Dynamics of quintessence models of dark energy with exponential coupling to dark matter

Tame Gonzalez, Genly Leon and Israel Quiros

Las Villas Central University, Santa Clara, Cuba

Received 9 September 2005, in final form 13 January 2006
Published 7 April 2006
Online at stacks.iop.org/CQG/23/3165

Abstract

We explore quintessence models of dark energy which exhibit non-minimal coupling between the dark matter and dark energy components of the cosmic fluid. The kind of coupling chosen is inspired by scalar–tensor theories of gravity. We impose a suitable dynamics of the expansion allowing us to derive exact Friedmann–Robertson–Walker solutions once the coupling function is given as input. Self-interaction potentials of single and double exponential types emerge as a result of our choice of the coupling function. The stability and existence of the solutions are discussed in some detail. Although, in general, models with appropriate interaction between the components of the cosmic mixture are useful for handling the coincidence problem, in the present study this problem cannot be avoided due to the choice of solution generating ansatz.

PACS numbers: 04.20.Jb, 04.20.Dw, 98.80.—k, 98.80.Es, 95.30.Sf, 95.35.+d

1. Introduction

At the present time, all available observational evidence suggests that the energy density of the universe today is dominated by a component with negative pressure of almost the same absolute value as its energy density. This mysterious component, which violates the strong energy condition (SEC), drives the present accelerated stage of cosmic evolution and is called ‘dark energy’ (DE). Many models have been investigated to account for this SEC-violating source of gravity. Among them, one of the most successful is a slowly varying scalar field, called ‘quintessence’ [1, 2].

Many models of quintessence assume that the background and the dark energy evolve independently, so their natural generalization are models with non-minimal coupling between both components. Although experimental tests in the solar system impose severe restrictions on the possibility of non-minimal coupling between the DE and ordinary matter fluids [3], due to the unknown nature of dark matter (DM), it is possible to have additional (non-gravitational) interactions between the DE and the DM components, without conflict with the
experimental data\(^1\). Since, models with non-minimal coupling imply interaction (exchange of energy) between the DM and the DE, these models provide new qualitative features for the coincidence problem \([5, 6]\). It has been shown, in particular, that a suitable coupling can produce scaling solutions that are free of the coincidence problem. The way in which the coupling is approached is not unique. In \([5]\), for instance, the coupling is introduced by hand. In \([6]\) the type of coupling is not specified at the beginning. Instead, the form of the interaction term is fixed by the requirement that the ratio of the energy densities of DM and quintessence have a stable (non-zero) equilibrium point that solves the problem of the cosmic coincidence. In \([7]\), a suitable interaction between the quintessence field and the DM leads to a transition from the matter domination era to an accelerated expansion epoch in the model of \([6]\). A model derived from the dilaton is studied in \([8]\). In this model, the coupling function is chosen as a Fourier expansion around some minimum of the scalar field.

A variety of self-interaction potentials have been studied in DE models to account for the desired evolution that fits the observational evidence. Among them, a single exponential is the simplest case. This type of potential leads to two possible late-time attractors in the presence of a barotropic fluid \([9, 10]\): (i) a scaling regime where the scalar field mimics the dynamics of the background fluid, i.e. the ratio between both DM and quintessence energy densities is a constant and (ii) an attractor solution dominated by the scalar field. Given that single exponential potentials can lead to one of the above scaling solutions, then it should follow that a combination of exponentials should allow for a scenario where the universe can evolve through a radiation–matter regime (attractor (i)) and, at some recent epoch, evolve into the scalar field dominated regime (attractor (ii)). Models with single and double exponential potentials have also been studied in \([11, 12]\).

The aim of the present paper is to investigate models with non-minimal coupling among the components of the cosmic fluid: the dust dark matter and the quintessence field (a scalar field model of dark energy). To specify the kind of coupling, we take as a Lagrangian model a scalar–tensor (ST) theory with Lagrangian written in Einstein frame variables. Otherwise, one might also consider the Lagrangian with additional non-gravitational coupling between the matter species as an effective theory.

In this paper, to derive solutions of the field equations of the model, we use an ansatz that makes possible the easy handling of the differential equations involved but at the cost, however, of losing the possibility of avoiding the coincidence problem through an appropriate choice of interaction between DM and DE (quintessence field). Actually, we impose the dynamics of the expansion by exploring a linear relationship between the Hubble parameter and the time derivative of the scalar field\(^2\). Using this relationship we can solve the field equations explicitly. However, as will be shown, unlike other models where the coincidence problem is solved (or smoothed out) through the choice of an appropriate interaction between the DM and the quintessence field, in the present investigation, solutions where the ratio between dark matter and quintessence energy densities is a constant \(\sim 1\) (the necessary requirement to avoid the coincidence problem) are unstable, so the coincidence problem will arise.

Here we consider a flat Friedmann–Robertson–Walker (FRW) universe filled with a mixture of two interacting fluids: a background of DM and the quintessence field. Since there are suggestive arguments showing that observational bounds on the ‘fifth’ force do not necessarily imply the decoupling of baryons from quintessence \([7]\), baryons are to be considered also as part of the background DM interacting with the quintessence field. In

\(^1\) It should be pointed out, however, that when the stability of dark energy potentials in quintessence models is considered, the dark matter–dark energy coupling is troublesome \([4]\).

\(^2\) This relationship is the simplest possible and, in the absence of any other information is, in a sense, the most natural since the Hubble parameter fixes the time scale. This argument was suggested to us by Diego Pavon.
other words, as in [7], we are considering the universal coupling of the quintessence field to all sorts of matter (radiation is excluded). Since the arguments given in the appendix of [7] to explain this possibility are also applicable in the cases of interest in the present study, we refer the interested reader to that reference for the details. However, we want to mention the basic arguments given therein: a possible explanation is through the ‘longitudinal coupling’ approach to inhomogeneous perturbations of the model. The longitudinal coupling involves energy transfer between matter and quintessence with no momentum transfer to matter, so that no anomalous acceleration arises. In consequence, this choice is not affected by observational bounds on the ‘fifth’ force exerted on the baryons. Other generalizations of the given approach could be considered that do involve an anomalous acceleration of the background due to its coupling to quintessence. However, due to the universal nature of the coupling, it could not be detected by differential acceleration experiments. Another argument given in [7] is that since the coupling chosen is of a phenomenological nature and its validity is restricted to cosmological scales (it depends on magnitudes that are only well defined in that setting), the form of the coupling at smaller scales remains unspecified. The requirements for the different couplings that could have a manifestation at these scales are that (a) they give the same averaged coupling at cosmological scales, and (b) they meet the observational bounds from local experiments. We complete these arguments by noting that they are applicable even if the coupling is not of phenomenological origin as in the present investigation where the kind of coupling chosen originates in a scalar–tensor theory of gravity.

The paper has been organized as follows. In section 2, the details of the model are given. The method used to derive FRW solutions is explained in section 3. We use the ‘two fluids’ approach: a background fluid of DM and a self-interacting scalar field (quintessence). Specific exponential couplings with different exponents, which are inspired in the ST theory and lead, correspondingly, to self-interaction potentials of the single exponential and double exponential class, are studied separately. In section 4, a study of the existence and stability of the solutions is presented. We point out that due to the choice of solution generating ansatz, the coincidence problem cannot be avoided since a constant ratio between the energy densities of the components of the cosmic mixture is never a stable attractor. In section 5, conclusions are given.

2. The model

We consider the following action, which is inspired by a scalar–tensor theory written in the Einstein frame, where the quintessence (scalar) field is coupled to the matter degrees of freedom:

\[ S = \int_M \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) + C_2(\phi)L_{\text{matter}} \right\}. \tag{1} \]

In this equation \( R \) is the curvature scalar, \( \phi \) is the scalar (quintessence) field, \( V(\phi) \) is the quintessence self-interaction potential, \( C_2(\phi) \) is the quintessence–matter coupling function, and \( L_{\text{matter}} \) is the Lagrangian density for the matter degrees of freedom. This action could be considered instead as an effective theory, implying additional non-gravitational interaction between the components of the cosmic fluid.

Although in the present study we will be concerned mainly with FRW spacetimes with flat spatial sections, for generality we write the FRW line element in the form

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \]
\[ d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2, \tag{2} \]
where $t$ is the cosmic time, $r, \theta, \phi$ are the spatial (radial and angular) coordinates and $k$ is the spatial curvature, which we take to be zero in this investigation. We use the system of units in which $8\pi G = c = \hbar = 1$.

The spacetime is filled with a background pressureless dark matter fluid and a quintessence field (the scalar field $\phi$). As already stated, the baryons (a subdominant component at present, but important in the past of the cosmic evolution) are included in the background of dark matter. In the introductory part of this paper, we commented on the possibility of a universal coupling of dark energy to all sorts of matter, including baryons (and excluding radiation).

The field equations that are derived from the action (1) are

\[ 3H^2 + \frac{3k}{a^2} = \rho_m + \frac{1}{2} \dot{\phi}^2 + V, \]  
\[ 2\dot{H} + 3H^2 + \frac{k}{a^2} = (1 - \gamma) \rho_m - \frac{1}{2} \dot{\phi}^2 + V, \]  
\[ \ddot{\phi} + 3H \dot{\phi} = -V_w + (\ln X)' \rho_m, \]

where we have introduced the reduced notation $X(\phi) = C(\phi)^{\frac{3(3\gamma - 2)}{2}}$. The parameter $\gamma$ is the barotropic index of the background fluid (DM). The ‘continuity’ equation for the background is

\[ \dot{\rho}_m + 3\gamma H \rho_m = -(\ln X)' \dot{\phi} \rho_m, \]

or, after integration,

\[ \rho_m = C M a^{-3\gamma} X^{-1}, \]

where $M$ is a constant of integration. In the former equations, the dot accounts for derivative with respect to the co-moving time $t$, while the prime denotes derivative with respect to $\phi$. We now proceed to derive exact solutions to the above field equations by fixing the dynamics of the expansion.

3. Deriving solutions

In order to derive exact analytic (flat; $k = 0$) solutions, one should either fix the dynamics of the cosmic evolution or fix the functional form of the self-interaction potential $V(\phi)$. In the present study we fix the dynamics of the model by imposing the constraint

\[ \dot{\phi} = \lambda H, \]

which involves the Hubble parameter and the square root of the kinetic energy of the quintessence field ($\lambda$ is an arbitrary constant parameter). As will be immediately shown, this relationship enables one to reduce the system of differential equations (3)–(5) to a single first-order differential equation, involving the self-interaction potential $V$ together with its derivative with respect to the scalar field variable $\phi$ and the corresponding derivative of the coupling function. Therefore, if one chooses further the form of the coupling function, the functional form of the self-interaction potential can be found by solving the corresponding differential equation. As will be shown, exponential coupling functions with different exponents represent the simplest situations to study in the present model. Correspondingly, the ansatz (8) implies that only self-interaction potentials of the exponential form (including their combination) can be considered. Integration of equation (8) implies that

\[ a = e^{\phi/\lambda}, \]
where the scale factor has been normalized so that the integration constant has been absorbed into it. After this, equation (7) can be rewritten in the following form:

\[ \rho_m(\phi) = M e^{-\frac{3}{2} \gamma \lambda \phi X(\phi)} \]

If one adds up equations (3) and (4), one obtains

\[ \dot{H} + 3H^2 = \frac{2 - \gamma}{2} \rho_m + V. \]  

Substituting (8) into (5), and comparing the resulting equation with (11), a differential equation relating \( V \) and the coupling function \( X \) (and their derivatives with respect to the scalar field variable \( \phi \)) can be obtained:

\[ \frac{dV}{d\phi} + \lambda V(\phi) = \rho_m(\phi) \left( \frac{d \ln X}{d\phi} - \frac{\lambda (\gamma - 2)}{2} \right). \]

Consider further equation (8) written in the form \( d\phi = \lambda d(\ln a) \). It is then worthwhile rewriting equation (12) in the form

\[ V' + \lambda^2 V = \left( \frac{X' X - \lambda (\gamma - 2)}{2a^{3\gamma}} \right) M e^{3\gamma \lambda \phi X^{-1}}, \]

where now the prime denotes derivative with respect to the variable \( N = \ln a \) and we have substituted \( \rho_m(\phi) \) from equation (10). Equation (3) can then be integrated in quadratures:

\[ \int \frac{d\phi}{\sqrt{M e^{-\frac{3}{2} \gamma \phi X^{-1}}(\phi) + V(\phi)}} = \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} (t + t_0), \]

or, if one introduces the time variable \( d\tau = e^{-\frac{3}{2} \gamma \phi X^{-1/2}} d\tau \),

\[ \int \frac{d\phi}{\sqrt{M e^{\frac{3}{2} \gamma \phi X(\phi)} V(\phi)}} = \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} (\tau + \tau_0). \]

In consequence, once the function \( X(\phi) \) (or \( X(a) \)) is given as input, one can solve equation (12) (or (13)) to find the functional form of the potential \( V(\phi) \) (or \( V(a) \)). The integral (14) (or (15)) can then be taken explicitly to obtain \( t = t(\phi) \) (or \( \tau = \tau(\phi) \)). By inversion we can obtain \( \phi = \phi(t) \) (or \( \phi = \phi(\tau) \)) so the scale factor can be given as the function of either the cosmic time \( t \) or the time variable \( \tau \) through equation (9).

### 3.1. Particular cases

We shall study separately the simplest situations that can be considered once the choice (8) is made. In both cases one deals with coupling functions of the exponential form \( X(\phi) = X_0 \exp(n\phi) \), where \( X_0 \) and \( n \) are constant parameters. If one chooses \( n = (2 - 3\gamma/2)/\sqrt{\omega + 3/2} \), where \( \omega \) is the Brans–Dicke coupling parameter, the action (1) corresponds to the Brans–Dicke theory written in the Einstein frame (EF). In this case the EF scalar field \( \phi \) is related to the Jordan frame scalar field \( \hat{\phi} \) through \( d\phi = d\phi/(\omega + 3/2) \). In general, once the dynamics (8) is imposed, this kind of coupling function leads to double and single exponential potentials depending on \( r \). The importance of this class of potential in cosmology has already been outlined in the introductory part of this paper.
3.1.1. Case A. Let us consider the simplest situation when, in equation (12),
\[ \frac{d \ln X}{d \phi} = \frac{\lambda (\gamma - 2)}{2}, \]
(16)
i.e. we are faced with an exponential coupling function of the form mentioned above with
\[ r = \lambda (\gamma - 2)/2 \] in the exponent. In this case equation (12) simplifies to
\[ \frac{dV}{d\phi} + \lambda V = 0, \]
(17)
which can be easily integrated to yield to a single exponential potential:
\[ V = V_0 e^{-\lambda \phi}. \]
(18)
In consequence, equation (15) can be written as
\[ \int \frac{dw}{\sqrt{w^2 + A^2}} = \mu (\tau + \tau_0), \]
(19)
where
\[ w = \exp \left( -\frac{\gamma (6 - \lambda^2)}{4\lambda} \phi \right), \]
\[ A^2 = V_0 X_0 / M \]
and
\[ \mu = \gamma \sqrt{M (\lambda^2 - 6)/8}, \]
so we have the explicit solution
\[ \phi(\tau) = \phi_0 + \ln \left( \sinh \left[ \mu (\tau + \tau_0) \right] ^{\frac{4}{\gamma (6 - \lambda^2)}} \right), \]
(20)
and consequently
\[ a(\tau) = a_0 \sinh \left[ \mu (\tau + \tau_0) \right] ^{\frac{4}{\gamma (6 - \lambda^2)}}. \]
(21)
The dimensionless density parameter (for the i-th component
\[ \rho_i / \Omega_1i = \rho_0 / 3 H^2 \]) and the Hubble expansion parameter can be given as functions of the redshift z also. Actually, if one considers that
\[ a(\tau) = a_0 / (1 + z), \]
where \( a_0 \equiv a(z = 0) \) (for the simplicity of the calculations we choose the normalization \( a_0 = 1 \)), then
\[ \Omega_m(z) = \frac{6 - \lambda^2}{6 A^2} \frac{(1 + z)^{3/2 + \gamma (6 - \lambda^2)/2 - \lambda^2}}{(1 + z)^{3/2 + \gamma (6 - \lambda^2)/2 - \lambda^2} + A^2}, \]
(22)
and
\[ H(z) = B \sqrt{\frac{1}{A^2 (1 + z)^{3/2 + \gamma (6 - \lambda^2)/2} + (1 + z)^{3/2}}}, \]
(23)
where
\[ B = \sqrt{2 V_0 / (6 - \lambda^2)}. \]
Note that \( \Omega_m \) is a maximum at \( z = \infty : \Omega_m(\infty) = (6 - \lambda^2) / 6 A^2. \nIn general one can write that at the epoch of nucleosynthesis \( \Omega_m(\infty) = (1 - \epsilon) \), where \( \epsilon \) is a very small number (the small fraction of the dark energy component during the nucleosynthesis epoch), \( \epsilon = [6(A^2 - 1) + \lambda^2] / 6 A^2. \) Taking into account the observational fact that, according to model-independent analysis of SNIa data [13], at present, \( (z = 0; \Omega_m(0) = 1/3, (22) \) can be rewritten as
\[ \Omega_m(z) = \frac{(1 - \epsilon)(1 + z)^{3/2 + \gamma (6 - \lambda^2)/2 - \lambda^2}}{(1 + z)^{3/2 + \gamma (6 - \lambda^2)/2 - \lambda^2} + (2 - 3 \epsilon)}, \]
(24)
where now the solution exhibits only two free parameters \( \lambda \) and \( \epsilon \) (the DM EOS parameter \( \gamma = 1 \)).
Other physical magnitudes of observational interest are the quintessence equation of state (EOS) parameter and the deceleration parameter, which are given by the expressions
\[ \omega_{\phi} = -1 + \frac{\lambda^2}{3 \Omega_{\phi}}, \]
(25)
\[ q = -1 + \frac{\lambda^2}{2} + \frac{3 \gamma}{2} \Omega_m, \]
(26)
respectively.
3.1.2. Case B. A second very simple choice is to consider, in equation (12),
\[
\frac{d}{d\phi}(\ln X(\phi)) = \text{const} = \alpha \quad \Rightarrow \quad X(\phi) = X_0 e^{\alpha \phi},
\]
(27)
where \(X_0\) is an integration constant. In this case, by integration of (12), one obtains
\[
V(\phi) = V_0 e^{-\lambda \phi} + W_0 e^{-(\alpha + 3\gamma / \lambda)\phi},
\]
(28)
where the constant
\[
W_0 = \frac{M}{2X_0} \frac{(\lambda (2 - \gamma) - 2\alpha)}{(\alpha + 3\gamma / \lambda - \lambda)}.
\]
(29)
We are faced with a self-interaction potential that is a combination of exponentials with different constants in the exponent. The usefulness of this kind of potential has already been explained in the introduction and will be briefly discussed later within the frame of the present model when we study the stability of the corresponding solution. If one introduces the variable \(y = e^{l \phi}\), equation (15) can be written as
\[
\int \frac{dy}{\sqrt{y^2 + b^2}} = -\frac{l}{2} \sqrt{\frac{2\lambda^2 X_0 V_0}{6 - \lambda^2 b^2}} \tau + \tau_0,
\]
(30)
where \(b^2 = (X_0 V_0)/(M + W_0 X_0)\) and \(l = \alpha + 3\gamma / \lambda - \lambda\). Integration of (30) yields
\[
\phi(\tau) = \phi_0 + \ln(\text{sinh}[\mu(\tau + \tau_0)])^{-2/l},
\]
(31)
and consequently
\[
a(\tau) = a_0(\text{sinh}[\mu(\tau + \tau_0)])^{-2/l},
\]
(32)
where \(\mu = -\frac{l}{2} \sqrt{\frac{2\lambda^2 X_0 V_0}{6 - \lambda^2 b^2}}\). On the other hand, by using the Friedmann equation (3) and inserting the potential (28), one obtains the Hubble parameter as a function of the quintessence field \(\phi\),
\[
H^2(\phi) = \frac{2V_0}{6 - \lambda^2} \left[b^2 e^{-(\alpha + 3\gamma / \lambda)\phi} + e^{-\lambda \phi}\right],
\]
(33)
where we have considered equation (10). In terms of the scale factor one has instead
\[
H^2(a) = \frac{2V_0}{6 - \lambda^2} \left[b^2 a^{-(\alpha + 3\gamma)} + a^{-\lambda^2}\right].
\]
(34)
Using the redshift variable \(z\) the above magnitudes can be written as follows:
\[
H^2(z) = \frac{2V_0 b^2}{6 - \lambda^2} \left[(z + 1)^{(\alpha + 3\gamma)} + \frac{(1 + z)^{\lambda^2}}{b^2}\right].
\]
(35)
In turn, the energy density of the DM can be written as \(\rho_m(z) = (M/X_0)(z + 1)^{\alpha + 3\gamma} = \rho_0(z + 1)^{\alpha + 3\gamma}\), so the DM dimensionless density parameter can be given as a function of \(z\) also:
\[
\Omega_m(z) = \frac{(6 - \lambda^2)\rho_0}{2V_0 b^2} \frac{(z + 1)^{\alpha + 3\gamma - \lambda^2}}{(z + 1)^{\lambda^2} + 1/b^2}.
\]
(36)
In order to constrain the parameter space of the solution, one should consider the model to fit the observational evidence of a universe with an early matter-dominated period and a former transition to dark energy dominance\(^3\). Therefore \(\lambda^2 - \alpha \lambda - 3\gamma < 0\), besides, as
\(^3\) Since we are dealing with models with non-minimal coupling between the quintessence and the matter fields, here we refer mainly to a model-independent analysis of SNIa observational data [13–15].
before, one can write $\Omega_m(\infty) = (1 - \epsilon)$, where $\epsilon$ is a small number (the small fraction of the dark energy component during the nucleosynthesis epoch). In correspondence, $b^2 = [(6 - \lambda^2)/(1 - \epsilon)](\rho_0/2V_0)$. Besides, if one considers, as before, that according to model-independent analysis of SNIa data [13], at present ($z = 0$); $\Omega_m(0) = 1/3$, then $1/b^2 = 2 - 3\epsilon$. After this equation (36) can be rewritten as

$$\Omega_m(z) = (1 - \epsilon) \frac{(z + 1)^{\alpha+3\gamma-\lambda^2}}{(z + 1)^{\alpha+3\gamma-\lambda^2} + (2 - 3\epsilon)},$$

so the solution depends on three free parameters: $\alpha, \lambda$ and $\epsilon$ (the barotropic index of the DM is fixed: $\gamma = 1$). These can be chosen so that the solution fits the observational data well. Other physical magnitudes of observational interest are the quintessence EOS parameter and the deceleration parameter in equations (25) and (26), respectively.

In figure 1, we show the evolution of both dimensionless DM and scalar field energy densities $\Omega_m$ and $\Omega_\phi$, respectively, versus $z$ for the single exponential potential (case A). The corresponding evolution for case B is very similar. In figure 2, the evolution of the dynamical quintessence EOS parameter versus $z$ is plotted for model A (single exponential potential), for three different values of the parameter $\lambda$ : 0.3 (thick solid curve), 1.41 (solid curve) and 2.24 (dashed curve), respectively. In all cases the free parameter $\epsilon$ is chosen to be $\epsilon = 0.01$. Note that only the curve with the smallest value of the parameter $\lambda$ (thick solid line) meets the requirements of the present observational data favouring a value of the EOS parameter $\omega_\phi \sim -1$.
Figure 3. We plot the evolution of the deceleration parameter $q$ versus $z$ for the model A (single exponential potential). As before, the following values of the free parameters $\epsilon = 0.01$ and $\lambda = 0.3$ have been chosen. Again, the curve with the smallest value of the parameter $\lambda$ is the one that fits better in the observational data set on the present stage of accelerated expansion.

Figure 4. We plot the dynamical EOS parameter of the quintessence versus $z$, for the model B (double exponential potential), for three different values of the parameter $\alpha$: 0.1 (thick solid line), 1 (solid line) and 5 (dashed line), respectively. In all cases the remaining free parameters are chosen to be $\epsilon = 0.01$ and $\lambda = 0.3$, respectively. At small redshift, the dependence upon $\alpha$ is only weak.

Note that only the curve with the smallest value of the parameter $\lambda$ (thick solid curve) meets the requirements of the present observational data favouring a value of the EOS parameter $\omega_\phi \sim -1$. Meanwhile, in figure 3, we plot the evolution of the deceleration parameter $q$ versus $z$ for the model A. Again, the curve with the smallest value of the parameter $\lambda$ is the one that fits better the observational data set on the present stage of accelerated expansion [16, 17].

Figures 4 and 5 show the behaviour of $\omega_\phi = \omega_\phi(z)$ and $q = q(z)$ for different values of the free parameter $\alpha$: 0.1 (thick solid curve), 1 (solid curve) and 5 (dashed curve). The free parameters $\epsilon$ and $\lambda$ have been fixed ad hoc, guided by the results of the study of the case A ($\epsilon = 0.01, \lambda = 0.3$). It is apparent that the present values of $\omega_\phi$ and $q$ do not depend on $\alpha$, so this parameter cannot be determined from the observational data on the present stage of accelerated expansion of the universe. This parameter could have impact if the model is applied to the early stages in the cosmic evolution.

4. Existence and stability of the solutions

We now turn to the study of the stability of the solutions found. In order to keep the study as general as possible, we do not specify any concrete model for the dark energy. We only require
that the DE EOS parameter $\omega_{de} \geq -1$. To this end we rewrite the field equations (3)–(5) in the following form:

the Friedmann equation (we include the most general situation with spatial curvature $k \neq 0$)

$$3H^2 + \frac{k}{a^2} = \rho_m + \rho_\phi, \tag{38}$$

the Raychaudhuri equation

$$2\dot{H} - \frac{2k}{a^2} = -(p_m + \rho_m + p_\phi + \rho_\phi), \tag{39}$$

the continuity equation for the quintessence field

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q, \tag{40}$$

and the continuity equation for the background matter

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q, \tag{41}$$

where the dot accounts for the derivative with respect to the cosmic time and $Q = -(\ln X)'\rho_m$ is the interaction term. Through this section the prime will denote derivative with respect to the new variable $N = \ln a$, which is related to the cosmic time through $dN = H dt$.

Let us introduce the following dimensionless phase space variables$^4$:

$$x \equiv \Omega_\phi, \quad y \equiv \Omega_{tot} = \Omega_\phi + \Omega_m. \tag{42}$$

After this, the Friedmann equation (38) can be rewritten in the following way:

$$y = 1 + \frac{k}{a^2H^2}. \tag{43}$$

The governing equations (38), (39), (40) and (41) can be written in terms of the phase space variables in the following way:

$$x' = (y - x)(\ln \chi)' + x(y - 1 + 3\omega_\phi(x - 1)) \tag{44}$$

$$y' = (y - 1)(y + 3\omega_\phi x).$$

The above equations represent an autonomous system of equations if $(\ln \chi)'$ and $\omega_\phi$ do not depend on $N$ explicitly. In the remaining part of this section, for simplicity, we assume that this is the case so that the system (44) is an autonomous one. Besides, we consider $\chi(a) = \chi_0 a^4$.

$^4$ See [18] for an alternative treatment of a related stability study.
Table 1. The properties of critical points for the case A.

| x   | y   | Existence | Stability                                      |
|-----|-----|-----------|------------------------------------------------|
| 1   | 1   | Always    | Stable node if $0 < \lambda < \sqrt{2}$; saddle if $\sqrt{2} < \lambda < \sqrt{6}$; unstable node otherwise |
| \( \frac{x^2}{\tau} \) | 1   | $0 < \lambda \leq \sqrt{6}$ | Unstable node                                      |
| \( \frac{x^2}{\tau} \) | \( \frac{y^2}{\tau} \) | Always | Saddle point if $0 < \lambda < \sqrt{2}$; stable node if $\lambda > \sqrt{2}$ |

Table 2. The eigenvalues for the case A.

| Point (x, y) | \( \lambda_1 \) | \( \lambda_2 \) |
|-------------|-----------------|-----------------|
| (1, 1)      | \frac{\lambda^2 - 2}{\lambda} | \lambda^2       |
| (\( \frac{x^2}{\tau} \), 1) | 3 - \frac{\lambda^2}{\lambda} | \frac{\lambda^2}{\lambda} |
| (\( \frac{y^2}{\tau} \), \( \frac{x^2}{\tau} \)) | \(-\frac{\lambda^2}{\lambda} \) | \(-\frac{\lambda^2}{\lambda} \) |

where \( \delta \) is some constant parameter. This choice of coupling function comprises many useful situations (including the solutions we have derived before) and it implies that \((\ln \chi)' = \delta\). We assume, also, that \( \omega_m = 0 \Rightarrow \gamma = 1 \), i.e. the background fluid is dust. For the flat space case \((k = 0)\), the system (44) should be complemented with the constraint equation,

\[ 0 \leq y - x \leq 1, \quad (45) \]

which follows from requiring that the positive dimensionless matter energy density parameter \( \Omega_m \leq 1 \).

The first step towards the study of the dynamics of the autonomous system (44) is to find its critical points \((x_c, y_c) \Rightarrow (x', y') = (0, 0)\). Then one can investigate their stability by expanding equations (44) in the vicinity of the critical points \(x = x_c + u, y = y_c + v\) (up to terms linear in the perturbations \(u, v\)):

\[ \begin{pmatrix} u' \\ v' \end{pmatrix} = \Lambda \begin{pmatrix} u \\ v \end{pmatrix}, \]

where \( \Lambda \) is the matrix of the coefficients in the expansion. The general solution for the evolution of the linear perturbations can be written as

\[ u = u_1 e^{\lambda_1 N} + u_2 e^{\lambda_2 N}, \]
\[ v = v_1 e^{\lambda_1 N} + v_2 e^{\lambda_2 N}, \]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of the matrix \( \Lambda \).

In table 1 we show the properties of the critical points (including existence and stability) for the case A, meanwhile, in table 2, the corresponding eigenvalues are given. In this case \( X = X_0 e^{-\lambda \phi/2} \). As seen, for the values of \( \lambda \) that make the model fit the observational data set \((0 \leq \lambda < 1)\), the first critical point \((x, y) = (1, 1) \Rightarrow \Omega_\phi = 1 \) (quintessence-dominated phase) is a stable attractor, while the scaling solution \((x, y) = (\lambda^2/6, 1) \Rightarrow \Omega_m/\Omega_\phi = 6/\lambda^2 - 1 > 1\), is always unstable. The phase portrait for this case is shown in figure 6. All of the trajectories in the phase plane \((x, y)\), diverge from the unstable point (matter-dominated scaling solution) and converge towards the attractor (quintessence-dominated) solution.

The properties of the critical points and the corresponding eigenvalues for case B, where \( X = X_0 e^{\alpha \phi} \), are shown in tables 3 and 4, respectively. It is apparent that for the relevant ranges of the free parameters \(0 \leq \lambda < 1\) and \(0 < \alpha < 3/\lambda - \lambda\), the quintessence-dominated solution (the first critical point \((x, y) = (1, 1)\)) is always a stable node. The (matter-dominated) scaling solution—the second critical point \((x, y) = (\lambda(\lambda + \alpha)/3, 1)\)—could be either a saddle \((1/\lambda < \alpha < 3/\lambda - \lambda)\) or, otherwise, an unstable node. The phase portrait in figure 7 shows that
Figure 6. The phase plane for case A ($\gamma = 1$, $\lambda = 0.3$). The critical point $(1, 1)$ is stable (a sink) so that the quintessence-dominated solution is the late time attractor. The scaling regime $(0.015, 1)$ is an unstable point. The saddle is located at $(0.045, 0.045)$. All the phase space trajectories diverge from the unstable point and converge towards the attractor.

Figure 7. The phase plane for case B ($\gamma = 1$, $\lambda = 0.3$ and $\alpha = 5.7$). The quintessence-dominated solution (point $(1, 1)$ in the phase plane) is an attractor. The scaling solution (critical point $(0.6, 1)$) is a saddle, while $(0.045, 0.045)$ is an unstable point. For different values of the parameter $\alpha$, the separation between the saddle (scaling phase) and the stable (quintessence-dominated) critical points varies.

Table 3. The properties of critical points for case B.

| x  | y  | Existence | Stability                                                                 |
|----|----|-----------|--------------------------------------------------------------------------|
| 1  | 1  | Always    | Stable node if $0 < \lambda < \sqrt{2}$ and $0 < \alpha < \frac{\sqrt{2}}{\lambda}$; saddle point if either $0 < \lambda < \sqrt{2}$ and $\alpha < \frac{\sqrt{2}}{\lambda}$, or $\sqrt{2} < \lambda < \sqrt{3}$ and $0 < \alpha < \frac{\sqrt{2}}{\lambda}$; unstable node otherwise |
| $\frac{1}{2}(\lambda + \alpha)$ | 1  | $0 < \lambda < \sqrt{3}$ | Saddle point if $0 < \lambda < \sqrt{2}$ and $\frac{1}{2} < \alpha < \frac{\sqrt{2} - \lambda}{\lambda}$; unstable node otherwise |
| $\frac{\lambda^2}{2}$ | $\frac{\lambda^2}{2}$ | Always | Stable node if $\lambda > \sqrt{2}$ and $0 < \alpha < \frac{1}{\lambda}$; unstable node if $0 < \lambda < \sqrt{2}$ and $\alpha > \frac{1}{\lambda}$; saddle point otherwise |

Table 4. The eigenvalues for case B.

| Point $(x, y)$ | $\lambda_1$ | $\lambda_2$ |
|---------------|-------------|-------------|
| $(1, 1)$      | $\lambda^2 - 2$ | $\lambda^2 + \lambda \alpha - 3$ |
| $(\lambda (\lambda + \alpha)/3, 1)$ | $3 - \lambda^2 - \lambda \alpha$ | $1 - \lambda \alpha$ |
| $(\frac{\lambda^2}{4}, \frac{\lambda^2}{4})$ | $\lambda \alpha - 1$ | $2 - \lambda^2$ |
Dynamics of quintessence models of dark energy with exponential coupling to dark matter

for the values of the free parameters chosen \((\lambda = 0.3, \alpha = 5.7)\), all the phase space trajectories diverge from the unstable node (the third critical point \((x, y) = (\lambda^2/2, \lambda^2/2)\) quintessence-dominated solution with curvature) and either converge towards the stable attractor solution dominated by the quintessence (first critical point), or are repelled by the saddle point (second critical point—the matter-dominated scaling solution). This result, which is generic for both solutions A and B, shows that, since the scaling (quintessence dominated) solution with \(\Omega_m/\Omega_\phi \lesssim 1\) is not even a critical point of the corresponding autonomous dynamical system (equations (44)), the coincidence problem could arise in the cases studied in the present investigation. In the following subsection we will show in a more definitive manner that this is indeed the case.

4.1. The coincidence problem

Let us investigate whether, in the situations of interest in the present study, the following question arises: why are the energy densities of the dark matter and of the dark energy of precisely the same order at present? For this purpose it is recommended to study the dynamics of the ratio [7]

\[ r = \frac{\rho_m}{\rho_\phi} = \Omega_m/\Omega_\phi, \tag{48} \]

with respect to the variable \(N \equiv \ln a\), that, as said before, is related to the cosmic time \(t\) through \(dN = H \, dt\). The following generic evolution equation holds for \(r\):

\[ r' = f(r), \tag{49} \]

where the prime denotes derivative with respect to the variable \(N\), and \(f\) is an arbitrary function (at least of the class \(C^1\)) of \(r\). One is then primarily interested in the equilibrium points of equation (49), i.e. those points \(r_{ei}\) at which \(f(r_{ei}) = 0\). After that one expands \(f\) in the neighbourhood of each equilibrium point; \(r = r_{ei} + \epsilon_i\), so that, up to terms linear in the perturbations \(\epsilon_i : f(r) = (df/dr)_{r_{ei}} \epsilon_i + O(\epsilon_i^2) \Rightarrow \epsilon_i' = (df/dr)_{r_{ei}} \epsilon_i\). This last equation can be integrated to yield the evolution of the perturbations

\[ \epsilon_i = \epsilon_{i0} \exp \left[ (df/dr)_{r_{ei}} N \right], \tag{50} \]

where \(\epsilon_{i0}\) are arbitrary integration constants. It is seen from (50) that only those perturbations for which

\[ (df/dr)_{r_{ei}} < 0 \tag{51} \]

decay with the time variable \(N\), and the corresponding equilibrium point is stable. The coincidence problem is evaded if the point \(\rho_m/\rho_\phi = r_{ei} \lesssim 1\) is stable.

Now, if we take into account the conservation equations (40) and (41), where \(Q = -\ln X' \rho_m\), and since \(\Omega_\phi = 1/(r + 1)\), implying that (see equation (25)) the quintessence EOS parameter \(\omega_\phi\) can be written as a function of \(r : \omega_\phi = -1 + \lambda^2(r + 1)/3\), then, for the cases under study here, the function \(f\) can be given by the expression

\[ f(r) = r[(\lambda^2 - \delta)(r + 1) - 3], \tag{52} \]

where \(\delta = \ln X' = n\lambda\). For case A, \(\delta = -\lambda^2/2\), while for case B, \(\delta = \lambda \alpha\). It can be easily checked that in both cases the only stable equilibrium point is that for which the ratio \(r_{ei} = 0\), i.e. it is the quintessence-dominated solution, so the coincidence problem cannot be avoided.
5. Conclusions

We have found a new parametric class of exact cosmological scaling solutions in a theory with general non-minimal coupling between the components of the cosmic mixture: the cold dark matter and the dark energy (the quintessence field). Baryons, although subdominant at present, in the past played a relevant role in cosmic evolution and could be included as part of the dark matter component in the present set-up. There are suggestive arguments showing that observational bounds on the ‘fifth’ force do not necessarily imply decoupling of baryons from quintessence [7].

To specify the general form of the coupling, we considered a scalar–tensor theory of gravity written in the Einstein frame. An alternative interpretation is to consider it as an effective theory, implying the additional non-gravitational interaction between the components of the cosmic fluid.

In order to derive exact flat FRW solutions to the model under study, we have assumed a linear relationship between the Hubble expansion parameter and the time derivative of the scalar field. Mathematically, this assumption allows us to reduce the original system of second-order differential equations to a single, first-order differential equation, involving only the self-interaction potential and the coupling function (together with their first derivatives with respect to the scalar field variable). However, the simplicity of the mathematical handling is at the cost of retaining the problem of the cosmic coincidence. In fact, the solution generating ansatz (8) implies that solutions where the ratio $\rho_m/\rho_\phi = \text{const} \sim 1$ are not stable. In the cases studied, only the quintessence-dominated solution is stable. Anyway, the assumed relationship between the square root of the scalar field kinetic energy and the Hubble parameter is the simplest possible and, in the absence of any other information is, in a sense, the most natural since the Hubble parameter fixes the time scale.

We have concentrated our study on the exponential class of coupling functions. The Brans–Dicke theory is a particular member in this class. Exponential coupling functions can lead to self-interaction potentials of the following class: (A) single exponential potential and (B) double exponential potential. The stability and existence of the solutions found have also been studied. For this purpose we have applied a fairly general method in which one does not need to specify any model for the dark energy. In both cases (case A and case B), the dynamical system exhibits three critical points. For the values of the free parameters that are allowed by the observational data, the scalar field (quintessence-) dominated solution is always an attractor, while the scaling (matter-dominated) solution can be either an unstable node or a saddle point.

We conclude that models with non-minimal coupling between the dark energy and the dark matter are easy to handle mathematically if one assumes a suitable dynamics. This is sometimes at the cost of retaining the problem of the coincidence. The present investigation could be complemented by the study of different classes of self-interaction potentials and/or the choice of dynamics allowing us to evade the coincidence problem, one of the motivations for studying models with interaction between the components of the cosmic mixture.

Acknowledgments

We thank Diego Pavon for useful comments and the MES of Cuba for financial support of this research.

References

[1] Kolda C and Lyth D H 1999 Phys. Lett. B 458 197–201 (Preprint hep-ph/9811375)
Dynamics of quintessence models of dark energy with exponential coupling to dark matter

[2] Sahni V 2002 *Class. Quantum Grav.* **19** 3435–48 (Preprint astro-ph/0202076)

[3] Sahni V 2002 *Class. Quantum Grav.* **19** 3435–48 (Preprint hep-th/0212290)

[4] Will C M 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press) (Preprint gr-qc/0103036)

[6] Amendola L and Tochini D 2002 *Phys. Rev.* D **66** 043519 (Preprint astro-ph/0203018)

[11] Rubano C and Scudellaro P 2002 *Gen. Rel. Grav.* **34** 307 (Preprint astro-ph/0103335)

[12] Arias O, Gonzalez T, Leyva Y and Quiros I 2003 *Class. Quantum Grav.* **20** 2563–78 (Preprint gr-qc/0307016)

[14] Freedman et al 2001 *Astrophys. J.* **553** 47

[15] John M 2004 *Astrophys. J.* **614** 1

[16] Turner M S 2002 *Int. J. Mod. Phys.* A **17S1** 180–96 (Preprint astro-ph/0202008)

[17] Riess A 2001 *Astrophys. J.* **560** 49–71 (Preprint astro-ph/0104455)

[19] Perlmutter S, Turner M S and White M 1999 *Phys. Rev. Lett.* **83** 670–3 (Preprint astro-ph/9901052)

[20] Sen A A and Sethi S 2002 *Phys. Lett.* B **532** 159–65 (Preprint gr-qc/0111082)

[21] Corasaniti P S and Copeland E J 2002 *Phys. Rev.* D **65** 043004 (Preprint astro-ph/0107378)

[22] Bean R, Hansen S H and Melchiorri A 2002 *Nucl. Phys. Proc. Suppl.* **110** 167 (Preprint astro-ph/0201127)

[23] Turner M S 1999 *Nucl. Phys. Proc. Suppl.* **72** 69 (Preprint astro-ph/9811366)

[24] Perlmutter S et al 1999 *Astrophys. J.* **517** 565 (Preprint astro-ph/9812133)

[25] Barreiro T, Copeland E J and Nunes N J 2000 *Phys. Rev.* D **61** 127301 (Preprint astro-ph/9910214)

Copeland E J, Nunes N J and Rosati F 2000 *Phys. Rev.* D **62** 123503 (Preprint astro-ph/0005222)