Abstract We show that a group presented by a labelled oriented tree presentation in which the tree has diameter at most three is an HNN extension of a finitely presented group. From results of Silver, it then follows that the corresponding higher dimensional ribbon knots admit minimal Seifert manifolds.

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1 Introduction

It is well known that every classical knot $k$ (knotted circle in $S^3$) bounds a compact orientable surface, known as a Seifert surface for the knot. A Seifert surface $\Sigma$ of minimal genus (among all Seifert surfaces for the given knot $k$) is called minimal, and satisfies the following property: the inclusion-induced map $\pi_1(\Sigma \setminus k) \to \pi_1(S^3 \setminus k)$ is injective.

For a higher dimensional knot, or more generally a knotted (closed, orientable) $n$–manifold $M$ in $S^{n+2}$, a Seifert manifold is defined to be a compact, orientable $(n+1)$–manifold $W$ in $S^{n+2}$, such that $\partial W = M$. A Seifert manifold $W$ for $M$ is defined to be minimal if the inclusion-induced map $\pi_1(W \setminus M) \to \pi_1(S^{n+2} \setminus M)$ is injective. In general, any $M$ will always admit Seifert manifolds, but not necessarily minimal Seifert manifolds. For example, Silver [13] has shown that, for any $n \geq 3$, there exist $n$–knots in $S^{n+2}$ with no minimal Seifert manifolds, and Maeda [9] has constructed, for all $g \geq 1$, a knotted surface of genus $g$ in $S^4$ that has no minimal Seifert manifold. Further examples of knotted tori in $S^4$ without minimal Seifert manifolds are constructed by Silver [16].

A theorem of Silver [14] says that, for $n \geq 3$, a knotted $n$–sphere $K$ in $S^{n+2}$ has a minimal Seifert manifold if and only if its group $G_K = \pi_1(S^{n+2} \setminus K)$ can be expressed as an HNN extension with a finitely presented base group. (It is standard that any higher knot group can be expressed as an HNN extension with a finitely generated base group.)
As Silver remarks, the proof of his theorem does not extend to the case $n = 2$. However, it remains a necessary condition for the existence of a minimal Seifert manifold that the group be an HNN extension with finitely presented base group. This applies also to knotted $n$–manifolds in $S^{n+2}$, a fact which is used implicitly by Maeda in the result mentioned above. It remains an open question whether every 2–knot in $S^4$ has a minimal Seifert manifold. This seems unlikely, however. For example Hillman [5], p. 139 shows that, provided the 3–dimensional Poincaré Conjecture holds, there is an infinite family of distinct 2–knots, all with the same group $G$, such that the commutator subgroup of $G$ is finite of order 3; and at most one of these knots can admit a minimal Seifert manifold.

In the present article we consider the case of higher dimensional ribbon knots, for which the existence of minimal Seifert manifolds is also an open question. Indeed, as we shall point out in the next section, higher ribbon knot groups are special cases of knot-like groups, in the sense of Rapaport [12], and Silver [15] has conjectured that every finitely generated HNN base for a knot-like group is finitely presented. It would therefore follow from Silver’s conjecture (and his Theorem) that every higher ribbon knot has a minimal Seifert manifold.

Now any higher ribbon knot group has a Wirtinger-like presentation that can be encoded in the form of a labelled oriented tree (LOT) [7]. Indeed the LOT encodes not only a presentation for the knot group, but the complete homotopy type of the knot complement. In [7] it was shown that, if the diameter of the tree is at most 3, then the group is locally indicable, and using this that the 2–complex model of the associated Wirtinger presentation is aspherical. A shorter proof of this fact is given in [8], where it is shown that the presentation is in fact diagrammatically aspherical.

In the present paper, we show that, under the same hypothesis on the diameter of the tree, the group is an HNN extension with finitely presented base group, and hence that the higher ribbon knot has a minimal Seifert manifold.

**Theorem 1.1** Let $\Gamma$ be a labelled oriented tree of diameter at most 3, and $G = G(\Gamma)$ the corresponding group. Then $G$ is an HNN extension with finitely presented base group.

**Corollary 1.2** Let $K$ be a ribbon $n$–knot in $S^{n+2}$, where $n \geq 3$, such that the associated labelled oriented tree has diameter at most 3. Then $K$ admits a minimal Seifert manifold.
The paper is arranged as follows. In section 2 we recall some basic definitions relating to LOTs and higher ribbon knots. In section 3 we prove some preliminary results about HNN bases for one-relator products of groups, which will allow us to simplify the original problem. In section 4 we reduce the problem to the study of minimal LOTs. In section 5 we construct a finitely generated HNN base $B$ for $G$, and describe a finite set of relators in these generators. In section 6 we prove some technical results about the structure of these relations, which we apply in section 7 to complete the proof of Theorem 1.1 by proving that this finite set is a set of defining relators for $B$. We close, in section 8, with a geometric description of our generators and relators for the HNN base, and a discussion of how this might be used to generalise Theorem 1.1.

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2 LOTs and higher ribbon knots

A labelled oriented tree (LOT) is a tree $\Gamma$, with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$, and initial and terminal vertex maps $\iota, \tau: E \to V$, together with an additional map $\lambda: E \to V$. For any edge $e$ of $\Gamma$, $\lambda(e)$ is called the label of $e$. In general, one can consider LOTs of any cardinality, but for the purposes of the present paper, every LOT will be assumed to be finite.

To any LOT $\Gamma$ we associate a presentation

$$\mathcal{P} = \mathcal{P}(\Gamma) : \langle V(\Gamma) \mid \iota(e)\lambda(e) = \lambda(e)\tau(e) \rangle$$

of a group $G = G(\Gamma)$, and hence also a 2-complex $K = K(\Gamma)$ modelled on $\mathcal{P}$. The 2-complex $K$ is a spine of a ribbon disk complement $D^4 \setminus k(D^2)$ [7], that is the complement of an embedded 2-disk in $D^4$, such that the radial function on $D^4$ composed with the embedding $k$ is a Morse function on $D^2$ with no local maximum. Conversely, any ribbon disk complement has a 2-dimensional spine of the form $K(\Gamma)$ for some LOT $\Gamma$.

By doubling a ribbon disk, we obtain a ribbon 2-knot in $S^4$, and by successively spinning we can obtain ribbon $n$-knots in $S^{n+2}$ for all $n \geq 2$. In each case the group of the knot is isomorphic to the fundamental group of the ribbon.
disk complement that we started with. Conversely, every ribbon \( n \)-knot (for \( n \geq 2 \)) can be constructed this way, so that higher ribbon knot groups and LOT groups are precisely the same thing.

Recall [12] that a group \( G \) is knot-like if it has a finite presentation with deficiency 1 (in other words, one more generator than defining relator), and infinite cyclic abelianisation. It is clear that every LOT group has these properties, so LOT groups are special cases of knot-like groups.

The diameter of a finite connected graph \( \Gamma \) is the maximum distance between two vertices of \( \Gamma \), in the edge-path-length metric. A key factor in our situation is the special nature of trees of diameter 3 or less. For any LOT \( \Gamma \) of diameter 0 or 1, it is easy to see that \( G(\Gamma) \) is infinite cyclic, so such LOTs are of little interest.

**Remark** Every tree of diameter 2 has a single non-extremal vertex. Every tree of diameter 3 has precisely 2 non-extremal vertices.

We recall from [7] that a LOT \( \Gamma \) is reduced if:

(i) for all \( e \in E \), \( \iota(e) \neq \lambda(e) \neq \tau(e) \);

(ii) for all \( e_1 \neq e_2 \in E \), if \( \lambda(e_1) = \lambda(e_2) \) then \( \iota(e_1) \neq \iota(e_2) \) and \( \tau(e_1) \neq \tau(e_2) \);

(iii) every vertex of degree 1 in \( \Gamma \) occurs as a label of some edge of \( \Gamma \).

For every LOT \( \Gamma \) there is a reduced LOT \( \Gamma' \) with the same group as \( \Gamma \), and the same or smaller diameter, so we may also restrict our attention to reduced LOTs.

A subgraph \( \Gamma' \) of a LOT \( \Gamma \) is admissible if \( \lambda(e) \in V(\Gamma') \) for all \( e \in E(\Gamma') \). If \( \Gamma' \) is connected and admissible, then it is also a LOT. A LOT is minimal if every connected admissible subgraph consists only of a single vertex.

If \( \Gamma \) is a LOT and \( A \subseteq V(\Gamma) \), we define the span of \( A \) (in \( \Gamma \)) to be the smallest subgraph \( \Gamma' \) of \( G \) such that:

(i) \( A \subseteq V(\Gamma') \); and

(ii) if \( e \in E(\Gamma) \) with \( \lambda(e) \in V(\Gamma') \) and at least one of \( \iota(e) \), \( \tau(e) \) belongs to \( V(\Gamma') \), then \( e \in E(\Gamma') \).

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We write \( \text{span}(A) \) for the span of \( A \), and say that \( A \) spans, or generates \( \Gamma' \) if \( \Gamma' = \text{span}(A) \). The following is essentially Proposition 4.2 of [7].

**Lemma 2.1** If \( \Gamma \) is a LOT spanned by \( A \), then \( \mathcal{P}(\Gamma) \) is Andrews–Curtis equivalent to a presentation with generating set \( A \). If \( \Gamma' \) is an admissible subgraph of \( \Gamma \) with \( V(\Gamma') \subseteq A \), then the presentation may be chosen to contain \( \mathcal{P}(\Gamma') \), and the Andrews–Curtis moves can be taken relative to \( \mathcal{P}(\Gamma') \).

**Corollary 2.2** If \( \Gamma \) is a LOT spanned by two vertices, then \( G(\Gamma) \) is a torsion-free one-relator group.

**Proof** Let \( A \) be a set of two vertices spanning \( \Gamma \). Then \( \mathcal{P}(\Gamma) \) is Andrews–Curtis equivalent to a presentation \( \langle A \mid R \rangle \). Since \( \mathcal{P}(\Gamma) \) has deficiency 1, the same is true of the equivalent presentation \( \langle A \mid R \rangle \). In other words \( |R| = 1 \), and \( G(\Gamma) \) is a one-relator group. But the abelianisation \( G_{ab} \) of \( G \) is infinite cyclic, so the relator \( r \in R \) cannot be a proper power, and so \( G \) is torsion-free.

We will require the following generalisation of Corollary 2.2. Recall that a one-relator product of two groups \( A, B \) is the quotient of the free product \( A \ast B \) by the normal closure of a single word \( w \), called the relator.

**Corollary 2.3** If \( \Gamma \) is a LOT spanned by \( V(\Gamma') \cup \{x\} \), where \( \Gamma' \) is an admissible subgraph of \( \Gamma \) and \( x \) is a vertex in \( V(\Gamma) \setminus V(\Gamma') \), then \( G(\Gamma) \) is a one-relator product of \( G(\Gamma') \) and \( \mathbb{Z} \), where the relator is not a proper power.

**Proof** Let \( A = V(\Gamma') \cup \{x\} \) and apply the Theorem. Then \( \mathcal{P}(\Gamma) \) is equivalent, relative to \( \mathcal{P}(\Gamma') \), to a presentation \( \mathcal{Q} \) with generating set \( A \) and containing \( \mathcal{P}(\Gamma') \). Now each of \( \mathcal{P}(\Gamma) \), \( \mathcal{P}(\Gamma') \) and \( \mathcal{Q} \) has deficiency 1. Moreover, \( \mathcal{Q} \) has one more generator than \( \mathcal{P}(\Gamma') \), so \( \mathcal{Q} \) also has one more defining relator than \( \mathcal{P}(\Gamma') \). It follows that \( G(\Gamma) \) is a one relator product of \( G(\Gamma') \) with the infinite cyclic group \( \langle x \rangle \). Finally, since the abelianisations of \( G(\Gamma) \), \( G(\Gamma') \) and \( \langle x \rangle \) are all infinite cyclic, it follows that the relator cannot be a proper power.

### 3 One-relator groups and one-relator products

The following result is merely a summary of some well-known properties of one-relator groups, which have useful applications to our situation. Recall that a group \( G \) is locally indicable if, for every nontrivial, finitely generated subgroup \( H \) of \( G \), there exists an epimorphism \( H \to \mathbb{Z} \).
Theorem 3.1 Let $G$ be a finitely generated one-relator group. Then

(i) $G$ is either a finite cyclic group, or an HNN extension of a finitely presented, one-relator group (with shorter defining relator);

(ii) if the defining relator of $G$ is not a proper power, then $G$ is locally indicable.

Proof See [11] and [3] respectively.

In order to complete the process of reducing ourselves to a simple special case, we require a generalisation of the above theorem to one-relator products. Suppose that $A$ and $B$ are locally indicable groups, and $N = N(w)$ is the normal closure in $A * B$ of a cyclically reduced word $w$ of length at least 2 that is not a proper power. Then the one-relator product $G = (A * B)/N$ is known [6] to be locally indicable. We show also that $G$ has a finitely presented HNN base, provided that $A$ and $B$ also have this property.

Theorem 3.2 Let $G = (A * B)/N(w)$ be a one-relator product of two finitely presented, locally indicable groups $A$ and $B$, each of which has a finitely presented HNN base. Suppose also that $G^{ab}$ is infinite cyclic, with each of the natural maps $A^{ab} \to G^{ab}$ and $B^{ab} \to G^{ab}$ an isomorphism. Then $G$ is a finitely presented, locally indicable group with a finitely presented HNN base.

Remark The condition on $G^{ab}$ in this theorem is unnecessary for the proof that $G$ has a finitely presented HNN base. It can be removed at the expense of a less straightforward proof. However the condition does hold for all the groups that we are considering in this paper, so there is no loss of generality for us in imposing that condition. The condition also ensures that $w$ cannot be a proper power, so that $G$ is locally indicable by the results of [6].

Proof A presentation for $G$ can be obtained by taking the disjoint union of finite presentations for $A$ and for $B$, and imposing the single additional relation $w = 1$. Hence $G$ is finitely presented. As pointed out in the remark above, $w$ cannot be a proper power, so $G$ is locally indicable by [6]. It remains only to prove that $G$ has a finitely presented HNN base.

Let

$$A = \langle A_0, a | a^{-1}ga = \alpha(g) \ (g \in A_1) \rangle$$

and

$$B = \langle B_0, b | b^{-1}hb = \beta(h) \ (h \in B_1) \rangle$$
be HNN presentations for $A$ and $B$ with finitely presented bases $A_0$ and $B_0$ respectively. Since $A$ and $B$ are finitely presented, it follows also that the associated subgroups $A_1$ and $B_1$ are finitely generated.

The commutator subgroup $G'$ of $G$ can be expressed in the form

$$\left(A' \ast B' \ast \langle c_n \ (n \in \mathbb{N})\rangle \right) \ast N\{w_n \ (n \in \mathbb{N})\},$$

where $c_n = a^{n+1}b^{-1}a^{-n}$ and $w_n = a^{-n}wa^n$.

Now $A'$ is an infinite stem product

$$\cdots (a^{-1}A_0a) \ast A_0 \ast (aA_0a^{-1}) \ast (a^{-1}A_1a) \ast A_1 \ast \cdots$$

Since $A_0$ is finitely presented and $A_1$ is finitely generated, the subgroup

$$\langle a^{-k}A_0a^k \rangle \ast \cdots \ast \langle a^{-k}A_1a^k \rangle \ast \cdots \ast \langle a^kA_0a^{-k} \rangle$$

is finitely presented for each $k$. Moreover it is also an HNN base for $A$. Replacing $A_0$ by this subgroup, for any sufficiently large $k$, we may assume that $w_0 \in A_0 \ast B' \ast \langle c_n \ (n \in \mathbb{N})\rangle$.

Similarly, possibly after replacing $B_0$ by a sufficiently large finitely presented HNN base for $B$, we may assume that $w_0 \in A_0 \ast B_0 \ast \langle c_n \ (n \in \mathbb{N})\rangle$. Now let $\mu$ and $\nu$ be the least and greatest indices $i$ such that $c_i$ occurs in $w_0$. (Note that at least one $c_i$ occurs in $w_0$, for otherwise $w_0 \in A_0 \ast B_0$, so $w \in A' \ast B'$, whence $G^{ab} \cong A^{ab} \times B^{ab} \not\cong \mathbb{Z}$, a contradiction.) Define $G_0 = (A_0 \ast B_0 \ast \langle c_\mu, \ldots, c_\nu \rangle) / N\{w_0\}$ and $G_1 = A_0 \ast B_0 \ast \langle c_\mu, \ldots, c_\nu-1 \rangle$, and observe that $G_0$ is a finitely presented HNN base for $G$, with associated subgroup $G_1$. \qed 

### 4 Reduction of the problem

Recall from section 2 that a LOT $\Gamma$ is minimal if it contains no admissible subtree with more than one vertex. In this section we reduce the proof of the main theorem to the case of a minimal LOT of diameter 3, using the results of section 3. The key point is that a non-minimal LOT can be obtained from a minimal admissible subtree by successively expanding to the span of the existing tree with one extra vertex. By Corollary 2.3, this construction corresponds at the group level to taking a one-relator product of a given group with an infinite cyclic group.
Lemma 4.1  Let $\Gamma$ be a LOT of diameter at most 3, containing a proper admissible subtree with more than one vertex. Then there is such an admissible subtree $\Gamma'$ and a vertex $x \in V(\Gamma) \setminus V(\Gamma')$ such that $\Gamma$ is spanned by $V(\Gamma') \cup \{x\}$.

Proof  Suppose first that some extremal vertex $x$ of $\Gamma$ does not occur as a label of any edge of $\Gamma$. In this case we take $\Gamma'$ to consist of $\Gamma$ with the vertex $x$ and the edge incident to $x$ removed. Clearly $\Gamma'$ is connected, so a subtree of $\Gamma$. Since $x$ is not the label of any edge in $E(\Gamma')$, it follows that $\Gamma'$ is admissible. Moreover $\Gamma$ is spanned by $V(\Gamma) = V(\Gamma') \cup \{x\}$, as required.

We may therefore assume that every extremal vertex of $\Gamma$ occurs at least once as the label of an edge of $\Gamma$.

Next suppose that $\Gamma$ has a proper admissible subtree that contains all the non-extremal vertices of $\Gamma$. Let $\Gamma'$ be a maximal such admissible subtree. The vertices in $V(\Gamma) \setminus V(\Gamma')$ are all extremal in $\Gamma$, so occur as labels of edges of $\Gamma$. But since $\Gamma'$ is admissible, no such vertex can be a label of an edge of $\Gamma'$. Since the finite sets $V(\Gamma) \setminus V(\Gamma')$ and $E(\Gamma) \setminus E(\Gamma')$ have the same cardinality, it follows that each vertex in $V(\Gamma) \setminus V(\Gamma')$ is the label of precisely one edge in $E(\Gamma) \setminus E(\Gamma')$. In turn, this edge has precisely one endpoint in $V(\Gamma) \setminus V(\Gamma')$, so we can define a permutation $\sigma$ on $V(\Gamma) \setminus V(\Gamma')$ by defining $\sigma(x)$ to be the extremal endpoint of the unique edge labelled $x$, for all $x \in V(\Gamma) \setminus V(\Gamma')$. Now fix some vertex $x \in V(\Gamma) \setminus V(\Gamma')$, let $t$ be the size of the orbit of $\sigma$ that contains $x$, and define $x_i = \sigma^i(x)$, $i = 1, \ldots, t$. Now $\Delta = \text{span}(V(\Gamma') \cup \{x\})$ contains the vertex $x = x_t$, together with any non-extremal vertex of $\Gamma$. Hence $\Delta$ contains the edge labelled $x_t$, and hence its endpoint $x_1$. Similarly $\Delta$ contains $x_2, \ldots, x_{t-1}$, as well as the edges labelled $x_1, \ldots, x_{t-1}$. On the other hand, The vertices $x_1, \ldots, x_t$, the edges labelled by them, and the vertices and edges of $\Gamma'$ together form an admissible subtree of $\Gamma$, which by maximality of $\Gamma'$ must be the whole of $\Gamma$. Hence $\Delta = \Gamma$, in other words $\Gamma$ is spanned by $V(\Gamma') \cup \{x\}$.

Finally, suppose that no proper admissible subtree of $\Gamma$ contains all the non-extremal vertices of $\Gamma$. In particular, $\Gamma$ must have more than one non-extremal vertex, so has diameter 3. By hypothesis, there is a proper admissible subtree $\Gamma'$ of $\Gamma$ that contains more than one vertex. Hence $\Gamma'$ contains precisely one of the two nonextremal vertices of $\Gamma$, say $u$. As an abstract graph, $\Gamma$ is the union of $\Gamma'$ with another tree $\Gamma''$, such that $\Gamma' \cap \Gamma'' = \{u\}$. Note that $\Gamma''$ contains both of the non-extremal vertices of $\Gamma$, so cannot be an admissible subtree, by hypothesis. Hence at least one edge $f$ of $\Gamma''$ is labelled by a vertex $a$ of $\Gamma'$ (other than $u$). Let $e$ be the edge of $\Gamma$ that joins the two non-extremal vertices $u, v$, and let $\Delta = \text{span}(V(\Gamma') \cup \{\lambda(e)\})$. Then $\Delta$ contains $\Gamma'$ and the edge $e$.  

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and hence \( v \), and hence the edge \( f \). Each extremal vertex of \( \Delta \) is the label of an edge of \( \Gamma \), and hence of \( \Delta \), since \( \Delta \) contains at least one endpoint (namely \( u \) or \( v \)) of every edge of \( \Gamma \). Moreover there are \( |E(\Gamma')| + 1 \) edges of \( \Delta \) labelled by the \( |V(\Gamma')| = |E(\Gamma')| + 1 \) vertices of \( \Gamma' \), so an easy counting argument shows that there must be at least \( |V(\Delta)| - 1 \) edges in \( \Delta \). In other words \( \Delta \) is a tree, so the whole of \( \Gamma \). In other words \( \Gamma \) is spanned by \( V(\Gamma') \cup \{ \lambda(e) \} \).

**Remark** If \( \Gamma \) is a minimal LOT of diameter 2, then the above argument still applies (to the subtree consisting of only the unique non-extremal vertex). In this case we see that the permutation \( \sigma \) is transitive, and that \( \Gamma \) is spanned by two vertices.

**Lemma 4.2** Let \( \Gamma \) be a minimal LOT of diameter 3, and let \( u, v \) be the two non-extremal vertices of \( \Gamma \). Then one of the following holds:

(i) One of \( u, v \) is a label in \( \Gamma \), and \( \Gamma \) is spanned by \( \{u, v\} \);

(ii) Some vertex \( a \) occurs twice as a label in \( \Gamma \), and \( \Gamma \) is spanned by \( \{a, u, v\} \).

**Proof** By minimality of \( \Gamma \), every extremal vertex of \( \Gamma \) occurs as a label. There are \( |V| - 2 \) extremal vertices, and \( |V| - 1 \) edges, so either one of \( u, v \) occurs as a label or some unique extremal vertex \( a \) occurs twice as a label. Note that every edge of \( \Gamma \) is incident to at least one of \( u, v \), so if \( u, v \in A \subset V \) then every edge labelled by a vertex of \( \text{span}(A) \) is an edge of \( \text{span}(A) \).

(i) Suppose that \( u \) occurs as a label, and let \( \Gamma' = \text{span}(\{u, v\}) \). If \( \Gamma' \) has \( k + 2 \) vertices \( u, v, x_1, \ldots, x_k \), then \( x_1, \ldots, x_k \) are all extremal in \( \Gamma \), so each of \( u, x_1, \ldots, x_k \) is a label of an edge of \( \Gamma \), which must therefore be an edge of \( \Gamma' \). Hence \( \Gamma' \) has at least \( k - 1 \) edges, so is connected. By minimality of \( \Gamma \) we have \( \Gamma = \Gamma' = \text{span}(\{u, v\}) \).

(ii) Suppose that an extremal vertex \( a \) appears twice as a label, and let \( \Gamma' = \text{span}(\{a, u, v\}) \). If \( \Gamma' \) has \( k + 3 \) vertices \( a, u, v, x_1, \ldots, x_k \), then each of \( x_1, \ldots, x_k \) is extremal, so the label of an edge of \( \Gamma \), while \( a \) is the label of 2 edges of \( \Gamma \). Each of these \( k + 2 \) edges is an edge of \( \Gamma' \), so \( \Gamma' \) is connected, and by minimality again we have \( \Gamma = \Gamma' = \text{span}(\{a, u, v\}) \). □

**Corollary 4.3** If \( \Gamma \) is either a minimal LOT of diameter 2, or a minimal LOT of diameter 3 in which no vertex occurs twice as a label, then \( G(\Gamma) \) is a locally indicable group with a finitely presented HNN base.
Proof By Lemma 4.2 or the remark following Lemma 4.1, \( \Gamma \) is spanned by two vertices. Hence \( G = G(\Gamma) \) is a 2-generator, one-relator group. Since \( G_{ab} \) is infinite cyclic, \( G \) is not finite, and the relator of \( G \) cannot be a proper power. The result follows immediately from Theorem 3.1.

Using the above results, we can reduce our problem to the case of a minimal LOT of diameter 3 that is not spanned by two vertices. In particular, some extremal vertex must occur twice as a label.

**Corollary 4.4** If the group of every reduced, minimal LOT of diameter 3 which is not spanned by two vertices is locally indicable with finitely presented HNN base, then the same is true for every LOT of diameter 3 or less.

Recall [7] that the initial graph \( I(\Gamma) \) of \( \Gamma \) is the graph with the same vertex and edge sets as \( \Gamma \), but with incidence maps \( i, \lambda \). Similarly the terminal graph \( T(\Gamma) \) of \( \Gamma \) has the same vertex and edges sets as \( \Gamma \), but incidence maps \( \lambda, \tau \). It was shown in [7] that the commutator subgroup of \( G(\Gamma) \) is locally free if either \( I(\Gamma) \) or \( T(\Gamma) \) is connected. (If \( I(\Gamma) \) and \( T(\Gamma) \) are both connected, then \( G(\Gamma)' \) is free of finite rank.) In particular, any finitely generated HNN base for \( G(\Gamma) \) is free, so automatically finitely presented.

Hence we can concentrate attention on the case of a minimal LOT \( \Gamma \) of diameter 3, not spanned by any two of its vertices, such that neither \( I(\Gamma) \) nor \( T(\Gamma) \) is connected. Our next result gives a detailed description of the structure of \( I(\Gamma) \). In particular it will show us that \( I(\Gamma) \) has precisely two connected components, one containing each of the nonextremal vertices of \( \Gamma \). A similar statement holds for \( T(\Gamma) \).

**Lemma 4.5** Let \( \Gamma \) be a minimal LOT of diameter 3, with nonextremal vertices \( u \) and \( v \), and an extremal vertex \( a \) that occurs twice as a label of edges of \( \Gamma \). Then:

(i) \( u \) and \( v \) are sources in \( I(\Gamma) \);

(ii) no vertex other than \( u \) or \( v \) is the initial vertex of more than one edge of \( I(\Gamma) \);

(iii) \( a \) is the terminal vertex of precisely two edges of \( I(\Gamma) \);

(iv) each vertex other than \( a, u, v \) is the terminal vertex of precisely one edge of \( I(\Gamma) \);

(v) any directed cycle in \( I(\Gamma) \) contains \( a \);

(vi) each component of \( I(\Gamma) \) contains at least one of \( u, v \);
(vii) \( I(\Gamma) \) has at most two connected components.

**Proof**  
(i) Since \( \lambda(e) \neq u \) for all \( e \in E(\Gamma) \), \( u \) is not the terminal vertex of any edge in \( I(\Gamma) \), in other words \( u \) is a source. Similarly \( v \) is a source in \( I(\Gamma) \).

(ii) Any vertex \( x \) of \( \Gamma \), with the exception of \( u \) and \( v \), is extremal in \( \Gamma \), so the initial vertex of at most one edge of \( \Gamma \). Hence \( x \) is also the initial vertex of at most one edge in \( I(\Gamma) \).

(iii) \( a = \lambda(e) \) for precisely two edges \( e \in E(\Gamma) \).

(iv) If \( x \in V(\Gamma) \setminus \{a, u, v\} \) then \( x = \lambda(e) \) for precisely one edge \( e \in E(\Gamma) \).

(v) Suppose \((e_1, e_2, \ldots, e_n)\) is a directed cycle in \( I(\Gamma) \). Then there are vertices \( x_1, \ldots, x_n \in V(\Gamma) \) with \( x_i = \iota(e_i) \) for all \( i \), \( \lambda(e_i) = x_{i+1} \) for \( i < n \), and \( \lambda(e_n) = x_1 \). Now each \( x_i \) is extremal since it occurs as a label. If no \( x_i \) is equal to \( a \) then we can remove the vertices \( x_1, \ldots, x_n \) and the edges \( e_1, e_2, \ldots, e_n \) from \( \Gamma \) to form a connected, admissible subgraph \( \Gamma' \) that contains at least three vertices \((a, u, v)\). This contradicts the minimality of \( \Gamma \), and so \( x_i = a \) for some \( i \), as claimed.

(vi) By (iv) if \( x \notin \{a, u, v\} \) then \( x \) is the terminal vertex in \( I(\Gamma) \) of a unique edge. If the initial vertex of this edge is not one of \( a, u, v \) then it also is the terminal vertex of a unique edge. Continuing in this way, we can construct a directed path that ends at \( x \), and either begins at one of \( a, u, v \) or contains a cycle. By (v) any directed cycle contains \( a \), so in any case we have a directed path from one of \( a, u, v \) to \( x \). It suffices therefore to find a path in \( I(\Gamma) \) from \( u \) or \( v \) to \( a \). But \( a \) is the terminal vertex in \( I(\Gamma) \) of precisely two edges, with initial vertices \( x_1 \) and \( x_2 \) say. Now apply the above argument to each of \( x_1, x_2 \). If there is a path from \( u \) or \( v \) to \( x_1 \) or \( x_2 \) then we are done. Otherwise there are directed paths from \( a \) to each of \( x_1, x_2 \). Neither \( u \) nor \( v \) can belong to these paths, since they are sources in \( I(\Gamma) \). But then from (ii) it follows that there is at most one directed path of any given length beginning at \( a \), whence \( x_1 = x_2 \), a contradiction. Hence there is a directed path in \( I(\Gamma) \) from \( u \) or \( v \) to \( a \), as claimed.

(vii) This follows immediately from (vi). \( \square \)

A similar result holds for \( T(\Gamma) \).

**Lemma 4.6** Let \( \Gamma \) be a minimal LOT of diameter 3, with nonextremal vertices \( u \) and \( v \), and an extremal vertex \( a \) that occurs twice as a label of edges of \( \Gamma \). Then:
(i) $u$ and $v$ are sinks in $T(\Gamma)$;
(ii) no vertex other than $u$ or $v$ is the terminal vertex of more than one edge of $T(\Gamma)$;
(iii) $a$ is the initial vertex of precisely two edges of $T(\Gamma)$;
(iv) each vertex other than $a, u, v$ is the initial vertex of precisely one edge of $T(\Gamma)$;
(v) any directed cycle in $T(\Gamma)$ contains $a$;
(vi) each component of $T(\Gamma)$ contains at least one of $u, v$;
(vii) $T(\Gamma)$ has at most two connected components.

**Corollary 4.7** Suppose that $\Gamma$ is a reduced, minimal LOT of diameter 3, which is not spanned by two vertices, and such that neither $I(\Gamma)$ nor $T(\Gamma)$ is connected. Then

(i) There is a unique extremal vertex $a$ of $\Gamma$ that is the label of two distinct edges of $\Gamma$. One of these edges has an extremal initial vertex, and the other has an extremal terminal vertex.

(ii) $I(\Gamma)$ has precisely two connected components, each containing one of the two nonextremal vertices $u, v$ of $\Gamma$.

(iii) There is a unique cycle in $I(\Gamma)$, which is either a directed cycle containing $a$, or consists of two directed paths (one of length 1, the other of length at least 2), from $u$ or $v$ to $a$.

(iv) $T(\Gamma)$ has precisely two connected components, each containing one of the two nonextremal vertices $u, v$ of $\Gamma$.

(v) There is a unique cycle in $T(\Gamma)$, which is either a directed cycle containing $a$, or consists of two directed paths (one of length 1, the other of length at least 2), from $a$ to $u$ or $v$.

(vi) The cycles in $I(\Gamma)$ and $T(\Gamma)$ are not both directed.

**Proof** (i) We already know that there is an extremal vertex $a$ occurring twice as a label, by Lemma 4.2, since otherwise $\Gamma$ can be spanned by two vertices. We also know that $a$ is unique, since every extremal vertex occurs at least once as a label. Now suppose that neither of the edges labelled $a$ has extremal initial vertex. The initial vertices of these two edges must be distinct, since $\Gamma$ is reduced, and so must be the two nonextremal vertices $u, v$ of $\Gamma$. But then there are edges of $I(\Gamma)$ from both $u$ and $v$ to $a$. Hence $u$ and $v$ belong to the same connected component of $I(\Gamma)$. By Lemma 4.5, (vi) it follows that $I(\Gamma)$ is connected, a contradiction.
A similar contradiction arises if neither edge has an extremal terminal vertex.

(ii) This is just a restatement of Lemma 4.5, (vi), together with the hypothesis that $I(\Gamma)$ is not connected.

(iii) Since $I(\Gamma)$ has the same vertex and edge sets as $\Gamma$, it has the same euler characteristic, namely 1. Since $I(\Gamma)$ has two components, it follows that $H_1(\Gamma) \cong \mathbb{Z}$, so there is a unique cycle in $I(\Gamma)$. If this cycle is directed, then it must contain $a$, by Lemma 4.5, (v). Otherwise it must contain at least two vertices at which the orientation of the edges of the cycle changes. This is possible only at a vertex which is either the initial vertex of at least two edges or the terminal vertex of at least two edges, and by Lemma 4.5 the only such vertices are $a, u, v$. Let us assume that $a$ is in the same component of $I(\Gamma)$ as $u$. Then the cycle must contain both $a$ and $u$, and indeed must consist of two directed paths from $u$ to $a$. By uniqueness of the cycle (or directly from Lemma 4.5), we see that there only two directed paths in $I(\Gamma)$ from $u$ to $a$. Moreover, precisely one of these paths is of length 1, since precisely one of the edges of $\Gamma$ labelled $a$ has a nonextremal initial vertex.

(iv) Similar to (ii).

(v) Similar to (iii).

(vi) If the cycle in $I(\Gamma)$ is directed, then there is an edge of $I(\Gamma)$ with initial vertex $a$, and so also there is an edge of $\Gamma$ with initial vertex $a$. Similarly, if the cycle in $T(\Gamma)$ is directed, then there is an edge of $\Gamma$ with terminal vertex $a$. Since $a$ is extremal in $\Gamma$, these cannot both occur.

5 Construction of the HNN base

In this section, we construct a presentation of a group that will turn out to be an HNN base for $G$. As a first step, we fix names for the various vertices of $\Gamma$. Throughout we make the following assumptions:

- $\Gamma$ is a minimal LOT of diameter 3, which cannot be spanned by fewer than three vertices.
- The non-extremal vertices of $\Gamma$ are $u$ and $v$.
- The unique vertex of $\Gamma$ that appears twice as a label is $a$.
- Of the edges labelled $a$, one has its initial vertex in $\{u, v\}$ and its terminal vertex extremal, while the other has its initial vertex extremal and its terminal vertex in $\{u, v\}$. 

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Neither $I(\Gamma)$ nor $T(\Gamma)$ is connected.

We know from Lemma 4.2 that $\Gamma$ is then spanned by $\{a, u, v\}$. Let $\Delta$ denote the subtree of $\Gamma$ whose vertex set is $\{a, u, v\}$. We give inductive definitions of two sequences $\{b_1, b_2, \ldots, b_P\}$ and $\{c_1, c_2, \ldots, c_Q\}$ of vertices of $\Gamma$, and two sequences $\{e_0, \ldots, e_P\}, \{f_0, \ldots, f_Q\}$ of edges of $\Gamma$ as follows.

Define $e_0$ to be the edge of $\Gamma$ whose label is $a$ and whose terminal vertex is in $\{u, v\}$. For $i \geq 0$, assume inductively that $e_i$ has been defined. If $e_i$ is an edge of $\Delta$, then we define $P = i$ and stop the construction of the sequences $\{b_1, b_2, \ldots, b_P\}$ and $\{e_0, \ldots, e_P\}$. Otherwise $e_i$ joins one of $\{u, v\}$ to an extremal vertex other than $a$, and we define $b_{i+1}$ to be that extremal vertex, and $e_{i+1}$ to be the unique edge of $\Gamma$ labelled by $b_{i+1}$.

Similarly, define $f_0$ to be the edge of $\Gamma$ whose label is $a$ and whose initial vertex is in $\{u, v\}$. For $i \geq 0$, assume inductively that $f_i$ has been defined. If $f_i$ is an edge of $\Delta$, then we define $Q = i$ and stop the construction of the sequences $\{c_1, c_2, \ldots, c_Q\}$ and $\{f_0, \ldots, f_Q\}$. Otherwise $f_i$ joins one of $\{u, v\}$ to an extremal vertex other than $a$, and we define $c_{i+1}$ to be that extremal vertex, and $f_{i+1}$ to be the unique edge labelled by $c_{i+1}$.

Note that the $P+Q+3$ vertices $\{u, v, a, b_1, \ldots, b_P, c_1, \ldots, c_Q\}$ and the $P+Q+2$ edges $\{e_0, \ldots, e_P, f_0, \ldots, f_Q\}$ together form an admissible subgraph of $\Gamma$, which has euler characteristic 1 and hence is connected, and hence by minimality of $\Gamma$ must be the whole of $\Gamma$. In other words

$$V = V(\Gamma) = \{u, v, a, b_1, \ldots, b_P, c_1, \ldots, c_Q\},$$

and

$$E = E(\Gamma) = \{e_0, \ldots, e_P, f_0, \ldots, f_Q\}.$$

We also introduce the following notation. For $i = 1, \ldots, P$, $x_i$ denotes the unique non-extremal vertex of $\Gamma$ (ie $x_i \in \{u, v\}$) incident with the edge $e_{i-1}$. For $i = 1, \ldots, Q$, $y_i$ denotes the unique non-extremal vertex of $\Gamma$ incident with the edge $f_{i-1}$. In other words, $x_i$ is the vertex adjacent to $b_i$ in $\Gamma$, and $y_i$ is the vertex adjacent to $c_i$.

**Lemma 5.1**

(i) If $x_2 = \ldots = x_P = u$, then $x_1 = v$ and $e_P$ is incident at $v$.

(ii) If $x_2 = \ldots = x_P = v$, then $x_1 = u$ and $e_P$ is incident at $u$.

(iii) If $y_2 = \ldots = y_Q = u$, then $y_1 = v$ and $f_Q$ is incident at $v$.

(iv) If $y_2 = \ldots = y_Q = v$, then $y_1 = u$ and $f_Q$ is incident at $u$. 

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Proof We prove (i). The other proofs are similar.

Suppose first that $x_1 = x_2 = \ldots = x_P = u$, and consider the subgraph $\Gamma_0 = \text{span}\{a, u\}$ of $\Gamma$. Since $\lambda(e_0) = a$ and $e_0$ is incident to $u$, we have $e_0 \in E(\Gamma_0)$, and since $b_1$ is an endpoint of $e_0$ we have $b_1 \in V(\Gamma_0)$. Similarly $e_1 \in E(\Gamma_0)$ and $b_2 \in V(\Gamma_0)$, and so on, until $e_P \in E(\Gamma_0)$. If $e_P$ is incident with $v$, then $v \in V(\Gamma_0)$, and since $\Gamma$ is spanned by $\{a, u, v\}$ it follows that $\Gamma = \Gamma_0$ is spanned by $\{a, u\}$, a contradiction. Otherwise, $e_P$ joins $a$ to $u$, in which case the vertices $a, u, p_1, \ldots, b_P$ and the edges $e_0, \ldots, e_P$ form an admissible subtree of $\Gamma$ of diameter two, which again is a contradiction.

Now suppose that $x_1 = v$ and $x_2 = \ldots = x_P = u$, and let $\Gamma_0 = \text{span}\{b_1, u\}$. Arguing as above, we see that $\Gamma_0$ contains the edges $e_1, \ldots, e_{P-1}$ and the vertices $u, b_1, \ldots, b_P$. If $e_P$ is not incident at $v$, then it joins $u$ to $a$, so $e_P$ and $a$ also belong to $\Gamma_0$. But then $e_0$ joins $b_1$ to $v$ and has label $a$, so we also have $v \in V(\Gamma_0)$. Hence $\Gamma = \Gamma_0$ since $\Gamma$ is spanned by $\{a, u, v\}$, and so $\Gamma$ is spanned by $\{b_1, u\}$, a contradiction. \hfill \square

We next subdivide each of the sequences $\{b_i\}, \{c_i\}$ into two subsequences, depending on the orientation of the edges labelled by these vertices. Specifically, let:

- $p(1), \ldots, p(s)$ be the sequence, in ascending order, of integers $i$ such that $0 < i \leq P$ and $b_i = \tau(e_{i-1})$;
- $p'(1), \ldots, p'(s')$ be the sequence, in ascending order, of integers $i$ such that $0 < i \leq P$ and $b_i = \iota(e_{i-1})$;
- $q(1), \ldots, q(t)$ be the sequence, in ascending order, of integers $i$ such that $0 < i \leq Q$ and $c_i = \iota(f_{i-1})$; and
- $q'(1), \ldots, q'(t')$ be the sequence, in ascending order, of integers $i$ such that $0 < i \leq Q$ and $c_i = \tau(f_{i-1})$.

For consistency of notation in what follows, we set $p(0) = p'(0) = q(0) = q'(0) = 0$.

Thus each $b_i$, for $i = 1, \ldots, P$, can be written uniquely as $b_{p(i)}$ or as $b_{p'(i)}$, and each $c_i$, for $i = 1, \ldots, Q$, can be written uniquely as $c_{q(i)}$ or as $c_{q'(i)}$.

This notation allows us to give a more precise description of the structure of the initial and terminal graphs of $\Gamma$. Specifically, $I(\Gamma)$ is constructed from the vertices $\{a, u, v\}$ by adding two edges.
together with directed chains

\[
\begin{align*}
x_{p(i)+1} & \quad b_{p(i)} & \quad b_{p(i-1)+2} & \quad b_{p(i-1)+1} \\
e_{p(i)} & \quad \ldots & \quad \ldots & \quad e_{p(i-1)+1}
\end{align*}
\]

for each \(i = 1, \ldots, s\), and

\[
\begin{align*}
y_{q'(i)+1} & \quad c_{q'(i)} & \quad c_{q'(i-1)+2} & \quad c_{q'(i-1)+1} \\
f_{q'(i)} & \quad \ldots & \quad \ldots & \quad f_{q'(i-1)+1}
\end{align*}
\]

for each \(i = 1, \ldots, t'\); and finally single edges

\[
\begin{align*}
x_{j+1} & \quad b_j \\
e_j & \quad \ldots
\end{align*}
\]

for \(p(s) < j \leq P\) and

\[
\begin{align*}
y_{j+1} & \quad c_j \\
f_j & \quad \ldots
\end{align*}
\]

for \(q'(t') < j \leq Q\).

In the above diagrams \(x_{P+1}\) and \(y_{Q+1}\) (which have not been defined) should be interpreted as \(\iota(e_P)\) and \(\iota(f_Q)\) respectively. Note that at most one of these is equal to \(a\). (This happens if and only if \(a\) is the initial vertex of its incident edge in \(\Gamma\).) All other \(x_j\) and \(y_j\) belong to \(\{u, v\}\).

If \(I(\Gamma)\) contains a directed cycle, for example, then this cycle must contain \(a\). From the above, we see that this can happen only if \(s = 1\), \(p(1) = P\), and \(x_{P+1} = a\).
The structure of $T(\Gamma)$ is entirely analogous, and similar remarks apply. We omit the details.

Now we are ready to construct a specific presentation for an HNN base for $G = G(\Gamma)$. Recall that $G$ is given by a finite presentation

$$\mathcal{P}(\Gamma) = \langle V(\Gamma) \mid \iota(e)\lambda(e) = \lambda(e)\tau(e), \ e \in E(\Gamma) \rangle.$$ 

Since $\Gamma$ is connected, we have $G^{ab} \cong \mathbb{Z}$, and the commutator subgroup $G'$ is the normal closure in $G$ of the subgroup $B = B(\Gamma)$ generated by the finite set $\{xy^{-1} \mid x, y \in V(\Gamma)\}$. A theorem of Bieri and Strebel [2] says that $G$ is an HNN extension of $B$ with stable letter $t$ (which can be taken to be any element of $V(\Gamma)$) and associated subgroups $A_0 = B \cap tBt^{-1}$ and $A_1 = B \cap t^{-1}Bt$:

$$G = \langle B, t \mid t^{-1}\alpha t = \phi(\alpha), \ \alpha \in A_0 \rangle,$$

where $\phi: A_0 \to A_1$ is the isomorphism induced by conjugation by $t$.

Clearly $B$ is finitely generated. It remains to prove that $B$ is finitely presentable, and we do this by constructing an explicit set of defining relators.

Recall that our assumptions on $\Gamma$ imply that each of $I(\Gamma)$ and $T(\Gamma)$ has precisely two connected components, with the vertices $u, v$ belonging to separate components in each case.

Let $F$ denote the subgroup of the free group on $V(\Gamma)$ generated by $\{xy^{-1} \mid x, y \in V(\Gamma)\}$.

Then $F$ is free of rank $|V(\Gamma)| - 1 = |E(\Gamma)|$, and any basis for $F$ can be chosen as a finite generating set for $B$. Rather than fix a specific basis for $F$, we proceed as follows. Let $K = K(\Gamma)$ be the maximal abelian cover of the 2–complex $K = K(\Gamma)$ associated to $\Gamma$ (which is the standard 2–complex model of the presentation $\mathcal{P}(\Gamma)$). Then since $K$ has a single 0–cell, we identify the 0–cells of $K$ with integers, via the isomorphism $H_1(K) \cong G^{ab} \cong \mathbb{Z}$. The 1–cells of $K$ with initial vertex $i \in \mathbb{Z}$ can be denoted $w_i$, where $w \in V(\Gamma)$, and each $w_i$ has terminal vertex $i+1 \in \mathbb{Z}$. Let $L$ be the 1–subcomplex of $K$ with 0–cells 0, 1 and 1–cells $\{w_0, w \in V(\Gamma)\}$. Then $F$ is naturally identified with $\pi_1(L, 0)$.

We also construct a graph $\hat{L}$ and an immersion $\pi: \hat{L} \to L$ as follows. $V(\hat{L}) = \{0, 1\} \times \{u, v\}$, $E(\hat{L}) = E(L)$, $\iota(w_0) = (0, x)$ where $x \in \{u, v\}$ belongs to the same component of $I(\Gamma)$ as $w$, and $\tau(w_0) = (1, y)$ where $y \in \{u, v\}$ belongs to the same component of $T(\Gamma)$ as $w$. The graph homomorphism $\pi$ is defined to be the identity map on edges, and is defined on vertices by $\pi(i, u) = \pi(i, v) = i$, $i = 0, 1$. It is not difficult to see that $\hat{L}$ is connected. Indeed, if the edge of
\( \Gamma \) between \( u \) and \( v \) has label \( w \), then the edges \( u, v, w \) of \( \hat{L} \) form a spanning tree. Since \( \pi \) is bijective on edges, it is an immersion, and hence injective on fundamental groups. Indeed, the fundamental group \( \hat{F} \) of \( \hat{L} \) embeds as a free factor of \( F = \pi_1(L) \) via \( \pi_* \), as we can see by the following construction: add an edge \( X \) to \( \hat{L} \) with \( \iota(X) = (0, u) \) and \( \tau(X) = (0, v) \), and an edge \( Y \) with \( \iota(Y) = (1, u) \), \( \tau(Y) = (1, v) \), to form a larger graph \( \tilde{L} \). The immersion \( \pi: \hat{L} \to L \) extends to a homotopy equivalence \( \pi: \tilde{L} \to L \) that shrinks the edge \( X \) to the vertex 0, and the edge \( Y \) to the vertex 1. Hence we have

\[
F = \pi_1(L) \cong \pi_1(\hat{L}) = \pi_1(\hat{L}) * \langle X, Y \rangle.
\]

Since the map \( \pi: \hat{L} \to L \) is bijective on edges, any path in \( L \) which lifts to a path in \( \hat{L} \) does so uniquely. Given a closed path \( C \) in \( L \) that lifts to a closed path \( \tilde{C} \) in \( \hat{L} \), we define two related paths in \( L \), namely the forward derivative \( \partial_+ C \) of \( C \) and the backward derivative \( \partial_- C \) of \( C \), as follows. For \( \partial_+ C \) we first fix a maximal subforest \( \Phi_I \) of \( I(\Gamma) \). Next, we cyclically permute \( \tilde{C} \) so that it begins and ends at one of the vertices \((1, u)\) or \((1, v)\). Hence \( \tilde{C} \) is a concatenation of length two subpaths of the form \( x^{-1}y \), where \( x, y \in E(\hat{L}) = V(\Gamma) \) belong to the same component of \( I(\Gamma) \). The next step is to replace each such subword \( x^{-1}y \) by the product

\[
(x^{-1}z_0)(z_0^{-1}z_1)\ldots(z_m^{-1}y),
\]

where \((x, z_0, z_1, \ldots, z_m, y)\) is the geodesic from \( x \) to \( y \) in \( \Phi_I \). We now have a concatenation of length 2 subwords of the form \( x^{-1}y \) where \( x \) and \( y \) are joined by an edge in \( \Phi_I \). This edge corresponds to an edge of \( \Gamma \), and hence to a defining relation in \( P(\Gamma) \) that can be written

\[
x^{-1}y = gh^{-1}
\]

for some \( g, h \in V(\Gamma) \). The final step is to replace each such word \( x^{-1}y \) by the corresponding word \( gh^{-1} \). The result is a closed path \( \partial_+ C \) in \( L \).

**Remarks**

(i) \( \partial_+ C \) depends on the choice of maximal forest \( \Phi_I \), and then is well-defined only up to cyclic permutation.

(ii) If \( C' \) is a cyclic permutation of \( C \), then \( C' \) also lifts to a closed path in \( \hat{L} \), so \( \partial_+ C' \) is defined. It is equal to (a cyclic permutation of) \( \partial_+ C \).

(iii) The definition of \( \partial_+ C \) does not depend on \( C \) being (cyclically) reduced. Indeed the insertion into \( C \) of a cancelling pair \( xx^{-1} \) may alter \( \partial_+ C \). However, the insertion of a cancelling pair \( x^{-1}x \) will not alter \( \partial_+ C \).

(iv) \( C \) and \( \partial_+ C \) are (freely) homotopic in \( \bar{K} \) (since the last part of the construction involves replacing a path \( x^{-1}y \) by a homotopic path \( gh^{-1} \)). In particular, if \( C \) is nullhomotopic in \( \bar{K} \), then so is \( \partial_+ C \).
(v) The unique lift of \( \partial_i C \) in \( \hat{L} \) does not contain the edge \( Y \).

The backward derivative \( \partial_C \) is defined similarly. This time we fix a maximal forest \( \Phi_T \) of \( T(\Gamma) \), and choose a cyclic permutation of \( \hat{C} \) beginning at \((0,u)\) or \((0,v)\), split \( \hat{C} \) into subpaths of the form \( xy^{-1} \) with \( x, y \) in the same component of \( T(\Gamma) \), and then use relations of \( \mathcal{P} \) corresponding to edges of \( \Phi_T \) to transform \( \hat{C} \). Remarks analogous to the above hold also for \( \partial_C \).

Now consider the unique cycle in \( T(\Gamma) \). If \( z_0, \ldots, z_m \) are the vertices of this cycle in cyclic order, define \( \hat{R}_0 \) to be the nullhomotopic path
\[
(z_m z_0^{-1})(z_0 z_1^{-1}) \cdots (z_m z_m^{-1})
\]
in \( \hat{L} \) and \( R_0 = \pi(\hat{R}_0) \) the corresponding nullhomotopic path in \( L \). Now define \( R_1 = \partial_+ R_0 \). If \( R_1 \) lifts to \( \hat{L} \) then define \( R_2 = \partial_- R_1 \), and so on. In this way we obtain either an infinite sequence \( R_1, R_2, \ldots \) of paths in \( L \), or a finite sequence \( R_1, \ldots, R_M \) such that \( R_M \) does not lift to \( \hat{L} \).

In a similar way, the unique cycle in \( I(\Gamma) \) determines a nullhomotopic closed path \( S_0 \) in \( L \) that lifts to \( \hat{L} \), so a sequence \( S_1, \ldots \) of closed paths in \( L \) (finite or infinite), such that \( S_i = \partial_- S_{i-1} \) for each \( i \geq 1 \), and if the sequence is finite with final term \( S_N \) then \( S_N \) does not lift to \( \hat{L} \).

**Lemma 5.2** The paths \( R_i \) and \( S_j \) are all nullhomotopic in \( \tilde{K} \).

**Proof** This follows by induction and Remark (iv) above, since \( R_0 \) and \( S_0 \) are nullhomotopic. \( \square \)

Now suppose that the sequence \( \{R_i\} \) contains at least \( m \) terms. We construct a 2–complex \( L_m \) as follows. The 1–skeleton of \( L_m \) is the subcomplex of \( \tilde{K} \) consisting of \( L \), together with the 0–cells \( 2, \ldots, m+1 \) and the 1–cells \( u_1, v_1, \ldots, u_m, v_m \). Then \( L_m \) has precisely \( m \) 2–cells attached to \( L \) using the paths \( R_1, \ldots, R_m \). We also consider the full subcomplex \( \tilde{K}_m \) of \( \tilde{K} \) on the set \( \{0,1,\ldots,m+1\} \) of 0–cells.

**Lemma 5.3** The 2–complexes \( L_m \) and \( \tilde{K}_m \) are homotopy equivalent.

**Proof** We argue by induction on \( m \), there being nothing to prove in the case \( m = 0 \). Let \( \gamma \) denote the covering transformation of \( \tilde{K} \) that sends a 0–cell \( n \in \mathbb{Z} \) to \( n+1 \). Note that the link of the 0–cell \( m+1 \) in \( \tilde{K}_m \) is naturally identifiable with the graph \( T(\Gamma) \). Let \( d \) be the unique edge in \( E(\Gamma) = E(T(\Gamma)) \) that does
By inductive hypothesis, $\bar{m}$ equivalent to $\bar{R}$ homotopy equivalent to the 2–complex obtained from $\bar{L}$ to $\bar{m}$ equivalent to a complex formed from $\bar{L}$ with boundary path $\bar{R}$ all these collapsing moves, we are left with a 2–complex $K$ 1–cell and a 2–cell that are incident at the 0–cell $m$. α this argument, we see that the boundary path of $K$ wedging on $m$ is a $\mathbb{Z}$ $K$. Each move in this collapsing process (removing a vertex and a n edge from $\bar{L}$ to a 2–complex formed from $\bar{L}$ with boundary path $\bar{R}$, . . . , $\bar{S}$, and the proof is complete. 

Remark An analogous result holds for the $S_j$. We omit the details, but will use this result implicitly in what follows.

Corollary 5.4 If $R_1, \ldots, R_m$ and $S_1, \ldots, S_n$ are all defined, then $m + n < |V(\Gamma)|$.

Proof By the Lemma and its analogue for the $S_j$, $\bar{K}_m$ is homotopy equivalent to a 2–complex formed from $L$ by attaching $m$ 2–cells and then wedging on $m$ circles; and $\gamma^{-n}(\bar{K}_n)$ is homotopy equivalent to a complex obtained from $L$ by adding $n$ 2–cells and then wedging on $n$ circles. Since $\gamma^{-n}(\bar{K}_{m+n}) = \gamma^{-n}(\bar{K}_n) \cup \bar{K}_m$, with $\gamma^{-n}(\bar{K}_n) \cap \bar{K}_m = \bar{K}_1 = L$, it follows that $\gamma^{-n}(\bar{K}_{m+n})$ is homotopy equivalent to a complex formed from $L$ by adding $m + n$ 2–cells and then wedging on $m + n$ circles. Hence $\beta_1(\bar{K}_{m+n}) \geq m + n$. Now $H_2(\bar{K}) = 0$, and $\bar{K}$ is a $\mathbb{Z}$–cover of $K$, so $H_2(\bar{K}) = 0$ by [1], Proposition 1. Hence also $H_2(\bar{K}') = 0$.
for any subcomplex $K' \subseteq K$. In particular $H_2(\overline{K}_{m+n}) = 0 = H_2(L)$. Since also $H_0(\overline{K}_{m+n}) = \mathbb{Z} = H_0(L)$ and $\chi(\overline{K}_{m+n}) = \chi(L) = 2 - |V(\Gamma)|$, it follows that

$$m + n \leq \beta_1(\overline{K}_{m+n}) = \beta_1(L) = |V(\Gamma)| - 1.$$  

**Corollary 5.5** Each of the sequences $\{R_i\}$ and $\{S_j\}$ are finite, and if the final terms are $R_M$ and $S_N$ respectively then $M + N < |V(\Gamma)|$.

We claim that the finite sequences $\{R_i\}$ and $\{S_j\}$ form a full set of defining relators for the HNN base $B$ of $G$, which completes the proof of our Theorem 1.1. In order to prove this claim, we need to derive some further information about the structure of the words $R_i$ and $S_j$.

**Remark** The definitions of $R_i$ and $S_i$ depend, *a priori*, on specific choices for the maximal forests $\Phi_T$ and $\Phi_I$ respectively. Suppose we were to choose a different maximal tree $\Phi'_I$ in $I(\Gamma)$, for example. Then geodesics in $\Phi_I$ and $\Phi'_I$ would differ at most by the unique cycle in $I(\Gamma)$. It follows from this that the resulting definitions of $\partial_+ C$, for any closed path $C$ in $L$ that lifts to $\hat{L}$, are equal modulo the normal closure of $S_1$. An easy induction shows that, for any $i$, the definitions of $S_i$ resulting from different choices of $\Phi_I$ are equal modulo the normal closure of $\{S_1, \ldots, S_{i-1}\}$. Hence our set of defining relators does not depend in an essential way upon the choices of maximal forests $\Phi_I$ and $\Phi_T$.

### 6 Structure of the relations

In this section we examine the structure of the proposed defining relators $R_i$ and $S_i$ of the HNN base $B$ for $G$. Recall that each of $R_i$ and $S_i$ is a closed path in the 2–complex $L$, and that we have a homotopy equivalence $\pi: \hat{L} \to L$, which restricts to an edge-bijective graph immersion on $\hat{L} = L\setminus\{X,Y\}$ and shrinks each of the 1–cells $X, Y$ to a point. Let $\hat{C}$ denote the unique (up to cyclic permutation) cyclically reduced closed path in $\hat{L}$ that maps to a given cyclically reduced closed path $C$ in $L$. Then $C$ lifts to $\hat{L}$ if and only if $\hat{C}$ is a path in $\hat{L}$, in which case $\hat{C}$ is the unique lift. By definition, each $R_i$ (resp $S_i$) is defined if and only if $R_{i-1}$ (resp $S_{i-1}$) lifts to $\hat{L}$. Hence $\hat{R}_i$ is a path in $\hat{L}$ for $1 \leq i \leq M - 1$, and $S_i$ is a path in $\hat{L}$ for $1 \leq i \leq N - 1$. Moreover, the path $\hat{R}_M$ involves $Y$ but not $X$, while the path $\hat{S}_N$ involves $X$ but not $Y$.

For any group $A$ and letter $Z$, we say that a word $w \in A \ast \langle Z \rangle$ is *positive* (resp *negative*) in $Z$ if only positive (resp negative) powers of $Z$ occur in $w$. We
say that \( w \) is strictly positive (resp. strictly negative) if in addition at least one positive (resp. negative) power of \( Z \) does occur in \( w \), in other words \( w \not\in A \).

We will concentrate our attention on the relators \( S_i \). The analysis of the \( R_i \) is entirely analogous.

We first treat the case where \( I(\Gamma) \) contains a directed cycle \( C \).

**Theorem 6.1** Suppose that the unique cycle \( C \) in \( I(\Gamma) \) is directed. Then:

- \( N = 1 \);
- \( \tilde{S}_1 \) is either strictly positive or strictly negative in \( X \);
- \( S_1 \) involves each of \( a, b_1, \ldots, b_P \) exactly once, and no \( c_j \);
- each of \( a, b_1, \ldots, b_P \) is an extremal source in \( \Gamma \).

**Proof** The vertex \( a \) is contained in \( C \), by Lemma 4.5, (v). Since \( \iota(f_0) \in \{ u, v \} \), \( f_0 \) is not an edge of \( C \), so the edge of \( C \) coming into \( a \) is \( e_0 \). Hence \( b_1 = \iota(e_0) \) is a vertex of \( C \), and since \( e_1 \) is the only edge with \( \lambda(e_1) = b_1 \), it is also an edge of \( C \), and so on. Hence each of \( b_1, \ldots, b_P \) are vertices of \( C \), \( \iota(e_P) = a \), and the edges of \( C \) are precisely \( e_P, \ldots, e_0 \) (in the order of the orientation of \( C \)). Each of the vertices of \( C \) is extremal in \( \Gamma \), and since it is the initial vertex of an edge of \( I(\Gamma) \) it is also the initial vertex of an edge of \( \Gamma \), ie a source in \( \Gamma \). Moreover

\[
S_0 = (a^{-1}b_P)(b_P^{-1}b_{P-1})\ldots(b_1^{-1}a),
\]

so

\[
S_1 = \partial_+ S_0 = (b_P\tau(e_P)^{-1})(b_{P-1}x_P^{-1})\ldots(b_1x_2^{-1})(ax_1^{-1}),
\]

where each \( x_i \in \{ u, v \} \).

Suppose that \( S_1 \) lifts to \( \hat{L} \). Then \( \tau(e_P) \) belongs to the same component of \( I(\Gamma) \) as \( b_{P-1}, x_P \) to the same component as \( b_{P-2} \), and so on. Since \( a, b_1, \ldots, b_P \) all belong to the same component of \( I(\Gamma) \), it follows that the \( x_i \) also all belong to the same component. But \( u \) and \( v \) belong to different components of \( I(\Gamma) \), and so the \( x_i \) are all equal, which contradicts Lemma 5.1.

Hence \( S_1 \) does not lift to \( \hat{L} \), and so \( N = 1 \). Moreover, by the above argument, some of the \( x_i \) belong to the opposite component of \( I(\Gamma) \) from \( a \). If \( a, u \) belong to the same component of \( I(\Gamma) \), this means that some of the \( x_i \) are equal to \( v \).

Then \( \tilde{S}_1 \) is formed from \( S_1 \) by replacing each occurrence of \( v^{-1} \) by \( v^{-1}X^{-1} \), and so \( \tilde{S}_1 \) is strictly negative in \( X \). Similarly, if \( a, v \) belong to the same component of \( I(\Gamma) \), then \( \tilde{S}_1 \) is strictly positive in \( X \). \( \square \)
For the rest of the section, we can assume that the cycle $C$ is not directed. Then $y_1 = \iota(f_0) = \iota(e_{p(1)}) \in \{u, v\}$. We may assume that $y_1 = u$. Then $C$ has the form

\[ u \xrightarrow{b_{p(1)}} \xrightarrow{b_{p(1)-1}} \xrightarrow{b_2} \xrightarrow{b_1} \]

Figure 1

For the purpose of defining forward derivatives, and hence the $S_i$, we fix $\Phi_I$ to be the maximal subforest of $I(\Gamma)$ obtained by removing the edge $f_0$ (the edge joining $u$ to $a$ in $C$).

For $k \leq \min(s, t' + 1)$, let $I_k(\Gamma)$ denote the subgraph of $\Phi_I$ consisting of the edges $\{e_i, 0 \leq i \leq p(k)\}$ and $\{f_i, 1 \leq i \leq q'(k - 1)\}$, together with all their incident vertices. Note that $I_k$ contains no more than two components, one contained in each component of $\Phi_I$. Hence whenever two vertices of $I_k$ belong to the same component of $\Phi_I$, then the geodesic between them is also contained in $I_k$.

\textbf{Theorem 6.2} Suppose that the cycle in $I(\Gamma)$ has the form shown in Figure 1. Then:

(i) Each $S_i$ can be written, up to cyclic permutation, in the form $aU_i a^{-1} V_i$, where $U_i$ is a word in $\{a, u, v, c_1, \ldots, c_{q'(i-1)+1}\}$; and $V_i$ is a word in $\{a, u, v, b_1, \ldots, b_{p(i)+1}\}$.

(ii) If $p(i) < P$, then $V_i$ contains a single occurrence of $b_{p(i)+1}$ and does not contain $a$.

(iii) If $q'(i-1) < Q$, then $U_i$ contains a single occurrence of $c_{q'(i-1)+1}$ and does not contain $a$.

(iv) Every letter occurring in $S_i$, other than $b_{p(i)+1}$ and $c_{q'(i-1)+1}$, is a vertex of the subgraph $I_i \subseteq I(\Gamma)$.
(v) If \( p(i) = P \) or \( q'(i - 1) = Q \) then \( i = N \).

**Proof** We prove this by induction on \( i \), the initial case being when \( i = 1 \). We have

\[
S_0 = (u^{-1}a)(a^{-1}b_1)(b_1^{-1}b_2)\ldots(b_{p(1)}^{-1}u),
\]

so

\[
S_1 = \partial_+ S_0 = (ac_1^{-1})(x_1a^{-1})(x_2b_1^{-1})\ldots(x_{p(1)}b_{p(1)-1}^{-1})(b_{p(1)+1}^{-1}b_{p(1)}^{-1})
\]

(if \( p(1) < P \)). The vertices \( a, u, b_1, \ldots, b_{p(1)} \) are contained in \( I_1 \), but not \( c_1, b_{p(1)+1} \). The first four statements of the result (for \( i = 1 \)) follow, setting \( U_1 = c_1^{-1}x_1 \) and

\[
V_1 = (x_2b_1^{-1})\ldots(x_{p(1)}b_{p(1)-1}^{-1})(b_{p(1)+1}^{-1}b_{p(1)}^{-1}).
\]

For the last statement, certainly \( Q > 0 = q'(0) \). Suppose that \( p(1) = P \) and \( i < N \). Then

\[
S_1 = (ac_1^{-1})(x_1a^{-1})(x_2b_1^{-1})\ldots(x_{p+1}b_{p+1}^{-1})(\tau(e_P)b_P^{-1})
\]

lifts to \( \tilde{L} \), so each of \( x_2, \ldots, x_P \) belongs to the same component of \( I(\Gamma) \) as \( a, b_1, \ldots, b_{p-1} \), in other words \( x_2 = \ldots = x_P = u \). By Lemma 5.1 we have \( x_1 = v \) and \( e_P \) incident with \( v \). But \( \imath(e_P) = u \) so \( \tau(e_P) = v \), which does not belong to the same component of \( I(\Gamma) \) as \( b_{p-1} \). It follows that \( S_1 \) does not, after all, lift to \( \tilde{L} \), a contradiction.

This completes the proof of the initial case of the induction.

Now assume inductively that \( i > 1 \) and the result is true for \( i-1 \). In particular, \( i-1 < N \), so \( p(i - 1) < P \) and \( q'(i - 2) < Q \). Hence \( U_{i-1} \) contains a single occurrence of \( c_{q'(i-2)+1} \), \( V_{i-1} \) contains a single occurrence of \( b_{p(i-1)+1} \), and every other letter occurring in \( S_{i-1} \) is a vertex of the subgraph \( I_{i-1} \) of \( I(\Gamma) \). Consider the construction of \( S_i = \partial_+ S_{i-1} \) from \( S_{i-1} \). We first write a suitable cyclic permutation of \( S_{i-1} \) as a product of length two subwords of the form \( g^{-1}h \). For all but two of these subwords, both \( g \) and \( h \) are vertices of \( I_{i-1} \). (There are precisely two exceptions, since the occurrences of \( b_{p(i-1)+1} \) and \( c_{q'(i-2)+1} \) in \( S_{i-1} \) are separated at least by an occurrence of \( a^{\pm 1} \).)

Suppose first that \( g, h \) are vertices of \( I_{i-1} \). The next step is to replace \( g^{-1}h \) by the product

\[
(g^{-1}z_1)(z_1^{-1}z_2)\ldots(z_t^{-1}h)
\]

where \( g, z_1, z_2, \ldots, z_t, h \) are the vertices on the geodesic from \( g \) to \( h \) in \( \Phi_I \). This geodesic is contained in \( I_{i-1} \), so each bracketed term here is \( (\imath(e)^{-1}\lambda(e))^{\pm 1} \) for
some edge $e$ of $I_{i-1}$. The final step is to replace this by $(\lambda(e)\tau(e)^{-1})^{\pm 1}$. Note that $\tau(e)$ is a vertex of $I_i$, and $\tau(e) \neq a$. Also, none of the intermediate vertices $z_j$ in the geodesic is equal to a, since a is an extremal vertex of $\Phi_I$. Note that, if $g^{-1}h$ is a subword of $U_{i-1}$, then all letters in the resulting subword of $S_i$ come from $\{u, v, e_1, \ldots, e_{g'(i-1)}\}$, while if it is a subword of $a^{-1}V_{i-1}a$ then all letters come from $\{a, u, v, b_1, \ldots, b_{p(i)}\}$.

A similar argument holds if, say $g = b_{p(i-1)+1}$. Here, however, the geodesic from $g$ to $h$ is not contained in $I_{i-1}$. It is the union of the geodesic from $b_{p(i-1)+1}$ to $z$ in $I_i$, where $z \in \{u, v\}$, with the geodesic (in $I_{i-1}$) from $z$ to $h$. Edges in $I_{i-1}$ give rise to length 2 subwords of $S_i$ consisting of letters which are vertices in $I_i$. The same is true for an edge $e_j$ from $b_j$ to $b_{j+1}$, for $p(i) < j < p(i)$. (The corresponding word is $x_jb_j^{-1}$. ) Finally, the edge $e_{p(i)}$ (from $b_{p(i)}$ to $z$) contributes a subword $\tau(e_{p(i)})b_{p(i)}^{-1}$. If $p(i) < P$ then $\tau(e_{p(i)}) = b_{p(i)+1}$; otherwise $\tau(e_{p(i)}) \in \{a, u, v\}$.

The analysis if $h = b_{p(i-1)+1}$, or if one of $g, h$ is $c_{q'(p-2)+1}$ is similar to the above.

Each of the two subwords $g^{-1}h$ of $S_{i-1}$ that contain the letter $a$ gives rise to a subword of $S_i$ containing an occurrence of $a$ with the same exponent. If $g = a$ then the subword begins $(x_1a^{-1}) \ldots$, while if $h = a$ then the subword ends \ldots $(ax_1^{-1})$. If $p(i) < P$ and $q'(i-1) < Q$ then this will be the only occurrence of $a$ in this subword of $S_i$.

Statements (i)–(iv) follow.

To prove (v), suppose for example that $i < N$ and $p(i) = P$. Another induction on $i$ shows that $x_2 = \ldots = x_P = u$. An argument similar to that given above in the initial case of the induction again gives rise to a contradiction: by Lemma 5.1, $\tau(e_P) = v$, which does not belong to the same component of $I(\Gamma)$ as $b_{P-1}$, so $S_i$ does not lift to $\hat{L}$ and $i = N$.

If $i < N$ and $q'(i-1) = Q$ then a similar argument applies. Here we can show that $y_1 = \ldots = y_Q = x_1 \in \{u, v\}$, which contradicts Lemma 5.1.

This result contains all the necessary information about $S_i$ if $i < N$. We now need to investigate further the structure of $\tilde{S}_N$, particularly as regards occurrences of $X$. Note that, up to cyclic permutation, we have $\tilde{S}_N = a\tilde{U}_Na^{-1}\tilde{V}_N$, by Theorem 6.2 (i).

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Lemma 6.3 Each of $\tilde{U}_N$, $\tilde{V}_N$ is either positive or negative in $X$.

Proof As indicated in the proof of Theorem 6.2, all of $V_N$, except for the part arising from the geodesic $\gamma$ from $b_{p(N)}(N-1)+1$ to $u$, consists of letters which are vertices in $I_{N-1}$. All of these vertices are in the same component of $I(\Gamma)$ as $u$. The part of $V_N$ arising from $\gamma$ is

$$[(x_{p(N-1)+b_{p(N)}^{-1}}(N-1)+1) \cdots (x_{p(N)}^{-1})(e_{p(N)}^{-1})]^{\pm 1},$$

or, if $\gamma$ passes through $a$ (ie if $\iota(e_{p(N)}) = a$):

$$[(x_{p(N-1)+b_{p(N-1)}^{-1}}(N-1)+1) \cdots (\tau(e_{p(N)}^{-1}))a^{-1}] \cdots (b_{p(N)}^{-1})]^{\pm 1}.$$  

The expression in square brackets is a product of terms $gh^{-1}$ with $h$ in the same component of $I(\Gamma)$ as $u$. To lift to $\tilde{L}$, we replace $h^{-1}g$ by $h^{-1}Xg$ whenever $g$ belongs to the same component of $I(\Gamma)$ as $v$ and $h$ to the same component as $u$, and by $h^{-1}X^{-1}g$ if $g$ belongs to the same component as $u$ and $h$ to the same component as $v$. Hence $\tilde{V}_N$ is either positive or negative in $X$.

A similar argument applies to $\tilde{U}_N$, replacing $u$ by $x_1$ in the above. □

We will also need to investigate possible occurrences of $a$ in $S_N$ other than those indicated in Theorem 6.2.

Lemma 6.4 The words $\tilde{U}_N$ and $\tilde{V}_N$ contain in total at most one occurrence of $a$.

Proof From the discussion in the proof of Lemma 6.3, the word $V_N$ (and hence also $\tilde{V}_N$) contains a single occurrence of $a$ if $e_{p(N)}$ is incident with $a$ in $\Gamma$, and no occurrence of $a$ otherwise. Similarly $U_N$ (and hence also $\tilde{U}_N$) contains a single occurrence of $a$ if $f_{q(N)}^{-1}$ is incident with $a$ in $\Gamma$, and no occurrence of $a$ otherwise. The result now follows from the fact that $a$ is extremal in $\Gamma$. □

7 Completion of the proof

Define

$$G_0 = \pi_1(\hat{L})/\{R_1, \ldots, R_{M-1}, S_1, \ldots, S_{N-1}\},$$

$$G_+ = (G_0 * \langle X \rangle)/\{\tilde{S}_N\},$$

$$G_- = (G_0 * \langle Y \rangle)/\{\tilde{R}_M\},$$

and

$$G_1 = (G_0 * \langle X, Y \rangle)/\{\tilde{R}_M, \tilde{S}_N\} \cong (\pi_1(L))/\{R_1, \ldots, R_M, S_1, \ldots, S_N\}.$$
Lemma 7.1 The group $G_0$ is free.

Proof By Theorems 6.1 and 6.2, and the analogous results for the $R_i$, the set of $M + N - 2$ distinct numbers $B = \{p(1) + 1, \ldots, p(N - 1) + 1, p'(0) + 1, \ldots, p'(M - 2) + 1\}$ has the property that each $j \in B$ is the greatest index of a $b$–letter occurring in a unique relator $R_i$ or $S_i$, and moreover that relator contains precisely one occurrence of $b_j$.

It follows that the 1–complex $L'$ obtained from $\hat{L}$ by removing the 1–cells $b_j$, $j \in B$ is connected, with fundamental group isomorphic to $G_0$. □

Lemma 7.2 The natural maps $G_0 \to G_+$ and $G_0 \to G_-$ are injective.

Proof We show that the map $G_0 \to G_+$ is injective. The proof of injectivity of $G_0 \to G_-$ is entirely analogous. Since $G_0$ is a free group and $G_+$ is a one-relator group $G_+ = (G_0 \ast \langle X \rangle) / \{\tilde{S}_N\}$, we need only show that $\tilde{S}_N$, regarded as a word in $(G_0 \ast \langle X \rangle)$, genuinely involves $X$. The result then follows from the Freiheitssatz for one-relator groups [10].

Consider the various possibilities for the structure of $\tilde{S}_N$. If the initial graph $I(\Gamma)$ contains a directed cycle, then $N = 1$ and $\tilde{S}_1$ is a strictly positive (or strictly negative) word in $X$, by Theorem 6.1. Thus $\tilde{S}_1$, regarded as a word in the free product $G_0 \ast \langle X \rangle$, is also strictly positive (or strictly negative) in $X$, and so genuinely involves $X$.

Suppose then that $I(\Gamma)$ does not contain a directed cycle. By Theorem 6.2 (i) and Corollary 6.3 we have (up to cyclic permutation) $\tilde{S}_N = a\tilde{U}_N a^{-1}\tilde{V}_N$, with each of $\tilde{U}_N$ and $\tilde{V}_N$ being either positive or negative in $X$. We also have $\tilde{S}_N$ definitely involving $X$, since otherwise $S_N$ would lift to $\hat{L}$.

If $X$ occurs in $\tilde{S}_N$ with nonzero exponent-sum, then occurrences of $X$ survive modulo the relators $R_1, \ldots, R_{M-1}, S_1, \ldots, S_{N-1}$, so we may assume that $X$ appears with exponent-sum zero. Thus one of $\tilde{U}_N$, $\tilde{V}_N$ is strictly positive, and the other is strictly negative, with precisely the same number of occurrences of $X^\pm 1$. We may rewrite $\tilde{S}_N$ (again, up to cyclic permutation) as

$$\tilde{S}_N = XA_1X \ldots A_tXW_1X^{-1}B_tX^{-1} \ldots B_1X^{-1}W_2$$

for some $t \geq 0$ and words $A_i, B_i$ and $W_1, W_2$ that do not involve $X$. If we can show that neither $W_1$ nor $W_2$ is equal to the identity element in $G_0$, then it will follow that the above expression for $\tilde{S}_N$ does not allow for cancellation of $X$–symbols, when reducing modulo the relators of $G_0$. The result will follow.
Now \( a \) occurs with exponent-sum zero in each of the relators \( R_1, \ldots, R_{M-1} \) and \( S_1, \ldots, S_{N-1} \) of the group \( G_0 \), by Theorem 6.2. If neither \( U_N \) nor \( V_N \) contains the letter \( a \), then each of \( W_1, W_2 \) contains precisely one occurrence of \( a \), and so has infinite order in \( G_0 \). In particular, they are nontrivial in \( G_0 \), as required.

This reduces us to the case where one of \( U_N, V_N \) involves the letter \( a \). By Corollary 6.4 we know that this can happen for only one of \( U_N, V_N \).

First suppose that \( a \) occurs in \( U_N \). Then \( q'(N - 1) = Q \) (and so also \( N > 1 \)). As in the proof of Corollary 6.3, the part of \( U_N \) that gives rise to occurrences of \( X \) comes from the geodesic \( \delta \) in \( \Phi_I \) from \( c_{q'(N-2)+1} \) to \( x_1 \). The relevant subword of \( U_N \) has the form:

\[
[(y_{q'(N-2)+2}c_{q'(N-2)+1}) \cdots (y_Qc_{Q-1})(\tau(f_Q)c_Q^{-1})]^{\pm 1},
\]

or, if \( \delta \) passes through \( a \):

\[
[(y_{q'(N-2)+2}c_{q'(N-2)+1}) \cdots (\tau(f_Q)c_Q^{-1})(x_1a^{-1}) \cdots (b_{p(1)+1}b_{p(1)}^{-1})]^{\pm 1}.
\]

The occurrences of \( X \) in \( \tilde{U}_N \) correspond to those \( y_j, j \geq q'(N - 2) + 2 \) that are not equal to \( x_1 \), and also from \( \tau(f_Q) \) if this is not in the same component of \( I(\Gamma) \) as \( x_1 \). In the case where \( \delta \) passes through \( a \), we see that, in \( \tilde{S}_N = a\tilde{U}_Na^{-1}\tilde{V}_N \) the \( a \)-letters that occur in the same \( W_i \) have the same exponent, and hence the \( W_i \) are both nontrivial in \( G_0 \), as required. In the other case, \( \tau(f_Q) = a \) and the unique occurrence of \( c_Q \) in \( \tilde{V}_N \) lies on the same side of all the \( X \)-letters as the unique occurrence of \( a \). Hence \( c_Q \) occurs (precisely once) in the same \( W_i \) that contains two \( a \)-letters. To prove that this \( W_i \) is nontrivial in \( G_0 \), it suffices to show that \( c_Q \) does not occur in any of the relators \( R_1, \ldots, R_{M-1} \) or \( S_1, \ldots, S_{N-1} \). But \( c_Q \) can occur in \( S_j \) \((j < N)\) only if \( j = N - 1 \) and \( q'(N - 2) = Q - 1 \), while \( c_Q \) can occur in \( R_j \) \((j < M)\) only if \( j = M - 1 \) and \( q(M - 1) = Q - 1 \). In either case \( y_2 = \ldots = y_Q = x_1 \) (since \( R_{M-1} \) and \( S_{N-1} \) lift to \( \hat{L} \) and \( f_Q \) joins \( a \) to \( x_1 \), which contradicts Lemma 5.1.

Suppose next that \( a \) occurs in \( V_N \). Then \( p(N) = P \). The occurrences of \( X \) in \( \tilde{V}_N \) arise as indicated in the proof of Corollary 6.3. The relevant subword of \( V_N \) has the form:

\[
[(x_{p(N-1)+2}b_{p(N-1)+1}) \cdots (x_Pb_{p-1})(\tau(e_P)b_P^{-1})]^{\pm 1},
\]

or, if \( \gamma \) passes through \( a \):

\[
[(x_{p(N-1)+2}b_{p(N-1)+1}) \cdots (\tau(e_P)b_P^{-1})(x_1a^{-1}) \cdots (b_{p(1)+1}b_{p(1)}^{-1})]^{\pm 1}.
\]
The occurrences of $X$ in $\tilde{V}_N$ correspond to those $x_j$, $j \geq p(N - 1) + 2$ in this subword that are equal to $v$, and also to $\tau(e_P)$ if $\tau(e_P) = v$. If $a = \tau(e_P)$ then since

$$\tilde{S}_N \sim a\tilde{U}_N^{-1}\tilde{V}_N \sim XA_1X \ldots A_tXW_1X^{-1}B_tX^{-1} \ldots B_1X^{-1}W_2$$

we see that the two $a$–letters that occur in the same $W_i$ have the same exponent, and hence both $W_i$ are nontrivial in $G_0$, as required.

If $a = \iota(e_P)$ then $\gamma$ passes through $a$. Assume for the moment that $x_1 = u$. Then the unique occurrence of $b_P$ in $\tilde{U}_N$ lies on the same side of all the $X$–letters as the unique occurrence of $a$. Hence the $W_i$ that contains two $a$–letters also contains a single occurrence of $b_P$. To prove that this $W_i$ is nontrivial in $G_0$, it suffices to show that $b_P$ does not occur in any of the relators $R_1, \ldots, R_{M-1}$ or $S_1, \ldots, S_{N-1}$ of $G_0$. But $b_P$ can occur in $S_j$ ($j < N$) only if $j = N - 1$ and $p(N - 1) = P - 1$, while if $b_P$ occurs in $R_j$ ($j < M$), then $j = M - 1$ and $p'(M - 2) = P - 1$. In either case $x_1 = \ldots = x_P = u$, contradicting Lemma 5.1.

This last argument does not apply if $x_1 = v$. In this case we still have $x_2 = \ldots = x_P = u$, and since $a = \iota(e_P)$ it follows from Lemma 5.1 that $\tau(e_P) = v$.

If, say, $W_1 = 1$ in $G_0$, then $A_t = vb_P^{-1}$ and $A_tW_1B_t = A_tB_t \neq 1$ in $G_0$, since this word contains a single occurrence of $b_P$, which by similar arguments to the above cannot occur in any of the relators of $G_0$. Hence no more than one pair of letters $X^\pm 1$ in $S_N$ can cancel modulo the relators of $G_0$, and so $S_N$, as a word in $G_0 * \langle X \rangle$, definitely involves $X$, as required.

This completes the proof of the Lemma. \hfill \square

**Corollary 7.3** The maps $G_\pm \to G_1$ are injective.

**Proof** The commutative square

\[
\begin{array}{ccc}
G_0 & \longrightarrow & G_+ \\
\downarrow & & \downarrow \\
G_- & \longrightarrow & G_1
\end{array}
\]

is a pushout, and the maps $G_0 \to G_\pm$ are injective by the lemma. Hence $G_1$ is the free product of $G_+$ and $G_-$, amalgamated over $G_0$. \hfill \square

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Let $L_+$ be the 1–complex obtained from $\hat{L}$ by identifying the 0–cells $(0, u)$ and $(0, v)$ to a single 0–cell 0. Then $L_+$ is homotopy equivalent to the subcomplex $\hat{L} \cup X$ of $\hat{L}$, and $G_+$ is a homomorphic image of the free group $\pi_1(\hat{L}) \ast \langle X \rangle$, which is naturally identifiable with $\pi_1(L_+)$. Let us fix the 0–cell 0 as a base-point for $L_+$, and consider the generating set

$$B_+ = \{ \theta e = \tau(e)\lambda(e)^{-1} ; e \in E(\Gamma) \}$$

for $\pi_1(L_+, 0)$. Note that $B_+$ is not a basis, since the unique cycle in $T(\Gamma)$ gives rise to a relation $R_0$ among the $\theta e$. However, this is the only relation, in the sense that $\pi_1(L_+, 0)$ has a one-relator presentation $\langle B_+ | R_0 \rangle$.

Similarly, if $L_-$ is obtained from $\hat{L}$ by identifying the 0–cells $(1, u)$ and $(1, v)$ to a single 0–cell 1, then $G_-$ is a homomorphic image of the free group $\pi_1(L_-, 1)$, which is generated by

$$B_- = \{ \phi e = \lambda(e)^{-1}\iota(e) ; e \in E(\Gamma) \}$$

modulo a single relator $S_0$ arising from the unique cycle in $I(\Gamma)$.

**Theorem 7.4** The correspondence $\theta e \leftrightarrow \phi e$ ($e \in E(\Gamma)$) induces a group isomorphism $G_+ \leftrightarrow G_-$. 

**Proof** The relation $R_0$ among the generators $B_+$ is precisely the nullhomotopic path $R_0$ in $L$, which lifts to $L_+$ (indeed to $\hat{L}$). Under the isomorphism $\Psi : F(B_+) \to F(B_-)$ induced by the map $\theta e \leftrightarrow \phi e$, this relation $R_0$ is mapped to $\partial_- R_0 = R_1$, which is a relation in $G_-$. Hence we have an induced homomorphism $\pi_1 L_+ \to G_-$. In order to show that this in turn induces a homomorphism $G_+ \to G_-$, we must show that each relation of $G_+$ is mapped to a relation of $G_-$. Each word $R_i$, $1 \leq i \leq M - 1$ is mapped under $\Psi$ to $\partial_+ R_i = R_{i+1}$, which is a relation in $G_-$. Similarly, for $1 \leq j \leq N$ we have $\Psi^{-1}(S_{j-1}) = \partial_- S_{j-1} = S_j$, so $\Psi(S_j) = S_{j-1}$, which is also a relation in $G_-$. Hence $\Psi$ induces a group homomorphism $G_+ \to G_-$, as claimed. Similarly $\Psi^{-1}$ induces a group homomorphism $G_- \to G_+$, and these homomorphisms are mutually inverse isomorphisms, by standard arguments.

**Corollary 7.5** $G(\Gamma)$ is isomorphic to an HNN extension of the finitely presented group $G_1$, with associated subgroups $G_\pm$.

**Proof** This is an easy exercise, given the isomorphism described in the previous lemma.

This completes the proof of our main result, Theorem 1.1.
8 Further remarks

In the proof of Theorem 1.1, we have relied heavily on one-relator theory to show that our HNN base $G_1$ is indeed defined by the relators $R_i$ and $S_i$. If we look at LOTs of larger diameter, we no longer have these tools at our disposal.

As long as $I(\Gamma)$ and $T(\Gamma)$ each have only two components (and hence only one cycle), a great deal of the proof goes through. Certainly the forward and backward derivatives give rise to two finite sequences $R_i$ and $S_i$ of relators for $G_1$, but in order to prove that these relations are sufficient to define $G_1$, we would need to prove a Freiheitssatz for the one-relator products $(G_0 \ast \langle X \rangle)/S_N$ and $(G_0 \ast \langle Y \rangle)/R_M$. In our case, we have used the combinatorics of the diameter 3 situation in a nontrivial way to show that $G_0$ is free and that $S_N$ properly involves $X$ (resp $R_M$ properly involves $Y$) modulo the relations of $G_0$, from which the Freiheitssatz follows.

It seems reasonable to conjecture in more generality that the HNN base $B$ for $G$, generated by $\{xy^{-1}, x, y \in V\}$ will be finitely presented. One may construct sets of relations on this generating set analogous to the $R_i$ and $S_i$ above, by repeatedly applying the forward derivative construction to nullhomotopic paths arising from closed paths in $I(\Gamma)$ (analogous to our $S_0$), and the backward derivative construction to nullhomotopic paths arising from closed paths in $T(\Gamma)$ (analogous to our $R_0$). Provided we restrict attention to simple closed paths, only finitely many relations arise in this way, and one can conjecture that these form a set of defining relators for $B$.

Before making this conjecture precise, let us first give a geometric interpretation of these relations. On the 2–complex $K = K(\Gamma)$ we define a track $T$ in the sense of Dunwoody [4] as follows: $T$ intersects each 1–cell in a single point, and each 2–cell in two arcs as in the diagram below.

![Figure 2](image-url)
The initial graph $I(\Gamma)$ is naturally embedded as a subgraph of the link of the 0-cell in $K$. Corresponding to a cycle
\[ C = (x_1, \ldots, x_n) \]
in $I(\Gamma)$ is a Dehn diagram $D_1$ over $P(\Gamma)$ with a single interior vertex (whose link maps isomorphically to $C$). We also have a nullhomotopic closed path
\[ S_0 = (x_1^{-1}x_2) \cdots (x_n^{-1}x_1) \]
in $K^{(1)}$. The boundary label of $D_1$ is $S_1 = \partial_+ S_0$. Moreover, if we regard $D_1$ as a map from the disc $D^2$ to $K$, then the track $T$ on $K$ induces a track on $D^2$. This track consists of a single circle in the interior of $D^2$, together with a collection of arcs, each connecting two adjacent track points on $\partial D^2$.

Now suppose that $S_1$ lifts to $\hat{L}$. Then the Dehn diagram $D_1$ can be extended to a diagram $D_2$ with boundary label $S_2 = \partial_+ S_1$, and so on. On any Dehn diagram arising in this way, the track induced by $T$ consists of a collection of concentric circles in the interior of $D^2$, together with a collection of arcs, each connecting two adjacent track points on $\partial D^2$.

Dual to the track $T$ is a flow on $K$, indicated on the boundary of the 2–cells by the arrows in Figure 2. The flow induced on $D^2$ by any of the Dehn diagrams obtained as above has only one singular point in the interior of $D^2$, which is a sink.

We can perform a similar construction for any cycle in $T(\Gamma)$. The boundary label of the resulting Dehn diagram is obtained by repeatedly applying the backward derivative operator to a nullhomotopic closed path in $K^{(1)}$. Again, the induced track on $D^2$ consists of a collection of concentric circles in the interior of $D^2$, together with a collection of arcs, each connecting two adjacent track points on $\partial D^2$. The induced flow has only one singular point in the interior of $D^2$, which is a source.

Let us define a Dehn diagram to be tame if the induced track on $D^2$ consists of a collection of concentric circles in the interior of $D^2$, together with a collection of arcs, each connecting two adjacent track points on $\partial D^2$. This is equivalent to the induced flow having only one singular point in the interior of $D^2$, which is either a sink or a source. It is not difficult to show that every tame Dehn diagram arises by the above construction from a cycle in $I(\Gamma)$ or $T(\Gamma)$, and that its boundary label is an alternating word in the generators $V(\Gamma)$ of $G(\Gamma)$.

**Conjecture 8.1** Let $B$ be the subgroup of $G(\Gamma)$ generated by the alternating words in $V(\Gamma)$. Then $B$ has a finite presentation in which the defining relators are the boundary labels of tame Dehn diagrams.
References

[1] J F Adams, A new proof of a theorem of WH Cockcroft, J. LMS 49 (1955) 482–488
[2] R Bieri, R Strebel, Almost finitely presented soluble groups, Comment. Math. Helv. 65 (1990) 243–254
[3] S D Brodskii, Equations over groups and groups with a single defining relator, Siberian Mathematical Journal 25 (1984) 231–251
[4] M J Dunwoody, The accessibility of finitely presented groups, Invent. Math. 81 (1985) 449–457
[5] J Hillman, 2-Knots and their Groups, Austral. Math. Soc. Lecture Series 5, Cambridge University Press (1989)
[6] J Howie, On locally indicable groups, Math. Z. 180 (1985) 231–251
[7] J Howie, On the accessibility of finitely presented groups, Invent. Math. 81 (1985) 449–457
[8] S V Ivanov, On asphericity of group presentations given by labelled oriented trees, preprint (1996)
[9] T Maeda, Knotted surfaces in the 4-sphere with no minimal Seifert manifolds, from: “Combinatorial and Geometric Group Theory” (A J Duncan, N D Gilbert and J Howie, editors), LMS Lecture Note Series 204, Cambridge University Press (1994) 239–246
[10] W Magnus, Über diskontinuierliche Gruppen mit einer definierenden Relation (der Freiheitssatz), J. reine angew. Math. 163 (1930) 141–165
[11] D I Moldavanski, Certain subgroups of groups with one defining relation (Russian), Sibirsk. Mat. Zh. 8 (1967) 1370–1384
[12] E S Rapaport, Knot-like groups, from: “Knots, Groups and 3-Manifolds” (L P Neuwirth, editor), Annals of Mathematics Studies, 84 Princeton University Press (1975) 119–133
[13] D Silver, Examples of 3-knots with no minimal Seifert manifolds, Math. Proc. Cambridge Phil. Soc. 110 (1991) 417–420
[14] D Silver, On the existence of minimal Seifert manifolds, Math. Proc. Cambridge Phil. Soc. 114 (1993) 103–109
[15] D Silver, HNN bases and high-dimensional knots, Proc. Amer. Math. Soc. 124 (1996) 1247–1252
[16] D Silver, Free group automorphisms and knotted tori in $S^4$, J. Knot Theory Ramifications (1997) 95–103

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