About Covariant Quartit Cloning Machines

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Abstract

The study of quantum cryptography and quantum entanglement has traditionally been based on two-level quantum systems (qubits) and more recently on three-level systems (qutrits). We investigate several classes of state-dependent quantum cloners for four-level systems (quartits). These results apply to symmetric as well as asymmetric cloners, so that the balance between the fidelity of the two clones can also be analyzed. We extend Cerf’s formalism for cloning states in order to derive cloning machines that remain invariant under certain unitary transformations. Our results show that a different cloner has to be used for two mutually unbiased bases which are related by a double Hadamard transformation, than for two mutually unbiased bases that are related by a Fourier transformation. This different cloner is obtained thanks to a redefinition of Bell states that respects the intrinsic symmetries of the Hadamard transformation.

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I. INTRODUCTION

Quantum cryptography aims at distributing a random key in such a way that the presence of an eavesdropper (traditionally called Eve) who monitors the quantum communication is revealed by the errors she introduces in the transmission (for a review, see e.g. [1]). To realize a quantum cryptographic protocol, it suffices that the key signal is encoded into quantum states that belong to incompatible bases, as in the original qubit protocol of Bennett and Brassard known as BB84 [2]. In past years, several qudit-based cryptographic protocols were shown to be more secure than their qubit-based counterparts [3, 4, 5, 6, 7, 8, 9]. This justifies the interest for studying the security of qudit protocols (see also Ref. [10]). In order to evaluate the security of such protocols against individual attacks (where Eve monitors the qudits separately or incoherently) we will consider a fairly general class of eavesdropping attacks that are based on (state-dependent) quantum cloning machines [11, 12, 13]. This will yield an upper bound on the acceptable error rate. Higher error rates do not allow secure communication, since in accordance with C-K theorem [14], a spy could in theory acquire all the information, if Alice and Bob restrict themselves to one way communication on the classical channel.

Quantum cloning is a concept that was first introduced in a seminal paper by Buzek and Hillery [15], where a universal (or state-independent) and symmetric cloning transformation was introduced for qubits. This transformation was later extended to higher-dimensional systems by Werner [16] but only in the special case of a universal (state-independent) cloner. In contrast, we will focus on non-universal (or state-dependent) cloners. Our starting point, summarized in Sec. III is a general characterization of asymmetric and state-dependent cloning transformations for $N$-level systems, as described in Refs. [12, 13]. In Sec. III we will establish the generality of this formalism. We adapt the formalism in Sec. IV such that it is invariant under certain unitary transformations. We propose in Sec. V an optical implementation for quantum key distribution that generalizes BB84 protocol [2]. In this new protocol, the signal is encoded in four-dimensional basis states, in two mutually unbiased bases [26]. The covariant cloning formalism developed in the previous sections will be applied in Sec. VI to clone these two mutually unbiased bases that are actually related by a double Hadamard transformation [10]. We will show that the optimal cloner requires a generalization of the Bell states well adapted to the particular problem under study, before
we conclude in Sec. VII.

II. STATE DEPENDENT CLONING FORMALISM

To explain the general cloning formalism it is convenient to look at the exchange of a quantum key in a different way. Suppose Alice and Bob exchange the maximally entangled state:

\[ |B_{00}⟩ = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k⟩ ⊕ |k⟩ \]  

(1)

The states \(|k⟩\) form a basis (\(|0⟩, |1⟩, ..., |N⟩\)) (called the computational basis) in the N-dimensional Hilbertspace. Alice has access to the first quantum, while Bob has access to the second. If Alice now projects her part of the state on \(|ψ^*⟩\), the resulting state is:

\[ (|ψ^*⟩⟨ψ^*| ⊗ \hat{I}) \sum_{k=0}^{N-1} |k⟩ ⊗ |k⟩ = |ψ^*⟩ \sum_{i=0}^{N-1} ⟨ψ^*|k⟩ ⊗ |k⟩ \]

\[ = |ψ^*⟩ \sum_{i=0}^{N-1} ⟨k|ψ⟩ ⊗ |k⟩ = |ψ^*⟩ ⊗ |ψ⟩ \]  

(2)

The result is a product state, and Bob has access to \(|ψ⟩\). We see that it makes no difference whether Alice creates and sends the state \(|ψ⟩\) to Bob, or whether Alice projects her part of a shared maximally entangled state on \(|ψ^*⟩\). Therefore, we will use this representation of the exchange of \(|ψ⟩\), as it makes the mathematical description of cloners more elegant. A spy’s attack would now consist of modifying the maximally entangled state when it was created or exchanged.

In the cloning formalism of N. Cerf [11, 12, 13], the maximally entangled state is modified into the \(N^4\) dimensional state \(|Ψ⟩_{R,A,B,C}\). The index ‘R’ denotes the reference state accessible to Alice. She will project it onto \(|ψ^*⟩\) to communicate with Bob (as she doesn’t know Eve has tampered with the maximally entangled state, she thinks that Bob will receive \(|ψ⟩\)). The index ‘A’ denotes the state that is accessible to Bob, and it will contain the first (imperfect) clone of \(|ψ⟩\). The index ‘B’ denotes the state of Eve that will contain the second clone. And finally ‘C’ is the state of the Cloning machine, and is also accessible by Eve. In the formalism of N. Cerf, \(|Ψ⟩_{R,A,B,C}\) has the form:

\[ |Ψ⟩_{R,A,B,C} = \sum_{m,n=0}^{N-1} a_{m,n} \langle B_{m,n}|_{R,A}|B_{m,-n}|_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n} \langle B_{m,n}|_{R,B}|B_{m,-n}|_{A,C} \]  

(3)
The states \(|B_{m,n}\rangle\) are generalized (Fourier) Bell states defined as \[27\]:

\[
|B_{m,n}\rangle_{R,A} = N^{-1/2} \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |k\rangle_R |k + m\rangle_A
\] (4)

In \[3\], \(a_{m,n}\) and \(b_{m,n}\) are normalized complex amplitudes:

\[
\sum_{m,n=0}^{N-1} |a_{m,n}|^2 = \sum_{m,n=0}^{N-1} |b_{m,n}|^2 = 1
\] (5)

We can easily calculate the inproducts of the basis states:

\[
(\langle B_{m,n}|_{R,A} \langle B_{m,-n}|_{B,C} (|B_{x,y}\rangle_{R,B} |B_{x,-y}\rangle_{A,C}) = N^{-1} \exp \left( \frac{2\pi i}{N} (nx - my) \right)
\] (6)

Using (6) and (3), we can easily show that \(a_{m,n}\) and \(b_{m,n}\) are dual under a Fourier transform:

\[
a_{m,n} = \frac{1}{N} \sum_{x,y=0}^{N-1} e^{2\pi i (nx-my)/N} b_{x,y}
\] (7)

\[
b_{m,n} = \frac{1}{N} \sum_{x,y=0}^{N-1} e^{2\pi i (nx-my)/N} a_{x,y}
\] (8)

Projecting the reference system of \(|\Psi\rangle_{R,A,B,C}\) onto \(|\psi^*\rangle\) yields (using \[3\]):

\[
\sum_{m,n=0}^{N-1} a_{m,n} U_{m,n} |\psi\rangle_A |B_{m,-n}\rangle_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n} U_{m,n} |\psi\rangle_B |B_{m,-n}\rangle_{A,C}
\] (9)

with:

\[
U_{m,n} = \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |k\rangle \langle k|
\] (10)

The error operators \(U_{m,n}\) shifts the state by \(m\) units (modulo \(N\)) in the computational basis, and multiplies it by a phase to shift its Fourier transform by \(n\) units (modulo \(N\)). Of course, \(U_{0,0} = I\), which corresponds to no error. Note that:

\[
(I \otimes U_{m,n}) |B_{0,0}\rangle = |B_{m,n}\rangle
\] (11)

The state (9) completely characterizes the cloning transformation \[28\]. To evaluate the quality of the clone \(A\) (\(B\)), we take the partial trace of over ‘\(B\)’ and ‘\(C\)’ (‘\(A\)’ and ‘\(C\)’) of the density operator associated with state (9). The resulting reduced density operator \(\rho_A\) and \(\rho_B\) of respectively the first and the second clone are:

\[
\rho_A = \sum_{m,n=0}^{N-1} |a_{m,n}|^2 |\psi_{m,n}\rangle \langle \psi_{m,n}|
\] (12)

\[
\rho_B = \sum_{m,n=0}^{N-1} |b_{m,n}|^2 |\psi_{m,n}\rangle \langle \psi_{m,n}|
\] (13)
where
\[ |\psi_{m,n}\rangle = U_{m,n} |\psi\rangle \quad (14) \]

The fidelity of the first clone when copying \(|\psi\rangle\) (i.e., the probability that there is a collapse onto this state) is equal to
\[ F_A = \langle\psi|\rho_A|\psi\rangle = \sum_{m,n=0}^{N-1} |a_{m,n}|^2 |\langle\psi|\psi_{m,n}\rangle|^2 \quad (15) \]

Focusing on the computational basis, we see that the fidelity of any state \(|k\rangle\) is equal to:
\[ F_A = \sum_{n=0}^{N-1} |a_{0,n}|^2 \quad (16) \]

We also define \(N - 1\) disturbances: we call \(D_i\) the probability that there is a collapse on \(|k + i\rangle\) if state \(|k\rangle\) was sent. The disturbances are (in the computational basis):
\[ D_i = \sum_{n=0}^{N-1} |a_{i,n}|^2 \quad \text{with } i \in \{1, 2, \ldots N - 1\} \quad (17) \]

The fidelities and the disturbances of the second clone are obtained by replacing \(a_{m,n}\) with \(b_{m,n}\) in (15) – (17). It is clear that the quality of the two clones depends on the amplitudes \(a_{m,n}\) and \(b_{m,n}\). Furthermore, there is a trade-off between the quality of the two clones. Suppose \(a_{m,n}\) is a peaked function (i.e., \(a_{m,n}\) is large for one value of \(\{m, n\}\) and small for the other values) leading to a high-fidelity of the first clone. As \(b_{m,n}\) is the Fourier transform of \(a_{m,n}\) it will be a rather flat function (i.e., all \(b_{m,n}\) will be almost equally large), leading to a rather low fidelity for the second clone. The balance between the quality of clones \(A\) and \(B\) can be expressed, in full generality, by an entropic no-cloning uncertainty relation that relates the probability distributions \(p_{m,n}\) and \(q_{m,n}\) [13]:
\[ H[p] + H[q] \geq \log_2(N^2) \quad (18) \]

where \(H[p]\) and \(H[q]\) denote the Shannon entropy of the discrete probability distributions \(p\) and \(q\) defined as
\[ p(m, n) = |a_{m,n}|^2 \quad (19) \]
\[ q(m, n) = |b_{m,n}|^2 \quad (20) \]

This inequality is actually a special case of a no-cloning uncertainty relation involving the losses of the channels that yield the two clones [17]. More refined uncertainty relations can be found that express the fact that the index \(m\) of output \(A\) is dual to the index \(n\) of output \(B\), and vice versa [13].
III. ABOUT THE GENERALITY OF CERF’S FORMALISM

One could object that the class of cloning machines considered in this paper is not the most general one. Indeed, we postulate from the beginning that the joint state $|\Psi\rangle_{R,A,B,C}$ has the form of Eq. (3) implying that the reduced density matrices of Alice and Bob $(R, A)$ and Eve $(B, C)$ are diagonal in the Bell basis. Although Eve is free to choose the basis that diagonalises her reduced density matrix, the most general state $|\Psi\rangle_{R,A,B,C}$ will not possess the property that $\text{Tr}_{B,C}|\Psi\rangle_{R,A,B,C}\langle\Psi|_{R,A,B,C}$ is diagonal in the Bell basis $|B_{m,n}\rangle_{R,A}$. We will call the states that have this property “Cerf”-states. Nearly all the (optimal) cloning machines that appeared in the literature can be unambiguously represented by a Cerf-state [8, 9, 18, 19, 20, 21]. We can guess the underlying physical reason if we note that the Bell states are invariant (up to a global phase) under a cyclic relabeling of the basis states. Indeed, let us consider the generator $C$ of cyclic permutations of the indices $l$ of the computational basis:

$$C.|l\rangle = |(l + 1) \mod 4\rangle \quad (21)$$

It is easy to check that the Bell states defined in Eq. (14) are mapped onto themselves under such a permutation (up to a global phase), and under all its powers. So, when cyclic permutations are a natural symmetry of the protocol, it is also natural that the Bell states defined in Eq. (14) play a privileged role in Eve’s attack: the symmetries of the effects reflect the symmetries of the causes as is well known in physics.

Now, we could ask the inverse question: “What is the most general state that is invariant under any cyclic relabeling of the basis states?” We shall show that when optimal qubit cloning machines are described by a $2^4$ dimensional pure state which is invariant under any cyclic relabeling of the basis states, such a state is a Cerf state. This shows (at least for qubits) that Cerf’s formalism for optimal cloning machines is more general than it could seem at first sight.

In two dimensions, only two permutations exist: the identity and the generator $C$ that switches the labels 0 and 1 (which is a parity operator: $C^2 = 1$). We shall now prove the following theorem:

**Theorem:**

Let us consider any qubit protocol. Let us consider the corresponding 16 dimensional pure state that is assumed to be optimal and to be invariant under any cyclic relabeling of
the basis states. Such a state is a Cerf-state so to say \( T_{R,B,C} |\Psi\rangle_{R,A,B,C} \langle \Psi|_{R,A,B,C} \) is diagonal in the Bell basis \( |B_{m,n}\rangle_{R,A} \).

**Proof:**

The four Bell states, when \( N = 2 \), are defined as follows:

\[
|B_{0,0}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |B_{1,0}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad (22)
\]
\[
|B_{0,1}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad |B_{1,1}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad (23)
\]

It is easy to check that the Bell states are eigenstates of the permutation operator \( C \) with the eigenvalues \( \pm 1 \). If we impose that the joint state \( |\Psi\rangle_{R,A,B,C} \) is invariant under the action of the permutation operator \( (C|\Psi\rangle_{R,A,B,C} = |\Psi\rangle_{R,A,B,C} \) ), then it belongs necessarily to the 8 dimensional eigenspace associated to the eigenvalue \( +1 \) of \( C \):

\[
|\Psi\rangle_{R,A,B,C} = \alpha_+ |B_{0,0}\rangle_{R,A} \otimes |B_{0,0}\rangle_{B,C} + \alpha_- |B_{0,1}\rangle_{R,A} \otimes |B_{1,1}\rangle_{B,C} \\
+ \beta_+ |B_{1,0}\rangle_{R,A} \otimes |B_{1,0}\rangle_{B,C} + \beta_- |B_{1,1}\rangle_{R,A} \otimes |B_{1,1}\rangle_{B,C} \\
+ \gamma_+ |B_{1,0}\rangle_{R,A} \otimes |B_{0,0}\rangle_{B,C} + \gamma_- |B_{1,1}\rangle_{R,A} \otimes |B_{0,0}\rangle_{B,C} \\
+ \delta_+ |B_{0,0}\rangle_{R,A} \otimes |B_{1,0}\rangle_{B,C} + \delta_- |B_{0,1}\rangle_{R,A} \otimes |B_{1,1}\rangle_{B,C} \quad (24)
\]

We see that the requirement that the full state is invariant under a permutation of the basis states reduces the generality of the joint state (which is usually described by 16 amplitudes). Here the amplitudes associated to states that are antisymmetric under the action of the permutation operator \( C \) are assumed to be equal to zero. Although Eq. (24) does not yet describe a Cerf-state, we shall proof that in the optimal case it does. In order to do so, let us first consider the reduced state shared by Alice and Bob:

\[
\rho_{R,A} = T_{R,B,C} |\Psi\rangle_{R,A,B,C} \langle \Psi|_{R,A,B,C} \\
= (\alpha_+ |B_{0,0}\rangle_{R,A} + \gamma_+ |B_{1,0}\rangle_{R,A}) (\alpha_+ \langle B_{0,0}|_{R,A} + \gamma_+ \langle B_{1,0}|_{R,A} ) \\
+ (\beta_+ |B_{1,0}\rangle_{R,A} + \delta_+ |B_{0,0}\rangle_{R,A}) (\beta_+ \langle B_{1,0}|_{R,A} + \delta_+ \langle B_{0,0}|_{R,A} ) \\
+ (\alpha_- |B_{0,1}\rangle_{R,A} + \gamma_- |B_{1,1}\rangle_{R,A}) (\alpha_- \langle B_{0,1}|_{R,A} + \gamma_- \langle B_{1,1}|_{R,A} ) \\
+ (\beta_- |B_{1,1}\rangle_{R,A} + \delta_- |B_{0,1}\rangle_{R,A}) (\beta_- \langle B_{1,1}|_{R,A} + \delta_- \langle B_{0,1}|_{R,A} ) \quad (25)
\]

It is worth noting that the non-diagonal components of the reduced density matrix \( \rho_{R,A} \) of the type \( |B_{i,j}\rangle \langle B_{m,n}| \) (for \( i \neq m \)) do not contribute to the statistics of the measurements performed by Alice and Bob in the computational basis. Indeed:
\[
\langle pq | B_{i,j} \rangle \langle B_{m,n} | pq \rangle = \frac{1}{2} \sum_{k,l} (-1)^{k+j+ln} \langle p | k \rangle \langle q | k+i \rangle \langle l | p \rangle \langle l+m | q \rangle
\] (26)
\[
= \frac{1}{2} \sum_{k,l} (-1)^{k+j+ln} \delta_{p,k} \delta_{q,k+i} \delta_{l,p} \delta_{l+m,q}
\] (27)
\[
= \frac{1}{2} (-1)^{p(j+n)} \delta_{q,p+i} \delta_{q,p+m}
\] (28)

which is 0 if \( m \neq i \). This means that everything happened, from the point of view of Alice and Bob, as if \( \rho_{R,A} \) was equal to the effective density operator \( \rho_{R,A}^{\text{eff}} \) defined as:

\[
\rho_{R,A}^{\text{eff}} = (|\alpha_+|^2 + |\delta_+|^2) |B_{0,0}\rangle \langle B_{0,0}| + (|\beta_+|^2 + |\gamma_+|^2) |B_{1,0}\rangle \langle B_{1,0}|
\]
\[
+ (|\alpha_-|^2 + |\delta_-|^2) |B_{0,1}\rangle \langle B_{0,1}| + (|\beta_-|^2 + |\gamma_-|^2) |B_{1,1}\rangle \langle B_{1,1}|
\] (29)

Let us evaluate the information possessed by Eve relatively to Alice. We denote \( P_{i,j}^E \) the probability that Eve simultaneously observes the output clone \( B \) to be in the \( i \)th detector and the cloning machine \( C \) in the \( j \)th detector and \( P_{k;i,j}^{AE} \) the probability that Alice’s \( k \)th detector fires while Eve simultaneously observes the output clone \( B \) in the \( i \)th detector and the cloning machine \( C \) in the \( j \)th detector. By direct computation, we obtain that:

\[
P_{0,0}^E = P_{1,1}^E = \frac{1}{2} (|\alpha_+|^2 + |\alpha_-|^2 + |\gamma_+|^2 + |\gamma_-|^2)
\] (30)
\[
P_{0,1}^E = P_{1,0}^E = \frac{1}{2} (|\beta_+|^2 + |\beta_-|^2 + |\delta_+|^2 + |\delta_-|^2)
\] (31)
\[
P_{0,0,0}^{AE} = 1 - P_{1,0,0}^{AE} = 1 - P_{0,1,1}^{AE} = P_{1,1,1}^{AE}
\]
\[
= \frac{1}{2} P_{0,0}^E + \frac{1}{2} \text{Re}(\alpha_+ \cdot \alpha_-^*) + \frac{1}{2} \text{Re}(\gamma_+ \cdot \gamma_-^*)
\] (32)
\[
P_{0,0,1}^{AE} = 1 - P_{1,0,1}^{AE} = 1 - P_{0,1,0}^{AE} = P_{1,1,0}^{AE}
\]
\[
= \frac{1}{2} P_{0,1}^E + \frac{1}{2} \text{Re}(\beta_+ \cdot \beta_-^*) + \frac{1}{2} \text{Re}(\delta_+ \cdot \delta_-^*)
\] (33)

Equations (30)–(33) show that the information gained by Eve during such an attack is certainly not superior to the information gained during an attack in which Eve chooses the Cerf state

\[
\frac{1}{\sqrt{P_1}} (\alpha_+ |B_{0,0}\rangle_{R,A} \otimes |B_{0,0}\rangle_{B,C} + \alpha_- |B_{0,1}\rangle_{R,A} \otimes |B_{0,1}\rangle_{B,C}
\]
\[
+ \beta_+ |B_{1,0}\rangle_{R,A} \otimes |B_{1,0}\rangle_{B,C} + \beta_- |B_{1,1}\rangle_{R,A} \otimes |B_{1,1}\rangle_{B,C})
\] (34)
with probability \( P_1 = |\alpha_+|^2 + |\alpha_-|^2 + |\beta_+|^2 + |\beta_-|^2 \) and the Cerf state
\[
\frac{1}{\sqrt{P_2}} (\gamma_+ \ket{B_{0,0}}_{R,A} \otimes \ket{B_{0,0}}_{B,C} + \gamma_- \ket{B_{0,1}}_{R,A} \otimes \ket{B_{0,1}}_{B,C}) \\
+ \delta_+ \ket{B_{1,0}}_{R,A} \otimes \ket{B_{1,0}}_{B,C} + \delta_- \ket{B_{1,1}}_{R,A} \otimes \ket{B_{1,1}}_{B,C})
\] (35)

with probability \( P_2 = |\gamma_+|^2 + |\gamma_-|^2 + |\delta_+|^2 + |\delta_-|^2 \). Furthermore, the two attacks lead to the same distribution of statistical results, as the effective density operator equals the density operator of the attack using (34) and (35). By assumption, the optimal (pure) Cerf state is superior to this second attack (which uses a mixture of Cerf states), which ends the proof. The generalization of this proof to arbitrary dimensions is given in [22].

IV. THE COVARIANT CLONING MACHINE

The formalism defined in the previous section is valid when Alice and Bob respectively encode and measure the signal in the computational basis \((\ket{0}, \ket{1}, \ket{2}, \ldots, \ket{N})\). Quantum cryptographic protocols impose that Alice must use at least another basis \(\tilde{\psi}\) \((\ket{\tilde{\psi}_0}, \ket{\tilde{\psi}_1}, \ket{\tilde{\psi}_2}, \ldots, \ket{\tilde{\psi}_N})\), with
\[
\langle i | \tilde{\psi}_j \rangle = A_{ij}
\] (36)

If Alice and Bob share the joint state \(\ket{B_{0,0}}\) and that Alice wishes to encode the signal in the \(\tilde{\psi}\) basis, she must project her component of \(\ket{B_{0,0}}\) into the conjugate basis (that we shall from now on denote the \(\tilde{\psi}^*\) basis) defined as follows: \(\langle i | \tilde{\psi}^*_j \rangle = A^*_{ij}\). This property is a direct consequence of Eq. (2). We now require that the computational basis is not privileged, and that the cloning machine clones the states of the computational and the \(\tilde{\psi}\) basis in the same manner. This implies that after projecting the reference (Alice’s) system onto the state \(\tilde{\psi}^*\) (which would result in an input state \(\tilde{\psi}\) of Bob, in the absence of an eavesdropper), the reduced density operators of Bob and Eve must be of the form:
\[
\rho_A = \sum_{m,n=0}^{N-1} p_{m,n} \ket{\bar{\psi}_{m,n}} \bra{\bar{\psi}_{m,n}}
\]
(37)
\[
\rho_B = \sum_{m,n=0}^{N-1} q_{m,n} \ket{\bar{\psi}_{m,n}} \bra{\bar{\psi}_{m,n}}
\]
(38)

where
\[
\ket{\bar{\psi}_{m,n}} = U_{m,n} \ket{\tilde{\psi}}
\]
(39)
and
\[ \tilde{U}_{m,n} = \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |\tilde{\psi}_{k+m}\rangle \langle \tilde{\psi}_k| . \] (40)

A possible corresponding joint state of the two clones and the cloning machine is (see also [8]):
\[ \sum_{m,n=0}^{N-1} a_{m,n} \tilde{U}_{m,n} |\tilde{\psi}_A\rangle |\tilde{B}_{m,n}^*\rangle_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n} \tilde{U}_{m,n} |\tilde{\psi}_B\rangle |\tilde{B}_{m,n}^*\rangle_{A,C} \] (41)

where
\[ |\tilde{B}_{m,n}\rangle = N^{-1/2} \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |\tilde{\psi}_k^*\rangle |\tilde{\psi}_{k+m}\rangle = (I \otimes \tilde{U}_{m,n}) |B_{0,0}\rangle \] (42)

Equation (42) is a generalization of the Bell States defined in (4). Indeed, if \( |\tilde{\psi}\rangle \) is the computational basis, (42) and (4) are equal, because the computational basis is real and \( |B_{m,n}^*\rangle = |B_{m,-n}\rangle \). The joint state of the reference \( R \), the two output clones (\( A \) and \( B \)), and the \( (N\text{-dimensional}) \) cloning machine \( C \) that corresponds with (41) is:
\[ \sum_{m,n=0}^{N-1} a_{m,n} |\tilde{B}_{m,n}\rangle_{R,A} |\tilde{B}_{m,n}^*\rangle_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n} |\tilde{B}_{m,n}\rangle_{R,B} |\tilde{B}_{m,n}^*\rangle_{A,C} \] (43)

With this choice[30], the requirement of covariance in the computational and the \( \tilde{\psi} \) bases imposes the extra-constraint:
\[ \sum_{m,n=0}^{N-1} a_{m,n} |B_{m,n}\rangle_{R,A} |B_{m,n}^*\rangle_{B,C} = \sum_{i,j=0}^{N-1} a_{i,j} \tilde{B}_{i,j}\rangle_{R,A} |\tilde{B}_{i,j}^*\rangle_{B,C} \] (44)

As the \( B_{m,n} \) states form an orthonormal basis, we can write:
\[ \sum_{m,n=0}^{N-1} a_{m,n} |B_{m,n}\rangle_{R,A} |B_{m,n}^*\rangle_{B,C} = \sum_{i,j} a_{i,j} \tilde{B}_{i,j}\rangle_{R,A} \left( \sum_{k,l} |B_{k,l}\rangle_{B,C} \right) |\tilde{B}_{i,j}^*\rangle_{B,C} \] (45)

Defining:
\[ V_{i,j,k,l} \equiv \langle B_{i,j} | \tilde{B}_{k,l} \rangle \] (46)
we get:

$$\sum_{m,n} a_{m,n} \delta_{(m,n),(k,l)} |B_{m,n}\rangle_{R,A} |B^*_{k,l}\rangle_{B,C}$$

(47)

$$= \sum_{i,j,k} a_{i,j} |B_{m,n}\rangle_{R,A} V_{m,n,i,j} |B^*_{k,l}\rangle_{B,C} V^*_{k,l,i,j}$$

(48)

$$= \sum_{m,n,i,p} a_{i,j} \delta_{(i,j),(p,q)} V_{m,n,i,j} V^*_{k,l,p,q} |B_{m,n}\rangle_{R,A} |B^*_{k,l}\rangle_{B,C}$$

(49)

Formally, this constraint can be expressed as a matrix relation:

$$\mathcal{V}A = AV$$

(50)

where $\mathcal{V}$ and $A$ are $N^2 \times N^2$ matrices defined as follows:

$$\mathcal{V}_{m,n,i,j} = V_{m,n,i,j}$$

(51)

and $A$ the diagonal matrix defined as

$$A_{i,j,p,q} = a_{i,j} \delta_{(i,j),(p,q)}$$

(52)

As $A$ is diagonal, (50) is extremely simple to solve:

$$\text{if } \mathcal{V}_{i,j,p,q} \neq 0 \text{ then } a_{i,j} = a_{p,q}$$

(53)

Therefore, to build a cloning machine that is invariant in two bases (the computational basis and the $\tilde{\psi}$ basis) we only need to compute the $N^4$ inproducts $\mathcal{V}_{i,j,k,l} = \langle B_{i,j} | \tilde{B}_{k,l} \rangle$. The solutions $a_{m,n}$ of Eq. (53) define the most general cloning machine that is invariant in the two bases.

The generalization in the case where none of the bases is the computational basis is straightforward. The condition (53) remains valid, as long as we construct the Bell states using (42) for the two bases.

V. QUANTUM CRYPTOGRAPHY WITH TWO COMPLEMENTARY QUARTIT BASES: A PROTOCOL BASED ON INTERFEROMETRIC COMPLEMENTARITY.

A quantum protocol for key distribution that implements quartit states can be conceived using the lateral shape of electromagnetic pulses. Consider figure 1. Each $\phi_{ij}$ represents the
same (lateral) field distribution, shifted in its own square, and coherent with respect to the others. In theory any distribution could be used, as long as two neighboring distributions do not overlap. In practice, these can be easily produced by a fan-out of a single laser source. In combination with an external modulator, that can switch off or switch on each channel, and a controllable π-phase retarder, we can easily realize the following two mutually unbiased bases:

\begin{align}
|0\rangle &= \frac{1}{\sqrt{2}} (|\phi_{11}\rangle + |\phi_{12}\rangle) \\
|1\rangle &= \frac{1}{\sqrt{2}} (|\phi_{11}\rangle - |\phi_{12}\rangle) \\
|2\rangle &= \frac{1}{\sqrt{2}} (|\phi_{21}\rangle + |\phi_{22}\rangle) \\
|3\rangle &= \frac{1}{\sqrt{2}} (|\phi_{21}\rangle - |\phi_{22}\rangle)
\end{align} (54)

and

\begin{align}
|0'\rangle &= \frac{1}{\sqrt{2}} (|\phi_{11}\rangle + |\phi_{21}\rangle) \\
|1'\rangle &= \frac{1}{\sqrt{2}} (|\phi_{11}\rangle - |\phi_{21}\rangle) \\
|2'\rangle &= \frac{1}{\sqrt{2}} (|\phi_{12}\rangle + |\phi_{22}\rangle) \\
|3'\rangle &= \frac{1}{\sqrt{2}} (|\phi_{12}\rangle - |\phi_{22}\rangle)
\end{align} (55)

so we have:

\begin{equation}
|\langle i | j' \rangle|^2 = \frac{1}{4}
\end{equation} (56)

If we represent the in-products between the first basis and the second one in a matrix form, we get:

\begin{equation}
H_{ij} = \langle i | j' \rangle = \frac{1}{2} \begin{pmatrix}
+1 & +1 & +1 & +1 \\
+1 & +1 & -1 & -1 \\
+1 & -1 & +1 & -1 \\
+1 & -1 & -1 & +1
\end{pmatrix}
\end{equation} (57)

Actually, this is (up to a permutation of the second and the third primed basis states) a double Hadamard transform [10, 23]. The simple Hadamard transform is well-known in
FIG. 2: Possible measuring apparatus, distinguishing between $|0\rangle$ and $|1\rangle$. The distances of the mirrors are tuned in such a way that in the horizontal (vertical) arm there is constructive (destructive) interference

quantum information [24]: it sends the qubit $|i\rangle$ ($i=0, 1$) onto $\frac{1}{\sqrt{2}}(|0\rangle + (-)^i|1\rangle)$. It is easy to check that if we identify the computational quartit basis with the product basis as follows: $|0^{\text{quart}}\rangle = |00\rangle$, $|1^{\text{quart}}\rangle = |10\rangle$, $|2^{\text{quart}}\rangle = |01\rangle$, and $|3^{\text{quart}}\rangle = |11\rangle$, then, when the qubits undergo a simple Hadamard transformation, quartits change according to Eq. (57) (up to a permutation of the labels 2 and 3 of the primed basis).

Therefore we shall from now on call it a Hadamard transform, and denote its elements $H_{ij}$ in accordance with Eq. (57). Figure 2 shows a relatively simple setup, consisting of a mirror, a 50/50 beam splitter and two photon counters, which can distinguish between the states. The mirror’s position is tuned to ensure that if two distributions enter the device, their fields are added in the horizontal branch and subtracted in the vertical one. So if we position the two input lines of this interferometer in the $\phi_{11}$ and $\phi_{12}$ channel, the photon counter in the horizontal(vertical) branch corresponds to a projective measurement on $|0\rangle(|1\rangle)$. In a similar way, and by rotating the setup (or the incoming channels) projections on the $|i'\rangle$-basis can be made.
VI. ESTIMATION OF THE SAFETY THRESHOLD.

A. Applying the covariant formalism to Fourier complementary bases

We will now apply the covariant formalism to find a cloner that clones equally well two quartit bases, that are Fourier transforms of each other. Cloning between such bases has been studied in the past \[5, 25\]. We would like to compare the results of our covariant cloner with the optimal cloners of the literature. We have the computational basis \{|0\}, \{|1\}, \{|2\}, \{|3\}\} and its Fourier transform \(|k\rangle = \frac{1}{2} \sum_{j=0}^{3} e^{2\pi i (kj/4)} |j\rangle\):

\[
|0\rangle' = \frac{1}{2} (|0\rangle + |1\rangle + |2\rangle + |3\rangle) \\
|1\rangle' = \frac{1}{2} (|0\rangle + i|1\rangle - |2\rangle - i|3\rangle) \\
|2\rangle' = \frac{1}{2} (|0\rangle - |1\rangle + |2\rangle - |3\rangle) \\
|3\rangle' = \frac{1}{2} (|0\rangle - i|1\rangle - |2\rangle + i|3\rangle) \tag{58}
\]

The \(|k\rangle\) basis plays the role of \(|\tilde{\psi}\rangle\) in Sec. \[IV\]. We denote the unitary transformation that connects the two bases as \(F\):

\[
F_{k,l} = \langle k|l\rangle = \frac{1}{2} e^{2\pi i (kj/4)} = \frac{1}{2} i^{kl} \tag{59}
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{60}
\]

Calculating the 16 \(\times\) 16 matrix \(\mathcal{V}_{i,j;k,l}\)(we write \(|k\rangle \equiv |k'\rangle\):

\[
\mathcal{V}_{m,n;k,l} = \langle B_{m,n} | \tilde{B}_{k,l}^* \rangle \tag{61}
\]

\[
= \frac{1}{4} \sum_{p,q=0}^{3} \exp \left( \frac{2\pi i}{2} (ql - pn) \right) \langle p|\tilde{q}^* \rangle \langle p + m|q + k \rangle \tag{62}
\]

\[
= \frac{1}{16} \sum_{p,q=0}^{3} \tilde{i}^{(ql-pn)} \tilde{i}^{-pq} \tilde{i}^{(p+m)(q+k)} \tag{63}
\]

\[
= \frac{1}{16} i^{mk} \sum_{p=0}^{3} \tilde{i}^{(k-n)p} \sum_{q=0}^{3} \tilde{i}^{(l+m)q} \tag{64}
\]

\[
= i^{mk} \delta_{l+m,4} \delta_{k,n} \tag{65}
\]
Remember that all additions are modulo 4. Applying (53) we get the condition:

\[ a_{m,n} = a_{k,l} \quad \text{if} \quad l + m = 0 \quad \text{and} \quad k = n \]  \quad (66)

leading to:

\[ a_{01} = a_{03} = a_{10} = a_{30} \quad \quad a_{02} = a_{20} \]  \quad (67)

\[ a_{21} = a_{12} = a_{23} = a_{32} \quad \quad a_{11} = a_{13} = a_{33} = a_{31} \]  \quad (68)

This corresponds to the following \( a_{mn} \) matrix:

\[
(a_{m,n}) = \begin{pmatrix}
    a & b & c & b \\
    b & d & f & d \\
    c & f & e & f \\
    b & d & f & d
\end{pmatrix}
\]  \quad (69)

If we desire to optimize the information of Eve, conditioned on Alice and Bob’s observations, this means that certain conditions of constructive interference must be fulfilled. Moreover, the disturbances must be the same because the transmission line is assumed to be isotropic. This leads eventually \[22\] to the condition that \( b = c \) and \( e = f \). So the \( a_{mn} \) matrix that is covariant in the two Fourier bases becomes:

\[
(a_{m,n}) = \begin{pmatrix}
    x & y & y & y \\
    y & z & z & z \\
    y & z & z & z \\
    y & z & z & z
\end{pmatrix}
\]  \quad (70)

which leads to the cloning state:

\[
|\Psi\rangle_{R,A,B,C} = \sum_{m,n=0}^{3} a_{m,n} |B_{m,n}\rangle_{R,A} |B_{m,n}^{*}\rangle_{B,C}
\]

\[= (v - 2x + y) |B_{0,0}\rangle_{R,A} |B_{0,0}^{*}\rangle_{B,C} + (x - y) \sum_{n=0}^{3} |B_{0,n}\rangle_{R,A} |B_{0,n}^{*}\rangle_{B,C}
\]

\[+ (x - y) \sum_{n=0}^{3} |B_{m,0}\rangle_{R,A} |B_{m,0}^{*}\rangle_{B,C} + y \sum_{m=0}^{3} |B_{m,n}\rangle_{R,A} |B_{m,n}^{*}\rangle_{B,C} \]  \quad (71)

Actually, it is easy to check \[8\] that, according to Eq. (42),

\[
|\tilde{B}_{m,n}\rangle_{R,A} = i^{-nm} |B_{-n,m}\rangle_{R,A}
\]  \quad (72)

\[
|\tilde{B}_{m,n}^{*}\rangle_{B,C} = i^{+nm} |B_{-n,m}^{*}\rangle_{B,C}
\]  \quad (73)
There exists thus a bijective relation between the Bell states expressed in the computational basis and those expressed in the Fourier basis. It helps to understand the invariance of the state $|\Psi^F_{R,A,B,C}\rangle$ in both bases. Thanks to this bijective relation, it is easy to show that:

$$|B_{0,0}\rangle_{R,A}|B_{0,0}\rangle_{B,C} = |\tilde{B}_{0,0}\rangle_{R,A}|\tilde{B}^*_{0,0}\rangle_{B,C}$$

$$\sum_{n=0}^{3} |B_{0,n}\rangle_{R,A}|B^*_{0,n}\rangle_{B,C} = \sum_{m=0}^{3} |\tilde{B}_{m,0}\rangle_{R,A}|\tilde{B}^*_{m,0}\rangle_{B,C}$$

$$\sum_{m=0}^{3} |B_{m,0}\rangle_{R,A}|B^*_{m,0}\rangle_{B,C} = \sum_{m=0}^{3} |\tilde{B}^*_{0,m}\rangle_{R,A}|\tilde{B}_{0,m}\rangle_{B,C}$$

$$\sum_{m=0}^{3} |B_{m,n}\rangle_{R,A}|B^*_{m,n}\rangle_{B,C} = \sum_{m=0}^{3} |\tilde{B}_{m,n}\rangle_{R,A}|\tilde{B}^*_{m,n}\rangle_{B,C}$$

A cloning machine with such a matrix was already studied in [5], although in that case the authors proposed the matrix without the covariant demand. [31] The optimum of this cloner (in four dimensions) and in the symmetric case (i.e. the cloner produces two clones of the same quality) is characterized by [5, 25]:

$$(a_{m,n}) = \frac{1}{4} \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(74)

The two clones have a fidelity of

$$F = \frac{3}{4}$$

(75)

and the three disturbances (error rates)

$$D_1 = D_2 = D_3 = \frac{1}{12}$$

(76)

As the three Disturbances are always equal with the matrix (70), the maximum fidelity will also be the maximum of the mutual information between Alice and Bob (or between Alice and Eve for the second clone):

$$I_{AB} = I_{AE} = 0.792$$

(77)

A theorem of Csiszár and Körner [14] shows that a secret key with $R$ bits can be generated through privacy amplification if:

$$R \geq I_{AB} - I_{AE}$$

(78)
It is therefore sufficient that $I_{AB} > I_{AE}$ in order to establish a secret key. If we restrict ourselves to one-way communication on the classical channel, (78) is also a necessary condition. Therefore, an error rate of 25% is the maximum that can be allowed between Alice and Bob.

As we noted in section [III] the Cerf states are invariant under any cyclic permutation of the labels of the basis states. Actually, the invariance under cyclic permutations of the indices of the computational (Fourier) basis plays a fundamental role in this approach. For instance, it is easy to check that the Bell state $|B_{m,n}\rangle_{X,Y}$ is obtained by projecting the state $|0, m\rangle$ onto the eigenspace of $C$ associated with the eigenvalue $i^{-n}$ ($C$ is defined in Eq. (21)). This projector is equal to
\[
\sum_{k=0}^{4-1} |B_{kn}\rangle \langle B_{kn}| = \frac{1}{4} \sum_{k=0}^{4-1} i^{nk} C^{k}
\]
where we made use of the fact that $C^4 = C^0 = 1$. Beside, one can check that the same Bell state, when it is expressed in the product bases $|\tilde{\psi}_k\rangle |\tilde{\psi}_{k'}\rangle$ (and $|\tilde{\psi}_{k}^*\rangle |\tilde{\psi}_{k'}^*\rangle$) is invariant under cyclic relabeling of the indices $k, k'$ of the type $k, k' \rightarrow k + 1, k' + 1$ (let us call these permutations $\tilde{C}$ and $\tilde{C}'$), for the eigenvalues $i^m$ and $i^{-m}$ respectively. We shall generalize this property in a forthcoming section.

**B. Estimation of the safety threshold in the usual formalism.**

Let us now study the security of the protocol based on interferometric complementarity using the covariant cloning machines described in the previous section. The condition ($\mathcal{V}_{ij;kl} \neq 0$, then $a_{ij} = a_{k,l}$) yields the $a_{mn}$ matrix:
\[
(a_{m,n}) = \begin{pmatrix}
a & b & c & b \\
b & d & b & d \\
c & b & e & b \\
b & d & b & d
\end{pmatrix}
\] (80)

The associated matrix $b_{mn}$ that we obtain according to Eq. (8) is:
\[
(b_{m,n}) = \frac{1}{4} \begin{pmatrix}
a + 8b + 2c + 2d + e & a - e & a - e & a - e \\
a - e & a - 2c + e & a - e & a - 2c + e \\
a + 2c - 4d + e & a - e & a - 4b + 2c + e & a - e \\
a - e & a - 2c + e & a - e & a - 2c + e
\end{pmatrix}
\] (81)
According to Eq. (16), one can check that the fidelity associated to such a cloning state is equal to \( F_A = |a|^2 + 2|b|^2 + |c|^2 \). Let us define the disturbance \( D_A^i \) \((i = 1, 2, 3)\) as the probability that when Alice measures the state \(|j\rangle\), Bob measures the state \(|j + i\rangle\). It is easy to check that

\[
D_A^i = \sum_{n=0}^{N-1} |a_{i,n}|^2
\]

(82)

As the transmission line is usually considered to be isotropic, we have \( D_A^1 = D_A^2 = D_A^3 \), yielding the condition that \( 2|d|^2 = |c|^2 + |e|^2 \). The cloning state defined by the following matrix \( a_{m,n}^{iso} \) obviously fulfills these constraints:

\[
(a_{m,n}^{iso}) = \begin{pmatrix}
a & b & c & b \\
b & c & b & c \\
c & b & c & b \\
b & c & b & c
\end{pmatrix}
\]

(83)

The fidelity and disturbances are:

\[
F_A^{iso} = |a|^2 + 2|b|^2 + |c|^2
\]

(84)

\[
D_A^{iso1} = D_A^{iso2} = D_A^{iso3} = 2|b|^2 + 2|c|^2
\]

(85)

The associated matrix \( b_{m,n}^{iso} \) that we obtain according to Eq. (8) is:

\[
(b_{m,n}^{iso}) = \frac{1}{4} \begin{pmatrix}
a + 8b + 7c & a - c & a - c & a - c \\
a - c & a - c & a - c & a - c \\
a - c & a - c & a - 8b + 7c & a - c \\
a - c & a - c & a - c & a - c
\end{pmatrix}
\]

(86)

with a fidelity:

\[
F_B^{iso} = |a|^2 + 2|b|^2 + |c|^2
\]

(87)

We can now maximize \( F_B^{iso} \) for a given \( F_A^{iso} \). This is shown in Fig. 3. For a “symmetric” cloner a fidelity of \( F_A^{iso} = F_B^{iso} = 0.7018 \) is obtained. This is significantly less than the limit threshold of 0.75 that was obtained by the optimal cloner that clones equally well two Fourier complementary bases in Ref. [5, 6]. The difference is that we are now looking at the cloning machine that clones equally well (and optimally) two Hadamard complementary bases. In order to convince ourselves that they are different cloning machines, it is useful
FIG. 3: Fidelity of the second clone as a function of the fidelity of the first clone. They equalize at $F = 0.7018$.

to recall that the optimal cloner that was found in Ref. [5, 6] is symmetric. If we impose this symmetry, $a_{m,n} = b_{m,n}$, at the present level, we get the universal or isotropic cloning machine [13]:

$$\begin{pmatrix} a & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{pmatrix}$$

(88)

The fidelity and disturbances are:

$$F = |a|^2 + 3|b|^2$$

(89)

$$D_1 = D_2 = D_3 = 4|b|^2$$

(90)

The second clone then has a $b_{mn}$ matrix defined as follows

$$\begin{pmatrix} \frac{a+b}{4} & \frac{a-b}{4} & \frac{a-b}{4} & \frac{a-b}{4} \\ \frac{a-b}{4} & \frac{a+b}{4} & \frac{a-b}{4} & \frac{a-b}{4} \\ \frac{a-b}{4} & \frac{a-b}{4} & \frac{a+b}{4} & \frac{a-b}{4} \\ \frac{a-b}{4} & \frac{a-b}{4} & \frac{a-b}{4} & \frac{a+b}{4} \end{pmatrix}$$

(91)

with a fidelity of $F' = \frac{3}{16}|a - b|^2 + \frac{1}{16}|a + 15b|^2$. Due to symmetry and normalization, we have

$$a = \frac{\sqrt{10}}{4}$$

(92)

$$b = \frac{\sqrt{10}}{20}$$

(93)
Note that the fidelity of the optimal symmetric universal (or isotropic) cloner in N dimensions \[13, 15, 16\] is
\[F = \frac{3 + N^2}{2^{N^2}},\]
which in 4 dimensions yields a fidelity of 70%. The optimal isotropic quartit cloner of \[5, 6\], which is slightly asymmetric, is characterized by a fidelity of 73.33\% according to Ref. \[6\], and not 75\% as in Eq. (75).

It is extremely puzzling that two mutually unbiased bases that are related by a Hadamard transformation cannot be cloned with the same fidelity as bases that are related through a Fourier transformation. We found a way to escape this paradox: it consists of a redefinition of the Bell states.

C. Definition of the Hadamard Bell states

It is worth noting that the optimal Fourier cloning state \(|\Psi^{F}_{R,A,B,C}\rangle\) of Eq. (71) is not invariant under arbitrary permutations. For instance
\[x. \sum_{m=0}^{4-1} |B_{m,0}\rangle_{R,A} |B_{m,0}\rangle_{B,C} = x. \sum_{k,i,l=0}^{4-1} |k\rangle_{R} |k + i\rangle_{A} |l\rangle_{B} |l + i\rangle_{C}\]
contains the component \(x.|0\rangle_{R}|1\rangle_{A}|2\rangle_{B}|3\rangle_{C}\) but not \(x.|0\rangle_{R}|1\rangle_{A}|3\rangle_{B}|2\rangle_{C}\). Therefore it is not invariant under permutations of the labels 2 and 3 of the basis states of the computational basis (note that in dimensions two and three, the corresponding state is invariant under such a relabeling).

Besides, it is easy to check by direct computation that the state \(|\Psi^{F}_{R,A,B,C}\rangle\) is not invariant in the bases described in Eqs. (54,55). Among others, the bijective relations described in Eqs. (72,73) are not valid when the bases are related through a Hadamard transformation. This is a novelty that only appears in 4 dimensions (or in higher dimensions), because for lower dimensions one can show that, up to a convenient redefinition of the phases of the basis vectors, two mutually unbiased bases are always the (discrete) Fourier transform of each other. Essentially this is due to the fact that when the sum of two (three) phases cancel out, these phases are unambiguously defined (up to a global phase). This is no longer true when four phases or more are considered.

If we want to apply the formalism for cloning machines developed throughout this paper in order to deal with the case of two mutually unbiased bases that are not a Fourier but a Hadamard transformation of each other, it is necessary to generalize the definition of a Bell state. In analogy with the “Fourier” cloning state \(|\Psi^{F}_{R,A,B,C}\rangle\) defined in Eq. (71), we are now looking for a “Hadamard” cloning state \(|\Psi^{H}_{R,A,B,C}\rangle\) state that is invariant in the two Hadamard bases, but that is not necessarily invariant under any permutation of the labels.
of the basis states. Moreover, we would be pleased if a bijective relation similar to those
described in Eqs. (12, 13) would relate the Bell states.

To understand how to get the Bell-states that are suited to clone the Hadamard states,
we have to look at the similarities between $F$ (see Eq. (60)) and $H$ (see equation (57)). They
are both symmetric matrices, and the elements of their columns (rows) form a group with
the product operator. Indeed we have

$$2F_{i,j} 2F_{i,k} = 2F_{i,j+k}$$

where the sum is (as before) modulo 4. We want to express the group structure of $H$ in a
similar way. Therefore we introduce a new ‘Hadamard sum’ operator:

$$i \oplus j = (i + j) \mod 4$$

except when both $i$ and $j$ are equal to 3 or 1

$$1 \oplus 1 = 3 \oplus 3 = 0 \quad 1 \oplus 3 = 3 \oplus 1 = 2$$

With this definition we can write as in (94):

$$2H_{i,j} 2H_{i,k} = 2H_{i,j \oplus k}$$

We have also shown in the section VI A that if we permute cyclically the labels of the
computational basis, the states of the Fourier basis were mapped onto themselves and vice
versa. This leads to the definition the Fourier Bell states:

$$B_{m,n}^F = \sum_{k=0}^{3} F_{k,n} C^k \otimes C^{k+m} |00\rangle = \sum_{k=0}^{3} F_{k,n} |k\rangle |k+m\rangle$$

with $C$ the operator that performs the cyclic permutation. The Hadamard states do not
have this cyclic symmetry. But there are other permutation symmetries in this case. We
define three permutations in the computational basis (the same permutations in the primed
basis are noted with a prime): $P_1$ switches 0 $\leftrightarrow$ 1 and simultaneously 2 $\leftrightarrow$ 3, $P_2$ switches
0 $\leftrightarrow$ 2 and simultaneously 1 $\leftrightarrow$ 3 and $P_3$ switches 0 $\leftrightarrow$ 3 and simultaneously 1 $\leftrightarrow$ 2. Together
with the identity (we will call $P_0 \equiv I$) these permutations form a commutative group. It is
obvious that $P_i$ ($P_i'$) maps the basis states of the primed (non-primed) basis onto themselves
(up to a global phase). Completely analogous with (98) we define the Hadamard Bell states:

\[ B_{m,n}^H = \sum_{k=0}^{3} H_{k,n} P_k \otimes P_{k \oplus m} |00\rangle = \sum_{k=0}^{3} H_{k,n} |k \oplus m\rangle \]  

(99)

The 16 Hadamard Bell states are contained in the following list. They form a maximally entangled orthonormal basis, and have still all the interesting properties that characterize Fourier Bell states (complementarity between Eve’s and Bob’s reduced density operators, interpretation of the cloning state in terms of error operators, . . .) as we shall show soon. The parities under the permutations \( P_1, P_3, P_1', \) and \( P_3' \) are given in parentheses for each state.

\[
|B_{0,0}^H\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)(+ + +'), \quad |B_{0,1}^H\rangle = \frac{1}{2}(|00\rangle + |11\rangle - |22\rangle - |33\rangle)(+ - +') \\
|B_{0,2}^H\rangle = \frac{1}{2}(|00\rangle - |11\rangle + |22\rangle - |33\rangle)(- + +'), \quad |B_{0,3}^H\rangle = \frac{1}{2}(|00\rangle - |11\rangle - |22\rangle + |33\rangle)(- + +') \\
|B_{1,0}^H\rangle = \frac{1}{2}(|01\rangle + |10\rangle + |23\rangle + |32\rangle)(+ + -'), \quad |B_{1,1}^H\rangle = \frac{1}{2}(|01\rangle + |10\rangle - |23\rangle - |32\rangle)(+ - +') \\
|B_{1,2}^H\rangle = \frac{1}{2}(|01\rangle - |10\rangle + |23\rangle - |32\rangle)(- + -'), \quad |B_{1,3}^H\rangle = \frac{1}{2}(|01\rangle - |10\rangle - |23\rangle + |32\rangle)(- + -') \\
|B_{2,0}^H\rangle = \frac{1}{2}(|02\rangle + |20\rangle + |13\rangle + |31\rangle)(+ - -'), \quad |B_{2,1}^H\rangle = \frac{1}{2}(|02\rangle - |20\rangle + |13\rangle - |31\rangle)(+ - -') \\
|B_{2,2}^H\rangle = \frac{1}{2}(|02\rangle + |20\rangle - |13\rangle - |31\rangle)(- - -'), \quad |B_{2,3}^H\rangle = \frac{1}{2}(|02\rangle - |20\rangle - |13\rangle + |31\rangle)(- - -') \\
|B_{3,0}^H\rangle = \frac{1}{2}(|03\rangle + |30\rangle + |12\rangle + |21\rangle)(+ - +'), \quad |B_{3,1}^H\rangle = \frac{1}{2}(|03\rangle - |30\rangle + |12\rangle - |21\rangle)(+ - +') \\
|B_{3,2}^H\rangle = \frac{1}{2}(|03\rangle - |30\rangle - |12\rangle + |21\rangle)(- - +'), \quad |B_{3,3}^H\rangle = \frac{1}{2}(|03\rangle + |30\rangle - |12\rangle - |21\rangle)(- - +')
\]

(100)

We can now easily show that we have a bijective relation similar to (12, 13):

\[
|B_{i,j}^H\rangle = \sum_{l=0}^{3} H_{i,j} |l, i \oplus l\rangle
\]

(101)

\[
= \sum_{k,l,m,p=0}^{3} H_{i,j} H_{i,m} H_{i\oplus l,p} |m', p\rangle = \sum_{k,l,m,p=0}^{3} 2H_{i,j} H_{i,m} H_{i,p} |m', p\rangle
\]

(102)

\[
= \sum_{l,m,p=0}^{3} H_{i,j} |m', (j \oplus m)\rangle = \sum_{m=0}^{3} 2H_{i,j} H_{i,m} |m', (j \oplus m)\rangle
\]

(103)

\[
|B_{j,i}^H\rangle = 2H_{i,j} |B_{j,i}^H\rangle
\]

(104)
Similarly, we can evaluate the in-product between \( |B_{i,j}^H \rangle_{R,A} |B_{i,j}^H \rangle_{B,C} \) and \(|B_{m,n}^H \rangle_{R,B} |B_{m,n}^H \rangle_{A,C} \) as follows:

\[
\langle B_{m,n}^H_{R,B} B_{m,n}^H_{A,C} | B_{i,j}^H_{R,A} B_{i,j}^H_{B,C} \rangle
\]

\[
= \sum_{o,p,l,k=0}^{3} H_{o,n} H_{p,n} H_{l,j} H_{k,j} \langle o, o \oplus m, p, p \oplus m | l, k \oplus i, k \oplus i \rangle
\]

\[
= \sum_{o,p,l,k=0}^{3} H_{o,n} H_{p,n} H_{l,j} H_{k,j} \delta_{o,l} \delta_{p,m} \delta_{l,i} \delta_{k,i}
\]

\[
= \sum_{l=0}^{3} H_{l \oplus i,n} H_{l,n} H_{l \oplus m,j} H_{l,j}
\]

\[
= \sum_{l=0}^{3} 4 H_{i,n} H_{l,n} H_{m,j} H_{l,j}
\]

\[
= H_{i,n} H_{m,j}
\]

Note that this defines a unitary transformation, since the states \(|B_{i,j}^H \rangle_{R,A} |B_{i,j}^H \rangle_{B,C} \) and \(|B_{m,n}^H \rangle_{R,B} |B_{m,n}^H \rangle_{A,C} \) are bases of the space that is invariant or symmetric (eigenvalues +) under the permutations \( P_i \) and \( P'_j \) \((i,j=0,1,2,3)\). Now that we know the in-products between these states, it is very easy to derive the following relation:

\[
|\Psi\rangle_{R,A,B,C}^H = \sum_{m,n=0}^{3} a_{m,n} |B_{m,n}^H \rangle_{R,A} |B_{m,n}^H \rangle_{B,C} = \sum_{m,n=0}^{3} b_{m,n} |B_{m,n}^H \rangle_{R,B} |B_{m,n}^H \rangle_{A,C}
\]

where \(a_{m,n}\) and \(b_{m,n}\) are two (complex) amplitude functions that are dual under a Hadamard transform (which generalizes the dual relation Eq. (8)):

\[
b_{m,n} = \sum_{x,y=0}^{3} H_{m,y} H_{n,x} a_{x,y}
\]

\[
a_{m,n} = \sum_{x,y=0}^{3} H_{m,y} H_{n,x} b_{x,y}
\]

Also the error operators, which for the Fourier cloner are defined as:

\[
U_{m,n}^F = 2 \sum_{k=0}^{3} F_{k,n} |k + m\rangle \langle k|
\]

are redefined as:

\[
U_{m,n}^H = 2 \sum_{k=0}^{3} H_{k,n} |k \oplus m\rangle \langle k|
\]
With these new definitions, the analysis of the cloner is analogous to the treatment of the cloner presented in Sec. VI A and in \[5, 6, 13\]. The reduced density operator the two clones are:

\[
\rho_A = \sum_{m,n} a_{m,n} |\psi_{m,n}^H\rangle \langle \psi_{m,n}^H| \quad (117)
\]

\[
\rho_B = \sum_{m,n} b_{m,n} |\psi_{m,n}^H\rangle \langle \psi_{m,n}^H| \quad (118)
\]

with

\[
|\psi_{m,n}^H\rangle = U_{m,n}^H |\psi\rangle \quad (119)
\]

The fidelities and disturbances are given by Eq. (16) and Eq. (17). Therefore, the optimal cloner will have the same amplitudes as in Sec. VI A and \[5, 6, 13\]: where

\[
(a_{m,n}) = (b_{m,n}) = (a'_{m,n}) = \begin{pmatrix}
  x & y & y & y \\
  y & z & z & z \\
  y & z & z & z \\
  y & z & z & z
\end{pmatrix} \quad (120)
\]

With these amplitudes, the state of Eq. (112) is the optimal state that clones the computational and the Hadamard bases equally well.

The reduced state obtained after tracing over Eve’s degrees of freedom exhibits no correlations when it is measured by Alice and Bob in non-correlated bases, so that it is impossible for them to establish the difference between this state and an unbiased noise.

The corresponding maximal admissible error rate (when attacks based on state-independent cloners are considered) can be shown as before to be equal to \( E_A = 1 - F_A = \frac{1}{2}(1 - \frac{1}{\sqrt{4}}) = 25.00\% \) (see Ref. \[5, 6\]). According to Csiszár and Körner theorem \[14\], the quantum cryptographic protocol above ceases to generate secret key bits precisely at this point, where Eve’s information matches Bob’s information.

\section{VII. CONCLUSION AND COMMENTS}

In this paper we investigated state dependent cloners that are suited to clone four-level systems (quartits). We have adapted the cloning formalism of N.Cerf \[12, 13\] in such a way that it is covariant under certain unitary transformations. We have used this protocol to
clone the states of two quartit bases that are related through a double Hadamard transform. Our results show that the protocol is not suited to clone such states, since the cloning state uses ‘Fourier’ Bell states which have different symmetries than the Hadamard bases that we want to clone. Therefore, we redefined the Bell states in such a way that they have the same symmetries as the states that we want to clone. The result is a cloner that clones the Hadamard bases equally well as the 'Fourier' cloner clones Fourier bases. Note that the approach is suited for an attack on all protocols that use Hadamard bases to encode the information as in Ref. [10, 23].

The question remains open whether better cloning machines exist. The safety of quantum cryptographic protocols depends crucially on the answer. It is out of the scope of the present paper to provide a definitive answer to this question, but it is worth mentioning that in the qubit \((N=2)\) and qutrit \((N=3)\) cases, the properties (optimal fidelity, upper bound on the error rate and so on) of the cloners derived in the literature following Cerf’s approach [5, 6, 8, 21] are equivalent to those obtained following more general approaches [9, 18, 19, 20].

Many constraints have to be fulfilled when Eve replaces the maximally entangled state shared by Alice and Bob by a clone: the clone must mimic the unbiased noise that would be observed by Alice and Bob in the absence of a spy. Therefore it must be independent on the basis of detection and on which state is detected among such a basis. Moreover it must mimic the correlations between different bases that would be observed in the presence of unbiased noise.

It is our belief that the present, constructive, approach, based on the intrinsic symmetries of the protocols under study, provides the most dangerous attack, taking account of all the constraints of the problem.

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informed (N. Cerf private communication) that N. Cerf and his coworkers obtained independently certain results similar to ours. Thanks to N. Cerf, S. Iblisdir, L-P Lamoureux, S. Massar and P. Navez for interesting discussions.

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By assumption, two orthonormal bases of an $N$-dimensional Hilbert space are said to be mutually unbiased if the norm of the scalar product between any two vectors belonging each to one of the bases is equal to $\frac{1}{\sqrt{N}}$.

The sum operator inside the kets is always defined modulo $N$.

This is actually the state that Eve would prepare if she intercepted a state $|\psi\rangle$ sent by Alice. This indeed shows that sending $|\psi\rangle$ to Bob, or using the maximally entangled state, is equivalent for what concerns cloning.

Consider the extreme case that $a_{0,0} = 1$ and the other $a_{m,n} = 0$. The first clone would in this case be perfect, with a fidelity of 1 and no disturbances ($D_i = 0$). The second clone would, however, be characterized by $b_{m,n} = \frac{1}{N}$ for all values of $m$ and $n$, leading to fidelities and disturbances $F = D_i = \frac{1}{N}$. This basically means that all values have an equal probability to be measured, independent of the value that was sent, making the clone worthless, as the transinformation between Alice and Bob is indeed 0.

This choice is unique up to arbitrary unitary transformations in the $N^2$ dimensional Hilbert space assigned to Eve ($B$ and $C$).

Of course this choice was not random. It was shown that such a matrix clones all the states that belong to the two Fourier bases with the same fidelity. However there is no guarantee that there are no other matrices which have this property. If we demand covariance of the cloner state, we have the assurance that (120) is the only solution.