Effects of asset frequency components on value-at-risk in emerging and developed markets: Analyses with MODWT and CWT wavelets

Milton Biage*  
Pierre Joseph Nelcide†

Abstract
This study measures the impacts of short, medium, and long-term cycles on daily value-at-risk estimates, and test the performance of some orthogonal filters in MODWT wavelets, and some CWT wavelets. We used the wavelets’ techniques to separate each frequency component, intrinsically, inserted in the series of shares returns, and analyzed the decomposition dynamics of the cyclic components since wavelet techniques distribute their decomposition frequency scales differently, for each level. The CWT wavelets have the flexibility to adjust the windows of the CWT, which allows different evolutions of the frequency range on the time support. Therefore, the characteristics of the effects of frequency components on VaR distributed differently for each wavelet kind. The FIGARCH(1,d,1) is used to model daily reconstructed returns series of twelve shares of Ibovespa and DJIA; then, applying the rolling window technique to establish the prediction of conditional variance. A set of back-testing was applied to confront the estimated VaRs, which demonstrated that the estimation of VaRs models is dynamically consistent. The results demonstrate that the dynamics of most shares traded on the BM&F BOVESPA and DJIA Index depends on the wavelet kind applied in the analyses. Nevertheless, the stock dynamics of the BM&F BOVESPA and DJIA no have significant differences in their dynamic evolution process, for any wavelets applied.

Keywords: FIGARCH, value-at-risk, wavelet decompositions, structural change, stock market, volatility models

JEL Codes: C52, C58, C61, G14, G15, G17

Submitted on 20 Oct 2018; Reviewed on 16 Feb 2019

*Department of Economics and International Relations, Federal University of Santa Catarina. Florianópolis, SC, CEP 88040-900, Brazil.
†Doctoral student of the Graduate Program in Economics, Federal University of Santa Catarina. Florianópolis, SC, CEP 88040-900, Brazil.

milton.biage@ufsc.br  pierre.joseph.nelcide@posgrad.ufsc.br
1 Introduction

The process of financial markets internationalization in the 1980s increased the demand for financial asset analysis instruments. Garbade (1986) created the VaR (value-at-risk) analysis. This mechanism for measuring credit risk was defined with purpose of strengthening the consistency and stability of the international banking system and minimizing competitive inequalities between bank assets internationally. Therefore, since then, numerous statistical procedures have been introduced in order to improve the VaR forecasts, due to their importance in the banking supervision process.

VaR refers to the worst outcome that is expected for a portfolio, over a predetermined period and with a certain level of confidence. In a simple way, the VaR is an empirical estimate of the distribution of asset returns, assuming that the return sample is normally distributed. However, the returns exhibit skewness and excess kurtosis, resulting in an underestimation or overestimation of the true VaR. Different types of distributions are used, such as the Skewed Student-t (Giot & Laurent, 2003), as a resource for capturing these effects and correcting them. However, the most common way applied to calculate the VaR is to estimate the conditional variances using GARCH models. The general understanding is that the VaRs models, estimated on the basis of the conditional variances, work better than others classical methods, estimated on the basis of the asset distribution tail.

The volatility information of the financial return contained in the historical return series can be analyzed in detail in terms of the time frequency decomposition of an underlying return series, using the wavelet technique (Percival & Walden, 2000; Gençay, Selçuk, & Whitcher, 2001). In this context, the return series are decomposed into different time scales comprised of particularly of short, medium and long-term trends, contained in the original series. These different scales can be captured explicitly through the Multiresolution Analysis (MRA), involving the underlying seasonalities of the series due to different levels (scales) of decomposition, e.g. “Monday Effects” (Alt, Fortin, & Weinberger, 2011), “Weekday Effects” (Doyle & Chen, 2009) and “Turn-of-the-Month Effects” (Khaled & Keef, 2012). It should also be emphasized that persistent memory’s effects on long-term will likewise be captured by the time frequency scales, as described by Powell, Shi, Smith, and Whaley (2009) and Tan, Chin, and Galagedera (2014). However, the advantage of wavelet decomposition for a financial series is to make it possible to establish an isolated evaluation of each specific time frequency scale. For example, assessing the effects of long-term or
short-term trend impacts on the volatility of financial time series in the absence of another scales, or in the absence of stochastic noise (Gallegati, 2012; Reboredo & Riveira-Castro, 2014). Thus, one of the goals of the study is to understand the impacts of different time frequencies on the performance of value-at-risk for shares of markets with different performance, then applying the technique of wavelet decomposition in the time series of returns.

To estimate the conditional variance for the times series, it will be applied the FIGARCH process because of the financial time series present evidence of long-term volatility in their returns. Baillie, Bollerslev, and Mikkelsen (1996) and Bollerslev and Mikkelsen (1996) introduced models to estimate of conditional variances of long-memory processes, extending the model of Bollerslev (1986). Fractionally integrated long memory models (FIGARCH) have received considerable interest in their ability to capture persistence in volatility. Moreover, it is also well known that long memory is easily confused with structural changes, since the slow decay of the autocorrelation function, which is typical of a time series with long memory, is also produced when a time series of short memory shows structural breaks (Boes & Salas-La Cruz, 1978; Hamilton & Susmel, 1994; Diebold & Inoue, 2001; Granger & Hyung, 2004; Gourieroux & Jasiak, 2001). However, it can be emphasized that structural breaks with small breaks in their structure, related to the changes in the intercept and/or slope, are easily captured as long memory effects by the FIGARCH model, since it identifies and adapts the estimates of the model by adjusting its memory fraction parameter, practically does not affecting the VaR predictions. In this context, some studies were performed to explaining the long memory effects in financial assets, among many others, the leverage effects in the economic cycle (Beltratti & Morana, 2006; Baillie & Morana, 2009), and disruptions in stock market volatility in response to monetary policy responding the conditions of economic cycle (Morana & Beltratti, 2004; Beltratti & Morana, 2006).

In this study it is assumed the basic idea that the volatility of most financial returns may be affected by any frequency scales, including their susceptibility to the occurrence of long memory effect. So, it will be applied a procedure to estimate conditional variance that makes possible to capture the volatility effects of any frequency scales, varying from shorter to long-term. In this case, the FIGARCH model is appropriated because it can adjust the memory effect parameter for any scale. This is necessary because the study will be conducted on reconstructed series that take-into-account only the components of decomposed frequencies that have shown substantial effects on VaR predictions. Thus,
considering assets of markets with different governability’s, the study contribution lies in the evaluation of the impacts of the time frequencies decomposed on the VaR estimates for each asset, regarding only those that have shown relevance in the VaR forecasts, discarding the other components that have caused bias in the VaR forecasts and/or shown insignificant effects.

To carry out this study was selected twelve daily returns of Ibovespa shares (BM&F BOVESPA), and twelve daily returns of shares of DJIA of the NYSE, with the purpose to compare the characteristics of the VaR estimates, respectively, for the share sets of these two stock exchanges. To obtain the decomposed series for each share returns’ sample, we applied some wavelets, such as the maximal overlap discrete wavelet transform (MODWT) technique, with the filters Haar, Daubechies (Db2), and Symlet (Sym2). It was also applied the Continuous Wavelet Transform (CWT) techniques, such as Morlet, Morse, second-derivative of gaussian (2-DOG), and fourth-derivative of gaussian (4-DOG) (see subsection 3.2). The decomposition of returns samples for each share was proceeded with eight levels to obtain the decomposed series for each share. Reconstructed returns’ series were generated from the frequencies decomposition series, as it is described in section 4. We applied the FIGARCH\(1,d,1\) model in the reconstructed returns’ series to model and establish the conditional variance forecast for a period-future given, applying the rolling window technique. The prediction of the value-at-risk was estimated using the future conditional variances, by Monte-Carlo methodologies. Then, the accuracy of the predicted was verified, counting the number of failures, based on a return series for out-of-sample forecasting, for each share (logically, those values of each share beyond the period used in the estimate of the FIGARCH model), thus obtaining the percentage of failures. Finally, the VaRs were contrasted with the back-testing measures to confirm their predictive failure capabilities.

The first objective in the study is verifying the dynamic behavior of frequency decompositions over the levels, for each wavelet kind analyzed. We also intend analyzing the influence of the decomposition dynamics of each wavelet over the performance of the VaR model estimates. Thus, it is considering two sets of shares from two financial markets, supposedly different. One is originating from a developed financial market (DJI-NYSE),\(^1\) with assumptions of lower effects of crises (domestic or international). This financial market develops their volatilities in short-term cycles and having the ability to overcome the

---

\(^1\)The Dow Jones Industrial Average. Shares included in the DJIA are traded on the NYSE, and a few are on the NASDAQ, the National Association of Securities Dealers Automated Quotations.
crises’ effects quickly. The other is a financial market of a developing country (Ibovespa-BM&F BOVESPA),\(^2\) with assumptions that it has more subject to the domestic and international crises’ effects, and may present difficulties to overcome the effects of crises, developing their volatilities in longer cycles. Therefore, it is expected in the study that, in general, DJI-NYSE shares and Ibovespa-BM&F BOVESPA shares respond to the volatility effects on different frequency scales.

A second objective of the study is to explore the effect of a relatively short sample,\(^3\) practically contrary to the procedures adopted in most of the studies present in the literature, in order to investigate how a VaR model under such condition may forecast failure. A large sample may better model the VaR, however, be less realistic because of the composition aspects of the observational and non-observational effects that interact in the process, over time. For example, the effects of failure concentration, due to crises, and larger samples can better absorb independence between failures since they can better mediate the probability of failure. However, a constructed VaR model involving long periods, incorporating concentration effects, may underestimate the proportion of failures, when confronted for failure forecast, with a short out-of-sample return series. Hoppe (1998) examined the question of the sample size and argued that the use of smaller samples would lead to more accurate VaR estimates than the larger ones. Likewise, Frey and Michaud (1997), in their study, supported the use of small sample sizes to capture structural changes over time due to changes in commercial behavior. In short, the choice of appropriate historical sample size, as well as a suitable model to predict volatility, should be considered as far from being solved.

It can be emphasized that the conditional heteroskedastic models reviewed throughout this study used samples larger than an eight-year period, during which many different observable and unobserved effects with different scales, possibly, could have interacted in the series’ volatility process. However, the series of return used in the conditional volatility analysis in the study was 890 samples (approximately three and a half years), 470 in the calibration process, 210 in the rolling window, and 210 in the forecasting process, much smaller than

\(^2\) BM&F BOVESPA is the official stock exchange of Brazil. The Ibovespa is a purported investment portfolio that currently comprises 55 stocks, representing the movement of the leading securities traded on BM&F BOVESPA.

\(^3\) Basel Committee defined that VaR should be calculated daily for Confidence Intervals of 99\%, using a historical series with approximately 250 samples, and a time horizon of ten working days (Basle Committee on Banking Supervision, 1996).
those used in similar studies.

The sequence of the article is structured as follows: section 2 presents the literature review; section 3 presents the theoretical mathematical scheme used in the study; section 4 presents the methodological procedure applied in the study; section 5 shows the results and analysis; and section 6 presents the conclusion.

2 Literature review

The use of wavelet decomposition has won many adepts in the analysis of economic and financial time series, from work of Gençay et al. (2001). Gençay and Selçuk (2004) investigated the relative performance of the value-at-risk (VaR) models, with the daily earnings (from Jan/1988 to Dez/1999) of stock markets in nine different emerging markets (Argentina, Brazil, Hong Kong, Indonesia, Korea, Mexico, Singapore, Taiwan and Turkey), using extreme value theory (EVT) to generate VaR estimates. The results obtained indicated that VaR models have different moment properties at their right and left tails. Huang (2011) proposed a BEKK-GARCH model with multiple resolutions, comprised of wavelet decompositions, to investigate the effects of spillover in financial markets, using daily returns for NASDAQ and TWSE-I\textsuperscript{4} indices, in the period from Jan/1998 to Dec/2004. In this analysis it was found that the predictive power of spillover effects spread unevenly on each time scale with patterns totally different from those revealed in the raw data level. The direction and magnitude of spillovers and volatility effects vary significantly with their time scales. In a global economy, shocks that occur in one market may spread to other markets. Martín-Barragán, B.Ramos, and Veiga (2015) used a wavelet-based approach to investigate the impacts of oil shocks and correlation shocks between stock and oil markets, for stock indexes in Germany, Japan, the United Kingdom and the United States, and the NYMEX\textsuperscript{5} oil futures contract settlement price, for the daily sampling period from Feb/1990 to Nov/2011. The results of which showed that oil shocks affect the correlation between both markets. The authors concluded that the evidence supports the inclusion of oil as a class of assets in asset allocation strategies.

Other authors have also used wavelet decomposition in research on the dynamic’s behavior of financial assets. Andries, Ihnatov, and Tiwari (2014) analyzed the dependencies between interest rates, stock prices and the exchange rate in India, for daily data from Jul/1997 and Dec/2010. In this study, were

\textsuperscript{4} Taiwan Stock Exchange Weighted Index.
\textsuperscript{5} New York Mercantile Exchange.
identified the patterns of co-movement of the specified variables, using the cross-spectral power, the coherence function and phase difference, for cross wavelet components. The results suggested that stock prices, exchange rates and interest rates are interdependent. But the stock price movements are behind, both in relation to the exchange rate and to interest rate fluctuations. The interrelationship between interest rates and stock price movements suggested that the stock market follows the signs that of interest rates. Chakrabarty, De, and Bandyopadhyay (2015) studied the nature and direction of shock transmissions and volatility among nine non-overlapping sector indices of the Bombay Stock Exchange (BSE), using data from April/2004 to April/2012, involving eight wavelet-based decomposition (from 2–4 days to 1–2 years), and the dynamic conditional correlation of MRA-EDCC GARCH models. The results revealed that the interaction volatility is dependent on scale, with variations in the magnitude and direction of spillover incidences. These conclusions suggest that a calibrated strategy for short-term investors may not be ideal for long-term investors, and vice versa. Berger (2016) examined the relevance of long-term seasonality to establish adequate VaR forecasts, via wavelet decomposition. The behavior of a set of financial return series of stocks listed in DJIA was evaluated, using daily market prices of 29 stocks, from Jan/2000 to Jul/2014. The MODWT wavelet was applied to decompose each series into eight scales, describing different trends of the underlying series. The conditional variance processes described by different memory schemes were simulated used a FIGARCH process. The study of daily market prices revealed the relevance of short-term fluctuations in the underlying time series to establish VaR forecasts. In particular, it was found that the long-term trend frequencies of the original series do not affect the accuracy of the VaR prediction statistics, which can be discarded when establishing VaR forecasts.

Tamoni (2011) study the effects of changes in risk on asset prices across different time horizons (timescales) and provide new insight into the dynamics of equity premia. It is found in the study that risk premia are weakly related to consumption volatility at short horizons (contrary to the Consumption-CAPM theory), whereas the long-run volatility strongly determines the long-run dynamics of expected stock returns. The empirical results emphasize the importance of simultaneously modeling consumption at multiple timescales and point to changing consumption volatility as an essential long-run priced factor. It was rebuilt the information of the original series, by using the multiresolution and temporal aggregation of the Haar wavelets analysis, to establish a collection
of details, at different resolutions. The time series is viewed as a cumulative sum of the detail variations of the return time series, as the smoother levels increase. The detail coefficients of the return series decompose the information from the original series into pieces associated with both time and scales. These coefficients are associated with the length scales’ changes, and the scaling coefficients are associated with the length scale average. Since the detail coefficients capture the variation of the time series at a given scale and interval of time, then, the approach models the detail coefficients directly, producing an adapted, non-anticipative, decomposition.

Ortu, Severino, Tamoni, and Tebaldi (2019) provide an analysis of orthogonal decomposition into uncorrelated components, Wold-type persistence-based for stationary time series, associated with different layers of persistence. This decomposition results from the application of the Wold Theorem on the Hilbert space (Baggett, Larsen, Packer, Raeburn, & Ramsay, 2010) for weakly stationary stochastic processes, an outcome of a multiresolution analysis of the wavelet decompositions of Hilbert spaces. The authors compare the Wold Decomposition results of economic time series, with the multiresolution approach results obtained by Discrete Haar Transform, a standard way to isolate phenomena with heterogeneous persistence in economic time series. Based on this comparison, it was shown that the Wold decomposition is appropriated to decompose across time scales, weakly stationary time series, with uncorrelated components associated with different degrees of persistence, intrinsic particularities in the variability of economic and financial time series. This scheme shows how to represent a stationary time series as a sum of orthogonal components, each one characterized by a specific level of persistence (or scale). The authors’ conclusion about the variance decomposition approach of time series, which each component in the analysis is associated with a specific scale, facilitates to provide an economic interpretation of the key driving factors, since different economic phenomena may operate at different frequencies. Price movements are related to the short-end of the spectrum, to political cycles at the medium-end, to technological and demographic changes at the long-end, and uncertainty regarding exhaustible energy resources and climate change at the very long-end.

Bandi, Perron, Tamoni, and Tebaldi (2019) introduce an analysis of future excess market returns onto past economic uncertainty. The modeling framework is that predictability is specified as a property of low-frequency components of both excess market returns and economic uncertainty. The authors consider that the transient effects with high frequencies hide the low-frequency components
of the series, which contain the economic relations of interest. Thus, the stock market returns and the economic uncertainty are modeled as aggregates of uncorrelated components operating over different frequencies, decomposing them on scale specific predictabilities (i.e., the returns and uncertainty are interpreted as linear aggregates of components operating over different frequencies). This covariance-stationary process was separated into orthogonal scales of specific components, based on Hilbert space theory (Baggett et al., 2010), a standard procedure of Wold representation. The stochastic innovation processes at any scale are shocks structured by Discrete Haar Transform. The conclusion is that the long-run past averages are not as easily interpretable. Hence, the role played by frequency decomposition applied in the study was to extract low-frequency information contained in the components and its translation into the long-run properties of the data. To the extent that market return data and the dividend yield, e.g., contain relevant information about long-run cash-flow and discount rate risks, the data aggregation on decomposition-properly components appear very well-suited to uncover this information.

Fractionally integrated GARCH models (FIGARCH) have been used intensively for the simulation of conditional variance. This class of models was introduced by Baillie et al. (1996). The conditional variance of this process is characterized by a decaying at a slow hyperbolic rate, due to the influence of lagged square innovations. Baillie et al. (1996) implemented a model to analyze the behavior of daily returns for the Deutsche Mark exchange rate, related to the US dollar, from March/1979 to Dec/1992, whose results highlight the efficiency of the FIGARCH to describe this kind of process. Jin and Frechette (2004) studied the effect of long memory on the daily volatilities of future returns of agricultural commodities. They adjusted the volatility of daily price time series for fourteen series of agricultural future prices, from January/1979 to Mar/2000. They found that the volatility series exhibit strong long-term dependence, which is an indicator of fractional integration. Belkhouja and Boutahary (2011) analyzed the TV-FIGARCH processes to model volatility. This model allows to explaining both the effects of long memory and the structural changes in the process of conditional variance. The TV-FIGARCH model was used to analyzed the behavior of two empirical examples involving the daily spot price of the crude oil and the daily index of the Standard and Poor 500 composite index (S&P 500). Both data series were from Jan/1990 to Dec/1999. The main finding was that the behavior of the series was not explained only by the existence of long memory in volatility, but also by deterministic changes in unconditional variance.
Tayefi and Ramanathan (2012) present a review of FIGARCH model. These authors state that these models are characterized as one of the best conditional heteroskedastic models to describe the volatility of stock market returns rates. They analyzed the performance of the FIGARCH models, using time series data from the Rupee exchange rates compared to the US dollar for the period from Jan/2000 to Jan/2011. The results of the model demonstrated the ability of the FIGARCH model to model series with long memory. Klein and Walther (2016, 2017), in order to accelerate the conditional variance estimation process with FIGARCH model, introduced the Fast Fourier Transform (FFT) to model the long-memory conditional variance. The implemented analyzes showed that the FFT approach offers a computational advantage for the representation of long memory models, specifically ARCH(∞), when using large data sets that are common in high frequency analyzes. To get robust results, different sample lengths were produced using an artificial data generating process.

3 Mathematical theoretical framework

In this section will be presented the scheme and notation of the value-at-risk model that will be used in the study, as well as the concepts of wavelet filtering, concepts of FIGARCH($p,d,q$) models and the Back-testing plans applied in the evaluation of the VaR estimates.

3.1 Value-at-risk

The value-at-risk, VaR, is a statistical procedure that allows estimating limits of losses or maximum gains of a series of returns, whose limits are given by the desired confidence levels. It is normal to use 95% or 99% confidence limits (Franke, Hardle, & Hafner, 2008, p.359). Therefore, to estimate the VaR model, let us denote $r_{t+h}$ as the return of an out-of-sample, at the time $t + h$, where $h$ represents the holding period (in the case of the study $h = 1$ day) of an asset (or a portfolio of assets) at time $t$. The ex-ante VaR for time $t$, at a coverage $\alpha$ rate, is denoted by $\text{VaR}_{t+h}(\alpha)|F_t$, conditioned to all information $F_t$ available at time $t$ (e.g. past returns and macroeconomic indicators).

In this study we will be estimating VaR models for long\(^6\) operation strategies (then at the quantile level $\alpha$). Therefore, we define the equation for the calculation

---

\(^6\) Long equity strategies are an investment strategy of taking long positions in stocks that are expected to appreciate and short positions in stocks that are expected to depreciate. 
the VaR at time $t$ as follow:

$$\alpha = P[r_{t+h} > \text{VaR}_{t+h}(\alpha) | F_t],$$  \hspace{1cm} (1)$$

where values $\text{VaR}_{t+h}(\alpha)$ are calculated with $(1 - \alpha) \times 100$ of confidence. Thus defined, the $\alpha$ reached by equation (1) indicates that there is $\alpha \times 100$ of chance that $r_{t+h}$ does not overtake the limit defined by $\text{VaR}_{t+h}(\alpha)$.

In this study, the return series, $r_{t+h}$, are interpreted in a scaling structure of location, as a function of $\text{VaR}_{t+h}(\alpha)$, as follows:

$$r_t = \mu_t + \sqrt{\sigma_{t+h}^2 z_{t,1-\alpha}} \approx \mathcal{N}(0,1),$$  \hspace{1cm} (2)$$

where $\mu_t$ is the average return at time $t$; $\sigma_{t+h}^2$, its conditional volatility at the time $t+h$; $z_{t,1-\alpha}$ the critical limits of the marginal distribution of the return series.

Then, considering that we are dealing with daily data, with characteristics of a distribution with a zero-trend scale (zero average), then equations (2) can be reduced to the following relation:

$$\text{VaR}_{t+h}(\alpha) = \sqrt{\sigma_{t+h}^2 z_{t,1-\alpha}} = \sigma_{t+h} z_{t,1-\alpha}. \hspace{1cm} (3)$$

The prediction of conditional volatility in equation (3) is previewed, in this study, by FIGARCH$(1,d,1)$ model. From now, considering that VaR will be generated for a holding period of 1 day ($h = 1$), then, $h$ will be substituted by 1 in equation (3).

The distribution to be assumed for the VaR calculations depends on the properties of the return series used in the VaR estimation process, however, it is usually assumed that $z_{t,1-\alpha}$ follow a Gaussian distribution, $\mathcal{N}(0,1)$. The normal distribution approximation is satisfactory, since it was applied the wavelet frequency decomposition of the returns, then, it is expected that the outliers and returns values concentrated around the mean (kurtosis and asymmetry effects), are found in the frequency scales discarded in the VaR estimation.

It is used in the study the MC method, generated by Monte-Carlo (stochastic series) realizations, following standard normal distribution, with average and standard deviation equal to the series $\text{VaR}_{t+1}(\alpha)$ estimated by equations (3). The advantage of the MC method is that it allows the generating of several stochastic series of realizations, making it possible to estimate the failure probabilities $\hat{p}$,
with greater reliability, since an average for each value of \( \hat{p} \) is established. It can be seen, the MC procedure allows to eliminating the asymmetry effects in the VaR estimation series, consequently, in the \( \hat{p} \) estimation.

Therefore, we can apply equations (3), above, to calculate the VaR, and obtain the failure probabilities \( \hat{p} \), by means of a violation sequence \( \{I_t\}_{t=1}^{n} \), for a VaR forecast sequence, \( \{\text{VaR}_{t|t-1}(\alpha)\}_{t=1}^{n} \) through the following relationships:

\[
I_t = 1, \quad \text{if} \quad r_t > \text{VaR}_{t|t-1}(\alpha),
I_t = 0, \quad \text{if} \quad r_t \leq \text{VaR}_{t|t-1}(\alpha),
\]

for \( t = 1, 2, \ldots, n \). \( (4) \)

In equation (4), \( r_t \) is an out-of-sample return series; \( n \) the number of elements in \( r_t \); \( I_t \) is the sequence of zeros and ones that allow the estimating of the \( \hat{p} \), respectively, by the following relationships:

\[
\hat{p} = \frac{\sum_{t=1}^{n} I_t}{n}.
\] \( (5) \)

### 3.2 Wavelets decomposition

#### 3.2.1 Wavelet families

The wavelet families include a large number of wavelets that can be used for both continuous and discrete analysis. The choice of wavelet is dictated by the signal characteristics and the nature of the application. We can consider some basic guidelines for deciding on whether to use a discrete or continuous wavelet transform. If the application requires an orthonormal transform (an orthogonal transformation preserves energy), it is recommended the use of the DWT, in conjunction with one of the orthogonal wavelet filters. The orthogonal wavelet filter families are Haar filter (a Daubechies Db1, useful for edge detection because it is Symmetric), Daubechies family filters (Db\( N \), with \( N = 1, 2, 3, \ldots, 45 \)), which present nonlinear phase and concentrated energy near the beginning of its support, and Symlet (Sym\( N \), with \( N = 2, 3, \ldots, 45 \)), less asymmetrical and almost linear phase; among others. For all these filters cited, \( N \) is the vanishing moments’ number, and \( K = 2N \) is the filter length (Daubechies, 1992; Domínguez-Jiménez & Ferreira, 2011).

Wavelet families vary in terms of several properties, as the extension of the wavelet support (in time and frequency) and rate of decay. Also, with the asymmetry of the wavelet, since the adequate signal reconstruction is associated with the symmetry property. The number of vanishing moments also affect the
wavelet behavior, as it increasing numbers of vanishing moments result in sparse representations for a large class of signals. And the wavelet regularity is also a need, since it provides sharper frequency resolution analysis, and allowing precise localization of signal transients (Domínguez-Jiménez & Ferreira, 2011).

If energy conservation is essential, it should be used compactly supported orthogonal wavelets. Except for the Haar wavelets, compactly supported orthogonal wavelets are not symmetrical, and they have nonlinear phase. In this case, the DWT can dissipate energy; but the maximal overlap discrete wavelet transform (MODWT) conserves the power, even if using compactly supported orthogonal wavelets. Wavelets with smaller support (time interval), such as HAAR, Db2, or Sym2, make it possible to find features closely spaced. Wavelets with more extensive support tend to have difficulty in detecting this kind of features. For data with sparsely spaced transients, it is recommended the use of the wavelets with more significant support. Finally, whether the primary goal is a detailed time-frequency (scale) analysis or precise localizations of signal transients, the CWT (Continuous Wavelets Transform) is adequate (Mallat, 2008; Gallegati & Semmler, 2014).

3.2.2 Wavelets Filters

The wavelets are defined by scaling and translating a mother function $\psi(t)$, set, respectively, for the DWT and maximal overlap discrete wavelet transform, MODWT, as:

\[
\frac{1}{\sqrt{2^j}} \psi\left(\frac{n-2^j m}{2^j}\right), \tag{6a}
\]

\[
\frac{1}{\sqrt{2^j}} \psi\left(\frac{n - m}{2^j}\right), \tag{6b}
\]

where $n$ is the scale number, and $m$ is the translation parameter (position), both are nonnegative integers.

Associated to this oscillating (high pass) wavelet function exists a non-oscillating (low pass) scaling function (a father wavelet $\varphi(t)$). The fast wavelet transform algorithm does not make use of the wavelet and scaling purposes, but of the filters that characterize their interactions by an inner product (Akansu, Serdijn, & Selesnick, 2010; Gençay, Selçuk, & Whitcher, 2002, pp.106–109),

---

7 Described in Continuous and Discrete Wavelet Transforms documentation, in MathWorks MATLAB-R2019b.
defined as:

\[ h[m] = \left\langle \psi\left(\frac{t}{2}\right), \varphi(t-m) \right\rangle, \]  

(7a)

\[ g[m] = \left\langle \varphi\left(\frac{t}{2}\right), \varphi(t-m) \right\rangle. \]  

(7b)

The estimation of the high pass filter \( h[m] \), and low pass \( g[m] \) depend on the definitions of the wavelet’s mother and the wavelet’s father functions, respectively, \( \psi(t) \) and \( \varphi(t) \). Thus, the filters are the main elements involved in the calculation of the DWT of a specific discrete-time signal \( f(t_i) \) of length \( M \), with \( i = 1, 2, \ldots, M - 1 \), since \( h[\cdot] \) and \( g[\cdot] \) are used to transform \( f(t_i) \) into its DWT. The scaling filter is defined as \( g_l = (-1)^{l+1}h_{L-1-l} \) (where \( L \) is the filter length), which has an inverse relationship with the filter, as \( h_l = (-1)^lg_{L-1-l} \). These wavelet filters must satisfy the three basic properties: \( \sum_{l=1}^{L-1} h_l = 0 \), \( \sum_{l=1}^{L-1} h_l^2 = 1 \), and \( \sum_{l=1}^{L-1} h_l h_{l+2k} = 0 \) (for all non-zero integers \( k \)); i.e., the wavelet filters sum to zero, they have unit energy, and they are orthogonal to its even shifts (Percival & Walden, 2000, p.69).

Daubechies wavelets extend the Haar wavelets by using more extended filters, producing smoother scaling functions and wavelets. How more significant is the filter size, \( K = 2N \), higher is the number \( N \) of vanishing moment, which allows compressing regular parts of the signal better; however, it also increases the size of the support of the wavelets that can be problematic in the region where the signal is singular (discontinuous). Thus, choosing the best wavelet is selecting the \( N \) that is adapted to a given class of signals.

Finally, the index number \( N \), in \( \text{Db}N \) filters and also in \( \text{symlets}N \), refers to the number of zero vanishing moments. Each wavelet has a number \( K = 2N \) of coefficients. A limited vanishing moment reduces the wavelets ability to encode, due to the phenomenon of scale leakage, and the lack of shift-invariance (Percival & Walden, 2000).

### 3.2.3 Maximal Overlap Discrete Wavelet Transform (MODWT)

The MODWT preserves the variance of the original series (Percival & Walden, 2000; Gençay et al., 2002). This property is essential for the estimation to be accurate. Another essential feature is that the MODWT presents shift-invariance, so when moving an observation in the rolling window, all the decomposed scales will move together (Berger, 2016). Once the variance of the original return series...
is preserved and there is a shift-invariance, we can estimate the volatility using reconstructed series from the decompositions.

To generate financial wavelet series, you must apply the filters on the return database. MODWT wavelet filters \( \tilde{h}_{j,l} \) and scaling filters \( \tilde{g}_{j,l} \) are obtained, respectively, from the DWT filters \( h_{j,l} \) and scaling filters \( g_{j,l} \) (equations (7a) and (7b)), as follows:

\[
\tilde{h}_{j,l} = h_{j,l} / 2^{j/2}, \\
\tilde{g}_{j,l} = g_{j,l} / 2^{j/2},
\]

where \( j = [1,2,\ldots,J] \) represents the decomposition levels with \( J \) being the highest level of decomposition, and \( l = [1,\ldots,L] \) is the size of the filter that is found intrinsically connected to the level \( J \) chosen, being that \( L \) is the largest scale length.

The values obtained for \( \tilde{g}_{j,l} \) and \( \tilde{h}_{j,l} \), through equations (8a) and (8b) above, are the MODWT wavelet filters generated by the transformations of the original DWT filters (equations (7b) and (8a)). Higher the level, lower the frequency of the wavelet function obtained (Gençay et al., 2002).

The wavelet coefficients, which will later generate the wavelet series, are obtained by the convolution of the returns \( r = \{ r_t, t = 1,2,\ldots,M-1 \} \) and the MODWT filters:

\[
\tilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} r_{t-l \mod M}, \quad \text{for } j = 1,2,\ldots,J \\
\tilde{V}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{g}_{j,l} r_{t-l \mod M}, \quad \text{for } j = 1,2,\ldots,J
\]

where \( L_j = (2^j - 1)(L - 1) + 1 \) is the length of the wavelet filter, associated with the scale \( \lambda_j \) (the corresponding lengths \( L \) for each filter kind are given in Table 1), and \( \mod N \) is the modulus operator, which indicates that is necessary to deal with a vector \( r_t \), with finite length in number of observations (Percival & Walden, 2000, p.121). Thus, we are implicitly assuming that it can be considered as a periodic.

As for each value of \( t \) the coefficients \( \tilde{V}_{j,t} \) and \( \tilde{W}_{j,t} \) are different sums, the representation of the convolutions given by (9a) and (9b) can be provided in
Table 1. Wavelet filters and the corresponding lengths of the filters ($L$).

| Wavelet filter | Length of the wavelet filter ($L$) | Wavelet filter | Length of the wavelet filter ($L$) |
|----------------|------------------------------------|----------------|------------------------------------|
| Haar (Db1)     | 2                                  | Sym2           | 4                                  |
| Db2            | 4                                  | Sym3           | 6                                  |
| Db3            | 6                                  | Sym4           | 8                                  |
| ...            | ...                                | ...            | ...                                |
| DbN            | $K$                                | SymN           | $K$                                |

matrix form as follows:

\[
\tilde{W}_j = \tilde{\omega}_j r, \quad (10a)
\]

\[
\tilde{V}_j = \tilde{V}_j r, \quad (10b)
\]

where $\tilde{\omega}_j$ and $\tilde{V}_j$ are matrices.

We can write relations (10a) and (10b) in matrix notation as

\[
\tilde{\omega}_j = \begin{bmatrix}
\tilde{h}_{j,0} & \tilde{h}_{j,N-1} & \tilde{h}_{j,N-2} & \cdots & \tilde{h}_{j,2} & \tilde{h}_{j,1} \\
\tilde{h}_{j,1} & \tilde{h}_{j,0} & \tilde{h}_{j,N-1} & \cdots & \tilde{h}_{j,3} & \tilde{h}_{j,2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{h}_{j,N-1} & \tilde{h}_{j,N-2} & \tilde{h}_{j,N-3} & \cdots & \tilde{h}_{j,1} & \tilde{h}_{j,0}
\end{bmatrix}, \quad (11a)
\]

and

\[
\tilde{V}_j = \begin{bmatrix}
\tilde{g}_{j,0} & \tilde{g}_{j,N-1} & \tilde{g}_{j,N-2} & \cdots & \tilde{g}_{j,2} & \tilde{g}_{j,1} \\
\tilde{g}_{j,1} & \tilde{g}_{j,0} & \tilde{g}_{j,N-1} & \cdots & \tilde{g}_{j,3} & \tilde{g}_{j,2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{g}_{j,N-1} & \tilde{g}_{j,N-2} & \tilde{g}_{j,N-3} & \cdots & \tilde{g}_{j,1} & \tilde{g}_{j,0}
\end{bmatrix}. \quad (11b)
\]

Finally, we can reconstruct the original series of returns by adding the wavelet coefficients, multiplied by the transpose of the matrices $\tilde{\omega}_j$ and $\tilde{V}_j$ (see footnote 6 at page 154), as follows:

\[
r = \sum_{j=1}^{J} \left( \tilde{\omega}_j^T \tilde{W}_j + \tilde{V}_j^T \tilde{V}_j \right) = \sum_{j=1}^{J} \tilde{D}_j + \tilde{S}_j. \quad (12)
\]
The reconstruction of the return series $r$, according to equation (12), makes it more and more precise, and approaches the original return series, as $J$ increases. As described in Berger (2016), the detail coefficients, at different levels, are given $\tilde{D}_j$ that describe the local details in the trend $j$, in which each $J$ represents a different frequency level. The revised version of the series $\tilde{S}_j$ represents the trend component. In our case, it is negligible.

### 3.2.4 Continuous Wavelet Transform (CWT)

The continuous wavelet transform (CWT) is an alternative approach to overcome the resolution problem. In the CWT analysis, the signal is multiplied by a wavelet function, a window function similar to the STFT (short-time Fourier transform). The CWT transform is computed separately for different segments of the time-domain signal. In the CWT, the window width is changed as the transform is calculated for every single spectral component (this the most significant characteristic of the CWT).

The CWT, in an interval $2D$, is an inner product of the signal to be transformed, $r(t)$ (single or complex-valued), and the basic window function (the wavelet function). The wavelet transforms of an integrable signal $r(t)$ concerning the wavelet $\Psi^{\tau,s}_r(t)$ is defined in the time domain, in the frequency domain, as (Olhede & Walden, 2002; Lilly, 2017; Addison, 2018):

$$
\Psi^{\tau,s}_r(t) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} r(t) \Psi^\ast_{\tau,s}(t) dt = \sqrt{s} \int_0^{\infty} \hat{r}(\omega) \hat{\psi}^\ast_{\tau,s}(s\omega) e^{i\omega t} d\omega, \quad (13)
$$

where $s \in [0, \infty)$ and $\tau \in (-\infty, \infty)$ are the scale (or dilation) and translation parameters, respectively; $\Psi^\ast_{\tau,s}(t)$ is the conjugate of the dilated and translated mother wavelet function $\Psi_{\tau,s}(t) = \psi((t - \tau)/s)$; $\hat{r}(\omega)$ and $\hat{\psi}^\ast_{\tau,s}(\omega)$ are, respectively, the Fourier transforms of $r(t)$ and $\Psi^\ast_{\tau,s}(t)$.

The frequency-domain form (13) is found by inserting $\int_{-\infty}^{\infty} e^{i\omega t} d\omega = 2\pi \delta(\omega)$, where $\delta(\omega)$ is the Dirac delta function. The scale variable $s$ specifies a stretching or compression of the wavelet in time. The rescaled frequency-domain wavelet $\Psi_{\tau,s}(\omega s)$ obtains a maximum at $\omega s = \Psi_{\tau,s}/s$, referred to here as the scale frequency. In equation (13), also in the frequency-domain form, we will consider only wavelets that vanish for negative frequencies; i.e., that have $\hat{\psi}^\ast_{\tau,s}(0) = 0$ for $\omega < 0$. Such wavelets are called analytic (Zhang, Ren, & Huang, 2003; Lilly & Olhede, 2010).
The wavelet function $\Psi_{\tau,s}(t)$ must have finite energy, such as

$$c_{\psi} = \int_{-\infty}^{\infty} |\Psi_{\tau,s}(t)|^2 \, dt < \infty,$$

or

$$c_{\psi} = \int_{0}^{\infty} \frac{\left|\hat{\psi}_{\tau,s}(\omega)\right|^2}{|\omega|} \, d\omega < \infty,$$

where $\hat{\psi}_{\tau,s}(\omega)$ is the Fourier transform of $\Psi_{\tau,s}(t)$ and $c_{\psi}$ is the admissibility constant. This condition implies that the wavelet has no zero-frequency component, i.e., $\Psi_{\tau,s}(0) = 0$ and $\hat{\psi}_{(j,k)}(0) = 0$.

Considering the time series $r(t)$ as the discrete sequence $t_k \equiv t(k\Delta t)$ where $\Delta t$ is the sampling interval. The discrete effects may be neglected, whether it is provided a scale $s$ sufficiently large compared with $\Delta t$. Thus, expressing that the scale discretization is $s = (s_0)^j$, and the translation discretization is $\tau = k(s_0)^j\tau_0$, where $s_0 > 1$ and $\tau_0 > 0$. Making $s_0 = 2$ and $\tau_0 = 1$, and using $s = 2^j$ and $\tau = 2^j\tau_0$, the discrete $\Psi_{k,j}(t)$ equation become $\Psi_{k,j}(t) = \psi(2^{-j}t - k)$, with $k,j \in \mathbb{Z}$, where $\mathbb{Z}$ is the integer’s numbers, varying in the interval of $(-\infty, +\infty)$.

The rebuilt of the series $r(t)$ is given by the inverse transform of the CWT (equation (13)), which can be written, in a discrete form as

$$r(t) = c_{\psi} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Psi_{X,j,k}^\psi(j,k)\Psi_{(j,k)}(2^{-j}t - k). \quad (14)$$

In this study, for continuous analysis in time-frequency scale, it is used three mother wavelets, commonly used in wavelet analysis in CWT: Analytic Morlet wavelet, Generalized Morse wavelet, and the $m$th order derivative of gaussian wavelets (DOG wavelet).

### The Morlet wavelet

Morlet wavelets are used in time and frequency domains for analyzing non-stationary time series data. The crucial parameter of Morlet wavelets is the width of the Gaussian that tapers the sine wave. The parameter width controls the trade-off between temporal precision and frequency precision (Cohen, 2019). The Morlet wavelet is formulated in the time and frequency domains (Holschneider, 1995; Lilly & Olhede, 2009), are respectively, as

$$\Psi_{\omega_0}(t) = a_{\omega_0}e^{-\frac{t^2}{2}}(e^{i\omega_0 t} - e^{-\omega_0^2t^2/2}) \quad (15a)$$

$$\Psi_{\omega_0}(\omega) = a_{\omega_0}e^{-\frac{1}{2}(s\omega-\omega_0)^2}(1 - e^{-s\omega_0}), \quad (15b)$$
where $i$ is the imaginary operator ($i = \sqrt{-1}$); $\omega_0$ is the carrier wave frequency; and $\omega$ is an angular frequency. When $\omega_0$ increases, more oscillations fit into the Gaussian window, and the wavelet becomes increasingly frequency-localized. The seconds terms introduced in (15a) and (15b), $e^{(\omega_0)^2/2}$ and $e^{-s\omega_0}$ are corrections necessary to enforce zero mean, while $a_{\omega_0}$ normalizes the wavelet amplitude, being $a_{\omega_0} = 1/\sqrt{\pi}$, and the wavelet scale $s = 2^j$ (Lilly & Olhede, 2009).

The analytic Morlet wavelet, defined in the Fourier domain, is a reduction of the equation (15b), neglecting the term of zero mean correction, $e^{-s\omega_0}$, and discarding the negative support, obtained $\hat{\psi}(s\omega) = \pi^{-1/4}e^{(s\omega-\omega_0)^2/2}U(s\omega)$, where $U(\omega)$ is the Heaviside step function: $U(\omega) = 1$ if $\omega > 0$; $U(\omega) = 0$ otherwise (Torrence & Compo, 1998). The neglecting the term of $e^{-s\omega_0}$ is possible since $\omega_0 \gg 0 \rightarrow \omega_0 > 5$ (Addison, 2018).

The non-analytic Morlet wavelet, defined in the Fourier domain, is a reduction of the equation (15b), in similar way of the analytic Morlet wavelet, maintaining the negative support, obtaining $\hat{\psi}(s\omega) = \pi^{-1/4}e^{(s\omega-\omega_0)^2/2}$; and the non-analytic Morlet wavelet with exact zero mean is precisely the equation (15b), reorganized as $\hat{\psi}(s\omega) = \pi^{-1/4}\left[e^{-\left(s\omega - \omega_0\right)^2/2} - \kappa_w e^{-\left(\omega_0\right)^2/2}\right]$, where $\kappa_w = e^{-\left(s\omega\right)^2/2}$ (Ashmead, 2010).

In the solution of these Morlet wavelets are assumed $\omega_0 = 6$, $s_0 = 2dt$ (where $dt$ is the sampling period), and $ds = 0.4875$.

$m$th order derivative of Gaussian Wavelets (DOG wavelet) In the Fourier domain, the $m$th order derivative of Gaussian wavelets is defined as (Ashmead, 2010)

$$\hat{\psi}(s\omega) = \frac{1}{\sqrt{\Gamma(m+1/2)}}(js\omega)^m e^{-(s\omega)^2/2},$$

(16)

where $\Gamma(\cdot)$ denotes the gamma function.

The $m$th order derivative must be an even order. The order of the derivative two is known as the Mexican hat wavelet. The rest of the conditions are the same for Morlet wavelet, showed in the end paragraph of the subsection above.

The Morse wavelet The family of functions termed generalized Morse wavelets are a family of precisely analytic wavelets whose Fourier transforms is supported only on the positive real axis. They are useful for analyzing modulated signals, which are signals with time-varying amplitude and frequency, and also helpful in analyzing localized discontinuities (Olhede & Walden, 2002). Morse wavelet is a two-parameter family of functions, $\Psi_{\beta,\gamma}(\omega)$, defined in the frequency domain for
\[ \beta \geq 0 \text{ and } \gamma > 0, \text{ as follows (Lilly & Olhede, 2009, 2012):} \]

\[ \Psi_{\beta,\gamma}(\omega) = U(\omega)a_{\beta,\gamma}\omega^{\beta}e^{-\omega^{\gamma}}, \quad \text{with } U(\omega) = \begin{cases} 1, & \text{if } \omega > 0, \\ 1/2, & \text{if } \omega = 0, \\ 0, & \text{if } \omega < 0, \end{cases} \]

(17)

where \( U(\omega) \) is the Heaviside step function, and \( a_{\beta,\gamma} = 2(e^{\gamma/\beta})^{\beta/\gamma} \) is a normalizing constant.

The constant \( e \) appearing in the numerator of \( a_{\beta,\gamma} \) is Euler’s number, \( e \approx 2.71828 \). The parameter \( \beta \) controls the low-frequency behavior, being viewed as a decay or compactness parameter; while \( \gamma \) characterizes the symmetry of the Morse wavelet, and \( \beta \gamma \) is the time-bandwidth product. The constant \( a_{\beta,\gamma} \) is the peak frequency. The choice of \( a_{\beta,\gamma} \) in (17) sets the maximum value of the frequency-domain wavelet to \( \Psi_{\beta,\gamma}(\omega) = 2 \) (Olhede & Walden, 2002; Lilly & Olhede, 2009).

### 3.3 GARCH and FIGARCH models

Fractionally integrated GARCH models (FIGARCH) were developed, to introduce flexibility to model data series that have both short and long memory characteristics (Baillie et al., 1996). FIGARCH\((p,d,q)\) model solves this problem, fractionalizing the integration of the innovations. In this model, \( p \) is the number of autoregressive components, \( q \) the number of moving-average components, and \( d \) a fragmentation parameter of memory effect. Its formula is written as follows:

\[ [1 - \alpha(L) - \beta(L)](1 - L)^d \epsilon_t^2 = \alpha_0 + [1 - \beta(L)]\nu_t. \]

(18)

In equation (18), \( \alpha(L) \) and \( \beta(L) \) are lag operators such that \( \alpha(L) = \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_p L^p \) and \( \beta(L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p \); \( \alpha_0 > 0 \), \( \alpha_i > 0 \), with \( i = 1,2,\ldots,p; \beta_j > 0 \), with \( j = 1,2,\ldots,q \); \( \nu_t = \epsilon_t^2 - \sigma_t^2 \); and \( \epsilon_t \) is a white noise process, with \( E(\epsilon_t) = 0 \) and \( \text{var}(\epsilon_t) = \alpha_0(1 - \alpha(L) - (L)^{-1}) \); \( \sigma_t^2 \) is the conditional variance to be estimated, and \( \nu_t = \epsilon_t^2 - \sigma_t^2 = (z_t^2 - 1)/\sigma_t^2 \) is uncorrelated \( \text{var}(z_t) = 1 \), and \( d \) is a fraction, is such that \( 0 < d < 1 \).

What characterizes FIGARCH\((p,d,q)\) is the fraction \( d \), which fragments the integration of the innovations. If \( d \approx 1 \), the FIGARCH model given by (18) resembles an IGARCH model, in which the impacts of past shocks squared are persistent (long memory model). If \( d \approx 0 \), the model (18) approaches the classical GARCH model, in which the impacts of the shocks decrease rapidly, as
it moves away from the shocks (short memory model). Therefore, FIGARCH works as a model that transits in the modeling between the GARCH model and the IGARCH, depending on the value of estimate \( d \).

Equation (18) will be reorganized, replacing in this equation \( \nu_t = \epsilon_t^2 - \sigma_t^2 \), as previously defined, and doing \( \varphi(L) = [1 - \alpha(L) - \beta(L)] \) so permitting to obtain:

\[
\sigma_t^2 = \alpha_0 (1 - \beta(L))^{-1} + \lambda(L) \epsilon_t^2,
\]  

(19)

where \( \lambda(L) = \left[ 1 - \varphi(L)(1 - \beta(L))^{-1}[1 - \beta(L)](1 - L)^d \right] \) and \( 0 < d < 1 \).

In the solution of (19), all roots \( \varphi(L) \) remain outside the unit circle, thus, for a FIGARCH\((p,d,q)\), \( \lambda(L) \) must have infinite elements, which is \( \lambda(L) = \lambda_1(L) + \lambda_2(L) + \lambda_3(L) + \cdots + \lambda_k(L) \), with \( k \rightarrow \infty \). The lambda operator measures the impact that past innovations will have on predictions of future volatility. Its value is defined by the parameters estimated by the FIGARCH method.

In order for equation (19) to be well defined, the conditional variance given in this equation is found in the \( \text{ARCH}(\infty) \) representation, which requires that the elements \( \lambda(L) \) must be non-negative, or being \( \lambda_k \geq 0 \), for \( k = 1,2,\ldots \).

For a FIGARCH\((1,d,1)\) model, \( \beta(L) = \beta_1, \alpha(L) = \alpha_1 \) and \( \varphi(L) = \varphi_1 \), and equation (19) becomes:

\[
\sigma_t^2 = \alpha_0 (1 - \beta_1)^{-1} + \lambda(L) \epsilon_t^2.
\]  

(20)

To estimate the elements of the operator \( \lambda(L) \), we first calculate the parameters \( \varphi_1, \alpha_1, \beta_1 \) and \( d \) by the FIGARCH\((1,d,1)\) model to be applied in this study, and then calculate the elements \( \lambda(L) \), by the following recurrence formula:

\[
\lambda_k = \beta_1 \lambda_{k-1} + \left( \frac{k - 1 - d}{k} - \varphi_1 \right) \delta_{d,k-1}, \quad \text{with} \ k = 2,\ldots,\infty. 
\]  

(21)

In the use of the initial process, when \( k = 1 \), the following recurrence formulas:

\[
\varphi_1 = (1 - \alpha_1 - \beta_1), 
\lambda_1 = (\varphi_1 - \beta_1 - d), 
\delta_{d,0} = 1.
\]

(22a)  
(22b)  
(22c)

In equation (21), \( \delta_{d,k} \) refers to the coefficients in the expansion series of
(1 − \mathcal{L})^d$, then, if it is used (22a), (22b) and (22c) as recurrence relations, $\delta_d(\mathcal{L})$ can be written as:

$$
\delta_d(\mathcal{L}) = \sum_{k=1}^{\infty} \delta_{d,k} \mathcal{L}^k.
$$

(23)

Finally, according to Tayefi and Ramanathan (2012, p.186), the FIGARCH model, given by equation (20), makes it possible to estimate the conditional variance by ARCH(\infty). Theoretically it should be expanded to \infty, but, in practice, this expansion should be limited to a large number of repetitions, specified here by $M$. Therefore, the predictions of conditional variances are made by lambda $M$ combinations and past innovations, without loss of generality, since, for very large $M$, the lambdas subsequently decrease, and become less and less significant for larger lags.

Thus, using the relation (23) in (21), and the result in equation (19), it is obtained as result, to $l$ steps ahead, the following equation:

$$
\sigma_t^2(l) = \alpha_0 (1 − \beta_1)^{-1} + \sum_{i=1}^{l-1} \lambda_i \sigma_t^2(l-i) + \sum_{k=0}^{M} \lambda_k + \epsilon_t^2.
$$

(24)

### 3.4 Rolling Window

After estimating the VaRs, they need to be tested to verify their reliability or use them on a day-to-day basis. So far, we have evaluated the future conditional variance for one day; then, the VaR predicted is limited for a one-day forecast. But for VaRs to be tested correctly, a much larger sample of VaR forecasts are needed. To overcome this problem, a rolling window, RW (Zivot & Wang, 2006), is applied. This procedure allows a significant increase in the number of future VaR forecasts, which will constitute a sample to test the performance of these VaRs, against an out-of-sample forecast-return series.

Considering that in this study, several predictions of future conditional volatility need to be established and, for that, the RW technique is applied to build them. First, the size of the rolling window was chosen, which in the study was $M = 470$ days. Then, the forecast horizon size of the VaRs was determined, which are daily with a one-day forecast horizon, was chosen. With this data, the structured rolling window in the study uses 470 past days to generate the predictions of a coming day, through equation (24). Therefore, the impacts of the $\lambda_i$ estimates, with $i = 1, 2, \ldots, M$, with $M = 470$, induce the intrinsic effects of the calibrated return series on the forecast of $\sigma_t^2(l+1)$, for a day ahead. With the prediction of the conditional volatility of day $t+1$, the rolling window
takes a step forward, thus finding $t + 1$, but aiming to estimate the conditional volatility in the time $t + 2$. This process is repeated as far as desired until the conditional variance is determined for the established forecast period. In this study, the process will be repeated 210 times to generate 210 days of predictions of conditional volatility for one day. These forecasts will be used in estimates of future VaRs.

For a better understanding of the RW process, when estimating the volatility one day ahead of a day $t$, located $l$ steps ahead of the end of the return series used in the calibration process, the rolling window uses all present data $p$ to 470 days before $t$, introducing the impacts of the larger ones $\lambda_i$, in the current stochastic noise term and the current time of $\sigma_t^2$, and subsequently, as can be seen in equation (24).

### 3.5 Back Testing

The Back-testings allow to emphasize the performance of the estimated models, providing conclusions that make possible highlighting the doubts about the risks of classifying models as accurate, or inaccurate. One can assure robustness in the back-testings’ performances if it is used a set of back-testings in the analysis of models. In practice, many metrics and statistical tests are used to identify the accuracies, in terms of estimates of failure probabilities. Frequently, the back-testings present distinct degrees of robustness and fragility in their estimate statistics, which suggest the use of more than one criterion to test the performance of a VaR.

The first test to be emphasized is the semaphore test, proposed by the Basle Committee on Banking Supervision (1996), based on the construction of a cumulative distribution function ($cdf$), from a binomial distribution, given by the following equation:

$$
cdf = F(x | n, p) = \sum_{i=0}^{x} \binom{n}{i} p^i (1 - p)^{n-i} I_{0,1,...,n}(i). \quad (25)
$$

In equation (25), $x$ is the number of attempts, with probability of obtaining up to $x$ successes; $n$ is the number of independent observations (the number of out-of-sample frameworks used in the VaR proof), with probability $p$ of successes, in which $n > 0$, $x \in [0,n]$; and $p \in [0,1]$. The indicator function, $I_{0,1,...,n}$, ensures that $x$ only adopts values in the intervals $[0,1,...,n]$.

From the $cdf$ result, three zones are defined in the semaphore test, as: 1) The “red” zone begins in the number of faults, corresponding to the probability equal...
to or above 99.99% of the cdf. It is unlikely that too many failures will come from a correct VaR model; ii) The “yellow” zone covers the number of failures where the probability is equal to or greater than 95%, but less than 99.99%, of the cdf. Even if there is a high number of violations, the violation count is not excessively high; and iii) Everything below the yellow zone (usually between 90% and less than 95% of the cdf), is the “green” zone. So, if you get very few failures with the VaR model, it falls into the green zone. Only many flaws lead to rejections of models, thus the red zone.

Kupiec (1995) introduced the probability of failure (POF) test, defined by a likelihood ratio function to test whether the likelihood of failure is synchronized with the implicit probability $p$ at the VaR confidence level. This unconditional coverage test, which is estimated considering the average of a violation sequence $\{I_t\}_{t=1}^n$, given by equation (4), where $I_t$ is a failure indicator variable, a chain of zeros and ones that allow the estimating of the $\hat{p}$, by equation (5). Since $I_t$ is a random variable, the values assumed by $\hat{p}$ will also be random for each time interval. Then, the null hypothesis assumes a failure rate equal to the expected rate, as follows:

$$H_0: E[I_t] = p \quad \text{and} \quad H_1: E[I_t] \neq p,$$

(26)

where $p = \alpha$ is the quantile level of VaR model; $x = \sum_{t=1}^n I_t$ is the number of failures; $E[I_t] = \hat{p} = x/n$ being the expected rate of failures; and $n$ the number of out-of-samples.

The statistic of the POF test is defined by:

$$LR_{POF} = -2 \times \log \left( \frac{(1-p)^{(n-x)}p^x}{(1-x/n)^{(n-x)}(x/n)^x} \right),$$

(27)

The statistic given by equation (27) is a chi-square variable with a degree of freedom $\chi^2(1)$. The VaR model fails the test if the likelihood ratio given in (27) exceeds a critical value.

Christoffersen (1998) proposed a test to measure whether the probability of a failure on a specific day depends on whether a failure occurred. This test measures the dependence of failures between consecutive days. This test covers both the violation rate and the independence of failures. If the model is exact, a failure today should not depend on whether or not a failure occurred the previous day.

The test statistic for failure independence is a likelihood ratio function,
written as follows:

\[ LR_{\text{IND}} = -2 \times \log \left( \frac{(1 - \pi_1)^{v_{00} + v_{10}} \pi_1^{v_{01} + v_{11}}}{(1 - \pi_0 v_{01} (1 - \pi_{11}) v_{10} \pi_{11})} \right), \]  

(28)

where \( v_{00} \) the number of periods without failures is followed by a period without failures; \( v_{10} \) the number of periods with failures, followed by a period without failures; \( v_{01} \) the number of periods without failures, followed by a period with failures; and \( v_{11} \) the number of periods with failures followed by a period with failures.

The parameters \( \pi_0, \pi_1, \) and \( \pi \) are defined, respectively, as follows:

\[ \pi_{01} = \frac{v_{01}}{v_{00} + v_{01}}, \]  

(29a)

\[ \pi_{11} = \frac{v_{11}}{v_{10} + v_{11}}, \]  

(29b)

\[ \pi_1 = \frac{v_{01} + v_{11}}{v_{00} + v_{01} + v_{10} + v_{11}}, \]  

(29c)

where \( \pi_{01} \) is the probability of having a failure in period \( t \), given there was no failure in the period \( t - 1 \); \( \pi_{11} \), the probability of having a failure in period \( t \), once a failure occurred in the period \( t - 1 \); \( \pi_1 \), the probability of having a failure in period \( t \).

The statistic \( LR_{\text{IND}} \), given in (28), is asymptotically distributed as a chi-square, with a degree of freedom \( \chi^2(1) \). This failure independence test statistic establishes, in the null hypothesis, that the failure probability must remain constant, in any series period considered, if the independence test is accepted; or that

\[ H_0: \hat{\rho}(I_t = 1 | r_{t-1}) - \hat{\rho}(I_t = 1) = 0, \]  

(30)

where \( I_t \) is the failure indicator, for a given interval forecast, for time \( t \), compared with one at time \( t - 1 \). It is defined equal 1 if failure condition in a region forecast for time \( t \) is similar to that one at time \( t - 1 \). Otherwise, it is zero.

Christoffersen (1998) combined its independence test (IND) with the Kupiec failures proportion test (POF) to establish a conditional coverage test. In this combined test is estimated to not only the failure rate, but also failures’ independence, that is conditional coverage (CC). It is as followed:

\[ LR_{\text{CC}} = LR_{\text{POF}} + LR_{\text{IND}}. \]  

(31)
The conditional coverage test is asymptotically distributed as a chi-square variable with two degrees of freedom, $\chi^2(2)$, stating that the failure probability must remain constant, in any series period considered, if the coverage test is accepted, as explained above, and equal to the coverage rate $\hat{p}$, given by $LR_{POP}$; this is

$$H_0: \hat{p}(I_t = 1 | r_{t-1}) - p = 0 \quad \text{and} \quad H_0: \hat{p}(I_t = 1 | r_{t-1}) - p \neq 0. \quad (32)$$

In summary, the conditional coverage test (CC) makes it possible to verify if a VaR model underestimates or overestimates the risk, essentially, due to the effects of concentration and independence\(^8\) of failures.

### 4 Methodological Procedure

The series of compound returns were estimated, taking the closing prices of the selected shares. The wavelet decomposition process was applied to the return series, using the methodologies presented in subsections 3.2.3 and 3.2.4, which were chosen eight levels’ decomposition. Therefore, there is a total of eight wavelet decomposition series with eight different levels for each component return. Figures 1 and 2 illustrate these series generated, respectively, for the decompositions obtained from the set of returns of the ABEV3 stock, in operation on the BM&F BOVESPA, and for the decompositions obtained for the KO stock of the Coca-Cola company, in NYSE. The graphs in Figure 1 where determined by the MODWT, in conjunction with the wavelet filters Db2, Sym2, Db3, and Haar; and the charts illustrated in Figure 2 were obtained by the CWT, using the analytical Morlet wavelet, generalized Morse wavelet, and the second and fourth-order derivatives of Gaussian wavelet (DOG wavelet). For both figures, each graph is a series of returns at a given level of decomposition $j$, which belongs to the interval $J = [1, 2, \ldots, 8]$.

It is clear from Figures 1 and 2 that as the level of decomposition increases, the series obtained will have longer wavelengths, however with amplitudes of oscillations smaller than the previous ones (see the vertical graphic scales). From this information, we conclude that the impacts of the higher-level decompositions (lower frequencies) on the variance of the original return series are smaller than

---

\(^8\) The independence test verifies the degree of concentration of the failures, considering that the failures do not always distribute in a homogeneous way over the period of time of the observations. Usually, in practice, the failures are often concentrated, demonstrating a dependency character, a factor that can change the number of violations, according to the size of the sample of observations (Godlewski & Merli, 2012).
of the Haar show a more substantial presence of noise than the components displayed by the other filters. Except for ... obtain the behaviors for the share decompositions, as shown in Figures (1) and (2).

Figure 1. Decomposition series of daily returns series for the ABEV3 share of BM & F-BOVESPA stock, in eight levels. In each figure, the chart at the top-left is the level \( j = 1 \), the graph at the top-right, is the level \( j = 2 \), and so on.
In general, the wavelet decompositions of the series of returns have an economic interpretation. Each level of detail can be seen as involving a cycle. The cycle period varies from level to level; the higher the level of decomposition is, the longer the cycle time interval. Because it is intuitive and enlightening, Table 2, below, shows the estimated contributions in the variance for each decomposition of the ABEV3 share of the BM & F-BOVESPA.

The values correspond to the intervals between the lengths of the orthogonal wavelet filters are obtained using the formula $j, 2^{j}, 2^{j+1}, \ldots, \lambda_{j}$, associated with the scale $2^{j+1} - 2$, and so on.

The estimated time scales' lengths for the CWT wavelets were obtained by dividing the generated frequency vector $\lambda_{j}$, $j=1,2,\ldots,8$, associated with the scale $2^{j+1} - 2$, and so on.

In Table 1. Thus, we are implicitly assuming that the length time interval for each filter is the following: for Haar filter is $2^{j}$, associated with the scale $2^{j}$, and so on.

The second column shows some specific Length of time scale via second-derivative of DOG wavelet, and via fourth-derivative of DOG wavelet. The second column shows some specific Length of time scale via second-derivative of DOG wavelet, and via fourth-derivative of DOG wavelet.

In each figure, the chart at the top-left is the level $j=1$, the graph at the top-right, is the level $j=2$, and so on.

**Figure 2.** Decomposition series of daily returns series for the NYSE's KO share, in eight levels. In each figure, the chart at the top-left is the level $j=1$, the graph at the top-right, is the level $j=2$, and so on.
the effects of the lower frequency decompositions with higher frequencies. This has a significant implication on this work, because if the amplitudes of the oscillations of the series of low frequencies are small, then the impacts on the VaR should also be minor. But, the low-frequency series are the long-term trends. So, a previously imagined conclusion is that for most actions, long-term oscillations are likely to have little effect on VaRs.

The decompositions showed in Figure 1, obtained from the series of returns of the ABEV3 stock, illustrate the decomposition behaviors, estimated by the filters Db2 (Figure 1a), Sym2 (Figure 1b), Db3 (Figure 1c), and Haar (Figure 1d). It is clearly remarked from these figures that there is no significant distinction among the corresponding components, estimated by the specified filters. Specifically, this characteristic is evident when comparing the results of the filters Db2 and Sym2. However, the decompositions’ results of the Db3 components show more significant smoothings, relative to that one indicated by the Db2 and Sym2 filters; and the decompositions’ results of the Haar show a more substantial presence of noise than the components displayed by the other filters. Except for this detail, all filters’ decompositions are very similar.

Figure 2 illustrates the decompositions obtained for the KO stock of the Coca-Cola company, in NYSE, estimated by using the generalized Morse wavelet (Figure 2a), analytical Morlet wavelet (Figure 2b), second-order derivative of DOG wavelet (Figure 2c), and fourth-order derivatives of DOG wavelet (Figure 2d). In the same way, as remarked in the graphs of Figure 1, we observe in the charts of the Figure 2 that there are no significant distinctions among the components estimated by the CWT wavelets—except that the decompositions of the generalized morse wavelet show less smoothed than one predicted by the other CWT wavelets. However, some specific details may be stressed if we compare the transition behaviors of components evaluated by the CWT with those ones obtained with orthogonal filters by MODWT. In particular, the trends estimated with the CWT wavelets appear more smoothed than those calculated by MODWT.

We applied the orthogonal filters to decompose the ABEV3 stock of the Ibovespa index, and the CWT wavelets to estimate the decompositions of the KO share of the DJA-NYSE stock. However, we can apply any of these techniques to decompose shares of Ibovespa or DJA-NYSE and obtain the behaviors for the share decompositions, as shown in Figures 1 and 2.

In general, the wavelet decompositions of the series of returns have an economic interpretation. Each level of detail can be seen as involving a cycle.
The cycle period varies from level to level; the higher the level of decomposition is, the longer the cycle time interval. Because it is intuitive and enlightening, Table 2 shows the estimated contributions in the variance for each decomposition of the ABEV3 share of the BM&F BOVESPA and KO of the NYSE stock, respectively.

In the first column, Table 2 presents the wavelet techniques applied in the frequency decomposition of the return series of the BM&F BOVESPA and NYSE stock. This column shows that we used the orthogonal wavelet filters Haar, Db2, Sym2, Db3, and Sym3; and the analytical Morlet wavelet, generalized Morse wavelet, and the second and fourth-order derivatives of DOG wavelet. The second column shows some specific Length of time scale, the intervals for the cycles. The cycles’ ranges are different for each filter kind since they have different scale lengths. The estimated time scales’ lengths for the CWT wavelets were obtained by dividing the generated frequency vector into eight intervals. The support range of CWT wavelets depends on the spacing between scales (in other words, the scale increment of the time-frequency is related to the window size of the wavelet). A large spacing between scales allows the timing scales to be distributed over all support extension of the return series, making possible a global analysis of the frequency distributions in the data, but shrinking the scales’ refinement and losing low-frequency oscillations. On the other way, a small spacing of scale enables the refinement of the data time support, but a detailed analysis of frequency only in part of the support; and significantly depreciating the coverage of large frequency scales. Thus, considering that our return series are daily, involving 780 days, we had to adjust the scale spacing to get full coverage of the series. This adjustment was particular for each type of CWT wavelet.

The third column shows the memory frequency effects, in the sense of economic interpretations of the period, in terms of cycle sustainability. Considering these interpretations and taking the length time scale distribution of the Haar filter (Figure 1a), we inferred that the decomposition $D_1$ incorporates the Stochastic noise, $D_2$ includes weekly cycles, with a time scale of 4 to 8 days. This time scale band captures the stock market “Monday Effects”. The detail

9 The values correspond to the intervals between the lengths of the orthogonal wavelet filters are obtained using the formula $L_j = (2^j - 1)(L - 1) + 1$, with $j = 1, 2, \ldots, 8$, associated with the scale $\lambda_j$ (the corresponding lengths $L$ for each filter kind are given in Table 1). Thus, we are implicitly assuming that the length time interval for each filter is the following: for Haar filter is $[2^j, 2^{j+1}]$; for Db2 and Sym2 are $[3 \times 2^j, -2, 3 \times 2^{j+1} - 2]$; and for Db3, and Sym3 are $[5 \times 2^j - 4, 5 \times 2^{j+1} - 4]$.
Table 2. Impact of the variances of the decomposition components on levels in the variance of the respective returns. An economic interpretation of these decompositions.

| Filter          | Length of time scale | Memory               | Contribution to variance of 'ABEV3' share | Contribution to variance of 'KO' share | Level of detail |
|-----------------|----------------------|----------------------|-----------------------------------------|--------------------------------------|----------------|
| Haar            | 2-4 days             | Stochastic noise     | 53.2%                                   | 50.8%                                 | D1             |
|                 | 4-8 days             | Short-term           | 26.4%                                   | 24.1%                                 | D2             |
|                 | 8-16 days            | Medium-Term          | 13.5%                                   | 13.5%                                 | D3             |
|                 | 16-32 days           | Medium-Term          | 5.3%                                    | 5.3%                                  | D4             |
|                 | 32-64 days           | Long-term            | 3.2%                                    | 3.2%                                  | D5             |
|                 | 64-128 days          | Long-term            | 1.2%                                    | 1.2%                                  | D6             |
|                 | 128-256 days         | Long-term            | 0.8%                                    | 0.8%                                  | D7             |
|                 | 256-512 days         | Trend                | 0.5%                                    | 0.5%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
|                  |                      |                      |                                         |                                       |                |
| D2 and Sym2      | 4-10 days            | Stochastic noise     | 53.2%                                   | 50.8%                                 | D1             |
|                 | 10-22 days           | Short-term           | 24.1%                                   | 24.1%                                 | D2             |
|                 | 22-46 days           | Medium-Term          | 13.6%                                   | 13.6%                                 | D3             |
|                 | 46-94 days           | Medium-Term          | 5.6%                                    | 5.6%                                  | D4             |
|                 | 94-190 days          | Long-term            | 3.4%                                    | 3.4%                                  | D5             |
|                 | 190-382 days         | Long-term            | 1.2%                                    | 1.2%                                  | D6             |
|                 | 382-766 days         | Trend                | 0.8%                                    | 0.8%                                  | D7             |
|                 |                      |                      | 0.3%                                    | 0.3%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
| D3 and Sym3      | 6-16 days            | Stochastic noise,    | 53.1%                                   | 50.9%                                 | D1             |
|                 | 16-36 days           | Short-term and Medium-Term | 24.1%                  | 24.1%                                 | D2             |
|                 | 36-76 days           | Long-term            | 13.7%                                   | 13.7%                                 | D3             |
|                 | 76-156 days          | Medium-Term          | 5.5%                                    | 5.5%                                  | D4             |
|                 | 156-316 days         | Long-term            | 2.4%                                    | 2.4%                                  | D5             |
|                 | 316-636 days         | Long-term            | 1.2%                                    | 1.2%                                  | D6             |
|                 | 636-1276 days        | Trend                | 0.8%                                    | 0.8%                                  | D7             |
|                 |                      |                      | 0.5%                                    | 0.5%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
| Morse           | 7.7-13.7 days        | Stochastic noise     | 44.7%                                   | 44.6%                                 | D1             |
|                 | 13.7-24.7 days       | Medium-term          | 22.7%                                   | 22.7%                                 | D2             |
|                 | 24.7-44.7 days       | Medium-Term          | 16.7%                                   | 16.7%                                 | D3             |
|                 | 44.7-80.8 days       | Medium-Term          | 7.7%                                    | 7.7%                                  | D4             |
|                 | 80.8-146.1 days      | Long-term            | 4.4%                                    | 4.4%                                  | D5             |
|                 | 146.1-264.1 days     | Long-term            | 1.9%                                    | 1.9%                                  | D6             |
|                 | 264.1-477.4 days     | Trend                | 1.3%                                    | 1.3%                                  | D7             |
|                 | 477.4-780.0 days     | Trend                | 0.5%                                    | 0.5%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
| Matrix          | 2.1-4.2 days         | Stochastic noise     | 53.9%                                   | 51.8%                                 | D1             |
|                 | 4.2-8.7 days         | Short-term           | 26.6%                                   | 26.6%                                 | D2             |
|                 | 8.7-18.0 days        | Medium-Term          | 12.6%                                   | 12.6%                                 | D3             |
|                 | 18.0-37.4 days       | Medium-Term          | 4.6%                                    | 4.6%                                  | D4             |
|                 | 33.4-77.7 days       | Medium-Term          | 2.4%                                    | 2.4%                                  | D5             |
|                 | 77.7-161.8 days      | Long-term            | 1.0%                                    | 1.0%                                  | D6             |
|                 | 161.4-336.1 days     | Long-term            | 0.8%                                    | 0.8%                                  | D7             |
|                 | 336.1-698.7 days     | Trend                | 0.1%                                    | 0.1%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
| Secondary wavelet| 7.9-13.5 days        | Stochastic noise     | 39.7%                                   | 41.7%                                 | D1             |
|                 | 13.5-23.8 days       | Medium-Term          | 18.7%                                   | 18.7%                                 | D2             |
|                 | 23.8-41.9 days       | Medium-Term          | 12.3%                                   | 12.3%                                 | D3             |
|                 | 41.9-73.7 days       | Medium-Term          | 7.7%                                    | 7.7%                                  | D4             |
|                 | 73.7-129.7 days      | Long-term            | 4.5%                                    | 4.5%                                  | D5             |
|                 | 129.7-228.4 days     | Long-term            | 4.1%                                    | 4.1%                                  | D6             |
|                 | 228.4-402.1 days     | Trend                | 2.3%                                    | 2.3%                                  | D7             |
|                 | 402.1-708.0 days     | Trend                | 2.6%                                    | 2.6%                                  | D8             |
|                 |                      |                      |                                         |                                       |                |
| Frequency wavelet| 5.9-10.5 days        | Stochastic noise     | 50.5%                                   | 49.7%                                 | D1             |
|                 | 10.5-19.3 days       | Medium-term          | 22.2%                                   | 22.2%                                 | D2             |
|                 | 19.3-35.6 days       | Medium-Term          | 11.5%                                   | 11.5%                                 | D3             |
|                 | 35.6-65.4 days       | Medium-Term          | 6.7%                                    | 6.7%                                  | D4             |
|                 | 65.4-120.4 days      | Long-Term            | 2.7%                                    | 2.7%                                  | D5             |
|                 | 120.4-221.6 days     | Long-term            | 2.2%                                    | 2.2%                                  | D6             |
|                 | 221.6-407.4 days     | Trend                | 1.1%                                    | 1.1%                                  | D7             |
|                 | 407.4-750.4 days     | Trend                | 0.0%                                    | 0.0%                                  | D8             |
level $D_5$ incorporates the effects of monthly cycles, and can achieve the “Turn-of-the-Month Effects”. In the time scale of the Haar filter, we also considered as medium scale, the time intervals that can capture oscillations, with cycles from half-month up to a month ($D_4$ and $D_5$). We define as the long-term fluctuations, the time intervals that capture oscillations of time length more extensive than a month (included the $D_6$ and $D_7$), repeatable in the series, fact that depends on the support of the return series, 780 days. Finally, we also considered as a trend the time intervals that capture oscillations, with a time scale length of the order of the support of the return series (ones comprise by only one oscillation cycle, and involving a non-zero mean, as $D_8$).

For the other orthogonal filters, besides Haar, and for some CWT wavelets, Table 2 shows the existence of more than one trend, because in these situations we consider the evolution of trends with oscillations of the order of the supports of the return series, and remnants of trends with supports larger than the return series (780 days).

It is also seen in Table 2, in the column of contributions to variance (third and fourth columns), where are shown, respectively, the percentage values of the shares ABEV3 of BM&F BOVESPA and KO of the NYSE, related to the variance of the total returns of the respective shares. Something apparent when comparing the contribution percent values of these two actions is that the lower frequency cycles do not contribute significantly to the variance of the respective total returns. They are the decompositions of short- and medium-term oscillations and stochastic noises that generate the significant parts of the return series variance. It is also observed that, in the third and fourth columns of Table 2, the sum of the variance decompositions of the levels is approximately 100%. Finally, In the last column, Table 2 presents the levels of oscillations, according to the wavelets’ decompositions, which are referenced by $D_j$, with $J$ comprising the integers in the interval $j = [1,2,\ldots,8]$.

The reconstructed series were generated by adding the decomposed components in sequential order from the lower level components until incorporating those of higher levels for each series. This rebuilt procedure for the decomposed components of the orthogonal filters are explained by equation (12), keeping the coefficient of detail $D_i$, and removing the coefficient $\tilde{S}_J$ (as already emphasized early, the impact of $\tilde{S}_J$ on series of daily returns is practically null). In equation (12), the detail levels for each action, given by $J$, is limited to eight, since the wavelet decomposition process was performed considering eight levels. For example, to generate a series considering only the sums of wavelet series of levels
being from one through \( J = 5 \), equation (12) would become \( r_{D_1 - D_5} = \sum_{i=1}^{J=5} D_i \).

Following this method, it was generated eight series of reconstructed returns, where \( J \) in equation (12) ranged from one to eight. The rebuilt procedure for the decomposed components of the CWT wavelets also follow the equation (12), despite \( \tilde{S}_J \neq 0 \), it is incorporated in the decomposition series, when estimating the inverse problem. Then, the rebuilt’ series already contains grated the detail \( D_i \), and the trend \( \tilde{S}_J \), if it exists on a series of daily returns, but it usually is practically null.

Figure 3 illustrates the impact that each rebuilt component causes on the variance of the original return series for the KO share. The figures’ sequence, in Figure 3, represents the estimates via the following wavelets: Db2, Db3, Haar, Morse, Morlex, and fourth derivative of DOG. In the abscissa axis are the levels’ representations of each reconstructed series, while in the ordinates are the respective mean-variance; also, for each reconstructed series. It is observed in this figure that the decompositions of lower levels (higher frequencies) hold the most significant parts of the floating energies (variance), and this contribution decreases as the decomposition level added increases (the oscillation frequencies decrease), making practically insignificant the variance contribution from level 5 (for almost every wavelet estimations) and annulling completely at level 8. As shown in Figure 3, the evolution process is, in detail, different for each wavelet applied in the rebuilt estimate, since each wavelet used for the estimation has a particular transition process among the levels of volatility. We can see from Figure 3 that the estimates for the Db2, Db3, and Morlet wavelets show steeper transition processes, i.e., with slopes in the steeper transition midline. It is observed that the reconstructions obtained with these wavelets already reach the maximum level of accumulated variance at the fifth reconstruction level. For reconstructions via the Haar and Morse wavelets, the transition process is less steep, reaching the sixth level of reconstruction, the maximum level of cumulative variance. Finally, the fourth derivative of the DOG wavelet presents an even less steep transition process, reaching only at the seventh level of reconstruction the maximum level of accumulated variance. These transition behaviors affect the evolution of VaR estimation along with the rebuilt components. We present estimates for only one DJI-NYSE share. However, the behavior of the reconstruction estimates using the specified wavelets remains similar for all BM&F BOVESPA stocks, and the analyzed DJI-NYSE shares.

In general, the result obtained in this study reinforces the conclusion presented by Berger (2016), in which the long-run cycles have little effect on the
Figure 3. Mean variances estimated for the rebuilt series for the NYSE's KO share. The figures, from the upper-left to the right, and from lower-left to the right, were estimates via the following wavelets: Db2, Db3, Haar, Morse, Morlex, and fourth-derivative of DOG.

variability of the series of daily returns. The reconstruction involving the wavelet decompositions, from level one to eight $D_{1-8}$ ($1 - 8$ in the horizontal legend of Figure 3), is precisely the representation of the original return series. Thus, the reconstruction series $D_{1-8}$ will be used as a proxy for the estimation of the original return series. Finally, it can be pointed out that the variance of a series of daily returns is preserved by its decomposition, via the wavelets’ algorithms, due to the energy preservation characteristic of these wavelet functions.

Predictions of the conditional volatilities present in rebuilt series were made for each reconstructed series, using the FIGARCH($1,d,1$) process. As already emphasized, this process presents good flexibility to deal with both long and short-term memory (Baillie et al., 1996). In total, four\textsuperscript{10} variations of the FIGARCH models were tested in the study, with FIGARCH($1,d,1$) showing the best results. It was used to estimate the parameters of each series of reconstructed returns, in a total of eight series for each action. For this purpose, the MFE\textsuperscript{11} library was used to estimate these parameters.

With the parameters estimated by the FIGARCH($1,d,1$) process, the procedure structured by equation (24) was applied to generate predictions of future conditional volatilities for one day, for each series of rebuilt returns. As predic-

\textsuperscript{10} The other three were FIGARCH($1,d,0$), FIGARCH($0,d,1$), and FIGARCH($0,d,0$).

\textsuperscript{11} The library was made available by Kevin Sheppard for MATLAB.
tions were made, using the FIGARCH of ARCH(∞) structure, this required that white noise series were constructed for each estimate and used the routine of the MFE library, to obtain the expansion of the Lambda parameter series. Finally, it made it possible to establish predictions of conditional variance, according to formula (26), using the rolling window process, as emphasized in subsection 3.5. In the generation of white noise, we used a stochastic series, generated by a normal distribution, applying a function of MATLAB 2019a, and using the mean and the standard deviation of the noise of the FIGARCH simulation process.

Figure 4 shows the conditional variances estimated by the FIGARCH(1,d,1) process for the ECOR3 stock of BM&F BOVESPA and for the PG share of the NYSE for the reconstituted components $D_1$, $D_{1-4}$ and $D_{1-8}$ (from left to right), estimated by Sym2 and Db2 wavelets. The charts in the first and second lines in the top correspond to the PG share of the NYSE. The graphs in the third and fourth lines are the estimations for the ECOR3 stock of the BM&F BOVESPA. The figure illustrates two colored lines, the black line characterizes the estimated conditional volatilities, for the reconstructed return series $D_1$, $D_{1-4}$ and $D_{1-8}$, using data for 680 days, applied in the calibration process of the parameters by FIGARCH(1,d,1). The red line is the one-day forecast using a 210-day Rolling Window\footnote{The rolling window was made following the routine presented by Klein and Walther (2016).} for the same series. It is observed in Figure 4 that the predictability of the conditional variance is satisfactory for both actions, demonstrating an evolutionary process very similar to the conditional variances estimated for the base calibration period of the model, thus preserving the energy of oscillations of the database used in the calibration. This behavior maintains for all the share reconstructed series, which characterizes the robustness of the prediction process of the volatility conditions of the respective series.

Figure 5 shows the estimates of the conditional volatilities estimated (black line) and predicted (red line) by FIGARCH(1,d,1) for the reconstituted components $D_1$, $D_{1-4}$ and $D_{1-8}$ (from left to right) of the ESTC3 stock of the BM&F BOVESPA. We are using the wavelets Morlex, Morse, Db3, and Haar. The charts in the first line at the top are the Molex estimations, in the second line (from the upper to lower) are the Db3 estimations, in the third line are the Morse estimations; finally, the graphs in the fourth line (lower charts) are, respectively, the Haar estimations. These figures were constructed to comparatively emphasize the volatility behaviors estimated by different wavelets. We used two CWT wavelets (Morlex and Morse) and two orthogonal filters, Db3 and Haar. It is clear from the graphs for the ESTC3 stock of the BM&F BOVESPA shown
in Figure 5 that the estimates and predictions of conditional volatilities are entirely similar. However, we analyzed the results of the conditional volatility estimates and forecasts for all stock sets (of the BM&F BOVESPA and NYSE), calculated by the wavelet techniques specified above, which showed quite similar to each other, without any marked discrepancy among the individual results. Therefore, in a general way, we can affirm any wavelet technique tested in this study can lead to satisfactory results in the value-at-risk to be estimated.

From the predictions of the conditional variances obtained, it was possible to calculate the VaRs for each series of reconstructed returns by the Monte Carlo (MC) method, which was presented in subsection 3.1. The Rolling Window generated 210 predictions of conditional variance for each reconstructed series of returns, whose values were used to estimate the standard deviations applied in the VaR calculations, according to equation (3). We used these estimated VaRs to test their qualities in the forecast of losses and gains for an out-of-sample of returns, according to the procedure given by Relations from (4) to (5). Finally, it applied a set of back-testing to evaluate the performance of the VaRs estimated in this study, whose concepts and procedure of tests are in subsection 3.5.

Figure 4. Conditional volatilities estimated (black line) and predicted (red line) by FIGARCH(1,d,1) for the reconstituted component $D_1$, $D_{1-4}$ and $D_{1-8}$ (from left to right). The charts in the first and second lines in the top are, respectively, the estimations of Sym2 and Db2, for the PG share of the NYSE. The graphs in the third and fourth lines are, respectively, the estimations of Sym2 and Db2, for the ECOR3 stock of the BM&F BOVESPA.
5 Results and analysis

Following the methodological procedure emphasized before, we selected twelve shares from Ibovespa of BM&F BOVESPA’s and twelve shares from NYSE’s DJIA. For each wavelet technique applied in the study (Haar, Db2, Db3, Sym2, Sym3, Morlex, Morse, second-derivative of DOG, and fourth-derivative of DOG), we take the eight levels of each wavelets’ decompositions and grouping them in extra new series. Firstly, creating new series, considering the decompositions in level one ($D_1$). We agglutinated after the levels one and two in news series called $D_{1-2}$, and the levels one, two, and three in the series $D_{1-3}$; and thus, subsequently, generating in total for each share, for each wavelets’ decompositions, eight new reconstructed series. Finally, from these series of reconstructed returns, the conditional volatilities were estimated with a reliability level of 99% using the MC method.

For each wavelets’ decompositions, for each stock, we estimated the conditional variances obtained from calibrated models using a sample of 680 daily data, with a rolling window of 470 points and using a series of 210 daily returns.

Figure 5. Conditional volatilities estimated (black line) and predicted (red line) by FIGARCH(1,$d$,1) for the reconstituted components $D_1$, $D_{1-4}$ and $D_{1-8}$ (from left to right) of the ESTC3 stock of the BM&F-BOVESPA. The charts in the first line at the top are, respectively, the estimations of the Morlex wavelet. The graphs in the second line (from the upper to lower) are, respectively, the estimations of the Db3 wavelet. The charts in the third line (from the upper to lower) are, respectively, the estimations of the Morse wavelet. The graphs in the fourth line (lower charts) are, respectively, the estimations of the Haar wavelet.
of out-of-sample. With the estimated conditional variances series, we calculated the VaR MC series. With the standard deviation and mean of this VaR series, we generated thirty-two Monte Carlo stochastic realizations; then, they were mediated at each time point for each VaR estimation. This final VaR series was used in the VaR qualification tests, taking a series of 210 out-of-sample return data.

The predictive capacity for all VaRs was tested by estimating the failure percentages and comparing them with the established significance level for each rebuilt return component. Then, back-testing was performed, as presented in subsection 3.5, in order to verify the statistical significance and the characteristics of these probabilities of estimated failures. Therefore, this subsection will present the results of estimated failure percentages and back-testing\textsuperscript{13} corresponding to the estimations by the MC method, performed with 99% reliability for all stocks included in the study. The share results of Ibovespa da BM&F BOVESPA were presented and analyzed in subsection 5.1, and the results of the stocks of DJIA of NYSE were presented and analyzed in subsection 5.2.

5.1 Results related to Ibovespa shares of BM&F BOVESPA

Table 3 present the values, in percentages of failures, obtained in the tests applied for MC VaR, with 99% confidence level (from now, denominated as MC VaR-99), by MODWT with Haar, involving twelve stocks selected from Ibovespa BM&F BOVESPA. We also present on Tables A-1 to A-4, in Appendix, the values of percentage failures for this set of Ibovespa shares, corresponding to the tests for MC VaR-99, obtained by the wavelets Morlex, Morse, fourth-derivative of DOG, and by MODWT with Deb2 filter. In each column of the tables cited in paragraphs above, between the second and the next to last, is the percentage values of failures ($\hat{p}$) of the reconstructed series, denominated, in the sequence of $D_{1-8}, D_{1-7}, \ldots, D_{1-2}, D_1$. In the next three lines below, $\hat{p}$, for each share, are the results of the back-testing, respectively, for the traffic light, TL, independence of failures, $LR_{\text{IND}}$, and conditional coverage, $LR_{\text{CC}}$. As emphasized, the reconstruction $D_{1-8}$ corresponds to the estimates for the rebuilt proxy of the original series and, the other columns present the values of the VaR failure percentages, as the decompositions of higher levels are gradually discarded.

The expectation is that, since VaRs were estimated with 99% confidence, the failure percentages should ideally be close to 1%. The tests were carried out

\textsuperscript{13} It was presented the following Back-testings: Semaphore test, independence of failures test, $LR_{\text{IND}}$, and the conditional coverage test, $LR_{\text{CC}}$. 

182 Brazilian Review of Econometrics 40(1) June 2020
Table 3. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Ibovespa BM&F BOVESPA shares, obtained by Haar wavelet filter.

|        | $D_{1,8}$ | $D_{1,7}$ | $D_{1,6}$ | $D_{1,5}$ | $D_{1,4}$ | $D_{1,3}$ | $D_{1,2}$ | $D_1$ |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------|
| **BRPA** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 1.429    | 0.952    | 1.429    | 0.952    | 1.429    | 2.857    | 2.857    |       |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Yellow)  | (Yellow)  | (Yellow)  |       |
| LR_{ind} | 0.00 1.00 0.00 0.99 0.00 0.99 0.99 0.00 1.000 0.1000 0.00 1.0000 0.00 1.0000 |       |
| LR_{cc} | 0.34 0.84 0.34 0.84 0.34 0.84 0.34 0.84 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **BRMK5** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.476    | 0.476    | 0.476    | 0.476    | 0.952    | 1.429    | 1.429    | 1.905   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **CBRO3** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.952    | 0.952    | 0.952    | 0.952    | 0.952    | 1.429    | 1.429    | 1.905   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **CECOR3** |          |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.952    | 0.952    | 0.952    | 0.952    | 0.952    | 1.429    | 1.429    | 1.905   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **CSTC3** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.952    | 0.952    | 0.952    | 0.952    | 0.952    | 2.857    | 2.857    | 3.810   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **GOMA4** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.952    | 0.952    | 0.952    | 0.952    | 0.952    | 2.381    | 2.381    | 3.810   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **IRENA3** |          |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.952    | 1.429    | 1.429    | 1.429    | 0.952    | 1.429    | 1.429    | 3.333   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 1.00 0.00 1.00 0.00 1.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.00 0.99 0.34 0.84 0.34 0.84 0.34 0.84 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **RAD13** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 0.476    | 0.476    | 0.476    | 0.476    | 0.476    | 0.952    | 1.429    | 2.381   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 0.99 0.00 0.99 0.00 0.99 0.99 0.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.72 0.69 0.72 0.69 0.72 0.69 0.72 0.69 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |
| **TIMP3** |           |           |           |           |           |           |           |       |
| $\bar{p}$ | 1.429    | 1.429    | 1.429    | 1.429    | 1.429    | 1.429    | 1.429    | 1.429   |
| TL (Green) | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   | (Green)   |       |
| LR_{ind} | 0.00 1.00 0.00 1.00 0.00 1.000 0.00 1.00 0.99 0.00 1.000 0.1000 0.00 1.0000 |       |
| LR_{cc} | 0.34 0.84 0.34 0.84 0.34 0.84 0.34 0.84 0.99 0.00 0.99 0.00 0.99 0.00 0.99 |       |

Brazilian Review of Econometrics 40(1) June 2020 183
on real assets, with a short forecast period (210 days). Then, one can expect that the results cannot be perfect since, in shorter return series of out-of-sample in the forecasting process, the amounts of failures in the \( \hat{p} \) average estimates will be smaller. With an estimated VaR with a 1% level, it is expected that, on the average, 21 failures occur in a sample of 210 days. However, for an out-of-sample of 1000, for VaR at 1%, it is expected ten failures. Therefore, if the number of out-of-sample samples is more extensive, in the calibration (and forecast) process, the estimates for \( \hat{p} \) will be better. More extensive out-of-sample series of returns allow diluting the effects of independence between failures and the failure concentration effects. These effects occur mainly as a result of atypical innovations, possibly stemming from economic and political crises that affect the financial markets, domestically and/or internationally, or stochastic innovations related to the fundamentals of stocks. One of the goals of the study is to verify the efficiency of the VaR model constructed in the forecasting process, relative to small series, using series of return data that approximate the size of those recommended by the Basle Committee on Banking Supervision.

Table 3 show the VaR-99 inferences for the estimations of MODWT-Haar for BM&F BOVESPA shares. Tables A-1 to A-4, in Appendix, show the VaR-99 inferences for the wavelets CWT-Morlex, CWT-Morse, MODWT-Db2, and CWT-fourth-average DOG, also for BM&F BOVESPA shares. These tables present the failure probabilities, and the respective backtesting’s estimated. From them, it can be stressed that there is a behavior pattern in the estimates of the probabilities of failure \( \hat{p} \) since if the higher levels (lower frequencies) of decompositions are removed from the rebuilt returns (with few exceptions, depending the wavelet technique applied), the number of failures increases. In a general way, this trend of growth is more gradual (or with few changes), between the components \( D_{1-8} \) and \( D_{1-3} \), and more accentuated between the rebuilt

---

14 The concentrations have a significant influence on the estimation of the probability of failure because if the daily variations of the financial assets are independent, the failures should distribute similarly over the period used in the forecasts and back-testings and converge to the expected probability. However, in practice, failures are often concentrated, and the risk of loss can no longer be independent of time (Godlewski & Merli, 2012).

15 The Basel Committee on Banking Supervision recommends the use of a database of the last twelve months, approximately 250 observations, to analyze the effects of back-testing.

16 Note that among the wavelet techniques analyzed in this study, we do not present the result tables for the estimates of the Db3, Sym3, Sym2, and Second-derivative of the DOG. Because of the estimates obtained with the Db3 and Sym3 did not meet (with few exceptions) the expected range of failures (0.476% \( \leq \hat{p} \leq 1.429\% \)), the estimates obtained with the Sym2 were very similar to those of the Db2. Second-Derivative of DOG also showed similar results to the Fourth-Derivative of DOG.

17 The highest decomposition level (level eight) is that one the lowest frequency.
components $D_{1-3}$ and $D_1$. This effect shows that the interactions between medium-term oscillations, with the short-term fluctuations, allow estimations more accurate for VaR for Ibovespa stocks. That pattern can be observed throughout all the tables presented for the Ibovespa shares, however, with a few differences that depend on the wavelet technique applied in the estimations. Understand these features has importance since the tendency of an increase in the estimated failures' number does not occur linearly. If the medium-term frequency decompositions are removed, the failures' number begins to grow significantly. For example, by removing the octave and seventh decompositions from the reconstructed series, the number of the failure remains practically stable concerning the original return ($D_{1-8}$). In fact, for the majority of shares (with few exceptions), the number of failures only begins to grow significantly when the decompositions of lower levels, with higher frequencies, are removed. The intermediate decompositions between levels four and six also have a minor effect in the quality of the VaRs with rare exceptions, because when these decompositions are removed from the series of reconstructed returns, the VaR failure numbers remain very little affected, not differing much from those obtained by the original returns.

The effects highlighted in the paragraph above can best be seen from the green traffic light specifications of the reconstructed series, asymptomatic evidence of the satisfactory accuracy failure numbers. We used these indicative of the green traffic light to build the information is presented in the last column in Table 3, for the estimations of MODWT-Haar filter; and in tables A-1 to A-4, for the estimations of the CWT-Morlex, CWT-Morse, and fourth-derivative of DOG, and for MODWT with Deb2 filter. The selected series, as showed in last column of these tables, emphasize the reconstructed series ranges that caused the smallest deviations in the failure estimates, included in the intervals specified ($1 \leq \text{failure number} \leq 3$, which produces for VaR-99 a confidence range of $98.57\% \leq IC \leq 99.52\%$ (or interval of significance level of $0.476\% \leq IC \leq 1.429\%$). $t$ is visible from this information that most of the stable reconstructed-series (with few exceptions) are included the first, second, and third decomposition (i.e., the components of higher frequencies), as highlighted in the previous paragraph.

We summarize in Table 4 a counting number that represents the distribution effects of the decompositions on the accuracy of the VaR-99, for the shares of Ibovespa. These numbers were estimated from the information in the last column in the tables 3, A-1 to A-4, and in a similar table mounted for the
Second derivative of DOG wavelet, which was not presented in this article.

The results presented in Table 4 emphasize that the number of estimated failures become stable (higher levels of independence and lower levels of concentration), for most shares, only when the first, second and/or third decompositions are included in the estimations. This behavior happens, primarily, due to the presence of the high-frequency components, whose oscillations approach cyclic behavior (short and medium-term time scales), and/or stochastic noise, due to the stochastic innovations exogenous to the financial system. This behavior is explicit for the wavelet estimations of the MODWT with Haar filter, MODWT with Db2 filter, CWT-Morlex, and CWT-Morse; this behavior is less significative for the estimations with the fourth-derivative of the DOG-CWT wavelet and, practically, does not prevail for the estimations with the second-derivative of the DOG-CWT wavelet.

The count number in Table 4 identifies the added rebuilt components that fit within the range of $0.476\% \leq \hat{p} \leq 1.429\%$. Table 4 shows some rebuilt series involving frequency components of medium and long-term (cyclical long-wavelength characteristics) and trends, features of the rebuilt series included in the ranges of $D_{1-4}$, $D_{1-5}$, $D_{1-6}$, $D_{1-7}$, and $D_{1-8}$ (among twelve shares, none is satisfied in Haar estimations, one in Db2, three in Morlex, two in Morse, five in fourth-derivative DOG, and nine in second-derivative DOG). The behavior of the BM&FBOVESPA shares included in these ranges specified is understandable since it is a financial market of a developing country. This kind of market is more susceptible to economic effects and political crises, of course, having more challenging to recover from these effects. Therefore, it is conjectured that shares that are more influenced by medium and long-term and trends exhibit less dynamic (efficient) behavior in the sense of Malkiel and Fama (1970) under the weak form of the efficient market hypothesis.\(^{18}\) Table 4 also indicates many shares analyzed from BM&FBOVESPA, involving higher frequency components in their rebuilt series, medium and short-term frequencies, $D_{1-2}$, and $D_{1-3}$. Among twelve shares, nine is satisfied in Haar estimations, none in Db2, six in Morlex, seven in Morse, six in fourth-derivative DOG, and three in second-derivative DOG. The rebuilt series in these ranges include weekly cycles and Monday effects. Finally, Table 4 shows a shares’ set of

\(^{18}\) The weak form of the efficient market hypothesis claims that prices reflect Implicit information in the past price sequence. The semi-strong hypothesis states that prices reflect all the relevant information that is publicly available, while the strong form of market efficiency states that the information known to any participant is reflected in the market prices of Malkiel and Fama (1970).
Table 4. Effects of decomposition components on the shares of Ibovespa, for VaR-99.

| Levels' rebuilt | Kind of wavelets | MODWT with Haar filter | MODWT with Db2 filter | CWT-Morlex |
|-----------------|------------------|------------------------|-----------------------|------------|
|                 | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers |
| $D_1 \rightarrow D_{1-8}$ | Stochastic noise (2-4 days) | Four shares | Short-term (4-10 days) | Eleven shares | Stochastic noise (2.1-4.2 days) | Three shares |
| $D_{1-2} \rightarrow D_{1-8}$ | Short-term (4-8 days) | Two shares | Medium-Term (10-22 days) | - | Short-term (4.2-8.7 days) | Five shares |
| $D_{1-3} \rightarrow D_{1-8}$ | Medium-Term (8-16 days) | Six shares | Medium-Term (22-46 days) | - | Medium-Term (8.7-18.0 days) | One share |
| $D_{1-4} \rightarrow D_{1-8}$ | Medium-Term (16-32 days) | - | Medium-Term (46-94 days) | One share | Medium-Term (18.0-37.4 days) | Two shares |
| $D_{1-5} \rightarrow D_{1-8}$ | Medium-Term (32-64 days) | - | Long-term (94-190 days) | - | Medium-Term (37.4-77.7 days) | One share |
| $D_{1-6} \rightarrow D_{1-8}$ | Long-term (64-128 days) | - | Long-term (190-382 days) | - | Long-term (77.7-161.4 days) | - |
| $D_{1-7} \rightarrow D_{1-8}$ | Long-term (128-256 days) | - | Trend (382-766 days) | - | Long-term (161.4-336.1 days) | - |
| $D_1 \rightarrow D_{1-8}$ | Trend (256-512 days) | - | Trend (time scale > 766 days) | - | Trend (336.1-698.7 days) | - |

| Levels' rebuilt | Kind of wavelets | CWT-Morse | CWT-fourth derivative | CWT-second derivative |
|-----------------|------------------|-----------|-----------------------|-----------------------|
|                 | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers |
| $D_1 \rightarrow D_{1-8}$ | Short-term (7.7-13.7 days) | Three shares | Short-term (5.9-10.5 days) | One share | Short-term (7.9-13.5 days) | - |
| $D_{1-2} \rightarrow D_{1-8}$ | Medium-Term (13.7-24.7 days) | Six shares | Medium-Term (10.5-19.3 days) | One share | Medium-Term (13.5-23.8 days) | One share |
| $D_{1-3} \rightarrow D_{1-8}$ | Medium-Term (24.7-44.7 days) | One share | Medium-Term (19.3-35.6 days) | Five shares | Medium-Term (23.8-41.9 days) | Two shares |
| $D_{1-4} \rightarrow D_{1-8}$ | Medium-Term (44.7-80.8 days) | One share | Medium-Term (35.6-65.4 days) | Four shares | Medium-Term (41.9-73.7 days) | Three shares |
| $D_{1-5} \rightarrow D_{1-8}$ | Long-term (80.8-146.1 days) | One share | Long-term (65.4-120.4 days) | One share | Long-term (73.7-129.7 days) | Two shares |
| $D_{1-6} \rightarrow D_{1-8}$ | Long-term (146.1-264.1 days) | - | Long-term (120.4-221.6 days) | - | Long-term (129.7-228.4 days) | One share |
| $D_{1-7} \rightarrow D_{1-8}$ | Trend (264.1-477.4 days) | - | Trend (221.6-407.4 days) | - | Trend (228.4-402.1 days) | One share |
| $D_1 \rightarrow D_{1-8}$ | Trend (477.4-780.0 days) | - | Trend (407.4-750.4 days) | - | Trend (402.1-708.0 days) | Two shares |
the BM&F BOVESPA, involving very short-term frequency (or stochastic noise) components in their rebuilt series, $D_1$. Among twelve shares, four is satisfied in Haar estimations, eleven in Db2, three in Morlex, three in Morse, one in fourth-derivative DOG, and none in second-derivative DOG. These shares are dominated by stochastic noise effects, following the “random path” hypothesis, according to Malkiel and Fama (1970), the returns data of these series tend to present a broader independence level and lower concentration indices, factors that determine the correlation effects between samples narrowly grouped.

Therefore, according to the added component distributions outlined in Table 4, it can be noted that wavelet techniques distribute their decomposition frequency scales differently, for each level. The time scale levels of the orthogonal filters are calculated, using the formula $L_j = (2^j - 1)(L - 1) + 1$, with $j = 1, 2, \ldots, 8$ (Table 1 give the corresponding lengths $L$ for each filter kind). Then the adjusting frequency for each level, for a specific kind of the orthogonal filter, changes in a way that we do not have control.

Nevertheless, we have the flexibility to adjust the windows of the CWT, fixing the spacing of the scales. Thus, we calibrated de wavelet windows to cover all the amplitude of the time support of the return series used in the study. However, different CWT wavelets adjust their scale distributions differently. Therefore, it is natural to have different evolutions of the time scale (or frequency range) on the time support, and different concentrations of the added rebuilt components satisfying the range $0.476\% \leq \hat{p} \leq 1.429\%$, as can see on Table 4.

The possibility of making a specific adjusting for the frequency windows for the CWT wavelets is an important characteristic. This property of the CWT makes it possible to adequate the frequency ranges of the wavelet windows to that one that makes up the specific characteristic of the return series. This flexibility makes it possible to find more appropriate frequency distribution to the VaR estimates, a fact not explored in this study.

Results of the backtests are presented in Table 3, and in tables A-1 and A-4. We observe in the results of these tables that, according to the backtests of Semaphore, the “green” sign is associated with the estimated VaRs models with few biases (with the failure probabilities into the interval $0.476\% \leq \hat{p} \leq 1.429\%$), for any wavelet technique used in the estimation; then, we consider these models as consistent with an exact VaR model. Therefore, we conjecture that, at least, the VaRs models applied to the reconstructed series, which the failure numbers estimated are included in the interval $0.476\% \leq \hat{p} \leq 1.429\%$, must be accepted as correct.
In Table 3, and in tables A-1 and A-4, in the lines specified by \( LR_{\text{ind}} \) (independence test) and \( LR_{\text{cc}} \) (conditional coverage test), in the first column under the indication of the reconstructed series (for example, \( D_{1-8} \)), they present the values of the statistics \( LR_{\text{ind}} \) and \( LR_{\text{cc}} \), respectively and, in the second column, the corresponding probability levels \( (1 - \hat{p}) \). Therefore, the results of those tables are that ones, whose the failure probabilities included in the interval \( 0.476\% \leq \hat{p} \leq 1.429\% \) present the p-values, corresponding to \( LR_{\text{ind}} \) and \( LR_{\text{cc}} \), more closely to 0.99, the VaR confidence interval. We can observe these situations in Table 3, and in tables A-1 and A-4, in which the p-values of independence are included in the range \([0.96, 0.99]\), and the p-values of concentration are included in the interval \([0.69\%, 0.99\%]\). These intervals prevail for any results of the wavelet techniques used in the estimations.

Therefore, the conclusions drawn from the results of these statistics of independence and conditional coverage tests are that when the failure probability approach of the exact significance level of the VaR-99, one percent, the failures show to be homogeneous (independent). In the same way, when the conditional coverage test approximates the confidence level applied in the VaR estimation, the failures probabilities can be considered to have no effects of concentration and being independent. The interval of the p-values of concentration are more spread for the “green” failures’ probabilities, included in range \([0.69\%, 0.99\%]\), than the p-values’ interval for the independence test, \([0.96, 0.99]\). Thus, for the BM&F BOVESPA shares, all “green” failure probabilities estimated are independent; however, only for that with \( \hat{p} = 0.952\% \), we can affirm that the conditional coverage test demonstrates no effects of concentration and independence.

### 5.2 Results related to the shares of DJIA index

Table 5, present the values, in percentages of failures, obtained in the tests applied for MC VaR-99, by MODWT with Haar, involving twelve shares selected from DJIA Index. We also present on tables A-5 to A-8, in Appendix, the values of percentage failures for this set of DJIA shares, corresponding to the tests for MC VaR-99, obtained by the wavelets Morlex, Morse, fourth-derivative of DOG, and by MODWT with Deb2 filter. These tables show the same presentation scheme for the tables shown earlier, referring to the Ibovespa shares of BM&F BOVESPA.

The estimates of failure probabilities for the DJIA stocks, \( \hat{p} \), follow with few differences, behavior patterns of the Ibovespa shares. In this case, the failures’
Table 5. VaR failure test results, with 99% confidence, for the return of each rebuilt series of DJIA shares, estimated by Haar filter wavelet.

|        | D_{1-8} | D_{1-7} | D_{1-6} | D_{1-5} | D_{1-4} | D_{1-3} | D_{1-2} | D_{1} |
|--------|---------|---------|---------|---------|---------|---------|---------|------|
| **MMM** |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.96    | 0.96    | 0.96    | 0.96    | 0.96    | 0.96    | 0.96 |
| L_{R_{wc}}  | 0.7     | 0.69    | 0.69    | 0.69    | 0.69    | 0.69    | 0.69    | 0.69 |
| **CAT**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **KO**    |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **XOM**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **HD**    |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **MCD**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **MRK**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **NKE**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **PE**    |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **PG**    |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **VZ**    |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| **WMT**   |         |         |         |         |         |         |         |      |
| L_{R_{ind}} | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
| L_{R_{wc}}  | 0.0     | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00 |
number does not depend on the medium and lower frequencies, with exceptions barely the results obtained by CWT-second-average DOG and CWT-fourth-average DOG. The inferences with the MODWT-Haar, CWT-Morlex, CWT-Morse, and MODWT-Db2 demonstrated inevitably that the failure probabilities, $\hat{p}$, are estimated by the rebuilt components $D_1$ and $D_1-2$, as we will observe latter.

We used the indicatives of the green traffic light to build the information are presented in the last column in Table 5, for the estimations of MODWT-Haar filter, and in tables A-5 to A-8, for the estimations of the CWT wavelets Morlex, CWT-Morse, and fourth-derivative of DOG, and for MODWT with Deb2 filter. In similar procedure applied for the analyses of the results for Ibovespa shares, the selected series for DJIA shares, as showed in last column of the tables 5, and A-5 to A-8, emphasize the reconstructed series ranges that caused the smallest deviations in the failure estimates, for VaR-99, included in the interval of significance level of $0.476\% \leq \hat{p} \leq 1.429\%$. For the estimations of MODWT-Haar filter, CWT-Morlex, CWT-Morse, and Deb2 filter, it is visible from this information that most of the stable reconstructed-series are included the first and second decomposition (i.e., the components of higher frequencies). In a counter way, from the estimations of second-derivative of DOG and fourth-derivative of DOG, it is visible that most of the stable reconstructed-series are included, in a scattered manner, in almost all levels of rebuilt added series. (i.e., involving the components of lower, medium and/or short-term frequencies).

We summarize in Table 6 a counting number that represents the distribution effects of the decompositions on the accuracy of the VaR-99, for the DJIA shares. These numbers were estimated from the information in the last column in tables 5 and A-5 to A-8, and in a similar table mounted for the Second derivative of DOG wavelet, which was not presented in this article. The results presented in Table 6 emphasize more clearly the conclusions presented in the paragraph above.

Results of the backtests are presented in Table 5, and in tables A-5 and A-7, for DJIA shares. We observe in these tables, according to the backtests of Semaphore, that the “green” sign is associated with the estimated VaRs models with few biases (with the failure probabilities into the interval $0.476\% \leq \hat{p} \leq 1.429\%$), for any wavelet technique used in the estimation. Then, we consider these models as consistent with an exact VaR model.

In Table 5 and in tables A-5 to A-8, in the lines specified by $LR_{ind}$ (independence test) and $LR_{ccc}$ (conditional coverage test), in the first column
Table 6: Effects of decomposition components on the shares of DJIA, for VaR-99.

| Levels’ rebuilt | Kind of wavelets | MODWT with Haar filter | MODWT with Db2 filter | CWT-Morlex |
|-----------------|------------------|------------------------|-----------------------|------------|
|                 | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers |
| $D_1$ to $D_{1-8}$ | Stochastic noise (2-4 days) | Nine shares | Short-term (4-10 days) | Eight shares | Stochastic noise (2.1-4.2 days) | Four shares |
| $D_{1-2}$ to $D_{1-8}$ | Short-term (4-8 days) | Three shares | Medium-Term (10-22 days) | Four shares | Short-term (4.2-8.7 days) | Eight shares |
| $D_{1-3}$ to $D_{1-8}$ | Medium-Term (8-16 days) | - | Medium-Term (22-46 days) | - | Medium-Term (8.7-18.0 days) | - |
| $D_{1-4}$ to $D_{1-8}$ | Medium-Term (16-32 days) | - | Medium-Term (46-94 days) | - | Medium-Term (18.0-37.4 days) | - |
| $D_{1-5}$ to $D_{1-8}$ | Medium-Term (32-64 days) | - | Long-term (94-190 days) | - | Medium-Term (37.4-77.7 days) | - |
| $D_{1-6}$ to $D_{1-8}$ | Long-term (64-128 days) | - | Long-term (190-382 days) | - | Long-term (77.7-161.4 days) | - |
| $D_{1-7}$ to $D_{1-8}$ | Long term (128-256 days) | - | Trend (382-766 days) | - | Long-term (161.4-336.1 days) | - |
| $D_{1-8}$ | Trend (256-512 days) | - | Trend (time scale>766 days) | - | Trend (336.1-698.7 days) | - |

| Levels’ rebuilt | Kind of wavelets | CWT-Morse | CWT-fourth derivativeDOG | CWT-second derivativeDOG |
|-----------------|------------------|-----------|--------------------------|--------------------------|
|                 | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers | Memory (Time scale) | Shares numbers |
| $D_1$ to $D_{1-8}$ | Short-term (7.7-13.7 days) | Nine shares | Short-term (7.9-13.5 days) | - | Short-term (7.9-13.5 days) | - |
| $D_{1-2}$ to $D_{1-8}$ | Medium-Term (13.7-24.7 days) | Three shares | Medium-Term (13.5-23.8 days) | Four shares | Medium-Term (13.5-23.8 days) | - |
| $D_{1-3}$ to $D_{1-8}$ | Medium-Term (24.7-44.7 days) | - | Medium-Term (23.8-41.9 days) | Three shares | Medium-Term (23.8-41.9 days) | Two shares |
| $D_{1-4}$ to $D_{1-8}$ | Medium-Term (44.7-80.8 days) | - | Medium-Term (41.9-73.7 days) | Three shares | Medium-Term (41.9-73.7 days) | One shares |
| $D_{1-5}$ to $D_{1-8}$ | Long-term (80.8-146.1 days) | - | Long-term (73.7-129.7 days) | One shares | Long-term (73.7-129.7 days) | Two shares |
| $D_{1-6}$ to $D_{1-8}$ | Long-term (146.1-264.1 days) | - | Long-term (129.7-228.4 days) | - | Long-term (129.7-228.4 days) | Four shares |
| $D_{1-7}$ to $D_{1-8}$ | Trend (264.1-477.4 days) | - | Trend (228.4-402.1 days) | - | Trend (228.4-402.1 days) | Two shares |
| $D_{1-8}$ | Trend (477.4-780.0 days) | - | Trend 402.1-708.0 days) | - | Trend 402.1-708.0 days) | One shares |
under the indication of the reconstructed series, they present the values of the statistics $LR_{\text{IND}}$ and $LR_{\text{CC}}$, respectively, and, in the second column, the corresponding probability levels $(1 - \hat{p})$. The failure probabilities included in the interval $0.476\% \leq \hat{p} \leq 1.429\%$ present the p-values, corresponding to $LR_{\text{IND}}$ and $LR_{\text{CC}}$, more closely to 0.99, the VaR confidence interval. We observe in Table 5 and tables A-5 to A-8 that the p-values of independence are included in the range $[0.96, 0.99]$, and the p-values of concentration are included in the interval $[0.69\%, 0.99\%]$. These intervals prevail for any results of the wavelet techniques used in the estimations.

The conclusions drawn from the tables regarding the VaR-99 estimates for the DJIA shares are the same as those previously described, regarding those presented for the VaR-99 estimates for the BM&FBOVESPA shares. The interval of the p-values of the conditional coverage tests is more spread for the “green” failures’ probabilities, included in the range $[0.69\%, 0.99\%]$, than the p-values’ interval for the independence test, $[0.96\%, 0.99\%]$. Thus, in the same way, as concluded for the BM&F BOVESPA shares, for the DJIA shares, all “green” failure probabilities estimated are also independent. However, only for that with $\hat{p} = 0.952\%$, we can affirm that the conditional coverage test demonstrates no effects of concentration and independence.

6 Conclusion

The main objective of this study was to measure the impacts of short, medium, and long-term cycles on daily value-at-risk estimates, and test the performance of some wavelets, as CWT-Morlex, CWT-Morse, CWT-second-derivative of DOG, CWT-fourth-derivative of DOG, MODWT-haar, MODWT-db2, MODWT-Sym2, MODWT-db3, and MODWT-Sym3. To measure the impacts of short, medium, and long-term cycles, we used the wavelets’ techniques to separate each frequency component, intrinsically, inserted in the series of shares returns analyzed in the study.

As one of the objectives of the study was to verify the efficiency of the wavelet techniques applied in the study, we analyzed the decomposition dynamics of the cyclic components. We check that wavelet techniques distribute their decomposition frequency scales differently, for each level. The time scale levels of the orthogonal filters adjust frequency for each level, using the corresponding time lengths $L$, which is different for each filter kind, in a way that we do not have control of the dynamic of the frequency distribution for the respective levels; thus, each filter has different frequency range for the levels. Nevertheless, the
CWT wavelets have the flexibility to adjust the windows of the CWT. However, each kind of CWT wavelet adjusts its scale distribution differently, which allows different evolutions of the time scale (or frequency range) on the time support. The possibility of making a specific adjusting for the frequency windows for the CWT wavelets is an important characteristic. This property of the CWT makes it possible to adequately the frequency ranges of the wavelets’ windows to that one that makes up the specific characteristic of the return series. This flexibility makes it possible to find more appropriate frequency distribution to the VaR estimates, a fact not explored in this study.

However, while wavelets have different decomposition dynamics, the conclusions of the characteristics of the effects of frequency components on VaR remain similar for all wavelets, at a minimum when using Haar, Db2, CWT-Morse, CWT-Morlex wavelets. The other CWT wavelets analyzed in the study, despite presenting a different evolution process from those mentioned, they also allow establishing a consistent analysis for the VaR estimates.

We analyze the dynamics of the assets of Ibovespa and DJIA for each rebuilt series. The VaR model was structured from conditional volatilities forecasts obtained from calibrated parameters using a FIGARCH(1, d, 1). Then, we apply the rolling window technique, which, according to Zivot and Wang (2006), makes it possible to increase the precision in the forecast of conditional volatilities significantly, and consequently obtain series of longer conditional volatilities and in a significant way, increase the series of estimates of daily VaRs. The structured models made it possible to analyze the VaR model dynamics, from which it was possible to demonstrate that the dynamics of most shares traded on the BM&F BOVESPA and DJIA Index depends on the wavelet kind applied in the analyses. Nevertheless, the stock dynamics of the BM&F BOVESPA and DJIA no have significant differences in their dynamic evolution process, for any wavelets applied.

Finally, the study performed a set of back-testings, which allow qualifying the accuracy of the probabilities’ failures estimated. The results of the traffic light approach backtesting demonstrated that the estimated VaRs classified as “green” are models that estimate with low biases and are highly likely to be consistent with the exact VaRs model. The possibility of mistakenly accepting such models as accurate is low. The failure independence and conditional concentration backtests demonstrated that the failure probability of those series of reconstructed returns, included in the interval $0.476\% \leq \hat{p} \leq 1.429\%$, are VaRs models acceptable, with failure time independence and little concentration-
Effects of asset frequency components on value-at-risk in emerging and developed markets: Analyses with MODWT and CWT wavelets

Therefore, with the potential to be used for risk analysis in financial markets.

References

Addison, P. S. (2018). Introduction to redundancy rules: The continuous wavelet transform comes of age. *Philosophical Transactions A*, 376(20170258). [http://dx.doi.org/10.1098/rsta.2017.0258](http://dx.doi.org/10.1098/rsta.2017.0258)

Akansu, A. N., Serdijn, W. A., & Selesnick, I. W. (2010). Wavelet transforms in signal processing: A review of emerging applications. *Physical Communication*, 3(1), 1–18. [http://dx.doi.org/10.1016/j.phycom.2009.07.001](http://dx.doi.org/10.1016/j.phycom.2009.07.001)

Alt, R., Fortin, I., & Weinberger, S. (2011). The Monday effect revisited: An alternative testing approach. *Journal of Empirical Finance*, 18(3), 447–460. [http://dx.doi.org/10.1016/j.jempfin.2011.04.002](http://dx.doi.org/10.1016/j.jempfin.2011.04.002)

Andreis, A. M., Ihnatov, I., & Tiwari, A. K. (2014). Analyzing time frequency relationship between interest rate, stock price and exchange rate through continuous wavelet. *Economic Modelling*, 41, 227–238. [http://dx.doi.org/10.1016/j.econmod.2014.05.013](http://dx.doi.org/10.1016/j.econmod.2014.05.013)

Ashmead, J. (2010). Morlet wavelets in quantum mechanics. *Quanta*, 1(1), 58–70. [http://dx.doi.org/10.12743/quanta.v1i1.5](http://dx.doi.org/10.12743/quanta.v1i1.5)

Baggett, L. W., Larsen, N. S., Packer, J. A., Raeburn, I., & Ramsay, A. (2010). Direct limits, multiresolution analyses, and wavelets. *Journal of Functional Analysis*, 258(8), 2714–2738. [http://dx.doi.org/10.1016/j.jfa.2009.08.011](http://dx.doi.org/10.1016/j.jfa.2009.08.011)

Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 74(1), 3–30. [http://dx.doi.org/10.1016/S0304-4076(95)01749-6](http://dx.doi.org/10.1016/S0304-4076(95)01749-6)

Baillie, R. T., & Morana, C. (2009). Modeling long memory and structural breaks in conditional variances: An adaptive FIGARCH approach. *Journal of Economic Dynamics & Control*, 33(8), 1577–1592. [http://dx.doi.org/10.1016/j.jedc.2009.02.009](http://dx.doi.org/10.1016/j.jedc.2009.02.009)

Bandi, F. M., Perron, B., Tamoni, A., & Tebaldi, C. (2019). The scale of predictability. *Journal of Econometrics*, 208(1), 120–140. [http://dx.doi.org/10.1016/j.jeconom.2018.09.008](http://dx.doi.org/10.1016/j.jeconom.2018.09.008)

Belkhouja, M., & Boutahary, M. (2011). Modeling volatility with time-varying FIGARCH models. *Economic Modelling*, 28(3), 1106–1116. [http://dx.doi.org/10.1016/j.econmod.2010.11.017](http://dx.doi.org/10.1016/j.econmod.2010.11.017)
Beltratti, A., & Morana, C. (2006). Breaks and persistency: Macroeconomic causes of stock market volatility. *Journal of Econometrics, 131*(1-2), 151–171. [http://dx.doi.org/10.1016/j.jeconom.2005.01.007](http://dx.doi.org/10.1016/j.jeconom.2005.01.007)

Berger, T. (2016, 7 29–31). On the impact of long-run seasonalities on daily value-at-risk forecasts. In *Word finance conference*, New York, USA. [https://www.world-finance-conference.com/papers_wfc2/282.pdf](https://www.world-finance-conference.com/papers_wfc2/282.pdf)

Böes, D. C., & Salas-La Cruz, J. D. (1978). Non stationarity of the mean and the Hurst phenomenon. *Water Resources Research, 14*(1), 135–143. [http://dx.doi.org/10.1029/WR014i001p00135](http://dx.doi.org/10.1029/WR014i001p00135)

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics, 31*(3), 307–327. [http://dx.doi.org/10.1016/0304-4076(86)90063-1](http://dx.doi.org/10.1016/0304-4076(86)90063-1)

Bollerslev, T., & Mikkelsen, H. O. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics, 73*(1), 151–184. [http://dx.doi.org/10.1016/0304-4076(95)01736-4](http://dx.doi.org/10.1016/0304-4076(95)01736-4)

Chakrabarty, A., De, A., & Bandyopadhyay, G. (2015). A wavelet-based MRA-EDCC-GARCH methodology for the detection of news and volatility spillover across sectoral indices: Evidence from the Indian financial market. *Global Business Review, 16*(1), 35–49. [http://dx.doi.org/10.1177/0972150914553506](http://dx.doi.org/10.1177/0972150914553506)

Christoffersen, P. (1998). Evaluating interval forecasts. *International Economic Review, 39*(4), 841–862. [http://dx.doi.org/10.2307/2527341](http://dx.doi.org/10.2307/2527341)

Cohen, M. X. (2019). A better way to define and describe Morlet wavelets for time-frequency analysis. *NeuroImage, 199*, 81–86. [http://dx.doi.org/10.1016/j.neuroimage.2019.05.048](http://dx.doi.org/10.1016/j.neuroimage.2019.05.048)

Daubechies, I. (1992). *Ten lectures on wavelets*. Philadelphia, PA: SIAM.

Diebold, F. X., & Inoue, A. (2001). Long memory and regime switching. *Journal of Econometrics, 105*(1), 131–159. [http://dx.doi.org/10.1016/S0304-4076(01)00073-2](http://dx.doi.org/10.1016/S0304-4076(01)00073-2)

Domínguez-Jiménez, M. E., & Ferreira, P. J. S. G. (2011). Some extremal properties of Daubechies filters and other orthonormal filters. *Signal Processing, 91*(1), 85–89. [http://dx.doi.org/10.1016/j.sigpro.2010.06.011](http://dx.doi.org/10.1016/j.sigpro.2010.06.011)

Doyle, J. R., & Chen, C. H. (2009). The wandering weekday effect in major stock markets. *Journal of Banking and Finance, 33*(8), 1388–1399. [http://dx.doi.org/10.1016/j.jbankfin.2009.02.002](http://dx.doi.org/10.1016/j.jbankfin.2009.02.002)

Franke, J., Hardle, W. K., & Häfner, C. M. (2008). *Statistical of financial markets: An introduction* (2nd ed.). Berlin-Heidelberg: Springer-Verlag.

Frey, R., & Michaud, P. (1997). *The effect of GARCH-type volatilities on prices and payoff-distributions of derivative assets: A simulation study*. [http://statmath.wu.ac.at/~frey/publications/garch-sim.pdf](http://statmath.wu.ac.at/~frey/publications/garch-sim.pdf)
Gallegati, M. (2012). A wavelet-based approach to test for financial market contagion. *Computational Statistics & Data Analysis, 56*(11), 3491–3497. [http://dx.doi.org/10.1016/j.csda.2010.11.003](http://dx.doi.org/10.1016/j.csda.2010.11.003)

Gallegati, M., & Semmler, W. (Eds.). (2014). *Wavelet applications in economics and finance*. Springer.

Garbade, K. (1986). *Assessing risk and capital adequacy for Treasury securities* [Topics in Money and Securities Markets, Vol. 22]. New York: Bankers Trust.

Gençay, R., & Selçuk, F. (2004). Extreme value theory and value-at-risk: Relative performance in emerging markets. *International Journal of Forecasting, 20*(2), 287–303. [http://dx.doi.org/10.1016/j.ijforecast.2003.09.005](http://dx.doi.org/10.1016/j.ijforecast.2003.09.005)

Gençay, R., Selçuk, F., & Whitcher, B. (2001). Differentiating intraday seasonalties through wavelet multi-scaling. *Physica A. Statistical Mechanics and its Applications, 289*(3-4), 543–556. [http://dx.doi.org/10.1016/S0378-4371(00)00463-5](http://dx.doi.org/10.1016/S0378-4371(00)00463-5)

Gençay, R., Selçuk, F., & Whitcher, B. (2002). *An introduction to wavelets and other filtering methods in finance and economics*. San Diego, CA: Academic Press.

Giot, P., & Laurent, S. (2003). Value-at-risk for long and short trading positions. *Journal of Applied Econometrics, 18*(6), 641–664. [http://dx.doi.org/10.1002/jae.710](http://dx.doi.org/10.1002/jae.710)

Godlewski, C., & Merli, M. (2012). *Gestion des risques et institutions financières* (3rd ed.). Pearson.

Gourieroux, C., & Jasiak, J. (2001). Memory and infrequent breaks. *Economics Letters, 70*(1), 29–41. [http://dx.doi.org/10.1016/S0165-1765(00)00346-3](http://dx.doi.org/10.1016/S0165-1765(00)00346-3)

Granger, C. W. J., & Hyung, N. (2004). Occasional structural breaks and long memory with an application to the S&P 500 absolute returns. *Journal of Empirical Finance, 11*(3), 399–421. [http://dx.doi.org/10.1016/j.jempfin.2003.03.001](http://dx.doi.org/10.1016/j.jempfin.2003.03.001)

Hamilton, J., & Susmel, R. (1994). Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics, 64*(1-2), 307–333. [http://dx.doi.org/10.1016/0304-4076(94)90067-1](http://dx.doi.org/10.1016/0304-4076(94)90067-1)

Holschneider, M. (1995). *Wavelets: An analysis tool*. Oxford, UK: Oxford University Press.

Hoppe, R. (1998). VAR and the unreal world. *Risk, 11*, 45–50.

Huang, S.-C. (2011). Wavelet-based multi-resolution GARCH model for financial spillover effects. *Mathematics and Computers in Simulation, 81*(11), 2529–2539. [http://dx.doi.org/10.1016/j.matcom.2011.04.003](http://dx.doi.org/10.1016/j.matcom.2011.04.003)
Biage and Neldicé

Jin, H. J., & Frechette, D. L. (2004). Fractional integration in agricultural futures price volatilities. *American Journal of Agriculture Economics, 86*(2), 432–443. http://dx.doi.org/10.1111/j.0002-5853.2004.00589.x

Khaled, M. S., & Keef, S. P. (2012). A note on the turn of the month and year effects in international stock returns. *The European Journal of Finance, 18*(6), 597–602. http://dx.doi.org/10.1080/1351847X.2011.617379

Klein, T., & Walther, T. (2016). On the application of fast fractional differencing in modeling long memory of conditional variance simulation study and rolling window estimations of crude oil. SSRN. http://dx.doi.org/10.2139/ssrn.2754102

Klein, T., & Walther, T. (2017). Fast fractional differencing in modeling long memory of conditional variance for high-frequency data. *Finance Research Letters, 22*, 274–279. http://dx.doi.org/10.1016/j.frl.2016.12.020

Kupiec, P. (1995). Techniques for verifying the accuracy of risk management models. *Journal of Derivatives, 3*(2), 73–84. http://dx.doi.org/10.3905/jod.1995.407942

Lilly, J. M. (2017). Element analysis: A wavelet-based method for analyzing time-localized events in noisy time series. *Proc. R. Soc. A, 473*. http://dx.doi.org/10.1098/rspa.2016.0776

Lilly, J. M., & Olhede, S. C. (2009). Higher-order properties of analytic wavelets. *IEEE Transactions on Signal Processing, 57*(1), 146–160. http://dx.doi.org/10.1109/TSP.2008.2007607

Lilly, J. M., & Olhede, S. C. (2010). On the analytic wavelet transform. *IEEE Transactions on Information Theory, 56*(8), 4135–4156. http://dx.doi.org/10.1109/TIT.2010.2050935

Lilly, J. M., & Olhede, S. C. (2012). Generalized Morse wavelets as a superfamily of analytic wavelets. *IEEE Transactions on Signal Processing, 60*(11), 6036–6041. http://dx.doi.org/10.1109/TSP.2012.2210890

Malkiel, B. G., & Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance, 25*(2), 383–417. http://dx.doi.org/10.1111/j.1540-6261.1970.tb00518.x

Mallat, S. (2008). *A wavelet tour of signal processing: The sparse way* (3rd ed.). Burlington, MA: Academic Press.

Martín-Barragán, B., B.Ramos, S., & Veiga, H. (2015). Correlations between oil and stock markets: A wavelet-based approach. *Economic Modelling, 50*, 212–227. http://dx.doi.org/10.1016/j.econmod.2015.06.010

Morana, C., & Beltratti, A. (2004). Structural change and long-range dependence involatility of exchange rates: Either, neither or both? *Journal of Empirical Finance, 11*(5), 629–658. http://dx.doi.org/10.1016/j.jempfin.2003.03.002
Effects of asset frequency components on value-at-risk in emerging and developed markets: Analyses with MODWT and CWT wavelets

Olhede, S. C., & Walden, A. T. (2002). Generalized Morse wavelets. *IEEE Transactions on Signal Processing, 50*(11), 2661–2670. [http://dx.doi.org/10.1109/TSP.2002.804066](http://dx.doi.org/10.1109/TSP.2002.804066)

Ortu, F., Severino, F., Tamoni, A., & Tebaldi, C. (2019). A persistence-based Wold-type decomposition for stationary time series. *Quantitative Economics, 11*(1), 203–230. [http://dx.doi.org/https://doi.org/10.3982/QE994](http://dx.doi.org/https://doi.org/10.3982/QE994)

Percival, D. B., & Walden, A. (2000). *Wavelet methods for time series analysis*. Cambridge: Cambridge University Press.

Powell, J. G., Shi, J., Smith, T., & Whaley, R. E. (2009). Political regimes, business cycles, seasonalities, and returns. *Journal of Banking & Finance, 33*(6), 1112–1128. [http://dx.doi.org/10.1016/j.jbankfin.2008.12.009](http://dx.doi.org/10.1016/j.jbankfin.2008.12.009)

Reboredo, J. C., & Riveira-Castro, M. A. (2014). Wavelet-based evidence of the impact of oil prices on stock returns. *International Review of Economics & Finance, 29*, 145–176. [http://dx.doi.org/10.1016/j.iref.2013.05.014](http://dx.doi.org/10.1016/j.iref.2013.05.014)

Tamoni, A. (2011). *The multi-horizon dynamics of risk and returns*. (Second version published in November 24, 2011) [http://dx.doi.org/10.2139/ssrn.2948595](http://dx.doi.org/10.2139/ssrn.2948595)

Tan, P. P., Chin, C. W., & Galagedera, D. U. A. (2014). A wavelet-based evaluation of time-varying long memory of equity markets: A paradigm in crisis. *Physica A: Statistical Mechanics and its Applications, 410*, 345–358. [http://dx.doi.org/10.1016/j.physa.2014.05.044](http://dx.doi.org/10.1016/j.physa.2014.05.044)

Tayefi, M., & Ramanathan, T. V. (2012). An overview of FIGARCH and related time series models. *Austrian Journal of Statistics, 41*(3), 175–196. [http://dx.doi.org/10.17713/ajs.v41i3.172](http://dx.doi.org/10.17713/ajs.v41i3.172)

Torrence, C., & Compo, G. P. (1998). A practical guide to wavelet analysis. *Bulletin of the American Meteorological Society (BAMS), 79*(1), 61–78. [http://dx.doi.org/10.1175/1520-0477(1998)079<0061:APGTWA>2.0.CO;2](http://dx.doi.org/10.1175/1520-0477(1998)079<0061:APGTWA>2.0.CO;2)

Zhang, Z. P., Ren, Z., & Huang, W. Y. (2003). A novel detection method of motor broken rotor bars based on wavelet ridge. *IEEE Transactions on Energy Conversion, 18*(3), 417–423. [http://dx.doi.org/10.1109/TEC.2003.815851](http://dx.doi.org/10.1109/TEC.2003.815851)

Zivot, E., & Wang, G. J. (2006). *Modeling financial time series with S_PLUS®* (2nd ed.). New York: Springer Science-Business Media.
## Appendix: Tables for VaR at 95% confidence level and complementary tables

**Table A.1**: MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Bovespa BMF BOVESPA shares, estimated by Morlex wavelet.

|        | D_{1,8} | D_{1,7} | D_{1,6} | D_{1,5} | D_{1,4} | D_{1,3} | D_{1,2} | D_{1} |
|--------|---------|---------|---------|---------|---------|---------|---------|------|
| **Rep.** |          |         |         |         |         |         |         |      |
| **BM1** | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM2** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM3** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM4** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM5** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM6** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM7** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |
| **BM8** | 0.476   | 0.476   | 0.476   | 0.476   | 1.429   | 1.429   | 1.429   | 1.905 |

- **LR**: Likelihood Ratio
- **p**: p-value

**Notes**: The results in parentheses are **Green** and in **Yellow** indicate the significance level of the test.
Table A.2. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Ibovespa BM&F BOVESPA shares, estimated with Morse wavelet.

| D_t^-# | D_t^-7 | D_t^-6 | D_t^-5 | D_t^-4 | D_t^-3 | D_t^-2 | D_t^-1 | D_t^-0 |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| BRAPA  | 0.476  | 0.476  | 0.476  | 1.429  | 1.429  | 1.429  | 1.905  |        |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.96   | 0.96   | 0.96   | 0.96   | 1.000  | 1.0    | 0.00   |
| LR_cc  | 0.7    | 0.69   | 0.7    | 0.69   | 0.69   | 0.3    | 0.642  | 0.3    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| BRES5  | 0.476  | 0.476  | 0.476  | 0.476  | 1.429  | 1.429  | 1.905  | 2.381  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.96   | 0.96   | 0.96   | 0.96   | 0.966  | 0.9    | 0.00   |
| LR_cc  | 0.7    | 0.69   | 0.7    | 0.69   | 0.69   | 0.7    | 0.696  | 0.7    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| BMSPS  | 0.952  | 0.952  | 0.952  | 1.429  | 1.429  | 0.952  | 1.905  |        |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 1.00   | 0.99   | 0.99   | 0.99   | 1.000  | 1.0    | 0.00   |
| LR_cc  | 0.3    | 0.94   | 0.94   | 0.94   | 0.94   | 1.57   | 0.504  | 0.3    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| CCPR3  | 0.476  | 0.476  | 0.476  | 0.476  | 0.952  | 1.429  | 1.429  | 1.905  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.96   | 0.96   | 0.96   | 0.96   | 0.966  | 0.9    | 0.00   |
| LR_cc  | 0.7    | 0.69   | 0.7    | 0.69   | 0.69   | 0.3    | 0.642  | 0.3    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| CDPR3  | 0.952  | 0.952  | 0.952  | 1.429  | 1.429  | 0.952  | 1.905  | 2.381  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.95   | 0.95   | 0.95   | 0.95   | 0.98   | 0.9    | 0.00   |
| LR_cc  | 0.0    | 0.95   | 0.95   | 0.95   | 0.95   | 0.3    | 0.34   | 0.3    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| GOJPA  | 0.476  | 0.476  | 0.476  | 0.476  | 0.952  | 1.429  | 1.429  | 1.905  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.95   | 0.95   | 0.95   | 0.95   | 0.98   | 0.9    | 0.00   |
| LR_cc  | 0.0    | 0.95   | 0.95   | 0.95   | 0.95   | 0.3    | 0.34   | 0.3    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| IBEN3  | 0.952  | 0.952  | 0.952  | 0.952  | 0.952  | 0.952  | 3.810  |        |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.9    | 0.00   |
| LR_cc  | 0.0    | 0.99   | 0.99   | 0.99   | 0.99   | 0.97   | 0.9    | 0.00   |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Red)  |
| MULT3  | 0.476  | 0.476  | 0.476  | 0.476  | 0.476  | 0.476  | 0.476  | 0.952  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
| LR_ind | 0.0    | 0.96   | 0.96   | 0.96   | 0.96   | 0.966  | 0.9    | 0.00   |
| LR_cc  | 0.7    | 0.69   | 0.7    | 0.69   | 0.69   | 0.696  | 0.696  | 0.6    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
| NATU3  | 0.952  | 0.952  | 0.952  | 1.429  | 1.429  | 1.905  | 1.905  | 2.381  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| LR_ind | 0.0    | 0.99   | 0.99   | 0.99   | 0.99   | 1.000  | 1.0    | 0.00   |
| LR_cc  | 0.0    | 0.99   | 0.99   | 0.99   | 0.99   | 0.842  | 0.8    | 1.372  |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Yellow)|        |
| RADU3  | 0.476  | 0.476  | 0.476  | 0.476  | 0.476  | 0.476  | 1.429  | 3.333  |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
| LR_ind | 0.0    | 0.96   | 0.96   | 0.96   | 0.96   | 0.966  | 0.9    | 0.00   |
| LR_cc  | 0.7    | 0.69   | 0.7    | 0.69   | 0.69   | 0.696  | 0.696  | 0.6    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
| TIMP3  | 1.429  | 1.429  | 1.429  | 1.429  | 1.429  | 1.429  | 1.429  |        |
| TL     | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
| LR_ind | 0.0    | 1.00   | 1.00   | 1.00   | 1.00   | 1.000  | 1.0    | 1.00   |
| LR_cc  | 0.3    | 0.84   | 0.3    | 0.84   | 0.3    | 0.642  | 0.3    | 0.8    |
|        | (Green)| (Green)| (Green)| (Green)| (Green)| (Green)| (Green)|        |
Table A-3. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Ibovespa BM&F BOVESPA shares, estimated with db2 wavelet filter.

|     | $\bar{p}$ | $\bar{q}$ | $\bar{r}$ | $\bar{p}^{\prime}$ | $\bar{q}^{\prime}$ | $\bar{r}^{\prime}$ |
|-----|-----------|-----------|-----------|---------------------|---------------------|---------------------|
| BRAF4 | 0.952 | 0.952 | 0.952 | 0.952 | 1.905 | 1.905 | 1.905 | 1.905 |
|       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Yellow) | (Yellow) | (Green) |
|       | $LR_{10}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 |
|       | $LR_{20}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 1.00 | 0.13 | 0.50 | 0.50 | 1.37 |
| BF53 | 1.905 | 1.905 | 1.905 | 1.905 | 1.429 | 1.429 | 0.952 | 0.952 |
|       | (Yellow) | (Yellow) | (Yellow) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 |
|       | $LR_{20}$ | 1.37 | 0.50 | 1.37 | 0.50 | 1.37 | 0.50 | 0.84 | 0.84 | 0.84 | 0.84 | 0.84 | 0.84 |
| BRM95 | 0.952 | 0.952 | 0.952 | 0.952 | 1.429 | 1.429 | 1.429 | 1.429 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 |
|       | $LR_{20}$ | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 |
| CCRG3 | 1.429 | 1.429 | 1.429 | 1.429 | 0.476 | 0.476 | 0.476 | 0.476 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 |
|       | $LR_{20}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 |
| CMRE3 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 |
|       | $LR_{20}$ | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 | 0.34 | 0.94 |
| ESTC3 | 1.905 | 2.381 | 2.381 | 2.381 | 2.381 | 0.952 | 0.952 | 0.476 |
|       | (Yellow) | (Yellow) | (Yellow) | (Yellow) | (Yellow) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 3.77 | 0.05 | 2.82 | 0.09 | 2.82 | 0.08 | 2.91 | 0.09 | 7.14 | 0.00 | 7.1 | 0.00 | 0.00 | 0.96 |
|       | $LR_{20}$ | 5.14 | 0.07 | 5.74 | 0.05 | 5.74 | 0.05 | 5.74 | 0.05 | 7.14 | 0.02 | 7.1 | 0.02 | 0.72 | 0.69 |
| GOAL4 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 | 0.952 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 |
|       | $LR_{20}$ | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 | 0.00 | 0.99 |
| LUB13 | 0.476 | 0.476 | 0.476 | 0.476 | 0.476 | 0.476 | 0.476 | 0.476 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.72 | 0.69 | 0.72 | 0.69 | 0.72 | 0.69 | 0.72 | 0.69 | 0.72 | 0.69 | 0.72 | 0.69 |
|       | $LR_{20}$ | 5.49 | 0.06 | 5.49 | 0.06 | 5.49 | 0.06 | 5.49 | 0.06 | 5.49 | 0.06 | 5.49 | 0.06 |
| MULT3 | 1.429 | 1.429 | 1.429 | 1.429 | 1.429 | 1.429 | 1.429 | 1.429 |
|       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
|       | $LR_{10}$ | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 | 0.00 | 1.00 |
|       | $LR_{20}$ | 0.34 | 0.84 | 0.34 | 0.84 | 0.34 | 0.84 | 0.34 | 0.84 | 0.34 | 0.84 | 0.34 | 0.84 |

Biage and Ncelde
Table A.4. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Ibovespa BM&F BOVESPA shares, estimated with fourth-derivative DG wavelet.

|     | D_{1,8} | D_{1,7} | D_{1,6} | D_{1,5} | D_{1,4} | D_{1,3} | D_{1,2} | D_{1} |
|-----|---------|---------|---------|---------|---------|---------|---------|-------|
| 0.000 | 0.592   | 0.592   | 1.429   | 1.429   | 1.429   | 2.857   | 6.667   | 14.286 |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.592   | 0.592   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.592   | 0.592   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.592   | 0.592   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.592   | 0.592   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.592   | 0.592   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
| 0.000 | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708   | 0.708  |
| 1.000 | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999   | 0.999  |
Table A.5. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of DJIA shares, estimated by Morlex wavelet.

|        | D_{1-g} | D_{1-7} | D_{1-6} | D_{1-5} | D_{1-4} | D_{1-3} | D_{1-2} | D_{1} |
|--------|---------|---------|---------|---------|---------|---------|---------|------|
| BRAP4  | \(\bar{p}\) 0.952 0.952 1.429 1.429 1.429 2.857 6.667 14.286 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Yellow) (Red) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
| BRFS3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 1.905 3.980 5.774 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Yellow) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| BM & F−BOVESPA | \(\bar{p}\) 0.952 0.952 0.952 0.952 0.952 0.05 0.95 2.857 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| CCRO3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 1.905 3.800 11.429 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| ECOR3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 1.905 3.800 10.476 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| ESTC3  | \(\bar{p}\) 0.476 0.476 0.952 0.952 0.952 1.905 2.857 9.048 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| GOAU4  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 0.476 0.952 6.190 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| IBN3   | \(\bar{p}\) 0.952 0.952 3.810 3.810 3.810 4.266 4.266 9.048 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| MULT3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 0.476 0.476 0.952 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| NATU3  | \(\bar{p}\) 0.952 0.952 0.952 1.429 1.429 1.429 2.857 4.762 12.381 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| RALI3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 0.476 0.952 4.719 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Red) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |
| TEMP3  | \(\bar{p}\) 0.476 0.476 0.476 0.476 0.476 0.476 0.952 1.905 3.333 |
|        | TL (Green) (Green) (Green) (Green) (Green) (Green) (Green) (Yellow) |
|        | \(L_{R_{ind}}\) 0.0 0.99 0.99 0.99 0.99 0.99 0.99 0.99 |
|        | \(L_{R_{ce}}\) 0.7 0.69 0.72 0.69 0.7 0.69 0.7 0.69 |

204 *Brazilian Review of Econometrics* 40(1) June 2020
Table A.4. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of Ibovespa BM &...
Table A-7. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of DJIA shares, estimated by Db2 filter wavelet.

|          | D_{1-8} | D_{1-7} | D_{1-6} | D_{1-5} | D_{1-4} | D_{1-3} | D_{1-2} | D_1     |
|----------|---------|---------|---------|---------|---------|---------|---------|---------|
| RBA0A    | 0.952   | 0.952   | 1.429   | 1.429   | 1.429   | 2.857   | 6.667   | 14.286  |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| RBA0S    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 9.048   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| BCR03    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 10.476  |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| ECOR3    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 9.048   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| ESTC3    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 9.048   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| GOA04    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 6.910   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| IBERI3   | 0.952   | 0.952   | 3.810   | 3.810   | 3.810   | 4.286   | 4.286   | 9.048   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| MUL03    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 4.762   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| NATU3    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 12.381  |
| TL       | (Green) | (Green) | (Green) | (Green) | (Yellow) | (Red)   | (Red)   | (Red)   |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| RAD03    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 17.619  |
| TL       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
| TIM03    | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 3.333   |
| TL       | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{red} | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   | 0.000   |
| LR_{ec}  | 0.000   | 0.099   | 0.000   | 0.099   | 0.000   | 0.000   | 0.000   | 0.000   |
Table A-8. MC VaR failure test results, with 99% confidence, for the return of each rebuilt series of DJIA shares, estimated by fourth-derivative DOG wavelet.

| Reconstructed Series | D_{1-8} | D_{1-7} | D_{1-6} | D_{1-5} | D_{1-4} | D_{1-3} | D_{1-2} | D_{1} |
|---------------------|---------|---------|---------|---------|---------|---------|---------|------|
| BDFA1               | 0.952   | 0.952   | 1.429   | 1.429   | 1.429   | 2.857   | 6.667   | 14.286|
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Yellow)| (Red)   | (Red) |
| LR_{ind}            | 0.0     | 0.997   | 0.0     | 1.00    | 1.00    | 1.00    | 1.00    | 1.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA5               | 0.952   | 0.952   | 0.952   | 0.952   | 1.429   | 1.429   | 2.857   | 9.048 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.999   | 0.0     | 0.99    | 0.99    | 0.99    | 0.98    | 1.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 1.00    | 0.0     | 0.99    | 0.99    | 0.99    | 0.98    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |
| BDFA3               | 0.952   | 0.476   | 0.476   | 0.476   | 0.476   | 1.905   | 3.810   | 11.291 |
| TL                  | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) | (Green) |
| LR_{ind}            | 0.0     | 0.98    | 0.0     | 0.98    | 0.98    | 0.98    | 0.96    | 0.00  |
| LR_{cc}             | 0.7     | 0.69    | 0.72    | 0.69    | 0.72    | 0.69    | 0.72    | 0.69  |