FREQUENCY AND PHASE NOISE IN NON-LINEAR
MICROWAVE OSCILLATOR CIRCUITS

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Abstract

We have developed a new methodology and a time-domain software package for the estimation of the oscillation frequency and the phase noise spectrum of non-linear noisy microwave circuits based on the direct integration of the system of stochastic differential equations representing the circuit. Our theoretical evaluations can be used in order to make detailed comparisons with the experimental measurements of phase noise spectra in selected oscillating circuits.

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I. INTRODUCTION

Electronically tuned microwave oscillators are key components used in a wide variety of microwave communications systems [1]. The phase of the output signal exhibits fluctuations in time about the steady state oscillations giving rise to Phase Noise a very important characteristic that influences the overall performance especially at higher microwave frequencies.

In order to understand the oscillator phase behaviour, a statistical model for a non-linear oscillating circuit has to be developed and presently, no accurate theoretical model for phase noise characterization is available because of the particularly difficult nature of this problem. This is due to the hybrid nature of non-linear microwave oscillator circuits where distributed elements (pertaining usually to the associated feeding or resonator circuits) and non-linear elements (pertaining usually to the amplifying circuit) have to be dealt with simultaneously [1].

The main aim of this report is to establish a theoretical framework for dealing with the noise sources and non-linearities present in these oscillators, introduce a new methodology to calculate the resonance frequency and evaluate the time responses (waveforms) for various voltages and currents in the circuit without or with the noise present. Once this is established, the phase noise spectrum is determined and afterwards the validity range of the model is experimentally gauged with the use of different types of microwave oscillators [2, 3]. This report is organised in the following way: Section II covers the theoretical analysis for the oscillating circuit, reviews noise source models and earlier approaches. Section III presents results of the theoretical analysis and highlights the determination of the resonance frequency for some oscillator circuits without noise. In section IV, phase noise spectra are determined for several oscillator circuits and section V contains the experimental results. The Appendix contains circuit diagrams and corresponding state equations for several non-linear oscillator circuits.

II. THEORETICAL ANALYSIS

In standard microwave analysis, it is difficult to deal with distributed elements in the time domain and difficult to deal with non-linear elements in the frequency domain. Non-linear microwave oscillator circuits have simultaneously non-linear elements in the amplifying part
and distributed elements in the resonating part [Non-linearity is needed since it is well known that only non-linear circuits have stable oscillations].

Before we tackle, in detail, the determination of the phase noise, let us describe the standard procedure for dealing with the determination of resonance frequency of non-linear oscillator circuits:

- The first step is to develop a circuit model for the oscillator device and the tuning elements. The equivalent circuit should contain inherently noiseless elements and noise sources that can be added at will in various parts of the circuit. This separation is useful for pinpointing later on the precise noise source location and its origin. The resulting circuit is described by a set of coupled non-linear differential equations that have to be written in a way such that a linear sub-circuit (usually the resonating part) is coupled to another non-linear sub-circuit (usually the oscillating part).

- The determination of the periodic response of the non-linear circuit.

- The third step entails performing small signal ac analysis (linearization procedure) around the operating point. The result of the ac analysis is a system matrix which is ill-conditioned since a large discrepancy of frequencies are present simultaneously (One has a factor of one million in going from kHz to GHz frequencies). The eigenvalues of this matrix have to be calculated with extra care due to the sensitivity of the matrix elements to any numerical roundoff.

We differ from the above analysis, by integrating the state equations directly with standard/non-standard Runge-Kutta methods adapted to the non-stiff/stiff system of ordinary differential equations. The resonance frequency is evaluated directly from the waveforms and the noise is included at various points in the circuit as Johnson or Shot noise.

This allows us to deal exclusively with time domain methods for the noiseless/noisy non-linear elements as well as the distributed elements. The latter are dealt with through an equivalence to lumped elements at a particular frequency.

As far as point 3 is concerned, the linearization procedure method is valid only for small-signal analysis whereas in this situation, we are dealing with the large signal case.

Previously, several methods have been developed in order to find the periodic response. The most well established methods are the Harmonic balance and the piecewise Harmonic
balance methods \( \mathbf{6} \). Schwab \( \mathbf{4} \) has combined the time-domain (for the non-linear amplifier part) with the frequency domain (for the linear resonating part) methods and transformed the system of equations into a boundary value problem that yields the periodic response of the system.

### III. TIME RESPONSES OF NON-LINEAR OSCILLATORS AND RESONANCE FREQUENCY DETERMINATION

For illustration and validation of the method we solve 6 different oscillator circuits (The Appendix contains the circuit diagrams and the corresponding state equations):

- The standard Van der Pol oscillator.
- The amplitude controlled Van der Pol oscillator.
- The Clapp oscillator.
- The Colpitts oscillator.
- Model I oscillator.
- Model II oscillator.

We display the time responses (waveforms) for various voltages and currents in the attached figures for each of the six oscillators. All oscillators reach periodic steady state almost instantly except the amplitude controlled Van der Pol (ACVDP) and the Colpitts circuits. For instance, we need, typically, several thousand time steps to drive the ACVDP circuit into the oscillatory steady state whereas several hundred thousand steps are required for the Colpitts circuit. Typically, the rest of the circuits studied reached the periodic steady state in only less a couple of hundred steps.

### IV. PHASE NOISE SPECTRUM EVALUATION

Once the oscillating frequency is obtained, device noise is turned on and its effect on the oscillator phase noise is evaluated. All the above analysis is performed with time domain simulation techniques.
Finally, Fourier analysis is applied to the waveform obtained in order to extract the power spectrum as a function of frequency. Very long simulation times (on the order of several hundred thousand cycles) are needed since one expects inverse power-law dependencies on the frequency \(1\).

We use a special Stochastic time integration method namely the 2S-2O-2G Runge-Kutta method developed by Klauder and Peterson, and we calculate the PSD (Power Spectral Density) from the time series obtained.

It is worth mentioning that our methodology is valid for any type of oscillator circuit and for any type of noise (Additive White as it is in Johnson noise of resistors, Mutiplicative and Colored or \(1/f^{\alpha}\) with \(\alpha\) arbitrary as it is for Shot noise stemming from junctions or imperfections inside the device). In addition, the approach we develop is independent of the magnitude of the noise. Regardless of the noise intensity we evaluate the time response and later on the power spectrum without performing any perturbative development whatsoever. Recently, Kartner \(7\) developed a perturbative approach to evaluate the power spectrum without having to integrate the state equations. His approach is valid for weak noise only and is based on an analytical expression for the power spectrum. Nevertheless one needs to evaluate numerically one Fourier coefficient \(g_{1,0}\) the spectrum depends on.

V. EXPERIMENTAL VERIFICATION

Microwave oscillators are realised using a very wide variety of circuit configurations and resonators. We plan to design, fabricate and test microstrip oscillators with GaAs MESFET devices with coupled lines and ring resonators \(5\). The measured phase noise of these oscillators will be compared with the theoretical prediction from the above analysis. We also plan to apply the above analysis to the experimental phase results obtained from various electronically tuned oscillators that have been already published in the literature \(1,2,3,4\).

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two additional circuits (Model I and II) to test the software.

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APPENDIX

A. Van Der Pol oscillator

State-Space Equations of Van der Pol oscillator:
\[
\frac{di_L}{dt} = \frac{V_c}{L} \quad (1)
\]
\[
\frac{dV_c}{dt} = \frac{1}{C}(-\mu V_c i_L^2 - i_L + \mu V_c) \quad (2)
\]

Define:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
\text{Def.} = 
\begin{bmatrix}
i_L \\
V_c
\end{bmatrix} \quad (3)
\]

Rewrite Equations 1 and 2 in state-space form:

\[
\dot{X}_1 = X_2 \quad (4)
\]
\[
\dot{X}_2 = \mu X_2(1 - X_1^2) - X_1 \quad (5)
\]

B. Amplitude controlled Van Der Pol oscillator

State-Space equations of Amplitude controlled Van der Pol oscillator:

\[
\frac{dV_c}{dt} = -\frac{1}{C}(i_L + Gu - G_a(z)u - i_r) \quad (6)
\]
\[
\frac{di_L}{dt} = \frac{u}{L} \quad (7)
\]
\[
\frac{dz}{dt} = \frac{1}{T}[(\frac{u}{V_0})^2 - z] \quad (8)
\]

where \(G_a(z) = G(1 - c(z - 1))\) is the voltage controlled conductance.

Choosing the variables:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
\text{Def.} = 
\begin{bmatrix}
u \\
\frac{i_L}{C} \\
z
\end{bmatrix} \quad (9)
\]
with the definitions $Q_0 = \frac{C_{\text{in}}}{G}$ and $i_{rn} = \frac{i}{C_{\omega_0}}$, $\gamma = \frac{1}{T_\omega}$ where $\omega_0$ is the oscillation frequency, we have:

$$\dot{X}_1 = -\frac{cX_1(X_3 - 1) + X_2}{Q_0} + i_{rn} \quad (10)$$

$$\dot{X}_2 = Q_0X_1 \quad (11)$$

$$\dot{X}_3 = -\gamma(X_3 - X_1^2) \quad (12)$$

C. Clapp oscillator

The parameters used are:

|                  | $R_s = 0\Omega$ | $R_E = 500\Omega$ | $R_A = 1000\Omega$ | $R_L = 1000\Omega$ |
|------------------|-----------------|-------------------|-------------------|-------------------|
| $C_A = 100pF$    | $C_E = 100pF$   | $C_{TE} = 3.5nF$  | $C_{TA} = 7nF$    |
| $\text{In addition, we have: } C_s = 25pF$ and $L_s = 1\text{mH.}$ |

State-Space Equations of Clapp oscillator:

$$\frac{dV_{CE}}{dt} = \frac{1}{C_E}[i_e + \left(\frac{V_{CTE} - V_{CE}}{R_E}\right)] \quad (13)$$

$$\frac{dV_{CTE}}{dt} = \frac{1}{C_{TE}}[-i_p - \left(\frac{V_{CTE} - V_{CE}}{R_E}\right)] \quad (14)$$

$$\frac{dV_{CTA}}{dt} = \frac{1}{C_{TA}}[i_p - \left(\frac{V_{CTA} - V_{CA}}{R_A}\right)] \quad (15)$$

$$\frac{dV_{CA}}{dt} = \frac{1}{C_A}[i_q + \left(\frac{V_{CTA} - V_{CA}}{R_A}\right) - \left(\frac{V_{CA}}{R_L}\right)] \quad (16)$$

$$\frac{di_p}{dt} = j_p \quad (17)$$

whereas:

$$\frac{dj_p}{dt} = -\frac{R_s j_p}{L_s} + \left(\frac{V_{CE} - V_{CTE}}{L_s R_E C_{TE}}\right) + \left(\frac{V_{CTA} - V_{CA}}{L_s R_A C_{TA}}\right)$$
\[ -i_p \left( \frac{1}{L_s C_{TA}} + \frac{1}{L_s C_s} + \frac{1}{L_s C_{TE}} \right) \]  

(18)

where:

\[ i_e = A_1 (e^{-\frac{V_{CE}}{\alpha_2}} - 1) \]  

(19)

and:

\[ i_q = \begin{cases} 
-B_1 B_2 & -V_{CA} \leq -B_2 \text{ or } V_{CA} \geq B_2 \\
-B_1 V_{CA} & -B_2 < V_{CA} < B_2 \\
B_1 B_2 & -V_{CA} \geq B_2 \text{ or } V_{CA} \leq -B_2 
\end{cases} \]

D. Colpitts oscillator

\[
\begin{array}{|c|c|c|}
\hline
R_1 & 350\Omega & R_2 = 110k\Omega & R_L = 500\Omega \\
L_1 = 10mH & L_2 = 30nH & C_1 = 10pF \\
C_2 = 940pF & C_3 = 2.7nF & C_4 = 1.5nF \\
\hline
\end{array}
\]

For the transistor:

\[
\begin{array}{|c|c|c|}
\hline
I_s = 2.7 \times 10^{-16}A & \beta_I = 5.5 & \beta_N = 140 \\
U_A = 15V & U_B = 4.3V & V_0 = 12V \\
\hline
\end{array}
\]

State-Space Equations of Colpitts oscillator:

\[
\begin{align*}
\frac{di_1}{dt} &= \frac{1}{L_1} (V_0 - V_1 - R_1 i_1) \\
\frac{di_2}{dt} &= \frac{1}{L_2} (V_1 - V_2) \\
\frac{dV_1}{dt} &= \frac{1}{C_1} [i_1 - i_2 - i_C + (\frac{V_4 - V_1}{R_L})] \\
\frac{dV_2}{dt} &= \frac{1}{C_2} [i_2 - i_B + (\frac{V_0 - V_2 - V_3}{R_2})] \\
\frac{dV_3}{dt} &= \frac{1}{C_3} [-i_B + (\frac{V_0 - V_2 - V_3}{R_2})] \\
\frac{dV_4}{dt} &= (\frac{V_1 - V_4}{R_L C_4}) 
\end{align*}
\]
Where the transistor currents are given by:

\[ i_C = i_{CE} - i_{BC} \]  
\[ i_B = i_{BE} + i_{BC} \]  
\[ i_{CE} = \frac{I_s}{Q_{bo}} \left( e^{\frac{u_{BE}}{u_T}} - e^{\frac{u_{BC}}{u_T}} \right) \]  
\[ i_{BE} = \frac{I_s}{\beta_N} \left( e^{\frac{u_{BE}}{u_T}} - 1 \right) \]  
\[ i_{BC} = \frac{I_s}{\beta_I} \left( e^{\frac{u_{BC}}{u_T}} - 1 \right) \]

Moreover, we have the additional relations:

\[ u_{BE} = V_2 + V_3 \]  
\[ u_{BC} = V_2 + V_3 - V_1 \]  
\[ \frac{Q_b}{Q_{bo}} = 1 + \frac{u_{BE}}{u_b} + \frac{u_{BC}}{u_a} \]

\[ i_C = I_s \left( e^{\frac{V_2+V_3}{u_T}} - e^{\frac{V_2+V_3-V_1}{u_T}} \right) \left( 1 + \frac{V_2}{u_b} + \frac{V_3-V_1}{u_a} \right) \]  
\[ - \frac{I_s}{\beta_I} \left( e^{\frac{V_2+V_3-V_1}{u_T}} - 1 \right) \]

\[ i_B = \frac{I_s}{\beta_N} \left( e^{\frac{V_2+V_3}{u_T}} - 1 \right) + \frac{I_s}{\beta_I} \left( e^{\frac{V_2+V_3-V_1}{u_T}} - 1 \right) \]

Define:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6
\end{bmatrix}
\equiv
\begin{bmatrix}
\frac{D_{ef.}}{V_1} \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\]
Rewrite Equations in state-space form:

\[
\begin{align*}
\dot{X}_1 &= \frac{1}{L_1}(V_0 - X_3 - R_1 X_1) \\
\dot{X}_2 &= \frac{1}{L_2}(X_3 - X_4)
\end{align*}
\]

\[
\dot{X}_3 = \frac{1}{C_1}[X_1 - X_2 + \frac{1}{R_L}(X_6 - X_3) + \frac{I_s}{\beta_I}(e^{\frac{X_4 + X_5 - X_3}{u_T}} - 1) - \frac{I_s}{\beta_I}(e^{\frac{X_4 + X_5 + X_3}{u_T}} - 1)]
\]

\[
\dot{X}_4 = \frac{1}{C_2}[X_2 + \frac{1}{R_2}(V_0 - X_4 - X_5) - \frac{I_s}{\beta_N}(e^{\frac{X_4 + X_5}{u_T}} - 1) - \frac{I_s}{\beta_I}(e^{\frac{X_4 + X_5 - X_3}{u_T}} - 1)]
\]

\[
\dot{X}_5 = \frac{1}{C_3}[\frac{1}{R_2}(V_0 - X_4 - X_5) - \frac{I_s}{\beta_N}(e^{\frac{X_4 + X_5}{u_T}} - 1) - \frac{I_s}{\beta_I}(e^{\frac{X_4 + X_5 - X_3}{u_T}} - 1)]
\]

\[
\dot{X}_6 = \frac{1}{R_L C_4}(X_3 - X_6)
\]

E. Model I

| Table |\r|---|---|---|
| \(V_0\) | 9V | \(R_1\) | 220kΩ | \(R_2\) | 1000Ω |
| \(R_3\) | 220kΩ | \(R_4\) | 2Ω | \(L\) | 10μH |
| \(C_1\) | 0.47μF | \(C_2\) | 200pF | \(C_3\) | 200pF |

For the transistor:
\[
\begin{array}{|c|c|c|}
\hline
I_s &= 2.7 \times 10^{-16} A \\
U_A &= 15V \\
U_B &= 4.3V \\
\hline
\text{Gain} &= \beta_I = 5.5 \\
\beta_N &= 140 \\
\hline
\end{array}
\]

State-Space Equations of Model I oscillator:

\[
\frac{di_4}{dt} = \frac{1}{L}(V_6 + V_7 - R_4i_4 - V_5)
\]

\[
\frac{dV_5}{dt} = \frac{i_4}{C_1}
\]

\[
\frac{dV_6}{dt} = \frac{1}{C_2} \left[ \left( \frac{V_6 - V_6 - V_7}{R_1} \right) - \left( \frac{V_6 + V_7}{R_3} \right) \\
- i_4 - i_B \right]
\]

\[
\frac{dV_7}{dt} = \frac{1}{C_3} \left[ \left( \frac{V_6 - V_6 - V_7}{R_1} \right) - \left( \frac{V_6 + V_7}{R_3} \right) \\
- i_4 - i_B + \frac{V_7}{R_2} \right]
\]

The currents and transistor voltages \(u_{BE}\) and \(u_{BC}\) are given by:

\[
i_E = i_B + i_C
\]

\[
u_{BE} = V_6
\]

\[
u_{BC} = V_6 + V_7 - V_0
\]

Therefore, \(i_B\) and \(i_C\) are given by:

\[
i_B = \frac{I_s}{\beta_N} (e^{u_T} - 1) + \frac{I_s}{\beta_I} (e^{\frac{V_6+V_7-V_0}{u_T}} - 1)
\]
\[ i_C = \frac{I_s}{(1 + \frac{V_6 - V_0}{u_b} + \frac{V_6 + V_7 - V_0}{u_a})} \left( e^{\frac{V_6}{u_T}} - e^{\frac{V_6 + V_7 - V_0}{u_T}} \right) \]
\[- \frac{I_s}{\beta I} \left( e^{\frac{V_6 + V_7 - V_0}{u_T}} - 1 \right) \]

(51)

Define:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix}
\overset{\text{Def.}}{=} 
\begin{bmatrix}
i_4 \\
V_3 \\
V_6 \\
V_7
\end{bmatrix}
\]

(52)

Rewrite equations in state-space form:

\[
\dot{X}_1 = \frac{1}{L} (X_3 + X_4 - X_2 - R_4 X_1)
\]

(53)

\[
\dot{X}_2 = \frac{X_1}{C_1}
\]

(54)

\[
\dot{X}_3 = \frac{1}{C_2} \left[ -X_1 + \frac{1}{R_1} (V_6 - X_3 - X_4) - \frac{1}{R_3} (X_3 + X_4) \right.
\]

\[
- \frac{I_s}{\beta N} \left( e^{\frac{X_3}{u_T}} - 1 \right) - \frac{I_s}{\beta I} \left( e^{\frac{X_3 + X_4 - V_0}{u_T}} - 1 \right) \]

(55)

\[
\dot{X}_4 = \frac{1}{C_3} \left[ \frac{1}{R_1} (V_6 - X_3 - X_4) - X_1 - \frac{1}{R_3} (X_3 + X_4) \right.
\]

\[
- \frac{X_4}{R_2} - \frac{I_s}{\beta I} \left( e^{\frac{X_3 + X_4 - V_0}{u_T}} - 1 \right)
\]

\[
+ \frac{I_s}{(1 + \frac{X_3}{u_b} + \frac{X_3 + X_4 - V_0}{u_a}) \left( e^{\frac{X_3}{u_T}} - e^{\frac{X_3 + X_4 - V_0}{u_T}} \right)} \]

(56)

F. Model II

\[
\begin{array}{|c|c|c|c|c|}
\hline
R_1 &=& 1000\Omega & R_2 &=& 82k\Omega \\
R_3 &=& 680\Omega & L &=& 100nH \\
C_1 &=& 33pF & C_2 &=& 33pF \\
C_3 &=& 10pF & C_4 &=& 0.1\mu F \\
\hline
\end{array}
\]
For the transistor:

\[
\begin{array}{|c|c|c|}
\hline
I_s & U_A & U_B \\
\hline
2.7 \times 10^{-16} A & 15 V & 4.3 V \\
\hline
Gain & \beta_I & \beta_N \\
\hline
5.5 & 140 \\
\hline
\end{array}
\]

State-Space Equations of Model II oscillator:

The transistor voltages and are given by:

\[
\frac{dV_1}{dt} = \frac{1}{C_1} \left[ \left( \frac{V_0 - V_1}{R_1} \right) - i_C - \left( \frac{V_1 - V_2}{R_2} \right) - \left( \frac{V_1 - V_2 - V_5}{R_3} \right) - i_6 \right]
\]

\[
(57)
\]

\[
\frac{dV_2}{dt} = \frac{1}{C_2} \left[ \left( \frac{V_1 - V_2}{R_2} \right) + \left( \frac{V_1 - V_2 - V_5}{R_3} \right) + i_6 - i_B \right]
\]

\[
(58)
\]

\[
\frac{dV_3}{dt} = \frac{i_6}{C_3}
\]

\[
(59)
\]

\[
\frac{di_6}{dt} = \frac{1}{L} (V_1 - V_2 - V_5)
\]

\[
(60)
\]

\[
\frac{dV_5}{dt} = \frac{1}{C_4} \left( \frac{V_1 - V_2 - V_5}{R_3} \right)
\]

\[
(61)
\]

The currents and transistor voltages \(u_{BE}\) and \(u_{BC}\) are given by:

\[
u_{BE} = V_2
\]

\[
u_{BC} = V_2 - V_1
\]

\[
(62)
\]

\[
(63)
\]

Therefore, \(i_B\) and \(i_C\) are given by:

\[
i_B = \frac{I_s}{\beta_N} \left( e^{\frac{V_2}{\alpha T}} - 1 \right) + \frac{I_s}{\beta_I} \left( e^{\frac{V_2 - V_1}{\alpha T}} - 1 \right)
\]

\[
(64)
\]
\[ i_C = \frac{I_s}{(1 + \frac{V_2}{u_b} + \frac{V_2-V_1}{u_a})} \left( e^{\frac{V_2}{u_T}} - e^{\frac{V_2-V_1}{u_T}} \right) \]
\[ \quad - \frac{I_s}{\beta_I} \left( e^{\frac{V_2-V_1}{u_T}} - 1 \right) \] 
\[ (65) \]

Define:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{bmatrix}
\overset{\text{Def.}}{=} 
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
i_6 \\
V_5
\end{bmatrix}
\]
\[ (66) \]

Rewrite equations in state-space form:

\[ \dot{X}_1 = \frac{1}{C_1} \left[ \frac{1}{R_1} (V_0 - X_1) - \frac{1}{R_2} (X_1 - X_2) - X_4 
\quad - \frac{1}{R_3} (X_1 - X_2 - X_5) + \frac{I_s}{\beta_I} \left( e^{\frac{x_2-x_1}{u_T}} - 1 \right) 
\right] 
\quad - \frac{I_s}{\beta_N} \left( e^{\frac{x_2}{u_T}} - 1 \right) - \frac{I_s}{\beta_I} \left( e^{\frac{x_2-x_1}{u_T}} - 1 \right) \]
\[ (67) \]

\[ \dot{X}_2 = \frac{1}{C_2} \left[ X_4 + \frac{1}{R_2} (X_1 - X_2) + \frac{1}{R_3} (X_1 - X_2 - X_5) 
\quad - \frac{I_s}{\beta_N} \left( e^{\frac{x_2}{u_T}} - 1 \right) - \frac{I_s}{\beta_I} \left( e^{\frac{x_2-x_1}{u_T}} - 1 \right) \right] \]
\[ (68) \]

\[ \dot{X}_3 = \frac{X_4}{C_3} \]
\[ (69) \]

\[ \dot{X}_4 = \frac{1}{L} (X_1 - X_2 - X_3) \]
\[ (70) \]

\[ \dot{X}_5 = \frac{1}{C_4 R_3} (X_1 - X_2 - X_5) \]
\[ (71) \]

**FIGURE CAPTIONS**
FIG. 1: Van der Pol oscillator circuit

FIG. 2: Amplitude controlled Van der Pol oscillator circuit

FIG. 3: Clapp oscillator circuit
FIG. 4: Colpitts oscillator circuit

FIG. 5: Model 1 oscillator circuit
FIG. 6: Model II oscillator circuit