Hermitian matrices and cohomology of Kähler varieties

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Abstract

We give some upper bounds on the dimension of the kernel of the cup product map $H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$, where $X$ is a compact Kähler variety without Albanese fibrations.

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Introduction

One of the special features of the Kähler geometry is the interplay between topology and linear algebra. The goal of this paper is to give some applications of results in advanced linear algebra appearing in [2] and [14] to obtain bounds on the dimension of the kernel of the cup product mapping

$$
\phi: \bigwedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C}),
$$

in the case where $X$ is a compact Kähler variety admitting no Albanese fibration.

Recall that the notion of no Albanese fibration, for an $n$–dimensional variety $X$ as before, can be given by requiring that, for any $k < n$ and any independent $\beta_1, \ldots, \beta_k \in H^0(\Omega^1(X))$, the product

$$
\beta_1 \wedge \cdots \wedge \beta_k \in H^0(\Omega^k(X)) = H^{k,0}(X)
$$

is not zero. Also remark that $\wedge^k_{i=1} \beta_i = 0$ gives rise to an integrable distribution and to a foliation with closed leaves; this allows to define a fibration (see [10] or later in this paper), hence the definition is consistent.

The main result we prove is:

Theorem 1 Let $X$ be a compact Kähler variety without Albanese fibrations, and let $\phi: \wedge^2 H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ be the cup product.

1. If $q \leq 2n - 1$, then $\phi$ is injective.

2. If $q = 2n$, then $\dim \ker \phi \leq 2c + 3$ where $q = 2c(2b + 1)$, and $b, c$ integers.

3. If $q = 5$ and $n = 2$, then $\dim \ker \phi \leq 14$.

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The previous bounds are achieved thanks to the Hodge decomposition that we shortly recall. We have
\[ H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}; \quad H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}, \]
hence \( \phi \) defines maps:

1. \( \phi^{2,0} : \Lambda^2 H^{1,0} \longrightarrow H^{2,0} \) (and its dual \( \phi^{0,2} \));
2. \( \phi^{1,1} : H^{1,0} \otimes H^{0,1} \longrightarrow H^{1,1}. \)

The role of the assumption of no Albanese fibration becomes clear when dealing with the estimate of \( \dim \ker \phi^{2,0} \) (remark, for instance, that there are not decomposable elements in the kernel).

We look, then, for some upper-bound for \( \kappa = \dim \ker \phi^{1,1} \). Set \( H^{1,1}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{R}) \) and call \( M \subset H^{1,0} \otimes H^{0,1} \) the subspace of forms invariant under complex conjugation. The space \( M \) is naturally identified with the space \( H^q \) of Hermitian \( q \times q \) matrices and \( \phi^{1,1} \) restricts to \( \phi^{1,1}_R : M \rightarrow H^{1,1}(X)_\mathbb{R} \); putting \( K = \ker \phi^{1,1}_R \) we have that \( \kappa = \dim K \). Assuming that \( X \) has no Albanese fibration and using a positivity argument we find restrictions on the signature and hence on the rank of the involved matrices: if \( A \in K \), \( A \neq 0 \), then the rank of \( A \) must be \( \geq 2n \). In particular when \( n = 2q \) then \( A \) is invertible and from \( \mathbb{R} \) we achieve a very good bound on \( \kappa \). Write \( q = 2^c(2b + 1) \), with \( b \) and \( c \) integers; we have \( \kappa \leq 2c + 1 \). The basic remark is that the eigenvalues of \( iA \), with \( i^2 = -1 \), are not real, so if \( v \in S^{2b-1} \subset \mathbb{C}^q \) the tangential projection of \( iAv \) on the unitary sphere defines a vector field not vanishing at any point. The result is then based on \( \mathbb{I} \).

Unfortunately when \( 2n < q \) no good bound seems to be known in general. However, at least for symmetric matrices, some very interesting works have been done; among the others, we suggest \( \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \). In a very strict sense this is a comeback of algebraic geometry. The basic idea is to consider the degeneracy locus \( \{ A : \text{rank}A \leq k \} \) of the matrices as real varieties, then to study its intersection with real subspaces. In particular in \( \mathbb{I} \mathbb{I} \), after the natural projectivization, an elegant Lefschetz fixed point argument is used on a suitable complex algebraic variety. This shows, in the case of \( 5 \times 5 \) real symmetric matrices of rank lower than 4, that the intersection is not empty. By performing the same kind of computations, for \( n = 2 \) and \( q = 5 \) we obtain \( \kappa \leq 8 \).

We would like to focus the importance of the mapping \( \phi \) in the case of compact Kähler varieties. On one side we have Castelnuovo-De Franchis-type theorem concerning Albanese fibrations (see \( \mathbb{I} \mathbb{I} \)). On the other side, formality theorems \( \mathbb{I} \mathbb{I} \mathbb{I} \) imply that the De Rham fundamental group \( \pi_1(X) \otimes \mathbb{C} \) is determined by \( \phi \) \( \mathbb{I} \). We also recall works of Campana (see \( \mathbb{I} \mathbb{I} \)) on Kähler nilpotent groups.

The paper is organized in the following way. In section 1 we recall the Albanese variety and the Castelnuovo-de Franchis theory as developed by Catanese \( \mathbb{I} \mathbb{I} \). In the second section we study the 1.1 real forms and their connection with hermitian matrices. This allows to prove the first two assertions of the above theorem. In section 3 we perform the computations necessary for the first case not covered in \( \mathbb{I} \).

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1 Variety of Albanese type

1.1 Albanese variety

Let $X$ be a complex compact Kähler variety of dimension $n$, and

$$H^{p,q} = H^q(X, \Omega^p)$$

be the Hodge spaces. We have the Hodge decomposition:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}.$$

Set $V = H^{1,0}$ and $H^{0,1} = \nabla$ its conjugate. Integration defines then

$$j : H_1(X, \mathbb{Z}) \longrightarrow V^*,$$

where $*$ stands for dual. Let

$$\text{Alb}(X) = V^*/j(H_1(X, \mathbb{Z}))$$

be the Albanese variety. The irregularity of $X$ is denoted by $q_X = \dim V = \dim A$.

The choice of a base point $p \in X$ defines the Albanese map

$$\alpha : X \longrightarrow \text{Alb}(X).$$

Definition 1.1.1. We say that $X$ is of Albanese type if $\alpha$ is generically finite. We say that $X$ is of Albanese strict type if, moreover, $\alpha$ is not surjective. That is:

$$\dim(\alpha(X)) = \dim(X) < q_X.$$

In the sequel we will assume $X$ of Albanese strict type. For our purposes, thanks to the result of Campana, we could also assume that $\alpha$ is generically one-to-one (see [9] and [11, Ch. 2, Sect. 4]).

1.2 Decomposable form and Albanese fibrations

We describe some Castelnuovo-de Franchis-type theorems. With the previous notations, the cohomology map induced by the Albanese morphism: $\alpha^* : H^k(A, \mathbb{C}) \to H^k(X, \mathbb{C})$ is a Hodge structure map. We can moreover make the identifications:

$$H^k(A, \mathbb{C}) \equiv \bigwedge^k H^1(X, \mathbb{C}), V = H^{1,0}(A) \equiv H^{1,0}(X) \text{ and } H^{p,q}(A) = \bigwedge^p V \otimes \bigwedge^q \nabla.$$

We have maps: $\alpha^{p,q} : \bigwedge^p V \otimes \bigwedge^q \nabla \to H^{p,q}$.

Definition 1.2.1. Given $s > 0$, we say that a (rational map) $f : X \to Y$ is an $s$-Albanese fibration if

1. $Y$ is of Albanese strict type;

2. $\dim X - \dim Y = s$.

When $s = n - 1$, $Y$ is a curve of genus $g > 1$ and $f$ is usually called an irregular pencil.

We have (see [10]):

Proposition 1.2.2. The following conditions are equivalent
1. X has no s-Albanese fibration for $s < n - k$

2. $\alpha^{k,0}$ is injective on the decomposable forms:

$$0 \neq \beta_1 \wedge \cdots \wedge \beta_k \in H^{k,0}(X),$$

$\beta_i$ independent.

Remark 1.2.3. More precisely, Fabrizio Catanese in [10] gave a one-to-one correspondence between fibrations of Albanese type and maximal isotropic subspaces of the first cohomology group of X.

2 Forms and matrices

2.1 Real 1.1 forms

As before, X is of strict Albanese type. We will consider in details the map:

$$\alpha^{1,1} \equiv \phi^{1,1} : H^{1,1}(A) \equiv V \otimes V \rightarrow H^{1,1}(X).$$

(1)

We set

$$\kappa = \dim \ker(\phi^{1,1}).$$

(2)

This is the (1.1) part of $\phi \equiv \alpha^2 : H^2(A, \mathbb{C}) \equiv \wedge^2 H^1(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C})$. Since $\alpha^2$ is a piece of a Hodge structure map, the kernel of $\alpha^2$ is defined over the rational numbers and a fortiori over the real numbers. We have then a map:

$$\alpha^{1,1}_R : H^{1,1}(A)_\mathbb{R} \rightarrow H^{1,1}(X)_\mathbb{R} \subset H^2(X, \mathbb{R})$$

and in particular $\kappa = \dim \ker(\alpha^{1,1}_R)$. We can identify $H^{1,1}(A)_\mathbb{R}$ with the sesquilinear forms on $V^*$. More explicitly, we fix a basis $\beta_j, j = 1, \ldots, q$ of V. An element of $H^{1,1}(A)_\mathbb{R}$ has the form $\Omega = i \sum_{j,s} a_{j,s} \beta_j \wedge \overline{\beta_s}, i^2 = -1, a_{j,s} \in \mathbb{C}: \overline{\Omega} = \Omega$, that is the matrix $A(\Omega) = (a_{j,s})$ is hermitian. Let $\mathbb{H}_q$ be the space of $q \times q$ hermitian matrices. We may define $\sigma : \mathbb{H}_q \rightarrow H^{1,1}(X)_\mathbb{R}$ as:

$$\sigma(A) = i \sum_{j,s} a_{j,s} \beta_j \wedge \overline{\beta_s},$$

(3)

where $A = (a_{j,s})$. We have $\kappa = \dim \ker \sigma$. In particular, the rank and the signature of a form $\Omega \in H^{1,1}(A)_\mathbb{R}$ are well defined.

Definition 2.1.1. If $\Omega \in H^{1,1}(A)_\mathbb{R}$ (respectively $A \in \mathbb{H}_q$) has signature $(r, s)$ (and rank $r + s \leq q$), we set $m(\Omega) = \min(r, s)$ (respectively $m(A) = \min(r, s)$).

The following proposition generalizes the elementary result explained in the introduction.

Proposition 2.1.2. Let $k < n$ be an integer, and assume that X has no $n - k$-Albanese fibration. Let $\Omega \in H^{1,1}(A)_\mathbb{R}, \Omega \neq 0$. If $m(\Omega) \leq k - 1$ then $\alpha^{1,1}(\Omega) \neq 0$.

Proof. Up to a change between $\Omega$ and $-\Omega$, we may assume $s = m(\Omega)$ where $(r, s)$ is the signature of $\Omega$, $s \leq k - 1 < n - 1$. We may find a basis $\beta_i$ of $V$ such that

$$\Omega = i \sum_{j=1}^r \beta_j \wedge \overline{\beta_j} - i \sum_{j=r+1}^{r+s} \beta_j \wedge \overline{\beta_j} = \Omega^+ - \Omega^-.$$
Setting $\varphi = \beta_{r+1} \wedge \cdots \wedge \beta_{r+s} \in H^{s,0}(X)$ and $\Theta = \varphi \wedge \overline{\varphi}$, we compute:

$$\Omega \wedge \Theta = i \sum_{j=1}^r \beta_j \wedge \overline{\beta}_j \wedge \Theta = (-1)^s i \sum_{j=1}^r \beta_j \wedge \beta_{r+1} \wedge \cdots \wedge \beta_{r+s} \wedge \beta_j \wedge \beta_{r+1} \wedge \cdots \wedge \beta_{r+s}.$$ 

Now, posing $\varphi_j = \beta_j \wedge \beta_{r+1} \wedge \cdots \wedge \beta_{r+s} \in H^{s+1,0}(X)$ and $\Theta_j = \varphi_j \wedge \overline{\varphi}_j$, we have:

$$\Omega \wedge \Theta = i(-1)^s \sum_j \Theta_j.$$ 

Assume by contradiction $\Omega \in \ker \alpha^{1,1}$. It follows that $\Omega \wedge \Theta = 0$ in $H^{s+1,s+1}(X)$. Fix $\omega = i \sum_{j=1}^q \beta_j \wedge \overline{\beta}_j$; this is the pull-back of a Kähler form on $A$ and is positive on a Zariski open set of $X$, since the Albanese map of $X$ is generically finite. We get

$$0 = \int_X \Omega \wedge \Theta \wedge \omega^{n-s-1} = \sum_j \int_X i(-1)^s \Theta_j \wedge \omega^{n-s-1}.$$ 

All terms have the same sign. It follows that

$$\Theta_j \wedge \omega^{n-s-1} = 0;$$

this forces $\Theta_j = 0$ and finally $\varphi_j = \beta_j \wedge \beta_{r+1} \wedge \cdots \wedge \beta_{r+s} = 0$. Since $s+1 \leq k$, we get a contradiction with Prop. 1.2.2.

We have then the following:

**Corollary 2.1.3.** Assume that $X$ has no Albanese fibration and $\Omega \in \ker(\alpha^{1,1})_\mathbb{R}$, $\Omega \neq 0$; then, $m(\Omega) \geq n - 1$ and rank$(\Omega) \geq 2n$.

### 2.2 Hermitian matrices

Let $\mathbb{H}_q$ be the space of the $q \times q$ hermitian matrices. Let $\mathbb{H}_{q,m} \subset \mathbb{H}_q$ be the subset of the matrices with rank bigger than $m - 1$:

$$\mathbb{H}_{q,m} \in \{A \in \mathbb{H}_q : \text{rank}(A) \geq m\}.$$ 

**Definition 2.2.1.** Let $V \subset \mathbb{H}_q$ be a real subspace. We say that $V$ has rank $\geq n$ ($\text{rank}(V) \geq n$) if $V \setminus \{0\} \subset \mathbb{H}_{q,n}$. We set $d_{q,n} = \max_{\text{rank}(V) \geq n} \dim V$.

**Proposition 2.2.2.**

1. If $q \leq m - 1$, then $d_{q,m} = 0$.

2. $d_{q,q} = 2c + 1$, where $q = 2^c(2b + 1)$, with $b$ and $c$ integers.

3. $d_{5,4} \leq 8$.

**Proof.** 1. Obvious.

2. The elements in $\mathbb{H}_{q,q}$ are given by invertible hermitian matrices. Then, we come back to the hermitian case in [2]. This result was obtained as a consequence of [1].

3. To be computed in the next section.

Recalling that $\kappa = \dim \ker \phi^{1,1} = \dim \ker \sigma$ (see [3], we have the following:
Proposition 2.2.3. Let \( n = \dim X, \ q = \dim H^{1,0}(X) \) and \( \kappa \) as before. Assume that \( X \) has no Albanese fibration; then \( \kappa \leq d_{q,2n} \).

Proof. It follows from 2.1.3.

Here we present some consequences of Prop. 2.2.2.

Corollary 2.2.4. If \( q \leq 2n - 1 \), then \( \alpha^2 : H^2(A, \mathbb{C}) \to H^2(X, \mathbb{C}) \) is injective.

Proof. Firstly one has that \( \alpha^{1,1} \equiv \phi^{1,1} \) is injective. Then consider \( \omega \in H^{2,0}(A) \) such that \( \alpha^*(\omega) = 0 \); we show that \( \omega = 0 \). Indeed, if it is not, we can find a basis \( \beta_i \) of \( V \) for which \( \omega = \sum_{i=1}^k \beta_i \wedge \beta_{i+k} \), with \( k < n \); taking the \( k - 1 \) form \( \phi = \wedge_{i=2}^k \beta_i \), we get \( \alpha^*(\omega \wedge \phi) = 0 \) and consequently \( \wedge_{i=1}^{k+1} \beta_i = 0 \) on \( X \). By 2.2.2 this would give an Albanese fibration contradicting our assumptions.

The previous result is essentially standard linear algebra. The first part of the following proposition is a consequence of the hard topological result of Adams [22].

Proposition 2.2.5. Assume than \( X \) has no Albanese fibration and \( q = 2n \). Write \( q = 2^s(2b + 1) \), \( b \) and \( c \) being integers. Then we have

1. \( \dim(\ker(\alpha^{1,1})) \leq 2c + 1 \) and
2. \( \dim(\ker(\alpha^{2,0})) \leq 1 \).

The two inequalities above can be unified by saying that \( \dim \ker(\alpha^2) \leq 2c + 3 \). Consequently it holds: \( b_2(X) \geq \dim \text{Im} \phi \geq q(2q - 1) - 2c - 3 \).

Proof. 1. From 2.2.3 and 2.2.2 we have \( \dim \ker \sigma \leq 2c + 1 \).

2. Arguing as in 21.13 the nontrivial forms in \( \ker \alpha^{2,0} \) must be of maximal rank \( n \). Representing them as anti-symmetric matrices, the elements in \( \ker \alpha^{2,0} \setminus \{0\} \) are invertible. This is a complex space, and it follows that \( \dim(\ker \alpha^{2,0}) \leq 1 \).

Proposition 2.2.6. Let \( X \) be a compact algebraic variety without irregular pencils whose fundamental group admits a presentation with \( \gamma \) generators and \( \rho \) relations.

1. If \( q = 2n \), then \( \rho - \gamma \geq q(2q - 3) - 2c - 3 \); moreover, if \( X \) is a surface i.e. \( n = 2 \), \( c_2(X) \geq 7 \).

2. If \( q = 5 \) and \( X \) is a surface, then \( b_2(X) \geq 31 \), hence \( \rho - \gamma \geq 31 \) and \( c_2(X) \geq 13 \).

Proof. Firstly recall that (see [3, Th. 1.1] and [2, Ch. 3]) the interplay between the cup product map and the fundamental group give rise to the following estimate: \( \rho - \gamma \geq \dim \text{Im}(\phi) - 2q \).

1. The estimate on \( \rho - \gamma \) follows directly from the previous remark and 2.2.5.

When \( q = 2n = 4 \), we have \( b_1(X) = b_3(X) = 8 \) and \( b_2(X) \geq 21 \).

2. To deduce \( b_2(X) \geq 31 \) one uses the third part of 2.2.2 together with the fact that \( \dim \text{Im} \phi^{2,0} \geq 2q - 3 = 7 \). The rest is as in the previous point.

Remark 2.2.7. In [3] the following estimate for a compact Kähler variety of any dimension with no irrational pencil is given: \( \rho - \gamma \geq 4q - 7 \).
The following result was our historical motivation:

**Corollary 2.2.8.** Let \( X \) be a minimal projective algebraic surface with \( q = 4 \) and \( p_g = \dim H^{2,0}(X) = 5 \). Let \( K \) be the canonical bundle of \( X \); then \( 16 \leq K^2 \leq 17 \).

**Proof.** In [13] it was proved that \( K^2 \geq 16 \) and that if \( X \) has an irregular pencil then \( K^2 = 16 \). When \( X \) has no irregular pencil Noether formula (see [7, 15]) \( K^2 + c_2(X) = 12\chi_{hol} = 12(p_g - q + 1) = 24 \) and the inequality \( c_2(X) \geq 7 \) forces then \( K^2 \leq 17 \). \qed

**Remark 2.2.9.** In the previous example the Miyaoka-Bogomolov-Yau inequality (see for instance [7]) gives only \( K^2 \leq 9\chi_{hol} = 18 \). No examples of surfaces with \( K^2 = 17 \) are actually known.

## 3 Real degeneracy loci.

The present section is entirely dedicated to the proof of the third point in 2.2.2. To this purpose, we adapt to hermitian matrices the work developed in [14] in the case of real symmetric matrices, by thinking to the degeneracy locus \( \{ A \text{ hermitian}, \ rank A \leq m \} \) as a real variety. The main theoretic tool proved in [14] is the following theorem, which is a consequence of the Hodge splitting and the Lefschetz fixed-point theorem:

**Theorem 3.1.** Let \( V(\mathbb{R}) \subset \mathbb{R}P^n \) be an algebraic variety whose complexification \( V \subset \mathbb{P}^n \) is an irreducible variety in codimension \( m \). Assume that the singular locus of \( V \) has codimension at least \( 2r + 1 \) in \( V \) and that, for a generic \( L \subset Gr(m + 2r + 1, \mathbb{C}^{n+1}) \), the (topological) Euler characteristic \( \chi(V \cap L) \) is odd. Then, for any \( P \subset Gr(m + 2r + 1, \mathbb{R}^{n+1}) \), \( V(\mathbb{R}) \cap P \neq \emptyset \).

In our case, by choosing \( V \) as the projectivization of the appropriate degeneracy locus (namely the \( 5 \times 5 \) complex matrices with rank 3 or less), and \( L \) as a generic 8-dimensional projective space, we show that the Euler characteristic of \( V \cap L \) is odd, hence deducing that in the space of \( 5 \times 5 \) hermitian matrices with rank not bigger than 3 there are 8 linearly independent elements. This statement is equivalent to say that \( d_{5,4} \leq 8 \). About the calculation of \( \chi(V \cap L) \), the arguments we use are pretty standard ones and essentially concern Schubert calculus of Grassmann spaces and Chern classes of projective bundles.

### 3.1 Chern classes of determinantal varieties.

Endow the space of complex \( q \times q \) matrices \( M_q(\mathbb{C}) \) with the real structure given by the (antiholomorphic) involution \( A \mapsto \overline{A}^t \) (where the bar stands for complex conjugation and \( t \) for transposition); with this choice, the real part of \( M_q(\mathbb{C}) \) is the space \( \mathbb{H}_q \) of hermitian matrices. Moreover, the real structure restricts to a real structure over the irreducible affine variety \( V^ q_{q,m} = \{ A \in M_q(\mathbb{C}) \mid \ rank(A) \leq m \} \) and its real part \( V^q_{q,m}(\mathbb{R}) \) is exactly the complement in \( \mathbb{H}_q \) of the set \( \mathbb{H}_{q,m+1} \) defined in the previous section.

Consider now the projectivizations \( \mathbb{P}^{q^2 - 1} = \mathbb{P}(M_q(\mathbb{C})) \) and \( V^q_{q,m} = \mathbb{P}(V^q_{q,m}) \), and set

\[
D_{q,m} = \min \{ \dim P \mid V^q_{q,m}(\mathbb{R}) \cap P \neq \emptyset \text{ for any linear } P \subset \mathbb{R}P^{q^2-1} \}.
\]

**Proposition 3.1.1.** Let \( d_{q,m} \) be as in the previous section. Then \( d_{q,m+1} = D_{q,m} \).
Proof. This should be clear since, by definition of $D_{q,m}$, there is a linear space of projective dimension $D_{q,m} - 1$ entirely included in $\mathbb{P}H_{q,m+1} = \mathbb{R}^q - V_{q,m}(\mathbb{R})$.

We would like, now, to make use of theorem 3.1: we set $V = V_{5,3} \subset \mathbb{P}^{24}$ with codimension 4 and degree 50; its singular locus is $V_{5,2}$ with codimension 5 in $V_{5,3}$ (see e.g. [5]). Our goal is showing that $D_{5,3} \leq 8$, hence the only thing to prove is that, given a generic $L = \mathbb{P}^{8} \subset \mathbb{P}^{24}$, one has:

$$
\chi(V_{5,3} \cap \mathbb{P}^{8}) \equiv 1 \mod (2).
$$

(4)

To this purpose, consider the following resolution of $V_{5,3}$ with a smooth variety $Y$:

$$
\begin{array}{ccc}
O(1) & \supset & O(1)|_Y \simeq O_{\mathbb{P}}(1) \\
\mathbb{P}(M_5(\mathbb{C})) \times Gr(3, \mathbb{C}^5) & \supset & Y = \{([A], W) \mid ImA \subset W\} \simeq \mathbb{P}(S^{\otimes 5}) \\
V_{5,3} & \overset{\pi_1}{\twoheadrightarrow} & \mathbb{P}(M_5(\mathbb{C})) \times Gr(3, \mathbb{C}^5) \\
& \overset{\pi_2}{\twoheadrightarrow} & Gr(3, \mathbb{C}^5).
\end{array}
$$

Here, $S \to Gr(3, \mathbb{C}^5)$ is the tautological bundle, $\mathbb{P}(S^5)$ the projective bundle associated to $S^{\otimes 5}$ and $O_{\mathbb{P}}(1)$ and $O(1)$ are the duals to the tautological bundles over their respective projective bundles. Hyperplane sections of $O(1)|_Y$ are exactly the same of those of $O_{\mathbb{P}}(1)$. Remark that, outside $\pi_1^{-1}(V_{5,2})$, $Y$ is formed by couples of type $([A], ImA)$ hence $\pi_1$ is generically 1 – 1; remark also that the isomorphism $Y \to \mathbb{P}(S^5)$ is given by the map $([A], W) \to A \in Hom(C^5, W) \simeq W \otimes \mathbb{C}^{*5} \simeq W^{\otimes 5}$.

For a vector bundle $F \to B$, let us denote its Chern classes by $c_i(F) \in H^{2i}(B, \mathbb{Z})$ and its Chern polynomial $\sum_{k=0}^{\infty} c_k(F)t^k$ by $c_t(F)$.

Computing $\chi(V_{5,3} \cap \mathbb{P}^{8})$ is the same as computing $\chi(Y \cap \bigcap_{i=1}^{16} H_i)$ with $H_i$ generic hyperplanes defined by sections of $O(1)$; since $Z = Y \cap \bigcap_{i=1}^{16} H_i$ is a smooth complex 4-dimensional manifold, denoting by $T_Z$ its tangent bundle and by $[Z] \in H_8(Z, \mathbb{Z})$ its fundamental class, we get

$$
\chi(Z) = c_4(T_Z)[Z].
$$

Following [14] and [15] we have:

**Proposition 3.1.2.** Let $i: Z \to Y$ be the embedding and $h = c_1(O(1)|_Y)$; then

$$
c_t(T_Y) = i^*c_t(T_Y|_Z)(1 + ht)^{-16}
$$

and, denoting by $e_4$ the coefficient of $t^4$ in $c_t(T_Y)(1 + ht)^{-16}$:

$$
\chi(Z) = h^{16}e_4[Y].
$$

(5)

Proof. The first equality is a consequence of the exact bundle sequence over $Z$:

$$0 \to T_Z \to i^*T_Y|_Z \to \bigoplus_{i=1}^{16} N_{H_i}|_Z \to 0$$

in which $N_{H_i}$ is the normal bundle of $H_i \subset Y$. The second one follows as in ([15] Ch. 9.2)) since (the restriction of) $h$ represents a submanifold in $Y \cap H_1 \cap \cdots \cap H_k$ for every $k$. 

\[\square\]
The diffeomorphism $Y \simeq \mathbb{P}(S^5)$ allows us to compute $c_t(T_Y)$ by means of the exact bundle sequences

$$0 \to T\mathbb{P}(S^5)/G \to T\mathbb{P}(S^5) \to p^*T_G \to 0$$
$$0 \to C \to S^5 \otimes \mathcal{O}_\mathbb{P}(1) \to Q \otimes \mathcal{O}_\mathbb{P}(1) = T\mathbb{P}(S^5)/G \to 0$$

where the first one comes from the projection $p : \mathbb{P}(S^5) \to G = G(3, \mathbb{C}^5)$ and the second one is the tautological sequence over $\mathbb{P}(S^5)$ tensorised with $\mathcal{O}_\mathbb{P}(1)$. Working out calculations we find (see [14]):

$$c_t(T\mathbb{P}(S^5)) = c_t(T_G)c_t(T\mathbb{P}(S^5)/G) = c_t(T_G) \left( \sum_{j=0}^{15} c_j(S^5)t^j(1 + ht)^{15-j} \right). \quad (6)$$

### 3.2 Computation of the Euler characteristic (modulo 2).

We now dispose of all the needed tools to complete the proof of the statement 4. Remark that universal coefficient theorem implies that $H^*(B, \mathbb{Z}) \otimes \mathbb{Z}_2 = H^*(B, \mathbb{Z}_2)$, where $B$ is either $G(3, \mathbb{C}^5)$ or $\mathbb{P}(S^5)$. Thanks to this, since we are only interested to the parity of our objects, we will perform any computation in $H^*(\cdot, \mathbb{Z}_2)$; moreover all polynomials in the Chern classes will be 1.1 truncated according to our real necessities.

Let us denote by $c_i$ the Chern classes of $S$.

**Proposition 3.2.1. Modulo 2, we have:**

1. $c_t(S^5) = 1 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6$;

2. $c_t(T_G) = 1 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + (c_6c_2 + c_3^2)t^6$;

3. The ring $H^*(\mathbb{P}(S^5), \mathbb{Z}_2)$ is $\mathbb{Z}_2[c_1, c_2, h]$ together with the relations

$$h^{15} = \sum_{j=1}^{15} c_j(S^5)h^{15-j} \quad c_1c_2^2 = 0 \quad c_1^4 + c_1^2c_2 + c_2^2 = 0 \quad (7)$$

**Proof.**

1. Follows directly from $c_t(S^5) = c_t(S)^5$.

2. Follows from the sequence $0 \to S \otimes \tilde{S} \to \tilde{S}^5 \to Q \otimes \tilde{S} = T_G \to 0$ where $\tilde{S}$ is the dual of $S$ and $c_t(S) = c_{-1}(S) = c_t(S)$ since we are working in $\mathbb{Z}_2$. Hence: $c_t(T_G) = c_t(S \otimes \tilde{S})c_t(S)^5$.

3. (For a more detailed treatment, see [8]).

The structure of the cohomology ring of a projective bundle $\mathbb{P}(E) \to B$ is well known: $H^*(\mathbb{P}(E)) \simeq H^*(B)[h]/(h^r + \sum_1^r c_i(E)h^{r-i})$ with $h$ the first Chern class of the canonical bundle over $\mathbb{P}(E)$ and $r = \text{rank}(E)$.

We have that $H^*(\text{Gr}(k, \mathbb{C}^n), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, \ldots, c_k]/(s_{n-k+1}, \ldots, s_n)$ where $c_i$ are the Chern classes of the tautological bundle $S \to \text{Gr}(k, \mathbb{C}^n)$ and $s_i$ those of the quotient bundle $\mathbb{C}^n/S$.

In our context $s_3 = c_1^3 + c_3$, $s_4 = c_1^4 + c_1^2c_2 + c_2^2$ and $s_5 = c_1^5 + c_1^3c_3 + c_1c_2^3$; the statement follows from the elimination of $c_3$ and simplification of the expression of $s_5$. 

$\square$
Let \( e_4 \) be the coefficient of \( t^4 \) in \( c_t(T_{\mathbb{P}(S^5)})(1 + ht)^{-16} \);

**Lemma 3.2.2.** \( e_4 = h^4 + c_1h^3 + c_1^2h^2 + c_2h^2 + c_1c_2h + c_4 \) modulo 2.

**Proof.** From equation (6) we have

\[
c_t(T_{\mathbb{P}(S^5)})(1 + ht)^{-16} = c_t(T_G)(1 + ht)^{-1} \left( \sum_{j=0}^{15} c_j(S^5) \left( \frac{t}{1 + ht} \right)^j \right);
\]

applying 3.2.1 and truncating polynomials to the 4th degree, this becomes

\[
(1 + c_1t + c_2t^2 + c_1^3t^3 + c_2^2t^4)(1 + ht + h^2t^2 + h^3t^3 + h^4t^4) \sum_{j=0}^{4} c_j(S^5)(t + ht^2 + h^2t^3 + h^3t^4)^j.
\]

The statement follows simply by working out calculations and by deleting couples of equal terms.

**Proposition 3.2.3.** The class \( h^{16}e_4 \) generates the top cohomology of \( H^*(\mathbb{P}(S^5), \mathbb{Z}_2) \).

**Proof.** The proof of this statement is almost entirely based on the third point of 3.2.1.

The second and third relation of (7) imply that \( H^{12}(Gr(3, \mathbb{C}^5), \mathbb{Z}_2) \) is generated by any degree 6 monomial in \( c_1, c_2 \), but \( c_1^2c_2^2 \).

We claim that \( H^{10}(\mathbb{P}(S^5), \mathbb{Z}_2) \) is generated by \( gh^{14} \) where \( g \) is any generator of \( H^{12}(Gr(3, \mathbb{C}^5), \mathbb{Z}_2) \). This can be seen directly by reducing the degree of \( h \) in any monomial of type \( m(c_1, c_2)h^{14+k} \) with the aid of the first relation of (7). As an example, consider \( c_1^2c_2h^{16} \):

\[
c_1^2c_2h^{16} = c_1^2c_2h^{15} = c_1^2c_2h(c_1h^{14} + c_2h^{13} + \text{higher order terms in } c_1, c_2) = c_1^3c_2h^{15} + c_1^2c_2h^{14} = c_1^3c_2(c_1h^{14} + \ldots) + 0 = c_1^4c_2h^{14}.
\]

Granting this, one finds (by analogous calculations) that the first and the three last terms in the expression of \( h^{16}e_4 \) given by 3.2.2 vanish, while the third others do not. Hence, \( h^{16}e_4 \) generates \( H^{10}(\mathbb{P}(S^5), \mathbb{Z}_2) \).

**Corollary 3.2.4.** If \( L \) is a generic 8-dimensional linear subspace of \( \mathbb{P}^{24} \), the euler characteristic \( \chi(L \cap V_{5,3}) \) is odd.

**Proof.** By equation (5):

\[
\chi(L \cap V_{5,3}) = \chi(Z) = h^{16}e_4[\mathbb{P}(S^5)]
\]

and the latter term is not zero modulo 2 thanks to 3.2.2 and Poincaré duality between \( H^{40}(\mathbb{P}(S^5), \mathbb{Z}) \) and \( H_{40}(\mathbb{P}(S^5), \mathbb{Z}) \).

**Remark 3.2.5.** The following matrix, depending on 7 parameters, shows that \( d_{5,4} = D_{5,3} \geq 7 \), since rank \( A \geq 4 \) unless \( \alpha = z = u = w = 0 \).

\[
A(\alpha, z, u, w) = \begin{pmatrix}
\alpha & z & u & w & 0 \\
z & \alpha & \bar{w} & -\bar{u} & 0 \\
u & -\bar{w} & -\bar{z} & \alpha & 0 \\
\bar{w} & -u & \bar{z} & -\alpha & z \\
0 & 0 & 0 & \bar{z} & 0
\end{pmatrix} \quad \alpha \in \mathbb{R}, \ z, u, w \in \mathbb{C}.
\]

To see this, let \( A_k \) be the submatrix obtained from \( A \) by elimination of the \( k \)-th row and column; \( A_3 \) is invertible unless \( z = 0 \) or \( |\alpha| = |z| \). When \( z = 0 \), \( A_5^2 = (\alpha^2 + |u|^2 + |w|^2)I \) (see also 2); when \( |\alpha| = |z| \), \( \det A_1 = |z|^2(|z|^2 + |w|^2) \).

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