A note on Wilson-’t Hooft operators

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Abstract

We find the basic ingredients required to compute the Operator Product Expansion of Wilson-’t Hooft operators in \( \mathcal{N} = 4 \) super-Yang-Mills theory with gauge group \( G = PSU(3) \). These include the geometry of certain moduli spaces of BPS configurations in the presence of ’t Hooft operators and vector bundles over them. The bundles arise in computing the OPE due to electric degrees of freedom in dyonic operators. We verify our results by reproducing the OPE of ’t Hooft operators predicted by S-duality.
1 Introduction

The famous S-duality conjecture [1] states that $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $G$ is isomorphic to $\mathcal{N} = 4$ SYM with the Langlands-dual gauge group $L^G$. This isomorphism maps Wilson loop operators [2] to ’t Hooft loop operators [3], [4] in the dual theory. Recalling that the product of Wilson loops is determined by the representation ring of $G$, S-duality conjecture predicts that product of ’t Hooft loops is controlled by the representation ring of $L^G$. This prediction has been verified in [5] based on the earlier mathematical result [6].

Yang-Mills theory also admits mixed Wilson-’t Hooft (WH) loop operators. As explained in [4], at zero $\theta$—angle they are labeled by elements of the set

$$\tilde{\Lambda}(G)/\mathcal{W} = \left(\Lambda_w(G) \oplus \Lambda_w(L^G)\right)/\mathcal{W},$$
where $\Lambda_w(G)$ is the weight lattice of $G$ and $W$ is the Weyl group (which is the same for $G$ and $L^G$). In $\mathcal{N} = 4$ SYM theory these mixed operators can be made supersymmetric preserving one quarter of the original supersymmetry.

In [7] we outlined an approach how to compute the product of WH loop operators for general group $G$ and actually computed it for $G = SU(2)$ and $G = PSU(2)$. Our approach uses the holomorphic-topological twist [8] of the $\mathcal{N} = 4$ SYM theory and the connection between BPS configurations in $\mathcal{N} = 4$ SYM theory in the presence of ’t Hooft operators and solutions of 3d Bogomolny equations with magnetic sources [5],[9]. For $G = SU(2)$ and $G = PSU(2)$ S-duality completely fixes the OPE of Wilson-’t Hooft loop operators and our results were in agreement with S-duality.

More recently alternative methods of computing OPEs of loop operators in a certain class [10] of $\mathcal{N} = 2$ SYM theories were proposed [11],[12],[13],[14],[15] using connection with 2d Conformal Field Theory [16]. The algebra of WH loop operators for gauge groups $SU(2)$ and $PSU(2)$ can be explicitly determined using these references.

For other gauge groups very little is known about the algebra of loop operators. There is some partial information [17] arising from conjectural connection with Toda CFT. As we clarify in Section 2 our approach [7] works for general $\mathcal{N} = 2$ theories. This gives an opportunity to compute OPEs of loop operators in these theories in our approach and compare with the forthcoming results from the alternative methods [18]. For example, it is interesting to find the complete algebra of loop operators in $\mathcal{N} = 2$ theories with gauge group $PSU(n)$ for $n > 2$.

The simplest non-trivial OPE of WH operators in $\mathcal{N} = 4$ SYM theory for $G = PSU(3)$, which is not predicted by S-duality, is

$$WT_{\mu,\nu} \times WT_{\mu,0} = WT_{2\mu,\nu} + \sum_j (\text{signs on the right side of (1))}$$

where magnetic charge $\mu = w_1(\bar{\mu} = w_2)$ is the highest weight of a fundamental (anti-fundamental) representation of $L^G = SU(3)$ and electric charge $\nu = aw_1 + bw_2$ is the highest weight of $G$, i.e. $a + 2b = 0 \mod 3$. The electric weights $\nu_j$ and signs $(-)^s_j$ on the right side of (1) are to be determined. As we already noted in [7], there could be minus signs on the right side of OPEs arising for the following reason. Loop operators can be promoted to line operators. While loop operators form a commutative ring, line operators form a monoidal category. We argued in [7] that the ring of loop operators can be thought of as the $K^0$-group of this category and in K-theory negative signs occur naturally.

As we review in Section 2 to compute (1) in our approach, we first need to determine the geometry of the moduli space $\mathcal{M}$ of 3d Bogomolny equations in $I \times \mathcal{C}$ with two sources, each characterized by magnetic charge $\mu$. Here $I$ is an interval and $\mathcal{C}$ is a Riemann surface, and boundary conditions at the two ends of $I$ are such that without any magnetic sources there is unique vacuum. $\mathcal{M}$ is obtained by blowing-up certain singular 4-fold $X_4$ which is the compactification of the moduli space of solutions of 3d Bogomolny equations in $I \times \mathcal{C}$ with a single source characterized by magnetic charge $2\mu$. The blow-up procedure produces exceptional divisor $D$ in $\mathcal{M}$. We further must write the appropriate metric on the bulk part $\mathcal{M}_{bulk}$ which
is obtained by removing from \(\mathcal{M}\) the vicinity of \(D\), i.e. the total space of the normal bundle of \(D\) in \(\mathcal{M}\).

The next step to determine the right side of (1) is to find vector bundles \(V\) over \(\mathcal{M}\) and \(V_{\text{bulk}}\) over \(\mathcal{M}_{\text{bulk}}\). These bundles arise in computing the OPE due to electric degrees of freedom in dyonic operators. Equipped with these vector bundles, one should compute cohomology groups

\[
H^p(\mathcal{M}, \Omega^q \otimes V) \quad \text{and} \quad H^p(\mathcal{M}_{\text{bulk}}, \Omega^q \otimes V_{\text{bulk}}).
\]

For compact space \(\mathcal{M}\) these are sheaf cohomology groups but for non-compact \(\mathcal{M}_{\text{bulk}}\) we are interested in \(L^2\) Dolbeault cohomology of the corresponding bundles. From these cohomology groups one will be able to determine the right side of equation (1). Namely the first term in (1) comes from the bulk part of the moduli space while the second sum is the so called bubbled contribution accounted by

\[
\sum_{p,q} (-1)^{p+q} \left( H^p(\mathcal{M}, \Omega^q \otimes V) - H^p(\mathcal{M}_{\text{bulk}}, \Omega^q \otimes V_{\text{bulk}}) \right).
\]

The existence of bubbled contribution is due to monopole bubbling [5] which occurs when the magnetic charge of the ’t Hooft operator decreases by absorbing a BPS monopole. This process is possible because the moduli space of solutions of 3d Bogomolny equations in the presence of magnetic source with charge \(2\mu\) is non-compact. For gauge group \(G = PSU(2)\) it was possible to write a complete \(PSU(2)\) invariant metric on the “bubbled geometry” and compute cohomology corresponding to the bubbled contribution using this metric [5], [7]. We found that for \(G = PSU(3)\) there is no complete \(PSU(3)\) invariant metric on the “bubbled geometry”. For this reason we adopt the procedure outlined above.

This note is organized as follows. In Section 2 we review our approach. We determine the geometry of \(\mathcal{M}\) in Section 3 and of \(\mathcal{M}_{\text{bulk}}\) in Section 4. As a non-trivial check of our results, in Section 5 we prove explicitly the OPE of ’t Hooft operators which is expected from S-duality:

\[
WT_{\mu,0} \times WT_{\mu,0} = WT_{2\mu,0} + WT_{\bar{\mu},0}.
\]

We compute \(L^2\) Dolbeault cohomology of \(\mathcal{M}\) in Appendix A and of \(\mathcal{M}_{\text{bulk}}\) in Appendix B and in Section 5 we show explicitly how principle \(SU(2)\) subgroup of \(LG = SU(3)\) acts on the cohomology of \(\mathcal{M}\) and \(\mathcal{M}_{\text{bulk}}\) in agreement with general facts about moduli spaces of BPS configurations in the presence of ’t Hooft operators [9]. Finally, in Section 6 we construct bundles \(V\) over \(\mathcal{M}\) and \(V_{\text{bulk}}\) over \(\mathcal{M}_{\text{bulk}}\) corresponding to \(\nu = aw_1 + bw_2\) with \(a + 2b = 0 \mod 3\). We will use the geometry of \(\mathcal{M}\) and \(\mathcal{M}_{\text{bulk}}\) together with these bundles to determine the right side of (1) in \(\mathcal{N} = 4\) SYM theory and in \(\mathcal{N} = 2\) SYM with \(N_f = 0\) in the future [10].

2 OPE of Wilson-’tHooft operators in \(\mathcal{N} = 4\) SYM: Review

In [7] we outlined an approach to study the Operator Product Expansion of Wilson-’t Hooft operators in \(\mathcal{N} = 4\) SYM theory with gauge group \(G\). The key step in our approach is to
use holomorphic-topological twist \[4\] of $\mathcal{N} = 4$ SYM theory on a manifold $C \times \Sigma$ where $C$ and $\Sigma$ are Riemann surfaces. It is convenient to treat $\mathcal{N} = 4$ SYM as $\mathcal{N} = 2$ SYM with a hypermultiplet in the adjoint representation. The theory has $SU(2)_R \times U(1)_N \times U(1)_B$ symmetry. The holonomy group is $U(1)_C \times U(1)_\Sigma$. One twists $U(1)_C$ action by a suitable linear combination of $U(1)_R \subset SU(2)_R$ and $U(1)_B$, and twists $U(1)_\Sigma$ by $U(1)_N$. The twisted theory is holomorphic-topological in a sense that correlators of various operators depend holomorphically on insertion points on $C$ and are completely independent of positions of the operators on $\Sigma$. The field content of the twisted theory depends on complex structures of $C$ and $\Sigma$. However, the dependence on the complex structure on $\Sigma$ can be eliminated \[4\].

Let $w$ and $z$ be a complex coordinate on $\Sigma$ and $C$ correspondingly. The twisted field theory has the following bosonic fields: the gauge field $A$, the adjoint Higgs field $q = \Phi_w dw \in K_\Sigma \otimes \text{ad}(E)$, the adjoint Higgs field $q = \Phi_w dw \in K_\Sigma \otimes \text{ad}(E)$, and the adjoint Higgs field $q = \Phi_w dw \in K_\Sigma \otimes \text{ad}(E)$. Here $K_\Sigma$ and $K_C$ are the pull-backs of the canonical line bundles of $\Sigma$ and $C$ to $\Sigma \times C$. We also define $\Phi_\bar{w} = \Phi_\bar{w}^\dagger$ and $q_{\bar{w}} = q_{\bar{w}}^\dagger$.

The fermionic fields are the “gauginos” $\lambda_w, \lambda_{\bar{w}}, \lambda_z, \lambda_{\bar{z}}, \lambda_{z\bar{w}}, \lambda_{z\bar{w}}$, and the “quarks” $\psi_w, \chi_w, \psi_{\bar{w}}, \chi_{\bar{w}}, \psi_{zw}, \chi_{zw}$, $\psi_{z\bar{w}}, \chi_{z\bar{w}}$. The fermions are all in the adjoint representation.

Let us recall how BRST-invariant loop operators look like in the twisted theory. If $\gamma$ is a closed curve on $\Sigma$ and $p$ is a point on $C$, the BRST invariant Wilson operator has the form:

$$W_R(\gamma, p) = \text{Tr}_R P \exp i \int_{\gamma \times p} A$$

Here $A_w = A_w + i \Phi_w$ and $A_{\bar{w}} = A_{\bar{w}} + i \Phi_{\bar{w}}$.

Next, the BRST invariant 't Hooft operator $W_T_{\mu,0}$ is a disorder operator prescribing the following singular behaviour for the fields near the support $\gamma$ which can be locally written as $\text{Re} w = 0, z = 0$:

$$F \sim *_3 d \left( \frac{\mu}{2r} \right) \Phi_w \sim \frac{\mu}{2r}. \quad (4)$$

where $\mu$ is in the Lie algebra of the gauge group and $r^2 = |z|^2 + (\text{Re} w)^2$ locally near $\gamma$.

Finally, there are mixed BRST invariant Wilson-'t Hooft loop operators which source both electric and magnetic fields. To describe them, one requires the singularity of fields as in \[4\] and inserts into the path-integral a factor

$$\text{Tr}_R P \exp i \int_{\gamma \times p} A$$

where $R$ is an irreducible representation of the stabilizer subgroup\[^{\dagger}\] $G_\mu \subset G$ of $\mu$.

Note that holomorphic-topological twist is well-defined for any $\mathcal{N} = 2$ super-conformal gauge theory for arbitrary choice of $C$ and $\Sigma$. To compute the OPE of a pair of Wilson-'t Hooft line operators we actually take $\Sigma = I \times \mathbb{R}$ where $I$ is an interval. In this case $\Sigma$ is flat and one does not twist along $\Sigma$. Hence one does not need the existence of non-anomalous $U(1)_N$

\[^{\dagger}\] $G_\mu = \{ g = e^t \in G : [t, \mu] = 0 \}$. 

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symmetry for $\Sigma = I \times \mathbb{R}$ and can apply our method of computing OPE to general $\mathcal{N} = 2$ gauge theories.

To compute the OPE of a pair of Wilson-\'t Hooft line operators we followed the same method as in [5]. Namely, we quantize the twisted gauge theory on a manifold $C \times I \times \mathbb{R}$, with suitable boundary conditions and with two insertions of Wilson-\'t Hooft operators that are sitting at the same point on $C$. The problem reduces to the supersymmetric quantum mechanics on the space of zero modes of the gauge theory.

The twisted theory is independent of gauge coupling [4] and semiclassical computation (at weak coupling) is exact. When quantizing the theory at weak coupling, the roles of Wilson and \'t Hooft operators are very different. \'t Hooft operators directly affect the equations for the BRST-invariant configurations whose solutions determine the moduli space, i.e. the space of bosonic zero modes. A Wilson operator corresponds to inserting an extra degree of freedom, which couples weakly to the gauge fields, and can be treated perturbatively.

As in [5], we choose boundary conditions so that in the absence of Wilson-\'t Hooft line operators the Hilbert space of the twisted gauge theory is one-dimensional. For explicit choice of such boundary conditions, see sections 5.2 and 5.3 in [7]. Let $\mathcal{M}$ be the moduli space of BPS configurations in the presence of \'t Hooft operators. As shown in [5] and [9], $\mathcal{M}$ is the moduli space of solutions of 3d Bogomolny equations with magnetic sources and can also be identified with the moduli space of Hecke modifications of a holomorphic vector bundle on $C$. The type of modifications is determined in terms of magnetic charges of \'t Hooft operators. We showed in [7] that in $\mathcal{N} = 4$ SYM theory, after holomorphic-topological twist, “gaugino” zero modes span anti-holomorphic tangent bundle $\overline{T}\mathcal{M}$, meanwhile “quark” zero modes span holomorphic tangent bundle $T\mathcal{M}$. Therefore the Hilbert space of the effective SQM is the space of $L^2$ sections of the vector bundle

$$\bigoplus_p \Lambda^p \left( T^*\mathcal{M} \oplus \overline{T}^*\mathcal{M} \right) = \bigoplus_{p,q} \Omega^{p,q}(\mathcal{M}).$$

We also showed that the BRST operator acts as the Dolbeault operator.

When electric degrees of freedom are switched on, i.e. one inserts Wilson-\'t Hooft operators as opposed to \'t Hooft operators, one has to read off vector bundle $\mathcal{V}$ over $\mathcal{M}$ from the electric charges of the operators. Then the Hilbert space of the effective SQM is the space of $L^2$ sections of the vector bundle

$$\mathcal{V} \otimes \left( \bigoplus_{p,q} \Omega^{p,q}(\mathcal{M}) \right),$$

and the BRST operator acts as the covariant Dolbeault operator.

To compute OPE of Wilson-\'t Hooft operators $WT_{\mu_1,\nu_1}WT_{\mu_2,\nu_2}$ (for $\mu_1$ and $\mu_2$ - minuscule representations of group $^LG$) in our approach, one should first find the moduli space $\mathcal{M}_{\mu_1+\mu_2}$ corresponding to magnetic charge $\mu_1 + \mu_2$. This space is non-compact and its compactification results in a singular manifold. Resolving the singularity one gets compact manifold $\mathcal{M}$. One should further excise the vicinity of the blown-up regions to get non-compact manifold $\mathcal{M}_{\text{bulk}}$. The next step is to construct vector bundles $\mathcal{V}$ on $\mathcal{M}$ and $\mathcal{V}_{\text{bulk}}$ on $\mathcal{M}_{\text{bulk}}$. The information about

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\(^2\)Weights of a minuscule representation form a single Weyl orbit. Moduli spaces of Hecke modifications corresponding to minuscule magnetic weights are smooth and compact.
these bundles is encoded in electric weights \( \nu_1, \nu_2 \). Then the bulk contribution to the zero mode Hilbert space is given by
\[
\sum_{p,q} (-)^{p+q} H^p(\mathcal{M}_{\text{bulk}}, \Omega^q \otimes \mathcal{V}_{\text{bulk}}),
\]
while the bubbled contribution is captured by
\[
\sum_{p,q} (-)^{p+q} \left( H^p(\mathcal{M}_{\text{total}}, \Omega^q \otimes \mathcal{V}) - H^p(\mathcal{M}_{\text{bulk}}, \Omega^q \otimes \mathcal{V}_{\text{bulk}}) \right).
\] (7)

To compute the OPE in \( \mathcal{N} = 2 \) SYM theory with \( N_f = 0 \) we simply note that “quark” zero modes are absent so in the Hilbert spaces (5) and (6) the sum goes only over \( p \) with \( q = 0 \). Therefore, the bubbled contribution is given by the sum similar to (7) but with \( q = 0 \).

### 3 Geometry of \( \mathcal{M} \)

Let us take gauge group \( G = PSU(3) \). In this section we find the moduli space \( \mathcal{M} \) of BPS configurations in \( \mathcal{N} = 4 \) SYM theory on \( R \times I \times C \) with two 't Hooft operators \( W_{\mu,0} \) inserted at points in \( I \times C \). Here \( \mu = w_1 \) is the highest weight of the fundamental representation of \( L^G = SU(3) \). As reviewed in Section 2, the boundary conditions at the ends of the interval \( I \) are chosen such that there is unique vacuum in the absence of 't Hooft operators.

Recall that the insertion of a t-Hooft operator can be viewed as a Hecke modification of a holomorphic vector bundle on \( C \). Let us first find out moduli space of Hecke modification corresponding to 't Hooft operator \( WT_{2\mu,0} \). Starting from holomorphic vector bundle \( E_- \) with sections
\[
s_a(z) = (e_a(z), f_a(z), h_a(z)) \quad a = 1, 2, 3
\]
the Hecke modification corresponding to \( 2\mu \) produces vector bundle \( E_+ \) with sections:
\[
s'_1 = z^{-2} s_1, \quad s'_2 = s_2, \quad s'_3 = s_3.
\]
Let us expand \( e_1(z) = u^1 + zv^1, \quad f_1(z) = u^2 + zv^2, \quad h_1(z) = u^3 + zv^3 \) where non-degeneracy requires that \( u^1, u^2, u^3 \) cannot vanish simultaneously. In this way we find the general local holomorphic section of \( E_+ \):
\[
s(z) = c(u^1 + zv^1, u^2 + zv^2, u^3 + zv^3) + d(u^1, u^2, u^3) + \ldots
\]
where \( \ldots \) stands for holomorphic functions and \( c, d \) are complex numbers.

There are identifications on parameters of the Hecke modification:
\[
(u^1, u^2, u^3, v^1, v^2, v^3) \sim t(u^1, u^2, u^3, v^1, v^2, v^3) \quad t \in \mathbb{C}^*
\] (8)
\[
v^a \sim v^a + wu^a \quad w \in \mathbb{C}^*
\] (9)

The three invariant combinations under (9) are
\[
y_a = \epsilon_{abc} u^b v^c.
\]
These have weight 2 under \( \mathbb{S} \) and satisfy
\[
\mu^a y_a = 0. 
\]

So we conclude that moduli space of the Hecke modification corresponding to \( WT_{2,0} \) is given by a hypersurface \( (10) \) in the weighted projective space \( W_{111222} \) with coordinates \( u^1, u^2, y_1, y_2, y_3 \) and with \( u^1 = u^2 = u^3 = 0 \) locus excluded (so that this moduli space is non-compact).

To find the bubbled contribution, we first compactify this moduli space by adding the locus \( \vec{u} = 0 \). The resulting space, let us denote it by \( X_4 \), is singular since the ambient \( W_{111222} \) is singular at \( \vec{u} = 0 \) and hypersurface \( (10) \) passes through this singularity. In fact \( X_4 \) near \( \vec{u} = 0 \) looks like \( \mathbb{C}^2/Z_2 \) fibered over \( \mathbb{P}^2_2 \).

We resolve this singularity by blowing up \( W_{111222} \)
\[
\mu^a u^b = \Lambda U^a U^b 
\]
with homogenous coordinates \( U^a \) on the exceptional \( \mathbb{P}^2_2 \). The weights under the two \( \mathbb{C}^* \) actions are
\[
\begin{array}{c|cccc|c|c|c}
 & U^1 & U^2 & U^3 & \Lambda & y_1 & y_2 & y_3 \\
new & 1 & 1 & 1 & -2 & 0 & 0 & 0 \\
old & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]  

Note that blow-up of \( W_{111222} \) gives 5-fold \( Y_5 \) which is \( \mathbb{P}^3 \) fibration over \( \mathbb{P}^2_1 \). Here \( U^1, U^2, U^3 \) are homogenous coordinates on the base \( \mathbb{P}^2_1 \) and \( \Lambda, y_1, y_2, y_3 \) are homogenous coordinates on the fiber \( \mathbb{P}^3 \). An exceptional divisor \( D \) in \( Y_5 \) is given by \( \Lambda = 0 \) and has topology of \( \mathbb{P}^2_2 \times \mathbb{P}^2_2 \).

The proper transform \( \mathcal{M} \) of \( X_4 \) under this blow-up is \( \mathbb{P}^2_1 \) fibration over \( \mathbb{P}^2_1 \). The 4-fold \( \mathcal{M} \) is defined by \( y_a U^a = 0 \) in the 5-fold \( Y_5 \). The exceptional divisor \( D \) intersects \( \mathcal{M} \) over 3-fold \( D \) which is \( \mathbb{P}^1 \) fibration over \( \mathbb{P}^2_2 \).

We can write the most general \( PSU(3) \) invariant Kähler form on \( \mathcal{M} \) as
\[
(-i) J = f_1(s) \mathcal{E}_1 \wedge \bar{\mathcal{E}}_1 + f_2(s) \mathcal{E}_2 \wedge \bar{\mathcal{E}}_2 + f_3(s) \mathcal{E}_3 \wedge \bar{\mathcal{E}}_3 + f_4(s) \mathcal{E}_4 \wedge \bar{\mathcal{E}}_4 
\]
where \( f_i(s) \) are functions of \( PSU(3) \) invariant \( s \) (which is also invariant under \( \mathbb{C}^* \times \mathbb{C}^* \) action):
\[
s = \frac{|\Lambda|^2 Y^2}{X}, \quad X = y_a \bar{y}_a, \quad Y = \bar{U}_a U^a.
\]

In \( (12) \) we used
\[
\mathcal{E}_1 = \partial s \quad \mathcal{E}_2 = \frac{y_a dU^a}{\sqrt{XY}} \quad \mathcal{E}_3 = \frac{\epsilon_{abc} U_a y_b dU^c}{X \sqrt{Y}} \quad \mathcal{E}_4 = \frac{\epsilon_{abc} \bar{y}_a U^b dU^c}{\sqrt{X} \sqrt{Y}}
\]
It is implied that \( \mathcal{E}_i \) are evaluated on a hypersurface \( y_a U^a = 0 \). Since \( \frac{U^a}{\sqrt{Y}} \) and \( \frac{y_a}{\sqrt{X}} \) are multiplied by phases under the corresponding \( \mathbb{C}^* \) actions, we note that
\[
\mathcal{E}_1 \in \Gamma\left( \Omega^{1,0}(0,0) \right), \quad \mathcal{E}_2 \in \Gamma\left( \Omega^{1,0}(1,1) \right), \quad \mathcal{E}_3 \in \Gamma\left( \Omega^{1,0}(-1,2) \right), \quad \mathcal{E}_4 \in \Gamma\left( \Omega^{1,0}(2,-1) \right).
\]
Here $\Omega^{1,0}(b, f)$ stands for $(1, 0)$ form with charges $b$ and $f$ under $C^*$ acting on the base and the fiber of the ambient 5-fold $Y_5$ respectively. The Kähler form must be invariant under the two $C^*$ actions. This is why there are no mixed terms such as $\mathcal{E}_i \wedge \overline{\mathcal{E}}_j$ with $i \neq j$ in the general expression (12).

Recall that the exceptional divisor $D$ in $\mathcal{M}$ is defined by $\Lambda = 0$. At $\Lambda = 0$, let us note that $y_a$ are homogenous coordinates on $\mathbb{P}^2 \vec{y}$. The Kähler form on $D$

$$J_D = C_1 J^{FS}_{\mathbb{P}^2 \overline{y}} + C_2 J^{FS}_{\mathbb{P}^2 \vec{y} | y_a U^a = 0}$$

(15)

can be expressed in terms of $\mathcal{E}_i$ using

$$(-i) J^{FS}_{\mathbb{P}^2 \overline{y}} = \mathcal{E}_2 \wedge \overline{\mathcal{E}}_2 + \mathcal{E}_4 \wedge \overline{\mathcal{E}}_4, \quad (-i) J^{FS}_{\mathbb{P}^2 \vec{y}} = \mathcal{E}_2 \wedge \overline{\mathcal{E}}_2 + \mathcal{E}_3 \wedge \overline{\mathcal{E}}_3.$$

Let us work in the patch $U^1 \neq 0$, $y_3 \neq 0$ and use inhomogenous coordinates

$$z_1 = \frac{U^2}{U^1}, \quad z_2 = \frac{U^3}{U^1}, \quad v = \frac{y_2}{y_3}, \quad \lambda = \frac{\Lambda(U^1)^2}{y_3}.$$

The hypersurface equation $y_a U^a = 0$ is solved in this patch as

$$y_1 = -y_3(z_2 + vz_1).$$

We will set $y_3 = 1$, $U^1 = 1$ to simplify the formulas in the rest of Section 3. In this patch we write $s = \frac{|\lambda|^2 y^2}{x}$ with

$$y = 1 + |z_1|^2 + |z_2|^2, \quad x = 1 + |v|^2 + |z_2 + vz_1|^2.$$

We find differentials explicitly in this patch:

$$\mathcal{E}_1 = s \left( \frac{d\lambda}{\lambda} - \frac{\partial x}{x} + 2 \frac{\partial y}{y} \right), \quad \mathcal{E}_2 = \frac{(vdz_1 + dz_2)}{\sqrt{xy}}$$

$$\mathcal{E}_3 = -\frac{\sqrt{y}}{x} \left( dv + \frac{(\bar{z}_1 - v\bar{z}_2)}{y} (vdz_1 + dz_2) \right), \quad \mathcal{E}_4 = \frac{1}{y\sqrt{x}} \left( \overline{\alpha}_2 dz_1 - \overline{\alpha}_1 dz_2 \right)$$

where we denote:

$$\overline{\alpha}_1 = \bar{v}(1 + |z_1|^2) + z_1 \bar{z}_2, \quad \overline{\alpha}_2 = 1 + |z_2|^2 + \bar{v}z_1 \bar{z}_2.$$

Covariant holomorphic differential $\nabla$ acts on $\omega \in \Gamma\left(\Omega^{1,0}(\mathcal{M}, (p, q))\right)$ as

$$\nabla \omega = \left( \partial - \frac{p}{2} \frac{\partial y}{y} - \frac{q}{2} \frac{\partial x}{x} \right) \omega$$

\footnote{Evaluated at $\Lambda = 0$ i.e. treating $y_a$ as homogenous coordinates on $\mathbb{P}^2 \overline{y}$}
We find:
\[ \nabla \mathcal{E}_1 = 0, \quad \nabla \overline{\mathcal{E}}_1 = \frac{1}{s} \mathcal{E}_1 \wedge \overline{\mathcal{E}}_1 + s \mathcal{E}_2 \wedge \overline{\mathcal{E}}_2 - s \mathcal{E}_3 \wedge \overline{\mathcal{E}}_3 + 2s \mathcal{E}_4 \wedge \overline{\mathcal{E}}_4 \]  
(17)
\[ \nabla \mathcal{E}_4 = 0, \quad \nabla \overline{\mathcal{E}}_4 = \mathcal{E}_3 \wedge \overline{\mathcal{E}}_2, \quad \nabla \mathcal{E}_2 = -\mathcal{E}_3 \wedge \mathcal{E}_4 \]  
(18)
\[ \nabla \mathcal{E}_3 = 0, \quad \nabla \overline{\mathcal{E}}_3 = -\mathcal{E}_4 \wedge \overline{\mathcal{E}}_2, \quad \nabla \overline{\mathcal{E}}_2 = 0. \]  
(19)

From \( dJ = 0 \) we find that \( f_2(s), f_3(s), f_4(s) \) are determined (up to two integration constants) in terms of \( f_1(s) \):
\[ f_2 = f_3 + f_4, \quad f_3' = -sf_1, \quad f_4' = 2sf_1. \]  
(20)

We assume the following asymptotics at \( s \to 0 \):
\[ f_1 \sim C_0 s^{-1}, \quad f_2 \sim C_1 + C_2 + C_0 s, \quad f_3(s) \sim C_2 - C_0 s, \quad f_4 \sim C_1 + 2C_0 s \]  
(21)
with \( C_0 > 0, C_1 > 0, C_2 > 0 \). This ensures that at \( \Lambda = 0 \) we find \( D \) with Kähler form (15).

Meanwhile, at \( s \to \infty \) we must choose the asymptotics
\[ f_1(s) \mapsto \frac{1}{2} A_0' s^{-3}, \quad f_2(s) \mapsto C'_0 - \frac{1}{2} A_0' s^{-1}, \quad f_3(s) \mapsto \frac{1}{2} A_0' s^{-1}, \quad f_4(s) \mapsto C'_0 - A_0' s^{-1} \]  
(22)
where \( A_0' > 0, C'_0 > 0 \).

This ensures that at \( s \to \infty \), i.e. as we go away from the exceptional divisor, we find flat space fibered over \( \mathbb{P}^2 \):
\[ (-iJ) = C'_0 (-iJ_{\mathbb{P}^2}^{FS}) + \frac{A_0'}{2y^2} \sum'_{a} \left( dw_a - 2 \frac{\partial y}{y} w_a \right) \wedge \left( d\bar{w}_a - 2 \frac{\bar{y}}{y} \bar{w}_a \right). \]

In this asymptotic regime \( \Lambda \neq 0 \) so we may introduce \( w_a = \frac{w_a}{\Lambda} \) and \( \sum'_{a} \) means that we use \( w_1 = -(w_2 z^1 + w_3 z^2) \).

For example, we may take the following functions with the right asymptotics:
\[ f_1 = \frac{C_2}{s(1 + s)^2}, \quad f_3 = \frac{C_2}{1 + s}, \quad f_4 = C_1 + \frac{2C_2 s}{1 + s}, \quad f_2 = f_3 + f_4 \]  
(23)
We compute \( L^2 \) Dolbeault cohomology of \( M \) using (23) in Appendix A. This information is used to verify that we identified the geometry of \( M \) correctly (see Section 5). The fact that we have two parameters \( C_1 \) and \( C_2 \) is justified since we show in Appendix A.2 that \( h^{1,1}(M) = 2 \).

4 Geometry of \( M_{\text{bulk}} \)

To describe bulk geometry we work in the patch \( \Lambda \neq 0 \) in \( \mathbb{P}^3 \) fiber and \( U^1 \neq 0 \) in \( \mathbb{P}^2 \) base of \( Y_5 \). The appropriate inhomogenous coordinates are
\[ t_a = \frac{y_a}{\Lambda} \quad a = 1, 2, 3; \quad z^1 = \frac{U^2}{U^1}, \quad z^3 = \frac{U^3}{U^1}. \]  

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In this patch we solve the equation \( y_a U^a = 0 \) as

\[
t_1 = -(z^1 t_2 + z^2 t_3).
\]

We can write \( PSU(3) \) invariant Kähler form on the bulk geometry \( \mathcal{M}_{\text{bulk}} \) as

\[
(-i)J_{\text{bulk}} = g_1(s) \mathcal{E}_1 \wedge \overline{\mathcal{E}}_1 + g_2(s) \mathcal{E}_2 \wedge \overline{\mathcal{E}}_2 + g_3(s) \mathcal{E}_3 \wedge \overline{\mathcal{E}}_3 + g_4(s) \mathcal{E}_4 \wedge \overline{\mathcal{E}}_4
\]

where \( g_i(s) \) are functions of \( PSU(3) \) invariant \( s \) which in this patch can be written as (we set \( U^1 = 1 \))

\[
s = \frac{y^2}{\bar{x}}, \quad \bar{x} = t_4 \bar{t}^3, \quad y = 1 + |z_1|^2 + |z_2|^2.
\]

In this patch (with \( U^1 = 1, \Lambda = 1 \)) the differentials have the form

\[
\mathcal{E}_1 = s \left( \frac{2 \partial y}{y} - \frac{\partial \bar{x}}{\bar{x}} \right), \quad \mathcal{E}_2 = \frac{t_2 dz^1 + t_3 dz^2}{\sqrt{xy}},
\]

\[
\mathcal{E}_3 = \frac{y^{1/2}}{\bar{x}} \left( t_2 dt_3 - t_3 dt_2 + \frac{(t_2 \bar{z}_2 - t_3 \bar{z}_1)(t_2 dz^1 + t_3 dz^2)}{y} \right), \quad \mathcal{E}_4 = \frac{\alpha_2 dz^1 - \bar{\alpha}_1 dz^2}{y \sqrt{\bar{x}}}
\]

where

\[
\alpha_1 = \bar{t}^2 + z^1(\bar{t}^2 \bar{z}_1 + \bar{t}_3 \bar{z}_2), \quad \bar{\alpha}_2 = \bar{t}^3 + z^2(\bar{t}^2 \bar{z}_1 + \bar{t}_3 \bar{z}_2).
\]

At \( s \to \infty \) (away from the blown-up region) the metric on the bulk geometry should coincide with the metric on total geometry \( \mathcal{M} \). This means that at \( s \to \infty \) we must choose the asymptotics

\[
g_1(s) \to \frac{1}{2} A'_0 s^{-3}, \quad g_2(s) \to C'_0 - \frac{1}{2} A'_0 s^{-1}, \quad g_3(s) \to \frac{1}{2} A'_0 s^{-1}, \quad g_4(s) \to C'_0 - A'_0 s^{-1}
\]

where \( A'_0 > 0, C'_0 > 0 \).

Meanwhile, at \( s \to 0 \) we choose

\[
g_1(s) \to \frac{1}{2} \frac{\hat{A}}{s^{3/2}}, \quad g_2(s) \to \hat{C} + \hat{A} s^{1/2}, \quad g_3(s) \to \hat{C} - \hat{A} s^{1/2}, \quad g_4(s) \to 2 \hat{A} s^{1/2},
\]

with \( \hat{C} > 0, \quad \hat{A} > 0 \).

We note that at \( s \to 0 \) (corresponding to \( \Lambda \to 0 \)) we may use coordinates \( y_a \) and \( u^a = \sqrt{X} U^a \) (i.e. coordinates before the blow-up). Moreover, \( y_a \) in this limit are homogenous coordinates on \( \mathbb{P}^2 \). So that

\[
s = \frac{\bar{y}^2}{X}, \quad X = y_a \bar{y}^a, \quad \bar{y} = u^a \bar{u}_a.
\]

We find at \( s \to 0 \)

\[
(-i)J_{\text{bulk}} = \frac{\hat{A}}{X^{1/2}} \sum_a \left( du^a - \frac{\partial X}{2X} u^a \right) \wedge \left( d\bar{u}^a - \frac{\partial \bar{X}}{2\bar{X}} \bar{u}^a \right) + \hat{C} \left( -i J^{FS}_{\bar{y} \bar{y}} \right),
\]

\[
\text{11}
\]
where $\sum_{a}'$ means that $u^a y_a = 0$ is implied. This correctly describes $\vec{u} = 0$ region in the moduli space before the blow-up, which was given by hypersurface $u^a y_a = 0$ in $\mathbb{C}^3/\mathbb{Z}_2$ fibered over $\mathbb{P}_y^2$.

We can, for example, make the following simple choice of functions $g_i(s)$ with the asymptotics (25) and (26):

$$
\begin{align*}
g_1 &= \frac{\hat{A}}{2} \frac{1}{s^{3/2}(1+s)^{3/2}}, \\
g_2 &= \hat{A} \left( 1 + \frac{s^{1/2}}{\sqrt{1+s}} \right), \\
g_3 &= \hat{A} \left( 1 - \frac{s^{1/2}}{\sqrt{1+s}} \right), \\
g_4 &= \frac{2\hat{A}}{s^{1/2}} \frac{s^{1/2}}{\sqrt{1+s}} 
\end{align*}
$$

(27)

so that $\tilde{C} = \hat{A}$ in (26) and $C'_0 = 2\hat{A}, A'_0 = \hat{A}$ in (25). We compute $L^2$ Dolbeault cohomology of $\mathcal{M}_{\text{bulk}}$ using (27) in Appendix B. This information is used to verify that we identified the geometry of $\mathcal{M}_{\text{bulk}}$ correctly (see Section 5). The fact that we have only one parameter $\hat{A}$ is justified since we show in Appendix B.2 that $h^{1,1}(\mathcal{M}_{\text{bulk}}) = 1$.

5 Consistency check

Let us make consistency check of our results with known general facts about moduli spaces of BPS configurations in the presence of ’t Hooft operators [9]. In Appendix A, we found the following non-zero cohomology groups for the total moduli space:

$$
\begin{align*}
H^{(0,0)}(\mathcal{M}) &\simeq H^{(4,4)}(\mathcal{M}) = \mathbb{V}_1, \\
H^{(1,1)}(\mathcal{M}) &\simeq H^{(3,3)}(\mathcal{M}) = \mathbb{V}_1 \oplus \mathbb{V}_1, \\
H^{(2,2)}(\mathcal{M}) &\simeq \mathbb{V}_1 \oplus \mathbb{V}_1 \oplus \mathbb{V}_1.
\end{align*}
$$

Here $\mathbb{V}_1$ is one-dimensional (singlet) representation of $PSU(3)$. Let us decompose all 9 harmonic forms which serve as basis vectors into three groups

- $1, J_{\text{tot}}, J_{\text{tot}}^2, J_{\text{tot}}^3, J_{\text{tot}}^4$
- $\omega^{(2)}_{(1,1)}, J_{\text{tot}} \wedge \omega^{(2)}_{(1,1)}, J_{\text{tot}}^2 \wedge \omega^{(2)}_{(1,1)}$
- $\omega^{(p.s.d)}_{(2,2)}$

where harmonic $(1,1)$ form $\omega^{(2)}_{(1,1)}$ is orthogonal to the Kähler form

$$J_{\text{tot}} \wedge *\omega^{(2)}_{(1,1)} = 0$$

and harmonic $(2,2)$ form $\omega^{(p.s.d)}_{(2,2)}$ is primitive and self-dual

$$J_{\text{tot}} \wedge \omega^{(p.s.d)}_{(2,2)} = 0 \quad \omega^{(p.s.d)}_{(2,2)} = *\omega^{(p.s.d)}_{(2,2)}.$$

This decomposition is consistent with the general fact that cohomology should transform in representations of the principle $SU(2)_{\text{principle}}$ subgroup of the dual group $L^G = SU(3)$ [9]. The Kähler form $J_{\text{tot}}$ plays the role of the raising operator of $SU(2)_{\text{principle}}$.  

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In Appendix B, for the bulk geometry we found:
\[ H^{(0,0)}(\mathcal{M}_{\text{bulk}}) \simeq H^{(4,4)}(\mathcal{M}_{\text{bulk}}) = V_1, \quad H^{(1,1)}(\mathcal{M}_{\text{bulk}}) \simeq H^{(3,3)}(\mathcal{M}_{\text{bulk}}) = V_1, \]
\[ H^{(2,2)}(\mathcal{M}_{\text{bulk}}) = V_1 \oplus V_1. \]

We decompose all 6 harmonic forms which serve as basis vectors into two groups

- 1, \( J_{\text{bulk}}, J_{\text{bulk}}^2, J_{\text{bulk}}^3, J_{\text{bulk}}^4 \)
- \( \omega^{(p.s.d)}_{(2,2)} \)

where harmonic (2,2) form \( \omega^{(p.s.d)}_{(2,2)} \) is primitive and self-dual

\[ J_{\text{bulk}} \wedge \omega^{(p.s.d)}_{(2,2)} = 0 \quad \omega^{(p.s.d)}_{(2,2)} = *\omega^{(p.s.d)}_{(2,2)}. \]

This decomposition is again consistent with [9] and \( J_{\text{bulk}} \) plays the role of the raising generator of \( SU(2)_{\text{principle}} \). Moreover, this corresponds precisely to the decomposition of representation \( 2\mu \) of \( LG = SU(3) \), which appears in the ‘t Hooft operator \( WT_{\mu,0} \) in the right side of the OPE [9], into representations with spin \( j = 2 \) and \( j = 0 \) under the principle \( SU(2)_{\text{principle}} \subset SU(3) \).

Comparing total and bulk cohomologies, we see that harmonic forms, which serve as a basis for bubbled contribution, are

\( \omega^{(2)}_{(1,1)}, J_{\text{tot}} \wedge \omega^{(2)}_{(1,1)}, J_{\text{tot}}^2 \wedge \omega^{(2)}_{(1,1)}. \)

This is a representation with spin \( j = 1 \) under \( SU(2)_{\text{principle}} \) which is consistent with the decomposition of representation \( \overline{\mu} \) of \( LG = SU(3) \), which appears in the ‘t Hooft operator \( WT_{\tau,0} \) in the right side of the OPE [9], under \( SU(2)_{\text{principle}} \subset SU(3) \). We conclude that both the cohomology groups in the bulk and the bubbled contribution are in agreement with S-duality prediction [9].

6 Vector bundles over \( \mathcal{M} \) and \( \mathcal{M}_{\text{bulk}} \)

In this section we construct the bundles \( \mathcal{V} \) over \( \mathcal{M} \) and \( \mathcal{V}_{\text{bulk}} \) over \( \mathcal{M}_{\text{bulk}} \) which appear in [2] and are required to compute the OPE [1].

Let us first identify the vector bundle \( \mathcal{V}_{a,b} \) over the base \( \mathbb{P}^2_\mathcal{U} \) which corresponds to the electric weight \( \nu = aw_1 + bw_2 \) (with \( a + 2b = 0 \) mod 3) in the Wilson-‘t Hooft operator \( WT_{\mu,\nu} \). Recall that \( \mu = w_1 \) breaks Lie algebra \( su(3) \) to \( su(2) \oplus u(1) \) and \( \nu \) tells us to look for a bundle in representation \( R_b \) with the highest weight \( b \) (number of boxes in the Young diagram) of \( SU(2) \) and with charge \( 2a + b \) under \( U(1) \).

Let us clarify this. We use the Chevalley basis of \( su(3) \):

\[ [h^1, E^{\pm \alpha_1}] = \pm 2E^{\pm \alpha_1}, \quad [h^2, E^{\pm \alpha_2}] = \pm 2E^{\pm \alpha_2}, \]
\[ [h^1, E^{\pm \alpha_2}] = \mp E^{\pm \alpha_2}, \quad [h^2, E^{\pm \alpha_1}] = \mp E^{\pm \alpha_1} \]

where states in a given irreducible representation \( \nu = aw_1 + bw_2 \) are labelled by eigenvalues of \( h^1 \) and \( h^2 \):

\[ h^1|a, b\rangle = a|a, b\rangle \quad h^2|a, b\rangle = b|a, b\rangle. \]

Acting on 3, we can represent raising operators corresponding to simple roots and Cartan generators of \( su(3) \) as

\[
E^{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad h^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E^{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

The operators \( h^2, E^{\pm \alpha_2} \) generate \( su(2) \) part of the Lie algebra of the unbroken group. Meanwhile, the generator for \( u(1) \) is

\[ J = (2h^1 + h^2), \quad J|a, b\rangle = (2a + b)|a, b\rangle \]

Note that the value of \( J \) on the weights of \( PSU(3) \) is always in \( 3\mathbb{Z} \) since \( a + 2b = 3n, n \in \mathbb{Z} \) implies \( 2a + b = 3m, m \in \mathbb{Z} \).

To write a connection on \( V_{a,b} \), we first find a connection on the principle \( SU(2) \times U(1) \) bundle over \( \mathbb{P}^2 \) from the metric on \( SU(3) \) group manifold viewed as \( SU(2) \times U(1) \) bundle over \( \mathbb{P}^2 \):

\[
\frac{ds^2_{SU(3)}}{4} = -\frac{1}{2} \text{Tr} \left[ (g^{-1}dg)^2 \right] = -\frac{1}{2} \left( \text{Tr} \left[ (G^{-1}dG)^2 \right] + \text{Tr} \left[ (dh^{-1})^2 \right] + 2\text{Tr} \left[ (G^{-1}dG dh^{-1}) \right] \right)
\]

with

\[ g = Gh \quad h = k \exp \left[ \frac{tJ}{2} \right] \quad k \in SU(2). \]

Here \( t \in [0,2\pi] \) is a coordinate on \( U(1) \) part of the fiber while for \( k \in SU(2) \) we define the forms \( (dk)k^{-1} = \frac{i}{2} \bar{\rho} \cdot \bar{\sigma} \), with \( \bar{\sigma} \) - Pauli matrices. Let us parametrize \( k \) in terms of the Euler angles:

\[ k = \exp \left[ i \frac{\psi}{2} \sigma^3 \right] \exp \left[ i \frac{\theta}{2} \sigma^1 \right] \exp \left[ i \frac{\phi}{2} \sigma^3 \right], \]

where \( \theta \in [0,\pi], \phi \in [0,2\pi], \psi \in [0,2\pi] \). Then, the forms are given by

\[ \rho^3 = d\psi + \cos \theta \, d\phi, \quad \rho^1 + i\rho^2 = e^{-i\psi}(d\theta + i \sin \theta \, d\phi). \]

Similar to [20], we parametrize the coset representative \( G \) as

\[
G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{b} & -\bar{c} \\ 0 & c & b \end{pmatrix} \begin{pmatrix} \cos \Upsilon & \sin \Upsilon & 0 \\ -\sin \Upsilon & \cos \Upsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

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where $|b|^2 + |c|^2 = 1$ and $\Upsilon \in [0, \frac{\pi}{2}]$. We use the following parametrization
\[ c = \sin \Theta e^{i \Phi_1}, \quad \bar{b} = \cos \Theta e^{i \Phi_2} \]
where $\Theta \in [0, \pi/2]$, $\Phi_1 \in [0, 2\pi]$, $\Phi_2 \in [0, 2\pi]$ and define forms $\vec{\xi}$ as
\[ D^{-1} dD = i \vec{\xi} \cdot \vec{\sigma}, \quad D = \begin{pmatrix} \bar{b} & -\bar{c} \\ c & b \end{pmatrix}. \]

Then we find
\[ \xi^3 = \sin^2 \Theta d \Phi_1 + \cos^2 \Theta d \Phi_2, \quad \xi^+ := \xi^1 + i \xi^2 = -ie^{i(\Phi_1 + \Phi_2)} \left( d\Theta + i \sin \Theta \cos (d\Phi_1 - d\Phi_2) \right). \]

The metric on $SU(3)$ in these coordinates is
\[ ds_{SU(3)}^2 = \frac{3}{4} (dt + A^0)^2 + \frac{1}{4} (\rho^3 + A^3)^2 + \frac{1}{4} |\rho^+ + A^+|^2 + ds_{\text{base}}^2 \]
Here the connection is
\[ A^0 = -\xi_3 \sin^2 \Upsilon, \quad A^3 = \xi^3 (1 + \cos^2 \Upsilon), \quad A^+ = 2 \xi^+ \cos \Upsilon \]
and the metric on the base
\[ ds_{\text{base}}^2 = (d\Upsilon)^2 + \sin^2 \Upsilon |\xi|^2 + \sin^2 \Upsilon \cos^2 \Upsilon \xi^2. \]

The coordinates $z_1, z_2$ used in Section 3 are expressed as
\[ z_1 = c \tan \Upsilon, \quad z_2 = \bar{b} \tan \Upsilon. \]

Note that
\[ ds_{\text{base}}^2 = \frac{(dz_1 \otimes \bar{d}z_1 + dz_2 \otimes \bar{d}z_2)}{y} - \frac{\partial y \otimes \bar{\partial} y}{y^2} \]
with $y = 1 + |z_1|^2 + |z_2|^2$. The Kähler form on the base $J_{\text{base}} = \frac{1}{2} J_{FS}$ where $[J_{FS}] = H$ with $H$ - the hyperplane class on $\mathbb{P}^2$.

In terms of $z_1, z_2$ we write the connection on the principle $U(1) \times SU(2)$ bundle over $\mathbb{P}^2$ as
\[ A^+ = \frac{2i(\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1)}{\sqrt{y}(y - 1)}, \quad A^3 = -\frac{(y + 1)}{y(y - 1)} \text{Im}(\partial y), \quad A^0 = \frac{\text{Im}(\partial y)}{y} \tag{29} \]

Therefore, the connection on the vector bundle $V_{a,b}$ over $\mathbb{P}^2$ corresponding to the electric weight $\nu = aw_1 + bw_2$ is
\[ A = A_{(1,0)} + A_{(0,1)}, \quad A_{(0,1)} = \frac{i(2a + b)}{2} \frac{\partial y}{y} + A^i T_i^R, \quad A_{(1,0)} = A_{(0,1)}^\dagger \tag{30} \]
with $A^iT_i^{R_b}$ - the $SU(2)$ connection in the representation $R_b$. This bundle is a tensor product

$$V_{a,b} = O(-(2a + b)H) \otimes \tilde{V}_b$$

where $H$ is a hyperplane class in $\mathbb{P}^2_U$ and the Chern classes of the vector bundle $\tilde{V}_b$ are

$$rk(\tilde{V}_b) = b + 1, \quad c_1(\tilde{V}_b) = 0, \quad c_2(\tilde{V}_b) = \left(\frac{-\kappa(R_b)}{2} \int_{\mathbb{P}^2} F^i \wedge F_i \right)H^2.$$  \hspace{1cm} (31)

Here we defined $\kappa(R)$ as

$$Tr R^{T_i T_j} = \kappa(R) \delta_{ij},$$

so that $\kappa(R_1) = \frac{1}{2}$.

We compute

$$F^3 = dA^3 + \frac{i}{2}A^- \wedge A^+, \quad F^+ = dA^+ + iA^- \wedge A^3$$

and

$$\int_{\mathbb{P}^2} F^i \wedge F_i = -24 \int sin^2\gamma \cos \gamma d\gamma \int \xi^1 \wedge \xi^2 \wedge \xi^3 = -4(2\pi)^2.$$  \hspace{1cm} (32)

Hence,

$$c_2(\tilde{V}_b) = 2\kappa(R_b)H^2.$$  \hspace{1cm} (33)

Since the bundle $\tilde{V}_b$ is the symmetric tensor product $S^b\tilde{V}_1$, it is crucial to understand the holomorphic structure of $\tilde{V}_1$. Let us write the $(0,1)$ part of the connection on $\tilde{V}_1$ as

$$A_{\bar{b}(0,1)} = i(\overline{\partial G}) G^{-1} \quad G = \begin{pmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} \beta & \alpha^{-1}(1 + \beta) \end{pmatrix}$$

where

$$\alpha = \frac{y^{1/4}}{(y-1)^{1/2}}, \quad \beta = -\frac{\bar{z}_1}{(y-1)z_2}.$$  \hspace{1cm} (34)

Let us use $G$, the transformation matrix from holomorphic to unitary gauge, to compute the norm of a section

$$|\psi_{\text{unit}}| = G \psi_{\text{hol}}, \quad \psi_{\text{hol}} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$  \hspace{1cm} (35)

We find

$$||\psi||^2 = \int \frac{dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2}{y^3} \left(\alpha^2 |\psi_1 + \psi_2|^2 + \alpha^{-2} |\beta(\psi_1 + \psi_2) + \psi_2|^2 \right).$$

Therefore, $H^0(\mathbb{P}^2, \tilde{V}_1) = \mathbb{C}$ since there is only one section with finite norm

$$\psi_{\text{hol}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$  \hspace{1cm} (36)
We further find $H^2(\mathbb{P}^2, \tilde{V}_1) = 0$ since general harmonic $(0,2)$ forms valued in $\tilde{V}_1$ are written as
\[
\psi_{\text{unit}} = d\bar{z}_1 \wedge d\bar{z}_2 \left( G^{-1} \right) \left( \frac{\psi_1}{\psi_2} \right)
\]
and the norm
\[
||\psi||^2 = \int dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \left( |\alpha^{-1}(1+\beta)\psi_1 - \alpha\psi_2|^2 + | - \alpha^{-1}\beta\psi_1 + \alpha\psi_2|^2 \right)
\]
diverges for any holomorphic $\psi_1, \psi_2$. Then from the holomorphic Euler characteristic $\chi(\mathbb{P}^2, \tilde{V}_1) = 1$ we find $H^1(\mathbb{P}^2, \tilde{V}_1) = 0$. We will use this information in [19] to identify $\tilde{V}_1$ as a certain well-known holomorphic vector bundle on $\mathbb{P}^2$.

If one would like to compute the OPE (1), the bundles $\mathcal{V}$ on $\mathcal{M}$ and $\mathcal{V}_{\text{bulk}}$ on $\mathcal{M}_{\text{bulk}}$, which appear in (2), are the pull-back of the vector bundle $\mathcal{V}_{a,b}$ over the base $\mathbb{P}^2_{\vec{U}}$ to $\mathcal{M}$ and $\mathcal{M}_{\text{bulk}}$ correspondingly. Indeed, recall that the total moduli space $\mathcal{M}$ is $\mathbb{P}^2$ fibration over $\mathbb{P}^2_{\vec{U}}$ where the base (the fiber) is the space of Hecke modifications corresponding to the first (the second) WH operator in the OPE. The bundles $\mathcal{V}$ and $\mathcal{V}_{\text{bulk}}$ are the pull-back from the base since only the first Wilson-’t Hooft operator in the left side of (1) carries non-zero electric weight.

Since $\mathcal{M}$ is compact we can take a connection on $\mathcal{V}$ to be the pull-back of the connection on the base $\mathbb{P}^2_{\vec{U}}$. We will clarify how to define a connection $\mathcal{V}_{\text{bulk}}$ on the non-compact $\mathcal{M}_{\text{bulk}}$ in [19] where we will present the computation of the OPE (1).

7 Conclusion

In this note we determined the basic ingredients required to compute the OPE (1) of Wilson-’t Hooft loop operators in $\mathcal{N} = 4$ SYM theory with gauge group $G = PSU(3)$. This work is an extension of our approach [7] which uses the holomorphic-topological twist [8] of the $\mathcal{N} = 4$ SYM theory and the connection between BPS configurations in $\mathcal{N} = 4$ SYM theory in the presence of ’t Hooft operators and solutions of 3d Bogomolny equations with magnetic sources [5],[9].

In Section 3 we found the compact moduli space $\mathcal{M}$ of BPS configurations in the theory on $R \times I \times \mathcal{C}$ with two ’t Hooft operators $W_{\mu,0}$ inserted at points in $I \times \mathcal{C}$. The $PSU(3)$ invariant Kähler form on $\mathcal{M}$ is written in (12) with functions $f_i(s)$ given in (23). We further determined the non-compact space $\mathcal{M}_{\text{bulk}}$ by removing from $\mathcal{M}$ the vicinity of the blown-up region corresponding to the bubbled contribution. The $PSU(3)$ invariant Kähler form on $\mathcal{M}_{\text{bulk}}$ is written in (24) with functions $g_i(s)$ given in (27).

We computed $L^2$ Dolbeault cohomology of $\mathcal{M}$ and $\mathcal{M}_{\text{bulk}}$ in Appendix A and Appendix B respectively. This allowed us to verify our results about geometry of these moduli spaces by making consistency check. Namely, we verified the OPE of ’t Hooft operators [3], predicted

\[\psi_{\text{unit}} = d\bar{z}_1 \wedge d\bar{z}_2 = d\bar{z}_2 \wedge d\bar{z}_1\] in solving $\mathbf{D}^\dagger \psi_{\text{unit}} = 0$.\]
by S-duality, by making explicit the action of principle SU(2) subgroup of the dual group \( L G = SU(3) \) on the cohomology. This is in agreement with general facts about moduli spaces of BPS configurations in the presence of ’t Hooft operators \([9]\).

We further determined the vector bundles \( \mathcal{V} \) and \( \mathcal{V}_{\text{bulk}} \) in Section 6. These bundles take into account electric degrees of freedom present in dyonic operators in the OPE \([11]\). We will compute the right side of (1) for \( N = 4 \) SYM and \( N = 2 \) SYM with \( N_f = 0 \) in the future \([19]\) and hope to compare with the forthcoming results from the alternative method \([18]\) based on the connection with 2d CFT.

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**A Appendix: \( L^2 \) Dolbeault Cohomology of \( \mathcal{M} \)**

Here we compute cohomology groups \( H^p(M, \Omega^p) \) for \( p = 0, 1, 2, 3, 4 \). It is clear that \( h^p(M, \Omega^q) = 0 \) for \( p \neq q \) as follows from the non-trivial transformation of the basic differentials \([14]\) under the two \( C^* \) actions \([11]\). We could have computed cohomology groups of \( \mathcal{M} \) simply using that \( \mathcal{M} \) is \( \mathbb{P}^2 \) fibration over \( \mathbb{P}^2 \). Instead, we chose to compute \( L^2 \) Dolbeault cohomology using Kähler form on \( \mathcal{M} \) to verify that we have correctly identified the geometry of \( \mathcal{M} \). Moreover, this allows us to directly compare with the \( L^2 \) Dolbeault cohomology of \( \mathcal{M}_{\text{bulk}} \), which we compute in Appendix B, and identify the bubbled contribution (see Section 5).

**A.1 Harmonic (0,0) and (4,4) forms**

To compute the volume of \( \mathcal{M} \), we introduce polar coordinates:

\[
\lambda = |\lambda| e^{i\phi}, \quad z_1 = r_1 e^{i\phi_1}, \quad z_2 = r_2 e^{i\phi_2}, \quad v = r_v e^{i\phi_v}
\]

and write

\[
x = a + b \cos \Phi, \quad a = 1 + t_2 + t_v(1 + t_1), \quad b = 2r_1 r_2 r_v, \quad \Phi = \phi_2 - \phi_1 - \phi_v, \quad y = 1 + t_1 + t_2
\]

\[
t_v = r_v^2, \quad t_1 = r_1^2, \quad t_2 = r_2^2.
\]

Using explicit expressions \([16]\) for differentials we compute

\[
\mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \mathcal{E}_4 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \mathcal{E}_4 = \frac{1}{2} s ds \wedge d\phi \wedge \frac{d\nu \wedge d\bar{\nu}}{x^2} \wedge \frac{dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2}{y^2}\quad (33)
\]
Then the volume of $\mathcal{M}$ is given by

$$\text{vol} \, \mathcal{M} = \frac{(2\pi)^4}{2} \int_0^\infty s \, ds \, f_1 \, f_2 \, f_3 \, f_4$$

where we used the following integrals over $\Phi$ and $t_v$:

$$\int_0^{2\pi} \frac{d\Phi}{(a + b \cos \Phi)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}, \quad a > b$$

$$\int_0^\infty \frac{t_v \, dt_v}{(\beta t_v^2 + 2\gamma t_v + \delta)^{3/2}} = \frac{1}{2} \frac{1}{y(1 + t)}$$

where

$$\beta = (1 + t_1)^2, \quad \gamma = 1 + t_1 + t_2 - t_1 t_2, \quad \delta = (1 + t_2)^2.$$ 

Using asymptotics (21) and (22) of $f_i(s)$, we find that volume form on $\mathcal{M}$ is convergent i.e. both $(0,0)$ form $(4,4)$ forms are square integrable.

### A.2 Harmonic (1,1) and (3,3) forms

General $PSU(3)$ invariant $(1,1)$ form is written as:

$$\omega = a_1 \mathcal{E}_1 \wedge \bar{\mathcal{E}}_1 + a_2 \mathcal{E}_2 \wedge \bar{\mathcal{E}}_2 + a_3 \mathcal{E}_3 \wedge \bar{\mathcal{E}}_3 + a_4 \mathcal{E}_4 \wedge \bar{\mathcal{E}}_4.$$ 

Taking the Hodge star operation we find:

$$*\omega = c_1 \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \mathcal{E}_4 \wedge \bar{\mathcal{E}}_2 \wedge \bar{\mathcal{E}}_3 \wedge \bar{\mathcal{E}}_4 + c_2 \mathcal{E}_1 \wedge \mathcal{E}_3 \wedge \mathcal{E}_4 \wedge \bar{\mathcal{E}}_1 \wedge \bar{\mathcal{E}}_3 \wedge \bar{\mathcal{E}}_4 + c_3 \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_4 \wedge \bar{\mathcal{E}}_1 \wedge \bar{\mathcal{E}}_2 \wedge \bar{\mathcal{E}}_4 +$$

$$c_4 \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \bar{\mathcal{E}}_1 \wedge \bar{\mathcal{E}}_2 \wedge \bar{\mathcal{E}}_3$$

where

$$c_i = -a_i B_i \quad i = 1, \ldots, 4$$

$$B_1 = \frac{f_2 f_3 f_4}{f_1}, \quad B_2 = \frac{f_1 f_3 f_4}{f_2}, \quad B_3 = \frac{f_1 f_2 f_4}{f_3}, \quad B_4 = \frac{f_1 f_2 f_3}{f_4}.$$ 

To be square-integrable, $\omega$ must satisfy

$$\int \omega \wedge *\omega = \frac{(2\pi)^4}{2} \int_0^\infty ds \, s \left( a_1^2 B_1 + a_2^2 B_2 + a_3^2 B_3 + a_4^2 B_4 \right) < \infty$$

We find that $\bar{\partial} \omega = 0$ implies:

$$a_3' = -sa_1, \quad a_4 = \# - 2a_3, \quad a_2 = a_3 + a_4$$

where $\#$ is an integration constant. Next, $\bar{\partial} * \omega = 0$ gives:

$$-sc_2 + sc_3 - 2sc_4 + c_1' = 0.$$
For the choice of asymptotics (21) at $s \to 0$ we find that all three solutions behave like constants at $s \to 0$, which ensures that each of them gives finite contribution to the norm of the solution from integrating around $s = 0$.

At $s \to \infty$ we use (22) to find general solution

$$a_3 \sim \delta_1 s^{-1} + \delta_2 s + \delta_3 s^{-2}$$

$$a_4 = -\delta_1 \frac{C_1^2 + 3C_1 C_2 + 2C_2^2}{C_2^2} - \delta_3 \frac{(C_1 + 2C_2)^2}{C_2^2} - 2a_3$$

One has to set $\delta_2 = 0$ to ensure convergence of the integral in the definition of the norm. We conclude that the vector space of harmonic square-integrable (1,1) forms is two dimensional. As a basis in this space, we can take the Kähler form $\omega_{(1,1)}^{(1)} = J_{tot}$ and the form $\omega_{(1,1)}^{(2)}$ orthogonal to $J_{tot}$ i.e. such that

$$J_{tot} \wedge \ast \omega_{(1,1)}^{(2)} = 0.$$  \hspace{1cm} (35)

Note that $\omega_{(1,1)}^{(1)}$ corresponds to $\delta_1 = -\delta_4 = C_2$ while $\omega_{(1,1)}^{(2)}$ to $\delta_1 = C_1 + 2C_2$, $\delta_3 = -C_1$.

By Serre duality the space of harmonic square-integrable (3,3) forms is also two dimensional. As a basis, we may take

$$\omega_{(3,3)}^{(1)} = J_{tot}^3, \quad \omega_{(3,3)}^{(2)} = J_{tot}^2 \wedge \omega_{(1,1)}^{(2)}$$

### A.3 Harmonic (2,2) forms

General $PSU(3)$ invariant (2,2) form is written as:

$$\omega = h_1 \mathcal{E}_1 \wedge \mathcal{E}_3 \wedge \mathcal{E}_1 \wedge \mathcal{E}_3 + h_2 \mathcal{E}_1 \wedge \mathcal{E}_4 \wedge \mathcal{E}_1 \wedge \mathcal{E}_4 + h_3 \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2 + h_4 \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \mathcal{E}_4 + h_5 \mathcal{E}_3 \wedge \mathcal{E}_4 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2 + h_6 \mathcal{E}_3 \wedge \mathcal{E}_4 \wedge \mathcal{E}_1 \wedge \mathcal{E}_2 + h_7 \mathcal{E}_2 \wedge \mathcal{E}_3 \wedge \mathcal{E}_1 \wedge \mathcal{E}_3 + h_8 \mathcal{E}_2 \wedge \mathcal{E}_4 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 + h_9 \mathcal{E}_2 \wedge \mathcal{E}_4 \wedge \mathcal{E}_1 \wedge \mathcal{E}_3$$

From $\partial \omega = 0$ we find:

$$sh_4' + h_4 + s(h_3 - h_2 - h_1) = 0, \quad h_6' - 2sh_1 + sh_2 + h_5 = 0,$$

$$h_7' - sh_1 + sh_3 - h_5 = 0, \quad h_8' - sh_2 - h_5 - 2sh_3 = 0.$$  \hspace{1cm} (36)

### A.3.1 Self-dual forms

Let us first look for self-dual $(2,2)$ forms solving (36)

$$h_4 = h_5, \quad h_1 = h_8 A_{24}, \quad h_2 = h_7 A_{23}, \quad h_3 = h_6 A_{34}$$  \hspace{1cm} (37)

where

$$A_{24} = \frac{f_1 f_3}{f_2 f_4}, \quad A_{23} = \frac{f_1 f_4}{f_2 f_3}, \quad A_{34} = \frac{f_1 f_2}{f_3 f_4}.$$

To be square-integrable, $\omega$ must satisfy

$$\int \omega \wedge \ast \omega = \frac{(2\pi)^4}{2} \int ds s \left( h_4^2 + h_6^2 A_{34} + h_7^2 A_{23} + h_8^2 A_{24} \right) < \infty.$$  \hspace{1cm} (38)
There is one obvious square-integrable solution
\[ \omega_{(2,2)} = J_{\text{tot}} \wedge J_{\text{tot}}. \]

Let us look for other solutions among primitive self-dual forms i.e. we impose \( J_{\text{tot}} \wedge \omega_{(2,2)} = 0 \) which, using self-duality (37), amounts to
\[ h_7 = -h_6 - f_3 H, \quad h_8 = -h_6 + f_4 H. \]

Then, equations (36) reduce to three ODEs for three functions \( h_4, h_6, H \):

\[ f_4 H' + 2sf_1 H - 2s(A_{34} - A_{24})h_6 - 2sf_4 A_{24} H = 0, \]
\[ h'_6 + sh_6(2A_{24} - A_{23}) + h_4 - sH(f_3 A_{23} + 2f_4 A_{24}) = 0, \]
\[ sh'_4 + h_4 + sh_6(A_{34} + A_{23} + A_{24}) + sH(f_3 A_{23} - f_4 A_{24}) = 0. \]

Using (22) we find general solution of (39) at \( s \to \infty \):
\[ h_4 = -\delta_1 s + \frac{(2\delta_2 + 3\delta_3)}{s^2}, \quad h_6 = \delta_1 s^2 + \frac{(\delta_2 + \delta_3)}{s}, \quad H = \frac{\delta_1}{2} s^2 - \frac{\delta_2}{s} + 2\delta_3 \]
where \( \delta_i \) are constants. We see that among primitive forms, there are two well-behaved at \( s \to \infty \) solutions obtained by choosing \( \delta_1 = 0 \).

Using (21) we find general solution of (39) at \( s \to 0 \):
\[ h_4 = \kappa_1 s^{-1} - \kappa_2 \frac{C_2(C_1 - C_2)}{C_1 + C_2} - 2\kappa_3 \frac{(C_1^2 + C_1 C_2 + C_2^2)}{C_1(C_1 + C_2)}, \]
\[ h_6 = -\kappa_1 \log(s) + \kappa_3 + a(\kappa_2, \kappa_3)s, \]
\[ H = -2 \frac{(C_1 + 2C_2)}{C_1(C_1 + C_2)} \kappa_1 s \log(s) + \kappa_2 + b(\kappa_2, \kappa_3)s \]
where \( a, b \) are linear combinations of constants \( \kappa_2 \) and \( \kappa_3 \). Recall that \( C_1, C_2 \) are Kähler moduli which appear in \( J_{\text{tot}} \) (23). Setting \( \kappa_1 = 0 \) leaves 2 solutions well-behaved at \( s \to 0 \).

We checked using Mathematica that each of the two good solutions of (39) at \( s \to 0 \) (parametrized by \( \kappa_2, \kappa_3 \)) interpolates at \( s \to \infty \) into a bad solution with \( \delta_1 \neq 0 \). There is a linear combination of the two good solutions at \( s \to 0 \) which interpolates into a good solution at \( s \to \infty \). Therefore the space of primitive self-dual harmonic (2,2) forms is one dimensional. In total, we conclude that the space of self-dual harmonic (2,2) forms on \( \mathcal{M} \) is two dimensional and we may choose as basis vectors \( \omega_{(2,2)}^{(1)} = J_{\text{tot}}^2 \) and \( \omega_{(2,2)}^{p.s.d} \) such that
\[ \omega_{(2,2)}^{p.s.d} = *\omega_{(2,2)}^{p.s.d} \quad \text{and} \quad \omega_{(2,2)}^{p.s.d} \wedge J_{\text{tot}} = 0. \]
A.3.2 Anti-self-dual forms

Let us now look for anti-self-dual (2,2) forms solving (36)

\[ h_4 = -h_5, \quad h_1 = -h_8 A_24, \quad h_2 = -h_7 A_23 \quad h_3 = -h_6 A_34. \]

There is an obvious solution:

\[ \omega_{a.s.d}^{(2)} = J_{tot} \wedge \omega^{(2)}_{(1,1)} \]

(41)

where the form \( \omega^{(2)}_{(1,1)} \) appeared in Section 3.2. This (1,1) form is orthogonal to \( J_{tot} \), see (35), which ensures the anti-self-duality of \( \omega_{a.s.d}^{(2)} \).

Let us prove that the space of harmonic square-integrable anti-self-dual (2,2) forms is one dimensional. Using (22) we find general solution at \( s \to \infty \)

\begin{align*}
    h_4 &= \delta_1 + \delta_3 s^{-3}, \quad h_6 = (\delta_1 + \delta_2) s + (\delta_4 - \delta_3) s^{-2} \\
    h_7 &= \delta_2 s + \delta_3 s^{-2}, \quad h_8 = -(3\delta_1 + 3\delta_2) s + (16 + \frac{1}{s^2}) \delta_4
\end{align*}

where \( \delta_i \) are constants. Setting \( \delta_1 = \delta_2 = 0 \) leaves 2 solutions well-behaved at \( s \to \infty \).

Meanwhile, with \( s \to 0 \) asymptotics (21) we find three well-behaved solutions

\begin{align*}
    h_4 &= \gamma h_6^{(0)} - \beta h_7^{(0)} - \alpha h_8^{(0)} + O(s) \\
    h_6 &= h_6^{(0)} + O(s), \quad h_7 = h_7^{(0)} + O(s), \quad h_8 = h_8^{(0)} + O(s).
\end{align*}

where \( h_i^{(0)} \) for \( i = 6, 7, 8 \) are independent constants. The fourth solution is not well-behaved:

\[ h_4 = \frac{\delta}{s} + O(ln(s)), \quad h_6, h_7, h_8 \sim O(ln(s)). \]

We use Mathematica to show that out of 3 solutions well-behaved at \( s \to 0 \) we can construct only one linear combination which also behaves well at \( s \to \infty \). More concretely, each of the three solutions, parametrized by \( h_k^{(0)} \) with \( k = 6, 7, 8 \), interpolates to a solution with non-zero \( \delta_1 \) and \( \delta_2 \) at large \( s \). We can construct only one linear combination of the three good solutions at \( s \to 0 \) which interpolates to a solution with \( \delta_1 = \delta_2 = 0 \) at \( s \to \infty \). We conclude that the space of square-integrable anti-self-dual harmonic (2,2) forms on \( \mathcal{M} \) is one dimensional with a basis vector (41).

### B Appendix: \( L^2 \) Dolbeault Cohomology of \( \mathcal{M}_{bulk} \)

Here we compute cohomology groups \( H^p(\mathcal{M}_{bulk}, \Omega^q) \) for \( p = 0, 1, 2, 3, 4 \). It is clear that \( h^p(\mathcal{M}_{bulk}, \Omega^q) = 0 \) for \( p \neq q \) as follows from the non-trivial transformation of the basic differentials (14) under the two \( C^* \) actions (11).
B.1 Harmonic (0,0) and (4,4) forms

The volume of $\mathcal{M}_{\text{bulk}}$ is computed in the same way as the volume of $\mathcal{M}$ with substitution $f_i(s) \mapsto g_i(s)$. We find
\[
\text{vol}_{\mathcal{M}_{\text{bulk}}} = \frac{(2\pi)^4}{2} \int_0^\infty s \, ds \, g_1 \, g_2 \, g_3 \, g_4.
\]

Using asymptotics (25) and (26) of $g_i(s)$ we find that volume form on $\mathcal{M}_{\text{bulk}}$ is convergent i.e. both (0,0) form (4,4) forms are in $L^2$.

B.2 Harmonic (1,1) and (3,3) forms

General $PSU(3)$ invariant (1,1) form is written as:
\[
\omega_{(1,1)} = a_1 \mathcal{E}_1 \wedge \mathcal{E}_1 + a_2 \mathcal{E}_2 \wedge \mathcal{E}_2 + a_3 \mathcal{E}_3 \wedge \mathcal{E}_3 + a_4 \mathcal{E}_4 \wedge \mathcal{E}_4.
\]

From $\overline{\partial} \omega_{(1,1)} = 0$ and $\overline{\partial}^\dagger \omega_{(1,1)} = 0$ we find
\[
a'_3 = -sa_1, \quad a_4 = # - 2a_3, \quad a_2 = a_3 + a_4
\]
\[
-\frac{\hat{B}_1}{s} a''_3 - \left( \frac{\hat{B}_1}{s} \right)' a'_3 + s(\hat{B}_2 + \hat{B}_3 + 4\hat{B}_4)a_3 = s#(\hat{B}_2 + 2\hat{B}_4)
\]
where $\hat{B}_i = \prod_{i \neq i} \frac{g_i(s)}{g_i(s)}$ and # is a constant. To be square-integrable, $\omega_{(1,1)}$ must satisfy
\[
\int \omega_{(1,1)} \wedge * \omega_{(1,1)} = \frac{(2\pi)^4}{2} \int_0^\infty ds \, s \left( a_1^2 \hat{B}_1 + a_2^2 \hat{B}_2 + a_3^2 \hat{B}_3 + a_4^2 \hat{B}_4 \right) < \infty \quad (42)
\]

There is an obvious square-integrable solution - the Kähler form on $\mathcal{M}_{\text{bulk}}$
\[
\omega_{(1,1)} = J_{\text{bulk}}
\]
but let us look for other solutions.

For the choice of asymptotics (26) at $s \mapsto 0$ we find that general solution has the form:
\[
a_3 = \kappa_1 s^{\frac{1}{2}} + \kappa_2 s^{-\frac{1}{2}} + \kappa_3
\]
with integration constants $\kappa_1, \kappa_2, \kappa_3$. There are two well-behaved solutions at $s \mapsto 0$ obtained by setting $\kappa_2 = 0$. The Kähler form $J_{\text{bulk}}$ corresponds to further taking $\kappa_3 = -\kappa_1 = \hat{A}$.

Meanwhile, using (25), general solution at $s \mapsto \infty$ has the form:
\[
a_3 = \gamma_1 s^{-1} + \gamma_2 s + \gamma_3 s^{-2}.
\]
There are two well-behaved solutions at $s \mapsto \infty$ obtained by setting $\gamma_2 = 0$.

Using Mathematica we checked that if $\kappa_3 \neq -\kappa_1$, then a good solution at $s \mapsto 0$ interpolates into a bad solution with $\gamma_2 \neq 0$ at $s \mapsto \infty$.

We conclude that the vector space of harmonic square-integrable (1,1) forms on $\mathcal{M}_{\text{bulk}}$ is one dimensional with a basis vector $J_{\text{bulk}}$. By Serre duality we also get that the space of harmonic square-integrable (3,3) forms is one dimensional with a basis vector $J_{\text{bulk}}^3$. 
B.3 Harmonic (2,2) forms

General $PSU(3)$ invariant (2,2) form is written as:

$$\omega_{(2,2)} = h_1 E_1 \wedge E_3 \wedge E_1 \wedge E_3 + h_2 E_1 \wedge E_4 \wedge E_1 \wedge E_4 + h_3 E_1 \wedge E_2 \wedge E_1 \wedge E_2 + h_4 E_1 \wedge E_2 \wedge E_3 \wedge E_4 + h_5 E_3 \wedge E_4 \wedge E_1 \wedge E_2 + h_6 E_3 \wedge E_4 \wedge E_3 \wedge E_4 + h_7 E_2 \wedge E_3 \wedge E_2 \wedge E_3 + h_8 E_2 \wedge E_4 \wedge E_2 \wedge E_4$$

From $\partial \omega = 0$ we find:

$$sh_4' + h_4 + s(h_3 - h_2 - h_1) = 0, \quad h_4' - 2sh_1 + sh_2 + h_5 = 0,$$

$$h_4' - sh_1 + sh_3 - h_5 = 0, \quad h_8' - sh_2 - h_5 - 2sh_3 = 0.$$

(43)

B.3.1 Self-dual forms

Let us first look for self-dual (2,2) forms solving (43)

$$h_4 = h_5, \quad h_1 = h_8 \hat{A}_{24}, \quad h_2 = h_7 \hat{A}_{23}, \quad h_3 = h_6 \hat{A}_{34}$$

(44)

where

$$\hat{A}_{24} = \frac{g_1 g_3}{g_2 g_4}, \quad \hat{A}_{23} = \frac{g_1 g_4}{g_2 g_3}, \quad \hat{A}_{34} = \frac{g_1 g_2}{g_3 g_4}.$$

To be square-integrable, $\omega$ must satisfy

$$\int \omega_{(2,2)} \wedge \ast \omega_{(2,2)} = \frac{(2\pi)^4}{2} \int ds \left( h_4^2 + h_6^2 \hat{A}_{34} + h_7^2 \hat{A}_{23} + h_8^2 \hat{A}_{24} \right) < \infty$$

(45)

There is one obvious square-integrable solution

$$\omega_{(2,2)} = J_{\text{balk}} \wedge J_{\text{balk}}.$$

Let us look for other solutions among primitive self-dual forms i.e. we impose $J_{\text{balk}} \wedge \omega = 0$ which, using self-duality (44), amounts to

$$h_7 = -h_6 - g_3 H, \quad h_8 = -h_6 + g_4 H.$$

Then, equations (43) reduce to three ODEs for three functions $h_4, h_6, H$:

$$g_4 H' + 2sg_1 H - 2s(\hat{A}_{34} - \hat{A}_{24}) h_6 - 2sg_4 \hat{A}_{24} H = 0,$$

$$h_6' + sh_6(2\hat{A}_{24} - \hat{A}_{23}) + h_4 - sH(g_3 \hat{A}_{23} + 2g_4 \hat{A}_{24}) = 0,$$

$$sh_4' + h_4 + sh_4\left( \hat{A}_{34} + \hat{A}_{23} + \hat{A}_{24} \right) + sH(g_3 \hat{A}_{23} - g_4 \hat{A}_{24}) = 0.$$

(46)

Using (25) we find general solution of (46) at $s \to \infty$:

$$h_4 = -\delta_1 s + \frac{(2\delta_2 + 3\delta_3)}{s^2}, \quad h_6 = \delta_1 s^2 + \frac{(\delta_2 + \delta_3)}{s} \quad H = \frac{\delta_1}{2} s^2 - \frac{\delta_2}{s} + 2\delta_3$$

24
where \( \delta_i \) are constants. We see that among primitive forms, there are two well-behaved at \( s \to \infty \) solutions obtained by choosing \( \delta_1 = 0 \).

Using (26) we find general solution of (46) at \( s \to 0 \):

\[
\begin{align*}
    h_4 &= -\frac{1}{2} \tilde{C}_1 s^{-2} - \frac{1}{2} \tilde{C}_2 s^{-\frac{3}{2}} - 3 \tilde{C}_3,
    &\quad h_6 = -\tilde{C}_1 s^{-1} + \frac{1}{2} \tilde{C}_2 s^{\frac{3}{2}} + \tilde{C}_3 (s^{\frac{3}{2}} + 2s),
    &\quad H = \tilde{C}_1 s^{-1} + \tilde{C}_2 s^{\frac{3}{2}} + \tilde{C}_3.
\end{align*}
\]

Setting \( \tilde{C}_1 = 0 \) leaves 2 solutions well-behaved at \( s \to 0 \).

We checked using Mathematica that each of the two good solutions of (46) at \( s \to 0 \) (parametrized by \( \tilde{C}_2, \tilde{C}_3 \)) interpolates at \( s \to \infty \) into a bad solution with \( \delta_1 \neq 0 \). There is a linear combination of the two good solutions at \( s \to 0 \) which interpolates into a good solution at \( s \to \infty \). Therefore the space of primitive self-dual harmonic (2,2) forms is one dimensional. In total, we conclude that the space of self-dual harmonic (2,2) forms on \( \mathcal{M}_{\text{bulk}} \) is two dimensional and we may choose as basis vectors \( \omega^{(1)}_{(2,2)} = J_{\text{bulk}}^2 \) and \( \omega^{p,s.d}_{(2,2)} \) such that

\[ \omega^{p,s.d}_{(2,2)} = *\omega^{p,s.d}_{(2,2)} \quad \text{and} \quad \omega^{p,s.d}_{(2,2)} \wedge J_{\text{bulk}} = 0. \]

### B.3.2 Anti-self-dual forms

Let us now look for anti-self-dual (2,2) forms solving (43)

\[
\begin{align*}
    h_4 &= -h_5, \\
    h_1 &= -h_8 \hat{A}_{24}, \\
    h_2 &= -h_7 \hat{A}_{23}, \\
    h_3 &= -h_6 \hat{A}_{34}
\end{align*}
\]

Using (25) we find general solution at \( s \to \infty \)

\[
\begin{align*}
    h_4 &= \delta_1 + \delta_3 s^{-3}, \\
    h_6 &= (\delta_1 + \delta_2)s + (\delta_4 - \delta_3)s^{-2} \\
    h_7 &= \delta_2 s + \delta_3 s^{-2}, \\
    h_8 &= -(3\delta_1 + 3\delta_2)s + (16 + \frac{1}{s^2})\delta_4
\end{align*}
\]

where \( \delta_i \) are constants. Setting \( \delta_1 = \delta_2 = 0 \) leaves 2 solutions well-behaved at \( s \to \infty \).

Using (26) we find general solutions at \( s \to 0 \):

\[
\begin{align*}
    h_4 &= \tilde{C}_1 + \tilde{C}_3 s^{-3/2}, \\
    h_6 &= 2(\tilde{C}_1 + \tilde{C}_2)s + \tilde{C}_4 s^{-1/2} \\
    h_7 &= \tilde{C}_2 (1 + s) + 3\tilde{C}_4 s^{-1/2}, \\
    h_8 &= -2(\tilde{C}_1 + \tilde{C}_2)s + (2\tilde{C}_3 + \tilde{C}_4)s^{-1/2}
\end{align*}
\]

where \( \tilde{C}_i \) are constants. Setting \( \tilde{C}_3 = \tilde{C}_4 = 0 \) leaves 2 solutions well-behaved at \( s \to 0 \).

We checked using Mathematica that there are no square integrable harmonic anti-selfdual (2,2) forms on \( \mathcal{M}_{\text{bulk}} \). Namely, each of the two good solutions at \( s \to \infty \) interpolates to a solution at \( s \to 0 \) with both \( \tilde{C}_3 \neq 0 \) and \( \tilde{C}_4 \neq 0 \). It is not possible to eliminate these divergent pieces at \( s \to 0 \) by any linear combination of the two good solutions at \( s \to \infty \).
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