Existence of ground state solutions for weighted biharmonic problem involving non linear exponential growth

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Abstract
In this article, we study the following problem

$$\Delta (w_\beta(x) \Delta u) = f(x, u) \text{ in } B, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B,$$

where $B$ is the unit ball of $\mathbb{R}^4$ and $w_\beta(x)$ a singular weight of logarithm type. The reaction source $f(x, u)$ is a radial function with respect to $x$ and it is critical in view of exponential inequality of Adams’ type. The existence result is proved by using the constrained minimization in Nehari set coupled with the quantitative deformation lemma and degree theory results.

Keywords  Weighted Sobolev space · Biharmonic operator · Critical exponential growth

Mathematics Subject Classification 35J20 · 49J45 · 35K57 · 35J60
1 Introduction and main results

In this paper, we consider the fourth order weighted elliptic equation:

\[
\begin{aligned}
(P) \quad & \Delta (w_\beta(x) \Delta u) = f(x, u) \text{ in } B \\
& u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B,
\end{aligned}
\]  

(1.1)

where \( B = B(0, 1) \) is the unit open ball in \( \mathbb{R}^d \), \( f(x, t) \) is a radial function with respect to \( x \), the weight \( w_\beta(x) \) is given by

\[ w_\beta(x) = \left( \log \frac{1}{|x|} \right)^\beta, \quad \beta \in (0, 1). \]  

(1.2)

In order to motivate our study, we first give a brief overview of the notion of exponential critical growth for problems of order superior or equal to 2 in dimension \( N \geq 2 \). We limit ourselves to Sobolev spaces \( W^{1,N}_0(\Omega) \) and \( W^{2,2}_0(\Omega) \). We give some examples of applications as we go along.

In dimension \( N \geq 2 \), the critical exponential growth is given by the well known Trudinger–Moser inequality \([26, 29]\)

\[
\sup_{f \in \Omega, |\nabla u|^N \leq 1} \int_\Omega e^{\alpha |u|^\frac{N}{N-1}} dx < +\infty \quad \text{if and only if} \quad \alpha \leq \alpha_N,
\]

where \( \alpha_N = \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \) with \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \). This last result opened the way to study second order problems under nonlinearities with exponential growths and in non-weighted Sobolev spaces. For instance, we cite the following problem in dimension \( N = 2 \)

\[-\Delta u = f(x, u) \text{ in } \Omega \subset \mathbb{R}^2\]

which have been studied considerably \([2, 18, 22, 24]\).

Later, the Trudinger–Moser inequality was improved to weighted inequalities \([7, 8]\). When the weight is of logarithmic type, Calanchi and Ruf \([9]\) extend the Trudinger–Moser inequality and proved the following results in the space \( W^{1,N}_{0,\text{rad}}(B, \rho) = cl \{ u \in C^\infty_{0,\text{rad}}(B) | \int_B |\nabla u|^N \rho(x) dx < \infty \} \):

**Theorem 1.1** \([8]\)

(i) Let \( \beta \in [0, 1) \) and let \( \rho \) given by \( \rho(x) = \left( \log \frac{1}{|x|} \right)^\beta \), then

\[
\int_B e^{\gamma |u|^\gamma} dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,\text{rad}}(B, \rho), \quad \text{if and only if} \quad \gamma \leq \gamma_{N,\beta}
\]

\[
= \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}
\]
and

$$\sup_{u \in W^{1,N}_{0,\text{rad}}(B, \rho)} \int_B e^{\alpha |u|^N} \  dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N,\beta} = N\left[\omega_{N-1}^{\frac{1}{N-1}}(1 - \beta)^{\frac{1}{N}}\right]^{\frac{1}{\beta}}$$

where $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$ and $N'$ is the Hölder conjugate of $N$.

(ii) Let $\rho$ given by $\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$, then

$$\int_B \exp\{e^{\alpha |u|^N}\} \  dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,\text{rad}}(B, \rho)$$

and

$$\sup_{u \in W^{1,N}_{0,\text{rad}}(B, \rho)} \int_B \exp\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^N}\} \  dx < +\infty \quad \Leftrightarrow \quad \beta \leq N.$$ 

These results paved the way for the study of second order weighted elliptic problems in dimension $N \geq 2$. We point out that recently, in the case, $V = 0$ or $V \neq 0$, Baraket et al. [5], Deng, Hu and Tang [13] and Calanchi et al. [10] have proved the existence of a nontrivial solution for the following boundary value problem

$$\begin{cases} 
-\text{div}(\sigma(x)\nabla u(x)) + V(x)|u|^{N-2}u = f(x, u) & \text{in } B \\
 u = 0 & \text{on } \partial B,
\end{cases}$$

where $B$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, the radial positive weight $\sigma(x)$ is of logarithmic type, the function $f(x, u)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{1}{N-1}}}\}$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved that there is a non-trivial solution to this problem using minimax techniques combined with Trudinger–Moser inequality.

Now we will give an overview of fourth order problems in relation to Adams’ inequalities. For bounded domains $\Omega \subset \mathbb{R}^d$, in [1, 27] the authors extended the Trudinger Moser inequality to the higher order space $W^{2,2}_0(\Omega)$ and obtained the so called Adams’ inequalities,

$$\sup_{u \in S} \int_{\Omega} (e^{\alpha u^2} - 1) dx < +\infty \quad \Leftrightarrow \quad \alpha \leq 32\pi^2$$

where

$$S = \{u \in W^{2,2}_0(\Omega) \mid (\int_{\Omega} |\Delta u|^2 dx)^{\frac{1}{2}} \leq 1\}.$$ 

These results allowed to investigate fourth-order problems with subcritical or critical nonlinearity involving continuous potential (see [11, 28]).
**Remark 1.1** The biharmonic equation in dimension $N > 4$

\[ \Delta^2 u = f(x, u) \text{ in } \Omega \subset \mathbb{R}^N, \]

where the nonlinearity $f$ has subcritical and critical polynomial growth of power less than $\frac{N+4}{N-4}$, have been extensively studied [6, 16, 19].

Before stating our results, let’s start by defining our functional space. Let $\Omega \subset \mathbb{R}^4$, be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function. We introduce the weighted Sobolev space

\[ W^{2,2}_0(\Omega, w) = cl \left\{ u \in C_0^\infty(\Omega) \mid \int_B w(x)|\Delta u|^2 dx < \infty \right\}. \]

We will focus on radial functions on the unit ball $B$ and consider the weighted subspace

\[ E = W^{2,2}_{0,rad}(B, w) = cl \left\{ u \in C_0^\infty(B) \mid \int_B w(x)|\Delta u|^2 dx < \infty \right\}, \]

endowed with the norm

\[ \|u\| = \left( \int_B w(x)|\Delta u|^2 dx \right)^{\frac{1}{2}}. \]

We note that this norm is issued from the product scalar

\[ \langle u, v \rangle = \int_B w(x) \Delta u \Delta v \, dx. \]

The choice of the weight induced in (1.2) and the space $E$ are also motivated by the following exponential inequality.

**Theorem 1.2** [30] Let $\beta \in (0, 1)$ and let $w_\beta$ given by (1.2), then

\[ \sup_{u \in W^{2,2}_{0,rad}(B, w_\beta)} \int_B e^{\alpha|u|^{\frac{2}{1-\beta}}} \Delta u^2 \, dx < +\infty \iff \alpha \leq \alpha_\beta = 4\left[8\pi^2(1-\beta)\right]^{\frac{1}{1-\beta}} \ (1.3) \]

This last result allowed the authors to study the problem (P) (see [15]). In this work, the nonlinearity is positive and the authors used the Ambrosetti-Rabinowitz Theorem [4]. They overcame the noncompactness of the energy function by imposing an asymptotic condition. Let $\gamma := \frac{2}{1-\beta}$, in view of inequality (1.3), we say that $f$
has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$,

$$\lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^p}} = 0, \quad \forall \alpha \text{ such that } \alpha_0 < \alpha \leq \alpha_\beta$$

$$\lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^p}} = +\infty, \quad \forall \alpha < \alpha_0 \leq \alpha_\beta.$$ (1.4)

Now, let’s state our assumptions. In this paper, we deal with problem $(P)$ under critical growth nonlinearities. Furthermore, we suppose that $f(x, t)$ satisfies the following hypothesis:

$(V_1)$ $f : B \times \mathbb{R} \to \mathbb{R}$ is continuous and radial in $x$.

$(V_2)$ There exist $p > 2$ such that we have

$$0 < p \ F(x, t) \leq tf(x, t), \forall (x, t) \in B \times \mathbb{R} \setminus \{0\}$$

where

$$F(x, t) = \int_0^t f(x, s)ds.$$ (2.1)

$(V_3)$ For each $x \in B, t \mapsto \frac{f(x, t)}{|t|}$ is increasing for $t \in \mathbb{R} \setminus \{0\}$.

$(V_4)$ $\lim_{t \to 0} \frac{|f(x, t)|}{|t|} = 0$.

$(V_5)$ There exists $C_p > 1$ such that

$$\text{sgn}(t)f(x, t) \geq C_p|t|^{p-1}, \quad \text{for all } (x, t) \in B \times \mathbb{R},$$

where $\text{sgn}(t) = 1$ if $t > 0$, $\text{sgn}(t) = 0$ if $t = 0$, and $\text{sgn}(t) = -1$ if $t < 0$.

We give an example of such nonlinearity. The nonlinearity $f(x, t) = C_p|t|^{p-2}t + |t|^{p-2}t \exp(\alpha_0|t|^{\gamma})$ satisfies the assumptions $(V_1), (V_2), (V_3), (V_4)$ and $(V_5)$.

We will consider the following definition of solutions.

**Definition 1.1** We say that a function $u \in E$ is a weak solution of the problem $(P)$ if

$$\int_B w_\beta(x) \Delta u \varphi dx = \int_B f(x, u)\varphi dx, \quad \forall \varphi \in E.$$ (1.6)

Let $J : E \to \mathbb{R}$ be the functional given by

$$J(u) = \frac{1}{2} \int_B w_\beta(x) |\Delta u|^2dx - \int_B F(x, u)dx,$$ (1.7)

where

$$F(x, t) = \int_0^t f(x, s)ds.$$
The energy functional $J$ is well defined and of class $C^1$ since there exist $a$, $C > 0$ positive constants and there exists $t_1 > 1$ such that

$$|f(x, t)| \leq C e^{a \cdot t^r}, \quad \forall |t| > t_1,$$

whenever the nonlinearity $f(x, t)$ is critical at $+\infty$.

It is standard to check that critical points of $J$ are precisely weak solutions of $(P)$. Moreover, we have

$$\langle J'(u), \varphi \rangle = J'(u) \varphi = \int_B \left( w_\beta(x) \Delta u \Delta \varphi \right) dx - \int_B f(x, u) \varphi \, dx, \quad \varphi \in E.$$

Our strategy consists in finding solutions which minimize the corresponding energy functional $J$ among the set of all solutions to problem $(P)$. To this end, we define the Nehari set as:

$$\mathcal{N} := \{ u \in E : \langle J'(u), u \rangle = 0, u \neq 0 \}.$$

In other words, we try to find a minimize of the energy functional $J$ over the following minimization problem,

$$m = \inf_{u \in \mathcal{N}} J(u).$$

To our best knowledge, there are no results for solutions to the weighted biharmonic equation with critical exponential nonlinearity on the weighted Sobolev space $E$.

Now, we give our main result as follows:

**Theorem 1.3** Let $f(x, t)$ be a function that has a critical growth at $+\infty$. Suppose that $(V_1)$, $(V_2)$, $(V_3)$, $(V_4)$ and $(V_5)$ are satisfied. The problem $(P)$ has a radial solution with minimal energy provided

$$C_p > \max \left\{ 1, \left( \frac{2p m_p}{(p - 2)} \left( \frac{2(a_0 + \delta)}{\alpha \beta} \right)^{1 - \beta} \right)^{\frac{p - 2}{2}} \right\}$$

where $\delta > 0$, $m_p = \inf_{u \in \mathcal{N}_p} J_p(u) > 0$,

$$J_p(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_B |u|^p \, dx$$

and

$$\mathcal{N}_p := \{ u \in E, u \neq 0 \text{ and } \langle J_p'(u), u \rangle = 0 \}.$$
applications in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see [12, 17, 25]. However many applications are generated by the weighted elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example [4, 23]).

This present work is organized as follows: in Sect. 2, we present some necessary preliminary knowledge about functional space and some preliminaries results. In Sect. 3, we give some technical key lemmas. In Sect. 4, we study an auxiliary problem which will be of great use to prove our main result. Section 5 is devoted to the proof of the Theorem 1.3.

Finally, we note that a constant $C$ may change from line to another and sometimes we index the constants in order to show how they change. Also, we shall use the notation $|u|_p$ for the norm in the Lesbegue space $L^p(B)$.

## 2 Weighted Sobolev space setting and preliminaries results

Let $\Omega \subset \mathbb{R}^4$, be a bounded domain in $\mathbb{R}^4$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$ $1 \leq p < \infty$, $W^{2,2}(\Omega, w)$, $W^{2,2}_0(\Omega, w)$ and some of their properties that will be used later.

We recall the standart Lebesgue space $L^p(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable}; \int_{\Omega} |u|^p \, dx < \infty\}$, endowed with the norm $|u|_p = \left(\int_{\Omega} |u|^p \, dx\right)^{\frac{1}{p}}$.

Following Drabek et al. [14] and Kufner [21], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable}; \, uw^{\frac{1}{p}} \in L^p(\Omega)\}$$

for any real number $1 \leq p < \infty$.

This is a Banach space (uniformly convex and reflexive for $p > 1$) equipped with the norm

$$\|u\|_{p, w} = \left(\int_{\Omega} w(x)|u|^p \, dx\right)^{\frac{1}{p}}.$$

For $m = 2$, let $w$ be a given family of weight functions $w_\tau, \, |\tau| \leq 2, \, w = \{w_\tau(x) \, x \in \Omega, \, |\tau| \leq 2\}$. In [14], the corresponding weighted Sobolev space was defined as

$$W^{2,2}(\Omega, w) = \{u \in L^2(\Omega) \text{ such that } D^\tau u \in L^2(\Omega) \text{ for all } |\tau| \leq 1, \, D^\tau u \in L^2(\Omega, w) \text{ for all } |\tau| = 2\}.$$
endowed with the following norm:

$$
\|u\|_{W^{2,2}(\Omega, w)} = \left( \sum_{|\tau| \leq 1} \int_{\Omega} |D^\tau u|^2 \, dx + \sum_{|\tau|=2} \int_{\Omega} |D^\tau u|^2 w(x) \, dx \right)^{\frac{1}{2}}.
$$

If we suppose also that $w(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{2,2}(\Omega, w)$ and we can introduce the space $W^{2,2}_0(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ in $W^{2,2}(\Omega, w)$. Moreover, the following embedding is compact

$$
W^{2,2}(\Omega, w) \hookrightarrow \hookrightarrow W^{1,2}(\Omega).
$$

Also, $(L^2(\Omega, w), \| \cdot \|_{L^2(\omega, w)})$ and $(W^{2,2}(\Omega, w), \| \cdot \|_{W^{2,2}(\omega, w)})$ are separable, reflexive Banach spaces provided that $w(x)^{-1} \in L^1_{loc}(\Omega)$. Then the space $E$ is a Banach and reflexive. The space $E$ is endowed with the norm

$$
\|u\| = \left( \int_B w_\beta(x) |\Delta u|^2 \, dx \right)^{\frac{1}{2}}
$$

which is equivalent to the following norm (see Lemma 1 below)

$$
\|u\|_{W^{2,2}_{0,rad}(B,w_\beta)} = \left( \int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B w_\beta(x) |\Delta u|^2 \, dx \right)^{\frac{1}{2}}.
$$

We also have the continuous embedding

$$
E \hookrightarrow L^q(B) \text{ for all } q \geq 1.
$$

Moreover, $E$ is compactly embedded in $L^q(B)$ for all $q \geq 1$. In fact, we have

**Lemma 1** Let $w_\beta$ be given by (1.2) and $u$ be a radially symmetric function, then

(i) function in $C_0^\infty(B)$. Then, we have

(i) [30]

$$
|u(x)| \leq \frac{1}{2\sqrt{2\pi}} \frac{|| \log(\frac{r}{|x|})||^{1-\beta} - 1 |^\frac{1}{2}}{\sqrt{1-\beta}} \int_B w_\beta(x) |\Delta u|^2 \, dx \\
\leq \frac{1}{2\sqrt{2\pi}} \frac{|| \log(\frac{r}{|x|})||^{1-\beta} - 1 |^\frac{1}{2}}{\sqrt{1-\beta}} \|u\|^2.
$$

(ii) The norms $\| \cdot \|$ and $\|u\|_{W^{2,2}_{0,rad}(B,w_\beta)} = \left( \int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B |\Delta u|^2 w_\beta(x) \, dx \right)^{\frac{1}{2}}$ are equivalent.
(iii) The following embedding is continuous

\[ E \hookrightarrow L^q(B) \quad \text{for all} \quad q \geq 1. \]

(iv) \( E \) is compactly embedded in \( L^q(B) \) for all \( q \geq 2 \).

**Proof** (i) see [30]
(ii) By Poincaré inequality, for all \( u \in W^{1,2}_{0,rad}(B) \)

\[ \int_B |u|^2 \leq C \int_B |\nabla u|^2. \]

Using the Green formula, we get

\[ \int_B |\nabla u|^2 = \int_B \nabla u \nabla u = -\int_B u \Delta u + \int_{\partial B} u \frac{\partial u}{\partial n} \leq |\int_B u \Delta u|. \]

By Young inequality, we get for all \( \varepsilon > 0 \)

\[ |\int_B u \Delta u| \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 + \frac{\varepsilon}{2} \int_B |u|^2 \leq \frac{1}{2\varepsilon} \int_B w_\beta(x)|\Delta u|^2 + \frac{\varepsilon}{2} \int_B |u|^2. \]

Hence

\[ (1 - \frac{\varepsilon}{2} C^2) \int_B |\nabla u|^2 \leq \frac{1}{2\varepsilon} \int_B w_\beta(x)|\Delta u|^2, \]

then,

\[ \int_B u^2 dx + \int_B |\nabla u|^2 dx + \int_B w_\beta(x)|\Delta u|^2 dx \leq C \int_B w_\beta(x)|\Delta u|^2 dx \leq C \|u\|^2. \]

Then (ii) follows, (iii) and (iv). Since \( w_\beta(x) \geq 1 \), then the following embedding are continuous and compact

\[ E \hookrightarrow W^{2,2}_{0,rad}(B, w_\beta) \hookrightarrow W^{2,2}_{0,rad}(B) \hookrightarrow L^q(B) \quad \forall q \geq 2 \]

and from (i), we have that \( E \hookrightarrow L^q(B) \) is continuous for all \( q \geq 1 \). This concludes the lemma.

\qed

3 Some technical lemmas

We begin by some key lemmas.
In the following we assume, unless otherwise stated, that the function \( f \) satisfies the conditions \((V_1)\) to \((V_4)\). Let \( u \in E \) with \( u \neq 0 \) a.e. in the ball \( B \), and we define the function \( \Upsilon_u : [0, \infty) \to \mathbb{R} \) as
\[
\Upsilon_u(t) = \mathcal{J}(tu).
\] (3.1)

It’s clear that \( \Upsilon_u'(t) = 0 \) is equivalent to \( tu \in \mathcal{N} \).

**Lemma 2** (i) For each \( u \in E \) with \( u \neq 0 \), there exists an unique \( t_u > 0 \), such that \( t_u u \in \mathcal{N} \). In particular, the set \( \mathcal{N} \) is nonempty and \( \mathcal{J}(u) > 0 \), for every \( u \in \mathcal{N} \).

(ii) For all \( t \geq 0 \) with \( t \neq t_u \), we have
\[
\mathcal{J}(tu) < \mathcal{J}(t_u u).
\]

**Proof.** (i)

Since \( f \) is critical, and from \((V_1)\) and \((V_4)\), for all \( \varepsilon > 0 \), there exist positive constants \( C_1 = C_1(\varepsilon) \) and \( C'_1 = C'_1(\varepsilon) \) such that
\[
f(x, t)t \leq \varepsilon |t|^2 + C_1 |t|^s \exp(\alpha |t|^\gamma) \quad \text{for all } \alpha > \alpha_0, s > 2.
\] (3.2)

and
\[
F(x, t) \leq \frac{1}{2} \varepsilon |t|^2 + C'_1 |t|^s \exp(\alpha |t| \gamma) \quad \text{for all } \alpha > \alpha_0, s > 2.
\] (3.3)

Now, given \( u \in E \) fixed with \( u \neq 0 \). From (3.3), for all \( \varepsilon > 0 \), we have
\[
\Upsilon_u(t) = \mathcal{J}(tu) \geq \frac{1}{2} t^2 \|u\|^2 - \int_B F(x, tu) tudx
\]
\[
\geq \frac{1}{2} t^2 \|u\|^2 - \frac{1}{2} \varepsilon t^2 \int_B |u|^2 dx - C'_1 \int_B |tu|^s \exp(\alpha \|u\| \gamma) dx
\]

Using the Hölder inequality, with \( a, a' > 1 \) such that \( \frac{1}{a} + \frac{1}{a'} = 1 \), and Sobolev embedding Lemma 1, we get
\[
\Upsilon_u(t) \geq \frac{1}{2} t^2 \|u\|^2 - C_2 \frac{1}{2} \varepsilon t^2 \|u\|^2 - C_1 \left( \int_B |tu|^{a'} dx \right)^{\frac{1}{a'}} \left( \int_B \exp(\alpha a \|u\| \gamma) dx \right)^{\frac{1}{a'}}
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2} \varepsilon C_2 \right) \|tu\|^2 - \left( \int_B \exp\left( \alpha a \|tu\|^{a'} \left( \frac{\|u\|}{\|u\|} \right)^{\gamma} \right) dx \right)^{\frac{1}{a'}} C_3 \|tu\|^s
\]

By \((1.3)\), the last integral is finite provided \( t > 0 \) is chosen small enough such that \( \alpha a \|tu\|^{a'} \leq \alpha_\beta \). Then,
\[
\Upsilon_u(t) \geq \left( \frac{1}{2} - \frac{1}{2} \varepsilon C_2 \right) \|tu\|^2 - C_4 \|tu\|^s \quad \text{with } \alpha a \|tu\|^{a'} \leq \alpha_\beta \text{ and } \alpha > \alpha_0
\]
holds. Choosing $\epsilon > 0$ such that $\frac{1}{2} - \frac{1}{2}\epsilon C_2 > 0$ and since $s > 2$, we obtain,

$$\Upsilon_u(t) > 0 \text{ for small } t > 0.$$  \hspace{1cm} (3.4)

Now, From $(V_2)$, we can derive that there exist $C_5, C_6 > 0$ such that

$$F(x, t) \geq C_5|t|^p - C_6.$$  \hspace{1cm} (3.5)

Then, by using (3.5), we get

$$\Upsilon_u(t) = J(tu) \leq \frac{1}{2}t^2\|u\|^2 - C_2|t|^p\|u\|^p - C_6|B|$$

Since $p > 2$, we get that

$$\Upsilon_u(t) \to -\infty \text{ as } t \to +\infty.$$  \hspace{1cm} (3.6)

Hence, from (3.4) and (3.6), there exists at least one $t_u > 0$ such that $\Upsilon'_u(t_u) = 0$, i.e. $t_u u \in \mathcal{N}$.

Now we will show the uniqueness of $t_u$. Let $s > 0$ such that $su \in \mathcal{N}$. Then we get $(\mathcal{J}'(tu), tu) = 0$, $(\mathcal{J}'(su), su) = 0$, and

$$\|su\|^2 = \int_B f(x, su)sudx$$  \hspace{1cm} (3.7)

$$\|tu\|^2 = \int_B f(x, tu)tudx$$  \hspace{1cm} (3.8)

Combining (3.7) and (3.8), we deduce that

$$0 = \int_B f(x, tu)\frac{u^2}{tu}dx - \int_B f(x, su)\frac{tu^2}{su}dx.$$  \hspace{1cm} (3.9)

It follows from $(V_4)$ that $t \mapsto \frac{f(x, t)}{t}$ is increasing for $t > 0$, which implies that $t_u = s$.

This completes the proof of (i).

(ii) Follows from (i), since $\mathcal{J}'(tu) = \max_{t \geq 0} \Upsilon_u(t)$.

In the sequel, we prove that sequences in $\mathcal{N}'$ cannot converge to 0. First, we show the following lemma.

**Lemma 3** Assume that $(V_1) - (V_4)$ hold. Then for any $u \in \mathcal{E}$ with $u \neq 0$ such that $(\mathcal{J}'(u), u) \leq 0$, the unique maximum point of $\Upsilon_u$ on $\mathbb{R}_+$ satisfies $0 < t_u \leq 1$.

**Proof** Since $t_u u \in \mathcal{N}$, we have

$$t_u^2\|u\|^2 = \int_B f(x, tu)tuudx.$$  \hspace{1cm} (3.9)
Furthermore, since \( \langle J'(u), u \rangle \leq 0 \), we have

\[
\| u \|^2 \leq \int_B f(x, u) u \, dx.
\]

Then by (3.9), we have

\[
(t_u^{-2} - 1) \| u \|^2 \geq \int_B \left( \frac{f(x, t_u u)}{t_u u} - \frac{f(x, u)}{u} \right) u^2 \, dx. \tag{3.10}
\]

Obviously, the left hand side of (3.10) is negative for \( t_u > 1 \) whereas the right hand side is positive, which is a contradiction. Therefore \( 0 < t_u \leq 1 \).

**Lemma 4** For all \( u \in \mathcal{N} \),

(i) there exists \( \kappa > 0 \) such that

\[ \| u \| \geq \kappa; \]

(ii) \( J(u) \geq \left( \frac{1}{2} - \frac{1}{p} \right) \| u \|^2 \)

**Proof.** (i) We argue by contradiction. Suppose that there exists a sequence \( \{ u_n \} \subset \mathcal{N} \) such that \( u_n \to 0 \) in \( E \). Since \( \{ u_n \} \subset \mathcal{N} \), then \( \langle J'(u_n), u_n \rangle = 0 \). Hence, it follows from (3.2), (3.3) and the radial Lemma 1 that

\[
\| u_n \|^2 = \int_B f(x, u_n) u_n \, dx \\
\leq \epsilon \int_B |u_n|^2 \, dx + C_1 \int_B |u_n|^s \exp(\alpha|u_n|^\gamma) \, dx \\
\leq \epsilon C_6 \| u_n \|^2 + C_1 \int_B |u_n|^s \exp(\alpha|u_n|^\gamma) \, dx \tag{3.11}
\]

Let \( a > 1 \) with \( \frac{1}{a} + \frac{1}{a'} = 1 \). Since \( u_n \to 0 \) in \( E \), for \( n \) large enough, we get

\[ \| u_n \| \leq \left( \frac{\alpha \beta}{aa} \right)^{\frac{1}{\gamma}} \]  

From Hölder inequality, (1.3) and again the radial Lemma 1, we have

\[
\int_B |u_n|^s \exp(\alpha|u_n|^\gamma) \, dx \leq \left( \int_B |u_n|^{\alpha a'} \, dx \right)^{\frac{1}{\alpha a}} \left( \int_B \exp(\alpha a u^+ \| u^+ \|^\gamma \left( |u^+| \| u^+ \| \right)^\gamma) \, dx \right)^{\frac{1}{\gamma a}} \\
\leq C_7 \left( \int_B |u_n|^{\alpha a'} \, dx \right)^{\frac{1}{\alpha a}} \leq C_8 \| u_n \|^s
\]

Combining (3.11) with the last inequality, for \( n \) large enough, we obtain

\[
\| u_n \|^2 \leq \epsilon C_6 \| u_n \|^2 + C_8 \| u_n \|^s \tag{3.12}
\]

Choose suitable \( \epsilon > 0 \) such that \( 1 - \epsilon C_6 > 0 \). Since \( 2 < s \), then (3.12) contradicts the fact that \( u_n \to 0 \) in \( E \).
(ii) Given \( u \in \mathcal{N} \), by the definition of \( \mathcal{N} \) and \((V_3)\) we obtain
\[
\mathcal{J}(u) = \mathcal{J}(u) - \frac{1}{p} \langle \mathcal{J}'(u), u \rangle
\]
\[
= \frac{1}{2} \| u_n \|^2 - \frac{1}{p} \| u_n \|^2 + \left( \int_{\mathcal{B}} \frac{1}{p} f(x, u)u - F(x, u)dx \right)
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \| u \|^2
\]

Lemma 4 implies that \( \mathcal{J}(u) > 0 \) for all \( u \in \mathcal{N} \). As a consequence, \( \mathcal{J} \) is bounded by below in \( \mathcal{N} \), and therefore \( m := \inf_{u \in \mathcal{N}} \mathcal{J}(u) \) is well-defined.

In the following lemma we prove that if the infimum of \( \mathcal{J} \) on \( \mathcal{N} \) is achieved in some \( u \in \mathcal{N} \), then \( u \) is a critical point of \( \mathcal{J} \).

**Lemma 5** If \( u_0 \in \mathcal{N} \) satisfies \( \mathcal{J}(u_0) = m \), then \( \mathcal{J}'(u_0) = 0 \).

**Proof** We argue by contradiction. We assume that \( \mathcal{J}'(u_0) \neq 0 \). By the continuity of \( \mathcal{J}'_\lambda \), there exist \( \iota, \delta \geq 0 \) such that
\[
\| \mathcal{J}'_\lambda(v) \|_E \geq \iota \text{ for all } v \text{ such that } \| v - u_0 \| \leq \delta.
\]

Let \( D = (1 - \tau, 1 + \tau) \subset \mathbb{R} \) with \( \tau \in \left( 0, \frac{\delta}{4\|u_0\|} \right) \) and define \( g : D \to E \), by
\[
g(\rho) = \rho u_0, \rho \in D
\]

By virtue of \( u_0 \in \mathcal{N}, \mathcal{J}(u_0) = m \) and Lemma 2, it is easy to see that
\[
\bar{m} := \max_{\delta D} \mathcal{J} \circ g < m \text{ and } \mathcal{J}(g(\rho)) < m, \forall \rho \neq 1.
\]

Let \( \epsilon := \min\{\frac{m - \bar{m}}{2}, \frac{\delta}{16}\} \), \( S_\delta := B(u_0, r), r \geq 0 \) and \( \mathcal{J}^a := \mathcal{J}^{-1}([m - \epsilon, m + \epsilon]) \cap S_\delta \).

According to the quantitative deformation Lemma [31], Lemma 2.3, there exists a deformation \( \eta \in C(E, E) \) such that:
1. \( \eta(v) = v \), if \( v \notin \mathcal{J}^{-1}([m - \epsilon, m + \epsilon]) \cap S_\delta \)
2. \( \eta \left( \mathcal{J}^{m+\epsilon} \cap S_\delta \right) \subset \mathcal{J}^{m-\epsilon} \)
3. \( \mathcal{J}(\eta(v)) \leq \mathcal{J}(v) \), for all \( v \in E \).

By Lemma 2 (ii), we have \( \mathcal{J}(g(\rho)) \leq m \). In addition, we have,
\[
\| g(\rho) - u_0 \| = \| (\rho - 1)u_0 \| \leq \frac{\delta}{4}, \forall \rho \in D.
\]

Then, \( g(\rho) \in S_\delta \) for \( \rho \in \bar{D} \). Therefore, it follows from (2) that
\[
\max_{\rho \in \bar{D}} \mathcal{J}(\eta(g(\rho))) \leq m - \epsilon.
\]
In the following, we prove that $\eta(g(D)) \cap N$ is nonempty. And in this case it contradicts (3.15) due to the definition of $m$. To do this, we first define

$$\bar{g}(\rho) := \eta(g(\rho)),$$

$$\Upsilon_0(\rho) = \langle J'(g(\rho)), u_0 \rangle,$$

and

$$\Upsilon_1(\rho) := \left( \frac{1}{\rho} \langle J'(\bar{g}(\rho)), (\bar{g}(\rho)) \rangle \right).$$

We have that for $\rho \in \bar{D}$, $J(g(\rho)) \leq \bar{m} < m - \varepsilon$. Then, $\bar{g}(\rho) = \eta(g(\rho)) = \rho u_0$. Hence,

$$\Upsilon_0(\rho) = \Upsilon_1(\rho), \forall \rho \in \bar{D} \quad (3.16)$$

On one hand, we have that $\rho = 1$ is the unique critical point of $\Upsilon_0$. So by degree theory, we get that $d^0(\Upsilon_0, J, 0) = 1$. On the other hand, from (3.16), we deduce that $d^0(\Upsilon_1, J, 0) = 1$. Consequently, there exists $\bar{\rho} \in D$ such that $\bar{g}(\bar{\rho}) \in N$. This implies that

$$m \leq J(\bar{g}(\bar{\rho})) = J(\eta(g(\bar{\rho}))).$$

This contradicts (3.15) and finish the proof of the Lemma.

### 4 Auxiliary problem

In this section, in order to prove our existence result, we consider the auxiliary problem

$$\begin{cases}
\Delta(w(x) \Delta u) = |u|^{p-2}u & \text{in } B \\
u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B,
\end{cases} \quad (4.1)$$

where $p$ is the constant that appears in the hypothesis $(V_5)$. We have associated to problem (4.1) the functional

$$J_p(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_B |u|^p \, dx$$

and the Nehari manifold

$$N_p := \{ u \in E, u \neq 0 \text{ and } \langle J'_p(u), u \rangle = 0 \}.$$ 

Let $m_p = \inf_{N_p} J_p(u) > 0$, we have the following results for $J_p$. ☛ Springer
Lemma 6 Given $u \in E, u \neq 0$, there exists a unique $t > 0$ such that $tu \in N_p$. In addition, $t$ satisfies

$$J_p(tu) = \max_{s \geq 0} J_p(su)$$

(4.2).

Proof Let $\gamma(s) = J_p(su) = \frac{1}{2}s^2\|u\|^2 - \frac{s^p}{p}|u|^p$, for $s > 0$. Since $p > 2$, we have that $\gamma(s) > 0$ for $s > 0$ small enough and $\gamma(s) \to -\infty$ as $s \to -\infty$. Hence, there exists a $t > 0$ satisfying (4.2). In particular, $tu \in N_p$. Moreover, $\gamma'(t) = 0$ if and only if $t = \left(\frac{\|u\|^2}{|u|^p}\right)^{\frac{1}{p-2}}$.

As a consequence, we have

Corollary 4.1 Let $u \in E, u \neq 0$. Then $u \in N_p$ if and only if $J_p(tu) = \max_{s \geq 0} J_p(su)$.

Lemma 7 For all $u \in N_p$,

(i) there exists $\kappa_0 > 0$ such that $\|u\| \geq \kappa_0$;

(ii) $J_p(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right)|u|^p$

Lemma 8 There exists $w_p \in N_p$ such that $J_p(w_p) = m_p$ and $m_p = \frac{p-2}{2p}|w_p|^p$.

Proof Let sequence $(w_n) \subset N_p$ satisfy $\lim_{n \to +\infty} J_p(w_n) = m_p$. It is clearly that $(w_n)$ is bounded by Lemma 7. Then, up to a subsequence, there exists $w_p \in E$ such that

$$w_n \rightharpoonup w_p \quad \text{in} \quad E,$$

$$w_n \to w_p \quad \text{in} \quad L^q(B), \quad \forall q \geq 2,$$

$$w_n \to w_p \quad \text{a.e. in} \quad B.$$  

(4.3)

We claim that $w_p \neq 0$. Suppose, by contradiction, $w_p = 0$. From the definition of $N_p$ and (4.3), we have that $\lim_{n \to +\infty} \|w_n\|^2 = 0$, which contradicts Lemma 7. Hence, $w_p \neq 0$.

From the lower semi continuity of norm and (4.3), it follows that

$$\|w_p\|^2 \leq \liminf_{n \to +\infty} \|w_n\|^2$$  

(4.4)

On the other hand, by using $(J_p'(w_n), w_n) = 0$ and (4.3), we have

$$\liminf_{n \to +\infty} \|w_n\|^2 = \liminf_{n \to +\infty} \int_B |w_n|^p dx = \int_B |w_p|^p dx.$$  

(4.5)

From (4.4) and (4.5) we deduce that $(J_p'(w_p), w_p) \leq 0$. Then, as in Lemma 3 this implies that there exists $s_u \in (0, 1]$ such that $s_u w_p \in N_p$. Thus, by the lower semi continuity of norm and (4.3), we get that
\[ m_p \leq J_p(s_u w_p) = J(s_u w_p) - \frac{1}{2} \langle J'_p(s_u w_p), s_u w_p \rangle \]
\[ = \left( \frac{1}{2} - \frac{1}{p} \right) s_u^p \int_B |w_p|^p \, dx \]
\[ \leq J_p(w_p) - \frac{1}{2} \langle J'_p(w_p), w_p \rangle \]
\[ = \frac{1}{2} \|w_p\|^2 - \frac{1}{p} \int_B |w_p|^p \, dx - \frac{1}{2} \|w_p\|^2 + \frac{1}{2} \int_B |w_p|^p \, dx \]
\[ \leq \lim inf_{n \to +\infty} \left[ \frac{1}{2} \|w_n\|^2 - \frac{1}{p} \int_B |w_n|^p \, dx - \frac{1}{2} \|w_n\|^2 + \frac{1}{2} \int_B |w_n|^p \, dx \right] \]
\[ \leq \lim inf_{n \to +\infty} \left[ J_p(w_n) - \frac{1}{2} \langle J'_p(w_n), w_n \rangle \right] = m_p. \]

Therefore, we get that \( J_p(w_p) = m_p \), which is the desired conclusion.

### 5 Proof of Theorem 1.2

Now, we will obtain an important estimate for the level \( m = \inf_{u \in \mathcal{N}} J(u) \), which will be a powerful tool in order to obtain an appropriate bound of the norm of a minimizing sequence for \( m \) in \( \mathcal{N} \).

#### Lemma 9

Assume that (\( V_1 \)) – (\( V_5 \)) and (1.6) are satisfied. It holds that

\[ m < \frac{p - 2}{2p} \left( \frac{\alpha \beta}{2(\alpha_0 + \delta)} \right)^{1-\beta}. \]  

(5.1)

**Proof** From Lemma 8, there exists \( w_p \in \mathcal{N}_p \) such that \( J_p(w_p) = m_p \) and \( J'_p(w_p) = 0 \). Consequently, we get

\[ \frac{1}{2} \|w_p\|^2 - \frac{1}{p} \int_B |w_p|^p \, dx = m_p \]  

(5.2)

and

\[ \|w_p\|^2 = \int_B |w_p|^p \, dx. \]  

(5.3)

By virtue of (\( V_5 \)) and (5.3), we have \( \langle J'(w_p), w_p \rangle \leq 0 \) which together with Lemma 3 yielding that there is a unique \( s \in (0, 1] \) such that \( sw_p \in \mathcal{N} \). Using (\( V_5 \)), (5.2) and (5.3), we obtain
\[ m \leq \mathcal{J}(sw_p) \]
\[ \leq \frac{s^2}{2} \|w_p\|^2 - \frac{C_p s^p}{p} |w_p|^p \]
\[ = \left( \frac{s^2}{2} - \frac{C_p s^p}{p} \right) |w_p|^p \]
\[ \leq \max_{\xi > 0} \left( \frac{\xi^2}{2} - \frac{C_p \xi^p}{p} \right) |w_p|^p \]

By some straightforward algebraic manipulations, we get
\[ m \leq C^{-2} p - 2 \frac{2}{2p} |w_p|^p. \] (5.4)

Note that by using (5.2), (5.3) and the fact that \( p > 2 \), we have
\[ \left( \frac{1}{2} - \frac{1}{p} \right) |w_p|^p = \frac{1}{2} \|w_p\|^2 - \frac{1}{p} |w_p|^p = m_p. \] (5.5)

Thus, by combining (5.4) and (5.5), we obtain
\[ m < C^{-2} p m_p. \] (5.6)

Therefore, by (1.6) and (5.6), we obtain that (5.1) holds.

The following result gives us some compactness properties of minimizing sequences.

**Lemma 10** If \( (u_n) \subset \mathcal{N} \) is a minimizing sequence for \( m \), then there exists \( u \in E \) such that
\[ \int_B f(x, u_n) u_n dx \to \int_B f(x, u) u dx \]
and
\[ \int_B F(x, u_n) dx \to \int_B F(x, u) dx. \]

**Proof** We prove the first limit, the second one is analogous. It is sufficient to prove that \( g(u_n(x)) \) is convergent in \( L^1(B) \), where \( g(u_n(x)) \) is defined by
\[ g(u_n(x)) := \epsilon |u_n|^q + C |u_n|^q \exp(\alpha(u_n)^) \geq f(x, u_n) u_n, \text{ for all } \alpha > \alpha_0, \text{ and, } q > 2. \]

First note that
\[ |u_n|^q \to |u|^q \text{ in } L^2(B). \] (5.7)
On the other hand, by \((V_2)\), we obtain that

\[
\begin{align*}
m &= \limsup_{n \to +\infty} J(u_n) = \limsup_{n \to +\infty} \left( \frac{1}{p} \langle J'(u_n), u_n \rangle \right) \\
&= \limsup_{n \to +\infty} \left( \frac{p - 2}{2p} \|u_n\|^2 + \frac{1}{p} \int_B \left( f(x, u_n) - pF(x, u_n) \right) dx \right) \\
&> \frac{p - 2}{2p} \limsup_{n \to +\infty} \|u_n\|^2, \tag{5.8}
\end{align*}
\]

which together with Lemma 9 gives that \(\limsup_{n \to +\infty} \|u_n\| < \frac{\alpha \beta}{2(\alpha_0 + \delta)}\).

Now choosing \(\alpha = \alpha_0 + \delta\), we have from Theorem 1.2 that

\[
\int_B \exp(2\alpha |u_n|^\gamma) dx \leq \int_B \exp \left( 2(\alpha_0 + \delta) \|u_n\|^\gamma \left( \frac{u_n}{\|u_n\|} \right)^\gamma \right) dx \\
\leq \int_B \exp \left( \alpha \beta \left( \frac{u_n}{\|u_n\|} \right)^\gamma \right) dx. \tag{5.9}
\]

Then it follows by (1.3) that there is \(M > 0\) such that

\[
\int_B \exp(2\alpha |u_n|^\gamma) dx \leq M. \tag{5.10}
\]

Since for a subsequence

\[
\exp(\alpha |u_n|^\gamma) \to \exp(\alpha |u|^\gamma) \text{ a.e in } B, \tag{5.11}
\]

from (5.10) and [20], Lemma 4.8, we get that

\[
\exp(\alpha |u_n|^\gamma) \to \exp(\alpha |u|^\gamma) \text{ in } L^2(B). \tag{5.12}
\]

Now using (5.7), (5.12) and [20], Lemma 4.8 again, we conclude that

\[
\int_B f(x, u_n) u_n dx \to \int_B f(x, u) u dx. \tag{5.13}
\]

In the sequel, we give an important result:

**Lemma 11** Assume that the conditions \((V_1), (V_2)\) and \((V_3)\) are satisfied. Then, for each \(x \in B\), we have

\[
 tf(x, t) - 2F(x, t) \text{ is increasing for } t > 0 \text{ and decreasing for } t < 0.
\]

In particular, \( tf(x, t) - 2F(x, t) > 0 \) for all \((x, t) \in B \times \mathbb{R} \setminus \{0\}\).
We claim that

Assume that $0 < s < t$. For each $x \in B$, we have

$$
t f(x, t) - 2F(x, t) = \frac{f(x, t)}{t^2} - 2F(x, s) + 2 \int_{s}^{t} f(x, v) dv
$$

$$
< \frac{f(x, t)}{t^2} - 2F(x, s) + \frac{f(x, s)}{s}(s^2 - t^2)
$$

$$
= sf(x, s) - 2F(x, s).
$$

The proof in the case $t < s < 0$ is similar. □

The assertion $tf(x, t) - 2F(x, t) > 0$ for all $(x, t) \in B \times \mathbb{R} \setminus \{0\}$ comes from (V2).

**Lemma 12** There exists $w_0 \in \mathcal{N}$ such that $J(w_0) = m$.

**Proof** Let sequence $(w_n) \subset \mathcal{N}$ satisfy $\lim_{n \to +\infty} J(w_n) = m$. It is clearly that $(w_n)$ is bounded by Lemma 7. Then, up to a subsequence, there exists $w_0 \in E$ such that

$$
\begin{align*}
  w_n &\to w_0 \quad \text{in } E, \\
  w_n &\to w_0 \quad \text{in } L^q(B), \quad \forall q \geq 2, \\
  w_n &\to w_0 \quad \text{a.e. in } B.
\end{align*}
$$

We claim that $w_0 \neq 0$. Suppose, by contradiction, $w_0 = 0$. From the definition of $\mathcal{N}$ and (5.14), we have that $\lim_{n \to +\infty} \|w_n\|^2 = 0$, which contradicts Lemma 4. Hence, $w_0 \neq 0$.

From the lower semi continuity of norm and (5.14), it follows that

$$
\|w_0\|^2 \leq \liminf_{n \to +\infty} (\|w_n\|^2) \quad (5.15)
$$

On the other hand, by using $\langle J'(w_n), w_n \rangle = 0$ and (5.14), we have

$$
\liminf_{n \to +\infty} (\|w_n\|^2) = \liminf_{n \to +\infty} \int_B f(x, w_n) w_n dx = \int_B f(x, w_0) w_0 dx. \quad (5.16)
$$

From (5.15) and (5.16) we deduce that $\langle J'(w_0), w_0 \rangle \leq 0$. Then, as in Lemma 3 this implies that there exists $s \in (0, 1]$ such that $sw_0 \in \mathcal{N}$. Thus, by the lower semi continuity of norm, Lemma 11 and Lemma 10, we get that

$$
m \leq J(sw_0) = J(sw_0) - \frac{1}{2} \langle J'(sw_0), sw_0 \rangle
$$

$$
= \frac{1}{2} \int_B \left( f(x, sw_0)sw_0 - 2F(x, sw_0) \right) dx
$$

$$
\leq \frac{1}{2} \int_B \left( f(x, w_0)w_0 - 2F(x, w_0) \right) dx
$$

$$
\leq \frac{1}{2} \int_B \left( f(x, w_0)w_0 - 2F(x, w_0) \right) dx
$$

$$
\leq \liminf_{n \to +\infty} \left[ \frac{1}{2} \|w_n\|^2 - \int_B F(x, w_n) dx - \frac{1}{2} \|w_n\|^2 + \frac{1}{2} \int_B f(x, w_n) w_n dx \right]
$$

$$
\leq \liminf_{n \to +\infty} \left[ J(w_n) - \frac{1}{2} \langle J'(w_n), w_n \rangle \right] = m.
$$
Therefore, we get that $\mathcal{J}(sw_0) = m$, which is the desired conclusion.

**Proof of Theorem 1.3** From Lemma 12 there exists $w_0$ such that $\mathcal{J}(w_0) = m$. Now, by Lemma 5, we deduce that $\mathcal{J}'(w_0) = 0$. So, $w_0$ is a solution for our problem $(P)$.

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