BOURGAIN-CHANG PROOF OF THE WEAK ERDŐS-SZEMERÉDI CONJECTURE

DMITRII ZHELEZOV

ABSTRACT. This is an exposition of the following ‘weak’ Erdős-Szemerédi conjecture for integer sets proved by Bourgain and Chang in 2004. For any $\gamma > 0$ there exists $\Lambda(\gamma) > 0$ such that for an arbitrary $A \subset \mathbb{N}$, if $|AA| \leq K|A|$ then

$$E_+(A) \leq K^\Lambda|A|^{2+\gamma}.$$ 

NOTATION

The following notation is used throughout the paper. The expressions $X \gg Y$, $Y \ll X$, $Y = O(X)$, $X = \Omega(Y)$ all have the same meaning that there is an absolute constant $c$ such that $|Y| \leq c|X|$. For a graph $G$, $E(G)$ denotes the set of edges and $V(G)$ denotes the set of vertices. If $X$ is a set then $|X|$ denotes its cardinality.

We write $A \approx B$ if $A \leq B \leq 2A$.

For sets of numbers $A$ and $B$ the sumset $A + B$ is the set of all pairwise sums $\{a + b : a \in A, b \in B\}$, and similarly $AB$, $A - B$ denotes the set of products and differences, respectively.

For a number $x$ and a set $Y$ the expression $xY$ denotes the set $\{xy : y \in Y\}$, and similarly for the additive shift $x + Y := \{x + y : y \in Y\}$.

If $G \subset A \times B$ is some graph then $A + B$ denotes the the restricted sumset $\{a + b : (a, b) \in G\}$.

For a vertex $v$ of $G$ we write $N_G(v)$ for the set of neighbors of $v$ in $G$. The subindex $G$ may be omitted if it is clear to which graph the vertex belongs.

The additive energy $E_+(A, B)$ is defined as the number of additive quadruples $(a_1, b_1, a_2, b_2)$ such that

$$a_1 + b_1 = a_2 + b_2.$$ 

We write $E_+(A)$ or simply $E(A)$ for $E_+(A, A)$.

For a set $A$ we write $1_A$ for the indicator function of $A$ and the convolution $f * g$ is defined as

$$f * g(x) := \sum_y f(y)g(x - y).$$
where the sum is taken over the support of $f$. In particular, one verifies that
\[
E_+(A, B) = \sum_x (1_A * 1_B)^2(x) = \|1_A * 1_B\|_2^2.
\]

1. Introduction

In the present exposition we give a slightly simplified proof of the result due to Bourgain and Chang [1] below.

**Theorem 1.** For any $\gamma > 0$ there exists $\Lambda(\gamma) > 0$ such that for an arbitrary $A \subset \mathbb{Z}$ if $|AA| \leq K|A|$ then
\[
E_+(A) \leq K^\Lambda |A|^{2+\gamma}.
\]

Bourgain and Chang actually proved a much stronger result which is as follows.

**Theorem 2.** Given $\gamma > 0$ and $q > 2$, there is a constant $\Lambda(\gamma, q)$ such that if $A \subset \mathbb{Z}$ is a finite set with $|A| = N; |AA| < KN$, then
\[
\left\| \sum_{n \in A} c_n e(n\theta) \right\|_{L_q(\mathbb{R}/\mathbb{Z})} \leq K^\Lambda N^{\gamma} \left( \sum_{n \in A} c_n^2 \right)^{1/2}
\]
with the usual notation $e(x) := \exp(2\pi ix)$.

In particular, taking $c_n := 1$ in Theorem 2 and expanding the $L_q$ with $q = 2k$, it follows that for an arbitrary $\gamma > 0$ and integer $k \geq 2$ there exists $\Lambda(\gamma, k)$ such that
\[
E_k(A) \leq K^{\Lambda(k, \gamma)} |A|^{k+\gamma},
\]
where $E_k$ is the $k$-energy, defined as
\[
E_k(A) := \| (a_1, \ldots, a_k, a'_1, \ldots, a'_k) : a_1 + \ldots + a_k = a'_1 + \ldots + a'_k \|.
\]

Finally, the pinnacle of [1] is the following infinite growth sum-product theorem.

**Theorem 3.** For any $b > 0$ there exists integer $k(b)$ with the property that
\[
|A + \ldots + A| + |A \ldots A| \geq |A|^b
\]
for any integer set $A$ ($A$ is taken $k$ times in the iterated sumset and the product set above).

This impressive result can be quickly deduced from Theorem 2, see Proposition 2 in [1].

The main purpose of this exposition is to present the combinatorial arguments of [1] in a way which is more familiar for mathematicians working in arithmetic combinatorics, in particular on problems related to the sum-product phenomena. The original paper exploits the machinery of trigonometric polynomials which
makes it somewhat hard to absorb for readers with little background in harmonic analysis. Since the weak Erdős-Szemerédi conjecture for real sets is still wide open, we hope that a better understanding of the Bourgain-Chang method may help to make progress on the real case or at least highlight the obstacles.

We are going to prove only Theorem 1 which is admittedly weaker then Theorem 2 or Theorem 3. Nevertheless, we have decided to pay such a price in order to streamline the exposition. While the machinery of trigonometric polynomials and $\Lambda_q$-constants turns out to be more robust, we believe that the proof of Theorem 1 presented below already contains all essential ingredients needed for the general case. The case of $E_+$, however, allows one to make all the arguments on the ‘physical side’ (basically, because the $L_2$ norm is invariant under the Fourier transform) which makes the proof purely combinatorial and elementary.

We therefore suggest using the current note as a warm-up or as a supplementary reading for [1]. We will occasionally skip some intermediate steps in the calculations which we think are routine, referring the reader to the original paper for details. Again, our motivation here is to give a somewhat informal sketch of the arguments and convince the reader that the whole setup must work. The details may then be filled by reading [1], which is an impeccable and rigorous piece. Any errors, gaps, inconsistencies or sloppy explanations are solely due to the author of this note.

2. Motivating examples

We start with some motivating examples. Let $Y$ be an integer set which can be decomposed as a disjoint union

$$Y = \bigcup_{i \in I} a_i X_i$$

with $I = 1, \ldots, N$, some distinct numbers $a_i \in \mathbb{Z}$ and integer sets $X_i$. Applying the Cauchy-Schwarz inequality twice, one can then write for the energy (all the summations are through the indices in $I$)

$$E^{1/2}_+(Y) = \left\| \sum_{i,j} 1_{a_i X_i} \ast 1_{a_j X_j} \right\|_2 \leq \sum_{i,j} \| 1_{a_i X_i} \ast 1_{a_j X_j} \|_2$$

$$= \sum_{i,j} E^{1/2}(a_i X_i, a_j X_j) \leq \sum_{i,j} E^{1/4}(X_i) E^{1/4}(X_j)$$

$$= (\sum_i E^{1/4}(X_i))^2 \leq N \sum_i E^{1/2}(X_i).$$

We can thus bound the energy $E(Y)$ by the energy of the constituents. Of course, without any prior knowledge about $X_i$’s and $a_i$’s the inequality above doesn’t give much, but one might hope that under certain conditions on $a_i, i \in I$
there is almost no additive interaction between the sets $a_iX_i$ and $a_jX_j$ in (2) and a better bound

$$E_{1/2}^{1/2} \left( \bigcup_{i \in I} a_iX_i \right) \leq \psi(|I|) \sum_i E_{1/2}^{1/2}(X_i)$$

(4)

holds with some sublinear function $\psi$. We make the following definition.

**Definition 1 (Separating sets).** A set $A \subset \mathbb{Z}$ is $\psi$-separating if the bound

$$E_{1/2}^{1/2} \left( \bigcup_{a \in A} \{aX_a\} \right) \leq \psi \sum_{a \in A} E_{1/2}^{1/2}(X_a)$$

(5)

holds for any collection of integer sets $X_a$ such that $(a, X_{a'}) = 1$ for any $a, a' \in A$.

**Remark 2.1.** The definition of separating sets seems to be related to the notion of decoupling in harmonic analysis extensively studied by Bourgain himself and coauthors in later works, see e.g. [2] and references therein. Indeed, for two sets $X$ and $Y$ the additive energy $E(X,Y)$ is equal to $\langle (\hat{1}_X)^2, (\hat{1}_Y)^2 \rangle$, so Theorem 2 can be seen as a decoupling result. In particular, (8) below is a manifestation of the fact that a family of bi-orthogonal functions exhibit $L_2$-decoupling, see Terry Tao’s blog [4] on decoupling for more details. We don’t know if there is a deeper connection between geometric decoupling and the sum-product phenomenon.

We record (3) for future use.

**Claim 2.1 (Trivial separation bound).** Any integer set $A$ is $|A|$-separating.

One may wonder if there exist sets with a good separation factor $\psi$. Here is the first motivating example when this is indeed the case. Let $p$ be a prime and $a_i = p^{k_i}$ for some $k_i \geq 0$ and

$$A = \bigcup_{1 \leq i \leq N} \{a_i\}.$$

Assume

$$Y = \bigcup_{1 \leq i \leq N} a_iX_i$$

for some integer sets $X_i$ with $(p, X_i) = 1$. If now $(y_1, y_2, y_3, y_4) \in Y^4$ with

$$y_1 + y_2 = y_3 + y_4$$

(6)

both sides must divide the same power of $p$, so at least two of the elements $y_1, y_2, y_3, y_4$ must be in the same slice $a_iX_i$ for some $i$. There are six possible cases for which two of $y_1, y_2, y_3, y_4$ belong to $a_iX_i$. Writing $r_{a_iX_i-a_iX_i}(x)$ (resp. $r_{Y-Y}(x)$) for the number of ways to represent $x$ as a difference of two elements in $a_iX_i$ (resp. $Y$), we can bound the number of quadruples (6) for given $i$ as either

$$\sum_x r_{a_iX_i-a_iX_i}(x)r_{Y-Y}(x)$$
or
\[
\sum_x r_{a_iX_i+a_iX_i}(x)r_{Y+Y}(x)
\]
(with the obvious modification of notation) depending on the case. Summing up, we have (± means either plus or minus depending on the case)
\[
E(Y) \leq \sum_i \sum_x r_{a_iX_i\pm a_iX_i}(x)r_{Y\pm Y}(x)
\]
\[
\leq \sum_i \left( \sum_x r_{a_iX_i\pm a_iX_i}(x) \right)^{1/2} \left( \sum_x r_{Y\pm Y}(x) \right)^{1/2}
\]
\[
= 6E^{1/2}(Y) \sum_i E^{1/2}(X_i),
\]
so
\[
E^{1/2}(Y) \leq 6 \sum_i E^{1/2}(X_i),
\]
which means that \( A \) is 6-separating.

In what follows it will be convenient to use the prime valuation map which is defined as follows. Let \( A \) be a rational set with the elements (after all possible cancellation in numerators and denominators) having prime factors in the set \( \{p_i\}, i \in I \). The map \( P_I : A \rightarrow \mathbb{Z}^I \) maps \( \prod_{i \in I} p_i^{\alpha_i} \) to \( (\alpha_1, \ldots, \alpha_{|I|}) \). It is clear, however, that since all our sets are finite there always exists a large enough index set \( I \) such that \( P_I \) is well-defined for all sets in question and is injective. We will therefore assume that this large index set is fixed and omit the subindex \( I \) in \( P_I \) when the actual index set is not important.

Let \( A \) be a finite-dimensional vector space \( V \). Recall that \( \text{rank}(A) \) is defined as the minimal \( d \) such that \( A \) is contained in an affine subspace of \( V \) of dimension \( d \). Next, define multiplicative dimension of a set \( A \) simply as \( \text{rank}(P(A)) \). Of course, \( \mathbb{Z}^I \) is not a linear space since \( \mathbb{Z} \) is not a field, so one should consider \( P(A) \) as a set naturally embedded into an ambient linear space over \( \mathbb{Q} \) (or \( \mathbb{R} \), which makes no difference in our case).

Recall the following lemma due to Freiman.

**Theorem 4** (Freiman’s Lemma, [5] Lemma 5.13). Let \( A \) be a finite subset of a finite-dimensional space \( V \) and suppose \( \text{rank}(A) = m \). Then
\[
|A + A| \geq (m + 1)|A| - \frac{m(m+1)}{2}.
\]

**Corollary 2.1.** Let \( A \subset \mathbb{Z} \). Assume \( |AA| \leq K|A| \). Then \( A \) is \( 6^K \)-separating.

**Proof.** Observe that \( P(AA) = P(A) + P(A) \) and thus \( P(A) \) is contained in an affine subspace of dimension at most \( K \) by Freiman’s Lemma. Then, by linear algebra, there exists an index set \( I \) of size at most \( K \) such that the map \( P_I : A \rightarrow \mathbb{Z}^I \) is
injective. In other words, there are at most $K$ primes $p_1, \ldots, p_{|I|}$ such that each $a \in A$ can be written as (the powers $\alpha_i$ depend on $a$)

$$a = x_a \prod_{i=1}^{|I|} p_i^{\alpha_i}$$

and $x_a, a \in A$ are all distinct. If we then take an arbitrary set $Y := \bigcup_{a \in A} aY_a$ with $(a, Y_{a'}) = 1$ for any $a, a' \in A$, we can expand

$$Y = \bigcup_{a \in A} x_a Y_a \prod_{i=1}^{|I|} p_i^{\alpha_i}.$$ 

By construction of the map $\mathcal{P}_I$ and the condition $(a, Y_{a'}) = 1$, we conclude that $x_a Y_a$ don’t have prime factors among $p_1, \ldots, p_{|I|}$ and thus we can repeatedly apply (8) $|I|$ times for each $p_i$. It follows that

$$E^{1/2}(Y) \leq 6^K \sum_i E^{1/2}(x_a Y_a) = 6^K \sum_i E^{1/2}(Y_a),$$

which means that $A$ is $6^K$-separating.

The argument above is due to Chang and immediately implies the Erdős-Szemerédi conjecture for small $K$.

**Theorem 5** (Chang, [3]). Assume $A \subset \mathbb{Z}$ with $|AA| \leq K|A|$. There is $c > 0$ such that

$$E_+(A) \leq c^K |A|^2.$$  

In particular there is $c > 0$ such that

$$|A + A| \geq c^{-K} |A|^2.$$  

**Proof.** Take $Y_a = \{1\}$ for each $a \in A$ in the argument above. The bound (10) follows from (9) by Cauchy-Schwarz. 

Chang’s theorem gives a non-trivial bound only in the regime $K \ll \log |A|$. Now we turn to the case when $K$ can be as large as some small power of $|A|$.  

Let us start with an heuristic argument which rests on somewhat unrealistic assumptions but reveals the structure of the upcoming proof. Assume that there is a way to decompose $\mathcal{P}(A)$ into a direct sum, such that

$$\mathcal{P}(A) = \mathcal{P}(A_1) \oplus \mathcal{P}(A_2)$$  

(11)
for some sets $A_1, A_2 \subset \mathbb{N}$. Since $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$ are orthogonal, we then have
\begin{equation}
|\mathcal{P}(A)| = |\mathcal{P}(A_1)||\mathcal{P}(A_2)|
\end{equation}
\begin{equation}
|\mathcal{P}(A) + \mathcal{P}(A)| = |\mathcal{P}(A_1) + \mathcal{P}(A_1)||\mathcal{P}(A_2) + \mathcal{P}(A_2)|
\end{equation}
If we define $K_1 := |A_1A_1|/|A_1|$ and $K_2 := |A_2A_2|/|A_2|$ then (12) and (13) give
\begin{equation}
K_1K_2|A| = K_1K_2|A_1||A_2| \leq |AA| \leq |A|
\end{equation}
so $K_1K_2 \leq K$.

Setting $|A| = N$, assume further that $|A_1| \approx |A_2| \approx N^{1/2}$ and, moreover, that $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$ can be iteratively decomposed further into direct sums in a similar way. In other words, we assume that for any $l \ll \log \log A$ there is a decomposition
\begin{equation}
\mathcal{P}(A) = \bigoplus_{i=1}^{2^l} \mathcal{P}(A_i)
\end{equation}
\begin{equation}
|A_i| \approx N^{1/2^l}
\end{equation}
which, iterating (14) and taking logarithms, gives
\begin{equation}
\sum_{i=1}^{2^l} \log K_i \leq \log K,
\end{equation}
where $K_i := |A_iA_i|/|A_i|$. Take $l = \lfloor \log \log K \rfloor$, fix an arbitrary (large) constant $C > 0$ and let
\begin{equation}
I := \{i : K_i > C\}.
\end{equation}
By (17) we have
\begin{equation}
|I| \leq \frac{1}{C} \log K,
\end{equation}
so the size of the set
\begin{equation}
A' := \prod_{i \in I} A_i
\end{equation}
is at most $N^{1/C}$ by (16). We can rewrite (16) as
\begin{equation}
A = \prod_{i \notin I} A_i A' = \bigcup_{a_i \in A_i} \left( \prod_{i \notin I} a_i \right) A',
\end{equation}
and an iterative application of Corollary 2.1 for $A_i, i \notin I$ gives (sum is over all elements in $\prod_{i \notin I} A_i$)
\begin{equation}
E^{1/2}(A) \leq \left( \prod_{i \notin I} 6^C \right) \sum_{A'} E^{1/2}(A') \leq K^{10C}|A'|^{3/2} \prod_{i \notin I} |A_i|.
\end{equation}
so
\begin{equation}
E(A) \leq K'^{20C}|A'||A|^2 \leq K'^{20C}N^{1/C}|A|^2.
\end{equation}
Taking $C > 0$ large enough we recover the claim of Theorem 1.

Of course, the assumption (11) is too strong to be true and one can easily come up with examples of sets which cannot be decomposed into a direct sum. However, in order to iterate Corollary 2.1 in (18) it suffices that the set $\mathcal{P}(A)$ "fibers" into sets with controlled separating constants, which is a much weaker assumption than (11).

The claim below illustrates this observation.

**Claim 2.2.** Assume that a set $A$ decomposes as

$$A = \bigcup_{b_i \in B} b_i C_i, \quad (19)$$

so that $(b_i, c_j) = 1$ for any $b_i \in B, c_j \in C_j$ (this means that $\mathcal{P}(b_i)$ and $\mathcal{P}(C_j)$ are orthogonal). Assume also that $B$ is $\psi_1$-separating and $C_i$ is $\psi_2$-separating for each $i$. Then $A$ is $\psi_1 \psi_2$-separating.

**Proof.** Let

$$Y = \bigcup_{a \in A} a X_a,$$

where $X_a$ are some sets with $(a, X_{a'}) = 1$ for $a, a' \in A$. Then we can write

$$Y = \bigcup_{b_i \in B} b_i \bigcup_{c_j \in C_i} c_j X_{i,j}.$$

Since $B$ is $\psi_1$-separating and each $C_i$ is $\psi_2$-separating we have

$$E^{1/2}(Y) \leq \psi_1 \sum_{b_i \in B} E^{1/2}(\bigcup_{c_j \in C_i} c_j X_{i,j}) \leq \psi_1 \psi_2 \sum_{i,j} E^{1/2}(X_{i,j}) = \psi_1 \psi_2 \sum_{a \in A} E^{1/2}(X_a),$$

and thus $A$ is $\psi_1 \psi_2$-separating. Note that we have used the fact

$$(b_i c_j, X_{i,j}) = 1$$

twice.

Note that the sets $C_i$ in (19) may depend on $b_i$, which makes it a more lax assumption than (11).

Another crucial ingredient in the model case above is the reduction of the doubling constants (14) which is then iterated $\approx \log \log K$ times so that most of the $K$’s shrink to the scale at which Corollary 2.1 gives non-trivial results. The next section is fully devoted to a technical lemma which is later used for a similar iterative scheme (see in particular (35)). It seems hard however to guarantee a full analog (14) to hold for arbitrary fibered sets decomposed as (19). Instead, Bourgain and Chang devised a scheme where the doubling constants are replaced with doubling constants along some graph of density $\delta$. By introducing an additional
parameter, the graph density $\delta$ (which, as one can check, never goes below $N^{-o(1)}$), they were able to close the induction.

3. Fibering Lemma

This section is devoted to the proof of a structural lemma which is a key ingredient in the inductive step. In fact, the proof works for subsets of linear spaces over any field $\mathbb{F}$. We assume the sets in question are subsets of $F^{[n]}$ with a coordinate basis $\{e_i\}_{i=1}^n$ which we assume fixed. For an index set $I \subset [n]$ there is a natural projection $\pi_I : F^{[n]} \to F^I$ which maps $(x_1, \ldots, x_n)$ to $\sum_{i \in I} x_i e_i$, that is, $\pi_I$ is the projection to the coordinates with indices in $I$.

When we add two elements $x \in F^I$ and $y \in F^J$ we treat them as elements in the ambient space $F^{I \cup J}$ filling the rest of coordinates with zeroes in the obvious way. When $I \cap J = \emptyset$ we write $x \oplus y$ for the sum to emphasize the orthogonality. This notation extends to sets in the obvious way.

**Definition 2** (Graph fibers). For a partition $I \cup J = [n]$ a bipartite graph $G \subset X \times Y \subset F^{[n]} \times F^{[n]}$ has a natural fibering

$$\bigcup_{(x,y) \in G} G_{x,y},$$

where the base graph $G_I$ is defined as

$$G_I \equiv \{(\pi_I(u), \pi_I(v)) : (u, v) \in G\} \subset \pi_I(X) \times \pi_I(Y)$$

and a fiber graph $G_{x,y} \subset \pi_J(X) \times \pi_J(Y)$ as

$$G_{x,y} \equiv \{(x', y') : (x \oplus x', y \oplus y') \in G\} \subset \pi_J(X) \times \pi_J(Y).$$

We will need another bit of notation to denote fibers. For a set $X$ and $x \in \pi_I(X)$ we write, following the original paper, $X(x)$ for the fiber over $x$. Namely, $X(x)$ is defined as

$$X(x) := \{x' \in \pi_I(X) : x \oplus x' \in X\}.$$

We will repeatedly use the following ”cheap regularity” lemma.

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1In fact, we need only the structure of a module. The results of this section will be applied later on only for subsets of $\mathbb{Z}^{[n]}$ (viewed as sets in the ambient linear space $\mathbb{Q}^{[n]}$). We have introduced $F$ here to emphasize that Lemma 2 works equally well when the ambient space is $\mathbb{F}_2^{[n]}$, say.
Lemma 1. Let $G$ be a graph on $X \times Y$ of size $\delta |X||Y|$. Then there exist $X' \subset X, Y' \subset Y$ and $G' \subset G$ such that

\begin{align*}
|N_{G'}(x)| & \geq \frac{\delta}{4} |Y| & (20) \\
|N_{G'}(y)| & \geq \frac{\delta}{4} |X| & (21) \\
|X'| & \geq \frac{\delta}{2} |X| & (22) \\
|Y'| & \geq \frac{\delta}{2} |Y| & (23) \\
|G'| & \geq \frac{\delta}{2} |X||Y| & (24)
\end{align*}

for any $x \in X', y \in Y'$. In particular, for any $A \subset X'$ and $B \subset Y'$

\begin{align*}
|(A \times Y') \cap G'| & \geq \frac{\delta}{4} |Y||A| & (25) \\
|(B \times X') \cap G'| & \geq \frac{\delta}{4} |X||B|. & (26)
\end{align*}

Proof. Remove from $X$ (resp $Y$) one by one all vertices with degree less than $\delta/4|Y|$ (resp. $\delta/4|X|$), until both $X$ and $Y$ contain only vertices of degree at least $\delta/2|Y|$ (resp. $\delta/4|X|$) in the remaining graph. Clearly, we cannot remove more than $\delta/2|X||Y|$ edges no matter how many vertices we remove. Take $X'$ and $Y'$ to be the sets of survived vertices in $X$ and $Y$ respectively and $G' := G \cap (X' \times Y')$. The bounds (22) and (23) follow immediately from (24).

Now we can formulate the main lemma of this Section. This is a key ingredient of the original proof and is of independent interest.

Lemma 2 (Finding a large subset with uniform fibers). Let $A_1, A_2 \subseteq F[n]$ be subsets of a linear space $V$ over a field $F$ of sizes $N_1, N_2$ respectively. Assume that for some $\delta > 0$ there is a graph $G \subset A_1 \times A_2$ with $|G| = \delta N_1 N_2$ such that

\[ |A_1 + A_2| \leq K N_1^{1/2} N_2^{1/2}. \]

Then for any partition $I \cup J = [n]$ there are sets $A'_1 \subset A_1, A'_2 \subset A_2$ and a subgraph $G' \subset G$ on $A'_1 \times A'_2$ with the following properties. There are numbers $M_1, m_1, M_2, m_2$ and absolute constants $c, C > 0$ with the properties below.

1. (Uniform fiber size) Define $M_1, M_2$ as

\[ |\pi_I(A'_1)| = M_1, \quad |\pi_I(A'_2)| = M_2. \quad (27) \]

There exist $m_1, m_2 > 0$ such that for any $x \in \pi_I(A'_1), y \in \pi_I(A'_2)$

\[ |A'_1(x)| \approx m_1, \quad |A'_2(y)| \approx m_2 \quad (28) \]

and
\[ M_1 m_1 \geq cN_1 \delta^2 \log^{-1}(K/\delta) \] (29)
\[ M_2 m_2 \geq cN_2 \delta^2 \log^{-1}(K/\delta) \] (30)
\[ m_1, m_2 \geq c\delta^{10} K^{-4} \max_{x \in \pi_I(A_1), y \in \pi_I(A_2)} \{|A_1(x)| + |A_2(y)|\}. \] (31)

(2) (Uniform graph fibering) There exist \( \delta_1, \delta_2 > 0 \) with
\[ \delta_1 \delta_2 > c \log^{-3}(\frac{K}{\delta})\delta. \] (32)
such that
\[ |G'_I| \geq \delta_1 M_1 M_2, \] (33)
and for any \((x, y) \in G'_I\)
\[ |G'_{(x,y)}| \geq \delta_2 m_1 m_2. \] (34)

(3) (Bounded doubling) There exist \( K_1, K_2 > 0 \) with
\[ K_1 K_2 \leq CK \log(K)\delta^{-2}. \] (35)
such that
\[ |\pi_I(A'_1) + \pi_I(A'_2)| = K_1(M_1 M_2)^{1/2} \] (36)
and for any \((x, y) \in G'_I\)
\[ |\pi_J(A'_1) + \pi_J(A'_2)| \approx K_2(m_1 m_2)^{1/2}. \] (37)

Proof. The proof proceeds in several steps. We will refine \( A_1, A_2 \) (thus abusing notation) such that eventually all the properties (1)-(3) are satisfied, while keeping track of the losses with respect to the quantities \( \delta, N_1, N_2 \) which are fixed and never updated. Without loss of generality we assume \( N_1 \geq N_2 \).

The constants \( c, C > 0 \) are always effective and absolute but may change in the course of the proof. One should think that \( c \) is ‘sufficiently small’ and \( C \) is ‘sufficiently large’ (though in principle one can evaluate suitable numerical values).
Applying Lemma 1 with $X = A_1, Y = A_2$ we assume that
\begin{align}
|A_1| &\geq \frac{\delta}{2}N_1 \\
|A_2| &\geq \frac{\delta}{2}N_2 \\
|A_1 \times A_2 \cap G| &\geq \frac{\delta}{2}N_1N_2 \\
\min_{x \in A_1} |x \times A_2 \cap G| &\geq \frac{\delta}{4}N_2 \\
\min_{y \in A_2} |A_1 \times y \cap G| &\geq \frac{\delta}{4}N_1.
\end{align}

**(Step 1.** (Regularizing the fibers of $A_2$))
Without loss of generality we assume that
\[n_1 := \max_{x \in \pi_I(A_1)} |A_1(x)| \geq \max_{y \in \pi_I(A_2)} |A_2(y)|.\]

Let $x \in \pi_I(A_1)$ such that $|A_1(x)| = n_1$. We have $|(x, A_1(x)) \times A_2 \cap G| \geq \frac{\delta}{4}|A_2|n_1$ so we choose using Lemma 1 a subset $A'_2 \subset A_2$ such that
\[|(x, A_1(x)) \times z \cap G| \geq \frac{\delta}{8}n_1\]
for any $z \in A'_2$. Also,
\begin{align}
|A'_2| &\geq \frac{\delta}{8}|A_2| \geq \frac{\delta^2}{16}N_2 \\
|A_1 \times A'_2 \cap G| &\geq \frac{\delta}{4}N_1|A'_2|,
\end{align}
since any vertex in $A'_2$ has degree at least $\frac{\delta}{4}N_1$.

We then claim that
\[|(x, A_1(x)) + A'_2| \geq \frac{\delta}{8}n_1|\pi_I(A'_2)|.\]

Indeed, let
\[\{z^{(i)}_I \oplus z^{(i)}_J\}_{i=1}^{\pi_I(A'_2)} \subset A'_2\]
be a collection of elements of size $|\pi_I(A'_2)|$ such that $z^{(i)}_I$ are all distinct. All the sums
\[(x, A_1(x)) + (z^{(i)}_I \oplus z^{(i)}_J) = (z^{(i)}_I + x) \oplus (z^{(i)}_J + A_1(x))\]
are distinct, and by (43) at least $\frac{\delta}{8}n_1|\pi_I(A'_2)|$ of them are in $(x, A_1(x)) + A'_2$.

Thus,
\[
\frac{K^2}{\delta} N_2 \geq K \sqrt{N_1 N_2} \\
\geq |A_1^G + A_2| \\
\geq |(x, A_1(x)) + A_2'| \geq \frac{\delta}{8} n_1 |\pi_1(A_2')|.
\] (47)

Now define
\[
\bar{A}_2 := \bigcup_{x \in \pi_1(A_2')} (x, A_2'(x)).
\] (48)

Clearly,
\[
|A_2' \setminus \bar{A}_2| \leq 10^{-4} \delta^5 K^{-2} n_1 |\pi_1(A_2')| \\
\leq 10^{-3} \delta^3 N_2 \leq \frac{\delta}{10} |A_2'|
\] (49)

so
\[
|A_1 \times \bar{A}_2 \cap G| \geq \frac{\delta}{4} N_1 |A_2'| - \frac{\delta}{10} N_1 |A_2'| \geq \frac{\delta}{10} N_1 |A_2'|
\] (50)

Now, by the dyadic pigeonhole principle there exists \(m_2\) with
\[
10^{-4} \delta^5 K^{-2} n_1 < m_2 < n_1
\] (51)

such that
\[
\bar{A}_2' := \bigcup_{m_2 \leq |A_2(x)| < 2m_2} (x, \bar{A}_2(x))
\] (52)

has size at least \(c \log^{-1} \left(\frac{K}{\delta}\right)|\bar{A}_2|\) for some \(c > 0\) (e.g. \(10^{-1}\) will do). We conclude
\[
|\bar{A}_2'| \geq c \frac{|\bar{A}_2|}{\log (K/\delta)} \geq c \frac{\delta^2}{\log (K/\delta)} N_2
\] (53)

and
\[
|A_1 \times \bar{A}_2' \cap G| \geq \frac{\delta}{4} N_1 |\bar{A}_2'|
\] (54)

\[
|\bar{A}_2'| \approx M_2 m_2,
\] (55)

with \(M_2\) defined as
\[
M_2 := |\pi_1(\bar{A}_2')|,
\] (56)

and \(m_2\) defined by (51), (52), (55).

**Step 2.** (Regularizing the fibers of \(A_1\))
Define $N_1' := |A_1|, N_2' := |\bar{A}_2'|$ and record
\[ N_1' \geq \frac{\delta}{2} N_1 \]  
\[ N_2' > c_2 \frac{\delta^2}{\log (K/\delta)} N_2 \]  

Let
\[ A_0 = \bigcup_{x \in \pi_I(A_1)} (x, A_1(x)). \]  

We want to show that
\[ |A_0 \times \bar{A}_2' \cap G| \leq \frac{\delta}{40} N_1 |\bar{A}_2'|. \]  

The argument is similar to the one of Step 1 with $n_1$ replaced by $m_2$. Assume
\[ |A_0 \times \bar{A}_2' \cap G| \geq \frac{\delta}{100} N_1 |\bar{A}_2'|. \]  

Then there is $\bar{y} \in \pi_I(\bar{A}_2')$ such that
\[ |A_0 \times (\bar{y}, \bar{A}_2'(\bar{y})) \cap G| \geq \frac{\delta}{100} N_1 m_2, \]  

since the vertex sets $\{(y, \bar{A}_2'(y)) : y \in \pi_I(\bar{A}_2')\}$ are disjoint and are of size $m_2$ each (within a factor of two). Next, let $A_0' \subset A_0$ be such that
\[ |z \times (\bar{y}, \bar{A}_2'(\bar{y})) \cap G| \geq \frac{\delta}{200} m_2. \]  

for each $z \in A_0'$. We clearly have
\[ |A_0'| \geq \frac{\delta}{200} N_1. \]  

Similarly to (47), denoting
\[ M := \max_{x \in \pi_I(A_0')} |A_0'(x)|, \]  

we write
\[ K N_1 \geq K \sqrt{N_1 N_2} \geq |A_1 \bigcup A_2| \geq |A_0' \bigcup (\bar{y}, \bar{A}_2'(\bar{y}))| \]  
\[ \geq \frac{\delta}{200} |\pi_I(A_0')| m_2 \geq \frac{\delta}{200} \frac{|A_0'|}{M} m_2 \]  
\[ \geq \frac{\delta^2}{4 \cdot 10^4} \frac{N_1 m_2}{M}, \]  

so
\[ M > 10^{-5} K^{-1} \delta^2 m_2, \]  

which contradicts (59).
If we now define

\[
\bar{A}_1 := \bigcup_{x \in \pi_I(A_1) \colon |A_1(x)| > 10^{-5} \delta^2 K^{-1} m_2} (x, A_1(x)),
\]

then by the preceding discussion and (54) we have

\[
|\bar{A}_1 \times \bar{A}_2' \cap G| \geq \frac{\delta}{8} N_1 |\bar{A}_2'|.
\]

Since \(10^4 K^2 \delta^{-5} m_2 > n_1 \geq |A_1(x)|\), by the dyadic pigeonhole principle (since the fibers are disjoint sets of vertices) there exist (the bounds below are somewhat weakened for easier bookkeeping)

\[
10^{-5} \delta^5 K^{-2} m_2 < m_1 < 10^5 K^2 \delta^{-5} m_2
\]

such that with

\[
\bar{A}_1' := \bigcup_{x \in \pi_I(A_1) \colon m_1 \leq |A_1(x)| < 2 m_1} (x, A_1(x)),
\]

and some \(c > 0\) (say \(10^{-2}\))

\[
|\bar{A}_1' \times \bar{A}_2' \cap G| \geq c \frac{\delta}{\log(K/\delta)} N_1 |\bar{A}_2'|.
\]

In particular,

\[
|\bar{A}_1'| \geq c \frac{\delta}{\log(K/\delta)} N_1.
\]

Finally, we define

\[
M_1 := |\pi_I(\bar{A}_1')|
\]

so that \(|\bar{A}_1'| \approx M_1 m_1\) and

\[
|\bar{A}_1'(x)| \approx m_1
\]

for each \(x \in \pi_I(\bar{A}_1')\).

**Step 3.** (Regularizing the graph fibers) We renew the definition of \(A_1 := \bar{A}_1'\) and \(A_2 := \bar{A}_2'\) so that

\[
|A_1 \times A_2 \cap G| \geq c \frac{\delta}{\log(K/\delta)} |A_1||A_2|
\]

\[
|A_1| \approx M_1 m_1 \geq c \frac{\delta}{\log(K/\delta)} N_1
\]

\[
|A_2| \approx M_2 m_2 \geq c \frac{\delta^2}{\log(K/\delta)} N_2
\]

and the fibers of \(A_1\) and \(A_2\) are approximately of size \(m_1\) and \(m_2\) respectively.
Recall that for \((x, y) \in \pi_I(A_1) \times \pi_I(A_2)\) we define the fiber graph \(G_{(x, y)}\) as
\[
G_{(x, y)} := \{(x', y') \in A_1(x) \times A_2(y) : (x \oplus x', y \oplus y') \in G\}.
\]
In particular, since we have regularized the fibers of \(A_1\) and \(A_2\), we have
\[
|G_{(x, y)}| \leq 4m_1 m_2.
\]
Let
\[
G'_I := \{(x, y) \in \pi_I(A_1) \times \pi_I(A_2) : |G_{(x, y)}| \geq \frac{c \delta}{16 \log(K/\delta)} m_1 m_2\}.
\]
It follows from (74) that
\[
\sum_{(x, y) \in G'_I} |G_{(x, y)}| \geq c \frac{\delta}{\log(K/\delta)} |A_1| |A_2|.
\]
By dyadic pigeonholing we can find \(\delta_2 \gg \frac{\delta}{\log(K/\delta)}\) such that with
\[
\tilde{G}'_I := \{(x, y) \in G'_I : \delta_2 m_1 m_2 \leq |G_{(x, y)}| < 2\delta_2 m_1 m_2\}.
\]
one has
\[
\sum_{(x, y) \in \tilde{G}'_I} |G_{(x, y)}| \geq c \frac{\delta}{\log^2(K/\delta)} |A_1| |A_2|.
\]
It then follows that
\[
|\tilde{G}'_I| \geq c \frac{\delta}{\delta_2 \log^2(K/\delta)} M_1 M_2.
\]

**Step 4.** (Regularizing the doubling constant)
Let \((x, y) \in \pi_I(A_1) \times \pi_I(A_2)\). We define
\[
K_+(G_{(x, y)}):= \frac{|A_1(x) + G_{(x, y)} A_2(y)|}{\sqrt{|A_1(x)||A_2(y)|}}
\]
to be the normalized doubling constant of the fibers along the fiber graph \(G_{(x, y)}\).
Define
\[
H := \{(x, y) \in \tilde{G}'_I : K_+(G_{(x, y)}) > C \log^3(K/\delta) \delta^{-10} K\}.
\]
We want to show that \(|H| < \frac{1}{10} |\tilde{G}'_I|\) provided \(C\) is large enough. Indeed, it is trivial that
\[
|\pi_I(A_1) + \pi_I(A_2)| \geq \frac{|H|}{\min(|\pi_I(A_2)|, |\pi_I(A_1)|)} \geq \frac{|H|}{\sqrt{M_1 M_2}}.
\]
so

\[
K \sqrt{N_1 N_2} \geq |A_1^G + A_2| \quad (83)
\]

\[
\geq \min_{(x,y) \in H} |A_1(x) + A_2(y)||\pi_I(A_1) + \pi_I(A_2)| \quad (84)
\]

\[
> C \log^3(K/\delta) \delta^{-10} K \sqrt{|A_1(x)||A_2(y)| H / \sqrt{M_1 M_2}} \quad (85)
\]

\[
\geq C \log^3(K/\delta) \delta^{-10} K \sqrt{m_1 M_1 m_2 M_2} \frac{|H|}{M_1 M_2} \quad (86)
\]

\[
\geq cCK \sqrt{N_1 N_2} \frac{|H|}{|G_I^G|}. \quad (87)
\]

Thus, by taking \( C \) large enough we can ensure that

\[
|H| \leq \frac{1}{10} |G_I^G| . \quad (88)
\]

Denote

\[
\bar{G}_I'' := \bar{G}_I' \setminus H . \quad (89)
\]

By dyadic pigeonholing there is \( K_2 \leq C \log^3(K/\delta) \delta^{-10} K \) such that

\[
|\{(x, y) \in \bar{G}_I' : K_2 \leq K_+(G_{(x,y)}) < 2K_2\}| \geq c \frac{|\bar{G}_I''|}{\log(K/\delta)} . \quad (90)
\]

Let \( G_{1,0} \subset \bar{G}_I'' \subset \pi_I(A_1) \times \pi_I(A_2) \) be the graph defined by such pairs. We then have

\[
K \sqrt{N_1 N_2} \geq |\pi_I(A_1)^{G_{1,0}} + \pi_I(A_2)| K_2 \sqrt{m_1 m_2} \quad (91)
\]

\[
\geq K_+(G_{1,0}) K_2 \sqrt{m_1 M_1 m_2 M_2} \quad (92)
\]

\[
\geq CK_+(G_{1,0}) K_2 \delta^{3/2} \sqrt{N_1 N_2} / \log(K/\delta) . \quad (93)
\]

so

\[
K_+(G_{1,0}) K_2 \leq c \delta^{-3/2} \log(K/\delta) K < C \delta^{-2} \log(K) K . \quad (94)
\]

It remains to sum up what we have achieved. By (81), (90) we have

\[
|\pi_I(A_1)| = M_1 \quad (95)
\]

\[
|\pi_I(A_2)| = M_2 \quad (96)
\]

\[
G_{1,0} \subset \pi_I(A_1) \times \pi_I(A_2) \quad (97)
\]

\[
|G_{1,0}| \geq c \frac{\delta}{\delta_2 \log^4(K/\delta)} M_1 M_2 \quad (98)
\]
For any \((x, y) \in G_{1,0}\) we have

\[
A_1(x) \approx m_1 \\
A_2(y) \approx m_2 \\
|G_{(x,y)}| \approx \delta_2 m_1 m_2
\]

\[
|A_1(x) + A_2(y)| \approx K_2 \sqrt{m_1 m_2}.
\]

with

\[
m_1 M_1 \geq c \frac{\delta}{\log(K/\delta)} N_1
\]

\[
m_2 M_2 \geq c \frac{\delta^2}{\log(K/\delta)} N_2
\]

Defining \(\delta_1 := c \delta_{\log^3(K/\delta)}\) and \(K_1 := K_+(G_{1,0})\) we also have

\[
\delta_1 \delta_2 > c \log^{-3}(K/\delta) \delta
\]

\[
K_1 K_2 \leq C K \log(K) \delta^{-2}.
\]

Finally, define

\[
G' := \{(x \oplus x', y \oplus y') : (x, y) \in G_{1,0}; (x', y') \in G_{(x,y)}\}
\]

so that \(G'_I = G_{1,0}\) and the proof is finished.

\[\square\]

4. Iteration scheme

In this section we will use Lemma 2 in order to setup an iteration scheme. At each step we have a pair of sets \((A_1, A_2)\) which correspond to a pair of additive sets \((A'_1, A'_2) := (\mathcal{P}(A_1), \mathcal{P}(A_2))\) and a graph \(G\) on \(A_1 \times A_2\), together with the data \((N, \delta, K)\) such that:

\[
|A_1||A_2| = N
\]

\[
|A_1 + A_2| \leq KN^{1/2}
\]

\[
|G| \geq \delta N
\]

Apart from that, the setup above is equipped with a pair of functions \(\psi(N, \delta, K)\), \(\phi(N, \delta, K)\) (which are called admissible in the original paper). The rather technical definition is below.

**Definition 3 (Admissible pair of functions).** A pair of functions \(\psi(N, \delta, K), \phi(N, \delta, K)\) is admissible if for arbitrary sets \(A_1, A_2 \subset \mathbb{Z}^n\) and a graph \(G\) on \(A_1 \times A_2\) satisfying (107)-(109) the following holds.

There is a graph \(G' \subset G\) such that
(i) Graph size is controlled by $\phi$:

$$|G'| \geq \phi(N, \delta, K)$$

(ii) Separation of $G'$-neighborhoods is controlled by $\psi$:

For any $a_1 \in A_1$ (resp. $a_2 \in A_2$) the $P$-preimage of the $G'$-neighborhood

$$P^{-1}[G'(a_1)] := P^{-1}\{a_2 \in A_2 : (a_1, a_2) \in G'\}.$$ 

(resp. of $G'(a_2)$) is $\psi(N, \delta, K)$-separating.

Note that by Claim 2.1, the pair $\psi(N, \delta, K) := N$; $\phi(N, \delta, K) := \delta N$ is trivially admissible with much room to spare.

The following lemma gives a Freiman-type pair of admissible functions which is better than non-trivial in the regime $K \ll \log N$ and will be used later to bootstrap the argument.

**Lemma 3** (Freiman-type admissible functions). There is an absolute constant $C > 0$ such that the pair of functions

$$\psi(N, \delta, K) := \min\{e^{(\frac{K}{\delta})C}, N\} \quad (110)$$

$$\phi(N, \delta, K) := \left(\frac{\delta}{K}\right)^{C} N \quad (111)$$

is admissible.

**Proof.** By the setup, we are given two sets $A_1, A_2$ of sizes $N_1, N_2$ and a graph $G$ of size $\delta N_1 N_2$ such that

$$|A_1^G + A_2| \leq K \sqrt{N_1 N_2} \quad (112)$$

Assume wlog that $N_1 \geq N_2$ and take $A := A_1 \cup A_2$, which is of size $\approx N_1$. Since by (112)

$$\frac{K^2}{\delta^2} N_2 \geq N_1$$

we have

$$|G| \gg \frac{\delta^3}{K^2} |A|^2$$

and

$$|A^G + A| \ll K |A|.$$ 

By a variant of the Balog-Szemeredi-Gowers theorem (see e.g. [5], Exercise 6.4.10) there is $A' \subset A$ such that $|A' + A'| < K'|A'|$ and $|G \cap (A' \times A')| > \delta' N_1^2$ with

$$\delta' > \left(\frac{\delta}{K}\right)^{C} \quad (113)$$

$$K' < \left(\frac{K}{\delta}\right)^{C} \quad (114)$$
By Theorem 4 any subset of $A'$ has rank at most $K'$ and by Corollary 2.1, the $P$-preimage of any subset of $A'$ is at most $c^{CK'}$-separating for some $C > 0$. Thus, taking $G' := G \cap (A' \times A')$ by (113) and (114) we verify that the pair (110), (111) is admissible.

The goal is to find a better pair of admissible functions. The lemma below implements the ‘induction on scales’ approach, which allows one to cook up a new pair $\phi_*(N, \cdot, \cdot), \psi_*(N, \cdot, \cdot)$ from a given pair of admissible functions, but taken at the smaller scale $\approx N^{1/2}$.

**Lemma 4.** Let $\psi$ and $\phi$ be an admissible pair of functions. Then for some absolute constant $C > 0$ the pair of functions

\[
\begin{align*}
\psi_*(N, \delta, K) &:= \max \{ \psi(N', \delta', K') \psi(N'', \delta'', K'') \} \\
\phi_*(N, \delta, K) &:= \min \{ \phi(N', \delta', K') \phi(N'', \delta'', K'') \},
\end{align*}
\]

is admissible. Here $\min$ and $\max$ is taken over the data $(N', \delta', K'), (N'', K'', \delta'')$ such that

\[
\begin{align*}
c\delta^7 \log^{-22} (K/\delta) N &\leq N'N'' N \\
N' + N'' &\leq C\delta^{-45} K^{11} N^{1/2} \\
K'K'' &\leq C\frac{K \log K}{\delta^{11}} \\
\delta'\delta'' &\geq c \log^{-6} (K/\delta) \delta.
\end{align*}
\]

**Proof.** Let $A_1, A_2 \subset \mathbb{Z}^{[n]}$ of sizes $N_1, N_2$ respectively, $G \subset A_1 \times A_2$ and suppose that the conditions (107)-(109) are satisfied with parameters $(N := N_1N_2, \delta, K)$. Our ultimate goal is to find a subgraph $G' \subset G$ of size at least $\phi_*(N, \delta, K)$ such that the $P$-preimage of any its neighbourhood is $\psi_*(N, \delta, K)$-separating. In order to achieve this, we will apply Lemma 2 and then use the hypothesis that the pair $\psi, \phi$ is admissible for much smaller sets.

Define a function $f(t)$ for $0 \leq t \leq n$ as

\[f(t) := \max \{|A_1(x)| + |A_2(y)|\}\]

where the maximum is taken over all $x \in \pi_{[t]}(A_1), y \in \pi_{[t]}(A_2)$. Clearly $f$ is decreasing, $f(0) = |A_1| + |A_2| \geq N^{1/2}$ and $f(n) = 0$ so there is $t'$ such that

\[f(t') \geq N^{1/4}\]

but

\[f(t' + 1) < N^{1/4}\]
together with a graph \( G' \subset A'_1 \times A'_2 \) such that

\[
A'_1 = \bigcup_{x \in \pi_I(A'_1)} (x', A'_1(x'))
\]

\[
A'_2 = \bigcup_{y \in \pi_I(A'_2)} (y', A'_2(y'))
\]

and the fibers \( A'_1(x'), A'_2(y') \) together with the fiber graphs \( G'_{x,y} \) are uniform as defined in the claim of Lemma 2. Note that it is possible that \( t' = 0 \), in which case the sets split trivially with \( \pi_I(A'_1) = \pi_I(A'_2) = \emptyset \).

Using the notation of Lemma 2 we have

\[
|\pi_I(A'_1) - \pi_I(A'_2)| \leq K_1(M_1M_2)^{1/2}.
\]

Since \( \phi, \psi \) is an admissible pair, there is \( G'' \subset G' \) of size at least \( \phi(M_1M_2, \delta_1, K_1) \) such that all \( \mathcal{P} \)-preimages of its vertex neighbourhoods are \( \psi(M_1M_2, \delta_1, K_1) \)-separating. Next, for each edge \( (x, y) \in G'' \), since it’s a subgraph of \( G' \), there is a graph \( G'_{x,y} \subset \pi_I(A'_1) \times \pi_I(A'_2) \) such that

\[
|A'_1(x) + A'_2(y)| \leq K_2(M_1M_2)^{1/2}
\]

Again, by admissibility of \( \phi, \psi \), there is \( G''_{(x,y)} \subset G'_{(x,y)} \) of size at least \( \phi(m_1m_2, \delta_2, K_2) \) such that all \( \mathcal{P} \)-preimages of its vertex neighbourhoods are \( \psi(m_1m_2, \delta_2, K_2) \)-separating.

Now define \( G'' \subset G \cap A'_1 \times A'_2 \) as

\[
G'' := \{(x \oplus x', y \oplus y') : (x, y) \in G'', (x', y') \in G'_{(x,y)}\}
\]

It is clear by construction that indeed all vertices of \( G'' \) belong to \( A'_1 \) and \( A'_2 \) respectively. Moreover, we have

\[
|G''| \geq \phi(M_1M_2, \delta_1, K_1)\phi(m_1m_2, \delta_2, K_2).
\]

Now let’s estimate the separating constant for the \( \mathcal{P} \)-preimage of a neighbourhood \( \mathcal{P}^{-1}[G''(u)] \) of some \( u \in V(G'') \). Without loss of generality assume that \( u \in A'_2 \) and \( u = v \oplus v' \). We can write

\[
G''(u) = \bigcup_{x \in G''(v)} \bigcup_{x' \in G''_{(x,v)}(v')} x \oplus x'
\]

Thus,

\[
\mathcal{P}^{-1}[G''(u)] = \bigcup_{x \in \mathcal{P}^{-1}[G''(v)]} \left\{ \bigcup_{x' \in \mathcal{P}^{-1}[G''_{(x,v)}(v')]} x' \right\}.
\]

Since \( G''_{(x,v)}(v') \) and \( G''_{(x,v)}(v') \) are orthogonal as linear sets we conclude that \( (x, x') = 1 \) for \( x \in \mathcal{P}^{-1}[G''(v)] \) and \( x' \in \mathcal{P}^{-1}[G''_{(x,v)}(v')] \). Thus, by Claim 2.2 and the
admissibility of $\phi, \psi$ applied to $G''_I$ and $G''_{(x,v)}$ we conclude that $P^{-1}[G''(u)]$ is at most $\psi(M_1 M_2, \delta_1, K_1)\psi(m_1 m_2, \delta_2, K_2)$-separating.

We now record the bounds for $\delta_2, K_2, m_1, m_2$ given by Lemma 2. We have

\begin{align*}
\delta_1 \delta_2 & > c \log^{-3}(K/\delta)\delta. \quad (127) \\
K_1 K_2 & < CK \log K \delta^{-2} \quad (128) \\
M_1 m_1 & > c \delta^2 \log^{-1}(K/\delta) N_1 \quad (129) \\
M_2 m_2 & > c \delta^2 \log^{-1}(K/\delta) N_2 \quad (130) \\
M_1 M_2 & < \frac{N_1 N_2}{m_1 m_2} < c \delta^{-20} K^8 N^{1/2}. \quad (132)
\end{align*}

In particular, we have

$M_1 M_2 < \frac{N_1 N_2}{m_1 m_2} < c \delta^{-20} K^8 N^{1/2}.$

We have thus verified the claim of the Lemma save for the fact that $m_1 m_2$ may well be larger than $N^{1/2}$. In order to further reduce the size we apply Lemma 2 again for each pair of sets $(A'_1(x), A'_2(y))$ such that $(x, y) \in G'_I$, stripping off only a single coordinate as explained below. Assume the base point $(x, y)$ is fixed henceforth.

Remark 4.1. The reader may keep in mind the following model case: $N_1 = N_2 = N^{1/2}$ and $A_1, A_2$ are three-dimensional arithmetic progressions

$$\left\{ [0, N^\alpha_x] e_1 + [0, N^{1-2\alpha}_1] e_2 + [0, N^\alpha_1] e_3 \right\},$$

for some small $\alpha > 0$. In this case $t' = 1, M_1 M_2 \approx N^\alpha$ and $m_1 m_2 \approx N^{1-\alpha}$. However, the fibers $A_1(x)$ and $A_2(y)$ are heavily concentrated on the second coordinate which makes the separating constant of the $P$-preimage small.

We split the coordinates $J = \{t' + 1, \ldots, n\}$ into

$$I' = \{t' + 1\}$$

and

$$J' = \{t' + 2, \ldots, n\}$$

and apply Lemma 2 for such a decomposition, the pair $(A'_1(x), A'_2(y))$ and the graph $G'_{(x,y)}$. We then get

$$A'_1 \subset A'_1(x), \ A''_1 \subset A'_2(y)$$

such that

\begin{align*}
A''_1 = \bigcup_{w \in \pi_{J'}(A''_1)} \{w, A''_1(w)\} \quad (133) \\
A''_2 = \bigcup_{z \in \pi_{J'}(A''_2)} \{z, A''_2(z)\} \quad (134)
\end{align*}
and the fibers $A_1''(w)$ and $A_2''(z)$ are of approximately the same size $l_1$ and $l_2$ respectively. Note again, that, say, the fiber $A_1''(w)$ may be trivial (e.g. $\{0\}$), which simply means that $l_1 \approx 1$.

Next, we have a graph $K \subset A_1'' \times A_2''$ with uniform fibers as defined in Lemma 2. Note that $l_1, l_2$ and $K$ depend on the base point $(x, y)$ which we assume is fixed.

The graph $K$ splits into the one-dimensional base graph $K' \subset \pi_{1'}(A_1'') \times \pi_{1'}(A_2'')$ and fiber graphs $K_{w,z}$ such that for $(w, z) \in K'_{1''}$

$$|A_1''(w) + K_{w,z} A_2''(z)| \leq K_3(l_1 l_2)^{1/2},$$

(135)

with

$$|A_1''(w)| \approx l_1$$

(136)

$$|A_2''(z)| \approx l_2$$

(137)

$$|K_{w,z}| \geq \delta_3 l_1 l_2.$$

(138)

The parameters $l_1, l_2, \delta_3, K_3$ as well as the sizes of $K_{1''}$ and $K_{w,z}$ are controlled by Lemma 2. By induction hypothesis, for each such a graph $K_{w,z}$ there is a subgraph $K'_{w,z} \subset K_{w,z}$ with

$$|K'_{w,z}| \geq \phi(l_1 l_2, \delta_3, K_3)$$

(139)

such that the $\mathcal{P}$-preimage of each neighborhood of $K'_{w,z}$ is $\psi(l_1 l_2, \delta_3, K_3)$-separating. Define $K' \subset K$ as

$$K' := \{(w \oplus w', z \oplus z') : (w, z) \in K_{1''}, (w', z') \in K_{w,z}'\}.$$  

(140)

The size of $K'$ is clearly at least $|K_{1''}| \phi(l_1 l_2, \delta_3, K_3)$. Next, the set of vertices of $K_{1''}$ all lie in a one-dimensional affine subspace, so combining (8) and Claim 2.2 one concludes that the $\mathcal{P}$-preimage of each neighborhood of $K'$ is $C \psi(l_1 l_2, \delta_3, K_3)$-separating with some absolute constant $C > 0$ (for the additive energy one can take $C = 6$ as in (8)). Summing up, we conclude that

$$\psi_{x,y} := C \psi(l_1 l_2, \delta_3, K_3); \quad \phi_{x,y} := |K_{1''}| \phi(l_1 l_2, \delta_3, K_3)$$

(141)

is admissible for the pair of sets $(A_1'(x), A_2'(y))$ and the graph $G_{x,y}$. In turn, substituting $\psi_{x,y}$ and $\phi_{x,y}$ into the argument leading to (124) and Claim 2.2, one concludes that

$$\psi(M_1 M_2, \delta_1, K_1) \max_{(x,y) \in G_{1'}} \psi_{x,y}; \quad \phi(M_1 M_2, \delta_1, K_1) \min_{(x,y) \in G_{1'}} \phi_{x,y}$$

(142)

is admissible for $(A_1, A_2)$ and the graph $G$. It remains to check that the quantities (142) can indeed be bounded as (115). By saying that (142) is admissible we mean that we can find a subgraph of $G$ of size at least

$$\psi(M_1 M_2, \delta_1, K_1) \cdot \max_{(x,y) \in G_{1'}} \psi_{x,y}.$$
such that the separating factors are most
\[
\phi(M_1 M_2, \delta_1, K_1) \cdot \min_{(x,y) \in \mathcal{G}_I'} \phi_{x,y}.
\]

Note that the quantities (142) do depend on the structure of \(A_1, A_2\). We are going to show, however, that they are uniformly bounded by (115) which are functions of \((N, \delta, K)\) only.

First, since \((x, y) \in \mathcal{G}_I'\) we have by (32)
\[
\delta_3 > c \log^{-3}(K_2/\delta_2)\delta_2.
\] (143)

By (35) and (32)
\[
\frac{K_2}{\delta_2} < \frac{C K \log(K) \log^3(K/\delta)}{\delta^3}
\] (144)
so
\[
\log(K_2/\delta_2) < C \log(K/\delta).
\] (145)

and
\[
\delta_1\delta_3 > c \log^{-3}(K/\delta)\delta_1\delta_2 > c \log^{-6}(K/\delta)\delta.
\] (146)

Next, by (35)
\[
K_3 \leq C K_2 \log(K_2)\delta_2^{-2}.
\] (147)
and by (32)
\[
\delta_2 > c \log^{-3}(K/\delta)\delta
\] (148)
\[
K_2 < C\delta^{-2}K \log K
\] (149)
so
\[
\log(K_2)\delta_2^{-2} \leq C \log^7(K/\delta)\delta^{-2}
\] (150)
\[
= C(\delta^7 \log^7(K/\delta))\delta^{-9} < C \log^7 K\delta^{-9}
\]
and
\[
K_1 K_3 \leq C K_1 K_2 \log(K_2)\delta_2^{-2} \quad \leq \quad C \frac{K \log^8 K}{\delta^{11}}
\] (151)

We turn to \(|\mathcal{K}_I'|\) and \(l_1 l_2\). We have by (29), (32), (33), (34) that
\[
|\mathcal{K}_I'| l_1 l_2 \geq c \log^{-3}(K_2/\delta_2)\delta_2(\delta_2^3 \log^{-2}(K_2/\delta_2))|A_1(x)||A_2(y)|
\geq c \log^{-5}(K/\delta)\delta_2^5 m_1 m_2
\] (148)
\[
\geq c \log^{-20}(K/\delta)\delta^5 m_1 m_2
\] (152)

Define
\[
N'' := \min\{N^{1/2}, \max\{l_1 l_2, c \log^{-20}(K/\delta)\delta^5 m_1 m_2\}\}
\] (153)
By our choice of $t'$ it follows that $l_1 l_2 \leq N''$. Thus, we may assume
\[
\frac{l_1 l_2}{N''} \phi(N'', \delta_3, K_3) \leq \phi(l_1 l_2, \delta_3, K_3).
\] (154)

Indeed, the function $\phi(\cdot, \delta_3, K_3)$ may always be taken sublinear since one can take a sparser graph if needed. For the same reason, defining
\[
N' := \frac{M_1 M_2 l_1 l_2}{N''} |K_{I'}|
\] (155)
we have by (152) that $M_1 M_2 \leq N'$ and
\[
\frac{M_1 M_2}{N'} \phi(N', \delta_1, K_1) \leq \phi(M_1 M_2, \delta_1, K_1),
\] (156)
so
\[
\phi(N', \delta_1, K_1) \phi(N'', \delta_3, K_3) \leq \frac{N'}{M_1 M_2} \phi(M_1 M_2, \delta_1, K_1) \frac{N''}{l_1 l_2} \phi(l_1 l_2, \delta_3, K_3)
\]
\[
= \phi(M_1 M_2, \delta_1, K_1) \phi_x(y) \] (157)

On the other hand,
\[
N' N'' = M_1 M_2 l_1 l_2 |K_{I'}| \geq c \log^{-20} (K/\delta) \delta^5 M_1 M_2 m_1 m_2
\] (158)
\[
\geq c \delta^7 \log^{-22} (K/\delta) N. \] (159)

Also, since
\[
m_1 m_2 > c \delta^{20} K^{-10} N^{1/2},
\]
it follows that
\[
c \delta^{45} K^{-11} N^{1/2} \leq N'' \leq N^{1/2}
\] and, since $N'' N' \leq N$,
\[
N' \leq C \delta^{-45} K^{11} N^{1/2},
\]
so
\[
N' + N'' \leq C \delta^{-45} K^{11} N^{1/2}. \] (160)

We now have all the estimates to finish the proof. The bounds (146), (151), (158), (160) verify that the parameters
\[
\delta' := \delta_1, \quad \delta'' := \delta_3
\]
\[
K' := K_1, \quad K'' := K_3
\] (161)

\[\text{In what follows } \phi(N, \delta, K) \text{ eventually will be of the form } N^{1-\tau + o(1)} \text{ for some } \tau > 0.\]
and $N', N''$ indeed satisfy the constraints (117). Next, it is trivial that $\psi(\cdot, \delta, K)$ can be taken monotone increasing in the first argument, so by (132) and (119)

$$\psi_*(N, \delta, K) \geq C\psi(\max\{N^{1/2}, \frac{N}{m_1m_2}\}, \delta_1, K_1)\psi(\min\{N^{1/2}, m_1m_2\}, \delta_3, K_3) \geq \psi(M_1M_2, \delta_1, K_1)\psi_{x,y}(162)$$

Also, (157) and (142) verify that

$$\phi_*(N, \delta, K) \leq \phi(N', \delta_1, K_1)\phi(N'', \delta_3, K_3) \leq \phi(M_1M_2, \delta_1, K_1)\phi_{x,y}. \quad (163)$$

It follows that the pair $(\psi_*, \phi_*)$ is indeed admissible since (162) and (163) hold for all base points $(x, y) \in G'_I$ and thus uniformly bound (142). □

5. A Better Admissible Pair

With Lemma 4 at our disposal we can start with the data $(N, \delta, K)$ and reduce the problem to the case of smaller and smaller $N$ and $K$ with reasonable losses in $\delta$. The process can be described by a binary tree where each node with the data $(N, \delta, K)$ splits into two children with the attached data being approximately equal $(N^{1/2}, \delta', K')$ and $(N^{1/2}, \delta', K'')$, with $K'K''$ roughly equal to $K$ and $\delta'\delta''$ roughly equal to $\delta$. Thus, when the height of the tree is about $\log \log K$, the $K$’s in the most of the nodes should be small enough so that Lemma 3 becomes non-trivial. Going from the bottom to the top we then recover an improved admissible pair of functions at the root node.

Lemma 5. For any $\gamma > 0$ there exists $C(\gamma) > 0$ such that the pair

$$\phi(N, \delta, K) := \left(\frac{\delta}{K}\right)^{C\log \log (K/\delta)} N \quad (164)$$

$$\psi(N, \delta, K) := \exp(\log (K/\delta)^{C/\gamma}) N^{\gamma} \quad (165)$$

is admissible.

Proof. Take an integer $t = 2^l$ to be specified later ($l$ is going to be the height of the tree and $t$ the total number of nodes). Each node then has an index $\nu \in \{0, 1\}^l$. We start with an admissible pair $\phi_0, \psi_0$ given by Lemma 3 at the bottom-most level. Going recursively from the leaves to the root, we have by Lemma 4 that for the levels $i = 1, \ldots, l$ the pairs

$$\psi_i := \max C\psi_{i-1}(N', \delta', K')\psi_{i-1}(N'', \delta'', K'') \quad (166)$$

$$\phi_i := \min \phi_{i-1}(N', \delta', K')\phi_{i-1}(N'', \delta'', K''), \quad (167)$$
are admissible (with the max and min taken over the set of parameters constrained by (117)). Thus, at the root node we have the admissible pair $\psi := \psi_{l-1}, \phi := \phi_{l-1}$ given by

$$
\psi(N, \delta, K) := C^{2l} \prod_{\nu \in \{0,1\}^l} \psi_0(N_\nu', \delta_\nu', K_\nu')
$$

$$
\phi(N, \delta, K) := \prod_{\nu \in \{0,1\}^l} \phi_0(N_\nu, \delta_\nu, K_\nu)
$$

for some data $(N_\nu, \delta_\nu, K_\nu)$ and (possibly different) $(N'_\nu, \delta'_\nu, K'_\nu)$ at the leaf nodes of the tree which attain the respective maxima and minima. For non-leaf tree nodes $\nu$, denoting by $\left\{\nu, 0\right\}$ and $\left\{\nu, 1\right\}$ the left and right child of $\nu$ respectively, one has

$$
c \delta_\nu^7 \log^{-22}(K_\nu/\delta_\nu) N_\nu \leq N_{\nu,0} N_{\nu,1} \leq N_\nu
$$

$$
N_{\nu,0} + N_{\nu,1} \leq C \delta_\nu^{-45} K_\nu^{11} N_\nu^{1/2}
$$

$$
K_{\nu,0} K_{\nu,1} \leq C K_\nu \log^8 K_\nu \quad \delta_{\nu,0} \delta_{\nu,1} \geq c \log^{-6}(K_\nu/\delta_\nu) \delta_\nu.
$$

and similarly for $(N'_\nu, \delta'_\nu, K'_\nu)$.

In what follows we assume that $N$ is large enough so that $\log K_\nu > C$ and $\log(\delta_\nu^{-1}) > c^{-1}$ and the constants $C, c$ can be swallowed by an extra power of $\log(K/\delta)$.

We have

$$
\log \frac{K_{\nu,0}}{\delta_{\nu,0}} + \log \frac{K_{\nu,1}}{\delta_{\nu,1}} < 15 \log \frac{K_\nu}{\delta_\nu}
$$

so for an arbitrary $1 < l' \leq l$

$$
\max_{\nu \in \{0,1\}^{l'}} \log \frac{K_\nu}{\delta_\nu} < 15' \log \frac{K}{\delta}.
$$

Next, it follows (see the original paper for more details) from (174) that

$$
\prod_{\nu \in \{0,1\}^{l'}} \delta_\nu > 15^{-6l'}2^{-l'} \log \left(\frac{K}{\delta}\right)^{-6l'} \delta
$$

and

$$
\prod_{\nu \in \{0,1\}^{l'}} K_\nu < 15^{50l'2} \log \left(\frac{K}{\delta}\right)^{50l'2} \delta^{-7l'} K
$$

and

$$
\prod_{\nu \in \{0,1\}^{l'}} N_\nu > 15^{-100l'2} \left(\log \frac{K}{\delta}\right)^{-200l'} \delta^{30l'} N.
$$
Taking \( l := \lfloor \log \log (K/\delta) \rfloor \)
and substituting (175), (176), (177) into (169) and (111) we get
\[
\prod_{\nu \in \{0, 1\}^l} \left( \frac{\delta_{\nu}}{K_{\nu}} \right)^C N_{\nu} \geq \left( \frac{\delta}{K} \right)^{C \log \log (K/\delta)} N
\]
for some suitable \( C > 0 \). The elementary but a bit tedious calculations can be found in the original paper. We note however, that it is natural to expect that the resulting function should look like (164). Ignoring \( \delta \)'s, we loose at each node at most by a multiplicative factor of \( \log^{-C} K \), totaling to the \( (\log K)^{-C^2 l} \) loss, which is approximately \( K^{-C \log \log K} \).

We now turn to \( \psi \). Again, we omit the details but only sketch the main idea of the calculation. For the sake of notation we use again \( (N_{\nu}, \delta_{\nu}, K_{\nu}) \) instead of \( (N'_{\nu}, \delta'_{\nu}, K'_{\nu}) \). The bounds above, however, still hold.

We split the data \( (N_{\nu}, \delta_{\nu}, K_{\nu}) \) into two parts, \( I \cup J = \{0, 1\}^l \), such that
\[
\frac{K_{\nu}}{\delta_{\nu}} < A : \nu \in I
\]
and
\[
\frac{K_{\nu}}{\delta_{\nu}} \geq A : \nu \in J,
\]
with the threshold \( A \) specified in due course.

By (175) and (176) it is easy to see that \( |J| \) is rather small:
\[
A^{|J|} \leq \prod_{\nu \in \{0, 1\}^l} \frac{K_{\nu}}{\delta_{\nu}} < 15^{100^l 2^l} \log \frac{K}{\delta} \delta^{-10l} K. \tag{178}
\]

Set \( t := 2^l \) and take
\[
\log A := 10^3 \gamma^{-1} t = \frac{10^3 \log \log (K/\delta)}{\gamma}.
\]

It follows from (178) that
\[
|J| \log A \leq 500tl + 100tl + 10tl + t < 10^3 tl,
\]
so
\[
\frac{|J|}{t} < \frac{10^3 l}{\log A} = \gamma.
\]

By (171) we have that at the bottom level each \( N_{\nu} \approx N^{\frac{1}{2}} \), so we can estimate by Lemma 3 (ignoring the logarithmic losses at each node, which one checks are acceptable)
\[
\psi \leq C^t \prod_{\nu \in \{0, 1\}^l} \min \{ e^{(\frac{K_{\nu}}{\delta_{\nu}})^C}, N_{\nu} \} \approx e^{|I| A^C N^{\frac{|I|}{t}}} < e^{|A^C N^{\frac{|J|}{t}}} < \exp (\log (K/\delta)^{C/\gamma}) N^\gamma.
\]
6. A strong admissible pair

Finally, in this section we will use Lemma 5 to get an even better pair of admissible functions.

**Lemma 6.** Given $0 < \tau, \gamma < 1/2$ there exist positive constants $A_i(\tau, \gamma), B_i(\tau, \gamma), i = 1, 2, 3$ such that

\[
\phi := K^{-A_1} \delta A_2 \log \log N e^{A_3(\log \log N)^2} N^{1-\tau} \tag{179}
\]

\[
\psi := K^{B_1} \delta B_2 \log \log N e^{-B_3(\log \log N)^2} N^\gamma \tag{180}
\]

are admissible.

**Proof.** The strategy of the proof is as follows. We start with already a not-so-bad admissible pair given by Lemma 5 and improve it by repeated application of Lemma 4.

The idea is that first we find a fixed threshold $\bar{N}(\tau, \gamma)$ such that the pair (179), (180) is either trivial or worse than that given by Lemma 5 if $N \leq \bar{N}$. One can achieve this by fine-tuning the constants $A_1, B_1$.

After such a bootstrapping we have an intermediate admissible pair, say $(\phi', \psi')$, defined by (179), (180) if $N \leq \bar{N}$ and by (164), (165) otherwise.

Next, we use Lemma 4 and prove that for a suitable choice of $A_2, A_3, B_2, B_3$ the following induction step holds.

Assume $(\phi_X, \psi_X)$ is an admissible pair defined as (179), (180) if $N \leq X$ and by (164), (165) otherwise. Then the pair $(\phi_X, \psi_X)$ defined as (179), (180) if $N \leq X^2$ and by (164), (165) otherwise, is also admissible.

Iterating, one then concludes that (179), (180) is admissible for all $(N, \delta, K)$.

Let us see how this induction scheme can be implemented. Starting with (179), (180) of Lemma 5 we should take $A_1, B_1$ and a threshold $\bar{N}(\delta, \gamma)$ such that for $N \leq \bar{N}$

\[
\left( \frac{\delta}{K} \right)^{C \log \log (K/\delta)} N > (179) \tag{181}
\]

\[
\exp(\log(K/\delta)^{C/\gamma}) N^\gamma < (180) \tag{182}
\]

For (179) it’s sufficient to take $A_1 = C \log \log \bar{N}$ with some $C(\tau) > 0$. (180) is more tricky, as later on it will be important that $B_3 > B_2 > B_1$. It suffices to guarantee that

\[
\log \left( \frac{K}{\delta} \right)^{\frac{\gamma}{4}} < \frac{\gamma}{4} \log N \tag{183}
\]

and

\[
e^{B_3(\log \log N)^2} < N^{\frac{\gamma}{2}}. \tag{184}
\]
The bound (183) does not hold only if $K/\delta$ is rather large,
\[
\frac{K}{\delta} > c^{\log^{c_B} N}
\]
for some $c(C, \gamma) > 0$. In this case it suffices to take $B_1$ so large that (180) $> N$ and thus trivially admissible. To this end it suffices to take
\[
B_1 := (\log \bar{N})^{1-c_B}
\]
and make the constraint that say
\[B_3, B_2 < 10B_1 \log \log \bar{N}.
\]

Summing up, we have found some fixed threshold $\bar{N}(\tau, \gamma)$ at which (179), (180) become admissible, with fixed $A_1, B_1$ and some freedom to define the constants $A_2, B_2, A_3, B_3$.

Now, assuming that $N', N''$ are at the scale so that (179), (180) are admissible with the data $(N', \delta', K')$; $(N'', \delta'', K'')$ we will show that (179), (180) are also admissible for the data $(N, \delta, K)$ with $N \approx N'N''$.

Assuming $B_1$ (or $\bar{N}$) is large enough we may assume that
\[
\frac{K}{\delta} < N^{10^{-3}}. \tag{185}
\]
as otherwise (180) $> N$ which is trivially admissible.

We need to estimate
\[
\psi(N', \delta', K') \psi(N'', \delta'', K'')
\]
from above and
\[
\phi(N', \delta', K') \phi(N'', \delta'', K'')
\]
from below in order to verify that (179), (180) are admissible for $(N, \delta, K)$. By (185), the constraints (117) can be relaxed to
\[
N \geq N'N'' > N \left(\frac{\delta}{\log N}\right)^{40} > N^{99/100} \tag{186}
\]
\[
N' + N'' < N^{1/2} \left(\frac{K}{\delta}\right)^{40} < N^{1/2+1/40} \tag{187}
\]
\[
\delta' \delta'' > \frac{\delta}{\log^6 N} \tag{188}
\]
\[
K'K'' < \delta^{-10} (\log N)^{10} K \tag{189}
\]

From (186) and (187) we have (with room to spare)
\[
N^{1/2-1/20} < N', N'' < N^{1/2+1/20} \tag{190}
\]
and assuming $N$ is large enough
\[
\frac{99}{100} \log \log N < \log \log N', \log \log N'' < \log \log N + \frac{20}{11}. \tag{191}
\]
With the constraints above, it suffices to verify (writing \(ll\) for \(\log \log\) like in the original paper) that
\[
(K''K''')^{-A_1}(\delta')^{A_2llN'}(\delta'')^{A_2llN''}e^{A_3(||N''|^2+||N'^2|)}(N''N')^{1-\tau}
\]
is indeed always bounded by (179). We can bound (192) by
\[
K^{-A_1}\delta^{A_2llN}e^{A_3(llN)^2}N^{1-\tau}u \cdot v
\]
where
\[
u = (\log N)^{-10A_1-6A_2llN-40}e^{\frac{A_3}{10}(llN)^2}
\]
\[
w = \delta^{14A_1-\log \frac{A_3}{10}A_2+40}.
\]
For suitable choices of \(A_2, A_3 > A_1\) both \(u, v > 1\) so (179) is admissible.
Similarly for (180) we have
\[
(K''K''')^{B_1}(\delta')^{B_2llN'}(\delta'')^{B_2llN''}e^{-B_3(||N''|^2+||N'^2|)}(N''N')^{1-\tau}
\]
\[
< K^{B_1}\delta^{B_2llN}e^{-B_3(llN)^2}N^{1-\tau}u \cdot v
\]
with
\[
u = (\log N)^{10B_1+6B_2llN}e^{-\frac{9}{10}B_3(llN)^2}
\]
\[
w = \delta^{14B_1+\log \frac{A_3}{10}B_2}.
\]
Again, by taking suitable \(B_3 > B_2 > B_1\) we make \(u, v < 1\) so (180) is admissible. It closes the induction on scales argument and finishes the proof.

\[\square\]

7. Finishing the proof

Proof. Let \(\gamma, \tau > 0\) be constants to be defined later. We start with \(A\) and its \(\mathcal{P}\)-image \(A\). Since \(|A\mathcal{A}| \leq K|A|\) we have \(|A + A| \leq K\). Define \(N_1 = N_2 = |A|\); \(N := N_1N_2 = |A|^2\) and take \(G\) to be the full graph \(A \times A\), so \(\delta = 1\). By Lemma 6 and the definition of admissible pairs, there is \(G' \subset G\) of size
\[
K^{-A_1}e^{A_3(\log \log N)^2}N^{1-\tau}
\]
so that for any vertex \(v \in V(G')\) the \(\mathcal{P}\)-preimage of \(N_{G'}(v)\) is
\[
K^{B_1}e^{-B_3(\log \log N)^2}N^{1-\gamma}
\]
separating. There is \(v \in V(G')\) such that
\[
|N_{G'}(v)| > K^{-A_1}e^{A_3(\log \log N)^2}N^{1/2-\tau}
\]
which means that there is \(A' \subset A\) of size at least
\[
K^{-A_1}|A|^{1-\tau}
\]
which is
\[
K^{B_1}e^{-B_3(\log \log N)^2}N^{\gamma}
\]
separating. In particular,
\[
E_+(\mathcal{A}') \leq K^{2B_1} e^{-2B_3 (\log \log N)^2} N^{2\gamma} |\mathcal{A}'|^2 \leq K^{2B_1} |\mathcal{A}|^{2+4\gamma}.
\]

(200)

It remains to show that if a fairly large subset $\mathcal{A}'$ has small energy then $\mathcal{A}$ itself has small energy. We formulate it as a separate combinatorial lemma in a slightly more general setting.

**Lemma 7.** Let $A_i, i = 1, \ldots, L$ be a family of sets such that
\[
A \subset \bigcup_{1 \leq i \leq L} A_i,
\]
and, moreover, each $a \in A$ is covered by at least $M$ sets $A_i$. Then
\[
E_+(A) \leq \frac{1}{M^4} \left( \sum_{1 \leq i \leq L} E_+(A_i)^{1/4} \right)^4.
\]

**Proof.** Since each $a \in A$ belongs to at least $M$ sets $A_i$, we have
\[
M^4E_+(A) \leq \sum_{1 \leq i,j,k,l \leq L} \sum_x 1_{A_i} \ast 1_{A_j}(x)1_{A_k} \ast 1_{A_l}(x).
\]

Applying the Cauchy-Schwarz inequality twice, we bound
\[
\sum_x 1_{A_i} \ast 1_{A_j}(x)1_{A_k} \ast 1_{A_l}(x) \leq \left( \sum_x 1_{A_i} \ast 1_{A_j}^2(x) \right)^{1/2} \left( \sum_x 1_{A_k} \ast 1_{A_l}^2(x) \right)^{1/2}
\]
\[
= E_+^{1/2}(A_i, A_j)E_+^{1/2}(A_k, A_l)
\]
\[
\leq E_+^{1/4}(A_i)E_+^{1/4}(A_j)E_+^{1/4}(A_k)E_+^{1/4}(A_l).
\]

Thus, after summing over the indices $1 \leq i,j,k,l \leq L$ one gets
\[
M^4E_+(A) \leq \left( \sum_{1 \leq i \leq L} E_+^{1/4}(A_i) \right)^4
\]
and the claim follows.

It remains to apply Lemma 7. Take $a \in \mathcal{A}$ and an arbitrary $a' \in \mathcal{A}'$. One can write
\[
a = \frac{a}{a'} a'.
\]

Thus, taking $A_\alpha := \alpha \mathcal{A}'$ and the covering
\[
A \subset \bigcup_{\alpha} A_\alpha
\]
with $\alpha \in \frac{A}{\mathcal{A}}$, we conclude that each $a \in \mathcal{A}$ is covered by at least $|\mathcal{A}'|$ sets $A_\alpha$. On the other hand, clearly
\[
E_+(A_\alpha) = E_+(\mathcal{A}') \leq K^{2B_1} |\mathcal{A}|^{2+\gamma}.
\]
Also, by the Plünnecke-Ruzsa inequality

\[ \frac{|A|}{|A'|} \leq \frac{|A|}{|A|} \leq K^2|A|. \]

Thus, applying Lemma 7 with \( M = |A'| \) and \( L = |A/A'| \), we get

\[ E_+(A) \leq K^8 \frac{|A|^4}{|A'|^4} K^{2B_1} |A|^{2+\gamma} \leq K^{2B_1+8+4A_1} |A|^{2+8\tau+\gamma}. \]

By taking \( \tau, \gamma \) small enough, we finish the proof of Theorem 1 since \( B_1, A_1 \) depend only on \( \tau \) and \( \gamma \). \( \square \)

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Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15., 1053 Budapest, Hungary

E-mail address: dzhelezov@gmail.com