Since \( g \in S_k^p [M, N, \mu] \), then, from (18),
\[
1 + \frac{1}{\mu} \left(\frac{z(\mathcal{F}_k^a g(z))^\prime}{\mathcal{F}_k^a g(z)} - 1\right) = \frac{1 + Mw(z)}{1 + Nw(z)}
\]
where \( w \) is the analytic function in \( D \), with \( w(0) = 0 \) and \( |w(z)| \leq |z| < 1 \forall z \in \overline{D} \).

Now, from the previous equality,
\[
z(\mathcal{F}_k^a g(z))^\prime = \frac{1 + (\mu(M - N) + N)w(z)}{1 + Nw(z)}.
\]

Now, we make use of relation (12), that is,
\[
z(\mathcal{F}_k^a g(z))^\prime = -\lambda(\mathcal{F}_k^a g(z)) + (\lambda + 1)\mathcal{F}_k^{a-1} g(z),
\]
For \(-1 < \lambda \leq 2 \), then, from (24), we have
\[
\mathcal{F}_k^{a-1} g(z) = \frac{\lambda + 1 + (\mu(M - N) + (\lambda + 1)N)w(z)}{(\lambda + 1)(1 + Nw(z))},
\]
which implies that
\[
|\mathcal{F}_k^{a-1} f(z)| \leq \frac{|z| \left(1 - |\psi(z)|^2\right) (1 + |N|)}{(1 - |z|^2) (\lambda + 1) - \| (M - N) + (\lambda + 1)N \| p)} + |\psi(z)| \left| \mathcal{F}_k^{a-1} g(z) \right|.
\]

Setting \( |z| = \rho \) and \( |\psi(z)| = c \), then inequality (33) leads to
\[
|\mathcal{F}_k^{a-1} f(z)| \leq \frac{\zeta(\rho, c) \left| \mathcal{F}_k^{a-1} g(z) \right|}{(1 - \rho^2) \left(\lambda + 1\right) - \| (M - N) + (\lambda + 1)N \| p},
\]
where
\[
\zeta(\rho, c) = \rho \left(1 - c^2\right) (1 + |N|) + c (1 - \rho^2) \left(\lambda + 1\right) - \mu (M - N) + (\lambda + 1)N \| p).
\]

Then, from (34),
\[
|\mathcal{F}_k^{a-1} f(z)| \leq \eta(\rho, c) \left| \mathcal{F}_k^{a-1} g(z) \right|,
\]
where
\[
\eta(\rho, c) = \frac{\zeta(\rho, c)}{(1 - \rho^2) \left(\lambda + 1\right) - \| (M - N) + (\lambda + 1)N \| p}.
\]

From relation (36), in order to prove our result, we need to determine
\[
|\mathcal{F}_k^a g(z)| \leq \frac{(\lambda + 1)(1 + N|z|) \left| \mathcal{F}_k^{a-1} g(z) \right|}{(\lambda + 1) - \| (M - N) + (\lambda + 1)N \| |z|}.
\]
It follows that \(v(\rho) \geq 0\forall \rho \in [0, \rho_0]\), where \(\rho_0 = \rho_0 (\mu, \lambda, M, N)\) is the smallest positive root of equation (22), which proves conclusion (21).

**Theorem 2.** Let the function \(f \in A\), and assume that \(g \in R_k^a (\mu)\). If \(F_k^a f (z)\) is majorized by \(F_k^a g (z)\) in \(D\), then

\[
|F_k^a f (z)| \leq |F_k^a g (z)| \forall |z| \leq \rho_1,
\]

where \(\rho_1\) is the smallest positive root of the equation

\[
(\epsilon^2 + \mu (\lambda + 1) - |\lambda|) |\lambda|^2 - 2 (1 + \mu) |\lambda| - \mu (\lambda + 1 - \epsilon^2) = 0,
\]

where \(a \geq 0, k \in \Phi, \mu \geq 0, -1 < \lambda \leq 2,\) and \(|\lambda| > \mu (\lambda + 1) + e\).

**Proof.** Since \(g \in R_k^a (\mu)\), then, from (19) and the subordination relation,

\[
\left| \frac{z(\mathcal{F}_k^a g(z)' - \mu z(\mathcal{F}_k^a g(z))'}{\mathcal{F}_k^a g(z)} - 1 \right| = e^w(z) (z \in D),
\]

where \(w\) is the analytic function in \(D\), with \(w(0) = 0\) and \(|w(z)| \leq |z| \leq 1, \forall z \in D\). Now, let

\[
W = \frac{z(\mathcal{F}_k^a g(z))'}{\mathcal{F}_k^a g(z)}.
\]

In (45), we have

\[
W - \mu |W - 1| = e^w(z),
\]

which implies that

\[
W - \mu (W - 1) e^{i\phi} = e^{w(z)}.
\]

Then, we have

\[
W = \frac{e^{w(z) - \mu e^{i\phi}}}{1 - \mu e^{i\phi}}.
\]

From (46 and 49), we have

\[
z(\mathcal{F}_k^a g(z))' = \frac{e^{w(z) - \mu e^{i\phi}}}{1 - \mu e^{i\phi}}
\]

Now, on using (12) in (50), for \(-1 < \lambda \leq 2\), we have the following:

\[
\mathcal{F}_k^a g(z) = \frac{e^{w(z) + \lambda (1 + \mu) e^{i\phi}}}{(\lambda + 1)(1 - \mu e^{i\phi})}.
\]

which implies that

\[
|\mathcal{F}_k^a g(z)| \leq \frac{(\lambda + 1)(1 + \mu)}{|\lambda| - \mu (\lambda + 1) - \epsilon^2} |\mathcal{F}_k^a f (z)|.
\]

Now, since \(F_k^a f (z)\) is majorized by \(F_k^a g (z)\) in \(D\), then we have

\[
F_k^a f (z) = \psi(z) F_k^a g(z).
\]

Differentiating the previous equality with respect to \(z\) and then multiplying by \(z\), we get

\[
z(\mathcal{F}_k^a f(z))' = z \psi(z) (\mathcal{F}_k^a g(z))' + z \psi'(z) (\mathcal{F}_k^a g(z)).
\]

On using relation (12), we have

\[
(\lambda + 1) \mathcal{F}_k^a f(z) = z \psi'(z) \mathcal{F}_k^a g(z) + (\lambda + 1) \psi(z) \mathcal{F}_k^a g(z).
\]

This implies

\[
(\lambda + 1) |\mathcal{F}_k^a f(z)| \leq |z| |\psi(z)| |\mathcal{F}_k^a g(z)| + |(\lambda + 1) |\psi(z)| |\mathcal{F}_k^a g(z)|.
\]

Thus, note that the Schwarz function \(\psi\) satisfies the inequality (see [21])

\[
|\psi(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}; \quad (z \in D).
\]

On using (52) and (57) in (56), we have

\[
|\mathcal{F}_k^a f(z)| \leq \frac{\zeta_1(\rho, c)}{(1 - \rho^2)(|\lambda| - \mu (\lambda + 1) - \epsilon^2)} |\mathcal{F}_k^a g(z)|.
\]

Setting \(|z| = \rho\) and \(|\psi(z)| = c (0 \leq c \leq 1)\), then inequality (58) leads to

\[
|\mathcal{F}_k^a f(z)| \leq \frac{\zeta_1(\rho, c)}{(1 - \rho^2)(|\lambda| - \mu (\lambda + 1) - \epsilon^2)} |\mathcal{F}_k^a g(z)|.
\]

where

\[
\zeta_1(\rho, c) = \rho (1 + \mu)(1 - c^2) + c (1 - \rho^2)(|\lambda| - \mu (\lambda + 1) - \epsilon^2).
\]

Then, from (59),

\[
|\mathcal{F}_k^a f(z)| \leq \eta_1(\rho, c) |\mathcal{F}_k^a g(z)|.
\]

Here,

\[
\eta_1(\rho, c) = \frac{\zeta_1(\rho, c)}{(1 - \rho^2)(|\lambda| - \mu (\lambda + 1) - \epsilon^2)}.
\]

From relation (61), in order to prove our result, we need to determine

\[
\rho_1 = \max\{\rho \in [0, 1]; \eta_1(\rho, c) \leq 1 \forall c \in [0, 1]\},
\]

\[
= \left[ \max\{\rho \in [0, 1]; G_1(\rho, c) \geq 0 \forall c \in [0, 1]\} \right],
\]

where

\[
G_1(\rho, c) = (1 - \rho^2)(1 - c)(|\lambda| - \mu (\lambda + 1) - \epsilon^2) - \rho (1 + \mu)(1 - c^2).
\]

A simple calculus shows that the inequality \(G_1(\rho, c) \geq 0\) is equivalent to

\[
u_1(\rho, c) = (1 - \rho^2)(|\lambda| - \mu (\lambda + 1) - \epsilon^2) - \rho (1 + \mu)(1 + c) \geq 0.
\]

However, the function \(\nu_1(\rho, c)\) takes its minimum value at \(c = 1\), that is,
\[
\min \{ u_1(\rho, c) \mid c \in [0, 1]\} = u_1(\rho, 1) = v_1(\rho),
\]
where
\[
v_1(\rho) = (1 - \rho^2)(|\lambda| - \rho(\lambda + 1) - \rho^2) - 2\rho(1 + \mu) = 0.
\]

Thus, note that the Schwarz function \( \phi \) satisfies the inequality (see [21])
\[
|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|}; \quad (z \in D).
\]

On using (71) and (78) in (77), we have
\[
|\mathcal{F}_k^{-1} f(z)| \leq \left( \frac{|z|(1 - |\psi(z)|^2)\sec \theta |}{(1 - |z|)^3(\lambda - (\lambda + 1)|\tan \theta - e^0| + |\psi(z)|)} \right) |\mathcal{F}_k^{-1} g(z)|.
\]

Setting \(|z| = \rho \) and \(|\psi(z)| = c \) \((0 \leq c \leq 1)\), then inequality (79) leads to
\[
|\mathcal{F}_k^{-1} f(z)| \leq \left( \frac{\zeta_2(\rho, c)}{(1 - \rho^2)(\lambda - (\lambda + 1)|\tan \theta - e^0|)} \right) |\mathcal{F}_k^{-1} g(z)|.
\]

Then, from (80),
\[
|\mathcal{F}_k^{-1} f(z)| \leq \eta_2(\rho, c) |\mathcal{F}_k^{-1} g(z)|,
\]
where
\[
\eta_2(\rho, c) = \frac{\zeta_2(\rho, c)}{(1 - \rho^2)(\lambda - (\lambda + 1)|\tan \theta - e^0|)}.
\]

From relation (82), in order to prove our result, we need to determine
\[
\rho^* = \max\{\rho \in [0, 1]; \eta_2(\rho, c) \leq \forall c \in [0, 1]\} = \max\{\rho \in [0, 1]; G_2(\rho, c) \geq \forall c \in [0, 1]\}
\]
where
\[
G_2(\rho, c) = (1 - \rho^2)(1 - c)(\lambda - (\lambda + 1)|\tan \theta - e^0| - \rho(1 + c)|\sec \theta| \geq 0.
\]

A simple calculus shows that the inequality \( G_2(\rho, c) \geq 0 \) is equivalent to
\[
\eta_2(\rho, c) = (1 - \rho^2)(1 - c)(\lambda - (\lambda + 1)|\tan \theta - e^0| - \rho(1 + c)|\sec \theta| \geq 0.
\]

However, the function \( \eta_2(\rho, c) \) takes its minimum value at \( c = 1 \), that is,
\[
\min \{ u_2(\rho, c) \mid c \in [0, 1]\} = u_2(\rho, 1) = v_2(\rho),
\]
where
\[
\zeta_2(\rho, c) = \frac{\zeta_2(\rho, c)}{(1 - \rho^2)(\lambda - (\lambda + 1)|\tan \theta - e^0|)}.
\]
Corollary 3. Let the function $f \in A$, and assume that $g \in S_0^k [M, N, \mu]$. If $\mathcal{P}^m f(z)$ is majorized by $\mathcal{P}^m g(z)$ in $D$, then
\[
|\mathcal{P}^{m-1} f(z)| \leq |\mathcal{P}^{m-1} g(z)| |z| \leq \rho_2,
\]
where $\rho_2$ is the smallest positive root of the equation
\[
|\mu(M - N) + 2\mu(N + 1)\rho^2 - 2(2 + \mu(M - N) + 2N)|\rho + 2 = 0,
\]
where $-1 \leq M < M \leq 1, \mu \in C^*$, and $2 \geq |\mu(M - N) + 2N|$.

3. Corollaries and Consequences

If we take the values of $k$ defined in (13) and (15), then the above theorems give the following corollaries.

Corollary 1. Let the function $f \in A$, and assume that $g \in S_0^k [M, N, \mu]$. If $\mathcal{P}^m f(z)$ is majorized by $\mathcal{P}^m g(z)$ in $D$, then
\[
(1 - \rho^2)^{(\lfloor \lambda \rfloor - (\lambda + 1)\tan \theta - \rho^2)} - 2(2 + \mu(M - N) + 2N)|\rho + (a - b)| = 0.
\]
(88)

It follows that $v_2(\rho) \geq 0$, $\rho \in [0, \rho^*]$, where $\rho^* = (\theta, \lambda)$ is the smallest positive root of (69), which proves conclusion (68).

Corollary 2. Let the function $f \in A$, and assume that $g \in S_0^k [M, N, \mu]$. If $\mathcal{P}^m f(z)$ is majorized by $\mathcal{P}^m g(z)$ in $D$, then
\[
|\mathcal{P}^{m-1} f(z)| \leq |\mathcal{P}^{m-1} g(z)| |z| \leq \rho_3,
\]
where $\rho_3$ is the smallest positive root of the equation
\[
(e^\theta - (a + b)^2 - 2(2 + \mu(M - N) + 2N)|\rho + (a - b)\theta = 0,
\]
where $a \geq 0, \mu \geq 0$, and $1 > 2\mu + e$.

Corollary 3. Let the function $f \in A$, and assume that $g \in T_k(\theta)$. If $\mathcal{P}^m f(z)$ is majorized by $\mathcal{P}^m g(z)$ in $D$, then
\[
|\mathcal{P}^{m-1} f(z)| \leq |\mathcal{P}^{m-1} g(z)| |z| \leq \rho_3',
\]
where $\rho_3'$ is the smallest positive root of the equation
\[
(e^\theta + 2\mu(\lambda - 1)\rho^2 - 2(2 + \mu(M - N) + 2N)|\rho + (1 - 2\tan \theta - \rho^2) = 0,
\]
where $a \geq 0, -(\pi/2) < \theta < (\pi/2)$, and $1 > 2\tan \theta + e$.

Corollary 4. Let the function $f \in A$, and assume that $g \in S_0^k [M, N, \mu]$. If $Q^m f(z)$ is majorized by $Q^m g(z)$ in $D$, then
\[
|Q^{m-1} f(z)| \leq |Q^{m-1} g(z)| |z| \leq \rho_4,
\]
where $\rho_4$ is the smallest positive root of the equation
\[
|\mu(M - N) + (a + b)N|\rho^3 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b)| = 0,
\]
where $-1 \leq M < M \leq 1, \mu \in C^*$, and $a > 1, b > -1$, and $(a + b) \geq |\mu(M - N) + (a + b)N|$.

Corollary 5. Let the function $f \in A$, and assume that $g \in R_0^k (\mu)$. If $Q^m f(z)$ is majorized by $Q^m g(z)$ in $D$, then
\[
|Q^{m-1} f(z)| \leq |Q^{m-1} g(z)| |z| \leq \rho_5,
\]
where $\rho_5$ is the smallest positive root of the equation
\[
(\rho^2 + (a + b)(\pi/2 - (a + b)\theta)^2 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b) = 0,
\]
where $a \geq 0, b > -1, \mu \geq 0$, and $|a + b - 1| = |(a + b) + e|$.

Corollary 6. Let the function $f \in A$, and assume that $g \in T_k(\theta)$. If $Q^m f(z)$ is majorized by $Q^m g(z)$ in $D$, then
\[
|Q^{m-1} f(z)| \leq |Q^{m-1} g(z)| |z| \leq \rho_5',
\]
where $\rho_5'$ is the smallest positive root of the equation
\[
(\rho^2 + (a + b)(\pi/2 - (a + b)\theta)^2 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b) = 0,
\]
where $a \geq 0, b > -1, \mu \geq 0$, and $|a + b - 1| = |(a + b) + e|$.

If we take $M = 1$ and $N = -1$, then Theorem 1, Corollary 1, and Corollary 4 give the following results.

Corollary 7. Let the function $f \in A$, and assume that $g \in S_0^k [1, -1, \mu]$. If $\mathcal{F}^m f(z)$ is majorized by $\mathcal{F}^m g(z)$ in $D$, then
\[
|\mathcal{F}^{m-1} f(z)| \leq |\mathcal{F}^{m-1} g(z)| |z| \leq \rho_0,
\]
where $\rho_0$ is the smallest positive root of the equation
\[
|2\mu - (1 + \lambda)(\pi - (\mu + 1)\rho^2 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b) = 0,
\]
where $k \in \Phi, \mu \in C^*, -1 < \lambda < 2, (\lambda - 1) \geq |2\mu - (1 + \lambda)|$.

Corollary 8. Let the function $f \in A$, and assume that $g \in S_0^k [1, -1, \mu]$. If $\mathcal{F}^m f(z)$ is majorized by $\mathcal{F}^m g(z)$ in $D$, then
\[
|\mathcal{F}^{m-1} f(z)| \leq |\mathcal{F}^{m-1} g(z)| |z| \leq \rho_2,
\]
where $\rho_2$ is the smallest positive root of the equation
\[
|\mu - (1 + \lambda)|\rho^2 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b) = 0,
\]
where $\mu \in C^*$ and $1 \geq |\mu - 1|$.

Corollary 9. Let the function $f \in A$, and assume that $g \in S_0^k [1, -1, \mu]$. If $Q^m f(z)$ is majorized by $Q^m g(z)$ in $D$, then
\[
|Q^{m-1} f(z)| \leq |Q^{m-1} g(z)| |z| \leq \rho_4,
\]
where $\rho_4$ is the smallest positive root of the equation
\[
|2\mu - (a + b)(\pi/2 - (2 + a + b)\theta)^2 - 2(2 + \mu(M - N) + (a + b)N)|\rho + (a + b) = 0,
\]
where $\mu \in C^*, a > 1, b > -1$, and $(a + b) \geq |(a + b) - (a + b)|$. 

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors have contributed equally to the paper.

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