Heavy Quark Effective Theory at Large Orders in $1/m$

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Abstract

The existing derivations of a heavy quark effective theory (HQET) are analyzed beyond the next-to-leading order in $1/m$. With one exception they are found to be incorrect. The problem is a wrong normalization of the heavy quark field in the effective theory. We argue that the correct effective theory should be given by a Foldy–Wouthuysen type field transformation to all orders in $1/m$. The renormalization of the resulting Lagrangian to order $1/m^2$ is performed including also effects arising through vacuum polarization. Our results for the anomalous dimensions disagree with the existing ones. Some applications are considered.

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1. Introduction

Interest in the physics of the mesons and baryons containing one heavy quark (c or b) has received a strong stimulus through the invention of the heavy quark effective theory (HQET) [1]. This allows for a systematic description of the symmetries appearing in the infinite quark mass limit and of their breaking by a finite quark mass. An important outcome of this theory concerns the possibility of accurately extracting the Kobayashi-Maskawa matrix element $V_{cb}$ in a model-independent way through a study of the weak semileptonic decay $\bar{B} \rightarrow D^* e \bar{\nu}$. This is made possible by Luke’s theorem [2], which asserts that some of the $1/m_c$-order corrections to the infinite mass limit predictions of the theory for this decay vanish when the c quark is produced at rest in the b quark’s reference frame. The next corrections come thus in at order $1/m_c^2$ and their estimation is clearly important for an assessment of the reliability of the $V_{cb}$ determination. The task of evaluating these corrections has been recently undertaken by Falk and Neubert [3], who made use for this of the HQET formalism developed in [4, 5]. The aim of this paper is to point out that these treatments are incorrect, already at order $1/m^2$. We show in Section 2 that the problem consists in a wrong normalization of the field in the effective theory. The correct treatment turns out to be the one given by the authors of [6]. However, due to the particular strategy of calculation adopted in [3], their results remain valid when the correct treatment is being used. In Sect.3 the matrix elements of current operators obtained with the two HQETs are directly compared and shown to differ at order $1/m^2$. A general presentation of the HQET as given in [6] to any order in $1/m$ is offered in Sect.4. The resulting effective Lagrangian to $O(1/m^2)$ is used in Sect.5 to sum up the leading logarithms of the heavy quark mass of the form $1/m^2 \log(m)$ in the various matrix elements of the theory. We compute also the quantum corrections appearing to this order from graphs containing one heavy quark inside closed loops, the analog of the Euler–Heisenberg Lagrangian relevant to the infrared behaviour of QCD. An additional complication which appears when light quarks are included is pointed out: light-quark operators are induced through renormalization. In Sect.6 applications of the formalism to the phenomenology of heavy hadrons are given.

2. Normalization of the field

Three distinct derivations of a heavy quark effective theory (HQET) from QCD have been proposed in the literature [4, 5, 6]. Two of them [4, 5], although phrased in different formalisms, are actually equivalent. The third one [6] gives different results from the first two, starting at order $O(1/m^2)$. The aims of this section are to explain the reasons for this disagreement and to show that the last theory is the correct one.
We start by first reviewing the results of the usual derivation of the HQET as given in [4]. The relation between the heavy quark field in QCD, \( Q(x) \), and that in the effective theory, \( h(x) \), is up to \( O(1/m^2) \)

\[
Q(x) = e^{-imv \cdot x} \left( 1 + \frac{1}{2m} \left( i \mathcal{D}_\perp + \frac{1}{4m^2} v \cdot \mathcal{D}_\perp \right) \right) h(x). 
\]

(1)

Here \( v_\mu \) is the velocity of the heavy quark, \( D_\mu = \partial_\mu + igA_\mu^a t^a \) and \( D^\perp = D_\mu - v_\mu v \cdot D \).

The heavy quark field \( h(x) \) satisfies \( \not{v} h = h \). The flavour-conserving vector current for quarks with the same velocity can be written by making use of (1) as

\[
\bar{Q} \gamma_\mu Q = \bar{h} \gamma_\mu h + \frac{i}{2m} \bar{h} (\gamma_\mu \mathcal{D}_\perp - \mathcal{D}_\perp \gamma_\mu) h 
\]

\[
+ \frac{1}{4m^2} \bar{h} (\gamma_\mu v \cdot \mathcal{D} \mathcal{D}_\perp + \mathcal{D}_\perp \gamma_\mu \mathcal{D}_\perp + \mathcal{D}_\perp v \cdot \mathcal{D} \gamma_\mu) h.
\]

(2)

The effective theory Lagrangian is

\[
\mathcal{L} = \bar{h} \left[ iv \cdot \mathcal{D} - \frac{1}{2m} \mathcal{D}_\perp^2 + \frac{i}{4m^2} \mathcal{D}_\perp v \cdot \mathcal{D} \mathcal{D}_\perp \right] h. 
\]

(3)

Let us consider the number operator for heavy quarks at rest \( (v_\mu = (1,0,0,0), \gamma_0 h(x) = h(x)) \):

\[
\hat{N} = \int d^3 \vec{x} (\bar{Q} \gamma_0 Q)(x) = \int d^3 \vec{x} \bar{h}(x) \left[ 1 + \frac{1}{4m^2} \mathcal{D}_\perp^2 \right] h(x). 
\]

(4)

Its expectation value in a hadronic state containing the heavy quark \( Q \), also at rest, is

\[
\frac{1}{2m_M} \langle M_{QCD}(v) | \int d^3 \vec{x} (\bar{Q} \gamma_0 Q)(x) | M_{QCD}(v) \rangle = \frac{1}{2m_M} \langle M_{HQET}(v) | \int d^3 \vec{x} \bar{h}_H(x) \left[ 1 + \frac{1}{4m^2} \mathcal{D}_\perp^2 \right] h_H(x) | M_{HQET}(v) \rangle. 
\]

(5)

In this relation the field \( h_H(x) \) is in the Heisenberg picture and its evolution is dictated by the full HQET Lagrangian (3). \( |M_{QCD}(v)\rangle \) is an eigenstate of the QCD Lagrangian and \( |M_{HQET}(v)\rangle \) is the corresponding eigenstate of the full HQET Lagrangian (3). Although they correspond to the same energy eigenvalue (to the order in \( 1/m \) we are considering), they are not necessarily identical and in fact, it will be shown in the next section that they are related in the rest frame of the hadron by an unitary transformation. The normalization convention for the states \( |M_{HQET}(v)\rangle \) is the usual one \( \langle M_{HQET}(\vec{p}) | M_{HQET}(\vec{p}') \rangle = 2E_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{p}') \).
The matrix elements on the r.h.s. of (5) can be written with the help of the Gell-Mann–Low formula [7] as power series in $1/m$. For example, the first term is equal to

$$\langle M_{\text{HQET}}(v) | \int d^3 \vec{x} (\bar{h}_H h_H)(x) | M_{\text{HQET}}(v) \rangle = \langle M(v) | \int d^3 \vec{x} (\bar{h}_I h_I)(x) | M(v) \rangle + \frac{1}{m} i \langle M(v) | T \int d^3 \vec{x} \int d^4 y \mathcal{L}_\infty(y) (\bar{h}_I h_I)(x) | M(v) \rangle + \cdots.$$  \hspace{1cm} (6)

Here, the field $h_I(x)$ is understood to be in the interaction picture where the term $\bar{h}(iv \cdot D)h$ in the Lagrangian (3) is the unperturbed Lagrangian and the rest is an interaction. That is, we work to all orders in $\alpha_s$; this is motivated by our desire to have hadrons containing one heavy quark in the asymptotic initial and final states. Another useful splitting of the Lagrangian (3) is the one which keeps only the term $\bar{h}_I (iv \cdot \partial) h_I$ as the free Lagrangian and treats the rest as a perturbation; this gives the usual double perturbative expansion in $\alpha_s$ and $1/m$ and is suited for studying the scattering of free quarks and gluons. $\mathcal{L}_1$ is the coefficient of $1/m$ in (3) and the states $|M(v)\rangle$ are eigenstates of the leading order term in the HQET Lagrangian.

It can be seen that there are two distinct sources of mass dependence in (5): a) the factor $1/m^2$ appearing explicitly in (5), which has a kinematic origin. It comes in through the very definition of a quantity of the theory, the number operator. b) powers of $1/m$ arising through application of the Gell-Mann–Low formula. These can be considered as having a dynamical origin, as they are due to the interaction terms in the Lagrangian (3).

In the final expression obtained after application of the Gell-Mann–Low formula to (5), the mass dependence is concentrated into the factors of $1/m$ standing in front of matrix elements like those on the r.h.s. of (6). The latter are mass-independent quantities, properties of the HQET in the infinite mass limit. A similar expansion as a power series in $1/m$ with constant coefficients can be written for any other matrix element of heavy quark currents. In fact, it is this separation of the mass dependence one of the properties which makes the HQET useful.

However, a closer look reveals some problems related to the form of the heavy quark number operator (4). Firstly, its expectation value in the unperturbed state $|M(v)\rangle$

$$\frac{1}{2m_M} \langle M(v) | \int d^3 \vec{x} \bar{h}_I(x) \left[ 1 + \frac{1}{4m^2} D_\perp^2 \right] h_I(x) | M(v) \rangle$$  \hspace{1cm} (7)

differs from unity. Usually, the scale of a quantized field is set by requiring that the eigenvalues of the particle number operator be integer numbers. In our case, the

1Neglecting the trivial $m$-dependence appearing through the normalization of the states.
matrix element (7) can be normalized to unity through the following field redefinition:

\[ h'(x) = 1 - \frac{1}{2} \left( \frac{i \not{D}_\perp}{2m} \right)^2 h(x). \] (8)

Secondly, even more disturbing, the number operator (4) is not a constant of motion for the leading order HQET Lagrangian:

\[ \frac{d}{dx_0} \int d^3x \bar{h}_I(x) \left[ 1 + \frac{1}{4m^2} \not{D}_\perp^2 \right] h_I(x) \neq 0. \] (9)

To be sure, it is conserved when the dynamics is given by the full Lagrangian (3). The point is that we do not know how to solve the problem corresponding to this Lagrangian (nor can useful statements be made about it). We are forced to resort to the use of perturbation theory, which expresses any quantity of interest in terms of matrix elements in the theory defined by the first term in (3), the “free” theory. For these matrix elements, use can be made of the symmetries of this theory in order to extract useful predictions.

Unfortunately, the nonconservation of the number operator (9) in the “free” theory has as a consequence the fact that the unperturbed states \( |M(v)\rangle \) upon which the perturbative expansion in \( 1/m \) is built, simply do not exist. This means that the theory defined by (1–3) does not lend itself to a consistent approximation scheme in powers of \( 1/m \).

Fortunately, the solution to the first problem, the field redefinition (8), happens to be also the solution for the second one. In terms of the new field \( h'(x) \), the Eqs.(1-3) read

\[ Q(x) = e^{-imv \cdot x} \left[ 1 + \frac{1}{2m} (i \not{D}_\perp) + \frac{1}{4m^2} \left( v \cdot \not{D} \not{D}_\perp - \frac{1}{2} \not{D}_\perp^2 \right) \right] h'(x), \] (10)

\[ \bar{Q} \gamma_\mu Q = \bar{h}' \gamma_\mu h' + \frac{i}{2m} \bar{h}' \left( - \not{p}_\perp \gamma_\mu + \gamma_\mu \not{p}_\perp \right) h' + \frac{1}{4m^2} \bar{h}' \left( \frac{1}{2} \not{p}_\perp^2 - \not{p}_\perp v \cdot \not{D} \right) \gamma_\mu - \not{p}_\perp \gamma_\mu \not{p}_\perp + \gamma_\mu \left( \frac{1}{2} \not{p}_\perp^2 - v \cdot \not{D} \not{p}_\perp \right) \right) h', \] (11)

\[ L' = \bar{h}' \left[ iv \cdot \not{D} - \frac{1}{2m} \not{p}_\perp^2 + \frac{i}{4m^2} \left( - \frac{1}{2} \not{p}_\perp^2 v \cdot \not{D} + \not{p}_\perp v \cdot \not{D} \not{p}_\perp - \frac{1}{2} v \cdot \not{D} \not{p}_\perp^2 \right) \right] h'. \] (12)

This is precisely what one directly obtains by applying the method described in [6] for deriving a heavy quark effective theory from QCD. The number operator (5) is

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2A similar transformation is sometimes made when deriving the Pauli equation to order \( 1/m^2 \). See e.g. [8].

3That is, the same is not true at higher orders in \( 1/m \).
written now simply as
\[ \hat{N} = \int d^3x (\vec{h}' h') (x) \]  
(13)
and is a constant of motion for the free theory (i.e., when keeping only the leading term in (12)).

Equally important, it stays conserved up to each order in the perturbative expansion in $1/m$ given by the Lagrangian (12). As discussed above, this is a prerequisite for the consistency of the approximation scheme. To see this to order $1/m$, we note that the equation of motion for the field $h'(x)$ can be written as
\[ \left[ i v \cdot \vec{D} - \frac{1}{2m} \vec{P}_\perp^2 \right] h'(x) = 0. \]  
(14)
This can be used to write
\[ \frac{d}{dt} \hat{N} = \int d^3x \left( \vec{h}' \cdot v h' + \vec{h}' \cdot \vec{D} h' \right) (x) \]  
(15)
\[ = \frac{i}{2m} \int d^3x \left( \vec{h}' \cdot \vec{P}_\perp^2 h' - \vec{h}' \cdot \vec{P}_\perp^2 h' \right) (x) = 0, \]
where an integration by parts has been performed, followed by the neglect of the surface terms at infinity.

The number operator for heavy quarks at rest is conserved also at order $1/m^2$ in the effective Lagrangian (12). This can be seen most easily by rewriting the $1/m^2$ term in the latter as
\[ \frac{1}{2} \vec{P}_\perp \left[ v \cdot \vec{D}, \vec{P}_\perp \right] + \frac{1}{2} \left[ \vec{P}_\perp, v \cdot \vec{D} \right] \vec{P}_\perp = \frac{ig}{2} \left( \vec{P}_\perp v^{\mu \nu} \gamma^\nu F_{\mu \nu} \cdot \vec{D} \right). \]  
(16)
This gives an equation of motion for $h'(x)$
\[ \left[ i v \cdot \vec{D} - \frac{1}{2m} \vec{P}_\perp^2 - \frac{g}{8m^2} \left( \vec{P}_\perp v^{\mu \nu} \gamma^\nu F_{\mu \nu} - v^{\mu \nu} \gamma^\nu F_{\mu \nu} \cdot \vec{P}_\perp \right) \right] h'(x) = 0, \]  
(17)
which does not contain, in the rest frame of $v$, any time–derivative acting on $h'(x)$, except in the leading–order term. The above procedure can be repeated therefore, step by step, for this case as well.

Everything which was said up to now refers to the number operator for heavy quarks at rest. However, it can be seen that similar results hold for the number operator for heavy quarks moving with an arbitrary velocity $v$. Heavy quark effective theory (even truncated to a finite order in $1/m$) is invariant under Lorentz transformations, provided the velocity $v$ is transformed at the same time. Furthermore the number operator is a Lorentz invariant and is the same in any reference frame. In particular, if it is conserved in one reference frame, it should also be conserved in any other reference frame.
3. Comparing matrix elements of the currents

We have shown in the preceding Section that the HQET (1-3) cannot be formulated as a consistent perturbative expansion in powers of $1/m$. In principle, this would suffice to reject it. However, we would like to present in this Section a different, more pragmatic point of view, which will permit a better understanding of the difference between the two HQETs, making at the same time the connection with the, perhaps more familiar, theory of the point field transformations in the Lagrangian formalism.

We will compare in the following the explicit predictions of the two HQETs, given by (1-3) and respectively by (10-12), for the matrix element of the current $\bar{c}\Gamma b$. Here $b, c$ are two different heavy quarks moving with velocities $v, v'$ and having masses $m_b, m_c$. By examining the form of the currents and Lagrangians, it is clear that a possible difference can only show up at order $O(1/m^2)$. This is equal to

$$\langle M_{HQET}'(v')(\bar{c}\Gamma b)(0)|M_{HQET}(v)\rangle_{(10-12)} - \langle M_{HQET}'(v')(\bar{c}\Gamma b)(0)|M_{HQET}(v)\rangle_{(1-3)} =$$

$$= \frac{1}{8m_c^2} \langle M'(v')|T\int d^4x \bar{h}_v^{(c)}(x) (\not{\vec{P}_\perp} v') \not{\vec{D}} + \not{\vec{v}} \cdot \not{\vec{D}} \not{\vec{P}_\perp}) h_v^{(c)}(x)(\bar{h}_v^{(c)} \Gamma h_v^{(b)})(0)|M(v)\rangle$$

$$+ \frac{1}{8m_b^2} \langle M'(v')|T\int d^4x \bar{h}_v^{(b)}(x) (\not{\vec{P}_\perp} v) \not{\vec{D}} + \not{\vec{v}} \cdot \not{\vec{D}} \not{\vec{P}_\perp}) h_v^{(b)}(x)(\bar{h}_v^{(c)} \Gamma h_v^{(b)})(0)|M(v)\rangle$$

$$- \frac{1}{8m_c^2} \langle M'(v')|(\bar{h}_v^{(c)} \not{\vec{P}_\perp} \Gamma h_v^{(b)})(0)|M(v)\rangle - \frac{1}{8m_b^2} \langle M'(v')|(\bar{h}_v^{(c)} \Gamma \not{\vec{P}_\perp} h_v^{(b)})(0)|M(v)\rangle.$$  

We can rewrite this as

$$- \frac{i}{8m_c^2} \langle M'(v')|T\int d^4x \left( \bar{h}_v^{(c)} \not{\vec{P}_\perp} \frac{d\mathcal{L}}{dh_v^{(c)}} + \frac{d\mathcal{L}}{dh_v^{(b)}} \not{\vec{P}_\perp} h_v^{(c)} \right)(x)(\bar{h}_v^{(c)} \Gamma h_v^{(b)})(0)|M(v)\rangle$$

$$- \frac{i}{8m_b^2} \langle M'(v')|T\int d^4x \left( \bar{h}_v^{(b)} \not{\vec{P}_\perp} \frac{d\mathcal{L}}{dh_v^{(b)}} + \frac{d\mathcal{L}}{dh_v^{(c)}} \not{\vec{P}_\perp} h_v^{(b)} \right)(x)(\bar{h}_v^{(c)} \Gamma h_v^{(b)})(0)|M(v)\rangle$$

$$- \frac{1}{8m_c^2} \langle M'(v')|(\bar{h}_v^{(c)} \not{\vec{P}_\perp} \Gamma h_v^{(b)})(0)|M(v)\rangle - \frac{1}{8m_b^2} \langle M'(v')|(\bar{h}_v^{(c)} \Gamma \not{\vec{P}_\perp} h_v^{(b)})(0)|M(v)\rangle,$$

where $\mathcal{L}$ denotes the sum of the “free” Lagrangians of the two heavy quarks:

$$\mathcal{L} = \bar{h}_v^{(c)} iv' \cdot Dh_v^{(c)} + \bar{h}_v^{(b)} iv \cdot Dh_v^{(b)}.$$  

The matrix elements (19) between hadronic states can be expressed with the help of the reduction formalism as the residues at the corresponding bound–state poles of the appropriate Green functions. For example, the first term in (19) can be obtained from the Green function

$$\langle 0|T(\bar{q} h_v^{(c)})(y) \int d^4x \left( \bar{h}_v^{(c)} \not{\vec{P}_\perp} \frac{d\mathcal{L}}{dh_v^{(c)}} + \frac{d\mathcal{L}}{dh_v^{(b)}} \not{\vec{P}_\perp} h_v^{(c)} \right)(x)(\bar{h}_v^{(c)} \Gamma h_v^{(b)})(0)|\bar{h}_v^{(b)} q)(z)|0\rangle$$  

(21)
where \((\bar{q}_h^{(c)})\) and \((\bar{h}^{(b)} q)\) are the interpolating fields of the outgoing and respectively, incoming hadrons. Only their flavor structure has been made explicit, the Lorentz and Dirac structure needed to endow them with the correct quantum numbers of the respective bound states are implicitly assumed. One can apply now the relation

\[
\langle 0|\mathcal{T}\left(F(\phi)\frac{d\mathcal{L}}{d\phi}\right)(x)Q_1(x_1)Q_2(x_2)\ldots|0\rangle = \sum_i \delta^{(4)}(x - x_i)\langle 0|\mathcal{T}Q_1(x_1)\ldots\left(F\frac{dQ_i}{d\phi}\right)(x_i)\ldots|0\rangle,
\]

(22)
to transform (21) into

\[
i\langle 0|\mathcal{T}(\bar{q}_h^{(c)}(y)(\bar{h}^{(c)}_v \bar{P}_\perp \Gamma \bar{h}^{(b)}_v)(0)(\bar{h}^{(b)}_v q)(z)|0\rangle + i\langle 0|\mathcal{T}(\bar{q} \bar{P}_\perp^2 \Gamma \bar{h}^{(c)}_v)(y)(\bar{h}^{(c)}_v \Gamma \bar{h}^{(b)}_v)(0)(\bar{h}^{(b)}_v q)(z)|0\rangle.
\]

(23)
The first term can easily be translated back into a matrix element between hadronic states, equal to

\[
i\langle M'(v')|(\bar{h}^{(c)}_v \bar{P}_\perp \Gamma \bar{h}^{(b)}_v)(0)|M(v)\rangle.
\]

(24)
In a completely analogous way, the second term in (19) can be transformed to the form

\[
i\langle 0|\mathcal{T}(\bar{q}_h^{(c)}(y)(\bar{h}^{(c)}_v \bar{P}_\perp q)(z)|0\rangle + i\langle 0|\mathcal{T}(\bar{q}_h^{(c)}(y)(\bar{h}^{(c)}_v \bar{P}_\perp \Gamma \bar{h}^{(b)}_v)(0)(\bar{h}^{(b)}_v q)(z)|0\rangle,
\]

(25)
of which the second term gives again a matrix element contributing in (18) a quantity

\[
i\langle M'(v')|(\bar{h}^{(c)}_v \bar{P}_\perp \Gamma \bar{h}^{(b)}_v)(0)|M(v)\rangle.
\]

(26)
Collecting now everything, one can see that the terms (24) and (26) exactly cancel the matrix elements of the two local terms in (18). The difference (18) of the matrix elements of the current \(\bar{c}\Gamma \bar{b}\) in the two HQETs is proportional to the residue at the corresponding bound–state poles of the Green function

\[
\frac{1}{8m_c^2}\langle 0|\mathcal{T}(\bar{q} \bar{P}_\perp^2 \Gamma \bar{h}^{(b)}_v)(0)(\bar{h}^{(b)}_v q)(z)|0\rangle + \frac{1}{8m_b^2}\langle 0|\mathcal{T}(\bar{q}_h^{(c)}(y)(\bar{h}^{(c)}_v \bar{P}_\perp \Gamma \bar{h}^{(b)}_v)(0)(\bar{h}^{(b)}_v q)(z)|0\rangle.
\]

(27)

\[^4\text{Given in} \ [9] \text{and} \ [10] \text{with a wrong coefficient on the right–hand side. The correct form can be found for example in} \ [11].\]
Next we observe that

$$p'^2 h^{(c)}_{v'} = \left( D^2 - (v' \cdot D)^2 + \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu} + ig v'_\mu \gamma_\nu F^{\mu\nu} \right) h^{(c)}_{v'} . \tag{28}$$

If we keep in this relation the terms containing only the heavy quark field\(^5\) the new interpolating field \((\bar{q}h^{(c)})\) → \((\bar{q}\partial_2^2 h^{(c)})\) has the same quantum numbers as the old one and is therefore equally qualified to represent the same hadronic bound states as the latter. A similar argument can be applied to the second term in (27). We are led therefore to conclude that the two Green functions in (27) have nonvanishing residues at the bound–state poles associated with the hadronic states \(|M(v)|\) and \(|M'(v')|\). Hence the two HQETs give different answers to order \(O(1/m^2)\).

In taking the difference (18), the matrix elements on the r.h.s. are taken between the same states. These are the physical states in the infinite–mass limit. On physical grounds these states exist and are unique. The reason for obtaining different results with the two HQETs should be clear: in passing from one theory to another, we have “forgotten” to make the change of variable (8) in the interpolating field for the mesons too. Once this is done, the contribution from the second term in (8) will cancel the matrix elements (27). But this means that the two interpolating fields \((\bar{q}h)\) and \((\bar{q}h')\) cannot be simultaneously good interpolating fields for the same states, to be used with their respective theories. That is, we have an alternative: i) Use the HQET (1-3) with the interpolating field \((\bar{q}h)\) for the heavy meson. Then, when working with the HQET (10-12), the correct interpolating field to be used should be

$$\mathcal{D}^{\alpha} h^{(c)}_{v'} = \frac{1}{8m^2} (\bar{q} \partial_2^2 h') . \tag{29}$$

Or, ii) Use the HQET (10-12) with the interpolating field \((\bar{q}h')\) and the HQET (1-3) with the interpolating field

$$\mathcal{D}^{\alpha} h = \frac{1}{8m^2} (\bar{q} \partial_2^2 h) . \tag{30}$$

Only one of these possibilities can be true (or none). We will prove in the next section that the alternative ii) is the correct one, by showing that the normalization of the interpolating field \((\bar{q}h')\) is the same as that of its QCD counterpart \((\bar{q}Q)\):

$$\langle 0 | (\bar{q}h'_H) | M_{HQET} \rangle = \langle 0 | (\bar{q}Q) | M_{QCD} \rangle . \tag{31}$$

The situation can be understood as a case of inapplicability of the well–known equivalence theorem\(^12\). According to this theorem, any field transformation \(\phi' = \phi + F(\phi)\), where \(F(\phi)\) contains only terms with at least two fields, leaves the physical

\(^5\)The terms containing at least two fields do not give a pole and vanish under multiplication with the LSZ factors.
content of the theory (the S–matrix) unchanged. In other words, any arbitrary field \( \phi' \) is as good as \( \phi \) as an interpolating field, as long as it has the same normalization:

\[
(0|\phi'(x)|\phi(p)) = (0|\phi(x)|\phi(p)) .
\]  

The condition on the field transformation \( F(\phi) \) has precisely the function of assuring that the new field is correctly normalized.

In our case, the two theories are related by the field transformation (8). It is easy to see that the above-mentioned condition on the field transformation is not satisfied for this case and therefore the two fields \( h(x) \) and \( h'(x) \) do not have the same normalization. We will show now that it is the field \( h'(x) \) which is correctly normalized.

4.Proof to all orders in \( 1/m \)

In this Section, after a short review of the results obtained in [6], we will take up the arguments of the preceding two Sections and will prove that they are true to all orders in \( 1/m \).

In the approach of [3], the heavy quark field \( h(x) \) in the effective theory is expressed in terms of the QCD field \( Q(x) \) as

\[
h(x) = h^{(+)}(x) + h^{(-)}(x)
\]  

with

\[
h^{(+)}(x) = \frac{1 + v'}{2} e^{imv \cdot x} \left( \cdots e^{-\frac{1}{2m^2} O^A_3 e^{-\frac{1}{2m^2} O^A_2 e^{-\frac{1}{2m} O^A_1}}} Q(x) \right)
\]  

\[
h^{(-)}(x) = \frac{1 - v'}{2} e^{-imv \cdot x} \left( \cdots e^{-\frac{1}{2m^2} O^A_3 e^{-\frac{1}{2m^2} O^A_2 e^{-\frac{1}{2m} O^A_1}}} Q(x) \right).
\]

The differential operators \( O^A_i \) are determined so that the Dirac Lagrangian

\[
\mathcal{L}_{\text{Dirac}} = \bar{Q}(i\not{D} - m)Q,
\]

does not contain, when expressed in terms of the fields \( h^{(\pm)}(x) \), any terms which couple \( h^{(+)}(x) \) with \( h^{(-)}(x) \). This is done by removing order by order in \( 1/m \), through successive field transformations, every term in the Lagrangian which anticommutes with \( v \). These represent the generalization of the “odd” operators in the usual Foldy–Wouthuysen procedure. The first few \( O^A_i \) are

\[
O^A_1 = i\not{D}_\perp
\]

\[
O^A_2 = -\frac{1}{2} (\not{D}_\perp \not{P}_\parallel + \not{P}_\parallel \not{D}_\perp)
\]

\[
O^A_3 = -i(\frac{1}{3} \not{D}^2 + \frac{1}{4} \not{P}_\perp \not{P}_\parallel + \frac{1}{2} \not{P}_\parallel \not{P}_\perp + \frac{1}{4} \not{P}^2 \not{D}_\perp).
\]
All $O_i^A$ satisfy $\{O_i^A, y\} = 0$ and contain, in the rest frame of $v$, only spatial derivatives which act on the $Q$ field in (34,35).

It can be seen that this is a generalization of the usual Foldy-Wouthuysen transformation which decouples the upper two components of the Dirac spinor from the lower two ones. In a similar way, the transformation (33-35) decouples those components of the Dirac spinor which correspond to a particle moving with velocity $v$ from its components corresponding to an antiparticle moving with the same velocity. In the rest frame of $v$ the usual Foldy-Wouthuysen transformation is recovered and the equation of motion for the Heisenberg field $h^{(+)}(x)$ becomes simply the Pauli equation. Motivated by the usual physical interpretation of the Foldy–Wouthuysen procedure, one can say in a loose way that heavy quark effective theory represents a “non-relativistic approximation about some arbitrary velocity $v$”. This has been formulated in a precise language to order $1/m^0$ by Grinstein \[13\] (in the rest frame of $v$).

The recursive algorithm for determining the operators $O_i^A$ and thereby the Lagrangian of the effective theory given in [6] is very simple and lends itself easily to implementation in a symbolic manipulation language (e.g. FORM [14]). The effective Lagrangian obtained in this way to order $O(1/m^3)$ reads (only for the particle field $h^{(+)}(x)$, which will be written simply as $h(x)$)

$$\mathcal{L} = \sum_{i=0}^{\infty} \frac{1}{m^3} \mathcal{L}_i$$

with

$$\mathcal{L}_0 = \bar{h}i\gamma^0 D\gamma^0 h$$

$$\mathcal{L}_1 = -\frac{1}{2} \bar{h} \left( D^2 - (v \cdot D)^2 + \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu} \right) h$$

$$\mathcal{L}_2 = -\frac{g}{8} (\bar{h}v^\mu t^a h) D^\nu F_{\mu\nu}^a + \frac{i g}{8} (\bar{h} \sigma^{\alpha\nu} v^\mu t^a h) D_\alpha F_{\mu\nu}^a + \frac{i g}{4} (\bar{h} \sigma^{\alpha\nu} v^\mu F_{\mu\nu} D_\alpha h)$$

$$\mathcal{L}_3 = \frac{1}{8} \bar{h} \left( D^2 - (v \cdot D)^2 + \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu} \right)^2 h.$$  

The equation of motion for the $h(x)$ field in the rest frame of the heavy quark is

$$i \frac{\partial h}{\partial t} = (g A^{0a} t^a + \frac{(\vec{P} - g \vec{A}^{a} t^a)^2}{2m} - \frac{g}{2m} \vec{\sigma} \cdot \vec{B}^a t^a - \frac{g}{8m^2} (\text{div} \vec{E}^a + g f_{abc} \vec{A}^b \cdot \vec{E}^c) t^a$$

$$- \frac{i g}{8m^2} \vec{\sigma} \cdot \text{rot} \vec{E}^a t^a - \frac{i g^2}{8m^2} f_{abc} \vec{\sigma} \cdot (\vec{A}^b \times \vec{E}^c) t^a - \frac{g}{4m^2} \vec{\sigma} \cdot \vec{E}^a t^a \times (\vec{P} - g \vec{A}^b t^b)$$

$$- \frac{1}{8m^3} \left[ (\vec{P}^a - g \vec{A}^a t^a)^2 - g \vec{\sigma} \cdot \vec{B}^a t^a \right]^2 + \frac{g^2}{8m^3} \left[ \vec{E}^a t^a \cdot \vec{E}^b t^b + \frac{i}{2} f_{abc} \vec{\sigma} \cdot (\vec{E}^a \times \vec{E}^b) t^c \right] h$$

$$+ \mathcal{O}(1/m^4),$$
with \( \vec{P} = -i\nabla \), \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf_{abc} A_\mu^b A_\nu^c \) and \( D_\lambda F_{\mu\nu}^a = \partial_\lambda F_{\mu\nu}^a - gf_{abc} A_\lambda^b F_{\mu\nu}^c \). The electric and magnetic color fields are defined as \( E_{ia} = F_{i0}^a \), \( B_{ia} = -\frac{1}{2} \epsilon^{ijk} F_{jk}^a \).

This agrees with the known form of the Pauli equation [15, 16], (though, to our knowledge, the complete \( O(1/m^3) \) contribution is new) although the effort involved in deriving it, even in QCD, is appreciably smaller than the one required by the traditional method using unitary transformations. In the Appendix are given the operators \( O_{A,5} \) which allow the determination of the HQET Lagrangian and of the Pauli equation up to \( O(1/m^5) \). The corresponding expressions have been calculated up to \( O(1/m^{11}) \) and will be presented elsewhere [17].

In Sect. 2 we have shown that the number operator in the HQET of [6] has the simple form shown in Eq. (13). An important consequence of this fact is that its expectation value in the unperturbed state \( |M(v)\rangle \) is equal to 1. Actually, this property is preserved to all orders in \( 1/m \). To see this, we note from (33-35) that the number operator can be written as

\[
\hat{N} = \int d^3x \bar{Q}\gamma_0 Q =
\int d^3x \bar{h}e^{imv \cdot x}\gamma_0 \left( \gamma_0 - \frac{1}{2m} \gamma_{\mu} O_{A,1}^\mu \gamma_{\mu} O_{A,2}^\mu \gamma_{\mu} O_{A,3}^\mu \gamma_{\mu} O_{A,4}^\mu \right) e^{imv \cdot x} h
\]

Here we have made use of the above-mentioned property of the differential operators \( O_{A,i}^\mu \) of containing only spatial derivatives in order to integrate by parts and drop the surface term. In the second step the relation \( \{O_{A,i}^\mu, \gamma_0\} = 0 \) has been repeatedly used (remember \( \gamma_0 = \gamma_0 \)). This proves the desired result.

Another crucial property of the HQET in [6] which was proven at order \( 1/m^2 \) in Sect. 2 is the fact that the number operator in the full theory is a constant of motion order by order in \( 1/m \). By making use of the preceding result concerning the form of the number operator, one can see that this is indeed the case to any order in \( 1/m \). The proof for this is similar to the one given in Eq. (15), where the main ingredient was the absence of the time derivatives in the HQET Lagrangian (in the rest frame of \( v \)). It has been proved in [3] that the method of constructing the effective Lagrangian does not introduce generally any \( v \cdot \partial \) derivatives on the heavy quark field. Any extra number of derivatives can be reexpressed in terms of \( F_{\mu\nu} \) and its derivatives. Thus, the conservation law of \( \hat{N} \) is guaranteed to hold at any order in \( 1/m \).

The conservation of the number operator order by order in \( 1/m \) has one interesting consequence, in that its expectation value remains equal to unity order by order in \( 1/m \). This follows from the nonrenormalization theorem for conserved quantities, according to which matrix elements of conserved operators have the same values as
in the noninteracting theory. We have just proved that the expectation value in the “free” theory is unity, so that this is also the value to each order in $1/m$. This is, of course, what one expects from the complete theory (QCD), and is reassuring for the consistency of our scheme.

The sequence of transformations (33-35) can be regarded as the effect of an unitary transformation on the Hilbert space of the initial (QCD) theory. Again, we consider the rest frame of $v$. The respective unitary transformation is, of course, the usual Foldy–Wouthuysen transformation in second–quantized form. We have

$$|M_{HQET}\rangle = U|M_{QCD}\rangle, \quad h = h^{(+)} + h^{(-)} = UQU^\dagger,$$

with

$$U = \exp\left[ -imx_0(N^{(+)}) - N^{(-)} \right] \cdots \exp\left[ \frac{1}{2m^2} \int d^3x Q'^\dagger Q' \right] \exp\left[ \frac{1}{2m} \int d^3x Q^\dagger Q \right].$$

Here

$$Q'(\vec{y}) = e^{\frac{1}{2m} \int d^3x Q^\dagger Q} Q(\vec{y}) e^{-\frac{1}{2m} \int d^3x Q^\dagger Q} = e^{-\frac{1}{2m} \int d^3x Q^\dagger Q} Q(\vec{y}),$$

$$Q''(\vec{y}) = e^{\frac{1}{2m^2} \int d^3x Q'^\dagger Q'} Q'(\vec{y}) e^{-\frac{1}{2m^2} \int d^3x Q'^\dagger Q'} = e^{-\frac{1}{2m^2} \int d^3x Q'^\dagger Q'} Q'(\vec{y}),$$

give the effect of the successive transformations on the heavy quark field operator and

$$h^{(+)}(\vec{y}) + h^{(-)}(\vec{y}) = e^{-imx_0 \int d^3x Q^{(\infty)} Q^{(\infty)}} Q^{(\infty)}(\vec{y}) e^{imx_0 \int d^3x Q^{(\infty)} Q^{(\infty)}} = e^{imv \cdot x} Q^{(\infty)}(\vec{y})$$

are the number operators for heavy particles and antiparticles, respectively. In these relations all the operators are taken at equal times and $Q^\dagger = Q\gamma_0$. The second equality in Eqs. (49-51) follows through the use of the canonical anticommutation relations for the heavy quark field. Here an important point is that these relations are left invariant by the successive unitary transformations.

We can see now that the relation anticipated in the previous Section

$$\langle 0|\bar{q}Q|M_{QCD}\rangle = \langle 0|\bar{q}h|M_{HQET}\rangle$$

holds indeed true. This shows that the field $h(x)$ defined by the sequence of transformations (49-51) has the same normalization as the QCD field $Q(x)$ and therefore
the results obtained with the HQET formulated in terms of it are the correct ones. On the other hand, as already remarked, the field corresponding to the usual HQET (1-3) has a different normalization.

One curious property of the effective Lagrangian (40) is the fact that the number operators \( N^+ \) and \( N^- \) for heavy quarks and respectively, antiquarks, are separately conserved to any order in \( 1/m \). This can be traced back to the invariance of the Lagrangian under the field transformations \( h \rightarrow \exp(i\alpha)h \) and \( h \rightarrow \exp(i\gamma\alpha)h \), the generators of which are related to the two number operators. But this is surprising since we know that in the full theory (QCD), it is only the difference of these two operators which is conserved (electric charge conservation), but not each of them. That is, HQET has one more integral of motion than the underlying theory. This seems to indicate the fact that the predictions of HQET do not approach arbitrarily close those of QCD when infinitely many orders in \( 1/m \) are taken into account. Namely, effects connected with heavy quark pair production are not described, to all orders in \( 1/m \). See also Ref. [18]. We mention that this limitation applies only to the tree-level HQET Lagrangian and it is possible that matching corrections might remedy this deficiency. If not, this would signal a limitation in the predictive power of HQET for very large orders in \( 1/m \).

5. Renormalization of the \( 1/m^2 \) order Lagrangian

There is a class of contributions at order \( \mathcal{O}(1/m^2) \) which are not described by the HQET Lagrangian (43). These arise in the full theory (QCD) from graphs containing one heavy quark in closed loops. They are therefore quantum effects which have not been accounted for in the tree level (classical) effective Lagrangian (40). Consequently, the missing contributions must be added by hand as gluonic operators with coefficients which have to be determined from a comparison (matching) with the full QCD graphs. For simplicity we treat only the case when no light quarks are present. There are three possible operators which can appear when inserted at zero total momentum:

\[
\mathcal{O}_0 = \frac{1}{2}(D^\mu F_{\mu\nu}^a)(D_\lambda F^{a\lambda\nu}) - \frac{1}{2}g(D^\mu F_{\mu
u}^a)(\bar{h}\gamma^\nu t^a h) \\
\mathcal{O}_1 = \frac{1}{4}F^a_{\mu\nu}D^2 F^{a\mu\nu} \\
\mathcal{O}_2 = \frac{1}{4}gf_{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}.
\]

The operator \( \mathcal{O}_0 \) vanishes when the equations of motion for the gluon field are used. By making use of the Bianchi identity

\[
D_\lambda F_{\mu\nu}^a + D_\mu F_{\nu\lambda}^a + D_\nu F_{\lambda\mu}^a = 0
\]
one can see that the following relation between these operators holds true, when 
inserted at zero momentum:
\[
\mathcal{O}_1 + 2\mathcal{O}_2 + \frac{1}{2} (D^\mu F^a_{\mu
u}) (D_\lambda F^{a\lambda\nu}) = 0.
\] (58)

The HQET Lagrangian (43) supplemented by these terms reads
\[
\mathcal{L}_2 = \frac{1}{m^2} \sum_{i=1}^4 c_i(\mu) \mathcal{O}_i.
\] (59)

Here we have denoted by \(\mathcal{O}_{3,4}\) the spin-symmetry violating terms in (43):
\[
\mathcal{O}_3 = \frac{ig}{8} (\bar{h}\sigma^{\nu\mu} i a h) D_\alpha F^a_{\mu\nu}
\] (60)
\[
\mathcal{O}_4 = \frac{ig}{4} (\bar{h}\sigma^{\nu\mu} F_{\mu\nu} D_\alpha h)
\] (61)

Additional \(\mathcal{O}(1/m^2)\) contributions also appear from double insertions of \(\mathcal{O}(1/m)\) terms in the effective Lagrangian:
\[
\mathcal{O}_5 = i \int d^4x T(\bar{h}(iD)^2 h)(x)(\frac{g}{2} \bar{h}\sigma \cdot Fh)(0)
\] (62)
\[
\mathcal{O}_6 = \frac{i}{2} \int d^4x T(\frac{g}{2} \bar{h}\sigma \cdot Fh)(x)(\frac{g}{2} \bar{h}\sigma \cdot Fh)(0)
\] (63)
\[
\mathcal{O}_7 = \frac{i}{2} \int d^4x T(\bar{h}(iD)^2 h)(x)(\bar{h}(iD)^2 h)(0)
\] (64)

The coefficients \(c_i(\mu)\) obey a renormalization group equation
\[
\mu \frac{d}{d\mu} c_i(\mu) + \gamma_{ij} c_j(\mu) = 0.
\] (65)

The anomalous dimension matrix \(\gamma_{ij}\) with \(i, j = 1 - 7\) has the form
\[
\hat{\gamma} = \frac{g^2}{16\pi^2} \begin{pmatrix}
-22 & -16 & 0 & 0 & 0 & 0 & 0 \\
9 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -12 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 48 & 48 & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (66)

The nonlocal operators \(\mathcal{O}_{5,6,7}\) are not required by the local ones as counterterms and therefore their coefficients \(c_{5,6,7}(\mu)\) evolve independently from the latter. Their
anomalous dimensions (the lower \(3 \times 3\) diagonal block) have been calculated previously [20, 10]. The renormalization of the dimension–6 gluon operator \(O_2\) has been studied in the presence of a light quark [22, 23] in connection with the renormalization of the corresponding condensate appearing in QCD sum rules. We agree with the results of these works for the case of a massless quark. However, when considering a HQET heavy quark, the renormalization is different. The diagrams to be calculated for the renormalization of \(O_1, O_2\) are shown in Fig.1, those for \(O_3, O_4\) in Fig.2 and in Fig.3. The necessary algebraic manipulations have been done with the help of FORM [14]. We have used for the gluon field a Feynman version of the background field gauge [21], where the combination \(gA_\mu\) is not renormalized. In general, some other operators will be needed as counterterms, which vanish when the equation of motion for the heavy quark is used:

\[
O_8 = \frac{ig}{2} \bar{h} \sigma \cdot (F(v \cdot D)h - \frac{ig}{2} \bar{h}(D \cdot v)\sigma \cdot Fh) \quad (67)
\]

\[
O_9 = i\bar{h} D^2(v \cdot D)h - i\bar{h}(D \cdot v)D^2 h \quad (68)
\]

\[
O_{10} = \bar{h}(iv \cdot D)^2 h \quad (69)
\]

Their contributions have to be properly separated. Even if these operators do not contribute to the hyperfine splittings, they do appear when considering corrections to the matrix elements of currents. Therefore, a complete study of the \(O(\alpha_s/m^2)\) order corrections should take them into account.

The problem of the renormalization of the \(O(1/m^2)\) effective Lagrangian has been treated previously by Lee [1]. Our results do not agree with his. For example, we find that the operators \(O_{3,4}\) are renormalized always in the combination \(O_3 + O_4\), whereas in [1] they appear together as \(2O_3 + O_4\) (denoted as \(\hat{O}_2\)). Unfortunately, this combination is not even hermitian (whereas \(O_3 + O_4\) is). Its expectation values on the other hand have a direct physical interpretation in terms of hyperfine level splittings which are clearly real quantities.

The coefficients \(c_i\) at the matching scale \(\mu = m\) are

\[
c_0(m) = -\frac{1}{4}, \quad c_1(m) = -\frac{1}{4} + \frac{g^2}{120\pi^2}
\]

\[
c_2(m) = -\frac{1}{2} + \frac{13g^2}{720\pi^2}, \quad c_3(m) = 1 \quad (70)
\]

\[
c_4(m) = 1, \quad c_5(m) = -\frac{1}{4}, \quad c_6(m) = \frac{1}{4}
\]

Here the leading terms of the coefficients \(c_{0,1,2}(m)\) have been determined by re-expressing the first term in (43) in terms of \(O_{0,1,2}\) by making use of (58). The contributions proportional to \(g^2\) in \(c_{1,2}(m)\) constitute the quantum corrections referred to at the beginning of this section. They have been obtained by computing
the QCD graphs in Fig.4, where the fermion in the loop is a dynamical one of mass \( m \).

To one-loop order, the combinations of \( O_{1,2} \) which are multiplicatively renormalized are

\[
\begin{align*}
O'_1 &= O_1 + 2O_2 \\
O'_2 &= 9O_1 + 8O_2
\end{align*}
\]

with the anomalous dimensions \( \gamma'_1 = -\frac{4\alpha_s^2}{27\pi^2}, \gamma'_2 = -\frac{7\alpha_s^2}{8\pi^2} \) and the initial conditions \( c'_1(m) = -\frac{1}{4} + \frac{23g^2}{400\pi^2}, c'_2(m) = -\frac{g^2}{720\pi^2} \). The matrix elements of \( O'_1 \) can be related via (58) to those of the first term in (43). Therefore, as far as its matrix elements are concerned, the latter operator can be considered as being multiplicatively renormalizable with the anomalous dimension \( \gamma'_1 \). Summing up the leading logarithms in \( c'_1(\mu) \) and \( c_{3,4}(\mu) \) with the help of the renormalization group equation (65) and the initial conditions (70) gives the effective Lagrangian \( \mathcal{L}_2 \) for an arbitrary \( \mu < m \):

\[
\mathcal{L}_2 = -\frac{\alpha_s(m)}{1800\pi} \left( \frac{\alpha_s(\mu)}{\alpha_s(m)} \right)^{\frac{3}{2}} O'_2 + \left( \frac{\alpha_s(\mu)}{\alpha_s(m)} \right)^{\frac{3}{2}} \frac{9}{8} (\bar{h}v^\mu t^a h) D^\mu F^a_{\mu\nu}
\]

\[
+ \left[ 2 \left( \frac{\alpha_s(\mu)}{\alpha_s(m)} \right)^{-\frac{3}{2}} - 1 \right] (O_3 + O_4).
\]

(73)

Here we have used the running of the coupling \( \alpha_s \) proper to a pure gauge field.

It is interesting to observe that the first term is related to what we would have obtained if we had “completely” integrated out the heavy quark. In the absence of any other (light) quarks, there is only one independent dimension-6 operator which can appear into an Euler–Heisenberg Lagrangian for the gluon field, which can be chosen to be \( O_2 \). Since now the gluon field is a free one, Eq. (58) gives \( O'_2 = -10O_2 \). Substituting this into (73) yields the well-known result [24]

\[
\mathcal{L}^{E-H} = \frac{g^2}{720\pi^2 m^2} O_2 + \mathcal{O}(1/m^4).
\]

(74)

Also, \( \gamma'_2 \) is identical with the anomalous dimension of \( O_2 \) for a pure gauge field [22].

However, keeping the first term in (73) is an inconsistent procedure, as there are contributions of comparable magnitude into the second term which were neglected. They are the non-leading logarithms in \( c'_1 \) of the form \( \alpha_s(\alpha_s \ln(\mu/m))^n \), whose summation by the renormalization group will require the knowledge of the anomalous dimension matrix (66) to two-loop order. This is somewhat a disappointing feature, since, in the view of the preceding paragraph, they constitute the genuine vacuum

\[\text{This equality is to be understood in the sense that the two operators have identical matrix elements.}\]
polarization contribution to the heavy quark effective Lagrangian. Nevertheless, such a two-loop calculation is beyond the scope of this paper.

Unfortunately, there are reasons to expect that the results for the anomalous dimensions (66) will be changed when light quarks are included. There are 19 new dimension–6 operators including both light- and heavy-quark fields and even purely light-quark operators which can be induced by renormalization in the presence of 3 massless quark species. Thus, there are 6 operators of the form

\[(\bar{q} q)(\bar{h} h), \quad (\bar{q} \gamma_\mu q)(\bar{h} v_\mu h), \quad (\bar{q} \sigma_{\mu\nu} q)(\bar{h} \sigma_{\mu\nu} h)\]

\[(\bar{q} t^a q)(\bar{h} t^a h), \quad (\bar{q} \gamma_\mu t^a q)(\bar{h} v_\mu t^a h), \quad (\bar{q} \sigma_{\mu\nu} t^a q)(\bar{h} \sigma_{\mu\nu} t^a h)\]

(75)

with the two light quark fields \(\bar{q} q, q = (u, d, s)\) in a SU(3)-singlet combination. There are 12 4-light-quark operators of the following two types

\[(\bar{q} \Gamma q)(\bar{q} \Gamma q), \quad (\bar{q} t^a q)(\bar{q} t^a q)\]

(76)

with two possibilities for each of them, corresponding to the two occurrences of 1 in the product of representations \((1\oplus 8)\otimes(1\oplus 8)\). Here \(\Gamma = (1, \gamma_\mu, \sigma_{\mu\nu})\). And finally, 1 operator of the type \(i\bar{q}\{F^{\mu\nu}, \gamma_\mu D_\nu\}q\). For a similar calculation in QED with one light and one heavy fermion, see [25]. Therefore the results of this section are mainly of academic interest.

6. Applications

In this section we discuss some of the implications of our results. One possible application would be to the hyperfine splittings of hadrons containing one heavy quark. As is well-known, in the infinite quark-mass limit \(m_Q \to \infty\), the states of such hadrons can be labelled by simultaneously specifying the angular momentum and parity \(s_\pi \ell \ell\) of its light degrees of freedom and the total angular momentum \(\vec{S} = \vec{s}_Q + \vec{s}_\ell\) (with \(\vec{s}_Q\) the heavy-quark spin). Since in this limit the spin of the heavy quark decouples, there exists a degeneracy between the two states with the same \(s_\ell^\pi\) but different \(S = s_\ell \pm \frac{1}{2}\) (for \(s_\ell \neq 0\)). The effect of a finite heavy quark mass is to lift this degeneracy, leaving us with the observed hadronic spectrum consisting of closely spaced doublets. The splittings within these doublets are mainly a \(1/m_Q\)-effect due to the spin-symmetry violating term in the effective Lagrangian \(L_1\) (42). At order \(1/m_Q^2\), an additional contribution is expected to appear from the corresponding terms into \(L_2\) containing \(\sigma_{\mu\nu}\). We can get some insight into the size of these contributions by examining the role played by these terms in the constituent quark model. In the rest frame of the heavy-quark, they can be seen to be responsible for the well-known spin-orbit interaction between the spin of the heavy quark and
its orbital angular momentum. Writing the spin-dependent part of the Breit-Fermi Hamiltonian for a pair of a heavy- and light-quarks as \[26\]

\[
H = (\alpha e_Q e_q + k\alpha_s)(H_{\text{spin-spin}} + H_{\text{spin-orbit}}) 
\]

with

\[
H_{\text{spin-spin}} = -\frac{8\pi}{3m_Q m_q} \delta^{(3)}(\vec{r})\vec{s}_Q \cdot \vec{s}_q 
+ \frac{1}{2|\vec{r}|^3 m_Q m_q} \left( \frac{\vec{s}_Q \cdot \vec{s}_q - 3(\vec{s}_Q \cdot \vec{r})(\vec{s}_q \cdot \vec{r})}{|\vec{r}|^2} \right) 
\]

\[
H_{\text{spin-orbit}} = \frac{1}{2|\vec{r}|^3 m_Q^2} \vec{r} \times \vec{p}_Q \cdot \vec{s}_q + \frac{1}{2|\vec{r}|^3 m_q^2} \vec{r} \times \vec{p}_q \cdot \vec{s}_Q 
- \frac{1}{|\vec{r}|^3 m_Q m_q} \vec{r} \times \vec{p}_Q \cdot \vec{s}_q + \frac{1}{|\vec{r}|^3 m_Q m_q} \vec{r} \times \vec{p}_Q \cdot \vec{s}_Q, 
\]

then it is the first term in \(H_{\text{spin-orbit}}\) which arises from the above-mentioned terms in \(L_2\). Here \(e_Q, m_Q, e_q, m_q\) are respectively, the heavy and light quark electric charges and masses and \(k\) is \(-\frac{4}{3}\) in mesons and \(-\frac{2}{3}\) in baryons. The constituent quark model suggests thus that the \(L_2\)-contribution to the mass splittings of s-wave mesons like \(m_{D^*} - m_D\) or \(m_{B^*} - m_B\) is expected to be very small, being exactly zero in the valence quark approximation. This provides an explanation for the fact that the observed hyperfine mass splittings of these mesons are well described by a simple \(1/m_Q\) law (including the leading order logarithmic corrections) [5]. It should be manifest on the other hand in the corresponding hyperfine splittings of the p-wave mesons which therefore might display larger deviations from a simple \(1/m_Q\) law. Even for this case, the spin-spin forces will probably give the dominant contribution.

The \(L_2\)-terms can be expected to play an important role for the case of the first excited negative-parity \(I = 0\) baryons with \(s_{\ell_{\pi}} = 1^-\). In a constituent quark model \[28, 29\] these states are described as having the light quark pair \((u, d)\) into a spatially symmetrical state and their total spin is zero. If the inter-quark forces are taken to be of a harmonic oscillator type, the relative angular momentum of the two light quarks is zero and the resulting diquark “looks” like a scalar particle which orbits about the heavy quark in a p-wave. When the heavy quark spin is added, two baryon states will result \(\Lambda^{\frac{3}{2}}-\) and \(\Lambda^{\frac{1}{2}}-\). The only term at \(\mathcal{O}(1/m_Q)\) responsible for the hyperfine splitting between these two states is the last one in \(H_{\text{spin-orbit}}\) \[79\]. One may thus

\[7\] Of course, there are also \(1/m^2\)-order contributions arising in second-order perturbation theory. If one takes the usual point of view according to which the Breit-Fermi Hamiltonian is only to be used as a first-order perturbation, they vanish (for a review see \[27\]).

\[8\] This statement is not specific to the harmonic force case. It is only based on the symmetry of the wave-function under a permutation of the two light quarks.
expect the $L_2$–term (the first one in (79)) to contribute an appreciable fraction to the hyperfine splitting of these baryons.

Another application of our formalism is connected to the possibility of a model-independent extraction of the Kobayashi-Maskawa matrix element $V_{cb}$ as described in [30]. This method is based on the absence of $1/m_c$ corrections to the form-factor $h_{A_1}(1)$ describing the matrix element of the axial current $A_\mu = \bar{c}\gamma_\mu\gamma_5b$ for the transition $\bar{B} \to D^*e\bar{\nu}$:

$$\langle D^*(v', \epsilon')|\bar{c}\gamma_\mu\gamma_5b|\bar{B}(v)\rangle = \frac{1}{\sqrt{m_Bm_{D^*}}}
\left[ h_{A_1}(v \cdot v')(1 + v \cdot v')\epsilon_{\mu}^{*} - h_{A_2}(v \cdot v')(\epsilon'_{\mu}^{*} \cdot v)v_\mu - h_{A_3}(v \cdot v')(\epsilon'_{\mu} \cdot v)v'_\mu \right].$$

In the infinite-mass limit, $h_{A_1}(1) = 1$ and the corrections to this result appear only at order $1/m_c^2$ [2]. It has been argued in [3] that these corrections amount to about 2%. As mentioned in the Introduction, the calculation of [3] is based upon the theory defined by (1) and (3). We have seen that these relations are incorrect and have to be replaced by (10) and, respectively (12). These changes imply a different form for the axial current in the HQET which contains now two new contributions:

$$\Delta A_\mu = -\frac{1}{8m_c^2} \frac{\tilde{h}^{(c)}_{\mu\gamma_5\gamma_5}}{P_\perp^2} + \frac{\epsilon_c}{8m_b^2} \frac{\tilde{h}^{(b)}_{\mu\gamma_5\gamma_5}}{P_\perp^2} + O(\epsilon^3),$$

proportional to $1/m_c^2$ and respectively $1/m_b^2$. It is natural to ask how this change will affect the predictions of [3]. It turns out that, due to the particular way of evaluating the $1/m^2$ corrections used in [3], they remain unchanged. A simple way to see this is presented in the following. For notations we refer to [3].

The form–factor $h_{A_1}(1)$ at the equal-velocity point $v \cdot v' = 1$ is equal to

$$h_{A_1}(1) = 1 + \epsilon_c^2 \ell_2(1) + \epsilon_b^2 \ell_1(1) + \epsilon_c \epsilon_b (m_2(1) + m_9(1)) + O(\epsilon^3),$$

with $\epsilon_Q = 1/(2m_Q)$ and $\ell_{1,2}$, $m_{2,9}$ are functions of $v \cdot v'$. Vector current conservation gives two conditions on the values of these functions at the equal-velocity point $v \cdot v' = 1$:

$$2\ell_1(1) + (m_1(1) - m_8(1)) = 0$$

$$2\ell_2(1) + (m_4(1) - m_{11}(1)) + (m_5(1) - m_{12}(1)) = 0.$$

The $O(1/m^2)$ correction to $h_{A_1}(1)$ is

$$h_{A_1}(1) - 1 = (\epsilon_c - \epsilon_b)[\epsilon_c \ell_2(1) - \epsilon_b \ell_1(1)] + \epsilon_c \epsilon_b \Delta$$

with

$$\Delta = \ell_1(1) + \ell_2(1) + m_2(1) + m_9(1).$$
The constants $\ell_{1,2}(1)$ are determined from a comparison with the predictions of the Isgur, Scora, Grinstein and Wise model. This determination is independent of the precise form of the HQET used. $\Delta$ can be rewritten by making use of the relations (83-84) as

$$\Delta = -\frac{1}{2}(m_1(1)-m_8(1)) - \frac{1}{2}(m_4(1)-m_{11}(1)) - \frac{1}{2}(m_5(1)-m_{12}(1)) + (m_2(1) + m_9(1)).$$ (87)

Now it can be observed that the correction (81) only affects the internal structure of the $\ell_{1-6}$ form–factors (see (4.34) in [3]) and has no effect on the structure of the functions $m_{1-24}$. Therefore the prediction for the value of the constant $\Delta$ resulting from (87) remains valid.

However, any calculation based on first principles of $\ell_{1-6}$, such as with the help of QCD sum rules or on lattice, should give an incorrect result when the HQET (1-3) is being used.

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**Appendix**

We give here the differential operators $O_{4,5}^A$, needed for constructing the HQET Lagrangian up to $O(1/m^5)$. They are

\[
O_4^A = \frac{5}{16}p_\perp^2 p_\parallel + \frac{3}{16}p_\perp^2 p_\parallel p_\perp + \frac{3}{16}p_\perp p_\parallel p_\perp^2 + \frac{1}{8}p_\perp p_\parallel^3 \\
+ \frac{5}{16}p_\parallel p_\perp^3 + \frac{3}{8}p_\parallel p_\perp p_\parallel^2 + \frac{3}{8}p_\parallel^2 p_\perp p_\parallel + \frac{1}{8}p_\parallel^3 p_\perp
\]

\[
O_5^A = i\left(\frac{1}{5}p_\perp^5 + \frac{7}{32}p_\perp^3 p_\parallel^2 + \frac{1}{4}p_\perp^2 p_\parallel p_\perp p_\parallel + \frac{3}{32}p_\perp^2 p_\parallel^2 p_\perp + \frac{3}{16}p_\perp p_\parallel p_\perp^2 p_\parallel + \frac{3}{16}p_\parallel p_\perp p_\parallel p_\perp^2 p_\perp + \frac{3}{32}p_\parallel^2 p_\perp^3 p_\parallel \\
+ \frac{1}{16}p_\parallel p_\perp^4 + \frac{3}{8}p_\parallel p_\perp^3 p_\parallel + \frac{3}{16}p_\parallel p_\perp^2 p_\parallel p_\perp + \frac{1}{4}p_\parallel p_\perp^2 p_\parallel p_\perp^2 + \frac{1}{16}p_\parallel^4 p_\perp + \frac{7}{32}p_\parallel^2 p_\perp^3 + \frac{3}{8}p_\parallel^2 p_\perp^2 p_\parallel + \frac{1}{4}p_\parallel^3 p_\perp p_\parallel + \frac{1}{16}p_\parallel^4 p_\perp\right)
\] (89)
Figure Captions

Fig. 1 Diagrams contributing to the anomalous dimension matrix of $O_1$ and $O_2$. (a) and (b) have a symmetry factor 1/2 and (f) appears together with another diagram obtained by reversing the direction of the fermion line. In (d-g) the blob can only be an insertion of $O_1$.

Fig. 2 Inserting $O_3$ or $O_4$ in heavy quark vertex functions. Diagram (e) has a symmetry factor 1/2. Three other diagrams contribute which are obtained from (a,b,d) by reversing the direction of the fermion line.

Fig. 3 Double insertions of $L_1$. The square and the circle may represent either different or identical terms in (42). To each diagram which is asymmetrical with respect to its central vertical axis, its mirror image should be added.

Fig. 4 Diagrams needed for the matching conditions (70).
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