MANIFOLDS WHICH ARE COMPLEX AND SYMPLECTIC BUT NOT KÄHLER

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Abstract. The first example of a compact manifold admitting both complex and symplectic structures but not admitting a Kähler structure is the renowned Kodaira-Thurston manifold. We review its construction and show that this paradigm is very general and is not related to the fundamental group. More specifically, we prove that the simply-connected 8-dimensional compact manifold of [17] admits both symplectic and complex structures but does not carry Kähler metrics.

1. Introduction

A complex manifold $M$ is a topological space modeled on open subsets of $\mathbb{C}^n$ and with change of charts being complex-differentiable (that is, biholomorphisms). Here we say that $n$ is the complex dimension of $M$. Complex manifolds are the objects that appear naturally in Algebraic Geometry: a projective variety is the zero locus of a collection of polynomials in the complex projective space $\mathbb{CP}^N$. When a projective variety is smooth and of complex dimension $n$, it is a complex manifold of dimension $n$.

A complex manifold $M$ of complex dimension $n$ is in particular a smooth differentiable manifold of real dimension $2n$. Multiplication by $i$ on each complex tangent space $T_pM$, $p \in M$, gives an endomorphism $J: TM \to TM$ such that $J^2 = -\text{Id}$. An endomorphism $J: TM \to TM$ with $J^2 = -\text{Id}$ is called an almost complex structure. For a complex manifold $M$, $J$ satisfies that the Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

vanishes, $N_J(X,Y) = 0$ for all vector fields $X, Y$. In this case, we say that the almost complex structure is integrable. The celebrated Newlander-Nirenberg theorem [30] says that an almost complex structure with $N_J = 0$ is equivalent to a complex structure. Hence, for a smooth manifold $M$ to admit a complex structure, we need to check if there exist almost complex structures (this is a topological question), and then to find an integrable one (this is an analytic problem).

Projective varieties have further geometric properties. The complex projective space $\mathbb{CP}^N$ has a natural hermitian metric, the Fubini-Study metric. This is the natural metric when one views $\mathbb{CP}^N$ as the homogeneous space $U(N+1)/U(1) \times U(N)$. Therefore a projective variety $M \subset \mathbb{CP}^N$ inherits this hermitian metric. Denote by $h$ the hermitian metric on $M$ and write $h = g + i\omega$, where $g(X,Y) = \text{Re}(h(X,Y))$ and $\omega(X,Y) = \text{Im}(h(X,Y)) = \text{Re}(-ih(X,Y)) = \text{Re}(h(JX,Y)) = g(JX,Y)$. Then $g$ is a Riemannian metric for which $J$ is an isometry ($g(JX,JY) = g(X,Y)$) and $\omega$ turns out to be skew-symmetric, hence it is a 2-form with $\omega(JX,JY) = \omega(X,Y)$ and $g(X,Y) = \omega(X,JY)$. We say that $\omega$ is the fundamental form of $(M,h)$. This 2-form is positive, in the sense that $d\omega = 0$. Therefore, for $\omega = \omega_{FS}$, it also holds $d\omega = 0$.

We say that $(M,h)$ is a Kähler manifold when $M$ is a complex manifold and the fundamental form $\omega$ satisfies $d\omega = 0$. A smooth projective variety is a Kähler manifold. Actually the converse holds when $[\omega] \in H^2(M,\mathbb{R})$ is an integral cohomology class, by Kodaira’s theorem [39].

A different weakening of the Kähler condition (forgetting $J$ but keeping $\omega$) is that of a symplectic structure. A symplectic structure on a smooth $2n$-dimensional manifold $M$ is given by a 2-form $\omega \in \Omega^2(M)$ that satisfies $\omega^n > 0$ (it gives the natural complex orientation). The Fubini-Study metric $h_{FS}$ has fundamental form $\omega_{FS} \in \Omega^2(\mathbb{CP}^N)$. It is easy to see, using the $U(N+1)$-invariance, that $d\omega_{FS} = 0$. Therefore, for $\omega = \omega_{FS}$, it also holds $d\omega = 0$.

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Figure 1. Diagram of the different classes of manifolds, including \( KT \)

\( \Omega^2(M) \) which is closed \((d\omega = 0)\) and non-degenerate \((\omega^n \text{ is nowhere zero})\). Let \( M \) be an even-dimensional manifold endowed with a complex structure \( J \) and a symplectic structure \( \omega \). Then \( J \) is said to be compatible with \( \omega \) if, for vector fields \( X, Y \) on \( M \), the bilinear form

\[
g(X, Y) = \omega(X, JY)
\]

is a Riemannian metric. Therefore a Kähler manifold is a symplectic manifold endowed with a compatible complex structure, and \( h = g + i\omega \) is the Kähler metric. The existence of a Kähler metric on a compact manifold constraints the topology. In particular, if \((M, J, \omega)\) is a compact Kähler manifold of dimension \( 2n \), then (see \[1, 12, 18, 39\])

1. the fundamental group \( \pi_1(M) \) belongs to a very restricted class of groups, called Kähler groups;
2. \( b_{2i-1}(M) \) is even for \( i = 1, \ldots, n \);
3. the Lefschetz map \( L^{n-p}: H^p(M; \mathbb{R}) \to H^{2n-p}(M; \mathbb{R}), a \mapsto [\omega]^{n-p} \wedge a \), is an isomorphism;
4. \( M \) is formal in the sense of Sullivan (see Section 2 for details).

So it is natural to ask if the classes of smooth manifolds admitting complex, symplectic and Kähler structures coincide under some topological constraints.

The lack of examples in symplectic geometry has been haunting this area of mathematics for many years now (pretty much since its \textit{début} as a discipline in its own). Indeed, the main source of examples of symplectic manifolds is Algebraic Geometry. This led to the belief that symplectic and Kähler conditions coincided in the compact case (see for instance \[21\]). There was a discrete breakthrough, in 1976, when Thurston \[38\] gave the first example of a compact symplectic manifold with no Kähler structure. Thurston’s example had already been discovered, as a complex manifold, by Kodaira during his work on the classification of compact complex surfaces \[23\]. We call it the \textit{Kodaira-Thurston manifold} \( KT \). Since \( KT \) is a compact complex and symplectic manifold without Kähler structure, we obtain

**Theorem 1.** There exist compact manifolds which admit complex and symplectic structures but carry no Kähler metrics.

This means that the complex and symplectic structures that \( KT \) admits cannot be compatible. The manifold \( KT \) is in the place shown in Figure 1.

The next natural question is whether some topological constraints may force the symplectic category to reduce to the Kähler one. Regarding the fundamental group, it is natural to look for simply connected symplectic compact manifolds. In \[28\], McDuff constructed a compact, simply connected, symplectic manifold with \( b_3 = 3 \), hence not Kähler. For a detailed study on the relationship between formality and Lefschetz property on symplectic manifolds, we refer to \[10\]. In \[9\], Bock constructed non formal symplectic manifolds with arbitrary Betti numbers.

The construction of simply connected symplectic non formal (compact) manifolds turned out to be a more difficult problem. In fact, it was conjectured in 1994 (see \[26\]) that a compact simply connected symplectic (compact) manifold should be formal: this is the so-called \textit{Lupton-Oprea conjecture} on the formalising tendency of a symplectic structure. This conjecture was proven false by Babenko and Taimanov in 2000 (see \[2\]). For every \( n \geq 5 \), they constructed an example of a simply connected, symplectic non formal compact manifold of real dimension \( 2n \). On the other hand, by a result of Miller
simply connected compact manifolds of dimension ≤ 6 are formal. Hence a remarkable gap in dimension 8 was left. This gap was filled by M. Fernández and the second author in 2008 (see [17]).

Here we shall prove that the manifold constructed in [17] admits a complex structure, thereby giving a new example fitting in the scheme of Theorem 1. The precise result is:

**Theorem 2.** There exists an 8-dimensional, compact, simply connected, symplectic and complex manifold which is non-formal and does not satisfy the Lefschetz property. In particular, it does not admit Kähler structures.

This paper is organized as follows. In Section 2 we recall the basics of rational homotopy theory and formality. In Section 3 we give a description of $KT$, construct explicit complex and symplectic structures on it and show that it carries no Kähler metric. In Section 4 we review the construction of the symplectic manifold $(\hat{M}, \hat{\omega})$ of [17]. This is constructed by resolving symplectically the singularities of a symplectic orbifold $(\tilde{M}, \tilde{\omega})$, a quotient of a compact symplectic nilmanifold $(M, \omega)$ by a certain $\mathbb{Z}_3$-action. In Section 5 we describe a complex structure $J$ on the orbifold $\tilde{M}$ and construct a complex resolution of singularities $(\tilde{M}, J)$. Finally, in Section 6 we show that $\tilde{M}$ and $\mathcal{M}$ are diffeomorphic.

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### 2. Formality

Formality is a property of the rational homotopy type of a space $X$. We present here a rough introduction, referring to [13, 15, 19] for more details. By space, we mean a connected CW complex of finite type (we allow a finite number of cells in each dimension) which is nilpotent (its fundamental group is nilpotent and acts nilpotently on higher homotopy groups). A space $X$ is rational if $\pi_i(X)$ is a rational vector space for every $i ≥ 1$ (recall that a nilpotent group has a well defined rationalization). The rationalization of a space $X$ is a rational space $X\mathbb{Q}$ together with a map $f: X → X\mathbb{Q}$ such that $f_∗: \pi_i(X) → \pi_i(X\mathbb{Q})$ is an isomorphism for every $i ≥ 1$. We identify two spaces if they have a common rationalization. By rational homotopy type of a space $X$ we mean the homotopy type of its rationalization. Quillen and Sullivan proposed two different approaches to capture the rational homotopy type of a space in an algebraic model, see [33, 37]. Here we review briefly Sullivan’s ideas.

A **commutative differential graded algebra** $(A, d)$ over a field $k$ of zero characteristic ($k$-cdga for short) is a graded algebra $A = \bigoplus_{i≥0} A^i$ which is graded commutative, together with a $k$-linear map $d: A^i → A^{i+1}$, the differential, which satisfies $d^2 = 0$ and which is a graded derivation, i.e., for homogeneous elements $a ∈ A^p$ and $b ∈ A^q$, $d(a \cdot b) = (da) \cdot b + (-1)^p a \cdot (db)$.

The cohomology of $(A, d)$, denoted $H^∗(A)$, is a $k$-cdga with trivial differential. A $k$-cdga is **connected** if $H^0(A) \cong k$.

The de Rham algebra $\Omega(M)$ of a smooth manifold $M$, together with the exterior differential, is an $\mathbb{R}$-cdga. The piecewise linear forms $APL(X)$ on a PL-manifold $X$, endowed with a suitable differential combining the exterior differential and the boundary of simplices, form a $\mathbb{Q}$-cdga (see [19]). There is a de Rham-type theorem for both cdga’s, hence we have isomorphisms

$$H^∗(\Omega(M)) \cong H^∗(M; \mathbb{R}) \quad \text{and} \quad H^∗(APL(X)) \cong H^∗(X; \mathbb{Q}).$$

Let $X$ be a space. The idea of Sullivan is to replace $APL(X)$ by another $\mathbb{Q}$-cdga, which has the same cohomological information as $APL(X)$ but is algebraically more tractable: the **minimal model**. A $k$-cdga $(A, d)$ is **minimal** if

- $A$ is the free graded algebra over a graded vector space $V = \oplus V^i$; this means that $A$ is the tensor product of the exterior algebra on the odd degree generators and the symmetric algebra on the even degree generators, $A = \text{Ext}(V^{\text{odd}}) \otimes \text{Sym}(V^{\text{even}})$. The standard notation is $A = \Lambda V$.
there exists a collection \( \{x_i\}_{i \in J} \) of generators of \( V \), indexed by a well-ordered set \( J \), such that \( |x_i| \leq |x_j| \) if \( i < j \) and the differential of a generator \( x_j \) is an element of \( \Lambda^{|x_j|} \). Here \( |\cdot| \) denotes the degree and \( V^{<\cdot} \) consists of the generators \( x_i \) with \( i < j \). Notice, in particular, that \( d \) does not have linear part.

We denote a minimal \( k \)-cdga by \((AV,d)\). A minimal model for a \( k \)-cdga \((A,d)\) is a minimal \( k \)-cdga \((AV,d)\) together with a \( k \)-cdga morphism \( \phi: (AV,d) \rightarrow (A,d) \) which induces an isomorphism in cohomology (such a morphism is called quasi-isomorphism).

We have the following fundamental result:

**Theorem 3** ([14], Theorem 14.12). Any connected \( k \)-cdga has a minimal model, which is unique up to isomorphism.

By definition, the rational minimal model of a space \( X \), \((AV_X,d)\), is the minimal model of the \( \mathbb{Q} \)-cdga \((A_{PL}(X),d)\). One can show that, when \( M \) is a smooth manifold, the real minimal model of \( M \) can be computed from the de Rham algebra \((\Omega(M),d)\). A central result in rational homotopy theory is the following:

**Theorem 4** ([37]). Two spaces have the same rational homotopy type if and only if their rational minimal models are isomorphic.

In particular, PL forms (resp. smooth forms) contain all the rational-homotopic (resp. real-homotopic) information of a space (smooth manifold). It is often difficult to know the whole de Rham algebra of a manifold; it would be very convenient if the (say, real) minimal model could be constructed directly from the de Rham cohomology. A space for which this happens is called formal. More precisely, a space \( X \) is formal if there exists a quasi-isomorphism \((AV_X,d) \rightarrow (H^*(X;\mathbb{Q}),0)\). In particular, the rational homotopy type of a formal space \( X \) is a formal consequence of its rational cohomology. Many spaces are known to be formal: compact Lie groups, \( H \)-spaces, symmetric spaces, ... For us, the relevant result is the following:

**Theorem 5** ([12]). A smooth compact manifold \( M \) admitting a Kähler structure is formal.

A very useful criterion for establishing formality is the following:

**Theorem 6** ([12], Theorem 4.1). Let \( X \) be a space and let \((AV_X,d)\) be its minimal model. Then \( X \) is formal if and only if we can write \( V_X = C \oplus N \) with \( d = 0 \) on \( C \) and \( d \) injective on \( N \), in such way that every closed element in the ideal generated by \( N \) in exact.

Let \((A,d)\) be a \( k \)-cdga and let \( H^*(A) \) be its cohomology. Let \( a \in H^{[a]}(A) \), \( b \in H^{[b]}(A) \) and \( c \in H^{[c]}(A) \) such that \( a \cdot b = b \cdot c = 0 \). Then \( a \cdot b \cdot c \) is zero for two reasons. Consequently, a difference element \((a,b,c) \in H^{[a]+[b]+[c]-1}(A)/\mathcal{J}\) can be formed, where \( \mathcal{J} \) is the ideal generated by \( a \) and \( c \) in \( H^*(A) \). Take cocycles \( \alpha, \beta, \gamma \in A \) representing \( a, b, c \) respectively. Then \( \alpha \cdot \beta = d\xi \) and \( \beta \cdot \gamma = d\eta \), hence \( \xi \cdot \gamma + (-1)^{[\alpha]+1}\alpha \cdot \eta \) is a closed \(([a] + [b] + [c] - 1)\)-form whose cohomology class is well defined modulo \( \mathcal{J} \). We set \((a,b,c) = [\xi \cdot \gamma + (-1)^{[\alpha]+1}\alpha \cdot \eta] \). Then \((a,b,c)\) is called the triple Massey product of the cohomology classes \( a, b, c \).

The definition of higher Massey products is as follows (see [23] [27]). Given \( a_i \in H^{[a_i]}(A) \), \( 1 \leq i \leq t \), \( t \geq 3 \), the Massey product \( \langle a_1, a_2, \ldots, a_t \rangle \) is defined if there are \( \alpha_{i,j} \in \mathcal{A} \), with \( 1 \leq i \leq j \leq t \), except for the case \((i,j) = (1,t)\), such that

\[
\alpha_i = [\alpha_{i,i}], \quad d\alpha_{i,j} = \sum_{k=i}^{j-1} (-1)^{[\alpha_{i,k}]} \alpha_{i,k} \wedge \alpha_{k+1,j}.
\]

Then the Massey product is

\[
\langle a_1, a_2, \ldots, a_t \rangle = \left\{ \sum_{k=1}^{t-1} (-1)^{[\alpha_{1,k}]} \alpha_{1,k} \wedge \alpha_{k+1,t} \right\} \subset H^{[\alpha_{1}+\cdots+[\alpha_t]-(t-2)]}(A),
\]

(2)
where the \( \alpha_{ij} \) are as in \([2]\). We say that the Massey product is trivial if \( 0 \in \langle a_1, a_2, \ldots, a_t \rangle \). Note that for \( (a_1, a_2, \ldots, a_t) \) to be defined it is necessary that both \((a_1, \ldots, a_{t-1})\) and \((a_2, \ldots, a_t)\) are defined and trivial.

**Proposition 7.** If \( X \) is formal then all (higher) Massey products of \( (\Lambda V_X, d) \) are zero.

**Proof.** The proof can be found in \([3]\). We shall give a simple proof for the case of triple and quadruple Massey products, which suffices for this paper.

As \( X \) is formal, Theorem \([6]\) guarantees that we can write \( V_X = C \oplus N \) with \( d = 0 \) on \( C \) and \( d \) injective on \( N \), in such way that every closed element in the ideal \( I(N) \) generated by \( N \) in exact. Note that there is a decomposition \( \Lambda V = \Lambda C \oplus I(N) \). Let \( a_i \in H^{[a_i]}(A) \), \( 1 \leq i \leq t \). By definition of Massey product, there are \( \alpha_{i,r} \in \Lambda V \) with \( a_i = [\alpha_{i,r}] \), and for each \( i < j \), \( (i, j) \neq (1, t) \), there are \( \alpha_{ij} \) with

\[
\begin{align*}
\alpha_{ij} &= \sum_{k=1}^{i-1} (-1)^{[a_i,k]} \alpha_{i,k} \wedge \alpha_{k+1,j}. \\
&= \sum_{k=1}^{i-1} (-1)^{[a_i,k]} \alpha_{i,k} \wedge \alpha_{k+1,j}.
\end{align*}
\]

Write \( \alpha_{ij} = \beta_{ij} + \eta_{ij} \) with \( \beta_{ij}, \eta_{ij} \in I(N) \). As \( d\alpha_{ij} = d\eta_{ij} \), we can use in the case of triple Massey products (that is, \( t = 3 \)), the elements \( \eta_{12} \) and \( \eta_{23} \). Then the triple Massey product \((\alpha_1, \alpha_2, \alpha_3)\) contains \((-1)^{[12]} \eta_{12} \eta_{033} + (-1)^{[11]} \alpha_{11} \eta_{23} \). Which is in \( I(N) \), hence exact.

In the case of quadruple Massey products (that is, \( t = 4 \)), we use \( \eta_{12}, \eta_{23}, \eta_{34} \) instead of \( \alpha_{12}, \alpha_{23}, \alpha_{34} \). The equation

\[
\begin{align*}
d\psi_{13} &= (-1)^{[12]} \alpha_{12} \alpha_{033} + (-1)^{[11]} \alpha_{11} \alpha_{23} \\
&= (-1)^{[12]} \eta_{12} \alpha_{033} + (-1)^{[11]} \alpha_{11} \eta_{23} + (-1)^{[12]} \beta_{12} \alpha_{033} + (-1)^{[11]} \alpha_{11} \beta_{23}
\end{align*}
\]

implies that \((-1)^{[12]} \eta_{12} \alpha_{033} + (-1)^{[11]} \alpha_{11} \eta_{23} \) is closed, hence exact (as it lives in \( I(N) \)). Write it as \( d\psi_{13} \) with \( \psi_{13} \in I(N) \). Analogously define \( \psi_{24} \). Thus the quadruple Massey product \((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) contains \((-1)^{[12]} \psi_{13} \alpha_{044} + (-1)^{[11]} \eta_{12} \eta_{34} + (-1)^{[11]} \alpha_{11} \psi_{24} \) which is in \( I(N) \), hence exact. \( \square \)

3. The Kodaira-Thurston manifold

The Kodaira-Thurston manifold can be described in various ways. For Kodaira, \( KT \) was a compact quotient of \( \mathbb{C}^2 \) by a certain group acting co-compactly. Thurston interpreted it as a symplectic \( T^2 \)-bundle over \( T^2 \). In this section we describe it as a nilmanifold, write down explicit symplectic and complex structures on \( KT \) and show that \( KT \) carries no Kähler metric.

A nilmanifold is a compact quotient of a simply connected, nilpotent Lie group \( G \) by a lattice \( \Gamma \). Since \( \Gamma \) is a subgroup of a nilpotent group, it is also nilpotent. The exponential map \( \exp: \mathfrak{g} \to G \) is a diffeomorphism, hence \( G \cong \mathbb{R}^n \) for some \( n \). Therefore, if \( N = \Gamma \setminus G \) is a compact nilmanifold, \( G \to N \) is the universal cover, \( \pi_1(N) \cong \Gamma \) and \( \pi_1(N) = 0 \) for \( i \geq 2 \). Hence a nilmanifold is a nilpotent space.

Nilmanifolds are interesting because they are a rich source of answers to many questions in different areas of Mathematics. As we already mentioned, \( KT \) was the first example of a compact symplectic non-Kähler manifold. From the point of view of complex geometry, there exist complex nilmanifolds for which the Fröhlicher spectral sequence is arbitrarily non-degenerate, see \([5]\).

Kähler nilmanifolds are very special:

**Theorem 8** (Benson-Gordon, Hasegawa \([3, 22]\)). Let \( N \) be a compact symplectic nilmanifold endowed with a Kähler structure. Then \( N \) is diffeomorphic to a torus.

Benson and Gordon proved that a symplectic nilmanifold \( N \) of dimension \( 2n \) for which the Lefschetz map \( L^{n-1}: H^1(N; \mathbb{R}) \to H^{2n-1}(N; \mathbb{R}) \) is an isomorphism is diffeomorphic to a torus. Hasegawa showed that a formal nilmanifold \( N \) is diffeomorphic to a torus. Notice, however, that there exists many examples of non-toral symplectic and complex nilmanifolds (see \([5, 20, 30]\)).
Let $H$ denote the Heisenberg group, i.e.

$$H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and let $H_\mathbb{Z}$ denote the subgroup of matrices with entries in $\mathbb{Z}$. Then $H$ is a nilpotent Lie group, diffeomorphic to $\mathbb{R}^3$, $H_\mathbb{Z} \subset H$ is a lattice and $N = H_\mathbb{Z} \backslash H$ is a compact nilmanifold. Let $G = H \times \mathbb{R}$ and $G_\mathbb{Z} = H_\mathbb{Z} \times \mathbb{Z}$. The Kodaira-Thurston manifold is $KT = G_\mathbb{Z} \backslash G$.

Let $\mathfrak{t}$ be a Lie algebra over a field $k$ of characteristic zero. The exterior algebra $\Lambda^\bullet \mathfrak{t}^*$ is endowed with a differential $d: \Lambda^p \mathfrak{t}^* \to \Lambda^{p+1} \mathfrak{t}^*$, defined as follows: $d: \mathfrak{t}^* \to \Lambda^2 \mathfrak{t}^*$ is the dualization of the bracket, i.e. $(da)(X,Y) = -\alpha([X,Y])$ if $\alpha \in \mathfrak{t}^*$ and $X, Y \in \mathfrak{t}$. $d$ is then extended to $\Lambda^\bullet \mathfrak{t}^*$ by imposing the graded Leibnitz rule: for $\alpha \in \Lambda^p \mathfrak{t}^*$ and $\beta \in \Lambda^q \mathfrak{t}^*$, $d(\alpha \wedge \beta) = (da) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$. The vanishing of $d^2$ is equivalent to the Jacobi identity in $\mathfrak{t}$. In the language of the Section 2, $(\Lambda^\bullet \mathfrak{t}^*, d)$ is a $k$-cdga, known as Chevalley-Eilenberg complex of $\mathfrak{t}$.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{g}^\ast$ be its dual. We identify tensors on $\mathfrak{g}$ and $\mathfrak{g}^\ast$ with left-invariant objects on $G$, which therefore descend to $KT$. It is easy to check that $\mathfrak{g}$ has a basis $\{X_1, X_2, X_3, X_4\}$ in which the only non-zero bracket is $[X_1, X_2] = -X_3$. Let $\langle x_1, x_2, x_3, x_4 \rangle$ be the dual basis of $\mathfrak{g}^\ast$. The only non-zero differential on $\mathfrak{g}^\ast$ is computed to be $dx_3 = x_1 \wedge x_2$.

The element $\omega = x_1 \wedge x_4 + x_2 \wedge x_3 \in \Lambda^2 \mathfrak{g}^\ast$ is closed and non-degenerate. By abuse of notation, we denote by $\omega$ the corresponding left-invariant symplectic structure on $KT$ as well. Thus $(KT, \omega)$ is a compact symplectic 4-manifold.

Recall that if $\mathfrak{t}$ is an even-dimensional Lie algebra, $J: \mathfrak{t} \to \mathfrak{t}$ is a complex structure if $J^2 = -\text{Id}$ and it satisfies the integrability condition

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0, \text{ for } X, Y \in \mathfrak{t}. \quad (4)$$

In our situation, define $J: \mathfrak{g} \to \mathfrak{g}$ by

$$J(X_1) = X_2, \quad J(X_2) = -X_1, \quad J(X_3) = X_4 \quad \text{and} \quad J(X_4) = -X_3.$$ 

A straightforward computation shows that (4) holds, hence $J$ is a complex structure on $\mathfrak{g}$. Again by abuse of notation, we denote by $J$ the corresponding left-invariant complex structure on $KT$. Thus $(KT, J)$ is a compact complex surface.

Let $N = \Gamma \backslash G$ be a compact nilmanifold. Considering $\Lambda^\bullet \mathfrak{g}^\ast$ as left-invariant forms on $N$, we have a natural inclusion $\iota: (\Lambda^\bullet \mathfrak{g}^\ast, d) \to (\Omega(N), d)$. By a result of Nomizu (see [31]), $\iota$ is a quasi-isomorphism, hence the de Rham cohomology of $N$ is isomorphic to the cohomology of the Chevalley-Eilenberg complex of $\mathfrak{g}$. In our case, three of the four generators of $\mathfrak{g}^\ast$ are closed, hence we get $b_1(KT) = 3$.

Since $KT$ has an odd Betti number which is odd, we see that it does not carry any Kähler metric. We also see explicitly that $KT$ does not satisfy the Lefschetz property. Indeed, take $[x_2] \in H^1(KT; \mathbb{R})$. Then $\mathcal{L}: H^1(KT; \mathbb{R}) \to H^1(KT; \mathbb{R})$ sends $[x_2]$ to $-[x_1 \wedge x_2 \wedge x_3] = [-d(x_3 \wedge x_4)] = 0$.

The Lie algebra $\mathfrak{g}$ is endowed with a complex structure $J$ and a symplectic structure $\omega$. Define a tensor $g: \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ by

$$g(X,Y) = \omega(X,JY), \quad X, Y \in \mathfrak{g}.$$ 

It is easy to see that the matrix of $g$ in the basis $\{X_1, X_2, X_3, X_4\}$ is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

$g$ is not a scalar product on $\mathfrak{g}$, hence the corresponding left-invariant tensor on $KT$ is not a Riemannian metric.

Let $M$ be a manifold endowed with a complex structure $J$ and a symplectic structure $\omega$. One could in principle relax condition (1) above and ask $J$ to be only tamed by $\omega$, which means $\omega(X,JX) > 0$.
for $X \in \mathfrak{X}(M)$. A symplectic manifold $(M, \omega)$ endowed with a tamed complex structure $J$ is called Hermitian-symplectic. There are no known examples of compact Hermitian-symplectic non Kähler manifolds.

We see that $(KT, J, \omega)$ is not Hermitian-symplectic. Indeed, $\omega(X_1, JX_1) = 0$. It is proved in [34, Theorem 2.12] that a compact nilmanifold endowed with a Hermitian-symplectic structure is actually Kähler. Hence we see that $KT$ does not carry any Hermitian-symplectic structure (not just left-invariant).

To see explicitly that $KT$ is non formal, we need to compute the minimal model of a nilmanifold.

**Theorem 9 ([22]).** Let $N = \Gamma\backslash G$ be a compact nilmanifold. Then $(\Lambda g^*, d)$ is the rational minimal model of $N$.

Since a nilmanifold is a nilpotent space, Theorem 4 holds and the rational homotopy of a compact nilmanifold is codified in the corresponding minimal model. Here $(\Lambda g^*, d)$ is a nilpotent Lie group manifold. Hermitian-symplectic $X$ model of nilmanifold is codified in the corresponding minimal model. Here (Λ $\mathfrak{g}$) we see that a compact nilmanifold endowed with a Hermitian-symplectic structure is actually Kähler. Hence the ideal generated by $N_a$ Take $C$ modulo the ideal generated in $\langle z, x_1 \rangle$. Massey product showing the importance of such manifolds in the whole theory.

In this section we recall the construction of a simply connected, 8-dimensional symplectic non-formal manifold performed in [17]. Although quite involved, this construction also starts with a nilmanifold, namely, set $M = Z_3 \times \mathbb{C}$. Also, let $M = Z_3 \backslash G$ be the discrete subgroup of matrices with entries in $\Gamma$. We let $G_\Gamma$ act on $G$ on the left and set $M = G_\Gamma\backslash G$. Then $M$ is a compact complex parallelizable nilmanifold. Notice that $M$ can be seen as a principal torus bundle

$$T^2 = \Gamma\backslash \mathbb{C} \hookrightarrow M \to T^6 = (\Gamma\backslash \mathbb{C})^3$$

using the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)$. $M$ is a complex version of the Kodaira-Thurston manifold.

We interpret $\mathbb{Z}_3$ as the group of cubic roots of unity and consider the right $\mathbb{Z}_3$-action $\rho: \mathbb{Z}_3 \times G \to G$ given, in terms of a generator $\zeta = e^{2\pi i/3}$, by

$$(\zeta, (u_1, u_2, u_3, u_4)) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4).$$

This action preserves the group operation on $G$ and the lattice, hence descends to an action of $\mathbb{Z}_3$ on $M$. Set $\tilde{M} = M/\mathbb{Z}_3$. Then $\tilde{M}$ is not smooth, it has 81 isolated quotient singularities.

A basis for left-invariant 1-forms on $G$ is given by

$$\mu = du_1, \quad \nu = du_2, \quad \theta = du_3 - u_2 du_1 \quad \text{and} \quad \eta = du_4$$
The action of $\mathbb{Z}_3$ on left-invariant 1-forms is given by

$$\rho^*\mu = \zeta\mu, \quad \rho^*\nu = \zeta\nu, \quad \rho^*\theta = \zeta^2\theta \quad \text{and} \quad \rho^*\eta = \zeta\eta.$$ 

The 2-form

$$\omega = i\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i\eta \wedge \bar{\eta}$$

(6)
on $M$ satisfies $\bar{\omega} = \omega$, so it is real. It is closed and satisfies $\omega^4 \neq 0$. Thus $\omega$ is a symplectic form. Notice also that

$$\rho^*\omega = \zeta^3(i\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i\eta \wedge \bar{\eta}) + \zeta^{-3}\bar{\nu} \wedge \bar{\theta} = \omega,$$

hence $\omega$ is $\mathbb{Z}_3$–invariant and descends to a symplectic form $\bar{\omega}$ on the quotient $\hat{M}$. Therefore $(\hat{M}, \bar{\omega})$ is a symplectic orbifold. In [17] a desingularization procedure for the symplectic orbifold is given, producing a symplectic manifold.

**Proposition 10** ([17], Propositions 2.1 and 2.3). There exists a smooth compact simply connected symplectic manifold $(\hat{M}, \bar{\omega})$ which is isomorphic to $(\hat{M}, \bar{\omega})$ outside a small neighborhood of the singular points.

In [17], it is shown that $(\hat{M}, \bar{\omega})$ is non formal, and also that it does not satisfy the Lefschetz property (see Remark 3.3 in [17]). The non-formality is seen in [17] via a modification of the Massey product, which are studied extensively in [11]. Here we shall see the non-formality of $\hat{M}$ by showing that there exists a non-zero quadruple Massey product. Transferring the Massey product from $\hat{M}$ to the desingularization $\tilde{M}$ follows the arguments of [17, Theorem 3.2] and it is quite standard.

The complex minimal model of $M$ is $\Lambda V_M = \Lambda(\mu, \nu, \theta, \eta, \bar{\mu}, \bar{\nu}, \bar{\theta}, \bar{\eta})$ with $d\theta = \mu \wedge \nu$ and $d\bar{\theta} = \bar{\mu} \wedge \bar{\nu}$. Our orbifold is $\tilde{M} = M/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts in the minimal model as $\alpha \mapsto (\zeta\mu, \zeta\nu, \zeta^2\theta, \zeta\eta)$. A model (that is, a $\mathbb{C}$-cdga quasi-isomorphic to its minimal model) for $M$ is given by $A = (\Lambda V_M)^{\mathbb{Z}_3}$. Easily

$$A^1 = 0,$$

$$A^2 = (\mu, \nu, \eta) \wedge (\bar{\mu}, \bar{\nu}, \bar{\eta}) \oplus (\mu \wedge \theta, \nu \wedge \eta, \bar{\mu} \wedge \bar{\theta}, \bar{\nu} \wedge \bar{\eta}, \eta \wedge \bar{\theta}, \theta \wedge \bar{\eta}),$$

$$A^3 = \Lambda^3(\mu, \nu, \eta, \bar{\eta}) \oplus \Lambda^3(\bar{\mu}, \bar{\nu}, \bar{\eta}, \theta).$$

With this, one can check that $H^3(A) = 0$.

Take now $a_1 = [\nu \wedge \eta], a_2 = [\mu \wedge \bar{\mu}], a_3 = [\mu \wedge \bar{\mu}]$ and $a_4 = [\eta \wedge \bar{\nu}]$. We shall compute $\langle a_1, a_2, a_3, a_4 \rangle$ and check that it does not contain the zero element. A Massey product $b \in \langle a_1, a_2, a_3, a_4 \rangle$ is computed according to formula (3). As $A^1 = 0$, it must be $\alpha_{11} = \nu \wedge \eta, \alpha_{22} = \alpha_{33} = \mu \wedge \bar{\mu}$ and $\alpha_{44} = \eta \wedge \bar{\nu}$. Then

$$\begin{align*}
\alpha_{12} &= -\theta \wedge \bar{\mu} \wedge \bar{\eta} + z_1, \\
\alpha_{13} &= \nu \wedge \bar{\eta} \wedge f_2 - f_1 \wedge \mu \wedge \bar{\mu} + w_1, \\
\alpha_{23} &= z_2, \\
\alpha_{24} &= \mu \wedge \bar{\mu} \wedge f_3 - f_2 \wedge \eta \wedge \bar{\nu} + w_2, \\
\alpha_{34} &= -\theta \wedge \mu \wedge \eta + z_3,
\end{align*}$$

where $z_1, z_2, z_3 \in A^3$ are closed, hence exact, thus $z_i = df_i$, with $f_i \in A^2$, and $w_1, w_2 \in A^4$ are closed. A computation gives

$$b = [a_{11} \wedge a_{24} - a_{12} \wedge a_{34} + a_{13} \wedge a_{44}] = [\theta \wedge \bar{\theta} \wedge \mu \wedge \bar{\mu} \wedge \eta \wedge \bar{\eta} + w_1 \wedge \eta \wedge \bar{\nu} + w_2 \wedge \nu \wedge \bar{\eta}].$$

To check that this is non-zero, we multiply by $[\nu \wedge \bar{\nu}]$. Then the terms with $w_1$ and $w_2$ cancel, so $b \wedge [\nu \wedge \bar{\nu}] \neq 0$, hence $b \neq 0$. 
5. The complex structure

In this section we describe the complex structure $J$ on $G$ in two equivalent ways, and we show that it descends to $M = G/T$ and also to the orbifold $\hat{M} = (G/T)/\mathbb{Z}_2$. Then we construct a complex resolution of singularities, which will give a smooth complex 4-fold $(\hat{M}, \hat{J})$.

Let us consider the group $G = H_C \times C$ above. $G$ can be realized as a complex Lie subgroup of $GL(5, C)$ by sending the pair $(A, u_4) \in H_C \times C$ to the matrix

$$
\begin{pmatrix}
1 & u_2 & u_3 & 0 & 0 \\
0 & 1 & u_1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & u_4 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

$GL(5, C)$ is an open subset of $C^{25}$, hence each tangent space $T_X GL(5, C) \cong C^{25}$, $X \in GL(5, C)$, inherits the standard complex structure of the ambient space, which is the multiplication by $i = \sqrt{-1}$. As a complex submanifold of $GL(5, C)$, $G$ inherits the same complex structure on each tangent space. This means that the complex structure on $G$ is multiplication by $i$ on each tangent space $T_g G$, $g \in G$. The left translations $L_g: G \to G$, $h \mapsto gh$, are holomorphic maps, since they are written as polynomials in local coordinates. This shows that $G$ is a complex parallelizable Lie group: the differential of $L_g$ is complex linear and a parallelization is given by moving $T_g G$ around. Let $J$ denote the complex structure on $G$ induced by the inclusion $G \hookrightarrow GL(5, C)$. The above considerations show that $J$ is left invariant.

Let us consider the tangent space $T_e G$, where $e \in G$ is the identity. There is an identification between the Lie algebra $\mathfrak{g}$ of $G$ and the vector space of left invariant holomorphic vector fields on $G$, endowed with the natural Lie bracket. The complex structure on $\mathfrak{g}$ is multiplication by $i$, and $\mathfrak{g}$ is a complex Lie algebra of dimension 4, described as follows:

$$
\mathfrak{g} = \{(Z_1, Z_2, Z_3, Z_4) \mid [Z_1, Z_2] = -Z_3\}.
$$

By identifying $\mathfrak{g}$ with $T_e G$, one has $T_g G = d_e L_g(\mathfrak{g})$, $\forall g \in G$. This shows again that the complex structure $J_g$ on $T_g G$ is multiplication by $i$, for every $g \in G$.

We go through the details of the construction of left invariant complex structure on $G$. Let $J_e$ denote the complex structure (i.e. multiplication by $i$) on $\mathfrak{g}$ and let $g \in G$ be a point. Define the complex structure $J_g: T_g G \to T_g G$ as

$$
J_g(X(g)) = d_e L_g(i x),
$$

where $X$ is a left invariant vector field on $G$ and $x \in \mathfrak{g}$ is such that $d_e L_g(x) = X(g)$. This defines $J$ as a smooth section of the bundle $End(TG)$. Let us show that $J^2 = -\text{Id}$. Indeed,

$$
J_g^2(X(g)) = J_g(J_g(X(g))) = d_e L_g(i(i x)) = -d_e L_g(x) = -X(g).
$$

Lemma 11. The (almost) complex structure defined above is left invariant.

Proof. We must prove that, for every $g \in G$, $(L_g)^* J = J$. So take $X(h) \in T_h G$. Then

$$
J_h(X(h)) = d_e L_h(i x),
$$

where $x \in \mathfrak{g}$ is the unique vector satisfying $d_e L_h(x) = X(h)$. On the other hand we have

$$
((L_g)^* J)(X(h)) = d_{gh} L_{g^{-1}} \circ (J_{gh}) \circ (d_h L_g(X(h)))
$$

$$
= d_{gh} L_{g^{-1}} \circ d_e L_{gh}(i x) = d_e L_h(i x)
$$

$$
= J_h(X(h)).
$$

\[\square\]

Lemma 12. The (almost) complex structure defined above is integrable.
Proof. This is trivial. Since $J$ is left invariant, it is enough to work in the Lie algebra $\mathfrak{g}$. But on $\mathfrak{g}$ the complex structure is multiplication by $i$, hence the Nijenhuis tensor
\[ N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,Y] = [X,Y] + i[iX,Y] + i[X,iY] - [iX,iY] = 0, \]
for $X, Y \in \mathfrak{g}$.

Lemma 13. The two complex structures on $G$ coincide.

Proof. It is enough to notice that the left translations are holomorphic maps, thus their differential is complex linear. Let $g \in G$ be a point and $X$ a left invariant vector field on $G$, such that $X(g) = d_eL_g(x)$, $x \in \mathfrak{g}$. Then
\[ iX(g) = id_eL_g(x) = d_eL_g(ix) = J_g(X(g)). \]

So far we know that the natural complex structure $J$ on the Lie group $G = H_\mathbb{C} \times \mathbb{C}$ is left invariant and it is multiplication by $i$ on each tangent space. As above, let $G_\Gamma \subset G$ be the subgroup of matrices whose elements belong to the lattice $\Gamma = \{a + bk \mid \zeta = e^{2\pi i/3}, a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Since $J$ is left invariant, it defines a complex structure on the quotient $M = G_\Gamma \backslash G$, which we denote again by $J$. Hence $(M, J)$ is a complex nilmanifold.

Next we show that $J$ is compatible with the $\mathbb{Z}_3$-action defined by $[5]$. The complex structure $J$ on $M$ is multiplication by $i$ at each tangent space $T_pM$, $p \in M$, since it comes from the complex structure on $G$. Let $\varphi: M \to M$ denote the $\mathbb{Z}_3$-action, and consider the map
\[ d_p\varphi: T_pM \to T_{\varphi(p)}M. \]

We claim that the map $\varphi$ can be lifted to a holomorphic action $\tilde{\varphi}$ of $\mathbb{Z}_3$ on $G$. By taking global coordinates $(u_1, u_2, u_3, u_4)$ on $G$, $\tilde{\varphi}$ sends the generator $\zeta \in \mathbb{Z}_3$ to the diagonal matrix $\text{diag}(\zeta, \zeta, \zeta^2, \zeta)$. Since $\tilde{\varphi}$ is linear, it coincides with its differential $d_p\tilde{\varphi}: T_pG \to T_{\tilde{\varphi}(p)}G$. This is clearly a complex linear map, i.e.
\[ d_p\tilde{\varphi} \circ J_g = J_{\tilde{\varphi}(g)} \circ d_p\tilde{\varphi}. \]
(7)

This proves the claim. Since the complex structure $J$ on $M$ is multiplication by $i$ on each tangent space, (7) shows that we can write
\[ d_p\varphi \circ J_p = J_{\varphi(p)} \circ d_p\varphi, \]
showing that the complex structure commutes with the $\mathbb{Z}_3$-action, hence descends to the quotient $\hat{M} = M/\mathbb{Z}_3$. We denote by $\hat{J}$ the complex structure on $\hat{M}$. Thus we have proved:

Proposition 14. Let $M = G_\Gamma \backslash G$ be as above and denote by $J$ the natural complex structure on $M$. Then $(\hat{M}, \hat{J})$ is a complex orbifold.

Remark 15. The complex nilmanifold $M$ is an example of an 8-dimensional non-simply connected complex, symplectic and non-Kähler manifold, the symplectic form being given by $[6]$. Indeed, $M$ is non-formal, hence it can not be Kähler. One can show that $(\hat{M}, \hat{J}, \hat{\omega})$ is simply connected. Therefore we have an example of an 8-dimensional simply connected complex and symplectic orbifold which is not Kähler. Indeed, one can see that $\hat{M}$ is not formal $[17]$, while Kähler orbifolds are formal $[6]$.

Proposition 16. There exists a smooth complex manifold $(\tilde{M}, \tilde{J})$ which is biholomorphic to $(\hat{M}, \hat{J})$ outside a neighborhood of a singular point.

Proof. Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$-action. Translating with an element $g \in G$, we can suppose that $p = (0,0,0,0)$ in our coordinates. Let $U \subset M$ be a neighborhood of $p$ and let $\phi: U \to B$ be a holomorphic local chart, given by the exponential map (by holomorphic we mean with respect to the complex structure $J$). Here $B = B_C(0,\varepsilon) \subset \mathbb{C}^4$. In these coordinates, the action of $\mathbb{Z}_3$ can be written as
\[ (u_1, u_2, u_3, u_4) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4). \]
The local model for the singularity is thus $B/\mathbb{Z}_3$. From now on, the desingularization process is formally analogous to that in [17]. We blow up $B$ at $p$ to obtain $\tilde{B}$. The point $p$ is replaced with a complex projective space $F = \mathbb{P}^3 = \mathbb{P}(T_pB)$ on which $\mathbb{Z}_3$ acts by
\[ [u_1 : u_2 : u_3 : u_4] \mapsto [\zeta u_1 : \zeta^2 u_2 : \zeta u_3 : u_4]. \]
Thus $\mathbb{Z}_3$ acts on the exceptional divisor $F$ with fixed locus $\{q\} \cup H$ where $q = [0 : 0 : 1 : 0]$ and $H = \{u_3 = 0\}$. Then one blows up $\tilde{B}$ at $q$ and $H$ to obtain $\tilde{\tilde{B}}$. The point $q$ is replaced by a projective space $H_1 \cong \mathbb{P}^3$. The normal bundle to $H \subset F \subset \tilde{B}$ is the sum of the normal bundle of $H$ in $\mathbb{P}^3$, which is $\mathcal{O}_{\mathbb{P}^2}(1)$, and the restriction to $H$ of the normal bundle of $F$ in $\tilde{B}$, which is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the second blow up replaces the projective plane $H$ with a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ defined as $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1))$. The strict transform of $F \subset \tilde{B}$ under the second blow up is the blow up $\tilde{F}$ of $F$ at $q$, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2 = H$, actually $\tilde{F} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The resulting situation is depicted in Figure 5.

![Figure 2. The second blow-up and the $\mathbb{Z}_3$-action](image)

The fixed point locus of the $\mathbb{Z}_3$-action on $\tilde{\tilde{B}}$ consists of the two disjoint divisors $H_1$ and $H_2$. According to [7], page 82, the quotient $\tilde{\tilde{B}}/\mathbb{Z}_3$ is a smooth Kähler manifold. This provides a complex resolution of the singularity $B/\mathbb{Z}_3$. Notice that the blowing up is performed with respect to the natural complex structure inherited from the ambient space. By resolving every singular point, we obtain a smooth complex manifold $(\overline{M}, \overline{J})$. 

\begin{proposition}
The complex manifold $(\overline{M}, \overline{J})$ is simply connected.
\end{proposition}

\begin{proof}
The proof is analogous to that of [17] Proposition 2.3.
\end{proof}

The desingularization process of Proposition 16 is completely similar to the symplectic resolution of [17] Proposition 2.1. However, the two blow ups are performed with respect to different complex structures. In the complex resolution, one uses the natural complex structure $\hat{J}$ of $\hat{M}$. In the symplectic resolution one uses a (local) complex structure obtained by using a Kähler model for a neighborhood of a fixed point which is not holomorphically equivalent to a local holomorphic chart for $\hat{J}$. Indeed, this Kähler model is obtained by performing the following change of coordinates in a small neighborhood of a fixed point of the action:
\begin{align*}
w_1 &= u_1 \\
w_2 &= \frac{1}{\sqrt{2}}(u_2 + iu_3) \\
w_3 &= \frac{1}{\sqrt{2}}(iu_2 - u_3) \\
w_4 &= u_4
\end{align*}

Certainly, this is not holomorphic with respect to the natural complex structure $\hat{J}$ on $\overline{M}$.

Locally, we have the following situation: on a small neighborhood $U$ of $0 \in \mathbb{C}^4$ (which is a fixed point of the $\mathbb{Z}_3$-action in suitable coordinates) we have two complex structures, $J_1$ and $J_2$. The two complex structures are different, because the change of variables which brings one to the other is not holomorphic. As a consequence, the two blow ups are different. In fact, the natural map that one would
constructs from one resolution to the other would not be even continuous. This becomes particularly clear when the blow up is interpreted as a symplectic cut, following Lerman and McDuff (see for instance [25]). The blow up of $\mathbb{C}^n$ at 0 can be thought of as removing a small ball of radius $\varepsilon$ centered at the origin and then collapsing the fibers of the Hopf fibration in the boundary of the remaining set. But the fibers of the Hopf fibration (i.e. the intersections of the boundary of the ball, which is a $S^{2n-1}$, with the “complex” lines through the origin) depend heavily on the complex structure of the ball.

6. Proof of the main Theorem

In this section we prove that the smooth manifolds which underly the two resolutions $\tilde{M}$ and $\bar{M}$ are diffeomorphic. This completes the proof of Theorem 2.

Proposition 18. The symplectic and the complex resolution of the orbifold $(\tilde{M}, \tilde{J}, \tilde{\omega})$ are diffeomorphic.

Proof. We work locally, in a small neighborhood of each fixed point. We construct a smooth map which is the identity outside this small neighborhood and that does the right job inside the neighborhood. The local model is thus a small ball $B_{\mathbb{C}^4}(0, \delta) \subset \mathbb{C}^4$ endowed with two different complex structures $J_1$ and $J_2$. There is a map $\Theta: B_{\mathbb{C}^4}(0, \delta) \to B_{\mathbb{C}^4}(0, \delta)$ which interchanges the two complex structures, namely

$$ \Theta^* J_1 = J_2. $$

Notice that $\Theta$ can be composed with biholomorphisms on the right and on the left, thus is not unique. If we take $J_1$ as the complex structure on the ball induced by the natural complex structure on $\tilde{M}$ and $J_2$ to be the complex structure associated to the local Kähler model used for the symplectic resolution, then $\Theta$ is given by (8). We introduce real coordinates $u_k = x_k + iy_k$ and $w_k = s_k + it_k$, $k = 1, 2, 3, 4$. In such coordinates, (8) is an automorphism of $\mathbb{R}^8$ written as

$$
\begin{align*}
    s_1 &= x_1 \\
    t_1 &= y_1 \\
    s_2 &= \frac{1}{\sqrt{2}}(x_2 + y_3) \\
    t_2 &= \frac{1}{\sqrt{2}}(y_2 + x_3) \\
    s_3 &= \frac{1}{\sqrt{2}}(y_2 - x_3) \\
    t_3 &= \frac{1}{\sqrt{2}}(x_2 - y_3) \\
    s_4 &= x_4 \\
    t_4 &= y_4
\end{align*}
$$

The associated matrix is

$$
\Theta = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The matrix $\Theta$ belongs to $\text{SO}(8)$. To construct the diffeomorphism we will find an isotopy $\{\Theta_t\}_{t \in [0,1]}$, such that $\Theta_0$ is the identity $\text{Id} \in \text{SO}(8)$ and $\Theta_1 = \Theta$. In this way we get a path of complex structures $J_{t+1} = \Theta_t^* J_1$ connecting $J_1$ and $J_2$. To do this we must produce a smooth path in $\text{SO}(8)$ between the identity matrix and $\Theta$, which is furthermore equivariant with respect to the $\mathbb{Z}_3$-action. In fact it is
enough to find a smooth \( \mathbb{Z}_3 \)-equivariant path in \( \text{SO}(4) \) connecting the identity to the matrix

\[
\theta = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]

In the coordinates \((s_2, t_2, s_3, t_3)\) spanning the \( \mathbb{R}^4 \) of interest, the \( \mathbb{Z}_3 \)-action can be written as

\[
\Psi = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

under the natural inclusion \( \text{U}(2) \hookrightarrow \text{SO}(4) \). We must ensure that the path \( \{ \Theta_s \} \subset \text{SO}(4) \) satisfies \( \Theta_s \circ \Psi = \Psi \circ \Theta_s \), for every \( s \in [0, 1] \). We do this explicitly. First notice that \( \Psi(0) = \text{Id} \) is the image, under the exponential map \( \exp: \mathfrak{so}(4) \to \text{SO}(4) \), of the matrix \( \frac{\pi}{4} Q \), where

\[
Q = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

The matrix \( \theta' \) is the image, under the exponential map \( \exp: \mathfrak{so}(4) \to \text{SO}(4) \), of the matrix \( \frac{\pi}{4} Q \), where

\[
Q = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

Thus a smooth path in \( \text{SO}(4) \) between the identity and \( \theta' \) is given by the image of the straight line in \( \mathfrak{so}(4) \) joining the zero matrix with \( Q \),

\[
\gamma: \ [0, \pi/4] \to \text{SO}(4) \quad s \mapsto \exp(sQ)
\]

One sees that, for every \( s \in [0, \pi/4] \), \( \gamma(s) \circ \Psi = \Psi \circ \gamma(s) \), hence \( \gamma(s) \) is \( \mathbb{Z}_3 \)-equivariant. Now consider the matrix \( P \). We juxtapose the following three paths in order to join \( P \) with the identity matrix:

\[
P_1(s) = \begin{pmatrix}
0 & 0 & \sin(\pi s/2) & \cos(\pi s/2) \\
0 & 0 & \cos(\pi s/2) & -\sin(\pi s/2) \\
\sin(\pi s/2) & \cos(\pi s/2) & 0 & 0 \\
\cos(\pi s/2) & -\sin(\pi s/2) & 0 & 0
\end{pmatrix}
\]

\[
P_2(s) = \begin{pmatrix}
\sin(\pi s/2) & 0 & \cos(\pi s/2) & 0 \\
0 & \sin(\pi s/2) & 0 & -\cos(\pi s/2) \\
\cos(\pi s/2) & 0 & -\sin(\pi s/2) & 0 \\
0 & -\cos(\pi s/2) & 0 & -\sin(\pi s/2)
\end{pmatrix}
\]

\[
P_3(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\cos(\pi s) & \sin(\pi s) \\
0 & 0 & -\sin(\pi s) & -\cos(\pi s)
\end{pmatrix}
\]

Again, a computation shows that \( P_i(s) \circ \Psi = \Psi \circ P_i(s) \), \( \forall s \in [0, 1] \), \( i = 1, 2, 3 \). Hence the path \( P(s) = P_1 \ast P_2 \ast P_3(s) \) satisfies \( P(0) = P \), \( P(1) = \text{Id} \) and is \( \mathbb{Z}_3 \)-equivariant. The path \( \theta(s) = P(1-s)\theta' \) satisfies \( \theta(0) = \theta' \) and \( \theta(1) = \theta \). Finally the path \( \Psi \circ \gamma \circ \theta \) satisfies \( \Psi \circ \gamma(0) = \text{Id} \) and \( \Psi \circ \gamma(1) = \theta \). However \( \Psi \) is not globally smooth, because at the concatenation points it is only continuous. To smooth it, we proceed as follows. Let \( 0 < s_1 < \ldots < s_{n-1} < s_n < 1 \) denote the points in which the resulting path has a cusp. Consider a smooth, increasing function \( h: [0, 1] \to [0, 1] \) such that there exist intervals \( \delta_i = (t_i - \varepsilon, t_i + \varepsilon) \), \( 0 < t_1 < \ldots < t_{n-1} < t_n < 1 \) with \( h(t) = s_i \) for \( t \in \delta_i \). Define a new path...
\[ \Theta_t = \Psi(t \cdot \Theta). \] Clearly \( \Psi \) and \( \Theta \) have the same image. Then \( \Theta_t \) is a smooth, \( \mathbb{Z}_3 \)-equivariant path in \( \text{SO}(4) \) connecting \( \theta \) with the identity matrix. Viewing it as a path in \( \text{SO}(8) \) we obtain the isotopy \( \Theta_t \) such that \( \Theta_0 = \text{Id} \) and \( \Theta_1 = \Theta \). Thus \( \Theta_t^* J_1 = J_1 \) and \( \Theta_t^* J_2 = J_2 \). We also endow the ball with the standard metric. Since \( \mathbb{Z}_3 \subset \text{SO}(8), \mathbb{Z}_3 \) acts by isometries.

We are ready to define the diffeomorphism between the two resolutions. Notice that the expression of the \( \mathbb{Z}_3 \)-action is the same in the two sets of coordinates \((u_1, \ldots, u_4)\) and \((w_1, \ldots, w_4)\). Thus when we blow up we get, in both cases, an exceptional divisor \( C \) where \( C \) is a cut-off function, i.e. a fixed hyperplane \( H \) that \( \Theta_1 \cdot J_1 \) and \( \Theta_2 \cdot J_2 \). Thus \( \Theta_t \) also lifts to the second blow-up, hence to a map between the two exceptional divisors. Let \( \rho: \mathbb{R} \to [0, 1] \) be a cut-off function, i.e. a \( C^\infty \) function, which is identically 0 on \((-\infty, 0]\) and identically 1 on \([1, \infty)\).

Using the metric on the ball, the diffeomorphism \( f \) can then be defined as follows:

\[
 f(x) = \begin{cases} 
  x & \text{if } |x| > \frac{2t}{3} \\
  \Theta_t(x) & \text{if } \frac{2}{3} < |x| < \frac{2t}{3} \\
  \Theta(x) & \text{if } |x| < \frac{2}{3}
\end{cases}
\]

where \( t = \rho(\frac{2t}{3} - |x|) \). By what we have said, \( f: \mathbb{M} \to \tilde{\mathbb{M}} \) lifts to a diffeomorphism \( \tilde{f}: \mathbb{M} \to \mathbb{M} \). \( \square \)

**Corollary 19.** The manifold \( \mathbb{M} \) is a simply connected, 8-dimensional, non-formal manifold that admits both complex and symplectic structures, but which carries no \( \text{Kähler} \) metric.

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