Maximising line subgraphs of diameter at most $t^*$

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Abstract

We wish to bring attention to a natural but slightly hidden problem, posed by Erdős and Nešetřil in the late 1980s, an edge version of the degree–diameter problem. Our main result is that, for any graph of maximum degree $\Delta$ with more than $\frac{1.5\Delta}{t}$ edges, its line graph must have diameter larger than $t$. In the case where the graph contains no cycle of length $2t + 1$, we can improve the bound on the number of edges to one that is exact for $t \in \{1, 2, 3, 4, 6\}$. In the case $\Delta = 3$ and $t = 3$, we obtain an exact bound. Our results also have implications for the related problem of bounding the distance-$t$ chromatic index, $t > 2$; in particular, for this we obtain an upper bound of $1.941\Delta^t$ for graphs of large enough maximum degree $\Delta$, markedly improving upon earlier bounds for this parameter.

Keywords: degree–diameter problem, strong cliques, distance edge-colouring

1 Introduction

Erdős in [9] wrote about a problem he proposed with Nešetřil:

“One could perhaps try to determine the smallest integer $h_t(\Delta)$ so that every $G$ of $h_t(\Delta)$ edges each vertex of which has degree $\leq \Delta$ contains two edges so that the shortest path joining these edges has length $\geq t$ . . . This problem seems to be interesting only if there is a nice expression for $h_t(\Delta)$.”

Equivalently, $h_t(\Delta) - 1$ is the largest number of edges inducing a graph of maximum degree $\Delta$ whose line graph has diameter at most $t$. Alternatively, one could consider this an edge version of the (old, well-studied, and exceptionally difficult) degree–diameter problem, cf. [3].

It is easy to see that $h_t(\Delta)$ is at most $2\Delta^t$ always, but one might imagine it to be smaller. For instance, the $t = 1$ case is easy and $h_1(\Delta) = \Delta + 1$. For $t = 2$, it was independently proposed by Erdős and Nešetřil [9] and Bermond, Bond, Paoli and Peyrat [2] that $h_2(\Delta) \leq 5\Delta^2/4 + 1$, there being equality for even $\Delta$. This was confirmed by Chung, Gyárfás, Tuza and Trotter [7]. For the case $t = 3$, we suggest the following as a “nice expression”.

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Conjecture 1. \( h_3(\Delta) \leq \Delta^3 - \Delta^2 + \Delta + 2, \) with equality if \( \Delta \) is one more than a prime power.

As to the hypothetical sharpness of this conjecture, first consider the point–line incidence graphs of finite projective planes of prime power order \( q \). Writing \( \Delta = q + 1 \), such graphs are bipartite, \( \Delta \)-regular, and of girth 6; their line graphs have diameter 3; and they have \( \Delta^3 - \Delta^2 + \Delta + 2 \) edges. At the expense of bipartiteness and \( \Delta \)-regularity, one can improve on the number of edges in this construction by one by subdividing one edge, which yields the expression in Conjecture 1. We remark that for multigraphs instead of simple graphs, one can further increase the number of edges by \( \left\lceil \frac{\Delta}{2} \right\rceil - 1 \), by deleting some arbitrary vertex \( v \) and replacing it with a multiedge of multiplicity \( \left\lfloor \frac{\Delta}{2} \right\rfloor \), whose endvertices are connected with \( \left\lfloor \frac{\Delta}{2} \right\rfloor \) and \( \left\lceil \frac{\Delta}{2} \right\rceil \) of the original \( \Delta \) neighbours of \( v \). This last remark contrasts to what we know for multigraphs in the case \( t = 2 \), cf. [4, 8].

Through a brief case analysis, we have confirmed Conjecture 1 in the case \( \Delta = 3 \).

Theorem 2. The line graph of any (multi)graph of maximum degree 3 with at least 23 edges has diameter greater than 3. That is, \( h_3(3) = 23 \).

For larger fixed \( t \), although we are slightly less confident as to what a “nice expression” for \( h_t(\Delta) \) might be, we believe that \( h_t(\Delta) = (1 + o(1)) \Delta^t \) holds for infinitely many \( \Delta \).

We contend that this naturally divides into two distinct challenges, the former of which appears to be more difficult than the latter.

Conjecture 3. For any \( \varepsilon > 0, h_t(\Delta) \geq (1 - \varepsilon) \Delta^t \) for infinitely many \( \Delta \).

Conjecture 4. For \( t \neq 2 \) and any \( \varepsilon > 0 \), \( h_t(\Delta) \leq (1 + \varepsilon) \Delta^t \) for all large enough \( \Delta \).

With respect to Conjecture 3 we mentioned earlier how it is known to hold for \( t \in \{1, 2, 3\} \). For \( t \in \{4, 6\} \), it holds due to the point–line incidence graphs of, respectively, a symplectic quadrangle with parameters \((\Delta - 1, \Delta - 1)\) and a split Cayley hexagon with parameters \((\Delta - 1, \Delta - 1)\) when \( \Delta - 1 = q \) is a prime power. For all other values of \( t \) the conjecture remains open. Conjecture 3 may be viewed as the direct edge analogue of an old conjecture of Bollobás [3]. That conjecture asserts, for any positive integer \( t \) and any \( \varepsilon > 0 \), that there is a graph of maximum degree \( \Delta \) with at least \((1 - \varepsilon) \Delta^t \) vertices of diameter at most \( t \) for infinitely many \( \Delta \). The current status of Conjecture 3 is essentially the same as for Bollobás’s conjecture: it is unknown if there is an absolute constant \( c > 0 \) such that \( h_t(\Delta) \geq c \Delta^t \) for all \( t \) and infinitely many \( \Delta \). For large \( t \) the best constructions we are aware of are (ones derived from) the best constructions for Bollobás’s conjecture.

Proposition 5. There is \( t_0 \) such that \( h_t(\Delta) \geq 0.629^t \Delta^t \) for \( t \geq t_0 \) and infinitely many \( \Delta \).

Proof. Canale and Gómez [4] proved the existence of graphs of maximum degree \( \Delta \), of diameter \( t' \), and with more than \( (0.6291 \Delta)^{t'} \) vertices, for \( t' \) large enough and infinitely many \( \Delta \). Consider this construction for \( t' = t - 1 \) and each valid \( \Delta \). Now in an iterative process arbitrarily add edges between vertices of degree less than \( \Delta \). Note that as long as there are at least \( \Delta + 1 \) such vertices, then for every one there is at least one other to which it is not adjacent. Thus by the end of this process, at most \( \Delta \) vertices have degree smaller than \( \Delta \), and so the resulting graph has at least \( \frac{1}{2}(0.6291 \Delta)^{t'} \Delta - \Delta^2 \) edges, which is greater than \( (0.629 \Delta)^t \) for \( t \) sufficiently large. Furthermore since the graph has diameter at most \( t - 1 \), its line graph has diameter at most \( t \). \( \square \)
By the above argument (which was noted in [17]), the truth of Bollobás's conjecture would imply a slightly weaker form of Conjecture 3; that is, with a leading asymptotic factor of 1/2. As far as we are aware, a reverse implication, i.e. from Conjecture 3 to some form of Bollobás's conjecture is not known.

Our main result is partial progress towards Conjecture 4 (and thus Conjecture 1).

**Theorem 6.** $h_t(\Delta) \leq \frac{3}{2}\Delta^t + 1$.

Theorem 6 is a result/proof valid for all $t \geq 1$, but as we already mentioned there are better, sharp determinations for $t \in \{1, 2\}$. We have also settled Conjecture 4 in the special case of graphs containing no cycle $C_{2t+1}$ of length $2t + 1$ as a subgraph.

**Theorem 7.** The line graph of any $C_{2t+1}$-free graph of maximum degree $\Delta$ with at least $\Delta^t$ edges has diameter greater than $t$.

For $t \in \{1, 2, 3, 4, 6\}$, this last statement is asymptotically sharp (and in its more precise formulation the result is in fact exactly sharp) due to the point–line incidence graphs of generalised polygons. The cases $t \in \{3, 4, 6\}$ are perhaps most enticing in Conjecture 4 and that is why we highlighted the case $t = 3$ first in Conjecture 1.

In order to discuss one consequence of our work, we can reframe the problem of estimating $h_t(\Delta)$ in stronger terms. Let us write $L(G)$ for the line graph of $G$ and $H^t$ for the $t$-th power of $H$ (where we join pairs of distinct vertices at distance at most $t$ in $H$). Then the problem of Erdős and Nešetřil framed at the beginning of the paper is equivalent to seeking optimal bounds on $|L(G)|$ subject to $G$ having maximum degree $\Delta$ and $L(G)^t$ inducing a clique. Letting $\omega(H)$ denote the clique number of $H$, our main results are proven in terms of bounds on the distance-$t$ edge-clique number $\omega(L(G)^t)$ for graphs $G$ of maximum degree $\Delta$. In particular, we prove Theorem 6 by showing the following stronger form.

**Theorem 8.** For any graph $G$ of maximum degree $\Delta$, it holds that $\omega(L(G)^t) \leq \frac{3}{2}\Delta^t$.

We should remark that Dębski and Śleszyńska-Nowak [17] announced a bound of roughly $\frac{3}{2}\Delta^t$. Note that the bound in Theorem 8 can be improved in the cases $t \in \{1, 2\}$: $\omega(L(G)) \leq \Delta + 1$ is trivially true, while $\omega(L(G)^2) \leq \frac{3}{2}\Delta^2$ is a recent result of Faron and Postle [10]. We also have a bound on $\omega(L(G)^t)$ analogous to Theorem 7 a result stated and shown in Section 2.

A special motivation for us is a further strengthened form of the problem. In particular, there has been considerable interest in $\chi(L(G)^t)$ (where $\chi(H)$ denotes the chromatic number of $H$), especially for $G$ of bounded maximum degree. For $t = 1$, this is the usual chromatic index of $G$; for $t = 2$, it is known as the strong chromatic index of $G$, and is associated with a more famous problem of Erdős and Nešetřil [9]; for $t > 2$, the parameter is referred to as the distance-$t$ chromatic index, with the study of bounded degree graphs initiated in [13]. We note that the output of Theorem 8 may be directly used as input to a recent result [11] related to Reed’s conjecture [15] to bound $\chi(L(G)^t)$. This yields the following.

**Corollary 9.** There is some $\Delta_0$ such that, for any graph $G$ of maximum degree $\Delta \geq \Delta_0$, it holds that $\chi(L(G)^t) < 1.941\Delta^t$.

**Proof.** By Theorem 8 and [11] Thm. 1.6, $\chi(L(G)^t) \leq \left[0.881(\Delta(L(G)^t) + 1) + 0.119\omega(L(G)^t)\right] \leq \left[0.881(2\Delta^t + 1) + 0.119 \cdot 1.5\Delta^t\right] < 1.941\Delta^t$ provided $\Delta$ is taken large enough. \(\square\)
For $t = 1$, Vizing’s theorem states that $\chi(L(G)) \leq \Delta + 1$. For $t = 2$, the current best bound on the strong chromatic index [11] is $\chi(L(G)^2) \leq 1.772\Delta^2$ for all sufficiently large $\Delta$. For $t > 2$, note for comparison with Corollary 9 that the local edge density estimates for $L(G)^t$ proved in [12] combined with the most up-to-date colouring bounds for graphs of bounded local edge density [11] yields only a bound of $1.999\Delta^t$ for all large enough $\Delta$. We must say though that, for the best upper bounds on $\chi(L(G)^t)$, $t > 2$, rather than bounding $\omega(L(G)^t)$ it looks more promising to pursue optimal bounds for the local edge density of $L(G)^t$, particularly for $t \in \{3, 4, 6\}$. We have left this to future study.

1.1 Terminology and notation

For a graph $G = (V,E)$, we denote the $i^{th}$ neighbourhood of a vertex $v$ by $N_i(v)$, that is, $N_i(v) = \{u \in V \mid d(u,v) = i\}$, where $d(u,v)$ denotes the distance between $u$ and $v$ in $G$. Similarly, we define $N_i(e)$ as the set of vertices at distance $i$ from an endpoint of $e$.

Let $T_{k,\Delta}$ denote a tree rooted at $v$ of height $k$ (i.e. the leaves are exactly $N_k(v)$) such that all non-leaf vertices have degree $\Delta$. Let $T^1_{k,\Delta}$ be one of the $\Delta$ subtrees starting at $v$, i.e. a subtree rooted at $v$ of height $k$ such that $v$ has degree 1, such that $N_k(v)$ only contains leaves and all non-leaf vertices have degree $\Delta$.

2 A bound on $\omega(L(G)^t)$ for $C_{2t+1}$-free $G$

In this section, we prove the following theorem.

**Theorem 10.** Let $t \geq 2$ be an integer. Let $G$ be a $C_{2t+1}$-free graph with maximum degree $\Delta$. Then $\omega(L(G)^t) \leq |E(T_{t,\Delta})|$. When $t \in \{2, 3, 4, 6\}$ equality can occur for infinitely many $\Delta$.

Since $|E(T_{t,\Delta})| \leq \Delta^t$, the expression is at most the bound desired for Conjecture [11] and thus this implies Theorem [7]. In fact, the expression matches the order of the point–line incidence graphs of generalised polygons when $t \in \{2, 3, 4, 6\}$, which are the examples for which equality holds. On the other hand, by subdividing one edge of any of these constructions, one can see in the cases $t \in \{2, 3, 4, 6\}$ that the result fails if we omit the condition of $C_{2t+1}$-freeness.

We note that Theorem [10] is a generalisation of a result in [15] which was specific to the case $t = 2$. It is also a stronger form of a result announced in [17] for bipartite graphs. One might wonder about excluding other cycle lengths, particularly even ones. Implicitly this was already studied in [14], in that the local sparsity estimations there imply the following statement: for any $t \geq 2$ and even $\ell \geq 2t$, $\omega(L(G)^t) = o(\Delta^t)$ for any $C_{2t}$-free graph of maximum degree $\Delta$. Similarly, it would be natural to pursue a similar bound as in Theorem [10] but for an excluded odd cycle length (greater than $2t + 1$), which was done for $t = 2$ in [15].

The bound in Theorem [10] is a corollary of the following proposition.

**Proposition 11.** For fixed $\Delta$ and $t$, let $G$ be a $C_{2t+1}$-free graph with maximum degree $\Delta$ and $H \subseteq G$ be a subgraph of $G$ with maximum degree $\Delta_H$. Let $v$ be a vertex with degree $d_H(v) = \Delta_H = j$ and let $u_1, u_2, \ldots, u_j$ be its neighbours. Suppose that in $L(G)^t$, every edge of $H$ is adjacent to $vu_i$ for every $1 \leq i \leq j$. Then $|E(H)| \leq |E(T_{t,\Delta})|$.

**Proof.** For fixed $\Delta$, let $H$ and $G$ be graphs satisfying all conditions, such that $|E(H)|$ is maximised. This can be done since $|E(H)|$ is upper bounded by say $j\Delta^t$. With respect to the graph $G$, we write $N_i = N_i(v)$ for $0 \leq i \leq t + 1$. We start proving a claim that makes work easier afterwards.
Claim 12. For any $1 \leq i \leq t$, $H$ does not contain any edge between two vertices of $N_i$.

Proof. Suppose it is not true for some $i \leq t - 1$. Take an edge $yz \in E(H)$ with $y, z \in N_i$. Construct the graph $H'$ with $V(H') = V(H) \cup \{y', z'\}$ and $E(H') = E(H) \setminus yz \cup \{yy', zz'\}$, where $y'$ and $z'$ are new vertices, and let $G'$ be the corresponding modification of $G$. Then $H' \subseteq G'$ also satisfies all conditions in Proposition 11 and $|E(H')| = |E(H)| + 1$, contradictory with the choice of $H$.

Next, suppose there is an edge $yz \in E(H)$ with $y, z \in N_i$. Take a shortest path from $u_1$ to $yz$, which is wlog a path $P_y$ from $u_1$ to $y$. Note that a shortest path $P_y$ from $v$ to $z$ will intersect $P_y$ since $G$ is $C_{2t+1}$-free. Let $w$ be the vertex in $V(P_y) \cap V(P_z)$ that minimises $d_G(w, z)$ and assume $w \in N_i$, i.e. $m = d_G(v, w)$ is the distance from $v$ to $w$. The condition $w \in V(P_y) \cap V(P_z)$ ensures that $d_G(w, z) = d_G(w, y)$. Furthermore note that $y$ and $z$ are interchangeable at this point, as both are at the same distance from $u_1$.

If $d_G(w, u_i) = m - 1$ for every $1 \leq i \leq j$, we can remove $yz$ again and add two edges $yy'$ and $zz'$ to get a graph $H'$ satisfying all conditions, leading to a contradiction again. This is the blue scenario illustrated in Figure 1.

In the other case there is some $1 < s \leq j$ such that $d_G(w, u_s) > m - 1$. Since $d_G(u_s, yz) = t - 1$, wlog $d_G(u_s, z) = t - 1$, there is a shortest path from $u_s$ to $z$ which is disjoint from the previously selected shortest path $P_y$ between $u_1$ and $y$. Hence together with the edges $u_1v, vu_s$ and $yz$, this forms a $C_{2t+1}$ in $G$, which again is a contradiction. This is sketched as the red scenario in Figure 1.

Figure 1: Sketch of two scenarios (red and blue) referred to in the proof of Claim 12.

For every $1 \leq m \leq t + 1$, let $A_m$ be the set of all vertices $x$ in $N_m$ such that $d_G(v, x) = d_G(u_i, x) + 1 = m$ for at least one index $1 \leq i \leq j$ and let $R_m = N_m \setminus A_m$. Also let $A_0 = \{v\}$. Let $A = \bigcup_{i=0}^{t+1} A_i$ and $R = \bigcup_{i=0}^{t+1} R_i$. We observe that a vertex in $R_{t+1}$ cannot be an endvertex of an edge of $H$. Indeed, assuming the contrary, the other endvertex of such an edge would be at distance $t - 1$ from every $u_i$, $1 \leq i \leq j$ and in particular would belong to $A_i$, leading to a contradiction. Also we observe that there are no edges in $H$ between $R_t$ and $A$. By definition of $R_t$, any vertex $r \in R_t$ has no neighbour in $A_i$ where $i < t$, nor does it have a neighbour in $A_t$ by Claim 12. To end with, an edge with endvertices in $R_t$ and $A_{t+1}$ is not connected to any edge $vu_s$, $1 \leq s \leq j$, in $L(G)^t$. As a consequence, the number of edges of $H$ which contain at least one vertex of $R$ can be upper bounded by $|R_1| \cdot \left(|E(T_{t, \Delta})| - 1\right)$, which equals

$$\left(\text{deg}(v) - \Delta_H\right) \left(\frac{1}{\Delta} |E(T_{t, \Delta})| - 1\right).$$

(1)
All other edges of $H$ are in the induced subgraph $H[A]$. We will now compute a bound on the number of those remaining edges.

We start with defining a weight function $w$ on the vertices $x$ in $A$ which will turn out to be useful. For every $x \in A_m$ where $1 \leq m \leq t$, we define $w(x)$ to be equal to the number of paths (in $G$) of length $m - 1$ between $x$ and $A_1$. Note that by definition $w(x)$ is at least equal to the number of vertices $u_i \in A_1$ with $d_G(x, u_i) = m - 1$ and by definition of $A_m$ this implies $w(x) \geq 1$. An equivalent recursive definition of $w$ is the following: we let $w(u_i) = 1$ for any $u_i \in A_1$ and for every vertex $x \in A_m$ where $m \geq 2$, we let

$$w(x) = \sum_{y \in A_{m-1} : xy \in E(G)} w(y).$$

We observe by induction that

$$\sum_{x \in A_m} w(x) \leq j(\Delta - 1)^{m-1}$$

for every $1 \leq m \leq t$. For $m = 1$ this is by definition of $A_1$ and $j$. For $m \geq 2$, we have by induction that

$$\sum_{x \in A_m} w(x) = \sum_{x \in A_m} \sum_{y \in A_{m-1} : xy \in E(G)} w(y)
= \sum_{y \in A_{m-1}} \sum_{x \in A_m : xy \in E(G)} w(y)
\leq \sum_{y \in A_{m-1}} (\Delta - 1)w(y)
\leq j(\Delta - 1)^{m-1}.$$

Let $A'_t = \{a \in A_t \mid w(a) < j\}$ and $A''_t = \{a \in A_t \mid w(a) \geq j\}$. We first count the edges that are incident to some fixed $a \in A'_t$. Note that $H$ contains no edges between $a$ and $A_{t+1}$ since for such an edge we would need that $a$ is connected by a path of length $t - 1$ to every $u_i, 1 \leq i \leq j$ and thus in particular we would have $w(a) \geq j$. By Claim 12, we also know that $a$ cannot be incident with an other vertex in $A_t$. So we only need to count the edges in $H$ between $a$ and $A_{t-1}$, and by definition of the weight function, this is bounded by $w(a)$.

On the other hand, for every $a \in A''_t$ there are at most $\Delta H = j \leq w(a)$ edges in $E(H)$ incident to $a$. Having proven that for every $a \in A_t$ there are at most $w(a)$ edges in $H[A]$ incident with $a$, we conclude (remembering 12) that there are at most $\sum_{x \in A_t} w(x) \leq j(\Delta - 1)^{t-1}$ edges in $E(H[A])$ having a vertex in $A_t$. Also we have for every $1 \leq m \leq t - 1$ that the number of edges between $A_{m-1}$ and $A_m$ is bounded by $j(\Delta - 1)^{m-1}$. Hence

$$|E(H[A])| \leq \sum_{m=1}^{t} j(\Delta - 1)^{m-1} = \frac{\Delta H}{\Delta} |E(T_t, \Delta)|.$$

Together with 1 on the number of edges that intersect $R$, this gives the result as $\deg(v) \leq \Delta$ by definition.

An inspection of the proof yields that the extremal graphs $H$ for Proposition 11 satisfy Claim 12, $R = \emptyset$ and for every $x \in A_m$ where $0 \leq m \leq t - 1$, there are exactly $\Delta - 1$ edges.
towards $A_{m+1}$. Hence such an extremal graph $H$ is exactly $T_{i,\Delta}$ where possibly some of its leaves are identified as one (as long as the maximum degree is still $\Delta$). Let us call such a graph a quasi-$T_{i,\Delta}$.

Next, we discuss some properties that should be satisfied by any graph that attains the bound of Theorem 10 (provided such a graph exists for the given values of $t$ and $\Delta$!). Let $H \subseteq G$ be a graph such that $E(H)$ is a clique in $L(G)^t$ and $v$ be a vertex of maximum degree in $H$, which maximises $|E(H)|$ among all choices for $G$ and $H$. Let $N_H(v) = \{u_1, \ldots, u_j\}$. Then in particular, in $L(G)^t$ every edge of $H$ is adjacent to every edge $vu_i$, for all $1 \leq i \leq j$.

So by Proposition 11 for every vertex $v$ of degree $\Delta$ we observe locally a quasi-$T_{i,\Delta}$ again, and in particular every neighbour of such a $v$ has degree $\Delta$ (for $t \geq 2$). So $H$ is $\Delta$-regular and in particular a connected component of $G$. So it is not a tree and hence has some girth. The girth is at least $2t$ (as for every vertex we locally have a quasi-$T_{i,\Delta}$), but it cannot be $2t + 1$ since $G$ is $C_{2t+1}$-free and it cannot be $2t + 2$ or more since $E(H)$ is a clique in $L(G)^t$. Also we observe that for every $a \in A^*_t$ the condition that $w(a) \geq j$ implies that it has $\Delta$ neighbours in $A_{i-1}$ as these all have a weight function equal to 1 and so it has no neighbours in $A_{t+1}$. Hence $H$ is a $\Delta$-regular graph with girth $2t$ and diameter $t$. In particular they need to be Moore graphs and consequently by 16 the extremal graphs are polygons when $t \geq 3$.

3 A general bound on $\omega(L(G)^t)$

When $H \subseteq G$ is a graph whose edges form a clique in $L(G)^t$, it implies in particular that all edges adjacent to a specific vertex $v$ are at distance at most $t - 1$ from all other edges. As $|E(T_{i,\Delta})| \leq \Delta^t$, the following proposition implies Theorem 8.

**Proposition 13.** For fixed $\Delta$ and $t$, let $G$ be a graph with maximum degree $\Delta$ and $H \subseteq G$ be a subgraph of $G$ with maximum degree $\Delta_H$. Let $v$ be a vertex with degree $d_H(v) = \Delta_H = j$ and let $u_1, u_2, \ldots, u_j$ be its neighbours. Suppose that in $L(G)^t$, every edge of $H$ is adjacent to $vu_i$ for every $1 \leq i \leq j$. Then $|E(H)| \leq \frac{3}{2}|E(T_{i,\Delta})|$.

**Proof.** We do this analogously to the proof of Proposition 11. For fixed $\Delta$, let $H$ and $G$ be graphs satisfying all conditions, such that $|E(H)|$ is maximized (which is again possible since $j\Delta$ is an upper bound for $|E(H)|$).

It suffices to show that $|E(H)| \leq \frac{3}{2}|E(T_{i,\Delta})|$. By the proof of Claim 12 we know that for any $1 \leq i \leq t - 1$, the set $N_i$ does not induce any edges of $H$ (but this is not necessarily true anymore for $N_t$).

Define $A_m, R_m$, the weight function $w$, $A^*_t$ and $A^*_t$ as has been done in the proof of Proposition 11.

As before, the number of edges that (are not induced by $N_i$ and) use at least one vertex of $R$ is bounded by 11. Also, we again have for every $1 \leq m \leq t - 1$ that the number of edges between $A_{m-1}$ and $A_m$ is bounded by $j(\Delta - 1)^{m-1}$. Furthermore, $R_t$ does not induce any edge of $H$, because such an edge would be at distance larger than $t$ from $vu_1$. Thus the number of edges of $H$ that are either disjoint from $A_t$, or join $A_t$ and $R \setminus R_t$, is at most

$$ (\Delta - j) \left( \frac{1}{\Delta} |E(T_{i,\Delta})| - 1 \right) + \sum_{m=1}^{t-1} j(\Delta - 1)^{m-1}. $$

(3)
We will derive that the remaining edges of $H$ (which all intersect $A_t$) can be bounded by a linear combination of the weight functions $w(a)$ of the vertices $a \in A_t$.

For every $a \in A_t$ there are at most $j \leq w(a)$ edges in $E(H)$ having $a$ as one of its endvertices. So let us now focus on the edges that intersect $A_t$.

We observe that there are no edges in $H$ between any $a \in A_t$ and $r \in R_t$, because there is some $u_t$ such that $d(a, u_t) \geq t$, which implies that $vu_t$ and $ar$ would be at distance larger than $t$. For the same reason $H$ has no edges between $A_t$ and $A_{t+1}$.

Finally, we want to count the edges between $A_{t-1}$ and $A_t$, as well as those that are induced by $A_t$. We will prove that their number is bounded by $\frac{3}{2} \sum_{a \in A_t} w(a)$.

For that, we need the following technical claim.

**Claim 14.** Let $j$ be fixed and assume $j > x \geq m > 0$ and $j > y \geq n > 0$ with $x + y \geq j$. Then

$$\frac{3x - m}{j - m} + \frac{3y - n}{j - n} \geq 1.$$

Equality occurs if and only $m = n = x = y = \frac{j}{2}$.

**Proof.** Multiplying both sides with the positive factor $2(j - m)(j - n)$, we need to prove that $3(x + y)j - 3xn - 3ym + 2mn \geq 2j^2$. For fixed $j, x$ and $y$ the left hand side is minimal when $m = x$ and $n = y$. This reduces to proving that $3(x + y)j - 4xy \geq 2j^2$. But this is true since

$$3(x + y)j - 4xy - 2j^2 = 0.25j^2 - (x + y - 1.5j)^2 + (x - y)^2$$

$$= (2j - (x + y)) \cdot (x + y - j) + (x - y)^2 \geq 0,$$

as $j \leq x + y < 2j$, i.e. $|x + y - 1.5j| \leq 0.5j$. \hfill \diamond

For every $a \in A_t$, let $m(a)$ denote the number of neighbours (in $H$) of $a$ in $A_{t-1}$ and let $q(a)$ denote the number of neighbours (in $H$) of $a$ in $A_t$. Furthermore, we define $f(a) = \frac{3w(a) - m(a)}{j - m(a)}$.

Suppose $H$ contains an edge $e$ between two vertices $a_1, a_2 \in A_t$. Then $w(a_1) + w(a_2) \geq j$, since $a_1a_2$ must be within distance $t - 1$ of each of $vu_1, vu_2, \ldots, vu_j$. Hence by Claim 14 (applied with $m = m(a_1), n = m(a_2), x = w(a_1)$ and $y = w(a_2)$), we obtain that $f(a_1) + f(a_2) \geq 1$ for every edge $a_1a_2$ of $H[A_t]$.

From this it follows that $|E(H[A_t])| \leq \sum_{a_1a_2 \in E(H[A_t])} f(a_1) + f(a_2)$. The right hand side can further be rewritten as $\sum_{a \in A_t} q(a) \cdot f(a)$. Since every vertex $a \in A_t$ has $q(a) \leq j - m(a)$ neighbours in $A_t$ and has $m(a)$ neighbours in $A_{t-1}$, we conclude that the number of edges of $H$ that are either induced by $A_t$ or join $A_t$ and $A_{t-1}$ is at most

$$\sum_{a \in A_t} ((j - m(a)) \cdot f(a) + m(a)) = \sum_{a \in A_t} \frac{3}{2}w(a).$$

Thus the number of edges in $E(H)$ using at least one vertex in $A_t$ is bounded by

$$\sum_{a \in A_t} \frac{3}{2}w(a) + \sum_{a \in A_t} w(a) \leq \sum_{x \in A_t} \frac{3}{2}w(x),$$
which (see the derivation of (2)) is at most $\frac{3}{2}j(\Delta - 1)t^{-1}$. Summing this and (3), we conclude that $H$ has fewer than $(\Delta - j) \left(\frac{1}{2}|E(T_i, \Delta)| - 1\right) + \frac{3}{2} \Delta \cdot |E(T_i, \Delta)| \leq \frac{3}{2}|E(T_i, \Delta)|$ edges. \hfill \Box

Note that the exact maximum in Proposition 13 is $\sum_{m=1}^{t-1} \Delta(\Delta - 1)^{m-1} + \frac{3}{2} \Delta(\Delta - 1)^{t-1}$ and this can be attained when $\Delta$ is even. For example when $t = 2$, the following example in Figure 2 shows that the blow-up of a $C_5$ is not extremal anymore when only taking into account the weaker conditions from Proposition 13.

![Figure 2: An extremal graph for Proposition 13 for $\Delta = 4$ and $t = 2$.](image)

4 Determination of $h_3(3)$

**Proof of Theorem 2.** Let $G = (V, E)$ be a graph of maximum degree 3 such that the line graph $L(G)$ of $G$ has diameter at most 3, i.e. $L(G)^3$ is a clique. If we can show that $G$ must have at most 22 edges, then the result is proven. Suppose to the contrary that $|E| \geq 23$. The proof proceeds through a series of claims that establish structural properties of $G$.

In each claim, we will estimate $|E|$ by performing a breadth-first search rooted at some specified edge $e$ up to distance 3. To avoid repetition, let us set out the notation we use each time. We write $e = uv$. Let $u_0$ and $u_1$ be the two neighbours of $u$ other than $v$ (if $u$ has degree 3). For $i \in \{0, 1\}$, let $u_{ij}$ and $u_{ik}$ be the two neighbours of $u_i$ other than $u$ (if $u_i$ has degree 3). For $i, j \in \{0, 1\}$, let $u_{ij0}$ and $u_{ij1}$ be the two neighbours of $u_{ij}$ other than $u_i$ (if $u_{ij}$ has degree 3). Similarly, define $v_i, v_{ij}, v_{ijk}$ for $i, j, k \in \{0, 1\}$.

**Claim 15.** $G$ contains no triangle, loop or multi-edge.

*Proof.* These 3 cases are straightforwardly bounded by the breadth-first search. If the edge $e$ is in a triangle, $|E| = |N_{L(G)^3}[e]| \leq |E(K_3)| + 2 \cdot |E(T^1_{3,2})| + |E(T^1_{2,3})| = 3 + 2 \cdot 1 + 3 = 8$. Analogously, if the edge $e$ is a loop, one obtains $|E| = |N_{L(G)^3}[e]| \leq 1 + 7 = 8$. If the edge $e$ has a parallel edge then $|E| = |N_{L(G)^3}[e]| \leq 2 + 2 \cdot 7 = 16$. \hfill \Box

**Claim 16.** $G$ is 3-regular, and so $|E|$ is divisible by 3.

*Proof.* If not, say, $v$ has degree at most 2, then, say, $v_1, v_{1j}, v_{1jk}$ are undefined, and so $|E| = |N_{L(G)^3}[e]| \leq 1 + 3 \cdot 7 = 22$, a contradiction. \hfill \Box

**Claim 17.** $G$ contains no 4-cycle.

*Proof.* If the edge $e$ is in a 4-cycle, then without loss of generality suppose $u_1 = v_0, v_0 = u_{11}$, and so on. Already $|E| = |N_{L(G)^3}[e]| \leq 4 + 2 \cdot 7 + 2 \cdot 3 = 24$ and by Claim 16 we have a contradiction if we can show that $|E|$ is 1 lower. So we may assume that $u_0, u_1, v_0, v_1, u_{00}, u_{01}, u_{10}, v_{01}, v_{10}, u_{11}$ are all distinct vertices and that the vertices $u_{00k}, u_{01k}, u_{10k}, v_{01k}, v_{10k}, v_{11k}$ (possibly not all distinct) are all at distance exactly 3 from $e$.  


Consider the edges \(u_0u_000, u_00u_001, u_01u_010\) and \(u_01u_011\). They are within distance 3 (in \(L(G)\)) from \(v_{01}\), so \(u_000, u_001, u_010\) and \(u_011\) all need to be adjacent to \(v_{01}\), leading to a contradiction as \(\deg v_{01} \leq 3\).

\[\triangle\]

**Claim 18.** \(G\) contains no 5-cycle.

**Proof.** If the edge \(e\) is in a 5-cycle, then without loss of generality suppose \(u_{11} = v_{00}, u_{111} = v_{000}\) and so on. Already \(|E| = |N_{L(G)}(e)| \leq 5 + 2 \cdot 7 + 2 \cdot 3 + 1 = 26\). Since 26 \(\geq |E| \geq 23\) and \(G\) is 3-regular by Claim 16, it follows that \(|E| = 24\) and \(|V| = \frac{2|E|}{3} = 16\).

Note first that \(N_2(e) = \{u_{00}, u_{01}, u_{10}, u_{11}, v_{01}, v_{10}, v_{11}\}\) are all distinct vertices or else \(|E|\) is already at most 23. Thus \(|N_2(e)| = 16 - 13 = 3\) and so (again using 3-regularity, and also the fact that \(N_3(e)\) must be an independent set) there are exactly 2 edges in the subgraph induced by \(N_2(e)\).

We divide our considerations into two cases. First, we assume \(u_{11}\) has a neighbour in \(N_2(e)\). By Claim 15 without loss of generality we can assume that this neighbour is \(u_{00}\) and thus \(u_{111} = v_{000} = u_{00}, u_{11} = u_{001}\) and so on. Since \(N_2(e)\) induces two edges, we can assume that \(v_{100}, v_{110} \in N_3(e)\). Note that the edge \(u_{00}u_{11}\) is within distance 3 (in \(L(G)\)) of both \(v_{10}v_{100}\) and \(v_{11}v_{110}\). It cannot be that \(u_{000}\) is equal to \(v_{10}\) or \(v_{11}\) or else one of the edges \(v_{10}v_{100}\) and \(v_{11}v_{110}\) remains too far from \(u_{00}u_{11}\) (taking Claim 17 into account). At this point, we note that \(v_{10}\) and \(v_{11}\) both need to be adjacent to \(u_{000}\), creating a \(C_4\) and hence leading to a contradiction.

Second, since we are not in the first case, \(u_{111} = v_{000}\) must be at distance exactly 3 from \(e\). The vertex \(u_{111}\) must have all of its 3 neighbours in \(N_2(e)\), one of which is \(u_{11}\). Keeping in mind that there is no four-cycle, we can therefore assume without loss of generality that \(u_{111} = v_{000} = v_{111}\).

Let us consider as a subcase the possibility that \(u_{01}\) and \(v_{11}\) are adjacent (the case \(v_{10}\) and \(u_{00}\) being adjacent, is done in exactly the same way), say, \(u_{010} = v_{11}\). Note that the edge \(u_{01}v_{11}\) is within distance 3 (in \(L(G)\)) of both \(u_{1}u_{10}\) and \(v_{0}v_{01}\). Since \(v_{11}\) has all its neighbours already fixed (and keeping in mind that \(N_2(e)\) induces only one edge other than \(u_{01}v_{11}\)), it can only be that \(v_{01}, v_{010}\) and \(v_{00}\) have a common neighbour in \(N_3(e)\). So without loss of generality, \(u_{011} = u_{100} = v_{010}\). But now, with only the free placement of \(u_{010}\), the only possibility to have \(u_{00}u_{000}\) within distance 3 (in \(L(G)\)) of both edges \(v_{10}v_{100}\) and \(v_{01}v_{010}\), is if \(u_{010}\) is equal to \(u_{10}\) or \(v_{01}\). But then we have already determined the two edges induced by \(N_2(e)\), none of which is incident to \(v_{10}\), so that both \(v_{100}\) and \(v_{010}\) must be in \(N_3(e)\), leading to the contradiction that \(|N_3(e)| = |\{u_{100}, u_{111}, v_{000}, v_{01}\}| \geq 4\).

We have thus shown that \(u_{01}v_{11}\) and \(v_{10}v_{00}\) are not present as an edge.

Let \(i \in \{0, 1\}\). The vertex \(u_{i01}\) is not adjacent to any vertex in \(\{u_{00}, u_{1}, v_{0}, v_{1}\}\) and so \(u_{01i}\) has to be adjacent to one of them to ensure that \(u_{11}u_{111}\) is within distance 3 (in \(L(G)\)) of \(u_{01i}u_{01i}\).

This implies \(u_{01i}\) has to be equal to \(v_{110}, u_{10}, u_{01}\) or \(v_{10}\) (taking Claims 15 and 17 into account). Incidentally, \(u_{01i}\) can also not be equal to \(v_{10}\), because in order for \(u_{01}v_{10}\) to be within distance 3 of \(u_{01}u_{011}\), we would need an edge between \(\{u_{01}, v_{10}\}\) and \(\{u_{00}, v_{11}\}\), which would either create a triangle or an edge that we already showed to be not present.

So \(u_{01i}\) has to be equal to \(v_{110}, u_{10}\) or \(v_{01}\), and symmetrically \(v_{10i}\) equals \(u_{01}, u_{10}\) or \(v_{01}\), for all \(i \in \{0, 1\}\). As there are only two edges in the graph induced by \(N_2(e)\), and both \(u_{01}\) and \(v_{10}\) are an endvertex of one of them, we may conclude without loss of generality that \(u_{010} = v_{110}\) and \(v_{01} = u_{011}\). Note that the edge \(v_{11}v_{110}\) is within distance 3 (in \(L(G)\)) of \(uu_{1}\).
and so $u_{10}v_{110}$ must be an edge. But then the distance between $u_{01}v_{110}$ or $u_{10}v_{110}$ and $vv_0$ is at least 4, a contradiction.

By the above claims, it only remains to consider $G$ being 3-regular and of girth at least 6. Let $e \in E$ be arbitrary. Then we have $|E| = |N_{L(G)}[e]| \leq 29$ and by Claim 16 we have a contradiction if we can show that $|E|$ is 6 lower. Since $|E| \geq 23$ and $G$ is 3-regular, we know $|V| = \left\lceil \frac{2|E|}{3} \right\rceil \geq 16$, hence there are at least $16 - (2 + 4 + 8) = 2$ vertices at distance 3 from $e$. Let $x$ and $y$ be vertices at distance 3 from $e$. We may assume without loss of generality that $x$ is adjacent to $u_{00}$, $u_{10}$, and $v_{00}$. Since the edge $vv_1$ is within distance 3 (in $L(G)$) of both edges $u_{00}x$ and $u_{10}x$, it follows (without loss of generality) that $u_{00}v_{10}$ and $u_{10}v_{11}$ are edges.

Since $y$ must satisfy similar constraints as $x$, and it cannot be adjacent to $u_{00}$ nor to $u_{10}$, there will be at least three edges between vertices in $N_2(e)$ and so similarly as before, we know that $|V| = \left\lceil \frac{2|E|}{3} \right\rceil \leq \left\lceil \frac{2(29-3)}{3} \right\rceil = 17$. Because every 3-regular graph has an even number of vertices, it follows that $|V| = 16$, so that in fact $x$ and $y$ are the only vertices in $N_3(e)$. From this and 3-regularity, we conclude that the subgraph induced by $N_2(e)$ must have exactly 5 edges. Since every edge between 2 vertices in $N_2(e)$ will be between some $u_{ij}$ and a $v_{k\ell}$ and $G$ is 3-regular, we know that $y$ is adjacent to exactly one of $u_{01}$ and $u_{11}$, wlog $u_{11}$. Similarly the two neighbours of $y$ of the form $v_{ij}$ are not a neighbour of $x$ and so $N(y)$ and $N(x)$ are disjoint. In particular $y$ cannot be adjacent to $v_{00}$ and so it has to be adjacent to $v_{01}$. The last neighbour of $y$ is either $v_{10}$ or $v_{11}$. If it is $v_{11}$, then to ensure that $uu_0$ is within distance 3 of both $yv_{01}$ and $yv_{11}$, we would need $u_{01}$ to be a neighbour of both $v_{10}$ and $v_{11}$, creating a four-cycle; contradiction. Thus the neighbours of $y$ must be $u_{11}$, $v_{01}$ and $v_{10}$.

So to ensure this, $u_{01}v_{01}$ is an edge as well.

We have now determined the whole graph, apart from two edges between $\{u_{01}, u_{11}\}$ and $\{v_{00}, v_{11}\}$. However, the edge $v_{00}u_{11}$ would create the five-cycle $xu_{10}u_{11}v_{00}$, while the edge $v_{00}u_{01}$ would yield the four-cycle $u_{01}v_{00}v_0v_{01}$. So in both cases, we get a contradiction, from which we conclude.

A brief inspection of the proof in Claim 16 yields that the extremal graph has exactly one vertex of degree 2 and 14 vertices of degree 3. Let the vertex of degree 2 be $w$. Let its two neighbours be $u$ and $v$ and then $u_i, v_i, u_{ij}, v_{ij}$ for $i, j \in \{0, 1\}$ are defined as before. Noting that every $v_{ij}$ has two neighbours of the form $v_{0k}$ and $v_{1k}$ where $k, \ell \in \{0, 1\}$, one can check that there is a unique extremal example with respect to Theorem 2, namely, the point–line incidence graph of the Fano plane, in which exactly one edge is subdivided.

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