Equal Sums of Like Powers
with Minimum Number of Terms

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Abstract
This paper is concerned with the diophantine system,
\[ \sum_{i=1}^{s_1} x_i^r = \sum_{i=1}^{s_2} y_i^r, \quad r = 1, 2, \ldots, k, \]
where \( s_1 \) and \( s_2 \) are integers such that the total number of terms on both sides, that is, \( s_1 + s_2 \), is as small as possible. We define \( \beta(k) \) to be the minimum value of \( s_1 + s_2 \) for which there exists a nontrivial solution of this diophantine system. We find nontrivial integer solutions of this diophantine system when \( k < 6 \), and thereby show that \( \beta(2) = 4, \quad \beta(3) = 6, \quad 7 \leq \beta(4) \leq 8 \) and \( 8 \leq \beta(5) \leq 10 \).

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1 Introduction

The Tarry-Escott problem of degree \( k \) consists of finding two distinct sets of integers \( \{x_1, x_2, \ldots, x_s\} \) and \( \{y_1, y_2, \ldots, y_s\} \) such that
\[ \sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \quad r = 1, 2, \ldots, k. \]

It is well-known that for a non-trivial solution of (1) to exist, we must have \( s \geq (k + 1) \) \cite[p. 616]{10}. Solutions of (1) with the minimum possible value of \( s \), that is, with \( s = k + 1 \), are known as ideal solutions of the problem.

This paper is concerned with finding solutions in integers of the related diophantine system,
\[ \sum_{i=1}^{s_1} x_i^r = \sum_{i=1}^{s_2} y_i^r, \quad r = 1, 2, \ldots, k, \]
where $s_1$ and $s_2$ are integers such that the total number of terms on both sides, that is, $s_1 + s_2$, is minimum.

Without loss of generality, we may take $s_1 \leq s_2$. A solution of the system of equations (2) will be said to be trivial if $y_i = 0$ for $s_2 - s_1$ values of $i$ and the remaining integers $y_i$ are a permutation of the integers $x_i$.

We define $\beta(k)$ to be the minimum value of $s_1 + s_2$ for which there exists a nontrivial solution of the diophantine system (2).

According to a well-known theorem of Frolov [10, p. 614], if $x_i = a_i$, $y_i = b_i$, $i = 1, 2, \ldots, s$ is any solution of the diophantine system (1), and $d$ is an arbitrary integer, then $x_i = a_i + d$, $y_i = b_i + d$, $i = 1, 2, \ldots, s$, is also a solution of (1). Taking $d = -a_1$, we immediately get a solution of (2) with $s_1 = s - 1$, $s_2 = s$. Thus, if an ideal solution of (1) is known for any specific value of $k$, then $\beta(k) \leq 2k + 1$.

Ideal solutions of (1) are known for $k = 2, 3, \ldots, 9$ ([1], [2], [3], [4], [7], pp. 52, 55-58], [9], [11, pp. 41-54], [12], [14]) as well as for $k = 11$ [5]. Thus, for these values of $k$, we have $\beta(k) \leq 2k + 1$, and in particular, we get

$$
\beta(2) \leq 5, \quad \beta(3) \leq 7, \quad \beta(4) \leq 9, \quad \beta(5) \leq 11.
$$

In this paper, we find parametric solutions of (2) when $2 \leq k \leq 5$, and show that,

$$
\beta(2) = 4, \quad \beta(3) = 6, \quad 7 \leq \beta(4) \leq 8, \quad 8 \leq \beta(5) \leq 10.
$$

2 A preliminary lemma

**Lemma 1:** If there exists a nontrivial solution of the diophantine system (2), then

$$
\max(s_1, s_2) \geq k + 1.
$$

Further, if $k \geq 4$, then,

$$
\min(s_1, s_2) \geq 2.
$$
Proof: Any solution of the diophantine system \(2\) with \(\max(s_1, s_2) < k+1\), implies the existence of a solution of \(2\) with \(s_1 = s_2 < k + 1\), and by a theorem of Bastien (as quoted by Dickson \(8\), p. 712), such a solution is necessarily trivial. Thus we must have the relation \(5\).

When \(k \geq 4\), if we assume a solution of the diophantine system \(2\) with \(s_1 = 1\), it is easily seen on eliminating \(x_1\) from the two relations obtained by taking \(r = 2\) and \(r = 4\) in \(2\) that the resulting condition can be satisfied only if all but one of the numbers \(y_i\) are 0, and the solution is necessarily trivial. Thus we must have the relation \(6\).

**Corollary 1:** For any arbitrary positive integer \(k\),

\[
\beta(k) \geq k + 2.
\]

Further, when \(k \geq 4\),

\[
\beta(k) \geq k + 3.
\]

Proof: We have trivially \(\min(s_1, s_2) \geq 1\). The corollary now follows immediately from the lemma.

We have already seen that if there exists a solution of the diophantine system \(2\) with \(s_1 = s_2\), Frolov’s theorem immediately gives another solution of \(2\) with \(s_1 = s - 1, s_2 = s\). Thus for \(s_1 + s_2\) to be minimum, we can always take \(s_1 < s_2\). Accordingly, we will henceforth always consider the diophantine system \(2\) with \(s_1 < s_2\).

We note that all the equations of the diophantine system \(2\) are homogeneous, and therefore, any solution of \(2\) in rational numbers may be multiplied through by a suitable constant to obtain a solution of \(2\) in integers.

### 3 Determination of \(\beta(k)\)

It is trivially true that \(\beta(1) = 3\). In the next four subsections, we will find solutions of the diophantine system \(2\) when \(k = 2, 3, 4\) and 5 respectively,
and prove the results stated in (4).

3.1 It follows from Cor. 1 that \( \beta(2) \geq 4 \). We will show that \( \beta(2) = 4 \) by solving the diophantine system (2) with \( s_1 = 1, s_2 = 3 \) and \( k = 2 \). On eliminating \( x_1 \) from the two equations of this diophantine system, we get,

\[
y_1y_2 + y_2y_3 + y_3y_1 = 0.
\]

The complete solution of Eq. (9) is readily obtained and this immediately yields the following simultaneous identities:

\[
(p^2 + pq + q^2)^r = (p^2 + pq)^r + (pq + q^2)^r + (-pq)^r, \quad r = 1, 2,
\]

where \( p \) and \( q \) are arbitrary parameters. This shows that \( \beta(2) = 4 \).

3.2 We now consider the diophantine system (2) when \( k = 3 \). It follows from Cor. 1 that \( \beta(3) \geq 5 \). We will prove that \( \beta(3) = 6 \).

If there exists a nontrivial solution of the system of equations,

\[
\sum_{i=1}^{s_1} x_i^r = \sum_{i=1}^{s_2} y_i^r, \quad r = 1, 2, 3,
\]

with \( s_1 < s_2 \), it follows from Lemma 1 that \( s_2 \geq 4 \). Thus the only case to consider when \( s_1 + s_2 = 5 \) is with \( s_1 = 1 \) and \( s_2 = 4 \) when the diophantine system (11) may be written as,

\[
x_1 = y_1 + y_2 + y_3 + y_4,
\]

\[
x_1^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2,
\]

\[
x_1^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3.
\]

Eliminating \( x_1 \) from Eqs. (12) and (13), we get the condition,

\[
(y_1 + y_2 + y_3)y_4 + y_1y_2 + y_1y_3 + y_2y_3 = 0,
\]

while eliminating \( x_1 \) from Eqs. (12) and (14), we get the condition,

\[
(y_1 + y_2 + y_3)y_4^2 + (y_1 + y_2 + y_3)^2y_4 + (y_2 + y_3)(y_1 + y_3)(y_1 + y_2) = 0.
\]

If \( y_1 + y_2 + y_3 = 0 \), it follows from (12) that \( x_1 = y_4 \) and now (13) implies that \( y_1 = 0, y_2 = 0, y_3 = 0 \), and we thus get a trivial solution of the system.
of equations (12), (13) and (14). If $y_1 + y_2 + y_3 \neq 0$, we eliminate $y_4$ from Eqs. (15) and (16), and get,

$$\begin{align*}
(y_1^2 + y_1y_2 + y_2^2)y_3^2 + y_1y_2(y_1 + y_2)y_3 + y_1^2y_2^2 &= 0.
\end{align*}$$

Eq. (17) can have a rational solution for $y_3$ only if its discriminant, that is, $-(3y_1^2 + 2y_1y_2 + 3y_2^2)y_1^2y_2^2$, is a perfect square. It follows that either $y_1$ or $y_2$, or both of them, must be 0, and in each case, it readily follows that the only solution of Eqs. (12), (13) and (14) is the trivial solution.

Since the diophantine system (11) has only the trivial solution when $s_1 + s_2 = 5$, it follows that,

$$\beta(3) > 5.$$  

We will now obtain nontrivial solutions of the diophantine system (11) with $s_1 = 2$ and $s_2 = 4$, that is, of the system of equations,

$$\begin{align*}
(19) \quad x_1 + x_2 &= y_1 + y_2 + y_3 + y_4, \\
(20) \quad x_1^2 + x_2^2 &= y_1^2 + y_2^2 + y_3^2 + y_4^2, \\
(21) \quad x_1^3 + x_2^3 &= y_1^3 + y_2^3 + y_3^3 + y_4^3.
\end{align*}$$

If $(a, b, c)$ is any Pythagorean triple satisfying the relation $a^2 + b^2 = c^2$, it is easily seen that a solution in integers of the simultaneous equations (19), (20) and (21) is given by $(x_1, x_2, y_1, y_2, y_3, y_4) = (c, -c, a, -a, b, -b)$.

Next we obtain a more general parametric solution of the simultaneous equations (19), (20) and (21). We will use a parametric solution of the simultaneous diophantine equations,

$$\begin{align*}
(22) \quad x + y + z &= u + v + w, \\
x^3 + y^3 + z^3 &= u^3 + v^3 + w^3,
\end{align*}$$

given in [5, Theorem 1, p. 61]. From this solution, on writing $x_1 = x$, $x_2 = y$, $y_1 = u$, $y_2 = v$, $y_3 = w$, $y_4 = -z$, we immediately derive the following solution of the simultaneous equations (19) and (21) in terms of arbitrary
parameters \( p, q, r \) and \( s \):

\[
\begin{align*}
  x_1 &= pq - pr + qr - (p - q - r)s, \\
  x_2 &= -pq + pr + qr + (p - q + r)s, \\
  y_1 &= pq + pr - qr + (p - q + r)s, \\
  y_2 &= pq - pr + qr + (p + q - r)s, \\
  y_3 &= -pq + pr + qr - (p - q - r)s, \\
  y_4 &= -pq + pr + qr - (p + q - r)s.
\end{align*}
\]  

(23)

Substituting the above values of \( x_i, y_i \) in (20), we get, after necessary transpositions, the following quadratic equation in \( s \):

\[
\begin{align*}
  2(p + q - r)^2 s^2 + (12p^2q - 4p^2r - 4pq^2 - 4pqr \\
  + 4pr^2 + 4q^2r - 4qr^2)s + 2(pq + pr - qr)^2 &= 0.
\end{align*}
\]  

(24)

On taking \( r = p+q \), the coefficient of \( s^2 \) in Eq. (24) vanishes, and we can then readily solve Eq. (24) for \( s \), and we thus obtain the following solution of the simultaneous equations (19), (20) and (21) in terms of arbitrary parameters \( p \) and \( q \):

\[
\begin{align*}
  x_1 &= (3p^4 - 2p^3q - p^2q^2 + q^4)q, \\
  x_2 &= (p^4 - p^2q^2 - 2pq^3 + 3q^4)p, \\
  y_1 &= (p^4 - p^2q^2 + 2pq^3 - q^4)p, \\
  y_2 &= 2pq(p - q)(p^2 - pq - q^2), \\
  y_3 &= -(p^4 - 2p^3q + p^2q^2 - q^4)q, \\
  y_4 &= 2pq(p - q)(p^2 + pq - q^2).
\end{align*}
\]  

(25)

As a numerical example, taking \( p = 2, q = 1 \), we get the solution,

\[
29r + 22r = 30r + 4r + (-3)r + 20r, \quad r = 1, 2, 3.
\]

We note that more parametric solutions of the system of equations (19), (20) and (21) can be obtained by solving Eq. (24) in different ways, for example, by choosing \( p, q, r \) such that the constant term in Eq. (24) vanishes and then solving this equation for \( s \), or by choosing \( p, q, r \) such that the discriminant of Eq. (24), considered as a quadratic equation in \( s \), becomes a perfect square, and then solving this equation for \( s \).
As we have obtained nontrivial solutions of the system of equations (19), (20) and (21), we get \( \beta(3) \leq 6 \), and on combining this result with (18), we get,

\[
\beta(3) = 6.
\]

**3.3** We will now obtain parametric solutions of the diophantine system,

\[
\sum_{i=1}^{3} x_i^r = \sum_{i=1}^{5} y_i^r, \quad r = 1, 2, 3, 4.
\]

We write,

\[
\begin{align*}
x_1 &= 4uv + w + 1, & x_2 &= -4uv + w - 1, \\
x_3 &= -8u^2 + 8uv + 4u - 2, & y_1 &= 4u - 2, \\
y_2 &= -4u, & y_3 &= 4uv + w - 1, \\
y_4 &= -4uv + w + 1, & y_5 &= -8u^2 + 8uv + 4u,
\end{align*}
\]

where \( u, v, w \) are arbitrary parameters.

It is readily verified that the values of \( x_i, y_i \) given by (28) satisfy Eq. (27) when \( r = 1 \) and \( r = 2 \). Further, on substituting these values of \( x_i, y_i \) in Eq. (27) and taking \( r = 3 \), we get the condition,

\[
4u^3 - 8u^2v + 4uv^2 - 4u^2 + 4uv - vw + u - v = 0.
\]

On solving Eq. (29), we get,

\[
w = (4u^3 - 8u^2v + 4uv^2 - 4u^2 + 4uv + u - v)/v.
\]

Finally, we substitute the values of \( x_i, y_i \) given by (28) in Eq. (27), and take \( r = 4 \), and use the value of \( w \) given by (30) to get the condition,

\[
u^2(2u - 1)^2 \{24uv^2 - 2(4u - 1)^2v + 3u(2u - 1)^2\} = 0.
\]

While equating the first two factors of Eq. (31) to 0 leads to trivial results, on equating the last factor to 0, we get a quadratic equation in \( v \) which will have a rational solution if its discriminant \( 4(-32u^4 + 32v^3 + 24u^2 - \)
16u + 1) becomes a perfect square. We thus have to solve the diophantine equation,

\[ t^2 = -32u^4 + 32u^3 + 24u^2 - 16u + 1. \]

Now Eq. (32) is a quartic model of an elliptic curve, and we use the birational transformation given by,

\[ t = (X^3 - 36X^2 + 36X - 72Y + 432)/(4X + Y - 12)^2, \]
\[ u = (X - 12)/(4X + Y - 12), \]

and,

\[ X = (4u^2 - 8u + t + 1)/(2u^2), \]
\[ Y = (8u^3 + 12u^2 - 4ut - 12u + t + 1)/(2u^3), \]

to reduce Eq. (32) to the Weierstrass model which is given by the cubic equation,

\[ Y^2 = X^3 - 36X. \]

It is readily seen from Cremona’s well-known tables that (35) is an elliptic curve of rank 1 and its Mordell-Weil basis is given by the rational point \( P \) with co-ordinates \((X, Y) = (-3, 9)\). There are thus infinitely many rational points on the elliptic curve (35) and these can be obtained by the group law. Using the relations (33), we can find infinitely many rational solutions of Eq. (32) and thus obtain infinitely many integer solutions of the diophantine system (27).

While the point \( P \) leads to a trivial solution of the diophantine system (27), the point \( 2P \) yields the solution,

\[ (-74)^r + 124^r + 78^r = 126^r + (-72)^r + (-20)^r + 70^r + 24^r, \quad r = 1, 2, 3, 4, \]

and the point \( 3P \) leads to the solution,

\[ (-40573)^r + 66494^r + 118981^r = (-15181)^r + 119510^r + 63756^r + (-37835)^r + 14652^r, \quad r = 1, 2, 3, 4. \]
In view of the above solutions of the diophantine system (27), it follows that \( \beta(4) \leq 8 \), and on combining with the result \( \beta(4) \geq 7 \) which follows from Cor. 1, we get,

\[ 7 \leq \beta(4) \leq 8. \]

3.4 We will now obtain parametric solutions of the diophantine system,

\[
\sum_{i=1}^{4} x_i^r = \sum_{i=1}^{6} y_i^r, \quad r = 1, 2, 3, 4, 5.
\]

We will use a parametric solution of the diophantine system,

\[
\sum_{i=1}^{6} X_i^r = \sum_{i=1}^{6} Y_i^r, \quad r = 1, 2, 3, 4, 5,
\]

to obtain two parametric solutions of the diophantine system (36).

A solution of the simultaneous equations \( \sum_{i=1}^{3} X_i^r = \sum_{i=1}^{3} Y_i^r, \quad r = 2, 4, \) in terms of arbitrary parameters \( m, n, x \) and \( y \), given in [4, p. 102], is as follows:

\[
\begin{align*}
X_1 &= (m + 2n)x - (m - n)y, & X_2 &= -(2m + n)x - (m + 2n)y, \\
X_3 &= (m - n)x + (2m + n)y, & Y_1 &= (m - n)x - (m + 2n)y, \\
Y_2 &= -(2m + n)x - (m - n)y, & Y_3 &= (m + 2n)x + (2m + n)y,
\end{align*}
\]

It immediately follows that a parametric solution of the diophantine system (37) is given by (38) and

\[
\begin{align*}
X_4 &= -X_3, & X_5 &= -X_2, & X_6 &= -X_1, \\
Y_4 &= -Y_3, & Y_5 &= -Y_2, & Y_6 &= -Y_1.
\end{align*}
\]

We choose the parameters \( x \) and \( y \) such that \( X_3 = 0 \) and immediately obtain the following parametric solution of the diophantine system (36):

\[
\begin{align*}
x_1 &= m^2 + mn + n^2, & x_2 &= m^2 + mn + n^2, \\
x_3 &= -m^2 - mn - n^2, & x_4 &= -m^2 - mn - n^2, \\
y_1 &= m^2 - n^2, & y_2 &= -m^2 - 2mn, \\
y_3 &= 2mn + n^2, & y_4 &= -2mn - n^2, \\
y_5 &= m^2 + 2mn, & y_6 &= -m^2 + n^2.
\end{align*}
\]
where \( m \) and \( n \) are arbitrary parameters.

To obtain a second solution of the diophantine system (36), we again use the parametric solution of the diophantine system (37) given by (38) and (39). We now choose the parameters \( x, y \) such that we get \( X_2 = X_3 \), and then apply the theorem of Frolov mentioned in the Introduction, taking \( d = -X_3 \). We thus get a solution of the diophantine system (36) which may be written as follows:

\[
\begin{align*}
    x_1 &= 3m^2 + 3mn + 3n^2, \quad x_2 = 2m^2 + 2mn + 2n^2, \\
    x_3 &= -m^2 - mn - n^2, \quad x_4 = 2m^2 + 2mn + 2n^2, \\
    y_1 &= 3m^2 + 3mn, \quad y_2 = -3mn, \\
    y_3 &= 3mn + 3n^2, \quad y_4 = 2m^2 - mn - n^2, \\
    y_5 &= 2m^2 + 5mn + 2n^2, \quad y_6 = -m^2 - mn + 2n^2,
\end{align*}
\]

(41)

where \( m \) and \( n \) are arbitrary parameters.

As a numerical example, taking \( m = 2, n = 1 \) in (41), we get the solution,

\[
21^r + 14^r + (-7)^r + 14^r = 18^r + (-6)^r + 9^r + 5^r + 20^r + (-4)^r, \quad r = 1, 2, 3, 4, 5.
\]

The two parametric solutions (40) and (41) of the diophantine system (36) are rather special since in both of them, the ratios \( x_i/x_j \) of the integers on the left-hand side are all fixed. We now show that there exist infinitely many other solutions of the diophantine system (36) that are not generated by these parametric solutions.
We write,

\begin{align*}
x_1 &= uv^2 + (6u^3 - 12u^2 + 32u - 32)v \\
&\quad + 9u^5 - 36u^4 - 336u^2 + 96u^3 + 240u, \\
x_2 &= (2u - 2)v^2 + (12u^3 - 48u^2 + 40u - 16)v \\
&\quad + 18u^5 - 126u^4 - 288u^2 + 264u^3 + 96, \\
x_3 &= -x_1, \quad x_4 = -x_2, \\
y_1 &= (2u - 2)v^2 + (12u^3 - 48u^2 + 48u)v \\
&\quad + 18u^5 - 126u^4 - 144u^2 + 288u^3 + 96u - 96, \\
y_2 &= uv^2 + (6u^3 - 12u^2 - 32u + 32)v \\
&\quad + 9u^5 - 36u^4 + 240u^2 - 96u^3 - 144u, \\
y_3 &= 2v^2 + (24u^2 - 40u + 16)v \\
&\quad + 54u^4 - 264u^3 + 96u + 192u^2 - 96, \\
y_4 &= -y_3, \quad y_5 = -y_2, \quad y_6 = -y_3.
\end{align*}

(42)

With these values of \(x_i, y_i\), it is readily seen that (36) is identically satisfied for \(r = 1, 3, 5\). Further,

\begin{align*}
\sum_{i=1}^{4} x_i^2 - \sum_{i=1}^{6} y_i^2 &= -8(9u^4 - 72u^3 + 24u^2 + 96u - 48 - v^2)^2, \\
\sum_{i=1}^{4} x_i^4 - \sum_{i=1}^{6} y_i^4 &= -8(9u^4 - 72u^3 + 24u^2 + 96u - 48 - v^2)^4.
\end{align*}

(43) (44)

It follows that a solution of the diophantine system (36) will be given by (42) if we choose \(u, v\) such that,

\begin{align*}
v^2 &= 9u^4 - 72u^3 + 24u^2 + 96u - 48.
\end{align*}

(45)

Now Eq. (45) represents the quartic model of an elliptic curve, and the birational transformation given by,

\begin{align*}
u &= (6X + 2Y - 12)/(3X - 24), \\
v &= (4X^3 - 96X^2 + 84X - 144Y + 832)/(3(X - 8)^2),
\end{align*}

(46)
and,

\[
X = \frac{(9u^2 - 36u + 3v + 4)}{8},
\]

\[
Y = \frac{(27u^3 - 162u^2 + 9uv + 36u - 18v + 72)}{16},
\]

reduces Eq. (45) to the Weierstrass form of the elliptic curve which is as follows:

\[
Y^2 = X^3 - 21X - 20.
\]

We again refer to Cremona’s database of elliptic curves, and find that (48) represents an elliptic curve of rank 1 and its Mordell-Weil basis is given by the rational point \(P\) with co-ordinates \((X, Y) = (-3, 4)\). There are thus infinitely many rational points on the elliptic curve (35) and these can be obtained by the group law. Using the relations (46), we can find infinitely many rational solutions of Eq. (45) and thus obtain infinitely many solutions of the diophantine system (36). While the point \(P\) leads to a trivial solution of the diophantine system (36), the point \(2P\) yields the solution,

\[
241r + 218r + (-241)r + (-218)r = 266r + 143r + 120r
\]

\[
+ (-266)r + (-143)r + (-120)r, \quad r = 1, 2, 3, 4, 5.
\]

The solutions of the diophantine system (36) obtained in this Section show that \(\beta(5) \leq 10\), and on combining with the result \(\beta(5) \geq 8\) which follows from Cor. 1, we get,

\[
8 \leq \beta(5) \leq 10.
\]

4 Concluding Remarks

It would be of interest to determine the precise values of \(\beta(4)\) and \(\beta(5)\). Further, it would be interesting to find integer solutions of the diophantine system (2) with \(k > 5\) and \(s_1 + s_2 < 2k + 1\).

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