S-matrix calculus using effective particles in the Fock space

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This article describes a method for calculating S-matrix elements using Hamiltonians obtained in the renormalization group procedure for effective particles. It is shown that the scattering amplitudes obtained using a canonical Hamiltonian $H_A$ with counterterms are the same as those obtained using a renormalized Hamiltonian for effective particles, $H_0$. The result is independent of the ultraviolet cutoff $\Delta$ and the renormalization-group parameter $\lambda$.

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INTRODUCTION

The field-theoretical approach to strong interactions based on the renormalization group procedure for effective particles (RGPEP) has developed considerably in recent years. Most progress has been made in bound-state problems [1, 2, 3, 4, 5, 6] and the structure of effective theories [7, 8]. This article deals with scattering processes [9] in the language of effective (i.e., constituent) particles. The issue is important because hadrons are understood in terms of their constituents and a precise definition of the constituents in quantum field theory is required for further progress. In particular, such a definition should be applicable in perturbative scattering theory and reproduce known results.

This article presents a perturbative description of the scattering of single physical particles (not their bound states) using effective Hamiltonians. I address two specific questions relating to this description. The first question concerns the removal of divergences. The standard perturbative description of scattering uses Feynman diagrams [10], based on a formal derivation of expressions for the S matrix. In the derived expressions one introduces a regularization of each divergent loop [11] and constructs counterterms that remove divergences. Covariant regularization simplifies the task of finding counterterms because one can use symmetries to limit their structure. Earlier works by Yan [12] followed a similar path within the context of light-front Hamiltonian theory [13]. In recent works by Ligterink and Bakker [14] formal covariant Feynman-diagram expressions are rewritten in a systematic way in terms of equivalent light-front expressions. See also [15, 16] in the context of planar diagrams. RGPEP introduces systematically regularized and renormalized Hamiltonians that are also applicable in bound-state equations at a later stage. Hamiltonians are regulated from the very beginning and counterterm operators are found before one considers a scattering matrix. The resulting renormalized theory may therefore lead to non-divergent non-perturbative predictions. The rule for finding the counterterms in the initial canonical Hamiltonian is that matrix elements of the effective Hamiltonians should not depend on the ultraviolet cutoff $\Delta$, when $\Delta \to \infty$. Since this way of constructing counterterms does not refer to any S matrix directly, we must ask whether the counterterms found using RGPEP secure a divergence-free S matrix.

The second question concerns the use of creation and annihilation operators for effective particles. These operators depend on the RGPEP parameter $\lambda$, the inverse of the size of the effective particles. However, $\lambda$ is a parameter of a unitary rotation of the Fock-space basis and, as such, should not influence physical results. The second question is thus whether the S matrix calculated using the effective Hamiltonian $H_0$ is independent of $\lambda$.

RGPEP

In local quantum field theories, canonical Hamiltonians usually lead to divergent results. It is necessary to introduce an ultraviolet cutoff $\Delta$ and construct counterterms in the regularized canonical Hamiltonian $H_A$ that remove the dependence of physical results on the regularization. RGPEP is based on the observation that if the Hamiltonian can be re-written as $H_0$, in which every interaction vertex contains a form factor $f_\lambda$ of width $\lambda$, then the values of observables predicted by the Hamiltonian $H_0$ will not depend on $\Delta$ provided that there is no explicit $\Delta$-dependence in matrix elements of $H_0$ [17, 18].

In RGPEP a family of unitarily equivalent effective-particle operators $a_\lambda^\dagger$ is defined for each bare-particle operator $a_\infty^\dagger$ [8]:

$$a_\lambda^\dagger = U_\lambda a_\infty^\dagger U_\lambda^\dagger. \quad (1)$$

Bare particles of the initial canonical theory correspond to $\lambda = \infty$. The same Hamiltonian operator can be expressed in terms of either basis:

$$H = H_0(a_\lambda) = H_A(a_\infty), \quad (2)$$

but with different coefficients. Vertices in $H_0$ contain the form factor $f_\lambda$. RGPEP equations can be solved in perturbation theory, leading to expressions for $U_\lambda$ and $H_0$ and allowing the ultraviolet structure of the counterterms in $H_A$ to be determined.
S MATRIX IN TERMS OF $H^\Delta$

The expression for the S matrix in a regularized light-front Hamiltonian theory can be derived using a procedure similar to the textbook derivation of Feynman diagrams [19]. The key steps are summarized below in order to facilitate further discussion.

It is assumed that the matrix elements of full interacting fields $\phi_m(x)$ (being combinations of bare-particle creation and annihilation operators $a_m^\dagger$ and $a_m$) can be approximated in the distant past by similar matrix elements of free fields $\phi_m(x)$ (being combinations of physical-particle creation and annihilation operators $a_m^\dagger$ and $a_m$):

$$\lim_{x^+\to-\infty} \langle \beta | \phi_m(x') | \alpha \rangle = \sqrt{Z^2} \lim_{x^+\to-\infty} \langle \beta | \phi_m(x') | \alpha \rangle,$$

where $|\alpha\rangle$ and $|\beta\rangle$ represent normalized packets of well-separated particles. For massive particles, the above limits for the light-front time $x^+\to-\infty$ are equivalent to the limits $x^0\to-\infty$.

Using this asymptotic condition, S-matrix elements can be written for all $p_i \neq q_j$ as:

$$\langle \text{out} | p_1 \cdots p_{n_1} | q_1 \cdots q_{n_2} \rangle_{\text{in}} = \left( \frac{i}{\sqrt{2\pi}} \right)^{n_1+n_2} \prod_{i=1}^{n_1} \int d^4x_i \times \prod_{j=1}^{n_2} \int d^4y_j e^{-i\mathbf{q}_j \cdot \mathbf{x}_j} \int \mathcal{D}\phi_\alpha (y_1) \cdots \Phi_m(y_n) \Phi_m(x_1) \cdots \Phi_m(x_n) |0\rangle \left\langle \mathbf{q}_j + m^2 \right\rangle e^{ip_m y_j},$$

where $T_{(+)}$ denotes ordering in the light-front “time” $x^+$, $d^4x = dx^+ dx^- dx_3$ and $m$ is the physical mass of one particle. The momenta in the exponents on the right-hand side have energy components that fulfill the dispersion relation with the physical mass. This is the light-front analogue of the Lehmann-Symanzik-Zimmermann (LSZ) formula [20]. When any $p_i$ is equal to any $q_j$, there are additional forward-scattering terms.

Assuming unitary equivalence of the fields $\phi_m$ and $\phi_m$:

$$a_m \rightarrow a_m^\dagger U^{-1}(x) a_{\mathbf{k}}(x) U(x),$$

Eq. (4) can be expanded in a perturbative series in powers of the renormalized canonical interaction Hamiltonian,

$$H^\Lambda(x^+) := H^\Delta(a_0) - H_0(a_0),$$

$$H_0(a_0) := \int [k] \frac{m^2 + k^2}{k^+} a_{\mathbf{k}}^\dagger a_{\mathbf{k}},$$

namely:

$$\langle 0 | T_{(+)} | \phi_m(x_1) \cdots \phi_m(x_n) | 0 \rangle = \langle 0 | T_{(+)} \left[ \phi_m(x_1) \cdots \phi_m(x_n) \exp \left( -i \int_{-\infty}^{+\infty} H^\Lambda(x^+) \frac{1}{2} dx^+ \right) \right] | 0 \rangle.$$

The above steps apply in the presence of regulators in the light-front Hamiltonians.

S MATRIX IN TERMS OF $\tilde{H}_\Lambda$

Instead of the field $\phi_m(x)$ used above, we may introduce an effective field $\phi_\Lambda(x)$ to represent the same physical situation. $\phi_\Lambda(x)$ is defined using the effective-particle creation operators $a_\mathbf{k}^\dagger$ as the Fourier coefficients. The evolution of both $\phi_m(x)$ and $\phi_\Lambda(x)$ is determined by the same evolution operator $H$ of Eq. (2), but their matrix elements have different asymptotic behaviors. Instead of (3), for $\phi_\Lambda$ we have:

$$\lim_{x^+\to-\infty} \langle \beta | \phi_\Lambda(x') | \alpha \rangle = \lim_{x^+\to-\infty} \sqrt{Z^2} \langle \beta | \phi_m(x') | \alpha \rangle.$$

The derivation of the LSZ formula for the same physical S-matrix element can be repeated using $\phi_\Lambda(x)$, leading to:

$$\langle \text{out} | p_1 \cdots p_{n_1} | q_1 \cdots q_{n_2} \rangle_{\text{in}} = \left( \frac{i}{\sqrt{2\pi}} \right)^{n_1+n_2} \prod_{i=1}^{n_1} \int d^4x_i \times \prod_{j=1}^{n_2} \int d^4y_j e^{-i\mathbf{q}_j \cdot \mathbf{x}_j} \int \mathcal{D}\phi_\alpha (y_1) \cdots \phi_\Lambda(y_n) \phi_m(x_1) \cdots \phi_m(x_n) |0\rangle \left\langle \mathbf{q}_j + m^2 \right\rangle e^{ip_m y_j}.$$

Substituting Eq. (1) into (5), we get:

$$a_{\mathbf{k}}^\dagger(x^+) = W_\Lambda^{-1}(x^+) a_{\mathbf{k}}(x^+) W_\Lambda(x^+),$$

where

$$W_\Lambda(x^+) = U(x^+) U_\Lambda^\dagger(x^+),$$

and $U_\Lambda(0) = U_\Lambda$ from Eq. (1). Eq. (11) is an analogue of (5) for effective-particle operators $a_\mathbf{k}$.

We may now repeat the steps from Eqs. (5)-(8) using $W_\Lambda(x^+)$ instead of $U(x^+)$. This leads to a similar perturbative expansion of the S matrix:

$$\langle 0 | T_{(+)} | \phi_m(x_1) \cdots \phi_m(x_n) | 0 \rangle = \langle 0 | T_{(+)} \left[ \phi_m(x_1) \cdots \phi_m(x_n) \exp \left( -i \int_{-\infty}^{+\infty} \tilde{H}_\Lambda(x^+) \cdot \frac{1}{2} dx^+ \right) \right] | 0 \rangle,$$

but with an interaction Hamiltonian defined as follows:

$$\tilde{H}_\Lambda = H_\Lambda(a_0) - H_0(a_0).$$

The results can be summarized in the following theorem: The same S matrix describing the scattering of physical particles can be obtained using either:

(i) A bare Hamiltonian $H^\Delta$ and representing the incoming/outgoing particles by bare-particle creation and annihilation operators $a_m^\dagger$ and $a_m$, or

(ii) An effective Hamiltonian $\tilde{H}_\Lambda$ and creation and annihilation operators for effective particles, $a_\mathbf{k}^\dagger$ and $a_\mathbf{k}$.
In each order of perturbation theory the result for the S matrix is the same, provided that the unitary relation between $a_{ee}$ and $a_{ee}$ (i.e., between $H^\Delta$ and $H_0^\Delta$) is fulfilled up to this order.

For (i), the $S$ matrix is obtained using the LSZ formula (4) with wave-function renormalization factors $Z^\Delta$ and perturbative expansion (8) in powers of the bare interaction Hamiltonian $H_0^\Delta$. For (ii), the $S$ matrix is obtained using LSZ formula (10) with different wave-function renormalization factors $Z_h$ and a perturbative expansion (13) in powers of the effective interaction Hamiltonian $H_0^{\Delta_f}$.

**EXAMPLE: TREE AMPLITUDE IN A SCALAR MODEL**

In order to illustrate the meaning of the preceding discussion using the simplest possible example, I consider here a Hamiltonian describing the interaction of three bosonic fields in 1+1 space-time dimensions:

$$H = H_0 + H_e$$

$$H_0 = \int [k](m^2/k^+)(a_{k,e}^+ a_{k,e} + e k_e e_{k,e} + q_{k,e}^+ q_{k,e})$$

$$H_\lambda = \int [k]k^+ \epsilon(\lambda^+ + k^+ - k^+)(\epsilon - 2\xi_1 \xi_2 \xi_3 + 2\xi_1 \xi_2 k^+ + H.c.)$$

where $[k] = dk \theta(k^+)/4\pi k^+$ and $[a_k, a_{k'}^+] = 4\pi k^+ \delta(k^- - p^-)$. Since this model is not divergent, the bare Hamiltonian does not require regularization and renormalization, and the comparison between the bare and effective descriptions is simple. Nevertheless, the Hamiltonian contains vertices of a structure similar to those present in realistic quantum field theories.

The effective Hamiltonian $H_\Delta$ is calculated using the RGPEP differential equations [8]. In the zeroth order in powers of the charges $e_q$ or $e_c$: $H_0 = H_0$.

In the first order, the effective Hamiltonian $H_\lambda^{(e)}$ is simply the sum of all bare vertices with form factors $f_k$:

$$H_\lambda^{(e)} = f_k$$

$$H_\lambda = f_k$$

(with $q_{k,e}^+ q_{k,e}$ and $e_{k,e} e_{k,e}$ operator structure, using the same conventions as in Eq. (17)) plus their Hermitian conjugates.

In the second-order, the only part of the effective Hamiltonian that contributes to the calculation below is the one with $q_{k,e}^+ q_{k,e}$, represented by the diagram:

$$H_\lambda^{(e)} = f_{ac} f_{abc}^{(2)}$$

In this expression, $f_{ac} = \exp(-ac^2/k^2)$ and $f_{abc}^{(2)} = (P_{ba}^+ ba + P_{bc}^+ bc) / (ba^2 + bc^2) / (f_{ba}f_{bc} - 1)$. $a$, $b$, and $c$ mark the left-most, intermediate, and right-most configurations of particles in interactions; combinations of letters denote the differences of the squares of free invariant masses, e.g., $ab = M_2^2 - M_1^2$ (see [8] for details of this notation).

In the order $e^2$, $H^\Delta$ leads to the following $S$ matrix for $ee \rightarrow qq$ scattering:

$$H_\lambda^{(e)} = \frac{1}{P_0 - H_0 + i\epsilon}$$

The same result is found when the effective Hamiltonian $H_\lambda$ is used. There are two contributions: one is from $H_\lambda^{(e)}$ acting twice:

$$H_\lambda^{(e)} = \frac{1}{P_0 - H_0 + i\epsilon}$$

and the other is from $H_\lambda^{(e)}$. When $H_\lambda^{(e)}$ contributes to the $S$ matrix, due to the energy conservation $f_{abc} \equiv 1$ and $f_{abc}^{(2)}$ simplifies to:

$$f_{abc}^{(2)} = \frac{P_{ba}^+}{ba} f_{ab} - 1$$

Thus on the energy shell:

$$H_\lambda^{(e)}_{abc} = (1 - f_{abc}^2) \frac{1}{P_0 - H_0} H_\lambda^{(e)}$$

For momenta distant from the pole, (22) and the $f_{ab}^2$ term in (24) cancel and the remainder reproduces the $S$ matrix obtained using the bare Hamiltonian, (21). For momenta close to the pole, $ab$ goes to zero (and $f_{ab} \rightarrow 1$) and the whole contribution to the pole comes from (22); the result for the residue in the pole is the same as that calculated using the bare Hamiltonian (21). The analytic structures of the amplitudes obtained using $H^\Delta$ and $H_\lambda$ are thus also the same when $\epsilon \rightarrow 0$.

This example also demonstrates the key difference between the general off-shell Hamiltonian term, e.g., (20), and the corresponding on-shell $S$-matrix element, e.g., (24): where $H$ is given, the $S$ matrix can be calculated, while the $S$ matrix alone is not sufficient to define a corresponding operator $H$. This difference is particularly important for diverging theories, where perturbative $S$-matrix renormalization is of little help in constructing well-defined non-perturbative bound-state equations.

**CONSEQUENCES OF THE THEOREM**

The counterterms found using RGPEP lead to a divergence-free $S$ matrix. When we calculate the scattering amplitude using $H_\lambda$, the results do not depend on $\Delta$ in the corresponding order of perturbation theory, when $\Delta \rightarrow \infty$, owing to the form factors in the Hamiltonian’s interaction terms. The theorem states that both $H_\lambda$ and $H^\Delta$ produce the same $S$-matrix
elements. Thus the scattering amplitude can be obtained using the renormalized canonical Hamiltonian $H^\Delta$ and the RG-PEP counterterms in $H^\Delta$ do indeed lead to results which are not divergent. This observation is not trivial, as the counterterms were found from conditions not directly related to the $S$-matrix formalism and the ultraviolet cutoff dependence of the $S$ matrix could originate from several different sources. For example, the factor $Z^\lambda$, appearing both in the LSZ formula (4) and in the full propagator in realistic theories is a divergent function of $\Delta$; the theorem, however, implies that $\Delta$-dependence will cancel.

The $S$ matrix calculated using the effective Hamiltonian $H_\theta$ is independent of $\lambda$. If we calculate a scattering amplitude using $H^\Delta$, the result does not depend on $\lambda$, since there is no such parameter in $H^\Delta$. However, the same result can be obtained using $H_\theta$. This means that the effective Hamiltonian $H_\theta$ leads to $\lambda$-independent results for the $S$ matrix in a given order of perturbation theory in an appropriately defined coupling constant $\epsilon_3$. The dependence of $\epsilon_3$ on $\lambda$ is calculated to the same order. Again, without the theorem this is not obvious: the wave-function renormalization factors for effective particles, $Z_{\lambda_i}$, depend on $\lambda$ and there are many terms in the effective Hamiltonian that do not appear in the canonical Hamiltonian.

**CONCLUSIONS**

In this article I have examined the applicability of RGPEP to $S$-matrix calculations. I have shown that both a renormalized canonical Hamiltonian $H^\Delta$ and the corresponding effective Hamiltonian $H_\theta$ give the same $S$ matrix. In the calculation using $H^\Delta$, physical states of colliding particles are expressed in terms of bare particles and their interactions are contained in the renormalized canonical Hamiltonian $H^\Delta$. In the calculation using $H_\theta$, the physical states are expressed in terms of effective particles and the effective interaction Hamiltonian $H_{eff}$ with form factors in all vertices is used. Since it is known that the effective-particle approach applies in the case of bound states [1], I conclude that a single approach based on RGPEP is applicable in the description of both scattering and bound states of effective, constituent particles.

A corollary of the equivalence of the canonical and effective $S$-matrix calculus is that RGPEP counterterms remove divergences from the $S$ matrix. (How this takes place in detail, at least for low orders, is an area for future investigation.) However, RGPEP does not give the finite parts of the counterterms – these are constrained by the requirement of covariance [21, 22, 23] (see also [24, 25, 26]).

A second corollary of the theorem is that the $S$ matrix calculated using $H_\theta$ is independent of $\lambda$. The example above exhibits this feature in a straightforward calculation, since the overall form factor $f_{ac}$ is equal to 1 (due to energy conservation). In higher orders, the calculation is less simple: the corresponding form factor $f_{\lambda}$ is no longer equal to 1, as $f_{\lambda}$ is defined using free energies, whereas energy denominators and energy conservation in $S$-matrix calculations correspond to the physical masses of colliding particles.

I do not analyze here the conceptual and technical difficulties associated with the so-called small-$x$ singularities in gauge theories (see [27, 28, 29, 30]). However, the general theorem I present also holds where there are cutoffs on small-plus-momentum fractions. It is likely that the theorem is also valid in gauge theories, although this requires verification.

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