Robertson-Schrödinger intelligent states for two-body Calogero model

M. Daoud

LPMC, Department of physics,
University Ibn Zohr, P.O.Box 28/S, Agadir
Morocco

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Abstract

Using the Gazeau-Klauder and Klauder-Perelomov coherent states, we derive the Robertson-Schrödinger intelligent states for two-body Calogero quantum system.
1 Introduction

Coherent states play an important role in quantum physics [1-3]. The term "coherent" itself originates in the current language of quantum optics (for instance, coherent radiation). It was introduced by Glauber in 1960 for quantum oscillator (see[1-3]). These states are closest to classical ones. Indeed, at the very beginning of quantum theory, Schrödinger was interested by states which restore the classical behaviour of the position operator for a quantum oscillator and show that the coherent states mediate a smooth transition from classical to quantum mechanics. The most important properties of such states are: (i) normalization, (ii) continuity in the labeling, (iii) over-completion and (iv) temporal stability. The properties (i)-(iv) are the minimal set of conditions of generalized coherent states [4-5]. They can be defined following three equivalent ways: (D1) eigenvectors of the annihilation operator, (D2) Unitary action of the Weyl-Heisenberg group on the vacuum, and (D3) Minimization of Heisenberg inequality.

More recently, coherent states associated with various exactly solvable quantum mechanics have been successfully studied [5-11] and was constructed following three unequivalents prescriptions which extend the definitions (D1), (D2) and (D3). The coherent states obtained in the (D1) scheme are called Gazeau-Klauder ones [5-7]. Using the (D2) definition, one obtain coherent states of Klauder-Perelomov kind [8-10]. The extension of third definition (D3) for quantum system other than harmonic oscillator leads to the minimization of the Robertson-Schrödinger uncertainty relation [12-13] (which generalizes the usual Heisenberg inequality). In this case, one obtain the so-called Robertson-Schrödinger or intelligent states [14-15]. In the present work, we are interested by the construction of intelligent states for the two-body-Calogero model. We give the explicit analytical representation of states which minimize the Robertson-Schrödinger uncertainty relation. Two equivalents representations are considered. The first one is related the Gazeau-Klauder coherent states and the second involves the states constructed à la Klauder-Perelomov. Note that, for shortness reasons, we will avoid the presentation of computation techniques which are more or less similar to ones used in the derivation of Intelligent states for other quantum systems(see [7-11]) This letter is organized as follows: The section 2 is devoted to a brief review of the two body Calogero system in order the fix the notations used through this work. The third section concern the algebraic derivation of Calogero states minimizing the Robertson-Schrödinger uncertainty relation. The analytical solutions of the Intelligent states, for the quantum system under consideration, are given in section 4. Concluding remarks close this letter.

2 The two-body Calogero model

The one dimensional harmonic oscillator

\[ H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2}x^2 \]  

(1)
still integrable if we add a $x^{-2}$ potential:

$$H_{cal} = a^+ a^- + \frac{1}{2} + \frac{\eta^2}{x^2}$$

(2)

where

$$a^\pm = \frac{1}{\sqrt{2}}(x \pm \frac{d}{dx})$$

(3)

are the usual harmonic creation and annihilation operators. The normalized eigenfunctions of the Hamiltonian $H_{cal}$ are

$$\Psi_n(x) = (-)^n \sqrt{\frac{2n!}{\Gamma(n+e_0)}} L_{n+1}^{e_0-1}(x^2) \exp\left(-\frac{x^2}{2}\right)$$

(4)

where $L_{n}^{e_0-1}$ are the Laguerre polynomials and $e_0 = 1 + \sqrt{\left(\frac{1}{4} + 2\eta^2\right)}$. The eigenvalues are given by:

$$H_{cal}\Psi_n(x) = (2n + e_0)\Psi_n(x)$$

(5)

The waves functions $\Psi_n(x)$ form a basis in the Hilbert space $\mathcal{H}$ of square integrable functions on the half axis $0 < x < \infty$. The raising and lowering operators are defined by

$$A^\pm = \frac{1}{2}((a^\pm)^2 - \frac{\eta^2}{x^2})$$

(6)

and act on the eigenstates $|\Psi_n\rangle$ as follows

$$A^+|\Psi_n\rangle = \sqrt{(n+1)(n+1+e_0)} e^{-i\beta f(n+1)}|\Psi_{n+1}\rangle$$

(7)

and

$$A^-|\Psi_n\rangle = \sqrt{n(n+e_0)} e^{i\beta f(n)}|\Psi_{n-1}\rangle.$$ 

(8)

The functions $f(n)$ are assumed to satisfy the following condition

$$f(n) + f(n-1) + ... + f(1) = e_n$$

(9)

which gives $f(1) = e_0 + 2$ and $f(n \neq 1) = 2$. Notice that the value 2 is exactly the spacing between two successive energy levels. The role of real parameter $\beta$ will be clarified, hereafter, when we will discuss the temporal stability of the coherent states. Note also that the state $\Psi_n(x)$ can be obtained by applying the operator $(A^+)^n$ on the ground state $\Psi_0(x)$.

3 Robertson-Schrödinger intelligent states: Algebraic derivation

We define two hermitains operators $A$ and $B$, in terms of the operators $A^+$ and $A^-$, as follows

$$A = \frac{1}{\sqrt{2}}(A^+ + A^-)$$

(10)
and

\[ B = \frac{1}{\sqrt{2}}(A^+ - A^-) \]  

(11) which satisfy the commutation relation

\[ [A, B] = i[A^+, A^-] = iH_{\text{cat}} \]  

(12) where \( H_{\text{cat}} \) is the Hamiltonian of the Calogero system. It is well known that for two hermitian operators satisfying the non canonical commutation relations (12), the covariances \((\Delta A)^2\) and \((\Delta B)^2\) satisfy the Robertson-Schrodinger uncertainty relation

\[ (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}(\langle H \rangle^2 + \langle F \rangle^2) \]  

(13) where the mean value of the operator \( F \) is defined by

\[ \langle F \rangle = i((\Delta A^+)^2 - (\Delta A^-)^2) \]  

(14) by means of the variances of the operators \( A^+ \) and \( A^- \).

The so-called Robertson-Schrödinger intelligent states are obtained when the equality in the uncertainty relation (13) are realized [12,13,14]. They satisfy the eigenvalues equation [14]

\[ ((1 - \lambda)A^+ + (1 + \lambda)A^-)|z, \beta, \lambda\rangle = 2z|z, \beta, \lambda\rangle \]  

(15) For the states solutions of the latter equation , the variances of \( A \) and \( B \) are

\[ (\Delta A)^2 = \frac{|\lambda|}{2} \sqrt{\langle H \rangle^2 + \langle F \rangle^2}, \]  

(16) and

\[ (\Delta B)^2 = \frac{1}{2|\lambda|} \sqrt{\langle H \rangle^2 + \langle F \rangle^2}. \]  

(17) Clearly, for \(|\lambda| = 1\), we have \((\Delta A)^2 = (\Delta B)^2\) and the states are called, in this case, the generalized coherent states. Expanding the states \(|z, \beta, \lambda\rangle\) in the \(|\Psi_n\rangle\) basis and using equations (7) and (8), one obtain the solutions of (15) for \( \lambda \neq -1 \). In compact form, they are given by (up to the normalization constant)

\[ |z, \beta, \lambda\rangle = U(z, \lambda)|\Psi_0\rangle \]  

(18) as the action of the operator

\[ U(z, \lambda) = \sum_{n=0}^{\infty} \left[ \left( \frac{2z}{1 + \lambda} \right) \frac{1}{A^+ A^-} A^+ \left( \frac{1 - \lambda}{1 + \lambda} \right) \frac{1}{A^+ A^-} (A^+) \right]^n \]  

(19) on the ground state \(|\Psi_0\rangle\) of the Calogero quantum system. Note that we have used similar computation techniques to ones used in obtaining the intelligent states for other exactly solvable quantum mechanical systems like for instance the Pöschl-Teller system [11]. The states (Eq.(18)), obtained by minimizing the Robertson-Schrödinger uncertainty relation, generalize the so-called Barut-Girardello coherent states. The latter, defined as the eigenvectors of the lowering operator \( A^- \), can be obtained by simply setting \( \lambda = 1 \) in (18) and (19). Their explicit expressions, in the \{|\Psi_n\rangle\} basis, and their properties will be given in the next section.
4 Robertson-Schrödinger Intelligent states: Analytical representations

Using the analytical representations of the Barut-Girardello and Klauder-Peremolov coherent states, we will construct the analytical representations of the Robertson-Schrödinger intelligent states. As we will see, the use of the analytical representation leads easily to the solutions of the eigenvalue equation (15). We start by giving the generalized intelligent states in the Barut-Girardello scheme.

4.1 Barut-Girardello analytic representation

The Barut-Girardello coherent states of two-body Calogero system are defined as eigenstates of the lowering operator $A^{-}$

$$A^{-}|z, \beta\rangle = z|z, \beta\rangle$$ (20)

Expanding the states $|z, \beta\rangle$ in the basis $\{|\Psi_n\rangle\}$ and using the action of the operator $A^{-}$, one obtain

$$|z\rangle = N(|z|) \sum_{n=0}^{\infty} \frac{z^n e^{-i\beta \epsilon_n}}{\sqrt{\Gamma(n+1)\Gamma(n+1+\epsilon_0)}}|\Psi_n\rangle$$ (21)

where the normalization constant $N(|z|)$ takes the form

$$N^2(|z|) = \frac{|z|^{\epsilon_0}}{I_0(2|z|)}$$ (22)

and $I_0(2|z|)$ is the modified Bessel function. Remark that states (18) coincides with ones expressed by (21) for $\lambda = 1$. The two-body Calogero system coherent states solve the unity

$$\int |z, \beta\rangle\langle z, \beta| = I_H$$ (23)

with respect the measure (computed using the inverse Mellin transform [16])

$$d\mu(z) = \frac{2}{\pi} I_0(2r)K_0(2r)rdrd\phi$$ (24)

where $z = re^{i\phi}$. The coherent states (22) are stable

$$e^{-iH_{cal}}|z, \beta\rangle = |z, \beta + t\rangle$$ (25)

This property is ensured by the presence of the phase factor $\beta$ in the definition of the actions of creation and annihilation operators (eqs. (7) and (8)). Due to the completion property (23), any state $|f\rangle$, of the Hilbert space, can be represented by an entire function

$$f(z, \beta) = \sqrt{\frac{I_0(2|z|)}{|z|^{\epsilon_0}}} \langle \bar{z}, \beta|f\rangle$$ (26)
In particular, the analytical representations of the vectors $|\Psi_n\rangle$ are

$$\mathcal{F}_n(z, \beta) = \frac{z^n e^{-i\beta \epsilon_n}}{\sqrt{\Gamma(n+1)\Gamma(n+1+\epsilon_0)}}$$  \hspace{1cm} (27)

The Ladder operators $A^+$ and $A^-$ are realized, in this representation, by

$$A^+ = z \hspace{1cm} (28)$$

$$A^- = z \frac{d^2}{dz^2} + e_0 \frac{d}{dz}$$  \hspace{1cm} (29)

This differential realization is an useful tool to determine the Roberton-Schrödinger intelligent states of the system under consideration. Indeed, introducing the analytic function

$$\Phi(z', \lambda)(z) = \sqrt{\frac{I_{\epsilon_0}(2|z|)}{|z|^\epsilon_0}} \langle \bar{z}, \beta|z', \lambda, \beta \rangle,$$  \hspace{1cm} (30)

the eigenvalues equation(15) is converted into the following second order differential equation

$$[(1 + \lambda)(z \frac{d^2}{dz^2} + e_0 \frac{d}{dz}) + (1 - \lambda)z - 2z']\Phi(z', \lambda)(z) = 0.$$  \hspace{1cm} (31)

For $\lambda \neq \pm 1$, the admissible solutions are given by

$$\Phi(z', \lambda)(z) = \exp \left( \pm \frac{\lambda - 1}{\lambda + 1}z \right) {}_1F_1 \left( \frac{e_0}{2} \pm z', e_0, \mp \sqrt{\frac{\lambda - 1}{\lambda + 1}z} \right)$$  \hspace{1cm} (32)

The upper and lower signs in Eq.(32) are equivalent because the confluent hyper-geometric function ${}_1F_1$ can be written in two equivalent forms thanks to Kummer’s transformation [16]. Due to the analytical properties of the hypergeometric functions, the squeezing parameter $\lambda$ satisfies the condition

$$Re(\lambda) > 0 \iff \frac{|\lambda - 1|}{|\lambda + 1|} < 1$$  \hspace{1cm} (33)

### 4.2 Klauder-Perelomov analytic representation

The set of Klauder-Perelomov coherent states is obtained from the lowest state $|\Psi_0\rangle$ as follows

$$|z, \beta\rangle = \exp(zA^+ - \bar{z}A^-)|\Psi_0\rangle = D(z)|\Psi_0\rangle$$  \hspace{1cm} (34)

which can be expanded as

$$|z, \beta\rangle = \sum_{n=0}^{\infty} \langle \Psi_n|D(z)|\Psi_0\rangle |\Psi_n\rangle$$  \hspace{1cm} (35)
To compute the matrix element occurring in the last equation, we used the actions of operators $A^+$ and $A^-$ on the states $|\Psi_n\rangle$ given by Eqs.(7) and (8). As result, we have

$$
\langle \Psi_n | D(z) | \Psi_0 \rangle = \sqrt{\frac{\Gamma(e_0 + n)}{\Gamma(e_0) \Gamma(n + 1)}} \frac{\kappa^n}{(1 + |\kappa|^2)^{e_0/2 + n/2}} e^{-i\beta e_n} \tag{36}
$$

where the variables $\kappa$ and $z$ are such that $\kappa = \frac{z \sinh(|z|)}{|z|}$. To get the identity resolution, it is more convenient to introduce the variable $\zeta = \kappa \frac{1 + |\kappa|^2}{2}^{1/2}$. The coherent states labeled now by the new variable $\zeta$ solve the identity

$$
\int d\mu(\zeta) |\zeta, \beta\rangle \langle \zeta, \beta| = \mathcal{I}_H \tag{37}
$$

where the measure is given by

$$
d\mu(\zeta) = \frac{e_0 - 1}{\pi} \frac{d^2 \zeta}{(1 - |\zeta|^2)^2} \tag{38}
$$

Then, any state $|\Psi_n\rangle$ is represented by the analytic function

$$
G_n(\zeta, \beta) = \zeta^n \frac{\Gamma(e_0 + n)}{\Gamma(e_0) \Gamma(n + 1)} e^{-i\beta e_n} \tag{39}
$$

The operators $A^+$ and $A^-$ act in the space of analytical functions as first-order differential operators

$$
A^+ = \zeta^2 \frac{d^2}{d\zeta^2} + e_0 \zeta \tag{40}
$$
$$
A^- = \frac{d}{d\zeta} \tag{41}
$$

Remark that the representations $\mathcal{F}_n(z, \beta)$ and $G_n(\zeta, \beta)$ are related via a transformation of Laplace type

$$
G_n(\zeta, \beta) = \frac{\zeta^{-e_0}}{\sqrt{\Gamma(e_0)}} \int_0^{+\infty} z^{e_0-1} \mathcal{F}_n(z, \beta) \exp(-z/\zeta) dz \tag{42}
$$

At this stage, we have the necessary tools to obtain the Robertson-Shrödinger states in the Klauder-Peremolov representation. Indeed, defining the following function

$$
\Phi(\zeta, \lambda, \lambda')(\zeta) = (1 - |\zeta|^2)^{-e_0/2} \langle \zeta', \beta | \zeta', \lambda, \beta \rangle, \tag{43}
$$

the eigenvalue equation (15) becomes

$$
\left[ (1 - \lambda) \zeta^2 + (1 + \lambda) \right] \frac{d}{d\zeta} + (1 + \lambda) e_0 \zeta - 2\zeta' \right] \Phi(\zeta, \lambda, \lambda')(\zeta) = 0. \tag{44}
$$

The functions $\Phi(\zeta, \lambda, \lambda')(\zeta)$, satisfying equation (44), should be analytic in the unit disk ($|\zeta| < 1$) [16] and are given by

$$
\Phi(\zeta, \lambda, \lambda')(\zeta) = \mathcal{N}^{-1/2} \left( 1 + \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^{-e_0/2 + \zeta'} / \lambda^{2-1} \left( 1 - \sqrt{\frac{\lambda - 1}{\lambda + 1}} \right)^{-e_0/2 - \zeta'} \tag{45}
$$
where \( \mathcal{N} \) is a normalization constant and \( \text{Re}(\lambda) > 0 \) (a condition imposed, here again, by the analytical properties of the obtained solutions). The states (32) and (45) are equivalents by virtue of Laplace transformation (42) relating Gazeau-Klauder and Klauder-Perelomov analytic representations.

5 Conclusion

In this short note, we gave the intelligent states which minimize Schrödinger-Robertson uncertainty relation for two body Calogero system. We constructed the Gazeau-Klauder coherent states, defined as the eigenstates of the lowering operator \( A^- \). We proved that they are normalizable, continuous, constitute an overcomplete set and are temporally stable. We introduced also the coherent states constructed following the group theoretical approach (Klauder-Perelomov method). It is shown that the states defined à la Klauder-Perelomov satisfy Gazeau-Klauder minimal set of conditions for generalized coherent states (i.e; noramlization, continuity in the labeling, resolution to identity and temporal stability). We established that the analytical representations of coherent states allow the derivation of intelligent states in an easy way. Other quantum systems, particularly those with continuous energy spectrum, are under investigation and we hope to report on them in the near future.

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