A NOTE ON DIRICHLET $L$-FUNCTIONS

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Abstract: We study the relation between the size of $L(1, \chi)$ and the width of the zero-free interval to the left of that point.

1. Introduction

Let $\chi(\text{mod } D)$ be a real, primitive character of conductor $D$ and

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

its Dirichlet $L$-function. We are interested in the size of $L(1, \chi)$. The Generalized Riemann Hypothesis implies

$$\left(\log \log D\right)^{-1} \ll L(1, \chi) \ll \log \log D .$$

Unconditionally, we have the easy upper bound

$$L(1, \chi) \ll \log D ,$$

but what one might expect to be the corresponding lower bound,

$$L(1, \chi) \gg (\log D)^{-1} ,$$

is not yet known to hold in general. We know that (1.2) does hold apart from rare exceptions.

There is a similar situation with respect to the location of the largest real zero, say $\beta$, of $L(s, \chi)$. Apart from rare exceptions one has the bound

$$1 - \beta \gg (\log D)^{-1} ,$$

and one expects that this always holds. Moreover, there is a close connection between the two phenomena and here we intend to study how close this is. In one direction this is clear, thanks to the following result of E. Hecke (see E. Landau [La]).

THEOREM 1. If (1.3) holds, then so does (1.2).

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In this note we make a modest step toward the reversal of this implication. It is easy to see from the equation

\[ L'(s, \chi) = - \sum_{n \leq D} \chi(n)(\log n)n^{-s} + O(D^{-\frac{3}{2}}(\log D)^2) , \]

valid if \( s - 1 \ll (\log D)^{-1} \), that

(1.4) \[ L'(s, \chi) \ll (\log D)^2 \]

there, and hence, by the mean-value theorem of differential calculus, that

(1.5) \[ 1 - \beta \ll (\log D)^{-3} \]

implies

(1.6) \[ L(1, \chi) \ll (\log D)^{-1} . \]

Using deeper arguments we can say a bit more.

**THEOREM 2.** If \( L(s, \chi) \) has a real zero \( \beta \) with

(1.7) \[ 1 - \beta \ll (\log D)^{-3} \log\log D , \]

then (1.6) holds.

It seems surprising not to be able to do better. One expects, in the case of an extremely small value of \( L(1, \chi) \), that \( \chi \) mimics the Möbius function and in such a situation \( L'(s, \chi) \), rather than being limited by (1.4), should be almost (though not quite) bounded, hence that (1.2) should imply something only slightly weaker than (1.3). The limitation in the bound of Theorem 2 comes from our imperfect knowledge about the complex zeros. At the end of this note we describe (in Theorem 3) how the bound can be substantially sharpened if we assume the Riemann Hypothesis holds apart from an exceptional real zero.

There is an extensive literature on the subject discussed in this note. We encourage the reader to obtain a broader perspective through the publications [GoSc], [Go], [Pi], [MV], [GrSo], [SaZa].

**2. Relations between \( \beta \) and \( L(1, \chi) \)**

From now on we assume that \( 1 - \beta \leq (3 \log D)^{-1} \). Denote \( \lambda = 1 * \chi \) and

\[ Z(s) = \zeta(s)L(s, \chi) = \sum_{1}^{\infty} \lambda(n)n^{-s} . \]
We evaluate the smoothly cropped sum

\[ S(x) = \sum_{n \leq x} \lambda(n) \left( 1 - \frac{n}{x} \right) n^{-\beta} \]

by contour integration of \( Z(s) \) as follows:

\[
S(x) = \frac{1}{2\pi i} \int_{(1)} Z(s + \beta) \frac{x^s}{s(s+1)} \, ds
= L(1, \chi) \frac{x^{1-\beta}}{(1-\beta)(2-\beta)} + O \left( x^{-1} D^\frac{1}{2} \log D \right).
\]

Hence,

**Proposition 2.1.** Assume \( L(s, \chi) \) has a real zero \( \beta \) with \( 1 - \beta \leq (3 \log D)^{-1} \). Then

\[ L(1, \chi) \simeq (1 - \beta) S(D). \]

Therefore, our problem reduces to the estimation of \( S(D) \). Note that trivially \( S(D) \gg 1 \) and this gives Hecke’s Theorem 1.

3. **Upper Bound for** \( S(D) \)

We have

\[ S(D) \ll \sum_{n \leq D} \frac{\lambda(n)}{n} \ll \prod_{p \leq D} \left( 1 + \frac{\lambda(p)}{p} \right). \]

We split this product into

\[ \prod_{p \leq B} \left( 1 + \frac{\lambda(p)}{p} \right) \leq \prod_{p \leq B} \left( 1 + \frac{2}{p} \right) \simeq (\log B)^2 \]

and

\[ \prod_{B < p \leq D} \left( 1 + \frac{\lambda(p)}{p} \right) \leq \exp \left( T(D)/ \log B \right), \]

where

\[ T(D) = \sum_{p \leq D} \frac{\lambda(p)}{p} \log p. \]

Up to now our estimates have been rather simple, but to estimate \( T(D) \) we require deeper tools, namely the Deuring-Heilbronn repulsion property of \( \beta \). Put

\[ \eta = 1/(1 - \beta) \log D \geq 3. \]
The repulsion property asserts that \( L(s, \chi) \) has no zeros other than \( \beta \) in the region

\[
\sigma > 1 - \frac{c \log \eta}{\log D(|t| + 1)}, \quad s = \sigma + it
\]

where \( c \) is an absolute positive constant (cf. Théorème 16 of E. Bombieri [B]).

A very nice way of using the repulsion property, together with a quite delicate estimate for the number of zeros of \( L(s, \chi) \) in small discs centered on \( \text{Re} \, s = 1 \) (cf. Ch. X, Lemma 2.1 of [Pr]), to bound sums of \( \lambda(p) \) over primes has been given by D.R. Heath-Brown. We borrow from his work the estimate (Lemma 3 of [H-B1])

\[
(3.6) \quad T(D) \ll (\log \eta)^{-\frac{1}{2}} \log D .
\]

Choosing \( \log B = (\log \eta)^{-\frac{1}{2}} \log D \) we obtain

\[
(3.7) \quad S(D) \ll (\log \eta)^{-1}(\log D)^2 .
\]

4. Conclusion

From (2.2) and (3.7) we get

\[
(4.1) \quad L(1, \chi) \ll \frac{1 - \beta}{\log \eta}(\log D)^2 .
\]

In particular, if \( \beta \) satisfies (1.7) then \( \eta \gg \log D \) and (4.1) gives (1.6).

5. Remarks Behind the Scenes

We take this opportunity to share some of our impressions about the nature of the arguments used in this note. The main issue is the question of how the exceptional zero is connected with the rarity of small primes which split in the quadratic field \( \mathbb{Q}(\sqrt{-1}D) \). A quick connection is displayed in the bound (see (24.20) of [FI4])

\[
(5.1) \quad \sum_{z < p \leq x} \lambda(p)p^{-1} \ll (1 - \beta) \log x ,
\]

valid for \( x > z \geq D^2 \). This shows that if

\[
(5.2) \quad 1 - \beta = o(1/\log D) ,
\]

then the splitting primes in the segments \( D^2 < p \leq D^A \) are very rare. One can easily deduce the same conclusion from the assumption (see (24.19) of [FI4])

\[
(5.3) \quad L(1, \chi) = o(1/\log D) .
\]
The deficiency of such primes is the driving force for finding prime numbers in many interesting sequences; cf. [H-B1], [H-B2], [FI3]. One of these is the proof by Heath-Brown that the existence of infinitely many exceptional zeros implies the existence of infinitely many twin primes. Another example is the implication to primes of the form \( p = a^2 + b^6 \).

In our series of papers [FI1–FI5] we used assumptions of type (5.3) rather than (5.2) and for those applications they serve the same purpose.

Note that (5.1) says nothing about splitting primes which are very small relative to the conductor \( D \). In all of these applications the rarity of small splitting primes was not needed, but for Theorem 2 it is essential. The current technology allows one to penetrate this territory, but only barely, due to the Deuring-Heilbronn repulsion property of the exceptional zero \( \beta \). Heath-Brown’s Lemma 3 of [H-B1] does just that!

We include, for curiosity, an alternative derivation of (2.2). We consider the function

\[
F(s) = (e^\gamma D)^s L(s + 1, \chi),
\]

which satisfies the conditions \( F(\beta - 1) = 0, F(0) = L(1, \chi), \)
\[
F'(0) = L(1, \chi)(\log D + \gamma) + L'(1, \chi)
\]
and \( F''(s) \ll (\log D)^3 \) if \( s - 1 \ll (\log D)^{-1} \). Hence, by the Taylor expansion of \( F(\beta - 1) \) at \( s = 0 \) we get

\[
L(1, \chi) = (1 - \beta)[L(1, \chi)(\log D + \gamma) + L'(1, \chi)]
\]
\[
+ O\big((1 - \beta)^2(\log D)^3\big).
\]

On the other hand, we have (cf. (22.109) of [IK])

\[
\sum_{n \leq D} \lambda(n)n^{-1} = L(1, \chi)(\log D + \gamma) + L'(1, \chi) + O\left(D^{-\frac{3}{4}} \log D\right).
\]

Combining (5.5) and (5.6) we obtain

\[
L(1, \chi) = (1 - \beta) \sum_{n \leq D} \lambda(n)n^{-1} + O\left((1 - \beta)(\log D)^3(1 - \beta D^{-\frac{3}{4}})\right).
\]

For our purposes, (5.7) and (2.2) amount to the same thing.

Note that, under the assumption (5.3) the formula (5.7) implies

\[
L'(1, \chi) \sim \sum_{n \leq D} \lambda(n)n^{-1},
\]
which can be compared with (22.117) of [IK].
There are infinitely many real primitive characters $\chi \pmod{D}$ with $D$ prime and $\chi(p) = 1$ for every $p < c \log D$. For such characters we have
\begin{equation}
\sum_{n \leq D} \lambda(n)n^{-1} \gg (\log \log D)^2 .
\end{equation}
Hence, if the largest real zero of $L(s, \chi)$ satisfies (3.5), we have
\begin{equation}
L(1, \chi) \gg (1 - \beta)(\log \log D)^2 .
\end{equation}
Therefore, (1.6) cannot hold for such special discriminants unless $L(s, \chi)$ has a zero $\beta$ with $1 - \beta \ll (\log D)^{-1}(\log \log D)^{-2}$.

6. The Ultimate Deuring-Heilbronn Phenomenon

In this final section we investigate the extent of improvement in the conclusion of Theorem 2 which could be obtained if one assumed the GRH for $L(s, \chi)$ apart from one real zero $\beta$. Let us assume that $\beta > \frac{3}{4}$ is the only zero of $L(s, \chi)$ in $\Re s > \frac{3}{4}$. We proceed along the lines of J. E. Littlewood [Li], beginning with the formula (cf. (5.58) of [IK])
\begin{equation}
- \frac{L'}{L}(\sigma, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n)(1 - \frac{n}{x})n^{-\sigma}
- \sum_\rho (\rho - \sigma)^{-1}(\rho - \sigma + 1)^{-1}x^{\rho - \sigma} + O(1 + x^{-\frac{1}{4}} \log D)
\end{equation}
which is valid for $1 \leq \sigma \leq 5/4$ with any $x \geq 1$, the implied constant being absolute. Here $\rho$ runs through the zeros of $L(s, \chi)$ in the critical strip. Separating $\rho = \beta$ and estimating the other terms trivially we find
\begin{equation}
- \frac{L'}{L}(\sigma, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n)n^{-\sigma}
- (\beta - \sigma)^{-1}(\beta - \sigma + 1)^{-1}x^{\beta - \sigma} + O(1 + x^{-\frac{1}{4}} \log D) .
\end{equation}
We take $x = (\log D)^4$ so the error term in (6.2) is bounded. Integrating (6.2) over $1 \leq \sigma \leq 5/4$ we obtain
\begin{equation*}
\log L(1, \chi) = \sum_{p \leq x} \frac{\chi(p)}{p} - \int_{1-\beta}^{5/4-\beta} x^{-t}t^{-1}(1-t)^{-1}dt + O(1) .
\end{equation*}
Up to a bounded error term, the integral is equal to
\begin{equation*}
\int_{1-\beta}^{5/4-\beta} x^{-t}t^{-1}dt = \int_0^{\infty} e^{-t}t^{-1}dt = -e^{-\vartheta} \log \vartheta + \int_0^{\infty} e^{-t}(\log t)dt ,
\end{equation*}
where \( \vartheta = (1 - \beta) \log x \). Here, we have \( e^{-\vartheta} \log \vartheta = (\log \vartheta)/(\vartheta + 1) + O(1) \) and the last integral is bounded. Hence,

\[
\log L(1, \chi) = \sum_{p \leq x} \chi(p)p^{-1} + (\log \vartheta)/(\vartheta + 1) + O(1).
\]

In other words

\[
L(1, \chi) \asymp \omega \exp \left( \sum_{p \leq \log D} \chi(p)p^{-1} \right)
\]

where

\[
\omega = \min \{1, (1 - \beta) \log \log D\}.
\]

This proves

**THEOREM 3.** Assuming that \( \beta > 3/4 \) is the only zero of \( L(s, \chi) \) in \( \text{Re } s > 3/4 \) we have (6.3). Hence

\[
\omega (\log \log D)^{-1} \ll L(1, \chi) \ll \omega (\log \log D).
\]

In particular, if \( 1 - \beta \ll (\log \log D)^{-1} \), then

\[
1 - \beta \ll L(1, \chi) \ll (1 - \beta) (\log \log D)^{2}.
\]

**References**

[B] E. Bombieri, Le Grand Crible dans la Théorie Analytique des Nombres, *Astérisque* 18, Soc. Math. France, 2ième ed., (Paris) 1987/1974.

[FI1] J.B. Friedlander and H. Iwaniec, Exceptional zeros and prime numbers in arithmetic progressions, *Int. Math. Res. Notices* 37 (2003), 2033–2050.

[FI2] J.B. Friedlander and H. Iwaniec, Exceptional zeros and prime numbers in short intervals, *Selecta Math.* 10 (2004), 61–69.

[FI3] J.B. Friedlander and H. Iwaniec, The illusory sieve, *Int. JNT.* 1 (2005), 459–494.

[FI4] J.B. Friedlander and H. Iwaniec, Opera de Cribro, *Amer. Math. Soc. Colloq. Pub.* 57 AMS (Providence), 2010.

[FI5] J.B. Friedlander and H. Iwaniec, Exceptional discriminants are the sum of a square and a prime, *Quart. J. Math.* 64 (2013), 1099–1107.

[Go] D. Goldfeld, An asymptotic formula relating the Siegel zero and class number of quadratic fields, *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.* (4) 2 (1975), 611–615.

[GoSc] D.M. Goldfeld and A. Schinzel, On Siegel’s zero, *Ann. Sc. Norm. Sup. Pisa, Cl. Sci.* (4) 2 (1975), 571–583.

[GrSo] A. Granville and K. Soundararajan, The distribution of values of \( L(1, \chi_d) \), *Geom. Funct. Anal.* 13 (2003), 992–1028.

[H-B1] D.R. Heath-Brown, Prime twins and Siegel zeros, *Proc. London Math. Soc.* (3) 47 (1983), 193–224.

[H-B2] D.R. Heath-Brown, Siegel zeros and the least prime in an arithmetic progression, *Quart. J. Math. Oxford* (2) 41 (1990), 405–418.
[IK] H. Iwaniec and E. Kowalski, Analytic Number Theory, *Amer. Math. Soc. Colloq. Pub.* 53 AMS (Providence), 2004.

[La] E. Landau, Über die Klassenzahl imaginär-quadratischer Zahlkörper, *Gött. Nachr.* (1918), 285–295.

[Li] J.E. Littlewood, On the class-number of the corpus $P(\sqrt{-k})$, *Proc. London Math. Soc.* (2) 27 (1928), 358–372.

[MV] H.L. Montgomery and R.C. Vaughan, Extreme values of Dirichlet $L$-functions at 1, *Number theory in progress Vol. 2*, (Zakopane-Kościelisko, 1997) pp. 1039–1052, de Gruyter (Berlin) 1999.

[Pi] J. Pintz, Elementary methods in the theory of $L$-functions, II On the greatest real zero of a real $L$-function, *Acta Arith.* XXXI (1976), 273–289.

[Pr] K. Prachar, Primzahlverteilung, *Grundlehren der Math. Wiss.* XCI Springer (Berlin) 1957.

[SaZa] P. Sarnak and A. Zaharescu, Some remarks on Landau-Siegel zeros, *Duke Math. J.* (2) 111 (2002), 495–507.

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