The Lindquist-Wheeler formulation of lattice universes

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This paper examines the properties of ‘lattice universes’ wherein point masses are arranged in a regular lattice on space-like hypersurfaces; open, flat, and closed universes are considered. The universes are modelled using the Lindquist-Wheeler approximation scheme, which approximates the space-time in each lattice cell by Schwarzschild geometry. It is shown that the resulting dynamics strongly resemble those of the Friedmann-Lemaître-Robertson-Walker (FLRW) universes. The cosmological redshifts for such universes are determined numerically, using a modification of Clifton and Ferreira’s approach, and they are found to closely resemble their FLRW counterparts, though with certain differences attributable to the ‘lumpiness’ in the underlying matter content.

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I. INTRODUCTION

Modern cosmology is founded upon the so-called Copernican principle, which posits that the universe ‘looks’ on average to be the same regardless of where one is in the universe or in which direction one looks. More formally, the Copernican principle states that Cauchy surfaces of the universe are homogeneous and isotropic, and this symmetry is expressed mathematically by requiring the universe’s metric to be of the form

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

where \( a(t) \) is the time-dependent conformal scale factor, and \( k \) is the curvature constant. The sign of \( k \) determines whether 3-spaces of constant \( t \) will be open, flat, or closed, with \( k < 0 \) corresponding to open universes, \( k = 0 \) to flat universes, and \( k > 0 \) to closed universes. This family of metrics is known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. For the metric to satisfy the Einstein field equations of general relativity, then \( a(t) \) must obey the Friedmann equations, which are

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \left( 8\pi \rho + \Lambda \right) - \frac{k}{a^2}, \]

\[ \frac{\ddot{a}}{a} = -\frac{4\pi}{3} \left( \rho + 3p \right) + \frac{\Lambda}{3}, \]

where \( \Lambda \) is the cosmological constant, and \( \rho \) and \( p \) are the energy density and pressure of the fluid that fills the space. For dust-filled \( \Lambda = 0 \) universes, the relationship between \( a \) and \( t \) is given by

\[ a = \left( \frac{9}{4}a_0 \right)^{1/3} t^{2/3}, \quad \text{for } k = 0, \]

\[ a = \frac{a_0}{2} (\cosh \eta - 1), \quad \text{for } k < 0, \]

\[ t = \frac{a_0}{2} (\sinh \eta - \eta), \]

as depicted in Fig. 1, where

\[ a_0 = \frac{8\pi \rho_0}{3}, \]

and \( \rho_0 \) is the energy density when \( a = 1 \). In the case of \( k > 0 \), the factor \( a_0 \) also corresponds to the maximum of \( a \).

FIG. 1. Plots of \( a \) versus \( t \) for open, flat, and closed dust-filled universes.

Observations however clearly show that the universe is not homogeneous and isotropic except at the coarsest of scales. Instead, matter is distributed predominantly in clusters and superclusters of galaxies with large voids in between, and the physics of such ‘lumpy’ universes is still not fully understood. To help illuminate what observable effects a non-homogeneous matter distribution might give rise to, we shall examine in this paper a universe that still possesses a high degree of symmetry, though not to the extent of the FLRW universes. Specifically, we shall consider the so-called lattice universes where the mat-
ter content on each Cauchy surface consists of identical point masses arranged into a regular lattice. We shall focus on lattices ‘constructed’ by tessellating a 3-space of constant curvature with identical regular polyhedral cells; the possible lattices obtainable from such a tessellation is summarised in Appendix A. To ‘construct’ a lattice universe, one of the lattices in Appendix A is selected, and all mass in each cell is ‘concentrated’ into a point at its centre. The metric consistent with such a matter distribution is expected to be invariant under the same symmetry transformations that leave the lattice invariant, symmetries which include discrete translation symmetries, discrete rotational symmetries, and reflection symmetries at the cell boundaries. In other words, the lattice universe should have a metric of the form
\[ ds^2 = -dt^2 + \gamma_{ab}^{(3)}(t, \mathbf{x}) \, dx^a \, dx^b, \]  
where \( \gamma_{ab}^{(3)}(t, \mathbf{x}) \) is the 3-dimensional metric for constant \( t \) hypersurfaces, and Latin indices \( a, b = 1, 2, 3 \) denote spatial co-ordinates only; the spatial metric \( \gamma_{ab}^{(3)}(t, \mathbf{x}) \) at constant \( t \) would possess the lattice symmetries. Effectively, the Copernican symmetries of FLRW universes have been reduced to just these symmetries.

However determining the 3-metric \( \gamma^{(3)}(t, \mathbf{x}) \) has so far proven intractable except in cases with a high degree of symmetry such as the FLRW universes themselves. Thus in 1957, Lindquist and Wheeler (LW) [1, 2] devised a method to approximate such lattice universes wherein each polyhedral lattice cell was approximated by a spherical cell with Schwarzschild geometry inside. For this approximation to work though, the masses are required to be of such magnitude and separation that the geometry around each can be reasonably approximated by Schwarzschild geometry; for example, should the masses get too close together, the deviation around each mass from Schwarzschild geometry would become too great, and the LW approximation would consequently break down. Lindquist and Wheeler focused only on the closed lattice universe, and they found that the dynamics of their model very closely resembled that of closed dust-filled FLRW universes. The resemblance improved as the total number \( N \) of masses in the universe increased, with exact correspondence attained in the limit \( N \to \infty \).

More recently, Clifton and Ferreira [3] have re-visited the LW construction and extended it in two notable ways. While Lindquist and Wheeler considered only closed lattice universes, Clifton and Ferreira were able to generalise the construction to flat and open universes. They were also able to extend the LW construction such that particles like photons could be propagated through the universe.

In this paper, we shall re-visit the modelling of the lattice universe using the LW approximation scheme with the CF extensions. We shall examine both the evolution of open, flat, and closed universes as well as the behaviour of photon redshifts in these universes. However, there will be several departures from Clifton and Ferreira’s approach. First, Clifton and Ferreira’s extension of the LW approximation was found to be unsuitable for studying closed lattice universes. Therefore, an alternative extension will be adopted that was actually proposed by Lindquist and Wheeler themselves at the end of their original paper on the LW model [1]. Secondly, though the method used for propagating photons across the lattice is strongly influenced by Clifton and Ferreira’s approach, this thesis will argue for several modifications that result in a new set of boundary conditions for propagating a photon across the interface between two contiguous cells.

The outline of this paper is as follows. The second section will explain the origins of the LW approximation and detail its construction; the CF generalisation to flat and open universes will also be presented. The third section will examine an analogue of the FLRW scale factor for the lattice universe. In particular, we shall show that this lattice scale factor behaves very similarly to its FLRW counterpart for universes of all three curvature types. For example, as Clifton and Ferreira noted, this lattice scale factor obeys a Friedmann-like equation as well, and we shall demonstrate that it has a form essentially identical to (2) for when \( \Lambda = 0 \). We shall also show that the closed lattice universe effectively reduces to the closed dust-filled FLRW universe in the limit that the number of lattice masses is taken to infinity. The latter half of this paper will examine the behaviour of redshifts in the lattice universe. The fourth section will briefly explain the manner by which redshifts are computed; the method used follows that of Clifton and Ferreira. The fifth section will explain the boundary conditions that are used to propagate photons across the boundary between two contiguous cells. The sixth section explains the numerical method used to simulate the propagation of photons through the lattice. The penultimate section presents the results of the simulations: it is shown that the redshifts of lattice universes greatly resemble their FLRW counterparts but with certain differences that can be attributed to the inhomogeneities of the lattice. The final section closes the paper with a discussion of possible directions in which this work might be extended.

In this paper, we shall use geometric units where \( G = c = 1 \).

## II. CONSTRUCTING THE LW APPROXIMATION

The LW approximation of lattice universes was inspired by Wigner and Seitz’s [3] method for approximating electronic wavefunctions in crystal lattices. This method approximates the polyhedral cell of a crystal lattice by a sphere of the same volume; any conditions that the wavefunction must satisfy on the original cell boundary get imposed on the spherical boundary instead. For instance in the original lattice, reflection symmetry at the cell boundary means that the wavefunction \( \Psi \) of the lowest energy free electron must satisfy
\[ \mathbf{n} \cdot \nabla \Psi = 0 \]
at the boundary, where the vector $\mathbf{n}$ is orthogonal to the boundary; in the Wigner-Seitz approximation, this same vanishing-derivative condition gets imposed instead on the spherical boundary; that is, $\Psi$ must now satisfy $\partial \Psi / \partial r = 0$ at the boundary; this is effectively assuming the electron potential within a cell to be spherically symmetric. The higher the symmetries of the original polyhedral cell, the closer the cell resembles a sphere, and the more accurate the results obtained from the Wigner-Seitz models. Indeed when applied to crystals where exact solutions are known, the Wigner-Seitz construction yields very accurate results.

In analogy with the Wigner-Seitz construction, Lindquist and Wheeler approximate each elementary cell of the lattice universe by a spherical cell and the metric inside each cell by the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

They have called this spherical cell the Schwarzschild-cell. Without knowing the true metric, one cannot directly assess how well is a polyhedral cell in the lattice universe approximated by a spherical one. However Lindquist and Wheeler have shown that polyhedra in a closed or flat 3-space of constant curvature are reasonably approximated by spheres of equal volume; hence for lattices in such 3-spaces, the polyhedral cells would be reasonably well-approximated by spherical cells. They therefore provide this as partial evidence that the approximation would probably be reasonable as well for the polyhedral cells in the lattice universe.

In contrast to the Wigner-Seitz lattice, the LW lattice is itself dynamical. A test particle sitting at the boundary between two Schwarzschild-cells will by symmetry always remain at the boundary. Yet like any other test particle in a Schwarzschild geometry, this test particle must also be radially falling towards the centre of one of the Schwarzschild-cells. And by the same reasoning, this particle is also radially falling towards the centre of the other Schwarzschild-cell. Therefore this test particle, and hence the cell boundary itself, is radially falling towards both cell centres simultaneously, as depicted in Fig. 2.

The boundary’s motion is hence given by the equation of motion for a radial time-like geodesic,

$$\left(\frac{dr}{d\tau}\right)^2 = E - 1 + \frac{2m}{r},$$

where $\tau$ is the proper time of a test particle following the geodesic and $E$ a positive constant of motion. It can be shown that $\sqrt{E}$ is the particle’s energy per unit mass at radial infinity. However this simultaneous free-fall of the boundary towards the two masses is actually the result of the two masses themselves falling towards each other under their mutual gravitational attraction. This mutual attraction of all the point-masses thereby gives rise to the expansion and contraction of the lattice itself, which manifests as the expansion and contraction of the Schwarzschild-cell boundary.

We can see from (10) that the value of $E = E_b$ at the boundary determines whether the cell boundaries will expand indefinitely or eventually re-collapse, and hence whether the underlying lattice universe is open, flat, or closed. For $E_b < 1$, the boundaries will expand until reaching a maximum radius of

$$r_{\text{max}} = \frac{2m}{1 - E_b}$$

before re-collapsing; this corresponds to a closed universe. For $E_b > 1$, the boundaries will expand indefinitely; this corresponds to an open universe. And for $E_b = 1$, the boundaries travel at the escape velocity and just reach radial infinity; this corresponds to a flat universe. Thus it is through this constant $E_b$ that Clifton and Ferreira generalise Lindquist and Wheeler’s work to flat and open universes in a natural manner.

In order to ‘glue’ the individual cells together into a lattice, we require that the 3-space of constant time in one Schwarzschild-cell mesh at the boundary with the corresponding 3-space of the neighbouring cell. As Lindquist and Wheeler have pointed out, two 3-spaces will mesh together if and only if they intersect their common boundary orthogonally. Surfaces of constant Schwarzschild time $t$ do not satisfy this meshing condition; instead, they intersect each other when they meet, as illustrated in Fig. 3.

Both Lindquist and Wheeler as well as Clifton and Ferreira devised new time co-ordinates that satisfied the meshing condition. The LW time co-ordinate is defined for closed universes only while the CF time co-ordinate can only be applied to flat and open universes. Both time co-ordinates are defined to equal the proper times of a congruence of radial time-like geodesics, which must include the geodesics of test particles co-moving with the boundary. Test particles following these geodesics will travel at the same velocity if at the same radius, thus forming freely falling shells. All geodesics’ clocks are calibrated to read identical proper time on some ini-
For boundaries following out-going trajectories, the CF time co-ordinate $\tau_{CF}$ is given by

$$d\tau_{CF} = \sqrt{E_b} \, dt - \frac{\sqrt{E_b - 1 + \frac{2m}{r}}}{(1 - \frac{2m}{r})} \, dr,$$

(12)

where $E_b$ is the same positive constant as $E$ in (10) for the boundary geodesic. The Schwarzschild metric \( 9 \) now becomes

$$ds^2 = -\frac{1}{E_b} \left( 1 - \frac{2m}{r} \right) d\tau_{CF}^2 - 2 \, \frac{1}{E_b} \, \sqrt{E_b - 1 + \frac{2m}{r}} \, dr \, dr + \frac{dr^2}{E_b} + r^2 d\Omega^2.$$  

The boundary’s equation of motion is still given by (10) but with the proper time $\tau$ replaced by $\tau_{CF}$, as it can be shown that $\tau_{CF}$ is identical to the boundary’s proper time. In CF co-ordinates, the 4-vector tangent to any trajectory satisfying (10) for $E = E_b$ is

$$u^a = \left( 1, \sqrt{E_b - 1 + \frac{2m}{r}}, 0, 0 \right).$$  

(14)

For an arbitrary vector $n$ tangent to a constant $\tau_{CF}$ surface,

$$n^a = (0, n^r, n^\theta, n^\phi),$$  

(15)

it can be shown that $u \cdot n = 0$ and hence that surfaces of constant $\tau_{CF}$ are orthogonal to all geodesics satisfying (10) for $E = E_b$, including particularly the boundary geodesics. Thus the meshing condition is satisfied, and thus $\tau_{CF}$ can serve as a cosmological time.

If we attempt to apply the CF co-ordinate system to closed universes though, we encounter problems. When the boundary is contracting, its radial velocity is given by the negative root of (10) instead, and as a result, the square-roots in (12) must change sign. However the combined co-ordinate patches do not correctly cover the interior of the Schwarzschild-cell. Rather, they leave an uncovered region centred on the boundary’s moment of maximum expansion, as illustrated in Fig. 5.

We shall denote by $\tau_{max}$ the boundary’s proper time at maximum expansion. For $\tau_{CF} < \tau_{max}$, surfaces of constant $\tau_{CF}$ are generated by the integral curves of (12) for $d\tau_{CF} = 0$, that is, by

$$\frac{dr}{dt} = \sqrt{E_b \left( 1 - \frac{2m}{r} \right)} \left( \frac{1}{E_b - 1 + \frac{2m}{r}} \right) > 0 \quad \forall r_b \geq r > 2m.$$  

(16)

This has a solution of the form $t = f(r) + t_0$ where $t_0$ is the Schwarzschild time co-ordinate of the surface when it passes through $r = r_0$ and $r_0$ is the Schwarzschild radial co-ordinate satisfying $f(r_0) = 0$. Thus on the $r$-$t$ plane, like the one shown in Fig. 5, all of these surfaces...
are identical apart from a horizontal shift corresponding to different $t_0$ constants. Increasing $t_0$ shifts the surface to the right. This implies the surface would intercept the boundary later in the boundary’s trajectory, and therefore $\tau_{\text{CF}}$ would increase as well. As we keep increasing $t_0$, we shall eventually reach a limiting surface $\tau_{\text{CF}} \rightarrow \tau_{\text{max}}$ from the right, and this surface we denote by $\tau_{\text{max}}^+$. The complete set of constant $\tau_{\text{CF}}$ surfaces will completely cover the interior of the cell without any overlaps or gaps if and only if surfaces $\tau_{\text{max}}^-$ and $\tau_{\text{max}}^+$ are identical. This implies that

$$f(r) + t_0^+ = -f(r) + t_0^-$$

$$\implies f(r) = \text{constant},$$

where $t_0^+$ and $t_0^-$ are the $t_0$ constants for $\tau_{\text{max}}^-$ and $\tau_{\text{max}}^+$, respectively. However we know that function $f(r)$ is not constant, so we have a contradiction.

Instead, we actually have a ‘gap’ between $\tau_{\text{max}}^-$ and $\tau_{\text{max}}^+$. Both surfaces meet at $(\tau_{\text{max}}, r_b(\tau_{\text{max}})) = (\tau_{\text{max}}, r_{\text{max}})$. If we move along $\tau_{\text{max}}^+$ into the cell, then decreasing $r$ would decrease $t$ because $dr/dt$ is always positive along this surface. If we move along $\tau_{\text{max}}^-$ into the cell, then decreasing $r$ would increase $t$ because $dr/dt$ is always negative along this surface. On the $r$-$t$ plane, $\tau_{\text{max}}^-$ moves to the left and $\tau_{\text{max}}^+$ to the right as $r$ decreases from $r_{\text{max}}$. Since $\tau_{\text{max}}^-$ is the rightmost of the $\tau_{\text{CF}} < \tau_{\text{max}}$ surfaces and $\tau_{\text{max}}^+$ the leftmost of the $\tau_{\text{CF}} > \tau_{\text{max}}$ surfaces, the region between $\tau_{\text{max}}^-$ and $\tau_{\text{max}}^+$ is not covered by any $\tau_{\text{CF}}$ surface and is hence a gap.

We shall therefore use the LW co-ordinate system, which covers closed cells correctly. This system is constructed from a congruence of closed geodesics that attain maximum radii at the same Schwarzschild time. Being a congruence, the geodesics have maximum radii spanning the entire range of $2m < r \leq r_b$, and from a generalised form of (11), the geodesics must therefore have different constants $E$. The geodesics’ clocks are calibrated to have identical proper time at maximum radii. The LW time co-ordinate $\tau_{\text{LW}}$ at any point is then defined to be the proper time of the geodesic passing through it. Lindquist and Wheeler also defined a new radial co-ordinate $\tilde{r}_{\text{LW}}$ that is constant along each geodesic and equals the geodesic’s maximum radius in Schwarzschild co-ordinates.

The co-ordinate transformation to $(\tau_{\text{LW}}, \tilde{r}_{\text{LW}})$ is then given implicitly by

$$t = t_0 \pm \left\{ \left[ \frac{\tilde{r}_{\text{LW}}}{2m} - 1 \right] \left[ \tilde{r}_{\text{LW}} - r \right] r \right\}^{1/2} + 2m \left( \frac{\tilde{r}_{\text{LW}}}{2m} - 1 \right)^{1/2} \left( \frac{\tilde{r}_{\text{LW}}}{2m} + 2 \right) \cos^{-1} \left( \frac{r}{\tilde{r}_{\text{LW}}} \right)^{1/2}$$

$$+ 2m \ln \frac{r^{1/2} \left( \tilde{r}_{\text{LW}}/2m - 1 \right)^{1/2} + (\tilde{r}_{\text{LW}} - r)^{1/2}}{r^{1/2} \left( \tilde{r}_{\text{LW}}/2m - 1 \right)^{1/2} - (\tilde{r}_{\text{LW}} - r)^{1/2}} \right\},$$

$$\tau_{\text{LW}} = \tau_0 \pm \left[ \frac{\tilde{r}_{\text{LW}}}{2m} \right]^{1/2} \left[ r^{1/2} (\tilde{r}_{\text{LW}} - r)^{1/2} + \tilde{r}_{\text{LW}} \cos^{-1} \left( \frac{r}{\tilde{r}_{\text{LW}}} \right)^{1/2} \right],$$

where $t_0$ and $\tau_0$ are the time co-ordinates at which the boundary attains its maximum expansion; the positive
signs are taken when the boundary is expanding and the

\[ ds^2 = -d\tau_{\text{LW}}^2 + \left( \frac{\tilde{r}_{\text{LW}} - r}{4 r \tilde{r}_{\text{LW}} (\tilde{r}_{\text{LW}}/2m - 1)} \right) \left[ \left( \frac{r}{\tilde{r}_{\text{LW}} - r} \right)^{1/2} + 3 \tilde{r}_{\text{LW}} \cos^{-1} \left( \frac{r}{\tilde{r}_{\text{LW}}} \right)^{1/2} \right]^2 d\tilde{r}_{\text{LW}}^2 + r^2 d\Omega^2, \tag{19} \]

where \( r \) is now a function of \( \tau_{\text{LW}} \) and \( \tilde{r}_{\text{LW}} \). From this metric, it is clear that surfaces of constant \( \tau_{\text{LW}} \) are always orthogonal to Lindquist and Wheeler’s congruence. Thus the meshing condition is also satisfied, and \( \tau_{\text{LW}} \) is also suitable for use as a cosmological time.

We shall henceforth drop any subscript labels from the cosmological time \( \tau \). In the case of the closed universe, \( \tau \) will denote \( \tau_{\text{LW}} \), while in the flat or open universe, it will denote \( \tau_{\text{CF}} \).

### III. The Cosmological Scale Factor and the Friedmann Equations

There is a clear analogy between the size of the Schwarzschild-cell \( r_b(\tau) \) and the FLRW scale factor \( a(t) \), as both provide a scale of their respective universe’s size. Indeed we expect the lattice universe’s scale factor \( a(\tau) \) should be a function of \( r_b \), that is \( a(\tau) = a(r_b(\tau)) \).\(^1\)

If we expand \( r_b \), then each cell of the lattice will expand by some scale \( \xi \), and therefore the lattice as a whole will expand by \( \xi \). We shall take \( a(\tau) \) to be related linearly to \( r_b(\tau) \), that is, \( a(r_b(\tau)) = \alpha r_b(\tau) \) for some constant \( \alpha > 0 \). As we shall see, \( a(\tau) \) then depends on \( \tau \) in a manner analogous to how \( a(t) \) depends on \( t \) for FLRW universes.

**FIG. 6.** Rescaling the cell size by \( \xi \) rescales the entire lattice universe by \( \xi \) as well.

Lindquist and Wheeler were able to derive a paramet-

\( ^1 \) We shall use \( a(t) \) when we refer to the FLRW scale factor and \( a(\tau) \) when referring to the lattice universe scale factor. The two functions are completely independent.
$r_b = \frac{r E_b}{2} (\cosh \eta - 1)$ for $E_b > 1$, \hfill (26) \\
$\tau = \frac{r E_b}{2} \sqrt{\frac{r E_b}{2m} (\sinh \eta - \eta)}$ \\
where $r E_b$ simply denotes $r E$ when $E = E_b$ and $\eta$ is simply a parametrisation. If we choose $\alpha$ to be \\
$\alpha = \sqrt{\frac{r E_b}{2m}}$, \hfill (27) \\
then the scale factor $a(\tau)$ is given by \\
$a = \frac{r E_b}{2} \sqrt{\frac{r E_b}{2m} (1 - \cos \eta)}$ for $0 < E_b < 1$, \hfill (28) \\
$a = \left( \frac{9}{4} r E_b \sqrt{\frac{r E_b}{2m}} \right) ^{1/3} \tau ^{2/3}$ for $E_b < 1$, \hfill (29) \\
$a = \frac{r E_b}{2} \sqrt{\frac{r E_b}{2m} (\cosh \eta - 1)}$ for $E_b > 1$. \hfill (30) \\
Comparing the relations just obtained for $a(\tau)$ and $\tau$ with their counterparts in $\text{(4)–(6)}$, we find that $a(\tau)$ for $\Lambda = 0$.

Making use of the factor $\alpha$ and $E_b$ are consistent with the relationship between the two quantities required by $\text{(27)}$. We shall summarise Lindquist and Wheeler’s arguments leading to their choices. Although they dealt only with closed universes, we shall generalise to flat and open universes as well.

We begin with Lindquist and Wheeler’s choice of $\alpha$. Note that hypersurfaces of constant $t$ in the FLRW metric $\text{(1)}$ correspond to 3-spaces of constant curvature. In the case of $k \neq 0$, the scale factor $a(t)$ can always be re-scaled to give the radius of curvature, in which case $k$ becomes $\pm 1$. We embed the same type of lattice as the lattice universe on such a hypersurface with the appropriate curvature.\hfill 3

We shall call this hypersurface the comparison hypersurface. We then approximate each polyhedral cell by a sphere of the same volume.\hfill 4

For closed hyper-spheres, we define $\psi$ to be the angle between the centre of the spherical cell and its boundary as measured from the centre of the hypersphere. Then $\psi$ is given implicitly by \\
$\frac{1}{N} = \frac{2\psi - \sin 2\psi}{2\pi}$, \hfill (35) \\
where $N$ is the total number of cells in the lattice. For hyperbolic spaces, we define $\chi$ analogously in terms of hyperbolic angles. If $r_0$ is the radius of one such spherical cell, then we can relate $r_0$ to the comparison hypersurface’s radius of curvature $a_{LW}$ by \\
$a_{LW} = \frac{r_0}{\chi(\psi)}$, \hfill (36) \\
where function $\chi(\psi)$ is given by\hfill 2

$$
\chi(\psi) = \begin{cases} 
\sin \psi & \text{for closed universes}, \\
1 & \text{for flat universes}, \\
\sinh \psi & \text{for open universes}. 
\end{cases}
$$

Lindquist and Wheeler identified the Schwarzschild-cell radius $r_b$ of the lattice universe with $r_0$ and used the

\footnotetext[2]{That is, for closed lattice universes, we use hyperspheres; for flat lattice universes, Euclidean space; and for open lattice universes, hyperbolic space.}

\footnotetext[3]{This is actually Clifton and Ferreira’s generalisation. Lindquist and Wheeler’s original condition was that the spheres occupy $N^{-1}$ of the comparison hypersphere’s total volume, where $N$ is the total number of cells. Such a condition clearly needs to be modified for flat and open universes, as both $N$ and the hypersurface volume are infinite in these cases.}

\footnotetext[4]{Note that when $k = 0$ in the FLRW metric $\text{(1)}$, we have complete freedom to make a re-scaling of the form $r \rightarrow \xi r$ and $a \rightarrow a/\xi$. Hence we can always choose a comparison hypersurface such that $a_{LW} = r_0$.}
corresponding \( a_{LW} \) as given by (36) to be the lattice universe’s scale factor; that is,

\[
a_{LW}(\tau) = \frac{r_b(\tau)}{\chi(\psi)}. \tag{38}
\]

Hence they have chosen \( \alpha = 1/\chi(\psi) \). According to (27), \( E_b \) would therefore correspond to

\[
E_b = \begin{cases} 
\cos^2 \psi & \text{for closed universes,} \\
1 & \text{for flat universes,} \\
\cosh^2 \psi & \text{for open universes.} 
\end{cases} \tag{39}
\]

However, Lindquist and Wheeler prescribed \( E_b \) independently of (27). They instead embedded the Schwarzschild-cell in the comparison hypersurface and required it to be in some sense tangent to the hypersurface. Their prescription for the embedding is as follows. Suppose we replaced each of the original spherical cells in the hypersurface with a Schwarzschild-cell from the lattice universe. Then we want to choose an \( E_b \) to make the Schwarzschild-cell tangent to the hypersurface. Lindquist and Wheeler formulate this tangency condition as follows. Take any great circle on the boundary of the original sphere and compare its circumference with that of the corresponding great circle on an infinitesimally smaller sphere. Depending on which hypersurface the cell is embedded in, the circumferences will obey the relation

\[
\frac{1}{2\pi} \frac{d(\text{circumference})}{d(\text{radial distance})} = \begin{cases} 
\frac{1}{2\pi} \frac{d(2\pi a_{LW} \sin \psi)}{a_{LW} d\psi} & \text{for closed hyperspheres,} \\
1 & \text{for flat hypersurfaces,} \\
\frac{1}{2\pi} \frac{d(2\pi a_{LW} \sinh \psi)}{a_{LW} d\psi} & \text{for open hypersurfaces,} 
\end{cases}
\]

as depicted by Fig. 7 to Fig. 9. Then for the Schwarzschild-cell to be tangent to the hypersphere, we also require that at its boundary,

\[
\frac{1}{2\pi} \frac{d(\text{circumference})}{d(\text{radial distance})} = \begin{cases} 
\cos \psi & \text{for closed universes,} \\
1 & \text{for flat universes,} \\
\cosh \psi & \text{for open universes.} 
\end{cases} \tag{40}
\]

From the Schwarzschild metric expressed in CF co-ordinates (13), we can see that

\[
\frac{1}{2\pi} \frac{d(\text{circumference})}{d(\text{radial distance})} = \sqrt{E_b}.
\]

and that \( d(\text{radial distance}) = dr_b/\sqrt{E_b} \). Thus,

\[
\frac{1}{2\pi} \frac{d(\text{circumference})}{d(\text{radial distance})} = \sqrt{E_b}.
\]

From the Schwarzschild metric in LW co-ordinates (19), it can also be shown, by making use of (18), that \((2\pi)^{-1}d(\text{circumference})/d(\text{radial distance})\) is the same.
Solving for $E_b$, we then obtain the same result as in (39).

We close this section with a few remarks about the closed LW model in the limit that $N \to \infty$. In this limit, the angle $\psi \to (3\pi/2N)^{1/3}$. If $M = Nm$ is the total mass of the universe, then

$$\lim_{N \to \infty} \alpha^3 m = \lim_{N \to \infty} \frac{M}{N \sin^3 \psi} = \frac{2M}{3\pi}.$$ 

Therefore the density $\tilde{\rho}_0$ for the lattice universe, as given by (39), becomes

$$\lim_{N \to \infty} \tilde{\rho}_0 = \frac{M}{2\pi^2}. \quad (41)$$

In closed FLRW space-time, a hypersphere of constant $t$ has volume $2\pi^2a(t)^3$. Since the FLRW $\rho_0$ is defined to be the density when $a(t) = 1$, then $\rho_0$ also equals $M/2\pi^2$. Therefore in the limit $N \to \infty$, the lattice universe density $\tilde{\rho}_0$ approaches its FLRW equivalent $\rho_0$, and hence the lattice universe factor $a_0$ approaches its FLRW equivalent as well. Consequently, $a(\tau)$ in (28) approaches $a(t)$ in (1), and $\tau$ in (24) becomes identical to $t$ in (1). We also note that $\tilde{\rho}$ in (34) becomes

$$\lim_{N \to \infty} \tilde{\rho} = \frac{M}{2\pi^2 a^3}, \quad (42)$$

which is identical to the FLRW energy density $\rho$. This implies that the CF Friedmann equation (33) would be identical to the FLRW Friedmann equation (2) for $k = 1$ and $\Lambda = 0$. Therefore, as the closed lattice universe approaches the continuum limit, it becomes completely identical to the closed dust-filled FLRW universe, as the matter content itself becomes effectively homogeneous and isotropic dust.

IV. REDSHIFTS IN THE LATTICE UNIVERSE

Cosmological redshifts, $1 + z_{FLRW}$, in FLRW universes are defined with respect to sources and observers that are co-moving with the universe’s expansion. The redshifts are given by

$$1 + z_{FLRW} = \frac{a_o}{a_e}, \quad (43)$$

where $a_o$ and $a_e$ are the cosmological scale factors at the moments of observation and emission, respectively.

We should like to find the lattice universe analogue to $z_{FLRW}$ so that we can compare redshifts in the two types of universes. In general, redshifts $1 + z$ are defined as the ratio of the photon frequency measured at the source to the frequency measured by the end observer. The closest analogy to co-moving sources and observers in the lattice universe would be sources and observers that are co-moving with respect to a constant $\tau$ surface. All such observers would be following radial geodesics that obey (10), with $E$ fixed to be $E_b$ in the flat and open universes but geodesic-dependent in the closed universe. If $u$ is the 4-velocity of an observer and $k$ the 4-velocity of a photon as it passes the observer, then the photon frequency measured by the observer would be $-u \cdot k$. For the lattice universe, the cosmological redshift is therefore given by

$$1 + z_{LW} = \frac{u_o \cdot k_o}{u_o \cdot k_e}, \quad (44)$$

where the subscripts $s$ and $o$ denote ‘source’ and ‘observer’, respectively. Clifton and Ferreira have constrained their consideration to observers that are co-moving with the photon’s source; that is, at any time $\tau$, the observer would be at the same radius as the source in their respective cells’ Schwarzschild co-ordinates; this has been illustrated in Fig. 10. To facilitate comparison with

![FIG. 10. A photon travels from radius $r_i$ at cosmological time $\tau_i$ in one cell to radius $r_f$ at time $\tau_f$ in the next, as indicated by the long solid arrow. We assume that the photon always passes through the boundary in the manner illustrated here: that is, the point of crossing is a point of tangency between the boundaries of the two cells. A photon is emitted at $(\tau_i, r_i)$ by a source travelling along a geodesic given by (45) and observed by an observer in another cell. Although in a different cell from the source, the observer is still ‘co-moving’ with the source; that is, for any $\tau$, the observer is always at the same radius as the source in their respective cells, and this requires the observer to travel along a geodesic with the same $E$ as the source.

Clifton and Ferreira’s results, we shall compute redshifts for the same set of observers.

Following Clifton and Ferreira’s example, we shall also use the lattice universe’s scale factor $a(\tau)$ at the moments of emission and observation to compute $z_{FLRW}$ for comparison; we thus re-express (43) as

$$1 + z_{FLRW} = \frac{r_b(\tau_o)}{r_b(\tau_e)}, \quad (45)$$

where we have made use of the fact that $a(\tau) = \alpha r_b(\tau)$.}
V. PROPAGATING PHOTONS ACROSS CELL BOUNDARIES

In order to propagate photons through the lattice universe, we must first specify what boundary conditions trajectories must satisfy whenever they pass from one cell into the next. Before discussing the boundary conditions though, we first note a difference between Clifton and Ferreira’s choice of boundary geometry and ours. Clifton and Ferreira converted from spherical cell boundaries back to polyhedral ones, deducing the polyhedral boundary velocity from the requirement that the spherical cell always have the same ‘Euclidean volume’ as the polyhedral cell. We shall instead continue to use spherical boundaries and propagate photons across boundaries in the manner illustrated in Fig. 10; that is, wherever a photon crosses, we always regard the point of crossing as a point of tangency between the two neighbouring cell boundaries. We assume that the boundaries are tangent not just in (3+1)-dimensional space-time as a whole, but also in each 3-dimensional constant-τ hypersurface, because we are treating the spherical boundaries as if they were approximately identical to the true polyhedral lattice cells. As Lindquist and Wheeler have argued, spherical boundaries should be a good approximation to the shape of polyhedral boundaries, with the approximation improving as the number of symmetries increases. Therefore any errors due to this approximation would be small to begin with. We further argue that although spherical cells may tile the lattice universe with gaps and overlaps, an arbitrary photon would on average travel through an equal number of gaps and overlaps such that the overall error approximately cancels out, as illustrated in Fig. 11. Finally, the idea of replacing polyhedral boundaries completely with a spherical approximation seems closer in spirit to the original idea of Wigner and Seitz. Therefore according to our choice of boundary geometry, if a photon exits its current cell at a radius \( r_1 = r_b \), we require it to enter the next cell at radius \( r_2 = r_b \). Owing to cell 2’s spherical symmetry, we are free to choose any \( \theta \) and \( \phi \) co-ordinate for the entry point.

We now present the conditions that photon trajectories must satisfy when propagated across boundaries. These conditions are applied locally at any pair of exit and entry points. Our conditions are founded upon Clifton and Ferreira’s principle that any physical quantity should be independent of which cell’s co-ordinate system an observer co-moving with the boundary may choose to use. In the context of photon trajectories, we require that

1) the photon frequency match across the boundary,

\[
u_1 \cdot k_1 = u_2 \cdot k_2, \tag{46}\]

2) and the projection of the photon’s 4-velocity onto the vector \( n^\tau \) orthogonal to the boundary match across the boundary\(^5\)

\[
n_1^\tau \cdot k_1 = -n_2^\tau \cdot k_2; \tag{47}\]

it can be shown that \( (n^\tau)^a = (0, \sqrt{E_b}, 0, 0) \) for both the LW and CF co-ordinate systems. There is a negative sign in the above equation because \( \mathbf{n}^\tau \) always points radially out of its respective cell, whereas the radial direction of \( \mathbf{k} \) would be out of one cell and into the next.

These conditions along with the normalisation \( \mathbf{k} \cdot \mathbf{k} = 0 \) are sufficient to deduce the components of \( \mathbf{k}_2 \) in terms of \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{k}_1 \).

\(^5\) Because of the normalisation condition, this second requirement is equivalent to requiring that the space-like projection of \( \mathbf{k} \) onto the boundary be identical in both cells.
The vectors \( \mathbf{u} \) and \( \mathbf{n}^r \) imply a decomposition of \( \mathbf{k} \) into the form
\[
\mathbf{k} = -(\mathbf{k} \cdot \mathbf{u}) \mathbf{u} + (\mathbf{k} \cdot \mathbf{n}^r) \mathbf{n}^r + \mathbf{k}^\Omega, \tag{48}
\]
where it can be shown that \( \mathbf{k}^\Omega \cdot \mathbf{u} = \mathbf{k}^\Omega \cdot \mathbf{n}^r = 0 \). We note that because of the cell’s spherical symmetry, we can always choose polar co-ordinates such that \( \mathbf{k} \) lies in the \( \theta = \pi/2 \) plane, thereby allowing us to suppress the \( \theta \) co-ordinate. In this case, \( \mathbf{k}^\Omega \) would take the form \((\mathbf{k}^\Omega)^2 = (k^\Omega)^2\). The above conditions imply that \((\mathbf{k}^\Omega)^2 = (k_2^\Omega)^2\), and by a suitable choice of the \( \phi \) co-ordinates, we can always make \( k_2^\Omega = k_2^\Omega \). We also note that there is no physical reason for the photon trajectory to refract when passing through a boundary, and therefore the \( \theta = \pi/2 \) planes of the two cells should be aligned.

Using the above decomposition and the two boundary conditions, we can therefore express the photon’s 4-velocity \( \mathbf{k}_2 \) in the new cell as
\[
\mathbf{k}_2 = -(k_1 \cdot \mathbf{u}_1) \mathbf{u}_2 - (k_1 \cdot \mathbf{n}_1^r) \mathbf{n}_2^r + \mathbf{k}_2^\Omega. \tag{49}
\]

VI. NUMERICAL IMPLEMENTATION OF THE LW MODEL

To numerically simulate photons propagating through the lattice universe, we have chosen to implement Williams and Ellis’ Regge calculus scheme \[2\] for Schwarzschild space-times. We shall use their method to discretise the space-time of each Schwarzschild-cell into Regge calculus blocks, and then employ their geodesic-tracing method to propagate photons and test particles through a cell, though with some modifications.

Under Williams and Ellis’ scheme, Schwarzschild space-time is discretised as follows. A grid is constructed in the Schwarzschild space-time such that along any particular gridline, only one Schwarzschild co-ordinate \((t, r, \theta, \phi)\) changes while the other three are held constant. The lines intersect at constant \( \Delta t, \Delta r, \Delta \theta, \) or \( \Delta \phi \) intervals, thus forming the edges of curved rectangular blocks. Each of these blocks gets mapped to flat rectangular Regge blocks such that the straight edges of the Regge blocks have the same lengths as the curved edges of the original blocks; Figure 12 illustrates an example of a curved Schwarzschild block with its corresponding Regge block. The Schwarzschild co-ordinates of the original blocks’ vertices \((t_i, r_j, \theta_k, \phi_l)\) are now taken over as labels for the Regge blocks’ vertices. As the Regge blocks are flat, the metric inside is simply the Minkowski metric.

Suppose a Regge block had vertices \((t_i, r_j, \theta_k, \phi_l)\) and \((t_{i+1}, r_{j+1}, \theta_{k+1}, \phi_{l+1})\); then we shall use the label \((t_i, r_j, \theta_k, \phi_l)\) to refer to this block as a whole. The edge-lengths for this block are
\[
d[(t_i, r_j, \theta_k, \phi_l), (t_{i+1}, r_{j+1}, \theta_{k+1}, \phi_{l+1})] = \sqrt{1 - \frac{2m}{r}} (t_{i+1} - t_i). \tag{50}
\]

Williams and Ellis propagate geodesics through Regge blocks on the principle that geodesics should follow straight lines both within a block and on crossing from one block into the next. There is an apparent refraction of the geodesic in crossing into a new block because the \((\tau, \rho, \psi)\) co-ordinate systems of the two blocks are not aligned; therefore the same tangent vector of a geodesic would be represented differently in different blocks’ co-ordinate systems. Williams and Ellis have demonstrated that their scheme successfully reproduces the orbits and redshifts of particles travelling in Schwarzschild spacetime \[7, 8\].

Referring to Fig. 12 we now summarise the rules for propagating geodesics from one block into the next. Suppose the particle is at position \((\tau_0, \rho_0, \psi_0)\) and traveling in direction \(k^\alpha = (k_1, k_2, k_3)\). It exits the block at \((\tau', \rho', \psi') = (\tau_0, \rho_0, \psi_0) + \lambda \mathbf{k}\), where the value of \(\lambda\) depends on the face exited.

\[6\] In \[2\], there was a missing factor of \(2m\) in front of the logarithm for this equation; this factor has been restored here.
FIG. 12. On the left is an example of the original Schwarzschild block with one angular co-ordinate suppressed, and on the right is the Regge block to which it is mapped. \((\tau, \rho, \psi)\) is a Minkowski co-ordinate system within the Regge block.

1) If the particle exits the block by the top face, then

\[
\lambda = \frac{\tilde{\tau} - \tau_0 + \rho_0 \tanh \beta_i}{k_1 - k_2 \tanh \beta_i},
\]

where \(\tilde{\tau} = (\tau_{i+1} + \tau_i)/2\) and \(\tanh \beta_i = (\tau_{i+1} - \tau_i)/2\rho_i\). In the new block, the particle’s new trajectory is given by applying to \(k\) the matrix

\[
\begin{pmatrix}
\cosh 2\beta_i & -\sinh 2\beta_i & 0 \\
-\sinh 2\beta_i & \cosh 2\beta_i & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

and the particle’s starting position is given by \((-\tilde{\tau}', \rho', \psi')\).

2) If the particle exits by the back face, then

\[
\lambda = \frac{\tilde{\psi} - \psi_0 + \rho_0 \tan \alpha_i}{k_3 - k_2 \tan \alpha_i},
\]

where \(\tilde{\psi} = (\psi_{i+1} + \psi_i)/2\) and \(\tan \alpha_i = (\psi_{i+1} - \psi_i)/2\rho_i\). In the new block, the particle’s trajectory \(k\) gets transformed by

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 2\alpha_i & \sin 2\alpha_i \\
0 & -\sin 2\alpha_i & \cos 2\alpha_i
\end{pmatrix},
\]

and the particle’s starting position is \((\tau', \rho', -\psi')\).  

3) If the particle exits by the front face, then

\[
\lambda = \frac{-\tilde{\psi} - \psi_0 - \rho_0 \tan \alpha_i}{k_3 + k_2 \tan \alpha_i},
\]

where \(\tilde{\psi}\) and \(\tan \alpha_i\) are the same as above. In the new block, the particle’s trajectory \(k\) gets transformed by the same rotation matrix as (55) but with angle \(-\alpha_i\) instead. The particle’s starting position in the new block is again \((\tau', \rho', -\psi')\), since \(\psi' < 0\) at the front face of the original block.

4) If the particle exits by the right/left face, then

\[
\lambda = \frac{\pm\rho_i - \rho_0}{k_2},
\]

and \(\rho'\) is simply \(\rho' = \pm\rho_i\). There is no refraction as the particle enters the next block. Its new starting position is \((\tau', -\rho_{i+1}, \psi')\) if entering the block to the right and \((\tau', \rho_{i-1}, \psi')\) if entering the block to the left.

However from numerical simulations, we have discovered an empirical relation for the tangent 4-vectors of radially out-going time-like geodesics. This has led us to introduce a correction to the above propagation rules. We have found that as a function of \(r_i\), these 4-vectors obey

\[
\tilde{u}_{\text{Regge}} = \begin{pmatrix}
\tilde{\tau} \\
\tilde{\rho} \\
\tilde{\psi}
\end{pmatrix} = \begin{pmatrix}
\frac{1 - 2m}{r_i - r_{\text{max}}} \\
\frac{2m}{r_i - r_{\text{max}}} \\
0
\end{pmatrix},
\]

where \(r_{\text{max}}\) is a constant of motion. Our numerical results supporting this have been presented in Appendix B.

This relation can be derived analytically by comparing \(u_{\text{Regge}}\) with its counterpart 4-vector in continuum Schwarzschild space-time. The tangent 4-vector of a radially out-going time-like geodesic in Schwarzschild space-

\footnote{In [2], there were a few errors in sign for this equation, which have been corrected here.}
time is
\[ u_{\text{Schwarzs}}^a = (\dot{t}, \dot{r}, \dot{\Omega}) = \left( \frac{\sqrt{1 - \frac{2m}{r_{\text{max}}}}}{1 - \frac{2m}{r}}, \frac{2m}{r} - \frac{2m}{r_{\text{max}}} \right)^{1/2}, 0 \right), \]
where \( r_{\text{max}} \) is also a constant of motion and is related to \( E \) in (10) by \( r_{\text{max}}^2 = 2m^2(1 - E) \). For in-going geodesics, the radial components of both \( u_{\text{Regge}} \) and \( u_{\text{Schwarzs}} \) would have an additional negative sign. If we take the scalar product of \( u_{\text{Regge}} \) and \( u_{\text{Schwarzs}} \) with the unit vector in the time direction, that is, with \( \hat{r}^a = (1, 0, 0) \) for Regge space-time and \( \hat{t}^a = (1 - 2m/r)^{-1/2}, 0, 0 \) for continuum Schwarzschild space-time, we have that
\[ \hat{t} \cdot u_{\text{Regge}} = \left( \frac{1 - \frac{2m}{r_{\text{max}}}}{1 - \frac{2m}{r_{\text{max}}}} \right)^{1/2}, \]
and that
\[ \hat{t} \cdot u_{\text{Schwarzs}} = \left( \frac{1 - \frac{2m}{r_{\text{max}}}}{1 - \frac{2m}{r}} \right)^{1/2}. \]
If we identify \( r_{\text{max}} \) with \( r_{\text{max}} \), then these two expressions are identical whenever \( r = r_{\text{max}} \) in the continuum space-time. Similarly, if we take the scalar product with the unit vector in the radial direction, that is, with \( \hat{r}^a = (0, 1, 0) \) for Regge space-time and \( \hat{r}^a = (0, (1 - 2m/r)^{1/2}, 0) \) for continuum space-time, we have that
\[ \hat{r} \cdot u_{\text{Regge}} = -\left( \frac{\frac{2m}{r_{\text{max}}} - \frac{2m}{r}}{1 - \frac{2m}{r_{\text{max}}}} \right)^{1/2}, \]
and that
\[ \hat{r} \cdot u_{\text{Schwarzs}} = -\left( \frac{\frac{2m}{r} - \frac{2m}{r_{\text{max}}}}{1 - \frac{2m}{r_{\text{max}}}} \right)^{1/2}. \]
Again if we identify \( r_{\text{max}} \) with \( r_{\text{max}} \), then the two expressions are also identical whenever \( r = r_{\text{max}} \) in the continuum space-time. Therefore provided \( r_{\text{max}} = \hat{r}_{\text{max}} \), we see that \( \hat{r}_{\text{max}} \) is indeed the Regge analogue of \( r_{\text{max}} \). Furthermore, the choice of \( r_{\text{max}} \) and \( \hat{r}_{\text{max}} \) determines whether the resulting particle orbit will be closed or open in the corresponding space-time. If \( 2m/r_{\text{max}} > 0 \), then the orbit in the continuum space-time will be closed and \( r_{\text{max}} \) would indeed be the maximum radius of the orbit. Similarly, we found that if \( 2m/r_{\text{max}} > 0 \), then the orbit in Regge space-time will also be closed, and the maximum radius would be \( \hat{r}_{\text{max}} \). If \( 2m/r_{\text{max}} = 0 \), then the geodesic will just reach spatial infinity in the continuum space-time, and \( (59) \) gives the particle’s escape velocity as a function of its radial position \( r > 2m \). Similarly, we show in Appendix \( \text{B} \) that when \( 2m/r_{\text{max}} = 0 \), then \( (58) \) gives the escape velocity for a test particle in the Regge space-time. Finally if \( 2m/r_{\text{max}} < 0 \), then the orbit in continuum space-time will be open. And if \( 2m/r_{\text{max}} < 0 \), then the orbit in Regge space-time will also be open, as we show in Appendix \( \text{B} \). We shall henceforth make the identification of \( r_{\text{max}} = \hat{r}_{\text{max}} \).

Inspired by this, we can generalise the expression for \( u_{\text{Regge}} \) to any geodesic in Regge Schwarzschild space-time. The Lagrangian for particles moving in continuum Schwarzschild space-time can be written as
\[ L = -\left( 1 - \frac{2m}{r} \right) \dot{t}^2 + \frac{\dot{r}^2}{(1 - \frac{2m}{r})^2} + r^2 \dot{\Omega}^2, \]
where the dot denotes differentiation with respect to some parameter denoted by \( \lambda \). Since \( 0 = \partial L/\partial t, \) we have a constant of motion \( E \), which we define by the relation
\[ E := \left( 1 - \frac{2m}{r} \right)^2 \dot{t}^2. \]
Similarly, since \( 0 = \partial L/\partial \Omega \), we have another constant of motion \( J \) defined by
\[ J := r^2 \dot{\Omega}. \]
These two constants correspond to the square of the particle’s energy per unit mass at radial infinity and to the angular momentum. For radial time-like geodesics, \( E \) here is the same as \( E \) in (10). In terms of these constants, the particle’s velocity \( v_{\text{Schwarzs}} \) can be expressed as
\[ v_{\text{Schwarzs}}^a = (\dot{t}, \dot{r}, \dot{\Omega}) = \left( \frac{\sqrt{E}}{(1 - \frac{2m}{r})}, \dot{r}, \frac{J}{r^2} \right), \]
which is clearly a function of \( r \) alone, since \( \dot{r} \) can be deduced from \( \dot{t} \) and \( \dot{\Omega} \) through the normalisation of \( v_{\text{Schwarzs}} \).

Let \( v_{\text{Regge}} \) denote the Regge analogue of \( v_{\text{Schwarzs}} \). If we assume that \( v_{\text{Regge}} \) and \( v_{\text{Schwarzs}} \) are related in a manner analogous to how \( u_{\text{Regge}} \) and \( u_{\text{Schwarzs}} \) are related, then we can deduce the components of \( v_{\text{Regge}} \) from \( v_{\text{Schwarzs}} \). Specifically, by equating \( \hat{r} \cdot v_{\text{Regge}} \) to \( \hat{t} \cdot u_{\text{Schwarzs}} \), we deduce \( \hat{r} \) to be
\[ \dot{r} = \frac{\sqrt{E}}{(1 - \frac{2m}{r})^{1/2}}. \]
Similarly by equating \( \hat{\psi} \cdot v_{\text{Regge}} \) to \( \hat{\Omega} \cdot u_{\text{Schwarzs}} \), we take \( \hat{\psi}^a = (0, 0, 1) \) and \( \hat{\Omega}^a = (0, 0, r^{-1}) \), we deduce \( \dot{\psi} \) to be
\[ \dot{\psi} = \frac{J}{r}. \]
Thus the components of \( \mathbf{v}_{\text{Regge}} \) are given by

\[
\mathbf{v}^a_{\text{Regge}} = \left( \frac{\sqrt{E}}{(1 - 2\nu^2)^{1/2}}, \frac{J}{\rho}, \frac{J}{r} \right),
\]

(63)

with \( \hat{\rho} \) deducible from the normalisation of \( \mathbf{v}_{\text{Regge}} \). Finally, it is easy to verify that \( \hat{\rho} \cdot \mathbf{v}_{\text{Regge}} \) and \( \hat{\mathbf{r}} \cdot \mathbf{v}_{\text{Schwarz}} \) are then consistent.

Our generalised expression for velocities can be used for both null and time-like geodesics following both radial and non-radial trajectories. For time-like radial geodesics, where \( J = 0 \), it can be shown that \( \mathbf{v}_{\text{Regge}} \) reduces to \( \mathbf{u}_{\text{Regge}} \). As we wish to simulate geodesics in continuum rather than Regge Schwarzschild-cells, we have introduced a correction to the Williams-Ellis scheme: whenever a geodesic crosses a right or left face, its tangent velocity is changed back to (63).

Having established the modified Williams-Ellis scheme, we now discuss its application to the Schwarzschild-cell. The boundary is simulated by propagating a test particle that is co-moving with the boundary. Whenever the photon reaches the same radial and time co-ordinates as the boundary particle, then at that point the photon exits one cell and enters the next, and we need to apply conditions (41) and (42) to determine the photon’s new trajectory. However vector components will differ from previously as we are now working in Regge space-time.

Let \( \mathbf{u}^a = (u^\tau, u^\rho, 0) \) be the boundary particle velocity and \( \mathbf{k}^a = (k^\tau, k^\rho, k^\psi) \) be the photon velocity at the boundary. We shall use subscripts 1 and 2 to indicate the cell being exited and the cell being entered, respectively. As in (43), we decompose the Regge vector \( \mathbf{k} \) into

\[
\mathbf{k} = \nu \mathbf{u} + \mathbf{n}^\rho + \mathbf{n}^\psi,
\]

(64)

where \( \nu = -\mathbf{u} \cdot \mathbf{k} \), \( (n^\psi)^a = (0, 0, n^\psi) \), and \( \mathbf{n}^\rho \) satisfies \( \mathbf{n}^\rho \cdot \mathbf{u} = \mathbf{n}^\rho \cdot \mathbf{n}^\psi = 0 \). If \( \hat{\mathbf{n}}^\rho \) is the normalised vector of \( \mathbf{n}^\rho \), that is \( \mathbf{n}^\rho = \hat{\mathbf{n}}^\rho \), then the components of \( \hat{\mathbf{n}}^\rho \) are

\[
(\hat{\mathbf{n}}^\rho)^a = (u^\rho, u^\tau, 0),
\]

as this satisfies all orthogonality relations required of \( \mathbf{n}^\rho \).

We then deduce \( n \) to be

\[
n = \hat{\mathbf{n}}^\rho \cdot \mathbf{k} = (u^\tau k^\rho - u^\rho k^\tau).
\]

(65)

Condition (41) implies that

\[
n_1 = n_2.
\]

(66)

Condition (42) in this context is equivalent to

\[
\mathbf{k}_1 \cdot \hat{\mathbf{n}}^\rho_1 = -\mathbf{k}_2 \cdot \hat{\mathbf{n}}^\rho_2,
\]

which implies that

\[
n_1 = -n_2.
\]

(67)

As at the end of Section VI these conditions imply that

\[
(n_1^\psi)^2 = (n_2^\psi)^2,
\]

and we again have freedom to choose polar co-ordinates such that

\[
n_1^\psi = n_2^\psi,
\]

(68)

which we shall henceforth assume to be the case. Using relation (64), conditions (41) and (42), relation (68), and decomposition (43), we can express the components of \( \mathbf{k}_2 \) as

\[
k_2^a = (\nu u_2^\rho + (u_1^\rho k_1^\tau - u_1^\tau k_1^\rho) u_2^\tau, \nu u_2^\rho + (u_1^\rho k_1^\tau - u_1^\tau k_1^\rho) u_2^\tau, k_1^\psi).
\]

(69)

We note that constant \( E \) in (43) will differ between \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \), but \( J \) will remain the same since \( k^\psi = J/r \) is identical on both sides of the boundary.

**VII. REDSHIFTS FROM LATTICE UNIVERSE SIMULATIONS**

We have simulated the propagation of photons through multiple Schwarzschild-cells for closed, flat, and open LW universes. Each time, a photon was propagated outwards in various directions from an initial radius of 10\( R \), \( R \) being the Schwarzschild radius. The initial direction of travel was given, in terms of block co-ordinates, by

\[
k^a = (\tau, \cos \theta_n, \sin \theta_n),
\]

where \( \tau > 0 \) was determined from the normalisation constraint \( \mathbf{k} \cdot \mathbf{k} = 0 \). The initial direction covered a range of \( \theta_n \), starting from the purely tangential direction of \( \theta_0 = \pi/2 \) and decreasing \( \theta_n \) until the direction was almost completely radial. Both LW and FLRW redshift factors would be computed whenever the photon, while travelling outwards again, passed an observer co-moving with the source.

All length-scales in our simulations have been specified in terms of \( R \). By simply re-scaling \( R \) in one set of results, we can readily obtain the results for an equivalent simulation where the only difference is the magnitude of \( R \). In particular, because redshifts are dimensionless quantities, they would not depend on the choice of \( R \), so we therefore made the arbitrary choice of setting \( R = 1 \) for all our simulations.

We have chosen the dimensions of our Regge block as follows. The angular length \( 2\psi \) was chosen so that \( \Delta \phi = 2\pi/(3 \times 10^{15}) \). For the closed universe, the radial length \( 2d_i \) was chosen to represent a fixed interval of \( \Delta r = 10^{-7} R \). For the flat and open universes, it was instead chosen to remain for blocks further away from the cell centre. This lengthening was implemented so as to increase computation speed with only a marginal expense to the accuracy: the underlying continuum Schwarzschild space-time becomes flatter as one moves further away from the centre, so a flat Regge block would approximate the region more accurately. Our exact method for determining the block’s length has been described in Appendix...
FIG. 13. A plot of $z_{LW}$ against $z_{FLRW}$ for 15 trajectories in a flat universe with an initial cell size of $a_0 = 3 \times 10^4 R$. The initial angle $\theta_n$ of the trajectory is given in the legend. Each trajectory was traced across 50 cells. A linear regression has been performed for each graph, with the first five data points excluded so as to focus only on the linear regime. The regression equations and corresponding root-mean-square of the residuals are listed above in order of decreasing $\theta_n$. 

$z_{LW} = 1.0994 z_{FLRW} + 0.1225$, RMS of residuals = $7.493 \times 10^{-4}$
$z_{LW} = 1.0696 z_{FLRW} + 0.0920$, RMS of residuals = $7.290 \times 10^{-4}$
$z_{LW} = 1.0400 z_{FLRW} + 0.0618$, RMS of residuals = $7.088 \times 10^{-4}$
$z_{LW} = 1.0108 z_{FLRW} + 0.0321$, RMS of residuals = $6.890 \times 10^{-4}$
$z_{LW} = 0.9823 z_{FLRW} + 0.0050$, RMS of residuals = $6.695 \times 10^{-4}$
$z_{LW} = 0.9547 z_{FLRW} + 0.0253$, RMS of residuals = $6.407 \times 10^{-4}$
$z_{LW} = 0.9281 z_{FLRW} + 0.0524$, RMS of residuals = $6.326 \times 10^{-4}$
$z_{LW} = 0.9028 z_{FLRW} + 0.0783$, RMS of residuals = $6.153 \times 10^{-4}$
$z_{LW} = 0.8789 z_{FLRW} + 0.1026$, RMS of residuals = $5.991 \times 10^{-4}$
$z_{LW} = 0.8567 z_{FLRW} + 0.1253$, RMS of residuals = $5.839 \times 10^{-4}$
$z_{LW} = 0.8362 z_{FLRW} + 0.1462$, RMS of residuals = $5.708 \times 10^{-4}$
$z_{LW} = 0.8177 z_{FLRW} + 0.1651$, RMS of residuals = $5.574 \times 10^{-4}$
$z_{LW} = 0.8012 z_{FLRW} + 0.1819$, RMS of residuals = $5.462 \times 10^{-4}$
$z_{LW} = 0.7870 z_{FLRW} + 0.1964$, RMS of residuals = $5.365 \times 10^{-4}$
$z_{LW} = 0.7750 z_{FLRW} + 0.2086$, RMS of residuals = $5.285 \times 10^{-4}$
between the first and second data points. LW and FLRW redshifts are shown. All four series start at the same data point, and there is a clear jump in the LW graphs.

FIG. 14. A plot of redshifts $z$ corresponding to the second data point of the $\text{LW}$ graphs for a photon starting instead with a position and trajectory between $\Delta r = 10\Delta R$ for the flat and open universes and simply $\Delta t = 10\Delta r$ for the closed universe.

In this section we shall present the redshift results of our simulations. We begin with the flat universe, for which $E = 1$. To investigate the effect of the initial cell size $r_{b0}$ on the redshifts, we have simulated flat universes for a range of initial sizes from $r_{b0} = 3 \times 10^4 R$ to $r_{b0} = 10^8 R$. Following the example of Clifton and Ferreira, we have chosen the largest initial size to approximate cells with Milky Way-like masses at their centres, as this is thought to best represent our actual universe. Depending on the initial cell size, $\Delta r_0$ ranged from $\Delta r = 10^{-5}$ for $r_{b0} = 3 \times 10^4 R$ to $\Delta r = 10^{-3}$ for $r_{b0} = 10^8 R$.

Figure 15 plots $z_{\text{LW}}$ against $z_{\text{FLRW}}$ for the universe where $r_{b0} = 3 \times 10^4 R$. Each trajectory was traced across 50 cells; only results for 15 trajectories are shown, although we simulated trajectories for 30 angles $\theta$, ranging from $\theta_0 = \pi/2$ to $\theta_{29} = 311\pi/3000$ in decrements of $41\pi/3000$. Apart from a brief curve at the start, each graph clearly demonstrates a strong linear relationship between $z_{\text{LW}}$ and $z_{\text{FLRW}}$, and the gradient of the line is different for each angle. This differs in several ways from the results obtained by Clifton and Ferreira. Their graphs showed more initial scatter, which varied depending on the initial angle of the trajectory, but their graphs would eventually converge upon a common mean graph as the photon passed through an increasing number of cells. Our graphs do not display such scatter nor any common mean. They also obtained a relation of $1+z_{\text{LW}} \approx (1+z_{\text{FLRW}})^{7/10}$ describing the mean behaviour, which stands in contrast to our linear relationship.

Although $\Delta r$ is no longer constant, it is constrained to be an integral multiple of a minimum interval $\Delta r_0$, a parameter we can freely specify, and the Regge grid is fixed so that its first set of blocks covers the region between $R + \Delta r_0$ and $R + 2\Delta r_0$. As we shall see below, $\Delta r_0$ was chosen according to which universe was being simulated. Finally, the temporal interval was chosen to be $\Delta t = 10\Delta r_0$ for the flat and open universes and simply $\Delta t = 10\Delta r$ for the closed universe.
FIG. 16. A plot of photon frequencies $\nu_{LW}$ against the radius of observation $r_{obs}$ for a photon travelling initially in the direction of (a) $\theta_0 = \pi/2$ and (b) $\theta_29 = 311\pi/3000$; the initial cell size is $r_{b0} = 3 \times 10^4 R$. Each graph represents the photon’s frequencies within a single cell; these are the frequencies that would be seen by a co-moving observer if the photon intercepted the observer at $r_{obs}$. The graphs appear, top to bottom, in the same order as in the legend, that is, from top to bottom, then left to right. The analytic frequencies are given by (70), and graphs for the first seven cells traversed are shown. Frequencies were also computed numerically by simulating the propagation of a photon across the first cell only; the corresponding graphs are shown and, in both cases, completely overlap with their analytic counterpart.
data point, and this was true of our simulations of all other universes as well. When we plotted the redshifts against the radius of observation for the $r_{bh} = 3 \times 10^4 R$ flat universe, as shown in Fig. [14], we found that the $z_{PLRW}$ graph would extend naturally outwards from the zero-redshift point at the starting radius of $10 R$; but the $z_{LW}$ graph would jump suddenly from the zero-redshift point to the next data point. Suppose we ignored our current zero-redshift data point and assumed the photon actually began its trajectory at the next data point; then when we re-calculated all subsequent redshifts based on our new initial frequency for the photon, we found that the resulting $z_{LW}$ graph progresses naturally from the new zero-redshift point to the next data point without any jumps, as shown in Fig. [17].

To investigate this initial jump further, we note that in universes using CF co-ordinates, the frequency $\nu_{LW}$ measured by a co-moving observer can be expressed as a function of the radius at which the measurement is made, that is, the radius $r_{obs}$ at which the photon and observer intercept. This implies the redshifts $z_{LW}$ can be expressed as a function of $r_{obs}$. Recall that the 4-vector of a co-moving observer is given by (63) in Regge Schwarzschild space-time, with $E = E_b$ and $J = 0$; this vector is a function of the observer’s radial position alone. The 4-vector of the photon is given by the same relation as well but with $E = E_{ph}$ and $J$ an arbitrary constant; as long as the photon does not cross into the next cell, $E_{ph}$ will be constant, so this vector is also a function of the photon’s radial position alone. When the observer and photon intercept, they will have the same radial position, that is, the same value for $r$, and we have denoted above this common $r$ by $r_{obs}$. By taking the scalar product of these two vectors, we obtain the measured frequency as a function depending only on the value of $r_{obs}$ alone,

$$\nu_{LW} = \frac{\sqrt{E_b E_{ph}}}{1 - \frac{2m}{r_{obs}}} - \left[ \left( \frac{E_b}{1 - \frac{2m}{r_{obs}}} - 1 \right) \left( \frac{E_{ph}}{1 - \frac{2m}{r_{obs}}} - \frac{J^2}{r_{obs}^2} \right) \right]^{1/2}.$$  

(70)

We note that if the photon were to cross into the next cell, this relation would still hold, but from the boundary conditions, constant $E_{ph}$ only would change. In Fig. [16] we have plotted this function for photons travelling at initial angles of $\pi/2$ and $311\pi/3000$ in the $r_{bh} = 3 \times 10^4 R$ flat universe. We have included plots for different $E_{ph}$ corresponding to the first few cells traversed. We have also simulated a photon’s propagation across the first cell for both trajectories, and for a selection of radii, we have computed the frequencies that a co-moving observer would measure if at those radii. These numerical results are also included in the figures, and we see that they agree very closely with their analytic counterparts. In Fig. [17] we show redshifts $z_{LW}$ instead against $r_{obs}$ for the $\theta_0 = \pi/2$ photon travelling through cell 1 only.

Based on these graphs, we make some speculations on the origin of the jump as well as the initial curve in the $z_{LW}$ vs $z_{PLRW}$ graphs. We note that for large $r_{obs}$, the frequency asymptotically approaches

$$\nu_{asym} = \sqrt{E_{ph}} \left( \sqrt{E_b} - \sqrt{E_b} - 1 \right) = \sqrt{E_{ph}} \quad \text{for} \quad E_b = 1.$$

Equivalently, the redshift asymptotically approaches

$$z_{asym} = \frac{\nu_c}{\sqrt{E_{ph}} \left( \sqrt{E_b} - \sqrt{E_b} - 1 \right)} - 1 = \frac{\nu_c}{\sqrt{E_{ph}}} - 1 \quad \text{for} \quad E_b = 1.$$

As the photon traversed ever more cells in our simulations, we found that it would intercept the co-moving observer at ever larger $r_{obs}$. So after enough cells, $z_{LW}$

---

8 We did check this result by simulating a photon starting at the same radius and direction as that of the original photon when it generated the first data point. The resulting graph was indeed identical to that of Fig. [16].
FIG. 18. A plot of $z_{LW}$ against $z_{F LRW}$ for 15 trajectories in the flat universe with initial cell size of $r_0 = 3 \times 10^4 R$. This is the equivalent plot to that of Fig. [13], but with the photon starting at $10^4 R$ instead. All regression lines now pass through the origin, and the initial curve is now absent.
should be very close to the asymptotic value $z_{asym}$. Also from our simulations, we found that $E_{ph}$ would decrease with each subsequent cell implying that $z_{asym}$ would increase, which is consistent with the $z_{LW}$ vs $z_{FLRW}$ graphs' positive gradients. Thus the asymptotic, large-radius behaviour of the redshifts is responsible for the linear behaviour of the $z_{LW}$ vs $z_{FLRW}$ graphs. The graphs' initial behaviour however must be explained by the behaviour of the redshifts at low $r_{obs}$, where the photon is still sufficiently close to the central mass to feel its influence strongly. We note that the frequencies always approach $\nu_{asym}$ from below, implying that the redshifts always approach $z_{asym}$ from above. This is consistent with the initial curve in the $z_{LW}$ vs $z_{FLRW}$ graphs, as this curve converges into the linear regime from above as well. At much smaller $r_{obs}$ however, the redshift may deviate much more significantly from the asymptotic value. In particular, the deviation of the zero-redshift point, where $r_{obs}$ is smallest, may be much greater than the deviation of the next data point, where $r_{obs}$ may have increased sufficiently for the redshift to be much closer to the asymptotic value. This would account for the sudden initial jump seen in the $z_{LW}$ vs $z_{FLRW}$ graphs. Therefore, the initial jump and the initial curve in the $z_{LW}$ vs $z_{FLRW}$ graphs can be attributed to the photon being closer to the central mass and thus feeling its effects more strongly.

To test our conjecture, we simulated photons in the same universe but starting from a large radius, specif-
FIG. 20. A plot of $z_{LW}$ against $z_{FLRW}$ for 12 trajectories starting at radius $10^7 R$ in the $r_{b_0} = 10^8 R$ universe. Each trajectory was traced across 15 cells. A linear regression has been performed for each graph, and the regression equations are listed in order of decreasing $\theta_n$. All regression lines pass through the origin, and the initial curve is now entirely absent. Compared to Fig. 18, the residuals here are even smaller.
tory would bring the photon closer to the central mass. A stronger influence on the photon, as a more radial trajectory becomes increasingly radial, the photon’s frequency determines the radial component $\dot{r}$.

Plot of $\theta$ vs $z_{\text{FLRW}}$ for the FLRW universe. Since $\cos(\theta_n)$ vs $z$ for the FLRW universe also applied to open universes. We simulated open universes for values of $E_0$ in the range of $1.1 \geq E_0 > 1$. In each simulation, the initial cell size was fixed to be $r_{b_0} = 3 \times 10^4$, and photons were propagated along 12 trajectories with initial angles ranging from $\theta_0 = \pi/2$ to $\theta_1 = 149\pi/1200$ in decrements of $41\pi/1200$. Depending on $E_0$, $\Delta \vartheta_0$ ranged from $10^{-5} R$ for $E_0 = 1$ to $10^{-3} R$ for $E_0 = 1.1$. Figure 22 shows a clear linear relationship between the gradients of $z_{\text{FLRW}}$ and $z_{\text{FLRW}}$ graphs and $\cos(\theta_n)$, or equivalently between the gradients and $\dot{\rho}_{\text{init}}/\vartheta_{\text{init}}$.

To see how the gradients of $z_{\text{FLRW}}$ vs $z_{\text{FLRW}}$ change with $E_0$, we have plotted the gradients against $E_0$ in Fig. 24 for a selection of trajectories. The graphs indicate that the gradients approach some asymptotic value as $E_0$ increases and that this asymptotic gradient decreases as the photon trajectory becomes more radial, consistent with the behaviour illustrated in Fig. 23.

Finally, we shall now discuss our simulation of the closed universe. We have chosen to focus on an LW universe built from the 600-cell Coxeter lattice described in Appendix A as it is the most finely subdivided closed Coxeter lattice possible. Using Eq. 35 and 39, we find that $E_0 \approx 0.96080152145$ for this universe, and the maximum cell size is therefore approximately $25.5 R$. Thus this universe is much smaller than the universes we have previously been considering, and the central mass’ influence would be much reduced. Figure 18 shows the resulting plot of $z_{\text{FLRW}}$ against $z_{\text{FLRW}}$. As hoped, all graphs now follow completely straight lines passing through the origin, with the initial jump and curve no longer present.

For flat universes with larger initial cell sizes, our simulations yielded results similar to those of the $r_{b_0} = 3 \times 10^4$ universe. Figure 19 depicts plots of $z_{\text{FLRW}}$ vs $z_{\text{FLRW}}$ for the $\theta_0 = \pi/2$ trajectory for universes with $r_{b_0}$ ranging from $10^5$ to $10^8$. All graphs again converged quickly to a straight line from above, including graphs, not shown, for the other trajectories. One noticeable development though was that the curve at the start of the graphs became more pronounced as the initial size increased. However when we simulated photons starting from a large radius again, the initial jump and curve again disappeared, as shown in Fig. 20 for photons starting from $r = 10^7 R$ in the $r_{b_0} = 10^8 R$ universe.

We also notice, in Fig. 18, that the gradients of $z_{\text{FLRW}}$ vs $z_{\text{FLRW}}$ graphs decrease with $\theta_n$. In fact, if we plot the gradients against $\cos(\theta_n)$, we find a strongly linear relationship between the two quantities, as shown in Fig. 21. As the figure shows, this is also true of the gradients for all the other flat universes we simulated. Since $\cos(\theta_n)$ determines the radial component $\dot{\rho}_{\text{init}}$ of the photon’s initial velocity, this implies that the gradients are linearly related to $\dot{\rho}_{\text{init}}/\vartheta_{\text{init}}$. As the photon’s initial velocity becomes increasingly radial, the photon’s frequency becomes increasingly blueshifted relative to the FLRW redshift. We believe this is due to the central mass’ stronger influence on the photon, as a more radial trajectory would bring the photon closer to the central mass.

Everything we noticed about redshifts in the flat universe also applied to open universes. We simulated open universes for values of $E_0$ in the range of $1.1 \geq E_0 > 1$. In each simulation, the initial cell size was fixed to be $r_{b_0} = 3 \times 10^4$, and photons were propagated along 12 trajectories with initial angles ranging from $\theta_0 = \pi/2$ to $\theta_1 = 149\pi/1200$ in decrements of $41\pi/1200$. Depending on $E_0$, $\Delta \vartheta_0$ ranged from $10^{-5} R$ for $E_0 = 1$ to $10^{-3} R$ for $E_0 = 1.1$. Figure 22 shows a clear linear relationship again between $z_{\text{FLRW}}$ and $z_{\text{FLRW}}$ for the $E_0 = 1.1$ universe. The regression equations’ non-zero intercepts indicate the presence again of a jump between the zero-redshift point and the first non-zero redshift point. Similar behaviour was seen for universes corresponding to other values of $E_0$. Figure 23 shows that for all $E_0$, there is again a clear negative linear relation between the gradients of $z_{\text{FLRW}}$ vs $z_{\text{FLRW}}$ graphs and $\cos(\theta_n)$, or equivalently between the gradients and $\dot{\rho}_{\text{init}}/\vartheta_{\text{init}}$.

as follows:

$\text{gradient} = -0.3477 \cos(\theta) + 1.0994$

$\text{RMS of residuals} = 7.01 \times 10^{-6}$

$\text{gradient} = -0.3380 \cos(\theta) + 1.0689$

$\text{RMS of residuals} = 2.11 \times 10^{-6}$

$\text{gradient} = -0.3322 \cos(\theta) + 1.0504$

$\text{RMS of residuals} = 2.16 \times 10^{-7}$

$\text{gradient} = -0.3234 \cos(\theta) + 1.0226$

$\text{RMS of residuals} = 2.11 \times 10^{-8}$

$\text{gradient} = -0.3104 \cos(\theta) + 0.9814$

$\text{RMS of residuals} = 1.99 \times 10^{-9}$

FIG. 21. A plot of the gradients of $z_{\text{FLRW}}$ vs $z_{\text{FLRW}}$ graphs against $\cos(\theta_n)$ for flat universes of different initial cell size $r_{b_0}$. A linear regression has been performed on each graph, and the regression equations are displayed in order of increasing $r_{b_0}$. 

The graph shows the following:

\begin{align*}
\text{gradient} &= -0.3477 \cos(\theta) + 1.0994 \\
\text{RMS of residuals} &= 7.01 \times 10^{-6} \\
\text{gradient} &= -0.3380 \cos(\theta) + 1.0689 \\
\text{RMS of residuals} &= 2.11 \times 10^{-6} \\
\text{gradient} &= -0.3322 \cos(\theta) + 1.0504 \\
\text{RMS of residuals} &= 2.16 \times 10^{-7} \\
\text{gradient} &= -0.3234 \cos(\theta) + 1.0226 \\
\text{RMS of residuals} &= 2.11 \times 10^{-8} \\
\text{gradient} &= -0.3104 \cos(\theta) + 0.9814 \\
\text{RMS of residuals} &= 1.99 \times 10^{-9}
\end{align*}
FIG. 22. A plot of $z_{LW}$ against $z_{FLRW}$ for 12 trajectories in the $E_b = 1.1$ universe. For this simulation, $\Delta r_0$ was $10^{-3} R$. The initial angle $\theta_n$ of the trajectory is given in the legend. Each trajectory was traced across 15 cells. A linear regression has been performed for each graph. The regression equations and corresponding root-mean-square of the residuals are listed above in order of decreasing $\theta_n$. 

$z_{LW} = 1.1109 z_{FLRW} + 0.0908$, RMS of residuals = 0.0353
$z_{LW} = 1.0732 z_{FLRW} + 0.0539$, RMS of residuals = 0.0341
$z_{LW} = 1.0361 z_{FLRW} + 0.0174$, RMS of residuals = 0.0328
$z_{LW} = 0.9997 z_{FLRW} - 0.0183$, RMS of residuals = 0.0317
$z_{LW} = 0.9647 z_{FLRW} - 0.0526$, RMS of residuals = 0.0305
$z_{LW} = 0.9313 z_{FLRW} - 0.0854$, RMS of residuals = 0.0294
$z_{LW} = 0.9000 z_{FLRW} - 0.1161$, RMS of residuals = 0.0283
$z_{LW} = 0.8711 z_{FLRW} - 0.1444$, RMS of residuals = 0.0274
$z_{LW} = 0.8450 z_{FLRW} - 0.1700$, RMS of residuals = 0.0265
$z_{LW} = 0.8220 z_{FLRW} - 0.1926$, RMS of residuals = 0.0257
$z_{LW} = 0.8022 z_{FLRW} - 0.2120$, RMS of residuals = 0.0251
$z_{LW} = 0.7869 z_{FLRW} - 0.2279$, RMS of residuals = 0.0245
is a plot for the flat universe, corresponding to $E = 1$ approaches an asymptotic value as $E \to 0$ for a selection of initial photon trajectories. Each graph approaches an asymptotic value as $E_b$ increases. We note that in the LW co-ordinates for the closed universe, we do not have freedom to choose the initial size of the cell; once the photon’s initial co-ordinates are chosen, then the cell boundary’s position is determined since both the boundary and the source must reach their respective maximum radii at the same time $\tau$. For this simulation, we chose to propagate photons along 30 trajectories ranging from $\theta_n = \pi/2$ to $\theta_{29} = 17\pi/150$ going in decrements of $2\pi/150$.

The simulation results were again similar to those for the flat and open universes but with several slight differences this time. The graphs of $z_{LW}$ against $z_{FLRW}$ again followed straight lines but the gradient was slightly different for when the photon was outgoing and when it was ingoing, as shown in Fig. 25. As the figure also shows, the graphs did not necessarily intersect the origin either. Furthermore, there was a subtle bend in the low $z$ end of several of the graphs for ingoing photons. We believe this bend to be analogous to the initial curve we saw in the flat universe graphs previously; it corresponds to redshifts measured at low $r_{obs}$ and is more pronounced in the more radial trajectories, that is, the smaller $\theta_n$ trajectories, which pass closer to the central mass. Thus, like the initial curve in the flat universe graphs, we believe the bend is caused by the central mass’ stronger influence at low $r_{obs}$. In Fig. 26a we show graphs of $z_{LW}$ against $z_{FLRW}$ for when the photons are outgoing,

\[
\text{gradient} = -0.3291 \cos(\theta) + 1.0407
\]
RMS of residuals = $6.57 \times 10^{-6}$

\[
\text{gradient} = -0.3321 \cos(\theta) + 1.0499
\]
RMS of residuals = $6.58 \times 10^{-6}$

\[
\text{gradient} = -0.3377 \cos(\theta) + 1.0678
\]
RMS of residuals = $6.53 \times 10^{-6}$

\[
\text{gradient} = -0.3407 \cos(\theta) + 1.0773
\]
RMS of residuals = $6.44 \times 10^{-6}$

\[
\text{gradient} = -0.3463 \cos(\theta) + 1.0950
\]
RMS of residuals = $6.03 \times 10^{-6}$

\[
\text{gradient} = -0.3478 \cos(\theta) + 1.0999
\]
RMS of residuals = $5.82 \times 10^{-6}$

\[
\text{gradient} = -0.3512 \cos(\theta) + 1.1109
\]
RMS of residuals = $7.13 \times 10^{-6}$

We note that the earlier analysis of the dependence of $\nu_{LW}$ on $r_{obs}$, given by function (70), would require some modification in this context. In particular, as we are now using LW co-ordinates, function (70) would change, with parameter $E_b$ replaced by a parameter $E_{lb}$ corresponding to the constant $E$ for the co-moving

FIG. 23. Plot of the gradients of $z_{LW}$ vs $z_{FLRW}$ graphs against $\cos(\theta_n)$ for open universes of different values of $E_b$. Also shown is a plot for the flat universe, corresponding to $E_b = 1$. A linear regression has been performed on each graph, and both the regression equation and the root-mean-square of the residuals are displayed in order of increasing $E_b$.

FIG. 24. A plot of the $z_{LW}$ vs $z_{FLRW}$ gradients against $E_b$ for a selection of initial photon trajectories. Each graph approaches an asymptotic value as $E_b$ increases.

\[
E_b = 1.00000 \quad E_b = 1.00005 \quad E_b = 1.00050 \quad E_b = 1.00100
\]

\[
E_b = 1.00001 \quad E_b = 1.00010 \quad E_b = 1.00100
\]
In fact, including all data points in the regression introduced with their FLRW counterparts in any universe and for universes. The LW redshifts generally increase linearly demonstrated several features common to redshifts in all LW starting radius.

In going ingoing photons, we excluded data other for when they are ingoing. To exclude the initial bend in the graphs for ingoing photons, we excluded data points corresponding to \( r_{\text{obs}} \) less than 10 \( R_c \), the photons’ starting radius\textsuperscript{1}\textsuperscript{1}. The figure shows that, as with the other universes, the gradient decreases as the photon’s trajectory becomes more radial, but now, the relationship between the gradients and \( \cos(\theta_n) \) is no longer linear.

Thus to summarise, our simulations have demonstrated several features common to redshifts in all LW universes. The LW redshifts generally increase linearly with their FLRW counterparts in any universe and for observer’s geodesic. The new function though would still have the same functional form as the original and therefore the same general behaviour. However since the radius of the observer’s geodesic is bounded, the new function’s domain is bounded from above by \( r_{\text{obs}} \leq 2m/(1 - E) \), otherwise the square-root in the function would turn imaginary; therefore there cannot be any large-\( r_{\text{obs}} \) asymptotic regime in the new function, and hence this aspect of our previous analysis is not transferable to the closed universe. Nevertheless, we believe that the general conclusion still applies, that the initial jump and the bend in \( z_{\text{LW}} \) vs \( z_{\text{FLRW}} \) graphs is caused by the central mass’ stronger influence at low \( r_{\text{obs}} \).

\textsuperscript{1}In fact, including all data points in the regression introduced some ‘jaggedness’ in the graph of ingoing gradients, especially towards the low \( \cos(\theta_n) \) regime. By excluding these points, the ‘jaggedness’ smoothed away, and we were left with a curve much more similar to that for the outgoing gradients.

![FIG. 25. A plot of \( z_{\text{LW}} \) against \( z_{\text{FLRW}} \) for two trajectories of photons, \( \theta_0 = \pi/2 \) and \( \theta_2 = 17\pi/150 \). Each graph is divided into an outgoing graph and an ingoing graph that meet at the point corresponding to redshift closest to maximum expansion of the universe. There is a slight bend in the blueshift regime of the ingoing \( \theta_2 = 17\pi/150 \) graph.](image)

and in Fig. 20\textsuperscript{1} we show the corresponding graphs for when the same photons are ingoing.

In Fig. 27 we have plotted the gradients of \( z_{\text{LW}} \) vs \( z_{\text{FLRW}} \) against \( \cos(\theta_n) \). Two graphs are shown, one for the gradients when the photons are outgoing, and the other for when they are ingoing. To exclude the initial bend in the graphs for ingoing photons, we excluded data points corresponding to \( r_{\text{obs}} \) less than 10 \( R_c \), the photons’ starting radius. The figure shows that, as with the other universes, the gradient decreases as the photon’s trajectory becomes more radial, but now, the relationship between the gradients and \( \cos(\theta_n) \) is no longer linear.

Thus to summarise, our simulations have demonstrated several features common to redshifts in all LW universes. The LW redshifts generally increase linearly with their FLRW counterparts in any universe and for any photon trajectory. When \( r_{\text{obs}} \) is small such that the central mass has a stronger influence, there is some deviation away from this linear behaviour. Any influence from the central mass can be suppressed by starting the photon at very large radii; \( z_{\text{LW}} \) then becomes completely proportional to \( z_{\text{FLRW}} \). The LW redshifts also generally decrease relative to their FLRW counterparts as the photon takes a more radial trajectory; this can also be attributed to the stronger influence of the central mass since a more radial trajectory would pass closer to it. Thus we see certain effects arising from the ‘lumpiness’ of the LW universe and which would not be present in the perfectly homogeneous and isotropic FLRW universe.

VIII. DISCUSSION

We have investigated the properties of the Lindquist-Wheeler universes, as we hope they might provide some insight into what observable effects might the ‘lumpy’ matter distribution of the actual universe yield. And although the LW universes are only approximations rather than exact solutions to the Einstein equations, we believe they model enough of the underlying physics to yield at least meaningful qualitative insights into the behaviour of the actual universe. Much of the LW universes’ dynamics bear strong resemblance to those of the matter-dominated FLRW universes. Additionally, photon redshifts in LW universes behaved roughly similarly to their FLRW counterparts. Yet there were also subtle direction-dependent effects in the redshifts due to the ‘lumpiness’ of the universe’s matter distribution.

Our investigation can be extended in several ways. It would be interesting to examine the optical properties of the LW universe, as this may have important implications for actual astrophysical observations. Clifton and Ferreira have already done this for their implementation of the LW universe, but we have used a different implementation from theirs, and this has led to certain differences in the behaviour of redshifts; thus there is reason to believe the optical properties may differ as well. It would also be interesting to extend our study to LW universes with non-zero cosmological constant. Clifton and Ferreira have already constructed an appropriate extension of the LW universe based on the Schwarzschild de-Sitter metric, and they have shown that the corresponding Friedmann-like equation strongly resembles its FLRW counterpart as well. It should be possible to include Clifton and Ferreira’s \( \Lambda \)-Schwarzschild-cells in our implementation and investigate the resulting model. Finally, our model has a still very idealised distribution of matter. Each mass is identical and distributed on a perfect lattice, which is clearly not the case in the actual universe. And all insights into what observable effects might the ‘lumpy’ universes, as we hope they might provide some insight into what observable effects might the ‘lumpy’ matter distribution of the actual universe yield. And although the LW universes are only approximations rather than exact solutions to the Einstein equations, we believe they model enough of the underlying physics to yield at least meaningful qualitative insights into the behaviour of the actual universe. Much of the LW universes’ dynamics bear strong resemblance to those of the matter-dominated FLRW universes. Additionally, photon redshifts in LW universes behaved roughly similarly to their FLRW counterparts. Yet there were also subtle direction-dependent effects in the redshifts due to the ‘lumpiness’ of the universe’s matter distribution.

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FIG. 26. A plot of $z_{\text{LW}}$ against $z_{\text{FLRW}}$ while photons are (a) outgoing and then (b) ingoing. Graphs for a selection of initial photon trajectories $\theta_n$ are shown. A linear regression was performed on each graph, and the regression equation is listed in order of decreasing $\theta_n$. In the ingoing case, the regression only includes redshifts measured at $r_{\text{obs}}$ equal to at least 10 $R$, the photon’s starting radius, so as to exclude points in the initial bend.

For (a) outgoing:

- $z_{\text{LW}} = 0.9990 z_{\text{FLRW}} + 0.0342$, RMS of residuals = $8.461 \times 10^{-4}$
- $z_{\text{LW}} = 0.9935 z_{\text{FLRW}} + 0.0251$, RMS of residuals = $1.141 \times 10^{-3}$
- $z_{\text{LW}} = 0.9866 z_{\text{FLRW}} + 0.0170$, RMS of residuals = $1.116 \times 10^{-3}$
- $z_{\text{LW}} = 0.9781 z_{\text{FLRW}} + 0.0105$, RMS of residuals = $7.763 \times 10^{-4}$
- $z_{\text{LW}} = 0.9665 z_{\text{FLRW}} + 0.0063$, RMS of residuals = $9.523 \times 10^{-4}$
- $z_{\text{LW}} = 0.9547 z_{\text{FLRW}} + 0.0024$, RMS of residuals = $7.152 \times 10^{-4}$
- $z_{\text{LW}} = 0.9404 z_{\text{FLRW}} + 0.0000$, RMS of residuals = $1.422 \times 10^{-3}$
- $z_{\text{LW}} = 0.9267 z_{\text{FLRW}} + 0.0032$, RMS of residuals = $9.969 \times 10^{-4}$
- $z_{\text{LW}} = 0.9085 z_{\text{FLRW}} + 0.0049$, RMS of residuals = $1.946 \times 10^{-3}$
- $z_{\text{LW}} = 0.8892 z_{\text{FLRW}} + 0.0085$, RMS of residuals = $8.650 \times 10^{-4}$

For (b) ingoing:

- $z_{\text{LW}} = 1.0271 z_{\text{FLRW}} - 0.0028$, RMS of residuals = $1.552 \times 10^{-3}$
- $z_{\text{LW}} = 1.0201 z_{\text{FLRW}} - 0.0098$, RMS of residuals = $6.617 \times 10^{-4}$
- $z_{\text{LW}} = 1.0128 z_{\text{FLRW}} - 0.0173$, RMS of residuals = $6.430 \times 10^{-4}$
- $z_{\text{LW}} = 1.0000 z_{\text{FLRW}} - 0.0260$, RMS of residuals = $1.085 \times 10^{-3}$
- $z_{\text{LW}} = 1.0008 z_{\text{FLRW}} - 0.0375$, RMS of residuals = $6.022 \times 10^{-4}$
- $z_{\text{LW}} = 0.9955 z_{\text{FLRW}} - 0.0491$, RMS of residuals = $9.149 \times 10^{-4}$
- $z_{\text{LW}} = 0.9891 z_{\text{FLRW}} - 0.0605$, RMS of residuals = $9.778 \times 10^{-4}$
- $z_{\text{LW}} = 0.9821 z_{\text{FLRW}} - 0.0718$, RMS of residuals = $1.498 \times 10^{-3}$
- $z_{\text{LW}} = 0.9700 z_{\text{FLRW}} - 0.0805$, RMS of residuals = $1.517 \times 10^{-3}$
- $z_{\text{LW}} = 0.9529 z_{\text{FLRW}} - 0.0870$, RMS of residuals = $1.710 \times 10^{-3}$
of such universes and the detailed investigation of their properties to future work.

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APPENDIX A: REGULAR LATTICES IN 3-SPACES OF CONSTANT CURVATURE

In this appendix, we shall list all possible lattices that cover 3-spaces of constant curvature with a single regular polyhedral cell. The cell is tiled to completely cover the 3-space without any gaps or overlaps. This tesselation problem has been thoroughly studied by Coxeter [9]. Clifton and Ferreira [3] have succinctly summarised Coxeter’s results relevant to our discussion, and we have presented their summary in Table I.

Table I. All possible lattices obtained by tessellating 3-spaces of constant curvature with a single regular polyhedron.

| Elementary cell shape | Number of cells at a lattice edge | Background curvature | Total cells in lattice |
|-----------------------|----------------------------------|----------------------|-----------------------|
| tetrahedron           | 3                                | +                    | 5                     |
| cube                  | 3                                | +                    | 8                     |
| tetrahedron           | 4                                | +                    | 16                    |
| octahedron            | 3                                | +                    | 24                    |
| dodecahedron          | 3                                | +                    | 120                   |
| tetrahedron           | 5                                | +                    | 600                   |
| cube                  | 4                                | 0                    | ∞                     |
| cube                  | 5                                | -                    | ∞                     |
| dodecahedron          | 4                                | -                    | ∞                     |
| dodecahedron          | 5                                | -                    | ∞                     |
| icosahedron           | 3                                | -                    | ∞                     |

Column 1 gives the lattice cell’s shape. Column 2 effectively determines the lattice’s structure: it indicates how many of Column 1’s elementary cells meet at any lattice edge. Column 3 indicates whether the background 3-space has positive, flat, or negative curvature. Column 4 gives the number of cells needed to cover the 3-space; only lattices on positively curved space can have a finite number of cells. To construct a closed lattice universe, one has six choices of lattices; for example, one can choose a 600-cell lattice where cells are equilateral tetrahedra and where five tetrahedra meet at any edge. To construct a flat lattice, one has but a single choice. To construct an open lattice, one has four.

APPENDIX B: RADIAL VELOCITIES IN REGGE SPACE-TIME

In this appendix, we shall present our numerical results supporting [68] to be the 4-velocity tangent to radial time-like geodesics in Regge Schwarzschild space-time. These geodesics include, most importantly, those of test particles co-moving with a Schwarzschild-cell boundary. We began by determining numerically the escape velocity for a test particle. We propagated a test particle radially outwards from a series of initial radii \( r_{\text{init}} \) and with a series of initial velocities \( \dot{r}_{\text{init}} \). We started \( \dot{r}_{\text{init}} \) from 0 and increased it until the maximum radius \( r_{\text{max}} \) attained by the particle was very large. For each \( r_{\text{init}} \), we then plotted \( R/r_{\text{max}} \) against its corresponding \( \dot{r}_{\text{init}} \), where \( R \) is the Schwarzschild radius. Both \( r_{\text{init}} \) and \( r_{\text{max}} \) refer to the lower Schwarzschild label of the block in which the particle is found. Our plots are shown in Fig. 28. Each
We have also looked at graphs for different $r_{\text{init}}$. Each graph has 100 data points. A quadratic regression has been performed on each graph, and the regression equations are ordered from $r_{\text{init}} = 8R$ to $512R$. The regressions were performed without a linear term; regression was also attempted with a linear term present, but it was found to be many orders of magnitude smaller than the other two terms. The simulation’s block parameters were $R = 31$, $\Delta t = 10\Delta r$, $\Delta r = R/10^3$, and $\Delta \phi = 2\pi/(3 \times 10^7)$.

![Figure 28](image.png)

**FIG. 28.** A plot of $R/r_{\text{max}}$ versus $\dot{r}_{\text{init}}$ for various $r_{\text{init}}$. Each graph has 100 data points. A quadratic regression has been performed on each graph, and the regression equations are ordered from $r_{\text{init}} = 8R$ to $512R$. The regressions were performed without a linear term; regression was also attempted with a linear term present, but it was found to be many orders of magnitude smaller than the other two terms. The simulation’s block parameters were $R = 31$, $\Delta t = 10\Delta r$, $\Delta r = R/10^3$, and $\Delta \phi = 2\pi/(3 \times 10^7)$.

![Figure 29](image.png)

**FIG. 29.** A plot of $(u^\rho)^2$ against $\dot{\rho}^2$ for various $r_{\text{max}} < 0$, where $u^\rho$ is given by [55]. Each plot consists of velocities computed at 100 different radii along the course of the test particle’s trajectory. A linear regression was performed on each graph, and the corresponding equations are ordered from $R/r_{\text{max}} = -1/11$ to $-1/91$. The graphs of the regressions completely overlap each other. The gradients only begin deviating from unity at the $10^{-5}$ order of magnitude. The simulation’s block parameters were $r_{\text{init}} = 2R$, $R = 21$, $\Delta t = \Delta r = R/10^3$, and $\Delta \phi = 2\pi/(3 \times 10^7)$.

graph is very well-fitted by a quadratic curve of the form

$$\frac{R}{r_{\text{max}}} = -A \dot{\rho}_{\text{init}}^2 + B.$$  

Moreover, the co-efficients $A$ and $B$ always satisfy the relations $A + B = 1$ and $B = R/r_{\text{init}}$ [54]. From these relations, we can therefore infer a relation for $\dot{\rho}$ corresponding to that given in [55], and the $\tau$ component follows from normalisation.

We have just provided numerical support for [58] for test particles following geodesics where $r_{\text{max}} \geq 0$, corresponding to closed orbits or orbits at the escape velocity. At this point, we conjectured that for open orbits, [58] would still apply but with $r_{\text{max}} < 0$. To test this, we propagated a test particle outwards with an initial velocity given by [58] but for $r_{\text{max}} < 0$, and we examined how the velocity evolved with $r$. Let us denote the ve-

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12 We have also looked at graphs for different $R$ and $r_{\text{init}}$, not shown, and they also conform to this pattern.
velocity of our simulated particle by \( \dot{\rho} \). Along the particle’s trajectory, we compared \( \dot{\rho}^2 \) against \( (u^\rho)^2 \) for the same radius, where \( u^\rho \) is given by the radial component of the Regge block. Our comparison is presented graphically in Fig. 29. In all cases, the graphs’ gradients were effectively unity and the constants effectively zero, thus indicating that the particle’s velocity does indeed obey (58). Indeed, this result was obtained even when the radial resolution was as low as \( \Delta r = R/10^5 \), as shown in Fig. 28. Therefore even if a particle began at the correct escape velocity for its initial radius, it would gradually travel faster than the correct escape velocity for its subsequent radii as it propagated along its trajectory.

We have just found that the initial velocity \( \dot{\rho}_{\text{init}} \) of a simulated particle must equal at least that of \( \dot{\rho}_{\text{max}} \) in order for the particle to ‘escape’. We next examined the long-term behaviour of the particle’s velocity \( \dot{\rho} \) as the particle propagated outwards from an initial velocity equaling at least the escape velocity. We found that \( \dot{\rho} \) would not stay equal to \( u^\rho \) as given by (58) but would instead decrease at a slower rate as a function of the particle’s radius. For example if we started a particle at the escape velocity at \( r_{\text{init}} = 2 \frac{R}{\rho} \) and propagated it outwards, then by the time it reached \( r = 3 \times 10^4 R \), there was a very significant discrepancy between \( \dot{\rho} \) and \( u^\rho \), as shown in Fig. 30, with \( \dot{\rho} \) being consistently larger than \( u^\rho \). This discrepancy reduced if we improved the radial resolution \( \Delta r \) of our Regge blocks, but it was still present even for resolutions as high as \( \Delta r = R/10^5 \). If instead, using the same method as previously, we numerically determined the escape velocity for a particle starting at \( r_{\text{init}} = 3 \times 10^4 R \), we again obtained the same quadratic behaviour between \( R/r_{\text{max}} \) and \( \dot{\rho} \) as that in Fig. 28, thus indicating the escape velocity should still correspond to (58). Indeed, this result was obtained even when the radial resolution was as low as \( \Delta r = R/10 \), as shown in Fig. 31 even though there is a huge discrepancy at this resolution between \( \dot{\rho} \) and \( u^\rho \) in Fig. 30. Therefore even if a particle began at the correct escape velocity for its initial radius, it would gradually travel faster than the correct escape velocity for its subsequent radii as it propagated along its trajectory.

![Fig. 30. A plot of \( 1/\dot{\rho} \) versus \( r/R \) comparing simulated particle velocities with \( u^\rho \). For the simulated particles’ graphs \( \dot{\rho} = \dot{\rho}_{\text{init}} \), while for the \( u^\rho \) graph, \( \dot{\rho} = u^\rho \). The simulation was run for various sizes \( \Delta r \) of the Regge block. For all simulations, the particle was propagated from \( r = 2 R \) to \( 3 \times 10^4 R \). The simulation’s other block parameters were \( R = 1 \), \( \Delta t = \Delta r \), and \( \Delta \phi = 2\pi/(3 \times 10^7) \).](image)

![Fig. 31. A plot of \( R/r_{\text{max}} \) versus \( \dot{\rho}_{\text{init}} \) for \( r_{\text{init}} = 3 \times 10^4 R \). A quadratic regression has been performed and the corresponding equation shown. The simulation’s block parameters were \( R = 1 \), \( \Delta t = 10\Delta r \), \( \Delta r = R/10 \), and \( \Delta \phi = 2\pi/(3 \times 10^7) \).](image)

13 Again, we have also simulated the cases of \( r_{\text{max}} = -R, -21 R, -41 R, -61 R, \) and \(-81 R \), not shown here, and they all display the same behaviour as in Fig. 29. The percentage difference seen between \( \dot{\rho}^2 \) and \( (u^\rho)^2 \) is also small but slowly increasing in all \( r_{\text{max}} \) that we looked at.
TABLE II. A list of \( \hat{\rho}^2 \) and \((u^\rho)^2\) at various radii, and the percentage difference between them. This data is for \( R/r_{\text{max}} = -91 \).

| \( r/R \) | \( \hat{\rho}^2 \) | \((u^\rho)^2\) | % difference |
|---------|-------------|-------------|-------------|
| 2       | 1.0219794649 | 1.02197802197802 | 0.00014118894047312 |
| 102     | 0.021001226724 | 0.0209988015595972 | 0.011547737528382 |
| 202     | 0.016021230625 | 0.0160188121184031 | 0.01509565191602 |
| 302     | 0.014350362849 | 0.0143477802197414 | 0.017969613713785 |
| 402     | 0.013512667536 | 0.0135101783683129 | 0.0184209940820187 |
| 502     | 0.013009455481 | 0.0130069512260618 | 0.0192494985037111 |
| 602     | 0.012673806084 | 0.012671189481995 | 0.0206457475160902 |
| 702     | 0.012433811049 | 0.0124312192608481 | 0.0208446802168433 |
| 802     | 0.012253604416 | 0.0122511667237755 | 0.0198936748870944 |

![Fig. 32](image_url)

FIG. 32. A plot of \( \hat{\rho} \) versus \( r/R \) comparing a simulated particle velocity with \( u^\rho \). The simulation began at \( r_{\text{init}} = 3 \times 10^4 \), but only data points from \( r = 2 R \) to \( 100 R \) are shown. The simulated velocity is always greater than or equal to \( u^\rho \). The simulation’s block parameters were \( R = 1 \), \( \Delta t = \Delta r = R \), and \( \Delta \phi = 2\pi/(3 \times 10^7) \).

change in velocity, but this change may not be enough to overcome the discrepancy already accumulated. By a similar argument, we expect the radial velocity of a radially in-going particle to decrease less quickly than \( u^\rho \) in \( 30 \), because if the particle crosses a \( +\rho \) face with the correct escape velocity, it will cross the neighbouring \( +\rho \) face with the same unchanged velocity, which would be less negative than \( u^\rho \) for the same radius; and indeed, this is what we see in Fig. 32. By improving the resolution of our Regge model, we slow the growth rate of the discrepancy, as we saw in Fig. 30; this is because our model becomes a better approximation to the underlying continuum Schwarzschild space-time, and its geodesics would therefore become more similar to those in the continuum space-time.

Thus combined with our analytic arguments provided at the end of Section VI, we conclude that (58) gives the correct velocity as a function of \( r \) for particles following radial time-like geodesics.

APPENDIX C: RADIAL LENGTHS OF SCHWARZSCHILD REGGE BLOCKS

In our simulations of the flat and open universes, our Regge blocks’ radial lengths were not constant but were increased as the blocks were further away from the centre. This decision was motivated by the fact that the underlying space-time being approximated becomes increasingly flat. We therefore attempted to maintain higher resolution while the curvature was high but then decrease the resolution with minimal impact on accuracy as the curvature decreased. This technique allowed us to perform larger-scale simulations within more attainable computation times. We now describe our method for specifying
the block’s radial length.

In Schwarzschild space-time, the radial distance between two radii \( r_i \) and some arbitrary \( r > 2m \) is given by

\[
d(r) = \left[ x \sqrt{1 - \frac{2m}{x}} - 2m \ln \left( \sqrt{\frac{x}{2m}} - \sqrt{\frac{x}{2m} - 1} \right) \right]^{r - r_i},
\]

When \( r = r_{i+1} \), this is identical to (31). Distances along ‘radial edges’ of Regge blocks approximate radial distances in Schwarzschild space-time by linearly interpolating between \( d(r_i) \) and \( d(r_{i+1}) \), as shown in Fig. 33.

Since \( d(r) \) is convex, the interpolation will always underestimate the true distance, and the error \( \varepsilon \) of the approximation is given by \( \varepsilon = \max \{ d(r) - \text{lin}(r) \} \), where \( \text{lin}(r) \) denotes the interpolating function.

![Schwarzschild space-time](image)

**FIG. 33.** The Regge block interpolation intersects the Schwarzschild graph at \( r = r_i \) and \( r = r_{i+1} \), where \( r_i < r_{i+1} \) and \( r_i \) corresponds to the origin of the graph; it interpolates the radial distance for only those values of \( r \) lying between these two points of intersection. Since the Schwarzschild distance function is convex, the interpolation will always underestimate the distance in this regime.

Our goal is as follows. Given some error-tolerance \( \varepsilon_{\text{tol}} \), we want our blocks to be as long as possible in the radial direction while still satisfying \( \varepsilon \leq \varepsilon_{\text{tol}} \). If we enter a new block from its left or right face, then we would need to re-calculate the block’s radial length \( h = r_{i+1} - r_i \) in accordance with our goal. It will always be the case that we know one of \( r_i \) or \( r_{i+1} \) depending on which face we entered by, and we would need to deduce the other quantity. However our computation must give consistent results for \( h \) regardless of which quantity we used. In our particular implementation, we have also chosen to impose the constraint that \( h \) must be some integer units of a minimal interval \( \Delta r_0 \), a parameter which we can freely specify.

The most natural approach would be to solve \( d'(r) = \text{lin}'(r) \), which maximises \( d(r) - \text{lin}(r) \), to obtain \( r = r_{\text{max}} \), the point at which the error is greatest. Also through its dependence on \( \text{lin}(r) \), \( r_{\text{max}} \) would be a function of \( h \). We would then set \( \varepsilon = \varepsilon_{\text{tol}} \) and solve \( d(r_{\text{max}}) - \text{lin}(r_{\text{max}}) = \varepsilon_{\text{tol}} \) for \( h \). However, this second equation can only be solved numerically, and from a programming point of view, it was easier to implement a different approach instead.

We note that for \( r > r_i \), the functions \( d(r) \) and \( \text{lin}(r) \) are bounded from above by

\[
d^{+}(r) = \sqrt{\frac{r_i}{r_i - 1}} (r - r_i) + d(r_i),
\]

since this is just the equation of the tangent to \( d(r) \) at \( r_i \). And also, for \( r > r_i \), the two functions are bounded from below by

\[
d^{-}(r) = (r - r_i) + d(r_i);
\]

this follows because the gradient of \( d(r) \) asymptotes to unity from above as \( r \to \infty \), which means \( d(r) \) will always increase more quickly than \( d^{-}(r) \). Therefore, after the two functions have intercepted at \( r = r_i \), \( d(r) \) will always be strictly greater than \( d^{-}(r) \). The function \( \text{lin}(r) \) also intercepts both \( d(r) \) and \( d^{-}(r) \) at \( r = r_i \). But it intercepts \( d(r) \) again at \( r_{i+1} > r_i \), a point where \( d(r) > d^{-}(r) \). Because \( \text{lin}(r) \) is a linear function, this interception at \( r_{i+1} \) implies that \( \text{lin}(r) \) must have a greater gradient than \( d^{-}(r) \), and therefore \( \text{lin}(r) \) must also be strictly greater than \( d^{-}(r) \) in the region \( r > r_i \).

Thus we can bound our error by \( \varepsilon \leq \max \{ d^{+}(r) - d^{-}(r) \} = d^{+}(r_{i+1}) - d^{-}(r_{i+1}) \). We shall therefore require that

\[
\varepsilon_{\text{tol}} \geq d^{+}(r_{i+1}) - d^{-}(r_{i+1}),
\]

and solve for \( r_{i+1} \) from this equation; this yields the solution

\[
h = \frac{\varepsilon_{\text{tol}}}{\sqrt{r_i / (r_i - 1)} - 1}.
\]

To get \( h \) as a function of \( r_{i+1} \) instead, we set \( r_i \) to be \( r_i = r_{i+1} - h \) and solve for \( h \) in the preceding equation; this yields

\[
h = \frac{\varepsilon_{\text{tol}}}{2(2\varepsilon_{\text{tol}} - 1) \left[ 2r_{i+1} + \varepsilon_{\text{tol}} - 2 + \left[ 4r_{i+1}(r_{i+1} - 1 - \varepsilon_{\text{tol}}) + \varepsilon_{\text{tol}}(4 + \varepsilon_{\text{tol}}) \right]^{1/2} \right]}.\]
As mentioned above, we want \( h \) to satisfy \( h = n \Delta r_0 \) for some integer \( n \). Therefore our actual \( h \) is obtained by taking \( n \) to be

\[
 n = \left\lfloor \frac{\varepsilon_{tol}}{\Delta r_0 \sqrt{r_i/(r_i - 1)} - 1} \right\rfloor \tag{C4}
\]

or

\[
 n = \left\lfloor \frac{\varepsilon_{tol}}{2\Delta r_0 (2\varepsilon_{tol} - 1)} \left\{ 2r_{i+1} + \varepsilon_{tol} - 2
\right.ight.
+ \left[ 4r_{i+1}(r_{i+1} - 1 - \varepsilon_{tol}) \right. \\
+ \left. \varepsilon_{tol}(4 + \varepsilon_{tol}) \right]^{1/2} \right\} \right\rfloor . \tag{C5}
\]

as appropriate, where \( \lfloor x \rfloor \) gives the greatest integer less than or equal to \( x \), and then substituting this \( n \) into \( h = n \Delta r_0 \).

Finally, we want to ensure that our algorithm would give the same \( h \) regardless of whether it used \( r_i \) or \( r_{i+1} \). To satisfy this requirement, whenever the program calculated \( r_{i+1} \) from \( r_i \), it would check to see if it could recover the same \( h \) using the new value for \( r_{i+1} \). If not, then it would decrement \( h \) by one unit of \( \Delta r_0 \) and check again, and it would continue decrementing until obtaining agreement. The program can only decrement if the error is to remain within tolerance. We had also required that \( h \) be at least \( \Delta r_0 \); thus the decrementing would stop if \( h \) reached this length.

**APPENDIX D: BOUNDARY CONDITIONS FOR SCHWARZSCHILD-CELLS OF UNEQUAL MASSES**

An observer sitting at the interface of two cell boundaries must observe the same physics regardless of which co-ordinate system the observer uses. In particular, the following two conditions must be satisfied:

1) the observer must measure the same proper time regardless of which cell’s metric is used;

2) and the observer must measure the same spatial distances locally along the boundary regardless of which cell’s metric is used.

However if the cells in the lattice are no longer identical, we would no longer have the same lattice symmetries as before. In particular, the exact location of the boundary between two unequal cells becomes less transparent, as it is no longer equidistant to the two cell centres. Recall that in the perfectly symmetric lattice, a test particle sitting on the boundary between two cells would by symmetry always remain at the boundary and yet fall simultaneously towards both cell centres. Although the symmetry is no longer present when the two cells are no longer identical, we shall still assume that test particles at the boundary fall simultaneously to both centres. This condition defines the location of the cell boundary for us; it is effectively defined to be where the two cells’ gravitational influences are equal. On one side of the boundary, the gravitational influence of one mass dominates, and we approximate the space-time by a Schwarzschild space-time centred on that mass. On the other side, the other mass dominates, and we approximate the space-time with a Schwarzschild space-time centred on that mass. Once again, we can understand this simultaneous free-fall of the boundary towards the two centres as actually the motion of the two masses towards each other under their mutual attraction. This mutual attraction gives rise to the expansion and contraction of the lattice itself, which manifests as the expansion and contraction of the cell boundary. Given this definition and our two boundary conditions, we can now derive a set of constraints that \( E_b \) and the cell radius \( r_b \) must satisfy.

Local spatial distances along the boundary are given by

\[
r_b \ d\Omega. \tag{D1}
\]

Since by assumption, particles co-moving with the boundary follow radial geodesics, then this distance evolves as

\[
dr_b \ d\Omega. \tag{D2}
\]

By condition (2) we require that

\[
r_{b1} \ d\Omega_1 = r_{b2} \ d\Omega_2, \tag{D3}
\]

where the numerical subscripts refer to cells 1 and 2; combined with condition (1) we additionally require that

\[
\frac{dr_{b1}}{d\tau} \ d\Omega_1 = \frac{dr_{b2}}{d\tau} \ d\Omega_2, \tag{D4}
\]

since \( dr_1 = dr_2 = d\tau \). Combining these two conditions, we obtain

\[
1 \frac{dr_{b1}}{r_{b1}} \ d\tau = \frac{1}{r_{b2}} \frac{dr_{b2}}{d\tau}, \tag{D5}
\]

and using (10), we can express this equivalently as

\[
1 \frac{1}{r_{b1}} \sqrt{E_{b1} - 1 + \frac{2m_1}{r_{b1}}} = \frac{1}{r_{b2}} \sqrt{E_{b2} - 1 + \frac{2m_2}{r_{b2}}}. \tag{D6}
\]

By condition (1) we require that \( \Delta r_1 = \Delta r_2 \). Recall that the proper time of a freely falling particle is given by equations (20) to (23). However, from the form of equations (21) to (22), it is clear that we cannot satisfy this condition unless both cells were of the same type, that is, both cells were open, flat, or closed. This constrains the cases we need to consider to just three, and we shall consider each in turn.
If $E_b = 1$ for both cells, then (21) and (23) imply that

$$\Delta \tau_1 = \Delta \tau_2,$$

$$\frac{2}{3} \sqrt{2m_1} r_{b_1}^{3/2} - \tau_0 = \frac{2}{3} \sqrt{2m_2} r_{b_2}^{3/2} - \tau_0,$$

where $\tau_0$ is a constant of integration. From this, we obtain the relation

$$r_{b_2} = \left( \frac{m_2}{m_1} \right)^{1/3} r_{b_1}, \quad (D7)$$

It can be checked that this relation also satisfies (D6) and hence condition (2) as well.

If $E_b < 1$ for both cells, then (20) and (23) imply that

$$\frac{2m_1}{(1 - E_{b_1})^{3/2}} = \frac{2m_2}{(1 - E_{b_2})^{3/2}},$$

and

$$\frac{2m_1}{(1 - E_{b_1})^{3/2}} \left[ \sqrt{\frac{1 - E_{b_2}}{2m_1}} r_{b_1}^{3/2} - \sinh^{-1} \sqrt{\frac{E_{b_1} - 1}{2m_1} r_{b_1}} \right] - \tau_0 = \frac{2m_2}{(1 - E_{b_2})^{3/2}} \left[ \sqrt{\frac{1 - E_{b_2}}{2m_2}} r_{b_2}^{3/2} - \sinh^{-1} \sqrt{\frac{E_{b_2} - 1}{2m_2} r_{b_2}} \right] - \tau_0.$$

As with the $E_b < 1$ case, the two sides of this equation can be made equal if we simultaneously equated

$$\frac{E_{b_1} - 1}{2m_1} r_{b_1} = \frac{E_{b_2} - 1}{2m_2} r_{b_2},$$

and

$$\frac{2m_1}{(E_{b_1} - 1)^{3/2}} = \frac{2m_2}{(E_{b_2} - 1)^{3/2}}.$$

This clearly leads to the same constraints as the $E_b < 1$ case.

We therefore find that for all cases, two neighboring cells must satisfy constraints (D7) and (D8) at the boundary. And when $m_1 = m_2$, we recover $E_{b_1} = E_{b_2}$ and $r_{b_1} = r_{b_2}$.
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