The spectrum of random matrices and integrable systems

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Consider a weight \( \rho(dz) := e^{-V(z)}dz \) on an interval \( F = [A, B] \subseteq \mathbb{R} \), with rational logarithmic derivative and subjected to the following boundary conditions:

\[
V' = \frac{g}{f} = \sum_0^\infty b_i z^i, \quad \lim_{z \to A,B} f(z) e^{-V(z)} z^k = 0 \quad \text{for all } k \geq 0.
\]

(1)

On the ensemble

\( \mathcal{H}_N = \{N \times N \text{ Hermitean matrices}\} \)

define the probability

\[
P(M \in dM) = c_N e^{-Tr V(M)} dM,
\]

(2)

with the Haar measure \( dM \) on \( \mathcal{H}_N \); the latter can be decomposed into a spectral part (radial part) and an angular part:

\[
dM := \prod_1^N dM_{ii} \prod_{1 \leq i < j \leq N} (dRe M_{ij} d \text{Im } M_{ij}) = \Delta^2(z) dz_1...dz_N \ dU,
\]

(3)

where \( \Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j) \) is the Vandermonde determinant. This lecture deals with the following probability for \( E \subseteq F \):

\[
P(\text{spectrum } M \in E, \text{ with } M \in \mathcal{H}_N) = \frac{\int_EN \Delta^2(z) \prod_1^N \rho(dz_k)}{\int_FN \Delta^2(z) \prod_1^N \rho(dz_k)}.
\]

(4)

As the reader can find out from the excellent book by Mehta [], it is well known that, if the probability \( P(M \in dM) \) satisfies the following two requirements: (i) invariance under conjugation by unitary transformations \( M \mapsto U M U^{-1} \), (ii) the random variables \( M_{ii}, \text{Re } M_{ij}, \text{Im } M_{ij}, 1 \leq i < j \leq N \) are independent, then \( V(z) \) is quadratic (Gaussian ensemble). For this ensemble and for very large \( N \), the probability \( P(\text{an eigenvalue } \in dz) \) tends to Wigner’s semi-circle distribution on the interval

\[\text{Appeared in: Group 21, Physical applications and Mathematical aspects of Geometry, Groups and Algebras, Vol.II, 835-852, Eds.:H.-D. Doebner, W. Scherer, C. Schulte, World scientific, Singapore, 1997.}\]
\[ [-\sqrt{2N}, \sqrt{2N}] \]. Upon normalizing the distribution such that the average spacing between eigenvalues equals one, one finds different limits for \( N \to \infty \), according to whether one considers the part of the spectrum near \( z = 0 \) (bulk scaling limit) or near the edge \( \sqrt{2N} \) (edge scaling limit). The first situation leads to the Sine kernel and the second one to the Airy kernel.

It is also known that the following probability can be expressed as a Fredholm determinant, for both, \( N \) finite and \( N \) infinite. A very sketchy proof of this fact will be exposed in this lecture. Therefore the real issue is to compute for \( E \subset F \),

\[
P(\text{exactly } k \text{ eigenvalues in } E) = \left. \frac{(-1)^k}{k!} \left( \frac{\partial}{\partial \lambda} \right)^k \det(I - \lambda K^E_N) \right|_{\lambda=1}
\]

for the kernel

\[
K^E(y, z) = K(y, z)I_E(z).
\]

Below are a few important examples of kernels, the first one being one side of the Christoffel-Darboux formula for classical orthogonal polynomials:

\begin{itemize}
  \item \( e^{-\frac{x}{2}(V(y)+V(z))} \sum_{k=1}^{N} p_{k-1}(y)p_{k-1}(z) \) ("Christoffel-Darboux" kernel)
  \item \( \frac{1}{2} \int_0^x (e^{-ixy} \pm e^{ixy})(e^{ixz} \pm e^{-ixz})dx = \frac{\sin x(y-z)}{y-z} \pm \frac{\sin x(y+z)}{y+z} \) (Sine kernel)
  \item \( \int_0^x A(x+y)A(x+z)dx \) (Airy kernel)
  \item \( \frac{1}{2} \int_0^x xJ_\nu(x\sqrt{y})J_\nu(x\sqrt{z})dx \) (Bessel kernel).
\end{itemize}

As a feature, common to all of them, note that, for instance, the orthogonal polynomials \( p_k(y) \), the exponential function \( e^{-ixy} \pm e^{ixy} \), the Airy function \( A(x+y) \) and the Bessel function \( J_\nu(x\sqrt{y}) \) are all eigenfunctions of second order problems.

Random matrices provide a model for excitation spectra of heavy nuclei at high excitations (Wigner [], Dyson [] and Mehta []). Since the energy levels are governed by the spectrum of a quantum mechanical Hamiltonian on a Hilbert space, it is reasonable upon truncation to assume that the Hamiltonian be represented by large matrices and, from a statistical mechanical point of view, by an ensemble of matrices, possibly satisfying some symmetry properties. In their analysis of nuclear experimental data, Porter and Rosenzweig [] observed that the occurrence of two levels, close to each other, is a rare event (level repulsion), showing that the spacing is not Poissonian, as one might expect from a naive point of view; this lead Wigner to his so-called surmise.

In their pioneering work, Jimbo, Miwa, Mori and Sato [] have shown some \( (p, q) \)-variables derived from the sine kernel satisfy a certain Neumann-like completely integrable finite-dimensional Hamiltonian system. Tracy and Widom [] have successfully
used functional-theoretical tools to compute the level spacing distributions for a more general class of kernels, always yielding Neumann-type systems. In the case where \( E \) is a semi-interval, they find a distribution whose density satisfies a Painlevé type equation. Deift, Its and Zhou [6] have used the Riemann-Hilbert approach to find the precise asymptotics for the distributions above. Random matrices have come up in the context of statistical mechanics and quantum gravity; see [7]. Random matrix ideas play an increasingly prominent role in mathematics: not only have they come up in the spacings of the zeroes of the Riemann zeta function, but their relevance has been observed in the chaotic Sinai billiard and, more generally, in chaotic geodesic flows; Sarnak [8] conjectures that chaos leads to the “spectral rigidity”, typical of the spectral distributions of random matrices, whereas the spectrum of an integrable system is random (Poisson)!

The main question of this lecture is the following: What is the connection of random matrices with integrable systems? Is this connection really useful? Remember an integrable system is a time evolution involving commuting vector fields, parametrized by times \( t_1, t_2, \ldots \). Celebrated examples are the KP hierarchy and the Toda lattice.

The approach in this lecture, based on joint work with M. Adler and T. Shiota [9], is novel and different from the previous studies, mentioned above. Here we get a system of partial differential equations directly for the probability (5) with \( k = 0 \), rather than a Neumann system for some complicated auxiliary variables. Moreover our PDE’s are nothing else but the KP hierarchy for which the \( t \)-partials, viewed as commuting operators, are replaced by non-commuting operators in the endpoints \( A_i \) of the interval \( E \) under consideration. When the boundary of \( E \) consists of one point, one recovers the Painlevé equations, found in [9], at least for some of the kernels (7); from our work, it also appears that some of the Painlevé equations can be viewed as the KP equation in non-commutative operators.

Invoking a different integrable system, H. Peng [11] has extended these methods to symmetric ensembles, instead of Hermitean ensembles. Symplectic ensembles will require the use of still a different integrable system. Picking the 2-Toda lattice leads to equations for the distribution of the spectrum associated with two coupled random matrices; see [12].

The first half of this lecture discusses the finite Hermitean matrix ensembles, which relate to discrete integrable models, while the second deals with infinite ensembles, which arise in the context of continuous systems. The first part serves, to a large extent, to understand and motivate the methods developed in the second part.

1 Finite ensembles at the crossroads of KP, Toda and Virasoro
1.1 Introducing time in the probability

On the ensemble $\mathcal{H}_n$, we consider probability (2), in which we introduce “time”, seemingly as an artifact:

$$P(M_n \in dM) = c_n(t)e^{-\text{Tr}V(M) + \sum t_i \text{Tr} M^i} dM$$

and thus

$$P(\text{spectrum } M_n \in E) = \int_E \Delta^2(z) \prod_{k=1}^n \rho_t(dz_k) = \frac{\tau_n(E,t)}{\tau_n(t)}, \quad (8)$$

where

$$\rho_t(dz) = e^{\sum t_i z_i} e^{-V(z)} dz = e^{\sum t_i z_i} \rho_0(dz). \quad (9)$$

1.2 $\tau_n(E,t)$ satisfies the KP hierarchy

Introduce orthonormal and (monic) orthogonal polynomials, $p_i(z) := p_i(E, t, z)$ and $\tilde{p}_i(z) := \tilde{p}_i(E, t, z)$ respectively, with regard to the density $\rho_t$, on $E \subset F \subseteq \mathbb{R}$,

$$\langle p_i, p_j \rangle_{E,t} = \delta_{ij}, \quad \langle \tilde{p}_i, \tilde{p}_j \rangle_{E,t} = h_i \delta_{ij}, \quad \tilde{p}_i = \sqrt{h_i} p_i. \quad (10)$$

The matrix $m_n$ of moments $\mu_{ij}(E, t)$, at time $t$

$$m_n(E, t) := (\mu_{ij}(E, t))_{0 \leq i, j \leq n-1} := \left(\langle z^i, z^j \rangle_{E,t}\right)_{0 \leq i, j \leq n-1}, \quad (11)$$

can be expanded in Schur polynomials $S_k(t)$, with coefficients given by the moments at time $t = 0$:

$$\mu_{ij}(E, t) = \int_E z^{i+j} e^{\sum t_k z_k} \rho(dz) = \sum_{k=0}^\infty S_k(t) \int_E z^{i+j+k} \rho(dz) = \sum_{k=0}^\infty S_k(t) \mu_{k+i+j}(E, 0). \quad (12)$$

Also, it is classically known (see Szegö []) that both $m_n$ and $p_n(z)$ have integral representations; these facts lead to the following identities:

$$\tau_n(E, t) = \int_{E^n} \Delta(u)^2 \prod_{k=1}^N \rho_t(du_k) = \det m_n(E, t) \quad (13)$$

$$= \det \left( \left( S_{j-i}(t) \right)_{0 \leq i, j \leq n-1} \left( \mu_{jk}(E, 0) \right)_{0 \leq j < \infty} \right) \quad (14)$$

and

$$\tilde{p}_n(E, t, z) = \frac{1}{\det m_n} \int_{E^n} \prod_{1 \leq j \leq \infty} (z - u_k) \Delta(u)^2 \prod_{k=1}^n \rho_t(du_k) \quad (15)$$

$$= z^n \frac{\tau_n(E, t - [z^{-1}])}{\tau_n(E, t)}, \quad \text{where } [\alpha] = \left( \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, ... \right). \quad (16)$$
using in the last equation
\[ e^{\sum_{i=1}^{\infty} t_i u^i} \left(1 - \frac{u}{z}\right) = e^{\sum_{i=1}^{\infty} \left(t_i - \frac{u}{z}\right) u^i}. \] (17)

Equation (14) shows that the integral \( \tau_n(E, t) \) is the determinant of the projection of an infinite-dimensional plane onto the positive Fourier modes \( H_+ = \text{span}\{1, z, z^2, \ldots\} \), as follows:
\[ e^{\sum_{i=1}^{\infty} t_i z^i} \left(\text{span}\left\{ \sum_{j=-\infty}^{n-1} z^j \mu_{n-1-j,k}(E, 0), k = 0, \ldots, n-1 \right\} \oplus z^n H_+ \right) \to H_. \] (18)

Then, Sato [] tells us that this projection satisfies Hirota bilinear identities (Plücker relations) and, in particular, the **KP-hierarchy**; but according to (13) this projection coincides with the integral \( \tau_n(E, t) \). Therefore \( \tau_n(E, t) \) satisfies for all \( n \geq 0 \) the KP-equations 

\[ \tau \frac{\partial^2 \tau}{\partial t_1 \partial t_k} - \frac{\partial \tau}{\partial t_k} \frac{\partial \tau}{\partial t_1} - \sum_{i+j=k+1, i,j \geq 0} S_i(\tilde{\rho}) \tau \cdot S_j(-\tilde{\rho}) \tau = 0, \quad k = 3, 4, \ldots \] (19)

### 1.3 The vector \((\tau_n(E, t))_{n \geq 0}\) satisfies the Toda lattice

Incidentally, equation (16) provides an elementary “classical” proof of Sato’s representation of a wave-function in terms of a \( \tau \)-function. Indeed, in view of equation (23) below, we have that the \( L^2 \)-norm \( \sqrt{h_k} \) of \( \tilde{\rho} \), as in (10), has the form \( h_k = \tau_{k+1}/\tau_k \), so that \( \tilde{p}_k = p_k \sqrt{h_k} = p_k \sqrt{\tau_{k+1}/\tau_k} \); this leads to Sato’s representation of the so-called “wave vector” in terms of the “\( \tau \)-vector” \( \tau(t) := (\tau_n(E, t))_{n \geq 0} \),

\[ \Psi(E, t, z) := (\Psi_k)_{k \geq 0} := e^{\frac{1}{2} \sum_{i=1}^{\infty} t_i z^i} (p_k(E, t, z))_{k \geq 0} \]
\[ = e^{\frac{1}{2} \sum_{i=1}^{\infty} t_i z^i} \left( z^k \frac{\tau_k(t - [z^{-1}])}{\sqrt{\tau_k(t) \tau_{k+1}(t)}} \right)_{k \geq 0} \]
\[ = e^{\frac{1}{2} \sum_{i=1}^{\infty} t_i z^i} S(t) \chi(z), \] (20)

where \( \chi(z) := (z^j)_{i \geq 0} \); this last equation holds, because \( p_k(t, z) \) is a triangular linear combination of monomials \( z^j \), represented by the (invertible) lower-triangular matrix \( S(t) \). Note \( z \chi = A \chi \), where \( A \) is the shift matrix \( A := (\delta_{i,j-1})_{i,j \geq 0} \); therefore

\[ z \Psi = L(t) \Psi, \quad \text{with} \quad L(t) = S(t) A S(t)^{-1} \text{ symmetric, tridiagonal matrix}. \] (21)

\[ z^2 S_i(\pm \tilde{\rho}) = S_i \left( \pm \frac{\partial}{\partial \tau}, \pm \frac{1}{2} \frac{\partial}{\partial \tau}, \pm \frac{1}{3} \frac{\partial}{\partial \tau}, \ldots \right), \quad \text{where} \quad S_i(t) \quad \text{are the elementary Schur polynomials}:
\]
\[ e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{0}^{\infty} S_k(t) z^k. \]
The orthonormality of the polynomials \( p_k(E, t, z) \) implies, by (10) and (20), the orthonormality of the \( \Psi_k(E, t, z) \)'s for the weight \( \rho_0(dz) \) and thus taking partials in \( t \), one finds:

\[
\frac{\partial}{\partial t_i} \int_E \Psi_k(E, t, z) \Psi_k(E, t, z) \rho_0(dz) = 0.
\]

Combining this relation with (20) and (21), one finds that \( \Psi \) and \( L \) evolve according to\(^3\):

\[
\frac{\partial \Psi}{\partial t} = \frac{1}{2} (L^n)_s \Psi \quad \text{and} \quad \frac{\partial L}{\partial t} = \frac{1}{2} [(L^n)_s, L]. \quad \text{(Toda lattice)} \tag{22}
\]

For a full account, see \[\]. The seeds for these facts were already present in the pioneering work of Bessis, Itzykson and Zuber \[\] and later in the work of Witten \[\].

### 1.4 The \( n \)-point correlation function

The methods in this subsection are adapted from Mehta \[\]. Return now to probability (8); upon expanding the products of the two determinants in the integral below, the denominator of (8) can be expressed as follows, using the orthogonality of the monic orthogonal polynomials \( \tilde{p}_k = \tilde{p}_k(F, t, z) \) on the full range \( F \):

\[
\int_{F^n} \Delta^2(z) \prod_{i=1}^n \rho_t(dz_i)
\]

\[
= \int_{F^n} \sum_{1 \leq i,j \leq n} \det(\tilde{p}_{i-1}(z_j)) \det(\tilde{p}_{k-1}(z_\ell)) \prod_{i=1}^n \rho_t(dz_n)
\]

\[
= \sum_{\pi, \pi' \in \sigma_n} (-1)^{\pi + \pi'} \prod_{k=1}^n \int_F \tilde{p}_{\pi(k)-1}(z_k) \tilde{p}_{\pi'(k)-1}(z_k) \rho_t(dz_k)
\]

\[
= n! \prod_{k=0}^{n-1} \int_F \tilde{p}_{k}^2(z) \rho_t(dz) = n! \prod_{k=0}^{n-1} h_k. \tag{23}
\]

Then, using the obvious fact \( \det(A^2) = \det(AA^\top) \), upon borrowing the exponentials from \( \rho_t(dz_i) \) and the \( h_i \) from the denominator, one computes the probability (8) in terms of the polynomials \( \tilde{p}(z) := \tilde{p}(F, t, z) \) and a kernel \( K_n(y, z) \):

\[
P(\text{spectrum } M_n \in E^n) = \frac{1}{n! \prod_{i=1}^n h_{i-1}} \int_{E^n} \det \left( \sum_{1 \leq j \leq n} \tilde{p}_{j-1}(z_k) \tilde{p}_{j-1}(z_\ell) \right) \prod_{1 \leq k, \ell \leq n} \rho_t(dz_i)
\]

\(^3\)with regard to the Lie algebra splitting of \( g\ell(\infty) \) into the algebras of skew-symmetric \( A_s \) and lower triangular (including the diagonal) matrices \( A_b \) (Borel matrices):

\[
g\ell(\infty) = D_s \oplus D_b = A = A_s + A_b.
\]
\[
\det(K_n(z_k, z_\ell))_{1 \leq k, \ell \leq n} \prod_{i=1}^{n} \rho_0(dz_i),
\]
(24)

where the kernel \( K_n \) is defined in terms of the wave function (20) for \( E = F \):
\[
K_n(y, z) := \sum_{j=1}^{n} \Psi_{j-1}(F, t, y)\Psi_{j-1}(F, t, z).
\]
(25)

The orthogonality relations \( \int_{F} \Psi_k(F, t, z)\Psi_\ell(F, t, z)\rho_0(dz) = \delta_{k\ell} \) lead to the reproducing property for the kernel \( K_n(y, z) \):
\[
\int_{F} K_n(y, z)K_n(z, u)\rho_0(dz) = K_n(y, u), \quad \int_{F} K_n(z, z)\rho_0(dz) = n.
\]
(26)

Upon replacing \( E_n \) by \( \prod_1^{k} dz_i \times F^{n-k} \) in (24), upon integrating out all the remaining variables \( z_{k+1}, ..., z_n \) and using the reproducing property (26), one finds the \( n \)-point correlation function
\[
P(\text{one eigenvalue in each } [z_i, z_i + dz_i], i = 1, ..., k)
\]
\[
= c_n \det(K_n(z_i, z_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} \rho_0(dz_i).
\]
(27)

Finally, by Poincaré’s formula for the probability \( P \left( \bigcup E_i \right) \), the probability that no spectral point of \( M \) belongs to \( E \) is given by a Fredholm determinant
\[
\det(I - \lambda A) = 1 + \sum_{k=1}^{\infty} (-\lambda)^k \int_{z_1 \leq ... \leq z_k} \det(A(z_i, z_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} \rho_0(dz_i),
\]
for the kernel \( A = K_n^E \), given by (25):
\[
P(\text{no spectrum } M \in E) = \det(I - K_n^E), \quad K_n^E(y, z) = K_n(y, z)I_E(z).
\]
(28)

1.5 Toda vertex operators and boundary-time Virasoro constraints

This subsection is taken from []; as a new ingredient, we introduced in [] the “vector vertex operator” for the Toda lattice,
\[
X(t, z) := \Lambda^{-1} \chi(z^2) e^{\sum_{i}^\infty t_i z_i} e^{-2 \sum_{i}^\infty \frac{z_i^2}{\ell_i}},
\]
(29)
acting on the vector \((\tau_n)_{n \geq 0}\); it can be viewed as a Darboux transformation, mapping a Toda solution into a new one. It satisfies the following simple “operator identity”[
\[
(\frac{-b^\ell+1}{\partial b} - a^\ell+1 \frac{\partial}{\partial a} + [J^{(2)}_\ell, \cdot]) \int_a^b dz X(t, z) = 0, \quad \ell \geq -1,
\]
(30)
\[
^4(J_\ell X\tau)_k = (J_\ell)_k (X\tau)_k, \quad \text{with } (X\tau)_k = z^{2(k-1)} e^{\sum_{i}^\infty t_i z_i} e^{-2 \sum_{i}^\infty \frac{z_i^2}{\ell_i}},
\]
where the vector operators\(^5\)

\[
J^{(1)}_k = \frac{\partial}{\partial t_k} + \frac{1}{2} (-k) t_k + \delta_{k,0} \text{diag}(..., -2, -1, 0, 1, 2, ...), \quad k \in \mathbb{Z}
\]

\[
J^{(2)}_k = \sum_{i+j=k} :J^{(1)}_i J^{(2)}_j:, \quad k \in \mathbb{Z}
\]

(31)

form the generators of the Heisenberg (oscillator) and Virasoro algebras respectively.

The Hirota bilinear identities for \(\tau\)-functions, already mentioned in section 1.2, also generate so-called Fay identities, familiar to algebraic geometers; they are quadratic relations between translates of \(\tau\)-functions. These Fay identities enable us to express the kernel \(K_n\), defined by (25), and the \(n\)-point correlation functions \(\det(K_n(z_i, z_j))\), in terms of vertex operators acting on the \(\tau\)-vector \(\tau(t) := (\tau_n(F, t))_{n \geq 0}\), underlying \(\Psi(F, t, z)\) as in (20). Namely,

\[
K_n(z, z) = \sum_{j=1}^n \Psi_{j-1}(t, z)^2 = \tau_n^{-1}(X(t, z)\tau(t))_n,
\]

(32)

and, more generally, using higher degree Fay identities:

\[
\det(K_n(z_i, z_j))_{1 \leq i, j \leq k} = \tau_n^{-1}\left(\prod_{i=1}^k X(t, z_i)\tau(t)\right)_n.
\]

(33)

Using the Neumann series in \(\lambda\) for \(\det(I - \lambda K_n^E)\), and setting \(E^c := F \setminus E\), it follows that

\[
P(\text{no spectrum } M_n \in E) = \frac{\tau_n(E^c, t)}{\tau_n(t)} = \frac{(e^{-\int_E X(t, z)e^{-V(z)}dz} \tau(t))_n}{\tau_n(t)}.
\]

(34)

Given the disjoint union

\[
E = \bigcup_{i=1}^r [A_{2i-1}, A_{2i}] \subset F \subset \mathbb{R},
\]

the vectors \(\tau(t) := \tau(F, t) := (\tau_n(F, t))_{n \geq 0}\) and \(\tau(E^c, t) := (\tau_n(E^c, t))_{n \geq 0}\) obey the following Virasoro-like constraints for \(m = -1, 0, 1, ...:\)

\[
\sum_{i \geq 0} \left( a_i J^{(2)}_{i+m} - b_i J^{(1)}_{i+m+1} \right) \tau(F, t) = 0.
\]

(35)

\[
\left( -\sum_{i=1}^{2r} A_i^{m+1} f(A_i) \frac{\partial}{\partial A_i} + \sum_{i \geq 0} \left( a_i J^{(2)}_{i+m} - b_i J^{(1)}_{i+m+1} \right) \right) \tau(E^c, t) = 0.
\]

(36)

\(^5\)the agreement is the following: \(\frac{\partial}{\partial t_k} = 0\) for \(k \leq 0\) and \(t_k = 0\) for \(k \geq 0\). Normal ordering :: means: pull the differentiation to the right, regardless of commutation relations.
A sketch of the proof of equations (35) and (36) goes as follows: besides the matrix $L$ satisfying $L\Psi = z\Psi$, we introduce a new matrix $M$, a matrix analogue of the Orlov-Schulman operator for KP, such that $M\Psi = (\partial/\partial z)\Psi$; of course, $[L, M] = 1$.

The boundary condition (1) for the weight $\rho(z)dz$ on $F$ implies the vanishing of the integral

$$\int_F \frac{\partial}{\partial z} \left( f(z)\Psi_k(z)\Psi_\ell(z)e^{-V(z)} \right) dz = 0. \tag{37}$$

Working out this integral and using the operators $L$ and $M$, one proves the skew-symmetry of the matrix $Q := Mf(L) + \frac{1}{2}(f' - g)(L)$, and hence $[Q, L^{m+1}]^\dagger$; remember $f$ and $g$ are the functions appearing in the weight (1); incidentally, the skew-symmetric matrix $Q$ actually satisfies the string equation $[L, Q] = f(L)$. We then use the ASV-correspondence between symmetry vector fields on the Toda wave functions and the Virasoro symmetries on the $\tau$-functions; this establishes equation (35). Equation (36) follows from (35), using (30) and taking into account the boundary of $E^c$. Observe the decoupling of the Virasoro constraints in (36) into a time-part and a boundary-part!

### 1.6 “Non-commutative” KP hierarchy for the probability distribution

Let now the rational function $V'(z)$, appearing in the weight (1), be of the following form:

$$V'(z) = \frac{g}{f} = \frac{b_0 + b_1 z}{a_0 + a_1 z + a_2 z^2}; \tag{38}$$

then the equations (36) and their powers enable us to extract all partial derivatives $\partial^{k_1+\ldots+k_\ell}\tau(E^c, t)/\partial t_1^{k_1}\ldots\partial t_\ell^{k_\ell}$, evaluated at $t = 0$ in terms of partial differential operators in the boundary points of the disjoint union $E$,

$$\mathcal{A}_k = \sum_{i=1}^{2r} A_i^{k-m} f(A)_i \frac{\partial}{\partial A_i} \quad k = 1, 2, 3, \ldots, \tag{39}$$

where $m = \max(\deg f - 1, \deg g)$. Declare $\mathcal{A}_k$ to be of homogeneous weight $k$.

When $a_2 = 0$ in $V'$, the probability $P_n(A_1, \ldots, A_{2r}) := P(\text{no spectrum } M_n \in E)$ satisfies the KP-hierarchy (19), but with the (commutative) partial derivatives in $t$ replaced by the (non-commutative) partial differential operators $\mathcal{A}_k$ defined by (39):

$$P_n \cdot \mathcal{A}_k A_1 P_n - \mathcal{A}_k P_n \cdot A_1 P_n - \sum_{i+j=k+1} S_i(\tilde{A}) P_n \cdot S_j(-\tilde{A}) P_n + \text{(terms of lower weight for } 1 \leq i \leq k) = 0, \quad k \geq 3. \tag{40}$$

When $a_2 \neq 0$ in $V'$, the $\mathcal{A}_k$’s in (36) are replaced by more complicated expressions.

6$[A, B]_\dagger = \frac{1}{2}(AB + BA)$

7$S_i(\pm \tilde{A}) := S_i(\pm A_1, \pm \frac{1}{2} A_2, \pm \frac{1}{3} A_3, \ldots)$ are the elementary Schur polynomials with $t_k$ replaced by $\frac{1}{k} A_k$. 

9
Examples: When $V(z) = z^2$, the $p_k(R, 0, z)$'s are Hermite polynomials and, for a semi-infinite interval $E = (A, \infty)$, the first equation of the system (40) reduces to the Painlevé IV equation for $\partial/\partial A \log \det(I - K_n^E)$. The case $V = z - \alpha \log z$ leads to Laguerre polynomials at $t = 0$ and to Painlevé V for $(A \partial/\partial A) \log \det(I - K_n^E)$. The Jacobi polynomials are also a special case of (38) and the first equation of the hierarchy (40) reduces to an unknown equation. These Painlevé equations were first obtained by Tracy and Widom [] and then by us [], as a special instance of the non-commutative KP.

2 Continuous kernels, KP and Virasoro

Here, we shall be dealing with continuous kernels, like the Sine, Airy or Bessel kernels listed in (7). How do we obtain PDE’s for the spectral distribution, using the theory of integrable systems ? The question is how to introduce time ? This question was resolved by Adler, Shiota and the author in []. In the finite situation, the kernel $K_N$ in (25) was given by a Christoffel-Darboux sum, based on eigenvectors of $L$:

$$K_n(y, z) := \sum_{j=1}^{n} \Psi_{j-1}(t, y) \Psi_{j-1}(t, z),$$  \hspace{1cm} \text{(41)}

with $\Psi := (\Psi_j)_{j \geq 0}$ an eigenvector of a second order problem and evolving in time according to the Toda lattice (see (21) and (22)):

$$z \Psi = L \Psi, \quad \frac{\partial \Psi}{\partial t_n} = \frac{1}{2} (L^{n}), \Psi, \quad \frac{\partial L}{\partial t_n} = \frac{1}{2} [(L^{n}), L].$$  \hspace{1cm} \text{(42)}

The continuous analogue goes as follows: consider now wave and adjoint wave functions $\Psi(x, t, z)$ and $\Psi^* (x, t, z)$, with $x \in \mathbb{R}$, $t \in \mathbb{C}^\infty$, $z \in \mathbb{C}$ satisfying the KP-hierarchy,

$$z \Psi = L \Psi, \quad \frac{\partial \Psi}{\partial t_n} = (L^{n})_{+} \Psi, \quad \frac{\partial L}{\partial t_n} = [(L^{n})_{+}, L],$$  \hspace{1cm} \text{(43)}

$$z \Psi^* = L^{\top} \Psi^*, \quad \frac{\partial \Psi^*}{\partial t_n} = -(L^{\top n})_{+} \Psi^*,$$

where $L$ is a pseudo-differential operator

$$L = D + a_{-1} D^{-1} + \ldots, \quad \text{with} \quad D = \frac{\partial}{\partial x}.$$

We consider the $p$-reduced KP hierarchy, i.e., the reduction to $L$’s such that $L^p = D^p + \ldots$ is a differential operator for some $p \geq 2$. The precise continuous analogue of (41), which relates to a second order problem, would be to choose $p = 2$. 

10
For the time being, we take \( p \geq 2 \) arbitrary. According to Sato’s theory, we have the following representation of \( \Psi \) and \( \Psi^* \) in terms of a \( \tau \)-function (see [1])

\[
\Psi(x, t, z) = e^{xz + \sum_{i=1}^{\infty} t_i z^i \frac{\tau(t - [z^{-1}])}{\tau(t)}} \quad \text{and} \quad \Psi^*(x, t, z) = e^{-xz - \sum_{i=1}^{\infty} t_i z^i \frac{\tau(t + [z^{-1}])}{\tau(t)}},
\]

(44)

for notation \([z^{-1}]\), see (16). Observe

\[
\Phi(x, t, z) := \sum_{\omega \in \zeta_p} a_{\omega} \Psi(x, t, \omega z) \quad \text{and} \quad \Phi^*(x, t, z) = \sum_{\omega \in \zeta_p} a_{\omega} \Psi^*(x, t, \omega z),
\]

(45)

are the most general solution of the spectral problems \( L^p \Phi = z^p \Phi \) and \( L^p \Phi^* = z^p \Phi^* \) respectively.

The continuous analogue of (41) is

\[
K_x(y, z) := \int x \, dx \, \Phi^*(x, t, y) \Phi(x, t, z), \quad \text{also} \quad K^E_x(y, z) := K_x(y, z) I_E(z),
\]

where \( \Phi \) and \( \Phi^* \) are given by (45), subjected to the condition \( \sum_{\omega \in \zeta_p} a_{\omega} b_{\omega} = 0 \). Introduce the \( p \)-reduced vertex operator

\[
Y(t, y, z) := \sum_{\omega, \omega' \in \zeta_p} c_{\omega\omega'} X(t, \omega y, \omega' z),
\]

which maps the space of \( p \)-reduced KP \( \tau \)-functions into itself, with

\[
X(t, y, z) := \frac{1}{z - y} e^{\sum_{i=1}^{\infty} (z^{i-1} - y^{i-1}) t_{i} e^{\sum_{i=1}^{\infty} (y^{i-1} - z^{i-1}) \frac{1}{t_i} \frac{\partial}{\partial t_i}}},
\]

see [1]. We now set \( c_{\omega\omega'} = a_{\omega} b_{\omega'} \) in the operator \( Y \) above; again using Fay identities, we obtain continuous analogues of (32),(33) and (34),

\[
K_x(y, z) := \frac{1}{\tau(t)} Y(t, y, z) \tau(t)
\]

\[
\det(K_x(y_i, z_j))_{1 \leq i, j \leq k} = \frac{1}{\tau} \prod_{i=1}^{k} Y(t, y_i, z_i) \tau
\]

\[
\det(I - \lambda K^E_x) = \frac{1}{\tau} e^{-\lambda \int_{E} dz Y(t, z, z) \tau}.
\]

If a \( \tau \)-function \( \tau(t) \) satisfies a Virasoro constraint\( ^8 \)(analogue of (35)):

\[
J^{(2)}_{kp} \tau = c_{kp} \tau \quad \text{for a fixed} \quad k \geq -1
\]

\[
^8 \zeta_p := \{ \omega \text{ such that } \omega^p = 1 \}
\]

\[
^9 J^{(1)}_k = \frac{\partial}{\partial t_k} + (-t) t_{-k}, \quad J^{(2)}_k = \sum_{i+j=k} : J^{(1)}_i J^{(1)}_j :;
\]

see footnote 4.
then, given a subset \( E = \bigcup_{i=1}^r [A_{2i-1}, A_{2i}] \subset \mathbb{R}_+ \), the Fredholm determinant of
\[
\tilde{K}_E^*(\lambda, \lambda') := \frac{1}{p} \prod_{i=1}^r k_{x,z}(z, z') I_E(\lambda'), \quad \lambda = z^p, \lambda' = z'^p
\]
satisfies the following constraint
\[
\left( -p \sum_{i=1}^{2r} A_i^{k+1} \frac{\partial}{\partial A_i} + \frac{1}{2} (J_{kp}^{(2)} - c_k) \right) \tau \det(I - \mu \tilde{K}_E^*) = 0.
\]
Notice again the decoupling of the boundary- and the time-parts, as in (36). We further proceed in the same way as in the discrete case: Under suitable assumptions on the initial conditions, analogous to (38), we can express all \( t \)-partials at \( t = 0 \) in terms of partial differential operators in the endpoints \( A_i \) of \( E \), provided \( p = 2 \); we then substitute those expressions in the KP hierarchy, thus leading to a hierarchy of PDE’s in terms of non-commutative operators in the \( A_i \). A precise statement due to Adler-Shiota-van Moerbeke [] proceeds as follows:

**Theorem.** Consider a first-order differential operator \( A \) in \( z \) of the form
\[
A = A_z = \frac{1}{2} z^{-m+1} \left( \frac{\partial}{\partial z} + V'(z) \right) + \sum_{i \geq 1} c_i z^{-2i},
\]
with
\[
V(z) = \frac{\alpha}{2} z + \frac{\beta}{6} z^3 \not= 0, \quad m = \text{deg} V' = 0 \text{ or } 2,
\]
and its “Fourier” transform
\[
A_x = \frac{1}{2} (x + V'(D)) D^{-m+1} + \sum_{i \geq 1} c_i D^{-2i}
\]
with \( D = \frac{\partial}{\partial x} \).

Let \( \Psi(x, z), x \in \mathbb{R}, z \in \mathbb{C} \) be a solution of the linear partial differential equation
\[
A_z \Psi(x, z) = A_x \Psi(x, z),
\]
with holomorphic \( (z^{-1}) \) initial condition at \( x = 0 \), subjected to the following differential equation for some \( a, b, c \in \mathbb{C} \),
\[
(aA^2 + bA + c) \Psi(0, z) = z^2 \Psi(0, z), \quad \text{with} \quad \Psi(0, z) = 1 + O(z^{-1}).
\]
Then
- \( \Psi(x, z) \) is a solution of a second order problem for some potential \( q(x) \)
\[
(D^2 + q(x)) \Psi(x, z) = z^2 \Psi(x, z).
\]
Given a subset \( E = \bigcup_{i=1}^{r} [A_{2i-1}, A_{2i}] \) and the kernel

\[
K^E_x(y, z) := I_E(z) \int^x \frac{\Phi(x, \sqrt{y})\Phi(x, \sqrt{z})}{2y^{1/4}z^{1/4}} dx,
\]

with

\[
\Phi(x, u) := \sum_{\omega = \pm 1} b_{\omega} e^{\omega V(u)} \Psi(x, \omega u),
\]

the Fredholm determinant \( \mathcal{P}(A_1, \ldots, A_{2r}) := \det(I - \lambda K^E_x) \) satisfies a hierarchy of bilinear partial differential equations\(^{10}\) in the \( A_i \) for odd \( n \geq 3 \):

\[
\mathcal{P} \cdot A_n A_1 \mathcal{P} - A_n \mathcal{P} \cdot A_1 \mathcal{P} - \sum_{i + j = n+1} S_i(\tilde{A}) \mathcal{P} \cdot S_j(-\tilde{A}) \mathcal{P} + \text{(terms of lower weight for } i) = 0,
\]

where the \( A_n \) are differential operators of homogeneous “weight” \( n \), defined by

\[
A_n = \sum_{i=1}^{2r} A_i \frac{n!}{2^{n-m}} \frac{\partial}{\partial A_i}, \quad n = 1, 3, 5, \ldots
\]

examples. **Airy kernel**: Given an entire function \( U(u) \) growing sufficiently fast at \( \infty \), consider its “Fourier” transform,

\[
F(y) = \int_{-\infty}^{\infty} e^{-U(u)+uy} du;
\]

define the associated function \( \rho(z) \) and the differential operator \( A \):

\[
\rho(z) = \frac{1}{\sqrt{2\pi}} e^{U(z)-zU'(z)} \sqrt{U''(z)} \text{ and } A := \rho(z) \frac{1}{U''(z)} \frac{\partial}{\partial z} \rho(z)^{-1}.
\]

Then the function

\[
\Psi(x, z) := \rho(z) F(x + U'(z))
\]

obeys, for a trivial reason,

\[
A \Psi(x, z) = \frac{\partial}{\partial x} \Psi(x, z),
\]

and, upon integration of \( \frac{\partial}{\partial u} e^{-U(u)+uy} \), one shows \( \Psi(x, z) \) also satisfies:

\[
\left( U'(\frac{\partial}{\partial x}) - x \right) \Psi(x, z) = U'(z) \Psi(x, z).
\]
Upon letting \( x \to 0 \) in (58) and using (57), we find
\[
U'(A)\Psi(0, z) = U'(z)\Psi(0, z). \tag{59}
\]
Choosing \( U(z) = z^3/3 \), the integral (54) becomes the Airy function; the operator \( A \) in (55) coincides with (46), for \( V(z) = \frac{2}{3}z^3 \), \( \alpha = 0 \), \( \beta = 4 \), \( m = 2 \):
\[
A := \frac{1}{2z} \left( \frac{\partial}{\partial z} + 2z^2 \right) - \frac{1}{4}z^{-2} \quad \text{and} \quad \hat{A} = D.
\]
Moreover (59) becomes equation (50) and, by stationary phase, satisfies the asymptotics requested in (50):
\[
A^2\Psi(0, z) = z^2\Psi(0, z), \quad \text{with} \quad \Psi(0, z) = 1 + O(z^{-1}) \quad \text{for} \quad z \not\to \infty.
\]
From (58), it follows that \( q(x) = -x \) in (51); the kernel (52) reduces to the Airy kernel mentioned in (7) and therefore its Fredholm determinant satisfies hierarchy (53) with \( \mathcal{A}_n = \sum_{i=1}^{2r} A_i^{\frac{1}{2r}} \frac{\partial}{\partial A_i} \). When \( E \) is a semi-infinite interval, the first PDE in the hierarchy (53) reduces to Painlevé II for \( (\partial/\partial A) \log \det(I - \hat{K}_E) \), as originally discovered in \[\text{[]}\]. Note that in this example, the \( \tau \)-function is the Kontsevich integral. For details on our methods, see \[\text{[]}\].

**Bessel kernel:** Pick \( m, V(z), A_z \) and \( \hat{A}_x \) in (46), (47) and (48) as follows,

\[
V(z) = -z, \quad \text{with} \quad m = 0, \quad A := A_z := \frac{1}{2}z\left( \frac{\partial}{\partial z} - 1 \right), \quad \hat{A} := \hat{A}_x := \frac{1}{2}(x - 1)\frac{\partial}{\partial x},
\]

and consider the partial differential equation (49) with initial condition (50):
\[
\left( 4A^2 - 2A - \nu^2 + \frac{1}{4} \right) \Psi(0, z) = z^2\Psi(0, z), \quad \text{with} \quad \Psi(0, z) = 1 + O(z^{-1}).
\]

The solution to this problem is given by\[\text{[]}\]
\[
\Psi(0, z) = B(z) = \varepsilon \sqrt{z} H_\nu(iz) = \frac{e^{z^{2\nu+\frac{1}{2}}}}{\Gamma(-\nu + \frac{1}{2})} \int_1^\infty \frac{z^{-\nu+\frac{1}{2}}e^{-uz}}{(u^2 - 1)^{\nu+\frac{1}{2}}} du = 1 + O\left( \frac{1}{z} \right)
\]
and
\[
\Psi(x, z) = e^{xz}B((1-x)z). \tag{60}
\]

The latter is an eigenfunction for the second order problem
\[
\left( D^2 - \frac{\nu^2 - \frac{1}{4}}{(x - 1)^2} \right) \Psi(x, z) = z^2\Psi(x, z).
\]

\[\text{[]}\]

\[\text{[]}\]

11 \( \varepsilon = i \sqrt{\frac{2}{\pi}} e^{i\pi\nu/2}, -\frac{1}{2} < \nu < \frac{1}{2} \).
Choosing in (52), \(b_+ = e^{-i\nu}/2\sqrt{\pi}, b_- = ib_+\), and integrating from 1 to \(x = i + 1\) leads to the Bessel kernel mentioned in (7). If \(E\) is a semi-infinite interval, the first equation of (53) reduces to the Painlevé V equation for \((-A\partial/\partial A) \log \det(I - KE)\), which was first done in \[.\] Specializing to \(\nu = \pm\frac{1}{2}\) leads to the sine kernels, mentioned in (7). Finally, the \(\tau\)-function associated with the wave function is a double Laplace-like transform; see \[.\] The details for the Bessel kernel can be found in \[.\]

**Acknowledgments**

The support of National Science Foundation # DMS 95-4-51179, Nato, FNRS and Francqui Foundation grants is gratefully acknowledged.

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