ELEMENTARY COUNTEREXAMPLES TO KODAIRA VANISHING IN PRIME CHARACTERISTIC

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Abstract. Using methods from the modular representation theory of algebraic groups one can construct [1] a projective homogeneous space for \( SL_4 \) in prime characteristic, which violates Kodaira vanishing. In this note we show how elementary algebraic geometry can be used to simplify and generalize this example.

Let \( X \) be a smooth projective variety of dimension \( m \) over an algebraically closed field of characteristic zero and let \( \mathcal{L} \) be an ample line bundle on \( X \). The Kodaira Vanishing Theorem [3] states that \( H^i(X, \mathcal{L}^{-1}) = 0 \) for \( i \neq m \). It is well known that the result is false in characteristic \( p > 0 \); Raynaud constructed [8] a smooth projective surface in positive characteristic with an ample line bundle for which Kodaira vanishing failed (Mumford [7] had earlier constructed a normal non-smooth projective surface counter-example). Raynaud posed two questions:

(1) Are there counter-examples where the vanishing fails for a very ample line bundle?

(2) Are there pairs \((X, \mathcal{L})\) (\( X \) smooth projective, \( \mathcal{L} \) ample) for which

\[
\chi_i(\mathcal{L} \otimes \omega_X) := h^i(X, \mathcal{L} \otimes \omega_X) - h^{i+1}(X, \mathcal{L} \otimes \omega_X) + \ldots
\]

is not always \( \geq 0 \)?

The first author, studying proper homogeneous spaces in characteristic \( p \) [4][5], answered both questions affirmatively using methods from modular representation theory of algebraic groups. The object of this note is to generalize, via elementary algebraic geometry, the simplest counterexample ([1], Example 4) answering question (1) without using Jantzen’s sum formula from modular representation theory. The simplest of our examples is as follows: let \( Y \) be the incidence correspondence of points lying on planes in projective three space \( \mathbb{P}(V) \). There is a natural bundle \( \mathcal{G} \) of rank 2 on \( Y \) such that the projectivization \( \mathbb{P}(\mathcal{G}) \) of \( \mathcal{G} \) is the variety of flags in \( V \). Now let \( X = \mathbb{P}(F^*\mathcal{G}) \) be the projectivization of \( F^*\mathcal{G} \) (the Frobenius pullback of \( \mathcal{G} \)). Then \( X \) can be embedded in \( \mathbb{P}(V) \times \mathbb{P}(\Lambda^2V^\vee) \times \mathbb{P}(V^\vee) \), and the line bundle \( \mathcal{L} = \mathcal{O}(1, 3, 1) \) is very ample on \( X \) and violates Kodaira vanishing (with \( H^5(X, \mathcal{L}^{-1}) \neq 0 \)).

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It would be interesting to see if such elementary calculations also yield examples to Raynaud’s second question. The example known so far [5] uses computer intensive calculations. Also the examples of this note are of dimension six or more (and with Picard group of rank three), whereas the example of Raynaud (and Mumford’s normal variety, earlier) is a surface. While the examples here are more elementary than Raynaud’s example, it would be nice to find other examples of smaller dimension (or with Picard group of rank two). Since it is only the penultimate $H^i(X, L^{-1})$ which is non-zero, this failure of Kodaira vanishing is not necessarily inherited by hyperplane sections.

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1. Preliminaries

1.1. Let $V$ be an $n + 1$-dimensional vector space over a field $k$. We will consider $\mathbb{P}(V)$, $\mathbb{P}(\wedge^2 V^\vee)$ and $\mathbb{P}(V^\vee)$. The tautological line (quotient) bundle of each will be denoted $\mathcal{O}(1,0,0)$, $\mathcal{O}(0,1,0)$ and $\mathcal{O}(0,0,1)$ respectively. So on $\mathbb{P}(V)$, there is an exact sequence of vector bundles

$$0 \to A \to V \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1,0,0) \to 0.$$ 

This identifies $H^0(\mathbb{P}(V), \mathcal{O}(1,0,0))$ with $V$. If we fix a basis $X_0, X_1, \ldots, X_n$ for $V$ and the dual basis $Y_0, Y_1, \ldots, Y_n$ for $V^\vee$, the dual of the above sequence gives the homomorphism

$$\mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1,0,0) \otimes_k V^\vee$$

which determines the global section $\sum X_i Y_i$.

1.2. Let $Y$ be a scheme, and let $E$ be a vector bundle of rank $r$ on $Y$. Let $X = \mathbb{P}(E)$ be the projectivized bundle. It comes with a morphism $\pi : X \to Y$, such that $X$ is smooth over $Y$ and $X$ has a tautological line bundle $\mathcal{O}_\pi(1)$ which appears in a sequence

$$0 \to F \to \pi^* E \to \mathcal{O}_\pi(1) \to 0.$$ 

$\omega_{X/Y}$ can be identified with $\wedge^r (\pi^* E) \otimes \mathcal{O}_\pi(-r)$ ([2], III, Ex 8.4).

1.3. Let $Y$ be a scheme over a field $k$ of characteristic $p \neq 0$. The absolute Frobenius morphism $F : Y \to Y$ is defined on the level of affine rings by mapping the function $a$ to $a^p$, and has the property that if $\mathcal{L}$ is a line bundle on $Y$, then $F^* \mathcal{L} \cong \mathcal{L}^p$, and if there is a homomorphism $\mathcal{O}_X \to \mathcal{L}$ defining a section $s$, it pulls back to a homomorphism $\mathcal{O}_X \to \mathcal{L}^p$ defining the section $s^p$ [6].

2. The Example

Let $V$ be a vector space of dimension $n + 1$ over a field $k$ of characteristic $p \neq 0$ where $n \geq 3$ and $p \geq n - 1$. On $\mathbb{P}(V)$ there is the sequence of bundles

$$0 \to A \to V \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1,0,0) \to 0.$$ 

Let $Y = \mathbb{P}(A^\vee)$, with the morphism $\alpha : Y \to \mathbb{P}(V)$. We have

$$0 \to G \to \alpha^* A^\vee \to \mathcal{O}(1) \to 0.$$
with $\mathcal{G}$ a vector bundle of rank $n - 1$. The surjection $V^\vee \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{A}^\vee$ induces an inclusion of $\mathbb{P}(V)$-schemes $Y \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(V^\vee)$.

Let $\pi_1, \pi_2$ be the two projections defined on $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$. There is the natural map on the product
\[
\pi_1^*(V^\vee \otimes_k \mathcal{O}_{\mathbb{P}(V)}(V)) \to \pi_2^*(\mathcal{O}(0, 0, 1)) \to 0
\]
and the composite of this map with the inclusion $\pi_1^*(\mathcal{O}(-1, 0, 0)) \hookrightarrow \pi_1^*(V^\vee \otimes_k \mathcal{O}_{\mathbb{P}(V)})$ defines a homomorphism $\pi_1^*(\mathcal{O}(-1, 0, 0)) \to \pi_2^*(\mathcal{O}(0, 0, 1))$. $Y$ is the zero scheme on the product $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ of the global section of $\mathcal{O}(1, 0, 1)$ induced by this homomorphism and this section can be seen to be just $\sum X_i Y_i$. So $Y$ has bihomogeneous coordinate ring
\[
k[X_0, \ldots, X_n; Y_0, \ldots, Y_n]/(\sum X_i Y_i)
\]
and has canonical line bundle $\omega_Y = \mathcal{O}(-n, 0, -n)$.

If $\beta : Y \to \mathbb{P}(V^\vee)$ is induced by projection $\pi_2$ on the second factor, we see that $\mathcal{O}_\alpha(1)$ is the pull-back of $\mathcal{O}(0, 0, 1)$. On $Y$ we have the commuting diagram of exact sequences:

\[
\begin{array}{ccccccccc}
0 & & 0 & & & & & & \\
& \uparrow & & \uparrow & & & & & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \alpha^* \mathcal{A}^\vee & \longrightarrow & \mathcal{O}(0, 0, 1) & \longrightarrow & 0 \\
& \uparrow & & \uparrow & & \| & & \| & & \\
0 & \longrightarrow & \mathcal{B} & \longrightarrow & V^\vee \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{O}(0, 0, 1) & \longrightarrow & 0 \\
& \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{O}(-1, 0, 0) & \longrightarrow & \mathcal{O}(-1, 0, 0) & & & & & & \\
& \uparrow & & \uparrow & & & & \uparrow & & \\
0 & & 0 & & & & & & \\
\end{array}
\]

The middle horizontal sequence is part of the Koszul complex, hence there is a surjection $\wedge^2 V^\vee \otimes \mathcal{O}(0, 0, -1) \to \mathcal{B} \to 0$.

Now consider $F^* \mathcal{G}'$ where $\mathcal{G}' = \mathcal{G} \otimes \mathcal{O}(0, 0, 1)$. Let $X = \mathbb{P}(F^* \mathcal{G}')$ and let $\pi : X \to Y$ be the projection. $X$ is smooth over $k$ and since there is a surjection $\wedge^2 V^\vee \otimes \mathcal{O}_Y \to F^* \mathcal{G}' \to 0$, there is an inclusion of $Y$-schemes
\[
X \hookrightarrow Y \times \mathbb{P}(\wedge^2 V^\vee)
\]
It is evident that via projection onto the second factor, $\mathcal{O}_\pi(1)$ can be identified with the pull-back of $\mathcal{O}(0, 1, 0)$. Hence the line bundle $\mathcal{O}(1, 1, 1)$ on $X$ is very ample. Since $\mathcal{O}(0, n - 1, 0)$ is globally generated on $X$, the line bundle $\mathcal{L} = \mathcal{O}(1, n, 1)$ is also very ample on $X$ ([2] II, Ex. 7.5.)
Claim: The very ample line bundle $L$ on the smooth variety $X$ (smooth over $k$ of dimension $3n - 3$) violates Kodaira vanishing.

Proof: We will compute $H^i(X, L^{-1})$, where $L = \mathcal{O}(1, 0, 1) \otimes O_\pi(n)$. Since $\omega_X = \omega_{X/k} = \omega_X^Y \otimes \pi^*(\omega_Y^*) = \wedge^{n-1} F^* G^* \otimes O_\pi(-n+1) \otimes (\mathcal{O}(0,0,-n) = \mathcal{O}(p,0,p(n-2)) \otimes O_\pi(-n+1) \otimes (\mathcal{O}(0,0,-n) = \mathcal{O}(p-n,0,p(n-2) - n) \otimes O_\pi(-n+1)$ we get

$$H^i(X, L^{-1}) = H^{3n-3-i}(X, L \otimes \omega_X)^\vee$$

$$= H^{3n-3-i}(X, \mathcal{O}(p-n+1,0,p(n-2) - n + 1) \otimes O_\pi(1)) ^\vee$$

$$= H^{3n-3-i}(Y, \mathcal{O}(p-n+1,0,p(n-2) - n + 1) \otimes F^* G') ^\vee$$

$$= H^{3n-3-i}(Y, \mathcal{O}(p-n+1,0,p(n-2) - n + 1 + p) \otimes F^* G) ^\vee$$

$$= H^{3n-3-i}(Y, \mathcal{O}(p-n+1,0,(p-1)(n-1)) \otimes F^* G) ^\vee$$

hence clearly 0 when $3n - 3 - i > 2n - 1$, the dimension of $Y$, i.e. when $i < n - 2$.

Let $M = \mathcal{O}(p-n+1,0,(p-1)(n-1))$ on $Y$. The Frobenius pull-back of the commuting diagram above, tensored with $M$ gives

$$\begin{array}{c}
0 \\
\uparrow \\
M \otimes F^* G \\
\uparrow \\
0 \rightarrow M \otimes F^* B \rightarrow V^\vee \otimes M \rightarrow M \otimes \mathcal{O}(0,0,p) \rightarrow 0.
\end{array}$$

By running along the exact sequence on $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$

$$0 \rightarrow \mathcal{O}(-1,0,-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_Y \rightarrow 0,$$

we see that $H^j(Y, M \otimes \mathcal{O}(-p,0,0)) = H^j(Y, \mathcal{O}(1-n,0,(p-1)(n-1)))$ is always zero, hence

$$H^i(X, L^{-1}) = H^{3n-3-i}(Y, M \otimes F^* B)^\vee.$$ 

$V^\vee \otimes \mathcal{M}$ and $\mathcal{M} \otimes \mathcal{O}(0,0,p)$ have only $H^0$ as nonzero cohomology ($H^0$ is non-zero since $p \geq n - 1$) hence $F^* B \otimes \mathcal{O}(p+1-n,0,(p-1)(n-1))$ has zero cohomology except possibly $H^0$ and $H^1$ which are to be studied by considering the map

$$A : V^\vee \otimes H^0(Y, \mathcal{O}(p+1-n,0,(p-1)(n-1))) \rightarrow H^0(Y, \mathcal{O}(p+1-n,0,(p-1)(n-1)+p)).$$

The homomorphism $A$ is given by the matrix $[Y_0^p, Y_1^p, \ldots, Y_n^p]$ and it is easy to see that it is not onto: Consider the element

$$t = Y_0^{p+1-n}Y_1^{p-1}Y_n^{p-1}.$$
which is well defined in $H^0(Y, \mathcal{O}(p + 1 - n, 0, (p - 1)(n - 1) + p))$ modulo multiples of $X_0Y_0 + X_1Y_1 + \cdots + X_nY_n$. No element in its equivalence class can be expressed as a sum of multiples of $Y_0^p, Y_1^p, \ldots, Y_n^p$, i.e. $t$ is not in the image of $A$.

It follows that with $3n - 3 - i = 1$, $H^i(X, \mathcal{L}^{-1}) \neq 0$. Hence $H^{3n-4}(X, \mathcal{L}^{-1}) \neq 0$, where $X$ has dimension $3n - 3$ and $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < 3n - 4$. □
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