Graded $K$-theory and Leavitt path algebras

Guido Arnone$^1$ · Guillermo Cortiñas$^1$

Received: 23 February 2022 / Accepted: 6 October 2022 / Published online: 10 November 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
Let $G$ be a group and $\ell$ a commutative unital $*$-ring with an element $\lambda \in \ell$ such that $\lambda + \lambda^* = 1$. We introduce variants of hermitian bivariant $K$-theory for $*$-algebras equipped with a $G$-action or a $G$-grading. For any graph $E$ with finitely many vertices and any weight function $\omega : E^1 \to G$, a distinguished triangle for $L(E) = L_\ell(E)$ in the hermitian $G$-graded bivariant $K$-theory category $kk^h_{G,gr}$ is obtained, describing $L(E)$ as a cone of a matrix with coefficients in $\mathbb{Z}[G]$ associated to the incidence matrix of $E$ and the weight $\omega$. In the particular case of the standard $\mathbb{Z}$-grading, and under mild assumptions on $\ell$, we show that the isomorphism class of $L(E)$ in $kk^h_{\mathbb{Z},gr}$ is determined by the graded Bowen–Franks module of $E$. We also obtain results for the graded and hermitian graded $K$-theory of $*$-algebras in general and Leavitt path algebras in particular which are of independent interest, including hermitian and bivariant versions of Dade’s theorem and of Van den Bergh’s exact sequence relating graded and ungraded $K$-theory.

Keywords Leavitt path algebras · Graded $K$-theory · Hermitian $K$-theory · Bivariant $K$-theory

Guido Arnone and Guillermo Cortiñas were partially supported by grant UBACyT 256BA from Universidad de Buenos Aires, PIP 2021-2023 GI 11220200100423CO from CONICET and PICT 2017-1395 from Agencia Nacional de Promoción Científica y Técnica. The first named author was supported by PhD fellowships, first from Universidad de Buenos Aires and then from CONICET. The second named author was supported by CONICET and partially supported by grant PGC2018-096446-B-C21 from the Spanish Ministerio de Ciencia e Innovación.

Guido Arnone garnone@dm.uba.ar
Guillermo Cortiñas gcorti@dm.uba.ar

$^1$ Departamento de Matemática-IMAS, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina

Guido Arnone

Guillermo Cortiñas
1 Introduction

A (directed) graph $E$ consists of sets $E^0$ of vertices and $E^1$ of edges and source and range maps $s, r : E^1 \to E^0$. A vertex $v \in E^0$ is a sink or an infinite emitter if the number $\#s^{-1}(v)$ of edges it emits is zero or infinite, and is regular otherwise. We say that $E$ is row-finite if it has no infinite emitters and finite if $E^0$ and $E^1$ are finite. We write $\text{reg}(E) \subset E^0$ for the subset of regular vertices. Let $A_E \in \mathbb{Z}^{\text{reg}(E) \times E^0}$ be the matrix whose $(v, w)$ entry is the number of edges from $v$ to $w$; put $I \in \mathbb{Z}^{E^0 \times \text{reg}(E)}$, $I_{v, w} = \delta_{v, w}$. Let $C_\infty = \langle \sigma \rangle$ be the infinite cyclic group; write $\mathbb{Z}[\sigma]$ for its group ring. The graded Bowen-Franks $\mathbb{Z}[\sigma]$-module of $E$ is

$$\text{BF}_{\text{gr}}(E) = \text{coker}(I - \sigma A_E^t).$$  \hspace{1cm} (1.1)

If $E^0$ is finite, we put $[1]_E = \sum_{v \in E^0} [v] \in \text{BF}_{\text{gr}}(E)$. Fix a commutative unital ring $\ell$ with involution and let $L(E)$ be the Leavitt path algebra of $E$ over $\ell$ [1], equipped with its canonical $\mathbb{Z}$-grading and involution. Write $K^\text{gr}_\ast(L(E))$ for the $K$-theory of graded, finitely generated projective $L(E)$-modules. The group $K^\text{gr}_0(L(E))$ is canonically a preordered $\mathbb{Z}[\sigma]$-module. If $E$ is row-finite, then there is a natural ordered $\mathbb{Z}[\sigma]$-module isomorphism (Corollary 5.4)

$$\text{BF}_{\text{gr}}(E) \otimes K_\ast(\ell) \xrightarrow{\sim} K^\text{gr}_\ast(L(E)),$$

which, when $E^0$ is finite, maps $[1]_E \otimes [1_\ell] \mapsto [1_{L(E)}]$. Let $E$ and $F$ be finite graphs. The graded classification conjecture [13, Conjecture 1] for finite graphs says that, when $\ell$ is a field, the existence of an order preserving $\mathbb{Z}[\sigma]$-module isomorphism $\text{BF}_{\text{gr}}(E) \sim \text{BF}_{\text{gr}}(F)$ mapping $[1]_E \mapsto [1]_F$ implies that $L(E) \cong L(F)$ as $\mathbb{Z}$-graded algebras. In this paper we begin the study of this conjecture in terms of the bivariant algebraic $K$-theory of algebras graded over a group $G$, $j_{G, \text{gr}} : G_{\text{gr}} - \text{Alg}_\ell \to kkG_{\text{gr}}$ of [12], and of a hermitian version $j^{h}_{G, \text{gr}} : G_{\text{gr}} - \text{Alg}_{\ell}^* \to kk^h_{G, \text{gr}}$ that we introduce here.

As in the ungraded case [9], to define $kk^h_{G, \text{gr}}$ we assume that there exists $\lambda \in \ell$ such that

$$\lambda + \lambda^* = 1. \hspace{1cm} (1.2)$$

In Theorem 13.1 we prove the following.

**Theorem 1.1** Let $\ell$ be a ring with involution satisfying (1.2), and let $E$ and $F$ be graphs with finitely many vertices. Write $L(E)$ and $L(F)$ for their Leavitt path $\ell$-algebras. Any $\mathbb{Z}[\sigma]$-module isomorphism $\xi : \text{BF}_{\text{gr}}(E) \xrightarrow{\sim} \text{BF}_{\text{gr}}(F)$ lifts to an isomorphism $j^{h}_{\mathbb{Z}_{\text{gr}}}(L(E)) \xrightarrow{\sim} j^{h}_{\mathbb{Z}_{\text{gr}}}(L(F))$ in $kk^h_{\mathbb{Z}_{\text{gr}}}$.

The exact meaning of lifting in the theorem above is made precise in Theorem 13.1. The forgetful functor $\text{Alg}_{\ell}^* \to \text{Alg}_{\ell}$ induces a functor $kk^h_{\mathbb{Z}_{\text{gr}}} \to kk_{\mathbb{Z}_{\text{gr}}}$, so under the hypothesis of Theorem 1.1 we also have an isomorphism $j_{\mathbb{Z}_{\text{gr}}}(L(E)) \cong j_{\mathbb{Z}_{\text{gr}}}(L(F))$. Further, we show that under mild additional assumptions on the homotopy algebraic $K$-theory $KH_{\ast}(\ell)$ of $\ell$, which are satisfied, for example, if $\ell$ is a field, a PID, or a

\[ \text{Springer} \]
noetherian regular local ring, the Bowen–Franks module classifies Leavitt path algebras in both $kk\mathbb{Z}_{gr}$ and $kk^h_{\mathbb{Z}_{gr}}$. In Theorem 13.2 we prove the following.

**Theorem 1.2** Assume that $\ell$ satisfies (1.2), that $KH_{-1}(\ell) = 0$ and that the canonical morphism $\mathbb{Z} \to KH_0(\ell)$ is an isomorphism. Then for each pair of graphs $E$ and $F$ with finitely many vertices, the following are equivalent:

(i) The algebras $L(E)$ and $L(F)$ are $kk^h_{\mathbb{Z}_{gr}}$-isomorphic.

(ii) The algebras $L(E)$ and $L(F)$ are $kk\mathbb{Z}_{gr}$-isomorphic.

(iii) The $\mathbb{Z}[\sigma]$-modules $\text{BF}_\mathbb{Z}(E)$ and $\text{BF}_\mathbb{Z}(F)$ are isomorphic.

We also obtain several other results about (hermitian) graded $K$-theory of $(\ast)$-algebras in general and Leavitt path algebras in particular which we think are of independent interest. Building upon work of Ara, Hazrat, Li and Sims in [3] and Preusser in [15], we show in Theorems 3.4 and 3.9 that if $R$ is a $(\ast)$-ring, graded over a group $G$, and having (self-adjoint) graded local units, then the (hermitian) graded $K$-theory of $R$ is the (hermitian) $K$-theory of the crossed product

$$K^G(R) = K^h(G \rtimes R), \quad K^{h,G}(R) = K^h(G \rtimes R).$$

(1.3)

The definition of $G \rtimes R$ is recalled in Subsection 2.5; the same formulas hold for homotopy algebraic and homotopy hermitian $K$-theory under no unitality assumptions (see Corollary 8.5). For example if $E$ is row-finite and $R = L(E)$ with its canonical $\mathbb{Z}$-grading, then $\mathbb{Z} \rtimes L(E) = L(\tilde{E})$ is the Leavitt path algebra of the universal covering of $E$ (Proposition 2.7). We use this to compute, for a row-finite graph $E$ and an algebra $R$ with local units, equipped with the trivial grading, the graded $K$-theory of the algebra $L(E) \otimes R$ and, in case $R$ is a $(\ast)$-algebra with self-adjoint graded local units, also its hermitian graded $K$-theory; we show in Corollary 5.4 that

$$K^G_{\text{gr}}(L(E) \otimes R) = \text{BF}_{\mathbb{Z}}(E) \otimes K^G_{\text{gr}}(R), \quad K^{h,\text{gr}}_{\text{gr}}(L(E) \otimes R) = \text{BF}_{\mathbb{Z}}(E) \otimes K^h_{\text{gr}}(R).$$

(1.4)

Again the same formulas hold for homotopy algebraic and homotopy hermitian $K$-theory without any unitality assumptions on $R$. We also consider gradings on $L(E)$ with values on a group $G$ for an arbitrary graph $E$. These are associated to functions $\omega : E^1 \to G$; we write $L_{\omega}(E)$ for $L(E)$ with the induced grading. There is a universal covering $\tilde{E} = (\tilde{E}, \omega)$ and again $G \rtimes L(E) = L(\tilde{E}, \omega)$. Let

$$A_{\omega} \in \mathbb{Z}[G]^{\text{reg}(E) \times E^0}, \quad (A_{\omega})_{v,w} = \sum_{v \rightarrow w} \omega(e).$$

(1.5)

Observe that for $G = \mathbb{Z}$ and $\omega$ the constant grading above, $A_{\omega} = \sigma A_E \in \mathbb{Z}[\sigma]^{\text{reg}(E) \times E^0}$. Define $\text{BF}_{\text{gr}}(E, \omega) := \text{coker}(I - A^t_{\omega})$. We show in Corollary 11.6 that if $R$ is a trivially $G$-graded, unital, regular supercoherent algebra, then there is an exact sequence

$$0 \to \text{BF}_{\text{gr}}(E, \omega) \otimes K_R(R) \to K^G_{\text{gr}}(L_{\omega}(E) \otimes R) \to \ker((I - A^t_{\omega}) \otimes K_{n-1}(R)) \to 0.$$  

(1.6)
If in addition $R$ is a $\ast$-algebra and 2 is invertible in $R$ we have a similar sequence for $G$-graded hermitian $K$-theory. Both (1.6) and its hermitian counterpart hold for homotopy $K$-theory for every trivially $G$-graded ($\ast$-) algebra $R$, where in the hermitian case we assume that the ground ring $\ell$ satisfies (1.2). We show in Proposition 9.1 that there is a ring isomorphism $kk^h_{G_{gr}}(\ell, \ell) = kk^h(\ell, \ell) \otimes \mathbb{Z}[G^{op}]$. In particular, if $E^0$ is finite, the matrix $I - A^t_0$ defines an element of $kk^h_{G_{gr}}(\ell^{reg}(E), \ell^{reg}(E_0))$, and Corollary 11.9 says that there is a distinguished triangle in $kk^h_{G_{gr}}$ which, omitting $j^h_0$, has the form

$$
\ell^{reg}(E) \xrightarrow{I - A^t_0} \ell^{E_0} \rightarrow L_\omega(E).
$$

(1.7)

Recall that a $G$-graded ring $A$ is strongly graded if $AgAh = Agh$ for all $g, h \in A$. A theorem of Dade characterizes strongly $G$-graded unital rings in terms of module categories; we obtain a hermitian variant of Dade’s theorem for $\ast$-rings in terms of module categories with duality (Theorem 4.1). We also show (Theorem 10.1) that if $B$ is a strongly $G$-graded $\ast$-algebra, then the canonical inclusion

$$
B_1 \subset G \overset{\ast}{\otimes} B
$$

(1.8)

is a $kk^h$-equivalence. For example if $A$ is any $\mathbb{Z}$-graded $\ast$-algebra, then $A[\iota, \iota^{-1}]$ is strongly $\mathbb{Z}$-graded. Using this together with (1.7) we show in Theorem 12.1 that there is a distinguished triangle in $kk^h$

$$
\mathbb{Z} \overset{1 - \mathbb{Z} \overset{\ast}{\otimes} \iota}{\rightarrow} \mathbb{Z} \overset{\ast}{\otimes} A \rightarrow A.
$$

(1.9)

As a consequence of (1.9) and of the homotopy hermitian $K$-theory version of (1.3), we obtain an exact sequence

$$
KH^h_{n+1}(A) \rightarrow KH^h_{n, Z_{gr}}(A) \rightarrow KH^h_{n, Z_{gr}}(A) \rightarrow KH^h_n(A)
$$

for any $n \in \mathbb{Z}$ and any $\ast$-algebra $A$, and a similar sequence for non-hermitian homotopy $K$-theory of any ring $A$. For sufficiently regular $A$, $K^h$ and $K$ may be substituted for $KH^h$ above; in particular we recover the classical Van den Bergh exact sequence of [18] for regular noetherian $A$, as well as a hermitian variant of his sequence (Corollary 12.2).

The rest of this article is organized as follows. In Sect. 2 we recall basic definitions, notations and properties for algebras equipped with an action of, or a grading over a group $G$. Section 3 contains the proof of the identities (1.3) in Theorems 3.4 and 3.9; the basic idea is to use the category isomorphism between graded $R$-modules and $G \overset{\ast}{\otimes} R$-modules due to Ara, Hazrat, Li and Sims [3] and the fact that the latter preserves finite generation, proved in Preusser’s article [15], and to check that the category isomorphism intertwines the relevant duality functors. As an application we also establish Theorem 4.1, which is a hermitian variant of Dade’s theorem. Section 5 is concerned with the proof of (1.4), obtained in Corollary 5.4 as a consequence of

Springer
the more general Theorem 5.3 which establishes a similar formula with \( K^h_n \) and \( K_n \) replaced by any functor \( H \) from \( \mathbb{Z} \)-graded \(*\)-rings to abelian groups satisfying some mild assumptions. We also prove in Theorem 5.6 that if \( \ell \) is a field, then \( K_{0\ell}^\text{gr} \) reflects injectivity of \( \mathbb{Z} \)-graded algebra homomorphisms \( L(E) \to R \) with \( E \) finite. Thus if \( \phi \) is such a homomorphism and \( K_{0\ell}^\text{gr} (\phi) \) is injective, then so must be \( \phi \). Section 6 concerns appropriate notions of stability for functors defined on the categories \( G - \text{Alg}^\ast \) and \( G_{\text{gr}} - \text{Alg}^\ast_{\ell} \) of \( G \)-graded \(*\)-algebras. Section 7 introduces triangulated categories \( \mathbb{K}_h \) and \( \mathbb{K}^h_{\text{gr}} \) and functors

\[
j^h_G : \text{Alg}^\ast \to k^h_G, \quad j^h_{G_{\text{gr}}} : \text{Alg}^\ast_{\ell} \to k^h_{G_{\text{gr}}}.
\]

Each of these functors satisfies a universal property, which essentially says that it is universal among excisive, homotopy invariant and stable homology theories. In Sect. 8 we show that the adjointness theorems proved by Ellis in [12] remain valid in the hermitian setting. In particular for a subgroup \( H \subset G \), the induction and restriction functors define an adjoint pair \( k^h_G \leftrightarrows k^h_H \) (Theorem 8.3) and the cross product functors induce inverse category equivalences \( k^h_G \cong k^h_{G_{\text{gr}}} \) (Theorem 8.1). Let \( KH^h \) be the homotopy hermitian \( K \)-theory of \([9, \text{Section 3}]\). In Sect. 9 we compute the coefficient ring \( k^h_{G_{\text{gr}}} (\ell, A) \). We show in Proposition 9.1 that composing the canonical isomorphisms between \( k^h \)-groups provided by Theorems 8.3 and 8.1 one obtains a ring isomorphism

\[
\mathbb{Z}[G^\text{op}] \otimes KH^h_0 (\ell) \iso k^h_{G_{\text{gr}}} (\ell, \ell).
\]

This leads to a left action of \( G \) by natural transformations on \( k^h_{G_{\text{gr}}} (\ell, A) \) for all \( A \in G_{\text{gr}} - \text{Alg}^\ast_{\ell} \); Lemma 9.2 shows this action agrees with that induced by the \( G \) action on \( G \hat{\times} A \) via the isomorphism \( k^h_{G_{\text{gr}}} (\ell, A) = k^h (\ell, G \hat{\times} A) \). When \( G \) is abelian, the tensor product of graded algebras is again \( G \)-graded; this together with the isomorphism (1.10) leads to an enrichment of \( k^h_{G_{\text{gr}}} \) over \( \text{mod} \mathbb{Z}(G) \), which is described by Proposition 9.4. Section 10 is devoted to the proof of Theorem 10.1, which says that if \( B \) is strongly \( G \)-graded, then the map (1.8) is a \( k^h \)-equivalence.

Let \( L(E) \cong C(E)/K(E) \) be the usual presentation of the Leavitt path algebra as a quotient of the Cohn algebra [1, Proposition 1.5.5]. Let \( \omega : E^1 \to G \) be a weight function and let \( L_\omega (E), C_\omega (E) \) and \( K_\omega (E) \) be the same algebras equipped with the associated \( G \)-gradings. In Sect. 11 we study the Cohn extension

\[
0 \to K_\omega (E) \to C_\omega (E) \to L_\omega (E) \to 0
\]

in \( k^h_{G_{\text{gr}}} \). Propositions 11.1 and 11.2 show that for any excisive, homotopy invariant and stable homology theory \( H \) of \( G \)-graded \(*\)-algebras which commutes with sums of sufficiently high cardinality we have \( H(K_\omega (E)) \cong H(\ell) (\text{reg}(E)) \) and \( H(C_\omega (E)) \cong H(\ell) (E^0) \). The map \( \xi : H(\ell) (\text{reg}(E)) \to H(\ell) (E^0) \) induced by the inclusion \( K_\omega (E) \subset \text{mod} \mathbb{Z}(G) \). Springer
$C_\omega(E)$ is computed in Theorem 11.5, and so we obtain a triangle

$$H(\ell)_{\text{reg}}(E) \xrightarrow{I - A_\omega^I} H(\ell)(E^0) \xrightarrow{\Delta} H(L_\omega(E)).$$

When $E^0$ is finite, we may take $H = j_G^R$; this gives triangle (1.7) (Corollary 11.9).

In Sect. 12 we apply Theorems 10.1 and 11.5 to prove Theorem 12.1 which establishes the distinguished triangle (1.9), and Corollary 12.2 which derives long exact sequences relating graded and ungraded (hermitian, homotopy) $K$-theory.

In Sect. 13 we use triangle (1.7) to prove Theorems 1.1 and 1.2 as Theorems 13.1 and 13.2.

2 Preliminaries on algebras, involutions, gradings, and actions

2.1 Algebras and involutions

A commutative, unital ring $\ell$ with involution $\ast$ will be fixed throughout the article. By an $\ell$-algebra we mean a symmetric $\ell$-bimodule $A$ together with an associative multiplication $A \otimes \ell A \to A$. An involution of an $\ell$-algebra is an additive morphism $\ast: A \to A$ such that

$$(a^\ast)^\ast = a, \quad (ab)^\ast = b^\ast a^\ast, \quad (\mu a)^\ast = \mu^\ast a^\ast$$

for all $a, b \in A$ and $\mu \in \ell$. An $\ast$-algebra is an $\ell$-algebra together with an involution; a $\ast$-morphism $f: A \to B$ between $\ast$-algebras is an $\ell$-algebra morphism such that $f(a^\ast) = f(a)^\ast$ for each $a \in A$. A two-sided ideal $I \triangleleft A$ is a $\ast$-ideal if in addition it is an $\ell$-submodule of $A$ such that $I^\ast \subset I$. The category of $\ast$-algebras together with $\ast$-morphisms will be denoted $\text{Alg}^\ast_\ell$. Tensor products of algebras are taken over $\ell$; we write $\otimes$ for $\otimes_\ell$. We also use $\otimes$ for tensor products of abelian groups, e.g. $K_0(R) \otimes K_0(S) = K_0(R) \otimes \mathbb{Z} K_0(S)$. We write $R \otimes \mathbb{Z} S$ for the tensor product of rings which are not necessarily $\ell$-algebras.

An element $u$ in a unital $\ast$-algebra $R$ is said to be unitary if $uu^\ast = u^\ast u = 1$. If $\epsilon \in R$ is both central and unitary, we say that $\phi \in R$ is $\epsilon$-hermitian if $\phi = \epsilon \phi^\ast$. An $\epsilon$-hermitian unit $\phi \in R^\ast$ gives rise to an involution

$$(\phi)^\ast: R \to R, \quad x \mapsto \phi^{-1}x^\ast \phi.$$  

We write $R^\phi$ for $R$ viewed as a $\ast$-algebra with this involution. Similarly, for a $\ast$-ideal $A \triangleleft R$ the notation $A^\phi$ will be employed for $A$ equipped with the involution induced from $R^\phi$.

Notation 2.1 An element $a$ in $\ast$-algebra $A$ is self-adjoint if $a = a^\ast$. A projection $p \in A$ is a self-adjoint idempotent element. Such an element determines a $\ast$-morphism $x \in \ell \mapsto px \in A$ which we will also call $p$.
2.2 Actions and gradings

From now on, we fix a group $G$, which we regard as a one object category. The category $G - \text{Alg}_\ell^*$ of $G$-$\ast$-algebras is the category of functors $G \to \text{Alg}_\ell^*$. A $G$-graded $\ast$-algebra is a $G$-graded $\ell$-algebra $A = \bigoplus_{g \in G} A_g$ equipped with an involution such that $A_g \subset A_{g^{-1}}$ for all $g \in G$. The degree of a homogeneous element $a \in A$ will be denoted $|a|$. We write $G_{gr} - \text{Alg}_\ell^*$ for the category of $G$-graded $\ast$-algebras together with $\ast$-homomorphisms that are homogeneous of degree 1 $\in G$. The letter $\mathfrak{A}$ will refer to either $G - \text{Alg}_\ell^*$ or $G_{gr} - \text{Alg}_\ell^*$.

**Remark 2.2** If $A$ is a $\ast$-algebra and $B$ a $G$-graded $\ast$-algebra, their tensor product $A \otimes_\ell B$ is $G$-graded by defining $|a \otimes b| = |b|$ for each $a \in A$, $b \in B$ with $b$ homogeneous. If both $A$ and $B$ are $G$-graded $\ast$-algebras and $G$ is abelian, then $A \otimes_\ell B$ can be made into a $G$-graded $\ast$-algebra by defining $|a \otimes b| = |a||b|$ for each pair of homogeneous elements $a \in A$, $b \in B$. When $G$ is not abelian, this assignment fails to be compatible with multiplication.

2.3 Matrix algebras

Let $X$ be a set and $A$ a $\ast$-algebra. Recall that Wagoner’s cone is the ring

$$C_X A := \{ f : X \times X \to A : |\text{supp } f(x, -)|, |\text{supp } f(-, x)| < \infty \ (\forall x \in X)\} \ (2.1)$$

with the operations given by the pointwise sum and the convolution product. Its canonical involution is defined to be $f^*(x, y) = f(y, x)^*$ for each $f \in C_X A$. It contains Karoubi’s cone

$$\Gamma_X A = \{ f : X \times X \to A : |\text{im } f| < \infty \text{ and } (\exists N \geq 1) \text{ s.t. } |\text{supp}(x, -)|, |\text{supp}(-, x)| \leq N(\forall x \in X)\}
\ (2.2)$$

as a $\ast$-subalgebra. We will also consider the $\ast$-ideal of $\Gamma_X A$ consisting of $X$-indexed finitely supported matrices

$$M_X A := \{ f : X \times X \to A : \text{supp}(f) \text{ is finite} \}.$$  

In the case $X = \mathbb{N}$ we use specific notation; we write $M_{\infty} A := M_\mathbb{N} A$.

**Convention 2.3** When $A = \ell$, we omit it from the notation in all matrix algebras defined above.

There is an isomorphism of $\ast$-algebras $M_X A \cong M_X \otimes_\ell A$; we write $\varepsilon_{x,y} \in M_X$ for the characteristic function of the pair $(x, y) \in X \times X$ and

$$\iota_X : A \to M_X A, \ a \mapsto \varepsilon_{x,x} \otimes a$$

for the inclusion in the diagonal entry corresponding to $x \in X$.

If $A$ is a $G$-$\ast$-algebra and $X$ is a $G$-set, then the algebras $C_X A$, $\Gamma_X A$ and $M_X A$ defined above are $G$-$\ast$-algebras with action $(g \cdot f)(x, y) = g \cdot f(g^{-1}x, g^{-1}y)$. The
analogue situation for $G$-graded $\ast$-algebras requires some preliminary definitions. A $G$-
graded set is a set $X$ together with a function $\omega : X \to G$, which we call a weight. A
morphism of $G$-graded sets $(X, \omega) \to (Y, \upsilon)$ is a map $f : X \to Y$ such that $\upsilon \circ f = \omega$.
We shall often drop the weight from our vocabulary, and say simply that $X$ is a graded
set; in this case we write $| |$ for the weight function of $X$.

**Definition 2.4** Let $B$ be a $G$-graded $\ast$-algebra and $X$ a $G$-graded set. Let $C_\ell^X B$
be the $\ell$-linear span of those elements $f \in C_X B$ such that $f(x, y)$ is homogeneous for each
$(x, y) \in X \times X$ and such that the function $(x, y) \mapsto |x||f(x, y)||y|^{-1}$ is constant
in $\text{supp}(f)$. The $\ast$-algebra $C_\ell^X B$ is $G$-graded; its homogeneous component of degree
$g \in G$ is

$$(C_\ell^X B)_g := \{ f \in C_\ell^X B : |x| \cdot |f(x, y)| = g|y| \ (\forall (x, y) \in \text{supp}(f)) \}.$$ Set $\Gamma_\ell^X B := C_\ell^X B \cap \Gamma_X B$; one checks that $\Gamma_\ell^X B \subset C_\ell^X B$ is a $G$-graded $\ast$-subalgebra.

### 2.4 Matricial stability

Let $X$ be a set and $\mathcal{C}$ a category; a functor $F : \mathcal{A} \to \mathcal{C}$ is called $M_X$-stable if it sends
each inclusion $\iota_x : A \to M_X A$ with $A \in \mathcal{A}$ and $x \in X$ to an isomorphism. By the
argument of [9, Lemma 2.4.1], this is equivalent to asking that the natural map $F(\iota_x)$
be an isomorphism for a fixed $x \in X$.

### 2.5 Crossed products

We now recall the definitions of crossed products for $G$-$\ast$-algebras and $G$-graded $\ast$-
algebras. We adopt the notations of [12]. Let $\ell[G]$ be the group algebra of $G$ and $A$
a $G$-$\ast$-algebra. The crossed product $A \rtimes G$ is the $\ell$-module $A \otimes \ell[G]$ together with
the multiplication defined by the rule $(a \rtimes g) \cdot (b \rtimes h) = a(g \cdot b) \rtimes gh$. Here $a \rtimes g$
is notation for the elementary tensor $a \otimes g$. The crossed product is a $G$-graded $\ast$-algebra
with homogeneous components $(A \rtimes G)_g = A \rtimes g = \text{span}_\ell \{ a \rtimes g : a \in A \}$ and
involution $(a \rtimes g)^* = g^{-1} \cdot a^* \rtimes g^{-1}$.

If $B$ is a $G$-graded $\ast$-algebra, we may regard it as a comodule algebra over the
(nonunital) Hopf algebra $\ell(G)$. We write $G \bowtie B = \ell(G) \bowtie B$ for the crossed product.
As an $\ell$-module, $G \bowtie B$ is just $\ell(G) \otimes B$; we write $\chi_g \in \ell(G)$ for the characteristic
function and put $\chi_g \bowtie b := \chi_g \otimes b$. As a $G$-$\ast$-algebra, $G \bowtie B$ embeds in $M_G B$ via
the identification $\chi_g \bowtie b = \delta_{g, |b|} \chi_g \bowtie b$ for $g \in G$ and homogeneous $b \in B$. Thus for
g, $h \in G$ and $b, c \in B$ with $b$ homogeneous, we have

$$(\chi_g \bowtie b)(\chi_h \bowtie c) = \delta_{h, |b|} \chi_g \bowtie bc,$$

$$\chi_g \bowtie b^* = \chi_{g^*} \bowtie b^*,$$

$$h \cdot (\chi_g \bowtie c) = \chi_{hg} \bowtie c.$$
Remark 2.5 In [3, Definition 2.1] the smash product $A\#G$ of a $G$-graded algebra $A$ is defined. One checks that, for the grading $A^g_{op} := A_{g-1}$ $(g \in G)$, the map

$$A\#G \rightarrow (G \hat{\otimes} A_{op})_{op}, \quad ap_g \mapsto \chi_{g-1} \hat{\otimes} a$$

is an $G$-algebra isomorphism.

Consider the functors

$$- \times G : G-\text{Alg}^* \leftrightarrow G_{gr-\text{Alg}}^* : G^\hat{\otimes} -.$$

(2.3)

Proposition 2.6 Let $A$ be a $G$-star-algebra and $B$ a $G$-graded star-algebra. We have a natural isomorphism of $G$-star-algebras $G \hat{\otimes} (A \times G) \cong M_G A$ and a natural isomorphism of $G$-graded star-algebras $(G \hat{\otimes} B) \times G \cong M_G B$.

Proof One checks that the natural isomorphisms of [12, Proposition 7.4] are $\ast$-homomorphisms.

2.6 Covering graphs

Let $G$ be a group. The covering of a graph $E$ associated to a weight $\omega : E^1 \rightarrow G$ is the graph $\tilde{E} = (E, \omega)$ with vertices and edges defined by

$$\tilde{E}^i = E \times G \quad (i = 0, 1).$$

Write $v_g = (v, g)$ and $e_g = (e, g)$ for each $v \in E^0, e \in E^1$; the source and range functions of $\tilde{E}$ are defined by

$$s(e_g) = v_g, \quad r(e_g) = r(e)_{g\omega(e)}.$$

Observe that left multiplication on the $G$ component gives an action of $G$ on $\tilde{E}$ by graph automorphisms. In particular $L(\tilde{E})$ is a $G$-algebra.

Proposition 2.7 Let $E$ be a graph, $\omega : E^1 \rightarrow G$ a weight, $L_\omega(E)$ the Leavitt path algebra equipped with its canonical involution and the $G$-grading induced by $\omega$, and $\tilde{E} = (E, \omega)$ the associated covering graph. There is an isomorphism of $G$-star-algebras $L\tilde{E} \rightarrow G \hat{\otimes} L_\omega(E)$.

Proof By [15, Example 7], this is a particular case of [15, Proposition 74].

3 Graded $K$-theory and crossed products

3.1 Graded $A$ modules versus $G \hat{\otimes} A$-modules

A $G$-graded ring $A$ has graded local units if for every finite subset $\mathcal{F} \subset A$ there exists a homogeneous idempotent $e$ such that $\mathcal{F} \subset eAe$. 

\(\text{Springer}\)
Remark 3.1 Recall that the set $E$ of homogeneous idempotents of a $G$-graded ring $A$ can be equipped with a partial order; concretely, we set $e \leq f$ whenever $ef = fe = e$. If $A$ has graded local units, then for any cofinal subset $U \subset E$ we have $A = \operatorname{colim}_{e \in U} eAe = \operatorname{colim}_{\chi \in E} e\chi e$ as $G$-graded rings. We call any such $U$ a set of graded local units of $A$.

In [3, Proposition 2.5] Ara, Hazrat, Li and Sims show that if $A$ is a $G$-graded ring that has graded local units, then there is an isomorphism between the categories of unital left $G$-graded $A$-modules and of unital left modules over the crossed product ring $A \# G$ of [3, Definition 2.1], which by Remark 2.5 is isomorphic to $(G \rtimes A^\text{op})^\text{op}$. Restating their result in terms of $G \rtimes A$, we obtain inverse isomorphisms of categories of unital right modules

$$\Psi : \text{mod}_{G^\text{gr}} A \leftrightarrow \text{mod}_{G \rtimes A} : \Phi$$

which we proceed to describe. The functor $\Psi$ sends a $G$-graded module $M$ to its underlying abelian group equipped with the $(G \rtimes A)$-action given by $m(\chi_g \rtimes a) = mga$ for each $g \in G, a \in A,$ and $m \in M$. For a homogeneous homomorphism $f$, $\Psi(f)$ is set to be same function $f$; a direct computation shows that the latter is $G \rtimes A$-linear. Fix a set $U$ of graded local units for $A$. We define the module $\Phi(N)$ for a $G \rtimes A$-module $N$ as the same abelian group equipped with the grading $\Phi(N)_g = \sum_{u \in U} N(\chi_g \rtimes u)$ for each $g \in G$. The equality $\Phi(N) = \bigoplus_{g \in G} \Phi(N)_g$ follows from the fact that $N$ is unital. The $A$-module action is given by $ma = m(\chi_g \rtimes a)$ for each $m \in \Phi(N)_g, a \in A$. If $f : M \rightarrow N$ is a morphism of $G \rtimes A$-modules, we set $\Phi(f)(m_g) = f(m)_g$. The categories $G^\text{gr}$-$\text{mod}_A$ and $\text{mod}_{G \rtimes A}$ are equipped with right actions of $G$ which we now describe. For $M \in G^\text{gr}$-$\text{mod}_A$ and $g \in G$, let $M[g]$ be the same $A$-module with the following grading

$$M[g]_h = M_{gh}.$$

(3.2)

Likewise, for $N \in \text{mod}_{G \rtimes A}$, let $N \cdot g$ be the same abelian group $N$ with right multiplication $\cdot_g$ defined as follows

$$x \cdot_g (\chi_s \rtimes a) = x \cdot (\chi_{gs} \rtimes a).$$

(3.3)

It is straightforward to check that $\Psi$ and $\Phi$ intertwine these actions.

By definition both compositions of $\Psi$ and $\Phi$ yield the respective identity functors. Moreover, both functors are exact; in particular, they preserve projective objects.

Proposition 3.2 Let $A$ be a $G$-graded ring with graded local units. Then the functors $\Psi$ and $\Phi$ of (3.1) send finitely generated modules to finitely generated modules.

Proof This is shown in the course of the proof of [15, Proposition 66].

Rem 3.3 Recall that the unitalization of a ring $A$ is the abelian group $\hat{A}_Z = A \oplus Z$ together with the multiplication rule $(a, k) \cdot (b, l) := (ab + al + bk, kl)$. If $A$ is $G$-graded, then so is $\hat{A}_Z$, with homogeneous components $(\hat{A}_Z)_g = A_1 \oplus Z$ and $(\hat{A}_Z)_g = A_g$ for $g \neq 1$. When $A$ is unital, $\hat{A}_Z \rightarrow A \times Z, (a, n) \mapsto (a + n \cdot 1, n)$ is a $G$-graded isomorphism.

\textcopyright Springer
Let $\text{Ring}$ be the category of rings and ring homomorphisms and $\text{Ring}_1 \subset \text{Ring}$ the subcategory of unital rings and unit preserving homomorphisms. A functor $F: G_{gr} \rightarrow \text{Ring}_1 \rightarrow \text{Ab}$ is additive if for each pair of $G$-graded $R, S \in \text{Ring}_1$ the canonical map $F(R \times S) \rightarrow F(R) \oplus F(S)$ is an isomorphism. For such an $F$, the functor $\hat{F}: G_{gr} \rightarrow \text{Ring} \rightarrow \text{Ab}$, $\hat{F}(A) := \ker(F(\hat{A}_Z) \xrightarrow{F(\pi_A)} F(\mathbb{Z}))$ extends $F$ up to natural isomorphism.

Since both unitalization and kernels preserve filtering colimits, if $F$ preserves filtering colimits then so does $\hat{F}$. Thus, given a $G$-graded ring $A$ with a set of (graded) local units $\mathcal{U}$ we have $\hat{F}(A) \cong \colim_{e \in \mathcal{U}} \hat{F}(eAe) \cong \colim_{e \in \mathcal{U}} F(eAe)$. In particular all of this applies to $F(A) = K_*^{G_{gr}}(A)$ which is defined for unital $A$ as the $K$-theory of the split-exact category $\text{Proj}_{G_{gr}}(A)$ of finitely generated projective $G$-graded $A$-modules, and we have

$$K_*^{G_{gr}}(A) = K_*(\text{Proj}_{G_{gr}}(A))$$

(3.4)

whenever $A$ has graded local units.

**Theorem 3.4** Let $A$ be a $G$-graded ring with graded local units. Then the functor $\Psi$ above induces $\mathbb{Z}[G]$-module isomorphisms $K_*^{G_{gr}}(A) \xrightarrow{\sim} K_*(G \hat{\otimes} A)$.

**Proof** Let $\mathcal{U} \subset A$ be a set of graded local units and let $\mathcal{F}(G)$ be the set of all finite subsets of $G$. Then $\{ \chi_F \hat{\otimes} e : F \in \mathcal{F}(G), e \in \mathcal{U} \}$ is a set of local units of $G \hat{\otimes} A$. Hence the theorem follows from Proposition 3.2 and Remark 3.3. □

**Remark 3.5** Let $R$ be a $G$-graded ring with graded local units. Recall from Subsection 2.5 that $G \hat{\otimes} R$ is a $G$-*$\mathrm{-}$algebra; write $\alpha_g: G \hat{\otimes} R \rightarrow G \hat{\otimes} R, (\alpha_g(\chi_s \hat{\otimes} a) = (\chi gs \hat{\otimes} a)$, for the algebra automorphism associated with left multiplication by $g \in G$. The functor $\beta_g$ that sends a unital right $G \hat{\otimes} R$-module $N$ to $N \cdot g$ as defined in (3.3) –which corresponds to shift grading (3.2) under the equivalence (3.1)– is naturally isomorphic to scalar extension along $\alpha_g^{-1}$. Thus the left and right $\mathbb{Z}[G]$-module structures induced by the $\alpha_g$ and the $\beta_g$ on $K_*(G \hat{\otimes} R)$ correspond to each other under the canonical involution $g \mapsto g^{-1}$.

### 3.2 Duality and hermitian $K$-theory

Let $A$ be a $G$-graded $*$-ring. If $M$ is a unital $G$-graded right $A$-module, its hermitian dual is the right module

$$M^* = \{ f \in \text{hom}_\mathbb{Z}(M, A) : f(xa) = a^*f(x) (\forall a \in A, x \in M)\}.$$

For each $d \in G$, consider the $\mathbb{Z}$-submodule

$$M^*_d := \{ f \in M^* : f(Mg) \subset A_{g^{-1}d} (\forall g \in G)\}.$$

The sum $\sum_{d \in G} M^*_d$ is always direct, but the inclusion $\bigoplus_{d \in G} M^*_d \subset M^*$ may be strict. Assume that $A$ has graded local units; we say that a $G$-graded $A$-module $M$
is finitely presented if there exist a homogeneous idempotent \( e \in A \), \( n, m \geq 1 \), \( g_1, \ldots, g_n, h_1, \ldots, h_m \in G \) and an exact sequence of homogeneous homomorphisms

\[
\bigoplus_{i=1}^n eA[g_i] \to \bigoplus_{i=1}^m eA[h_i] \to M \to 0.
\]

If \( M \) is finitely presented, then we have the equality

\[
M^* = \bigoplus_{d \in G} M_d^*.
\]

Hence \( M^* \) is a \( G \)-graded module in this case.

**Lemma 3.6** Let \( A \) be a \( G \)-graded \( * \)-ring with self-adjoint graded local units and \( M \) a finitely presented graded unital right \( A \)-module. There is a natural isomorphism \( \eta_M : \Psi(M^*) \to \Psi(M)^* \).

**Proof** As in Subsection 2.5, we regard \( \hat{G} \ltimes A \) as \( * \)-subalgebra of \( MG_A \); we write \( \pi_{g,h}(x) \in A \) for the \((g, h)\)-entry of \( x \in \hat{G} \ltimes A \). For \( \alpha \in \Psi(M^*) \) and \( \beta \in \Psi(M)^* \), set

\[
\alpha^\sharp(x) = \sum_{g,h \in G} \chi_g \ltimes \alpha_h(x_g), \quad \beta^\flat(y) = \sum_{g,h \in G} \pi_{g,h}(\beta(y))
\]

One checks that \( \alpha^\sharp \in \Psi(M)^* \) and \( \beta^\flat \in \Psi(M)^* \) and that the map \((-)^\sharp : \Psi(M^*) \to \Psi(M)^* \) is a \( \hat{G} \ltimes A \)-linear isomorphism with inverse \((-)^\flat \). \( \square \)

Let \( A \) be a \( G \)-graded \( * \)-ring with graded local units. If \( M \) is a unital, finitely generated and projective \( G \)-graded right \( A \)-module, then the canonical map

\[
\text{can} : M \to M^{**}, \quad \text{can}(x)(f) = f(x)^*
\]

is a homogeneous isomorphism. Hence the triple \((\text{Proj}_{G_{gr}}(A), *, \text{can})\) with the split-exact sequences as conflations, is an exact category with duality in the sense of [16, Definition 2.1]. If \( A \) is unital, we write

\[
K^h_{*,G_{gr}}(A) = G\text{W}_*(\text{Proj}_{G_{gr}}(A), *, \text{can})
\]

for its Grothendieck-Witt groups. We extend \( K^h_{*,G_{gr}} \) to all, not necessarily unital \( G \)-graded \( * \)-rings as in Remark 3.3. If \( A \) is a \( G \)-graded \( * \)-ring with self-adjoint graded units, then it can be written as a filtering colimit of unital \( G \)-graded rings with involution and so its \( G \)-graded hermitian \( K \)-theory groups coincide with the Grothendieck-Witt groups of \((\text{Proj}_{G_{gr}}(A), *, \text{can})\). Similarly, the hermitian \( K \)-theory groups of \( \hat{G} \ltimes A \) are the Grothendieck-Witt groups of the split exact category with duality \((\text{Proj}(\hat{G} \ltimes A), *, \text{can}')\). Here \( \text{can}' \) is the natural transformation between a \( G \ltimes A \)-module and its double hermitian dual.
Lemma 3.7 Let $A$ be a $G$-graded $\ast$-ring with self-adjoint graded local units. Let $\text{can}$ and $\text{can}'$ be as above. Then for each unital $G$-graded $A$-module $M$ the following diagram is commutative.

$$
\begin{array}{ccc}
\Psi(M) & \xrightarrow{\text{can}_{\Psi(M)}} & \Psi(M)^{**} \\
\Psi(\text{can}_M) \downarrow & & \downarrow \eta_M^{**} \\
\Psi(M^{**}) & \xrightarrow{\eta_{M'}} & \Psi(M^*)
\end{array}
$$

In other words, $(\Psi, \eta)$ is a form functor $(\text{Proj}_{G,\text{gr}}(A), \ast, \text{can}) \to (\text{Proj}(G \hat{\otimes} A), \ast, \text{can}')$ in the sense of [16, Definition 3.2].

**Proof** It suffices to see that given an element $m \in \Psi(M)$ which is homogeneous as an element of $M$, the maps $(\eta_M^{**})_{\Psi(M)}(m)$ and $(\eta_M^{**})_{\Psi(can)}(m)$ coincide when evaluating them at each element $\alpha \in \Psi(M^*)$ which is homogeneous as an element of $M^*$. Indeed, given $m \in M_s$ and $\alpha \in M_t^*$ we have

$$
(\eta_M^{**})_{\Psi(M)}(m)(\alpha) = \text{can}_{\Psi(M)}(m)\eta_M(\alpha)(\alpha^*(m))^* = (\chi_s \hat{\otimes} \alpha(m))^* = \chi_t \hat{\otimes} \alpha(m)^*
$$

and

$$
(\eta_M^{**})_{\Psi(can)}(m)(\alpha) = \text{can}(m)^*(\alpha) = \chi_t \hat{\otimes} \Psi(\text{can})(m)(\alpha) = \chi_t \hat{\otimes} \alpha(m)^*.
$$

Remark 3.8 Let $A$ and $B$ be exact categories with duality. An exact form functor $(F, \varphi) : A \to B$ nonsingular if $\varphi$ is an isomorphism, in which case it induces a homomorphism between the Grothendieck-Witt groups of $A$ and $B$ [16, Section 3.1]. If $F$ is moreover an isomorphism, then by definition of form functor composition [16, Definition 3.2] the (strict) inverse $F^{-1}$ can be equipped with a non-singular form functor structure in such a way that $F$ and $F^{-1}$ induce inverse isomorphisms on Grothendieck-Witt groups as defined in [16, Definition 4.12].

Theorem 3.9 Let $A$ be a $G$-graded $\ast$-ring with self-adjoint graded local units. Then the morphisms (3.1) induce $\mathbb{Z}[G]$-module isomorphisms

$$
K^{h, G, gr}_*(A) \xrightarrow{\sim} K^{h}_*(G \hat{\otimes} A).
$$

**Proof** A colimit argument similar to that used to show (3.4) proves that the left- and right-hand side of (3.7) are the Grothendieck-Witt groups of the exact categories with duality $(\text{Proj}_{G, gr}(A), \ast, \text{can})$ and $(\text{Proj}(G \hat{\otimes} A), \ast, \text{can}')$. The functor $\Psi$ of (3.1) induces an isomorphism between these Grothendieck-Witt groups, by Proposition 3.2, Lemmas 3.6 and 3.7 and Remark 3.8. 

$\square$
Corollary 3.10 Let A be a G-graded ∗-ring and let $\tilde{A}_\mathbb{Z}$ be its unitalization. Then for all $n \in \mathbb{Z}$ we have

$$K_h^{G_{gr}}(A) = \ker \left( K_h^n(G \hat{\otimes} \tilde{A}_\mathbb{Z}) \to K_h^n(\mathbb{Z}^{(G)}) \right).$$

If furthermore $n \leq 0$, then $K_h^{G_{gr}}(A) = K_h^n(G \hat{\otimes} A)$.

Proof The general formula follows from Theorem 3.9 and the fact that

$$K_h^{G_{gr}}(A) = \ker \left( K_h^n(\tilde{A}_\mathbb{Z}) \to K_h^n(\mathbb{Z}) \right).$$

The second assertion is a consequence of the first and of the fact that hermitian $K$-theory satisfies excision in non-positive dimensions. □

3.3 Homotopy hermitian graded $K$-theory

Let $A$ be a $G$-graded ∗-algebra, $P A = \ker(\text{ev}_0 : A[t] \to A)$, $\Omega A = \ker(\text{ev}_1 : P A \to A)$. By Corollary 3.10, hermitian graded $K$-theory satisfies excision in nonpositive degrees. In particular the path extension

$$0 \to \Omega A \to P A \xrightarrow{\text{ev}_1} A \to 0$$

gives rise to a connecting map $K_h^{G_{gr}}(A) \to K_h^{G_{gr}}(\Omega A)$ for each $n \leq 0$. Set

$$KH_n^{G_{gr}}(A) = \text{colim}_{r \geq 0} K_h^{G_{gr}}(\Omega^{n+r} A). \quad (3.8)$$

It follows from Corollary 3.10 and the definition of $KH^n$ [9, Section 3] that

$$K_h^{G_{gr}}(A) = KH_n^{G_{gr}}(G \hat{\otimes} A). \quad (3.9)$$

for every $G$-graded ∗-ring $A$ and every $n \in \mathbb{Z}$.

3.4 Free involutions

If $R$ is a $G$-graded ring, the ring $\text{inv}(R) = R \oplus R^{op}$ equipped with the involution $(a, b)^* = (b, a)$ has a compatible $G$-grading, with $\text{inv}(R)_g = R_g \oplus R_g^{op} = R_g \oplus R_g^{-1}$. There is an isomorphism $G \hat{\otimes} \text{inv}(R) \xrightarrow{\sim} \text{inv}(G \hat{\otimes} R)$ given by the assignment $\chi_g \hat{\otimes} (x, y) \mapsto (\chi_g \hat{\otimes} x, \chi_g |_{y} \hat{\otimes} y)$ for homogenous $x, y \in R$.

Proposition 3.11 Let $R$ be a $G$-graded unital ring. There is a natural isomorphism

$$K_{G_{gr}}^*(R) \cong K_{h,G_{gr}}^*(\text{inv}(R)).$$

Proof Via the isomorphism $\text{Proj}_{G_{gr}}(\text{inv}(R)) \cong \text{Proj}_{G_{gr}}(R) \times \text{Proj}_{G_{gr}}(R^{op})$, the duality functor of $\text{Proj}_{G_{gr}}(\text{inv}(R))$ can be naturally identified with the endofunctor of $G \hat{\otimes}$ Springer
consider the inverse equivalences

\[
\text{Proj}_{G_{\text{gr}}} (R) \times \text{Proj}_{G_{\text{gr}}} (R^{\text{op}})
\]
given by \((P, Q)^{\dagger} = (Q^\vee, P^\vee)\) where \((-)^\vee\) denotes the non-hermitian dual. Writing \(\text{ev}_M : M \to M^{\vee\vee}, \text{ev}_M(x)(f) = f(x)\), the natural transformation can: \(\text{Proj}_{G_{\text{gr}}} (\text{inv}(R)) \to \text{Proj}_{G_{\text{gr}}} (\text{inv}(R))\) is identified with \(\text{ev} \times \text{ev}\).

In view of Remark 3.8 and [16, Proposition 4.7], it suffices to see that the Grothendieck-Witt groups of \((\text{Proj}_{G_{\text{gr}}} (R) \times \text{Proj}_{G_{\text{gr}}} (R^{\text{op}}), \dagger, \text{ev} \times \text{ev})\) coincide with those of the hyperbolic category \(H(\text{Proj}_{G_{\text{gr}}} (R))\) of [16, Section 3.5]. The latter consists of the category \(\text{Proj}_{G_{\text{gr}}} (R) \times \text{Proj}_{G_{\text{gr}}} (R^{\text{op}})\) together with the duality functor \((P, Q)^* = (Q, P)\) and the identity natural transformation \(\text{id} : 1 \Rightarrow **.\) We now consider the inverse equivalences

\[
F := \text{id} \times (-)^\vee : \text{Proj}_{G_{\text{gr}}} (R) \oplus \text{Proj}_{G_{\text{gr}}} (R^{\text{op}}) \leftrightarrow \text{Proj}_{G_{\text{gr}}} (R) \oplus \text{Proj}_{G_{\text{gr}}} (R^{\text{op}}) : \text{id} \times (-)^\vee =: G.
\]

We promote these inverse equivalences to non-singular form functors by means of the natural transformations

\[
\varphi : F \circ \dagger \Rightarrow * \circ F, \quad \varphi_{(P, Q)} = (1_Q^\vee, \text{ev}_P) : (Q^\vee, P^{\vee\vee}) \to (Q^\vee, P)
\]

and

\[
\psi : G \circ * \Rightarrow \dagger \circ G, \quad \psi_{(P, Q)} = (\text{ev}_Q, 1_P^\vee) : (Q, P^\vee) \to (Q^{\vee\vee}, P^\vee).
\]

Let \(\star\) be as in [16, bottom of page 113]; the form functor compositions \((F \circ G, \varphi \star \psi)\) and \((G \circ F, \psi \star \varphi)\) yield

\[
F \circ G = \text{id} \times (-)^{\vee\vee}, \quad (\varphi \star \psi)_{(P, Q)} = (\text{ev}_Q, \text{ev}_P),
\]

\[
G \circ F = \text{id} \times (-)^{\vee\vee}, \quad (\psi \star \varphi)_{(P, Q)} = (\text{ev}_Q^\vee, \text{ev}_P^\vee).
\]

One checks that \(\zeta : F \circ G \Rightarrow \text{id}, \zeta_{(P, Q)} = (1_P, \text{ev}_Q)\) and \(\xi : \text{id} \Rightarrow G \circ F, \xi_{(P, Q)} = (1_P, \text{ev}_Q)\) are natural isomorphisms of form functors as defined in [16, Section 3.3]. This implies that the morphisms induced by \(F\) and \(G\) at the level of Grothendieck-Witt groups are mutually inverse; see [17, Section 2.8, Lemma 2] and [17, Section 2.10, Proposition 2].

Remark 3.12 By Proposition 3.11, the isomorphism \(K_{\ast}^{G_{\text{gr}}} (A) \cong K_{\ast} (G \langle G \rangle A)\) of Theorem 3.4 can be recovered from Theorem 3.9 applied to \(\text{inv}(R)\).

4 Hermitian Dade theorem

Recall that a \(G\)-graded ring \(R\) is strongly graded if for every \(g, h \in G\), we have \(R_g \cdot R_h = R_{gh}\). Dade proves in [11, Theorem 2.8] that a unital ring \(R\) is strongly graded if and only if the functors

\[
(-)_1 : G_{\text{gr}} - \text{mod}_R \to \text{mod}_{R_1} \quad \text{and} \quad \otimes_{R_1} R : \text{mod}_{R_1} \to G_{\text{gr}} - \text{mod}_R \quad \text{(4.1)}
\]

are mutually inverse category equivalences.
The categories above, equipped with the natural transformation $\varphi_M : M \to M^{**}$ are categories with duality as defined in [16, Definition 3.1]. Similarly, the functors (4.1) equipped with the following transformations are form functors in the sense of [16, Definition 3.2].

$$\varphi_M : (M^*)_1 \mapsto (M_1)^*, \quad \varphi_M(f) = f|_{M_1}^{R_1};$$
$$\psi_N : N^* \otimes_{R_1} R \to (N \otimes_{R_1} R)^*, \quad \psi_N(f \otimes r)(n \otimes s) = s^* f(m)r. \quad (4.2)$$

With these definitions in place, we have the following hermitian version of Dade’s theorem.

**Theorem 4.1** Let $R$ be a unital $G$-graded $*$-ring. The following are equivalent:

(i) The ring $R$ is strongly graded.

(ii) The form functors given by (4.1) and (4.2) are mutual inverse equivalences of categories with duality.

**Proof** By the non-hermitian version of the theorem the functors $(-)_1$ and $- \otimes_{R_1} R$ are mutual inverses if and only $R$ is strongly graded; see e.g. [14, proof of Theorem 1.5.1]. A straightforward computation shows that when $R$ is strongly graded the natural isomorphisms $(N \otimes_{R_1} R)_1 \cong N$ and $M_1 \otimes_{R_1} R \cong M$ given in loc. cit. are in fact natural isomorphisms of form functors. $\Box$

**Remark 4.2** Let $R$ be as in Theorem 4.1. The composite $\Psi \circ (- \otimes_{R_1} R) : \text{Proj}(R_1) \to \text{Proj}(G \bowtie R)$ is naturally isomorphic to the functor induced by the ring morphism $r \in R_1 \mapsto \chi_1 \bowtie r \in G \bowtie R$ by means of the natural isomorphism

$$\eta_M : M \otimes_{R_1} R \to M \otimes_{R_1} (G \bowtie R), \quad m \otimes r \mapsto m \otimes \chi_1 \bowtie r.$$

**Example 4.3** Let $R, S \in G_{gr} \text{-Alg}_\ell$; if either $R$ is trivially graded or $G$ is abelian, then by Remark 2.2, $R \otimes S$ is again $G$-graded. If moreover $R$ and $S$ are unital, and $S$ is strongly graded, then so is $R \otimes S$. In particular this applies to $S = \ell[G]$, so $R[G]$ is strongly graded. Moreover, $R[G]_1 = \bigoplus_{g \in G} R_g \otimes g^{-1} \cong R$, so by Theorem 4.1, we have

$$K_*^{h,G_{gr}}(R[G]) = K_*^{h}(R).$$

## 5 Graded $K$-theory of Leavitt path algebras

Let $E$ be a (directed) graph and $L(E)$ its Leavitt path algebra over $\ell ([11])$. In this section we show that if $E$ is row-finite, then its Bowen-Franks $\mathbb{Z}[\sigma]$-module (1.1) together with the (hermitian) $K$-theory of $\ell$ completely characterize the graded $K$-theory of $L(E)$. We write

$$\text{sink}(E) = \{ v \in E^0 : s^{-1}(v) = \emptyset \},$$

$\bowtie$ Springer
for the sets of sinks of a graph $E$. Observe that if $E$ is row-finite, we have $E^0 = \text{reg}(E) \sqcup \text{sink}(E)$. A graph $E$ will be said to be regular if $E^0 = \text{reg}(E)$.

**Remark 5.1** Recall that if $E$ is a regular graph, then for each $n \geq 1$ and $v, w \in E^0$ the $(v, w)$-entry of $A_E^n$ equals the number of paths of length $n$ that start at $v$ and end at $w$. By regularity, any path $\alpha$ of length $k \in \mathbb{N}$ can be extended to a path of length $k + 1$ by an edge of $s^{-1}(r(\alpha))$. Thus, in a regular graph there exist paths of every possible integer length. In particular, the adjacency matrix of a regular graph cannot be nilpotent.

Recall that an exact sequence of abelian groups is *pure exact* if it remains exact upon tensoring with any abelian group.

**Lemma 5.2** Let $E$ be a row-finite graph. Then

i) The sequence of abelian groups

$$0 \rightarrow \mathbb{Z}[\sigma]^{\text{reg}(E)} I - \sigma A_E^i \rightarrow \mathbb{Z}[\sigma]^{\text{E}^0} \rightarrow \text{BF}_{\text{gr}}(E) \rightarrow 0$$

is pure exact.

ii) If $E$ is finite, then $\text{BF}_{\text{gr}}(E) \neq 0$.

**Proof** The sequence of part i) is pure exact because it is the colimit over $n$ of the split-exact sequences

$$0 \rightarrow \bigoplus_{i=-n}^{n} \mathbb{Z}^{\text{reg}(E)} I - \sigma A_E^i \rightarrow \bigoplus_{i=-n}^{n+1} \mathbb{Z}^{\text{E}^0} \rightarrow \bigoplus_{i=-n}^{n+1} \mathbb{Z}^{\text{sink}(E)} I^{-1} - \sigma A_E^i \rightarrow 0.$$

Next assume that $E$ is finite. To prove ii), by contradiction, assume that $I - \sigma \cdot A_E^i$ is surjective. Since $\mathbb{Z}[\sigma]$ is an integral domain, by rank considerations

$$\# \text{reg}(E) = \text{rk} \mathbb{Z}[\sigma]^{\text{reg}(E)} \geq \text{rk} \mathbb{Z}[\sigma]^{\text{E}^0} = \# E^0.$$

Hence $E$ must be regular. In particular, the matrix $\sigma I - A_E^i$ is square; let $\chi_{A_E}(\sigma) = \det(\sigma I - A_E^i) \in \mathbb{Z}[\sigma]$ be its characteristic polynomial.

Using once again the fact that $\mathbb{Z}[\sigma]$ is an integral domain, the surjectivity of $\sigma I - A_E^i$ implies that its determinant

$$ \det(I - \sigma A_E^i) = \sigma^{\# E^0} \cdot \chi_{A_E}(\sigma^{-1})$$

must be invertible in $\mathbb{Z}[\sigma]$. It follows that $\chi_{A_E}(\sigma)$ is also invertible, and therefore it is a power of $\sigma$; by degree considerations we must have $\chi_{A_E}(\sigma) = \sigma^{\# E^0}$. The Cayley–Hamilton theorem now implies that $A_E$ is nilpotent, contradicting the regularity of $E$ as per Remark 5.1. This concludes the proof. 

$\square$
In the next theorem we write $\tilde{E}$ for the covering associated to the constant function $\omega : E^1 \to \mathbb{Z}$, $\omega(e) = 1$ for all $e \in E^1$.

**Theorem 5.3** Let $E$ be a row-finite graph and let $H : \text{Alg}^a \to \text{Ab}$ be an $M_\infty$-stable, additive functor that preserves filtering colimits. Then there is a $\mathbb{Z}[\sigma]$-module isomorphism

$$H(L(\tilde{E})) \cong BF_{gr}(E) \otimes_{\mathbb{Z}} H(\ell).$$

**Proof** Because every row-finite graph is the filtering colimit of its finite complete subgraphs [2, Lemma 3.1], we may assume that $E$ is finite. The canonical action of the generator $\sigma$ of $\mathbb{Z}$ on $\tilde{E}$ induces a $*$-algebra automorphism $s : L(\tilde{E}) \to L(\tilde{E})$ such that $s(v_n) = v_{n+1}$, $s(e_n) = e_{n+1}$ ($v \in \tilde{E}^0$, $e \in \tilde{E}^1$). For $n \in \mathbb{N}_0$, let $E_n$ be the full subgraph of $\tilde{E}$ on the vertices $E_n^0 = \{v_k : v \in \tilde{E}^0, |k| \leq n\}$ and observe that $s$ can be (co-)restricted to a map $s_n : L(E_n) \to L(E_{n+1})$ for all $n \geq 0$. Since $\tilde{E} = \bigcup_{n \geq 1} E_n$, we have $L(\tilde{E}) = \operatorname{colim}_{n \geq 0} L(E_n)$ and thus

$$H(L(\tilde{E})) \cong \operatorname{colim}_{n \geq 0} H(L(E_n)).$$

The action induced by $s$ carries over to the colimit by means of the automorphism associated to the family $\{H(s_n)\}_{n \geq 0}$. Observe that each graph $E_n$ is acyclic. We shall use the fact that for an acyclic graph $F$ and the set $P_v(F)$ of paths ending in $v \in \text{sink}(F)$, there is a $*$-isomorphism $L(F) \cong \bigoplus_{v \in \text{sink}(F)} M_{P_v}(F)$. This is proved in [1, Theorem 2.6.17] under the assumption that $\ell$ is a field, but the proof does not use it; see also [5, Proposición 2.5.1]. By additivity and $M_\infty$-stability of $H$, we have an isomorphism $H(L(E_n)) \cong \mathbb{Z}^{\text{sink}(E_n)} \otimes_{\mathbb{Z}} H(\ell)$. Observe that

$$\text{sink}(E_n) = (\text{sink}(E) \times \{i \in \mathbb{Z} : |i| \leq n\}) \sqcup (\text{reg}(E) \times \{n\}).$$

The transition map $H(\ell) \otimes \mathbb{Z}^{\text{sink}(E_n)} \to H(\ell) \otimes \mathbb{Z}^{\text{sink}(E_{n+1})}$ is induced by the inclusion \(\mathbb{Z}^{\text{sink}(E) \times \{i \in \mathbb{Z} : |i| \leq n\}} \subset \mathbb{Z}^{\text{sink}(E) \times \{i \in \mathbb{Z} : |i| \leq n+1\}}\) and by

$$(v, n) \mapsto \sum_{w \in r(s^{-1}(v))} A_E(v, w)(w, n + 1) \quad (5.1)$$

on $\mathbb{Z}^{\text{reg}(E) \times \{n\}}$. It follows that \(\operatorname{colim}_{n \in \mathbb{N}_0} \mathbb{Z}^{\text{sink}(E_n)} \otimes_{\mathbb{Z}} H(\ell) \cong BF_{gr}(E) \otimes_{\mathbb{Z}} H(\ell). \)

In what follows we write $K^{gr}$ and $K^{h, gr}$ for $K^{Z_{gr}}$ and $K^{h, Z_{gr}}$.

**Corollary 5.4** Let $E$ be a row-finite graph and $R$ a $*$-algebra. If $R$ has local units and trivial $\mathbb{Z}$-grading, then there are $\mathbb{Z}[\sigma]$-module isomorphisms

$$K^{gr}_n(L(E) \otimes R) \cong BF_{gr}(E) \otimes_{\mathbb{Z}} K_n(R), \quad K^{h, gr}_n(L(E) \otimes R) \cong BF_{gr}(E) \otimes_{\mathbb{Z}} K^{h}_n(R).$$

**Proof** We have isomorphisms $K^{gr}_n(L(E) \otimes R) \cong K_n(L(\tilde{E}) \otimes R)$ and $K^{h, gr}_n(L(E) \otimes R) \cong K^{h}_n(L(\tilde{E}) \otimes R)$ by Proposition 2.7 and Theorems 3.4 and 3.9. The result now follows from Theorem 5.3. \(\square\)
Lemma 5.5 Let $E$ be a finite graph and $\text{cl} : \mathbb{Z}^{E_0} \to \text{BF}_{\text{gr}}(E)$, $\text{cl}(v) = [v]$, the canonical map. We have $\ker(\text{cl}) \cap \mathbb{N}^{E_0} = 0$.

Proof Write $\text{BF}_{\text{gr}}(E) \cong \text{colim}_N \text{BF}(E_n) \cong \text{colim}_N \mathbb{Z}^{\text{sink}(E_n)}$ as in the proof of Theorem 5.3. The transition maps $\text{cl}_n : \mathbb{Z}^{\text{sink}(E_n)} \to \mathbb{Z}^{\text{sink}(E_{n+1})}$ are as described in the proof of Theorem 5.3; in particular they are given by matrices with nonnegative coefficients and no zero columns. Hence $\text{cl}_n(N^{\text{sink}(E_n)}) \subset N^{\text{sink}(E_{n+1})}$ and $N^{\text{sink}(E_n)} \cap \ker(\text{cl}_n) = 0$. It follows that $\ker(\text{cl}) \cap \mathbb{N}^{E_0} = 0$. $\square$

Theorem 5.6 Let $E$ be a finite graph and $f : L(E) \to R$ a $\mathbb{Z}$-graded algebra homomorphism. Assume that $\ell$ is a field. If $K_0^{\text{gr}}(f)$ is a monomorphism, then $f$ is a monomorphism.

Proof Let $v \in E_0$. Because $\text{BF}_{\text{gr}}(E) \ni [v] \neq 0$ by Lemma 5.5, the fact that $K_0^{\text{gr}}(f)$ is a monomorphism implies that $[f(v)] = K_0^{\text{gr}}(f)([v])$ is nonzero. In particular, we have $f(v) \neq 0$ for all $v \in E_0$, and so the result follows from the graded uniqueness theorem for Leavitt path algebras (see e.g. [1, Theorem 2.2.15]). $\square$

6 G-Stability

Standing assumption 6.1 From now on, we will assume the existence of an element $\lambda \in \ell$ satisfying (1.2).

Convention 6.2 Fix an infinite set $\mathcal{X}$ of cardinality greater or equal than that of $G \sqcup \mathbb{N}$. We will write $\mathcal{S}$ for the categories of $G$-sets or $G$-graded sets of cardinality less or equal than $\#\mathcal{X}$, and $\mathcal{G} \to \text{Sets}$ for the forgetful functor. If $X \in \mathcal{S}$ and $A \in \mathcal{A}$ we equip $M_X A$ with the $G$-action or grading induced by the inclusion into $C_X A$ or $C^o_X A$. If $X$ is just a set, then unless specified otherwise, we will regard it as an object of $\mathcal{S}$ with the trivial $G$-action or grading.

We say that a functor is matricially stable to mean that it is $M_X$-stable as defined in Subsection 2.4.

Remark 6.3 If $R$ is a unital $*$-algebra and $X$ is any set, then both $C_X R$ and $C^o_X R$ are also unital with unit $1 = \sum_{x \in X} \varepsilon_{x,x}$. Note that if $R$ is $G$-graded then this element belongs to $G^o_X R \subset C^o_X R$, hence both of these $*$-algebras are unital as well.

We define an ideal embedding in $\mathcal{A}$ to be a monomorphism $i : A \to R$ such that $R$ is unital and $i(A)$ is a $G$-invariant (resp. homogenous) $*$-ideal of $R$. Let $R$ be a unital algebra in $\mathcal{A}$ and $X \in \mathcal{S}$. In the equivariant case, we set $C^o_X R = C_X R$. Let $\epsilon$ be a central unitary; in the equivariant case, assume $\epsilon$ is fixed by $G$; in the $G$-graded case, assume it is homogeneous of degree 1. An element $\phi \in C^o_X R$ is $\epsilon$-invariant if it is $\epsilon$-hermitian and fixed by $G$ (resp. and homogeneous of degree 1). We are now in position to define the notion of $G$-stability.
Definition 6.4 A functor $F: \mathcal{A} \to \mathcal{C}$ is $G$-stable if for any pair of sets $X, Y \in \mathcal{S}$, any ideal embedding $i: A \to R$ and any two invariant $\epsilon$-hermitian elements $\phi \in C_X^\circ R$, $\psi \in C_Y^\circ R$, $F$ sends the inclusion

$$(M_X A)^\phi \to (M_{X \sqcup Y} A)^{\phi \oplus \psi}$$

to an isomorphism.

Put

$h_{\pm} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This element is $1$-hermitian; write $M_{\pm} := M^h_{\pm}$. For each $A$ in $\mathcal{A}$, we write $\iota_+: A \to M_{\pm} A$ for the upper-left corner inclusion. A functor $F: \mathcal{A} \to \mathcal{C}$ is $\iota_+-$stable if the natural transformation $F(\iota_+): F \to F \circ M_{\pm}$ is an isomorphism.

The following proposition says that under the hypothesis of $\iota_+-$stability one may omit considering the $\epsilon$-invariant elements $\phi$ and $\psi$ in the definition above.

Proposition 6.5 An $\iota_+-$stable functor $F: \mathcal{A} \to \mathcal{C}$ is $G$-stable if and only if for each $A$ in $\mathcal{A}$ it sends each inclusion

$$M_X A \to M_{X \sqcup Y} A$$

induced by a pair of sets $X, Y \in \mathcal{S}$ to an isomorphism.

Proof The argument of [9, Proposition 2.4.4] shows this.

The proof of the following lemma is a straightforward verification.

Lemma 6.6 Let $A$ a $G$-algebra and $B$ a $G$-graded $*$-algebra.

(i) If $X$ is a $G$-set, then the linear map

$$M_{G} M_X A \ni \epsilon_{x,t} \otimes \epsilon_{x,y} \otimes a \mapsto \epsilon_{s}^{-1} \epsilon_{x,t}^{-1} \epsilon_{s} \otimes \epsilon_{x,t} \otimes a \in M_{X} M_{G} A$$

(6.1)

is a $G-*$-algebra isomorphism, natural with respect to both $X$ and $A$.

(ii) If $X$ is a $G$-graded set, then the linear map

$$M_{G} M_X B \ni \epsilon_{x,t} \otimes \epsilon_{x,y} \otimes b \mapsto \epsilon_{x,y} \otimes \epsilon_{s|x|,t|y|} \otimes b \in M_{X} M_{G} B$$

(6.2)

is a $G$-graded $*$-algebra isomorphism, natural with respect to both $X$ and $B$.

As a consequence of Lemma 6.6, we obtain the following hermitian analogue of [12, Proposition 3.1.9], which will useful for constructing the equivariant and graded versions of hermitian bivariant $K$-theory.

Proposition 6.7 Let $F: \mathcal{A} \to \mathcal{C}$ be a matricially stable, $\iota_+-$stable functor. Then $F(M_{G}(-))$ has the same stability properties and is moreover $G$-stable.
Remark 6.8 Consider the $G$-set $G \sqcup \star$, where $G$ acts trivially on $\star$. For each $G$-$\ast$-algebra $A$, the inclusions $\star \subset G \sqcup \star \supset G$ yield a zig-zag of inclusions

$$A \to M_{G \sqcup \star} A \leftarrow M_G A$$

relating $A$ and $M_G A$. Observe that any $G$-stable functor maps (6.3) to an isomorphism.

In the graded setting, for each $G$-graded $\ast$-algebra $B$ the $G$-graded set inclusion $\{1\} \subset G$ yields a $G$-graded $\ast$-algebra inclusion

$$\iota_1 : B \to M_G B,$$

which any $G$-stable functor sends to an isomorphism.

Corollary 6.9 Let $F : \mathcal{A} \to C$ be a matricially stable, $\iota_+\text{-stable functor. The following statements are equivalent:}$

(i) The functor $F$ is $G$-stable.

(ii) For each $A$ in $\mathcal{A}$ the functor $F$ sends the morphism (6.4) (resp. the zig-zag (6.3)) to an isomorphism.

(iii) The functors $F$ and $F \circ M_G$ are naturally isomorphic.

Proof The implication (i)$\Rightarrow$(ii) follows from Remark 6.8. The fact that (ii)$\Rightarrow$(iii) is trivial. The implication (iii)$\Rightarrow$(i) is a consequence of Proposition 6.7, together with the fact that $G$-stability is preserved under natural isomorphisms. \hfill $\Box$

7 Hermitian graded bivariant $K$-theory

In this section we introduce bivariant $K$-theory categories for $G - \text{Alg}_\ell^\ast$ and $G_{gr} - \text{Alg}_\ell^\ast$. For this purpose we first adapt the notions of polynomial homotopy invariance, extensions and excisive homology theories from [9] and [12] to the present context.

Let $A$ be an object of $\mathcal{A}$. We view the polynomial ring $A[t] = A \otimes \mathbb{Z}[t]$ as an object of $\mathcal{A}$ by considering the trivial action or grading on $t$. A functor $F : \mathcal{A} \to C$ is (polynomially) homotopy invariant if it sends each inclusion $A \hookrightarrow A[t]$ of an algebra into its ring of polynomials to an isomorphism.

An extension in $\mathcal{A}$ is an exact sequence

$$0 \to A \overset{i}{\to} B \overset{p}{\to} C \to 0.$$  \hfill (7.1)

We say that the category of $\ell[G]$-modules or $G$-graded $\ell$-modules is the underlying category of $\mathcal{A}$ and we denote it $\mathcal{A}$; writing $U : \mathcal{A} \to \mathcal{A}$ for the canonical forgetful functor, we say that (7.1) is weakly split if $U(p)$ has a section.

Write $\mathcal{E}$ for the class of weakly split extensions of $\mathcal{A}$. Let $\mathcal{T}$ be a triangulated category with inverse suspension functor $\Omega$. An excisive homology theory on $\mathcal{A}$ with values in $\mathcal{T}$ is a functor $H : \mathcal{A} \to \mathcal{T}$ together with a family of maps $\{ \partial_E : \Omega H(C) \to H(A) \}_{E \in \mathcal{E}}$...
compatible with morphisms of extensions ([8, Section 6.6]) such that for each $E \in \mathcal{E}$ there is a distinguished triangle

$$\Omega H(C) \xrightarrow{\phi_E} H(A) \to H(B) \to H(C).$$

Applying the analogue of the construction of [12, Section 2] to bivariant hermitian $K$-theory [9], we obtain an excisive homology theory

$$j^h_\mathfrak{A} : \mathfrak{A} \to kk^h_\mathfrak{A}$$

that is $M_X$-stable, $\iota_+$-stable and homotopy invariant. Moreover, any other excisive homology theory $H : \mathfrak{A} \to T$ which satisfies these stability properties factors uniquely through $j^h_\mathfrak{A}$ via a triangle functor.

**Definition 7.1** We define *equivariant bivariant hermitian $K$-theory* as the category whose objects are $G$-$\ast$-algebras and the morphisms are given by

$$kk^h_G(A, B) := kk^h_{G-\text{Alg}}^\ast(M_G A, M_G B),$$

with the composition rule of $kk^h_{G-\text{Alg}}^\ast$. There is a canonical functor

$$j^h_G : G-\text{Alg}^\ast \to kk^h_G$$

defined as the identity on objects, and that sends an equivariant map $f : A \to B$ to the image of $M_G f : M_G A \to M_G B$ via $j^h_{G-\text{Alg}}^\ast$, viewed as an arrow of $kk^h_G$.

Likewise, we define *$G$-graded hermitian bivariant $K$-theory* as the category whose objects are $G$-graded $\ast$-algebras and its morphisms are given by

$$kk^h_{G_{\text{gr}}}(A, B) := kk^h_{G_{\text{gr}}-\text{Alg}}^\ast(M_G A, M_G B).$$

There is an analogous functor

$$j^h_{G_{\text{gr}}} : G_{\text{gr}}-\text{Alg}^\ast \to kk^h_{G_{\text{gr}}}$$

in the graded setting, which is the identity on objects and maps a $G$-graded morphism $f : A \to B$ to the image of $M_G f : M_G A \to M_G B$ via $j^h_{G_{\text{gr}}-\text{Alg}}^\ast$, viewed as an arrow of $kk^h_{G_{\text{gr}}}$. 

With these definitions in place, the following theorems can be adapted from [12, Theorem 4.1.1, Theorem 4.2.1] by means of Proposition 6.7 and the universal property of $j^h_\mathfrak{A}$.

**Theorem 7.2** (cf. [12, Theorem 4.1.1]) The functor $j^h_G : G-\text{Alg}^\ast \to kk^h_G$ is initial in the category of $G$-stable, homotopy invariant, $M_X$-stable and $\iota_+$-stable excisive homology theories. 

\[ \square \] Springer
Theorem 7.3 (cf. [12, Theorem 4.2.1]) The functor \( j^h_G : \text{Alg}_\ell^* \to \text{kk}^h_G \) is initial in the category of \( G \)-stable, homotopy invariant, \( M_X \)-stable and \( \iota_+ \)-stable excisive homology theories.

8 Adjointness theorems

In this section we show that the adjointness theorems of [12] that relate algebraic bivariant \( K \)-theory with its equivariant and homogenous counterparts can be extended to the hermitian context.

Composing the crossed product functors (2.3) with the canonical functors to each of the bivariant \( K \)-theory categories and using their universal properties we obtain functors

\[
- \ltimes G : \text{kk}^h_G \leftrightarrow \text{kk}^h_{G_{\text{gr}}} : G_{\text{gr}}^{-} \to \text{kk}^h_{G_{\text{gr}}}
\]

By Proposition 2.6 the composite functors \( G_{\text{gr}} \ltimes (\cdot) \ltimes G \) and \( (G_{\text{gr}} \ltimes \cdot) \ltimes G \) are naturally isomorphic to the endofunctors \( M_G(\cdot) : \text{Alg}_\ell^* \to \text{Alg}_\ell^* \) and \( M_G(\cdot) : \text{Alg}_\ell^* \to \text{Alg}_\ell^* \) respectively. Consequently, the algebraic analogue of Baaj-Skandalis’ duality proved in [12] extends to the hermitian setting as follows.

Theorem 8.1 (cf. [12, Theorem 7.6]) The functors \(- \ltimes G \) and \( G_{\text{gr}} \ltimes \cdot \) extend to inverse triangle equivalences

\[
\begin{array}{ccc}
\text{kk}^h_G & \overset{- \ltimes G}{\longrightarrow} & \text{kk}^h_{G_{\text{gr}}} \\
G_{\text{gr}} \ltimes & \text{kk}^h_G & \text{kk}^h_{G_{\text{gr}}} \ltimes
\end{array}
\]

The next adjointness result concerns induction and restriction. Given a subgroup \( H \leq G \) and a \( G \)-\( \ast \)-algebra \( A \) we will write \( \text{res}^G_H(A) \) for the \( H \)-\( \ast \)-algebra resulting from restricting the \( G \)-action on \( A \) to an \( H \)-action. Write \( \pi : G \to G/H \) for the canonical quotient function, and set

\[
\text{ind}^G_H(A) = \{ f \in A^{(G)} : |\pi(\text{supp}(f))| < \infty, \ f(g) = h \cdot f(gh) \ (\forall g \in G, h \in H) \}.
\]

This is a \( G \)-\( \ast \)-algebra with product given by pointwise multiplication, and involution and action defined by \( f^\ast(s) := f(s)^\ast \) and \( g : f(s) = f(g^{-1}s) \) for each \( f \in \text{ind}^G_H(A) \) and \( g, s \in G \) respectively. We also define \( \text{ind}^G_H(\varphi)(f) = \varphi \circ f \) for each \( H \)-\( \ast \)-algebra morphism \( \varphi : A \to B \).

Example 8.2 If \( A \) is a \( G \)-\( \ast \)-algebra such that the restricted \( H \)-action is trivial, then

\[
\text{ind}^G_H \text{res}^G_H(A) \cong A^{(G/H)}.
\]
Notice, in particular, that
\[ \text{ind}_G^H \text{res}_H^G (\ell) \cong \ell^{(G)} = G \hat{\times} \ell \quad (8.1) \]

Both \( \text{res}_H^G \) and \( \text{ind}_H^G \) extend to triangulated functors
\[ \text{res}_H^G : \text{kk}^h_G \longrightarrow \text{kk}^h_H : \text{ind}_H^G. \]

The following theorem can be adapted from \[12\] by verifying that the algebra maps involved preserve involutions.

**Theorem 8.3** (cf. \[12, Theorem 6.1.4\]) For each subgroup \( H \leq G \) there exists an adjunction
\[
\begin{array}{ccc}
\text{kk}^h_H & \cong & \text{kk}^h_G \\
\downarrow \text{ind}_H^G & & \downarrow \text{res}_H^G \\
& \text{kk}^h_H & \\
\end{array}
\]

**Remark 8.4** The proof of \[12, Theorem 6.1.4\] makes clear that when applying Theorem 8.3 with \( H = \{1\} \), the map \( \text{kk}^h_G (A^{(G)}, B) \longrightarrow \text{kk}^h (A, B) \) is induced by forgetting the \( G \)-action and precomposing with the inclusion \( \iota_1 : a \in A \mapsto a \cdot \chi_1 \in A^{(G)} \).

In the next corollary and elsewhere we write \( KH^h \) for homotopy hermitian \( K \)-theory, as defined in \[9, Section 3\].

**Corollary 8.5** For each \( G \)-graded \( * \)-algebra \( A \), there is an isomorphism of abelian groups
\[ \text{kk}^h_{Gr} (\ell, A) \cong KH^h_{0,Gr} (A). \]

**Proof** We have the following sequence of natural isomorphisms, where we have used Theorem 8.1 at the first step, (8.1) and Theorem 8.3 at the second, the natural isomorphism \( \text{kk}^h (\ell, -) \cong KH^h_0 \) of \[9, Proposition 8.1\] at the third, and (3.9) at the last.
\[ \text{kk}^h_{Gr} (\ell, A) \cong \text{kk}^h_G (\ell^{(G)}, G \hat{\times} A) \cong \text{kk}^h (\ell, G \hat{\times} A) \cong KH^h_0 (G \hat{\times} A) = KH^h_{0,Gr} (A). \]

The groups \( KH^h_{n,Gr} (A) \) for \( n \neq 0 \) can be interpreted as \( \text{kk}^h_{Gr} (\ell, -) \) applied to \( \Omega^n A \) for \( n > 0 \) and to the iterated Karoubi suspension \( \Sigma^{-n} A \) if \( n < 0 \), see \[8, Subsection 4.7\] and \[8, Corollary 6.4.2\].

**Remark 8.6** Both the algebraic analogue of the Green-Julg theorem proved in \[12\] and the bivariant version of Karoubi’s 12-term exact sequence given in \[9\] can be adapted to the present context. As in the theorems above, the proofs consist in verifying that all algebras and maps involved preserve the extra structure considered.
9 The coefficient ring \( kk_h^{G_{gr}}(\ell, \ell) \) and \( G \)-action on graded \( K \)-theory

We have the following chain of isomorphisms of abelian groups, where we use agreement between \( KH_0 \) and \( kk_h(\ell, -) \) [9, Proposition 8.1] for the first and third isomorphisms, additivity of \( KH_0 \) for the second, Theorem 8.1 for the fourth and identity (8.1) and Theorem 8.3 for the last

\[
\mathbb{Z}[G] \otimes kk_h(\ell, \ell) \cong \bigoplus_{g \in G} \mathbb{Z}_g \otimes KH_0^h(\ell) 
\cong KH_0^h(\ell(G)) \cong kk_h(\ell, \ell(G)) \cong kk_h^G(\ell, \ell(G)) \cong kk_h^{G_{gr}}(\ell, \ell). \tag{9.1}
\]

Observe that both the domain and codomain of the composite isomorphism are rings; the next proposition says that the map is an anti-homomorphism.

**Proposition 9.1** The composite (9.1) is a ring isomorphism

\[
\mathbb{Z}[G^{op}] \otimes kk_h(\ell, \ell) \sim \longrightarrow kk_h^G(\ell, \ell), \tag{9.2}
\]

and maps

\[
g \otimes j^h_{G_{gr}}(id_\ell) \mapsto m_g := j^h_{G_{gr}}(1_1) \circ j^h_{G_{gr}}(G_g). \tag{9.3}
\]

**Proof** The last isomorphism in the composite (9.1) comes from a linear functor, so it is a ring homomorphism. Hence it suffices to show that the composite of all maps but the last

\[
\mathbb{Z}[G^{op}] \otimes \mathbb{Z} kk_h^h(\ell, \ell) \sim \longrightarrow kk_h^G(\ell(G), \ell(G)) \tag{9.4}
\]

is a ring homomorphism. This is immediate upon checking that (9.4) maps an elementary tensor \( g \otimes [1] \) to the \( kk_h \) class of \( \rho_g : \ell(G) \rightarrow \ell(G), \rho_g(\chi_s) = \chi_{sg}, \) and an elementary tensor \( 1 \otimes \xi \) with \( \xi \in KH_0^h(\ell) = kk_h(\ell, \ell) \) to \( \xi(G) = \text{ind}_G(\xi). \)

Next, one checks, using the explicit formula of [12, Proposition 7.4], that under the isomorphism \( G \cong \ell(G) \cong MG \) of Proposition 2.6, \( G \cong G \) identifies with the graded isomorphism \( R_g : MG \rightarrow MG, R_g(\varepsilon_s, t_g) = \varepsilon_{sg, tg}. \) Thus the isomorphism \( kk_h^G(\ell(G), \ell(G)) \cong kk_h^G(\ell, \ell) \) sends the class of \( \rho_g \) to \( j^h_{G_{gr}}(1_1) \circ j^h_{G_{gr}}(R_g1_1) = j^h_{G_{gr}}(1_1) \circ j^h_{G_{gr}}(G_g). \)

It follows from Proposition 9.1 that we have a left action

\[
\cdot : G \times kk_h(\ell, A) \rightarrow kk_h(\ell, A), \quad g \cdot \xi = \xi \circ m_g.
\]

In addition, \( G \) acts on \( G \cong A \) by automorphisms; this gives a second action

\[
\cdot' : G \times kk_h(\ell, G \cong A) \rightarrow kk_h(\ell, G \cong A).
\]
Using Theorems 8.3 and 8.1 we obtain an isomorphism
\[ \mu : \text{kh}_{\text{Gr}} (\ell, A) \sim \text{kh} (\ell, G \hat{\otimes} A) \] (9.6)

**Lemma 9.2** The actions \( \cdot \) and \( \cdot' \) correspond to each other under the isomorphism (9.6): we have \( \mu(g \cdot \xi) = g \cdot' \mu(\xi) \) for all \( g \in G, \xi \in \text{kh}_{\text{Gr}} (\ell, A) \).

**Proof** By the proof of Proposition 9.1, \( G \hat{\otimes} m_g \) is the class of the endomorphism \( \rho_g \) of (9.5). Hence it suffices to show that for \( \xi \in \text{kh}_{\text{Gr}} (\ell, A) \), we have
\[ g \cdot' (G \hat{\otimes} \xi) = (G \hat{\otimes} \xi) \circ \rho_g. \] (9.7)

It is clear that (9.7) holds whenever \( \xi \) is the class of a homogeneous projection \( p \in MG M_{\infty} \). From this one obtains that (9.7) also holds when \( \xi \) comes from a quasi-homomorphism \( \ell \Rightarrow MG M_{\infty} A \otimes MG M_{\infty} A \) in \( G_{\text{gr}} - \text{Alg}^*_{\ell} \). The general case follows from the latter case using Corollary 8.5 together with (3.9), Definition 3.8 and naturality.

By the same argument as in [9, Lemma 6.2.13], the tensor product of \( G\text{-*}-\text{algebras} \) extends to a product on the bivariant \( K \)-theory category \( \text{kh}_{G_{\text{gr}}} \). The graded case follows similarly when the group is abelian (see Remark 2.2) and we have the following.

**Lemma 9.3** Let \( G \) be an abelian group and let \( A_1, A_2, B_1, B_2 \in G_{\text{gr}} - \text{Alg}^*_{\ell} \). There is a natural, associative, bilinear product
\[ \otimes : \text{kh}_{G_{\text{gr}}} (A_1, A_2) \times \text{kh}_{G_{\text{gr}}} (B_1, B_2) \rightarrow \text{kh}_{G_{\text{gr}}} (A_1 \otimes \ell B_1, A_2 \otimes \ell B_2), \]
\[ \xi \otimes \eta := (\xi \otimes B_2) \circ (A_1 \otimes \eta). \]

which is compatible with composition and satisfies \( j_{G_{\text{gr}}}^h (f \otimes g) = j_{G_{\text{gr}}}^h (f) \otimes j_{G_{\text{gr}}}^h (g) \).

**Proposition 9.4** Let \( G \) be an abelian group, \( g \in G, A, B \in G_{\text{gr}} - \text{Alg}^*_{\ell} \) and \( \xi \in \text{kh}_{G_{\text{gr}}} (A, B) \). Omitting \( j_{G_{\text{gr}}}^h \), we have a commutative diagram in \( \text{kh}_{G_{\text{gr}}}^h \)
\[
\begin{array}{ccc}
A & \overset{\xi \otimes g}{\longrightarrow} & B \\
\downarrow \iota_g & & \downarrow \iota_g \\
M_G A & \overset{\iota_g^{-1}}{\longrightarrow} & A
\end{array}
\]

**Proof** Immediate from Proposition 9.1 and Lemma 9.3.

**10 Bivariant Dade theorem**

In this section we show that for a strongly graded unital \( \text{*}-\text{algebra} \) \( R \), the map \( R_1 \rightarrow G \hat{\otimes} R, r \mapsto \chi_1 \hat{\otimes} r \), is a \( \text{kh}^h \)-equivalence. In view of Theorem 3.9 and Remark 4.2,
Let $S\in\text{Alg}_G^*$ and $E \subset S$ a set of nonzero orthogonal projections of cardinality $\#E \leq \#X$. Assume that

$$U = \left\{ \sum_{e \in F} e : E \ni F \text{ finite} \right\}$$

is a set of local units for $S$. Further assume that $E$ contains a full projection $e_1$. Then the inclusion $e_1Se_1 \to S$ is a $kk^h$-equivalence.

**Proof** Because the elements of $E$ are orthogonal projections and $U$ is a set of local units, we have $S = \bigoplus_{e,f \in E} eSe$. Thus for $C_E$ as in (2.1) and $p = \sum_{e \in E} e \otimes e \in C_E S$, we have $S \cong pM_E Sp$. By hypothesis, for each $e \in E$ there are a finite row matrix $r_e \in (eSe_1)^{(1) \times (1)}$ and a finite column matrix $c_e \in (e_1Se)^{((1) \times (1))}$ such that $r_e c_e = e$; in particular we may choose $r_{e_1} = c_{e_1} = e_{1,1} e_{1,1}$. Let $r, c \in C_{\mathbb{N} \times E^*} S$, $r = \sum_{n,e} \varepsilon_{(1,e),(1,e)}(r_e)_{1,n}$ and $c = \sum_{n,e} \varepsilon_{(1,e),(1,e)}(c_e)_{n,1}$. We have $rc = \sum_{e \in E} \varepsilon_{(1,e),(1,e)} e$, which is the image of $p$ under the inclusion $\alpha : C_E S \subset C_{\mathbb{N} \times E}$ induced by $E \to \mathbb{N} \times E$, $e \mapsto (1, e)$. Let $\lambda$ be as in (1.2) and set

$$u = \begin{bmatrix} \lambda^* c + \lambda r^* & c - r^* \\ \lambda \lambda^* (c - r^*) & \lambda c + \lambda^* r^* \end{bmatrix} \in M_{\pm} C_{\mathbb{N} \times E} S.$$

For $A \in \text{Alg}_G^*$, let $\iota^- : A \to M_{\pm} A$ be the lower right corner inclusion. One checks that $u^* u = \iota_+ (\alpha(p)) + \iota_-(\alpha(p))$. Set

$$R = e_1 Se_1, \quad T = \alpha(p) M_\infty M_E S \alpha(p).$$

The composite

$$\phi : S \cong T \xrightarrow{\iota_+} M_{\pm} T \xrightarrow{\text{ad}(u)} M_{\pm} M_\infty M_E R$$
is a $\ast$-algebra homomorphism. One checks that the composite of $\phi$ and the inclusion $\text{inc} : R \subset S$ is the corner inclusion $t_+ \otimes t_1 \otimes t_{e_1}$ and is thus a $kk^h$ equivalence by matricial and hermitian stability. The composite $M_\pm M_\infty M_F(\text{inc}) \circ \phi$ maps $a \mapsto \text{ad}(u)(t_+(a))$, and is thus a $kk^h$-equivalence by [6, Lemma 8.12] and hermitian stability of the functor $j^h$. This finishes the proof. 

**Proof of Theorem 10.1** Set $S := G \rtimes R$, $\mathcal{E} = \{ \chi_g \rtimes 1 : g \in G \}$ and $e_g = \chi_g \rtimes 1$. Because $R$ is strongly graded, for every $g \in G$ there exist $n \geq 1$, $y \in R_1^{1 \times n}$ and $y' \in R_g^{n \times 1}$ such that $yy' = 1$. Then $x = \chi_g \rtimes y := (\chi_g \rtimes y_1, \ldots, \chi_g \rtimes y_n) \in (\chi_g \rtimes R_{g^{-1}})^{1 \times n} = e_g S^{1 \times n} e_1$ and $x' = x \rtimes y'$ satisfy $e_g = xx'$. Now apply Lemma 10.3.

**Remark 10.4** We do not know whether the converse of Theorem 10.1 holds. However it is straightforward to show that the key property that we use in the proof, namely, that the projection $\chi_1 \rtimes 1 \in G \rtimes R$ is full, is equivalent to the grading of $R$ being strong.

**11 A distinguished triangle for Leavitt path algebras in $kk^h_{G_{gr}}$**

Let $E$ be a graph; write $C(E)$ and for its Cohn algebra. For $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$, put

$$m_v := \sum_{e \in s^{-1}(v)} ee^*, \quad q_v = v - m_v \in C(E). \quad (11.1)$$

Recall that by definition we have a short exact sequence (1.11) where $\mathcal{K}(E) = \langle q_v : v \in \text{reg}(E) \rangle$.

Let $E$ be a graph and $\omega : E^1 \to G$ a weight; it induces a $G$-grading on $C(E)$ by defining $|e| = \omega(e)$, $|e^*| = \omega(e)^{-1}$ and $|v| = 1$ for each $v \in E^0$, $e \in E^1$. Moreover, the ideal $\mathcal{K}(E)$ is homogeneous with respect to this grading, as it is generated by homogenous elements of degree 1. In particular, the Leavitt path algebra $L(E)$ inherits a $G$-grading associated to $\omega$. We write $\mathcal{K}_{\omega}(E)$, $C_{\omega}(E)$ and $L_{\omega}(E)$ for the algebras $\mathcal{K}(E)$, $C(E)$ and $L(E)$ equipped with these $G$-gradings. It follows from [1, Proposition 1.5.11] that (1.11) is $\ell$-split and, moreover, that the section preserves the $G$-gradings induced by $\omega$.

Fix a graph $E$; a homology theory $H : G_{gr} \to \text{Alg}_{\ell}^*$ is called $E$-stable if it is $\ell$-stable and $G$-stable in the sense of Convention 6.2 with respect to $G$-graded sets of cardinality no higher than that of $\mathcal{X} = E^0 \sqcup E^1 \sqcup \mathbb{N} \sqcup G$.

If $I$ is a set, $\mathcal{A}$ an additive category and $F : G_{gr} \to \text{Alg}_{\ell}^* \to \mathcal{A}$ a functor, we say that $F$ is $I$-additive if first of all direct sums of cardinality $\leq \# I$ exist in $\mathcal{A}$ and second of all the map

$$\bigoplus_{j \in J} F(A_j) \to F(\bigoplus_{j \in J} A_j)$$
is an isomorphism for any family \( \{ A_j : j \in J \} \subset G_{\text{gr}} \setminus \text{Alg}_G^* \) with \( \# J \leq \# I \).

**Proposition 11.1** Let \( E \) be a graph and \( \omega : E^1 \to G \) a weight. Consider the graded \(*\)-homomorphism

\[
q : \ell^{(\text{reg}(E))} \to K_\omega(E), \quad \chi_v \mapsto q_v.
\]

Let \( A \) be an additive category and \( F : G_{\text{gr}} \setminus \text{Alg}_G^* \to A \) a \( \text{reg}(E) \)-additive, \( E \)-stable functor. Then \( F(q) \) is an isomorphism.

**Proof** For each \( v \in E^0 \), write \( P_v \) for the set of paths that end at \( v \). We may view any path in \( E \) as homogeneous element of \( C_\omega(E) \); this equips \( P_v \) with a \( G \)-grading, and the canonical \(*\)-algebra isomorphism

\[
\bigoplus_{v \in \text{reg}(E)} M_{P_v} \xrightarrow{\sim} K_\omega(E), \quad \varepsilon_{\alpha, \beta} \mapsto \alpha q_v \beta^*, \quad (v \in E^0, \alpha, \beta \in P_v)
\]

is \( G \)-graded. Since \( q \) factors through (11.3) via the sum of inclusions \( \bigoplus_{v \in \text{reg}(E)} M_{P_v} \), and since \( F \) is \( \text{reg}(E) \)-additive by hypothesis, it suffices to show that \( F(\ell) \) is an isomorphism for every \( v \in \text{reg}(E) \). This follows from \( E \)-stability, as the morphism in question is induced by the \( G \)-graded set inclusion \( \{ v \} \subset P_v \) and \( \# P_v \leq \# (E^0 \sqcup E^1 \sqcup \mathbb{N}) \).

\( \square \)

**Proposition 11.2** Let \( E \) be a graph and \( \omega : E^1 \to G \) a weight. Consider the \( G \)-graded \(*\)-homomorphism

\[
\varphi : \ell^{(E^0)} \to C_\omega(E), \quad \chi_v \mapsto v.
\]

Let \( H : G_{\text{gr}} \setminus \text{Alg}_G^* \to T \) be a homotopy invariant, excisive, \( E \)-stable and \( E^0 \)-additive homotopy theory. Then \( H(\varphi) \) is an isomorphism.

**Proof** The proof of this proposition consists in verifying that the argument of [7, Theorem 4.2] for the ungraded case, together with the necessary adaptations to the hermitian context [6, Theorem 3.2] follow through, once we equip all algebras and morphisms with their corresponding \( G \)-gradings induced by \( \omega \). The ungraded, non-hermitian case is based upon the usage of quasimorphisms, which can be adapted to the present context from [10, Proposition 3.3] in the same way. The homotopies given in [7, Theorem 4.2] do not preserve involutions, hence they need to be adapted according to [6, Theorem 3.2]. One checks that moreover these elementary polynomial homotopies are \( G \)-graded. To conclude, we record the \( G \)-grading of the algebras involved in the proof, for which all maps of [7, Section 4] are \( G \)-graded. Write \( \mathcal{P} \) for the set of paths of the graph \( E \), equipped by the \( G \)-grading induced by \( \omega \). In [7, proof of Theorem 4.2, part I] an algebra \( C_m(E) \) is introduced, together with a injective \(*\)-homomorphism \( \rho : C_m(E) \to \Gamma \mathcal{P} \). One checks that the image of \( \rho \) lies in \( C_0^\mathcal{P} \) and equips \( C_m(E) \) with the induced \( G \)-grading. The subalgebra \( \mathfrak{A} \) defined in [7, proof of Theorem 4.2, part II] is a homogeneous ideal of \( M_{\mathcal{P}} C_\omega(E) \) and thus carries a canonical \( G \)-grading. In the same proof a crossed product \( \mathfrak{A} \rtimes \rho C_m(E) \) is considered, together with an ideal \( J \subset \mathfrak{A} \rtimes \rho C_m(E) \). As an \( \ell \)-module the former coincides with \( \mathfrak{A} \oplus C_m(E) \); the \( G \)-grading given by \( (\mathfrak{A} \rtimes \rho C_m(E))_g = \mathfrak{A}_g \oplus C_m(E)_g \) makes it into a \( G \)-graded \(*\)-algebra for which the ideal \( J \) is homogeneous. \( \square \)
Lemma 11.3 Let $E$ be a graph, $\omega : E^1 \to G$ a weight, and $\tilde{E} = (\tilde{E}, \omega)$. For each $v \in E^0$ and $g \in G$, the isomorphism

$$kk^h_{G_{gr}}(\ell, C_\omega(E)) \cong kk^h_G(\ell(G), C_\omega(E)) = kk^h_G(\ell(G), C(\tilde{E})) \cong kk^h(\ell, C(\tilde{E}))$$

maps $g \cdot j^h_{G_{gr}}(v)$ to $j^h(vg)$.

Proof Apply Lemma 9.2 to the projection $v \in C_\omega(E)$.

Remark 11.4 Let $H : G_{gr} \to Alg^*_\ell \to T$ be a homotopy invariant, $E$-stable, excisive homology theory. Then by the universal property of $j^h_{G_{gr}}$ (Theorem 7.3), there is a ring homomorphism $\bar{H} : kk^h_{G_{gr}}(\ell, \ell) \to \text{End}_T(\ell)$. Composing it with the map of Proposition 9.1, we obtain a ring homomorphism $\mathbb{Z}[G^{op}] \to \text{End}_T(H(\ell))$.

In particular, if $H$ is $E^0$-additive, then $I - A^t_\omega$ defines a homomorphism $H(\ell)^{\text{reg}(E)} \to H(\ell)^{E^0}$ in $T$.

Theorem 11.5 Let $E$ be a graph, $G$ a group, $\omega : E^1 \to G$ a weight, and $A_\omega \in \mathbb{Z}[G]^{\text{reg}(E) \times E^0}$ the weighted adjacency matrix of (1.5). Let $H : G_{gr} \to Alg^*_\ell \to T$ be a homotopy invariant, $E$-stable, excisive and $E^0$-additive homology theory. Then $H$ maps the Cohn extension to a distinguished triangle in $T$ of the form

$$H(\ell)^{\text{reg}(E)} \xrightarrow{I - A^t_\omega} H(\ell)^{E^0} \to H(L_\omega(E)).$$

Proof Let $\text{inc} : K_\omega(E) \to C_\omega(E)$ be the inclusion map. By Propositions 11.2 and 11.1 together with extension (1.11), for $\xi = H(\varphi)^{-1} \circ H(\text{inc} \circ q)$ we have a distinguished triangle

$$H(\ell)^{\text{reg}(E)} \xrightarrow{\xi} H(\ell)^{E^0} \to H(L_\omega(E)).$$

The proof consists in proving that $\xi$ is multiplication by $I - A^t_\omega$ by equivalently showing that $H(\text{inc} \circ q) = H(\varphi) \circ (I - A^t_\omega)$. By the additivity hypothesis on $H$, it suffices to prove that for each inclusion $\iota_v : \ell \hookrightarrow \chi_v \ell \subseteq \ell^{\text{reg}(E)}$ we have that $H(\text{inc} q_{tv}) = H(\varphi)(I - A^t_\omega)H(\iota_v)$. By the universal property of $j^h_{G_{gr}}$, we are reduced to proving that

$$j^h_{G_{gr}}(q_v) = j^h_{G_{gr}}(v) - \sum_{e \in \iota_v^{-1}(v)} \omega(e) \cdot j^h_{G_{gr}}(r(e)).$$

By Lemma 11.3, this is equivalent to having

$$j^h(q_{v_1}) = j^h(v_1) - \sum_{e \in \iota_{v_1}^{-1}(v_1)} j^h(r(e) \omega(e)).$$
in \(kk^h(\ell, C(\tilde{E}))\), which follows from [6, proof of Theorem 3.4] applied to \(v_1 \in \tilde{E}^0\). \(\square\)

Let \(E\) be a graph, \(G\) a group, and \(\omega : E^1 \to G\) a weight. The graded Bowen-Franks and dual Bowen-Franks \(G\)-modules associated to \(\omega\) are

\[
\text{BF}_{gr}(E, \omega) = \ker(I - A^I_{1,\omega}), \; \text{BF}^\vee_{gr}(E, \omega) = \ker(I^I - A_{\omega}).
\]

Observe that for \(G = \mathbb{Z}\) and \(c_n\) the constant weight \(c_n(e) = n \in \mathbb{Z}\) for all \(e \in E^1\), \(\text{BF}_{gr}(E, c_1) = \text{BF}_{gr}(E)\) is the \(\mathbb{Z}[\sigma]\)-module of (1.1), while \(\text{BF}_{gr}(E, c_0) = \text{BF}(E)\) is the usual Bowen-Franks group.

**Corollary 11.6** Let \(R \in \text{Alg}_\ell^\ast\); equip \(R\) with the trivial \(G\)-grading and \(L_{\omega}(E) \otimes R\) with the tensor product grading.

i) For each \(n \in \mathbb{Z}\) there is a short exact sequence

\[
0 \to \text{BF}_{gr}(E, \omega) \otimes KH_n(R) \to KH_n^{Gr}(L_{\omega}(E) \otimes R) \to \ker((I - A^I_{1,\omega}) \otimes KH_{n-1}(R)) \to 0. \tag{11.7}
\]

For \(R \in \text{Alg}_\ell^\ast\), the exact sequence (11.7) for \(KH_{h,Gr}^\ast\) also holds.

ii) If \(R\) is regular supercoherent and \(G\) and \(E\) are countable, then the canonical maps \(KH_*(R) \to KH_*(R)\) and \(KH_{h,Gr}^\ast(L_{\omega}(E) \otimes R) \to KH_{h,Gr}^\ast(L_{\omega}(E) \otimes R)\) are isomorphisms. If in addition \(R \in \text{Alg}_\ell^\ast\) and \(2\) is invertible in \(\ell\), then we also have \(KH^h_*(R) \sim KH^h_*(R)\) and \(KH_{h,Gr}^\ast(L_{\omega}(E) \otimes R) \sim KH_{h,Gr}^\ast(L_{\omega}(E) \otimes R)\).

**Proof** The second assertion of part i) follows from Theorem 11.5 applied to \(KH_{h,Gr}^\ast(- \otimes R)\), using that, as \(R\) is trivially graded, \(KH_{h,Gr}^\ast(R) = \mathbb{Z}[G] \otimes KH_{h}^\ast(R)\); the first is the particular case with \(\text{inv}(R)\) substituted for \(R\). By [6, Lemma 4.3], if \(F\) is a countable graph and \(R\) is unital and regular supercoherent then \(L(F) \otimes R\) is \(K\)-regular, and even \(K^h\)-regular if \(2\) is invertible in \(R\). In particular this applies when \(F\) is the covering graph of \(E\) or the edgeless graph on one vertex. Combining this with part i) and using Theorems 3.4 and 3.9 and Proposition 2.7, we obtain the remaining assertions. \(\square\)

**Remark 11.7** If \(G\) is abelian, then \(- \otimes R : Gr_{\ell} \to Gr_{\ell} - \text{Alg}_{\ell}\) is defined for every \(R \in Gr_{\ell} - \text{Alg}_{\ell}\), and a similar argument as that of Corollary 11.6 shows that there is an exact sequence

\[
0 \to \text{BF}_{gr}(E, \omega) \otimes \mathbb{Z}[G] \to KH_{n}^{Gr}(R) \to KH_{n-1}^{Gr}(R) \to 0. \tag{11.8}
\]

The rest of Corollary 11.6, including the hermitian and regular supercoherent versions of (11.8) also extend to nontrivially graded \(R\) in the obvious way, provided that \(G\) is abelian.

We remark that in the particular case \(G = \mathbb{Z}\) and \(\omega(e) = 1\) for all \(e \in E^1\), then \(I - A^I_{1,\omega} = I - \sigma A^I\) is injective by Lemma 5.2 and the rightmost nonzero term of (11.8) is \(\text{Tor}_1^{\mathbb{Z}[\sigma]}(\text{BF}_{gr}(E), KH_{n-1}^{Gr}(R))\).
Remark 11.8 It follows from Corollary 11.6 that there is a canonical homomorphism of $\mathbb{Z}[G]$-modules

$$\text{can} : BF_{\text{gr}}(E, \omega) \to KH_0^{h,G_{\text{gr}}}(L_\omega(E)), \quad \text{can}(x) = x \otimes [1] \quad (11.9)$$

The map can is an isomorphism whenever $KH_0^h(\ell) = \mathbb{Z}$ and $KH_{-1}^h(\ell) = 0$. Such is the case, for example, when $\ell = \text{inv}(\ell_0)$ and $\ell_0$ is a field or a PID, and we have $K_0^{G_{\text{gr}}}(L_\omega(E)) = BF_{\text{gr}}(E, \omega)$.

Corollary 11.9 Assume that $E^0$ is finite. Then there is a distinguished triangle in $kk_{G_{\text{gr}}}^h$

$$\ell_{\text{reg}}(E) \xrightarrow{I - \sigma I^0} \ell E^0 \to L_\omega(E).$$

Proof Apply Theorem 11.5 to $H = j_{G_{\text{gr}}}^h$.

Remark 11.10 Applying Corollary 11.9 to the unique weight with codomain the trivial group, we recover the distinguished triangle for $L(E)$ in hermitian bivariant $K$-theory given in [6, Theorem 3.6], and by [9, Example 6.2.11], also that in algebraic bivariant $K$-theory proved in [7, Proposition 5.2].

Let $E$ be a graph with finitely many vertices and let $\text{can} : BF_{\text{gr}}(E, \omega) \to KH_0^{h,G_{\text{gr}}}(L(E))$ be as in (11.9). For each $G$-graded $*$-algebra $A$ we have a map

$$\text{ev} : kk_{G_{\text{gr}}}^h(L(E), A) \to \text{hom}_{\mathbb{Z}[G]}(BF_{\text{gr}}(E, \omega), KH_0^{h,G_{\text{gr}}}(A)), \xi \mapsto KH_0^{h,G_{\text{gr}}}(\xi) \circ \text{can}. \quad (11.10)$$

Corollary 11.11 Let $E$ be a graph with finitely many vertices and $\omega : E^1 \to G$ a weight. For each $G$-graded $*$-algebra $A$, there is an exact sequence

$$0 \to KH_1^{h,G_{\text{gr}}}(A) \otimes_{\mathbb{Z}[G]} BF_{\text{gr}}^\vee(E, \omega)$$

$$\to kk_{G_{\text{gr}}}^h(L_\omega(E), A) \xrightarrow{\text{ev}} \text{hom}_{\mathbb{Z}[G]}(BF_{\text{gr}}(E, \omega), KH_0^{h,G_{\text{gr}}}(A)) \to 0.$$

Proof Apply the argument of [6, Theorem 12.2] to the triangle of Corollary 11.9 and the free $\mathbb{Z}[G]$-module presentations

$$\mathbb{Z}[G]_{\text{reg}}(E) \xrightarrow{I - A^0} \mathbb{Z}[G]E^0 \to BF_{\text{gr}}(E, \omega) \to 0,$$

$$\mathbb{Z}[G]E^0 \xrightarrow{I^0 - A_{\omega}} \mathbb{Z}[G]_{\text{reg}}(E) \to BF_{\text{gr}}^\vee(E, \omega) \to 0.$$
12 AVandenBerghtrianglein\(kk^h\)

In [18, Theorem on page 1563] Van den Bergh proves that the graded \(K\)-theory of a \(\mathbb{Z}\)-graded noetherian regular ring \(R\) is related to its ungraded \(K\)-theory by means of an exact sequence

\[
\cdots \to K_{n+1}(R) \to K_n^{\mathbb{Z}_{gr}}(R) \overset{1-\sigma}{\longrightarrow} K_n^{\mathbb{Z}_{gr}}(R) \overset{\text{forg}}{\longrightarrow} K_n(R) \to \cdots
\]

(12.1)

Here \(\sigma\) is the map induced by degree shift and forg is the map induced by the forgetful functor \(\text{Proj}_{\mathbb{Z}_{gr}}(R) \to \text{Proj}(R)\).

**Theorem 12.1** Let \(A \in \mathbb{Z}_{gr} - \text{Alg}^\ast\). Then there is a distinguished triangle in \(kk^h\)

\[
\mathbb{Z} \hat{\otimes} A \overset{1-\mathbb{Z} \hat{\otimes} \sigma}{\longrightarrow} \mathbb{Z} \hat{\otimes} A \longrightarrow A
\]

**Proof** Corollary 11.9 applied to \(G = \mathbb{Z}\), the graph \(\mathcal{R}_1\) consisting of a single loop \(e\), and the weight \(\omega(e) = 1\), yields a distinguished triangle in \(kk^h_{\mathbb{Z}_{gr}}\)

\[
\ell \overset{1-\sigma}{\longrightarrow} \ell \rightarrow \ell[t, t^{-1}]
\]

Tensor this triangle with \(A\), apply the functor \(\mathbb{Z} \hat{\otimes} - : kk^h_{\mathbb{Z}_{gr}} \to kk^h_{\mathbb{Z}_{gr}}\) followed by the forgetful functor \(kk^h_{\mathbb{Z}} \to kk^h\), and use Corollary 10.2. \(\square\)

**Corollary 12.2** Let \(R \in \mathbb{Z}_{gr} - \text{Alg}_\ell\), \(S \in \mathbb{Z}_{gr} - \text{Alg}_\ell^+\) and \(n \in \mathbb{Z}\). Then there are long exact sequences

\[
K_{H_{n+1}}(R) \longrightarrow K_{H_n}^{\mathbb{Z}_{gr}}(R) \overset{1-\sigma}{\longrightarrow} K_{H_n}^{\mathbb{Z}_{gr}}(R) \longrightarrow K_{H_n}(R)
\]

(12.2)

\[
K_{H_{n+1}}^h(S) \longrightarrow K_{H_n}^{h,\mathbb{Z}_{gr}}(S) \overset{1-\sigma}{\longrightarrow} K_{H_n}^{h,\mathbb{Z}_{gr}}(S) \longrightarrow K_{H_n}^h(S)
\]

(12.3)

If \(R\) is regular supercoherent, we may substitute \(K\) and \(K^{\mathbb{Z}_{gr}}\) for \(K_{H}\) and \(K_{H}^{\mathbb{Z}_{gr}}\) in (12.2). If \(S\) is regular supercoherent and 2 is invertible in \(S\), we may substitute \(K^h\) and \(K^{h,\mathbb{Z}_{gr}}\) for \(K_{H}^h\) and \(K_{H}^{h,\mathbb{Z}_{gr}}\) in (12.3).

**Proof** Observe that (12.2) is the particular case \(S = \text{inv}(R)\) of (12.3). We prove (12.3).

By excision, we can assume that \(R\) is unital. Applying \(kk^h(\ell, -)\) to the triangle of Theorem 12.1 we obtain (12.3). The assertions about the regular supercoherent case follow from Corollary 11.6 and hermitian Dade’s theorem 4.1. \(\square\)

**Remark 12.3** The exact sequences of Corollary 12.2 are similar to that of Van den Bergh (12.1), except perhaps in the map going from graded to ungraded \(K\)-theory; we shall presently see that the two maps are equivalent up to appropriate identifications.
Let $R$ be unital ring and $\text{inc}: R \subset R[t, t^{-1}]$ the canonical inclusion. The forgetful functor $\mathbb{Z}_{\text{gr}} - \text{mod}_R \to \text{mod}_R$ is naturally equivalent to the composite of the scalar extension along $\text{inc}$ with the functor $\mathbb{Z}_{\text{gr}} - \text{mod}_{R[t, t^{-1}]} \to \text{mod}_R, M \mapsto M_0$. Under the equivalence (3.1) the latter functor corresponds to $\text{mod}_{\mathbb{Z}} \to \mathbb{R} \to \text{mod}_R, M \mapsto M \cdot \chi_0$, which is left inverse to the scalar extension along $a \mapsto \chi_0 \circ a$ (see Remark 4.2). Summing up, the map $\text{forg}$ in (12.1) corresponds to that in Corollary 12.2, which is the composite of those induced by the inclusion $R \subset R[t, t^{-1}]$ and by the $kk$-inverse of the inclusion as the homogeneous component of degree zero coming from Corollary 10.2.

13 A graded classification result

We now proceed to classify Leavitt path algebras as objects in $kk^h_{\mathbb{Z}_{\text{gr}}}$ in terms of the Bowen-Franks $\mathbb{Z}[\sigma]$-modules of their associated graphs.

Given graphs $E$ and $F$ with finitely many vertices, the map $\text{ev}$ of (11.10) associates to each morphism $L(E) \to L(F)$ in $kk^h_{\mathbb{Z}_{\text{gr}}}$ a $\mathbb{Z}[\sigma]$-module homomorphism $BF_{\text{gr}}(E) \to KH^h_{0, \mathbb{Z}_{\text{gr}}}(L(F))$. This assignment is defined in terms of the comparison morphism $\text{can}: BF_{\text{gr}}(E) \to KH^h_{0, \mathbb{Z}_{\text{gr}}}(L(E))$ of (11.9). We remark that the latter is an isomorphism whenever $KH_0(\ell) = \mathbb{Z}$, by Corollary 5.4.

If we start with a morphism $\xi: BF_{\text{gr}}(E) \to BF_{\text{gr}}(F)$, we can obtain a map $BF_{\text{gr}}(E) \to KH^h_{0, \mathbb{Z}_{\text{gr}}}(L(F))$ by postcomposing with $\text{can}$. The lemma below shows that if $\xi$ is an isomorphism, then it can be lifted to an isomorphism $\tilde{\xi}: L(E) \to L(F)$ in $kk^h_{\mathbb{Z}_{\text{gr}}}$, in the sense that $\text{can} \circ \tilde{\xi} = \text{ev}(\tilde{\xi})$.

**Theorem 13.1** Let $E$ and $F$ be graphs with finitely many vertices. If $\xi: BF_{\text{gr}}(E) \to BF_{\text{gr}}(F)$ is a $\mathbb{Z}[\sigma]$-module isomorphism, then there exists an isomorphism $\tilde{\xi}: L(E) \to L(F)$ in $kk^h_{\mathbb{Z}_{\text{gr}}}$ such that $\text{ev}(\tilde{\xi}) = \text{can} \circ \tilde{\xi}$.

**Proof** By Lemma 5.2, we have length one free $\mathbb{Z}[\sigma]$-module resolutions of $BF_{\text{gr}}(E)$ and $BF_{\text{gr}}(F)$ which guarantee that the argument of [6, Lemma 6.4] can be carried out under the present hypotheses.

**Theorem 13.2** Assume that $KH_{-1}(\ell) = 0$ and that the canonical morphism $\mathbb{Z} \to KH_0(\ell)$ is an isomorphism. Let $E$ and $F$ be graphs with finitely many vertices. Then the following are equivalent:

(i) The algebras $L(E)$ and $L(F)$ are $kk^h_{\mathbb{Z}_{\text{gr}}}$-isomorphic.
(ii) The algebras $L(E)$ and $L(F)$ are $kk_{\mathbb{Z}_{\text{gr}}}$-isomorphic.
(iii) The $\mathbb{Z}[\sigma]$-modules $BF_{\text{gr}}(E)$ and $BF_{\text{gr}}(F)$ are isomorphic.

**Proof** The fact that (i) implies (ii) amounts to applying the forgetful functor $kk^h_{\mathbb{Z}_{\text{gr}}} \to kk_{\mathbb{Z}_{\text{gr}}}$. On the other hand, the assumptions on $\ell$ imply that $kk_{\mathbb{Z}_{\text{gr}}}(\ell, \ell) \cong \mathbb{Z}[\sigma]$; this together with the case $G = \mathbb{Z}, \omega = c_1$ of Corollary 11.9 shows that $kk_{\mathbb{Z}_{\text{gr}}}(\ell, L(E)) \cong BF_{\text{gr}}(E)$ and likewise for the analogue statement replacing $E$ by $F$. It follows that (ii) implies (iii). Finally, (iii) implies (i) by Theorem 13.1.
Acknowledgements A good part of the material of Sects. 6, 7, 8, 9, 11 and 13 first appeared in the diploma thesis of the first author [4].

Funding Funding for this article came from the sources stated on the title page, and include the University of Buenos Aires, Argentine government agencies CONICET and Agencia Nacional de Promoción Científica y Tecnológica, and the Spanish Ministerio de Ciencia e Innovación.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Code Availability Not applicable.

Declarations

Conflict of interest The authors state that there is no conflict of interest.

References

1. Abrams, G., Ara, P., Siles Molina, M.: Leavitt path algebras, Lecture Notes in Math, vol. 2008. Springer (2017)
2. Ara, P., Moreno, M.A., Pardo, E.: Nonstable K-theory for graph algebras. Algebr. Represent. Theory 10(2), 157–178 (2007). https://doi.org/10.1007/s10468-006-9404-z
3. Ara, P., Hazrat, R., Huanhuan, L., Aidan, S.: Graded Steinberg algebras and their representations. Algebra Number Theory 12(1), 131–172 (2018). https://doi.org/10.2140/ant.2018.12.131
4. Arnone G.: Álgebras de Leavitt y K-teoría bivariante hermitiana graduada, Diploma Thesis, Buenos Aires (2021). http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2021/Arnone.pdf
5. Cortiñas G.: Álgebra II+1/2, Cursos y seminarios de matemática, Serie B, vol. Fascículo 13, Departamento de matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires
6. Cortiñas G.: Classifying Leavitt path algebras up to involution preserving homotopy, Math. Ann., posted on (2022). https://doi.org/10.1007/s00208-022-02436-2
7. Cortiñas, G., Montero, D.: Algebraic bivariant K-theory and Leavitt path algebras. J. Noncommut. Geom. 15(1), 113–146 (2021). https://doi.org/10.4171/jncg/397
8. Cortiñas, G., Thom, A.: Bivariant algebraic K-theory. J. Reine Angew. Math. 610, 71–123 (2007)
9. Cortiñas, G., Vega, S.: Bivariant hermitian K-theory and Karoubi’s fundamental theorem. J. Pure Appl. Algebra 226(12), 107124 (2022). https://doi.org/10.1016/j.jpaa.2022.107124
10. Cuntz, J., Meyer, R., Rosenberg, J.M.: Topological and bivariant K-theory, Oberwolfach Seminars, vol. 36. Birkhäuser Verlag, Basel (2007)
11. Dade, E. C.: Group-graded rings and modules. Math. Z. 174(3), 241–262 (1980). https://doi.org/10.1007/BF01161413
12. Ellis, E.: Equivariant algebraic kk-theory and adjointness theorems. J. Algebra 398, 200–226 (2014). https://doi.org/10.1016/j.jalgebra.2013.09.023
13. Hazrat, R.: The graded Grothendieck group and the classification of Leavitt path algebras. Math. Ann. 355(1), 273–325 (2013). https://doi.org/10.1007/s00208-012-0791-3
14. Hazrat, R.: Graded rings and graded Grothendieck groups, London mathematical society lecture note series, vol. 435. Cambridge University Press, Cambridge (2016)
15. Preussler, R.: Leavitt path algebras of hypergraphs. Bull. Braz. Math. Soc. (N.S.) 51(1), 185–221 (2020). https://doi.org/10.1007/s00574-019-00150-3
16. Schlichting, M.: Hermitian K-theory of exact categories. J. K-Theory 5(1), 105–165 (2010). https://doi.org/10.1017/is009010017jkt075
17. Schlichting, M.: The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. Invent. Math. 179(2), 349–433 (2010). https://doi.org/10.1007/s00222-009-0219-1
18. Van den Bergh, M.: A note on graded K-theory. Comm. Algebra 14(8), 1561–1564 (1986). https://doi.org/10.1080/00927878608823384
Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.