Crossed Bimodules over Rings and Shukla Cohomology

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Abstract
In this paper we present some applications of Ann-category theory to classification of crossed bimodules over rings, classification of ring extensions of the type of a crossed bimodule.

AMS Subject Classification: 18D10, 16E40, 16S70
Keywords: Ann-category, crossed bimodule, obstruction, ring extension, ring cohomology

1 Introduction
Crossed modules over groups were introduced by J. H. C. Whitehead [15]. A crossed module over a group $G$ with kernel a $G$-module $M$ represents an element in the cohomology $H^3(G, M)$ [8]. The results on group extensions of the type of a crossed module were also represented by the cohomology of groups [6].

Later, H-J. Baues [2] introduced crossed modules over $k$-algebras. Crossed modules over $k$-algebras which are $k$-split with the same kernel $M$ and cokernel $B$ were classified by Hochschild cohomology $H^3_{	ext{Hoch}}(B, M)$ [3].

In [4] the field $k$ is replaced by a commutative ring $K$, and crossed modules over $K$-algebras were called crossed bimodules. In particular, if $K = \mathbb{Z}$ one obtains crossed bimodules over rings.

Crossed modules over groups can be defined over rings in a different way under the name of $E$-systems. The notion of an $E$-system is weaker than that of a crossed bimodule over rings.

Crossed modules over groups are often studied in the form of $G$-groupoids [5], or strict 2-groups [11]. From this point, we represent $E$-systems in the form of strict Ann-categories (also called strict 2-rings). Hence, one can use the results on Ann-category theory to study crossed bimodules over rings.

The plan of this paper is, briefly, as follows. Section 2 is dedicated to review definitions and some basic facts concerning Ann-categories. In Section 3, we introduce the concept of an $E$-system and prove that there is an isomorphism between the category of regular $E$-systems and that of crossed bimodules over rings. The relation among these concepts and crossed $C$-modules in the sense of T. Porter [10] is also discussed. The next section is devoted to showing a categorical equivalence of the category of $E$-systems and a subcategory of the category of strict Ann-categories, which is an extending of the result of R. Brown and C. Spencer [5].
The group extensions of the type of a crossed module were dealt with by R. Brown and O. Mucuk [6]. The similar results for $\partial$-extensions by an algebra $R$ were done by H.-J. Baues and T. Pirashvili [4] in a particular case. In Section 5 we solve this problem for ring extensions of the type of an E-system by Shukla cohomology groups. Our classification result contains the result in [4] when $R$ is a ring.

2 Ann-categories

We state a minimum of necessary concepts and facts of Ann-categories and Ann-functors (see [11]).

A Gr-category (or a categorical group) is a monoidal category in which all objects are invertible and the background category is a groupoid. A Picard category (or a symmetric categorical group) is a Gr-category equipped with a symmetry constraint which is compatible with associativity constraint.

Definition 1. An Ann-category consists of
i) a category $A$ together with two bifunctors $\oplus, \otimes : A \times A \rightarrow A$;
ii) a fixed object $0 \in \text{Ob}(A)$ together with natural isomorphisms $a_+, c_0$ such that $(A, \oplus, a_+, c_0, (0, 0, 0))$ is a Picard category;
iii) a fixed object $1 \in \text{Ob}(A)$ together with natural isomorphisms $a_1, l_1, r_1$ such that $(A, \otimes, a_1, (1, 1, 1))$ is a monoidal category;
iv) natural isomorphisms $L, R$ given by

$$L_{A,X,Y} : A \otimes (X \oplus Y) \rightarrow (A \otimes X) \oplus (A \otimes Y),$$
$$R_{X,Y,A} : (X \oplus Y) \otimes A \rightarrow (X \otimes A) \oplus (Y \otimes A)$$

such that the following conditions hold:
(Ann - 1) for $A \in \text{Ob}(A)$, the pairs $(L^A, \tilde{L}^A), (R^A, \tilde{R}^A)$ defined by

$$L^A = A \otimes -$$
$$\tilde{L}_{X,Y} = L_{A,X,Y}$$
$$R^A = - \otimes A$$
$$\tilde{R}^A_{X,Y} = R_{X,Y,A}$$

are $\oplus$-functors which are compatible with $a_+$ and $c$;
(Ann - 2) for all $A, B, X, Y \in \text{Ob}(A)$, the following diagrams commute

$$\begin{array}{ccc}
(AB)(X \oplus Y) & \xrightarrow{a_{A,B,X \oplus Y}} & A(B(X \oplus Y)) \\
& \xrightarrow{id_A \otimes \tilde{L}^B} & \xrightarrow{\tilde{L}^A} \xrightarrow{L^A} A(BX \oplus BY)
\end{array}$$

$$\begin{array}{ccc}
(AB)X \oplus (AB)Y & \xrightarrow{a_{A,B,X \oplus Y} \oplus a_{A,B,Y}} & A(BX) \oplus A(BY)
\end{array}$$

$$\begin{array}{ccc}
(X \oplus Y)(BA) & \xrightarrow{a_{X \oplus Y,B,A}} & ((X \oplus Y)B)A \\
& \xrightarrow{\tilde{R}^B \oplus \tilde{R}^A} & \xrightarrow{\tilde{R}^A} (XB \oplus YB)A
\end{array}$$

$$\begin{array}{ccc}
X(BA) \oplus Y(BA) & \xrightarrow{a_{X,B,A \oplus Y,B,A}} & (XB)A \oplus (YB)A
\end{array}$$
\[(A \oplus Y)B \xrightarrow{\alpha_{A,X \oplus Y,B}} A((X \oplus Y)B) \xrightarrow{id_A \otimes R_B} A(XB \oplus YB)\]

\[(AX \oplus AY)B \xrightarrow{\beta_B} (AX)B \oplus (AY)B \xrightarrow{\alpha_{A,X \oplus Y,B}} A(XB) \oplus A(YB)\]

\[(A \oplus B)X \oplus (A \oplus B)Y \xrightarrow{R_X \oplus R_Y} (A \oplus B)(X \oplus Y) \xrightarrow{\alpha_{A,X \oplus Y,B}} A(X \oplus Y) \oplus B(X \oplus Y)\]

\[(AX \oplus BX) \oplus (AY \oplus BY) \xrightarrow{v} (AX \oplus AY) \oplus (BX \oplus BY)\]

where \(v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \rightarrow (U \oplus Z) \oplus (V \oplus T)\) is a unique morphism constructed from \(\oplus, a, c, id\) of the symmetric monoidal category \((A, \oplus)\); (Ann - 3) for the unit \(1 \in \text{Ob}(A)\) of the operation \(\otimes\), the following diagrams commute:

\[
\begin{align*}
1(X \oplus Y) & \xrightarrow{L^1} 1X \oplus 1Y \\
X \oplus Y & \xrightarrow{1_X \oplus 1_Y} X \oplus Y \\
X1 \oplus Y1 & \xrightarrow{R^1} X1 \oplus Y1
\end{align*}
\]

An Ann-category \(A\) is regular if its symmetry constraint satisfies the condition \(c_{X,X} = id\), and strict if all of its constraints are identities.

**Example 1.** Let \(A = (A, \oplus)\) be a Picard category whose unity and associativity constraints are identities. Denote by \(\text{End}(A)\) a category whose objects are symmetric monoidal functors from \(A\) to \(A\) and whose morphisms are \(\oplus\)-morphisms. Then, \(\text{End}(A)\) is a Picard category together with the operation \(\oplus\) on monoidal functors and on morphisms. In this \(\oplus\)-category, the unity and associativity constraints are identities, the commutativity constraint is given by

\[(c_{F,G})_X = c_{F_X,G_X}, \quad X \in \text{Ob}(A), \quad F, G \in \text{End}(A)\]

The operation \(\otimes\) on \(\text{End}(A)\) is naturally defined being the composition of functors. Then, \(\text{End}(A)\) together with two operations \(\oplus, \otimes\) is an Ann-category in which the left distributivity constraint is given by

\[(\sigma_{F,G,H})_X = F_{GX,HX}, \quad X \in \text{Ob}(A),\]

and other constraints are identities (for details, see [1]).

**Example 2.** Let \(R\) be a ring with an unit and \(M\) be an \(R\)-bimodule. The pair \(I = (R, M)\) is a category whose objects are elements of \(R\) and whose morphisms are automorphisms \((r, a) : r \rightarrow r, r \in R, a \in M\). The composition of morphisms is given by the addition in \(M\). Two operations \(\oplus\) and \(\otimes\) on \(I\) is defined by

\[x \oplus y = x + y, \quad (x, a) + (y, b) = (x + y, a + b)\]
\[ x \otimes y = xy, \ (x, a) \otimes (y, b) = (xy, xb + ay). \]

The constraints of \( I \) are identities, except for left distributivity and commutativity constraints which are given by

\[
\mathcal{L}_{x,y,z} = (\bullet, \lambda(x, y, z)) : x(y + z) \to xy + xz, \\
\mathcal{C}^+_{x,y} = (\bullet, \eta(x, y)) : x + y \to y + x,
\]

where \( \lambda : R^3 \to M, \eta : R^2 \to M \) are functions satisfying the appropriate coherence conditions.

Here are standard consequences of the axioms of an Ann-category.

**Lemma 1.** For every Ann-category \( A \) there exist uniquely isomorphisms

\[
\hat{L}^A : A \otimes 0 \to 0, \quad \hat{R}^A : 0 \otimes A \to 0,
\]

where \( A \in \text{Ob}(A) \), such that \( \oplus \)-functors \((\hat{L}^A, \hat{L}, \hat{L}^A)\) and \((\hat{R}^A, \hat{R}, \hat{R}^A)\) are compatible with unit constraints \( (0, g, d) \).

It is easy to see that if \((A, \oplus)\) and \((A', \oplus)\) are Gr-categories, then every \( \oplus \)-functor \((F, \bar{F}, F_*) : A \to A'\), which is compatible with associativity constraints, is a monoidal functor. Thus, we state the following definition.

**Definition 2.** Let \( A \) and \( A' \) be Ann-categories. An Ann-functor \((F, \bar{F}, \tilde{F}, F_*) : A \to A'\) consists of a functor \( F : A \to A' \), natural isomorphisms

\[
\bar{F}_{X,Y} : F(X \oplus Y) \to F(X) \oplus F(Y), \quad \tilde{F}_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y),
\]

and an isomorphism \( F_* : F(1) \to 1' \) such that \((F, \bar{F})\) is a symmetric monoidal functor for the operation \( \oplus \), \((F, \tilde{F}, F_*)\) is a monoidal functor for the operation \( \otimes \), and the following diagrams commute

\[
\begin{array}{ccc}
F(X(Y \oplus Z)) & \xrightarrow{\bar{F}} & F.X.F(Y \oplus Z) \\
F((X \oplus Y)Z) & \xrightarrow{\bar{F}} & F(X \oplus Y).FZ \\
F((X \otimes Y)Z) & \xrightarrow{\bar{F} \otimes \bar{F}} & (F(X \otimes Y).FZ) \\
\end{array}
\]

These diagrams are called the compatibility of the functor \( F \) with the distributivity constraints.

An Ann-morphism (or a homotopy)

\[
\theta : (F, \bar{F}, \tilde{F}, F_*) \to (F', \bar{F}', \tilde{F}', F'_*)
\]

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between Ann-functors is an $\oplus$-morphism, as well as an $\otimes$-morphism.

If there exist an Ann-functor $(F', \tilde{F}', F'_s) : \mathcal{A}' \to \mathcal{A}$ and Ann-morphisms $F' \tilde{F} \sim id_{\mathcal{A}}$, $FF' \sim id_{\mathcal{A}'_s}$, we say that $(F, \tilde{F}, F_s)$ is an Ann-equivalence, and $\mathcal{A}, \mathcal{A}'$ are Ann-equivalent.

For an Ann-category $\mathcal{A}$, the set $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ is a ring with two operations $+, \times$ induced by the functors $\oplus, \otimes$ on $\mathcal{A}$, and the set $M = \pi_1 \mathcal{A} = Aut(0)$ is a group with the composition denoted by $\circ$. Moreover, $M$ is a $R$-bimodule with the actions

$$sa = \lambda_X(a) \quad as = \rho_X(a)$$

for $X \in s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$, and $\lambda_X, \rho_X$ satisfy

$$\lambda_X(a) \circ \tilde{L}^X = \tilde{L}^X \circ (id \otimes a) : X.0 \to 0,$$
$$\rho_X(a) \circ \tilde{R}^X = \tilde{R}^X \circ (a \otimes id) : 0.X \to 0.$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_\mathcal{A}$ of $\mathcal{A}$ via the structure transport (for details, see [11]). The objects of $S_\mathcal{A}$ are the elements of the ring $\pi_0 \mathcal{A}$. A morphism is an isomorphism $(s, a) : s \to s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0 \mathcal{A}$, choose an object $X_s \in \text{Ob}(\mathcal{A})$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \to X_s$ such that $i_{X_s} = id_{X_s}$. We obtain two functors

$$\begin{align*}
G : \mathcal{A} \to S_\mathcal{A} \\
G(X) = [X] = s \\
G(X \xrightarrow{f} Y) = (s, \gamma^{-1}_X(i_Y fi^{-1}_X)),
\end{align*}$$

and

$$\begin{align*}
H : S_\mathcal{A} \to \mathcal{A} \\
H(s) = X_s \\
H(s, u) = \gamma_X(u),
\end{align*}$$

for $X, Y \in s, f : X \to Y$, and $\gamma_X$

$$\gamma_X(a) = g_X \circ (a \oplus id) \circ g^{-1}_X.$$  \hspace{1cm} (1)

Two operations on $S_\mathcal{A}$ are given by

$$\begin{align*}
s \oplus t &= G(H(s) \oplus H(t)) = s + t, \\
s \otimes t &= G(H(s) \otimes H(t)) = st, \\
(s, a) \oplus (t, b) &= G(H(s, a) \oplus H(t, b)) = (s + t, a + b), \\
(s, a) \otimes (t, b) &= G(H(s, a) \otimes H(t, b)) = (st, sb + at),
\end{align*}$$

for $s, t \in \pi_0 \mathcal{A}, a, b \in \pi_1 \mathcal{A}$. Clearly, they do not depend on the choice of the representative $(X_s, i_X)$.

The constraints in $S_\mathcal{A}$ are defined by sticks. A stick of $\mathcal{A}$ is a representative $(X_s, i_X)$ such that

$$\begin{align*}
i_{0 \otimes X} &= g_X, \\
i_{X_s \oplus 0} &= d_X, \\
i_{1 \otimes X} &= 1_X, \\
i_{X_s \otimes 1} &= r_X, \\
i_{0 \otimes X} &= \tilde{L}^X, \\
i_{X_s \otimes 0} &= \tilde{L}^X.
\end{align*}$$
The unit constraints in $S_A$ are $(0, id, id)$ and $(1, id, id)$. The family of the rest ones, $h = (\xi, \eta, \alpha, \lambda, \rho)$, is defined by the compatibility of the constraints $a_+, c, a, L, R$ of $A$ with the functor $H$ and isomorphisms

$$\tilde{H} = i^{-1}_{X_s \otimes X_t}, \tilde{\tilde{H}} = \tilde{i}^{-1}_{X_s \otimes X_t}. \tag{2}$$

Then $(H, \tilde{H}, \tilde{\tilde{H}}) : S_A \to A$ is an Ann-equivalence. Besides, the functor $G : A \to S_A$ together with isomorphisms

$$\tilde{G}_{X,Y} = G(i_X \oplus i_Y), \tilde{\tilde{G}}_{X,Y} = G(i_X \otimes i_Y) \tag{3}$$

is also an Ann-equivalence. We refer to $S_A$ as an Ann-category of type $(R, M)$, and $(H, \tilde{H}, \tilde{\tilde{H}}), (G, \tilde{G}, \tilde{\tilde{G}})$ are canonical Ann-equivalences. The family of constraints $h = (\xi, \eta, \alpha, \lambda, \rho)$ of $S_A$ is called a structure of the Ann-category of type $(R, M)$.

Mac Lane [7] and Shukla [14] cohomology groups at low dimensions are used to classify Ann-categories and regular Ann-categories, respectively. A structure $h$ of the Ann-category $S_A$ is an element in the group of Mac Lane 3-cocycles $Z^3_{\text{MacL}}(R, M)$. In the case when $A$ is regular, $h \in Z^3_{\text{Shu}}(R, M)$.

**Proposition 2** (Proposition 11 [11]). Let $A$ and $A'$ be Ann-categories.

i) Every Ann-functor $(F, \tilde{F}, \tilde{\tilde{F}}) : A \to A'$ induces an Ann-functor $S_F : S_A \to S_{A'}$ of type $(p, q)$, where

$$p = F_0 : \pi_0 A \to \pi_0 A', \quad [X] \mapsto [FX],$$

$$q = F_1 : \pi_1 A \to \pi_1 A', \quad u \mapsto \gamma^{-1}_{F_0}(Fu),$$

for $\gamma$ is a map given by the relation (1).

ii) $F$ is an equivalence if and only if $F_0, F_1$ are isomorphisms.

iii) The Ann-functor $S_F$ satisfies

$$S_F = G' \circ F \circ H,$$

where $H, G'$ are canonical Ann-equivalences.

Let $S = (R, M, h), S' = (R', M', h')$ be Ann-categories. Since $\tilde{F}_{x,y} = (\bullet, \tau(x, y)), \tilde{\tilde{F}}_{x,y} = (\bullet, \nu(x, y))$, then $g_F = (\tau, \nu)$ is a pair of maps associated with $(\tilde{F}, \tilde{\tilde{F}})$, we thus can regard an Ann-functor $F : S \to S'$ as a triple $(p, q, g_F)$. It follows from the compatibility of $F$ with the constraints that

$$q_s h - p^* h' = \partial(g_F),$$

where $q_s, p^*$ are canonical homomorphisms,

$$Z^3_{\text{MacL}}(R, M) \xrightarrow{q_s} Z^3_{\text{MacL}}(R, M') \xrightarrow{p^*} Z^3_{\text{MacL}}(R', M').$$

Further, two Ann-functors $(F, g_F), (F', g_{F'})$ are homotopic if and only if $F = F'$, that is, they are the same type of $(p, q)$, and there exists a function $t : R \to M'$ such that $g_{F'} = g_F + \partial t$.
We denote by 
\[ \text{Hom}^{\text{Ann}}_{(p,q)}[\mathcal{S}, \mathcal{S}'] \]
the set of homotopy classes of Ann-functors of type \((p, q)\) from \(\mathcal{S}\) to \(\mathcal{S}'\).

Let \(F : \mathcal{S} \to \mathcal{S}'\) be an Ann-functor of type \((p, q)\), then the function
\[ k = q^*h - p^*h' \in Z^3_{\text{MacL}}(R, M') \quad (4) \]
is called an obstruction of \(F\).

**Theorem 3** (Theorem 4.4, 4.5 [13]). A functor \(F : \mathcal{S} \to \mathcal{S}'\) of type \((p, q)\) is an Ann-functor if and only if its obstruction \(k\) vanishes in \(H^3_{\text{MacL}}(R, M')\). Then, there exists a bijection
\[ \text{Hom}^{\text{Ann}}_{(p,q)}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H^2_{\text{MacL}}(R, M') (= H^2_{\text{Shu}}(R, M')). \]

### 3 Crossed bimodules over rings and regular E-systems

The results on crossed bimodules can be found in [2, 3, 4, 9]. We shall show a characteristic of crossed bimodules when the base ring \(K\) is the ring of integers \(\mathbb{Z}\). Based on this characteristic, we can establish the relation between crossed bimodules over rings and Ann-category theory in the next section.

**Definition [9].** A crossed bimodule is a triple \((B, D, d)\), where \(D\) is an associative \(K\)-algebra, \(B\) is a \(D\)-bimodule and \(d : B \to D\) is a homomorphism of \(D\)-bimodules such that
\[ d(b)b' = bd(b'), \quad b, b' \in B. \quad (5) \]

A morphism \((k_1, k_0) : (B, D, d) \to (B', D', d')\) of crossed bimodules is a pair \(k_1 : B \to B', k_0 : D \to D'\), where \(k_1\) is a group homomorphism, \(k_0\) is a \(K\)-algebra homomorphism such that
\[ k_0d = d'k_1 \quad (6) \]
and
\[ k_1(xb) = k_0(x)k_1(b), \quad k_1(bx) = k_1(b)k_0(x), \quad (7) \]
for all \(x \in D, b \in B\).

The condition (7) shows that \(k_1\) is a homomorphism of \(D\)-bimodules, where \(B'\) is a \(D\)-bimodule with the action \(xb' = k_0(x)b', \quad b'x = b'k_0(x)\).

Below, the base ring \(K\) is the ring of integers \(\mathbb{Z}\), and a crossed bimodule \((B, D, d)\) is called a crossed bimodule over rings. Thus, \(D\) is a ring with unit.

In order to introduce the concept of an E-system, we now recall some terminologies due to Mac Lane [7]. The set of all bimultiplications of a ring \(A\) is a ring denoted by \(M_A\). For each element \(c\) of \(A\), a bimultiplication \(\mu_c\) is defined by
\[ \mu_c a = ca, \quad a\mu_c = ac, \quad a \in A. \]
we call \( \mu_c \) an inner bimultiplication. Then \( C_A = \{ c \in A | \mu_c = 0 \} \) is called the bicenter of \( A \).

The bimultiplications \( \sigma \) and \( \tau \) are permutable if for every \( a \in A \),

\[
\sigma(a\tau) = (\sigma a)\tau, \quad \tau(a\sigma) = (\tau a)\sigma.
\] (8)

We now introduce the main concept of the present paper which can be seen as a version of the concept of a crossed module over rings.

**Definition 3.** An \( E \)-system is a quadruple \( (B, D, d, \theta) \), where \( d : B \to D \), \( \theta : D \to M_B \) are the ring homomorphisms such that the following diagram commutes

\[
\begin{array}{ccc}
B & \xrightarrow{d} & D \\
\downarrow{\mu} & & \downarrow{\theta} \\
M_B & & \\
\end{array}
\] (9)

and the following relations hold for all \( x \in D, b \in B \),

\[
d(\theta_x b) = x.d(b), \quad d(b\theta_x) = d(b).x.
\] (10)

An \( E \)-system \( (B, D, d, \theta) \) is regular if \( \theta \) is a 1-homomorphism (a homomorphism carries the identity to the identity), and the elements of \( \theta(D) \) are permutable.

A morphism \( (f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta') \) of \( E \)-systems consists of ring homomorphisms \( f_1 : B \to B' \), \( f_0 : D \to D' \) such that

\[
f_0d = d'f_1
\] (11)

and \( f_1 \) is an operator homomorphism, that is,

\[
f_1(\theta_x b) = \theta'_{f_0(x)}f_1(b), \quad f_1(b\theta_x) = f_1(b)\theta'_{f_0(x)}.
\] (12)

In this paper, an \( E \)-system \( (B, D, d, \theta) \) is sometimes denoted by \( B \xrightarrow{d} D \), or \( B \to D \).

**Example 3.** If \( B \) is a two-sided ideal in \( D \), then \( (B, D, d, \theta) \) is a regular \( E \)-system, where \( d \) is an inclusion, \( \theta : D \to M_B \) is given by the bimultiplication type, that is,

\[
\theta_x b = xb, \quad b\theta_x = bx, \quad x \in D, b \in B.
\]

**Example 4.** Let \( D \) be a ring, \( B \) be a \( D \)-bimodule, \( 0 : B \to D \) is the zero homomorphism of \( D \)-bimodules. \( B \) can be considered as a ring with zero multiplication defined by \( b.b' = 0(b)b' = b0(b') = 0 \), for all \( b, b' \in B \). Then, \( (B, D, 0, \theta) \) is a regular \( E \)-system, where \( \theta \) is given by the action of \( D \)-bimodules.

**Example 5.** Let \( B \) be a ring, \( M_B \) be the ring of bimultiplications of \( B \), and \( \mu : B \to M_B \) be the homomorphism which carries an element \( b \) in \( B \) to an inner bimultiplication of \( B \). Then \( (B, M_B, \mu, id) \) is an \( E \)-system. In general, this \( E \)-system is not regular.

Standard consequences of the axioms of an \( E \)-system are as below.

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Proposition 4. Let \((B, D, d, \theta)\) be an \(E\)-system.

i) \(\text{Ker} \ d \subset C_B\).

ii) \(\text{Im} \ d\) is an ideal in \(D\).

iii) The homomorphism \(\theta\) induces a homomorphism \(\varphi : D \to M_{\text{Ker} d}\) given by 
\[
\varphi_x = \theta_x|_{\text{Ker} d}.
\]

iv) \(\text{Ker} \ d\) is a \(\text{Coker} \ d\)-bimodule with the actions 
\[
sa = \varphi_x(a), \quad as = (a)\varphi_x, \quad a \in \text{Ker} d, \ x \in s \in \text{Coker} d.
\]

To state the relation between regular \(E\)-systems and crossed bimodules over rings, one recalls the following definition.

Definition 4. A functor \(\Phi : C \to C'\) is an isomorphism of categories if it is bijective on objects and on morphism sets.

Theorem 5. The categories of regular \(E\)-systems and of crossed bimodules over rings are isomorphic.

Proof. Let \(sB = (B, D, d, \theta)\) be a regular \(E\)-system. The abelian additive group \(B\) is a \(D\)-bimodule with the actions 
\[
xb = \theta_x b, \quad bx = b\theta_x,
\]
for \(x \in D, b \in B\). It is then easy to check that the axioms of a crossed bimodule hold. For example, the relation (13) follows from the relation (12),
\[
d(b)b' = \theta_{d(b)}(b') = \mu_b(b') = bb' = b\mu_{b'} = b\theta_{d(b')} = bd(b'),
\]
since \(\mu_b, \mu_{b'}\) are inner bimultiplications of the ring \(B\). Besides, the regularity of the \(E\)-system \((B, D, d, \theta)\) is necessary and sufficient for the two-sided module \(B\) to be a \(D\)-bimodule.

Conversely, if \(cB = (B, D, d)\) is a crossed bimodule then \(B\) has a ring structure with the multiplication
\[
b \ast b' := d(b)b' = bd(b'), \quad b, b' \in B.
\]

Clearly, \(d : B \to D\) is a ring homomorphism since for all \(b, b' \in B\), 
\[
d(b \ast b') = d(d(b)b') = d(b)d(b').
\]
The map \(\theta : D \to M_B\) is defined by the \(D\)-bimodule actions (13). Then, \(\theta\) is a homomorphism with image in \(M_B\), the elements of \(\theta(D)\) are permutable since \(B\) is a \(D\)-bimodule. The homomorphism \(\theta\) satisfies the condition (11) since \(d\) is a homomorphism of bimodules. Thus, the correspondence \(sB \mapsto cB\) is bijective on objects.

Now, if \((f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')\) is a morphism of \(E\)-systems, it is then clear that \((f_1, f_0)\) satisfies the relation (10).
Further, for all \( x \in D, b \in B \), one has
\[
f_1(xb) = f_1(\theta_f(x)b) \quad \theta_{f_0(x)}f_1(b) = f_0(x)f_1(b) = xf_1(b).
\]
Similarly, one obtains \( f_1(bx) = f_1(b)x \). This means that the pair \((f_1, f_0)\) is a morphism of crossed bimodules.

Conversely, let \((k_1, k_0) : (B, D, d) \rightarrow (B', D', d')\) be a morphism of crossed bimodules. We show that \(k_1\) is a ring homomorphism. According to the determination of the multiplication on the ring \(B\), we have
\[
k_1(b \ast b') = k_1(d(b)b') = k_0(d(b))k_1(b') = d'(k_1(b))k_1(b') = k_1(b)k_1(b'),
\]
for all \( b, b' \in B \). Besides, the pair \((k_1, k_0)\) also satisfies (12).

By the above proposition, the notion of an E-system can be seen as a weaken version of the notion of a crossed bimodule over rings.

We now discuss the relationship among the above concepts and the concept of a crossed module of \(D\)-structures in the category \(C\) of \(\Omega\)-groups (see [10]). For convenience, such a crossed module is called a crossed \(C\)-module. T. Porter proved that there is an equivalence between the category of crossed \(C\)-modules and that of internal categories in \(C\). A crossed \(C\)-module can be described as follows.

**Proposition 6** (Proposition 2 [10]). Given a \(D\)-structure on \(B, d : B \rightarrow D\) is a crossed \(C\)-module if and only if the following conditions are satisfied for all \(b, b_1, b_2 \in B, x \in D, * \in \Omega_2 \subset \Omega\)
\[
\begin{align*}
(i) & \quad d((-x) \cdot b) = -x + d(b) + x, \\
(ii) & \quad (-d(b_1)) \cdot b_2 = -b_1 + b_2 + b_1, \\
(iii) & \quad d(b_1) \ast b_2 = b_1 \ast b_2 = b_1 \ast d(b_2), \\
(iv) & \quad \begin{cases} 
        d(xb) = x \ast d(b) \\
        d(bx) = d(b) \ast x.
\end{cases}
\end{align*}
\]

Here \(\ast\) is a binary operation which is not the group operation \(+\), the actions \(x \cdot b, x \ast b\) are given by
\[
x \cdot b = s(x) + b - s(x), \\
x \ast b = s(x) \ast b,
\]
where \(s\) is the morphism in the split exact sequence
\[
0 \rightarrow B \xrightarrow{i} E \xrightarrow{\pi} D \rightarrow 0.
\]

To establish the link between these crossed \(C\)-modules and crossed modules over rings, we take \(C\) to be a category whose objects are rings. The morphisms of \(C\) are ring homomorphisms which are not necessarily 1-homomorphisms.

**Proposition 7.** Every crossed \(C\)-module is a crossed bimodule over rings.
Proof. Let \( d : B \to D \) be a crossed \( C \)-module. Then \( d \) is a ring homomorphism, and \( D \) acts on \( B \) by
\[
xb = s(x)b, \ bx = bs(x), \ x \in D, b \in B.
\] (15)
The map \( \theta : D \to M_B \) is given by
\[
\theta_x(b) = xb, (b)\theta_x = bx.
\]
Since \( s \) is a ring homomorphism, so is \( \theta \). The relation (10) follows from the condition (iii). Indeed, for \( b, b' \in B \)
\[
(\theta d)(b)(b') = \theta_{db}(b') = (db)b' = bb' = \mu_b(b').
\]
It follows from (iv) that \( d(\theta_x(b)) = d(xb) = xd(b)b \). This means the relation (10) holds, and therefore \((B, D, d, \theta)\) is an E-system. \( \square \)

One can see that a crossed \( C \)-module \( d : B \to D \) satisfies most of the conditions of a crossed bimodule over rings. We first see that \( B \) is a \( D \)-bimodule with the action (15). By (iv), the ring homomorphism \( d : B \to D \) is a \( D \)-bimodule. The relation (15) follows directly from the condition (iii). Note that the ring \( D \) is not necessarily unitary and if it has a unit, the ring \( B \) is not assumed to be a unitary \( D \)-bimodule. These investigations show that the concept of a crossed \( C \)-module can be seen as a weaken version of the concept of a crossed bimodule over rings.

Remark. Since \( C \) can be any of categories of \( \Omega \)-groups, use of crossed \( C \)-modules has resulted in various contexts. However, in each particular case there is a certain restriction. For example, by Proposition 3 [10] \( \text{Kerd} \) is singular; while for crossed modules over groups, (or crossed modules over rings) \( \text{Kerd} \) is a subgroup in the center (or the bicenter) of \( B \).

Since rings with unit are not \( \Omega \)-groups, one can not seek a relation among the category of crossed \( C \)-modules, cohomology of algebras and cohomology of rings.

4 Strict Ann-categories and E-systems

Crossed modules over groups are often studied in the form of strict 2-groups (see [11, 31]). In this section, we prove that E-systems and strict Ann-categories are equivalent.

For every E-system \((B, D, d, \theta)\) we can construct a strict Ann-category \( \mathcal{A} = \mathcal{A}_{B \to D} \), called the Ann-category associated to the E-system \((B, D, d, \theta)\), as follows. One sets
\[
\text{Ob}(\mathcal{A}) = D,
\]
and for two objects \( x, y \) of \( \mathcal{A} \),
\[
\text{Hom}(x, y) = \{b \in B / y = d(b) + x\}.
\]
The composition of morphisms is given by

\[(x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).\]

Two operations \(\oplus, \otimes\) on objects are given by the operations \(+, \times\) on the ring \(D\). For the morphisms, we set

\[(x \xrightarrow{b} y) \oplus (x' \xrightarrow{b'} y') = (x + x' \xrightarrow{b+b'} y + y'),\]
\[(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (x x' \xrightarrow{bb'+b\theta_{x,}\theta_{x'}b'} yy').\]

Based on the definition of an E-system, it is easy to verify that \(A\) is an Ann-category with the strict constraints.

Conversely, for every strict Ann-category \((A, \oplus, \otimes)\) one can define an E-system \(C_A = (B, D, d, \theta)\). Indeed, let

\[D = \text{Ob}(A), \quad B = \{0 \xrightarrow{b} x \mid x \in D\}.\]

Then, \(D\) is a ring with two operations \(x + y = x \oplus y, \quad xy = x \otimes y,\)

and \(B\) is a ring with two operations \(b + c = b \oplus c, \quad bc = b \otimes c.\)

The homomorphisms \(d : B \to D\) and \(\theta : D \to M_B\) are defined by

\[d(0 \xrightarrow{b} x) = x,\]
\[\theta_y(0 \xrightarrow{b} x) = (0 \xrightarrow{id_x \otimes b} xy),\]
\[(0 \xrightarrow{b} x)\theta_y = (0 \xrightarrow{b \otimes id_x} xy).\]

The quadruple \((B, D, d, \theta)\) defined as above is an E-system.

In the following lemmas, let \(A_{B \to D}\) and \(A_{B' \to D'}\) be Ann-categories associated to E-systems \((B, D, d, \theta)\) and \((B', D', d', \theta')\), respectively.

**Lemma 8.** Let \((f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')\) be a morphism of E-systems.

i) There is a functor \(F : A_{B \to D} \to A_{B' \to D'}\) defined by

\[F(x) = f_0(x), \quad F(b) = f_1(b), \quad x \in \text{Ob}(A_{B \to D}), b \in \text{Mor}(A_{B \to D}).\]

ii) The functor \(F\) together with isomorphisms \(\tilde{F}_{x, y} : F(x + y) \to Fx + Fy, \tilde{F}_{x, y} : F(xy) \to FxFy\) is an Ann-functor if \(\tilde{F}_{x, y}\) and \(\tilde{F}_{x, y}\) are constants in \(\text{Kerd'}\) and for all \(x, y \in D\) the following conditions hold:

\[\theta'_{F_{x, y}}(\tilde{F}) = (\tilde{F})\theta'_{F_{x, y}} = \tilde{F}, \quad (16)\]
\[\theta'_{F_{x, y}}(\tilde{F}) = (\tilde{F})\theta'_{F_{x, y}} = \tilde{F} + \tilde{F}. \quad (17)\]
Then, we say that $F$ is an Ann-functor of form $(f_1, f_0)$.

Proof. i) Every element $b \in B$ can be considered as a morphism $(0 \rightarrow db)$ in $\mathcal{A}_{B \rightarrow D}$. Then, $(F0 \xrightarrow{F(b)} F(db))$ is a morphism in $\mathcal{A}_{B' \rightarrow D'}$. By the construction of the Ann-category associated to an E-system, $F$ is a functor.

ii) We define the natural isomorphisms

$$\tilde{F}_{x,y} : F(x + y) \rightarrow F(x) + F(y), \tilde{F}_{x,y} : F(xy) \rightarrow F(x)F(y)$$

such that $F = (F, \tilde{F}, \tilde{F})$ becomes an Ann-functor. First we see that

$$F(x) + F(x') = F(x + x'),$$

so $d'(\tilde{F}_{x,x'}) = 0$. Analogously, $d'(\tilde{F}_{x,x'}) = 0$, thus

$$\tilde{F}_{x,x'}, \tilde{F}_{x,x'} \in \text{Ker}d' \subset C_{B'}.$$

(18)

Now, for two morphisms $(x \xrightarrow{b} y)$ and $(x' \xrightarrow{b'} y')$ in $\mathcal{A}_{B \rightarrow D}$, we have:

- $F(b \oplus b') = F(x + x' \xrightarrow{b + b'} y + y') = (f_0(x + x') \xrightarrow{f_1(b + b')} f_0(y + y'))$.

$$F(b) \oplus F(b') = (f_0(x) \xrightarrow{f_1(b)} f_0(y)) \oplus (f_0(x') \xrightarrow{f_1(b')} f_0(y'))$$

$$= (f_0(x) + f_0(x') \xrightarrow{f_1(b) + f_1(b')} f_0(y) + f_0(y')).$$

Since $f_1$ is a ring homomorphism, one obtains

$$F(b \oplus b') = F(b) \oplus F(b').$$

(19)

By (18) and (19), the commutativity of the diagram

$$\begin{array}{ccc}
F(x + x') & \xrightarrow{\tilde{F}_{x,x'}} & F(x) + F(x') \\
\downarrow F(b \oplus b') & & \downarrow F(b) \oplus F(b') \\
F(y + y') & \xrightarrow{\tilde{F}_{y,y'}} & F(y) + F(y')
\end{array}$$

follows from $\tilde{F}_{x,x'} = \tilde{F}_{y,y'}$.

- $F(b \oplus b') = F(xx' \xrightarrow{bb' + b_\theta x + \theta_x b'} yy') = (f_0(xx') \xrightarrow{f_1(bb' + b_\theta x + \theta_x b')} f_0(yy'))$.

$$F(b) \oplus F(b') = (f_0(x) \xrightarrow{f_1(b)} f_0(y)) \oplus (f_0(x') \xrightarrow{f_1(b')} f_0(y'))$$

$$= (f_0(x)f_0(x') \xrightarrow{f_1(b)f_1(b') + f_1(b)\theta_{f_0(x') + \theta_{f_0(x)} f_1(b')}} f_0(y)f_0(y')).$$
By (12), \( f_1(\theta x b') = \theta' f_0(x) f_1(b') \) and \( f_1(b \theta x') = f_1(b) \theta' f_0(x') \), hence
\[
F(b \otimes b') = F(b) \otimes F(b'). \tag{21}
\]

By (18) and (21), the commutativity of the diagram
\[
\begin{array}{ccc}
 F(xx') & \xrightarrow{\tilde{F}_x} & F(x)F(x') \\
 F(b \otimes b') \downarrow & & \downarrow F(b) \otimes F(b') \\
 F(yy') & \xrightarrow{\tilde{F}_y} & F(y)F(y')
\end{array}
\tag{22}
\]
follows from \( \tilde{F}_{x,x'} = \tilde{F}_{y,y'} \). The equalities (16) and (17) come from the compatibility of \((F, \tilde{F})\) with the associativity constraint and the distributivity ones, respectively.

An Ann-functor \( F \) is single if \( F(0) = 0' \), \( F(1) = 1' \) and \( \tilde{F}, \tilde{F}' \) are constants. Then we state the converse of Lemma 8.

**Lemma 9.** Let \((F, \tilde{F}, \tilde{F}'): A_{B \rightarrow D} \rightarrow A_{B' \rightarrow D'}\) be a single Ann-functor. Then, there is a morphism of \( E \)-systems \((f_1, f_0): (B \rightarrow D) \rightarrow (B' \rightarrow D')\), where
\[
f_1(b) = F(b), \quad f_0(x) = F(x),
\]
for \( b \in B, x \in D \).

**Proof.** Since \( F(0) = 0', F(1) = 1' \) and \( \tilde{F}, \tilde{F}' \) are constants, it is easy to see that \( \tilde{F}, \tilde{F}' \) are in \( \text{Ker} d' \). By the determination of a morphism in \( A_{B' \rightarrow D'} \),
\[
F(x + y) = F(x) + F(y), \quad F(xy) = F(x)F(y),
\]
so \( f_0 \) is a ring homomorphism.

Since \( \tilde{F} \) is a constant in \( \text{Ker} d' \), the commutative diagram (20) implies
\[
F(b \otimes b') = F(b) \otimes F(b').
\]
This means that \( f_1(b + b') = f_1(b) + f_1(b') \).

Since \( \tilde{F} \) is a constant in \( \text{Ker} d' \), the commutative diagram (22) implies
\[
F(b \otimes b') = F(b) \otimes F(b').
\]

By the definition of \( \otimes \),
\[
f_1(bb') + f_1(b \theta x') + f_1(\theta x b') = f_1(b) f_1(b') + f_1(b) \theta' f_0(x') + \theta' f_0(x) f_1(b'). \tag{23}
\]
In this relation, taking \( b = 0 \) and then \( b' = 0 \) yield
\[
f_1(\theta x b') = \theta' f_0(x) f_1(b'), \quad f_1(b \theta x') = f_1(b) \theta' f_0(x').
\]
Thus, \( (12) \) holds. Then, the equation \( (23) \) turns into \( f_1(bb') = f_1(b)f_1(b') \), that is, \( f_1 \) is a ring homomorphism. The rule \( (11) \) also holds. Indeed, for all morphisms \( (x \overset{b}{\rightarrow} y) \) in \( A_{B \rightarrow D} \), \( y = d(b) + x \). It follows that

\[
f_0(y) = f_0(d(b) + x) = f_0(d(b)) + f_0(x).
\]

Besides, \( (f_0(x) \overset{f_1(b)}{\rightarrow} f_0(y)) \) is a morphism in \( A_{B' \rightarrow D'} \), so

\[
f_0(y) = d'(f_1(b)) + f_0(x).
\]

Thus, \( f_0(d(b)) = d'(f_1(b)) \) for all \( b \in B \).

**Lemma 10.** Two Ann-functors \( (F, \tilde{F}, \tilde{F}'), (G, \tilde{G}, \tilde{G}') : A_{B \rightarrow D} \rightarrow A_{B' \rightarrow D'} \) of the same form are homotopic.

**Proof.** Suppose that \( F \) and \( G \) are two Ann-functors of form \( (f_1, f_0) \). By Lemma \( (8) \), \( \tilde{F}', \tilde{G} \) are constants. We prove that \( \alpha = \tilde{G} - \tilde{F} \) is a homotopy between \( F \) and \( G \). It is easy to check the naturality of \( \alpha \) and the compatibility of \( \alpha \) with the addition. Besides, \( \alpha \) is compatible with the multiplication. In other words, the following diagram

\[
\begin{array}{ccc}
F(xy) & \xrightarrow{\tilde{F}} & F(x)F(y) \\
\downarrow{\alpha} & & \downarrow{\alpha \otimes \alpha} \\
G(xy) & \xrightarrow{\tilde{G}} & G(x)G(y)
\end{array}
\]

commutes. Indeed, by Lemma \( (8) \)

\[
\tilde{G} - \tilde{F} = (\theta'_{Fx}(\tilde{G}) - \tilde{G}) - (\theta'_{Fx}(\tilde{F}) - \tilde{F})
\]

\[
= \theta'_{Fx}(\alpha) - \alpha.
\]

Since \( \alpha \in \ker d' \subset C_{B'} \), so

\[
\alpha \otimes \alpha = \alpha \cdot \alpha + (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha)
\]

\[
= (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha).
\]

For \( y = 0 \), or \( x = 0 \) we have

\[
\alpha \otimes \alpha = (\alpha)\theta'_{Gy} = \theta'_{Gx}(\alpha).
\]

Thus,

\[
\tilde{G} - \tilde{F} = \alpha \otimes \alpha - \alpha,
\]

that is, \( (24) \) holds.

Two Ann-functors \( (F, \tilde{F}, \tilde{F}') \) and \( (G, \tilde{G}, \tilde{G}') \) are strong homotopic if they are homotopic and \( F = G \). By Lemma \( (10) \), one obtains the following fact.
Corollary 11. Two Ann-functors \( F, G : A_{B \rightarrow D} \rightarrow A_{B' \rightarrow D'} \) are strong homotopic if and only if they are of the same form.

We write \( \text{Annstr} \) for the category of strict Ann-categories and their single Ann-functors. We can define the strong homotopy category \( \text{HoAnnstr} \) to be the quotient category with the same objects, but morphisms are strong homotopy classes of single Ann-functors. We write \( \text{Hom}_{\text{Annstr}}[A, A'] \) for the homsets of the homotopy category, that is,

\[
\text{Hom}_{\text{Annstr}}[A, A'] = \frac{\text{Hom}_{\text{Annstr}}(A, A')}{\text{strong homotopies}}
\]

Denote \( \text{ESyst} \) the category of E-systems, we obtain the following result which is an extending of Theorem 1 [5]

Theorem 12 (Classification Theorem). There exists an equivalence

\[
\Phi : \text{ESyst} \rightarrow \text{HoAnnstr}
\]

\[
(B \rightarrow D) \mapsto A_{B \rightarrow D}
\]

\[
(f_1, f_0) \mapsto [F]
\]

where \( F(x) = f_0(x), F(b) = f_1(b), \) for \( x \in \text{Ob}A, b \in \text{Mor}A. \)

Proof. By Corollary 11, the correspondence \( \Phi \) on homsets,

\[
\text{Hom}_{\text{ESyst}}(B \rightarrow D, B' \rightarrow D') \rightarrow \text{Hom}_{\text{Annstr}}[A_{B \rightarrow D}, A_{B' \rightarrow D'}],
\]

is an injection. By Lemma 9, every single Ann-functor \( F : A_{B \rightarrow D} \rightarrow A_{B' \rightarrow D'} \) determines a morphism of E-systems \( (f_1, f_0) \), and clearly \( \Phi(f_1, f_0) = [F] \), thus \( \Phi \) is surjective on homsets.

Let \( C_A \) be an E-system associated to a strict Ann-category \( A \). By the construction of an Ann-category associated to an E-system, \( \Phi(C_A) = A \) (rather than an isomorphism). Hence, \( \Phi \) is an equivalence of categories.

\[ \square \]

5 Ring extensions of the type of an E-system

In this section we consider the ring extensions of the type of an E-system, which are analogous to the group extensions of the type of a crossed module [4].

Definition 5. Let \((B, D, d, \theta)\) be an E-system. A ring extension of \( B \) by \( Q \) of type \( B \rightarrow D \) is a diagram of ring homomorphisms

\[
\begin{array}{ccccccc}
0 & \rightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \rightarrow & 0, \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & B & \xrightarrow{d} & D & & & & \\
\end{array}
\]

where the top row is exact, the quadruple \((B, E, j, \theta')\) is an E-system where \( \theta' \) is given by the bimultiplication type, and the pair \((id, \varepsilon)\) is a morphism of E-systems.
Two extensions of $B$ by $Q$ of type $B \xrightarrow{d} D$ are said to be equivalent if there is a morphism of exact sequences

\[
0 \rightarrow B \xrightarrow{i} E \xrightarrow{p} Q \rightarrow 0, \quad E \xrightarrow{\varepsilon} D \tag{25}
\]

and $\varepsilon' \eta = \varepsilon$. Obviously, $\eta$ is an isomorphism.

In the diagram

\[
\begin{array}{ccc}
E : & 0 & \rightarrow B \xrightarrow{i} E \xrightarrow{p} Q \rightarrow 0, \quad E \xrightarrow{\varepsilon} D \\
& \downarrow \varepsilon & \downarrow \psi & \downarrow \varepsilon' \\
& B \xrightarrow{d} D \rightarrow \text{Coker} \end{array}
\]

where $q$ is a canonical projection, since the top row is exact and $q \circ \varepsilon \circ \eta = q \circ d = 0$, there is a ring homomorphism $\psi : Q \rightarrow \text{Coker}d$ such that the right hand side square commutes. Moreover, $\psi$ depends only on the equivalence class of the extension $E$. Our purpose is to study the set $\text{Ext}_{B \rightarrow D}(Q, B, \psi)$ of equivalence classes of extensions of $B$ by $Q$ of type $B \rightarrow D$ inducing $\psi$. The results use the obstruction theory of Ann-functors

Let $A = A_{B \rightarrow D}$ be the Ann-category associated to an E-system $B \rightarrow D$. Clearly, $\pi_0 A = \text{Coker}d$, $\pi_1 A = \text{Kerd}$ and therefore the reduced Ann-category $S_A$ is of form

$S_A = (\text{Coker}d, \text{Kerd}, k)$

where $k \in H^3_{\text{Shu}}(\text{Coker}d, \text{Kerd})$ since $A$ and $S_A$ are regular Ann-categories. The homomorphism $\psi : Q \rightarrow \text{Coker}d$ induces an obstruction,

$\psi^* k \in Z^3_{\text{Shu}}(Q, \text{Kerd}), \tag{27}$

which plays a fundamental role to state Theorem $[13]$. This is the main result of this section, an extending of Theorem 5.2 $[6]$. Besides, a particular case of a regular E-system when $Q = \text{Coker}d$ and $\psi = \text{id}_{\text{Coker}d}$ is a $\partial$-extension $[4]$, so our result contains Theorem 4.4.2 $[4]$.

**Theorem 13.** Let $(B, D, d, \theta)$ be a regular E-system, $\psi : Q \rightarrow \text{Coker}d$ be a ring homomorphism. Then, the vanishing of $\psi^* k$ in $H^3_{\text{Shu}}(Q, \text{Kerd})$ is necessary and sufficient for there to exist a ring extension of $B$ by $Q$ of type $B \rightarrow D$ inducing $\psi$. Further, if $\psi^* k$ vanishes then there is a bijection

$\text{Ext}_{B \rightarrow D}(Q, B, \psi) \leftrightarrow H^2_{\text{Shu}}(Q, \text{Kerd}).$

The first assertion is based on the following lemmas.
Lemma 14. For every Ann-functor \((F, \tilde{F}, \bar{F}) : \text{Dis}Q \to \mathcal{A}\) there exists an extension \(E_F\) of \(B\) by \(Q\) of type \(B \to D\) inducing \(\psi : Q \to \text{Coker}d\).

Such extension \(E_F\) is called an associated extension to Ann-functor \(F\).

Proof. By Proposition \(\ddagger\) \((F, \tilde{F}, \bar{F})\) induces an Ann-functor \(K : \text{Dis}Q \to S_\mathcal{A}\) of type \((\psi, 0)\). Let \((H, \tilde{H}, \bar{H}) : S_\mathcal{A} \to \mathcal{A}\) be a canonical Ann-functor defined by the stick \((x_s, i_x)\). By \(\ddagger\), we have

\[
H(s) = x_s, \quad H(s, b) = b, \quad \tilde{H}_{s,r} = -i_{x_s + x_r}, \quad \bar{H}_{s,r} = -i_{x_s + x_r}.
\]

Also by Proposition \(\ddagger\) \((F, \tilde{F}, \bar{F})\) is homotopic to the composition \(\text{Dis}Q \xrightarrow{K} S_\mathcal{A} \xrightarrow{H} \mathcal{A}\).

So one can choose \((F, \tilde{F}, \bar{F})\) being this composition. By the determination of \(HK\) and \(HK\),

\[
\tilde{F}_{u,v} = f(u, v) = f'(u, v) - i_{x_s + x_r}, \quad (28)
\]

\[
F_{u,v} = g(u, v) = g'(u, v) - i_{x_s - x_r} \in B, \quad (29)
\]

where \(u, v \in Q, s = \psi(u), r = \psi(v), f'(u, v) = \tilde{K}_{u,v}, g'(u, v) = \bar{K}_{u,v}\). By the compatibility of \((F, \tilde{F}, \bar{F})\) with the strict constraints of \(\text{Dis}Q\) and \(\mathcal{A}\), the functions \(f\) and \(g\) are the “normal” ones satisfying

\[
f(u, v + t) + f(v, t) - f(u, v) = f(u + v, t) = 0, \quad (30)
\]

\[
f(u, v) = f(v, u), \quad (31)
\]

\[
\theta_{Fu} g(v, t) - g(uv, t) + g(u, vt) - g(u, v) \theta_{Ft} = 0, \quad (32)
\]

\[
g(u, v + t) - g(u, v) - g(u, t) + \theta_{Fu} f(v, t) - f(uv, ut) = 0, \quad (33)
\]

\[
g(u + v, t) - g(u, t) - g(v, t) + f(u, v) \theta_{Ft} - f(u, vt) = 0. \quad (34)
\]

The function \(\varphi : Q \to M_B\) defined by

\[
\varphi(u) = \theta_{Fu} = \theta_{x_s} (s = \psi(u))
\]

satisfies the relations

\[
\varphi(u) + \varphi(v) = \mu_{f(u,v)} + \varphi(u + v), \quad (35)
\]

\[
\varphi(u) \varphi(v) = \mu_{g(u,v)} + \varphi(uv). \quad (36)
\]

We only prove the relation \(\ddagger\), the proof of \(\ddagger\) follows from \(\ddagger\) in the same way. Since \(f'(u, v) = \tilde{K}_{u,v} \in \text{Ker}d\), then by Proposition \(\ddagger\) \(f'(u, v) \in C_B\). By \(\ddagger\), one has \(\mu_{f(u,v)} = \mu(-i_{x_s + x_r})\). Thus,

\[
\varphi(u) + \varphi(v) = \theta_{x_s} + \theta_{x_r} = \theta_{x_s + x_r}
\]

\[
= \theta[d(-i_{x_s + x_r}) + x_{s+r}] = \theta[d(-i_{x_s + x_r})] + \theta_{x_{s+r}}
\]

\[
= \mu(-i_{x_s + x_r}) + \varphi(u + v) \Rightarrow \mu_{f(u,v)} + \varphi(u + v).
\]
Since the family of functions \((\varphi, f, g)\) satisfies the relations \([30] - [36]\), we have a crossed product \(E_0 = [B, \varphi, f, g, Q]\), that means \(E_0 = B \times Q\), and two operations are

\[
(b, u) + (b', u') = (b + b' + f(u, u'), u + u'),
\]

\[
(b, u)(b', u') = (b.b' + b\varphi(u') + \varphi(u)b' + g(u, u'), uu').
\]

The set \(E_0\) satisfies the axioms of a ring, in which note that the associativity for the multiplication in \(E_0\) holds if and only if the E-system \(B \rightarrow D\) is regular. Indeed, one can calculate the triple products as follows:

\[
[(b, u)(b', u')](b'', u'') = ((bb')b'' + b\varphi(u')\varphi(u'') + [\varphi(u)b']\varphi(u'')
\]

\[
+ g(u, u')\varphi(u'') + \varphi(uu')b'' + g(uu', u''), (uu')u''),
\]

\[
(b, u)[(b', u')(b'', u'')] = (b(b'b'') + b\varphi(u'u'') + \varphi(u)[b'\varphi(u'')]
\]

\[
+ \varphi(u)\varphi(u)b'' + \varphi(u)g(u, u') + g(u, u'u''), u(u'u''),
\]

By \([22], [20]\), associative law for the multiplication in \(B, Q\), and commutative law for the addition in \(B\), especially by the relation \([8]\), \([\varphi(u)b'][\varphi(u'')] = \varphi(u)[b'\varphi(u'')]\), we get the associative law for product in \(E_0\). Then, there is an exact sequence of ring homomorphisms

\[
\mathcal{E}_F: \quad 0 \rightarrow B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \rightarrow 0,
\]

where \(j_0(b) = (b, 0); \ p_0(b, u) = u, \ b \in B, u \in Q\). Since \(j_0(B)\) is a two-sided ideal in \(E_0\), \(B \xrightarrow{j_0} E_0\) is an E-system, where \(\theta_0 : E_0 \rightarrow M_B\) is given by the bimultiplication type.

We define a ring homomorphism \(\varepsilon : E_0 \rightarrow D\) by

\[
\varepsilon(b, u) = db + x_{\varphi(u)}, \ (b, u) \in E_0,
\]

where \(x_{\varphi(u)}\) is a representative of \(u\) in \(D\). We show that the pair \((id_B, \varepsilon)\) satisfies the rules \([11], [12]\). Clearly, \(\varepsilon \circ j_0 = d\). Besides, for all \((b, u) \in E_0, c \in B\),

\[
\theta_0(b, u)(c) = j_0^{-1}([b, u)(c, 0)] = bc + \varphi(u)c,
\]

\[
\theta_\varepsilon(b, u)(c) = \theta_{db + x_{\varphi(u)}}c = bc + \varphi(u)c.
\]

Thus, \(\theta_0(b, u)(c) = \theta_\varepsilon(b, u)(c)\). Analogously, \(c\theta_0(b, u) = c\theta_\varepsilon(b, u)\). So \((id_B, \varepsilon)\) is a morphism of E-systems, that is, one has an extension \([20]\), where \(E\) is replaced by \(E_0\).

For all \(u \in Q\) we have \(q\varepsilon(0, u) = q(x_{\varphi(u)}) = \psi(u)\), then the extension \(\mathcal{E}_F\) induces \(\psi : Q \rightarrow \text{Coker } d\).

**The proof of Theorem [13]**
Proof. Let us recall that \(\mathcal{A}\) is the Ann-category associated to the regular E-system \(B \xrightarrow{\delta} D\). Then, its reduced Ann-category is \(S_{\mathcal{A}} = (\text{Coker} d, \text{Ker} d, k)\), where \(k \in \mathbb{Z}_{3}^{3}\). Then, its reduced Ann-category is \(S_{\mathcal{A}} = (\text{Coker} d, \text{Ker} d, k)\), and by Proposition 3, we have an associated extension \(\mathcal{E}_F\).

Conversely, suppose that there is an extension as in the diagram (2.6). Let \(\mathcal{A}'\) be the Ann-category associated to the E-system \(B \rightarrow E\). By Proposition 2, there is an Ann-functor \(F : \mathcal{A}' \rightarrow \mathcal{A}\). Since the reduced Ann-category of \(\mathcal{A}'\) is \(\text{Dis}_Q\), so by Proposition 2, \(F\) induces an Ann-functor of type \((\psi, 0)\) from \(\text{Dis}_Q\) to \((\text{Coker} d, \text{Ker} d, k)\). Now, by Proposition 3, the obstruction of the pair \((\psi, 0)\) must vanish in \(H_{3}^{3}(\text{Shu}(Q, \text{Ker} d))\), that is, \(\psi^* k = 0\).

The final assertion of Theorem 13 follows from the next theorem.

Theorem 15 (Schreier Theory for ring extensions of the type of an E-system). There is a bijection

\[ \Omega : \text{Hom}^{\text{Ann}}_{(\psi, 0)}[\text{Dis}_Q, \mathcal{A}] \rightarrow \text{Ext}_{B \rightarrow D}(Q, B, \psi). \]

Proof. Step 1: The Ann-functors \((F, \tilde{F}, \tilde{F}')\), \((F', \tilde{F}', \tilde{F}')\) are homotopic if and only if their corresponding associated extensions \(\mathcal{E}_F, \mathcal{E}_{F'}\) are equivalent.

Let two Ann-functors \(F, F' : \text{Dis}_Q \rightarrow \mathcal{A}\) be homotopic by a homotopy \(\alpha : F \rightarrow F'\). Then, by the definition of an Ann-morphism, the following diagrams commute

\[
\begin{array}{c}
F(u + v) \xrightarrow{F'(u + v)} F(u) + F(v) \\
\downarrow \alpha_{u+v} \quad \quad \downarrow \alpha_{u+v} \quad \quad \downarrow \alpha_{u+v} \\
F'(u + v) \xrightarrow{F'(u) + F'(v)} F'(u) + F'(v),
\end{array}
\]

\[
\begin{array}{c}
F(uv) \xrightarrow{\tilde{F}'(uv)} F'(uv) \\
\downarrow \alpha_{uv} \quad \downarrow \alpha_{uv} \quad \downarrow \alpha_{uv} \\
F'(uv) \xrightarrow{\tilde{F}'(u) \cdot F'(v)} F'(u) \cdot F'(v).
\end{array}
\]

By the definition of the operation \(\otimes\) on \(\mathcal{A}\),

\[ \alpha_u \otimes \alpha_v = \alpha_u \alpha_v + \alpha_u \theta_{F_v} + \theta_{F_u} \alpha_v. \]

Then, since \(f(u, v) = \tilde{F}_{u,v}, f'(u, v) = \tilde{F}'_{u,v}, g(u, v) = \tilde{F}_{u,v}, g'(u, v) = \tilde{F}'_{u,v}\), we have

\[ f'(u, v) - f(u, v) = \alpha_u - \alpha_{u+v} + \alpha_v, \quad (37) \]

\[ g'(u, v) - g(u, v) = \alpha_u \alpha_v + \alpha_u \theta_{F_v} + \theta_{F_u} \alpha_v - \alpha_{u+v}. \quad (38) \]
Now, we set

\[ \alpha^* : E_F \to E_{F'}, \]
\[ (b, u) \mapsto (b - \alpha_u, u). \]

Note that \( \theta_{F^u} = \mu_{\alpha_u} + \theta_{F^u} \), and by the relations (37), (38), the correspondence \( \alpha^* \) is an isomorphism. Besides, the diagram (24) commutes in which \( E \) and \( E' \) are replaced by \( E_F \) and \( E_{F'} \), respectively.

Finally, \( \varepsilon^* \alpha^* = \varepsilon \). Indeed, since \( \alpha : F \to F' \) is a homotopy, then \( Fu \equiv x_{\psi(u)} = F'u \). Thus \( x_{\psi(u)} = d(\alpha_u) + x_{\psi(u)} \), or \( d(\alpha_u) = 0 \). Hence,

\[ \varepsilon^* \alpha^*(b, u) = \varepsilon'(b - \alpha_u, u) = d(b) + x_{\psi(u)} = \varepsilon(b, u). \]

That means two extensions \( \mathcal{E}_F \) and \( \mathcal{E}_{F'} \) are equivalent.

Conversely, if \( \mathcal{E}_F \) and \( \mathcal{E}_{F'} \) are equivalent, there exists a ring isomorphism \( (b, u) \mapsto (b - \alpha_u, u) \). Then, we have a homotopy \( \alpha : F \to F' \) by retracing our steps.

**Step 2: \( \Omega \) is a surjection.**

Let \( \mathcal{E} \) be an extension \( E \) of \( B \) by \( Q \) of type \((B, D, d, \theta)\) inducing \( \psi : Q \to \text{Coker } d \) (see the commutative diagram (26)). We prove that \( \mathcal{E} \) is equivalent to an extension \( \mathcal{E}_F \) which is associated to an Ann-functor \((F, \overline{F}, \tilde{F}) : \text{Dis}Q \to \mathcal{A} \).

Let \( \mathcal{A}' = \mathcal{A}_{B \to E} \) be the Ann-category associated to the E-system \((B, E, j, \theta')\).

By Lemma 8, the pair \((id_{\mathcal{B}}, \varepsilon)\) in the diagram (26) determines a single Ann-functor \((K, \overline{K}, \tilde{K}) : \mathcal{A}' \to \mathcal{A} \).

Since \( \pi_0 \mathcal{A}' = Q, \pi_1 \mathcal{A}' = 0 \), the reduced Ann-category \( S_{\mathcal{A}'} \) is nothing else but the Ann-category \( \text{Dis}Q \). Choose a stick \((e_u, i_e), e \in E, u \in Q\), of \( \mathcal{A}' \) (that is, \( \{e_u\} \) is a representative of \( Q \) in \( E \)). By (2), the canonical Ann-functor \((H', \overline{H}', \tilde{H}') : \text{Dis}Q \to \mathcal{A}' \) is given by

\[ H'(u) = e_u, \overline{H}'_{u,v} = -ie_{u+e_v} = g'(u, v), \tilde{H}'_{u,v} = -ie_{u,e_v} = h'(u, v). \]

The composition \( F = K \circ H' \) is an Ann-functor \( \text{Dis}Q \to \mathcal{A} \), where

\[ F(u) = \varepsilon(e_u), \overline{F}_{u,v} = \overline{H}'_{u,v} = g'(u, v), \tilde{F}_{u,v} = \tilde{H}'_{u,v} = h'(u, v). \]

According to the proof of Theorem 13 we construct an extension \( \mathcal{E}_F \) of the crossed product \( E_0 = [B, \varphi, g', h', Q] \) which is associated to \((F, \overline{F}, \tilde{F})\).

We now prove that \( \mathcal{E} \) and \( \mathcal{E}_F \) are equivalent, that is, there is a commutative diagram

\[
\begin{array}{c}
\mathcal{E}_F : & 0 & \overset{j_0}{\longrightarrow} & B & \overset{p_0}{\longrightarrow} & E_0 & \overset{p_0}{\longrightarrow} & Q & \overset{0}{\longrightarrow} & \text{Dis}Q & \overset{\varepsilon_0}{\longrightarrow} & D \\
\mathcal{E} : & 0 & \overset{j}{\longrightarrow} & B & \overset{p}{\longrightarrow} & E & \overset{p}{\longrightarrow} & Q & \overset{0}{\longrightarrow} & \text{Dis}Q & \overset{\varepsilon}{\longrightarrow} & D \\
\end{array}
\]

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and \( \varepsilon \eta = \varepsilon_0 \).

Indeed, since every element of \( E \) can be written uniquely as \( b + e_u, b \in B \), we can define a map

\[
\eta : E_0 \to E, \quad (b, u) \mapsto b + e_u.
\]

We next verify that \( \eta \) is a ring isomorphism. The representatives \( e_u \) have the following properties

\[
\varphi(u)c = \theta'_{e_u}c, \quad c\varphi(u) = c\theta'_{e_u}, \quad c \in B, \quad (39)
\]

\[
e_u + e_v = -ie_u + e_{u+v} = g'(u, v) + e_{u+v}, \quad (40)
\]

\[
e_u,e_v = -ie_u,e_v + e_{u,v} = h'(u, v) + e_{u,v}. \quad (41)
\]

(The relation (39) holds since the pair \((\text{id}_B, \varepsilon)\) is a morphism of \( E \)-systems. The relations (40), (41) hold thanks to the definition of a morphism in \( A' \).) Now, we have

\[
\eta[(b, u) + (c, v)] = \eta(b + c + g'(u, v), u + v) = b + c + g'(u, v) + e_{u+v} = b + c + e_u + e_v = (b + e_u) + (c + e_v) = \eta(b, u) + \eta(c, v). \quad (40)
\]

\[
\eta[(b, u)(c, v)] = \eta(bc + b\varphi(v) + \varphi(u)c + h'(u, v), uv)
\]

\[
= bc + b\varphi(v) + \varphi(u)c + h'(u, v) + e_{uv} \quad (39) \quad (41)
\]

\[
= bc + b\theta'_{e_u} + \theta'_{e_u}c + e_ue_v
\]

\[
= bc + b.e_u + e_u.c + e_ue_v
\]

\[
= (b + e_u).(c + e_v) = \eta(b, u).\eta(c, v). \quad (39) \quad (41)
\]

Finally, choose the representative \( e_u \) such that \( \varepsilon(e_u) = x_{\psi(u)} \) (since it follows from (26) that \( q(\varepsilon(e_u)) = \psi p(e_u) = \psi(u) \)). Thus,

\[
\varepsilon \eta(b, u) = \varepsilon(b + e_u) = \varepsilon(b) + \varepsilon(e_u) = d(b) + x_{\psi(u)} = \varepsilon_0(b, u),
\]

that is, \( E \) and \( \mathcal{E}_F \) are equivalent.

\[
\text{Acknowledgement} \quad \text{The authors are much indebted to the referee, whose useful observations greatly improved our exposition.}
\]
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