LOG-SOBOLEV INEQUALITIES: DIFFERENT ROLES OF RIC AND HESS

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Let \( P_t \) be the diffusion semigroup generated by \( L := \Delta + \nabla V \) on a complete connected Riemannian manifold with \( \text{Ric} \geq -(\sigma^2 \rho^2_0 + c) \) for some constants \( \sigma, c > 0 \) and \( \rho_0 \) the Riemannian distance to a fixed point. It is shown that \( P_t \) is hypercontractive, or the log-Sobolev inequality holds for the associated Dirichlet form, provided \( -\text{Hess}_V \geq \delta \) holds outside of a compact set for some constant \( \delta > (1 + \sqrt{2})\sigma \sqrt{d-1} \). This indicates, at least in finite dimensions, that Ric and \(-\text{Hess}_V\) play quite different roles for the log-Sobolev inequality to hold. The supercontractivity and the ultracontractivity are also studied.

1. Introduction. Let \( M \) be a \( d \)-dimensional completed connected noncompact Riemannian manifold and \( V \in C^2(M) \) such that

\[
Z := \int_M e^{V(x)} \, dx < \infty,
\]

where \( dx \) is the volume measure on \( M \). Let \( \mu(dx) = Z^{-1} e^{V(x)} \, dx \). Under (1.1) it is easy to see that \( H_0^{2,1}(\mu) = W^{2,1}(\mu) \), where \( H_0^{2,1}(\mu) \) is the completion of \( C^1_0(M) \) under the Sobolev norm \( \| f \|_{2,1} := \mu(f^2 + |\nabla f|^2)^{1/2} \), and \( W^{2,1}(\mu) \) is the completion of the class \( \{ f \in C^1(M) : f + |\nabla f| \in L^2(\mu) \} \) under \( \| \cdot \|_{2,1} \). Then the \( L \)-diffusion process is nonexplosive and its semigroup \( P_t \) is uniquely determined. Moreover, \( P_t \) is symmetric in \( L^2(\mu) \) so that \( \mu \) is \( P_t \)-invariant. It is well known by the Bakry–Emery criterion (see [4]) that

\[
\text{Ric} - \text{Hess}_V \geq K
\]

for some constant \( K > 0 \) implies the Gross log-Sobolev inequality [14],

\[
\mu(f^2 \log f^2) := \int_M f^2 \log f^2 \, d\mu \leq C \mu(|\nabla f|^2),
\]

\[
\mu(f^2) = 1, \, f \in C^1(M)
\]
for $C = 2/K$. This result was extended by Chen and the author [9] to the situation that $\text{Ric} - \text{Hess}_V$ is uniformly positive outside a compact set. In the case that $\text{Ric} - \text{Hess}_V$ is bounded below, sufficient concentration conditions of $\mu$ for (1.3) to hold are presented in [1, 19, 20]. Obviously, in a condition on $\text{Ric} - \text{Hess}_V$ the Ricci curvature and $-\text{Hess}_V$ play the same role.

What can we do when $\text{Ric} - \text{Hess}_V$ is unbounded below? It seems very hard to confirm the log-Sobolev inequality with the unbounded below condition of $\text{Ric} - \text{Hess}_V$. Therefore, in this paper we try to clarify the roles of $\text{Ric}$ and $-\text{Hess}_V$ in the study of the log-Sobolev inequality. Let us first recall the gradient estimate of $P_t$, which is a key point in the above references to prove the log-Sobolev inequality.

Let $x_t$ be the $L$-diffusion process starting at $x$, and let $v \in T_xM$. Due to Bismut [6] and Elworthy–Li [11], under a reasonable lower bound condition of $\text{Ric} - \text{Hess}_V$, one has

$$\langle \nabla P_t f, v \rangle = \mathbb{E} \langle \nabla f(x_t), v_t \rangle, \quad t > 0, f \in C_0^1(M),$$

where $v_t \in T_{x(t)}M$ solves the equation

$$D_t v_t := \lim_{t \to 0} \frac{1}{t} \frac{d}{dt} \langle \nabla f(x_t), v_t \rangle = -(\text{Ric} - \text{Hess}_V)(v_t)$$

for $\langle \nabla P_t f, v \rangle = \mathbb{E} \langle \nabla f(x_t), v_t \rangle, \quad t > 0, f \in C_0^1(M)$.

Thus, for the gradient of $P_t$, which is a short distance behavior of the diffusion process, a condition on $\text{Ric} - \text{Hess}_V$ appears naturally.

On the other hand, however, $\text{Ric}$ and $-\text{Hess}_V$ play very different roles for long distance behaviors. For instance, let $\rho_o$ be the Riemannian distance function to a fixed point $o \in M$. If $\text{Ric} \geq -k$ and $-\text{Hess}_V \geq \delta$ for some $k \geq 0, \delta \in \mathbb{R}$, the Laplacian comparison theorem implies

$$L \rho_o \leq \sqrt{k(d-1)} \coth(\sqrt{k/(d-1)} \rho_o) - \delta \rho_o.$$ 

Therefore, for large $\rho_o$, the Ric lower bound leads to a bounded term while that of $-\text{Hess}_V$ provides a linear term. The same phenomena appears in the formula on distance of coupling by parallel displacement (cf. [3], (2.3), (2.4)), which implies the above Bismut–Elworthy–Li formula by letting the initial distance tend to zero (cf. [15]). Here, $k \geq 0$ is essential for our framework, since the manifold has to be compact, if $\text{Ric}$ is bounded below by a positive constant.

Since the log-Sobolev inequality is always available on bounded regular domains, it is more likely a long-distance property of the diffusion process. So, $\text{Ric}$ and $-\text{Hess}_V$ should take different roles in the study of the log-Sobolev inequality. Indeed, it has been observed by the author [20] that (1.3) holds for some
\( C > 0 \), provided \( \text{Ric} \) is bounded below and \( -\text{Hess}_V \) is uniformly positive outside a compact set. This indicates that for the log-Sobolev inequality, the positivity of \( -\text{Hess}_V \) is a dominative condition, which allows the Ricci curvature to be bounded below by an arbitrary negative constant, and hence, allows \( \text{Ric} - \text{Hess}_V \) to be globally negative on \( M \).

The first aim of this paper is to search for the weakest possibility of curvature lower bound for the log-Sobolev inequality to hold under the condition

\[
-\text{Hess}_V \geq \delta \quad \text{outside a compact set}
\]

for some constant \( \delta > 0 \). This condition is reasonable as the log-Sobolev inequality implies \( \mu(e^{\lambda \rho_o^2}) < \infty \) for some \( \lambda > 0 \) (see, e.g., [2, 17]).

According to the following Theorem 1.1 and Example 1.1, we conclude that under (1.4) the optimal curvature lower bound condition for (1.3) to hold is

\[
\inf_M (\text{Ric} + \sigma^2 \rho_o^2) > -\infty
\]

for some constant \( \sigma > 0 \), such that \( \delta > (1 + \sqrt{2})\sigma \sqrt{d-1} \). More precisely, let \( \theta_0 > 0 \) be the smallest positive constant, such that for any connected complete noncompact Riemannian manifold \( M \) and \( V \in C^2(M) \), such that \( Z := \int_M e^V \rho_o^2 \, dx < \infty \), the conditions (1.4) and (1.5) with \( \delta > \sigma \theta_0 \sqrt{d-1} \), implies (1.3) for some \( C > 0 \). Due to Theorem 1.1 and Example 1.1 below, we conclude that

\[
\theta_0 \in [1, 1 + \sqrt{2}].
\]

The exact value of \( \theta_0 \) is however unknown.

**Theorem 1.1.** Assume that (1.4) and (1.5) hold for some constants \( c, \delta, \sigma > 0 \) with \( \delta > (1 + \sqrt{2})\sigma \sqrt{d-1} \). Then (1.3) holds for some \( C > 0 \).

**Example 1.1.** Let \( M = \mathbb{R}^2 \) be equipped with the rotationally symmetric metric

\[
ds^2 = dr^2 + (re^{kr^2})^2 d\theta^2,
\]

under the polar coordinates \((r, \theta) \in [0, \infty) \times S^1 \) at 0, where \( k > 0 \) is a constant, then (see, e.g., [13])

\[
\text{Ric} = -\frac{\left(\frac{d^2}{dr^2}(re^{kr^2})\right)}{re^{kr^2}} = -4k - 4k^2 r^2.
\]

Thus, (1.5) holds for \( \sigma = 2k \). Next, take \( V = -k \rho_o^2 - \lambda(\rho_o^2 + 1)^{1/2} \) for some \( \lambda > 0 \). By the Hessian comparison theorem and the negativity of the sectional curvature, we obtain (1.4) for \( \delta = 2k \). Since \( d = 2 \) and

\[
e^V \, dx = re^{-\lambda(1+r^2)^{1/2}} \, dr \, d\theta,
\]
one has $Z < \infty$ and $\delta = 2k = \sigma \sqrt{d - 1}$. But the log-Sobolev inequality is not valid since by Herbst’s inequality it implies $\mu(e^{r\rho_o^2}) < \infty$ for some $r > 0$, which is, however, not the case due to (1.6). Since in this example one has $\delta > \sigma \theta \sqrt{d - 1}$ for any $\theta < 1$, according to the definition of $\theta_0$, we conclude that $\theta_0 \geq 1$.

Following the line of [19, 20], the key point in the proof of Theorem 1.1 will be a proper Harnack inequality of type

$$(P^t f(x))^\alpha \leq C_\alpha (t, x, y) P^t f^\alpha(y), \quad t > 0, x, y \in M,$$

for any nonnegative $f \in C_b(M)$, where $\alpha > 1$ is a constant and $C_\alpha \in C((0, \infty), M^2)$ is a positive function. Such an inequality was established in [19] for $\text{Ric} - \text{Hess}_V$ bounded below and extended in [3] to a more general situation with $\text{Ric}$ satisfying (1.5).

The Harnack inequality presented in [3] contains a leading term $\exp[\rho(x, y)^4]$, which is, however, too large to be integrability w.r.t. $\mu \times \mu$ under our conditions. So, to prove Theorem 1.1, we shall present a sharper Harnack inequality in Section 3 by refining the coupling method introduced in [3] (see Proposition 3.1 below). This inequality, together with the concentration of $\mu$ ensured by (1.4) and (1.5), will imply the hypercontractivity of $P_t$. To establish this new Harnack inequality, some necessary preparations are presented in Section 2.

Finally, in the same spirit of Theorem 1.1, the supercontractivity and ultracontractivity of $P_t$ are studied in Section 4 under explicit conditions on $\text{Ric}$ and $-\text{Hess}_V$.

2. Preparations. We first study the concentration of $\mu$ by using (1.4) and (1.5), for which we need to estimate $L_{\rho_o}$ from above according to [5] and references within.

**Lemma 2.1.** If (1.4) and (1.5) hold, then there exists a constant $C_1 > 0$ such that

$$(L_{\rho_o})^2 \leq C_1 (1 + \rho_o) - 2(\delta - \sigma \sqrt{d - 1}) \rho_o^2$$

holds outside $\text{cut}(o)$, the cut-locus of $o$. If moreover $\delta > \sigma \sqrt{d - 1}$ then $Z < \infty$ and $\mu(e^{\lambda \rho_o^2}) < \infty$ for all $\lambda < \frac{1}{2} (\delta - \sigma \sqrt{d - 1})$.

**Proof.** By (1.5) we have $\text{Ric} \geq -(c + \sigma^2 \rho_o^2)$ for some constant $c > 0$. By the Laplacian comparison theorem this implies that

$$\Delta \rho_o \leq \sqrt{(c + \sigma^2 \rho_o^2)/(d - 1)} \coth[\sqrt{(c + \sigma^2 \rho_o^2)/(d - 1)} \rho_o]$$

holds outside $\text{cut}(o)$. Thus, outside $\text{cut}(o)$ one has

$$\Delta \rho_o^2 \leq 2 \rho_o \sqrt{(c + \sigma^2 \rho_o^2)/(d - 1)} \coth[\sqrt{(c + \sigma^2 \rho_o^2)/(d - 1)} \rho_o] + 2$$

$$\leq 2d + 2 \rho_o \sqrt{(c + \sigma^2 \rho_o^2)/(d - 1)},$$

outside $\text{cut}(o)$. Therefore, we conclude that

$$(L_{\rho_o})^2 \leq C_1 (1 + \rho_o) - 2(\delta - \sigma \sqrt{d - 1}) \rho_o^2$$

holds outside $\text{cut}(o)$. The proof is completed.
where the second inequality follows from the fact that

\[ r \cosh r \leq (1 + r) \sinh r, \quad r \geq 0. \]

On the other hand, for \( x \notin \text{cut}(o) \) and \( U \) the unit tangent vector along the unique minimal geodesic \( \ell \) form \( o \) to \( x \), by (1.4) there exists a constant \( c_1 > 0 \) independent of \( x \) such that

\[ \langle \nabla V, \nabla \rho_o \rangle (x) = \langle \nabla V, U \rangle (o) + \int_0^{\rho_o(x)} \text{Hess}_V(U, U)(\ell_s) \, ds \leq c_1 - \delta \rho_o(x). \]

Combining this with (2.2) we prove (2.1).

Finally, let \( \delta > \sigma \sqrt{d - 1} \) and \( 0 < \lambda < \frac{1}{2} (\delta - \sigma \sqrt{d - 1}) \). By (2.1) we have

\[
Le^{\lambda \rho_o^2} \leq e^{\lambda \rho_o^2} (C_1(1 + \rho_o) - 2(\delta - \sigma \sqrt{d - 1})\rho_o^2 + 4\lambda \rho_o^2)
\leq c_2 - c_3 \rho_o^2 e^{\lambda \rho_o^2}
\]

for some constants \( c_2, c_3 > 0 \). By [5], Proposition 3.2, this implies \( Z < \infty \) and

\[
\int_M \rho_o^2 e^{\lambda \rho_o^2} \, d\mu \leq \frac{C_2}{c_3} < \infty.
\]

**Lemma 2.2.** Let \( x_t \) be the \( L \)-diffusion process with \( x_0 = x \in M \). If (1.4) and (1.5) hold with \( \delta > \sigma \sqrt{d - 1} \), then for any \( \delta_0 \in (\sigma \sqrt{d - 1}, \delta) \) there exists a constant \( C_2 > 0 \) such that

\[
\mathbb{E} \exp \left[ \frac{(\delta_0 - \sigma \sqrt{d - 1})^2}{4} \int_0^T \rho_o(x_t)^2 \, dt \right] 
\leq \exp \left[ C_2 T + \frac{1}{4} (\delta_0 - \sigma \sqrt{d - 1}) \rho_o(x)^2 \right], \quad T > 0, x \in M.
\]

**Proof.** By Lemma 2.1, we have

\[
L \rho_o^2 \leq C - 2(\delta_0 - \sigma \sqrt{d - 1}) \rho_o^2
\]

outside cut(\( o \)) for some constant \( C > 0 \). Then the Itô formula for \( \rho_o(x_t) \) due to Kendall [16] implies that

\[
d \rho_o^2(x_t) \leq 2 \sqrt{2} \rho_o(x_t) \, db_t + [C - 2(\delta_0 - \sigma \sqrt{d - 1}) \rho_o^2(x_t)] \, dt
\]

holds for some Brownian motion \( b_t \) on \( \mathbb{R} \). This implies that the \( L \)-diffusion process is nonexplosive so that

\[
T_n := \inf \{ t \geq 0 : \rho_o(x_t) \geq n \} \to \infty
\]

as \( n \to \infty \). Indeed, (2.3) implies that

\[
n \mathbb{P}(T_n \leq t) \leq \mathbb{E} \rho_o(x_t \wedge T_n)^2 \leq \rho_o(x)^2 + Ct, \quad n \geq 1, t > 0.
\]
Hence, $\mathbb{P}(T_n \leq t) \to 0$ as $n \to \infty$ for any $t > 0$. This implies $\lim_{n \to \infty} T_n = \infty$ a.s.

For any $\lambda > 0$ and $n \geq 1$, it follows from (2.3) that
\[
\mathbb{E}\exp \left[ 2\lambda (\delta_0 - \sigma \sqrt{d-1}) \int_0^{T \wedge T_n} \rho_o^2(x_t) \, dt \right] 
\leq e^{\lambda \rho_o^2(x) + C \lambda T} \mathbb{E}\exp \left[ 2\sqrt{2} \lambda \int_0^{T \wedge T_n} \rho_o(x_t) \, db_t \right] 
\leq e^{\lambda \rho_o^2(x) + C \lambda T} \left( \mathbb{E}\exp \left[ 16\lambda^2 \int_0^{T \wedge T_n} \rho_o^2(x_t) \, dt \right] \right)^{1/2},
\]
where in the last step we have used the inequality
\[
\mathbb{E}e^{M_t} \leq \left( \mathbb{E}e^{2\langle M \rangle_t} \right)^{1/2}
\]
for $M_t = 2\sqrt{2} \int_0^{T \wedge T_n} \rho_o(X_s) \, db_s$. This follows immediately from the Schwartz inequality and the fact that $\exp[2M_t - 2\langle M \rangle_t]$ is a martingale. Thus, taking
\[
\lambda = \frac{1}{8} (\delta_0 - \sigma \sqrt{d-1}),
\]
we obtain
\[
\mathbb{E}\exp \left[ \frac{1}{4} (\delta_0 - \sigma \sqrt{d-1})^2 \int_0^{T \wedge T_n} \rho_o^2(x_t) \, dt \right] 
\leq \exp \left[ \frac{1}{4} (\delta_0 - \sigma \sqrt{d-1}) \rho_o^2(x) + C_2 T \right]
\]
for some $C_2 > 0$. Then the proof is completed by letting $n \to \infty$. $\Box$

Finally, we recall the coupling argument introduced in [3] for establishing the Harnack inequality of $P_t$.

Let $T > 0$ and $x \neq y \in M$ be fixed. Then the $L$-diffusion process starting from $x$ can be constructed by solving the following Itô stochastic differential equation:
\[
d_I x_t = \sqrt{2} \Phi_t \, dB_t + \nabla V(x_t) \, dt, \quad x_0 = x,
\]
where $d_I$ is the Itô differential on manifolds introduced in [12] (see also [3]), $B_t$ is the $d$-dimensional Brownian motion, and $\Phi_t$ is the horizontal lift of $x_t$ onto the orthonormal frame bundle $O(M)$.

To construct another diffusion process $y_t$ starting from $y$ such that $x_T = y_T$, as in [3], we add an additional drift term to the equation (as explained in [3], Section 3, we may and do assume that the cut-locus of $M$ is empty)
\[
d_I y_t = \sqrt{2} P_{x,y} \Phi_t \, dB_t + \nabla V(y_t) \, dt + \xi_t U(x_t, y_t) 1_{[t < \tau]} \, dt, \quad y_0 = y,
\]
where \( P_{x_t,y_t} \) is the parallel transformation along the unique minimal geodesic \( \ell \) from \( x_t \) to \( y_t \), \( U(x_t, y_t) \) is the unit tangent vector of \( \ell \) at \( y_t \), \( \xi_t \geq 0 \) is a smooth function of \( x_t \) to be determined, and
\[
\tau := \inf \{ t \geq 0 : x_t = y_t \}
\]
is the coupling time. Since all terms involved in the equation are regular enough, there exists a unique solution \( y_t \). Furthermore, since the additional term containing \( 1_{\{ t < \tau \}} \) vanishes from the coupling time on, one has \( x_t = y_t \) for \( t \geq \tau \) due to the uniqueness of solutions.

**Lemma 2.3.** Assume that (1.4) and (1.5) hold with \( \delta \geq 2\sigma \sqrt{d-1} \). Then there exists a constant \( C_3 > 0 \) independent of \( x, y \) and \( T \) such that \( x_T = y_T \) holds for \( \xi_t := C_3 + 2\sigma \sqrt{d-1} \rho_o(x_t) + \frac{\rho(x,y)}{T} \).

**Proof.** According to Section 2 in [3], we have
\[
d\rho(x_t, y_t) = \{ I(x_t, y_t) + \langle \nabla V, \nabla \rho(\cdot, y_t) \rangle(x_t) \\
+ \langle \nabla V, \nabla \rho(x_t, \cdot) \rangle(y_t) - \xi_t \} dt, \quad t < \tau,
\]
where
\[
I_Z(x_t, y_t) = \sum_{i=1}^{d-1} \int_0^{\rho(x_t,y_t)} \left( |\nabla U J_i|^2 - \langle R(U, J_i)U, J_i \rangle(\ell_s) \right) ds
\]
for \( R \) the Riemann curvature tensor, \( U \) the unit tangent vector of the minimal geodesic \( \ell : [0, \rho(x_t, y_t)] \rightarrow M \) from \( x_t \) to \( y_t \), and \( \{ J_i \}_{i=1}^{d-1} \) the Jacobi fields along \( \ell \), which, together with \( U \), consist of an orthonormal basis of the tangent space at \( x_t \) and \( y_t \) and satisfy
\[
J_i(y_t) = P_{x_t,y_t} J_i(x_t), \quad i = 1, \ldots, d - 1.
\]
By (1.5) we take a constant \( c \geq 0 \) such that \( \text{Ric} \geq -(c + \sigma^2 \rho_o^2) \). Letting
\[
K(x_t, y_t) = \sup_{\ell([0, \rho(x_t, y_t)])} \{ c + \sigma^2 \rho_o^2 \},
\]
we obtain from Wang [21], Theorem 2.14 (see also [7, 8]), that
\[
I(x_t, y_t) \leq 2\sqrt{K(x_t, y_t)(d-1)} \tanh \left[ \frac{\rho(x_t,y_t)}{2} \sqrt{K(x_t, y_t)/(d-1)} \right].
\]
Moreover, by (1.4) there exist two constants \( r_0, r_1 > 0 \) such that \( -\text{Hess}_V \geq \delta \) outside \( B(o, r_0) \) but \( \leq r_1 \) on \( B(o, r_0) \), where \( B(o, r_0) \) is the closed geodesic ball
at \( o \) with radius \( r_0 \). Since the length of \( \ell \) contained in \( B(o, r_0) \) is less than \( 2r_0 \), we conclude that
\[
(\nabla V, \nabla \rho(x_t, y_t)) + (\nabla V, \nabla \rho(x_t, y_t))
\]
\[
= \int_0^{\rho(x_t, y_t)} \text{Hess}_V(U, U)(\ell_s) \, ds \leq 2r_0r_1 - (\rho(x_t, y_t) - 2r_0)^+ \delta
\]
\[
\leq c_1 - \delta \rho(x_t, y_t)
\]
for some constant \( c_1 > 0 \). Combining this with (2.4), (2.5) and
\[
\xi_t = C_3 + 2\sqrt{d-1} \rho_o(x_t) + \frac{\rho(x, y)}{T},
\]
we arrive at
\[
d \rho(x_t, y_t) \leq \left\{ 2\sqrt{K(x_t, y_t)(d-1)} + c_1 - \delta \rho(x_t, y_t)
\right.
\]
\[
\left. - C_3 - 2\sigma \sqrt{d-1} \rho_o(x_t) - \frac{\rho(x, y)}{T} \right\} dt
\]
for \( t < \tau \). Noting that
\[
\sqrt{K(x_t, y_t)} \leq (c + \sigma^2[\rho_o(x_t) + \rho(x_t, y_t)]^2)^{1/2}
\]
\[
\leq \sqrt{c} + \sigma[\rho_o(x_t) + \rho(x_t, y_t)],
\]
and \( \delta \geq 2\sigma \sqrt{d-1} \), one has
\[
2\sqrt{K(x_t, y_t)(d-1)} - \delta \rho(x_t, y_t) - 2\sigma \sqrt{d-1} \rho_o(x_t) \leq 2\sqrt{c(d-1)}.
\]
Thus, when \( C_3 \geq c_1 + 2\sqrt{c(d-1)} \) we have
\[
d \rho(x_t, y_t) \leq -\frac{\rho(x, y)}{T} \, dt, \quad t < \tau,
\]
so that
\[
0 = \rho(x_\tau, y_\tau) \leq \rho(x, y) - \int_0^\tau \frac{\rho(x, y)}{T} \, dt = \frac{T - \tau}{T} \rho(x, y),
\]
which implies that \( \tau \leq T \) and hence, \( x_T = y_T \). \( \square \)

3. Harnack inequality and proof of Theorem 1.1. We first prove the following Harnack inequality using results in Section 2.

**Proposition 3.1.** Assume that (1.4) and (1.5) hold with \( \delta > (1 + \sqrt{2})\sigma \times \sqrt{d-1} \). Then there exist \( C > 0 \) and \( \alpha > 1 \) such that
\[
(P_T f(y))^\alpha \leq (P_T f^{\alpha}(x)) \exp \left[ \frac{C}{T} \rho(x, y)^2 + C(T + \rho_o(x)^2) \right]
\]
holds for all \( x, y \in M \), \( T > 0 \) and nonnegative \( f \in C_b(M) \).
PROOF. According to Lemma 2.3, we take
\[ \xi_t = C_3 + 2\sigma \sqrt{d - 1} \rho_o(x_t) + \frac{\rho(x, y)}{T}, \]
such that \( \tau \leq T \) and \( x_T = y_T \). Obviously, \( y_t \) solves the equation
\[ d_I y_t = \sqrt{2} \Phi_t d \tilde{B}_t + \nabla V(y_t) \, dt \]
for \( \tilde{B}_t := P_{x_t, y_t} \Phi_t \) being the horizontal lift of \( y_t \), and \( \tilde{B}_t \) solving the equation
\[ d \tilde{B}_t = d B_t + \frac{1}{\sqrt{2}} \Phi_t^{-1} \xi_t U(x_t, y_t) 1_{\{t < \tau\}} \, dt. \]
By the Girsanov theorem and the fact that \( \tau \leq T \), the process \( \{ \tilde{B}_t : t \in [0, T]\} \) is a \( d \)-dimensional Brownian motion under the probability measure \( R \mathbb{P} \) for
\[ R := \exp \left[ -\frac{1}{\sqrt{2}} \int_0^\tau \langle P_{x_t, y_t} \Phi_t d B_t, \xi_t U(x_t, y_t) \rangle - \frac{1}{4} \int_0^\tau \xi_t^2 \, dt \right]. \]
Thus, under this probability measure \( \{ y_t : t \in [0, T]\} \) is generated by \( L \). In particular, \( P_T f(y) = \mathbb{E}[f(y_T) R] \). Combining this with the H"older inequality and noting that \( x_T = y_T \), we obtain
\[ P_T f(y) = \mathbb{E}[f(y_T) R] = \mathbb{E}[f(x_T) R] \leq (P_T f^\alpha(x))^{1/\alpha} (\mathbb{E} R^{\alpha/(\alpha - 1)})^{(\alpha - 1)/\alpha}. \]
That is,
\[ (P_T f(y))^\alpha \leq (P_T f^\alpha(x)) (\mathbb{E} R^{\alpha/(\alpha - 1)})^\alpha. \]
(3.2)
Since for any continuous exponential integrable martingale \( M_t \) and any \( \beta, p > 1 \), the process \( \exp[\beta p M_t - \beta^2 p^2 \langle M_t \rangle] \) is a martingale, by the H"older inequality one has
\[ \mathbb{E} e^{\beta M_t - (\beta^2/2) \langle M_t \rangle} = \mathbb{E}\left[e^{\beta M_t - (\beta^2/2) \langle M_t \rangle} e^{(\beta(2p-1)/2) \langle M_t \rangle}\right] \leq \mathbb{E}\left[e^{(\beta p(p-1)/(2(p-1))) \langle M_t \rangle}\right]^{(p-1)/p}. \]
(3.3)
By taking \( \beta = \alpha/(\alpha - 1) \) we obtain
\[ (\mathbb{E} R^{\alpha/(\alpha - 1)})^\alpha \leq \left\{ \mathbb{E} \exp \left[ \frac{p \alpha(\alpha - 1) + 1}{8(p - \alpha - 1)^2} \int_0^T \xi_t^2 \, dt \right] \right\}^{(\alpha - 1)(p-1)/p}, \quad p > 1. \]
(3.4)
Since \( \delta > (1 + \sqrt{2}) \sigma \sqrt{d - 1} \), we may take \( \delta_0 \in ((1 + \sqrt{2}) \sigma \sqrt{d - 1}, \delta) \), small \( \epsilon' > 0 \) and large \( C_4 > 0 \), independent of \( T, x \) and \( y \), such that
\[ \xi_t^2 = \left( C_3 + 2\sigma \sqrt{d - 1} \rho_o(x_t) + \frac{\rho(x, y)}{T} \right)^2 \leq (1 - \epsilon') \left[ C_4 + \frac{C_4 \rho(x, y)^2}{T^2} + 2(\delta_0 - \sigma \sqrt{d - 1})^2 \rho_o(x_t)^2 \right] \]
holds. Moreover, since
\[
\lim_{p \downarrow 1} \lim_{\alpha \uparrow \infty} \frac{p\alpha(p\alpha - \alpha + 1)}{8(p-1)(\alpha-1)^2} = \frac{1}{8},
\]
there exist \(p, \alpha > 1\) such that
\[
\frac{p\alpha(p\alpha - \alpha + 1)}{8(p-1)(\alpha-1)^2} \int_0^T \xi_t^2 \, dt
\leq C_4T + \frac{C_4\rho(x,y)^2}{T} + \frac{(\delta_0 - \sigma \sqrt{d-1})^2}{4} \int_0^T \rho_o(x_t)^2 \, dt.
\]
Combining this with (3.4) and Lemma 2.2, we obtain
\[
(\mathbb{E} R^{\alpha/(\alpha-1)})^{\alpha-1} \leq \exp \left[ C_5T + \frac{C_5\rho(x,y)}{T} + (\delta_0 - \sigma \sqrt{d-1})^2 \right], \quad T > 0, x \in M,
\]
for some constant \(C_5 > 0\). This completes the proof by (3.2).

**Proof of Theorem 1.1.** By Proposition 3.1, let \(\alpha > 1\) and \(C > 0\) such that (3.1) holds. Since \(\delta > \sigma \sqrt{d-1}\), we may take \(T > 0\) such that
\[
\frac{C}{T} \leq \epsilon := \frac{1}{8}(\delta - \sigma \sqrt{d-1}).
\]
Then for any nonnegative \(f \in C_b(M)\) with \(\mu(f^\alpha) = 1\), since \(\mu\) is \(P_T\)-invariant, it follows from (3.1) that
\[
1 = \int_M P_T f^\alpha(x) \mu(dx) \geq (P_T f(y))^\alpha \int_M e^{-\epsilon\rho(x,y)^2-\epsilon(1+\rho_o(x))^2} \mu(dx)
\geq (P_T f(y))^\alpha \int_{\{\rho_o \leq 1\}} e^{-\epsilon(1+\rho_o(y))^2-2C} \mu(dx)
\geq \epsilon'(P_T f(y))^\alpha \exp[-2\epsilon\rho_o(y)^2], \quad y \in M,
\]
for some constant \(\epsilon' > 0\). Thus,
\[
\int_M (P_T f(y))^{2\alpha} \mu(dy) \leq \frac{1}{\epsilon'} \int_M e^{4\epsilon\rho_o(y)^2} \mu(dy) < \infty,
\]
according to Lemma 2.1. This implies that
\[
\|P_T\|_{L^{\alpha}(\mu) \rightarrow L^{2\alpha}(\mu)} < \infty.
\]
Therefore, the log-Sobolev inequality (1.3) holds for some constant \(C > 0\), due to the uniformly positively improving property of \(P_t\) (see [20], proof of Theorem 1.1, and [1]).
4. Supercontractivity and ultracontractivity. Recall that $P_t$ is called supercontractive if $\|P_t\|_{2\to4} < \infty$ for all $t > 0$ while ultracontractive if $\|P_t\|_{2\to\infty} < \infty$ for all $t > 0$ (see [10]). In the present framework these two properties are stronger than the hypercontractivity: $\|P_t\|_{2\to4} \leq 1$ for some $t > 0$, which is equivalent to (1.3) due to Gross [14].

**Proposition 4.1.** Under (1.4) and (1.5), $P_t$ is supercontractive if and only if $\mu(\exp[\lambda \rho_o^2]) < \infty$ for all $\lambda > 0$, while it is ultracontractive if and only if $\|P_t \exp[\lambda \rho_o^2]\|_{\infty} < \infty$ for all $t, \lambda > 0$.

**Proof.** The proof is similar to that of [18], Theorem 2.3. Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$. By (3.1) for $\alpha = 2$ and noting that $\mu$ is $P_t$-invariant, we obtain

$$1 \geq (P_T f(y))^2 \int_M \exp \left[-\frac{C}{T} \rho(x, y)^2 - C(T + \rho_o(x)^2)\right] \mu(dx)$$

$$\geq (P_T f(y))^2 \exp \left[-\frac{2C}{T} \rho_o(y)^2 + 1 - C(T+1)\right] \mu(B(o, 1)).$$

Hence, for any $T > 0$ there exists a constant $\lambda_T > 0$ such that

$$|P_T f| \leq \exp[\lambda_T (1 + \rho_o^2)], \quad T > 0, \mu(f^2) = 1.$$  \hfill (4.1)

(1) If $\mu(e^{\lambda \rho_o^2}) < \infty$ for any $\lambda > 0$, (4.1) yields that

$$\|P_T\|_{2\to4}^4 \leq \mu(e^{4\lambda_T (1+\rho_o^2)}) < \infty, \quad T > 0.$$  

Conversely, if $P_t$ is supercontractive then the super log-Sobolev inequality (cf. [10])

$$\mu(f^2 \log f^2) \leq r \mu(|\nabla f|^2) + \beta(r), \quad r > 0, \mu(f^2) = 1,$$

holds for some $\beta : (0, \infty) \to (0, \infty)$. By [2] (see also [17, 18]), this inequality implies $\mu(e^{\lambda \rho_o^2}) < \infty$ for all $\lambda > 0$.

(2) By (4.1) and the semigroup property,

$$\|P_T\|_{2\to\infty} \leq \|P_{T/2} e^{\lambda_T/2 (1+\rho_o^2)}\|_{\infty} < \infty, \quad T > 0,$$

provided $\|P_t e^{\lambda \rho_o^2}\|_{\infty} < \infty$ for any $t, \lambda > 0$. Conversely, since the ultracontractivity is stronger than the supercontractivity, it implies that $e^{\lambda \rho_o^2} \in L^2(\mu)$ for any $\lambda > 0$ as explained above. Therefore,

$$\|P_t e^{\lambda \rho_o^2}\|_{\infty} \leq \|P_t\|_{2\to\infty} \|e^{\lambda \rho_o^2}\|_{2} < \infty, \quad \lambda > 0.$$
Then the proof is completed. □

To derive explicit conditions for the supercontractivity and ultracontractivity, we consider the following stronger version of (1.4):

\[(4.2) \quad -\text{Hess}v \geq \Phi \circ \rho_o \quad \text{holds outside a compact subset of } M\]

for a positive increasing function \(\Phi\) with \(\Phi(r) \uparrow \infty\) as \(r \uparrow \infty\). We then aim to search for reasonable conditions on positive increasing function \(\Psi\) such that

\[(4.3) \quad \text{Ric} \geq -\Psi \circ \rho_o\]

implies the supercontractivity and/or ultracontractivity.

**Theorem 4.2.** If (4.3) and (4.2) hold for some increasing positive functions \(\Phi\) and \(\Psi\) such that

\[(4.4) \quad \lim_{r \to \infty} \Phi(r) = \lim_{r \to \infty} \frac{(\int_0^r \Phi(s) \, ds)^2}{\Phi(r)} = \infty,\]

\[(4.5) \quad \sqrt{\Psi(r+t)(d-1)}\]

\[\leq \theta \int_0^r \Phi(s) \, ds + \frac{1}{2} \int_0^{t/2} \Phi(s) \, ds + C, \quad r, t \geq 0,\]

for some constants \(\theta \in (0, 1/(1 + \sqrt{2}))\) and \(C > 0\). Then \(P_t\) is supercontractive. Furthermore, if

\[(4.6) \quad \int_1^\infty \frac{ds}{\sqrt{s} \int_0^{\sqrt{s}} \Phi(u) \, du} < \infty,\]

then \(P_t\) is ultracontractive. More precisely, for

\[\Gamma_1(r) := \frac{1}{\sqrt{r}} \int_0^{\sqrt{r}} \Phi(s) \, ds, \quad \Gamma_2(r) := \int_r^\infty \frac{ds}{\sqrt{s} \int_0^{\sqrt{s}} \Phi(u) \, du}, \quad r > 0,\]

(4.6) implies

\[(4.7) \quad \|P_t\|_{2 \to \infty} \leq \exp \left[ c + \frac{c}{t} (1 + \Gamma_1^{-1}(c/t) + \Gamma_2^{-1}(t/c)) \right] < \infty, \quad t > 0,\]

for some constant \(c > 0\) and

\[\Gamma_1^{-1}(s) := \inf\{t \geq 0 : \Gamma_1(t) \geq s\}, \quad s \geq 0.\]
PROOF. (a) Replacing \( c + \rho^2 \) by \( \Psi \circ \rho \) and noting that \( \text{Hess}_V \leq -\Phi \circ \rho \) for large \( \rho \), the proof of Lemma 2.1 implies

\[
L^2 \rho^2 \leq c_1 (1 + \rho) - 2 \rho \left( \int_0^{\rho} \Phi(s) \, ds - \sqrt{\Psi \circ \rho}(d - 1) \right)
\]

for some constant \( c_1 > 0 \). Combining this with (4.5) and noting that \( \frac{1}{\rho} \times \int_0^{\rho} \Phi(s) \, ds \to \infty \) as \( \rho \to \infty \), we conclude that for any \( \lambda > 0 \),

\[
Le^{\lambda \rho^2} \leq C - \frac{2 \lambda \rho^2}{1 + \sqrt{2}} e^{\lambda \rho^2} \int_0^{\rho} \Phi(s) \, ds + 4 \lambda^2 \rho^2 e^{\lambda \rho^2}
\]

\[
\leq C + C(\lambda) - \lambda \rho^2 \exp \left[ \frac{\lambda}{\Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)} \right] \leq \exp \left[ \frac{4 \lambda + 2 \lambda \Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)}{\Gamma_1^{-1}(4(1 + \sqrt{2})^2 \lambda)} \right] < \infty.
\]

Therefore, (1.1) holds and

\[
\mu(e^{\lambda \rho^2}) < \infty, \quad \lambda > 0.
\]

(b) By (4.5), (4.8) and Kendall’s Itô formula [16] as in the proof of Lemma 2.2, we have

\[
d \rho^2(x_t) \leq 2 \sqrt{2} \rho(x_t) \, db_t + \left( C_1 - \frac{2 \sqrt{2} \rho(x_t)(1 + \varepsilon)}{1 + \sqrt{2}} \int_0^{\rho(x_t)} \Phi(s) \, ds \right) dt
\]

for some constants \( \varepsilon, C_1 > 0 \), where \( x_t \) and \( b_t \) are in the proof of Lemma 2.2. Let

\[
\varphi(r) = \int_0^r \frac{ds}{\sqrt{s}} \int_0^{\sqrt{s}} \Phi(u) \, du, \quad r \geq 0.
\]

We arrive at

\[
d \varphi \circ \rho^2(x_t) \leq 2 \sqrt{2} \rho(x_t) \varphi' \circ \rho^2(x_t) \, db_t + 4 \rho^2(x_t) \varphi'' \circ \rho^2(x_t) dt
\]

\[
+ \varphi' \circ \rho^2(x_t) \left( C_1 - \frac{2 \sqrt{2} \rho(x_t)(1 + \varepsilon)}{1 + \sqrt{2}} \int_0^{\rho(x_t)} \Phi(s) \, ds \right) dt.
\]
From (4.4) we see that
\[
\frac{\rho_o \varphi'' \circ \rho_o^2}{\varphi' \circ \rho_o^2 \int_0^{\rho_o} \Phi(s) \, ds} \leq \frac{\Phi \circ \rho_o}{2(\int_0^{\rho_o} \Phi(s) \, ds)^2},
\]
which goes to zero as \( \rho_o \to \infty \). Then there exists a constant \( C_2 > C_1 \) such that
\[
d\varphi \circ \rho_o^2(x_t) \leq 2\sqrt{2} \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right) \, db_t + C_2 \, dt - \frac{2\sqrt{2}}{1 + \sqrt{2}} \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right)^2 \, dt.
\]
This implies that for any \( \lambda > 0 \),
\[
\mathbb{E} \exp \left[ \frac{2\sqrt{2} \lambda}{1 + \sqrt{2}} \int_0^T \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right)^2 \, dt \right] \leq e^{C_2 \lambda T + \lambda \varphi \circ \rho_o^2(x)} \mathbb{E} \exp \left[ 2\sqrt{2} \lambda \int_0^T \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right) \, db_t \right] \leq e^{C_2 \lambda T + \lambda \varphi \circ \rho_o^2(x)} \left( \mathbb{E} \exp \left[ 16\lambda^2 \int_0^T \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right)^2 \, dt \right] \right)^{1/2}.
\]
Taking
\[
\lambda = \frac{\sqrt{2}}{8(1 + \sqrt{2})},
\]
we arrive at
\[
\mathbb{E} \exp \left[ \frac{1}{2(1 + \sqrt{2})^2} \int_0^T \left( \int_0^{\rho_o(x_t)} \Phi(s) \, ds \right)^2 \, dt \right] \leq e^{2C_2 T + \varphi \circ \rho_o^2(x) \sqrt{2}/8(1 + \sqrt{2})}.
\]

(c) Let \( \gamma : [0, \rho(x_t, y_t)] \to M \) be the minimal geodesic from \( x_t \) to \( y_t \), and \( U \) its tangent unit vector. By (4.2), there exists a constant \( C_3 > 0 \) such that
\[
\langle \nabla V, \nabla \rho(\cdot, y_t) \rangle(x_t) + \langle \nabla V, \nabla \rho(x_t, \cdot) \rangle(y_t)
\]
\[
= \int_0^{\rho(x_t, y_t)} \text{Hess}_V(U_s, U_s) \, ds \leq C_3 - \int_0^{\rho(x_t, y_t)/2} \Phi(s) \, ds.
\]
To understand the last inequality, we assume, for instance, that \( \rho_o(x_t) \geq \rho_o(y_t) \) so that by the triangle inequality,
\[
\rho_o(y_s) \geq \rho_o(x_t) - s \geq \rho(x_t, y_t)/2 - s, \quad s \in [0, \rho(x_t, y_t)/2].
\]
For the coupling constructed in Section 3, one concludes from \((4.14)\) and the proof of Lemma 2.3 that
\[
d\rho(x_t, y_t) \leq \begin{cases} 
2\sqrt{K(x_t, y_t)(d-1)} + C_4 \\
- \int_0^{\rho(x_t, y_t)/2} \Phi(s) ds - \xi_t \end{cases} dt,
\]
holds for some constant \(C_4 > 0\), where
\[
K(x_t, y_t) := \sup_{\ell([0,\rho(x_t, y_t)])} \Psi \circ \rho_0 \leq \Psi(\rho_0(x_t) + \rho(x_t, y_t)),
\]
and \(\ell\) is the minimal geodesic from \(x_t\) to \(y_t\). Combining \((4.5)\) and \((4.15)\), we obtain
\[
d\rho(x_t, y_t) \leq \begin{cases} 
C_4 + 2\theta \int_0^{\rho_0(x_t)} \Phi(s) ds - \xi_t \\
- \frac{\rho(x, y)}{T} \end{cases} dt,
\]
where \(\xi_t = C_4 + 2\theta \int_0^{\rho_0(x_t)} \Phi(s) ds + \frac{\rho(x, y)}{T}\).

So, taking
\[
\xi_t = C_4 + 2\theta \int_0^{\rho_0(x_t)} \Phi(s) ds + \frac{\rho(x, y)}{T},
\]
we arrive at
\[
d\rho(x_t, y_t) \leq -\frac{\rho(x, y)}{T} dt, \quad t < \tau.
\]
This implies \(\tau \leq T\), and hence \(x_T = y_T\) a.s.

Combining \((4.5)\) with \((3.4)\) and \((3.5)\) we conclude that for the present choice of \(\xi_t\) there exist \(\alpha, p, C_5 > 1\) such that
\[
(E \mathcal{R}^{\alpha/(\alpha - 1)})^{p/(p - 1)} \leq E \exp \left[ \frac{1}{2(1 + \sqrt{2})^2} \int_0^T \left( \int_0^{\rho(x_t)} \Phi(s) ds \right)^2 dt 
\right. 
+ C_5 T + \frac{C_5}{T} \rho(x, y)^2 \bigg].
\]
Combining this with \((4.13)\) and \((3.2)\) we obtain
\[
(P_T f(y))^\alpha \leq (P_T f^{\alpha}(x)) \exp \left[ C T + \frac{C}{T} \rho(x, y)^2 + C \varphi \rho^{2}(x) \right]
\]
holds for some \(\alpha, C > 1\), any positive \(f \in C_b(M)\) and all \(x, y \in M, T > 0\).

(d) For any positive \(f \in C_b(M)\) with \(\mu(f^{\alpha}) = 1\), \((4.16)\) implies that
\[
(P_T f(y))^\alpha \int_{B(o, 1)} \exp \left[ -CT - \frac{C}{T} \rho(x, y)^2 - C \varphi^2(x) \right] \mu(dx) \leq 1.
\]
Therefore, there exists a constant \(C' > 0\) such that
\[
(P_T f(y))^\alpha \leq \exp \left[ C'(1 + T) + \frac{C'}{T} \rho(y)^2 \right], \quad y \in M, T > 0.
\]
Combining this with (4.11) we obtain
\[ \| P_T \|_{\alpha \to p\alpha} < \infty, \quad T > 0, \ p > 1. \]
This is equivalent to the supercontactivity by the Riesz–Thorin interpolation theorem and \( \| P_t \|_{1 \to 1} = 1 \). Thus, the first assertion holds.

(e) To prove (4.7), it suffices to consider \( t \in (0, 1] \) since \( \| P_t \|_{2 \to \infty} \) is decreasing in \( t > 0 \). So, below we assume that \( T \leq 1 \). By (4.17) and the fact that \( (P_{2T}f)^\alpha \leq P_T (P_T f)^\alpha \), we have
\[ (4.18) \quad \| P_{2T} \|_{\alpha \to \infty} \leq \| P_T e^{2C' \rho_0^2 / T} \|_{\infty} e^{C'(1+T)}, \quad T > 0. \]
Therefore, by the Riesz–Thorin interpolation theorem and \( \| P_t \|_{1 \to 1} = 1 \), for the ultracontractivity it suffices to show that
\[ (4.19) \quad \| P_T e^{\lambda \rho_0^2} \|_{\infty} < \infty, \quad \lambda, T > 0. \]
Since \( \Phi \) is increasing, it is easy to check that
\[ \eta(r) := \sqrt{r} \int_0^{\sqrt{r}} \Phi(s) \, ds, \quad r \geq 0, \]
is convex, and so is \( s \mapsto s\eta(\log s / \lambda) \) for \( \lambda > 0 \). Thus, it follows from (4.9) and the Jensen inequality that
\[ h_{\lambda, x}(t) := \mathbb{E} e^{\lambda \rho_0^2 (x_t)} < \infty, \quad x_0 = x \in M, \ \lambda, t > 0, \]
and
\[ \frac{d^+}{dt} h_{\lambda, x}(t) \leq C + C(\lambda) - \lambda h_{\lambda, x}(t) \eta(\lambda^{-1} \log h_{\lambda, x}(t)), \quad t > 0. \]
This implies (4.19), provided (4.6) holds. This can be done by considering the following two situations:

(1) Since \( h_{\lambda, x}(t) \) is decreasing provided \( \lambda h_{\lambda, x}(t) \eta(\lambda^{-1} \log h_{\lambda, x}(t)) > C + C(\lambda) \), if
\[ \lambda h_{\lambda, x}(0) \eta(\lambda^{-1} \log h_{\lambda, x}(0)) \leq 2C + 2C(\lambda), \]
then
\[ h_{\lambda, x}(t) \leq \sup \{ r \geq 1 : \lambda r \eta(\lambda^{-1} \log r) \leq 2C + 2C(\lambda) \} \leq \frac{1}{\lambda} (2C + 2C(\lambda)) + C'' \]
for some constant \( C'' > 0 \).

(2) If \( \lambda h_{\lambda, x}(0) \eta(\lambda^{-1} \log h_{\lambda, x}(0)) > 2C + 2C(\lambda) \), then \( h_{\lambda, x}(t) \) is decreasing in \( t \) up to
\[ t_\lambda := \inf \{ t \geq 0 : \lambda h_{\lambda, x}(t) \eta(\lambda^{-1} \log h_{\lambda, x}(t)) \leq 2C + 2C(\lambda) \}. \]

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Indeed,
\[
\frac{d^+}{dt} h_{\lambda,x}(t) \leq -\frac{\lambda}{2} h_{\lambda,x}(t) \eta(\lambda^{-1} \log h_{\lambda,x}(t)), \quad t \leq t_\lambda.
\]
Thus,
\[
\int_0^\infty \frac{dr}{r h_{\lambda,x}(T \wedge t_\lambda) r \eta(\lambda^{-1} \log r)} \geq \frac{\lambda}{2} (T \wedge t_\lambda).
\]
This is equivalent to
\[
\Gamma_2(\lambda^{-1} \log h_{\lambda,x}(T \wedge t_\lambda)) \geq \frac{1}{2} (T \wedge t_\lambda).
\]
Hence,
\[
h_{\lambda,x}(T \wedge t_\lambda) \leq \exp[\lambda (1/2) (T \wedge t_\lambda)].
\]
Since it is reduced to case (1) if \( T > t_\lambda \) by regarding \( t_\lambda \) as the initial time, in conclusion we have
\[
\sup_{x \in M} h_{\lambda,x}(T) \leq \max \left\{ \exp[\lambda (1/2) (T/2)], \ C'' + \frac{1}{\lambda} (2C + 2C(\lambda)) \right\}.
\]
Therefore, (4.7) follows from (4.18), (4.10) with \( \lambda = 2C'/T \), and the Riesz interpolation theorem. ∎

Finally, we note that a simple example for conditions in Theorem 4.2 to hold is
\[
\Phi(s) = s^{\alpha-1}, \quad \Psi(s) = \varepsilon s^{2\alpha}
\]
for \( \alpha > 1 \) and small enough \( \varepsilon > 0 \). In this case \( P_t \) is ultracontractive with
\[
\| P_t \|_{2 \to \infty} \leq \exp\left[ c (1 + t^{-(\alpha+1)/(\alpha-1)}) \right], \quad t > 0,
\]
for some \( c > 0 \).

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