Counting conjectures and $e$-local structures in finite reductive groups

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Abstract

We prove new results in generalised Harish-Chandra theory providing a description of the so-called Brauer–Lusztig blocks in terms of information encoded in the $\ell$-adic cohomology of Deligne–Lusztig varieties. Then, we propose new conjectures for finite reductive groups by considering geometric analogues of the $\ell$-local structures that lie at the heart of the local-global counting conjectures. For large primes, our conjectures coincide with the counting conjectures thanks to a connection established by Broué, Fong and Srinivasan between $\ell$-structures and their geometric counterpart. Finally, using the description of Brauer–Lusztig blocks mentioned above, we reduce our conjectures to the verification of Clifford theoretic properties expected from certain parametrisations of generalised Harish-Chandra series.

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1 Introduction

Over the past few decades the research in representation theory of finite groups has been driven by the pursuit of an explanation of the so-called local-global principle. This states that for each prime number $\ell$ dividing the order of a finite group $G$, the $\ell$-modular representation theory of $G$ is largely determined by the $\ell$-local structure of the group $G$. The local-global principle is supported by numerous conjectural evidences including the McKay Conjecture, the Alperin–McKay Conjecture, Alperin’s Weight Conjecture and Brauer’s Height Zero Conjecture among others.

In the 1990s, extending a connection made by Knörr and Robinson between local-global conjectures and the Brown complex associated to chains of $\ell$-subgroups, Dade introduced a new conjecture known as Dade’s Projective Conjecture. This provides a unifying statement which implies all of the local-global conjectures mentioned above [Dad92], [Dad94]. More recently, Dade’s Projective Conjecture has been reduced to the verification of the so-called inductive condition for Dade’s Conjecture for finite quasi-simple groups [Spä17]. This inductive condition can be stated in terms of Späth’s Character Triple Conjecture (see [Spä17, Conjecture 6.3]).

When considering large primes in non-defining characteristic, work of Broué, Fong and Srinivasan shows that the $\ell$-local structure of a finite reductive group and the associated Brown complex can be seen as a shadow of geometric objects arising from the underlining linear algebraic group (see Section 7.2). Building on this idea, in this paper we propose new conjectures for finite reductive groups that can be seen as geometric realisations of the local-global counting conjectures (see Sections 5.1 and Section 5.2). These new conjectures imply the counting conjectures under the assumptions considered above (see Section 7.3). Remarkably, our conjectures can be explained within the framework of generalised Harish-Chandra theory and, in fact, they reduce to the verification of Clifford theoretic properties expected from certain parametrisation of generalised Harish-Chandra series. In order to prove these results, we first prove new modular representation theoretic results for finite reductive groups by studying the decomposition of certain virtual representations constructed from the $\ell$-adic cohomology of Deligne–Lusztig varieties.
1.1 \( e \)-Harish-Chandra theory

Let \( G \) be a connected reductive group defined over an algebraic closure \( \mathbb{F} \) of a finite field of characteristic \( p \), \( F : G \rightarrow G \) a Frobenius endomorphism endowing \( G \) with an \( \mathbb{F}_q \)-structure for some power \( q \) of \( p \) and \( G^F \) the finite reductive group consisting of the \( \mathbb{F}_q \)-rational points. Fix a prime number \( \ell \) not dividing \( q \) and denote by \( e \) the multiplicative order of \( q \) modulo \( \ell \) (modulo 4 if \( \ell = 2 \)). All modular representation theoretic notions are considered with respect to the prime \( \ell \). Let \( (G^*, F^*) \) be in duality with \( (G, F) \). Blocks of finite reductive groups have been parametrised by work of Fong–Srinivasan [FS82, FS85], Broué–Malle–Michel [BMM93], Cabanes–Enguehard [CE94, CE99], Enguehard [Eng00] and Kessar–Malle [KM13, KM15]. Given this parametrisation, we then need to understand the distribution of characters into such blocks. For this purpose, recall that the set \( \text{Irr}(G^F) \) of irreducible characters of \( G^F \) admits a partition

\[
\text{Irr}(G^F) = \bigsqcup_{B,s} \text{Irr}(B) \cap \mathcal{E}(G^F, [s])
\]

where \( B \) runs over the set of Brauer \( \ell \)-blocks of \( G^F \), \( s \) runs over the set of semisimple elements in \( G^* \) up to (rational) conjugation and \( \mathcal{E}(G^F, [s]) \) is the rational Lusztig series associated to \( s \). Using a terminology introduced by Broué, Fong and Srinivasan, we call each non-empty intersection

\[
\mathcal{E}(G^F, B, [s]) := \text{Irr}(B) \cap \mathcal{E}(G^F, [s])
\]

a Brauer–Lusztig block. In particular, each \( \ell \)-block \( B \) is a union of Brauer–Lusztig blocks and therefore, in order to understand the distribution of characters into \( \ell \)-blocks, we need to describe Brauer–Lusztig blocks. Our first main result provides such a description in terms of \( e \)-Harish-Chandra series defined in terms of the \( \ell \)-adic cohomology of Deligne–Lusztig varieties.

**Theorem A.** Assume Hypothesis \([4.1]\) and let \( \mathcal{E}(G^F, B, [s]) \) be a Brauer–Lusztig block. Then

\[
\mathcal{E}(G^F, B, [s]) = \bigsqcup_{(L, \lambda)} \mathcal{E}(G^F, (L, \lambda)),
\]

where the union runs over the \( G^F \)-conjugacy classes of \((e, s)\)-cuspidal pairs \((L, \lambda)\) (see Definition \([3.1]\)) such that \( b_l(\lambda)G^F = B \) via Brauer induction of \( \ell \)-blocks.

We point out that Hypothesis \([4.1]\) is satisfied in most of the cases of interest and, in particular, whenever \([G, G]\) is simply connected with no irreducible rational components of type \( ^2E_6(2), E_7(2), E_8(2) \) while considering \( \ell \in \Gamma(G, F) \) with \( \ell \geq 5 \) (see Remark \([4.2]\)). Moreover, in this case Brauer’s induction of blocks is defined (see the discussion preceding Lemma \([4.6]\)).

From the perspective of \( e \)-Harish–Chandra theory, Theorem \([A]\) can be seen as an extension of results of Cabanes–Enguehard (see [CE99 Theorem 4.1]) to \( e \)-cuspidal pairs associated to \( \ell \)-singular semisimple elements. In addition, Theorem \([A]\) provides a generalisation of [BMM93 Theorem 3.2 (1)] to non-unipotent characters. This provides a uniform formulation for \( e \)-Harish-Chandra theory by considering arbitrary \( e \)-cuspidal pairs. In fact in Corollary \([4.12]\) we show that for every \( e \geq 1 \) the set of irreducible characters of \( G^F \) is partitioned into \( e \)-Harish-Chandra series provided that there exists a good prime \( \ell \) for which \( e \) is the order of \( q \) modulo \( \ell \). It is worth pointing out that this latter statement does not depend on the choice of the prime \( \ell \) while, relying on block theoretic techniques,
On the way to prove Theorem A we obtain two results that are of independent interest. First in Corollary 4.9 we prove an extension of [CE99, Theorem 2.5] that shows how Deligne–Lusztig induction preserves the decomposition into ℓ-blocks. More precisely, under Hypothesis 4.1 we show that for every ℓ-block \( b_\ell \) of an \( e \)-split Levi subgroup \( L \) of \( G \) the irreducible constituents of the virtual representations obtained via Deligne–Lusztig induction from any \( \lambda \in \text{Irr}(b_\ell) \) belong to a unique \( ℓ \)-block \( b_G \) of \( G_F \). Secondly in Corollary 4.11 we give a partial solution to a conjecture (see [CE99, Notation 1.11] and Conjecture 3.2) introduced by Cabanes and Enguehard on the transitivity of a certain relation \( \leq_e \) defined on the set of \( e \)-pairs (see also Proposition 4.5).

As an immediate consequence of Theorem A we obtain the above-mentioned description of all the characters in any given \( ℓ \)-block by considering the union over all conjugacy classes of semisimple elements of \( G^{+F} \). Namely, for every \( ℓ \)-block \( B \) of \( G^F \) there is a partition

\[
\text{Irr}(B) = \bigsqcup_{(L, \lambda)} \mathcal{E}(G^F, (L, \lambda)),
\]

where the union runs over the \( G^F \)-conjugacy classes of \( e \)-cuspidal pairs \( (L, \lambda) \) such that \( bl((\lambda))^{G^F} = B \) via Brauer induction of \( ℓ \)-blocks (see Theorem 4.13). Given this partition, the next natural step to understand the distribution of characters into \( ℓ \)-blocks of finite reductive groups is to find a parametrisation of the characters in each \( e \)-Harish-Chandra series \( \mathcal{E}(G^F, (L, \lambda)) \). Inspired by classical Harish-Chandra theory and by results of Broué, Malle and Michel for unipotent characters (see [BMM93, Theorem 3.2]), we propose a parametrisation for arbitrary \( e \)-Harish-Chandra series which is additionally compatible with Clifford theory and with the action of automorphisms. This Clifford theoretic compatibility is expressed via \( G^F \)-block isomorphisms of character triples as defined in [Spä17, Definition 3.6].

**Parametrisation B.** Let \( G, F, ℓ, q \) and \( e \) be as above and consider an \( e \)-cuspidal pair \( (L, \lambda) \) of \( G \). There exists a defect preserving \( \text{Aut}_F(G^F)_{(L, \lambda)} \)-equivariant bijection

\[
\Omega^G_{(L, \lambda)} : \mathcal{E}(G^F, (L, \lambda)) \to \text{Irr}(N_G(L)^F|\lambda)
\]

such that

\[
(X_\vartheta, G^F, \vartheta) \sim_{G^F} (N_{X_\vartheta}(L), N_{G^F}(L), \Omega^G_{(L, \lambda)}(\vartheta))
\]

in the sense of [Spä17, Definition 3.6] for every \( \vartheta \in \mathcal{E}(G^F, (L, \lambda)) \) and where \( X := G^F \rtimes \text{Aut}_F(G^F) \).

It is the author’s belief that existence of the above parametrisation should provide an explanation for the validity of the inductive conditions for the local-global conjectures for finite reductive groups in non-defining characteristic. As a matter of fact, similar bijections have been used in [MS16] to verify the McKay Conjecture for the prime \( ℓ = 2 \) and then in [Ruh22] to prove the Alperin–McKay Conjecture and Brauer’s Height Zero Conjecture for \( ℓ = 2 \). Regarding the validity of the above parametrisation, in [Ros22b] we show that in order to obtain Parametrisation B it is enough to verify certain requirements on the extendibility of characters of \( e \)-split Levi subgroups. These also appear in the proofs of the inductive conditions for the McKay, the Alperin–McKay and the Alperin Weight
conjectures and the checking of these requirements is part of an ongoing project in representation theory of finite reductive groups (see [NS16], [CS17a], [CS17b], [CS19], [BS20], [Spä21], [Bro22]).

1.2 Counting conjectures via $e$-local structures

Our next aim is to provide a geometric realisation of the local-global principle for finite reductive groups by replacing $\ell$-local structures with $e$-local structures. For this purpose we first need some notation. We keep $G$, $F$, $q$, $\ell$ and $e$ as previously defined. Denote by $\mathcal{L}_e(G, F)_{>0}$ the set of non-trivial descending chains of $e$-split Levi subgroups $\sigma = \{G = L_0 > L_1 \cdots > L_n\}$ and define the length of $\sigma$ as $|\sigma| := n > 0$. We denote by $G^F_\sigma$ the stabiliser of a chain $\sigma$ under the action of $G^F$. Then, as explained in Section 5.1, we can associate to every $\ell$-block $b$ of $G^F_\sigma$ an $\ell$-block $R^G_{G_\sigma}(b)$ of $G^F$. For every $\ell$-block $B$ of $G^F$ and $d \geq 0$, we denote by $k^d(B_\sigma)$ the number of irreducible characters $\vartheta$ of $G^F_\sigma$ with $\ell$-defect $d$ and such that the $\ell$-block of $\vartheta$ in $G^F_\sigma$ correspond to $B$ via $R^G_{G_\sigma}$. Moreover, let $k^d(B)$ and $k^d(\ell)$ be the number of irreducible and $e$-cuspidal characters respectively, with $\ell$-defect $d$ and belonging to the $\ell$-block $B$. Our first conjecture proposes a formula to count the number of characters of any given defect in any $\ell$-block in terms of $e$-local data.

**Conjecture C.** Let $B$ be an $\ell$-block of $G^F$ and $d \geq 0$. Then

$$k^d(B) = k^d_e(B) + \sum_{\sigma} (-1)^{|\sigma|+1} k^d(B_\sigma)$$

where $\sigma$ runs over a set of representatives for the action of $G^F$ on $\mathcal{L}_e(G, F)_{>0}$.

Conjecture 5.1 provides a geometric form of Dade’s Conjecture for finite reductive groups. In fact, in Section 7 we show that the two statements coincide when the prime $\ell$ is large for $G$ (see Proposition 7.10). We expect this connection to hold for good primes as well and we suspect that this could be connected to the existence of a homotopy equivalence between the Brown complex and the simplicial complex associated to $\mathcal{L}_e(G, F)_{>0}$. In the spirit of Conjecture C in Section 5 we introduce a statement (see Conjecture 5.2) which relates to Alperin’s Weight Conjecture. As for the Alperin–McKay Conjecture and its inductive condition, at least for large primes, we identify a geometric counterpart in Parametrisation $\mathbb{B}$ (see Proposition 7.11 and Proposition 7.15).

The numerical phenomena proposed in Conjecture C as well as in the local-global counting conjectures are believed to be consequences of a deeper underlying theory. For blocks with abelian defect groups, Broué has suggested a structural explanation which predicts the existence of certain derived equivalences [Bro90]. In a different direction, work of Isaacs, Malle and Navarro [IMN07], followed by Navarro and Späth [NS14 Theorem 7.1] (see also [Ros22c]) and Späth [Spä17 Conjecture 1.2], suggests a description of Clifford theoretic properties hidden behind the counting conjectures by studying certain relations on sets of character triples. Exploiting this second approach, we now introduce a more conceptual background for Conjecture C by introducing $G^F$-block isomorphisms of character triples in this context.

In Section 5.2, for every $\ell$-block $B$, $d \geq 0$ and $\epsilon = \pm$, we introduce a set $\mathcal{L}^d(B)_\epsilon/G^F$ consisting of $G^F$-orbits of quadruples $\omega$. Each $\omega \in \mathcal{L}^d(B)_\epsilon/G^F$ determines a $G^F$-orbit $\omega^*$ of pairs $(\sigma, \vartheta)$ consisting of a chain $\sigma \in \mathcal{L}_e(G, F)$ and a certain irreducible character $\vartheta$ of $G^F_\sigma$. With this notation, we can now present our second conjecture.
**Conjecture D.** For every $\ell$-block $B$ of $G^F$ and $d \geq 0$, there is an $\Aut_F(G^F)_B$-equivariant bijection

$$\Lambda : \mathcal{L}^d(B)_+ \backslash G^F \to \mathcal{L}^d(B)_- \backslash G^F$$

such that

$$(X_{\sigma, \vartheta}, G^F_\sigma, \vartheta) \sim_{G^F} (X_{\rho, \chi}, G^F_\rho, \chi)$$

in the sense of [Spä17, Definition 3.6] for every $\omega \in \mathcal{L}^d(B)_+/\!\!\!/G^F$, any $(\sigma, \vartheta) \in \omega^*$, any $(\rho, \chi) \in \Lambda(\omega)^*$ and where $X := G^F \rtimes \Aut_F(G^F)$.

As a first application of Theorem A, we establish a connection between Conjecture C and Conjecture D. More precisely, we show that for a fixed $\sigma \in \mathcal{L}^d(G^F) > 0$ the number $k^d(B_\sigma)$ coincides with the number of quadruples $\omega$ for which there is a character $\vartheta \in \Irr(G^F_\sigma)$ such that $(\sigma, \vartheta) \in \omega^*$ and where $\sigma = \{G\}$ is the trivial chain. In particular we see that Conjecture D implies Conjecture C.

**Theorem E.** Assume Hypothesis 4.1 and consider an $\ell$-block $B$ and $d \geq 0$. If Conjecture D holds for $B$ and $d \geq 0$, then Conjecture C holds for $B$ and $d \geq 0$.

As a result of the inductive conditions for the local-global counting conjectures, Conjecture D provides a more conceptual explanation for the numerical phenomenon proposed by Conjecture C and, in particular, yields a description of the Clifford theoretic properties naturally arising in this context. Furthermore, in Section 7 we show that Conjecture D implies Späth’s Character Triple Conjecture and the inductive condition for Dade’s Conjecture for large primes under suitable assumptions (see Proposition 7.13 and Corollary 7.14).

Remarkably, Conjecture D can be explained within the framework of $\epsilon$-Harish-Chandra theory. Our final theorem shows that Conjecture D (and hence Conjecture C) is a consequence of Parametrisation E via an application of Theorem A. In what follows, we say that Parametrisation E holds for $(G, F)$ at the prime $\ell$ if it holds for every $\epsilon$-cuspidal pair $(L, \lambda)$ of $G$ where $q$ is the prime power associated to $F$ and $\epsilon$ is the order of $q$ modulo $\ell$. Moreover, we refer the reader to Section 6 for the definition of irreducible rational components (see Definition 6.4).

**Theorem F.** Assume Hypothesis 4.1 and suppose that $G$ is simply connected with Frobenius endomorphism $F$. If Parametrisation E holds at the prime $\ell$ for every irreducible rational component $(H, F)$ of every $\epsilon$-split Levi subgroup of $G$, then Conjecture D holds for the prime $\ell$.

As a consequence of Theorem E, Theorem F and the results obtained in [Ros22b], our conjectures are now reduced to the verification of technical requirements on character extendibility from $\epsilon$-split Levi subgroups in finite reductive groups of irreducible rational type.

### 1.3 Reader’s guide

The paper is organised as follows. Section 2 contains the main notation and preliminary results on finite reductive groups. In Section 3 and more precisely in Proposition 3.6 we study the transitivity of a certain relation defined on the set of $\epsilon$-pairs and provide a solution to a fundamental case of a conjecture proposed by Cabanes–Enguehard in [CE99, Notation 1.11]. This is then extended to the
a correspondence between the set of $G$ given by $F$ and the set of Levi subgroups of $F$ with respect to a choice of an $x$ by the elements $\text{Dynkin diagram of } \Delta F$ in an $G$ with an $F$ closure $\Phi$. In this case, there exists a bijection $\text{Aut}_{\Phi}(G^F)$ to arbitrary $\ell$-singular case in Corollary 4.11. Furthermore, in Proposition 4.5 we prove the conjecture inside $G$ connected type such that $\alpha \in \Phi$ for every $G$ connected type such that $\alpha \in \Phi$. Thereby, up to inner automorphisms of $G$, any Frobenius endomorphism $F$ and an $F^*$-stable maximal torus $T$ of $G^*$. In this case, there exists a bijection $L \rightarrow L^*$ between the set of Levi subgroups of $G$ containing $T$ and the set of Levi subgroups of $G^*$ containing $T^*$ (see [CE04, p.123]). This bijection induces a correspondence between the set of $F$-stable Levi subgroups of $G$ and the set of $F^*$-stable Levi subgroups of $G^*$. Moreover, it is compatible with the action of $G^F$ and $G^{*F^*}$.

2 Preliminaries

Throughout this paper, $G$ is a connected reductive linear algebraic group defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic $p$ and $F : G \rightarrow G$ is a Frobenius endomorphism endowing $G$ with an $\mathbb{F}_q$-structure for a power $q$ of $p$. We denote by $(G^*, F^*)$ a group in duality with $(G, F)$ with respect to a choice of an $F$-stable maximal torus $T$ of $G$ and an $F^*$-stable maximal torus $T^*$ of $G^*$. In this case, there exists a bijection $L \rightarrow L^*$ between the set of Levi subgroups of $G$ containing $T$ and the set of Levi subgroups of $G^*$ containing $T^*$ (see [CE04, p.123]). This bijection induces a correspondence between the set of $F$-stable Levi subgroups of $G$ and the set of $F^*$-stable Levi subgroups of $G^*$. Moreover, it is compatible with the action of $G^F$ and $G^{*F^*}$.

2.1 Automorphisms

Let $G$ and $F$ be as above. If $\sigma : G \rightarrow G$ is a bijective morphism of algebraic groups satisfying $\sigma \circ F = F \circ \sigma$, then the restriction of $\sigma$ to $G^F$, which by abuse of notation we denote again by $\sigma$, is an automorphism of the finite group $G^F$. We denote by $\text{Aut}_{\Phi}(G^F)$ the set of those automorphisms of $G^F$ obtained in this way. As mentioned in [CS13, Section 2.4], a morphism $\sigma \in \text{Aut}_{\Phi}(G^F)$ is determined by its restriction to $G^F$ up to a power of $F$. It follows that $\text{Aut}_{\Phi}(G^F)$ acts on the set of $F$-stable closed connected subgroups $H$ of $G$. In particular, for any $F$-stable closed connected subgroup $H$ of $G$, there is a well defined set $\text{Aut}_{\Phi}(G^F)_H$ whose elements are the restrictions to $G^F$ of those morphisms $\sigma$ as above that stabilize $H$. When $G$ is a simple algebraic group of simply connected type such that $G^F/Z(G^F)$ is a non-abelian simple group, then we have $\text{Aut}_{\Phi}(G^F) = \text{Aut}(G^F)$ (see [GLS98, Section 1.15] and the comments in [CS13, Section 2.4]).

Assume now that $G$ is simple of simply connected type. Fix a maximally split torus $T_0$ contained in an $F$-stable Borel subgroup $B_0$ of $G$. This choice corresponds to a set of simple roots $\Delta \subseteq \Phi := \Phi(G, T_0)$. For every $\alpha \in \Phi$ consider a one-parameter subgroup $x_\alpha : G_\alpha \rightarrow G$. Then $G$ is generated by the elements $x_\alpha(t)$, where $t \in G_\alpha$ and $\alpha \in \pm \Delta$. Consider the field endomorphism $F_0 : G \rightarrow G$ given by $F_0(x_\alpha(t)) := x_\alpha(t^p)$ for every $t \in G_\alpha$ and $\alpha \in \Phi$. Moreover, for every symmetry $\gamma$ of the Dynkin diagram of $\Delta$, we have a graph automorphism $\gamma : G \rightarrow G$ given by $\gamma(x_\alpha(t)) := x_{\gamma(\alpha)}(t)$ for every $t \in G_\alpha$ and $\alpha \in \pm \Delta$. Then, up to inner automorphisms of $G$, any Frobenius endomorphism $F$
defining an $\mathbb{F}_q$-structure on $G$ can be written as $F = F_0^m \gamma$, for some symmetry $\gamma$ and $m \in \mathbb{Z}$ with $q = p^m$ (see [MT11 Theorem 22.5]). One can construct a regular embedding $G \leq \bar{G}$ in such a way that the Frobenius endomorphism $F_0$ extends to an algebraic group endomorphism $F_0 : \bar{G} \to \bar{G}$ defining an $\mathbb{F}_p$-structure on $G$. Moreover, every graph automorphism $\gamma$ can be extended to an algebraic group automorphism of $\bar{G}$ commuting with $F_0$ (see [MS16 Section 2B]). If we denote by $\mathcal{A}$ the group generated by $\gamma$ and $F_0$, then we can construct the semidirect product $\bar{G}^F \rtimes \mathcal{A}$. Finally, we define the set of \textit{diagonal automorphisms} of $G^F$ to be the set of those automorphisms induced by the action of $G^F$ on $G^F$. If $G^F / Z(G^F)$ is a non-abelian simple group, then the group $\bar{G}^F \rtimes \mathcal{A}$ acts on $G^F$ and induces all the automorphisms of $G^F$ (see, for instance, the proof of [Spä12 Proposition 3.4] and of [CS19 Theorem 2.4]).

We conclude this section by recalling an important property that is needed in Section 6.

\textbf{Lemma 2.1.} Let $G$, $\bar{G}$, $F$ and $\mathcal{A}$ as in the above paragraph and suppose that $G^F$ is the universal covering group of $G^F / Z(G^F)$. Let $Z \leq Z(G^F)$ and denote by $(\bar{G}^F \mathcal{A})_Z$ the normaliser of $Z$ in $\bar{G}^F \mathcal{A}$. Then

$$C_{(\bar{G}^F \mathcal{A})_Z / Z}(G^F / Z) = Z(\bar{G}^F) / Z$$

and the canonical map $(\bar{G}^F \mathcal{A})_Z \to \text{Aut}(G^F / Z)$ induces an isomorphism

$$(\bar{G}^F \mathcal{A})_Z / Z(\bar{G}^F) \simeq \text{Aut}(G^F / Z).$$

\textbf{Proof.} By the above paragraph, we know that $\bar{G}^F \mathcal{A} / C_{\bar{G}^F \mathcal{A}}(G^F) \simeq \text{Aut}(G^F)$ and therefore, using the fact that $C_{\bar{G}^F \mathcal{A}}(G^F) = Z(\bar{G}^F)$ (see the argument used in [Spä12 Proposition 3.4 (a)], [CS19 Theorem 2.4] and ultimately [GLS98 Theorem 2.5.1]), we obtain $(\bar{G}^F \mathcal{A})_Z / Z(\bar{G}^F) \simeq \text{Aut}(G^F)_Z$. Then, by [GLS98 Corollary 5.1.4 (b)], it follows that

$$(\bar{G}^F \mathcal{A})_Z / Z(\bar{G}^F) \simeq \text{Aut}(G^F)_Z \simeq \text{Aut}(G^F / Z).$$

On the other hand, since

$$\text{Aut}(G^F / Z) \simeq \frac{(\bar{G}^F \mathcal{A})_Z / Z}{C_{(\bar{G}^F \mathcal{A})_Z / Z}(G^F / Z)},$$

the third isomorphism theorem yields the desired isomorphism. $\square$

\subsection{2.2 Good primes and $e$-split Levi subgroups}

For the rest of this section we consider the following setting.

\textbf{Notation 2.2.} Let $G$ be a connected reductive linear algebraic group defined over an algebraic closure $\overline{\mathbb{F}}$ of a finite field of characteristic $p$ and $F : G \to G$ a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $G$, for a power $q$ of $p$. Consider a prime $\ell$ different from $p$ and denote by $e$ the multiplicative order of $q$ modulo $\ell$ (modulo 4 if $\ell = 2$). All blocks are considered with respect to the prime $\ell$. 8
In what follows we make some restrictions on the prime $\ell$. First, recall that $\ell$ is a good prime for $G$ if it is good for each simple factor of $G$, while the conditions for the simple factors are as follows:

\[ A_n : \text{every prime is good} \]
\[ B_n, C_n, D_n : \ell \neq 2 \]
\[ G_2, F_4, E_6, E_7 : \ell \neq 2, 3 \]
\[ E_8 : \ell \neq 2, 3, 5. \]

We say that $\ell$ is a bad prime for $G$ if it is not a good prime. Next, we introduce the set of primes $\Gamma(G, F)$ from [CE94 Notation 1.1].

**Definition 2.3.** We denote by $\gamma(G, F)$ the set of primes $\ell$ such that: $\ell$ is odd, $\ell \neq p$, $\ell$ is good for $G$ and $\ell$ doesn’t divide $[Z(G)^F : \Phi^*(G)^F]$. Let $(G^*, F^*)$ be in duality with $(G, F)$ and set $\Gamma(G, F) := (\gamma(G, F) \cap \gamma(G^*, F^*)) \setminus \{3\}$ if $G^*_\text{ad}$ has a component of type $^3D_4(q^n)$ and $\Gamma(G, F) := \gamma(G, F) \cap \gamma(G^*, F^*)$ otherwise.

**Remark 2.4.** Notice that, if $\ell \in \Gamma(G, F)$, then $\ell \in \Gamma(G^*, F^*)$ and $\ell \in \Gamma(H, F)$ for every $F$-stable connected reductive subgroup $H$ of $G$ containing an $F$-stable maximal torus of $G$ (see [CE04 Proposition 13.12]). In particular, if $\ell \in \Gamma(G, F)$ and $L$ is an $F$-stable Levi subgroup of $G$, then $\ell \in \Gamma(L, F)$.

Using [CE04 Table 13.11], we describe the primes $\ell \in \Gamma(G, F)$ when $G$ is simple of simply connected type with Frobenius endomorphism $F$ defining an $F_q$-structure on $G$:

\[ A_n(q) : \ell + 2q(n + 1, q - 1), \]
\[ B_n(q), C_n(q), D_n(q), 2D_n(q) : \ell \neq 2, p \]
\[ ^3D_4(q), G_2(q), F_4(q), E_6(q), ^2E_7(q) : \ell \neq 2, 3, p \]
\[ E_8(q) : \ell \neq 2, 3, 5, p. \]

As a consequence, if a connected reductive group $G$ has no simple components of type $A$, then $\ell \in \Gamma(G, F)$ if and only if $\ell$ is good for $G$ and $\ell \neq p$.

In this paper we make use of the terminology of Sylow $\Phi_e$-theory introduced in [BM92] (see also [BMM93]). For a set of positive integers $E$, we say that an $F$-stable Levi subgroup $T$ of $G$ is a $\Phi_E$-torus if its order polynomial is of the form $P_T(T, F) = \prod_{n \in E} \Phi_n^{a_n}$ for some integers $a_n$ and where $\Phi_n$ denotes the $n$-th cyclotomic polynomial (see [CE04 Definition 13.3]). The centralisers of $\Phi_E$-tori are called $E$-split Levi subgroups. If $E = \{e\}$, we call $\Phi_{\{e\}}$-tori and $\{e\}$-split Levi subgroups simply $\Phi_e$-tori and $e$-split Levi subgroups respectively. When $\ell \in \Gamma(G, F)$, some significant consequences on the structure of $e$-split Levi subgroups can be drawn.

**Lemma 2.5.** Let $L$ be an $F$-stable Levi subgroup of $G$.

(i) Let $E$ be a set of positive integers. Then $L$ is $E$-split if and only if $L = C_G(Z^e(L)_{\Phi_E})$.

(ii) Set $E_{q, \ell} := \{e \cdot \ell^m \mid m \in \mathbb{N}\}$. If $L = C_G^{\ast}(Z^e(L)^{\ell})$, then $L$ is $E_{q, \ell}$-split. The converse holds provided that $\ell \in \Gamma(G, F)$.
Proof. The first statement follows directly from the definition. In fact, since $Z^o(L)$ is a torus, we deduce that $Z^o(L)_{\Phi_E}$ is a $\Phi_E$-torus and therefore $C_G(Z^o(L)_{\Phi_E})$ is $E$-split. Conversely, assume that $L$ is $E$-split. Then there exists a $\Phi_E$-torus $T$ such that $L = C_G(T)$. Since $T$ is abelian, we deduce that $T \leq Z(L)$. Then, as $T$ is connected, we have $T \leq Z^o(L)$ and therefore $T \leq Z^o(L)_{\Phi_E}$ because $T = T_{\Phi_E}$. By [DM91 Proposition 1.21], we conclude that $L = C_G(Z^o(L)) \leq C_G(Z^o(L)_{\Phi_E}) \leq C_G(T) = L$. For the second statement see [CE04 Proposition 13.19].

**Lemma 2.6.** Let $\ell \in \Gamma(G, F)$ and consider an $\ell$-subgroup $Y$ of $G^F$. Then:

(i) $C_G(Y)^F = C_G^o(Y)^F$;

(ii) if $Y$ is abelian, then $Y \leq Z^o(C_G^o(Y))$;

(iii) if $Y$ is abelian, then $C_G^o(Y)$ is an $E_q,\ell$-split Levi subgroup of $G$.

Proof. First notice that $C_G^o(Y)$ is a Levi subgroup of $G$ by [CE04 Proposition 13.16 (ii)]. The first statement is [CE04 Proposition 13.16 (i)]. Assume now that $Y$ is abelian and notice that $L \leq C_G(Y)^F = C_G^o(Y)^F$. Then $Y \leq Z(C_G^o(Y))$. By (i) we know that $C_G^o(Y)$ is a Levi subgroup of $G$ and hence $\ell \in \Gamma(C_G^o(Y), F)$ by Remark 2.5. In particular $\ell$ does not divide $[Z(C_G^o(Y))^F : Z^o(C_G^o(Y))^F]$ and so $Y \leq Z^o(C_G^o(Y))$. Now, $L := C_G^o(Y)$ is an $F$-stable Levi subgroup with $Y \leq Z^o(L)$. Then $Y \leq Z^o(L)^\ell \leq Z^o(L)$ and [DM91 Proposition 1.21] implies that $L = C_G^o(Z^o(L)) \leq C_G^o(Z^o(L)^\ell) \leq C_G^o(Y) = L$. By Lemma 2.5 it follows that $L$ is $E_q,\ell$-split in $G$.

We say that a prime $\ell$ is large for $(G, F)$ if there exists a unique integer $e_0$ such that $\Phi_{e_0}$ divides the order polynomial $P_{(G, F)}$ and $\ell$ divides $\Phi_{e_0}(q)$ (see [BMM93 Definition 5.1] and [Mal14 Section 2.1]). In this case we also say that $\ell$ is $(G, F, e_0)$-adapted (see [BMM93 Definition 5.3]). When $G$ is semisimple, then $e_0$ coincides with $e$. Observe that if $\ell$ is large, then the Sylow $\ell$-subgroups of $G^F$ are abelian [MT11 Theorem 25.14]. Moreover, if $\ell$ is large for $G$, then $\ell \in \Gamma(G, F)$ by [Mal14 Lemma 2.1].

For any finite $\ell$-group $H$ and positive integer $n$ we define the subgroup $\Omega_n(H) := \langle h \in H \mid h^{\ell^n} = 1 \rangle$. In particular, when $H$ is abelian, $\Omega_1(H)$ is the largest $\ell$-elementary abelian subgroup of $H$.

**Proposition 2.7.** Suppose that $\ell$ is large for $G$ and consider an $e$-split Levi subgroup $L$ of $G$ and an $\ell$-subgroup $Y$ of $G^F$. Then:

(i) if $\ell$ is $(G, F, e_0)$-adapted for some integer $e_0$, then $C_G^o(Y)$ is an $e_0$-split Levi subgroup of $G$;

(ii) if $S := Z^o(L)_{\Phi_e}$, then $S \leq Z(G)$ if and only if $S^F \leq Z(G)$ if and only if $\Omega_1(S^F) \leq Z(G)$;

(iii) $L = C_G^o((Z^o(L)_{\Phi_e})^F) = C_G^o(\Omega_1((Z^o(L)_{\Phi_e})^F))$.

Proof. By [CE04 Proposition 13.16 (ii)] and Lemma 2.6 (ii) we know that $C_G^o(Y)$ is an $F$-stable Levi subgroup and $Y \leq Z^o(C_G^o(Y))$. Then [BMM93 Proposition 2.4 (1)] implies that $C_G^o(Y)$ is an $e_0$-split Levi subgroup. This proves (i).

Next, set $S := Z^o(L)_{\Phi_e}$. It is enough to show that $S \not\leq Z(G)$ implies $\Omega_1(S^F) \not\leq Z(G)$. Since $S$ is a $\Phi_e$-torus, we have $S \not\leq Z(G)$ if and only if $S \not\leq Z^o(G)_{\Phi_e}$ while, using the fact that $\ell$ is large, we deduce that $Z(G)^F = (Z^o(G)_{\Phi_e})^F$ and therefore $\Omega_1(S^F) \not\leq Z(G)$ if and only if $\Omega_1(S^F) \not\leq Z^o(G)_{\Phi_e}$. Hence, in order to obtain (ii) we need to show that $S \not\leq Z^o(G)_{\Phi_e}$ implies $\Omega_1(S^F) \not\leq Z^o(G)_{\Phi_e}$.
Assume that $S \not\leq Z^a(G)\Phi_e =: T_e$ and consider the canonical morphism $\pi_e : G \to G/T_e$. Observe, that $T_e \leq S$ and that $S$ and $\pi_e(S) \neq 1$ are $\Phi_e$-tori. If $\ell^\alpha$ is the largest power of $\ell$ dividing $\Phi_e(q)$, then $T^F_e$ is the direct product of copies of $C_{\ell^\alpha}$ for every $\Phi_e$-torus $T$ (see [BM92, Proposition 3.3]). Let $y \in \pi_e(S)^F$ be an element of order $\ell^\alpha$. Proceeding as in the proof of [CE04] Lemma 13.17 (i) and noticing that $(S(T_e))^F = S^F/T^F_e$, we deduce that $\pi_e(S)^F = \pi_e(S^F_e)$ and hence there exists $x \in S^F_e$ such that $\pi_e(x) = y$. Now, the order of $y$ divides the order of $x$ while the order of $x$ must be less or equal than $\ell^\alpha$ by the description of $S^F_e$ given above. We conclude that $x$ has order $\ell^\alpha$. Then $s := x^{\ell^{\alpha-1}} \in \Omega_1(S^F)$ and $\pi_e(s) = y^{\ell^{\alpha-1}} \neq 1$. This shows that $\Omega_1(S^F) \not\leq T_e = Z^a(G)\Phi_e$.

To prove (iii), we proceed by induction on the dimension of $G$. Set $S := Z^a(L)\Phi_e$ and notice that $L = C_G(S) \leq C^o_G(S^F) \leq C^o_G(\Omega_1(S^F))$. We need to show that $K := C^o_G(\Omega_1(S^F)) \leq L$. Observe that $K$ is a Levi subgroup of $G$ by [CE04] Proposition 13.16 (ii). If $S \leq Z(G)$, then $K = G = L$. Therefore, we can assume $S \not\leq Z(G)$. By the above paragraph, we obtain $\Omega_1(S^F) \not\leq Z(G)$ and therefore $\dim(K) < \dim(G)$. Noticing that $\ell$ is large for $K$ and that $L$ is an $e$-split Levi subgroup of $K$, the inductive hypothesis yields $L = C^o_K(\Omega_1(S^F))$. The result follows by noticing that $C^o_K(\Omega_1(S^F)) = K$.

### 2.3 Deligne–Lusztig induction and blocks

Let $G, F, q, \ell$ and $e$ be as in Notation 2.2 and consider an $F$-stable Levi subgroup of a (not necessarily $F$-stable) parabolic subgroup $P$ of $G$. By tensoring with a $(G^F, L^F)$-bimodule arising from the $\ell$-adic cohomology of Deligne–Lusztig varieties, Deligne–Lusztig [DL76] (in the case where $L$ is a maximal torus) and Lusztig [Lus76] (in the general case) have defined a map

$$R^G_{L\leq P} : \mathbb{Z}Irr(L^F) \to \mathbb{Z}Irr(G^F)$$

with adjoint

$$^*R^G_{L\leq P} : \mathbb{Z}Irr(G^F) \to \mathbb{Z}Irr(L^F)$$

that we call Deligne–Lusztig induction and restriction respectively. Notice that these maps are often referred to simply as Lusztig induction and restriction, and the terms Deligne–Lusztig induction and restriction are only used when considering the case of a maximal torus. Nonetheless we believe that the contribution of Deligne should be acknowledged. It is expected that the map $R^G_{L\leq P}$ does not depend on the choice of the parabolic subgroup $P$ and this would, for instance, follow from the Mackey formula which has been proved wherever $G^F$ does not have components of type $2E_6(2)$, $E_7(2)$ or $E_8(2)$ [BM11]. For simplicity, we just write $R^G_P$ when the results are known not to depend on the choice of $P$. Similar remarks apply for Deligne–Lusztig restriction.

Recall that $(L, \lambda)$ is an $e$-cuspidal pair of $(G, F)$ (or simply of $G$ when no confusion arises) if $L$ is an $e$-split Levi subgroup of $G$ and $\lambda \in Irr(L^F)$ satisfies $^*R^G_{L\leq M\leq Q}(\lambda) = 0$ for every $e$-split Levi subgroup $M < L$ and every parabolic subgroup $Q$ of $L$ containing $M$ as Levi complement.

To fix our notation, we now review the parametrisation of blocks given in [CE99]. Let’s assume $\ell \geq 5$ with $\ell \geq 7$ if $G$ has a component of type $E_8$. Then for every $B \in Bl(G^F)$ there exists a unique $e$-cuspidal pair $(L, \lambda)$ up to $G^F$-conjugation such that $\lambda$ lies in a rational Lusztig series associated with an $\ell$-regular semisimple element and all the irreducible constituents of $R^G_{L\leq P}(\lambda)$ belongs to the block $B$ for every parabolic subgroup $P$ of $G$ having $L$ as Levi complement. In this case we write $B = b_{G^F}(L, \lambda)$. Moreover, [CE99] Theorem 2.5 implies that $b(\lambda)^{G^F} = B$ whenever $\ell \in \Gamma(G, F)$ (see Lemma 2.5). See [KM15] for a generalisation of these results to all primes.
3 \((e, \ell')\)-pairs and transitivity

In this section we provide new evidence for a conjecture proposed by Cabanes and Enguehard in [CE99 Notation 1.11]. Consider \(G, F, q, \ell\) and \(e\) as in Notation 2.2. We start by defining the notion of \(e\)-pair and \((e, s)\)-pair.

**Definition 3.1.** An \(e\)-pair of \((G, F)\) (or simply of \(G\) when no confusion arises) is a pair \((L, \lambda)\) where \(L\) is an \(e\)-split Levi subgroup of \(G\) and \(\lambda \in \text{Irr}(L^F)\). For any semisimple element \(s \in G^{*F^e}\), we say that an \(e\)-pair \((L, \lambda)\) is an \((e, s)\)-pair of \((G, F)\) if \(\lambda \in \mathcal{E}(L^F, [s'])\) for some \(s' \in L^{*F^e}\) that is \(G^{*F^e}\)-conjugate to \(s\). Finally, we say that \((L, \lambda)\) is an \((e, \ell')\)-pair if it is an \((e, s)\)-pair for some \(\ell\)-regular semisimple element \(s \in G^{*F^e}\).

If \(\mathcal{P}_e(G, F)\) is the set of \(e\)-pairs of \((G, F)\), then there exists a binary relation on \(\mathcal{P}_e(G, F)\) denoted by \(\leq_e\) (see [CE99 Notation 1.11]). Namely, \((L, \lambda) \leq_e (K, \kappa)\) provided that \(L \leq K\) are \(e\)-split Levi subgroups of \(G\) and there exists a parabolic subgroup \(P\) of \(K\) containing \(L\) as a Levi complement such that \(\kappa\) is an irreducible constituent of the generalised character \(R^K_L(\lambda)\). Noticing that Deligne–Lusztig induction sends characters to generalised characters, we observe that the relation \(\leq_e\) might not be transitive at first glance. We denote by \(\ll_e\) the transitive closure of \(\leq_e\). Since \(\mathcal{P}_e(G, F)\) is finite, we deduce that two \(e\)-pairs \((L, \lambda)\) and \((K, \kappa)\) satisfy \((L, \lambda) \ll_e (K, \kappa)\) if and only if there exist a finite number of \(e\)-pairs \((L_i, \lambda_i)\), with \(i = 1, \ldots, n\), such that

\[(L, \lambda) \leq_e (L_1, \lambda_1) \leq_e \cdots \leq_e (L_n, \lambda_n) \leq_e (K, \kappa).\]

With this notation, a pair \((L, \lambda)\) is \(e\)-cuspidal if and only if it is a minimal element in the poset \((\mathcal{P}_e(G, F), \ll_e)\). We denote by \(\mathcal{C}_e(G, F)\) the subset of \(\mathcal{P}_e(G, F)\) consisting of \(e\)-cuspidal pairs. Observe that, by [CE04 Proposition 15.7] the relations \(\leq_e\) and \(\ll_e\) restrict to the set of \((e, s)\)-pairs for every \(s \in G^{*F^e}_{ss}\). A minimal element in the induced poset of \((e, s)\)-pairs is called \((e, s)\)-cuspidal.

The following conjecture has been proposed in [CE99 Notation 1.11] and is inspired by [BMM93 Theorem 3.11].

**Conjecture 3.2** (Cabanes–Enguehard). The relation \(\leq_e\) is transitive and therefore coincides with \(\ll_e\).

In this section, we show that this conjecture holds when considering \((e, \ell')\)-cuspidal pairs in groups of simply connected type under certain assumptions on \(\ell\). Before proceeding with the proof of this result, we point out an important consequence of Conjecture 3.2. Let \((L, \lambda)\) be an \(e\)-pair of \(G\). If Conjecture 3.2 holds, then

\[\{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\} = \mathcal{E}(G^F, (L, \lambda)),\]

where \(\mathcal{E}(G^F, (L, \lambda))\) is the \(e\)-Harish-Chandra series determined by \((L, \lambda)\), that is the set of irreducible constituents of \(R^L_G(\lambda)\) for every parabolic subgroup \(P\) of \(G\) having \(L\) as a Levi complement. In addition, if Deligne–Lusztig induction does not depend on the choice of a parabolic subgroup, then

\[\{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\} = \text{Irr}(R^G_L(\lambda)),\]

where we recall that, for any finite group \(X\) and \(\chi \in \mathbb{Z}\text{Irr}(X)\), we denote by \(\text{Irr}(\chi)\) the set of irreducible constituent of \(\chi\). Because this remark is used multiple times in Section 4 we introduce the following condition.
**Condition 3.3.** Consider $G$, $F$, $q$, $\ell$ and $e$ as in Notation 2.2 and assume that Deligne–Lusztig induction does not depend on the choice of parabolic subgroups and 

$$\{\kappa \in \text{Irr}(K^F) \mid (L, \lambda) \ll_e (K, \kappa)\} = \text{Irr}(R^K_L(\lambda))$$

for every $F$-stable Levi subgroup $K$ of $G$ and every $(e, \ell')$-cuspidal pair $(L, \lambda)$ of $K$.

Observe that Conjecture 3.2 is known for $(e, 1)$-pairs by [BMM93, 3.11] while Condition 3.3 has been proved for $G$ simple of exceptional simply connected type and good primes in [Hol22, Theorem 1.1]. Exceptional simple groups and bad primes have been considered in [KM13, Theorem 1.4]. Moreover Condition 3.3 is known to hold for groups with connected centre and good primes $\ell \geq 5$ by [Eng13, Proposition 2.2.4]. Proposition 3.6 below extends these results and shows that Condition 3.3 holds for every connected reductive group $G$ with $[G, G]$ simply connected and good primes $\ell \geq 5$. In the next section we extend this result to $e$-pairs associated with $\ell$-singular semisimple elements (see Corollary 4.11). Moreover, in Proposition 4.5 we prove Conjecture 3.2 inside $e$-Harish-Chandra series associated to certain semisimple elements. Notice that our proof does not depend on [Eng13] in any way.

**Lemma 3.4.** Let $L$ be an $e$-split Levi subgroup of a connected reductive group $G$ and consider $G_0 := [G, G]$ and $L_0 := L \cap G_0$.

(i) Let $\lambda_0 \in \text{Irr}(L_0^F)$ and $\chi_0 \in \text{Irr}(G_0^F)$. If $(L_0, \lambda_0) \leq_e (G_0, \chi_0)$ and $\chi \in \text{Irr}(G^F \mid \chi_0)$, then there exists $\lambda \in \text{Irr}(L^F \mid \lambda_0)$ such that $(L, \lambda) \leq_e (G, \chi)$.

(ii) Let $\lambda \in \text{Irr}(L^F)$ and $\chi \in \text{Irr}(G^F)$. If $(L, \lambda) \leq_e (G, \chi)$ and $\lambda_0 \in \text{Irr}(\lambda_{L_0}^F)$, then there exists $\chi_0 \in \text{Irr}(\chi_{G_0}^F)$ such that $(L_0, \lambda_0) \leq_e (G_0, \chi_0)$.

**Proof.** First observe that $L_0$ is an $e$-split Levi subgroup of $G_0$. By [GM20, Proposition 3.3.24] (see also the proof of [GM20, Corollary 3.3.25]) and since $G = Z^o(G)G_0$, it follows that

$$R_L^G \circ \text{Ind}_{L_0}^{G_0} = \text{Ind}_{G_0}^{G_0} \circ R_{L_0}^{G_0} \circ \text{Res}_{L_0}^{L} \circ \text{Ind}_{L_0}^{G_0}$$ (3.1)

and

$$R_{L_0}^{G_0} \circ \text{Res}_{G_0}^{G} = \text{Res}_{L_0}^{L} \circ R_{L_0}^{G_0} \circ \text{Res}_{G_0}^{G} \circ R_{L_0}^{G_0} \circ \text{Res}_{L_0}^{L}$$ (3.2)

Suppose first that $(L_0, \lambda_0) \leq_e (G_0, \chi_0)$ and consider $\chi \in \text{Irr}(G_0^F \mid \chi_0)$. Then $\chi$ is an irreducible constituent of $\text{Ind}_{G_0}^{G_0}(R_{L_0}^{G_0}(\lambda_0))$ and by (3.1) we can find $\lambda \in \text{Irr}(L^F \mid \lambda_0)$ such that $(L, \lambda) \leq_e (G, \chi)$.

Suppose now that $(L, \lambda) \leq_e (G, \chi)$ and let $\lambda_0$ be an irreducible constituent of $\lambda_{L_0}^F$. Since Deligne–Lusztig induction and restriction are adjoint with respect to the usual scalar product, we deduce that $\lambda_0$ is an irreducible constituent of $\text{Res}_{L_0}^{L} \circ \text{Res}_{G_0}^{G}(\chi_{G_0}^F)$. By (3.2) there exists $\chi_0 \in \text{Irr}(\chi_{G_0}^F)$ such that $\lambda_0$ is a constituent of $\text{Res}_{L_0}^{L_0}(\chi_0)$ and therefore $(L_0, \lambda_0) \leq_e (G_0, \chi_0)$.

The following result shows that Condition 3.3 holds when $G$ has only components of classical types and $\ell \geq 5$ or when $G$ is simple, $K = G$ and $\lambda$ lies in a rational Lusztig series associated with a quasi-isolated element. Recall that a semisimple element $s$ of a reductive group $G$ is called quasi-isolated if $C_G(s)$ is not contained in any proper Levi subgroup of $G$. 


Lemma 3.5. Let $G$ be connected reductive, $\chi \in \text{Irr}(G^F)$ and consider an $e$-cuspidal pair $(L, \lambda) \ll_e (G, \chi)$, where $\lambda \in E(L^F, [s])$ for some $\ell$-regular semisimple element $s \in L^*F^*$. Suppose that $\ell \geq 5$ is good for $G$ and that the Mackey formula holds for $(G, F)$. If either $G$ has only components of classical types and $F$ does not induce the triality automorphism on components of type $D_4$ or $G$ is simply connected and $s$ is quasi-isolated in $G^*$, then $(L, \lambda) \leq_e (G, \chi)$.

Proof. Consider a regular embedding $i : G \to \tilde{G}$. By applying [BMM93, 3.11] together with [GM20] Theorem 4.7.2 and Corollary 4.7.8] to $\tilde{G}$, it follows that Conjecture 3.2 holds in $\tilde{G}$ unless $s$ is quasi-isolated in $G$ and $G$ is simply of connected type $E_6$ or $E_7$ or $F^F = 3D_4(q)$. However, in these excluded cases the result holds by [Hol22, Theorem 1.1] and we can therefore assume that Conjecture 3.2 holds in $\tilde{G}$. Now, the result follows by applying [CE99, Proposition 5.2].

We can now prove the main result of this section. Recall that for a connected reductive group $G$, we say that $G$ is simply connected if the semisimple group $[G, G]$ is simply connected.

Proposition 3.6. Let $G$ be a simply connected reductive group, $\chi \in \text{Irr}(G^F)$ and consider an $(e, \ell')$-cuspidal pair $(L, \lambda) \ll_e (G, \chi)$. If $\ell \geq 5$ is good for $G$ and the Mackey formula holds for $(G, F)$, then $(L, \lambda) \leq_e (G, \chi)$.

Proof. Let $(G^*, F^*)$ be dual to $(G, F)$ and let $L^*$ be the $e$-split Levi subgroup of $G^*$ corresponding to $L$. Consider an $\ell$-regular semisimple element $s \in L^*F^*$ such that $\lambda \in E(L^F, [s])$ and notice that $\lambda \in E(G^F, [s])$ because $(L, \lambda) \ll_e (G, \chi)$ (see [CE04, Proposition 15.7]). By induction on $\dim(G)$, we show that $s$ is quasi-isolated in $G^*$. Suppose that $G_1$ is a proper $F$-stable Levi subgroup of $G$ such that $C_{G^*}(s) \leq G_1^*$. Observe that $G_1$ is simply connected by [MT11] Proposition 12.14. Set $L_1^* := C_{G^*}(Z^*(L^*)\Phi_e) = C_{G^*}(Z^*(L^*)\Phi_e) \cap G_1^* = L^* \cap G_1^*$ and notice that its dual $L_1 \leq L$ is an $e$-split Levi subgroup of $G_1$ and that $C_{G^*}(s) \leq L^* \cap G_1^* = L_1^*$. By [CE04, Theorem 8.27] there exist unique $\lambda_1 \in E(L_1^F, [s])$ and $\chi_1 \in E(G_1^F, [s])$ such that $\lambda = \pm R_{G_1}^{L_1}(\lambda_1)$ and $\chi = \pm R_{G_1}^{G_1}(\chi_1)$. Since $(L, \lambda) \ll_e (G, \chi)$, it follows by the transitivity of Deligne–Lusztig induction that $(L_1, \lambda_1) \ll_e (G_1, \chi)$.

By a similar argument also shows that $\lambda_1$ is $e$-cuspidal. Since $\dim(G_1) < \dim(G)$, we obtain $(L_1, \lambda_1) \leq_e (G_1, \chi_1)$. This shows that $\chi_1$ is an irreducible constituent of $R_{L_1}^{G_1}(\lambda_1)$ and, because all constituents of $R_{L_1}^{G_1}(\lambda_1)$ are contained in $E(G_1^F, [s])$ and $R_{G_1}^{G_1}$ induces a bijection between $E(G_1^F, [s])$ and $E(G^F, [s])$ (see [CE04, Theorem 8.27]), we conclude that $\chi$ is an irreducible constituent of $\pm R_{G_1}^{G_1}(R_{L_1}^{G_1}(\lambda_1)) = \pm R_{G_1}^{G_1}(\lambda_1)$.

Hence $(L, \lambda) \leq_e (G, \chi)$ and we may assume that $s$ is quasi-isolated in $G^*$.

Let $G_0 := [G, G]$ and $L_0 := L \cap G_0$. By assumption, there exist $e$-split Levi subgroups $L_i$ of $G$ containing $L$ and characters $\lambda_i \in \text{Irr}(L_i^F)$ such that $(L, \lambda) \leq_e (L_1, \lambda_1) \leq_e \cdots \leq_e (G, \chi)$. If we define $L_{i,0} := L_i \cap G_0$, then a repeated application of Lemma 3.4 yields characters $\lambda_0 \in \text{Irr}(\lambda_{L_i^F})$, $\lambda_{i,0} \in \text{Irr}(\lambda_{L_{i,0}^F})$ and $\chi_0 \in \text{Irr}(\chi_{G_{0,F}^*})$ such that $(L_0, \lambda_0) \leq_e (L_{1,0}, \lambda_{1,0}) \leq_e \cdots \leq_e (G_0, \chi_0)$. Then $(L_0, \lambda_0) \ll_e (G_0, \chi_0)$ with $(L_0, \lambda_0)$ an $(e, \ell')$-cuspidal pair. Moreover, if the result is true for $G_0$, then $(L_0, \lambda_0) \leq_e (G_0, \chi_0)$ and using Lemma 3.4 we find $\lambda' \in \text{Irr}(L^F \mid \lambda_0)$ such that $(L, \lambda') \leq_e (G, \chi)$. Then [CE99, Theorem 4.1] shows that $\lambda'' = \lambda$, for some $g \in N_G(L^F)$, and hence $(L, \lambda) = (L, \lambda') \leq_e (G, \chi) = (G, \chi)$. Notice that the inclusion $G_0 \to G$ induces a dual morphism $G^*_0 \to G^*_0$ and that, if $s \in G_{0,ss}^*$ is quasi-isolated in $G^*$, then the corresponding element $s_0 \in G_{0,ss}^*$ is...
quasi-isolated in $G_0^*$ by [Bon05 Proposition 2.3]. Without loss of generality we can hence assume $G = [G, G]$.

Now, $G$ is a direct product of simple algebraic groups $H_1, \ldots, H_n$ (see [Mar91 Proposition 1.4.10]). The action of $F$ induces a permutation on the set of simple components $H_i$. For every orbit of $F$ we denote by $G_j$, $j = 1, \ldots, t$, the direct product of the simple components in such an orbit. Then $G_j$ is $F$-stable and

$$G^F = G_1^F \times \cdots \times G_t^F.$$ 

Define $L_j := L \cap G_j$ and observe that $L_j$ is an $e$-split Levi subgroup of $G_j$ and that

$$L^F = L_1^F \times \cdots \times L_j^F.$$ 

Then we can write $\chi = \chi_1 \times \cdots \times \chi_t$ and $\lambda = \lambda_1 \times \cdots \lambda_t$, with $\chi_j \in \text{Irr}(G_j^F)$ and $\lambda_j \in \text{Irr}(L_j^F)$. Since $R_L^G = R_{L_1}^G \times \cdots \times R_{L_t}^G$ (see [DM91 Proposition 10.9 (ii)]), eventually considering intermediate $e$-split Levi subgroups, the fact that $(L, \lambda) \ll_e (G, \chi)$ implies that $(L_j, \lambda_j) \ll_e (G_j, \chi_j)$ for every $j$. Noticing that $G_j^{s \cdot F_*} = G_1^{s \cdot F_*} \times \cdots \times G_t^{s \cdot F_*}$, we can write $s = s_1 \times \cdots \times s_t$ for some $\ell$-regular semisimple elements $s_j \in G_j^{s \cdot F_*}$. Moreover, since $s$ is quasi-isolated in $G^*$, it follows that $s_j$ is quasi-isolated in $G_j^*$. Without loss of generality, we may thus assume that $F$ is transitive on the set of simple components $H_i$ or equivalently that $t = 1$.

Now, consider a simple component $H$ of $G$ and observe that there are isomorphisms

$$G^F \simeq H^{F^n} \quad (3.3)$$

and

$$G^{s \cdot F_*} \simeq H^{s \cdot F^n} \quad (3.4)$$

where $n$ is the number of simple components $H_i$ of $G$. Let $M := L \cap H$ and notice that $M$ is an $e$-split Levi subgroup of $(H, F^n)$ and that the isomorphism from (3.3) restricts to an isomorphism

$$L^F \simeq M^{F^n}. \quad (3.5)$$

Let $\psi \in \text{Irr}(H^{F^n})$ correspond to $\chi \in \text{Irr}(G^F)$ via the isomorphism (3.3) and similarly $\mu \in \text{Irr}(M^{F^n})$ correspond to $\lambda \in \text{Irr}(L^F)$ via the isomorphism (3.5). Since $(L, \lambda) \ll_e (G, \chi)$, we deduce that $(M, \mu) \ll_e (H, \psi)$. Moreover, as $s$ is quasi-isolated in $G^*$, it follows that the semisimple element $t \in H^{s \cdot F^n}$ obtained via the isomorphism (3.4) is quasi-isolated in $H^*$. Finally, Lemma 3.5 implies that $(M, \mu) \preceq_e (H, \psi)$ and we hence conclude that $(L, \lambda) \preceq_e (G, \chi)$. □

Since the hypotheses of the above proposition are inherited by Levi subgroups, it follows that Condition 3.3 holds whenever $G$ is a simply connected reductive group.

**Corollary 3.7.** Let $G$ be a simply connected reductive group. Then Condition 3.3 holds for every $F$-stable Levi subgroup $K$ of $G$ and every $(e, \ell')$-cuspidal pair $(L, \lambda)$ of $K$. 

15
4 Brauer–Lusztig blocks and $e$-Harish-Chandra series

In this section we prove Theorem A and hence obtain a description of the distribution of characters into blocks for finite reductive groups in non-defining characteristic under suitable assumptions on $\ell$.

As already mentioned in Section 2.3 under certain assumptions on $\ell$, the results of [CE99, Theorem 4.1] show that for every block $B \in \text{Bl}(G^F)$ there exists a unique $G^F$-conjugacy class of $e$-cuspidal pairs $(L, \lambda)$ such that $\lambda \in \mathcal{E}(L^F,[s])$ for some $\ell$-regular semisimple element $s \in L^*F^*$ and every irreducible constituent of $R^G_L \leq P(\lambda)$ is contained in $\text{Irr}(B)$ for every parabolic subgroup $P$ of $G$ containing $L$ as Levi complement. [CE99, Theorem 4.1] also provides a characterisation of the set of so-called $\ell'$-characters in the block $B$ as

$$
\mathcal{E}(G^F, \ell') \cap \text{Irr}(B) = \left\{ \chi \in \text{Irr}(G^F) \left| (L, \lambda) \ll_e (G, \chi) \right. \right\}.
$$

(4.1)

On the other hand, [BM89, Theorem 2.2] shows that

$$
\text{Irr}(B) = \bigoplus_{t} \mathcal{E}(G^F, B,[st]),
$$

where $t$ runs over the elements of $C_{G^*}(s)/^eF^*$ up to conjugation and $\mathcal{E}(G^F, B,[st])$ is a Brauer–Lusztig block (see Definition 4.15). In particular, in order to obtain all the characters in $\text{Irr}(B)$, we have to describe the Brauer–Lusztig blocks $\mathcal{E}(G^F, B,[st])$. Now, by using Corollary 3.7 the equality (4.1) can be restated as

$$
\mathcal{E}(G^F, B,[s]) = \mathcal{E}(G^F, (L, \lambda))
$$

showing that those Brauer–Lusztig blocks associated with $\ell$-regular semisimple elements coincide with $e$-Harish-Chandra series. Our aim is to provide a similar description for arbitrary Brauer–Lusztig blocks and remove the restriction on $s$ being $\ell$-regular.

4.1 $e$-Harish-Chandra series and $\ell$-blocks

Throughout this section we assume the following conditions.

**Hypothesis 4.1.** Let $G$, $F : G \to G$, $q$, $\ell$ and $e$ be as in Notation 2.2. Assume that:

(i) $\ell \in \Gamma(G, F)$ with $\ell \geq 5$ and the Mackey formula hold for $(G, F)$;

(ii) Condition 3.3 holds for $(G, F)$.

Observe that the Mackey formula and Condition 3.3 are expected to hold for any connected reductive group. Moreover, it follows from Remark 2.4 that Hypothesis 4.1 is inherited by $F$-stable Levi subgroups.

**Remark 4.2.** Suppose that $[G, G]$ is of simply connected type and has no irreducible rational component of type $^2E_6(2)$, $E_7(2)$, $E_8(2)$ and consider $\ell \in \Gamma(G, F)$ with $\ell \geq 5$. Then Hypothesis 4.1 is satisfied. In fact, under our assumption, the Mackey formula holds by [BM11] while Condition 3.3 holds by Corollary 3.7.
We now start working towards a proof of Theorem A. First, we show how to associate to every $(e, s)$-pair an $(e, s')\text{-}pair$ via Jordan decomposition. This can be used to extend some of the results of [CE99] from $(e, l')\text{-}pairs$ to arbitrary $e\text{-}pairs$.

**Lemma 4.3.** Assume Hypothesis [J] (i). If $(L, \lambda)$ is an $(e, s)$-pair of $G$ with $s \in L^*$, then there exists an $E_q, l\text{-}split$ Levi subgroup $G(s_\ell)$ of $G$, an $(e, s')\text{-}pair$ $(L(s_\ell), \lambda(s_\ell))$ of $G(s_\ell)$ and a linear character $\lambda_{\ell}$ of $L(s_\ell)$ such that $\lambda = \epsilon L \epsilon L(s_\ell) R^L(s_\ell)(\lambda(s_\ell) \cdot \lambda_{\ell})$. In particular $L(s_\ell)$ is an $E_q, l\text{-}split$ Levi subgroup of $G$.

**Proof.** Under our assumptions, Lemma 2.6 (iii) implies that $C^Q_{G^*}(s_{\ell})$ is an $E_q, l\text{-stable}$ Levi subgroup of $G^*$. By [Hol22] Proposition 2.12 we know that $C^Q_{L^*}(s_{\ell})$ is an $e\text{-split}$ Levi subgroup of $C^Q_{\Gamma^*}(s_{\ell})$. As $\ell \in \Gamma(G, F)$, Remark 2.4 implies that $\ell \in \Gamma(L^*, F^*)$ and therefore $C^Q_{L^*}(s_{\ell}) = C^Q_{L^*}(s_{\ell})^{F}$ by Lemma 2.6 (i). Recalling that $s_{\ell}$ is a power of $s$, it follows that $C^Q_{L^*}(s_{\ell}) C^Q_{L^*}(s_{\ell})^{F} \subseteq C^Q_{L^*}(s_{\ell}) C^Q_{L^*}(s_{\ell})^{F} = C^Q_{L^*}(s_{\ell})$. Let $G(s_{\ell})$ be an $E_q, l\text{-split}$ Levi subgroup of $G$ in duality with $C^Q_{G^*}(s_{\ell})$ and $L(s_{\ell})$ an $e\text{-split}$ Levi subgroup of $G(s_{\ell})$ in duality with $C^Q_{L^*}(s_{\ell})$. By [CE04] Proposition 8.26 and [CE04] Theorem 8.27 there exists a unique character $\lambda(s_{\ell}) \in \mathcal{E}(L(s_\ell), [s_{\ell}])$ such that

$$\lambda = \epsilon L \epsilon L(s_{\ell}) R^L_L(s_{\ell})(\lambda_{\ell} \cdot \lambda_{s_{\ell}}),$$

where $\lambda_{\ell}$ is the linear character corresponding to $s_{\ell} \in Z(C^Q_{L^*}(s_{\ell})^{F})$. To conclude, notice that $L(s_{\ell})$ is an $E_q, l\text{-split}$ Levi subgroup of $G(s_{\ell})$. Since $G(s_{\ell})$ is $E_q, l\text{-split}$ in $G$ we deduce that $L(s_{\ell})$ is $E_q, l\text{-split}$ in $G$.

Next, we show that the relation $\ll$ is preserved under the construction of Lemma 4.3. Moreover we consider $e\text{-cuspiality}$ and a property related to the Jordan criterion (J) introduced by Cabanes and Enguehard (see [CE99] Proposition 1.10). The final part of the following statement could be seen as a partial converse of [CE99] Proposition 1.10.

**Lemma 4.4.** Assume Hypothesis [J] (i). Let $(L, \lambda)$ and $(K, \kappa)$ be two $(e, s)$-pairs and consider the corresponding $(e, s')\text{-}pairs$ $(L(s_{\ell}), \lambda(s_{\ell}))$ and $(K(s_{\ell}), \kappa(s_{\ell}))$ given by Lemma 4.3. Then $(L, \lambda) \ll (K, \kappa)$ if and only if $(L(s_{\ell}), \lambda(s_{\ell})) \ll (K(s_{\ell}), \kappa(s_{\ell}))$. In particular $(L, \lambda) \ll (K, \kappa)$ if and only if $(L(s_{\ell}), \lambda(s_{\ell})) \ll (K(s_{\ell}), \kappa(s_{\ell}))$. Moreover $(L, \lambda)$ is $e\text{-cuspial}$ in $G$ if and only if $(L(s_{\ell}), \lambda(s_{\ell}))$ is $e\text{-cupsial}$ in $G(s_{\ell})$ and $Z^e(L)_{\phi_e} = Z^e(L(s_{\ell}))_{\phi_e}$.

**Proof.** First, using [Hol22] Proposition 2.12 notice that every $e\text{-split}$ Levi subgroup of $G(s_{\ell})$ is of the form $M(s_{\ell})$ for some $e\text{-split}$ Levi subgroup $M$ of $G$. Then, since $\ll$ is a pre-order on a finite poset, proceeding by induction it is enough to show that $(L, \lambda) \ll (K, \kappa)$ if and only if $(L(s_{\ell}), \lambda(s_{\ell})) \ll (K(s_{\ell}), \kappa(s_{\ell}))$.

Suppose first that $(L, \lambda) \ll (K, \kappa)$ and assume without loss of generality that $s \in L^*$. We know that $\kappa$ is an irreducible constituent of $R^K_L(\lambda)$. By the transitivity of Deligne–Lusztig induction (see [GM20] Theorem 3.3.6), we have

$$R^K_L(\lambda) = \epsilon L \epsilon L(s_{\ell}) R^K_L(s_{\ell})(\lambda_{s_{\ell}} \cdot \lambda(s_{\ell})) = \epsilon L \epsilon L(s_{\ell}) R^K_L(s_{\ell})(R^K(s_{\ell})(\lambda_{s_{\ell}} \cdot \lambda(s_{\ell}))).$$
Moreover, by [CE04 Proposition 15.7], every irreducible constituent of \( R^{K(s)}_{L(s)}(\tau \cdot \lambda(s)) \) is contained in \( E(K(s))^F, [s] \). Then, since

\[
\epsilon^{K(s)}_{K(s)} : E(K(s))^F, [s] \to E(K, [s])
\]

is a bijection, we deduce that \( \tau \cdot \kappa(s) \) is an irreducible constituent of \( R^{K(s)}_{L(s)}(\tau \cdot \lambda(s)) \). It follows that \( \kappa(s) \) is an irreducible constituent of \( R^{K(s)}_{L(s)}(\lambda(s)) \) (see [Bon06, 10.2]). The reverse implication follows from a similar argument.

Next suppose that \( (L, \lambda) \) is \( e \)-cuspidal in \( G \) and let \( (M_{s,t}, \mu_{s,t}) \ll_e (L(s_{t}), \lambda(s_{t})) \). By [Hol22 Proposition 2.12] we can find an \( e \)-split Levi subgroup \( M \) of \( G \) such that \( M_{s,t} = M(s_{t}) \) and we define \( \mu := \epsilon_{M(s_{t})} \epsilon_{M} \epsilon_{M_{s,t}}(\mu_{s,t} \tau_{s}) \). Then \( (M_{s,t}, \mu_{s,t}) = (M(s_{t}), \mu(s_{t})) \) and the above paragraph implies \( (M, \mu) \ll_e (L, \lambda) \). Since \( (L, \lambda) \) is \( e \)-cuspidal, we deduce that \( (M, \mu) = (L, \lambda) \) and therefore \( (M_{s,t}, \mu_{s,t}) = (L(s_{t}), \lambda(s_{t})) \). This shows that \( (L(s_{t}), \lambda(s_{t})) \) is \( e \)-cuspidal. Furthermore if \( Z^e(L_{s_{t}})_{\phi_{s_{t}}} < Z^e(L(s_{t}))_{\phi_{s_{t}}} \), then \( L(s_{t})_{\phi_{s_{t}}} = C_{G}^{\circ}(Z^e(L(s_{t}))_{\phi_{s_{t}}}) \) is an \( e \)-split Levi subgroup of \( G \) strictly contained in \( L \) and containing \( L(s_{t}) \). In this case \( (L(s_{t}), \lambda(s_{t}))_{\phi_{s_{t}}} \leq (L, \lambda) \) for \( \lambda(s_{t})_{\phi_{s_{t}}} := \epsilon_{L(s_{t})} \epsilon_{L(s_{t})} \epsilon_{L(s_{t})} R^{L(s_{t})}_{L(s_{t})}(\tau_{s_{t}}) \cdot \lambda(s_{t})) \) which is a contradiction. Hence \( Z^e(L)_{\phi_{s_{t}}} = Z^e(L(s_{t}))_{\phi_{s_{t}}} \).

Conversely, assume that \( (L(s_{t}), \lambda(s_{t})) \) is \( e \)-cuspidal in \( G(s_{t}) \) and that \( Z^e(L)_{\phi_{s_{t}}} = Z^e(L(s_{t}))_{\phi_{s_{t}}} \). Let \( (M, \mu) \ll_e (L, \lambda) \) and observe that \( (M(s_{t}), \mu(s_{t})) \ll_e (L(s_{t}), \lambda(s_{t})) \). Since \( (L(s_{t}), \lambda(s_{t})) \) is \( e \)-cuspidal, it coincides with \( (M(s_{t}), \mu(s_{t})) \). In particular \( M(s_{t}) = L(s_{t}) \) and, recalling that \( Z^e(M) \leq Z^e(M(s_{t})) \) and hence \( Z^e(M)_{\phi_{s_{t}}} \leq Z^e(M(s_{t}))_{\phi_{s_{t}}} \), we deduce that \( L = C_{G}^{\circ}(Z^e(L)_{\phi_{s_{t}}} = C_{G}^{\circ}(Z^e(L(s_{t})_{\phi_{s_{t}}}) = C_{G}^{\circ}(Z^e(M(s_{t}))_{\phi_{s_{t}}}) \leq C_{G}^{\circ}(Z^e(M)_{\phi_{s_{t}}}) \). Thus \( L = M \) and \( (L, \lambda) \) is \( e \)-cuspidal.

The argument used in the proofs of Lemma 4.3 and Lemma 4.4 can be used to obtain the following result of independent interest.

**Proposition 4.5.** Let \( s \in G^{sF^s} \) be a semisimple element such that \( o(s) \) is not divisible by bad primes and \( (o(s), |Z(G)^F : Z^e(G)^F|) = 1 \). Then Conjecture 32 holds in \( E(G^F, [s]) \) for every \( e \geq 1 \).

**Proof.** Let \( (L, \lambda) \ll_e (K, \kappa) \) be \( (e, s) \)-pairs. Replacing \( s \) with a \( G^F \)-conjugate, we can assume that \( s \in L^s \). Since no bad prime divide \( o(s) \), a repeated application of [CE04 Proposition 13.16 (i)] shows that \( C_{G}^{s}(s) \) is an \( F \)-stable Levi subgroup. Moreover, since \( (o(s), |Z(G)^F : Z^e(G)^F|) = 1 \), it follows that \( C_{G}^{s}(s) \ll \left( C_{G}^{s}(s) \right)^F \) (see [DM20 Lemma 11.2.1 (iii)]). Now proceeding as in Lemma 4.3 we construct unipotent \( e \)-pairs \( (L(s), \lambda(s)) \) and \( (K(s), \kappa(s)) \) of \( G(s) \). Arguing as in Lemma 4.4 we deduce from \( (L, \lambda) \ll_e (K, \kappa) \) that \( (L(s), \lambda(s)) \ll_e (K(s), \kappa(s)) \). Applying [BMM93 Theorem 3.11] we get \( (L(s), \lambda(s)) \ll_e (K(s), \kappa(s)) \) and again proceeding as in the proof of Lemma 4.4 we conclude that \( (L, \lambda) \ll_e (K, \kappa) \). □

The following lemma is a fundamental ingredient to understand the distribution of characters into blocks. The proof is based on an idea used first in [CE94] in order to deal with unipotent blocks. Notice that, if \( \ell \in \Gamma(G, F) \) and \( L \) is an \( e \)-split Levi subgroup of \( G \), then \( L^F = C_{G}^{F}(Q) \) for some abelian \( \ell \)-subgroup \( Q \leq G^F \) by Lemma 2.3 together with Lemma 2.6(i). Therefore, block induction from \( L^F \) to \( G^F \) is defined by [Nav98 Theorem 4.14].
Lemma 4.6. Assume Hypothesis \([4.7](i)\). Let \((K, \kappa)\) be an \((e, s)\)-pair of \(G\) and consider the \((e, s_{\ell}')\)-pair \((K(s_{\ell}'), \kappa(s_{\ell}'))\) given by Lemma \([4.3]\). Consider an \((e, s_{\ell}')\)-cuspidal pair \((L, \lambda)\) of \(K(s_{\ell})\) such that \(bl(\kappa(s_{\ell})) = b_{K(s_{\ell})}^L(L, \lambda)\). Then \(bl(\kappa) = bl(\lambda)^K\).

Proof. Since \(K(s_{\ell})\) is an \(E_{\ell}\)-split Levi subgroup of \(K\), \([CE99\text{ Theorem } 2.5]\) implies that all irreducible constituents of \(R^K_{K(s_{\ell})}(\kappa(s_{\ell}))\) are contained in a unique block \(b\) of \(K^F\). Moreover, under our assumption, Lemma \([2.5]\) implies that \(K(s_{\ell}) = C_{K}(Z(K(s_{\ell}))^F)\) and therefore \(b = bl(\kappa(s_{\ell}))^{K^F}\). Similarly \(bl(\kappa(s_{\ell})) = bl(\lambda)^{K^F}\) and so \(b = bl(\lambda)^{K^F}\) by the transitivity of block induction. It remains to show that \(b = bl(\kappa)\). In order to do so, we apply Brauer’s second Main Theorem (see \([CE04\text{ Theorem } 5.8]\)). Then, it suffices to show that \(d^1(R_{K(s_{\ell})}^K(\kappa(s_{\ell})))\) has an irreducible constituent in \(bl(\kappa)\). By \([CE04\text{ Proposition } 21.4]\) and since \(R_{K(s_{\ell})}^K\) and \(^eR_{K(s_{\ell})}^K\) are adjoined, it follows that

\[
\begin{align*}
    d^1(R_{K(s_{\ell})}^K(\kappa(s_{\ell}))) &= R_{K(s_{\ell})}^K(d^1(\kappa(s_{\ell}))) \\
    &= R_{K(s_{\ell})}^K(d^1(\tilde{s}_{\ell} \cdot \kappa(s_{\ell}))) \\
    &= c_{K\ell} R_{K(s_{\ell})}^K d^1(\kappa).
\end{align*}
\]

Since by Brauer’s second Main Theorem \(d^1(\kappa) \in \mathbb{N}\text{Irr}(bl(\kappa))\), the proof is now complete. \(\Box\)

As a corollary we deduce that the construction given in Lemma \([4.3]\) preserves the decomposition of characters into blocks.

Corollary 4.7. Assume Hypothesis \([4.7](i)\). Let \(L\) be an \(e\)-split Levi subgroup of \(G\) and consider \(s \in L_{s_{\ell}^F}^F\). For \(i = 1, 2\), let \(\lambda_i \in \mathcal{E}(L^F, [s])\) and consider \(\lambda_i(s_{\ell}) \in \mathcal{E}(L(s_{\ell})^F, [s_{\ell}'])\) given by Lemma \([4.3]\). If \(\lambda_1(s_{\ell})\) and \(\lambda_2(s_{\ell})\) are in the same block of \(L(s_{\ell})^F\), then \(\lambda_1\) and \(\lambda_2\) are in the same block of \(L^F\).

Proof. Let \(c\) be the block of \(L(s_{\ell})\) containing \(\lambda_1(s_{\ell})\) and \(\lambda_2(s_{\ell})\) and consider an \(e\)-cuspidal pair \((M, \mu)\) such that \(c = b_{L(s_{\ell})}^M(M, \mu)\). Then, Lemma \([4.6]\) implies that \(bl(\lambda_1) = bl(\mu)^L = bl(\lambda_2)\). \(\Box\)

The next result can be seen as an extension of \([CE99\text{ Theorem } 2.5]\) to \((e, s)\)-pairs with \(s\) not necessarily \(\ell\)-regular.

Proposition 4.8. Assume Hypothesis \([4.7](i)\). Let \(K\) be an \(e\)-split Levi subgroup of \(G\) and \((L, \lambda)\) an \(e\)-pair of \(K\). Then there exists a block \(b\) of \(K^F\) such that \(R^K_L(\lambda) \in \mathbb{Z}\text{Irr}(b)\). Moreover \(b = bl(\lambda)^K\).

Proof. Let \(s \in L_{s_{\ell}^\ell}^F\) such that \((L, \lambda)\) is an \((e, s)\)-pair. Consider the \((e, s_{\ell}')\)-pair \((L(s_{\ell}), \lambda(s_{\ell}))\) given by Lemma \([4.3]\). By \([CE99\text{ Theorem } 2.5]\), there exists a block \(b(s_{\ell})\) of \(K(s_{\ell})\) such that \(R_{L(s_{\ell})}^K(\lambda(s_{\ell})) \in \mathbb{Z}\text{Irr}(b(s_{\ell}))\). Furthermore \(b(s_{\ell}) = bl(\lambda(s_{\ell}))^{K(s_{\ell})}\) by Lemma \([4.7]\) If we denote by \(\tilde{s}_{\ell} \cdot b(s_{\ell})\) the block of \(K(s_{\ell})\) consisting of those characters of the form \(\tilde{s}_{\ell} \cdot \xi\), for \(\xi \in \text{Irr}(b(s_{\ell}))\), then

\[
R_{L(s_{\ell})}^K(\lambda(s_{\ell})) = \tilde{s}_{\ell} \cdot R_{L(s_{\ell})}^K(\lambda(s_{\ell})) \in \mathbb{Z}\text{Irr}(\tilde{s}_{\ell} \cdot b(s_{\ell})).
\]

By Corollary \([4.7]\) and \([4.8]\), it follows that there exists a unique block \(b\) of \(K^F\) such that

\[
R_{L}^K(\lambda) = R_{L}^K(\tilde{s}_{\ell}) \cdot R_{L(s_{\ell})}^K(\lambda(s_{\ell})) = R_{L(s_{\ell})}^K(\tilde{s}_{\ell} \cdot \lambda(s_{\ell})) \in \mathbb{Z}\text{Irr}(b).
\]
Next, set \( c := \text{bl}(\lambda(s_\ell)) \). Consider an \((e, \ell')\)-cuspidal pair \((M, \mu)\) such that \( c = b_{L(s_\ell)}^F(M, \mu) \).

Since \( c = \text{bl}(\mu)L(s_\ell)F \) and \( b(s_\ell) = eK(s_\ell)F \) it follows that

\[
b(s_\ell) = eK(s_\ell)F = \text{bl}(\mu)K(s_\ell)F = b_{K(s_\ell)}F(M, \mu).\]

Now, Lemma 4.6 implies that \( \text{bl}(\lambda) = \text{bl}(\mu)LF \) and that \( b = \text{bl}(\mu)K\). We conclude that \( b = \text{bl}(\mu)K^F = (\text{bl}(\mu)L^F)K^F = \text{bl}(\lambda)K^F \) and this concludes the proof.

The next corollary is basically a restatement of Proposition 4.3.

**Corollary 4.9.** Assume Hypothesis 4.7(i), let \( L \) be an \( e\)-split Levi subgroup of \( G \) and \( b_L \) an \( \ell \)-block of \( L^F \). Then there exists a unique \( \ell \)-block \( b_G \) of \( G^F \) such that for every \( \lambda \in \text{Irr}(b_L) \), all irreducible constituents of \( R_{L,G}^\ell(\lambda) \) lie in \( b_G \). Moreover \( b_G = b_L^F \) via Brauer induction.

Finally, we show that for every \( e\)-pair \((K, \kappa)\) there exists a unique \( e\)-cuspidal pair \((L, \lambda)\) up to \( K^F\)-conjugation satisfying \((L, \lambda) \leq_e (K, \kappa)\). Observe that our next results also extends Proposition 3.6 to \( e\)-pairs associated with \( \ell\)-singular semisimple elements.

**Proposition 4.10.** Assume Hypothesis 4.4. Let \((L, \lambda) \leq_e (K, \kappa)\) be \( e\)-pairs of \( G \) such that \((L, \lambda)\) is \( e\)-cuspidal. Then \((L, \lambda) \leq_e (K, \kappa)\). Moreover, if \((L', \lambda')\) is another \( e\)-cuspidal pair of \( G \) satisfying \((L', \lambda') \leq_e (K, \kappa)\), then \((L, \lambda) \leq (L', \lambda')\) are \( K^F\)-conjugate.

**Proof.** Assume that \( \lambda \in \mathcal{E}(L^F, [s]) \) for a semisimple element \( s \in L^{*F} \) and consider the \((e, s_\ell')\)-pairs \((L(s_\ell), \lambda(s_\ell))\) and \((K(s_\ell), \kappa(s_\ell))\) given by Lemma 4.3. Applying Lemma 4.4 together with our assumption, we deduce that \((L(s_\ell), \lambda(s_\ell)) \leq_e (K(s_\ell), \kappa(s_\ell))\) and that \((L(s_\ell), \lambda(s_\ell))\) is \( e\)-cuspidal in \( G(s_\ell) \). Now, Condition 4.3 shows that \((L(s_\ell), \lambda(s_\ell)) \leq_e (K(s_\ell), \kappa(s_\ell))\) and we obtain \((L, \lambda) \leq_e (K, \kappa)\) by applying Lemma 4.4 one more time.

Next consider another \((e, s)\)-cuspidal pair \((L', \lambda') \leq_e (K, \kappa)\). Let \( \lambda' \in \mathcal{E}(L', [s'])\) and notice that \( s \) and \( s' \) are \( K^{*F}\)-conjugate by [CE04] Proposition 15.7. Replacing \((L', \lambda')\) with a \( K\)-conjugate we may assume that \( s = s' \). As before consider the \((e, s_\ell')\)-cuspidal pair \((L'(s_\ell), \lambda'(s_\ell))\) and observe that \((L'(s_\ell), \lambda'(s_\ell)) \leq_e (K(s_\ell), \kappa(s_\ell))\). By [CE99] Theorem 4.1 it follows that \((L(s_\ell), \lambda(s_\ell))\) and \((L'(s_\ell), \lambda'(s_\ell))\) are \( K(s_\ell)^F\)-conjugate. Since \( s_\ell \) is \( K(s_\ell)^F\)-invariant, we deduce that \((L(s_\ell), s_\ell \cdot \lambda(s_\ell))\) and \((L'(s_\ell), s_\ell \cdot \lambda'(s_\ell))\) are \( K(s_\ell)^F\)-conjugate. Write

\[
(L(s_\ell), \lambda(s_\ell) \cdot s_\ell) = (L'(s_\ell), \lambda'(s_\ell) \cdot s_\ell)^x \tag{4.3}
\]

for some \( x \in K(s_\ell)^F \). An application of [CE99] Proposition 1.10 with respect to the \( e\)-cuspidal characters \( \lambda \) of \( L^F \) and \( \lambda(s_\ell) \cdot s_\ell \) of \( L(s_\ell)^F \) shows that \( Z^\circ(L^*)_{\Phi_e} = Z^\circ(C_{L^*}^\circ(s))_{\Phi_e} \) and \( Z^\circ(C_{L^*}^\circ(s))_{\Phi_e} = Z^\circ(C_{L^*}^\circ(s_\ell))_{\Phi_e} \). However \( C_{L^*}^\circ(s) \leq C_{L^*}^\circ(s_\ell) \) and therefore we obtain \( Z^\circ(L^*)_{\Phi_e} = Z^\circ(C_{L^*}^\circ(s))_{\Phi_e} \). This shows that \( L \) is the smallest \( e\)-split Levi subgroup of \( K \) containing \( L(s_\ell) \). Since \( L^F \) is an \( e\)-split Levi subgroup of \( K \) containing \( L'(s_\ell)^x = L(s_\ell) \) we deduce that \( L^F = L \). Now, (4.3) implies \((L, \lambda) = (L', \lambda')^x\). □

Proposition 4.10 shows in particular that Conjecture 3.2 holds when working above \( e\)-cuspidal pairs.
Corollary 4.11. Assume Hypothesis\ref{Hyp4.1}. Then the relation $\leq_e$ and $\ll_e$ from Conjecture\ref{Conj3.2} coincide on the subset $\mathbb{C}P_e(G, F) \times P_e(G, F) \subseteq P_e(G, F) \times P_e(G, F)$.

Proof. If $((L, \lambda), (K, \kappa)) \in \mathbb{C}P_e(G, F) \times P_e(G, F)$, then Proposition\ref{Prop4.10} implies that $(L, \lambda) \leq_e (K, \kappa)$ if and only if $(L, \lambda) \ll_e (K, \kappa)$.

As an immediate consequence of Proposition\ref{Prop4.10} we deduce that the set $\text{Irr}(K^F)$ is a disjoint union of $e$-Harish-Chandra series. This should be compared with the classical Harish-Chandra theory (see \cite[Corollary 3.1.17]{GM20}) and with the analogous result for unipotent characters \cite[Theorem 4.6.20]{GM20}. These two results, can be recovered by considering $(1, s)$-pairs and $(e, 1)$-pairs respectively.

Corollary 4.12. Let $G$ be a connected reductive group with a Frobenius endomorphism $F$ defining an $\mathbb{F}_q$-structure on $G$ and consider an integer $e \geq 1$. For every $e$-split Levi subgroup $K$ of $G$ there is a partition

$$\text{Irr}(K^F) = \bigsqcup_{(L, \lambda)} \mathcal{E}(K^F, (L, \lambda)),$$

where the union runs over a $K^F$-transversal in the set of $e$-cuspidal pairs of $K$, provided that there exists a prime $\ell$ such that Hypothesis\ref{Hyp4.1} is satisfied with respect to $(G, F, q, \ell, e)$.

Observe that although the proof of the above corollary depends on the choice of a certain prime $\ell$, its statement does not. This is due to the fact that our result is obtained as a consequence of $\ell$-modular representation theoretic techniques. Nonetheless, a partition of characters into $e$-Harish-Chandra series is expected to hold without the restrictions considered here.

Next, combining Corollary\ref{Cor4.12} and Proposition\ref{Prop4.8} we can describe all the characters in the blocks of $K^F$ in terms of $e$-Harish-Chandra series.

Theorem 4.13. Assume Hypothesis\ref{Hyp4.1}. Let $K$ be an $e$-split Levi subgroup of $G$ and $b$ a block of $K^F$. Then

$$\text{Irr}(b) = \bigsqcup_{(L, \lambda)} \mathcal{E}(K^F, (L, \lambda)),$$

where the union runs over the $K^F$-conjugacy classes of $e$-cuspidal pairs $(L, \lambda)$ of $K$ such that $\text{bl}((\lambda)K^F) = b$.

Proof. Proposition\ref{Prop4.8} shows that $\mathcal{E}(K^F, (L, \lambda)) \subseteq \text{Irr}(b)$ for every $e$-cuspidal pair $(L, \lambda)$ such that $\text{bl}((\lambda)K^F) = b$. On the other hand, if $k \in \text{Irr}(b)$, then by Corollary\ref{Cor4.12} there exists an $e$-cuspidal pair $(L, \lambda)$ of $K$ such that $k \in \mathcal{E}(K^F, (L, \lambda))$. Moreover, applying Proposition\ref{Prop4.8} once more, it follows that $b = \text{bl}(\kappa) = \text{bl}((\lambda)K^F)$. Finally, the union is disjoint by Proposition\ref{Prop4.10}.

In the following remark we compare Theorem\ref{Thm4.13} and \cite[Theorem (iii)]{CE94}.

Remark 4.14. Let $B$ be a unipotent block and consider a unipotent $e$-cuspidal pair $(L_0, \lambda_0)$ such that $B = b_{G^F}(L_0, \lambda_0)$. In this situation, \cite[Theorem (iii)]{CE94} shows that every irreducible character of $B$ occurs in some $R_{G^F}(\chi_s \cdot \mathbb{S})$ where $s \in G^{*F}$ is an $\ell$-element, $\chi_s \in \mathcal{E}(G(s)^F, [1])$ and $(L_s, \lambda_s) \preceq_e (G(s), \chi_s)$ for some $e$-cuspidal pair $(L_s, \lambda_s)$ of $G(s)$ satisfying $(L_s, \lambda_s) \sim (L_0, \lambda_0)$ (see \cite[Definition 3.4]{CE94}). As usual, here we denote by $G(s)$ a Levi subgroup in duality with
Lemma 4.3. Moreover, applying Lemma 4.4 we deduce that $\epsilon$ implies that $\mathcal{E}(L^F, [s])$, then $s$ is an $\ell$-element by [BM89]. Consider the unipotent $e$-pairs $(L(s), \lambda(s))$ and $(G(s), \chi(s))$ of $G(s)$ given by Lemma 4.3. By definition, $\chi$ occurs in $R^G_{G(s)}(\chi(s) \cdot \delta)$. Next, applying Lemma 4.3 we deduce that $(L(s), \lambda(s)) \leq e$ $(G(s), \chi(s))$ and that $(L(s), \lambda(s))$ is $e$-cuspidal in $G(s)$. By [CE94] Proposition 3.5 (iii) there exists a unipotent $e$-cuspidal pair $(M, \mu)$ of $G$ such that $(L(s), \lambda(s)) \sim (M, \mu)$. Proceeding as at the end of the proof of [CE94] Theorem 4.4 (iii), we conclude that $(M, \mu)$ is $G^F$-conjugate to $(L_0, \lambda_0)$ and by replacing $(L, \lambda)$ with a $G^F$-conjugate we may assume $(L(s), \lambda(s)) \sim (L_0, \lambda_0)$ as required by [CE94] Theorem (iii).

Conversely, given a unipotent $e$-cuspidal pair $(L_s, \lambda_s) \leq (G(s), \chi_s)$ and an irreducible constituent $\chi$ of $R^G_{G(s)}(\chi_s \cdot \delta)$ as in [CE94] Theorem (iii), we obtain an $e$-cuspidal pair $(L, \lambda)$ of $G$ such that $bl(\lambda)^{G^F} = B$ and $\chi \in \mathcal{E}(G^F, (L, \lambda))$ as follows. Consider $L := C^G_G(Z^G(L_s)_{\Phi_e})$ and define $\lambda := e_{L(s), \lambda} \epsilon_{1, R^L_{L(s)}} (\lambda_s \cdot \delta)$. Then $(L(s), \lambda_s)$ coincides with the pair $(L(s), \lambda(s))$ obtained from $(L, \lambda)$ as in Lemma 4.3. Moreover, $Z^G(L(s))_{\Phi_e} = Z^G(L)_{\Phi_e}$ and since $(L_s, \lambda_s)$ is $e$-cuspidal in $G(s)$ Lemma 4.4 implies that $(L, \lambda)$ is an $e$-cuspidal pair of $G^F$. Under our assumption, $(G(s), \chi_s) = (G(s), \chi(s))$ and the relation $(L_s, \lambda_s) \leq (G(s), \chi_s)$ implies $(L, \lambda) \leq (G, \chi)$ thanks to Lemma 4.4. This shows that $\chi \in \mathcal{E}(G^F, (L, \lambda))$. Finally, since by assumption $\chi \in \text{Irr}(B)$ we conclude that $bl(\lambda)^{G^F} = B$ by applying Corollary 4.9.

### 4.2 Brauer–Lusztig blocks

We now extend Theorem 4.13 in order to obtain Theorem A. To start, following Broué, Fong and Srinivasan, we define the Brauer–Lusztig blocks of $G^F$.

**Definition 4.15** (Broué–Fong–Srinivasan). A Brauer–Lusztig block of $G^F$ is any non-empty set of the form

$$\mathcal{E}(G^F, B, [s]) := \mathcal{E}(G^F, [s]) \cap \text{Irr}(B),$$

where $B$ is a block of $G^F$ and $s$ is a semisimple element of $G^*F^*$. In this case, we say that $(G, B, [s])$ is the associated **Brauer–Lusztig triple** of $G^F$. Moreover, we denote by $\mathcal{B}(G, F)$ the set of all Brauer–Lusztig triples of $G^F$. We also define the set

$$\mathcal{B}^\forall(G, F) := \bigsqcup_{L \in G} \mathcal{B}(L, F),$$

where $L$ runs over all $\ell$-split Levi subgroups of $G$.

Next, assume $\ell \in \Gamma(G, F)$. Recall from the discussion preceding Lemma 4.6 that, for every $\ell$-split Levi subgroup $L$ of $G$ and $b \in \text{Bl}(L^F)$, the Brauer induced block $b^{G^F}$ is defined. Then, we can introduce a partial order relation on $\mathcal{B}^\forall(G, F)$ by defining

$$(L, b, [s]) \leq (K, c, [t])$$

if $L \leq K$, $b^{K^F} = c$ and the semisimple elements $s$ and $t$ are conjugate by an element of $K^{*F^*}$. If $(L, b, [s])$ is a minimal element of the poset $(\mathcal{B}^\forall(G, F), \leq)$, then we say that $(L, b, [s])$ is a **cuspidal** Brauer–Lusztig triple.
In the next lemma we compare the relation \( \leq \) on Brauer–Lusztig triples with the relations \( \ll_c \) and \( \leq_e \) on \( e \)-pairs.

**Lemma 4.16.** Assume Hypothesis \( \text{[L]} \) Let \( L \) and \( K \) be e-split Levi subgroups of \( G \) and consider semisimple elements \( s \in L^{s_{F^s}} \) and \( t \in K^{s_{F^s}} \).

(i) Let \( \lambda \in \mathcal{E}(L^F, b, [s]) \) and \( \kappa \in \mathcal{E}(K^F, c, [t]) \). If \( (L, \lambda) \ll_c (K, \kappa) \), then \( (L, b, [s]) \subseteq (K, c, [t]) \).

(ii) Let \( \lambda \in \mathcal{E}(L^F, b, [s]) \). If \( (L, b, [s]) \) is cuspidal, then \( (L, \lambda) \) is e-cuspidal.

(iii) If \( (L, b, [s]) \subseteq (K, c, [t]) \), then for every \( \lambda \in \mathcal{E}(L^F, b, [s]) \) there exists \( \kappa \in \mathcal{E}(K^F, c, [t]) \) such that \( (L, \lambda) \leq_e (K, \kappa) \).

**Proof.** We start by proving (i). Let \( (L, \lambda) \ll_c (K, \kappa) \). By \([CE04, \text{Proposition 15.7}]\), we may assume \( s = t \) and it is enough to show that \( bl(\lambda)^{K^F} = bl(\kappa) \). To see this, choose an e-cuspidal pair \( (M, \mu) \ll_e (L, \lambda) \) and notice that \( (M, \mu) \ll_e (K, \kappa) \). By Proposition 4.10 we deduce that \( (M, \mu) \leq_e (L, \lambda) \) and \( (M, \mu) \leq_e (K, \kappa) \). Then, Proposition 4.8 implies that \( bl(\lambda) = bl(\mu)^{L^F} \) and \( bl(\kappa) = bl(\mu)^{K^F} \). By the transitivity of block induction, we conclude that \( bl(\kappa) = bl(\lambda)^{K^F} \).

This proves (i) and (ii) is an immediate consequence. In fact, if \( (L, b, [s]) \) is a cuspidal Brauer–Lusztig triple and we consider an e-cuspidal \( (M, \mu) \ll_e (L, \lambda) \), then (i) shows that \( (M, bl(\mu), [r]) \subseteq (L, b, [s]) \), where \( \mu \in \mathcal{E}(M^F, [r]) \). It follows that \( L = M \) and that \( (L, \lambda) = (M, \mu) \) is e-cuspidal.

Finally, let \( (L, b, [s]) \subseteq (K, c, [t]) \) and consider \( \lambda \in \mathcal{E}(L^F, b, [s]) \). Let \( \kappa \) be an irreducible constituent of \( R^{K^F}_c(\lambda) \) so that \( (L, \lambda) \leq_e (K, \kappa) \). We need to show that \( \kappa \in \mathcal{E}(K^F, c, [t]) \). By \([CE04, \text{Proposition 15.7}]\) we have \( \kappa \in \mathcal{E}(K^F, [s]) = \mathcal{E}(K^F, [t]) \). Moreover, applying Proposition 4.8 we obtain \( bl(\kappa) = bl(\lambda)^{K^F} = bK^F = c \). We conclude that \( \kappa \in \mathcal{E}(K^F, c, [t]) \).

Finally, we are able to prove the main result of this section which provides a slightly more general version of Theorem A.

**Theorem 4.17.** Assume Hypothesis \( \text{[L]} \) Let \( (K, c, [t]) \in \mathcal{B}^c(G, F) \). Then

\[
\mathcal{E}(K^F, c, [t]) = \bigsqcup_{(L, \lambda)} \mathcal{E}(K^F, (L, \lambda)),
\]

where the union runs over the \( K^F \)-conjugacy classes of \( (e, t) \)-cuspidal pairs \( (L, \lambda) \) of \( K \) with \( \lambda \in \mathcal{E}(L^F, [s_\lambda]) \) such that \( (L, bl(\lambda), [s_\lambda]) \subseteq (K, c, [t]) \).

**Proof.** Consider an \( e \)-cuspidal pair \( (L, \lambda) \) such that \( (L, bl(\lambda), [s]) \subseteq (K, c, [t]) \), where \( s \in L^{s_{F^s}} \) and \( \lambda \in \mathcal{E}(L^F, [s]) \). Since \( s \) and \( t \) are \( K^{s_{F^s}} \)-conjugate, \([CE04, \text{Proposition 15.7}]\) implies that \( \mathcal{E}(K^F, (L, \lambda)) \subseteq \mathcal{E}(K^F, [t]) \). Moreover, using the fact that \( c = bl(\lambda)^{K^F} \), Proposition 4.8 shows that the \( e \)-Harish-Chandra series \( \mathcal{E}(K^F, (L, \lambda)) \) is contained in \( \text{Irr}(c) \). This shows that the union on the right hand side of (4.4) is contained in the Brauer–Lusztig block \( \mathcal{E}(K^F, c, [t]) \). Moreover the union is disjoint by Proposition 4.10. To conclude, let \( \kappa \in \mathcal{E}(K^F, c, [t]) \) and notice that there exists an \( e \)-cuspidal pair \( (L, \lambda) \) of \( K \) such that \( \kappa \in \text{Irr}(R_c^{K^F}(\lambda)) \) by Corollary 4.12. If \( \lambda \in \mathcal{E}(L^F, [s]) \), then \( s \) and \( t \) are \( K^{s_{F^s}} \)-conjugate by \([CE04, \text{Proposition 15.7}]\). Moreover, \( c = bl(\kappa) = bl(\lambda)^{K^F} \) by Proposition 4.8. It follows that \( (L, bl(\lambda), [s]) \subseteq (K, c, [t]) \). \( \square \)
As we have mentioned before, the results obtained by Cabanes and Enguehard have been extended to all primes by Kessar and Malle in [KM15] and the reader might wonder why we are not considering this more general situation. Unfortunately, many of the techniques used in this section fail for bad primes and a different proof needs to be found in this case.

### 4.3 Defect zero characters and $\ell$-cuspidal pairs

Recall that for an irreducible character $\chi$ of a finite group $G$, the $\ell$-defect of $\chi$ is the non-negative integer $d(\chi)$ defined by $\ell^{d(\chi)}\chi(1)_{\ell} = |G|_{\ell}$. Our next result shows a necessary condition for an $\ell$-cuspidal character of $G$ to be of $\ell$-defect zero.

**Proposition 4.18.** Suppose that $\ell$ is large for $G$ and $(G, F, e)$-adapted with $Z(G^*)_{F^*}^{\ell} = 1$. If $\chi$ is an $\ell$-cuspidal pair of $G$, then $\chi$ has $\ell$-defect zero. In particular $\chi \in \mathcal{E}(G^*, \ell')$.

**Proof.** Let $s \in G^*_{F^*}$ such that $\chi \in \mathcal{E}(G^*, [s])$. By Jordan decomposition (see [GM20] Theorem 2.6.22 and Remark 2.6.26), $\chi$ corresponds to a unique $\chi(s) \in \mathcal{E}(C_{G^*}(s)^F, 1)$ lying over some unipotent character $\chi^o(s) \in \mathcal{E}(C_{G^*}(s)^F, 1)$. Notice that

$$
\chi(1)_{\ell} = \frac{|G^F|_{\ell}}{|C_{G^*}(s)^F|_{\ell}} \chi(s)(1)_{\ell}.
$$

Since $\ell$ is large for $G$ we deduce that $\ell$ does not divide $|Z(G^*)_{F^*} : Z(G^*)_{F^*}|$ by [Mal14] Lemma 2.1. Now, since $|C_{G^*}(s)^F : C_{G^*}(s)^F|$ divides $|Z(G^*)_{F^*} : Z(G^*)_{F^*}|$ by [DM20] Lemma 11.2.1 (iii), Clifford’s theorem implies

$$
\chi(1)_{\ell} = \frac{|G^F|_{\ell}}{|C_{G^*}(s)^F|_{\ell}} \chi^o(s)(1)_{\ell}.
$$

(4.5)

Set $H := C_{G^*}^o(s)$ and notice that, by [CE94] Theorem (ii), the block $b(l(\chi^o(s)))$ has defect group $D \in \text{Syl}_e(C_H^o([H, H])_{F^*})$. Since $H = Z^o(H)[H, H]$, it follows that $C_H^o([H, H]) = Z^o(H)$. Thus $D \leq Z(H)_{F^*} \leq Z(H_{F^*})$ and, using [Nav93] Theorem 9.12, we obtain

$$
\chi^o(s)(1)_{\ell} = |H_{F^*} : D|_{\ell}.
$$

(4.6)

This implies

$$
\chi^o(s)(1)_{\ell} = |H_{F^*} : Z^o(H)_{F^*}|_{\ell}.
$$

(4.6)

Combining (4.5) and (4.6) we see that it is enough to show that $Z := Z^o(H)_{F^*}^{\ell} = 1$. To do so, observe that $Z^o(G^*)_{\Phi_e} = Z^o(H)_{\Phi_e}$ by [CE99] Proposition 1.10. In particular, for every $e$-split Levi subgroup $K^*$ of $G^*$ containing $H$, we have $K^* = G^*$. Notice that $H \leq C_{G^*}(Z)$ and that $C_{G^*}(Z)$ is an $e$-split Levi subgroup of $G^*$ by Proposition 2.7 (i). Therefore $C_{G^*}(Z) = G^*$ and $Z \leq Z(G^*)_{F^*}^{\ell} = 1$. This shows that $\chi$ has defect zero. To conclude, we notice that $\chi$ is the only character in its block $B$ while using [Hi690] we obtain $\text{Irr}(B) \cap \mathcal{E}(G^*, \ell') \neq \emptyset$. It follows that $\chi \in \mathcal{E}(G^*, \ell')$. $\square$
5 New conjectures for finite reductive groups

For any prime number \( \ell \), it is expected that the \( \ell \)-modular representation theory of a finite group is strongly determined by its \( \ell \)-local structure. This belief is supported by numerous results and conjectural evidences. If \( G^F \) is a finite reductive group defined over \( \mathbb{F}_q \) with \( \ell \) not dividing \( q \), then its \( \ell \)-local structure is closely related to its \( e \)-local structure where \( e \) is the order of \( q \) modulo \( \ell \) (modulo 4 if \( \ell = 2 \)). For instance, under suitable restrictions on the prime \( \ell \), every \( e \)-split Levi subgroup \( L \) of \( G \) gives rise to an \( \ell \)-local subgroup of \( G^F \). Namely, \( L^F = C_{G^F}(Z(L)^F) \) is the centraliser of an \( \ell \)-subgroup in \( G^F \) (see Lemma 2.5 and Lemma 2.6 (i)). Using this idea, we can then try to determine a link between the \( \ell \)-modular representation theory of \( G^F \) and the \( e \)-local structure of \( G \). In this section we propose new conjectures that can be seen as analogues for finite reductive groups of the so-called counting conjectures. In Section 7 we compare our statements with the counting conjectures and show that they coincide whenever the prime \( \ell \) is large enough.

5.1 Counting characters \( e \)-locally

Let \( G \), \( F \), \( q \), \( \ell \) and \( e \) be as in Notation 2.2. Denote by \( \mathcal{L}(G, F) \), or simply by \( \mathcal{L}(G) \) when \( F \) is clear from the context, the set of descending chains of \( e \)-split Levi subgroups \( \sigma = \{ G = L_0 > L_1 \cdots > L_n \} \) and define the length of \( \sigma \) as \( |\sigma| := n \). We denote by \( L(\sigma) \) the smallest term of the chain \( \sigma \). Consider the subset \( \mathcal{L}(G)_{>0} \) of all chains of positive length. Notice that \( G^F \) acts on \( \mathcal{L}(G) \) and on \( \mathcal{L}(G)_{>0} \) and denote by \( G^F_{\sigma} \) the stabiliser in \( G^F \) of any chain \( \sigma \).

Next, using [CE99, Theorem 2.5] and [KM15, Theorem 3.4], we associate to every block \( b \) of \( G^F_{\sigma} \) a uniquely determined block of \( G^F \). In order to apply [KM15, Theorem 3.4], we assume that \( G \) is an \( F \)-stable Levi subgroup of a simple simply connected group whenever \( \ell \) is bad for \( G \). Let \( L := L(\sigma) \) be the smallest term of \( \sigma \) and consider a block \( b_L \) of \( L^F \) covered by \( b \). By [CE99, Theorem 2.5] and [KM15, Theorem 3.4], we deduce that there exists a unique block of \( G^F \), denoted by \( R^G_{L_F} (b_L) \) (see [CE99, Notation 2.6]), containing all irreducible constituents of \( R^G_{L_F} (\lambda) \) for every parabolic subgroup \( P \) of \( G \) with Levi complement \( L \) and every \( \lambda \in \text{Irr}(b_L) \cap \mathcal{E}(L^F, \ell^\prime) \). Since \( b_L \) is uniquely determined by \( b \) up to \( G^F_{\sigma} \)-conjugation, the block

\[
R^G_{G^F_{\sigma}} (b) := R^G_{L_F} (b_L)
\]

of \( G^F \) is well defined. Now, for every \( \ell \)-block \( B \) of \( G^F \) and \( d \geq 0 \), we denote by \( k^d(B_\sigma) \) the cardinality of the set

\[
\text{Irr}^d(B_\sigma) := \{ \vartheta \in \text{Irr}(G^F_{\sigma}) \mid d(\vartheta) = d, R^G_{G^F_{\sigma}} (bl(\vartheta)) = B \}.
\]

Moreover, we denote by \( k^d(B) \) the cardinality of the set \( \text{Irr}^d(B) \) consisting of the irreducible characters of \( B \) with defect \( d \) and by \( k^d_e(B) \) be the number of \( e \)-cuspidal characters in \( \text{Irr}^d(B) \).

With the above notation, we can present our conjecture which proposes a formula to count the number of characters of a given defect in a block in terms of \( e \)-local data. Notice that, since \( e \)-cuspidal characters are minimal with respect to the \( e \)-structure of \( G \), they should be interpreted as \( e \)-local objects. Here, we use the term \( e \)-local structure to indicate the collection of (proper) \( e \)-split Levi subgroups of \( G \) together with their normalisers and their intersections.
Conjecture 5.1. Let $B$ be an $\ell$-block of $G^F$ and $d \geq 0$. Then

$$k^d(B) = k^d_c(B) + \sum_{\sigma} (-1)^{|\sigma|+1} k^d(B_{\sigma})$$

where $\sigma$ runs over a set of representatives for the action of $G^F$ on $L(G)_{>0}$.

Our statement can be considered as an adaptation of Dade’s Conjecture to finite reductive groups. In fact, in Section 7 we show the two statements coincide when the prime $\ell$ is large for $G$ (see Proposition 7.10). We also notice that by considering the contribution of characters of any defect, we can state a weak version of Conjecture 5.1 which could be interpreted as an analogue of Alperin’s Weight Conjecture to finite reductive groups. In Section 7 we show that Conjecture 5.2 is equivalent to Alperin’s Weight Conjecture for finite reductive groups and large primes. Indeed, in this case these two statements coincide when the prime $\ell$ is large.

Conjecture 5.2. Let $B$ be an $\ell$-block of $G^F$. Then

$$k(B) = k^d_c(B) + \sum_{\sigma} (-1)^{|\sigma|+1} k(B_{\sigma})$$

where $\sigma$ runs over a set of representatives for the action of $G^F$ on $L(G)_{>0}$.

In Section 7 we show that Conjecture 5.2 is equivalent to Alperin’s Weight Conjecture for finite reductive groups and large primes. Indeed, in this case these two statements coincide with both Conjecture 5.1 and Dade’s Conjecture (see Proposition 7.10). As it was the case the Knörr–Robinson reformulation of Alperin’s Weight Conjecture, it is natural to ask whether the alternating sums presented in Conjecture 5.1 and Conjecture 5.2 can be expressed as the Euler characteristic of a chain complex. Such a complex has been constructed by using Bredon cohomology for Alperin’s Weight Conjecture (see [Sym05] and [Lin05]).

Remark 5.3. Assume that $k^d_c(B) \neq 0$ for some $d \geq 0$ and let $\chi$ be an $e$-cuspidal character in $\text{Irr}^d(B)$. If $\ell$ is large for $G$ and $(G, F, e)$-adapted with $Z(G^+)_{\ell^*} = 1$, then Proposition 4.18 implies that $\chi$ has defect zero and therefore $d = 0$ and $\text{Irr}^d(B) = \text{Irr}(B) = \{\chi\}$. As a consequence $k^d(B) = k^d_c(B) = 1 = k(B) = k_c(B)$ and, we must have

$$\sum_{\sigma} (-1)^{|\sigma|} k^d(B_{\sigma}) = 0.$$

We claim that in this case we even have $k(B_{\sigma}) = 0$ for every $\sigma \in L(G)_{>0}$. In fact, if $\sigma$ has smallest term $L : L(\sigma) < G$ and $k^d(B_{\sigma}) > 0$, then there exists a block $b_L$ of $L^F$ such that $L(\sigma)$ contains all constituent of $R^G_L(\lambda)$ for any $\lambda \in \text{Irr}(b_L) \cap E(L^F, \ell')$. Since $\chi$ is the only character in $B$, it follows that $(L, \lambda) \preceq_e (G, \chi)$. Since $\chi$ is $e$-cuspidal, this is a contradiction and hence $k^d(B_{\sigma}) = 0$ for every $\sigma \in L(G)_{>0}$. Notice that, if $\ell$ is $(G, F, e_0)$-adapted for some $e_0 \neq e$, then $G$ has no proper $e$-split Levi subgroups and Conjecture 5.1 holds trivially.

5.2 Introducing $G^F$-block isomorphisms of character triples

Counting conjectures for finite groups provide numerical evidence that is believed to be consequence of an underlying structural theory. In this regard, Broué’s Abelian Defect Group Conjecture
proposes a structural explanation when considering the case of blocks with abelian defect groups. In a similar fashion, although perhaps on a more superficial level, the introduction by Isaacs, Malle and Navarro \cite{MN07} of the so-called inductive conditions for the counting conjectures has initiated a study of stronger conjectures which suggest a way to control Clifford theory via the use of relations on the set of character triples. This idea has been exploited further in \cite{NS14} Theorem 7.1 for the Alperin–McKay Conjecture (see also \cite{Ros22c} for the simpler McKay Conjecture) and in \cite{Spä17} Conjecture 1.2 for Dade’s Conjecture. In this section, we introduce $G^F$-block isomorphisms of character triples in the context of Conjecture 5.1 and Conjecture 5.2. The equivalence relation on character triples that we consider here is denoted by $\sim_{G^F}$ and has been introduced in \cite{Spä17} Definition 3.6. We refer the reader to that paper for further details. Before proceeding further, we mentioned that in order to obtain the conditions on defect groups necessary to define $G^F$-block isomorphisms of character triples we must assume $\ell \in \Gamma(G,F)$.

Recall from Section 3 that, for a finite reductive group $G$, we denote by $CP_e(G,F)$ the set of all $e$-cuspidal pairs $(L,\lambda)$ of $(G,F)$ and by $CP_e(G,F)_e$ the subset of $e$-cuspidal pairs $(L,\lambda)$ with $L < G$. When $\ell \in \Gamma(G,F)$ and $B$ is a block of $G^F$, we define the subsets $CP_e(B)$ and $CP_e(B)_e$ consisting of those pairs $(L,\lambda)$ in $CP_e(G,F)$ and $CP_e(G,F)_e$ respectively such that $bl(\lambda)^{G^F} = B$. Recall that block induction is defined in this case as explained in the discussion preceding Lemma 4.6. As in \cite{BMM93} Definition 2.18, let $AbIrr(L^F)$ be the set of (linear) characters of $L^F$ containing $[L,L]^F$ in their kernel and, for a fixed character $\lambda \in Irr(L^F)$, define

$$Ab(\lambda) := \{\eta | \eta \in AbIrr(L^F)\}$$

and

$$E(G^F, (L, Ab(\lambda))) := \bigcup_{\lambda' \in Ab(\lambda)} E(G^F, (L, \lambda')).$$

By \cite{Bon06} Proposition 12.1, observe that if $\lambda$ is $e$-cuspidal then every character in $Ab(\lambda)$ is $e$-cuspidal. Finally, for every $B \in Bl(G^F)$, $d \geq 0$ and $e \in \{+, -\}$ we define

$$L^d(B)_e := \left\{ (\sigma, M, Ab(\mu), \vartheta) \left| \sigma \in L(G), e \in CP_e(B) \text{ with } M \leq L(\sigma), \vartheta \in Irr(B), \ell | \ell (\sigma)^F, (M, Ab(\mu)) \right. \right\},$$

where $L(G)_e$ is the subset of $L(G)$ consisting of those chains $\sigma$ satisfying $(-1)^{|\sigma|} = e$ and the set $Irr(B, L(G), (M, Ab(\mu)))$ consists of those characters $\vartheta \in Irr(G^{F})$ lying over some character in $L(G)^F, (M, Ab(\mu))$ and such that $d(\vartheta) = d$ and $bl(\vartheta)^{G^F} = B$. Notice that the group $G^F$ acts by conjugation on $L^d(B)_e$ and denote by $L^d(B)_e/G^F$ the corresponding set of $G^F$-orbits. As usual, for $(\sigma, M, Ab(\mu), \vartheta) \in L^d(B)_e$, we denote the corresponding $G^F$-orbit by $(\sigma, M, Ab(\mu), \vartheta)$. Moreover, for every $\omega \in L^d(B)_e/G^F$ we denote by $\omega^*$ the $G^F$-orbit of pairs $(\sigma, \vartheta)$ such that $(\sigma, M, Ab(\mu), \vartheta) \in \omega$ for some $e$-cuspidal pair $(M, \mu)$.

With the notation introduced above, we can now present a more conceptual framework for the conjectures presented in the previous section by considering bijections inducing $G^F$-block isomorphisms of character triples.
Conjecture 5.4. Let $\ell \in \Gamma(G, F)$ and consider a block $B$ of $G^F$ and $d \geq 0$. There exists an $\text{Aut}_F(G^F)_B$-equivariant bijection

$$\Lambda : \mathcal{L}^d(B)_+ \langle G^F \to \mathcal{L}^d(B)_- \langle G^F$$

such that

$$\left(X_{\sigma, \vartheta}, G_{\sigma, \vartheta} \right) \sim_G \left(X_{\rho, \chi}, G_{\rho, \chi} \right)$$

for every $\omega \in \mathcal{L}^d(B)_+ \langle G^F$, any $(\sigma, \vartheta) \in \omega^*$, $(\rho, \chi) \in \Lambda(\omega)^*$ and where $X := G^F \times \text{Aut}_F(G^F)$.

In analogy with the inductive conditions for the counting conjectures, the above statement should be understood as a version of Conjecture 5.1 compatible with Clifford theory and with the action of automorphisms. Although not completely satisfactory from a structural point of view, Conjecture 5.4 suggest a deeper explanation for the numerical phenomena proposed in Conjecture 5.1. By considering the contribution given by characters of any defect, we could introduce $G^F$-block isomorphisms in the context of Conjecture 5.2. In Section 7, we show that Conjecture 5.4 implies Späth’s Character Triple Conjecture for finite reductive groups and large primes (see Proposition 7.13).

We now explain in more details the connection between Conjecture 5.1 and Conjecture 5.4. First, we provide a more explicit description of the blocks and irreducible characters of stabilisers of chains.

Lemma 5.5. Consider a chain of $e$-split Levi subgroups $\sigma \in \mathcal{L}(G)$ with final term $L := L(\sigma)$. If $\ell \in \Gamma(G, F)$, then:

(i) every block of $G_{\sigma}^F$ is $L^F$-regular (see [Nav98, p.210]). In particular, for $b \in \text{Bl}(L^F)$, the induced block $bG_{\sigma}^F$ is defined and is the unique block of $G_{\sigma}^F$ that covers $b$;

(ii) if $\vartheta \in \text{Irr}(G_{\sigma}^F)$, then $b\vartheta G_{\sigma}^F$ is defined and $R_{G_{\sigma}^F}(b\vartheta) = b\vartheta G_{\sigma}^F$;

(iii) assume Hypothesis 4.4. There is a partition of the irreducible characters of $G_{\sigma}^F$ given by

$$\text{Irr}(G_{\sigma}^F) = \bigsqcup_{(M, \mu)/\sim} \text{Irr}(G_{\sigma}^F \mid \mathcal{E}(L^F, (M, \mu)))$$

where the union runs over the $e$-cuspidal pairs $(M, \mu)$ of $L$ up to $G_{\sigma}^F$-conjugation.

Proof. To prove the first statement, set $X := Z^e(L)_{\ell}^F$ and observe that $L^F = C_{G_{\sigma}^F}(X)^F = C_G(X)^F$ by Lemma 2.5 and Lemma 2.6(i). In particular $X \leq O_{\ell}(G_{\sigma}^F)$. If $B \in \text{Bl}(G_{\sigma}^F)$ has defect group $D$, then $X \subseteq D$ by [Nav98, Theorem 4.8]. Thus $C_{G_{\sigma}^F}(D) \leq C_{G_{\sigma}^F}(X) = L^F$ and [Nav98, Lemma 9.20] shows that $B$ is $L^F$-regular. In particular, if the block $B$ covers $b \in \text{Bl}(L^F)$, then $B = bG_{\sigma}^F$ by [Nav98, Theorem 9.19]. This proves (i). Moreover, by [CEF99, Theorem 2.5] we know that $R_{G_{\sigma}^F}(b) = bG_{\sigma}^F$ and so, if $\vartheta \in \text{Irr}(G_{\sigma}^F)$ and $b\vartheta$ covers $b$, we deduce that $R_{G_{\sigma}^F}(b\vartheta) = R_{G_{\sigma}^F}(b) = bG_{\sigma}^F$.

Next, as $\text{Irr}(L^F)$ is the union of the $e$-Harish-Chandra series $\mathcal{E}(L^F, (M, \mu))$ by Corollary 4.12 we deduce that every character $\chi \in \text{Irr}(G_{\sigma}^F)$ lies over some character of an $e$-Harish-Chandra series $\mathcal{E}(L^F, (M, \mu))$, where $(M, \mu)$ is an $e$-cuspidal pair of $L$. To conclude we have to show that, if $(M', \mu')$ is another $e$-cuspidal pair of $L$, then $\text{Irr}(G_{\sigma}^F \mid \mathcal{E}(L^F, (M, \mu)))$ and $\text{Irr}(G_{\sigma}^F \mid \mathcal{E}(L^F, (M', \mu'))) are G_{\sigma}^F$-conjugate. Suppose that $\chi$ is a
character belonging to the intersection of $\text{Irr}(G^F \mid \mathcal{E}(L^F, (M, \mu)))$ and $\text{Irr}(G^F \mid \mathcal{E}(L^F, (M', \mu'))).$

Let $\psi \in \mathcal{E}(L^F, (M, \mu))$ and $\psi' \in \mathcal{E}(L^F, (M', \mu'))$ lie below $\chi$ and consider $g \in G^F$ such that $\psi = \psi'^g$. Then, $\psi \in \mathcal{E}(L^F, (M, \mu)) \cap \mathcal{E}(L^F, (M', \mu'))$ and Corollary 4.12 implies that $(M, \mu) = (M', \mu')^g$, for some $x \in L^F$. Since $gx \in G^F$ the proof is now complete. 

We can now prove Theorem 5.6 as a consequence of the above considerations.

**Theorem 5.6.** Assume Hypothesis 4.1 and consider $B \in \text{Bl}(G^F)$ and $d \geq 0$. If Conjecture 5.4 holds for $B$ and $d$, then Conjecture 5.1 holds for $B$ and $d$.

**Proof.** If there exists a bijection between the sets $\mathcal{L}^d(B)_c/G^F$ and $\mathcal{L}^d(B)_{-}/G^F$, then

$$
\sum_{(\sigma, M, \text{Ab}(\mu))/G^F} (-1)^{|\sigma|} \left| \text{Irr}^d \left( B_\sigma \mid \mathcal{E} \left( (L(\sigma)^F, (M, \text{Ab}(\mu))) \right) \right) \right| = 0 \tag{5.1}
$$

where the sum runs over $G^F$-conjugacy classes of triples $(\sigma, M, \text{Ab}(\mu))$ with $\sigma \in \mathcal{L}(G)$ and $(M, \mu) \in \mathcal{CP}_e(B)_c$ with $M \leq L(\sigma)$. By Lemma 5.5(ii)-(iii) we deduce that

$$
\sum_{(M, \text{Ab}(\mu))/G^F} \left| \text{Irr}^d \left( B_\sigma \mid \mathcal{E} \left( (L(\sigma)^F, (M, \text{Ab}(\mu))) \right) \right) \right| = k^d(B_\sigma) \tag{5.2}
$$

whenever $\sigma \in \mathcal{L}(G)_{>0}$. On the other hand, since we are only considering $(M, \mu) \in \mathcal{CP}_e(B)_c$, the contribution given by the trivial chain $\{G\}$ is

$$
\sum_{(M, \text{Ab}(\mu))/G^F} \left| \text{Irr}^d \left( B \mid \mathcal{E} \left( (G^F, (M, \text{Ab}(\mu))) \right) \right) \right| = k^d(B) - k^d_c(B). \tag{5.3}
$$

Using (5.2) and (5.3), we deduce that (5.1) may be rewritten as

$$
0 = \sum_{\sigma/G^F} (-1)^{|\sigma|} \sum_{(M, \text{Ab}(\mu))/G^F} \left| \text{Irr}^d \left( B_\sigma \mid \mathcal{E} \left( (L(\sigma)^F, (M, \text{Ab}(\mu))) \right) \right) \right|
= k^d(B) - k^d_c(B) + \sum_{\sigma \in \mathcal{L}(G)_{>0}/G^F} (-1)^{|\sigma|} k^d(B_\sigma)
$$

and therefore Conjecture 5.1 holds for $B$ and $d$. 

In the following concluding remark we discuss the definition of the sets of quadruples $\mathcal{L}^d(B)_c$ and show that some of the conditions imposed above are redundant.

**Remark 5.7.** If $(M, \mu) \in \mathcal{CP}_e(B)$ and $\mu' \in \mathcal{Y}(\mu)$, then we have $\mathcal{Y}(\mu) = \mathcal{Y}(\mu')$ although it might happen that $(M', \mu') \notin \mathcal{CP}_e(B)$. On the other hand, let $\sigma \in \mathcal{L}(G)$ with last term $L(\sigma)$ and consider an $e$-cuspidal pair $(M, \mu)$ of $L(\sigma)$. If $\vartheta \in \text{Irr}(G^F \mid \mathcal{E}(L(\sigma)^F, (M, \mathcal{Y}(\mu))))$ and $\text{bl}(\vartheta)^G = B$, then there exists $\mu' \in \mathcal{Y}(\mu)$, so that $\mathcal{Y}(\mu) = \mathcal{Y}(\mu')$, such that $(M, \mu') \in \mathcal{CP}_e(B)$. In fact, there exists $\mu' \in \mathcal{Y}(\mu)$ such that $\vartheta \in \text{Irr}(G^F \mid \mathcal{E}(L(\sigma)^F, (M, \mathcal{Y}(\mu))))$. By Proposition 4.8 every character of $\mathcal{E}(L(\sigma)^F, (M, \mu'))$ is contained in $\text{bl}(\mu')^{L(\sigma)^F}$. Then, applying Lemma 5.5(i) and using the transitivity of block induction, it follows that $\text{bl}(\vartheta) = (\text{bl}(\mu')^{L(\sigma)^F})^{G^F} = \text{bl}(\mu)^{G^F}$. We deduce that $\text{bl}(\mu')^{G^F} = \text{bl}(\mu)^{G^F} = B$ and hence $(M, \mu') \in \mathcal{CP}_e(B)$. In particular, we have

$$
\mathcal{L}^d(B)_c = \left\{ (\sigma, M, \mathcal{Y}(\mu), \vartheta) \mid \sigma \in \mathcal{L}(G)_{>0}, (M, \mu) \in \mathcal{CP}_e(L(\sigma)^F), \vartheta \in \text{Irr}^d(G^F \mid \mathcal{E}(L(\sigma)^F, (M, \mathcal{Y}(\mu)))) \text{ with } \text{bl}(\vartheta)^G = B \right\}.
$$
5.3 A parametrisation of $e$-Harish-Chandra series

In section 4 we have shown how to describe the characters in a block of a finite reductive group in terms of $e$-Harish-Chandra theory. More precisely, Theorem 4.13 shows that the set of characters of a block $B$ of $G^F$ is the disjoint union of $e$-Harish-Chandra series $E(G^F, (L, \lambda))$ associated to certain $e$-cuspidal pairs $(L, \lambda)$. The next natural step to understand the distribution of characters in the block $B$ is to find a parametrisation of the characters in each series. Inspired by the results of [BMM93] and by classical Harish-Chandra theory, we propose a parametrisation of the series $E(G^F, (L, \lambda))$ in terms of data analogue to the one encoded in the relative Weyl group $W_G(L, \lambda)^F$. At the same time, this parametrisation suggests an explanation for the Clifford theoretic and cohomological requirements imposed by the inductive conditions for the counting conjectures.

**Parametrisation 5.8.** Let $\ell \in \Gamma(G, F)$ and consider an $e$-cuspidal pair $(L, \lambda)$ of $G$. There exists a defect preserving $\operatorname{Aut}_F(G^F)_{(L, \lambda)}$-equivariant bijection

$$\Omega_{(L, \lambda)}^G : E(G^F, (L, \lambda)) \to \operatorname{Irr}(N_G(L)^F | \lambda)$$

such that

$$(X_\vartheta, G^F, \vartheta) \sim_{G^F} \left(N_{X_\vartheta}(L), N_{G^F}(L), \Omega_{(L, \lambda)}^G(\vartheta)\right)$$

for every $\vartheta \in E(G^F, (L, \lambda))$ and where $X := G^F \rtimes \operatorname{Aut}_F(G^F)$.

We say that Parametrisation 5.8 holds for $(G, F)$ at the prime $\ell$ if it holds for every $e$-cuspidal pair $(L, \lambda)$ of $G$ where $q$ is the prime power associated to $F$ and $e$ is the order of $q$ modulo $\ell$.

As we have said before, the above parametrisation should provide an explanation for the inductive conditions for the counting conjectures for finite reductive groups. Analogously, in Section 6 we show that Conjecture 5.4 and hence Conjecture 5.1 and Conjecture 5.2 holds once we assume the existence of Parametrisation 5.8 (see Theorem 6.13). We are then left to prove Parametrisation 5.8. The results of [Ros22b] shows that the bijections $\Omega_{(L, \lambda)}^G$ can be constructed once we assume certain technical conditions on the extendibility of characters of $e$-split Levi subgroups. These conditions also appear in the proofs of the inductive conditions for the McKay, the Alperin–McKay and the Alperin Weight conjectures (see [MT16], [CS17a], [CS17b], [CS19], [RS20], [Spä21], [Bro22]).

**Remark 5.9.** To conclude this section we derive an interesting consequence of Parametrisation 5.8. For every $e$-cuspidal pair $(L, \lambda)$ of $G$ and any $d \geq 0$ we denote by $k^d(G^F, (L, \lambda))$ the number of characters $\chi \in E(G^F, (L, \lambda))$ with $d(\chi) = d$ and by $k^d(N_G(L)^F, \lambda)$ the number of characters $\psi \in \operatorname{Irr}(N_G(L)^F | \lambda)$ with $d(\psi) = d$. Since the bijection $\Omega_{(L, \lambda)}^G$ preserves the defect of characters, assuming Parametrisation 5.8 we obtain

$$k^d(G^F, (L, \lambda)) = k^d(N_G(L)^F, \lambda) \quad (5.4)$$

On the other hand, under Hypothesis 4.11 the partition given by Theorem 4.13 implies that

$$k^d(B) = k^d_c(B) + \sum_{(L, \lambda)} k^d(G^F, (L, \lambda)) \quad (5.5)$$

for any block $B$ of $G^F$ and where $(L, \lambda)$ runs over a set of representatives for the action of $G^F$ on...
Then, combining (5.4) and (5.5) we obtain
\begin{equation}
k^d(B) = k_c^d(B) + \sum_{(L,\lambda)} k^d\left(NG(L)^F, \lambda\right)
\end{equation}

where as before \((L,\lambda)\) runs over a set of representatives for the action of \(G^F\) on \(\mathcal{CP}_e(B)^<_c\). The formula given in (5.6) suggests another way of counting the number \(k^d(B)\) in terms of \(e\)-local data. In particular, if we believe Conjecture 5.1, then under the above hypothesis we must have
\begin{equation}
\sum_{(L,\lambda)} k^d\left(NG(L)^F, \lambda\right) = \sum_\sigma (-1)^{|\sigma|+1} k^d(B_\sigma)
\end{equation}

where \((L,\lambda)\) and \(\sigma\) run over a set of representatives for the action of \(G^F\) on \(\mathcal{CP}_e(B)^<_c\) and on \(\mathcal{L}(G)_{>0}\) respectively.

6 Counting characters via \(e\)-Harish-Chandra theory

The results obtained in Section 4 together with the parametrisation proposed in Section 5.3 constitute crucial properties of \(e\)-Harish-Chandra theory. As an application of these powerful tools we show that Conjecture 5.4, and hence Conjecture 5.1 and Conjecture 5.2, holds if we assume Parametrisation 5.8 (see Theorem 6.13). First, we prove some preliminary showing how to lift isomorphisms of character triples.

6.1 Bijections and \(N\)-block isomorphic character triples

The following proposition is an adaptation of [Ros22a, Proposition 2.10] to finite reductive groups. Recall that, for \(Y \triangleleft X\) and \(S \subseteq \text{Irr}(Y)\), we denote by \(\text{Irr}(X/S)\) the set of irreducible characters of \(X\) whose restriction to \(Y\) has an irreducible constituent contained in \(S\). Moreover, we define \(X_S := \{x \in X \mid S^x = S\}\).

**Proposition 6.1.** Let \(K \leq G \leq A\) be finite groups with \(G \triangleleft A\), consider \(A_0 \leq A\), and set \(H_0 := H \cap A_0\) for every \(H \leq A\). Consider \(S \subseteq \text{Irr}(K)\) and \(S_0 \subseteq \text{Irr}(K_0)\) and suppose there exists \(K \leq V \leq X \leq N_A(K)\) and \(U \leq X_0\) such that:

(i) \(V \leq X_S\). Moreover, if \(x \in X\) and \(S \cap S^x \neq \emptyset\), then \(x \in V\);

(ii) \(U \leq X_0, S_0\). Moreover, if \(x \in X_0\) and \(S_0 \cap S_0^x \neq \emptyset\), then \(x \in K_0U\);

(iii) \(V = KU\).

Assume there exists a \(U\)-equivariant bijection
\[
\Psi : S \rightarrow S_0
\]
such that
\[
(X_{\vartheta}, K, \vartheta) \sim_K \left(X_{0, \vartheta}, K_0, \Psi(\vartheta)\right)
\]
for every \(\vartheta \in S\). If \(K \leq J \leq X \cap G\) and \(C_X(Q) \leq X_0\) for every radical \(\ell\)-subgroup \(Q\) of \(J_0\), then there exists an \(N_U(J)\)-equivariant bijection
\[
\Phi_J : \text{Irr}(J|S) \rightarrow \text{Irr}(J_0|S_0)
\]
such that

\[ (N_X(J, \chi, J, \chi) \sim_J (N_{X_0}(J, \chi, J_0, \Phi_J(\chi))) \]

for every \( \chi \in \text{Irr}(J \mid S) \).

\textbf{Proof.} Consider an \( N_U(J) \)-transversal \( S \) in \( S \) and define \( S_0 := \{ \Psi(\vartheta) \mid \vartheta \in S \} \). Since \( \Psi \) is \( U \)-equivariant, it follows that \( S_0 \) is an \( N_U(J) \)-transversal in \( S_0 \). For every \( \vartheta \in S \), with \( \vartheta_0 := \Psi(\vartheta) \in S_0 \), we fix a pair of projective representations \( (\mathcal{P}(\vartheta), \mathcal{P}(\vartheta_0)) \) giving \( (X_\vartheta, K, \vartheta) \sim_K (X_\vartheta, K_1, \vartheta_0) \).

Now, let \( T \) be an \( N_U(J) \)-transversal in \( \text{Irr}(J \mid S) \) such that every character \( \chi \in T \) lies above a character \( \vartheta \in S \) (this can be done by the choice of \( S \)). Moreover, using Clifford’s theorem together with hypotheses (i) and (iii), it follows that every \( \chi \in T \) lies over a unique \( \vartheta \in S \).

For \( \chi \in T \) lying over \( \vartheta \in S \), let \( \psi \in \text{Irr}(J_\vartheta \mid \vartheta) \) be the Clifford correspondent of \( \chi \) over \( \vartheta \). Set \( \vartheta_0 := \Psi(\vartheta) \in S_0 \) and consider the \( N_U(J_\vartheta) \)-equivariant bijection \( \sigma_J : \text{Irr}(J_\vartheta \mid \vartheta) \to \text{Irr}(J_{0, \vartheta} \mid \vartheta_0) \) induced by our choice of projective representations \( (\mathcal{P}(\vartheta), \mathcal{P}(\vartheta_0)) \). Let \( \psi_0 := \sigma_J(\psi) \). Observe that \( J_{0, \vartheta_0} = J_{0, \vartheta} \). To see this, notice that \( U_\vartheta = U_{\vartheta_0} \) since \( \Psi \) is \( U \)-equivariant and that \( J_{0, \vartheta_0} \leq K_0U \) by (ii) above. Therefore \( J_{0, \vartheta_0} \leq J_{0, \vartheta} \). On the other hand, since \( (J \cap U)_\vartheta = (J \cap U)_{\vartheta_0} \) because \( \Psi \) is \( U \)-equivariant and noticing that \( J_{0, \vartheta} \leq J_0 \cap V = K_0(J \cap U) \) by using (iii), it follows that \( J_{0, \vartheta} \leq J_{0, \vartheta_0} \).

Now \( \Phi_J(\chi) := \psi^{\vartheta_0} \) is irreducible by the Clifford correspondence. We define

\[ \Phi_J(\chi^x) := \Phi_J(\chi^x) \]

for every \( \chi \in T \) and \( x \in N_U(J) \). This defines an \( N_U(J) \)-equivariant bijection \( \Psi : \text{Irr}(J \mid S) \to \text{Irr}(J_0 \mid S_0) \).

To prove the condition on character triples, consider \( \chi \in \text{Irr}(J \mid S) \), \( \vartheta \in \text{Irr}(\chi_K \cap S) \), \( \psi \in \text{Irr}(J_\vartheta \mid \vartheta) \) and \( \vartheta_0 := \Psi(\vartheta) \), \( \psi_0 := \sigma_J(\psi) \) and \( \chi_0 := \Phi_J(\chi) \) as in the previous paragraph. Since \( (X_\vartheta, K, \vartheta) \sim_K (X_\vartheta, K_1, \vartheta_0) \), [Ros22a Proposition 2.9 (ii)] implies that

\[ (N_{X_\vartheta}(J_\vartheta, \chi_\vartheta, J_\vartheta, \psi) \sim_{J_\vartheta} (N_{X_0\vartheta}(J_\vartheta, \psi, J_{0, \vartheta}, \psi_0)) \]

and, because \( N_X(J_\vartheta) \leq N_{X_\vartheta}(J_\vartheta) \), [Spa17 Lemma 3.8] implies

\[ (N_X(J_\vartheta, \vartheta, \psi, J_\vartheta, \psi) \sim_{J_\vartheta} (N_{X_0}(J_\vartheta, \psi, J_{0, \vartheta}, \psi_0)). \] (6.1)

To conclude, observe that by hypothesis we have

\[ C_{N_X(J_\chi)}(Q) \leq N_{X_\chi}(J_\chi) \]

for every \( \chi_0 \in \text{Irr}(J_0 \mid S) \) and \( Q \in \delta(\text{bl}(\chi_0)) \) and therefore we can apply [Ros22a Proposition 2.8] which, together with (6.1), yields

\[ (N_X(J_\chi, J_\chi, J_\chi) \sim_J (N_{X_0}(J_\chi, J_0, \chi_0)). \]

The proof is now complete. \( \Box \)

\textbf{Remark 6.2.} Consider the setup of Proposition 6.1. Then, the bijection \( \Phi_J \) is defect preserving if and only if \( \Psi \) is defect preserving.
Proof. For \( \chi \in \text{Irr}(J \mid \mathcal{S}) \), let \( \psi \) be the Clifford correspondent of \( \chi \) over some \( \vartheta \in \text{Irr}(\chi_K) \cap \mathcal{S} \) and let \( \psi_0 := \sigma_{\vartheta,0}(\psi) \) and \( \vartheta_0 := \Psi(\vartheta) \). If \( \chi_0 := \Phi_{\vartheta}(\chi) = \psi_0^{\vartheta_0} \), then \( d(\chi) = d(\psi) \) and \( d(\chi_0) = d(\psi_0) \). By [Ros22a, Proposition 2.9 (iii)] we deduce that \( d(\psi) - d(\psi_0) = d(\vartheta) - d(\vartheta_0) \). \( \square \)

In [Spä17] Lemma 3.8 (c) it is shown that \( N \)-block isomorphisms of character triples are compatible with the action of inner automorphisms. It is straightforward to extend this compatibility to arbitrary automorphisms.

**Lemma 6.3.** Suppose that \( (H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2) \) with \( G = H_1 N = H_2 N \). If \( \gamma \in \text{Aut}(G) \), then \( (H_1^\gamma, M_1^\gamma, \vartheta_1^\gamma) \sim_{N\gamma} (H_2^\gamma, M_2^\gamma, \vartheta_2^\gamma) \).

**Proof.** The claim follows directly from the definition of \( \sim_N \) (see [Spä17] Definition 3.6]). \( \square \)

### 6.2 Proof of Theorem F

We now start working towards a proof of Theorem F. In order to apply the results on \( e \)-Harish-Chandra theory obtain in Section 4, we assume throughout this section that Hypothesis 4.1 holds (see [MT11, Proposition 12.14]). By [Mar91, Proposition 1.4.10], we deduce that \( \text{Aut}_{\mathcal{F}}(K_0^F)_{(L_0, \lambda_0)} \)-equivariant bijection

\[
\Omega_{K_0}^{K_0} : \mathcal{E}(K_0^F, (L_0, \lambda_0)) \to \text{Irr}(N_{K_0}^F(L_0)^F | \lambda_0)
\]

such that

\[
(Y_0, K_0^F, \vartheta) \sim_{K_0^F} \left( N_{Y_0}^F(L_0), N_{K_0}^F(L_0), \Omega_{(L_0, \lambda_0)}^{K_0} \right)
\]

for every \( \vartheta \in \mathcal{E}(K_0^F, (L_0, \lambda_0)) \) and where \( Y := K_0^F \times \text{Aut}_{\mathcal{F}}(K_0^F) \).

**Proposition 6.5.** Assume Hypothesis and suppose that \( G \) is simply connected. Consider an \( e \)-split Levi subgroup \( K \) of \( G \) and suppose that Parametrisation holds at the prime \( \ell \) for every irreducible rational component of \( (K, F) \). Let \( K_0 := [K, K] \) and consider an \( e \)-cuspidal pair \( (L_0, \lambda_0) \) of \( K_0 \). Then there exists a defect preserving \( \text{Aut}_{\mathcal{F}}(K_0^F)_{(L_0, \lambda_0)} \)-equivariant bijection

\[
\Omega_{K_0}^{K_0} : \mathcal{E}(K_0^F, (L_0, \lambda_0)) \to \text{Irr}(N_{K_0}^F(L_0)^F | \lambda_0)
\]

such that

\[
(Y_0, K_0^F, \vartheta) \sim_{K_0^F} \left( N_{Y_0}^F(L_0), N_{K_0}^F(L_0), \Omega_{(L_0, \lambda_0)}^{K_0} \right)
\]

for every \( \vartheta \in \mathcal{E}(K_0^F, (L_0, \lambda_0)) \) and where \( Y := K_0^F \times \text{Aut}_{\mathcal{F}}(K_0^F) \).

**Proof.** Since \( G \) is simply connected, we deduce that \( K_0 \) is a simply connected semisimple group (see [MT11 Proposition 12.14]). By [Mar91 Proposition 1.4.10], \( K_0 \) is the direct product of simple algebraic groups \( K_1, \ldots, K_n \) and the action of \( F \) induces a permutation on the set of simple components \( K_i \). For every orbit of \( F \) we denote by \( H_j, j = 1, \ldots, t \), the direct product of the simple components in such an orbit. Then \( H_j \) is \( F \)-stable and

\[
K_0^F = H_1^F \times \cdots \times H_t^F,
\]
where by abuse of notation we denote the restriction of $F$ to $H_j$ again by $F$. Observe that the $(H_j, F)$’s are the irreducible rational components of $(K, F)$. Define $M_j := L_0 \cap H_j$ and observe that $M_j$ is an $e$-split Levi subgroup of $H_j$ and that

$$L_0^F = M_1^F \times \cdots \times M_t^F.$$  

Then, we can write $\lambda_0 = \mu_1 \times \cdots \times \mu_t$ with $\mu_1 \in \text{Irr}(M_1^F)$. As $R_{L_0}^{K_0} = R_{M_1}^{H_1} \times \cdots \times R_{M_l}^{H_l}$ (see [DM91, Proposition 10.9 (ii)]), it follows that $(M_j, \mu_j)$ is an $e$-cuspidal pair of $H_j$ for every $j = 1, \ldots, t$ and, using our assumption, there exist bijections

$$\Omega_{(M_j, \mu_j)}^H : \mathcal{E} (H_j^F, (M_j, \mu_j)) \to \text{Irr} (N_{H_j}(M_j)^F | \mu_j)$$  

as in Parametrisation [5.8]. Since $R_{L_0}^{K_0} = R_{M_1}^{H_1} \times \cdots \times R_{M_l}^{H_l}$, we deduce that $\mathcal{E}(K_0^F, (L_0, \lambda_0))$ coincides with the set of characters of the form $\vartheta_1 \times \cdots \times \vartheta_t$ with $\psi_j \in \mathcal{E}(H_j^F, (M_j, \mu_j))$, while it is not hard to see that $\text{Irr}(N_{K_0}(L_0)^F | \lambda_0)$ coincides with the set of characters of the form $\xi_1 \times \cdots \times \xi_t$ with $\xi_j \in \text{Irr}(N_{H_j}(M_j)^F | \mu_j)$. Hence, we obtain a bijection

$$\Omega_{(L_0, \lambda_0)}^{K_0} : \mathcal{E}(K_0^F, (L_0, \lambda_0)) \to \text{Irr}(N_{K_0}(L_0)^F | \lambda_0)$$

$$\vartheta_1 \times \cdots \times \vartheta_t \mapsto \Omega_{(M_1, \mu_1)}^{H_1}(\vartheta_1) \times \cdots \times \Omega_{(M_l, \mu_l)}^{H_l}(\vartheta_t).$$

We now show that $\Omega_{(L_0, \lambda_0)}^{K_0}$ satisfies the required properties.

First, consider the partition $\{1, \ldots, t\} = \bigsqcup_l A_l$ given by $j, k \in A_l$ if there exists a bijective morphism $\varphi : H_j \to H_k$ commuting with $F$ such that $\varphi(M_j, \mu_j) = (M_k, \mu_k)$. Fix $j_l \in A_l$. By Lemma 6.3 we may assume without loss of generality that

$$K_0^F = \bigtimes_l H_{A_l}^F$$

and

$$\Omega_{(L_0, \lambda_0)}^{K_0} = \bigtimes_l \Omega_{(M_1, \mu_1)}^{H_{A_l}}$$

where $H_{A_l} := H_{A_l}^{(M_1)}$, $M_{A_l} := M_{A_l}^{(M_1)}$, $\mu_{A_l} = \mu_{A_l}^{(M_1)}$ and $\Omega_{(M_1, \mu_1)}^{H_{A_l}} := (\Omega_{(M_1, \mu_1)}^{(H_{A_l})})^{(\mu_1)}$. Fix $\vartheta = \times_l \vartheta_{A_l}$, with $\vartheta_{A_l} \in \mathcal{E}(H_{A_l}^F, (M_{A_l}, \mu_{A_l}))$, and write $\xi := \Omega_{(L_0, \lambda_0)}^{K_0}(\vartheta) = \times_l \xi_{A_l}$ with $\xi_{A_l} = \Omega_{(M_1, \mu_1)}^{H_{A_l}}(\vartheta_{A_l})$. Then, noticing that $\text{Aut}_F(K_0^F) = \bigtimes_l \text{Aut}_F(H_{A_l}^F)$, by [Spa17, Theorem 5.1] it is enough to check that

$$\left( Y_{A_l, \vartheta_{A_l}, H_{A_l}^F, \vartheta_{A_l}} \right) H_{A_l}^F \left( N_{H_{A_l}}(M_{A_l}), N_{H_{A_l}}(M_{A_l})^F, \xi_{A_l} \right)$$

where $Y_{A_l} := H_{A_l}^F \rtimes \text{Aut}_F(H_{A_l}^F)$. To prove (6.3), observe that $\vartheta_{A_l}$ is $\text{Aut}_F(H_{A_l}^F)_{(M_{A_l}, \mu_{A_l})}$-conjugate to a character of the form $x_u \vartheta_u$ such that for every $u, v$ we have either $\vartheta_u = \vartheta_v$ or $\vartheta_u$ and $\vartheta_v$ are not $\text{Aut}_F(H_{A_l}^F)$-conjugate. By Lemma 6.3 we may assume without loss of generality that $\vartheta_u = \times_u \nu^{m_u}_u$, where for every $u \neq v$ the characters $\nu_u$ and $\nu_v$ are distinct and not $\text{Aut}_F(H_{A_l})$-conjugate while $m_u$ are non-negative integers such that $|A_l| = \sum_u m_u$. Then

$$\text{Aut}_F(H_{A_l}^F) \vartheta_{A_l} = \bigtimes_u (\text{Aut}_F(H_{A_l})_{\nu_u} : S_{m_u})$$  

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Proof. Suppose that \( s \) holds. Moreover, notice that \( Y \) and define the set \( \text{Levi subgroup} \) for every \( \ell \). We now prove an easy lemma which we use to combine bijections \( \Omega_{K_0}^{(L_0, \lambda_0)} \) given by Proposition \( 6.5 \).

**Lemma 6.6.** Let \( X \leq Y \leq Z \) be finite groups with \( X, Y \leq Z \) and \( Y/X \) abelian. Consider \( \eta \in \text{Irr}(Y) \) and define the set \( \mathcal{Y} := \{ \eta \nu \mid \nu \in \text{Irr}(Y/X) \} \). If \( z \in Z \) and \( \mathcal{Y}^z \cap \mathcal{Y} \neq \emptyset \), then \( \mathcal{Y}^z = \mathcal{Y} \).

**Proof.** Suppose that \( \eta \nu \in \mathcal{Y}^z \cap \mathcal{Y} \), then there exists \( \nu_1 \in \text{Irr}(Y/X) \) such that \( \eta \nu = (\eta \nu_1)^z \). Since \( Y/X \) is abelian we deduce that \( \eta^z = \eta (\nu_1)^{-1} \). Now, if \( \eta \nu_2 \in \mathcal{Y} \), then \( (\eta \nu_2)^z = \eta^z (\nu_1)^{-1} \nu_2 \). Noticing that \( (\nu_1)^{-1} \nu_2 \in \text{Irr}(Y/X) \), we conclude that \( \mathcal{Y}^z \in \mathcal{Y} \) and the result follows.

For every \( \ell \)-split Levi subgroup \( L_0 \) of a connected reductive group \( K_0 \) and every subset \( \mathcal{Y}_0 \subset \text{Irr}(L_0) \) of \( \ell \)-cuspidal characters, we define \( \mathcal{E}(K_0^F, (L_0, \mathcal{Y}_0)) \) to be the union of the \( \ell \)-Harish-Chandra series \( \mathcal{E}(K_0^F, (L_0, \nu)) \) where \( \nu \in \mathcal{Y}_0 \).

**Corollary 6.7.** Assume Hypothesis 4.1 and suppose that \( G \) is simply connected. Consider an \( \ell \)-split Levi subgroup \( K \) of \( G \) and suppose that Parametrisation 5.8 holds at the prime \( \ell \) for every irreducible rational component of \( (K, F) \). Let \( (L, \lambda) \) be an \( \ell \)-cuspidal pair of \( K \), set \( K_0 := [K, K] \) and \( L_0 := L \cap K_0 \) and consider \( \lambda_0 \in \text{Irr}(L_0^F) \). Define \( \mathcal{Y}_0 := \{ \lambda_0 \xi \mid \xi \in \text{Irr}(L_0^F/[L, L]^F) \} \). Then there exists a defect preserving \( \text{Aut}_F(K_0^F)_{K, L, \mathcal{Y}_0} \)-equivariant bijection

\[
\Psi_{K_0}^{(L_0, \lambda_0)} : \mathcal{E}(K_0^F, (L_0, \mathcal{Y}_0)) \to \text{Irr}(\text{N}_{K_0}(L_0)^F | \mathcal{Y}_0)
\]

such that

\[
(Y_\vartheta, K_0^F, \vartheta) \sim_{K_0^F}(N_{Y_\vartheta}(L_0), N_{K_0}(L_0)^F, \Psi_{K_0}^{(L_0, \lambda_0)}(\vartheta))
\]

for every \( \vartheta \in \mathcal{E}(K_0^F, (L_0, \mathcal{Y}_0)) \) and where \( Y := K_0^F \rtimes \text{Aut}_F(K_0^F) \).

**Proof.** First observe that for \( \lambda_0 \xi \in \mathcal{Y}_0 \) the pair \( (L_0, \lambda_0 \xi) \) is \( \ell \)-cuspidal in \( K_0 \) (see [Bon06] Proposition 12.1)). Moreover, notice that \( L = Z(K)L_0 \) and therefore \( N_K(L_0) = N_K(L) \). Let \( T \) be an \( N_{K_0}(L)_0 \rtimes \text{Aut}_F(G^F)_{K, L, \lambda_0 \xi} \)-transversal in \( \mathcal{Y}_0 \). For each \( \lambda_0 \xi \in T \) consider an \( \text{Aut}_F(G^F)_{K, L, \lambda_0 \xi} \)-transversal \( T_{\lambda_0 \xi} \) in \( \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi)) \) and define \( \mathcal{T} \) as the union of the sets \( T_{\lambda_0 \xi} \) with \( \lambda_0 \xi \in T \).

We claim that \( \mathcal{T} \) is an \( \text{Aut}_F(G^F)_{K, L, \mathcal{Y}_0} \)-transversal in

\[
\mathcal{E}(K_0^F, (L_0, \mathcal{Y}_0)).
\]

Let \( \chi \in \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi)) \) with \( \xi \in \text{Irr}(L_0^F/[L, L]^F) \) and consider the unique \( \lambda_0 \xi \in \mathcal{T} \) such that \( (\lambda_0 \xi)^{x,y} = \lambda_0 \xi^x \) for some \( x \in N_{K_0}(L)_0 \) and \( y \in \text{Aut}_F(G^F)_{K, L, \mathcal{Y}_0} \). Noticing that \( \chi^y = \chi^{x,y} \in \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi)) \) we can find a unique \( \xi^{y}_z \in \mathcal{T}_{\lambda_0 \xi} \) such that \( \chi^{y,z} = \xi^{y}_z \) for some \( z \in \text{Aut}_F(G^F)_{K, L, \lambda_0 \xi} \). By Lemma 6.6 we obtain \( \text{Aut}_F(G^F)_{K, L, \lambda_0 \xi} \leq \text{Aut}_F(G^F)_{K, L, \mathcal{Y}_0} \) and hence \( \mathcal{T} = \text{Aut}_F(G^F)_{K, L, \mathcal{Y}_0} \). Next, for \( i = 1, 2 \) consider \( \chi_i \in T_{\lambda_0 \xi_i} \), with \( \lambda_0 \xi_i \in T \) such that \( \chi_1 = \chi_2^{y} \) with \( y \in \text{Aut}_F(G^F)_{K, L, \mathcal{Y}_0} \). In particular \( \chi_1 \in \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi_i)) \cap \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi_2)^y) \) and Proposition 4.10 implies that \( \lambda_0 \xi_1 = \lambda_0 \xi_2 \)."
Define $(\lambda_0 \xi_2)^{yx}$ for some $x \in N_{K_0}(L)^F$. Moreover, Lemma [6.6] yields $x \in N_{K_0}(L)^F$ and by the choice of $T$ it follows that $\lambda_0 \xi_1 = \lambda_0 \xi_2$. Now $yx \in \text{Aut}_K(G^F)_{K,L,\lambda_0 \xi_1}$ satisfies $\chi_1 = \chi_2^{yx}$ and the choice of $T_{\lambda_0 \xi_2}$ implies that $\chi_1 = \chi_2$. This proves the claim.

Next, using Proposition 6.5, for every $\lambda_0 \xi \in T$, $\chi \in T_{\lambda_0 \xi}$ and $x \in \text{Aut}_K(G^F)_{K,L,\lambda_0}$ we define

$$\psi_{K_0, (L_0, \lambda_0)}^F \left( \chi \right) := \Omega_{K_0, (L_0, \lambda_0)}^F \left( \chi \right).$$

Noticing that $\psi_{K_0, (L_0, \lambda_0)}^F \left( T \right)$ is an $\text{Aut}_K(G^F)_{K,L,\lambda_0}$-transversal in

$$\text{Irr} \left( N_{K_0}(L)^F \big| \lambda_0 \right)$$

we deduce that $\psi_{K_0, (L_0, \lambda_0)}^F$ is an $\text{Aut}_K(G^F)_{K,L,\lambda_0}$-equivariant bijection. The remaining properties follow directly from the corresponding properties of the bijections $\Omega_{K_0, (K_0, \lambda_0 \xi)}^F$ given by Proposition 6.5.

Using [Spä17, Theorem 5.3] we rewrite the relations on character triples given by Corollary 6.7 replacing $K_0 \times \text{Aut}_E(K_0^F)$ with $(G^F \times \text{Aut}_E(G^F))_K$.

**Corollary 6.8.** Consider the setup of Corollary 6.7. Then

$$\left( X_{\vartheta}, K_6^F, \vartheta \right) \sim_{K_0^F} \left( N_{X_{\vartheta}}(L_0), N_{K_0}(L_0), \psi_{K_0, (L_0, \lambda_0)}^F \left( \vartheta \right) \right)$$

for every $\vartheta \in \mathcal{E}(K_0^F, (L_0, \lambda_0))$ and where $X := (G^F \times \text{Aut}_E(G^F))_K$.

**Proof.** Fix $\vartheta$ as in the statement, let $Y := K_0^F \times \text{Aut}_E(K_0^F)$ and consider the canonical maps

$$\epsilon : Y_{\vartheta} \to \text{Aut}(K_0^F)$$

and

$$\varpi : X_{\vartheta} \to \text{Aut}(K_0^F).$$

Define $U := \varpi^{-1}(\epsilon(X_{\vartheta})) \leq Y_{\vartheta}$. By Corollary 6.7 we know that

$$\left( U_{\vartheta}, K_0^F, \vartheta \right) \sim_{K_0^F} \left( N_{U_{\vartheta}}(L_0), N_{K_0}(L_0), \psi_{K_0, (L_0, \lambda_0)}^F \left( \vartheta \right) \right)$$

and applying [Spä17, Lemma 3.8] we obtain

$$\left( U_{\vartheta}, K_0^F, \vartheta \right) \sim_{K_0^F} \left( N_{U_{\vartheta}}(L_0), N_{K_0}(L_0), \psi_{K_0, (L_0, \lambda_0)}^F \left( \vartheta \right) \right).$$

Now [Spä17, Theorem 5.3] implies that

$$\left( X_{\vartheta}, K_0^F, \vartheta \right) \sim_{K_0^F} \left( N_{X_{\vartheta}}(L_0), N_{K_0}(L_0), \psi_{K_0, (L_0, \lambda_0)}^F \left( \vartheta \right) \right)$$

and this concludes the proof.

Our next goal is to lift the bijection $\psi_{K_0, (L_0, \lambda_0)}^F$ to a similar bijection $\psi_{K, (L, \lambda)}^F$. To do so we need the following preliminary result.
Lemma 6.9. Consider the setup of Corollary 6.7 and recall that $\text{Ab}(\lambda) = \{\lambda \eta \mid \eta \in \text{AbIrr}(L^F)\}$. Then

$$\text{Irr}(K^F \mid \mathcal{E}(K_0^F, (L_0, \gamma_0))) = \mathcal{E}(K^F, (L, \text{Ab}(\lambda)))$$

and

$$\text{Irr}(N_K(L)^F | \gamma_0) = \text{Irr}(N_K(L)^F | \text{Ab}(\lambda)).$$

Proof. Let $\lambda_0 \xi \in \gamma_0$ and consider $\chi \in \text{Irr}(K^F \mid \mathcal{E}(K_0^F, (L_0, \lambda_0 \xi)))$. Since $L^F/[L, L]^F$ is abelian, $\xi$ has an extension $\xi \in \text{AbIrr}(L^F)$. By [GM20 Corollary 3.3.25] and [Isa76 Problem 5.3] we obtain

$$\text{Ind}_{K_0^F}^{K^F}(R_{L_0}(\lambda_0 \xi)) = \text{R}_{L_0}^{K^F}(\text{Ind}_{L_0}^{K}(\lambda_0 \xi)) = \text{R}_{L_0}^{K}(\text{Ind}_{L_0}^{K}(\lambda_0 \xi)).$$

Then, by [Isa76 Problem 6.2] there exists $\eta \in \text{Irr}(L^F/L_0^F)$ such that $\chi \in \mathcal{E}(K^F, (L, \lambda \eta \xi))$ with $\eta \xi \in \text{AbIrr}(L^F)$. Conversely, assume that $\chi \in \mathcal{E}(K^F, (L, \lambda \eta))$ with $\lambda \eta \in \text{Ab}(\lambda)$. Applying [GM20 Corollary 3.3.25], we obtain

$$\text{Res}_{K_0^F}^{K^F}(R_{L_0}^{K}(\lambda \eta)) = R_{L_0}^{K}(\text{Res}_{L_0}^{K}(\lambda \eta)).$$

By Clifford’s theorem we deduce that $\text{Res}_{K_0^F}^{K^F}(\chi)$ has an irreducible constituent $R_{L_0}^{K}(\lambda_0 \xi)$ for some $g \in L^F$ and $\xi := \eta L_0^F \in \text{Irr}(L_0^F/[L, L]^F)$. This proves the first equality.

Next, consider $\psi \in \text{Irr}(N_K(L)^F \mid \lambda \eta)$ with $\lambda \eta \in \text{Ab}(\lambda)$. Since $\lambda \eta$ lies above $\lambda_0 \xi$, with $\xi := \eta L_0^F \in \text{Irr}(L_0^F/[L, L]^F)$, we deduce that $\psi \in \text{Irr}(N_K(L)^F \mid \gamma_0)$. Conversely, suppose that $\psi \in \text{Irr}(N_K(L)^F \mid \lambda_0 \xi)$ with $\lambda_0 \xi \in \gamma_0$ and consider an extension $\eta_1 \in \text{AbIrr}(L^F)$ of $\xi$. By [Isa76 Problem 5.3 and Problem 6.2], we conclude that there exists $\eta_2 \in \text{Irr}(L^F/L_0^F)$ such that $\psi$ lies above $\lambda \eta_1 \eta_2$. Since $\eta := \eta_1 \eta_2 \in \text{AbIrr}(L^F)$ the result follows. \hfill \Box

Corollary 6.10. Assume Hypothesis 4.1 and suppose that $G$ is simply connected, let $K$ be an $e$-split Levi subgroup of $G$ and suppose that Parametrisation 5.8 holds at the prime $\ell$ for every irreducible rational component of $(K, F)$. Let $(L, \lambda)$ be an $e$-cuspidal pair of $K$ and consider $\text{Ab}(\lambda)$ as defined in Section 5.2. Then there exists a defect preserving $\text{Aut}_{\mathbb{F}}(G^F)_{K, L, \text{Ab}(\lambda)}$-equivariant bijection

$$\Psi_{(K, \lambda)} : \mathcal{E}(K^F, (L, \text{Ab}(\lambda))) \rightarrow \text{Irr}(N_K(L)^F \mid \text{Ab}(\lambda))$$

such that

$$(X, \vartheta, K^F, \theta) \sim_{K^F}(N_{X^\vartheta}(L), N_K(L)^F, \Psi_{(K, \lambda)}(\theta))$$

for every $\vartheta \in \mathcal{E}(K^F, (L, \text{Ab}(\lambda)))$ and where $X := (G^F \rtimes \text{Aut}_{\mathbb{F}}(G^F))_K$.

Proof. Define $K_0 := [K, K]$, $L_0 := L \cap K_0$, fix an irreducible constituent $\lambda_0$ of $L_0^F$ and set $\gamma_0 := \{\lambda_0 \xi \mid \xi \in \text{Irr}(L_0^F/[L, L]^F)\}$. We apply Proposition 6.1 with $A := G^F \rtimes \text{Aut}_{\mathbb{F}}(G^F)$, $A_0 := N_A(L)$, $K := K_0^F$, $K_0 = N_{K_0}(L)^F = N_{K_0}(L_0)^F$, $G := G^F$, $X := (G^F \rtimes \text{Aut}_{\mathbb{F}}(G^F))_K$, $S := \mathcal{E}(K_0^F, (L_0, \gamma_0))$, $S_0 := \text{Irr}(N_{K_0}(L_0)^F \mid \gamma_0)$, $V := (G^F \rtimes \text{Aut}_{\mathbb{F}}(G^F))_{K, S}$ and $U := (G^F \rtimes \text{Aut}_{\mathbb{F}}(G^F))_{K, L, \gamma_0}$. Observe that assumptions (ii) and (iii) of Proposition 6.1 are satisfied by Proposition 4.10 and Lemma 6.6. Consider the bijection between $S$ and $S_0$ given by Corollary 6.7 and Corollary 6.8. In order to apply Proposition 6.1 with $J := K^F$ we need to show that $C_X(Q) \leq X_0$.
for every radical $\ell$-subgroup $Q$ of $J_0 = N_K(L)^F$. By Lemma 2.5(ii), we know that $L = C_G^e(E)$ with $E := Z^e(L)^F$ and hence $E \leq O_e(N_K(L)^F)$. Since $Q$ is a radical $\ell$-subgroup of $J_0$, it follows that $E \leq Q$ (see [Dad92 Proposition 1.4]) and therefore $C_X(Q) \leq C_X(E) \leq N_X(E) = N_X(L) = X_0$. We can thus apply Proposition 6.1 together with Corollary 6.7, Corollary 6.8, and Lemma 6.9 to obtain an $\text{Aut}_F(G^F)_{K,L,\chi_0}$-equivariant bijection

$$
\psi^K_{\chi_0} : \mathcal{E}(K^F, (L, \text{Ab}(\chi))) \to \text{Irr}(N_K(L)^F | \text{Ab}(\chi))
$$

such that

$$(X_{\psi^K_{\chi_0}}, K^F, \psi) \sim_K (N_X(L), N_K(L)^F, \psi^K_{\chi_0} / \chi, \psi)$$

for every $\chi \in \text{Irr}(H \mid \mathcal{E}(K^F, (L, \text{Ab}(\chi))))$ and where $X = (G^F \times \text{Aut}_F(G^F))_K$. □

Now, applying Proposition 6.1, we show how to lift the bijection given by Corollary 6.1 to a bijection

$$
\Omega_{\chi_0}^{K_H} : \text{Irr}(H \mid \mathcal{E}(K^F, (L, \text{Ab}(\chi)))) \to \text{Irr}(N_H(L) \mid \text{Ab}(\chi))
$$

for every $K^F \leq H \leq N_G(K)^F$. Notice that if $\sigma \in \mathcal{L}(G)$ has final term $K$ and $H := G_\sigma^F$, then the above bijections gives a parametrisation of the characters considered in the definition of the set $\mathcal{L}^e(B)_e$ of Conjecture 5.4.

**Proposition 6.11.** Consider the setup of Corollary 6.1 and let $K^F \leq H \leq N_G(K)^F$. Then there exists a defect preserving $\text{Aut}_F(G^F)_{H,K,L,\chi_0}$-equivariant bijection

$$
\Omega_{\chi_0}^{K_H} : \text{Irr}(H \mid \mathcal{E}(K^F, (L, \text{Ab}(\chi)))) \to \text{Irr}(N_H(L) \mid \text{Ab}(\chi))
$$

such that

$$(N_X(H), \chi, H) \sim_H (N_X(H,L), N_H(L), \psi)$$

for every $\chi \in \text{Irr}(H \mid \mathcal{E}(K^F, (L, \text{Ab}(\chi))))$ and where $X = (G^F \times \text{Aut}_F(G^F))_K$.

**Proof.** We apply Proposition 6.1 to the bijection given by Corollary 6.1. We consider $A := G^F \times \text{Aut}_F(G^F)$, $G := G^F$, $K := K^F$, $A_0 := N_A(L)$, $X := N_A(K)$, $S := \mathcal{E}(K^F, (L, \text{Ab}(\lambda)))$, $S_0 := \text{Irr}(N_K(L)^F \mid \text{Ab}(\lambda))$, $U := X_{0,\chi_0}(\lambda)$, $V := X_S$ and $J := H$. By Proposition 4.10 and Lemma 6.6 we deduce that conditions (ii) and (iii) of Proposition 6.1 hold. Next, let $Q$ be a radical $\ell$-subgroup of $N_H(L)$. Set $E := Z^e(L)^F$ and notice that under our assumptions $L = C_G^e(E)$ by Lemma 2.5. Then $E \leq O_e(N_H(L)) \leq Q$ because $Q$ is radical and we conclude that $C_X(Q) \leq C_X(E) \leq N_X(L) = X_0$. We can therefore apply Proposition 6.1 to obtain an $\text{Aut}_F(G^F)_{H,K,L,\chi_0}$-equivariant bijection $\Omega_{\chi_0}^{K_H}$ as in the statement. Moreover $\Omega_{\chi_0}^{K_H}$ preserves the defect of characters by Remark 6.2. □

**Remark 6.12.** It is worth pointing out that, in the previous results, the assumption that $K$ is an $e$-split Levi subgroup of $G$ can be weakened by only requiring $K$ to be an $F$-stable Levi subgroup of $G$. 38
We can finally prove Theorem \[6.13\].

**Theorem 6.13.** Assume Hypothesis\[4.12\] and suppose that \(G\) is simply connected. If Parametrisation\[5.8\] holds at the prime \(\ell\) for every irreducible rational component of any \(e\)-split Levi subgroup of \((G, F)\), then Conjecture\[5.4\] holds with respect to the prime \(\ell\).

**Proof.** Consider an \(\ell\)-block \(B\) of \(G^F\), \(d \geq 0\) and define \(A := \text{Aut}_F(G^F)\) and \(X := G^F \rtimes A\). Let \(T_{1,+}\) be an \(A_B\)-transversal in the set

\[
S_{1,+} := \{ (\sigma, M, Ab(\mu)) \mid \sigma \in \mathcal{L}(G)_+, (M, \mu) \in CP_e(B)_\prec \text{ with } M \leq L(\sigma) \}
\]

and fix an \(A_{(\sigma, M, Ab(\mu))}\)-transversal \(T_{2,+}^{(\sigma, M, Ab(\mu))}\) in \(\text{Irr}^d(B_\sigma \mid E(L(\sigma)^F, (M, Ab(\mu))))\) for each \((\sigma, M, Ab(\mu)) \in T_{1,+}\). By our choices

\[
T_+: = \left\{ (\sigma, M, Ab(\mu), \vartheta) \mid (\sigma, M, Ab(\mu)) \in T_{1,+}, \vartheta \in T_{2,+}^{(\sigma, M, Ab(\mu))} \right\}
\]

is an \(A_B\)-transversal in \(\mathcal{L}^d(B)_+/G^F\).

We now fix \((\sigma, M, Ab(\mu)) \in S_{1,+}\). If \(L(\sigma) = M\), then define \(\rho\) to be the chain obtained by deleting \(L(\sigma)\) from \(\sigma\). Since \((M, \mu) \in CP_e(B)_\prec\), we have \(M < G\) and hence the chain \(\rho\) is non-empty. On the other hand if \(M < L(\sigma)\), then define \(\rho\) to be the chain obtained by adding \(M\) to \(\sigma\). In this case the last term \(L(\rho)\) of \(\rho\) coincides with \(M\). This construction yields an \(A_B\)-equivariant bijection

\[
\Delta: S_{1,+} \to S_{1,-}
\]

where

\[
S_{1,-} := \{ (\rho, N, Ab(\nu)) \mid \rho \in \mathcal{L}(G)_-, (N, \nu) \in CP_e(B)_\prec \text{ with } N \leq L(\rho) \}.
\]

In particular, the image \(T_{1,-} := \Delta(T_{1,+})\) is an \(A_B\)-transversal in \(S_{1,-}\). Moreover, notice that if \(\Delta((\sigma, M, Ab(\mu))) = (\rho, N, Ab(\nu))\), then we have \((N, Ab(\nu)) = (M, Ab(\mu))\) and

\[
A_{(\sigma, M, Ab(\mu))} = A_{(\rho, N, Ab(\nu))}.
\] (6.4)

Next, consider \((\sigma, M, Ab(\mu)) \in T_{1,+}\) and \((\rho, M, Ab(\mu)) := \Delta((\sigma, M, Ab(\mu))) \in T_{1,-}\). Assume first that \(L(\sigma) = M\). By Proposition\[6.11\] applied with \(H = G^F\), we obtain a bijection

\[
\Omega_{(\rho, M, Ab(\mu))}^{L(\rho), G^F_{\sigma}}: \text{Irr}(G^F_\rho \mid E(L(\rho)^F, (M, Ab(\mu)))) \to \text{Irr}(N_{G^F_\sigma}(M) \mid Ab(\mu)).
\]

Since \(M = L(\sigma)\), notice that \(N_{G^F_\sigma}(M) = G^F_\sigma\) and that \(\text{Irr}(G^F_\sigma \mid E(L(\sigma)^F, (M, Ab(\mu)))) = \text{Irr}(N_{G^F_\sigma}(M) \mid Ab(\mu))\). We define

\[
T_{2,-}^{(\rho, M, Ab(\mu))} := \left( \Omega_{(\rho, M, Ab(\mu))}^{L(\rho), G^F_{\sigma}} \right)^{-1}(T_{2,+}^{(\sigma, M, Ab(\mu))}).
\]

Similarly, if \(M < L(\sigma)\), then Proposition\[6.11\] applied with \(H = G^F_\sigma\) yields a bijection

\[
\Omega_{(M, Ab(\mu))}^{L(\sigma), G^F_{\sigma}}: \text{Irr}(G^F_\sigma \mid E(L(\sigma)^F, (M, Ab(\mu)))) \to \text{Irr}(N_{G^F_\sigma}(M) \mid Ab(\mu)).
\]
To do so, applying \([Ros22a, Lemma 2.11]\), it is enough to check that First we show that It remains to show that Noticing that the main result of \([KM13]\) implies that Conjecture 5.1 and Dade’s Conjecture are equivalent.

In this section we show that if \(\ell\) is large, \((G, F, e)\)-adapted and \(Z(G^*)_{\ell}^{F^*} = 1\), then Conjecture 5.1 is equivalent to Dade’s Conjecture. Moreover, since in this case we are dealing with blocks of abelian defect, the main result of \([KM13]\) implies that Conjecture 5.1 and Dade’s Conjecture are equivalent.
to Conjecture 5.2 and Alperin’s Weight Conjecture (in the Knörr–Robinson reformulation) respectively. Notice that the above equivalences are not merely logical equivalences but holds block by block. Furthermore, under the same assumptions, we show that the Alperin–McKay Conjecture holds for a block $B$ of $G^F$ if and only if

$$k(B) = \sum_{(L, \lambda)} k(N_G(L)^F, \lambda)$$

where $(L, \lambda)$ runs over a set of representatives for the action of $G^F$ on $CP_e(B)$ and $k(N_G(L)^F, \lambda)$ is the number of characters of $\text{Irr}(N_G(L)^F | \lambda)$. Since the Alperin–McKay Conjecture holds in this case as a consequence of [BMM93, Theorem 5.24], the above equality provides evidence for the validity of Parametrisation 5.8 as discussed in Remark 5.9.

We need to make a small remark: in the discussion following [BMM93, Theorem 5.24] the authors claim that Dade’s Conjecture holds for finite reductive groups with respect to large primes as a consequence of Broué’s Isotypy Conjecture. Unfortunately, although Broué’s Isotypy Conjecture logically implies Dade’s Conjectures, it is not known whether this implication holds block by block (see also [Lin18, Remark 10.7.2]). Therefore, at the time of writing Dade’s Conjecture remains open for finite reductive groups and large primes. On the other hand, the connection established here with Conjecture 5.1 could lead to a proof of such a result.

Under the above assumptions, we also show that Conjecture 5.4 implies Späth’s Character Triple Conjecture and, when $G^F/Z(G^F)$ is a nonabelian simple group with universal covering group $G^F$, the inductive condition for Dade’s Conjecture. Finally, we show that Parametrisation B implies the inductive Alperin–McKay condition.

### 7.1 Statements of the conjectures

Let $G$ be a finite group and $\ell$ a prime. We denote by $\text{Irr}^d(G)$ the set of irreducible characters $\chi \in \text{Irr}(G)$ with $\ell$-defect $d(\chi) = d$ and define $\text{Irr}^d(B) := \text{Irr}^d(G) \cap \text{Irr}(B)$ for any $\ell$-block $B$ of $G$. If $D$ is a defect group of $B$, then we define the defect of $B$ as the integer $d(B)$ such that $|D| = \ell^{d(B)}$.

The set $\text{Irr}_0(B) := \text{Irr}^d(B)(B)$ consists of the characters of height zero in $B$. We denote by $k(B)$, $k^d(B)$ and $k_0(B)$ the number of characters in the sets $\text{Irr}(B)$, $\text{Irr}^d(B)$ and $\text{Irr}_0(B)$ respectively. In [Alp76], Alperin introduced the following blockwise version of the McKay Conjecture.

**Conjecture 7.1** (Alperin–McKay Conjecture). Let $G$ be a finite group and $B$ an $\ell$-block of $G$ with defect group $D$. Then

$$k_0(B) = k_0(b)$$

where $b$ is the Brauer correspondent of $B$ in $N_G(D)$.

Inspired by [IMN07] and by the inductive Alperin–McKay condition for quasi-simple groups introduced in [Spä13], a stronger version of the above conjecture has been considered in [NS14].
Conjecture 7.2 (Inductive Alperin–McKay condition). Let $G$ be a finite group and $B$ an $\ell$-block of $G$ with defect group $D$ and Brauer correspondent $b$ in $N_G(D)$. If $G \leq X$, then there exists an $X_{D,B}$-equivariant bijection
\[ \Omega : \text{Irr}_0(B) \rightarrow \text{Irr}_0(b) \]
such that
\[ (X, G, \chi) \sim_G (N_X(D)_G, N_G(D), \Omega(\chi)) \]
for every $\chi \in \text{Irr}_0(B)$.

The main result of [NS14] shows that Conjecture 7.2 reduces to quasi-simple groups and, together with [KM13], implies Brauer’s Height Zero Conjecture.

Now, consider the set $\mathcal{P}(G)$ of $\ell$-chains of $G$ with initial term $O_\ell(G)$. These are the $\ell$-chains $\mathbb{D} = \{D_0 = O_\ell(G) < D_1 < \cdots < D_n\}$ where $D_i$ is an $\ell$-subgroup of $G$ and $n$ is a non-negative integer. If we denote by $|\mathbb{D}|$ the integer $n$, called the length of $\mathbb{D}$, then we obtain a partition of $\mathcal{P}(G)$ into the sets $\mathcal{P}(G)_+$ and $\mathcal{P}(G)_-$ consisting of $\ell$-chains of even and odd length respectively. Notice that $G$ acts by conjugation on the sets $\mathcal{P}(G), \mathcal{P}(G)_+$ and $\mathcal{P}(G)_-$ and we denote by $G_\mathbb{D} = \cap_i N_G(D_i)$ the stabiliser in $G$ of $\mathbb{D} \in \mathcal{P}(G)$ and by $\mathcal{P}(G)/G$ a set of representatives for the $G$-orbits on $\mathcal{P}(G)$. For any $\ell$-block $B$ of $G$, we set $\text{Irr}(B_\mathbb{D}) = \{\vartheta \in \text{Irr}(G_\mathbb{D}) \mid \text{bl}(\vartheta)^G = B\}$ and denote its cardinality by $k(B_\mathbb{D})$. Notice that the induced block $\text{bl}(\vartheta)^G$ is well defined according to [KR89, Lemma 3.2]. Moreover, $k^d(B_\mathbb{D})$ denotes the cardinality of $\text{Irr}^d(B_\mathbb{D}) := \text{Irr}^d(G_\mathbb{D}) \cap \text{Irr}(B_\mathbb{D})$ for any $d \geq 0$. Then Knörr-Robinson reformulation of Alperin’s Weight Conjecture (see [Alp87] and [KR89, Theorem 3.8 and Theorem 4.6]) can be stated as follows.

Conjecture 7.3 (Alperin’s Weight Conjecture). Let $G$ be a finite group such that $O_\ell(G) \leq Z(G)$. Then
\[ \sum_{\mathbb{D} \in \mathcal{P}(G)/G} (-1)^{|\mathbb{D}|} k(B_\mathbb{D}) = 0 \]
for every $\ell$-block $B$ of $G$ with defect groups strictly containing $O_\ell(G)$.

Inspired by the reformulation introduced by Knörr and Robinson, in [Dad92] (see also [Dad94]) Dade proposed a refinement of Alperin’s Weight Conjecture by considering characters of any fixed defect.

Conjecture 7.4 (Dade’s Conjecture). Let $G$ be a finite group such that $O_\ell(G) \leq Z(G)$. Then
\[ \sum_{\mathbb{D} \in \mathcal{P}(G)/G} (-1)^{|\mathbb{D}|} k^d(B_\mathbb{D}) = 0 \]
for every $\ell$-block $B$ of $G$ with defect groups strictly containing $O_\ell(G)$ and any $d \geq 0$.

Next, we recall the statements of the Character Triple Conjecture and of the inductive condition for Dade’s Conjecture. For $\epsilon \in \{+, -\}$ and $B$ an $\ell$-block of $G$, define
\[ C^d(B)_{\epsilon} := \{(\mathbb{D}, \vartheta) \mid \mathbb{D} \in \mathcal{P}(G)_{\epsilon}, \vartheta \in \text{Irr}^d(B_\mathbb{D})\} \]
Observe that $G$ acts on $C^d(B)_{\epsilon}$ and denote by $[(\mathbb{D}, \vartheta)]$ the $G$-orbit of $(\mathbb{D}, \vartheta) \in C^d(B)_{\epsilon}$ and by $C^d(B)_{\epsilon}/G$ a set of representatives for the $G$-orbits on $C^d(B)_{\epsilon}$. The following statement has been
proposed by Späth’s in [Spä17, Conjecture 6.3] and can be seen as an analogue of Conjecture 7.2 to Dade’s Conjecture.

**Conjecture 7.5** (Späth’s Character Triple Conjecture). Let $G$ be a finite group such that $O_{e}(G) \leq Z(G)$ and consider a block $B \in \mathcal{B}(G)$ with defect groups strictly larger than $O_{e}(G)$. Suppose that $G \leq X$. Then, for every $d \geq 0$, there exists an $X_B$-equivariant bijection

$$\Omega : C^d(B)_+ / G \to C^d(B)_- / G$$

such that

$$(X_{\mathcal{D}, \vartheta}, G_{\mathcal{D}}, \vartheta) \sim_G (X_{\mathcal{E}, \chi}, G_{\mathcal{E}}, \chi)$$

for every $(\mathcal{D}, \vartheta) \in C^d(B)_+$ and $(\mathcal{E}, \chi) \in \Omega((\mathcal{D}, \vartheta))$.

We now show that the Character Triple Conjecture plays the role of an inductive condition for Dade’s Conjecture. Notice that we could also consider a version of the above statement in the context of Alperin’s Weight Conjecture (Conjecture 7.3) by removing the restriction on the defect of characters, however the reduction theorem for Alperin’s Weight Conjecture does not take into account the Knörr–Robinson reformulation and deals with Brauer characters and weights instead.

**Lemma 7.6.** Let $S$ be a non-abelian simple group with universal covering group $\overline{S}$ and consider $B \in \mathcal{B}(\overline{S})$ with non-central defect groups. Then the following are equivalent:

1. the inductive condition for Dade’s Conjecture (in the sense of [Spä17, Definition 6.7]) holds for $B$;
2. for each $Z \leq Z(G)$, Conjecture 7.5 holds with respect to $G := \overline{S} / Z$, $X := G \rtimes \text{Aut}(G)$ and every block of $G$ dominated by $B$;
3. there exists a bijection $\Omega : C^d(B)_+ / \overline{S} \to C^d(B)_- / \overline{S}$ as in Conjecture 7.3 satisfying $\ker(\vartheta_{Z(\overline{S})}) = \ker(\chi_{Z(\overline{S})}) =: Z$ and

$$(X_{\mathcal{D}, \vartheta} / Z, \overline{S}_{\mathcal{D}} / Z, \overline{\vartheta}) \sim_{\overline{S}} (X_{\mathcal{E}, \chi} / Z, \overline{S}_{\mathcal{E}} / Z, \overline{\chi})$$

(7.1)

for every $(\mathcal{D}, \vartheta) \in C^d(B)_+$ and $(\mathcal{E}, \chi) \in \Omega((\mathcal{D}, \vartheta))$ and where $\overline{\vartheta}$ and $\overline{\chi}$ correspond, via inflation of characters, to $\vartheta$ and $\chi$ respectively and $X := \overline{S} \rtimes \text{Aut}(\overline{S})$.

**Proof.** This is just a restatement of [Spä17, Proposition 6.8].

The main theorem of [Spä17] shows that if the above conditions are satisfied for every simple group, then Dade’s Conjecture holds for every finite group. Späth’s results have been improved in [Ros] where Conjecture 7.3 is shown to reduce to quasi-simple groups. The requirement presented in (7.1) is more restrictive than the one stated in Conjecture 7.5. In fact, $G$-block isomorphisms of character trusses can be lifted from quotients with respect to central subgroups (see [Spä17, Corollary 4.4]). The above lemma tells us that proving the inductive condition for Dade’s Conjecture (as defined in [Spä17, Definition 6.7]) for a non-abelian simple group $S$ is equivalent to show that Conjecture 7.5 holds, with this more restrictive requirement (7.1), for its universal covering group $\overline{S}$ with respect to $\overline{S} \leq \overline{S} \rtimes \text{Aut}(\overline{S})$. This remark is helpful since, in the majority of cases, the universal covering group of a simple group of Lie type is a finite reductive group $G^F$ where $G$ is a simple simply connected algebraic group with a Frobenius endomorphism $F$.  

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7.2 An idea of Broué, Fong and Srinivasan

For every finite group $G$, recall that the set $\mathcal{E}(G)$ of $\ell$-elementary abelian chains of $G$ (starting at $\mathcal{O}_{\ell}(G)$) consists of those chains $E = \{E_0 = \mathcal{O}_{\ell}(G) < E_1 < \cdots < E_n\}$ such that $E_n/\mathcal{O}_{\ell}(G)$ is $\ell$-elementary abelian. Consider $G$, $F$, $q$, $\ell$ and $e$ as in Notation 2.2.

Definition 7.7 (Broué–Fong–Srinivasan). Let $E$ be an $\ell$-elementary abelian subgroup of $G^F$. Then $E$ is said to be good if

$$E = \Omega_1\left(\mathcal{O}_\ell\left(Z^\circ(G^\ell)(E)^F\right)\right),$$

and bad otherwise. An $\ell$-elementary abelian chain $E \in \mathcal{E}(G^F)$ is said to be good if $E_i$ is good for every $i$, while it is bad otherwise. The set of good and bad $\ell$-elementary abelian chains of $G^F$ is denoted by $\mathcal{E}_g(G^F)$ and $\mathcal{E}_b(G^F)$ respectively.

When $\ell$ is large and $(G, F, e)$-adapted there exists a bijection between chains of $e$-split Levi subgroups of $G$ and good $\ell$-elementary abelian chains of $G^F$. Recall from Section 2.1 that every automorphism $\alpha \in \text{Aut}_F(G^F)$ extends to a bijective endomorphism of $G$ commuting with $F$. Then $\text{Aut}_F(G^F)$ acts on the set of $F$-stable closed connected subgroups of $G$.

Lemma 7.8. Suppose that $\ell$ is large, $(G, F, e)$-adapted and $\mathcal{O}_\ell(G^F) = 1$. Then the maps

$$\mathcal{L}(G) \to \mathcal{E}_g(G^F)$$

$$\sigma = (L_i) \mapsto E = \left(\Omega_1\left(\mathcal{O}_\ell\left(Z^\circ(L_i)^F\right)\right)\right)$$

and

$$\mathcal{E}_g(G^F) \to \mathcal{L}(G)$$

$$E = (E_i) \mapsto \sigma := \left(\mathcal{C}_G^\alpha(E_i)\right)$$

are mutually inverse $\text{Aut}_F(G^F)$-equivariant length preserving bijections.

Proof. First, consider a chain of $e$-split Levi subgroups $\sigma = (G = L_0 > \cdots > L_n)$. Since $\ell$ is large for $G$, Proposition 2.7(iii) implies that $E_i := \Omega_1(\mathcal{O}_{\ell}(Z^\circ(L_i)^F))$ is a good $\ell$-elementary abelian subgroup such that $L_i = \mathcal{C}_G^\circ(E_i)$. Since $L_i > L_{i+1}$, this also shows that $E_i < E_{i+1}$ for every $i = 0, \ldots, n - 1$. Moreover, as $\mathcal{O}_\ell(G^F) = 1$, we deduce that $E_0 = \mathcal{O}_\ell(G^F)$. On the other hand, if $D = (\mathcal{O}_\ell(G^F) = D_0 < \cdots < D_n)$ is a good $\ell$-elementary abelian chain, then all terms $D_i$ are elementary abelian (since $\mathcal{O}_\ell(G^F) = 1$) and Proposition 2.7(i) shows that $K_i := \mathcal{C}_G^\circ(D_i)$ is an $e$-split Levi subgroup. Furthermore $D_i = \Omega_1(\mathcal{O}_\ell(Z^\circ(K_i)^F))$, because $D_i$ is good in the sense of Definition 7.7 and $K_0 = G$. As a consequence, since $D_i < D_{i+1}$, we obtain that $K_i > K_{i+1}$ for every $i = 0, \ldots, n - 1$. It follows that the above maps are inverses of each other and preserve the length of chains. To show that the maps are $\text{Aut}_F(G^F)$-equivariant, observe that $\Omega_1(\mathcal{O}_\ell(Z^\circ(L)^F))^\alpha = \Omega_1(\mathcal{O}_\ell(Z^\circ(L^\alpha)^F))$ and $\mathcal{C}_G^\circ(E)^\alpha = \mathcal{C}_G^\circ(E^\alpha)$ for every $e$-split Levi subgroup $L$ of $G$, every $\ell$-elementary abelian abelian subgroup $E$ of $G^F$ and every $\alpha \in \text{Aut}_F(G^F)$.

Next, we show that there exists a self inverse $\text{Aut}_F(G^F)$-equivariant bijection on the set of bad $\ell$-elementary abelian chains.
Lemma 7.9. If $\ell \in \Gamma(G, F)$ and $O_{\ell}(G^F) = 1$, then there exists an \text{Aut}_F(G^F)$-equivariant bijection
\[ \mathfrak{c}_b(G^F) \to \mathfrak{c}_b(G^F) \]
such that, if $E$ is mapped to $E'$, then $|E| = |E'| \pm 1$.

Proof. Let $E = (E_0 < \cdots < E_n) \in \mathfrak{c}_b(G^F)$ and set $D_i := \Omega_i \left( O_{\ell}( \mathbb{Z}^0(C_G(E_i))^F \right))$. Since $O_{\ell}(G^F) = 1$, notice that $E_i$ is elementary abelian so that $E_i \leq D_i$ by Lemma 2.6 (iii) and hence $C_G^0(D_i) \leq C_G^0(E_i)$. On the other hand, as $D_i \leq \mathbb{Z}^0(C_G^0(E_i))^F$, we have $C_G^0(E_i) \leq C_G^0(D_i)$. Thus $C_G^0(E_i) = C_G^0(D_i)$ and we conclude that $D_i$ is a good $\ell$-elementary abelian subgroup. Now, since $E$ is a bad chain, there exists a maximal index $j$ such that $E_j < D_j$. If $j = n$, then we define $E'$ by adding $D_n$ to the chain $E$. Assume $j < n$. In this case we claim that $D_j \leq E_{j+1}$ and we define $E'$ to be the chain obtained from $E$ by adding or removing $D_j$ to $E$ if $D_j < E_{j+1}$ or $D_j = E_{j+1}$ respectively. To prove the claim, notice that $E_{j+1} \leq C_G(D_{j+1})^F \leq C_G(E_j)^F = C_G^0(E_j)^F$ by Lemma 2.6 (i). As $D_j$ centralises $C_G^0(E_j)$, we deduce that $D_j \leq C_G^0(E_{j+1})^F = C_G^0(E_j)^F$ and that $D_j$ centralises $C_G^0(E_{j+1})$. Thus $D_j \leq \mathbb{Z}(C_G^0(E_{j+1}))$. Observing that $C_G^0(E_{j+1})$ is an $F$-stable Levi subgroup of $G$ (see [CE04 Proposition 13.16]) we obtain $\ell \in \Gamma(C_G^0(E_{j+1}), F)$ by Remark 2.4 and hence $D_j \leq \mathbb{Z}(C_G^0(E_{j+1}))$. This proves our claim that $D_j \leq D_{j+1} = E_{j+1}$. \hfill \square

7.3 New and old conjectures for large primes

Using the results described in the previous section, we can now show the equivalence between the Conjectures proposed in Section 5.1 and the local-global counting conjectures for finite reductive groups with respect to large primes. First, we recall that building on ideas of Bouc, Quillen, Thevenaz and Webb, [KR89 Proposition 3.3] and [Spa17 Proposition 6.10] show that it is no loss of generality to restrict our attention to $\ell$-elementary abelian chains when working with Conjecture 7.4 and Conjecture 7.5 respectively.

Proposition 7.10. Let $\ell$ be large, $(G, F, e)$-adapted and suppose that $O_{\ell}(G^F) = 1 = \mathbb{Z}(G^*)^\ell$. Then the following are equivalent for any $\ell$-block $B$ of $G^F$ with non-trivial defect:

(i) Conjecture 5.1 holds for $B$;

(ii) Conjecture 5.2 holds for $B$;

(iii) Alperin’s Weight Conjecture holds for $B$ (see Conjecture 7.3);

(iv) Dade’s Conjecture holds for $B$ (see Conjecture 7.4).

Proof. Since $\ell$ is large, the block $B$ has abelian defect and therefore [KM13] implies that Conjecture 5.1 is equivalent to Conjecture 5.2 while Conjecture 7.3 is equivalent to Conjecture 7.4. We show that Conjecture 5.1 is equivalent to Conjecture 7.4. As explained above, [KR89 Proposition 3.3] implies that Conjecture 7.4 holds for $B$ if and only if
\[ \sum_{D \in \mathfrak{c}_{b}(G^F)/G^F} (-1)^{|D|} k^d(B_{D}) = 0. \]  \tag{7.2} \]
On the other hand, using Lemma 7.9 we deduce that the contribution given by considering bad $\ell$-elementary abelian chains is zero and hence (7.2) is equivalent to
\[ \sum_{D \in \mathfrak{c}_{b}(G^F)/G^F} (-1)^{|D|} k^d(B_{D}) = 0. \]  \tag{7.3} \]
Next, consider the bijection described in Lemma 7.8 and suppose that \( \sigma \in \mathcal{L}(G) \) corresponds to \( D \in \mathfrak{E}_g(G^F) \). Since the bijection is \( \text{Aut}_e(G^F) \)-equivariant, it follows that \( G^F_D = G^F_\sigma \) and therefore \( k^d(B_D) = k^d(B_\sigma) \). Then (7.3) is equivalent to

\[
0 = \sum_{\sigma \in \mathcal{L}(G)/G^F} (-1)^{|\sigma|} k^d(B_\sigma) = \sum_{\sigma \in \mathcal{L}(G)/\sigma_0/G^F} (-1)^{|\sigma|} k^d(B_\sigma) + k^d(B)
\]

which, since \( k^d_\sigma(B) = 0 \) (see Proposition 4.18), holds if and only if Conjecture 5.1 holds for \( B \). □

Next, we reformulate the Alperin–McKay Conjecture in terms of \( e \)-local structures. Recall that for any \( e \)-pair \( (L, \lambda) \) of \( G \) we denote by \( k(N_G(L)^F, \lambda) \) the number of characters of \( N_G(L)^F \) lying above \( \lambda \)

**Proposition 7.11.** Assume Hypothesis 4.4 with \( \ell \) large and \((G, F, e)\)-adapted and consider an \( \ell \)-block \( B \) of \( G^F \). Then the Alperin–McKay Conjecture (Conjecture 7.1) holds for \( B \) if and only if

\[
k(B) = \sum_{(L, \lambda)} k(N_G(L)^F, \lambda)
\]

where \((L, \lambda)\) runs over a set of representatives for the action of \( G^F \) on \( CP_e(B) \). Moreover, this is equivalent to (5.6) whenever \( B \) has non-trivial defect and \( Z(G^*)^F_\ell = 1 \).

**Proof.** Let \( D \) be a defect group of \( B \) and consider its Brauer correspondent \( b \) in \( N_{G^F}(D) \). Notice that \( D \) is abelian since \( \ell \) is large and hence \( k(B) = k_0(B) \) and \( k(b) = k_0(b) \) (see [KM13]). To prove the first statement we need to show that

\[
k(b) = \sum_{(L, \lambda)} k(N_G(L)^F, \lambda)
\]

where \((L, \lambda)\) runs over a set of representatives for the action of \( G^F \) on \( CP_e(B) \).

Let \((L, \lambda_0)\) be an \((e, \ell')\)-cuspidal pair of \( G \) such that \( B = b_{G^F}(L, \lambda_0) \) and notice that \( N_{G^F}(D) = N_G(L)^F \) by [CE99 Lemma 4.16]. We claim that for every \((M, \mu) \in CP_e(B)\) the \( e \)-split Levi subgroup \( M \) is \( G^F \)-conjugate to \( L \). In fact, consider a semisimple element \( s \in M^{F*} \) such that \( \mu \in \mathcal{E}(M^F, [s]) \) and let \((M(s_\ell), \mu(s_\ell))\) be the \( e \)-cuspidal pair of \( G(s_\ell) \) given by Lemma 4.3 (see also Lemma 4.4). Under our assumptions \( G(s_\ell) \) is an \( e \)-split Levi subgroup of \( G \) (see Proposition 2.7(i)) and therefore \((M(s_\ell), \mu(s_\ell))\) is an \( e \)-cuspidal pair of \( G \). Moreover, Lemma 4.6 implies that \((M(s_\ell), \mu(s_\ell)) \in CP_e(B)\) since \((M, \mu) \in CP_e(B)\). It follows that \( B = b_{G^F}(M(s_\ell), \mu(s_\ell)) \) and hence \( M(s_\ell) \) is \( G^F \)-conjugate to \( L \) by [CE99 Theorem 4.1]. On the other hand \( M(s_\ell) \) is an \( e \)-split Levi subgroup of \( M \) such that \( C_M^*(s) = M(s_\ell)^{F*} \). Since \( \mu \) is \( e \)-cuspidal, [CE99 Proposition 1.10] implies that \( M(s_\ell) = M \) and we conclude that \( M \) is \( G^F \)-conjugate to \( L \).

Fix an \( N_G(L)^F \)-transversal \( \mathcal{T}' \) in the set of \( e \)-cuspidal characters \( \lambda \in \text{Irr}(L^F) \) such that \( bl(\lambda)G^F = B \). By the above paragraph we deduce that \( \mathcal{T} := \{(L, \lambda) \mid \lambda \in \mathcal{T}'\} \) is an \( N_G(L)^F \)-transversal in \( CP_e(L, F) \cap CP_e(B) \) and so, applying Lemma 5.5 together with Brauer’s first main theorem, we get

\[
\text{Irr}(b) = \bigsqcup_{(L, \lambda) \in \mathcal{T}} \text{Irr}(N_G(L)^F | \lambda).
\]
Now, (7.5) follows by noticing that $T$ is also a $G^F$-transversal in $CP_e(B)$.

To conclude, assume that $B$ has non-trivial defect and that $Z(G^*)^F_\ell = 1$. In this case, Proposition 4.18 implies that no $e$-cuspidal character belong to the block $B$. In particular $CP_e(B) = CP_e(B)_\prec$ and $k^d(B) = 0$ for every $d \geq 0$. Furthermore, the main result of [KM13] shows that $k(B) = k^d(B)$ and that $k^d(N_G(L)^F, \lambda) = k(N_G(L)^F, \lambda)$ for every $(L, \lambda) \in CP_e(B)$. Now, (7.4) is equivalent to (5.6), that is,

$$k^d(B) = k^d(B) + \sum_{(L, \lambda)} k^d(N_G(L)^F, \lambda)$$

where $(L, \lambda)$ runs over a set of representatives for the action of $G^F$ on $CP_e(B)_\prec$. \hfill \Box

By Remark 5.9 (see 5.6), the following corollary provides evidence for Parametrisation 5.8.

**Corollary 7.12.** Assume Hypothesis 4.1 with $\ell$ large and $(G, F, e)$-adapted and consider an $\ell$-block $B$ of $G^F$. Then

$$k(B) = \sum_{(L, \lambda)} k(N_G(L), \lambda)$$

where $(L, \lambda)$ runs over a set of representatives for the action of $G^F$ on $CP_e(B)$. Moreover, 5.6 holds whenever $B$ has non-trivial defect and $Z(G^*)^F_\ell = 1$.

**Proof.** By [BMM93 Theorem 5.24] and the subsequent discussion, under our assumptions the Alperin–McKay Conjecture holds for any block $B$ of $G^F$. Then the result follows from Proposition 7.11. \hfill \Box

Next, we consider the connection between Conjecture 5.4 the Character Triple Conjecture and the inductive condition for Dade’s Conjecture.

**Proposition 7.13.** Assume Hypothesis 4.1 with $\ell$ large, $(G, F, e)$-adapted and such that $O_\ell(G^F) = 1 = Z(G^*)^F_\ell$. Consider an $\ell$-block $B$ of $G^F$ with non-trivial defect and fix $d \geq 0$. If Conjecture 5.4 holds for $B$ and $d$, then the Character Triple Conjecture (Conjecture 7.5) holds for $B$ and $d$.

**Proof.** Consider $(E, \vartheta) \in C^d(B)_\prec$. By [Spa17 Proposition 6.10] we may assume that $E$ is an $\ell$-elementary abelian chain. If $E$ is a bad $\ell$-elementary abelian chain (see Definition 7.7), then we define

$$\Omega\left((E, \vartheta)\right) := (E', \vartheta),$$

where $E'$ is the chain corresponding to $E$ via the bijection given by Lemma 7.9. Notice in this case that $G^F_{E'} = G^F_E$ and therefore that $(E', \vartheta) \in C^d(B)_\succ$. Assume that $E$ is a good $\ell$-elementary abelian chain and consider the corresponding chain of $e$-split Levi subgroups $\sigma$ given by Lemma 7.8. Since the map $\mathbb{D} \mapsto \sigma$ is $Aut_E(G^F)$-equivariant, we know that $G^F_E = G^F_{E'}$. Recall that $L(\sigma)$ denotes the final term of $\sigma$. By Lemma 5.5(iii), there exists an $e$-cuspidal pair $(M, \mu)$ of $L$, unique up to $G^F_{\sigma}$-conjugation, such that $\vartheta \in \text{Ir}^d(G^F_{\sigma} | E(L^F, (M, \mu)))$. Then, as $\text{bl}(\vartheta)^G = B$, we have $\vartheta \in \text{Ir}^d(B_\sigma | E(L(\sigma)^F, (M, Ab(\mu))))$.

Next, we claim that $(M, \mu) \in CP_e(B)_\prec$, so that $(\sigma, M, Ab(\mu), \vartheta) \in \mathcal{L}^d(B)_\prec$. First, observe that every character of $E(L^F, (M, \mu))$ is contained in the block $\text{bl}(\mu)^L = B$ by Proposition 4.3. Then, applying
Lemma 7.5(i) and using the transitivity of block induction, it follows that $bl(\vartheta) = (\Omega^{\mu})^{G^F} = \Omega^{(\mu)^{G^F}}$. Since $(\mathbb{D}, \vartheta) \in C^d(B)_+$, we deduce that $\Omega^{(\mu)^{G^F}} = B$ and hence $(\mathbb{M}, \mu) \in CP_e(B)$. Furthermore, as $B$ has non-trivial defect, Proposition 4.18 shows that $M < G$ and so $(\mathbb{M}, \mu) \in CP_e(B)_\circ$. This proves our claim.

Now $(\sigma, \mathbb{M}, Ab(\mu), \vartheta) \in L^d(B)_+$ and we choose $(\rho, N, Ab(\nu), \chi) \in \Lambda(\mathbb{M}, Ab(\mu), \vartheta)$ where $\Lambda$ is the bijection given by Conjecture 5.4. If $\mathbb{D}$ is the good $\ell$-elementary abelian chain corresponding to $\rho$ via the bijection given by Lemma 7.8 then $(\mathbb{D}, \chi) \in C^d(B)_-$ and we define

$$\Omega \left( \left( \mathbb{E}, \vartheta \right) \right) := (\mathbb{D}, \chi).$$

Since $(\mathbb{M}, \mu)$ is unique up to $G^F$-conjugation while $\Lambda$ and the bijections given by Lemma 7.8 and Lemma 7.9 are equivariant, we conclude that $\Omega$ is a well defined $\text{Aut}_F(G^F)$-equivariant bijection. Moreover, using the property on character triples of $\Lambda$ it is immediate to show that $\Omega$ satisfies the analogous properties required by Conjecture 7.5 with respect to $X := G^F \rtimes \text{Aut}_F(G^F)$. This completes the proof.

To obtain the inductive condition for Dade’s Conjecture we apply Lemma 7.6. Recall that, apart from a few exceptions, the universal covering group of a (non-abelian) simple group of Lie type is a finite reductive group $G^F$ with $G$ simple and simply connected. Moreover, in this case, $\text{Aut}_F(G^F) = \text{Aut}(G^F)$ (see [LS98, 1.15] and [CS13, 2.4]).

**Corollary 7.14.** Consider the set up of Proposition 7.13 and suppose that $G^F / Z(G^F)$ is a non-abelian simple group with universal covering group $G^F$ where $G$ is simple and simply connected. If Conjecture 5.4 holds for $B$ and $d$, then the inductive condition for Dade’s Conjecture holds for $B$ and $d$.

**Proof.** Consider the bijection $\Omega : C^d(B)_+/G^F \rightarrow C^d(B)_-/G^F$ constructed in the proof of Proposition 7.13 and fix $(\mathbb{D}, \vartheta) \in C^d(B)_+$ and $(\mathbb{E}, \chi) \in \Omega((\mathbb{D}, \vartheta))$. Since $\Omega$ satisfies the requirements of Conjecture 7.5 we know that

$$(X_{\mathbb{D}, \vartheta}, G^F_{\mathbb{D}, \vartheta}) \sim_{G^F} (X_{\mathbb{E}, \chi}, G^F_{\mathbb{E}, \chi})$$

with $X := G^F \rtimes \text{Aut}_F(G^F)$ and applying [Spä17, Lemma 3.4] we obtain $Z := \text{Ker}(\vartheta_{Z(G^F)}) = \text{Ker}(\chi_{Z(G^F)})$. Because, under our assumption, $Z(G^F)$ has order coprime to $\ell$, it follows from [Spä17, Corollary 4.5] (see also Lemma 2.1) that

$$(X_{\mathbb{D}, \vartheta}/Z, G^F_{\mathbb{D}, \vartheta}/Z, \overline{\vartheta}) \sim_{G^F/Z} (X_{\mathbb{E}, \chi}/Z, G^F_{\mathbb{E}, \chi}/Z, \overline{\chi})$$

where $\overline{\vartheta}$ and $\overline{\chi}$ correspond to $\vartheta$ and $\chi$ respectively via inflation of characters. Now, the result follows from Lemma 7.6 after noticing that $\text{Aut}(G^F) = \text{Aut}_F(G^F)$ under our hypothesis.

To conclude we show that Parametrisation 5.8 implies the inductive Alperin–McKay condition (Conjecture 7.2).

**Proposition 7.15.** Assume Hypothesis 4.1 with $\ell$ large and $(G, F, e)$-adapted and consider an $\ell$-block $B$ of $G^F$. If Parametrisation 5.8 holds with respect to every $(L, \lambda) \in CP_e(B)$, then the inductive Alperin–McKay condition (Conjecture 7.2) holds for $B$ with respect to $G^F \leq G^F \rtimes \text{Aut}_F(G^F)$.
Proof. Let $D$ be a defect group of $B$ and consider its Brauer correspondent $b$ in $N_{GF}(D)$. Notice that $\text{Irr}_0(B) = \text{Irr}(B)$ and that $\text{Irr}_0(b) = \text{Irr}(b)$ by [KMI13] since $D$ is abelian under our assumption. Let $(L, \lambda_0)$ be an $(e, \ell')$-cuspidal pair of $G$ such that $B = b_{GF}(L, \lambda)$ and observe that $N_{GF}(D) = N_G(L)^F$ and that $\text{Aut}_G(F)_D = \text{Aut}_F(G^F)_D$ by [CEF9]. Fix an $\text{Aut}_F(G^F)_{L,B}$-transversal $T'$ in the set of $e$-cuspidal characters $\lambda \in \text{Irr}(L^F)$ such that $bl(\lambda)_{GF} = B$. Then, proceeding as in the proof of Proposition 7.11 we deduce that $T := \{(L, \lambda) \mid \lambda \in T'\}$ is an $\text{Aut}_F(G^F)_{L,B}$-transversal in $\mathcal{C}_{\ell}(L, F) \cap \mathcal{C}_{\ell}(B)$ and a $G^F \text{Aut}_F(G^F)_{L,B}$-transversal in $\mathcal{C}_{\ell}(B)$.

Next, for every $(L, \lambda) \in T$ we choose an $\text{Aut}_F(G^F)_{(L,\lambda)}$-transversal $T_{(L,\lambda)}$ in $\mathcal{E}(G^F, (L, \lambda))$. We claim that

$$T_B := \bigcup_{(L, \lambda) \in T} T_{(L, \lambda)}$$

is an $\text{Aut}_F(G^F)_{L,B}$-transversal in $\text{Irr}(B)$. First suppose that $\chi \in \text{Irr}(B)$ and let $(M, \mu) \in \mathcal{C}_{\ell}(B)$ such that $\chi \in \mathcal{E}(G^F, (M, \mu))$ (see Theorem 4.13). By the choices made in the previous paragraph, there exists $(L, \lambda) \in T$ such that $(M, \mu)^g = (L, \lambda)$ for some $g \in G^F$ and $x \in \text{Aut}_F(G^F)_{L,B}$. It follows that $\chi^x = \chi^{g^x} \in \mathcal{E}(G^F, (L, \lambda))$ and hence we find $\chi_0 \in T_{(L, \lambda)}$ such that $\chi_0^y = \chi_0$ for some $y \in \text{Aut}_F(G^F)_{(L,\lambda)}$. Noticing that $\text{Aut}_F(G^F)_{(L,\lambda)} \leq \text{Aut}_F(G^F)_{L,B}$, this shows that $\chi$ is $\text{Aut}_F(G^F)_{L,B}$-conjugate to an element in $T_B$. On the other hand suppose that $(L, \lambda) \in T$, $\chi, \psi \in T_{(L, \lambda)}$ with $\chi = \psi^x$ for some $x \in \text{Aut}_F(G^F)_{L,B}$. Then $(L, \lambda)^{xg} = (L, \lambda)$ for some $g \in G^F$ by Proposition 4.10 and we deduce that $\chi = \psi^x = \psi^{xg}$ with $xy \in \text{Aut}_F(G^F)_{(L,\lambda)}$. By the choice of $T_{(L,\lambda)}$, this shows that $\chi = \psi$ and our claim is proved.

By assumption, for every $(L, \lambda) \in T$, there is an $\text{Aut}_F(G^F)_{(L,\lambda)}$-equivariant bijection

$$\Omega^G_{(L,\lambda)} : \mathcal{E}(G^F, (L, \lambda)) \to \text{Irr}(N_G(L)^F | \lambda)$$

and hence $T_{(L,\lambda)} := \Omega^G_{(L,\lambda)}(T_{(L,\lambda)})$ is an $\text{Aut}_F(G^F)_{(L,\lambda)}$-transversal in $\text{Irr}(N_G(L)^F | \lambda)$. We claim that

$$T_b := \bigcup_{(L, \lambda) \in T} T_{(L, \lambda)}$$

is an $\text{Aut}_F(G^F)_{L,B}$-transversal in $\text{Irr}(B)$. If $\vartheta \in \text{Irr}(b)$, arguing as in the proof of Proposition 7.11 we deduce that $\vartheta \in \text{Irr}(N_G(L)^F | \mu)$ for some $(L, \mu) \in \mathcal{C}_{\ell}(L, F) \cap \mathcal{C}_{\ell}(B)$ via an application of Lemma 5.5 and Brauer’s first main theorem. Let $(L, \lambda) \in T$ and $x \in \text{Aut}_F(G^F)_{L,B}$ such that $(L, \mu)^x = (L, \lambda)$. Then $\vartheta^x \in \text{Irr}(N_G(L)^F | \lambda)$ and we can find $\vartheta_0 \in T_{(L, \lambda)}$ such that $\vartheta^y = \vartheta_0$ for some $y \in \text{Aut}_F(G^F)_{(L,\lambda)}$. Therefore $\vartheta$ is conjugate to $\vartheta_0$ via an element of $\text{Aut}_F(G^F)_{L,B}$. On the other, let $(L, \lambda) \in T$, $\vartheta, \eta \in T_{(L, \lambda)}$ and $x \in \text{Aut}_F(G^F)_{L,B}$ such that $\vartheta^x = \eta$. By Clifford’s theorem there exists $g \in N_G(L)^F$ such that $\lambda^g = \lambda$ and hence $\vartheta = \vartheta^g = \eta$ for some $xg \in \text{Aut}_F(G^F)_{(L,\lambda)}$. This shows that $\vartheta = \eta$ by the choice of $T_{(L,\lambda)}$ and so we obtain our claim.

Since $T_B$ and $T_b$ are in bijection, we obtain an $\text{Aut}_F(G^F)_{L,B}$-equivariant bijection between the sets $\text{Irr}(B)$ and $\text{Irr}(b)$ by setting

$$\Omega(\chi^x) := \Omega^G_{(L,\lambda)}(\chi)^x$$
for every \((L, \lambda) \in T, \chi \in \mathcal{T}^{(L, \lambda)}_B\) and \(x \in \text{Aut}_F(G^F)_{L,B}\). Recalling that \(N_G(L)^F = N_{G^F}(D)\) and \(\text{Aut}_F(G^F)_{L,B} = \text{Aut}_F(G^F)_{D,B}\), we deduce that the equivalence of character triples required by Conjecture [7,2] follow from the analogue properties given by Parametrisation [5,8].

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