We report a clear evidence of atomic fractals in the nonlinear motion of a two-level atom in a standing-wave microcavity. Fractal-like structures, typical for chaotic scattering, are numerically found in the dependencies of outgoing positions and momenta of scattered atoms on their ingoing values and in the dependence of exit times of cold atoms on their initial momenta in the generic semiclassical models of cavity QED (1) with atoms in a far-detuned amplitude (phase)-modulated standing wave and (2) with coupled atomic external and internal degrees of freedom. Tiny interplay between all the degrees of freedom in the second model is responsible for trapping atoms even in a very short microcavity. It may lead simultaneously, at least, to two kinds of atomic fractals, a countable fractal (a set of initial momenta generating separatrix-like atomic trajectories) and a seemingly uncountable fractal with a set of momenta generating infinite walkings of atoms inside the cavity.

PACS numbers: 42.50.Vk, 05.45.Df

1. Cavity quantum electrodynamics (QED) with cold atoms is a rapidly growing field of atomic physics and quantum optics studying the atom-photon interaction in cavities [1]. Recent experiments [2, 3] succeeded in exploration coupled external atomic center-of-mass, internal atomic, and field dynamics under condition of strong-coupling between a single cold atom and a single-mode field in a high-finesse optical microcavity. Methods to monitoring the atomic motion in real time have been realized experimentally. They open a new way to study the very foundations of the matter-field interaction and for other purposes.

The basic model Hamiltonian of the interaction of a two-level atom with a single-mode standing-wave field in an ideal cavity is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} \hbar \omega_a \hat{\sigma}_z + \hbar \omega_f (\hat{a}^\dagger \hat{a} + \frac{1}{2}) - \hbar \Omega_0 f(t) (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) \cos(k_f \hat{x}),$$

(1)

where $\hat{x}$ and $\hat{p}$ are the atomic position and momentum operators, $\hat{\sigma}$ the atomic Pauli operators, $\hat{a}$ and $\hat{a}^\dagger$ the field-mode operators. We incorporate a function of time $f(t)$ in (1) to describe a possible modulation of the standing-wave amplitude in the framework of the basic Hamiltonian. The strongly coupled atom-field system (1) is a highly nonlinear one and is known to possess a rich variety of qualitatively different dynamics including chaos. The full Hamiltonian (1) for an atom, to be placed in a far-detuned modulated standing wave, can be reduced to an effective non-autonomous Hamiltonian of a “structure-less” atom with one and half degree of freedom which is a paradigm for quantum chaos in atomic optics [1, 3, 4, 5, 6]. It has been shown in [3, 4] that semiclassical chaos in the sense of extremal sensitivity to small changes in initial conditions is possible without any modulation, i.e. with an autonomous dynamical system with three degrees of freedom, the field, external, and internal atomic ones. In this Letter we report with both the models a clear evidence of a new property of the atom-field dynamics, atomic fractals in cavity QED. In the strong-coupling limit, one neglects dissipation in all the degrees of freedom that may be justified by available microcavities with very large quality factors $Q \gtrsim 10^6$ [4, 5] or/and operating with far-detuned atoms.

2. Scattering of atoms by a far-detuned modulated standing wave. Following a scheme of experiments on scattering of atoms by light [10], suppose that a monokinetic atomic beam propagates at an angle to the direction $y$ which is perpendicular to the standing-wave axis $x$. The field is assumed to be uniform in the $y$-direction. In a reference frame moving with a constant velocity in the $y$-direction there remains only the transverse atomic motion which can be considered classically if the atomic momentum is much larger than the photon momentum. So the atom is treated as a point particle subject to a position-dependent force and is thus deflected. One measures atomic positions $x$ and momenta $p$ after passing the interaction zone. For sufficiently large detuning between the field and atomic frequencies, one can adiabatically eliminate the excited-state amplitude of an atom [4] and derive a dimensionless effective Hamiltonian for the center-of-mass motion

$$H_{e\text{ff}} = \frac{\omega_r p^2}{2} - \varepsilon (1 - \sin \beta \tau) \sin^2 x,$$

(2)

where $x = k_f \hat{x}$ and $p = <\hat{p}> / \hbar k_f$ are the scaled expectation values for the atomic position and momentum, $\omega_r = \hbar k_f^2 / m \Omega_0$ and $\beta$ the scaled recoil and modulation frequencies, $\varepsilon$ and $\tau = \Omega_0 t$ the scaled well depth and time, respectively.

Solving the respective Hamilton equations we compute outgoing atomic positions $x_{out}$ at $\tau = 500$ as a function of ingoing positions $x_0$ with $\omega_r = 0.001$, $\varepsilon = 0.625$, and...
$\beta = 0.07$. Atoms in the beam are supposed to be initially in the ground state, uniformly distributed along the $x$-axis with the same value of the initial momentum $p_0 = 40.1$ and strongly detuned with $\delta = (\omega_f - \omega_0)/\Omega_0 = 16$. Chaotic atomic transversal motion in the modulated standing wave results in a self-similar structure of the scattering function which reveals itself under successive magnifications (see Fig. 1). Let us consider the initial interval of $x_0$ (that may be identified with a cavity length) as a scattering region. Chaos implies that there exist atomic trajectories that never escape this region, and so there exist atoms never leaving the cavity. A set of initial positions corresponding to nonescaping trajectories forms a fractal. We have found singularities computing the time atoms need to go out off the initial interval (to leave the cavity). Similar results with appearing fractals are found in the following ranges of the control parameters: $\Omega \sim 2 \times 10^{-3}$, $N \lesssim 10^2$, and $|\delta| \lesssim 1.5$. These magnitudes seem to be reasonable with available optical microcavities in which the strength of the atom-field coupling may reach the values of the order of $\Omega_0 \simeq 2 \pi \cdot 10^8$ Hz, and the conditions of strong coupling are fulfilled for both the internal and external degrees of freedom.

3. Fractals in the atom-field dynamics. We wish to illustrate a generic effect of fractals in cavity QED and use the full Hamiltonian (1) without any modulation, i.e. $f(t) = 1$. A comparatively large average number of photons is supposed in a single-mode cavity to sustain atom-field Rabi oscillations. By using the following expectation values: $x, p, z = < \hat{x}, \hat{p}, \hat{z} >$, $u = < \hat{a}^2 \hat{\sigma}_- + \hat{\sigma}_+ >$, and $v = i < \hat{a}^2 \hat{\sigma}_- - \hat{\sigma}_+ >$, we translate the Heisenberg equations with the Hamiltonian (1) into the closed nonlinear semiclassical system (3)

$$
\dot{x} = \omega_r p, \ \dot{p} = -u \sin x, \ \dot{u} = \delta v, \ \dot{z} = -2v \cos x,
$$

$$
\dot{v} = -\delta u + [(2N - 1)z - 3z^2/2 + 1/2] \cos x,
$$

where dot denotes a derivative with respect to $\tau$. The dimensionless control parameters are the recoil frequency $\omega_r$, the detuning $\delta$, and the number of excitations $N = < \hat{a}^2 \hat{\sigma}_- + \hat{\sigma}_+ >$. The integral of motion, $E = \omega_r p^2/2 - u \cos x - \delta z/2$, reflects the conservation of energy. At exact resonance, $\delta = 0$, the energy of the atom-photon interaction $u$ is conserved, and the system becomes integrable with solutions describing regular atomic center-of-mass motion in a potential well or over potential hills and atom-field Rabi oscillations modulated by the translational motion. Out of resonance, the atomic translational motion is described by the equation for nonlinear parametric pendulum, $\ddot{x} + \omega_r u(\tau) \sin x = 0$, that has been analytically shown to produce weak chaos even in the case of the simplest harmonic modulation $u(\tau) \sim \cos \Omega \tau$ caused by the Rabi oscillations with a constant frequency $\Omega = \sqrt{\delta^2 + 4v}$. In fact, the Rabi oscillations is an amplitude- and frequency-modulated signal which may induce pronounced erratic motion of the atomic center-of-mass inside the cavity. The motion is very complicated nearby the unperturbed separatrix of the nonlinear pendulum where the period of oscillations goes to infinity. In this zone small changes in frequency, caused by respective small changes in energy, may lead to dramatic changes in phase which are the ultimate reason of exponential instability of atomic motion in a periodic standing wave. The speculations above have been confirmed in our numerical experiments where positive values of the maximal Lyapunov exponent have been found in the following ranges of the control parameters: $\omega_r \gtrsim 10^{-3}$, $N \lesssim 10^2$, and $|\delta| \lesssim 1.5$. These magnitudes seem to be reasonable with available optical microcavities in which the strength of the atom-field coupling may reach the values of the order of $\Omega_0 \simeq 2 \pi \cdot 10^8$ Hz, and the conditions of strong coupling are fulfilled for both the internal and external degrees of freedom.

Fractals in the dependencies $x_{\text{out}}(x_0)$ and $p_{\text{out}}(p_0)$ with atoms being transversely injected into a stationary stand-
The exit time $T$, corresponding to both smooth and unresolved $p_0$ intervals, increases with increasing the magnification factor. It follows that there exist atoms never reaching the detectors in spite of the fact that they have no obvious energy restrictions to leave the cavity. Tiny interplay between chaotic external and internal dynamics prevents these atoms from leaving the cavity. The similar phenomenon in chaotic scattering is known as dynamical trapping. Different kinds of atomic trajectories, which are computed with the system $(\beta)$, are shown in FIG. 3b. A trajectory with the number $m$ transverses the central node of the standing-wave before being detected $m$ times and is called $m$-th trajectory. There are also special separatrix-like $mS$-trajectories following which atoms in infinite time reach the stationary points $x = \pm \pi n (n = 0, 1, 2, ...)$, $p = 0$, transversing $m$ times the central node. These points are the anti-nodes of the standing wave where the force acting on atoms is zero. In difference from separatrix motion in the resonant system ($\delta = 0$) with the initial atomic momentum $p_{cr}$, a detuned atom can asymptotically reach one of the stationary points after transversing the central node $m$ times. The smooth $p_0$ intervals in the first-order structure in FIG. 3a correspond to atoms transversing once the central node and reaching the right detector. The unresolved singular points in the first-order structure with $T = \infty$ at the border between the smooth and unresolved $p_0$ intervals are generated by the $1S$-trajectories. Analogously, the smooth and unresolved $p_0$ intervals in the second-order structure in FIG. 3b correspond to the 2-nd order and the other trajectories, respectively, with singular points between them corresponding to the $2S$-trajectories and so on.

There are two different mechanisms of generation of infinite detection times, namely, dynamical trapping with infinite oscillations ($m = \infty$) in a cavity and the separatrix-like motion ($m \neq \infty$). The set of all initial momenta generating the separatrix-like trajectories is a countable fractal. Each point in the set can be specified as a vector in a Hilbert space with $m$ integer nonzero components. One is able to prescribe to any unresolved interval of $m$-th order structure a set with $m$ integers, where the first integer is a number of a second-order structure to which trajectory under consideration belongs in the first-order structure, the second integer is a number of a third-order structure in the second-order structure mentioned above, and so on. Such a number set is analogous.
FIG. 4: Exit time distributions in the regimes of strong chaos with $\delta = 0.4$ (crosses) and weak chaos with $\delta = 0.05$ (circles) with the same initial conditions and the other control parameters, as in FIG. 3.

to a directory tree address: "\<a subdirectory of the root directory>\<a subdirectory of the 3-rd level>\<a subdirectory of the 2-nd level>\". Unlike the separatrix fractal, the set of all initial atomic momenta leading to dynamically trapped atoms with $n = \infty$ seems to be uncountable.

The dependence of the maximal Lyapunov exponent of the set (3) on the detuning $\delta$ have been found almost the same as compared with a slightly different version of (3) considered in [8]. It shows a deep gorge around $\delta = 0$, maxima around $|\delta| \sim 0.5$, and falls to zero at $|\delta| \gtrsim 1.5$. We collect an exit time statistics, by counting atoms with different initial momenta reaching the detectors, in the regimes of comparatively strong chaos ($\delta = 0.4$) and weak chaos ($\delta = 0.05$). The plots of the respective histograms of exit times are shown in FIG. 4.

The probability distribution function $P(T)$ with almost resonant atoms decays rapidly and demonstrates a single local maximum around $T \sim 140$. In the regime of strong chaos, $P(T)$ demonstrates a few local peaks (the first one around $T \sim 140$) and a long tail up to $T \sim 500$.

4. In summary, fractals in single-atom standing-wave cavity QED typically arise in the nonlinear models of the atom-field interaction which are chaotic in the semiclassical limit. It has been shown as for the non-autonomous Hamiltonian system (3) with one and half degree of freedom describing the nonlinear motion of a "structureless" atom in a far detuned amplitude (phase)-modulated standing-wave field, as for the autonomous system (3) with three degrees of freedom including internal and external atomic and field ones. The fractals of the types to be found in this Letter should appear also in the center-of-mass motion of an ion confined in a Paul trap and interacting with a laser field [13]. Different effective Hamiltonians have been introduced to describe this interaction. The simplest one [13] is of a periodically driven linear oscillator which is chaotic in the classical limit [13].

We have considered the semiclassical chaotic scattering. In quantum dynamics there are no individual atomic trajectories. It implies lacking of singularities in scattering functions, but classical fractals should manifest themselves in fluctuations of corresponding quantum $S$-matrices whose structures for generic quantum scattering models have been shown to be described by random matrices [7]. We plan to treat the quantum-classical correspondence in chaotic scattering of atoms by a standing wave in the future.

We thank an anonymous referee for valuable comments. This work was supported by the Russian Foundation for Basic Research under Grant Nos. 02-02-17796, 02-02-06840, and 02-02-06841.

[1] Cavity Quantum Electrodynamics, Advances in Atomic, Molecular, and Optical Physics, Supplement 2, ed. by P.R. Berman (Academic, San-Diego, 1994); Phys. Scr. T76, 1 (1998).
[2] C.J. Hood, T.W. Lynn, A.C. Doherty, A.S. Parkins, and H.J. Kimble, Science 287, 1447 (2000).
[3] P.W.H. Pineske, T. Fischer, P. Maunz, and G. Rempe, Nature (London) 404, 365 (2000).
[4] R. Graham, M. Schlautmann, and P. Zoller, Phys. Rev. A 45, 19 (1992).
[5] F.L. Moore, J.C. Robinson, C. Bharucha, P.E. Williams, and M.G. Raizen, Phys. Rev. Lett. 73, 2974 (1994).
[6] H. Ammann, R. Gray, I. Shvarchuck, and N. Christensen, Phys. Rev. Lett. 80, 4111 (1998).
[7] W.K. Hensinger et al, Nature 412, 52 (5 July, 2001).
[8] S.V. Prants and V.Yu. Sirotkin, Phys. Rev. A 64, 033412 (2001).
[9] S.V. Prants and L.E. Kon’kov, JETP Letters 73, 180 (2001) [Pis’ma Zh. Eksp. Teor. Fiz. 73, 200 (2001)].
[10] P.L. Gould, G.A. Ruff, D.E. Pritchard, Phys. Rev. Lett. 56, 827 (1986); T. Sleator, T. Pfau, V. Balykin, O. Carnal, and J. Mlynek, Phys. Rev. Lett. 68, 1996 (1992).
[11] S.V. Prants, JETP Letters 75, 63 (2002) [Pis’ma Zh. Eksp. Teor. Fiz. 75, 71 (2002)].
[12] E. Ott, Chaos in dynamical systems (Cambridge University Press, Cambridge, England, 1993).
[13] C.F. Hillermeier, R. Blümel, and U. Smilansky, Phys. Rev. A 45, 3486 (1992).
[14] D.J. Wineland et al, J. Res. Natl. Stand. Technol. 103, 259 (1998).
[15] D.I. Kamenev and G.P. Berman, Quantum chaos: a harmonic oscillator in monochromatic wave (Rinton Press, Inc., Princeton New Jersey, USA, 2001).
[16] G.M. Zaslavsky, R.Z. Sagdeev, D.A. Usikov, and A.A. Chernikov. Weak Chaos and Quasiperiodic Patterns (Cambridge University Press, Cambridge, England, 1991).
[17] R. Blümel, and U. Smilansky, Phys. Rev. Lett. 60, 477 (1988).