THE WITTEN GENUS AND VERTEX ALGEBRAS

Pokman Cheung

February 15, 2010

Abstract

This article is the first report of an ongoing project aimed at finding a geometric interpretation of the Witten genus and other tmf classes. Section 2 reviews the sheaves of chiral differential operators $D^M_{\mathcal{M},\xi}$ over a complex manifold $M$, including their construction, obstructions and relation with the Witten genus of $M$. In section 3, the structure of $D^M_{\mathcal{M},\xi}$ as a sheaf of vertex algebras is reorganized in terms of $\mathcal{O}_M$-modules. This invokes the notion of a differential graded (dg) vertex algebroid. The construction of $D^M_{\mathcal{M},\xi}$ is due to Gorbounov, Malikov and Schechtman, and so is the notion of a vertex algebroid; the dg version is first introduced here. Section 4 contains the main result, namely the construction of a sheaf of dg conformal vertex algebras that provides a fine resolution of $D^M_{\mathcal{M},\xi}$. This ‘infinite dimensional Dolbeault complex’ plays a role for the Witten genus similar to that of the Dolbeault complex for the Todd genus.

§1. BACKGROUND & OVERVIEW OF THE PAPER

In this section, all manifolds are compact unless specified otherwise.

In a study of connections between topology and physics, Witten considered various types of conformal field theory associated to a manifold $M$. Among them, one type $^1$ can only be constructed when $M$ is spin and $p_1(M) = 0$. The torus partition function of this conformal field theory is up to a constant factor a modular form of weight $\frac{1}{2} \dim M$. The $q$-expansion of the modular form is given by

$$W(M) = \int_M \hat{A}(TM) \, ch \left( \bigotimes_{k=1}^{\infty} \text{Sym}^{q} \left( TM \otimes \mathbb{C} \right) \right) \cdot \prod_{k=1}^{\infty} (1 - q^k)^{\dim M}. \quad (1.1)$$

[Wit87] This is the Witten genus of $M$. Unfortunately, physicists’ construction of the conformal field theory and their reasoning for the modularity of (1.1) involve the ill-defined path integral. From a mathematical point of view, the expression in (1.1) makes sense whenever $M$ is oriented but is not a priori modular. Zagier gave a mathematical proof that

$$W(M) \text{ is a modular form over } \mathbb{Z} \text{ when } M \text{ is string.} \quad (1.2)$$

$^2$ [Zag88] However, a deeper understanding of this result is desired. Guidance has been provided by an analogy with the $\hat{A}$-genus. Defined for any oriented manifold $M$, $\hat{A}(M)$ is a priori a rational number but

$$\hat{A}(M) \text{ is an integer when } M \text{ is spin.} \quad (1.3)$$

It is well known that the search for an explanation of (1.3) largely motivated the invention of topological $K$-theory and the discovery of the Atiyah-Singer index theorem.

$^1$ $N = 1/2$ sigma model with only right-moving fermions.

$^2$ Let $\lambda \in H^4(B\text{Spin}; \mathbb{Z}) \cong \mathbb{Z}$ denote the generator twice which is $p_1$, and also the corresponding characteristic class for spin vector bundles. A spin vector bundle or manifold is said to be string if its $\lambda$ vanishes. Moreover, for $n \geq 1$, there is a topological group $\text{String}_n$ that admits a homomorphism into $\text{Spin}_n$ such that the induced map $B\text{String}_n \to B\text{Spin}_n$ is the homotopy fiber of $\lambda|_{B\text{Spin}_n} : B\text{Spin}_n \to K(\mathbb{Z}, 4)$. For specific models of $\text{String}_n$, see [ST04, BCSS05, Hen06]. A string structure on a spin vector bundle or manifold is a lifting of its structure group from $\text{Spin}_n$ to $\text{String}_n$. 

A topological explanation of (1.3) consists of two ingredients: $KO$-theory and a $KO$-orientation of spin vector bundles. The orientation can be regarded as a map of ring spectra from $M_{\text{Spin}}$ to $KO$. The induced map in homotopy defines a $\pi_*(KO)$-valued bordism invariant for spin manifolds $M$. If the 2-torsions in $\pi_*(KO)$ are ignored (or equivalently, if $\dim M$ is divisible by 4), this invariant equals $A(M)$, proving that it is an integer. A topological explanation of (1.2) consists of similar ingredients: the ring spectrum $\text{tmf}$ and the $\sigma$-orientation

$$\sigma : M_{\text{String}} \to \text{tmf}.$$ (1.4)

The construction of $\text{tmf}$ and the $\sigma$-orientation is the culmination of the combined work of many people. [LRS93, AHS01, Hop02, AHR08] Modularity is built into $\text{tmf}$ in the sense that $\pi_*(\text{tmf})$ maps into the ring of modular forms and it is an isomorphism away from 6. The map induced by (1.4) in homotopy defines a $\pi_*(\text{tmf})$-valued bordism invariant for string manifolds $M$. Once 6 is inverted, this invariant equals $W(M)$, which is therefore a modular form. [Hop02, AHR08]

Of course, (1.3) also has a geometric explanation. It is closely related to the interpretation of $KO$-theory in terms of Clifford module bundles. Namely, the $KO$-orientation of a spin manifold is represented by the symbol of its Clifford-linear Dirac operator, and the associated $\pi_*(KO)$-valued invariant by the kernel of this operator. [LM89] In contrast, $\text{tmf}$ lacks a description similar to that of $K$- or $KO$-theory using vector bundles. In light of the physical origin of the Witten genus, a geometric explanation of (1.2) and a related description of $\text{tmf}$ seem to entail a mathematical definition of conformal field theory.

Vertex algebras provide a mathematical approach to conformal field theory [Kac98, FB04] and may therefore provide a geometric meaning of the Witten genus and $\text{tmf}$. The first result of this kind was due to Gorbounov, Malikov and Schechtman. Here is an outline of the part of their work in [GMS00] related to the Witten genus. The space of chiral differential operators $D_{\sigma}^{ch}(U)$ over an open set $U$ in $\mathbb{C}^d$ is a basic example of a conformal vertex algebra (§2.2-§2.3). To patch these local objects into a sheaf over a complex manifold $M$, it turns out to require the choice of some geometric data $\xi$ (§2.6) whose existence requires $c_1(M) = c_2(M) = 0$ rationally (§2.7-§2.8). Denote the resulting sheaf of conformal vertex algebras by $D_{M,\xi}^{ch}$. Its cohomology $H^*(D_{M,\xi}^{ch})$ is a conformal vertex superalgebra with character (§2.1)

$$\text{char } H^*(D_{M,\xi}^{ch}) = W(M) : (\text{a constant factor})$$ (1.5)

where the constant factor depends only on $d$ (§2.11). This provides a geometric meaning of the Witten genus for a particular class of complex manifolds.

The author’s present goal is to extend the result (1.5) and obtain an interpretation of the Witten genus for a larger class of manifolds that includes string manifolds. To motivate the first step that has been taken, again consider an analogy. For complex manifolds, the appropriate analogue of the Witten genus is the Todd genus. By the Hirzebruch-Riemann-Roch theorem, the Todd genus of a complex manifold $M$ has the following closely related interpretations

$$\text{Td}(M) = \text{Euler characteristic of the sheaf of holomorphic functions } \mathcal{O}_M$$

= Euler characteristic of the Dolbeault complex $(\mathcal{E}^{0,*}(M), \bar{\partial})$

= super dimension of the kernel of the operator $\bar{\partial} + \bar{\partial}^*$ on $\mathcal{E}^{0,*}(M);$  

among them, it is the last one that has a generalization to all spin$^c$ manifolds by the Atiyah-Singer index theorem. [LM89] In the case of the Witten genus, each sheaf of chiral differential operators $D_{M,\xi}^{ch}$ plays the role of $\mathcal{O}_M$. In fact, its lowest-weight component is $(D_{M,\xi}^{ch})_0 = \mathcal{O}_M$ (§2.4, §2.6). The analogy with the Todd genus suggests that we find an analogue of the Dolbeault complex. More precisely, for each $D_{M,\xi}^{ch}$, we would like to construct

a sheaf of dg conformal vertex algebras such that

i. it provides a fine resolution of the sheaf of conformal vertex algebras $D_{M,\xi}^{ch}$, and

ii. its weight-zero component is the Dolbeault resolution of $\mathcal{O}_M$  

$^3$The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is defined with respect to a hermitian metric on $M$.  

3
The construction of such an object, which is outlined below, is the main result of this article.

At first glance, there are two difficulties in building the desired object (1.6). First, we have to deal with an infinite number of sheaves — i.e. the weight components \(D^{\text{ch}}_{M,\xi} \), \(k \geq 0 \) — as well as an infinite number of morphisms between these sheaves given by the vertex algebra structure. Second, all positive weight components are not \(\mathcal{O}_M\)-modules (§2.10). Both difficulties are tackled using vertex algebroids. The notion of a vertex algebroid was also introduced by Gorbounov, Malikov and Schechtman [GMS04] 4

Roughly speaking, given a vertex algebra, the part of its structure involving only the two lowest weights are also modified appropriately (§3.1-§3.2). The resulting algebraic notion is called a vertex algebroid. The forgetful functor

\[
\text{category of vertex algebras} \xrightarrow{\text{forget}} \text{category of vertex algebroids}
\]

has a left adjoint that defines the vertex algebra ‘freely generated’ by a vertex algebroid (§3.3). For example, the space of chiral differential operators \(\mathcal{D}^{\text{ch}}(U)\) over \(U \subset \mathbb{C}^d\) is freely generated by a vertex algebroid defined in terms of \(\mathcal{O}(U)\), \(\Omega^1(U)\), \(\mathcal{T}(U)\) — i.e. the holomorphic functions, 1-forms and vector fields on \(U\) (§3.4). The above discussion has a similar dg version. For concreteness, a dg vertex algebroid consists of data of the form

\[
(A^*, \Omega^*, \tau, \Delta : \tau^* \to \Omega^{*+1}, *, : A^* \otimes \tau^* \to \Omega^*, \{ \}^0 : \tau^* \otimes \tau^* \to A^*, \{ \}^1 : \tau^* \otimes \tau^* \to \Omega^*)
\]

satisfying a host of axioms. In particular, \(A^2\) is a dg ring, \(\Omega^*, \tau^*\) are \(A^*\)-modules, \(\Delta\) is a chain map of degree one, and \(*, \{ \}^0, \{ \}^1\) are null homotopies of chain maps constructed from \(\Delta\) (§3.8).

The construction of the desired object (1.6) is summarized in figure 1. 5 The diagram starts with the dg vertex algebra \(\hat{C}_*^*(D^{\text{ch}}_{M,\xi})\). 6 The top row follows from the comment about \(\mathcal{D}^{\text{ch}}(U)\) in the previous paragraph (§3.11). The right column is a sequence of quasi-isomorphisms of dg vertex algebroids (§4.1). The first quasi-isomorphism extends \(\hat{\Delta}, \hat{*}, \{ \}^0, \{ \}^1\) from Čech complexes to Čech-Dolbeault complexes such that \(\hat{\Delta}\) has Čech degree one and Dolbeault degree zero (§4.3). The second quasi-isomorphism modifies \(\hat{\Delta}\) by a homotopy into a map \(\Delta\) with Čech degree zero and Dolbeault degree one; the other three maps are also modified appropriately (§4.4). This step requires the choice of some geometric data (\(\nabla, H\)) to be explained below. The third quasi-isomorphism restricts \(\hat{\Delta}, \hat{*}, \{ \}^0, \{ \}^1\) to global sections (§4.8). Finally, the bottom arrow defines a new dg vertex algebra \(\Gamma(\mathcal{E}^\text{ch},^*,\nabla, H)\). As the notation suggests, it is in fact the global sections of a sheaf of dg vertex algebras \(\mathcal{E}^\text{ch},^*,\nabla, H\) (§4.8-§4.9). This object has the properties stated in (1.6). 7 In particular, there is a fine resolution

\[
0 \xrightarrow{i} \mathcal{D}^\text{ch}_{M,\xi} \xrightarrow{\text{forget}} \mathcal{E}^\text{ch},^*,\nabla, H \tag{1.7}
\]

where \(i\) is a morphism of sheaves of conformal vertex superalgebras (§4.8-§4.9, §4.11).

The following are some features in the construction of \(\mathcal{E}^\text{ch},^*,\nabla, H\). (i) The input data include a connection \(\nabla\) on \(TM\) and a 3-form \(H\) satisfying certain conditions, most notably \(dH = -\text{Tr}(R \wedge R)\), where \(R\) is the

4The author apologizes for using a different set of notations. The structure maps denoted by \(*, \{ \}^0, \{ \}^1\) in this article (§3.1) are respectively equal to the maps \(-\gamma, (\cdot, \cdot), -c + \frac{1}{2} \phi \circ (\cdot, \cdot)\) in [GMS04].

5Notations: \(\hat{C}_*^*(-)\) are Čech complexes of a finite good open cover; \(\mathcal{O}_M, \Omega^1_M, TM\) are the sheaves of holomorphic functions, 1-forms and vector fields; \(\mathcal{E}^0_*\) is the Dolbeault resolution of \(\mathcal{O}_M\); tensor products are taken over \(\mathcal{O}_M\).

6In fact, the Čech complex valued in a sheaf of vertex algebras is only a dg vertex algebra up to homotopy (§3.7). Similarly, every term in the right column of figure 1 except the last is only a dg vertex algebra up to homotopy.

7The steps taken to construct this object may seem devious. However, even though the presheaf of vertex algebroids induced by \(\mathcal{D}^{\text{ch}}_{M,\xi}\) is defined in terms of \(\mathcal{O}_M\)-modules, it contains collections of maps that do not form morphisms of sheaves (§3.5) and hence do not extend easily to maps between Dolbeault resolutions. Instead, those data define non-chain maps between Čech complexes, namely \(*, \{ \}^0, \{ \}^1\) in the first dg vertex algebroid in figure 1 (§3.11). Their extensions to Čech-Dolbeault complexes respect the Dolbeault but not the Čech differential (§4.3). They are then modified into maps \(*, \{ \}^0, \{ \}^1\) that respect the Čech but not the Dolbeault differential (§4.4). In particular, \(*, \{ \}^0, \{ \}^1\) define morphisms of sheaves between Dolbeault resolutions (§4.8).
### Figure 1: Construction of the chiral Dolbeault complex.

Curvature of $\nabla$ (§4.5–§4.7). A choice of $(\nabla, H)$ is essentially equivalent to a choice of the data $\xi$ needed to construct $D^{\text{ch}}_{M, \xi}$ (§4.10). For $(\nabla, H)$ to exist, a necessary condition is $c_1(M) = c_2(M) = 0$ rationally, which is also sufficient in the case $M$ is Kähler (§4.12). (ii) While the definition of each $D^{\text{ch}}_{M, \xi}$ depends necessarily on local coordinates (§2.6), $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$ has a global description (§4.8). (iii) The differentials on the weight-zero and -one components are respectively the ordinary Dolbeault operator on $\mathcal{E}^{0,*}_M$ and a ‘deformed’ Dolbeault operator (§4.9)

\[
\begin{bmatrix}
\bar{\partial} & \bar{\Delta} \\
0 & \partial
\end{bmatrix}
\]  

on $(\Omega^1_M \oplus T_M) \otimes \mathcal{E}^{0,*}_M$, $(\alpha, X) \mapsto (\bar{\partial} \alpha + \bar{\Delta} X, \partial X)$.  

(iv) It follows from the resolution (1.7) that $\Gamma(\mathcal{E}^{\text{ch}*}_{M, \nabla, H})$ computes $H^*(D^{\text{ch}}_{M, \xi})$. This provides a new interpretation of the conformal vertex superalgebra $H^*(D^{\text{ch}}_{M, \xi})$ and hence, by (1.5), the Witten genus

\[
\text{char } H^*(\Gamma(\mathcal{E}^{\text{ch}*}_{M, \nabla, H})) = W(M) \cdot (\text{a constant factor}).
\]  

In view of the connection between conformal vertex (super)algebras and conformal field theory, (1.8) goes some way to ‘explaining’ the modularity of $W(M)$.

This paragraph is added in February 2010. After completing the work reported here, the author has realized that the construction of $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$ is somewhat ad hoc and it is not clear how to generalize the interpretation of the Witten genus in (1.8). Another construction $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$ that also fulfils (1.6) is given in [Che10]. Roughly speaking, it is a sheaf of ‘smooth’ chiral differential operators on the cs-manifold $\Pi \overline{T M}$, equipped with a derivation associated to an odd vector field. While $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$ embeds quasi-isomorphically into $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$, the latter belongs to a more general and systematic framework. In fact, as a sheaf of conformal vertex superalgebras $\mathcal{E}^{\text{ch}*}_{M, \nabla, H}$ is defined whenever $M$ is almost complex and $c_1(M) = c_2(M) = 0$ rationally (but the derivation may not be order-two in general). On the other hand, some of the ideas in [Che10] came from this work. For a discussion of future directions, see also [Che10].

The author would like to thank Matthew Ando, Victor Kač, Haynes Miller, Stephan Stolz, Peter Teichner and his Ph.D. advisor Ralph Cohen for their useful comments as well as their continual interest and encouragement in this project. The notion of a vertex algebroid was first brought to the author’s attention by Haynes Miller. The author worked out most of the content of this article when he was a Moore Instructor at MIT before completing it at the Max-Planck-Institut für Mathematik; he is grateful for the generous support from both institutes.
\section{Chiral Differential Operators}

This section reviews the construction of chiral differential operators on complex manifolds as well as their relation with the Witten genus. The presentation here somewhat differs from \cite{GMS00}.

\subsection{2.1}

In this article, we adopt the definition of a vertex superalgebra and a conformal vertex superalgebra in \cite{FB04}. This in particular means the space state of a vertex algebra always has a \textit{Z}+\textit{grading}, referred to as weight. (See also \cite{Kac98}.) The \textit{character} of a conformal vertex superalgebra \(V\), when defined, is the formal power series

\[
\text{char}\ V = \sum_k q^{k-c/24}(\dim V^\text{even}_k - \dim V^\text{odd}_k) \in q^{-c/24} \mathbb{Z}[q]
\]

where \(V^\text{even}_k\), \(V^\text{odd}_k\) are the even and odd parts of the weight-\(k\) component, and \(c\) is the central charge.

\subsection{2.2}

Let \(W\) be the unital associative \(\mathbb{C}\)-algebra with generators \(a_{i,n}, b^*_n\) for \(n \in \mathbb{Z}\), \(i = 1, \ldots, d\), and relations

\[
[a_{i,n}, b^*_m] = \delta^i_j \delta_{n,-m}, \quad [a_{i,n}, a_{j,m}] = 0 = [b^*_n, b^*_m].
\]

The subalgebra \(W_+ \subset W\) generated by \(a_{i,n}, n \geq 0\), and \(b^*_n, n > 0\), is commutative and hence admits a trivial representation \(\mathbb{C}\). The induced representation \(W \otimes_{W_+} \mathbb{C}\) of \(W\) has the structure of a conformal vertex algebra. Let us denote it by \(\mathcal{D}^ch_{\text{alg}}(\mathbb{C}^d)\) and define its structure below.

A general element of \(\mathcal{D}^ch_{\text{alg}}(\mathbb{C}^d)\) is a linear combination of elements of the form

\[
a_{i_1 n_1} \cdots a_{i_k n_k} b^{i_1}_{m_1} \cdots b^{i_{\ell}}_{m_{\ell}} f
\]

(2.1)

where \(n_1 \leq \cdots \leq n_k < 0\), \(m_1 \leq \cdots \leq m_\ell < 0\) \((k, \ell \geq 0)\), and \(f \in \mathbb{C}[b^*_0, \ldots, b^*_d]\). The vacuum is 1. The vertex operators of the elements \(a_{i,-1}\) and \(b^*_0\) are respectively

\[
a_i(z) = \sum_{n \in \mathbb{Z}} a_{i,n} z^{-n-1} \quad \text{and} \quad b^i(z) = \sum_{n \in \mathbb{Z}} b^i_n z^{-n}.
\]

The axioms of a vertex algebra then determine all other vertex operators as well as the infinitesimal translation operator. The conformal element is

\[
a_{1,-1} b^*_{-1} + \cdots + a_{d,-1} b^*_{-1}
\]

(2.2)

with central charge 2\(d\). The weight zero subspace is

\[
\mathcal{D}^ch_{\text{alg}}(\mathbb{C}^d)_0 = \mathbb{C}[b^*_0, \ldots, b^*_d] = \{\text{algebraic functions on } \mathbb{C}^d\}
\]

and the operators \(a_{i,n}\) and \(b^*_n\) change weight by \(-n\). In particular, the element (2.1) has weight \(-n_1 - \cdots - n_k - m_1 - \cdots - m_\ell\).

\subsection{2.3}

According to \S\ 3.4, there are ‘holomorphic analogues’ of \(\mathcal{D}^ch_{\text{alg}}(\mathbb{C}^d)\). More precisely, for each open set \(U \subset \mathbb{C}^d\), there is a conformal vertex algebra \(\mathcal{D}^ch(U)\) whose weight zero subspace is

\[
\mathcal{D}^ch(U)_0 = \mathcal{O}(U) = \{\text{holomorphic functions on } U\}
\]

and whose elements are linear combinations of those of the form (2.1) but with \(f \in \mathcal{O}(U)\). This defines a sheaf of conformal vertex algebras \(\mathcal{D}^ch\) over \(\mathbb{C}^d\).
2.4. Let \( \varphi = (\varphi^1, \ldots, \varphi^d) : U \to V \) be a biholomorphism between two open sets in \( \mathbb{C}^d \) and \( g_\varphi \) its matrix derivative, i.e. \((g_\varphi)^j_i = \partial_j \varphi^i\). Define a matrix-valued 1-form and a (scalar) 3-form as follows
\[
\theta_\varphi = g_\varphi^{-1} \cdot \partial g_\varphi, \quad WZ_\varphi = \frac{1}{3} \text{Tr} (\theta_\varphi \wedge \theta_\varphi \wedge \theta_\varphi).
\]
Notice that \( \partial \theta_\varphi = -\theta_\varphi \wedge \theta_\varphi \) and \( \partial WZ_\varphi = 0 \).

Consider the problem of finding all isomorphisms of conformal vertex algebras \( \mathcal{D}^{ch}(V) \to \mathcal{D}^{ch}(U) \) whose weight-zero component is \( \varphi^* : \mathcal{O}(V) \to \mathcal{O}(U) \), i.e. \( b_0^i \mapsto \varphi^i \). Such an isomorphism is determined by the images of the weight-one elements \( a_i, \ldots, a_{i-1} \). The result is stated below. Firstly, we obtain an isomorphism of vertex algebras (not necessarily preserving the conformal elements) if and only if \( a_i, \ldots, a_{i-1} \) are sent to
\[
a_{j,1}(g_\varphi^{-1})^j_i + \frac{1}{2} b_{i-1} \left\{ \xi_{jk} + \text{Tr} [\theta_\varphi(\partial_j) \cdot \theta_\varphi(\partial_k)] \right\} (g_\varphi^{-1})^k_i \quad (2.3)
\]
where \( \xi_{ij} \in \mathcal{O}(U) \), \( \xi_{ji} = -\xi_{ij} \) and the holomorphic 2-form \( \xi = \sum_{i < j} \xi_{ij} db_0^i \wedge db_0^j \) satisfies the equation
\[
\partial \xi = WZ_\varphi.
\]
According to the Poincaré lemma for the \( \partial \)-operator, since \( \partial WZ_\varphi = 0 \), such \( \xi \) exists after possibly replacing \( U \) and \( V \) by smaller open sets. Label this isomorphism by
\[
\varphi_\xi^* : \mathcal{D}^{ch}(V) \to \mathcal{D}^{ch}(U).
\]
Secondly, \( \varphi_\xi^* \) preserves the conformal element (2.2) if and only if
\[
\text{Tr} \theta_\varphi = 0.
\]

2.5. Given isomorphisms \( (\varphi_1)_\xi^* : \mathcal{D}^{ch}(V) \to \mathcal{D}^{ch}(U) \) and \( (\varphi_2)_\xi^* : \mathcal{D}^{ch}(W) \to \mathcal{D}^{ch}(V) \) of vertex algebras, they compose as follows
\[
(\varphi_1)_\xi^* \circ (\varphi_2)_\xi^* = (\varphi_2 \varphi_1)_\xi^*, \quad \eta = \xi_1 + \varphi_1^* \xi_2 + \sigma_{\varphi_2, \varphi_1}
\]
where \( \sigma_{\varphi_2, \varphi_1} = \text{Tr} (\theta_1 \wedge g_\varphi^{-1} \cdot \theta_2 g_\varphi) \) is a 2-form on \( U \).

2.6. Let \( M^d \) be a compact, complex manifold. Choose a finite good (in the sense of [BT82]) open cover \( \mathcal{U} = \{U_1, \ldots, U_N\} \) of \( M \) with holomorphic coordinate charts \( \varphi_\alpha : U_\alpha \to \mathbb{C}^d, \alpha = 1, \ldots, N \). Let
\[
W_\alpha = \varphi_\alpha(U_\alpha), \quad W_{\alpha \beta} = \varphi_\alpha(U_\alpha \cap U_\beta) \quad \text{and} \quad W_{\alpha \beta \gamma} = \varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)
\]
whenever \( U_\alpha \cap U_\beta \neq \emptyset \) and \( U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \). For the coordinate transformations
\[
\varphi_{\beta \alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : W_{\alpha \beta} \to W_{\beta \alpha}
\]
we shall write \( g_{\beta \alpha}, \theta_{\beta \alpha}, WZ_{\beta \alpha}, \sigma_{\gamma \beta \alpha} \) respectively for \( g_{\varphi_{\beta \alpha}}, \theta_{\varphi_{\beta \alpha}}, W_{\varphi_{\beta \alpha}}, \sigma_{\varphi_{\gamma \beta \alpha}} \).

Assign to each \( U_\alpha \) the sheaf \( \mathcal{D}^{ch}(W_{\alpha \beta}) \). To glue these into a sheaf of conformal vertex algebras over \( M \), it follows from §2.4 that we need to choose on each \( W_{\alpha \beta} \) a holomorphic solution \( \xi_{\beta \alpha} \) to
\[
d\xi_{\beta \alpha} = \partial \xi_{\beta \alpha} = WZ_{\beta \alpha}
\]
in order to define an isomorphism \( (\varphi_{\beta \alpha})_\xi^* : \mathcal{D}^{ch}(W_{\beta \alpha}) \to \mathcal{D}^{ch}(W_{\alpha \beta}) \) of vertex algebras. This is possible, perhaps after a refinement of \( \mathcal{U} \). The following conditions must be satisfied:
\[
\varphi_{\beta \alpha}^* \xi_{\gamma \beta} - \xi_{\gamma \alpha} + \xi_{\beta \alpha} + \sigma_{\gamma \beta \alpha} = 0 \quad \text{on each } W_{\alpha \beta \gamma}
\]
\[
\text{Tr} \theta_{\beta \alpha} = 0 \quad \text{on each } W_{\alpha \beta}
\]
Indeed, (2.6) is equivalent to the cocycle condition \((\varphi_{\gamma\alpha})^* = (\varphi_{\beta\alpha})^* \circ (\varphi_{\gamma\beta})^*\) in view of (2.4), and
(2.7) ensures that the conformal elements from \(U_\alpha\) agree over intersections, according to the last statement in §2.4. These conditions allow us to construct a sheaf of conformal vertex algebras on \(M\) that restricts to \(D^\text{ch}|_{U_\alpha}\) on each \(U_\alpha\). Denote this sheaf by \(\mathcal{D}^\text{ch}_{M,\xi}\) where \(\xi\) stands for the Čech cochain \(\{\xi_{\beta\alpha}\}\).  

2.7. Denote by \(\Omega^p_M\) the sheaf of holomorphic \(p\)-forms on \(M\) and \(\Omega^p_{M,c}\) its subsheaf of closed forms. Suppose \(\xi = \{\xi_{\beta\alpha}\}\) satisfying (2.5) have been chosen but (2.6)-(2.7) do not necessarily hold.

Consider the Čech cochain
\[
\{\varphi^\beta_{\alpha} \xi_{\gamma\beta} - \xi_{\gamma\alpha} + \xi_{\beta\alpha} + \sigma_{\gamma\beta\alpha}\}_{\alpha\beta\gamma} \in \check{C}^2(\mathfrak{U}, \Omega^2_{M,c}).
\] (2.8)

Indeed, (2.8) consists of closed forms in view of (2.5) and (2.4). It follows from the associativity of (2.4) that (2.8) is a cocycle. Keeping the coordinate charts \(\varphi_{\alpha}\) fixed, one can obtain the equations in (2.6) after a modification of \(\xi\) if and only if (2.8) is a coboundary of closed 2-forms. Therefore the obstruction to constructing a sheaf of (not necessarily conformal) vertex algebras \(\mathcal{D}^\text{ch}_{M,\xi}\) as described in §2.6 is the class of (2.8) in \(H^2(\Omega^2_{M,c})\). This class does not depend on \(\xi\).

Now consider the Čech cochain
\[
\{\text{Tr } \theta_{\beta\alpha}\}_{\alpha\beta} \in \check{C}^1(\mathfrak{U}, \Omega^1_{M,c}).
\] (2.9)

It is easy to verify that (2.9) indeed consists of closed forms and is a cocycle. Furthermore, one can obtain the equations in (2.7) after a modification of the coordinate charts \(\varphi_{\alpha}\) if and only if (2.9) is a coboundary of closed 1-forms. Therefore given a sheaf of vertex algebras \(\mathcal{D}^\text{ch}_{M,\xi}\) as constructed above, its obstruction to having a global conformal element is the class of (2.9) in \(H^1(\Omega^1_{M,c})\).

2.8. Denote by \(E^p_M\) the sheaf of smooth \(p\)-forms on \(M\) and \(E^p_{M,c}\) its subsheaf of closed forms. Choose a connection on \(TM\) and let \(\Gamma_\alpha\) be the connection 1-form on \(W_\alpha\) associated to the coordinate chart \(\varphi_\alpha\). The curvature operator is locally given by \(R_\alpha = d\Gamma_\alpha + \Gamma_\alpha \wedge \Gamma_\alpha\). Define a 3-form on \(W_\alpha\) by
\[
CS(\Gamma_\alpha) = \text{Tr } (\Gamma_\alpha \wedge R_\alpha) - \frac{1}{3} \text{Tr } (\Gamma_\alpha \wedge \Gamma_\alpha \wedge \Gamma_\alpha).
\]

Consider the computation in the Čech-de Rham double complex \(\check{C}^*(\mathfrak{U}, E^*_M)\) shown in figure 2: it shows that (2.8) as an element of \(\check{C}^2(\mathfrak{U}, E^2_M)\) is cohomologous to \(\{-\text{Tr } (R_\alpha \wedge R_\alpha)\}_\alpha \in \check{C}^0(\mathfrak{U}, E^1_M)\) in the total Čech-de Rham complex. By Chern-Weil theory, the latter represents \(8\pi^2 ch_2(M)\). In other words, the obstruction class represented by (2.8) maps to \(8\pi^2 ch_2(M)\) under the following composition
\[
H^2(\Omega^2_{M,c}) \overset{\approx}{\longrightarrow} H^2(E^2_{M,c}) \overset{\approx}{\longrightarrow} H^4(M; \mathbb{C})
\]
where the first arrow is induced by an inclusion of sheaves and the second by the inclusion of the Čech complex valued in \(E^2_{M,c}\) into the total Čech-de Rham complex.

Similarly, the computation in figure 3 shows that (2.9) as an element of \(\check{C}^1(\mathfrak{U}, E^1_M)\) is cohomologous to \(\{-\text{Tr } R_\alpha\}_\alpha \in \check{C}^0(\mathfrak{U}, E^2_M)\). The latter represents \(2\pi i ch_1(M)\). In other words, the obstruction class represented by (2.9) maps to \(2\pi i ch_1(M)\) under the following composition
\[
H^1(\Omega^1_{M,c}) \overset{\approx}{\longrightarrow} H^1(E^1_{M,c}) \overset{\approx}{\longrightarrow} H^2(M; \mathbb{C})
\]
where the first arrow is again induced by an inclusion of sheaves and the second by the inclusion of the Čech complex valued in \(E^1_{M,c}\) into the total Čech-de Rham complex.

\footnote{For each open set \(U \subset M\), an element of \(\mathcal{D}^\text{ch}_{M,\xi}(U)\) is a collection of elements of \(\mathcal{D}^\text{ch}(\varphi_\alpha(U \cap U_\alpha))\) that agree over intersections via \((\varphi_{\beta\alpha})^*\). The finiteness of \(\mathfrak{U}\) ensures that \(\mathcal{D}^\text{ch}_{M,\xi}(U)\) satisfies all the axioms of a vertex algebra.}

\footnote{The total differential of the Čech-de Rham complex is \(\delta + (-1)^p d\), where \(\delta\) is the Čech differential, \(d\) the de Rham differential and \(p\) the Čech degree.
2.9. Assume that both obstructions discussed in §2.7 vanish. The cochains $\xi$ satisfying (2.5) and (2.6) form an affine space modeled on the space of cocycles in $\check{C}^1(\mathcal{U}, \Omega^2_{M, \text{cl}})$. Given two such cochains $\xi, \xi'$, the sheaves of conformal vertex algebras $\mathcal{D}^{\text{ch}}_{M, \xi}, \mathcal{D}^{\text{ch}}_{M, \xi'}$ are isomorphic over $M$ if and only if there exist closed holomorphic 2-forms $\delta_{\alpha}$ on $W_{\alpha}$ such that all diagrams of the form

\[
\begin{array}{ccc}
\mathcal{D}^{\text{ch}}(W_{\beta\alpha}) & \xrightarrow{\text{id}} & \mathcal{D}^{\text{ch}}(W_{\beta\alpha}) \\
\mathcal{D}^{\text{ch}}(W_{\alpha\beta}) & \xrightarrow{\text{id}_{\beta\alpha}} & \mathcal{D}^{\text{ch}}(W_{\alpha\beta}) \\
\end{array}
\]

commute. By (2.4), this is equivalent to the equations

\[\xi'_{\beta\alpha} - \xi_{\beta\alpha} = \xi_{\alpha} - \varphi_{\beta\alpha}^* \xi_{\beta}\]

Namely, $\xi, \xi'$ differ by a coboundary in $\check{C}^1(\mathcal{U}, \Omega^2_{M, \text{cl}})$. Therefore the isomorphism classes of sheaves of conformal vertex algebras $\mathcal{D}^{\text{ch}}_{M, \xi}$ form an affine space modeled on $\oplus_{\beta} H^1(\Omega^2_{M, \text{cl}})$. This does not depend on the open cover $\mathcal{U}$ or the coordinate charts $\varphi_{\alpha}$.

2.10. Let $(\mathcal{D}^{\text{ch}}_{M, \xi})_k$ be the weight-$k$ subsheaf of $\mathcal{D}^{\text{ch}}_{M, \xi}$, $k \geq 0$. At weight zero, we have

\[\mathcal{D}^{\text{ch}}_{M, \xi}(U_{\alpha}) = \mathcal{O}_M\]  (2.10)

Denote by $\mathcal{T}_M$ the sheaf of holomorphic vector fields on $M$. There is a short exact sequence

\[
0 \longrightarrow \Omega^1_M \longrightarrow (\mathcal{D}^{\text{ch}}_{M, \xi})_1 \longrightarrow \mathcal{T}_M \longrightarrow 0 \tag{2.11}
\]

The inclusion of $\Omega^1_M$ sends the 1-forms $d\varphi_{\alpha}^i$ on $U_{\alpha}$ to $b^i_{\alpha} \in \mathcal{D}^{\text{ch}}(W_{\alpha}) = \mathcal{D}^{\text{ch}}_{M, \xi}(U_{\alpha})$. The projection to $\mathcal{T}_M$ sends $a_{i,-1} \in \mathcal{D}^{\text{ch}}_{M, \xi}(U_{\alpha})$ to the vector fields $\partial / \partial \varphi_{\alpha}^i$ on $U_{\alpha}$. The way $a_{i,-1}$ transform between different coordinate neighborhoods, as described in (2.3), indicates that the short exact sequence (2.11) does not
split and \((D_{M,\xi}^{ch})_{1}\) is not an \(O_M\)-module. In general, each \((D_{M,\xi}^{ch})_{k}\) admits a filtration whose associated graded space is an \(O_M\)-module.\(^1\) For example, \((D_{M,\xi}^{ch})_{2}\) can be filtered as follows, with the indicated successive quotients

\[
\begin{array}{cccccccc}
0 & \text{Sym}^2\Omega^1_M & F^1 & \Omega^1_M & \mathcal{T}_M \oplus \Omega^1_M & F^2 & \mathcal{T}_M & F^3 & \text{Sym}^3\mathcal{T}_M & (D_{M,\xi}^{ch})_{2} \\
\end{array}
\quad \quad \text{(2.12)}
\]

2.11. The cohomology of a sheaf of conformal vertex algebras such as \(D_{M,\xi}^{ch}\) is a conformal vertex super-algebra with the same central charge. The character of \(H^* (D_{M,\xi}^{ch})\) is given by

\[
\text{char } H^* (D_{M,\xi}^{ch}) = q^{-d/12} \sum_{k=0}^{\infty} q^k \chi ((D_{M,\xi}^{ch})_k)
\]

where \(\chi (-)\) denotes the Euler characteristic. Because of (2.10) and (2.11), we have

\[
\chi ((D_{M,\xi}^{ch})_0) = \chi (O_M), \quad \chi ((D_{M,\xi}^{ch})_1) = \chi (\Omega^1_M) + \chi (\mathcal{T}_M).
\]

In general, filtrations like (2.12) allow us to express \(\chi ((D_{M,\xi}^{ch})_k)\) in terms of the Euler characteristics of \(O_M\)-modules. Using the Hirzebruch-Riemann-Roch formula, we obtain

\[
\text{char } H^* (D_{M,\xi}^{ch}) = q^{-d/12} \chi \left( \bigotimes_{i=1}^{\infty} \text{Sym}^i (\Omega^1_M \oplus \mathcal{T}_M) \right) = \int_M e^{2 \chi (M) \frac{W(TM)}{\eta(q)^{2d}}}
\]

where \(\text{Sym}^i (-) = \sum_{n=0}^{\infty} t^n \text{Sym}^n (-)\) and \(\eta(q)\) is the Dedekind \(\eta\)-function. According to \(\S 2.8\), \(\chi (M) = 0\) and hence

\[
\text{char } H^* (D_{M,\xi}^{ch}) = \frac{W(M)}{\eta(q)^{2d}}.
\quad \text{(2.13)}
\]

To summarize, given a compact complex manifold \(M\), if the obstruction classes described in \(\S 2.7\) vanish, each sheaf of conformal vertex algebras \(D_{M,\xi}^{ch}\) provides a geometric interpretation of the Witten genus \(W(M)\).

\section{Differential Graded Vertex Algebroids}

The sheaf of vertex algebras \(D_{M,\xi}^{ch}\) is not an \(O_M\)-module (\(\S 2.10\)), but in this section we reorganize its data in terms of \(O_M\)-modules using the notion of vertex algebroids introduced in [GMS04].

3.1. Given a vertex algebra \(V = \bigoplus_{k \geq 0} V_k\), the part of its structure involving only \(V_0\) and \(V_1\) consists of the following element and maps

\[
1 \in V_0, \quad \partial : V_0 \to V_1, \quad (i+j-k-1) : V_i \times V_j \to V_k \quad \text{for } i, j, k = 0 \text{ or } 1
\quad \text{(3.1)}
\]

satisfying a number of identities. Consider \((-1) : V_0 \times V_0 \to V_0\) and \((-1) : V_0 \times V_1 \to V_1\) for example. The first map makes \(A := V_0\) a commutative ring with unit 1. The second map does not make \(V_1\) an \(A\)-module but induces \(A\)-module structures on

\[
\Omega := A \cdot (-1) (\partial A) \subset V_1 \quad \text{and} \quad \mathcal{T} := V_1 / \Omega.
\]

\(^1\)The weight-\(k\) monomials in \(a_{i-1}, b_{i-1}\) admit a partial ordering with the property that each isomorphism \(\varphi_E^i\) sends a monomial to a sum of monomials of the same or lower order. Therefore the weight-\(k\) monomials from all \(D_{M,\xi}^{ch}(U_a)\) equal to or lower than a particular order span a subsheaf. These subsheaves form the described filtration of \((D_{M,\xi}^{ch})_k\).
Choose a splitting \( s : \mathcal{T} \to V_1 \) of the projection so as to obtain an identification of vector spaces
\[
\Omega \oplus \mathcal{T} \cong V_1, \quad (\alpha, X) \mapsto \alpha + s(X).
\] (3.2)
In terms of this identification, \((-1) : A \times V_1 \to V_1\) takes the form
\[
f_{(-1)}(\alpha, X) = (f\alpha + f \ast X, fX) \quad \text{where} \quad f \ast X := f_{(-1)}(s(X) - s(fX)).
\] (3.3)
Similarly we can rephrase the other maps in (3.1) and the identities they satisfy in terms of \( A, \Omega \) and \( \mathcal{T} \). The resulting data fall into two types. The first type of data are independent of \( s\):
- \((A, 1)\) is a commutative \( \mathbb{C}\)-algebra with unit and \( \Omega, \mathcal{T}\) are \( A\)-modules;
- there is an \( A\)-derivation \( \partial : A \to \Omega \) whose image generates \( \Omega \) as an \( A\)-module;
- \( \mathcal{T} \) is also a Lie algebra (with Lie bracket \([ \ ]\));
- there is an \( A\)-linear Lie algebra homomorphism \( \mathcal{T} \to \text{End } A\) (denoted \( X \mapsto X\));
- there is a \( \mathbb{C}\)-linear Lie algebra homomorphism \( \mathcal{T} \to \text{End } \Omega\) (denoted \( X \mapsto L_X\));
- the \( \mathcal{T}\)-actions on \( A \) and \( \Omega \) are \( \partial\)-equivariant;
- the \( \mathcal{T}\)-actions on \( A, \Omega \) and \( \mathcal{T}\) (via \( X \mapsto [X, -]\)) satisfy the Leibniz rule with respect to \( A\)-multiplications;
- there is an \( A\)-bilinear pairing \( \langle \cdot, \cdot \rangle : \Omega \times \mathcal{T} \to A \) satisfying \( \langle \partial f, X \rangle = Xf \).

The second type of data include three \( \mathbb{C}\)-bilinear maps depending on \( s\)
\[
* : A \times \mathcal{T} \to \Omega, \quad \{ \}_{0} : \mathcal{T} \times \mathcal{T} \to A, \quad \{ \}_{1} : \mathcal{T} \times \mathcal{T} \to \Omega,
\] (3.5)
and satisfying the following identities
\[
\{X,Y\}_0 = \{Y,X\}_0 \quad \partial\{X,Y\}_0 = \{X,Y\}_1 + \{Y,X\}_1 \\
(fg)*X - f*(gX) - f(g*X) = -(fX)\partial g -(Xf)\partial f \\
\{X,fY\}_0 = f\{X,Y\}_0 - \{f*Y,X\} - XYf \\
\{X,fY\}_1 = f\{X,Y\}_1 - L_X(f*Y) + (Xf)f*Y + f*[X,Y] \\
X\{Y,Z\}_0 - \{\{X,Y\}_1, Z\} - \{X,\{Y,Z\}_0\} = (\{X,Y\}_1, Z) + (\{X,Z\}_1, Y) \\
L_X\{Y,Z\}_1 - L_Y\{X,Z\}_0 + \{X,\{Y,Z\}_1\} - \{Y,\{X,Z\}_0\} - \{\{X,Y\}, Z\}_1 \\
= \partial(\{X,Y\}, Z)_1
\] (3.6)
for \( f, g \in A \) and \( X, Y, Z \in \mathcal{T} \).

In general, any triple \((A, \Omega, \mathcal{T})\) as in (3.4) is called an extended Lie algebroid. The entire collection of data in (3.4)-(3.6) is called a vertex algebroid, denoted as \((A, \Omega, \mathcal{T}, *, \{ \}_0, \{ \}_1)\). According to the above discussion, every vertex algebra (equipped with a ‘splitting’) gives rise to a vertex algebroid.

3.2. Consider a homomorphism of vertex algebras \( \Phi : V \to V' \). Suppose \((A, \Omega, \mathcal{T}, *, \{ \}_0, \{ \}_1)\) and \((A', \Omega', \mathcal{T}', *, '\{ \}_0', '\{ \}_1')\) are the vertex algebroids associated to \( V \) and \( V' \) with respect to some splittings \( s : \mathcal{T} \to V_1 \) and \( s' : \mathcal{T}' \to V_1' \). Now \( \Phi \) induces in the obvious way a map of triples
\[
\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')
\] (3.7)
and, in terms of the identification (3.2), \( \Phi|_{V_1} : V_1 \to V_1' \) takes the form
\[
\Phi(\alpha, X) = (\varphi\alpha + \Delta(X), \varphi X) \quad \text{where} \quad \Delta(X) := \Phi s(X) - s'(\varphi X).
\] (3.8)
While (3.7) respects the extended Lie algebroid structures on the nose, it only respects the rest of the vertex algebroid structures up to terms linear in \( \Delta \), namely
\[
\varphi f \ast' \varphi X = \varphi(f \ast X) + \Delta(fX) - (\varphi f)\Delta(X) \\
\{\varphi X, \varphi Y\}_0 = \varphi\{X,Y\}_0 - \{\Delta(X), \varphi Y\} - \{\Delta(Y), \varphi X\} \\
\{\varphi X, \varphi Y\}_1 = \varphi\{X,Y\}_1 - L_{\varphi X}\Delta(Y) + L_{\varphi Y}\Delta(X) - \partial(\Delta(X), \varphi Y) + \Delta([X,Y])
\] (3.9)
\footnote{The last two maps are defined by \( \{X,Y\}_0 = s(X)_{(1)} s(Y) \) and \( \{X,Y\}_1 = s(X)_{(0)} s(Y) - s([X,Y]). \)
for \( f \in A \) and \( X, Y \in T \).

The discussion above motivates us to define a \textit{morphism of vertex algebroids} as a pair
\[
(\varphi, \Delta) : (A, \Omega, T, \ast, \{ \}, \{ \}) \rightarrow (A', \Omega', T', \ast', \{ \}, \{ \})
\]
consisting of a map of triples as in (3.7) that respects the extended Lie algebroid structures, as well as a map \( \Delta : T \rightarrow \Omega' \) that satisfies (3.9). Composition of morphisms is given by
\[
(\varphi', \Delta') \circ (\varphi, \Delta) = (\varphi' \varphi, \varphi' \Delta + \Delta' \varphi).
\]

Together with \S\,3.1, this completes the description of the category of vertex algebroids \( \void \) and a forgetful functor \( G : \void' \rightarrow \void \), where \( \void' \) is the category of vertex algebras (equipped with ‘splitting’). \footnote{A morphism in \( \void' \) between two objects \( V, V' \) is simply a vertex algebra homomorphism between them.}

### 3.3. The forgetful functor \( G : \void' \rightarrow \void \)

The forgetful functor \( G : \void' \rightarrow \void \) has a left adjoint \( F : \void \rightarrow \void' \). The composition \( GF \) is naturally isomorphic to the identity on \( \void \). Given a vertex algebroid \( (A, \Omega, T, \ast, \{ \}, \{ \}) \), let us briefly describe its \textit{freely generated vertex algebra} \( V = F(A, \Omega, T, \ast, \{ \}, \{ \}) \).

The weight-zero and -one components of \( V \) are
\[
V_0 = A, \quad V_1 = \Omega \oplus T.
\]

Throughout this discussion, we always have \( f, g \in A, \alpha \in \Omega \) and \( X, Y \in T \), regarded as elements of \( V \). For any \( u \in V \), denote its vertex operator by \( u(z) \). In particular, \( 1(z) = \text{id} \) and \( (\partial f)(z) = \partial_z f(z) \). The vertex operators of \( A \) and \( \Omega \) commute among themselves and with each other. Normally ordered products with \( f(z) \) are given by
\[
f(z)g(z) = (fg)(z), \quad f(z)\alpha(z) = (f\alpha)(z), \quad f(z)X(z) = (fX)(z) + (f \ast X)(z).
\]

On the other hand, OPEs with \( X(z) \) have the following singular parts
\[
\begin{align*}
X(z)f(w) & \sim \frac{(f X)(w)}{z - w} \\
X(z)\alpha(w) & \sim \frac{(\alpha X)(w)}{(z - w)^2} + \frac{(L \alpha)(w)}{z - w} \\
X(z)Y(w) & \sim \frac{\{X, Y\}_0(w)}{(z - w)^2} + \frac{\{X, Y\}_1(w)}{z - w}
\end{align*}
\]

The vertex algebroid axioms guarantee the consistency of the above relations, i.e. their combinations do not lead to any nontrivial constraints. Subject to these relations, the fields \( f(z), \alpha(z), X(z) \) generate \( V \) as a vertex algebra.

### 3.4. Consider the vertex algebra \( \mathcal{D}_{\text{alg}}(\mathbb{C}^d) \) defined in \S\,2.2. Its induced extended Lie algebroid is
\[
\{ \text{algebraic functions on } \mathbb{C}^d \}, \{ \text{algebraic } 1\text{-forms on } \mathbb{C}^d \}, \{ \text{algebraic vector fields on } \mathbb{C}^d \}
\]
equipped with the usual differential on functions, Lie bracket on vector fields, Lie derivation of functions and 1-forms by vector fields, and pairing between 1-forms and vector fields. Choose the splitting
\[
s : \{ \text{algebraic vector fields on } \mathbb{C}^d \} \rightarrow \mathcal{D}_{\text{alg}}(\mathbb{C}^d)_1, \quad X \mapsto a_i X^i
\]
where \( X^i \) are the components of \( X \), i.e. \( X = X^i \partial_i \). The three maps in (3.5) are then given by
\[
f \ast X = -X^i \partial_i f \db, \quad \{X, Y\}_0 = -(\partial_i X^i)(\partial_j Y^j), \quad \{X, Y\}_1 = -(\partial_i \partial_j X^i)(\partial_k Y^j) \db^k.
\]
This vertex algebroid freely generates $\mathcal{D}^\text{ch}_{\text{alg}}(\mathbb{C}^d)$. Now if we replace the algebraic functions, 1-forms and vector fields on $\mathbb{C}^d$ by holomorphic ones on an open set $U \subset \mathbb{C}^d$, the same formulae in (3.11) define a new vertex algebroid. Denote its freely generated vertex algebra by $\mathcal{D}^\text{ch}(U)$.

Consider an isomorphism of vertex algebras $\varphi^*_\xi : \mathcal{D}^\text{ch}(V) \to \mathcal{D}^\text{ch}(U)$ as described in §2.4. Its induced isomorphism of vertex algebroids is given by

\[ (\varphi^*, \Delta_{\varphi, \xi}) : (O(V), \Omega^1(V), T(V), \ast, \{ \}^0_0, \{ \}^1_1) \to (O(U), \Omega^1(U), T(U), \ast, \{ \}^0_0, \{ \}^1_1) \]  \hspace{1cm} (3.12)

where $\varphi^*$ denotes pullback along a biholomorphism $\varphi : U \to V$ and $\Delta_{\varphi, \xi}$ sends $X \in T(V)$ to

\[ \Delta_{\varphi, \xi}(X) = -\partial_i(\varphi^* X)^j (\theta^j)_i - \frac{1}{2} \text{Tr} [\theta_{\varphi}(\varphi^*X) \cdot \theta_{\varphi}] - \frac{1}{2} \xi \varphi^*X \xi \in \Omega^1(U). \]  \hspace{1cm} (3.13)

Recall the notations in §2.4.

3.5. Given a presheaf of vertex algebras $V = \bigoplus_{k \geq 0} V_k$, the part of its structure involving only $V_0$ and $V_1$ is equivalent to the following set of data: For each open set $U$, there is a vertex algebroid

\[ (A(U), \Omega(U), T(U), \ast_U, \{ \}^0_U, \{ \}^1_U) \]  \hspace{1cm} (3.14)

where a splitting $s_U : T(U) \to V_1(U)$ has been chosen to obtain an identification of vector spaces

\[ \Omega(U) \oplus T(U) \cong V_1(U), \quad (\alpha, X) \mapsto \alpha + s_U(X) \]

and define $\ast_U, \{ \}^0_U, \{ \}^1_U$ e.g. as in (3.3). Given $V \subset U$, there is a morphism of vertex algebroids

\[ (\varphi_{V,U}, \Delta_{V,U}) : (A(U), \Omega(U), T(U), \ast_U) \to (A(V), \Omega(V), T(V), \ast) \]  \hspace{1cm} (3.15)

where $\Delta_{V,U} : T(U) \to \Omega(V)$ is defined in terms of $s_U$, $s_V$ as in (3.8). Given $W \subset V \subset U$, we have

\[ \varphi_{W,U} = \varphi_{W,V} \varphi_{V,U}, \quad \Delta_{W,U} = \varphi_{W,V} \Delta_{V,U} + \Delta_{W,V} \varphi_{V,U} \]  \hspace{1cm} (3.16)

according to the composition law (3.10). In general, any collection of data as described in (3.14)-(3.16) is called a presheaf of vertex algebroids.

The structures of the extended Lie algebroids $(A(U), \Omega(U), T(U))$ are respected on the nose by the restriction maps $\varphi_{V,U}$ and hence define morphisms of presheaves. For example, there are morphisms of presheaves of $C$-vector spaces

\[ A \times A \to A, \quad A \times \Omega \to \Omega, \quad A \times T \to T \]  \hspace{1cm} (3.17)

making $A$ a presheaf of unital commutative $C$-algebras and $\Omega, T$ both $A$-modules. On the other hand, in view of (3.9), the maps $\ast_U, \{ \}^0_U, \{ \}^1_U$ from various $U$ do not collaborate to define morphisms of presheaves; their ‘global’ meanings are not yet clear.

3.6. Before continuing, we introduce some shorthand notations for Čech complexes. Given a presheaf of $C$-vector spaces $S$, let $\check{C}^*(S)$ be the Čech complex of a fixed open cover $\Omega = \{U_\alpha\}_{\alpha \in I}$ valued in $S$.

Subscripts like ‘$0 \cdots p$’ will stand for a nonempty open set of the form $U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$, where $\alpha_0, \ldots, \alpha_p \in I$. For example, given $a \in \check{C}^p(S)$, the notation ‘$a_{0 \cdots p}$’ refers to the value of $a$ on some $U_{\alpha_0 \cdots \alpha_p}$. Bilinear maps between Čech complexes will always be defined via the following Eilenberg-Zilber map $\otimes_{ex} : \check{C}^*(S) \otimes \check{C}^*(T) \to \check{C}^*(S \otimes T)$

\[ (a \otimes_{ex} b)_{0 \cdots p+q} = r^S_{0 \cdots p+q,0 \cdots p}(a_{0 \cdots p}) \otimes r^T_{0 \cdots p+q,0 \cdots p}(b_{0 \cdots p+q}), \quad a \in \check{C}^p(S), \ b \in \check{C}^q(T) \]  \hspace{1cm} (3.18)

where e.g. $r^S_{0 \cdots p+q,0 \cdots p}$ denotes the restriction map in $S$ associated to an inclusion of the form $U_{\alpha_0 \cdots \alpha_p} \subset U_{0 \cdots p+q}$. While (3.18) is strictly associative, it is only graded symmetric up to homotopy. This is why the Čech complexes studied below carry only ‘homotopy versions’ of various algebraic structures.
3.7. Continuing with the discussion in §3.5, we consider the Čech complex $\tilde{C}^*(V)$ valued in a presheaf of vertex algebras $\mathcal{V}$. Denote the restriction homomorphisms in $\mathcal{V}$ by $\Phi_{V,U}$. The local vacua, i.e. the vacua of $\mathcal{V}(U_a)$, constitute a cocycle in $\tilde{C}^0(\mathcal{V})$. The local infinitesimal translation operators form a chain endomorphism of $\tilde{C}^*(\mathcal{V})$. The local vertex operators define a bilinear chain map as follows: Given $a \in C^p(\mathcal{V})$ and $b \in C^q(\mathcal{V})$, let $Y(a,z)b \in \tilde{C}^{p+q}(\mathcal{V})[[z,z^{-1}]]$ be given by
\[
[Y(a,z)b]_{0\ldots p+q} = Y(\Phi_{0\ldots p,q,0\ldots p}(a_0\ldots p),z)(\Phi_{0\ldots p+q,p+q}(b_0\ldots p+q)).
\]

These data satisfy the axioms of a vertex superalgebra except that $Y(a,z)b$ may not be in $\tilde{C}^*(\mathcal{V})(z)$ unless $U$ is finite, and the locality axiom holds only up to homotopy even assuming $U$ finite.

Let us define a dg vertex algebra to be a 4-tuple
\[
(V^*, \delta : V^* \to V^*, Y(\cdot, \cdot), \delta^*: V^* \otimes V^* \to V^*((\cdot, \cdot)))
\]
where each $V_k^*$ is a cochain complex, $1$ is a cocycle, $\delta$ and $Y(\cdot, \cdot)$ are degree-preserving chain maps, such that the data form a vertex superalgebra. Given a presheaf of vertex algebras $\mathcal{V}$, the Čech complex $\tilde{C}^*(\mathcal{V})$ in the case of a finite $U$ is an example of a homotopy version of a dg vertex algebra.

3.8. Given a dg vertex algebra $V^* = \bigoplus_{k \geq 0} V_k^*$, we reorganize the part of its structure involving only $V_0^*$ and $V_1^*$. Regarded simply as a vertex superalgebra, $V^*$ induces a vertex superalgebroid
\[
(A^*, \Omega^*, \mathcal{T}^*, *, \{ \}_{0}, \{ \}_{1})
\]
where $A^* := V_0^*$, $\Omega^*$, $\mathcal{T}^*$ are now cochain complexes and a degree-preserving splitting $s : \mathcal{T}^* \to V_1^*$ has been chosen to obtain an identification of graded vector spaces
\[
\Omega^* \oplus \mathcal{T}^* \cong V_1^*, \hspace{1cm} (\alpha, X) \mapsto \alpha + s(X).
\]

The structure maps of the extended Lie superalgebroid $(A^*, \Omega^*, \mathcal{T}^*)$ are all chain maps. Let us call such an object a dg extended Lie algebroid. In terms of (3.19), the differential $d : V_1^* \to V_1^*$ takes the form
\[
d(\alpha, X) = (d\alpha + \Delta(X), dX) \hspace{1cm} \text{where} \hspace{1cm} \Delta(X) := ds(X) - s(dX).
\]
Since $d^2 = 0$, we have $d\Delta + \Delta d = 0$. On the other hand, $(-1)^{\mathcal{T}^*} : A^* \otimes V_1^* \to V_1^*$ takes the form
\[
d(-1)(\alpha, X) = (f\alpha + f \ast X, fX) \hspace{1cm} \text{where} \hspace{1cm} f \ast X := f(-1)s(X) - s(fX)
\]
just like (3.3). The fact that $(-1)$ is a chain map translates into the equation (for $f \in A^p$
\[
d(f \ast X) - (df) \ast X = (-1)^p f \ast (dX) = -\Delta(fX) + (-1)^p f\Delta(X)
\]
i.e. $\Delta$ is $A^*$-linear up to the homotopy $\ast$. Similarly $\{ \}_{0}, \{ \}_{1}$ are also defined using $s$ (see footnote 11) and are null homotopies of chain maps constructed from $\Delta$, namely (for $X \in \mathcal{T}^p, Y \in \mathcal{T}^q$
\[
d\{X,Y\}_0 - \{dX,Y\}_0 - (-1)^p\{X,dY\}_0 = \langle \Delta(X), Y \rangle + (-1)^p\langle Y, \Delta(X) \rangle
\]
\[
d\{X,Y\}_1 - \{dX,Y\}_1 - (-1)^p\{X,dY\}_1 = (-1)^pL_X\Delta(Y) - (-1)^{(p+1)}L_Y\Delta(X) + \partial(\Delta(X), Y) - \Delta([X,Y])
\]

Motivated by the above discussion, we define a dg vertex algebroid as a collection of data
\[
(A^*, \Omega^*, \mathcal{T}^*, \Delta : \mathcal{T}^* \to \Omega^{*+1}, *, A^* \otimes \mathcal{T}^* \to \Omega^*, \{ \}_{0} : \mathcal{T}^* \otimes \mathcal{T}^* \to A^*, \{ \}_{1} : \mathcal{T}^* \otimes \mathcal{T}^* \to \Omega^*)
\]
where $(A^*, \Omega^*, \mathcal{T}^*)$ is a dg extended Lie algebroid, $\Delta$ is a degree-one chain map and $*, \{ \}_{0}, \{ \}_{1}$ are degree-preserving maps, such that the data besides $\Delta$ form a vertex superalgebroid and the four maps satisfy (3.22)-(3.24). The above discussion describes how (resp. a homotopy version of) a dg vertex algebra induces (resp. a homotopy version of) a dg vertex algebroid.

---

Footnotes:
13. This is an example of a bilinear map between Čech complexes defined via the Eilenberg-Zilber map (3.18).
14. $V^* = \bigoplus_{k \geq 0} V_k^*$ has two compatible gradings, called ‘degree’ and ‘weight.’ The differential in the chain complex structure has degree one and preserves the weight. The weight is part of the vertex superalgebra structure (§2.1).
15. In fact, $H^*(\Delta)$ are the connecting maps in the cohomology long exact sequence induced by $0 \to \Omega^* \to V_1^* \to \mathcal{T}^* \to 0$. 
3.9. Consider again a presheaf of vertex algebras $\mathcal{V}$ and its induced presheaf of vertex algebroids as described in §3.5. The local sections $s_U$ provide an identification of graded vector spaces

$$\hat{C}^*(\Omega) \oplus \hat{C}^*(\mathcal{T}) \cong \hat{C}^*(\mathcal{V}_1), \quad (\alpha, X) \mapsto \alpha + s(X)$$

where $s(X)_{0:p} = s_{0:p}(X_{0:p})$, $p \geq 0$. Applying the discussion in §3.8 to $V^* = \hat{C}^*(\mathcal{V})$, we obtain a homotopy version of a dg vertex algebroid

$$(\hat{C}^*(\mathcal{A}), \hat{C}^*(\Omega), \hat{C}^*(\mathcal{T}), \Delta, *, \{ \}_0, \{ \}_1).$$

Not surprisingly, this homotopy dg vertex algebroid can be expressed entirely in terms of the presheaf of vertex algebroids, i.e. the data in (3.14)-(3.16). Let us demonstrate this for $\Delta$ and $\ast$. Denote the restriction maps in $\mathcal{V}$ by $\Phi_{V,U}$, and those in $\mathcal{A}$, $\Omega$, $\mathcal{T}$ by $\phi_{V,U}$. Keep in mind the definitions of $\Delta_{V,U}$ and $\ast_U$ according to (3.8) and (3.3). Given $X \in \hat{C}^p(\mathcal{T})$, it follows from (3.20) that

$$\Delta(X)_{0:p+1} = \sum_{i=0}^{p+1} (-1)^i \Phi_{0:p+1,0:\ldots,0:i} \left( s_{0:i} \cdot (X_{0:i} + 1) \right) - s_{0:p+1} \left( \sum_{i=0}^{p+1} (-1)^i \phi_{0:p+1,0:\ldots,0:i} \left( X_{0:i} - 1 \right) \right)$$

$$= \sum_{i=0}^{p+1} (-1)^i \Delta_{0:p+1,0:\ldots,0:i} \left( X_{0:i} - 1 \right). \quad (3.25)$$

Given $f \in \hat{C}^p(\mathcal{A})$ and $X \in \hat{C}^q(\mathcal{T})$, it follows from (3.21) that

$$(f \ast X)_{0:p+q} = \phi_{0:p+q,0:q} (f_{0:p}) (-1) \Phi_{0:p+q,p+q} \left( s_{p+q} (X_{p+q}) \right) - s_{0:p+q} \left( \phi_{0:p+q,0:q} (f_{0:p}) \phi_{0:p+q,p+q} (X_{p+q}) \right)$$

$$= \phi_{0:p+q,0:q} (f_{0:p}) \left( \sum_{i=0}^{p+1} (-1)^i \phi_{0:p+1,0:q} \left( X_{p+q} \right) \right) - \phi_{0:p+q,0:q} (f_{0:p}) \left( \sum_{i=0}^{q+1} (-1)^i \phi_{0:q+1,0:q} \left( X_{p+q} \right) \right)$$

$$= \phi_{0:p+q,0:q} (f_{0:p}) \Delta_{0:p+q,p+q} (X_{p+q}) + \phi_{0:p+q,0:q} (f_{0:p}) \phi_{0:p+q,p+q} (X_{p+q}). \quad (3.26)$$

There are similar formulae for $\{ \}_0$ and $\{ \}_1$. The fact that these formulae define a homotopy dg vertex algebroid can be verified using only the structure of the presheaf of vertex algebroids.

3.10. The discussions in the last few subsections can be summarized in a commutative diagram:

$$\text{presheaf of vertex algebras} \xrightarrow{\text{§3.7}} \text{homotopy dg vertex algebra}$$

$$\downarrow \text{§3.5} \quad \downarrow \text{§3.8} \quad \downarrow \text{§3.9} \quad \downarrow \text{homotopy dg vertex algebroid}$$

---

\[16\] For $X \in \hat{C}^p(\mathcal{T})$, $Y \in \hat{C}^q(\mathcal{T})$, we have

$$\langle (X, Y)_{0:p+q} = \langle \beta, X' \rangle + \langle \alpha, Y'' \rangle + \{ X', Y'' \}_{0:0:p+q} \rangle$$

$$\langle (X, Y)_{1:0:p+q} = L_{X'} \beta - L_{Y''} \alpha + \partial(\alpha, Y'') + \{ X', Y'' \}_{1:0:p+q} \rangle$$

where $X' = f_{0:p+q,0:q}(X_{0:q})$, $Y'' = f_{0:p+q,p+q}(Y_{p+q})$, $\alpha = \Delta_{0:p+q,0:0}(X_{0:q})$, $\beta = \Delta_{0:p+q,p+q}(Y_{p+q})$. 

14
3.11. Consider the sheaf of vertex algebras $\mathcal{D}_{\mathcal{M},\xi}^{ch}$ constructed in §2.6. Let the fixed open cover now be the finite good cover $U = \{U_1, \ldots, U_N\}$ of $M$ used in §2.6. It follows from the goodness of $U$ and the filtrations discussed in §2.10 that $\mathcal{D}_{\mathcal{M},\xi}^{ch}$ is acyclic over the open sets $U_{\alpha_0 \ldots \alpha_p}$. Hence $H^1(\mathcal{D}_{\mathcal{M},\xi}^{ch})$ can be computed from the homotopy dg vertex algebra $\mathcal{C}^*(\mathcal{D}_{\mathcal{M},\xi}^{ch})$. On the other hand, since each $\mathcal{D}_{\mathcal{M},\xi}^{ch}(U_{\alpha_0 \ldots \alpha_p})$ is freely generated by a vertex algebroid, all information of $\mathcal{C}^*(\mathcal{D}_{\mathcal{M},\xi}^{ch})$ can be recovered from the induced homotopy dg vertex algebroid (§3.4, §3.9)

\[
(\mathcal{C}^*(\mathcal{O}_M), \mathcal{C}^*(\Omega^1_M), \mathcal{C}^*(T_M), \Delta, \ast, \{ \hat{1}_0, \hat{1}_1 \}).
\]

(3.27)

This fulfills the goal of reorganizing the data of $\mathcal{D}_{\mathcal{M},\xi}^{ch}$ in terms of $\mathcal{O}_M$-modules.

In order to compute (3.27) explicitly, recall the coordinate charts $\varphi_{\alpha} : U_{\alpha} \to \mathbb{C}^d$ and let $W_{\alpha_0 \ldots \alpha_p} = \varphi_{\alpha_0}(U_{\alpha_0 \ldots \alpha_p})$. In subsequent computations, we will identify

\[
\begin{align*}
\mathcal{D}_{\mathcal{M},\xi}^{ch}(U_{\alpha_0 \ldots \alpha_p}) &\quad \text{with} \quad \mathcal{O}_M(U_{\alpha_0 \ldots \alpha_p}), \\
\Omega^1_M(U_{\alpha_0 \ldots \alpha_p}) &\quad \text{with} \quad \Omega^1(W_{\alpha_0 \ldots \alpha_p}), \\
T_M(U_{\alpha_0 \ldots \alpha_p}) &\quad \text{with} \quad T(W_{\alpha_0 \ldots \alpha_p}).
\end{align*}
\]

(3.28)

For example, the component $X_{0 \ldots p}$ of a Čech cochain $X \in \check{C}^p(T_M)$ will be understood as a holomorphic vector field over $W_{\alpha_0 \ldots \alpha_p}$. Consider the restriction homomorphisms in $\mathcal{D}_{\mathcal{M},\xi}^{ch}$

\[
\Phi_{0 \ldots p, i_0 \ldots i_k} = \left\{ \begin{array}{ll}
(\varphi_{i_0 0})_{i_0 0}^*, & i_0 > 0 \\
\text{inc}_{i_0}, & i_0 = 0
\end{array} \right.
\]

where $0 \leq i_0 < \cdots < i_k \leq p$ and ‘inc’ is an inclusion of open subsets within $\mathbb{C}^d$. According to (3.12), the induced morphism of vertex algebroids is

\[
(\varphi_{0 \ldots p, i_0 \ldots i_k}^*, \Delta_{0 \ldots p, i_0 \ldots i_k}) = \left\{ \begin{array}{ll}
(\varphi_{i_0 0}^*, \Delta_{i_0 0}), & i_0 > 0 \\
(\text{res}_0), & i_0 = 0
\end{array} \right.
\]

(3.29)

where $\Delta_{i_0 0} = \Delta_{\varphi_{i_0 0}, \xi_{i_0 0}}$ is given by (3.13) and res = inc$^*$. In the definition of (3.27), the maps

\[
\hat{\Delta} : \check{C}^*(T_M) \to \check{C}^{*+1}(\Omega^1_M), \quad \hat{\ast} : \check{C}^*(\mathcal{O}_M) \otimes \check{C}^*(T_M) \to \check{C}^*(\Omega^1_M)
\]

are given by (3.25) and (3.26). In view of (3.29), those formulae now read

\[
\hat{\Delta}(X)_{0 \ldots p+1} = \Delta_{10}(X_{1 \ldots p+1})
\]

(3.30)

\[
(f \ast X)_{0 \ldots p+q} = f_{0 \ldots p} \ast \varphi_{p0}^*(X_{p \ldots p+q}) + f_{0 \ldots p} \Delta_{p0}(X_{p \ldots p+q})
\]

(3.31)

where $\ast$ is given by (3.11). The other two maps $\{ \hat{1}_0, \hat{1}_1 \}$ can be computed in a similar way.  

\footnote{It follows from footnote 16 and (3.29) that

\[
\begin{align*}
\langle X, Y \rangle_{0 \ldots p+q} &\equiv \{ X_{0 \ldots p}, \varphi_{p0}^*(Y_{p \ldots p+q}) \}_{0} + \langle \Delta_{p0}(Y_{p \ldots p+q}), X_{0 \ldots p} \rangle \\
\langle X, Y \rangle_{1 \ldots p+q} &\equiv \{ X_{0 \ldots p}, \varphi_{p0}^*(Y_{p \ldots p+q}) \}_{1} + L_{X_{0 \ldots p}}[\Delta_{p0}(Y_{p \ldots p+q})]
\end{align*}
\]

where $\{ \}_{0}, \{ \}_{1}$ (on the right hand sides) are given by (3.11).}

\section{An Infinite Dimensional Dolbeault Complex}

In this section, we construct a sheaf of dg conformal vertex algebras which provides a fine resolution of $\mathcal{D}_{\mathcal{M},\xi}^{ch}$ and whose weight-zero component is the Dolbeault resolution of $(\mathcal{D}_{\mathcal{M},\xi})_0 = \mathcal{O}_M$. 


4.1. Consider a chain homomorphism of dg vertex algebras \( \Phi : (V^*, d) \to (V'^*, d') \). Suppose
\[
(A^*, \Omega^*, T^*, \Delta, *, \{1\}, \{0\}, \{1\}), \quad (A'^*, \Omega'^*, T'^*, \Delta', *, \{1\}, \{0\}, \{1\})
\]
are the dg vertex algebras associated to \( V^* \) and \( V'^* \) (§3.8). Regarded simply as a homomorphism of vertex superalgebras, \( \Phi \) induces a morphism of vertex superalgebroids (see §3.2)
\[
(\varphi, h) : (A^*, \Omega^*, T^*, *, \{1\}, \{0\}, \{1\}) \to (A'^*, \Omega'^*, T'^*, *, \{1\}, \{0\}, \{1\}).
\]
Since \( \Phi \) is a chain map, it follows that \( \varphi \) is also a chain map and, by (3.8) and (3.20)
\[
d' h - h d = \varphi \Delta - \Delta' \varphi \tag{4.1}
\]
i.e. \( \varphi \) respects \( \Delta \) and \( \Delta' \) up to the homotopy \( h \).

In general, define a morphism of dg vertex superalgebroids
\[
(\varphi, h) : (A^*, \Omega^*, T^*, *, \{1\}, \{0\}, \{1\}) \to (A'^*, \Omega'^*, T'^*, *, \{1\}, \{0\}, \{1\})
\]
as a morphism between the underlying vertex superalgebroids such that \( \varphi \) is a chain map and (4.1) is satisfied. It is a (quasi-)isomorphism if \( \varphi \) is a (quasi-)isomorphism.

4.2. Our goal is to construct a dg vertex algebroid quasi-isomorphic to (3.27) but with the Čech complexes replaced by Dolbeault complexes. This will be carried out via Čech-Dolbeault complexes.

Given an \( \mathcal{O}_M \)-module \( \mathcal{M} \), it admits a fine resolution
\[
0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{E}_M \overset{1 \otimes \partial}{\longrightarrow} \mathcal{M} \otimes \mathcal{E}_M^{0,1} \overset{1 \otimes \partial}{\longrightarrow} \mathcal{M} \otimes \mathcal{E}_M^{0,2} \longrightarrow \cdots
\]
where \( \mathcal{E}_M^{p,q} \) denotes the sheaf of smooth \((p, q)\)-forms and tensor products are taken over \( \mathcal{O}_M \). Depending on the context, the Čech-Dolbeault complex \( \hat{C}^*(\mathcal{M} \otimes \mathcal{E}_M^{0,*}) \) will be understood as one of the following: (i) the double complex whose degree-(\(p, q\)) term is \( \hat{C}^p(\mathcal{M} \otimes \mathcal{E}_M^{0,q}) \) and whose differentials are the Čech operator \( \partial \) and the Dolbeault operator \( \bar{\partial} \), or (ii) the total complex of the double complex just described with the differential \( D = \partial + (-1)^p \bar{\partial} \), where \( p \) is the Čech degree. There are quasi-isomorphic embeddings
\[
\hat{C}^*(\mathcal{M}) \hookrightarrow \hat{C}^*(\mathcal{M} \otimes \mathcal{E}_M^{0,*}) \hookleftarrow \Gamma(\mathcal{M} \otimes \mathcal{E}_M^{0,*}) \tag{4.2}
\]
respectively into the degree-(\(0, 0\)) and -(0, \(\ast\)) terms.

Let \( I \) denote a sequence of integers of the form \( 0 \leq i_1 < \cdots < i_q \leq d \) and \( d \bar{b}^j = \bar{d}b^j \wedge \cdots \wedge \bar{d}b^{s_j} \).\(^{18}\)
For any smooth \((0, q)\)-form \( \omega \) on \( \mathbb{C}^d \) (resp. \( U_a \)), let \( \omega_I \) denote its coefficient of \( d \bar{b}^j \) (resp. \( d \bar{b}^j \)), i.e. \( \omega = \omega_I d \bar{b}^j \).

We continue to use the shorthand notations introduced in §3.6 (with the same open cover used in §3.11) and the identifications (3.28).

4.3. In this subsection, we replace (3.27) with a quasi-isomorphic dg vertex algebroid defined using Čech-Dolbeault complexes. First, we have a sheaf of dg extended Lie algebroids \( \mathcal{E}_M \otimes \mathcal{E}_M^{0,*} \) whose derivation is \( \partial \otimes 1 \) and whose bilinear structures are given by the bilinear structures on \( \mathcal{O}_M \otimes \mathcal{E}_M^{0,*} \). For example, the \( \mathcal{T}_M \otimes \mathcal{E}_M^{0,*} \)-action on \( \mathcal{E}_M^{0,*} \) reads
\[
Y_{\eta} := (Y_I \eta_J) \bar{d} \bar{\phi}_I \wedge d \phi_J \quad \text{on} \quad U_a
\]
which is independent of the choice of holomorphic coordinates \( \phi_{\alpha} \). Applying \( \hat{C}^*(-) \) yields a dg extended Lie superalgebroid made up of Čech-Dolbeault complexes. Then, the last four components of (3.27) extend to maps between those Čech-Dolbeault complexes. In particular, the two extensions
\[
\bar{\Delta} : \hat{C}^*(\mathcal{T}_M \otimes \mathcal{E}_M^{0,*}) \to \hat{C}^{*+1}(\mathcal{O}_M \otimes \mathcal{E}_M^{0,*}), \quad \bar{\delta} : \hat{C}^*(\mathcal{E}_M^{0,*}) \otimes \hat{C}^*(\mathcal{T}_M \otimes \mathcal{E}_M^{0,*}) \to \hat{C}^1(\mathcal{O}_M \otimes \mathcal{E}_M^{0,*})
\]
\(^{18}\)From here on, the coordinates of \( \mathbb{C}^d \) are written as \( b^j \) instead of \( b_0^j \).
are respectively the degree-(1, 0) and bidegree-preserving maps given by the following generalizations of formulae (3.30) and (3.31)

\[
\begin{align*}
(\tilde{\Delta}X)_{0\ldots p+1} &= \Delta_{10}(X_{1\ldots p+1,I}) \otimes d\varphi^I_0 \\
(\omega \ast X)_{0\ldots p+q} &= [\omega_{0\ldots p,I} \ast \varphi^I_0(X_{p\ldots p+q,J}) + \omega_{0\ldots p,I} \Delta_{p0}(X_{p\ldots p+q,J})] \otimes d\bar{\varphi}^J_{p0}
\end{align*}
\]  

(4.3) (4.4)

19 The Čech-Dolbeault versions of $\{\tilde{1}\}_{0}$ and $\{\tilde{1}\}_{1}$ are defined similarly (see footnote 17). It is easy to check that these data constitute a homotopy dg vertex algebroid

\[
(\bar{\mathcal{C}}^*(\mathcal{E}^0_{M}), \bar{\mathcal{C}}^*(\Omega^1_M \otimes \mathcal{E}^0_{M}), \bar{\mathcal{C}}^*(\mathcal{T}_M \otimes \mathcal{E}^0_{M}), \bar{\Delta}, \ast, \{\tilde{1}\}_{0}, \{\tilde{1}\}_{1}).
\]  

(4.5)

In particular, $D\bar{\Delta} + \Delta D = 0$ and equations (3.22)-(3.24) hold with $D$, $\bar{\Delta}$, $\ast$, $\{\tilde{1}\}_{0}, \{\tilde{1}\}_{1}$ in place of $d$, $\Delta$, $\ast$, $\{0\}, \{1\}$. There is a quasi-isomorphism

\[
(\text{inc}, 0) : (3.27) \rightarrow (4.5)
\]  

(4.6)

where inc are embeddings of the first type in (4.2).

4.4. Now we would like to construct another homotopy dg vertex algebroid

\[
(\bar{\mathcal{C}}^*(\mathcal{E}^0_{M}), \bar{\mathcal{C}}^*(\Omega^1_M \otimes \mathcal{E}^0_{M}), \bar{\mathcal{C}}^*(\mathcal{T}_M \otimes \mathcal{E}^0_{M}), \bar{\Delta}, \ast, \{\tilde{1}\}_{0}, \{\tilde{1}\}_{1})
\]  

(4.7)

such that $\bar{\Delta}$ has degree $(0, 1)$ and there is an isomorphism

\[
(id, h) : (4.5) \rightarrow (4.7)
\]  

(4.8)

composed of the identity on the Čech-Dolbeault complexes and a bidegree-preserving map

\[
h : \bar{\mathcal{C}}^*(\mathcal{T}_M \otimes \mathcal{E}^0_{M}) \rightarrow \bar{\mathcal{C}}^*(\Omega^1_M \otimes \mathcal{E}^0_{M}).
\]

If such an isomorphism exists, it will follow from (4.1) that

\[
\bar{\Delta} - \bar{\Delta} = Dh - hD
\]  

(4.9)

and from the graded version of (3.9) that (with deg $X = (p, q)$, deg $Y = (r, s)$)

\[
\begin{align*}
\omega \ast X &= \omega \ast X + h(\omega \wedge X) - \omega \wedge h(X) \\
\{X, Y\}_{0} &= \{X, Y\}_{0} - \langle h(X), Y \rangle - (-1)^{(p+q)(r+s)}\langle h(Y), X \rangle \\
\{X, Y\}_{1} &= \{X, Y\}_{1} - L_X h(Y) - (-1)^{(p+q)(r+s)}L_Y h(X) - \partial \langle h(X), Y \rangle + h([X, Y])
\end{align*}
\]  

(4.10)

In fact, we will define $\bar{\Delta}$, $\ast$, $\{\tilde{1}\}_{0}, \{\tilde{1}\}_{1}$ using (4.9), (4.10) and an appropriate choice of $h$. It is then a formal consequence of these formulae that (4.7) is indeed a homotopy dg vertex algebroid.

4.5. The degree-(1, 0) and -(0, 1) components of equation (4.9) are

\[
\begin{align*}
\Delta &= \delta h - h\delta, \\
\bar{\Delta} &= (-1)^{p+1}(\bar{\delta}h - h\bar{d})
\end{align*}
\]  

(4.11)

where $p$ is the Čech degree. Assume that $h$ is of the form (with deg $X = (p, *)$)

\[
h(X)_{0\ldots p} = h_0(X_{0\ldots p,I}) \otimes d\bar{\varphi}^I
\]  

(4.12)

\[\Delta_{\beta\alpha} = \Delta_{\beta\alpha} \ast \bar{\delta} \alpha, \text{ still defined by (3.13), is now viewed as an operator on smooth vector fields.}\]
for some $\mathfrak{g}$-linear first-order differential operators $h_\alpha$ on $W_\alpha$. Then by (4.3), the first equation in (4.11) boils down to

$$\Delta_{\beta \alpha} = \varphi^*_\beta \alpha \circ h_\beta - h_\alpha \circ \varphi^*_\beta \alpha$$  \hspace{1cm} (4.13)

where $\Delta_{\beta \alpha} = \Delta_{\varphi_\beta \alpha, \xi_{\beta \alpha}}$ is defined in (3.13). Furthermore, it follows from a computation that any solution to (4.13) is of the form

$$h_\alpha(Y) = \partial_i Y^j \left( \Gamma_\alpha \right)^i_j + \frac{1}{2} \text{Tr} \left[ \Gamma_\alpha(Y) \cdot \Gamma_\alpha \right] - \frac{1}{2} \ell_Y B_\alpha \quad \text{in } W_\alpha$$  \hspace{1cm} (4.14)

where $\Gamma_\alpha = [(\Gamma_\alpha)^i_j]$ is a smooth, matrix-valued $(1,0)$-form on $W_\alpha$ such that

$$g^{-1}_{\beta \alpha} \cdot \varphi^*_\beta \alpha \Gamma_\beta \cdot g_{\beta \alpha} - \Gamma_\alpha = -\theta_{\beta \alpha} \quad \text{in } W_{\alpha \beta}$$  \hspace{1cm} (4.15)

and $B_\alpha$ is a smooth $(2,0)$-form on $W_\alpha$ such that

$$\varphi^*_\beta \alpha B_\beta - B_\alpha = \xi_{\beta \alpha} + \text{Tr} \left[ \theta_{\beta \alpha} \wedge \Gamma_\alpha \right] \quad \text{in } W_{\alpha \beta}.$$  \hspace{1cm} (4.16)

The existence of such $\{\Gamma_\alpha\}$ and $\{B_\alpha\}$ is addressed below.

### 4.6. A collection $\{\Gamma_\alpha \in \mathcal{E}^1(W_\alpha) \otimes \mathfrak{g}(d)\}$ satisfying (4.15) is precisely a set of local connection 1-forms on $TM$ that are compatible with the coordinate transformations $\varphi_{\beta \alpha}$. Also, if $\{\Gamma_\alpha\}$ satisfy (4.15), so do their $(1,0)$-parts. This proves the existence of $\{\Gamma_\alpha\}$ with the desired properties.

Given a hermitian metric on $M$, there is a unique unitary connection on $TM$ whose connection 1-forms are of pure type $(1,0)$. [Wel80] This is sometimes called the Chern connection. From here on, $\nabla$ will always denote the Chern connection with respect to a fixed hermitian metric, and $\{\Gamma_\alpha\}$ its connection $(1,0)$-forms. The curvature form $R$ associated to $\nabla$ is of pure type $(1,1)$.

### 4.7. Consider again the computations in the Čech-de Rham complex shown in figure 2. By (2.6), the degree-(1,2) cochain in the diagram is $\delta$-closed, hence also $\delta$-exact as each row is acyclic. In other words, there exist $\{B_\alpha \in \mathcal{E}^2(W_\alpha)\}$ that satisfy (4.16). Also, if $\{B_\alpha\}$ satisfy (4.16), so do their $(2,0)$-parts. This proves the existence of $\{B_\alpha\}$ with the desired properties.

Observe that the cochain $\{dB_\alpha - CS(\Gamma_\alpha)\}$ is $\delta$-closed. Let $H$ be the global 3-form such that

$$H|_{U_\alpha} = \varphi^*_\alpha(dB_\alpha - CS(\Gamma_\alpha)).$$

By construction, $H$ has only $(3,0)$- and $(2,1)$-parts, and $dH = -\text{Tr} \left( R \wedge R \right)$. In fact, figure 2 shows that the existence of the following are equivalent: \hspace{1cm} 20

- $\{\xi_{\beta \alpha} \in \mathcal{E}^{2,0}(W_{\alpha \beta})\}$ satisfying $d\xi_{\beta \alpha} = WZ_{\beta \alpha}$ and (2.6)
- $\{B_\alpha \in \mathcal{E}^{2,0}(W_\alpha)\}$ such that $\{dB_\alpha - CS(\Gamma_\alpha)\}$ glue into a global 3-form
- $H \in \mathcal{E}^{3,0}(M) \oplus \mathcal{E}^{2,1}(M)$ satisfying $dH = -\text{Tr} \left( R \wedge R \right)$

### 4.8. Now that we have solved the first equation in (4.11), we define the homotopy dg vertex algebroid (4.7) using (4.10) and the second equation in (4.11). Since $\Delta$ has degree $(0,1)$, the equation $D\Delta + \Delta D = 0$ implies that $\bar{\Delta}$ anticommutes with $\delta$ and thus is induced by a morphism of sheaves

$$\bar{\Delta} : \mathcal{T}_M \otimes \mathcal{E}^{0,*}_M \to \mathcal{O}^1_M \otimes \mathcal{E}^{0,*+1}_M.$$  

\hspace{1cm} 20 This is easy to show directly. More conceptually, if $H^*(\Omega^2_M)$ is computed using the fine resolution

$$\cdots \to \mathcal{E}^{2,1}_M \to \mathcal{E}^{3,0}_M \oplus \mathcal{E}^{1,1}_M \to \mathcal{E}^{4,0}_M \oplus \mathcal{E}^{2,2}_M \to \cdots$$

figure 2 shows that the first obstruction class studied in §2.7 is represented by the global $(2,2)$-form $-\text{Tr} \left( R \wedge R \right)$.  

18
Axioms (3.22)-(3.24) for (4.11) similarly imply that \( \tilde{*}, \{ \tilde{0}, \{ \tilde{1} \} \) are induced by morphisms of sheaves
\[
\tilde{*} : E^{0,*}_M \times (T_M \otimes E^{0,*}_M) \to \Omega^1_M \otimes E^{0,*}_M \\
\{ \tilde{0} \} : (T_M \otimes E^{0,*}_M) \times (T_M \otimes E^{0,*}_M) \to E^{0,*}_M \\
\{ \tilde{1} \} : (T_M \otimes E^{0,*}_M) \times (T_M \otimes E^{0,*}_M) \to \Omega^1_M \otimes E^{0,*}_M
\]

These operators define a sheaf of dg vertex algebroids
\[
(E^{0,*}_M, \Omega^1_M \otimes E^{0,*}_M, T_M \otimes E^{0,*}_M, \tilde{\Delta}, \tilde{*}, \{ \tilde{0}, \{ \tilde{1} \})
\]
whose structure is strict, i.e. not merely up to homotopy. \(^{21}\) There is a quasi-isomorphism
\[
(\text{inc}, 0) : \Gamma(4.17) \to (4.7)
\]
where inc are embeddings of the second type in (4.2).

Let us obtain formulae for \( \tilde{\Delta}, \tilde{*}, \{ \tilde{0}, \{ \tilde{1} \)}. First restrict to degree (0, 0). By (4.11) and (4.12)
\[
\tilde{\Delta}(X)_\alpha = [-\partial h_\alpha(X_\alpha) + h_\alpha(\partial_i X_\alpha)] \otimes d\bar{\beta}.
\]

On the other hand, by (4.4), (4.10), (4.12) as well as footnote 17, we have
\[
(f \tilde{*} X)_\alpha = f_\alpha \ast X_\alpha + h_\alpha(f_\alpha X_\alpha) - f_\alpha h_\alpha(X_\alpha) \\
\{(X, Y)_0\}_\alpha = \{X_\alpha, Y_\alpha\}_0 - \langle h_\alpha(X_\alpha), Y_\alpha \rangle - \langle h_\alpha(Y_\alpha), X_\alpha \rangle \\
\{(X, Y)_1\}_\alpha = \{X_\alpha, Y_\alpha\}_1 - L_{X_\alpha} h_\alpha(Y_\alpha) + L_{Y_\alpha} h_\alpha(X_\alpha) - \partial \langle h_\alpha(X_\alpha), Y_\alpha \rangle + h_\alpha([X_\alpha, Y_\alpha])
\]

Applying (4.14) and (3.11) yields invariant expressions for (4.19)-(4.20). To write them down, pick a local frame \( e_1, \ldots, e_d \) of \( TM \), let \( e^1, \ldots, e^d \) be its dual frame of \( T^*M \), and associate to every vector field \( X \) a section of \( \text{End} \, T M \) as follows
\[
\tilde{\nabla}X := \nabla X + T(X, -).
\]

Also, repeated indices will be summed over. The invariant expression for (4.19) is given by
\[
\tilde{\Delta}(X) = \left[ \text{Tr} \left( \tilde{\nabla}X \circ R_{e_i \bar{e}_j} \right) + \frac{1}{2} H(X, e_i, e_j) \right] e^i \otimes \bar{e}^j
\]
where \( R \) is the curvature (1,1)-form and \( H \) is the 3-form constructed in §4.7. The invariant expressions for (4.20) are given by
\[
f \tilde{*} X = -(\nabla_{e_i} \partial f)(X) e^i \\
\{X, Y\}_0 = -\text{Tr} \left( \tilde{\nabla}X \circ \tilde{\nabla}Y \right) \\
\{X, Y\}_1 = \left[ -\text{Tr} \left( \nabla_{e_i} \tilde{\nabla}X \circ \tilde{\nabla}Y \right) + \frac{1}{2} H(X, Y, e_i) \right] e^i
\]

Now consider higher Dolbeault degrees. For a smooth function \( f \), smooth vector fields \( X, Y \), and anti-holomorphic \((0,*)\)-forms \( \omega, \eta \), we have
\[
\tilde{\Delta}(X \otimes \omega) = \tilde{\Delta}(X) \wedge \omega \\
(f \tilde{*} \omega)(X \otimes \eta) = (f \tilde{*} X) \otimes (\omega \wedge \eta)
\]

\(^{21}\)This is expected because e.g. at weight zero, while the Cech complexes \( C^*(\mathcal{O}_M), C^*(\mathcal{E}^{0,*}_M) \) are graded commutative only up to homotopy, the Dolbeault complex \( \mathcal{E}^{0,*}_M \) is strictly graded commutative.
which are well-defined because (4.21)-(4.22) are $\overline{O}_M$-linear.

According to (4.20), there are morphisms of sheaves of vertex algebroids

$$\left(\phi^*_\alpha, \phi^*_\alpha h_\alpha \right) : \left( O_{W_{\alpha}}, \Omega^1_{W_{\alpha}}, T_{W_{\alpha}}, *, \{ \bar{0}, \bar{1} \} \right) \rightarrow (E_M, \Omega^1_M \otimes E_M, T_M \otimes E_M, \bar{*}, \{ \bar{0}, \bar{1} \}) \quad (4.24)$$

Furthermore, it follows from (4.13) that there are commutative diagrams

$$\begin{align*}
(\Omega^1_{W_{\alpha}}, T_{W_{\alpha}}, *, \{ \bar{0}, \bar{1} \}) & \rightarrow (E_M, \Omega^1_M \otimes E_M, T_M \otimes E_M, \bar{*}, \{ \bar{0}, \bar{1} \}) \\
(\phi^*_\alpha, \Delta_{\beta \alpha}) & \rightarrow (\phi^*_\beta, \phi^*_\beta h_\beta)
\end{align*} \quad (4.25)$$

4.9. Consider the sheaf of dg vertex algebras freely generated by (4.17)

$$(\xi^{ch}_{M,H}, \bar{\partial}^{ch}) := F(\xi^{0*}_{M}, \Omega^1_M \otimes \xi^{0*}_{M}, T_M \otimes \xi^{0*}_{M}, \Delta, \bar{*}, \{ \bar{0}, \bar{1} \}). \quad (4.26)$$

For a description of its vertex superalgebra structure, see §3.3. The differential at weight zero is

$$\bar{\partial}^{ch}_{0} = \bar{\partial} \text{ on } \xi^{0*}_{M}. \quad (4.27)$$

On the other hand, by (3.20) the differential at weight one is a ‘deformed’ Dolbeault operator

$$\bar{\partial}^{ch}_{1} = \begin{bmatrix} \bar{\partial} & \Delta \\ 0 & \partial \end{bmatrix} \text{ on } (\Omega^1_M \otimes T_M) \otimes \xi^{0*}_{M}, \quad (\alpha, X) \mapsto (\bar{\partial} \alpha + \Delta(X), \partial X). \quad (4.27)$$

Notice that (4.26) depends on $H$ via the definition of $\Delta$ and $\{ \bar{1} \}$ in (4.21)-(4.23). Recall the sequence of quasi-isomorphisms (4.6), (4.8) and (4.18). Since $\mathcal{C}^*(\mathcal{D}^{ch}_{M,\xi})$ and $\Gamma(\mathcal{E}^{ch}_{M,H})$ are freely generated by quasi-isomorphic dg vertex algebroids, we may expect that they are quasi-isomorphic dg vertex algebras and hence both compute $H^*(\mathcal{D}^{ch}_{M,\xi})$. This will be argued more carefully below.

Applying the free functor $F$ to (4.24) and (4.25) yields morphisms of sheaves of vertex algebras

$$i_{\alpha} := F(\phi^*_\alpha, \phi^*_\alpha h_\alpha) : \mathcal{D}^{ch}_{W_{\alpha}} \rightarrow \mathcal{E}^{ch,0}_{M,H} \quad (4.28)$$

and commutative diagrams of the form (recall the last paragraph of §3.4)

$$\begin{align*}
\mathcal{D}^{ch}_{W_{\alpha}} & \rightarrow \mathcal{E}^{ch,0}_{M,H} \\
(\phi^*_\alpha)_{\beta \alpha} & \rightarrow \mathcal{E}^{ch,0}_{M,H}
\end{align*} \quad (4.29)$$

This is equivalent to a morphism of sheaves of vertex algebras $i : \mathcal{D}^{ch}_{M,\xi} \rightarrow \mathcal{E}^{ch,0}_{M,H}$ (§2.6). Composition with the inclusion $\mathcal{E}^{ch,0}_{M,H} \rightarrow \mathcal{E}^{ch,*}_{M,H}$ then defines a fine resolution

$$0 \rightarrow \mathcal{D}^{ch}_{M,\xi} \rightarrow \mathcal{E}^{ch,*}_{M,H} \quad (4.29).$$
Indeed, the weight-zero component, namely the ordinary Dolbeault resolution, is exact. The weight-one component, which is described in (4.27), fits into the commutative diagram in figure 4 in which exactness of the top row, bottom row and all columns implies that of the middle row. Applying this argument repeatedly to filtrations like (2.12) proves exactness at all higher weights. The resolution (4.29) leads to the following isomorphism of vertex superalgebras

$$H^* (\Gamma (\mathscr{E}_{M, \text{ch},*}^0), \partial^\text{ch}) \cong H^* (\mathcal{D}_{M, \xi}^\text{ch}).$$

(4.30)

**4.10.** Consider an isomorphism of sheaves of dg vertex algebras between $\mathscr{E}_{M, \text{ch},*}^0$ and $\mathscr{E}_{M, \text{ch},*}^1$, over the identity on $M$. This is equivalent to an isomorphism between the associated sheaves of dg vertex algebroids

$$(\text{id}, \beta) : (\mathscr{E}_{M, \text{ch},*}^0, \Omega_M^1 \otimes \mathcal{E}_{M, \text{ch},*}^0, T_M \otimes \mathcal{E}_{M, \text{ch},*}^0, \Delta, \bar{s}, \{ \tilde{1}_0 \}, \{ \tilde{1}_1 \})$$

$$\rightarrow (\mathscr{E}_{M, \text{ch},*}^0, \Omega_M^1 \otimes \mathcal{E}_{M, \text{ch},*}^0, T_M \otimes \mathcal{E}_{M, \text{ch},*}^0, \Delta', \bar{s}, \{ \tilde{1}_0 \}, \{ \tilde{1}_1 \})$$

where $\beta : T_M \otimes \mathcal{E}_{M, \text{ch},*}^0 \rightarrow \Omega_M^1 \otimes \mathcal{E}_{M, \text{ch},*}^0$ is a degree-preserving operator. According to (4.1) and (3.9), $\beta$ has to satisfy precisely the following conditions

$$\bar{\partial} [\beta (X)] - \beta (\bar{\partial} X) = \Delta (X) - \Delta' (X)$$

(4.31)

$$\beta (\omega \wedge X) - \omega \wedge \beta (X) = 0$$

(4.32)

$$\langle \beta (X), Y \rangle + (-1)^{pq} \langle \beta (Y), X \rangle = 0$$

(4.33)

$$L_X \beta (Y) - (-1)^{pq} L_Y \beta (X) + \partial (\beta (X), Y) - \beta ([X, Y]) = \{ X, Y \}_{1} - \{ X, Y \}_{1}$$

(4.34)

where $\omega, X, Y$ are respectively sections of $\mathcal{E}_{M, \text{ch},*}^0$, $T_M \otimes \mathcal{E}_{M, \text{ch},*}^0$ and $T_M \otimes \mathcal{E}_{M, \text{ch},*}^0$. By (4.32), $\beta$ is determined by its component in degree zero. Then (4.32) and (4.33) together imply that

$$\tilde{\beta} (X, Y) := \langle \beta (X), Y \rangle$$

is a smooth $(2,0)$-form. Using formulae (4.21)-(4.22), conditions (4.31) and (4.34) can be rewritten as

$$\bar{\partial} \tilde{\beta} = \frac{1}{2} (H - H')^{2,1}, \quad \partial \tilde{\beta} = \frac{1}{2} (H - H')^{3,0}.$$

21
Hence \( \mathcal{E}^{ch,*}_{M,H}, \mathcal{E}^{ch,*}_{M,H'} \) are isomorphic over the identity if and only if \( H, H' \) differ by the de Rham derivative of a smooth \((2, 0)\)-form. \(^{22}\)

4.11. Now we address the conformal structure of \( \mathcal{E}^{ch,*}_{M,H} \). First let us rephrase condition (2.7) in global terms. Recall §4.6. According to figure 3, (2.7) implies that the cochain \( \{ \text{Tr} \Gamma_\alpha \} \) is \( \delta \)-closed. Let \( A \) be the global \((1,0)\)-form such that

\[
A|_{U_\alpha} = \varphi_\alpha^*(\text{Tr} \Gamma_\alpha).
\]

Then \( dA = \text{Tr} R \). In fact, the existence of the following are equivalent: \(^{23}\)

- coordinate charts \( \varphi_\alpha \) satisfying (2.7)
- coordinate charts \( \varphi_\alpha \) such that \( \{ \text{Tr} \Gamma_\alpha \} \) glue into a global form
- \( A \in \mathcal{E}^{1,0}(M) \) satisfying \( dA = \text{Tr} R \)

For each \( \alpha \), denote by \( \nu_\alpha \) the local conformal element (2.2) of \( \mathcal{D}^{ch,M}(U_\alpha) = \mathcal{D}^{ch}(W_\alpha) \). By a computation using (4.28) and (4.14), the image of \( \nu_\alpha \) in \( \mathcal{E}^{ch,0}_{M,H}(U_\alpha) \) equals

\[
\iota_\alpha(\nu_\alpha) = \sum_{i=1}^{d} d\left( \frac{\partial}{\partial \varphi_\alpha^*} \right)^{-1}(d\varphi_\alpha^* \Gamma_\alpha) + \frac{1}{2} \text{Tr} \left[ (\varphi_\alpha^* \Gamma_\alpha) - (\varphi_\alpha^* \Gamma_\alpha) \right].
\]

It can be checked that (4.35) is a conformal element of the larger vertex superalgebra \( \mathcal{E}^{ch,0}_{M,H}(U_\alpha) \) with the same central charge \( 2d \). By assumption, as \( \alpha \) varies, the local sections (4.35) glue into a global one. Denote this global section as well as any of its restrictions by \( \nu \).

Consider the Virasoro field \( L(z) \) associated to \( \nu \). Let \( f \) be a smooth function and \( X \) a smooth vector field on \( M \). OPEs with \( L(z) \) have the following singular parts

\[
L(z)f(w) \sim \frac{\partial_w f(w)}{z-w},
\]

\[
L(z)X(w) \sim \frac{[\text{Tr} \nabla X - A(X)](w)}{(z-w)^2} + \frac{X(w)}{(z-w)^2} + \frac{\partial_w X(w)}{z-w}.
\]

Since \( \nu \) is in the image of the morphism \( i \) in (4.29)

\[
\bar{\partial}^{ch}(\nu) = 0 \quad \Rightarrow \quad [\bar{\partial}^{ch}, L(z)] = 0.
\]

Therefore \( \nu \) induces a conformal structure on the cohomology of \( (\Gamma(\mathcal{E}^{ch,*}_{M,H}), \bar{\partial}^{ch}) \), making (4.30) an isomorphism of conformal vertex superalgebras. By (2.13), we have

\[
\text{char} H^*(\Gamma(\mathcal{E}^{ch,*}_{M,H}), \bar{\partial}^{ch}) = \frac{W(M)}{\eta(q)^{2d}}
\]

which provides a new geometric interpretation of the Witten genus.

\(^{22}\)Therefore by the resolution in footnote 20, the isomorphism classes of \( \{ \mathcal{E}^{ch,*}_{M,H} \} \) form an affine space modeled on \( H^1(\Omega^2_M) \).

See also §2.9.

\(^{23}\)If \( H^*(\Omega^1_M) \) is computed using the fine resolution

\[
\begin{array}{ccccccc}
0 & \Omega^1_M & \mathcal{E}^{1,0}_M & \mathcal{E}^{2,0}_M & \mathcal{E}^{1,1}_M & \mathcal{E}^{1,2}_M & \cdots \end{array}
\]

figure 3 shows that the second obstruction class studied in §2.7 is represented by the global \((1,1)\)-form \( \text{Tr} R \).
4.12. Let us finish with a comment on the geometric conditions stated in §4.7 and §4.11:

(i) $\exists H \in E^{1,0}(M) \oplus E^{2,1}(M)$ such that $dH = -\text{Tr} (R \wedge R)$
(ii) $\exists A \in E^{1,0}(M)$ such that $dA = \text{Tr} R$

Recall that (i) is required for the construction of either $\mathcal{D}^{ch}_{M,\xi}$ or $\mathcal{E}^{ch}_{M,H}$, and (ii) is required for a global conformal structure. In the case $M$ is Kähler, they are equivalent to the following topological conditions

(i) $\exists H \in E^3(M)$ such that $dH = -\text{Tr} (R \wedge R)$
(ii) $\exists A \in E^1(M)$ such that $dA = \text{Tr} R$

For example, assume (ii). Since $R$ has no $(0,2)$-part, $\bar{\partial} A^{0,1} = 0$. Suppose $\omega$ is the Dolbeault harmonic representative of $[A^{0,1}] \in H^{0,1}(M)$. Hence $A^{0,1} - \omega = \bar{\partial} f$ for some $f \in \mathcal{E}(M)$. Let $A' = A - \omega - df$. By construction $A'$ has no $(0,1)$-part. Since $\omega$ is also de Rham harmonic, $dA' = dA = \text{Tr} R$. This proves (ii).

The proof of (i) $\Rightarrow$ (i) is similar. By Chern-Weil theory, (i) and (ii) are equivalent to the vanishing of $c_2(M)$ and $c_1(M)$ respectively.

REFERENCES

[AHR08] M. Ando, M. J. Hopkins and C. Rezk, Multiplicative orientations of KO-theory and of the spectrum of topological modular forms, http://www.math.uiuc.edu/~mando/node1.html

[AHS01] M. Ando, M. J. Hopkins and N. Strickland, Elliptic spectra, the Witten genus and the theorem of the cube, Invent. Math. 146 (2001), no.3, 595–687.

[BCSS05] J. Baez, A. Crans, D. Stevenson and U. Schreiber, From loop groups to 2-groups, arXiv: math.QA/0504123.

[BT82] R. Bott and L. W. Tu, Differential forms in algebraic topology, Springer-Verlag, New York, 1982.

[Che10] P. Cheung, Chiral differential operators and topology, in preparation.

[FB04] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, 2nd Ed., American Mathematical Society, Providence, RI, 2004.

[GMS00] V. Gorbounov, F. Malikov and V. Schechtman, Gerbes of chiral differential operators, Math. Res. Lett. 7 (2000), no. 1, 55-66.

[GMS04] V. Gorbounov, F. Malikov and V. Schechtman, Gerbes of chiral differential operators II. Vertex algebroids, Invent. Math. 155 (2004), no. 3, 605-680.

[Hen06] A. Henriques, Integrating $L_\infty$ algebras, arXiv: math.QA/0603563.

[Hop02] M. J. Hopkins, Algebraic topology and modular forms in: Proceedings of the International Congress of Mathematicians, Vol. I, 291–317, Higher Ed. Press, 2002.

[Kac98] V. Kac, Vertex algebras for beginners, 2nd Ed., American Mathematical Society, Providence, RI, 1998.

[LRS93] P. Landweber, D. Ravenel and R. Stong, Periodic cohomology theories defined by elliptic curves in: The Čech centennial, 317–337, Contemp. Math., 181, Amer. Math. Soc., 1995.

[LM89] H. B. Lawson and M.-L. Michelsohn, Spin geometry, Princeton Univ. Press, Princeton, 1989.

[ST04] S. Stolz and P. Teichner, What is an elliptic object? in: Topology, geometry and quantum field theory, 247–343, Cambridge Univ. Press, 2004.

[Wel80] R. O. Wells, Differential analysis on complex manifolds, 2nd Ed., Spring-Verlag, New York, 1980.

[Wit87] E. Witten, Elliptic genera and quantum field theory, Comm. Math. Phys. 109 (1987), no. 4, 525–536.

[Zag88] D. Zagier, Note on the Landweber-Stong elliptic genus, in: Elliptic curves and modular forms in algebraic topology, Lec. Notes in Math., vol. 1326, 216–224, Springer, Berlin, 1988.

Max-Planck-Institut für Mathematik, 53111 Bonn, Germany
Email address: pokman@mpim-bonn.mpg.de