Induced pseudoscalar form factor of the nucleon at two-loop order
in chiral perturbation theory

N. Kaiser
Physik Department T39, Technische Universität München, D-85747 Garching, Germany

Abstract

We calculate the imaginary part of the induced pseudoscalar form factor of the nucleon $G_P(t)$ in the framework of two-loop heavy baryon chiral perturbation theory. The effect of the calculated three-pion continuum on the pseudoscalar constant $g_P = (m_\mu/2M)G_P(t = -0.877 m_\pi^2)$ measurable in ordinary muon capture $\mu^-p \rightarrow \nu_\mu n$ turns out to be negligibly small. Possible contributions from counterterms at two-loop order are numerically smaller than the uncertainty of the dominant pion-pole term proportional to the pion-nucleon coupling constant $g_{\pi N} = 13.2 \pm 0.2$. We conclude that a sufficiently accurate representation of the induced pseudoscalar form factor of the nucleon at low momentum transfers $t$ is given by the sum of the pion-pole term and the Adler-Dothan-Wolfenstein term:

$$G_P(t) = \frac{4g_{\pi N}Mf_\pi}{(m_\pi^2 - t)} - \frac{2g_AM^2\langle r_A^2 \rangle}{3},$$

where $\langle r_A^2 \rangle = (0.44 \pm 0.02) \text{fm}^2$ the axial mean square radius of the nucleon.

PACS: 11.30.Rd, 12.20.Ds, 12.38.Bx, 12.39.Fe, 14.20.Dh.

To be published in: The Physical Review C (2003), Brief Report

The structure of the nucleon as probed by charged weak currents is encoded in two form factors, the axial and the pseudoscalar ones. To be specific consider the matrix element of the isovector axial current between nucleon states:

$$\langle N(p + k) | \bar{q} \gamma^\nu \gamma_5 \tau_a q | N(p) \rangle = \bar{u}(p + k) \left[ \gamma^\nu G_A(t) + \frac{k^\nu}{2M} G_P(t) \right] \gamma_5 \tau_a u(p),$$

where $t = k^2$ denotes the Lorentz-invariant squared momentum transfer and $u(p)$ stands for a Dirac-spinor. $M = 938.92 \text{MeV}$ is the (average) nucleon mass. The form in eq.(1) follows from Lorentz-covariance, isospin conservation and the discrete symmetries C, P and T. $G_A(t)$ is called the axial form factor of the nucleon and $G_P(t)$ the induced pseudoscalar form factor of the nucleon. While experimentally much attention has been focussed on the first one, the latter is generally believed to be well understood in terms of pion-pole dominance as indicated from ordinary muon capture experiments $\mu^-p \rightarrow \nu_\mu n$ (see e.g. refs.[1, 2, 3]). The pseudoscalar coupling constant $g_P$ as measured in ordinary muon capture is defined via:

$$g_P = \frac{m_\mu}{2M}G_P(t_\mu), \quad t_\mu = \frac{M_p^2 m_\mu}{M_p + m_\mu} - M_p m_\mu = -0.877 m_\mu^2 = -0.502 m_\pi^2,$$

with $t_\mu$ the Lorentz-invariant squared momentum transfer if the proton and muon are initially at rest. $m_\mu = 105.66 \text{MeV}$ is the muon mass, $M_p = 938.27 \text{MeV}$ the proton mass, $M_n = 939.57 \text{MeV}$ the neutron mass and $m_\pi = 139.57 \text{MeV}$ the charged pion mass.

Chiral perturbation theory allows to calculate systematically the corrections to the dominant pion-pole term in $G_P(t)$ (see eq.(7) below). At one-loop order this correction is uniquely expressed in terms of the mean square axial radius of the nucleon $\langle r_A^2 \rangle$ by making use of the chiral Ward identity of QCD [4]. Exactly the same term, derived originally by Adler,
Dothan and Wolfenstein [5], is also found in the small scale expansion of ref.[6] where additional diagrams with intermediate $\Delta(1232)$-isobars contribute. While the one-loop prediction for the pseudoscalar coupling constant $g_P = 8.4 \pm 0.2$ [4, 6] is consistent with the earlier result of the Saclay experiment $g_P = 8.7 \pm 1.9$ [2] a reanalysis [1] of that experiment using the modern world average of the muon mean life time gives the enhanced value $g_P = 10.6 \pm 2.7$. For further details on that and the conflicting situation concerning radiative muon capture $\mu^- p \rightarrow \nu_\mu n \gamma$ see the recent review of Gorringe and Fearing [1] and also ref.[3].

![Diagram of two-loop diagrams](image)

**Fig.1:** Two-loop diagrams contributing to the imaginary part of the induced pseudoscalar form factor of the nucleon $G_P(t)$. Dashed and solid lines denote pions and nucleons, respectively. The wiggly line symbolizes the external isovector axial source. The combinatoric factor of the first four diagrams is 1/6 and the next four graphs have the combinatoric factor 1/2. The last two diagrams scale as $g_A^3$ whereas the other eight graphs scale as $g_A$.

The purpose of the present short paper is to investigate the two-loop corrections to the induced pseudoscalar form factor $G_P(t)$ in order to clarify whether these could affect (num-
ically) the theoretical prediction for $g_P$ used so far. The essentially new feature at two-loop order is a nonvanishing imaginary part $\text{Im} G_P(t)$ for $t > 9m_\pi^2$ which originates from the (direct and indirect) coupling of the isovector axial current to the three-pion intermediate state. The pertinent ten topologically distinct two-loop diagrams generated by leading order vertices of the effective chiral Lagrangian $\mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi NN}^{(1)}$ are shown in Fig. 1. The Feynman rules for the relevant interaction vertices can be found in appendix A of ref. [7].

Let us now turn to the evaluation of the imaginary part $\text{Im} G_P(t)$ from the two-loop diagrams shown in Fig. 1. Application of the Cutkosky cutting rules gives the spectral function $\text{Im} G_P(t)$ as an integral of the product of axial source $\rightarrow 3\pi$ and $3\pi \rightarrow NN$ transition amplitude over the Lorentz-invariant three-pion phase space. Some details about these techniques can be found in refs. [8, 9] where the same method has been used to calculate the (two-loop) spectral functions of the isoscalar electromagnetic nucleon form factors and the $3\pi$-exchange nucleon-nucleon potential. The choices $\epsilon \cdot k = 0$ and $\epsilon^\nu = k^\nu$ for the polarization vector $\epsilon^\nu$ of the external isovector axial source allow a separation of $\text{Im} G_A(t)$ and $\text{Im} G_P(t)/4M^2$. Alternatively, one can use projection operators and tensorial integrals over the $3\pi$-phase space can be reduced to scalar ones (see eq. (13) in ref. [8]). The pertinent three-body phase space integrals are most conveniently performed in the three-pion center-of-mass frame. The corresponding on-mass-shell four-momenta of the three pions read in this frame: $k_1 = (\omega_1, k_1), k_2 = (\omega_2, k_2)$ and $k_3 = (\sqrt{t} - \omega_1 - \omega_2, -k_1 - k_2)$. The mass-shell condition $k_3^2 = m_\pi^2$ determines the cosine of the angle between $k_1$ and $k_2$ (called $z$) as:

$$z k_1 k_2 = \omega_1 \omega_2 - \sqrt{t}(\omega_1 + \omega_2) + \frac{1}{2}(t + m_\pi^2), \quad k_{1,2} = \sqrt{\omega_{1,2}^2 - m_\pi^2}.$$

The ten diagrams in Fig. 1 fall into two classes. The first eight diagrams carrying the common prefactor $g_A/f_\pi^4$ give rise altogether to the following contribution to the imaginary part of the induced pseudoscalar form factor of the nucleon:

$$\text{Im} G_P^{(1)}(t) = \frac{g_{\pi NN} M}{(2\pi f_\pi)^3} \int_{z^2 < 1} d\omega_1 d\omega_2 \left\{ \frac{1}{18} - \frac{m_\pi^4}{12(t - m_\pi^2)^2} + \frac{4\omega_1^2 - m_\pi^2}{6t} + \frac{\omega_1^2(3m_\pi^2 - t)}{(t - m_\pi^2)^2} + \frac{2m_\pi^2 \omega_1 \omega_2 z k_2}{t(t - m_\pi^2) k_1} \right\}.$$

Here, $f_\pi = 92.4$ MeV denotes the pion decay constant and we have employed the Goldberger-Treiman relation: $g_{\pi NN} f_\pi = g_A M$. The inequality $z^2 < 1$ determines the kinematically allowed region in the $\omega_1 \omega_2$-plane (which is bounded by a cubic curve) together with the obvious kinematical constraints $m_\pi < \omega_1, \omega_2 < \sqrt{t} - 2m_\pi$ and $2m_\pi < \omega_1 + \omega_2 < \sqrt{t} - m_\pi$. Furthermore, one derives from the last two diagrams in Fig. 1 which are proportional to $g_A^2/f_\pi^4$ the following contribution to the imaginary part $\text{Im} G_P(t)$:

$$\text{Im} G_P^{(3)}(t) = \frac{g_{\pi NN} M g_A^2}{(2\pi f_\pi)^3} \int_{z^2 < 1} d\omega_1 d\omega_2 \left\{ \left( m_\pi^2 - \sqrt{t} \omega_1 \right) \left( z + \frac{k_2}{k_1} \frac{\arccos(-z)}{\sqrt{1 - z^2}} \right) \right. \left. + \frac{k_1^2}{3} + \frac{t}{9} + \frac{m_\pi^2}{t - m_\pi^2} \left( \frac{7}{8} \sqrt{t} - \omega_1 - \omega_2 \right) \left[ 2\omega_1 \frac{zk_2}{k_1} + \sqrt{t} \right] \right. \left. + \left( (t + m_\pi^2)(4\omega_1 - \sqrt{t}) - 4\sqrt{t}\omega_1 \omega_2 \right) \frac{\arccos(-z)}{2k_1 k_2 \sqrt{1 - z^2}} \right\}.$$

In the chiral limit $m_\pi = 0$ the total (two-loop) spectral function $\text{Im} G_P(t)$ shows a simple linear $t$-dependence of the form:

$$\text{Im} G_P(t)|_{m_\pi = 0} = -\frac{4M^2}{t} \text{Im} G_A(t)|_{m_\pi = 0} = \frac{g_{\pi NN} M t}{9(8\pi f_\pi)^3} \left[ 1 - g_A^2 \left( 1 + \frac{64\pi^2}{35} \right) \right] \approx -\frac{3t}{M^2}.$$


The first part of this equation follows from the fact that the combination $G_A(t) + t G_P(t)/4M^2$ is the form factor of the divergence of the isovector axial current, which vanishes in the chiral limit $m_\pi = 0$ (QCD chiral Ward identity). The (two-loop) result for $\text{Im} G_A(t)|_{m_\pi=0}$ has been taken over from eq.(27) in ref.[8].

In Fig. 2 we show by the full line the total imaginary part $\text{Im} G_P(t)$ calculated from eqs.(4,5) after division by a factor $t$. The horizontal dashed line in Fig. 2 indicates the asymptotic behavior of $\text{Im} G_P(t)/t$ for $t \to \infty$.

![Graph](image)

Fig.2: The spectral function $\text{Im} G_P(t)$ of the induced pseudoscalar form factor of the nucleon divided by $t$. The horizontal dashed line indicates its asymptotic form obtained by taking the chiral limit $m_\pi = 0$.

With the help of the spectral function $\text{Im} G_P(t)$ the complete two-loop representation of the induced pseudoscalar form factor of the nucleon can be written as:

$$G_P(t) = \frac{4g_{\pi N}Mf_\pi}{m_\pi^2 - t} - \frac{2}{3}g_A M^2 \langle r_A^2 \rangle + \frac{M^2}{(2\pi f_\pi)^4}(\zeta_0 m_\pi^2 + \zeta_1 t) + \frac{t^2}{\pi} \int_{0m_\pi^2}^{\infty} dt' \frac{\text{Im} G_P(t')}{t^2(t' - t + i0^+)}.$$ (7)

The first two terms are the well-known pion-pole term and Adler-Dothan-Wolfenstein term [4, 5]. The parameters $g_{\pi N}, f_\pi, m_\pi, \langle r_A^2 \rangle$ etc. are to be understood as the physical ones including their individual one- and two-loop chiral corrections. These (not explicitly calculated) two-loop renormalization effects come along with the real parts of the diagrammatic amplitudes. Note that the dispersion integral in eq.(7) requires two subtractions because of the asymptotic linear growth of the imaginary part $\text{Im} G_P(t)$. The third term in eq.(7) involving the two dimensionless low-energy constants $\zeta_0$ and $\zeta_1$ subsumes all polynomial contributions which arise from (tadpole-type) loop diagrams and possible chiral-invariant counter terms (beyond renormalizing the Adler-Dothan-Wolfenstein term). The prefactor of this term is chosen such that the negative mass dimension of the counterterm coupling strength is accounted for by appropriate powers of the chiral symmetry breaking scale $\Lambda_\chi = 2\sqrt{2}\pi f_\pi \simeq 0.82$ GeV. Based on naturalness arguments one expects that the dimensionless low-energy constants $\zeta_{0,1}$ are of order one. Indeed the same
considerations applied to the Adler-Dothan-Wolfenstein term give for the analogous dimensionless low-energy constant at one-loop order: $-(2\pi f^2) g_A (r_A^2) / 3 = -g_A (4\pi f^2 / M_A)^2 \simeq -1.63$. Here, we have inserted the value of the axial dipole mass $M_A = (1.03 \pm 0.02)$ GeV as extracted in ref.[10] from (quasi)-elastic neutrino and antineutrino scattering experiments.

Let us now turn to numerical results. From the twice-subtracted dispersion integral in eq.(7) one gets a tiny contribution to the pseudoscalar coupling constant of $\delta g_P \simeq -1 \cdot 10^{-5}$. The extreme smallness of this number comes partly from the proximity of the chosen subtraction point $t_0 = 0$ to $t_\mu = -0.502 m_\pi^2$. Nevertheless, when varying the subtraction point (via the substitution $t^2 / t^2 \rightarrow (t - t_0)^2 / (t' - t_0)^2$ in eq.(7)) in the broad range $-24 m_\pi^2 < t_0 < 9 m_\pi^2$ the contribution of the dispersion integral to the pseudoscalar coupling constant stays in magnitude smaller than one percent, $0 > \delta g_P > -1 \cdot 10^{-2}$. Note that a change of the subtraction point $t_0$ is equivalent to changes of the low-energy constants $\zeta_{0,1}$ parameterizing the polynomial piece in eq.(7). Since we cannot accurately determine the coefficients $\zeta_{0,1}$ we turn here the argument around and ask only for some upper bound. For example, in order to cause a correction of $\delta g_P = 0.09$, corresponding to a relative 1% change of $g_{\pi N}$ in the pion-pole term, the relation $2\zeta_0 - \zeta_1 \approx 21$ must hold. Such values of $\zeta_{0,1}$ exceed the expectation from naturalness already by one order of magnitude.

The delta-nucleon mass-splitting $\Delta = 293$ MeV introduces another small scale to the problem. The systematic power scheme inherent to the small scale expansion of refs.[3, 6] ensures however that the contribution to $G_P(t)$ from two-loop diagrams with intermediate $\Delta(1232)$-excitations is a homogeneous function of degree one in the three variables $(t, m_\pi^2, \Delta^2)$. Consequently, possible negative powers of $\Delta$ get always overcompensated by two more powers of the numerically smaller scales $m_\pi$ and/or $\sqrt{|t_\mu|}$ in $G_P(t_\mu)$.

One may therefore conclude that all two-loop corrections to the pseudoscalar coupling constant $g_P$ are numerically insignificant. A sufficiently accurate representation of $G_P(t)$ at low momentum transfers $t$ is given by the sum of the pion-pole term and the Adler-Dothan-Wolfenstein term. Using for the $\pi N$-coupling constant $g_{\pi N} = 13.2 \pm 0.2$ [11, 12] which is consistent with recent results from $\pi N$-dispersion relation analyses [12] and $\langle r_A^2 \rangle = 12 / M_A^2 = (0.44 \pm 0.02)$ fm$^2$ [10] for the axial mean square radius one gets in this case:

$$g_P = 8.3 \pm 0.2. \quad (8)$$

The major theoretical uncertainty of $g_P$ comes obviously from the $\pi N$-coupling constant $g_{\pi N}$ entering the dominant pion-pole term.

Let us finally consider the form factor of the nucleon related to the divergence of the isovector axial current:

$$G_A(t) + \frac{t}{4M^2} G_P(t) = \frac{g_A m_\pi^2}{m_\pi^2 - t} D(t). \quad (9)$$

The prefactor on the right hand side expresses the vanishing of this form factor in the chiral limit $m_\pi = 0$ as well as the presence of a pion-pole contribution. The imaginary part $\text{Im} D(t)$ (at two-loop order) can be easily constructed from the expressions for $\text{Im} G_P(t)$ given here in eqs.(4,5) as well as from the formula for $\text{Im} G_A(t)$ written in eq.(26) of ref.[8]. As a further non-trivial result we give here only the spectral function $\text{Im} D(t)$ in the chiral limit:

$$\text{Im} D(t)|_{m_\pi = 0} = \frac{2t^2}{9\pi^3 (8f^2)^4} \left[ 1 + g_A^2 \left( 5 + \frac{68\pi^2}{35} \right) \right] \simeq \frac{t^2}{g_A M^4}, \quad (10)$$

which may be useful for some quick order of magnitude estimates.

In summary we have calculated in this work the imaginary part of the induced pseudoscalar form factor of the nucleon $\text{Im} G_P(t)$ at two-loop order in heavy baryon chiral perturbation
theory. Two-loop corrections to the pseudoscalar coupling constant $g_P$ measurable in ordinary muon capture $\mu^−p \rightarrow \nu_\mu n$ are numerically unimportant in view of the present uncertainty of the pion-nucleon coupling constant $g_{\pi N}$.

I thank T. Hemmert for useful discussions.

References

[1] T. Gorringe and H. W. Fearing, ”Induced pseudoscalar coupling of the proton weak interaction”, nucl-th/0206039; and references therein.
[2] G. Bardin et al., Phys. Lett. B104, 320 (1981).
[3] V. Bernard, T.R. Hemmert and Ulf-G. Meißner, Nucl. Phys. 686, 290 (2001); and references therein.
[4] V. Bernard, N. Kaiser and Ulf-G. Meißner, Phys. Rev. D50, 6899 (1994).
[5] S.L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966); L. Wolfenstein, in ”High energy physics and nuclear structure”, edited by S. Devons (Plenum, New York, 1970), p. 661.
[6] V. Bernard, H.W. Fearing, T.R. Hemmert and Ulf-G. Meißner, Nucl. Phys. 635, 121 (1998).
[7] V. Bernard, N. Kaiser and Ulf-G. Meißner, Int. J. Mod. Phys. E4, 193 (1995).
[8] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. A611, 429 (1996).
[9] N. Kaiser, Phys. Rev. C61 014003 (2000); C62 024001 (2000); C63 044010 (2001).
[10] A. Liesenfeld et al., Phys. Lett. B468, 20 (1999).
[11] B. Loiseau and T.E.O. Ericson, Physica Scripta T87, 53 (2000); and references therein.
[12] M.M. Pavan, R.A. Arndt, I.I. Strakovsky and R.L. Workman, Physica Scripta T87, 65 (2000).