Regularization path-following methods with the trust-region updating strategy for linear complementarity problems

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Abstract In this article, we consider the regularization path-following method with the trust-region updating strategy for the linear complementarity problem. Moreover, we prove the global convergence of the new method under the standard assumptions without the condition of the priority to feasibility over complementarity. Numerical results show that the new method is robust and efficient for the linear complementarity problem, especially for the dense linear complementarity problem. And it is more robust and faster than some state-of-the-art solvers such as the built-in subroutines PATH and MILES of the GAMS v28.2 (2019) environment. The computational time of the new method is about 1/3 to 1/10 of that of PATH for the dense linear complementarity problem.

Keywords Continuation Newton method · regularization · trust-region updating strategy · complementarity · path-following method

Mathematics Subject Classification (2010) 90C33 · 65K05 · 65L05 · 65L20

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1 Introduction

In this article, we are mainly concerned with the linear complementarity problem as follows:

\[ y = Mx + q, \quad x_i y_i = 0, \quad i = 1, 2, \ldots, n, \quad x \geq 0, \quad y \geq 0, \]  

(1)

where \( q \in \mathbb{R}^n \) is a vector and \( M \) is an \( n \times n \) positive semi-definite matrix. For the linear complementarity problem (1), there are many practical applications such as the equilibrium of forces [17] and the economic equilibrium problem [43, 50]. And the solutions of many problems such as the linear programming and the convex quadratic programming can be obtained by solving it [8, 56]. Furthermore, there are many efficient methods to solve it such as the Lemke’s complementary pivoting algorithm (MILES) [8, 32, 50], the path-following methods [20, 55, 56, 59] and their mixture method (PATH) [14, 15].

In this paper, we consider another path-following method based on the Newton flow with nonnegative constraints, which is the variant of the primal-dual path-following method. In order to improve its robustness and efficiency, we use the regularization technique to avoid the singularity of the Jacobian matrix, and adopt the trust-region updating strategy to adjust the time step adaptively. Firstly, we construct the regularization Newton flow with nonnegative constraints for the linear complementarity problem (1) based on the primal-dual path-following method. Then, we use the implicit Euler method and the linear approximation of the quadratic function to obtain the regularization path-following method for following the trajectory of the Newton flow. Finally, we adopt the trust-region updating strategy to adjust the time step adaptively. The advantage of this adaptive adjustment strategy compared with the line search method is that it does not require the condition of the priority to feasibility over complementarity of the conventional path-following method [55, 59] when we prove its global convergence.

The rest of this article is organized as follows. In the next section, we consider the regularization path-following method and the adaptive trust-region updating strategy for the linear complementarity problem. In section 3, we analyze the global convergence of the new method under the standard assumptions without the condition of the priority to feasibility over complementarity. In section 4, we compare the new method with two state-of-the-art solvers, i.e. PATH [14, 15, 48] and MILES (a Mixed Inequality and nonLinear Equation Solver) [43, 44, 50] for sparse problems and dense problems, test matrices of which come from the linear programming subset of NETLIB [45]. The new method is coded with the MATLAB language and executed in MATLAB (R2020a) environment [42]. PATH and MILES are executed in the GAMS v28.2 (2019) environment [21]. Numerical results show that the new method is robust and efficient for solving the linear complementarity problem. It is more robust and faster than PATH and MILES for the dense problems. Finally, some discussions are given in section 5. \( \| \cdot \| \) denotes the Euclidean vector norm or its induced matrix norm throughout the paper.
2 Regulation path-following methods

2.1 The continuous Newton flow

For convenience, we rewrite the linear complementarity problem (1) as the following nonlinear system of equations with nonnegativity constraints:

\[
F(z) = \begin{bmatrix} y - (Mx + q) \\ XYe \end{bmatrix} = 0, \ (x, y) \geq 0, \ z = (x, y),
\]

(2)

where \(X = \text{diag}(x), \ Y = \text{diag}(y)\) and all components of vector \(e\) equal one. It is not difficult to know that the Jacobian matrix \(J(z)\) of \(F(z)\) has the following form:

\[
J(z) = \begin{bmatrix} -M & I \\ Y & X \end{bmatrix}.
\]

(3)

From the second block \(XYe = 0\) of equation (2), we know that \(x_i = 0\) or \(y_i = 0\) \((i = 1 : n)\). Thus, the Jacobian matrix \(J(z)\) of equation (3) may be singular, which leads to numerical difficulties near the solution of the nonlinear system (2) for the Newton method or its variants. In order to overcome this difficulty, we consider its perturbed system [2, 53] as follows:

\[
F_{\mu}(z) = F(z) - \begin{bmatrix} 0 \\ \mu e \end{bmatrix} = 0, \ z = (x, y) > 0, \ \mu > 0.
\]

(4)

It is not difficult to verify that the Jacobian matrix \(J(z)\) defined by equation (3) is nonsingular when \(M\) is a positive semi-definite matrix and \((x, y) > 0\) (see Lemma 5.9.8, p. 469 in [8]). Thus, by using the implicit theorem and the inequality \((a - b)(c - d) \leq |ac - bd| \ (a > 0, b > 0, c > 0, d > 0)\), the perturbed system (4) has a unique solution when \(M\) is a positive definite matrix (see Theorem 5.9.13, p. 471 in [8]) for its detailed proof). The solution \(z(\mu)\) of the perturbed system (4) defines the central path, and \(z(\mu)\) approximates the solution \(z^*\) of the nonlinear system (2) when \(\mu\) tends to zero [8, 56].

If the damped Newton method is applied to the perturbation system (4) [30, 46], we have

\[
z_{k+1} = z_k - \alpha_k J(z_k)^{-1} F_{\mu}(z_k),
\]

(5)

where \(J(z_k)\) is the Jacobian matrix of \(F_{\mu}(z)\). We regard \(z_{k+1} = z(t_k + \alpha_k), \ z_k = z(t_k)\) and let \(\alpha_k \to 0\), then we obtain the continuous Newton flow with nonnegativity constraints [4, 11, 13, 39, 53] of the perturbed system (4) as follows:

\[
\frac{dz(t)}{dt} = -J(z)^{-1} F_{\mu}(z), \ z = (x, y) > 0.
\]

(6)
Actually, if we apply an iteration with the explicit Euler method [51] for the continuous Newton flow (6), we obtain the damped Newton method (5).

Since the Jacobian matrix \( J(z) = F'(z) \) may be singular, we reformulate the continuous Newton flow (6) as the following general formula for the linear complementarity problem (2):

\[
-M \frac{dx(t)}{dt} + \frac{dy(t)}{dt} = -r_q(x, y), \quad (7)
\]

\[
Y \frac{dx(t)}{dt} + X \frac{dy(t)}{dt} = -(XYe - \mu(t)e), \quad (x, y) > 0, \quad (8)
\]

where the residual \( r_q(x, y) = y - (Mx + q) \). The continuous Newton flow (7)-(8) has some nice properties. We state one of them as the following property 2.1 [5, 37–39, 52].

**Property 2.1** Assume that \( (x(t), y(t)) \) is the solution of the continuous Newton flow (7)-(8), then \( r_q(x(t), y(t)) \) converges to zero and \( x_i(t)y_i(t) (i = 1, 2, \ldots, n) \) converge to zero when \( 0 \leq \mu(t) \leq \sigma \min_{1 \leq i \leq n} \{x_i(t)y_i(t)\} \) \( (0 < \sigma < 1) \) and \( t \to \infty \). That is to say, for every limit point \( (x^*, y^*) \) of \( (x(t), y(t)) \), it is also a solution of the linear complementarity problem (2). Furthermore, \( x(t) \) and \( y(t) \) keep positive values when the initial point \( (x_0^i, y_0^i) > 0 (i = 1, 2, \ldots, n) \).

**Proof.** Assume that \( z(t) \) is the continuous solution of the continuous Newton flow (7)-(8), then we have

\[
\frac{d}{dt}r_q(x, y) = -M \frac{dx}{dt} + \frac{dy}{dt} = -r_q(x, y), \quad (9)
\]

\[
\frac{d}{dt}(XYe) = X \frac{dy}{dt} + Y \frac{dx}{dt} = -(XYe - \mu(t)e). \quad (10)
\]

Consequently, from equations (9)-(10) and \( 0 \leq \mu(t) \leq \sigma \min_{1 \leq i \leq n} \{x_i(t)y_i(t)\} \), we obtain

\[
r_q(x(t), y(t)) = r_q(x_0^i, y_0^i) \exp(-t), \quad (11)
\]

\[
-XYe \leq \frac{d}{dt}(XYe) \leq -(1 - \sigma)XYe. \quad (12)
\]

From equations (11)-(12), we know that \( r_q(x(t), y(t)) \) converges to zero with the linear rate of convergence when \( t \) tends to infinity. Furthermore, from equation (12) and the Gronwall inequality [25], we have

\[
x_0^i y_0^i \exp(-t) \leq x_i(t)y_i(t) \leq x_0^i y_0^i \exp(-(1 - \sigma)t), \quad i = 1, 2, \ldots, n. \quad (13)
\]

Consequently, from equation (13), we know that \( x_i(t)y_i(t) \geq 0 (i = 1, 2, \ldots, n) \) and \( \lim_{t \to \infty} x_i(t)y_i(t) = 0 (i = 1, 2, \ldots, n) \).

From equation (13), we know that \( x_i(t)y_i(t) > 0 (t \geq 0) \) when \( (x_0^i, y_0^i) > 0 \). If we have \( x_i(T) < 0 \) or \( y_i(T) < 0 \) for a finite value \( T > 0 \), there exists \( \bar{t} \in (0, T) \) such that \( x_i(\bar{t}) = 0 \) or \( y_i(\bar{t}) = 0 \). Consequently, we have \( x_i(\bar{t})y_i(\bar{t}) = 0 \), which contradicts \( x_i(t)y_i(t) > 0 \). Thus, we have \( (x(t), y(t)) > 0 \) for all \( t > 0 \). Therefore, if the solution
$(x(t), y(t))$ of the continuous Newton flow (7)-(8) belongs to a compact set, there exists a limit point $(x^*, y^*)$ when $t$ tends to infinity, and this limit point $(x^*, y^*)$ is also a solution of the linear complementarity problem (2).

Remark 2.1 The inverse $J(z)^{-1}$ of the Jacobian matrix $J(z)$ can be regarded as the pre-conditioner of $F_\mu(z)$ such that the every element $z_i(t)$ of $z(t)$ has roughly the same rate of convergence and it mitigates the stiffness of the ODE (6) [37, 38]. This property is very useful since it makes us adopt the explicit ODE method to follow the trajectory of the Newton flow (6) efficiently.

2.2 The regularization path-following method

From property 2.1, we know that the continuous Newton flow (7)-(8) has the global convergence. However, when the Jacobian matrix $J(z)$ is singular or nearly singular, the ODE (7)-(8) is the system of differential-algebraic equations [3, 7, 26] and its trajectory can not be efficiently solved by the general ODE method [6,29] such as the backward differentiation formulas (the built-in subroutine ode15s.m of the MATLAB environment [42, 51]). Thus, we need to construct the special method to solve this problem. Furthermore, we expect that the new method has the global convergence as the homotopy continuation methods [2, 47] and the fast rate of convergence as the traditional optimization methods. In order to achieve these two aims, we consider the continuation Newton method and the trust-region updating strategy for problem (7)-(8).

We apply the implicit Euler method [3, 7] to the continuous Newton flow (7)-(8), then we obtain

$$-M\frac{x^{k+1}-x^k}{\Delta t_k} + \frac{y^{k+1}-y^k}{\Delta t_k} = -r_q(x^{k+1}, y^{k+1}),$$

(14)

$$y^{k+1}\frac{x^{k+1}-x^k}{\Delta t_k} + x^{k+1}\frac{y^{k+1}-y^k}{\Delta t_k} = -\left(X^{k+1}Y^{k+1}e - \mu(l_{k+1})e\right).$$

(15)

Since equation (15) is a nonlinear system, it is not directly solved, and we seek for its explicit approximation formula. We replace $Y^{k+1}$ and $X^{k+1}$ with $Y^k$ and $X^k$ in the left-hand side of equation (15), respectively. And we substitute $X^{k+1}Y^{k+1}e$ with its linear approximation $X^kY^ke + \frac{\Delta x^k}{\Delta t_k}(Y^k\Delta x^k + X^k\Delta y^k)$ in the right-hand side of equation (15). We set $\mu(l_{k+1}) = \sigma_k\mu$. Then, we obtain the continuation Newton method (one of path-following methods) as follows:

$$-M\Delta x^k + \Delta y^k = -r_q^k,$$

(16)

$$Y^k\Delta x^k + X^k\Delta y^k = -r_e^k,$$

(17)

$$x^{k+1} = x^k + \frac{\Delta t_k}{1+\Delta t_k}\Delta x^k, \quad y^{k+1} = y^k + \frac{\Delta t_k}{1+\Delta t_k}\Delta y^k,$$

(18)
where \(r_k^q = r_q(x^k, y^k), r_k^c = X_k y^k - \sigma_k \mu_k e, 0 < \sigma_{\min} \leq \sigma_k \leq \sigma_{\max} < 1\) and the perturbation parameter \(\mu_k\) is selected as follows:

\[
\mu_k = \frac{(x^k)^T y^k + \| r_k^q \|^2}{2n}.
\] (19)

The selection of \(\mu_k\) in equation (19) is slightly different to the traditional selection \(\mu_k = \frac{(x^k)^T y^k}{n}\) [55, 56, 59]. According to our numerical experiments, the selection of \(\mu_k\) in equation (19) can improve the robustness of the path-following method, in comparison to the traditional selection selection \(\mu_k = \frac{(x^k)^T y^k}{n}\).

**Remark 2.2** If we set \(\alpha_k = \Delta t_k / (1 + \Delta t_k)\) in equation (18), we obtain the damped Newton method (the primal-dual path-following method) (5). However, from the view of the ODE method, they are different. The damped Newton method (5) is derived from the explicit Euler method applied to the continuous Newton flow (7)-(8). Its time step \(\alpha_k\) is restricted by the numerical stability [26, 51]. That is to say, for the linear test equation \(dx/dt = -\lambda x (\lambda > 0)\), its time step \(\alpha_k\) is restricted by the stable region \(|1 - \lambda \alpha_k| \leq 1\). Therefore, the large step \(\alpha_k\) can not be adopted in the steady-state phase. The continuation Newton method (16)-(18) is derived from the implicit Euler method applied to the continuous Newton flow (7)-(8) and the linear approximation of \(F_{\mu_k(z_k)}(z_{k+1})\), and its time step \(\Delta t_k\) is not restricted by the numerical stability for the linear test equation. Therefore, the large time step \(\Delta t_k\) can be adopted in the steady-state phase, and the continuation Newton method (16)-(18) mimics the Newton method. Consequently, it has the fast rate of convergence near the solution \(z^*\) of the nonlinear system (2). The most of all, the substitution \(\alpha_k\) with \(\Delta t_k / (\Delta t_k + 1)\) is favourable to adopt the trust-region updating strategy for adaptively adjusting the time step \(\Delta t_k\) such that the continuation Newton method (16)-(18) accurately follows the trajectory of the continuation Newton flow (7)-(8) in the transient-state phase and achieves the fast rate of convergence in the steady-state phase.

When the diagonal matrix \(X^k\) is positive definite, from equations (16)-(17), \(\Delta x^k\) and \(\Delta y^k\) can be solved by the following two subsystems:

\[
\begin{align*}
(M + (X^k)^{-1} Y^k) \Delta x^k &= r^q_k - (X^k)^{-1} r^c_k, \\
\Delta y^k &= M \Delta x^k - r^c_k.
\end{align*}
\] (20) (21)

When matrix \(M\) is positive semi-definite and \((\dot{x}^k, \dot{y}^k) > 0\), the left hand-side matrix is positive definite. Thus, the system (20) can be solved by the partial pivoting Gaussian elimination method (see pp. 125-130, [24]).

### 2.3 The time-stepping control and the initial point selection

Another issue is how to adaptively adjust the time step \(\Delta t_k\) at every iteration. A popular and efficient time-stepping control is based on the trust-region updating strategy [9, 12, 27, 33–41, 58]. Its main idea can be described as follows. Firstly, we construct a merit function reflecting the feasibility \(r_q(x, y) = 0\) and the complementarity

...
We define the predicted reduction \( \text{Pred}_k \) and the actual reduction \( \text{Ared}_k \) as follows:

\[
\text{Pred}_k = \phi(x^k, y^k) - m_k(\alpha_k) = \alpha_k \left( \|r_q^k\|_2 - \langle y^k, T\Delta x^k \rangle - \langle x^k, T\Delta y^k \rangle \right),
\]

\[
\text{Ared}_k = \phi(x^k, y^k) - \phi(x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k)
= \alpha_k \left( \|r_q^k\|_2 - \langle y^k, T\Delta x^k \rangle - \langle x^k, T\Delta y^k \rangle \right) - \alpha_k^2 \langle \Delta x^k, T\Delta y^k \rangle.
\]

Then, we enlarge or reduce the time step \( \Delta t_{k+1} \) at every iteration according to the following ratio:

\[
\rho_k = \frac{\text{Ared}_k}{\text{Pred}_k} = \frac{\|r_q^k\|_2 - \langle y^k, T\Delta x^k \rangle - \langle x^k, T\Delta y^k \rangle - \alpha_k \langle \Delta x^k, T\Delta y^k \rangle}{\|r_q^k\|_2 - \langle y^k, T\Delta x^k \rangle - \langle x^k, T\Delta y^k \rangle},
\]

where \( \alpha_k = \Delta t_k/(1 + \Delta t_k) \). A particular adjustment strategy is given as follows:

\[
\Delta t_{k+1} = \begin{cases} 
2\Delta t_k, & \text{if } \rho_k \geq \eta_2 \text{ and } (x^{k+1}, y^{k+1}) > 0, \\
\Delta t_k, & \text{if } \eta_1 \leq \rho_k < \eta_2 \text{ and } (x^{k+1}, y^{k+1}) > 0, \\
\frac{1}{2}\Delta t_k, & \text{others,}
\end{cases}
\]

where the constants \( \eta_1, \eta_2 \) are selected as \( \eta_1 = 0.25, \eta_2 = 0.75 \), respectively. We set

\[
(x^{k+1}, y^{k+1}) = (x^k, y^k) + \frac{\Delta t_k}{1 + \Delta t_k} (\Delta x^k, \Delta y^k),
\]

When \( \rho_k \geq \eta_a \) and \( (x^{k+1}, y^{k+1}) > 0 \), we accept the trial step, otherwise we discard the trial step and set

\[
(x^{k+1}, y^{k+1}) = (x^k, y^k),
\]

where \( \eta_a \) is a small positive number such as \( \eta_a = 1.0 \times 10^{-6} \).
Remark 2.3 This new time-stepping control based on the trust-region updating strategy has some advantages compared to the traditional line search strategy. If we use the line search strategy and the damped Newton method (5) to follow the trajectory $z(t)$ of the continuous Newton flow (6), in order to achieve the fast rate of convergence in the steady-state phase, the time step $\alpha_k$ of the damped Newton method is tried from 1 and reduced by the half with many times at every iteration. Since the linear model $F_{\sigma_k\mu_k}(z_k) + J(z_k)\Delta z_k$ may not approximate $F_{\sigma_k\mu_k}(z_k + \Delta z_k)$ well in the transient-state phase, the time step $\alpha_k$ will be small. Consequently, the line search strategy consumes the unnecessary trial steps in the transient-state phase. However, the selection of the time step $\Delta t_k$ based on the trust-region updating strategy (26)-(27) can overcome this shortcoming.

In order to ensure that the algorithm works well for the general linear complementarity problem, the initial point selection is also a key point. We select the starting point $(x^0, y^0)$ as follows:

$$x^0 = 10 \cdot e, \quad y^0 = Mx^0 + q, \quad y^0_i = \begin{cases} v^0_i, & \text{if } v^0_i > 0, \\ 10^{-3}, & \text{otherwise.} \end{cases} \quad (30)$$

In order to improve the stability of the algorithm, we add a small regularization item $\nu I$ to matrix $M$ [8, 22, 28, 54] when $\mu_k$ is not small, where $\mu_k$ is defined by equation (19). Specifically, we adopt the following strategy as the regularizer $M_\nu$ of matrix $M$:

$$M_\nu = \begin{cases} M + \nu I, & \text{if } \mu_k \geq \nu, \\ M, & \text{otherwise}, \end{cases} \quad (31)$$

where $\nu = 10^{-3}$.

According to the above discussions, we give the detailed descriptions of the path-following method and the trust-region updating strategy for the monotone linear complementarity problem (1) in Algorithm 1.

3 Convergence analysis

In this section, we analyze the global convergence of Algorithm 1. Without loss of generality, we assume $M_\nu = M$ in the following analysis. Namely, we do not discriminate between $M_\nu$ and $M$. Similarly to the analysis techniques of the references [55, 57, 59], we firstly construct an auxiliary sequence $(u^k, v^k)$ as follows:

$$u^{k+1} = u^k + \alpha_k(x^k + \Delta x^k - u^k),$$

$$v^{k+1} = v^k + \alpha_k(y^k + \Delta y^k - v^k), \quad (32)$$

$$v^{k+1} = v^k + \alpha_k(y^k + \Delta y^k - v^k), \quad (33)$$
Algorithm 1 Regularization path-following methods with the trust-region updating strategy for linear complementarity problems (The RPFMTr method)

**Input:**
- matrix \( M \in \mathbb{R}^{n \times n} \) and vector \( q \in \mathbb{R}^{n} \) for the problem: \( y = Mx + q, x_{y_i} = 0 \) \((i = 1:n)\), \( x \geq 0, y \geq 0 \).

**Output:** the linear complementarity solution: \((solx, soly)\).

1. Initialize parameters: \( \eta_1 = 10^{-6}, \eta_2 = 0.25, \eta_3 = 0.75, \varepsilon = 10^{-6}, \Delta t_0 = 10^{-2}, \text{bigMfac} = 10, v = 10^{-3}, \sigma_0 = 0.5, \text{maxit} = 600. \)
2. Initialize \( x^0 = \text{bigMfac} \cdot \text{ones}(n, 1) \); \( y^0 = Mx^0 + q; y^0(y^0 < 0) = 10^{-3} \).
3. Regularize matrix: \( M_0 = M + uf \).
4. Set flag\_success\_trialstep = 1, itc = 0, k = 0.
5. while (itc < maxit) do
6. if (flag\_success\_trialstep == 1) then
7. Set itc = itc + 1.
8. Compute \( r^k = y^k - (Mx^k + q) \); \( \mu_k = (\| r^k \| + (x^k)^T y^k)/(2n) \).
9. Compute \( \text{Resk} = \max \{ \| x^k \|, \| y^k - (Mx^k + q) \| \} \).
10. if (Resk < \varepsilon) then
11. break;
12. end if
13. Set \( \sigma_k = \min \{ \sigma_0, \mu_k \} \).
14. Compute \( r^k = x^k, y^k = -\sigma_k \mu_k \cdot \text{ones}(n, 1) \).
15. By solving the linear system (20)-(21), we obtain \( \Delta x^k \) and \( \Delta y^k \).
16. end if
17. Set \( (x^k, y^k + \Delta y^k) = (x^k, y^k) + \frac{dy}{dx} \cdot (\Delta x^k, \Delta y^k) \).
18. Compute the ratio \( \rho_k \) from equation (36) and adjust \( \Delta t_{k+1} \) according to the formula (27).
19. if (\( \rho_k \geq \eta_3 \)) \& \& (\( x^k, y^k + \Delta y^k > 0 \)) then
20. Accept the trial point \((x^{k+1}, y^{k+1})\); Set flag\_success\_trialstep = 1.
21. if (\( \| x^{k+1} - x^k \| < 0.1 \)) then
22. Set \( \sigma_{k+1} = 0.5 \).
23. else
24. Set \( \sigma_{k+1} = 0.1 \).
25. end if
26. if (\( \mu_k < v \)) then
27. Set \( M_0 = M \).
28. end if
29. else
30. Set \( (x^{k+1}, y^{k+1}) = (x^k, y^k) \); flag\_success\_trialstep = 0.
31. end if
32. Set \( k = k + 1 \).
33. end while
34. Return \((solx, soly) = (x^k, y^k)\).

where \((\Delta x^k, \Delta y^k)\) are solved by equations (16)-(17) and \((u^0, v^0) = (x^0, y^0)\) satisfies the feasibility \( r_q(u^0, v^0) = 0 \). Then, it is not difficult to verify

\[
\begin{align*}
  r_q(u^k, v^k) = v^k - (Mu^k + q) &= 0, \\
  \Delta x^{k+1} - u^{k+1} &= (1 - \alpha_k)(x^k - u^k) \geq 0, \\
  y^{k+1} - v^{k+1} &= (1 - \alpha_k)(y^k - v^k) \geq 0,
\end{align*}
\]

where \( \alpha_k = \Delta t_k / (1 + \Delta t_k) \).
Meanwhile, in order to obtain the global convergence, we need to enforce the condition specified below. Firstly, we select a constant $\gamma \in (0, 1)$ that satisfies

$$0 < \gamma \leq \frac{\mu_0}{\min_{1 \leq i \leq n} \{\|y^0_i\|} = \frac{(x^0)^T y^0 + \|r_q^0\|}{2n \min_{1 \leq i \leq n} \{\|y^0_i\|}.$$  \hfill (37)

Then, we select $\alpha_k$ such that

$$x^k_i(\alpha) y^k_i(\alpha) \geq \gamma \mu_k(\alpha), \quad i = 1, 2, \ldots, n \quad \hfill (38)$$

holds for all $\alpha \in (0, \alpha_c] \subset (0, 1]$, where $x^k(\alpha), y^k(\alpha), r_q^k(\alpha)$ and $\mu_k(\alpha)$ are defined by

$$x^k(\alpha) = x^k + \alpha \Delta x^k, \quad y^k(\alpha) = y^k + \alpha \Delta y^k,$$

$$r_q^k(\alpha) = r_q(x^k(\alpha), y^k(\alpha)), \quad \mu_k(\alpha) = \frac{(x^k)^T y^k + \|r_q^k(\alpha)\|}{2n}. \quad \hfill (40)$$

Condition (38) is to prevent iterations from prematurely getting too close to the boundary of the positive quadrant and its restriction on $\gamma$ is very mild. In the practice, it can be selected to be very small.

In order to establish the main global convergence of Algorithm 1, similarly to the results [55, 59], we prove the following several technique lemmas.

**Lemma 3.1** When $(x^k(\alpha), y^k(\alpha))$ satisfies the condition (38), where $(\Delta x^k, \Delta y^k)$ is the solution of equation (16)-(17), we have

$$(x^k(\alpha), y^k(\alpha)) \geq 0.$$  \hfill (41)

Furthermore, when $x^k_i(\alpha) y^k_i(\alpha) \geq \gamma \mu_k$ ($i = 1, 2, \ldots, n$) and $\alpha$ satisfies

$$0 \leq \alpha \leq \min \left\{1, \frac{1 - \gamma/2}{1 + \gamma/2 \|\Delta x^k, \Delta y^k\|} \right\},$$  \hfill (42)

the inequality (38) holds.

**Proof.** By summing two sides of the condition (38), we obtain

$$(x^k(\alpha))^T y^k(\alpha) \geq n \gamma \mu_k(\alpha) = \frac{1}{2} \gamma \left((x^k(\alpha))^T y^k(\alpha) + (1 - \alpha)\|r_q^k\|\right)$$

$$\geq \frac{1}{2} \gamma (x^k(\alpha))^T y^k(\alpha). \quad \hfill (43)$$

Consequently, from equation (43) and $0 < \gamma < 1$, we obtain $(x^k(\alpha))^T y^k(\alpha) \geq 0$. By substituting it into the condition (38), we have $x^k_i(\alpha) y^k_i(\alpha) \geq 0$ ($i = 1, 2, \ldots, n$). From equation (17), we obtain

$$\alpha \sigma_i \mu_k = \alpha x^k_i y^k_i + \alpha \Delta x^k_i + \alpha y^k_i \Delta x^k_i$$

$$= (\alpha - 2)x^k_i y^k_i + y^k_i (\alpha) + x^k_i y^k_i, \quad i = 1, 2, \ldots, n. \quad \hfill (44)$$
If \((x_i^k(\alpha), y_i^k(\alpha)) < 0\) or \(x_i^k(\alpha) = 0, y_i^k(\alpha) < 0\) or \(x_i^k(\alpha) < 0, y_i^k(\alpha) = 0\), by combining it with \((x_i^k, y_i^k) \geq 0\), it contradicts equation (44). Therefore, we obtain \((x_i^k(\alpha), y_i^k(\alpha)) \geq 0\) \((i = 1, 2, \ldots, n)\), which proves equation (41).

From equations (16)-(17), we have

\[
x_i^k(\alpha)y_i^k(\alpha) - \gamma \mu_k(\alpha) = x_i^k + \alpha(x_i^k\Delta y_i^k + y_i^k\Delta x_i^k) + \alpha^2 \Delta x_i^k\Delta y_i^k
\]

\[
- \gamma \mu_k - \frac{1}{2n} \gamma \alpha \left((x_i^k)^{T}\Delta y_i^k + (y_i^k)^{T}\Delta x_i^k - \|r_i^k\| + \alpha(\Delta x_i^k)^{T}\Delta y_i^k\right)
\]

\[
= x_i^k + \alpha(x_i^k\Delta y_i^k + y_i^k\Delta x_i^k) + \alpha^2 \Delta x_i^k\Delta y_i^k
\]

\[
- \gamma \mu_k - \frac{1}{2n} \gamma \alpha \left(n\sigma_i\mu_k - (x_i^k)^{T}\Delta y_i^k - \|r_i^k\| + \alpha(\Delta x_i^k)^{T}\Delta y_i^k\right)
\]

\[
= (1 - \alpha)x_i^k + \alpha\left(\sigma_i\mu_k \left(1 - \frac{Y}{2}\right) + \alpha\left(\Delta x_i^k\Delta y_i^k - \frac{Y}{2n}(\Delta x_i^k)^{T}\Delta y_i^k\right)\right)
\]

\[
\geq (1 - \alpha)x_i^k + \alpha\left(\sigma_i\mu_k \left(1 - \frac{Y}{2}\right) - \alpha \left(1 + \frac{Y}{2}\right) \|\Delta X^k\|\right). \tag{45}
\]

where \(\Delta X^k = \text{diag}(\Delta x_i^k)\). By substituting \(x_i^k \geq \gamma \mu_k\) and \(0 < \alpha \leq 1\) into equation (45), we have

\[
x_i^k(\alpha)y_i^k(\alpha) - \gamma \mu_k(\alpha) \geq (1 - \alpha)x_i^k + \alpha\left(\sigma_i\mu_k \left(1 - \frac{Y}{2}\right) - \alpha \left(1 + \frac{Y}{2}\right) \|\Delta X^k\|\right). \tag{46}
\]

Therefore, when \(\alpha\) satisfies equation (42), from equation (46), we obtain

\[
x_i^k(\alpha)y_i^k(\alpha) - \gamma \mu_k(\alpha) \geq 0, \quad i = 1, 2, \ldots, n,
\]

which gives the inequality (38). \(\square\)

**Lemma 3.2** Assume that \(\{(x^k, y^k)\}\) is generated by Algorithm 1. Then, for any feasible solution \((\bar{x}, \bar{y})\) of the linear complementarity problem (2), there exists a positive constant \(C_1\) such that

\[
(y^k)^{T}(x^k - u^k) + (x^k)^{T}(y^k - v^k) \leq C_1 \tag{47}
\]

holds for all \(k = 0, 1, \ldots\). Furthermore, if \((\bar{x}, \bar{y})\) is strictly feasible, \(\{(x^k, y^k)\}\) is bounded.

**Proof.** From equation (34), we know that \((u^k, v^k)\) satisfies the feasibility. By combining it with the semi-definite positivity of \(M\), we have

\[
(\bar{x} - u^k)^{T}(\bar{y} - v^k) = (\bar{x} - u^k)^{T}M(\bar{x} - u^k) \geq 0. \tag{48}
\]

Consequently, from equation (48), we have

\[
0 \leq (\bar{x} - u^k)^{T}(\bar{y} - v^k)
\]

\[
= (\bar{x} - x^k + (x^k - u^k))(\bar{y} - y^k + (y^k - v^k))
\]

\[
= x^T(y^k - v^k) - (x^k)(y^k - v^k) + y^T(\bar{x} - x^k) - y^T(x^k - u^k)
\]

\[
+ (\bar{x} - u^k)^{T}(y^k - v^k) + y^T(u^k - x^k) - (y^k)^{T}((x^k - u^k). \tag{49}
\]
From equations (35)-(36), we have
\[ 0 \leq x^k - u^k \leq x^0 - u^0, \quad 0 \leq y^k - v^k \leq y^0 - v^0. \] (50)

By substituting equation (50) into equation (49), from \((\hat{x}^k, y^k) \geq 0, (\bar{x}, \bar{y}) \geq 0\) and \(\phi(\bar{x}^k, y^k) \leq \phi(x^0, y^0)\), we obtain
\[
(y^k)^T (x^k - u^k) + (x^k)^T (y^k - v^k) \leq \bar{x}^T (y^k - v^k) + \bar{y}^T (x^k - u^k) + \bar{x}^T \bar{y} \\
+ (x^k)^T y^k + (x^k - u^k)^T (y^k - v^k) \\
\leq \bar{x}^T (y^0 - v^0) + \bar{y}^T (x^0 - u^0) + \phi(x^k, y^k) + (x^0 - u^0)^T (y^0 - v^0) + \bar{x}^T \bar{y} \\
\leq \bar{x}^T (y^0 - v^0) + \bar{y}^T (x^0 - u^0) + \phi(x^k, y^k) + (x^0 - u^0)^T (y^0 - v^0) + \bar{x}^T \bar{y} \\
= \bar{x}^T (y^0 - v^0) + \bar{y}^T (x^0 - u^0) + (x^0 - u^0)^T (y^0 - v^0) + 2\mu_0 + \bar{x}^T \bar{y}. \] (51)

We set
\[ C_1 = (y^0 - v^0)^T (x^0 - u^0) + \bar{y}^T (x^0 - u^0) + \bar{x}^T (y^0 - v^0) + \bar{x}^T \bar{y} + 2\mu_0. \]

Then, from (51), we obtain the inequality (47).

When \((\bar{x}, \bar{y})\) is strictly feasible, from inequalities (49)-(50), we have
\[
\bar{x}^T x^k + \bar{y}^T y^k \leq \bar{x}^T (y^0 - v^0) + \bar{y}^T (x^0 - u^0) + \phi(x^k, y^k) + (x^0 - u^0)^T (y^0 - v^0) \\
\leq \bar{x}^T (y^0 - v^0) + \bar{y}^T (x^0 - u^0) + \phi(x^k, y^k) + (x^0 - u^0)^T (y^0 - v^0) + 2\mu_0 + (x^0 - u^0)^T (y^0 - v^0) \triangleq C_2, \] (52)

where we use the property \(\phi(x^k, y^k) \leq \phi(x^0, y^0) = 2\mu_0\) in the third inequality. Consequently, from equation (52) and \((\bar{x}, \bar{y}) > 0\), we have
\[ 0 \leq x_i^k \leq \frac{C_2}{\min_{1 \leq i \leq n} \{\bar{y}_i\}}, \quad 0 \leq y_i^k \leq \frac{C_2}{\min_{1 \leq i \leq n} \{\bar{x}_i\}}, \quad i = 1, 2, \ldots , n, \]
which proves the boundedness of \((x^k, y^k)\). \(\square\)

**Lemma 3.3** Assume that \(\{(x^k, y^k)\}\) is generated by Algorithm 1 and satisfies the condition (38). Then, when \(\mu_k \geq \varepsilon > 0\), there exists a positive constant \(\omega^*\) such that
\[
\|D^k \Delta x^k\|^2 + \|(D^k)^{-1} \Delta y^k\|^2 \leq (\omega^*)^2, \] (53)

holds for all \(k = 0, 1, \ldots \), where \(D^k = \text{diag}((y^k, x^k)^{1/2}), (D^k)^{-1} = \text{diag}((x^k, y^k)^{1/2})\) and \((\Delta x^k, \Delta y^k)\) is the solution of equations (16)-(17).

**Proof.** We denote
\[
\omega_k = \left(\|D^k \Delta x^k\|^2 + \|(D^k)^{-1} \Delta y^k\|^2\right)^{1/2}. \] (54)

Then, from equation (54), we have
\[
\|D^k \Delta x^k\|_\infty \leq \|D^k \Delta x^k\| \leq \omega_k, \quad \|(D^k)^{-1} \Delta y^k\|_\infty \leq \|(D^k)^{-1} \Delta y^k\| \leq \omega_k. \] (55)
From equation (34), we know that \((u^k, v^k)\) is feasible. By combining it with equation (16), we obtain
\[
y^k + \Delta y^k - v^k = M(x^k + \Delta x^k - u^k).
\] (56)

Therefore, from equation (56) and the semi-definite positivity of \(M\), we have
\[
(y^k + \Delta y^k - v^k)^T (x^k + \Delta x^k - u^k) = (x^k + \Delta x^k - u^k)^T M(x^k + \Delta x^k - u^k) \geq 0.
\] (57)

From equations (35)-(36), we know
\[
x^0 - u^0 \geq x^k - u^k, \quad y^0 - v^0 \geq y^k - v^k.
\]

By substituting them into equation (57), we obtain
\[
(x^0 - u^0)^T (y^0 - v^0) + (y^k - v^k)^T \Delta x^k + (x^k - u^k)^T \Delta y^k + (\Delta x^k)^T \Delta y^k \geq 0.
\] (58)

By using the inequality \(|x^T y| \leq ||x|| ||y|| \leq ||x||_1 ||y||\), from equation (55), we have
\[
\left\| (y^k - v^k)^T \Delta x^k \right\| \leq \left\| (D^k)^{-1} (y^k - v^k)^T (D^k \Delta x^k) \right\| \\
\leq \left\| (D^k)^{-1} (y^k - v^k) \right\| \left\| D^k \Delta x^k \right\| \leq \left\| (D^k)^{-1} (y^k - v^k) \right\|_1 \omega_k.
\] (59)

From the condition (38) and \(y^k - v^k \geq 0\), we have
\[
(x_i^k/y_i^k)^{1/2} (y_i^k - v_i^k) = x_i^k (y_i^k - v_i^k)/(x_i^k y_i^k)^{1/2} \leq x_i^k (y_i^k - v_i^k)/(\gamma \mu_k)^{1/2}.
\] (60)

By substituting equation (60) into equation (59), we obtain
\[
\left\| (y^k - v^k)^T \Delta x^k \right\| \leq \left( \sum_{i=1}^n (x_i^k/y_i^k)^{1/2} (y_i^k - v_i^k) \right) \omega_k \\
\leq (x^k)^T (y^k - v^k) \omega_k/(\gamma \mu_k)^{1/2}.
\] (61)

Similarly to the proof of equation (61), we obtain
\[
\left\| (x^k - u^k)^T \Delta y^k \right\| \leq (y^k)^T (x^k - u^k) \omega_k/(\gamma \mu_k)^{1/2}.
\] (62)

By substituting inequalities (47) and (61)-(62) into inequality (58), we obtain
\[
(\Delta x^k)^T \Delta y^k \geq -(x^0 - u^0)^T (y^0 - v^0) - \left\| (y^k - v^k)^T \Delta x^k \right\| - \left\| (x^k - u^k)^T \Delta y^k \right\| \\
\geq -(x^0 - u^0)^T (y^0 - v^0) - ((y^k)^T (x^k - u^k) + (x^k)^T (y^k - v^k)) \omega_k/(\gamma \mu_k)^{1/2} \\
\geq -(x^0 - u^0)^T (y^0 - v^0) - (C_1/(\gamma e)^{1/2}) \omega_k.
\] (63)
From equation (17) and the condition (38), we have
\[ \|D^k \Delta x^k\|^2 + \|D^k \Delta y^k\|^2 + 2(\Delta x^k)^T \Delta y^k = \|D^k \Delta x^k + (D^k)^{-1} \Delta y^k\|^2 \]
\[ = (\sigma_k \mu_k)^2 \sum_{i=1}^{n} \frac{1}{x_i^k} y_i^k + (x^k)^T y^k - 2 \sigma_k \mu_k n \]
\[ \leq n \sigma_k^2 \mu_k / \gamma + \phi(x^k, y^k) + 2 \sigma_k \mu_k n \leq n \mu_0 \left( \sigma_{\text{max}}^2 / \gamma + 2 - 2 \sigma_{\text{min}} \right). \] (64)

By substituting inequality (63) into inequality (64), we obtain
\[ q(\omega_k) \triangleq \omega_k^2 - 2 \left( C_1 / (\gamma \varepsilon)^{1/2} \right) \omega_k - \zeta \leq 0, \] (65)
where the positive constant \( \zeta \) is defined by
\[ \zeta = 2(x^0 - u^0)^T (y^0 - v^0) + n \mu_0 \left( \sigma_{\text{max}}^2 / \gamma + 2 - 2 \sigma_{\text{min}} \right). \] (66)
The quadratic function \( q(\omega) \) is convex and has a unique positive root at
\[ \omega^* = C_1 / (\gamma \varepsilon)^{1/2} + \sqrt{C_1^2 / (\gamma \varepsilon) + \zeta}. \]
This implies
\[ \omega_k \leq \omega^*. \] (67)
Consequently, we obtain inequality (53). \( \square \)

In order to prove the global convergence of Algorithm 1, we need to estimate the positive lower bound of \( \Delta t_k (k = 0, 1, \ldots) \).

**Lemma 3.4** Assume that \( \{(x^k, y^k)\} \) is generated by Algorithm 1 and satisfies the condition (38). Then, when \( \mu_k \geq \varepsilon > 0 \), there exists a positive constant \( \delta_{\Delta t} \) such that
\[ \Delta t_k \geq \frac{1}{2} \delta_{\Delta t} > 0 \] (68)
holds for all \( k = 0, 1, \ldots \).

**Proof.** From inequality (53), we have
\[ |\Delta x^k \Delta y^k| = \left| (x^k / y^k)^{1/2} \Delta x^k \right| \left| (y^k / x^k)^{1/2} \Delta y^k \right| \leq \|D^k \Delta x^k\| \|D^k \Delta y^k\| \]
\[ \leq \frac{1}{2} \omega_k^2 \leq \frac{1}{2} (\omega^*)^2, \quad i = 1, 2, \ldots, n, \]
which gives
\[ \|\Delta X^k \Delta y^k\|_\infty \leq \frac{1}{2} (\omega^*)^2, \quad k = 0, 1, 2, \ldots. \] (69)
Consequently, we have
\[ \frac{\sigma_k \mu_k}{\|\Delta X^k \Delta y^k\|_\infty} \geq \frac{2 \sigma_{\text{min}} \varepsilon}{(\omega^*)^2}, \quad k = 0, 1, 2, \ldots. \] (70)
We denote
$$\alpha^*_\mu = \min \left\{ 1, \frac{1 - \gamma/2}{1 + \gamma/2} \left( \frac{2\sigma_{\text{min}} \varepsilon}{\omega^*} \right)^2 \right\}. \quad (71)$$

Then, when $\Delta t_k$ satisfies
$$\Delta t_k \leq \frac{\alpha^*_\mu}{1 - \alpha^*_\mu}, \quad (72)$$
from equations (70)-(71) and (42), we know that the condition (38) holds, where
$$\alpha_k = \Delta t_k / (1 + \Delta t_k).$$

From equation (53) and the Cauchy-Schwartz inequality $|x^Ty| \leq \|x\| \|y\|$, we have
$$|(\Delta x^k)^T \Delta y^k| = |(D^k \Delta x^k)^T ((D^k)^{-1} \Delta y^k)| \leq \|D^k \Delta x^k\| \|((D^k)^{-1} \Delta y^k)\|$$
$$\leq \frac{1}{2} \left( \|D^k \Delta x^k\|^2 + \|((D^k)^{-1} \Delta y^k)\|^2 \right) \leq \frac{1}{2} (\omega^*)^2. \quad (73)$$

From equation (17), we have
$$\langle x^k \rangle^T \Delta y^k + \langle y^k \rangle^T \Delta x^k = n\sigma_k \mu_k - \langle x^k \rangle^T y^k. \quad (74)$$

By substituting equations (73)-(74) into equation (26), we obtain
$$\rho_k = \frac{\Lambda_{\text{red}}}{\Lambda_{\text{red}}} = 1 - \frac{\alpha_k (\Delta x^k)^T \Delta y^k}{\|r^k_o\|^2 + \langle x^k \rangle^T y^k - n\sigma_k \mu_k}$$
$$= 1 - \frac{\alpha_k (\Delta x^k)^T \Delta y^k}{2 - \sigma_k \mu_k} \geq 1 - \frac{(\omega^*)^2}{2(2 - \sigma_{\text{max}})} \varepsilon \alpha_k. \quad (75)$$

We denote
$$\alpha^*_\rho = \min \left\{ 1, \frac{2(2 - \sigma_{\text{max}})(1 - \eta_2) \varepsilon}{(\omega^*)^2} \right\}. \quad (76)$$

Then, when $\Delta t_k$ satisfies
$$\Delta t_k \leq \frac{\alpha^*_\rho}{1 - \alpha^*_\rho}, \quad (77)$$
from equations (75)-(76), we know $\rho_k \geq \eta_2$, where
$$\alpha_k = \Delta t_k / (1 + \Delta t_k).$$

We denote
$$\delta_{\Delta t} \triangleq \min \left\{ \frac{\sigma^*_\mu}{1 - \sigma^*_\mu}, \frac{\alpha^*_\rho}{1 - \alpha^*_\rho}, \Delta t_0 \right\}. \quad (78)$$

Assume that $K$ is the first index such that $\Delta t_K \leq \delta_{\Delta t}$ holds. Then, from equations (72)-(78), we know that the condition (38) holds and $\rho_K \geq \eta_2$. Consequently, according to the time-stepping adjustment scheme (27), $\Delta t_{K+1}$ will be enlarged. Therefore, $\Delta t_k \geq (1/2)\delta_{\Delta t}$ holds for all $k = 0, 1, 2, \ldots$.

By using the estimation results of Lemma 3.4, we prove that $\{\mu_k\}$ converges to zero when $k$ tends to infinity as follows.
Theorem 3.1 Assume that \( \{(x^k, y^k)\} \) is generated by Algorithm 1 and satisfies the condition (38). Then, we have
\[
\lim_{k \to \infty} \mu_k = 0. \tag{79}
\]

Proof. We prove the result (79) by contradiction. Assume that there exists a positive constant \( \varepsilon \) such that
\[
\mu_k \geq \varepsilon > 0 \tag{80}
\]
holds for all \( k = 0, 1, \ldots \). Then, according to Algorithm 1 and Lemma 3.4, we know that there exists an infinite subsequence \( \{(x^{k_l}, y^{k_l})\} \) such that
\[
\phi(x^{k_l}, y^{k_l}) - \phi(x^{k_l} + \alpha_{k_l} \Delta x^{k_l}, y^{k_l} + \alpha_{k_l} \Delta y^{k_l}) \geq \eta_1 \tag{81}
\]
holds for all \( l = 0, 1, 2, \ldots \), where \( \alpha_{k_l} = \Delta t_{k_l} / (1 + \Delta t_{k_l}) \). Otherwise, all steps are rejected after a given iteration index, then the time step will keep decreasing, which contradicts equation (68).

From equations (68), (75) and (81), we have
\[
\phi(x^{k_l}, y^{k_l}) - \phi(x^{k_l} + \alpha_{k_l} \Delta x^{k_l}, y^{k_l} + \alpha_{k_l} \Delta y^{k_l}) \\
\geq \frac{\eta_1 \Delta t_{k_l}}{1 + \Delta t_{k_l}} (2 - \sigma_{k_l}) \eta \mu_{k_l} \geq \frac{\eta_1 \delta_t / 2}{1 + \delta_t / 2} (2 - \sigma_{\max}) \eta \mu_{k_l}. \tag{82}
\]

Therefore, from equation (82) and \( \phi(x^{k+1}, y^{k+1}) \leq \phi(x^k, y^k) \), we have
\[
\phi(x^0, y^0) \geq \phi(x^0, y^0) - \lim_{k \to \infty} \phi(x^k, y^k) = \sum_{k=0}^{\infty} \left( \phi(x^k, y^k) - \phi(x^{k+1}, y^{k+1}) \right) \\
\geq \sum_{l=0}^{\infty} \left( \phi(x^{k_l}, y^{k_l}) - \phi(x^{k_l+1}, y^{k_l+1}) \right) \geq \frac{\eta_1 \delta_t / 2}{1 + \delta_t / 2} \sum_{l=0}^{\infty} (2 - \sigma_{\max}) \eta \mu_{k_l}. \tag{83}
\]

Consequently, from equation (83), we obtain
\[
\lim_{l \to \infty} \mu_{k_l} = 0,
\]
which contradicts the assumption \( \mu_k \geq \varepsilon > 0 \) for all \( k = 0, 1, \ldots \). Therefore, we have
\[
\lim_{k \to \infty} \inf \mu_k = 0. \tag{84}
\]

Since \( \mu_k = \phi(x^k, y^k) / (2n) \) and \( \{\phi(x^k, y^k)\} \) is monotonically decreasing, we know that \( \{\mu_k\} \) is monotonically decreasing. By combining it with equation (84), we know that the result (79) is true. \( \square \)
Remark 3.1 In the analysis framework of the global convergence, in comparison to that of the known path-following algorithm, we do not need to select $\alpha_k$ such that the condition of the priority to feasibility over complementarity
\[
\|r^k(\alpha)\| \leq (x^k(\alpha))^T y^k(\alpha) \frac{\|y^0\|}{(x^0)^T y^0}
\]  \tag{85}
holds for all $\alpha \in (0, \alpha_k] \subset (0, 1]$.

Theorem 3.2 Assume that $\{(x^k, y^k)\}$ is the sequence of iterations generated by Algorithm 1 and satisfies the condition (38). If the linear complementarity problem has a strictly feasible solution $(\bar{x}, \bar{y})$, $\{(x^k, y^k)\}$ exists a limit point $(x^*, y^*)$ and this limit point satisfies the linear complementarity equation (1).

Proof. Since the linear complementarity problem has a strictly feasible solution, from Lemma 3.2, we know that $\{(x^k, y^k)\}$ is bounded. Consequently, the sequence $\{(x^k, y^k)\}$ exists a convergence subsequence $\{(x^{k_l}, y^{k_l})\}$ and we denote its limit point as $(x^*, y^*)$. By combining it with Theorem 3.1, we obtain
\[
\lim_{l \to \infty} \mu_{k_l} = \frac{1}{2n} \left( \lim_{l \to \infty} (x^{k_l})^T y^{k_l} + \lim_{l \to \infty} \|y^{k_l} - (Mx^{k_l} + q)\| \right) = \frac{1}{2n} \left( (x^*)^T y^* + \|y^* - (Mx^* + q)\| \right) = 0. \tag{86}
\]
Since $(x^{k_l}, y^{k_l}) \geq 0$, we have $(x^*, y^*) \geq 0$. By substituting it into equation (86), we conclude
\[
y^* - (Mx^* + q) = 0, \quad (x^*)^T y^* = 0, \quad (x^*, y^*) \geq 0.
\]
It implies that $(x^*, y^*)$ is the solution of the linear complementarity problem (1). \hfill \square

4 Numerical experiments

In this section, some numerical experiments are conducted to test the performance of RPFMTr (Algorithm 1) for the linear complementarity problems (LCPs), in comparison to two state-of-the-art commercial solvers, i.e. PATH [15, 18, 19, 48] and MILES (a Mixed Inequality and nonLinear Equation Solver) [43, 44, 50]. RPFMTr (Algorithm 1) is coded with the MATLAB language and executed in MATLAB (R2020a) environment [42]. PATH and MILES are executed in the GAMS v28.2 (2019) environment [21]. All numerical experiments are performed by a HP notebook with the Intel quadcore CPU and 8Gb memory. MILES solves the LCP based on the Lemke’s almost complementary pivoting algorithm [8, 32, 50]. PATH is a solver based on the path-following procedure [14, 15], the Fisher’s non-smooth regularization technique [20] and the Lemke’s pivoting algorithm [8, 32]. PATH and MILES are two robust and efficient solvers for the complementarity problems and used in many general modelling systems such as AMPL (A Mathematical Programming Language) [1]
and GAMS (General Algebraic Modelling System) [21]. Therefore, we select these
two solvers as the basis for comparison.

The construction approach of test problems is described as follows. Firstly, we
generate a test matrix $M$ with the following structure:

$$M = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix},$$  
(87)

where the test matrix $A$ comes from the linear programming subset of NETLIB [45].
It is easy to verify that the test matrix $M$ defined by equation (87) is semi-definite
positive. Then, we generate two complementarity vectors $x$ and $y$ as follows:

$$x = [1 \ 0 \ 1 \ 0 \ \cdots \ \cdot \ 1 \ 0]^T, \quad y = [0 \ 1 \ 0 \ 1 \ \cdots \ \cdot \ 0 \ 1]^T.$$  
(88)

Finally, by using these two vectors $x, y$ defined by equation (88) and the test matrix
$M$ defined by equation (87), we generate the following test vector $q$:

$$q = y - Mx.$$  
(89)

The scales of test problems vary from dimension 78 to 40216. And the termination
conditions of Algorithm 1 (RPFMTr), PATH and MILES for finding a solution of the
nonlinear system (2) are

$$\|y - (Mx + q)\|_\infty \leq 10^{-6}, \quad \|Xy\|_\infty \leq 10^{-6}, \quad (x, y) \geq 0,$$

where $X = \text{diag}(x)$.

According to the manual of GAMS [19], an LCP will be generated a list of all
variables appearing in the equations found in the model statement, and the number
of equations equals the number of variables. In other words, for an LCP solved in the
GAMS environment, its matrix $M$ should not include the row with all zeros. In order
to avoid this error, we add $\varepsilon = 1.0 \times 10^{-6}$ to the first element of the row or column
with all zeros in the test sub-matrix $A$ of matrix $M$ defined by equation (87).

Most of matrices $A$ coming from the linear programming subset of NETLIB [45]
are sparse. In order to test the performance of RPFMTr, PATH and MILES for the
dense LCP further, we add a small random perturbation to the elements of the sub-
matrix $A$ in equation (87) and generate the dense test matrix $M$ to replace the test
matrix $M$ in equation (87) as follows:

$$\overline{M} = \begin{bmatrix} 0 & -A^T \\ A\varepsilon & 0 \end{bmatrix}, \quad A\varepsilon = A + \text{rand}(m_A, n_A) \ast \varepsilon, \quad [m_A, n_A] = \text{sizes}(A),$$  
(90)

where $\varepsilon = 10^{-3}$ and matrix $A$ comes from the linear programming subset of NETLIB
[45].

Numerical results of the sparse test problems are arranged in Tables 1-3 and Fig-
ures 1-2. And numerical results of the dense test problems are arranged in Tables 4-6.
and Figures 3-4. “major” in the fourth column of Tables 1-6 represents the number of
the linear mixed complementarity problems solved by a pivotal Lemke’s method of
PATH [19]. “minor” in the fourth column of Tables 1-6 represents the number of
pivots performed per major iteration of PATH [19]. “grad” in the fourth column of Ta-
bles 1-6 represents the cumulative number of Jacobian evaluations used in PATH [19].
“major” in the sixth column of Tables 1-6 represents the number of Newton iterations
of MILES [44]. “pivots” in the sixth column of Tables 1-6 represents the number of
Lemke pivots of MILES [44]. “refactor” in the sixth column of Tables 1-6 represents
the number of re-factorizations in the LUSOL solver [23], which is called by MILES
for solving the linear systems of equations.

From Tables 1-6 and Figures 1-4, we find that RPFMTr and PATH work well for
the sparse LCPs and the dense LCPs. However, MILES is not robust for solving the
sparse LCPs or the dense LCPs. For the sparse LCPs, PATH performs better than
RPFMTr and MILES from Tables 1-3 and Figures 1-2. For the dense LCPs, from
Tables 4-6, we find that PATH and MILES fail to solve 5 problems and 57 problems
of 73 test problems, respectively. RPFMTr solves all the sparse test LCPs and the
dense test LCPs. Furthermore, from Tables 4-6 and Figures 3-4, we find that the
computational time of RPFMTr is about 1/2 to 1/10 of that of PATH for the dense
LCPs. Therefore, RPFMTr is a robust and efficient solver for the LCPs, especially
for the dense LCPs.
Fig. 1: Iterations of RPFMTr, PATH and MILES for sparse LCPs.

Fig. 2: CPU (s) of RPFMTr, PATH and MILES for sparse LCPs.
## Table 1: Numerical results of small-scale sparse LCPs (no. 1-27).

| Problems (n*a) | RPMTY | PATH | MILES |
|---------------|-------|------|-------|
|               | steps (time) | Terr | major+minor+grad (time) | Terr | major+spos+refactor (time) | Terr |
| Exam. 1 lp250-47 (n = 2697) | 49 (4.13) | 8.84e-07 | 8+949+23 (0.30) | 5.82e-08 | 1+1684+24 (1.34) | 7.48e-12 |
| Exam. 2 lp_additive (n = 194) | 45 (0.03) | 9.61e-07 | 4+80+9 (0.02) | 4.44e-12 | 1+1+3 (0.02) | 4.65e-14 |
| Exam. 3 lp_diffo (n = 78) | 41 (0.01) | 6.20e-07 | 3+5+7 (0.02) | 1.01e-10 | 1+1+3 (0.00) | 3.91e-14 |
| Exam. 4 lp_egg (n = 1103) | 44 (0.19) | 9.85e-07 | 11+413+16 (0.08) | 2.87e-09 | 1+1+6 (0.05) | 3.74e-12 |
| Exam. 5 lp_egg2 (n = 1274) | 46 (0.84) | 8.24e-07 | 10+303+13 (0.03) | 6.96e-10 | 1+959+6 (0.06) | 5.08e-12 |
| Exam. 6 lp_egg3 (n = 1274) | 47 (0.50) | 6.64e-07 | 9+239+12 (0.05) | 2.41e-07 | 1+959+6 (0.06) | 9.37e-12 |
| Exam. 7 lp_bundm (n = 777) | 51 (0.27) | 5.85e-07 | 11+613+14 (0.05) | 3.31e-07 | 1+1+8 (0.06) | 1.19e-12 |
| Exam. 8 lp_beacond (n = 468) | 50 (0.17) | 6.06e-07 | 8+124+17 (0.03) | 3.76e-09 | 1+1+6 (0.03) | 5.46e-12 |
| Exam. 9 lp_blend (n = 188) | 46 (0.03) | 6.12e-07 | 9+143+12 (0.02) | 6.50e-08 | 1+1+4 (0.02) | 1.96e-12 |
| Exam. 10 lp_buil1 (n = 2229) | 50 (1.24) | 8.60e-07 | 12+727+17 (0.50) | 5.22e-11 | 1+1+10 (0.27) | 1.62e-12 |
| Exam. 11 lp_borec3d (n = 967) | 48 (0.13) | 8.63e-07 | 11+423+16 (0.03) | 6.99e-07 | 1+1+6 (0.03) | 1.31e-12 |
| Exam. 12 lp_brandy (n = 523) | 49 (0.31) | 7.42e-07 | 10+384+13 (0.06) | 2.84e-10 | 1+1+6 (0.03) | 1.31e-12 |
| Exam. 13 lp_capri (n = 753) | 46 (0.16) | 9.07e-07 | 11+573+15 (0.03) | 7.18e-07 | 1+1+6 (0.05) | 2.60e-12 |
| Exam. 14 lp_copoly (n = 4491) | 51 (3.37) | 8.74e-07 | 4+25+14 (0.05) | 1.90e-10 | 1+1+13 (1.19) | 1.82e-12 |
| Exam. 15 lp_degen2 (n = 1201) | 42 (0.60) | 8.53e-07 | 4+447+13 (0.11) | 1.71e-07 | 1+1+10 (0.14) | 2.22e-16 |
| Exam. 16 lp_degen3 (n = 4107) | 47 (5.85) | 6.85e-07 | 4+185+26 (0.42) | 3.00e-07 | 1+1+1 (0.13) | 0.00e-00 |
| Exam. 17 lp_k226 (n = 695) | 54 (0.28) | 9.05e-07 | 9+412+17 (0.03) | 2.94e-07 | 1+1+7 (0.05) | 5.86e-12 |
| Exam. 18 lp_etamacro (n = 1216) | 49 (0.73) | 6.31e-07 | 6+440+15 (0.05) | 1.87e-12 | 1+58+6 (0.05) | 6.82e-12 |
| Exam. 19 lp_fBB2000 (n = 1552) | 64 (1.27) | 6.05e-07 | 15+1111+20 (0.17) | 1.39e-07 | 100+72624+770 (13.80) | 1.78e+01 (failed) |
| Exam. 20 lp_finnis (n = 1561) | 47 (0.26) | 6.06e-07 | 8+313+16 (0.03) | 5.24e-09 | 1+1+9 (0.14) | 4.55e-13 |
| Exam. 21 lp_Jr1d (n = 1073) | 65 (1.16) | 6.45e-07 | 8+671+12 (0.24) | 2.34e-11 | 1+1+3 (0.03) | 1.75e-10 |
| Exam. 22 lp_gamgabgs (n = 3015) | 43 (0.49) | 8.88e-07 | 4+98+21 (0.05) | 4.95e-07 | 1+1+11 (0.47) | 2.40e-14 |
| Exam. 23 lp_gifld_pmc (n = 1776) | 61 (0.28) | 7.85e-07 | 18+1250+21 (0.17) | 3.80e-11 | 1+776+13 (0.25) | 3.64e-12 |
| Exam. 24 lp_grow15 (n = 945) | 33 (0.24) | 7.23e-07 | 7+703+9 (0.13) | 8.66e-07 | 1+927+6 (0.08) | 3.33e-14 |
| Exam. 25 lp_grow22 (n = 1386) | 33 (0.42) | 7.27e-07 | 8+1054+10 (0.23) | 3.86e-11 | 1+1403+8 (0.13) | 1.20e-14 |
| Exam. 26 lp_joth (n = 519) | 52 (0.20) | 6.05e-07 | 13+263+17 (0.03) | 8.42e-07 | 1+1+4 (0.05) | 9.10e-13 |
| Exam. 27 lp_matusos (n = 2812) | 68 (2.80) | 6.18e-07 | 13+3222+14 (0.50) | 2.16e-09 | 1+776+20 (1.13) | 2.95e-09 |
Table 2: Numerical results of small-scale sparse LCPs (no. 28-53).

| Problems (n×n) | RPFMTTr | PATH | MILES |
|----------------|---------|------|-------|
|                | steps (time) | major+minor+grad (time) | major+pivots+refactor (time) | Terr |
| Exam. 28 lp_modsc1 (n = 2307) | 42 (0.46) | 5×10^{-7} (0.05) | 1×10^{-9} (0.22) | 8.70e-14 |
| Exam. 29 lp_pentrol (n = 2131) | 67 (2.62) | 12×10^{-8} (0.42) | 1×10^{-9} (0.44) | failed |
| Exam. 30 lp_pilot4a (n = 3207) | 78 (7.43) | 14×10^{-8} (1.81) | 1×10^{-9} (1.11) | (35.88) |
| Exam. 31 lp_app8 (n = 2544) | 43 (4.78) | 9×10^{-7} (1.31) | 1×10^{-9} (27.30) | 2.13e-14 |
| Exam. 32 lp_recipe (n = 295) | 53 (0.04) | 8×10^{-3} (0.00) | 1×10^{-9} (0.00) | 5.12e-13 |
| Exam. 33 lp_clc50u (n = 128) | 40 (0.01) | 3×10^{-4} (0.05) | 1×10^{-3} (0.00) | 1.33e-15 |
| Exam. 34 lp_chagr7 (n = 314) | 42 (0.03) | 1×10^{-2} (0.02) | 1×10^{-1} (0.00) | 1.07e-14 |
| Exam. 35 lp_chagr25 (n = 1142) | 42 (0.08) | 4×10^{-3} (0.02) | 1×10^{-2} (0.00) | 1.07e-14 |
| Exam. 36 lp_cxfm1 (n = 930) | 52 (0.51) | 1×10^{-2} (0.00) | 1×10^{-2} (0.00) | 2.62e-12 |
| Exam. 37 lp_cxfm2 (n = 1860) | 52 (0.61) | 1×10^{-2} (0.00) | 1×10^{-2} (0.00) | 1.00e-11 |
| Exam. 38 lp_cxfm3 (n = 2790) | 53 (1.14) | 1×10^{-2} (0.27) | 1×10^{-2} (0.70) | 7.38e-12 |
| Exam. 39 lp_chagrop (n = 854) | 40 (0.40) | 5×10^{-1} (0.03) | 1×10^{-1} (0.00) | 6.22e-15 |
| Exam. 40 lp_shell (n = 2313) | 45 (0.40) | 9×10^{-4} (0.38) | 1×10^{-1} (0.24) | 0.00e-00 |
| Exam. 41 lp_chap04l (n = 2568) | 50 (1.06) | 1×10^{-2} (0.00) | 1×10^{-2} (0.00) | 1.81e-14 |
| Exam. 42 lp_chap06s (n = 1998) | 49 (0.57) | 1×10^{-2} (0.24) | 1×10^{-2} (0.00) | 1.81e-14 |
| Exam. 43 lp_chap08s (n = 3245) | 47 (0.91) | 1×10^{-2} (0.44) | 1×10^{-2} (0.41) | 5.68e-14 |
| Exam. 44 lp_chap12s (n = 4020) | 48 (0.91) | 1×10^{-2} (0.52) | 1×10^{-2} (0.53) | 4.17e-14 |
| Exam. 45 lp_vsierra (n = 3962) | 63 (1.35) | 1×10^{-2} (0.53) | 1×10^{-2} (0.72) | 4.37e-11 |
| Exam. 46 lp_standgub (n = 1744) | 60 (0.70) | 4×10^{-1} (0.02) | 1×10^{-1} (0.00) | 1.16e-13 |
| Exam. 47 lp_lutf (n = 995) | 61 (0.69) | 9×10^{-3} (0.15) | 1×10^{-2} (0.11) | 3.82e-11 |
| Exam. 48 lp_wood1p (n = 2839) | 53 (6.00) | 7×10^{-3} (0.15) | 1×10^{-2} (0.16) | 5.08e-09 |
| Exam. 49 lp_box1 (n = 492) | 39 (0.03) | 7×10^{-3} (0.02) | 1×10^{-3} (0.02) | 1.11e-16 |
| Exam. 50 lp_aplex2 (n = 602) | 41 (0.07) | 9×10^{-3} (0.03) | 1×10^{-3} (0.03) | 4.44e-15 |
| Exam. 51 lp_box72a (n = 412) | 40 (0.03) | 7×10^{-4} (0.00) | 1×10^{-4} (0.00) | 0.00e-00 |
| Exam. 52 lp_box30a (n = 404) | 40 (0.03) | 6×10^{-3} (0.00) | 1×10^{-3} (0.00) | 0.00e-00 |
| Exam. 53 lp_mondou2 (n = 916) | 38 (0.09) | 6×10^{-3} (0.08) | 1×10^{-3} (0.03) | 0.00e-00 |
### Table 3: Numerical results of large-scale sparse LCPs (no. 54–78).

| Problems (n*a) | RPFMTr | PATH | MILES |
|---------------|--------|------|-------|
|               | steps (time) | major+minor+grad (time) | max+pr+refactor (time) |
| Exam. 54 lp_3080l3 | 49 (7.56) | 9.4588×10^{-3} (0.30) | 1.52e-10 | 414.101144×10^{-3} (1015.16) | 6.19e+01 (failed) |
| Exam. 55 lp_bul2 (n = 6810) | 47 (4.83) | 12.4597×10^{-1} (1.47) | 2.72e-11 | 1+1+27 (6.84) | 1.24e-12 |
| Exam. 56 lp_crc (n = 10764) | 53 (5.14) | 12.5755×10^{-1} (2.22) | 4.09e-07 | 1+1+42 (22.98) | 1.48e-12 |
| Exam. 57 lp_crc (n = 9479) | 50 (4.50) | 11.3604×10^{-1} (1.34) | 1.77e-09 | 1.496×10^{-1} (18.56) | 1.59e-12 |
| Exam. 58 lp_cycle (n = 5274) | 59 (5.21) | 17.4532×10^{-2} (2.58) | 8.67e-09 | 100+1.976×10^{6} (461.45) | 7.13e+01 (failed) |
| Exam. 59 lp_6cube (n = 6599) | 56 (48.11) | 5.941×10^{-1} (0.53) | 7.37e-07 | 1+1+3 (10.75) | 2.73e-12 |
| Exam. 61 lp_greenea (n = 7990) | 50 (4.27) | 12.3899×10^{-1} (2.91) | 1.01e-09 | 1000+7.407×10^{4} (513.08) | 9.90e+01 (failed) |
| Exam. 62 lp_greeneb (n = 7990) | 50 (14.48) | 12.3899×10^{-1} (2.91) | 1.01e-09 | 1000+7.407×10^{4} (514.13) | 9.90e+01 (failed) |
| Exam. 63 lp_kas_07 (n = 6028) | 42 (1.09) | 8.4404×10^{-1} (0.98) | 9.71e-09 | 1+1+21 (3.31) | 0.00e-00 |
| Exam. 64 lp_kas_02 (n = 10699) | 38 (2.66) | 5.326×10^{-1} (3.27) | 7.09e-09 | 1+1+47 (24.70) | 0.00e-00 |
| Exam. 65 lp_kas_07 (n = 8710) | 59 (59.97) | 10.3082×10^{-1} (2.30) | 6.47e-07 | 1000+1.21×10^{5} (498.08) | 1.00e+01 (failed) |
| Exam. 66 lp_kas_12 (n = 12048) | 45 (223.00) | 7.9138×10^{-1} (205.47) | 8.54e-07 | 124+1.594×10^{4} (1064.42) | 1.00e+01 (failed) |
| Exam. 67 lp_kas_08 (n = 5141) | 48 (1.87) | 10.1676×10^{-1} (0.80) | 5.38e-07 | 1+1+12 (1.31) | 5.68e-14 |
| Exam. 68 lp_kas_12 (n = 6684) | 47 (1.88) | 14.8950×10^{-1} (4.24) | 1.22e-10 | 1+1+16 (3.03) | 4.17e-14 |
| Exam. 69 lp_gauss (n = 4086) | 44 (4.72) | 6.5907×10^{-1} (6.92) | 4.99e-08 | 98+2.995×10^{5} (1011.18) | 2.66e+00 (failed) |
| Exam. 70 lp_woodw (n = 9516) | 85 (201.33) | 10.4715×10^{-1} (9.89) | 1.78e-09 | 100+2.32×10^{4} (735.08) | 2.00e+01 (failed) |
| Exam. 71 lp_bndy (n = 15551) | 44 (50.77) | 4.614×10^{-1} (1.28) | 1.20e-07 | 1+1+39 (34.14) | 9.06e-08 |
| Exam. 72 lp_gran (n = 5183) | 81 (7.50) | 15.1988×10^{-1} (1.27) | 8.14e-07 | 1000+7.03×10^{6} (74.33) | 5.68e+01 (failed) |
| Exam. 73 lp_greenea (n = 7989) | 50 (11.91) | 11.3173×10^{-1} (2.94) | 4.68e-07 | 1000+3.03×10^{6} (921.31) | 6.72e+01 (failed) |
| Exam. 74 lp_kas_11 (n = 36043) | 45 (16.37) | 9+2.83×10^{-1} (1.72) | 2.65e-08 | 1+1+42 (1081.75) | 0.00e-00 |
| Exam. 75 lp_os_07 (n = 26185) | 45 (451.93) | 4+9.54×10^{-1} (2.64) | 1.77e-11 | 1+1+23 (78.39) | 3.87e-09 |
| Exam. 76 lp_gds_06 (n = 39322) | 45 (115.16) | 5+6.809×10^{-1} (43.47) | 9.00e-11 | 1+1+240 (1885.67) | 0.00e-00 |
| Exam. 77 lp_gds_15 (n = 28605) | 46 (824.18) | 5+20.05×10^{-1} (1800.75) | 2.11e+01 (1410.55) | 1.20e+01 (failed) |
| Exam. 78 lp_nockl3 (n = 40216) | 54 (11.08) | 21+6.23×10^{-1} (5.69) | 2.78e-12 | 2+47.76×10^{-1} (1169.09) | 5.14e+01 (failed) |
Fig. 3: Iterations of RPFMTr, PATH and MILES for dense LCPs.

Fig. 4: CPU (s) of RPFMTr, PATH and MILES for dense LCPs.
| Problems (n*a) | PFMTr | PATH | MILES |
|---------------|-------|------|-------|
| Exam. 1 lp | 56 (16.83) | 1243457+22 (200.42) | 100621000+600 (132.30) | 3.56e+02 (failed) |
| Exam. 2 lp litt | 43 (0.26) | 8+203+10 (0.03) | 1+327+3 (0.02) | 1.38e-12 |
| Exam. 3 lp litt | 40 (0.03) | 9+73+11 (0.03) | 1+77+2 (0.08) | 2.22e-14 |
| Exam. 4 lp litt | 53 (2.78) | 123+248+17 (5.20) | 10026000+600 (47.64) | 1.43e+00 (failed) |
| Exam. 5 lp litt | 54 (2.80) | 10+1071+13 (4.61) | 1+132+8 (3.30) | 3.58e-09 |
| Exam. 6 lp litt | 44 (1.64) | 10+1053+13 (4.58) | 1+1335+8 (2.25) | 3.42e-10 |
| Exam. 7 lp litt | 51 (0.78) | 1+46+64+17 (0.45) | 9.99e-07 | 1.76e+02 (failed) |
| Exam. 8 lp litt | 44 (0.03) | 13+258+15 (0.03) | 4.73e-08 | 1.83e-09 |
| Exam. 9 lp litt | 52 (9.88) | 13+308+33 (56.92) | 1.65e-09 | 7.76e-01 |
| Exam. 10 lp litt | 50 (0.40) | 13745+18 (0.81) | 8.51e-08 | 1.05e-11 |
| Exam. 11 lp litt | 53 (0.34) | 13+697+16 (0.53) | 2.73e-07 | 7.27e-01 |
| Exam. 12 lp litt | 52 (0.75) | 10+761+19 (1.45) | 2.37e-09 | 2.10e+01 |
| Exam. 13 lp litt | 44 (42.87) | 9+3102+16 (650.25) | 3.36e-07 | 5.33e+00 (failed) |
| Exam. 14 lp litt | 49 (2.32) | 120+1582+14 (7.38) | 2.83e-07 | 2.47e-09 |
| Exam. 15 lp litt | 45 (35.17) | 9+4003+30 (388.39) | 9.91e-07 | 9.50e+01 (failed) |
| Exam. 16 lp litt | 55 (0.65) | 14+953+17 (1.14) | 1.45e-10 | 3.32e+01 |
| Exam. 17 lp litt | 51 (2.46) | 7+640+18 (6.08) | 5.85e-07 | 2+120+9 (1.17) | 1.36e-08 |
| Exam. 18 lp litt | 70 (5.76) | 14+2172+21 (15.77) | 2.09e-08 | 1.09e-05 (failed) |
| Exam. 19 lp litt | 46 (3.87) | 10+1583+18 (14.41) | 1.92e-07 | 5.84e+01 (failed) |
| Exam. 20 lp litt | 54 (2.09) | 8+681+13 (0.53) | 3.46e-10 | 1+152+4 (0.06) | 5.64e-09 |
| Exam. 21 lp litt | 40 (14.89) | 8+1910+28 (111.67) | 6.15e-08 | 8+10446+147 (1003.45) | 9.53e+00 |
| Exam. 22 lp litt | 64 (7.15) | 14+16640+20 (40.06) | 2.69e-09 | 1.26e+03 |
| Exam. 23 lp litt | 42 (1.17) | 8+856+10 (2.91) | 9.04e-13 | 1+141 (3.02) | 1.44e-09 |
| Exam. 24 lp litt | 41 (2.65) | 9+153+11 (12.94) | 4.57e-08 | 100+2600+400 (18.05) | 1.66e+01 (failed) |
| Exam. 25 lp litt | 46 (0.29) | 11+408+18 (0.48) | 3.33e-11 | 100+47400+500 (13.56) | 4.44e-02 |
| Exam. 26 lp litt | 69 (21.44) | 15+4178+17 (193.91) | 1.04e-08 | 100+14400+300 (35.28) | 2.35e+04 (failed) |
Table 5: Numerical results of small-scale dense LCPs (no. 28-53).

| Problems (n*a) | RPFTMTri (steps (time)) | PATH (major/minorgrad (time)) | MILES (major/primary-refactor (time)) |
|---------------|-------------------------|-------------------------------|--------------------------------------|
| Exam. 28 lph_modock1 (n = 2307) | 46 (9.37) | 13*3281*16 (89.69) | 100*34000*500 (33.89) | 9.03e+00 (failed) |
| Exam. 29 lph_pertold (n = 2131) | 60 (10.21) | 16*2738*24 (66.41) | 100*4200*400 (16.06) | 2.88e+04 (failed) |
| Exam. 30 lph_pilotja (n = 3207) | 76 (32.29) | 18*4063*29 (272.11) | 100*71396*779 (233.45) | 5.77e+06 (failed) |
| Exam. 31 lph_dip8 (n = 2544) | 42 (10.69) | 12*3423*14 (116.20) | 100*23000*400 (29.77) | 1.18e+01 (failed) |
| Exam. 32 lph_recipe (n = 295) | 42 (0.09) | 10*2401*13 (0.16) | 100*5144*595 (4.11) | 1.91e+01 (failed) |
| Exam. 33 lph_ec50u (n = 128) | 42 (0.02) | 5*92*7 (0.02) | 1+135*2 (0.02) | 7.11e-14 |
| Exam. 34 lph_ecagr7 (n = 314) | 41 (0.07) | 12*3911*14 (0.16) | 2+452*8 (0.11) | 1.77e-13 |
| Exam. 35 lph_ecagr25 (n = 1142) | 41 (1.72) | 9*646*19 (3.58) | 100*53500*700 (33.66) | 1.09e+01 (failed) |
| Exam. 36 lph_cfx1m1 (n = 930) | 54 (1.37) | 12*1260*16 (2.67) | 100*102400*1462 (75.61) | 2.23e+02 (failed) |
| Exam. 37 lph_cfx2m2 (n = 1860) | 58 (7.08) | 15*2686*19 (29.86) | 100*104383*802 (94.95) | 2.35e+02 (failed) |
| Exam. 38 lph_cfx3m3 (n = 2790) | 61 (18.68) | 13*3868*17 (135.97) | 100*27200*500 (46.27) | 3.08e+02 (failed) |
| Exam. 39 lph_cropton (n = 854) | 40 (0.79) | 13*1094*15 (2.64) | 100*65472*800 (31.31) | 1.38e+00 (failed) |
| Exam. 40 lph_shell (n = 2133) | 50 (10.06) | 13*3371*15 (85.95) | 100*56400*497 (45.75) | 1.17e+02 (failed) |
| Exam. 41 lph_ship04l (n = 2506) | 42 (10.56) | 12*4243*15 (96.83) | 100*31600*500 (61.70) | 1.51e+02 (failed) |
| Exam. 42 lph_ship04s (n = 1908) | 42 (5.50) | 12*3170*15 (32.59) | 100*20000*500 (32.59) | 5.69e+01 (failed) |
| Exam. 43 lph_ship08s (n = 3245) | 43 (18.97) | 12*5907*15 (279.95) | 100*70070*700 (162.72) | 2.38e+02 (failed) |
| Exam. 44 lph_ship12s (n = 4020) | 42 (30.77) | 9*3805*21 (333.53) | 77*205665*1472 (1002.61) | 1.97e+02 (failed) |
| Exam. 45 lph_siptera (n = 3962) | 47 (33.55) | 12*4009*16 (319.80) | 100*5000*300 (35.28) | 9.99e+04 (failed) |
| Exam. 46 lph_standgub (n = 1744) | 53 (5.65) | 12*2064*16 (19.86) | 100*94960*792 (146.17) | 2.44e+01 (failed) |
| Exam. 47 lph_sipt (n = 961) | 61 (1.72) | 15*1404*18 (3.91) | 100*24300*939 (62.39) | 7.78e+03 (failed) |
| Exam. 48 lph_woodlp (n = 2839) | 64 (20.82) | 58*3749*64 (473.97) | 100*2700*300 (10.44) | 7.27e+05 (failed) |
| Exam. 49 lph_box1 (n = 492) | 30 (0.20) | 13*657*15 (0.56) | 2+2115*24 (0.66) | 2.67e-15 |
| Exam. 50 lph_boxl (n = 602) | 43 (0.45) | 12*627*14 (0.95) | 100*64600*798 (25.13) | 1.48e+02 (failed) |
| Exam. 51 lph_ex72a (n = 412) | 34 (0.12) | 13*570*15 (0.41) | 1+867*6 (0.14) | 3.13e-14 |
| Exam. 52 lph_ex73a (n = 404) | 33 (0.12) | 13*552*15 (0.38) | 100*6739*600 (9.94) | 1.50e-03 (failed) |
| Exam. 53 lph_mondou2 (n = 916) | 38 (0.94) | 11*773*13 (2.36) | 2+1151*9 (0.63) | 1.55e-14 |
Table 6: Numerical results of large-scale dense LCPs (no. 54-73).

| Problems (n*a) | RPFMTr | PATH | MILES |
|---------------|--------|------|-------|
|               | steps | (time) | major/minor/grad | steps | (time) | major/pivots/refactor | Terr |
| Exam. 54 lp_30baub3 | 45 (1385.62) | 5.38e-07 | 5.8283e+08 | (18049.72) | 1.92e+01 | 2.8254e+04 | (1432.94) | 1.32e+03 | failed |
| Exam. 55 lp_len2 | 55 (155.40) | 8.21e-07 | 11.8765e+18 | (4549.92) | 3.93e-08 | 4.3e+04 | (1009.19) | 9.22e+01 | failed |
| Exam. 56 lp_cnc8 | 61 (583.43) | 6.36e-07 | 4.3835e+17 | (18099.73) | 7.38e+01 | 1.84e+09 | (1003.36) | 1.51e+03 | failed |
| Exam. 57 lp_ge9 | 59 (372.39) | 8.33e-07 | 7.8042e+17 | (18009.97) | 2.68e-00 | 1.40e+09 | (1904.88) | 7.55e+00 | failed |
| Exam. 58 lp_cycle | 62 (94.96) | 5.76e-07 | 1.2478e+17 | (2322.98) | 2.77e-07 | 1.00e+10 | (148.55) | 2.07e+03 | failed |
| Exam. 59 lp_hcube | 54 (141.97) | 6.46e-07 | 1.06e+16 | (725.33) | 5.15e-07 | 8.0e+07 | (1909.78) | 2.19e+03 | failed |
| Exam. 60 lp_fl4 | 59 (550.98) | 7.06e-07 | 8.337e+13 | (185.28) | 2.82e-09 | 2.1474e+04 | (131.11) | 3.78e-10 | failed |
| Exam. 61 lp_greenbea | 50 (223.83) | 8.66e-07 | 1.2419e+16 | (7877.61) | 5.44e-09 | 1.00e+10 | (373.86) | 1.14e+02 | failed |
| Exam. 62 lp_greenbeb | 50 (225.27) | 9.43e-07 | 1.241203e+16 | (10248.31) | 2.66e-08 | 1.00e+10 | (575.45) | 1.14e+02 | failed |
| Exam. 63 lp_ken07 | 42 (96.95) | 8.68e-07 | 8.407e+12 | (2922.75) | 1.13e-07 | 5.6e+06 | (1008.28) | 5.01e+03 | failed |
| Exam. 64 lp_ken08 | 48 (437.80) | 7.48e-07 | 8.4386e+12 | (15408.33) | 1.82e-08 | 1.00e+10 | (503.61) | 1.86e+03 | failed |
| Exam. 65 lp_pole87 | 58 (307.63) | 5.54e-07 | 1.1488e+23 | (2329.50) | 8.07e-10 | 5.6e+08 | (1014.17) | 9.99e+02 | failed |
| Exam. 66 lp_pole12p1 | 48 (722.82) | 8.45e-07 | 3.6579e+16 | (18016.27) | 2.73e+01 | 1.00e+10 | (771.77) | 2.14e+01 | failed |
| Exam. 67 lp_pole12p1 | 47 (66.37) | 9.12e-07 | 1.2407e+15 | (1537.80) | 1.03e-09 | 1.00e+10 | (463.21) | 4.34e+02 | failed |
| Exam. 68 lp_pole12p1 | 47 (124.64) | 7.67e-07 | 1.241202e+15 | (7256.50) | 1.51e-07 | 2.9e+10 | (1001.00) | 7.53e+02 | failed |
| Exam. 69 lp_trans | 45 (339.85) | 6.61e-07 | 1.06e+10 | (10715.00) | 2.30e-08 | 2.3e+10 | (1022.70) | 8.80e+00 | failed |
| Exam. 70 lp_woodw | 71 (494.92) | 7.61e-07 | 1.14e+10 | (18080.28) | 1.35e-05 | 1.00e+10 | (340.14) | 2.06e+02 | failed |
| Exam. 71 lp_windy | 53 (125.55) | 7.46e-07 | 1.48e+06 | (1811.72) | 3.12e+02 | 9.11e+06 | (1025.28) | 2.06e+02 | failed |
| Exam. 72 lp_gran | 60 (81.64) | 5.32e-07 | 1.44552e+13 | (1450.42) | 1.29e-08 | 5.0e+10 | (719.47) | 2.35e+03 | failed |
| Exam. 73 lp_greenbea | 50 (211.05) | 5.37e-07 | 1.241205e+16 | (12504.91) | 7.46e-09 | 4.7e+10 | (1006.56) | 1.13e+02 | failed |

5 Conclusions

In this paper, we give the regularization path-following method with the trust-region updating strategy (RPFMTr) for the linear complementarity problem. Meanwhile, we prove the global convergence of the new method under the standard assumptions without the condition of the priority to feasibility over complementarity. Nu-
Numerical results show that RPFMTr is a robust and efficient solver for the linear complementarity problem, especially for the dense linear complementarity problem. Furthermore, it is more robust and faster than some state-of-the-art solvers such as PATH [15, 18, 19, 48] and MILES [43, 44, 50] (the built-in subroutines of the GAMS v28.2 (2019) environment [21]). The computational time of RPFMTr is about 1/3 to 1/10 of that of PATH for the dense linear complementarity problem. Therefore, RPFMTr is an alternating solver for the linear complementarity problem and worth exploring further for the nonlinear complementarity problem.

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Availability of data and material (data transparency): If it is requested, we will provide the test data.

Code availability (software application or custom code): If it is requested, we will provide the code.

References

1. AMPL, A Mathematical Programming Language, http://www.ampl.com, 2022.
2. E. L. Allgower and K. Georg, Introduction to Numerical Continuation Methods, SIAM, Philadelphia, 2003.
3. U. M. Ascher and L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, Philadelphia, 1998.
4. O. Axelsson and S. Sysala, Continuation Newton methods, Comput. Math. Appl. 70 (2015), 2621-2637.
5. F. H. Branin, Widely convergent method for finding multiple solutions of simultaneous nonlinear equations, IBM J. Res. Dev. 16 (1972), 504-521.
6. J. C. Butcher and Z. Jackiewicz, Construction of high order diagonally implicit multistage integration methods for ordinary differential equations, Appl. Numer. Math. 27 (1998), 1-12.
7. K. E. Brenan, S. L. Campbell, and L. R. Petzold, Numerical solution of initial-value problems in differential-algebraic equations, SIAM, Philadelphia, 1996.
8. R. W. Cottle, J.-S. Pang and R. E. Stone, The Linear Complementarity Problem, SIAM, Philadelphia, 2009.
9. A. R. Conn, N. Gould, and Ph. L. Toint, Trust-Region Methods, SIAM, Philadelphia, 2009.
10. M. T. Chu and M. M. Lin, Dynamical system characterization of the central path and its variants- a revisit, SIAM J. Appl. Dyn. Syst. 10 (2011), 887-905.
11. D. F. Davidenko, On a new method of numerical solution of systems of nonlinear equations (in Russian), Dokl. Akad. Nauk SSSR 88 (1953), 601-602.
12. P. Deuflhard, Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms, Springer, Berlin, 2004.
13. P. Deuflhard, H. J. Pesch, and P. Rentrop, A modified continuation method for the numerical solution of nonlinear two-point boundary value problems by shooting techniques, Numer. Math. 26 (1975), 327-343.
14. S. P. Dirkse, *Robust Solution of Mixed Complementarity Problems*, PhD thesis, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, 1994.
15. S. Dirkse and M. C. Ferris, *The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems*, Optim. Methods Softw. 5 (1995), 123-156.
16. E. J. Doedel, *Lecture notes in numerical analysis of nonlinear equations*, in *Numerical Continuation Methods for Dynamical Systems*, B. Krauskopf, H.M. Osinga, and J. Galán-Vioque, eds., Springer, Berlin, 2007, pp. 1-50.
17. K. Erleben, *Velocity-based shock propagation for multibody dynamics animation*, ACM Trans. Graph. 26 (2):12 (2007), https://doi.org/10.1145/1243980.1243986.1.
18. M. C. Ferris and T. S. Munson, Complementarity problems in GAMS and the PATH solver, J. Econ. Dyn. Control 24 (2000), 165-188.
19. M. C. Ferris and T. S. Munson, *The manual of PATH*, GAMS Corporation, https://www.gams.com/latest/docs/S_PATH.html, 2022.
20. A. Fischer, *A special Newton-type optimization method*, Optimization 24 (1992), 269-284.
21. GAMS v28.2, GAMS Corporation, https://www.gams.com/, 2019.
22. A. Gana, *Studies in the Complementarity Problem*, Ph.D. dissertation, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI, 1982.
23. P. Gill, W. Murray, M. A. Saunders, M. H. Wright, *Maintaining LU-factors of a general sparse matrix*, Linear Algebra Appl. 88-89 (1991), 239-270.
24. G. H. Golub and C. F. Van Loan, *Matrix Computation*, 4th ed., The John Hopkins University Press, Baltimore, 2013.
25. T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. 20 (1919), pp. 292-296, JFM 47.0399.02 (https://zbmath.org/?format=complete&q=an:47.0399.02), JSTOR 1967124 (https://www.jstor.org/stable/1967124), MR 1502565 (https://www.ams.org/mathscinet-getitem?mr=1502565).
26. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*, 2nd ed., Springer, Berlin, 1996.
27. D. J. Higham, *Trust region algorithms and timestep selection*, SIAM J. Numer. Anal. 37 (1999), 194-210.
28. P. C. Hansen, *Regularization Tools: A MATLAB package for analysis and solution of discrete ill posed problems*, Numer. Algorithms 6 (1994), 23-29.
29. Z. Jackiewicz, S. Tracogna, *A general class of two-step Runge-Kutta methods for ordinary differential equations*, SIAM J. Numer. Anal. 32 (1995), 1390-1427.
30. C. T. Kelley, *Solving Nonlinear Equations with Newton’s Method*, SIAM, Philadelphia, 2003.
31. C. T. Kelley, *Numerical methods for nonlinear equations*, Acta Numer. 27 (2018), 207-287.
32. C. E. Lemke and J. T. Howson, Jr. *Equilibrium points of bimatrix games*, Journal of the Society for Industrial and Applied Mathematics 12 (1964), 413-423.
33. X.-L. Luo, *A second-order pseudo-transient method for steady-state problems*, Appl. Math. Comput. 216 (2010), 1752-1762.
34. X.-L. Luo, *A dynamical method of DAEs for the smallest eigenvalue problem*, J. Comput. Sci. 3 (2012), 113-119.
35. X.-L. Luo, H. Xiao, J.-H. Lv and S. Zhang, *Explicit pseudo-transient continuation and the trust-region updating strategy for unconstrained optimization*, Appl. Numer. Math. 165 (2021), 290-302, available at http://doi.org/10.1016/j.apnum.2021.02.019.
36. X.-L. Luo, J.-H. Lv and G. Sun, *Continuation method with the trusty time-stepping scheme for linearly constrained optimization with noisy data*, Optim. Eng. 23 (2022), 329-360, available at http://doi.org/10.1007/s10898-020-09590-z.
37. X.-L. Luo, H. Xiao and J.-H. Lv, *Continuation Newton methods with the residual trust-region time-stepping scheme for nonlinear equations*, Numer. Algorithms 89 (2022), 223-247, available at http://doi.org/10.1007/s11075-021-00112-x.
38. X.-L. Luo and H. Xiao, *Generalized continuation Newton methods and the trust-region updating strategy for the underdetermined system*, J. Sci. Comput. 88 (2021), article 56, published online at http://doi.org/10.1007/s10915-021-01025-y, or available at http://arxiv.org/abs/2006.07568, pp. 1-21, March 15, 2021.
40. X.-L. Luo and H. Xiao, Continuation Newton methods with deflation techniques and quasi-genetic evolution for global optimization problems, arXiv preprint available at http://arxiv.org/abs/2107.13864, or Research Square preprint available at https://doi.org/10.21203/rs.3.rs-110277/v1, July 30, 2021. Software available at https://teacher.bupt.edu.cn/zzcg/41406/list/index.htm.

41. X.-L. Luo and H. Xiao, The regularization continuation method with an adaptive time step control for linearly constrained optimization problems, submitted to Applied Numerical Mathematics, the second round of review, arXiv preprint available at http://arxiv.org/abs/2106.01122, July 7, 2021.

42. MATLAB v9.8.0 (R2020a), The MathWorks Inc., http://www.mathworks.com, 2020.

43. L. Mathiesen, Computation of economic equilibria by a sequence of linear complementarity problems, Math. Program. 23 (1985), 144-162.

44. T. F. Rutherford, The manual of MILES, GAMS Corporation, https://www.gams.com/latest/docs/S_MILES.html, 2022.

45. MATLAB collection of linear programming problems, available at http://www.netlib.org.

46. J. Nocedal and S. J. Wright, Numerical Optimization, Springer, Berlin, 1999.

47. J. M. Ortega and W. C. Rheinboldt, Iteration Solution of Nonlinear Equations in Several Variables, SIAM, Philadelphia, 2000.

48. M. C. Ferris et al., PATH solver 5.0.00, software available at https://pages.cs.wisc.edu/~ferris/path.html, 2019.

49. M. J. D. Powell, Convergence properties of a class of minimization algorithms, in Nonlinear Programming 2, O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, eds., Academic Press, New York, 1975, pp. 1-27.

50. T. F. Rutherford, Extension of GAMS for complementarity problems arising in applied economic analysis, J. Econ. Dyn. Control 19 (1995), 1209-1324.

51. L. F. Shampine, I. Gladwell, and S. Thompson, Solving ODEs with MATLAB, Cambridge University Press, Cambridge, 2003.

52. K. Tanabe, Continuous Newton-Raphson method for solving an underdetermined system of nonlinear equations, Nonlinear Anal. 3 (1979), 495-503.

53. K. Tanabe, Centered Newton method for mathematical programming, in System Modeling and Optimization: Proceedings of the 13th IFIP conference, vol. 113 of Lecture Notes in Control and Information Systems, Springer, Berlin, 1988, pp. 197-206.

54. V. Venkateswaran, An algorithm for the linear complementarity problem with a P_0-matrix, SIAM J. Matrix Anal. Appl. 14 (1993), 967-977.

55. S. J. Wright, An infeasible-interior-point algorithm for linear complementarity problems, Math. Program. 67 (1994), 29-51.

56. S. J. Wright, Primal-dual Interior Point Methods, SIAM, Philadelphia, 1997.

57. Q. Xu, On the New Linear Programming Algorithms-New Sequential Penalty Function Method and Two Point Barrier Function Method (in Chinese), Ph.D. thesis, Institute of Nuclear Technology, Tsinghua University, Beijing, China, 1991.

58. Y. X. Yuan, Recent advances in trust region algorithms, Math. Program. 151 (2015), 249-281.

59. Y. Zhang, On the convergence of a class of infeasible interior-point methods for the horizontal linear complementarity problem, SIAM J. Optim. 4 (1994), 208-227.