Weakly non-radiative radial solutions to 3D energy subcritical wave equations

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Abstract

In this work we consider the energy subcritical 3D wave equation
\[ \partial_t^2 u - \Delta u = \pm |u|^{p-1} u \]
and discuss its (weakly) non-radiative solutions, i.e. the solutions defined in an exterior region \{(x, t) : |x| > |t| + R\} with \( R \geq 0 \) satisfying
\[ \lim_{t \to \pm \infty} \int_{|x| > |t| + R} (\|\nabla u(x, t)\|^2 + |u_t(x, t)|^2) \, dx = 0. \]

It has been known that any radial weakly non-radiative solution to the linear wave equation is a multiple of \( 1/|x| \). In addition, any radial weakly non-radiative solutions \( u \) to the energy critical wave equation must possess a similar asymptotic behaviour, i.e. \( u(x, t) \simeq C/|x| \) when \( |x| \) is large. In this work we give examples to show that radial weakly non-radiative solutions to energy subcritical equation \( 3 < p < 5 \) may possess a much different asymptotic behaviour. However, a radial weakly non-radiative solution \( u \) with initial data in the critical Sobolev space \( H^{s_p} \times H^{s_{p-1}}(\mathbb{R}^3) \) must coincide with a \( C^2 \) solution \( W \) to the elliptic equation \(-\Delta W = -|W|^{p-1} W\) so that \( u(x, t) \equiv W(x) \simeq C/|x| \) when \( |x| \) is large.

1 Introduction and Main Results

1.1 Background and topics

The channel of energy plays an important role in the study of radial wave equation in recent years. This method is first considered in 3-dimensional case in Duyckaerts-Kenig-Merle [1] and then in 5-dimensional case in Kenig-Lawrie-Schlag [9]. Its application includes all of the following: the proof of solution resolution conjecture of energy critical wave equation with radial data in 3-dimensional case by Duyckaerts-Kenig-Merle [3] and in odd dimensions \( d \geq 5 \) by the same authors [5]; the scattering of radial, bounded solutions to 3-dimensional wave equations in energy supercritical case by Duyckaerts-Kenig-Merle [4] and in energy subcritical case by Shen [13], and many more. Let us make a brief review on the energy channel property of solutions to wave equations.

Linear equation Assume that the dimension \( d \geq 3 \) is odd. Let \( u \) be a solution to the free wave equation (not necessarily radially symmetric):
\[ \partial_t^2 u - \Delta u = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R} \]
Then by Duyckaerts-Kenig-Merle [2] we have
\[
\lim_{t \to \pm \infty} \int_{|x|>|t|} |\nabla_x u(x, t)|^2 \, dx + \lim_{t \to -\infty} \int_{|x|>|t|} |\nabla_x u(x, t)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla_x u(x, 0)|^2 \, dx.
\]
Here \(\nabla_x u = (\nabla_x u, u_t)\). Thus the only possible non-radiative solution, i.e. the free wave satisfying
\[
\lim_{t \to \pm \infty} \int_{|x|>|t|} |\nabla_x u(x, t)|^2 \, dx = 0,
\]
must be zero solution. Weakly non-radiative solutions to free wave equation in the radial setting, i.e. radial free waves satisfying
\[
\lim_{t \to \pm \infty} \int_{|x|>|t|+R} |\nabla_x u(x, t)|^2 \, dx = 0
\]
for a positive constant \(R\) are also well-understood. Let us first introduce a few notations, which will be used through this work, before we give the results proved by Kenig et al. [10]. Fix \(R \geq 0\). We define \(\mathcal{H}_R\) to be the space consisting of the restrictions of all \(L^2(\mathbb{R}^d)\) functions to the exterior region \(\{x : |x| > R\}\), with the norm
\[
\|u_0(0)\|_{\mathcal{H}_R} = \inf \left\{ \|u_0(0)\|_{L^2(\mathbb{R}^d)} : (u'(x), u_1(x)) = (u(0), u_1(0)), |x| > R \right\}.
\]
We also define \(\mathcal{H}_{rad,R}\) to be the subspace of \(\mathcal{H}_R\) consisting of radial functions. Because the choice
\[
(u_0(x), u_1(x)) = \begin{cases} (u_0(x), u_1(x)), & |x| > R; \\ (u_0(R), 0), & |x| \leq R \end{cases}
\]
clearly minimize \(\|u_0(0), u_1(0)\|_{\mathcal{H}_R}\) in the definition of \(\mathcal{H}_R\) norm above if \((u_0, u_1)\) are radial, we know in the radial case \(\|u_0(0), u_1(0)\|_{\mathcal{H}_{rad,R}} = \|\nabla u_0, u_1\|_{L^2(\{x:|x|>R\})}\). Thus \(\mathcal{H}_{rad,R}\) is a Hilbert space with pairing
\[
\langle (u_0, u_1), (\tilde{u}_0, \tilde{u}_1) \rangle_{\mathcal{H}_{rad,R}} = \int_{|x|>R} (\nabla u_0(x) \cdot \nabla \tilde{u}_0(x) + u_1(x) \tilde{u}_1(x)) \, dx.
\]
It was proved in Kenig et al. [10] that a radial free wave satisfies (11) if and only if the restriction of its initial data to the exterior region \(\{x : |x| > R\}\) is contained in a \((d-1)/2\)-dimensional subspace \(P(R)\) of \(\mathcal{H}_{rad,R}\):
\[
P(R) = \text{Span} \left\{ (r^{2k_1-d}, 0), (0, r^{2k_2-d}) : k_1, k_2 \in \mathbb{N}, k_1 < \frac{d+2}{4}, k_2 < \frac{d}{4} \right\}
\]
We use the notation \(\Theta_k, k = 1, 2, \cdots, (d-1)/2\) for the generators of \(P(R)\) given above. Here the lower index \(k\) is assigned by the identity \(\|\Theta_k\|_{\mathcal{H}_{rad,R}} = c_k/R^{k-1/2}\). In addition, for any radial free waves \(u\) we have
\[
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x|>|t|+R} |\nabla_x u(x, t)|^2 \, dx = \|\Pi_{P(R)\perp}(u(0), u_t(0))\|_{\mathcal{H}_{rad,R}}^2.
\]
Here \(\Pi_{P(R)\perp}\) is the orthogonal projection in \(\mathcal{H}_{rad,R}\) on the orthogonal subspace \(P(R)\perp\). Please note that \(P(R)\) is a one-dimensional space with generator \((1/r, 0)\) if \(d = 3\).

**Energy critical nonlinear equation** The weakly non-radiative solutions to energy critical, focusing wave equation in all odd dimensions \(d \geq 3\)
\[
\partial_t^2 u - \Delta u = +|u|^{s-1} u
\]
has also been discussed by Duyckaerts, Kenig and Merle. In summary we have (Please see [3] for 3-dimensional case and [6] for higher dimensional case \(d \geq 5\))
Interpretation

The difference between energy critical and subcritical cases is not as surprising as at the first glance. In the energy critical case, when we consider the solution in an exterior region \( \{(x,t) : |x| > |t| + R\} \) with a large \( R > 0 \), the linear wave operator dominates the wave propagation of data because this part of solution coincides with a solution with a small energy. In the energy subcritical case, although we still know that the solution in an exterior region \( \{(x,t) : |x| > |t| + R\} \) with a large \( R > 0 \) carries a small amount of energy, this does not means that linear wave operator dominates the propagation, even in the defocusing case, because the energy space \( \dot{H}^1 \times L^2 \) is no longer the critical Sobolev space of this Cauchy problem. However, if we know the propagation is dominated by the linear part, we may still prove a similar result as in the energy critical case. In this work we prove that if the initial data \((u_0, u_1)\) is in the critical Sobolev space \( H^{s_p} \times H^{s_p-1}(\mathbb{R}^3) \), then any weakly non-radiative solution to (CP1) must coincide with a stationary solution in the exterior region \( \{(x,t) : |x| > |t| + R\} \). Before we may give the precise statement of our main theorems, we need to define exterior solutions to (CP1) in a suitable way.

\footnote{Please see [4] for the definition of a solution in an exterior region}
1.2 Exterior and non-radiative solutions

Fix \( R_0 \geq 0 \). Let \( \mathcal{H}_{R_0} \) be the space of plane waves defined in last subsection. We also define a space

\[
X(I) = L^{\frac{2p}{p-1}}(I; L^2_p(\{|x| > |t| + R_0\}))
\]

for a time interval \( I \). Given any \( T > 0 \) and initial data \((u_0, u_1) \in \mathcal{H}_{R_0}\), we define \( \chi_{R_0} \) to be the characteristic function of the exterior region \( \{(x, t) : |x| > |t| + R_0\} \), use the notation \( F(u) = \zeta|u|^{p-1}u \) and consider a transformation from \( X([0, T]) \) to itself

\[
T u = S_L(t)(u_0, u_1) + \int_0^t \sin \left( (t-\tau)\sqrt{-\Delta} \right) \chi_{R_0} F(u) d\tau, \quad |x| > |t| + R_0.
\]

Here \( S_L(t) \) is the linear wave propagation operator. In other words, \( T u \) is the solution \( \tilde{u} \) to linear wave equation \( \partial_t^2 \tilde{u} - \Delta \tilde{u} = \chi_{R_0} F(u) \) in the exterior region with initial data \((\tilde{u}_0, \tilde{u}_1) \). We may apply Strichartz estimates and obtain (An almost complete version of Strichartz estimates can be found in Ginibre-Velo \[7\]. For reader’s convenience we put their results in 3-dimensional case in Section \[2\])

\[
\|T(u)\|_{X([0, T])} \leq C_p \left( \|u_0\|_{\mathcal{H}_{R_0}} + \|\chi_{R_0} F(u)\|_{L^1_s L^2_p([0, T] \times \mathbb{R}^3)} \right)
\]

\[
\leq C_p \left( \|u_0\|_{\mathcal{H}_{R_0}} + T^{\frac{5-2p}{2p}} \|\chi_{R_0} F(u)\|_{L^\infty_s L^p([0, T] \times \mathbb{R}^3)} \right)
\]

\[
= C_p \left( \|u_0\|_{\mathcal{H}_{R_0}} + T^{\frac{5-2p}{2p}} \|u\|_{X([0, T])} \right);
\]

and

\[
\|T(u) - T(\tilde{u})\|_{X([0, T])} \leq C_p \|\chi_{R_0} F(u) - \chi_{R_0} F(\tilde{u})\|_{L^1_s L^2_p([0, T] \times \mathbb{R}^3)}
\]

\[
\leq pC_p T^{\frac{5-2p}{2p}} \left( \|u\|_{X([0, T])}^{\frac{5-2p}{2p}} + \|\tilde{u}\|_{X([0, T])}^{\frac{5-2p}{2p}} \right) \|u - \tilde{u}\|_{X([0, T])}.
\]

Thus the transformation \( T \) is a contraction map from the complete metric space \( \{u \in X([0, T]) : \|u\|_{X([0, T])} < 2C_p \|u_0\|_{\mathcal{H}_{R_0}} \} \) to itself if

\[
T < C(p) \|u_0\|_{\mathcal{H}_{R_0}}^{\frac{2(p-1)}{p}}.
\]

We may apply a classic fixed-point argument to obtain the local existence and uniqueness of solutions to (CP1) in exterior regions. More details about this type of argument can be found in \[8\].

Definition 1.2 (Exterior solutions). We say a function \( u \) defined in the exterior region \( \{(x, t) : |x| < |t| + R_0, -T_+ < \tau < T_+\} \) is an exterior solution to (CP1) with initial data \((u_0, u_1) \in \mathcal{H}_{R_0}\), if \( u \in X(I) \) for all bounded closed intervals \( I \subset (-T_-, T_-) \), so that

\[
u = S_L(t)(u_0, u_1) + \int_0^t \sin \left( (t-\tau)\sqrt{-\Delta} \right) \chi_{R_0} F(u) d\tau, \quad |x| > |t| + R_0.
\]

Remark 1.3. We may define initial data \((u_0, u_1) \in \mathcal{H}_{R_0}\) so that \((\hat{u}_0, \hat{u}_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\). The right hand side of (5) then becomes a function \( \hat{u} \) defined for all \((x, t) \in \mathbb{R}^3 \times (-T_-, T_-) \) so that \((\hat{u}(\cdot, t), \hat{u}_t(\cdot, t)) \in C((-T_-, T_+); \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))\) and \( \hat{u} \in L_{loc}^{\frac{2p}{p-1}}(-T_-, T_+) \times \mathbb{R}^3 \). Thus an exterior solution defined above is always the restriction of such a function in the exterior region. We also have \( F(\hat{u}), \chi_{R_0} F(u) \in L^1_{loc} L^2((-T_-, T_+) \times \mathbb{R}^3)\).

Proposition 1.4. Given any initial data \((u_0, u_1) \in \mathcal{H}_{R_0}\), there is a unique exterior solution to (CP1) with a maximal lifespan \((-T_-, T_+)\) with

\[
|T_-|, |T_+| \geq C(p) \|u_0\|_{\mathcal{H}_{R_0}}^{\frac{2(p-1)}{p}}.
\]

One may define \((u_0, u_1) \in \mathcal{H}_{R_0}\) in any way so that \((u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) and consider the solution \( \tilde{u} \) to linear wave equation \( \partial_t^2 \tilde{u} - \Delta \tilde{u} = \chi_{R_0} F(u) \) with these initial data. By finite speed propagation of wave equation, the choice of \((u_0, u_1)\) does not affect the value of \( \tilde{u} \) in the exterior region.
Non-radiative solutions

Now we may define (weakly) non-radiative solutions.

Definition 1.5 (Non-radiative solutions). We say a function $u$ defined in the exterior region \( \{ (x,t) : |x| > |t| + R_0 \} \) is an $R_0$-weakly non-radiative solution to (CP1) with initial data $(u_0, u_1) \in \mathcal{H}_{R_0}$, if $u$ is an exterior solution to (CP1) in Definition 1.2 so that

$$\lim_{t \to \pm \infty} \int_{|x| > |t| + R_0} |\nabla_{x,t} u(x,t)|^2 dx = 0.$$ 

In particular, we call $u$ a non-radiative solution if $u$ satisfies the conditions above with $R_0 = 0$.

Global existence of defocusing equations

If initial data $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ comes with a finite energy

$$E = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p + 1} |u_0(x)|^{p+1} \right) dx < +\infty,$$

then we may combine a local theory (with $X(I) = L^{\frac{6p}{5p-6}} L^{2p}(I \times \mathbb{R}^3)$) defined for the whole space $\mathbb{R}^3$ instead of exterior region \( \{ x : |x| > |t| + R_0 \} \) and the energy conservation law to conclude that

Proposition 1.6. Given any $(u_0, u_1) \in (\dot{H}^1 \cap L^{p+1})(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, there exists a unique solution to (CP1) in the defocusing case in the whole space-time $\mathbb{R}^3 \times \mathbb{R}$.

Remark 1.7. The restriction of a solution given in Proposition 1.6 to the exterior regions is of course an exterior solution to (CP1). Thus given any initial data $(u_0, u_1) \in \mathcal{H}_{R_0}$ with $\|u_0\|_{L^{p+1}(x \in \mathbb{R}^3; |x| > R_0)} < +\infty$, the corresponding exterior solution to (CP1) in the defocusing case must be the restriction of a finite-energy solution of (CP1) in the whole space-time to the exterior region thus globally defined in time.

1.3 Main results

Examples of weakly non-radiative solutions

According to the definition given above, a direct calculation shows that

$$u(x,t) = \left[ \frac{2(p-3)}{(p-1)^2} \right]^{\frac{1}{2}} |x|^{-\frac{3}{2}}$$

is an $R$-weakly non-radiative solution to (CP1) in the focusing case for all $R > 0$. The angle $\theta$ between $(u(x,0), u_1(x,0))$ and $(1/|x|, 0)$ in the space $\mathcal{H}_{rad,R}$ is determined by

$$\cos \theta = \frac{\langle (|x|^{-2/3},0), (|x|^{-1},0) \rangle_{\mathcal{H}_{rad,R}}}{\| (|x|^{-2/3},0) \|_{\mathcal{H}_{rad,R}} \| (|x|^{-1},0) \|_{\mathcal{H}_{rad,R}}} = \frac{1}{2} \sqrt{(5-p)(p-1)} \in (0,1).$$

Thus $\theta \in (0, \pi/2)$ is a positive angle independent of $R$. Please note that $u(x,t)$ is NOT a non-radiative solution to (CP1) because the initial data $u(x,0) \notin \dot{H}^1(\mathbb{R}^3)$. In the defocusing case we may give a more interesting example:

Theorem 1.8. There exists a radial $C^2$ solution $u$ to (CP1) in the defocusing case defined in the exterior region \( \{ (x,t) : |x| > |t| \} \) with nonzero initial data $(0, u_1)$ so that $u$ is an $R$-weakly non-radiative solution to (CP1) for any $R > 0$.

Remark 1.9. This is clear that $\langle (0,u_1), (1/|x|,0) \rangle_{\mathcal{H}_{rad,R}} = 0$ for all $R > 0$, i.e. initial data $(0,u_1)$ is orthogonal to $(1/|x|,0)$ in $\mathcal{H}_{rad,R}$. But the example we give is not a non-radiative solution to (CP1), because $u_1 \notin L^2(\mathbb{R}^3)$ although $u_1 \in L^2(\{ x : |x| > R \})$ for all $R > 0$. Please see section 3 for more details.
Theorem 1.13. with initial data 

stationary solutions to (CP1) with similar asymptotic behaviour to $1/|x|$, the single generator of $P(R)$ in dimension 3. The focusing case (Proposition 1.10) has been considered in Shen [13]. The defocusing case (Proposition 1.11) is discussed in Section 6.

Proposition 1.10. The elliptic equation $-\Delta u = |u|^{p-1}u(x)$ has a radial solution $u^+ \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ so that $u^+ \notin \dot{H}^s_x(\mathbb{R}^3)$ and 

$$|u^+(x) - 1/|x|| \lesssim |x|^{2-p}, \quad |\nabla u^+(x)| \lesssim 1/|x|^2, \quad |x| \gg 1.$$ 

(6)

Proposition 1.11. The elliptic equation $-\Delta u = -|u|^{p-1}u(x)$ has a radial solution $u^- \in C^\infty(\{x : |x| > R_\infty\})$ so that $u^-$ has the same asymptotic behaviour as in (6) and the blow-up

$$\lim_{|x| \to (R_\infty)^+} u^-(x) = +\infty, \quad R_\infty > 0.$$

Remark 1.12. The function $u^+ \notin \dot{H}^1(\mathbb{R}^3)$, otherwise we might combine $u^+ \in \dot{H}^1(\mathbb{R}^3)$ with the asymptotic behaviour of $u^+$ to obtain $u \in \dot{W}^{1,\infty}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_x(\mathbb{R}^3)$. This contradicts the already known fact $u^+ \notin \dot{H}^s_x$. The defocusing case is similar. According to Lemma 2.2, a radial $\dot{H}^1(\mathbb{R}^3)$ function $u(x)$ can never blow up when $|x|$ approaches a positive number. Thus $u^-$ is not the restriction of any radial $\dot{H}^1(\mathbb{R}^3)$ function.

We may define a family of radial stationary solutions to (CP1) by rescaling 

$$U^C(x, t) = U^C_C(x) = \begin{cases} C^{-\frac{2-s}{2}}u^+(x/C) & \text{if } C > 0; \\ 0 & \text{if } C = 0; \\ -|C|^{-\frac{2-s}{2}}u^-(x/C) & \text{if } C < 0; \end{cases}$$

and 

$$U^-_C(x, t) = U^-_C(x) = \begin{cases} C^{-\frac{2-s}{2}}u^+(x/C) & |x| > C^\frac{s-1}{2-2s}R_\infty, \quad C > 0; \\ 0 & |x| > 0, \quad |x| > 0, \quad C = 0; \\ -|C|^{-\frac{2-s}{2}}u^-(x/C) & |x| > |C|^\frac{s-1}{2-2s}R_\infty, \quad C < 0. \end{cases}$$

The solutions $U^+_C(x, t)$ are $R$-weakly non-radiative for any $R > 0$. The solution $U^-_C(x, t)$ is $R$-weakly non-radiative for any $R > |C|^\frac{s-1}{2-2s}R_\infty$. We recall the behaviour of $U^\pm$ near infinity, conduct a simple calculation and obtain 

$$|U^\pm_C(x, t) - C/|x|| \lesssim |x|^{2-p}, \quad |x| \gg 1.$$ 

Please note that these solutions are NOT non-radiative unless $C = 0$ because we have $U^\pm_C(x) \notin \dot{H}^1(\mathbb{R}^3)$ for $C \neq 0$.

Weakly non-radiative solution in $\dot{H}^s \times \dot{H}^s_x$. The second main result of this work is that any weakly non-radiative solutions to (CP1) in the critical Sobolev space must coincide with a stationary solution as given above. For convenience we define $\mathcal{H}^s_R$ to be the space of restrictions of all $(\dot{H}^1 \cap \dot{H}^s_x) \times (L^2 \cap \dot{H}^s_x)$ functions in the exterior region $\{x \in \mathbb{R}^3 : |x| > R\}$.

Theorem 1.13. Assume $R > 0$. Let $u$ be a radial $R$-weakly non-radiative solution to (CP1) with initial data $(u_0, u_1) \in \mathcal{H}^s_R$. Then there exist a constant $C$, so that $u(x, t) = U^\pm_C(x, t)$ in the exterior region $\{(x, t) : |x| > |t| + R\}$. In the defocusing case, the constant $C$ above satisfies $|C| < (R/R_\infty)^\frac{s-1}{2-2s}$.

Since a non-radiative solution to (CP1) is $R$-weakly non-radiative for all $R > 0$, we may apply Theorem 1.13 with $R \to 0^+$ and utilize the fact $U^\pm_C(x, 0) \notin \dot{H}^1(\mathbb{R}^3)$ for $C \neq 0$ to obtain
Corollary 1.14. If \( u \) is a radial non-radiative solution to (CP1) with initial data \((u_0, u_1) \in (H^1(\mathbb{R}^3) \cap H^{s_p}(\mathbb{R}^3)) \times (L^2(\mathbb{R}^3) \cap H^{s_p-1}(\mathbb{R}^3))\), then \( u(x,t) \equiv 0 \) for all \((x,t) \) with \(|x| > |t|\).

Remark 1.15. Initial data \((u_0, u_1) \in H_R\) are contained in the space \(H^s_R\) as long as they coincide with some \(H^{s_p} \times H^{s_p-1}\) data near the infinity. In fact, if \((u_0, u_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) and \((u_0', u_1') \in H^{s_p}(\mathbb{R}^3) \times H^{s_p-1}(\mathbb{R}^3)\), we have

\[
(u_0(x), u_1(x)) = (u_0'(x), u_1'(x)), \quad |x| > R_1.
\]

then we may write \((u_0, u_1) = (1 - P_{2R_1})(u_0, u_1) + P_{2R_1}(u_0', u_1')\). Here \(P\) is a center cut-off operator as defined in Lemma 2.1. Since \(P_{2R_1}\) is a bounded operator from \(H^s\) to itself for \( s \in [-1,1]\), we have \(P_{2R_1}(u_0, u_1) \in H^{s_p} \times H^{s_p-1}\) and \((1 - P_{2R_1})(u_0, u_1) \in H^1 \times L^2\). The latter is also compactly supported, thus

\[
(1 - P_{2R_1})(u_0, u_1) \in \dot{W}^{\frac{3(p-1)}{p+1}, \frac{3(p-1)}{p+1}} \times L^{\frac{3(p-1)}{p+1}} \hookrightarrow \dot{H}^{s_p} \times \dot{H}^{s_p-1}.
\]

In summary we have \((u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}\).

2 Preliminary Results

2.1 Technical Lemma

Lemma 2.1 (center cut-off operator). Given a constant \( s \in [-1,1]\). Fix a smooth radial cut-off function \( \phi : \mathbb{R}^3 \to [0,1] \) satisfying

\[
\phi(x) = \begin{cases} 
0, & |x| \leq 1/2; \\
1, & |x| \geq 1.
\end{cases}
\]

We define an operator \( P_R f = \phi(x/R)f \). Here \( R > 0 \) is a positive constant. Then \( P_R \) is a bounded operator from \(H^s(\mathbb{R}^3)\) to \(H^s(\mathbb{R}^3)\), whose operator norm \( \|P_R\|_{H^s \to H^s} \) is independent of \( R \). In addition, if \( f \in H^s(\mathbb{R}^3) \), then \( \|P_R f\|_{H^s(\mathbb{R}^3)} \to 0 \) as \( R \to +\infty \).

Proof. We first show \( P_R \) is a bounded operator. By dilation it suffices to consider the case \( R = 1 \). A basic calculation shows that \( P_1 \) is a bounded operator from \(\dot{H}^1\) to itself, and from \(L^2\) to itself. An interpolation then gives the boundedness for all \( s \in [0,1] \). By duality the operator \( P_1 \) is also bounded for \( s \in [-1,0]\). Next we show the limit \( \|P_R f\|_{\dot{H}^s} \to 0 \) as \( R \to +\infty \). By the uniform boundedness of \( P_R \), it suffices to show that this limit holds for \( f \) in a dense subset of \(\dot{H}^s\). Now we may finish the proof by observing that this limit clearly holds for \( f \in \mathcal{C}_c^\infty(\mathbb{R}^3) \).

Lemma 2.2 (See Lemma 3.2 of [11]). Let \( u \in \dot{H}^s(\mathbb{R}^3) \) be a radial function, \( 1/2 < s < 3/2 \). Then we have the following pointwise estimate

\[
|u(x)| \lesssim \frac{\|u\|_{\dot{H}^s(\mathbb{R}^3)}}{|x|^{3/2-s}}.
\]

Lemma 2.3. If \( u \in \dot{H}^1(\mathbb{R}^3) \) be a radial function. Then we have

\[
|u(x)| \lesssim |x|^{-1/2} \left( \frac{1}{4\pi} \int_{|y|>|x|} |\nabla u(y)|^2 dy \right)^{1/2}.
\]

\(^3\)More precisely, the identity holds in the sense of distribution.
Proof. If \( u \) is smooth and compactly supported, we have
\[
|u(x)| = \left| \int_{|x|}^{\infty} u_r(r) dr \right| \leq \left( \int_{|x|}^{\infty} \frac{1}{r^2} dr \right)^{1/2} \left( \int_{|x|}^{\infty} r^2 |u_r(r)|^2 dr \right)^{1/2}
\]
\[
= |x|^{-1/2} \left( \frac{1}{4\pi} \int_{|y|>|x|} |\nabla u(y)|^2 dy \right)^{1/2}
\]
A standard smooth approximation and cut-off technique then deals with the general case. \( \square \)

**Lemma 2.4.** Let \( u \in \dot{H}^1(\mathbb{R}^3) \) be radial. Then given any \( R > 0 \), the one-variable function \( w(r) = ru(r) \) satisfies
\[
\int_{R}^{\infty} |w_r(r)|^2 dr = \frac{1}{4\pi} \int_{|x|>R} |\nabla u(x)|^2 dx - R |u(R)|^2.
\]
Proof. We may apply the identity \( w_r(r) = ru_r + u \) and calculate
\[
\int_{R}^{R'} |w_r(r)|^2 dr = \int_{R}^{R'} (r^2 u_r + 2ru_r + u^2) dr
\]
\[
= \int_{R}^{R'} |w_r|^2 dr + \int_{R}^{R'} \partial_r(ru^2) dr
\]
\[
= \frac{1}{4\pi} \int_{R<x<R'} |\nabla u(x)|^2 dx + R' |u(R')|^2 - R |u(R)|^2.
\]
Finally we make \( R' \to +\infty \) and finish the proof. Here we need to use the following fact (see Lemma A.7 of [13]): If \( u \) is a radial \( \dot{H}^1 \) function, then \( R' |u(R')|^2 \to 0 \) as \( R' \to \infty \). \( \square \)

This immediately gives

**Corollary 2.5.** Let \( u \) be a radial \( R \)-weakly non-radiative solution to (CP1). Then the function \( w(r,t) = ru(r,t) \) satisfies
\[
\lim_{t \to \pm\infty} \int_{|t|+R} (|w_r(r,t)|^2 + |w_r(r,t)|^2) dr = 0.
\]

### 2.2 Local theory in critical Sobolev spaces

**Proposition 2.6** (Generalized Strichartz estimates, see [7]). Let \( 2 \leq q_1, q_2 \leq \infty \), \( 2 \leq r_1, r_2 < \infty \) and \( \rho_1, \rho_2, s \in \mathbb{R} \) be constants with
\[
1/q_i + 1/r_i \leq 1/2, \quad i = 1, 2; \quad 1/q_1 + 3/r_1 = 3/2 - s + \rho_1; \quad 1/q_2 + 3/r_2 = 1/2 + s + \rho_2.
\]
Assume that \( u \) is the solution to the linear wave equation
\[
\left\{ \begin{array}{l}
\partial_t u - \Delta u = F(x,t), \\
u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3); \\
\partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3).
\end{array} \right.
\]
Then we have
\[
\|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_{\xi}^\rho u\|_{L^q(0,T;L^r(\mathbb{R}^3))} \leq C \left( \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_{\xi}^{\rho_2} F(x,t)\|_{L^q(0,T;L^r(\mathbb{R}^3))} \right).
\]
Here the coefficients \( \hat{q}_2 \) and \( \hat{r}_2 \) satisfy \( 1/q_2 + 1/\hat{q}_2 = 1, 1/r_2 + 1/\hat{r}_2 = 1 \). The constant \( C \) does not depend on \( T \) or \( u \).
Combining suitable Strichartz estimates with a fixed-point argument, we have the following scattering theory with small data in the critical Sobolev space.

**Proposition 2.7** (Scattering with small initial data). There exists a constant \( \delta = \delta(p) > 0 \), so that if the initial data satisfy \( \| (u_0, u_1) \|_{H^p \times H^{p-1}} < \delta \), then the corresponding solution \( u \) to (CP1) exists globally in time and scatters with \( \| (u(\cdot, t), u_1(\cdot, t)) \|_{H^p \times H^{p-1}} < 2\delta \).

### 3 Examples of weakly non-radiative solutions

In this section we prove that energy subcritical wave equation in the defocusing case admits weakly non-radiative solutions that are orthogonal to \( (r^{-1}, 0) \) in the energy space, i.e. we prove Theorem [L8].

**Reduction to ODE** The \( C^2 \) solution we construct is in the form of \( u(x, t) = |x|^{-2/(p-1)} f(t/|x|) \) with initial data \( (u_0, u_1) = (0, a|x|^{-2/(p-1)-1}) \). Here \( f \) is a \( C^2 \) function defined on \((-1, 1)\) and \( a > 0 \) is a parameter. Please note \( u_1 \notin L^2(\mathbb{R}^3) \) but \( u_1 \in L^2(\{x \in \mathbb{R}^3 : |x| > R\}) \) for any \( R > 0 \). We may use polar coordinates and the notation \( \beta = 2/(p-1) \) to write \( u(r, t) = r^{-\beta} f(t/r) \) for convenience. A straightforward calculation shows

\[
\begin{align*}
    u_t &= r^{-\beta-2} f''(t/r); \\
    u_r &= -\beta r^{-\beta-1} f(t/r) - tr^{-\beta-2} f'(t/r); \\
    u_{rr} &= \beta(\beta+1)r^{-\beta-2} f(t/r) + (2\beta+2)tr^{-\beta-3} f'(t/r) + t^2r^{-\beta-4} f''(t/r); \\
    \Delta u &= u_{rr} + (2/r)u_r = \beta(r^{-\beta-2} f(t/r) + 2\beta tr^{-\beta-3} f'(t/r) + t^2r^{-\beta-4} f''(t/r)); \\
    |u|^{p-1} u &= r^{-\beta-2} |f(t/r)|^{p-1} f(t/r).
\end{align*}
\]

We plug these in the defocusing wave equation \( \partial_t^2 u - \Delta u = -|u|^{p-1} u \) and obtain

\[
r^{-\beta-2} \left[ \left( 1 - \frac{t^2}{r^2} \right) f'' \left( \frac{t}{r} \right) - 2\beta \cdot \frac{t}{r} f' \left( \frac{t}{r} \right) + \beta(1 - \beta) f \left( \frac{t}{r} \right) + \left| f \left( \frac{t}{r} \right) \right|^{p-1} f \left( \frac{t}{r} \right) \right] = 0.
\]

Therefore \( f \) satisfies the ordinary differential equation

\[
\begin{align*}
    \left\{ \begin{array}{ll}
    (1 - x^2)f''(x) - 2\beta xf'(x) + \beta(1 - \beta)f(x) + |f(x)|^{p-1}f(x) = 0, & x \in (-1, 1); \\
    f(0) = 0, & f'(0) = a.
\end{array} \right.
\]

Each solution \( f \) to (7) gives a solution \( u(x, t) = |x|^{-2/(p-1)} f(t/|x|) \) to the defocusing wave equation defined on \((x, t) : |t| < |x|\). Some useful properties of solutions to the initial value problem (7) are summarized in the following proposition. We postpone its proof until Section [S] of this work since it is irrelevant to our main topics.

**Proposition 3.1.** Let \( \beta \in (1/2, 1), \gamma > 0 \) and \( p > 1 \) be constants. The solutions to the ordinary differential equation

\[
\begin{align*}
    \left\{ \begin{array}{ll}
    (1 - x^2)f''(x) - 2\beta xf'(x) + \gamma f(x) + |f(x)|^{p-1}f(x) = 0, & x \in (-1, 1); \\
    f(0) = 0, & f'(0) = a;
\end{array} \right.
\]

satisfy the following properties

(i) The solutions \( f(x) \) are classic solutions defined for all \( x \in (-1, 1) \); i.e. \( f \in C^2((-1, 1)) \).

(ii) We have continuous dependence of \( f(x) \) on initial value \( a \) up to the endpoints, i.e. we may define \( f(x) \) at \( x = \pm 1 \) so that \( f(x) \) becomes a continuous function of \( (x, a) \in [-1, 1] \times \mathbb{R} \). In addition, we have a uniform upper bound \( |f(x)| \lesssim_{\beta, \gamma, p} |a| \).
Weakly non-radiative solutions

Now let us choose a positive parameter $a$ as in part (iv) and consider the solution $u(x, t) = |x|^{-2/(p-1)}f(t/|x|)$. According to part (ii) of Proposition 3.1 we have a uniform upper bound $|u(x, t)| \lesssim |x|^{-2/(p-1)}$ for all $|x| > t$. A simple calculation shows that $u \in X(\mathbb{R}) = L^{\frac{2p}{p-1}}(\mathbb{R}; L^2(\{x : |x| > |t| + R\}))$ for any $R > 0$. Thus $u$ is always an exterior solution to (CP1) in the exterior region $\{(x, t) : |x| > |t| + R\}$. Our choice of initial value $a$ guarantees that both $f$ and $f'$ are bounded, therefore we have the following estimates for any $r = |x| > t$:

$$|u_r| \lesssim r^{-2/(p-1)-1} |f(t/r)| + |t| r^{-2/(p-1)-2} |f'(t/r)| \lesssim r^{-2/(p-1)-1};$$

$$|u_t| \lesssim r^{-2/(p-1)-1} |f'(t/r)| \lesssim r^{-2/(p-1)-1}.$$ 

Thus

$$\int_{|x| > |t|} (|\nabla u(x, t)|^2 + |u_t(x, t)|^2) \, dx = 4\pi \int_{|t|}^{\infty} \left( |u_r(r, t)|^2 + |u_t(r, t)|^2 \right) r^2 \, dr \lesssim \int_{|t|}^{\infty} r^{-4/(p-1)} \, dr \lesssim |t|^{-\frac{4}{p-4}}.$$ 

This vanishes as $|t| \to \infty$. As a result, $u$ is an $R$-weakly non-radiative solution to (CP1) for any $R > 0$.

4 Weakly non-radiative solutions in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$

In this section we give a proof of Theorem 1.13. The general idea comes from Duyckaerts-Kenig-Merle. In the author’s previous work [13] the same idea is used to deal with soliton-like minimal blow-up solutions $v$ obtained via the compactness-rigidity argument, whose trajectory $\{(v(t), v_x(t)) : t \in \mathbb{R}\}$ is pre-compact in both spaces $\dot{H}^1 \times L^2$ and $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. A soliton-like minimal blow-up solution is clearly a special case of non-radiative solutions. In this work we improve the argument so that it works for all $R$-weakly non-radiative solution with initial data in $\dot{H}^{s_p}_R$.

4.1 Asymptotic behaviour of non-radiative solutions

The lemmata in the subsection describe behaviour of weakly non-radiative radial solutions $u(x, t)$ to (CP1) with initial data in $\dot{H}^{s_p}_R$ when $x$ is sufficiently large.

Lemma 4.1. Let $u$ be a radial, $R_0$-weakly non-radiative solution to (CP1) with initial data $(u_0, u_1) \in \dot{H}^{s_p}_R$. Then given any $\varepsilon > 0$, there exists a large radius $R_* = R_*(\varepsilon, u) > 0$, so that the inequality $|u(r, t)| \leq \varepsilon r^{-2/(p-1)}$ holds for all $r > \max\{|t| + R_0, R_*\}$.
\textbf{Proof.} Given any small positive constant \( \varepsilon < \varepsilon(p) \), we may choose a large radius \( R \) so that
\[
\|P_R(u_0, u_1)\|_{H^p \times H^{p-1}} < \varepsilon.
\]
Here \( P_R \) is the center cut-off operator defined in Lemma 2.1. Let \( u^{(R)} \) be the solution to (CP1) with initial data \( P_R(u_0, u_1) \). Scattering theory with small initial data (Proposition 2.7) then guarantees that \( u^{(R)} \) is globally defined in time and satisfies
\[
\|\left(u^{(R)}(\cdot, t), u_{t}^{(R)}(\cdot, t)\right)\|_{H^p(\mathbb{R}^3) \times H^{p-1}(\mathbb{R}^3)} < 2\varepsilon, \quad \forall t \in \mathbb{R}.
\]
By finite speed of propagation and Lemma 2.2, we have \( r, t \) for all \((r, t)\) with \( r \) sufficiently large radius in the lighter grey region of figure 1, then
\[
|w(r, t)| \leq |w(|t| + R)| + \int_{r}^{|t| + R} |w_r(r', t)|dr'.
\]
In the final step we apply Lemma 2.3. The latter term in the final line above has an upper bound independent of \((r, t)\) by our non-radiative assumption and Remark 1.3. Thus there exists a sufficiently large radius \( R_0 > 0 \), so that if \( \max\{R_0, |t| + R_0\} < r < |t| + R \), i.e. the point \((r, t)\) is in the lighter grey region of figure 1, then
\[
|w(r, t)| \leq 3C_p\varepsilon r^{-2/(p-1)} \Rightarrow |u(r, t)| \leq 3C_p\varepsilon r^{-2/(p-1)}.
\]
Combining this with the case \( r \geq |t| + R \), we finish the proof.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{regions.png}
\caption{Illustration of regions in the proof of Lemma 4.1}
\end{figure}

Lemma 4.2. Assume that $R \geq R_0 > 0$. Let $u$ be a radial, $R_0$-weakly non-radiative solution to (CP1) satisfying $\beta \geq 2/(p-1)$

$$|u(r,t)| \leq \varepsilon r^{-\beta}, \quad r > \max\{|t| + R_0, R\}.$$ 

Then we have

(a) The function $v_+(r,t) = (\partial_t - \partial_r)(ru)$ satisfies the identity

$$v_+(r,t) = -\zeta \int_1^\infty (t' - t + r)|u|^{p-1}u(t' - t + r, t')dt'$$

for almost everywhere $r > |t| + R_0$.

(b) The inequality $|\partial_t (ru)|, |ru_1| \leq 2e^{2p_r-\beta}$ hold for almost everywhere $r > \max\{|t| + R_0, R\}$.

Proof. Let $w(r,t) = ru(r,t)$ solves the one-dimensional wave equation $w_{tt} - w_{rr} = \zeta r|u|^{p-1}u$. Therefore $v_+(r,t) = w_t(r,t) - w_r(r,t)$ satisfies

$$\frac{d}{dt}v_+(t - t_1 + r, t) = \zeta(t - t_1 + r)|u|^{p-1}u(t - t_1 + r, t), \quad t > t_1, \ r > |t_1| + R_0;$$

$$v_+(t_2 - t_1 + r, t_2) - v_+(r, t_1) = \int_{t_1}^{t_2} \zeta(t - t_1 + r)|u|^{p-1}u(t - t_1 + r, t)dt, \quad t_2 > t_1, \ r > |t_1| + R_0. \quad (8)$$

Our assumption on decay of $u$ implies that the absolute value of integrand satisfies

$$(t - t_1 + r)|u(t - t_1 + r, t)|^p \leq \varepsilon^p(t - t_1 + r)^{1-\beta}, \quad t > \max\{t_1, R - R_0\}, \ r > |t_1| + R_0.$$

Thus if we fix $t_1$, then the right hand side integral of (8) converges uniformly for all $r > |t_1| + R_0$ as $t_2 \to +\infty$. Combining this uniform convergence, our non-radiative assumption and Corollary 2.5, we may make $t_2 \to +\infty$ and obtain an identity

$$v_+(r, t_1) = -\int_{t_1}^\infty \zeta(t - t_1 + r)|u|^{p-1}u(t - t_1 + r, t)dt, \quad \text{in } L^2_{loc}\{r : r > |t_1| + R_0\}.$$

This proves part (a). If $r > \max\{|t| + R_0, R\}$, we may use the conclusion of part (a), and plug the decay assumption $u(r,t) \leq \varepsilon r^{-\beta}$ in the right hand integral to conclude

$$|v_+(r,t)| \leq 2e^{2p_r-\beta}, \quad a.e. \ r > \max\{|t| + R_0, R\}.$$

We may prove a similar inequality about $v_-(r,t) = (\partial_t + \partial_r)(ru)$ in the same manner. Combining these two inequalities we finish the proof. \qed

Lemma 4.3. Assume that $R_0 > 0$ and $R \geq \max\{1, R_0\}$. Let $u$ be a radial, $R_0$-weakly non-radiative solution to (CP1) satisfying

$$|u(r,t)| \leq \varepsilon r^{-\beta}, \quad r > \max\{|t| + R_0, R\}$$

for a sufficiently small constant $\varepsilon < \varepsilon_0(p)$ and $\beta \in [2/(p-1), 3/p]$. Then there exists a large radius $R_1 = R_1(p, R)$ and a small constant $\kappa = \kappa(p) > 0$ so that

$$|u(r,t)| \leq \varepsilon r^{-\beta - \kappa}, \quad r > \max\{|t| + R_0, R_1\}.$$

Proof. Let us define $(n = 0, 1, 2, \cdots)$

$$a_n = \sup\{r^n|u(r,t)| : t \in \mathbb{R}, \ r > \max\{|t| + R_0, 2^n R\}\} \leq \varepsilon.$$
Thus we have \(|u(r,t)| \leq a_n r^{-\beta}\) for all \((r,t)\) with \( r > \max\{|t| + R_0, 2^n R_0\} \). Given any \((r,t)\) with \( r > \max\{|t| + R_0, 2^n R_0\} \), we may utilize Lemma 4.3 and verify that \(w(r,t) = ru(r,t)\) satisfies

\[
|w(r,t)| \leq |w(r/2,0)| + |w(r,0) - w(r/2,0)| + |w(r,t) - w(r,0)|
\]

\[
\leq |w(r/2,0)| + \int_{r/2}^{r} |w_t(r',0)|dr' + \left| \int_{0}^{t} w_t(r,t')dt' \right|
\]

\[
\leq (r/2)^{1-\beta}a_{n-1} + 4a_{n-1}r^{3-\beta}.
\]

Thus we have

\[
a_n = \sup\{r^{3-1}|w(r,t)| : t \in \mathbb{R}, r > \max\{|t| + R, 2^n R\}\}
\]

\[
\leq \sup\left\{(1/2)^{1-\beta}a_{n-1} + 4a_{n-1}r^{3-2(p-1)} : t \in \mathbb{R}, r > \max\{|t| + R, 2^n R\}\right\}
\]

\[
\leq (1/2)^{1-3/p + 4\varepsilon p^{-1}}a_{n-1} \leq \lambda a_{n-1}.
\]

Here we may choose an arbitrary constant \( \lambda = \lambda(p) \in ((1/2)^{1-3/p}, 1) \) and determine \( \varepsilon = \varepsilon(p) \) accordingly. Therefore we have \( a_n \leq \varepsilon \lambda^n \). As a result, given any \((r,t)\) with \( r > \max\{|t| + R_0, R\} \), we may choose \( n = \max\{n \in \mathbb{Z} : 2^n R < r\} \) and find an upper bound

\[
r^{\beta} |u(r,t)| \leq a_n \leq \varepsilon \lambda^n \leq \varepsilon \lambda^{\log_2(r/2R)} = \varepsilon (2R)^{\log_2(1/\lambda)} r^{-\log_2(1/\lambda)}
\]

Thus we may choose an arbitrary constant \( \kappa = \kappa(p) \in (0, \log_2(1/\lambda)) \) and determine \( R_1 = R_1(R, p) \) accordingly so that the inequality \(|u(r,t)| \leq \varepsilon r^{-\beta - \kappa}\) holds for all \((r,t)\) with \( r > \max\{|t| + R_0, R_1\} \).

**Proposition 4.4.** Assume that \( R_0 > 0 \) and \( R \geq \max\{1, R_0\} \). Let \( u \) be a radial, \( R_0\)-weakly non-radiative solution to (CP1) satisfying

\[
|u(r,t)| \leq \varepsilon r^{-2/(p-1)}, \quad r > \max\{|t| + R_0, R\}
\]

for a sufficiently small positive constant \( \varepsilon < \varepsilon_0(p) \). Then there exists two constants \( C \in \mathbb{R} \) and \( R' > 1 \) so that

\[
|u(r,t) - C/r| \lesssim r^{2-p}, \quad \forall r > \max\{|t| + R_0, R'\};
\]

\[
|u_r(r,t) + C/r^2| + \left| u_t(r,t) \right| \lesssim r^{1-p}, \quad \forall \ a.e. \ r > \max\{|t| + R_0, R'\}.
\]

**Proof.** First of all, we gain better decay estimates of \( u \) by induction. Application of Lemma 4.3 multiple times leads to a finite sequence \( R < R_1 < R_2 < \cdots < R_n \) with \( 2/(p-1) + (n-1)\kappa \leq 3/p < 2/(p-1) + n\kappa \) and

\[
|u(r,t)| \leq \varepsilon r^{-2/(p-1)-\kappa}, \quad \forall r > \max\{|t| + R_0, R_j\}, \quad j = 1, 2, \ldots, n.
\]

For convenience we define \( \beta = 2/(p-1) + n\kappa > 3/p \) and apply Lemma 4.2

\[
|w_r(r,t)|, \ |w_t(r,t)| \leq 2\varepsilon p r^{2-p}\beta, \quad \forall \ a.e. \ r > \max\{|t| + R_0, R_n\}.
\]

Since \( 2 - p\beta < -1 \), the function \( w(r,t) \) converges as \( r \to +\infty \) for all fixed \( t \in \mathbb{R} \). In addition, this limit is independent of \( t \) because of the decay estimate of \( w_t \). Therefore there exists a constant \( C \), so that

\[
|w(r,t) - C| \lesssim r^{3-\beta}, \quad r > \max\{|t| + R_0, R_n\}.
\]

It immediately follows that \(|w(r,t)| \lesssim 1 \Rightarrow |u(r,t)| \lesssim r^{-1} \). We then apply Lemma 4.2 again and obtain

\[
|w_r(r,t)| + |w_t(r,t)| \lesssim r^{2-p} \quad a.e. \Rightarrow |w(r,t) - C| \lesssim r^{3-p}, \quad \forall r > \max\{|t| + R_0, R_n\}.
\]

Finally we rewrite these inequalities in term of \( u \) and finish the proof. \( \square \)
4.2 Coincidence of Non-radiative Solutions

In this subsection we show that two weakly non-radiative radial solutions with the same asymptotic behaviour as $r \to \infty$ must be exactly the same.

**Lemma 4.5.** Assume that $R' > R_0 > 0$. Let $u$ and $\tilde{u}$ be two radial, $R_0$-weakly non-radiative solutions to (CP1) so that $u(r, t) = \tilde{u}(r, t)$ if $r > |t| + R'$. Then the identity $u(r, t) = \tilde{u}(r, t)$ also holds if $r > |t| + R_0$.

**Proof.** Let us define

$$R = \min \{ r' \geq R_0 : u(r, t) = \tilde{u}(r, t) \text{ if } r > |t| + r' \} \leq R'.$$

It suffices to show $R = R_0$. If $R > R_0$, we define a function $g$ for $\delta \in (0, R - R_0)$. Functions $v_\pm, w$ below be defined as in the proof of Lemma 4.2. $\tilde{v}_\pm, \tilde{w}$ are derived from $\tilde{u}$ in the same manner.

$$g(\delta) = \sup_{t \in \mathbb{R}} \left\{ \int_{|t| + R - \delta}^{\infty} |w_r(r, t) - \tilde{w}_r(r, t)|^2 + |w_t(r, t) - \tilde{w}_t(r, t)|^2 \, dr \right\}$$

$$= \sup_{t \in \mathbb{R}} \left\{ \int_{|t| + R - \delta}^{R} |w_r(r, t) - \tilde{w}_r(r, t)|^2 + |w_t(r, t) - \tilde{w}_t(r, t)|^2 \, dr \right\}$$

We always have $g(\delta) < +\infty$ by our non-radiative assumption, Remark 4.3 and Lemma 2.4. By the same argument as in Lemma 4.2, the following identity holds for any time $t_2 > t_1$

$$[v_+(t_2 - t_1 + r, t_2) - \tilde{v}_+(t_2 - t_1 + r, t_2)] - [v_+(r, t_1) - \tilde{v}_+(r, t_1)]$$

$$= \int_{t_1}^{t_2} \zeta(t - t_1 + r) [|u|^{p-1} u(t - t_1 + r) - |\tilde{u}|^{p-1} \tilde{u}(t - t_1 + r)] \, dt, \text{ in } L^p(J(t_1, \delta)).$$

For convenience we use the notation $J(t_1, \delta) = [|t_1| + R - \delta, |t_1| + |t|]$. Please see figure 2 for an illustration of the integral path involved. By considering the limits of both sides of (9) in the space $L^p(J(t_1, \delta))$ as $t_2 \to +\infty$, we obtain an identity

$$\tilde{v}_+(r, t_1) - v_+(r, t_1) = \int_{t_1}^{\infty} \zeta(t - t_1 + r) [|u|^{p-1} u(t - t_1 + r) - |\tilde{u}|^{p-1} \tilde{u}(t - t_1 + r)] \, dt. \quad (10)$$
The limit of the left hand side is relatively easy. We only need to recall the non-radiative assumption, apply Corollary 2.3 and obtain

$$\lim_{t_2 \to +\infty} \left( \|v_+(t_2 - t_1 + r, t_2)\|_{L^2(J(t_1, \delta))} + \|\tilde{v}_+(t_2 - t_1 + r, t_2)\|_{L^2(J(t_1, \delta))} \right) = 0.$$  

In order to evaluate the limit of the right hand side we first give upper bounds of $u$, $\tilde{u}$ as well as $w - \tilde{w}$. We recall Remark 2.3 our non-radiative assumption and apply Lemma 2.3 to obtain

$$|u(r, t)| \leq M_r r^{-1/2}, \quad r > |t| + R_0; \quad M = \sup_{t \in \mathbb{R}} \left( \frac{1}{4\pi} \int_{|x| > |t| + R_0} |\nabla u(x)|^2 dx \right)^{1/2} < +\infty.$$  

$$|\tilde{u}(r, t)| \leq \tilde{M} r^{-1/2}, \quad r > |t| + R_0; \quad \tilde{M} = \sup_{t \in \mathbb{R}} \left( \frac{1}{4\pi} \int_{|x| > |t| + R_0} |\nabla \tilde{u}(x)|^2 dx \right)^{1/2} < +\infty.$$  

Similarly we have

$$|w(r, t) - \tilde{w}(r, t)| \leq \tilde{C} r^{-1/2}, \quad r > |t| + R_0; \quad \tilde{C} = \sup_{t \in \mathbb{R}} \left( \frac{1}{4\pi} \int_{|x| > |t| + R_0} |\nabla (w - \tilde{w})(x)|^2 dx \right)^{1/2} < +\infty.$$  

We may also find an upper bound of $w - \tilde{w}$ at $(t - t_1 + r, t)$ with $r \in J(t_1, \delta)$ and $t > t_1$

$$|(w - \tilde{w})(t - t_1 + r, t)| \leq \left( (w - \tilde{w})(t - t_1 + r + \delta, t) + \int_{t - t_1 + r}^{t - t_1 + r + \delta} |w_r(r', t) - \tilde{w}_r(r', t)| dr' \right)^{1/2} \leq \delta^{1/2} \left( \int_{t - t_1 + r}^{t - t_1 + r + \delta} |w_r(r', t) - \tilde{w}_r(r', t)|^2 dr' \right)^{1/2} \leq \delta^{1/2} g(\delta)^{1/2}.$$  

Here $t - t_1 + r + \delta \geq t - t_1 + (|t_1| + R - \delta) + \delta \geq |t| + R$, thus $|(w - \tilde{w})(t - t_1 + r + \delta, t)| = 0$. We also have $t - t_1 + r \geq |t| + R - \delta$ thus the integral of $|w_r - \tilde{w}_r|^2$ is dominated by $g(\delta)$. Combining these two estimates we may find an upper bound of the integrand in the right hand side of (10) as below. Please note that all the functions are evaluated at $(t - t_1 + r, t)$ unless specified otherwise.

$$\int J(t - t_1 + r) [p|u|^{p-1}u - |\tilde{u}|^{p-1}\tilde{u}] \leq p\|u - \tilde{u}\|^{p-1} \left( (w - \tilde{w})(t - t_1 + r, t) + \int_{t - t_1 + r}^{t - t_1 + r + \delta} |w_r(r', t) - \tilde{w}_r(r', t)| dr' \right)^{1/2} \leq p(M + \tilde{M})^{-1/2} g(\delta)^{1/2}.$$  

Here $\frac{p-1}{2} > 1$. Thus as $t_2 \to \infty$, the integral in the right hand side of (10) converges to that of (11) uniformly for all $r \in J(t_1, \delta)$. This immediately gives the convergence in $L^2(J(t_1, \delta))$. Next we substitute the integrand in (11) by its upper bound given above and obtain

$$\left| \tilde{v}_+(r, t_1) - v_+(r, t_1) \right| \leq \int_{t_1}^{\infty} p(M + \tilde{M})^{-1/2} \left( (w - \tilde{w})(t - t_1 + r, t) + \int_{t - t_1 + r}^{t - t_1 + r + \delta} |w_r(r', t) - \tilde{w}_r(r', t)| dr' \right)^{1/2} dt \leq \frac{2p}{p-3} (M + \tilde{M})^{-1/2} g(\delta)^{1/2} r -\frac{\epsilon_2}{3}.$$  

Thus

$$\int_{|t_1| + R - \delta}^{|t_1| + R} \left| \tilde{v}_+(r, t_1) - v_+(r, t_1) \right|^2 dr \leq \int_{|t_1| + R - \delta}^{|t_1| + R} \frac{4p^2}{(p - 3)^2} (M + \tilde{M}) g(\delta) R_0^{-2} dr = C(p, M, \tilde{M}, R_0) g(\delta).$$  

Similarly we have

$$\int_{|t_1| + R - \delta}^{|t_1| + R} \left| \tilde{v}_-(r, t_1) - v_-(r, t_1) \right|^2 dr \leq C(p, M, \tilde{M}, R_0) g(\delta).$$  

Since these inequalities hold for all $t_1 \in \mathbb{R}$, we have

$$2g(\delta) = \sup_{t \in \mathbb{R}} \int_{|t| + R - \delta}^{|t| + R} (|v_- - \tilde{v}_-(r, t)|^2 + |v_+ - \tilde{v}_+(r, t)|^2) dr \leq 2C(p, M, \tilde{M}, R_0) g(\delta).$$  

This means $g(\delta) = 0$ for sufficiently small $\delta > 0$, which implies that $w(r, t) = \tilde{w}(r, t)$ for all $(r, t)$ with $r > |t| + R - \delta$, thus gives a contradiction. \hfill \Box
Proposition 4.6. Let u and \( \tilde{u} \) be two radial, \( R_0 \)-weakly non-radiative solutions to (CP1). In addition, there exists a large radius \( R' > \max \{ R_0, 1 \} \) and a constant \( C > 0 \) so that

\[
|u(r, t)| + |\tilde{u}(r, t)| \leq \frac{C}{r}, \quad r > \max \{ |t| + R_0, R' \};
\]

\[
\lim_{r \to +\infty} |ru(r, t) - r\tilde{u}(r, t)| = 0, \quad \forall t \in \mathbb{R}.
\]

Then \( u(r, t) \equiv \tilde{u}(r, t) \) for all \( (r, t) \) with \( r > |t| + R_0 \).

Proof. Without loss of generality we assume \( R_0 > 0 \). Otherwise we first prove the identity for \( r > |t| + R \) with small positive numbers \( R > 0 \) and then let \( R \to 0^+ \). We first apply Lemma 4.2 on both \( u \) and \( \tilde{u} \) to obtain \( (v_\pm, w) \) defined as in the proof of Lemma 4.2, \( \tilde{v}_\pm, \tilde{w} \) are derived from \( \tilde{u} \) in the same manner.

\[
v_+(r, t) - \tilde{v}_+(r, t) = \zeta \int_t^{\infty} (t' - t + r) \left[ |\tilde{u}|^{p-1}\tilde{u}(t' - t + r, t') - |u|^{p-1}u(t' - t + r, t') \right] dt' \quad (11)
\]

Now let us assume \( |\tilde{u}(r, t) - u(r, t)| \leq Cr^{-\beta} \) for some constant \( \beta \geq 1 \) and all \( r > \max \{ R_0 + |t|, R' \} \). (This holds for \( \beta = 1 \)) Then we immediately have

\[
|v_+(r, t) - \tilde{v}_+(r, t)| \leq \int_t^{\infty} (t' - t + r) \cdot p[C(t' - t + r)^{-1}]^{p-1}|u(t' - t + r, t') - \tilde{u}(t' - t + r, t')| dt' 
\]

\[
\leq \frac{p}{p-3+\beta}C^{p-\beta}r^{p-3+\beta}.
\]

Similarly we may prove an inequality regarding \( v_- \) and \( \tilde{v}_- \). By \( v_\pm = w_t \mp w_r \) and \( \tilde{v}_\pm = \tilde{w}_t \mp \tilde{w}_r \) we have

\[
|w_r(r, t) - \tilde{w}_r(r, t)| \leq \frac{p}{p-3+\beta}C^{p-\beta}r^{p-3+\beta}, \quad \forall a.e. r > \max \{ R_0 + |t|, R' \}
\]

Combining this with our assumption on the limit of \( w - \tilde{w} \) as \( r \to \infty \), we have

\[
|w(r, t) - \tilde{w}(r, t)| \leq \frac{p}{(p-3+\beta)(p-4+\beta)}C^{p-\beta}r^{p-4+\beta}, \quad r > \max \{ R_0 + |t|, R' \};
\]

\[
|u(r, t) - \tilde{u}(r, t)| \leq \frac{p}{(p-2)(p-3)}C^{p-\beta}r^{p-\beta}, \quad r > \max \{ R_0 + |t|, R' \}.
\]

Without loss of generality we may assume (otherwise we may enlarge \( R' \))

\[
\frac{p}{(p-2)(p-3)}C^{p-1}(R')^{-\frac{p-1}{2}} < 1.
\]

Thus we have \( |u(r, t) - \tilde{u}(r, t)| \leq Cr^{-\frac{p-2}{2}} \) if \( r > \max \{ R_0 + |t|, R' \} \). By induction we have \( |u(r, t) - \tilde{u}(r, t)| \leq Cr^{-\frac{p-2}{2}n-\beta} \) for all \( n = 0, 1, 2, \cdots \) and \( r > \max \{ R', |t| + R_0 \} \). This means \( u(r, t) \equiv \tilde{u}(r, t) \) for all \( r > \max \{ R', |t| + R_0 \} \). Finally we apply Lemma 4.5 to conclude that \( u(r, t) \equiv \tilde{u}(r, t) \) for all \( r > |t| + R_0 \).

4.3 Proof of Theorem 1.13

Now let us assume that \( u \) is a radial, \( R_0 \)-weakly non-radiative solution to (CP1) with initial data \( (u_0, u_1) \in H^{R_0}_{\mu} \). By Lemma 4.1, given \( \varepsilon > 0 \), there exists a large radius \( R_* = R_*(u, \varepsilon) \), so that \( |u(r, t)| \leq Cr^{-2/(p-1)} \) holds for all \( r > \max \{ |t| + R_0, R_1 \} \). This enables us to apply Proposition 4.4 and obtain two constants \( C \in \mathbb{R}, R' > 1 \) so that

\[
|u(r, t) - C/r| \leq \varepsilon r^{2-p}, \quad \text{if } r > \max \{ |t| + R_0, R' \}.
\]
We claim that the constant $C$ also satisfies $|C|^p R_- < R_0$ in the defocusing case. If this were false, i.e. $R_C^- \leq |C|^p R_- \geq R_0 > 0$, then we might apply Proposition 1.3 on $u$ and $U_C^-$ in the region $\{(x, t) : |x| > |t| + R_C^- + \varepsilon\}$ with an arbitrary $\varepsilon > 0$. We obtain $u(x, t) = U_C^-(x)$ if $|x| > |t| + R_C^- + \varepsilon$. By making $\varepsilon \to 0^+$ we have $u(x, t) = U_C^-(x)$ for all $(x, t)$ with $|x| > |t| + R_C^-$. Thus $u_0(x) = u(x, 0) = U_C^-(x)$ blows up when $|x| \to (R_C^-)^+$. This gives a contradiction, thanks to Lemma 2.3. Finally we are able to apply Proposition 4.6 on $u$ and $U_C^-$ in the region $\{(x, t) : |x| > |t| + R_0\}$ to conclude that $u(x, t) = U_C^-(x)$ whenever $|x| > t + R_0$ and finish the proof.

5 Appendix A: Ordinary Differential Equations

In this section we consider the ordinary differential equation

$$\begin{cases}
(1 - x^2)f''(x) - 2\beta x f'(x) + \gamma f(x) + |f(x)|^{p-1} f(x) = 0, & x \in (-1, 1); \\
f(0) = 0, \quad f'(0) = a;
\end{cases}$$

and prove Proposition 5.1. If we use the notation $\lesssim$ in the proof below, then the implicit constant depends on $\beta, \gamma, p$ unless specified otherwise.

5.1 Global existence

Classic theory of ordinary differential equations guarantees that the solution $f$ is $C^2$ and defined in a maximal interval $(-\delta, \delta)$ for some $\delta \in (0, 1]$. In order to verify $\delta = 1$ we only need to show that $f(x)$ and $f'(x)$ are both bounded in the interval $(-\delta, \delta)$ if $\delta < 1$. This can be done by a semi-conservation law. We may multiply both sides of equation (12) by $(1 - x^2)^{2\beta-1} f'(x)$ and obtain

$$\frac{d}{dx} \left[ \frac{1}{2}(1 - x^2)^{2\beta}|f'(x)|^2 + (1 - x^2)^{2\beta-1} P(f(x)) \right] = -2(2\beta - 1)x(1 - x^2)^{2\beta-2} P(f(x)).$$

Here the potential $P$ is defined by

$$P(y) = \frac{\gamma}{2}|y|^2 + \frac{1}{p+1}|y|^{p+1} \geq 0.$$ 

The derivative above is nonpositive if $x > 0$ and nonnegative if $x < 0$. Thus we always have

$$\frac{1}{2}(1 - x^2)^{2\beta}|f'(x)|^2 + (1 - x^2)^{2\beta-1} P(f(x)) \leq \frac{a^2}{2}. \quad (13)$$

This immediately gives the boundedness of $f'(x)$ and $f(x)$, as well as the continuous dependence of $f(x)$ and $f'(x)$ on parameter $a$, as long as $x$ is away from the endpoints $\pm 1$. Before we conclude this subsection, we also give another semi-conservation law for future use. We may multiply the original equation by $f'(x)$ and obtain

$$\frac{d}{dx} \left[ \frac{1}{2}(1 - x^2)|f'(x)|^2 + P(f(x)) \right] = (2\beta - 1)x|f'(x)|^2.$$

Therefore we can find a lower bound regarding $f(x)$ and $f'(x)$.

$$\frac{1}{2}(1 - x^2)|f'(x)|^2 + P(f(x)) \geq \frac{a^2}{2}, \quad \forall x \in (-1, 1). \quad (14)$$
5.2 Continuity of $f(x)$ at the endpoints

Now let us consider the behaviour of $f(x)$ when $x \to 1^-$. The behaviour of $f(x)$ when $x \to -1^+$ is similar because $f$ is an odd function. First of all, the inequality (13) implies

$$|f'(x)| \leq |a|(1-x)^{-\beta}, \quad x > 0.$$  

Since $\beta < 1$, we know the limit $f(1) = \lim_{x \to 1^-} f(x)$ is well-defined. In addition, if we fix $x_0 \in (0, 1)$, then we have

- The function $f$ is a continuous function of $(x, a) \in [0, x_0] \times \mathbb{R}$.
- The upper bound of $f'(x)$ given above also implies

$$\sup_{x_1, x_2 \in [x_0, 1]} |f(x_1) - f(x_2)| \leq \int_{x_0}^1 |f'(y)| dy \lesssim_\beta |a|(1-x_0)^{1-\beta}. \quad (15)$$

These immediately give the continuity of $f$ on $(x, a) \in [0, 1] \times \mathbb{R}$. We may also combine (13) and (15) (with $x_0 = 1/2$) to give an upper bound

$$\max_{x \in [0, 1]} |f(x)| \lesssim |a|. \quad (16)$$

5.3 Asymptotic behaviour of $f'(x)$ at endpoints

In order to investigate the asymptotic behaviour of $f'(x)$ as $x \to 1^-$, we calculate (We always assume $x \geq 0$ in this subsection, the property of $f(x)$ for negative $x$ can be obtained by symmetry)

$$\frac{d}{dx} [(1-x^2)^{-\beta} f'(x)] = (1-x^2)^{-\beta-1} [(1-x^2)f''(x) - 2\beta xf'(x)] = -(1-x^2)^{-\beta-1} P'(f(x)). \quad (17)$$

Here we use the equation (12) again. Since $|(1-x^2)^{-\beta-1} P'(f(x))| \lesssim (1-x)^{\beta-1} P'(|a|)$ is integrable in $[0, 1]$, we have a well-defined limit $G = \lim_{x \to 1^-} (1-x^2)^{-\beta} f'(x)$ with

$$G - (1-x^2)^{-\beta} f'(x) = -\int_x^1 (1-y)^{-\beta-1} P'(f(y)) dy; \quad (18)$$

$$|G - (1-x^2)^{-\beta} f'(x)| \lesssim \int_x^1 (1-y)^{-\beta-1} P'(|a|) dy \lesssim (1-x)^{\beta} P'(|a|); \quad (19)$$

$$|f'(x) - (1-x^2)^{-\beta} G| \lesssim C a P'(|a|), \quad x \in [0, 1). \quad (20)$$

We may combine (19) with the fact that $f'(x_0)$ depends continuously on parameter $a$ for a fixed $x_0 \in (0, 1)$ to conclude that $G$ is a continuous function of $a$. Now let us have a more careful look at the asymptotic behaviour of $f'(x)$ near 1. According to (15) and (16), we have

$$|P'(f(x)) - P'(f(1))| \lesssim P'(|a|)|f(x) - f(1)| \lesssim P'(|a|)(1-x)^{1-\beta} \lesssim P'(|a|)(1-x)^{1-\beta}. \quad (21)$$

We combine this with (18) and obtain

$$\left| G - (1-x^2)^{-\beta} f'(x) + \int_x^1 (1-y)^{-\beta-1} P'(f(1)) dy \right| \leq \int_x^1 (1-y)^{-\beta-1} |P'(f(y)) - P'(f(1))| dy$$

$$\lesssim \int_x^1 (1-y)^{-\beta-1}(1-y)^{1-\beta} P'(|a|) dy$$

$$\lesssim P'(|a|)(1-x).$$

Thus we have

$$\left| f'(x) - G(1-x^2)^{-\beta} - \frac{1}{2\beta} P'(f(1)) \right| \lesssim P'(|a|)(1-x)^{1-\beta}. \quad (21)$$

Finally we claim that $G$ and $f(1)$ can never be zero at the same time unless $a = 0$. In fact, if $G = f(1) = 0$, the estimate above implies that $f'(x) \to 0$ as $x \to 1^-$. Our semi-conservation law (14) then gives $a = 0.$
5.4 Extreme values of \( f \)

We prove part (iv) of Proposition 5.1 by considering the extreme values of \( f \) on \((0, 1)\). We have

**Proposition 5.1.** The solution \( f \) to ordinary differential equation (12) satisfies

(a) Given \( a > 0 \), there are finitely many points \( x \in (0, 1) \) so that \( f'(x) = 0 \). All of these are local maxima or minima. We use the notation \( N(a) \) for the number of local extreme points.

(b) Let \( a = a_0 \) be a positive parameter so that \( G \doteq \lim_{x \to 1^+} (1 - x^2)^\beta f'(x) \neq 0 \). Then \( N(a) \) is a constant in a small neighbourhood of \( a_0 \).

(c) When \( a > 0 \) is large, we have a lower bound \( N(a) \geq a^\frac{2}{\beta+1} \).

We temporarily postpone the proof of Proposition 5.1 and first show why part (iv) of Proposition 5.1 is a direct consequence of it.

**Proof of part (iv)** First of all, the approximation formula

\[
\left| f'(x) - G(1 - x^2)^{-\beta} - \frac{1}{2\beta} P'(f(1)) \right| \lesssim P'(|a|)(1 - |x|)^{1-\beta}
\]

given in part (iii) implies that \( \sup_{x \in (-1,1)} |f'(x)| < +\infty \) is equivalent to \( G = 0 \). If there were only a finite number of \( a \)'s so that \( G = 0 \), then \( N(a) \) would be a constant for all sufficiently large \( a > 0 \), by part (b) of Proposition 5.1. However, this contradicts with part (c).

**Proof of Proposition 5.1.** Let us start by part (a). If \( f'(x) = 0 \) for some \( x \in (0, 1) \), then \( f''(x) \neq 0 \). Otherwise we have \( f(x) = 0 \) thus \( f \equiv 0 \). Thus \( x \) must be either a maximum, if \( f''(x) < 0 \), or a minimum, if \( f''(x) > 0 \). This also implies that all these extreme points are isolated from each other. Thus it suffices to show these points cannot accumulate around the endpoints 0, 1. The case of \( x = 0 \) is trivial since \( f \in C^2 \) and we have assumed \( f(0) = a > 0 \). The case \( x = 1 \) can be dealt with by the approximation formula of \( f'(x) \) near \( x = 1 \) given in (21). Please note that at least one of \( G \) and \( f(1) \) is nonzero. Now let us prove part (b). Let \( x_1 < x_2 < \cdots < x_n \) be all extreme points of \( f \) in \((0, 1)\) when \( a = a_0 \). We can always choose a sufficiently small positive constant \( \varepsilon \) so that

\[
0 < x_1 - \varepsilon < x_1 + \varepsilon < x_2 - \varepsilon < x_2 + \varepsilon < \cdots < x_n - \varepsilon < x_n + \varepsilon < 1 - \varepsilon < 1
\]
satisfy

\[
\inf_{x \in [0, x_1-\varepsilon]} f'(x) > 0; \quad \inf_{x \in [x_i+\varepsilon, x_{i+1}-\varepsilon]} |f'(x)| > 0, \quad i = 1, 2, \cdots, n-1; \quad \inf_{x \in [x_n+\varepsilon, 1-\varepsilon]} |f'(x)| > 0;
\]

\[
\inf_{x \in [x_i-\varepsilon, x_i+\varepsilon]} |f''(x)| > 0, \quad f'(x_i-\varepsilon)f'(x_i+\varepsilon) < 0, \quad i = 1, \cdots, n; \quad \left| 1 - (1-\varepsilon)^2 \right|^{-\beta}|G| > C_0 P'(|a|).
\]

Here the constant \( C_0 = C_0(\beta, \gamma, p) \) is the one in (20). The final inequality above guarantees that \( f'(x) \neq 0 \) for all \( x \in [1-\varepsilon, 1] \). The continuous dependence of \( f(x) \), \( f'(x) \), \( f''(x) \) away from \( x = \pm 1 \) and \( G \) on parameter \( a \) then guarantees that all the inequalities above also hold for parameters \( a \) in a small neighbourhood of \( a_0 \). It immediately follows that there is exactly one extreme point in each interval \([x_i-\varepsilon, x_i+\varepsilon] \) but none elsewhere. Finally we prove part (c). We consider an interval \( I = [y_1, y_2] \subseteq (0, 1/2) \) which does not contain an extreme point or zero of \( f(x) \). According to the semi-conservation law (13), either \( |f'(x)| > |a|/2 \) or \( |f(x)| > |a|^{2/(p+1)}/2 \) holds for any \( x \in (-1, 1) \). Thus

\[
|I| \leq \left| \{ x \in I : |f'(x)| > a/2 \} \right| + \left| \{ x \in I : |f(x)| > a^{2/(p+1)}/2 \} \right|.
\]

Next we observe the following facts
• None of \( f(x) \) and \( f'(x) \) may change its sign in \( I \) by our assumption on the interval \( I \);
• The inequalities \( |f(x)| \lesssim a^{2/(p+1)} \) and \( |f'(x)| \lesssim a \) hold for all \( x \in [0,1/2] \) by semi-conservation law \([13]\).

These help to give upper bounds of the terms in the right hand side of \((22)\):
\[
\left| \{ x \in I : |f'(x)| > a/2 \} \right| \leq \frac{2}{a} \left| \int_{y_1}^{y_2} f'(x) dx \right| = \frac{2}{a} |f(y_1) - f(y_2)| \lesssim a^{-\frac{3\gamma}{p+1}};
\]
and (we utilize \((17)\) below)
\[
\left| \{ x \in I : |f(x)| > a^{2/(p+1)/2} \} \right| \leq \left| \{ x \in I : (1 - x^2)^{\beta-1}|P'(f(x))| > a^{2\beta/(p+1)/2} \} \right|
\lesssim a^{-\frac{2\gamma}{p+1}} \left| \int_{y_1}^{y_2} (1 - x^2)^{\beta-1} P'(f(x)) dx \right|
= a^{-\frac{2\gamma}{p+1}} |(1 - y_1^2)\beta f'(y_1) - (1 - y_2^2)\beta f'(y_2)|
\lesssim a^{-\frac{\gamma}{p+1}}.
\]

In summary we have \(|I| \lesssim a^{-\frac{\gamma}{p+1}}\), i.e. there is a constant \( C = C(\beta, \gamma, p) \), so that \(|I| \leq Ca^{-\frac{\gamma}{p+1}}\). Because there is at least one extreme point between any two zeros of \( f \), it immediately follows that any interval \( I \subset (0,1/2) \) with \(|I| > 2Ca^{-\frac{\gamma}{p+1}}\) must contain at least an extreme point. This finishes the proof. \(\square\)

6 Appendix B: Elliptic Equation

In this section we consider the elliptic equation \(-\Delta U = \zeta |U|^{p-1} U\). This gives stationary solutions to (CP1). The case with \( \zeta = +1 \) has been discussed in the author’s previous work \([13]\). We still need to deal with the case \( \zeta = -1 \) and prove Proposition \([14]\). We will follow roughly the same argument as in the case \( \zeta = +1 \), thus we omit some details of proof and focus on the difference of these two cases.

Transformation to one-dimensional case We define \( z(|x|) = |x|U(x) \) and consider the equation \( z(r) \) satisfies
\[
z''(r) = \frac{|z(r)|^{p-1} z(r)}{rp^{-1}}, \quad r > 0.
\]
The existence of \( z \) near infinity (i.e. for \( r \in (R,\infty) \) with a large \( R > 0 \)) with prescribed asymptotic behaviour \( z(+\infty) = 1 \) and \( z'(+\infty) = 0 \) then follows a fixed point argument. We then solve \( z(r) \) backward by a standard ODE theory. This argument is exactly the same as in the focusing case. Please see Section 9 of \([13]\) for details. The only difference is that \( z(r) \) can no longer be defined for all \( r > 0 \) but blows up at \( r = R_- > 0 \) in the current setting. We will discuss this blow-up phenomenon in details.

Monotonicity When \( r \) is sufficiently large, we know \( z(r) > 1 \), \( z'(r) < 0 \), \( z''(r) > 0 \). A continuity argument then verifies that all these inequalities still hold in the whole lifespan of \( z \). Thus \( z(r) \) is either defined for all \( r > 0 \) or blows up to \(+\infty\) at some point \( r = R_- \). We will show that the former can never happen.
Iteration of lower bounds. Because $z(r) > 1$, we have $z''(r) = |z(r)|^{p-1}z(r)/r^{p-1} \geq r^{-(p-1)}$. We may integrate, use $z'(\infty) = 0$ and obtain
\[ z'(r) \leq -\frac{1}{p-2} r^{-(p-2)}. \]
We integrate again, use $z(\infty) = 1$ and obtain
\[ z(r) \geq \frac{1}{(p-2)(p-3)} r^{-(p-3)} + 1 \geq \frac{1}{(p-2)(p-3)} r^{-(p-3)}. \]
We may iterate this argument and obtain a family of lower bounds
\[ z(r) \geq \frac{r^{-\beta_k}}{c_k}, \quad k = 0, 1, 2, \ldots \]
Here the coefficients are defined by induction
\[ \beta_{k+1} = p\beta_k + (p-3), \quad c_{k+1} = (p\beta_k + p-3)(p\beta_k + p-2)c_k^2, \quad (\beta_0, c_0) = (0, 1). \]
We may give an explicit formula $\beta_k = \frac{(p-3)(p^k-1)}{p-1}$. We also have
\[ \ln c_{k+1} = p\ln c_k + \ln(p\beta_k + p-3) + \ln(p\beta_k + p-2) \leq p\ln c_k + 2(k+1)\ln p \]
\[ \Rightarrow \ln c_{k+1} + (k+3)\ln p \leq p[\ln c_k + (k+2)\ln p] \]
\[ \Rightarrow \ln c_k + (k+2)\ln p \leq 2p^k\ln p. \]
Thus we have
\[ \ln z(r) \geq \beta_k \ln(1/r) - \ln c_k \geq \left(\frac{p-3}{p-1}\right) \ln \left(\frac{1}{r} - 2\ln p\right) \left(p^k - 1\right) - 2\ln p, \quad \forall k = 0, 1, 2, \ldots \]
This implies that $z(r)$ cannot be defined for $r < p^{2(p-1)/(p-3)}$ otherwise the inequality above fails when $k \to +\infty$.

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