FREE POLYNILPOTENT GROUPS AND
THE MAGNUS PROPERTY

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ABSTRACT. Motivated by a classic result for free groups, one says that a group $G$ has the Magnus property if the following holds: whenever two elements generate the same normal subgroup of $G$, they are conjugate or inverse-conjugate in $G$.

It is a natural problem to find out which relatively free groups display the Magnus property. We prove that a free polynilpotent group of any given class row has the Magnus property if and only if it is nilpotent of class at most 2. For this purpose we explore the Magnus property more generally in soluble groups, and we produce new techniques, both for establishing and for disproving the property. We also prove that a free centre-by-(polynilpotent of given class row) group has the Magnus property if and only if it is nilpotent of class at most 2.

On the way, we display $2$-generated nilpotent groups (with non-trivial torsion) of any prescribed nilpotency class with the Magnus property. Similar examples of finitely generated, torsion-free nilpotent groups are hard to come by, but we construct a $4$-generated, torsion-free, class-$3$ nilpotent group of Hirsch length $9$ with the Magnus property. Furthermore, using a weak variant of the Magnus property and an ultraproduct construction, we establish the existence of metabelian, torsion-free, nilpotent groups of any prescribed nilpotency class with the Magnus property.

1. Introduction

A group $G$ has the Magnus property if the following holds: whenever $g, h \in G$ generate the same normal subgroup $\langle g \rangle^G = \langle h \rangle^G$, the element $g$ is already conjugate in $G$ to $h$ or to $h^{-1}$. Magnus [Mag30] established this property for free groups, using his “Freiheitssatz”. The Magnus property is a first-order property, in the sense of model theory; consequently all groups with the same elementary theory as free groups have the Magnus property. During the last two decades, the Magnus property has been explored and established for various classes of groups using different techniques, e.g., for fundamental groups of closed surfaces [BS08], direct products of free groups [KK16, Fel21], and certain amalgamated products [Fel19].

Groups with the Magnus property are typically torsion-free and ‘big’, for instance, in the sense that they do not satisfy any non-trivial law, viz. any non-trivial identical relation. Even so free abelian groups possess the Magnus property, for obvious reasons, and certain crystallographic groups with the Magnus property were manufactured.
in [KK16]. In conjunction with Magnus’ classic result, this prompts a natural question for relatively free groups, viz. \( V \)-free groups for any given variety \( V \) of groups.

**Problem A.** Let \( V \) be a variety of groups, viz. the class of all groups satisfying each one of a given set of laws. Which \( V \)-free groups have the Magnus property?

By basic considerations, it is enough to settle the question for relatively free groups of finite rank, viz. on finitely many free generators, because for each variety of groups \( V \) there is a precise cut-off point \( \delta_V \in \mathbb{N}_0 \cup \{ \infty \} \) for the ranks of \( V \)-free groups with the Magnus property; see Corollary 2.3. As mentioned above, for the variety \( U \) of all groups and for the variety \( A \) of abelian groups, we know that every free group and every \( A \)-free group has the Magnus property: hence \( \delta_U = \delta_A = \infty \). Likewise, it is easy to see that the variety \( A_m \) of abelian groups of exponent \( m \) (that is, \( m \) or dividing \( m \)) has \( \delta_{A_m} = \infty \) if \( m \in \{ 1, 2, 3, 4, 6 \} \), and \( \delta_{A_m} = 0 \) otherwise.

Perhaps it is natural to concentrate first on varieties of exponent zero, viz. varieties \( V \) such that \( x^m \) is not a universal law in \( V \)-groups, for any \( m \in \mathbb{N} \). Prominent examples of this kind are the varieties \( N_c \) of all polynilpotent groups of class row \( c \), for any given length \( l \in \mathbb{N} \) and class tuple \( c = (c_1, \ldots, c_l) \in \mathbb{N}^l \). We recall that a group \( G \) belongs to \( N_c \) if the term \( \gamma_{(c_1+1, \ldots, c_l+1)}(G) \) of its iterated lower central series vanishes; here \( \gamma_{(1)}(G) = \gamma_1(G) = G \) and inductively we set

\[
\gamma_{(c_1+1, \ldots, c_l+1)}(G) = \gamma_{c_l+1}(\gamma_{(c_1+1, \ldots, c_{l-1}+1)}(G)), \quad \text{for } l > 1,
\]

where \( \gamma_{(c_1+1)}(G) = \gamma_{c_1+1}(G) = [\gamma_{c_1}(G), G] \) is the \( (c_1 + 1) \)th term of the ordinary lower central series of \( G \). For instance, for \( l = 1 \) and \( c = (c) \) the variety \( N_c \) consists of all nilpotent groups of class at most \( c \); for \( l \in \mathbb{N} \) and \( c = (1, \ldots, 1) \in \mathbb{N}^l \), the variety \( N_c \) consists of all soluble groups of derived length at most \( l \). For free polynilpotent groups, we resolve Problem A completely.

**Theorem 1.1.** Let \( G \) be an \( N_c \)-free group of rank \( d \), i.e., a free polynilpotent group of class row \( c \) that is freely generated by \( d \) elements, where \( d, l \in \mathbb{N} \) and \( c \in \mathbb{N}^l \).

Then \( G \) has the Magnus property if and only if \( G \) is nilpotent of class at most 2; equivalently, if and only if \( d = 1 \) or \( c \in \{(1), (2)\} \).

The proof uses the notion of basic witness pairs for not having the Magnus property; which are defined in Lemma 2.6. The starting point of the proof is that the restricted wreath product \( C_\infty \wr C_\infty \) admits such witness pairs; see Proposition 4.1. Almost as a by-product, we obtain the following similar result, for further varieties.

**Theorem 1.2.** Let \( G \) be a free centre-by-\( N_c \) group of rank \( d \), where \( d, l \in \mathbb{N} \) and \( c \in \mathbb{N}^l \). Then \( G \) has the Magnus property if and only if \( G \) is nilpotent of class at most 2; equivalently, if and only if \( d = 1 \) or \( c = (1) \).

Since the Magnus property is a first-order property, one may wonder about groups with the same elementary theory as free polynilpotent groups. Groups that are elementarily equivalent to free nilpotent groups were considered in [MS09, MS11]. We remark that \( N_c \)-free groups are torsion-free, while free centre-by-\( N_c \) groups may involve central torsion of exponent 2; compare with [Kuz82, Stö89].

In order to prove Theorems 1.1 and 1.2 we explore the Magnus property in more general groups, and we produce new techniques, both for establishing and for disproving the property. In Proposition 2.4 we provide a useful sufficient criterion under which the Magnus property passes to factor groups; for instance, if \( G \) has the
we see that every torsion-free, class-

Example 3.8

Example 3.11

Theorem 1.3

we construct explicitly a

provides an explicit family of finitely generated, nilpotent

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\[ (x, y) \in G \times X, \text{viz. the smallest normal subgroup containing } x, y. \]

Throughout, we use left-normed commutators; for instance, we write \([x, y, z] = [[x, y], z].\) A similar convention applies to iterated Lie commutators in associated Lie rings. For \(X \subseteq G\) we denote by \((X)^G = \{x^g \mid x \in X, g \in G\}\) the normal closure of \(X\) in \(G,\) viz. the smallest normal subgroup containing \(X.\) For a singleton \(X = \{x\}\) we use the shorter notation \((x)^G.\)

We denote by \(Z(G)\) the centre of \(G,\) and we write \(Z_i(G),\) \(i \in \mathbb{N}_0,\) for the terms of the upper central series of \(G.\) The iterated lower central series and, in particular, the lower central series \(\gamma_i(G), i \in \mathbb{N},\) were already discussed above.

Suppose that \(\mathcal{V}\) is a non-trivial variety of groups. The rank of a \(\mathcal{V}\)-free group \(G\) is the cardinality of a \(\mathcal{V}\)-free generating set for \(G.\) We use the term sparingly and no confusion with other common notions of rank, such as Prüfer rank should arise.

For \(m \in \mathbb{N} \cup \{\infty\}\) we write \(C_m\) to denote a cyclic group of order \(m.\)

2. Preliminaries and auxiliary results

We recall that the Magnus property is a first-order property in the sense of model theory; indeed, sometimes it is useful to rephrase it for a group \(G\) as follows:

\[(\text{MP}) \quad \forall k, l \in \mathbb{N}_0 \quad \forall g, h \in G \quad \forall m_1, \ldots, m_k \in \{1, -1\} \quad \forall v_1, \ldots, v_k \in G \quad \forall n_1, \ldots, n_l \in \{1, -1\} \quad \forall w_1, \ldots, w_l \in G : \]

\[
\left( h = \prod_{i=1}^k (g^{m_i})^{v_i} \land g = \prod_{j=1}^l (h^{n_j})^{w_j} \right) \implies \left( \exists v \in G : g^v = h \lor g^{-v} = h^{-1} \right),
\]
where the quantifier over the integers \(k,l\) can be eliminated by passing to a countable collection of sentences in the first-order language of groups. For short we say that \(G\) is an \(\text{MP-group}\) if \(G\) has the Magnus property.\(^1\)

We recall that, if \(\mathcal{P}\) is any property of groups, then a group \(G\) is \textit{locally a} \(\mathcal{P}\)-\textit{group} if each finite subset of \(G\) is contained in a \(\mathcal{P}\)-subgroup of \(G\). If \(\mathcal{P}\) is inherited by subgroups, this is equivalent to the requirement that each finitely generated subgroup of \(G\) has \(\mathcal{P}\). The proof of the following lemma is routine, using (\text{MP}).

\textbf{Lemma 2.1.} Every locally MP-group is an MP-group.

Of course, the Magnus property does not generally pass from a group to its subgroups or quotients. Nevertheless there are interesting situations, where this happens. We record a simple, but useful observation.

\textbf{Lemma 2.2.} Let \(H \leq G\) be a retract of a group \(G\), that is \(G = H \rtimes N\) for a normal complement \(N \triangleleft G\). If \(G\) is an MP-group then so is \(H\).

For each variety of groups \(\mathbb{V}\) we set
\[
\delta_\mathbb{V} = \sup\{d \in \mathbb{N}_0 \mid \mathbb{V}\text{-free groups of rank } d \text{ are MP-groups}\} \in \mathbb{N}_0 \cup \{\infty\}.
\]

\textbf{Lemmata 2.1 and 2.2} already have a useful consequence for relatively free groups.

\textbf{Corollary 2.3.} Let \(\mathbb{V}\) be a variety of groups. If \(\delta_\mathbb{V} = \infty\), then every \(\mathbb{V}\)-free group is an MP-group. If \(\delta_\mathbb{V} < \infty\), then a \(\mathbb{V}\)-free group is an MP-group if and only if it has rank at most \(\delta_\mathbb{V}\).

The next result is less obvious, if not surprising; in particular, it provides a powerful handle to deal with free nilpotent and, more generally, free polynilpotent groups.

\textbf{Proposition 2.4.} Let \(G\) be an MP-group, and let \(N \triangleleft G\) such that for each \(g \in G \setminus N\) the \(\subseteq\)-partially ordered set of normal subgroups
\[
\Omega_{gN} = \{\langle gz \rangle^G \mid z \in N\}
\]
satisfies the minimal condition. Then \(G/N\) is an MP-group.

\textit{Proof.} Let \(g,h \in G\) such that their images in \(G/N\) have the same normal closure, in other words such that \(\langle g \rangle^G \equiv_N \langle h \rangle^G\). If \(g \equiv_N 1\), also \(h \equiv_N 1\), and they are conjugate to one another modulo \(N\). Now suppose that \(g \not\equiv_N 1\). Choose \(k,l \in \mathbb{N}, m_1,\ldots,m_k,n_1,\ldots,n_l \in \mathbb{Z}\) and \(v_1,\ldots,v_k,w_1,\ldots,w_l \in G\) such that
\[
\prod_{i=1}^k (g^{m_i})^{v_i} \equiv_N h \quad \text{and} \quad \prod_{j=1}^l (h^{n_j})^{w_j} \equiv_N g.
\]
Since \(\Omega_{gN}\) satisfies the minimal condition, we find \(g_{\min} \in gN\) such that \(\langle g_{\min}\rangle^G\) is \(\subseteq\)-minimal among all subgroups of the form \(\langle y \rangle^G\) for \(y \in gN\). Consider
\[
h_0 = \prod_{i=1}^k (g_{\min}^{m_i})^{v_i} \equiv_N h \quad \text{and} \quad g_0 = \prod_{j=1}^l (h_{0}^{n_j})^{w_j} \equiv_N g.
\]
These elements satisfy \(\langle g_0 \rangle^G \subseteq \langle h_0 \rangle^G \subseteq \langle g_{\min}\rangle^G\); hence, by the minimal choice of \(g_{\min}\), we conclude that \(\langle g_0 \rangle^G = \langle h_0 \rangle^G\). Since \(G\) has the Magnus property, there exists \(v \in G\) such that \(g_0^v = h_0\) or \(g_0^v = h_0^{-1}\), hence
\[
g_0^v \equiv_N g_0^v \equiv_N h_0 \equiv_N h \quad \text{or, similarly,} \quad g_0^v \equiv_N h_0^{-1}.
\]
\[\square\]

\(^1\)The terms “\(M\text{-group}\)” and “Magnus group” are unfortunately already in use with other meanings. For lack of better alternatives, we have settled for “\(\text{MP-group}\)".
We record some immediate consequences, which are quite remarkable.

**Corollary 2.5.** Let $G$ be an MP-group, $\Omega_G = \{(g)^G \mid g \in G\}$, and let $N \trianglelefteq G$.

1. If $\Omega_G$ satisfies the minimal condition, then $G/N$ is an MP-group.
2. If $N$ is finite, then $G/N$ is an MP-group.

In particular, if $G$ is a finitely generated nilpotent group with the Magnus property then $G$ modulo its torsion subgroup $\tau(G)$ is a finitely generated, torsion-free nilpotent group with the Magnus property; in contrast, the finite group $\tau(G)$ does not in general inherit the Magnus property; compare with Example 3.8.

The following sufficient criterion turns out to be useful for rejecting the Magnus property and gives rise to the notion of basic witness pairs which is to play a key role.

**Lemma 2.6.** Let $G$ be a group, and let $g \in G$ and $v \in [G, G] \setminus \{[g, w] : w \in G\}$ be such that $g^2 \not\equiv_{[G, G]} 1$ and $(g)^G = (gv)^G$. Then $G$ is not an MP-group.

**Proof.** From $gv \equiv_{[G, G]} g \not\equiv_{[G, G]} g^{-1}$ it follows that $g$ and $gv$ are not inverse-conjugate in $G$. They are not conjugate to one another either, as $gv \not\equiv g[w, v] = gw$ for all $w \in G$. Thus $(g)^G = (gv)^G$ shows that $G$ does not have the Magnus property. \qed

For short we say that $(g, v)$ is a basic $\neg$(MP)-witness pair for $G$, viz. a witness pair for $G$ not having the Magnus property, if $g, v$ satisfy the conditions in Lemma 2.6. Part (1) of the following lemma is straightforward; compare with Lemma 2.2. Part (2) is established by following the proof of Proposition 2.4 as per contrapositive.

**Lemma 2.7.** Let $G$ be a group, and let $N \trianglelefteq G$.

1. If $G = H \times N$ splits over $N$, then every basic $\neg$(MP)-witness pair for $H$ is also a basic $\neg$(MP)-witness pair for $G$.
2. Suppose that $N \subseteq [G, G]$ and that $g, v \in G$ are such that their images modulo $N$ form a basic $\neg$(MP)-witness pair $(\bar{g}, \bar{v})$ for $G/N$. If $\Omega_{gN} = \{(gz)^G : z \in N\}$ satisfies the minimal condition, then $(g, v)$ lifts to a basic $\neg$(MP)-witness pair $(g_0, v_0)$ for $G$, with $g_0 \equiv_N g$ and $v_0 \equiv_N v$.

Another useful tool is the co-centraliser of an element $g$ in a group $G$, that is

$$C_G^*(g) = \{[g, w] : w \in G\} \trianglelefteq G;$$

this group is closely related to the normal closure $(g)^G$ and thus of interest to us.

**Lemma 2.8.** Let $G$ be a group, and let $g \in G$. Then

1. $C_G^*(g) \trianglelefteq G$ and $(g)^G = [g] C_G^*(g)$; in particular, $(g)^G / C_G^*(g)$ is cyclic;
2. $C_G^*(g) = [g] C_G^*(g)$, i.e., $C_G^*(g)$ is the smallest normal subgroup $N$ of $G$ such that $N \subseteq [g]^G$ and $G$ acts trivially by conjugation on $[g]^G/N$;
3. $C_G^*(h) \leq C_G^*(g)$ for all $h \in (g)^G$;
4. if $h \in G$ with $(g)^G = (h)^G$ then $C_G^*(h) = C_G^*(g)$.

**Proof.** We set $X = \{[g, w] : w \in G\}$ so that $C_G^*(g) = (X)$ and $(g)^G = \{[g] \cup X\}$. From the identity $[g, w]^{-1} = [g, w]^{-1}$, for $w, v \in G$, we deduce that $C_G^*(g) \trianglelefteq G$. This establishes (1) and (2). Claims (3) and (4) are immediate consequences of (2). \qed

### 3. Locally nilpotent groups

Clearly, if $G$ is an MP-group then so is its centre $Z(G)$. We are interested in sufficient conditions so that the factor group $G / Z(G)$ inherits the Magnus property. First we observe a useful feature of co-centralisers in torsion-free, locally nilpotent groups.
Lemma 3.1. Let $G$ be a torsion-free, locally nilpotent group. Let $g \in G \setminus Z(G)$. Then the co-centraliser of $g$ satisfies
\[
\langle g \rangle^G \cap Z(G) \leq C^*_G(g).
\]

Proof. Choose $v \in G$ such that $[g,v] \neq 1$. For every $z \in \langle g \rangle^G \cap Z(G)$ there exist finitely many elements $w_1, \ldots, w_n \in G$ such that $z \in \langle g^{w_1}, \ldots, g^{w_n} \rangle$. If the claim holds true for the nilpotent group $H = \langle g, v, w_1, \ldots, w_n \rangle \leq G$, we conclude that $z \in C^*_H(g) \leq C^*_G(g)$. Thus we may assume without loss that $G$ is nilpotent.

Let $c$ denote the nilpotency class of $G$, and let us fix the position where $g$ makes its appearance within the upper central series: $g \in Z_{i+1}(G) \setminus Z_i(G)$ for suitable $i \in \{1, \ldots, c-1\}$. Since $G$ is torsion-free, so is $G/Z_i(G)$. Hence we deduce from $C^*_G(g) \leq [Z_{i+1}(G),G] \leq Z_i(G)$ that $\langle g \rangle^G = \langle g \rangle \times C^*_G(g)$ and consequently $C^*_G(g) = \langle g \rangle^G \cap Z_i(G)$. This implies $\langle g \rangle^G \cap Z(G) \leq C^*_G(g)$. \hfill \square

With this insight we are ready to deal with torsion-free, class-2 nilpotent groups.

Proposition 3.2. Let $G$ be a torsion-free, class-2 nilpotent group. Then $G$ has the Magnus property.

Proof. Let $g, h \in G$ such that $\langle g \rangle^G = \langle h \rangle^G$. If $g \in Z(G)$, then $\langle g \rangle^G = \langle h \rangle^G = \langle h \rangle$ and, because $G$ is torsion-free, we conclude that $g \in \{h,h^{-1}\}$.

Now suppose that $g \notin Z(G)$. Since $G/Z(G)$ is torsion-free abelian, we deduce from $\langle g \rangle Z(G) = \langle h \rangle Z(G)$ that $g \equiv_{Z(G)} h^{\pm 1}$; replacing $g$ by its inverse if necessary, we may assume without loss that $g \equiv_{Z(G)} h$; hence $g^{-1}h \in Z(G) \cap \langle g \rangle^G \leq C^*_G(g)$ by Lemma 3.1. This implies $g^{-1}h = \prod_{i=1}^{k} [g,v_i]^{e_i}$ for suitable $k \in \mathbb{N}_0$, $v_1, \ldots, v_k \in G$ and $e_1, \ldots, e_k \in \{1,-1\}$. Since $G$ has nilpotency class 2, we obtain
\[
h = g \prod_{i=1}^{k} [g,v_i]^{e_i} = g \left[ g \prod_{i=1}^{k} v_i^{e_i} \right] = g^v \quad \text{for } v = \prod_{i=1}^{k} v_i^{e_i}.
\]
\hfill \square

Example 3.3. It would be interesting to complement Proposition 3.2 by characterising (or even classifying) finite, class-2 nilpotent groups with the Magnus property. Each such group is necessarily a $\{2,3\}$-group and hence a direct product $G = P \times Q$ of its Sylow-2 and its Sylow-3 subgroup, each of which is again an MP-group.

However, it does not seem easy to give a succinct characterisation of finite, class-2 nilpotent 2- or 3-groups, in terms of canonical subgroups or quotients. A halfway practical criterion for 3-groups is the following: a finite, class-2 nilpotent 3-group $G$ has the Magnus property if and only if (i) $Z(G)$ and $G/Z(G)$ are elementary abelian and (ii) for every $(Z(G)$-coset of an) element $g$ of order 3 there exists (a $Z(G)$-coset of) an element $h$ such that $[g,h] = g^3$.

With this criterion it is, for instance, easy to see that the class-2 nilpotent group
\[
G = \langle t, a, b \mid t^3 = a^9 = b^9 = [a,b] = [a,t][b^3 = [b,t]a^3 = 1]\]
satisfies $Z(G) = [G,G] \cong C^2_3$ and $G/Z(G) \cong C^3_3$, but does not have the Magnus property; for instance, $a$ and $a^4$ have the same normal closure $\langle a, b^3 \rangle$, but are neither conjugate nor inverse-conjugate to one another. Incidentally, examples of such kind illustrate that the condition of torsion-freeness is not redundant in Lemma 3.1.

Lemma 3.4. Let $G$ be a group and let $g \in G \setminus Z(G)$ be such that Eq. (3.1) holds. Then every $z \in \langle g \rangle^G \cap Z(G)$ satisfies $\langle gz \rangle^G = \langle g \rangle^G$. 
Proof. Let $z \in \langle g \rangle^G \cap Z(G)$. Clearly, $gz \in \langle g \rangle^G$ and it remains to show that $g \in \langle gz \rangle^G$. Since $z \in \langle g \rangle^G \cap Z(G) \subseteq C_G^*(g)$, there exist $k \in \mathbb{N}, v_1, \ldots, v_k \in G$ and $e_1, \ldots, e_k \in \{1, -1\}$ such that $z = \prod_{i=1}^k [g, v_i]^{e_i}$. Since $z$ is central, we deduce that $z = \prod_{i=1}^k [gz, v_i]^{e_i} \in C_G^*(gz) \subseteq \langle gz \rangle^G$, and consequently $g \in \langle gz \rangle^G$. \hfill $\square$

Lemma 3.5. Let $G$ be an MP-group, and suppose that Eq. (3.1) holds for each $g \in G \setminus Z(G)$. Then $G/\gamma(G)$ is an MP-group.

Proof. Lemma 3.4 shows that, for each $g \in G \setminus Z(G)$, any two distinct elements of $\Omega_{gZ(G)}$ are $\subseteq$-incomparable. Thus Proposition 2.4 applies. \hfill $\square$

From Lemma 3.1 and Lemma 3.5 we see that within the class of torsion-free, locally nilpotent groups the Magnus property passes from $G$ to $G/\gamma(G)$; this is a useful insight for a future characterisation (or even classification) of finitely generated, torsion-free nilpotent groups with the Magnus property.

Corollary 3.6. Let $G$ be a torsion-free, locally nilpotent group. If $G$ is an MP-group then so is $G/\gamma(G)$.

For the next result we recall the notion of a basic $\neg$-(MP)-witness pair in the wake of Lemma 2.6.

Proposition 3.7. Let $G$ be a free class-c nilpotent group of rank at least 2, where $c \in \mathbb{N}_{\geq 3}$. Then there exists a basic $\neg$-(MP)-witness pair for $G$.

Proof. The free class-c nilpotent group of rank 2 is a retract of $G$; by part (1) of Lemma 2.7 we may suppose that $G = \langle x, y \rangle$ is freely generated by two elements. Furthermore, by part (2) of Lemma 2.7 and induction on $c$ we may suppose that $c = 3$. In accordance with Witt's formula, the non-trivial sections of the lower central series of $G$ are: $G/\gamma_2(G) = \langle \bar{x}, \bar{y} \rangle \cong C_{\infty} \times C_{\infty}$, $\gamma_2(G)/\gamma_3(G) = \langle \bar{y}, \bar{x} \rangle \cong C_{\infty}$ and $Z(G) = \gamma_3(G) = \langle [y, x, x], [y, x, y] \rangle \cong C_{\infty} \times C_{\infty}$.

Let $v = [y, x, y] \in Z(G)$. Clearly, $x$ and $xy$ have the same normal closure in $G$, namely $\langle x \rangle^G = \langle x \rangle \gamma_2(G) = \langle xv \rangle^G$. Moreover, $x$ has infinite order modulo $[G, G]$. It remains to prove that $[x, w] \neq v$ for all $w \in G$.

Let $w \in G$. For $[x, w] = v$ it is necessary that $[x, w] \in Z(G)$ and consequently $w \in \langle x \rangle \gamma_2(G)$. But $w = z \prod_{i=1}^n [y, x]^m$ for $m, n \in \mathbb{Z}$ gives

$$[x, w] = [x, [y, x]^m] = [y, x]^{-n} \neq [y, x, y] = v.$$

The following straightforward example illustrates that there are finitely generated, nilpotent MP-groups (with non-trivial 3-torsion) of any prescribed nilpotency class.

Example 3.8. For $c \in \mathbb{N}$ the 2-generated group $G = \langle t, a \mid a^{3^c} = [a, t]a^{-3} = 1 \rangle \cong C_{\infty} \times C_{3^c}$ is nilpotent of class $c$ and an MP-group.

Indeed, let $g, h \in G$ with $\langle g \rangle^G = \langle h \rangle^G$. If $g = h = 1$ then there is nothing to show. Now suppose that $g$ and $h$ are non-trivial. There are unique parameters $l, m \in \mathbb{Z}$ with $0 \leq m < 3^c$ such that $g = t^la^m$. Put $n = n(g) = 1 + \min\{v_3(l), v_3(m)\} \in \mathbb{N}$, where $v_3(k)$ denotes the 3-adic valuation of an integer $k$. Lemma 2.8 and a routine calculation show that

$$M = C^*_G(g) = C^*_G(h) = \langle a^{3^n} \rangle = \{[g, w] \mid w \in G \}.$$
further details are given at the end of Section 5, where a related group $H$ is considered. It suffices to show that $g \equiv_M h$ or $g \equiv_M h^{-1}$.

If $l = 0$, then $n = 1 + 3b(m)$ and $(g)^G = (g) = a^{2^{n-1}}$ gives $(g)^G/M = (h)^G/M \cong C_3$; it follows that $g \equiv_M h$ or $g \equiv_M h^{-1}$. If $l \neq 0$, then $g$ and $h$ generate the same infinite cyclic subgroup modulo $M$, and thus $g \equiv_M h$ or $g \equiv_M h^{-1}$.

It would be interesting to construct finitely generated, nilpotent MP-groups of prescribed nilpotency class that are torsion-free. In fact, it is already a challenge to construct an explicitly a finitely generated, torsion-free, class-3 nilpotent MP-group, as we do below. In principle, Corollary 3.6 suggests that one proceeds by induction on the nilpotency class.

**Lemma 3.9.** Let $G$ be a group such that $Z(G)$ and $G/Z(G)$ are MP-groups. Suppose that for every $g \in G \setminus Z(G)$, the set $\{[g, w] \mid w \in G\}$ contains $(g)^G \cap Z(G)$. Then $G$ is an MP-group.

**Proof.** Suppose that $g, h \in G$ have the same normal closure in $G$. If $g \in Z(G)$ then $h \in Z(G)$; furthermore $(g) = (h)$ implies $g \in \{h, h^{-1}\}$, because $Z(G)$ is an MP-group.

Now suppose that $g \notin Z(G)$. Since $G/Z(G)$ is an MP-group, there exist $v \in G$ and $e \in \{-1, 1\}$ such that $g^v \equiv_{Z(G)} h^e$, and hence $g^v z = h^e$ for suitable $z \in (g)^G \cap Z(G)$. Choose $w \in G$ such that $g^w = gz$; then $g^{vv} = (gz)^v = g^v z = h^e$. \qed

**Corollary 3.10.** Let $G$ be a torsion-free group such that $G/Z(G)$ is an MP-group. If $C^G_G \cap Z(G) \subseteq \{[g, w] \mid w \in G\}$ for every $g \in G \setminus Z(G)$, then $G$ is an MP-group.

**Proof.** Since $G$ is torsion-free, $Z(G)$ is an MP-group. Let $g \in G \setminus Z(G)$. Since $G/Z(G)$ is torsion-free, we deduce that $(g)^G \cap Z(G) = C^G_G \cap Z(G)$. Thus the claim follows from Lemma 3.9. \qed

**Example 3.11.** We use Corollary 3.10 to construct a 4-generated, torsion-free, class-3 nilpotent MP-group of Hirsch length 9.

Let $F = \langle \hat{x}, \hat{y}, \hat{z}, \hat{w} \rangle$ be a free class-3 nilpotent group on four generators and consider

$$G = \langle x, y, z, w \rangle = F / \langle \{[\hat{z}, \hat{y}], [\hat{w}, \hat{z}]\} \cup R \rangle^F,$$

where $x, y, z, w$ denote the images of $\hat{x}, \hat{y}, \hat{z}, \hat{w}$ and

$$R = \left\{ \begin{array}{c}
[\hat{y}, \hat{x}, \hat{x}],
[\hat{z}, \hat{x}, \hat{x}],
[\hat{w}, \hat{x}, \hat{x}][\hat{z}, \hat{x}, \hat{y}]^{-1},
[\hat{y}, \hat{x}, \hat{w}],
[\hat{z}, \hat{x}, \hat{z}],
[\hat{w}, \hat{x}, \hat{z}],
[\hat{w}, \hat{y}, \hat{y}],
[\hat{w}, \hat{y}, \hat{z}] \end{array} \right\} .$$

Clearly, $G$ is a 4-generated nilpotent group of class at most 3. The precise structure of $G$ can be determined as follows. The collection process, subject to the initial ordering $\hat{x} < \hat{y} < \hat{z} < \hat{w}$, yields a Hall basis for $F$ consisting of $4 + 6 + 20 = 30$ basic commutators; for instance, see [CMZ17, Section 3.1.3]. The relators in $R$ simply tell us to cancel or identify certain basic commutators of degree 3. The additional relators $[\hat{z}, \hat{y}]$ and $[\hat{w}, \hat{z}]$ are basic commutators of degree 2 to be cancelled; they also tell us to cancel the basic commutators $[\hat{z}, \hat{y}, \hat{y}], [\hat{z}, \hat{y}, \hat{z}], [\hat{w}, \hat{z}], [\hat{w}, \hat{z}, \hat{w}]$ of degree 3. The relations $[z, y, x] = [w, z, x] = [w, z, y] = 1$, which also come with the relators $[\hat{z}, \hat{y}]$ and $[\hat{w}, \hat{z}]$, are already consequences of the relators in $R$ and the Witt identity. For instance, $[z, y, x] = [z, y, x][y, x, z][z, x, y]^{-1} = [z, y, x][y, x, z][x, z, y] = 1$. In this way we see that $G$ is torsion-free of nilpotency class 3 and admits a poly-$C_\infty$ basis $x, y, z, w, [y, x], [z, x], [w, x], [w, y], [y, x, z] = [z, x, y] = [w, x, x] = [w, y, w]$. 
such that
\[ G/\gamma_2(G) = \langle x, y, z, w \rangle \cong C_4^2, \quad \gamma_2(G)/\gamma_3(G) = \langle [y, x], [z, x], [w, x], [w, y] \rangle \cong C_4 \]
and
\[ Z = Z(G) = \gamma_3(G) = \langle [z, x, y] \rangle \cong C_\infty. \]
In particular, \( G \) has Hirsch length 9. The group commutator induces a bi-additive map \( \beta : \gamma_2(G)/Z \times G/\gamma_2(G) \to Z \) whose values on pairs of basis elements are given by the following commutator table:

| \([x, y]\) | \(x\) | \(y\) | \(z\) | \(w\) |
|-----------|--------|--------|--------|--------|
| \([y, x]\) | \([y, x, x] = 1\) | \([y, x, y] = 1\) | \([y, x, z] = [z, x, y]\) | \([y, x, w] = 1\) |
| \([z, x]\) | \([z, x, x] = 1\) | \([z, x, y] = 1\) | \([z, x, z] = 1\) | \([z, x, w] = 1\) |
| \([w, x]\) | \([w, x, x] = [z, x, y]\) | \([w, x, y] = 1\) | \([w, x, z] = 1\) | \([w, x, w] = 1\) |
| \([w, y]\) | \([w, y, x] = 1\) | \([w, y, y] = 1\) | \([w, y, z] = 1\) | \([w, y, w] = [z, x, y]\) |

The underlined entry is the only one that perhaps still requires a short explanation:
\[ [w, y, x] = [w, y, x][y, x, w][w, x, y]^{-1} = [w, y, x][y, x, w][x, w, y] = 1, \]
using again relators from \( R \) and the Witt identity. The table shows that \( \beta \) is a perfect pairing between \( \gamma_2(G)/Z \) and \( G/\gamma_2(G) \).

It remains to verify that \( G \) is an MP-group. By Proposition 3.2, the quotient \( G/Z \) has the Magnus property. By Corollary 3.10 it suffices to check that \( C^e_r(g) \cap Z \subseteq \{ [g, v] \mid v \in G \} \) for all \( g \in G \setminus Z \).

If \( g \in \gamma_2(G) \) then \( g \equiv Z \langle [y, x]^{m_1}[z, x]^{m_2}[w, x]^{m_3}[w, y]^{m_4} \rangle \) for suitable \( m_1, \ldots, m_4 \in \mathbb{Z} \); since \( \beta \) is a perfect pairing, this implies that \( C^e_r(g) = \langle [z, x, y]^n \rangle = \{ [g, v] \mid v \in G \} \) for \( n = \gcd(m_1, m_2, m_3, m_4) \). Now suppose that \( g \notin \gamma_2(G) \). Since \( G \) is nilpotent of class 3, the commutator identities
\[ [g, v]^{-1} = [g, v^{-1}]_{\in Z} \quad \text{and} \quad [g, v][g, w] = [g, wv]_{\in Z} [g, v, w]^{-1}, \]
for \( v, w \in G \), hold. From these we deduce that every \( h \in C^e_r(g) \cap Z \) is of the form \( h = h_1 h_2 \) with
\[ h_1 \in \{ [g, v] \mid v \in G \} \cap Z \quad \text{and} \quad h_2 \in \langle [g, w_1, w_2] \mid w_1, w_2 \in G \rangle \leq Z. \]
Observe that \( g \equiv \gamma_2(G) \langle x^{m_1}y^{m_2}z^{m_3}w^{m_4} \rangle \) for suitable \( m_1, \ldots, m_4 \in \mathbb{Z} \), and put \( n = \gcd(m_1, m_2, m_3, m_4) \). Since \( \beta \) is a perfect pairing, we deduce as previously that
\[ \langle [g, w_1, w_2] \mid w_1, w_2 \in G \rangle = \{ [g, w_1, w_2] \mid w_1, w_2 \in G \} = \langle [z, x, y]^n \rangle = \{ [g, w] \mid w \in \gamma_2(G) \}. \]
Writing \( h_1 = [g, v] \) and \( h_2 = [g, w] \), we obtain \( h = h_1 h_2 = [g, v][g, w]^v = [g, wv] \).

4. Polynilpotent groups

In this section we prove Theorems 1.1 and 1.2. To achieve this, we show first that the restricted wreath product \( C_\infty \wr C_\infty \) does not have the Magnus property.

**Proposition 4.1.** There is a basic \( \neg \)-(MP)-witness pair for the group \( C_\infty \wr C_\infty \).

**Proof.** We realise the wreath product as the group \( G = \langle t \rangle \rtimes A \cong C_\infty \wr C_\infty \), where \( A = \mathbb{Z}[T^{\pm 1}] = \mathbb{Z}[T, T^{-1}] \) is written additively and the action of \( t \) on \( A \) by conjugation.
is given by multiplication by $T$. The commutator of elements $x = t^m a$ and $y = t^n b$, with $m, n \in \mathbb{Z}$ and $a, b \in A$, is
\[ [x, y] = [t^m a, t^n b] = -a - b \cdot T^m + a \cdot T^n + b = a \cdot (T^m - 1) - b \cdot (T^n - 1). \]
In particular, $[G, G] = (T - 1)\mathbb{Z}[T^\pm 1] \leq A$ is the equal to the ideal of the ring $\mathbb{Z}[T^\pm 1]$ generated by $(T - 1)$. If $x = t^m a \in G \setminus A$ then $\langle x \rangle^G = \langle x \rangle \times C^*_G(x)$ and
\[ (4.1) \quad C^*_G(x) = I_x = a \cdot (T - 1) \mathbb{Z}[T^\pm 1] + (T^m - 1) \mathbb{Z}[T^\pm 1] \]
is the ideal of the ring $\mathbb{Z}[T^\pm 1]$ generated by $a \cdot (T - 1)$ and $T^m - 1$.

Choose a prime $p \geq 5$ and consider the ring of integers $\mathcal{O} = \mathbb{Z}[\zeta] \cong \mathbb{Z}[T]/\Phi_p \mathbb{Z}[T]$ of the $p$th cyclotomic field, where $\zeta$ denotes a primitive $p$th root of unity and $\Phi_p$ the $p$th cyclotomic polynomial. It is well known that $\mathcal{O}/(p - 1)\mathcal{O} \cong \mathbb{F}_p$ is a field with $p$ elements. By the Dirichlet unit theorem, the torsion-free rank of the unit group $\mathbb{Z}^\times$ is $(p - 1)/2 - 1 \geq 1$. Consider the $(p - 1)^{st}$ power $\nu$ of an element of infinite order, for instance, the power of a cyclotomic unit such as $\nu = (\zeta + 1)^{p-1}$, and write $\nu = f(\zeta)$ for a suitable polynomial $f \in 1 + (T - 1)\mathbb{Z}[T]$. Since $\nu$ has infinite order in $\mathcal{O}^\times$, we deduce that
\[ (4.2) \quad f \not\equiv_{T^{p-1}} T^m \quad \text{for all } n \in \mathbb{Z}. \]
Furthermore, we observe that $\nu^{-1}$ can be written in a similar form: $\nu^{-1} = \bar{f}(\zeta)$ for suitable $f \in 1 + (T - 1)\mathbb{Z}[T]$. The embedding of rings
\[ \mathbb{Z}[T^\pm 1]/(T^p - 1) \hookrightarrow \mathbb{Z}[T]/(T - 1) \times \mathbb{Z}[T]/\Phi_p \mathbb{Z}[T] \cong \mathbb{Z} \times \mathcal{O} \]
shows that
\[ (4.3) \quad f \cdot \bar{f} \equiv_{T^{p-1}} 1. \]

Now consider $g = t^p e \in G$, where $e \in A$ denotes the constant polynomial 1, and $v = f - 1 \in (T - 1)\mathbb{Z}(T) = [G, G] \subseteq A$. Clearly, $g$ has infinite order modulo $A \supseteq [G, G]$. Commutators $[g, w]$, with $w = t^p b \in G$ for $n \in \mathbb{Z}$ and $b \in \mathbb{Z}[T^\pm 1]$, are of the form
\[ (T^m - 1) - b \cdot (T^p - 1) \equiv_{T^{p-1}} T^m - 1; \]
thus Eq. (4.2) shows that $v \not\in \{[g, w] \mid w \in G\}$. It remains to prove that $\langle g \rangle^G = \langle gv \rangle^G$, and for this it is enough to show that $v = f - 1 \in (T - 1)\mathbb{Z}[T^\pm 1]$ lies in $I_g \cap I_{gv}$. Indeed, the general description provided in Eq. (4.1) yields directly
\[ I_g = (T - 1) \mathbb{Z}[T^\pm 1] + (T^p - 1) \mathbb{Z}[T^\pm 1] = (T - 1)\mathbb{Z}[T^\pm 1], \]
and Eq. (4.3) implies that
\[ I_{gv} = f \cdot (T - 1) \mathbb{Z}[T^\pm 1] + (T^p - 1) \mathbb{Z}[T^\pm 1] = (T - 1)\mathbb{Z}[T^\pm 1]. \]

\textbf{Example 4.2.} It easy to produce explicit examples of elements $g$ and $v$ in the proof of Proposition 4.1, for which the claims could be checked directly by computation. For $p = 5$ one may take $\nu = (1 + \zeta)^4 = 1 + (\zeta - 1)f_0(\zeta)$ with $f_0 = 3 + 2T - T^2 - 2T^3$ and $\nu^{-1} = 1 + (\zeta - 1)f_0(\zeta)$ with $f_0 = -T + T^2 - 2T^3$. A routine calculation yields
\[ (1 + (T - 1)f_0(T))(1 + (T - 1)f_0(T)) \]
\[ = 1 + (T - 1)(3 + T - 4T^3) + (T - 1)^2(-3T + T^2 - 3T^3 - 3T^4 + 4T^6) \]
\[ \equiv_{T^{p-1}} 1 + (T - 1)(3 + T - 4T^3) + (T - 1)^2(3 + 4T + 4T^2) = 1. \]
Next we would like to use Proposition 4.1 to prove that, for instance, the free metabelian group $G$ of rank 2 does not have the Magnus property. If $C_{\infty} \lhd C_{\infty}$ was a retract of $G$, then Lemma 2.2 would immediately yield the desired conclusion. But, in fact, $C_{\infty} \lhd C_{\infty}$ is not a retract of $G$. Assume, for a contradiction, that $G = H \ltimes K$ with $H \cong C_{\infty} \lhd C_{\infty}$. Then $[G, G] = [H, H] \ltimes [K, G]$ implies $C_{\infty} \ltimes C_{\infty} \cong G/[G, G] \cong H/[H, H] \ltimes K/[K, G] \cong C_{\infty} \ltimes C_{\infty} \ltimes K/[K, G]$, hence $[K, G] = K$. But $G$ is residually a finite nilpotent group; compare with [Gru57]. Thus there is a finite-index normal subgroup $N \leq G$ such that $G/N$ is nilpotent and $K \not\subset N$. This gives $1 \neq KN/N \leq G/N$ with $[KN/N, M/N] = KN/N$, a contradiction.

In the proof of the next proposition, which deals more generally with free abelian-by-(class-c nilpotent) groups, we by-pass the obstacle that $C_{\infty} \lhd C_{\infty}$ is not a retract.

**Proposition 4.3.** Let $c \in \mathbb{N}$, and let $G$ be an $N_{(c,1)}$-free group of rank 2 viz. a free abelian-by-(class-c nilpotent) group that is freely generated by two elements. Then there exists a basic $-(\text{MP})$-witness pair for $G$.

**Proof.** Let $H = \langle \bar{x}, \bar{g} \rangle = G/\gamma_{c+1}(G)$, the free class-c nilpotent group of rank 2, and let $R = \mathbb{Z}H$ denote the associated integral group ring. Let $V = eR \oplus fR$ be the free right $R$-module on two free generators. The Magnus embedding

$$G \hookrightarrow H \ltimes V, \quad w \mapsto \begin{pmatrix} \bar{w} & 0 \\ v_w & 1 \end{pmatrix}$$

allows us to realise the relatively free group $G$ as a group of $2 \times 2$ matrices with entries $\bar{w} \in H$ and $v_w \in V$; compare with [Wil10, § 2.1]. In this embedding, the matrices

$$x = \begin{pmatrix} \bar{x} & 0 \\ e & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \bar{g} & 0 \\ f & 1 \end{pmatrix}.$$ 

constitute free generators of the relatively free group $G$.

Let us consider the iterated commutator

$$z = \left[y, x_1, \ldots, x_c \right] \in \gamma_{c+1}(G).$$

We observe that $z \neq 1$; for instance, one can deduce $z \notin \gamma_{c+2}(G)$ from the fact that $z$ is a basic commutator in the Hall collection process, or carry out an explicit calculation as below. It follows that the subgroup $\langle x, z \rangle = \langle x \rangle \rtimes A \leq G$, with $\langle x \rangle \cong C_{\infty}$ and $A = \langle x^m \mid m \in \mathbb{Z} \rangle \leq \gamma_{c+1}(G)$ free abelian, is isomorphic to the wreath product $C_{\infty} \wr C_{\infty}$; employing a transversal for the subgroup $\langle \bar{x} \rangle \cong C_{\infty}$ of $H$, we can regard $V$ as a free $\mathbb{Z}\langle \bar{x} \rangle$-module (of infinite rank) and, accordingly, the non-trivial element $z$ of this module is moved about freely by the action of $x$ which is simply multiplication by the scalar $\bar{x} \in \mathbb{Z}\langle \bar{x} \rangle$.

Following the proof of Proposition 4.1, we consider elements

$$g = x^pz \quad \text{and} \quad v = z^{f(x)-1},$$

where $f \in 1 + (T - 1)\mathbb{Z}[T]$ is carefully chosen such that $g$ and $h = gv$ are not conjugate in $\langle x, z \rangle \cong C_{\infty} \wr C_{\infty}$, but generate the same normal closure in this subgroup and hence in $G$. Clearly, $g \equiv_{[G, G]} x^p \neq_{[G, G]} 1$ has infinite order in $G/[G, G]$ and $v \in [\langle x, z \rangle, \langle x, z \rangle] \subseteq [G, G]$. It suffices to prove that $v \neq [g, w]$, or equivalently $gw \neq h$, for all $w \in G$.

As in the proof of Proposition 4.1, let $\mathcal{O} = \mathbb{Z}[\zeta] \cong \mathbb{Z}[T]/\Phi_p\mathbb{Z}[T]$ denote the ring of integers of the $p^{th}$ cyclotomic field, with $\zeta$ a primitive $p^{th}$ root of unity; let $\nu =$...
Finally, we conjugate

\[
x^p = \begin{pmatrix} \bar{x}^p \\ 0 \end{pmatrix}, \quad \text{where } v_{x^p} = \mathbf{e} \Phi_p(\zeta) = 0,
\]

\[
z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{where } v_z = \mathbf{f}(\zeta - 1)^c.
\]

From this we continue to see that

\[
g = \begin{pmatrix} \bar{x}^p \\ 0 \end{pmatrix}, \quad \text{where } v_{g^\vartheta} = v_z = \mathbf{f}(\zeta - 1)^c,
\]

\[
h = \begin{pmatrix} \bar{x}^p \\ 0 \end{pmatrix}, \quad \text{where } v_{h^\vartheta} = (v_z^\vartheta) f(\zeta) = \mathbf{f}(\zeta - 1)^c \nu.
\]

Finally, we conjugate \( g \) by an arbitrary element \( w \in G \) to obtain

\[
\begin{pmatrix} \bar{w}^\vartheta \\ v_{gw} \\ v_g \end{pmatrix} = g^w = \begin{pmatrix} \bar{w} \\ v_w \\ 1 \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}^p \\ 0 \end{pmatrix} \begin{pmatrix} \bar{w} \\ v_g \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{w}^{-1} \bar{x}^p \bar{w} \\ -v_w \bar{w}^{-1} \bar{x}^p \bar{w} + v_g \bar{w} + v_w \\ 1 \end{pmatrix}
\]

and, for \( \bar{w}^n = \zeta^m \) with suitable \( m \in \{0, 1, \ldots, p - 1\} \), it follows that

\[
(v_{gw}^\vartheta) = (v_{w}^\vartheta) \begin{pmatrix} 1 - \zeta^p \\ v_g \end{pmatrix} + (v_g^\vartheta) \zeta^m = \mathbf{f}(\zeta - 1)^c \zeta^m.
\]

By construction, \( \nu \in O^\times \) has infinite order so that \( (\zeta - 1)^c \nu \neq (\zeta - 1)^c \zeta^m \). Comparison with our computations for \( h \) and \( g^w \) yields \( v_{h^\vartheta} \neq v_{g^w^\vartheta} \), and hence \( h \neq g^w \). \( \square \)

We require another variant of Propositions 3.7 and 4.3. First, we record a proposition which is perhaps folklore; we include a proof for completeness.

**Proposition 4.4.** Let \( G = \langle x, y \rangle \) be an \( \mathcal{N}_c \)-free group of rank 2, where \( l \in \mathbb{N}_{\geq 2} \) and \( c \in \mathbb{N}^l \). Let \( n \in \mathbb{N} \). Then the centraliser of \( x^n \) in \( G \) is \( C_G(x^n) = \langle x \rangle \).

**Proof.** We write \( \mathbf{c} = (c_1, \ldots, c_l) \), \( \mathbf{c}' = (c_1, \ldots, c_{l - 1}) \) and put

\[
K = \gamma_{(c_1 + 1, \ldots, c_{l - 1} + 1)}(G) \leq G
\]

so that \( G/K \) is an \( \mathcal{N}_c \)-free group of rank 2 and \( K \) is a free class-\( c_l \)-nilpotent group.

**Step 1.** First we argue by induction on \( l \) that \( C_G(x^n) = \langle x \rangle C_K(x^n) \). Indeed, it suffices to fill in the base of the induction. Suppose that \( l = 2 \), and put \( c = c_1 \) so that \( K = \gamma_{c + 1}(G) \). We observe that \( H = \langle \bar{x}, \bar{y} \rangle = G/\gamma_{c + 2}(G) \) is a free class-\((c + 1)\)-nilpotent group of rank 2. It suffices to show that \( C_H(\bar{x}^n) = \langle \bar{x} \rangle \gamma_{c + 1}(H) \).

Clearly, \( C_H(\bar{x}^n) \subseteq \langle \bar{x} \rangle \gamma_2(H) \). Hence it is enough to show that, for each \( k \in \{2, \ldots, c\} \), the homomorphism of abelian groups

\[
\gamma_k(H)/\gamma_{k + 1}(H) \to \gamma_{k + 1}(H)/\gamma_{k + 2}(H),
\]

\[
w \gamma_{k + 1}(H) \mapsto [\bar{x}^n, w] \gamma_{k + 2}(H) = [\bar{x}, w]^n \gamma_{k + 2}(H)
\]

is injective. We may interpret the torsion-free sections \( L_k = \gamma_k(H)/\gamma_{k + 1}(H) \) as the first few homogeneous components of the free Lie ring \( L = \bigoplus_{i=1}^{\infty} L_i \) on \( \bar{x}, \bar{y} \), the images of \( x, y \) in \( L_1 \). Furthermore, we may think of \( L \) as a Lie subring (generated by \( \bar{x}, \bar{y} \))
of the commutation Lie ring on a free associative ring $A$, where $A$ is freely generated by non-commuting indeterminates $\bar{x}, \bar{y}$; compare with [CMZ17, Chap. 3]. The free ring $A$ admits a natural $\mathbb{N}_0$-grading $A = \bigoplus_{i=0}^\infty A_i$, by means of the total $\{\bar{x}, \bar{y}\}$-degree function, which in turn induces the natural $\mathbb{N}$-grading $L = \bigoplus_{i=1}^\infty L_i$ of $L$.

Fix $i \in \mathbb{N}_{\geq 2}$ and let $a \in L_i = L \cap A_i$ be a non-zero homogeneous Lie element of degree $i$. We are to show that the Lie commutator $[\bar{x}, a]_{\text{Lie}}$ is non-zero. The monomials in $\bar{x}, \bar{y}$ of degree $i$ form a $\mathbb{Z}$-basis for the component $A_i$; we order them lexicographically

\[ \bar{x}^i < \bar{x}^{i-1}\bar{y} < \bar{x}^{i-2}\bar{y}^2 < \ldots < \bar{y}^{i-1}\bar{x} < \bar{y}^i, \]

and proceed similarly with the monomials of degree $i + 1$. Suppose that $a$, expressed as a $\mathbb{Z}$-linear combination of monomials, has leading term $m \ u(\bar{x}, \bar{y})$ for $m \in \mathbb{Z} \setminus \{0\}$ and $u(\bar{x}, \bar{y})$ the smallest monomial occurring with non-zero coefficient. Since $a \in L$ is a Lie element, we deduce that $u(\bar{x}, \bar{y}) \neq \bar{x}^i$ and hence the Lie commutator $[\bar{x}, a]_{\text{Lie}} = \bar{x}a - a\bar{x} \in L_{i+1} \subseteq A_{i+1}$ is non-zero with leading term $m \ \bar{x}u(\bar{x}, \bar{y})$.

**Step 2.** It remains to prove that $C_K(x^n) = 1$. Put $c = c_l$. Let $L = \bigoplus_{k=1}^\infty L_k$ denote the free class-$c$ nilpotent Lie ring associated to $K$ and its lower central series; thus $L_k \cong \gamma_k(K)/\gamma_{k+1}(K)$ for $1 \leq k \leq c$ as a free $\mathbb{Z}$-module, and the Lie commutator of two homogeneous elements is induced by the group commutator, as in Step 1 above. Extension of scalars yields the free class-$c$ nilpotent $\mathbb{Q}$-Lie algebra $L = \bigoplus_{k=1}^\infty L_k$, with $L_k = \mathbb{Q} \otimes_{\mathbb{Z}} L_k$ for each $k$. Clearly, conjugation by $x$ induces an automorphism $\xi$ of $L$ which respects the natural grading. It suffices to prove that, for each $k$, the only element of $L_k$ fixed by $\xi^n$ is 0.

Put $H = G/K$ and $R = \mathbb{Z} H$. The action of $G$ on $L$ factors through $H$, and $\bar{x} \in H$ generates an infinite cyclic subgroup. The Magnus embedding for the group $G/[K, K]$ shows that the $\mathbb{Z}(\bar{x})$-module $L_1$ embeds into a free $\mathbb{Z}(\bar{x})$-module (of infinite rank); compare with the proof of Proposition 4.3. Thus, the $\mathbb{Q}(\bar{x})$-module $L_1$ embeds into a free $\mathbb{Q}(\bar{x})$-module. Observe that $\mathbb{Q}(\bar{x})$ is just the ring of Laurent polynomials over $\mathbb{Q}$ and, in particular, a principal ideal domain. Therefore $L_1$ is itself a free $\mathbb{Q}(\bar{x})$-module, with $\mathbb{Q}(\bar{x})$-basis $e_1, e_2, \ldots$, say. Notice that $L_1$ admits the $\mathbb{Q}$-basis

\[ f_{i, m} = e_i \bar{x}^m, \quad i \in \mathbb{N} \text{ and } m \in \mathbb{Z}; \]

these basis elements are at the same time free generators of the free class-$c$ nilpotent $\mathbb{Q}$-Lie algebra $L$.

Now fix $k \in \{1, \ldots, c\}$ and consider the action of $\xi$ on $L_k$. We observe that $L_k$ is the $\mathbb{Q}$-span of the iterated Lie commutators

\[ F_{\bar{x}, m} = [f_{i_1, m_1}, f_{i_2, m_2}, \ldots, f_{i_k, m_k}]_{\text{Lie}}, \]

where $\bar{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ and $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. Furthermore, we understand the action of $\xi$ and hence of iterates $\xi^r$, $r \in \mathbb{N}$, on these Lie commutators:

\[ F_{\bar{x}, m} \xi^r = F_{\bar{x}, m} \xi^{\bar{i}, m} \cdot (r, r, \ldots, r). \]

Let $v \in L_k \setminus \{0\}$, written as a $\mathbb{Q}$-linear combination

\[ v = \sum_{\bar{i} \in \mathbb{N}^k, \bar{m} \in \mathbb{Z}^k} v(\bar{i}, \bar{m}) F_{\bar{x}, \bar{m}}, \]

where $v: \mathbb{N}^k \times \mathbb{Z}^k \to \mathbb{Q}$ is such that its ‘support’ $S = \{(\bar{i}, \bar{m}) \in \mathbb{N}^k \times \mathbb{Z}^k \mid v(\bar{i}, \bar{m}) \neq 0\}$ is finite. Also the ‘fine support’ in the second coordinate

\[ S_{\text{fine}} = \bigcup_{(\bar{i}, \bar{m}) \in S} \{m_1, \ldots, m_k\} \subseteq \mathbb{Z}, \]
Proposition 4.3. For this purpose we compare with the proof of Proposition 4.4. The generators modulo $V = \mathbb{Z}$ are non-trivial; in the additive notation, it corresponds to $\bar{\mathbb{Z}} \{\}$. is a set of representatives for the equivalence classes $R$ constitute an "equal or inverse to one another", then the elements $z, z_2$ in $K$ and their images $\bar{z}_1, \bar{z}_2$ modulo $[K, K]$ yield $\mathbb{Z}$-linear independent generators of $K^ab$; this follows, for instance, from the fact that $z_1$ and $z_2$ are basic commutators and form part of a Hall basis for $\gamma_{c+1}(G)/\gamma_{c+2}(G)$ and $\gamma_{c+2}(G)/\gamma_{c+3}(G)$, respectively, where $\gamma_{c+3}(G) \geq \gamma_{2c+2}(G) \geq [K, K]$. Hence the group commutator $z = [z_1, z_2] \in [K, K] \sim \{1\}$ is non-trivial; in the additive notation, it corresponds to $\bar{z}_1 \wedge \bar{z}_2 \in K^ab \wedge K^ab \sim \{0\}$. The action of $G$ on $K^ab$ and on $[K, K] \cong K^ab \wedge K^ab$ factors through $H = G/K$; concretely, $z^w = [z_1, z_2]^w = [\bar{z}_1^w, \bar{z}_2^w]$ translates to $(\bar{z}_1 \wedge \bar{z}_2)^w = (\bar{z}_1^w \wedge \bar{z}_2^w)$ for $w \in G$ with image $\bar{w} \in H$.

Let $R = \mathbb{Z}H$ denote the integral group ring associated to $H = \langle \bar{x}, \bar{y} \rangle$, and let $V = eR \oplus fR$ denote the free right $R$-module of rank 2. We compose reduction modulo $[K, K]$ with the Magnus embedding for $G/[K, K]$ to obtain a homomorphism $\eta: G \to G/[K, K] \hookrightarrow H \times V, \quad w \mapsto \begin{pmatrix} \bar{w} & 0 \\ v_w & 1 \end{pmatrix}$; compare with the proof of Proposition 4.3. The generators $x, y$ of $G$ are mapped to $x\eta = \begin{pmatrix} \bar{x} & 0 \\ e & 1 \end{pmatrix}$ and $y\eta = \begin{pmatrix} \bar{y} & 0 \\ f & 1 \end{pmatrix}$.

The exterior square $V \wedge V$ of the $\mathbb{Z}$-module $V$ is an $R$-module via the diagonal action. In fact, $V \wedge V$ is a free $R$-module (of countably infinite rank); if $H_0 \subseteq H \sim \{1\}$ is a set of representatives for the equivalence classes $\{\bar{w}, \bar{w}^{-1}\} \subseteq H \sim \{1\}$ of the relation "equal or inverse to one another", then the elements $e \wedge ew$, (for $w \in H_0$), $f \wedge fw$ (for $w \in H_0$), $e \wedge fw$ (for $w \in H$) constitute an $R$-basis for $V \wedge V$. Since the $R$-module $K^ab$ embeds into $V$, the $R$-module $K^ab \wedge K^ab$ embeds into the free $R$-module $V \wedge V$. A similar argument as in
the proof of Proposition 4.3 shows that the subgroup \( \langle x, z \rangle = \langle x \rangle \times \langle z^{m} \mid m \in \mathbb{Z} \rangle \) is isomorphic to the wreath product \( C_{\infty} \wr C_{\infty} \). As before, we consider elements
\[
g = x^{p}z \quad \text{and} \quad v = z^{f(x)-1},
\]
where \( f \in 1+(T-1)\mathbb{Z}[T] \) is such that \( g \) and \( gv \) are not conjugate in \( \langle x, z \rangle \), but generate the same normal closure in this subgroup and hence in \( G \). Clearly, \( g \equiv_{[G,G]} x^{p} \not\equiv_{[G,G]} 1 \) has infinite order in \( G/[G,G] \) and \( v \in [K,K] \subseteq [G,G] \). It suffices to prove that \( [g, w] \neq v \) for all \( w \in G \). Reduction modulo \([K,K] \) shows that for \( [g, w] = v \) it would be necessary that \( [x^{p}, w] \equiv_{[K,K]} 1 \); hence by Proposition 4.4 we only need to prove that \([g, w] \neq v \) for \( w \in [K,K] \).

As before let \( \mathcal{O} = \mathbb{Z}[\zeta] \) denote the ring of integers of the \( p^{th} \) cyclotomic field, with \( \zeta \) a primitive \( p^{th} \) root of unity; by construction, \( \nu = f(\zeta) \in \mathcal{O}^\times \) has infinite order. We consider the ring of Laurent polynomials \( \mathcal{O}[Y^{\pm 1}] = \mathcal{O}[Y, Y^{-1}] \). The natural projection of rings \( \pi: R \rightarrow \mathcal{O}[Y^{\pm 1}] \) specified by \( x^{n} = \zeta \) and \( y^{n} = Y \) induces a \( \pi \)-equivariant projection \( \vartheta: V \rightarrow \mathcal{O}[Y^{\pm 1}] \otimes \mathcal{O}[Y^{\pm 1}] = \bar{V} \), from the free \( R \)-module \( V \) onto a free \( \mathcal{O}[Y^{\pm 1}] \)-module \( \bar{V} \) on \( \mathcal{O} \).

It is straightforward to work out that
\[
z_{1}\eta = \begin{pmatrix} 1 \\ v_{z_{1}} \end{pmatrix}^{\vartheta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{where} \quad v_{z_{1}}^{\vartheta} = \hat{e}(1-Y)(\zeta-1)^{c-1}+\hat{f}(\zeta-1)^{c}.\]
\[
z_{2}\eta = \begin{pmatrix} 1 \\ v_{z_{2}} \end{pmatrix}^{\vartheta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{where} \quad v_{z_{2}}^{\vartheta} = \hat{e}(1-Y)^{2}(\zeta-1)^{c-1}+\hat{f}(1-Y)(\zeta-1)^{c}.\]

Restriction of scalars turns \( \bar{V} \) into a free \( \mathcal{O} \)-module, with an \( \mathcal{O} \)-basis consisting of
\[
\hat{e}Y^{m} (m \in \mathbb{Z}), \quad \hat{f}Y^{n} (n \in \mathbb{Z}).
\]
Thus the exterior square \( \hat{V} \wedge_{\mathcal{O}} \hat{V} \) over \( \mathcal{O} \) is a free \( \mathcal{O} \)-module, with \( \mathcal{O} \)-basis
\[
\hat{e}Y^{m} \wedge \hat{e}Y^{n} \quad \text{(for} \ m < n \text{),} \quad \hat{f}Y^{m} \wedge \hat{f}Y^{n} \quad \text{(for} \ m < n \text{),} \quad \hat{e}Y^{m} \wedge \hat{f}Y^{n}, \quad \text{where} \ m, n \in \mathbb{Z};
\]
below we express elements of \( \hat{V} \wedge_{\mathcal{O}} \hat{V} \) with respect to this \( \mathcal{O} \)-basis, keeping track of the coefficients of the basis element \( \hat{e}Y \wedge \hat{e}Y^{2} \).

Next we consider the composition \( \psi = \vartheta \circ \varphi \) of the \( \pi \)-equivariant morphism
\[
\hat{\varphi}: [K,K] \cong K^{ab} \wedge K^{ab} \hookrightarrow V \wedge V \rightarrow \hat{V} \wedge \hat{V}
\]
with the canonical homomorphism of \( \mathbb{Z} \)-modules \( \varphi: \hat{V} \wedge \hat{V} \rightarrow \hat{V} \wedge \hat{V} \). A routine computation yields
\[
z^{\psi} = [z_{1}, z_{2}]^{\vartheta \circ \varphi} = ([v_{z_{1}}^{\vartheta} \wedge v_{z_{2}}^{\vartheta}]^{\varphi})^{\vartheta}
\]
\[
= \left( (\hat{e}(1-Y)+\hat{f}(\zeta-1)) \wedge (\hat{e}(1-Y)^{2}+\hat{f}(Y-1)(\zeta-1)) \right)(\zeta-1)^{2c-2} = \ldots + (\hat{e}Y \wedge \hat{e}Y^{2})(-(\zeta-1)^{2c-2}) + \ldots,
\]
where on the far right-hand side we only display the \( \hat{e}Y \wedge \hat{e}Y^{2} \)-term. From this we deduce that
\[
v^{\psi} = (z^{f(x)-1})^{\psi} = z^{\psi}(f(\zeta)-1) = \ldots + (\hat{e}Y \wedge \hat{e}Y^{2})(-(\zeta-1)^{2c-2}(\nu-1)) + \ldots,
\]
where we again only record the \( \hat{e}Y \wedge \hat{e}Y^{2} \)-term to see that \( v^{\psi} \) is non-zero.

Finally, we deduce that \([g, w] \neq v \) for all \( w \in [K,K] \) from
\[
[g, w]^{\psi} = [x^{p}z, w]^{\psi} = [x^{p}, w]^{\psi} = (w^{1-x^{p}})^{\psi} = w^{\psi}(1-\zeta^{p}) = 0 \neq v^{\psi}. \quad \square
\]
Proof of Theorem 1.1. The group \( G \) is an \( N_c \)-free group of rank \( d \), for parameters \( d, l \in \mathbb{N} \) and \( c \in \mathbb{N}^l \). If \( d = 1 \) or \( c \in \{(1),(2)\} \) then \( G \) is nilpotent of class at most 2; in this situation Proposition 3.2 implies that \( G \) has the Magnus property.

Now suppose that \( d \geq 2 \) and that \( c \not\in \{(1),(2)\} \). We are to show that \( G \) does not have the Magnus property, and by Corollary 2.3 we may suppose that \( d = 2 \). If \( l = 1 \) then \( G \) is an \( N_{(c)} \)-free group with \( c = c_1 \geq 3 \), and Proposition 3.7 shows that \( G \) does not have the Magnus property. Likewise, if \( l = 2 \) and \( c_2 \in \{1,2\} \), then \( G \) is an \( N_{(c,1)} \)-free or \( N_{(c,2)} \)-free group of rank 2 for \( c = c_1 \), and \( G \) does not have the Magnus property by Propositions 4.3 and 4.5.

Thus, we may suppose that we are in none of these special circumstances. We write \( c = (c_1, \ldots, c_l) \in \mathbb{N}^l \) and distinguish two cases.

Case 1: \( c_l \geq 3 \). In this case \( l \geq 2 \) and we put \( N = \gamma_{(c_1+1,\ldots,c_{l-1}+1)}(G) \). We note that \( G/N \) is an \( N_{c'} \)-free group of rank 2, for \( c' = (c_1, \ldots, c_{l-1}) \), while \( N \) is a free class-\( c \) nilpotent group with \( c = c_l \geq 3 \) of countably infinite rank.

Case 2: \( c_l \in \{1,2\} \). In this case \( l \geq 3 \) and we put \( N = \gamma_{(c_1+1,\ldots,c_{l-2}+1)}(G) \). We note that \( G/N \) is an \( N_{c'} \)-free group of rank 2, for \( c' = (c_1, \ldots, c_{l-2}) \), while \( N \) is an \( N_{(c,1)} \)-free or \( N_{(c,2)} \)-free group with \( c = c_{l-1} \) of countably infinite rank.

In any case, Propositions 3.7, 4.3 and 4.5 and part (1) of Lemma 2.7 provide \( g, v \in N \) such that \( (g,v) \) is a basic \(-(\text{MP})\)-witness pair for \( N \). Clearly, \( g \) and \( h = gv \) have the same normal closure \( \langle g \rangle^G = \langle h \rangle^G \) in \( G \), and it suffices to prove that \( g \) and \( h \) are neither conjugate nor inverse-conjugate to one another in \( G \).

For a contradiction, assume that \( g^w \in \{h, h^{-1}\} \) for some \( w \in G \). We put \( H = G/N \) and \( R = ZH \). Observe that \( G/[N,N] \) is a free abelian-by-\( N_{c'} \) group of rank 2. The Magnus embedding for this group yields an embedding of the \( R \)-module \( N_{ab} = N/[N,N] \) into a free \( R \)-module. Our assumption yields \( v_g \bar{w} \bar{w} \in \{v_g, -v_g\} \), hence \( v_g(\bar{w}-1) = 0 \) or \( v_g(\bar{w}+1) = 0 \), where \( v_g \) denotes the image of \( g \) in \( N_{ab} \), regarded as a module element, and \( \bar{w} \) denotes the image of \( w \) in \( H \subseteq R \). We observe that \( g \not\in [N,N] \) implies that \( v_g \neq 0 \). The group \( H \) is right-orderable and thus the group ring \( R \) has no zero-divisors; see [Pas77, Ch. 13, Thm. 1.11], where the result is attributed to Bovdi [Bov60]. This implies \( \bar{w}-1 = 0 \) or \( \bar{w}+1 = 0 \) in \( R \). From \( \bar{w} \in H \) we see that \( \bar{w} \neq -1 \). Hence \( \bar{w} = 1 \) and \( w \in N \), in contradiction to the initial choice of \( g \) and \( h = gv \) which precludes that they are conjugate in \( N \).

Finally, we extend Theorem 1.1 to yet another class of relatively free groups. For any \( d, l \in \mathbb{N} \) and \( c = (c_1, \ldots, c_l) \in \mathbb{N}^l \), the free centre-by-\( N_c \) group of rank \( d \) can be constructed as the quotient \( F/\langle [\gamma_{(c_1+1,\ldots,c_{l+1})}, F] \rangle \) of an absolutely free group of rank \( d \).

**Proposition 4.6.** Let \( G \) be a free centre-by-\( N_c \) group of rank 2, where \( c \in \mathbb{N}^l \) with \( l \in \mathbb{N}_{\geq 2} \). Then there exists a basic \(-(\text{MP})\)-witness pair for the group \( G \).

**Proof.** Write \( G = \langle x, y \rangle \), with free generators \( x, y \), and \( c = (c_1, \ldots, c_l) \). Let \( Z = \gamma_{(c_1+1,\ldots,c_{l+1})}(G) \subseteq [G,G] \). Consider \( g = x \) and any \( v \in Z \setminus \{1\} \). Clearly, \( g \) has infinite order modulo \([G,G]\). It remains to show that \( \langle g \rangle^G = \langle gv \rangle^G \) and \( v \not\in \{[g,w] \mid w \in G\} \).

Since \( v \in [G,G] \subseteq \langle x \rangle^G \), we find \( k \in \mathbb{N}, c_1, \ldots, c_k \in \{1,-1\} \) and \( w_1, \ldots, w_k \in G \) such that \( v = \prod_{i=1}^{k} (x^{e_i})^{w_i} \). From \( x^{w_{i+1}} = x^{w_i} \cdot x^{e_i} \) we conclude that \( \prod_{i=1}^{k} (x^{e_i})^{w_i} = \prod_{i=1}^{k} ((xv)^{e_i})^{w_i} \in \langle g \rangle^G \cap \langle gv \rangle^G \).
and hence $\langle g \rangle^G = \langle gw \rangle^G$.

Next assume, for a contradiction, that $[g, w] = v$ for some $w \in G$. Then $[x, w] \equiv_Z 1$, and Proposition 4.4 implies $w = x^m z$, for suitable $m \in \mathbb{Z}$ and $z \in \mathbb{Z}$. This gives $[g, w] = [x, x^m z] = 1 \neq v$, a contradiction. □

Proof of Theorem 1.2. The group $G$ is a free centre-by-$N_c$ group $G$ of rank $d$, for parameters $d, l \in \mathbb{N}$ and $c \in \mathbb{N}^d$. If $d = 1$ or $c = (1)$ then $G$ is nilpotent of class at most 2; in this situation Proposition 3.2 implies that $G$ has the Magnus property.

Now suppose that $d \geq 2$ and that $c \neq (1)$. We are to show that $G$ does not have the Magnus property, and by Corollary 2.3 we may suppose that $d = 2$. If $l = 1$ then $G$ is an $N_{c+1}$-free group with $c = c_1 \geq 2$, and Proposition 3.7 shows that $G$ does not have the Magnus property. If $l \geq 2$ then Proposition 4.6 shows that $G$ does not have the Magnus property. □

5. Torsion-free nilpotent groups with the Magnus property

In this section we establish Theorem 1.3. Let $c \in \mathbb{N}$. In order to construct a torsion-free, nilpotent $\text{MP}$-group of prescribed nilpotency class $c$ we aim to build an ultraproduct $G = (\prod_{p \in P} G_p)/\sim_\mathcal{U}$ of suitable finite $p$-groups $G_p$, where $P = \mathbb{P}_{>2}$ denotes the set of all odd primes, and to appeal to Łoś’s theorem.

Being class-$c$ nilpotent, not having $q$-torsion for a given prime $q$, and possessing the Magnus property are first-order properties; thus it would suffice to construct a family of finite nilpotent $\text{MP}$-groups $G_p$, $p \in P$, such that each group has nilpotency class $c$ and such that for any prime $q$ there are only finitely many $p \in P$ with $q \mid |G_p|$. However, finite nilpotent $\text{MP}$-groups are necessarily $\{2, 3\}$-groups. More generally, every group with the Magnus property is inverse semi-rational, that is, every pair of elements generating the same subgroup (not necessarily normal) is already a pair of conjugate or inverse-conjugate elements. Finite groups with this property can be characterised using character theory. Chillag and Dolfi [CD10] establish that all finite soluble inverse semi-rational groups are $\{2, 3, 5, 7, 13\}$-groups. Consequently, no family as described above exists. However, we can salvage our strategy by considering a variant of the Magnus property.

For convenience we use $[k, l]_\mathbb{Z} = \{m \in \mathbb{Z} \mid k \leq m \leq l\}$, for $k, l \in \mathbb{Z}$, as a short notation for intervals in $\mathbb{Z}$. Suppose that $G_p$, $p \in P$, is a family of groups with the following properties:

(i) for each $p$, the group $G_p$ is a metabelian finite $p$-group of nilpotency class $c$;
(ii) there exists a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that, for each $p \in P$, the group $G_p$ satisfies the following uniform, but ‘weak’ Magnus property:

\[
\forall g, h \in G_p \quad \forall N \in \mathbb{N} \quad \forall k, l \in [0, N]_\mathbb{Z} \quad \forall e_1, \ldots, e_k \in \{1, -1\} \quad \forall v_1, \ldots, v_k \in G_p \quad \forall d_1, \ldots, d_l \in \{1, -1\} \quad \forall w_1, \ldots, w_l \in G_p : \\
\quad (wM_f) \quad h = \prod_{i=1}^k (g^{e_i})^{v_i} \quad g = \prod_{j=1}^l (h^{d_j})^{w_j} \\
\quad \quad \implies \quad (\exists r, s \in [-f(N), f(N)]_\mathbb{Z} \quad \exists v, w \in G_p : \\
\quad \quad \quad g = (h^r)^v \quad h = (g^s)^w ).
\]

We observe that the quantifier over the integer $N$ can be eliminated by passing to a countable collection of sentences in the first-order language of groups; the quantifiers over $k, l$ are purely for convenience and can be eliminated directly.
Let $U$ be a non-principal ultrafilter on the index set $P$. Then, by Loš’s theorem, the ultraproduct
\[ \mathcal{G} = \left( \prod_{p \in P} G_p \right) / U \]
is a metabelian, torsion-free, class-$c$ nilpotent group satisfying the uniform ‘weak’ Magnus property (\textbf{wM}_f); compare with [CK90, Thm. 4.1.9]. But, since $\mathcal{G}$ is torsion-free nilpotent, the latter implies that $\mathcal{G}$ has the Magnus property. Indeed, suppose that $g, h \in \mathcal{G}$ are such that $\langle \dot{g} \rangle^G = \langle \dot{h} \rangle^G$. If $g = 1$ then $h = 1$, and $g = h = 0$ so that $g$ and $h$ are certainly conjugate. Now suppose that $g \neq 1$. Then (wM_f) yields $r, s \in \mathbb{Z}$ and $v, w \in \mathcal{G}$ such that $g = (h^r)^v$ and $h = (g^s)^w$, thus
\[ g = (h^r)^v = \left( (g^s)^w \right)^v = (g^{rs})^{vw}. \]

Consider the upper central series $1 = Z_0(\mathcal{G}) \triangleleft Z_1(\mathcal{G}) \triangleleft \ldots \triangleleft Z_c(\mathcal{G}) = \mathcal{G}$ of the nilpotent group $\mathcal{G}$. Since $\mathcal{G} \neq 1$, we find $i \in [1, c]_{\mathbb{Z}}$ such that $g \in Z_i(\mathcal{G}) \setminus Z_{i-1}(\mathcal{G})$. Since $Z_i(\mathcal{G})/Z_{i-1}(\mathcal{G})$ is torsion-free, $g$ generates an infinite cyclic group modulo $Z_{i-1}(\mathcal{G})$ and the congruence $g \equiv Z_{i-1}(\mathcal{G}) g^{rs}$ implies that $rs = 1$. Thus $r \in \{1, -1\}$, and $g = (h^r)^v$ is conjugate to $h$ or to $h^{-1}$ in $\mathcal{G}$.

Finally, since all relevant properties of $\mathcal{G}$, including the ordinary Magnus property, are expressible in terms of first-order sentences, the Löwenheim–Skolem theorem [CK90, Cor. 2.1.4] shows that there exists a countable MP-group $\mathcal{G}$ which is metabelian, torsion-free and nilpotent of class precisely $c$. This establishes Theorem 1.3.

It remains to construct the family of groups $G_p$, $p \in P$, with the properties (i) and (ii) described above. We give one rather concrete construction. Fix an odd prime $p$, and consider
\[ G = G_p = \langle t, a \mid [a, t] = a_p, t^{p^{c-1}} = a_p^{c} = 1 \rangle. \]
Clearly, $G = \langle t \rangle \ltimes \langle a \rangle$ is metacyclic, with $\langle t \rangle \cong C_{p^{c-1}}$ and $\langle a \rangle \cong C_{p^c}$. It is easy to work out the lower central series:
\[ \gamma_i(G) = G \quad \text{and} \quad \gamma_{i+1}(G) = \langle a^{p^i} \rangle \quad \text{for } i \in \mathbb{N}; \]
in particular, $G$ has nilpotency class $c$. In order to check the ‘weak’ Magnus property, we make use of the following lemma, which is heuristically a torsion analogue of Proposition 3.2.

**Lemma 5.1.** Let $G$ be a finite nilpotent group such that $C_G^+(x) = \{ [x, w] \mid w \in G \}$ for every $x \in G$. Then $G$ has the ‘weak’ Magnus property (wM_f) for $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto n$.

**Proof.** Let $g, h \in G$ and suppose that $k, l \in \mathbb{N}_0$ and
\[ h = \prod_{i=1}^{k} (g^{e_i})^{v_i} \quad \text{and} \quad g = \prod_{j=1}^{l} (h^{d_j})^{w_j} \]
for suitable $e_1, \ldots, e_k, d_1, \ldots, d_l \in \{1, -1\}$ and $v_1, \ldots, v_k, w_1, \ldots, w_l \in G$. In particular, this implies that $\langle g \rangle^G = \langle h \rangle^G$. By symmetry, it suffices to show that there exist $w \in G$ and $s \in [-l, l]_{\mathbb{Z}}$ such that $h = (g^s)^w$.

If $g = 1$ then $h = 1$, and no further explanations are necessary. Now suppose that $g \neq 1$, and write $M = C_G^+(g) = C_G^+(h) \subseteq G$; see Lemma 2.8. From Eq. (5.2) we deduce that $h \equiv_M g^s$, for some $s \in [-k, k]_{\mathbb{Z}}$.

We claim that $\langle g \rangle = \langle g^s \rangle$. For this it is enough to show that $p \nmid s$ for every prime $p$ that divides the order of $g$. The finite nilpotent group $G$ is the direct product of its Sylow subgroups; let $\bar{x}$ denote the image of $x \in G$ under the canonical projection onto
the Sylow $p$-subgroup of $G$. Then $\bar{g} \neq 1$ implies that $g \in Z_i(\overline{G}) \setminus Z_{i-1}(\overline{G})$, for suitable $i \in \mathbb{N}$, hence $\overline{M} \leq Z_{i-1}(\overline{G})$ and $\bar{g} \not\in \overline{M}$. Consequently, $\langle \bar{g} \rangle \overline{M} = \langle \bar{g} \rangle \equiv \langle h \rangle \overline{G} = (\bar{g}^s) \overline{M}$ implies $p \nmid s$.

Using our general assumption on cocentralisers in $G$, we deduce from $\langle g \rangle = (g^s)$ that $M = C_G^*(g^s) = \{[g^s, w] \mid w \in G\}$. Therefore $h \equiv_M g^s$ shows that there exists $w \in G$ such that $h = (g^s)^w$. \hfill $\square$

It remains to verify that the condition on cocentralisers in Lemma 5.1 applies to the concrete groups $G = G_p$, defined in (5.1). Recall that $p > 2$. Actually, it is convenient to check the required property for the compact $p$-adic analytic group

$$H = \langle t, a \mid [a, t] = a^p \rangle_{\text{pro-}p} \cong (1 + p\mathbb{Z}_p) \ltimes \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers and the multiplicative group of one-units $1 + p\mathbb{Z}_p = (1 + p)^\lambda \mid \lambda \in \mathbb{Z}_p$ acts naturally on the additive group $\mathbb{Z}_p$. The group $H$ maps naturally onto $G$, with kernel $\langle (1^\nu, a^\nu) \rangle$, and it is easy to see that the condition on cocentralisers that we are interested in is inherited by factor groups.

Let $h \in H$. For $h = 1$, it is clear that $\{[h, y] \mid y \in H\} = \{1\}$ is closed under multiplication. Now suppose that $h \neq 1$. Then $h$ is of the form $h = t^a a^m$ for uniquely determined $\lambda, \mu \in \mathbb{Z}_p$, not both equal to 0. Easy computations show:

$$\{[h, a^\nu] \mid \nu \in \mathbb{Z}_p\} = \{a^{(-1 + p)\lambda + 1}\nu} \mid \nu \in \mathbb{Z}_p\} = \{a^\sigma \mid \sigma \in p^{1 + v_p(\lambda)} \mathbb{Z}_p\},$$

$$\{[h, t^\nu] \mid \nu \in \mathbb{Z}_p\} = \{a^{(-1 + p)\nu + \lambda}\nu} \mid \nu \in \mathbb{Z}_p\} = \{a^\sigma \mid \sigma \in p^{1 + v_p(\mu)} \mathbb{Z}_p\},$$

where $v_p: \mathbb{Z}_p \to \mathbb{N}_0 \cup \{\infty\}$ denotes the $p$-adic valuation map.

Put $m = 1 + \min\{v_p(\lambda), v_p(\mu)\}$. As $A_m = \{a^\sigma \mid \sigma \in p^m \mathbb{Z}_p\}$ is a closed normal subgroup of $H$, the well-known commutator identity $[a, bc] = [a, c][a, b]c$ for arbitrary group elements $a, b, c$ shows that

$$\{[h, y] \mid y \in H\} = A_m \trianglelefteq H$$

is indeed closed under multiplication.

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