Joint Angle Estimation Error Analysis and 3-D Positioning Algorithm Design for mmWave Positioning System

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Abstract—This article presents a comprehensive framework for jointly analyzing the angle estimation error and designing a 3-D positioning algorithm for an Internet of Things (IoT) millimeter wave (mmWave) positioning system. Initially, the azimuth and elevation Angles of Arrival (AoA) at the anchors are estimated by applying the 2-D discrete Fourier transform (2D-DFT) algorithm. The angle estimation error is then analyzed in terms of probability density functions (PDFs) by utilizing the properties of the 2D-DFT algorithm and employing challenging derivations and linear approximations. The analysis reveals that the resulting angle estimation error is non-Gaussian, distinguishing it from previous studies. Next, the complex expression of the PDF for the AoA estimation error is simplified using the first-order linear approximation of triangle functions. Subsequently, a complex expression for the variance is derived based on the obtained PDF. Specifically, the variance for the azimuth estimation error is integrated separately according to the different nonzero intervals of the obtained PDF. Additionally, the closed-form expressions of the variances are formulated using generalized hypergeometric series. Finally, the two-stage weighted least square (TSWLS) algorithm is employed to estimate the 3-D position of the mobile user (MU) using the estimated AoA and the obtained non-Gaussian variance. Extensive simulation results confirm the non-Gaussian nature of the derived angle estimation error and demonstrate the superiority of the proposed framework.

Index Terms—Angles of Arrival (AoA), Internet of Things (IoT), millimeter wave (mmWave), non-Gaussian, positioning.

I. INTRODUCTION

A S AN essential requirement in the context of the sixth-generation (6G) Internet of Things (IoT) wireless networks, high-accuracy positioning services play a crucial role in a wide range of applications [1]. These applications include automated driving vehicles [2], smart factories [3], and virtual reality [4]. For instance, it is projected that the 2023 will witness the rise of automated driving vehicles with positioning accuracy at the decimeter level [5]. However, the prevailing global positioning system (GPS) [6] falls short of meeting the stringent positioning accuracy requirements of these burgeoning applications [7]. Consequently, network-based positioning systems are emerging as a promising IoT technique for the future [8].

In current IoT wireless positioning networks, the millimeter wave (mmWave) technique has been proved to be effective in providing highly accurate estimation of channel parameters, including channel gain, time delays, and Angle of Arrival (AoA) [9]. These parameters play a crucial role in estimating the position of the mobile user (MU). Consequently, there is significant interest in exploring positioning algorithms for mmWave positioning systems. Typically, positioning algorithms involve two key steps.

1) Channel parameters are acquired through various estimation methods [10], [11], [12], [13].

2) Positioning algorithms are designed based on these acquired parameters. Specifically, complex nonlinear geometric relationships between the channel parameters and position coordinates are initially derived [14]. Subsequently, the estimation error of the channel parameters and the nonlinear geometric relationships are jointly utilized to derive nonlinear equations [15]. Finally, the position of the MU is determined by solving these equations using iterative or noniterative algorithms. In the aforementioned positioning steps, the estimation error of the channel parameters directly impacts the positioning accuracy. Different methods for estimating various channel parameters result in different types of parameter estimation errors [16].
Accurate estimation of channel parameters is crucial for enhancing the accuracy of positioning in IoT mmWave systems. Nguyen and Ghrayeb [17], Alkhateeb et al. [18], and Gao et al. [19] proposed mmWave channel estimation schemes using compressive sensing (CS) with a hybrid architecture, but they are computationally complex and rely on the restricted isometry property. High-resolution subspace-based algorithms like multiple signal classification (MUSIC), estimation of signal parameters via rotational invariance technique (ESPRIT) and their variants are known for precise angle estimation [20], [21], [22]. Extensive research has been done on their applications in massive multiple-input multiple-output (MIMO) and full-dimension MIMO systems for angle estimation [23], [24]. However, conventional MUSIC and ESPRIT methods are unsuitable for mmWave communications due to their high-computational complexity during the singular value decomposition (SVD) operation and their reliance on blind estimation without utilizing the training sequence available in wireless communication systems.

Due to the high accuracy of positioning, mmWave positioning systems have attracted extensive research attention. However, because of the large number of antennas used in the mmWave systems, the high-computational complexity becomes a major obstacle for practical design. To reduce the high-computational complexity due to a large number of antennas, [25] proposed a novel channel compression method for the mmWave positioning systems. To further improve the accuracy of localization, Wen et al. [26] presented a novel tensor-based method for channel estimation that allows estimation of mmWave channel parameters in a nonparametric form. The method is able to accurately estimate the channel, even in the absence of a specular component. Due to the mobility of the MU, in [27], the successive localization and beamforming scheme was proposed to estimate the long-term MU location and the instantaneous channel state information for the mmWave MIMO communications. Shahmansoori et al. [28] proposed a joint heuristic beam selection and user position and orientation tracking approach. However, the distribution of the estimation error of the channel parameters, which directly reflects the impact of the channel parameter estimation on the positioning accuracy, has not been taken into account when designing the positioning algorithm.

Furthermore, researchers have investigated various aspects of IoT localization systems. Sellami et al. [29] focused on localization using mmWaves and a massive multiple-antenna configuration, specifically examining the impact of channel estimation errors on accuracy. In [30], a unified framework based on factor graphs was proposed for high-accuracy indoor localization, efficiently integrating ranging and fingerprinting techniques. Su et al. [31] explored underwater localization using a mobile acoustic array network and linear frequency modulated (LFM) signals to enhance scalability and feasibility. Additionally, [32] introduced an AI-applied UWB positioning system that enhanced the performance by classifying channel conditions using the channel impulse response (CIR) of the received signal.

Besides, there are also some researchers focused on the performance analysis of the mmWave positioning systems. In [33], the position and orientation error bounds were derived for the single-anchor two-way positioning system. Shahmansoori et al. [34] derived the Cramér-Rao bound (CRB) on position and rotation angle estimation uncertainty from mm-wave signals from a single transmitter, in the presence of scatterers. In [35], error bounds were derived for uplink and downlink 3-D mmWave systems. However, most of them did not model the estimation error of the channel parameters, and some of them assumed that the error of the channel parameters follows the Gaussian distribution, which is inconsistent with practical scenarios. In practice, the distribution of estimation error of these parameters depends on the practical estimation methods (e.g., the 2-D discrete Fourier transform (2D-DFT) algorithm [36]), which may not follow the Gaussian distribution. Therefore, comparing the estimation error with the CRLB is not a practical approach for evaluating the performance.

Therefore, existing positioning algorithms and performance analysis may not be applicable when considering practical channel parameter estimation methods. However, it is a new challenge to model the estimation error of the estimated channel parameters in terms of distribution when considering the practical channel estimation methods. Furthermore, it is crucial and practical to derive the closed-form solution of the MU’s position when considering the estimation error of the channel parameters.

Against the above background, we propose a complete framework to jointly analyze the angle estimation and design the 3-D positioning algorithm for the IoT mmWave positioning system. To model the angle estimation error, we adopt the 2D-DFT estimation method as an example and leverage the geometric relationship between the AoA and their triangle functions. Furthermore, to derive the closed-form solution of the MU’s position, we derive the variance of the angle estimation error and apply the two-stage weighted least square (TWSKL) algorithm. Our main contributions are summarized as follows.

1) We propose a comprehensive framework to jointly analyze the angle estimation and design the 3-D positioning algorithm for the IoT mmWave positioning system. Specifically, we first estimate the azimuth and the elevation AoA at the anchors by applying the 2D-DFT algorithm. Then, we derive the closed-form expression of the probability density function (PDF) of the estimation error. We further derive the variance of the estimation error based the PDF. Finally, by using the estimated AoA and the derived variance, we apply the TWSLS algorithm to estimate the 3-D position of the MU.

2) We obtain closed-form expressions for the estimated azimuth and elevation AoA using the 2D-DFT method. Our analysis reveals that the angle estimation error at the anchors depends on the search grid, anchor panel size, and the number of antennas used by the anchors.

3) The angle estimation error, including both azimuth and elevation, is characterized by a non-Gaussian PDF. We derive the PDF by utilizing the geometric relationship between the AoA and the corresponding triangle functions. To simplify the complex geometric expression, we
employ a first-order linear approximation of the triangle function. Specifically, we develop an algorithm to derive and approximate the PDF expression for the azimuth angle estimation error.

4) Using the derived PDF of the estimation error, we analytically derive the variance of the angle estimation error. For the azimuth estimation error, which has three distinct nonzero intervals in its PDF, we calculate the integral separately for each interval to obtain the variance. This variance is then incorporated into the WLS algorithm to obtain the closed-form 3-D position of the MU.

5) Simulation results verify the accuracy of the derived results and demonstrate the superiority of the proposed framework. We observe that the variance decreases with the number of antennas, which means that increasing the number of anchor antennas improves the estimation accuracy.

The remainder of this article is organized as follows. The system model for the mmWave positioning system is described in Section II. The procedures of estimating angles are given in Section III. Section IV derives the PDF in more details. Section V calculates the variance. The positioning algorithm is given in Section VI. Simulation results are given in Section VII. Section VIII concludes the work of this article.

**Notations:** Small and upper bold-face letters denote column vectors and matrices, respectively; the superscripts * stand for the conjugate transpose, transpose, inverse, and pseudoinverse of a matrix, respectively; the superscripts , , are extracted from the diagonal components of ;\{·\} denotes a vector whose \(i\)th entry of ; vec(\(A\)) denotes the vectorization of \(A\); and \(E\{·\}\) denotes the statistical expectation.

II. SYSTEM MODEL AND FRAMEWORK

Consider an IoT mmWave time division duplex (TDD) 3-D positioning system, where an MU sends pilot signals to the anchors to locate the MU. We assume that there are \(I\) anchors, each of which is equipped with a uniform planar array (UPA) with \(N_y, z = N_y \times N_z\) antennas, where \(N_y\) and \(N_z\) denote the numbers of antennas along the y-axis and z-axis, respectively. The MU is equipped with a single antenna.

As shown in Fig. 1, the anchors are placed parallel to the y-o-z plane with the center located at \(s_i = [x_i, y_i, z_i]^T\), \(i \in [1, \ldots, I]\). The true location of the MU is \(q = [x_q, y_q, z_q]^T\). The estimated location of the MU is denoted as \(\hat{q} = [\hat{x}_q, \hat{y}_q, \hat{z}_q]^T\). Generally, once the anchors are deployed, the coordinate \(s_i\) are known and invariant. In order to locate the MU, we need to obtain the estimated \(\hat{q}\).

A. Channel Model

Assuming that the number of propagation paths between the MU and the \(i\)th anchor is \(N_i\), the AoA of the \(n\)th path from the MU to the \(i\)th anchor can be decomposed into the elevation angle \(0 \leq \Theta_{n,i} \leq \pi\) in the vertical direction, and the azimuth angle \(0 \leq \Phi_{n,i} \leq \pi\) in the horizontal direction, respectively.

As a result, the array response vector at the \(i\)th anchor of the \(n\)th path can be expressed as

\[
\mathbf{a}(\Theta_{n,i}, \Phi_{n,i}) = \mathbf{a}_c(\Phi_{n,i}) \otimes \mathbf{a}_d(\Theta_{n,i}, \Phi_{n,i})
\]

where \(\otimes\) denotes the Kronecker product. Moreover, we have

\[
\mathbf{a}_c(\Phi_{n,i}) = \begin{bmatrix}
1, e^{-j2\pi d_i \sin \theta_{n,i} \lambda_c / \pi}, \ldots, e^{-j2\pi (N_z-1) d_i \sin \theta_{n,i} \lambda_c / \pi}
\end{bmatrix}^T
\]

and

\[
\mathbf{a}_d(\Theta_{n,i}, \Phi_{n,i}) = \begin{bmatrix}
1, e^{-j2\pi d_i \cos \theta_{n,i} \cos \phi_{n,i} / \pi}, \ldots, e^{-j2\pi (N_z-1) d_i \cos \theta_{n,i} \cos \phi_{n,i} / \pi}
\end{bmatrix}^T
\]

where \(d_i\) and \(\lambda_c\) denote the distance between the antennas of the anchors and the carrier wavelength, respectively. Then, the channel between the MU and the \(i\)th anchor, denoted as \(\mathbf{h}_i\), can be modeled as

\[
\mathbf{h}_i = \sum_{n=1}^{N_i} \alpha_{n,i} \mathbf{a}(\Theta_{n,i}, \Phi_{n,i})
\]

\[
= \alpha_{1,i} \mathbf{a}(\Theta_{1,i}, \Phi_{1,i}) + \sum_{n=2}^{N_i} \alpha_{n,i} \mathbf{a}(\Theta_{n,i}, \Phi_{n,i})
\]

where \(\alpha_{n,i}\) denotes the complex channel gain of the \(n\)th path and the \(i\)th anchor. Moreover, \(\alpha_{1,i}, \Theta_{1,i}, \phi_{1,i}\) denote the complex channel gain, the elevation AoA, and the azimuth AoA of the line-of-sight (LoS) path, respectively. As we can see from (4), channel components of \(\mathbf{h}_i\) can be categorized into two types, namely, LoS and non-LoS (NLoS). LoS path component is the direct path between the anchors and the MU, NLoS path component consists of the paths between the anchors and the MU reflected by scatterers, e.g., walls, human bodies, etc. Moreover, according to [37], the complex channel gain of LoS is given by

\[
\alpha_{1,i} = \frac{\lambda_c e^{-j2\pi d_{1,i}}}{4\pi d_{1,i}}
\]
Algorithm 1 Joint Angle Estimation Error Analysis and 3-D Positioning Framework

1: Estimate \((\Theta_{1,i}, \Phi_{1,i})\) by using the 2D-DFT estimation algorithm;
2: Derive the PDF of the angle estimation error;
3: Derive the closed-form expression of the variance of the angle estimation error;
4: Estimate the 3-D position of the MU by using the variance of the angle estimation error.

B. Received Signal Model

Accordingly, the received signal from the MU to the \(i\)th anchor at time slot \(t\) could be expressed as

\[
y_i(t) = h_i\sqrt{p} s_i(t) + n(t)
\]

where \(s_i(t)\) denotes the transmitted pilot signal of the MU and \(n(t) \in \mathbb{C}^{N_y \times 1}\) is additive white Gaussian noise (AWGN) following the distribution of \(\mathcal{CN}(0, \sigma^2 I)\). Moreover, \(p\) is the transmit power of the MU.

Besides, since the NLoS path component varies fast and its contribution to the channel is marginal, especially in the mmWave band, we are more interested in the LoS path component \([38], [39], [40]\). Thus, we consider the NLoS path component as a noise component and intend to estimate the LoS path component in this article. Therefore, the received signal can be further written as

\[
y_i(t) = \hat{h}_i\sqrt{p} s_i(t) + \tilde{n}(t)
\]

where \(\hat{h}_i = \alpha_{1,i} a(\Theta_{1,i}, \Phi_{1,i})\) and \(\tilde{n}(t) = \sum_{n=2}^{N_y} \alpha_{n,i} a(\Theta_{n,i}, \Phi_{n,i})\sqrt{p} s_i(t) + n(t)\).

C. Framework

In the existing works concerning the positioning algorithm design, for tractability, the estimation error of channel parameters is assumed to be additive zero-mean complex Gaussian noise \([41]\). However, in practice, the distribution of these channel parameters depends on the practical estimation methods, which may not follow the Gaussian distribution. To investigate the impact of the practical estimation error of the channel parameters, we propose to design a comprehensive framework to jointly analyze the angle estimation error and estimate the 3-D position of the MU.

Based on the 2D-DFT angle estimation technique \([12]\), the angle estimation error analysis and 3-D positioning algorithm design is investigated in this article. First, we apply the 2D-DFT algorithm to estimate \((\Theta_{1,i}, \Phi_{1,i})\) by using the received signal \(y(t)\) in (7). Then, based on the estimation of the AoA, we first derive the PDF of the angle estimation error, based on which the closed-form expression of the variance of the angle estimation error is derived. Finally, we apply the TSWLS algorithm to estimate the 3-D position of the MU by using the estimated AoA and the derived variance.

The details of the proposed framework are summarized in Algorithm 1. The descriptions of each step of the proposed framework will be introduced in the following sections.

III. ANGLE ESTIMATION

The first step of the proposed framework is to estimate \((\Theta_{1,i}, \Phi_{1,i})\) by using the received signal \(y(t)\) in (7). Hence, in this section, we apply the 2D-DFT algorithm \([12]\) to estimate the AoA at the anchors.

A. Initial Angle Estimation

According to the expression of \(a(\Theta_{1,i}, \Phi_{1,i})\) in (1), \(a(\Theta_{1,i}, \Phi_{1,i})\) can be further derived as

\[
a(\Theta_{1,i}, \Phi_{1,i}) = \vec{a}(\Theta_{1,i}, \Phi_{1,i}) \end{align*}
\]

To estimate the AoA at the \(i\)th anchor, we define two DFT matrices \(F_{Ni}\) and \(F_{Nz}\), elements of which are written as \(\left[F_{Ni}^{(j)}\right]_{b_i b_i'} = e^{-j(2\pi/NI) b_i b_i'}\) \((b_i, b_i' = 0, 1, \ldots, N_i - 1)\) and \(\left[F_{Nz}^{(j)}\right]_{q_i q_i'} = e^{-j(2\pi/Nz) q_i q_i'}\) \((q_i, q_i' = 0, 1, \ldots, N_z - 1)\), respectively. Meanwhile, let us define \(u_{1,i} = (2\pi d_i \sin \Theta_{1,i} / \lambda_c)\) and \(v_{1,i} = (2\pi d_i \sin \Phi_{1,i} / \lambda_c)\). Then, we define the normalized 2D-DFT of the matrix \(A(\Theta_{1,i}, \Phi_{1,i})\) in (8) as \(\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i}) = F_{Ni} A(\Theta_{1,i}, \Phi_{1,i}) F_{Nz}^\dagger\), whose \((b_i, q_i)\)th element is calculated as

\[
\begin{align*}
\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i}) & \in \mathbb{C}^{N_i \times N_z} \\
& = \sum_{n_i=0}^{N_i-1} \sum_{n_z=0}^{N_z-1} \left[A(\Theta_{1,i}, \Phi_{1,i})\right]_{n_i n_z} e^{-j2\pi \left(\frac{n_i b_i}{N_i} - \frac{n_z q_i}{N_z}\right)} \\
& = e^{j\frac{N_i - 1}{2} \left(\frac{u_{1,i} - 2\pi b_i}{N_i}\right)} e^{j\frac{N_z - 1}{2} \left(\frac{v_{1,i} - 2\pi q_i}{N_z}\right)} \\
& \times \frac{\sin\left(\pi b_i - \frac{N_i u_{1,i}}{2}\right)/N_i}{\sin\left(\frac{\pi b_i - \frac{N_i u_{1,i}}{2}}{N_i}\right)} \cdot \frac{\sin\left(\pi q_i - \frac{N_z v_{1,i}}{2}\right)/N_z}{\sin\left(\frac{\pi q_i - \frac{N_z v_{1,i}}{2}}{N_z}\right)}.
\end{align*}
\]

When the number of antennas becomes infinite, i.e., \(N_i \to \infty, N_z \to \infty\), there always exist some integers \(b_n = (N_i u_{1,i}/2\pi), q_n = (N_z v_{1,i}/2\pi)\) such that \(\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i})\left|_{b_n q_n} = 1\), while the other elements are all zero. Therefore, all power is concentrated on the \((b_n, q_n)\)th element and \(\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i})\) is a sparse matrix. However, the anchor size could not be infinitely large, thus \(N_i u_{1,i}/2\pi\) and \(N_z v_{1,i}/2\pi\) may not be integers in general, which leads to the channel power leakage from the \((b_n, q_n)\)th element to its adjacent element. However, \(\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i})\) can still be approximated as a sparse matrix with the most power concentrated around the \((b_n, q_n)\)th element. Therefore, the peak power position of \(\tilde{A}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i})\) is still useful for estimating the AoA at the anchor. Then, the initial estimation is derived as follows:

\[
\begin{align*}
\hat{\Theta}_{1,i}^{\text{ini}} \cos \hat{\Phi}_{1,i}^{\text{ini}} &= \frac{\lambda_c b_n}{N_i d_i} \\
\sin \hat{\Phi}_{1,i}^{\text{ini}} &= \frac{\lambda_c q_n}{N_z d_i},
\end{align*}
\]

where \(\hat{\Theta}_{1,i}^{\text{ini}}\) and \(\hat{\Phi}_{1,i}^{\text{ini}}\) denote the initial estimated angles at the anchor.
B. Fine Angle Estimation

The resolution of the estimated \( \hat{\Phi}_{1,i} \) and \( \cos \hat{\Phi}_{1,i} \) is limited by the half of the DFT interval. In order to improve the estimation accuracy, angle rotation is provided to solve the mismatch issue in this section [36].

Let us define the angle rotation matrix of \( A(\Theta_{1,i}, \Phi_{1,i}) \) as \( A^{\circ}(\Theta_{1,i}, \Phi_{1,i}) \), expressed as

\[
A^{\circ}(\Theta_{1,i}, \Phi_{1,i}) = U_{N}(\hat{\Theta}_{1,i})A(\Theta_{1,i}, \Phi_{1,i})U_{N}(\hat{\Phi}_{1,i})
\]

where the diagonal matrices \( U_{N}(\hat{\Theta}_{1,i}) \) and \( U_{N}(\hat{\Phi}_{1,i}) \) are given by

\[
U_{N}(\hat{\Theta}_{1,i}) = \text{Diag}\{1, e^{i\hat{\Theta}_{1,i}}, \ldots, e^{i(N_{s} - 1)\hat{\Theta}_{1,i}}\}
\]

\[
U_{N}(\hat{\Phi}_{1,i}) = \text{Diag}\{1, e^{i\hat{\Phi}_{1,i}}, \ldots, e^{i(N_{s} - 1)\hat{\Phi}_{1,i}}\}
\]

with \( \hat{\Theta}_{1,i} \in [-\pi/N_s, \pi/N_s] \) and \( \hat{\Phi}_{1,i} \in [-\pi/N_s, \pi/N_s] \) being the angle rotation parameters. By using the angle rotation operation, the \((b_i, q_i)\)th element of the 2D-DFT of the rotated matrix \( A^{\circ}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i}) \) is calculated as

\[
\begin{align*}
[A^{\circ}_{\text{DFT}}(\Theta_{1,i}, \Phi_{1,i})]_{b_i q_i} & = \sum_{n_i=0}^{N_s - 1} \sum_{n=0}^{N-1} [A(\Theta_{1,i}, \Phi_{1,i})]_{b_i q_i} \\
& = \sum_{n_i=0}^{N_s - 1} \sum_{n=0}^{N-1} e^{-j2\pi \left( \frac{kn_i}{N_s} + \frac{qn_i}{N} \right)} e^{j2\pi \left( \frac{\hat{\Theta}_{1,i} - n_{s} \hat{\Theta}_{1,i}}{2N} \right) - \frac{j2\pi \hat{\Phi}_{1,i}}{N} \\
& = e^{j\frac{N_s-1}{2} \left( n_i + \hat{\Theta}_{1,i} - \frac{2n_i \hat{\Theta}_{1,i}}{N} \right)} e^{j\frac{N_s-1}{2} \left( \hat{\Phi}_{1,i} - \frac{2n \hat{\Phi}_{1,i}}{N} \right)} \\
& \times \sin \left( \pi \frac{b_i - N_{d_{s} / 2}}{N_s} - \frac{n_i \hat{\Theta}_{1,i}}{N} \right) \\
& \times \sin \left( \pi \frac{q_i - N_{d_i / 2}}{N_s} - \frac{n_i \hat{\Phi}_{1,i}}{N} \right)
\end{align*}
\]

where \( \hat{\Theta}_{1,i} \) and \( \hat{\Phi}_{1,i} \) could be optimized with the 1-D search. Then, we could obtain the estimated results as

\[
\begin{align*}
\cos \hat{\Theta}_{1,i} &= \frac{\lambda_c n_i}{N_s d_i} - \frac{\lambda_c \hat{\Theta}_{1,i}}{2\pi d_i} \\
\sin \hat{\Phi}_{1,i} &= \frac{\lambda_c q_i}{N_s d_i} - \frac{\lambda_c \hat{\Phi}_{1,i}}{2\pi d_i}
\end{align*}
\]

where \( \hat{\Theta}_{1,i} \) and \( \hat{\Phi}_{1,i} \) denote the final estimated azimuth and elevation AoA after angle rotation, respectively. Furthermore, \( \hat{\Theta}_{1,i} \) and \( \hat{\Phi}_{1,i} \) could be expressed as

\[
\begin{align*}
\hat{\Phi}_{1,i} &= \arcsin \left( \frac{\lambda_c n_i}{N_s d_i} - \frac{\lambda_c \hat{\Theta}_{1,i}}{2\pi d_i} \right) \\
\hat{\Theta}_{1,i} &= \arccos \left( \frac{\lambda_c q_i}{N_s d_i} - \frac{\lambda_c \hat{\Phi}_{1,i}}{2\pi d_i} \right)
\end{align*}
\]

Furthermore, the estimation of the AoA (the elevation angle \( \Theta_{n,i} \) and the azimuth angle \( \Phi_{n,i} \)) at the \( i \)th anchor can be also formulated as

\[
\begin{align*}
\hat{\Theta}_{1,i} &= \Theta_{1,i} + \hat{\Theta}_{1,i} \\
\hat{\Phi}_{1,i} &= \Phi_{1,i} + \hat{\Phi}_{1,i}
\end{align*}
\]
Algorithm 2 Algorithm of Deriving the PDF of $\Phi_{1,i}$

1. Derive the PDF of $\cos \Phi_{1,i}$ by using the PDF of $\tilde{Y}_i$ in (20);
2. Derive the PDF of $\cos \Theta_{1,i}$ for two cases by using the PDF of $\cos \Phi_{1,i}$ in (33) and the PDF of $\cos \Phi_{1,i} \cos \Theta_{1,i}$ in (35);
3. Derive the PDF of $\tilde{\Theta}_{1,i}$ based on the two cases of the PDF of $\cos \Theta_{1,i}$ in (44) and (45).

As we have $\tilde{\Phi}_{1,i} = \Phi_{1,i} + \tilde{\Phi}_{1,i}$, the CDF of the estimation error $\tilde{\Phi}_{1,i}$ is calculated as

$$F_{\tilde{\Phi}_{1,i}}(\tilde{\phi}_{1,i}) = Pr(\tilde{\Phi}_{1,i} \leq \tilde{\phi}_{1,i}) = Pr(\Phi_{1,i} - \Phi_{1,i} \leq \tilde{\phi}_{1,i}) = Pr(\Phi_{1,i} \geq \Phi_{1,i} - \tilde{\phi}_{1,i}) = 1 - Pr(\Phi_{1,i} \leq \Phi_{1,i} - \tilde{\phi}_{1,i}) .$$

(23)

Define $a_{1,i} = \arcsin(\tilde{Y}_i + a_i)$ and $a_{2,i} = \arcsin(\tilde{Y}_i - a_i)$. Based on (23), the PDF of $\tilde{\Phi}_{1,i}$ is written as

$$f_{\tilde{\Phi}_{1,i}}(\tilde{\phi}_{1,i}) = \frac{\partial F_{\tilde{\Phi}_{1,i}}(\tilde{\phi}_{1,i})}{\partial \tilde{\phi}_{1,i}} = f_{\Phi_{1,i}}(\Phi_{1,i} - \tilde{\phi}_{1,i})$$

$$= \begin{cases} \frac{\cos(\tilde{\phi}_{1,i} - \tilde{\phi}_{1,i})}{2\pi a_i}, & \tilde{\phi}_{1,i} - a_{1,i} \leq \tilde{\phi}_{1,i} \leq \tilde{\phi}_{1,i} - a_{2,i} \quad (24) \\ 0, & \text{others.} \end{cases}$$

Moreover, as the estimation techniques are relatively mature, it is assumed that the estimation error is very small. Therefore, we have $\sin(\tilde{\phi}_{1,i}) \approx \tilde{\phi}_{1,i}$ and $\cos(\tilde{\phi}_{1,i}) \approx 1$. Consequently, we have the following approximation:

$$\cos(\tilde{\Phi}_{1,i} - \tilde{\phi}_{1,i}) = \cos \tilde{\Phi}_{1,i} \cos \tilde{\phi}_{1,i} + \sin \tilde{\Phi}_{1,i} \sin \tilde{\phi}_{1,i} \approx \cos \tilde{\Phi}_{1,i} + \tilde{\phi}_{1,i} \sin \tilde{\Phi}_{1,i} .$$

(25)

To simplify the variance calculation of $\tilde{\Phi}_{1,i}$ in the subsequent section, we approximate the PDF of $\tilde{\Phi}_{1,i}$ as

$$f_{\tilde{\Phi}_{1,i}}(\tilde{\phi}_{1,i}) \approx \begin{cases} \cos(\tilde{\phi}_{1,i} + \tilde{\phi}_{1,i}) \sin \tilde{\phi}_{1,i}, & \tilde{\phi}_{1,i} - a_{1,i} \leq \tilde{\phi}_{1,i} \leq \tilde{\phi}_{1,i} - a_{2,i} \quad (26) \\ 0, & \text{others.} \end{cases}$$

B. PDF of $\tilde{\Theta}_{1,i}$

In this section, we derive the PDF of the estimation error $\tilde{\Theta}_{1,i}$. As the derivations are complicated, we summarize the main procedure in Algorithm 2.

1) PDF of $\cos \Phi_{1,i}$: First of all, we need to derive the PDF of $\cos \Phi_{1,i}$, so that the PDF of $\cos \Theta_{1,i}$ could be calculated.

Let $X_i = \cos \Phi_{1,i}$, which can be expressed as a function of $\tilde{Y}_i$ as follows:

$$X_i = \cos \Phi_{1,i} = \sqrt{1 - \sin^2 \Phi_{1,i}} = \sqrt{1 - Y_i^2}$$

$$= \sqrt{1 - (\tilde{Y}_i - \tilde{Y}_i)^2} .$$

(27)

By defining the estimation of $X_i$ as $\hat{X}_i = \cos \Phi_{1,i} = X_i + \tilde{X}_i$, the estimation error $\tilde{X}_i$ could be calculated as

$$\tilde{X}_i = \hat{X}_i - \sqrt{1 - (\tilde{Y}_i - \tilde{Y}_i)^2} .$$

(28)

However, the expression (28) is complicated and thus challenging to derive a compact form of the PDF of $\tilde{X}_i$. Fortunately, since the value of $\tilde{Y}_i$ is relatively small, we can approximate $\hat{X}_i$ in (28) by using the Taylor expansion, which is given by

$$\hat{X}_i = \sqrt{1 - Y_i^2} - \frac{1}{2} \left(1 - (\tilde{Y}_i)^2\right)^{1/2}$$

$$\approx \sqrt{1 - Y_i^2} - \left[1 - \left(\tilde{Y}_i^2\right)^{1/2} + \frac{1}{2} \tilde{Y}_i^2 \right]$$

$$= -\frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} \tilde{Y}_i .$$

(29)

Utilizing (29), the CDF of $\tilde{X}_i$ could be calculated as

$$F_{\tilde{X}_i}(\tilde{x}_i) = Pr(\tilde{X}_i \leq \tilde{x}_i) \approx Pr\left(\frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} \tilde{x}_i \leq \tilde{Y}_i \right)$$

$$= Pr(\tilde{Y}_i \leq -\frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} \tilde{x}_i) = 1 - Pr\left(\tilde{Y}_i \leq -\frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} \tilde{x}_i \right) .$$

(30)

Then, by using (30), the PDF of $\tilde{X}_i$ could be calculated as

$$f_{\tilde{X}_i}(\tilde{x}_i) = \frac{\partial F_{\tilde{X}_i}(\tilde{x}_i)}{\partial \tilde{x}_i} = f_{\tilde{Y}_i}(\tilde{Y}_i) \left(\frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}}\right)$$

$$= \begin{cases} \frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} f_{\tilde{Y}_i}(\tilde{Y}_i) a_i, & \tilde{x}_i \leq \tilde{x}_i \leq \tilde{x}_i a_i \\ 0, & \text{others.} \end{cases}$$

(31)

Furthermore, by using $\hat{X}_i = X_i + \tilde{X}_i$, the CDF of $X_i$ can be calculated as

$$F_{X_i}(x_i) = Pr(X_i \leq x_i) = Pr(\hat{X}_i - \tilde{X}_i \leq x_i)$$

$$= Pr(\hat{X}_i \geq \hat{X}_i - x_i) = 1 - Pr(\hat{X}_i \leq \hat{X}_i - x_i) .$$

(32)

By using (31) and (32), the PDF of $X_i$ could be written as

$$f_{X_i}(x_i) = \frac{\partial F_{X_i}(x_i)}{\partial x_i} = f_{\tilde{X}_i}(\hat{X}_i - x_i)$$

$$= \begin{cases} \frac{\tilde{Y}_i}{\sqrt{1 - Y_i^2}} f_{\tilde{Y}_i}(\tilde{Y}_i) a_i, & \tilde{x}_i \leq \tilde{x}_i \leq \tilde{x}_i a_i \\ 0, & \text{others.} \end{cases}$$

(33)

2) PDF of $\cos \Theta_{1,i}$: By using the PDF of $\cos \Phi_{1,i}$ and $\cos \Theta_{1,i}$, we can derive the PDF of $\cos \Theta_{1,i}$ as follows.

First, from the above section, we know that $\tilde{\Theta}_{1,i}$ could be obtained by using the 1-D search in the interval of $[-(\pi/2), (\pi/2)]$. Similarly, we assume that there are $S_1$ grids points in the interval $[-(\pi/2), (\pi/2)]$ and $s_{1,i} \in \{1, \ldots, S_1\}$ is the optimal point. Therefore, the optimal solution for the 1-D search is $\tilde{\Theta}_{1,i} = (2\pi s_{1,i}/N_1 S_1)$, and the estimated $\cos \Phi_{1,i} \cos \Theta_{1,i}$ is given by

$$\hat{Z} = \cos \Theta_{1,i} \cos \Phi_{1,i} = \frac{\lambda_c b_{mi}}{N_1 d_1} - \frac{\lambda_c \tilde{\Theta}_{1,i}}{2\pi d_r}$$

$$= \frac{\lambda_c b_{mi}}{N_1 d_1} - \frac{\lambda_c s_{1,i}}{N_1 d_1 S_1} .$$

(34)

By using (34) and the nature of the 1-D search method, the real value of $Z_i = \cos \Theta_{1,i} \cos \Phi_{1,i}$ follows the uniform distribution within the region of $[\hat{Z} - b_i, \hat{Z} + b_i]$, where $b_i = (\lambda_c / [2N_1 d_1 S_1])$. The PDF of $Z_i$ is thus given by

$$f_{Z_i}(z_i) = \begin{cases} \frac{1}{2b_i}, & \hat{Z}_i - b_i \leq z_i \leq \hat{Z}_i + b_i \\ 0, & \text{others.} \end{cases}$$

(35)
By denoting the estimation error of $Z_i$ as $\tilde{Z}_i$, we have $\hat{Z}_i = Z_i + \tilde{Z}_i$. Then, the distribution of $\tilde{Z}_i$ is given by

$$f_{\tilde{Z}_i}(\tilde{z}_i) = \begin{cases} \frac{1}{2b_i}, & -b_i \leq \tilde{z}_i \leq b_i \\ 0, & \text{others.} \end{cases}$$  \hfill (36)

Next, for simplicity, let us denote $U_i = \cos \Theta_{1,i}$. Then, by utilizing the definition of $Z_i$ below (34) and $X_i$ above (27), we have $Z_i = U_iX_i$. By combining the PDF of $X_i$ in (33) with the PDF of $Z_i$ in (35), we can derive the PDF of $U_i$ as follows:

$$f_U(u_i) = \int_{\tilde{Z}_i} x_f(x_i)f_{\tilde{Z}_i}(u_i)dx.$$  \hfill (37)

According to (35), it is observed that $f_U(u_i)$ is nonzero when $\hat{Z}_i - b_i \leq u_i x \leq \hat{Z}_i + b_i$, which determines the PDF of $U_i$. Therefore, we need to discuss the different conditions according to the nonzero intervals of $f_U(u_i)$ and $f_{\tilde{Z}_i}(x)$ in the following. For notation brevity, we denote $\alpha_{1,i} = \hat{X}_i - (\hat{Y}_i/\hat{X}_i)a_i$, $\alpha_{2,i} = \hat{X}_i + (\hat{Y}_i/\hat{X}_i)a_i$, $\beta_{1,i} = \hat{Z}_i - b_i$ and $\beta_{2,i} = \hat{Z}_i + b_i$.

**Condition 1:** If $\alpha_{1,i} < (\beta_{1,i}/u_i) \leq \alpha_{2,i} < (\beta_{2,i}/u_i)$, the integral interval of $x$ in (37) can be recast as $[\beta_{1,i}/u_i, \alpha_{2,i}]$, thereby yielding the PDF of $U_i$ as

$$f_U(u_i) = \int_{\beta_{1,i}/u_i}^{\alpha_{2,i}} \frac{1}{2b_i} \cdot \frac{\hat{X}_i}{4\hat{a}_i\hat{Y}_i} \cdot xdx = \frac{\hat{X}_i}{8\hat{a}_i\hat{b}_i\hat{Y}_i} \left( \alpha_{2,i}^2 - \left( \frac{\beta_{1,i}}{u_i} \right)^2 \right).$$  \hfill (38)

Furthermore, the interval of $u_i$ is given by

$$\left[ \frac{\beta_{1,i}}{\alpha_{2,i}}, \frac{\beta_{1,i}}{\alpha_{1,i}} \right] \cap \left( -\infty, \frac{\beta_{2,i}}{\alpha_{2,i}} \right).$$  \hfill (39)

Based on (39), it is necessary to compare $(\beta_{1,i}/\alpha_{1,i})$ with $(\beta_{2,i}/\alpha_{2,i})$, so that the interval of $u_i$ can be further determined. If $(\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i})$ holds, the interval can be rewritten as $[(\beta_{1,i}/\alpha_{1,i}), (\beta_{1,i}/\alpha_{1,i})]$. Otherwise, the interval is $[(\beta_{1,i}/\alpha_{1,i}), (\beta_{2,i}/\alpha_{2,i})]$.

**Condition 2:** If $\alpha_{1,i} \leq \alpha_{1,i} < (\beta_{1,i}/u_i) \leq \alpha_{2,i}$, the integral interval of $x$ in (37) can be recast as $[\alpha_{1,i}, (\beta_{1,i}/u_i)]$, thus the PDF of $U_i$ is given by

$$f_U(u_i) = \int_{\alpha_{1,i}}^{\beta_{1,i}/u_i} \frac{1}{2b_i} \cdot \frac{\hat{X}_i}{4\hat{a}_i\hat{Y}_i} \cdot xdx = \frac{\hat{X}_i}{8\hat{a}_i\hat{b}_i\hat{Y}_i} \left( \beta_{1,i}^2 - \alpha_{1,i}^2 \right).$$  \hfill (40)

The interval of $u_i$ is given by

$$\left[ \frac{\beta_{1,i}}{\alpha_{2,i}}, \frac{\beta_{1,i}}{\alpha_{1,i}} \right] \cap \left[ \frac{\beta_{1,i}}{\alpha_{1,i}}, +\infty \right).$$  \hfill (41)

As a result, if $(\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i})$ holds, the interval can be further recast as $[(\beta_{2,i}/\alpha_{2,i}), (\beta_{2,i}/\alpha_{2,i})]$. Otherwise, the interval can be derived as $[(\beta_{1,i}/\alpha_{1,i}), (\beta_{2,i}/\alpha_{1,i})]$.

**Condition 3:** If $(\beta_{1,i}/u_i) \leq \alpha_{1,i} < \alpha_{2,i} < (\beta_{2,i}/u_i)$, the integral interval of $x$ in (37) can be derived as $[\alpha_{1,i}, \alpha_{2,i}]$. Thus, the PDF of $U_i$ is

$$f_U(u_i) = \int_{\alpha_{1,i}}^{\alpha_{2,i}} \frac{1}{2b_i} \cdot \frac{\hat{X}_i}{2a_iY_i} \cdot xdx = \frac{\hat{X}_i}{4\hat{a}_i\hat{b}_i\hat{Y}_i} \int_{\alpha_{1,i}}^{\alpha_{2,i}} xdx.$$  \hfill (42)
Furthermore, we consider the indoor positioning system in this article, where the anchors are supposed to be mounted on the wall. Hence, we have \( \Theta_{1,i} \in (0, \pi) \). Thus, \( \Theta_{1,i} \) decreases monotonically with \( U_i \). According to the PDF of \( U_i \) in the aforementioned two cases, we can derive the PDF of \( \Theta_{1,i} \) in the following.

Case 1: When \( (\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i}) \), according to (44), we can derive the PDF of \( \Theta_{1,i} \), which is given by (48), shown at the bottom of the page.

Case 2: When \( (\beta_{2,i}/\alpha_{2,i}) < (\beta_{1,i}/\alpha_{1,i}) \), we can derive the PDF of \( \Theta_{1,i} \), which is given by (49), shown at the bottom of the page.

4) PDF of \( \Theta_{1,i} \): By using the CDF and PDF of \( \Theta_{1,i} \) in (46), (48), and (49), we can calculate the CDF and PDF of \( \Theta_{1,i} \) as follows.

First, based on \( \hat{\Theta}_{1,i} = \Theta_{1,i} + \hat{\Theta}_{1,i} \), the CDF of \( \Theta_{1,i} \) can be derived as follows:

\[
F_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) = \Pr(\hat{\Theta}_{1,i} \leq \Theta_{1,i}) = \Pr(\Theta_{1,i} - \hat{\Theta}_{1,i} \leq \hat{\Theta}_{1,i}) = \Pr(\Theta_{1,i} \geq \hat{\Theta}_{1,i} - \hat{\Theta}_{1,i}) = 1 - \Pr(\Theta_{1,i} \leq \hat{\Theta}_{1,i} - \hat{\Theta}_{1,i}).
\]

Then, based on the above two cases of \( f_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) \) in (48) and (49), the PDF of \( \Theta_{1,i} \) can be derived accordingly by using (50).

Case 1: When \( (\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i}) \), by using (48), the PDF of \( \Theta_{1,i} \) is calculated as (51), shown at the bottom of the page.

Case 2: When \( (\beta_{2,i}/\alpha_{2,i}) < (\beta_{1,i}/\alpha_{1,i}) \), we can derive the PDF of \( \Theta_{1,i} \) by using (49), which is given by (52), shown at the bottom of the next page.

To facilitate the error analysis, we now aim to derive the approximation of \( f_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) \) in this article. As we have assumed that the estimation error is very small, we have \( \sin \hat{\Theta}_{1,i} \approx \sin \hat{\Theta}_{1,i} \) and \( \cos \hat{\Theta}_{1,i} \approx 1 \), leading to

\[
\sin(\Theta_{1,i} - \hat{\Theta}_{1,i}) \approx \sin \hat{\Theta}_{1,i} - \hat{\Theta}_{1,i} \cos \hat{\Theta}_{1,i},
\]

\[
\cos(\Theta_{1,i} - \hat{\Theta}_{1,i}) \approx \cos \hat{\Theta}_{1,i} + \hat{\Theta}_{1,i} \sin \hat{\Theta}_{1,i}.
\]

Using the approximations in (53), we can derive the approximation of \( f_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) \) according to the above two cases. Furthermore, by denoting \( \sin \hat{\Theta}_{1,i} = \hat{Y}_i \cos \hat{\Theta}_{1,i} = \hat{U}_i \), \( B_{1,i} = \hat{\Theta}_{1,i} - \arccos(\hat{\beta}_{1,i}/\hat{\alpha}_{1,i}) \), \( B_{2,i} = \hat{\Theta}_{1,i} - \arccos(\hat{\beta}_{1,i}/\hat{\alpha}_{1,i}) \), the expression could be further simplified in the following.

Case 1: When \( (\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i}) \), based on (51), we can derive the approximation of \( f_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) \), which is written as (54), shown at the bottom of the next page.

Case 2: When \( (\beta_{2,i}/\alpha_{2,i}) < (\beta_{1,i}/\alpha_{1,i}) \), as we have (52), we could derive the approximation of \( f_{\Theta_{1,i}}(\hat{\Theta}_{1,i}) \), which is written as (55), shown at the bottom of the next page.
V. VARIANCE OF ANGLE ESTIMATION ERROR

In this section, we aim to calculate the variance of $\tilde{\phi}_{1,i}$ and $\tilde{\Theta}_{1,i}$ by using the PDF in the above section, which will be used for the 3-D position estimation in the next section.

A. Variance of $\tilde{\phi}_{1,i}$

In this section, we provide the variance expression of $\tilde{\phi}_{1,i}$. Based on the PDF of $\Phi_{1,i}$ in (26), the variance of $\tilde{\phi}_{1,i}$ can be calculated as

$$D(\tilde{\phi}_{1,i}) = E\left[\tilde{\phi}_{1,i}^2\right] - \left(E\left[\tilde{\phi}_{1,i}\right]\right)^2$$

$$= \frac{1}{2\pi} \left(\frac{\phi_{i,1}^3}{3} \cos \Phi_{1,i} + \frac{\phi_{i,1}^4}{4} \sin \Phi_{1,i}\right)\bigg|_{\Phi_{1,i} = \phi_{1,i}}^{} - \frac{1}{4\pi^2} \left(\frac{\phi_{i,1}^2}{2} \cos \Phi_{1,i} + \frac{\phi_{i,1}^3}{3} \sin \Phi_{1,i}\right)^2 \bigg|_{\Phi_{1,i} = \phi_{1,i}}^{}$$

(56)

where $f(x)|_{x_1}^{x_2} = f(x_1) - f(x_2)$ and $E[\tilde{\phi}_{1,i}]$ denotes the expectation of $\tilde{\phi}_{1,i}$.

B. Variance of $\tilde{\Theta}_{1,i}$

In this section, we aim to derive the variance of $\tilde{\Theta}_{1,i}$. Different from the PDF of $\Phi_{1,i}$, the PDF of $\Theta_{1,i}$ is more complicated. Therefore, we need to analyze the variance according to the aforementioned two cases as follows.

Case 1: When $(\beta_{2,i}/\alpha_{2,i}) > (\beta_{1,i}/\alpha_{1,i})$, based on the PDF of $\tilde{\Theta}_{1,i}$ in (54), the variance of $\tilde{\Theta}_{1,i}$ is derived as

$$D(\tilde{\Theta}_{1,i}) = \int_{D_{1,i}} \tilde{\Theta}_{1,i} f_{\tilde{\Theta}_{1,i}}(\tilde{\Theta}_{1,i}) d\tilde{\Theta}_{1,i} - \left(\int_{D_{1,i}} \tilde{\Theta}_{1,i} f_{\tilde{\Theta}_{1,i}}(\tilde{\Theta}_{1,i}) d\tilde{\Theta}_{1,i}\right)^2$$

(57)

For $D_{1,i}$ in (57), we divide it into three different nonzero intervals that can be expressed as

$$D_{1,i} = D_{11,i} + D_{12,i} + D_{13,i}$$

(58)

where $D_{11,i}$, $D_{12,i}$ and $D_{13,i}$ are the integral expressions in the intervals of $[B_{1,i}, B_{2,i}]$, $[B_{2,i}, B_{3,i}]$ and $[B_{3,i}, B_{4,i}]$, respectively. The expressions of $D_{11,i}$, $D_{12,i}$, and $D_{13,i}$ are given in Appendix A.

$$f_{\tilde{\Theta}_{1,i}}(\tilde{\Theta}_{1,i}) = \frac{\partial F_{\tilde{\Theta}_{1,i}}(\tilde{\Theta}_{1,i})}{\partial \tilde{\Theta}_{1,i}} = f_{\Theta_{1,i}}(\tilde{\Theta}_{1,i} - \tilde{\Theta}_{1,i})$$

(52)

$$= \frac{\tilde{X}_i \sin(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}{8a_i b_i Y_i} \left[\frac{\phi_{2,i}}{2} - \frac{\beta_{2,i}}{\alpha_{2,i} \cos(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}\right]^2$$

$$+ \frac{\tilde{X}_i \tilde{X}_i \sin(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}{8a_i b_i Y_i} \left[\frac{\phi_{2,i}}{2} - \frac{\beta_{2,i}}{\alpha_{2,i} \cos(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}\right]^2$$

$$+ \frac{\tilde{X}_i \tilde{X}_i \sin(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}{8a_i b_i Y_i} \left[\frac{\phi_{2,i}}{2} - \frac{\beta_{2,i}}{\alpha_{2,i} \cos(\tilde{\Phi}_{1,i} - \tilde{\Theta}_{1,i})}\right]^2$$

$$0,$$
For $D_{2,i}$ in (57), it is the expectation of $\hat{\theta}_{1,i}$, which is denoted by $E(\hat{\theta}_{1,i})$. It can be divided into three different nonzero intervals that are given by 
\[
D_{2,i} = D_{21,i} + D_{22,i} + D_{23,i}
\]
where $D_{21,i}$, $D_{22,i}$, and $D_{23,i}$ are the integral expressions in the intervals of $[B_{1,i}, B_{2,i})$, $[B_{2,i}, B_{3,i})$, and $[B_{3,i}, B_{4,i}]$, respectively. The expressions of $D_{21,i}$, $D_{22,i}$, and $D_{23,i}$ are given in Appendix B.

Case 2: When $(\beta_{2,i}/\alpha_{2,i}) < (\beta_{1,i}/\alpha_{1,i})$, according to the PDF of $\hat{\theta}_{1,i}$ in (55), the variance of $\hat{\theta}_{1,i}$ can be calculated as 
\[
D'(\hat{\theta}_{1,i}) = \frac{2}{\pi} \frac{\sin \theta_{1,i}}{\cos^2 \theta_{1,i}} \left( \frac{\sin \theta_{1,i}}{\cos \theta_{1,i}} \right) d\theta_{1,i}
\]
(60) 
\[
D'_{2,i} = D'_{21,i} + D'_{22,i} + D'_{23,i}
\]
where $D'_{21,i}$, $D'_{22,i}$, and $D'_{23,i}$ are the integral expressions in the intervals of $[B_{1,i}, B_{2,i})$, $[B_{2,i}, B_{3,i})$, and $[B_{3,i}, B_{4,i}]$. The expressions of $D'_{21,i}$, $D'_{22,i}$, and $D'_{23,i}$ are given in Appendix C.

For $D'_{2,i}$, we divide it into three different nonzero intervals that are given by 
\[
D'_{2,i} = D'_{11,i} + D'_{12,i} + D'_{13,i}
\]
where $D'_{11,i}$, $D'_{12,i}$, and $D'_{13,i}$ are the integral expressions in the intervals of $[B_{1,i}, B_{2,i})$, $[B_{2,i}, B_{3,i})$, and $[B_{3,i}, B_{4,i}]$. The expressions of $D'_{11,i}$, $D'_{12,i}$, and $D'_{13,i}$ are given in Appendix D.

VI. 3-D POSITION ESTIMATION

Using the estimated AoA in Section III and the variance of the estimation error in Section V, we aim to derive the expression of the estimation of the MU’s 3-D position. First, we have the AoA at the $i$th anchor $\hat{\theta}_{1,i}$ and $\hat{\phi}_{1,i}$ given in (16). For the sake of illustration, we collect all the estimated AoA in the following vectors:
\[
\hat{\Theta} = \Phi + \Theta
\]
(63) 
\[
\hat{\Phi} = \Phi + \Phi
\]
where 
\[
\Theta = \begin{bmatrix} \hat{\theta}_{L,1}, \ldots, \hat{\theta}_{L,J} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \hat{\phi}_{L,1}, \ldots, \hat{\phi}_{L,J} \end{bmatrix}
\]
\[
\hat{\Theta} = \begin{bmatrix} \hat{\theta}_{L,1}, \ldots, \hat{\theta}_{L,J} \end{bmatrix}, \quad \hat{\Phi} = \begin{bmatrix} \hat{\phi}_{L,1}, \ldots, \hat{\phi}_{L,J} \end{bmatrix}
\]
(64)

In the existing works, for tractability, $\hat{\Theta}$ and $\hat{\Phi}$ are assumed to be the additive complex Gaussian noise with zero mean. However, according to the angle estimation error analysis in our previous section, the PDF of the estimation error $\Phi$ should be modeled as (26), while the PDF of the estimation error $\hat{\Theta}$ should be modeled as (54) or (55). Furthermore, the covariance matrices can be written as 
\[
Q_{\Theta} = \text{diag}\left[ \sigma_{\Theta_{1,1}}^2, \ldots, \sigma_{\Theta_{J,J}}^2 \right]
\]
\[
Q_{\Phi} = \text{diag}\left[ \sigma_{\Phi_{1,1}}^2, \ldots, \sigma_{\Phi_{J,J}}^2 \right]
\]
(65)
where $\sigma_{\Theta_{1,1}}^2$ and $\sigma_{\Phi_{1,1}}^2$ denote the variance of $\hat{\theta}_{1,i}$ and $\hat{\phi}_{1,i}$, respectively. $\sigma_{\Phi_{1,1}}^2$ can be calculated according to (56), and $\sigma_{\Theta_{1,1}}^2$ can be calculated according to (57) or (60).

Accordingly, we propose to derive the closed-form expression of the MU’s estimated 3-D position. First, we can derive the pseudolinear equations [42] based on the estimated AoA as follows:
\[
\hat{g}_{\Theta}^T s_i - \hat{g}_{\Theta}^T q \approx -\hat{\theta}_{1,i} d_{1,i} \cos \hat{\phi}_{1,i}
\]
\[
\hat{g}_{\Phi}^T s_i - \hat{g}_{\Phi}^T q \approx -\hat{\phi}_{1,i} d_{1,i}
\]
(66)
where $d_{1,i}$ denotes the distance between the $i$th anchor and the MU, and we have 
\[
\hat{g}_{\Theta} = \begin{bmatrix} -\cos \hat{\Theta}_{1,i}, \sin \hat{\Theta}_{1,i}, 0 \end{bmatrix}^T
\]
\[
\hat{g}_{\Phi} = \begin{bmatrix} -\sin \hat{\Phi}_{1,i}, \sin \hat{\Phi}_{1,i}, \cos \hat{\phi}_{1,i}, -\cos \hat{\phi}_{1,i} \end{bmatrix}^T
\]
(67)
As a result, we can derive the following compact form of equations as:
\[
\hat{h} - \hat{G} q = B z
\]
(68)
where 
\[
\hat{h} = \begin{bmatrix} 1^T (\hat{G}_{\Theta} \odot S), 1^T (\hat{G}_{\Phi} \odot S) \end{bmatrix}^T, \quad \hat{G} = \begin{bmatrix} \hat{G}_{\Theta}^T, \hat{G}_{\Phi}^T \end{bmatrix}^T
\]
\[
B = [B_1, B_2]^T, \quad z = [\Theta^T, \Phi^T]^T
\]
\[
\hat{G}_{\Theta} = [\hat{g}_{\Theta 1}, \ldots, \hat{g}_{\Theta J}]^T, \quad \hat{G}_{\Phi} = [\hat{g}_{\Phi 1}, \ldots, \hat{g}_{\Phi J}]^T
\]
\[
S = [s_1, \ldots, s_J]^T, \quad B_1 = [B_{\Theta}, O]^T, B_2 = [O, B_{\Phi}]^T
\]
\[
B_{\Theta} = -\text{diag}\left[ d_{L,1} \cos \hat{\phi}_{L,1}, \ldots, d_{L,J} \cos \hat{\phi}_{L,J} \right]^T
\]
\[
B_{\Phi} = -\text{diag}\left[ d_{L,1}, \ldots, d_{L,J} \right]^T.
\]
(69)

Based on (68), we can apply the TSWLS algorithm [42] to derive the closed-form expression of the MU’s estimated position, which is written as 
\[
\hat{q} = \left( \hat{G}^T W \hat{G} \right)^{-1} \hat{G}^T W \hat{h}
\]
(70)
where $W = BQB^T$ and $Q = \text{diag}(Q_{\Theta}, Q_{\Phi})$. The details of the derivation can be found in [42].

VII. SIMULATION RESULTS

This section presents simulation results to validate the accuracy of our derivations and approximations. In our simulation, we consider a TDD mmWave multiple input single output (MISO) channel from the MU to the anchors. Moreover, the MU, the anchors are assumed to be placed in a 3-D area. The locations of four anchors are $s_i = [2, 20, 3]^T$. 

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s_2 = [-12, 15, 50]^T, s_3 = [-10, -6, -8]^T, and s_4 = [10, 6, -20]^T. It is assumed that the interantenna spacing of UPA at the anchors is \( d_r = \lambda_c / 2 \). There are ten paths between the MU and the anchors. The following results are obtained by averaging over 10,000 random estimation error realizations. Unless otherwise stated, we assume \( S = S_1 \times S_2 = 64 \times 64 \) for rotation angle search grids, and the SNR is assumed to be 10 dB. The positioning accuracy is assessed in terms of the mean-square error (MSE) and the bias.

Fig. 2 and 3 illustrate the PDF of estimation errors. It is assumed that the anchor size is \( N_y = N_z = 16 \). It can be observed from Figs. 2 and 3 that our derived results match well with the simulation results, which verify the accuracy of our derived results and confirm that the angle estimation error is non-Gaussian.

Fig. 4 presents the PDF of \( \tilde{\Theta}_{1,i} \) in cases of Gaussian and non-Gaussian. As depicted in Fig. 4, the simulation results demonstrate that as the anchor size increases, the PDF of error based on the Gaussian distribution (referred to as “Gaussian”) gradually approaches the PDF of actual errors (referred to as “Simulation”). In contrast, the PDF derived from our theoretical analysis (referred to as “Theory”) consistently approximates the true error PDF across all scenarios. This finding indicates that our derived PDF offers a broader applicability compared to the Gaussian distribution-based error model.

Figs. 5 and 6 display the variance \( \sigma_{\tilde{\Theta}_{1,i}}^2 \) of estimation error \( \tilde{\Theta}_{1,i} \) and the variance \( \sigma_{\tilde{\Phi}_{1,i}}^2 \) of estimation error \( \tilde{\Phi}_{1,i} \) as the functions of anchor size \( N_y(N_z) \), respectively. Figs. 5 and 6 display the variance \( \sigma_{\tilde{\Theta}_{1,i}}^2 \), which show that the theoretical results coincide with the simulation results, which validates the correctness of the derived results. Moreover, it is observed that the variances of \( \tilde{\Theta}_{1,i} \) and \( \tilde{\Phi}_{1,i} \) decrease with the anchor size, which means that increasing the number of antennas could improve the estimation accuracy.

Fig. 7 clearly demonstrates that the variances in both azimuth and elevation angles decrease gradually with increasing SNR. However, in low-SNR conditions, the impact of noise on angle estimation is more pronounced. As the SNR increases, the two curves approach a plateau, indicating that the influence of SNR on positioning performance diminishes progressively. This observation highlights the crucial role of

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1In this article, the positions of the anchors are fixed for the proposed IoT mmWave positioning system. However, exploring the optimal geometric configuration of the anchors for the IoT mmWave positioning system would be an intriguing avenue for future work.
SNR in angle estimation accuracy, particularly in challenging environments characterized by low-SNR levels. In such conditions, the presence of noise can significantly affect the accuracy of angle estimation. However, as the SNR improves, the impact of noise becomes less significant, leading to more stable and reliable angle estimates.

Fig. 8 compares the MSE of the proposed framework aided by two anchors with that aided by three anchors when the size of anchors increases from \( N_y = N_z = 1 \) to \( N_y = N_z = 20 \). As shown in the figure, the MSE decreases with the anchor size as expected. Furthermore, it is shown that the MSE of two anchors is larger than that of three anchors. It implies that increasing the number of anchors can significantly improve the positioning accuracy of the proposed framework.

In [43], the position of the MU is derived by using the geometry relationship between the estimated AOA and the 3-D position, which is denoted as the geometry algorithm in this article. Fig. 8 also presents the positioning performance comparison of the proposed framework with the geometry algorithm. It is seen from Fig. 8 that the proposed framework outperforms the geometry algorithm even with only two anchors.

The results in Fig. 9 show a significant gap between the curves for \( N_y = N_z = 1 \) and \( N_y = N_z = 2 \). However, the improvement from \( N_y = N_z = 4 \) to \( N_y = N_z = 8 \) is relatively small. This suggests that the impact of anchor size diminishes when \( N_y(N_z) \) exceeds 4, providing insights for optimizing positioning performance. By carefully balancing the dimensions of anchor antennas and the number of anchors, we can enhance localization while reducing costs and energy consumption, contributing to greener IoT deployments. The simulation results in Figs. 10 and 11 illustrate the MSE and the bias performance of three algorithms under varying SNR conditions. As the SNR increases, all algorithms exhibit improved positioning performance, particularly in low-SNR scenarios. However, beyond a certain threshold, typically around 20 to 30 dB, the impact of noise becomes less significant, resulting in negligible changes in the MSE and the bias for all three algorithms. Notably, the geometry algorithm, used as a benchmark, demonstrates relatively consistent performance throughout.

As shown in Fig. 10, when comparing the performance in terms of MSE, the geometry algorithm, which relies on geometric relationships for positioning estimates, shows approximately 10 times higher MSE compared to the WLS algorithm.
and TSWLS algorithms. On the other hand, the TSWLS algorithm consistently outperforms the WLS algorithm, achieving approximately 5 times lower MSE. These results strongly demonstrate the superior performance of our proposed TSWLS algorithm within the framework. It outperforms both the geometry algorithm and the WLS algorithm, providing more accurate and reliable positioning estimations, particularly in challenging SNR conditions. These findings reinforce the effectiveness and superiority of our proposed TSWLS algorithm within the framework. It significantly outperforms both the geometry algorithm and the WLS algorithm, providing more accurate and reliable positioning estimations, especially in challenging SNR conditions.

In the context of the bias comparison depicted in Fig. 11, the geometry algorithm demonstrates a discrepancy over ten times larger than that of the TSWLS algorithm. Conversely, the TSWLS algorithm exhibits a performance enhancement of approximately two to three times when compared to the WLS algorithm. These findings highlight the substantial disparity in bias among the algorithms, emphasizing the superior performance of TSWLS over Geometry and WLS algorithms.

Fig. 10. MSE comparison of the proposed framework with other algorithms versus SNR (dB).

Fig. 11. Bias comparison of the proposed framework with other algorithms versus SNR (dB).

VIII. CONCLUSION

In this article, we designed a comprehensive framework to analyze the angle estimation error and design the 3-D positioning algorithm for the mmWave system. First, we estimated the AoA at the anchors by applying the 2D-DFT algorithm. Based on the property of the 2D-DFT algorithm, the angle estimation error was analyzed in terms of PDF. We then simplified the intricate geometric expression of the error PDF by employing the first-order linear approximation of triangle functions. We also derived the variance expression of the error using the error PDF and theoretically derived the variance from the error PDF. Finally, we applied the WLS algorithm to estimate the 3-D position of the MU using the estimated AoA and the obtained non-Gaussian variance. Extensive simulation results confirmed that the derived angle estimation error was non-Gaussian and demonstrated the superiority of the proposed framework.

APPENDIX A

DERIVATION OF $D_{1,i}$

First, we derive the expression of $D_{1,i}$ as

$$D_{1,i} = \frac{\hat{X}_{i,2}^2}{8a_i b_i \hat{Y}_i} \int_{B_{1,i}}^{B_{2,i}} \left( \hat{V}_i \hat{\theta}_i^3 - \hat{U}_i \hat{\theta}_i^3 \right) d\hat{\theta}_1,i$$

$$- \frac{\hat{X}_{i,1}^2}{8a_i b_i \hat{Y}_i} \int_{B_{1,i}}^{B_{2,i}} \left( \hat{V}_i \hat{\theta}_i^2 - \hat{U}_i \hat{\theta}_i^2 \right) d\hat{\theta}_1,i$$

$$= \frac{\hat{X}_{i,2}^2}{8a_i b_i \hat{Y}_i} \left[ \int_{0}^{B_{2,i}} \left( \hat{V}_i \hat{\theta}_i^2 - \hat{U}_i \hat{\theta}_i^2 \right) d\hat{\theta}_1,i \right]$$

$$- \int_{0}^{B_{2,i}} \frac{\hat{V}_i \hat{\theta}_i^3}{\hat{U}_i} d\hat{\theta}_1,i$$

$$= \frac{\hat{X}_{i,2}^2}{8a_i b_i \hat{Y}_i} \left( \int_{0}^{B_{2,i}} \left( \hat{V}_i \hat{\theta}_i^2 - \hat{U}_i \hat{\theta}_i^2 \right) d\hat{\theta}_1,i \right)$$

$$- \int_{0}^{B_{2,i}} \frac{\hat{V}_i \hat{\theta}_i^3}{\hat{U}_i} d\hat{\theta}_1,i$$

where $\int_{0}^{\infty} [x^{\mu-1}/(1 + \beta x)^{\nu}] = \sum_{i=0}^{\infty} F_i(v, \mu; 1 + \mu; -\beta \mu)$ is a generalized hypergeometric series [44].
Finally, $D_{12,i}$ in (58) is derived as
\[
D_{12,i} = \int_{B_{1,i}} \frac{\hat{X}_i \left( \hat{V}_i - \hat{\theta}_{1,i} \hat{U}_i \right)}{2b} \hat{\theta}_{1,i}^2 d\hat{\theta}_{1,i}
\]
\[
= \frac{\hat{X}_i}{2b_i} \left( \hat{V}_i \hat{\theta}_{1,i}^3 + \hat{U}_i \hat{\theta}_{1,i}^4 \right) \bigg|_{B_{1,i}}.
\]  
(72)

Finally, $D_{13,i}$ in (58) is given by
\[
D_{13,i} = \left[ \left( -\frac{\hat{X}_i \beta_2^2}{8a_i b_i \hat{Y}_i} \hat{U}_i^2 \right) \frac{B_{3,i}}{4} - 2 F_1 \left( 2, 4; 5; -\hat{V}_i \hat{U}_i \right) \right]
\]
\[
+ \frac{\hat{X}_i \beta_2^2 \hat{V}_i}{8a_i b_i \hat{Y}_i} \frac{B_{3,i}}{2} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_1 \hat{U}_i \right)
\]
\[
- \left( -\frac{\hat{X}_i \beta_2^2}{8a_i b_i \hat{Y}_i} \frac{B_{3,i}}{4} - 2 F_1 \left( 2, 4; 5; -\hat{V}_i \hat{B}_3 \hat{U}_i \right) \right]
\]
\[
+ \frac{\hat{X}_i \beta_2^2 \hat{V}_i}{8a_i b_i \hat{Y}_i} \frac{B_{3,i}}{3} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_3 \hat{U}_i \right)
\]
\[
- \frac{\hat{X}_i \alpha_{1,i}^2}{8a_i b_i \hat{Y}_i} \left( \hat{U}_i \hat{\theta}_{1,i}^4 + \hat{V}_i \hat{\theta}_{1,i}^3 \right) \bigg|_{B_{1,i}}.
\]  
(73)

**APPENDIX B**
**DERIVATION OF $D_{2,i}$**

First, $D_{21,i}$ in (59) is derived as
\[
D_{21,i} = \frac{\hat{X}_i \alpha_{1,i}^2}{8a_i b_i \hat{Y}_i} \left( -\frac{\hat{U}_i \hat{\theta}_{1,i}^4}{3} + \hat{V}_i \hat{\theta}_{1,i}^3 \right) \bigg|_{B_{1,i}}
\]
\[
- \left[ \left( -\frac{\hat{X}_i \beta_{1,i}^2}{8a_i b_i \hat{Y}_i} \hat{U}_i^2 \right) \frac{B_{2,i}}{4} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_2 \hat{U}_i \right) \right]
\]
\[
+ \frac{\hat{X}_i \beta_{1,i}^2 \hat{V}_i}{8a_i b_i \hat{Y}_i} \frac{B_{2,i}}{2} - 2 F_1 \left( 2, 2; 3; -\hat{V}_i \hat{B}_2 \hat{U}_i \right)
\]
\[
- \left( -\frac{\hat{X}_i \beta_{1,i}^2}{8a_i b_i \hat{Y}_i} \hat{U}_i^2 \right) \frac{B_{2,i}}{3} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_3 \hat{U}_i \right)
\]
\[
+ \frac{\hat{X}_i \beta_{1,i}^2 \hat{V}_i}{8a_i b_i \hat{Y}_i} \frac{B_{2,i}}{3} - 2 F_1 \left( 2, 2; 3; -\hat{V}_i \hat{B}_3 \hat{U}_i \right)
\]
\[
\frac{\hat{X}_i \alpha_{1,i}^2}{8a_i b_i \hat{Y}_i} \left( \hat{U}_i \hat{\theta}_{1,i}^4 + \hat{V}_i \hat{\theta}_{1,i}^3 \right) \bigg|_{B_{1,i}}.
\]  
(74)

Then, $D_{22,i}$ in (59) is written as
\[
D_{22,i} = \int_{B_{2,i}} \frac{\hat{X}_i \left( \hat{V}_i - \hat{\theta}_{1,i} \hat{U}_i \right)}{2b_i} \hat{\theta}_{1,i} \hat{\theta}_{1,i} \bigg|_{B_{2,i}}
\]
\[
= \frac{\hat{X}_i}{2b_i} \left( \hat{V}_i \hat{\theta}_{1,i}^3 + \hat{U}_i \hat{\theta}_{1,i}^4 \right) \bigg|_{B_{2,i}}.
\]  
(75)

Finally, $D_{23,i}$ in (59) can be formulated as
\[
D_{23,i} = \left[ \left( -\frac{\hat{X}_i \beta_{2,i}^2}{8a_i b_i \hat{Y}_i} \hat{U}_i^2 \right) \frac{B_{3,i}}{4} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_4 \hat{U}_i \right) \right]
\]
\[
+ \frac{\hat{X}_i \beta_{2,i}^2 \hat{V}_i}{8a_i b_i \hat{Y}_i} \frac{B_{3,i}}{2} - 2 F_1 \left( 2, 2; 3; -\hat{V}_i \hat{B}_4 \hat{U}_i \right)
\]
\[
- \left( -\frac{\hat{X}_i \beta_{2,i}^2}{8a_i b_i \hat{Y}_i} \hat{U}_i^2 \right) \frac{B_{3,i}}{3} - 2 F_1 \left( 2, 3; 4; -\hat{V}_i \hat{B}_3 \hat{U}_i \right)
\]
\[
- \frac{\hat{X}_i \alpha_{1,i}^2}{8a_i b_i \hat{Y}_i} \left( \hat{U}_i \hat{\theta}_{1,i}^4 + \hat{V}_i \hat{\theta}_{1,i}^3 \right) \bigg|_{B_{2,i}}.
\]  
(79)
APPENDIX D

DERIVATION OF $D'_{2,i}$

First, $D'_{21,i}$ in (82) is formulated as

$$D'_{21,i} = \frac{\hat{X}_i \alpha_i \beta_i}{8 a_i b_i y_i} \left( -\frac{\hat{U}_i \beta_i^3}{3} + \frac{\hat{V}_i \beta_i^2}{2} \right) B_{3,i} - \left[ \left( -\frac{\hat{X}_i \beta_i^2 \beta_i}{8 a_i b_i y_i \hat{U}_i} \right) B_{3,i} + \frac{\hat{X}_i \beta_i^2 \hat{V}_i}{8 a_i b_i \hat{U}_i} + \frac{B_{3,i}}{2} \right] F_1(2, 3, 2; 3, 4; -\frac{\hat{V}_i \beta_i}{\hat{U}_i}) \right].$$

Then, $D'_{22,i}$ in (82) is written as

$$D'_{22,i} = \left( \frac{\hat{X}_i \beta_i^2 \beta_i}{2 a_i \hat{Y}_i \hat{U}_i} \right) B_{2,i} F_1(2, 3; 2; 3, 4; -\frac{\hat{V}_i \beta_i}{\hat{U}_i}) + \left( \frac{\hat{X}_i \beta_i \hat{V}_i}{2 a_i \hat{Y}_i \hat{U}_i} \right) B_{2,i} F_1(2, 3; 2; 3; -\frac{\hat{V}_i \beta_i}{\hat{U}_i}) \right].$$

Finally, $D'_{23,i}$ in (82) is obtained as

$$D'_{23,i} = \left[ \frac{\hat{X}_i \beta_i^2}{8 a_i b_i y_i} \left( -\frac{\hat{U}_i \beta_i^3}{3} + \frac{\hat{V}_i \beta_i^2}{2} \right) B_{2,i} - \left( \frac{\hat{X}_i \beta_i^2 \beta_i}{8 a_i b_i \hat{Y}_i \hat{U}_i} \right) B_{2,i} + \frac{\hat{X}_i \beta_i \hat{V}_i}{8 a_i b_i \hat{Y}_i} \right] F_1(2, 3; 2; 3, 4; -\frac{\hat{V}_i \beta_i}{\hat{U}_i}) \right].$$
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