GRADIENT ESTIMATES FOR STOKES AND NAVIER-STOKES SYSTEMS WITH PIECEWISE DMO COEFFICIENTS

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ABSTRACT. We study stationary Stokes systems in divergence form with piecewise Dini mean oscillation coefficients and data in a bounded domain containing a finite number of subdomains with $C^{1, Dini}$ boundaries. We prove that if $(u, p)$ is a weak solution of the system, then $(Du, p)$ is bounded and piecewise continuous. The corresponding results for stationary Navier-Stokes systems are also established, from which the Lipschitz regularity of the stationary $H^1$-weak solution in dimensions $d = 2, 3, 4$ is obtained.

1. Introduction

In this paper, we consider stationary Stokes systems with variable coefficients

\[
\begin{aligned}
\mathcal{L}u + \nabla p &= D_\alpha f_\alpha \quad \text{in } \mathcal{D}, \\
\text{div } u &= g \quad \text{in } \mathcal{D}.
\end{aligned}
\]

(1.1)

The differential operator $\mathcal{L}$ is in divergence form acting on column vector valued functions $u = (u^1, \ldots, u^d)^\top$ as follows:

\[
\mathcal{L}u = D_\alpha (A^{\alpha\beta} D_\beta u),
\]

(1.2)

where we use the Einstein summation convention over repeated indices. The domain $\mathcal{D}$ is bounded in $\mathbb{R}^d$ which consists of a finite number of disjoint subdomains and the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ can have jump discontinuities along the boundaries of the subdomains. As is well known, such a system is partly motivated by the study of composite materials with closely spaced interfacial boundaries. We refer the reader to [23, 17] for Stokes flow over composite spheres. Moreover, it can be used to model the motion of inhomogeneous fluids with density dependent viscosity and multiple fluids with interfacial boundaries; see [18, 22, 1, 9] and the references therein. Another direction is the study of stress concentration in high-contrast composites with densely packed inclusions whose material properties differ from that of the background. In [2], Ammari et al. investigated the stress concentration of Stokes flow between adjacent circular cylinders.

There is a large body of literature concerning regularity theory for partial differential equations arising from the problems of composite materials. For the theory of elliptic equations/systems in divergence form, we refer the reader to Chipot et al. [5], Li-Vogelius [19], Li-Nirenberg [20], Dong-Li [11], and the references...
therein. In particular, $W^{1,\infty}$ and piecewise $C^{1,\delta'}$-estimates were obtained by Li-Nirenberg [20] for elliptic systems with piecewise $C^\delta$ coefficients in a domain which consists of a finite number of disjoint subdomains with $C^{1,\mu}$ boundaries, $0 < \mu \leq 1$ and $0 < \delta' \leq \min\{\delta, \frac{\mu}{2(1+\mu)}\}$. The results in [20] were extended by the second and third named authors [14] to the system with piecewise Dini mean oscillation (DMO) coefficients and subdomains having $C^{1,\text{Dini}}$ boundaries. They also established piecewise $C^{1,\delta'}$-estimate for solutions under the same conditions and with $0 < \delta' \leq \min\{\delta, \frac{\mu}{1+\mu}\}$. See also [13] for the corresponding results for parabolic systems. For further related results, one can refer to [8, 12] for parabolic and elliptic systems with partially Dini or Hölder continuous coefficients and [6] for Stokes systems with partially Dini mean oscillation coefficients.

Inspired by the work mentioned above, we are interested in gradient estimates for Stokes systems with piecewise DMO coefficients. The goal of this paper consists of two aspects. We first extend the results in [14] for elliptic systems to the stationary Stokes systems (1.1). Precisely, we show in Theorem 2.3 that if the coefficients and data are of piecewise Dini mean oscillation and the boundaries of subdomains are $C^{1,\text{Dini}}$, then for every weak solution $(u, p) \in W^{1,q}(\Omega) \times L^q(\Omega)$ to (1.1), $q \in (1, \infty)$, $Du$ and $p$ are locally bounded and piecewise continuous. As an application, we obtain piecewise Hölder continuity for $Du$ and $p$ under Hölder regularity assumptions on the coefficients and the boundaries of the subdomains. We remark that the corresponding estimates are independent of the distance between subdomains so that the boundaries of more than two subdomains can touch at some points. We also prove a local $W^{1,q}$-estimate for $W^{1,1}$-weak solutions in Corollary 2.7 by exploiting the argument in [4, 3] combined with Theorem 2.3.

Second, we consider the stationary Navier-Stokes systems

$$
\begin{cases}
\mathcal{L} u + \nabla p + u^\alpha D_\alpha u = D_\alpha f_\alpha & \text{in } \Omega, \\
\text{div } u = g & \text{in } \Omega.
\end{cases}
$$

We obtain any $W^{1,q}$-solution is Lipschitz and piecewise $C^1$, where $q \in [d/2, \infty)$; see Theorem 2.9 for the details. This result can be applied to $H^{1}$-weak solution to stationary Navier-Stokes systems with piecewise Dini mean oscillation coefficients in dimensions $d = 2, 3, 4$. Related work can be found in [15], in which the author considered the Laplace operator and proved the smoothness of every weak solution for $d = 4$ provided the data are good enough.

Let us briefly describe our arguments based on Campanato’s approach. Such approach was used in [16, 21] and further developed in [8, 10, 6, 14]. The key point is to show the mean oscillations of $Du$ and $p$ in balls vanish in a certain order as the radii of balls go to zero. Recalling the nature of the domain and the coefficients, $Du$ and $p$ are discontinuous in one direction, say, $x^d$, which is the main challenge in this paper. We overcome this difficulty by choosing a coordinate system according to the geometry of the subdomains and then using the weak type-$(1,1)$ estimates obtained in [6, Lemma 3.4] to control the $L^{1/2}$-mean oscillations of $Du$ and the linear combinations $A^{\beta\gamma} D_\beta u + p e_\gamma - f_d$; see Lemma 3.1 for the details. We point out that the proof in our case is more involved than that in [6] since our arguments and estimates depend on the coordinate system, and also more involved than that in [14] because of the pressure term and the divergence equation in the Stokes systems (1.1). For example, in the proof of local boundedness of $Du$ and $p$ (see Step 4 in the proof of Theorem 2.3), an additional difficulty appears from the
pressure term on the right-hand side after the localization. For this, we adapt a
delicate approximation argument and the fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we fix our notation,
introduce function spaces and assumptions on the domain, coefficients, and data,
and then state our main results, Theorem 2.3 for stationary Stokes systems and
Theorem 2.9 for stationary Navier-Stokes systems. In Section 3, we provide the
proofs of the main theorems.

2. Assumptions and main results

We first fix some notation used throughout the paper. We use \( x = (x', x^d) \)
to denote a generic point in the Euclidean space \( \mathbb{R}^d \), where \( d \geq 2 \) and \( x' = (x^1, \ldots, x^{d-1}) \in \mathbb{R}^{d-1} \). We also write \( y = (y', y^d) \) and \( x_0 = (x_0', x_0^d) \), etc. For \( r > 0 \), we denote
\[
B_r(x) = \{ y \in \mathbb{R}^d : |y - x| < r \}, \quad B'_r(x') = \{ y' \in \mathbb{R}^{d-1} : |y' - x'| < r \}.
\]
We often write \( B_r \) and \( B'_r \) instead of \( B_r(0) \) and \( B'_r(0) \), respectively. For \( k \in \{1, \ldots, d\} \), we use \( e_k \) to denote the \( k \)-th unit vector in \( \mathbb{R}^d \).

Let \( \Omega \) be a domain in \( \mathbb{R}^d \). For \( q \in (0, \infty] \), we define
\[
\tilde{L}^q(\Omega) = \{ f \in L^q(\Omega) : (f)_{\Omega} = 0 \},
\]
where \( (f)_{\Omega} \) is the average of \( f \) over \( \Omega \), i.e.,
\[
(f)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} f \, dx.
\]
For \( q \in [1, \infty] \), we denote by \( W^{1,q}(\Omega) \) the usual Sobolev space and by \( W_0^{1,q}(\Omega) \) the completion of \( C_0^\infty(\Omega) \) in \( W^{1,q}(\Omega) \), where \( C_0^\infty(\Omega) \) is the set of all infinitely differentiable functions with a compact support in \( \Omega \). We say that a function \( \omega : [0, 1] \to [0, \infty) \) is a Dini function if it is monotonically increasing and satisfies
\[
\int_0^1 \frac{\omega(t)}{t} \, dt < +\infty.
\]
We also say that a function \( f \) defined on \( \Omega \) is Dini continuous if the function \( \varrho_f : [0, 1] \to [0, \infty) \) given by
\[
\varrho_f(t) = \sup_{x, y \in \Omega, |x - y| \leq t} |f(x) - f(y)|
\]
is a Dini function.

2.1. Assumptions on the domain. Before we state our assumptions on the domain,
we recall the definition of a domain having a \( C^{1,\text{Dini}} \) boundary.

Definition 2.1. Let \( \Omega \) be a domain in \( \mathbb{R}^d \). We say that \( \Omega \) has a \( C^{1,\text{Dini}} \) boundary if there exist a constant \( R_0 \in (0, 1] \) and a concave Dini function \( \varrho_0 \) such that the following holds. For any \( x_0 = (x_0', x_0^d) \in \partial \Omega \), there exist a \( C^1 \) function \( \chi : \mathbb{R}^{d-1} \to \mathbb{R} \) and a coordinate system depending on \( x_0 \) such that
\[
\varrho_{\nabla_x' \chi}(t) \leq \varrho_0(t) \quad \text{for all} \quad t \in [0, R_0]
\]
and that in the new coordinate system, we have
\[
|\nabla_x' \chi(x_0')| = 0.
\]
and
\[ \Omega \cap B_{R_0}(x_0) = \{ x \in B_{R_0}(x_0) : x^d > \chi(x') \}. \] (2.1)

In this paper, we always assume that \( D \) is a bounded domain in \( \mathbb{R}^d \) containing \( M \) subdomains \( D_1, \ldots, D_M \) such that

i. \( D_M = D \setminus \left( \bigcup_{i=1}^{M-1} D_i \right) \),

ii. for \( i, j \in \{1, \ldots, M-1\} \) with \( i \neq j \), we have either
\[ D_i \subset D_j \quad \text{or} \quad D_i \cap D_j = \emptyset, \] (2.2)

iii. for \( i \in \{1, \ldots, M-1\} \), \( D_i \) has a \( C^{1, \text{Dini}} \) boundary as in Definition 2.1 with the same constant \( R_0 \) and Dini function \( g_0 \).

Our assumptions on the domain, which look a bit different from those in \[14\] are in fact identical. Precisely, by disjointing the subdomains \( D_1, \ldots, D_{M-1} \), one can understand \( D \) as a domain containing \( M \) disjoint subdomains \( D_1, \ldots, D_M \) such that

i. \( D_M = D_M \).

ii. any point in \( D \) belongs to the boundaries of at most two of the subdomains.

iii. for \( i \in \{1, \ldots, M-1\} \), \( D_i \) has a \( C^{1, \text{Dini}} \) boundary in an appropriate sense.

Among the above two expressions of the nature of the domain, the second is useful to describe the regularity conditions on the coefficients and data, which may have jump discontinuities across the interfacial boundaries; see Section 2.2. On the other hand, the first expression is convenient to explain the regularity of the boundaries by using Definition 2.1. Because the disjointed subdomains \( D_i \) in the second expression may have “narrow” regions, (2.1) is not guaranteed with the same constant \( R_0 \) independent of the distance between subdomains. For example, if \( M = 3 \), \( D_1 := B_{1/2-\varepsilon} \), \( D_2 := B_{1/2} \setminus B_{1/2-\varepsilon} \), and \( D_3 := B_1 \setminus B_{1/2} \), then when we explain the regularity of \( \partial D_2 \) via Definition 2.1 we need to take \( R_0 \) to be less than \( \varepsilon \) which is the distance between \( D_1 \) and \( D_3 \). That is why we added “appropriate sense” in the condition iii’. In the following, we will use the notation \( D_i \) introduced above to denote the subdomains.

We end this subsection with a remark that the condition (2.2) can be relaxed to
\[ D_i \subset D_j \quad \text{or} \quad D_i \cap D_j = \emptyset, \]
so that the boundaries of more than two subdomains touch at some points; see Remark 2.3.

2.2. Assumptions on the coefficients and data. We assume that the coefficients \( A^{\alpha\beta} \) of the operator \( L \) in (1.2) are bounded and satisfy the strong ellipticity condition, that is, there exists \( \nu \in (0, 1) \) such that
\[ |A^{\alpha\beta}(x)| \leq \nu^{-1}, \quad \sum_{\alpha, \beta=1}^d A^{\alpha\beta}(x) \xi_{\beta} \cdot \xi_{\alpha} \geq \nu \sum_{\alpha=1}^d |\xi_{\alpha}|^2 \] (2.3)
for any \( x \in \mathbb{R}^d \) and \( \xi_{\alpha} \in \mathbb{R}^d \), \( \alpha \in \{1, \ldots, d\} \). We also assume that the coefficients and data are of piecewise Dini mean oscillation satisfying Definition 2.2 below in the domain \( D \) containing \( M \) disjoint subdomains \( D_1, \ldots, D_M \) as in Section 2.1.
**Definition 2.2.** Let $f \in L^1(D)$. We say that $f$ is of piecewise Dini mean oscillation in $D$ if there exists a Dini function $\omega_f$ such that for any $x_0 \in D$ and $r \in (0,1]$ satisfying $B_r(x_0) \subset D$, we have

$$\int_{B_r(x_0)} |f(x) - \hat{f}(x)| \, dx \leq \omega_f(r), \quad (2.4)$$

where $\hat{f} = \hat{f}_{x_0,r}$ is a piecewise continuous function on $B_r(x_0)$ given by

$$\hat{f}(x) = \int_{B_r(x_0) \cap \mathcal{D}_i} f(y) \, dy \quad \text{if} \quad x \in B_r(x_0) \cap \mathcal{D}_i.$$

Our definition of a function of piecewise Dini mean oscillation is equivalent to the definition in [14], where the piecewise mean oscillation is measured by taking the infimum over the set of all piecewise constant functions.

**2.3. Main results.** The main results of this paper are as follows.

**Theorem 2.3.** Let $D$ be a bounded domain in $\mathbb{R}^d$ containing $M$ disjoint subdomains $D_1, \ldots, D_M$ with $C^{1, \text{Dini}}$ boundaries as in Section 2.1. Also, let $q \in (1, \infty)$ and $(u,p) \in W^{1,q}(D)^d \times L^q(D)$ be a weak solution of

$$\begin{aligned}
\mathcal{L}u + \nabla p &= D_\alpha f_\alpha, \quad \text{in} \; D, \\
\text{div} u &= g, \quad \text{in} \; D,
\end{aligned} \quad (2.5)$$

where $f_\alpha \in L^\infty(D)^d$ and $g \in L^\infty(D)$. If $A^{\alpha\beta}, f_\alpha$, and $g$ are of piecewise Dini mean oscillation in $D$ satisfying Definition 2.2, then for any $D' \subset D$, we have

$$(u,p) \in W^{1,\infty}(D')^d \times L^\infty(D')$$

and

$$(u,p) \in C^1(\overline{D}_i \cap D')^d \times C(\overline{D}_i \cap D'), \quad i \in \{1, \ldots, M\}.$$

If we further assume that there exist $\gamma_0 \in (0,1)$ and $K > 0$ such that

$$g_0(r) \leq Kr^{\gamma_0}, \quad \omega_{A^{\alpha\beta}}(r) + \omega_{f_\alpha}(r) + \omega_g(r) \leq Kr^{\gamma_0} \quad (2.6)$$

for all $r \in (0, R_0]$, then

$$(u,p) \in C^{1,\gamma_0}(\overline{D}_i \cap D')^d \times C^{\gamma_0}(\overline{D}_i \cap D'), \quad i \in \{1, \ldots, M\}.$$

Related to the theorem above, we have a few remarks.

**Remark 2.4.** Upper bounds of the $L^\infty$-norms and the modulus of continuity of $Du$ and $p$ can be found in the proof of the theorem; see Section 3.1. Note that these upper bounds are independent of the distance between the subdomains. Thus our results can be applied to the case when the boundaries of more than two subdomains touch at some points.

In the middle of the proof, we also proved that for any $x_0 \in D'$, there exists a coordinate system associated with $x_0$ such that the certain linear combinations

$$D_x u \quad \text{and} \quad A^{d\beta} D_\beta u + p e_d - f_d$$

are continuous at $x_0$. Moreover, if (2.6) holds, then they are Hölder continuous with the same exponent $\gamma_0$.

**Remark 2.5.** The condition (2.6) holds provided that the subdomains $D_i$ have $C^{1,\gamma_0/(1-\gamma_0)}$ boundaries and that $A^{\alpha\beta}, f_\alpha$, and $g$ are in $C^{\gamma_0}(\overline{D}_i)$ for each $i \in \{1, \ldots, M\}$. 
Remark 2.6. By the same reasoning as in [6, Remark 2.4], one can extend the results in Theorem 2.3 to weak solutions of the system

\[
\begin{align*}
Lu + \nabla p &= D_\alpha f_\alpha + f & \text{in } & D, \\
\text{div } u &= g & \text{in } & D,
\end{align*}
\]

where \( f \in L^s(D)^d \) with \( s > d \). The corresponding upper bounds of the \( L^\infty \)-norms and the modulus of continuity of \( Du \) and \( p \) can be found in Remark 3.2 at the end of Section 3.1.

In the corollary below, we present the \( W^{1,q} \)-estimate for \( W^{1,1} \)-weak solutions, which follows from Theorem 2.3, the solvability results in [6], and the argument in Brezis [4] (see also [3, Appendix]). One may refer to the proof of [6, Theorem 2.5], where the authors proved the \( W^{1,q} \)-estimate for \( W^{1,1} \)-weak solutions to the Stokes system with partially Dini mean oscillation coefficients.

Corollary 2.7. Let \( D \) be a bounded domain in \( \mathbb{R}^d \) containing \( M \) disjoint subdomains \( D_1, \ldots, D_M \) as in Section 2.1. Also, let \((u, p) \in W^{1,1}(D)^d \times L^1(D)\) be a weak solution of (2.5), where \( f_\alpha \in L^q(D)^d \) and \( g \in L^q(D) \) with \( q \in (1, \infty) \). If \( A^{\alpha, \beta}, f_\alpha, \) and \( g \) are piecewise Dini mean oscillation in \( D \) satisfying Definition 2.2, then for \( D' \subset D \), we have

\[
(u, p) \in W^{1,q}(D')^d \times L^q(D')
\]

with the estimate

\[
\|u\|_{W^{1,q}(D')} + \|p\|_{L^q(D')} \leq N\left(\|u\|_{W^{1,1}(D)} + \|p\|_{L^1(D)} + \|f_\alpha\|_{L^q(D)} + \|g\|_{L^q(D)}\right),
\]

where the constant \( N \) depends only on \( d, \nu, M, R_0, \rho_0, \omega_{A^{\alpha, \beta}}, q, \) and \( \text{dist}(\partial D, D') \).

Remark 2.8. From Corollary 2.7, the results in Theorem 2.3 still hold under the assumption that \((u, p) \in W^{1,1}(D)^d \times L^1(D)\).

We also consider stationary Navier-Stokes systems with piecewise Dini mean oscillation coefficients.

Theorem 2.9. Let \( D \) be a bounded domain in \( \mathbb{R}^d \) containing \( M \) disjoint subdomains \( D_1, \ldots, D_M \) with \( C^{1, \text{Dini}} \) boundaries as in Section 2.1. Also, let \( q \geq d/2 \) and \((u, p) \in W^{1,q}(D)^d \times L^q(D)\) be a weak solution of

\[
\begin{align*}
Lu + \nabla p + u^\alpha D_\alpha u &= D_\alpha f_\alpha & \text{in } & D, \\
\text{div } u &= g & \text{in } & D,
\end{align*}
\]

where \( f_\alpha \in L^\infty(D)^d \) and \( g \in L^\infty(D) \). If \( A^{\alpha, \beta}, f_\alpha, \) and \( g \) are of piecewise Dini mean oscillation in \( D \) satisfying Definition 2.2, then for any \( D' \subset D \), we have

\[
(u, p) \in W^{1,\infty}(D')^d \times L^\infty(D')
\]

and

\[
(u, p) \in C^1((\overline{D}_i \cap D')^d \times C((\overline{D}_i \cap D')^d), \quad i \in \{1, \ldots, M\}.
\]

If we further assume (2.6), then

\[
(u, p) \in C^{1,\gamma_0}((\overline{D}_i \cap D')^d \times C^{\gamma_0}((\overline{D}_i \cap D')^d, \quad i \in \{1, \ldots, M\}.
\]

As a corollary of Theorem 2.9, when \( d = 2, 3, 4 \), any \( H^1 \)-weak solution to the stationary Navier-Stokes system with piecewise Dini mean oscillation coefficients is Lipschitz.
3. Proofs of main theorems

Throughout this paper, we use the following notation.

Notation 3.1. For nonnegative (variable) quantities $A$ and $B$, we denote $A \lesssim B$ if there exists a generic positive constant $C$ such that $A \leq CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant $C$ on the parameters $a$ and $b$.

3.1. Proof of Theorem 2.3. We begin the proof with the following observation. Under the assumptions on the domain $D$ with a scaling whose parameter depends only on $d, R_0, \varrho_0$, and $\text{dist}(\partial D, \mathcal{D}^r)$, we may suppose that for any $x_0 \in \mathcal{D}^r$, there exist $C^{1,\text{Dini}}$ functions $\chi_i : \mathbb{R}^{d-1} \to \mathbb{R}$, $i \in \{1, \ldots, \ell\}$ for some $\ell < M$, and a coordinate system such that the following properties hold in the new coordinate system (called the coordinate system associated with $x_0$):

(A1) We have that
$$g \nabla_{x'} \chi_i (r) \leq \varrho_0 (r)$$
for all $r \in [0, R_0]$ and $i \in \{1, \ldots, \ell\}$, and that
$$\chi_0 (x') < \chi_1 (x') < \cdots < \chi_\ell (x') < \chi_{\ell + 1} (x')$$
for all $x' \in B'(x_0)$, where we have adopted the notation $\chi_0 \equiv x_d^0 - 1$ and $\chi_{\ell + 1} \equiv x_d^0 + 1$.

(A2) $B_1(x_0) \subset \mathcal{D}$ and $B_1(x_0)$ is divided into $\ell + 1$ disjoint subdomains
$$\hat{D}_i := \{ x \in B_1(x_0) : \chi_{i-1} (x') < x^d < \chi_i (x') \}, \quad i \in \{1, \ldots, \ell + 1\}.$$ 

Here, in an appropriate sense one may think of $\hat{D}_i$ as $\mathcal{D}_i \cap B_1(x_0)$. Moreover, $x_0 \in \hat{D}_{i_0} \cup \partial \hat{D}_{i_0}$ for some $i_0 \in \{1, \ldots, \ell + 1\}$, the closest point on $\partial \hat{D}_{i_0}$ to $x_0$ is $(x'_0, \chi_{i_0} (x'_0))$, and $\nabla_{x'} \chi_{i_0} (x'_0) = 0'$.

Throughout this proof, we shall use the following notation and properties in the coordinate system associated with $x_0$ satisfying (A1) and (A2).

(B1) For $i \in \{1, \ldots, \ell + 1\}$, we denote
$$\Omega_i = \{ x \in B_1(x_0) : \chi_{i-1} (x'_0) < x^d < \chi_i (x'_0) \}.$$ 

By [14] Lemma 2.3, there exists $R_1 = R_1 (R_0, \varrho_0) \in (0, R_0]$ such that for any $r \in (0, R_1]$,
$$r^{-d} |(\hat{D}_i \setminus \Omega_i) \cap B_r(x_0)| \lesssim_{d, M, \varrho_0} \varrho_1 (r), \quad (3.1)$$
where $\varrho_1$ is a Dini function derived from $\varrho_0$.

(B2) Let $f$ be of piecewise Dini mean oscillation in $\mathcal{D}$ satisfying Definition 2.2 with a Dini function $\omega_f$. For $r \in (0, R_1]$, we define piecewise continuous functions $\tilde{f} = \tilde{f}_{x_0, r}$ and $\check{f} = \check{f}_{x_0, r}$ in $B_r(x_0)$ by
$$\tilde{f}(x) = \int_{\hat{D}_i \cap B_r(x_0)} f(y) \, dy \quad \text{if} \quad x \in B_r(x_0) \cap \hat{D}_i$$
and
$$\check{f}(x) = \int_{\hat{D}_i \cap B_r(x_0)} f(y) \, dy \quad \text{if} \quad x \in B_r(x_0) \cap \Omega_i, \quad (3.2)$$
for all $r \in (0, R_1]$. 

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where \( \hat{f} \) is indeed a function of \( x^d \). Since \( \hat{f} \equiv \tilde{f} \) in \( B_r(x_0) \cap \tilde{D}_{i} \cap \Omega_i \), by (3.1), we have

\[
\int_{B_r(x_0)} |\hat{f} - \tilde{f}| \, dx = \frac{1}{|B_r|} \sum_{i=1}^{\ell+1} \int_{(\tilde{D}_i \cap \Omega_i) \cap B_r(x_0)} |\hat{f} - \tilde{f}| \, dx 
\lesssim \|f\|_{L^\infty(B_{r}(x_0))} \varrho_1(r).
\]

From this together with (2.3), it follows that

\[
\int_{B_r(x_0)} |f - \hat{f}| \, dx \lesssim D, M, \varrho_1 \omega f(r) + \|f\|_{L^\infty(B_{r}(x_0))} \varrho_1(r). \tag{3.3}
\]

(B3) We set

\[ U = A^{d\beta} D_\beta u + p e_d - f_d. \]

For \( y \in D \) and \( r > 0 \) with \( B_r(y) \subset B_1(x_0) \), we define

\[ \Phi(x_0, y, r) = \inf_{\Theta \in \mathbb{R}^{d \times d}} \left( \int_{B_r(y)} \|D_{x^d} u, U\|_2 \, dx \right)^2, \]

where we used the subindex \( x_0 \) to indicate that the function is defined in the coordinate system associated with \( x_0 \).

To prove Theorem 2.3, we will use the following decay estimates.

Lemma 3.1. Let \( x_0 \in D' \), \( r \in (0, R_1] \), and \( \gamma \in (0, 1) \). Then under the same hypotheses of Theorem 2.3 with an additional assumption that \( Du \) and \( p \) are locally bounded, there exists \( N = N(d, \nu, M, g_0, \gamma) > 0 \) such that the following assertions hold.

(i) For any \( \rho \in (0, r) \), we have

\[
\Phi(x_0, y, \rho) \leq N \left( \frac{\rho}{r} \right)^\gamma \Phi(x_0, y, r) + N \|Du\|_{L^\infty(B_r(x_0))} \left( \hat{\omega}_{\alpha_\beta}(\rho) + \hat{g}_1(\rho) \right) 
+ N \left( \|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))} \right) \hat{g}_1(\rho) 
+ N \left( \hat{\omega}_{f_\alpha}(\rho) + \hat{\omega}_g(\rho) \right). \tag{3.4}
\]

(ii) For any \( y \in B_{r/2}(x_0) \) and \( \rho \in (0, r/2] \) such that \( B_{\rho}(y) \subset \tilde{D}_{i_1} \) for some \( i_1 \in \{1, \ldots, \ell + 1\} \), we have

\[
\Phi(x_0, y, \rho) \leq N \left( \frac{\rho}{r} \right)^\gamma \Phi(y, r/2) + N \|Du\|_{L^\infty(B_{r/2}(y))} \left( \hat{\omega}_{\alpha_\beta}(\rho) + \hat{g}_1(\rho) \right) 
+ N \left( \|f_\alpha\|_{L^\infty(B_{r/2}(y))} + \|g\|_{L^\infty(B_{r/2}(y))} \right) \hat{g}_1(\rho) 
+ N \left( \hat{\omega}_{f_\alpha}(\rho) + \hat{\omega}_g(\rho) \right). \tag{3.5}
\]

In the above, \( \hat{\omega}_\bullet \) and \( \hat{g}_1 \) are Dini functions derived from \( \omega_\bullet \) and \( g_1 \), respectively, as formulated in (3.10).

Proof. We may assume that \( x_0 = 0 \) for simplicity of notation. For a given function \( f \), we denote by \( \hat{f} = \hat{f}(x^d) \) the piecewise constant function in \( B_r \) defined as in (3.2).

We first prove the assertion (i). Let \( \mathcal{L}_0 \) be an elliptic operator given by

\[ \mathcal{L}_0 u = D_\alpha (A^{\alpha_\beta} D_\beta u), \]

and set

\[ u_e = u - \int_{-1}^{x^d} u_0 \, ds, \quad p_e = p - p_0, \]
where \( u_0 = (u_0^1, \ldots, u_0^d) \) and \( p_0 \) are functions of \( x^d \) satisfying
\[
\begin{align*}
u_0^d &= \bar{g}, \quad \bar{A}^d u_0 + p_0 e_d = \bar{f}_d.
\end{align*}
\]
Then \((u_e, p_e)\) satisfies
\[
\begin{align*}
\begin{cases}
\mathcal{L}_0 u_e + \nabla p_e = D_\alpha F_\alpha & \text{in } B_r, \\
\text{div } u_e = G & \text{in } B_r,
\end{cases}
\end{align*}
\]
where \( F_\alpha = (\bar{A}^\alpha \beta - A^\alpha \beta) D_\beta u + f_\alpha - \bar{f}_\alpha \) and \( G = g - \bar{g} \). We decompose
\[
(u_e, p_e) = (v, p_1) + (w, p_2),
\]
where \((v, p_1) \in W^{1,2}_0(B_r/4) \times L^2(B_r)\) is a unique weak solution of
\[
\begin{align*}
\begin{cases}
\mathcal{L}_0 v + \nabla p_1 = D_\alpha (I_{B_r/4} F_\alpha) & \text{in } B_r, \\
\text{div } v = I_{B_r/4} G - (I_{B_r/4} G)_{B_r} & \text{in } B_r,
\end{cases}
\end{align*}
\]
Here, \( I_{B_r/4} \) is the characteristic function. Then by [6, Lemma 3.4] with scaling and relabeling the coordinate axes, we have for all \( t > 0 \) that
\[
\left| \{ x \in B_{r/4} : |Dv(x)| + |p_1(x)| > t \} \right| \lesssim d \frac{1}{t} \int_{B_{r/4}} (|F_\alpha| + |G|) \, dx.
\]
This implies that (c.f. [6, Eq. (4.5)]
\[
\left( \int_{B_{r/4}} (|Dv| + |p_1|)^{\frac{1}{2}} \, dx \right)^2 \lesssim \int_{B_{r/4}} (|F_\alpha| + |G|) \, dx.
\] (3.7)
On the other hand, since \((w, p_2)\) satisfies
\[
\begin{align*}
\begin{cases}
\mathcal{L}_0 w + \nabla p_2 = 0 & \text{in } B_{r/4}, \\
\text{div } w = (I_{B_r/4} G)_{B_r} & \text{in } B_{r/4},
\end{cases}
\end{align*}
\]
by [6, Eq. (3.7)], we obtain
\[
\left( \int_{B_{\kappa r}} |D_{x^r} w - (D_{x^r} u)_{B_{\kappa r}}|^{\frac{1}{2}} + |W - (W)_{B_{\kappa r}}|^{\frac{1}{2}} \, dx \right)^2 \lesssim \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left( \int_{B_{r/4}} |(D_{x^r} w, W) - \Theta \Phi| \, dx \right)^2
\] (3.8)
for any \( \kappa \in (0, 1/8] \), where \( W = \bar{A}^d \beta D_\beta w + p_2 e_d \).
Now we set
\[
U_e = \bar{A}^d \beta D_\beta u + p_e e_d,
\]
and observe that
\[
D_{x^r} u_e = D_{x^r} u, \quad U - U_e = (A^d \beta - \bar{A}^d \beta) D_\beta u - (f_d - \bar{f}_d).
\] (3.9)
By (3.6)–(3.8) and the triangle inequality, we have
\[
\left( \int_{B_{kr}} |D_x u - (D_x w)_{B_{kr}}|^{\frac{1}{2}} + |w - (W)_{B_{kr}}|^{\frac{1}{2}} \, dx \right)^2 \\
\lesssim \left( \int_{B_{kr}} |D_x u - (D_x w)_{B_{kr}}|^{\frac{1}{2}} + |w - (W)_{B_{kr}}|^{\frac{1}{2}} \, dx \right)^2 \\
+ \left( \int_{B_{kr}} (|Dv| + |p_1|)^{\frac{1}{2}} \, dx \right)^2
\]
\[
\leq \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left( \int_{B_{\rho/4}} |(D_x u, W) - \Theta|^{\frac{1}{2}} \, dx \right)^2 + \kappa^{-2d} \int_{B_{\rho/4}} (|F_u| + |G|) \, dx
\]
\[
\leq \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left( \int_{B_{\rho}} |(D_x u, U_e) - \Theta|^{\frac{1}{2}} \, dx \right)^2 + \kappa^{-2d} \int_{B_{\rho}} (|F_u| + |G|) \, dx.
\]

From this together with (3.3) and (3.9), we get
\[
\Phi_0(0, \kappa r) \leq N_0 \kappa \Phi_0(0, r) + N_0 \kappa^{-2d} \|Du\|_{L^\infty(B_r)} (\omega_{A^{\alpha}}(r) + \varrho_1(r))
\+
N_0 \kappa^{-2d} (\|f_a\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)}) \varrho_1(r) + N_0 \kappa^{-2d} (\omega_{f_a}(r) + \omega_g(r)),
\]
where \(N_0 = N_0(d, \nu, M, \varrho_0) > 0\). Fix \(\kappa \in (0, 1/8]\) small enough so that \(N_0 \kappa^{1-\gamma} \leq 1\). Then,
\[
\Phi_0(0, \kappa r) \leq \kappa^{\gamma} \Phi_0(0, r) + N \|Du\|_{L^\infty(B_r)} (\omega_{A^{\alpha}}(r) + \varrho_1(r))
\+
N (\|f_a\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)}) \varrho_1(r) + N (\omega_{f_a}(r) + \omega_g(r)),
\]
where \(N = N(d, \nu, M, \varrho_0, \gamma) > 0\). Let \(\tilde{\omega}_*\) and \(\tilde{\varrho}_0\) be Dini functions defined by
\[
\tilde{\omega}_*(r) = \sum_{i=1}^\infty \kappa^{\gamma i} (\omega_*(\kappa^{-i} r) |\kappa^{-i} r < 1| + \omega_*(1) |\kappa^{-i} r \geq 1|),
\]
\[
\tilde{\varrho}_1(r) = \sum_{i=1}^\infty \kappa^{\gamma i} (\varrho_1(\kappa^{-i} r) |\kappa^{-i} r < 1| + \varrho_1(1) |\kappa^{-i} r \geq 1|),
\]
where we used the Iverson bracket notation, i.e., \([P] = 1\) if \(P\) is true and \([P] = 0\) otherwise. By iterating and using the fact that
\[
\sum_{i=1}^j \kappa^{\gamma(i-1)} \omega_*(\kappa^{j-i} r) \leq \kappa^{\gamma \gamma} \tilde{\omega}_*(\kappa^{j} r), \quad j \in \{1, 2, \ldots\},
\]
we obtain
\[
\Phi_0(0, \kappa^j r) \leq \kappa^{\gamma j} \Phi_0(0, r) + N \|Du\|_{L^\infty(B_r)} \left( \tilde{\omega}_{A^{\alpha}}(\kappa^j r) + \tilde{\varrho}_1(\kappa^j r) \right)
\+
N \left( \|f_a\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)} \right) \tilde{\varrho}_1(\kappa^j r) + N \left( \tilde{\omega}_{f_a}(\kappa^j r) + \tilde{\omega}_g(\kappa^j r) \right),
\]
which also obviously holds for \(j = 0\). Finally, for \(\rho \in (0, r]\), by taking the nonnegative integer \(j\) such that \(\kappa^j < \rho/r \leq \kappa^j\) and using (3.11) with \(\rho\) in place of \(\kappa^j r\), we get the desired estimate.

Next, we prove the assertion (ii). For a given function \(f\), we define
\[
\hat{f} = \int_{B_{kr}(y)} f(x) \, dx.
\]
Notice from the definition of $U$ that for any $\Theta_\beta \in \mathbb{R}^d$ and $\theta \in \mathbb{R}$, we have

$$|U - \Theta_0|^2 \leq |(A^{d\beta} - \hat{A}^{d\beta})D_\beta u|^2 + |\hat{A}^{d\beta}(D_\beta u - \Theta_\beta)|^2 + |p - \theta|^2 + |f_d - \hat{f}_d|^2,$$

where $\Theta_0 = \hat{A}^{d\beta}\Theta_\beta + \theta e_d - \hat{f}_d$, in the coordinate system associated with $x_0$. By averaging the above inequality on $B_\rho(y)$, taking the square, and using (2.4) (with the fact that $B_\rho(y)$ is contained in a subdomain), we obtain

$$\left(\int_{B_\rho(y)} |U - \Theta_0|^2 dx \right)^2 \lesssim \left(\int_{B_\rho(y)} |D_\beta u - \Theta_\beta|^2 + |p - \theta|^2 dx \right)^2 + \|Du\|_{L^\infty(B_\rho(y))}\omega_{A^{d\beta}}(\rho) + \omega_{f_d}(\rho).$$

From this we get

$$\Phi_{x_0}(y, \rho) \lesssim \Psi(y, \rho) + \|Du\|_{L^\infty(B_\rho(y))}\omega_{A^{d\beta}}(\rho) + \omega_{f_d}(\rho),$$

(3.12)

where

$$\Psi(y, \rho) := \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_\rho(y)} |D_\beta u - \Theta|^2 + |p - \theta|^2 dx \right)^2.$$

Note that $\Psi(y, \rho)$ is independent of coordinate systems.

We now control the quantity $\Psi(y, \rho)$ in the coordinate system associated with $y$. Using (2.4) and the relation

$$D_d u^d = g - \sum_{i=1}^{d-1} D_i u^i,$$

(3.13)

we have

$$\inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_\rho(y)} |D_d u^d - \Theta|^2 dx \right)^2 \lesssim \sum_{i=1}^{d-1} \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_\rho(y)} |D_i u^i - \Theta|^2 dx \right)^2 + \int_{B_\rho(y)} |g - \hat{g}| dx \lesssim \Phi_g(y, \rho) + \omega_g(\rho).$$

(3.14)

Note that

$$\sum_{j=1}^{d-1} A_{ij}^{dd} D_d u^j = U^i - \sum_{j=1}^d \sum_{\beta=1}^{d-1} A_{ij}^{d\beta} D_\beta u^j - A_{ij}^{dd} D_d u^d + f_d^i, \quad i \in \{1, \ldots, d-1\},$$

(3.15)

where, by the ellipticity condition on $A^{\alpha\beta}$, $(A_{ij}^{d\beta})_{i,j=1}^{d-1}$ is nondegenerate. Hence,\n
$$\mathcal{X} = \mathcal{YZ},$$

where

$$\mathcal{X} = (D_d u^1, \ldots, D_d u^{d-1})^\top, \quad \mathcal{Y} = ((A_{ij}^{d\beta})_{i,j=1}^{d-1})^{-1},$$

$$\mathcal{Z} = (Z^1, \ldots, Z^{d-1})^\top, \quad Z^i = U^i - \sum_{j=1}^d \sum_{\beta=1}^{d-1} A_{ij}^{d\beta} D_\beta u^j - A_{ij}^{dd} D_d u^d + f_d^i.$$

Since

$$|\mathcal{X} - \tilde{\mathcal{Y}}\varnothing| \leq |(\mathcal{Y} - \hat{\mathcal{Y}})\mathcal{Z}| + |\hat{\mathcal{Y}}(\mathcal{Z} - \varnothing)|, \quad \forall \varnothing \in \mathbb{R}^{d-1},$$
we see that
\[
\inf_{\vartheta \in \mathbb{R}^{d-1}} \left( \int_{B_\rho(y)} |X - \vartheta|^\frac{2}{d} \, dx \right)^2
\leq \left( \int_{B_\rho(y)} |\mathcal{Y} - \hat{\mathcal{Y}}|^\frac{2}{d} \, dx \right) \|\mathcal{Z}\|_{L^\infty(B_\rho(y))} + \inf_{\vartheta \in \mathbb{R}^{d-1}} \left( \int_{B_\rho(y)} |\mathcal{Z} - \vartheta|^\frac{2}{d} \, dx \right)^2
\leq \Phi_y(y, \rho) + (\|Du\|_{L^\infty(B_\rho(y))} + \|f_\alpha\|_{L^\infty(B_\rho(y))}) \omega_{\mathcal{A}_\rho}(\rho) + \omega_{f_\alpha}(\rho) + \omega_y(\rho) =: K_0.
\]

From this together with (3.14), we get
\[
\inf_{\Theta \in \mathbb{R}^{d\times d}} \left( \int_{B_\rho(y)} |Du - \Theta|^\frac{2}{d} \, dx \right)^2 \lesssim K_0.
\]

By the relation
\[
p = U^d - \sum_{j=1}^{d} \sum_{\beta=1}^{d} A_{d\beta} D_\beta u^j + f_d^d,
\]
we also have
\[
\inf_{\theta \in \mathbb{R}} \left( \int_{B_\rho(y)} |p - \theta|^\frac{2}{d} \, dx \right)^2 \lesssim K_0.
\]

Combining these inequalities, we obtain that \(\Psi(y, \rho) \lesssim K_0\), which together with (3.12) gives \(\Phi_{x_0}(y, \rho) \lesssim K_0\). We finish the proof of the assertion (ii) by applying (3.4) with \(y\) and \(r/2\) in place of \(x_0\) and \(r\), to bound \(K_0\) by the right-hand side of (3.5).

We are ready to prove Theorem 2.3.

**Proof of Theorem 2.3** We adapt the arguments in the proof of [14, Theorem 1.1]. Let \(\tilde{\omega}_*\) and \(\tilde{\varrho}_1\) be the Dini functions derived from \(\omega_*\) and \(\varrho_1\), respectively, as formulated in (3.10) with a fixed \(\gamma \in (0, 1)\). We denote
\[
\mathcal{F}(r) = \int_0^r \frac{\tilde{\omega}_{f_\alpha}(t) + \tilde{\varrho}_y(t)}{t} \, dt.
\]

For given \(y \in \mathcal{D}\) and \(\rho > 0\) with \(B_\rho(y) \subset B_1(x_0)\), we let \(\Theta_{x_0}(y, \rho) \in \mathbb{R}^{d\times d}\) be such that
\[
\Phi_{x_0}(y, \rho) = \left( \int_{B_\rho(y)} |(D_{x_0}'u, U) - \Theta_{x_0}(y, \rho)|^\frac{2}{d} \, dx \right)^2.
\]

We divide the proof into four steps. In the first step, we will derive an a priori \(L^\infty\)-estimate for \((Du, p)\) under the assumption that \((Du, p)\) is locally bounded. We then obtain an estimate of the modulus of continuity of \((D_{x_0}'u, U)\) in the second step, from which the piecewise continuity of \((Du, p)\) follows. In the third step, we shall derive an a priori estimate of the modulus of continuity of \((Du, p)\) under the additional condition (2.6). In the last step, we shall show that \((Du, p)\) is indeed locally bounded by using the technique of flattening the boundary and a fixed point argument combined with partial Schauder estimates for Stokes systems.

**Step 1.** Let \(r \in (0, R_1]\). Note that Lemma 3.1 (i) implies
\[
\lim_{i \to \infty} \Phi_{x_0}(x_0, \kappa^i r) = 0
\]
Since \( \lim_{i \to \infty} \Theta_{x_0}(x_0, \kappa^i r) = (D_{x'} u(x_0), U(x_0)) \), we obtain the following for a.e. \( x_0 \in \mathcal{D}' \), where \( \kappa \in (0, 1/8] \) is the constant from the proof of Lemma 3.1. Thus, using the assumption that \( Du \) and \( p \) are bounded, we have

\[
\lim_{i \to \infty} \Theta_{x_0}(x_0, \kappa^i r) = (D_{x'} u(x_0), U(x_0))
\]

for a.e. \( x_0 \in \mathcal{D}' \), in the coordinate systems associated with \( x_0 \) satisfying (A1) and (A2). By the same iteration argument that led to Eq. (4.10), we have

\[
|(D_{x'} u(x_0), U(x_0)) - \Theta_{x_0}(x_0, r)| \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(x_0, \kappa^i r).
\]

Since

\[
|\Theta_{x_0}(x_0, r)| \lesssim r^{-d} \left( \|D_{x'} u\|_{L^1(B_r(x_0))} + \|U\|_{L^1(B_r(x_0))} \right),
\]

by Lemma 3.1 (i) and the fact that

\[
\sum_{i=0}^{\infty} \varphi_i(r) \lesssim \int_0^r \frac{\varphi_i(t)}{t} \, dt, \quad \sum_{i=0}^{\infty} \varphi_i(r) \lesssim \int_0^r \frac{\varphi_i(t)}{t} \, dt,
\]

we obtain

\[
|D_{x'} u(x_0)| + |U(x_0)| \lesssim_{d, \nu, M, \eta, \gamma} \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\varphi_i(t)}{t} \, dt + r^{-d} \left( \|D_{x'} u\|_{L^1(B_r(x_0))} + \|U\|_{L^1(B_r(x_0))} \right) + \left( \|f_0\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))} \right) \left( 1 + \int_0^r \frac{\varphi_i(t)}{t} \, dt \right) + N_0 \mathcal{F}(r),
\]

From this together with the fact that

\[
|Du| + |p| \lesssim_{d, \nu} |D_{x'} u| + |U| + |f_d| + |g|,
\]

we get

\[
|Du(x_0)| + |p(x_0)| \leq N_0 \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\varphi_i(t)}{t} \, dt + N_0 r^{-d} \left( \|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))} \right) + N_0 \mathcal{F}(r),
\]

where \( N_0 = N_0(d, \nu, M, \eta, \gamma) \). Taking \( r_0 \in (0, R_1] \) sufficiently small so that

\[
N_0 \int_0^{r_0} \frac{\varphi_i(t)}{t} \, dt \leq \frac{1}{3^d},
\]

we have

\[
|Du(x_0)| + |p(x_0)| \leq 3^{-d} \|Du\|_{L^\infty(B_r(x_0))} + N_0 r^{-d} \left( \|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))} \right) + N_0 \mathcal{F}(r) + N \mathcal{F}(r),
\]

for all \( r \in (0, r_0] \). Note that the above inequality holds for a.e. \( x_0 \in \mathcal{D}' \) and does not depend on coordinate systems. Therefore, by the same iteration argument that led to Eq. (4.16), we obtain the following \( L^\infty \)-estimate for \( Du \) and \( p \):

\[
\|Du\|_{L^\infty(B_{r/2}(x_0))} + \|p\|_{L^\infty(B_{r/2}(x_0))} \leq N r^{-d} \left( \|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))} \right) + N \mathcal{F}(r) + N \mathcal{F}(r).
\]
where \( x_0 \in \mathcal{D}' \) and \( r \in (0, R_1] \) with \( B_r(x_0) \subset \mathcal{D}' \). In the above, \( N \) depends only on \( d, \nu, M, g_0, \omega_{A^{\alpha \beta}}, \) and \( \gamma \).

**Step 2.** Let \( x_0 \in \mathcal{D}' \) and \( r \in (0, R_1] \) with \( B_r(x_0) \subset \mathcal{D}' \), and fix a coordinate system associated with \( x_0 \) satisfying (A1) and (A2). We claim that

\[
\| (D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0)) \| \lesssim r^{-d} (\| Du \|_{L^1(B_r(x_0))} + \| p \|_{L^1(B_r(x_0))}) \mathcal{E}(|x_0 - y_0|)
\]

(3.20)

for any \( y_0 \in B_{r/4}(x_0) \), where

\[
\mathcal{E}(|x_0 - y_0|) := \left( \frac{|x_0 - y_0|}{r} \right)^\gamma + \int_0^{[x_0 - y_0]} \omega_{A^{\alpha \beta}}(t) + \tilde{g}_1(t) \, dt.
\]

Let \( y_0 \in B_{r/4}(x_0) \) and \( \rho := |x_0 - y_0| \). We consider the following two cases:

\( B_\rho(y_0) \subset \hat{D}_{i_0}, B_\rho(y_0) \not\subset \hat{D}_{i_0} \).

**Case 1.** \( B_\rho(y_0) \subset \hat{D}_{i_0} \). By the triangle inequality, we have

\[
| (D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0)) | \lesssim I_1 + I_2,
\]

where

\[
I_1 = |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, \rho)| + |(D_{x'}u(x_0), U(x)) - \Theta_{x_0}(x_0, \rho)|
\]

and

\[
I_2 = |(D_{x'}u(y_0), U(y_0)) - \Theta_{y_0}(y_0, \rho)| + |(D_{x'}u(y_0), U(x)) - \Theta_{y_0}(y_0, \rho)|.
\]

Note that by (3.17), we have

\[
I_1 \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(x_0, \kappa^i \rho).
\]

It follows from Lemma 3.1 (ii) that

\[
\lim_{i \to \infty} \Phi_{x_0}(y_0, \kappa^i \rho) = 0.
\]

Then by replicating a similar argument that used in (3.17), we obtain

\[
I_2 \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(y_0, \kappa^i \rho).
\]

Therefore, by Lemma 3.1 (iii), and (3.19), we get (3.20).

**Case 2.** \( B_\rho(y_0) \not\subset \hat{D}_{i_0} \). In this case, for simplicity of notation, we assume that \( y_0 = 0 \). Suppose that \( 0 \in \hat{D}_{i_1} \cup \partial \hat{D}_{i_1} \) for some \( i_1 \in \{1, \ldots, \ell + 1\} \) and denote by \( \tilde{y}_0 \) the closest point on \( \partial \hat{D}_{i_1} \) to the origin. We also denote \( \tilde{x}_0 = (x'_0, \tilde{x}_0(x'_0)) \), which is the closest point on \( \partial \hat{D}_{i_0} \) to \( x_0 \). Since \( |\tilde{y}_0| < \rho \) and \( |\tilde{x}_0 - x_0| < 2\rho \), we have

\[
|\tilde{x}_0 - \tilde{y}_0| \leq |\tilde{x}_0 - x_0| + |x_0| + |\tilde{y}_0| < 4\rho < r \leq R_1.
\]

(3.21)
Let
\[ y = \Lambda x, \quad x = \Lambda^{-1} y = \Gamma y, \]
where \(\Lambda\) is a \(d \times d\) rotation matrix from the coordinate systems associated with \(x_0\) to a coordinate system associated with the origin. Then by (3.21) and the same argument as in [14, pp. 2465–2466], we see that
\[ |I - \Gamma| \lesssim \varrho_1(4\rho), \]
where \(I\) is the \(d \times d\) identity matrix. From the definition of \(\varrho_1\) and (3.18), it follows that
\[ |I - \Gamma| \lesssim \varrho_1(\rho) \lesssim \int_0^\rho \frac{\varrho_1(t)}{t} dt. \quad (3.22) \]

Now we set
\[ v(y) = \Lambda u(x), \quad \pi(y) = p(x), \]
which satisfies
\[
\begin{align*}
\left\{& D_\alpha (A^{\alpha\beta} D_\beta v) + \nabla \pi = D_\alpha F_\alpha, \\
& \div v = G,
\right.
\end{align*}
\]
where
\[ A^{\alpha\beta}(y) = \Lambda (\Lambda^{\alpha k} \Lambda^{\beta l} A_{kl}(x)) \Gamma, \]
\[ (F_1, \ldots, F_d)(y) = \Lambda (f_1, \ldots, f_d)(x) \Gamma, \quad G(y) = g(x). \]

We also denote
\[ V = A^{d\beta} D_\beta v + \pi e_d - F_d. \]

By the triangle inequality, we have
\[
| (D_{x'} u(x_0), U(x_0)) - (D_{x'} u(0), U(0)) | ^{1 \over 2} \\
\leq | (D_{x'} u(x_0), U(x_0)) - \Theta x_0(x_0, \rho) | ^{1 \over 2} + | (D_{x'} u(x_0), U(x_0)) - \Theta x_0(x_0, \rho) | ^{1 \over 2} \\
+ | \Gamma(D_{y'} v(0), V(0)) - \Gamma \Theta_0(0, \rho) | ^{1 \over 2} + | \Gamma(D_{y'} v(\Lambda x), V(\Lambda x)) - \Gamma \Theta_0(0, \rho) | ^{1 \over 2} \\
+ | (D_{x'} u(0), U(0)) - \Gamma(D_{y'} v(0), V(0)) | ^{1 \over 2} \\
+ | (D_{x'} u(x), U(x)) - \Gamma(D_{y'} v(\Lambda x), V(\Lambda x)) | ^{1 \over 2} 
\]
for any \(x \in B_\rho(x_0) \cap B_\rho(0)\), where \(\Gamma(D_{y'} v, V) := (\Gamma D_{y'} v, \Gamma V)\). Taking the average over \(x \in B_\rho(x_0) \cap B_\rho(0)\) and then taking the square, we obtain that
\[ | (D_{x'} u(x_0), U(x_0)) - (D_{x'} u(0), U(0)) | \lesssim J_1 + J_2 + J_3, \quad (3.23) \]
where
\[ J_1 = | (D_{x'} u(x_0), U(x_0)) - \Theta x_0(x_0, \rho) | + \Phi x_0(x_0, \rho), \]
\[ J_2 = | (D_{y'} v(0), V(0)) - \Theta_0(0, \rho) | + \Phi_0(0, \rho), \]
\[ J_3 = \text{ess sup} \quad | (D_{x'} u(x), U(x)) - \Gamma(D_{y'} v(\Lambda x), V(\Lambda x)) |. \]

Note that \(J_1\) and \(J_2\) can be estimated by Lemma 5.11(i), (3.18), and (3.19) in the same way as in Case 1. For the estimate of \(J_3\), we observe that
\[ D_{x'} u(x) - \Gamma D_{y'} v(\Lambda x) = D_{x'} u(x) - \Gamma D_{y'} v(\Lambda x) \mathbf{I}_0 = D_{x'} u(x) (I - \Gamma) \mathbf{I}_0, \]
where \(\mathbf{I}_0 = (I_0^{\alpha\beta})\) is a \(d \times (d - 1)\) matrix with
\[ I_0^{\alpha\beta} = \delta_{\alpha\beta} \text{ for } \alpha, \beta = 1, \ldots, d - 1; \quad I_0^{d\beta} = 0 \text{ for } \beta = 1, \ldots, d - 1, \]
and
\[ U(x) - \Gamma V(\Lambda x) = (1 - \Lambda^k \alpha) A^{\alpha \beta}(x) D_{\beta} u(x) \]
\[ + p(x)(I - \Gamma)c_d + (f_1, \ldots, f_d)(x)(I - \Gamma)^d, \]
where \((I - \Gamma)^d\) is the \(d\)th column of \(I - \Gamma\). Hence by (3.18) and (3.22), we have
\[ J_3 \lesssim \left( \| Du \|_{L^\infty(B_{r/4}(x_0))} + \| p \|_{L^\infty(B_{r/4}(x_0))} + \| f_0 \|_{L^\infty(B_r(x_0))} \right) \int_0^\rho \frac{\hat{\rho}_1(t)}{t} dt 
\lesssim r^{-d} \left( \| Du \|_{L^1(B_{r/2}(x_0))} + \| p \|_{L^1(B_r(x_0))} \right) \int_0^\rho \frac{\hat{\rho}_1(t)}{t} dt 
+ \left( \| f_0 \|_{L^\infty(B_r(x_0))} + \| g \|_{L^\infty(B_r(x_0))} \right) \int_0^\rho \frac{\hat{\rho}_1(t)}{t} dt \]
+ \( F(r) \int_0^\rho \frac{\hat{\rho}_1(t)}{t} dt \).

Using this together with the estimates \( J_1 \) and \( J_2 \), we get (3.20) from (3.23).

Note that the piecewise continuity of \((Du, p)\) follows from the estimate (3.20) combined with the fact that the coefficients and data are piecewise continuous. Indeed, by using the relations (3.13), (3.15), and (3.16), and using the triangle inequality, we have that
\[ |Du^d(x_0) - Du^d(y_0)| \leq |Du^d(x_0) - Du^d(y_0)| + |Du^d(y_0) - Du^d(y)|, \]
\[ |\mathcal{X}(x_0) - \mathcal{X}(y_0)| \leq |\mathcal{X}(x_0) - \mathcal{X}(y_0)| + |\mathcal{X}(y_0) - \mathcal{X}(y)|, \]
\[ |p(x_0) - p(y_0)| \leq |p(x_0) - p(y_0)| + |p(y_0) - p(y)|, \]
where \( \mathcal{X} = (D_{\alpha} u^1, \ldots, D_{\alpha} u^{d-1})^T \), and
\[ |(Du(x_0), p(x_0)) - (Du(y_0), p(y_0))| \]
\[ \leq N r^{-d} (\| Du \|_{L^1(B_{r/2}(x_0))} + \| p \|_{L^1(B_r(x_0))}) \left( |\mathcal{E}(x_0 - y_0)| + |A^{\alpha \beta}(x_0) - A^{\alpha \beta}(y_0)| \right) \]
\[ + N (\| f_0 \|_{L^\infty(B_r(x_0))} + \| g \|_{L^\infty(B_r(x_0))}) \left( |\mathcal{E}(x_0 - y_0)| + |A^{\alpha \beta}(x_0) - A^{\alpha \beta}(y_0)| \right) \]
\[ + N \mathcal{F}(r) \left( |\mathcal{E}(x_0 - y_0)| + |A^{\alpha \beta}(x_0) - A^{\alpha \beta}(y_0)| \right) + N \mathcal{F}(x_0 - y_0) \]
\[ + N |f_0(x_0) - f_0(y_0)| + N \| g(x_0) - g(y_0) \| \]
(3.24)
for any \( x_0, y_0 \in \Omega' \) and \( r \in (0, R_1) \) satisfying \( y_0 \in B_{r/4}(x_0) \subset B_r(x_0) \subset \Omega' \), which gives the piecewise continuity of \((Du, p)\).

**Step 3.** In this step, we derive the corresponding estimate of (3.24) under the additional stronger (2.6). We again let \( x_0 \in \Omega' \) and \( r \in (0, R_1) \) with \( B_{r/4}(x_0) \subset \Omega' \), and fix a coordinate system associated with \( x_0 \) satisfying (A1) and (A2). To present the precise dependence of the constant in the estimates, we assume that
\[ g_0(r) \leq K_0 r^{-\omega_0}, \quad \omega_{A^{\alpha \beta}}(r) \leq K_0 r^{\gamma_0}, \quad \omega_f(r) + \omega_g(r) \leq K_1 r^{\gamma_0} \]
(3.25)
for some constants \( K_0, K_1 > 0 \). Thus if \( f_0 \) and \( g \) are in \( C^{\gamma_0}(\Omega) \) for each \( i \in \{1, \ldots, M\} \), then \( K_1 \) can be regarded as
\[ \max_{1 \leq i \leq M} \left\{ |f_0|_{C^{\gamma_0}(\Omega)} + |g|_{C^{\gamma_0}(\Omega)} \right\}. \]
From [19, Lemma 5.1] it follows that for any $r \in (0, R_1]$, 
\[ r^{-d}|(\mathcal{D}_1 \setminus \Omega_i) \cap B_r(x_0)| \lesssim_{d,M,K_0,\gamma_0} r^{\gamma_0} =: q_1(r). \]

Hence we have 
\[ \tilde{\omega}_{A_{\alpha\beta}}(r) + \tilde{g}_1(r) \lesssim_{d,M,K_0,\gamma_0} r^{\gamma_0} \]
and 
\[ \tilde{\omega}_{f_{\alpha}}(r) + \tilde{\omega}_g(r) \lesssim_{d,M,K_0,\gamma_0} K_1 r^{\gamma_0}. \]

Therefore by [32,4] with $\gamma = \frac{1+\gamma_0}{2}$, we conclude that 
\[ |(Du(x_0), p(x_0)) - (Du(y_0), p(y_0))| \]
\[ \leq Nr^{-d}\left( \|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))} \right) \left( \frac{|x_0 - y_0|^{\gamma_0}}{r^{\gamma_0}} + |A_{\alpha\beta}^\beta(x_0) - A_{\alpha\beta}^\beta(y_0)| \right) 
+ N\left( \|f_{\alpha}\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))} \right) \left( \frac{|x_0 - y_0|^{\gamma_0}}{r^{\gamma_0}} + |A_{\alpha\beta}^\beta(x_0) - A_{\alpha\beta}^\beta(y_0)| \right) 
+ NK_1 \left( |x_0 - y_0|^{\gamma_0} + |A_{\alpha\beta}^\beta(x_0) - A_{\alpha\beta}^\beta(y_0)| \right) 
+ N|f_{\alpha}(x_0) - f_{\alpha}(y_0)| + N|g(x_0) - g(y_0)|, \]
(3.26)
where $N = N(d,\nu,M,K_0,\gamma_0)$. We can see from (3.26) that if $x_0$ and $y_0$ are in the same subdomain, then the estimate of the modulus of continuity of $(Du, p)$ is established.

**Step 4.** In this last step, we prove the local boundedness of $(Du, p)$. We first observe that 
\[ (Du, p) \in L^q_{\text{loc}}(\mathcal{D}) \times L^q_{\text{loc}}(\mathcal{D}) \quad \text{for any} \quad q < \infty. \]
(3.27)
Indeed, since $(u, p)$ satisfies (2.3), where the coefficients $A_{\alpha\beta}$ are of variably partially small bounded mean oscillation (variably partially BMO) satisfying [3] Assumption 2.2 (i) for any $\rho > 0$ and the data $f_{\alpha}, g$ are bounded, by applying a local version of [3, Theorem 2.4] combined with a bootstrap argument, we get (3.27).

Due to the regularity result in [6], where the authors proved $W^{1,\infty}$-estimates for solutions to Stokes systems with (partially) Dini mean oscillation coefficients in a ball, it suffices to show that for $x_0 = (x_0', x_0^d) \in \partial D_i$, $i \in \{1, \ldots, M-1\}$, there is a neighborhood of $x_0$ in which $(Du, p)$ is bounded. Recall that $x_0$ belongs to the boundaries of at most two of the subdomains. Thus we can find a small $r_0 > 0$ and a $C^{1,\text{Dini}}$ function, say $\chi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, such that $B_{r_0}(x_0)$ is divided into two disjoint subdomains separated by $\chi$ and $|\nabla_x \chi(x_0^d)| = 0$ in a coordinate system. Here, we choose $r_0$ small enough so that 
\[ |\nabla_{x'} \chi(x')| \leq \mu_0 \quad \text{if} \quad |x' - x_0'| \leq r_0, \]
(3.28)
where $\mu_0 > 0$ is a constant to be chosen below. Without loss of generality, we assume that $x_0 = (0', 0)$ and $\chi(0') = 0$. For sufficiently small $\varepsilon > 0$, we let $\chi_{\varepsilon}$ be a standard mollification of $\chi$ with respect to $x'$. We also let $\phi \in C_0^\infty(B_1)$ be a smooth non-negative function with unit integral, and define piecewise mollifications of $A_{\alpha\beta}$ by 
\[ A_{\varepsilon}^{\alpha\beta}(x) = \int_{B_{r_0}(x_\varepsilon)} \phi_{\varepsilon}(x_\varepsilon - y) A_{\alpha\beta}^\beta(y) \, dy = \int_{B_{r_0}(x_\varepsilon)} \phi_{\varepsilon}(y) A_{\alpha\beta}^\beta(x_\varepsilon - y) \, dy, \]
where $\phi_{\varepsilon}(x) = \varepsilon^{-d} \phi(x/\varepsilon)$ and 
\[ x_\varepsilon = \begin{cases} x + \lambda \varepsilon e_d & \text{if} \quad x^d > \chi_{\varepsilon}(x'), \\ x - \lambda \varepsilon e_d & \text{if} \quad x^d < \chi_{\varepsilon}(x'). \end{cases} \]
Here $\lambda$ is large enough, say $\lambda = \mu_0 + 1$. Similarly, we define $f_{a, \varepsilon}$ and $g_\varepsilon$. Then the piecewise mollifications are piecewise Dini mean oscillation in $B_{r_0}$ with $$\omega_{\bullet}(r) \leq \omega_{\bullet}(r).$$

Let $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ be the weak solution in $W_0^{1,2}(B_{r_0})^d \times L^2(B_{r_0})$ to the problem

$$
\begin{align*}
&D_{\alpha}(A_{\varepsilon}^{\alpha\beta} D_{\beta} \tilde{u}_\varepsilon) + \nabla \tilde{p}_\varepsilon = D_{\alpha}(f_{a, \varepsilon} - f_{a, \varepsilon}) + D_{\alpha}((A_{\varepsilon}^{\alpha\beta} - A^{\alpha\beta}) D_{\beta} u), \\
&\text{div } \tilde{u}_\varepsilon = g - g_\varepsilon - (g - g_\varepsilon)_{B_{r_0}}.
\end{align*}
$$

(3.29)

Since $f_{a, \varepsilon} \to f_a$ in $L^2$, $g_\varepsilon \to g$ in $L^2$, and $A_{\varepsilon}^{\alpha\beta} \to A^{\alpha\beta}$ a.e., by the dominated convergence theorem, the right-hand sides of (3.29) go to zero in $L^2$ as $\varepsilon \to 0^+$. By the $W^{1,2}$-estimate, we see that

$$
\|D\tilde{u}_\varepsilon\|_{L^2(B_{r_0})} + \|\tilde{p}_\varepsilon\|_{L^2(B_{r_0})} \to 0 \quad \text{as } \varepsilon \to 0^+,
$$

and thus, there is a subsequence, still denoted by $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$, such that $|D\tilde{u}_\varepsilon| + |\tilde{p}_\varepsilon| \to 0$ a.e. in $B_{r_0}$.

Now we set $(u_\varepsilon, p_\varepsilon) = (u - \tilde{u}_\varepsilon, p - \tilde{p}_\varepsilon) \in W^{1,2}(B_{r_0})^d \times L^2(B_{r_0})$, which satisfies

$$
\begin{align*}
&D_{\alpha}(A_{\varepsilon}^{\alpha\beta} D_{\beta} u_\varepsilon) + \nabla p_\varepsilon = D_{\alpha} f_{a, \varepsilon}, \\
&\text{div } u_\varepsilon = g_\varepsilon + (g - g_\varepsilon)_{B_{r_0}}
\end{align*}
$$

(3.30)

in $B_{r_0}$. By the same reasoning as in (3.27), it holds that

$$(Du_\varepsilon, p_\varepsilon) \in L^q_{\text{loc}}(B_{r_0})^{d \times d} \times L^q_{\text{loc}}(B_{r_0})$$

for any $q < \infty$.

We shall prove that $(Du_\varepsilon, p_\varepsilon)$ is bounded near the origin so that (3.19) can be applied to the above system, which gives uniform $L^\infty$-estimate of $(Du_\varepsilon, p_\varepsilon)$. To this end, we fix $\varepsilon > 0$ and let

$$
y = \Lambda(x) = (x', x^d - \chi_\varepsilon(x')), \quad x = \Lambda^{-1}(y) = (y', y^d + \chi_\varepsilon(y')).
$$

Then $(v(y), \pi(y)) = (u_\varepsilon(x), p_\varepsilon(x))$ satisfies

$$
\begin{align*}
&D_{\alpha}(A^{\alpha\beta} D_{\beta} v) + \nabla \pi = D_{\alpha} F_{\alpha} + D_d(\pi b), \\
&\text{div } v = G + D_d v \cdot b
\end{align*}
$$

(3.31)

in $B_{r_1}$ with a sufficiently small $r_1 > 0$ so that $\overline{B_{r_1}} \subseteq \Lambda(B_{r_0})$, where

$$
A^{\alpha\beta}(y) = D_{\alpha} \Lambda^{\beta} D_{\beta} \Lambda^\alpha f_k(x), \quad F_{\alpha}(y) = D_{\alpha} \Lambda^\alpha f_k(x),
$$

$$
G(y) = g_\varepsilon(x) + (g - g_\varepsilon)_{B_{r_0}/2}, \quad b(y) = (D_{2} \chi_\varepsilon(y'), \ldots, D_{d-1} \chi_\varepsilon(y'), 0).
$$

Note that the coefficients and data are of partially Dini mean oscillation in $B_{r_1}$ except $\pi b$ and $D_d v \cdot b$, which are only known to be in $L^q(B_{r_1})$ for $q < \infty$. Thus we are not able to apply the result in (3.31) directly. To overcome this difficulty, we use the following fixed point argument.

Let $\eta$ be an infinitely differentiable function in $\mathbb{R}^d$ such that

$$
0 \leq \eta \leq 1, \quad \eta \equiv 1 \quad \text{in } B_{r_1/2}, \quad \text{supp } \eta \subset B_{r_1}.
$$

Then we see that $(\eta v, \eta \pi)$ satisfies

$$
\begin{align*}
&D_{\alpha}(A^{\alpha\beta} D_{\beta} (\eta v)) + \nabla (\eta \pi) = D_{\alpha} \tilde{F}_{\alpha} + \tilde{F} + D_d(\eta \pi b), \\
&\text{div } (\eta v) = \tilde{G} + D_d(\eta v) \cdot b
\end{align*}
$$

(3.32)

in $B_{r_1}$, where

$$
\tilde{F}_{\alpha} = \eta F_{\alpha} + A^{\alpha\beta} D_{\beta} \eta v, \quad \tilde{F} = A^{\alpha\beta} D_{\alpha} \eta D_{\beta} v - D_{\alpha} \eta F_{\alpha} - D_d \eta \pi b + \nabla \eta \pi,
$$

$$
\tilde{G} = G + D_d \eta \pi b - \eta \pi b.
$$
\( \tilde{G} = \eta G + \nabla \eta \cdot v - D_d \eta v \cdot b. \)

For each positive integer \( k \), let \( (v^{(k)}, \pi^{(k)}) \) be the weak solution in \( W^{1,2}_0(B_{r_1})^d \times \tilde{L}^2(B_{r_1}) \) to the problem
\[
\begin{aligned}
D_\alpha (A^{\alpha \beta} D_\beta v^{(k)}) + \nabla \pi^{(k)} &= D_\alpha \tilde{F}_\alpha + \tilde{F} + D_d (\pi^{(k-1)}b), \\
\text{div} v^{(k)} &= \tilde{G} + D_d v^{(k-1)} \cdot b - (\tilde{G} + D_d v^{(k-1)} \cdot b)_{B_{r_1}}
\end{aligned}
\]
in \( B_{r_1} \), where \( (v^{(0)}, \pi^{(0)}) = (0, 0) \). By applying the \( W^{1,2} \)-estimate to
\[
(v^{(k+1)} - v^{(k)}, \pi^{(k+1)} - \pi^{(k)})
\]
(3.33)
and using (3.28), we have
\[
\| D\varphi^{(k+1)} - D\varphi^{(k)} \|_{L^2(B_{r_1})} + \| \pi^{(k+1)} - \pi^{(k)} \|_{L^2(B_{r_1})} \\
\leq N_0 \| (D\varphi^{(k)} - D\varphi^{(k-1)})b \|_{L^2(B_{r_1})} + N_0 \| (\pi^{(k)} - \pi^{(k-1)})b \|_{L^2(B_{r_1})}
\]
(3.34)
where the constant \( N_0 \) is independent of \( \varepsilon \) and \( \{(v^{(k)}, \pi^{(k)})\} \). We take \( r_0 \) sufficiently small so that (3.25) holds with \( \mu_0 = 1/(2N_0) \). Then by the fixed point theorem, there exists
\[
(v^*, \pi^*) = (v^*_\varepsilon, \pi^*_\varepsilon) \in W^{1,2}_0(B_{r_1})^d \times \tilde{L}^2(B_{r_1})
\]
such that as \( k \to \infty \),
\[
v^{(k)} \to v^* \quad \text{in} \quad W^{1,2}_0(B_{r_1}), \quad \pi^{(k)} \to \pi^* \quad \text{in} \quad L^2(B_{r_1})
\]
and that in \( B_{r_1} \),
\[
\begin{aligned}
D_\alpha (A^{\alpha \beta} D_\beta v^*) + \nabla \pi^* &= D_\alpha \tilde{F}_\alpha + \tilde{F} + D_d (\pi^*b), \\
\text{div} v^* &= \tilde{G} + D_d v^* \cdot b - (\tilde{G} + D_d v^* \cdot b)_{B_{r_1}}
\end{aligned}
\]
(3.35)
From (3.32) and (3.35), it follows that in \( B_{r_1} \),
\[
\begin{aligned}
D_\alpha (A^{\alpha \beta} D_\beta ((\eta v - v^*)) + \nabla (\eta \pi - (\eta \pi)_{B_{r_1}} - \pi^*) &= D_d ((\eta \pi - \pi^*)b), \\
\text{div} (\eta v - v^*) &= D_d (\eta v - v^*) \cdot b + (\tilde{G} + D_d v^* \cdot b)_{B_{r_1}}
\end{aligned}
\]
Note that \( D_d b = 0 \) and \( (\tilde{G} + D_d v^* \cdot b)_{B_{r_1}} = 0 \). Hence by the \( W^{1,2} \)-estimate with the smallness of \( b \), we obtain that
\[
\eta v = v^*, \quad \eta \pi - (\eta \pi)_{B_{r_2}} = \pi^*.
\]
Next, let \( \rho_0 \in (0, r_0] \) be small enough so that
\[
|\nabla x' \chi(x')| \leq \mu_1 \quad \text{if} \quad |x'| \leq \rho_0,
\]
(3.36)
where \( \mu_1 \) is a constant to be chosen below. We also let \( \rho_1 \in (0, r_1] \) such that \( B_{\rho_1} \subset \Lambda(B_{\rho_0}) \). Since \( A^{\alpha \beta}, \tilde{F}_\alpha, \) and \( \tilde{G} \) are partially Hölder continuous with respect to \( y' \), \( \tilde{F}_d \in L^\infty(B_{\rho_1}) \), and \( \tilde{F} \in L^q(B_{\rho_1}) \) for all \( q < \infty \), by applying [6, Theorem 2.2 (b) and Remark 2.4] combined with covering and scaling arguments, we obtain that
\[
(D\varphi^{(1)}, \pi^{(1)}) \in L^\infty(B_{\rho})^{d \times d} \times L^\infty(B_{\rho}) \quad \text{for all} \quad \rho < \rho_1.
\]
Moreover,
\[
A^{\alpha \beta} D_\beta v^{(1)} + \pi^{(1)} c_d \in C^\delta_x(B_{\rho})^d, \quad D_x v^{(1)} \in C^\delta(B_{\rho})^{d \times (d - 1)},
\]
from which we get
\[(Dv^{(1)}, \pi^{(1)}) \in C^\delta_{x'}(B\rho_d \times C^\delta_{x'}(B\rho) \quad \text{for all } \delta \in (0, 1).\]
Repeating this procedure, we see that
\[(Dv^{(k)}, \pi^{(k)}) \in \{L^\infty(B\rho_d \times L^\infty(B\rho)) \cap (C^\delta_{x'}(B\rho_d) \times C^\delta_{x'}(B\rho))\]
for any positive integer \(k\). Hence, from the estimates in the proof of [6 Theorem 2.2 (b)] applied to \([3.33]\) with covering and scaling arguments, we deduce that for any \(0 < s < \rho < \rho_1\),
\[
\|Dv^{(k+1)} - Dv^{(k)}\|_{L^\infty(B\rho)} + \|\pi^{(k+1)} - \pi^{(k)}\|_{L^\infty(B\rho)}
\]
\[
+ (\rho - s)^d \left( [Dv^{(k)} - Dv^{(k-1)}]_{C^\delta_x(B\rho)} + [\pi^{(k)} - \pi^{(k-1)}]_{C^\delta_x(B\rho)} \right)
\]
\[
\leq N_1(\rho - s)^d \left( \|Dv^{(k)} - Dv^{(k-1)}\|_{L^1(B\rho)} + \|\pi^{(k)} - \pi^{(k-1)}\|_{L^1(B\rho)} \right)
\]
\[
+ N_1(\rho - s)^d \left( [Dv^{(k)} - Dv^{(k-1)}]_{C^\delta_x(B\rho)} + [\pi^{(k)} - \pi^{(k-1)}]_{C^\delta_x(B\rho)} \right)
\]
\[
\leq \mu_0 N_0 N_1(\rho - s)^{-d} \left( \|Dv^{(k)} - Dv^{(k-1)}\|_{L^2(B\rho)} + \|\pi^{(k)} - \pi^{(k-1)}\|_{L^2(B\rho)} \right)
\]
\[
+ \mu_1 N_1 \left( [Dv^{(k)} - Dv^{(k-1)}]_{C^\delta_x(B\rho)} + [\pi^{(k)} - \pi^{(k-1)}]_{C^\delta_x(B\rho)} \right),
\]
where we used (3.25), (3.34), and (3.36) in the second inequality. Note that the constant \(N_1\) is independent of \(\{v^{(k)}, \pi^{(k)}\}\), but it may depend on \(\varepsilon\). By choosing \(\rho_0\) sufficiently small, which (and also \(\rho_1\)) may depend on \(\varepsilon\), and following a standard iteration argument, we get uniform \(L^\infty\) bounds of \(Dv^{(k)}\) and \(\pi^{(k)}\) in \(B_{\rho_1/2}\). Thus the functions
\[Dv(y) = Dv^*(y), \quad \pi(y) = \pi^*(y),\]
and hence \(Du(x)\) and \(p(x)\) are bounded in a neighborhood of the origin with a radius depending also on \(\varepsilon\). It is easy to check that the same argument as above still works at every point near the origin, for instance, in \(B_{\rho_0/2}\), where \(\rho_0\) is the constant from the beginning of this step, which is independent of \(\varepsilon\). Therefore,
\[(Du, p) \in L^\infty(B_{\rho_0/2})^{d \times d} \times L^\infty(B_{\rho_0/2}).\]
Now we can apply the a priori estimate in Step 1 to (3.30) to get uniform \(L^\infty\)-bounds of \((Du, p)\), and then, take the limit \(\varepsilon \to 0^+\) to obtain the boundedness of the limit function \((Du, p)\) in \(B_{\rho_0/2}\). The theorem is proved. \(\square\)

We conclude the proof of Theorem 2.3 with the following remark.

Remark 3.2. As mentioned in Remark 2.6, the regularity results in Theorem 2.3 can be extended to weak solutions of
\[
\begin{align*}
\begin{cases}
\mathcal{L} u + \nabla p &= D_\alpha f_\alpha + f & \text{in } D, \\
\text{div } u &= g & \text{in } D,
\end{cases}
\end{align*}
\]
where \(f \in L^s(D)\) with \(s > d\). In this case, the upper bounds of the \(L^\infty\)-norm of \((Du, p)\) and the modulus of continuity of \((D_x u, U)\) can be derived as follows.
Let $x_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ such that $B_r(x_0) \subset \mathcal{D}'$. Due to the solvability of the divergence equation (see, for instance, [7, Lemma 3.1]), there exist $h_\alpha \in W^{1,q}(B_r(x_0))^d$, $\alpha \in \{1, 2, \ldots, d\}$, such that

$$
\sum_{\alpha=1}^{d} D_\alpha h_\alpha = f \quad \text{in } B_r(x_0)
$$

and

$$(h_\alpha)_{B_r(x_0)} = 0, \quad \|Dh_\alpha\|_{L^r(B_r(x_0))} \lesssim_d \|f\|_{L^r(B_r(x_0))}.$$

Then $(u, p)$ satisfies

$$
\begin{aligned}
\{ L u + \nabla p &= D_\alpha(f_\alpha + h_\alpha) \quad \text{in } B_r(x_0), \\
\div u &= g \quad \text{in } B_r(x_0),
\end{aligned}
$$

where, by both Morrey and Poincaré inequalities,

$$
r^{1-d/s}[h_\alpha]_{C^{1-\theta/s}(B_r(x_0))} + \|h_\alpha\|_{L^\infty(B_r(x_0))} \lesssim r^{1-d/s} \|f\|_{L^r(B_r(x_0))}.
$$

Thus by the same argument as in the proof of Theorem 2.3 with a fixed $\gamma \in (1 - \frac{4}{s}, 1)$, we have

$$
\|Du\|_{L^\infty(B_{r/2}(x_0))} + \|p\|_{L^\infty(B_{r/2}(x_0))} \\
\leq Nr^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \\
+ N [f_\alpha]_{L^\infty(B_r(x_0))} + g]_{L^\infty(B_r(x_0))} + N \mathcal{F}(r) + Nr^{1-d/s} \|f\|_{L^r(B_r(x_0))},
$$

where $N = N(d, \nu, M, \eta, \omega_{\lambda, \nu}, s)$. Moreover, for $y_0 \in B_{r/4}(x_0)$, we obtain that

$$
\begin{aligned}
\|Du(x_0) - Du(y_0)\| &
\leq Nr^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \mathcal{E}(\|x_0 - y_0\|) \\
+ N [f_\alpha]_{L^\infty(B_r(x_0))} + g]_{L^\infty(B_r(x_0))} \mathcal{E}(\|x_0 - y_0\|) \\
+ N \mathcal{F}(r) + Nr^{1-d/s} \|f\|_{L^r(B_r(x_0))} \mathcal{E}(\|x_0 - y_0\|) + N \mathcal{F}(\|x_0 - y_0\|) \\
+ N \|f\|_{L^r(B_r(x_0))} \|x_0 - y_0\|^{1-d/s},
\end{aligned}
$$

3.2. Proof of Theorem 2.3. Note that $(u, p)$ satisfies

$$
\begin{aligned}
\{ L u + \nabla p &= D_\alpha f_\alpha + f \quad \text{in } \mathcal{D}, \\
\div u &= g \quad \text{in } \mathcal{D},
\end{aligned}
$$

where $f = -u^\alpha D_\alpha u$. We consider two cases.

Case 1. $q > d$. In this case, by the Morrey–Sobolev embedding theorem, we see that $f \in L^q_{\text{loc}}(\mathcal{D})$. Thus the theorem follows from Remark 2.6 applied to a slightly shrunk domain.

Case 2. $q \leq d$. From the first case, it suffices to improve the regularity of $Du$ from $L^q$ to $L^q_{\text{loc}}$ for some $s > d$. Let $x_0 \in \mathcal{D}$. We may assume that $x_0 = 0$ and $B_1 \subset \mathcal{D}$ after translating and scaling the coordinates.

We first derive an a priori estimate for $(Du, p)$ under the assumption that $(u, p) \in W^{1,q'}(B_1)^d \times L^{q'}(B_1)$, where $q^*$ is the Sobolev conjugate of $q$, i.e., $q^* = dq/(d-q)$ when $q < d$ and $q^* \in (q, \infty)$ is arbitrary when $q = d$. Let $\eta$ be an infinitely differentiable function in $\mathbb{R}^d$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{1/2}, \quad \text{supp} \eta \subset B_1, \quad |\nabla \eta| \lesssim_d 1.$$
We define an elliptic operator $\tilde{\mathcal{L}}$ by
\[
\tilde{\mathcal{L}}u = D_\alpha (\tilde{A}^{\alpha\beta} D_\beta u),
\]
where $\tilde{A}^{\alpha\beta} = \eta A^{\alpha\beta} + \nu (1 - \eta) \delta^{\alpha\beta} I$. Here, $\nu$ is the constant from $[23]$, $\delta^{\alpha\beta}$ is the Kronecker delta symbol, and $I$ is the $d \times d$ identity matrix. Note that $\tilde{A}^{\alpha\beta}$ and $\Omega = B_1$ satisfy $[3]$ Assumption 2.2 ($\rho$) for any $\rho > 0$. Therefore, the $W^{1,q^*}$-estimate in $[3]$ Theorem 2.4 is available for $\tilde{\mathcal{L}}$ on $\Omega = B_1$.

Now, for $r, R$ with $0 < r < R \leq 1/2$, let $\zeta = \zeta_{r,R}$ be an infinitely differentiable function in $\mathbb{R}^d$ such that
\[
0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \text{ in } B_r, \quad \text{supp } \zeta \subset B_R, \quad |\nabla \zeta| \lesssim (R - r)^{-1}.
\]
Then $(v, \pi) = (\zeta u, \zeta p) \in W_0^{1,q}(B_1)^d \times L^q(B_1)$ satisfies
\[
\begin{aligned}
\tilde{\mathcal{L}} v + \nabla \pi &= F + D_\alpha F_\alpha \quad \text{in } B_1, \\
\text{div } v &= G \quad \text{in } B_1,
\end{aligned}
\tag{3.37}
\]
where
\[
F = D_\alpha \zeta A^{\alpha\beta} D_\beta u + \nabla \zeta p - D_\alpha \zeta f_\alpha - \zeta u^{\alpha} D_\alpha u,
\]
\[
F_\alpha = A^{\alpha\beta} u D_\beta \zeta + \zeta f_\alpha, \quad G = \nabla \zeta \cdot u + \zeta g.
\]
Observe that $F_\alpha \in L^q(B_1)^d$, $G \in L^q(B_1)$, and
\[
\begin{aligned}
\|F\|_{L^q(B_1)} &\lesssim\| \zeta \cdot (R - r)^{-1} (\|D_u\|_{L^q(B_1)} + \|p\|_{L^q(B_1)}) \\
&\quad + R (R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_1)} + \|u\|_{L^{q^*}(B_1)} \|Du\|_{L^{q^*}(B_1)}.\end{aligned}
\]

Then by the $W^{1,q}$-solvability in $[3]$ Theorem 2.4, (3.37) also have a unique solution $(\tilde{v}, \tilde{\pi}) \in W_0^{1,q}(B_1)^d \times L^{q^*}(B_1)$, which is also in $W_0^{1,q}(B_1)^d \times L^q(B_1)$. By the uniqueness of $W_0^{1,q}(B_1)^d \times L^q(B_1)$ solutions, we get $(\tilde{v}, \tilde{\pi}) = (v, \pi - (\pi)_{B_1})$. By applying the $W^{1,q^*}$-estimate in $[3]$ Theorem 2.4 to (3.37) and using the above inequality, we obtain that
\[
\begin{aligned}
\|Du\|_{L^{q^*}(B_r)} + \|p\|_{L^{q^*}(B_r)} &\leq N_0 \|f_\alpha\|_{L^{q^*}(B_1)} + \|G\|_{L^{q^*}(B_1)} \\
&\leq N_0 (R - r)^{-1} \|Du\|_{L^q(B_r)} + \|p\|_{L^q(B_r)} + N_0 (R - r)^{-1} \|u\|_{L^{q^*}(B_r)} \\
&\quad + N_0 (R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_1)} + N_0 \|g\|_{L^{q^*}(B_1)} + N_0 \|u\|_{L^{q^*}(B_1)} \|Du\|_{L^{q^*}(B_r)},
\end{aligned}
\]
where $N_0 = N_0 (d, \nu, M, R_0, q_0, \omega_{A^{\alpha\beta}}, q)$. From the triangle and Hölder’s inequalities, it follows that
\[
\begin{aligned}
\|Du\|_{L^{q^*}(B_r)} + \|p\|_{L^{q^*}(B_r)} &\leq \|Du\|_{L^{q^*}(B_r)} + \|\pi - (\pi)_{B_1}\|_{L^{q^*}(B_1)} + N_1 \|\pi\|_{L^q(B_1)} \\
&\leq \|Du\|_{L^{q^*}(B_r)} + \|\pi - (\pi)_{B_1}\|_{L^{q^*}(B_1)} + N_1 \|\pi\|_{L^q(B_1)}.
\end{aligned}
\]

Then by taking $R_2 \in (0, 1/2)$ so that
\[
N_0 \|u\|_{L^q(B_{R_2})} \leq \varepsilon \coloneqq \frac{1}{8},
\]
we have
\[
\begin{aligned}
\|Du\|_{L^{q^*}(B_{R_2})} &+ \|p\|_{L^{q^*}(B_{R_2})} \\
\leq (N_0 + 1) (R - r)^{-1} \|Du\|_{L^q(B_{R_2})} + \|p\|_{L^q(B_{R_2})} + N_0 (R - r)^{-1} \|u\|_{L^{q^*}(B_{R_2})} \\
&\quad + N_0 (R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_1)} + N_0 \|g\|_{L^{q^*}(B_{R_2})} + \varepsilon \|Du\|_{L^{q^*}(B_{R_2})}.
\end{aligned}
\]
Note that the above inequality holds for all $r, R$ with $0 < r < R \leq R_2$. Therefore, by the well-known iteration argument, we conclude the following a priori estimate for $(Du, p)$:
\[
\|Du\|_{L^q(B_{R/2})} + \|p\|_{L^q(B_{R/2})} \\
\lesssim R^{-1} \left( \|Du\|_{L^s(B_R)} + \|p\|_{L^s(B_R)} \right) \\
+ R^{-1} \|u\|_{L^s(B_R)} + \|f_a\|_{L^s(B_R)} + \|g\|_{L^s(B_R)} \\
\] (3.38)
for all $R \in (0, R_2]$.

We are ready to prove
\[
Du \in L^s_{\text{loc}}(D)^{d \times d} \quad \text{for some } s > d. \\
\] (3.39)

From (3.38) and a standard approximation argument, one can show that $Du \in L^q_{\text{loc}}(D)^{d \times d}$. This yields (3.39) when $d/2 < q \leq d$ because $q^s > d$. On the other hand, if $q = d/2$, then since $Du \in L^q_{\text{loc}}(D)^{d \times d}$ for all $q_1 \leq d$, by applying the above regularity result again, we get (3.39). We have thus proved the regularity results in the theorem. The corresponding upper bounds of the $L^\infty$-norm of $(Du, p)$ and the modulus of continuity of $(D_x u, U)$ can be derived as in Remark 3.2. \qed

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