THE GEOMETRY OF THE HILTON SPLITTING

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Abstract. One of the important theorems in homotopy theory is the Hilton Splitting Theorem which states: there is an isomorphism $H = \oplus_{\gamma \in \Gamma} H_\gamma$ from the $m$-th homotopy group of the wedge of a number of spheres to the direct sum of the $m$-th homotopy groups of some spheres, see [Hi]. In this paper we will construct geometrically all Hilton homomorphisms $H_\gamma$ and prove a family of sharper symmetry relations between linking coefficients which desuspend and generalize the relations of Kervaire [Ke], Haefliger and Steer [Ha,St].

Keywords: basic Whitehead products, Hilton homomorphisms, framed links, linking coefficients.

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1. introduction

Let $X'$ be a connected smooth manifold without boundary and let $X = X' \times \mathbb{R}$ be of dimension $m \geq 2$. We denote by $k_1, k_2, \ldots, k_r$ some natural numbers greater than 1. A framing of a $k$-codimensional submanifold in $X \times \mathbb{R}$ is a trivialization of the normal vector bundle, or equivalently, an ordered set $(u_1, u_2, \ldots, u_k)$ of $k$ linearly independent normal vector fields. A $(k_1, k_2, \ldots, k_r)$-link is a disjoint union $M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \subset X \times \mathbb{R}$ of closed, framed submanifolds of codimensions $k_1, k_2, \ldots, k_r$. We denote the bordism group of such framed links by $\text{FL}_{k_1, k_2, \ldots, k_r}(X)$ which is isomorphic to the homotopy group $[\Sigma X \vee \bigvee_{i=1}^r S^{k_i}]$ via the Pontryagin-Thom construction, where $\Sigma X_C$ is the suspension of the one point compactification of $X$. Denote by $\iota_i : S^{k_i} \hookrightarrow \bigvee_{i=1}^r S^{k_i}$ the inclusions and by $\Gamma$ a system of basic Whitehead products in $\iota_1, \iota_2, \ldots, \iota_r$. So we have the Hilton isomorphism (generalized by Milnor, see [Mi])

$$H = \oplus_{\gamma \in \Gamma} H_\gamma : \text{FL}_{X}^{k_1, k_2, \ldots, k_r} \longrightarrow \oplus_{\gamma \in \Gamma} \text{FL}_{X}^{q(\gamma)+1},$$

where $q(\gamma)$ is the height of $\gamma$, see [Hi]. Let $p_\gamma$ be the projection from the direct sum onto the factor $\text{FL}_{X}^{q(\gamma)+1}$ corresponding to $\gamma$, then $H_\gamma = p_\gamma \circ H$. The homomorphism $\gamma_*$, induced by $\gamma$, embeds this factor into $\text{FL}_{X}^{k_1, k_2, \ldots, k_r}$. Clearly, the Hilton homomorphisms $H_\gamma$ are characterized by

(a) $H_\gamma \circ \gamma_* = id$, for $\gamma \in \Gamma$;
(b) $H_\gamma \circ \gamma_*' = 0$, for $\gamma, \gamma' \in \Gamma$ and $\gamma' \neq \gamma$.

As the main result of this paper we will prove a family of symmetry relations between linking coefficients and construct geometrically all Hilton homomorphisms $H_\gamma$. It is a
classical subject to construct or interpret homotopical invariants by means of differential topology, for example by using the well known Pontryagin-Thom construction and transversal intersections of submanifolds. Hopf invariants and Hilton homomorphisms are of particular interest. For example, if the $H_\gamma$’s are already constructed, then for a given element in the homotopy group of the wedge of spheres we can compute its Hilton splitting. Kervaire [Ke] gave a geometrical description of the stable Whitehead-Hopf invariant and proved a symmetry relation between linking coefficients in high dimensions. Haefliger and Steer [Ha,St] constructed the suspension of $H_\gamma$ corresponding to $\gamma = [\iota_1, \iota_2]$ and obtained a further symmetry relation between linking coefficients. Boardman and Steer [Bo,St] defined the Hopf ladder and presented a geometrical discussion. Koschorke and Sanderson applied immersion theory to this topic in [K,S 1] and [K,S 2].

This work is also strongly motivated by the close connection to homotopy theory of link maps. For example, Koschorke [Ko 1] and [Ko 3] generalized the $\mu$-invariants of Milnor by using his geometrical interpretation of some stable Hilton homomorphisms which in fact makes some computations possible. Hilton splitting also played an important role in the study of the different homotopy behaviour of link maps in $S^m$ and $\mathbb{R}^m$, see [Ka 1] and [Ka 2] of Kaiser.

In the case $X \times \mathbb{R} \cong \mathbb{R}^3$ and $k_1 = k_2 = \cdots = k_r = 2$ it is well known that the elements in the factors $\pi_3(S^3) \cong \mathbb{Z}$ in the Hilton splitting of $\pi_3(\vee_{i=1}^r S^2_i)$ can be interpreted as linking numbers. Sanderson [Sa 2] gave a geometrical isomorphism from $\pi_4(S^2 \vee S^2)$ to $\mathbb{Z}_2^3 \oplus \mathbb{Z}^2$ by using intersections with Seifert surfaces. This isomorphism takes the form of the Hilton splitting but is different from it, see the author’s dissertation [Wa]. In general cases the geometry of the Hilton splitting is unknown up to date, because of its complicated algebraic topological nature.

This paper is organized as follows. We introduce in §2 a new construction, call it the $\tau$-construction, and establish its basic properties. Our $\tau$-construction desuspends the one of Haefliger and Steer in [Ha,St]. As an application of our construction we prove a family of sharper symmetry relations between linking coefficients in §3. The $\tau$-construction leads to the definition of the $\tau$-reduction in §4, all Hilton homomorphisms are geometrically constructed there by means of the $\tau$-reductions. We work in the category of smooth manifolds.

We have extracted the materials in §2 and §3 from the author’s dissertation [Wa]. So it is a great pleasure to express my gratitudes to my supervisor Professor U. Koschorke as well as Professor U. Kaiser for many helpful discussions and encouragements. I am grateful to Professor M. Heusener for nice talks. Thanks also to Dr. Pho who helped me use xfig.
2. THE $\tau$-CONSTRUCTION

Let $M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \subset X \times \mathbb{R}$ be a $(k_1, k_2, \ldots, k_r)$-link and $1 \leq i \neq j \leq r$. We construct now a framed submanifold $Z = \tau(M_j, M_i) \subset X \times \mathbb{R}$ as follows. Let $W_j = M_j \times [0, 1]$, and let $W_i \subset X \times \mathbb{R} \times [0, 1]$ be a framed bordism of $M_i$ such that $M_j \times \{1\}$ and $W_i \cap X \times \mathbb{R} \times \{1\}$ are separated by some $X_t = X \times \{t\} \times \{1\}$, see Fig.1.

Denote the naturally framed intersection of $W_i$ and $W_j$ by $\bar{Z}$ and let $u_{k_j}$ be the last vector field in the framing of $M_j$. For $\varepsilon > 0$ small enough we deform first $W_j$ to $M_j \times [0, \varepsilon]$ and then rotate at every point $x \in M_j$ the interval $[0, \varepsilon]$ to $u_{k_j}$ through the angle $\pi/2$. By doing this we have isotoped $\bar{Z}$ to a submanifold $Z = \tau(M_j, M_i) \subset X \times \mathbb{R} \times \{0\}$ which is naturally framed, because the isotopy induces a homotopy of the normal vector bundles and during the isotopy $u_{k_j}$ is deformed to the negative direction of the interval $[0, 1]$. See Fig.1 again. We call this construction of $Z = \tau(M_j, M_i) \subset X \times \mathbb{R} \times \{0\}$ the $\tau$-construction, which desuspends the one of Haefliger and Steer [Ha,St].

If one changes the roles of $M_i$ and $M_j$, namely takes $W_i$ to be the cylinder $M_i \times [0, 1]$ and takes $W_j$ to be a framed bordism, then one will get another framed submanifold $\tau(M_i, M_j) \subset X \times \mathbb{R} \times \{0\}$. Denote by $\tau[\ast, \ast]$ the framed bordism class. According to [Ha,St] it holds $E\tau[M_j, M_i] = E\tau[M_i, M_j]$ up to sign, where $E$ denotes the suspension homomorphism. But, as we will see later in this section, $\tau[M_j, M_i] \neq \tau[M_i, M_j]$, even if up to involution (namely an automorphism $u$ of the target group with the property $u \circ u = \text{id}$).

![Figure 1.](image-url)

Because $\tau(M_j, M_i)$ (or $\tau(M_i, M_j)$) lies in a small neighbourhood of $M_j$ (or $M_i$), it is disjoint from all components of the original link. This interesting fact makes it possible to get a new link

$$M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau(M_j, M_i) \subset X \times \mathbb{R}$$

from the old one. Denote by $[\ ]$ the bordism class of a framed submanifold or link.
\textbf{Theorem 2.1.} The assignment
\[ M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \to M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau(M_j, M_i) \]
gives a well defined injective homomorphism \( \tau_{ji} \)

\begin{center}
\begin{tikzpicture}
\node (X) at (0,0) {$FL_X^{k_1,k_2,\ldots,k_r}$};
\node (Y) at (3,0) {$FL_X^{k_1,k_2,\ldots,k_r+1}$};
\node (Z) at (3,-3) {$FL_X^{k_r+1}$};
\draw[->] (X) to node[above] {\( \tau_{ji} \)} (Y);
\draw[->] (Y) to node[right] {\( \text{proj} \)} (Z);
\draw[->] (X) to node[above left] {\( \tau_{ji}^p \)} (Z);
\end{tikzpicture}
\end{center}

where \( k_{r+1} = k_i + k_j - 1 \). In particular, \( \tau_{ji}^p = \text{proj} \circ \tau_{ji} \) is a well defined invariant.

\textbf{Proof.} Let \( I_0 = [0,1], I_1 = [1,2], I_2 = [-1,0] \) and let \( N_1 \sqcup N_2 \sqcup \cdots \sqcup N_r \subset X \times \mathbb{R} \times I_0 \) be a framed bordism from \( M'_1 \sqcup M'_2 \sqcup \cdots \sqcup M'_r \) to \( M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \). We perform \( \tau(M_j, M_i) \) in \( X \times \mathbb{R} \times I_1 \) using the cylinder \( W_j \) and similarly we perform \( \tau(M'_j, M'_i) \) in \( X \times \mathbb{R} \times I_2 \) using the cylinder \( W'_j \) (the negative orientation of \( I_2 = [-1,0] \) is used), see Fig.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}
Consider now the framed submanifolds \( V_j = W_j' \cup N_j \cup W_j \) and \( V_i = W_i' \cup N_i \cup W_i \). For \( t \in \mathbb{R} \) let \( \mathbb{R}_t^- = \{ x \in \mathbb{R} | x < t \} \) and similarly \( \mathbb{R}_t^+ \). We may assume that there is some \( t \in \mathbb{R} \) such that
\[
V_j \subset X \times \mathbb{R}_t^- \times [-1, 2]
\]
\[
A = W_i' \cap X \times \mathbb{R} \times \{-1\} \subset X \times \mathbb{R}_t^+ \times \{-1\},
\]
\[
B = W_i \cap X \times \mathbb{R} \times \{2\} \subset X \times \mathbb{R}_t^+ \times \{2\}.
\]
This means that \( V_j \) and the boundary of \( V_i \), namely the union of \( A \) and \( B \), are separated by \( X_t'' = X \times \{t\} \times [-1, 2] \), see Fig.2 again. Because we are working in \( X \times \mathbb{R} = X' \times \mathbb{R}^2 \), this can always be satisfied by isotoping \( V_i \) without changing the framed intersection
\[
V_j \cap V_i = Z \sqcup Z' = (W_j \cap W_i) \sqcup (W_j' \cap W_i').
\]

\[\begin{align*}
\text{Figure 3.}
\end{align*}\]

Let \( I = [0, 1] \), \( Q_j = V_j \times I \) and \( Q_i \subset X \times \mathbb{R} \times [-1, 2] \times I \) be a framed bordism of \( V_i \) such that
\[
\partial Q_i = V_i \cup A \times I \cup B \times I \cup V_i'
\]
and such that \( Q_j, V_i' \) are separated by \( X_t'' = X \times \{t\} \times [-1, 2] \times I \), where \( V_i' \) is the boundary part of \( Q_i \) lying in \( X \times \mathbb{R} \times [-1, 2] \times \{1\} \), see Fig.3. Such a manifold \( Q_i \) always exists.

Let \( \bar{Q} = Q_j \cap Q_i \) be the naturally framed intersection, its boundary is \( \partial \bar{Q} = \bar{Z} \sqcup \bar{Z}' \). Just because of
\[
Q_j = (M_j' \times [-1, 0] \times I) \cup (N_j \times I) \cup (M_j \times [1, 2] \times I)
\]
there is an isotopy of \( Q_j \) which deforms \( Q_j \) to \( N_j \times I \) and is smooth at least in a small neighbourhood of \( \bar{Q} \subset Q_j \). For example, for any \( x \in M_j \) one can easily isotope
\{x\} \times [1 - \varepsilon, 2] \times [0, 1] \text{ to } \{x\} \times [1 - \varepsilon, 1] \times [0, 1] \text{ by using the trick in Fig.4. Note that collars of } M_j, M_j' \subset N_j \text{ are used to perform this isotopy.}

Figure 4.

So we can isotope \(\bar{Q}\) smoothly to a framed submanifold \(\bar{Q}' \subset N_j \times I\). Let \(\tilde{u}_{kj}\) be the last normal vector field in the framing of \(N_j\). Just like in the \(\tau\)-construction we deform \(N_j \times I\) to \(N_j \times [0, \varepsilon]\) and then rotate the positive \(I\)-direction to \(\tilde{u}_{kj}\). By doing this we have isotoped \(\bar{Q}'\) to a framed submanifold \(Q \subset X \times \mathbb{R} \times [0, 1] \times \{0\}\). Now it is easy to see that \(Q\) is a framed bordism between \(\tau(M_j, M_i)\) and \(\tau(M_j', M_i')\), and \(Q\) is also disjoint from \(N_1 \sqcup N_2 \sqcup \cdots \sqcup N_r\), for it lies in a small neighbourhood of \(N_j\). The desired framed bordism is given by \(N_1 \sqcup N_2 \sqcup \cdots \sqcup N_r \sqcup Q\).

So \(\tau_{ji}\) is a well defined map. The assumption \(X = X' \times \mathbb{R}\) guarantees it is also a homomorphism. Other assertions follow easily.

Note that the homomorphism \(\tau_{ji}\) in the theorem is independent of the choice of the vector field used to rotate \(M_j \times [0, \varepsilon]\), because one can always rotate one vector field to another. Theorem 2.1 implies that we can perform the \(\tau\)-construction successively to get further well-defined invariants. For example, for any \(1 \leq k \leq r\) we have \(\tau(M_k, \tau(M_j, M_i))\). This is a very important property of our \(\tau\)-construction. To understand this, note that in the Haefliger-Steer construction we see only one geometrical obstruction (the framed intersection \(\bar{Z}\)) of a framed link of two components from being the trivial link (in the sense the two components are not linked), in contrast the Hilton splitting says there are many other obstructions; our iterated \(\tau\)-invariants are surely related to such further obstructions.

Let \(M \subset X \times \mathbb{R}\) be a closed, framed submanifold. A suitably framed Seifert surface of \(M\) is a compact, framed submanifold \(F \subset X \times \mathbb{R}\) with boundary \(M\) such that the framing of \(M\) as boundary is homotopic to the original framing. In this case we say \(M\) is \(S\)-framed. Note that two \(S\)-framings of \(M\) are not necessarily homotopic. The first part of the following lemma is directly to see and the second part follows by a simple discussion of fibre-wise embeddings, so we omit the proof.
Lemma 2.2. (i) If $M_i$ has a suitably framed Seifert surface $F_i$, then the following two framed links

$$M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau(M_j, M_i), \quad M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup (M_j \cap F_i)^{sh}$$

are framed bordant at least up to involution of the framing of the last component, where $(M_j \cap F_i)^{sh}$ is a small shift of $M_j \cap F_i$ along the framing of $M_j$.

(ii) Let $L_1 \subset \mathbb{R}^n$ and $L_2 \subset \mathbb{R}^{n'}$ be $(k_1, k_2, \ldots, k_r)$-links with components $M_i$ and $M'_i$ respectively, $Z, Z' \subset X \times \mathbb{R}$ be disjoint, closed and framed submanifolds of codimensions $n$ or $n'$. By means of fibre-wise embeddings we get a new $(k_1, k_2, \ldots, k_r)$-link $L \subset X \times \mathbb{R}$ with components $M_i = Z \times M_i \sqcup Z' \times M'_i$. If $M_i$ and $M'_i$ are framed zerobordant for some $i$, then we can perform the $\tau$-construction so that the following holds

$$\tau_{ji}(L) = Z \times \tau_{ji}(L_1) \sqcup Z' \times \tau_{ji}(L_2).$$

The special case $Z' = \phi$ is also useful.

The inclusions $\iota_i : S^{k_i} \hookrightarrow \vee_{i=1}^r S^{k_i}$ and the Whitehead products $[\iota_i, \iota_j]$ can be geometrically interpreted as framed points or as $S$-framed Hopf links, via Pontryagin-Thom construction. Iteratedly we can represent every Whitehead product $\gamma$ in $\iota_1, \ldots, \iota_r$ by a $(k_1, k_2, \ldots, k_r)$-link in $\mathbb{R}^{q(\gamma)+1}$, where $q(\gamma)$ is the height of $\gamma$. We are now ready to identify the homomorphism $\tau_{ji}^p$ in Theorem 2.1 with the Hilton homomorphism corresponding to $\gamma = [\iota_i, \iota_j]$.

Theorem 2.3. Let $\gamma = [\iota_i, \iota_j] \in \Gamma$ be a basic Whitehead product. It holds $\tau_{ji}^p = H_\gamma$ up to involution.

Proof. We show that up to involution $\tau_{ji}^p$ satisfies the properties (a) and (b) in §1 which characterize the Hilton homomorphisms. For $\gamma' = \iota_k \in \Gamma$ property (b) is trivial. Assume that $\gamma' \in \Gamma$ is of weight $\geq 2$ and let $L = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \subset \mathbb{R}^{q(\gamma')+1}$ be the framed link representing $\gamma'$. Each component $M_i$ is clearly framed zerobordant, so Lemma 2.2 reduces (a) and (b) to the following

(a’) $\tau[M_j, M_i] = \pm 1$, if $\gamma = \gamma' = [\iota_i, \iota_j],$
(b’) $\tau[M_j, M_i] = 0$, if $\gamma' \neq \gamma = [\iota_i, \iota_j].$

Let $\gamma' = [\iota', \iota_j']$ be of weight 2. If the pair $(i', j') \neq (i, j)$, then $M_i$ or $M_j$ is the empty, (b’) follows easily; if $(i', j') = (i, j)$, then $M_i \sqcup M_j$ is an $S$-framed Hopf link and all other components are the empty, (a’) follows by Lemma 2.2.

Let $\gamma' = [\alpha, \beta]$ be of weight $\geq 3$. We have the following formula

$$(1) \quad M_k = S^{l_k-1} \times M_k(\beta) \sqcup S^{l_k-1} \times M_k(\alpha),$$

where $S^{l_k-1} \sqcup S^{l_k-1}$ is an $S$-framed Hopf link and $M_k(\alpha), M_k(\beta)$ are the components of the links representing $\alpha$ and $\beta$, $1 \leq k \leq r$. The weight of $\beta$ is at least 2, so $M_i(\beta)$ is
framed zerobordant. If \( \alpha \neq \nu_i \), then \( M_i(\alpha) \) is also framed zerobordant. We use Lemma 2.2 again and obtain
\[
(2) \quad \tau(M_j, M_i) = S^{l_1-1} \times \tau(M_j(\beta), M_i(\beta)) \sqcup S^{l_2-1} \times \tau(M_j(\alpha), M_i(\alpha)).
\]
By inductive assumption for \( \alpha \) and \( \beta \) we see that at least one of \( \tau(M_j(\alpha), M_i(\alpha)) \) and \( \tau(M_j(\beta), M_i(\beta)) \) is framed zerobordant, say the first. By means of the fibre-wise embedding of the framed zerobordism one can easily prove that \( \tau(M_j, M_i) \) is framed bordant to \( S^{l_1-1} \times \tau(M_j(\beta), M_i(\beta)) \), and which is clearly framed zerobordant, for \( S^{l_1-1} \) is \( S \)-framed and therefore framed zerobordant. (\( b' \) follows).

Now let \( \alpha = \nu_i \). It holds then \( \gamma' = \left[ \nu_i, [\nu_i, \ldots, [\nu_i, \nu_j'] \ldots] \right] \) according to the definition of basic Whitehead products, where \( i \) \( \geq \nu_i \geq \cdots \geq \nu_i < \nu_j \). If \( \nu_j \neq \nu_i \), then \( \nu_j \) does not appear in \( \gamma' \), because \( \nu_i < \nu_j \). This means \( M_j = \phi \) and (\( b' \)) follows. So let \( \nu_j = \nu_i \).

We assume now \( \nu_i = \nu_i = \cdots = \nu_i \), otherwise the argument is essentially the same.

Denote by \( w \) the weight of \( \gamma' \). The link representing \( \gamma' \) is given by
\[
(3) \quad M_i = S_{w-1}^{k_1-1} \times S_{w-2}^{k_1-1} \times \cdots \times S_2^{k_1-1} \times S_{w-1}^{k_1-1} \sqcup \\
S_{w-1}^{k_1-1} \times S_{w-2}^{k_1-1} \times \cdots \times S_{k_1+k_2-2}^{k_1-1} \sqcup \\
\cdots \sqcup \\
S_{w-1}^{k_1-1} \times S_{w-2}^{k_1-1} \times \cdots \times S_{(w-3)(k_1-1)+k_2-1} \sqcup \\
S_{w-1}^{w-2}(k_1-1)+k_2-1 \sqcup \\
N_{i,1} \sqcup \cdots \sqcup N_{i,w-1},
\]
where \( S^{(k-1)(k_1-1)+k_2-1} \sqcup S_{k}^{k_1-1} \) are framed Hopf links, \( 1 \leq k \leq w-1 \). The products are given by fibre-wise embeddings. All other components are the empty.

Let \( e \) be the last vector in the standard base of \( \mathbb{R}^{q(\gamma')-1} \). We may assume \( M_j \subset \mathbb{R}^{q(\gamma')} \times \{0\} \subset \mathbb{R}^{q(\gamma')+1} \) and that \( e \) is just the last vector field in the framing of \( M_j \). This implies the following: the small shifts \( \hat{Q}_1 \) and \( \hat{Q}_2 \) of any \( Q_1, Q_2 \subset M_j \) along \( e \) through distances \( d_1 < d_2 \) are separated by \( \mathbb{R}^{q(\gamma')} \times \{(d_1 + d_2)/2\} \).

Obviously, every \( N_{i,k} \) bounds some suitably framed Seifert surface \( F_{i,k} \), \( 1 \leq k \leq w-1 \).

In \( \mathbb{R}^{q(\gamma')+1} \times [0,1] \) one can push them into different heights \( a_1 < a_2 < \cdots < a_{w-1} \) (with boundaries fixed) to get a framed zerobordism of \( M_i \) which will be used to perform \( \tau(M_j, M_i) \).

Let \( \{pt\} \) be a set consisting of a single point and cosider
\[
Q_k = S_{w-1}^{k_1-1} \times \cdots \times S_{k+1}^{k_1-1} \times \{pt\} \times S_{k-1}^{k_1-1} \times \cdots \times S_1^{k_1-1} \subset M_j.
\]
It is not difficult to see \( \tau(M_j, M_i) = \sqcup_{k=1}^{w-1} \hat{Q}_k \), where the framed submanifolds \( \hat{Q}_k \) are small shifts of the \( Q_k \)’s through distances \( d_1 < d_2 < \cdots < d_{w-1} \) along \( e \) (or equivalently along the framing). Clearly, every \( \hat{Q}_k \) bounds a suitably framed Seifert surface and the discussion above shows they are separated. (\( b' \) follows.)
Example 2.4. Our invariant $\tau_{ij}^p$ is asymmetric, namely $\tau_{ji}^p \neq \tau_{ij}^p$, even if up to involution. Let $r = 2$ and consider the Whitehead products

$$
\gamma = [\iota_2, [\iota_1, [\iota_1, \iota_2]]], \quad \gamma_1 = [\iota_1, [\iota_2, [\iota_1, \iota_2]]], \quad \gamma_2 = [[\iota_1, \iota_2], [\iota_1, \iota_2]] = [\iota_1, \iota_2] \circ [\iota, \iota],
$$

where $\iota$ is the identity of $S^{k_1+k_2-1}$. Denote the framed links representing $\gamma, \gamma_1, \gamma_2$ by $L, L_1$ and $L_2$. From the Jacobi-identity follows $L = \pm L_1 \pm L_2$.

Let $\Gamma, \Gamma'$ be two systems of basic Whitehead products in $\iota_1 \prec \iota_2$ and $\iota_2 \prec \iota_1$ respectively, then $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$. From Theorem 2.3 we have $\tau_{2,1}^p[L] = 0 = \tau_{1,2}^p[L_1]$ and

$$
\tau_{1,2}^p[L_2] = \pm \tau_{1,2}^p[L_1] \pm \tau_{1,2}^p[L_2] = \pm [\iota, \iota] \neq 0
$$

if $k_1+k_2-1 \neq 1, 3, 7$, according to a well known result of G. Whitehead and F. Adams, see for example [Ad]. The statement is proved. This example shows that our $\tau$-construction catches what is lost in the Haefliger-Steer construction due to the suspension.

By using the $\tau$-construction successively we can do the following:

(i) for many basic Whitehead products construct homomorphisms $h'_\gamma$ with property (a) in §1, but we can not guarantee the property (b), so these homomorphisms may be different from the corresponding Hilton homomorphisms;

(ii) construct all the Hilton homomorphisms if the basic Whitehead products of weight at least 4 are not involved in the Hilton splitting (using this we can re-prove the Jacobi-identity);

for details see Kapitel 3 in the author’s dissertation [Wa].

3. Symmetry relations between linking coefficients

As an application of the $\tau$-construction we prove here a family of symmetry relations between linking coefficients. Our argument is based on some beautiful ideas in [Ha,St].

Let $M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \subset \mathbb{R}^m \times \{1\}$, $W_j = M_j \times [0,1]$, and $W_i \subset \mathbb{R}^m \times [0,1]$ be as in the $\tau$-construction. If one isotopes the intersection $W_i \cap W_j$ into $\mathbb{R}^m \times \{0\}$ (instead of into $\mathbb{R}^m \times \{1\}$) and then project it to $\mathbb{R}^m \times \{1\}$, one obtains a framed submanifold $\tau'(M_j, M_i)$. By rotating the negative direction of $[0,1]$ to $-u_{k_j}$ one gets another framed submanifold $\tau''(M_j, M_i)$. We have the following fact

**Fact:** The framed links

$$
M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau(M_j, M_i),
$$

$$
M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau'(M_j, M_i),
$$

$$
M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup \tau''(M_j, M_i)
$$

are framed bordant at least up to involution of the framing of the last component. For the first and third links rotating $u_{k_j}$ to $-u_{k_j}$ in the plane spanned by $u_{k_j}$ and any other vector field $u$ in the framing of $M_j$; for the first and second links rotate $u_{k_j}$ through the angle $\pi$ in the plane spanned by $u_{k_j}$ and $u$ with the middle point of $u_{k_j}$ fixed. This fact will be used later in the proof of Theorem 3.1.

Let $\gamma = [\tau_j, [\cdots, [\tau_{j_2}, [\tau_{j_1}, \tau_1]] \cdots]]$ be any Whitehead product in $\iota_1, \cdots, \iota_r$, such that $\iota_1$ appears in $\gamma$ exactly one time in the given position. Define

$$\mu_{\gamma}^\tau : \pi_*(S^{k_1} \vee \cdots \vee S^{k_r}) \longrightarrow \pi_*(S^{q(\gamma)+1})$$

as the framed bordism class of

$$Z_\gamma = \tau(\tau(\cdots \tau(\tau(M_1, M_{j_1}), M_{j_2}), \cdots, M_{j_{r-1}}, M_{j_r})).$$

Let $L = S^{p_0} \sqcup S^{p_1} \sqcup \cdots \sqcup S^{p_r} \subset S^{n+1}$ be a smoothly embedded spherical link with $p_0, p_1, \cdots, p_r \leq n - 2$. From the well known $(n-1)$-homotopy equivalence we obtain the linking coefficients

$$\lambda_0 \in \pi_{p_0}(S^{k_1} \vee S^{k_2} \vee \cdots \vee S^{k_r}),$$

$$\lambda_1 \in \pi_{p_1}(S^{k_1'} \vee S^{k_2'} \vee \cdots \vee S^{k_r'}),$$

where $k_1 = n - p_1, k_1' = n - p_0$, and $k_i = k_i' = n - p_i$ for $2 \leq i \leq r$.

**Theorem 3.1.** Let $E$ denote the suspension homomorphism. It holds

$$E^{n+1-p_0} \mu_\gamma^\tau(\lambda_0) = \pm E^{n+1-p_1} \mu_\gamma^\tau(\lambda_1).$$

**Proof.** I. Let $I = I_1 = [0, 1]$. We assume first that following links are zero $h$-cobordant

$$L_0 = S^{p_0} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r},$$

$$L_1 = S^{p_1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}.$$  

The spheres in $L_0$ bound disjoint, framed Seifert surfaces $V_0, V_{0,2}, \cdots, V_{0,r}$; by the same token the spheres in $L_1$ bound disjoint, framed Seifert surfaces $V_1, V_{1,2}, \cdots, V_{1,r}$. In addition, we can suppose that for $2 \leq i \leq r$ the framed submanifolds

$$V_{0,i} \times \{0\} \sqcup S^{p_i} \times I \sqcup V_{1,i} \times \{1\}$$

bound suitably framed Seifert surfaces $W_{0,1} \subset S^{n+1} \times I$, for details see §3 in [Ha,St].

$\lambda_0$ and $\lambda_1$ are represented by the following framed links in $S^{p_0}$ and $S^{p_1}$ respectively

$$M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r = (S^{p_0} \sqcup V_1) \sqcup (S^{p_0} \sqcup V_{1,2}) \cdots \sqcup (S^{p_0} \sqcup V_{1,r}),$$

$$M'_1 \sqcup M'_2 \sqcup \cdots \sqcup M'_r = (S^{p_1} \sqcup V_0) \sqcup (S^{p_1} \sqcup V_{0,2}) \cdots \sqcup (S^{p_1} \sqcup V_{0,r});$$

see [Ha,St] again.

From $V_0 \sqcup V_1$ we get a framed bordism $W_{0,1} \subset S^{n+1} \times I$ between $E^{n+1-p_0}[M_1]$ and $E^{n+1-p_1}[M'_1]$. To see this note that $M_1$ lies in a ball $D^{p_0} \subset S^{p_0}$, and we can isotope this
ball to the standard embedding in \( S^{n+1} \) and homotope its framing to the standard one, this implies the boundary part \( M_1 \) of \( W_{0,1} \) represents \( E^{n+1-p_0}[M_1] \) up to sign. For the other boundary part it is completely similar. Clearly, the symmetry relation of Kervaire [Ke] is desuspended.

We can obviously arrange \( W_{0,1} \) so that it is disjoint from \( V_{0,i} \times I \) and \( V_{1,i} \times I \) for \( 2 \leq i \leq r \). We embed \( W_{0,1} \) in the natural way into \( S^{n+1} \times I \) and so \( W_{0,1} \times I \) into \( S^{n+1+1} \times I \times I \). Consider now the framed intersection

\[
Q_1' = W_{0,1} \times I_1 \cap W_{0,1}^{j_1} \times I \subset S^{n+1} \times I \times I.
\]

According to the construction we see directly \( Q_1' \subset S^{n+1} \times I \times (0,1) \). It holds in addition

\[
\partial Q_1' = Z_1' \sqcup Z_2' = (M_1 \times \{0\} \times I_1 \cap W_{0,1}^{j_1} \times \{0\}) \sqcup (M'_1 \times \{1\} \times I_1 \cap W_{0,1}^{j_1} \times \{1\}).
\]

Just as in the \( \tau \)-construction we isotope \( Q_1' \), using the last normal vector field \( v \) in the framing of \( W_{0,1} \), to a framed submanifold \( Q_1 \subset S^{n+1} \times I \times \{0\} \) which lies in a small tubular neighbourhood of \( W_{0,1} \), \( \partial Q_1 = Z_1 \sqcup Z_2 \). Up to homotopy of the framing we can assume that \( v \) restricts to the last vector fields in the framings of \( M_1 \) and \( M'_1 \) (considered as submanifolds of \( S^{p_0} \) and \( S^{p_1} \) respectively). This means \( Z_1 \) lies in \( S^{p_0} \) and is just \( \tau(M_1, M_{j_1}) \) up to involution of the framing. In fact the intersection

\[
M_1 \times \{0\} \times I_1 \cap W_{0,1}^{j_1} \times \{0\}
\]

is just the transversal intersection

\[
M_1 \times \{0\} \times I_1 \cap (S^{p_0} \times \{0\} \times I_1 \cap W_{0,1}^{j_1} \times \{0\})
\]

considered in \( S^{p_0} \times \{0\} \times I_1 \), in particular \( S^{p_0} \times \{0\} \times I_1 \cap W_{0,1}^{j_1} \times \{0\} \) is a framed zero-bordism of \( M_{j_1} \) under the assumption that the sublinks \( L_0 \) and \( L_1 \) of \( L \) are zero \( h \)-cobordant. The same is true for \( Z_2 \) (the fact at the beginning of this section is used here). So, considered in \( S^{n+1} \) the framed submanifolds \( Z_1, Z_2 \) represent

\[
\pm E^{n+1-p_0} \tau[M_1, M_{j_1}], \quad \pm E^{n+1-p_1} \tau'[M'_1, M'_{j_1}]
\]

respectively and \( Q_1 \) is the desired framed bordism. The case \( \gamma = [t_{j_1}, t_1] \) follows. Note that the symmetry relation of Haefliger and Steer is desuspended one time.

Let now \( \gamma = [t_{j_1}, \gamma'] \) be as at the beginning of this section. Assume inductively that the assertion for \( \gamma' \) is true and the corresponding framed bordism \( W_{\gamma'} \) lies in a small tubular neighbourhood of \( W_{0,1} \) and considered in \( S^{p_0} \) or \( S^{p_1} \) its two boundary parts represent \( \mu_{\gamma'}(\lambda_0) \) and \( \mu_{\gamma'}(\lambda_1) \) respectively. In particularly, this means that \( W_{\gamma'} \) is disjoint from \( V_{0,j_1} \times I \) and \( V_{1,j_1} \times I \). The assertion for \( \gamma \) follows, if we replace \( j_1 \) and \( W_{0,1} \) in the argument above by \( j_i \) and \( W_{\gamma'} \) respectively.
II. Let $D^{p_0+1}$, $D^{p_1+1}$ be two disjoint balls in $S^{n+1}$ which are disjoint from all components of the link $L$. Define

$$L_0' = \partial D^{p_0+1} \sqcup S^{p_1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}$$

$$L_1' = S^{p_0} \sqcup \partial D^{p_1+1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}$$

$$L_{0,1}' = \partial D^{p_0+1} \sqcup \partial D^{p_1+1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}.$$ 

If the condition in part I is not satisfied, namely if the sublinks $L_0$ and $L_1$ of $L$ are not zero $h$-cobordant, then consider the connected sum

$$L' = L - L_0' - L_1' + L_{0,1}'.$$ 

$L'$ satisfies clearly the just mentioned condition. For example, by forgetting the $p_0$-dimensional sphere in $L'$ we obtain

$$(S^{p_1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}) - (S^{p_1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}) -$$

$$(\partial D^{p_1+1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r}) + (\partial D^{p_1+1} \sqcup S^{p_2} \sqcup \cdots \sqcup S^{p_r})$$

which is evidently zero $h$-cobordant. Denote by

$$\lambda_0, \lambda_0^{(1)}, \lambda_0^{(2)}, \lambda_0^{(3)}, \lambda_0^{(4)} \in \pi_{p_0}(S^{k_1} \lor \cdots \lor S^{k_r})$$

the elements given by $L, L', L_0', L_1' \text{ and } L_{0,1}'$ respectively. It holds $\lambda_0^{(2)} = \lambda_0^{(4)} = 0$ and therefore $\lambda_0^{(1)} = \lambda_0 - \lambda_0^{(3)}$. Because the first component of the link representing $\lambda_0^{(3)}$ is the empty, we see $\mu^*_\tau(\lambda_0^{(1)}) = \mu^*_\tau(\lambda_0)$. $\mu^*_\tau(\lambda_1^{(1)}) = \mu^*_\tau(\lambda_1)$ follows by the same token. We finish the proof by using part I.

We will obtain more symmetry relations if we replace the pair $(0, 1)$ by any $(i, j)$ with $0 \leq i \neq j \leq r$. We do not know the exact relationship between our symmetry relations and those of Turaev in [Tu] and those of Koschorke in [Ko 3]. We presume that our relations can in general not be desuspended, because the framed manifolds like $W_{0,1}$ and $W_{0,1}'$, which we have used, take their place very naturally in the sphere $S^{n+1}$.

4. THE $\tau$-REDUCTION

In this section we define first the $\tau$-reductions by using $\tau$-constructions and then construct all the Hilton homomorphisms geometrically by means of $\tau$-reductions.

Let $\gamma$ be a Whitehead product in $t_1, \cdots, t_r$ and $1 \leq i < j \leq r$. If we replace all $[t_i, t_j]$ and $[t_j, t_i]$ in $\gamma$ by $t_{r+1}$, then we get a new Whitehead product $\tau^S_{ji}(\gamma)$ in $t_1, \cdots, t_r, t_{r+1}$. We call $\tau^S_{ji}$ a symbolic reduction. Note that $\tau^S_{ji}(\gamma)$ is generally not a basic Whitehead product even if $\gamma$ is.
We will construct by geometrical means a homomorphism \( \tau_{ji}^R \) such that the following diagram commutes for some Whitehead products \( \gamma \). We call \( \tau_{ji}^R \) a \( \tau \)-reduction. In the diagram \( k_{r+1} = k_i + k_j - 1 \).

\[
\begin{array}{ccc}
FL_{X}^{k_1, \ldots, k_r} & \xrightarrow{\tau_{ji}^R} & FL_{X}^{k_1, \ldots, k_r, k_{r+1}} \\
\gamma^* & & \downarrow \gamma^* \\
FL_X^{q(\gamma)+1} & & \\
\end{array}
\]

Fix \( m \), the dimension of \( X \), and the codimensions \( k_1, \ldots, k_r \). All these numbers are supposed to be \( \geq 2 \). Then there is a \( w_0 \) such that \( FL_X^{q(\gamma)+1} = 0 \) holds for all Whitehead products \( \gamma \) of weight greater than \( w_0 \). We define for \( 2 \leq w \leq w_0 \) and \( 1 \leq i < j \leq r \) a homomorphism \( \tau_{ji}^w \) as follows. Consider a framed \((k_1, \ldots, k_r, k_{r+1})\)-link \( M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \sqcup M_{r+1} \) and define

\[
Z'_w = \tau(M_i, \cdots, \tau(M_i, M_{i+1}^{sh})), \\
Z_w = \tau(M_j, Z'_w),
\]

where \( M_{i+1}^{sh} \) is a small shift of \( M_i \) along the framing and \( (w - 2) \) \( \tau \)-constructions are used to get \( Z'_w \). If \( w = 2 \) let \( Z'_w = M_i \). Let \( \gamma_w = \{\tau_i, [\tau_i, \tau_j, \ldots] \} \) be of weight \( w \) and let \( L_w, L_w^S \) be the framed links in \( \mathbb{R}^{q(\gamma_w)+1} \) representing \( \gamma_w \) and \( \tau_{ji}^S (\gamma_w) \) respectively. The framed submanifold \( Z_w \) is of codimension \( q(\gamma_w) + 1 \), so we can embed the sum \(- (L_w \sqcup \phi) + L_w^S \) fibre-wise in a small tubular neighbourhood of \( Z_w \) to get a new framed link

\[
M'_1 \sqcup M'_2 \sqcup \cdots \sqcup M'_r \sqcup M'_{r+1} = Z_w \times (- (L_w \sqcup \phi) + L_w^S).
\]

Note, by some suitable conventions of the framings involved we can guarantee that for the link representing \( \gamma_w \) it holds strictly \( Z_w = +1 \) (that \( Z_w \) is a single point is proved in Lemma 4.3), from now on we assume this has been done. We define now

\[
\tau_{ji}^w (M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1}) = \bar{M}_1 \sqcup \cdots \sqcup \bar{M}_r \sqcup \bar{M}_{r+1} \\
= (M_1 \sqcup M'_1) \sqcup \cdots \sqcup (M_{r+1} \sqcup M'_{r+1}).
\]

**Lemma 4.1.** The assignment \( \tau_{ji}^w \) above gives a well defined homomorphism

\[
\tau_{ji}^w : FL_{X}^{k_1, \ldots, k_r, k_{r+1}} \longrightarrow FL_X^{k_1, \ldots, k_r, k_{r+1}}.
\]

**Proof.** The \( \tau \)-constructions, fibre-wise embeddings and fusion of components are or induce well defined homomorphisms. \( \square \)
Definition 4.2. Let \( \text{incl}_* : FL_X^{k_1, \ldots, k_r} \to FL_X^{k_1, \ldots, k_r, k_{r+1}} \) be the inclusion, where \( k_{r+1} = k_i + k_j - 1 \). We define
\[
\tau_{ji}^R = \tau_{ji}^2 \circ \tau_{ji}^3 \circ \cdots \circ \tau_{ji}^w \circ \text{incl}_*
\]
and call it a \( \tau \)-reduction.

We observe the following: if \( M_i \) is framed zerobordant, then the link \( M_i \sqcup M_i^{sh} \) is framed zerobordant and therefore every \( Z'_w \) defined as above is framed zerobordant.

Lemma 4.3. Let \( \gamma_w, L_w \) and \( L^S_w \) be as above. It holds \( \tau_{ji}^R(L_w) = L^S_w \).

Proof. The link \( L_w \) is given by (3) and (4) in the proof of Theorem 2.3. Part I of this proof is heavily based on the following observations:

(i) Every \( N_{i,\lambda} \subset M_i \) lies in a \( q(\gamma_w) \)-dimensional subspace of \( \mathbb{R}^{q(\gamma_w)+1} \) and has therefore a (constant) vector in \( \mathbb{R}^{q(\gamma_w)+1} \) as the last normal vector field in its framing (at least up to homotopy of the framing), \( 1 \leq \lambda \leq w - 1 \).

(ii) Every sub-product \( P_{i,\lambda}^{sub} \subset N_{i,\lambda} \) containing the factor \( S^{(\lambda-1)(k_i-1)+k_j-1} \) bounds a suitably framed Seifert surface \( F_{i,\lambda}^{sub} \). If \( \lambda_1 < \lambda_2 \) then we have \( F_{i,\lambda_1}^{sub} \cap F_{i,\lambda_2}^{sub} = \emptyset \). If \( \lambda_1 = \lambda_2 \) we may shift one of the two sub-products slightly along the framing and see that the same is true according to (i).

I. We prove first the following assertion by induction: evaluated on \( L_w \) it holds \( Z'_w = \phi \) for \( w > w \) and \( \{pt\} \) is a set consisting of a single point
\[
Z'_w = \{pt\} \times \cdots \times \{pt\} \times S^{k_j-1},
\]
which is a small shift of the obvious submanifold in \( N_{i,1} \). For \( w = 2 \) it is trivial, so assume \( w \geq 3 \). Let \( \hat{M}_i \) be the components of the link representing \( \gamma_w \). According to (3) and (4) in the proof of Theorem 2.3, \( \hat{M}_i = \hat{N}_{i,1} \sqcup \cdots \sqcup \hat{N}_{i,w-2} \) and
\[
M_i = S_{w-1}^{k_i-1} \times \hat{M}_i \sqcup S^{(w-2)(k_i-1)+k_j-1},
\]
\[
M_j = S_{w-1}^{k_j-1} \times \hat{M}_j.
\]
All other components are the empty. Using the observations we get
\[
Z'_3 = \tau(M_i, M_i^{sh})
\]
\[
= S^{k_i-1} \times \hat{Z}'_3 \sqcup \big( \cup_{\lambda_1=1}^{w-2} \tau(N_{i,\lambda_1}, N_{i,w-1}^{sh}) \big),
\]
where \( \hat{Z}'_3 = \tau(\hat{M}_i, \hat{M}_i^{sh}) \) and \( N_{i,w-1}^{sh} \) is a small shift of \( N_{i,w-1} \). We use the observation again and get
\[
Z'_4 = S^{k_i-1} \times \hat{Z}'_4 \sqcup \big( \cup_{\lambda_2=1}^{w-3} \cup_{\lambda_1 \geq \lambda_2}^{w-2} \tau(N_{i,\lambda_2}, \tau(N_{i,\lambda_1}, N_{i,w-1}^{sh})) \big).
\]
Just repeat this until we get $Z'_w$. $S^{k-1}_{w-1} \times Z'_w = \phi$ is obvious (using the induction assumption). Denote by $\Lambda$ the condition

$$1 \leq \lambda_{w-2} < \lambda_{w-3} < \cdots < \lambda_2 < \lambda_1 \leq w - 2.$$ 

The other part of $Z'_w$ is given by

$$\sqcup \Lambda \tau(N_{i, \lambda_{w-2}}, \tau(\cdots, \tau(N_{i, \lambda_2}, \tau(N_{i, \lambda_1}, N^{sh}_{i, w-1})) \cdots))$$

$$= \tau(N_{i, 1}, \tau(\cdots, \tau(N_{i, w-3}, \tau(N_{i, w-2}, N^{sh}_{i, w-1})) \cdots))$$

$$= \{pt\} \times \cdots \times \{pt\} \times S^{k_j - 1}.$$ 

For $\bar{w} > w$ we see $Z'_{\bar{w}} = \phi$ immediately.

II. From part I we obtain $Z_w = \{pt\}$ with positive sign, and $Z_{\bar{w}} = \phi$ if $\bar{w} > w$. This means

$$\tau^{w+1}_{ji} \circ \cdots \circ \tau^w_{ji} \circ incl_*(L_w) = L_w \sqcup \phi.$$ 

From the definition of $\tau^w_{ji}$ we also have

$$\tau^w_{ji}(L_w \sqcup \phi) = (L_w \sqcup \phi) + \{pt\} \times (-S^1_{w} \sqcup \phi + L^S_{w}) = L^S_w.$$ 

Because the $j$-th component of $L^S_w$ is empty it follows $\tau^2_{ji} \circ \cdots \circ \tau^1_{ji}(L^S_w) = L^S_w$. This shows $\tau^R_{ji}(L_w) = L^S_w$.

We hope that the background of the definition of $\tau^R_{ji}$ is more or less presented in the proof of this lemma. Recall formula (5). We use the negative part $-(L_w \sqcup \phi)$ to eliminate what troubles us and use the part $L^S_w$ to get what we desire. We show next that the $\tau$-reductions fit in the commutative diagram at the beginning of this section for some Whitehead products.

**Lemma 4.4.** Let $\Gamma$ be a system of basic Whitehead products in $\iota_1 < \iota_2 < \cdots < \iota_r$. For all basic Whitehead products $\gamma \in \Gamma$ it holds $\tau^R_{ji} \circ \gamma_* = (\tau^S_{ji}(\gamma))_*$.  

**Proof.** If $\gamma$ is of weight 1 then the statement is trivial. Let the weight of $\gamma$ be greater than 1 and let $L_\gamma = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r$ be the framed link representing $\gamma$. Because every component $M_i$ is framed zero-bordant we need only to show $\tau^R_{ji}(L_\gamma) = L^S_\gamma$ according to Lemma 2.2, where $L^S_\gamma$ represents $\tau^S_{ji}(\gamma)$. If $\gamma$ is of weight 2 the statement follows easily. Let $\gamma = [\alpha, \beta]$ be of weight $\geq 3$. Then formula (1) in the proof of Theorem 2.3 holds. According to the definition $\beta$ has weight at least 2, therefore all components $M_k(\beta)$ of the framed link $L_\beta$ representing $\beta$ are framed zero-bordant. If $\alpha \neq \iota_1$ then the first component $M_1(\alpha)$ of the link $L_\alpha$ representing $\alpha$ is also framed zero-bordant. Using Lemma 2.2 we get the following formula similar to (2) in the proof of Theorem 2.3

$$\tau^R_{ji, 1}(L_\gamma) = S^{l_1-1} \times \tau^R_{ji, 1}(L_\beta) \sqcup S^{l_2-1} \times \tau^R_{ji, 1}(L_\alpha).$$ 

Under the inductive assumption for $\alpha$ and $\beta$ the assertion follows from this formula.
If \( \alpha = \iota_1 \) then the only possibility is \( \gamma = [\iota_1, [\iota_1, \cdots [\iota_1, \iota_j] \cdots] \) according to the definition of basic Whitehead products. If \( \iota_j' \neq \iota_j \) then \( \iota_j \) does not appear in \( \gamma \), for \( \iota_1 < \iota_j \). This means the \( j \)-th component \( M_j \) is the empty and the statement follows. If \( \iota_j' = \iota_j \) then Lemma 4.3 applies. \( \square \)

We define now an ordered sequence \( T^S \) of symbolic reductions as follows. Let

\[
\iota_1 < \cdots < \iota_r < \gamma_1 < \cdots < \gamma_n < \gamma_{n+1} < \cdots
\]

be the sequence of basic Whitehead products in \( \Gamma \). If \( \gamma_1 = [\iota_{i_1}, \iota_{j_1}] \) then the first reduction in \( T^S \) is \( \tau_{j_1, i_1}^S \) determined by \( \gamma_1 \). After this reduction we get

\[
\iota_1 < \cdots < \iota_r < \gamma_1 < \gamma_{1} < \cdots < \gamma_{n} < \gamma_{n+1} < \cdots
\]

Now \( \gamma_{j_2} = [\iota_{i_2}, \iota_{j_2}] \) is a Whitehead product in \( \iota_1, \cdots, \iota_r, \iota_{r+1} \) of weight 2 from which we obtain the second reduction \( \tau_{j_2, i_2}^S \) in \( T^S \). After doing this \( n \)-times one gets

\[
\iota_1 < \cdots < \iota_r < \cdots < \iota_{r+n} < \gamma_{n+1} < \gamma_{n+2} < \cdots < \gamma_m < \gamma_{m+1} < \cdots
\]

It is not difficult to show that for all \( k \geq 1 \) the Whitehead products \( \gamma_{n+k} \) are different and are of weight \( \geq 2 \), and \( \gamma_{n+1} = [\iota_{i_{n+1}}, \iota_{j_{n+1}}] \) is of weight 2. So we define the \( (n+1) \)-th reduction to be \( \tau_{j_{n+1}, i_{n+1}}^S \). Defined in this way \( T^S \) reduces the original sequence to the following

\[
\iota_1 < \cdots < \iota_r < \cdots < \iota_{r+n} < \iota_{r+n+1} < \cdots
\]

Note that if the numbers \( m, k_1, \cdots, k_r \) are fixed, then the sequence of basic Whitehead products \( \gamma_i \) and the sequence \( T^S \) of reductions are both finite.

**Definition 4.5.** Define \( T^R \) to be the sequence of \( \tau \)-reductions determined by \( T^S \). We denote by \( T^S_n \) and \( T^R_n \) the first \( n \) reductions in \( T^S \) and \( T^R \) respectively.

**Theorem 4.6.** Let \( \Gamma \) be a system of basic Whitehead products in \( \iota_1 < \cdots < \iota_r \), such that the conditions \( \alpha_1 < \alpha_2 \) and \([\alpha_1, \beta], [\alpha_2, \beta] \in \Gamma \) together imply \([\alpha_1, \beta] < [\alpha_2, \beta] \). Then the diagram

\[
\begin{array}{ccc}
FL_X^{k_1, \cdots, k_r} & \xrightarrow{T^R} & FL_X^{k_1, \cdots, k_r, k_{r+n(\gamma)}, \cdots} \\
\gamma \downarrow & & \downarrow p_\gamma \\
FL_X^{q(\gamma)+1} & \xrightarrow{id} & FL_X^{q(\gamma)+1} \\
\end{array}
\]

commutes for all \( \gamma \in \Gamma \), where \( k_{r+n(\gamma)} = q(\gamma) + 1 \), \( p_\gamma \) is the obvious projection, \( \iota_{r+n(\gamma)} = T^S(\gamma) \) and we have assumed \( \gamma \) is the \( (r + n(\gamma)) \)-th basic Whitehead product in \( \Gamma \). In particular, \( \Delta_\gamma = p_\gamma \circ T^R \) is exactly the Hilton homomorphism \( H_\gamma \).
Proof. If the first statement is true then one can easily check that $\Delta_\gamma$ satisfies the properties (a) and (b) in §1. For example

$$\Delta_\gamma \circ \gamma_*= p_\gamma \circ T^R \circ \gamma_* = p_\gamma \circ (\tau_{r+n(\gamma)})_* = \text{id}.$$  

(b) follows easily. This shows $H_\gamma = \Delta_\gamma$.

For the first statement we need to show $T^R_k \circ \gamma_* = (T^S_k(\gamma))_*$ for any $k$. Because $\tau_1$ is the first basic Whitehead product in $\Gamma$ the first reduction is $\tau_{j_1,1}$. So the case $k = 1$ follows from Lemma 4.4. Assume inductively that the statement is true for all natural numbers $\leq k$. We will prove the case $k + 1$ by induction on the weight $w$ of $T^S_k(\gamma)$.

Let $\tau^R_j$ be the $(k + 1)$-th reduction. If $w = 1$ the assertion is trivial. Let $w \geq 2$ and denote by $L^k_\gamma$ the framed link representing $T^S_k(\gamma)$. The components of this link are framed zerobordant. By Lemma 2.2 we just need to show $\tau^R_j(L^k_\gamma) = L^k_{\gamma}$. If $w = 2$ this is not difficult to see. Let then $T^S_k(\gamma) = [T^S_k(\alpha), T^S_k(\beta)]$ to be of weight $\geq 3$, and $\gamma = [\alpha, \beta] \in \Gamma$. Then we have

$$M_i(\gamma) = S^{l_1-1} \times M_i(\beta) \sqcup S^{l_2-1} \times M_i(\alpha),$$

where $M_i(\alpha), M_i(\beta), M_i(\gamma)$ are components of the links $L^k_\alpha, L^k_\beta, L^k_\gamma$ representing $T^S_k(\alpha), T^S_k(\beta)$ and $T^S_k(\gamma)$ respectively. According to the definition of $T^S$ and the basic Whitehead products we know $T^S_k(\beta)$ is at least of weight $2$ and therefore the components of the corresponding link are framed zerobordant. If $T^S_k(\alpha) \neq t_i$ then $M_i(\alpha)$ is also framed zerobordant. Using Lemma 2.2 again we obtain

$$\tau^R_{j_1}(L^k_\gamma) = S^{l_1-1} \times \tau^R_{j_1}(L^k_\beta) \sqcup S^{l_2-1} \times \tau^R_{j_1}(L^k_\alpha).$$

The statement for $\gamma$ now follows from this formula under the inductive assumption for $\alpha$ and $\beta$. If $T^S_k(\alpha) = t_i$ then

$$T^S_k(\gamma) = [t_i, [t_{i_1}, \ldots [t_{i_t}, t_{j'}] \ldots ]],$$

with $t_i \geq t_{i_1} \geq \cdots \geq t_{i_t} < t_{j'}$ and $t_i < t_{j}$, according to the construction of $T^S$. So if $t_{j'} \neq t_{j}$ then $t_{j}$ does not appear in $T^S_k(\gamma)$ and the assertion follows trivially. Let $t_{j'} = t_{j}$ and assume $t_i > t_{i_t}$. Denote the original basic Whitehead products of $[t_{i_t}, t_{j}]$ and $[t_i, t_{j}]$ by $\gamma_1, \gamma_2$ respectively. Then $\gamma_1 < \gamma_2$, according to the condition on $\gamma$ in the theorem. This means $\tau^R_{j_1} \gamma_i$ is the $k'$-th reduction with $k' < k + 1$. But after this $k'$-th reduction there is no appearance of $[t_{i_t}, t_{j}]$. So $t_i = t_{i_1} = \cdots = t_{i_t}$ is the only possibility. The statement follows now from Lemma 4.3. 

The restriction in Theorem 4.6 on the system of basic Whitehead products is not necessary, but the sequences $T^S$ and $T^R$ should be adjusted as follows. Let $\Gamma$ be any system of basic Whitehead products in $\tau_1 < \tau_2 \cdots < \tau_r$. We give the elements of $\Gamma$ a new order $\prec$ which respects the weights and has the following property: if $\alpha_1, \alpha_2, [\alpha_1, \beta], [\alpha_2, \beta]$ are basic Whitehead products in $\Gamma$ and $\alpha_1 < \alpha_2$, then $[\alpha_1, \beta] < [\alpha_2, \beta]$. 

\[\Box\]
It is easily seen that such an order exists. Using this new order of $\Gamma$ we get a sequence $T^S(\Gamma)$ of symbolic reductions and the corresponding sequence $T^R(\Gamma)$ of $\tau$-reductions. If $\gamma$ is the $(r + n(\gamma))$-th and $(r + n'(\gamma))$-th basic Whitehead product in $\Gamma$ with respect to the old order $< $ and the new order $\prec$ respectively, then the $(r + n'(\gamma))$-th reduction in $T^S(\Gamma)$ is the one after which $\gamma$ is just reduced to $\tau_{r+n(\gamma)}$ (not $\tau_{r+n'(\gamma)}$). Note that if $\gamma$ and $\gamma'$ are basic Whitehead products of the same weight and if the first $k$ reductions $T^S_k(\Gamma)$ already reduce $\gamma$ to weight 1, then the weight of $T^S_k(\Gamma)(\gamma')$ must $\leq 2$.

**Proposition 4.7.** The result in Theorem 4.6 still holds for any system $\Gamma$ of basic Whitehead products in $\tau_1 < \tau_2 \cdots < \tau_r$, if we replace the sequences $T^S$, $T^R$ there by the sequences $T^S(\Gamma)$ and $T^R(\Gamma)$.

**Proof.** Except following changes the proof remains the same.

Let $\tau^R_{ji}$ be the $(k+1)$-th reduction and $T^S_k(\Gamma)(\gamma) = [T^S_k(\Gamma)(\alpha), T^S_k(\Gamma)(\beta)]$ be of weight $\geq 3$, $\gamma = [\alpha, \beta] \in \Gamma$. We see $T^S_k(\Gamma)(\beta)$ may be of weight 1, in this case $T^S_k(\Gamma)(\alpha)$ must be of weight 2 according to the definitions of basic Whitehead products and the sequence of symbolic reductions. So $T^S_k(\Gamma)(\gamma) = [[\tau_{i_1}, \tau_{i_2}], \tau_{i_3}]$. Let $\tau_{i_3} = \tau_{i_1}$, then no one of $\tau_{i_1}, \tau_{i_2}$ can be $\tau_j$, because, if $\alpha_1, \alpha_2, \gamma'$ are the original basic Whitehead products of $\tau_{i_1}, \tau_{i_2}$ and $\tau_j$, then we have $w(\alpha_1) < w(\alpha) < w(\beta) \leq w(\gamma')$ according to the definition of the basic Whitehead products. So the assertion follows trivially. Other cases (where $\tau_{i_3} \neq \tau_{i_1}$) can be easily checked.

If the weights of both $T^S_k(\Gamma)(\alpha)$ and $T^S_k(\Gamma)(\beta)$ are at least 2, or if $T^S_k(\Gamma)(\alpha)$ is of weight 1 but is different from $\tau_i$ and $T^S_k(\Gamma)(\beta)$ is of weight at least 2, then the statement follows by induction as in the proof of Theorem 4.6.

So assume the weight of $T^S_k(\Gamma)(\beta)$ is at least 2 and $T^S_k(\Gamma)(\alpha) = \tau_i$, then

$$T^S_k(\Gamma)(\gamma) = [\tau_i, [T^S_k(\Gamma)(\beta_1), T^S_k(\Gamma)(\beta_2)]]$$

Because $\alpha \geq \beta_1$, the weight of the original basic Whitehead product of $[\tau_i, \tau_j]$ is clearly greater than the weight of $\beta_1$; according to the definitions of $\prec$ and $T^S(\Gamma)$, a basic Whitehead product can only be symbolically reduced to weight 1 when all the basic Whitehead products of smaller weight are already reduced to weight 1. This means $T^S_k(\Gamma)(\beta_1) = \tau_{i_1}$ must be of weight 1. Therefore

$$T^S_k(\Gamma)(\gamma) = [\tau_i, [\tau_{i_1}, \cdots, [\tau_{i_t}, \tau_{j'}]\cdots]].$$

Now let $\alpha, \beta''', \alpha_1, \cdots, \alpha_t, \beta''$ be the original basic Whitehead products corresponding to $\tau_i, \tau_j, \tau_{i_1}, \cdots, \tau_{i_t}, \tau_{j'}$. Because

$$\beta''' > \alpha \geq \alpha_1 \geq \cdots \geq \alpha_t < \beta''',$$

if $\tau_{j'} \neq \tau_j$ then no one of $\tau_i, \tau_{i_1}, \cdots, \tau_{i_t}$ can be $\tau_j$, the statement follows trivially.
Let now \( t_{j'} = t_j \). If \( t_{i'} \neq t_i \), then \([\alpha_{t_i}, \beta] < [\alpha, \beta]\) with respect to the new order, because \( \alpha_t < \alpha \) with respect to the original order. By the definition of \( T^S(\Gamma) \) this means \( \tau^S_{j,i} \) is the \( k' \)-th reduction in \( T^S(\Gamma) \) with \( k' < k + 1 \). But after this \( k' \)-th reduction \([t_{i'}, t_j]\) can not appear in \( T_k^S(\Gamma)(\gamma) \) for any \( \gamma \). So the only possibility is

\[
T_k^S(\Gamma)(\gamma) = [t_{i'}, [t_{i'}, \cdots, [t_{i'}, t_j] \cdots]].
\]

The assertion follows now from Lemma 4.3. \( \square \)

Note that, if \( T_n^S(\Gamma) \) already reduces \( \gamma \) to weight 1, then \( H_{\gamma} = \Delta^m_{\gamma} = p_{\gamma} \circ T_n^R(\Gamma) \); and if \( \tau^R_{j,i} \) is the \((k+1)\)-th reduction and at least one of \( t_i \) and \( t_j \) does not appear in \( T_k^S(\Gamma)(\gamma) \), we can eliminate \( \tau^R_{j,i} \) from \( T_n^R(\Gamma) \). Therefore if the weight of \( \gamma \) is \( w \), we need exactly \( w - 1 \) \( \tau \)-reductions to get \( H_{\gamma} \).

**Example 4.8.** Let \( r = 2 \) and consider

\[
l_1 < l_2 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5 < \gamma_6 < \cdots
\]

where \( \gamma_1 = [l_1, l_2], \gamma_2 = [l_1, [l_1, l_2]], \gamma_3 = [l_2, [l_1, l_2]], \gamma_4 = [l_1, [l_1, [l_1, l_2]]] \). Then the reductions \( T_4^S = (\tau^S_{2,1}, \tau^S_{3,1}, \tau^S_{3,2}, \tau^S_{4,1}) \) reduce the sequence above to

\[
l_1 < l_2 < l_3 < l_4 < l_5 < l_6 < \gamma_5^4 < \gamma_6^4 \cdots
\]

So, the Hilton homomorphisms corresponding to \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) are given by

\[
p_3 \circ \tau^R_{2,1}, \quad p_4 \circ \tau^R_{3,1} \circ \tau^R_{2,1}, \quad p_5 \circ \tau^R_{3,2} \circ \tau^R_{2,1}, \quad p_6 \circ \tau^R_{4,1} \circ \tau^R_{3,1} \circ \tau^R_{2,1},
\]

where \( p_3, p_4, p_5 \) and \( p_6 \) are the obvious projections.

Let \( B_i \) be a closed connected smooth manifold and \( \xi_i \) be a differential vector bundle over \( B_i, i = 1, 2, \cdots, r \). We may consider links \( M_1 \sqcup \cdots \sqcup M_r \) in \( X \times \mathbb{R} \) such that the normal bundle of \( M_i \) is classified by a bundle map into \( \xi_i \oplus \varepsilon \), where \( \varepsilon \) is the trivial line bundle. We call such links \((\xi_1, \cdots, \xi_r)\)-links. For the bordism group of such links we have the Hilton-Milnor splitting. Note that to perform our \( \tau \)-construction we need only one normal vector field, so \( \tau \)-construction can easily be generalized to \((\xi_1, \cdots, \xi_r)\)-links. Some basic things concerning this have been done in the author’s dissertation [Wa]. We presume there are no essential difficulties to generalize the discussions in this section to the mentioned case above.

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