Collisionless dynamics of the condensate predicted in the random phase approximation

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From the microscopic theory, we derive a number conserving quantum kinetic equation, valid for a dilute Bose gas at any temperature, in which the binary collisions between the quasi-particles are mediated by phonon-like excitations (called “condensation”). This different approach starts from the many-body Hamiltonian of a Bose gas and uses, in an appropriate way, the generalized random phase approximation. As a result, the collision term of the kinetic equation contains higher order contributions in the expansion in the interaction parameter. This different expansion shows up that a scattering involves the emission and the absorption of a phonon-like excitation. The major interest of this particular mechanism is that, in a regime where the condensate is stable, the collision process between condensed and non condensed particles is totally blocked due to a total annihilation of the mutual interaction potential induced by the condensate itself. As a consequence, the condensate is not constrained to relax and can be superfluid. Furthermore, a Boltzmann-like H-theorem for the entropy exists for this equation and allows to distinguish between dissipative and non dissipative phenomena (like vortices). We also illustrate the analogy between this approach and the kinetic theory for a plasma, in which the excitations correspond precisely to a plasmon. Finally, we show the equivalence of this theory with the non-number conserving Bogoliubov theory at zero temperature.

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I. INTRODUCTION

A. Superfluidity and H-theorem

A lot of studies have been devoted to the theoretical understanding of statistical and dynamical properties of a weakly interacting Bose condensed gas. In particular, many works have been accomplished on the derivation of quantum kinetic equations (QKE) that govern the evolution of the condensate fraction together with its thermal excitations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In a simple microscopic model of an homogeneous gas, one can describe the condensate by the atoms that populate the lowest ground state of energy and the thermal excitations by the atoms contained in the excited energy levels. The population of atoms in each level evolves according to the probability of scattering between the atoms. In the case of a uniform gas, the so-called Uehling-Uhlenbeck quantum kinetic equation (UUQKE) is a non-linear integral equation which describes the detailed balance of the population transfer of atoms for each mode of the wave-vector through a binary collision term [12]. This term depends linearly on the scattering differential cross section and nonlinearly on the mode population. It has also the remarkable properties to allow the QKE to obey conservation laws. These laws guarantee that the total number, the total momentum and the total kinetic energy of particles are preserved during the collision processes. More striking is the law stating that the production of entropy must be always positive, guaranteeing that the system obeys the second law of thermodynamics and, consequently, that it is always dissipative. This important requirement is known as the Boltzmann H-theorem established for a classical gas.

The UUQKE has been derived in a weak coupling approximation, valid for a diluted gas for which the kinetic energy is much higher than the potential energy. The collision term is indeed a second order expansion in the interaction potential between atoms leading to an expression of the differential cross section in the Born approximation.

One could use such a QKE in a regime below the critical point of condensation. In particular, for an inhomogeneous gas in a trap potential, the kinetic equation describes the evolution of the Wigner function and must contain additional terms, taking into account the free propagation of the atoms, the influence of the external potential and the Hartree-Fock mean field. The conservation laws are still valid and allow to express the hydrodynamic equations, including the equation for the local entropy production.

Despite these consistencies, the resulting QKE suffers from the lack of understanding of an important phenomenon: superfluidity [13]. If this phenomenon really exists for a dilute gas [14], then the second order theory must necessarily be revisited as it does not take into account the frictionless motion. Indeed according to the H-theorem, the homogeneous gas with a zero total momentum evolves towards a statistical equilibrium state characterized by the Bose-Einstein particle number distribution \( n_{\vec{k}}^{eq} = 1/\exp(\beta(\epsilon_{\vec{k}} - \mu)) - 1 \) where \( \vec{k} \) is the wave-vector, \( \beta = 1/k_B T \) the inverse temperature, \( \mu \) the chemical potential, and \( \epsilon_{\vec{k}} = k^2/2m \) the particle kinetic energy respectively (\( \hbar = 1 \)). For \( \mu \to 0 \) and \( \vec{k} \to 0 \), the ground state has a macroscopic population \( n_0 = 1/\exp(-\beta \mu) - 1 \) having no relative velocity with the non condensed part of the gas. A non zero relative velocity corresponds to a non equilibrium situation and collisions between condensed and non condensed atoms will irrecoverably damp this velocity towards zero.

From this observation, we conclude that the second or-
der theory is no longer valid as far as superfluidity is concerned. Obviously, at zero temperature, only the potential energy is the dominant contribution since the condensate is at rest and no thermal excitation subsists. Moreover, other indications confirm that an improved description of a weakly interacting Bose gas requires a higher order analysis in the interaction parameter. Among them, let us mention that the entropy of the condensate must be zero or close to zero while the H-theorem predicts an entropy $S_0 \sim \log n_0$ corresponding to a system in the Grand Canonical ensemble with large statistical fluctuations of the particle number compared to his average value $\delta n_0/n_0 \sim 1$. This is due to the Bose enhancement factor which stimulates the collision rate in a huge manner as long as a condensed particle is involved in the process. On the other hand, the equilibrium statistical approach based on the partition function formalism indicates that the presence of a small interaction lowers considerably the fluctuations to an amount irrelevant in the thermodynamic limit $\delta n_0/n_0 \to 0$.

Landau was the first to give an explanation of the superfluidity mechanism. He found a necessary but not sufficient condition for which this phenomenon happens. He showed that a superfluid interacting with an external body (for example the wall) cannot release its energy, unless it evolves with a velocity higher than a critical one. The argument is based on the impossibility to satisfy the momentum-energy conservation requirement because the excitation emitted by the superfluid has a phonon-like dispersion relation.

\section{Beyond the second order perturbation theory}

Attempts to analyze contribution coming from higher terms have been carried out with some success \cite{17,18} for a review). At zero temperature, Bogoliubov has calculated corrections to the ground state energy. The result is non analytic in the interaction parameter, due to infrared divergencies, and is obtained through a re-summation of an infinite number of contributions. Moreover, the theory predicts the existence of a phonon-like excitation, necessary in order to justify the Landau mechanism for superfluidity.

Using Green function techniques, Beliaev improved the description of the phonon like excitation namely by calculating its damping rate. Later on, Hugenholtz and Pines (HP) demonstrated that this excitation must necessarily be gapless. If at zero temperature, the theory seems well understood and widely accepted, for finite temperature it seems rather controversial. Popov was one of the first to extent the Bogoliubov and Beliaev works using the Matsubara formalism at finite temperature. However, this approach is essentially valid for a weakly depleted gas and thus for temperatures much below the critical one $T_c$. To take into account a strong depletion, Girardeau derived the Hartree-Fock-Bogoliubov (HFB) mean field equations which unfortunately, have, the strong inconvenient of having a gap in the quasi-particle energy spectrum.

Since then, many attempts, have been made, in order to suppress this gap and “rescue” the HFB equations, with the purpose to go beyond the Bogoliubov phonon-like dispersion relation. A very popular one is the so-called “Popov approximation” which consists in suppressing the product of anomalous term in the HFB equations. Another consists in the renormalization of the HFB equations but at the price of making the approximation of a weakly depleted Bose gas. Consequently, these approaches are only strictly valid for temperatures much below $T_c$. A derivation of a well-defined theory, which is able to describe the weakly interacting Bose gas for any regime of temperature, is still an open problem and remains to be established (see the conclusion of \cite{25}). By well-defined, we mean that the expansion in the small interaction parameter must be valid, whether the condensate is strongly depleted or not.

Nevertheless, using these approaches, QKE have been derived taking into account these higher order effects in the collision term. While some authors use the dispersion relation resulting from the Bogoliubov theory \cite{22}, others use the one resulting form the renormalized HFB theory \cite{21}. Superfluidity can be achieved in these models. Indeed, the resulting QKE can preserve detailed balance even for metastable states, in which a non zero relative velocity persists. In this situation, the Bogoliubov dispersion relation becomes asymmetric - due to a Lagrange multiplier representing the relative velocity - and remains non negative, as long as this velocity does not exceed the sound velocity. Otherwise, the dispersion relation is negative and the gas becomes unstable. Although these models represent a considerable amount of work, they do not yet provide a full account of the conservation laws and the H-theorem that might be deduced from their QKE.

However, let us mention that a QKE has been proposed valid in principle for temperatures close to $T_c$. The main problem of this approach is that the dispersion relation has a gap and thus fails to explain the superfluidity.

In general, in all these models, collisions between condensed and non condensed atoms are possible and their rate is huge because of the Bose enhancement factor in the same way as in the UUQKE. If we follow the same reasoning as above and if an H-theorem really exists, we might expect a condensate entropy of the same order of magnitude $S_0 \sim \log n_0$. As a consequence, the particle number fluctuations of the condensed mode are also huge, in contradiction with estimations made from the partition function formalism.

\section{The random phase approximation approach}

The present paper is devoted to provide a possible alternative explanation to the problem of superfluidity in a diluted Bose gas. For this purpose a different QKE
is derived in the random phase approximation (RPA) in which higher order terms in the interaction parameter are retained in the binary collision term (in some work, the RPA QKE refers to the UUQKE and thus has a different meaning from this paper [2]). Furthermore, in comparison with other approaches, the QKE is valid for any regime of temperature below and above $T_c$ since, under no circumstances, the approximation of weak depletion has been used.

The RPA is commonly used to describe the collective excitations in the quantum plasma of an electron liquid [27]. The idea behind this approximation is to neglect contribution containing average over pair of field operators that are not oscillating in phase but rather randomly. In the terminology of optics, we neglect contribution that are not phase matching. These contributions come essentially from averages over pairs of different modes which oscillate with a relative random phase. The reason for using the RPA in a diluted plasma is that, we expect that the excitations propagate over a sufficiently long time that non phase-matching terms are destroyed by interference. By analogy with optics, we make somehow the far field approximation. The analysis of collective excitations in plasma using RPA reveals that the coulombian interaction potential between the electrons is shielded at long distance, due to the dynamic dielectric function. In fact, these results generalize the Debye theory predicting the screening of the potential between a charged ion due to the presence of other surrounding ions. As an extension, the RPA includes also the dynamical aspects of these charged particles.

A QKE for the plasma has been derived [28, 29]. It predicts that the coulombian interaction potential for the cross section is shielded precisely by this dynamic dielectric factor. The classical version of this equation is known as the BGL kinetic equation (Balescu-Guerney-Lenard) [12]. For deriving it, Balescu uses a re-summation of the so-called ring diagrams, which is another approach equivalent to the RPA [28]. Later on, Wyld and Pines established a connection between the QKE and the plasma theory [29]. In their approach, the shielded potential results from a more subtile dynamical mechanism, in which the two electrons must emit and reabsorb an intermediate plasmon during their interaction. The dispersion relation of the plasmon corresponds precisely to the collective mode of the plasma and its decay rate to the Landau damping.

In this paper, an identical type of approach will be reproduced in the case of a dilute Bose gas. It is already known that the RPA allows to recover the Bogoliubov results for the ground state energy but with one important difference [18]. Namely, in the RPA, there is no need of a spontaneous breaking of the symmetry $U(1)$ but rather, during the derivation, the total particle number is kept conserved. Note that a number-conserving formalism has been already used to derive a QKE [7]. But this formalism is based on a $1/N$ expansion [30] and thus differs from the one in the RPA [18].

These considerations allow to derive a number conserving QKE in which the interaction potential is also modified by the presence of a dynamic dielectric function resulting from collective excitations. However, in contrast to a plasma, this function presents a strange behavior according to the kind of quasi-particles interaction it mediates. When the interaction involves a particle of the macroscopic condensate, it has the effect to totally annihilate the potential, forbidding any binary collision to occur.

This surprising result allows to explain why a metastable condensate cannot decay into a state of lower energy. Simply, it is forbidden for a particle of the condensate to scatter with the excited particle and to become thus excited. No exchange of particle can occur between the two fluids. This unexpected phenomenon is a consequence of the collective behavior induced by the presence of a macroscopic condensate. In short, collision cannot happen because an induced mean field force, generated by the condensate, compensates exactly the interaction force felt by the condensed and non condensed particle. The net result is the absence of an effective interaction force preventing any scattering. In other words, the dielectric function used to attenuate the binary interaction potential becomes simply infinite.

This phenomenon of “collision blockade” also influences the mechanism of condensate formation. Since particle cannot be exchanged with the macroscopic condensate, other mechanisms must be found. A closer analysis reveals that the collision blockade happens only when the Bose gas is stable. By stable we mean that the collective oscillations are always damped. Precisely, the Landau damping plays this role in both a plasma and a condensate as long as we are close to equilibrium [12, 31]. However, for some non equilibrium situation, it has been shown for a plasma that the oscillations can grow exponentially leading to an instability [32]. Such behavior happens also for a Bose gas which becomes thus unstable. For example, we will show that this is the case when the relative velocity between the normal and superfluid is higher than the speed of sound. In such unstable regime, the picture of a collision blockade is no longer valid and the QKE gets more complicated. Such more sophisticated QKE can describe the exchange of particle with the condensate [32, 33] and provides an alternative explanation for the condensate formation [34, 35, 36]. Another possibility is to exchange them on the edge region where the condensate is not macroscopically populated so that scattering can occur. But this requires also a really specific analysis. Therefore, the present QKE derived here will not address the important issue of particle exchange with the condensate.

Assuming that collision processes are local in space, the derivation can be extended for a weakly inhomogeneous Bose gas. In this way, we recover the generalized Gross-Pitaevskii equation for the condensate, but with the difference that the absence of a binary collision term in this equation has now a clear justification. When the
normal cloud is in a local thermal equilibrium, no dissipation exists anymore and we can derive a set of coupled equations defining the superfluid regime for any finite temperature below $T_c$.

The paper is divided as follows. We first begin by a heuristic approach of the “collision blockade” mechanism in section 2. In section 3, we derive the QKE for a homogeneous Bose gas. We review the simple RPA in which only the Hartree or direct term is considered and the generalized RPA (GRPA) in which both Hartree and exchange of an intermediate boson (condensate) are included. In section 4, we show how the number conserving RPA theory allows to recover the Bogoliubov results for the ground state energy. Finally, section 5 is devoted to conclusions and perspectives.

II. THE “COLLISION BLOCKADE” PHENOMENON

A. Preliminary definitions

We start from the Hamiltonian:

$$H = \sum_k \epsilon_k c_k^\dagger c_k + \sum_{k,k',q} \frac{U_q}{2mV} c_{k+q}^\dagger c_{k'-q}^\dagger c_k c_{k'}$$  \hspace{1cm} (1)$$

$c_k^\dagger$ and $c_k$ are the creation and annihilation operators obeying the commutation relations $[c_k, c_q^\dagger] = \delta_{k,k'}$, $[c_k, c_{k'}] = [c_k^\dagger, c_{k'}^\dagger] = 0$. $\epsilon_k = \frac{E_k}{mV}$ is the kinetic energy of the particle. $U_q$ is the interaction potential expressed in Fourier transform. Since we are concerned only with low energy binary collisions in the channel $l = 0$, it is common to replace the potential $U_q$ by a pseudo-potential or contact potential that we treat in the Born approximation 17:

$$U_q = \frac{4\pi a}{m} \frac{4\pi a}{mV} \sum_{q'} \frac{1}{\epsilon_{k+q} + \epsilon_{k'-q} - \epsilon_k - \epsilon_{k'}}$$  \hspace{1cm} (2)$$

where $a$ is the scattering length. Usually we consider only the first linear term in the scattering length. The second term is ultra-violet divergent and is only present to renormalize the theory by removing eventual high energy divergencies. In what follows we will concentrate on a repulsive contact interaction $a > 0$.

The so-called annihilation and creation operators of the particle-hole pair are of interest:

$$\rho_{k, q} = c_k^\dagger c_{k+q}, \quad \rho_{k, q}^\dagger = c_{k+q}^\dagger c_k$$  \hspace{1cm} (3)$$

They represent an excitation of momentum $q$ created from a particle which transfers its momentum from $k$ to $k + q$. The kinetic energy transfer for this excitation is given by $\omega_{k, q} = \epsilon_{k+q} - \epsilon_k$. In particular, we define the density fluctuation operator $\rho_q = \sum_k \rho_{k, q} = \rho_{-q}^\dagger$, the number operator $\hat{n}_q = \rho_{k, 0}$ and the total number $\hat{N} = \sum_k \hat{n}_k$. In terms of these operators, the Hamiltonian becomes:

$$H = \sum_k \frac{K^2}{2m} \hat{n}_k + \sum_q \frac{U_q}{2V} (\rho_q^\dagger \rho_q - \hat{N})$$  \hspace{1cm} (4)$$

B. Heuristic approach

Assume a macroscopic condensate containing $n_{k_s}$ particles evolving with momentum $k_s$. In the absence of excitations, there are no fluctuations of the density operator i.e. $\langle \rho_q^{\dagger} \rangle = \delta_{q, 0} n_{k_s}$. Suppose that an external potential is turned on creating locally fluctuations of the density of the condensate. Expressing these perturbations in Fourier space, we can characterize the external potential $\phi_{ext}(q, \omega)$ and the density fluctuations $\delta n(q, \omega) = \langle \rho_q \rangle = \langle \rho_q^{\dagger} \rangle$ in terms of its wave-vector $q$ and frequencies $\omega$ components. The Hamiltonian created by this external potential is given by $H_{ext}(t) = \sum_q n_q \phi_{ext}^{\dagger}(q, \omega) + c.c.$ The linear response to this potential is given by the formula:

$$\Delta n(q, \omega) = \chi(q, \omega) \phi_{ext}(q, \omega)$$  \hspace{1cm} (5)$$

where $\chi(q, \omega)$ is the susceptibility function. This response function can be calculated formally by treating $H_{ext}(t)$ as a perturbation in the first order. Without the presence of a binary interaction potential between the particle, this function is given by 27:

$$\chi_0(q, \omega) = \frac{n_{k_s}}{\omega^2 - \omega_{k_s, q} + i0^+} - \frac{n_{k_s}}{\omega^2 - \omega_{k_s, -q} + i0^+}$$  \hspace{1cm} (6)$$

It represents the transition amplitude for a condensed particle to increase its kinetic energy by an amount $\omega_{k, q} = \epsilon_{k+q} - \epsilon_k$ minus the amplitude for an excited particle to be transferred to the condensate releasing the energy $-\omega_{k_s, -q} = \epsilon_{k_s} - \epsilon_{k_s}$. An infinitesimal quantity $i0^+$ has been added to ensure the convergence and is due to the adiabatic switching process of external perturbation. In the presence of the interaction, the condensate acquires a self-interaction potential energy given by $U_{02} n_{k_s}^2/(2V)$. Moreover, according to the RPA, the presence of fluctuation density changes the potential energy.
by the amount $\delta H_{\text{int}} = \sum_q \rho_q^0 e^{-i\omega t} \delta \phi(q, \omega) + c.c.$ where $\delta \phi(q, \omega) = (U_q / V) (\delta \rho_q)$ is the potential induced by the presence of the density fluctuations. Consequently, the global response of the system becomes:

$$
\delta n(q, \omega) = \chi_0(q, \omega) (\delta \phi(q, \omega) + \delta \phi(q, \omega)) = \chi_0(q, \omega) (\delta \phi(q, \omega) + (U_q / V) \delta n(q, \omega))
$$

Comparison between (5) and (7) allows to deduce:

$$
\chi(q, \omega) = \frac{\chi_0(q, \omega)}{1 - (U_q / V) \chi_0(q, \omega)}
$$

The total potential created inside the system $\phi_{\text{tot}}(q, \omega) = \phi_{\text{ext}}(q, \omega) + \delta \phi(q, \omega)$ is related to the external potential through the dynamical dielectric function:

$$
\tilde{K}(q, \omega) = \frac{\phi_{\text{ext}}(q, \omega)}{\phi_{\text{tot}}(q, \omega)} = 1 - \frac{U_q}{V} \chi_0(q, \omega)
$$

In the electromagnetism language, the gradient of $\phi_{\text{ext}}(q, \omega)$ corresponds to the electrical displacement, while the gradient of $\phi_{\text{tot}}(q, \omega)$ corresponds to the electric field. These are respectively the external and total field force acting on the particle. The dielectric function usually has the effect of attenuating the external field force by means of an induced force so that the total field force is smoothed out. In particular, we notice that this function is infinitely resonant for frequencies $\omega = \omega_{k_s, q} - \omega_{k_s, q}$ which means that the induced potential exactly compensates the external one. For these frequencies, the external potential affects the density fluctuations but has no local influence anymore on the condensate particle itself. This important observation is at the origin of the collision blockade phenomenon.

Indeed, assume that the source of this finite external potential results from the transition of an excited particle of momentum $k'$ towards another excited state with a momentum $k' - q$. The released energy during this process is $\omega = -\omega_{k_s, q}$. According to the Fermi golden rule, a collision occurs between the condensed and non condensed ingoing particles if the transfer energy are equal $-\omega_{k_s, q} = \omega_{k_s, q}$ or equivalently the total energy is conserved $\epsilon_{k_s} + \epsilon_{k'} = \epsilon_{k_s + q} + \epsilon_{k' - q}$. But then, the dielectric function becomes infinitely resonant $\tilde{K}(q, \omega) \to \infty$ causing the total potential $\phi_{\text{tot}}(q, \omega) \to 0$. Thus the interaction potential felt by the condensate particle becomes non-existent, since the induced mean field cancels the external potential generated by the excited particle. That precise phenomenon is responsible for the absence of an effective scattering process since, without interaction potential, no scattering amplitude can appear. The same effect happens when the scattering involves an outgoing particle in the condensate whereas, in such a case, the transfer energy $-\omega_{k_s, q} - \omega_{k_s, q}$ causes the infinite resonance. Thus, these giant resonances have the effect to protect the condensate particle from scattering with the others. We must notice that this mechanism works only if the condensate population is macroscopic, otherwise, no induced potential is generated in the thermodynamic limit since, for $n_{k_s} / V \to 0$, $\tilde{K}(q, \omega) \to 1$. As we shall see in the next sections, a more elaborated model confirms this prediction in the more general case of a non equilibrium Bose gas, in which the condensate can be strongly depleted.

### III. KINETIC THEORY IN THE RPA

Many methods have been developed to derive kinetic equations for a dilute Bose gas [11, 26]. In order to arrive rapidly to the final result, instead of using complicated many body techniques, we base the derivation on an operatorial method developed in [27] which appeared to be a simpler way to reach results without loss of generality. The approximations are the followings: 1) homogeneous Bose gas, 2) thermodynamic limit, 3) generalized RPA, 4) instantaneous collisions (Markovian QKE), 5) no fragmentation of the condensate (only one macroscopic population mode).

#### A. The random phase approximation

In this subsection, we give a brief overview of the RPA developed in [27]. This approximation has been used quite extensively for the quantum electron liquid for describing the screening effect.

The dynamic of $\rho_{k,q}$ is given by the Heisenberg equation motion. Using the relation:

$$
[\rho_{k,q}, \rho_{k', q'}] = \delta_{k,k'} \delta_{q,q'} - \delta_{k,q'} \delta_{k',q} - \delta_{k,q} \delta_{k',q'}\quad(10)
$$

we find

$$
i \frac{\partial}{\partial t} \rho_{k,q} = [\rho_{k,q}, H] = (\epsilon_{k+q} - \epsilon_{k}) \rho_{k,q} + \sum_{q'} \frac{U_{q}}{2V} [\rho_{k+q,q'} - \rho_{k,q,q'} - \rho_{k+q',q} + \rho_{k+q',q} \rho_{k+q',q}^\dagger] + \quad(11)
$$

where the brackets refer to an anti-commutator. Two cases in the RPA are generally considered in systems close to equilibrium [27]. The simple RPA, in which only the Hartree or direct terms contribute, and the generalized RPA, in which the Fock or exchange terms are also retained. We shall concentrate on the second approximation, since for a contact potential Hartree and Fock terms are identical. Only for the condensate-condensate interaction, the Fock term does not appear. In the GRPA, we consider that operator $\rho_{k,q}$ with non zero momentum transfer $\mathbf{q} \neq 0$ gives a negligible contribution in comparison to $\hat{n}_k$, as it involves different modes oscillating with a random relative phase.

The procedure to get the non equilibrium RPA equations is as follows. For a momentum transfer $\mathbf{q} = 0$ the Eq.(11) is kept unchanged. For $\mathbf{q} \neq 0$, however, we keep among all terms those combinations of creation and annihilation operators involving product of $\rho_{k', q}$ and $\hat{n}_{k'}$ for
all possible values of $k'$ and $k''$, and neglect those combinations that cannot be written in this form. These removed contributions are quadratic in the operator $\rho_{k',q'}$ with $q' \neq q, 0$. In this approximation, only contributions conserving the momentum transfer are relevant, the others coupling the various $\rho_{k,q}$ with different momentum transfers are neglected. We expect that the excitations of momentum $q$ propagate, without interfering with the others, over an enough long time determined by the dilution of the gas, that (this would correspond to the far field limit). The result is for $q = 0$

$$i \frac{\partial}{\partial t} \hat{n}_k = \sum_{q'} \frac{U_{q'}^2}{2V} [\rho_{k,q'} - \rho_{k'-q,q'} - \rho_{k',q'}] +$$  \hspace{0.5cm} (12)

and for $q \neq 0$

$$i \frac{\partial}{\partial t} \rho_{k,q} = (\epsilon_{k+q} - \epsilon_k) \rho_{k,q} +$$

$$+ \sum_{k'} \left[ \frac{U_{k-k'} q}{2V} \hat{n}_{k'} - \frac{U_{k-k'} q}{2V} \hat{n}_{k'+q', \rho_{k,q}} \right] +$$

$$+ \sum_{k'' \neq k} \left[ \frac{U_{k-k''} q}{2V} \hat{n}_{k''} - \frac{U_{k-k''} q}{2V} \hat{n}_{k'+q', \rho_{k,q}} \right] +$$

$$- \sum_{k' \neq k+q} \left[ \frac{U_{k-k'} q}{2V} \hat{n}_{k+q} - \frac{U_{k-k'} q}{2V} \hat{n}_{k'+q', \rho_{k,q}} \right] +$$  \hspace{0.5cm} (13)

Eq. (12) is an integral operational equation linear in $\rho_{k,q}$. We can linearize this equation by averaging all possible bilinear contributions. Since the homogeneity of the gas imposes $\langle \rho_{k,q} \rangle = 0$ if $q \neq 0$, we are left with the average on the number operator $\langle \hat{n}_k \rangle = n_k$ and we obtain an integral equation which possesses the same structure as Eq. (5.183) p. 318 of [27]:

$$\left[ \frac{\partial}{\partial t} - (\epsilon_{k+q} - \epsilon_k - HFA) \right] \rho_{k,q} = (n_k - n_{k+q}) \sum_{k'' \neq k} \frac{U_{k-k''} q}{V} \rho_{k'', q} +$$

$$+ n_k \sum_{k' \neq k-q} \frac{U_{k-k'} q}{V} \rho_{k', q} - n_{k+q} \sum_{k'' \neq k+q} \frac{U_{k-k''} q}{V} \rho_{k'', q}$$  \hspace{0.5cm} (14)

Indeed, in the term containing the bracket, we recognize the difference $\epsilon_{k,q} = \epsilon_{k+q} - \epsilon_k - HFA$ between the quasi-particle energy calculated in Hartree-Fock approximation (HFA):

$$\epsilon_{k}^{HFA} = \frac{k^2}{2m} + \sum_{k''} \frac{U_0}{V} n_{k''} + \sum_{k'} \frac{U_{k-k'} q}{V} n_{k'}.$$

On the other hand, in the integral terms, both Hartree and Fock terms are present. However, Eq. (14) shows some differences: firstly, since we are dealing with bosons, the Fock terms have the opposite sign compared to fermions; secondly, since a macroscopic condensate might appear, we should be cautious not to count twice terms between the same modes. We have avoided this difficulty by excluding carefully in the sum over the modes those leading to a double counting and which appear only in the integral terms. Finally, another difference is that $n_k$ is still a function depending on the time, generalizing in this way the RPA approach for systems in non equilibrium.

For comparison, Zaremba et al. [2] have derived a QKE in an approximation which would correspond to a different equation for $\rho_{k,q}$. As we shall see below, their analysis is equivalent to removing the integral terms in (14) and to approximating the quasi-particle energy by

$$\epsilon_{k}^{ZNG} = \frac{k^2}{2m} + \sum_{k'} \frac{U_0}{V} n_{k'} + \sum_{k' \neq k} \frac{U_{k-k'}}{V} n_{k'}$$  \hspace{0.5cm} (16)

We notice immediately that they have avoided a double counting in the exchange term. As a consequence, in the particular case of a contact potential and for $k_0 = 0$, the energy difference that they obtained has a gap:

$$\epsilon_{k}^{ZNG} - \epsilon_{0}^{ZNG} = \frac{k^2}{2m} + \frac{4\pi a n_0}{m} \frac{k_0}{m} \rightarrow \frac{4\pi a n_0}{m}$$  \hspace{0.5cm} (17)

Clearly, for $k \rightarrow 0$ and if the mode $k_s = 0$ is macroscopically populated, this finite gap cannot be neglected in the scattering energy spectrum. In our approach, however, this gap does not appear anymore in $\epsilon_{k}^{HFA}$.

The two operational equations (12) and (14) are the equations leading to the QKE.

### B. Collective and scattering excitations

It is instructive to analyze the frequency spectrum solution of the eigenvalue problem given by Eq. (14). In an electron liquid, the linear equation possesses two kinds of eigenvectors: (i) the scattering solutions involving the presence of only one particle and (ii) the collective solutions involving the presence of many particles.

For reasons we will explain below, let us concentrate on the problem of calculating the impulse response or dielectric propagator $\mathcal{U}_q(k,k_1, t)$ to an initial particle having a momentum $k_1$. We replace the operator $\rho_{k,q}$ by this function and we substitute the interaction potential by its first order expression in $\epsilon_{k}^{HFA}$. After simplifications, the integral equation for this function is:

$$\left[ \frac{\partial}{\partial t} - (\epsilon_{k+q} - \epsilon_k - HFA) \right] \mathcal{U}_q(k,k_1, t) =$$

$$= \frac{4\pi a}{m} \left( n_k - n_{k+q} \right) \sum_{k' \neq k} \mathcal{U}_q(k', k_1, t) +$$

$$+ \frac{n_k}{V} \sum_{k' \neq k-q} \mathcal{U}_q(k', k_1, t) - \frac{n_{k+q}}{V} \sum_{k' \neq k+q} \mathcal{U}_q(k', k_1, t)$$  \hspace{0.5cm} (18)

with the initial condition:

$$\mathcal{U}_q(k, k_1, t = 0) = \delta_{k,k_1}$$  \hspace{0.5cm} (19)
If $n_k$ varies slowly in time and thus can be considered as constant during the evolution of the impulsion response, Eq. (14) can be solved exactly in the thermodynamic limit. Similar equations have been solved in the field of plasma physics. For this purpose, we define the Laplace transform as:

$$
\mathcal{U}_q(k, k_1, \omega) = \int_0^\infty dt e^{i(\omega + i\theta_0)t} \mathcal{U}_q(k, k_1, t)
$$

(20)

A $0_+$ has been added in order to ensure convergence of the integral. If we suppose that $k_s$ is the wave-vector for the superfluid mode, then we can make the decomposition between a normal component and components involving the superfluid mode:

$$
\mathcal{U}_q(k, k_1, \omega) = \tilde{\mathcal{U}}_q(k, k_1, \omega) + \delta_{k,k_s} \mathcal{U}_q(k_s, k_1, \omega) + \delta_{k,k_s-q} \mathcal{U}_q(k_s-q, k_1, \omega)
$$

(21)

Also, we distinguish the normal mode population $n'_k = (1 - \delta_{k,k_s})n_k$ from the condensed mode. Plugging this decomposition into (14) and neglecting some $n'$ by taking the thermodynamic limit, we obtain for the superfluid modes:

$$
[\omega + i0_+ - \frac{k_s q}{m} - \frac{q^2}{2m}] \tilde{\mathcal{U}}_q(k_s, k_1, \omega) = i\delta_{k_s, k_1}
$$

$$+ \frac{4\pi a}{m} \frac{n_{k_s}}{V} \left( \sum_{k'} 2\tilde{\mathcal{U}}_q(k', k_1, \omega) + \mathcal{U}_q(k_s, k_1, \omega) \right)
$$

(22)

and for the normal component:

$$
[\omega + i0_+ - \frac{k q}{m} - \frac{q^2}{2m}] \tilde{\mathcal{U}}_q(k_s - q, k_1, \omega) = i\delta_{k_s-q,k_1}
$$

$$- \frac{4\pi a}{m} \frac{n_{k}}{V} \left( \sum_{k'} 2\tilde{\mathcal{U}}_q(k', k_1, \omega) + \mathcal{U}_q(k_s, k_1, \omega) \right)
$$

$$+ \mathcal{U}_q(k_s - q, k_1, \omega)
$$

(23)

and

$$
[\omega + i0_+ - \frac{k q}{m} - \frac{q^2}{2m}] \mathcal{U}_q(k_s - q, k_1, \omega) = i\delta_{k,q,k_s}
$$

$$+ \frac{8\pi a (n_k - n_{k+q})}{m} \left( \sum_{k'} \tilde{\mathcal{U}}_q(k', k_1, \omega) + \mathcal{U}_q(k_s, k_1, \omega) \right)
$$

$$+ \mathcal{U}_q(k_s - q, k_1, \omega)
$$

(24)

The prime in the sum excludes terms involving the condensed mode. This close set of equations is solved in the Appendix A. For $k \neq k_s, k_s - q$, the scattering solutions are $\omega = \epsilon_{k+q} - \epsilon_k$. For excitations involving a superfluid mode $k = k_s, k_s - q$, the presence of interaction with the macroscopic condensed transforms the scattering solutions into collective solutions of the discriminant equation:

$$
\Delta(q, \omega) = K_n(q, \omega) [(\omega + i0_+ - \frac{k_q}{m})^2 - \epsilon^B_q]^2 + (K_n(q, \omega) - 1) \frac{8\pi a n_k q^2}{m V} m = 0
$$

(25)

where

$$
\epsilon^B_q = \sqrt{c^2 q^2 + \left( \frac{q^2}{2m} \right)^2}
$$

is the Bogoliubov excitation energy,

$$
c = \sqrt{\frac{4\pi a n_k}{m^2 V}}
$$

(27)

is the sound velocity and

$$
K_n(q, \omega) = 1 - \frac{8\pi a}{m V} \sum_{k'} \frac{n_k' - n_{k+q}'}{k_q m - q^2/2m}
$$

(28)

is the dynamic dielectric function of the normal fluid. Eq. (25) allows to find a dispersion relation for any function $n_k$ possibly in non thermodynamic equilibrium. In this sense, it generalizes the dispersion relation that is obtained equivalently from the density fluctuations response formalism for a gas at equilibrium. Note in (38) the non number conserving Belaev formalism has been used to derive the dispersion relation up to the next order beyond the Bogoliubov theory. The solution can be put in the complex form: $\omega = \omega_0 - i\gamma_q$ where $\gamma_q$ corresponds to the Landau damping. For the particular case of a weakly depleted Bose gas, we can solve analytically Eq. (25). In that case, we can approximate (31):

$$
K_n(q, \omega) \simeq 1 + i\text{Im}K_n(q, \omega)
$$

(29)

Eq. (30) is obtained considering in a first approximation that the imaginary term can be neglected. We find that the real part corresponds to the Bogoliubov spectrum:

$$
\omega_q \simeq \frac{k_q}{m} \pm \epsilon^B_q
$$

(30)

The imaginary part corresponds to the Landau damping and can be calculated perturbatively assuming $|\gamma_q| \ll \omega_q$ which is the case for a weakly depleted condensate. We find, up to the first order,

$$
\gamma_q \simeq \text{Im}K_n(q, \omega_q) \frac{4\pi a n_k q^2}{m V} \frac{k_q}{m}
$$

(31)

By analogy with plasma physics, we say that a Bose gas is stable provided that $\gamma_q \geq 0$ and unstable otherwise. In a stable condensate the collective oscillations are damped, while in a unstable condensate they are amplified exponentially.

For the case of thermal equilibrium with $v_n = 0$, $\mu = 0$ and a temperature close to zero, $\text{Im}K_n(q, \omega_q)$ is calculated in appendix B and is a positive function for $\omega_q \geq 0$ and negative otherwise. Thus, from (31) the stability condition could be written as:

$$
\frac{\epsilon^B_q}{m} \frac{k_q}{m} \geq 0
$$

(32)
We can check that this inequality is fulfilled for all values of the momentum at the condition that $|k/m| \leq c$. In other words, the condensate is stable if its velocity relative to the normal fluid is much less than the sound velocity. Anticipating the next subsections, this condition corresponds to the Landau criterion for superfluidity for a weakly depleted Bose gas. More generally, the occurrence of instability depends on the form of $n_k$ which influences the spectrum obtained from \[34\]. In the case of a plasma, this problem has been studied long time ago \[32, 33\] .

Finally, in the absence of a macroscopic condensate i.e. $n_{k=1}/V \to 0$, we recover the two scattering solutions for the superfluid:

$$\omega_{n_{k=1}=0} = \frac{|k|}{m} \pm \frac{q^2}{2m}$$

and the collective solution is given by

$$\mathcal{K}_n(q, \omega) = 0$$

According to \[31\], at equilibrium, this collective solution contains only an imaginary part and thus no collective oscillation can be observed in the system. Therefore, any collective oscillation results specifically from the condensation which transforms the scattering solution of the condensed mode into a collective solution.

C. Derivation of the kinetic equation

From the results of the previous section, we are now ready to derive a generalized Boltzmann like equation for the Bose condensed gas. We define the spatial correlation function as the average $\langle \rho_{k=1-q}\rho_{k=1}\rangle$. Using Eq. \[12\], we derive the following equation for the correlation function:

$$i\frac{\partial}{\partial t} + \frac{(k' - k).q}{m} - \frac{q^2}{m} \langle \rho_{k=1-q}\rangle = 4\pi a \sum_{k=1}^{\infty} \delta_{k,k'}\langle \rho_{k=1-q}\rangle + 2\pi a \sum_{q=1}^{\infty} [n_k(2 - \delta_{k,k'}(\omega_q + 1) + n_{k'=1-q}(n_{k'=1-q}+1)n_k + (1 + n_{k'=1-q})n_{k=1-q})]$$

This function can be decomposed as a non interacting part and an interacting part \[12\]:

$$\langle \rho_{k=1-q}\rangle = (n_k + 1)n_k\delta_{k,0} + n_{k=n_{k=1}}g_{q}(k,k)$$

where $g_{q}(k,k')$ represents the correlation function due to the interactions. For $q \neq 0$ or $k \neq k'$, the non interacting part follows the Wick’s decomposition and since the system is homogeneous $\langle \rho_{k}\rangle = \delta_{k,0}n_k$. For $q = 0$ or $k = k'$, however, it reduces to the quadratic average population that we choose to be $\langle n_k^2 \rangle = n_k^2$.

Would we have used the Wick’s decomposition in that case then $\langle n_k^2 \rangle = 2n_k^2$ which corresponds to non zero particle number fluctuations $\langle \delta^2 n_k \rangle$ and the total particle number would display fluctuations as well. Indeed, since $\langle n_k\delta_{k,k'} \rangle = n_k n_k'$ for $k \neq k'$, we calculate $\langle \delta^2 N \rangle = \sum_k n_k^2$. In the presence of condensation, these fluctuations are huge i.e. of the same order of the average value \[\hat{N}\]. Consequently, in an isolated system where the total particle number is conserved, this unphysical situation must be excluded.

Taking the average over each side of Eq. \[12\], a comparison with \[34\] allows to deduce:

$$\frac{\partial}{\partial t}n_k = \sum_{q=0}^{\infty} \frac{2\pi a}{mV} (g_{q}(k,k') - g_{q}(k-k',q')) + g_{q}(k',q) - g_{q}(k,k+q)$$

On the other hand, inserting this definition into Eq. \[35\] and using \[37\], we obtain

$$Q_q(k,k') = \frac{8\pi a}{mV} [(n_k - n_{k=1-q})(n_{k'=1-q} + 1)n_k + (n_{k'-1-q} + 1)n_{k=1-q}]$$

where

$$Q_q(k,k') = \frac{8\pi a}{mV} [(n_k - n_{k=1-q})(n_{k'=1-q} + 1)n_k + (n_{k'-1-q} + 1)n_{k=1-q}]$$

is the inhomogeneous term. To get \[38\], we eliminate terms involving delta functions which will give negligible contribution to the QKE in the thermodynamic limit. Both Eq. \[37\] and Eq. \[38\] form a close set in which $g_{q}(k,k')$ must be eliminated in order to get a kinetic equation for $n_k$. This elimination is done following an analog procedure to that of Ichimaru \[10\]. We assume that the correlations due to the interactions are non-existent for $t \to -\infty$. This requirement is usual in kinetic theory and allows to provide the following initial condition $g_{q}(k,k')\mid_{t=\to -\infty} = 0$. Also, we assume that $n_k$ evolves on a much more long time scale than the duration of a collision $g_{q}(k,k')$ and so is considered as constant in solving Eq. \[38\]. This approximation amounts to claiming that the binary collision process is instantaneous in comparison with the time associated to the relaxation of the system. Inspired by the previous subsections and by \[12\], we can check that the solution, satisfying both Eq. \[38\] and the initial condition, is expressed in terms of
the dielectric propagator as:

\[ g_q(k, k', t) = -i \int_0^\infty dt' \sum_{k_1, k_1'} \mathcal{U}_q(k, k_1, t') \mathcal{U}_q(k', k_1', t') Q_q(k_1, k_1', t - t') \]  

(40)

We have inserted the explicit time dependence in the functions. If the creation of such correlations is much faster in comparison with the relaxation time for \( n_k \), \( Q_q(k_1, k_1', t - t') \approx Q_q(k_1, k_1', t) \) and we obtain a Markovian equation. We re express the dielectric propagator in Fourier transform according to,

\[ \mathcal{U}_q(k, k_1, t) = \frac{1}{2\pi} \int_C d\omega e^{-i\omega t} \mathcal{U}_q(k, k_1, \omega) \]  

(41)

where the contour \( C \) extends from \(-\infty\) to \(+\infty\) along a path in the upper half of the \( \omega \) plane in such a way that all the singularities lie below it. The substitution allows to carry out successively integrations over \( t' \) and \( \omega' \) by closing the contour in the upper half plane in order to get:

\[ g_q(k, k') = \int_\infty^{\infty} d\omega \sum_{k_1, k_1'} \mathcal{U}_q(k, k_1, \omega) \mathcal{U}_q(k', k_1', -\omega) Q_q(k_1, k_1', t) \]  

(42)

Calculations in appendix C allow to find an explicit expression for \( q_q(k, k') \) in terms of the one particle distribution function, provided the Landau damping factor is always positive. The substitution into \((47)\) allows finally to get the GRPA kinetic equation for a stable Bose gas:

\[ \frac{\partial}{\partial t} n_k = C_k[n_k; k_s] = C_k[n_k; k_s] + \tilde{C}_k[n_k; k_s] \]  

(43)

where we define the collision terms as a functional of \( n_k \) and a function of \( k_s \). The first term describes the collision rate between particles of the normal fluid:

\[ C_k[n_k; k_s] = \sum_{q, q'} \left( \frac{\delta_{q, q'}}{\mathcal{K}(q, q_k - q_k)} \right)^2 (1 - \delta_{q, q_k - q_k}) \pi \delta(\epsilon_{q + q} + \epsilon_{q' - q} - \epsilon_q) \pi \delta(\epsilon_{q + q} + \epsilon_{q' - q} - \epsilon_q) \left[ n_{q + q} n_{q' - q} (n_k + 1)(n_k' + 1) - n_k n_{k'} (n_{q + q} + 1)(n_{q' - q} + 1) \right] \]  

(44)

The second term describes the collision rate between condensed and non condensed particles, the condensed particle is either the input or the output state in the scattering process:

\[ \tilde{C}_k[n_k; k_s] = \sum_{q, q'} \left( \frac{\delta_{q, q'}}{\mathcal{K}(q, q_k - q_k)} \right)^2 \left( \delta_{q, q_k - q_k} + \delta_{q + q_k, q_k} + \delta_{q_k, q_k} + \delta_{q' - q_k, q_k} \right) \pi \delta(\epsilon_{q + q} + \epsilon_{q' - q} - \epsilon_q) \pi \delta(\epsilon_{q + q} + \epsilon_{q' - q} - \epsilon_q) \left[ n_{q + q} n_{q' - q} (n_k + 1)(n_k' + 1) - n_k n_{k'} (n_{q + q} + 1)(n_{q' - q} + 1) \right] \]  

(45)

In the first expression \((44)\), the contact potential has been replaced by an effective one depending on the transfer particle energy \( \omega = \epsilon_{q + q_k} - \epsilon_{q_k} \):

\[ U_{q, q_k}^{\text{eff}}(\omega) = \frac{8\pi a}{\mathcal{K}(q, q_k)} \]  

(46)

The correcting term is the dynamic dielectric constant:

\[ \mathcal{K}(q, q_k) \]

\[ = \frac{8\pi a m v}{m v} \left( \omega + i0_+ - \frac{k_{q, q_k} a^2}{m v} \right) - \frac{\Delta(q, q_k)}{m v} \left( \omega + i0_+ - \frac{k_{q, q_k} a^2}{m v} \right) \]

(47)

In the second expression \((44)\), we define another dynamical dielectric function:

\[ \tilde{\mathcal{K}}(q, q_k) = \frac{\Delta(q, q_k)}{m v} \left( \omega + i0_+ - \frac{k_{q, q_k} a^2}{m v} \right)^2 \]

(48)

This result generalizes \((45)\) in the case where \( n_k' \neq 0 \). Plugging this expression into Eq.\((45)\), the energy conservation imposes the two choices \( \omega = \frac{k_{q, q_k} a^2}{2m} \pm \frac{\Delta(q, q_k)}{2m} \) when a particle of the condensate participates to the collision process. But, with such an energy transfer and for a macroscopic population of the condensate, this dynamic dielectric function gets infinite:

\[ \tilde{\mathcal{K}}(q, q_k) \bigg|_{\omega = \frac{k_{q, q_k} a^2}{2m} \pm \frac{\Delta(q, q_k)}{2m}} \rightarrow \infty \]

(49)

leading to \( \tilde{C}_k[n_k; k_s] \rightarrow 0 \). In this situation, surprisingly, no collision involving the condensate particle occurs. As a consequence, in a uniform Bose gas, no transfer of particle is possible between the condensate and the normal component. Indeed, due to the absence of Fock interaction term in the GRPA, the effective potential has a different expression when a particle of the condensate is involved in a scattering. As said in section 2, a potential induced by the condensate compensates exactly the po-
ternal created by the excited particle susceptible to scatter. As a consequence, the dielectric function suppresses the effectiveness of the potential which thus becomes invisible to the condensate and totally shelters it from collision. Furthermore, let us note that this shielding remains unchanged for any momentum \( \mathbf{k}_s \) which preserves the stability of the condensate. If the Landau damping factor \( \gamma_q \) becomes negative, then an instability occurs and therefore Eq. (41) and Eq. (45) are not longer valid, as well as all considerations concerning collision blockade. In that case, a more elaborated derivation must  be carried out that could allow particle exchange with the condensate. The expressions (C6) and (C7) must be recalculated taking into account the instability [32, 33].

In this way, the GRPA kinetic equation provides a different understanding of the superfluidity phenomenon as it describes the motion of two independent fluids that cannot transfer particle through collisions. Since \( \frac{\partial}{\partial t} n_{\mathbf{k}} = 0 \), \( n_{\mathbf{k}} \) is really a constant of motion independent of the dynamics of the normal fluid.

The collision blockade is the consequence of a higher order expansion in the interaction parameter. Since \( K(q, \omega) \) is really a constant of motion independent of the dynamics of the normal fluid.

The collision blockade is the consequence of a higher order expansion in the interaction parameter. Since \( K(q, \omega) \) remains unchanged for any momentum \( \mathbf{k}_s \) and that the fermion now becomes a boson, the only difference is that the coulombian potential becomes a confinement potential and that the fermion now becomes a boson.

In the RPA, a supplementary term coming from the exchange effect \( 4\pi a \frac{m V}{m} \) ensures a reasonable value for the effective potential guaranteeing an efficient collision rate, even with particle in the energy levels close to the condensate level.

### D. Properties of the RPA collision term

The collision term exhibits a number of remarkable properties analog to those encountered by other kinetic equations [10]. These allow to establish the particle number, momentum and energy conservation laws as well as the Boltzmann H-theorem. Since the condensed particles do not participate to the collision process, these concerns only the excited particles of the normal fluid. Three of them can be stated as:

\[
\sum_k \epsilon_k C_k[n_k^r; \mathbf{k}_s] = 0 \tag{51}
\]

\[
\sum_k k \epsilon_k C_k[n_k^r; \mathbf{k}_s] = 0 \tag{52}
\]

\[
\sum_k \epsilon_k^2 C_k[n_k^r; \mathbf{k}_s] = 0 \tag{53}
\]

These three integral relations are checked easily by dividing the collision term in four equal parts. After carrying out the successive changes of the integration variables \( \mathbf{k} \leftrightarrow \mathbf{k}' \), \( q \leftrightarrow -q \) on the second term, \( \mathbf{k} \leftrightarrow \mathbf{k}' - \mathbf{q}, \mathbf{k}' \leftrightarrow \mathbf{k} + \mathbf{q} \) on the third term and a successive combination of these two variable changes in the fourth term, these four terms cancel each other if we use the relation \( K(q, \omega) = K^*(q', -\omega) \). As a consequence, for a uniform Bose gas, the total particle number, the total momentum and the total kinetic energy, associated with the normal component are independent of the time i.e. \( \frac{d}{dt} \sum_k n_k^r = 0 \), \( \frac{d}{dt} \sum_k \epsilon_k n_k^r = 0 \) and \( \frac{d}{dt} \sum_k \epsilon_k^2 n_k^r = 0 \). These properties together with the conservation of \( n_{\mathbf{k}} \) imply that the total energy in the Hartree-Fock approximation is also conserved [2] i.e.:

\[
\frac{d}{dt} \sum_k \epsilon_k n_k^r + 4\pi a \frac{m V}{m} (N^2 - \frac{1}{2} n_{\mathbf{k}_s}^2) = 0 \tag{54}
\]

Another crucial property is the Boltzmann H-theorem. The entropy, due to thermal excitations, has the expression:

\[
S = \sum_k \left[ (1 + n_k^r) \ln(1 + n_k^r) - n_k^r \ln n_k^r \right] \tag{55}
\]
The time evolution of the entropy is always positive. A similar derivation using the change of variables allows to calculate a positive production of entropy:

$$\frac{dS}{dt} = \sum_{q, k, k'} \frac{8\pi a}{mV} \frac{1}{K(q, c_{k+q} - c_k)} \left[ \frac{\pi}{4} \delta(\epsilon_{k+q} + \epsilon_{k'-q} - \epsilon_k - \epsilon_{k'}) \ln \left( \frac{n'_k n'_{k'-q} (n'_k + 1)(n'_{k'} + 1)}{n_k n'_k (n'_k + 1)(n'_{k'} + 1)} \right) \right] \geq 0$$ (56)

The positivity is a consequence of the mathematical relation $\ln(x/y)(x - y) \geq 0$. The equality is achieved for $x = y$. From Eq. (56) we deduce that the entropy always increases until the system reaches a stationary equilibrium distribution. This occurs when the production of entropy becomes zero. In that situation,

$$\frac{n'_{k+q} n'_{k'-q}}{(n'_{k+q} + 1)(n'_{k'-q} + 1)} = \frac{n'_k n'_{k'}}{(n'_k + 1)(n'_{k'} + 1)}$$ (57)

This relation holds only for the Bose-Einstein distribution function [12],

$$n'_k = n'_q = \frac{1}{\exp[\beta(\epsilon_k - \mathbf{v}_n - \mu)] - 1}$$ (58)

where $\mathbf{v}_n$ is the average velocity of the normal component. In this way, the inverse temperature $\beta$ and the chemical potential $\mu$ are defined as the free parameters of the equilibrium solution. Although similar, the properties of the thermodynamic equilibrium predicted by the H-theorem in the RPA are really different from the one predicted by the calculation of the ensemble partition functions of an ideal Bose gas [13]. The superfluid and normal components can move relatively with two different velocities $\mathbf{v}_s$ and $\mathbf{v}_n$. The relative difference $\mathbf{v}_s - \mathbf{v}_n$ and $\mu$ are subject to the only constraint of stability which gives a limitation. As said in the previous subsection, for $\mathbf{v}_n = 0$, and the temperature close to zero, and using [13] in [13], this constraint is realized provided that $\mu = 0$ and that the relative velocity does not exceed the sound velocity (see Eq. [12]). Otherwise, the condensate becomes unstable. Thus, we recover the Landau criterion for the weakly interacting Bose gas. Note that this result differs from the ZNG approach since it corresponds to an equilibrium fugacity of the normal fluid equal to one (see Eq. (44) in [2]).

On the contrary, the statistical equilibrium ensemble formalism for an ideal Bose gas imposes a zero relative velocity and a chemical potential close to zero when condensation occurs [13]. This contradiction might be explained if we remind that in this formalism we postulate an equal a priori probability of any possible configuration of the gas [12, 13]. This basic assumption of equilibrium statistical mechanics originates from the observation that, over a long time range, the collision process will mix statistically these configurations. But, since in the GRPA

E. Condensate formation

The kinetic equation [14] does not explain the condensate formation through, for example, evaporative cooling [34, 35, 36]. If we start indeed from an initially condensed gas with a Bose-Einstein momentum distribution whose tail has been cut, then according to the RPA model, the irreversible evolution process towards equilibrium will undergo a strange behavior. Instead of populating the lowest energy level which is forbidden, the excited particles will be transferred to the lowest excited levels in which a macroscopic population can eventually appear. In that situation, we are led to a fragmentation of the condensate into at least two energy levels. But such two macroscopic states contradict the assumption made at the beginning that only one macroscopic state does exist. A more elaborated model can take into account more than one level macroscopically populated. Although this more general description must not be excluded, for energetic reason, this phenomenon is not likely to happen [41].

A fragmented condensate has a much higher potential energy than a non-fragmented one due to the presence of a Fock interaction energy between two fragmented parts. To explain the condensate formation and if we exclude fragmentation, we must assume either that the Bose gas must be instable due to a far non-equilibrium situation or that it must be inhomogeneous with regions of strong depletion. Indeed, in those regions - like the edge of the gas - the condensate population is not macroscopic anymore and so collisions between the two fluids may happen.
IV. EXTENSION TO A WEAKLY INHOMOGENEOUS BOSE GAS

A. Inhomogeneous equations

The extension of the previous equations to inhomogeneity is important in order to understand how the evolution of the condensate and the thermal excitations are coupled through mean field forces. In what follows, we assume that the gas is weakly inhomogeneous i.e. most of the quasi-particles collide inside a sufficiently small volume that can be considered as homogeneous. To quantify the level of acceptable inhomogeneity, we divide the volume $V$ of the gas into small cubic volume $\Omega$. The edge $l$ of each volume $\Omega = l^3$ is adjusted in such a way to be infinitesimal so that the gas is homogeneous inside it, but big enough to contain a large amount of particle whose dynamic obeys still locally the homogeneous equation. In the literature [12], $l$ is referred as the hydrodynamic scale and is estimated from the formula $l(r,t) = \rho(r,t)/\nabla_r \rho(r,t)$ where $\nabla_r \rho(r,t)$ is the normal fluid density and $r$ is the position in space of the small volume. This length scale must be much greater than the mean free path which can be estimated more less as $1/(\rho \sigma)$ where $\sigma = 8\pi a^2$ is the total cross section [2].

In this infinitesimal volume, we define the local condensate wave function

$$\Psi(r,t) = \sqrt{n_{\text{c}}(r,t)} e^{i \theta(r,t)}$$

which corresponds to the eigenfunction of the density matrix with the highest macroscopic eigenvalue [13]. The local momentum of the condensate depends also on the position and the time through the relation

$$k_{\text{c}}(r,t) = \nabla_r \theta(r,t)$$

Up to a constant phase, the amplitude and the local momentum of the condensate characterize fully $\Psi(r,t)$. In this volume, we define also the local particle number distribution or Wigner function $n_{\text{c}}'(r,t)$ (for clarity, explicit dependence of time has been added). Also the particles feel an external local potential due to the gradients of the particle density and of the potential energy. As a consequence, the kinetic equations must be modified in order to take into account locally the modifications inside $\Omega$. If we consider that the excited particle moves classically, then the dynamic of transfer is given by the Liouville operator acting on the distribution function:

$$\left\{ \epsilon_k(r,t), n_{\text{c}}'(r,t) \right\} = \nabla_k \epsilon_k(r,t) \nabla_k n_{\text{c}}'(r,t) - \nabla_r \epsilon_k(r,t) \nabla_k n_{\text{c}}'(r,t)$$

The condensate particles however move locally according to the quantum wave function, which obeys to the generalized Gross-Pitaevskii equation with the effective potential [2]:

$$\epsilon_k(r,t) = V_{\text{ext}}(r) + 8\pi a \frac{\left| \Psi(r,t) \right|^2}{mV} + 2 \sum_k n_k'(r,t)$$

This energy density varies slowly in space. The gradients produced by these variations govern the dynamic of the gas particle between each small volume $\Omega$. The local energy per excited particle with a momentum $k$ is given by the derivative of the local energy density with respect to the particle number [2]:

$$\epsilon_k(r,t) = \frac{dE}{dn_k'(r,t)} = \frac{k^2}{2m} + V_c(r,t)$$

where we define the effective potentials felt by the condensate and the particle

$$V_c(r,t) = V_{\text{ext}}(r) + \frac{8\pi a}{mV} \left| \Psi(r,t) \right|^2 + 2 \sum_k n_k'(r,t)$$

Between each infinitesimal volume, particles are transferred due to the gradients of the particle density and of the potential energy. As a consequence, the kinetic equations for weakly inhomogeneous and stable Bose gas in the region of condensation. We find two coupled set of equations, one for the condensate, the other for the normal fluid:
\[
\frac{\partial}{\partial t} \Psi(r, t) = \left[ -\frac{\nabla^2}{2m} + V_c(r, t) \right] \Psi(r, t) \tag{66}
\]

\[
\frac{\partial}{\partial t} n_k'(r, t) = \left[ -\frac{k}{m} \nabla r + \nabla_r V_c(r, t), \nabla_k \right] n_k'(r, t) + C_k[n_k'(r, t); k_n(r, t)] \tag{67}
\]

With the exception of the collision term, the Eq. (66) and (67) are identical to the kinetic equations formulated by Zaremba et al. [2]. The inhomogeneous kinetic equations satisfy locally the conservation laws in these regions of highly populated condensate. We can derive indeed a conservation equation for the local particle number \( N(r, t) = \sum_k n_k(r, t) \), the local momentum \( P(r, t) = \sum_k k n_k(r, t) \) and the total local energy \( \mathcal{E}(r, t) \) [2]. It is also easy to verify that the production of the local entropy \( S(r, t) \) is always positive. [12]

The local production of entropy stops when we reach the local equilibrium:

\[
n_{k'}^{eq}(r, t) = \frac{1}{\exp[\beta(r, t)(\epsilon_{k'} - k\mathbf{v}_n(r, t) - \mu(r, t))] - 1} \tag{68}
\]

where \( \beta(r, t) \), \( \mathbf{v}_n(r, t) \) and \( \mu(r, t) \) are now local functions of the position and the time.

A gap shows up between the energies of the condensed and non condensed particles since \( V_c(r, t) < V_{\text{ext}}(r) \). This is not a problem as long as in the region of the gap the transfer of particle is forbidden. The transfer of particle is only possible in the region where there is no gap i.e. when the condensate population is not macroscopic. Therefore, on the basis of these non-equilibrium considerations, the Hartree-Fock model showing up this forbidden gap does not suffer from any inconsistency related to the conservation of energy.

**B. The superfluid universe at finite temperature**

Using the expression (68) as the solution of the kinetic equation, we can find a set of equations describing the non dissipative motion of the condensed gas at finite temperature. If we assume that \( \mathbf{v}_n(r, t) = 0 \) then the substitution of (68) in Eq. (67) imposes that \( n_{k'}^{eq}(r, t) \) must be stationary in time, that \( \beta(r, t) = \beta \) is a constant and that \( \mu(r, t) = -V_c(r, t) + \mu_c \), where \( \mu_c \) is a chemical potential independent of the position controlling the number of excited particles. As a consequence, after carrying out the integration over the momentum, the local number of excited particle is given by the closed equation:

\[
N_{c}^{eq}(r) = \sum_{k} n_{k'}^{eq}(r) = V \left( \frac{m}{2\pi\beta} \right)^{3/2} g_{3/2} \left( e^{\beta(\mu_c - V_{\text{ext}}(r) - \frac{3}{8}\mu_c(|\Psi(r, t)|^2 + N_{c}^{eq}(r))}} \right) \tag{69}
\]

where \( g_i(x) = \sum_{j=1}^{\infty} j^{-i}x^j \). Since the system is stationary, the macroscopic condensate wave function has the form:

\[
\Psi(r, t) = e^{-i\mu_c t}\Psi_c(r) \tag{70}
\]

where \( \mu_c \) is the chemical potential of controlling the population of the condensate particle. The substitution of this form into Eq. (66) produces:

\[
\left[ -\frac{\nabla^2}{2m} + V_{\text{ext}}(r) + \frac{4\pi a}{mV} (|\Psi_c(r)|^2 + 2N_{c}^{eq}(r)) \right] \Psi_c(r) = \mu_c \Psi_c(r) \tag{71}
\]

The coupled set of equations (69) and (71) describe locally all the non dissipative or superfluid phenomena. These are valid provided the gas is stable. As an example, they describe the superfluid moving with a different velocity than the normal fluid. In that case when \( V_{\text{ext}}(r) = 0 \), the solution is a plane wave function:

\[
\Psi_c(r) = \left.e^{-i\beta \mathbf{k}_c \cdot \mathbf{r}} \right|^\alpha \tag{72}
\]

and \( N_{c}^{eq}(r) \) is a constant. More generally, any non dissipative complex structure at finite temperature, like vortices, should be a solution of (69) and (71).

In contrast to previous studies, non-dissipative phenomena are not a requirement or assumption of a model but rather a prediction of the GRPA kinetic theory. Another difference is that the chemical potential for both fluids \( \mu_c \) and \( \mu_e \) must not be necessarily identical, contrary to the equilibrium statistical mechanics which imposes equality [12]. In our example, \( \mu_e = (8\pi a N)/(mV) \neq \mu_c = k_c^2/(2m) + 4\pi a(2N_c + n_{k_c})/(mV) \) at the difference of [2] where \( \mu_e = \mu_c \) for \( \mathbf{k}_c \) = 0.

**V. ANALOGY WITH PLASMON THEORY**

The derivation leading to the QKE for a Bose gas is similar to the one leading to the QKE equation for plasma physics [25, 29] except that, in these equations, the collective excitations are derived in the SRPA. In plasma physics, the Coulomb interactions potential for the collision process is screened beyond the Debye wavelength. The dynamic dielectric function displays this screening and removes the singularity in the Coulomb potential in the long wavelength limit. Static screening is well known since the Debye theory. However, for dynamic screening, Wyld and Pines have proposed an interpretation in terms
of the plasmon theory. This theory is an alternative version of the kinetic theory of a plasma which emphasizes the role played by the collective modes. According to their work, the effective potential interaction is the result of a plasmon mediating the interaction with the quasi-particles. In other words, during a collision process, a quasi-particle emits an intermediate plasmon on the energy shell with the momentum transfer \( \mathbf{q} \). Later on, this plasmon is eventually absorbed by another quasiparticle which acquires a new momentum and a new energy. The energy spectrum for this plasmon corresponds precisely to the frequency spectrum of the collective excitations in a plasma.

In the case of a Bose condensed gas, the collective excitations spectrum corresponds to the Bogoliubov frequency spectrum for low temperature. This property suggests the interpretation that these phonon-like excitations play also the role of mediators during collisions between quasi-particles. Thus, the Bose condensation has the effect to transform the scattering modes, used for collision involving condensed particle, into a collective mode used for mediating the interaction between non-condensed particles.

The plasmon theory has been derived using the theory of quantum electrodynamics. In principle, for a Bose gas, a similar approach must be carried out taking into account that, at a more fundamental level, the interaction potential originates from processes involving the absorption and the emission of photons. In this paper, we shall not rederive a similar theory but rather recover heuristically the analogy with the plasmon theory.

Following this approach, this intermediate excitation behaves like a particle characterized by its own distribution function \( f_q \) and that we shall call by analogy "condenson". The energy spectrum of the condenson is given by the zeroes of the dynamic dielectric function i.e. by the solution of \( \Delta(q, \omega) = 0 \). We consider the simple case of an homogenous condensate at rest \( k_s = 0 \) and weakly depleted. In that case, the solution is the complex number \( \omega = \omega_q - i\gamma_q \), where the real part represents the energy spectrum \( \omega_q = \epsilon_q^B \) and the imaginary part represents the decay rate of the condenson given by Eq. (61). Then, looking at Eq.(7) and Eq.(8) in [29], we can write two coupled equations, one for the dynamic evolution of the quasi-particle and the other for the evolution of the condenson. They describe the time rate change of these quantities due to the emission and absorption of a condenson by a quasi-particle:

\[
\frac{\partial}{\partial \tau} n'_k = \frac{8\pi a}{mV} \sum_q \frac{8\pi a q^2}{m^2 V \omega_q} \left[ \pi \delta(\omega_q + \epsilon_k - \epsilon_{k+q}) \left( (n'_k + 1)n'_{k+q}(f_q + 1) - n_k(n_k' + 1)f_q \right) + \pi \delta(\omega_q - \epsilon_k + \epsilon_{k-q}) \left( (n'_k + 1)n'_{k-q}f_q - n_k(n_k' + 1)(f_q + 1) \right) \right] \tag{73}
\]

\[
\frac{\partial}{\partial \tau} f_q = -2\gamma_q f_q + \frac{8\pi a}{mV} \frac{8\pi a q^2}{m^2 V \omega_q} \sum_k \pi \delta(\omega_q - \epsilon_k + \epsilon_{k-q})n'_k(n'_k + 1) \tag{74}
\]

Similarly to the reasoning of Wyld and Pines, the kinetic equation (73) can be rederived from Eq.(63) and Eq.(24) by eliminating adiabatically \( f_q \). We assume that the time derivative in Eq. (24) is zero and the elimination allows to obtain an equation for \( n'_k \) only. Eq. (13) is recovered, provided that the condensate is weakly depleted. Under these circumstances, the square of the effective potential has a narrow Lorentzian shape whose two peaks and widths correspond to the condenson energy and decay rate respectively (see [24]). It plays the role of a propagator for the mediated interaction. This supposes that the dielectric function can be approximated around the resonant frequencies as (see Eq.(25) in [24] for comparison):

\[
\mathcal{K}(q, \omega) = \left\{ \begin{array}{c}
\frac{m^2 V \omega_q}{\pi \gamma_q} (\omega - \omega_q + i\gamma_q), \quad \omega \sim \omega_q \\
-\frac{m^2 V \omega_q}{\pi \gamma_q} (\omega + \omega_q - i\gamma_q), \quad \omega \sim -\omega_q
\end{array} \right. \tag{75}
\]

leading to the approximation:

\[
\frac{1}{|\mathcal{K}(q, \omega)|^2} = \frac{\pi}{\gamma_q \left( \frac{4\pi a q^2}{m^2 V \omega_q} \right)^2} \left( \delta(\omega - \omega_q) + \delta(\omega + \omega_q) \right) \tag{76}
\]

These coupled equations suggest that a weakly interacting Bose condensed gas is in reality composed of two kind of bosonic particles: the quasi-particles that become the real particles in the limit of an ideal Bose gas and the condensons that result from a mediation (see Fig.1). At thermal equilibrium, the various particles obey to the Bose-Einstein statistics:

\[
n'_k = n'^q_k = \frac{1}{\exp \left[ \beta(\epsilon_k - \mu) \right] - 1} \tag{77}
\]

and

\[
f_q = \frac{1}{\exp (\beta \omega_q) - 1} \tag{78}
\]

In this way, the condenson distribution function has the same form as the excitations distribution function pre-
dicted by Bogoliubov \cite{18}. But there is an important difference however:

In the Bogoliubov theory, the Bogoliubov excitations correspond to the quasi-particle. On the contrary, in the GRPA model, the Bogoliubov excitations correspond to the condensons and, in this respect, are not the quasi-particles.

The QKEs proposed by \cite{1, 6, 7, 26} suggest instead that the Bogoliubov excitations correspond to the quasi-particle. We must be cautious with this interpretation since the GRPA model is not an exact one. It might be predicted by Bogoliubov \cite{18}. But there is an important derivation for the ground state energy and the structure factor.

Indeed, the correlation function defined in \cite{27} allows to calculate the correction to the energy but also to the total particle momentum distribution $N_k$. This new distribution differs from the quasi-particles population distribution $n_k$. We start initially from the distribution without interaction $N_k = \delta_{k,0}n_0$ where only the condensed mode $k_a = 0$ is populated. Then, we switch on adiabatically the interaction and the correlations appear in a form given by \cite{19}. As a consequence, the condensate becomes weakly depleted and the total kinetic and potential energy gets modified.

These corrections can be calculated directly the static structure factor which, using \cite{30} and for $q \neq 0$, is given by:

$$NS(q) = \langle \rho_q^\dagger \rho_q \rangle = \sum_{k,k'} (n_k + 1)n_{k+q} + \sum_{q} g_q(k,k')(79)$$

where $N = \sum_{k} N_k = \sum_{k} n_k$. An expression for this integral can be found in the appendix C. In contrast, Nozières and Pines express the structure factor in terms of the susceptibility function \cite{8} of the ground state as \cite{26}:

$$NS(q) = \langle \rho_q^\dagger \rho_q \rangle = -\int_0^\infty \frac{d\omega}{\pi} \text{Im} \chi(q, \omega) \quad (80)$$

At zero temperature, the limit $n'_k \rightarrow 0$ must be taken carefully. It implies that the Landau damping $\gamma_{\omega} \rightarrow 0$ but this parameter is different from 0. The limit 0 to 0 must be carried out before the limit $\gamma_{\omega} \rightarrow 0$ and this procedure can be done by adjusting adequately the infinitesimal value of 0 and $n'_k$. In particular, for a normal fluid in equilibrium, this amounts to prescribing to take the limit 0 to zero and then after taking the limit $\beta \rightarrow \infty$ in Eq. (81). The physical reason for such a procedure is that the presence of infinitesimal fraction of excited particles will force the system to create the correct equilibrium correlation. In these conditions and using Eq.(24) and Eq. (31), we can re express for $k_a = 0$:

$$\Delta(q, \omega) \rightarrow (\omega + i\gamma_{q})^2 - \epsilon_{q}^2 \quad (81)$$

Also, we note the relation:

$$\text{Im}A'(q, \omega) = \frac{-mV}{8\pi a \exp(\beta\omega) - 1} \text{Im} K_n(q, \omega) \quad (82)$$

Inserting these results and \cite{31} into \cite{18}, setting to zero the real part of $K(q, \omega)$ and $A'(q, \omega)$, and setting $A_0(q, \omega) = N/(\omega + i0^+ + q^2/(2m))$ then Eq. (79) becomes:

$$S(q) = 1 - \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\omega - (\omega + i\gamma_{q})^2 - \epsilon_{q}^2}{|\omega + i\gamma_{q}|^2 - \epsilon_{q}^2} \times \left(\omega - \frac{q^2 e^{\beta \omega} + 1}{2m e^{\beta \omega} - 1}\right) \text{Im} K_n(q, \omega) \quad \beta \rightarrow \infty$$

$$\gamma_{q} \rightarrow 0$$

$$\beta \rightarrow \infty$$

$$\frac{q^2}{2me^\beta} \quad (83)$$

VI. EQUIVALENCE WITH THE BOGOLIUBOV THEORY

The RPA model presents important differences compared to any previous model for finite temperature. However for zero temperature, it allows to recover the next order correction for the ground state energy and the static structure factor usually obtained from the Bogoliubov theory. Such a derivation has already been made by Nozières and Pines for an electron gas using the density response formalism. They were also able to apply this formalism to the condensate \cite{18}. The nice result of this section is that, precisely, the formalism developed in this paper allows to recover exactly the same results for
In order to get [S3], the integration has been carried out transforming the Lorentzian factors into delta functions and using [S1]. This result is identical to that obtained from the Bogoliubov theory. This result can also be obtained substituting [3] into [S1] and carry out the integration over \( \omega \). Instead of evaluating the total energy and the correction to the particle number distribution directly, some useful tricks allow to avoid some complex calculation [2]. The total ground state energy is a functional of the interaction \( a \) and the dispersion relation \( \epsilon_k \) (in what follows, we consider only the first term in [2]). It can be expressed in terms of the matrix element \( E(a, \epsilon_k) = \langle \psi | H | \psi \rangle \) where \( | \psi \rangle \) is the ground state depending also on \( a \) and \( \epsilon_k \). Using the property that \( | \psi \rangle \) is a normalized eigenfunction, the first derivative with respect to these parameters produces:

\[
\frac{\partial E(a, \epsilon_k)}{\partial a} = E_{\text{int}}(a, \epsilon_k) \tag{84}
\]

\[
\frac{\partial E(a, \epsilon_k)}{\partial \epsilon_k} = N_k \tag{85}
\]

The total interaction energy can be expressed in terms of the structure factor:

\[
E_{\text{int}}(a, \epsilon_k) = \frac{2\pi a}{mV} \left[ N(N-1) + N \sum_{q \neq 0} (S(q) - 1) \right] \tag{86}
\]

At zero temperature and without interaction, all the particles are in the ground state and thus \( E(0, \epsilon_k) = N\epsilon_0 \). With this initial condition, an integration over \( a \) allows to calculate the total energy:

\[
E(a, \epsilon_k) = N\epsilon_0 + \int_0^a \frac{da'}{a'} E_{\text{int}}(a', \epsilon_k)
\]

\[
+ \left( \frac{2\pi aN}{V} \right)^2 \sum_{q \neq 0} \frac{2m}{q^2} \tag{87}
\]

where a supplementary term coming from [2] has been added to remove ultra-violet divergencies. The structure factor can be recalculated for any arbitrary dispersion relation \( \epsilon_k \). Plugging this new expression into [S6] and carrying out the integral over \( a \), we obtain:

\[
E(a, \epsilon_k) = N\epsilon_0 + \frac{2\pi aN^2}{mV} + \frac{1}{2} \sum_{q \neq 0} \left[ \left( \frac{2\pi aN}{V} \right)^2 \frac{m}{q^2} + \sqrt{\Delta \epsilon_q^2 + \frac{8\pi aN}{mV}\Delta \epsilon_q} - \Delta \epsilon_q - \frac{4\pi aN}{mV} \right] \tag{88}
\]

where \( \Delta \epsilon_q = (\epsilon_q + \epsilon_0 - a\epsilon_q)/2 \). In the usual case, \( \Delta \epsilon_q = q^2/2m \) and an integration over the momentum \( q \) allows to find the ground state energy [15]:

\[
E(a) = \frac{2\pi aN^2}{mV} \left[ 1 + \frac{128}{15} \frac{\alpha V}{\pi V} \right] \tag{89}
\]

Finally, the derivative with respect to \( \epsilon_k \) gives the first correction to the momentum particle distribution. We get for \( k \neq 0 \):

\[
N_k = \left. \frac{\partial E(a, \epsilon_k)}{\partial \epsilon_k} \right|_{\epsilon_k = \epsilon_k^0} = \frac{1}{4} \left( \frac{k^2}{2m} + \frac{4\pi aN}{\epsilon_k^0 m^2} - 1 \right) \tag{90}
\]

and for \( k = 0 \):

\[
N_0 = N - \sum_{k \neq 0} N_k = N \left[ 1 - \frac{32}{3} \sqrt{\frac{a^2N}{\pi V}} \right] \tag{91}
\]

From our number conserving approach, we recover the well-known results usually obtained from the Bogoliubov approach at zero temperature, i.e. the first order energy correction in [S9] and the first order correction to the number of excited particles [60] showing a depletion of the condensate particle. In principle, the present method can be extended to finite temperature, since the relations [S1] and [S3] are valid also for any excited state labeled by the occupation number \( n_k \). The only problem remains the explicit calculation of \( \langle \rho^4_q \rangle \).

VII. CONCLUSIONS AND PERSPECTIVES

We attempted to reexamine the kinetic theory of the weakly interacting and stable Bose condensed gas with the objective to explain superfluidity for any temperature and, ultimately, to be able to distinguish between its dissipative and superfluid behaviors. A different QKE, taking into account some higher order terms in the interaction parameter, has been derived from the microscopic theory and has the merit to predict a collision blockade between condensed and non condensed particle. More precisely, the condensate remains invisible to any thermal quasi-particle due to an induced force, which acts as a “coarse-graining”, removing any local force necessary for binary scattering. In this way a metastable state can be built locally with a non-zero relative velocity. For the homogeneous gas, a microscopic derivation has been carried out using the same expansion procedure as that for deriving the QKE for a quantum plasma. The only difference is that our approach takes into account the exchange term which is of the same order of magnitude than the direct term. For the weakly inhomogeneous gas, the derivation needs some supplementary justifications from first principles, in particular, for the way to obtain the generalized Gross-Pitaevskii equation from a number conserving approach, but also for the limit of use of the collision terms [11] and [15].

The generalized RPA is the basic approximation which is at the origin of the prediction of the collision blockade phenomenon. It is not only a method to achieve results but has also the deep meaning that average contributions that are not oscillating in phase can be neglected for diluted gas. The RPA does not serve only to predict the QKE; it allows to build a genuine alternative theory that can be compared with all previous approaches. The main advantages of this theory are: 1) it is number...
of the one particle Green function have a richer structure; namely one gapless pole which represents the Bogoliubov or condensate excitation spectrum and another pole which may have a gap and which represents the quasi-particle spectrum. In this way, we recover the compatibility with the HP theorem. Secondly, in principle, the calculation must start with the real potential and not the effective contact potential. A resummation of all ladder diagrams should allow to reexpress the effective interaction in terms of the $T$ matrix for the binary collision. Then the limit of low energy allows to deduce the expression. If rather the $T$ many body matrix is considered, then the effective interaction could depend also non linearly on the quasi-particle number. Thirdly, the assumption that the gas is weakly homogeneous might not be true, due to a sharp trap potential realized in real experiment. The theory must be improved by decomposing the one particle Wigner function in terms of its eigenfunctions and by deriving an equation of motion for each of them.

APPENDIX A: SOLUTION OF THE EQUATION OF MOTION FOR THE EXCITATIONS

The system and can be solved exactly. We isolate the dielectric propagator on the left hand side of each equation. After summing over the superfluid and normal modes separately and using, we obtain after rearranging:

\[
\begin{align*}
\frac{1}{\Delta(q, \omega)} & \left[ (\omega + i0_+ - \frac{k_s \cdot q}{m})^2 - \epsilon_q^2 \right] \left( \mathcal{U}_q(k_s, k_1, \omega) + \mathcal{U}_q(k_s - q, k_1, \omega) \right) - \frac{8\pi n a_q}{mV} q^2 \sum_{k'} \tilde{u}_q(k', k_1, \omega) = \\
& i[\omega + i0_+ - \frac{k_s \cdot q}{m} + \frac{q^2}{2m}] \delta_{k_s, k_1} + i[\omega + i0_+ - \frac{k_s \cdot q}{m} - \frac{q^2}{2m}] \delta_{k_s - q, k_1}
\end{align*}
\]

(A1)

\[
(K_n(q, \omega) - 1) \left( \mathcal{U}_q(k_s, k_1, \omega) + \mathcal{U}_q(k_s - q, k_1, \omega) \right) + K_n(q, \omega) \sum_{k'} \tilde{u}_q(k', k_1, \omega) = i \frac{1 - \Pi_{k_1}}{\omega + i0_+ - \frac{k_s \cdot q}{m} - \frac{q^2}{2m}}
\]

(A2)

where we define the function $\Pi_k = \delta_{k_s, k} + \delta_{k_s - q, k}$. Using the definition, the solution of this coupled set of equations gives:

\[
\mathcal{U}_q(k_s, k_1, \omega) + \mathcal{U}_q(k_s - q, k_1, \omega) = i \frac{K_n(q, \omega) \Pi_{k_1}}{\Delta(q, \omega)} \left[ (\omega + i0_+ - \frac{k_s \cdot q}{m})^2 - \frac{q^2}{2m} \right] + \frac{8\pi n a_q q^2}{mV} \left( 1 - \Pi_{k_1} \right)
\]

(A3)

and

\[
\sum_{k'} \tilde{u}_q(k', k_1, \omega) = i \frac{(1 - K_n(q, \omega)) \left[ (\omega + i0_+ - \frac{k_s \cdot q}{m})^2 - \frac{q^2}{2m} \right] \Pi_{k_1} + (\omega + i0_+ - \frac{k_s \cdot q}{m})^2 - \epsilon_q^2 \right] \left( 1 - \Pi_{k_1} \right)}{\Delta(q, \omega) \left[ \omega + i0_+ - \frac{k_s \cdot q}{m} - \frac{q^2}{2m} \right]}
\]

(A4)
Consequently, the definitions 48 and 19 allow to write

\[
\sum_{k'} U_q(k', k_1, \omega) = \frac{i}{\omega + i0^+ - \frac{k_1 q}{m} - \frac{q^2}{2m}} \left( 1 - \frac{\Pi_{k_1}}{\kappa(q, \omega)} + \frac{\Pi_{k_1}}{\kappa(q, \omega)} \right)
\]  

(A5)

Plugging these results into the r.h.s. of 22, 23 and 24, the integral terms do not appear anymore and we get the solution:

\[
U_q(k, k_1, \omega) = \frac{i}{\omega + i0^+ - \frac{k q}{m} - \frac{q^2}{2m}} \left[ \frac{8\pi a}{mV} (\langle n_k - n_{k+q} \rangle \kappa_{k,k_1}^{-1}(q, \omega)) \right]
\]

(A6)

where we define the reverse dielectric function

\[
\kappa_{-1,k,k_1}^{-1}(q, \omega) = \frac{(1 - \Pi_k)(1 - \Pi_{k_1})}{\kappa(q, \omega)} + \frac{(1 - \Pi_k)\Pi_{k_1} + \Pi_k(1 - \Pi_{k_1}) + \Pi_k\Pi_{k_1}, (1 - \kappa_n(q, \omega)/2)}{\kappa(q, \omega)}
\]

(A7)

This function can be rewritten in matrix notation, where we distinguish in the column and the row the channel \( k_s \) and \( k_s - q \) from the other channels:

\[
\kappa_{-1,k,k_1}^{-1}(q, \omega) \equiv \begin{pmatrix} \kappa_{-1,k_1}^{-1}(q, \omega) & (1 - \kappa_n(q, \omega)/2) \kappa_{-1,k_1}^{-1}(q, \omega) \end{pmatrix}
\]

(A8)

These results are in agreement with 31 but with the difference that we have included the exchange term which doubles the interaction strength.

**APPENDIX C: CALCULATION OF THE COLLISION TERM**

As said in section 3, the calculation in the SRPA is easy since it presents strong resemblance with the plasma gas. The extension to GRPA is more complicated because the condensed and normal modes cannot be treated on the same footing anymore, as long as the interaction energy per particle is not the same. Nevertheless, the method developed by Ichimaru for a classical plasma can be adapted straightforwardly. Here we shall give the intermediate steps of the calculations.

Before doing so, few symmetry properties are interesting:

\[
g_q(k, k') = g_q^*(k - q, k' + q)
\]

(C1)

\[
\left( \kappa_{-1,k,k_1}^{-1}(q, \omega) \right)^* = \kappa_{-1,k,k_1}^{-1}(-q, -\omega)
\]

(C2)

The first property can be checked from the definition 46 and the third one is straightforward. Using the first property and 37, the collision term defined in Eq. (48) can be expressed in terms of the imaginary part of the correlation function:

\[
G_k^T (n_k; k_s) = \sum_{q, k'} \frac{4\pi a}{mV} \text{Im} (g_q(k, k' - q - k))
\]

(C3)

Let us define:

\[
A_q(k, \omega) = \sum_k (1 - \Pi_k) \frac{(n_k + 1)n_{k+q}}{\omega + i0^+ - \frac{k q}{m} - \frac{q^2}{2m}}
\]

(C4)
\[
A_0(q, \omega) = \sum_k \Pi_k \frac{(n_k + 1)n_k + q}{\omega + i0^+ - \frac{kq}{m} - \frac{q^2}{2m}} \quad \text{(C5)}
\]

With these definitions together with \(\mathbf{A5}, \mathbf{A6}\) and \(\mathbf{A9}\),

\[
\sum_q \sum_{k,k'} (1 - \Pi_k) g_q(k, k') = -\sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega + i0^+ - \frac{kq}{m} - \frac{q^2}{2m}} \left\{ \frac{(n_k + 1)n_{k+q} + q}{|\mathcal{K}|^2} + \frac{8\pi a}{mV} (n_k - n_{k+q}) \right\} \quad \text{(C6)}
\]

\[
\sum_k g_q(k, k') = -\sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega + i0^+ - \frac{kq}{m} - \frac{q^2}{2m}} \left\{ \frac{(n_k + 1)n_{k+q} + q}{|\mathcal{K}|^2} + \frac{8\pi a}{mV} (n_k - n_{k+q}) \right\} \quad \text{(C7)}
\]

Equivalently \(\sum_{q,k'} g_q(k_s - q, k')\) is obtained by substituting \(k_s\) by \(k - q\) in \(\mathbf{C7}\). The domain of integration over \(\omega\) can be extended in the complex plane. Since the various reverse dielectric functions in \(\mathbf{A8}\) converge to unity when \(|\omega| \to \infty\), the integrand goes to zero in this limit faster than \(1/\omega\). As a consequence, the integral over \(\omega\) can be carried out by closing the contour either in the upper half plane or in the lower half plane. Since we assume that the Bose gas is in a stable regime, the poles of the dielectric functions have their imaginary part in the lower half of the complex plane, while the complex conjugate of these functions have their imaginary part in the upper half plane. Thus in \(\mathbf{C6}\) and in \(\mathbf{C7}\), contributions in which poles only lie in one of these half planes are canceled since, in that case, the contour of integration can be chosen in such a way that no pole is surrounded. On the other hand, in order to simplify the Eqs. \(\mathbf{C6}\) and \(\mathbf{C7}\), we can also add arbitrary contributions in which poles lie only in one of these half planes. Taking into account these considerations, we arrive at:

\[
\sum_q \sum_{k,k'} (1 - \Pi_k) g_q(k, k') = -\sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega + i0^+ - \frac{kq}{m} - \frac{q^2}{2m}} \left\{ \frac{(n_k + 1)n_{k+q} + q}{|\mathcal{K}|^2} + \frac{8\pi a}{mV} (n_k - n_{k+q}) \right\} \quad \text{(C8)}
\]

\[
\sum_q g_q(k, k') = -\sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega + i0^+ - \frac{kq}{m} - \frac{q^2}{2m}} \left\{ \frac{(n_k + 1)n_{k+q} + q}{|\mathcal{K}|^2} + \frac{8\pi a}{mV} (n_k - n_{k+q}) \right\} \quad \text{(C9)}
\]

These expressions can be further simplified if we neglect some irrelevant infinitesimal terms proportional to \(0_+\). Up to these infinitesimal term and using the change of variable \(k' = k + q\) and the formula \(\text{Im}(1/(x + i0^+)) = \pi \delta(x)\), we notice that for the two different limits \(n_{k_s}/V \to 0\) or \(n_{k_s}/V\) finite:

\[
\text{Im}\mathcal{K}(\omega) = \text{Im}\mathcal{K}_n(\omega) = \frac{8\pi a}{mV} \sum_{k'} (n_{k'} - n_{k'}) \pi \delta(\omega - \frac{k'q}{m} + \frac{q^2}{2m}) \quad \text{(C10)}
\]
\[ \text{Im} \left( \frac{K(q, \omega)}{K^+(q, \omega)K(q, \omega)} \right) = \frac{\text{Im}K(q, \omega)}{|K^+(q, \omega)K(q, \omega)|} \] (C11)

\[ \text{Im} \left( \frac{A_0(q, \omega)}{|K(q, \omega)|^2} \right) = \frac{\text{Im}A_0(q, \omega)}{|K(q, \omega)K^+(q, \omega)|} \] (C12)

Finally, it remains to take the imaginary part of these expressions, integrate them over \( \omega \) and plug the results into (C12). The integration is easy since it involves only delta functions. After rearranging terms we notice that, in Eq.(C13), the term proportional to \( 1 - K_n(q, \omega)/2 \) will not contribute. Note that, concerning the term \( g_q(k - q, k') \), we must carry out the change of variable \( q \to -q \) and \( k' - q \to k' \). In this way, we obtain Eq.(12) and Eq.(15).

Using (A5), a similar reasoning allows also to find the expression:

\[ \sum_{k, k'} g_q(k, k') = -\int_{\infty}^{-\infty} \frac{d\omega}{2\pi i} 2\text{Im} \left[ \frac{1}{|K(q, \omega)|} \left( \frac{1}{K^+(q, \omega)} - 1 \right) A'(q, \omega) + \frac{1}{K(q, \omega)} \left( \frac{1}{K^+(q, \omega)} - 1 \right) A_0(q, \omega) \right] \] (C13)

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