Central limit theorems via Stein’s method for randomized experiments under interference

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Abstract

Controlling for interference through design and analysis can consume both engineering resources and statistical power, so it is of interest to understand the extent to which estimators and confidence intervals constructed under the SUTVA assumption are still valid in the presence of interference. Toward this end, Sävje et al. (2017) provide laws of large numbers for standard estimators of the average treatment effect under a limited form of interference characterized by an interference dependence graph. In this paper we link that view of interference to the dependency graph version of Stein’s method. We prove a central limit theorem for a variant of the difference-in-means estimator if the $o(n)$ restriction on average dependency degree in Sävje et al. (2017) is replaced by an $o(n^{1/4})$ constraint on the maximal dependency degree. We then provide a central limit theorem that can handle interference that exists between all pairs of units, provided the interference is approximately local. The asymptotic variance admits a decomposition into two terms: (a) the variance that is expected under no-interference and (b) the additional variance contributed by interference. The results arise as an application of two flavors of Stein’s method: the dependency graph approach and the generalized perturbative approach.

Keywords: causal inference, dependency graph, normal approximation, SUTVA

1 Introduction

In randomized experiments it is standard to assume that units do not interfere with each other (Cox 1958). Such an assumption of no-interference is also known as individualistic treatment response (Manski 2013) and is a key part of the stable unit treatment value assumption (SUTVA) (Rubin 1974, 1980). However, in many social, medical, and online settings it is common that the no-interference assumption fails to hold (Rubin 1990; Rosenbaum 2007; Hudgens and Halloran 2008; Walker and Muchnik 2014; Aral 2016; Taylor and Eckles 2017).

There has been a wealth of recent research into methods for handling interference in randomized experiments. In some cases it is possible to make reasonable structural assumptions about the nature of interference. The most well-studied such assumption, known as partial interference, is the case in which individuals can be partitioned naturally into groups, like households or schools, such that interference may exist arbitrarily between individuals within the same group but not between individuals of different groups (Sobel 2000; Hudgens and Halloran 2008). Partial interference is often paired with an additional exchangeability assumption known as stratified interference, which assumes that the potential outcomes are only a function of the number of within-group treated individuals and not the identity of those individuals. In this setting, sizable contributions have been made regarding how best to use two-stage randomized designs or random saturation designs to estimate a variety of direct and indirect effects and, more generally, dose-response curves (VanderWeele and Tchetgen 2011; Tchetgen and VanderWeele 2012; Liu and Hudgens 2014; Baird et al. 2016; Basse et al. 2017).

The case of general or arbitrary interference is more difficult. Generally, researchers proceed by proposing a set of exposure conditions that inform the interference pattern (Manski 2013). For example, one might...
assume that interference is local in nature. This idea is the basis for local interference assumptions such as the neighborhood treatment response assumption, which assumes that the potential outcomes of unit $i$ are constant conditional on all treatments in a local neighborhood of $i$ \cite{ugander2013,dynverse,castilla2017}. \cite{aronow2017} provide unbiased estimators and randomization-based inference under the assumption that the exposure model is correctly specified. \cite{choi2017} studies the case where the treatment effects are monotone, and \cite{sussman2017} develop unbiased estimators under neighborhood interference response for various parametric assumptions. In graph cluster randomization, the researcher attempts to reduce bias by using a clustered experimental design in which the clusters have been determined using a graph clustering algorithm designed to minimize edge cuts in a suitably chosen graph \cite{ugander2013,dynverse,castilla2017,interference}. In online settings, in which a standard experimental platform has been operationalized, two-stage or clustered designs may be edge use cases and so may require significant engineering effort to set up. Such experiments also sacrifice statistical power if it turns out the interference was weak or non-existent. Therefore, it is of interest to practitioners to be able to tell when controlling for interference is necessary, and when it is appropriate to use standard estimators constructed under the no-interference assumption. This is especially pertinent in the case of general interference, when there may be no clearly observable structures in the data to indicate whether interference is present. One option is to develop hypothesis tests for testing for spillover or interference effects \cite{aronow2012,athey2017,basse2017}. Another study that attempts to move the literature in this direction, and the one that is most relevant to the present work, is that of \cite{savje2017}. In that paper, the authors develop a framework for studying the behavior of standard estimators under a weak form of interference. They characterize interference based on the notion of a interference dependence graph, which defines an edge between two units $i$ and $j$ if there exists some unit $k$ (which is possibly $i$ or $j$) whose treatment affects the responses of both $i$ and $j$. They establish consistency results for various estimators and experimental designs under the restriction that the average degree of the interference dependence graph grows at rate $o(n)$. Beyond consistency, it is desirable to know whether estimators satisfy a central limit theorem so that valid asymptotic inference can be performed. This paper makes two contributions toward this goal. First, we demonstrate that the interference dependence graph of \cite{savje2017} is equivalent to the dependency graph introduced by \cite{chen1975}, used in a variant of Stein’s method for bounding distances between random variables. We show that in a Bernoulli randomized experiment, a central limit theorem exists for the Horvitz-Thompson version of the difference-in-means estimator if one is willing to constrain the maximal degree of the dependency graph at rate $o(n^{1/4})$, rather than the average degree at rate $o(n)$.

In practice the dependency graph may be quite dense, and may even have edges between every single pair of units in the population. As an example of how this may occur, consider the time-dynamic model studied in \cite{ugander2013} for experiments conducted on a social network. In this model, similar in spirit to the linear-in-means model of Manski (1993) for capturing endogenous social effects, individuals observe the responses of other individuals and use that information to inform their actions in the following time period. Interference thus spreads through the network over time and, provided the network is connected, eventually creates long-range dependencies between all pairs of nodes. Therefore any local model of interference, such as the neighborhood treatment response condition, does not apply. Our second contribution is to propose a notion of approximate local interference that allows for long-range dependencies. We use a more general form of Stein’s method to prove a central limit theorem in a setting where all units may interfere with all other units, but the potential outcome functions are such that most (but not necessarily all) of the interference is constrained to be local.

We find that the asymptotic variance of the difference-in-means estimator can be decomposed into two pieces: (a) the variance that results from conditioning on the standard potential outcomes $Y_i^{(0)}$ and $Y_i^{(1)}$, which would have been the true variance under SUTVA; and (b) the additional variation of $Y_i^{(0)}$ and $Y_i^{(1)}$ resulting from interference. If the additional variation due to interference is sufficiently large then standard confidence intervals may be anticonservative.

Our technical results rely heavily on Stein’s method, a flexible family of approaches for bounding distances between functions of random variables. As such, it can be used for proving central limit theorems when it is difficult or impossible to make stronger assumptions such as independence or the existence of identically distributed random variables. Stein’s method develops from the seminal paper \cite{stein1972}, which provides a bound for the error in the normal approximation of a sum of random variables with a certain dependency.
structure. We provide a short summary of the relevant literature here, but for a longer exposition on the historical development of Stein’s method we refer the reader to the surveys found in [Ross 2011] and [Chatterjee 2014].

The theory of dependency graphs, as a particular version of Stein’s method, was developed in [Chen 1975, Stein 1980, Baldi and Rinott 1989, Chen and Shao 2004] and is used for establishing limit theorems when dependence is exactly contained within a small neighborhood of variables. The dependency graph method is also similar in spirit to the idea of $m$-dependence for sequences of random variables; see for example [Hoeffding and Robbins 1948, Berk 1973, Romano and Wolf 2000]. Section 2 of Chatterjee (2014) summarizes the main idea of the dependency graph approach.

Classical versions of Stein’s method have the property that the random variables need to satisfy some “nice” condition—in the case of dependency graphs, that the degree is limited. The papers [Chatterjee 2008, 2009] develop a more general version of Stein’s method that Chatterjee (2014) calls the generalized perturbative method. This approach formalizes the idea that exact independence is really not too different from approximate independence when it comes to establishing limiting distributional results. Using this technology we are able to show that asymptotic normality still holds when there exists a weak form of long-range dependencies, even if the induced dependency graph is dense. In short, if units technically share a dependency edge but this dependence is sufficiently weak, then we may view them as essentially independent of each other.

Our results are of primary interest to two audiences. First, for practitioners, we contribute to a growing characterization in the literature of understanding when interference is a practical concern and when specialized estimators and robust confidence intervals are needed. Second, for researchers seeking to establish technical results for causal estimators under interference, our work demonstrates how Stein’s method can be a useful machinery for handling the complicated dependencies that often appear among statistical objects in interference problems.

The remainder of the paper is organized as follows. In Section 2, we define notation and assumptions. Section 3 discusses a central limit theorem framed in the language of dependency graphs, and Section 4 provides a central limit theorem that can handle weak, long-range interference. In Section 5 we provide simulations and Section 6 concludes. Proofs are provided in the appendix.

2 Setup

We work within the potential outcomes framework, or Rubin causal model [Neyman 1923, Rubin 1974]. Consider a population of $n$ units indexed on the set $[n] = \{1, \ldots, n\}$ and let $W = (W_1, \ldots, W_n) \in \{0, 1\}^n$ be a random vector of binary treatments. For every individual $i$ and realized vector of treatments $w \in \{0, 1\}^n$, we posit the existence of a fixed potential outcome $Y_i(w)$. Note that the potential outcomes are functions of the entire treatment vector and not just the treatment of unit $i$. We make no parametric restrictions on the form of the potential outcomes.

For $w = 0, 1$, let $Y_i^{(w)}$ be the conditional random variable defined by

$$Y_i^{(w)} = Y_i(W) | (W_i = w).$$

The quantity $Y_i^{(w)}$ represents the potential outcome under the scenario in which unit $i$ is exposed to the treatment condition $W_i = w$. It is a random quantity, as it may vary depending on the treatment assignments assigned to the other units. If the no-interference assumption is true, then conditioning on $W_i$ removes all randomness in $Y_i(W)$, and so $Y_i^{(w)}$ are degenerate random variables and hence reduce to the standard (fixed) potential outcomes.

An alternative view of $Y_i^{(w)}$ is perhaps more interpretable. Let $w_{-i}$ denote the vector of $n - 1$ elements obtained by removing the $i$-th element from $w$, and partition the vector $w$ into the direct or ego treatment $w_i$ and the indirect treatment $w_{-i}$. Then we may index the potential outcomes by these two arguments, writing $Y_i(w_i, w_{-i})$ instead of $Y_i(w)$. Now define

$$Y_i^{(w)} = Y_i(w, W_{-i})$$

for $w = 0, 1$. Evidently, expressions (1) and (2) are equivalent.
The conceptual advantage of viewing the potential outcomes in this way is that we can get a handle on the variation that exists before and after conditioning on the direct effect. A situation in which such conditioning removes most of the variance can be viewed as a scenario in which “SUTVA approximately holds,” even if strictly speaking SUTVA is violated.

If SUTVA fails to hold, the standard average treatment effect is undefined. We follow Sävje et al. (2017) and focus on the estimand that they call the expected average treatment effect (EATE)

\[ \tau = \frac{1}{n} \sum_{i=1}^{n} E[Y_i^{(1)} - Y_i^{(0)}]. \]

As noted by Sävje et al. (2017), \( \tau \) is the natural relaxation of the standard average treatment effect in the sense that they coincide whenever SUTVA holds. It may be viewed as an expected direct effect, where the marginalization is taken over the indirect treatment assignments. In other words, we first consider difference-in-means estimators that are typically used when SUTVA holds, “even if strictly speaking SUTVA is violated.

Regardless of whether or not the no-interference assumption holds, one of \( Y_i^{(0)} \) and \( Y_i^{(1)} \) is still unobserved. Let \( Y_i = Y_i(W) = W_i Y_i^{(1)} + (1 - W_i) Y_i^{(0)} \) denote the observed outcome. Let \( N_1 = \sum_{i=1}^{n} W_i \) and \( N_0 = \sum_{i=1}^{n} (1 - W_i) \) denote the within-group sample sizes. We study the behavior of the difference-in-means estimator

\[ \hat{\tau} = \frac{1}{N_1} \sum_{i=1}^{n} W_i Y_i - \frac{1}{N_0} \sum_{i=1}^{n} (1 - W_i) Y_i. \]

Note that this estimator is well-defined even when SUTVA is violated, as \( Y_i \) is simply the observed outcome. Sävje et al. (2017) study a wider class of estimators, namely the design-based Horvitz-Thompson and Hájek estimators that are typically used when \( P(W_i = 1) \) varies with \( i \) (Horvitz and Thompson 1952; Hájek 1971). The difference-in-means estimator is a special case of the Hájek estimator, coinciding when the assignment probabilities are the same for every unit. For simplicity of exposition our analysis focuses on the difference-in-means estimator and an experimental design in which the assignment probabilities are constant across units. We briefly discuss generalizations at the end of this paper.

In order to obtain asymptotic results, we follow the standard finite population regime (Freedman 2008a, b; Lin 2013; Abadie et al. 2017; Sävje et al. 2017) in which we have access to a sequence of finite populations indexed by size \( n \). Each population is comprised of its own treatments and outcomes, and the only randomness within each population is induced by the treatment assignment vector \( W_n \). Let \( W_{n,i} \) and \( Y_{n,i} \) denote the appropriate triangular arrays of random variables. We consider the sequence of estimands defined by

\[ \tau_n = \frac{1}{n} \sum_{i=1}^{n} E[Y_{n,i}^{(1)} - Y_{n,i}^{(0)}], \]  

which are the EATEs for each population. The corresponding sequence of difference-in-means estimators is given by

\[ \hat{\tau}_n = \frac{1}{N_1} \sum_{i=1}^{n} W_{n,i} Y_{n,i} - \frac{1}{N_0} \sum_{i=1}^{n} (1 - W_{n,i}) Y_{n,i}. \]

Our goal is to study the limiting behavior of \( (\hat{\tau}_n - \tau_n) \), subject to appropriate scaling.

Throughout this paper we will make use of the following regularity conditions. The first two conditions, overlap and uniformly bounded fourth moments, are standard regularity conditions for asymptotic analysis of regression estimators of treatment effects.

**Assumption 1** (Design and overlap). \( P(W_{n,i} = 1) = \pi_n \) independently. The sequence of assignment probabilities satisfies \( \pi_n \to \pi \), where the limiting treatment proportion \( \pi \in (0, 1) \) is bounded away from 0 and 1.

**Assumption 2** (Bounded fourth moments). \( E[|Y_{n,i}|^k] \) are uniformly bounded by a constant for all \( i, n \) and all \( k \leq 4 \).
We also assume existence of the following limits of the potential outcome moments.

**Assumption 3** (Existence of limits). Let \( \bar{Y}_n^{(1)} = n^{-1} \sum_{i=1}^n Y_{n,i}^{(1)} \) and \( \bar{Y}_n^{(0)} = n^{-1} \sum_{i=1}^n Y_{n,i}^{(0)} \). The following limits exist:

\[
\begin{align*}
\sigma_1^2 &:= \lim_{n \to \infty} E \left[ \frac{1}{n} \sum_{i=1}^n (Y_{n,i}^{(1)} - \bar{Y}_n^{(1)})^2 \right] \\
\sigma_0^2 &:= \lim_{n \to \infty} E \left[ \frac{1}{n} \sum_{i=1}^n (Y_{n,i}^{(0)} - \bar{Y}_n^{(0)})^2 \right] \\
\sigma_{01} &:= \lim_{n \to \infty} E \left[ \frac{1}{n} \sum_{i=1}^n (Y_{n,i}^{(1)} - \bar{Y}_n^{(1)})(Y_{n,i}^{(0)} - \bar{Y}_n^{(0)}) \right] \\
\sigma_r^2 &:= \lim_{n \to \infty} n \text{Var} \left[ \bar{Y}_n^{(1)} - \bar{Y}_n^{(0)} \right],
\end{align*}
\] (5)

The quantities inside the expectation are “population” quantities in the sense that they do not involve the treatment assignment, and would be computable if we were able to observe \( Y_{n,i}^{(1)} \) and \( Y_{n,i}^{(0)} \) for every unit. Because of interference they may be random, which is why the expectation is needed. A consequential implication of the assumption that \( \sigma_r^2 \) exists (equation (5)) is that the population difference of means \( \bar{Y}_n^{(1)} - \bar{Y}_n^{(0)} \) is consistent at a \( n^{1/2} \) rate of convergence. It is possible that limiting results are still achievable even when the difference of means converges at a slower rate, but we do not address this case in this paper.

### 3 A dependency graph central limit theorem

In order for central limit theorems to exist for \( (\hat{r}_n - \tau_n) \), the observed outcomes \( Y_i \) need to be “sufficiently independent.” One way to enforce this constraint is to directly require that enough pairs of units are completely independent. This idea is formalized via the following definition.

**Definition 1.** Let \( \{X_i\}_{i=1}^n \) be a collection of random variables on the nodes \([n]\) of a graph \( D \). Then \( D \) is a dependency graph if for any two disjoint sets of nodes \( A, B \subset [n] \) such that no edge in \( D \) crosses between \( A \) and \( B \), the sets \( \{X_i\}_{i \in A} \) and \( \{X_i\}_{i \in B} \) are independent.

The method of dependency graphs is a classical way of characterizing dependence in collections of random variables; see for example [Baldi and Rinott, 1989]. Dependency graphs are not necessarily unique; the complete graph always satisfies Definition 1, for example. In this paper we work with the dependency graph that is minimal in the sense that it has the fewest number of edges satisfying the definition.

In order to characterize interference between units, we consider dependency graphs on the collection of observed outcomes. For each population \( n \), let \( D_n \) denote the dependency graph on the set of random variables \( \{Y_{n,i}\}_{i=1}^n \). Because the outcomes are defined as functions of the treatment vector, units \( i \) and \( j \) are connected in this dependency graph if (a) the treatment of \( i \) affects the response of \( j \), (b) the treatment of \( j \) affects the response of \( i \), or (c) the responses of both \( i \) and \( j \) are affected by the treatment of some third unit. Thus we see that the dependency graph corresponds to the notion of interference dependence considered in [Savje et al., 2017], via the edge definition in Definition 5 of that paper.

Given a dependency graph defined on a collection of random variables, we can take advantage of bounds from the literature on Stein’s method. Such bounds characterize the Wasserstein distance between sums of random variables and a Gaussian random variable. Recall that the Wasserstein metric between probability measures \( \mu \) and \( \nu \) is

\[ d_W(\mu, \nu) = \sup \left\{ \left| \int h(x) d\mu(x) - \int h(x) d\nu(x) \right| : h \text{ is } 1\text{-Lipschitz} \right\}, \]

where a function \( h \) is 1-Lipschitz if it satisfies \( |h(x) - h(y)| \leq |x - y| \). In this paper we are concerned only with controlling the Wasserstein distance between \( \mu \) and a standard Gaussian random variable. For any random variable \( S \), denote the distance to Gaussianity as

\[ d_W(S) = d_W(\mu, \nu), \]
where $\mu$ is the law of $S$ and $\nu$ is the law of a standard Gaussian random variable, having density
\[
\frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]
We rely on the following dependency graph bound, which we state as a lemma.

**Lemma 1.** ([Ross 2011, Theorem 3.6]) Let $X_1, \ldots, X_n$ be a collection of random variables such that $E[X_i] < \infty$ and $E[X_i^3] < \infty$. Let $\sigma^2 = \text{Var}(\sum_i X_i)$ and $S = \sum_i X_i$. Let $d$ be the maximal degree of the dependency graph of $(X_1, \ldots, X_n)$. Then for constants $C_1$ and $C_2$ which do not depend on $n$, $d$ or $\sigma^2$,
\[
dW(S/\sigma) \leq C_1 \frac{d^3/2}{\sigma^2} \left( \sum_{i=1}^n E[X_i^3] \right)^{1/2} + C_2 \frac{d^2}{\sigma^3} \sum_{i=1}^n E[X_i]^3. \tag{6}
\]

From here, we see that one can define an appropriate choice of $X_i$ such that $S$ is the desired treatment effect estimator, and then provide conditions so that the right-hand side converges to zero. However, because the difference-in-means estimator is written as $\hat{\tau}_n = \sum_{i=1}^n \left[ W_{n,i}Y_{n,i} \right] - \frac{(1 - W_{n,i})Y_{n,i}}{N_0}$, each $X_i$ would involve the random sample sizes $N_0$ and $N_1$, which both depend on the treatment assignments of all $n$ units. Therefore, the dependency graph on $\{X_i\}_{i=1}^n$ unfortunately is complete, even if the dependency graph on $\{Y_{n,i}\}_{i=1}^n$ is not, and Lemma 1 is not applicable to the difference-in-means estimator.

As a result, in this section we restrict ourselves to studying a modified form of the difference-in-means estimator, defined by
\[
\tilde{\tau}_n = \sum_{i=1}^n \left[ \frac{W_{n,i}Y_{n,i}}{n\pi} - \frac{(1 - W_{n,i})Y_{n,i}}{n(1 - \pi)} \right]. \tag{7}
\]
The estimator $\tilde{\tau}_n$ is a Horvitz-Thompson [Horvitz and Thompson 1952] variant of the difference-in-means estimator $\hat{\tau}_n$, and uses the population sample sizes $n\pi_n$ and $n(1 - \pi)$ in the denominator in place of the empirical sample sizes $N_1$ and $N_0$. Though there is little advantage to using $\tilde{\tau}_n$ over $\hat{\tau}_n$ in practice, it is still instructive for seeing how the dependency graph method works. Results are provided for the difference-in-means estimator $\tilde{\tau}_n$ in Section 4.

We now define a limited interference condition that constrains the structure of the dependency graph. The metric that we use to measure the extent of interference for a collection of random variables is the maximal degree of the dependency graph.

**Assumption 4** (Local interference). $d_n = o(n^{1/4})$.

This assumption is a local interference assumption in the sense that it requires all interference for a given unit to come from a small number of other units. For comparison, consider the restricted interference assumption ([Sävje et al. 2017, Assumption 2]), which requires the average degree of the dependency graph to be of order $o(n)$. Our Assumption 4 is stronger, but it still allows the amount of interference to grow with $n$. By restricting the maximal degree rather than the average degree, we can apply Lemma 1.

Under the notion of local dependence defined in Assumption 4, we obtain the following asymptotic normality result for the Horvitz-Thompson estimator:

**Theorem 1.** Let $\tau_n$ and $\tilde{\tau}_n$ be defined as in equations (3) and (7). Under regularity conditions (Assumptions 4 and 3) and the restricted dependency degree condition (Assumption 4), $\sqrt{n}(\tilde{\tau}_n - \tau_n)$ is asymptotically Gaussian:
\[
\sqrt{n}(\tilde{\tau}_n - \tau_n) \Rightarrow N(0, \sigma^2),
\]
where
\[
\sigma^2 = \lim_{n \to \infty} n \text{Var}(\tilde{\tau}_n).\]
We arrive at Theorem 1 by defining an appropriate choice for $X_i$ and evaluating the variance $\sigma^2$, which allows us to control the bound in Lemma 1. The full proof is provided in the appendix.

A curious feature of Stein’s method is that it allows one to make statements about the asymptotic behavior of random objects without calculating an explicit expression for the variance. Because our primary interest is not in the Horvitz-Thompson estimator $\tilde{\tau}_n$, we skip calculating the limiting variance $\sigma^2$, but is not hard to express it in terms of the moments defined in Assumption 3. For the difference-in-means estimator in Section 3 we provide an explicit characterization of the limiting variance.

4 A central limit theorem for approximate local interference

There are two drawbacks of relying on the dependency graph approach for studying treatment effect estimators. It does not allow us to study estimators like the difference-in-means estimator that use empirical sample sizes, and it requires exact local interference (Assumption 4). In this section we discuss how these issues can be overcome. Rather than require most pairs of nodes to be exactly independent, we only require approximate independence, which allows long-range interference as long as it is not too strong.

To describe the main idea, developed in Chatterjee (2008), let $X = (X_1, \ldots, X_n) \in \mathcal{X}^n$ be a random vector of independent random variables on a measure space $\mathcal{X}$ and let $f : \mathcal{X}^n \to \mathbb{R}$ be a scalar-valued measurable function. The objective is to bound the distance to normality of $S := f(X)$. To do so, we characterize the behavior of $f$ when $X$ is “perturbed” by replacing some components of $X$ by independent copies. Let $X' = (X'_1, \ldots, X'_n)$ denote an independent copy of $X$. For every $A \in [n]$ let $X^A$ be the vector where the entries corresponding to $A$ are replaced by elements of $X'$, defined componentwise as

$$X^A_i = \begin{cases} X'_i & \text{if } i \in A \\ X_i & \text{if } i \not\in A. \end{cases}$$

Now define

$$\Delta_if = f(X) - f(X'), \quad i \in [n],$$
$$\Delta_if^A = f(X^A) - f(X^{A \cup \{i\}}), \quad A \subset [n], i \not\in A,$$

where we have made a notational simplification by writing $X^i$ instead of $X^{\{i\}}$ and $X^{A \cup \{i\}}$ instead of $X^{A \cup \{i\}}$.

The quantities $\Delta_if$ and $\Delta_if^A$ can be viewed as discrete derivatives, because they measure the change in the function $f$ in response to perturbations of $X$. If perturbations in $X$ act upon $f$ mostly independently by coordinate, then we expect the resulting value $f(X)$ to be approximately normal. We can now state the following normal approximation theorem, which is the main result in Chatterjee (2008). We state it as a lemma.

Lemma 2. Let $X = (X_1, \ldots, X_n)$ be a vector of independent real-valued random variables, and let $S = f(X)$. Suppose $\mathbb{E}[S] = 0$ and $\sigma^2 := \mathbb{E}[S^2] < \infty$. Define

$$T = \frac{1}{2} \sum_{i=1}^n \sum_{A \subset [n] \setminus \{i\}} \frac{\Delta_if \Delta_if^A}{n(n-1)}$$

Then $\mathbb{E}T = \sigma^2$ and

$$d_W(S/\sigma) \leq \frac{1}{\sigma^2} \left| \text{Var}(\mathbb{E}(T|S)) \right|^{1/2} + \frac{1}{2\sigma^2} \sum_{i=1}^n \mathbb{E}[|\Delta_if|^3].$$

It is more convenient to study a version of Lemma 2 that characterizes the Wasserstein distance in terms of local dependencies. This corollary is essentially a variant of Corollary 3.2 in Chatterjee (2014).

Corollary 1. Let all variables be defined as in Lemma 2. For every $i, j$, let $c_{i,j}$ be a constant such that for all $A \in [n] \setminus \{i\}$ and $B \in [n] \setminus \{j\}$,

$$\text{Cov}(\Delta_if, \Delta_if^A, \Delta_jf, \Delta_jf^B) \leq c_{i,j}.$$
\[ d_{\mathcal{W}}(S/\sigma) \leq \frac{1}{2\sigma^2} \left( \sum_{i,j=1}^{n} c_{i,j} \right)^{1/2} + \frac{1}{2\sigma^3} \sum_{i=1}^{n} E[|\Delta_i f|^3]. \] (8)

One may gain intuition for Corollary 1 by considering the case of a dependency graph on \( X \), in which an upper bound can be provided for the number of covariance terms \( c_{i,j} \) that can be nonzero. Let \( d_n \) denote the maximal degree of the dependency graph and \( N_i \) denote the neighborhood of unit \( i \). Consider the function \( f(X) = \sum_{i=1}^{n} X_i \), so that \( \Delta_i f = \sum_{r \in N_i} \Delta_i X_r \), where \( \Delta_i X_r = X_r - X_r^i \) is the effect on unit \( r \) of perturbing unit \( i \). Now consider the covariance between the discrete derivatives for units \( i \) and \( j \), which can be calculated as

\[
\text{Cov}(\Delta_i f, \Delta_j f) = \sum_{r \in N_i} \sum_{s \in N_j} \text{Cov}(\Delta_i X_r, \Delta_j X_s) = \sum_{r \in N_i \cap N_j} \text{Cov}(\Delta_i X_r, \Delta_j X_r) \leq C d_n \mathbb{1}(|N_i \cap N_j| > 0),
\]

where \( C \) is a constant that does not depend on \( n \) or \( d_n \). In other words, the covariance is always of order \( d_n \), but is exactly zero whenever the neighborhoods of \( i \) and \( j \) do not intersect. Now, for every unit \( i \), the number of units \( j \) such that \( |N_i \cap N_j| > 0 \) is at most \( d_n^2 \). Therefore the total number of covariances that can be nonzero is \( nd_n^2 \), and so the total magnitude of those covariances is \( Cn d_n^3 \). Assuming the variance \( \sigma^2 \) is of order \( n \), and noting that \( \Delta_i f^A \) is of order at most \( d_n \), the quantity

\[
\frac{1}{2\sigma^2} \left( \sum_{i,j=1}^{n} c_{i,j} \right)^{1/2}
\]

can be made small in the limit if \( d_n \) grows sufficiently slowly.

We return now to the problem of obtaining a limiting result for \( \hat{\tau}_n \). Define the sequence of functions

\[ f_n(W_n) = \sqrt{n}(\hat{\tau}_n - \tau_n) = \sqrt{n} \sum_{i=1}^{n} \left[ \frac{W_{n,i}}{N_i} - \frac{1 - W_{n,i}}{N_0} \right] Y_{n,i}(W_n) - \sqrt{n} \tau_n. \]

Since the treatment vector \( W_n \) is comprised of independent Bernoulli(\( \tau_n \)) random variables and is the sole source of randomness in \( f_n \), Corollary 1 is applicable provided we define appropriate constraints on the behavior of \( f_n \) under perturbations of the treatment vector.

Let \( W_{n,i}^i \) denote an independent copy of \( W_{n,i} \) and let \( W_n^i \) the resulting treatment vector obtained by swapping out \( W_{n,i} \) for \( W_{n,i}^i \) in \( W_n \), defined componentwise as

\[ W_{n,j}^i = \begin{cases} W_{n,j} & \text{if } j \neq i \\ W_{n,j}^i & \text{if } j = i \end{cases}. \]

Let

\[ Y_{n,j}^i = Y_{n,r}(W_n^i) \]

denote the resulting response of unit \( r \) when \( i \) is perturbed, and define

\[ \Delta_i Y_{n,r} = Y_{n,r} - Y_{n,r}^i \]

to be the change in \( Y_{n,r} \) when \( W_i \) is replaced with an independent copy. Furthermore, let

\[ N_i' = N_1 + W_{n,i}' - W_{n,i} \]
\[ N_0' = n - N_1' \]
denote the adjusted sample sizes.

The following lemma, which we state without proof, characterizes the discrete derivative \( \Delta_i f_n \).

**Lemma 3.** Let \( W_{n,i}', N_i', N_0', \) and \( Y_{n,r}' \) be defined as above. Then for every \( i \in [n] \), the discrete derivative can be written as

\[ \Delta_i f_n = \sqrt{n} \left( A_{n,i} + \sum_{r \neq i} B_{n,i,r} \right), \] (9)
where

\[
A_{n,i} = \left[ \frac{W_{n,i}}{N_1} - \frac{W'_{n,i}}{N'_1} \right] Y^{(1)}_{n,i} - \left[ \frac{1 - W_{n,i}}{N_0} - \frac{1 - W'_{n,i}}{N'_0} \right] Y^{(0)}_{n,i}
\]

\[
B_{n,i,r} = \left[ \frac{W_{n,r}}{N_1} Y_{n,r} - \frac{W'_{n,r}}{N'_1} Y'_{n,r} \right] - \left[ \frac{1 - W_{n,r}}{N_0} Y_{n,r} - \frac{1 - W'_{n,r}}{N'_0} Y'_{n,r} \right].
\]

Notice that \(A_{n,i}\) describes the change for unit \(i\) (the direct effect) and \(B_{n,i,r}\) describes the effect that perturbing the treatment of unit \(i\) has on the response of unit \(r\).

We make two assumptions that constrain the behavior of \(\Delta_i Y_{n,r}\), which in turn allows us to handle \(\Delta_i f_n\).

**Assumption 5 (Restricted interference).** (a) For every unit \(i\), the total amount of interference that results from perturbing the treatment of unit \(i\) is bounded in \(n\):

\[
\sum_{r \neq i} |\Delta_i Y_{n,r}| = O_p(1).
\]

(b) The following global covariance constraints hold:

(i)

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |\text{Cov}(Y_{n,i}, Y_{n,j})| = o(n^2).
\]

(ii)

\[
\sum_{i=1}^{n} \sum_{r \neq i} \sum_{j \neq i} |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,j})| = o(n^2).
\]

(iii)

\[
\sum_{i=1}^{n} \sum_{r \neq i} \sum_{j \neq i} \sum_{s \neq j} \sum_{s \neq r} |\text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s})| = o(n^2).
\]

Collectively, Assumption 5 defines a set of regularity conditions for interference. Part (a) states that changing the treatment of each unit \(i\) can only have a limited effect collectively on the responses of the other units. Part (b) is concerned with limits on the total amount of dependence across all pairs of units. Part (i) states that the overall responses are not too dependent. Part (ii) states that the effect of perturbing \(i\) on \(r\) is mostly independent of the behavior of unit \(j\). Part (iii) states that the effect of perturbing \(i\) on \(r\) and the effect of perturbing \(j\) on \(s\) are mostly independent, when \(r\) and \(s\) are distinct from \(i\) and \(j\) and each other.

To see why these assumptions may be reasonable, consider the case where SUTVA holds. Then since there is no interference, \(\sum_{r \neq i} |\Delta_i Y_{n,r}| = 0\) so part (a) is satisfied. Similarly, the expressions in part (ii) and (iii) of (b) are also exactly zero. Finally, \(\text{Cov}(Y_{n,i}, Y_{n,j}) = 0\) whenever \(i \neq j\) so that the expression in part (i) of (b) is \(O(n)\).

**Assumption 6 (Approximate local interference).** For every integer \(k, n \geq k\), and units \(i\) and \(r\), there exists subsets of integers \(N_{k,n,i}^\text{out}, N_{k,n,r}^\text{in} \subset [n]\) of size \(k = |N_{k,n,i}^\text{out}| = |N_{k,n,r}^\text{in}|\) such that \(r \in N_{k,n,i}^\text{out}\) if and only if \(i \in N_{k,n,r}^\text{in}\). Furthermore, they satisfy

\[
\sum_{r \in N_{k,n,i}^\text{out}} |\Delta_i Y_{n,r}| \leq \delta_k
\]

\[
\sum_{i \in N_{k,n,r}^\text{in}} |\Delta_i Y_{n,r}| \leq \delta_k,
\]

where \(\delta_k \geq 0\) is a sequence of numbers tending to 0 as \(k \to \infty\).
Assumption 6 is a statement of “approximately local interference” rather than “exact local interference.” It says that for every unit $i$, we can find a small collection of other units such that the collection is responsible for most of the interference coming in and out of unit $i$. The “out” neighborhood of $i$, $\mathcal{N}_{k,n,i}^\text{out}$, corresponds to the units whose responses are greatly affected by perturbing the treatment of $i$, and the “in” neighborhood of $r$, $\mathcal{N}_{k,n,i}^\text{in}$, corresponds to the units whose treatments have a great influence on the response of $r$. These two neighborhoods can be comprised of the same units, but they need not be.

One way that Assumption 6 is automatically satisfied is if there is no interference or the units lie on a dependency graph with bounded degree, in which case any sequence $\delta_k$ that is eventually zero suffices. However, it also allows for long-range interference of the sort that a sparse dependency graph does not permit. One may conceptualize the existence of two different dependency graphs defined on the collection of units, one capturing strong interference and the other capturing weak interference. The weak interference network may be arbitrarily dense but we require the strong interference network to be sufficiently sparse. For example, in a social network, it may be reasonable to believe that there is long-range interference between all pairs of units, but any meaningful interference involving unit $i$ is limited to members of the immediate neighborhood of $i$ in the network.

We now state the main result. In comparison to Theorem 1, it replaces a restriction on the dependency graph, Assumption 4, with the approximate local interference requirements, Assumptions 5 and 6. It also is a statement about the difference-in-means estimator $\tau_n$ rather than the Horvitz-Thompson estimator $\tilde{\tau}_n$.

**Theorem 2.** Let $\tau_n$ and $\tilde{\tau}_n$ be defined as in equations (3) and (1), and assume that the regularity conditions (Assumptions 1-3) hold. Assume further that the outcome functions are constrained according to Assumptions 5 and 6. Then $\sqrt{n}(\tilde{\tau}_n - \tau_n)$ is asymptotically Gaussian:

$$\sqrt{n}(\tilde{\tau}_n - \tau_n) \Rightarrow N(0, \sigma^2).$$

The limiting variance $\sigma^2$ has the form

$$\sigma^2 := \lim_{n \to \infty} n \text{Var}(\tilde{\tau}_n) = \frac{1 - \pi}{\pi} \sigma^2_1 + \frac{\pi}{1 - \pi} \sigma^2_0 + 2\sigma_{01} + \sigma^2_2, \quad (10)$$

where the quantities $\sigma^2_1$, $\sigma^2_0$, $\sigma_{01}$, and $\sigma^2_2$ are defined in Assumption 3, and $\pi = \lim_{n \to \infty} P(W_i = 1)$ is the limiting treatment proportion.

The asymptotic variance takes the form of a variance decomposition based on conditioning on the potential outcomes $Y_{n,i}^{(0)}$ and $Y_{n,i}^{(1)}$. To see this, denote the $\sigma$-field generated by the potential outcomes as

$$\mathcal{F}_n := \{Y_{n,i}^{(w)} : i \in [n], w \in \{0, 1\}\}. \quad (11)$$

Then by the law of total variance,

$$\text{Var}(\tilde{\tau}_n) = \mathbb{E}[\text{Var}(\tilde{\tau}_n|\mathcal{F}_n)] + \text{Var}[\mathbb{E}(\tilde{\tau}_n|\mathcal{F}_n)].$$

This decomposition is evident in the asymptotic variance $\sigma^2$, as

$$\lim_{n \to \infty} n \mathbb{E}[\text{Var}(\tilde{\tau}_n|\mathcal{F}_n)] = \frac{1 - \pi}{\pi} \sigma^2_1 + \frac{\pi}{1 - \pi} \sigma^2_0 + 2\sigma_{01} \quad (12)$$

and

$$\lim_{n \to \infty} n \text{Var}[\mathbb{E}(\tilde{\tau}_n|\mathcal{F}_n)] = \sigma^2_2. \quad (13)$$

The first three terms, (12), form the standard asymptotic variance of the difference-in-means estimator under no-interference, in which $Y_{n,i}^{(0)}$ and $Y_{n,i}^{(1)}$ are fixed quantities (Freedman 2008a; Lin 2013). The last term, (13), captures the additional variation of the “total population” average treatment effect, which is variation that remains even if we were able to observe $Y_{n,i}^{(0)}$ and $Y_{n,i}^{(1)}$ for every unit. This decomposition implies that if $\sigma^2_2$ stabilizes to a non-zero value, then confidence intervals constructed under the SUTVA assumption will only provide correct coverage conditional on $\mathcal{F}_n$, and will fail to account for the fact that $Y_{n,i}^{(0)}$ and $Y_{n,i}^{(1)}$ may exhibit additional variation under the experimental design.
5 Simulations

This section is devoted to two sets of simulations that are designed to illuminate some of the practical implications of our theoretical findings. The first simulation involves tests of Gaussianity and considers situations in which asymptotic inference may be invalid. The second simulation involves the variance decomposition provided by equations (12) and (13) and considers situations in which inference built under the SUTVA assumption may be invalid. For both simulations we use the same set of networks and generative response model.

In order to replicate as closely as possible the structural characteristics observed in real-world networks, we use five empirical networks from the Facebook100 dataset, an assortment of complete online friendship networks for one hundred colleges and universities collected from a single-day snapshot of Facebook in September 2005. A detailed analysis of the social structure of these networks was given in Traud et al. (2012). The five schools used are the California Institute of Technology, Haverford College, Amherst College, Michigan Technological University, and Wake Forest University; the only reason for the selection of these five particular schools was so as to produce a rough stratification of population sizes. For each school, we use the largest connected component only. Table 1 contains basic summary statistics for the networks used.

| network       | nodes | edges  | avg. degree | avg. pairwise dist. | diameter |
|---------------|-------|--------|-------------|---------------------|----------|
| Caltech       | 762   | 16651  | 43.70       | 2.34                | 6        |
| Haverford     | 1446  | 59589  | 82.42       | 2.23                | 6        |
| Amherst       | 2235  | 90954  | 81.39       | 2.40                | 7        |
| Michigan Tech | 3745  | 81901  | 43.74       | 2.84                | 7        |
| Wake Forest   | 5366  | 279186 | 104.06      | 2.51                | 9        |

Table 1: Summary statistics for the five networks used in the simulation.

We use a simple response model that allows us to control the amount of dependence exhibited among the observations. For network $G$ and nodes $i$ and $j$, let $\tilde{Z}_{\rho,i,j} = 1$ if nodes $i$ and $j$ are exactly $\rho$ units apart in graph $G$, and then define

$$Z_{\rho,i} = \left( \sum_j \tilde{Z}_{\rho,i,j} \right)^{-1} \sum_j \tilde{Z}_{\rho,i,j} W_j$$

to be the proportion of units which are exactly distance $\rho$ from $i$ that receive the treatment. Then we model the outcome as

$$Y_i^{(w)} = \alpha_i^{(w)} + \rho_{\text{max}} \sum_{\rho=1}^{\rho_{\text{max}}} \beta_{\rho}^{(w)} Z_{\rho,i}$$

for $w = 0, 1$. The intercept $\alpha_i = (\alpha_i^{(0)}, \alpha_i^{(1)})$ captures a heterogeneous direct effect. The maximum distance parameter $\rho_{\text{max}}$ is an integer ranging from 0 to the diameter of the graph. By $\rho_{\text{max}} = 0$ we mean the summation is omitted entirely, so that $Y_i^{(w)} = \alpha_i^{(w)}$, and there is no spillover effect and hence no interference. When $\rho_{\text{max}} = 1$, each unit is subject to a direct effect and a spillover effect governed by coefficient $\beta_1^{(w)}$ and the proportion $Z_{1,i}$ of neighbors of $i$ receiving the treatment. Analogously, higher values of $\rho_{\text{max}}$ admit more distant sources of interference.

We model the coefficient vector as decaying exponentially in the graph distance,

$$\beta_{\rho}^{(1)} = 2\gamma^\rho, \quad \beta_{\rho}^{(0)} = \gamma^\rho,$$

for a decay parameter $\gamma \in (0, 1)$. Therefore, each node receives a direct effect $\alpha_i^{(1)} - \alpha_i^{(0)}$ and an indirect effect

$$\sum_{\rho=1}^{\rho_{\text{max}}} \gamma^\rho Z_{\rho,i}.$$

We control the amount of dependence by varying the parameters $\rho_{\text{max}}$, which explicitly controls the structure of the dependency graph, and $\gamma$, which controls the rate at which spillover effects dissipate as they travel through the network.
5.1 Tests of normality

We first compute normality test statistics for a variety of parameter configurations. We vary the decay rate \( \gamma \), and the maximum dependency distance \( \rho_{\text{max}} \). The parameter values we use are \( \gamma \in \{0.5, 0.9, 0.99\} \), and \( \rho_{\text{max}} \in \{2, 6\} \). The maximum value \( \rho_{\text{max}} = 6 \) was used because all networks have diameter at least 6. For every parameter configuration and each network, we generate 10 instances of the direct effect. The direct effect values \( \alpha_i^{(1)} \) and \( \alpha_i^{(0)} \) are sampled from independent exponential distributions with different means; the treatment group has mean 1/0.3 and the control group has mean 2.

For each of the 10 instances, we sample 500 draws of the treatment vector as independent Bernoulli(0.5) variables, and compute the outcomes and resulting difference-in-means estimate. We report the test statistic and \( p \)-value of the Shapiro-Wilk (SW) test for normality \cite{Shapiro1965}. This produces 10 \( p \)-values for each network and parameter configuration, one for each instance of the direct effect. Note that we use these \( p \)-values purely for exploratory purposes and do not require nor attempt multiple comparison control.

The results are displayed in Figure 1. Recall that \( p \)-values are uniform under the null hypothesis that the difference-in-means statistics are normally distributed, so that configurations in which most of the \( p \)-values are small may indicate a departure from normality. The scenarios representing the greatest amount of interference are those in which the indirect effect is allowed to propagate over a long distance (\( \rho_{\text{max}} = 6 \)) and in which the indirect effect does not decay much (\( \gamma = 0.99 \)) as it travels across the network. We see that the \( p \)-values are smallest for these configurations. For all networks, the value of \( \gamma \) needs to be quite large in order to cause serious problems; \( p \)-values appear to be roughly uniform for a dissipation rate of \( \gamma = 0.5 \) even when \( \rho_{\text{max}} = 6 \). This supports the claim that having a sparse dependency graph is not necessary for asymptotic normality, since for these networks, the induced dependency graph when \( \rho_{\text{max}} = 6 \) is either complete or nearly complete. Departures from normality also seem to be sensitive to the particular network structure; the Caltech and Michigan Tech networks seem to be quite well-behaved even under the strongest regimes of interference (\( \gamma = 0.99 \) and \( \rho_{\text{max}} = 6 \)).

5.2 Variance decompositions

For this simulation we explore the relationship between the strength of interference and the resulting variance components. We focus on the Caltech network, which, based on the previous simulation, appears to have a Gaussian distribution even under strong regimes of interference. We draw a single set of \( \alpha_i^{(0)} \) and \( \alpha_i^{(1)} \) using the same distribution as the previous simulation, with exponential distributions of mean 1/0.3 for the treatment group and mean 2 for the control group. We vary the maximum distance \( \rho_{\text{max}} \) from 0 to 5 and the decay parameter \( \gamma \) from 0.1 to 0.9 in increments of 0.1. For each parameter configuration, we draw 10,000 iterates of the treatment vector \( W \) as iid Bernoulli(0.5), and recompute the potential outcomes \( Y_i^{(1)} \) and \( Y_i^{(0)} \) each time. We then use the potential outcomes to compute the variance components

\[
\begin{align*}
\sigma_1^2 &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (Y_i^{(1)} - \bar{Y}^{(1)})^2 \right] \\
\sigma_0^2 &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (Y_i^{(0)} - \bar{Y}^{(0)})^2 \right] \\
\sigma_{01} &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (Y_i^{(1)} - \bar{Y}^{(1)})(Y_i^{(0)} - \bar{Y}^{(0)}) \right] \\
\sigma_\tau^2 &= n \text{Var} \left[ \bar{Y}^{(1)} - \bar{Y}^{(0)} \right],
\end{align*}
\]

where the expectation and variance are computed as finite population moments over the 10,000 simulation replicates. We also compute the observed variance \( \sigma_{\text{DM}}^2 \) of the difference-in-means estimator.

The SUTVA (conditional) variance is

\[
\sigma_{\text{SUTVA}}^2 = \sigma_1^2 + \sigma_0^2 + 2\sigma_{01}.
\]

12
Figure 1: (top) Every data point is a $p$-value for the Shapiro-Wilk test against a Gaussian reference distribution. Each panel represents a different network and dependency distance $\rho_{\text{max}}$ combination. The panels with orange points correspond to $\rho_{\text{max}} = 2$ (less interference) and those with blue points correspond to $\rho_{\text{max}} = 6$ (more interference). The vertical axis contains the three levels of the decay rate $\gamma$, ranging from $\gamma = 0.5$ (less interference) to $\gamma = 0.99$ (more interference). The nominal cutoff values 0.1 (vertical dotted line) and 0.01 (vertical dashed line) are highlighted for reference. (bottom) The same plot but using a logarithmic scale for the horizontal axis.
Figure 2: Variance ratios for the Caltech network. Each panel represents a different maximum distance $\rho_{\text{max}}$. The horizontal axis is the decay rate $\gamma$ and the vertical axis marks the variance ratios. The horizontal dotted line marks the baseline, which is a ratio of one. The orange values are the expected variance ratios $(\sigma^2_{\text{SUTVA}} + \sigma^2_{\tau})/\sigma^2_{\text{SUTVA}}$, and the blue values are the observed variance ratios $\hat{\sigma}^2_{DM}/\sigma^2_{\text{SUTVA}}$. The upper-left most panel, $\rho_{\text{max}} = 0$, is the case when SUTVA is true. The markup is greatest when $\rho_{\text{max}}$ and $\gamma$ are both large. Note that the horizontal axis starts at 100% and that the greatest observed ratio is about a 60% increase in the variance over SUTVA.

We display the ratio of the expected true variance of $\hat{\tau}$ to the conditional variance, $(\sigma^2_{\text{SUTVA}} + \sigma^2_{\tau})/\sigma^2_{\text{SUTVA}}$, as well as the observed ratio, $\hat{\sigma}^2_{DM}/\sigma^2_{\text{SUTVA}}$. The resulting ratios are displayed in Figure 2. The observed variances mostly track the expected variances. Under SUTVA, $Y_{i}^{(0)}$ and $Y_{i}^{(1)}$ exhibit no additional variation so the observed variance appears to match $\sigma^2_{\text{SUTVA}}$. As we allow units to influence units farther away in the graph, the variance ratio grows. The discrepancy is not too large for fast decaying interference ($\gamma < 0.5$) but for $\gamma$ close to 1.0 it can be drastic. When $\rho_{\text{max}} = 5$ and $\gamma = 0.6$ the observed variance is only 7.4% larger than $\sigma^2_{\text{SUTVA}}$, but for $\rho_{\text{max}} = 5$ and $\gamma = 0.9$ the observed discrepancy is 60.1%.

6 Discussion

In this work we have developed a framework for obtaining asymptotic results for causal estimators in randomized experiments in the presence of interference. We contextualize the work of Sävje et al. (2017) within Stein’s method and obtain asymptotic normality results.

Our two main results—one constraining the dependency degree (Theorem 1) and another placing conditions on how the treatment variables interact with the response variables (Theorem 2)—highlight two general ways one may proceed for handling arbitrary interference. The dependency graph approach follows a motif found in the interference literature of relying on local interference assumptions, such as the neighborhood treatment response condition. Such assumptions, which enforce a sort of “sparsity of interference,” are often viewed as implausible yet necessary for tractability of results. However, we have shown that progress is possible under certain dense regimes of interference. Our result is still restrictive in the sense that it requires a fairly strong condition of approximate locality. It is possible that more general advancements can be made by characterizing the behavior of the object $T$ in the main perturbative theorem (Lemma 2).

As discussed in Section 2 the definition of $\sigma^2_{T}$ (equation 5) in Assumption 3 means that we have
restricted ourselves to studying rate-optimal scenarios. A useful extension would be to establish similar limiting results for the case when the difference-in-means $\bar{Y}_n^{(1)} - \bar{Y}_n^{(0)}$ converges at a slower-than-$\sqrt{n}$ rate. A related issue is efficiency in the presence of interference. The semiparametric efficiency bound (Hahn 1998) that serves as the basis for efficient estimation of average treatment effects in observational studies is reliant on independent units. A characterization of similar semiparametric efficiency bounds for various levels of interference is a prerequisite for understanding whether efficient estimators remain optimal under interference.

We have also not addressed the question of variance estimation. The main difficulty in providing valid variance estimates is estimation of the additional variance component $\sigma^2 \tau$, which is not identified without making stronger assumptions. Appropriate assumptions are likely to be problem specific, requiring more information about the specific nature of interference. We prefer to think of variance estimation as a separate problem that warrants further study.

Finally, our work is a novel application of Stein’s method. From a technical standpoint, our results demonstrate that tools from that literature can be used for establishing theoretical results for causal estimators under interference. By overlaying the interference framework on top of Stein’s method we are able to sidestep more complicated calculations or detailed assumptions about the structure of interference. We have not addressed other statistical objects such as more general estimators and designs, or different estimands including the global treatment effect that compares all units in treatment to all units in control. Since interference at its core involves handling a dependent collection of random variables, we suspect that Stein’s method may be useful for understanding those settings as well.

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A Proof of Theorem 1

In this section we prove Theorem 1 restated here.

**Theorem 1.** Let $\tau_n$ and $\tilde{\tau}_n$ be defined as in equations (3) and (7). Under regularity conditions (Assumptions 1-3) and the restricted dependency degree condition (Assumption 4), $\sqrt{n}(\tilde{\tau}_n - \tau_n)$ is asymptotically Gaussian:

$$\sqrt{n}(\tilde{\tau}_n - \tau_n) \Rightarrow N(0, \sigma^2),$$

where

$$\sigma^2 = \lim_{n \to \infty} n \Var(\tilde{\tau}_n).$$

**Proof.** Decompose the Horvitz-Thompson estimator (7) as $\tilde{\tau}_n = \sum_{i=1}^{n} \tilde{\tau}_{n,i}$ where

$$\tilde{\tau}_{n,i} = \frac{1}{n} \left[ W_{n,i} Y_{n,i}^{(1)} - \frac{(1 - W_{n,i}) Y_{n,i}^{(0)}}{1 - \pi} \right].$$

First, by conditioning on the $\sigma$-field defined in equation (11),

$$n \Var(\tilde{\tau}_n) = n \sum_{i=1}^{n} \Var(\tilde{\tau}_{n,i})$$

$$= n \sum_{i=1}^{n} \mathbb{E}\left[ \Var(\tilde{\tau}_{n,i} | \mathcal{F}_n) \right] + \Var(\mathbb{E}(\tilde{\tau}_{n,i} | \mathcal{F}_n))$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{E}[(Y_{n,i}^{(1)})^2 + (Y_{n,i}^{(0)})^2] + \Var(Y_{n,i}^{(1)} - Y_{n,i}^{(0)}) \right).$$

By Assumption 3, we see that the variance stabilizes.

Now let $X_i = \sqrt{n}(\tilde{\tau}_{n,i} - \mathbb{E}(\tilde{\tau}_{n,i}))$ and denote $\sigma^2 = n \Var(\tilde{\tau}_n)$. Then

$$S = S_n = \sum_{i=1}^{n} X_i = \sqrt{n}(\tilde{\tau} - \tau)/\sigma.$$ 

By the uniform moment bound (Assumption 2), $X_i = O_p(n^{-1/2})$, so for large enough $n$ there exist constants $C_1$ and $C_2$ such that

$$\left( \sum_{i=1}^{n} \mathbb{E}[X_i^4] \right)^{1/2} \leq C_1 n^{-1/2}$$

and

$$\sum_{i=1}^{n} \mathbb{E}|X_i|^3 \leq C_2 n^{-1/2}.$$ 

We can now apply Lemma 1, which establishes for fixed $n$ that

$$d_W(S_n) \leq C_1 \frac{d_{n,1}^{3/2}}{n^{1/2}} + C_2 \frac{d_{n,2}^2}{n^{1/2}},$$

where we have ignored the $\sigma^2$ term because it stabilizes. Therefore $d_W(S_n) \to 0$ whenever $d_n = o(n^{1/4})$, which is the constraint we have placed on the dependency graph (Assumption 4). Hence $S_n$ converges to a standard Gaussian random variable.
B \hspace{1cm} \textbf{Proof of Theorem 2}

B.1 \hspace{1cm} \textbf{Proof of Corollary 1}

We first prove the version of Lemma 2 for local dependencies.

**Corollary 1.** Let all variables be defined as in Lemma 2. For every $i, j$, let $c_{i,j}$ be a constant such that for all $A \in [n] \setminus \{i\}$ and $B \in [n] \setminus \{j\}$,

$$\text{Cov}(\Delta_i f \Delta_i f^A, \Delta_j f \Delta_j f^B) \leq c_{i,j}.$$  

Then

$$d_W(S/\sigma) \leq \frac{1}{2\sigma^2} \left( \sum_{i,j=1}^{n} c_{i,j} \right)^{1/2} + \frac{1}{2\sigma^3} \sum_{i=1}^{n} E[|\Delta_i f|^3].$$

**Proof.** Notice that

$$\text{Var}(E(T|S)) \leq \text{Var} T \leq \frac{1}{4} \sum_{i,j=1}^{n} \sum_{A \subset [n] \setminus \{i\}} \sum_{B \subset [n] \setminus \{j\}} \frac{\text{Cov}(\Delta_i f \Delta_i f^A, \Delta_j f \Delta_j f^B)}{n^2(|A|)|B|} \leq \frac{1}{4} \sum_{i,j=1}^{n} \sum_{A \subset [n] \setminus \{i\}} \sum_{B \subset [n] \setminus \{j\}} \frac{c_{i,j}}{n^2(|A|)|B|} = \frac{1}{4} \sum_{i,j=1}^{n} c_{i,j}.$$  

Applying Lemma 2 completes the proof. \qed

B.2 \hspace{1cm} \textbf{Lemmas}

The lemmas in this section focus on getting a handle on the discrete derivative. Throughout this section, $C_1, C_2, C_3, \ldots$ indicate numerical constants that do not depend on $n$, and their values may change from line to line.

**Lemma 4.** Let $A_{n,i}$ and $B_{n,i,r}$ be defined as in Lemma 3 and assume the regularity conditions (Assumptions 3). For all $i, j, r, \text{ and } s$,

$$|\text{Cov}(A_{n,i}, A_{n,j})| \leq \frac{C_1}{n^2} |\text{Cov}(Y_{n,i}, Y_{n,j})| \quad (14)$$

$$|\text{Cov}(B_{n,i,r}, A_{n,j})| \leq \left( \frac{C_1}{n^2} + \frac{C_2}{n^3} \right) |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,j})| + \frac{C_3}{n^3} |\text{Cov}(Y_{n,r}, Y_{n,j})| \quad (15)$$

$$|\text{Cov}(B_{n,i,r}, B_{n,j,s})| \leq \left( \frac{C_1}{n^2} + \frac{C_2}{n^3} \right) |\text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s})| + \left( \frac{C_3}{n^3} + \frac{C_4}{n^4} \right) |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,s})|$$

$$\quad + \left( \frac{C_5}{n^3} + \frac{C_6}{n^4} \right) |\text{Cov}(Y_{n,r}, \Delta_j Y_{n,s})| + \frac{C_7}{n^4} |\text{Cov}(Y_{n,r}, Y_{n,s})|, \quad (16)$$

where the $C_k$ are constants, not necessarily the same from line to line.

**Proof.** Note that $B_{n,i,r}$ can be written as

$$B_{n,i,r} = \left[ \frac{W_{n,r}}{N_1} - \frac{1 - W_{n,r}}{N_0} \right] \Delta_i Y_{n,r} + W_{n,r} Y_{n,i} \frac{W_{n,i} - W_{n,i}'}{N_1 N_1'} - \left( 1 - W_{n,r} \right) Y_{n,r} \frac{W_{n,i} - W_{n,i}'}{N_0 N_0'}.$$  

Equation (14) follows from examining the form of $A_{n,i}$ and noting that $N_1 = O_p(n)$ and $N_0 = O_p(n)$. For equation (15), note

$$|\text{Cov}(B_{n,i,r}, A_{n,j})| \leq \frac{C_1}{n^2} |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,j})| + \frac{C_2}{n^3} |\text{Cov}(Y_{n,i}, Y_{n,j})|$$

$$\quad = \frac{C_1}{n^2} |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,j})| + \frac{C_2}{n^3} (|\text{Cov}(Y_{n,r}, Y_{n,j})| + |\text{Cov}(Y_{n,r} - Y_{n,r}, Y_{n,j})|),$$

where $\sigma = \sqrt{\text{Var}(E(T|S))}$. \qed
which gives equation \((15)\). Similarly, we have

\[
| \text{Cov}(B_{n,i,r}, B_{n,j,s}) | \leq \frac{C_1}{n^2} | \text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s}) | + \frac{C_2}{n^3} | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,j}^i) | \\
+ \frac{C_3}{n^3} | \text{Cov}(Y_{n,r}^i, \Delta_j Y_{n,s}) | + \frac{C_4}{n^4} | \text{Cov}(Y_{n,r}^i, Y_{n,s}^j) |
\]

and rewriting \(Y_{n,r}^i = Y_{n,r} - \Delta_i Y_{n,r}\) and \(Y_{n,s}^j = Y_{n,s} - \Delta_j Y_{n,s}\) gives equation \((10)\). \(\square\)

**Lemma 5.** Under the regularity conditions (Assumptions 1-3), there exist constants \(C_1\) through \(C_5\) such that

\[
\frac{1}{n} \sum_{i,j} | \text{Cov}(\Delta_i f_n, \Delta_j f_n) | \leq \frac{C_1}{n^2} \sum_{i,j} | \text{Cov}(Y_{n,i}, Y_{n,j}) | + \left( \frac{C_2}{n^2} + \frac{C_3}{n^3} \right) \sum_{i,j} \sum_{r \neq i} | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,j}) | \\
+ \left( \frac{C_4}{n^2} + \frac{C_5}{n^3} \right) \sum_{i,j} \sum_{r \neq i, s \neq j} | \text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s}) |.
\]

**Proof.** By expanding the form of the discrete derivative \((4)\), we have

\[
\frac{1}{n} | \text{Cov}(\Delta_i f_n, \Delta_j f_n) | = | \text{Cov}(A_{n,i}, A_{n,j}) | + \sum_{r \neq i} | \text{Cov}(B_{n,i,r}, A_{n,j}) | \\
+ \sum_{s \neq j} | \text{Cov}(A_{n,i}, B_{n,j,s}) | + \sum_{r \neq i, s \neq j} | \text{Cov}(B_{n,i,r}, B_{n,j,s}) |.
\]

By summing over \(i\) and \(j\) substituting the bounds from Lemma 4, the right-hand side above is bounded above by

\[
\frac{1}{n^2} \sum_{i,j} \left[ C_1 | \text{Cov}(Y_{n,i}, Y_{n,j}) | + \sum_{r \neq i} \left[ \left( C_2 + \frac{C_3}{n} \right) | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,j}) | + \frac{C_4}{n} | \text{Cov}(Y_{n,r}, Y_{n,j}) | \right] \\
+ \sum_{s \neq j} \left[ \left( C_5 + \frac{C_6}{n} \right) | \text{Cov}(Y_{n,i}, \Delta_j Y_{n,s}) | + \frac{C_7}{n} | \text{Cov}(Y_{n,i}, Y_{n,s}) | \right] \\
+ \sum_{r \neq i} \sum_{s \neq j} \left[ \left( C_8 + \frac{C_9}{n} \right) | \text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s}) | + \left( \frac{C_{10}}{n} + \frac{C_{11}}{n^2} \right) | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,s}) | \\
+ \left( \frac{C_{12}}{n} + \frac{C_{13}}{n^2} \right) | \text{Cov}(Y_{n,r}, \Delta_j Y_{n,s}) | + \frac{C_{14}}{n^2} | \text{Cov}(Y_{n,r}, Y_{n,s}) | \right]. \right]
\]

We now exploit the symmetry in the summations and combine terms to give the desired result. \(\square\)

**Lemma 6.** Under the regularity conditions (Assumptions 1-3 and 3),

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{r \neq i} | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,i}) | = o(1).
\]

**Proof.** By the uniform moment bound there exists a constant \(C\) such that

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{r \neq i} | \text{Cov}(\Delta_i Y_{n,r}, Y_{n,i}) | \leq \frac{C}{n^2} \sum_{i=1}^{n} \sum_{r \neq i} E|\Delta_i Y_{n,r}|.
\]

The result follows from the fact that \(\sum_{r \neq i} |\Delta_i Y_{n,r}| = O_p(1)\). \(\square\)

**Lemma 7.** Under the regularity conditions (Assumptions 1-3 and 2) and approximate local interference (Assumption 2),

\[
\frac{1}{n^2} \sum_{i,j} \sum_{r \neq i, s \neq j} | \text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s}) | = o(1).
\]
Proof. Fix $k$ and let $n \geq k$. Denote $\Delta_{n,r}^{i,j} = \text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,r})$. We proceed by partitioning the sum depending on whether $r$ belongs to the neighborhoods of $i$ and $j$ as defined in Assumption [6]. That is, we can write

$$
\frac{1}{n^2} \sum_{i,j} \Delta_{n,r}^{i,j} \leq \frac{1}{n^2} \sum_{i,j} \left[ \sum_{r \in N_{k,n,i}^{\text{out}}} \sum_{r \in N_{k,n,j}^{\text{out}}} |\Delta_{n,r}^{i,j}| + \sum_{r \not\in N_{k,n,i}^{\text{out}}} \sum_{r \not\in N_{k,n,j}^{\text{out}}} |\Delta_{n,r}^{i,j}| + \sum_{r \not\in N_{k,n,i}^{\text{out}}} \sum_{r \in N_{k,n,j}^{\text{out}}} |\Delta_{n,r}^{i,j}| + \sum_{r \in N_{k,n,i}^{\text{out}}} \sum_{r \not\in N_{k,n,j}^{\text{out}}} |\Delta_{n,r}^{i,j}| \right]
$$

Now, by Assumption [6] each of the inner sums of the last three terms is bounded above by $\delta_k$. For the first term, the sum is zero whenever the intersection of $N_{k,n,i}^{\text{out}}$ and $N_{k,n,j}^{\text{out}}$ is empty, and of order $k$ otherwise. Therefore,

$$
\frac{1}{n^2} \sum_{i,j} \sum_{r \not\in i,j} |\Delta_{n,r}^{i,j}| \leq C \frac{k^4}{n} + 3\delta_k
$$

Since $\delta_k \to 0$ as $k \to \infty$, the proof is finished by taking $n \to \infty$ first and then $k \to \infty$.

**Lemma 8.** In addition to the regularity conditions (Assumptions [5]), assume that Assumption [6] (approximate local independence) holds. Then for all $i \in [n]$ and $A \in [n] \setminus \{i\}$,

$$
|\Delta_i f_n| = O_p(n^{-1/2}) \quad (17)
$$

$$
|\Delta_i f_n^A| = O_p(n^{-1/2}). \quad (18)
$$

**Proof.** By a similar argument as in Lemma [5]

$$
\text{E}(\Delta_i f_n)^2 \leq n \left[ \text{Var}(A_{n,i}) + \sum_{r \not\in i} \text{Cov}(A_{n,i}, B_{n,i,r}) + \sum_{r \not\in i} \sum_{s \not\in i} \text{Cov}(B_{n,i,r}, B_{n,i,s}) \right]
$$

$$
\leq \frac{C_1}{n} \text{Var}(Y_{n,i}) + \left( \frac{C_2}{n^2} + \frac{C_4}{n^2} \right) \sum_{r \not\in i} \text{Cov}(Y_{n,i}, \Delta_i Y_{n,r}) + \left( \frac{C_4}{n} + \frac{C_5}{n^2} \right) \sum_{r \not\in i} \sum_{s \not\in i} \text{Cov}(\Delta_i Y_{n,r}, \Delta_i Y_{n,s})
$$

$$
\leq \frac{C_1}{n} + \left( \frac{C_2}{n} + \frac{C_3}{n^2} \right) \sum_{r \not\in i} \text{E}|\Delta_i Y_{n,r}| + \left( \frac{C_4}{n} + \frac{C_5}{n^2} \right) \sum_{r \not\in i} \sum_{s \not\in i} \text{E}|\Delta_i Y_{n,r} \Delta_i Y_{n,s}| \text{E}(\Delta_i f_n)^2
$$

$$
= \frac{C_1}{n} + \left( \frac{C_2}{n} + \frac{C_3}{n^2} \right) \sum_{r \not\in i} \text{E}|\Delta_i Y_{n,r}| + \left( \frac{C_4}{n} + \frac{C_5}{n^2} \right) \left( \sum_{r \not\in i} \text{E}|\Delta_i Y_{n,r}| \right)^2.
$$

The whole right-hand side is then $O(n^{-1})$ by the fact that $\sum_{r \not\in i} |\Delta_i Y_{n,r}| = O_p(1)$ (Assumption [6]). Then (17) follows from Markov’s inequality. Equation (18) immediately follows because $\Delta_i f_n^A$ is equal in distribution to $\Delta_i f_n$. 

B.3 Proof of main theorem

We are now ready to prove Theorem 2, restated here.

**Theorem 2.** Let \( \tau_n \) and \( \hat{\tau}_n \) be defined as in equations (3) and (4), and assume that the regularity conditions (Assumptions 1-3) hold. Assume further that the outcome functions are constrained according to Assumptions 5 and 6. Then \( \sqrt{n}(\hat{\tau}_n - \tau_n) \) is asymptotically Gaussian:

\[
\sqrt{n}(\hat{\tau}_n - \tau_n) \Rightarrow N(0, \sigma^2).
\]

The limiting variance \( \sigma^2 \) has the form

\[
\sigma^2 := \lim_{n \to \infty} n \text{Var}(\hat{\tau}_n) = \frac{1 - \pi}{\pi} \sigma_1^2 + \frac{\pi}{1 - \pi} \sigma_0^2 + 2\sigma_01 + \sigma_\tau^2,
\]

where the quantities \( \sigma_1^2, \sigma_0^2, \sigma_01, \) and \( \sigma_\tau^2 \) are defined in Assumption 3, and \( \pi = \lim_{n \to \infty} P(W_i = 1) \) is the limiting treatment proportion.

**Proof.** We first compute the limiting variance \( \sigma^2 := \lim_{n \to \infty} n \text{Var}(\hat{\tau}_n) \). Let \( F_n \) be the \( \sigma \)-field defined by equation (11). By conditioning on \( F_n \) we have

\[
\text{Var}(\hat{\tau}_n) = \mathbb{E} \left[ \text{Var} \left( \hat{\tau}_n | F_n \right) \right] + \text{Var} \left[ \mathbb{E} \left( \hat{\tau}_n | F_n \right) \right].
\]

Now,

\[
\text{Var}(\hat{\tau}_n | F_n) = \mathbb{E} \left[ \sum_{i=1}^{n} \frac{W_{n,i}Y_{n,i}^{(1)} - (1 - W_{n,i})Y_{n,i}^{(0)}}{N_1} \right] - \mathbb{E} \left[ \frac{W_{n,i}Y_{n,i}^{(0)}}{N_0} \right]^2 \left( Y_{n,i}^{(1)}, Y_{n,i}^{(0)} \right)
\]

is the usual variance of a difference-in-means estimator under SUTVA, i.e. fixed potential outcomes. This is known to be (see for example Lin 2013)

\[
\lim_{n \to \infty} n \mathbb{E} \left[ \text{Var}(\hat{\tau}_n | F_n) \right] = \frac{1 - \pi}{\pi} \sigma_1^2 + \frac{\pi}{1 - \pi} \sigma_0^2 + 2\sigma_01.
\]

For the second term, we have \( \mathbb{E} \left[ \hat{\tau}_n | F_n \right] = \bar{Y}_n^{(1)} - \bar{Y}_n^{(0)} \), so

\[
\lim_{n \to \infty} n \text{Var} \left[ \mathbb{E} \left( \hat{\tau}_n | F_n \right) \right] = \sigma_\tau^2
\]

by Assumption 3. This produces the variance expression (10).

Since the variance term \( \sigma^2 \) of expression (8) in Corollary 1 stabilizes, it is sufficient to show

\[
\lim_{n \to \infty} \left( \sum_{i,j} c_{i,j} \right)^{1/2} = 0 \quad \text{and} \quad \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E} |\Delta_i f_n| = 0.
\]

Since \( |\Delta_i f_n| = O_p(n^{-1/2}) \) by equation (18) of Lemma 8 there exists a constant \( C \) such that

\[
|\text{Cov}(\Delta_i f_n, \Delta_j f_n^A, \Delta_j f_n \Delta_j f_n^B)| \leq \frac{C}{n} |\text{Cov}(\Delta_i f_n, \Delta_j f_n)|.
\]

Then by Lemma 5 there exist constants \( c_{i,j} \geq 0 \) such that

\[
|\text{Cov}(\Delta_i f_n, \Delta_j f_n^A, \Delta_j f_n \Delta_j f_n^B)| \leq c_{i,j}
\]

and

\[
\sum_{i,j} c_{i,j} \leq \frac{C_1}{n^2} \sum_{i,j} |\text{Cov}(Y_{n,i}, Y_{n,j})| + \frac{C_2}{n^2} \left[ \sum_{i=1}^{n} \sum_{r \neq i} |\text{Cov}(\Delta_i Y_{n,r}, Y_{n,i})| + \sum_{i=1}^{n} \sum_{j \neq i} \sum_{r \neq j} |\text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,j})| \right] + \frac{C_3}{n^2} \left[ \sum_{i,j} \sum_{r \neq i} |\text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,r})| + \sum_{i,j} \sum_{s \neq j} \sum_{r \neq s} |\text{Cov}(\Delta_i Y_{n,r}, \Delta_j Y_{n,s})| \right].
\]
Each of the five terms in the bound captures a different relationship among the responses and discrete derivatives. The first term measures a global covariance structure which tends to zero by Assumption 5. The third and fifth terms concern covariances among distinct actors, which are also negligible by Assumption 5. The second and fourth terms are the only ones that include elements measuring strong interference. These two terms are handled by Lemmas 6 and 7 respectively. So we conclude

$$\lim_{n \to \infty} \left( \sum_{i,j} c_{i,j} \right)^{1/2} = 0.$$  

Finally, by equation 17 of Lemma 8, $E|\Delta_i f_n|^3 = O(n^{-3/2})$. Hence

$$\sum_{i=1}^{n} E|\Delta_i f_n|^3 = O(n^{-1/2})$$

and so tends to zero.  \(\square\)
## Tables of simulation results

| Network school | Nodes | Parameters | SW statistic | SW p-values |
|----------------|-------|------------|--------------|-------------|
|                |       | $p_{\text{max}}$ | $\gamma$ | average | avg | min | max |
| Caltech 762    | 2     | 0.5        | 0.996       | 0.355 0.038 0.669 |
| Caltech 762    | 2     | 0.9        | 0.996       | 0.373 0.061 0.688 |
| Caltech 762    | 6     | 0.5        | 0.997       | 0.438 0.081 0.827 |
| Caltech 762    | 6     | 0.9        | 0.997       | 0.569 0.016 0.943 |
| Caltech 762    | 6     | 0.99       | 0.998       | 0.688 0.285 0.915 |
| Haverford 1446 | 2     | 0.5        | 0.996       | 0.331 0.110 0.698 |
| Haverford 1446 | 2     | 0.9        | 0.997       | 0.586 0.014 0.958 |
| Haverford 1446 | 2     | 0.99       | 0.997       | 0.496 0.056 0.913 |
| Haverford 1446 | 6     | 0.5        | 0.996       | 0.406 0.010 0.904 |
| Haverford 1446 | 6     | 0.9        | 0.957       | 0.000 0.000 0.000 |
| Haverford 1446 | 6     | 0.99       | 0.928       | 0.000 0.000 0.000 |
| Amherst 2235   | 2     | 0.5        | 0.996       | 0.309 0.027 0.937 |
| Amherst 2235   | 2     | 0.9        | 0.997       | 0.581 0.167 0.955 |
| Amherst 2235   | 2     | 0.99       | 0.996       | 0.455 0.014 0.951 |
| Amherst 2235   | 6     | 0.5        | 0.997       | 0.576 0.013 0.938 |
| Amherst 2235   | 6     | 0.9        | 0.991       | 0.011 0.000 0.066 |
| Michigan Tech 3745 | 2 | 0.5 | 0.997 | 0.649 0.057 0.985 |
| Michigan Tech 3745 | 2 | 0.9 | 0.996 | 0.506 0.001 0.828 |
| Michigan Tech 3745 | 2 | 0.99 | 0.996 | 0.403 0.026 0.835 |
| Michigan Tech 3745 | 6 | 0.5 | 0.997 | 0.506 0.091 0.829 |
| Michigan Tech 3745 | 6 | 0.9 | 0.996 | 0.443 0.011 0.886 |
| Michigan Tech 3745 | 6 | 0.99 | 0.997 | 0.528 0.116 0.968 |
| Wake Forest 5366 | 2 | 0.5 | 0.996 | 0.497 0.008 0.853 |
| Wake Forest 5366 | 2 | 0.9 | 0.997 | 0.591 0.089 0.976 |
| Wake Forest 5366 | 2 | 0.99 | 0.996 | 0.372 0.015 0.876 |
| Wake Forest 5366 | 6 | 0.5 | 0.997 | 0.550 0.040 0.933 |
| Wake Forest 5366 | 6 | 0.9 | 0.979 | 0.000 0.000 0.002 |
| Wake Forest 5366 | 6 | 0.99 | 0.975 | 0.000 0.000 0.000 |

Table 2: Summary of Shapiro-Wilk p-values from Simulation 1. Average, minimum, and maximum are taken over the 10 instances of the response.
| Parameters | Variances | Ratios to SUTVA |
|------------|-----------|----------------|
| $\rho_{\text{max}}$ | $\gamma$ | SUTVA | expected | observed | SUTVA | expected | observed |
| 0 | 0.1 | 14.770 | 14.770 | 15.228 | 1.000 | 1.031 |
| 0 | 0.2 | 15.205 | 15.205 | 15.570 | 1.000 | 1.024 |
| 0 | 0.3 | 14.690 | 14.690 | 14.680 | 1.000 | 0.999 |
| 0 | 0.4 | 15.382 | 15.382 | 16.208 | 1.000 | 1.054 |
| 0 | 0.5 | 14.478 | 14.478 | 14.714 | 1.000 | 1.016 |
| 0 | 0.6 | 14.321 | 14.321 | 14.164 | 1.000 | 0.989 |
| 0 | 0.7 | 16.674 | 16.674 | 16.521 | 1.000 | 0.991 |
| 0 | 0.8 | 17.574 | 17.574 | 17.623 | 1.000 | 1.003 |
| 0 | 0.9 | 16.453 | 16.453 | 16.440 | 1.000 | 0.999 |
| 1 | 0.1 | 12.717 | 12.722 | 12.758 | 1.000 | 1.003 |
| 1 | 0.2 | 14.845 | 14.864 | 15.094 | 1.001 | 1.017 |
| 1 | 0.3 | 14.694 | 14.736 | 14.954 | 1.003 | 1.018 |
| 1 | 0.4 | 14.282 | 14.360 | 14.034 | 1.005 | 0.983 |
| 1 | 0.5 | 12.739 | 12.856 | 12.906 | 1.009 | 1.013 |
| 1 | 0.6 | 16.073 | 16.251 | 16.346 | 1.011 | 1.017 |
| 1 | 0.7 | 13.262 | 13.497 | 13.655 | 1.018 | 1.030 |
| 1 | 0.8 | 15.247 | 15.546 | 15.867 | 1.020 | 1.041 |
| 1 | 0.9 | 14.324 | 14.713 | 14.528 | 1.027 | 1.014 |
| 2 | 0.1 | 16.199 | 16.205 | 16.198 | 1.000 | 1.000 |
| 2 | 0.2 | 17.088 | 17.113 | 16.990 | 1.001 | 0.994 |
| 2 | 0.3 | 15.325 | 15.386 | 15.373 | 1.004 | 1.003 |
| 2 | 0.4 | 14.046 | 14.168 | 14.283 | 1.009 | 1.017 |
| 2 | 0.5 | 15.489 | 15.703 | 15.868 | 1.014 | 1.024 |
| 2 | 0.6 | 16.697 | 17.040 | 17.247 | 1.021 | 1.033 |
| 2 | 0.7 | 17.665 | 18.186 | 18.088 | 1.029 | 1.024 |
| 2 | 0.8 | 15.419 | 16.159 | 16.219 | 1.048 | 1.052 |
| 2 | 0.9 | 14.598 | 15.631 | 15.846 | 1.071 | 1.085 |
| 3 | 0.1 | 14.095 | 14.100 | 14.678 | 1.000 | 1.041 |
| 3 | 0.2 | 14.267 | 14.292 | 14.359 | 1.002 | 1.006 |
| 3 | 0.3 | 15.799 | 15.863 | 15.755 | 1.004 | 0.997 |
| 3 | 0.4 | 13.442 | 13.581 | 13.517 | 1.010 | 1.006 |
| 3 | 0.5 | 12.762 | 13.013 | 13.302 | 1.020 | 1.042 |
| 3 | 0.6 | 16.095 | 16.519 | 16.592 | 1.026 | 1.031 |
| 3 | 0.7 | 14.900 | 15.595 | 15.410 | 1.047 | 1.034 |
| 3 | 0.8 | 18.009 | 19.103 | 19.158 | 1.061 | 1.064 |
| 3 | 0.9 | 14.031 | 15.690 | 15.607 | 1.118 | 1.112 |
| 4 | 0.1 | 12.765 | 12.771 | 12.980 | 1.000 | 1.017 |
| 4 | 0.2 | 13.902 | 13.927 | 14.105 | 1.002 | 1.015 |
| 4 | 0.3 | 15.799 | 15.866 | 15.638 | 1.004 | 0.990 |
| 4 | 0.4 | 15.210 | 15.352 | 15.541 | 1.009 | 1.022 |
| 4 | 0.5 | 14.311 | 14.601 | 14.962 | 1.020 | 1.046 |
| 4 | 0.6 | 16.144 | 16.755 | 16.893 | 1.038 | 1.046 |
| 4 | 0.7 | 15.692 | 16.942 | 17.043 | 1.080 | 1.086 |
| 4 | 0.8 | 16.461 | 19.185 | 19.188 | 1.165 | 1.166 |
| 4 | 0.9 | 18.759 | 24.566 | 24.343 | 1.310 | 1.298 |
| 5 | 0.1 | 14.747 | 14.752 | 14.928 | 1.000 | 1.012 |
| 5 | 0.2 | 13.261 | 13.286 | 13.006 | 1.002 | 0.981 |
| 5 | 0.3 | 15.400 | 15.467 | 15.409 | 1.004 | 1.001 |
| 5 | 0.4 | 14.980 | 15.127 | 14.462 | 1.010 | 0.965 |
| 5 | 0.5 | 14.784 | 15.094 | 14.978 | 1.021 | 1.013 |
| 5 | 0.6 | 14.554 | 15.221 | 15.635 | 1.046 | 1.074 |
| 5 | 0.7 | 14.374 | 15.951 | 16.093 | 1.110 | 1.120 |
| 5 | 0.8 | 16.307 | 20.078 | 20.575 | 1.231 | 1.262 |
| 5 | 0.9 | 15.546 | 24.286 | 24.894 | 1.562 | 1.601 |

Table 3: Table of variances for the Caltech network from Simulation 2.