Zitterbewegung in Bogoliubov’s System

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I. INTRODUCTION

The phenomenon “Zitterbewegung” (ZB), a quivering motion of a free relativistic particle, has drawn many researchers’ attention since its theoretical prediction by Schrödinger in 1930 [1]. Nonetheless, the high requirement on measuring precision has ever since defied a direct observation. This triggered proposals [2–6] using some other more experimentally accessible systems to simulate this relativistic quantum effect. For instance, a quantum simulation [7] of the (1 + 1)-dimensional Dirac’s equation could be attributed to an interference between the positive- and negative-energy components of wave functions. Such understanding is based on the single-particle interpretation of the relativistic quantum mechanics. On the other hand, in the quantum field theory which allows braiding relation, we find that the ZB in Bogoliubov’s system possesses an amplitude of order $10^{-15}m$ and a period of order $10^{-16}s$, comparing with an amplitude of order $10^{-12}m$ and a period of order $10^{-21}s$ for the electron. Thus, the larger period (the lower frequency) of Bogoliubov’s quasiparticle renders a promising observation of the ZB in $^3He – B$ comparably simpler than that of an electron.

II. BOGOLIUBOV’S SYSTEM AND ZITTERBEWEGUNG

The Hamiltonian of Bogoliubov’s system for quasiparticles in $^3He – B$ can be expressed as [17–19]

$$H_B = m(\vec{p})\beta + c\vec{p} \cdot \vec{\alpha},$$

where $m(\vec{p}) = \vec{p}^2/2m – \mu$, $m$ the mass of $^3He$, $\vec{p}$ the momentum, $\mu$ the chemical potential, $c = \Delta_B/k_F\hbar$, $\hbar$ the Planck constant, $\Delta_B$ the equilibrium order parameter, and $k_F$ the Fermi momentum. $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ are $4 \times 4$ Hermitian matrices satisfying

$$\alpha_i^2 = \beta^2 = 1,$$

$$\alpha_i \beta + \beta \alpha_i = 0,$$

$$\alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik},$$

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with $\delta_{i,k} = 1$ for $i = k$, and $\delta_{i,k} = 0$ for $i \neq k$. We use the Pauli representation

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $I$ is a $2 \times 2$ unit matrix, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a Pauli matrices vector with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

This Hamiltonian (1) becomes “relativistic” in the limit $m c^2 \gg \mu$, where it tends asymptotically toward the Dirac Hamiltonian. However in a real $^3$He $- B$, one has an opposite limit $mc^2 \ll \mu$.

The wave equation of a Bogoliubov’s quasiparticle can be written in the form

$$i \hbar \frac{d \Psi(\vec{x}(t), t)}{dt} = H |\Psi(\vec{x}(t), t)|,$$

where $|\Psi(\vec{x}(t), t)|$ is a four-component wavefunction. The velocity of the particle is defined and calculated as

$$\frac{d \vec{x}(t)}{dt} = \frac{1}{i \hbar} [\vec{x}(t), H] = \left( \frac{\vec{p}}{m} \beta(t) + c\alpha(t) \right). \quad (6)$$

We should work out $\alpha(t)$ first, and it also follows the Heisenberg equation:

$$\frac{d \alpha(t)}{dt} = \frac{1}{i \hbar} [\alpha(t), H] = \frac{2}{i \hbar} (\alpha(t) H - c \vec{p}). \quad (7)$$

Because the momentum $\vec{p}$ and the Hamiltonian $H$ are constants of motion, this equation can be integrated easily:

$$\alpha(t) = c \vec{p} H^{-1} + (\alpha(0) - c \vec{p} H^{-1}) e^{-2iHt/\hbar}. \quad (8)$$

Similarly,

$$\beta(t) = m(\vec{p}) H^{-1} + (\beta(0) - m(\vec{p}) H^{-1}) e^{-2iHt/\hbar}. \quad (9)$$

Substituting (8) and (9) into Eq. (6), we obtained

$$\vec{x}(t) = \vec{x}(0) + \left( c^2 + \frac{m(\vec{p})}{m} \right) \vec{p} H^{-1} t \\
+ \frac{i \hbar}{2} \left( c\alpha(0) + \frac{m(\vec{p})}{m} \beta(0) - \left( c^2 + \frac{m(\vec{p})}{m} \right) \vec{p} H^{-1} \right) e^{-2iHt/\hbar} - \frac{1}{H}.$$ (10)

Through this part of the computation, we can see that the operators $\vec{p}$ and $H$ depend on time in a nontrivial way. This adds a second term representing a rapidly oscillating motion of the quasiparticles over the conventional velocity operator $\vec{v}$. This result shows that the quasiparticle exhibits a similar structure as a relativistic electron, and the centroid of the wave-packet

$$Z = \langle i \hbar \left( c\alpha(0) + \frac{m(\vec{p})}{m} \beta(0) - \left( c^2 + \frac{m(\vec{p})}{m} \right) \vec{p} H^{-1} \right) e^{-2iHt/\hbar} - \frac{1}{H} \rangle,$$

represents a rapid oscillatory motion, i.e., the position-Zitterbewegung, whose amplitude is of order $v_F \hbar 2E \sim 10^{-15} m$ (with $v_F = \frac{\hbar}{m} k_F$ and $E$ is the energy), and the period is of order $\frac{\hbar}{2} \sim 10^{-16} s$, respectively.

Energy and Eigenfunction.— Next, we need to obtain the energy and eigenfunction of the Hamiltonian described by the equation (1). It is convenient to express the eigenfunction in the form

$$|\psi\rangle = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} e^{i \vec{k} \cdot \vec{x}} = \begin{pmatrix} \phi \\ \varphi \end{pmatrix} e^{i \vec{k} \cdot \vec{x}}, \quad (10)$$

with $\phi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\varphi = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$. Now we define a dichotomous-valued operator

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad (11)$$

so that

$$\Sigma = \frac{\hbar}{2} \Sigma \quad (12)$$

is the spin momentum operator. It is easily to see the set of observables $\{ H, \vec{p}, \Sigma, \vec{p} \}$ commute with one another. Since

$$\Sigma \cdot \vec{p} |\psi\rangle = \xi |\psi\rangle = \pm \hbar k |\psi\rangle, \quad (13)$$

we have

$$\hbar \vec{\sigma} \cdot \vec{k} \phi = \xi \phi, \quad \hbar \vec{\sigma} \cdot \vec{k} \varphi = \xi \varphi. \quad (14)$$

Here $\phi$ and $\varphi$ are differed by a constant coefficient, so we focus on $\phi$. We have

$$\begin{pmatrix} k_3 - \frac{\xi}{\hbar} \\ k_1 - ik_2 \\ k_1 + ik_2 \\ -k_3 + \frac{\xi}{\hbar} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \quad (15)$$

It is easy to get the solutions:

for $\lambda = +\hbar k$, $\frac{u_1}{u_2} = \frac{k_3 + k}{k_1 + ik_2} = -\frac{k_1 - ik_2}{k_3 + k}$, (16)

for $\lambda = -\hbar k$, $\frac{u_1}{u_2} = \frac{k_3 - k}{k_1 + ik_2} = -\frac{k_1 - ik_2}{k_3 + k}$. (17)
with $k = |\vec{k}|$.

On the other hand, we substitute (10) into Bogoliubov’s Hamiltonian $H$, (i.e. $H\psi = E\psi$),

$$
\begin{pmatrix}
m(p) - E & c\sigma \cdot \vec{p} \\
c\sigma \cdot \vec{p} & -m(p) - E
\end{pmatrix}
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix}
e^{i\vec{k} \cdot \vec{r}} = 0.
$$

(18)

Because $\phi, \varphi$ can not all be 0, i.e.

$$
\left|\begin{array}{ll}
m(p) - E & c\sigma \cdot \vec{p} \\
c\sigma \cdot \vec{p} & -m(p) - E
\end{array}\right| = 0,
$$

(19)

then we obtain the spectra

$$
E = \pm \sqrt{\eta^2 + c^2\hbar^2 k^2} \equiv E_{\pm},
$$

(20)

with $\eta = \hbar^2 k^2 / 2m - \mu$. Accordingly,

for $E = E_+$, $\phi = \frac{c\hbar \cdot \vec{k}}{\sqrt{\eta^2 + c^2\hbar^2 k^2} - \eta} \varphi$,

for $E = E_-$, $\phi = -\frac{c\hbar \cdot \vec{k}}{\sqrt{\eta^2 + c^2\hbar^2 k^2} + \eta} \varphi$.

From the equations (13) and (14), we know that the eigenvalue of $\vec{\sigma} \cdot \vec{k}$ is $\pm k$. Without explicit normalization, we list all solutions in the following:

(i) when the eigenvalues of $\{H, \vec{p}, \vec{\Sigma} \cdot \vec{p}\}$ are $(E_+, \hbar k, h\bar{k})$, the eigenfunction is

$$
|\psi_1\rangle = \frac{1}{N_1} \begin{pmatrix} k_1 - ik_2 \\ k - k_3 \\ \epsilon_1(k_1 - ik_2) \end{pmatrix} e^{i\vec{k} \cdot \vec{x}},
$$

where $\epsilon_1 = \left(\sqrt{\eta^2 + c^2\hbar^2 k^2} - \eta\right) / c\hbar k$.

(ii) when the eigenvalues of $\{H, \vec{p}, \vec{\Sigma} \cdot \vec{p}\}$ are $(E_-, \hbar \bar{k}, -h\bar{k})$, the eigenfunction is

$$
|\psi_2\rangle = \frac{1}{N_2} \begin{pmatrix} k_1 - ik_2 \\ -k - k_3 \\ -\epsilon_2(k_1 - ik_2) \end{pmatrix} e^{i\vec{k} \cdot \vec{x}},
$$

where $\epsilon_2 = \epsilon_1$.

(iii) when the eigenvalues of $\{H, \vec{p}, \vec{\Sigma} \cdot \vec{p}\}$ are $(E_-, \hbar k, h\bar{k})$, the eigenfunction is

$$
|\psi_3\rangle = \frac{1}{N_3} \begin{pmatrix} k_1 - ik_2 \\ k - k_3 \\ -\epsilon_3(k_1 - ik_2) \end{pmatrix} e^{i\vec{k} \cdot \vec{x}},
$$

where $\epsilon_3 = \left(\sqrt{\eta^2 + c^2\hbar^2 k^2} + \eta\right) / c\hbar k$.

(iv) when the eigenvalues of $\{H, \vec{p}, \vec{\Sigma} \cdot \vec{p}\}$ are $(E_-, \hbar \bar{k}, -h\bar{k})$, the eigenfunction is

$$
|\psi_4\rangle = \frac{1}{N_4} \begin{pmatrix} k_1 - ik_2 \\ -k - k_3 \\ \epsilon_4(k_1 - ik_2) \end{pmatrix} e^{i\vec{k} \cdot \vec{x}},
$$

where $\epsilon_4 = \epsilon_3$.

**Position-Zitterbewegung.** — To study the ZB, it is necessary to measure $\langle \vec{p} \rangle$, the expectation value of the position operator of the quasiparticles. We introduce the energy projection operators $\Gamma_\pm = \frac{1}{2} (1 \pm \Lambda)$ with $\Lambda = \frac{H}{\hbar^2}$ and $E_p = \sqrt{(\frac{k^2 \hbar^2}{2m} - \mu)^2 + c^2 \hbar^2 k^2}$. These operators have the following properties:

$$
\Gamma_+ |\psi_+\rangle = |\psi_+\rangle,
\Gamma_- |\psi_-\rangle = |\psi_-\rangle,
\Gamma_+ |\psi_-\rangle = \Gamma_- |\psi_+\rangle = 0
$$

(21)

where $\psi_\lambda$ is the positive-energy solution and the negative-energy solution. $\lambda = \pm 1$ is the eigenvalue of the energy projection operators $\Gamma_\pm$. It can be shown that

$$
[\Gamma_\pm, \vec{\alpha}] = \pm \frac{1}{2E_p} [H, \vec{\alpha}]
$$

(22)

in additional,

$$
H\Gamma_\pm = \pm \hbar E_\pm \Gamma_\pm.
$$

(23)

Since

$$
[\Gamma_\pm, \vec{\alpha}] = 0,
$$

(24)

after some simple calculation, we find that

$$
\Gamma_\pm (\vec{\alpha}(0) - c\vec{\rho}H^{-1}) \frac{e^{-2iHT/\hbar}}{2H} \Gamma_\pm = 0.
$$

(25)

Likewise,

$$
\Gamma_\pm \left(\beta - m(\vec{\bar{k}})H^{-1}\right) \frac{e^{-2iHT/\hbar}}{2H} \Gamma_\pm = 0.
$$

(26)

Then we have the relationship

$$
\Gamma_\pm \left[ (c\vec{\alpha}(0) + \frac{\vec{\bar{p}}}{m}\beta(0) - \left( c^2 + \frac{m(\bar{p})}{m} \right) \vec{\rho}H^{-1} \right] \times H^{-1} e^{-2iHT/\hbar} \Gamma_\pm = 0.
$$

That is to say, the oscillatory motion vanishes if the wavepacket is a superposition of positive-energy solution only, i.e. is of the form

$$
\left| \Psi \right\rangle = \sum \int A(\vec{p}) \psi_\lambda d\vec{p}
$$

(27)
with $\lambda = +1$, or of oscillatory-energy solutions only $\lambda = -1$. It follows that the oscillatory motion is due to interference between the positive- and negative-energy solutions which are normally required to form a wave-packet, since neither set alone constitutes a complete set of functions.

For the superposition state $|\Psi\rangle = \sin \theta |\psi_+\rangle + \cos \theta |\psi_-\rangle$, one may calculate the position-Zitterbewegung as

$$Z = \frac{m(k)\hbar^2 k^2 - E^2_+}{E^+_+} \frac{c\hbar}{mkE_+} k \sin 2\theta \sin \left( \frac{2E_+t}{\hbar} \right)$$

(28)

for $|\Psi\rangle = \sin \theta |\psi_+\rangle + \cos \theta |\psi_-\rangle$. This is the expectation value of oscillatory motion, that is an interference effect between the positive and negative-energy parts. It does not appear in the case that spinors consist entirely of positive-energy (negative-energy) parts.

Spin-Zitterbewegung. — Moreover, we have a look at the spin of the Bogoliubov’s system, and see whether it has the same properties as the ZB. The spin operator satisfies the Heisenberg equation

$$\frac{dS(t)}{dt} = \frac{1}{\hbar} \left[ S(t), H \right]$$

$$= -c\alpha(0) \times \vec{p} e^{-2iHt/\hbar},$$

(29)

which can be integrated easily, so that

$$\vec{S}(t) = \vec{S}(0) - \frac{i\hbar}{2} (c\alpha(0) \times \vec{p}) \frac{e^{-2iHt/\hbar} - 1}{H}.$$ 

(30)

There has a rapid oscillatory motion, i.e., spin-Zitterbewegung as

$$Z_{\text{spin}} = \frac{i\hbar}{2} (c\alpha(0) \times \vec{p}) \frac{e^{-2iHt/\hbar}}{H}. $$

(31)

Similar to the above analysis, we observe that the expectation values of spin for some types of initial superpositions of positive- and negative-energy wavefunctions strangely vanish. That is, $\langle \psi_1|\vec{S}|\psi_i\rangle = 0$, $\langle \psi_1|\vec{S}|\psi_2\rangle = 0$, $\langle \psi_3|\vec{S}|\psi_4\rangle = 0$, $\langle \psi_1|\vec{S}|\psi_4\rangle \neq 0$, $\langle \psi_2|\vec{S}|\psi_3\rangle \neq 0$.

III. CONCLUSION

To summarize, we have discussed the position-ZB in Bogoliubov’s system and shown that, besides classical uniform motion, the centroid of the wave-packet has a rapid oscillatory motion. The expectation value of the rapid oscillatory motion has been obtained, indicating an interference between the positive- and negative-energy wavefunctions. The ZB of Bogoliubov’s quasiparticle has a frequency dramatically lower than that of a free Dirac’s electron, rendering a promising observation comparably simpler. We have also discussed the spin-ZB in Bogoliubov’s system in the end.

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