

EQUIVARIANT K- THEORY OF CENTRAL EXTENSIONS AND TWISTED EQUIVARIANT K- THEORY: $SL_3 \mathbb{Z}$ AND $St_3 \mathbb{Z}$.

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Abstract. In this work, we compare twisted Equivariant K-theory of $SL_3 \mathbb{Z}$ with untwisted equivariant K-theory of a central extension $St_3 \mathbb{Z}$. We compute all twisted equivariant K-theory groups of $SL_3 \mathbb{Z}$, and compare with previous work on the equivariant K-theory of $BSt_3 \mathbb{Z}$ by Tezuka and Yagita.

Using a universal coefficient theorem by the authors, the computations explained here give the domain of Baum-Connes assembly maps landing on the topological K-theory of twisted group C*-algebras related to $SL_3 \mathbb{Z}$, for which a version of KK-Theoretic Duality studied by Echterhoff, Emerson and Kim is verified.

1. Introduction

In this note, we compare versions of twisted equivariant K-theory with respect to a discrete group $G$, and untwisted equivariant K-theory of a universal central extension of $G$.

Given a discrete group $G$, a proper $G$-CW complex $X$ and a cohomology class $\alpha$ in the third Borel cohomology group $H^3(X \times_G EG, \mathbb{Z})$, twisted equivariant K-theory, denoted by $^{\alpha}K^*_G(X)$ was defined in [BEJU14].

Specializing to the classifying space $EG$ of proper actions of $G$ and performing the Borel construction $EG \times_G EG$ gives a model for $BG$ and thus all twistings agree with elements in the cohomology groups $H^3(BG, \mathbb{Z})$.

In the case of a discrete group $G$ (compare [Moo64], [Moo68]), a class $\alpha \in H^3(BG, \mathbb{Z}) = H^2(BG, S^1)$ determines a central extension

$$1 \to S^1 \to \tilde{G}_\alpha \to G \to 1.$$  

The space $EG$ with the $\tilde{G}_\alpha$-action given by precomposition with $p_\alpha$ is a model for the classifying space of proper actions of $\tilde{G}_\alpha$, denoted by $E\tilde{G}_\alpha$. We compare the abelian groups $K^{*}_{\tilde{G}_\alpha}(EG_\alpha)$ and $^{\alpha}K^{*}_{G}(EG)$.

We pay specific attention to the groups $SL_3 \mathbb{Z}$ and $St_3 \mathbb{Z}$, related by a central extension of the form

$$1 \to \mathbb{Z}/2 \to St_3 \mathbb{Z} \to SL_3 \mathbb{Z} \to 1.$$  

The integral cohomology of both groups $St_3 \mathbb{Z}$ and $SL_3 \mathbb{Z}$ has been extensively studied in [Sou78], where also a model for the classifying space for proper actions $ESL_3 \mathbb{Z}$ was constructed. In third degree, the cohomology groups are finitely generated, 2-torsion, and generated by classes $u_1$, $u_2$ in the case of $SL_3 \mathbb{Z}$ and a single class $w_1$ in the case of $St_3 \mathbb{Z}$.

We describe the restriction of the classes $u_1$ and $u_2$ to the cohomology of finite subgroups of $SL_3 \mathbb{Z}$ in Section 5 where also the relation to the generating class $w_1$.

Date: August 19, 2014.

Key words and phrases. Twisted Equivariant K-Theory, Bredon Cohomology, Baum-Connes Conjecture with coefficients, Twisted Group C*-algebras, KK-Theoretic duality. 2010 Math Subject classification: Primary: 19L64, Secondary: 19L50, 19K33, 19L47.
is stated. We follow these classes to their restrictions on finite subgroups of \( St_3 \mathbb{Z} \),
which are covers 2 to 1 of finite subgroups of \( SL_3 \mathbb{Z} \).

It turns out that the torsion class \( u_1 + u_2 \) represents the central extension
\[
1 \to \mathbb{Z}/2\mathbb{Z} \to St_3 \mathbb{Z} \overset{\rho}{\to} SL_3 \mathbb{Z} \to 1,
\]
and its restriction to finite groups \( H \leq G \) gives a model for Schur covering groups
of \( H \):
\[
1 \to \mathbb{Z}/2\mathbb{Z} \to p^{-1}(H) \to H,
\]
(However, more finite subgroups appear in \( St_3 \mathbb{Z} \) that are not a Schur covering group
for any finite group of \( SL_3 \mathbb{Z} \)).

Thus, a cocycle representing \( u_1 + u_2 \) and the central extension satisfy the hypotheses
of the following Theorem (4.4)

**Theorem.** Let \( G \) be a discrete group and let \( \alpha \in \mathbb{Z}^2(G; S^1) \) be a cocycle taking
values in \( \mathbb{Z}/n\mathbb{Z} \). Consider the extension associated to \( \alpha \)
\[
1 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\rho} G \xrightarrow{\alpha} 1.
\]

Denote by \( \tilde{E}G \) a model for the classifying space of proper actions and notice that
the action of \( G_\alpha \) via \( \rho \) on \( \tilde{E}G \) exhibits the later space as a model for \( \tilde{E}G_\alpha \).

Then, the map \( \rho \) gives an isomorphism of abelian groups between the Bredon cohomology
groups of \( \tilde{E}G \) with coefficients in the \( \alpha \)-twisted representation ring and
the Bredon cohomology groups of \( \tilde{E}G_\alpha \) with coefficients in the 1-central group
representation Bredon module \( \alpha \) (defined in 4.2). In symbols,
\[
H^*(\tilde{E}G; R_\alpha^G) \overset{\rho^*}{\to} H^*(\tilde{E}G_\alpha; R_\alpha^G).
\]

We use the (Bredon) cohomological description to feed a spectral sequence constructed
to compute twisted equivariant \( K \)-theory which was constructed in [BV14].
The input of the Spectral sequence are Bredon Cohomology groups with coefficients
in twisted representations, as briefly introduced in Section 2. The spectral sequence is
taken to collapse at the \( E_2 \)-term and the twisted equivariant \( K \)-theory groups are
determined.

- (Theorem 6.1) The twisted equivariant \( K \)-theory groups with respect to \( u_1 \) are as follows:
  \[
u_1 K^0_{SL_3}(\tilde{E}SL_3) \cong \mathbb{Z}^{\oplus 13}, \quad u_1 K^1_{SL_3}(\tilde{E}SL_3) = 0.\]

- (Theorem 6.3) The twisted equivariant \( K \)-theory groups with respect to \( u_2 \) are as follows:
  \[
u_2 K^0_{SL_3}(\tilde{E}SL_3) \cong \mathbb{Z}^{\oplus 7}, \quad u_2 K^1_{SL_3}(\tilde{E}SL_3) = 0.\]

- (Theorem 6.9) The twisted equivariant \( K \)-theory groups with respect to \( u_1 + u_2 \) are as follows:
  \[
u_1 + u_2 K^0_{SL_3}(\tilde{E}SL_3) \cong \mathbb{Z}^{\oplus 5}, \quad u_1 + u_2 K^1_{SL_3}(\tilde{E}SL_3) \cong \mathbb{Z}/2\mathbb{Z}.\]

Using the Universal Coefficient Theorem for Bredon cohomology with coefficients
in twisted representations, Theorem 1.13 in [BV14], the previous groups are verified
to be isomorphic to some equivariant \( K \)-Homology groups with coefficients defined
in terms of Kasparov \( KK \)-Theory groups in Section 4. This extends and generalizes
work by Sánchez-García in [SG08] in the untwisted setting.

A version of the Baum-Connes Conjecture with coefficients, [CE01] relates these
groups to the topological \( K \)-theory of twisted group \( C^* \)-algebras. We see that the
input of the Baum Connes map with coefficients given by the twistings \( u_1, u_2 \) and
\( u_1 + u_2 \) satisfy a version of KK-theoretic Duality studied in [EEK08], and verified
in [BV14] for the twist \( u_1 \).
The result is interpreted in terms of twisted equivariant \( K \)-theory of the classifying space \( BSL_3 \mathbb{Z} \) using results by Tezuka and Yagita [TY92], the Atiyah-Segal Completion Theorem 4.4 on page 611 of [LO01]. This work is organized as follows: In Section 2, we introduce Bredon (Co)-homology, focusing on coefficients in twisted representations. In Section 3, we review spectral sequences relating Bredon cohomology groups to versions of twisted equivariant \( K \)-theory. Section 4 deals with the proof of Theorem 4.4 relating twisted equivariant \( K \)-theory and untwisted \( K \)-theory which is equivariant with respect to a central extension coding the twist. Section 5 describes cohomological information determining the twists, as well as some misunderstandings in the literature concerning the universal central extension of \( SL_3 \mathbb{Z} \) and \( St_3 \mathbb{Z} \), see 5.3. Section 6 deals with the computations in Bredon cohomology. Finally, Section 7 gives interpretations of the results as computations of twisted equivariant \( K \)-homology related to versions with coefficients of the Baum-Connes Conjecture, as well as computations of the complex \( K \) theory of the classifying space \( BSt_3 \mathbb{Z} \) by Tezuka and Yagita.

Acknowledgements. The first author thanks the support of a CONACYT Postdoctoral fellowship. The second author thanks the support of a UNAM Postdoctoral Fellowship.

The first author thanks Prof. Pierre de la Harpe for enlightening correspondence related to the difference between \( St_3 \mathbb{Z} \) and the universal central extension of \( SL_3 \mathbb{Z} \).

Both authors thank an anonymous referee for making crucial suggestions about both the presentation and the mathematical content of this note, particularly the suggestion of the material in section 4, which helped the authors to identify a mistake in a previous version of this work.

2. Bredon (co)-homology

We recall briefly some definitions relevant to Bredon homology and cohomology, see [MV03] for more details. Let \( G \) be a discrete group. A \( G \)-CW-complex is a CW-complex with a \( G \)-action permuting the cells and such that if a cell is sent to itself, this is done by the identity map. We call the \( G \)-action proper if all cell stabilizers are finite subgroups of \( G \).

**Definition 2.1.** A model for \( E \mathcal{G} \) is a proper \( G \)-CW-complex \( X \) such that for any proper \( G \)-CW-complex \( Y \) there is a unique \( G \)-map \( Y \to X \), up to \( G \)-homotopy equivalence.

One can prove that a proper \( G \)-CW-complex \( X \) is a model of \( E \mathcal{G} \) if and only if the subcomplex of fixed points \( X^H \) is contractible for each finite subgroup \( H \subseteq G \). It can be shown that classifying spaces for proper actions always exist.

Let \( \text{Or}_{\mathcal{F}_G}(G) \) be the orbit category of finite subgroups of \( G \); a category with one object \( G/H \) for each finite subgroup \( H \subseteq G \) and where morphisms are given by \( G \)-equivariant maps. There exists a morphism \( \phi : G/H \to G/K \) if and only if \( H \) is conjugate in \( G \) to a subgroup of \( K \).

**Definition 2.2 (Bredon chain complex).** Let \( X \) be a proper \( G \)-CW-complex. The contravariant functor \( \underline{C}_\ast(X) : \text{Or}_{\mathcal{F}_G}(G) \to \mathbb{Z} - \text{CHCOM} \) assigns to every object \( G/H \) the cellular \( \mathbb{Z} \)-chain complex of the \( H \)-fixed point complex \( \underline{C}_\ast(X^H) \cong C_\ast(\text{Map}_G(G/H, X)) \) with respect to the cellular boundary maps \( \partial_\ast \).

We will use homological algebra to define Bredon cohomology. A contravariant coefficient system is a contravariant functor \( M : \text{Or}_{\mathcal{F}_G}(G) \to \mathbb{Z} - \text{MODULES} \). Given a contravariant coefficient system \( M \), the Bredon cochain
module $C^n_G(X; M)$ is defined as the abelian group of natural transformations of functors defined on the orbit category $\mathcal{C}_G(X) \to M$. In symbols,

$$C^n_G(X; M) = \text{Mor}_{\text{funct}(\mathcal{C}_G(X), \mathbb{Z} - \text{MODULES})}(\mathcal{C}_G(X), M)$$

Given a set $\{e_\lambda\}$ of orbit representatives of the n-cells of the $G$-CW complex $X$, and isotropy groups $H_\lambda$ in $G$ of the cells $e_\lambda$, the abelian groups $C^n_G(X, M)$ satisfy:

$$C^n_G(X, M) = \bigoplus_\lambda \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/H_\lambda))$$

with one summand for each orbit representative $e_\lambda$. They afford a differential $\delta^n : C^n_G(X, M) \to C^{n+1}_G(X, M)$ determined by $\delta_\alpha$, and pullback maps $M(\phi) : M(G/H_\mu) \to M(G/H_\nu)$ for morphisms $\phi : G/H_\lambda \to G/H_\mu$.

**Definition 2.3** (Bredon cohomology). Let $M$ be a contravariant coefficient system. The Bredon cohomology groups with coefficients in $M$, denoted by $H^n_G(X, M)$ are the abelian groups of the cochain complex $(C^n_G(X, M), \delta^n)$.

A covariant coefficient system is a covariant functor $N : \text{Or}_{\mathcal{F}N}(G) \to \mathbb{Z} - \text{MODULES}$. Let $N$ be a covariant coefficient system and $X$ be a proper $G$-CW-complex. Dually to the cohomological situation, one can define the Bredon homology groups with coefficients in $N$. We denote these by $H^n_G(X, N)$. Details can be found in pages 14-15 of [MV03].

**Bredon (co)-homology with coefficients in twisted representations.**

**Definition 2.4.** Let $K$ be a finite subgroup in the discrete group $G$. Let $V$ be a complex vector space and $S^1$ be the unit circle in the complex numbers. Given a cocycle $\alpha : K \times K \to S^1$ representing a class in $H^2(BK, S^1) \cong H^3(BK, \mathbb{Z})$, an $\alpha$-twisted representation is a function to the general linear group of $V$, $P : K \to \text{Gl}(V)$ satisfying:

$$P(e) = 1$$

$$P(x)P(y) = \alpha(x, y)P(xy).$$

The Grothendieck group of isomorphism classes of $\alpha$-twisted representations is called the $\alpha$-twisted representation group and it is denoted by $R_{\alpha}(K)$

Two $\alpha, \alpha'$-twisted representations are isomorphic if the cocycles $\alpha, \alpha'$ are cohomologous in $H^2(BK, S^1)$.

**Definition 2.5.** Let $H$ be a finite group and let $\alpha \in Z^2(H, S^1)$ be a cocycle. Recall that the $\alpha$-twisted Complex group algebra $\mathbb{C}^\alpha H$ is generated as a complex vector space by the elements $\{h \mid h \in H\}$. The multiplication is given by the following formula on representatives:

$$h_1h_2 = \alpha(h_1, h_2)h_1h_2,$$

and extended $\mathbb{C}$-linearly to define a complex algebra structure on $\mathbb{C}^\alpha H$.

It is a consequence of Theorem 3.2 in page 112, Volume 2, part 1 of [Kar94], that the $K_0$ group of the $\alpha$-twisted complex group algebra $\mathbb{C}^\alpha H$ agrees with the $\alpha$-twisted representation group $R_{\alpha}(H)$.

We define a contravariant and a covariant coefficient system for the family $\mathcal{F}_G = \mathcal{F}N$ of finite subgroups agreeing on objects by using the $K_0$-group of the twisted group algebra, using restriction to define the contravariant functoriality, and using induction to define the covariant functoriality.
Definition 2.6. Let $G$ be a discrete group and let $\alpha \in Z^2(G, S^1)$ be a cocycle. Let $i : H \to G$ be an inclusion of a finite subgroup $H$.

Define $\mathcal{R}_\alpha$ on objects $G/H$ by

$$\mathcal{R}_\alpha(G/H) := K_0(C^\ast_\alpha(H)) \cong R_{\tau(\alpha)}(H).$$

Let $\phi : G/H \to G/K$ be a $G$-equivariant map, we denote by $\mathcal{R}_\alpha(\phi) : R_{\alpha}(H) \to R_{\alpha}(K)$ the induction of $\alpha$-twisted, representations for the covariant functor. For the contravariant functor, we denote by $\mathcal{R}_\alpha(\phi) : R_{\alpha}(K) \to R_{\alpha}(H)$ the restriction of $\alpha$-twisted representations.

Definition 2.7. Let $G$ be a discrete group, let $X$ be a proper $G$-CW complex, and let $\alpha \in Z^2(G, S^1)$ be a cocycle. The $\alpha$-twisted Bredon (co)-homology groups of $X$ are the Bredon (co)-homology groups with respect to the functors described in Definition 2.6.

Remark 2.8. Notice the role of the family of finite groups in definition 2.7. More generally, one can define Bredon (co)-homology groups for a family $\mathcal{F}$ of subgroups which contains the isotropy groups of a $G$-CW complex $X$, and a functor $\mathcal{F} \to \mathcal{Z} \text{- MODULES}$. Since we are dealing with proper actions on $G$-CW complexes, we can concentrate on Bredon cohomology for the family of finite subgroups.

3. Spectral sequences for Twisted Equivariant $K$-Theory.

Twisted equivariant $K$-theory for proper and discrete actions has been defined in a variety of ways. For a torsion cocycle $\alpha \in Z^2(G, S^1)$, it is possible to define it in terms of finite dimensional, so called $\alpha$-twisted vector bundles, as for example in [Dwy08]. This is not possible for twistings of infinite order, and the general approach of [BEJU14] or $C^\ast$-algebraic methods are needed.

Definition 3.1. Let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle of order $n$ for the discrete group $G$, with associated central extension

$$0 \to \mathbb{Z}/n \to G_\alpha \to G.$$

An $\alpha$-twisted vector bundle is a finite dimensional $G_\alpha$-equivariant complex vector bundle such that $\mathbb{Z}/n$ acts by multiplication with a primitive $n$-th root of unity. The $\alpha$-twisted, $G$-equivariant $K$-theory groups $\alpha K^n_0(X)$ are defined as the Grothendieck groups of the isomorphism classes of $\alpha$-twisted vector bundles over $X$.

Given a proper $G$-CW complex $X$, define $\alpha K^n_0(X)$ as the kernel of the induced map

$$\alpha K^n_0(X \times S^n) \xrightarrow{incl^\ast} \alpha K^n_0(X).$$

The $\alpha$-twisted equivariant $K$-theory catches information relevant to the class of twistings coming from the torsion part of the group cohomology of the group, in the sense that the $K$-groups are zero for cocycles representing non-torsion classes. In contrast, the approach discussed in [BEJU14] overcomes this difficulty.

As noted in [BEUV13], there is a spectral sequence connecting the $\alpha$-twisted Bredon cohomology and the $\alpha$-twisted equivariant $K$-theory of finite proper $G$-CW complexes. When the twisting is given by a torsion element of $H^3(BG, \mathbb{Z})$, this spectral sequence is a special case of the Atiyah-Hirzebruch spectral sequence for untwisted $G$-cohomology theories constructed by Davis and Lück [DL98]. In particular, it collapses rationally.

Theorem 3.2 ([BEUV13]). Let $X$ be a finite proper $G$-CW complex for a discrete group $G$, and let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle. Then there is a spectral sequence with...
\[ E_{p,q}^* = \begin{cases} H^p_G(X, \mathcal{R}_\alpha^q) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases} \]

so that \( E_{p,q}^* \Rightarrow \sigma K_G^{p+q}(X) \).

4. \( S^1 \)-central extensions and torsion cocycles.

**Definition 4.1.** Let \( 1 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \tilde{H} \rightarrow H \) be a central extension. Let \( k \) be a natural number with \( 0 \leq k \leq n \). Let \( V \) be a complex vector space. A \( k \)-central \( \tilde{H} \) is a map \( \tilde{H} \rightarrow GL(V) \), where the generator \( t \in \mathbb{Z}/n\mathbb{Z} \) acts by multiplication by \( e^{2\pi ik/n} \).

**Definition 4.2.** The \( k \)-central representation group of \( \tilde{H} \), denoted by \( R_k(\tilde{H}) \), is the Grothendieck group of isomorphism classes of \( k \)-central representations of \( \tilde{H} \).

The \( k \)-central representation group is a contravariant coefficient system. Given a discrete group \( \tilde{G} \), we denote by \( R^?_k(\tilde{G}) \) the functor
\[
\tilde{G}/\tilde{H} \mapsto R^?_k(\tilde{H}).
\]

**Lemma 4.3.** Let \( G \) be a discrete group and let \( \alpha \in Z^2(G; S^1) \) be a torsion cocycle of order \( n \). Then,
- There exists a cocycle \( \gamma \) with values on \( \mathbb{Z}/n \subset S^1 \), which is cohomologous to \( \alpha \).
- There exists a central extension of the form

\[
1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.
\]

With the property that for each finite group \( H \leq G \), the \( 1 \)-central representation group \( R_1(\rho^{-1}(H)) \) is isomorphic to the \( \alpha \mid_H \)-twisted representation ring \( \alpha R(H) \) as an abelian group.
- Moreover, this extends to a natural transformation of contravariant functors defined over the orbit category of \( G \),

\[
T: \alpha R_1^? \cong R_1^? \circ \rho
\]

which consists of group isomorphisms on each orbit.

**Proof.**
- Let \( \alpha \in Z^2(G; S^1) \) be a torsion cocycle of order \( n \). Then \( \alpha^n \) is cohomologous to the trivial cocycle, i.e there is a cochain \( t \in C^1(G, S^1) \) with \( \alpha^n = \delta t \).
  Define a cochain \( u \in C^1(G, S^1) \) by \( u(g) = (t(g))^{-\frac{1}{n}} \). The cocycle \( \gamma = \alpha \cdot \delta u \) is again torsion of order \( n \). The cocycle \( \gamma \) takes values in \( \mathbb{Z}/n\mathbb{Z} \) and it is cohomologous to \( \alpha \).
- We use the \( \mathbb{Z}/n\mathbb{Z} \)-valued cocycle \( \gamma \) to define a group structure on the set \( G \times \mathbb{Z}/n\mathbb{Z} \), and obtain a central extension of \( G \) by \( \mathbb{Z}/n\mathbb{Z} \), denoted by \( G_\alpha \) (the notation being justified by the fact that \( \alpha \) is cohomologous to \( \gamma \)).
  Let \( \sigma \) be the generator of \( \mathbb{Z}/n\mathbb{Z} \) and \( 0 \leq i \leq n - 1 \). The multiplication on the group \( G_\alpha \) is given on elements \((g, \sigma^i)\) by

\[
(g, \sigma^i) \cdot (h, \sigma^j) = (gh, \alpha(g, h)\sigma^{j+i}),
\]

thus defining a central extension
\[
1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.
\]
• Let $H$ be a finite subgroup of $G$. Given a torsion cocycle $\alpha$, consider the central extension

$$1 \to \mathbb{Z}/n\mathbb{Z} \to G_\alpha \xrightarrow{\rho} G \to 1,$$

which was constructed in the previous part.

Let $\beta : H \to GL(V)$ be an $\alpha$-representation. We define the 1-central representation $T(H)$ as the isomorphism class determined by the map $\rho^{-1}(H) \to GL(V)$ defined on elements $(h, t) \in H \times \mathbb{Z}/n\mathbb{Z}$ by $\cdot t(h, t))$, where we consider $t \in \mathbb{Z}/n\mathbb{Z} \subset S^1$ and $\cdot$ denotes complex multiplication.

This defines a group homomorphism

$$T : \alpha \mathcal{R}(G/H) \to \mathcal{R}_1(G_\alpha/\rho^{-1}(H)).$$

An inverse to the homomorphism $T$ is given by assigning to the 1-central representation $\epsilon : \tilde{H} \to GL(V)$ the projective representation $\kappa : H \to GL(V)$ given by $\kappa(h) = \epsilon(h, 1)$. One checks that this is a $\gamma$-representation, where $\gamma$ is the cocycle with values on $\mathbb{Z}/n\mathbb{Z}$ constructed in the first part.

• Let $H$ and $K$ be finite subgroups of $G$. The map $\rho : G_\alpha \to G$ defines a functor

$$\text{Or}_{FIN}(G) \xrightarrow{\rho^*} \text{Or}_{FIN}(G_\alpha)$$

between the orbit categories with respect to the family of finite subgroups.

We will analyze the behaviour of the functor $T$ with respect to restrictions.

Let $\phi : G/H \to G/K$ be a $G$-equivariant map. Recall that such a map is determined up to $G$-conjugacy by an inclusion $H \to K$ of finite subgroups of $G$.

Given an $\alpha$-projective representation $\beta : H \to GL(V)$, the following diagram is commutative

$$\begin{array}{ccc}
K & \xrightarrow{\rho} & H \\
\downarrow & & \downarrow \beta \\
\rho^{-1}(K) & \xrightarrow{\rho^{-1}(H)} & \rho^{-1}(H)
\end{array}$$

where the unlabelled arrow denote inclusions.

Hence, the functor $T$ is compatible with restrictions and thus defines a natural transformation of contravariant functors over the orbit category. $\square$

**Theorem 4.4.** Let $G$ be a discrete group and let $\alpha \in Z^2(G; S^1)$ be a cocycle taking values in $\mathbb{Z}/n\mathbb{Z}$. Consider the extension associated to $\alpha$

$$1 \to \mathbb{Z}/n\mathbb{Z} \to G_\alpha \xrightarrow{\rho} G \to 1.$$

Denote by $EG$ a model for the classifying space of proper actions and notice that the action of $G_\alpha$ via $\rho$ on $EG$ exhibits the latter space as a model for $EG_\alpha$.

Then, the map $\rho$ gives an isomorphism of abelian groups between the Bredon cohomology groups of $EG$ with coefficients in the $\alpha$-twisted representation ring and the Bredon cohomology groups of $EG_\alpha$ with coefficients in the 1-central group representation group. In symbols,

$$H^*(EG; R^G_\alpha) \xrightarrow{\rho^*} H^*(EG_\alpha; R^G_1).$$
Proof. Fix a \( G \)-cellular structure of \( EG \). Associate to each equivariant cell in \( EG \) of the form \( G/H \times D^n \) a cell in \( EG_\alpha \) of the form \( G_\alpha/\rho^{-1}(H) \times D^n \).

Consider the cellular cochain complex of \( EG \). In degree \( n \), it has the form

\[
C^n_G(EG, M) = \bigoplus \Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], \alpha^* R(G/H_\lambda)).
\]

From [3] this term is isomorphic (via \( \rho^* \)) to

\[
C^n_G(EG, M) = \bigoplus \Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], R_1(G_\alpha/\rho^{-1}(H_\lambda)))
\]

and the isomorphism commutes with the cellular boundary, thus determining a chain isomorphism

\[
C^*(EG; \alpha^* R) \cong C^*(EG_\alpha; R_1^\alpha),
\]

which induces an isomorphism in Bredon cohomology. \( \square \)

Corollary 4.5. Let \( G \) be a discrete group and let \( \alpha \in \mathbb{Z}^2(G; S^1) \) be a cocycle taking values in \( \mathbb{Z}/n\mathbb{Z} \). Consider the extension associated to \( \alpha \)

\[
1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.
\]

Then, there exists an isomorphism of abelian groups

\[
\rho K^*_G(EG) \cong K^*_G(EG_\alpha),
\]

between the \( \alpha \)-twisted, \( G \)-equivariant \( K \)-Theory and the untwisted \( G_\alpha \)-equivariant \( K \)-theory of the classifying spaces for proper actions \( EG = EG_\alpha \).

Proof. From Theorem 4.4, the Bredon cohomology groups are all isomorphic. The spectral sequence [3.2] lets us conclude the desired isomorphism. \( \square \)

5. Twisting in \( SL_3 \mathbb{Z} \) and \( St_3 \mathbb{Z} \).

The cohomology of \( SL_3 \mathbb{Z} \). We recall the analysis of the cohomology of \( SL_3 \mathbb{Z} \) in [BV14]. Soulé proved in [Sou78] that the integral cohomology of \( SL_3 \mathbb{Z} \) only consists of 2 and 3-torsion. The 3-primary part is isomorphic to the graded algebra

\[
\mathbb{Z}[x_1, x_2]/(3x_1, 3x_2)
\]

with both generators in degree 4.

The two-primary component is isomorphic to the graded algebra

\[
\mathbb{Z}[u_1, \ldots, u_7]
\]

with respective degrees 3, 3, 4, 4, 5, 6, 6, subject to the relations

\[
2u_1 = 2u_3 = 4u_3 = 4u_4 = 2u_5 = 2u_6 = 2u_7 = 0,
\]

\[
u_2u_1 = u_2u_4 = u_2u_5 = u_2u_6 = u_2u_7 = 0,
\]

\[
u_2^2 + u_2u_2^2 = u_3u_4 + u_1u_5 = u_3u_6 + u_3u_7 = u_3u_8 + u_3^2 = 0,
\]

\[
u_1u_0 + u_4u_5 = u_0u_1 + u_0u_2 + u_0u_3 = u_5u_6 + u_5u_7 = 0.
\]

The twisting in equivariant \( K \)-theory are given by classes in \( H^2(SL_3 \mathbb{Z}, \mathbb{Z}) \) all of which are 2-torsion. For this reason, we shall restrict to the two-primary component (we indicate this with the subscript (2)) in integral cohomology. In order to have a local description of these classes, we describe the cohomology of some finite subgroups inside \( SL_3 \mathbb{Z} \).

The finite groups of \( SL_3 \mathbb{Z} \) include \( S_4 \), the symmetric group in four letters, \( D_4 \), the dihedral group of order 8, the dihedral group of order 12, \( D_6 \), as well as the group of order two denoted by \( C_2 \).
Theorem 4 in page 14 of [Sou78], gives the following result: For all \(n \in \mathbb{N}\) there exists an exact sequence of abelian groups

\[ 0 \to H^n(SL_3\mathbb{Z})_{(2)} \xrightarrow{\phi} H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \xrightarrow{\delta} H^n(D_4) \oplus H^n(C_2) \to 0 \]

where \(\phi\) is given by restrictions (see [Sou78][2.1(b), Cor.]) and \(\delta\) by the system of embeddings

\[
\begin{array}{c}
\text{SL}_3\mathbb{Z} \\
\downarrow i_2 \\
D_4 \\
\downarrow i_1 \\
S_4 \\
\downarrow j_1 \\
C_2 \\
\downarrow j_2 \\
S_4
\end{array}
\]

If \(R\) is as in Proposition 4 in [Sou78], the image of the morphism \(\phi : H^*(\text{SL}_3\mathbb{Z})_{(2)} \to H^*(S_4)_{(2)} \oplus (i_1)^{-1}(R)\), is the set of elements \((y, z)\) such that \(j_2^*(y) = j_1^*(z)\). From the paper of Soulé, we know that \(H^*(S_4)_{(2)} = \mathbb{Z}[y_1, y_2, y_3]\), with \(2y_1 = 2y_2 = 4y_3 = y_1^3 + y_2^3 + y_3^3 = 0\), and, \((i_1)^{-1}(R) = \mathbb{Z}[z_1, z_2, z_3]\), with \(2z_1 = 4z_2 = 2z_3 = z_1^3 + z_2z_3 = 0\). Furthermore \(j_2^*(y_1) = t, j_2^*(y_2) = t^2, j_1^*(z_1) = 0, j_1^*(z_2) = t^2\), and \(j_1^*(z_3) = 0\). Then the elements \(u_1 = y_2, u_2 = z_1, u_3 = y_1^2 + z_2, u_4 = y_1^3 + y_2, u_5 = y_1y_2, u_6 = y_1y_3 + y_3^2\) and \(u_7 = z_3\) generate \(\phi(H^*(\text{SL}_3\mathbb{Z})_{(2)})\).

In \(H^3\) the above discussion can be summarized in the following diagram

\[ \langle u_1, u_2 \rangle = H^3(\text{SL}_3\mathbb{Z}) \]

\[ \langle z_1 \rangle \subseteq H^3(S_4) \]

\[ \langle x_3 \rangle \subseteq H^3(D_4) \]

\[ \langle y_2 \rangle \subseteq H^3(S_4) \]

\[ \langle y_2 \rangle \subseteq H^3(D_4) \]

\[ \langle y_2 \rangle \subseteq H^3(\mathbb{Z}) \]

In the Following section we will give explicit generators and analyze the depicted embeddings in \(\text{SL}_3\mathbb{Z}\).

**The cohomology of \(St_3\mathbb{Z}\).** The following result was published as Theorem 8, page 17 in [Sou78], see also section 4 in [TY92], page 92 for a more precise account.

**Theorem 5.2.**

- There exists a 3 torsion cohomology class \(\xi \in H^4(St_3\mathbb{Z}, \mathbb{Z})\) such that, for any \(St_3\mathbb{Z}\)-module \(A\), the cup product by \(\xi\) induces an isomorphism

\[ \cdot \cup \xi : H^k(St_3\mathbb{Z}, A) \to H^{k+4}(St_3\mathbb{Z}, A) \]

as soon as \(k > 3\) and \(k > 0\) when \(A\) is constant.

- The ring \(H^*(St_3\mathbb{Z}, \mathbb{Z})\) is generated by elements \(w_1, w_2, w_3\) with respective degrees 3, 4, 4, submitted to the defining relations \(2w_1 = 4w_2 = 16w_3 = w_1^2 = w_1w_2 = w_2w_3 = 0\). Hence \(H^1(St_3\mathbb{Z}, \mathbb{Z}) = H^2(St_3\mathbb{Z}, \mathbb{Z}) = 0, H^3(St_3\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2, H^4(St_3\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/16 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3\).
The cohomology of $St_3\mathbb{Z}$ is seen to be completely determined by the classes $w_1, w_2, w_3$, as well as the periodicity class $\xi$. The classes $w_i$ restrict non-trivially to some specific generators of the cohomology of finite subgroups. We will analyze briefly how they relate to the generating classes $u_1, u_2$ of $H^3(SL_3\mathbb{Z}, \mathbb{Z})$. This is a summary of the discussion in Lemma 9, and the proof of Theorem 8 in [Sou78].

The group $St_3(\mathbb{R})$ acts as a central extension $1 \to \mathbb{Z}/2 \to St_3(\mathbb{R}) \to SL_3(\mathbb{R}) \to 1$, which restricts to a central extension of lattices $1 \to \mathbb{Z}/2 \to St_3\mathbb{Z} \to SL_3\mathbb{Z} \to 1$.

The maximal compact subgroups of $St_3(\mathbb{R})$, respectively $SL_3(\mathbb{R})$, are Spin$_3$, respectively SO(3). Hence, all finite subgroups of $St_3\mathbb{Z}$ are contained in Spin$_3$, which is homeomorphic to the 3-dimensional sphere, thus the cohomology of all finite subgroups in $St_3\mathbb{Z}$ is 4-periodic. This is the origin of the periodicity class $\xi$.

The class $w_1$ restricts nontrivially under a system of inclusions of finite groups

$$
\begin{array}{ccc}
S_4^* & \to & S_4^* \\
D_4 & \to & \\
\end{array}
$$

which covers the inclusions $i_1, i_2 : D_4 \to S_4$ in $SL_3\mathbb{Z}$.

Thus, $u_1$ maps to $w_1$, and $u_2$ maps to the trivial class under the map induced by the universal cover $St_3\mathbb{Z} \to SL_3\mathbb{Z}$ in cohomology.

**Remark 5.3.** [The universal central extension of $SL_n\mathbb{Z}$ and $St_n\mathbb{Z}$.]

In the early literature on the Steinberg group (particularly Steinberg's Yale notes [Ste68]), there is an unfortunate identification of $St_3\mathbb{Z}$ with the universal central extension of $SL_3\mathbb{Z}$. This mistake has been repeated in the literature [Sou78], 2.7 and [BrillH74], Example IV.

Denote by $\widehat{SL_n}(\mathbb{Z})$ the universal central extension of $SL_n(\mathbb{Z})$. It fits in an exact sequence

$$1 \to H_2(SL_n(\mathbb{Z}), \mathbb{Z}) \to \widehat{SL_n}(\mathbb{Z}) \to SL_n(\mathbb{Z}).$$

While there is an identification of $St_n(\mathbb{Z})$ with $\widehat{SL_n}(\mathbb{Z})$ for $n \geq 5$, Van der Kallen [vdK75] computes the Schur Multiplier $H_2(G, \mathbb{Z})$ for $G$ $SL_3\mathbb{Z}$ and $SL_4(\mathbb{Z})$, being in both cases isomorphic to Klein’s Four group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus, the universal central extension defining $St_3\mathbb{Z}$,

$$1 \to \mathbb{Z}/2\mathbb{Z} \to St_3\mathbb{Z} \to SL_3\mathbb{Z},$$

and the one defining $\widehat{SL_3}\mathbb{Z}$

$$1 \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \widehat{SL_3}\mathbb{Z} \to SL_3\mathbb{Z}$$

are not the same. We thank Prof. Pierre De la Harpe for pointing this fact to us on personal correspondence, leading to the correction of a mistake in a previous version of this note.

6. **Twisted K-theory of $SL_3\mathbb{Z}$**

We use the following notations: $\{1\}$ denotes the trivial group, $C_n$ the cyclic group of $n$ elements, $D_n$ the dihedral group with $2n$ elements and $S_n$ the symmetric group of permutations on $n$ objects.

There are four twistings for $SL_3\mathbb{Z}$ up to cohomology, namely $0, u_1, u_2, u_1 + u_2$, continuing the work started in [BV14], we will calculate the twisted K-theory for the twistings $u_2$ and $u_1 + u_2$.

From diagram 5.1 one can see that the class $u_2$ restricts nontrivially to two copies of $S_4$ corresponding to the stabilizer of the vertices $v_1$ and $v_2$. We recall the $SL_3\mathbb{Z}$-CW-complex structure of $\mathbb{E}SL_3\mathbb{Z}$ as is given in [Sou78]. The labels $O$, $Q$, $\cdots$
$M$, $N$, $P$ of the vertices refer to the Figure 2 of [SG08], where also Soulé’s matrices $g_1, \ldots, g_{14}$ are recalled.

| vertices | 2-cells |
|----------|---------|
| $v_1$ | $O$ $g_2$, $g_3$ $S_4$ $t_1$ $OQM$ $g_2$ $C_2$ |
| $v_2$ | $Q$ $g_4$, $g_5$ $D_6$ $t_2$ $QM'N$ $g_1$ $[1]$ |
| $v_3$ | $M$ $g_6$, $g_7$ $S_4$ $t_3$ $MN'P$ $g_{12}, g_{14}$ $C_2 \times C_2$ |
| $v_4$ | $N$ $g_6$, $g_8$ $D_4$ $t_4$ $OQN'P$ $g_5$ $C_2$ |
| $v_5$ | $P$ $g_5$, $g_9$ $S_4$ $t_5$ $OOM'P$ $g_6$ $C_2$ |

| edges | 3-cells |
|-------|---------|
| $e_1$ | $OQ$ $g_2$, $g_5$ $C_2 \times C_2$ $T_1$ $g_1$ $[1]$ |
| $e_2$ | $OM$ $g_6$, $g_{10}$ $D_3$ |
| $e_3$ | $OP$ $g_6$, $g_5$ $D_3$ |
| $e_4$ | $QM$ $g_2$ $C_2$ |
| $e_5$ | $QN'$ $g_5$ $C_2$ |
| $e_6$ | $MN$ $g_6$, $g_{11}$ $C_2 \times C_2$ |
| $e_7$ | $M'P$ $g_6$, $g_{12}$ $D_4$ |
| $e_8$ | $N'P$ $g_5$, $g_{13}$ $D_4$ |

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The generating elements referred to above are the same as in [BV14].

**The twisting $u_1$.** The following theorem was proved in [BV14]:

**Theorem 6.1.**

$$u_1 K^0_{SL_3Z}(E SL_3Z) \cong \mathbb{Z}^{\oplus 13},$$

$$u_1 K^1_{SL_3Z}(E SL_3Z) = 0.$$

**The twisting $u_2$.** In order to determine the twisted K-theory, we calculate Bredon cohomology.

**Determination of $\Phi_1$.** In order to determine the morphism $\Phi_1$, we need to recall the projective character tables of the groups where $u_2$ restricts non trivially.

Here we denote by $z$ the generator of the central copy of $\mathbb{Z}_2$. The linear character table of a Schur covering group $S^*_4$ is obtained on page 254, volume 3 of [Kar94] by considering the group with presentation

$$S^*_4 = \langle h_1, h_2, h_3, z \mid h_i^2 = (h_j h_{j+i})^3 = (h_k h_l)^2 = z, z^2 = [z, h_i] = 1 \rangle$$

$$1 \leq i \leq 3, j = 1, k \leq l - 2$$

and the central extension

$$1 \to \langle z \rangle \to S^*_4 \xrightarrow{f} S_4 \to 1$$
given by \( f(h_i) = g_i \), as well as the choice of representatives of regular conjugacy classes as below.

| \( S^*_4 \) | \( e \) | \( z \) | \( h_1 \) | \( h_1h_3 \) | \( h_1h_2 \) | \( h_1h_2h_3 \) | \( h_1h_2h_3z \) |
|------------|----|----|-----|------|------|-------|------|
| \( \epsilon_1 \) | 1  | 1  | 1   | 1    | 1    | 1     | 1    |
| \( \epsilon_2 \) | 1  | 1  | -1  | 1    | 1    | -1    | -1   |
| \( \epsilon_3 \) | 2  | 2  | 0   | 2    | -1   | -1    | 0    |
| \( \epsilon_4 \) | 3  | 3  | 1   | -1   | 0    | 0     | -1   |
| \( \epsilon_5 \) | 3  | 3  | 1   | -1   | 0    | 0     | 1    |
| \( \epsilon_6 \) | 2  | -2 | 0   | 0    | 1    | -1    | \( \sqrt{2} \) |
| \( \epsilon_7 \) | 2  | -2 | 0   | 0    | 1    | -1    | -\( \sqrt{2} \) |
| \( \epsilon_8 \) | 4  | -4 | 0   | 0    | -1   | 1     | 0    |

where the first five lines are characters associated to \( S_4 \), and \( \epsilon_6 \) is the Spin representation.

We take the following presentation of Dihedral groups, \( D_n = \langle g_i, g_j \rangle = \langle g_i, g_j | g_i^2 = g_j^2 = (g_i g_j)^n = 1 \rangle \).

The dihedral group of order six has trivial 3 dimensional integer cohomology. Thus its projective representations do agree with the linear ones. The dihedral subgroups with \( n \) even in \( SL_3\mathbb{Z} \) are \( C_2 \times C_2 \) and \( D_4 \).

The following is the linear character table for \( D_n \):

| \( D_n \) | \( (g_i, g_j)^k \) | \( (g_i g_j)^k \) |
|-----------|-------------------|-------------------|
| \( \xi_1 \) | 1 1               | 1 1               |
| \( \xi_2 \) | 1               | -1               |
| \( \xi_3 \) | -1^k 1^k         | -1^k 1^k         |
| \( \xi_4 \) | -1^k -1^k        | -1^k+1 -1^k+1    |

\( \phi_p \) is given by \( 2 \cos(2\pi pk/n) \) and \( \eta \) is defined as \( 2 \cos(2\pi pk/n) \).

where \( 0 \leq k \leq n-1 \), \( p \) varies from 1 to \((n/2) - 1 \) (\( n \) even) or \((n-1)/2 \) (\( n \) odd) and the hat denotes a representation which only appears in the case \( n \) even. The group \( D^*_2 \) is isomorphic to the eight elements quaternion group, and a linear character table is given by

| \( D^*_2 \) | \( z \) | \( \{h_1, h_1^{-1}\} \) | \( \{h_3, h_3^{-1}\} \) | \( \{h_1h_3, (h_1h_3)^{-1}\} \) |
|------------|-----|----------------------|----------------------|----------------------|
| \( \eta_1 \) | 1   | 1                    | 1                    | 1                    |
| \( \eta_2 \) | 1   | 1                    | -1                   | -1                   |
| \( \eta_3 \) | 1   | -1                   | 1                    | -1                   |
| \( \eta_4 \) | 1   | 1                    | -1                   | 1                    |
| \( \eta_5 \) | 2   | -2                   | 0                    | 0                    |

A Schur cover of \( D_4 \) can be taken as \( D_8 = \langle a, x | a^4 = x^2 = e, xax^{-1} = a^{-1} \rangle \), whose character table is:

| \( D_8 \) | \( e \) | \( a^4 (= z) \) | \( a^2 \) | \( a \) | \( a^3 (= az) \) | \( x \) | \( ax \) |
|-----------|----|----------------|------|------|----------------|-----|------|
| \( \lambda_1 \) | 1  | 1              | 1    | 1    | 1              | 1   | 1    |
| \( \lambda_2 \) | 1  | 1              | 1    | 1    | -1             | -1  | -1   |
| \( \lambda_3 \) | 1  | 1              | 1    | -1   | 1              | 1   | -1   |
| \( \lambda_4 \) | 1  | 1              | 1    | 1    | -1             | -1  | 1    |
| \( \lambda_5 \) | 2  | 2              | -2   | 0    | 0              | 0   | 0    |
| \( \lambda_6 \) | 2  | -2             | 0    | \( \sqrt{2} \) | -\( \sqrt{2} \) | 0   | 0    |
| \( \lambda_7 \) | 2  | -2             | 0    | -\( \sqrt{2} \) | \( \sqrt{2} \) | 0   | 0    |

The relevant inclusions among stabilizers are the following. We give a conjugacy representative appearing in the corresponding character table when necessary.
Using the above inclusions and elementary calculations with characters, particularly the rectification procedure, Theorem 1.7 in [BV14], we obtain a matrix of size 34 × 33 representing the morphism $\Phi_1$. The matrices representing the restrictions among stabilizers are the following. The signs corresponding to the coboundary map as in [SG68].

| $v_1$ | $e_1$ | $e_2$ | $e_3$ |
|-------|-------|-------|-------|
| $e_1$ | $-$1 0 0 0 | $-$1 0 0 | $-$1 0 0 |
| $e_2$ | 0 $-$1 0 0 | 0 $-$1 0 | 0 $-$1 0 |
| $e_3$ | $-$1 $-$1 0 0 | 0 0 $-$1 | 0 0 $-$1 |
|       | $-$1 0 $-$1 $-$1 | $-$1 0 $-$1 | $-$1 0 $-$1 |
|       | 0 $-$1 $-$1 $-$1 | 0 $-$1 $-$1 | 0 $-$1 $-$1 |

| $v_2$ | $e_1$ | $e_4$ | $e_5$ |
|-------|-------|-------|-------|
| $e_1$ | 1 0 0 | 0 $-$1 | 0 $-$1 |
| $e_4$ | 0 1 0 | 0 $-$1 | 0 $-$1 |
| $e_5$ | 0 0 1 | 0 $-$1 | 0 $-$1 |
|       | 0 0 1 | $-$1 0 | $-$1 0 |
|       | 0 0 1 | $-$1 0 | $-$1 0 |
|       | 1 1 0 | $-$1 $-$1 | $-$1 $-$1 |

| $v_3$ | $e_2$ | $e_4$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|
| $e_2$ | 0 0 1 | 1 1 | $-$1 | $-$1 0 |
| $e_4$ | 0 0 1 | 1 1 | $-$1 | 0 $-$1 |
| $e_6$ | 0 0 1 | 1 1 | $-$1 | 0 $-$1 |
| $e_7$ | 1 1 1 | 2 2 | $-$2 | $-$1 $-$1 |
The elementary divisors of the matrix representing the morphism $\phi$ is 1 repeated 12 times. The rank of this matrix is 12.

**Determination of $\Phi_2$.** The relevant inclusions among stabilizers are the following. We give a conjugacy representative appearing in the corresponding character table when necessary.

| $\mathcal{S}_6$ | $\mathcal{S}_5$ | $\mathcal{S}_4$ |
|-----------------|-----------------|-----------------|
| $v_4$ 1 1 1 -1 0 |
| 1 1 1 0 -1 |
| $v_5$ 0 0 1 1 0 |
| 0 0 1 0 1 |
| 1 1 1 1 1 |

Using the above inclusions, an elementary calculation yields a matrix of size $33 \times 12$ representing the morphism $\Phi_2$. The matrices representing the restrictions among stabilizers are the following.

| $t_1$ | $t_4$ | $t_4$ | $t_5$ | $t_5$ | $t_4$ | $t_5$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | 1 0 1 0 |
| 0 1 0 1 |
| 0 1 1 0 |
| 1 0 0 1 |
| $e_2$ -1 0 1 0 |
| 0 -1 0 1 |
| -1 -1 1 1 |
| $e_3$ -1 0 -1 0 |
| 0 -1 0 -1 |
| -1 -1 -1 -1 |

| $t_2$ | $t_3$ | $t_4$ | $t_5$ |
|-------|-------|-------|-------|
| $e_4$ 1 0 1 |
| 0 1 1 |
| $e_5$ -1 1 0 |
| -1 0 1 |
| $e_6$ 2 1 0 0 0 |
| $e_7$ -1 1 1 |
| -1 1 1 |
| $e_8$ 1 1 1 |
| 1 1 1 |
The elementary divisors of the matrix representing the morphism $\phi$ is 1, repeated 7 times. The rank of this matrix is 7.

**Determination of $\Phi_3$.** The morphism $\Phi_3$ is given by blocks which are represented by the following matrices

\[
\begin{pmatrix}
T_1 & T_1 \\
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
T_1 & T_1 \\
1 & -1 \\
-2 & 1
\end{pmatrix}
\begin{pmatrix}
T_1 & T_1 \\
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
T_1 & T_1 \\
1 & -1 \\
1 & -1
\end{pmatrix}
\]

We have the Bredon cochain complex

\[0 \to \mathbb{Z}^{\oplus 19} \xrightarrow{\Phi_1} \mathbb{Z}^{\oplus 19} \xrightarrow{\Phi_2} \mathbb{Z}^{\oplus 8} \xrightarrow{\Phi_3} \mathbb{Z} \to 0.\]

Using the information concerning ranks and elementary divisors of $\Phi_1^{u_2}$, we obtain

\[(6.2) \quad H^0_{SL_3 \mathbb{Z}}(E SL_3 \mathbb{Z}, R_{u_2}) = 0, \text{ if } p > 0, \quad H^0_{SL_3 \mathbb{Z}}(E SL_3 \mathbb{Z}, R_{u_2}) \cong \mathbb{Z}^{\oplus 7}.\]

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in section 3.2 collapses at level 2 and we conclude

**Theorem 6.3.**

\[u_2 K^0_{SL_3 \mathbb{Z}}(E SL_3 \mathbb{Z}) \cong \mathbb{Z}^{\oplus 7},\]

\[u_2 K^1_{SL_3 \mathbb{Z}}(E SL_3 \mathbb{Z}) = 0.\]

**The twisting $u_1 + u_2$.** Now we continue with the calculation of $u_1 + u_2 K_{SL_3 \mathbb{Z}}(E SL_3 \mathbb{Z})$. Notice that the classes $u_1$ and $u_2$ are disjoint, i.e. they do not restrict simultaneously to a non-zero element in the cohomology of any subgroup of $SL_3 \mathbb{Z}$. This observation and diagram [5.1] lead to the following.

**Remark 6.4.** The matrix $\Phi_1^{u_1 + u_2}$ corresponding to the twisting $u_1 + u_2$ can be obtained as:

\[
\begin{array}{cccccccc}
\v_1 & u_1 & u_2 & u_3 & u_4 & u_5 \\
\v_2 & 0 & 0 & 0 & 0 & u_6 \\
\v_3 & 0 & 0 & 0 & 0 & 0 \\
\v_4 & 0 & 0 & 0 & u_2 & u_2 \\
\v_5 & 0 & 0 & 0 & u_2 & u_2 \\
\end{array}
\]

Where a $u_i$ in position $(j,k)$ means that we take the corresponding submatrix of $\Phi_1^{u_i}$ associated to the inclusion $stab(e_j) \to stab(v_k)$.

This matrix has size $14 \times 16$ and it has elementary divisors $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and its rank is 9.

**Remark 6.5.** The matrix $\Phi_2^{u_1 + u_2}$ corresponding to the twisting $u_1 + u_2$ can be obtained as:
Where an $u_i$ in position $(j, k)$ means that we take the corresponding submatrix of $\Phi_{u_i}$ associated to the inclusion $\text{stab}(t_j) \to \text{stab}(e_k)$ ($u_0$ denotes the trivial cocycle).

This matrix has size $16 \times 8$ and it has 1 as elementary divisor 7 times and its rank is 7.

Finally the matrix $\Phi_{u_1 + u_2}$ corresponding to the twisting $u_1 + u_2$ is the same as the matrix $\Phi_{u_2}$.

We have the following cochain complex

$$
\begin{array}{cccccc}
0 & \to & \mathbb{Z}^{14} & \varphi^{u_1 + u_2} & \to & \mathbb{Z}^{16} \\
\end{array}
$$

Using the data of $\Phi_{u_1 + u_2}$ concerning ranks and elementary divisors we obtain

(6.6) $\quad H^p_{\text{SL}_3 \mathbb{Z}}(\mathbb{E} \text{SL}_3 \mathbb{Z}, \mathcal{R}_{u_1 + u_2}) = 0$, if $p > 1$,

(6.7) $\quad H^0_{\text{SL}_3 \mathbb{Z}}(\mathbb{E} \text{SL}_3 \mathbb{Z}, \mathcal{R}_{u_1 + u_2}) \cong \mathbb{Z}^{\oplus 5}$

(6.8) $\quad H^1_{\text{SL}_3 \mathbb{Z}}(\mathbb{E} \text{SL}_3 \mathbb{Z}, \mathcal{R}_{u_1 + u_2}) \cong \mathbb{Z}/2\mathbb{Z}$.

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in section 3.2 collapses at level 2 and we conclude

**Theorem 6.9.**

\[ u_1 + u_2 K^0_{\text{SL}_3 \mathbb{Z}}(\mathbb{E} \text{SL}_3 \mathbb{Z}) \cong \mathbb{Z}^{\oplus 5}, \]

\[ u_1 + u_2 K^1_{\text{SL}_3 \mathbb{Z}}(\mathbb{E} \text{SL}_3 \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}. \]

### 7. Applications

**Twisted equivariant $K$-Homology and the Baum-Connes Conjecture.**

The Baum-Connes Conjecture [BCH94], [MV03] predicts for a discrete group $G$ the existence of an isomorphism

$\mu_\iota : K^G_i(\mathbb{E}G) \to K_i(C^*_r(G))$

given by the (analytical) assembly map, where $C^*_r(G)$ is the reduced $C^*$-algebra of the group $G$.

More generally, given any $G$-$C^*$-Algebra, the Baum-Connes conjecture with coefficients predicts an isomorphism given by an assembly map

$\mu_\iota : K^G_i(\mathbb{E}G, A) \to K_i(A \rtimes G)$.

Where $K^G_i(\mathbb{E}G, A)$ is defined in terms of equivariant and bivariant $KK$-groups ,

\[ K^G_i(\mathbb{E}G, A) = \varinjlim_{G-\text{compact} X \subset \mathbb{E}G} K K^*_r(C_0(X), A) \]

and $A \rtimes G$ denotes the crossed product $C^*$-algebra, $X \subset \mathbb{E}G$ is a cocompact subcomplex. See [CE01], [Ech08] for more details.
Definition 7.1. Let $G$ be a discrete group. Given a cocycle $\omega \in Z^2(G, S^1)$, an $\omega$-representation on a Hilbert space $\mathcal{H}$ is a map $V : G \to U(\mathcal{H})$ satisfying $V(s)V(t) = \omega(s, t)V(st)$.

Consider the quotient map $U(\mathcal{H}) \to PU(\mathcal{H}) = U(\mathcal{H})/S^1$. Recall that the group $PU(\mathcal{H})$ is the outer automorphism group $Out(K)$ of the $C^*$-algebra of compact operators on $\mathcal{H}$, denoted by $K$. The cocycle $\omega$ defines in this way an action of $G$ on $K$. This algebra is denoted by $K_\omega$.

Let $G$ be a discrete group with a finite model for $EG$. Let $\omega \in Z^2(G, S^1)$ be a cocycle and assume that the bredon cohomology groups $H^*(EG, K^-\omega)$ are concentrated in degrees 0 and 1.

Then, the Universal Coefficient Theorem for Bredon cohomology, Theorem 1.13 in [BV14] identifies the Bredon homology groups $H_*(EG, K_\omega)$ with the bredon cohomology groups $H^*_G(EG, K^-\omega)$. By inspecting the Bredon cohomology groups computed in 6.6 [CZ] the hypotheses of Corollary 7.3 in [BV14] are satisfied for the twistings $u_2$ and $u_1 + u_2$. This gives a duality isomorphism

$$\omega K^*_G(EG) \to K^C_G(EG, K_{-\omega}).$$

Similar forms of Poincaré duality for proper and twisted actions have been studied by Echterhoff, Emerson and Kim in [ELK08] (Theorem 3.1) under assumptions concerning the Baum-Connes conjecture, particularly the validity of the Dirac-Dual-Dirac Method for the group $G$.

Theorem 7.2. The equivariant $K$-homology groups with coefficients in the $G$-$C^*$ algebra $K_\omega$ are given as follows:

- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_1}) \cong \mathbb{Z}^{\oplus 13}$,
- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_1}) = 0$.
- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_2}) \cong \mathbb{Z}^{\oplus 7}$,
- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_2}) = 0$.
- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_1 + u_2}) \cong \mathbb{Z}^{\oplus 5}$,
- $K^G_{S^1\mathbb{Z}}(ESL_3\mathbb{Z}, K_{u_1 + u_2}) \cong \mathbb{Z}/2\mathbb{Z}$.

Relation to the work of Tezuka and Yagita. In the case of finite order twists given by cocycles $\alpha \in Z^2(G, S^1)$, the finite dimensional, $\alpha$-twisted vector bundle model of twisted equivariant $K$-theory is related to untwisted equivariant $K$-theory groups in a way we will describe below.

Recall that given a normalized torsion cocycle $\alpha$, there exists a central extension

$$1 \to \mathbb{Z}/n \to G \to G \to 1$$

Let $X$ be a $G$-connected $G$-CW complex. The $\alpha$-Twisted $K$-theory groups are seen to agree with the abelian group of $G_\alpha$-equivariant, complex vector bundles for which the generator of $\mathbb{Z}/n$ acts by complex multiplication by $e^{2\pi i n}$. There is a splitting

$$K^G_{G_\alpha}(X) \cong \bigoplus_{V \in Irr(\mathbb{Z}/n)} K^G_{G_\alpha}(X, V),$$

where $K^G_{G_\alpha}(X, V)$ is the subgroup of $G_\alpha$-equivariant, complex vector bundles for which the action of the central $\mathbb{Z}/n$ on each fiber restricts to the irreducible
representation $V$, and the definition is extended to other degrees using the remarks following definition [3.1].

Given a discrete group $G$ and a normalized torsion cocycle $\alpha$, Theorem 3.4 in [Dwy08] proves that the groups $\alpha K^*_G(X)$ extend to a $\mathbb{Z}/2$-graded equivariant cohomology theory on the category of finite, proper $G$-CW pairs. This theory restricts to equivariant $K$-theory [LO01] in the case of a trivial cocycle. The groups $\alpha K^*_G(X)$ have a natural graded $K^*_G(X)$-module structure.

The multiplicative structure on the graded ring $K^*_G(E_G)$ is well known. Recall the definition of the augmentation ideal $I_G = \ker(K^0_G(E_G) \xrightarrow{i^*} K^0_G(EG_0) \xrightarrow{} K^0_G(E_G_0))$, where $EG_0 \rightarrow EG$ denotes the inclusion of the 0th-skeleton and the last map is the restriction map associated to the trivial group $\{e\} \subset G$.

The following result is a generalization of the Atiyah-Segal completion Theorem and it is proved in part b) of 4.4 in [LO01], page 611.

**Theorem 7.4.** Let $EG$ be the classifying space for proper actions.

- if $EG$ has the homotopy type of a finite $G$-CW complex, then there is an isomorphism $K^*(BG) \cong K^*_G(EG)_{I_G}$, where the right hand side denotes the completion with respect to the ideal $I_G$.

Specializing to the case of $St_3\mathbb{Z}$, the topological $K$-theory ring $K^*(BSt_3\mathbb{Z})$ is known after computations by Tezuka and Yagita using Brown Peterson spectra and its Conner-Floyd isomorphism, Corollary 4.7 in page 93 of page [TY92], which we recall:

**Theorem 7.5.** Localized at the prime 2, the topological $K$-theory of $BSt_3\mathbb{Z}$ is given as follows:

- $K^0(BSt_3\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)}$
- $K^1(BSt_3\mathbb{Z}) = \mathbb{Z}_2$,

where $\mathbb{Z}_{(2)}$ is the localization at 2, and $\mathbb{Z}_2$ denotes the 2-adical completion of the integers.

Putting together Theorems 6.9, 7.4, and 7.5 one obtains:

**Corollary 7.6.** The completion of the equivariant $K$-theory groups $K^*_G(ESt_3\mathbb{Z})$ computed in 6.2 with respect to the augmentation ideal $I_{St_3\mathbb{Z}}$ is given as follows:

- $K^0_G(ESt_3\mathbb{Z})_{I_{St_3\mathbb{Z}}} = \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)}$
- $K^1_G(ESt_3\mathbb{Z})_{I_{St_3\mathbb{Z}}} = \mathbb{Z}_2$.

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