Quantum filtering for multiple measurements driven by fields in single-photon states*

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Abstract—In this paper, we derive the stochastic master equations for quantum systems driven by a single-photon input state which is contaminated by quantum vacuum noise. To improve estimation performance, quantum filters based on multiple-channel measurements are designed. Two cases, namely diffusive plus Poissonian measurements and two diffusive measurements, are considered.

Key words: single-photon state, quantum filtering, homodyne detection, photon-counting, continuous-time stochastic master equation, Wiener process, quantum trajectories.

1. INTRODUCTION

When light interacts with a quantum system, e.g., a two-level atom or an optical cavity, partial system information can be transferred to the output light. The output light may be measured, say via homodyne detection, to produce photocurrent upon which the state of quantum system can be conditioned. The stochastic evolution of the conditional system state is usually called quantum trajectory. Quantum filter can be designed to estimate these trajectories [3], [4], [11], [13], [17], [19].

In quantum optics, the quantum filtering problem is known under the names of stochastic master equation and quantum trajectory theory [5], [11], [19]. It was first developed by Belavkin within a framework of continuous non-demolition quantum measurement [3], [4]. The formalism of quantum filtering for Gaussian input fields, including the vacuum state, coherent state, squeezed state and thermal state, have been considered and well studied [8], [11], [15], [19]. With a variety of experimental architectures, such as cavity quantum electrodynamics (QED) [14], circuit QED [9] and quantum dots in semiconductors [20], nonclassical states of light have also been discussed in connection with quantum networks. A range of nonclassical states, single-photon states and coherent states have been considered in [13]. Particularly, the master equations and stochastic master equations are presented for an arbitrary quantum system probed by a continuous-mode single-photon input field. As an application, the conditional dynamics for the cross phase modulation in a doubly resonant cavity are considered in [6], where both homodyne detection and photon-counting measurements are simulated for a cavity driven by a single-photon input field. The interaction of a two-level atom with a propagating mode single-photon in free space has been discussed in the literature, see e.g., [18]. The dependence of the atomic excitation probability on the temporal and spectral features of single-photon pulse shapes and coherent states pulse shapes are also considered in [18].

In real physical experiments, there may be limitations for the case of single measurements due to the existence of noise. To circumvent this imperfection, quantum filtering problem with multiple output fields has been developed using quantum trajectory theory with multi-input-multi-output (MIMO) quantum feedback [7]. A finite dimensional discrete-time Markov model in the cases of perfect and imperfect measurements are described in [2]. For the state estimations used in the feedback scheme, the quantum filters are discussed and a general robustness property for perfect and imperfect measurements are proved. An experimental implementation has been conducted by using the photon box and closed-loop simulations are also presented [16]. The observed system in [1] is assumed to be governed by a continuous-time stochastic master equation driven by Wiener and Poisson processes. Particularly, the incompleteness and errors in measurements have been taken into account and the measurement imperfections are modeled by a stochastic matrix.

In this paper, we extend the single-photon filtering framework proposed in [6], [13] by including imperfect measurements. More specifically, we study the case when the output light field is corrupted by a vacuum noise. We show how to design filters based on multiple measurements to achieve desired estimation performance. Two scenarios are studied, 1) homodyne plus homodyne detection and 2) homodyne plus photon-counting detection.

The paper is organized as follows. In section 2 we review some basic theory such as open quantum systems, series products and quantum filtering. In section 3 we propose quantum filters for the cases of joint homodyne detection and photon-counting measurements and both homodyne de-
The commutator is defined to be 
\[ [A, B] = AB - BA \]
and by Itô calculus, we can find the following evolution equations (QSDE)
\[
dU(t) = \left\{ (S - I)d\Lambda(t) + \frac{1}{2}L^*L + iH \right\} dt \quad (2.2)
\]
where \( U(0) = I \).

In the Heisenberg picture, we can derive the evolution of \( j_t(X) = X(t) \) as
\[
dj_t(X) = j_t([Z_t, X] S) d\Lambda(t) + j_t(\{L^*, X\} S) dB(t) + j_t(\{S^*, X, L\} B^*(t)) dB(t)
\]
\[ + j_t(\{S^*, X, S\} B^*(t)) d\Lambda(t). \quad (2.3) \]

The output fields are defined by
\[
B_{out}(t) = U^\dagger(t)(I_{\text{out}} \otimes B(t)) U(t),
\]
\[
\Lambda_{out}(t) = U^\dagger(t)(I_{\text{out}} \otimes \Lambda(t)) U(t),
\]
and by Itô calculus, we can derive the following evolution equations
\[
dB_{out}(t) = S(t) dB(t) + L(t) dt,
\]
\[
d\Lambda_{out}(t) = S^*(t) d\Lambda(t) S^T(t) + S^*(t) dB(t) L^T(t) + S^*(t) L^T(t) dt. \quad (2.4)\]

The dynamical evolution can be described by a unitary operator \( U(t) \) on the tensor product Hilbert space \( H_S \otimes H_F \)

\[
\frac{d}{dt} U(t) = \left\{ (S - I) d\Lambda(t) + \frac{1}{2}L^*L + iH \right\} dt.
\]

\[
dU(t) = \left\{ (S - I) d\Lambda(t) + \frac{1}{2}L^*L + iH \right\} dt \quad (2.2)
\]

and

\[
dj_t(X) = j_t([Z_t, X] S) d\Lambda(t) + j_t(\{L^*, X\} S) dB(t).
\]

\[
dj_t(X) = j_t([Z_t, X] S) d\Lambda(t) + j_t(\{L^*, X\} S) dB(t) + j_t(\{S^*, X, L\} B^*(t)) dB(t)
\]

\[
\quad + j_t(\{S^*, X, S\} B^*(t)) d\Lambda(t). \quad (2.3)
\]

\[
B_{out}(t) = U^\dagger(t)(I_{\text{out}} \otimes B(t)) U(t),
\]

\[
\Lambda_{out}(t) = U^\dagger(t)(I_{\text{out}} \otimes \Lambda(t)) U(t),
\]

\[
dB_{out}(t) = S(t) dB(t) + L(t) dt,
\]

\[
d\Lambda_{out}(t) = S^*(t) d\Lambda(t) S^T(t) + S^*(t) dB(t) L^T(t) + S^*(t) L^T(t) dt. \quad (2.4)
\]

\[
Y(t) = U^\dagger(t)(I_{\text{out}} \otimes \Lambda(t)) U(t). \quad (2.6)
\]

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\]

\[
Y(t) = \Lambda_{out}(t) = U^\dagger(t)(I_{\text{out}} \otimes \Lambda(t)) U(t).
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\]

Both of the measurements satisfy the following commutation relations \( [Y(s), Y(t)] = 0, 0 \leq s \leq t \).

\[
B \quad \text{The Concatenation and Series Products}
\]

1) Concatenation Product [12]: Given two systems \( G_1 = (S_1, L_1, H_1) \) and \( G_2 = (S_2, L_2, H_2) \), we define the concatenation product to be the system \( G_1 \oplus G_2 \) by
\[
G_1 \oplus G_2 = \left( \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H_1 + H_2 \right). \quad (2.7)
\]

Fig. 1. Concatenation product.
2) Series Product [12]: Given two systems \( G_1 = (S_1, L_1, H_1) \) and \( G_2 = (S_2, L_2, H_2) \) with the same number of field channels, see Fig. 2 we define the series product \( G_2 \parallel G_1 \) by \( G_2 \parallel G_1 = \left( S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \text{Im}\{L_2^* S_2 L_1\} \right) \).

C. Quantum Filtering

The quantum conditional expectation is defined by
\[
\hat{X}(t) = \pi_i(X) = \mathbb{E}[j_i(X)|\mathcal{Y}],
\]
where \( \mathcal{Y} \) is generated by \( \{Y(s) : 0 \leq s \leq t\} \). Generally the quantum filtering problem is about minimizing the least mean-squares estimate \( \mathbb{E}\{(\hat{X}(t) - j_i(X))^2\} \) of system observables \( j_i(X) \) based on the past measurement information \( \mathcal{Y} \). Furthermore, we note that the set of observables \( \{Y(s) : 0 \leq s \leq t\} \) is self-commuting \( [Y(t), Y(s)] = 0 \), \( s \leq t \). The quantum conditional expectation is well-defined since it satisfies the non-demolition property \( [X(t), Y(s)] = 0 \), \( s \leq t \).

3. QUANTUM FILTER FOR MULTIPLE MEASUREMENTS

In this section, we mainly present the stochastic master equations for a quantum system interacting with a single-photon state, see Fig. 3. The single-photon state is defined in subsection 3-A, the extended system is briefly reviewed and the relation between expectation for system and for extended system is discussed in subsection 3-B. The general quantum filter for multiple compatible measurements is introduced in subsection 3-C. For the single-photon input state, we derive the filtering equations in subsection 3-D and 3-E. An illustrating example is given in subsection 3-E.

A. Continuous-mode Single-photon State

The creation operator for a photon with wave packet \( \tilde{\xi}(t) \) in time domain is defined as
\[
B^*(\tilde{\xi}) = \int_0^\infty \tilde{\xi}(t) b^*(t)dt,
\]
with the normalization condition \( \int_0^\infty |\tilde{\xi}(t)|^2 dt = 1 \). Then the single-photon state is given by
\[
|1_\xi\rangle = B^*(\tilde{\xi})|0\rangle.
\] (3.10)

The original departure of the master and filter equations between single-photon input and the vacuum case are given by the following identities
\[
dB(t)|1_\xi\rangle = \tilde{\xi}(t)dt|0\rangle,
\]
\[
d\lambda(t)|1_\xi\rangle = \tilde{\xi}(t)dB^\dagger(t)|0\rangle.
\] (3.11)

B. The Extended System

We construct a quantum signal generating filter \( M = (S_M, L_M, H_M) \), which is usually called ancilla. Cascading this ancilla system model \( M \) with the quantum system \( G \), then we get the extended system \( G_T = G \parallel M \). Since the extended system \( G_T \) is driven by vacuum input, the master equation and quantum filter follow from the known result in subsection 3-C with the parameters for the extended system \( G_T \) [13]. The interaction with the vacuum input is given by
\[
(S_M, L_M, H_M) = (I, \lambda(t)\sigma_-, 0),
\] (3.12)
where \( \sigma_- \) is the lowing operator from the upper state \( |\uparrow\rangle \) to the ground state \( |\downarrow\rangle \). It means that the atom decays into its ground state at some stage, creating a single photon in the output. The signal model will output the desired single-photon state \( |1_\xi\rangle \) since we can choose the coupling strength \( \lambda(t) \)
\[
\lambda(t) = \frac{\tilde{\xi}(t)}{\sqrt{w(t)}},
\] (3.13)
where \( w(t) = \int_0^t |\tilde{\xi}(s)|^2 ds \).

By using the cascade connection formalism, we have the extended system \( G_T \)
\[
G_T = (S, L + \lambda(t)S\sigma_-, H + \lambda(t)\text{Im}\{L^\dagger S\sigma_\}) \cdot
\] (3.14)

Let \( \bar{U}(t) \) be the unitary operator for the joint ancilla, system and field spaces. The following equality can be shown (see [13] for more details)
\[
\mathbb{E}_{\eta}(X(t)) = \mathbb{E}_{\eta_0}(\bar{U}^\dagger(t)(I \otimes X \otimes I)\bar{U}(t)),
\] (3.15)
with initial state \( |\uparrow\rangle \otimes |\eta\rangle \otimes |0\rangle \) for arbitrary operator \( X(t) \) of the system \( G \).

C. Quantum Filter for Multiple Measurements Driven by Vacuum Input

To derive the quantum filter for system driven by a single-photon input state, we firstly introduce the result of multiple measurements with vacuum state input.

**Lemma 3.1**: ([10, Theorem 3.2]) Let \( \{Y_{ij}, i = 1, 2, \ldots, N\} \) be a set of \( N \) compatible measurement outputs for a quantum
system $G$. With vacuum initial state, the corresponding joint measurement quantum filter is given by

$$d\tilde{X} = \pi_t[\mathcal{L}_G(X_t)]dt + \sum_{i=1}^{N} \beta_{t,i} dW_{t,i}, \quad (3.16)$$

where $dW_{t,i} = dY_{t,i} - \pi_t(dy_{t,i})$ is a martingale process for each measurement output and $\beta_{t,i}$ is the corresponding gain given by

$$\zeta_t = \pi_t(X_t dY_{t,i}^T - \pi_t(dy_{t,i}^T) + \pi_t(L_t X_t S_t dB_t Y_{t,i}^T),$$

$$\Sigma = \pi_t(dy_{t,i} Y_{t,i}^T), \quad \beta = \Sigma^{-1} \zeta,$$

(3.17)

where $\Sigma$ is assumed to be non-singular.

Remark 3.1: A general measurement equation, which is a function of annihilation, creation and conservation processes in the output field, is defined as [10]

$$dY(t) = F^* dB_{\text{out}}^*(t) + F dB_{\text{out}}(t) + G \text{diag}(dX_{\text{out}}(t)). \quad (3.18)$$

Particularly, a combination of homodyne detection and photon-counting measurement is given by

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.19)$$

D. Quantum Filter for Joint Homodyne and Photon-counting

Detections Driven by Single-photon State

Suppose that the system is in an initial state $\rho_0 = |\eta\rangle\langle\eta|$ and the single-photon input state is $|1_2\rangle$. For a given system operator $X$, we define the expectation

$$\omega_{jk}^{\eta}(X) = \mathbb{E}_{j,k}[j(X) = \langle\eta|j\rangle|X\rangle\langle\eta|\phi_k], \quad j,k = 0,1,$$

where $\phi_j = \{0\}$, $j = 0$; $|1_2\rangle$, $j = 1$.

The quantum filter for the conditional expectation for the system $G$ driven by a single-photon field is given by

$$\pi_t^{11}(X) = \mathbb{E}_{\eta} \xi_t[X(t)|Y(s), 0 \leq s \leq t], \quad (3.20)$$

and the quantum filter for the extended system $G_T = G \otimes M$ driven by vacuum input is defined as

$$\pi_t(A \otimes X) = \mathbb{E}_{\eta_0} [U(t)^\dagger(A \otimes X) U(t)|I \otimes Y(s), 0 \leq s \leq t],$$

(3.21)

where $A$ is an ancilla operator and $X$ is a system operator.

The whole system $G$ with the measurements in Fig. 3 can be depicted as shown in Fig. 4 $G_1 = (S, L, H)$ is the original system $G$, which has been connected with a signal model (ancilla) $M = (I, \lambda(t)I, 0)$. By introducing a second open quantum system $G_2 = (1, 0, 0)$, we concatenate the vacuum noise into our system. The last open quantum system is a beam splitter $G_3 = (S_h, 0, 0)$, where

$$S_h = \begin{bmatrix} \sqrt{1-r^2} e^{i\theta} & re^{i(\theta + \frac{\pi}{2})} \\ re^{i(\theta - \frac{\pi}{2})} & \sqrt{1-r^2} e^{i\theta} \end{bmatrix}, \quad 0 \leq r \leq 1. \quad (3.22)$$

By the concatenation and series product, the whole system $G$ is given by

$$G = G_3 \circ (G_1 \circ M) \circ G_2 = (S_t, L_t, H_t), \quad (3.23)$$

where $S_t = S_h \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}$, $L_t = \begin{bmatrix} L + \lambda(t) S_{\sigma^+} & 0 \\ 0 & 0 \end{bmatrix}$, $H_t = H + \lambda(t) \Im\{L^T S_{\sigma^-}\}$.

Furthermore, the Lindblad superoperator $\mathcal{L}_G(A \otimes X)$ for the whole system $G$ can be expressed in the following form

$$\mathcal{L}_G(A \otimes X) = A \otimes \mathcal{L}_G X + (\mathcal{L}_G A \otimes X + L^T_M A \otimes S^T X, L)$$

$$+ ALM \otimes [L^T, X] S + L^T_M ALM \otimes (S^T XS - X), \quad (3.24)$$

where $A$ is any operator of ancilla and $X$ is the system operator.

In what follows, we denote $B_{i,j}$, which is a vacuum state, is the input of signal model $M$ and $B_{i,j}$ is the vacuum noise for system $G_2$, then the total input, together with gauge process for the whole system $G$ are given by

$$B_i = \begin{bmatrix} B_{i,j} \\ B_{i,j} \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \Lambda_i \Lambda_i, \Lambda_i \Lambda_i \\ \Lambda_i \Lambda_i, \Lambda_i \Lambda_i \end{bmatrix}. \quad (3.25)$$

By the evolution of output fields (2.23), the measurements stochastic equations are given by

$$dY_{1,j} = \sqrt{1-r^2} \begin{bmatrix} e^{i\theta}(L + SL_M) + e^{-i\theta}(L^T + L^T_M S^T) \\ + e^{i\theta} S_{d_i} + e^{-i\theta} S^T d_{i_j} \end{bmatrix} dt$$

$$+ e^{i\theta} S_{d_i} + e^{-i\theta} S^T d_{i_j} + ir \begin{bmatrix} e^{i\theta} d_{i_j} - e^{-i\theta} d_{i_j} \end{bmatrix}, \quad (3.26)$$

and

$$dY_{2,j} = r^2 \begin{bmatrix} S_{d_i} S^T + (L + SL_M) S^T d_{i_j} + S(L^T + L^T_M S^T) d_{i_j} \\ + (L^T + L^T_M S^T)(L + SL_M) dt \end{bmatrix} + (1-r^2) d_{i_j}$$

$$+ ir \sqrt{1-r^2} \begin{bmatrix} S_{d_i} S^T - S^T d_{i_j} + (L + SL_M) d_{i_j} \\ - (L^T + L^T_M S^T) d_{i_j} \end{bmatrix}, \quad (3.27)$$

where $dY_{1,j}$ is the first channel with homodyne detection and $dY_{2,j}$ is the second channel with photon-counting measurement. Then the expectation and correlation of the measure-
Theorem can be derived as
\[
\pi_t(dy_{1,t}) = \sqrt{1-r^2} \rho \left[ e^{i\theta} (L + SL_M) + e^{-i\theta} (L^\dagger + L_M^\dagger S^\dagger) \right] dt,
\]
\[
\pi_t(dy_{2,t}) = \pi_t(dy_{2,t}) = r^2 \rho \left[ (L^\dagger + L_M^\dagger S^\dagger)(L + SL_M) \right] dt,
\]
\[
\pi_t(dy_{1,t}) = dt,
\]
\[
\pi_t(dy_{1,t}, dy_{2,t}) = \pi_t(dy_{2,t}, dy_{1,t}) = 0.
\]

Thus, the corresponding gain \( \beta = [\beta_1, \beta_2] \) can be calculated by (3.17)
\[
\beta_1 = \sqrt{1-r^2} e^{i\theta} \rho (A \otimes XL + AL_M \otimes XS)
+ \sqrt{1-r^2} e^{-i\theta} \rho (A \otimes L^\dagger XL + L_M^\dagger A \otimes S^\dagger X)
- \sqrt{1-r^2} \rho \rho (A \otimes X)
\times \pi_t \left[ e^{i\theta} (L + SL_M) + e^{-i\theta} (L^\dagger + L_M^\dagger S^\dagger) \right],
\]
\[
\beta_2 = \left[ \pi_t(L^\dagger L + L_M^\dagger S^\dagger L + L_M^\dagger SL_M) \right]^{-1}
\times \pi_t (A \otimes L^\dagger XL + L_M^\dagger A \otimes S^\dagger XL + AL_M \otimes L^\dagger XS)
+ L_M^\dagger AL_M \otimes S^\dagger XS) - \pi_t (A \otimes X),
\]
where \( A \) is any ancilla operator and \( X \) is the system operator.

Finally, by Lemma 3.1, the joint measurement quantum filter for the whole system \( G \) driven by vacuum input state is given by
\[
d\pi_t(A \otimes X) = \pi_t(A \otimes X) dt + \beta_1 [dy_{1,t} - \pi_t(dy_{1,t})]
+ \beta_2 [dy_{2,t} - \pi_t(dy_{2,t})],
\]
where the Lindblad superoperator \( \mathcal{L}_G(A \otimes X) \) is defined in (3.29).

If we define [13]
\[
\pi^{jk}_t(X) = \frac{\pi_t(Q_{jk} \otimes X)}{w_{jk}}, \quad j, k = 0, 1,
\]
where \( Q_{jk} \) and \( w_{jk} \) are given by
\[
Q_{jk} = \begin{bmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{bmatrix} = \begin{bmatrix} \sigma_x & \sigma_x & \sigma_y & \sigma_y \\ \sigma_x & \sigma_x & \sigma_y & \sigma_y \\ \sigma_y & \sigma_y & \sigma_x & \sigma_x \\ \sigma_y & \sigma_y & \sigma_x & \sigma_x \end{bmatrix},
\]
\[
w_{jk} = \begin{bmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{bmatrix} = \begin{bmatrix} w(t) & \sqrt{w(t)} \\ \sqrt{w(t)} & 1 \end{bmatrix},
\]
we obtain the following theorem which presents the quantum filter for the system \( G \) driven by single-photon state [12].

**Theorem 3.1:** Let \( \{Y_{i,t}, i = 1, 2\} \) be a combination of homodyne detection and photon-counting measurement for a quantum system \( G \). With single-photon input state, the quantum filter for the conditional expectation in the Heisenberg picture is given by (3.36). Here,
\[
K_t := e^{i\theta} \pi^{11}_t(L) + e^{-i\theta} \pi^{11}_t(L^\dagger)
+ e^{i\theta} \pi^{01}_t(S^\dagger) \xi^*(t) + e^{-i\theta} \pi^{10}_t(S) \xi(t),
\]
\[
\nu_t := \pi^{11}_t(L^\dagger L) + \pi^{01}_t(S^\dagger L) \xi^*(t)
+ \pi^{10}_t(L^\dagger S) \xi(t) + \pi^{00}_t(I) [\xi(t)]^2,
\]
\[
W_t := \pi^{11}_t(L^\dagger L) + \pi^{01}_t(S^\dagger L) \xi^*(t)
+ \pi^{10}_t(L^\dagger S) \xi(t) + \pi^{00}_t(I) [\xi(t)]^2,
\]
\[
\nu_t := \pi^{11}_t(L^\dagger L) + \pi^{01}_t(S^\dagger L) \xi^*(t)
+ \pi^{10}_t(L^\dagger S) \xi(t) + \pi^{00}_t(I) [\xi(t)]^2.
\]

E. Quantum Filter for Both Homodyne Detection Measurements

In this subsection, we will derive the filter equations for the case of joint homodyne-homodyne measurements, see Fig. 5. Here, the general measurement equation (3.18), we choose \( F = I \), \( G = 0 \). Then, the measurements stochastic equations are given by (3.24) and
\[
dY_{2,t} = \sqrt{1-r^2} \left( e^{i\theta} dB_{1,t} + e^{-i\theta} dB_{1,t}^\dagger \right)
+ i \left[ e^{i\theta} (L + SL_M) - e^{-i\theta} (L^\dagger + L_M^\dagger S^\dagger) \right] dt
+ e^{i\theta} S dB_{1,t} - e^{-i\theta} S^\dagger dB_{1,t}^\dagger,
\]
\[
dW_t = \sqrt{1-r^2} \left( e^{i\theta} dB_{1,t} + e^{-i\theta} dB_{1,t}^\dagger \right)
+ i \left[ e^{i\theta} (L + SL_M) - e^{-i\theta} (L^\dagger + L_M^\dagger S^\dagger) \right] dt
+ e^{i\theta} S dB_{1,t} - e^{-i\theta} S^\dagger dB_{1,t}^\dagger,
\]
\[
\nu_t = \left[ e^{i\theta} (L + SL_M) - e^{-i\theta} (L^\dagger + L_M^\dagger S^\dagger) \right] dt
+ e^{i\theta} S dB_{1,t} - e^{-i\theta} S^\dagger dB_{1,t}^\dagger,
\]
\[
\pi_t = e^{-i\theta} \pi^{11}_t(L) + e^{i\theta} \pi^{11}_t(L^\dagger)
+ e^{i\theta} \pi^{01}_t(S^\dagger) \xi^*(t) + e^{-i\theta} \pi^{10}_t(S) \xi(t),
\]
\[
\pi_t(dy_{1,t}) = \pi_t(dy_{2,t}) = \pi_t(dy_{2,t}) = dt,
\]
\[
\pi_t(dy_{1,t}, dy_{2,t}) = \pi_t(dy_{2,t}, dy_{1,t}) = 0.
\]
Given by $dY$, measurement. Thus, the corresponding gain expectation in the Heisenberg picture is given by (3.47).

Then, in the case of both channels are with homodyne detection measurement. The corresponding gain expectation in the Heisenberg picture is given by (3.47).

\[
\begin{align*}
\text{Corollary 3.2:} & \quad \text{Let } Y_{i,j}, i = 1,2 \text{ be the two homodyne detection measurements for a quantum system } G. \text{ With single-photon input state, the quantum filter for the conditional expectation in the Heisenberg picture is given by (3.47).}
\end{align*}
\]

Here,

\[
\begin{align*}
K_{1,j} & = e^{i\theta} \pi_{11}^{(1)}(L) + e^{-i\theta} \pi_{11}^{(1)}(L^\dagger) \\
& + e^{-i\theta} \pi_{01}^{(1)}(S^\dagger) \pi_{11}^{(1)}(S) + e^{i\theta} \pi_{10}^{(1)}(S) \pi_{11}^{(1)}(S^\dagger), \\
K_{2,j} & = e^{i\theta} \pi_{11}^{(1)}(L) + e^{-i\theta} \pi_{11}^{(1)}(L^\dagger) \\
& - e^{-i\theta} \pi_{01}^{(1)}(S^\dagger) \pi_{11}^{(1)}(S) + e^{i\theta} \pi_{10}^{(1)}(S) \pi_{11}^{(1)}(S^\dagger),
\end{align*}
\]

the Wiener processes $W_1(t)$ and $W_2(t)$ are given by

\[
\begin{align*}
dW_1(t) & = dY_{1,j} - \sqrt{1 - r^2} K_{1,j} dt, \\
dW_2(t) & = dY_{2,j} - i r K_{2,j} dt.
\end{align*}
\]

respectively. We have $\pi_{01}^{(1)}(X) = \pi_{01}^{(1)}(X^\dagger)$, the initial conditions are $\pi_{11}^{(1)}(X) = \pi_{00}^{(0)}(X) = \langle \eta, \eta \rangle$, $\pi_{10}^{(1)}(X) = \pi_{01}^{(0)}(X) = 0.$

By the filter equations (3.47) and $\pi_{jk}^{(1)}(X) = \text{Tr}[(\rho^{(j)}(t)^\dagger X)]$, we also have the quantum filter in the Schrödinger picture.

\textbf{Corollary 3.2:} With the two homodyne detection measurements, the quantum filter for the system $G$ driven by single-photon input state in the Schrödinger picture is given
by (3.51). Here,

\[
K_{1,t} = e^{-i\theta_{1} t} \text{Tr}[L^1 \rho^{11}(t)] + e^{i\theta_{1} t} \text{Tr}[L \rho^{11}(t)]
+ e^{i\theta_{1} t} \text{Tr}[S \rho^{01}(t)] \xi(t),
\]

\[
K_{2,t} = e^{-i\theta_{1} t} \text{Tr}[L^1 \rho^{11}(t)] - e^{i\theta_{1} t} \text{Tr}[L \rho^{11}(t)]
- e^{i\theta_{1} t} \text{Tr}[S \rho^{01}(t)] \xi(t) + e^{i\theta_{1} t} \text{Tr}[S \rho^{01}(t)] \xi^*(t),
\]

(3.52)

and the initial conditions are \( \rho^{11}(0) = \rho^{00}(0) = |\eta\rangle \langle \eta|, \rho^{10}(0) = \rho^{01}(0) = 0. \)

### F. Simulation Results

Here we apply the filter equations derived in subsection 3-B to the problem of exciting a two-level atom with a continuous-mode single-photon, [13]. This system can be parameterized as follows. The scattering is \( S = I, \) the coupling operator is \( L = \kappa \sigma_{-} \) with coupling strength \( \kappa = 1. \)

The atom is taken to be in the ground state initially \(|g\rangle \langle g|\) with the Hamiltonian \( H = 0. \) The wave packet \( \xi(t) \) for the single-photon is given by

\[
\xi(t) = \left( \frac{\Omega^2}{2\pi} \right)^{1/4} \exp \left[ -\frac{\Omega^2}{4} (t-t_0)^2 \right],
\]

(3.53)

where \( t_0 \) is the peak arrival time and \( \Omega \) is the frequency bandwidth of the wave packet.

Now we choose \( \Omega = 1.46 \) and wish to calculate the exciting probability for the atom as a function of time. The exciting probability for quantum filtering equations is given by

\[
P_{e}^c(t) = \text{Tr}[\rho^{11}(t)|e\rangle \langle e|]
\]

(3.54)

where \( \rho^{11}(t) \) is the solution to (3.51).
Moreover, showing the stability of multi-photon filtering is in the perspective of our research.

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