Robust transitivity of singular hyperbolic attractors

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Abstract
Singular hyperbolicity is a weakened form of hyperbolicity that has been introduced for vector fields in order to allow non-isolated singularities inside the non-wandering set. A typical example of a singular hyperbolic set is the Lorenz attractor. However, in contrast to uniform hyperbolicity, singular hyperbolicity does not immediately imply robust topological properties, such as the transitivity. In this paper, we prove that on an open and dense subset of the space of $C^1$ vector fields of a compact manifold, any singular hyperbolic attractors is robustly transitive.

1 Introduction

In 1963 Lorenz [21] studied some polynomial ordinary differential equations in $\mathbb{R}^3$ and he found a strange attractor with the help of computers. By trying to understand the chaotic dynamics in Lorenz’ systems, [2,18,19] constructed some geometric abstract models which are called geometrical Lorenz attractors: these are robustly transitive non-hyperbolic chaotic attractors with singularities in three-dimensional manifolds.

In order to study attractors containing singularities for general vector fields, Morales–Pacifico–Pujals [26] first gave the notion of singular hyperbolicity in dimension 3. This notion can be adapted to the higher dimensional case, see [10,12,22,31]. In the absence of singularity, the singular hyperbolicity coincides with the usual notion of uniform hyperbolicity; in that case it has many nice dynamical consequences: spectral decomposition, stability, probabilistic description and so on. However there also exist open classes of vector fields exhibiting singular hyperbolic attractors with singularity: the geometrical Lorenz attractors are such examples. In order to have a description of the dynamics of general flows, we thus need to
develop a systematic study of singular hyperbolicity in the presence of singularity. This paper contributes to that goal.

We do not expect to be able to describe the dynamics of arbitrary vector fields. Instead, we consider the Banach space $\mathcal{X}^r(M)$ of all $C^r$ vector fields on a compact manifold $M$ without boundary and focus on a subset $\mathcal{G}$ which is dense and as large as possible. A successful approach consists of considering subsets that are $C^1$-residual (i.e. containing a dense $G_δ$ subset with respect to the $C^1$-topology), but this does not immediately allow us to handle smoother systems. For that reason it is useful to work with subsets $\mathcal{G} \subset \mathcal{X}^1(M)$ that are $C^1$-open and $C^1$-dense and to address for each dynamical property the following question:

**Knowing that a given property holds on a $C^1$-residual subset of vector fields, is it satisfied on a $C^1$-open and dense subset?**

Note that if a property holds on a $C^1$-open and $C^1$-dense subset of $\mathcal{X}^1(M)$, then for any $r > 1$, it holds on an open set of $\mathcal{X}^r(M)$ which is $C^1$-dense, but not a priori $C^r$-dense.

**Precise setting** Given a vector field $X \in \mathcal{X}^1(M)$, the flow generated by $X$ is denoted by $(ϕ_t^X)_{t \in \mathbb{R}}$, and sometimes by $(ϕ_t)$ if there is no confusion. A point $σ$ is a *singularity* of $X$ if $X(σ) = 0$. A point $p$ is *periodic* if it is not a singularity and there is $T > 0$ such that $ϕ_T(p) = p$. We denote by $\text{Sing}(X)$ the set of singularities and by $\text{Per}(X)$ the set of periodic orbits of $X$. The union $\text{Crit}(X) := \text{Sing}(X) \cup \text{Per}(X)$ is the set of *critical elements* of $X$.

We will mainly discuss the recurrence properties of the dynamics. An invariant compact set $Λ$ is *transitive* if it contains a point $x$ whose positive orbit is dense in $Λ$. More generally $Λ$ is *chain-transitive* if for any $ε > 0$ and $x, y \in Λ$, there exists $x_0 = x, x_1, \ldots, x_n = y$ in $Λ$ and $t_0, t_1, \ldots, t_{n-1} \geq 1$ such that $d(ϕ_{t_i}(x_i), ϕ_{t_{i+1}}(x_{i+1})) < ε$ for each $i = 0, \ldots, n-1$. A compact invariant set $Λ$ is said to be a *chain-recurrence class* if it is chain-transitive, and is not a proper subset of any other chain-transitive compact invariant set. The chain-recurrence classes are pairwise disjoint.

Among invariant sets, important ones are those satisfying an attracting property. An invariant compact set $Λ$ is an *attracting set* if there exists a neighborhood $U$ of $Λ$ such that $\cap_{t > 0} ϕ_t(U) = Λ$ and an *attractor* if it is a transitive attracting set. More generally $Λ$ is *Lyapunov stable* if for any neighborhood $V$ there exists a neighborhood $U$ such that $ϕ_t(U) \subset V$ for all $t > 0$.

We will also study the stability of these properties. For instance a set $Λ$ is *robustly transitive* if there exists neighborhoods $U$ of $Λ$ and $V$ of $X$ in $\mathcal{X}^1(M)$ such that for any $Y \in τ(U)$, the maximal invariant set $\cap_{t \in \mathbb{R}} ϕ_t^X(V)$ is transitive, and coincides with $Λ$ when $Y = X$.

As said before, singular hyperbolicity is a weak notion of hyperbolicity that has been introduced in order to characterize some robust dynamical properties. A compact invariant set $Λ$ is *singular hyperbolic* if for the flow $(ϕ_t)$ generated by either $X$ or $–X$, there are a continuous $Dϕ_t$-invariant splitting $T_Λ M = E^{ss} \oplus E^{cu}$ and $T > 0$ such that for any $x \in Λ$:

- $E^{ss}$ is contracted: $\|Dϕ_T|_{E^{ss}(x)}\| \leq 1/2$.
- $E^{ss}$ is dominated by $E^{cu}$: $\|Dϕ_T|_{E^{ss}(x)}\| \|Dϕ_{–T}|_{E^{cu}(ϕ_T(x))}\| \leq 1/2$.
- $E^{cu}$ is area-expanded: $|\det Dϕ_{–T}|_P \leq 1/2$ for any two-dimensional plane $P \subset E^{cu}(x)$.

Note that if $Λ$ is a singular hyperbolic set, and $Λ \cap \text{Sing}(X) = ∅$, then $Λ$ is a hyperbolic set.

Some robust properties or generic assumptions imply singular hyperbolicity. In dimension 3, robustly transitive sets are singular hyperbolic [26] and any generic vector field $X \in \mathcal{X}^1(M)$ far from homoclinic tangencies supports a global singular hyperbolic structure [13,14]. In higher dimensions, the transitive attractors of a generic vector field $X \in \mathcal{X}^1(M)$ satisfying the star property are singular hyperbolic [29].
Statement of the results  It is well known that for uniformly hyperbolic sets:

- chain-transitivity and local maximality (i.e. the set coincides with the maximal invariant set in one of its neighborhoods) imply robust transitivity;
- Lyapunov stability implies that the set is an attracting set.

We do not know whether these properties extend to general singular hyperbolic sets but this has been proved in [28] for $C^1$-generic vector fields: generically Lyapunov stable chain-recurrence classes which are singular hyperbolic are transitive attractors. We show that this holds robustly.

**Theorem A** There is an open and dense set $\mathcal{U} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{U}$, any singular hyperbolic Lyapunov stable chain-recurrence class $\Lambda$ of $X$ is a robustly transitive attractor.

Theorem A improves an older result by Morales–Pacifico [25] which gave sufficient conditions for the robust transitivity of 3-dimensional singular hyperbolic attractors with one singularity.

Let us mention that a more general notion of hyperbolicity, called *multi-singular hyperbolicity* has been recently introduced in order to characterize star vector fields, in [10] (see also [12]). In contrast to Theorem A above, a multi-singular chain-recurrence class of a $C^1$-generic vector field may be isolated and not robustly transitive [15].

Under the setting of Theorem A, we have a more accurate description of the singular hyperbolic attractors in Theorem A. Two hyperbolic periodic orbits $\gamma_1$ and $\gamma_2$ are *homoclinically related* if $W^s(\gamma_1)$ intersects $W^u(\gamma_2)$ transversely and $W^s(\gamma_2)$ intersects $W^u(\gamma_1)$ transversely. The *homoclinic class* $H(\gamma)$ of a hyperbolic periodic orbit $\gamma$ is the closure of the union of the periodic orbits that are homoclinically related to $\gamma$. This is a transitive invariant compact set. In dimension 3, any singular hyperbolic transitive attractor is a homoclinic class by [7] and [4, Theorem 6.8]; we show that this also holds in higher dimension for vector fields in a dense open set.

**Theorem B** There is an open and dense set $\mathcal{U} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{U}$, any singular hyperbolic Lyapunov stable chain-recurrence class $\Lambda$ of $X$ (not reduced to a singularity) is a homoclinic class (in particular the set of periodic points is dense in $\Lambda$).

Moreover, any two periodic orbits contained in $\Lambda$ are hyperbolic and homoclinically related.

In dimension 3, we know more properties of chain-recurrence classes with singularities for generic systems. This gives the following consequence (we do not need to assume that the class is Lyapunov stable):

**Corollary C** When $\dim M = 3$, there is an open and dense set $\mathcal{U} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{U}$, any singular hyperbolic chain-recurrence class is robustly transitive. Unless the class is an isolated singularity, it is a homoclinic class.

Theorem A in fact solves Conjecture 7.5 in [8]: a $C^1$-generic three-dimensional flow has either infinitely many sinks or finitely many robustly transitive attractors (hyperbolic or singular hyperbolic ones) whose basins form a full Lebesgue measure set of $M$. With [23], only the robustness was unknown before this paper.
Further discussions It is natural to expect that the previous results hold for an arbitrary singular hyperbolic chain-recurrence class, also in higher dimension:

**Question 1** Does there exist (when \( \dim(M) \geq 4 \)) a dense and open subset \( U \subset X^1(M) \) such that for any \( X \in U \), any singular hyperbolic chain-recurrence class is robustly transitive? Is a homoclinic class?

This would imply in particular that if \( X \in U \) is singular hyperbolic (i.e. each of its chain-recurrence classes is singular hyperbolic), then it admits only finitely many chain-recurrence classes.

One can also study stronger forms of recurrence. It is known that for a \( C^r \)-generic vector field, each homoclinic class is topologically mixing, see [1] and one may wonder if this holds robustly.

**Question 2** Does there exist an open and dense subset \( U \subset X^1(M) \) such that for any \( X \in U \), any singular hyperbolic transitive attractor is robustly topologically mixing.

The answer is positive in the case of non-singular transitive attractors [16]. Also [3] proves that \( X^2(M) \) contains a \( C^2 \)-dense and \( C^2 \)-open subset of vector fields such that any singular hyperbolic robustly transitive attractor with \( \dim(E^{cu}) = 2 \) is robustly topologically mixing.

2 Preliminaries

2.1 Chain-recurrence classes and homoclinic classes

Let us consider a \( C^1 \) vector field \( X \) and a compact subset \( K \) of \( M \). We will use the following notations. The orbit of a point \( x \) under the flow of \( X \) is denoted by \( \text{Orb}(x) \). Two injectively immersed submanifolds \( W_1 \) and \( W_2 \) are said to intersect transversely at a point \( x \in W_1 \cap W_2 \) if \( T_x W_1 + T_x W_2 = T_x M \). The set of transverse intersections is denoted by \( W_1 \prec W_2 \).

**Hyperbolicity** A compact invariant set \( \Lambda \) of a vector field \( X \) is **hyperbolic** if there is a continuous invariant splitting

\[
T_\Lambda M = E^{ss} \oplus \mathbb{R} X \oplus E^{uu},
\]

and constants \( C, \lambda > 0 \) such that for any \( x \in \Lambda \) and any \( t \geq 0 \),

\[
\| D\varphi_t|_{E^{ss}(x)} \| \leq Ce^{-\lambda t} \quad \text{and} \quad \| D\varphi_{-t}|_{E^{uu}(x)} \| \leq Ce^{-\lambda t}.
\]

Note that \( \dim \mathbb{R} X(x) = 0 \) when \( x \) is a singularity.

To each point \( x \) in a hyperbolic set \( \Lambda \) is associated a strong stable manifold \( W^{ss}(x) \) and a strong unstable manifold \( W^{uu}(x) \) that are tangent to \( E^{ss} \) and \( E^{uu} \), respectively. For a hyperbolic critical element \( \gamma \) we introduce the stable and unstable sets \( W^s(\gamma) \) and \( W^u(\gamma) \). Note that:

- for a hyperbolic singularity \( \sigma \), we have \( W^s(\sigma) = W^{ss}(\sigma) \) and \( W^u(\sigma) = W^{uu}(\sigma) \);
- for a hyperbolic periodic orbit \( \gamma \), the sets \( W^s(\gamma) \) and \( W^u(\gamma) \) are injective immersed submanifolds and

\[
W^s(\gamma) = \bigcup_{x \in \gamma} W^{ss}(x), \quad W^u(\gamma) = \bigcup_{x \in \gamma} W^{uu}(x).
\]
Homoclinic classes The homoclinic class of a hyperbolic periodic orbit \( \gamma \) is
\[
H(\gamma) = W^s(\gamma) \cap W^u(\gamma).
\]

The set \( H(\gamma) \) can also be defined in another way. Two hyperbolic periodic orbits \( \gamma_1 \) and \( \gamma_2 \) are said to be homoclinically related if \( W^s(\gamma_1) \cap W^u(\gamma_2) \) and \( W^s(\gamma_2) \cap W^u(\gamma_1) \) are non-empty. The Birkhoff–Smale theorem implies that this is an equivalence relation and that \( H(\gamma) \) is the union of the hyperbolic periodic orbits that are homoclinically related with \( \gamma \). Moreover \( H(\gamma) \) is a transitive set. See [27] and [4, Section 2.5.5].

Chain-recurrence classes If \( \sigma \) is a critical element of \( X \), we denote by \( C(\sigma) \) the chain-recurrent class of \( X \) that contains \( \sigma \).

Singular hyperbolicity As with uniform hyperbolicity, singular hyperbolicity is robust: there exist a \( C^1 \)-neighborhood \( \mathcal{U} \) of \( X \) and a neighborhood \( U \) of \( \Lambda \) such that for any \( Y \in \mathcal{U} \), any \( \varphi^1 \)-invariant compact set \( \Lambda' \) contained in \( U \) is singular hyperbolic.

In this case, any point \( x \in \Lambda \) admits a strong stable manifold \( W^{ss}(x) \) tangent to \( E^{ss}(x) \). Moreover one can choose a neighborhood \( W^{ss}_{loc}(x) \) of \( x \) in \( W^{ss}(x) \) which is an embedded disc which varies continuously for the \( C^1 \)-topology with respect to \( x \) and to the vector field.

2.2 Genericity

We recall several generic results. Some notations and definitions are given first.

- For a hyperbolic critical element \( \gamma \) of \( X \), its hyperbolic continuation will be denoted by \( \gamma_Y \) for \( Y \) that is \( C^1 \)-close to \( X \).
- A compact invariant set \( \Lambda \) is locally maximal if there is a neighborhood \( U \) of \( \Lambda \) such that \( \Lambda = \cap_{t \in \mathbb{R}} \varphi_t(U) \).

Proposition 2.1 There is a dense \( G_\delta \) set \( \mathcal{G} \) in \( \mathcal{X}^1(M) \) such that for any \( X \in \mathcal{G} \), we have:

1. \( X \) is Kupka–Smale: each critical element of \( X \) is hyperbolic; the intersections of the stable manifold of one critical element with the unstable manifold of another critical element are all transverse.
2. For any \( \sigma \in \text{Crit}(X) \), the map \( Y \mapsto C(\sigma_Y) \) is continuous at \( X \) in the Hausdorff topology.
3. If a chain-recurrence class contains a hyperbolic periodic orbit \( \gamma \), then it coincides with the homoclinic class \( H(\gamma) \) of \( \gamma \).
4. If a chain-recurrence class of a critical element \( \sigma \) is locally maximal, then it is robustly locally maximal: there are a neighborhood \( \mathcal{U} \) of \( X \) in \( \mathcal{X}^1(M) \) and a neighborhood \( U \) of \( C(\sigma) \) such that \( C(\sigma_Y) \) is the maximal invariant set in \( U \) for any \( Y \in \mathcal{U} \).
5. For any critical elements \( \gamma_1, \gamma_2 \) in a same chain-recurrence class \( C(\gamma_1) \) and satisfying \( \dim W^u(\gamma_2) \geq \dim W^u(\gamma_1) \), any neighborhood \( U_x \) of a point \( x \in C(\gamma_1) \cap W^s_{loc}(\gamma_1) \) contains a point \( y \in W^u(\gamma_2) \cap W^s_{loc}(\gamma_1) \).
6. Any hyperbolic periodic orbits in the same chain-recurrence class and with the same stable dimension are homoclinically related.
7. For \( \gamma \in \text{Crit}(X) \), if \( C(\gamma) \) contains the unstable manifold \( W^u(\gamma) \), then it is Lyapunov stable and \( W^u(\gamma) = C(\gamma) \). Moreover \( W^u(\gamma_Y) \subset C(\gamma_Y) \) still holds for any \( Y \) that is \( C^1 \)-close to \( X \).
8. If \( \sigma \in \text{Sing}(X) \) has unstable dimension equal to one, then either \( C(\sigma) = \{ \sigma \} \) or \( C(\sigma) \) is Lyapunov stable.

The properties in Proposition 2.1 are well known. We give some comments:
– Item 1 is the classical Kupka–Smale theorem [20,30].
– Item 2 follows from the upper semi-continuity of the chain-recurrence class $C(\sigma_Y)$ with respect to the vector field: the continuity holds at generic points.
– Item 3 and Item 4 have been proved for diffeomorphisms in [9, Remarque 1.10 and Corollaire 1.13]. The proofs for flows are similar.
– Item 5 is an application of the connecting lemma in [9]. For any point $x$ as in the statement, by using the connecting lemma, there is a point $y$ close to $x$ such that $y \in W^u(\gamma_2) \cap W^s_{loc}(\gamma_1)$ for $Y$ close to $X$. Then one can apply a Baire argument to conclude.
– Item 6 is a consequence of Item 5.
– Item 7 and 8 are applications of the connecting lemma for pseudo-orbits in [9], see also [17, Lemmas 3.13, 3.14 and 3.19].

We know the following theorem from [28, Corollary C] and [6, Theorem 1.1] (and previously [24, Theorem D] in the three-dimensional case).

**Theorem 2.2** There is a dense $G_\delta$ set $\mathcal{G} \subset \chi^1(M)$ such that for any vector field $X \in \mathcal{G}$, if $C(\sigma)$ is a singular hyperbolic Lyapunov stable chain-recurrence class (and not reduced to $\sigma$), then $C(\sigma)$ contains periodic orbits and is an attractor.

The following proposition gives some open and dense properties for chain-recurrence classes.

**Proposition 2.3** There is an open and dense set $\mathcal{U} \subset \chi^1(M)$ such that any $X \in \mathcal{U}$ has a neighborhood $\mathcal{U}_X$ with the following property. For each $\sigma \in \text{Sing}(X)$ and $Y \in \mathcal{U}_X$,

1. $W^u(\sigma_X) \subset C(\sigma_X) \Leftrightarrow W^u(\sigma_Y) \subset C(\sigma_Y)$;
2. $C(\sigma_X)$ is non-trivial (i.e. not reduced to $\{\sigma_X\}$) if and only if $C(\sigma_Y)$ is non-trivial.

**Proof** We take the dense $G_\delta$ set $\mathcal{G}$ provided by Proposition 2.1 and we consider the subset $\mathcal{G}_n \subset \mathcal{G}$ of vector fields with exactly $n$ singularities. Consider $X \in \mathcal{G}_n$ and denote the set of its singularities by $\{\sigma_1, \ldots, \sigma_n\}$.

Note that $\text{Closure}(W^u(\gamma_X))$ varies lower semi-continuously with respect to the vector field $X$ and $C(\gamma_X)$ varies upper semi-continuously with respect to the vector field $X$. So, for any (hyperbolic) critical element $\gamma$ of $X$, if $W^u(\gamma_X) \cap C(\gamma) \neq \emptyset$, then there is a neighborhood $\mathcal{U}_X$ such that $W^u(\gamma_Y) \cap C(\gamma) \neq \emptyset$. Similarly, if $C(\sigma_X) = \{\sigma_X\}$, there exists a neighborhood $\mathcal{U}_X$ such that $C(\sigma_Y)$ is contained in a small neighborhood of $C(\sigma_X)$. Since $\sigma_X$ is hyperbolic, $C(\sigma_Y) = \{\sigma_Y\}$.

Together with Items 2 and 7 of Proposition 2.1, for each $\sigma_i$, there is an open set $\mathcal{U}_{i,X}$ such that

1. either for any $Y \in \mathcal{U}_{i,X}$ we have $W^u(\sigma_i,Y) \cap C(\sigma_i,Y) \neq \emptyset$, or for any $Y \in \mathcal{U}_{i,X}$ we have $W^u(\sigma_i,Y) \subset C(\sigma_i,Y)$;
2. $C(\sigma_i)$ is non-trivial if and only if $C(\sigma_i,Y)$ is non-trivial for any $Y \in \mathcal{U}_{i,X}$.

By reducing $\mathcal{U}_{i,X}$ if necessary for each $i$, any singularity of $Y \in \bigcap_{i=1}^n \mathcal{U}_{i,X}$ is the continuation of a singularity of $X$. Now we take

$$\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n, \quad \text{where} \quad \mathcal{U}_n = \bigcup_{X \in \mathcal{G}_n} (\bigcap_{i=1}^n \mathcal{U}_{i,X}).$$

It is clear that $\mathcal{G} \subset \mathcal{U}$. Thus $\mathcal{U}$ is dense. Now we check that the open set $\mathcal{U}$ has the required properties. For any vector field $Y \in \mathcal{U}$, there is $n \in \mathbb{N}$ and a vector field $X \in \mathcal{G}_n$ such that $Y \in \bigcap_{i=1}^n \mathcal{U}_{i,X}$. Thus, any singularity $\sigma_Y$ of $Y$ is a continuation of a singularity $\sigma_{i,X}$ of $X$. By the choice of $\mathcal{U}_{i,X}$, we have

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1. \(W^u(\sigma_Y) \subset C(\sigma_Y)\) if and only if \(W^u(\sigma_Z) \subset C(\sigma_Z)\) for any \(Z \in \bigcap_{i=1}^n \mathcal{U}_i, x\).
2. \(C(\sigma_Y)\) is non-trivial if and only if \(C(\sigma_{i,Z})\) is non-trivial for any \(Z \in \bigcap_{i=1}^n \mathcal{U}_i, x\).

This implies the proposition.

2.3 Robust properties of singular hyperbolic attractors

**Proposition 2.4** Assume that \(C(\sigma)\) is a transitive singular hyperbolic attractor of a vector field \(X\) containing a hyperbolic singularity \(\sigma\) (and not reduced to \(\sigma\)). Then there is a neighborhood \(\mathcal{U}\) of \(X\) such that the continuation \(C(\sigma_Y)\) of any \(Y \in \mathcal{U}\) satisfies:

1. \(C(\sigma_Y)\) admits a singular hyperbolic splitting \(T_{C(\sigma_Y)}M = E^{ss}_Y \oplus E^{au}_Y\) for \(Y\) such that \(E^{ss}_Y\) is \(D\varphi_T\)-contracted and \(E^{cu}_Y\) is \(D\varphi_T\)-area-expanded for some \(T > 0\).
2. For each \(x \in C(\sigma_Y)\), we have that \(Y(x) \subset E^{cu}_Y(x)\).
3. The stable space \(E^s_Y(\rho)\) of any singularity \(\rho \in C(\sigma_Y)\) has a dominated splitting \(E^s_Y(\rho) = E^{ss}_Y(\rho) \oplus E^{cs}_Y(\rho)\) such that \(\dim E^{ss}_Y = 1\). Moreover \(W^{ss}_Y(\rho) \subset C(\sigma_Y) = \{\rho\}\).

**Proof** We first prove Item 1 for the vector field \(X\). Let us recall a result from [11, Lemma 3.4]: for a transitive set \(\Lambda\) with a dominated splitting \(T\Lambda M = E \oplus F\) we have

- either, \(X(x) \in E(x)\) for any point \(x \in \Lambda\),
- or, \(X(x) \in F(x)\) for any point \(x \in \Lambda\).

Now we consider a transitive singular hyperbolic attractor \(C(\sigma)\). Assume by contradiction that it has a singular hyperbolic splitting \(T_{C(\sigma)}M = E^s \oplus E^u\) such that \(E^u\) is expanded. Since it is an attractor, the strong unstable manifold of \(\sigma\) is contained in \(C(\sigma)\). For a point \(z \in W^{uu}(\sigma)\), we have \(X(z) \in E^{uu}(z)\). Thus, for any point \(x \in C(\sigma)\), we have \(X(x) \in E^{uu}(x)\). But \(\mathbb{R} \times X\) cannot be uniformly expanded everywhere and one gets a contradiction. Thus the singular hyperbolic splitting on \(C(\sigma)\) has the form \(T_{C(\sigma)}M = E^{ss} \oplus E^{cu}\) and Item 1 is proved for \(X\). Since the singular hyperbolic splitting is robust, and the chain-recurrence class varies upper semi-continuously, \(C(\sigma_Y)\) admits the same kind of splitting for any \(Y\) close to \(X\). This proves Item 1.

Now we prove Items 2 and 3 for the vector field \(X\). By using the result in [11] again, if for some regular point \(x\), we have \(X(x) \in E^{ss}(x)\), then this property holds for every point. But \(\mathbb{R} \times X\) cannot be uniformly contracted and one gets a contradiction again. Thus Item 2 for \(X\) is proved. Since for any point \(x \in C(\sigma)\), we have \(X(x) \in E^{cu}(x)\), the strong stable manifold \(W^{ss}(\sigma)\) cannot contain a regular orbit. Thus \(W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}\). On the other hand, the stable manifold \(W^s(\sigma)\) contains regular orbits of \(C(\sigma)\) since \(C(\sigma)\) is transitive. Thus \(\dim E^s(\sigma) > \dim E^{ss}(\sigma)\). If \(\dim E^s(\sigma) > \dim E^{ss}(\sigma) + 1\), then the dimension of the invariant space \(E^c(\sigma) := E^s(\sigma) \cap E^{cu}(\sigma)\) is at least 2. But this space is not area-expanded, since this would contradict the definition of singular hyperbolicity. Thus, we have \(\dim E^c(\sigma) = 1\). The proof of Item 3 for \(X\) is complete.

Now we give the proof of Item 3 for any \(Y\) close to \(X\). Since \(C(\sigma)\) is an attractor of \(X\), there is a neighborhood \(U\) of \(C(\sigma)\) such that \(U \cap \text{Sing}(X) \subset C(\sigma)\). Consequently, for \(Y\) close to \(X\), any singularity \(\rho_Y \in C(\sigma_Y)\) is a continuation of some singularity \(\rho_X\) in \(C(\sigma)\). Since Item 3 holds for \(\rho_X\), for \(Y\) close to \(X\) the singularity \(\rho_Y\) still has a dominated splitting \(E^s_Y(\rho) = E^{ss}_Y(\rho) \oplus E^{cs}_Y(\rho)\) with \(\dim E^{ss}_Y = 1\). It also holds for every singularity \(\rho_Y \in C(\sigma_Y)\).

Let us assume by contradiction that there is a sequence of vector fields \((X_n) \to X\) and a singularity \(\rho_X \in C(\sigma)\) such that

\[W^{ss}_{X_n}(\rho_{X_n}) \cap C(\sigma_{X_n}) \setminus \{\rho_{X_n}\} \neq \emptyset.\]
Thus, there is $\varepsilon_0$ that is independent of $n$ such that the local manifold $W^{ss}_{X_n,\varepsilon_0}(\rho X_n)$ of size $\varepsilon_0$ intersects $C(\sigma X_n)$ for each $n \in \mathbb{N}$. Since the homoclinic class $C(\sigma Y)$ is upper semi-continuous with respect to $Y$, we have that the boundary of $W^{ss}_{E_\varepsilon}(\rho)$ intersects $C(\sigma)$. This contradicts Item 3 for $X$. Thus the proof of Item 3 of this proposition is complete.

Let us assume by contradiction that Item 2 fails: this means that there is a sequence of vector fields $X_n \to X$ and a sequence of points $x_n \in C(\sigma X_n)$ such that $X_n(x_n)$ is not contained in $E^{cu}_{X_n}(\gamma)$. From the dominated splitting $E^s \oplus E^c$, by replacing $x_n$ by a backward iterate, one can assume furthermore that the angle between $\mathbb{R}.X_n(x_n)$ and $E^{ss}_{X_n}(x_n)$ is less than $1/n$. In particular, the angles between $\mathbb{R}.X_n(x_n)$ and $E^{cu}_{X_n}(x_n)$ are uniformly bounded from below.

**Claim** The points $x_n$ avoid a uniform neighborhood of the singularities $\rho X_n$.

**Proof of the Claim** We argue by contradiction. If this does not hold, we can extract a subsequence such that $(x_n)$ converges to a singularity $\rho X$. Since $X_n(x_n) \notin E^{cu}_{X_n}(x_n)$, the point $x_n$ does not belong to the local unstable manifold of $\rho X_n$. Taking $\delta > 0$ small, there exists $t_n > 0$ satisfying $d(\varphi_{-t_n}^X(x_n), \rho X_n) = \delta$. Choosing the smallest $t_n$, one gets $d(\varphi_{-t_n}^X(x_n), \rho X_n) \leq \delta$ for $t \in [0, t_n]$.

Up to extracting a new subsequence, there exists a limit $z = \lim_{n \to \infty} \varphi_{-t_n}^X(x_n)$. Since $\lim_{n \to \infty} d(x_n, \rho X_n) = 0$, one has $\lim_{n \to \infty} t_n = \infty$. By continuity of the flow, $d(\varphi_t(z), \rho X) \leq \delta$ for any $t \geq 0$. By hyperbolicity of $\rho X$, $z$ belongs to the local stable manifold of $\rho X$.

Since the splitting $E^s \oplus E^c$ is dominated and the angle between $\mathbb{R}.X_n(x_n)$ and $E^{cu}_{X_n}(x_n)$ is bounded away from zero, the angle between $\mathbb{R}.X_n(\varphi_{-t_n}^X(x_n))$ and $E^{ss}_{X_n}(\varphi_{-t_n}^X(x_n))$ tends to $0$ as $n \to +\infty$. By the continuity of the vector field, $X(z) \in E^{ss}(z)$. Inside the local stable manifold of $\rho X$, this implies that $z$ is contained in the strong stable manifold of $\rho X$.

As a limit of points in $C(\sigma X_n)$, the point $z$ belongs to the chain-recurrence class $C(\sigma X)$. We have thus obtained a point in $C(\sigma X) \cap W^{ss}(\rho X) \setminus \{\rho X\}$. This contradicts Item 3.

By the claim above, the points $x_n$ are far from the singularities. One can thus assume that $(x_n)$ converges to a regular point $x \in C(\sigma)$. By taking the limit, we have $X(x) \in E^{ss}(x)$. This contradicts Item 2 for $X$. Thus Item 2 holds for any $Y$ close to $X$.  

### 3 Density of the unstable manifold

This section is devoted to the following result which will be used to prove the density of the unstable manifold.

**Theorem 3.1** There is a dense $G_\delta$ set $\mathcal{J}$ in $\mathcal{X}^1(M)$ such that for any $X \in \mathcal{J}$, for any singular hyperbolic Lyapunov stable chain-recurrence class $C(\sigma)$ of $X$, and for any hyperbolic periodic orbit $\gamma$ in $C(\sigma)$, there exists a neighborhood $\mathcal{U}_X$ of $X$ with the following property.

For any $Y \in \mathcal{U}_X$ and for any $x \in C(\sigma Y)$, either $x$ belongs to the unstable manifold of a singularity or $W^{ss}_{loc,Y}(x) \cap W^u(\gamma) \cap \{x\}$ is transversely.

Let $\mathcal{J}$ be a dense $G_\delta$ subset in $\mathcal{X}^1(M)$ given by Proposition 2.1. Let $X \in \mathcal{J}$. We first prove some preliminary lemmas.

**Lemma 3.2** For any singularity $\rho \in C(\sigma)$, and for any $x \in (W^s(\rho) \cap C(\sigma)) \setminus \{\rho\}$, the submanifolds $W^{ss}_{loc,Y}(x)$ and $W^u(\gamma)$ have a transverse intersection point.
Proof Consider \( x \in (W^s(\rho) \cap C(\sigma)) \setminus \{\rho\} \). Without loss of generality, we may assume that \( x \) belongs to \( W_{\text{loc}}^u(\rho) \).

By Item 5 of Proposition 2.1, there exists a transverse intersection point \( y \) between \( W_{\text{loc}}^u(\rho) \) and \( W^u(\gamma) \) near \( x \). The flow preserves \( W^u(\gamma) \), \( W^s(\rho) \) and the strong stable foliation of \( W^s(\rho) \); moreover since the strong stable foliation is one-codimensional inside \( W^s(\rho) \) and since \( x, y \) are two points close in \( W^s(\rho) \setminus W^s(\rho) \), there exists \( t \in \mathbb{R} \) such that \( \varphi_t(W_{\text{loc}}^{ss}(y)) = W_{\text{loc}}^{ss}(x) \). In particular, \( \varphi_t(y) \) is a transverse intersection point between \( W_{\text{loc}}^{ss}(x) \) and \( W^u(\gamma) \). That point can be chosen arbitrarily close to \( x \) in \( W_{\text{loc}}^{ss}(x) \), hence belongs to \( W_{\text{loc}}^{ss}(x) \). \( \square \)

Since the unstable manifold \( W^u(\gamma_Y) \) and the leaves of the strong stable foliation \( W_{\text{loc,Y}}^{ss}(x) \) vary continuously in the \( C^1 \) topology with respect to \( x \) and \( Y \), a compactness argument gives:

**Corollary 3.3** For any singularity \( \rho \in C(\sigma) \), let \( K_{\rho} \) be a compact subset of \( W^s(\rho) \cap C(\sigma) \setminus \{\rho\} \). Then there exists a neighborhood \( V_{\rho} \) of \( K_{\rho} \) and a \( C^1 \)-neighborhood \( U_{\rho} \) of \( X \) such that for any \( Y \in U_{\rho} \) and any \( y \in V_{\rho} \cap C(\gamma_Y) \), the submanifolds \( W_{\text{loc,Y}}^{ss}(y) \) and \( W^u(\gamma_Y) \) have a transverse intersection point.

**Lemma 3.4** For any invariant compact set \( K_{\text{reg}} \subset C(\sigma) \setminus \text{Sing}(X) \), there exists a neighborhood \( V_{\text{reg}} \) of \( K_{\text{reg}} \) and a \( C^1 \)-neighborhood \( U_{\text{reg}} \) of \( X \) such that for any \( Y \in U_{\text{reg}} \) and any \( y \in V_{\text{reg}} \cap C(\gamma_Y) \), the submanifolds \( W_{\text{loc,Y}}^{ss}(y) \) and \( W^u(\gamma_Y) \) have a transverse intersection point.

**Proof** Since \( C(\sigma) \) is singular hyperbolic, the invariant compact set \( K_{\text{reg}} \subset C(\sigma) \setminus \text{Sing}(X) \) is hyperbolic. By the shadowing lemma, and a compactness argument, there exist finitely many periodic orbits \( \gamma_1, \ldots, \gamma_\ell \) in an arbitrarily small neighborhood of \( K_{\text{reg}} \) such that for any \( y \in K_{\text{reg}} \), the submanifolds \( W_{\text{loc,Y}}^{ss}(y) \) intersects transversally some \( W^u(\gamma_i) \). Moreover each \( \gamma_i \) is in the chain-recurrence class of a point of \( K_{\text{reg}} \), hence is contained in \( C(\sigma) \). By Item 6 of Proposition 2.1, \( \gamma_i \) is homoclinically related to \( \gamma \). The inclination lemma then implies that for any \( y \in K_{\text{reg}} \), the submanifolds \( W_{\text{loc,Y}}^{ss}(y) \) intersects transversally \( W^u(\gamma) \). As before this property is \( C^1 \)-robust. \( \square \)

**Proof of Theorem 3.1** Let \( \rho_1, \ldots, \rho_\ell \) be the singularities contained in \( C(\sigma) \). For each of them, one chooses a compact set \( K_i \subset W^s(\rho_i) \cap C(\sigma) \setminus \{\rho_i\} \) which meets each orbit of \( W^s(\rho_i) \cap C(\sigma) \setminus \{\rho_i\} \). Corollary 3.3 associates open sets \( V_i \) and \( U_i \). There exists a neighborhood \( O_i \) of \( \rho_i \) such that any point \( z \in O_i \setminus W_{\text{loc}}^u(\rho_i) \) has a backward iterate in a compact subset of \( V_i \).

Let \( K_{\text{reg}} \) be the maximal invariant set of \( C(\sigma) \setminus \bigcup_i (V_i \cup O_i) \). Lemma 3.4 associates open sets \( U_{\text{reg}} \) and \( V_{\text{reg}} \). By construction, any point \( z \in C(\sigma) \) either belongs to some \( W^u(\rho_i) \), has a backward iterate in a compact subset of some \( V_i \), or belongs to \( K_{\text{reg}} \). Since \( C(\sigma_Y) \) varies upper semi-continuously with \( Y \), this property is still satisfied for \( Y \) in a \( C^1 \)-neighborhood \( U_{\text{return}} \) of \( X \).

We set \( U_X = U_1 \cap \cdots \cap U_\ell \cap U_{\text{reg}} \cap U_{\text{return}} \). For any \( Y \in U_X \) and any \( x \in C(\sigma) \) which does not belong to the unstable manifold of a singularity \( \rho_Y \), there exists a backward iterate \( \varphi_{-t}^Y(x) \) which belongs either to some \( V_i \) or to \( V_{\text{reg}} \). In both cases, \( W_{\text{loc,Y}}^{ss}(\varphi_{-t}^Y(x)) \) intersects \( W^u(\gamma_Y) \) transversely. This concludes the proof. \( \square \)

4 Density of the stable manifold

This section is devoted to the proof of the following result which will be used to prove the density of the stable manifold.
Theorem 4.1 There is a dense $G_δ$ set $\mathcal{S}$ in $X^1(M)$ such that for any $X \in \mathcal{S}$, for any singular hyperbolic transitive attractor $C(\sigma)$ of $X$, and for any hyperbolic periodic orbit $γ$ in $C(\sigma)$, there exist a neighborhood $U$ of $C(\sigma)$ and a neighborhood $U_X$ of $X$ with the following property.

For any $Y \in U_X$, the stable manifold $W^s_γ(γ_Y)$ is dense in $U$. Moreover, for any periodic orbit $γ' \subset C(\sigma_Y)$, the set of transverse intersections $W^s_γ(γ') \cap W^s_γ(γ_Y)$ is dense in $W^s_γ(γ')$.

The set $\mathcal{S}$ is the dense $G_δ$ subset of vector fields satisfying Items 1 and 3 of Proposition 2.1. In the whole section, we fix $X \in \mathcal{S}$ and $C(\sigma)$, $γ$ as in the statement of Theorem 4.1.

4.1 Center-unstable cone fields

We consider the singular hyperbolic splitting $T_{C(\sigma)}M = E^{ss} \oplus E^{cu}$ on $C(\sigma)$ (see Item 1 of Proposition 2.4). These bundles may be extended as a continuous (but maybe not invariant) splitting $T_{U_0}M = \tilde{E}^{ss} \oplus \tilde{E}^{cu}$ on a neighborhood $U_0$ of $C(\sigma)$. For $x \in U_0$ and $α > 0$ we define the center-unstable cone

$$c_α(x) = \{v \in T_xM : v = v^s + v^c, \ v^s \in \tilde{E}^{ss}(x), \ v^c \in \tilde{E}^{cu}(x), \ \|v^c\| ≤ α\|v^s\|\}.$$

Since the splitting is uniformly dominated, we have the following lemma.

Lemma 4.2 For any $α > β > 0$, there are $T > 0$, a $C^1$-neighborhood $U_1$ of $X$ and a neighborhood $U_1 \subset U_0$ of $C(\sigma)$ such that for any $Y \in U_1$ and for any orbit segment $ϕ^Y_{[0,t]}(x) \subset U_1$ with $t ≥ T$,

$$Dϕ^Y_t(c_α(x)) \subset c_β(ϕ^Y_t(x)).$$

Proof The proof for orbit segments of the vector field $X$ inside the class $C(\sigma)$ follows from the dominated splitting, is standard and is omitted. The conclusion extends to neighborhoods of $X$ and $C(\sigma)$ by continuity of the cone fields and of $Dϕ^Y_t(x)$, $t \in [T, 2T]$, with respect to $(x, t, Y)$. □

In the following we fix $α > 0$ small and set $C = c_α$. Note that (by increasing $T$, reducing $U_1$, $U_1$ and using singular hyperbolicity) the following property holds: for any $Y \in U_1$, any orbit segment $ϕ^Y_{[0,t]}(x) \subset U_1$ with $t ≥ T$ and any 2-dimensional subspace $P \subset C_α(x)$,

$$|\det(Dϕ^Y_t)|_P | ≥ 2. \quad (1)$$

The inner radius $r$ of a submanifold $Γ'$ is the supremum of $R > 0$ such that the exponential map $B(0, R) \subset T_xΓ → Γ$ with respect to the metric induced on $Γ$ by the Riemannian metric of $M$ is well defined and injective for some $x \in Γ$, where $B(0, R) \subset T_xΓ$ is the ball $\{v \in T_xΓ : \|v\| < R\}$. Denote by $B(x, R)$ the image of $B(0, R) \subset T_xΓ$ by the exponential map from $T_xΓ$ to $Γ$. Note that $Γ$ always contains a submanifold $Γ' \subset Γ$ with the same inner radius $r$ as $Γ$ and with diameter smaller than or equal to $2r$: indeed we consider a sequence of balls $B(x_k, r_k) \subset Γ$ with $r_k → r$ and consider a limit point $x$ for $(x_k)$; the submanifold $Γ'$ is the ball $B(x, r) \subset Γ$.

Theorem 4.1 is a consequence of the next proposition.

Proposition 4.3 Under the setting of Sect. 4.1, there exists $ε > 0$, a $C^1$-neighborhood $U_1 \subset U_1$ of $X$ and a neighborhood $U_2$ of $C(\sigma)$ with the following property.

For any $Y \in U_2$ and any submanifold $Γ \subset U_2$ of dimension $\dim(E^{cu})$, tangent to the center-unstable cone field $C$ and satisfying $Y(x) \in T_xΓ$ for each $x \in Γ$ (i.e., $Γ$ is a family of
pieces of orbits of the flow), there is \( t > 0 \) such that \( \varphi_{t,0}^y(\Gamma) := \cup_{s \in [0,t]} \varphi_s^y(\Gamma) \) contains a submanifold tangent to \( \mathcal{C} \), with dimension \( \dim(E^{cu}) \) and inner radius larger than \( \varepsilon \).

The proof of this proposition is postponed to the end of the Sect. 4.

**Proof of Theorem 4.1** Since \( C(\sigma) \) is a transitive attractor, it is the chain-recurrence class of \( \gamma \). By Item 3 of Proposition 2.1, \( C(\sigma) \) coincides with the homoclinic class \( H(\gamma) \) of \( \gamma \). Hence, the stable manifold of \( \gamma \) is dense in \( C(\sigma) \), and intersects transversally any submanifold of dimension \( \dim(E^{cu}) \) tangent to \( \mathcal{C} \) of inner radius \( \varepsilon \) and which meets a small neighborhood \( U_3 \subset U_2 \) of \( C(\sigma) \). By continuation of the stable manifold, this property is still satisfied for vector fields \( Y \) in a small \( C^1 \)-neighborhood \( \mathcal{U}_X \subset \mathcal{U}_2 \) of \( 
abla \). Since \( C(\sigma) \) is an attractor, one may reduce \( \mathcal{U}_X \), choose a smaller neighborhood \( U \subset U_3 \) of \( C(\sigma \chi) \) and assume that \( \varphi(t) U \subset U_3 \) for any \( t > 0 \) and \( Y \in \mathcal{U}_X \). Let \( O \) be a small isolating neighborhood of \( \text{Sing}(X) \). By Item 2 of Proposition 2.4, \( X \) is tangent to \( \mathcal{C} \) on \( C(\sigma \chi) \setminus O \). Up to reducing the neighborhoods \( U \) and \( \mathcal{U}_2 \), one can thus require that the vector fields \( Y \in \mathcal{U}_2 \) are tangent to \( \mathcal{C} \) on \( U \setminus O \).

In order to check that \( W^s_Y(\gamma_Y) \) is dense in \( U \) for any \( Y \in \mathcal{U}_X \), we choose an arbitrary non-empty open subset \( V \subset U \setminus O \) and a submanifold \( \Gamma \subset V \) as in the statement of Proposition 4.3: such a submanifold can be built by choosing a small disc \( \Sigma \subset V \) of dimension \( \dim(E^{cu}) - 1 \) tangent to \( \mathcal{C} \) and transverse to \( \nabla \); then we set \( \Gamma = \cup_{|t| < \delta} \varphi(t)(\Sigma) \) for \( \delta \) close to 0; since \( Y \) is tangent to \( \mathcal{C} \) on \( U \setminus O \), the submanifold \( \Gamma \) is tangent to \( \mathcal{C} \) as required. From the choice of \( U \) and Proposition 4.3, there exists \( t > 0 \) such that \( \varphi(t)(\Gamma) \) has inner radius larger than \( \varepsilon \), intersects \( U_3 \), is still tangent to \( \mathcal{C} \), hence meets \( W^s_Y(\gamma_Y) \) transversally. This shows that \( W^s_Y(\gamma_Y) \) intersects \( V \) as required. Let us now consider an open set \( V \subset O \). Since \( O \) is isolating for \( Y \), there exists a forward iterate \( \varphi(t)(V) \) which meets \( U \setminus O \). In particular \( W^s_Y(\gamma_Y) \) meets \( \varphi(t)(V) \), hence \( V \). We have proved that \( W^s_Y(\gamma_Y) \) is dense in \( U \).

For the last part of the Theorem, one considers a small open subset of \( W^u_Y(\gamma_Y') \). It is a submanifold \( \Gamma \) as considered above, hence it intersects transversally \( W^s_Y(\gamma_Y) \). \( \square \)

### 4.2 Cross sections, holonomies

Let \( Y \) be a vector field whose singularities \( \rho \) are hyperbolic. One associates to each of them their local stable and unstable manifold \( W^s_{Y,\text{loc}}(\rho), W^u_{Y,\text{loc}}(\rho) \).

**Definition 4.4** A cross section of \( Y \) is a one-codimensional submanifold \( D \subset M \setminus \text{Sing}(Y) \) such that there exists \( \alpha > 0 \) and a compact subset \( \Delta \subset D \) satisfying:

- \( \overline{D} \cap \text{Sing}(X) = \emptyset \) and for each \( x \in D \), the angle between \( Y(x) \) and \( T_x D \) is larger than \( \alpha \),
- the interior of \( \Delta \) in \( D \) intersects any forward orbit in \( M \setminus \cup \rho W^s_{Y,\text{loc}}(\rho) \) and any backward orbit in \( M \setminus \cup \rho W^u_{Y,\text{loc}}(\rho) \),
- For each singularity \( \rho \), the submanifold \( D \) intersects \( W^s_{Y,\text{loc}}(\rho) \) along a fundamental domain contained in the interior of \( \Delta \); the intersection is a one-codimensional sphere inside \( W^s_{Y,\text{loc}}(\rho) \) which meets each orbit in \( W^s_{Y,\text{loc}}(\rho) \) exactly once; similarly \( D \) intersects \( W^u_{Y,\text{loc}}(\rho) \) along a fundamental domain contained in the interior of \( \Delta \).

Such a compact set \( \Delta \subset D \) is called a core of the cross section \( D \).

A submanifold \( \Sigma \) is said to be a Poincaré section of the vector field \( Y \) if it does not contain any singularity and \( T_x \Sigma \oplus Y(x) = T_x M \) for any \( x \in \Sigma \).
Lemma 4.5 There exists a cross section $D$ for $Y$.

**Proof** For any singularity $\rho$, one considers the fundamental domains $\Delta^s(\rho)$, $\Delta^u(\rho)$ in $W^s_{Y,loc}(\rho)$ and $W^u_{Y,loc}(\rho)$ respectively; these are one-codimensional submanifolds in the local stable and unstable manifolds. One chooses two small Poincaré sections $\Sigma^s(\rho)$ and $\Sigma^u(\rho)$ containing $\Delta^s(\rho)$ and $\Delta^u(\rho)$ in their interiors, respectively.

For each singularity $\rho$, there exists a small neighborhood $O_\rho$, such that the forward (resp. backward) orbit of any point $x \in O_\rho$ (which does not belong to the local stable or unstable manifold of $\rho$) intersects $\Sigma^u(\rho)$ (resp. $\Sigma^s(\rho)$). Then we consider the maximal invariant set $\Lambda_0$ in $M \setminus \bigcup O_\rho$. For each point $x \in \Lambda_0$, let $\Sigma^+_x$, $\Sigma^-_x$ be small Poincaré sections containing $\varphi_{-1}(x)$ and $\varphi_1(x)$; for any point close to $x$, the forward (resp. backward) orbit intersects $\Sigma^+_x$ (resp. $\Sigma^-_x$). By the compactness of $\Lambda_0$ and the fact $\Lambda_0$ contains no singularity, one can find finitely many Poincaré sections $\Sigma_1$, $\Sigma_2$, $\ldots$, $\Sigma_k$ and a neighborhood $U_0$ of $\Lambda_0$ such that the forward and backward orbits of any point $x \in U_0$ intersect $\bigcup_{1 \leq i \leq k} \Sigma_i$. We re-index the sections $\{\Sigma^s(\rho), \Sigma^u(\rho)\}_{\rho \in \text{Sing}(X)}$ by $\Sigma_{k+1}, \ldots, \Sigma_{\ell}$. The forward (resp. backward) orbit of any point which is not contained in the local stable and unstable manifold of singularities intersect the interiors of one of these sections.

The union of the sections $\Sigma_i$ may not be a submanifold and has to be modified. Note that these sections can be chosen so that their closures are submanifolds with boundary. We first perturb them so that they intersect transversally. We then explain how to make them disjoint. Each intersection $\Sigma_i$, $\Sigma_j$ for $1 \leq i < j \leq \ell$ is a submanifold $N$. We choose $s \neq 0$ small such that $\varphi_s(N)$ does not intersect $\Sigma_i$ for any $1 \leq i \leq k$ and find a small one-codimensional transverse section $\Sigma_{i,j}$ containing $\varphi_s(N)$ in its interior such that $\Sigma_{i,j}$ does not intersect any other sections. Let $T_{i,j}$ be a compact neighborhood of $N$ inside $\Sigma_i \cap \Sigma_j$. Now we replace $\Sigma_i$ and $\Sigma_j$ by the three new sections: $\Sigma_{i,j}, \Sigma_i \setminus \bigcup_{|t|<|s|} \varphi_t(T_{i,j})$ and $\Sigma_j \setminus \bigcup_{|t|<|s|} \varphi_t(T_{i,j})$ and obtain a new set of Poincaré sections. After a finite number of steps, we obtain disjoint Poincaré sections $\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_L$. At last we choose some Poincaré sections $\tilde{\Sigma}_1, \tilde{\Sigma}_2, \ldots, \tilde{\Sigma}_L$ such that $\tilde{\Sigma}_i$ contains $\Sigma'_i$ in its interior and $\tilde{\Sigma}_i$ is contained in a small neighborhood of $\Sigma'_i$, for each $1 \leq i \leq L$. Thus, $\tilde{\Sigma}_1, \tilde{\Sigma}_2, \ldots, \tilde{\Sigma}_L$ are still pairwise disjoint. Now one takes $\Delta = \bigcup_{1 \leq i \leq L} \Sigma'_i$ and $D = \bigcup_{1 \leq i \leq L} \tilde{\Sigma}_i$. 

Fig. 1 Construction of the cross sections close to a singularity
By the construction of $D$, it is clear that the first item and the third item of Definition 4.4 are satisfied. It remains to check the second item of Definition 4.4. For any point $x \in M \backslash \bigcup_{\rho} W^s_{Y,loc}(\rho)$, one distinguishes if $\omega(x)$ contains some singularity or not.

When $\omega(x)$ contains some singularity $\rho_0$ but $\omega(x) \neq \{\rho_0\}$, the forward orbit of $x$ has arbitrarily large iterates close to $\Delta^s(\rho)$ and outside $W^s(\rho)$. Since $\Delta^s(\rho)$ is contained in the interior of the Poincaré section $\Sigma^s(\rho)$, the forward orbit of $x$ intersects the interior of $\Sigma^s(\rho)$, and hence of $\Delta$. When $\omega(x) = \{\rho_0\}$, since $x \in M \backslash \bigcup_{\rho} W^s_{Y,loc}(\rho)$, the forward orbit of $x$ intersects the fundamental domain $\Delta^s(\rho_0)$, hence the interior of $\Sigma^s(\rho)$ and of $\Delta$.

When $\omega(x)$ contains no singularity, $\omega(x) \subset \Lambda_0$ and the forward orbit of $x$ intersects the interior of some $\Sigma_i$. Hence by the construction, the forward orbit of $x$ intersects the interior of $\Delta$.

The property for backward orbits is proved analogously. \(\square\)

We now fix the cross section $D$ for $Y$. Note that it still satisfies the definition of cross section (with the same core $\Delta$) for vector fields $Z$ that are $C^1$-close to $Y$. We also consider an open set $U \subset M$.

**Definition 4.6** A holonomy for $(Y, D, U)$ is a $C^1$-diffeomorphism $\pi: V \rightarrow \tilde{V}$ between some open sets $V, \tilde{V} \subset D \cap U$ such that there exists a continuous function $t: V \rightarrow (0, +\infty)$ satisfying for each $x \in V$:

- $\pi(x) = \varphi_{t(x)}(x) \in \tilde{V}$,
- $\varphi_s(x) \in U$ for any $s \in [0, t]$.

The time $t$ is called the transition time of $x$ for $\pi$. (We do not require $t$ to be the first return time to $\tilde{V}$.)

**Construction and continuation of holonomies** One can construct a holonomy when two open subsets $V_0, \tilde{V}_0 \subset D$, an open set $W \subset U$ and an open interval $I$ in $(0, +\infty)$ satisfy the following property:

(H) For any $x \in V_0$, for any piece of orbit $\{\varphi^Y_s(x), s \in [0, t]\}$ in $W$ with $t \in I$, at most one point $\varphi^Y_s(x)$ with $s \in [0, t] \cap I$ belongs to $\tilde{V}_0$.

Precisely, one considers all the connected pieces of orbits contained in $W$ which meet both $V_0$ and $\tilde{V}_0$ at points $x \in V_0$ and $\varphi^Y_{t(x)}(x) \in \tilde{V}_0$ for some $t \in I$. The domain $V$ (resp. the image $\tilde{V}$) of the holonomy is the set of such points $x$ (resp. $\varphi^Y_{t(x)}(x)$) and we set $\pi(x) = \varphi^Y_{t(x)}(x)$. By property (H), the map $\pi$ is well defined. Since the flow is transverse to $D$ and since $V_0, \tilde{V}_0, I$ are open, the implicit function theorem implies that the sets $V$ and $\tilde{V}$ are open in $D$. If the property (H) still holds for vector fields $Z$ that are $C^1$-close to $Y$, this construction defines also holonomies $\pi^Z: V^Z \rightarrow \tilde{V}^Z$ for $Z$, called continuations of $\pi$.

The next proposition shows that the dynamics can be “covered” by a finite collection of holonomies admitting continuations. Since we may want to localize the dynamics, one considers an attracting invariant compact set $\Lambda$ with an attracting neighborhood $U$: there exists $t_0 > 0$ such that the orbit $(\varphi^Y_s(x))_{s > t_0}$ of any point $x \in U$ is contained in $U$ and accumulates on a subset of $\Lambda$.

**Proposition 4.7** Let us consider for $Y$ a cross section $D$ with a core $\Delta$, an attracting set $\Lambda$ with an attracting neighborhood $U$ and $T' > 0$. Then, there exist a finite collection of holonomies $\pi^Y_i: V^Y_i \rightarrow \tilde{V}^Y_i$, $i = 1, \ldots, \ell$ for $(Y, D, U)$, a $C^1$-neighborhood $\mathcal{U}$ of $Y$, a neighborhood $\mathcal{O}$ of $\Lambda \cap \Delta$ in $D$, and $\varepsilon_0 > 0$ such that for each $Z \in \mathcal{U},$
(i) the holonomies admit continuations \( \pi_i^Z : V_i^Y \to \tilde{V}_i^Z \),
(ii) \( O \setminus \bigcup_{\rho} W^s_{\text{loc}}(\rho Z) \subset \bigcup_{i} V_i^Z \) and \( \bigcup_{i} \tilde{V}_i^Z \subset \Delta \),
(iii) the transition time of each holonomy \( \pi_i^Z \) is bounded from below by \( T' \),
(iv) if a set \( A \subset D \setminus \bigcup_{\rho} W^s_{\text{loc}}(\rho Z) \) is contained in the \( \varepsilon_0 \)-neighborhood of \( \Lambda \cap \Delta \) and has diameter smaller than \( \varepsilon_0 \), then \( A \) is contained in a \( V_i^Z \).

**Proof** The holonomies are built in two different ways depending if the points are close to \( W^s_{\text{loc}}(\text{Sing}(Y)) \) or not.

**Claim** For any \( \rho \in \Lambda \cap \text{Sing}(Y) \), there exist a neighborhood \( O_{\rho} \) of \( W^s_{\text{loc}}(\rho) \cap \Lambda \cap \Delta \) in \( D \) and a holonomy \( \pi \) admitting continuations for any \( Z \) close to \( Y \), such that the domain \( V^Z \) of \( \pi^Z \) contains \( O_{\rho} \setminus W^s_{\text{loc}}(\rho Z) \), the image \( \tilde{V}^Z \) is contained in \( \Delta \) and the transition time is bounded from below by \( T' \).

**Proof** By definition of the cross section, the sets \( \Delta^s = W^s_{\text{loc}}(\rho) \cap \Delta \) and \( \Delta^u = W^u_{\text{loc}}(\rho) \cap \Delta \) are compact fundamental domains in the local stable and unstable manifolds. Let \( V_0 \) and \( \tilde{V}_0 \) be neighborhoods of \( \Lambda \cap \Delta^s(\rho) \) and \( \Lambda \cap \Delta^u(\rho) \) in \( D \) and let \( W \subset U \) be a small neighborhood of \( \Lambda \cap (W^s_{\text{loc}}(\rho) \cup W^u_{\text{loc}}(\rho)) \). A piece of orbit in \( W \) intersects \( \tilde{V}_0 \) at most once, hence the property (H) is satisfied for the time interval \( I = (0, +\infty) \) and any \( Z \) close to \( Y \). As a consequence, this defines a holonomy \( \pi : V^Z \to \tilde{V}^Z \) which admits a continuation for vector fields \( Z \) that are \( C^1 \)-close to \( Y \).

For any \( Z \) close to \( Y \), the intersection \( \Delta \cap W^s_{\text{loc}}(\rho Z) \) is contained in a small neighborhood \( O_{\rho} \) of \( \Delta^s \) in \( D \) and \( V^Z \) contains \( O_{\rho} \setminus W^s_{\text{loc}}(\rho Z) \). Moreover by definition of the cross sections, \( W^u_{\text{loc}}(\rho) \cap \Lambda \cap \Delta \) is contained in the interior of \( \Delta \) in \( D \), hence \( \tilde{V}^Z \subset \Delta \). We may have chosen \( W \) small enough so that the transition times of the holonomies are larger than \( T' \). \( \square \)

**Claim** For each \( x \in (\Lambda \cap \Delta) \setminus \cup_{\rho} W^s_{\text{loc}}(\rho) \), there exist a neighborhood \( O_x \) of \( x \) in \( D \) and a holonomy \( \pi \) admitting continuations for any \( Z \) close to \( Y \), such that the domain \( V^Z \) of \( \pi^Z \) contains \( O_x \), the image \( \tilde{V}^Z \) is contained in \( \Delta \) and the transition time is bounded from below by \( T' \).

**Proof** By the second item of the Definition 4.4 of the cross section \( D \) with core \( \Delta \), there exists \( t > T' \) such that \( \phi_t(x) \) belongs to the interior of \( \Delta \). We choose small open neighborhoods \( V_0 \subset D \) and \( \tilde{V}_0 \subset \Delta \) in \( D \), a small neighborhood \( W \subset U \) of the piece of orbit \( \{ \phi_s(x), s \in [0, t]\} \) \( \subset \Lambda \) and a time interval \( (t - \delta, t + \delta) \) for \( \delta > 0 \) small. This defines a holonomy, admitting a continuation for \( Z \) close to \( Y \) with transition time bounded from below by \( T' \). \( \square \)

By the claims above, for any point \( x \in \Delta \), one can find a small neighborhood \( O_x \) and a small neighborhood \( \mathcal{U}_x \) of \( Y \) such that a holonomy map and its continuation can be defined on \( O_x \) for \( Z \subset \mathcal{U}_x \) with transition time larger than \( T' \). By compactness of \( \Lambda \cap \Delta \), we can select a finite numbers of neighborhoods \( \{O_i\} \) and a finite collection of neighborhoods \( \mathcal{U}_i \) of \( Y \) such that the union \( O = \bigcup O_i \) contains \( \Lambda \cap \Delta \) and the continuations of the holonomy maps on \( O_i \) can be defined for vector fields in \( \mathcal{U}_i \). The holonomies associated to the \( O_i \) admit continuations for vector fields \( Z \) in a \( C^1 \)-neighborhood \( \mathcal{U} = \bigcap \mathcal{U}_i \) of \( Y \) and satisfy Items (i), (ii) and (iii). Considering a compact neighborhood of \( \Lambda \cap \Delta \) in \( D \) that is contained in \( O \) and a Lebesgue number of the covering \( \{O_i\} \), one gets \( \varepsilon_0 > 0 \) such that if \( A \subset D \) has diameter smaller than \( \varepsilon_0 \) and is contained in the \( \varepsilon_0 \)-neighborhood of \( \Lambda \cap \Delta \), then \( A \) is included in some open set \( O_i \); Item (iv) follows. \( \square \)

As in Sect. 4.1, we consider a center-unstable cone field on an open neighborhood \( U_1 \) of a singular hyperbolic chain-recurrence class \( C(\sigma) \).
**Definition 4.8** Let $Z$ be a vector field $C^1$-close to $Y$. A cu-section of $Z$ is a submanifold $N \subset D$ with dimension $\dim(E^{cu}) - 1$ such that $T_x N \oplus Z(x) \subset C(x)$ for each $x \in N$.

The forward invariance of the cone field (see Lemma 4.2) implies:

**Lemma 4.9** Let us consider open sets $V_0, \tilde{V}_0$ and $W \subset U_1$ satisfying (H) and the associated holonomy $\pi$. Then the image by $\pi$ of a cu-section $N$ is still a cu-section.

The minimal norm of a linear map $A$ between two euclidean spaces is defined by $m(A) := \inf \{ \|Av\| : \text{ unit vector } v \}$.

**Definition 4.10** A holonomy $\pi : V \rightarrow \tilde{V}$ is $10$-expanding if there exists $\chi > 10$ such that for any cu-section $N \subset V$, the derivative $D\pi|_N$ has minimal norm larger than $\chi$ with respect to the induced metrics on $N$ and $\pi(N)$.

The definition of the singular hyperbolicity (in particular condition (1)) and the uniform transversality between $D$ and the vector field imply:

**Lemma 4.11** Let us consider a cross section $D$ and an open neighborhood $U_1$ of a singular hyperbolic chain-recurrence class $C(\sigma)$ as in Sect. 4.1. There exists $T' > 0$ such that any holonomy $\pi : V \rightarrow \tilde{V}$ for $(Y, D, U_1)$ whose transition times are bounded below by $T'$ is 10-expanding.

**Proof** Note that $D$ is disjoint from singularities. For any $x \in D$, the angle between $T_x D$ and $X(x)$ is larger than $\alpha$. By reducing $\alpha$ if necessary, one can assume furthermore that $\|X(x)\| \geq \alpha$, for all $x \in D$.

Now we take a unit vector $v \in T_x N$, where $N$ is a cu-section in $D$. Assume that $\pi(x) = \varphi_t(x) \in D$ with $t > T'$. We consider the parallelograms $P_x$ generated by $v$ and $X(x)$ and $P_{\pi(x)}$ generated by $D\pi(v)$ and $X(\varphi_t(x))$. By the area expansion, there is a constant $c > 0$ such that

$$\text{Area}(P_{\pi(x)}) \geq c \cdot 2^{T'/T} \text{Area}(P_x).$$

So we have that

$$D\pi(v) \geq \frac{\text{Area}(P_{\pi(x)})}{\|X(\varphi_t(x))\|} \geq \frac{c \cdot 2^{T'/T}}{\|X\|} \cdot \frac{\text{Area}(P_x)}{\|X\|} \cdot \|v\| \cdot |X(x)| \cdot \sin(\alpha) \geq \frac{c \cdot \alpha \cdot \sin(\alpha) \cdot 2^{T'/T}}{\|X\|}.$$

Thus $\|D\pi(v)\| > 10$ when $T'$ large enough.

\[\square\]

### 4.3 Proof of the Proposition 4.3

Let us consider the neighborhoods $U_1$ of $C(\sigma_X)$, $\cup_1$ of $X$ and the cone field $\mathcal{C}$ satisfying condition (1). Let us consider a cross section $D$ for $X$ with a core $\Delta \subset D$ (as given by Lemma 4.5). One can always replace $\mathcal{C}$ and $U_1$ by forward iterates under $\varphi^X$, so that $\mathcal{C}$ is arbitrarily close to the bundle $E^{cu}$ on $C(\sigma_X)$.

**Claim** (Up to replacing $\mathcal{C}$ and $U_1$ by forward iterates), for each singularity $\rho$ one can assume that $\mathcal{C}$ is transverse to $D \cap W^s_{\text{loc}}(\rho)$ at points of $C(\sigma)$.

**Proof** For each singularity $\rho$, the intersection $D \cap W^s_{\text{loc}}(\rho)$ is a one-codimensional sphere in $W^s_{\text{loc}}(\rho)$ transverse to $X$. Since $E^{cu} \cap T W^s_{\text{loc}}(\rho) = \mathbb{R}X$ at regular points of $C(\sigma)$, one deduces that at points of $D \cap W^s_{\text{loc}}(\rho) \cap C(\sigma)$, the submanifold $D \cap W^s_{\text{loc}}(\rho)$ is transverse to $E^{cu}$, hence to $\mathcal{C}$.

\[\square\]
Since $C(\sigma_X)$ is an attractor, it admits a neighborhood $U \subset U_1$ which is attracting such that $\Delta := C(\sigma_X)$ is the maximal invariant set in $U$. By the claim above, up to reducing the neighborhoods $U$ and $\mathcal{U}_1$, one can require that for any $Y \in \mathcal{U}_1$ the cone field $\mathcal{C}$ is transverse to $D \cap W^s_{loc}(\rho_Y) \cap U$ for each singularity $\rho_Y \in U$ of $Y$. In particular, there exists $\varepsilon_1 > 0$ such that any cu-section $N$ of $Y \in \mathcal{U}_2$ with diameter smaller than $\varepsilon_1$ intersects $W^s_{loc}(\rho_Y)$ in at most one point.

Let $T' > 0$ be given by Lemma 4.11. By Proposition 4.7, there exists a finite collection of holonomies $\pi_i^Y : V_i^Y \to \bar{V}_i^Y$, $i = 1, \ldots, \ell$, defined for any vector field $Y$ in a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of $X$ and whose transition times are bounded from below by $T'$. The union $\bigcup_i V_i^Y$ covers a uniform neighborhood $O$ of $C(\sigma_X) \cap \Delta$. Since $C(\sigma_X)$ is an attractor for $\phi^X$ and by our choice of the cross section $D$ and of $\Delta$, one can reduce the neighborhood $\mathcal{U}_2$ of $X$ and choose a neighborhood $\mathcal{U}_2 \subset U$ of $C(\sigma_X)$ such that for any $Y \in \mathcal{U}_2$ and any $x \in \mathcal{U}_2$:

- the forward orbit of $x$ is contained in $U$,
- $x$ belongs to the stable manifold of a singularity $\rho_Y$ or has a forward iterate by $\phi^Y$ which belongs to the interior of $\Delta$ in $D$.

Moreover there exists $\varepsilon_0$ satisfying Item (iv) of Proposition 4.7. We can always reduce $\varepsilon_0$ to be smaller than $\varepsilon_1$.

The angle between the cross section $D$ and the vector field is bounded away from zero by $\alpha$. Hence if $\varepsilon_0$ is reduced enough and $\varepsilon > 0$ is chosen small, then for any cu-section $N \subset \Delta$ with inner radius $\varepsilon_0$ and diameter less than $3 \varepsilon_0$, the set $\bigcup \{\phi^Y_i(N) : |s| \leq \varepsilon_0\}$ is a submanifold with inner diameter larger than $\varepsilon$.

Let us consider $Y \in \mathcal{U}_2$ and $\Gamma$ be a submanifold of dimension $\dim(E^{cu})$ as in the statement of Proposition 4.3. Note that $C(\sigma_Y)$ is not a sink (it contains $\gamma_Y$), hence $\sigma_Y$ has a non-trivial unstable space. From Item 3 of Proposition 2.4, this implies that the bundle $E^{cu}$ has dimension at least 2. From the area expansion (1), the stable manifolds of the singularities are meager in $\Gamma$. By definition of the cross section and of its core $\Delta$, one can thus find $x \in \Gamma$ having a forward iterate $\phi^Y_0(x)$ in $\bigcup_i V_i^Y \setminus \bigcup_i W^s(\rho_Y)$. By construction and forward invariance the set $\Gamma' := \bigcup_{|t-0|<\delta} \phi^Y_t(\Gamma)$ is still a submanifold tangent to $\mathcal{C}$ if $\delta > 0$ is small enough. Since $Y(z) \in T_1 \Gamma'$, the intersection $N := \Gamma' \cap D$ is a cu-section. See Fig. 2.

We inductively build a sequence of cu-sections $N_n \subset D \setminus \bigcup_i W^s_{loc}(\rho_Y)$ with diameter smaller than $\varepsilon_0$ and contained in the orbit $\bigcup_{t>0} \phi^Y_t(N)$ of $N$. We denote by $r_n$ their inner radius. As explained in Sect. 4.1, we can reduce $N_n$ without reducing the inner radius in such a way that the diameter of $N_n$ is smaller than or equal to $2 r_n$.

Since by definition a cu-section is transverse to $W^s_{loc}(\rho_Y)$, one can choose $0 \subset N \setminus W^s_{loc}(\rho_Y)$. The cu-section $N_n$ is then built inductively from $N_{n-1}$ by:

(a) Choosing a domain $V_i^Y$ intersecting $N_{n-1}$: if $N_{n-1}$ is disjoint from $W^s_{loc}(\rho_Y)$ for any singularity $\rho_Y$, then by Item (iv) of Proposition 4.7, there exists a domain $V_i^Y$ which contains $N_{n-1}$; otherwise, there exists a singularity $\rho_Y$ such that $N_{n-1} \setminus W^s_{loc}(\rho_Y)$ is contained in a domain $V_i^Y$.

**Claim** The inner radius of $N_{n-1} \cap V_i^Y$ is larger than $r_{n-1}/3$.

**Proof** If $N_{n-1} \subset V_i^Y$, there is nothing to prove. Otherwise by our choice of $\varepsilon_1 > \varepsilon_0$, the intersection $N_{n-1} \cap W^s_{loc}(\rho_Y)$ contains only one point. Hence the inner radius of $N_{n-1} \setminus W^s_{loc}(\rho_Y)$ is larger than one third of the inner radius of $N_{n-1}$. See Fig. 3. \(\square\)

(b) Considering the image $A_n := \pi_i^Y(N_{n-1} \cap V_i^Y)$ and the inner radius $r_n$ of $A_n$. 

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Fig. 2 Construction of the cu-section $\mathcal{N}$

Fig. 3 Construction of $\mathcal{N}_n$ from $\mathcal{N}_{n-1}$

(c) Choosing $\mathcal{N}_n$ as a subset of $\mathcal{A}_n$ having the same inner radius as $\mathcal{A}_n$ and satisfying $\text{Diam}(\mathcal{N}_n) < 3r_n$ (as explained in Sect. 4.1). By Lemma 4.9, $\mathcal{N}_n$ is a cu-section.

By Lemma 4.11, the holonomies are 10-expanding, hence a ball of radius $r$ in $\mathcal{N}_{n-1} \cap V^Y_i$ is mapped on a subset of $\mathcal{A}_n$ whose inner radius is larger than $10r$. By definition of $r_n$ one gets $10r < r_n$. One can choose $r$ close to the inner radius of $\mathcal{N}_{n-1} \cap V^Y_i$ and the Claim above gives $r_{n-1}/3 < r_n/10$. This implies that the sequence of radii increases until $r_n \geq \varepsilon_0$ (and then the construction stops).

By our choice of $\varepsilon, \varepsilon_0$, the set $\cup \{\phi^Y_i(\mathcal{N}_n) : |s| \leq \varepsilon_0\}$ contains a submanifold tangent to $\mathcal{C}$ with inner diameter larger than $\varepsilon > 0$. This ends the proof of the Proposition 4.3 and of Theorem 4.1.
5 The proofs of main theorems

We will first prove Theorem A in a generic setting.

**Theorem A’** There is a dense $G_δ$ set $♯_X \in X(M)$ such that for any $X \in ♯_X$, any singular hyperbolic Lyapunov stable chain-recurrence class $C(γ)$ of $X$ is robustly transitive.

More precisely, there are neighborhoods $∪_X$ of $X$ and $U$ of $C(γ)$ such that the maximal invariant set of $U$ for any $Y \in ∪_X$ is an attractor which coincides with $C(σ_Y)$. If it is not an isolated singularity, it is a homoclinic class.

**Proof** We consider a dense $G_δ$ set $♯_X \in X(M)$ whose elements satisfy the conclusions of Proposition 2.1 and Theorems 2.2, 3.1, 4.1.

By Theorem 2.2, we know that $C(σ)$ is in fact an attractor for $ϕ^X$, hence it is locally maximal. By Item 4 of Proposition 2.1, $C(σ)$ is robustly locally maximal, i.e. there are a neighborhood $∪_X$ of $X$ and a neighborhood $U$ of $C(σ)$ such that $C(σ_Y)$ is the maximal invariant set in $U$ for any $Y \in ∪_X$. Since $C(σ)$ is an attractor, we can assume that $ϕ^X(U) \subset U$ for some $T > 0$ and that the same holds for any $Y$ close to $X$. As a consequence, up to reducing $∪_X$ and $U$, the class $C(σ_Y)$ is also an attractor for $Y \in ∪_X$. It remains to prove that $C(σ_Y)$ is a homoclinic class (if $C(σ)$ is not trivial).

By Theorem 2.2, and up to reducing $∪_X$, there exists a periodic orbit $γ \subset C(σ)$ such that $γ_Y \subset C(σ_Y)$ for any $Y \in ∪_X$. Furthermore, the conclusions of Theorems 3.1 and 4.1 hold. Consider a point $x \in C(σ_Y)$. We will show that $x$ is accumulated by transverse homoclinic points of $γ_Y$.

**Claim** Any neighborhood $V_x$ of $x$ intersects $W^u_Y(γ_Y)$.

**Proof of the Claim** If $α(x)$ is not a single singularity, then by Theorem 3.1, the unstable manifold $W^u_Y(γ_Y)$ intersects $W^{ss}_{loc,Y}(ϕ^Y_{−t}(x))$ for any $t > 0$. By choosing $t > 0$ large, one deduces that $W^u_Y(γ_Y)$ intersects $W^{ss}_{loc,Y}(x)$ at a point arbitrarily close to $x$.

Let us assume now that $α(x)$ is a single singularity $ρ_Y$. By Item 5 of Proposition 2.1, the unstable manifold of $γ_Y$ intersect $W^s_Y(ρ_Y)$ transversely. Hence by the inclination lemma, there is a point $y \in V_x \cap W^s_Y(γ_Y)$. The claim is verified in both cases.

By Theorem 4.1, the transverse intersection points between $W^u_Y(γ_Y)$ and $W^s_Y(ρ_Y)$ are dense in $W^u_Y(γ_Y)$. This concludes that $C(σ_Y)$ is a homoclinic class. Since a homoclinic class is transitive, we have that $C(σ_Y)$ is transitive.

We can now conclude the proof of the main results.

**Proof of Theorems A and B** We first notice that these theorems hold for singular hyperbolic attractors which do not contain any singularity and for an isolated hyperbolic singularity: in these cases the attractors are uniformly hyperbolic and the proof is classical. In the following we only consider non-trivial classes which contain at least one singularity.

Let $♯_X$ be a dense $G_δ$ subset of $X(M)$ satisfying the conclusions of Theorem A’, Propositions 2.3, 2.1 and Theorems 3.1, 4.1. For $X \in ♯_X$, the singularities are hyperbolic and finite and there exists a neighborhood $∪_X$ where the singularities admit a continuation, satisfy Proposition 2.3 and such that (as in Theorem A’, 3.1 and 4.1) the following property holds: if $C(σ)$ is a (non-trivial) Lyapunov stable chain-recurrence class of a singularity $σ$ for $X$, then there exists a periodic orbit $γ \subset C(σ)$ such that for any $Y \in ∪_X$:

- the continuation $C(σ_Y)$ is a transitive attractor and coincides with the homoclinic class $H(γ_Y)$.
– for any $x \in C(\sigma_Y)$ which does not belong to the unstable manifold of a singularity, there exists a transverse intersection between $W^{ss}_Y(x)$ and $W^u_Y(\gamma_Y)$,
– for any periodic orbit $\gamma' \in C(\sigma_Y)$, there exists a transverse intersection between $W^s_Y(\gamma_Y)$ and $W^u_Y(\gamma'\gamma$).

We define the dense open set
\[
\mathcal{U} = \bigcup_{X \in \mathcal{G}} \mathcal{U}_X.
\]

Now we will verify that Theorem A holds in this open dense set $\mathcal{U}$.

Take $Y \in \mathcal{U}$, then there is a vector field $X \in \mathcal{G}$ such that $Y \in \mathcal{U}_X$. Consider a singular hyperbolic Lyapunov stable chain-recurrence class $C(\sigma_Y)$ of $Y$: it has the property that $W^u_Y(\sigma_Y) \subset C(\sigma_Y)$. Hence by the first property of Proposition 2.3, we have that $W^u_X(\sigma_X) \subset C(\sigma_Y)$. Hence $C(\sigma_X)$ is Lyapunov stable by Item 7 of Proposition 2.1. By Theorem A’, $C(\sigma_Y)$ is a robustly transitive attractor since $Y \in \mathcal{U}_X$. Hence Theorem A holds.

Since $Y \in \mathcal{U}_X$, we also conclude that $C(\sigma_Y) = H(\gamma_Y)$ is a homoclinic class and contains a dense subset of periodic points. Moreover, for any periodic orbit $\gamma' \subset C(\sigma_Y)$, there is a transverse intersection between $W^s_Y(\gamma_Y)$ and $W^u_Y(\gamma'\gamma_Y)$. Considering any $x \in \gamma'\gamma_Y$, by Theorem 3.1, there also exists a transverse intersection between $W^{ss}_Y(x)$ and $W^u_Y(\gamma_Y)$ because $x$ is not contained in the unstable manifolds of singularities. The periodic orbits $\gamma_Y$ and $\gamma'\gamma_Y$ are thus homoclinically related. Being homoclinically related is a transitive relation on hyperbolic periodic orbits (by the inclination lemma), hence any two periodic orbits in $C(\sigma_Y)$ are homoclinically related. This concludes the proof of Theorem B. $\square$

**Proof of Corollary C** Now we assume that $\dim M = 3$. We introduce the same open dense set $\mathcal{U}$ as the one built in the proof of Theorems A and B. Consider $Y \in \mathcal{U}$ and a singular hyperbolic chain-recurrence class $C$. If it is an isolated singularity or if it does not contain any singularity, then it is a uniformly hyperbolic set. Hence it is robustly transitive and a homoclinic class (by the shadowing lemma).

We can now assume that $C$ is non-trivial (i.e. it is not reduced to a single critical element) and that it contains a singularity $\sigma_Y$. Since $\dim M = 3$ and $C$ is non-trivial, the stable dimension of $\sigma_Y$ is 2 or 1. Assume for instance that it is 2 (when it is 1 we replace $X$ by $-X$). By construction of $\mathcal{U}$, there is a generic vector field $X \in \mathcal{G}$ with a neighborhood $\mathcal{U}_X$ such that $Y \in \mathcal{U}_X$. Since $C = C(\sigma_Y)$ is non-trivial, by Proposition 2.3, Item 2, $C(\sigma_X)$ is non-trivial. Hence by Item 8 of Proposition 2.1, $C(\sigma_X)$ is Lyapunov stable. By definition of $\mathcal{U}_X$ (introduced in the proof of Theorems A and B), the class $C(\sigma_Y)$ is a homoclinic class (hence transitive) for any $Y$ that is $C^1$-close to $X$. $\square$

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