A note on quantum error correction with continuous variables

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We demonstrate that continuous-variable quantum error correction based on Gaussian ancilla states and Gaussian operations (for encoding, syndrome extraction, and recovery) can be very useful to suppress the effect of non-Gaussian error channels. For a certain class of stochastic error models, reminiscent of those typically considered in the qubit case, quantum error correction codes designed for single-channel errors may enhance the transfer fidelities even when errors occur in every channel employed for transmitting the encoded state. In fact, in this case, the error-correcting capability of the continuous-variable scheme turns out to be higher than that of its discrete-variable analogues.

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I. INTRODUCTION

Quantum error correction is an essential tool in order to protect fragile quantum states whenever they are subject to errors in a quantum computation protocol, including unwanted decoherence effects due to interactions with the environment [1]. In the case of systems described by discrete quantum variables such as qubits, the theory of quantum error correction codes (QECC) is very advanced. Various codes have been proposed to correct arbitrary types of errors. In these codes, a signal state is encoded into a larger Hilbert space, typically including additional ancilla systems, resulting in an encoded, multipartite entangled state. Local errors on some subsystems can then be corrected, as the quantum information remains undisturbed in the global structure of the total encoded state. The simplest qubit QECC are designed such that only single-qubit errors can be corrected, i.e., errors occurring in a single channel, assuming that the subsystems of the encoded state are sent through individual, independent channels. Provided the error probabilities of such single-channel errors are below a certain threshold (and hence, multiple-channel errors are highly unlikely), a protocol based upon QECC leads to a better performance than an unencoded scheme.

In the case of systems described by continuous quantum variables, the situation is not so clear. Although there are proposals of QECC in the regime of continuous variables [2, 3, 4], the applicability of these codes, and to what extent they are useful, has not been fully understood. For example, encoding an arbitrary signal state into a nine-mode wavepacket code would enable one to correct an arbitrary single-mode error, including Gaussian or non-Gaussian errors (where a Gaussian error maps a Gaussian state back to a Gaussian state). The whole protocol would nonetheless rely upon only Gaussian ancilla states (squeezed states) and Gaussian operations (beam splitter transformations, homodyne detection, and feedforward; online squeezing transformations are not needed [4]). This would lead to very efficient implementations of QECC. However, similar to the qubit case, do these codes also improve the performance of quantum information protocols in the presence of independent, multiple-channel errors? Or do we have to strictly (and rather artificially) assume that only a single mode in a single channel is affected by an error, while all the other channels are ideal? The answer to this question will certainly depend on the error (or noise) model taken into account. In fact, quite intuitively, the transmission of an optical mode in a Gaussian state subject to photon loss cannot be enhanced using the Gaussian QECC, as the losses would occur in every channel of the encoded state and the corresponding, realistic noise model would lack the stochastic nature of the most common qubit channels. More generally, it has been proven that Gaussian QECC are not useful to protect Gaussian states against Gaussian errors [5, 6].

Here, we will focus on a simple Gaussian three-mode repetition code which can correct arbitrary “x-errors”, i.e., x-displacements and any other errors decomposable into x-displacements (including non-Gaussian errors). A code for correcting arbitrary errors including non-commuting x and p-errors is obtainable, for instance, by concatenating the three-mode code into a nine-mode code [4]. We find that for a certain class of (non-Gaussian) error models, the deterministic three-mode QECC protocol enables one to achieve a significant improvement of fidelity compared to the direct transmission of the signal state. Such error models may describe, for instance, free-space channels with atmospheric fluctuations causing beam jitter, as considered recently for various non-deterministic distillation protocols [8, 9, 10] (see also [11]).

II. STOCHASTIC ERROR MODELS

Let us consider the following error model. The input state described by the Wigner function $W_{in}$ is transformed into a new state $W_{error}$ with probability $\gamma$; it re-
mains unchanged with probability $1 - \gamma$. Thus, we have

$$W_{\text{out}}(x, p) = (1 - \gamma)W_{\text{in}}(x, p) + \gamma W_{\text{error}}(x, p).$$  \hspace{1cm} (1)

In general, the Wigner functions $W_{\text{in}}$ and $W_{\text{error}}$ may describe arbitrary quantum states. Note that even in the case of two Gaussian states, $W_{\text{in}}$ and also $W_{\text{error}}$, the resulting state $W_{\text{out}}$ is no longer Gaussian. Thus, this channel model describes a certain, simple form of non-Gaussian errors. It is a generalization of the “erasure” channel considered in Ref. [9], where a coherent-state input is displaced to a vacuum state with an error probability of $\gamma$; otherwise the coherent state leaves the channel untouched.

Similarly, as an example, we will now consider a coherent-state input, $|\alpha_1 \rangle = |\bar{x}_1 + i\bar{p}_1 \rangle$, described by the Wigner function,

$$W_{\text{in}}(x_1, p_1) = \frac{2}{\pi} \exp[-2(x_1 - \bar{x}_1)^2 - 2(p_1 - \bar{p}_1)^2].$$  \hspace{1cm} (2)

Moreover, for simplicity, we assume that the effect of the error is just an $x$-displacement by $\bar{x}_2$ such that

$$W_{\text{error}}(x_1, p_1) = W_{\text{in}}(x_1 - \bar{x}_2, p_1).$$  \hspace{1cm} (3)

Note that more general errors, including non-Gaussian $x$-errors, could be considered as well. The sign of the displacement error shall be fixed and known, e.g., without loss of generality, $\bar{x}_2 > 0$.

### III. Encoding and Transmission

Now in order to encode the input state, we use two ancilla modes, each in a single-mode $x$-squeezed vacuum state, represented by

$$W_{\text{anc}}(x_k, p_k) = \frac{2}{\pi} \exp[-2e^{x_k^2/2} - 2e^{-p_k^2/2}],$$  \hspace{1cm} (4)

with squeezing parameter $r$ and $k = 2, 3$. The total three-mode state before encoding is

$$W(\alpha_1, \alpha_2, \alpha_3) = W_{\text{in}}(x_1, p_1)W_{\text{anc}}(x_2, p_2)W_{\text{anc}}(x_3, p_3),$$  \hspace{1cm} (5)

with $\alpha_j = x_j + ip_j$, $j = 1, 2, 3$. The encoding may be achieved by applying a “tritter”, i.e., a sequence of two beam splitters with transmittances $1: 2$ and $2: 1$. The total, encoded state will be an entangled three-mode Gaussian state with Wigner function,

$$W_{\text{enc}}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{2}{\pi}\right)^3 \times \exp\left\{-2\left[\frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) - \bar{x}_1\right]^2 - \frac{2}{3} e^{-2r} \left[(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_1 - p_3)^2\right] - 2\left[\frac{1}{\sqrt{3}}(p_1 + p_2 + p_3) - \bar{p}_1\right]^2 - \frac{2}{3} e^{2r} \left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_1 - x_3)^2\right]\right\}.$$  \hspace{1cm} (6)

Note that this encoding procedure does not require any online implementations of continuous-variable CNOT gates $\ref{2}, \ref{3}$; combining the signal mode with the offline squeezed ancilla modes at a sequence of beam splitters is sufficient $\ref{4}$.

Now we send the three modes through individual channels where each channel acts independently upon every mode as described by Eq. (1) with $W_{\text{error}}$ corresponding to an $x$-displacement by $\bar{x}_2$. As a result, the three noisy channels will turn the encoded state into the following three-mode state,

$$W_{\text{enc}}(\alpha_1, \alpha_2, \alpha_3) = (1 - \gamma)^3 W_{\text{enc}}(\alpha_1, \alpha_2, \alpha_3) + \gamma(1 - \gamma)^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_1, \alpha_2, \alpha_3) + \gamma(1 - \gamma)^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_2, \alpha_2, \alpha_3) + \gamma(1 - \gamma)^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_3, \alpha_2, \alpha_3) + \gamma^2(1 - \gamma)^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_1, x_2 - \bar{x}_2 + ip_2, \alpha_3) + \gamma^2(1 - \gamma)^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_1, x_3 - \bar{x}_2 + ip_3, \alpha_3) + \gamma^2 W_{\text{enc}}(x_1 - \bar{x}_2 + ip_1, x_2 - \bar{x}_2 + ip_2, x_3 - \bar{x}_2 + ip_3, \alpha_3).$$  \hspace{1cm} (7)

Note that we assumed the same $x$-displacements in every channel. Let us now consider the decoding procedure and how to extract the error syndromes by using homodyne detections on the ancilla modes.

### IV. Decoding and Syndrome Extraction

The decoding procedure now simply means inverting the tritter, which results in

$$W_{\text{dec}}(\alpha_1, \alpha_2, \alpha_3) = (1 - \gamma)^3 W_{\text{in}}(x_1, p_1)W_{\text{anc}}(x_2, p_2)W_{\text{anc}}(x_3, p_3) + \gamma(1 - \gamma)^2 W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2, p_1\right) \times W_{\text{anc}}\left(x_2 - \sqrt{\frac{2}{3}} \bar{x}_2, p_2\right) W_{\text{anc}}(x_3, p_3) + \gamma(1 - \gamma)^2 W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2, p_1\right) \times W_{\text{anc}}\left(x_2 + \frac{1}{\sqrt{6}} \bar{x}_2, p_2\right) W_{\text{anc}}(x_3 - \frac{1}{\sqrt{2}} \bar{x}_2, p_3) + \gamma(1 - \gamma)^2 W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2, p_1\right) \times W_{\text{anc}}\left(x_2 + \frac{1}{\sqrt{6}} \bar{x}_2, p_2\right) W_{\text{anc}}(x_3 + \frac{1}{\sqrt{2}} \bar{x}_2, p_3)\,$$  \hspace{1cm} (8)
\[ +\gamma^2(1 - \gamma)W_{\text{in}}(x_1 - \frac{2}{\sqrt{3}} \bar{x}_2, p_1) \]
\[ \times W_{\text{anc}}\left(x_2 - \frac{1}{\sqrt{6}} \bar{x}_2, p_2\right)W_{\text{anc}}(x_3 - \frac{1}{\sqrt{2}} \bar{x}_2, p_3) \]
\[ +\gamma^2(1 - \gamma)W_{\text{in}}(x_1 - \frac{2}{\sqrt{3}} \bar{x}_2, p_1) \]
\[ \times W_{\text{anc}}\left(x_2 - \frac{1}{\sqrt{6}} \bar{x}_2, p_2\right)W_{\text{anc}}(x_3 + \frac{1}{\sqrt{2}} \bar{x}_2, p_3) \]
\[ +\gamma^2(1 - \gamma)W_{\text{in}}\left(x_1 - \frac{2}{\sqrt{3}} \bar{x}_2, p_1\right) \]
\[ \times W_{\text{anc}}\left(x_2 + \frac{2}{3} \bar{x}_2, p_2\right)W_{\text{anc}}(x_3, p_3) \]
\[ +\gamma^3W_{\text{in}}(x_1 - \sqrt{3} \bar{x}_2, p_1) \]
\[ \times W_{\text{anc}}(x_2, p_2)W_{\text{anc}}(x_3, p_3). \]

By looking at this state, we can easily see that x-homodyne detections of the ancilla modes 2 and 3 (the syndrome measurements) will almost unambiguously identify in which channel a displacement error occurred and how many modes were subject to a displacement error. The only ambiguity comes from the case of an error occurring in every channel at the same time (with probability \(\gamma^3\)), which is indistinguishable from the case where no error at all happens. In both cases, the two ancilla modes are transformed via decoding back into the two initial single-mode squeezed vacuum states. All the other cases, however, can be identified, provided the initial squeezing \(r\) is sufficiently large such that the displacements \(\propto \bar{x}_2\), originating from the errors, can be resolved in the ancilla states. This will always be possible when \(r \to \infty\) and/or \(\bar{x}_2 \gg 1\). Note that even without squeezing, \(r = 0\), all those cases can be distinguished, provided that \(\bar{x}_2 \gg 1\) holds. However, in this case, the recovery displacement operations will result in larger excess noises. Perfect recovery with unit fidelity requires infinite squeezing. Similarly, small displacements \(\bar{x}_2\) will require sufficiently large squeezing to be resolved, \(e^{-2r}/4 < \bar{x}_2\).

The recovery operation, i.e., the final phase-space displacement of mode 1 depends on the syndrome measurement results for modes 2 and 3 which are consistent with either undisplaced squeezed vacuum states (‘0’) or squeezed vacua displaced in either ‘+’ or ‘−’ \(x\)-direction. The syndrome results for modes 2 and 3 corresponding to the eight possibilities for the errors occurring in the three channels are \((0,0)\) for no error at all, \((+,0)\) for an error in channel 1, \((-,+\) for an error in channel 2, \((-,+)\) for an error in channel 3, \((+,+)\) for errors in channels 1 and 2, \((+,−)\) for errors in channels 1 and 3, \((−,+)\) for errors in channels 2 and 3, and, again, \((0,0)\) for errors occurring in all three channels.

The (unnormalized) conditional states of mode 1 depending on the syndrome measurement results \(x_2\) and \(x_3\), including suitable feedforward operations (i.e., displacements of mode 1 using the measured results), can be obtained by integrating Eq. \(\text{(8)}\) over \(p_2\) and \(p_3\),

\[ W_{\text{cond}}(\alpha_1|x_2, x_3) \]
\[ \propto \left(1 - \gamma\right)^3 \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-e^{-2r} x_2^2} \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-e^{-2r} x_3^2} \]
\[ \times W_{\text{in}}(x_1, p_1) \]
\[ +\gamma(1 - \gamma)^2 \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_2 - \sqrt{2/3} x_3)^2} \]
\[ \times \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r} x_3^2} \]
\[ \times W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2 - \frac{1}{\sqrt{2}} e^r x_2, p_1\right) \]
\[ +\gamma(1 - \gamma)^2 \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_2 + \bar{x}_2/\sqrt{6})^2} \]
\[ \times \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_3 - \bar{x}_2/\sqrt{7})^2} \]
\[ \times W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2 + \frac{2}{\sqrt{3}} \bar{x}_3, p_1\right) \]
\[ +\gamma(1 - \gamma)^2 \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_2 - \bar{x}_2/\sqrt{6})^2} \]
\[ \times \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_3 - \bar{x}_2/\sqrt{7})^2} \]
\[ \times W_{\text{in}}\left(x_1 - \frac{1}{\sqrt{3}} \bar{x}_2 - \frac{2}{\sqrt{3}} \bar{x}_3, p_1\right) \]
\[ +\gamma^2(1 - \gamma) \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_2 - \bar{x}_2/\sqrt{6})^2} \]
\[ \times \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_3 + \bar{x}_2/\sqrt{7})^2} \]
\[ \times W_{\text{in}}\left(x_1 - \frac{2}{\sqrt{3}} \bar{x}_2 - 2 \frac{\sqrt{2}}{3} \bar{x}_3, p_1\right) \]
\[ +\gamma^2(1 - \gamma) \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r}(x_2 + \sqrt{2/3} x_3)^2} \]
\[ \times \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r} x_3^2} \]
\[ \times W_{\text{in}}\left(x_1 - \frac{2}{\sqrt{3}} \bar{x}_2 - \sqrt{2} x_2, p_1\right) \]
\[ +\gamma^3 \sqrt{\frac{2}{\pi \epsilon e^{-2r}}} e^{-2e^{-2r} x_3^2} \]
\[ \times W_{\text{in}}\left(x_1 - \sqrt{3} \bar{x}_2, p_1\right). \]
We observe that in almost all cases, the feedforward operations turn mode 1 back into the initial state up to some Gaussian-distributed excess noise depending on the degree of squeezing used for the encoding. The only case for which no correction occurs is when errors appear in every channel at the same time, at a probability of $\gamma^3$. In this case, the initial state remains uncorrected, with an $x$-displacement error of $\sqrt{3}\bar{x}_2$.

In the limit of infinite squeezing, $r \to \infty$, the Gaussian distribution functions in Eq. (9) become delta functions. As a result, the ensemble output state of mode 1 upon averaging over all syndrome measurement results $x_2$ and $x_3$ (by integrating over $x_2$ and $x_3$) becomes

$$(1 - \gamma^3)W_{\text{in}}(x_1, p_1) + \gamma^3W_{\text{in}}\left(x_1 - \sqrt{3}\bar{x}_2, p_1\right). \quad (10)$$

We see that a fidelity of at least $1 - \gamma^3$ can be achieved (assuming $\bar{x}_2 \gg 1$; for small $\bar{x}_2$, the fidelity would exceed $1 - \gamma^3$, but those smaller $\bar{x}_2$ may be too hard to detect at the syndrome extraction, depending on the degree of squeezing, see below). This result implies that the encoded scheme performs better than the unencoded scheme (direct transmission with $F_{\text{direct}} = 1 - \gamma$) for any $0 < \gamma < 1$. In other words, by employing the quantum error correction protocol, the error probability can be reduced from $\gamma$ to $\gamma^3$. The continuous-variable scheme, in this model, is more efficient than the analogous qubit repetition code, and it does not require error probabilities $\gamma < 1/2$ as for the case of qubit bit-flip errors [1].

Consider now the regime $e^{-2r}/4 < \bar{x}_2 < 1/4$, corresponding to small displacements below the vacuum limit. The resulting displacements can only be resolved provided the squeezing is large enough. In the limit of infinite squeezing $r \to \infty$, arbitrarily small shifts can be detected and perfectly corrected (with zero excess noise in the output states corresponding to unit fidelity). In the regime $\bar{x}_2 \gg 1$, corresponding to large shifts, even zero squeezing in the ancilla modes (i.e., vacuum ancilla states) is sufficient for error identification. For $r = 0$ and $\bar{x}_2 \gg 1$, the syndrome measurements still provide enough information on the location of the error and, to some extent, on the size of the error. However, for this case of “classical error correction”, the recovery displacements lead to finite excess noises in the output state after error correction, originating from the vacuum ancilla states.

V. CONCLUSIONS

Using the simple example of a three-wavepacket repetition code, we demonstrated that for certain stochastic error models, the continuous-variable, Gaussian protocol (based on offline squeezing, beam splitter transformations, and homodyne detection) leads to a significant improvement of fidelity even when the errors occur in every channel. In this case, the errors correspond to errors in one variable, e.g., $x$-displacements or any errors decomposable into $x$-displacements. The appropriate error model is reminiscent of the most typical qubit channels such as a bit-flip channel. In the continuous-variable regime, these types of stochastic errors map a Gaussian signal state into a non-Gaussian state represented by a discrete, incoherent mixture of the input state with a Gaussian (or a non-Gaussian) state; thus, circumventing the recent nogo result on Gaussian QECC [2].

It turns out that the fidelity gain through encoding compared to direct transmission without encoding is larger than that for the analogous qubit scheme. In fact, for the three-qubit bit-flip repetition code, the eight-dimensional, physical Hilbert space can be divided into only four orthogonal, logical qubit subspaces, corresponding to the four cases of no error at all and a bit flip on any one of the three qubits (corresponding to two classical syndrome bits). In the continuous-variable case, even those events with errors occurring on two of the three modes simultaneously can be unambiguously identified and corrected, since more error subspaces are available and a correspondingly larger amount of syndrome information. Possible applications and extensions of the scheme considered here are stochastic error models with quadratic or even cubic or higher-order $x$-errors, gain optimizations for the recovery displacements with finite squeezing, higher-level repetition codes in one variable, and non-commuting (“truly quantum”) errors in both $x$ and $p$, requiring more complex codes than just three modes.

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