Solutions to the mean kings problem: higher-dimensional quantum error-correcting codes

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Abstract

Mean king's problem is a kind of quantum state discrimination problems. In the problem, we try to discriminate eigenstates of noncommutative observables with the help of classical delayed information. The problem has been investigated from the viewpoint of error detection and correction. We construct higher-dimensional quantum error-correcting codes against error corresponding to the noncommutative observables. Any code state of the codes provides a way to discriminate the eigenstates correctly with the classical delayed information.

Keywords: Mean king's problem, quantum error-correcting codes

1 Introduction

Mean king's problem is a kind of quantum state discrimination problems formulated by Vaidman, Aharanov, and Albert [1]. The problem is often told as a tale of a king and a physicist Alice. At first, Alice prepares a quantum bit (qubit)-system in an initial state. King performs a measurement with one of observables $\sigma_x$, $\sigma_y$, $\sigma_z$ and obtains an outcome. After the king's measurement, Alice performs a measurement and obtains an outcome. After the Alice's measurement, king reveals the observables he has measured. Then, she guesses the king’s outcome by using her outcome and the classical delayed information from king. A solution to the problem is defined as a pair of the initial state and the Alice's measurement such that she can guess the king's outcome correctly. A solution has been shown by making use of a bipartite qubits-system in a Bell state [1]. Then, Alice keeps one of the qubits and king performs the measurement on the other one. Her measurement derived from Aharonov-Bergman-Lebowitz rule [2] is performed on the bipartite qubits-system.

The mean king’s problem is considered in several settings. Most naturally, it is considered that king employs measurements constructed from a complete set of mutually unbiased bases (MUBs) [3] [4]. In the setting, when a bipartite system is prepared in a maximally entangled state, the existence of a solution to the problem depends on the existence of a complete set of orthogonal Latin squares [5]. Note that the complete sets of MUBs and orthogonal Latin squares exist in prime-power dimension. For arbitrary dimension, it has been shown that solutions always exist when Alice is allowed to employ a positive operator valued-measure (POVM) measurement [6]. Nonexistence of the solutions to the problem has been shown when Alice cannot prepare a bipartite system [7] [8].

By investigating the problem from the viewpoint of error detection and correction, a solution to the problem by using quantum error-correcting codes has been shown [9]. A quantum error-correcting code is defined as a subspace of a Hilbert space and a quantum state in the code is called a code state. Roughly
speaking, correction of error on the state is realized with discrimination of added error states and performing
an appropriate quantum operation effectively. Applying the context of quantum error-correcting codes to the
problem, Alice prepares a code state, which is in a quantum error-correcting code against error corresponding
to the king’s measurements, as an initial state. Then, she can guess the king’s outcome by discriminating
error with the help of the classical delayed information from king. In the previous works, since the specific
quantum state, e.g., the maximally entangled state, is considered for the initial state, it is not clear how large
solution space is. On the other hand, any code state of the code is considered for the initial state in this
method (we recall that the code is the subspace of the whole space). However, the general construction of
such quantum error-correcting codes under any problem setting is not known.

In this paper, we show higher-dimensional quantum error-correcting codes. Here, higher-dimensional
means that the dimension of the code is greater than or equal to 2. On a bipartite system, which consists of
different dimensional local systems, and a multipartite system (this case is outside the purview of the setting
as stated above), we provide constructions of such quantum codes based on some properties of a pair of an
entangled state and a measurement. Then, Alice can guess the king’s outcome correctly by using any
code state of the codes as an initial state if the pair of the state and the measurement provides a solution
to the problem. This implied that we can find more large solution space in the context of the solution by
using quantum error-correcting codes if the there exists the solution. We also show some examples of the
higher-dimensional codes in the case that the king’s measurements are performed on the qubit system.

This paper is organized as follows. In Sec. 2, we review quantum error-correcting codes and the solution
to the mean king’s problem by using quantum error-correcting codes. In Sec. 3 and Sec. 4, we show
the constructions of higher-dimensional quantum error-correcting codes in the bipartite system and the
multipartite system, respectively. We also show some examples of the constructed codes on the qubits
system. Finally, in Sec. 5, we summarize this paper.

2 Review of Solutions Using Quantum Error-Correcting Codes

In this section, we review the basics of quantum error-correcting codes and introduce the solution to the
mean king’s problem using quantum error-correcting codes [9].

We regard $d$-dimensional Hilbert spaces in the same light as $d$-level quantum systems and call a 2-level
quantum system a quantum bit (qubit) system. In this paper, we treat a quantum operation described by a
trace nonincreasing completely positive (CP) map as adding error to a quantum system $\mathcal{H}$, $\epsilon : S(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$
is a trace nonincreasing if and only if there exist operators $(E_k)_k$ satisfying $\epsilon(\rho) = \sum_k E_k^\dagger \rho E_k$ for $\rho \in S(\mathcal{H})$ and
$\sum_k E_k^\dagger E_k \leq I$, where $S(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ are the sets of density operators and linear operators on $\mathcal{H}$, respectively.
This representation of $\epsilon$ $(E_k)_k$ is called the Kraus representation. We identify a trace nonincreasing CP map
$\epsilon$ with its Kraus representation $(E_k)_k$. Furthermore, we call $\epsilon$ (or $(E_k)_k$) an error in the context of quantum
codes. A $d^\prime$-dimensional subspace of a $d$-dimensional Hilbert space is called a $(d,d^\prime)$ quantum code and a
unit vector in the subspace is called a code state. We omit the notation $(d,d^\prime)$ when the context is clear.
A $(d,d^\prime)$ quantum code $C$ is called a $(d,d^\prime)$ quantum error-correcting code against an error $\epsilon$ if there exists a
trace-preserving completely positive map $R$ such that $R(\epsilon(\rho)) \propto \rho$ holds for any $\rho \in S(C)$. Such $R$ is called
a recovery. The general condition for the existence of quantum error-correcting codes was given by Knill and
Laflamme [10]. Let $C$ be a $(d,d^\prime)$ quantum code and $(E_k)_k$ an error. There exists a recovery $R$ for $C$ to be
a quantum error-correcting code against $(E_k)_k$ if and only if $P E_k^\dagger E_k P = \alpha_{kk^\prime}$ holds, where $(\alpha_{kk^\prime})_{k,k^\prime}$ is a
positive matrix and $P$ denotes the projection onto $C$.

We review the solution to the mean king’s problem using quantum error correcting codes. The problem
has been summarized as follows:
1. Alice prepares a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_K$ in an initial state, where $\mathcal{H}_A$ (resp. $\mathcal{H}_K$) is a $d_A$-dimensional (resp. $d_K$-dimensional) Hilbert space.

2. King performs one of measurements $M^{(J)}_i \ (J = 1, 2, \ldots, n)$ on $\mathcal{H}_K$, which are described by measurement operators $(M^{(J)}_i)_{i=1}^m$, and obtains an outcome $i$.

3. Alice performs a measurement described by a POVM $\mathcal{P}$ on the bipartite system and obtains an outcome $k$.

4. King reveals the measurement type $J$ he has performed.

5. Immediately, Alice guesses $i$ by using $k$ and $J$.

In the above setting, a solution to the problem is defined as a pair of the initial state and the Alice’s measurement such that she can guess the king’s outcome correctly. Then, the following theorem was given.

**Theorem 1** (Theorem 3 in Ref. [9]) Let $C \subset \mathcal{H}_A \otimes \mathcal{H}_K$ be a $(d_A d_K, d')$ quantum code and $P$ the projection onto $C$. If there exist $l$ tuple operators on $\mathcal{H}_K$ $(L_k)_{k=1}^l$ with $\sum_k L_k^\dagger L_k \leq I$ and nonempty index sets $X^{(J,i)} \subset \{ 1, 2, \ldots, l \} \ (J = 1, 2, \ldots, n : i = 1, 2, \ldots, m)$ satisfying the following conditions:

$$X^{(J,i)} \cap X^{(J,i')} = \emptyset \ \forall J, i, i', \tag{1}$$

$$I \otimes M^{(J)}_i = \sum_{k \in X^{(J,i)}} f^{(J,i)}_k I \otimes L_k \text{ on } C \ \forall J, i, \tag{2}$$

$$P(I \otimes L_k)^\dagger (I \otimes L_{k'}) P = \lambda_k \delta_{kk'} P \tag{3}$$

for some $\lambda_k \geq 0$ and $f^{(J,i)}_k \in C$. Then,

(i) Alice can guess the king’s outcome correctly by using any code state in $C$ as an initial state,

(ii) $C$ is a quantum error-correcting code against an error $(I \otimes L)_{k}$.

An outline of the proof of Theorem 1 is as follows. (i) a subspace which contains a state after the king’s measurement is uniquely determined by $(L_k)_{k \in X^{(J,i)}}$ from Eq. (2) and such subspaces are orthogonal from Eq. (3), $X^{(J,i)}$ which contains $k$ uniquely exists for $J$ from Eq. (1). Then, Alice can guess the king’s outcome by using her outcome and $J$ when she performs a measurement to distinguish the subspaces (i.e., this measurement is described by a projection valued measure (PVM), which consists of the projections onto the subspaces). (ii) It is straightforward from Eq. (3) and the Knill-Laflamme theorem as stated above.

Here, we give one points. Theorem 1 says that there exists a solution to the problem if there exists a quantum error-correcting code against the error $(I \otimes L)_{k}$ satisfying Eqs. (1), (2), and (3) for the king’s measurement. However, the state-change caused by the error (quantum operation) $(I \otimes L)_{k}$ defer from the measurement process of the measurement operators $(M^{(J)}_i)_i$. The state-change of $|\psi\rangle \in C$ against the error $(I \otimes L)_{k}$ is described by $|\psi\rangle\langle\psi| \mapsto \sum_k (I \otimes L_k)|\psi\rangle\langle\psi|(I \otimes L_k)^\dagger =: \rho'$. The post-state is recovered by a map $R(\rho') \propto |\psi\rangle\langle\psi|$. At that time, error detection is not necessarily required. On the other hand, in the context of mean king’s problem, the initial state $|\psi\rangle \in C$ is changed by a king’s measurement described by measurement operators $(M^{(J)}_i)_i$ with an outcome $i$: $|\psi\rangle\langle\psi| \mapsto (I \otimes M^{(J)}_i)|\psi\rangle\langle\psi|(I \otimes M^{(J)}_i)^\dagger/p_i =: \sigma$.

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1 Let $(M^{(J)}_i)_i$ be a collection of measurement operator on a Hilbert space $\mathcal{H}$. The postmeasurement state from the $\rho \in S(\mathcal{H})$ is given by $M^{(J)}_i \rho M^{(J)}_i^\dagger/\text{tr}M^{(J)}_i \rho M^{(J)}_i^\dagger$ and the probability to get the outcome $i$ is $\text{tr}M^{(J)}_i \rho M^{(J)}_i^\dagger$. 

3
where $p_i = \text{tr}(\mathbb{I} \otimes M_i^{(J)}) \langle \psi | (\mathbb{I} \otimes M_i^{(J)}) | \psi \rangle$ is the probability to get the outcome $i$. Then, we observe $\sigma = (\sum_{k \in X(I,J)} \mathbb{I} \otimes L_k) | \psi \rangle (\mathbb{I} \otimes M_i^{(J)}) | \psi \rangle / p_i$. Therefore, Alice can guess the king’s outcome $i$ with an outcome of the measurement to distinguish $(\mathbb{I} \otimes L_k) | \psi \rangle_k$ and $J$.

A “reverse” statement of Theorem 1 was given. Let $\mathcal{H}_A = \mathcal{H}_K := \mathcal{H}$ (dim $\mathcal{H}$ = $d$) and an entangled state (in the form of the Schmidt decomposition):

$$|\Psi_\eta\rangle := \sum_{j=0}^{d-1} \eta_j |\psi_j\rangle \otimes |\phi_j\rangle \quad \eta_j > 0, \sum_{j=0}^{d-1} \eta_j^2 = 1$$

be prepared with orthonormal bases $\{|\psi_j\rangle\}_j$ and $\{|\phi_j\rangle\}_j$ of $\mathcal{H}$. Let $\mathcal{P} := \{|p_k\rangle \langle p_k|\}_{k=1}^{d^2}$ be a PVM on $\mathcal{H}_A \otimes \mathcal{H}_K$ with an orthonormal basis $\{|p_k\rangle\}_{k=1}^{d^2}$.

**Theorem 2** (Theorem 5 in Ref. [9]) If $|\Psi_\eta\rangle$ and $\mathcal{P}$ provide a solution to the mean king’s problem, there exists a quantum operation $(L_k)_{k=1}^{d^2}$ on $\mathcal{H}_K$ and index sets $X^{(J,i)}$ satisfying the following conditions,

$$X^{(J,i)} \cap X^{(J,i')} = \emptyset \quad \forall J, i \neq i',$$

$$M_i^{(J)} = \sum_{k \in X^{(J,i)}} f_k^{(J,i)} L_k \quad \forall J, i,$$

$$(\mathbb{I} \otimes L_k |\Psi_\eta\rangle \otimes L_{k'} |\Psi_\eta\rangle) = \frac{\alpha}{d} \delta_{kk'}$$

for some $\alpha > 0$ and $f_k^{(J,i)} \in \mathbb{C}$.

From Theorem 1 the initial state is a code state of a $(d^2, 1)$ quantum error-correcting code spanned by $|\Psi_\eta\rangle$ against the error $(\mathbb{I} \otimes L_k)_k$. The error satisfies $\sum_k L_k L_k^\dagger \leq \mathbb{I}$ when $\alpha = \min \{\eta_j\}_j$ from Lemma 4 in the previous work [9].

### 3 Higher-Dimensional Quantum Error-Correcting Codes in Bipartite Systems

To construct higher-dimensional quantum error-correcting codes such that a pair of any code state of the codes and the corresponding measurement provides a solution to the mean king’s problem, we will utilize Theorem 1 and Theorem 2 effectively. Let $\mathcal{H}_A'$ be a quantum system in place of $\mathcal{H}_A$ satisfying dim $\mathcal{H}_A' =: d_A \geq d = \text{dim} \mathcal{H}_K$. Let

$$|\Psi_{\eta,l}\rangle := \sum_{j=0}^{d-1} \eta_j |\xi_{d(l-1)+j}\rangle \otimes |\phi_j\rangle \in \mathcal{H}_A' \otimes \mathcal{H}_K,$$

where $l = 1, 2, \ldots, \lfloor \frac{d_A}{d} \rfloor := \max \{s \in \mathbb{Z} \mid s \leq \frac{d_A}{d} \}$, and $\{|\xi_i\rangle\}_{i=0}^{d_A-1}$ is an orthonormal basis of $\mathcal{H}_A'$. Then, we obtain the following theorem.

**Theorem 3** There exists a $(d_A d, \lfloor \frac{d_A}{d} \rfloor)$ quantum error-correcting code spanned by $\{|\Psi_{\eta,l}\rangle\}_{l=1}^{\lfloor \frac{d_A}{d} \rfloor} \subset \mathcal{H}_A' \otimes \mathcal{H}_K$ such that Alice can guess the king’s outcome by using any code state of the code as an initial state if the pair of $|\Psi_\eta\rangle \in \mathcal{H}_A \otimes \mathcal{H}_K$ and $\mathcal{P}$ provides a solution to the problem.
Then, a pair of a Bell state and also exist index sets listed in Table 1 satisfying Eqs. (5), (6), and (7). In particular, for the error (C Theorem 1), there exist operators (Lk) and the corresponding Alice’s measurement is a solution to the problem. Therefore, from Theorem 3, we obtain a higher dimensional quantum error-correcting code by using Theorem 3. This example originates from APPENDIX A in the previous work [9]. King’s measurements are fixed as follows:

Here, we give a more specific application example of Theorem 2 in a qubits-system to construct a higher dimensional quantum error-correcting code by using Theorem 3. This example originates from APPENDIX A in the previous work [1].

Proof. From Theorem 2 if the pair of P and |Ψη⟩ ∈ HA ⊗ HK is a solution to the mean king’s problem, there exist the index sets X(j,i) and the quantum operation (Lk)k=1 on HK satisfying Eqs. (8), (9), and (10).

From Eq. (10), we observe

$$\langle 1 \otimes L_k | η, ν \rangle = \sum_{j,j' \geq 0} \eta_j \eta_{j'} \langle ξ_{d(l-1)+j} | ξ_{d(l'-1)+j'} \rangle \langle L_k \phi_j | L_k \phi_{j'} \rangle$$

$$= \sum_{j,j' \geq 0} \eta_j \eta_{j'} \delta_{l,l'} \delta_{j,j'} \langle L_k \phi_j | L_k \phi_{j'} \rangle$$

$$= \frac{α}{d} δ_{kk'} \delta_{ll'}$$

for the error (1 ⊗ Lk)k satisfying \( \sum_k L_k^4 L_k = 1 \). Let C be a \( (d_A d, \left[ \frac{d_A}{d} \right] ) \) quantum code spanned by \{ |Ψη, ν⟩ \}_{i=1}^4 \ ⊂ HA ⊗ HK. Then, C, X(j,i), and (Lk)k satisfy Eqs. (10), (11), and (12). Therefore, from Theorem 1 C is the (d_Ad, [d_A/d]) quantum error-correcting code against the error (1 ⊗ Lk)k and a pair of any code state of C and the corresponding Alice’s measurement is a solution to the problem.

In the previous work [5], for the problem when king employs measurements described by MUBs, the authors showed a solution which consists of a maximal entangled state and a PVM measurement constructed from an orthonormal basis. Therefore, from Theorem 3 we obtain a higher dimensional quantum error-correcting code from the existence of the solution.

Here, we give a more specific application example of Theorem 2 in a qubits-system to construct a higher dimensional quantum error-correcting code by using Theorem 3. This example originates from APPENDIX A in the previous work [9].

King’s measurements are fixed as follows: \( M^{(1)} := (M^{(1)}_{1} := |+⟩⟨+|, M^{(1)}_{2} := |−⟩⟨−|), M^{(2)} := (M^{(2)}_{1} := |+⟩′⟨+|, M^{(2)}_{2} := |−⟩′⟨−|), M^{(3)} := (M^{(3)}_{1} := |0⟩⟨0|, M^{(3)}_{2} := |1⟩⟨1|), \) where \( |0⟩ := (1, 0)^T, |1⟩ := (0, 1)^T, |+⟩ := \frac{1}{\sqrt{2}} (1, 1)^T, |−⟩ := \frac{1}{\sqrt{2}} (1, −1)^T, \) and \( |−⟩ := \frac{1}{\sqrt{2}} (1, i)^T, \). Then, a pair of a Bell state \( |Ψ⟩ := \frac{1}{\sqrt{2}} (|00⟩ + |11⟩) ∈ HA ⊗ HK \) is a solution to the problem. For the solution, from Theorem 2 there exist operators (Lk)k with \( \sum_k L_k^4 L_k = 1 \):

\[
L_1 := \frac{1}{4} \begin{pmatrix} 2 & 1 - i \\ 1 + i & 0 \end{pmatrix}, \quad L_2 := \frac{1}{4} \begin{pmatrix} 2 & -1 + i \\ -1 - i & 0 \end{pmatrix},
\]

\[
L_3 := \frac{1}{4} \begin{pmatrix} 0 & 1 + i \\ 1 - i & 2 \end{pmatrix}, \quad L_4 := \frac{1}{4} \begin{pmatrix} 0 & -1 - i \\ -1 + i & 2 \end{pmatrix}
\]

(8)

and also exist index sets listed in Table 1 satisfying Eqs. (5), (9), and (10). In particular,

\[
M^{(1)}_1 = L_1 + L_3, \quad M^{(2)}_1 = L_1 + L_4, \quad M^{(3)}_1 = L_1 + L_2,
\]

\[
M^{(1)}_2 = L_2 + L_4, \quad M^{(2)}_2 = L_2 + L_3, \quad M^{(3)}_2 = L_3 + L_4,
\]

(9)

and

\[
\langle 1 \otimes L_k | η, ν \rangle = \frac{1}{4} δ_{kk'}
\]

2 We have this state to substitute \( η_j = 1/√d \) for any j in Eq. (4).

3 This basis is listed in the previous work [5].

4 This basis is listed in the previous work [1].
From Theorem 3, a quantum code spanned by \(|\Psi_l\rangle := \frac{1}{\sqrt{2}}(|\xi_{2(l-1)}\rangle \otimes |0\rangle + |\xi_{2l-1}\rangle \otimes |1\rangle)\}_{l=1}^{d_A} \subset \mathbb{C}^{d_A} \otimes \mathbb{C}^2$, where \(d_A \geq 2\) and \(|\xi_i\rangle\}_{i=0}^{d_A-1}\) is an orthonormal basis of \(\mathbb{C}^{d_A}\), is a \((2d_A, \lfloor \frac{d_A}{2} \rfloor)\) quantum error-correcting code against the error \((I \otimes L_k)\) such that a pair of any code state of this code and a corresponding Alice’s measurement is a solution to the problem.

| Table 1: Index sets \(X^{(J,i)}\) \((J = 1, 2, 3; i = 1, 2)\) |
|-----------------|-----------------|-----------------|
| \(J\) \(i\) \(X^{(J,i)}\) | \(J\) \(i\) \(X^{(J,i)}\) | \(J\) \(i\) \(X^{(J,i)}\) |
| 1 1 \{1, 3\} | 2 1 \{1, 4\} | 3 1 \{1, 2\} |
| 1 2 \{2, 4\} | 2 2 \{2, 3\} | 3 2 \{3, 4\} |

4 Higher-Dimensional Quantum Error-Correcting Codes in Multipartite Systems

In this section, by considering multipartite systems, we try to construct a more higher-dimensional quantum error-correcting code. Here, we also utilize the previously mentioned quantum operation \((L_k)_k\) and index sets \(X^{(J,i)}\) satisfying Eqs. (5), (6), and (7) in Theorem 2 for the solution \(|\Psi_\eta\rangle\) and \(P\).

We will start from modifying the setting of the problem as follows. Let us consider that Alice can prepare a multipartite system \(H \otimes n\) \((\dim H = d, n \geq 2)\) and gives any \(l\)-th system to king, then she keeps the leftover system in secret. Remark that this case is outside the purview of the formulated problems. King performs one of the measurements \((M_i^{(J)})_i\) \((J = 1, 2, \ldots, n)\) on the \(l\)-th system. After the king’s measurement, Alice performs a POVM measurement on the multipartite system \(H \otimes n\). The following state is considered as an initial state:

\[
|\Psi_{\eta_1, \ldots, \eta_n}\rangle := \sum_{j=0}^{d-1} \eta_j |\phi_{j \oplus i_1}\rangle \otimes |\phi_{j \oplus i_2}\rangle \otimes \cdots \otimes |\phi_{j \oplus i_n}\rangle, \tag{10}
\]

where \(i_u \in \{0, 1, \ldots, d-1\}\) and \(j \oplus i_u := j + i_u \mod d\). This state is inspired by generalized Greenberger-Horne-Zeilinger (GHZ) states \([11, 12]\).

Lemma 4

\[
\langle \tilde{L}_k \Psi_{\eta_1, \ldots, \eta_n} | \tilde{L}_k' \Psi_{\eta_1, \ldots, \eta_n} \rangle = \frac{\alpha}{d} \delta_{kk'} \tag{11}
\]

holds, where \(\tilde{L}_k := I \otimes \cdots \otimes I \otimes L_k \otimes I \otimes \cdots \otimes I\). And

\[
\langle \tilde{L}_k \Psi_{\eta_1, \ldots, \eta_n} | \tilde{L}_k' \Psi_{\eta_1', \ldots, \eta_n'} \rangle = 0 \tag{12}
\]

holds if there exist \(s\) and \(t\) \((s \neq t\) and \(s, t \neq l\)) such that \(i_s \oplus i_s' \neq i_t \oplus i_t'\), where \(i_u \oplus i_u' := i_u - i_u' \mod d\).  

\[\text{From Theorem 1, a quantum code spanned by the Bell state is a (4,1) quantum error-correcting code against the same error.}\]
Proof. We observe
\[
\langle \tilde{L}_k \Psi_{i_1, \ldots, i_n}, \tilde{L}_k \Psi_{i'_1, \ldots, i'_m} \rangle = \sum_{j,j'} \eta_j \eta_{j'} \langle \phi_j \otimes i_n | \phi_{j'} \otimes i'_n \rangle \cdots \langle L_k \phi_{j \otimes i} | \phi_{j' \otimes i'} \rangle \cdots \langle \phi_{j \otimes i_{n-2}} | \phi_{j' \otimes i'_{n-2}} \rangle.
\] (13)

From Eq. 7, the right hand side of Eq. 13 = \sum_{j} \eta_j^2 \langle L_k \phi_j | L_k' \phi_j \rangle = \sum_{j} \eta_j^2 \langle L_k \phi_j | \phi_j \rangle = \frac{\eta^2}{d} \delta_{kk'}
when \( i_s = i'_s \) for any \( s \). This implies that Eq. 13 holds.

We show that \( i_s \otimes i'_s = i_t \otimes i'_t \) holds if there exist \( j \) and \( j' \) such that \( \langle \phi_{j \otimes i_s} | \phi_{j' \otimes i'_s} \rangle \neq 0 \) for fixed \( s \) and \( t \neq t \). Since \{\{\phi_j\}\}, \( j \) is the orthonormal basis, \( j \otimes i_s = j' \otimes i'_s \) and \( j \otimes i_t = j' \otimes i'_t \) holds if \( \langle \phi_{j \otimes i_s} | \phi_{j' \otimes i'_s} \rangle \neq 0 \). This implies that \( j + i_s - (j' + i'_s) = a_{ss'}d \leftrightarrow i_s - i'_s = a_{ss'}d + j' - j \)}
and \( j + i_t - (j' + i'_t) = a_{tt'}d \leftrightarrow i_t - i'_t = a_{tt'}d + j' - j \) hold for some integers \( a_{ss'} \) and \( a_{tt'} \). Then \( i_s \otimes i'_s = i_t \otimes i'_t \) holds. Therefore, if there exist \( s \) and \( t \) such that \( i_s \otimes i'_s = i_t \otimes i'_t \), \( \langle \phi_{j \otimes i_s} | \phi_{j' \otimes i'_s} \rangle \langle \phi_{j \otimes i_t} | \phi_{j' \otimes i'_t} \rangle = 0 \) for any \( j \) and \( j' \) which implies that Eq. 13 is equal to 0. \( \blacksquare \)

Theorem 5 There exists a \((d^n, g \geq 2)\) quantum error-correcting code constructed from the states described by Eq. 10 such that Alice can guess the king’s outcome by using any code state of the code as an initial state under the above setting if the pair of \(|\Psi_\eta\rangle \in H_A \otimes H_K \) and \( P \) provides a solution to the problem.

Proof. From Theorem 2 there exist \((L_k)_k \) and \( X^{(J,i)} \) satisfy Eqs. 1, 6, and 7 if the pair of \(|\Psi_\eta\rangle \) and \( P \) provides a solution. We also observe
\[
\tilde{M}_i^{(J)} = \sum_{k \in X^{(J,i)}} \tilde{L}_k, \quad X^{(J,i)} \cap X^{(J,i')} = \emptyset, \quad \text{and} \quad \sum_k \tilde{L}_k = 1,
\] (14)

where \( \tilde{M}_i^{(J)} := \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \tilde{M}_i^{(J)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \).

Let \( S \) be a set of the states described by Eq. 10 such that \( \langle \tilde{L}_k \Psi_{i_1, \ldots, i_n}, \tilde{L}_k \Psi_{i'_1, \ldots, i'_m} \rangle = 0 \) holds for any pair of different states in \( S \) and let \( g \) be the number of the elements of \( S \). By observing the inner product, we find that Eq. 11 holds for any state in \( S \) and \( g \geq 2 \) from Lemma 1. Let \( \tilde{C} \) be a \((d^n, g)\) quantum code spanned by \( S \) and \( P \) the projection onto \( \tilde{C} \). Then,
\[
\tilde{P} \tilde{L}_k \tilde{L}_k' \tilde{P} = \frac{\alpha}{d} \delta_{kk'} \tilde{P}
\] (15)
holds.

We regard the \( t \)-th system and the leftover system in the same light as \( H_K \) and \( H_A \), respectively. Then, \((\tilde{L}_k)_k \) and \( X^{(J)} \) satisfy Eqs. 1, 2, and 3 for \( \tilde{C} \) from Eqs. 13 and 14. Therefore, from Theorem 1 it is straightforward that \( \tilde{C} \) is a \((d^n, g)\) quantum error-correcting code and a pair of any code state of \( \tilde{C} \) and a corresponding Alice’s measurement are a solution to the problem. \( \blacksquare \)

We give a \((2^n, 2^n-2)\) quantum error-correcting code constructed from GHZ states as an application example of Theorem 5 in a multipartite qubits-system. Let \( H_{\otimes n} = (\mathbb{C}^2)^{\otimes n} \) be a multipartite qubits-system prepared by Alice, \((\tilde{M}_i^{(J)})_{i=1,2,3} \) described by Eq. 9 be the measurements employed by King, \((L_k)_k \) listed as Table 1 be the corresponding quantum operation and the index sets, respectively. Then, \( \tilde{M}_i^{(J)} = \sum_{k \in X^{(J,i)}} \tilde{L}_k, \quad X^{(J,i)} \cap X^{(J,i')} = \emptyset, \quad \text{and} \quad \sum_k \tilde{L}_k = 1 \)
hold.

We will try to construct a quantum code spanned by GHZ states defined as
\[
|\Psi_{i_1, \ldots, i_n}\rangle := \frac{1}{\sqrt{2}}(|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle + |\bar{i}_1\rangle \otimes |\bar{i}_2\rangle \otimes \cdots \otimes |\bar{i}_n\rangle),
\]
\footnote{We have this state to substitute \( d = 2, \eta_j = 1/\sqrt{2} \) in Eq. 10.}
where \( i_u \in \{0, 1\} \) and \(|i_u\rangle := |i_u + 1\rangle\). Then, \( \langle \tilde{L}_k \Psi_{i_1, \ldots, i_n} | \tilde{L}_k \Psi_{i_1, \ldots, i_n} \rangle = \frac{1}{4} \delta_{kk'} \) holds. Let \( \tilde{S}_{ghz} \) be a set of the GHZ states such that \( \langle \tilde{L}_k \Psi_{i_1, \ldots, i_n} | \tilde{L}_k \Psi_{i_1, \ldots, i_n} \rangle = 0 \) for any pair of different states in \( \tilde{S}_{ghz} \). By observing the inner product, we find that the number of the elements of \( \tilde{S}_{ghz} \) is \( 2^n - 2 \). Let \( \tilde{C}_{ghz} \) be a \((2^n, 2^n - 2)\) quantum code spanned by \( \tilde{S}_{ghz} \) and \( \tilde{P}_{ghz} \) the projection onto \( \tilde{C}_{ghz} \). Then, \( \tilde{P}_{ghz} \tilde{L}_k \tilde{L}_k^\dagger \tilde{P}_{ghz} = \frac{1}{4} \delta_{kk'} \tilde{P}_{ghz} \) holds. In the same way as the proof of Theorem 5, it is straightforward that \( \tilde{C}_{ghz} \) is a \((2^n, 2^n - 2)\) quantum error-correcting code and a pair of any code state of \( \tilde{C}_{ghz} \) and a corresponding Alice’s measurement is a solution from Theorem 1.

5 Summary

The solution to the mean king’s problem by using quantum error-correcting codes has been shown in the previous work. In the solution, Alice can guess the king’s outcomes correctly when she utilize any code state of the code as an initial state and a measurement to discriminate error corresponding to the king’s measurement. However, the general construction of such codes under any problem setting is not known. In this paper, we showed the constructions of higher dimensional quantum error-correcting codes based on the solution \(|\Psi_\eta\rangle\) and \(P\) in both of the bipartite systems and the multipartite systems. The dimension of our constructed codes is greater than or equal to 2 and it is recalled that a code state is defined as a pure state of a quantum code. This implies that we can find more large solution space in the context of the solution using quantum error-correcting codes if \(|\Psi_\eta\rangle\) and \(P\) provide a solution to the mean king’s problem.

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