Decentralized Measurement Feedback Stabilization of Large-scale Systems via Control Vector Lyapunov Functions

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Abstract

This paper studies the problem of decentralized measurement feedback stabilization of nonlinear interconnected systems. As a natural extension of the recent development on control vector Lyapunov functions, the notion of output control vector Lyapunov function (OCLVF) is introduced for investigating decentralized measurement feedback stabilization problems. Sufficient conditions on (local) stabilizability are discussed which are based on the proposed notion of OCLVF. It is shown that a decentralized controller for a nonlinear interconnected system can be constructed using these conditions under an additional vector dissipation-like condition. To illustrate the proposed method, two examples are given.

Keywords: Decentralized control, control Lyapunov function, vector dissipativity

1. Introduction

Large-scale system modeling has been an accepted approach to the investigation of complex dynamical systems that consist of, or can be partitioned into a set of interconnected subsystems. One of the most common feedback control design strategies for such systems is the decentralized control strategy ([15, 17, 20]). Considerable efforts have been made in the literature to develop manageable analysis and control design algorithms to reduce the computation complexity of the existing methodologies. One such effort relates to the notion of vector Lyapunov function ([2, 16]). This notion has been extensively used in the analysis and control design of large-scale systems, see [13, 14, 15, 17, 20, 21]. For more recent results on vector Lyapunov functions, we refer the reader to [8, 9, 12, 18, 27]. A wide range of applications of the method of vector Lyapunov functions to real world problems arising in the areas of aerospace engineering, power systems, economics, immunology, can be found in [14, 20, 21].

The Lyapunov function approach dominates in the system analysis and control theory. However in general, the construction of a suitable (scalar) Lyapunov function for general nonlinear systems is not a trivial task, especially when the system has a complex structure. In view of this, the vector Lyapunov function approach is often considered as a viable alternative to the scalar Lyapunov method in situations involving complex systems, see [9, 20] for instances. As a generalization of the standard scalar Lyapunov function...
function methodology, the method of vector Lyapunov functions offers potentially more flexible strategies to cope with complexity of dynamical systems because it imposes different, potentially less rigid requirements on the system components, see [14]. Specifically, Lyapunov functions constructed for individual subsystems of a large-scale system only need to have certain dissipation properties. In addition, a so-called comparison system of a reduced dimension should have certain stability property which will confirm the corresponding stability property of the original composite system by the well-known comparison principle.

A recent development in the area of vector Lyapunov functions is concerned with the notion of a control vector Lyapunov function and the methodology of state-feedback stabilization based on this notion, see [18]. The work in [18] is an extension of the control Lyapunov function approach originating in [11], also see [6], [24], [25]. Compared with these results, this paper further extends the method of control vector Lyapunov functions to investigate problems of measurement feedback decentralized stabilization when the complete system state information is not available.

The main contribution of this paper is summarized as follows. A notion of OCVLF is introduced and is used to formulate sufficient conditions for decentralized stabilization. We further show that, when the system has certain additional vector dissipation properties, a constructive stabilizing control solution can be obtained. From the theoretical viewpoint, our contribution broadens the use of the method of vector Lyapunov functions in the decentralized control design of large-scale nonlinear systems.

The first result of this paper relates the existence of an OCVLF for a nonlinear system to the existence of the partition of unity of a certain set. In general, this makes the derivation of the stabilizing output feedback control laws difficult in practice, because of the lack of systematic methods to carry out the partition of unity. Therefore, unlike the state feedback case in [18], the computational tractability of this extension is a critical issue. This paper shows that this issue can be circumvented in a situation where the control input for each subsystem admits a special decomposition into a pair of separate input channels, cf. [24]. Specifically, we show that in this case, the decentralized control design with an OCVLF is constructive provided the control laws differ in the partition of unity of a certain set. In general, this makes the derivation of the stabilizing output feedback control laws difficult in practice, because of the lack of systematic methods to carry out the partition of unity.

This paper is organized as follows. Section 2 describes the class of systems under consideration and presents the formulation of the stabilization problem for a class of large-scale interconnected nonlinear systems that admit a certain decomposition structure. In Section 3, sufficient conditions for measurement feedback based decentralized stabilization of this class of large-scale interconnected systems are presented. To illustrate the proposed design method, two examples are given in Section 4. Section 5 provides concluding remarks.

**Notation & Definition:** \( ||x|| \) is the Euclidean norm in \( \mathbb{R}^n \) for \( x \in \mathbb{R}^n \). \( \mathbb{R}_+^n \) denotes the set of vectors with all the components being nonnegative real numbers. In particular, \( \mathbb{R}_+ \) denotes the set of nonnegative real numbers. For a pair of vectors \( x, x' \in \mathbb{R}^n \), \( x < x' \) (\( x \leq x' \) respectively) means \( x_i < x'_i \) (\( x_i \leq x'_i \) respectively) for each \( i = 1, \cdots, n \). That is, \( x \preceq x' \) if and only if \( x' - x \in \mathbb{R}_+^n \). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) is said to be positive definite if \( f(x) > 0 \) for \( x \neq 0 \) and \( f(0) = 0 \). The class of \( k \) times continuously differentiable functions from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is denoted by \( C^k[\mathbb{R}^n, \mathbb{R}^m] \), and the class of Lipschitz continuous functions is denoted by \( L[\mathbb{R}^n, \mathbb{R}^m] \). Also, \( C[\mathbb{R}^n, \mathbb{R}^m] \) is the class of continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). We use \( \mathcal{W} \) to denote the class of quasimonotone nondecreasing functions \( w(z) \in C[\mathbb{R}^n, \mathbb{R}_+] \). Recall [14] that a function \( w : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is quasimonotone nondecreasing if for each \( i = 1, \cdots, n \), \( w_i(z') \leq w_i(z'') \) for any two points \( z', z'' \in \mathbb{R}^n \) satisfying \( z'_i = z''_i \) and \( z'_i \leq z''_i \). For two functions \( f : \mathbb{R}^m \rightarrow \mathbb{R}_+^l, g : \mathbb{R}^n \rightarrow \mathbb{R}^m \), the notation \( \circ \) denotes the function composition, i.e., \( (f \circ g)(x) = f(g(x)) \), simply denoted by \( f \circ g(x) \). Given a differentiable function \( W : \mathbb{R}^n \rightarrow \mathbb{R}_+^1 \) and a function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the notation \( L_f W(x) \) refers to the Lie derivative \( (L_f W)(x) = \sum_{i=1}^n \frac{\partial W}{\partial z_i} f_i(x) \), simply denoted by \( L_f W(x) \).
2. Problem Formulation

Consider a large-scale control-affine system $\mathcal{S}$ described by the equations

$$\mathcal{S} : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

(1)

consisting of $n$ subsystems described by

$$\mathcal{S}_i : \begin{cases} \dot{x}_i = f_i(x) + g_i(x)u_i \\ y_i = h_i(x_i), \quad i = 1, \cdots, n. \end{cases}$$

Here, the system state is $x = [x_1^T, \cdots, x_n^T]^T \in \mathbb{R}^n$ with $x_i \in \mathbb{R}^{n_i}$, the control input is $u = [u_1^T, \cdots, u_n^T]^T \in \mathbb{R}^m$ with $u_i \in \mathbb{R}^{m_i}$, the measurement output is $y = [y_1^T, \cdots, y_n^T]^T \in \mathbb{R}^l$ with $y_i \in \mathbb{R}^{l_i}$. It is assumed that $f_i, g_i, h_i$ are all smooth functions of appropriate dimensions and

$$f(x) = [f_1(x)^T, \cdots, f_n(x)^T]^T$$

$$g(x) = \text{diag}(g_1(x), \cdots, g_n(x))$$

$$h(x) = [h_1(x_1)^T, \cdots, h_n(x_n)^T]^T$$

with $f(0) = 0$ and $h(0) = 0$.

The large-scale system $\mathcal{S}$ will be referred to as the composite system with a decomposition $\{\mathcal{S}_i\}_{i=1}^n$. In view of the system structure, the subsystems $\mathcal{S}_i$ are interconnected through the functions $f_i(x)$. To highlight that the initial conditions for the system (1) are within the closed ball $S_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, with a fixed real number $\rho > 0$, we will use the notation $\mathcal{S}(S_\rho)$ for the composite system (1).

Remark 1. It is worth noting that dynamics of each subsystem $\mathcal{S}_i$ are coupled to other subsystems through functions $f_i, g_i$ being dependent on states external to $\mathcal{S}_i$. On the other hand, its output $h_i$ does not depend on external dynamics $x_j, j \neq i$. This model is consistent with our objective in this paper which is to develop a methodology for decentralized stabilization, where the feedback law for each subsystem $\mathcal{S}_i$ is based on its local outputs reflecting dynamics $x_i$ of this subsystem. Problems where subsystem measurements depend on external states $x_j, j \neq i$, are usually regarded as distributed control problems. This is for example the case in multi-agent cooperative control problems, where locally available measurements reflect a relative state of the subsystem with respect to its neighbors.

The stabilization problem for the system $\mathcal{S}(S_\rho)$ considered in this paper is defined as follows.

**Problem 1 (Decentralized stabilization).** For the given composite system $\mathcal{S}(S_\rho)$ with the decomposition $\{\mathcal{S}_i\}_{i=1}^N$, we aim to find a decentralized controller

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \Gamma_1 \circ h_1(x_1) \\ \vdots \\ \Gamma_n \circ h_n(x_n) \end{bmatrix}$$

(2)

that asymptotically stabilizes the composite system $\mathcal{S}$ at the origin. The structure of the desired decentralized controller is illustrated in Figure 1.
3. Measurement Feedback Decentralized Stabilization

3.1. Preliminaries

We begin with presenting the notion of decentralized output feedback stabilizability and the associated notion of OCVLF to be used in this paper.

**Definition 1.** The system \( \mathcal{S}(S_\rho) \) with a decomposition \( \{\mathcal{S}_i\}_{i=1}^n \) is said to be decentralized output feedback stabilizable (w.r.t. \( S_\rho \)) if there exists a decentralized controller of the form (2) that solves Problem 7.

Before giving the definition of an OCVLF, we introduce the following notation. For each \( i = 1, \cdots, n \) and a given set \( Q \subset \mathbb{R}^n \), let

\[
K_i(Q) = \{ y_i \in \mathbb{R}^l_i : y_i = h_i(x_i), \ x \in Q \}
\]

\[
\tilde{K}_i(y_i) = \{ x \in S_\rho : h_i(x_i) = y_i \}.
\]

Note that since each \( h_i \) is smooth, then \( K_i(S_\rho) \) is a compact set and for each fixed \( y_i \in K_i(S_\rho) \), \( \tilde{K}_i(y_i) \) is also a compact set. Obviously, \( K_i(\tilde{K}_i(y_i)) = \{ y_i \} \).

**Definition 2.** The system \( \mathcal{S}(S_\rho) \) with a decomposition \( \{\mathcal{S}_i\}_{i=1}^n \) is said to have an OCVLF triple \( \{V, \Lambda, S_\rho\} \), where \( V \in C^1[S_\rho, \mathbb{R}_+] \) and \( \Lambda \in L[\mathbb{R}_+, \mathbb{R}_+] \cap W \) with \( \Lambda(0) = 0 \), if for each \( i = 1, \cdots, n \),

(i) \( V_i : \mathbb{R}^n \to \mathbb{R}_+ \) is positive definite.

(ii) For each \( y_i \in K_i(S_\rho) \setminus \{0\} \), there is a vector \( u_i \in \mathbb{R}^m_i \) such that

\[
L_{f_i} V_i(x) + L_{g_i} V_i(x) \cdot u_i < \Lambda_i \circ V(x), \ \forall x \in \tilde{K}_i(y_i).
\]

(iii) For every \( x \in \tilde{K}_i(0) \setminus \{0\} \), \( L_{f_i} V_i(x) < \Lambda_i \circ V(x) \).

(iv) The trivial solution of the following system

\[
\dot{z} = \Lambda(z), \quad z(0) \in \mathbb{R}_+^n
\]

is asymptotically stable (cf. Definition 1.6.1 in [14]).
Remark 2. The properties in conditions (ii) and (iii) are analogous to the corresponding properties of scalar Lyapunov functions, and are fundamental for stabilizability by output-feedback control. The significance of these conditions is that they formulate the output-feedback stabilizability property of the system in terms of properties of individual subsystems. Such a formulation has proved useful in the linear case and the case of nonlinear systems of Lur’e type where constructive conditions to verify these properties were found [27, 28]. Later in the paper, we present a constructive condition of vector dissipativity which addresses this question in part in the nonlinear setting.

Remark 3. In the above definition, the system (3) is known as the comparison system for $S(\mathcal{S}_\rho)$. The key idea of the method of vector Lyapunov functions is to reduce the stability analysis to find a comparison system whose stability implies that of the original system. Recently, a number of results have been developed in the literature to facilitate the analysis of comparison systems of the form (3) associated with large-scale interconnected systems. In particular, small-gain criteria have been developed that serve this purpose; e.g., see [4, 13, 19]. In the light of these results, in this paper the comparison system will be assumed to be given.

Comparing the form of the comparison system (3) with that in [14, 18], we note that comparison systems could be chosen to have a more general form, e.g., to be time-varying or trajectory-dependent. However in this more general case, the stability of the comparison system would need to be carefully addressed in certain uniform sense. From this viewpoint, the results in this paper could be further extended, at least on a case by case basis. For simplicity, we will restrict attention to the class of time-invariant comparison systems (3), and will use a corresponding comparison principle to be given later in Lemma 1.

The proofs of our main results are analogous to the proofs of the similar results in [24, 25], where the problem of centralized measurement feedback control was studied using a scalar Lyapunov function in the case where $S_\rho$ is a small set. For the sake of completeness, we give the full proofs to show all the extensions. The following lemmas will be used in the derivation of the results of the paper. The first lemma establishes a comparison principle used in this paper; see also Theorem 1.6.1 in [14] or Theorem 1 in [8]. The second lemma is concerned with the existence of a partition of unity; see page 52 in [7]. The last lemma shows that property (ii) of the OCVLF defined in Definition 2 holds in a small neighbourhood of the point $y_i^\star$, using the same control $u_i^\star$. This property can be regarded as certain ‘robustness’ of control $u_i^\star$ under small perturbations of $y_i^\star$. This robustness property will allow us to select a countable set of control actions $u_i^\star$, from which a smooth in $y_i$ control law, except possibly at the origin, will be constructed using Lemma 2.

Lemma 1. Consider a nonlinear autonomous system described by
\[
\dot{x} = F(x), \quad x(0) = x_0
\]
where the state is $x \in \mathbb{R}^n$, $F \in C^1[\mathbb{R}^n, \mathbb{R}^n]$, $F(0) = 0$ and $x_0$ is the initial value. Suppose that

(i) $V \in C^1[S_\rho, \mathbb{R}^n]$ and $v(x) = \sum_{i=1}^n V_i(x)$ is positive definite;

(ii) For all $x \in S_\rho$, $\dot{V}(x) \leq \Lambda \circ V(x)$, where $\Lambda \in L[\mathbb{R}^n, \mathbb{R}^n] \cap \mathcal{W}$ and $\Lambda(0) = 0$.

Also, consider the comparison system defined by
\[
\dot{z} = \Lambda(z), \quad z(0) \in \mathbb{R}^n.
\]
Then the asymptotic stability property of $z = 0$ of the comparison system (5) implies the asymptotic stability of $x = 0$ of the system (4).
Recall that for a given set $X \subset \mathbb{R}^n$, an open set $U_i$ is said to be a relatively open subset with respect to $X$ if $U_i = X \cap P_i$ where $P_i$ is some open set in $\mathbb{R}^n$. In this paper, where it causes no confusion, we will refer to relative open sets as open sets. A collection of sets $\{U_\alpha\}$ covers a set $X$ if $X$ is contained in the union $\bigcup_\alpha U_\alpha$. An open covering of $X$ is a collection of open sets $\{U_\alpha\}$ which covers $X$.

**Lemma 2.** Let $X$ be an arbitrary subset of $\mathbb{R}^n$. For any countable covering of $X$ by relatively open subsets $\{U_i\}_{i=1}^\infty$, there exists a sequence of smooth functions $\{\theta_i(x)\}_{i=1}^\infty$ on $X$, as a partition of unity subordinate to the open cover $\{U_i\}_{i=1}^\infty$, such that

(i) $0 \leq \theta_i(x) \leq 1$ for all $x \in X$ and all $i \geq 1$.

(ii) Each $x \in X$ has a neighborhood on which all but finitely many functions $\theta_i(x)$ are identically zero.

(iii) Each function $\theta_i$ is identically zero except on some closed set contained in one of the $\{U_i\}_{i=1}^\infty$.

(iv) For each $x \in X$, $\sum \theta_i(x) = 1$.

**Lemma 3.** For each pair $(y_i^*, u_i^*)$ with $y_i^* \neq 0$ satisfying condition (ii) in Definition 2, there exists an open ball $B_{y_i^*}$ centered at $y_i^*$ such that

$$L_h V_i(x) + L_g V_i(x) \cdot u_i^* < \Lambda_i \circ V(x), \quad \forall x \in \bar{K}_i(B_{y_i^*})$$

where $\bar{K}_i(B_{y_i^*}) = \{x \in S_\rho : h_i(x_i) \in B_{y_i^*}\}$.

**Proof.** Fix $y_i^* \neq 0$ and $(y_i^*, u_i^*)$ satisfying condition (ii) in Definition 2 and introduce the set

$$\bar{K}_i(y_i^*) = \{x \in S_\rho : \text{dist}(x, \bar{K}_i(y_i^*)) \leq \varepsilon\}$$

where $\varepsilon > 0$ is a constant and $\text{dist}(x, \bar{K}_i(y_i^*)) \triangleq \min_{s \in \bar{K}_i(y_i^*)} ||x - s||$. Since $y_i^* 
eq 0$, due to continuity of $h_i(x_i)$ one can select a sufficiently small $\varepsilon_1 > 0$ to ensure that

$$0 \notin K_i(\bar{K}_i(y_i^*)) \quad \forall \varepsilon < \varepsilon_1.$$  

(7)

Observe that for any sufficiently small $\varepsilon > 0$, $\bar{K}_i(y_i^*) \cap S_\rho$ is a compact set. To establish this it suffices to show that $\bar{K}_i(y_i^*) \cap S_\rho$ is a closed set. Consider a converging sequence $x_k \in \bar{K}_i(y_i^*) \cap S_\rho$, $\lim_{k \to \infty} x_k = x$. Since $x_k \in \bar{K}_i(y_i^*) \cap S_\rho$, then there exists $\bar{x}_k \in \bar{K}_i(y_i^*)$ such that $||x_k - \bar{x}_k|| \leq \varepsilon$. Also, since $\bar{K}_i(y_i^*)$ is compact, a converging subsequence $\{\bar{x}_{k_l}\}$ can be extracted from $\{\bar{x}_k\}$. Let $\bar{x}$ be the limit point of $\{\bar{x}_{k_l}\}$. We have

$$||x - \bar{x}|| \leq ||x - x_k|| + ||x_k - \bar{x}_k|| + ||\bar{x}_k - \bar{x}|| \leq \varepsilon + ||x - x_k|| + ||\bar{x}_k - \bar{x}||.$$  

Letting $k_l \to \infty$ leads us to conclude that $||x - \bar{x}|| \leq \varepsilon$. Since $\bar{x} \in \bar{K}_i(y_i^*)$, this implies $x \in \bar{K}_i(y_i^*)$. Also, $x \in S_\rho$ since the latter set is a compact. Thus, $x \in \bar{K}_i(y_i^*) \cap S_\rho$, which confirms that $\bar{K}_i(y_i^*) \cap S_\rho$ is closed. Hence, it is compact, because $\bar{K}_i(y_i^*) \cap S_\rho \subseteq S_\rho$, and the latter set is bounded.

From now, let us fix $\varepsilon \in (0, \varepsilon_1)$ such that $\bar{K}_i(y_i^*)$ is compact.

By assumption, the function $\alpha_i(x) \triangleq L_h V_i(x) + L_g V_i(x) \cdot u_i^* - \Lambda_i \circ V(x)$ is continuous on the compact set $S_\rho$, therefore it is continuous on its compact subsets $\bar{K}_i(y_i^*)$ and $\bar{K}_i(y_i^*)$. This observation leads to the
following conclusions. Firstly, $\alpha_i(x)$ attains its maximum value on $\tilde{K}_i(y_i^*)$. This implies that there exists $\delta_{i^*} > 0$ such that

$$\alpha_i(x) < -\delta_{i^*} \quad \forall x \in \tilde{K}_i(y_i^*).$$

Secondly, $\alpha_i(x)$ is uniformly continuous on $\tilde{K}_i(y_i^*)$, by the Heine-Cantor Theorem. This allows us to ascertain the existence of a sufficiently small $\varepsilon(\delta_{i^*}) > 0$ such that $|x - \bar{x}| \leq \varepsilon(\delta_{i^*})$, $x, \bar{x} \in \tilde{K}_i(y_i^*) \cap S_\rho$, implies

$$|\alpha_i(x) - \alpha_i(\bar{x})| \leq \frac{\delta_{i^*}}{2}.$$ 

Since $\delta_{i^*}$ and hence $\varepsilon(\delta_{i^*}) > 0$ can be chosen to be arbitrarily small, one can always ensure that $\varepsilon(\delta_{i^*}) < \varepsilon$. Therefore, if we select an arbitrary $\bar{x} \in \tilde{K}_i(y_i^*)$, then for any $x$ such that $|x - \bar{x}| \leq \varepsilon(\delta_{i^*}) < \varepsilon$, it follows that

$$\alpha_i(x) \leq -\frac{\delta_{i^*}}{2} < 0.$$ 

That is, inequality (8) holds for any $x \in \tilde{K}_i(\varepsilon(\delta_{i^*}))(y_i^*)$.

To complete the proof, we now show that an open ball $B_i(y_i^*)$ can be selected with the property that $h_i(x_i) \in B_i(y_i^*)$ implies $x \in \tilde{K}_i(\varepsilon(\delta_{i^*}))(y_i^*)$. We prove this by contradiction. Suppose such a ball does not exist, and hence for an arbitrarily small $\nu > 0$ there exists $x_i$ such that $||h_i(x_i) - y_i^*|| < \nu$ and $||x_i - x|| > \varepsilon(\delta_{i^*})$, $\forall x \in \tilde{K}(y_i^*)$. The second condition means that $x_i \notin \tilde{K}_i(\varepsilon(\delta_{i^*}))(y_i^*)$.

Let us consider a sequence of radii $\nu_l = \frac{1}{l} \to 0$ as $l \to \infty$, and let $x_{y_l}$ be the corresponding sequence of points satisfying (9). Since $\{x_{y_l} : l = 1, 2, \cdots\} \subset S_\rho$ and the latter set is compact, then a converging subsequence can be extracted from $\{x_{y_l} : l = 1, 2, \cdots\}$, which we again denote $\{x_{y_l} : l = 1, 2, \cdots\}$ and lim$_{l \to \infty} x_{y_l} = x^0$. Due to continuity of $h_i$, we then have lim$_{l \to \infty} h_i(x_{y_l}) = h_i(x^0)$, and also $||h_i(x_{y_l}) - y_i^*|| < \frac{1}{l} \to 0$. Thus, $h_i(x^0) = y_i^*$ due to uniqueness of the limit point. This implies that $x^0 \in \tilde{K}(y_i^*)$. Therefore, it must follow from the second condition (9) that $||x_{y_l} - x^0|| > \varepsilon(\delta_{i^*})$ for all $l$. However, we have previously established that $\{x_{y_l} : l = 1, 2, \cdots\}$ converges to $x^0$. This contradiction shows that there exists a ball $B_i(y_i^*)$ centered at $y_i^*$, of sufficiently small radius, with the property

$$\tilde{K}_i(B_i(y_i^*)) = \{x \in \mathbb{R}^n : h_i(x_i) \in B_i(y_i^*) \} \subseteq \tilde{K}_i(\varepsilon(\delta_{i^*}))(y_i^*).$$

Thus, we conclude that (6) holds.

3.2. Decentralized Stabilization using an OCVLF

**Theorem 4.** The system $\mathcal{S}(S_\rho)$ with the decomposition $\mathcal{S} = \mathcal{S}(S_\rho)$ is decentralized output feedback stabilizable by a smooth (except possibly at the origin) output feedback controller $u = \Gamma(y)$, if there exists an OCVLF triple $\{V, \Lambda, S_\rho\}$ for this system.

**Proof.** Suppose there is an OCVLF triple $\{V, \Lambda, S_\rho\}$ for $\mathcal{S}(S_\rho)$. Then for each $i = 1, \cdots, n$, consider the component $V_i(x_i)$ of $V(x)$. From condition (ii) of Definition 2 and by Lemma 3, it follows that for each $y_i \in K_i(S_\rho)\{0\}$, there exist a vector $u_i$ and an open ball $B_{y_i}$ centered at $y_i$ such that (6) holds.
We now note that the set $K_i(S_p) \setminus \{0\}$ endowed with the Euclidean metric is a metric space. Also, this metric space is separable. Indeed, the set $K_i(S_p)$ is compact, since $S_p$ is compact and $h_i$ is continuous. Therefore $K_i(S_p)$ is separable, i.e., it contains a dense subset. Removing, if necessary, the zero element from this dense subset yields a dense subset for $K_i(S_p) \setminus \{0\}$.

Furthermore, the collection of balls $\{B_{y_i} : y_i \in K_i(S_p) \setminus \{0\}\}$ forms an open covering for the separable metric space $K_i(S_p) \setminus \{0\}$. According to Theorem 2 (Lindelöf) on page 94 of [3], it is possible to extract a countable covering $\{B_{j_i}^* : y_i^* \in K_i(S_p) \setminus \{0\}\}$ such that

$$K_i(S_p) \setminus \{0\} \subset \bigcup_{j=1}^{\infty} B_{j_i}.$$  

Here, $B_{ij} \triangleq B_{j_i}^*$ is an open ball centered at $y_i^* \in K_i(S_p) \setminus \{0\}$ satisfying (6). Clearly, the condition (10) remains true when each open set $B_{ij}$ is replaced with its relative open version $B_{ij} \cap (K_i(S_p) \setminus \{0\})$, which we also denote $B_{ij}$.

We now observe that the conditions of Lemma 2 are satisfied for the set $X = K_i(S_p) \setminus \{0\}$ and its relatively open covering $\{B_{ij}\}_{j=1}^{\infty}$. According to Lemma 2, there exists a sequence of smooth functions $(\psi_{ij}(y_i))^\infty_{j=1}$ with properties (i) to (iv) stated in that lemma. In particular, we note that $\psi_{ij}(y_i) = 0$ for $y_i \notin B_{ij}$, according to claim (iii) of that lemma.

Next, define the mapping $\Gamma : \mathbb{R}^l \mapsto \mathbb{R}^m$ with $\Gamma_i : \mathbb{R}^l \mapsto \mathbb{R}^{mu}$ described by

$$\Gamma_i(y_i) = \begin{cases} 0, & \text{for } y_i = 0; \\ \sum_{j=1}^{\infty} u_{i_j}^* \psi_{ij}(y_i), & \text{otherwise} \end{cases}$$  

(11)

where $u_{i_j}^*$ are vectors corresponding to the centers $y_i^*$ of the covering sets $B_{ij}$. By virtue of property (iv) of Lemma 2, $\Gamma_i(y_i)$ is well defined and smooth on $K_i(S_p) \setminus \{0\}$, because according to (ii), the sum in (11) contains a finite number of addends for each $y_i \in K_i(S_p) \setminus \{0\}$.

Now let us fix $x \in S_p$, $x \neq 0$, and consider $y_i = h_i(x_i)$. Suppose $y_i \neq 0$, then

$$L_{fi} V_i(x) + L_{gi} V_i(x) \cdot \Gamma_i(y_i)$$

$$= \left( \sum_{j=1}^{\infty} \psi_{ij}(y_i) \right) L_{fi} V_i(x) + L_{gi} V_i(x) \cdot \sum_{j=1}^{\infty} u_{i_j}^* \psi_{ij}(y_i)$$

$$= \sum_{j=1}^{\infty} L_{fi} V_i(x) \psi_{ij}(y_i) + \sum_{j=1}^{\infty} L_{gi} V_i(x) \cdot u_{i_j}^* \psi_{ij}(y_i)$$

$$= \sum_{j \in B_{ij}} \left( L_{fi} V_i(x) + L_{gi} V_i(x) \cdot u_{i_j}^* \right) \psi_{ij}(y_i)$$

$$+ \sum_{j \in \overline{B_{ij}}} \left( L_{fi} V_i(x) + L_{gi} V_i(x) \cdot u_{i_j}^* \right) \psi_{ij}(y_i)$$

$$= \sum_{j \in B_{ij}} \left( L_{fi} V_i(x) + L_{gi} V_i(x) \cdot u_{i_j}^* \right) \psi_{ij}(y_i).$$

Here we used claim (iii) of Lemma 2, stating that $\psi_{ij}(y_i) = 0$ for $y_i \notin B_{ij}$.

Next, we observe that by definition, the inclusion $y_i = h_i(x_i) \in B_{ij}$ implies $x \in \overline{K_i(B_{ij})}$. Therefore $x \in \cap_{j: y_i \in B_{ij}} \overline{K_i(B_{ij})}$. Also, according to Lemma 3, for all $j$ such that $y_i \in B_{ij}$

$$L_{fi} V_i(x) + L_{gi} V_i(x) \cdot u_{i_j}^* < \Lambda_i \circ V(x) \text{ since } x \in \overline{K_i(B_{ij})}.$$
This allows us to conclude that
\[
L_f V_i(x) + L_y V_i(x) \cdot \Gamma_i(y_i) < \sum_{j: y_j = h_i(x)} \Lambda_j \circ V(x) \cdot \varphi_j(y_i) 
\leq \Lambda_i \circ V(x).
\]

We have shown that for any \( x \in \tilde{K}_i(K_i(S_\rho) \setminus \{0\}) \)
\[
L_f V_i(x) + L_y V_i(x) \cdot \Gamma_i(y_i) < \Lambda_i \circ V(x).
\]  
(12)

Also, from condition (iii) of Definition 2, we have for any \( x \in \tilde{K}_i(0) \setminus \{0\} \)
\[
L_f V_i(x) + L_y V_i(x) \cdot \Gamma_i(0) = L_f V_i(x) < \Lambda_i \circ V(x).
\]  
(13)

From (12) and (13), along the trajectory of \( \dot{x}_i = f_i(x) + g_i(x) \Gamma_i(y_i), \) \( V_i(x_i) \) satisfies
\[
\dot{V}_i(x_i) < \Lambda_i \circ V(x), \quad \forall x \in S_\rho \setminus \{0\}.
\]  
(14)

Therefore, together with the trivial case \( x = 0, V(x) \) satisfies the inequality \( \dot{V}(x) \leq \Lambda \circ V(x) \) for all \( x \in S_\rho \).

By condition (iv) of Definition 2 and using Lemma 1, we conclude that the origin of the closed-loop system is asymptotically stable.

To further characterize continuity properties of the decentralized feedback controller of Theorem 4, we give the following definition.

**Definition 3.** The system \( \mathcal{S}(S_\rho) \) with the decomposition \( \{\mathcal{S}_i\}_{i=1}^n \) satisfies the (decentralized) small control property if for each \( i = 1, \ldots, n \), there exists a continuous positive definite function \( \mu_i(y_i) \in \mathbb{R}_+ \), with the following property: For each \( y_i \in K_i(S_\rho) \setminus \{0\} \), there exists some \( u_i \in \mathbb{R}_+^m \) such that \( \|u_i\| < \mu_i(y_i) \) and
\[
L_f V_i(x) + L_y V_i(x) \cdot u_i < \Lambda_i \circ V(x), \quad \forall x \in \tilde{K}_i(y_i).
\]  
(15)

In (15) and elsewhere, we acknowledge that due to the fact that \( y_i \neq 0 \) in Definition 3, \( x = 0 \) is not contained in the set \( \tilde{K}_i(y_i) \).

We are now in a position to present a sufficient condition for the existence of a decentralized output feedback stabilizing controller expressed as a continuous function.

**Theorem 5.** The system \( \mathcal{S}(S_\rho) \) is decentralized output feedback stabilizable by a continuous decentralized output feedback control law \( \Gamma(y) \), if there is an OCVLF triple \( \{V, \Lambda, S_\rho\} \) for this system and moreover the small control property in the sense of Definition 3 holds.

**Proof.** First, we make the following observation. From (15) in Definition 3, for each \( y_i \in K_i(S_\rho) \setminus \{0\} \) with its corresponding \( u_i \), there exists an open ball \( B_i = B_{\gamma_i} \) centered at \( y_i \) and satisfying the following conditions:

(i) \( L_f V_i(x) + L_y V_i(x) \cdot u_i < \Lambda_i \circ V(x), \forall x \in \tilde{K}_i(B_i) \setminus \{0\} \), and

(ii) \( u_i \) satisfies \( \|u_i\| < \mu_i(y_i) \), \( \forall y_i \in B_i \).
Indeed, since the function \( \mu_i(\cdot) \) in Definition 3 is continuous, for each \( y_i \in K_i(S,\rho) \setminus \{0\} \), it follows from the condition \( \|u_i\| < \mu_i(y_i) \) that there exists a sufficiently small open ball \( U_{l_i}(y_i) \) centered at \( y_i \) such that \( \|u_i\| < \mu_i(y_i) \) for all \( y_i \in U_{l_i} \). Also, by condition (15) and Lemma 3 there exists an open ball \( U_{l_2} \) which is also centered at \( y_i \) and such that \( L_{q_i} V_i(x) + L_{q_i} V_i(x) \cdot u_i < \Lambda_i \circ V(x), \ \forall x \in \bar{K}_i(U_{l_2}) \).

Observe that both balls are centered at \( y_i \). This leads us to conclude that by choosing the smallest ball among \( U_{l_1}, U_{l_2} \) as \( B_i \), we will ensure the satisfaction of both properties (i) and (ii) stated at the beginning of the proof.

Next, in the same manner as was done in the proof of Theorem 4, a sequence of open balls \( \{B_j = B_i^{\psi_j}\}_{j=1}^{\infty} \) can be selected which satisfy (10) and also satisfy the corresponding versions of conditions (i) and (ii); that is, for every \( j \), there exists \( u_{ij}^{*} \) such that

\[
(i') \quad L_{q_i} V_i(x) + L_{q_i} V_i(x) \cdot u_{ij}^{*} < \Lambda_i \circ V(x), \ \forall x \in \bar{K}_i(B_i^{\psi}) \setminus \{0\};
\]

\[
(ii') \quad u_{ij}^{*} \text{ satisfies } \|u_{ij}^{*}\| < \mu_i(y_i), \ \forall y_i \in B_i^{\psi}.
\]

Using the above selected balls \( \{B_i^{\psi}\}_{i=1}^{\infty} \) and Lemma 2 we can now construct the controller \( \Gamma(y) \) with \( \Gamma_j : \mathbb{R}^k \mapsto \mathbb{R}^m \) defined in (11). It remains to show that the function \( \Gamma_i(y_i) \) is continuous. The stability of the closed-loop system can be shown by the same argument as in the proof of Theorem 4.

It follows from (11) that for any \( y_i \in K_i(S,\rho) \setminus \{0\} \)

\[
\Gamma(y_i) = \sum_{j:y_j \in B_j} u_{ij}^{*} \psi_j(y_i) + \sum_{j:y_j \notin B_j} u_{ij}^{*} \psi_j(y_i).
\]

(16)

Using statement (iii) of Lemma 2 concerning the partition of unity subordinate to the covering \( \{B_i^{\psi}\}_{i=1}^{\infty} \), we conclude that the second sum vanishes. On the other hand, for all \( j \) such that \( y_j \in B_i^{\psi} \), we have established that \( \|u_{ij}^{*}\| < \mu_i(y_i) \). Therefore,

\[
\|\Gamma_i(y_i)\| \leq \sum_{j:y_j \in B_j} \|u_{ij}^{*}\| \psi_j(y_i) \leq \mu_i(y_i)
\]

(17)

where the second inequality follows from the fact that \( \sum_{j:y_j \in B_j} \psi_j(y_i) \leq \sum_{j=1}^{\infty} \psi_j(y_i) = 1 \). Hence, \( \Gamma_i(y_i) \) is continuous at \( y_i = 0 \) and \( \Gamma_i(0) = 0 \) because \( \mu_i(y_i) \) is continuous and positive definite at \( y_i = 0 \). Furthermore, according to Lemma 2 the functions \( \psi_j \) are smooth. This implies that \( \Gamma_j(y_i) \) is continuous in \( K_i(S,\rho) \) and so is \( \Gamma(y) \). The proof is complete.

3.3. Constructive Decentralized Stabilization

As pointed out in the introduction, the controller synthesis based on partitioning the unity is not constructive due to the lack of regular efficient methods to compute a suitable precise partition. This comment serves as a motivation for the material in this section, which is focused on another synthesis procedure. Here, we specialize the result of Theorem 4 to a certain class of large-scale systems, which admits a decomposition of the control input into a pair of separate input channels as shown in Figure 2.

The introduced condition below is closely related to the vector dissipativity theory, see [8]. The main advantage of the approach undertaken in this section is that the decentralized design proposed here does not rely on the partitioning the unity. It is constructive if a vector dissipation-like condition is satisfied in addition to an existing OCVLF. Specifically, we will assume a candidate control law for the first channel that assists the construction of the second one.

The mathematical class of systems amenable to this design method is stated in the following condition which formulates the required vector dissipation-like property of the system.
Condition 1. The system $\mathcal{S}(\rho)$ has a decomposition $\{\mathcal{S}_i\}_{i=1}^N$, and its input $u_i$ is decomposed into $u_i = (u_{i1}, u_{i2})^T$, $u_{ij} \in \mathbb{R}^{m_j}$, $j = 1, 2$ and $m_{i1} + m_{i2} = m_i$ for $i = 1, \ldots, n$ (one of the components can be of zero dimension). Furthermore, there exist functions $V_i \in \mathbb{R}^{n_i}$, such that the closed-loop system is asymptotically stable at the origin. Due to the above mentioned vector dissipativity property and condition (i), this controller $u$ satisfies, along the trajectory of $\mathcal{S}_i$, the following dissipation-like inequality

$$\dot{V}_i(x_i) \leq W_i(x, u_{i1}) + p_{i1}(y_i, u_{i1}) + p_{i2}(y_i)u_{i2}$$  \hspace{1cm} (18)

for some function $W_i(x, u_{i1}) \in \mathbb{R}$ with $W_i(0, 0) = 0$ and smooth functions $p_{i1}(y_i, u_{i1}) \in \mathbb{R}$ with $p_{i1}(0, 0) = 0$ and $p_{i2}(y_i) \in \mathbb{R}^{m_i}$.

Here and in the following, along $\mathcal{S}_i$, it means $\dot{V}_i(x) = L_f V_i(x) + L_{\rho_i} V_i(x) \cdot u_i$.

(iii) There exists a smooth control law $u_{i1} = \phi_{i1}(y_i)$ vanishing at the origin such that

$$W_i(x, \phi_{i1}(y_i)) \leq \Lambda_i \circ V(x), \ \forall x \in S_\rho.$$  \hspace{1cm} (19)

(iv) $p_{i2}(y_i, \phi_{i1}(y_i)) \leq 0$ for any $y \in \{y_i \in K_i(S_\rho) : p_{i2}(y_i) = 0\}$.

Condition [1] alludes to a two-step controller design procedure for the system whose subsystems have a two-channel structure shown in Figure 2. First, using the function $W$ defined in (ii), a control low $u_{i1} = \phi_{i1}(y_i)$ is obtained to satisfy condition (19). This ensures that the composite system resulting from closing the inner feedback loop is vector dissipative with respect to the vector supply rate

$$S(u_{i2}, y_i) = p_{i1}(y_i, \phi_{i1}(y_i)) + p_{i2}(y_i)u_{i2}$$

see Definition 6 in [8] for the definition of the notion of vector dissipativity. This property is a natural extension of the corresponding scalar dissipativity property, see [11]. Next, at the second step, the design of $u_{i2}$ is carried out to ensure that, for any $y_i \neq 0$

$$S(\phi_{i2}(y_i), y_i) = p_{i1}(y_i, \phi_{i1}(y_i)) + p_{i2}(y_i)\phi_{i2}(y_i) < 0.$$  \hspace{1cm} (19)

Due to the above mentioned vector dissipativity property and condition (i), this controller $u_{i2} = \phi_{i2}(y_i)$ will ensure that the closed-loop system is asymptotically stable at the origin.
Remark 4. Note that this procedure and Condition 1 can also be adapted for a special case where the inner loop in Figure 2 is absent, and \( u_i = u_{i2} \). In this case, conditions (ii) and (iii) can be combined into one condition
\[
V_i(x_i) \leq \Lambda_i \circ V(x) + p_{i1}(y_i) + p_{i2}(y_i)u_i
\]
which imposes on the system the vector dissipativity requirement with the vector supply rate
\[
S(u_i, y_i) = p_{i1}(y_i) + p_{i2}(y_i)u_i.
\]

The following theorem crystallizes the above discussion in the form of a concrete stabilization algorithm.

Theorem 6. The composite system \( \mathcal{S}(S_p) \) is decentralized output feedback stabilizable if Condition 1 is satisfied. Furthermore, one stabilizing controller for this system is given by
\[
u_i = \phi_i(y_i) = \begin{bmatrix} \phi_{i1}(y_i) \\ \phi_{i2}(y_i) \end{bmatrix},
\]
(20)

In (20), \( \phi_{i1}(y_i) \) is the function with properties (iii), (iv) of Condition 1 and \( \phi_{i2}(y_i) \) is defined as follows
\[
\phi_{i2}(y_i) = \begin{cases} 0, & \text{if } y_i = 0; \\ \varphi_i(\tilde{p}_{i1}, p_{i2}, \sigma_i) \cdot p_{i2}^\top, & \text{otherwise} \end{cases},
\]
(21)

where \( \tilde{p}_{i1}(y_i) = p_{i1}(y_i, \phi_{i1}(y_i)) \), \( \sigma_i : \mathbb{R}^k \mapsto \mathbb{R}^k \) is a smooth nonnegative design function vanishing at \( y_i = 0 \), and
\[
\varphi_i(\tilde{p}_{i1}, p_{i2}, \sigma_i) = \begin{cases} 0, & \text{if } p_{i2}(y_i) = 0; \\ -\frac{\tilde{p}_{i1} + \sigma_i}{\|p_{i2}\|}, & \text{otherwise}. \end{cases}
\]
(22)

Proof. Consider the closed-loop system composed of \( \mathcal{S}(S_p) \) and (20). From conditions (ii) to (iv) of Condition 1 for each \( i = 1, \cdots, n \), \( V_i(x) \) satisfies, along the trajectory of \( \mathcal{S}_i \) with \( u_i = \phi_i(y_i) \), the following inequality
\[
\dot{V}_i(x) \leq W_i(x, \phi_{i1}(y_i)) + p_{i1}(y_i, \phi_{i1}(y_i)) + p_{i2}(y_i)\phi_{i2}(y_i)
\]
\[
\leq \begin{cases} \Lambda_i \circ V(x), & \text{if } y_i = 0 \text{ or } p_{i2}(y_i) = 0; \\ \Lambda_i \circ V(x) - \sigma_i(y_i), & \text{otherwise} \end{cases}
\]
\[
\leq \Lambda_i \circ V(x).
\]
Therefore, we have \( \dot{V}(x) \leq \Lambda \circ V(x) \). Then, using condition (i) of Condition 1 and the comparison principle, we conclude that \( \mathcal{S}(S_p) \) is decentralized output feedback stabilizable.

Remark 5. The function \( \sigma \) in Theorem 6 is the design parameter which provides a certain freedom at the second stage of the design algorithm. For example, the flexibility in selecting the design function \( \sigma_i(y_i) \) allows us to adopt the Sontag formula to obtain \( u_{i2} \) (see (23)). Indeed, selecting the design function \( \sigma_i(y_i) \) to be
\[
\sigma_i(y_i) = \sqrt{\tilde{p}_{i1}^2 + \|p_{i2}\|^2}
\]
(23)
yields the aforementioned controller \( u_{i2} \) for \( \mathcal{S}(S_p) \). The complete control law is then given by equations (21), (22), and (23). Alternatively, \( \sigma_i(y_i) \) can be chosen to be equal to zero.
Remark 6. We also note that the proposed theory can be extended to include vector storage functions $V_i$ as components of $V$. In this case, each inequality in Condition 1 needs to be understood as a coordinate-wise inequality. In some situations, particularly for large-scale systems composed of structured subsystems (i.e. when the system has a nested structure), such an extension may offer some convenience. Therefore, our treatment utilizing a scalar storage function for each subsystem is quite general.

4. Examples

We now present examples to illustrate the theory developed in the preceding sections. In the first example, we illustrate the situation where, while the OCVLF $V$ exists, output feedback stabilizability does not automatically follow using a trivial Lyapunov function $V = \sum_{i=1}^{n} V_i(x_i)$. The second example illustrates the construction of a decentralized controller by applying Theorem 6 with a vector storage function for each subsystem.

Example 1. Consider the following system
\[
\begin{align*}
\dot{x}_1 &= -x_1 + 2x_1x_3^2 \\
\dot{x}_2 &= x_1 - x_2 - x_1x_3 + u_2 \\
\dot{x}_3 &= x_1x_2 - x_3.
\end{align*}
\] (24)

In order to demonstrate that this system fits into the framework of our theory, we will treat each equation as a subsystem and define subsystems outputs to be $y_i = x_i, i = 1, 2, 3$. Also, we will assume that $u_1 = u_3 \equiv 0$.

We now demonstrate that the function
\[
V(x) = [V_1, V_2, V_3]^T, \quad V_i = \frac{1}{2}x_i^2, \quad i = 1, 2, 3
\] (25)
is a valid OCVLF for the system (24) in the sense of Definition 2 while
\[
\tilde{V}(x) = \sum_{i=1}^{3} V_i
\] (26)
fails to satisfy the conditions for output-feedback stabilizability in [26].

Suppose that $S_\rho = \{x \in \mathbb{R}^3 : \|x\| \leq 2\}$ in this example. It can be readily verified that in $S_\rho \setminus \{0\}$ the functions (25) satisfy the following conditions
\[
\begin{align*}
\dot{V}_1 &= -x_1^2 + 2x_1^2x_3^2 \\
&\leq -2V_1 + 8V_3 \\
&< -(2 - \epsilon)V_1 + \epsilon V_2 + (8 + \epsilon)V_3, \\
\dot{V}_2 &= x_2x_1 - x_2^2 - x_1x_2x_3 + x_2u_2 \\
&\leq \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_2^2 + 2x_2^2 + \frac{1}{2}x_3^2 + x_2u_2 \\
&= V_1 + 3V_2 + V_3 + x_2u_2 \\
&< (1 + \epsilon)V_1 + (3 + \epsilon)V_2 + (1 + \epsilon)V_3 + x_2u_2, \\
\dot{V}_3 &= -x_3^2 + x_1x_2x_3 \\
&\leq 4V_2 - V_3 \\
&< \epsilon V_1 + (4 + \epsilon)V_2 - (1 - \epsilon)V_3.
\end{align*}
\]
Letting \( u_2 = -\kappa y_2 \) yields the vector inequality

\[
\dot{V} < \Lambda V, \quad \Lambda = \begin{bmatrix}
-2 + \epsilon & \epsilon & 8 + \epsilon \\
1 + \epsilon & 3 + \epsilon - \kappa & 1 + \epsilon \\
\epsilon & 4 + \epsilon & -1 + \epsilon
\end{bmatrix}.
\]

Take \( \kappa = 33, \epsilon = 0.001 \). Then, \( -\Lambda \) is an M-matrix and \( \Lambda \) is Hurwitz. This observation verifies conditions (ii) and (iv) of Definition 2.

Next we verify condition (iii). For \( i = 1 \) and \( y_1 = x_1 = 0 \), we have \( \bar{K}_1(0) \setminus \{0\} = \{ x \in \mathbb{R}^3 : x_1 = 0, 0 < x_2^2 + x_3^2 \leq 4 \} \) and \( V_1(x_1) = 0 \) for all \( x \in \bar{K}_1(0) \setminus \{0\} \). This ensures that

\[
\dot{V}_1 = 0 < \Lambda_1 V, \quad \forall x \in \bar{K}_1(0) \setminus \{0\}
\]

where \( \Lambda_1 \) denotes the first row of the matrix \( \Lambda \). The cases \( i = 2, 3 \) are considered in a similar manner. Hence condition (iii) is satisfied, which leads us to conclude that the function \((25)\) is an OCVLF.

However, it can be seen that \((26)\) satisfies, along \((24)\)

\[
\dot{V} = -x_1^2 + 2x_1^2x_2^2 - x_3^2 + x_1y_2x_3 + y_2(x_1 - y_2 - x_1x_3) + y_2u.
\]

Clearly when \( y_2 = 0 \), we have \( \dot{V} = 0 \) at \( x_1 = x_3 = 1 \), that is, the function \((26)\) fails to satisfy Condition C) in \((26)\). Hence, the function \((26)\) cannot be used as a scalar control Lyapunov function for output-feedback stabilization of the system \((24)\).

**Example 2.** Adopted from [5], consider a network of controlled Lorenz-type systems described by

\[
\mathcal{S} : \begin{cases}
\dot{x}_{i1} = w_1(x_{i2} - x_{i1}) \\
\dot{x}_{i2} = w_3x_{i4} - x_{i2} - x_{i1}x_3 + u_i + \sigma_i \sum_{j=1, j \neq i}^n Hx_j \\
\dot{x}_{i3} = x_{i1}x_2 - w_2x_{i3} \\
y_i = x_{i2}, \quad i = 1, \ldots, n
\end{cases}
\]

where \( x_i = [x_{i1}, x_{i2}, x_{i3}]^T \) is the state variable of the \( i \)-th subsystem, \( u_i \) is the control input, \( H \) is a coupling matrix, and \( \sigma_i > 0 \) is the coupling strength. The problem here is to stabilize the system at the origin \( x = 0 \).

In this example, we focus on a \( y \)-coupled type network (see Remark 1 in [5]), i.e., we assume

\[
\sigma_i \sum_{j=1, j \neq i}^n Hx_j = \sigma_i \sum_{j=1, j \neq i}^n y_j.
\]

Also, we take \( w_1 = 10, w_2 = \frac{8}{3} \) and \( w_3 = 28 \). We now verify Condition [1] this will ensure that Theorem [6] is applicable to this problem.

First, for each subsystem, we construct a vector storage function. Similar to [30], notice that the \((x_{i1}, x_{i3})\) subsystem

\[
\begin{cases}
\dot{x}_{i1} = w_1(x_{i2} - x_{i1}) \\
\dot{x}_{i3} = x_{i1}x_2 - w_2x_{i3}
\end{cases}
\]

admits a Lyapunov-like function of the form

\[
V_{i1} = \frac{1}{2} x_{i1}^2 + \frac{1}{4} x_{i3}^2 + \frac{1}{2} x_{i3}^2.
\]
which satisfies the dissipation inequality
\[ \dot{V}_{i} \leq -c_{i1} V_{i1} + c_{i2}(y_{i}^{2} + y_{i}^{4}) \]  
for some real numbers \( c_{i1}, c_{i2} > 0 \). Then, we define a vector storage function candidate to be
\[ V_{i} = \begin{bmatrix} V_{i1} \\ V_{i2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} x_{i1}^{2} + \frac{1}{4} x_{i4}^{4} + \frac{1}{2} x_{i3}^{2} \\ y_{i}^{2} \end{bmatrix}. \]
Using (28), it can be established that for all \( x \in S_{\rho} \)
\[ \dot{V}_{i} \leq -c_{i1} V_{i1} + c_{i2}' V_{i2} + y_{i} u_{i} + \sum_{j=1, j \neq i}^{n} \sigma_{ij} V_{j2} \]
where \( \sigma_{ij}, c_{i2}' \) are some appropriately chosen positive real numbers. This condition can be written in the form of inequality (18) in Condition 1. To show that define \( W_{i} = W_{i}(x) \in \mathbb{R}^{2} \) as follows
\[ W_{i} = \begin{bmatrix} -c_{i1} & c_{i2}' \\ \frac{c_{i4}}{2} & -2k_{i} + c_{i2}' \end{bmatrix} \begin{bmatrix} V_{i1} \\ V_{i2} \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{j=1, j \neq i}^{n} \sigma_{ij} V_{j2} \end{bmatrix}. \]
Notice that the control input of each subsystem in this example is one-dimensional, that is, \( u_{i} \) is a zero-dimensional vector as explained in Remark 4. For that reason, \( W_{i} \) is defined as a function of \( x \) only. Also, let \( p_{i1} = k_{i} y_{i}^{2} \) and \( p_{i2} = y_{i} \). With these definitions, (29) can be written in the form of inequality (18):
\[ V_{i}(x_{i}) \leq W_{i}(x, 0) + \begin{bmatrix} 0 \\ p_{i1}(y_{i}, 0) \end{bmatrix} + \begin{bmatrix} 0 \\ p_{i2}(y_{i}, u_{i}) \end{bmatrix}. \]
This verifies conditions (ii) and (iv) of Condition 1.
Next, let us consider the remaining conditions (i) and (iii) of Condition [1]. It can be seen that the above defined function $W_i$ satisfies inequality [19], where $\Lambda(z)$ is a linear function $\Lambda_z$,

$$\Lambda = \begin{bmatrix}
  -c_{11} & c'_{12} & \cdots & 0 & 0 \\
  -2k_1 + c'_{12} & -c_{12} & \cdots & 0 & \varpi_{1n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & -c_{ni} & c'_{n2} \\
  0 & \varpi_{nn} & \cdots & c_{ni} & -2k_n + c'_{n2}
\end{bmatrix}.$$ 

This verifies condition (iii). Also, the matrix $-\Lambda$ is an M-matrix and $\Lambda$ can be made Hurwitz by choosing the constants $k_i$ to be sufficiently large [10]. Specifically, the matrix $\Lambda$ is Hurwitz if

$$k_i > \frac{c_{ii}}{2} + c'_{i2} + \sum_{j=1, j\neq i}^{n} \varpi_{ij}.$$ 

This selection of $k_i$ verifies property (i) of Condition [1].

We have verified that the candidate OCVLF

$$V = (V_1, \cdots, V_n)^T$$

satisfies Condition [1]. Hence, we conclude from Theorem [6] that any controller of the form

$$u_i = \phi_{ij}(y_i) = \begin{cases} 
0, & \text{if } y_i = 0; \\
-\frac{1}{y_i} \cdot (k_i y_i^2 + \sigma_i) \cdot y_i, & \text{otherwise}
\end{cases}$$

is a decentralized stabilizing controller for the system in this example. In particular, choosing $\sigma_i = 0$ yields a special controller as follows

$$u_i = -k_i y_i$$

which is clearly smooth in $y_i$ at the origin.

To confirm these findings, the system [27] was simulated with the controller obtained using $\sigma_i$ specified in [23] and also with the controller [32]. Simulation results for these controllers are shown in Figures 3 and 4 respectively. In both cases, the system parameters were selected to be $\rho = 2$, $k_i = 30$, $\varpi_i = 1$, $n = 3$, and the initial condition $(0.9, 0.1, 0.6; -0.6, 0.8, -0.5; -0.5, 0.7, 0.4)$ was chosen. From these two plots, one can observe that the first controller has a better performance, in particular, a better settling time, thanks to selecting the design function $\sigma_i$ given by [23].

To conclude this example, we point out that the proposed OCVLF design procedure may potentially be used to solve some other problems in [5] involving other types of coupling, as well as global stabilization or synchronization problems. This issue is left for future research.

5. Conclusion

A problem of decentralized stabilization of large-scale systems via static measurement feedback has been studied in this paper using the method of output control vector Lyapunov functions. We have proved a general result relating stabilizability of a large-scale system to the existence of such a function. We also extended the results in [18, 24, 25] to the case of output-feedback decentralized stabilization. A constructive design was then presented based on a vector dissipation-like condition. The proposed method has been applied to decentralized control of a network consisting of a set of coupled Lorenz-type systems.
Figure 4: State responses of the closed-loop system with $\sigma_i = 0$.

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