On \( H \)-irregularity strengths of \( G \)-amalgamation of graphs

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Abstract

A simple graph \( G = (V(G), E(G)) \) admits an \( H \)-covering if every edge in \( E(G) \) belongs at least to one subgraph of \( G \) isomorphic to a given graph \( H \). Then the graph \( G \) admitting \( H \)-covering admits an \( H \)-irregular total \( k \)-labeling \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) if for every two different subgraphs \( H' \) and \( H'' \) isomorphic to \( H \) there is \( wt_f(H') \neq wt_f(H'') \), where \( wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) \) is the associated \( H \)-weight. The minimum \( k \) for which the graph \( G \) has an \( H \)-irregular total \( k \)-labeling is called the \textit{total \( H \)-irregularity strength} of the graph \( G \).

In this paper, we obtain the precise value of the total \( H \)-irregularity strength of \( G \)-amalgamation of graphs.

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1. Introduction

All graphs we consider are simple and finite. For a given graph \( G \) denote \( V(G), E(G), \Delta(G) \) and \( \delta(G) \) as its sets of vertices and edges, the maximum and minimum degree, respectively.

In [12], Chartrand et al. introduced labelings of the edges of a graph \( G \) with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices.
Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph $G$ is known as the minimum $k$ for which $G$ has an irregular assignment using labels at most $k$. The exact value of $s(G)$ is known only for some special classes of graphs, e.g. complete graphs [12], graphs with the components being paths and cycles [4, 18], or some families of trees [5]. The lower bound on the $s(G)$ is given by the inequality

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + 1 - i}{i},$$

where $n_i$ denotes the number of vertices of degree $i$. In the case of $d$-regular graphs of order $n$ it reduces to

$$s(G) \geq \frac{n + d - 1}{d}.$$

The conjecture stated in [12] says that the value of $s(G)$ is for every graph equal to the above lower bound plus some constant not depending on $G$. The best bound of this form is currently due to Kalkowski, Karonski and Pfender. Namely, the authors in [17] have proved that $s(G) \leq 6 \left\lceil n/\delta \right\rceil < 6n/\delta + 6$. Currently Majerski and Przybyło [19] proved that $s(G) \leq (4 + o(1))n/\delta + 4$ for graphs with minimum degree $\delta \geq \sqrt{n} \ln n$.

For a given vertex labeling $h : V(G) \to \{1, 2, \ldots, k\}$ the associated weight of an edge $xy \in E(G)$ is $w_h(xy) = h(x) + h(y)$. Such a labeling $h$ is called *edge irregular* if for every two different edges $xy$ and $x'y'$ there is $w_h(xy) \neq w_h(x'y')$. The minimum $k$ for which the graph $G$ has an edge irregular $k$-labeling is called the *edge irregularity strength* of $G$ and denoted by $es(G)$. The notion of the edge irregularity strength was defined by Ahmad et al. in [1]. There is estimated the lower bound as follows

$$es(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G) \right\}.$$  \hspace{1cm} (1)

In [1] are determined the exact values of the edge irregularity strength for paths, stars, double stars and for Cartesian product of two paths.

For a given total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated total vertex-weight of a vertex $x$ is

$$wt_f(x) = f(x) + \sum_{xy \in E(G)} f(xy)$$

and the associated total edge-weight of an edge $xy$ is

$$wt_f(xy) = f(x) + f(xy) + f(y).$$

In [9] a total $k$-labeling $f$ is defined to be an *edge* (respectively, *vertex*) *irregular total $k$-labeling* of the graph $G$ if for every two distinct edges $xy$ and $x'y'$ respectively, distinct vertices $x$ and $y$ of $G$ there is $wt_f(xy) \neq wt_f(x'y')$ (respectively, $wt_f(x) \neq wt_f(y)$).

The minimum $k$ for which the graph $G$ has an edge (respectively, vertex) irregular total $k$-labeling is called the *total edge* (respectively, *vertex*) *irregularity strength* of the graph $G$ and denoted by $tes(G)$ (respectively, $tvs(G)$).

The following lower bound on the total edge irregularity strength of a graph $G$ is given in [9].

$$tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$  \hspace{1cm} (2)
Ivančo and Jendrůf [14] posed a conjecture that for arbitrary graph $G$ different from $K_5$ the total edge irregularity strength equals to the lower bound (2). This conjecture has been verified for complete graphs and complete bipartite graphs in [15, 16], for the categorical product of two cycles and two paths in [3, 2], for generalized Petersen graphs in [13], for generalized prisms in [10], for corona product of a path with certain graphs in [22] and for large dense graphs with $\left(\left|E(G)\right| + 2\right)/3 \leq (\Delta(G) + 1)/2$ in [11].

The bounds for the total vertex irregularity strength are given in [9] as follows.

$$\left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1. \quad (3)$$

Przybyło [23] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for $r$-regular graphs. This was then improved by Anholcer et. al in [6] by the following way

$$\text{tvs}(G) \leq 3 \left\lceil \frac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4. \quad (4)$$

Recently Majerski and Przybyło in [20] based on a random ordering of the vertices proved that if $\delta(G) \geq \left(|V(G)|\right)^{0.5} \ln |V(G)|$, then

$$\text{tvs}(G) \leq \frac{(2 + o(1))|V(G)|}{\delta(G)} + 4. \quad (5)$$

Combining previous modifications of the irregularity strength, Marzuki, Salman and Miller [21] introduced a new irregular total $k$-labeling of a graph $G$ called totally irregular total $k$-labeling, which is required to be at the same time vertex irregular total and also edge irregular total. The minimum $k$ for which a graph $G$ has a totally irregular total $k$-labeling is called the total irregularity strength of $G$ and is denoted by $\text{ts}(G)$. In [21] there are given upper and lower bounds for the parameter $\text{ts}(G)$. Ramdani and Salman in [24] determined the exact values of the total irregularity strength for several Cartesian product graphs.

Motivated by the irregularity strength and the total edge (respectively, vertex) irregularity strength of a graph $G$, Ashraf et al. in [7, 8] introduced new parameters, total (respectively, edge and vertex) $H$-irregularity strengths, as a natural extension of the parameters $s(G)$, $es(G)$, $tes(G)$ and $\text{tvs}(G)$.

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering. Note, that in this case all subgraphs of $G$ isomorphic to $H$ must be in the $H$-covering.

Let $G$ be a graph admitting $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $f$, we define the associated $H$-weight as

$$\text{wt}_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

A total $k$-labeling $f$ is called to be an $H$-irregular total $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $\text{wt}_f(H') \neq \text{wt}_f(H'')$. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has

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an $H$-irregular total $k$-labeling. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength, that is $\text{ths}(G, K_2) = \text{tes}(G)$.

For the subgraph $H \subseteq G$ under the edge labeling $g : E(G) \to \{1, 2, \ldots, k\}$ (respectively, the vertex labeling $h : V(G) \to \{1, 2, \ldots, k\}$) the associated $H$-weight is $wt_g(H) = \sum_{e \in E(H)} g(e)$ (respectively, $wt_h(H) = \sum_{v \in V(H)} h(v)$).

Such edge labeling $g$ (respectively, vertex labeling $h$) is called to be an $H$-irregular edge (respectively, vertex) $k$-labeling of the graph $G$ if for every two different subgraphs $H'$ and $H''$ isomorphic to $H$ there is $wt_g(H') \neq wt_g(H'')$ (respectively, $wt_h(H') \neq wt_h(H'')$). The edge (respectively, vertex) $H$-irregularity strength of a graph $G$, denoted by $\text{ehs}(G, H)$ (respectively, $\text{vhs}(G, H)$), is the smallest integer $k$ such that $G$ has an $H$-irregular edge (respectively, vertex) $k$-labeling.

Note, that $\text{vhs}(G, H) = \infty$ if there exist two subgraphs in $G$ isomorphic to $H$ that have the same vertex sets.

Let $G_i$, $i = 1, 2, \ldots, n$, be finite graphs containing a graph $G$ as a subgraph. The graph $G$ we will call a connector. The $G$-amalgamation of graphs $G_1, G_2, \ldots, G_n$ denoted by $\text{Amal}(G_i, G)$ is a graph obtained by taking all $G_i$’s and identifying their connectors $G$. If all graphs $G_i$, $i = 1, 2, \ldots, n$, are isomorphic to a given graph $G$ we will use the notation $\text{Amal}(G, G, n)$. Note that if $G = K_1$ then this operation is known as a vertex-amalgamation and if $G = K_2$ then it is called an edge-amalgamation.

In this paper we will study the total (respectively, edge and vertex) $G$-irregularity strengths of the graph $\text{Amal}(G, G, n)$ when $\text{Amal}(G, G, n)$ contains exactly $n$ subgraphs isomorphic to $G$ and we prove that the exact values of the total (respectively, edge and vertex) $G$-irregularity strengths of the investigated family of graphs equals to the lower bounds.

2. Lower bounds

Let $G$ be a graph admitting $H$-covering. Let $H^S_m = (H_1^S, H_2^S, \ldots, H_m^S)$ be the set of all subgraphs of $G$ isomorphic to $H$ such that the graph $S$, $S \not\cong H$, is their maximum common subgraph. Thus $V(S) \subset V(H_i^S)$ and $E(S) \subset E(H_i^S)$ for every $i = 1, 2, \ldots, m$. In [8] was given the lower bound of the total $H$-irregularity strength if the subgraphs isomorphic to $H$ share some elements.

**Theorem 2.1.** [8] Let $G$ be a graph admitting an $H$-covering. Let $S_i$, $i = 1, 2, \ldots, z$, be all subgraphs of $G$ such that $S_i$ is a maximum common subgraph of $m_i$, $m_i \geq 2$, subgraphs of $G$ isomorphic to $H$. Then

$$\text{ths}(G, H) \geq \max \left\{ \left[ 1 + \frac{m_1 - 1}{|V(H/S_1)| + |E(H/S_1)|} \right], \ldots, \left[ 1 + \frac{m_z - 1}{|V(H/S_z)| + |E(H/S_z)|} \right] \right\}.$$

Next theorem proved in [7] gives the lower bound of the edge (vertex) $H$-irregularity strength of a graph.
**Theorem 2.2.** [7] Let \( G \) be a graph admitting an \( H \)-covering. Let \( S_i, i = 1, 2, \ldots, z \), be all subgraphs of \( G \) such that \( S_i \) is a maximum common subgraph of \( m_i, m_i \geq 2 \), subgraphs of \( G \) isomorphic to \( H \). Then

\[
\text{ehs}(G, H) \geq \max \left\{ 1 + \frac{m_1 - 1}{|E(H/S_1)|}, 1 + \frac{m_2 - 1}{|E(H/S_2)|}, \ldots, 1 + \frac{m_z - 1}{|E(H/S_z)|} \right\},
\]

\[
\text{vhs}(G, H) \geq \max \left\{ 1 + \frac{m_1 - 1}{|V(H/S_1)|}, 1 + \frac{m_2 - 1}{|V(H/S_2)|}, \ldots, 1 + \frac{m_z - 1}{|V(H/S_z)|} \right\}.
\]

Immediately form previous theorems we obtain the following result.

**Theorem 2.3.** If the graph \( \text{Amal}(G, \mathcal{G}, n) \) contains exactly \( n \) subgraphs isomorphic to \( G \) then

\[
\text{ths}(\text{Amal}(G, \mathcal{G}, n), G) \geq 1 + \left[ \frac{n-1}{|V(G)|+|E(G)|-|V(G)|-|E(G)|} \right],
\]

\[
\text{ehs}(\text{Amal}(G, \mathcal{G}, n), G) \geq 1 + \left[ \frac{n-1}{|E(G)|-|E(G)|} \right],
\]

\[
\text{vhs}(\text{Amal}(G, \mathcal{G}, n), G) \geq 1 + \left[ \frac{n-1}{|V(G)|-|V(G)|} \right].
\]

### 3. Upper bounds

In this section we prove that the lower bounds presented in Theorem 2.3 are also the upper bounds. First we prove the corresponding result for the total \( G \)-irregularity strength of the graph \( \text{Amal}(G, \mathcal{G}, n) \).

**Theorem 3.1.** If the graph \( \text{Amal}(G, \mathcal{G}, n) \) contains exactly \( n \) subgraphs isomorphic to \( G \) then

\[
\text{ths}(\text{Amal}(G, \mathcal{G}, n), G) = 1 + \left[ \frac{n-1}{|V(G)|+|E(G)|-|V(G)|-|E(G)|} \right].
\]

**Proof.** Let the graph \( \text{Amal}(G, \mathcal{G}, n) \) contains exactly \( n \) subgraphs isomorphic to \( G \). Let us denote by the symbols \( x_i^j, i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, s \), where \( s = |V(G)|+|E(G)|-|V(G)|-|E(G)| \), the elements (vertices and edges) of the graph \( \text{Amal}(G, \mathcal{G}, n) \) from the \( i \)th copy \( G^i \) that are not elements of the connector \( \mathcal{G} \).

We define a total labeling \( f \) of \( \text{Amal}(G, \mathcal{G}, n) \) such that

\[
f(x) = 1 \quad \text{if} \quad x \in V(\mathcal{G}) \cup E(\mathcal{G}),
\]

\[
f(x_i^j) = \begin{cases} 
\frac{i-1}{s} + 1 & \text{if} \quad i \equiv 1 \pmod{s}, \ 1 \leq i \leq n, \ j = 1, 2, \ldots, s, \\
\frac{i-1}{s} + 2 & \text{if} \quad i \not\equiv 1 \pmod{s}, \ 2 \leq i \leq n, \ j = 1, 2, \ldots, i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1, \\
\frac{i-1}{s} + 1 & \text{if} \quad i \not\equiv 1 \pmod{s}, \ 2 \leq i \leq n, \ j = i - \left\lfloor \frac{i-1}{s} \right\rfloor s, i - \left\lfloor \frac{i-1}{s} \right\rfloor s + 1, \ldots, s.
\end{cases}
\]

If \( n \equiv 1 \pmod{s} \) then the maximal used label is

\[
\left\lfloor \frac{n-1}{s} \right\rfloor + 1 = \left\lfloor \frac{n-1}{s} \right\rfloor + 1.
\]

If \( n \not\equiv 1 \pmod{s} \) then the maximal used label is

\[
\left\lfloor \frac{n-1}{s} \right\rfloor + 2 = \left( \left\lfloor \frac{n-1}{s} \right\rfloor - 1 \right) + 2 = \left\lfloor \frac{n-1}{s} \right\rfloor + 1.
\]
Thus $f$ is $(\left\lceil (n - 1)/s \right\rceil + 1)$-labeling.

In the light of Theorem 2.3 it suffices to prove that the $G$-weights are distinct.

For the weights of graphs $G^i$, $i = 1, 2, \ldots, n$, we get the following

$$wt_f(G^i) = \sum_{x \in V(G^i) \cup E(G^i)} f(x) = \sum_{x \in V(G) \cup E(G)} f(x) + \sum_{j=1}^s f(x^i_j) = \sum_{x \in V(G) \cup E(G)} 1 + \sum_{j=1}^s f(x^i_j).$$

If $i \equiv 1 \pmod s$ then

$$wt_f(G^i) = |V(G)| + |E(G)| + \sum_{j=1}^s \left( \frac{i-1}{s} + 1 \right) = |V(G)| + |E(G)| + \left( \frac{i-1}{s} + 1 \right) s$$

$$= |V(G)| + |E(G)| + s - 1 + i.$$

For $i \not\equiv 1 \pmod s$ we get

$$wt_f(G^i) = |V(G)| + |E(G)| + \sum_{j=1}^s f(x^i_j) + \sum_{j=i-\left\lfloor \frac{i-1}{s} \right\rfloor}^s f(x^i_j)$$

$$= |V(G)| + |E(G)| + \sum_{j=1}^{i-1} \left( \left\lfloor \frac{i-1}{s} \right\rfloor + 2 \right) + \sum_{j=i-\left\lfloor \frac{i-1}{s} \right\rfloor}^s \left( \left\lfloor \frac{i-1}{s} \right\rfloor + 1 \right)$$

$$= |V(G)| + |E(G)| + \left( i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1 \right) \left( \left\lfloor \frac{i-1}{s} \right\rfloor + 2 \right)$$

$$+ \left( s - i + \left\lfloor \frac{i-1}{s} \right\rfloor s + 1 \right) \left( \left\lfloor \frac{i-1}{s} \right\rfloor + 1 \right)$$

$$= |V(G)| + |E(G)| + s \left\lfloor \frac{i-1}{s} \right\rfloor + 2 \left( i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1 \right) + \left( s - i + \left\lfloor \frac{i-1}{s} \right\rfloor s + 1 \right)$$

$$= |V(G)| + |E(G)| + s \left\lfloor \frac{i-1}{s} \right\rfloor + s + i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1$$

$$= |V(G)| + |E(G)| + s - 1 + i.$$

Combining the previous facts we get that for every $i, i = 1, 2, \ldots, n$, holds

$$wt_f(G^i) = |V(G)| + |E(G)| + s - 1 + i.$$

Thus the set of $G$-weights consists of consecutive integers.

For the edge $G$-irregularity strengths of the graph $Amal(G, \mathcal{G}, n)$ we get the following result.
Theorem 3.2. If the graph $Amal(G, \mathcal{G}, n)$ contains exactly $n$ subgraphs isomorphic to $G$ then

$$ehs(Amal(G, \mathcal{G}, n), G) = 1 + \left\lfloor \frac{n-1}{|E(G)| - |E(\mathcal{G})|} \right\rfloor.$$ 

Proof. Let the graph $Amal(G, \mathcal{G}, n)$ contains exactly $n$ subgraphs isomorphic to $G$. Let us denote by the symbols $e^i_j, i = 1, 2, \ldots, n, j = 1, 2, \ldots, s$, where $s = |E(G)| - |E(\mathcal{G})|$, the edges of the graph $Amal(G, \mathcal{G}, n)$ from the $i$th copy $G^i$ that are not edges of the connector $\mathcal{G}$.

We define an edge labeling $g$ of $Amal(G, \mathcal{G}, n)$ such that

$$g(e) = 1 \quad \text{if} \ e \in E(\mathcal{G}),$$

$$g(e^i_j) = \begin{cases} \frac{i-1}{s} + 1 & \text{if} \ i \equiv 1 \pmod{s}, 1 \leq i \leq n, j = 1, 2, \ldots, s, \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 2 & \text{if} \ i \not\equiv 1 \pmod{s}, 2 \leq i \leq n, j = 1, 2, \ldots, i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1, \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 1 & \text{if} \ i \not\equiv 1 \pmod{s}, 2 \leq i \leq n, j = i - \left\lfloor \frac{i-1}{s} \right\rfloor s, i - \left\lfloor \frac{i-1}{s} \right\rfloor s + 1, \ldots, s. \end{cases}$$

Using similar arguments as in the proof of Theorem 3.1 we prove that under the edge labeling $g$ the induced $G$-weights are distinct. \qed

For the vertex version we have

Theorem 3.3. If the graph $Amal(G, \mathcal{G}, n), |V(G)| \neq |V(\mathcal{G})|$, contains exactly $n$ subgraphs isomorphic to $G$ then

$$vhs(Amal(G, \mathcal{G}, n), G) = 1 + \left\lfloor \frac{n-1}{|V(G)| - |V(\mathcal{G})|} \right\rfloor.$$ 

Proof. Let the graph $Amal(G, \mathcal{G}, n), |V(G)| \neq |V(\mathcal{G})|$, contains exactly $n$ subgraphs isomorphic to $G$. Let us denote by the symbols $v^i_j, i = 1, 2, \ldots, n, j = 1, 2, \ldots, s$, where $s = |V(G)| - |V(\mathcal{G})|$, the vertices of the graph $Amal(G, \mathcal{G}, n)$ from the $i$th copy $G^i$ that are not vertices of the connector $\mathcal{G}$.

It is easy to see that the vertex labeling $h$ of $Amal(G, \mathcal{G}, n)$ defined below has the desired properties.

$$h(v) = 1 \quad \text{if} \ v \in V(\mathcal{G}),$$

$$h(v^i_j) = \begin{cases} \frac{i-1}{s} + 1 & \text{if} \ i \equiv 1 \pmod{s}, 1 \leq i \leq n, j = 1, 2, \ldots, s, \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 2 & \text{if} \ i \not\equiv 1 \pmod{s}, 2 \leq i \leq n, j = 1, 2, \ldots, i - \left\lfloor \frac{i-1}{s} \right\rfloor s - 1, \\ \left\lfloor \frac{i-1}{s} \right\rfloor + 1 & \text{if} \ i \not\equiv 1 \pmod{s}, 2 \leq i \leq n, j = i - \left\lfloor \frac{i-1}{s} \right\rfloor s, i - \left\lfloor \frac{i-1}{s} \right\rfloor s + 1, \ldots, s. \end{cases}$$

Immediately from Theorem 3.1 we obtain the result for the total edge irregularity strength of a star $K_{1,n}$, that was proved in [9].

Corollary 3.1. [9] Let $n$ be a positive integer, then

$$tes(K_{1,n}) = 1 + \left\lfloor \frac{n-1}{2} \right\rfloor.$$
Proof. Let \( n \) be a positive integer, then

\[
\text{ths}(\text{Amal}(K_2, K_1, n), K_2) = \text{ths}(K_{1,n}, K_2) = \text{tes}(K_{1,n}) = 1 + \left\lceil \frac{n-1}{|V(K_2)|+|E(K_2)|-|V(K_1)|-|E(K_1)|} \right\rceil
\]

\[
= 1 + \left\lceil \frac{n-1}{2+1-1-0} \right\rceil = 1 + \left\lceil \frac{n-1}{2} \right\rceil.
\]

The friendship graph is a finite graph with the property that every two vertices have exactly one neighbor in common. Friendship graph \( f_n \) can be obtained as a collection of \( n \) triangles with a common vertex. The generalized friendship graph \( f_{m,n} \) is a collection of \( n \) cycles (all of order \( m \)), meeting at a common vertex. Immediately from the previous theorems we get the following.

**Corollary 3.2.** Let \( m, n \) be positive integers, \( m \geq 3 \) and \( n \geq 1 \). Then

\[
\text{ths}(f_{m,n}, C_m) = 1 + \left\lceil \frac{n-1}{2m-1} \right\rceil,
\]

\[
\text{ehs}(f_{m,n}, C_m) = 1 + \left\lceil \frac{n-1}{m} \right\rceil,
\]

\[
\text{vhs}(f_{m,n}, C_m) = 1 + \left\lceil \frac{n-1}{m-1} \right\rceil.
\]

4. **Conclusion**

In the paper we studied the total (respectively, edge and vertex) \( G \)-irregularity strengths of the graph \( \text{Amal}(G, G, n) \) when \( \text{Amal}(G, G, n) \) contains exactly \( n \) subgraphs isomorphic to \( G \). We estimated the lower bounds of the total (respectively, edge and vertex) \( G \)-irregularity strengths and proved that the exact values of these parameters for the amalgamation of graphs equal to the lower bounds.

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