Representation of G-martingales as stochastic integrals with respect to the G-Brownian motion *

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Abstract

The objective of this paper is to derive a representation of symmetric G-martingales as stochastic integrals with respect to the G-Brownian motion. For this end, we first study some extensions of stochastic calculus with respect to G-martingales under the sublinear expectation spaces.

Keywords: G-expectation; G-Brownian motion; G-martingales; representation; stochastic calculus.

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1 Introduction

Motivated by uncertainty problems, risk measures and superhedging in finance, Peng has introduced recently a new notion of nonlinear expectation, the so-called G-expectation (cf. [6], [7], [9]), which, unlike the classical linear one, is not associated with the linear but a nonlinear heat equation. The G-expectation represents a special case of general nonlinear expectations \( \hat{E} \) which importance stems from the fact that they are related to risk measures \( \rho \) in finance by the relation \( \hat{E}[X] = \rho(-X) \), where \( X \) runs the class of contingent claims. Although the G-expectations represent only a special case, their importance inside the class of nonlinear expectations reflects in the law of large numbers and central limit theorem under nonlinear expectation, proven by Peng [8] and [10]. Together with the notion of G-expectation Peng also introduced the related G-normal distribution and the G-Brownian motion. The G-Brownian motion is a stochastic process which, under the G-expectation, has independent increments which are G-normally distributed. Moreover, in [7] Peng developed an Itô calculus for the G-Brownian motion.

A celebrated result of Lévy [5] and Doob [3] states that a continuous classical martingale \( M \) is a Brownian motion if and only if its quadratic variation process is the deterministic function \( \langle M \rangle_t = t, \ t \geq 0 \). Recently, the martingale characterization of the G-Brownian motion has been obtained by Xu [11]. The objective of the present paper is to extend this characterization and to investigate a representation of symmetric G-martingales as stochastic integrals with respect to the G-Brownian motion in the framework of the sublinear expectation spaces. In order to

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obtain this, we first study the stochastic integral with respect to a larger class of symmetric G-martingales $M$. This generalizes considerably the recent works by Xu [11], in which the process \( \{M_t^2 - t\}_{t \in [0,T]} \) has been supposed to be a G-martingale. We discuss even more general case in our another paper.

However, there are several difficulties for studying the martingale characterization of the G-Brownian motion: firstly, in contrast to the classical Brownian motion the G-Brownian motion is not defined on a given probability space but only on a nonlinear expectation space. The nonlinear expectation $\hat{E}[]$ can be represented as the supremum over the linear expectations $E_P[]$, where $P$ runs a certain class of probability measures which are not mutually equivalent. Secondly, the quadratic variation process $\langle B \rangle$ of the G-Brownian motion is a random process which satisfies the relation $\sigma dt \leq d\langle B \rangle_t \leq \sigma dt$, q.s., $t \geq 0$. Thirdly, related with their absence or restriction is the applicability of some well known tools in the classical case (i.e. localization with stopping times, the dominated convergence theorem).

Our paper is organized as follows: Section 2 introduces the necessary notations and it gives a short recall of some elements of the G-stochastic analysis, which will be used in what follows. Moreover, the notion of G-martingales will be introduced. In Section 3 we extend the stochastic calculus with respect to G-martingales to a larger class of G-martingales and study related properties. Moreover, a downcrossing inequality for G-supermartingales is obtained. Finally, section 4 investigates the representation of G-martingales as stochastic integrals with respect to G-Brownian motion. This leads to the main result of this paper.

## 2 Notations and preliminaries

In this section, we introduce the G-framework which was established by Peng [6], and which we will need in what follows.

Let $\Omega$ be a given nonempty set and $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ such that if $x_1, \ldots, x_n \in \mathcal{H}$ then $\varphi(x_1, \ldots, x_n) \in \mathcal{H}$, for each $\varphi \in C_{l,lip}(\mathbb{R}^m)$. Here $C_{l,lip}(\mathbb{R}^m)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|,$$

for all $x, y \in \mathbb{R}^m$, for some $C > 0$ and $n \in \mathbb{N}$, both depending on $\varphi$. The space $\mathcal{H}$ is considered as a set of random variables.

**Definition 2.1** A Sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(i) **Monotonicity:** If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$.

(ii) **Constant preserving:** $\hat{E}[c] = c$, for all $c \in \mathbb{R}$.

(iii) **Self-dominated property:** $\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y]$.

(iv) **Positive homogeneity:** $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Remark 2.2** The sublinear expectation space can be regarded as a generalization of the classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the linear expectation associated with $\mathbb{P}$.

**Definition 2.3** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}$, is said to be independent under $\hat{E}$ of another random vector $X = (X_1, \ldots, X_m), X_i \in \mathcal{H}$, if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^{m+n})$ we have

$$\hat{E}[^{\hat{E}[\varphi(X,Y)]] = \hat{E}[\hat{E}[\varphi(x,Y)]_{x=X}].$$
\textbf{Definition 2.4 (G-normal distribution)} Let be given two reals $\underline{\sigma}, \overline{\sigma}$ with $0 \leq \underline{\sigma} \leq \overline{\sigma}$. A random variable $\xi$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called $G_{\underline{\sigma}, \overline{\sigma}}$-normal distributed, denoted by $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $\varphi \in C_{l,lip}(\mathbb{R})$, the following function defined by

$$u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

is the unique continuous viscosity solution with polynomial growth of the following parabolic partial differential equation:

$$\begin{cases}
\partial_t u(t, x) = G(\partial_{xx}^2 u(t, x)), & (t, x) \in [0, \infty) \times \mathbb{R}, \\
u(0, x) = \varphi(x).
\end{cases}$$

Here $G = G_{\underline{\sigma}, \overline{\sigma}}$ is the following sublinear function parameterized by $\underline{\sigma}$ and $\overline{\sigma}$:

$$G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-), \quad \alpha \in \mathbb{R}$$

(Recall that $\alpha^+ = \max\{0, \alpha\}$ and $\alpha^- = -\min\{0, \alpha\}$).

For simplicity, we suppose that $\overline{\sigma}^2 = 1$ and $\underline{\sigma}^2 = \sigma_0^2$, $\overline{\sigma}^2 \in [0, 1]$, in the following paper.

Throughout this paper, we let $\Omega = C_0(\mathbb{R}^+)$ be the space of all real valued continuous functions $(\omega_t)_{t \in \mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left[ \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \right] \wedge 1, \quad \omega^1, \omega^2 \in \Omega.$$

For each $T > 0$, we consider the following space of random variables:

$L_{ip}^0(\mathcal{F}_T) := \left\{ X(\omega) = \varphi(\omega_{t_1}, \ldots, \omega_{t_m}) \mid t_1, \ldots, t_m \in [0, T], \text{ for all } \varphi \in C_{l,lip}(\mathbb{R}^m), m \geq 1 \right\}.$

Obviously, it holds that $L_{ip}^0(\mathcal{F}_t) \subseteq L_{ip}^0(\mathcal{F}_0)$, for all $t \leq T < \infty$. We notice that $X, Y \in L_{ip}^0(\mathcal{F}_t)$ implies $X \cdot Y \in L_{ip}^0(\mathcal{F}_t)$ and $|X| \in L_{ip}^0(\mathcal{F}_t)$. We further define

$$L_{ip}^0(\mathcal{F}) = \bigcup_{n=1}^{\infty} L_{ip}^0(\mathcal{F}_n).$$

We will work on the canonical space $\Omega$ and set $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$.

We now introduce a sublinear expectation $\mathbb{E}$ defined on $\mathcal{H}_T^0 = L_{ip}^0(\mathcal{F}_T)$ as well as on $\mathcal{H}^0 = L_{ip}^0(\mathcal{F})$. For this, we consider the function $G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-), a \in \mathbb{R}$, and we apply the following procedure: for each $X \in \mathcal{H}^0$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$$

for some $m \geq 1, \varphi \in C_{l,lip}(\mathbb{R}^m)$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_m < \infty$, we set

$$\begin{align*}
\mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] \\
= \mathbb{E}[\varphi(\sqrt{t_1 - t_0} \xi_1, \sqrt{t_2 - t_1} \xi_2, \ldots, \sqrt{t_m - t_{m-1}} \xi_m)],
\end{align*}$$

where $(\xi_1, \xi_2, \cdots, \xi_m)$ is an $m$-dimensional G-normal distributed random vector in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ such that $\xi_i \sim \mathcal{N}(0, \sigma_0^2)$ and $\xi_{i+1}$ is independent of $(\xi_1, \cdots, \xi_i)$, for every $i = 1, 2, \cdots, m$. 

\[3\]
The related conditional expectation of $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$ under $\mathcal{H}_{t_j}$ is defined by

$$
\mathbb{E}[X|\mathcal{H}_{t_j}] = \mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_j}]
$$

where

$$
\psi(x_1, x_2, \ldots, x_j) = \mathbb{E}[\varphi(x_1, x_2, \ldots, x_j, \sqrt{t_{j+1} - t_j} \xi_{j+1}, \ldots, \sqrt{t_m - t_{m-1}} \xi_m)],
$$

$(x_1, x_2, \ldots, x_j) \in \mathbb{R}^j, 0 \leq j \leq m$.

For $p \geq 1$, $\|X\|_p = \mathbb{E}[|X|^p]^{\frac{1}{p}]$, $X \in L^0_{ip}(\mathcal{F})$, defines a norm on $L^0_{ip}(\mathcal{F})$. Let $\mathcal{H} = L^p_G(\mathcal{F})$ (resp. $\mathcal{H}_t = L^p_G(\mathcal{F}_t)$) be the completion of $L^0_{ip}(\mathcal{F})$ (resp. $L^0_{ip}(\mathcal{F}_t)$) under the norm $\| \cdot \|_p$. Then the space $(L^p_G(\mathcal{F}), \| \cdot \|_p)$ is a Banach space and the operators $\mathbb{E}[\cdot]$, $\mathbb{E}[\cdot|\mathcal{H}_t]$ can be continuously extended to the Banach space $L^p_G(\mathcal{F})$. Moreover, we have $L^p_G(\mathcal{F}_t) \subseteq L^p_G(\mathcal{F}_T) \subset L^p_G(\mathcal{F})$, for all $0 \leq t \leq T < \infty$.

**Definition 2.5** The expectation $\mathbb{E} : L^p_G(\mathcal{F}) \mapsto \mathbb{R}$ defined through the above procedure is called $G$-expectation.

**Proposition 2.6** For all $t, s \in [0, \infty)$, we list the properties of $\mathbb{E}[\cdot|\mathcal{H}_t]$ that hold for all $X, Y \in L^p_G(\mathcal{F})$:

(i) If $X \geq Y$, then $\mathbb{E}[X|\mathcal{H}_t] \geq \mathbb{E}[Y|\mathcal{H}_t]$;

(ii) $\mathbb{E}[\eta|\mathcal{H}_t] = \eta$, for all $\eta \in L^p_G(\mathcal{F}_t)$;

(iii) $\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t] \leq \mathbb{E}[X - Y|\mathcal{H}_t]$;

(iv) $\mathbb{E}[\eta X|\mathcal{H}_t] = \eta^+ \mathbb{E}[X|\mathcal{H}_t] - \eta^- \mathbb{E}[-X|\mathcal{H}_t]$, for all $\eta \in L^p_G(\mathcal{F}_t)$;

(v) If $\mathbb{E}[Y|\mathcal{H}_t] = -\mathbb{E}[-Y|\mathcal{H}_t]$, then $\mathbb{E}[X + Y|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t] + \mathbb{E}[Y|\mathcal{H}_t]$;

(vi) $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \mathbb{E}[X|\mathcal{H}_{t \wedge s}]$, and, in particular, $\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]] = \mathbb{E}[X]$.

For $p \geq 1$ and an arbitrary but fixed time horizon $0 < T < \infty$, we now consider the following space of step processes:

$$
M^0_{pG}(0, T) = \left\{ \eta : \eta = \sum_{j=0}^{n-1} \xi_j I_{[t_j, t_{j+1})}, 0 = t_0 < t_1 < \cdots < t_n = T, \xi_j \in L^p_G(\mathcal{F}_{t_j}), j = 0, \cdots, n-1, \text{for all } n \geq 1 \right\},
$$

and we define the following norm in $M^0_{pG}(0, T)$:

$$
\| \eta \|_p = \left( \mathbb{E} \left[ \int_0^T |\eta|^p dt \right] \right)^{\frac{1}{p}} = \left( \mathbb{E} \left[ \sum_{j=0}^{n-1} |\xi_{t_j}|^p(t_{j+1} - t_j) \right] \right)^{\frac{1}{p}}.
$$

Finally, we denote by $M^0_G(0, T)$ the completion of $M^0_G(0, T)$ under the norm $\| \cdot \|_p$.

**Definition 2.7** A process $B = \{B_t, t \geq 0\}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a $G$-Brownian motion if $\{B_t, t \geq 0\} \subset \mathcal{H}$ and the following properties are satisfied:
(i) $B_0 = 0$;

(ii) For each $t, s \geq 0$, the difference $B_{t+s} - B_t$ is $\mathcal{N}(0, [\sigma^2_0 s, s])$-distributed and is independent of $(B_{t_1}, \cdots, B_{t_n})$, for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

**Remark 2.8** The canonical process $(B_t)_{t \geq 0}$ in $(\Omega, \mathcal{H})$, $\Omega = C_0(\mathbb{R}_+)$, endowed with the G-expectation $\hat{\mathbb{E}}$ is a G-Brownian motion.

**Remark 2.9** In [6], [7] and [9], Peng established a stochastic calculus of Itô’s type with respect to the G-Brownian motion and its quadratic variation process. Peng derived an Itô’s formula and moreover, he obtained the existence and uniqueness of the solution to stochastic differential equations with Lipschitz coefficients driven by G-Brownian motion.

In [4], Hu and Peng obtained the representation theorem of G-Expectations as follows.

**Proposition 2.10** Let $\hat{\mathbb{E}}_f$ be G-expectation. Then there exists a weekly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X], \text{ for all } X \in \mathcal{H},$$

where $E_P[\cdot]$ is the linear expectation with respect to $P$.

The authors of [4] also introduced the associate Choquet capacity

$$c(A) = \sup_{P \in \mathcal{P}} P(A), A \in \mathcal{B}(\Omega).$$

We have the following proposition.

**Proposition 2.11**

(i) $0 \leq c(A) \leq 1$, for all $A \subset \Omega$.

(ii) If $A \subset B$, then $c(A) \leq c(B)$.

(iii) If $\{A_n\}_{n=1}^\infty$ is an increasing sequence in $\mathcal{B}(\Omega)$ and $A_n \uparrow A$, then $c(A) = \lim_{n \to \infty} c(A_n)$.

**Definition 2.12** A set $A$ is polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set.

As in the classical stochastic analysis, the definition of a modification of a process plays an important role.

**Definition 2.13** Let $I$ be a set of indexes, and $\{X_t\}_{t \in I}$ and $\{Y_t\}_{t \in I}$ two processes indexed by $I$. We say that $Y$ is a modification of $X$ if for all $t \in I$, $X_t = Y_t$ q.s.

Finally, we recall the definition of a G-martingale introduced by Peng [9].

**Definition 2.14** A process $M = \{M_t, t \geq 0\}$ is called a G-martingale (respectively, G-supermartingale, and G-submartingale) if for each $t \in [0, \infty)$, $M_t \in L_G^1(F_t)$ and for each $s \in [0, t]$, we have

$$\hat{\mathbb{E}}[M_t|\mathcal{H}_s] = M_s, \text{ (respectively } \leq M_s, \text{ and } \geq M_s \text{) q.s.}$$

A process $M = \{M_t, t \geq 0\}$ is called a symmetric G-martingale, if $M$ and $-M$ are G-martingales.
3 Stochastic integrals of G-martingales

In this sections, we study the stochastic integrals of G-martingales and related properties, which will be important in next section.

Let $p \geq 1$ and $T > 0$ be an arbitrarily fixed time horizon. Let $\{A_t, t \in [0, T]\}$ be a continuous and increasing process such that for all $t \in [0, T], A_t \in \mathcal{H}_t, A_0 = 0$ and $\mathbb{E}[A_T] < \infty$. We first consider the following space of step processes:

$$M_{G}^{p,0} (0, T) = \left\{ \eta : \eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}, 0 = t_0 < t_1 < \cdots < t_n = T, \xi_{t_j} \in L^p_{G}(\mathcal{F}_{t_j}), j = 0, \cdots, n-1, \text{for all } n \geq 1 \right\},$$

and we define the following norm in $M_{G}^{p,0} (0, T)$:

$$\| \eta \|_p = \left( \mathbb{E} \left[ \int_0^T |\eta_t|^p \, dA_t \right] \right)^{\frac{1}{p}} = \left( \mathbb{E} \left[ \sum_{j=0}^{n-1} |\xi_{t_j}|^p (A_{t_{j+1}} - A_{t_j}) \right] \right)^{\frac{1}{p}}.$$

We denote by $M_{G, A}^2 (0, T)$ the completion of $M_{G}^{p,0} (0, T)$ under the norm $\| \cdot \|_p$, and we introduce the following space of G-martingales related with $A$:

$$\mathcal{M} = \left\{ M | M \text{ is a continuous symmetric G-martingale such that } M^2 - A \text{ is a G-supermartingale} \right\}.$$

We will see later that $\mathcal{M} \subset M_{G, A}^2 (0, T)$.

**Definition 3.1** For any $M \in \mathcal{M}$ and $\eta \in M_{G}^{2,0} (0, T)$ of the form $\eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}(t)$, we define

$$I(\eta) = \int_0^T \eta_t \, dM_t = \sum_{j=0}^{n-1} \xi_{t_j} (M_{t_{j+1}} - M_{t_j}).$$

**Proposition 3.2** For all $M \in \mathcal{M}$, the mapping $I : M_{G}^{2,0} (0, T) \to L^2_{G}(\mathcal{F}_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_{G, A}^2 (0, T) \to L^2_{G}(\mathcal{F}_T)$. Moreover, for all $\eta \in M_{G, A}^2 (0, T)$, the process $\left\{ \int_0^t \eta_s \, dM_s \right\}_{t \in [0, T]}$ is a symmetric G-martingale and

$$\mathbb{E} \left[ \left| \int_0^T \eta_t \, dM_t \right|^2 \right] \leq \mathbb{E} \left[ \int_0^T |\eta_t|^2 \, dA_t \right]. \quad (3.1)$$

**Proof:** From $M$ is a symmetric G-martingale and $M^2 - A$ is a G-supermartingale it follows that, for all $0 \leq s \leq t \leq T$,

$$\mathbb{E}[(M_t - M_s)^2 - (A_t - A_s)|\mathcal{H}_s] = \mathbb{E}[M_t^2 - M_s^2 - 2M_s(M_t - M_s) - (A_t - A_s)|\mathcal{H}_s] = \mathbb{E}[M_t^2 - M_s^2 - (A_t - A_s)|\mathcal{H}_s] \leq 0.$$

For $\eta \in M_{G}^{2,0} (0, T)$ of the form $\eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}(t)$, we have

$$\mathbb{E} \left[ \left| \int_0^T \eta_t \, dM_t \right|^2 \right] = \mathbb{E} \left[ \left| \sum_{j=0}^{n-1} \xi_{t_j} (M_{t_{j+1}} - M_{t_j}) \right|^2 \right] = \mathbb{E} \left[ \sum_{j=0}^{n-1} \xi_{t_j}^2 (M_{t_{j+1}} - M_{t_j})^2 \right].$$
Consequently, (3.1) holds for all $\eta \in M^2_G(0,T)$. We then can continuously extend the above inequality to the case $\eta \in M^2_{G,A}(0,T)$ and obtain (3.1).

For $\eta \in M^2_{G,A}(0,T)$, there exists a sequence of $\eta^n \in M^2_G(0,T)$ of the form $\eta^n = \sum_{j=0}^{n-1} \xi_{t,j}(t_{j+1}^n - t_j)$, $\xi_{t,j} \in L^2_G(\mathcal{F}_{t,j})$ such that

$$\hat{E}[\int_0^T (\eta^n - \eta_u) dM_u^2] \to 0, \text{ as } n \to \infty.$$ 

Let $0 \leq s \leq t \leq T$. Without loss of generality, we assume that $t_i \leq s < t_{i+1} < t$, for some $0 \leq i \leq n-1$. Then we have

$$\hat{E}\left[\int_0^t \eta^n_u dM_u^2 \big| \mathcal{H}_s\right] = \hat{E}\left[\sum_{j=0}^{n-1} \xi_{t,j}(M_{t,j+1}^n - M_{t,j}^n) \big| \mathcal{H}_s\right] = \sum_{j=0}^{i-1} \xi_{t,j}(M_{t,j+1}^n - M_{t,j}^n) + \xi_{t,i}(M_s - M_{t_i}) = \int_0^s \eta^n_u dM_u.$$ 

Consequently, $\int_0^t \eta^n_u dM_u^2$ is a G-martingale. Moreover,

$$\hat{E}[\|\hat{E} \int_0^t \eta_u dM_u^2 \big| \mathcal{H}_s\| - \int_0^s \eta_u dM_u^2] = \hat{E}[\|\hat{E} \int_0^t \eta_u dM_u^2 - \hat{E} \int_0^t \eta^n_u dM_u^2 + \hat{E} \int_0^t \eta^n_u dM_u^2 - \hat{E} \int_0^s \eta^n_u dM_u^2\|] \leq 2\hat{E}[\|\int_0^t (\eta_u - \eta^n_u) dM_u^2\| + \|\int_0^s (\eta_u - \eta_u^n) dM_u^2\|]$$
Consequently, \( \{ \int_0^t \eta_u dM_u, t \in [0, T] \} \) is a symmetric G-martingale. The proof is complete. 

For \( 0 \leq s \leq t \leq T \) and \( \eta \in M^2_{G,A}(0, T) \), we denote

\[
\int_s^t \eta_u dM_u = \int_0^T I_{[s,t]}(u) \eta_u dM_u.
\]

It is now straightforward to see that we have the following properties of the stochastic integral of G-martingales.

**Proposition 3.3** Let \( 0 \leq s < r \leq t \leq T \). For all \( M \in \mathcal{M} \) and \( \theta, \eta \in M^2_{G,A}(0, T) \), we have

(i) \( \int_s^r \eta_u dM_u = \int_s^t \eta_u dM_u + \int_r^t \eta_u dM_u \);

(ii) \( \int_s^t (\eta_u + \alpha \theta_u) dM_u = \int_s^t \eta_u dM_u + \alpha \int_s^t \theta_u dM_u \), for all \( \alpha \) bounded random variable in \( L^p_G(\mathcal{F}_s) \);

(iii) \( \hat{E}[X + \int_r^T \eta_u dM_u | \mathcal{H}_s] = \hat{E}[X | \mathcal{H}_s] \), for all \( X \in L^p_G(\mathcal{F}) \).

For proving the continuity of the stochastic integral regarded as a process, we need the following Doob inequality for symmetric G-martingale.

**Theorem 3.4** If \( X \) is a right-continuous symmetric G-martingale running over an interval \([0, T]\) of \( \mathbb{R} \), then for every \( p > 1 \) such that \( X_T \in L^p_G(\mathcal{F}) \),

\[
\hat{E}[\sup_{0 \leq t \leq T} |X_t|^p] \leq (\frac{p}{p - 1})^p \hat{E}[|X_T|^p].
\]

**Proof:** By Remark 3.10 there exists a weekly compact family of probability measures \( \mathcal{P} \) on \((\Omega, \mathcal{B}(\Omega))\) such that \( \hat{E}[X] = \max_{P \in \mathcal{P}} E_P[X] \), for all \( X \in \mathcal{H} \), where \( E_P[\cdot] \) is the linear expectation with respect to \( P \).

For all \( 0 \leq t \leq T \), let \( \mathcal{F}_t^B = \sigma\{B_s, s \leq t\} \). For any \( 0 \leq s \leq t \leq T \) and any positive \( \xi \in L^p_{L^2_G}(\mathcal{F}_s) \), we have

\[
\hat{E}[(X_t - X_s)\xi] = \hat{E}[\xi(\hat{E}[X_t | \mathcal{H}_s] - X_s)] = 0.
\]

On the other hand,

\[
\hat{E}[(X_t - X_s)\xi] = \max_{P \in \mathcal{P}} E_P[(X_t - X_s)\xi] \geq E_P[(X_t - X_s)\xi] = E_P[\xi(E_P[X_t | \mathcal{F}_s^B] - X_s)],
\]

\[ \rightarrow 0, \text{ as } N \rightarrow \infty. \]
then we have $E_P[X_t|\mathcal{F}_s^B] \leq X_s$, P-a.s., for all $P \in \mathcal{P}$. By the same argument but this time with negative $\xi \in L_{G}^{-\infty}(\mathcal{F}_s)$, we can prove that $E_P[X_t|\mathcal{F}_s^B] \geq X_s$, P-a.s., for all $P \in \mathcal{P}$. Therefore $E_P[X_t|\mathcal{F}_s^B] = X_s$, P-a.s., for all $P \in \mathcal{P}$. Thus $X$ is a $P$-martingale and from the classical Doob’s inequality it follows that

$$E_P[\sup_{0 \leq t \leq T} |X_t|^p] \leq \left(\frac{p}{p-1}\right)^p E_P[|X_T|^p], \text{ for all } P \in \mathcal{P}.$$ 

Therefore,

$$\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |X_t|^p] \leq \left(\frac{p}{p-1}\right)^p \hat{\mathbb{E}}[|X_T|^p].$$

The proof is complete. □

We now give a downcrossing inequality for $G$-supermartingales. Let $a, b$ be two positive constants such that $a < b$. Let $\pi_n = \{0 = t_0 < \cdots < t_n = T\}$ be a partition of the interval $[0, T]$. We define $D_{n}^b[X, n]$ the number of downcrossing of $[a, b]$ by $\{X_t\}_{t=0}^n$.

**Theorem 3.5** Let $X$ be a positive $G$-supermartingale and $0 = t_0 \leq \cdots \leq t_n = T$ be a strictly increasing sequences. Then for all real positive numbers $a$ and $b$ such that $a < b$,

$$\hat{\mathbb{E}}[D_n^b[X, n]] \leq \frac{1}{b-a} \hat{\mathbb{E}}[X_0 \wedge b].$$

**Proof:** For any $0 \leq s \leq t \leq T$, from the first part of the proof of Theorem 3.4, we know that $E_P[X_t|\mathcal{F}_s^B] \leq X_s$, P-a.s., for all $P \in \mathcal{P}$. From the classical downcrossing inequality for supermartingales (cf. [2]) it follows that

$$E_P[D_{n}^b[X, n]] \leq \frac{1}{b-a} E_P[X_0 \wedge b], \text{ for all } P \in \mathcal{P}.$$ 

Therefore,

$$\hat{\mathbb{E}}[D_n^b[X, n]] \leq \frac{1}{b-a} \hat{\mathbb{E}}[X_0 \wedge b].$$

The proof is complete. □

**Theorem 3.6** For all $M \in \mathcal{M}$ and $\eta \in M_{G,A}^2(0, T)$, there exists a q.s. continuous version of stochastic integral

$$\int_0^t \eta_s dM_s, \quad 0 \leq t \leq T,$$

i.e. there exists a continuous process $Y = \{Y_t\}_{t \in [0, T]}$ in the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that

$$c(Y_t \neq \int_0^t \eta_s dM_s) = 0, \text{ for all } t, \ 0 \leq t \leq T.$$

**Proof:** We use $\pi^n = \{0 = t^n_0 < t^n_1 \cdots < t^n_n = T\}$ to denote a partition of $[0, T]$ such that $\max\{t_{i+1}^n - t_i^n, 0 \leq i \leq n - 1\} \rightarrow 0$, as $n \rightarrow \infty$.

For any $\eta \in M_{G,A}^2(0, T)$, there exists a sequence of $\eta^n \in M_{G}^{2,0}(0, T), n \geq 1$, of the form

$$\eta^n = \sum_{j=0}^{n-1} \xi^n_j I_{[t^n_j, t^n_{j+1})}(t),$$
where $\xi^n_i \in L^2_G(\mathcal{F}^n_j), 0 \leq i \leq n - 1$, such that
\[
\mathcal{E}\left[ \int_0^T |\eta^n - \eta^m|^2 dA_t \right] \to 0, \text{ as } n \to \infty.
\]

We put $X^n_t = \int_0^t \eta^n_s dM_s = \sum_{j=0}^{n-1} \xi^n_j (M^n_{j+1,t} - M^n_{j,t}),$ for all $i = 1, \cdots, n.$ Then $X^n$ is a continuous G-martingale and
\[
\mathcal{E}[X^n_t|\mathcal{H}_s] = -\mathcal{E}[-X^n_t|\mathcal{H}_s] = X^n_s, \text{ for all } s \in [0,t].
\]

For any $\lambda > 0$, by Markov inequality for capacity (see Lemma 13 in [1]) as well as Theorem 3.4 we have
\[
c\left( \{ \sup_{0 \leq t \leq T} |X^n_t - X^n_t| \geq \lambda \} \right) \leq \frac{1}{\lambda^2} \mathcal{E}\left[ \sup_{0 \leq t \leq T} |X^n_t - X^n_t|^2 \right] \leq \frac{4}{\lambda^2} \mathcal{E}[|X^n_T - X^n_T|^2],
\]
and thanks to Proposition 3.2 it follows that
\[
c\left( \{ \sup_{0 \leq t \leq T} |X^n_t - X^n_t| \geq \lambda \} \right) \leq \frac{4}{\lambda^2} \mathcal{E}\left[ \int_0^T |\eta^n_s - \eta^m_s|^2 dA_t \right] \to \infty,
\]
as $n, m \to \infty.$ Hence, we can choose a subsequence $n_k \uparrow \infty$ such that
\[
c\left( \{ \sup_{0 \leq t \leq T} |X^{n_k+1}_t - X^n_t| \geq 2^{-k} \} \right) < 2^{-k}, k \geq 1,
\]
and from the Borel-Cantelli lemma for the capacity (see Lemma 5 in [1]) we obtain
\[
c\left( \{ \sup_{0 \leq t \leq T} |X^{n_k+1}_t - X^n_t| \geq 2^{-k}, \text{ for infinitely many } k \} \right) = 0.
\]

Hence, there exists a random integer $k_1$ such that
\[
\sup_{0 \leq t \leq T} |X^{n_k+1}_t - X^n_t| < 2^{-k}, \text{ q.s., for all } k > k_1.
\]

This proves that the process $X$ converges uniformly in $t \in [0,T]$ q.s.. We denote the limit of $X^n_{n_k}$ by $Y$. Thanks to the quasi-sure uniform convergence it is a process whose paths are continuous. On the other hand, the sequence $\{X^n_k, k \geq 1\}$ converges to $\int_0^t \eta_s dM_s$ in $L^2_G(\mathcal{F}),$ for all $t \in [0,T].$ Thus,
\[
Y_t = \int_0^t \eta_s dM_s, \text{ q.s., for all } t, 0 \leq t \leq T,
\]
and so $Y$ is a continuous modification of the integral process. The proof is complete. \qed

The following very useful Lemma was established by Peng [7].

Lemma 3.7 Let $X, Y \in L^1_G(\mathcal{F})$ be such that $\mathcal{E}[Y|\mathcal{H}_s] = -\mathcal{E}[-Y|\mathcal{H}_s],$ for $s \geq 0.$ Then we have
\[
\mathcal{E}[X + Y|\mathcal{H}_s] = \mathcal{E}[X|\mathcal{H}_s] + \mathcal{E}[Y|\mathcal{H}_s].
\]

In particular, if $\mathcal{E}[Y] = -\mathcal{E}[-Y] = 0,$ then we have
\[
\mathcal{E}[X + Y] = \mathcal{E}[X] + \mathcal{E}[Y].
\]

Now we give the Burkholder-Davis-Gundy inequality for the stochastic integral with respect to G-martingales.
\textbf{Theorem 3.8} For every \( q > 0 \), there exist a positive constant \( C_q \) such that, for all \( M \in \mathcal{M} \) and all \( \eta \in M^2_{\mathcal{G},A}(0,T) \),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} | \int_0^t \eta_s dM_s |^{2q} \right] \leq C_q \mathbb{E} \left[ ( \int_0^T \eta_s^2 dA_s )^q \right].
\]

\textbf{Proof:} Let \( M \in \mathcal{M} \). Then for all \( P \in \mathcal{P} \), \( M \) is a continuous \( P \)-martingale and \( M^2 - A \) is a continuous \( P \)-supermartingale. Let \( \langle M \rangle^P \) denote the quadratic variation process of \( M \) under \( P \), i.e., the unique continuous, \( P \)-predictable increasing process \( \langle M \rangle^P \) such that \( \langle M \rangle^P_0 = 0 \) and \( M^2 - \langle M \rangle^P \) is a \( P \)-martingale. Then

\[
\langle M \rangle^P - A = (M^2 - A) - (M^2 - \langle M \rangle^P), \langle M \rangle^P_0 - A_0 = 0,
\]

is a continuous \( P \)-supermartingale and of finite variation.

Thanks to Doob-Meyer decomposition theorem, we have

\[
\langle M \rangle^P - A = N^P - B^P, N^P_0 - B^P_0 = 0,
\]

where \( N^P \) is a continuous \( P \)-martingale and \( B^P \) is a \( P \)-predictable, continuous increasing process. Therefore, \( \langle M \rangle^P - A + B^P \) is a continuous \( P \)-martingale and of finite variation. Consequently,

\[
\langle M \rangle^P - A + B^P = 0, P - a.s., i.e.,
\]

\[
d\langle M \rangle^P_t \leq dA_t, t \geq 0, P - a.s.. \tag{3.2}
\]

Let \( \eta \in M^2_{\mathcal{G},A}(0,T) \). Then for any \( 0 \leq s \leq t \leq T \), by the proof of Theorem 3.4, we know that \( E_P[\int_0^t \eta_s dM_s | \mathcal{F}_s] = \int_0^s \eta_t dM_t, \) \( P \)-a.s., for all \( P \in \mathcal{P} \). From the classical Burkholder-Davis-Gundy inequalities, for every \( q > 0 \), there exist a positive constant \( C_q \) such that

\[
E_P \left[ \sup_{t \in [0,T]} | \int_0^t \eta_s dM_s |^{2q} \right] \leq C_q E_P \left[ ( \int_0^T \eta_s^2 d\langle M \rangle^P_s )^q \right],
\]

and from (3.2)

\[
E_P \left[ \sup_{t \in [0,T]} | \int_0^t \eta_s dM_s |^{2q} \right] \leq C_q E_P \left[ ( \int_0^T \eta_s^2 dA_s )^q \right],
\]

Therefore,

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} | \int_0^t \eta_s dM_s |^{2q} \right] \leq C_q \mathbb{E} \left[ ( \int_0^T \eta_s^2 dA_s )^q \right].
\]

The proof is complete. \( \square \)

Let \( \pi^n = \{ 0 = t^n_0 < t^n_1 < \cdots < t^n_n = T \} \) with \( |\pi^n| \to 0 \), as \( n \to \infty \), be a partition of the interval \([0,T]\). In the following of this section, we assume that the process \( A \) satisfies the following assumption:

\[
\mathbb{E}[A_T^2] < \infty, \quad \text{and for all } \{ \pi^n \}_{n \geq 1} \text{ sequence of partition of } [0,T] \text{ such that } |\pi^n| \to 0, \text{ as } n \to \infty,
\]

\[
\mathbb{E} \left[ \sum_{i=0}^{n-1} (A_{t^n_i} - A_{t^n_{i+1}})^2 \right] \to 0, n \to \infty.
\]

\textbf{Proposition 3.9} Let \( M \in \mathcal{M} \). Then the quadratic variation of \( M \) exists and

\[
\langle M \rangle_t = M_t^2 - 2 \int_0^t M_s dM_s, \text{ for all } t \geq 0.
\]
We define \( M^{2}_{t} = \sum_{i=0}^{n-1} [M^{2}_{t_{i+1}^{n} \wedge t} - M^{2}_{t_{i}^{n} \wedge t}] \) for any continuous mapping, and thus, can be continuously extended to \( M \). Then
\[
M^{2}_{t} = 2 \sum_{i=0}^{n-1} M^{n}_{t_{i}^{n} \wedge t}[M^{n}_{t_{i+1}^{n} \wedge t} - M^{n}_{t_{i}^{n} \wedge t}] + \sum_{i=0}^{n-1} [M^{n}_{t_{i+1}^{n} \wedge t} - M^{n}_{t_{i}^{n} \wedge t}]^{2}.
\]

Thanks to Theorem 3.8, we have
\[
\mathbb{E} \left[ \int_{0}^{T} (M_{s} - M_{s}^{n})^{2}dA_{s} \right] \leq C \mathbb{E} \left[ \sum_{i=0}^{n-1} (M_{s} - M_{s}^{n})^{2} \left( A_{t_{i+1}^{n}} - A_{t_{i}^{n}} \right) \right] 
\leq C \mathbb{E} \left[ \sum_{i=0}^{n-1} \sup_{s \in [t_{i}^{n}, t_{i+1}^{n}]} (M_{s} - M_{s}^{n})^{2} \left( A_{t_{i+1}^{n}} - A_{t_{i}^{n}} \right) \right] 
\leq C \mathbb{E} \left[ \sum_{i=0}^{n-1} \left( A_{t_{i+1}^{n}} - A_{t_{i}^{n}} \right)^{2} \right] \to 0, \text{ as } n \to \infty.
\]

Therefore,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_{0}^{t} (M_{s} - M_{s}^{n})dM_{s} \right|^{2} \right] \leq C \mathbb{E} \left[ \int_{0}^{T} (M_{s} - M_{s}^{n})^{2}dA_{s} \right] \to 0, \text{ as } n \to \infty. \tag{3.4}
\]

Consequently, the first term of (3.3) converges to the stochastic integral \( 2 \int_{0}^{t} M_{s}dM_{s} \), then the quadratic variation of \( M \) exists and is equal to
\[
\langle M \rangle_{t} := \lim_{n \to \infty} \sum_{i=0}^{n-1} [M^{n}_{t_{i+1}^{n} \wedge t} - M^{n}_{t_{i}^{n} \wedge t}]^{2} = M^{2}_{t} - 2 \int_{0}^{t} M_{s}dM_{s}.
\]

The proof is complete. \( \square \)

By Theorem 3.8 and Proposition 3.9, we have

**Remark 3.10** For all \( t \in [0, T] \), we have \( \langle M \rangle^{P}_{t} = \langle M \rangle_{t}, \text{ P - a.s.}, \text{ for all } P \in \mathcal{P} \).

**Definition 3.11** Let \( M \in \mathcal{M} \). Then for all \( \eta \in M^{1,0}_{G}(0, T) \) of the form \( \eta_{t} = \sum_{j=0}^{n-1} \xi_{t_{j}}I_{[t_{j}, t_{j+1})} \) we define
\[
I(\eta) = \int_{0}^{T} \eta_{t}d\langle M \rangle_{t} = \sum_{j=0}^{n-1} \xi_{t_{j}}(\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_{j}}).
\]

We have the following proposition.

**Proposition 3.12** For any \( M \in \mathcal{M} \), the mapping \( I : M^{1,0}_{G}(0, T) \to L^{1}_{G}(\mathcal{F}_{T}) \) is a linear continuous mapping, and thus, can be continuously extended to \( I : M^{1}_{G,A}(0, T) \to L^{1}_{G}(\mathcal{F}_{T}) \). Moreover, for all \( \eta \in M^{1}_{G,A}(0, T) \) we have
\[
\mathbb{E} \left[ \left| \int_{0}^{T} \eta_{t}d\langle M \rangle_{t} \right| \right] \leq \mathbb{E} \left[ \int_{0}^{T} |\eta_{t}|dA_{t} \right]. \tag{3.5}
\]
Proposition 3.13

If \( \eta \) Thus, (3.5) holds for all \( \eta \) to the case

Consequently,

Proof: From \( M \) is a symmetric G-martingale and \( M^2 - A \) is a G-supermartingale it follows that, for all \( 0 \leq s \leq t \leq T \),

\[
\mathbb{E}[\langle M \rangle_t - \langle M \rangle_s - (A_t - A_s)|\mathcal{H}_s] = \mathbb{E}[M_t^2 - M_s^2 - 2 \int_s^t M_r dM_r - (A_t - A_s)|\mathcal{H}_s] = \mathbb{E}[M_t^2 - M_s^2 - (A_t - A_s)|\mathcal{H}_s] \leq 0.
\]

For \( \eta \in M_G^{1,0}(0,T) \) of the form \( \eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j,t_{j+1})}(t) \), we have

\[
\mathbb{E}\left[ \int_0^T \eta_t d\langle M \rangle_t \right] = \mathbb{E}\left[ \sum_{j=0}^{N-1} \xi_{t_j} (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j}) \right] \\
\leq \mathbb{E}\left[ \sum_{j=0}^{N-1} |\xi_{t_j}| (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j}) \right] \\
\leq \mathbb{E}\left[ \sum_{j=0}^{N-1} |\xi_{t_j}| (\langle M \rangle_{t_{j+1}} - A_{t_{j+1}} + A_{t_j}) \right] + \mathbb{E}\left[ \sum_{j=0}^{N-1} |\xi_{t_j}| (A_{t_{j+1}} - A_t) \right],
\]

where

\[
\mathbb{E}\left[ \sum_{j=0}^{N-1} |\xi_{t_j}| (\langle M \rangle_{t_{j+1}} - A_{t_{j+1}} + A_{t_j}) \right] \\
\leq \mathbb{E}\left[ \sum_{j=0}^{N-2} |\xi_{t_j}| (\langle M \rangle_{t_{j+1}} - A_{t_{j+1}} + A_{t_j}) \right] + \mathbb{E}[\langle M \rangle_{t_N} - \langle M \rangle_{t_{N-1}} - A_{t_{N}} + A_{t_{N-1}}|\mathcal{H}_{t_{N-1}}] \\
\leq \mathbb{E}\left[ \sum_{j=0}^{N-2} |\xi_{t_j}| (\langle M \rangle_{t_{j+1}} - A_{t_{j+1}} + A_{t_j}) \right] \leq \cdots \leq 0.
\]

Consequently,

\[
\mathbb{E}\left[ \int_0^T \eta_t d\langle M \rangle_t \right] \leq \mathbb{E}\left[ \sum_{j=0}^{N-1} |\xi_{t_j}| (A_{t_{j+1}} - A_{t_j}) \right] = \mathbb{E}\left[ \int_0^T |\eta_t| dA_t \right].
\]

Thus, (3.5) holds for all \( \eta \in M_G^{1,0}(0,T) \). We then can continuously extend the above inequality to the case \( \eta \in M_G^{1,0}(0,T) \) and prove (3.5). The proof is complete. \( \square \)

Now we can prove the following proposition by the same argument as in (7).

Proposition 3.13 If \( M \in \mathcal{M} \), \( X \in L^1_G(F) \) and \( \xi \in L^2_G(F_s) \), then for all \( 0 \leq s \leq t < \infty \)

\[
\mathbb{E}[X + \xi(\langle M \rangle_t - \langle M \rangle_s)] = \mathbb{E}[X + \xi(M_t - M_s)^2] = \mathbb{E}[X + \xi(M_t^2 - M_s^2)].
\]

Moreover, we have the following isometry property.

Proposition 3.14 If \( M \in \mathcal{M} \) and \( \eta \in M_G^{2,0}(0,T) \), then

\[
\mathbb{E}\left[ \left( \int_0^T \eta_t dM_t \right)^2 \right] = \mathbb{E}\left[ \int_0^T \eta_t^2 d\langle M \rangle_t \right].
\]
Proof: For \( \eta \in M_{2,0}^2(0,T) \) of the form \( \eta_t = \sum_{j=0}^{n-1} \xi_{t_j}I_{[t_j,t_{j+1})}(t) \) a straightforward argument gives
\[
\hat{E}\left[ \int_0^T |\eta_t dM_t|^2 \right] = \hat{E}\left[ \sum_{j=0}^{n-1} \xi_{t_j}^2 (M_{t_{j+1}} - M_{t_j})^2 \right] = \hat{E}\left[ \sum_{j=0}^{n-1} \xi_{t_j}^2 (M_{t_{j+1}} - M_{t_j})^2 \right].
\]

Thanks to Proposition 3.13 we have
\[
\hat{E}\left[ \int_0^T |\eta_t dM_t|^2 \right] = \hat{E}\left[ \sum_{j=0}^{n-1} \xi_{t_j}^2 (\langle M \rangle_{t_{j+1}} - \langle M \rangle_{t_j}) \right] = \hat{E}\left[ \int_0^T \eta_t^2 d\langle M \rangle_t \right].
\]

Thus, Proposition 3.14 holds for all \( \eta \in M_{2,0}^2(0,T) \). Finally, Proposition 3.13 allows to extend continuously the above inequality to all \( \eta \in M_{2,A}^2(0,T) \), and thus, yields the desired result. The proof is complete.

Now we give another kind of the Burkholder-Davis-Gundy inequalities for the stochastic integral with respect to G-martingales.

Theorem 3.15 For every \( p > 0 \), there exist two positive constants \( c_p \) and \( C_p \) such that, for all \( M \in \mathcal{M} \) that \( M \in M_{2,A}^2(0,T) \) and all \( \eta \in M_{2,A}^2(0,T) \),
\[
c_p \hat{E}\left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right] \leq \hat{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t \eta_s dM_s \right|^p \right] \leq C_p \hat{E}\left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right].
\]

Proof: For any \( 0 \leq s \leq t \leq T \), by the proof of Theorem 3.4, we know that \( \int_0^t \eta_r dM_r \) is a continuous \( P \)-martingale, for all \( P \in \mathcal{P} \). Thanks to Remark 3.10 we know that \( \langle M \rangle_t^{(P)} = \langle M \rangle_t, P \text{ a.s.} \) all \( t \in [0,T] \). From the classical Burkholder-Davis-Gundy inequalities it follows that
\[
c_p E_P \left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right] \leq E_P \left[ \sup_{t \in [0,T]} \left| \int_0^t \eta_s dM_s \right|^p \right] \leq C_p E_P \left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right],
\]

where two constants \{0} \leq c_p \leq C_p \ato depend on \( p \). Therefore,
\[
c_p \hat{E}\left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right] \leq \hat{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t \eta_s dM_s \right|^p \right] \leq C_p \hat{E}\left[ \left( \int_0^T \eta^2 d\langle M \rangle_s \right)^p \right].
\]
The proof is complete.

Theorem 3.16 For all \( t \geq 0 \). Let \( M \in \mathcal{M} \). If \( f \in M_{2,A}^2(0,t) \) is a bounded process, then the quadratic variation process of \( X_t := \int_0^t f_s dM_s \) exists and
\[
\langle X \rangle_t = \int_0^t f_s^2 d\langle M \rangle_s.
\]

Proof: For \( f \in M_{2,A}^2(0,t) \), there exists an \( f^n \) of the form \( f^n_s = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j,t_{j+1})}(s) \), where \( \xi_{t_j} \in L^2_G(\mathcal{F}_{t_j}) \), \( 0 \leq i \leq n - 1 \), such that
\[
\hat{E}\left[ \int_0^t (f_s - f^n_s)^2 dM_s \right] \leq \hat{E}\left[ \int_0^t |f_s - f^n_s|^2 dA_s \right] \to 0, \text{ as } n \to \infty.
\]
For any $\varepsilon > 0$ and $0 \leq s \leq t$, we have

$$\hat{E}[\int_s^t f_r^2 dA_r - \int_s^t (f_r^n)^2 dA_r] \leq (1 + \frac{1}{\varepsilon})\hat{E}[\int_s^t |f_r - f_r^n|^2 dA_r] + \varepsilon\hat{E}[\int_s^t f_r^2 dA_r]$$

$$\rightarrow \varepsilon\hat{E}[\int_s^t f_r^2 dA_r], \text{ as } n \to \infty.$$ 

Therefore,

$$\hat{E}[\int_s^t f_r^2 dA_r - \int_s^t (f_r^n)^2 dA_r] \to 0, \text{ as } n \to \infty.$$ 

Since $M^2 - A$ is a G-supermartingale, we have

$$\hat{E}\left[(\int_s^t f_r^n dM_r)^2 - \int_s^t (f_r^n)^2 dA_r | \mathcal{H}_s\right]$$

$$= \hat{E}\left[(\sum_{j=0}^{n-1} \xi_{j+1}^n (M_{r+1} - M_{r}) - M_{r} + \sum_{j=0}^{n-1} \xi_{j+1}^n (A_{r+1} - A_{r}) - A_{r}) | \mathcal{H}_s\right]$$

$$= \hat{E}\left[(\sum_{j=0}^{n-1} \xi_{j+1}^n (M_{r+1} - M_{r}) - M_{r})^2 - \sum_{j=0}^{n-1} \xi_{j+1}^n (A_{r+1} - A_{r}) - A_{r}) | \mathcal{H}_s\right]$$

$$= \hat{E}\left[(\sum_{j=0}^{n-1} \xi_{j+1}^n (M_{r+1} - M_{r}) - M_{r})^2 - \sum_{j=0}^{n-1} \xi_{j+1}^n (A_{r+1} - A_{r}) - A_{r}) | \mathcal{H}_s\right]$$

$$\leq \sum_{j=0}^{n-1} \hat{E}(\xi_{j+1}^n (M_{r+1} - M_{r}) - M_{r})^2 - \sum_{j=0}^{n-1} \hat{E}(A_{r+1} - A_{r}) - A_{r}) | \mathcal{H}_s \leq 0.$$ 

For all $\varepsilon > 0$, from the above inequalities it follows that

$$\hat{E}\left[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s\right]$$

$$\leq \hat{E}\left[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s\right] - \hat{E}\left[(\int_s^t f_r^2 dM_r)^2 - \int_s^t (f_r)^2 dA_r | \mathcal{H}_s\right]$$

$$+ \hat{E}\left[(\int_s^t f_r^2 dM_r)^2 - \int_s^t (f_r)^2 dA_r | \mathcal{H}_s\right]$$

$$= \hat{E}\left[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s\right] - \hat{E}\left[(\int_s^t f_r^2 dM_r)^2 - \int_s^t (f_r)^2 dA_r | \mathcal{H}_s\right]$$

$$\leq \hat{E}\left[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s\right] - \hat{E}\left[(\int_s^t f_r^2 dM_r)^2 - \int_s^t (f_r)^2 dA_r | \mathcal{H}_s\right]$$

$$\leq \hat{E}\left[(\int_s^t f_r dM_r)^2 - (\int_s^t f_r^2 dM_r)^2\right] + \hat{E}\left[(\int_s^t f_r^2 dA_r - \int_s^t (f_r)^2 dA_r\right]$$

$$\leq (1 + \frac{1}{\varepsilon})\hat{E}\left[(\int_s^t f_r - f_r^2)^2 dM_r\right] + \varepsilon\hat{E}\left[(\int_s^t f_r dM_r)^2\right] + \hat{E}\left[(\int_s^t f_r^2 dA_r - \int_s^t (f_r)^2 dA_r\right]$$

$$\rightarrow \varepsilon\hat{E}\left[(\int_s^t f_r dM_r)^2\right], \text{ as } n \to \infty.$$ 

Therefore,

$$\hat{E}\left[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s\right] = 0,$$
and which yields
\[ \hat{E} \left[ (\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r \mid \mathcal{H}_s \right] \leq 0, \text{ q.s.} \]

From the above inequality and Proposition 3.13 it follows that for all \(0 \leq s \leq t\)
\[
\hat{E}[X_t^2 - \int_0^t f_r^2 dA_r | \mathcal{H}_s] = \hat{E}[(X_s + \int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s]
= \hat{E}[X_s^2 + 2X_s \int_s^t f_r dM_r + (\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s]
= X_s^2 - \int_0^s f_r^2 dA_r + \hat{E}[(\int_s^t f_r dM_r)^2 - \int_s^t f_r^2 dA_r | \mathcal{H}_s] \leq X_s^2 - \int_0^s f_r^2 dA_r.
\]

Consequently, \(X^2 - \int_0^t f_r^2 dA_r\) is a G-supermartingale.

From Proposition 3.2 we know that \(X\) is a symmetric G-martingale. Then from Proposition 3.11 it follows that the quadratic variation process of \(X\) exists. Therefore
\[
\hat{E}[\langle X \rangle_t - \int_0^t f_r^2 d\langle M \rangle_s] \leq \hat{E}[\langle X \rangle_t - \sum_{i=0}^{n-1} (X_{t_{i+1}^n} - X_{t_i^n})^2]
+ \hat{E}[\sum_{i=0}^{n-1} (X_{t_{i+1}^n} - X_{t_i^n})^2 - \sum_{i=0}^{n-1} \xi_{t_i^n}^2 (M_{t_{i+1}^n} - M_{t_i^n})^2]
+ \hat{E}[\sum_{i=0}^{n-1} \xi_{t_i^n}^2 (M_{t_{i+1}^n} - M_{t_i^n})^2 - \sum_{i=0}^{n-1} \xi_{t_i^n}^2 (\langle M \rangle_{t_{i+1}^n} - \langle M \rangle_{t_i^n})]
+ \hat{E}[\sum_{i=0}^{n-1} \xi_{t_i^n}^2 (\langle M \rangle_{t_{i+1}^n} - \langle M \rangle_{t_i^n}) - \int_0^t f_r^2 d\langle M \rangle_s] \leq I_1 + I_2 + I_3 + I_4.
\]

As \(n \to \infty\), \(I_1 \to 0\), \(I_4 \to 0\). Now we prove \(I_2 \to 0\), \(I_3 \to 0\), as \(n \to \infty\).

\[
I_2 = \hat{E}[\sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} f_s dM_s)^2 - \sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} f_s^2 dM_s)^2]
\leq \hat{E}\left[\sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} (f_s - f_{s_i^n}) dM_s)^2 + \epsilon \sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} f_s dM_s)^2\right]
\leq \epsilon (1 + \frac{1}{\epsilon}) \hat{E}\left[\sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} (f_s - f_{s_i^n}) dM_s)^2\right] + \epsilon \sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} f_s dM_s)^2
\leq (1 + \frac{1}{\epsilon}) \epsilon \hat{E}\left[\sum_{i=0}^{n-1} (\int_{t_i^n}^{t_{i+1}^n} (f_s - f_{s_i^n}) dM_s)^2\right] + \epsilon \hat{E}\left[\int_0^t (f_s - f_{s_i^n})^2 dM_s\right] + \epsilon \hat{E}\left[\int_0^t f_s^2 dM_s\right]
\]

Thanks to Proposition 3.2 we have
\[
I_2 \leq (1 + \frac{1}{\epsilon}) \hat{E}\left[\int_0^t (f_s - f_{s_i^n})^2 dA_s\right] + \epsilon \hat{E}\left[\int_0^t f_s^2 dA_s\right]
\to \epsilon \hat{E}\left[\int_0^t f_s^2 dA_s\right], \text{ as } n \to \infty,
\]
where $M^n_s = \sum_{j=0}^{n-1} M^n_{t_j} I_{(t^n_j, t_{j+1}^n)}(s)$.

From Proposition 3.2 and the properties of $G$-expectation it follows that

$$
\hat{E}[\sum_{i=0}^{n-1} \xi_i^2 ((M^n_{t_{i+1}} - M^n_t)^2 - \sum_{i=0}^{n-1} \xi_i^2 ((M^n_{t_{i+1}} - M^n_t)^2)
= 4\hat{E}[\sum_{i=0}^{n-1} \xi_i^2 (M^n_s - M^n_{t_i})^2]
= 4\hat{E}[\sum_{i=0}^{n-1} \xi_i^2 (\int_{t_i}^{t_{i+1}} (M^n_s - M^n_{t_i})dM^n_s)^2]
\leq 4C\hat{E}[\int_0^t (M^n_s - M^n_t)^2] \to 0, \text{ as } n \to \infty.
$$

Therefore $I_3 \to 0$, as $n \to \infty$.

By the above inequality, we have

$$
\hat{E}[\langle X \rangle_t - \int_0^t f_s^2 d\langle M \rangle_s] \leq \varepsilon \int_0^t \hat{E}[f_s^2] dA_s.
$$

Thus,

$$
\langle X \rangle_t = \int_0^t f_s^2 d\langle M \rangle_s, \text{ q.s.}
$$

We obtain the desired result. The proof is completed. $\square$

**Proposition 3.17** For a fixed $T \geq 0$, $M$ is a symmetric $G$-martingale, $M^2 - A$ is a $G$-martingale and for $0 \leq \sigma_0^2 \leq 1$, $-(M^2 - \sigma_0^2 A)$ is a $G$-martingale, if $f \in M_{G,A}^1(0,T)$, then the process

$$
X_t := \int_0^t f_s d\langle M \rangle_s - 2 \int_0^t G(f_s) dA_s, \text{ } t \in [0,T]
$$

is a decreasing $G$-martingale.

**Proof:** It is easy to check that $X$ is a decreasing $G$-martingale. We use $\pi = \{0 = t^n_0 < t^n_1 \cdots < t^n_n = T\}$ to denote a partition of $[0,T]$ such that $\max\{t^n_i - t^n_{i-1}, 0 \leq i \leq n-1\} \to 0$, as $n \to \infty$.

For $f \in M_{G,A}^1(0,T)$, there exists an $f^n$ of the form $f^n_t = \sum_{j=0}^{n-1} \xi_j I_{(t^n_j, t^n_{j+1})}(t)$, where $\xi_j \in L^1_G(F^n_j), 0 \leq i \leq n-1$. Let

$$
X^n_t := \sum_{i=0}^{n-1} \xi_i^n ((M^n)_{t_{i+1}} - (M^n)_{t_i}) - 2 \sum_{i=0}^{n-1} G(\xi_i^n)(A^n_{t_{i+1} \wedge t} - A^n_{t_i \wedge t}),
$$

where $t \in [0,T]$. 
For $0 \leq s \leq t \leq T$. Without loss of generality, we suppose that $t_{k-1}^n \leq s \leq t_k^n \leq t \leq t_{k+1}^n$, for some $k = 1, \ldots, n - 1$. Thus,

$$
\hat{E}[X_t^n|\mathcal{H}_t^n] = \hat{E}[\xi_t^n M_t^n + \langle \xi_t^n \rangle |\mathcal{H}_t^n]
$$

...
\[ \rightarrow 0, \text{ as } n \rightarrow \infty. \]

Thus,

\[ \hat{E}[X_t|\mathcal{H}_s] = X_s, \text{ q.s. for all } 0 \leq s \leq t \leq T. \]

The proof is complete. \[ \square \]

We can easily get the following corollary, which was established by Peng [7].

**Corollary 3.18** If \( f \in M^1_G(0, T) \), then the process

\[ \left\{ \int_0^t f_s d\langle B \rangle_s - 2 \int_0^t G(f_s) ds, \; t \in [0, T] \right\} \]

is a G-martingale.

**Remark 3.19** With respect to a linear expectation, if \( X \) is a continuous martingale with finite variation, then \( X \) is a constant. But it is not true in G-stochastic analysis. We give an example as follows.

\( \{\langle B \rangle_t - t\}_{t \geq 0} \) is a continuous G-martingale with finite variation. But \( \{\langle B \rangle_t - t\}_{t \geq 0} \) is not a constant. It is a decreasing stochastic process.

### 4 Representation of G-martingales as stochastic integrals with respect to G-Brownian motion

In this section, we investigate a representation of G-martingales as stochastic integrals with respect to G-Brownian motion. The result of this section will play an important role in the study of stochastic differential equations driven by G-Brownian motion.

The following martingale characterization of G-Brownian motion was established by Xu [11].

**Lemma 4.1** A process \( M \in M^2_G(0, T) \) is a G-Brownian motion with a parameter \( 0 < \sigma_0 \leq 1 \) if

(i) \( M \) is a symmetric G-martingale;

(ii) For any \( t \geq 0 \), \( M^2_t - t \) is a G-martingale;

(iii) For any \( t \geq 0 \), \( \hat{E}[-M^2_t] = -\sigma_0^2 t \);

(iv) \( M \) is continuous, which means for every \( \omega \in \Omega \), \( M(t, \omega) \) is continuous.

**Remark 4.2** It can be easily show that we do not need the assumption \( M \in M^2_G(0, T) \) in Lemma 4.1 in our framework. Indeed, one can use the argument (3.4) in which \( A_t \) is replaced by \( t \).

The following representation of G-martingales as stochastic integrals with respect to G-Brownian motion is the main result in this section.

**Theorem 4.3** Let \( 0 < \sigma_0 \leq 1 \) and \( f \in M^2_G(0, T) \) be such that \( \hat{E}[\int_0^T |f_s|^4 ds] < \infty \). Moreover, if there exists a constant \( C \) (small enough) such that \( 0 < C \leq |f| \), then the following statements

(i) \( M \) is a symmetric G-martingale and \( \left\{ M^2_t - \int_0^t f^2_s ds \right\}_{t \in [0, T]} \) are G-martingales;

(ii) There exists a G-Brownian motion \( B \) such that \( M_t = \int_0^t f_s dB_s, \; \text{for all } t \in [0, T] \).

Recall that \( G(\alpha) = \frac{1}{2}(\alpha^+ - \sigma_0^2 \alpha^-), \; \alpha \in \mathbb{R} \).
Proof: We first prove (i) \(\Rightarrow\) (ii).

For all \(0 \leq t \leq T\), we use \(\pi^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_n = t\}\) to denote a partition of \([0,t]\) such that \(\max\{t^n_{i+1} - t^n_i, 0 \leq i \leq n - 1\} \to 0\), as \(n \to \infty\).

Since \(f \in M^2_G(0,T)\), there exists a \(f^n\) of the form \(f^n_s = \sum_{j=0}^{n-1} \xi^n_{t^n_j} I_{[t^n_j,t^n_{j+1})}(s)\), \(\xi^n_{t^n_j} \in L^2_G(F^n_t)\), such that

\[
\hat{E}\left[\int_0^t |f_s - f^n_s|^2 ds\right] \to 0, \text{ as } n \to \infty. \tag{4.1}
\]

Thanks to \(0 < C \leq |f|\), we have

\[
\frac{1}{C^2} \hat{E}\left[\int_0^t |f_s - (f^n_s)|^2 ds\right] \to 0, \text{ as } n \to \infty.
\]

Let

\[X_t := \int_0^t \frac{dM_s}{f_s}.\]

Then by Proposition 3.2 and Theorem 3.6, we know that \(X\) is a symmetric G-martingale and continuous. Now we prove that \(\{X^2_t - t\}_{t \in [0,T]}\) is a G-martingale and \(\hat{E}[X^2_t] = -\sigma^2_G t\), for all \(t \in [0,T]\).

From the assumptions of \(f\) as well as Proposition 3.9 it follows that the quadratic variation of \(M\) exists.

By inequality (4.1), we have, for any \(\varepsilon > 0\),

\[
\hat{E}\left[\int_0^t |f_s^2 - (f^n_s)^2| ds\right] \leq (1 + \frac{1}{\varepsilon}) \hat{E}\left[\int_0^t |f_s - f^n_s|^2 ds\right] + \varepsilon \hat{E}\left[\int_0^t |f_s|^2 ds\right]
\]

\[
\to \varepsilon \hat{E}\left[\int_0^t |f_s|^2 ds\right], \text{ as } n \to \infty.
\]

Therefore,

\[
\hat{E}\left[\int_0^t |f_s^2 - (f^n_s)^2| ds\right] \to 0, \text{ as } n \to \infty,
\]

and

\[
\hat{E}\left[\left|\int_0^t \frac{d\langle M\rangle_s}{f^2_s} - \int_0^t \frac{d\langle M\rangle_s}{(f^n_s)^2}\right|\right]
\]

\[
\leq \frac{1}{C^2} \hat{E}\left[\int_0^t |f_s^2 - (f^n_s)^2| ds\right] \to 0, \text{ as } n \to \infty. \tag{4.2}
\]

From the subadditivity of the G-expectation it follows that

\[
\hat{E}[-X^2_t] = \hat{E}[-\left(\int_0^t \frac{dM_r}{f^2_r}\right)^2]
\]

\[
\leq \hat{E}[-\left(\int_0^t \frac{dM_r}{f^2_r}\right)^2] + \hat{E}[-\left(\int_0^t \frac{dM_r}{f^2_r} - \int_0^t \frac{dM_r}{f^n_r}\right)^2]
\]

\[
+ 2\hat{E}[-\left(\int_0^t \frac{dM_r}{f^n_r}\right)(\int_0^t \frac{dM_r}{f_r} - \int_0^t \frac{dM_r}{f^n_r})]
\]

\[\to 0, \text{ as } n \to \infty.\]
By the inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\hat{E}[ -X_t^2 ] \leq \hat{E}[ -( \int_0^t dM^2_r / f^2_r ) ] + \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM^2_r - \int_0^t dM_r / f^2_r )^2 ] + \varepsilon \hat{E}[ ( \int_0^t dM_r / f^2_r )^2 ]$$

$$= \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (M^2_{i+1} - M^2_i) ] + \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM_r - \int_0^t dM_r / f^2_r )^2 ] + \varepsilon \hat{E}[ ( \int_0^t d(M)_s / (f^2_s )^2 ]$$

$$\leq \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (M^2_{i+1} - M^2_i - \sigma^2 \int_{t_i}^{t_{i+1}} f^2_s ds) ] + \sigma^2_0 \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 \int_{t_i}^{t_{i+1}} f^2_s ds ]$$

$$+ \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM_r - \int_0^t dM_r / f^2_r )^2 ] + \varepsilon \hat{E}[ ( \int_0^t d(M)_s / (f^2_s )^2 ]$$

$$\leq \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (M^2_{i+1} - M^2_i - \sigma^2 \int_{t_i}^{t_{i+1}} f^2_s ds) ] + \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM_r - \int_0^t dM_r / f^2_r )^2 ]$$

$$+ \sum_{i=0}^{n-1} \xi_i^2 \int_{t_i}^{t_{i+1}} (f^2_s )^2 ds ] + \sigma^2_0 \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 \int_{t_i}^{t_{i+1}} (f^2_s )^2 ds ]$$.

Thanks to (i), we obtain

$$\hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (M^2_{i+1} - M^2_i - \sigma^2 \int_{t_i}^{t_{i+1}} f^2_s ds) ] = 0.$$

Therefore, from inequalities (4.1) and (4.2) we have

$$\hat{E}[ -X_t^2 ] \leq \sigma^2_0 \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (\int_{t_i}^{t_{i+1}} f^2_s ds - \int_{t_i}^{t_{i+1}} (f^2_s )^2 ds ) ] - \sigma^2_0 t$$

$$+ \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM_r - \int_0^t dM_r / f^2_r )^2 ] + \varepsilon \hat{E}[ ( \int_0^t d(M)_s / (f^2_s )^2 ]$$

$$\leq \frac{1}{\varepsilon} \hat{E}[ ( \int_0^t dM_r - \int_0^t dM_r / f^2_r )^2 ] + \frac{\varepsilon}{C^2} \hat{E}[ (M)_T ] + \frac{\sigma^2_0}{C^2} \hat{E}[ \int_0^t |f^2_s - (f^2_s )^2 | ds ] - \sigma^2_0 t$$

$$\to -\sigma^2_0 t + \frac{\varepsilon}{C^2} \hat{E}[ (M)_T ], \text{ as } n \to \infty.$$

On the other hand, from the subadditivity of the G-expectation it follows that

$$\hat{E}[ -X_t^2 ] \geq \hat{E}[ -( \int_0^t dM^2_r / f^2_r ) ] - \hat{E}[ ( \int_0^t dM^2_r - \int_0^t dM_r / f^2_r )^2 ]$$

$$- 2\hat{E}[ ( \int_0^t dM^2_r ) ( \int_0^t dM_r / f^2_r - \int_0^t dM^2_r ) ]$$

$$\geq \hat{E}[ -( \int_0^t dM^2_r / f^2_r ) ] - (1 + \frac{1}{\varepsilon}) \hat{E}[ ( \int_0^t dM^2_r - \int_0^t dM_r / f^2_r )^2 ] - \varepsilon \hat{E}[ ( \int_0^t dM^2_r / f^2_r )^2 ]$$

$$= \hat{E}[ -\sum_{i=0}^{n-1} \xi_i^2 (M^2_{i+1} - M^2_i) ] - (1 + \frac{1}{\varepsilon}) \hat{E}[ ( \int_0^t dM^2_r - \int_0^t dM_r / f^2_r )^2 ] - \varepsilon \hat{E}[ ( \int_0^t dM^2_r / f^2_r )^2 ].$$
\[ \mathbb{E}[-X_t^2] \geq -(1 + \frac{1}{\varepsilon})\mathbb{E}[\int_0^t \frac{dM_r}{f_r} - \int_0^t \frac{dM_r}{f^n_r}]^2 - \varepsilon \mathbb{E}[\int_0^t \frac{dM_r}{f_r}]^2 \]

From (i) and the inequalities (4.1) and (4.2) again it follows that

\[ \mathbb{E}[-X_t^2] \geq -(1 + \frac{1}{\varepsilon})\mathbb{E}[\int_0^t \frac{dM_r}{f_r} - \int_0^t \frac{dM_r}{f^n_r}]^2 - \varepsilon \mathbb{E}[\int_0^t \frac{dM_r}{f_r}]^2 - \sigma_0^2 t \]

Thus, \( \mathbb{E}[-X_t^2] = -\sigma_0^2 t \), for all \( t \in [0, T] \).

Now we prove that \( \{X_s^2 - t\}_{t \in [0, T]} \) is a G-martingale. Let \( 0 \leq s \leq t \leq T \). Then

\[
\mathbb{E}[X_t^2 - X_s^2 | \mathcal{H}_s] = \mathbb{E}[\int_s^t \frac{dM_r}{f_r}^2 + 2(\int_s^t \frac{dM_r}{f_r})(\int_0^s \frac{dM_r}{f_r}) | \mathcal{H}_s] = \mathbb{E}[\int_s^t \frac{d(M_r^r)}{f_r^2} | \mathcal{H}_s] \leq t - s.
\]

On the other hand, from the subadditivity of G-expectation again it follows that

\[
\mathbb{E}[X_t^2 - X_s^2 | \mathcal{H}_s] \geq \mathbb{E}[\int_s^t \frac{dM_r}{f_r}^2 | \mathcal{H}_s] - \varepsilon \mathbb{E}[\int_s^t \frac{dM_r}{f_r} - \int_s^t \frac{dM_r}{f^n_r}]^2 | \mathcal{H}_s] 
\]

By virtue of (i), we have

\[
\mathbb{E}[X_t^2 - X_s^2 | \mathcal{H}_s] \geq (1 - \varepsilon)\mathbb{E}[\sum_{i=0}^{n-1} \xi_i (M^{i+1}_{t_{i+1}} - M^2_{t_{i+1}}) | \mathcal{H}_s] - \varepsilon \mathbb{E}[\int_s^t \frac{dM_r}{f_r} - \int_s^t \frac{dM_r}{f^n_r}]^2 | \mathcal{H}_s]
\]
From inequalities (4.1) and (4.2) it follows that

\[
- (1 - \varepsilon)\hat{E}[-\sum_{i=0}^{n-1} \xi_i t_i^n f_s^n dr | \mathcal{H}_s] < \frac{1 - \varepsilon}{C^2} \hat{E} \left[ \int_s^t \left( f_s^n - f_r^n \right)^2 dr \right] + \frac{1}{\varepsilon} \hat{E} \left[ \int_s^t \frac{dM_r}{f_r} - \int_s^t \frac{dM_r}{f^n_r} \right]^2 \rightarrow - (1 - \varepsilon) (t - s), as n \rightarrow \infty.
\]

Therefore,

\[
\hat{E}[-\hat{E}[X_t^n - X_s^n | \mathcal{H}_s] + (t - s)] \leq 0.
\]

The above inequality and (4.3) yields

\[
\hat{E}[X_t^n - t | \mathcal{H}_s] = X_s^n - s, q.s., for all 0 \leq s \leq t \leq T,
\]

which means \(\{X_t^n - t\}_{t \in [0, T]}\) is a G-martingale. Consequently, from Lemma 4.1 and Remark 4.2 we know that \(X\) is a G-Brownian motion with a parameter \(\sigma_0\).

Now we prove \((ii) \Rightarrow (i)\). From Peng [7] we know that \(M\) is a symmetric G-martingale. Now put

\[
Y_t := M_t^2 - \int_0^t f_s^n ds, for all t \in [0, T].
\]

We use \(\pi = \{0 = t_0^n < t_1^n \cdots < t_n^n = T\}\) to denote a partition of \([0, T]\) such that \(\max\{t_{i+1}^n - t_i^n, 0 \leq i \leq n - 1\} \rightarrow 0, as n \rightarrow \infty\).

For \(f \in M^n_G(0, T)\), there exists an \(f^n\) of the form \(f_t^n = \sum_{j=0}^{n-1} \xi_t, \mathcal{I}_{t_j, t_{j+1}}(t)\), where \(\xi_t \in L^2_G(\mathcal{F}_t)\), \(0 \leq i \leq n - 1\), such that

\[
\hat{E} \left[ \int_0^T |f_s - f_s^n|^2 ds \right] \rightarrow 0, as n \rightarrow \infty.
\]

Then we have

\[
\hat{E} \left[ \int_0^T f_s^2 ds - \int_0^T (f_s^n)^2 ds \right] \rightarrow 0, as n \rightarrow \infty,
\]

and

\[
\hat{E} \left[ \int_0^T f_s^n dB_s \right]^2 - (\int_0^T f_s^2 dB_s)^2 \rightarrow 0, as n \rightarrow \infty.
\]
Let
\[ Y^n_t := (\int_0^t f^n_r dB_r)^2 - \int_0^t (f^n_r)^2 dr, \text{ for all } t \in [0, T]. \]

Let \( 0 \leq s \leq t \leq T \). We suppose that \( s = t^n_k \leq \cdots \leq t^n_{k+l-1} \leq t \leq t^n_{k+l} \), for some \( k, l = 1, \cdots, n-1 \) such that \( k+l \leq n \). Thus,
\[
\hat{E}[Y^n_t | \mathcal{H}_{t^n_{k+l-1}}] = \hat{E}[\left( \int_0^t f^n_r dB_r \right)^2 - \int_0^t (f^n_r)^2 dr | \mathcal{H}_{t^n_{k+l-1}}]
\]
\[
= \hat{E}[\left( \int_0^{t^n_{k+l-1}} f^n_r dB_r \right)^2 - \int_0^{t^n_{k+l-1}} (f^n_r)^2 dr + (\xi^n_{t^n_{k+l-1}})^2(\xi^n_{t^n_{k+l-1}})^2 - (\xi^n_{t^n_{k+l-1}})^2(t - t^n_{k+l-1}) | \mathcal{H}_{t^n_k}]
\]
\[
= (\int_0^{t^n_{k+l-1}} f^n_r dB_r)^2 - \int_0^{t^n_{k+l-1}} (f^n_r)^2 dr + \hat{E}[(\xi^n_{t^n_{k+l-1}})^2(\xi^n_{t^n_{k+l-1}})^2 - (\xi^n_{t^n_{k+l-1}})^2(t - t^n_{k+l-1}) | \mathcal{H}_{t^n_k}]
\]
\[
= (\int_0^{t^n_{k+l-1}} f^n_r dB_r)^2 - \int_0^{t^n_{k+l-1}} (f^n_r)^2 dr
\]
\[
= Y^n_{t^n_{k+l-1}}.
\]

Therefore,
\[
\hat{E}[Y^n_t | \mathcal{H}_s] = \hat{E}[\hat{E}[Y^n_t | \mathcal{H}_{t^n_{k+l-1}}] | \mathcal{H}_s] = \hat{E}[Y^n_{t^n_{k+l-1}} | \mathcal{H}_s] = \cdots = Y^n_s.
\]

Consequently,
\[
\hat{E}[|\hat{E}[Y_t | \mathcal{H}_s] - Y_s|] \leq \hat{E}[|\hat{E}[Y^n_t | \mathcal{H}_s] - \hat{E}[Y^n_t | \mathcal{H}_s]|] + \hat{E}[|\hat{E}[Y^n_t | \mathcal{H}_s] - Y^n_s|] + \hat{E}[|Y^n_s - Y_s|]
\]
\[
\leq \hat{E}[|Y^n_t - Y_t|] + \hat{E}[|Y^n_s - Y_s|].
\]

From inequalities (4.3) and (4.6) it follows that
\[
\hat{E}[|\hat{E}[Y_t | \mathcal{H}_s] - Y_s|] \leq \hat{E}[|Y^n_t - Y_t|] + \hat{E}[|Y^n_s - Y_s|]
\]
\[
\to 0, \text{ as } n \to \infty.
\]

Thus,
\[
\hat{E}[Y_t | \mathcal{H}_s] = Y_s, \text{ q.s. for all } 0 \leq s \leq t \leq T.
\]

In the similar argument we can prove that \(-M^2 + \sigma_0^2 \int_0^t f^2_s ds\) is a G-martingale. The proof is complete. \(\Box\)

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