FUNCTIONALITY AND SPECIAL VALUES OF $L$-FUNCTIONS

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Abstract. This is a semi-expository article concerning Langlands functoriality and Deligne's conjecture on the special values of $L$-functions. The emphasis is on symmetric power $L$-functions associated to a holomorphic cusp form, while appealing to a recent work of Mahnkopf on the special values of automorphic $L$-functions.

Contents

1. Introduction 1
2. Symmetric powers and functoriality 3
3. Deligne's conjecture for symmetric power $L$-functions 3
3.1. Symmetric power $L$-functions 4
3.2. Deligne's conjecture 5
3.3. Critical points 6
4. Dihedral calculations 7
5. Representations with cohomology 9
5.1. Cohomological representations of GL$_n$(R) 9
5.2. Functoriality and cohomological representations 13
6. Special values of $L$-functions of GL$_n$: The work of Mahnkopf 16
6.1. General remarks on functoriality and special values 16
6.2. The main results of Mahnkopf 16
7. A conjecture on twisted $L$-functions 18
References 19

1. Introduction

Langlands functoriality principle reduces the study of automorphic forms on the adelic points of a reductive algebraic group to those of an appropriate general linear group. In particular, every automorphic $L$-function on an arbitrary reductive group must be one for a suitable GL$_n$. One should therefore be able to reduce the study of special values of an automorphic $L$-function to those of a principal $L$-function of Godement and Jacquet on GL$_n$. 

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While the integral representations of Godement and Jacquet do not seem to admit a cohomological interpretation, there is a recent work of J. Mahnkopf [27] [28] which provides us with such an interpretation for certain Rankin–Selberg type integrals. In particular, modulo a nonvanishing assumption on local archimedean Rankin–Selberg product $L$-functions for forms on $\text{GL}_n \times \text{GL}_{n-1}$, he defines a pair of periods, which seem to be in accordance with those of Deligne [9] and Shimura [43]. This work of Mahnkopf is quite remarkable and requires the use of both Rankin–Selberg and Langlands–Shahidi methods in studying the analytic (and arithmetic) properties of $L$-functions. His work therefore brings in the theory of Eisenstein series to play an important role. In §6 we briefly review this work of Mahnkopf.

This article is an attempt to test the philosophy— to study the special values of $L$-functions while using functoriality— by means of recent cases of functoriality established for symmetric powers of automorphic forms on $\text{GL}_2$ [17] [21]. While a proof of the precise formulae in the conjectures of Deligne [9] still seem to be out of reach, we expect to be able to prove explicit connections between the special values of symmetric power $L$-functions twisted by Dirichlet characters and those of the original symmetric power $L$-functions using this work of Mahnkopf. These relations are formulated in this paper as Conjecture 7.1 which also follows from the more general conjectures of Blasius [3] and Panchiskin [33], although the heuristics underlying our conjecture are quite different.

A standard assumption made in the study of special values of $L$-functions is that the representations (to which are attached the $L$-functions) are cohomological. This is the case in Mahnkopf’s work. A global representation being cohomological is entirely determined by the archimedean components. For representations which are symmetric power lifts of a cusp form on $\text{GL}_2$ we have the following fact. Consider a holomorphic cusp form on the upper half plane of weight $k$. This corresponds to a cuspidal automorphic representation, which is cohomological if $k \geq 2$, and any symmetric power lift, if cuspidal, is essentially cohomological. See Theorem 5.5 (If the weight $k = 1$ then the representation is not cohomological, and furthermore none of the symmetric power $L$-functions have any critical points.) In §5 we review representations with cohomology in the case of $\text{GL}_n$.

We recall the formalism of Langlands functoriality for symmetric powers in §2. We then review Deligne’s conjecture for the special values of symmetric power $L$-functions in §3 and give a brief survey as to which cases are known so far. In §4 we sketch a proof of the conjecture for dihedral representations. We add that in the dihedral case the validity of the conjecture in known to experts.

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2. SYMMETRIC POWERS AND FUNCTORIALITY

In this section we recall the formalism of Langlands functoriality especially for symmetric powers. We will be brief here as there are several very good expositions of the principle of functoriality; see for instance [8, Chapter 2].

Let $F$ be a number field and let $\mathbb{A}_F$ be its adèles ring. We let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$, by which we mean that, for some $s \in \mathbb{R}$, $\pi \otimes | \cdot |^s$ is an irreducible summand of $L^2_{cusp}(GL_2(F) \backslash GL_2(\mathbb{A}_F), \omega)$ the space of square-integrable cusp forms with central character $\omega$. We have the decomposition $\pi = \bigotimes_v \pi_v$ where $v$ runs over all places of $F$ and $\pi_v$ is an irreducible admissible representation of $GL_2(F_v)$.

The local Langlands correspondence for $GL_2$ (see [25] and [24] for the $p$-adic case and [23] for the archimedean case), says that to $\pi_v$ is associated a representation $\sigma(\pi_v) : W_{F_v}^\prime \rightarrow GL_2(\mathbb{C})$ of the Weil–Deligne group $W_{F_v}^\prime$ of $F_v$. (If $v$ is infinite, we take $W_{F_v}^\prime = W_{F_v}$.) Let $n \geq 1$ be an integer. Consider the $n$-th symmetric power of $\sigma(\pi_v)$ which is an $n+1$ dimensional representation. This is simply the composition of $\sigma(\pi_v)$ with $Sym^n : GL_2(\mathbb{C}) \rightarrow GL_{n+1}(\mathbb{C})$. Appealing to the local Langlands correspondence for $GL_{n+1}$ ([14], [15], [23], [24]) we get an irreducible admissible representation of $GL_{n+1}(F_v)$ which we denote as $Sym^n(\pi_v)$. Now define a global representation of $Sym^n(\pi)$ of $GL_{n+1}(\mathbb{A}_F)$ by

$$Sym^n(\pi) := \bigotimes_v Sym^n(\pi_v).$$

Langlands principle of functoriality predicts that $Sym^n(\pi)$ is an automorphic representation of $GL_{n+1}(\mathbb{A}_F)$, i.e., it is isomorphic to an irreducible subquotient of the representation of $GL_{n+1}(\mathbb{A}_F)$ on the space of automorphic forms [4, §4.6]. If $\omega_\pi$ is the central character of $\pi$ then $\omega_\pi^{n(n+1)}$ is the central character of $Sym^n(\pi)$. Actually it is expected to be an isobaric automorphic representation. (See [8, Definition 1.1.2] for a definition of an isobaric representation.) The principle of functoriality for the $n$-th symmetric power is known for $n = 2$ by Gelbart–Jacquet [11]; for $n = 3$ by Kim–Shahidi [21]; and for $n = 4$ by Kim [17]. For certain special forms $\pi$, for instance, if $\pi$ is dihedral, tetrahedral, octahedral or icosahedral, it is known for all $n$ (see [18] [34]).

3. DELIGNE’S CONJECTURE FOR SYMMETRIC POWER $L$-FUNCTIONS

Deligne’s conjecture on the special values of $L$-functions is a conjecture concerning the arithmetic nature of special values of motivic $L$-functions at critical points.
The definitive reference is Deligne’s article [9]. We begin by introducing the symmetric power $L$-functions, which are examples of motivic $L$-functions, and then state Deligne’s conjecture for these $L$-functions.

3.1. Symmetric power $L$-functions. Let $\varphi \in S_k(N, \omega)$, i.e., $\varphi$ is a holomorphic cusp form on the upper half plane, for $\Gamma_0(N)$, of weight $k$, and nebentypus character $\omega$. Let $\varphi(z) = \sum_{n=1}^{\infty} a_n q^n$ be the Fourier expansion of $\varphi$ at infinity. We let $L_f(s, \varphi)$ stand for the completed $L$-function associated to $\varphi$ and let $L_f(s, \varphi)$ stand for its finite part. Assume that $\varphi$ is a primitive form in $S_k(N, \omega)$. By primitive, we mean that it is an eigenform, a newform and is normalized such that $a_1(\varphi) = 1$. In a suitable right half plane the finite part $L_f(s, \varphi)$ is a Dirichlet series with an Euler product

$$L_f(s, \varphi) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p L_p(s, \varphi),$$

where, for all primes $p$, we have

$$L_p(s, \varphi) = (1 - a_p p^{-s} + \omega(p)p^{k-1-2s})^{-1} = (1 - \alpha_{p,\varphi} p^{-s})^{-1}(1 - \beta_{p,\varphi} p^{-s})^{-1},$$

with the convention that if $p | N$ then $\beta_{p,\varphi} = 0$. Let $\eta$ be a Dirichlet character modulo an integer $M$. We let $S_f$ stand for the set of primes dividing $NM$ and let $S = S_f \cup \{\infty\}$. For any $n \geq 1$, the twisted partial $n$-th symmetric power $L$-function of $\varphi$ is defined as

$$L^S(s, \text{Sym}^n \varphi, \eta) = \prod_{p \notin S} L_p(s, \text{Sym}^n \varphi, \eta),$$

where, for all $p \notin S$, we have

$$L_p(s, \text{Sym}^n \varphi, \eta) = \prod_{i=0}^{n} (1 - \eta(p)\alpha_{p,\varphi}^i \beta_{p,\varphi}^{n-i} p^{-s})^{-1}.$$

If $M = 1$ and $\eta$ is (necessarily) trivial then we denote the corresponding $L$-function as $L^S(s, \text{Sym}^n \varphi)$.

Using the local Langlands correspondence the partial $L$-function can be completed by defining local factors $L_p(s, \text{Sym}^n \varphi, \eta)$ for $p \in S$ and the completed $L$-function, which is a product over all $p$ including $\infty$, will be denoted as $L(s, \text{Sym}^n \varphi, \eta)$. The Langlands program predicts that $L(s, \text{Sym}^n \varphi, \eta)$, which is initially defined only in a half plane, admits a meromorphic continuation to the entire complex plane and that it has all the usual properties an automorphic $L$-function is supposed to have. This is known for $n \leq 4$ from the works of several people including Hecke, Shimura, Gelbart–Jacquet, Kim and Shahidi. It is also known for all $n$ for cusp forms of a special type, for instance, if the representation corresponding to the cusp form is dihedral or the other polyhedral types. (The reader is referred to the same references as in the last paragraph of the previous section.)
3.2. Deligne’s conjecture. Let \( \varphi \) be a primitive form in \( S_k(N, \omega) \). Let \( M(\varphi) \) be the motive associated to \( \varphi \). This is a rank two motive over \( \mathbb{Q} \) with coefficients in the field \( \mathbb{Q}(\varphi) \) generated by the Fourier coefficients of \( \varphi \). (We refer the reader to Deligne \cite{Deligne} and Scholl \cite{Scholl} for details about \( M(\varphi) \).) The \( L \)-function \( L(s, M(\varphi)) \) associated to this motive is \( L(s, \varphi) \).

Given the motive \( M(\varphi) \), and given any complex embedding \( \sigma \) of \( \mathbb{Q}(\varphi) \), there are nonzero complex numbers, called Deligne’s periods, \( c^\pm(M(\varphi), \sigma) \) associated to it. For simplicity of exposition we concentrate on the case when \( \sigma \) is the identity map, and denote the corresponding periods as \( c^\pm(M(\varphi)) \). We note that this “one component version” of Deligne’s conjecture that we state below is not as strong as the full conjecture and so we are dealing with a simplistic situation.

Similarly, for the symmetric powers \( \text{Sym}^n(M(\varphi)) \), we have the corresponding periods \( c^\pm(\text{Sym}^n(M(\varphi))) \). In \cite{Deligne} Proposition 7.7 the periods for the symmetric powers are related to the periods of \( M(\varphi) \). The explicit formulae therein have a quantity \( \delta(M(\varphi)) \) which is essentially the Gauss sum of the nebentypus character \( \omega \) and is given by

\[
\delta(M(\varphi)) \sim (2\pi i)^{1-k} g(\omega) := (2\pi i)^{1-k} \sum_{u=0}^{c-1} \omega_0(u)\exp(-2\pi i u/c),
\]

where \( c \) is the conductor of \( \omega \), \( \omega_0 \) is the primitive character associated to \( \omega \), and by \( \sim \) we mean up to an element of \( \mathbb{Q}(\varphi) \). We will denote the right hand side by \( \delta(\omega) \). For brevity, we will denote \( c^\pm(M(\varphi)) \) by \( c^\pm(\varphi) \). Recall \cite{Deligne} Definition 1.3 that an integer \( m \) is critical for any motivic \( L \)-function \( L(s, M) \) if both \( L_\infty(s, M) \) and \( L_\infty(1-s, M^\vee) \) are regular at \( s = m \). We now state Deligne’s conjecture \cite{Deligne} Section 7] on the special values of the symmetric power \( L \)-functions.

**Conjecture 3.1.** Let \( \varphi \) be a primitive form in \( S_k(N, \omega) \). There exist nonzero complex numbers \( c^\pm(\varphi) \) such that

1. If \( m \) is a critical integer for \( L_f(s, \text{Sym}^{2l+1} \varphi) \), then

\[
L_f(m, \text{Sym}^{2l+1} \varphi) \sim (2\pi i)^{m(l+1)} c^\pm(\varphi) \delta(\omega)^{l(l+1)/2},
\]

where \( \pm = (-1)^m \).

2. If \( m \) is a critical integer for \( L_f(s, \text{Sym}^{2l} \varphi) \), then

\[
L_f(m, \text{Sym}^{2l} \varphi) \sim \begin{cases} (2\pi i)^{m(l+1)} c^+(\varphi) \delta(\omega)^{l(l+1)/2} & \text{if } m \text{ is even}, \\ (2\pi i)^{m} c^+(\varphi) \delta(\omega)^{l(l-1)/2} & \text{if } m \text{ is odd}. \end{cases}
\]

By \( \sim \) we mean up to an element of \( \mathbb{Q}(\varphi) \).

We wish to emphasize that in the original conjecture of Deligne, the numbers \( c^\pm \) are periods, but in this paper they are just a couple of complex numbers in terms of which the critical values of the symmetric power \( L \)-functions can be expressed.
It adds some clarity to write down explicitly the statement of the conjecture for the \( n \)-th symmetric power, in the special cases \( n = 1, 2, 3, 4 \), and while doing so we also discuss about how much is known in these cases.

Let \( m \) be a critical integer for \( L_f(s, \varphi) \). Then Conjecture 3.1 takes the form

\[
(3.2) \quad L_f(m, \varphi) \sim (2\pi i)^m c^\pm(\varphi),
\]

where \( \pm = (-1)^m \). In this context, the conjecture is known and is a theorem of Shimura [42] [43]. Shimura relates the required special values to quotients of certain Petersson inner products, whose rationality properties can be studied.

Let \( m \) be a critical integer for \( L_f(s, \text{Sym}^2 \varphi) \). Then Conjecture 3.1 takes the form

\[
(3.3) \quad L_f(m, \text{Sym}^2 \varphi) \sim \begin{cases} (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))\delta(\omega) & \text{if } m \text{ is even,} \\ (2\pi i)^m(c^+(\varphi)c^-(\varphi)) & \text{if } m \text{ is odd.} \end{cases}
\]

The conjecture is known in this case and is due to Sturm [44] [45]. Sturm uses an integral representation for the symmetric square \( L \)-function due to Shimura [41].

Let \( m \) be a critical integer for \( L_f(s, \text{Sym}^3 \varphi) \). Then Conjecture 3.1 takes the form

\[
(3.4) \quad L_f(m, \text{Sym}^3 \varphi) \sim (2\pi i)^{2m}(c^+(\varphi)^3c^-(\varphi))\delta(\omega),
\]

where \( \pm = (-1)^m \). The conjecture is known in this case and is due to Garrett and Harris [10]. The main thrust of that paper is to prove a theorem on the special values of certain triple product \( L \)-functions \( L(s, \varphi_1 \times \varphi_2 \times \varphi_3) \). Deligne’s conjecture for motivic \( L \)-functions predicts the special values of such triple product \( L \)-functions, for which an excellent reference is Blasius [2]. Via a standard argument, the case \( \varphi_1 = \varphi_2 = \varphi_3 = \varphi \), gives the special values of the symmetric cube \( L \)-function for \( \varphi \). This was reproved by Kim and Shahidi [19] emphasizing finiteness of these \( L \)-values which follows from their earlier work [20].

Let \( m \) be a critical integer for \( L_f(s, \text{Sym}^4 \varphi) \). Then Conjecture 3.1 takes the form

\[
(3.5) \quad L_f(m, \text{Sym}^4 \varphi) \sim \begin{cases} (2\pi i)^{3m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega)^3 & \text{if } m \text{ is even,} \\ (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega) & \text{if } m \text{ is odd.} \end{cases}
\]

In general the conjecture is not known for higher \( (n \geq 4) \) symmetric power \( \text{Sym} \)-functions. Although, if \( \varphi \) is dihedral, then we have verified the conjecture for any symmetric power; see [41].

We remark that a prelude to this conjecture was certain calculations made by Zagier [47] wherein he showed that such a statement holds for the \( n \)-th symmetric power \( \text{Sym} \)-function, with \( n \leq 4 \), of the Ramanujan \( \Delta \)-function.

3.3. Critical points. As recalled above, an integer \( m \) is critical for any motivic \( L \)-function \( L(s, M) \) if both \( L_\infty(s, M) \) and \( L_\infty(1-s, M^\vee) \) are regular at \( s = m \). For example, if \( M = \mathbb{Z}(0) = H^0(\text{Point}) \) be the trivial motive, then \( L(s, M) \) is the Riemann zeta function \( \zeta(s) \) [9, §3.2]. Then \( L_\infty(s, M) = \pi^{-s/2}\Gamma(s/2) \). It is an easy exercise to see that an integer \( m \) is critical for \( \zeta(s) \) if \( m \) is an even positive integer or an odd negative integer. More generally, as in Blasius [2], one can calculate the
critical points for any motivic $L$-function in terms of the Hodge numbers of the corresponding motive. For the specific $L$-functions at hand, namely the symmetric power $L$-functions, one explicitly knows the $L$-factors at infinity \[29\] using which it is a straightforward exercise to calculate the critical points. In the following two lemmas we record the critical points of the $n$-th symmetric power $L$-function associated to a modular form $\varphi$. (For more details see \[34\].)

**Lemma 3.6.** Let $\varphi$ be a primitive cusp form of weight $k$. The set of critical integers for $L_f(s, \text{Sym}^{2r+1}\varphi)$ is given by integers $m$ with

$$r(k - 1) + 1 \leq m \leq (r + 1)(k - 1).$$

**Lemma 3.7.** Let $\varphi$ be a primitive cusp form of weight $k$. The set of critical integers for $L_f(s, \text{Sym}^r\varphi)$ is given below.

1. If $r$ is odd and $k$ is even, then

$$\{(r-1)(k-1)+1, \dots, r(k-1) - 1, \text{ or } r(k-1)+1, r(k-1)+3, \dots, (r+1)(k-1)\}.$$

2. If $r$ and $k$ are both odd, then

$$\{(r-1)(k-1)+1, \dots, r(k-1)-1, \text{ or } r(k-1)+2, r(k-1)+4, \dots, (r+1)(k-1)\}.$$

3. If $r$ and $k$ are both even, then

$$\{(r-1)(k-1)+2, \dots, r(k-1)-1, \text{ or } r(k-1)+2, r(k-1)+4, \dots, (r+1)(k-1)-1\}.$$

4. If $r$ is even and $k$ is odd, then

$$\{(r-1)(k-1)+1, \dots, r(k-1)-1, \text{ or } r(k-1)+2, r(k-1)+4, \dots, (r+1)(k-1)\}.$$

**Remark 3.8.** Here are some easy observations based on the above lemmas.

1. If $k = 1$ then $L_f(s, \text{Sym}^n\varphi)$ does not have any critical points for any $n \geq 1$. In particular, this is the case if $\varphi$ is a cusp form which is tetrahedral, octahedral or icosahedral $\[34\]$.

2. If $k = 2$ then $L_f(s, \text{Sym}^n\varphi)$ has a critical point if and only if $n$ is not a multiple of 4; further $L_f(s, \text{Sym}^{2r+1}\varphi)$ has exactly one critical point $m = r + 1$; and if $r$ is odd $L_f(s, \text{Sym}^r\varphi)$ has two critical points $r, r + 1$. This applies in particular for symmetric power $L$-functions of elliptic curves.

3. Let $m$ be a critical integer for $L_f(s, \text{Sym}^r\varphi)$. Then $m$ is even if and only if $m$ is to the right of the center of symmetry.

4. **Dihedral calculations**

A primitive form $\varphi$ is said to be *dihedral* if the associated cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$, denoted $\pi(\varphi)$, is the automorphic induction of an idèle class character, say $\chi$, of a quadratic extension $K/\mathbb{Q}$. This is denoted as $\pi(\varphi) = \text{AI}_{K/\mathbb{Q}}(\chi)$. (Since $\varphi$ is a holomorphic modular form, in this situation, $K$ is necessarily an imaginary quadratic extension.) In $\[34\]$ we give a proof of Deligne’s conjecture for the special values of symmetric power $L$-functions for such dihedral forms using
purely the language of L-functions. (See the last paragraph of this section.) In this section we summarize the main results of those calculations while referring the reader to \cite{34} for all the proofs.

Recall from Remark 3.8 that if the weight \( k = 1 \) then there are no critical integers for \( L_f(s, \text{Sym}^n \varphi) \). It is easy to see \cite{34} that if \( \pi(\varphi) = \text{AI}_{K/Q}(\chi) \) and some nonzero power of \( \chi \) is Galois invariant (under the Galois group of \( K/Q \)) then \( k = 1 \). Hence we may, and henceforth shall, assume that for every nonzero integer \( n \), \( \chi^n \) is not Galois invariant. The following lemma describes the isobaric decomposition of a symmetric power lifting of a dihedral cusp form.

**Lemma 4.1.** Let \( \chi \) be an idèle class character of an imaginary quadratic extension \( K/Q \); assume that \( \chi^n \) is not Galois invariant for any nonzero integer \( n \). Let \( \chi_Q \) denote the restriction of \( \chi \) to the idèles of \( Q \). Then we have

\[
\begin{align*}
\text{Sym}^{2r}(\text{AI}_{K/Q}(\chi)) &= \bigoplus_{a=0}^{r-1} \text{AI}_{K/Q}(\chi^{2r-a} \chi^a) \bigoplus \chi_Q', \\
\text{Sym}^{2r+1}(\text{AI}_{K/Q}(\chi)) &= \bigoplus_{a=0}^{r} \text{AI}_{K/Q}(\chi^{2r+1-a} \chi^a),
\end{align*}
\]

where \( \chi' \) is the nontrivial Galois conjugate of \( \chi \).

Note that every isobaric summand above is either cuspidal or is one dimensional. This lemma can be recast in terms of \( L \)-functions. For an idèle class character \( \chi \) of an imaginary quadratic extension \( K/Q \), we let \( \varphi_\chi \) denote the primitive cusp form such that \( \pi(\varphi_\chi) = \text{AI}_{K/Q}(\chi) \). If \( \varphi_\chi \in S_k(N, \omega) \) then \( \omega \omega_K = \chi_Q \), where we make the obvious identification of classical Dirichlet characters and idèle class characters of \( Q \), and \( \omega_K \) denotes the quadratic idèle class character of \( Q \) associated to \( K \) via global class field theory.

**Lemma 4.2.** The symmetric power \( L \)-functions of \( \varphi_\chi \) decompose as follows:

\[
\begin{align*}
L_f(s, \text{Sym}^{2r} \varphi_\chi) &= L_f(s - r(k - 1), (\omega \omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k - 1), \varphi_\chi^{2(r-a)}, \omega^a) \\
&= L_f(s - r(k - 1), (\omega \omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k - 1), \varphi_\chi^{2(r-a)}, (\omega \omega_K)^a).
\end{align*}
\]

\[
\begin{align*}
L_f(s, \text{Sym}^{2r+1} \varphi_\chi) &= \prod_{a=0}^{r} L_f(s - a(k - 1), \varphi_\chi^{2(r-a)+1}, \omega^a) \\
&= \prod_{a=0}^{r} L_f(s - a(k - 1), \varphi_\chi^{2(r-a)+1}, (\omega \omega_K)^a).
\end{align*}
\]

We can now use the results of Shimura \cite{42} \cite{43} and classical theorems on special values of abelian (degree 1) \( L \)-functions for the factors on the right hand side of the above decompositions to prove Deligne’s conjecture on the special values of
$L_f(s, \text{Sym}^n \varphi_\chi)$. The proof is an extended exercise in keeping track of various constants after one has related the periods of the cusp form $\varphi_{\chi^n}$ to the periods of $\varphi_\chi$. We state this as the following theorem.

**Theorem 4.3** (Period relations for dihedral forms). For any positive integer $n$ we have the following relations:

1. $c^+(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n$, 
2. $c^-(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n g(\omega_K)$,

where $\sim$ means up to an element of $\mathbb{Q}(\chi)$—the field generated by the values of $\chi$, and $g(\omega_K)$ is the Gauss sum of $\omega_K$.

As mentioned in the introduction, Deligne’s conjecture for dihedral forms is known. This is because Deligne’s main conjecture is known if one considers only the motives as those attached to abelian varieties and the category used is that defined by using absolute Hodge cycles for morphisms. The theorem above can therefore be deduced from the literature. However, the proof that we have is entirely in terms of $L$-functions: by considering the Rankin–Selberg $L$-function for $\varphi_{\chi^n} \times \varphi_\chi$. We also use some nonvanishing results for $L$-functions in the course of proving this theorem.

5. **Representations with cohomology**

In the study of special values of $L$-functions, if the $L$-function at hand is associated to a cuspidal automorphic representation, then a standard assumption made on the representation is that it contributes to cuspidal cohomology. This cohomology space admits a rational structure and certain periods are obtained by comparing this rational structure with a rational structure on the Whittaker model of the representation at hand. This approach to the study of special values is originally due to Harder [12] and since then pursued by several authors and in particular by Mahnkopf [28].

The purpose of this section, after setting up the context, is to record Theorem 5.5 which says that the $n$-th symmetric power lift of a cohomological cusp form on $\text{GL}_2$, if cuspidal, contributes to cuspidal cohomology of $\text{GL}_{n+1}$. This theorem is essentially due to Labesse and Schwermer [26]. We then digress a little and discuss the issue of functoriality and a representation being cohomological.

### 5.1. Cohomological representations of $\text{GL}_n(\mathbb{R})$

In this section we set up the context of cohomological representations. This is entirely standard material; we refer the reader to Borel-Wallach [6] and Schwermer [40] for generalities on the cohomology of representations.

We let $G_n = \text{GL}_n$ and $B_n$ be the standard Borel subgroup of upper triangular matrices in $G_n$. Let $T_n$ be the diagonal torus in $G_n$ and $Z_n$ be the center of $G_n$. We denote by $X^+(T_n)$ the dominant (with respect to $B_n$) algebraic characters of $T_n$. For $\mu \in X^+(T_n)$ let $(\rho_\mu, M_\mu)$ be the irreducible representation of $G_n(\mathbb{R})$ with highest
weight $\mu$. The Lie algebra of $G_n(\mathbb{R})$ will be denoted by $\mathfrak{g}_n$. We let $K_n = O_n(\mathbb{R})Z_n(\mathbb{R})$ and $K_n^o$ be the topological connected component of the identity element in $K_n$.

Let $\text{Coh}(G_n, \mu)$ be the set of all cuspidal automorphic representations $\pi = \otimes'_{p \leq \infty} \pi_p$ of $\text{GL}_n(\mathbb{A}_\mathbb{Q})$ such that

$$H^*(\mathfrak{g}_n, K_n^o; \pi_{\infty} \otimes \rho_\mu) \neq (0).$$

By $H^*(\mathfrak{g}_n, K_n^o; -)$ we mean relative Lie algebra cohomology. We recall the following from [6, §I.5.1]: Given a $(\mathfrak{g}_n, K_n)$ module $\sigma$, one can talk about $H^*(\mathfrak{g}_n, K_n^o; \sigma)$ as well as $H^*(\mathfrak{g}_n, K_n; \sigma)$. Note that $K_n/K_n^o \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $H^*(\mathfrak{g}_n, K_n^o; \sigma)$ and by taking invariants under this action we get $H^*(\mathfrak{g}_n, K_n; \sigma)$.

Observe that a global representation being cohomological is entirely a function of the representation at infinity. There are two very basic problems, one local and the other global, which has given rise to an enormous amount of literature on this theme.

1. The local problem is to classify all irreducible admissible representations $\pi_\infty$ of $G_n(\mathbb{R})$ which are cohomological, i.e., $H^*(\mathfrak{g}_n, K_n^o; \pi_{\infty} \otimes \rho_\mu) \neq (0)$, and for such representations to actually calculate the cohomology spaces.

2. The global problem is to construct global cuspidal representations whose representation at infinity is cohomological in the above sense.

The reader is referred to Borel-Wallach [6] as a definitive reference for the local problem. For the purposes of this article we discuss the solution of the local problem for tempered representations of $\text{GL}_n(\mathbb{R})$. To begin, we record a very simplified version of [6, Theorem II.5.3] and [6, Theorem II.5.4].

**Theorem 5.1.** Let $G$ be a reductive Lie group. Let $K$ be a maximal compact subgroup adjoined with the center of $G$. Discrete series representations of $G$ (if they exist) are cohomological and have nonvanishing cohomology only in degree $\dim(G/K)/2$.

We have suppressed any mention of the finite dimensional coefficients because Wigner’s Lemma [6, Theorem I.4.1] gives a necessary condition for the infinitesimal character, and nonvanishing cohomology of a representation pins down the finite dimensional representation. Here is a well known example illustrating this theorem.

**Example 5.2.** Let $G = G_2(\mathbb{R})$ and $K = K_2 = O_2(\mathbb{R})Z_2(\mathbb{R})$. For any integer $l \geq 1$, we let $M_l$ denote the irreducible representation of $G$ of dimension $l$ which is the $(l - 1)$-th symmetric power of the standard two dimensional representation. Let $D_l$ be the discrete series representation of lowest weight $l + 1$. (If we take a weight $k$ holomorphic cusp form then the representation at infinity is $D_{k-1}$.) The Langlands parameter of $D_l$ is $\text{Ind}_{W_\mathbb{R}}^{G_2}(\chi_l)$, where $W_\mathbb{R}$ is the Weil group of $\mathbb{R}$, and $\chi_l$ is the character of $\mathbb{C}^*$ sending $z$ to $(z/|z|)^l$. The representation $D_l$ is cohomological; more precisely, we have

$$H^q(\mathfrak{g}, K; (D_l \otimes | \cdot |^{-l-1/2}) \otimes M_l) = \begin{cases} \mathbb{C} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$
SPECIAL VALUES OF L-FUNCTIONS

See [46] Proposition I.4 (1) for instance. To compare our notation to the notation therein, take \( h = l + 1, \ a = l - 1, \ \epsilon = 0 \) and put \( d = (h, a, \epsilon) \). Then our \( M_l \) is the \( r[d] \) of [46] and our \( D_l \otimes | \cdot |_{\mathbb{R}}^{(l-1)/2} \) is the \( \pi[d] \) of [46]. See [26] §2.1 for an SL_2 version of this example.

It is a standard fact that relative Lie algebra cohomology satisfies a Künneth rule [6] §I.1.3. Using this one can see that if \( G \) is a product of \( m \) copies of GL_2(\mathbb{R}) then the representation \( D_{l_1} \otimes \cdots \otimes D_{l_m} \) is, up to twisting by a suitable power of \( | \cdot |_{\mathbb{R}}, \) cohomological with respect to the finite dimensional coefficients \( M_{l_1} \otimes \cdots \otimes M_{l_m}. \)

We now recall, very roughly, a version of Shapiro’s lemma for relative Lie algebra cohomology. Consider a parabolically induced representation. The cohomology of the induced representation can be described in terms of the cohomology of the inducing representation. (See [6] Theorem III.3.3, (ii) for a precise formulation.)

We can now give a reasonably complete picture for tempered representations of \( G \). For \((w,l) \in \mathcal{L}_0(G_n)\), define the parabolically induced representation \( J(w,l) \) by

\[
J(w,l) = \text{Ind}_{G}^{G_{n}}((D_{l_1} \otimes | \cdot |^{w/2}_{\mathbb{R}}) \otimes \cdots \otimes (D_{l_{n/2}} \otimes | \cdot |^{w/2}_{\mathbb{R}}))
\]

if \( n \) is even, and

\[
J(w,l) = \text{Ind}_{G}^{G_{n}}((D_{l_1} \otimes | \cdot |^{w/2}_{\mathbb{R}}) \otimes \cdots \otimes (D_{l_{(n/2)-1/2}} \otimes | \cdot |^{w/2}_{\mathbb{R}}) \otimes | \cdot |^{w/2}_{\mathbb{R}})
\]

if \( n \) is odd. It is well known that, up to the twist \( | \cdot |^{w/2}_{\mathbb{R}}, \) the representations \( J(w,l) \) are irreducible tempered representations of \( G_n \) [23] §2.

Now we describe the finite dimensional coefficients. Let \( X^+_0(T_n) \) stand for all dominant integral weights \( \mu = (\mu_1, \ldots, \mu_n) \) satisfying the purity condition that there is an integer \( w \), called the weight of \( \mu \), such that \( \mu_i + \mu_{n-i+1} = w \). The sets \( \mathcal{L}_0^+(G_n) \) and \( X^+_0(T_n) \) are in bijection via the map \((w,l) \mapsto \mu = w/2 + 1/2 - \rho_n \) where \( \rho_n \) is half the sum of positive roots for GL_n. Let \( w_n \) be the Weyl group element of \( G_n \) of longest length and let \( \mu^\vee = -w_n \cdot \mu \). Then \( \mu^\vee \simeq (\rho_\mu)^\vee \) is the contragredient of \( \rho_\mu \).

Assume that the pair \((w,l)\) corresponds to \( \mu \) as above. Using Example 5.2 on the cohomology of discrete series representations, and appealing to the Künneth rule and Shapiro’s lemma as recalled above, one can conclude that

\[
H^0(\mathfrak{g}_n, K_n^\circ; (J(w,l) \otimes \text{sgn}^1) \otimes M_{\mu^\vee}) = (0)
\]
unless the degree \( q \) is in the so-called cuspidal range \( b_n \leq q \leq t_n \), where the bottom degree \( b_n \) is given by

\[
b_n = \begin{cases} 
\frac{n^2}{4} & \text{if } n \text{ is even}, \\
\frac{(n^2 - 1)}{4} & \text{if } n \text{ is odd},
\end{cases}
\]

and the top degree \( t_n \) is given by

\[
t_n = \begin{cases} 
\frac{((n + 1)^2 - 1)}{4} - 1 & \text{if } n \text{ is even}, \\
\frac{(n + 1)^2}{4} - 1 & \text{if } n \text{ is odd},
\end{cases}
\]

and finally that the dimension of \( H^q(\mathfrak{g}_n, K_n; (J(w, l) \otimes \text{sgn}) \otimes M_{l^s}) \) is 1 if \( q = b_n \) or \( q = t_n \). The exponent \( t \) of the sign character \( \text{sgn} \) is in \( \{0, 1\} \). If \( n \) is even, \( t \) plays no role since \( J(w, l) \otimes \text{sgn} = J(w, l) \). If \( n \) is odd, \( t \) is determined by the weight of \( \mu \) and the parity of \((n - 1)/2\), due to considerations of central character (Wigner’s lemma).

To complete the picture one notes that, given \( M_{\mu} \), there is, up to twisting by the sign character, only one irreducible, unitary (up to twisting by \(|\cdot|_R^{w/2}\)) generic representation with nonvanishing cohomology with respect to \( M_{\mu} \) and this representation is a suitable \( J(w, l) \) as above.

**Remark 5.3.** Let \( \pi \) be a cohomological cuspidal algebraic (\cite{8} §1.2.3) automorphic representation of \( G_n(\mathbb{A}_\mathbb{Q}) \) then the representation \( \pi_{\infty} \) at infinity has to be a \( J(w, l) \) for some \( (w, l) \in L_0^G(G_n) \). This can be seen as follows. Since \( \pi \) is cuspidal and algebraic, by the purity lemma \cite{8} Lemme 4.9], we get that the parameter of \( \pi_{\infty} \) is pure. Since it is cohomological the finite dimensional coefficients has a highest weight \( \mu \) which is also pure, i.e., \( \mu \in X_0^+(T_n) \). Further, \( \pi_{\infty} \) being generic and essentially unitary implies that it is a \( J(w, l) \) as above.

**Example 5.4.** To illuminate this picture we work through the above recipe for the case of a holomorphic cusp form. (We use the notation introduced in Example 5.2 and the previous sections.) Let \( \varphi \in S_k(N, \omega) \) and consider the cuspidal automorphic representation \( \pi = \pi(\varphi) \otimes |\cdot|^s \). Then \( \pi_{\infty} = \pi(\varphi)_{\infty} \otimes |\cdot|^s \mid_{\mathbb{R}} = D_{k-1} \otimes |\cdot|^s \mid_{\mathbb{R}} \).

1. If \( k \) is even, then the representation \( \pi_{\infty} \) is a \( J(w, l) \) exactly when \( w = 2s \in \mathbb{Z} \) and \( w + k - 1 \) is odd. Hence \( s \in \mathbb{Z} \) and \( \pi_{\infty} = J(2s, (k - 1), -(k - 1)) \).

The corresponding dominant weight \( \mu \) is \((s + (k - 2)/2, s - (k - 2)/2)\). The representation \( M_{\mu^s} \) is \( M_{k-1} \otimes \text{det}^{-s-(k-2)/2} \). (For a dominant weight \( \mu = (\mu_1, \mu_2) \in \mathbb{Z}^2 \) the rational representation \( M_{\mu^s} \) is \( M_{\mu_1-\mu_2+1} \otimes \text{det}^{-\mu_1} \).) Using the fact that \( \text{det} = \text{sgn} \otimes |\cdot|_{\mathbb{R}} \) and that \( D_{k-1} \otimes \text{sgn} = D_{k-1} \), we get

\[
\pi_{\infty} \otimes M_{\mu^s} = (D_{k-1} \otimes |\cdot|^s \mid_{\mathbb{R}}) \otimes (M_{k-1} \otimes \text{det}^{-s-(k-2)/2}) = (D_{k-1} \otimes |\cdot|^{-(k-2)/2}) \otimes M_{k-1},
\]

which has nontrivial \((\mathfrak{g}, K)\)-cohomology (see Example 5.2).

2. If \( k \geq 3 \) is odd, then \( \pi_{\infty} \) is a \( J(w, l) \) exactly when \( w = 2s \) is an odd integer. Letting \( s = 1/2 + r \), with \( r \in \mathbb{Z} \), we have \( \pi_{\infty} = J(2r + 1, (k - 1, -(k - 1))) \).
The corresponding dominant weight $\mu$ is $(r + (k - 1)/2, r + 1 - (k - 1)/2)$. The representation $M_{\mu^r}$ is $M_{k-1} \otimes (\det)^{-r-(k-1)/2}$. In this case we get

$$\pi_{\infty} \otimes M_{\mu^r} = (D_{k-1} \otimes | \cdot |^{1/2+r}) \otimes (M_{k-1} \otimes (\det)^{-r-(k-1)/2})$$

$$= (D_{k-1} \otimes | \cdot |^{-(k-2)/2}) \otimes M_{k-1},$$

which has nontrivial $(g,K)$-cohomology as mentioned before. We have excluded the case $k = 1$, because, firstly, the representation at infinity is not cohomological, and secondly, any symmetric power $L$-function of a weight one form has no critical points.

We finally remark that in both cases, the condition $\pi_{\infty}$ being a $J(w,1)$ is exactly the condition which ensures that the representation $\pi = \pi(\varphi) \otimes | \cdot |^s$ is regular algebraic in the sense of Clozel [S §1.2.3 and §3.4].

5.2. Functoriality and cohomological representations. Now we turn to the global problem, namely, to construct a cuspidal automorphic representation whose representation at infinity is cohomological. The specific theorem we are interested in is the following.

**Theorem 5.5.** Let $\varphi \in S_k(N, \omega)$ with $k \geq 2$. Let $n \geq 1$. Assume that $\text{Sym}^n(\pi(\varphi))$ is a cuspidal representation of $GL_{n+1}(\mathbb{A}_\mathbb{Q})$. Let

$$\Pi = \text{Sym}^n(\pi(\varphi)) \otimes \xi \otimes | \cdot |^s$$

where $\xi$ is any idèle class character such that $\xi_{\infty} = \text{sgn}^{s_1}$, with $\epsilon \in \{0, 1\}$, and $| \cdot |$ is the adèlic norm. We suppose that $s$ and $\epsilon$ satisfy:

1. If $n$ is even, then let $s \in \mathbb{Z}$ and $\epsilon \equiv n(k - 1)/2 \pmod{2}$.
2. If $n$ is odd, then let $s \in \mathbb{Z}$ if $k$ is even, and we let $s \in 1/2 + \mathbb{Z}$ if $k$ is odd. We impose no condition on $\epsilon$.

Then $\Pi \in \text{Coh}(G_{n+1}, \mu^r)$ where $\mu \in X_0^+(T_{n+1})$ is given by

$$\mu = \left(\frac{n(k - 2)}{2} + s, \frac{(n - 2)(k - 2)}{2} + s, \ldots, \frac{-n(k - 2)}{2} + s\right) = (k - 2)\rho_{n+1} + s.$$

(Recall that $\rho_{n+1}$ is half the sum of positive roots of $GL_{n+1}$. ) In other words, the representation $\text{Sym}^n(\pi(\varphi)) \otimes \xi \otimes | \cdot |^s$, with $\xi$ and $s$ as above, contributes to the cohomology of the locally symmetric space $GL_{n+1}(\mathbb{Q}) \backslash GL_{n+1}(\mathbb{A}_\mathbb{Q})/K_fK_{n+1,\infty}$ with coefficients in the local system determined by $\rho_{\mu^r}$, where $K_f$ is a deep enough open compact subgroup of $GL_{n+1}(\mathbb{A}_{\mathbb{Q}, f})$. (Here $\mathbb{A}_{\mathbb{Q}, f}$ denotes the finite adeles of $\mathbb{Q}$.)

**Proof.** See Labesse–Schwermer [26] Proposition 5.4 for an $\text{SL}_n$-version of this theorem. When $k = 2$, the theorem has also been observed by Kazhdan, Mazur and Schmidt [16] pp.99).

We sketch the details in the case when $n = 2r$ is even (the case when $n$ is odd being absolutely similar.) The proof follows by observing that the representation at infinity of $\text{Sym}^n(\pi(\varphi))$ is the representation of $GL_{n+1}(\mathbb{R})$ whose Langlands parameter is
\[ \text{Sym}^n(\text{Ind}_{\mathbb{C}^*}^\mathbb{R}(\chi_{k-1})) \text{ where } \chi_{k-1}(z) = (z/|z|)^{k-1}. \] It is a pleasant exercise to calculate a symmetric power of a two dimensional induced representation, after doing which one gets that the representation \( \Pi_\infty \) is given by

\[
\Pi_\infty = \text{Ind}_{P_2,...,2,1}^{Gn+1} (D_{2r(k-1)} \otimes \cdots \otimes D_{2(k-1)} \otimes \text{sgn}^{r(k-1)} \otimes \xi_\infty \otimes | \cdot |^s_R).
\]

We deduce that \( \Pi_\infty \) is a \( J(w, l) \) (which, as mentioned before, is equivalent to \( \Pi \) being regular algebraic) exactly when \( w = 2s \in \mathbb{Z}, r(k-1) + \epsilon \) is even,

\[ l = (2r(k-1), \ldots, 2(k-1), 0, -2(k-1), \ldots, -2r(k-1)) = 2(k-1)\rho_{n+1}, \]

and \( w + l \) is even which implies that \( w \) is even. These conditions are satisfied by the hypothesis in the theorem. The weight \( \mu \) is determined by \( \mu = w/2 + 1/2 - \rho_{n+1} \) and the first part of the theorem follows from the discussion in the previous section.

Finally, the relation with the cohomology of locally symmetric spaces follows as in [26, §1] or [28, §3.2].

**Corollary 5.6.** Let \( \varphi \in S_k(N, \omega) \). Assume that \( k \geq 2 \) and that \( \varphi \) is not dihedral. Then, up to twisting by a quadratic character, \( \text{Sym}^n(\pi(\varphi)) \) for \( n = 2, 3, 4 \), contributes to cuspidal cohomology.

**Proof.** Other than the above theorem one appeals to the main result of [22].

One might view this theorem as an example of the possible dictum that a functorial lift of a cohomological representation is cohomological. This has been used in many instances to construct global representations which contribute to cuspidal cohomology. However, there are several instances where a functorial lift of a cohomological representation is not cohomological. On the other hand, it is an interesting question to ask if the converse of the above dictum is true, namely, if a lift is cohomological, then whether the preimage, is a fortiori cohomological? We end the section by a series of examples illustrating some of these principles. But before doing so, we would like to draw the reader’s attention to a conjecture of Clozel [7, §1] which is motivated by the ideas of Labesse and Schwermer [26]. It roughly states that given a tempered cohomological representation at infinity, one can find a global cuspidal automorphic representation whose representation at infinity is the given one.

**Example 5.7.** The following is a sampling of results—which by no means is to be considered exhaustive—where functoriality is used to produce cohomological representations.

1. Labesse and Schwermer [26] proved the existence of nontrivial cuspidal cohomology classes for \( \text{SL}_2 \) and \( \text{SL}_3 \) over any number field \( E \) which contains a totally real number field \( F \) such that \( F = F_0 \subset F_1 \subset \cdots \subset F_n = E \) with each \( F_{i+1}/F_i \) either a cyclic extension of prime degree or a non-normal cubic extension. The functorial lifts used were base change for \( \text{GL}_2 \) and the
symmetric square lifting of Gelbart and Jacquet. This was generalized for $\text{SL}_n$ over $E$, in conjunction with Borel [5]; with the additional input of base change for $\text{GL}_n$.

(2) Motivated by [26], Clozel [7] used automorphic induction and proved the existence of nontrivial cuspidal cohomology classes for $\text{SL}_{2n}$ over any number field.

(3) Rajan [35], also motivated by [26], proved the existence of nontrivial cuspidal cohomology classes for $\text{SL}_1(D)$ for a quaternion division algebra $D$ over a number field $E$, with $E$ being an extension of a totally real number field $F$ with solvable Galois closure. Other than base change, he used the Jacquet–Langlands correspondence.

(4) Ash and Ginzburg [1, §4] have commented on a couple of examples of cuspidal cohomology classes for $\text{GL}_4$ over $\mathbb{Q}$. The first is by lifting from $\text{GSp}_4$ to $\text{GL}_4$ a weight 3 Siegel modular form. The second is to use automorphic induction from $\text{GL}_2$ over a quadratic extension.

(5) Ramakrishnan and Wang [38] used the lifting from $\text{GL}_2 \times \text{GL}_3 \rightarrow \text{GL}_6$, due to Kim and Shahidi, to construct cuspidal cohomology classes of $\text{GL}_6$ over $\mathbb{Q}$.

In almost all the above works, functoriality is used to construct cuspidal representations, and in doing so, one exercises some control over the representations at infinity to arrange for them to be cohomological.

**Example 5.8.** We construct an example to show that a functorial lift of a cohomological representation need not be cohomological. For an even integer $k$, take two weight $k$ holomorphic cusp forms $\varphi_1$ and $\varphi_2$, and let $\pi_i = \pi(\varphi_i)$ for $i = 1, 2$. By Example 5.4 we have that both $\pi_1$ and $\pi_2$ are cohomological representations. Put $\Pi = \pi_1 \boxtimes \pi_2$ (see Ramakrishnan [36]). Choose the forms $\varphi_1$ and $\varphi_2$ such that $\Pi$ is cuspidal; this can be arranged by taking exactly one of them to be dihedral, or by arranging that $\pi_1$ is not $\pi_2 \otimes \chi$ for any character $\chi$, by virtue of [37, Theorem 11.1].

It is easy to see that $\Pi_\infty$ is given by

$$\Pi_\infty = \text{Ind}_{G_2,1,1}^{G_4(R)}(D_{2(k-1)} \otimes \text{sgn} \otimes \mathbb{I}),$$

where $\mathbb{I}$ is the trivial representation of $\mathbb{R}^*$. Observe that $\Pi_\infty$ is not a $J(w, 1)$ and hence is not cohomological by applying Remark 5.3 (Note that $\Pi$, as it stands, is not algebraic, but we can replace $\Pi$ by $\pi_1 \boxtimes T \pi_2$ (see [3] Definition 1.10)) and make it algebraic; this replaces $\Pi_\infty$ by $\Pi_\infty \otimes | \cdot |_{\mathbb{R}}^{1/2}$. However, note that if we took $\varphi_1$ and $\varphi_2$ to be in general position (unequal weights) then the lift $\Pi$ would be cohomological. Similarly, it is possible to construct such an example for the lifting from $\text{GL}_2 \times \text{GL}_3$ to $\text{GL}_6$.

**Example 5.9.** We would like to mention that in the converse direction the $\text{GL}_2 \times \text{GL}_2$ to $\text{GL}_4$ lifting is well behaved. Now let $\varphi_i$ have weight $k_i \geq 1$, for $i = 1, 2$, and assume
without loss of generality that $k_1 \geq k_2$. With $\Pi = \pi(\varphi_1) \boxtimes \pi(\varphi_2)$ we have

$$\Pi_\infty = \text{Ind}_{G_2(\mathbb{R})}^{G_4(\mathbb{R})} (D_{k_1+k_2-2} \otimes D_{k_1-k_2}).$$

Suppose $\Pi_\infty$ is cohomological, i.e., is a $J(w,1)$, then we would have $k_1 + k_2 - 2 > k_1 - k_2 > 0$, which implies that $k_1 > k_2 \geq 2$, and hence both $\pi(\varphi_1)$ and $\pi(\varphi_2)$ are cohomological.

**Example 5.10.** It seems to be well known that given a cuspidal representation $\pi$ of $\text{GL}_4(\mathbb{A}_\mathbb{Q})$, its exterior square lift $\wedge^2(\pi)$, which by Kim [17] is an automorphic representation of $\text{GL}_6(\mathbb{A}_\mathbb{Q})$, is never cohomological.

### 6. Special values of $L$-functions of $\text{GL}_n$: The work of Mahnkopf

**6.1. General remarks on functoriality and special values.** This section is a summary of some recent results due to Joachim Mahnkopf [27] [28]. In this work he proves certain special values theorems for the standard $L$-functions of cohomological cuspidal automorphic representations of $\text{GL}_n$. In principle one can appeal to functoriality and this work of Mahnkopf to prove new special values theorems. For example, given a cusp form $\varphi \in S_k(N, \omega)$, let $\pi(\varphi)$ denote the cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$. Functoriality predicts the existence of an automorphic representation $\text{Sym}^n(\pi(\varphi))$ of $\text{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$. (See [24]) Then it is easy to check that

$$L(s, \text{Sym}^n(\pi(\varphi))) = L(s + n(k - 1)/2, \text{Sym}^n\varphi),$$

where the left hand side is the standard $L$-function of $\text{Sym}^n(\pi(\varphi))$. Using Mahnkopf’s work for the function on the left, one can hope to prove a special values theorem for the function on the right. This is fine in principle, but there are several obstacles to overcome before it can be made to work.

**6.2. The main results of Mahnkopf [28].** Let $\mu \in X_0^+(T_n)$ and let $\pi \in \text{Coh}(G_n, \mu)$. We let $L(s, \pi) = \prod_{p \leq \infty} L(s, \pi_p)$ be the standard $L$-function attached to $\pi$. Any character $\chi_\infty$ of $\mathbb{R}^*$ is of the form $\chi_\infty = \epsilon_\infty | \cdot |^m$ for a complex number $m$. We say $\chi_\infty$ is critical for $\pi_\infty$ if

1. $m \in 1/2 + \mathbb{Z}$ if $n$ is even, and $m \in \mathbb{Z}$ if $n$ is odd; and
2. $L(\pi_\infty \otimes \chi_\infty, 0)$ and $L(\pi_\infty \otimes \chi_\infty^{-1}, 1)$ are regular values.

We say $\chi : \mathbb{Q}^* \backslash \mathbb{A}_\mathbb{Q}^* \to \mathbb{C}^*$ is critical for $\pi$ if $\chi_\infty$ is critical for $\pi_\infty$. Let $\text{Crit}(\pi)$ stand for all such characters $\chi$ which are critical for $\pi$. Let $\text{Crit}(\pi)_{<}$ stand for all $\chi \in \text{Crit}(\pi)$ such that if $\chi_\infty = \epsilon_\infty | \cdot |^m$ then $m \leq (1 - \text{wt}(\mu))/2$.

Let $\pi \in \text{Coh}(G_n, \mu)$ and let $\chi \in \text{Crit}(\pi)$. Let $\chi_\infty = \epsilon_\infty | \cdot |^m$. Given $\mu = (\mu_1, \ldots, \mu_n) \in X^+(T_n)$ choose a $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in X^+(T_{n-1})$ such that

1. $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n$; and
2. $\lambda_{n/2} = -m + 1/2$ if $n$ is even, and $\lambda_{(n+1)/2} = -m$ if $n$ is odd.
Proposition 1.1 of [27] says that such a $\lambda$ exists. Let $P$ be the standard parabolic subgroup of $G_{n-1}$ of type $(n-2,1)$ and let $W^P$ be a system of representatives for $W_{M_P}\backslash W_{G_{n-1}}$. Let $\hat{w} \in W^P$ be given by

$$\hat{w} = \begin{pmatrix} 1 & 2 & \cdots & \left[\frac{n}{2}\right] - 1 & \left[\frac{n}{2}\right] + 1 & \cdots & n - 1 \\ 1 & 2 & \cdots & \left[\frac{n}{2}\right] - 1 & n - 1 & \cdots & n - 2 \end{pmatrix}.$$ 

Define the weight $\mu' = (\hat{w}(\lambda+\rho_{n-1})-\rho_{n-1})|_{T_{n-2}} \in X^+(T_{n-2})$ where $T_{n-2}$ is embedded in $T_{n-1}$ as $t \mapsto \text{diag}(t,1)$.

**Theorem 6.1** (Theorem 5.4 in Mahnkopf [28]). Let $\mu \in X_0^+(T_n)$ be regular and let $\pi \in \text{Coh}(\text{GL}_n, \mu^\vee)$. Let $\mu' \in X^+(T_{n-2})$ be as above and $\pi' \in \text{Coh}(\text{GL}_{n-2}, \mu')$; if $n$ is odd then $\pi'$ has to satisfy a parity condition. We have

1. $\text{Crit}(\pi) \subseteq \text{Crit}(\pi') \subseteq$.

2. Let $\chi \in \text{Crit}(\pi) \subseteq$, with $\chi_\infty = \epsilon_\infty | \cdot |^m$. There exists a collection of complex numbers $\Omega(\pi, \pi', \epsilon_\infty) \in \mathbb{C}^* / \mathbb{Q}(\pi')$ such that for any finite extension $E / \mathbb{Q}(\pi')$ the tuple $\{\Omega(\pi, \pi', \epsilon_\infty)\}_{\epsilon_\infty \in \text{Hom}(E, \mathbb{C})} \in (E \otimes \mathbb{C})^*/(\mathbb{Q}(\pi')^*)^*$ is well defined. There exists a complex number $P_\mu(m)$, depending only on $\mu$ and $m$, subject to Assumption 6.2 below, such that for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ and almost all $\chi$ as above, we have

$$\left(\Omega(\pi, \pi', \epsilon_\infty) \frac{\mathbf{g}(\chi) \mathbf{G}(\eta) P_\mu(m)}{\Omega(\pi, \chi, \epsilon_\infty)} \frac{L(\pi \otimes \chi, 0)}{L(\pi' \otimes \chi, 0)}\right)^\sigma = \mathbf{g}(\chi^\sigma) \mathbf{G}(\eta^\sigma) P_\mu(m) \frac{L(\pi^\sigma \otimes \chi^\sigma, 0)}{L(\pi'^\sigma \otimes \chi^\sigma, 0)},$$

where $\eta$ is a certain auxiliary character and $\mathbf{G}(\eta)$ a certain product of Gauss sums associated to $\eta$.

The above theorem is valid only under the following assumption.

**Assumption 6.2.** $P_\mu(m) \neq 0$.

The quantity $P_\mu(m)$ is the value at $s = 1/2$ of an archimedean Rankin–Selberg integral attached to certain cohomological choice of Whittaker functions. Mahnkopf proves a necessary condition for this nonvanishing assumption [28, 8.6]. At present this seems to be a serious limitation of this technique. It is widely believed that this assumption is valid and it has shown up in several other works based on the same, or at any rate similar, techniques. See for instance Ash–Ginzburg [1], Kazhdan–Mazur–Schmidt [10] and Harris [13]. It is an important technical problem to be able to prove this nonvanishing hypothesis.

The proof of the above theorem combines both the Langlands–Shahidi and the Rankin–Selberg methods of studying $L$-functions. One considers the pair of representations $\pi \times \text{Ind}_{P}^{G_{n-1}}(\pi' \otimes \chi)$ of $G_n(\mathbb{A}_\mathbb{Q}) \times G_{n-1}(\mathbb{A}_\mathbb{Q})$ and carefully chooses a cusp form $\phi \in \pi$ and an Eisenstein series $E$ corresponding to a section in $\text{Ind}_{P}^{G_{n-1}}(\pi' \otimes \chi)$. To this pair $(\phi, E)$ a certain Rankin–Selberg type zeta integral [28, 2.1.2], which has a cohomological interpretation, computes the quotient of $L$-functions appearing in the theorem.
The theorem roughly says that the special values of a standard \( L \)-function for \( \text{GL}_n \) are determined in terms of those of a standard \( L \)-function for \( \text{GL}_{n-2} \). This descent process terminates since we know the special values of \( L \)-functions for \( \text{GL}_1 \) and \( \text{GL}_2 \), and we get the following theorem; see [28, §5.5] for making the right choices in the induction on \( n \).

**Theorem 6.3** (Theorem A in Mahnkopf [28]). Assume that \( \mu \in X_0^+(T_n) \) is regular and let \( \pi \in \text{Coh}(G_n, \mu^\vee) \). Let \( \chi \in \text{Crit}(\pi) \leq 1 \). To \( \pi \) and \( \chi_\infty \) is attached \( \Omega(\pi, \chi_\infty) \in \mathbb{C} \) such that for all but finitely many such \( \chi \) we have

\[
\left( \frac{g(\chi)^{[n/2]}\mathcal{G}(\eta)}{\Omega(\pi, \chi_\infty)} L(\pi \otimes \chi \eta, 0) \right)^\sigma = \frac{g(\chi_\sigma)^{[n/2]}\mathcal{G}(\eta_\sigma)}{\Omega(\pi_\sigma, \chi_\infty)} L(\pi_\sigma \otimes \chi_\sigma \eta_\sigma, 0),
\]

where \( \eta \) is a certain auxiliary character and \( \mathcal{G}(\eta) \) a certain product of Gauss sums associated to \( \eta \). Moreover, write \( \chi_\infty = \epsilon_\infty(\chi)^\ell \). There are periods \( \Omega_n(\pi) \in \mathbb{C}^* \) if \( n \) is even, and \( \Omega(\pi) \in \mathbb{C}^* \) if \( n \) is odd, and a collection \( P_\mu \in \mathbb{C} \), such that \( \Omega(\pi, \chi_\infty) = P_\mu \Omega(\pi) \) if \( n \) is odd, and \( \Omega(\pi, \chi_\infty) = P_\mu \epsilon(\chi_\infty)(\pi) \) if \( n \) is even.

Note that Theorem 6.3 since it uses Theorem 6.1 also depends on Assumption 6.2.

7. A CONJECTURE ON TWISTED \( L \)-FUNCTIONS

The periods \( c^+ \) and \( c^- \) which appear in Deligne’s conjecture are motivically defined. (See Deligne [9, (1.7.2)].) On the other hand, the periods which appear in the work of Harder, and also Mahnkopf, have an entirely different origin, namely, they come by a comparison of rational structures on cuspidal cohomology on the one hand and a Whittaker model for the representation, on the other. See Harder [12, p. 81] and Mahnkopf [28, §3.4]. It is not at the moment clear how one might explicitly compare these different periods attached to the same object. (See also Remark (2) in Harder’s paper [12, p. 85].)

However, one might ask if these different periods behave in the same manner under twisting. Here is a simple example to illustrate this. Let \( \chi \) be an even Dirichlet character. Let \( m \) be an even positive integer. Such an \( m \) is critical for \( L(s, \chi) \). It is well known [31] Corollary VII.2.10 that

\[
L_f(m, \chi) \sim_{Q(\chi)} (2\pi i)^m \text{g}(\chi).
\]

By \( \sim_{Q(\chi)} \) we mean up to an element of the (rationality) field \( Q(\chi) \) generated by the values of \( \chi \). Now let \( \eta \) be possibly another even Dirichlet character. Applying the result to the character \( \chi \eta \), and using [42] Lemma 8], we get

\[
L_f(m, \chi \eta)/L_f(m, \chi) \sim_{Q(\chi)Q(\eta)} \text{g}(\eta).
\]

Observe that the period, namely the \((2\pi i)^m\), does not show up, and we have the relation that the special value of the twisted \( L \)-function and the original \( L \)-function differ, up to rational quantities, by the Gauss sum of the twisting character.
Another example along these lines which follows from Shimura [43] is the following. Let \( \varphi \in S_k(N, \omega) \) and let \( \eta \) be an even Dirichlet character. For any integer \( m \), with \( 1 \leq m \leq k - 1 \), we have

\[
L_f(m, \varphi, \eta) \sim_{Q(\varphi)Q(\eta)} g(\eta)L_f(m, \varphi).
\]

The point being that, in the above relation, the periods \( c^\pm(\varphi) \) do not show up, and so the definition of these periods is immaterial. (One can rewrite this relation entirely in terms of periods of the associated motives and it takes the form \( c^\pm(M(\varphi) \otimes M(\eta)) \sim g(\eta)c^\pm(M(\varphi)) \), the notation being obvious.)

Even if one cannot prove a precise theorem on special values of \( L \)-functions in terms of these—motivically or otherwise defined—periods, one can still hope to prove such period relations. Sometimes such period relations are sufficient for applications; see for instance Murty–Ramakrishnan [30] where such a period relation is used to prove Tate’s conjecture in a certain case.

With this motivation, we formulate the following conjecture on the behavior of the special values of symmetric power \( L \)-functions under twisting by Dirichlet characters.

**Conjecture 7.1.** Let \( \varphi \in S_k(N, \omega) \) be a primitive form. Let \( \eta \) be a primitive Dirichlet character.

1. Suppose \( \eta \) is even, i.e., \( \eta(-1) = 1 \). Then the critical set for \( L_f(s, \text{Sym}^n \varphi, \eta) \) is the same as the critical set for \( L_f(s, \text{Sym}^n \varphi) \), and if \( m \) is critical, then

\[
L_f(m, \text{Sym}^n \varphi, \eta) \sim g(\eta)^{(n+1)/2}L_f(m, \text{Sym}^n \varphi),
\]

unless \( n \) is even and \( m \) is odd (to the left of center of symmetry), in which case we have

\[
L_f(m, \text{Sym}^n \varphi, \eta) \sim g(\eta)^{n/2}L_f(m, \text{Sym}^n \varphi).
\]

2. Suppose \( \eta \) is odd, i.e., \( \eta(-1) = -1 \), and \( n \) is even. Then, if \( m \) is critical for \( L_f(s, \text{Sym}^n \varphi, \eta) \), then either \( m + 1 \) or \( m - 1 \) is critical for \( L_f(s, \text{Sym}^n \varphi) \). For such an \( m \) to the right of the center of symmetry we have

\[
L_f(m, \text{Sym}^n \varphi, \eta) \sim ((2\pi i)^+g(\eta))^{n/2+1}L_f(m \pm 1, \text{Sym}^n \varphi),
\]

and if \( m \) is to the left of the center of symmetry, we have

\[
L_f(m, \text{Sym}^n \varphi, \eta) \sim ((2\pi i)^-g(\eta))^{n/2}L_f(m \pm 1, \text{Sym}^n \varphi).
\]

3. Suppose \( \eta \) is odd, i.e., \( \eta(-1) = -1 \), and \( n \) is odd. Then the critical set for \( L_f(s, \text{Sym}^n \varphi, \eta) \) is the same as the critical set for \( L_f(s, \text{Sym}^n \varphi) \). Let \( k \geq 3 \). If \( m \) is critical for \( L_f(s, \text{Sym}^n \varphi, \eta) \), then either \( m + 1 \) or \( m - 1 \) is critical for \( L_f(s, \text{Sym}^n \varphi) \), and for such an \( m \)

\[
L_f(m, \text{Sym}^n \varphi, \eta) \sim ((2\pi i)^\pm g(\eta))^{(n+1)/2}L_f(m \pm 1, \text{Sym}^n \varphi).
\]
In all the three cases $\sim$ means up to an element of $\mathbb{Q}(\phi)\mathbb{Q}(\eta)$.

Now we elaborate on the heuristics on which we formulated the above conjecture. For $n = 1$ and $n = 2$ this is contained in the theorems of Shimura [42, 43] and Sturm [44, 45] respectively. For $n = 3$, using results on triple product $L$-functions for which Blasius [2] is a convenient reference and using Garrett–Harris [10, §6], one can verify that the above conjecture is true. Further, for $n \geq 4$ and if $\phi$ is dihedral, i.e., $\pi(\phi) = \text{AI}_K/K(\chi)$, then the conjecture follows by applying the known cases of $n = 1, 2$ to each summand in the isobaric decomposition in Lemma 4.1. Observe that the exponent $\lceil (n+1)/2 \rceil$ appearing in the conjecture is the number of summands in the isobaric decomposition.

Although we have not checked this, our conjecture should follow from the more general conjectures of Blasius [3, Conjecture L.9.8] and Panchiskin [33, Conjecture 2.3] on the behavior of periods of motives twisted by Artin motives.

It appears that the authors may be able to prove a weaker version of Conjecture 7.1 and so really prove a relation amongst appropriate periods, using Theorem 6.3 of Mahnkopf; at least in the case when $\text{Sym}^n(\pi(\phi))$ is known to exist as a cuspidal automorphic representation.

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SPECIAL VALUES OF \( L \)-FUNCTIONS

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