A HARDY SPACE ANALYSIS OF THE BÁEZ-DUARTE CRITERION FOR THE RH

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Abstract. In this article, methods from sub-Hardy Hilbert spaces such as the de Branges-Rovnyak spaces and local Dirichlet spaces are used to investigate Báez-Duarte’s Hilbert space reformulation of the Riemann hypothesis (RH).

Introduction

A classical reformulation of the Riemann hypothesis by Nyman and Beurling (see [3], [16]) says that all the non-trivial zeros of the \( \zeta \)-function lie on the critical line \( \text{Re}(s) = 1/2 \) if and only if the characteristic function \( \chi_{(0,1)} \) belongs to the closed linear span in \( L^2((0,1)) \) of the set \( \{ f_\lambda : 0 \leq \lambda \leq 1 \} \), where \( f_\lambda(x) = \{ \lambda/x \} - \lambda \{ 1/x \} \) (here \( \{ x \} \) is the fractional part). Almost fifty years later a remarkable strengthening of this result by Báez-Duarte [5] shows that we may replace \( \lambda \in (0,1) \) by \( \lambda = 1/\ell \) for \( \ell \geq 2 \). There is an equivalent version of the Báez-Duarte criterion in the weighted sequence space \( \ell^2_\omega \) with inner product given by

\[
\langle x, y \rangle = \sum_{n=1}^{\infty} \frac{x(n)\overline{y(n)}}{n(n+1)}
\]

for sequences \( x, y \in \mathcal{H} \) (see [7, page 73]). For each \( k \geq 2 \), let \( r_k \) denote the sequence defined by \( r_k(n) = k\{n/k\} \). Then the Báez-Duarte criterion may be stated as follows:

**Theorem 1.** The RH is true if and only if \( 1 := (1,1,1,\ldots) \) belongs to the closure of the linear span of \( \{ r_k : k \geq 2 \} \) in \( \ell^2_\omega \).

The plan of the paper is the following. Let \( \mathcal{N} \) denote the linear span of the functions

\[
h_k(z) = \frac{1}{1-z} \log \left( \frac{1+z+\ldots+z^{k-1}}{k} \right)
\]

for \( k \geq 2 \), which all belong to the Hardy space \( H^2 \) (see Lemma [1]). In Section 2 a unitary equivalent version of Theorem 1 for the Hardy space \( H^2 \) is presented. In particular, the RH holds if and only if the constant 1 belongs to the closure of \( \mathcal{N} \) in \( H^2 \) (see Theorem [9]). Section 3 introduces a multiplicative semigroup of weighted composition operators \( \{ W_n : n \geq 1 \} \) on \( H^2 \) and shows that the constant 1 (appearing in Theorem [6]) may be replaced by any cyclic vector for \( \{ W_n : n \geq 1 \} \) in \( H^2 \). It follows that the RH is equivalent to the density of \( \mathcal{N} \) in \( H^2 \) (see Theorem [8]). Section 4 proves that \( (I-S)\mathcal{N} \) is dense in \( H^2 \), where \( S \) is the shift operator on \( H^2 \) (see Theorem [9]). This central result has the following remarkable consequence.

**Key words and phrases.** Riemann hypothesis, Hardy space, Dirichlet space, de Branges-Rovnyak space, Dilation completeness problem.
That \( \mathcal{N} \) is dense in \( H^2 \) with respect to the compact-open topology (see Theorem 10). Since convergence in \( H^2 \) implies convergence in the compact-open topology, this may be viewed as a weak form of RH. Section 5 shows that \( \mathcal{N}^\perp \) is in a sense small by proving that
\[
\mathcal{N}^\perp \cap D_{\delta_1} = \{0\}
\]
where \( D_{\delta_1} \) is the local Dirichlet space at 1 (which is dense in \( H^2 \)), and in particular that \( \mathcal{N}^\perp \) contains no function holomorphic on a neighborhood of the closed unit disk \( \overline{D} \) (see Theorem 12). Section 6 shows that the cyclic vectors for \( \{W_n : n \geq 1\} \) in \( H^2 \) are properly embedded into the set of all 2-periodic functions \( \phi \) on \( (0, \infty) \) having the property that the span of its dilates \( \{\phi(nx) : n \geq 1\} \) is dense in \( L^2(0,1) \) (see Theorem 13). The characterization of all such \( \phi \) is a famous open problem known as the Periodic Dilation Completeness Problem.

1. Background

1.1. The Hardy-Hilbert space. We denote by \( D \) and \( T \) the open unit disk and the unit circle respectively. A holomorphic function \( f \) on \( D \) belongs to the Hardy-Hilbert space \( H^2 \) if
\[
||f||_{H^2} = \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} < \infty.
\]
The space \( H^2 \) is a Hilbert space with inner product
\[
\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b}_n,
\]
where \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are the Maclaurin coefficients for \( f \) and \( g \) respectively. Similarly \( H^\infty \) denotes the space of bounded holomorphic functions defined on \( D \). For any \( f \in H^2 \) and \( \zeta \in T \), the radial limit \( f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta) \) exists m.a.e. on \( T \), where \( m \) denotes the normalized Lebesgue measure on \( T \).

1.2. A weighted Bergman space. Let \( A \) be the Hilbert space of analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) defined on \( D \) for which the inner product is given by
\[
\langle f, g \rangle := \sum_{n=0}^{\infty} \frac{a_n \overline{b}_n}{(n+1)(n+2)}.
\]
There also exists an area integral form of the corresponding \( A \)-norm given by
\[
||f||_A^2 = \int_D |f(z)|^2 (1 - |z|^2) dA(z)
\]
where \( dA \) is the normalized area measure on \( D \). Comparing (0.1) with (1.1) shows that the map
\[
\Psi : (x(1), x(2), \ldots) \mapsto \sum_{n=0}^{\infty} x(n+1) z^n
\]
is a canonical isometric isomorphism of \( \ell^2_\omega \) onto \( A \).
The text \[12\] is a modern reference for such weighted Bergman spaces. Also if $\text{Hol}(\mathbb{D})$ is the space of all holomorphic functions on $\mathbb{D}$ and $T : \text{Hol}(\mathbb{D}) \to \text{Hol}(\mathbb{D})$ is an operator defined by

\[
Tg(z) := \frac{\left(1 - z\right)g(z)}{1 - z},
\]

then $T$ restricted to $H^2$ is an isometric isomorphism onto $A$ (see Lemma 7.2.3 \[13\]). Hence $\Phi := T^{-1} \circ \Psi$ is an isometric isomorphism of $\ell^2_\omega$ onto $H^2$. Therefore to obtain a reformulation of the Báez-Duarte Theorem in $H^2$, we need to calculate $\Psi \Phi_1$ and $\Phi_{r_k}$ for $k \geq 2$. But to do so we shall need some results about local Dirichlet spaces.

1.3. Generalized Dirichlet spaces. Let $\mu$ be a finite positive Borel measure on $\mathbb{T}$, and let $P \mu$ denote its Poisson integral. The generalized Dirichlet space $D_\mu$ consists of $f \in H^2$ satisfying

\[
D_\mu(f) := \int_\mathbb{D} |f'(z)|^2 P\mu(z)\,dA(z) < \infty.
\]

Then $D_\mu$ is a Hilbert space with norm $\|f\|_{D_\mu}^2 := \|f\|^2_2 + D_\mu(f)$. If $\mu = \delta_\zeta$, then $D_{\delta_\zeta}$ is the classical Dirichlet space. If $\mu = \delta_\zeta$ is the Dirac measure at $\zeta \in \mathbb{T}$, then $D_{\delta_\zeta}$ is called the local Dirichlet space at $\zeta$ and in particular

\[
D_{\delta_\zeta}(f) = \int_\mathbb{D} |f'(z)|^2 \frac{1 - |\zeta|^2}{|z - \zeta|^2} \,dA(z).
\]

The recent book \[13\] contains a comprehensive treatment of local Dirichlet spaces and the following result establishes a criterion for their membership.

**Theorem 2.** (See \[13\] Thm. 7.2.1) Let $\zeta \in \mathbb{T}$ and $f \in \text{Hol}(\mathbb{D})$. Then $D_{\delta_\zeta}(f) < \infty$ if and only if

\[
f(z) = a + (z - \zeta)g(z)
\]

for some $g \in H^2$ and $a \in \mathbb{C}$. In this case $D_{\delta_\zeta}(f) = \|g\|^2_2$ and

\[
a = f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta).
\]

Each local Dirichlet space $D_{\delta_\zeta}$ is a proper subspace of $H^2$ and it has the distinctive property that evaluation at the boundary $f \mapsto f^*(\zeta)$ is a bounded linear functional \[13\] Thm. 8.1.2 (ii)].

1.4. The de Branges-Rovnyak spaces. Given $\psi \in L^\infty(\mathbb{T})$, the corresponding Toeplitz operator $T_\psi : H^2 \to H^2$ is defined by

\[
T_\psi f := P_+ (\psi f)
\]

where $P_+ : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2$. Clearly $T_\psi$ is a bounded operator on $H^2$ with $\|T_\psi\| \leq \|\psi\|_{L^\infty}$. If $h \in H^\infty$, then $T_h$ is simply the operator of multiplication by $h$ and its adjoint is $T_h^*$. Given $b$ in the closed unit ball of $H^\infty$, the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of $H^2$ under the operator $(I - T_h T_h^*)^{1/2}$. A norm is defined on $\mathcal{H}(b)$ making $(I - T_h T_h^*)^{1/2}$ a partial isometry from $H^2$ onto $\mathcal{H}(b)$. If $b \equiv 0$ then $\mathcal{H}(b) = H^2$, and if $b$ is inner then $\mathcal{H}(b) = (bH^2)^\perp$ is the model subspace of $H^2$. The recent two-volume work \[9\] \[10\] is an encyclopedic reference for these spaces.

The general theory of $\mathcal{H}(b)$ spaces divides into two distinct cases, according to whether $b$ is an extreme point or a non-extreme point of the unit ball of $H^\infty$. 


We shall only be concerned with the non-extreme case which is best illustrated by the next result (see [9, Chapter 6] and [18, Sects. IV-6 and V-1]).

**Theorem 3.** Let \( b \in H^\infty \) with \( ||b||_{H^\infty} \leq 1 \). The following are equivalent:

1. \( b \) is a non-extreme point of the unit ball of \( H^\infty \),
2. \( \log(1 - |b|^2) \in L^1(\mathbb{T}) \),
3. \( \mathcal{H}(b) \) contains all functions holomorphic in a neighborhood of \( \overline{\mathbb{D}} \).

When \( b \) is non-extreme there exists a unique outer function \( a \in H^\infty \) such that \( a(0) > 0 \) and \( |a^*|^2 + |b|^2 = 1 \) a.e. on \( \mathbb{T} \). In this situation \((b, a)\) is usually called a pair and the function \( b/a \) belongs to the Smirnov class \( N^+ \) of quotients \( p/q \) where \( p, q \in H^\infty \) and \( q \) is an outer function. That all \( N^+ \) functions arise as the quotient of a pair associated to a non-extreme function was shown by Sarason (see [17]).

In [17], Sarason also demonstrated how \( \mathcal{H}(b) \) spaces appear naturally as the domains of some unbounded Toeplitz operators. Let \( \varphi \) be holomorphic in \( \mathbb{D} \) and \( T_\varphi \) the operator of multiplication by \( \varphi \) on the domain

\[
\text{dom}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}.
\]

Then \( T_\varphi \) is a closed operator, and \( \text{dom}(T_\varphi) \) is dense in \( H^2 \) if and only if \( \varphi \in N^+ \) (see [17, Lemma 5.2]). In this case its adjoint \( T_\varphi^* \) is also densely defined and closed. In fact the domain of \( T_\varphi^* \) is a de Branges-Rovnyak space.

**Theorem 4.** *(See [17, Prop. 5.4]*) Let \( \varphi \) be a nonzero function in \( N^+ \) with \( \varphi = b/a \), where \( (b, a) \) is the associated pair. Then \( \text{dom}(T_\varphi^*) = \mathcal{H}(b) \).

If \( \varphi \) is a rational function in \( N^+ \) the corresponding pair \((b, a)\) is also rational (see [17, Remark. 3.2]). Recently Constara and Ransford [8] characterized the rational pairs \((b, a)\) for which \( \mathcal{H}(b) \) is a generalized Dirichlet space.

**Theorem 5.** *(See [8, Theorem 4.1]*) Let \((b, a)\) be a rational pair and \( \mu \) a finite positive measure on \( \mathbb{T} \). Then \( \mathcal{H}(b) = \mathcal{D}_\mu \) if and only if

1. the zeros of \( a \) on \( \mathbb{T} \) are all simple, and
2. the support of \( \mu \) is exactly equal to this set of zeros.

These ideas will be used in Section 5 to investigate the orthogonal complement of the functions \( \{h_k : k \geq 2\} \) in \( H^2 \).

2. The Báez-Duarte criterion in \( H^2 \)

The first main objective is to obtain a unitary equivalent version of Báez-Duarte’s theorem (Theorem 1) in \( H^2 \) upon which to base the rest of our analysis.

**Theorem 6.** For each \( k \geq 2 \), define

\[
h_k(z) = \frac{1}{1 - z} \log \left( \frac{1 + z + \ldots + z^{k-1}}{k} \right).
\]

Then the Riemann hypothesis holds if and only if the constant 1 belongs to the closed linear span of \( \{h_k : k \geq 2\} \) in \( H^2 \).

In order prove this, we must that \(-1 = \Phi 1 \) and \( h_k = \Phi r_k \) for \( k \geq 2 \), where \( \Phi := T^{-1} \circ \Psi : L^2_{\omega} \to H^2 \) is an isometric isomorphism (see subsection 1.2). We first find \( R := \Psi 1 \) and \( R_k := \Psi r_k \), which belong to the weighted Bergman space \( \mathcal{A} \). Then

\[
R(z) = \frac{1}{1 - z}, \quad R_k(z) = \frac{1}{1 - z} [\log(1 + z + \ldots + z^{k-1})]'.
\]
for each \( k = 2, 3, \ldots \) (note that \( R_1 \equiv 0 \)). The expression for \( R \) is trivial. For \( R_k \)
we first note that the sequence \( r_k(n) = k \{n/k\} \) is periodic with \( k \) distinct integer
terms \( \{1, 2, \ldots, k-1, 0, \ldots\} \). Hence collecting terms with common coefficients gives

\[
R_k(z) = \sum_{n=0}^{\infty} z^{nk} + 2 \sum_{n=0}^{\infty} z^{nk+1} + \ldots + (k-1) \sum_{m=1}^{\infty} m \sum_{n=0}^{\infty} z^{nk+m-1} = \sum_{m=1}^{k-1} m \frac{z^{m-1}}{1-z^k} \sum_{n=1}^{m} m z^{m-1} = \frac{1}{1-z} \left[ \frac{(1+z+\ldots+z^{k-1})'}{1+z+\ldots+z^{k-1}} \right]
\]

(2.1) \( = \frac{1}{1-z} \log(1+z+\ldots+z^{k-1})' \).

Next we calculate \( T^{-1}R \) and \( T^{-1}R_k \) in \( H^2 \). It is easy to see that \( T(-1) = R \)
and hence \(-1 = \Phi 1\). But finding the \( T^{-1}R_k \) is not as straightforward because \( T \)
is not injective on \( \text{Hol}(\mathbb{D}) \).

**Lemma 7.** For each non-zero \( c \) and integer \( k \geq 2 \), define the function

\[
h_{k,c}(z) = \frac{1}{1-z} \log \left( \frac{1+z+\ldots+z^{k-1}}{c} \right).
\]

Then \( Th_{k,c} = R_k \) for each \( c \), but \( h_{k,c} \in H^2 \) if and only if \( c = k \).

**Proof.** Let \( s_k(z) := \log(1+z+\ldots+z^{k-1}) \) for \( k \geq 2 \). Since \( R_k \) belongs to \( A \) for \( k \geq 2 \), by (1.2), (1.5) and (2.1) we have

\[
D_{\delta_1}(s_k) = \int_0^\infty \left| \log(1+z+\ldots+z^{k-1}) \right|^2 \frac{1-|z|^2}{|z-1|^2} dA(z)
\]

\[
= \int_\mathbb{D} |R_k(z)|^2 (1-|z|^2) dA(z)
\]

\[
= ||R_k||_A^2 < \infty.
\]

Therefore \( s_k \) belongs to the local Dirichlet space \( D_{\delta_1} \). By Theorem 2 there exists \( f_k \in H^2 \) such that \( s_k(z) = s_k^*(1) + (z-1)f_k(z) = \log k + (z-1)f_k(z) \) and it follows immediately that

\[
f_k(z) = \frac{1}{z-1} \log \left( \frac{1+z+\ldots+z^{k-1}}{k} \right).
\]

Hence \( h_{k,k} = -f_k \in H^2 \). Since clearly \( Th_{k,c} = R_k \) for each non-zero \( c \) and \( T \) is
injective on \( H^2 \), therefore \( c = k \) is the only value for which \( h_{k,c} \in H^2 \). \( \square \)

Therefore with \( h_k := h_{k,k} \) for all \( k \geq 2 \) this concludes the proof of Theorem 6.

### 3. A Weighted Composition Semigroup

In [2] Bagchi showed that in addition to Theorem 1 the RH is equivalent to the
density of \( \text{span}\{r_k : k \geq 2\} \) in \( \ell^2 \). A key ingredient in his proof is a multiplicative
semigroup of operators which leave \( \text{span}\{r_k : k \geq 2\} \) invariant (see [2, Theorem 7]). The relation of invariant subspaces of semigroups to the RH has been evident
since the thesis of Nyman [16] (see also [6] and [15]). In our \( H^2 \) case, a semigroup
of weighted composition operators makes an appearance.
For each $n \geq 1$, let $W_n$ be defined on $H^2$ by
\begin{equation}
W_n f(z) = (1 + z + \ldots + z^{n-1}) f(z^n) = \frac{1 - z^n}{1 - z} f(z^n).
\end{equation}

Note that each $W_n$ is bounded on $H^2$, $W_1 = I$ and $W_m W_n = W_{mn}$ for each $m, n \geq 1$. Hence $\{W_n : n \geq 1\}$ is a multiplicative semigroup on $H^2$. Now if we write
\begin{equation}
h_k(z) = \frac{1}{1 - z} \left( \log(1 - z^k) - \log(1 - z) - \log k \right)
\end{equation}
then it is easy to see that $W_n h_k = h_{nk} - h_n$ for all $k, n \geq 1$ (where $h_1 \equiv 0$). Hence the closure of $\text{span}\{h_k : k \geq 2\}$ is invariant under $\{W_n : n \geq 1\}$. A vector $f \in H^2$ is called a cyclic vector for an operator semigroup $\{S_n : n \geq 1\}$ if $\text{span}\{S_n f : n \geq 1\}$ is dense in $H^2$. Hence the following combines Bagchi’s result and a generalization of Theorem [9].

**Theorem 8.** The following statements are equivalent

1. The Riemann hypothesis,
2. the closure of $\text{span}\{h_k : k \geq 2\}$ contains a cyclic vector for $\{W_n : n \geq 1\}$,
3. $\text{span}\{h_k : k \geq 2\}$ is dense in $H^2$.

**Proof.** The equivalence $(1) \iff (3)$ is just Bagchi’s result transferred to $H^2$ via the isomorphism $\Phi : L^2_\omega \to H^2$. The implication $(1) \to (2)$ follows from Theorem [8] and the fact that $1$ is a cyclic vector for the semigroup $\{W_n : n \geq 1\}$. Indeed $(W_n 1)(z) = 1 + z + \ldots + z^{n-1}$ for all $n \geq 1$ so $\text{span}\{W_n 1 : n \geq 1\}$ contains all analytic polynomials and is hence dense in $H^2$. Finally $(2) \to (3)$ because if the closure of $\text{span}\{h_k : k \geq 2\}$ contains a cyclic vector $f \in H^2$, then it also contains the dense manifold $\text{span}\{W_n f : n \geq 1\}$ by the invariance of $\text{span}\{h_k : k \geq 2\}$ under $\{W_n : n \geq 1\}$. $\square$

In Section 6, we shall see that characterizing the cyclic vectors for $\{W_n : n \geq 1\}$ is intimately related to another famous open problem known as the *Periodic Dilation Completeness Problem* (see [11] and [15]).

4. **The density of $\text{span}\{(I - S) h_k : k \geq 2\}$ in $H^2$**

Let $S = T_\omega$ be the shift operator on $H^2$. Since $I - S$ has dense range (because $I - S^*$ is injective), therefore $\text{span}\{(I - S) h_k : k \geq 2\}$ is dense in $H^2$ under the RH by Theorem [8]. Proving that this statement is unconditionally true is the main objective of this section and it will play a central role in the rest of this work.

**Theorem 9.** The span of $\{I - S) h_k : k \geq 2\}$ is dense in $H^2$.

Since convergence in $H^2$ implies uniform convergence on compact subsets of $\mathbb{D}$, we obtain a weak version of the RH.

**Theorem 10.** The span of $\{h_k : k \geq 2\}$ is dense in $H^2$ with the compact-open topology.

**Proof.** The formal inverse of $I - S$ is the Toeplitz operator $T_\varphi$ of multiplication by the function $\varphi(z) = \frac{1}{1 - z}$. Although $T_\varphi$ is unbounded on $H^2$ (otherwise Theorem [9] would imply the RH), it is still continuous on $H^2$ with the compact-open topology. Therefore the result follows immediately from Theorem [9]. $\square$
Define the multiplicative operator semigroup \( \{T_n : n \geq 1\} \) on \( H^2 \) by
\[
T_n f(z) = f(z^n).
\]
(4.1) Then by (3.1) and (4.1) it is easily seen that
\[
(4.2) T_n(I - S) = (I - S) W_n \quad \forall n \geq 1.
\]
Recall that \( \text{span}\{h_k : k \geq 2\} \) is invariant under \( \{W_n : n \geq 1\} \) (see Section 4), and hence (4.2) implies that \( \text{span}\{(I - S)h_k : k \geq 2\} \) is invariant under \( \{T_n : n \geq 1\} \). So to prove Theorem 9 it is enough to prove that the closure of \( \text{span}\{(I - S)h_k : k \geq 2\} \) contains a cyclic vector for \( \{T_n : n \geq 1\} \). And the cyclic vector we consider is \( 1 - z \).

Indeed, if \( f \in H^2 \) is orthogonal to each \( T_n(1 - z) = 1 - z^n \) then \( \hat{f}(0) = \hat{f}(n) \) for all \( n \geq 1 \) and hence \( f \equiv 0 \). Hence the next result completes the proof of Theorem 9.

**Lemma 11.** The series \( \sum_{k=2}^{\infty} \frac{\mu(k)}{k} (I - S)h_k \) converges to \( 1 - z \) in \( H^2 \), where \( \mu \) is the Möbius function.

Recall that the Möbius function is defined on \( \mathbb{N} \) by \( \mu(k) = (-1)^s \) if \( k \) is the product of \( s \) distinct primes, and \( \mu(k) = 0 \) otherwise. In the proof we shall need the Prime Number Theorem in the equivalent forms
\[
\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\mu(k) \log k}{k} = -1
\]
(see [11, Thm. 4.16] and [14, p. 185, Exercise 16]).

**Proof.** It is enough to prove that
\[
\left\| \sum_{k=2}^{n} \frac{\mu(k)}{k} (I - S)h_k + z - 1 \right\|_{H^2} \to 0
\]
as \( n \to \infty \). Since \( (I - S)h_k(z) = \log(1 - \frac{z^k}{1}) - \log(1 - z) - \log k \) (see (4.2)), we get
\[
\sum_{k=2}^{n} \frac{\mu(k)}{k} (I - S)h_k(z) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(1 - \frac{z^k}{1}) - \sum_{k=1}^{n} \frac{\mu(k)}{k} \log(1 - z) + \log k.
\]

First note that the last sum on the right of (4.5) tends 1 as \( n \to \infty \) by (3.3). Writing the first sum as a double sum after noting that \( \log(1 - \frac{z^k}{1}) = -\sum_{j=1}^{\infty} \frac{z^{jk}}{j} \), interchanging the order of summation and using the basic identity \( \sum_{d|j} \mu(d) = \frac{1}{j} \) if \( j \geq 1 \) [11, Thm. 2.1] ([x] denotes the integer part of \( x \)), we get
\[
\sum_{k=1}^{n} \frac{\mu(k)}{k} \log(1 - \frac{z^k}{1}) = -\sum_{k=1}^{n} \frac{\mu(k)}{k} \sum_{j=1}^{\infty} \frac{z^{jk}}{j} = -\sum_{k=1}^{n} \mu(k) \sum_{j=1}^{\infty} \frac{z^{jk}}{jk}
\]
\[
= -\sum_{j=1}^{\infty} \frac{z^j}{j} \sum_{d|j, 1 \leq d \leq n} \mu(d) = -\sum_{j=1}^{\infty} \frac{z^j}{j} \sum_{d|j} \mu(d) - \sum_{j=n+1}^{\infty} \frac{z^j}{j} \sum_{d|j, 1 \leq d \leq n} \mu(d)
\]
\[
= -\sum_{j=1}^{n} \frac{z^j}{j} \left[ \frac{1}{j} \right] - \sum_{j=n+1}^{\infty} \frac{z^j}{j} \sum_{d|j} \mu(d) - \frac{z^n}{n} \sum_{d|n} \mu(d) = -z - \phi_n(z).
\]
(4.6)
Therefore by (1.5) and (1.6), we will prove (4.4) once we prove that \( ||\phi_n||_{H^2} \rightarrow 0 \) as \( n \rightarrow \infty \). Since

\[
\phi_n(z) = \sum_{j=n+1}^{\infty} z^j \sum_{1 \leq d \leq n} \mu(d)
\]

and if \( \sigma(n) \) denotes the number of divisors of \( n \), then it follows that

\[
| \sum_{d|j} \mu(d) | \leq \sum_{d|j} 1 = \sigma(j).
\]

The function \( \sigma \) satisfies the relation \( \sigma(n) = o(n^\epsilon) \) for every \( \epsilon > 0 \) \([11, \text{p. 296}]\). In particular, \( \sigma(n) \lesssim n^\epsilon \) for some \( 0 < \epsilon < \frac{1}{2} \), and therefore by (4.7)

\[
||\phi_n||_{H^2}^2 \leq \sum_{j=n+1}^{\infty} \frac{\sigma(j)^2}{j^2} \lesssim \sum_{j=n+1}^{\infty} j^{2\epsilon - 2} \rightarrow 0
\]

as \( n \rightarrow \infty \). This proves (4.4) and hence the lemma. \( \square \)

5. Functions orthogonal to \( \{h_k : k \geq 2\} \)

The RH is equivalent to \( \{h_k : k \geq 2\} \perp \) being trivial \( \{0\} \) (see Theorem 3). The main result of this section shows that \( \{h_k : k \geq 2\} \parallel \) is indeed in a sense very small.

**Theorem 12.** We have

\[
\{h_k : k \geq 2\} \perp \cap \mathcal{D}_{\delta_1} = \{0\}
\]

where \( \mathcal{D}_{\delta_1} \) is the local Dirichlet space at 1. In particular \( \{h_k : k \geq 2\} \perp \) contains no function holomorphic on a neighborhood of the closed unit disk \( \overline{D} \).

The key idea is to use the formal inverse \( T_\varphi \) of \( I - S \), where \( \varphi(z) = \frac{1}{1-z} \) is clearly an \( N^+ \) function. Then there is a pair \((b,a)\) associated with \( \varphi \) where

\[
a(z) = \frac{\gamma(1-z)}{(\gamma+1)z}
\]

and \( \gamma = \frac{1+\sqrt{5}}{2} \) is the golden ratio (see [17, page 284]). Therefore by Theorem 4, Theorem 5 and (5.1) we immediately see that

\[
\text{dom}(T_\varphi^*) = \mathcal{H}(b) = \mathcal{D}_{\delta_1}
\]

where \( T_\varphi^* \) is the adjoint of \( T_\varphi \) (see subsection 1.4).

**Proof.** Let \( g_k := (I - S)h_k \) for each \( k \geq 2 \) and note that \( \text{span}\{g_k : k \geq 2\} \) is dense in \( H^2 \) by Theorem 9. Also note \( g_k \in \text{dom}(T_\varphi) \) because \( h_k = T_\varphi g_k \) and by (1.6).

Now let \( p \) be an element in \( \{h_k : k \geq 2\} \perp \cap \text{dom}(T_\varphi^*) \). Hence for each \( k \geq 2 \), we have

\[
\langle T_\varphi^* p, g_k \rangle = \langle p, T_\varphi g_k \rangle = \langle p, h_k \rangle = 0.
\]

Therefore \( T_\varphi^* p \equiv 0 \). But this implies that \( p \equiv 0 \), because

\[
\langle p, T_\varphi f \rangle = \langle T_\varphi^* p, f \rangle = 0
\]

for each \( f \in \text{dom}(T_\varphi) \) and the range of \( T_\varphi \) is all of \( H^2 \) (it is the domain of \( I - S \)). Hence \( \{h_k : k \geq 2\} \parallel \cap \text{dom}(T_\varphi^*) = \{0\} \), (5.2) and Theorem 3 complete the proof. \( \square \)
6. The Periodic Dilation Completeness Problem PDCP

The PDCP asks which 2-periodic functions $\phi$ on $(0, \infty)$ have the property that
\[ \text{span}\{\phi(nx) : n \geq 1\} \]
is dense in $L^2(0,1)$. In this case we shall just say that $\phi$ is a \textit{PDCP function}.

This difficult open problem was first considered independently by Wintner [19] and Beurling [4]. See [11] and [15] for beautiful modern treatments. The main result of this section shows that the cyclic vectors for $\{W_n : n \geq 1\}$ in $H^2$ (see Theorem 8) are properly embedded into the PDCP functions.

**Theorem 13.** There exists an injective linear map $V : H^2 \to L^2(0,1)$ such that if $f$ is a cyclic vector for $\{W_n : n \geq 1\}$ in $H^2$, then $Vf$ is a PDCP function.

The function $Vf \in L^2(0,1)$ is defined on the whole real line by extending it as an odd 2-periodic function.

**Proof.** Recall that the semigroups $\{W_n : n \geq 1\}$ and $\{T_n : n \geq 1\}$ satisfy the relation
\[ T_n(I - S) = (I - S)W_n \quad \forall \ n \geq 1, \]
where $I - S$ has dense range in $H^2$ (see (4.2)). It follows that if $\text{span}\{W_n f : n \geq 1\}$ is dense in $H^2$ for some $f \in H^2$, then $\text{span}\{T_n(I - S)f : n \geq 1\}$ must also be dense.

So $f \mapsto (I - S)f$ maps cyclic vectors for $\{W_n : n \geq 1\}$ to cyclic vectors for $\{T_n : n \geq 1\}$. Let
\[ H^2_0 := \{f \in H^2 : f(0) = 0\} = H^2 \ominus \mathbb{C} \]
and note that $H^2_0$ is a reducing subspace for $T_n$ since $T_n \mathbb{C} \subset \mathbb{C}$ and $T_n H^2_0 \subset H^2_0$. Denote by $P$ the orthogonal projection of $H^2$ onto $H^2_0$. It follows that if $f$ is a cyclic vector for $\{T_n : n \geq 1\}$ in $H^2$ then $Pf$ is a cyclic vector for $\{T_n : n \geq 1\}$ restricted to $H^2_0$.

Therefore
\[ P(I - S) : H^2 \to H^2_0 \]
maps cyclic vectors for $\{W_n : n \geq 1\}$ into cyclic vectors for $\{T_n : n \geq 1\}$ restricted to $H^2_0$. Finally there is a unitary operator $U : H^2_0 \to L^2(0,1)$ such that $f$ is cyclic for $\{T_n : n \geq 1\}$ in $H^2_0$ if and only if $Uf$ is a PDCP function (see [15] page 1707). In fact, it is defined by
\[ (6.1) \quad U : z^k \mapsto e_k(x) := \sqrt{2} \sin(\pi k x) \]
for each $k \geq 1$, where $(e_k)_{k \geq 1}$ is an orthonormal basis for $L^2(0,1)$. Therefore the operator
\[ (6.2) \quad V := UP(I - S) : H^2 \to L^2(0,1) \]
maps cyclic vectors for $\{W_n : n \geq 1\}$ into PDCP functions. It is injective since $\text{Ker}(P) = \mathbb{C}$ and the inverse image of $\mathbb{C}$ under $I - S$ is $\{0\}$. \qed

Finally, we show that not all PDCP functions belong to the range of $V$ (6.2). Wintner [19] showed that for $\text{Re}(s) > 1/2$ the function
\[ f_s(x) = \sum_{k \geq 1} k^{-s} \sqrt{2} \sin(\pi k x) \]
is a PDCP function. We give an independent proof that $f_1$ is a PDCP function and that it does not belong to the range of $V$.

**Theorem 14.** $f_1$ is a PDCP function that does not belong to the range of $V$. 
Proof. Let \( L(z) := \log(1-z) = -\sum_{k \geq 1} z^k / k \) and note that \( U(-L) = f_1 \) (see \( (6.1) \)). Hence it is enough to prove that \( L \) is a cyclic vector for \( \{T_n : n \geq 1\} \) in \( H_0^2 \). Note that since \( (I - S)h_k(z) = \log(1 - z^k) - \log(1 - z) - \log k \) we have
\[
P(I - S)h_k = T_kL - T_1L
\]
and hence
\[
(6.3) \quad P(\text{span}\{(I - S)h_k : k \geq 2\}) \subset \text{span}\{T_nL : n \geq 1\}.
\]
By Theorem [3], the left side of \( (6.3) \) is dense in \( H_0^2 \) and hence \( L \) is cyclic. Therefore \( f_1 \) is a PDCP function. To prove that \( f_1 \) is not in the range of \( V \), we show that \( L \) is not in the range \( P(I - S) \). The functions mapped onto \( L \) by \( P \) are of the form \( \alpha + L \) for some \( \alpha \in \mathbb{C} \). But \( \alpha + L \) does not belong to \( (I - S)H^2 \) because \( L^*(1) \) does not exist and \( f^*(1) = 0 \) for all \( f \in (I - S)H^2 \) (see Theorem [4]).

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