A new class of compact high sensitive tiltmeter based on the UNISA folded pendulum mechanical architecture

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Abstract. We present the Extended Folded Pendulum Model (EFPM), a model developed for a quantitative description of the dynamical behavior of a folded pendulum generically oriented in space. This model, based on the Tait-Bryan angular reference system, highlights the relationship between the folded pendulum orientation in the gravitational field and its natural resonance frequency. This model validated by tests performed with a monolithic UNISA Folded Pendulum, highlights a new technique of implementation of folded pendulum based tiltmeters.

1. Introduction
The innovative architecture of the UNISA class of folded pendulums (based on the classical Watt’s Linkage) allow the implementation of compact, robust and light sensors, characterized by high sensitivity and large band, especially in the low frequency region, configurable as seismometers, velocimeters and/or accelerometers, capable to operate also in harsh, ultra-high-vacuum and cryogenic environments (UHV) \cite{1, 2, 3, 4}. Implemented in different materials and with different readouts to satisfy the requirements of a large variety of applications aimed both to control and monitoring (earthquake engineering, geophysics, civil engineering, etc.) \cite{5, 6, 7, 8, 9}, these devices appear to be very interesting also as sensors for applications to interferometric detectors of gravitational waves of the present and next generation \cite{10}. But despite the high quality of the present implementations, a relevant problem still remains unsolved: the output signal is always a combination of linear and angular signals \cite{1}. This ambiguity requires relevant assumptions on the output signal, although often very close to the real situation, like, for example, that one of the two signals is much higher than the other one or that the two signals have different bands \cite{11, 12}.

In this paper, we demonstrate that it is possible to obtain the angular signal by reinterpreting the output signal on the basis of the Extended Folded Pendulum Model (EFPM). This model, developed for a quantitative description of the dynamical behavior of a folded pendulum generically oriented in the gravitational field, highlights the relationship between its orientation and natural resonance frequency. The EFPM, based on the Tait-Bryan angular reference system and validated by the results of tests performed with a monolithic UNISA Folded Pendulum, opens the way to new interesting applications of the folded pendulum as tiltmeter.
2. Generalized Folded Pendulum Model

Although the folded pendulum dynamical behavior can be described with enough accuracy only with numerical approaches, the Lagrangian two-dimensional analytical model of Bertolini [14], based on the simplified Liu et al. model [13] and generalized by Barone et al. [1, 2] to describe the dynamics of a generically oriented folded pendulums on a vertical plane, is generally sufficient for a synthetic analysis. The generalized model requires the definition of two reference systems, the first, xyz, integral to the folded pendulum, characterized by the x axis parallel to the direction of motion of the central mass and by the y axis parallel to the hinges (or to the flexures) rotation axes, while the second one, XYZ, is characterized by the plane XY perpendicular to the direction of the gravitational acceleration vector, \(\vec{g}\) (Figure 1). In the generalized folded pendulum model [1, 2], the xyz reference system can rotate around the Y (Y \(\equiv\) y) axis of a generic angle \(\beta\), positive for a counterclockwise rotation, so that an horizontal folded pendulum is described by a rotation angle \(\beta = 0\) rad, while a vertical one by \(\beta = \pi/2\) rad. This model, schematized in Figure 1 for \(\beta = 0\), consists of two vertical arms of equal length, \(l_p\), connected to one side to a single support (frame) by means of two hinges, forming a simple pendulum of mass \(m_{p}\), and an inverted pendulum of mass \(m_{p_{i}}\). The two pendulums masses are concentrated in their centers of mass, \(P_{s}\) and \(P_{i}\), respectively, positioned in \(l_{0} = l_{p}/2\). The other sides of the arms are connected (in \(C_{s}\) and \(C_{i}\), respectively) to a bar of mass, \(m_{c}\), and length, \(l_{d}\), by means of two other hinges. The mass of the central bar is modeled with two equivalent masses \(m_{c_{s}}\) and \(m_{c_{i}}\), being

\[
m_{c} = m_{c_{s}} + m_{c_{i}}
\]

(1)

whose values depend on the position of the center of mass on the central bar, \(l_{m}\) (being \(l_{m} < l_{d}\)), measured with respect to the pivot point, \(C_{s}\), according to the relations (\(\beta = 0\)) [2]

\[
m_{c_{s}} = m_{c} \left( 1 - \frac{l_{m}}{l_{d}} \right) \quad m_{c_{i}} = m_{c} \left( \frac{l_{m}}{l_{d}} \right)
\]

(2)

Taking into account that the UNISA Folded Pendulum is monolithic and that its pivot points are elastic flexure joints, then for a generic orientation (\(\beta \neq 0\)) the angle describing the central rod rest position, \(\theta_{o}\), is determined by the equilibrium position determined by the force component due to the component of the equivalent gravitational acceleration along the x-axis direction, \(g_{x}\), and by the reaction due to the elastic force generated by the folded pendulum flexure joints, and described by the following equation

\[
M_{eq} l_{c} g_{x}(\beta) - k_{\theta_{o}} \theta_{o} = 0
\]

(3)

being \(\vec{g}(\beta)\), the gravitational acceleration vector, expressed by

\[
\vec{g}(\beta) = g_{x}(\beta) \vec{i} - g_{z}(\beta) \vec{k} = g \sin \beta \vec{i} - g \cos \beta \vec{k}
\]

(4)

A change of orientation, quantified by the rotation angle, \(\beta\), produces a displacement of the centre of the mass \(m_{c}\) of a quantity \(\Delta l_{\theta_{o}}\), that, can be expressed at the first order by the trigonometric relation

\[
\Delta l_{\theta_{o}}(\beta) \approx l_{c} \theta_{o}(\beta) = \frac{M_{eq} g_{eq}(\beta)}{K_{c_{eq}}}
\]

(5)

that, in turn, produces changes in \(m_{c_{s}}\) and \(m_{c_{i}}\), now functions of the angle \(\beta\)

\[
m_{c_{s}}(\beta) = m_{c_{s}} - \frac{m_{c}}{l_{d}} \Delta l_{\theta_{o}}(\beta) \quad m_{c_{i}}(\beta) = m_{c_{i}} + \frac{m_{c}}{l_{d}} \Delta l_{\theta_{o}}(\beta).
\]

(6)

The solution of the lagrangian equations including these effects, albeit simplified, shows that
the folded pendulum can be still effectively described as a second order system [1], with small deflection angles, $\theta$, evaluated around $\theta_0$. In fact, assuming the folded pendulum frame rigidly connected to the ground, that is $x_f(t) = x_g(t)$, being $x_g(t)$ the component of the ground motion parallel to the direction of the inertial mass, then the folded pendulum transfer function is expressed as

$$H(s) = \frac{X_c(s) - X_g(s)}{X_g(s)} = \frac{X_{output}(s)}{X_g(s)} = \frac{1}{s^2 + \frac{\omega^2}{Q(\omega_o)}s + \omega_o^2}$$

(7)

where $Q(\omega_o)$ is the global mechanical quality factor, $A_c$ is the parameter related to the center of percussion effects and $\omega_o$ is the angular frequency [1], expressed, in absence of losses, by

$$\omega_o(\beta) = \sqrt{\left(\frac{(m_p - m_p) \frac{l_p^2}{2l_c} + (m_c - m_c)}{(m_p + m_p) \frac{l_p^2}{3l_c} + (m_c + m_c)}\right) - 2m_c M_{eq} g_{eq}(\beta) \frac{g_{eq}(\beta)}{l_c} + \frac{h_0}{l_c^2}}$$

$$\sqrt{\frac{K_{eq}(\beta) + K_{eq}}{M_{eq}}}$$

(8)

being $k_\theta$ the sum of the contributions of the elastic constants of all the folded pendulum flexures [1]. $K_{eq}(\beta)$ and $K_{eq}(\beta)$ are instead the equivalent elastic constants of the folded pendulum, modeled as a spring-mass equivalent oscillator, defined, respectively, as global equivalent elastic constant and gravitational equivalent elastic constant,

$$K_{eq}(\beta) = \frac{k_\theta l_c^2}{l_c^2}$$

$$K_{eq}(\beta) = \left[ \left( m_p-s m_p \right) \frac{l_p^2}{2l_c} + \left( m_c-m_c \right) - 2m_c M_{eq} g_{eq}(\beta) \frac{g_{eq}(\beta)}{l_c} \right] \frac{g_{eq}(\beta)}{l_c}$$

(9)

while $M_{eq}$ is the equivalent mass, defined as

$$M_{eq} = \left( m_p + m_p \right) \frac{l_p^2}{3l_c^2} + \left( m_c + m_c \right) = \left( m_p + m_p \right) \frac{l_p^2}{3l_c^2} + m_c$$

(10)

3. The Extended Folded Pendulum Model

In the EFPM the folded pendulum orientation with respect to the local horizontal is quantified by the relative orientation between the $xyz$ and $XYZ$ reference systems, defined by the Tait-Bryan angles: a rotation angle around $x$ axis (roll angle, $\alpha$), a rotation angle around $y$ axis (pitch angle, $\beta$) and a rotation angle around $z$ axis (yaw angle, $\gamma$). The rotation sequence, chosen for
the EFPM model [15] is the $\beta \alpha \gamma$ sequence (pitch-roll-yaw), shown in Fig. 2 and described by the following rotation matrix:

$$R = \begin{pmatrix}
\cos \gamma \cos \beta + \sin \gamma \sin \alpha \sin \beta & \sin \gamma \cos \alpha & -\cos \gamma \sin \beta + \sin \gamma \sin \alpha \cos \beta \\
-\sin \gamma \cos \beta + \cos \gamma \sin \alpha \sin \beta & \cos \gamma \cos \alpha & \sin \gamma \sin \beta + \cos \gamma \sin \alpha \cos \beta \\
\cos \alpha \sin \beta & -\sin \alpha & \cos \alpha \cos \beta
\end{pmatrix} \quad (11)$$

being the folded pendulum horizontally positioned when $\alpha = 0$, $\beta = 0$. In the same way, the gravitational acceleration vector, $\vec{g}$, is expressed in the reference system integral to the folded pendulum, $xyz$, as

$$\vec{g} = \begin{pmatrix}
\cos \gamma \sin \beta - \sin \gamma \sin \alpha \cos \beta \\
-\sin \gamma \sin \beta - \cos \gamma \sin \alpha \cos \beta \\
-\cos \alpha \cos \beta
\end{pmatrix} g \quad (12)$$

being $(0,0,-g)$ the acceleration gravity vector in the $XYZ$ reference frame. Therefore, the gravitational acceleration can be split into three components changing according to the folded pendulum orientation with respect to the $XYZ$ reference system. In particular, for a folded pendulum oriented as shown in Fig. 1, constrained to maintain $\gamma = 0$ the components of the gravitational acceleration along the $x$-axis, $y$-axis and $z$-axis, are respectively

$$g_x(\beta) = g \sin \beta \quad g_y(\alpha, \beta) = -g \sin \alpha \cos \beta \quad g_z(\alpha, \beta) = -g \cos \alpha \cos \beta \quad (13)$$

where $g_x(\beta)$ acts along the central mass direction of motion, generating a force opposed by the elastic reaction of the joints; $g_y(\alpha, \beta)$ acts along the direction perpendicular to the motion of the central mass, forcing the joints along a direction parallel to their rotation axes and generating their reaction as constraints; $g_z(\alpha, \beta)$ is the effective component of the gravitational acceleration to used in the simplified Lagrangian model.

The change of orientation of the folded pendulum has a direct effect on its potential energy. The solution of the new lagrangian equations lead to a more general expression or the angular frequency, $\omega$, that now depends on both $\alpha$ and $\beta$ angles, that is

$$\omega(\alpha, \beta) = \sqrt{\frac{(m_{p_x} - m_{p_y}) \frac{l_y}{2c} + (m_{c_x} - m_{c_y}) - 2m_{p_z} \frac{M_{eq} \sin \beta}{l_x}}{l_x}} \cos \alpha \cos \beta + \frac{k_x}{l_x} \quad (14)$$

being $K_{eq}$ now function of both the roll angle, $\alpha$, and of the pitch angle, $\beta$.

The most evident consequence is that a folded pendulum angular displacement determines a change of its natural resonance frequency, described, for $\gamma = 0$, by Eq. 14 for small, static or slowly variable angular displacements.

In figure 3 the resonance frequency for a standard folded pendulum is shown as a function of both roll angle, $\alpha$, and pitch angle, $\beta$. Let us now analyze two limit cases: angular rotation in presence of only roll ($\beta = 0$) (i.e. rotation around the $x$-axis) and angular rotation in presence of the only pitch ($\alpha = 0$) (i.e. rotation around the $y$-axis).

In the first case the component of the gravitational acceleration, $g_y(\alpha, 0)$, is always perpendicular to the motion of the central mass and, by definition, totally compensated by the constraint reaction of the joints. Therefore, the resonance frequency is an even function the resonance frequency (referred to $\alpha = 0$): a measurement of the resonance frequency allows to obtain the roll angle module, but not the rotation verse. This problem can be solved in different ways [15], the easiest one requiring the use of only one of the two branches of this function, i.e. either $\alpha > 0$ or $\alpha < 0$, positioning the folded pendulum rotated of a known roll angle, $\alpha_0 \neq 0$.

In second case, the rotation occurs along the $y$ axis, perpendicular to the motion of the central mass, so that the component of the gravitational acceleration, $g_z(0, \beta)$, perpendicular to the motion of the central mass, changes with the pitch angle, $\beta$, generating a monotone resonance frequency function with respect to the folded pendulum horizontal orientation.
4. Tests of the Extended Folded Pendulum Model

The validation of the EFPM has been performed using a monolithic UNISA Folded Pendulum of 134 mm × 140 mm × 40 mm (Model GC12) (Fig. 4), implemented from an Aluminum metal block (Alloy 7075-T6) with precision mechanical milling machining while the eight elliptical joints, characterized by ellipticity ratio of 16/5 and minimum thickness of 100 μm, have been implemented with electro-discharge machining (EDM). The arms of the pendulum (of length 81.5 mm, distant from each other by 102 mm and spaced apart by 6.5 mm from the central mass and the frame) have been designed to minimize mass and inertia momentum without loss of stiffness and symmetry, characterised by a measured quality factor of $Q > 1500$ in air at $\approx f_0 = 700$ mHz and by an optical lever readout with a position sensing photodiode (PSD) [1]. The signal, acquired with a digital acquisition system at a sampling frequency of 1 kHz, pre-filtered by a numerical pass band filter, is then processed to get the resonance sampling frequency. The measurements have been performed in air positioning the prototype on a platform, whose orientation could be changed acting on suitably positioned piezoelectric actuators, optimized to generate independently roll and pitch angles. A step signal has been used to change the angular position of the platform in order to provide, at the same time, also

![Figure 3](image_url) **Figure 3.** Example of resonance frequency for a typical folded pendulum as function of roll angle, $\alpha$, and pitch angle, $\beta$.

![Figure 4](image_url) **Figure 4.** Example of monolithic folded pendulum used for the EFPM test - Model GC12.

![Figure 5](image_url) **Figure 5.** Resonance Frequency vs. roll angle, $\alpha$, for model GC12.

![Figure 6](image_url) **Figure 6.** Resonance Frequency vs. pitch angle, $\beta$, for model GC12.
the mechanical signal necessary to force the central mass oscillation. Two series of angular displacement measurements are reported, carried out with the folded pendulum positioned as described in the previous sections ($\gamma = 0$), in the above described limit cases of pure roll ($\beta = 0$) and pure pitch ($\alpha = 0$), as reported in Fig. 5 and in Fig. 6. In Fig. 5 the trend of the GC12 natural resonance frequency is shown for both positive and negative roll angles, demonstrating that, as predicted by the EFPM, the resonance frequency is an even function and in very good agreement with the theoretical model. In Fig. 6, instead, the trend of the GC12 natural resonance frequency is shown for both positive and negative pitch angles, demonstrating again that, as predicted by the EFPM, the folded pendulum natural resonance frequency is a monotone function of the pitch angle and in very good agreement with the theoretical model.

5. Conclusions
We have presented a simplified version of the EFPM, a model capable to predict the folded pendulum dynamical behavior in the gravitational field, highlighting the relationship between its orientation and the natural resonance frequency. The EFPM, based on the Tait-Bryan angular reference system and validated by the results of tests performed with a monolithic UNISA Folded Pendulum, opens the way to new interesting applications of the folded pendulum as tiltmeter.

References
[1] Barone F, Giordano G 2015 Mechanical Accelerometers, J. Webster (ed.) Wiley Encyclopedia of Electrical and Electronics Engineering John Wiley & Sons, Inc. doi: 10.1002/047134608X.W8280.
[2] Barone F, Giordano G 2017 The UNISA folded pendulum: A very versatile class of low frequency high sensitive sensors Measurement doi:10.1016/j.measurement.2017.09.001.
[3] Barone F, Giordano G Low frequency folded pendulum with high mechanical quality factor, and seismic sensor utilizing such a folded pendulum Pat. Num.: IT1394612, EP2452169, JP 5409912, RU 2518587, AU2010209796, US8,950,263, Canada pending.
[4] Barone F, Giordano G, Acernese F Low frequency folded pendulum with high mechanical quality factor in vertical configuration, and vertical seismic sensor utilizing such a folded pendulum Pat. Num.: IT1405600, EP2643711, AU201427104, JP5981530, RU2589944, US9256000, Canada pending.
[5] Acernese F, De Rosa R, Giordano G, Romano R and Barone F 2008 Mechanical monolithic horizontal sensor for low frequency seismic noise measurement. Rev. Sci. Instrum. 79 074501 doi:10.1063/1.2943415.
[6] Acernese F, Giordano G, Romano R, De Rosa R and Barone F 2010 Tunable mechanical monolithic sensor with interferometric readout for low frequency seismic noise measurement Nucl Instrum. and Meth. A 617 457 ISSN:0016-9407 doi:10.1016/j.nima.2009.10.112.
[7] Barone F, Giordano G, Acernese F and Romano R 2015 Watt’s linkage based large band low frequency sensors for scientific applications, Nucl Instrum. and Meth. A 827 187 doi:10.1016/j.nima.2015.11.015.
[8] Barone F, Giordano G, Acernese F, Romano R 2016 Tunable mechanical monolithic sensors for large band low frequency monitoring and characterization of sites and structures Proc. SPIE 9986 99860C ISBN:97815156063769 doi:10.1117/12.2242080.
[9] Barone F, Giordano G 2017 Watt’s linkage based high sensitivity large band monolithic seismometers and accelerometers for geophysics and seismology Proc. of 16th WCEE Santiago, Chile.
[10] Barone F, Giordano G, Acernese F, Romano R, Gennai A, Passuello D, Boschi V, Cerretani G, Passaquieti R 2016 Large band low frequency sensors based on Watts linkage for future generations of interferometric detectors GWADW 2016 Isola d-Elba, Italy.
[11] Wu S, Fan S and Luo J, 2002 Folded pendulum tiltmeter Rev. Sci. Instrum. 73 2150 doi:10.1063/1.1469676.
[12] Takamori A, Bertolini A, DeSalvo R, Ayara A, Kanazawa T, Shinohara M 2011 Novel compact tiltmeter for ocean bottom and other frontier observations Meas. Sci. Technol. 22 1115091 doi:10.1088/0957-0233/22/11/115091.
[13] Liu J F, Ju L, Blair D G 1997 Vibration isolation performance of an ultra low frequency folded pendulum resonator Phys. Lett. A 228 243 doi:10.1016/S0375-9601(97)0105-9.
[14] Bertolini A 2001 High Sensitivity Accelerometers for Gravity Experiments Ph.D Thesis, LIGO P0100009-00-Z.
[15] Barone F, Giordano G, Acernese F, Method for the measurement of angular and/or linear displacements utilizing one or more folded pendula Pat. Num.: IT1425605, Europe, Japan, USA, Canada pending.