SU($N_c$) gauge theories at deconfinement

Biagio Lucini

College of Science, Swansea University, Singleton Park, Swansea SA2 8PP, UK

Antonio Rago

School of Computing and Mathematics, University of Plymouth, Plymouth PL4 8AA, UK

Enrico Rinaldi

SUPA and The Tait Institute, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK

Abstract

The deconfinement transition in SU($N_c$) Yang–Mills is investigated by Monte Carlo simulations of the gauge theory discretised on a spacetime lattice. We present new results for $4 \leq N_c \leq 8$ (in particular, for $N_c = 5$ and $N_c = 7$), which are analysed together with previously published results. The increased amount of data, the improved statistics and simulations closer to the continuum limit provide us with better control over systematic errors. After performing the thermodynamic limit, numerical results for the ratio of the critical temperature $T_c$ over the square root of the string tension $\sqrt{\sigma}$ obtained on lattices with temporal extensions $N_t = 5, 6, 7, 8$ are extrapolated to the continuum limit. The continuum results at fixed $N_c$ are then extrapolated to $N_c = \infty$. We find that our data are accurately described by the formula $T_c/\sqrt{\sigma} = 0.5949(17) + 0.458(18)/N_c^2$. Possible systematic errors affecting our calculations are also discussed.

Keywords: SU($N_c$) Yang-Mills Theories, Large $N_c$ limit, Deconfinement Transition, Lattice Gauge Theories.

1. Introduction

In recent years, various lattice studies have been performed for SU($N_c$) gauge theories in the ’t Hooft limit (see e.g. Refs. [1, 2] for a review). As a result, on the one hand the old idea that the bulk of the physics is shared between SU(3) and the simpler SU($\infty$) [3,4] has been confirmed; on the other hand, solid bases have been provided for gauge-string duality studies aiming at describing QCD-like theories (see e.g. Ref. [4] for an early review of this field).

On the lattice, one of the main areas of activity has been the finite temperature regime [7–24]. In particular, it has been shown that the deconfinement temperature can be determined with very good accuracy [7,9,17,18,21]. This suggests to use the deconfinement temperature as the physical scale in large $N_c$ limit studies of observables at fixed lattice spacing [25,29], which are often a useful intermediate step before performing the continuum extrapolation. The main motivation of this work is to complement existing results on the deconfinement phase transition by providing the value of the (pseudo-)critical coupling at various temporal extensions for the gauge groups SU(5) and SU(7), which have not been investigated at finite temperature before. Some of these results have already been used in our study of glueball masses at large $N_c$ at the critical coupling for a
temporal extension \( N_t \) of six lattice spacings \[28\].

There, the inclusion of the \( N_c = 5 \) and \( N_c = 7 \) data allowed us to increase the precision of the large \( N_c \) extrapolation of the masses. The results of that study suggest that knowing the critical large \( N \) data allowed us to increase the precision of the deconfined phase. The same criterion in terms of trips (system at criticality to perform at least one million measurements, in order for the relations in SU(8), in most cases consisting of at least one million measurements, in order for the system at criticality to perform at least 8 round trips (tunnellings) between the confined and the deconfined phase. The same criterion in terms of number of tunnellings has been used to determine the statistics for all the new simulations discussed in this paper. This should remove any bias related to a possible loss of ergodicity in the critical region.

The rest of this letter is organised as follows. In Sect. 2 we describe the system under investigation, define the observables we study and provide numerical results for an estimator of the coupling at which the deconfinement phase transition takes place at fixed \( N_c \), \( N_t \) and \( N_s \). Sect. 3 deals with the thermodynamic limit of the critical couplings. Sect. 4 is devoted to the continuum extrapolation of the critical temperature in units of the string tension at fixed \( N_c \). The large \( N_c \) limit of the latter quantity is discussed in Sect. 5. Finally, in Sect. 6 we summarise our results and briefly discuss possible future directions.

2. The phase transition

Our calculation follows the method exposed e.g. in Refs. [7, 8], which we will briefly summarise below. We consider a SU\((N_c)\) gauge theory described by the Wilson action

\[ S = \beta \sum_{\mu,\nu > 0} \left( 1 - \frac{1}{N_c} \text{Re} \, \text{Tr} \left( U_{\mu\nu}(i) \right) \right), \tag{1} \]

where \( U_{\mu\nu}(i) \) is the path-ordered product of the links \( U_{\mu}(i) \in \text{SU}(N_c) \) around the lattice plaquette identified by the point \( i \) and the directions \( \mu \) and \( \nu \). \( \beta \) is defined as \( \beta = 2N_c/g_0^2 \), with \( g_0 \) the bare gauge coupling. The finite temperature regime is realised by considering the system on a lattice of volume \( L_s^3 \times L_t \), where \( L_s = aN_c \) and \( L_t =aN_t \), \( a \) being the lattice spacing, with \( N_t \ll N_s \). Periodic boundary conditions are imposed in all directions. The temperature of the system is then given by \( T = 1/L_t \) and for fixed \( N_t \) it becomes a function of \( \beta \) only, through the dependence of the lattice spacing \( a \) on the gauge coupling. To find the value of the coupling at which the deconfining transition takes place, at fixed \( N_c \), we compute the spatial average of the temporal Polyakov loop

\[ \bar{l}_p = \frac{1}{N_cN_s^3} \sum_{\vec{x}} \text{Tr} \left( \prod_{t=0}^{N_t-1} U_t(\vec{x}, t) \right), \tag{2} \]
where $\vec{x}$ and $t$ are the components of the Euclidean four-vector $i$ respectively in the spatial and in the temporal directions (the latter corresponding to the dimension of size $L_t$). The deconfinement phase transition can be seen as a transition from the phase symmetric under the center symmetry $\mathbb{Z}_N$ (the confined phase), to the phase in which this symmetry is spontaneously broken. $\bar{l}_p$ is the order parameter of the deconfinement phase transition. In addition, we study the four-volume average of the plaquette $\bar{u}_p$, defined as

$$\bar{u}_p = \frac{1}{N_c N_t N_s^3} \sum_{i,\mu>\nu} \text{Re Tr}(U_{\mu\nu}(i)). \quad (3)$$

At fixed volume, we define the coupling $\beta_c(N_s, N_t)$ corresponding to the deconfinement temperature by looking at the peak of the susceptibility of the modulus of $\bar{l}_p$:

$$\chi_l = N_s^3 \left( \langle |\bar{l}_p|^2 \rangle - \langle |\bar{l}_p| \rangle^2 \right). \quad (4)$$

Our calculation is meant to complement the results already present in the literature \cite{7,8,13,21}. Since calculations at $N_c$ larger than 8 become quite expensive \cite{21}, we focused our attention to lower $N_c$. For $N_c < 4$, very precise results are already available. For $N_c = 4$ and $N_c = 6$, previous studies already attained a good level of precision at $N_t = 5, 6, 8$. We hence studied the case $N_t = 7$, which has not been investigated before. The addition of these calculations helps to improve the extrapolation of the corresponding $\beta_c$ to the continuum limit ($N_t = \infty$). Calculations at $N_c = 5, 7$ have not been performed before. Hence, most of our numerical effort is devoted to the determination of the critical temperature $T_c$ for SU(5) and SU(7). Finally, for completeness, we have performed a new, high statistics numerical investigation of SU(8) (comparable to that of \cite{21}), which has enabled us to perform a more robust large $N_c$ extrapolation.

SU($N_c$) gauge theories for $N_c \geq 5$ have a bulk phase transition at some value $\beta_B$ of the coupling constant. The continuum physics is realised for $\beta > \beta_B$. For SU(5), $\beta_B \approx 16.655$ \cite{13}. We have determined $\beta_B$ for SU(7), which turns out to be around 33.246. The request that the system at criticality be in the continuum regime is fulfilled if $N_t \geq 5$. As expected, this is the same bound on $N_t$ already found for $N_c = 4, 6, 8$. At fixed $N_c$, $N_s$ and $N_t$ we have computed $\chi_l$ (see Eq. (4)) for about 10 $\beta$s in the critical region, in a range that covers both the confined and the deconfined phase. For each calculation, we have used a combination of overrelaxation and heatbath updates with ratio 4:1. The number of sweeps has been chosen in such a way that at least eight tunnellings were observed. In fact, in most of the cases we observed 12–15 tunnellings for the largest lattices. The statistics for each gauge group, spatial and temporal sizes is reported in Tab. 1.

Always at fixed $N_c$, $N_s$ and $N_t$, using the points that are closer to the critical $\beta$ (typically five or six), we have reweighted $\chi_l$ using the Ferrenberg-Swendsen procedure \cite{30}, as illustrated for one set of parameters in Fig. 1. In particular, the reweighted data have been generated for several different bootstrap samples of the original simulations in order to give an unbiased estimate of the statistical error. One $\beta$ value corresponding to the location of the maximum is chosen for each bootstrap sample (which is sufficiently well-behaved for a unique choice to be made). The central

| $N_c$ | $N_t$ | $N_s$ | $N_{\text{meas}} \times 10^6$ |
| ---- | ---- | ---- | ------------------ |
| 4   | 7    | 22   | 12                |
| 5   | 5    | 8,10,12,14,16 | 3                  |
| 6   | 14   | 22   | 2                 |
| 7   | 16   | 22   | 6                 |
| 8   | 18   | 22   | 5                 |
| 6   | 7    | 20   | 10                |
| 7   | 5    | 8,10,11,12 | 2                  |
| 6   | 10,12 | 3      |
| 7   | 11,12,13,14 | 6     |
| 8   | 14   | 6    | 6                 |

Table 1: Simulated volumes $N_s$ for different gauge groups SU($N_c$) and different temporal lengths $N_t$. An approximate counting of the total number of measurements on each lattice is also shown. The results on these new lattices complement and improve the study of Ref. \cite{13}.
value and the standard deviation of the gaussian distribution of these β values determine our best estimate for the critical coupling β_c. The procedure described above allows us to avoid choosing the range for a quadratic fit approximating χ_l and therefore gives a more reliable and robust result.

3. Thermodynamic limit

For fixed N_c and fixed N_t, the critical coupling β_c(N_t) is the thermodynamic limit of β_c(N_s, N_t). Since for N_c ≥ 3 the phase transition is first order \[7, 8\], the extrapolation is performed according to the ansatz

\[ \beta_c(\infty, N_t) = \beta_c(N_s, N_t) + h(N_t) \frac{N_s^3}{N_t^3}, \]

where only the leading volume correction is taken into account and the value of the coefficient h(N_t) depends on the lattice spacing \[13\]:

\[ h(N_t) = \frac{h_0}{K(\beta_c(N_t))} + \mathcal{O}(a^2), \]

with

\[ K(\beta_c(N_t)) = \frac{d \ln a(\beta)}{d \beta} \bigg|_{\beta=\beta_c(N_s, N_t)}. \]

The procedure to evaluate h(N_t) can be described as follows. At first we obtain the coefficient h(N_t = 5) (corresponding to our largest lattice spacing) directly from a finite–size scaling (FSS) analysis using a wide range of volumes V = N_s^3. We then use our own data for the beta function \[\beta(\beta)\] obtained from the interpolation of the string tension over a large set of couplings in order to estimate h(N_t) at N_t = 6, 7, 8, with higher order corrections \[\mathcal{O}(a^2)\] accounted for by a 15% error increase on h(N_t) \[13\].

The determination of the beta function requires measuring zero-momentum correlators of Polyakov loops at zero temperature (torelons). In order to extract am_t, the mass of the loop in lattice units, we look at the large separation exponential decay of those correlators, which is controlled by am_t itself (see e.g. Ref. \[25\]). For SU(N_c) gauge groups with N_c = 2, 3, 4, 6, 8, detailed measurements of torelon masses am_t are already available on a wide range of coupling constants \[25\]. From the mass of such states extracted using Polyakov loops of length N_s in units of the lattice spacing, we obtain the string tension a\sqrt{\sigma} by solving the equation

\[ am_t(N_s) = a^2 \sigma N_s - \frac{\pi}{3N_s} - \frac{\pi^2}{18N_s^3} \frac{1}{a^2 \sigma}, \]

where the last two terms immediately derive from the effective theory describing the low–energy dynamics of confining strings in the SU(N_c) theory \[31\]. If we keep σ fixed to its continuum value, the numerical data for a\sqrt{\sigma} give us the variation of a as a function of β.

In Tab. \[2\] we summarise the string tension measured in high statistics simulations on large symmetric lattices L^4 at the reported values of β. For the computation of σ, we have used Eq. \[8\]. Since previous lattice calculations only used the leading correction \[-\frac{\pi}{3N_c}\], we have verified that the insertion of the next-to-leading correction, whose universal character has been discovered only recently \[31, 32\], does not affect the numerical results within errors. We have calculated for the first time the behaviour of the string tension in SU(5) and SU(7) as a function of the bare coupling. This allows us to have a precise estimate
Table 2: SU(5) and SU(7) string tensions on hypercubic lattices $L^4$ at the reported values of $\beta$. The string tension is extracted from the mass of the lightest torelon state of length $L$. SU(4), SU(6) and SU(8) string tensions at the critical coupling for $N_t = 7$ are also shown.

| $N_c$ | $L$ | $\beta$ | $a\sqrt{\sigma}$ |
|-------|-----|---------|------------------|
| 4     | 14  | 10.9415| 0.2314(11)       |
| 5     | 10  | 16.8762| 0.3352(17)       |
| 12    | 17.1070| 0.2755(10)|           |
| 14    | 17.22 | 0.2553(74)|           |
| 14    | 17.3371| 0.23649(53)|           |
| 16    | 17.44 | 0.20935(58)|           |
| 16    | 17.556| 0.20710(53)|           |
| 18    | 17.66 | 0.19386(40)|           |
| 6     | 14  | 25.1707| 0.2379(9)        |
| 7     | 10  | 33.5465| 0.27981(96)      |
| 12    | 34.22 | 0.25950(75)|           |
| 14    | 34.4397| 0.23997(75)|           |
| 16    | 34.63 | 0.22435(47)|           |
| 16    | 34.8295| 0.21010(68)|           |
| 18    | 35.00 | 0.2011(10)|            |
| 8     | 14  | 44.0955| 0.2426(6)        |

Table 3: SU(5), SU(7) and SU(8) values of the critical coupling for the corresponding temporal extent and in the infinite volume limit. Simulations are performed on $N_t^3 \times N_t$ lattices and the values $\beta_c(\infty, N_t)$ are obtained from Eq. (5). For SU(4) and SU(6) we also report our novel estimate of $\beta_c$ at fixed $N_t = 7$.

Ref. 21, at the lattice spacing corresponding to $\beta_c$ for $N_t = 5$ the theory might still be affected by artefacts related to the nearby bulk phase transition. In the SU(7) gauge theory at $N_t = 5$, for which the bulk phase transition is very close to the finite temperature transition, we have explicitly checked whether the determination of $h(N_t)$ at that lattice spacing is significantly affected by lattice artefacts. To this purpose, we performed a FSS analysis on lattices with a larger temporal extension ($N_t = 7$), where the deconfinement transition is pushed to a weaker coupling. Both procedures give infinite volume estimates for $\beta_c$ that are compatible within the statistical uncertainty of our simulations.

4. Continuum extrapolation

Before we can take the continuum and the large $N_c$ limit, the location of the deconfinement transition we found in the previous section needs to be translated into a physical temperature $T_c$. In addition, for the continuum limit, it proves convenient to use dimensionless quantities, since at the leading order these have corrections that are
quadratic in the lattice spacing. Using the string tension $\sqrt{\sigma}$ to set the scale of pure gauge lattice simulations gives good control over systematic errors in the continuum extrapolation. For this reason, we study the continuum limit of the deconfinement temperature in units of the square root of $\sigma$, $T_c/\sqrt{\sigma}$. This is determined for each $N_c$ using only the leading $O(a^2)$ correction [8]:

$$\left. \frac{T_c}{\sqrt{\sigma}} \right|_{a=0} = \left. \frac{T_c}{\sqrt{\sigma}} \right|_{a} + \delta a^2 \sigma, \quad (9)$$

where at fixed lattice spacing

$$\left. \frac{T_c}{\sqrt{\sigma}} \right|_{a} = \frac{1}{N_t a(\beta_c)\sqrt{\sigma}} \quad (10)$$

and $\delta$ is a numerical coefficient of order one. We performed the continuum limit of $T_c/\sqrt{\sigma}$ according to Eq. (9) and using four different lattice spacings. The precision we achieve on the ratio $T_c/\sqrt{\sigma}$ is mostly determined by the precision of the string tension, since this latter quantity is affected by a relative error larger than that of $\beta_c$.

The availability of an additional lattice spacing in the asymptotic scaling region for the continuum extrapolation can help us identifying possible systematic effects due to the inclusion of the coarsest point. In Fig. 2 we show the continuum limit of the deconfinement temperature for SU(5) and SU(7), which is the key original contribution of this work. Fits with and without the coarsest lattice point give compatible results for SU(7) with a $\chi^2$ per d.o.f. above three disfavouring the former. For SU(5) the situation is similar, but the fit with all points has an acceptable $\chi^2$. This is also true for $N_c = 4, 6, 8$. Our conclusion is that if there is any systematic effect in extrapolating the ratio $T_c/\sqrt{\sigma}$ to the $a = 0$ limit including points measured on $N_t = 5$ lattices, this effect is significantly smaller than the statistical error.

In Tab. 4 we summarise the continuum limit values of $T_c/\sqrt{\sigma}$ that we used to obtain the SU($\infty$) limit. Only for $N_c = 7$ we discard the coarsest lattice point in the continuum limit and show the result for the fit obtained using $N_t > 5$. Since all the other gauge groups have well behaved extrapolations with a low $\chi^2$ even including the $N_t = 5$ point, in those cases we perform the fit using results at all the available values of $N_t$.

5. Large $N_c$ extrapolation

According to large $N_c$ arguments, the large $N_c$ limit of $T_c/\sqrt{\sigma}$ can be expressed as a power series in $1/N_c^2$.

$$\left. \frac{T_c}{\sqrt{\sigma}} \right|_{N_c} = \left. \frac{T_c}{\sqrt{\sigma}} \right|_{N_c=\infty} + \frac{c}{N_c^2} + O(N_c)^{-4}, \quad (11)$$

where $c$ is a numerical constant of order one. Our best fit for the large $N_c$ deconfinement temperature according to Eq. (11) is shown in Fig. 3. We extrapolate keeping only the $1/N_c^2$ correction to the planar limit, a procedure that has been shown to work very well down to $N_c = 2$ [13]. Including also SU(2) and SU(3) data from Ref. [8], we obtain

$$T_c/\sqrt{\sigma} = 0.5949(17) + 0.458(18)/N_c^2, \quad (12)$$

with good $\chi^2$/d.o.f. = 1.18. Discarding $N_c = 2, 3$ worsens the quality of the fit without changing the fitted parameters within the quoted error. Our value for the SU($\infty$) deconfinement temperature in units of the string tension is compatible with previous results reported in Ref. [13], but the relative accuracy has increased approximately by a factor of 2. This is due to a better control over the continuum extrapolation for $N_c \geq 4$ and to the inclusion of SU(5) and SU(7) data. Note that
the more precise result is still compatible with the finite $N_c$ value being accounted for by the leading $1/N_c^2$ correction only.

In order to assess the robustness of our result, we tested it against possible systematic errors in the continuum extrapolation. A different set of continuum values was created in the following way: for each $N_c \geq 4$, results of continuum fits with and without the $N_t = 5$ point were merged together such that the error accounted for the whole possible range of values, while the middle point of the error bar was taken as the central value. The estimates we obtained are fully compatible with Eq. (12). Fitting the large $N_c$ behaviour of points obtained by extrapolating to the continuum limit results for $N_t \geq 6$ for all $N_c \geq 4$ (except for $N_c = 5$, where if we consider only points at $N_t > 5$ the errors resulting from the continuum fit are anomalously small) also gives compatible results.

6. Conclusions

We have determined numerical values of the ratio $T_c/\sqrt{\sigma}$ for gauge groups SU(5) and SU(7). We have used the new data together with results for $N_c = 2, 3, 4, 6, 8$ already available in the literature [7, 8, 13] (supplemented with calculations at an additional lattice spacing for $N_c = 4, 6$ and with calculations with increased statistics for $N_c = 8$) to reanalyse the large $N_c$ limit of this quantity, for which it turns out that only the leading $1/N_c^2$ correction is needed to extrapolate finite $N_c$ results in the range $2 \leq N_c \leq 8$. This had been already observed in previous simulations. We obtain an accurate large $N_c$ limit that improves by a factor of two the precision of previous calculations. At the same time, we investigated possible finite lattice spacing artefacts. Our analysis lead us to the conclusion that for $4 \leq N_c \leq 8$, it is safe to extrapolate to the continuum limit from $N_t = 5$, as done in [7, 8].

In order to obtain a further noticeable improvement, it is likely that gauge groups with $N_c \geq 8$ need to be investigated. However, since the strength of the first order deconfinement transition grows with $N_c$, reliable Monte Carlo studies of those systems will crucially require algorithms that mitigates substantially the exponential suppression in the spatial volume of the tunnelling rate between the confined and the deconfined phases at criticality, like for instance the multicanonical algorithm [34].

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