Frequency Domain Analysis and Applications for Fractional-order Control Systems

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Abstract. This paper is concerned with the frequency domain analysis for fractional-order control systems. By Bode diagrams and Nyquist contour, the relationship of frequency properties between fractional-order systems and integer-order ones is found. A method of judging fractional-order system transfer functions from their frequency properties is provided.

1. Introduction
Real systems are more or less affected by non-integer orders. Systems such as materials having memory and hereditary effects and dynamical processes including mass diffusion and heat conduction, etc [1], need fractional-order models to obtain more precise math models and control performances. In the last two decades, great efforts have been made to bring fractional-order systems into control theory. Podlubny found time domain analytical solutions of fractional differential equations [2], but it’s hard to analyze because the expression with infinite terms does not converge in steady state [1]. Many authors have changed their minds to study time domain approximate expression and its efficient algorithms [6] [7]. Some results have been made in time domain analysis, but they are always complicated to realize.

For finding a more convenient method, we analyzed such systems properties in complex and frequency domain and got the clear relationship of frequency properties between fractional-order systems and integer-order ones. Meanwhile, we got several steps for fractional-order systems transfer functions from frequency response diagrams, which expanded the range of identifiable systems.

2. Fractional Calculus and Fractional-order Control Systems

2.1. Definition of Fractional Calculus
Orders of fractional calculus are real numbers. The fractional-order differential arithmetic operator is:

\[
_{a}D_{t}^{\alpha} = \begin{cases} 
\frac{d^{\alpha}}{dt^{\alpha}} & \Re(\alpha) > 0 \\
1 & \Re(\alpha) = 0 \\
\frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} f(\tau)d\tau, & \Re(\alpha) < 0 
\end{cases}
\]  

where \(a\) is a complex number and is defaulted as a real one in this paper.

As for \(_{a}D_{t}^{\alpha} f(t)\), there are three different definitions[3], of which M.Caputo’s approach can use Laplace transform formulas directly.

\[
_{a}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} f^{(n)}(\tau)(t-\tau)^{n-\alpha-1}, (n-1 < \alpha < n)
\]  

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2.2. Math Description of Fractional-order Control Systems

Consider a fractional-order SISO control system, we have its closed loop transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\overline{G}(s)}{1 + \overline{G}(s)H(s)} = \frac{b_n s^{\beta_n} + b_{n-1} s^{\beta_{n-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}$$

(3)

where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$, $\beta_n > \beta_{n-1} > \cdots > \beta_0 \geq 0, a_k, b_k (k=0, 1, 2, \cdots, n)$ are constants.

The time domain description can be expressed by M. Caputo’s fractional derivatives:

$$a_n D^\alpha \dot{y}(t) + a_{n-1} D^\alpha \dot{y}(t) + \cdots + a_0 D^\alpha y(t) = b_n D^\beta u(t) + b_{n-1} D^\beta \dot{u}(t) + \cdots + b_0 D^\beta \ddot{u}(t)$$

(4)

where $D^\alpha \equiv \frac{d^\alpha}{dt^\alpha}$.

3. Frequency Analysis for Fractional-order Control Systems

We take closed loop transfer function of fractional-order systems as:

$$G(s) = e^{-st} \cdot \frac{b_n s^{\beta_n} + b_{n-1} s^{\beta_{n-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} = e^{-st} \cdot K \cdot \prod_{i=1}^{m_n} \frac{(s^{\beta_i} - z_i)}{s^{\alpha_i - \beta_i} \cdot \prod_{i=1}^{m_0} (s^{\mu_i} - p_i)}$$

(5)

Where, $K = b_n / a_n, \beta_n = \beta_0 + \sum_{j=1}^{m_n} \mu_j, \alpha_n = \alpha_0 + \sum_{j=1}^{m_0} \mu_j$. And the efficient solution of each product factor must meet the uniqueness conditions1. So, we need to study the frequency properties of fractional-order basic terms more generalizing than integer-order ones.

3.1. Frequency Analysis for Fractional-order Basic Terms

We conclude fractional-order basic terms are in three categories: fractional-order pure differential term, fractional-order derivative term and fractional-order integral term. The value range of the frequency discussed below is real domain.

The first category is fractional-order pure derivative term. The expression in the complex domain is:

$$G(s) = s^\mu$$

(6)

where $\mu$ is an arbitrary real number. It has derivative function when $\mu$>0 and integral when $\mu$<0. It degrades to a constant gain, whose value equals to 1, if $\mu$=0. Here, we let $-1 \leq \mu \leq 1$. Take $s = j\omega$ to (6) and obtain the frequency expression, we have:

$$G(j\omega) = (j\omega)^\mu = \omega^\mu (\cos \frac{\pi}{2} + j \sin \frac{\pi}{2})^\mu = \omega^\mu (\cos \frac{\pi}{2} \mu + j \sin \frac{\pi}{2} \mu)$$

(7)

Magnitude and phase properties are:

$$|A(\omega)| = |\omega^\mu|$$

(8)

$$\theta(\omega) = \frac{\pi}{2} [\mu - (1 - \frac{\omega}{|\omega|})]$$

(9)

$$20 \log |A(\omega)| = 20 \log |\omega^\mu| = 20 \mu \log |\omega|$$

(10)

From (7) (9) (10), we obtain the Nyquist contour, magnitude and phase diagrams of this term.

Lemma 1: The Nyquist contour of fractional-order pure derivative term $s^\mu$ is a beeline with its slope $\pi\mu/2$, which can be obtained by anticlockwise rotation of the real axis in the complex plane $\pi\mu/2$ around the origin according to variation of the fractional order $\mu$. The magnitude decibel-log-frequency diagram is a beeline with its slope $20\mu$ when $\omega > 0$ and $-20\mu$ when $\omega < 0$. It crosses the 0dB line at $\omega = \pm 1$. The phase log-frequency diagram is independent with frequency. Under a certain order $\mu$, the phase is $\pi\mu/2$ when $\omega > 0$ and $-\pi + \pi\mu/2$ when $\omega < 0$.

1. Jifeng Wang, Yuankai Li, “Frequency Domain Stability Criteria for Fractional-order control
Figure 1. Bode diagrams of $s^\mu$

Figure 1 describes the frequency properties of a pure derivative term with its order $\mu = 0.73$. The slope of the magnitude diagram is ±14.6dB/sec. Its phase is 1.147 when $\omega > 0$ and -1.995 when $\omega < 0$. The Nyquist contour is shown in Figure 2 and the slope is 1.147.

The second category name is fractional-order derivative term. Their complex domain expression is:

$$G(s) = T_s s^\mu + 1, \mu \in [0,1]$$  \hspace{1cm} (11)

where $T_s = A_c \xi \theta_a = A_c (\cos \theta_a + j \sin \theta_a)$, $A_c > 0, \theta_a \in [0,2\pi], 0 < \mu \leq 1$. The frequency domain expression is:

$$G(j\omega) = T_s (j\omega)^\mu + 1 = 1 + \omega^\mu A_c \cos(\theta_a + \frac{\pi}{2} \mu) + j\omega^\mu A_c \sin(\theta_a + \frac{\pi}{2} \mu)$$  \hspace{1cm} (12)

$$A(\omega) = \sqrt{[1 + \omega^\mu A_c \cos(\theta_a + \frac{\pi}{2} \mu)]^2 + [\omega^\mu A_c \sin(\theta_a + \frac{\pi}{2} \mu)]^2}$$  \hspace{1cm} (13)

$$\theta(\omega) = \tan^{-1} \frac{\omega^\mu A_c \sin(\theta_a + \frac{\pi}{2} \mu)}{1 + \omega^\mu A_c \cos(\theta_a + \frac{\pi}{2} \mu)}$$  \hspace{1cm} (14a)

Phase properties are determined by $\omega^\mu A_c$ and $\theta_a + \pi \mu / 2$. When $\omega \to \infty$, $\omega^\mu A_c \to \infty$, we have:

$$\lim_{\omega \to \infty} \theta(\omega) = \lim_{\omega \to \infty} \tan^{-1} \frac{\omega^\mu A_c \sin(\theta_a + \frac{\pi}{2} \mu)}{1 + \omega^\mu A_c \cos(\theta_a + \frac{\pi}{2} \mu)} = \lim_{\omega \to \infty} \frac{\sin(\theta_a + \frac{\pi}{2} \mu)}{\cos(\theta_a + \frac{\pi}{2} \mu)} = \theta_a + \frac{\pi}{2} \mu \mp \Theta$$  \hspace{1cm} (14b)

Where $\Theta \in [0,2\pi)$. We named it limit phase. Therefore, the phase curve starts from the origin and approaches to $\Theta$. Especially, $\theta(\omega) = \frac{1}{2}(\theta_a + \frac{\pi}{2} \mu)$ when $\omega^\mu A_c = 1$. Because $\Theta$ varies in four quadrants of the complex plane, the asymptote of phase diagram has four different instances as in Figure 3.

As for the magnitude properties:

$$20 \log A(\omega) = 20 \log \sqrt{(\omega^\mu A_c)^2 + 2\omega^\mu A_c \cos(\theta_a + \frac{\pi}{2} \mu) + 1}$$  \hspace{1cm} (15)

$$20 \log A(\omega) \approx \begin{cases} 20 \log 1 = 0 & |\omega^\mu A_c| \leq 1 \\ 20 \log \sqrt{(\omega^\mu A_c)^2} = 20 \log |\omega^\mu A_c| = 20 \log A_c + 20 \mu \log |\omega| & |\omega^\mu A_c| > 1 \end{cases}$$  \hspace{1cm} (16)

The frequency $\omega_b = A_c^{-\frac{1}{\mu}}$ makes $\omega^\mu A_c = 1$ is named break frequency. Through (16), we got the approximate decibel magnitude diagram. What Figure 3 shows are the real curves.

Approximate diagrams have error and are affected by the break frequency $\omega_b$ and the limit phase $\Theta$. Meanwhile, $\omega_b$ is affected by both $A_c$ and $\mu$. In the upper part of Figure 4, $A_c = 3, \mu = 0.73, \Theta$ takes...
four quadrant angles separately. In the lower part, \( A_u = 3, \mu = 0.45 \). The peak value of amplitude error moves along the frequency axis with variation of \( \omega_b \), and along the amplitude axis when \( \Theta \) changes.

The Nyquist contour is shown in Figure 5. The points of the curve are in a circle cluster:

\[
(x - 1)^2 + y^2 = (\omega^\mu A_u)^2
\]  

Figure 3. Bode diagrams of \( G(s) = T_\omega s^\mu + 1 \)  

Figure 4. Amplitude error of approximate diagrams

Lemma 2: The magnitude diagram of fractional-order derivative term \( G(s) = T_\omega s^\mu + 1 \) can be represented by 0dB line when \( |\omega| < \omega_b = A_u^{-1/\mu} \), by a beeline with the slope \( -20\mu \) when \( \omega > \omega_b \) and by a beeline with its slope \( -20\mu \) if \( \omega < -\omega_b \). The phase diagram starts from the origin and approaches to limit phase \( \Theta = \theta_u + \pi\mu / 2 \) in positive frequency domain, from \( \Theta = -\pi \) to the origin in negative frequency domain. At the break frequency \( \omega_b \), the phase values \( \frac{1}{2}(\theta_u + \frac{\pi}{2} \mu) \). The Nyquist contour is a beeline obtained by anticlockwise rotation of real axis \( \theta_u + \pi\mu / 2 \) around the \((1,0)\) point.

The third category is fractional-order integral term. We have the following expressions:

\[
G(s) = \frac{1}{T_\omega s^\mu + 1}, \mu \in [0,1]
\]  

\[
G(j\omega) = \frac{1}{T_\omega (j\omega)^\mu + 1} = \frac{1}{1 + \omega^\mu A_u \cos(\theta_u + \frac{\pi}{2} \mu) + j\omega^\mu A_u \sin(\theta_u + \frac{\pi}{2} \mu)}
\]  

\[
A(\omega) = \frac{1}{\sqrt{[1 + \omega^\mu A_u \cos(\theta_u + \frac{\pi}{2} \mu)]^2 + [\omega^\mu A_u \sin(\theta_u + \frac{\pi}{2} \mu)]^2}}
\]

\[
\theta(\omega) = -\tan^{-1} \frac{\omega^\mu A_u \sin(\theta_u + \frac{\pi}{2} \mu)}{1 + \omega^\mu A_u \cos(\theta_u + \frac{\pi}{2} \mu)}
\]

Hence, the logarithm frequency properties are minus functions of those of fractional-order derivative term. Such is described in Figure 6. The shape of the Nyquist contour needs entire frequency response, which means \( \omega \in (-\infty, \infty) \). Firstly, there is always an asymptote with any parameter variation. Secondly, we study the effects of \( \omega_b \) and \( \Theta \). As for certain \( \Theta \), we get a type of Nyquist contour with \( \omega_b \) variation. If \( \Theta \) varies in the range \([0,2\pi)\), shape of the Nyquist curve has corresponding change. In Figure 7, we lay out all the twelve different shapes.

Lemma 3: The magnitude diagram of fractional-order integral term \( G(s) = \frac{1}{T_\omega s^\mu + 1} \) can be represented by 0dB line when \( |\omega| < \omega_b = A_u^{-1/\mu} \), by a beeline with the slope \( -20\mu \) when \( \omega > \omega_b \) and by a beeline with its slope \( -20\mu \) if \( \omega < -\omega_b \). The phase diagram starts from the origin and approaches the limit phase \( \Theta = -(\theta_u + \pi\mu / 2) \) in the positive frequency domain, from \( \Theta = \pi \) to the origin in negative
frequency domain. At the break frequency $\omega_b$, the phase values $-\frac{1}{2}(\theta_a + \frac{\pi}{2} \mu)$. The Nyquist contour starts along the asymptote with its slope $-\theta_a - \frac{\pi}{2} \mu$ from the origin and approaches the point $(1,0)$ in negative frequency domain, begins at the point $(1,0)$ and approaches to origin along the same direction of the asymptote in the positive frequency domain.

![Figure 5. Nyquist contour of $G(s) = Ts^\alpha + 1$](image)

![Figure 6. Bode diagrams of $G(s) = \frac{1}{Ts^\alpha + 1}$](image)

![Figure 7. Nyquist contour of $G(s) = \frac{1}{Ts^\alpha + 1}$](image)

3.2. Synthesis of Frequency Characteristics for Fractional-order Control Systems

Through the above, we can get the frequency properties of arbitrary fractional-order systems. To show the approach is reversible, we provide the steps to ascertain the transfer function by frequency characteristics.

1) Get frequency response diagrams of fractional-order system through frequency identification.
2) If the phase diagram has excursion $\Delta \varphi$, the system contains a delay term $e^{-sT}$ and $\varphi = \Delta \varphi / \omega$.
3) The slope of the first segment of the magnitude diagram equals to -$20 \mu_b$ dB/dec where $\mu_b$ is the order of fractional-order pure derivative or integral terms.
4) At the point $\omega = 0$, the initial segment or its prolonged line value is $20 \lg K$. $K$ is the system gain, and this initial segment or its prolonged line crosses 0dB line at $\omega = 0$ which equals to $k^{-1/\mu_b}$.
5) Orders $\mu_i$ of the other terms $(Ts^\alpha + 1)^\mu_i$ can be ascertained by the variation of slope at those break frequency $\omega_i$. We can get $A_i$ and $\theta_{ai}$ from the following two equations:

Break frequency:

$$\omega_i = A_i^{1/\mu_i}$$  \hspace{1cm} (22)

Phase at break frequency:

$$\psi_i = -\frac{\pi}{2} \mu_i + \sum_{j=1}^{m} f(\omega_j, \theta_{aj}) + \sum_{k=1}^{n} f(\omega_k, \theta_{ak}) + \Delta \varphi$$  \hspace{1cm} (23)
where \( i = 1, 2, \cdots, m + n \), \( T_{ii} = A_{ii} (\cos \theta_i + j \sin \theta_i) \), and
\[
f(\omega_i, \theta_i) = \pm \tan^{-1} \frac{\omega_i^{\mu_i} A_{ii} \sin(\theta_i + \frac{\pi}{2} \mu_i)}{1 + \omega_i^{\mu_i} A_{ii} \cos(\theta_i + \frac{\pi}{2} \mu_i)}.
\]
takes positive sign in derivative terms.

Finally, we have the frequency domain transfer function of the fractional-order system:
\[
G(s) = K \frac{1}{s^n} \prod_{i=1}^{n} \frac{1}{(T_{ii} s^{\mu_i} + 1)} \cdot e^{-\tau_i}
\]
(24)

4. Conclusion
For fractional-order control systems in the time domain, both the approaches and the expressions are complicated. They often need large scale mathematically approximate algorithms. To find a fast and convenient analysis method to receive system properties, the authors started from the frequency domain, concluded three categories of fractional-order basic terms, studied their frequency characteristics and Nyquist contours, researched the effects of their parameters to system performances. To apply this extend frequency method into practice easily, we provided concise steps for getting fractional-order system transfer functions from frequency characteristics. The frequency method for fractional-order systems still belongs to classical control theory. It’s available and efficient in analysis, synthesis and adjustment for common SISO systems.

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