TEST-BED SIMULATIONS OF COLLISIONLESS SELF-GRAVITATING SYSTEMS USING THE SCHRÖDINGER METHOD

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ABSTRACT

The Schrödinger method is a novel approach for modeling numerically self-gravitating, collisionless systems that may have certain advantages over N-body and phase-space methods. In particular, smoothing is part of the dynamics and not just the force calculation. This paper describes test-bed simulations which illustrate the viability of the Schrödinger method. We develop the techniques necessary to handle "hot" systems as well as spherically symmetric systems. A number of experiments are performed, and direct comparisons are made to results obtained using a simple shell code. We demonstrate that the method can adequately model a stable, equilibrium star cluster by constructing and then evolving a Plummer sphere. We also follow the evolution of a system from nonequilibrium initial conditions as it attempts to reach a state of virial equilibrium. Finally, we make a few remarks concerning the dynamics of axions and other bosonic dark matter candidates. The Schrödinger method, in principle, provides an exact treatment of these fields. However, such "scalar field" simulations are feasible and warranted only if the de Broglie wavelength of the particle is comparable to the size of the system of interest, a situation that is almost certainly not the case for axions in the Galaxy. The dynamics of axions is therefore no different from that of any other system of collisionless particles. We challenge recent claims in the literature that axions in the Galaxy form soliton stars.

Subject headings: dark matter — elementary particles — galaxies: kinematics and dynamics — methods: numerical

1. INTRODUCTION

Recently Widrow & Kaiser (1993) have developed a new numerical method for modeling self-gravitating, collisionless systems. Known as the Schrödinger method (SM), this approach allows one to follow the evolution of such systems on a three-dimensional Eulerian grid and provides an alternative to both N-body and phase-space methods.

Collisionless systems are ones in which the constituent particles move under the influence of the mean gravitational potential generated by all of the other particles. The state of a collisionless system is specified by a distribution function \( f = f(x, v, t) \) which gives the density of particles in phase space as a function of time. The function \( f \) is treated as a continuous fluid whose evolution is governed by the coupled Vlasov and Poisson equations (see e.g., Binney & Tremaine 1987):

\[
\frac{\partial f}{\partial t} + \nabla_x \left( \frac{v f}{2} \right) = 0 \quad (1)
\]

\[
\nabla^2 V = 4\pi G \int d^3v f \quad (2)
\]

The Vlasov-Poisson pair must be solved numerically for most time-dependent systems of interest. One approach is to evolve \( f \) directly in phase space, though efforts along these lines have had limited success primarily because of the large number of phase-space dimensions typically involved. In contrast, N-body or particle methods (see e.g., Hockney & Eastwood 1988) have been used successfully in virtually all astrophysical problems involving gravitational dynamics.

The number of particles employed in an N-body simulation is typically many orders of magnitude less than the actual number of particles in the physical system one is modeling. This discrepancy can lead to unphysical effects due to "particle noise" such as two-body relaxation. Smoothing techniques alleviate this problem but at the cost of spatial resolution, and care must be taken to choose a technique appropriate to the problem one is solving (Earn & Sellwood 1995). On the other hand, phase-space methods avoid these difficulties by explicitly constructing the smooth distribution function.

The SM attempts to circumvent the problems of both phase-space and N-body methods by encoding phase-space information in a continuous position space function that we call \( \psi \). Given an initial distribution function at time \( t_i \), we can find the distribution function at a later time \( t_f \) by the following general procedure:

\[
f(x, v, t_i) \xrightarrow{.M} \psi(x, t_i) \xrightarrow{DEM} \psi(x, t_f) \xrightarrow{.M^*} f(x, v, t_f)
\]

(3)

Here DEM is the dynamical equation of motion for \( \psi \), \( .M \) maps \( f \) to \( \psi \), and \( .M^* \) maps \( \psi \) to \( f \). Ideally, \( .M \) would be an invertable map with \( .M^* = .M^{-1} \), in which case the procedure would yield an exact solution to the Vlasov-Poisson pair.

With the SM, \( \psi \) is formally identified as a complex Schrödinger field obeying the coupled Schrödinger and Poisson equations:

\[
\frac{i \hbar}{2m} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + mV \psi \quad (4)
\]

\[
\nabla^2 V = 4\pi G \rho \quad (5)
\]

\( .M^* \) is the coherent state or Husimi transform (Husimi 1940), essentially the square of a windowed Fourier transform, and \( .M \) is a procedure to sample the initial distribution function (to be discussed below).
In \( N \)-body simulations, fictitious “superparticles” are used to provide a statistical representation of the distribution function. Imagine dividing position space into cells smaller than the scales of interest, but larger than the inter-particle spacing. The number of particles in each cell gives the (coarse-grained) density field, while their velocity distribution provides the remaining information contained in \( f \).

The resolution in position space relative to velocity space can be controlled by adjusting the cell size. However, the size of a phase space resolution element, \( \Delta \Omega \), is set by the number of particles \( N \) in the simulation: \( \Delta \Omega \approx n_{\Omega}/N \). Here, \( \Omega \) is the volume in phase space filled by the system (six-dimensional in general; three-dimensional for spherically symmetric systems with angular momentum (see below)), and \( n \) is the number of particles required in each resolution element in order to have a reasonably accurate estimate of \( f \) at a given phase-space point.

The SM shares some of these features. The de Broglie wavelength \( \lambda_{\text{DB}} \) for \( \psi \) enters as a model parameter and is chosen to be as small as possible, in general a few grid spacings. Position space is divided into regions smaller than the scales of interest by larger than \( \lambda_{\text{DB}} \), so that the WKB approximation applies. We can then think of \( \psi \) in this region as the superposition of plane waves, each of which represents different velocity streams in the distribution function. The correspondence principle guarantees that the “acceleration” of each stream is given by Newton’s Law. Since \( \psi \) is a continuous function, there is no problem with particle noise. Moreover, it is a function only of position rather than the phase-space coordinates and is therefore easier to work with. Formally, the resolution in phase space is set by the uncertainty principle. In practice it is set by the number of grid points \( N_{\Omega} \): \( \Delta \Omega \approx n_{\Omega}/N_{\Omega} \), where \( n_{\Omega} \) is the number of grid points across a typical de Broglie wavelength. We therefore expect the SM to be competitive with particle-mesh simulations in terms of resolution in phase space as a function of the number of grid points.

In a previous Letter, Widrow & Kaiser (1993) outlined the SM and presented simple illustrative simulations. In this paper we expand on that original work demonstrating, in a more quantitative way, the validity of the method. In particular we develop a new technique for handling “hot” systems (e.g., virialized systems) as well as systems with spherical symmetry. In \( \S \) 2 we review the SM and describe some of the new techniques implemented in this paper. Section 3 presents results of test-bed simulations including (1) the evolution of test particles in a spherically symmetric, background potential, (2) the evolution of self-gravitating particles with uniform angular momentum and large, random, initial radial velocities, (3) the stability of a fully self-coupled Plummer sphere, and (4) the equilibration of a destabilized Plummer sphere. Simulations (1) and (2) illustrate the basic features of the SM and its extension to spherical symmetry. The results from these simulations are compared with results obtained using a spherical \( N \)-body code, i.e., a shell code. The purpose of simulations (3) and (4) is to demonstrate quantitatively the ability of the method to handle complex spherical systems, i.e., systems that have a nontrivial distribution of angular momentum. In \( \S \) 4 we make a few remarks concerning recent work on axion dynamics and, in particular, challenge claims made in the literature that suggest that the “field” nature of axions is important for understanding their behavior in the Galaxy. Section 5 offers a summary and some concluding remarks.

## 2. \textbf{SCHRÖDINGER METHOD}

### 2.1. From \( \psi \) to \( \mathcal{F} \)

We use the coherent-state (Husimi 1940) representation to construct a phase-space distribution function \( \mathcal{F}(x, v) \) from \( \psi \). Mathematically, this is just the absolute square of a windowed Fourier transform. However, the physical content of this mapping is clearer when expressed in bra-ket notation:

\[
\mathcal{F}(x, v) = |\langle n(x, v) | \psi \rangle|^2 \equiv |\Psi(x, v)|^2 .
\]

Here \( |\eta(x, v)\rangle \) is the wavepacket for a single “particle” centered on the phase-space point \( (x, v) \),

\[
|\eta(x, v) \rangle | x' \rangle = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \frac{1}{\pi \eta^3} e^{-\frac{(x-x')^2}{2 \hbar^2} - imo \cdot x'/\hbar} .
\]

Equation (6) can be seen as a prescription for decomposing \( \psi \) into Gaussian wavepackets with velocity \( v \) and position \( x \). One can show, directly from the Schrödinger equation, that \( \mathcal{F} \) satisfies

\[
\frac{\partial \mathcal{F}}{\partial t} = \sum_i \left( \frac{\partial V}{\partial x_i} \mathcal{F} - v_i \mathcal{F}_{x_i} \right) + \left[ O\left( \frac{\eta^2}{L^2} \right) + O\left( \frac{\lambda_{\text{DB}}^2}{\eta^2} \right) \right] \frac{\partial \mathcal{F}}{\partial t}
\]

(8)

(see, e.g., Skodje, Rohrs, & van Buskirk 1989), where \( L \) is the length scale over which the system varies and \( \lambda_{\text{DB}} \sim |\psi/\nabla \psi| \approx h/mv \) is the typical de Broglie wavelength of \( \psi \). This equation reduces approximately to the Vlasov equation, provided \( \lambda_{\text{DB}} \ll \eta \ll L \).

Moments of the distribution function are calculated in the usual way (e.g., \( \bar{v}^a = \int d^3v v^a \mathcal{F}/\int d^3v \mathcal{F} \)). In particular, the density field \( \rho \), which is the source term for the potential \( V \), is given by

\[
\rho = \int d^3v \mathcal{F} = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \int d^3x e^{-\frac{(x-x')^2}{2 \hbar^2}} |\psi(x')|^2 .
\]

(9)

Evidently \( \rho \) is found by applying a Gaussian filter to \( |\psi|^2 \), in effect, removing the high-frequency modes. Of course, we are interested in the potential \( V \sim \nabla^2 \rho \), and, since the inverse Laplacian also removes high-frequency modes, we can safely replace \( \rho \) with \( |\psi|^2 \) in equation (5). In practice we calculate \( V \) from \( \rho \) using a fast Fourier transformation (FFT). Particle-mesh simulations also use FFT to calculate the force from \( \rho \), and so the CPU time for this part of the calculation will be the same for the two methods.

### 2.2. From \( f \) to \( \psi \)

We begin by sampling the distribution function, compiling a list of phase-space points as much as we would in an \( N \)-body simulation. The initial \( \psi \) is then taken to be the (incoherent) superposition of wavepackets centered on each of these points:

\[
|\psi \rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |\eta(x_j, v_j)\rangle .
\]

(10)

The \( |\eta(x_j, v_j)\rangle \) are given by equation (7). The phase factor \( e^{i\phi} \) is included to ensure that the wavepackets will add incoherently. While the physical motivation for this prescription is clear (we are just summing up particles), a more
mathematical motivation is the completeness of $|\eta(x, v)\rangle$:

$$\int d^3x \int d^3v \langle \eta(x, v) \rangle \langle \eta(x, v)| = I .$$

(11)

To see that this form provides a valid representation of $f$ we compute $F(x, v) = \mathcal{N}(\mathcal{L}f)$:

$$F(x, v) = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \langle \eta | \eta_k \rangle \langle \eta_j | \eta \rangle$$

(12)

$$= \frac{1}{N} \sum_{jk} e^{-(x-x_j)^2/4\eta^2} e^{-(x-x_k)^2/4\eta^2}$$

$$\times e^{-m_2q^2((v-v_j)^2/2\hbar^2)} e^{-m_2q^2((v-v_k)^2/2\hbar^2)} \Gamma_{jk} ,$$

(13)

where

$$\Gamma_{jk} = e^{-2m_2q^2((v_j-v_k))/\hbar} e^{-2m_2q^2((v_j-v_k))/\hbar}$$

$$\times e^{2i(m_2q^2/v_j - m_2q^2/v_k)/2} .$$

(14)

Splitting this into a summation over equal and unequal indices we find

$$F = \frac{1}{N} \sum_{j=1}^{N} e^{-(x-x_j)^2/4\eta^2} e^{-m_2q^2((v-v_j)^2/2\hbar^2)} + \mathcal{F}_1 ,$$

(15)

where $\mathcal{F}_1$ is a summation over the interference terms and will tend to be small. It is instructive to compare this result with the usual $N$-body distribution function:

$$f(x, v) = \frac{1}{N} \sum_{j=1}^{N} \delta(x-x_j)\delta(v-v_j) .$$

(16)

In the limit where the interference terms vanish, equation (15) becomes equation (16) with the $\delta$-functions smoothed into Gaussians.

The discussion above reveals a key distinction between $N$-body methods and the SM: in an $N$-body simulation, smoothing is implemented only in the force calculation. The particles in the SM are themselves treated as extended objects which evolve as would a distribution of particles (see below). Indeed, the particles/wavepackets in the initial distribution mix with one another and do not keep their identity as do particles in an $N$-body simulation. We therefore have, in effect, a form of dynamical smoothing.

2.3. Spherical Symmetry

We now specialize to spherical symmetry, where phase space has three dimensions; radial coordinate $r$, radial velocity $v_r$, and angular momentum $j = r(v_r^2 + v_\theta^2)^{1/2}$. The Vlasov and Poisson equations take the form

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \left( \frac{j^2}{r^3} - \frac{\partial V}{\partial r} \right) \frac{\partial f}{\partial v_r} = 0$$

(17)

$$\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 4\pi G \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dj^2 f .$$

(18)

The derivation of equation (17) assumes implicitly that $j \neq 0$. (The Jacobian of the transformation from $(x, v)$ to $(r, v_r, j)$ is singular for $j = 0$.) This in general does not present a problem, so long as $j$ is treated as a continuous variable. However, for numerical work, $j$ is discretized and $j = 0$ must be treated with care. Of course, $j$ can be viewed as a label since no differential operation with respect to $j$ is ever performed. We can therefore write $f(r, v_r, j) = f_j(r, v_r)$ and treat the evolution of the different $f_j$'s separately.

For radial orbits ($j = 0$), we write

$$\tilde{f}_0(r, v_r) \equiv \frac{F(r, v_r)}{r^2} \delta(v_r)\delta(v_\theta) ,$$

(19)

where $F$ satisfies the following equation:

$$\frac{\partial F}{\partial t} + v_r \frac{\partial F}{\partial r} - \frac{\partial V}{\partial r} \frac{\partial F}{\partial v_r} = 0 .$$

(20)

The Poisson equation is now

$$\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} = 4\pi G \int_{-\infty}^{\infty} dv_r \left( F + \pi \sum_{j \neq 0} f_j \right) ,$$

(21)

where the sum over $j^2$ is the discretized version of the integral over $j^2$ in equation (18).

It is straightforward to extend the SM method to spherically symmetric systems with angular momentum. We construct a $\psi_j$ for each $f_j$ in the initial distribution function: for $j = 0$ we have

$$\psi_0(r, v_r) = C \int_{-\infty}^{\infty} dr e^{-(r-r')^2/2\eta^2} e^{-imrv_r/r^4} \psi(r')$$

(22)

where $\psi_0$ obeys the equation

$$i\hbar \frac{\partial \psi_0}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_0}{\partial r^2} + \frac{mV}{r} \psi_0 \bigg|_{r=0} = 0 .$$

(23)

For $j \neq 0$,

$$\psi_j(r, v_r) = C \int_{-\infty}^{\infty} dr e^{-(r-r')^2/2\eta^2} e^{-imrv_r/r^4} \phi_j(r')$$

(24)

and the Poisson equation becomes

$$\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} V = 4\pi G \rho ,$$

(27)

where

$$\rho = |\psi_0|^2 + \frac{\pi}{r^2} \sum_{j \neq 0} |\phi_j|^2 .$$

(28)

Equation (28) emphasizes an important point: the $j = 0$ phase-space map gives a density, while the $j \neq 0$ map produces $dM(j)/dr$.

2.4. Numerical Preliminaries

It is convenient to write our equations in terms of the dimensionless quantities $y = x/L$, $\tau = t/T$, $\chi = \psi/(\rho_0)^{1/2}$, and $U = T^2/L^2 V$ where $L$ and $\rho_0$ are the characteristic size and density of the system of interest. We then have

$$i \frac{\partial \chi}{\partial \tau} = -\frac{\alpha}{2} V^2 \chi + \frac{1}{\alpha} U \chi$$

(29)

$$V^2 U = 3\pi^2 \frac{\beta^2}{32} \beta^2 \chi^* ,$$

(30)

where $\alpha \equiv \hbar T/\mu L^2$ and $\beta^2 \equiv 32G\rho_0 T^2/3\pi$. We therefore have three dimensionless parameters, $\alpha$, $\beta$, and $\eta/L$, at our
disposal. $\beta$ determines the choice of timescale: $T = \beta T_d$, where $T_d = (3\pi/32RGd)^{1/2}$ is the dynamical time. As we will now show, $\alpha$ determines the size of the phase-space region accessible in the simulation, while $\eta/L$ sets the relative resolution in position and velocity space. For simplicity, consider one coordinate or two phase-space dimensions. Suppose that our system comfortably fits inside a region of size $L$ in units of $L$. We discretize position space by choosing $y_j = jL/N$ where $N$ is the number of gridpoints. Further suppose that the system, when viewed in velocity space, comfortably fits inside a region of size $\gamma$ in units of $L/T_d$. The lattice spacing in velocity space is connected to the number of gridpoints. Further, $\gamma \approx 5$ works well for the Plummer sphere simulations below we take $\gamma \approx 5$ and $\beta \approx 2\pi$.

The SM has the desirable feature that many quantities of interest can be calculated without having to actually construct the full phase-space distribution function. We have already seen an example of this when calculating $\rho$. In general, if the function does not depend on velocity, the velocity integral will collapse into a $\delta$-function:

$$\langle Q \rangle = \int d^3 x d^3 v Q(x, v) \mathcal{F}(x, v)$$

$$= (2\pi \eta^2)^{-3/2} \int d^3 x d^3 x' e^{-2(\alpha - \alpha')^2/\eta^2} Q(x) | \psi(x') |^2$$

$$\cong \int d^3 x Q(x) | \psi(x) |^2.$$ (33)

In the case of velocity moments, we can replace the $v^n$ term with a corresponding derivative, perform a change of coordinates, and integrate by part to get

$$\langle v^n \rangle = \left(-i \frac{\alpha}{\beta} \right)^n \int d^3 x$$

$$\times \left\{ \frac{\partial^n}{\partial s^n} \left[ e^{-s^2/4\eta^2} \psi \left( \frac{x + s \hat{n}}{2} \right) \psi^* \left( \frac{x - s \hat{n}}{2} \right) \right] \right\} |_{s=0},$$ (34)

where $\hat{n}$ is the unit vector in the $j$ direction. For the important case of $\langle v^2 \rangle$, this expression reduces to

$$\langle v^2 \rangle \approx - \left( \frac{\alpha}{\beta} \right)^2 \int d^3 x \{ \psi(x) \nabla^2 \psi^*(x) \},$$ (35)

where we have neglected a term proportional to $\psi \psi^*$ and have integrated by parts. This is just the usual quantum mechanical result $\langle v^2 \rangle = \langle \mathcal{P}^2 \rangle / m^2$.

Applying this to the case of spherical symmetry we then find

$$\langle Q \rangle \approx \sum \int d r Q(r) | \psi(r) |^2$$

$$\langle v_i^2 \rangle \approx - \sum \left( \frac{\hbar}{m} \right)^2 \int d r \psi^* r \left( \frac{d^2 \psi(r)}{dr^2} \right).$$ (37)

One can, of course, resort to the “brute force” calculation of more complicated quantities,

$$\langle Q \rangle = \sum \int d r \int d v Q(r, v, j) \mathcal{F}(r, v).$$ (38)

The drawback of this approach is that it requires an $O(N^2 \ln N)$ calculation for each $j$ plane. These calculations can be very time-consuming, and it is time well spent looking for a shortcut of the kind just presented.

3. TEST-BED SIMULATIONS

3.1. Test Particles

As a first concrete example of the SM, we will consider the motion of “test particles” in a fixed Plummer sphere potential,

$$V(r) = - \frac{GM}{r^{1/2}}.$$ (39)

This is a useful diagnostic and also allows one to observe some of the main features of the method in a straightforward manner. The SM analogue of a test particle is a single coherent-state wavepacket centered at some chosen point in phase space. Here we will examine the motion of two particles with angular momentum $j = 0.5$ and zero initial radial velocity. The resulting initial wave function is then just the sum of two Gaussians centered at the particle positions. The system is evolved by numerically solving equations (25) and (39), using the algorithm presented in Visscher (1991). This is an explicit algorithm and is at least a factor of 3 faster than the usual implicit ones. Figure 1 shows the particles moving in the Plummer sphere potential. These plots help to show the physical connection between wave function, $\psi$, and the Schrödinger distribution function, $\mathcal{F}$. The particle positions are given by the peaks of the wave function, while the velocity information is encoded in the high-frequency modes of $\psi$. From these plots one can see that the particles do follow the expected trajectories, justifying our choice of a spherical phase-space map.

These plots also allow an interesting comparison of the SM and $N$-body methods at a fundamental level. When one follows the evolution of an $N$-body superparticle, it effectively carries with it a piece of phase space that has a fixed size and shape. When modeling systems in which mixing occurs, these phase-space blocks lead to unphysical coarseness of the distribution function. Evolving Schrödinger particles, on the other hand, carry with them a piece of phase space that has a variable shape and fixed mass. The motion of the test particles in Figure 1 illustrates this point. The individual Schrödinger particles spread out along the phase-space orbit in the same fashion as an equivalent distribution of point particles would. The conclusion is that fiducial SM particles can themselves take part in the phase-mixing process.

We now examine the evolution of an initially “cold” (i.e., single-velocity stream) distribution of test particles. The corresponding wave function is given by the square root of the desired density multiplied by an appropriate phase factor:

$$\psi(x) = \sqrt{\rho(x)} e^{-i\theta(x)},$$ (40)

where $\nabla \theta = mv'$. This leads to the phase-space distribution

$$\mathcal{F}(x, v) \sim \rho(x) e^{-\nu^2 (mv' - \nabla \theta)^2}.$$ (41)
Compare this expression with the actual distribution function for a cold system:

\[
f(x, v) = \rho(x)\delta[v - v'(x)]. \tag{42}\]

The \(\delta\)-function has been replaced by a Gaussian of width \(\sim h/\eta\). For our cold \((v_r = 0)\) test particles, \(\theta\) is constant. We choose the initial wave function to reproduce the mass distribution of the Plummer sphere in the \(j = 0.5\) plane:

\[
\psi(r, t) = \sqrt{\int_{-\infty}^{\infty} dv_r f(r, v_r, j^2) \bigg|_{j=0.5}}, \tag{43}\]

where \(f\) is the Plummer sphere distribution function (see below).
Figure 2 shows the evolution of the system in phase space found using the SM with 8192 grid points. For comparison, we present results obtained using a simple spherical N-body or shell code with 8192 particles in Figure 3. Agreement between the two methods is excellent. Compare, for example, the positions of the turnaround radii for different streams in the distribution function. There are of course important differences in how the two methods follow the evolution of the system. In particular, while the shell code preserves the one-dimensional nature of the initial distribution, the SM, which must share resolution in both position and momentum, produces a fuzzy thread in phase space.

We next consider the evolution of a self-gravitating system of particles. As before, we assume all of the particles have the same angular momentum, here $j = 0.9$. However, in this case, the initial radial velocities are chosen at random and are significant as compared with the characteristic velocities of the gravitational potential. To be precise, the initial position and radial momentum for each particle are chosen at random from Gaussian distributions.

We use 32,768 grid points for this experiment, making it our largest run. These phase-space coordinates are fed directly into the shell code. The initial wave function for the SM is constructed using the coherent state summation technique discussed above with this same set of coordinates.

Figures 4 and 5 present the results of the SM and shell code simulations, respectively. Figure 5 is actually a gray-scale plot of the distribution function made by binning particles in $200 \times 200$ phase-space cells. The size of these cells is chosen to correspond roughly to the size of the resolution elements in the SM as set by $\eta$ but is also what one would choose based on the number of particles in the simulation. Once again, we find very good agreement between the two methods.

The CPU time/time step for both shell code and SM scale as $N \log N$. In the SM this arises because we must Fourier transform the density in order to calculate the potential. In the shell code, this comes about because we must sort the shells by radius in order to calculate the force on each shell. Running on a Silicon Graphics workstation, with $N = 32,768$, the CPU time/time step is roughly 1.4 s for the SM and 1.6 s for the shell code. The time step in the SM is set by the requirement that $t^* / \Delta t \ll 1$, where $t^*$ is the change in $t$ over a single time step. This implies that $\Delta t \propto \Delta r$, where $\Delta r$ is the grid spacing. A similar scaling would result in the shell code simulation if we required that shells only move about one intershell distance in each time step. This appears too conservative and, in fact, for the experiments discussed here, the time step for the shell code was about a factor of 10 greater than that for the SM.

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**Fig. 2.—** Cold distribution. The evolution of a cold Schrödinger distribution is followed as it begins to phase mix and virialize in a fixed Plummer sphere potential. The initial phase-space distribution corresponds to the $j = 0.5$ plane of a Plummer sphere with the velocity distribution suppressed.
3.2. Equilibrium Initial Conditions: The Plummer Sphere

Our first full-scale test-bed calculation is of an equilibrium star cluster, specifically a Plummer sphere. The distribution function for this model is given by

$$f(E) = \begin{cases} \frac{A}{|E|^{7/2}} & \text{if } E < 0 \\ 0 & \text{otherwise} \end{cases}, \quad (44)$$

where

$$E = \frac{1}{2} \left( \frac{v_r^2}{r^2} + j^2 \right) - \frac{GM}{L} \left( 1 + \frac{r^2}{L^2} \right)^{-1/2}, \quad (45)$$

is the energy per unit mass, $A = 256/35\pi^4 L^2/G^5 M^5$, and $M$ is the total mass of the system.

To begin the construction of the Plummer sphere, we slice the three-dimensional phase space into $N_j$ planes of constant angular momentum. Each of these $(r, v_r)$ planes has $N^2$ grid points and will be mapped into a radial wave function, $\psi(r)$, that has $N$ grid points. For the simulations that follow, we set $N = 1024$ and $N_j = 25$, with no $j = 0$ plane. (The distribution function is sharply peaked in $j^2$ at the origin, and it is difficult to integrate $\mathcal{P}$ with a small value for $N_j$ consistently. The lack of a $j = 0$ plane is numerically consistent with a large value of $N_j$, where the $j = 0$ plane would have a very small weight.) The planes are spaced equally in $j$, although in general we can select planes at arbitrary values of $j$. We sample the set of $N_j \times N^2$, $(r, v_r)$, phase space points using the usual N-body technique (see, e.g., Hénon 1973). Figure 6 illustrates this sampling method. Here the set of sampled phase-space points are shown for the $j = 0.5$ and $j = 1.0$ planes. With these points in hand, one can algorithmically compute the coherent state summation to produce the set of $N_j$ initial wave functions. Figure 7 shows the results of this construction. Here a typical phase-space plane is shown, along with the generating wave function and the mass and velocity distributions for that plane. One should note that with the above choice of $\eta$, the $r$ and $v_r$ resolutions are very similar. If $\eta$ were increased (corresponding to a larger smoothing window), the mass distribution would become smoother while the velocity distribution would become more jagged. In terms of the Schrödinger particles of Figure 1, the unit phase-space particle would transform from a circle to an ellipse, becoming shrunken in the $v_r$ direction and elongated in the $r$ direction. All of these effects would be transposed for $r$ and $v_r$ if $\eta$ were, instead, to be decreased.

The system is evolved by solving the coupled Schrödinger-Poisson system using the Visscher algorithm along with a potential-solving routine based on Eastwood & Brownrigg (1979). Care must be taken to obtain the correct boundary condition at infinity when solving the
Poission equation on a finite grid. The routine that was implemented makes use of "padding" to eliminate the effects of the grid boundary (at the expense of doubling the number of grid points). Figure 8 shows the mass and velocity distributions of the system in its initial state and after being evolved through 10 dynamical times. From these plots alone the system does appear to be stable. A quantitative measure of this is the virialization of the system: for a perfect Plummer sphere \( 2T + W = 0 \), where \( T \) is the total kinetic energy

\[
T = \frac{1}{2} \left( \sum_{j=1}^{j} \left( \frac{1}{r^2} \right) \right) |\psi_j|^2 dr ,
\]

and \( W \) is the potential energy of the system. The ratio \( 2T/|W| \) was calculated during the evolution of the system over the course of 50 dynamical times. Figure 9 shows the evolution of \( 2T/|W| \) for the system, with a similar curve obtained using a standard tree code (Barnes & Hut 1986), with 1024 particles, for reference. As the two systems begin to relax, they oscillate with comparable amplitudes. However, given that the SM incorporates perfect spherical symmetry and has 25 times more grid points than the tree code has particles, this is not an outstanding result. Our Plummer sphere is stable, but noisy. As we have already seen, the SM produces good results for a single \( j \) plane distribution. The Plummer sphere noise is most likely due to our method of discretizing its three-dimensional phase space.

3.3. Nonequilibrium Initial Conditions

As we saw in the last section, the Plummer sphere can be well modeled with only 25 \( j \) planes. But how well can this method model a perturbed system? One can imagine a situation where the numerical representation of a system is sufficient near equilibrium but is too coarse to accurately model an evolving system. Therefore, in order to test the true dynamics of the spherical SM, we performed a destabilized Plummer sphere simulation (Rasio, Shapiro, & Teukolsky 1989 use this system to test their phase-space code). The system was destabilized by reducing the \( v_r \) components of all the sampled Schrödinger particles by a factor of 1/8\(^{1/2} \), giving \( 2T/|W| = 17/24 \approx 0.708 \). This was also done with a tree code simulation, again with \( N = 1024 \) particles. Figure 10 shows the evolution of \( 2T/|W| \) for both of these simulations. As can be seen, both undergo the expected bounce, and then begin to virialize and settle into a new equilibrium. This result shows that the spherical SM can be used to
model hot systems that are out of equilibrium and have a three-dimensional phase space.

4. AXION DYNAMICS AND THE FORMATION OF SOLITON STARS

The SM is applicable to any collisionless system regardless of what form the constituent particles take. (The same holds for N-body and phase-space methods.) This is particularly relevant for studies of galactic halos and large-scale structure where the dominant component of the mass density, the so-called dark matter, is in some unknown form. Dark matter candidates range from astrophysical compact objects such as black holes and brown dwarfs to elementary particles such as neutralinos. Still, the dynamics of the system is independent of nature’s choice.

There is at least one exception to the above argument. If dark matter is in the form of a very light scalar field, then quantum-mechanical effects will enter into the dynamical equations of motion. In this case, the Schrödinger-Poisson pair provides the exact equations of motion (for nonrelativistic motion) for the system. The mass \( m \) is now a true physical parameter determined by particle physics theory. Of course, so long as the Compton wavelength for the field is much less than the scales of interest in the problem, the field will behave like collisionless matter.

Recently there has been considerable interest in soliton stars (for a review, see Jetzer 1992). These objects are essentially self-gravitating compact objects made up of bosonic fields. The interest in soliton stars has been generated largely by the conjecture that dark matter may be bosonic. Indeed, one of the most popular dark matter candidates is the axion, a pseudo–Nambu-Goldstone boson that arises in the Peccei & Quinn (1977) solution to the strong CP problem (Weinberg 1978; Wilczek 1978). Axions that are
The initial wave function of a Plummer sphere plane, along with the associated phase-space distribution, $\mathcal{F}(r, v)$. The mass and velocity distributions were calculated directly from $\mathcal{F}$. The solid curves are the model distributions.

Figure 7—Equilibrium Plummer sphere. The initial wave function of a Plummer sphere $j$ plane, along with the associated phase-space distribution, $\mathcal{F}(r, v)$. The mass and velocity distributions were calculated directly from $\mathcal{F}$. The solid curves are the model distributions.

viable (there are constraints on the axion mass from cosmology, stellar evolution, and supernova studies; see, e.g., Kolb & Turner 1990) have a mass $\sim 10^{-5}$ eV and therefore a de Broglie wavelength in a galactic halo of $\sim 10$ m. All this suggests that axions behave like any other dark matter candidate, i.e., like collisionless matter. However, Seidel & Suen (1994) suggest that axions can form soliton stars. The question of whether this is indeed the case, or whether axions, like any other form of collisionless matter, form virialized clumps of matter (as in the axion miniclusters of Hogan & Rees 1988) is of fundamental importance in understanding whether or not axions are a viable dark matter candidate. Axions can annihilate into photons ($AA \rightarrow \gamma\gamma$), provided the density is high enough. In the axion miniclusters of Hogan & Rees (1988), the density is $10^7$ g cm$^{-3}$ and annihilation is unimportant. Alternatively, the density in axion stars can be as high as $10^{24}$ g cm$^{-3}$. At these densities, the annihilation is very efficient, making the configuration unstable.

The question, of course, is whether axion stars ever form in the early universe. A diffuse axion cloud must lose both angular momentum and energy if it is to form an axion star. Seidel & Suen (1994) find that a collapsing cloud can lose energy by radiating scalar material, a process they call gravitational cooling. However, they only briefly mention the issue of angular momentum and carry out simulations of purely radial collapse. In these simulations the field collapses quickly, and an excited soliton star forms at the center and then settles quickly by radiating scalar matter.

In what follows, we argue that any reasonable amount of angular momentum will prevent the object from reaching the stage where gravitational cooling can take place. Consider an initial cloud of mass $M$, radius $R$, binding energy $E_i \sim GM^2/R$, and angular momentum $J$. A useful quantity is the dimensionless parameter

$$\lambda \equiv J \left( \frac{E_i}{GM^2} \right)^{1/2},$$

which is roughly the square root of the ratio between the rotational energy and the binding energy. As an object collapses, $\lambda \propto R^{-1/2}$, provided there is no loss of angular momentum. We expect that this will indeed be the case, at least in the initial collapse of the axion cloud. In order for the final object to be gravitationally bound, $\lambda_f \lesssim 1$. We therefore require $\lambda(R/R_f)^{1/2} \lesssim 1$, where $R_f \approx \hbar^2/\mu^2GM$ is the radius of the final compact object. The bound is then

$$\lambda_i \lesssim 10^{-12} \left( \frac{10^{-5} \text{ eV}}{m} \right) \left( \frac{M_\odot}{M} \right)^{2/3} \left( \frac{\rho}{\rho_{\text{halo}}} \right)^{1/6},$$

where $\rho_{\text{halo}}$ is the density of the halo.
where $\rho_{\text{halo}} = 0.008 \ M_\odot \ pc^{-3}$ is characteristic of the density of dark matter in the Galaxy. This is an incredibly small value for $\lambda$. It is widely believed that galaxies acquire angular momentum during formation from tidal interactions with nearby mass distributions. Typical values from both analytic and numerical work suggest that $\lambda \approx 0.01$–0.1 for galaxy-sized objects (see, e.g., Peebles 1969; Efstathiou & Jones 1979). The conclusion is that axionic matter, if indeed it does make up the dark halo, will have too much angular momentum to ever reach the densities required for gravitational cooling and soliton star formation.

5. SUMMARY AND CONCLUSION

This paper describes test-bed simulations of collisionless, self-gravitating matter using the SM. Included are several improvements over the earlier work by Widrow & Kaiser (1993). In particular we describe how to set up hot (e.g., virialized) systems. We also develop the techniques required to handle spherically symmetric systems with angular momentum, which may be useful for studying violent relaxation and equilibrium star clusters. Of course, the method is not restricted to problems with high degrees of symmetry. Our focus on spherically symmetric systems is mainly for illustrative purposes and, in hindsight, may not have been the best choice because of the special features of spherical coordinates.

As discussed above, we expect the SM to be as efficient as a particle-mesh code in following the evolution of a system in phase space, although the SM generally requires a small time step ($\Delta t \ll \Delta r$) for stability. One obvious improvement would be to use an adaptive grid and adaptive time step to do the simulation. This could in principle make the method competitive with the widely used tree code of Barnes & Hut (1986), or perhaps even particle-mesh codes. (See, e.g., Sellwood 1996 for a comparison of the two methods.)

The SM can be adapted to a wide variety of problems that have, as their central equation, Vlasov. Widrow & Kaiser (1993) ran two-dimensional cosmological simulations with results that were in excellent agreement with those from $N$-body simulations. The method can also be applied to problems in electrodynamics that require information about the phase-space structure of particles, thereby providing an alternative to particle-in-cell codes and Eulerian-Vlasov codes (see, e.g., Manfredi 1995). In an extension to relativistic systems, Widrow (1996) has shown that the SM (actually the Klein-Gordon equation) can be used to simulate collisionless systems in general relativity.

$N$-body techniques have been used in virtually all areas of astrophysics that require numerical simulations of collisionless matter. However, despite years of research, there are still questions that arise over the validity and applicability of these simulations. Most of the questions center on...
smoothing, the technique used to construct a smooth density field from a system of discrete particles (Earn & Sellwood 1995). The SM provides an alternative where the superparticles of the N-body experiment are replaced by a continuous field. While the field may look like the superposition of particle-like wavepackets (indeed, this is how we set up our initial conditions), these wavepackets spread out in phase space as the system evolves. We therefore have a type of dynamical smoothing. Whether this dynamical smoothing has any real advantages over N-body methods remains to be seen.

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