Altitude Terrain Guarding and Guarding Uni-Monotone Polygons

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Abstract
We show that the problem of guarding an $x$-monotone terrain from an altitude line and the problem of guarding a uni-monotone polygon are equivalent. We present a polynomial time algorithm for both problems, and show that the cardinality of a minimum guard set and the cardinality of a maximum witness set coincide. Thus, uni-monotone polygons are perfect; this result also extends to monotone mountains.

1 Introduction

Both the Art Gallery Problem (AGP) and the 1.5D Terrain Guarding Problem (TGP) are well known problems in Computational Geometry. In the AGP, we are given a polygon $P$ in which we have to place the minimum number of point-shaped guards, such that they cover all of $P$. In the 1.5D TGP, we are given an $x$-monotone chain of line segments in $\mathbb{R}^2$, the terrain $T$, on which we have to place a minimum number of point-shaped guards, such that they cover $T$.

Both problems have been shown to be NP-hard: Krohn and Nilsson [3] proved the AGP to be hard even for monotone polygons by a reduction from MONOTONE 3SAT, and King and Krohn [2] established the NP-hardness of both the discrete and the continuous TGP (with guards restricted to the terrain vertices or guards located anywhere on the terrain) by a reduction from PLANAR 3SAT.

The problem of guarding a uni-monotone polygon (an $x$-monotone polygon with a single horizontal segment as one of its two chains) and the problem of guarding a terrain with guards placed on a horizontal line above the terrain appear to be problems somewhere between the 1.5D TGP and the AGP in monotone polygons. We show that, surprisingly, both problems allow for a polynomial time algorithm: a simple sweep.

Moreover, we are able to construct a maximum witness set of the same cardinality as the minimum guard set for uni-monotone polygons. Hence, we establish the first non-trivial class of perfect polygons (the only earlier results concerned “rectilinear visibility” [6] and “staircase visibility” [4]).

One application of guarding a terrain with guards placed on a horizontal line above the terrain, the Altitude Terrain Guarding Problem (ATGP), comes from the idea of using

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drones to surveil a complete geographical area. Usually, these drones will not be able to fly arbitrarily high, which motivates to cap the allowed height for guards (and without this restriction a single sufficiently high guard above the terrain will be enough). Of course, eventually we are interested in working in two dimensions and a height, the 2.5D ATGP. One dimension and height, the ATGP, is a natural starting point to develop techniques for a 2.5D ATGP. However, the 2.5D ATGP—in contrast to the 1.5D ATGP—is NP-hard by a straight-forward reduction from the (2D) AGP: we construct a terrain such that we carve out a hole for the polygon’s interior and need to guard it from the altitude line at the “original” height, then we do need to find the minimum guard set for the polygon.

Roadmap. In Section 2 we formally introduce our problems and necessary definitions, and show some basic properties of our problems. In Section 3 we present our algorithm, prove that it computes an optimal guard set and that uni-monotone polygons are perfect, extend that result to monotone mountains, and show how we can yield a runtime of $O(n^2 \log n)$. Finally, we conclude in Section 4.

2 Notation and Preliminaries

Definition 2.1. A polygon $P$ is $x$-monotone if any line orthogonal to the $x$-axis has a simply connected intersection with $P$. A uni-monotone polygon $P$ is an $x$-monotone polygon, such that one of its two chains is a single horizontal segment.

W.l.o.g. we will assume the single horizontal segment to be the upper chain for the remainder of this paper; we denote this segment by $H$.

The lower chain of $P$, $LC(P)$, is defined by its vertices $V(P) = \{v_1, \ldots, v_n\}$ and has edges $E(P) = \{e_1, \ldots, e_{n-1}\}$ with $e_i = v_i v_{i+1}$. Unless specified otherwise, $n := |V(P)|$. Due to uni-monotonicity the vertices of $P$ are totally ordered w.r.t. their $x$-coordinates.

Definition 2.2. A point $p \in P$ sees or covers $q \in P$ if and only if $pq$ is fully contained in $P$. $V_P(p)$ is the visibility polygon ($V_P$) of $p$ in $P$ with $V_P(p) := \{q \in P \mid p \text{ sees } q\}$. For $G \subset P$ we abbreviate $V_P(G) := \bigcup_{g \in G} V_P(g)$.

Definition 2.3. A terrain $T$ is an $x$-monotone chain of line segments in $\mathbb{R}^2$ defined by its vertices $V(T) = \{v_1, \ldots, v_n\}$ that has edges $E(T) = \{e_1, \ldots, e_{n-1}\}$ with $e_i = v_i v_{i+1}$. Unless specified otherwise, $n := |V(T)|$. $v_i$ and $v_{i+1}$ are the vertices of the edge $e_i$, and $\text{int}(e_i) := e_i \setminus \{v_i, v_{i+1}\}$ is its interior. Due to monotonicity the points on $T$ are totally ordered w.r.t. their $x$-coordinates. For $p, q \in T$, we write $p \leq q$ ($p < q$) if $p$ is (strictly) left of $q$, i.e., has a (strictly) smaller $x$-coordinate.

Definition 2.4. An altitude line $A$ for a terrain $T$ is a horizontal segment located above $T$ (that is, the $y$-coordinate of all vertices is smaller than the $y$-coordinate of $A$), with the leftmost point vertically above $v_1$ and the rightmost point vertically above $v_n$, see Figure 2. The points on $A$ are totally ordered as well w.r.t. their $x$-coordinates, and we adapt the same notation as for two points on $T$: for $p, q \in A$, we write $p \leq q$ ($p < q$) if $p$ is (strictly) left of $q$, i.e., has a (strictly) smaller $x$-coordinate.
Definition 2.5. A point \( p \in \mathcal{A} \) sees or covers \( q \in T \) if and only if \([pq]\) is nowhere below \( T \). \( \mathcal{V}_T(p) \) is the visibility region of \( p \) with \( \mathcal{V}_T(p) := \{q \in T \mid p \text{ sees } q\} \). For \( G \subseteq \mathcal{A} \) we abbreviate \( \mathcal{V}_T(G) := \bigcup_{g \in G} \mathcal{V}_T(g) \). We also define the visibility region for \( p \in T \): \( \mathcal{V}_p := \{q \in \mathcal{A} \mid p \text{ sees } q\} \).

Definition 2.6. For an edge \( e \in P \) or \( e \in T \) the strong visibility polygon is the set of points that see all of \( e \) and is defined as \( \mathcal{V}_p^s(e) := \{p \in P : \forall q \in e \ p \text{ sees } q\} \) and \( \mathcal{V}_T^s(e) := \{p \in \mathcal{A} : \forall q \in e \ p \text{ sees } q\} \). The weak visibility polygon of an edge \( e \) is the set of points that see at least one point on \( e \) and is defined as \( \mathcal{V}_p^w(e) := \{p \in P : \exists q \in e \ p \text{ sees } q\} \) and \( \mathcal{V}_T^w(e) := \{p \in \mathcal{A} : \exists q \in e \ p \text{ sees } q\} \).

Definition 2.7 (Altitude Terrain Guarding Problem). In the Altitude Terrain Guarding Problem (ATGP), abbreviated ATGP(\( T, \mathcal{A} \)), we are given a terrain \( T \) and an altitude line \( \mathcal{A} \). A guard set \( G \subseteq \mathcal{A} \) is optimal w.r.t. ATGP(\( T, \mathcal{A} \)) if \( G \) is feasible, that is, \( T \subseteq \mathcal{V}_T(G) \), and \( \|G\| = \text{OPT}(T, \mathcal{A}) := \min\{|C| \mid C \subseteq \mathcal{A} \text{ is feasible w.r.t. ATGP}(T, \mathcal{A})\} \).

Definition 2.8 (Art Gallery Problem). In the Art Gallery Problem (AGP), abbreviated AGP(\( G, W \)), we are given a polygon \( P \) and sets of guard candidates and witnesses \( G, W \subseteq P \). A guard set \( C \subseteq G \) is optimal w.r.t. AGP(\( G, W \)) if \( C \) is feasible, that is, \( W \subseteq \mathcal{V}_P(C) \), and \( |C| = \text{OPT}(G, W) := \min\{|C| \mid C \subseteq G \text{ is feasible w.r.t. AGP}(G, W)\} \). In general, we want to solve the AGP for \( G = P \) and \( W = P \), that is, AGP(\( P, P \)).

Definition 2.9. A set \( W \subseteq P \) (\( W \subseteq T \)) is a witness set if \( \forall \ w_i \neq w_j \in W \) we have \( \mathcal{V}_p(w_i) \cap \mathcal{V}_p(w_j) = \emptyset \). A maximum witness set \( W_{\text{opt}} \) is a witness set of maximum cardinality, \( |W_{\text{opt}}| = \max\{|W| \mid \text{witness set } W\} \).

Definition 2.10. A polygon class \( \mathcal{P} \) is perfect if the cardinality of an optimum guard set and the cardinality of a maximum witness set coincide for all polygons \( P \in \mathcal{P} \).

The following two lemmas show that for guarding uni-monotone polygons we only need guards on \( \mathcal{H} \), and coverage of \( LC(P) \) is sufficient to guarantee coverage of the entire polygon. Hence, the Altitude Terrain Guarding Problem (ATGP) and the Art Gallery Problem (AGP) in uni-monotone polygons are equivalent.

Lemma 2.11. Let \( P \) be a uni-monotone polygon, with optimal guard set \( G \). Then there exists a guard set \( G^\mathcal{H} \) with \( |G| = |G^\mathcal{H}| \) and \( g \in \mathcal{H} \forall g \in G^\mathcal{H} \). That is, if we want to solve the AGP for a uni-monotone polygon, w.l.o.g. we can restrict our guards to be located on \( \mathcal{H} \).

Proof. Consider any optimal guard set \( G \), let \( g \in G \) be a guard not located on \( \mathcal{H} \). Let \( g^\mathcal{H} \) be the point located vertically above \( g \) on \( \mathcal{H} \). Let \( p \in \mathcal{V}_P(g) \) be a point seen by \( g \).

Figure 1: A terrain \( T \) in black and an altitude line \( \mathcal{A} \) in red.
Let \( P \) be a uni-monotone polygon, let \( G \) be a guard set with \( g \in G \) that covers \( LC(P) \), that is, \( LC(P) \subseteq V_P(G) \). Then \( G \) covers all of \( P \), that is, \( P \subseteq V_P(G) \).

Proof. Assume there is a point \( p \in P \), \( p \notin LC(P) \) with \( p \notin V_P(G) \). Consider the point \( p^{LC} \), which is located vertically below \( p \) on \( LC(P) \). Let \( g \in G \) be a guard that sees \( p^{LC} \) (as \( p^{LC} \in LC(P) \) and \( LC(P) \subseteq V_P(G) \), there exists at least one such guard, possibly more than one guard in \( G \) covers \( p^{LC} \), see Figure 3). \( LC(P) \) does not intersect the line \( p^{LC}g \), and because \( P \) is uni-monotone the triangle \( \Delta(g, p, p^{LC}) \) is empty, hence, \( g \) sees \( p \); a contradiction.

Consequently, the ATGP and the AGP for uni-monotone polygons are equivalent; we will only refer to the ATGP in the remainder of this paper, with the understanding that all our results can be applied directly to the AGP for uni-monotone polygons.

The following lemma shows a general property of guards on the altitude line, which we will use (in parts implicitly) in several cases; it essentially says that if a guard cannot see a point to its right, no guard to its left will help him by covering this point:

**Lemma 2.13.** Let \( g \in A, p \in T, g < p \). If \( p \notin V_T(g) \) then \( \forall g' < g, g' \in A : p \notin V_T(g') \).
Figure 3: A uni-monotone polygon $P$. The guard $g \in G$ sees $p^{LC}$ the point on $LC(P)$ vertically below $p$. $LC(P)$ does not intersect $p^{LC}g$ and $P$ is uni-monotone, hence, $g$ sees $p$.

Figure 4: If $p \notin V_T(g)$ for $g \in A, p \in T, g < p$, then $p \notin V_T(g')$ for $g' \in A, g' < g$: if $g'$ sees $p$, the gray triangle $\Delta(g', p, p^A)$ is empty, which leads to a contradiction, as then also $g$ could see $p$.

Proof. Assume $g' \in A, g' < g$ could see $p$, that is, $p \in V_T(g')$, see Figure 4 for an illustration of the proof. Then $g'p$ lies on or over $T$, and the triangle $\Delta(g', p, p^A)$, with $p^A$ being the point located vertically above $p$ on $A$, is empty. We have $g' < g < p$, and as $x(p) = x(p^A)$ we have $g' < g < p^A$. Hence, $g'p$ is fully contained in the triangle $\Delta(g', p, p^A)$, and lies on or over $T$, that is, $g$ sees $p$, a contradiction. $\square$

Before we present our algorithm, we conclude this section with an observation that clarifies that guarding a terrain from an altitude is intrinsically different from terrain guarding, where the guards have to be located on the terrain itself. We repeat (and extend) a definition from [1]:

**Definition 2.14.** For a feasible guard cover $C$ of $T$ ($C \subset T$ for terrain guarding and $C \subset A$ for terrain guarding from an altitude), an edge $e \in E$ is critical w.r.t. $g \in C$ if $C \setminus \{g\}$ covers some part of, but not all of the interior of $e$. If $e$ is critical w.r.t. some $g \in C$, we call $e$ critical edge.

That is, $e$ is critical if and only if more than one guard is responsible for covering its interior.

$g \in C$ is a left-guard (right-guard) of $e_i \in E$ if $g < v_i$ ($v_{i+1} < g$) and $e_i$ is critical w.r.t. $g$. We call $g$ a left-guard (right-guard) if it is a left-guard (right-guard) of some $e \in E$.

**Observation 2.15.** For terrain guarding we have: Let $C$ be finite and cover $T$, then no $g \in C \setminus V(T)$ is both a left- and a right-guard, see Friedrichs et al. [1]. However, for guarding a terrain from an altitude, a guard $g$ on $A$ may be responsible to cover critical edges both to its left and to its right, that is, guards may be both a left- and a right-guard, see Figure 5.
Figure 5: A terrain shown in black and an altitude line \( A \) shown in red. Four guards, \( g_1, \ldots, g_4 \), of an optimal guard cover are shown as points. The green and the blue guard are both responsible for covering a critical edge both to their left and to their right: \( g_2 \) for both \( e_i \) and \( e_j \), and \( g_3 \) for both \( e_j \) and \( e_k \).

3 Sweep Algorithm

Our algorithm is a sweep, and informally it can be described as follows:

- We start with an empty set of guards, \( G = \emptyset \), and at the leftmost point of \( A \); all edges \( E(T) \) are completely unseen.
- We sweep along \( A \) from left to right and place a guard \( g_i \) (and add \( g_i \) to \( G \)) whenever we could no longer see all of an edge \( e' \) if we would move more to the right.
- We compute the visibility polygon of \( g_i \), \( V_T(g_i) \), and for each edge \( e = \{v, w\} \) partially seen by \( g_i \) (\( v \notin V_T(g_i), w \in V_T(g_i) \)), we split the edge, and only keep the open interval that is not yet guarded.
- Thus, whenever we insert a new guard \( g_i \) we have a new set of “edges” \( E_i(T) \) that are still completely unseen, and \( \forall f \in E_i(T) \) we have \( f \subseteq e \in E(T) \).
- We continue placing new guards until \( T \subseteq V_T(G) \).
- As we can define a witness set of \( |G| \) our guard set is optimal: we place a point witness on \( e' \) at the point \( p \) we would lose coverage of, if we had not placed guard \( g_i \).

In the remainder of this section we

- Describe how we split partly covered edges in Subsection 3.1
- Formalize our algorithm in Subsection 3.2
- Prove that our guard set is optimal, and how that proves that uni-monotone polygons are perfect in Subsections 3.3 and 3.4
- Show how that results extends to monotone mountains in Subsection 3.5
- Show how we can efficiently preprocess our terrain, and that we obtain an algorithm runtime of \( O(n^2 \log n) \) in Subsection 3.6.
is not yet guarded, see for example Figure 7(a), we identify the mark, \( m \), with mark points and mark the points, \( \text{vertex to the leftmost vertex; for each vertex we shoot a ray to all other vertices to its left} \). For each vertex we shoot a ray \( r \) and shoot a ray \( r \) to all other vertices to its left and shoot a ray \( r \) to all other vertices to its left.

\[ r \text{ includes } \text{we need to add to } C \text{ the opening point, and the soft opening point of an edge } e \text{ includes } \text{the two terrain vertices that defined the ray hitting the terrain at } (\text{it can see at least parts of } e \text{, a guard } g \text{ with } p_e^c \leq g < p_e^c \text{ can see parts, but not all of } e \text{, a guard } g \text{ with } p_e^c \leq g \leq p_e^c \text{ can see the complete edge } e \text{, and a guard } g \text{ with } g > p_e^c \text{ cannot see all of } e. \]

\[ 3.1 \text{ How to Split the Partly Seen Edges} \]

For each edge \( e \in E(T) \), in the initial set of edges we need to determine the point \( p_e^c \) that closes the interval on \( A \) from which all of \( e \) is visible. We denote the set of points \( p_e^c \forall e \in E(T) \) as the set of closing points \( C \), that is, \( C = \bigcup_{e \in E(T)} \{ p_e^c \in A : (e \subseteq V_T(p_e^c)) \land (e \not\subseteq V_T(p) \forall p > p_e^c, \ p \in A) \} \). The points in \( C \) are the rightmost points on \( A \) in the strong visibility polygon of the edge \( e \), for all edges. Analogously, we define the set of opening points \( O \) : for each edge the leftmost point \( p_e^c \) on \( A \), such that \( e \subseteq V_T(p_e^c) \), \( O = \bigcup_{e \in E(T)} \{ p_e^c \in A : (e \subseteq V_T(p_e^c)) \land (e \not\subseteq V_T(p) \forall p < p_e^c, \ p \in A) \} \). For each edge \( e \) the point in \( O \) is the leftmost point on \( A \) in the strong visibility polygon of \( e \).

Moreover, whenever we place a new guard, we need to split partly seen edges to obtain the new, completely unseen, possibly open, interval, and determine the point on \( A \) where we would lose coverage of this edge (interval). That is, whenever we split an edge we need to add the appropriate point to \( C \).

To be able to easily identify whether an edge \( e \) of the terrain needs to be split due to a new guard \( g \), we define the set of “soft openings” \( S \): the leftmost point on \( A \) in the weak visibility polygon of \( e \) (if \( g \) is to the right of this point (and to the left of the closing point) it can see at least parts of \( e \)). We define \( S = \bigcup_{e \in E(T)} \{ p_e^c \in A : (\exists q \in e, q \in V_T(p_e^c)) \land (\exists q \in e, q \in V_T(p) \forall p < p_e^c, \ p \in A) \} \). See Figure 6 for an illustration of the closing point, the opening point, and the soft opening point of an edge \( e \).

So, how do we preprocess our terrain such that we can easily identify the point on \( A \) that we need to add to \( C \) when we split an edge? We make an initial sweep from the rightmost vertex to the leftmost vertex; for each vertex we shoot a ray to all other vertices to its left and mark the points, mark points, where these rays hit the edges of the terrain. This leaves us with \( O(n^2) \) preprocessed intervals. For each mark point \( m \) we store the rightmost of the two terrain vertices that defined the ray hitting the terrain at \( m \), let this terrain vertex be denoted by \( v_m \). Note that for each edge \( e_j = \{v_j, v_{j+1}\} \) with \( v_{j+1} \) convex vertex, this includes \( v_{j+1} \) as a mark point.

Whenever the placement of a guard \( g \) splits an edge \( e \), such that the open interval \( e' \subset e \) is not yet guarded, see for example Figure 7(a), we identify the mark, \( m_{e'} \) to the right of \( e' \) and shoot a ray \( r \) from the right endpoint of \( e' \) through \( v_{m_{e'}} \) (the one we stored with \( m_{e'} \)). The intersection point of \( r \) and \( A \) defines our new closing point \( p_{e'}^c \), see Figure 7(b).
Figure 7: The terrain $T$ is shown in black, the altitude line $A$ is shown in red. The orange lines show the rays from the preprocessing step, their intersection points with the terrain define the mark points. Assume the open interval $e'$, shown in light green, is still unseen. To identify the closing point for $e'$ we identify the mark to the right of $e'$, $m_{e'}$, and shoot a ray $r$, shown in dark green, from the right end point of $e'$ through $v_{m_{e'}}$. The intersection point of $r$ and $A$ defines our new closing point $p_{c'e'}$.

3.2 Algorithm Pseudocode

The pseudocode for our algorithm is presented in Algorithm [1].

3.3 Minimum Guard Set

Lemma 3.1. The set $G$ output by Algorithm [1] is feasible, that is, $T \subseteq V_T(G)$.

Proof. Assume there is a point $p \in T$ with $p \notin V_T(G)$. $p \in e$ for some edge $e \in E(T)$. As $p$ is not covered, there exists no guard in $G$ in the interval $[p_{c_{e'}}, p_{c_e}]$ on $A$. Thus, $p_{c_e}$ is never the event point that defines the placement of a guard in lines 6,7. Moreover, as $\exists g_i: p_{e'} \leq g_i \leq p_{c_e}$, $e$ is never completely deleted from $E_g$ in lines 10–12. Consequently, for some $i$ we have $p_{e'} > g_i$ and $g_i \geq p_{c_e}$ (lines 14–22). As $p \notin V_T((G)$, we have $p \in e' \subset e$.

Again, because $p \notin V_T(G)$, $\exists g_j \in [p_{e'}, p_{c_e}] \subset A$, $j \geq i$. Due to line 6 no guard may be placed to the left of $p_{c_e}$, hence, there is no guard placed in $[p_{e'}, b]$ (where $b$ is the right end point of $A$). That is, $e'$ is never deleted from $E_g$, a contradiction to $G$ being the output of Algorithm [1].

To show optimality, we show that we can find a witness set $W$ with $|W| = |G|$. Given any witness set $W$ and a guard set $G$, $|W| \leq |G|$ holds. Hence, if we have equality, we can show that $G$ is minimum. We will place a witness for each guard Algorithm [1] places. First, we need two auxiliary lemmas:

Lemma 3.2. Let $c \in C$ be the closing point in line 6 of Algorithm [1] that enforces the placement of a guard $g_i$, and let $c$ be the closing point for a complete edge (and not just an edge interval). Then there exists an edge $e_j = \{v_j, v_{j+1}\} \in E(T)$ for which $c$ is the closing point, such that $v_{j+1}$ is a reflex vertex.

Proof. Assume for no edge $e_j v_{j+1}$ is a reflex vertex, pick the rightmost edge $e_j$ with $v_{j+1}$ being a convex vertex for which $c$ is the closing point. Let $E_c \subseteq E_g$ be the set of edges (and edge intervals) for which $c$ is the closing point ($e_j \in E_c$). As $c = p_{c_{e_j}}$ is the closing point that defines the placement of a guard we have $p_{c_{e_j}} > c \forall e \in E_g \setminus E_c$ (all other active closing
INPUT : Terrain $T$, altitude line $A$, its leftmost point $a$, sets $C, O, S$ of closing, opening, and soft opening points for all edges in $T$, all ordered from left to right.

OUTPUT: An optimal guard set $G$.

1. $E_g = E(T)$ // set of edges that still need to be guarded
2. $i := 1$
3. $g_0 := a$ // the point on $A$ before the first guard is $a$, $g_0$ is NOT a guard
4. while $E_g \neq \emptyset$ // as long as there are still unseen edges
do
  1. Sweep right from $g_{i-1}$ along $A$ until the first closing point $c \in C$ is hit
  2. Place $g_i$ on $c$, $G = G \cup \{g_i\}$, $i := i + 1$
  3. for all $e \in E_g$ // $g_i \neq p_e^c$ by construction
do
    if $p_e^c \leq g_i \leq p_e^{c'}$ then
      $E_g = E_g \setminus \{e\}$ // if all of $e$ is seen, delete it from $E_g$
      $C = C \setminus \{p_e^{c'}\}$ // and delete the closing point from the event queue
    else if $p_e^{c'} > g_i$ then
      if $p_e^c \leq g_i$ // if $g_i$ can see the right point of $e$
        then
          Shoot a visibility ray from $g_i$ onto $e$, let the intersection point be $r_e$
          // all points on $e$ to the right of $r_e$ (incl. $r_e$) are seen
          Identify the mark $m_e$ immediately to the right of $r_e$ on $e$
          Shoot a ray $r$ from $r_e$ through $v_{m_e}$
          Let $p_e^{c'}$ be the intersection point of $r$ and $A$ // $p_e^{c'}$ is the closing point for the still unseen interval $e' \subseteq e$
          $C = C \cup \{p_e^{c'}\} \setminus \{p_e^{c'}\}$ // insert and delete, keeping queue sorted
          $E_g = E_g \cup \{e'\} \setminus \{e\}$

Algorithm 1: Optimal Guard Set for ATGP

points are to the right of $c$). Because $v_{j+1}$ sees $c$: $\angle(v_j, v_{j+1}, c) \leq \angle(v_j, v_{j+1}, v_{j+2}) < 180^\circ$. Because $\angle(v_j, v_{j+1}, c) = \angle(v_j, v_{j+1}, v_{j+2})$ would imply that $e_j$ and $e_{j+1}$ are a single edge, we only consider the case $\angle(v_j, v_{j+1}, c) < \angle(v_j, v_{j+1}, v_{j+2})$. See Figure 8(a) for an illustration of this case. Let $q$ be the closing point for $e_{j+1}$. Then the two triangles $\Delta(v_j, v_{j+1}, c)$ and $\Delta(v_{j+1}, v_{j+2}, q)$ are empty (and we have $c \geq v_{j+1}$ and $q \geq v_{j+2}$). Because $T$ is $x$-monotone also the triangle $\Delta(c, q, v_{j+1})$ is empty, hence, $q \in V^*_T(e_j)$, a contradiction to $c$ being $e_j$’s closing point.

Lemma 3.3. Let $c \in C$ be the closing point in line 6 of Algorithm 1 that enforces the placement of a guard $g_i$, and let $c$ be the closing point for a complete edge (and not just an edge interval). Then there exists an edge $e_j = \{v_j, v_{j+1}\} \in E(T)$ for which $c$ is the closing
Figure 8: (a) If \( \angle(v_j, v_{j+1}, c) < \angle(v_j, v_{j+1}, v_{j+2}) \), the triangles \( \triangle(v_j, v_{j+1}, c) \), \( \triangle(v_{j+1}, v_{j+2}, q) \) (shown in light gray) and the triangle \( \triangle(c, q, v_{j+1}) \) (shown in dark gray) are empty. Hence, \( c \) is not the closing point for \( e_j \). (b) Placement of the witness in case \( c \) is only defined by edge intervals: we pick the rightmost such edge interval \( e' \), we have \( e' = [v_j, q] \) for some point \( q \in e_j, q \neq v_{j+1} \), and we place a witness at \( q' \).

Figure 9: Cases from the proof of Lemma 3.3: If \( v_j \) is a convex (a) or reflex (b) vertex of the chain \( g, v_j, v_{j+1} \).

point, such that \( v_{j+1} \) is a reflex vertex, and \( v_j \) is a convex vertex.

Proof. By Lemma 3.2 we know that there exists an edge \( e_j = \{v_j, v_{j+1}\} \in E(T) \) for which \( c \) is the closing point, such that \( v_{j+1} \) is a reflex vertex. Assume that for all these edges \( v_j \) is a reflex vertex as well. Then \( c \) cannot be the closing point for \( e_{j-1} \), and there exists a guard \( g \) with \( g < c \) that monitors \( (p, v_j] \subset e_{j-1} \). Hence, the triangle \( \Delta(g, p, v_j) \) is empty. We distinguish whether the chain \( g, v_j, v_{j+1} \) has \( v_j \) as a convex or a reflex vertex.

If \( v_j \) is a convex vertex of this chain, see Figure 9(a), then also the triangle \( \Delta(g, p, v_{j+1}) \) is empty. Thus, \( g \) also monitors \( e_j \), a contradiction, as \( e_j \) is not an edge for which the closing point \( c \) is still in the queue.

If \( v_j \) is a reflex vertex of this chain, see Figure 9(b), there has to exist a vertex \( w \), \( w > v_{j+2} > v_{j+1} \), that blocks the sight from any point to the right of \( c \) to \( v_{j+1} \) and makes \( c \) the closing point. Then all of the terrain between \( v_{j+1} \) and \( w \) lies completely below the line segment \( \overline{v_{j+1}, w} \). Hence, \( c \) cannot see \( v_{j+2} \) (in fact it cannot see \( (v_{j+1}, v_{j+2}] \subset e_{j+1} \)). As \( v_j \) is a reflex vertex of the chain \( g, v_j, v_{j+1} \), \( g \) cannot see \( v_{j+2} \) either. Thus, the closing point for \( e_{j+1} \) is still in the queue, and to the left of \( c \), a contradiction to \( c \) being the closing point that is chosen in line 6 of Algorithm 1.

Now we can define our witness set:

Lemma 3.4. Given the set \( G \) output by Algorithm 1, we can find a witness set \( W \) with \( |W| = |G| \).
Proof. We consider the edges or edge intervals, which define the closing point \( c \in C \) that leads to a placement of guard \( g_i \) in lines 6,7 of Algorithm 1.

If \( c \) is defined by some complete edge \( e_j \in E(T) \), let \( E_c \subseteq E_g \) be the set of edges for which \( c \) is the closing point. We pick the rightmost edge \( e_j \in E_c \) such that \( v_j \) is a convex vertex and \( v_{j+1} \) is a reflex vertex, which exists by Lemma 3.3 and choose \( w_i = v_j \).

Otherwise, that is, if \( c \) is only defined by edge intervals, we pick the rightmost such edge interval \( e' \subset e_j \). Then \( e' = [v_j, q] \) for some point \( q \in e_j, q \neq v_{j+1} \), and we place a witness at \( q' \), a point \( \varepsilon \) to the left of \( q \) on \( T \): \( w_i = q' \), see Figure S(b).

We define \( W = \bigcup_{i=1}^{G} w_i \). By definition \( |W| = |G| \), and we still need to show that \( W \) is indeed a witness set.

Let \( S_i \) be the strip of all points with \( x \)-coordinates between \( x(g_{i-1}) + \varepsilon' \) and \( x(g_i) \). Let \( p_T \) be the vertical projection of a point \( p \) onto \( T \), and \( p_A \) the vertical projection of \( p \) onto \( A \). \( S_i = \{ p \in \mathbb{R}^2 : (x(g_{i-1}) + \varepsilon' \leq x(p) \leq x(g_i)) \land (y(p) \leq y(p_T)) \} \). See Figure 10 for an illustration of these strips.

We show that \( V_T(w_i) \subseteq S_i \forall i \), hence, \( V_T(w_k) \cap V_T(w_\ell) = \emptyset \forall w_k \neq w_\ell \in W \), which shows that \( W \) is a witness set.

If \( w_i = v_j \) for an edge \( e_j \in E(T) \), \( V_T(w_i) \) contains the guard \( g_i \), but no other guard: If \( g_{i-1} \) could see \( v_j \), we have \( \angle(g_{i-1}, v_j, v_{j+1}) \leq 180^\circ \) because \( v_j \) is a convex vertex, thus, \( g_{i-1} \) could see all of \( e_j \), a contradiction to \( e_j \in E_g \).

Moreover, assume \( w_i \) could see some point \( p \) with \( x(p) \leq x(g_{i-1}) \). The terrain does not intersect the line \( \overline{p_T} \), and because the terrain is monotone the triangle \( \Delta(w_i, p, g_{i-1}) \) would be empty, a contradiction to \( g_{i-1} \) not seeing \( w_i \).

If \( w_i = q' \) for \( e' = [v_j, q] \), again \( V_T(w_i) \) contains the guard \( g_i \), but no other guard: If \( g_{i-1} \) could see \( w_i, q \) would not be the endpoint of the edge interval, a contradiction.

Moreover, assume \( w_i \) could see some point \( p \) with \( x(p) \leq x(g_{i-1}) \). Again, the terrain does not intersect the line \( \overline{w_i} \), and because the terrain is monotone the triangle \( \Delta(w_i, p, g_{i-1}) \) would be empty, a contradiction.

\( \Box \)

Theorem 3.5. The set \( G \) output by Algorithm 1 is optimal.

Proof. To show that \( G \) is optimal, we need to show that \( G \) is feasible and that \( G \) is minimum, that is, \( |G| = \text{OPT}(T,A) := \min\{|C| \mid C \subseteq A \text{ feasible w.r.t. ATGP}(T,A)\} \). Feasibility follows from Lemma 3.1 and by Lemma 3.4 we can find a witness set \( W \) with \( |W| = |G| \), hence, \( G \) is minimum.

\( \Box \)
Figure 11: An example where for $O(n)$ guards each guard needs to shoot $O(n)$ (colored) rays to compute mark points to its right, yielding a lower bound of $O(n^2)$ for this approach.

3.4 Uni-monotone Polygons are Perfect

In the proof for Lemma 3.4 we showed that for the ATGP there exists a maximum witness set $W \subset T$ and a minimum guard set $G \subset A$ with $|W| = |G|$. By Lemmas 2.11 and 2.12 the ATGP and the AGP for uni-monotone polygons are equivalent. Thus, also for a uni-monotone polygon $P$ we can find a maximum witness set $W' \subset LC(P) \subset P$ and a minimum guard set $G \subset H \subset P$ with $|W'| = |G|$. This yields:

Theorem 3.6. Uni-monotone polygons are perfect.

3.5 Guarding Monotone Mountains

We considered the Art Gallery Problem (AGP) in uni-monotone polygons, for which the upper polygonal chain is a single horizontal edge. There exist a similar definition of polygons: that of monotone mountains by O’Rourke [5]. A polygon $P$ is a monotone mountain if it is a monotone polygon for which one of the two polygonal chain is a single line segment (which in contrast to a uni-monotone polygon does not have to be horizontal). All our proofs also apply to monotone mountains, hence, we have:

Corollary 3.7. Monotone mountains are perfect.

3.6 Algorithm Runtime

The preprocessing step to compute the mark points costs $O(n^2)$, based on these we can compute the closing points for all edges of the terrain. Similarly, we compute the mark points from the left to compute the opening points (using the left vertex of an edge to shoot the ray) and the soft opening points (using the right vertex of an edge to shoot the ray).

Then, whenever we insert a guard (of which we might add $O(n)$), we need to shoot up to $O(n)$ rays, see Figure 11 which altogether costs $O(n^2 \log n)$. Similarly, for each of the intersection points $r_e$, we need to shoot a ray through $v_{me}$. This gives a total runtime of $O(n^2 \log n)$.

3.6.1 Improved Preprocessing Step using Convex Hulls

In fact, we do not need to shoot a ray from the rightmost vertex to all the vertices to its left and so on, in the preprocessing step described in Subsection 3.1. We stepwise build the
Figure 12: The convex hull of all terrain vertices to the right of e is shown in gray. The two orange CH edges are candidates only for the intersection with e, once e’s left vertex is added to the CH (the dashed edge is added, and the two orange edges are deleted), and we proceed to the left, they can never define a mark point again.

convex hull (CH) of the terrain vertices from the right, and only the terrain vertices on this CH are candidates for any rays intersecting with a terrain edge to the left of this CH (if we shoot a ray from a CH vertex through a terrain vertex within the CH, this ray can never intersect with an edge to the left of the CH), see Figure 12.

Thus, we obtain at most $n$ mark points on all edges in $E(T)$, that is, an amortized constant number of mark points per edge. Moreover, this process directly outputs the mark points in the right order. If we assume that the terrain vertices are given in order, the preprocessing step that stepwise builds the CH of the terrain vertices from the right and computes the mark points costs $O(n)$. Similarly, we build up a convex hull from the left to compute all the opening points for the terrain edges.

However, the improvement for the preprocessing step does not lead to an improved asymptotic runtime.

4 Conclusion and Discussion

We presented a polynomial-time algorithm for guarding a 1.5D terrain from an altitude line (the ATGP) and for the art gallery problem in uni-monotone polygons and monotone mountains. The preprocessing takes $O(n)$, the runtime of our algorithm is $O(n^2 \log n)$. We show that the ATGP and the AGP in uni-monotone polygons are equivalent. We prove optimality of our guard set by placing a maximum witness set (packing witnesses) of the same cardinality. Hence, we establish that both uni-monotone polygons and monotone mountains are perfect.

Currently, when we place a new (of $O(n)$) guard, we shoot up to $O(n)$ rays, and then shoot another ray from the intersection point of ray and terrain through the vertex stored with the corresponding mark point. Possibly, this process and, hence, the algorithm runtime can be improved. However, our focus was on showing perfectness and that the problems actually allow for polynomial-time algorithms.

In our algorithm, we compute the optimal guard set for a given altitude line $A$, the question at which heights $a_h$ of $A$ the number of guards in the minimum guard set changes is open. That is, what are the heights $a_{h_i}$ for altitude lines $A_i$ such that $\text{OPT}(T, A_{i+1}) > \text{OPT}(T, A_i)$ and $\text{OPT}(T, A_i) < \text{OPT}(T, A) \forall$ altitude lines $A$ at height $a_{h_i} + \varepsilon$?

Moreover, while guarding a 2.5D terrain from an altitude plane above the terrain is NP-hard, it would be interesting to find approximation algorithms for that case.
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