Elementarity in Open Systems
- Unstable Particle Decay in Medium -

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ABSTRACT

The problem of unstable particle decay is discussed to show how elementarity of a subsystem immersed in an infinitely larger environment is lost. The decay law, when the same kind of particles as decay product make up a thermal medium, is worked out in detail. The relic abundance of unstable particles does not suffer from the Boltzmann suppression crucial at very low temperatures, because the off shell contribution not considered in the Boltzmann approach, becomes dominant at low temperatures. The short-time behavior of the non-decay probability is also clarified, which is important to discuss physical relevance of the non-observation of nucleon decay. Two powerful methods in this respect are the operator and the path integral approach, both of which are reviewed.

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I Introduction

Ideal elementary processes rarely occur. They presumably never occur except in carefully prepared laboratory experiments. It might be useful to recall that even processes usually considered elementary should actually be regarded taking place in complex environments. This has to do with how one separates a system in question from a surrounding environment. For instance, the beta decay, the fundamental weak process $\text{n} \rightarrow \text{p} + \text{e} + \bar{\nu}_e$, when it occurs in nuclei, is compounded by effects of nuclear strong interaction with the rest of nucleons. The most spectacular of this kind is how nucleon decay proceeds. The extremely weak process of baryon number violating process at the quark-lepton level must inevitably occur in the hadronic environment, a nucleon. A repeated question on the nucleon decay raised in the past is how the extremely slow process of the lifetime of order $10^{31}$ years or even larger is modified by strong interaction having the time scale of order, $10^{-23}$ seconds. In this respect one must deal with the short time limit of the decay law, and one must face the fact that the exponential decay law is not exact in quantum mechanics.[1]

In many cases of analysis on a physical system, one focusses on a small subsystem put in an infinitely large closed system. Behavior of the subsystem then exhibits an apparent loss of elementarity due to interaction with the larger environment over which we have no control. Phrased somewhat differently, one often asks how a quantum system evolves with time under influence of an environment described by some mixed state. When one has a clear idea of how to separate the small system in question from the environment, the essential part of our problem becomes how elementary processes are modified in dissipative medium. Dissipation reflects the fact that we have ignorance of the larger environment. We believe that the problem should be analyzed from the first principles of quantum mechanics, at least at the conceptual level. In condensed matter physics the problem of this kind is generally known as quantum Brownian motion or quantum dissipation.

It has become increasingly clear to us that the bulk of the past works on quantum dissipation relies on the simplified form of quantum friction, namely dissipation which is local in time. This corresponds to the exponential decay law when one examines the fate of some initial excitation including the unstable particle. This approximation is excellent in the most dominant phase of dissipation. But it fails both in early time and very late time behavior of the quantum system immersed in dissipative medium. Some formalism exists to deal with more general nonlocal dissipation, but they are
not very useful to the problems we would like to address. Indeed, some of our new results have not been recognized in the past.

The basic idea taken by most approaches in the past [2], and also the one I use here, is that existence of a continuously infinite number of environment variables coupled to a finite number of subsystem variables is the essential part of dissipation. In this view the origin of dissipation is that once something flows from the small system to the infinite environment, then it practically never returns to the small system. Dissipation, or at least something recognized as such, occurs when one does not and cannot make a measurement on the environment. It is well known that even for a pure quantum system the entropy, a measure of dissipation, is nonvanishing once one traces out a part of the quantum system. Thus one postulates that detailed modeling of the environment and its coupling to the small system should be unimportant to the dissipative behavior of a small system. After integrating out environment variables, one should have only a few phenomenological parameters to describe the dissipation, ultimately obtainable from experiments. Despite this phenomenological nature one should base all arguments on rigorous quantum physics, and modeling is inevitable.

The simplest, yet the most fundamental model of quantum dissipation is harmonic oscillator coupled to infinitely many oscillators that make up a bath in a mixed state. As an ideal limit of the mixed state one can also consider a pure quantum state. The nonlinearity not considered here is presumably important to restore the complete equilibrium of the small plus the environment system. But our interest is solely in the behavior of the small system, which is modified by interaction with a larger environment, and we are not interested in the environment part. For this discussion the linear approximation should be adequate.

The problem of how a small system behaves under influence of a larger environment has been investigated in a variety of approaches. Two powerful methods to analyze this problem are the quantum Langevin equation, and the path integral approach [4]. Both approaches have merits and demerits, but when combined, they become very powerful. I shall first give a fundamental result from the operator approach, which makes clear the meaning of some basic functions frequently used later. In the second part of this lecture I also explain the path integral approach. Our main results are in our two papers already in print [5], [6].

In this lecture I shall exclusively discuss one single problem of this rather broad
subject. It is the decay process in medium. The problem we set up is described as follows. Suppose that one would like to know how the decay of unstable particles proceeds when they decay into two particles that also make up the bulk of the environment. It is not clear how the decay law known in the pure quantum case persists or is modified. For instance, is the well known time dependence \[1\], namely the exponential law followed by the power law decay modified? An indication of substantial modification to this decay law is suggested by the following consideration. Some amount of parent particles is clearly left behind in medium, even much later than the decay lifetime, because in thermal environment even a heavier parent particle may be created by energetic daughter particles of smaller mass. If this is the case, what is the fraction of the parent particles left behind? We would like to answer these questions and elucidate the basic law by using a rigorous result of the general formalism. I shall first give a fairly self-contained discussion of the general formalism, because in some literature an unnecessary approximation of the local friction is made at the level of the general formalism, obscuring the validity of results obtained when the general formalism is applied to some specific problem. Our result applied to the unstable particle decay in thermal medium indeed casts a serious doubt on the familiar result based on the Boltzmann equation that uses quantities on the mass shell.

II The Model

In the present approach one models the environment by a continuously infinite set of harmonic oscillators of some arbitrary spectrum and couples it to the subsystem via a bilinear term. As will be explained below, the two particle states of decay product may be taken to be the environment oscillator of this kind. Let the subsystem variable in question be denoted by \(q\) and the environment variable by \(Q(\omega)\). For simplicity we assume that the subsystem has one degree of freedom, but it should be evident to extend it to any finite number of degrees of freedom. The Lagrangian of our problem consists of three parts:

\[
L = L_q[q] + L_Q[Q] + L_{\text{int}}[q, Q].
\] (1)

We take for the system-environment interaction the bilinear term:

\[
L_q = \frac{1}{2} \dot{q}^2 - V(q),
\] (2)
\[
L_Q = \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \left( \dot{Q}^2(\omega) - \omega^2 Q^2(\omega) \right), 
\]
\[
L_{\text{int}} = -q \int_{\omega_c}^{\infty} d\omega c(\omega) Q(\omega). 
\]

Here \(\omega_c\) taken to be positive is the smallest of the environment frequency spectrum, and \(c(\omega)\) describes the strength distribution of environment-system interaction. In the present work we only consider the harmonic oscillator for the small system:

\[
V(q) = \frac{1}{2} \omega_0^2 q^2. 
\]

The model of environment including its interaction to the subsystem is characterized by the quantity,

\[
r(\omega) = \frac{c^2(\omega)}{2\omega}, 
\]

which we called the response weight. This quantity is fundamental to the rest of our discussion. The simplest response weight familiar in condensed matter physics is given in terms of three parameters, a threshold \(\omega_c\), an index \(\alpha\) and a strength \(c\),

\[
r(\omega) = c (\omega - \omega_c)^\alpha. 
\]

The other hidden parameter here is the cutoff frequency, \(\Omega\), above which the response weight vanishes, or it is nonvanishing only for \(\omega_c < \omega < \Omega\). In condensed matter physics the gapless case of \(\omega_c = 0\) is especially popular, and the cases of \(\alpha = 1, < 1, > 1\), are called the Ohmic, sub-Ohmic, and super-Ohmic dissipation.

Our main interest is in the unstable particle decay. In this case the form of the response weight is more complicated than those given above. Let us first discuss how one identifies the environment variable \(Q(\omega)\) when the same species of particles as the decay product make up the environment. We take a relativistic field theory of Yukawa type of decay interaction:

\[
L_{\text{int}} = \frac{\mu}{2} \varphi \chi^2, 
\]

where \(\varphi\) is the decaying parent and \(\chi\) the daughter particle, with \(\mu\) the coupling of mass dimension. We shall use the capital letter \(M\) for the mass of the parent \(\varphi\) and \(m\) for the mass of the daughter \(\chi\). The threshold of the decay in vacuum is \(\omega_c = \sqrt{k^2 + 4m^2}\), but this is modified in thermal environment, as will be made clear shortly. The decaying particle, or the Fourier component of its field operator
$q_k = \int d^3x \varphi(\vec{x}) e^{i\vec{k} \cdot \vec{x}},$ couples to the two-body operator of $\chi$’s, and we identify the environment variable as

$$c_k(\omega) Q_k(\omega) = \frac{\mu}{2} \int d^4x \chi^2(x) e^{-i\vec{k} \cdot \vec{x} + i\omega x_0}. \quad (9)$$

The translational invariance makes each $\vec{k}$-mode independent. It is crucial to realize that each momentum state of the parent particle couples to a continuously infinite number of two particle states of daughter particles. The approximation, implicit here, that two particle $\chi$ states are made of independent, non-interacting two particle states is equivalent to that we neglect the final state interaction of $\chi$’s. We shall omit the vector notation such that $\vec{k} \rightarrow k$, and when confusion does not arise, we also omit the mode $k$ altogether.

For definiteness, we take a thermal environment of temperature $T = 1/\beta$. The response weight $r(\omega)$ is then calculable using the technique of the finite temperature field theory. It is the discontinuity or the imaginary part of the self-energy $\Pi(\omega, \vec{k})$ of $\varphi$ field in thermal medium:

$$r(\omega) = \frac{1}{2i \pi^2} \left( \Pi(\omega - i0^+) - \Pi(\omega + i0^+) \right) \equiv -\frac{1}{\pi} \Im \Pi(\omega). \quad (10)$$

Let us explain some details of this calculation of the response weight in the subthreshold region of $|\omega| < k$. The discontinuity is readily calculable from the imaginary-time formalism and it is given by

$$-\Im \Pi(\omega) = \frac{\mu^2}{16\pi k} \int_{-\omega_-}^{\infty} dE \left( n(E) - n(E + \omega) \right), \quad (11)$$

$$\omega_\pm = \frac{\omega}{2} \pm \frac{k}{2} \sqrt{1 - \frac{4m^2}{\omega^2 - k^2}}, \quad (12)$$

where

$$n(E) = \frac{1}{e^{\beta E} - 1} \quad (13)$$

is the Planck distribution function of $T = 1/\beta$. Since

$$n(E) - n(E + \omega) = n(E)(1 + n(E + \omega)) - n(E + \omega)(1 + n(E)), \quad (14)$$

the imaginary part (\textsuperscript{1}) for $|\omega| < k$ is a sum of the two contributions, $\chi + \varphi \rightarrow \chi$ and its inverse process that is allowed to occur in thermal medium. Note that $\varphi$ can be off the mass shell: $\omega^2 - \vec{k}^2 \neq$ the $\varphi$ mass$^2$. The factor $1 + n$ represents the effect of stimulated boson emission.
On the other hand, for $\omega > \sqrt{k^2 + 4m^2}$ relevant physical processes are $\varphi \leftrightarrow \chi + \chi$. Since

$$(1 + n(E))(1 + n(E + \omega)) - n(E)n(E + \omega) = 1 + n(E) + n(E + \omega),$$

the imaginary part of the self-energy is given by

$$-\Im\Pi(\omega) = \frac{\mu^2}{32\pi k} \int_{\omega - \omega}^{\omega + \omega} dE \left(1 + 2n(E)\right),$$

$$\to \frac{\mu^2}{32\pi} \sqrt{1 - \frac{4m^2}{\omega^2 - k^2}} \left(1 + 2e^{-\beta\omega/2}\right).$$

(15)

The last limit is valid as $\omega \to \infty$. When divided by $\omega$, the first term here $\Gamma(\omega) = \Im\Pi(\omega)/\omega$ is $\approx \mu^2/(32\pi \omega)$, which is the decay rate of $\varphi \to \chi\chi$ in vacuum, including the effect of prolonged lifetime at $\omega \gg m_\varphi$. [8]

The result of computation is now summarized. For $\omega > \sqrt{k^2 + 4m^2}$ the response weight is [5]

$$r(\omega) = \frac{\mu^2}{32\pi^2} \left(\sqrt{1 - \frac{4m^2}{\omega^2 - k^2}} + \frac{2}{k\beta} \ln \frac{1 - e^{-\beta\omega_+}}{1 - e^{-\beta|\omega_-|}}\right).$$

(18)

For $0 < \omega < k$ only the second term in the bracket of Eq.(18) contributes. Note that $r(\omega) \to$ a constant ($\approx M/\pi \times$ decay rate in the rest frame of $\varphi$) as $\omega \to \infty$.

We note that there is a gap between $k$ and $\sqrt{k^2 + 4m^2}$ for which $r(\omega) = 0$. Existence of the gap is important in discussing the analytic property of some basic functions that appear later. However, the location of the gap in thermal medium differs from that in vacuum. The gap ceases to exist for the massless case $m = 0$.

III Diagonalization of the total system

A typical question one frequently asks with regard to the system behavior is how a pure state of the $q$ system evolves in time in environment described by a mixed state such as the thermal one of $Q(\omega)$’s. We shall first answer this by giving explicit time dependent operator solution for $q(t), p(t) = \dot{q}(t)$ that is written in terms of initial values, $q_i, p_i, Q_i(\omega), P_i(\omega)$. With this, one can clearly express correlators such as $\langle q(t_1)q(t_2) \rangle$ in terms of the probability distribution of these values in any initial state. Closest to our present approach is the classic work of Ullersma [9], with an important difference of the presence of a gap in the environment spectrum, which
yields different behaviors of the correlators. The gapless case is applied for instance to phonons in medium, but in the case of our interest such as the unstable particle decay in medium, a gap exists if the mass of daughter particle is finite.

The model itself, a harmonic system coupled to an infinite number of harmonic oscillators, frequently appears in many idealized physical problems. We utilize in the present investigation the exact solution of the Hamiltonian eigenvalue problem to this system, which might also be useful in other contexts. With this exact eigen operator the Heisenberg time evolution becomes evident. A great advantage of this way of solving the present problem is that one can employ a full analogy to the scattering problem, especially the analyticity based on elastic unitarity. What happens is that an isolated spectrum of the subsystem above the two particle threshold becomes unstable due to the interaction with the environment, and the single particle pole enters into the second Riemann sheet below the cut real axis. Due to the structure of our model, the unitarity relation is saturated by the elastic two-body state. The system then becomes integrable.

We first consider a related problem of diagonalization of the infinite dimensional matrix of the potential part \( V \):

\[
V = \frac{1}{2} \begin{pmatrix} q & Q(\omega) \end{pmatrix} V \begin{pmatrix} q \\ Q(\omega') \end{pmatrix}, \quad V = \begin{pmatrix} \omega_0^2 & c(\omega') \\ c(\omega) & \omega^2 \delta(\omega - \omega') \end{pmatrix},
\]

written in matrix notation. The matrix element labels \( \omega \) and \( \omega' \) are to be integrated over here. What we are doing here is not a diagonalization of the operator itself, which will be dealt with later.

The potential matrix may be decomposed into the two parts as \( V = V_0 + V' \), where \( V_0 \) consists of the environment part alone,

\[
V_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega^2 \delta(\omega - \omega') \end{pmatrix},
\]

\[
V' = |S_1\rangle \langle S_1| - |S_2\rangle \langle S_2|, \quad |S_1\rangle = \begin{pmatrix} \omega_0 \\ c(\omega) \end{pmatrix}, \quad |S_2\rangle = \begin{pmatrix} 0 \\ \frac{c(\omega)}{\omega_0} \end{pmatrix},
\]

\[
\langle S_i|S_i\rangle = \omega_0^2 \delta_{ii} + \frac{1}{\omega_0^2} \int_{\omega_c}^{\infty} d\omega \, 2\omega \, r(\omega).
\]

We may note a trivial relation:

\[
|S_i\rangle \langle S_i| = \langle S_i|S_i\rangle \mathcal{P}_i, \quad \mathcal{P}_i^2 = \mathcal{P}_i,
\]
where $\mathcal{P}_i$ is a projection operator onto a one-dimensional subspace. The fact that the nontrivial part $\mathcal{V}'$ is finite dimensional is the reason this system is solvable.

The eigenvector of diagonalized $\mathcal{V}$ is given with the aid of some analytic function. First, one defines the proper self-energy $\mathcal{G}(z)$ and the full propagator $F(z)$ by

$$\mathcal{G}(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{r(\omega)}{z-\omega},$$

$$F(z) = \frac{1}{-z^2 + \omega_0^2 + 2\pi \mathcal{G}(z)},$$

using an extended response weight, $r(-\omega) = -r(\omega)$ for $\omega < -\omega_c$. The function $F(z)$ has cuts along the real axis, $\omega > \omega_c$ and $\omega < -\omega_c$. With the assumption of $\omega_c > 0$ there is a gap between the two cuts. The following discontinuity relation holds:

$$F(\omega + i0^+) - F(\omega - i0^+) = i2\pi r(\omega)F(\omega + i0^+)F(\omega - i0^+) \equiv i2\pi H(\omega),$$

along this cut. The frequency renormalization, corresponding to the mass renormalization in field theory, is necessary when the high frequency behavior of the response weight is $r(\omega) \to \text{constant}$. The renormalized frequency is

$$\omega_R^2 = \omega_0^2 + \delta \omega^2 = \omega_0^2 - 2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega)}{\omega}.$$  

When this happens, one has to subtract a term in the continuous $\omega$ integral involving $r(\omega)$ and replace the bare $\omega_0^2$ by $\omega_R^2$. We shall not indicate this procedure in further presentation, because it is fairly straightforward.

The eigenvector of $\mathcal{V}$ is then given by

$$|\Psi(\omega)\rangle = -c(\omega)F(\omega - i0^+) |0\rangle + |\omega\rangle + c(\omega)F(\omega - i0^+) \int_{\omega_c}^{\infty} d\omega' \frac{c(\omega')}{\omega'^2 - \omega^2 + i0^+} |\omega'\rangle.$$  

The notation here is such that $|\omega\rangle$ and $|0\rangle$ are eigenvectors of $\mathcal{V}_0$:

$$\left( \mathcal{V}_0 - \omega^2 \right) |\omega\rangle = 0, \quad \mathcal{V}_0 |0\rangle = 0.$$  

This diagonalization involves a complex phase such as

$$e^{i\varphi(\omega)} = \frac{F(\omega - i0^+) / |F(\omega - i0^+)|}{\mathcal{G}(\omega) / |\mathcal{G}(\omega)|}.$$  

Actually one can show that these are overall phases of vectors and by removing them one achieves diagonalization of the real symmetric matrix $\mathcal{V}$ by a real orthogonal transformation.
The matrix diagonalization precisely parallels the operator diagonalization. The canonical transformation from the original to the Hamiltonian eigen operator is given by

\[ \tilde{Q}(\omega) = Q(\omega) - c(\omega) F(\omega - i0^+) \left( q - \int_{\omega_c}^{\infty} d\omega' \frac{c(\omega')}{\omega'^2 - \omega^2 + i0^+} Q(\omega') \right), \quad (31) \]

\[ \tilde{P}(\omega) = P(\omega) - c(\omega) F(\omega - i0^+) \left( p - \int_{\omega_c}^{\infty} d\omega' \frac{c(\omega')}{\omega'^2 - \omega^2 + i0^+} P(\omega') \right), \quad (32) \]

\[ q = - \int_{\omega_c}^{\infty} d\omega c(\omega) F^*(\omega - i0^+) \tilde{Q}(\omega), \quad (33) \]

\[ p = - \int_{\omega_c}^{\infty} d\omega c(\omega) F^*(\omega - i0^+) \tilde{P}(\omega), \quad (34) \]

\[ Q(\omega) = \tilde{Q}(\omega) + c(\omega) \int_{\omega_c}^{\infty} d\omega' \frac{c(\omega')}{\omega^2 - \omega'^2 - i0^+} \tilde{Q}(\omega'), \quad (35) \]

\[ P(\omega) = \tilde{P}(\omega) + c(\omega) \int_{\omega_c}^{\infty} d\omega' \frac{c(\omega')}{\omega^2 - \omega'^2 - i0^+} \tilde{P}(\omega'). \quad (36) \]

Here we find it more convenient to use

\[ \tilde{Q}(\omega) \equiv e^{i\varphi(\omega)} \overline{Q}(\omega) \]

instead of the true eigen operator \( \overline{Q}(\omega) \), by retaining those \( \omega \) dependent phases.

Assuming the canonical commutation for the original variables, one can verify that diagonal variables obey the correct form of the commutation relation;

\[ [\overline{Q}(\omega), \overline{P}(\omega')] = i \delta(\omega - \omega') \]

etc. It can be proved that with the specified phases of \( \tilde{Q}(\omega), \tilde{P}(\omega) \) the original variables are all hermitian, as required.

The overlap probability of the two vectors, or equivalently the overlap between \( q \) and \( \overline{Q}(\omega) \), is

\[ | \langle 0 | \Psi(\omega) \rangle |^2 = c^2(\omega) | F(\omega - i0^+) \rangle^2 = 2 \omega H(\omega), \quad (37) \]

\[ H(\omega) = \frac{r(\omega)}{(\omega^2 - \omega_0^2 - \Pi(\omega))^2 + (\pi r(\omega))^2}, \quad (38) \]

with

\[ \Pi(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{r(\omega')}{\omega - \omega'}. \quad (39) \]

The analytic function \( F(z) \) contains the crucial information with regard to the behavior of the small subsystem. The system frequency \( \omega_0 \) originally given in the...
Lagrangian is modified by the interaction with the environment. The shifted frequency is determined by the singularity of the self-energy function $F(z)$. With a condition,

$$f(\omega_c^2) > \omega_c^2,$$

$$f(\lambda) \equiv \lambda + \mathcal{R}F^{-1}(\sqrt{\lambda} + i0^+) ,$$

(in this equation $-$ indicates the limit from below), it can be shown that there is no singularity except the branch cut starting from the threshold, $\omega > \omega_c$ and $\omega < -\omega_c$, in the first Riemann sheet. The original pole at $\omega = \omega_0$ thus moves into the second Riemann sheet whose location is given by

$$z^2 - \omega_0^2 - 2\pi G(z) + 2\pi ir(z) = 0 .$$

The imaginary part of this location gives the decay rate of any initial configuration of the system, as will be made evident below.

**IV Time evolution**

With the diagonal variable derived, it is easy to write the Heisenberg operator solution at any time $t$ in terms of the initial operator values. For instance,

$$q(t) = -\int_{\omega_c}^{\infty} d\omega c(\omega) F^*(\omega - i0^+) \tilde{Q}(\omega,t),$$

$$\tilde{Q}(\omega,t) = \cos(\omega t) \tilde{Q}_i(\omega) + \frac{\sin(\omega t)}{\omega} \tilde{P}_i(\omega).$$

The initial values $\tilde{Q}_i, \tilde{P}_i$ are then rewritten in terms of the original variables.

After some straightforward calculation one finds [3] that

$$q(t) = p_i g(t) + q_i \dot{g}(t) - \int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( h^*(\omega,t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right).$$

$$p(t) = p_i \dot{g}(t) + q_i \ddot{g}(t) - \int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( k^*(\omega,t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right).$$

We have introduced

$$g(t) = 2 \int_{\omega_c}^{\infty} d\omega H(\omega) \sin(\omega t)$$

$$= 2 \int_{\omega_c}^{\infty} d\omega \frac{r(\omega) \sin(\omega t)}{(\omega^2 - \omega_0^2 - \Pi(\omega))^2 + (\pi r(\omega))^2},$$

$$h(\omega,t) = \int_0^t d\tau g(\tau) e^{-i\omega \tau},$$

$$k(\omega,t) = \int_0^t d\tau \dot{g}(\tau)e^{-i\omega \tau} = g(t)e^{-i\omega t} + i\omega h(\omega,t).$$
Furthermore, $b_i(\omega) = (\sqrt{\omega}Q(\omega) + iP(\omega)/\sqrt{\omega})/\sqrt{2}$ is the annihilation operator for environment harmonic oscillators.

The important time dependence is governed by the function $g(t)$. It is useful to clarify the physical significance of the function $g(t)$ in detail. For this purpose let us first derive a local form of the quantum Langevin equation. Using the explicit form of solution, one gets after eliminating initial $q_i, p_i$ dependence,

\[
\frac{d^2q}{dt^2} + \Omega^2(t) q + C(t) \frac{dq}{dt} = -\Omega^2(t) f_q - C(t) f_p - \dot{f}_p ,
\]

\[
\Omega^2(t) = \frac{\dot{g} \ddot{g} - \dot{g}^2}{g \ddot{g} - \dot{g}^2} , \quad C(t) = \frac{\dot{g} \ddot{g} - \dot{g} \dddot{g}}{g \ddot{g} - \dot{g}^2} ;
\]

\[
f_q = \int_{\infty}^{\omega_c} d\omega \sqrt{r(\omega)} \left( h^*(\omega, t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right) ,
\]

\[
f_p = \int_{\infty}^{\omega_c} d\omega \sqrt{r(\omega)} \left( k^*(\omega, t) e^{-i\omega t} b_i(\omega) + (\text{h.c.}) \right) .
\]

The quantities that determine the subsystem behavior in this equation are the time dependent friction ($C(t)$) and the time dependent frequency squared ($\Omega^2(t)$), which incorporate environmental effects. Both of these are locally determined from $g(t)$. Thus $g(t)$ describes an average behavior of the system variable disregarding the random force from environment. On the other hand, $f_q$ and $f_p$ give the random force from the environment.

The system variable has been determined in terms of the initial operator values of both the system and the environment variables. Dependence on the system initial values $p_i, q_i$ are given by the function $g(t)$. Both $g(t)$ and $\dot{g}(t)$ can be shown to obey integro-differential equation of the following form ($y = g$ or $\dot{g}$),

\[
\frac{d^2y}{dt^2} + \omega_0^2 y + 2 \int_0^t d\tau \alpha_1(t - \tau) y(\tau) = 0 ,
\]

\[
\alpha_1(\tau) = -\frac{i}{2} \int_{-\infty}^{\infty} d\omega r(\omega) e^{-i\omega \tau} = -\int_{\omega_c}^{\omega} d\omega r(\omega) \sin(\omega \tau) .
\]

The two functions, $g(t)$ and $\dot{g}(t)$, differ in their boundary conditions: $g(0) = 0 , \dot{g}(0) = 1$.

The characteristic behavior of the function $g(t)$ is that it decreases first exponentially and then finally by an inverse power of time, as can be seen in the following way. Using the discontinuity formula, one may rewrite the $\omega$ integration containing

\[
H(\omega) = \left( F(\omega + i0^+) - F(\omega - i0^+) \right) / (2\pi i) ,
\]
along the real axis into the $F(z)$ integration, the complex $z$ running both slightly above and below the cuts. The factor $\sin(\omega t)$ is replaced by $\Im e^{-i\omega t}$ in this procedure. A half of this complex contour can be deformed into the second sheet, and one thereby encounters simple poles in the second sheet. We assume for simplicity that there exists only a single pole in the nearby second sheet. The integral for $g(t)$ may then be expressed as the sum of the pole contribution (at $z = z_0$ with $\Im z_0 < 0$) in the second sheet and the contribution parallel to the imaginary axis passing through $z = \omega_c$, both in the first (I) and in the second (II) sheet:

$$\begin{align*} g(\tau) &= \Im \left( K e^{-i\Re z_0 \tau} e^{i\omega_0 \tau} \right) + \Im \left[ \frac{e^{i\omega_0 \tau}}{\pi} \int_0^\infty dy e^{-y\tau} \left( F_I(\omega_c + iy) - F_{II}(\omega_c + iy) \right) \right], \quad (57) \end{align*}$$

with $K^{-1} = z_0 - \pi\gamma G'(z_0) + i\pi \gamma'(z_0)$.

The pole contribution given by the first term describes the exponential decay which usually lasts very long during the most important phase of the decay period, while at very late times the rest of contribution gives the power law decay. In order to explain this late time behavior, let us take the response weight, $r(\omega) = c (\omega - \omega_c)^\alpha$, for $\omega_c < \omega < \Omega$. We assume the parameters in the range of $\Omega \gg \omega_c$. The late time behavior of $g(t)$ is derived from the continuous part of $H(\omega)$ integration and is given by

$$g(t) \approx \frac{2c}{\bar{\omega}^4} \Gamma(\alpha + 1) \frac{\cos(\omega_c t + \frac{\pi}{2} \alpha)}{t^{\alpha + 1}},$$

where $\bar{\omega}$ is the pole mass assumed to obey $\bar{\omega} \gg$ the decay rate, the imaginary part of the pole location. Thus, the power $-\alpha - 1$ of $g(t) \propto t^{-\alpha - 1}$ is related to the threshold behavior of the response weight.

One can estimate the transient time $t_*$ from the exponential period to the power period by equating the two formulas of $g(t)$ in their respective ranges, to obtain

$$t_* \approx \frac{1}{\gamma} \ln \left( \frac{\bar{\omega}^3}{2c\Gamma(\alpha + 1) \gamma^{\alpha + 1}} \right),$$

with $\gamma = -\Im z_0$ the decay rate. For a very small $c$ the factor inside the logarithm becomes large ($\propto c^{-2\alpha - 3}$), and by the time $t_*$ the initial population has decreased like

$$e^{-2\gamma t_*} \propto c^{4\alpha + 6}.$$
It may thus be claimed that the power law behavior is difficult to observe. But we shall later show that this may not be so in cosmology.

We may call the approximation that neglects the non-pole contribution for \( g(t) \) as the resonance approximation since this approximation is equivalent to taking a Breit-Wigner form for \( H(\omega) \),

\[
H(\omega) \approx \frac{1}{\pi} \frac{\eta \omega}{(\omega^2 - \omega_R^2)^2 + \eta^2 \omega^2},
\]

and then integrating in Eq.(58) for \( g(t) \) in the entire range of \(-\infty < \omega < \infty\) without considering the threshold effect. It may also be called the local friction approximation since the equation (55) for \( g(t) \) simply reduces to

\[
\ddot{g} + \omega_R^2 g + \eta \dot{g} = 0
\]

in this case. The solution of this equation is

\[
g(t) \approx \frac{1}{\omega_R} \sin(\omega_R t) e^{-\frac{\eta}{2} t},
\]

for \( \eta \ll \omega_R \). This approximation is practically very useful in many cases, but there are physical effects such as the final abundance of unstable particles in thermal medium that cannot be explained in this approximation.

A short-time behavior of quantum dissipation is neither well described by the resonance approximation. We compare a precise estimate of the time dependent friction \( C(t) \) and the frequency \( \Omega^2(t) \) to the approximate one. Assuming a smooth limit of the function \( g(t) \), one has

\[
C(t) \sim -\frac{t^3}{3} \left( g^{(3)}(0)^2 - g^{(5)}(0) \right),
\]

\[
\Omega^2(t) \sim -g^{(3)}(0),
\]

as \( t \to 0^+ \). On the other hand, the pole approximation, \( g(t) \approx \frac{1}{\omega} \sin(\bar{\omega} t) e^{-\gamma t} \), gives \( C(t) \approx 2\gamma \), \( \Omega^2(t) \approx \bar{\omega}^2 + \gamma^2 \).

More seriously, the pole approximation violates the positivity of the reduced density matrix:

\[
\frac{d}{dt} \text{tr} \rho^2 \approx 2\gamma \text{tr} \rho^2.
\]

This relation implies that if the initial subsystem is in a pure quantum state with \( \text{tr} \rho^2 = 1 \), it evolves into a state with \( \text{tr} \rho^2 > 1 \). Because \( \text{tr} \rho = 1 \), this cannot
be satisfied unless some diagonal element of the density matrix is negative, whose absolute value is larger than 1.

A great advantage of the operator approach is that one may explicitly work out various correlators. For instance,

\[ \langle q(t_1)q(t_2) \rangle = -\frac{i}{2} g(t_1 - t_2) + \int_0^{t_1} d\tau \int_0^{t_2} ds \, g(t_1 - \tau) \alpha_R(\tau - s) g(t_2 - s) \]

\[ + g(t_1)g(t_2) \langle p_i^2 \rangle + \dot{g}(t_1)\dot{g}(t_2) \langle q_i^2 \rangle + \left( g(t_1)\dot{g}(t_2) + \dot{g}(t_1)g(t_2) \right) \frac{1}{2} \left( p_i q_i + q_i p_i \right), \quad (67) \]

\[ \alpha_R(\tau) = \int_{\omega_c}^\infty d\omega \, \langle 2n_i(\omega) + 1 \rangle \frac{\cos(\omega \tau)}{2\omega}. \quad (68) \]

Coincident time limits are evaluated from these, resulting in

\[ \langle q^2(t) \rangle = \int_{\omega_c}^\infty d\omega \, \langle 2n_i(\omega) + 1 \rangle \, r(\omega) \, |h(\omega, t)|^2 \]

\[ + g^2(t) \langle p_i^2 \rangle + \dot{g}^2(t) \langle q_i^2 \rangle + g(t)\dot{g}(t) \langle p_i q_i + q_i p_i \rangle, \quad (69) \]

\[ \langle p^2(t) \rangle = \int_{\omega_c}^\infty d\omega \, \langle 2n_i(\omega) + 1 \rangle \, r(\omega) \, |k(\omega, t)|^2 \]

\[ + \dot{g}^2(t) \langle p_i^2 \rangle + \dot{g}^2(t) \langle q_i^2 \rangle + \ddot{g}(t) \langle p_i q_i + q_i p_i \rangle, \quad (70) \]

\[ \frac{1}{2} \langle q(t)p(t) + p(t)q(t) \rangle = \int_{\omega_c}^\infty d\omega \, \langle 2n_i(\omega) + 1 \rangle \, r(\omega) \, h(\omega, t)k^*(\omega, t) \]

\[ + \ddot{g}(t)g(t) \langle p_i^2 \rangle + \ddot{g}(t) \ddot{g}(t) \langle q_i^2 \rangle + \left( \dot{g}^2(t) + g(t)\ddot{g}(t) \right) \frac{1}{2} \left( p_i q_i + q_i p_i \right). \quad (71) \]

It is an important feature of these formulas that the initial state dependence is clearly separated from the rest of physics as

\[ \langle 2n_i(\omega) + 1 \rangle, \quad \langle q_i^2 \rangle, \quad \langle p_i^2 \rangle, \quad \langle p_i q_i + q_i p_i \rangle. \]

## V Relic abundance of unstable particles in thermal medium

We shall apply the general result thus obtained to the unstable particle decay. As already mentioned, in this context the system coordinate \( q_\vec{k} \) refers to the Fourier component of the field operator of decaying particle \( \varphi \). In this section we discuss the number operator given by

\[ n(t) \equiv \frac{1}{2} \left( \frac{p^2(t)}{\omega} + \dot{\omega} q^2(t) \right) - \frac{1}{2}. \quad (72) \]
Here the pole mass $\omega$ is chosen as the reference frequency to define the number operator from the Hamiltonian of the decaying field.

We first note that the asymptotic limit of the operator is given by

$$q(t) \to -\int_{\omega_c}^{\infty} d\omega \sqrt{r(\omega)} \left( F^*(\omega - i0^+) e^{-i\omega t} b_1(\omega) + (\text{h.c.}) \right),$$  \hspace{1cm} (73)

$$p(t) \to i \int_{\omega_c}^{\infty} d\omega \omega \sqrt{r(\omega)} \left( F^*(\omega - i0^+) e^{-i\omega t} b_1(\omega) - (\text{h.c.}) \right),$$  \hspace{1cm} (74)

due to $h(\omega, \infty) = F(\omega - i0^+)$, which may readily be proved. Noting that

$$r(\omega)|h(\omega, \infty)|^2 = H(\omega) = \frac{|\langle 0|\Psi(\omega) \rangle|^2}{2\omega},$$  \hspace{1cm} (75)

$$r(\omega)|k(\omega, \infty)|^2 = \omega^2 H(\omega) = \frac{\omega^2}{2} |\langle 0|\Psi(\omega) \rangle|^2,$$  \hspace{1cm} (76)

one has the asymptotic values,

$$\langle q^2(\infty) \rangle = \int_{\omega_c}^{\infty} d\omega \frac{|\langle 0|\Psi(\omega) \rangle|^2}{2\omega} (2n_i(\omega) + 1),$$  \hspace{1cm} (77)

$$\langle p^2(\infty) \rangle = \int_{\omega_c}^{\infty} d\omega \frac{\omega}{2} |\langle 0|\Psi(\omega) \rangle|^2 (2n_i(\omega) + 1).$$  \hspace{1cm} (78)

The overlap probability of the original subsystem variable with the true eigenvalue of the entire system $|\langle 0|\Psi(\omega) \rangle|^2$ is thus fundamental to these and subsequent formulas. The quantity $\langle 2n_i(\omega) + 1 \rangle$ refers to the number density of the environment bilinear fields. We may take the value in thermal medium,

$$\langle 2n_i(\omega) + 1 \rangle = \coth\left(\frac{\beta\omega}{2}\right).$$  \hspace{1cm} (79)

In the infinite time limit the occupation number is then

$$n(\infty) = \frac{1}{2} \int_{\omega_c}^{\infty} d\omega \coth\left(\frac{\beta\omega}{2}\right) \left(\bar{\omega} + \frac{\omega^2}{\bar{\omega}}\right) H(\omega) - \frac{1}{2}.$$  \hspace{1cm} (80)

When an precise form of the overlap $H(\omega) = \frac{|\langle 0|\Psi(\omega) \rangle|^2}{2\omega}$ is used, this formula gives a reliable relic abundance of unstable particles.

Let us first check that in some limit this formula gives the familiar formula for the abundance. When the pole term dominates, or equivalently one approximates $H(\omega)$ by the Breit-Wigner function, then the temperature dependent part of the occupation number defined by

$$n^\beta = n(\infty, \beta) - n(\infty, 0) = \int_{\omega_c}^{\infty} d\omega \frac{1}{e^{\beta\omega} - 1} \left(\bar{\omega} + \frac{\omega^2}{\bar{\omega}}\right) H(\omega),$$  \hspace{1cm} (81)
has the factor $e^{-\bar{\omega}/T}$ at low temperatures.

In the high temperature regime the pole approximation is excellent. But this approximation is not good at low temperatures. Indeed, let us examine a typical example by taking again the form of $r(\omega) = c(\omega - \omega_c)^\alpha$, with $0 < \alpha < 1$ in the range of $\omega_c < \omega < \Omega$ ($\Omega \gg \omega_c$) and with $\bar{\omega} \gg \text{Max} (\omega_c, T)$. The result is

$$n^3 \approx \frac{c}{\bar{\omega}^3} \Gamma(\alpha + 1) e^{-\beta \omega_c \frac{T^{\alpha+1}}{\alpha+1}},$$

(82)

where $\Gamma$ is the Euler's gamma function. This shows that instead of the exponential suppression at low temperatures what is left in medium after the decay has a power-law behavior of temperature dependence ($\propto T^{\alpha+1}$).

An implication of this behavior to the unstable particle decay, as will be made more explicit shortly, is that the remnant fraction in thermal medium does not suffer from the Boltzmann suppression factor at temperatures even much lower than the mass of the unstable particle. That this is possible is due to that the conventional approach using the approximate Boltzmann-like equation is based on S-matrix elements computed on the mass shell, while the true quantum mechanical equation may contain quantities off the mass shell. As is well known, the Green's function, which is quantum mechanically more fundamental than the S-matrix element, does contain important contributions off the mass shell. What is called virtual intermediate states in elementary quantum mechanics gives rise to the off-shell contribution. Exact treatment of the problem such as ours has indeed contributions off the mass shell, and moreover the off-shell contribution makes up the dominant part of behaviors at low temperatures.

Let us further apply these general considerations to the decay of unstable particle; $\varphi \rightarrow \chi + \chi$. Since we focus on the late time behavior, the initial state dependence disappears: in particular, whether the parent particle is or is not in thermal equilibrium with the rest of medium is not important. We shall limit our discussion here to the decay that occurs when the parent $\varphi$ becomes non-relativistic,

$$\omega_k \sim M + \frac{\vec{k}^2}{2M} \gg T,$$

(83)

with $M$ the $\varphi$ mass. This condition is relevant in interesting cosmological problems of the neutron decay at the time of nucleosynthesis and GUT $X$ boson decay at baryogenesis.[11]
Since we already know the general expression for the relic abundance, what remains to be done is the mode sum over the momentum $\vec{k}$ of the unstable particle. First, when the pole approximation is valid, the number density is

$$n \approx \int \frac{d^3k}{(2\pi)^3} e^{-\beta(M+k^2/2M)} = \left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}. \quad (84)$$

This is the familiar Boltzmann suppressed formula.

This is a bad approximation at low temperatures, $T \ll M$. To derive a precise formula, one has to integrate both over $\omega$ and $\vec{k}$, which is difficult to do analytically. As an illustration, take a constant response weight $r(\omega) = r(\infty)$, hoping that the asymptotic region of $r(\omega)$ dominates. Then,

$$n \approx \frac{1}{2\pi^2} \frac{r(\infty)}{M^3} \int_0^\infty dk k^2 \int_k^\infty d\omega \frac{1}{e^{\omega/T} - 1} = \frac{1}{90} \frac{r(\infty) T^4}{M^3}. \quad (85)$$

Although this is not a precise calculation, it nevertheless gives a correct temperature dependence.

We numerically computed all terms including the logarithmic factor in $r(\omega)$ along with $O[m^2]$ corrections. It turns out that the total contribution is ten times larger than the analytic result above: in the $m \to 0$ limit,

$$n \approx 10^{-2} \frac{\mu^2 T^4}{M^3}. \quad (86)$$

The main part of this large contribution comes from $|\omega| < k$. With a dimensionless constant introduced by $\mu = gM$, this gives, relative to the photon number density ($= \frac{2\zeta(3)}{\pi^2} T^3$),

$$\frac{n}{T^3} \approx 10^{-12} \left(\frac{g}{G_F m_N^2}\right)^2 \frac{T}{M}. \quad (87)$$

We wrote the ratio here using the numerical value $G_F$, the weak interaction constant of mass dimensions $-2$ ($G_F m_N^2 \approx 10^{-5}$).

One may estimate the equal time temperature $T_{eq}$ at which the power contribution becomes equal to the Boltzmann suppressed number density, to give

$$\frac{T_{eq}}{M} \approx \frac{1}{30}, \quad \frac{n}{T_{eq}^3} \approx 10^{-13}, \quad (88)$$

taking as an example $\mu = 10^{-5} M$, the weak interaction strength. This number is in an interesting range that may affect nucleosynthesis, but we should keep in mind that we did not work out the relevant three body decay, $n \to p + e + \bar{\nu}_e$. 

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We shall mention another application of immediate interest in cosmology; the heavy $X$ boson decay for GUT baryogenesis. It has been argued \cite{11} that there exists a severe mass bound of order,

$$m_X > O[\alpha_X m_{pl}] \approx 10^{16}\text{GeV},$$  \hspace{1cm} (89)

to block the inverse process of the $X$ boson decay so that generation of the baryon asymmetry proceeds with sufficient abundance of parent $X$ particles. The usual estimate of the mass bound mentioned above is however based on the on-shell Boltzmann equation. More appropriate formula in this estimate is our relic number density,

$$n_X \approx O[10^{-2}] \frac{g^2_X T^4}{m_X}.$$  \hspace{1cm} (90)

(In a more realistic estimate one should consider the $X$ boson decay into quarks and leptons. But for an order of magnitude estimate difference in statistics is not important.) With the GUT coupling of $g^2_X/4\pi = 1/40$, the equal temperature is roughly

$$T_{eq} \approx \frac{M}{2}.$$  \hspace{1cm} (91)

Thus, already at temperature of about half of the $X$ mass the Boltzmann suppressed formula is replaced by the power formula. The kinematical condition Eq.\text{(89)} for baryogenesis must be reconsidered in view of our off-shell formula.

\textbf{VI Path integral method}

The basic idea of the influence functional method \cite{12} is that one is interested in the behavior of the subsystem alone and traces out the environment variable altogether in the path integral formula. Furthermore, one directly deals with the probability instead of the amplitude. This way one can compute the reduced density matrix that describes a state of the small system incorporating effects of interacting environment. We discuss this method here, simply because this approach gives the non-decay probability of unstable particle which is difficult to deal with in the operator approach so far used.

We define the influence functional by convoluting with the initial state of the environment. To do so we assume for technical reasons that initially we may take
an environment state uncorrelated with the system. The influence functional is thus obtained after integrating out the environment variables:

\[ \mathcal{F}[q(\tau), q'(\tau)] \equiv \int \mathcal{D}Q(\tau) \int \mathcal{D}Q'(\tau) \int dQ_i \int dQ'_i \int dQ_f \int dQ'_f \]
\[ \cdot \delta(Q_f - Q'_f) K(q(\tau), Q(\tau)) K^*(q'(\tau), Q'(\tau)) \rho_i(Q_i, Q'_i), \] (92)

\[ K(q(\tau), Q(\tau)) = \exp \left( iS_0[Q] + iS_{\text{int}}[q, Q] \right), \] (93)

\[ S_0[Q] + S_{\text{int}}[q, Q] = \int_0^t d\tau \left( L_Q[Q] + L_{\text{int}}[q, Q] \right). \] (94)

The influence functional is a functional of the entire path of the system \( q(\tau) \) and its conjugate path \( q'(\tau) \).

\[ \rho_i(Q_i, Q'_i) = \sum_n w_n \psi_n^*(Q_i') \psi_n(Q_i), \quad (0 \leq w_n \leq 1), \] (95)

is the initial density matrix of the environment, which can be any mixture of pure quantum state \( n \) with the probability \( w_n \). What deserves to be stressed is that one does not observe the final state of the environment, hence integration with respect to the final values of \( Q_f = Q'_f \) is performed here.

Once the influence functional is known, one may compute the transition probability and any physical quantities of the \( q \)–system by convoluting dynamics of the system under study. For instance, the transition probability is given, with introduction of the density matrix \( \rho^{(R)} \), by

\[ \int dq_f \int dq'_f \psi^*_f(q_f) \rho^{(R)}(q_f, q'_f) \psi_f(q'_f), \] (96)

\[ \rho^{(R)} = \int \mathcal{D}q(\tau) \int \mathcal{D}q'(\tau) \int dq_i \int dq'_i \]
\[ \cdot \psi^*_i(q'_i) \psi_i(q_i) \mathcal{F}[q(\tau), q'(\tau)] e^{iS_q[q] - iS_q[q']}, \] (97)

where \( \psi_i, f \)'s are wave functions of the initial and the final \( q \)–states, and \( S_q[q] \) is the action of the \( q \)–system.

The form of the influence functional is dictated by general principles such as probability conservation and causality. Feynman and Vernon found a closed quadratic form consistent with these,

\[ \mathcal{F}[q(\tau), q'(\tau)] = \exp \left[ -\int_0^t d\tau \int_0^t ds \left( \xi(\tau) \alpha_R(\tau - s) \xi(s) + i \xi(\tau) \alpha_I(\tau - s) X(s) \right) \right], \] (98)

with \( \xi(\tau) = q(\tau) - q'(\tau), \quad X(\tau) = q(\tau) + q'(\tau). \) (99)
Thus two real functions $\alpha_i(\tau)$ are all we need to characterize the system-environment interaction. These are defined here in the range of $\tau \geq 0$. The fact that $\alpha_i$ depends on the difference of time variables, $\tau - s$, is due to the assumed stationarity of the environment. The Feyman-Vernon formula is valid for general $L_Q[Q]$ and $L_q[q]$, not limited to the harmonic oscillator model if the interaction $L_{\text{int}}[q, Q]$ is bilinear.

The correlation kernels appear in the influence functional as a form of the nonlocal interaction and they are the dissipation $\alpha_I$ and the noise $\alpha_R$. The dissipation kernel $\alpha_I$ thus computed agrees with the one defined in Eq.(56). Let us now specialize to the case of the oscillator bath of temperature $T = 1/\beta$, which is described for a single oscillator of frequency $\omega$ by

$$\rho_\beta(Q, Q') = \left( \frac{\omega}{\pi \coth(\beta\omega/2)} \right)^{1/2} \cdot \exp \left[ -\frac{\omega}{2 \sinh(\beta\omega)} \left( (Q^2 + Q'^2) \cosh(\beta\omega) - 2QQ' \right) \right].$$
(100)

The dissipation kernel is then

$$\alpha_R(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \ coth(\frac{\beta\omega}{2}) r(\omega) e^{-i\omega\tau}.$$ (101)

Combined together, it gives the real-time thermal Green’s function:

$$\alpha(\tau) \equiv \alpha_R(\tau) + i\alpha_I(\tau) = \sum_k c_k^2 \ tr \left( \rho_\beta T \left[ Q(\omega_k, \tau) Q(\omega_k, 0) \right] \right),$$ (102)

$$\alpha(\omega) \equiv \int_{-\infty}^{\infty} d\tau \alpha(\tau) e^{i\omega\tau} = i \sum_k c_k^2 \left( \frac{1}{\omega^2 - \omega_k^2 + i\epsilon} - \frac{2\pi i}{e^{\beta\omega_k} - 1} \delta(\omega^2 - \omega_k^2) \right).$$ (103)

As noted already, these are given in terms of the response weight $r(\omega)$, and are governed by the analytic function $\overline{G}(z)$.

For the system dynamics we further assume a single harmonic oscillator of frequency $\omega_0$. In the path integral approach integration over the sum variable $X(\tau)$ is trivial in this case, since both the local part and the nonlocal action above are linear in this variable:

$$\frac{i}{2} \int_0^t \left( \dot{\xi}(\tau)\dot{X}(\tau) - \omega_0^2 \xi(\tau)X(\tau) \right)$$

$$- \int_0^t d\tau \int_0^\tau ds \left( \xi(\tau)\alpha_R(\tau - s)\xi(s) + i \xi(\tau)\alpha_I(\tau - s)X(s) \right).$$ (104)

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Thus result of the path integration of the system variable $X(\tau)$ gives the classical integro-differential equation for $\xi(\tau)$:

$$\frac{d^2\xi}{d\tau^2} + \omega_0^2 \xi(\tau) + 2 \int_\tau^t ds \xi(s) \alpha_I(s - \tau) = 0.$$  \hspace{1cm} (105)

The end result of the $\xi$ path integral then contains an integral of the form,

$$- \int_0^t d\tau \int_0^\tau ds \xi(\tau) \alpha_R(\tau - s) \xi(s),$$  \hspace{1cm} (106)

using the classical solution $\xi(\tau)$ with specified boundary conditions, $\xi(0) = \xi_i$, $\xi(t) = \xi_f$.

In the local approximation often used the dissipation kernel is taken to have the form of

$$\alpha_I(\tau) = \delta \omega^2 \delta(\tau) + \eta \delta'(\tau),$$  \hspace{1cm} (107)

with $\delta \omega^2$ representing the frequency shift and the $\eta$ term the local friction. This choice enables one to solve the $\xi$ equation (105) by elementary means. On the other hand, the noise kernel is usually given by the response weight of the form,

$$r(\omega) = \frac{\eta \omega f\left(\frac{\omega}{\Omega}\right)}{\pi},$$  \hspace{1cm} (108)

with $\omega f(x)$ some cutoff function and $\Omega$ a high frequency cutoff. The cutoff is needed to tame the high frequency integral of $\alpha_R(t)$. The simplest cutoff function $f(x) = \theta(1 - x)$ gives an approximate form of $\alpha_I(t)$ with the friction $\eta$ and

$$\delta \omega^2 \approx - \frac{2}{\pi} \eta \Omega.$$  \hspace{1cm} (109)

The rest of deduction uses the Laplace transform, and we shall be brief, leaving technical details to our original paper. Solution of the integro-differential equation (105) is, using $g(\tau)$ defined by eq.(107), given as

$$\xi(\tau) = \xi_i \frac{g(t - \tau)}{g(t)} + \xi_f \left( \dot{g}(t - \tau) - \frac{g(t - \tau) \dot{g}(t)}{g(t)} \right),$$  \hspace{1cm} (110)

with the dot denoting derivative.

The reduced density matrix of the quantum system at any time is obtained from the action written in terms of the boundary values, $\mathcal{S}_d(\xi_f, X_f; \xi_i, X_i)$, by
convoluting with the initial density matrix of the thermal environment. This action
is computed as

\[ iS = -\frac{U}{2} \xi_f^2 - \frac{V}{2} \xi_i^2 - W \xi_i \xi_f + \frac{i}{2} X_f \dot{\xi}_f - \frac{i}{2} X_i \dot{\xi}_i, \tag{111} \]

\[ U = 2 \int_0^t \int_0^\tau ds \ z(\tau) \alpha_R(\tau - s) \ z(s), \tag{112} \]

\[ V = 2 \int_0^t \int_0^\tau ds \ y(\tau) \alpha_R(\tau - s) \ y(s), \tag{113} \]

\[ W = \int_0^t \int_0^\tau ds \ (y(\tau) z(s) + y(s) z(\tau)) \alpha_R(\tau - s), \tag{114} \]

\[ y(\tau) = \frac{g(t - \tau)}{g(t)}, \tag{115} \]

\[ z(\tau) = \dot{g}(t - \tau) - g(t - \tau) \frac{\dot{g}(t)}{g(t)}, \tag{116} \]

\[ \dot{\xi}(\tau) = -\xi_i \frac{\dot{g}(t - \tau)}{g(t)} - \xi_f \left( \ddot{g}(t - \tau) - \frac{\dot{g}(t) \dot{g}(t)}{g(t)} \right). \tag{117} \]

For further discussion we take as the initial state a product of the thermal states, a system of temperature \( T_0 = 1/\beta_0 \) and an environment of temperature \( T = 1/\beta \). We may take \( T_0 = T \) when we apply to the decay process of excited level initially in thermal equilibrium. On the other hand, in the limit of \( T_0 \to 0 \) it describes the ground state of the system harmonic oscillator.

After a series of straightforward Gaussian integration we find the reduced density matrix as a function of \( X_f \) and \( \xi_f \), of the form,

\[ \rho^{(R)}(X_f, \xi_f) = 2 \sqrt{\frac{A}{\pi}} \exp[-AX_f^2 - BX_f^2 + iCX_f\xi_f], \tag{118} \]

\[ A = \frac{1}{8I_1}, \quad B = \frac{1}{2} \left( I_3 - \frac{I_2^2}{I_1} \right), \quad C = \frac{I_2}{2I_1}, \tag{119} \]

\[ I_1 = \mathcal{I} \left[ |h(\omega, t)|^2 \right] + \frac{1}{2\bar{\omega}} \coth\left( \frac{\beta_0 \bar{\omega}}{2} \right) (\dot{g}^2 + \bar{\omega} g^2), \tag{120} \]

\[ I_2 = \Re \mathcal{I} \left[ h(\omega, t) k^*(\omega, t) \right] + \frac{1}{2\bar{\omega}} \coth\left( \frac{\beta_0 \bar{\omega}}{2} \right) \dot{g} (\dot{g} + \omega g), \tag{121} \]

\[ I_3 = \mathcal{I} \left[ |k(\omega, t)|^2 \right] + \frac{1}{2\bar{\omega}} \coth\left( \frac{\beta_0 \bar{\omega}}{2} \right) (\dot{g}^2 + \bar{\omega} \dot{g}). \tag{122} \]

Here \( \bar{\omega} \) is a reference frequency taken as that of the initial system state, and equated here to the frequency at the pole. If one so desires, either the renormalized \( \omega_R \) or the bare \( \omega_0 \) may be taken as another choice. If we imagine a situation in which the small system was added to a large environment at some time, its mutual interaction being absent prior to the initial time, then it is appropriate to take \( \omega_0 \) as the reference
frequency. Since dependence on the initial state dies away quickly as time passes, the choice of the initial reference is not crucial for determining the behavior of states at late times. Both of $h(\omega,t)$ and $k(\omega,t)$ are already defined in the preceding subsection. The density matrix $\rho^{(R)}$ from which any physical quantity at time $t$ can be computed has explicitly been given by the discontinuity, $H(\omega)$ or $r(\omega)$.

The basic quantities that appear in the reduced density matrix are related to expectation values of the coordinate and the momentum operators at the same moment by

$$\langle q^2 \rangle = \frac{1}{8A} = I_1,$$

$$\langle p^2 \rangle = 2B + \frac{C^2}{2A} = I_3,$$

$$\langle \frac{1}{2} (qp + pq) \rangle = \frac{C}{4A} = I_2. \quad (125)$$

Thus one may write the density matrix as

$$\rho^{(R)}(X_f, \xi_f) = \sqrt{\frac{1}{2\pi \langle q^2 \rangle}} \exp[-\frac{1}{8\langle q^2 \rangle} X_f^2 - \left(\frac{\langle p^2 \rangle}{2} - \frac{\langle qp + pq \rangle^2}{8\langle q^2 \rangle}\right) \xi_f^2 + i\frac{\langle qp + pq \rangle}{4\langle q^2 \rangle} X_f \xi_f]. \quad (126)$$

The reduced density matrix is thus characterized by expectation values of quadratic operators, just as in the case of pure Gaussian system without the environmental effect.

It is sometimes useful to transform the density matrix in the configuration space to the Wigner function $f_W(x,p)$,

$$f_W(x,p) \equiv \int_{-\infty}^{\infty} d\xi \rho^{(R)}(2x, \xi) e^{-i\xi p} \quad (127)$$

$$= \sqrt{\frac{4A}{B}} \exp[-4Ax^2 - \frac{(p - 2C x)^2}{4B}]. \quad (128)$$

The Wigner function is expected to give the probability distribution in the phase space $(x,p)$ when the semi-classical picture is valid. Expectation value of the number operator, namely the occupation number, in terms of the reference frequency, equated to the pole location $\bar{\omega}$ here, is calculated most easily from the Wigner function:

$$\langle n \rangle \equiv \langle -\frac{1}{2\bar{\omega}} \frac{d^2}{dq^2} + \frac{\bar{\omega}}{2} q^2 - \frac{1}{2} \rangle = \frac{B}{\bar{\omega}} + \frac{C^2}{4\bar{\omega} A} + \frac{\bar{\omega}}{16A} - \frac{1}{2}$$

$$= \frac{1}{2\bar{\omega}} (I_3 + \bar{\omega}^2 I_1) - \frac{1}{2}. \quad (129)$$
It consists of two terms except the trivial $\frac{1}{2}$, the term $\bar{\omega}/(16A)$ from the Gaussian width of the diagonal density matrix element and the rest from the kinetic term $-\frac{d^2}{dq^2}$. This formula of course agrees with that of the previous derivation in the operator method.

VII Short-time behavior of non-decay probability

As an application of the influence functional method, I shall discuss the short-time behavior of the decay probability. This is an interesting problem from the point of the nucleon decay, as mentioned in Introduction. I shall describe some fundamental aspects of this problem, and leave details to our forthcoming paper.\[13\]

We take as the initial state of the subsystem the first excited state of harmonic oscillator that may be considered as a one-particle state of unstable particle:

$$|i\rangle = a^\dagger_\omega |0\rangle, \quad \rho_i(q,q') = (\frac{\bar{\omega}}{\pi})^{1/2} 2\omega qq' e^{-\bar{\omega}(q^2+q'^2)/2}. \quad (130)$$

After some straightforward computation one obtains the reduced density matrix for this case:

$$\rho^R = 2\sqrt{\frac{A}{\pi}} \left( F_0 + F_1 X_2^2 + F_2 \xi_f^2 + i F_3 X_f \xi_f \right) \exp[-A X_f^2 - B \xi_f^2 + i C X_f \xi_f], \quad (131)$$

$$A = \frac{1}{8I_1}, \quad B = \frac{1}{2}(I_3 - \frac{I_2}{I_1}), \quad C = \frac{I_2}{2I_1}, \quad (132)$$

$$F_0 = \frac{1}{I_1} \mathcal{I}[|h(\omega,t)|^2], \quad F_1 = \frac{1}{8\omega I_1^2} (g^2 + \bar{\omega}^2 g^2), \quad (133)$$

$$F_2 = -\frac{1}{2\omega I_1^2} \left( I_1^2 \left( \ddot{g}^2 + \bar{\omega}^2 g^2 \right) + I_2^2 \left( \ddot{g}^2 + \bar{\omega}^2 g^2 \right) - 2I_1 I_2 \ddot{g} \left( \ddot{g} + \bar{\omega}^2 g \right) \right), \quad (134)$$

$$F_3 = \frac{1}{2\omega I_1^2} \left( \mathcal{I}[|h(\omega,t)|^2] \ddot{g} \left( \ddot{g} + \bar{\omega}^2 g \right) - \Re \mathcal{I}[h(\omega,t)k(\omega,t)^*] \left( \ddot{g}^2 + \bar{\omega}^2 g^2 \right) \right). \quad (135)$$

The quantities $I_i$ are defined in Eq.[123].

The non-decay probability, or sometime called the survival probability, is defined as the overlap between this density matrix and that of the first excited state written by the final variables, $q_f, q'_f$. This leads to

$$\frac{\bar{\omega}}{4} \sqrt{A\bar{\omega}} \left( \left( A + \frac{\bar{\omega}}{4} \right) \left( B + \frac{\bar{\omega}}{4} \right) + \frac{C^2}{4} \right)^{-5/2} \quad (136)$$
\[
\cdot \left[ \left( (\mathcal{A} + \bar{\omega})(\mathcal{B} + \bar{\omega}) + \frac{C^2}{4} \right) (\mathcal{F}_0(\mathcal{B} - \mathcal{A}) + \mathcal{F}_1 - \mathcal{F}_2) \\
+ \frac{3}{2}(\mathcal{B} - \mathcal{A}) \left( \mathcal{F}_1(\mathcal{B} + \bar{\omega}) + \mathcal{F}_2(\mathcal{A} + \bar{\omega}) - \mathcal{F}_3 \frac{C}{2} \right) \right].
\] (137)

The survival probability must be derived after dividing the disconnected contribution that corresponds to the ground to the ground transition,

\[
\left[ I_1 I_3 - I_2^2 + \frac{1}{4} + \frac{\bar{\omega}}{2} I_1 + \frac{I_3}{2\bar{\omega}} \right]^{-1/2}.
\] (138)

For the short-time behavior this gives a leading term for the non-decay probability of the form,

\[
P_{1\rightarrow 1} \approx 1 - \frac{1}{\bar{\omega}} \int_{\omega_c}^{\infty} d\omega \, r(\omega) \coth\left(\frac{\beta\omega}{2}\right) |k(\omega, t)|^2.
\] (139)

One is very much interested in whether the second term is proportional to time\(^2\) or not.\[14\] Naively,

\[
|k(\omega, t)|^2 \rightarrow \dot{g}(0)^2 t^2 = t^2,
\] (140)

as \(t \rightarrow 0^+\), but one must be careful, because the limit may not be exchangeable with the \(\omega\) integration when the integral is conditionally convergent.

Since

\[
\coth\left(\frac{\beta\omega}{2}\right) = 1 + \frac{2}{e^{\beta\omega} - 1},
\]

the temperature dependent term is given by an absolutely convergent \(\omega\) integral, for which one may exchange the \(t \rightarrow 0^+\) limit and the integral. Thus, effect dependent on the environment gives a term of order \(t^2\) at short times.

On the other hand, the environment independent term differs, depending on whether

\[
\int_{\omega_c}^{\infty} d\omega \, r(\omega)
\]

is finite or not. If it is finite as is the case in many applications in condensed matter physics, the survival probability contains terms of order \(t^2\). But if it approaches a constant, \(r(\omega) \rightarrow r(\infty)\), then a more careful computation gives

\[
P_{1\rightarrow 1} \approx 1 - \frac{\pi r(\infty)}{\bar{\omega}} t.
\] (141)
The result thus seems to suggest that the short time limit in our boson decay model gives the short time limit of the exponential decay,

\[ e^{-\Gamma t} \approx 1 - \Gamma t. \]  \hspace{1cm} (142)

But, there are many details worth of further investigation which will be dealt with elsewhere.\[13\]

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