The origin of neutrino mass: stations along the path of cognition

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Abstract. I propose to give an outline based on partial answers to questions beneath ‘the origin of neutrino mass’:
a) building on the limiting uncurved structure of 1 time + 3 space dimensions, what is the full extent of space-time dimensions and their meaning in quantum gravity?
b) what is the origin and nature of spin-\(\frac{1}{2}\) fermions?
c) if charge-like and orientation-like gauging is related, then what is the explanation for 3 colors and 3 families along the path of unification?

1. There does not exist a symmetry —within the standard model including gravity and containing only chiral spin-\(\frac{1}{2}\) 16 families of SO(10)— which could enforce the vanishing of neutrino mass(es)

The divergence of the current associated to the global charge \(B - L\) for three standard model families of 15 base fields (in the left chiral basis), removing to infinite mass the 16th components (\(N\)) pertaining to one full 16-representation of SO(10) [spin (10)]

\[
(f)\gamma = \begin{pmatrix} u^1 & u^2 & u^3 & \nu \\ d^1 & d^2 & d^3 & e^- \end{pmatrix} \begin{pmatrix} N^\dagger & \hat{u}^3 & \hat{u}^2 & \hat{u}^1 \end{pmatrix} \begin{pmatrix} e^\mu \end{pmatrix} \quad \text{(1)}
\]

and admitting a gravitational background field, is anomalous in this minimal neutrino flavor embedding, i.e. the global symmetry is broken by winding gravitational fields [6].

\[
j_\psi(B - L)|_{3 \times 15} = \sum_{\text{families}} \left[ (u^*)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (u)\gamma^c - (\hat{u}^*)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (\hat{u})\gamma^c \right.
\]

\[
\left. + (d^*)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (d)\gamma^c - (\hat{d}^*)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (\hat{d})\gamma^c \right]
\]

\[
- (e^-)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (e^-)\gamma^c + (e^+)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (e^+)\gamma^c
\]

\[
- (\nu)^{\alpha \dot{c}} (\sigma_\mu)_{\alpha \dot{c}} (\nu)\gamma^c \right] e_\psi^\mu \quad \text{(2)}
\]

where \(g_\psi = e_\psi^\mu \eta_{\mu \nu} e_\nu^\nu\): metric; \(e_\psi^\mu\): vierbein; * hermitian operator conjugation; \((u^*)^{\alpha \dot{c}} \equiv (u^{\dot{a} c})^*\); \(\eta_{\mu \nu} = \text{diag}(1, -1, -1, -1)\): tangent space metric; \(c (\bar{c})\): color (anticolor); and \(c = 1, 2, 3\).

The contribution of charged fermions (pairs) \(q, \bar{q}\): \(e^+\) can be combined to vector currents (Dirac doubling) \(\bar{\psi} \gamma_\mu q; \bar{\epsilon} \gamma_\mu e\) with \(q \rightarrow u, d, c, s, t, b; e \rightarrow e^-, \mu^-, \tau^-\). The anomalous Ward
identity for the $B - L$ current (-density) defined in eq. (2) takes the form
\[ d^4x \sqrt{|g|} D^a j_\nu(B - L)|_{3 \times 15} = 3 \tilde{A}_1(X) \]
\[ \tilde{A}_1(X) = -\frac{1}{144} \text{tr} X^2; \quad (X)_{ab} = \frac{1}{2\pi} \text{tr} dx^a \wedge dx^b (R^a_{\nu})_{\nu} \]
\[ (R^a_{\nu})_{\nu} : \{ \begin{array}{l} \text{Riemann curvature tensor} \\ \text{mixed components: } \hat{q}_a \rightarrow \text{tangent space} \\ \hat{q}_a \rightarrow \text{covariant space} \end{array} \] (3)

Before discussing the extension $j_\nu(B - L)|_{3 \times 15} \rightarrow j_\nu(B - L)|_{3 \times 16}$, which renders the latter current conserved, let us define the quantities appearing in eq. (3):
\[ (R^a_{\nu})_{\nu} = e^a_{\mu} e_{\nu} (R^a_{\nu})_{\nu}; \quad e_{\nu} = \eta_{\nu \nu} e^\nu_{\nu} \]
\[ (R^a_{\nu})_{\nu} = (\partial_\nu \Gamma_\mu - \partial_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\nu)_{\nu} \]
\[ (\Gamma^a_{\nu})_{\nu} : \text{matrix valued (} GL(4, \mathbb{R}) \text{)} \text{ connection; minimal here} \] (4)

In eq. (3), $\tilde{A}(X \rightarrow \lambda) = \frac{1}{2\lambda} / \sinh(\frac{1}{2\lambda})$ denotes the Atiyah-Hirzebruch character or $\tilde{A}$-genus [7] with its integral over a compact, euclidean signatured closed manifold $M_4$, capable of carrying on SO4-spin structure, becomes the index of the associated elliptic Dirac equation
\[ \int \tilde{A}(X_E) = n_R - n_L = \text{integer} \] (5)

In eq. (5), $n_{R,L}$ denote the numbers of right- and left-chiral solutions of the Dirac equation on $M_4$. The index $E \rightarrow X_E$ shall indicate the euclidean transposed curvature 2-form, and is adapted here to physical curved and uncurved space time.

For the latter case the first relation in eq. (3) yields the integrated form —in the limit of infinitely heavy $N_F$ (eq. (1))—
\[ \Delta_{R-L} n_\nu = \int d^4x \sqrt{|g|} D^\mu j_\nu^{B-L(15)} = 3\Delta n(\tilde{A}) \] (6)

In eq. (6), $\Delta_{R-L} n_\nu$ denotes the difference of right-chiral ($\hat{\nu}$) and left-chiral ($\nu$) flavors between times $t \rightarrow \pm \infty$.

Here a subtlety arises precisely because the number of families on the level of $G_{SM}$ is odd and the light neutrino flavors are not ‘Dirac-doubled’, which according to eq. (6) could potentially lead to a change in fermion number being odd, which violates the rotation by $2\pi$ symmetry, equivalent to $\Theta^2 \left( CPT^2 \right)$, unless
\[ \Delta n(\tilde{A}) = \text{even} \quad (\sqrt{\text{for dim} = 4 \text{mod 8}}) \] (7)

We now turn to the SO(10)-inspired cancellation of the gravity-induced anomaly, giving rise to the completion of neutrino flavors to 3 families of 16-plets, sometimes called ‘right-handed’ neutrino flavors, denoted $N^*$ in the left-chiral basis in eq. (1) [8].
\[ j_\nu(B - L)|_{3 \times 15} \rightarrow j_\nu(B - L)|_{3 \times 16} \] (8)

1 $\tilde{\nu}_a \equiv \varepsilon_{a \beta} (\nu^*)$; $\varepsilon = i\sigma_2$ (2nd Pauli matrix) stands for the left-chiral neutrino fields transformed to the right-chiral basis.

2 The obviously nontrivial relation between the compact Euclidean and noncompact asymptotic and locality-restricted form of the index theorem involves not clearly formulated boundary conditions.

\[ j_g(B - L)|_{3 \times 15} \rightarrow j_g(B - L)|_{3 \times 16} = \]
\[ \sum_{families} \left[ (u^*)^{\alpha c} (\sigma_\mu)_{\alpha \gamma} (u)^{\gamma c} - (\tilde{u}^*)^{\alpha c} (\sigma_\mu)_{\alpha \gamma} (\tilde{u})^{\gamma c} + (d^*)^{\alpha c} (\sigma_\mu)_{\alpha \gamma} (d)^{\gamma c} - (\tilde{d}^*)^{\alpha c} (\sigma_\mu)_{\alpha \gamma} (\tilde{d})^{\gamma c} - (e^-)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (e^-)^{\gamma \gamma} + (e^+)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (e^+)^{\gamma \gamma} - (\nu)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (\nu)^{\gamma \gamma} + (\bar{N})^{\gamma \alpha} (\sigma_\mu)_{\alpha \gamma} (\bar{N})^{\gamma \gamma} \right] e_\mu^\nu \]  

(9)

where \( g_{\mu \nu} = \epsilon_\mu^\nu \eta_{\mu \nu} \): metric; \( \epsilon_\mu^\nu \): vierbein; *: hermitian operator conjugation; \( (u^*)^{\alpha c} \equiv (u^{\dagger c})^* \); \( \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1) \): tangent space metric; \( c \): color and anticolor; and \( c = 1, 2, 3 \rightarrow D^\mu j_g(B - L)|_{3 \times 16} = 0 \).

Let me illustrate the triple-doubling inherent in the elimination of the anomaly in the covariant divergence of \( j_g(B - L)|_{3 \times 15} \) in eq. (2), as seen through the left-chiral basis, repeating only the \( \nu, \bar{N} \) components of the \( B - L \) current in eq. (9)

\[ j_g(B - L)|_{3 \times 16} = \sum_{families} \left[ - (\nu)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (\nu)^{\gamma \gamma} + (\bar{N})^{\gamma \alpha} (\sigma_\mu)_{\alpha \gamma} (\bar{N})^{\gamma \gamma} \right] \]  

(10)

| \( B - L \) | \( \nu_F^\gamma \) | \( \bar{N}_F^\gamma \) |
|-----------|--------|--------|
| \( 1 \)   | 1      | +1     |

2. There does not exist a symmetry—within the standard model including gravity and containing only chiral 16 families of \( SO(10) \)—enforcing the vanishing of neutrino mass(es), yet there exist chiral extensions, which accomplish this

I briefly describe one such extension. It consists of replacing in each family the \( SO(10) \)-induced \( N_F \) flavors by four alternative (sterile) \( X_{J = 2, 3, 4, 5} \) flavors—singlets under the electroweak gauge group with genuinely chiral \( B - L \) charges—changing the structure in eq. (10) to

\[ j_g(B - L)|_{3 \times 19} = \sum_F \left[ - (\nu)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (\nu)^{\gamma \gamma} + \sum_{J=2}^{5} (\chi)_J (X_J)^{\alpha \gamma} (\sigma_\mu)_{\alpha \gamma} (X_J)^{\gamma \gamma} \right] \]  

(11)

| \( B - L = (\chi)_J \) | \( \nu_F^\gamma \) | \( X_2^\gamma_F \) | \( X_3^\gamma_F \) | \( X_4^\gamma_F \) | \( X_5^\gamma_F \) |
|-------------------|--------|--------|--------|--------|--------|
| \( 1 \)           | -1     | -5     | -9     | 7      | 8      |

The genuinely chiral couplings \( (\chi)_J = [-1, -5, -9; 7, 8] \) for neutrino flavors as shown in eq. (11) with 5 chiral base flavors merit some comments:

1) a sequence of charges \( (\chi)_J, J = 1, \ldots, N \) with respect to the left-chiral basis—to be specific—shall be called genuinely chiral, if none of the charges vanishes and no pairs of opposite charge \( \pm(\chi) \) are admitted.
2) the absence of an anomaly of the associated chiral current, of the form given for neutrino flavors in eqs. (2), (8) and (11) including also gravitational fields leads in 4 dimensions to the two conditions

\[ \sum_j N_j (\chi)_j = 0, \quad \sum_j [ (\chi)_j ]^3 = 0 \]  \hspace{1cm} (12)

3) there does not exist a genuinely chiral set \( \{ (\chi)_j \} \) for \( N < 5 \). For \( N = 3, 4 \), it is equivalent to show that the two equations

\begin{align*}
A + B &= C + D, \quad A^3 + B^3 = C^3 + D^3 \\
A &= x - a, \quad B = x + a, \quad C = x - b, \quad D = x + b
\end{align*}

\hspace{1cm} (13)

have no solution, satisfying the conditions for genuine chirality.

4) There are infinitely many solutions for \( N \geq 5 \), with chiral charges relatively irrational as well as rational. For integer values and \( N = 5 \) with the norm \( |(\chi)| = \sum |(\chi)_j| \) the solution with smallest norm is unique up to an overall change of sign

\[ (\chi)_j = \{-1, -5, -9; 7, 8\} \]  \hspace{1cm} (14)

2.1. Some conclusions from sections 1 and 2

C1) The oscillation phenomena indicate clearly, that a genuinely chiral extension of \( B - L \) to a conserved, global symmetry, generating a continous U1 group of transformations, is not involved.

C2) On the other hand, the binary code of a (minimally) supposed unifying gauge group SO or spin(10) could, if \( B - L \) is not gauged, equivalently generate a global symmetry of the vectorlike nature. The latter however would allow neutrino mass through the (electroweak doublet-singlet) pairing

\[ -L = \mu_{FG} N_\gamma^F \nu^G + h.c.; \quad F, G = 1, 2, 3 \text{ family} \]  \hspace{1cm} (15)

without symmetry restrictions on the mass matrix \( \mu_{FG} \) in eq. (15).

C3) Then however the question arises, why the mass matrix \( \mu \), involving the scalar doublet(s) within the electroweak gauge group, also generating masses of charged spin-\( \frac{1}{2} \) fermions, gives rise to very small physical neutrino masses. Thus we follow the hypothesis that SO(10) is gauged and that it is the large mass scale of the gauge boson associated with \( B - L \) in particular, which distinguishes neutrino flavors [9–11].

3. The Majorana logic [12] and mass from mixing – setting within the ‘tilt to the left’ or ‘seesaw’ of type I (···)

Within the subgroup decompositions of SO(10) the ‘tilt to the left’ does not appear obvious

\[ \begin{array}{ccc}
\text{spin}(10) & \times & \text{spin}(4) \\
\text{spin}(6) \cong \text{SU}4 & \times & \text{SU2}_L \times \text{SU2}_R \\
\text{lepton number as 4th color} & \downarrow & \downarrow \\
\text{SU3}_e \times \text{U1}_{B-L} & \times & \text{SU2}_L \times \text{U1}_{I_{3R}} \\
\text{Q.e.m.}/e = I_{3L} + I_{3R} + \frac{1}{2}(B - L)
\end{array} \]  \hspace{1cm} (16)

\[ ^3 \text{It is due to Paul Frampton, on a beautiful morning in 1993, along the coastal range above the Mediterranean Sea near Cassis, France.} \]
The large scale breaking of gauged $B - L$ or ‘tilt to the left’ was not an issue in refs. [9–11] and brings about a definite mass from mixing’ scenario [14, 15] to which we turn below.

3.1. The Majorana logic characterized by $N_F$

Here we consider the alternative subgroup decomposition

$$\text{spin}(10) \rightarrow \text{SU}5 \times \text{U1}_{J_5}$$

(17)

Among the 3 generators of spin(10) commuting with SU3c, $I_{3L}, I_{3R}, B - L$ and forming part of the Cartan subalgebra of spin(10) there is one combination, denoted $J_5$ in eq. (17), commuting with its largest unitary subgroup SU5.

The 16 representation in the left-chiral basis displays the charges pertinent to $J_5$ normalized to integer values modulo an overall sign —as in the discussion of genuinely chiral U1-charges in eq. (14)— but here referring to $N = 16$.

While the Majorana logic indeed opens a ‘path’ to trace the origin of the ‘tilt to the left’, the origin of three families remains unexplained at this stage. The associative Clifford algebras $\{\Gamma_{p,q}; \mathbb{C}\} \supset \{\Gamma_{p,q}; \mathbb{R}\}$ are constructed in sections 7.2 – 7.5 and Appendices A, B as supplementary material to the present outline. The variables $p, q$ denote time-like ($p$) and spacelike ($q$) dimensions of space-time.

Figure 1 shows the repartition of real (Maj-r) and complex (Maj-c) character of irreducible associative, real (Majorana) Clifford algebras with their mod 8 property relative to $q - p$ [5].

Figure 1. The complex and real Majorana representations MajCR($p, q$).

These representations form the roots of the ‘Majorana logic’ discussed below.

$$ (f)^\dagger = \begin{pmatrix} u^1 & u^2 & u^3 & \nu \\ d^1 & d^2 & d^3 & \nu^- \end{pmatrix} \bigg| \begin{pmatrix} N & \tilde{u}^3 & \tilde{u}^2 & \tilde{u}^1 \\ e^+ & \tilde{d}^3 & \tilde{d}^2 & \tilde{d}^1 \end{pmatrix} \bigg)^{J_5 \rightarrow L}$$

(18)

$$J_5 \rightarrow \begin{pmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & -3 \end{pmatrix} \bigg| \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & -3 & -3 & -3 \end{pmatrix}$$
The assignment of $J_5$—charges in eq. (18) follows from the fermionic oscillator representation of the spin (2$n$) associated $\Gamma$ algebra through $n$ such oscillators and the associated embedding $\text{spin}(10) \supset \text{SU5}$ [16] for $n = 5$ here [17]
\[
\{ a_s, a_t^\dagger \} = \delta_{st}; \quad s, t = 1, 2 \cdots, n; \quad \{ a_s, a_t \} = 0 = \{ a_s^\dagger, a_t^\dagger \} \rightarrow 
\]
\[
J_n = \sum_{s=1}^{n} \left( a_s a_s^\dagger - a_s a_s^\dagger \right) = 2\hat{n} - n \cdot 2^n \cdot 2^n; \quad \hat{n} = \sum_{s=1}^{n} a_s a_s^\dagger
\]

The eigenvalues ($X$) and multiplicities ($\#$) of $J_n$

| ($X$)   | $n$ | $n-2$ | $n-4$ | $\cdots$ | $-n+2$ | $-n$ |
|--------|-----|-------|-------|----------|--------|------|
| ($\#$) |     |       |       |          |        |      |
|        | (2) | (1)   | (0)   | (n-2)    | (n-1)  | (n)  |

The orthogonal series for $n$ even $\leftrightarrow$ real ($\text{spin}(8)$, $\text{spin}(12) \cdots$) has another decomposition within the associated $\Gamma$ algebra, than the one with $n$ odd $\leftrightarrow$ complex ($\text{spin}(10)$, $\text{spin}(14) \cdots$). We give here the explicit numbers according to eq. (20) for $n = 5$, i.e. $\text{spin}(10)$

| ($X$)   | 5   | 3    | 1    | $-1$   | $-3$   | $-5$ |
|--------|-----|------|------|--------|--------|------|
| ($\#$) |     |       |       |        |        |      |
|        | (5) | (5)   | (5)   | (5)    | (5)    | (5)  |
| SU5   | \{1\} | \{5\} | \{10\} | \{10\} | \{5\} | \{T\} |

The subset of states in eq. (21) ($X$) = $\{5, 1, -3\}$ forms the 16 representation of spin 10, while ($X$) = $\{3, -1, -5\}$ the complex conjugate $\overline{16}$. This opens the 'path' of linking the 'tilt to the left' with a substructure based on the primary in strength breakdown of the local gauged chargelike symmetry associated with
\[
J_5 = -4I_{3R} + 3(B - L)
\] (22)

$J_5$ as defined through integer eigenvalues ($X$) given in eqs. (18) and (21) is normalized differently from the other Cartan subalgebra charges $I_{3L}, I_{3R}, B - L$
\[
|QC|^2 = \sum_{\{16\}} (QC(f))^2, \quad |I_{3L}|^2 = 2, \quad |I_{3R}|^2 = 2
\]
\[
|B - L|^2 = \frac{16}{3}, \quad |J_5|^2 = 80
\] (23)

The consequence as far as neutrino-mass and mixing is concerned follows from identifying the $J_5$ direction with a major axis of primary spontaneous gauge-symmetry breaking, bringing about the ‘tilt to the left’ from eq. (15).
\[
\mathcal{H}_M = \mu_{FG} \mathcal{N}^F \nu^G + \text{h.c.} + \mathcal{H}_M
\]
\[
\mathcal{H}_M = \frac{1}{2} M_{FG} \mathcal{N}^F \mathcal{N}^G + \text{h.c.}; \quad F, G = 1, 2, 3
\]
\[
M_{FG} = M_{GF} : \text{ complex arbitrary otherwise; } |M| \gg |\mu|
\] (24)

It is the primary breakdown along the direction of $J_5$ which contrary to all ‘mirror complexes’ brings on the level of (pseudo-) scalar fields to the foreground the complex bosonic 126 and $\overline{126}$ representations of SO10
\[
\mathcal{H}_M \leftarrow \Phi \overline{126}FG \left( f_{a16F}, f_{b16G} \right) \mathcal{C} \left( \begin{array}{c}
126 \\
16 \\
a \\
b 
\end{array} \right) + \text{h.c.}
\] (25)
In eq. (25), \( C \left( \begin{array}{c|cc}
126 & 16 & 16 \\
\xi & a & b
\end{array} \right) \) denotes the coupling coefficients, projecting the symmetric product of two 16-representations of spin(10) to the 126 representation of SO(10). The 126 complex representation of SO(10) is singled out by the value of \( J_3 \) of \( 10 = 2 \times 5 \). The relatively complex conjugate representations 126 \( \oplus 3 \) of SO(10) are contained in the real, reducible fivefold antisymmetric tensor representation of SO(10) decomposing into the irreducible pair upon the duality conditions

\[
\left\{ A_1 A_2 \cdots A_5 \right\}; \quad A_1 \cdots A_5 = 1, 2, \ldots, 10
\]

\[
\varepsilon [A_1 A_2 \cdots A_5] = \text{sgn} \begin{pmatrix} 1 & 2 & \cdots & 5 \\ \pi_1 & \pi_2 & \cdots & \pi_5 \end{pmatrix} t^{[A_1 A_2 A_3]}
\]

\[
\frac{1}{\sqrt{2}} \varepsilon_{A_1 \cdots A_5 B_1 \cdots B_5} t_{\pm}^{[B_1 B_2 \cdots B_5]} = (\pm i) t^{[A_1 A_2 \cdots A_5]}
\]

\[
\varepsilon_{A_1 \cdots A_5 A_6 \cdots A_{10}} = \text{sgn} \begin{pmatrix} 1 & 2 & \cdots & 10 \\ \pi_1 & \pi_2 & \cdots & \pi_{10} \end{pmatrix} \varepsilon_{A_1 \cdots A_5 A_6 \cdots A_{10}}
\]

\[
\varepsilon_{12 \cdots 10} = 1
\]

Within the complex spin \((2 \nu = 4 \tau + 2), \tau = 2, 3, \cdots\) series \(- \tau = 2 \leftrightarrow \text{spin}(10)\) — the relatively complex conjugate spinorial pair of representations with dimension \(4 \tau \leftarrow 16(64, \cdots)\) and the complex selfdual-antiselfdual pair of representations with dimension \(\frac{1}{2} \left( \begin{array}{c}
4 \tau + 2 \\
2 \tau + 1
\end{array} \right) \leftarrow 126(11 \cdot 12 \cdot 13 = 1716, \cdots)\) are intrinsically related for \(\tau = 2, 3, 4, \cdots\).

3.2. Some conclusions and questions from section 3

Q1) Is it enough to consider the primary breakdown and its characteristic, the ‘tilt to the left’ concerning 3 families, as due essentially to spin(10), which is the lowest simple spin group along the complex orthogonal chain?

It has been argued interestingly by Feza Gursey and collaborators \[18\], that it is the chain of exceptional groups which encode intrinsically the number 3, which in turn underlies the 3 as the number of (left-chiral) families as well as the strong interaction gauge group SU\(3_c\).

A1) I think the answer is to the affirmative, since all higher gauge groups, including the exceptional chain and especially E8, but also spin(14), \[18\] do not explain the #3 of families, rather generate together with even the apparently correct 3 families —for E8— also mirror families —3 for E8, and powers of 2 for the orthogonal chain with \(\tau \geq 3\). The tentative conclusion remains, that the structure of families has to be explained outside spin(10) and also outside larger unifying gauge groups containing spin(10), whereas the origin of neutrino mass is layed out by the lowest member of the complex orthogonal chain \(\rightarrow\text{spin}(10)\).

C4) The two apparently different phenomena of a) ‘tilt to the left’ and b) baryon number violation are intrinsically associated with the unusual sequence of (pseudo)scalar fields generating primary breakdown. We use the notation (eq. (17))

\[
\text{spin}(10) \rightarrow SU5 \times U1_b \rightarrow SU3_c \times SU2_L \times U1_Y = G_{s.m.}
\]

\[
\begin{align*}
[16] &= \{1\}_{+5} + \{10\}_{+1} + \{\overline{5}\}_{-3} \\
[\overline{16}] &= \{1\}_{-5} + \{\overline{10}\}_{-1} + \{5\}_{+3}
\end{align*}
\]

\[
\{\overline{5}\}_{-3} = \left( \begin{array}{c}
(3, 1)_{+\frac{1}{2}} \\
(1, 2)_{-\frac{1}{2}}
\end{array} \right)_{-3}
\]
4. Mass from mixing for light $\nu$ flavors or 'seesaw'
Having outlined the 'fault-lines' (c.f. figure 2) of primary and secondary breakdown of charge-like gauge interactions, let me turn to some general consequence for neutrinos, light and heavy. To this end we take up eq. (24).

As the (p)scalar $[126, \overline{126}]$ representations are singled out through their major role in the primary breakdown along the $J_5$ direction (eq. (28)) we locate the $SU_2 \times U_1 Y \rightarrow [(1, 3)_{-1} [-6]$ triplet therein (seesaw of type II [1]). The complete decomposition of all $f \times f'$ couplings is given in Appendix E, from which we display eq. (E.16) as eq. (30) below. The two relatively hermitian conjugate triplets and their e.m. charges are

$$[126] \subset (T^0, T^-, T^{--}) \leftrightarrow (\overline{T}^0, \overline{T}^+, \overline{T}^{++}) \subset [\overline{126}]$$

$$\overline{T} = T^*$$

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**Figure 2.** #Q-1082 Lizard Carving within Polished Quartz Crystal from Russia. Detail in carving is excellent. The fault lines are visible. 2.5” long, from [http://www.bestcrystals.com/](http://www.bestcrystals.com/)
We complete the classification of the $[126]$ (p)scalar multiplet (eq. (E.15) in Appendix E)

\[
\begin{align*}
[126] & \quad \{\overline{15}\}_{-6} \quad \left[ (\overline{5}, 1) + \frac{1}{3} [3, 2] - \frac{1}{3} [1, 3] - [1] \right] \\
[126] [10] \quad \{45\}_{-2} & \quad \left[ \left( \frac{1}{2} \right) (6, 1) + \frac{1}{3} [2] + \left( \frac{1}{3} \right) (8, 2) - \frac{1}{2} [2] \right] + \left( \frac{1}{3} \right) (1, 2) - \frac{1}{2} [2] \\
[10] [126] [120] & \quad \{45\}_2 \quad \left[ (1, 3) + \frac{1}{2} [2] + (3, 2) + \frac{1}{2} [2] \right] \\
[126] & \quad \{50\}_2 \quad \left[ \left( \frac{1}{3} \right) (6, 3) + \frac{1}{2} [2] + \left( \frac{1}{2} \right) (8, 2) - \frac{1}{3} [2] \right] + \left( \frac{1}{3} \right) (3, 1) - \frac{1}{3} [2] + (\overline{5}, 1) - \frac{1}{2} [2] + (1, 1) + [2] \right]
\end{align*}
\]

We complete the classification of the $[126]$ (p)scalar multiplet (eq. (E.15) in Appendix E)

\[
\begin{align*}
[10] [126] & \quad \{5\}_{-2} \quad \left[ (3, 1) - \frac{1}{3} [3] + (1, \overline{2}) + \frac{1}{3} [3] \right] \\
[10] [126] [120] & \quad \{\overline{5}\}_2 \quad \left[ (\overline{3}, 1) + \frac{1}{2} [3] + (1, 2) - \frac{1}{2} [2] \right] \\
[126] & \quad \{1\}_0 \quad \left[ (1, 1) + [1] \right] \\
[126] [120] & \quad \{10\}_6 \quad \left[ (3, 2) - \frac{1}{6} [6] + (\overline{3}, 1) - \frac{1}{3} [6] + (1, 1) + [1] \right] \\
[120] & \quad \{\overline{10}\}_6 \quad \left[ (3, \overline{3}) + \frac{1}{6} [6] + (3, 1) + \frac{1}{3} [6] + (1, 1) - [1] \right]
\end{align*}
\]

While a direct $\nu \nu$ mass term could be induced by a *small* vacuum expected value

\[
\langle \Omega | \overline{T}^0 | \Omega \rangle \rightarrow \mathcal{H}_{\nu \nu} = \frac{1}{2} m_F \nu^F \nu^G + h.c.
\]

we do not consider this (p)scalar hierarchy of v.e.v. in the following; by hypothesis, that primary and e.w. breaking is associated with *one* (p)scalar v.e.v. for *one* representation of SO(10): $[126], [45], [10]$ respectively.
4.1. ‘Mass from mixing’ only [14, 15]

The mass matrix for the 6 neutrino flavors forming 3 families of [16] spin(10) representations, whence considered in the left-chiral basis takes the reduced form (eq. (24))

\[
\mathcal{H}_M = \frac{1}{2} \nu^j \mathcal{M}_{jk} \nu^k + \text{h.c.} ; \quad j, k = 1, \ldots, 6
\]

\[
\mathcal{M} = \begin{pmatrix}
0 & \mu^T \\
\mu & \mathcal{M}
\end{pmatrix}
\]

\[
\mu \leftrightarrow y_{FG} \langle \Omega \phi_0^* | 10 \rangle N_F^* [16] \nu^G [16]
\]

\[
M \leftrightarrow Y_{FG} \langle \Omega \phi_0^* | 126 \rangle N_F^* [16] N^*G [16]
\]

The 0\text{\texttimes}3 entry in \(\mathcal{M}\) is the consequence of our hypothesis \(\mathcal{H}_{\nu \nu} = 0\) in eq. (32). This is potentially fruitful ground for applying discrete symmetries to the (p)scalar self interactions.

Chiral fermionic structure ensures positive physical eigenvalues, for arbitrary complex \(\mu\) and symmetric but otherwise arbitrary \(M\). This would similarly guarantee positive masses for scalars, for (p)scalar mass from mixing, only in a supersymmetric setting.

I proceed reviewing properties of mixing and the mass relation following from the structure of \(\mathcal{M}\) as defined in eq. (33).

\textbf{‘seesaw’}

The relative ‘size’ of \(\mu\) and \(M\) defines the ‘mass from mixing’ situation and segregates 3 heavy neutrino flavors from the 3 light ones:

\[
\sqrt{||\mu||} \ll ||M||
\]

\[
||\mu||^2 = tr \mu \mu^\dagger , \quad ||M||^{-2} = tr M^{-1} M^{-1}
\]
Diagonalization of \( \mathcal{M} \)

We use the generic expansion parameter \( \vartheta = ||\mu||/||M|| \ll 1 \) and define a unitary \( 6 \times 6 \) matrix \( U \) with the property

\[
\mathcal{M} = U \mathcal{M}_{\text{diag}} U^T \rightarrow \mathcal{M}_{\text{diag}} = \mathcal{M}_{\text{diag}}(m_1, m_2, m_3; M_1, M_2, M_3) \]

\[
0 \leq m_1 \leq \cdots \leq M_3, \quad m_3 \ll M_1
\]

and \( U = TU_0; \quad T^{-1}MT^{-1T} = \mathcal{M}_{\text{bl.diag.}} \rightarrow \)

\[
= \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = U_0 \mathcal{M}_{\text{diag}} U_0^T
\]  

(35)

The matrix \( T \) in eq. (35) describes the mixing of light and heavy flavors, determined from a \( 3 \times 3 \) submatrix \( t \). 

\[
T = \begin{pmatrix} (1 + tt^\dagger)^{-1/2} & (1 + tt^\dagger)^{-1/2}t \\ -t^\dagger(1 + tt^\dagger)^{-1/2} & (1 + t^\dagger t)^{-1/2} \end{pmatrix}
\]  

(36)

The upper left \( 3 \times 3 \) block of \( T \) (eq. (36)) \( (1 + tt^\dagger)^{-1/2} \) causes the \( (3 \times 3) \) mixing matrix governing oscillations of light (anti)neutrino’s to deviate from unitarity, i.e. it becomes subunitary, but by a tiny amount since as we will discuss below

\[
||t||^2 = \sum_{kl}^3 |t_{kl}|^2 = O(10^{-21})
\]  

(37)

The matrix \( t \) in eq. (36) is rendered diagonal through two unitary \( 3 \times 3 \) matrices \( u \) and \( w \).

\[
t = u(tan a_{\text{diag}})w^{-1}; \quad a_{\text{diag}} = a_{\text{diag}}(a_1, a_2, a_3)
\]

\[
0 \leq a_k \leq \pi/2; \quad a_k \ll \pi/2 \quad \text{for} \quad \vartheta = ||\mu||/||M|| \ll 1
\]  

(38)

\( t \) is determined from the quadratic equation

\[
t = \mu^T M^{-1} - t \mu^T M^{-1}
\]  

(39)

which can be solved recursively setting

\[
t_{n+1} = \mu^T M^{-1} - t_n \mu^T M^{-1} \quad ; \quad t_0 = 0, \quad t_1 = \mu^T M^{-1},
\]

\[
t_2 = t_1 - \mu^T M^{-1} \mu^T \mu^T M^{-1} M^{-1}, \quad \cdots
\]  

(40)

The sequence defined in eq. (40) is convergent for \( \vartheta = ||\mu||/||M|| < 1 \). \( u, w \) in eq. (38) contain all 9 CP violating phases, pertaining to \( T \). \( t = u(tan a_{\text{diag}})w^{-1} \) defined in eq. (39) and its determining equation, repeated below

\[
t = \mu^T M^{-1} - t \mu^T M^{-1}
\]  

(39)

\[
\text{lead to block diagonal form of } \mathcal{M}_{\text{bl.diag.}}.
\]

---

4. To account for inverted hierarchy, the order of the light masses can be accordingly permuted.

5. In eq. (38), \( a_{\text{diag}} \) defines the three (real) heavy-lightmixing angles \( a_{1,2,3} \), which without loss of generality can be chosen in the first quadrant, but which are small for \( \vartheta = ||\mu||/||M|| \ll 1 \).
\[ \mathcal{M}_{\text{bl.diag.}} = T^{-1}MT^{-1T} ; \mathcal{M}_{\text{bl.diag.}} = \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix} \]  

(41)

(\mathcal{M}_1, \mathcal{M}_2) \text{ (eq. (41)) become}^6

\[ \begin{align*}
\mathcal{M}_1 &= (1 + tt^\dagger)^{-1/2} \left[ -t\mu - \mu^T t^T + tMtt^T \right] (1 + tt^\dagger)^{-1/2T} \\
\mathcal{M}_2 &= (1 + t^\dagger t)^{-1/2} \left[ \mu^T + t^\dagger \mu T + M \right] (1 + t^\dagger t)^{-1/2T} \\
\rightarrow \mathcal{M}_1 &= -t\mathcal{M}_2t^T
\end{align*} \]  

(42)

It follows from the assumptions detailed in footnote 6, that \( \text{Det} t \neq 0 \) and hence the heavy-light mixing angles \( a_{1,2,3} > 0 \) defined in eq. (38) are strictly bigger than 0.

The lowest approximation, \( t \rightarrow t_1 \) and \( \mathcal{M}_2 \rightarrow M \), yields the first nontrivial approximation of the light neutrino mass matrix in second order mixing

\[ \mathcal{M}_1 \sim \mathcal{M}_1^{(2)} = -\mu^T M^{-1}\mu \]  

(43)

5. Generic mixing and mass estimates

We introduce the arithmetic mean measure for \( 3 \times 3 \) matrices \( A \), not to be confused with the norms \( ||.|| \) defined in eq. (34)

\[ |A| = |\text{Det} A|^{1/3} \]  

(45)

Equation (42) then implies

\[ \begin{align*}
|\mathcal{M}_1| / |\mathcal{M}_2| &= |t|^2 \\
|\mathcal{M}_1| &= |m_{\text{diag}}| = (m_1m_2m_3)^{1/3} \\
|\mathcal{M}_2| &= |M_{\text{diag}}| = (M_1M_2M_3)^{1/3}
\end{align*} \]  

(46)

We consider the arithmetic mean of the light and heavy neutrino masses and the corresponding ‘would be’ masses if \( \mu \) and \( \mu^T \) would be the only parts of the full \( 6 \times 6 \) mass matrix \( \mathcal{M} \)

\[ \overline{\mu} = (m_1m_2m_3)^{1/3}, \quad \overline{M} = (M_1M_2M_3)^{1/3} \]

\[ \mu = u_{\mu} m_{\text{diag}}(\mu_1, \mu_2, \mu_3) v_\mu^{-1}, \quad \overline{\mu} = (\mu_1, \mu_2, \mu_3)^{1/3} \]  

(47)

6 In the scenario adopted here, we further assume \( \text{Det} M \neq 0 \) and \( \text{Det} \mu \neq 0 \).
Then beyond eq. (46) there is one more (exact) relation
\[
\hat{t} = (\tan a_1 \tan a_2 \tan a_3)^{1/3} = |t| \\
|\mu|^2 = |M_1| |M_2| \rightarrow \frac{m}{\mu} = \hat{t}, \quad \frac{m}{M} = \hat{t}^2
\]
(48)
or equivalently \( m = \hat{t} \mu \) \( \rightarrow \quad m/M = \hat{t}^{-1} \mu \)

The estimates below are based on the assumption that the scalar doublets (2) are part of a complex (p)scalar multiplet in [10] of SO10. It follows that at the unification scale we have
\[
\mu = \mu^T = \mu_u
\]
(49)

We shall use the relation at a scale near 100 GeV
\[
\mu \sim \frac{1}{3}(\mu_u)
\]
(50)
The factor \( \frac{1}{3} \) accounts for the color rescaling reducing the (colored) up-quark mass matrix from the unification scale down to 100 GeV. Using the definitions in eq. (47) and the quark masses \( m_u \sim 5.25 \text{ MeV}, m_c \sim 1.25 \text{ GeV} \) and \( m_t \sim 172.5 \text{ GeV} \)
\[
\bar{m}_u = (m_u m_c m_t)^{1/3} \sim 1 \text{ GeV} \rightarrow \bar{\mu} \sim \frac{1}{3} \text{GeV}
\]
(51)

Further lets approximate the mass square differences obtained from the combined neutrino oscillation measurements by
\[
\Delta m_{12}^2 \sim 10^{-4} \text{eV}^2 \quad , \quad \Delta m_{23}^2 \sim 2.510^{-2} \text{eV}^2
\]
(52)

Finally ‘pour fixer les idées’, I set the lowest light neutrino mass \( \sim 1 \text{ meV} \) and assume hierarchical (123) light masses. This implies
\[
m_1 \sim 1 \text{ meV}, m_2 \sim 10 \text{ meV} \\
m_3 \sim 50 \text{ meV} \rightarrow \bar{m} \sim 8 \text{ meV}
\]
(53)

and
\[
\hat{t} = \frac{m}{\mu} \sim 2.510^{-11}, \quad \hat{t}^2 \sim 6.010^{-22} \\
\hat{M} = \frac{m}{\hat{t}} \sim 1.410^{10} \text{ GeV}
\]
(54)

6. Conclusions, questions and outlook

C5 The origin of neutrino mass can be understood within the specific structure of \( \text{spin}(10) \) as charge-like gauge group. Boson fields appear to correspond to the full set of local \( f(x)f'(x) \) binary products with \( f, f' \subset [16] \oplus [\bar{16}] \).

This brings us to a starting point along the path of unification of gravitational and charge like gauge groups
\[
G_0 \quad = \quad \text{spin}(1, 3) \quad \otimes \quad \text{spin}(0, 10) \quad \downarrow \quad \text{gauging} \quad \downarrow \quad \text{gauging} \\
\text{orientation} \quad \text{charges} \quad \text{space-time} \quad \wedge \Lambda^1_3 \quad \times \quad \Lambda^0_{10}
\]
(55)
Q1 Is the geometric association of spin \((0, 10)\) in eq. (55) indicating internal space-like coordinates extending the geometric origin of spin \((1, 3)\) from space-time?  

Q2 What is the nature of coordinates in extended space-time?  

Proposed structures for general \(\bigwedge \bigwedge |X\), within superstring theories

\[
X \subset \bigwedge \bigwedge \begin{array}{c|c}
\{ x^\mu, \theta_\alpha, \overline{\theta}_i \} & x^\mu : \text{c-numbers} \\
\text{base superspace} & \theta, \overline{\theta} : \text{Grassmann variables} \\
\{ \widehat{x}^\mu, \widehat{f}^A, \widehat{f}^*B \} & \widehat{x}^\mu : \text{bosonic q-numbers} \\
\text{target superspace} & \widehat{f}^A, \widehat{f}^*B : \text{fermionic q-numbers}
\end{array}
\]

but the question addressed is more general and may not necessarily concern a space \(\bigwedge \bigwedge |X\), endowed with a supersymmetric structure.

Outlook The pathways of Nature, entangled indeed make tremble the doubtful who may not proceed yet build on assurance acquired to feed the hope to discover those road signs to read.

7. Complementary material

7.1. From Lorentzian \((p,q)\) to conformal groups \([3]\)

\[
M_{\mu\nu} \sim i (x_\mu \partial_\nu - x_\nu \partial_\mu) ; \quad \partial_\alpha x_\beta = \eta_{\alpha\beta} \mid_{\text{p-time,q-space}}
\]

\[
[M_{\mu\nu}, M_{\sigma\tau}] = i \begin{cases}
+ \eta_{\mu\sigma} M_{\nu\tau} - \eta_{\nu\tau} M_{\mu\sigma} \\
- \eta_{\mu\tau} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\tau}
\end{cases} \to \text{Lie} (SO(p,q)) \tag{57}
\]

Given an associative \((p,q)\) Clifford algebra \(\Gamma\)

\[
\{ \gamma_\mu, \gamma_\nu \} = 2 \eta_{\mu\nu} q_{4} \to \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]
\]

\[
s_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} \to s_{\mu\nu} \sim M_{\mu\nu} \to \text{spin}(p,q;\Gamma) \tag{58}
\]

\[
[s_{\mu\nu}, s_{\sigma\tau}] = i \begin{cases}
+ \eta_{\mu\tau} s_{\nu\sigma} - \eta_{\nu\sigma} s_{\mu\tau} \\
- \eta_{\mu\sigma} s_{\nu\tau} + \eta_{\nu\tau} s_{\mu\sigma}
\end{cases}
\]

Completing the conformal Lie algebra with conformal infinitesimal boosts

\[
K_\mu \sim i (2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu) ; \quad D \sim ix^\alpha \partial_\alpha \tag{59}
\]

with the commutation relations

\[
[K_\mu, K_\nu] = 0 ; \quad [P_\mu, K_\nu] = 2i (\eta_{\mu\nu} D - M_{\mu\nu})
\]

\[
K_\mu K_\nu = \begin{cases}
4 \left[ x_\mu x^\alpha \partial_\alpha, x_\nu x^\beta \partial_\beta \right] - 2 \left[ x_\mu x^\alpha \partial_\alpha, x_\nu x^\beta \partial_\beta \right] \\
\left[ x^2 \partial_\mu, x^2 \partial_\nu \right] - 2 \left[ x_\mu x^\alpha \partial_\alpha, x^2 \partial_\nu \right]
\end{cases}
\]

\[
= - \begin{cases}
4 \left( x_\mu x^\alpha \left[ \partial_\alpha, x_\nu x^\beta \partial_\beta \right] - \left[ x_\mu x^\alpha \partial_\alpha, x_\nu x^\beta \partial_\beta \right] \partial_\alpha \right) \\
- 2 \left( x^2 \left[ \partial_\mu, x_\nu x^\beta \partial_\beta \right] - \left[ x_\mu x^\beta \partial_\beta, x^2 \right] \partial_\mu - \mu \leftrightarrow \nu \right) \\
+ \left( x^2 \left[ \partial_\mu, x^2 \partial_\nu \right] - \left[ x^2 \partial_\nu, x^2 \right] \partial_\mu \right) \\
- 2 \left( -x^2 x_\nu \partial_\mu - \mu \leftrightarrow \nu \right) \\
+ 2 \left( x^2 x_\nu \partial_\mu - \mu \leftrightarrow \nu \right)
\end{cases} = 0 \tag{60}
\]

\[\text{I cite here just one reference: Élie Cartan, ‘Sur une classe remarquable d’espaces de Riemann’, Bull. Soc. Math. de France 54 (1926) 214, and 55 (1927) 114 [2].}\]
\[ [P_\mu, K_\nu] = - \left[ \partial_\mu, 2 x_\nu x^\alpha \partial_\alpha - x^2 \partial_\nu \right] \]
\[ = - \left\{ \left[ \partial_\mu, 2 x_\nu x^\alpha \right] \partial_\alpha \right\} - 2 (\eta_{\mu\nu} x^\alpha \partial_\alpha + x_\nu \partial_\mu - x_\mu \partial_\nu ) \]
\[ = 2i (\eta_{\mu\nu} - M_{\mu\nu}) \]

This can be completed to the Lie algebra of SO(2,0) for completeness we also verify the commutation rules as well as
\[ M_{\mu4} = \frac{1}{2} \left( m^{-1} P_\mu - m K_\mu \right) \]
\[ M_{\mu5} = \frac{1}{2} \left( m^{-1} P_\mu + m K_\mu \right) \]
\[ [M_{\mu4}, M_{\mu5}] = \frac{1}{4} \left( [P_\mu, K_\nu] + \mu \leftrightarrow \nu \right) = i \eta_{\mu\nu} D \]
\[ M_{45} = - D \]

For completeness we also verify the commutation rules
\[ [M_{\mu4(5)}, M_{\mu4(5)}] = \mp \frac{1}{4} \left\{ \left[ P_\mu, K_\nu \right] \right\} \]
\[ = \pm i M_{\mu\nu} = i \left( \begin{array}{cc} -\eta_{44} & \eta_{45} \\ \eta_{54} & -\eta_{55} \end{array} \right) M_{\mu\nu} \sqrt{\eta} \]
as well as
\[ [m^{-1} \partial_\mu, x^\beta \partial_\beta] = 1 m^{-1} \partial_\mu \]
\[ 2 m x_\mu x^\alpha \partial_\alpha , x^\beta \partial_\beta] = 2 \left\{ x_\mu x^\alpha \left[ \partial_\alpha, x^\beta \partial_\beta \right] \right\} + \left\{ m^{-1} \partial_\mu, x^\beta \partial_\beta \right\} \partial_\alpha \]
\[ = 2 \left\{ m^{-1} \partial_\mu, x^\beta \partial_\beta \right\} \partial_\alpha \]
\[ = (-1) 2 m x_\mu x^\alpha \partial_\alpha \]
\[ m x_\mu x^\alpha \partial_\alpha , x^\beta \partial_\beta] = m \left\{ x^2 \left[ \partial_\mu, x^\beta \partial_\beta \right] \right\} + \left\{ x^2, x^\beta \partial_\beta \right\} \partial_\mu \]
\[ = (-1) m x_\mu x^\alpha \partial_\alpha \]

Commutation with \( x^\beta \partial_\beta \) returns the mass dimension. Thus eq. (64) becomes
\[ [M_{\mu4(5)}, M_{45}] = \frac{1}{2} \left\{ \left[ \begin{array}{c} m^{-1} \partial_\mu \\ \pm (2 m x_\mu x^\alpha \partial_\alpha - m x^2 \partial_\mu) \end{array} \right] \right\} \]
\[ = - i \frac{1}{2} \left\{ \left[ \begin{array}{c} m^{-1} P_\mu \pm m K_\mu \end{array} \right] \right\} \]
\[ = - i M_{\mu5(4)} = i \left( \begin{array}{c} \eta_{44} M_{\mu5} \\ -\eta_{55} M_{\mu4} \end{array} \right) \sqrt{\eta} \]
7.2. Details of Lorentzian \((p, q) \times P_{\mu} \rightarrow \text{conformal} \ (p + 1, q + 1)\) - extension for the Majorana setting: \(p = 1, \ q = 3\)

It becomes clear from the derivations in section 7.1 that the extension from the motion group in \(d = p + q\) dimensions with \(p\) time- and \(q\) space-signatures follows the same rules for all \(p, q\). The extended group structure becomes simple and forms the conformal group \(\text{SO}(p + 1, q + 1)\). For the corresponding \(\Gamma\) - algebra extension we discuss here just the characteristic Majorana setting inherent to \(p = 1, \ q = 3\), illustrating the induced extension: \(\text{spin}(p, q) \rightarrow \text{spin}(p + 1, q + 1)\). It will become clear after the above comparison of extensions, that they are not in any way related.

To study the Majorana representations of signatured (and associative) Clifford algebras (eq. (58)), it is necessary to adopt a real form of the Dirac equation, i.e. to pass from the matrices and conventions \(\gamma_{\mu} \rightarrow \Gamma_{\mu} = i\gamma_{\mu}\) and \(\eta_{\mu\nu} \rightarrow -\eta_{\mu\nu}\), which satisfy the relations

\[
\begin{align*}
\eta_{\mu\nu} &= \text{diag} \ (1_p \times 1; \ -1_q \times -1) \\
\{\gamma_{\mu}, \gamma_{\nu}\} &= 2\eta_{\mu\nu} \quad \{\gamma_{\mu}, \gamma_{\nu}\} \\
\{\Gamma_{\mu}, \Gamma_{\nu}\} &= 2(\eta_{\mu\nu}) \quad \Gamma_{\mu} = i\gamma_{\mu} \\
-\eta_{\mu\nu} &= \text{diag} \ (1_p \times -1; \ -1_q \times 1) \\
\end{align*}
\]

(67)

The full \(\Gamma\) algebra over the complex numbers is the same for all space-time signatures. It shall be denoted \(\{d\Gamma; \mathbb{C}\}\) and identified with its unique —modulo (inner) automorphisms— irreducible matrix representation.

\[
\dim \{d\Gamma; \mathbb{C}\} = 2^{[\frac{d}{2}]} : \ d = p + q ; \ [\frac{d}{2}] = \begin{cases} \frac{d}{2} & \text{for } d \text{ even} \\ \frac{d + 1}{2} & \text{for } d \text{ odd} \end{cases}
\]

(68)

If within \(\{d\Gamma; \mathbb{C}\}\) and given \(p, q\) signatures the \(\Gamma_{\mu}\) matrices (eq. (67)) can be chosen real, we deal with a Majorana representation, discussed for \(p = 1, \ q = 3\) in appendix A.

Equipped with the Majorana representation \(\{\Gamma_{p=1,q=3}; \mathbb{R}\}\) we extend it to \(\{\Gamma_{p=2,q=4}; \mathbb{R}\}\) below, keeping in mind that there is no Majorana representation for general \(p, q\) values. Also care must be taken in the numbering of coordinates beyond the four pertaining to \(d = 1 + 3\). We continue the enumeration of space-time dimensions, always starting with extended time followed by extended 3-space, as follows

\[
x^0 = t, \ x^k, k = 1, 2, 3 \cdots q; \ x^{q+r}, r = 1, \cdots , p - 1 ; \ \text{for } p=2, \ q=4 \rightarrow
\]

\[
x^0 \ x^1 \ x^2 \ x^3 \ x^4 \ \ ; \ x^5
\]

\[+ \quad + \quad - \quad - \quad - \quad + \]

(69)

In order to distinguish the space-time dimensionality and its associated \(\Gamma\) - algebra we shall use (or substitute) the notation

\[
\Gamma_{\mu} \rightarrow_d \Gamma_{\mu} : \ d = p + q , \ \mu = 0, 1 \cdots d - 1
\]

(70)

Thus \(\Gamma_5, \gamma_{5R(L)}\) (eq. (A.2)) for general \(d, p, q\) and general basis become

\[
d\Gamma_{d+1} = d (\Gamma_0\Gamma_1 \cdots \Gamma_{d-1})
\]

(71)

In the product in eq. (71) the prefix \(d\) is not repeated for brevity.
7.3. Product representation for the Clifford algebra \( \{d_1 + d_2 \Gamma; \mathbb{C}\} \) \[4\]

We consider two Clifford algebras corresponding to even dimensions \( d_1, d_2 \) respectively

\[
d_1 = 2\nu_1 : d_1 (\Gamma_0, \cdots \Gamma_{d_1-1}) \quad D = d_1 + d_2
\]

\[
d_2 = 2\nu_2 : d_2 (\Sigma_{0}, \cdots \Sigma_{d_2-1}) \quad \dim \{d_j \Gamma; \mathbb{C}\} = 2\nu_j = N_j \quad j = 1, 2
\]

Then we construct the direct product representation of \( \{D \Gamma; \mathbb{C}\} \) in the following way (two ways)

\[
d \Gamma_\alpha = d_1 \Gamma_\alpha \otimes \mathbb{R} N_2 \times N_2 ; \quad \alpha = 0, \cdots d_1 - 1
\]

\[
d \Gamma_{d_1+\beta} = d_1 \Gamma_{d_1+1} \otimes d_2 \Sigma_\beta ; \quad \beta = 0, \cdots d_2 - 1
\]

\[
d \Gamma_{D+1} = (d_1 \Gamma_{d_1+1})^{1+d_2} \otimes (d_2 \Gamma_{d_2+1})
\]

As long as we work over the field \( \mathbb{C} \), signatures i.e. the value of the squares

\[
(D \Gamma x)^2 = - \eta_{xx} [\text{sum}] \quad (D \sigma_x) \quad x = 0, \cdots, D - 1
\]

\[
(D \sigma_{D+1})^2 = (D \sigma_{D+1}) \quad (D \Gamma_{D+1}) = (D \sigma_{D+1})\quad \text{for any one of assigned parities}
\]

\[
(D \sigma_{D+1})\quad \text{for } x = 0, \cdots, D - 1, D + 1
\]

are immaterial. Nevertheless the ‘straight’ product definition of \( d \Gamma_{d+1} \) for even in eq. (71) implies, always within even \( D = d_1 + d_2, d_1, d_2 \), by eqs. (72), (73), recursively

\[
(D \sigma_{D+1}) = (d_1 \sigma_{d_1+1}) \quad (d_2 \sigma_{d_2+1})
\]

The suffix \( \Pi \) of the signature \( (D \sigma_{D+1}) \) in eqs. (74) and (75) shall indicate that this quantity depends on the chosen form of the product representation.

It becomes obvious that, if the \( d \sigma_x \) parities are assigned the direct product composition defined in eq. (73), this may not be compatible from \( d_1 \& d_2 \) to \( D \leftrightarrow d_1 \& d_2 \).

To this end we compute, for generic even \( d \), the quantities \( d \sigma_{d+1} \) for any one of assigned parities \( d \sigma_x \) for \( x = 0 \cdots d - 1 \), maintaining the ‘strict’ product representation of \( d \Gamma_{d+1} \) as defined in eq. (71)

\[
\left( \prod_{x=0}^{d-1} (d \Gamma_x) \right)^2 = \left( \prod_{x=0}^{d-1} (d \Gamma_x) \right)^2 \sigma_{rev}(d)
\]

\[
\sigma_{rev}(d) = (-1)^{(1+2+\cdots+d-1)} = (-1)^d \quad d = 2\nu
\]

Hence, continuing to work over \( \mathbb{C} \), we can upon multiplication and/or rearrangement in ordering of individual \( d \Gamma_x; x = 0, \cdots, d - 1 \) elements with appropriate powers of \( i \), assign arbitrary signatures and \( d \sigma_x \) parities, yielding a signature \( (p, q); p + q = d \rightarrow \)

\[
d \sigma_{d+1}(p, q) = (-1)^{\frac{1}{2}(p-q)} = (-1)^{\frac{1}{2}(q-p)} = d \sigma_{d+1}(q, p)
\]

It is the symmetry with respect to exchange of \( p \leftrightarrow q \) signatures, which renders the direct product, defined in eq. (73) non-symmetric yet consistent with eq. (75). However the assigned \( d_1, d_2 \sigma_x \) parities or signatures are not directly transferred to the ordered product \( \{D \Gamma_y; y = 0, \cdots, D - 1\} \)

\[
D \sigma_\alpha = (d_1 \sigma_\alpha)
\]

\[
D \sigma_\alpha = (d_1 \sigma_{d_1+1}) \quad (p_1, q_1) \quad (d_2 \sigma_\alpha)
\]

\[
\rightarrow (p, q) \quad D \quad \text{as defined in eq. (71)}
\]

\[
\rightarrow (p, q) \quad (D)_{\Pi} = \left\{ \begin{array}{ll}
(p_1 + p_2, q_1 + q_2) & \text{for } q_1 - p_1 = 0 \mod 4 \\
(p_1 + q_2, q_1 + p_2) & \text{for } q_1 - p_1 = 2 \mod 4
\end{array} \right.
\]

\[17\]
7.4. Reduction of a Majorana representation $\{\Gamma_{p,q}; \mathbb{R}\} \rightarrow \{\Gamma_{p-1,q-1}; \mathbb{R}\}$

Given a Majorana representation $\{\Gamma_{p,q}; \mathbb{R}\}$ with $p,q \geq 1$ we single out the last two real matrices for each signature respectively, using here the reordered numbering

\[
\{\Gamma_{-1}, \cdots, \Gamma_{p-1}; \Gamma_{1}^{+}, \cdots, \Gamma_{q}^{+}\}
\]

\[
\{\Gamma_{p}, \Gamma_{q}\} : \left\{(\Gamma_{p}^{-})^{2}, (\Gamma_{q}^{+})^{2}\right\} = (\mathbb{I})_{2^{\nu} \times 2^{\nu}} \{ -1, +1 \} ; 2\nu = p + q
\]

Next we consider the product

\[
\Pi = \Gamma_{p}^{-}\Gamma_{q}^{+}
\]

and the projectors $Pr_{\pm}$ defined in eq. (81)

\[
Pr_{\pm} = \frac{1}{2} (\mathbb{I}_{2^{\nu} \times 2^{\nu}} + \Pi) ; \left\{ Pr_{\pm}^{2} = Pr_{\pm} \right\}
\]

\[
Pr_{\mp} + Pr_{\pm} = (\mathbb{I})_{2^{\nu} \times 2^{\nu}}
\]

As a consequence of the real nature of the matrices $\Gamma_{p,q}$ the projectors $Pr_{\pm}$ defined in eq. (81) are real symmetric and thus hermitian matrices, projecting on two orthogonal subspaces $S_{\pm}$ of dimension $2^{\nu-1}$ respectively. Furthermore these projectors commute with the remaining $\Gamma$ matrices

\[
\{\Gamma_{1}^{-}, \cdots, \Gamma_{p-1}^{-}; \Gamma_{1}^{+}, \cdots, \Gamma_{q-1}^{+}\}
\]

\[
[Pr_{\pm}, \Gamma_{k}^{-}] = 0 \quad \text{for} \quad k = 1, \cdots, p - 1
\]

\[
[Pr_{\pm}, \Gamma_{j}^{+}] = 0 \quad \text{for} \quad j = 1, \cdots, q - 1
\]

It follows that the projected matrices

\[
\hat{\Gamma}_{k}^{-} = \Gamma_{k}^{-} Pr_{\pm} \quad \text{for} \quad k = 1, \cdots, p - 1
\]

\[
\hat{\Gamma}_{j}^{+} = \Gamma_{j}^{+} Pr_{\pm} \quad \text{for} \quad j = 1, \cdots, q - 1
\]

form – for either sign of $Pr_{\pm}$ separately – irreducible representations over $\mathbb{R}$

\[
\rightarrow \{\Gamma_{\hat{p},\hat{q}}; \mathbb{R}\} \text{ with } \hat{p} = p - 1, \hat{q} = q - 1; \hat{p} + \hat{q} = 2(\nu - 1).
\]

7.5. The two base sets of Majorana representations $\{\Gamma_{p=2\nu-\nu=0}; \mathbb{R}(-)\}$ and $\{\Gamma_{\nu=0,\nu=2\nu}; \mathbb{R}(+)\}$

We give the case $p = 2\nu^{-} = d^{-}; q = 0$ a label $(-)$ and conversely $q = 0; q = 2\nu^{+}$ the label $(+)$. It now follows from eq. (B.11)

\[
p = d^{-} \quad q = 0 \quad M(-) = 4M_{(1)}^{d^{-}} + 4M_{(2)}^{d^{-}}
\]

\[
p = 0 \quad q = d^{+} \quad M(+) = 4M_{(3)}^{d^{+}} + 4M_{(2)}^{d^{+}}
\]

\[
d^{\pm} = 2\nu^{\pm} \text{ even}
\]

From eq. (D.6) we obtain

\[
4M_{(i)}^{2\nu} = 2^{\nu-1} \left( 2^{\nu-1} + F [2\nu - 2(l)] \right)
\]

Combining eqs. (84) and (85) we obtain
The quantities $M(\mp)$ represent the number of antisymmetric $2\nu \times 2\nu$ matrices forming the full Clifford algebras $\{\Gamma_{p=2\nu^- \cdot q=0} : \mathbb{R}(-)\}$ and $\{\Gamma_{p=0,q=2\nu^+} : \mathbb{R}(+)\}$.

Comparing the two relations for $M(\mp)$ in eq. (86) it follows using the labels $p(=2\nu^-)$ for $\{\Gamma_{p=2\nu^- \cdot q=0} : \mathbb{R}(-)\}$ and $q(=2\nu^+)$ for $\{\Gamma_{p=0,q=2\nu^+} : \mathbb{R}(+)\}$

\[
F(p-2) + F(p-4) = -1 \quad \text{for} \quad (-) \quad \rightarrow \quad p = 6, 8
\]
\[
F(q-6) + F(q-4) = -1 \quad \text{for} \quad (+) \quad \rightarrow \quad q = 2, 8
\]  

In order to illustrate the solutions to eq. (87) I display the function $F(j); j = \text{even}$ from eq. (D.5) below

\[
F(j) = F(-j) = F(j + 8)
\begin{align*}
    j & \mid -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\
    F & \mid 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1
\end{align*}
\]  

Acknowledgments

I thank Émilie Passemar for her constructive interest in the material discussed here and for a careful reading of the texts as they evolved. Further thanks are due to the organizers of the Discete08’ meeting in Valencia for creating a stimulating and warm atmosphere enabling interesting discussions.

Appendix A. The Majorana representation of $\{\Gamma_{p,q} : \mathbb{R}\}$ for $p = 1, q = 3$.

If for a given signature $p,q$ a Majorana representation over the real numbers $\{d\Gamma ; \mathbb{C}\}$ exists, this representation shall be denoted $\{\Gamma_{p,q} : \mathbb{R}\}$. For $p = 1, q = 3$ the (left- and right-) chiral basis over the field $\mathbb{C}$ corresponds to the $\Gamma_{\mu}$ matrices

\[
\Gamma_{\mu}^{(x)} = \begin{pmatrix}
0 & i\sigma_{\mu} \\
0 & 0
\end{pmatrix}; \quad \Gamma_{\mu} = (\sigma_{0}; \sigma_{k}) \equiv \Gamma_{\mu}^{(x)} ; \quad k = 1, 2, 3
\]

\[
\sigma_{0} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}; \quad \sigma_{1} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}; \quad \sigma_{2} = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}; \quad \sigma_{3} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]  

\[
\Gamma_{\mu} = \eta_{\mu\nu} \Gamma_{\nu}, \quad \Gamma_{5} = \Gamma_{0}\Gamma_{1}\Gamma_{2}\Gamma_{3}, \quad \Gamma_{5}^{2} = -\mathbb{I}_{4\times4} |_{p=1q=3}; \quad \text{in any basis}
\]

In the chiral basis we have

\[
\Gamma_{5}^{(x)} = i\begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix} = i\gamma_{5R} = -i\gamma_{5L}
\]  

In the Majorana basis, $\Gamma_{5}^{(\text{Major})}$ is real, antisymmetric. In the chiral basis the substrate of the spinor (a 4-dimensional column ‘vector’) is of the form

\[
\begin{pmatrix}
\tilde{\varphi}_{\alpha} \\
\tilde{\psi}_{\beta}
\end{pmatrix}^{(x)} ; \quad \alpha,\delta,\tilde{\gamma},\delta = 1, 2; \quad \tilde{\varphi}_{\gamma} = -\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
with the (Majorana-) reality condition $\phi_0 = \psi_0$. We introduce two component and 2 by 2 matrix notation

$$\begin{align*}
(\varphi_0, \psi_0) & \rightarrow \varphi, \psi; \quad \bar{\varepsilon} \delta & \rightarrow \bar{\varepsilon} \equiv \varepsilon^{-1} = -\varepsilon \\
\varepsilon = i\sigma_2 = & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{align*}$$

(A.4)

The representation for a spinor (eq. (A.3)) in the chiral basis becomes

$$\begin{pmatrix} \varphi \\ \varepsilon^{-1} \psi^* \end{pmatrix}^{(x)}; \quad \text{with the Majorana condition } \varphi = \psi$$

(A.5)

The Majorana basis obtains whence $\varphi, \psi$ in eqs. (A.3) and (A.5) are identified and then decomposed into real and imaginary parts (component by component)

$$\varphi = \psi = x + iy; \quad x = \frac{1}{2} (\varphi + \varphi^*); \quad y = \frac{1}{2i} (\varphi - \varphi^*)$$

(A.6)

The action of $\Gamma^{(x)}_\mu$ (eq. (A.1)) then becomes

$$\Gamma^{(x)}_\mu \begin{pmatrix} x + iy \\ \varepsilon^{-1} (x - iy) \end{pmatrix} = \begin{pmatrix} i\sigma_\mu \varepsilon^{-1} (x - iy) \\ i\bar{\sigma}_\mu (x + iy) \end{pmatrix} = \begin{pmatrix} x' + iy' \\ \varepsilon^{-1} (x' - iy') \end{pmatrix}_\mu$$

(A.7)

Thus we obtain, working out the action of $\Gamma^{(x)}_\mu$ (eq. (A.7)), separately for each $\mu$

$$\begin{align*}
x'_0 & = -\varepsilon y \\
y'_0 & = -\varepsilon x \\
x'_1 & = \sigma_3 y \\
y'_1 & = \sigma_3 x \\
x'_2 & = x \\
y'_2 & = -y \\
x'_3 & = -\sigma_1 y \\
y'_3 & = -\sigma_1 x
\end{align*}$$

(A.8)

The Majorana representation of $\{\Gamma_{p=1,q=3; \mathbb{R}}\}$ constructed in eq. (A.8) allows inner automorphisms

$$\Gamma'_\mu = R \Gamma^{(Maj)}_\mu R^{-1}; \quad R : \text{real}_{4 \times 4}, \quad \{R|DetR = 1\} \simeq SL(4, \mathbb{R})$$

$$\text{dim} (SL(4, \mathbb{R})) = 15$$

(A.9)

forming the special linear (real, simple, noncompact) group in 4 dimensions. We include $\Gamma^{(x)}_5$ (eq. (A.2)) and transform it to the Majorana representation, using eq. (A.7)

$$\begin{align*}
\Gamma^{(x)}_5 \begin{pmatrix} x + iy \\ \varepsilon^{-1} (x - iy) \end{pmatrix} & = \begin{pmatrix} i (x + iy) \\ -i\varepsilon^{-1} (x - iy) \end{pmatrix} = \begin{pmatrix} x' + iy' \\ \varepsilon^{-1} (x' - iy') \end{pmatrix}_5 \\
& = \begin{pmatrix} y + ix \\ \varepsilon^{-1} (-y - ix) \end{pmatrix}
\end{align*}$$

(A.10)

$$\begin{align*}
x'_5 & = -y \\
y'_5 & = x \\
\Gamma^{(Maj)}_5 & = \begin{pmatrix} 0 & -\mathbf{q} \\ \mathbf{q} & 0 \end{pmatrix}
\end{align*}$$
Appendix B. The Majorana representation \{\Gamma_{p,q}: \mathbb{R}\} counting of \((p)\) antisymmetric real \(\Gamma_x\) matrices

Let’s first reorder the usual numbering of \(\Gamma\) matrices such that for signature \((p,q)\) we have

\[
0 \to 1, \ldots, p-1 \to p; \quad p \to 1, \ldots, p+q-1 \to q
\]

\[
(\Gamma^-_r) = (-1)^{\frac{s}{2}} \nu \times 2^\nu; \quad r = 1, \ldots, p; \quad d = 2\nu = p + q
\]

\[
(\Gamma^+_s) = (+1)^{\frac{s}{2}} \nu \times 2^\nu; \quad s = 1, \ldots, q
\]

(B.1)

Given a Majorana representation the first level \(\Gamma^+\) matrices can be brought to symmetric, the \(\Gamma^-\) matrices to antisymmetric form. The level \(\lambda\) product of \(\Gamma\) matrices is thus of the form

\[
\Pi^\lambda_s = \left( \Gamma^-_{r_1} \cdots \Gamma^-_{r_e} \right) \left( \Gamma^+_{s_1} \cdots \Gamma^+_{s_\sigma} \right); \quad 1 \leq r_1 \cdots \leq r_e \leq p
\]

\[
1 \leq s_1 \cdots \leq s_\sigma \leq q
\]

\[
\lambda = \rho + \sigma; \quad 0 \leq \lambda \leq d
\]

(B.2)

The signature of any member of \(\Pi^\lambda_s\) is

\[
s(\pi) = s(\lambda; \rho, \sigma) = (-1)^{\rho} \times (-1)^{\frac{1}{2}\lambda^2}\]

\[
\lambda = \rho + \sigma; \quad 0 \leq \lambda \leq d; \quad 0 \leq \rho \leq p; \quad 0 \leq \sigma \leq q
\]

(B.3)

The factor \((-1)^{\frac{1}{2}\lambda^2}\) segregates \(\lambda\) into the four classes mod 4

\[
(-1)^{\frac{1}{2}\lambda^2} = \begin{cases} 
+1 & \text{for } \lambda = \begin{cases} 
4m + 0 \to (\lambda)_0 \\
4m + 1 \to (\lambda)_1 \\
4m + 2 \to (\lambda)_2 \\
4m + 3 \to (\lambda)_3 
\end{cases} \\
-1 & \text{for } \lambda = \begin{cases} 
4m + 0 \to (\lambda)_0 \\
4m + 1 \to (\lambda)_1 \\
4m + 2 \to (\lambda)_2 \\
4m + 3 \to (\lambda)_3 
\end{cases} 
\end{cases}
\]

\[
m = 0, 1 \cdots; \quad 0 \leq \lambda \leq 2\nu
\]

(B.4)

The signature \(s(\lambda; \rho, \sigma)\) defined in eq. (B.3) then separates the (integer) indices further

\[
s(\lambda; \rho, \sigma) = \begin{cases} 
+1 & \text{for } \rho = \begin{cases} 
en even \& (\lambda)_{0\&1} \\
o odd \& (\lambda)_{2\&3} 
\end{cases} \\
-1 & \text{for } \rho = \begin{cases} 
en even \& (\lambda)_{2\&3} \\
o odd \& (\lambda)_{0\&1} 
\end{cases} 
\end{cases}
\]

(B.5)

Hence the power \(M\) of the set \(S_- = \{\rho, \sigma \mid s(\lambda; \rho, \sigma) = -1\}\) is the number of antisymmetric \(2\nu \times 2\nu\) matrices

\[
M = \sum_{S_-} \binom{p}{\rho} \binom{q}{\sigma} = 2^{\nu-1}(2\nu - 1); \quad p + q = 2\nu
\]

(B.6)

\[
S_- = \begin{cases} 
\rho \text{ even } & (\lambda)_{2\&3} \\
\rho \text{ odd } & (\lambda)_{0\&1} 
\end{cases}; \quad \lambda = \rho + \sigma; \quad 0 \leq \rho \leq p; \quad 0 \leq \sigma \leq q
\]
Equation (B.6) only holds provided a Majorana representation \{\Gamma_{p,q}; \mathbb{R}\} exists, thereby yielding a nontrivial condition. We illustrate this for \( p = 2, q = 0 \)

\[
q = 0 \rightarrow \lambda = \varrho \rightarrow \\
\varrho \text{ even } \& (\lambda)_{2k:3} \rightarrow \varrho = 2 \\
\varrho \text{ odd } \& (\lambda)_{0k:1} \rightarrow \varrho = 1 \rightarrow M = 3 \\
\nu = 1 \rightarrow 2^{\nu-1}(2^{\nu} - 1) = 1 \neq M
\]

The set \( S_- \) defined in eq. (B.6) can also be classified according to the mod 4 classes of \( \rho, \sigma; (\rho), (\sigma) \) separately

\[
S_- = \left\{ \begin{array}{ccc}
(\rho) & (\sigma) & (\rho) \\
0 & 2 & 0 \\
1 & 0 & 1 \\
2 & 0 & 2 \\
3 & 1 & 3
\end{array} \right\}
\]

Hence the calculation of \( M \) involves a selected sum over the pair of mod 4 class sums

\[
M_{(\rho)(\sigma)}^{p q} = \sum_{r,s} \left( \begin{array}{c} p \\ \frac{4r + (\rho)}{4} \\
q \\
\frac{4s + (\sigma)}{4} \end{array} \right)
\]

with \( \{ 4r + (\rho) \leq p \}
\]
\( \{ 4s + (\sigma) \leq q \}

The double sums for \( M_{(\rho)(\sigma)}^{p q} \) in eq. (B.9) factorize

\[
M_{(\rho)(\sigma)}^{p q} = M_{(\rho)}^{p} M_{(\sigma)}^{q} \\
M_{(\tau)}^{n} = \sum_{u=0}^{4u+(\tau) \leq n} \left( \begin{array}{c} n \\
4u + (\tau) \end{array} \right)
\]

\[
M_{(\tau)}^{n} = 0 \text{ for } n < (\tau)
\]

If the condition(s) \( 4u + (\tau) \leq n \) cannot be satisfied, i.e. for \( n < (\tau) \) the mod 4 sum \( M_{(\tau)}^{n} \) has to be set to zero, as indicated in eq. (B.10). The factorized forms thus yield for \( M \) (eq. (B.6))

\[
M = \begin{bmatrix}
M_{(0)}^{p} & M_{(2)}^{q} + M_{(3)}^{q} \\
+M_{(1)}^{p} & M_{(0)}^{q} + M_{(3)}^{q} \\
+M_{(2)}^{p} & M_{(0)}^{q} + M_{(1)}^{q} \\
+M_{(3)}^{p} & M_{(1)}^{q} + M_{(2)}^{q}
\end{bmatrix}
\]

Appendix C. mod 4 sums of binomials and powers of 2 – Pascal’s triangle [5]

The triangle of Pascal is shown in figure C1.
The mod 4 sums of binomial coefficients $M_n^{\tau}$ defined in eq. (B.10) shall be endowed with the prefix 4 for clarity of notation

$$M_n^{\tau} \rightarrow 4M_n^{\tau} = \sum_{4u+(\tau)\leq n} \binom{n}{4u+(\tau)} ; \; (\tau) = 0, 1, 2, 3$$  \hspace{1cm} (C.1)

The periodicity structure \{mod4\} can be mapped on the powers of the fourth roots of 1 (over $\mathbb{C}$)

$$r^{(\tau)} = i^{(\tau)} = \exp \left( \frac{2\pi i (\tau)}{4} \right) ; \; (\tau) = 0, 1, 2, 3 \rightarrow$$

$$k : r^{(\tau)} \rightarrow (r^{(\tau)})^k = (r^{(\tau)})^{(k)} = x^{(\tau)(k)} ; \; (\tau), (k) = 0, 1, 2, 3$$  \hspace{1cm} (C.2)

I display the mod 4 multiplication table for the quantity $(r^{(\tau)}(k))$ below

| $(\tau)$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| 0      | 0 | 0 | 0 | 0 |
| 1      | 0 | 1 | 2 | 3 |
| 2      | 0 | 2 | 0 | 2 |
| 3      | 0 | 3 | 2 | 1 |

It follows that the inverse matrix to $x^{(\tau)(k)}$ denoted $y^{(k)(\tau)}$ filters out the mod 4 sums in conjunction with any generating function given by a power series $G(z) = \sum_{k=0}^{\infty} G_k z^k$

$$y^{(k')(\tau)} x^{(\tau)(k)} = \delta^{(k')(k)}$$

$$y^{(k)(\tau)} = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
i & -1 & 1 & -i \\
\end{pmatrix} = \frac{1}{4} (i)^{- (k)(\tau)}$$  \hspace{1cm} (C.3)

**Appendix C.1. mod 4 filtering**

We define the following set of generating functions associated with a given generating function $G(z)$

$$G(z) = \sum_{k=0}^{\infty} a_k z^k \rightarrow$$

$$G^{(\tau)}(z) = G(i^{(\tau)} z) ; \; (\tau) = 0, 1, 2, 3$$  \hspace{1cm} (C.4)

$$G^{(\tau)}(z) = \sum_{k=0}^{\infty} i^{(\tau)(k)} a_k z^k$$  \hspace{1cm} (C.5)

**Figure C.1.** Pascal’s triangle $\binom{n}{k}$ for $n = 1, 2, \cdots, 16.$
In eq. (C.5) the quantities \( x(\tau)(k) \) defined in eq. (C.2) appear, multiplying the k-th not the (k)-th term in the power series for \( G(\tau)(z) \). It follows using \( y \), the inverse of \( x \) defined in eqs. (C.2) and (C.4), setting

\[
\tilde{G}^{(l)}(z) = \sum_{(\tau)} y(l)(\tau) G(\tau)(z) ; \quad k = 4u + (k); \quad u = 0, 1, \ldots
\]

Before generating the mod 4 sums of binomial coefficients we have to settle a subtle case in the definition of \( 4^M^0 \), which occurs through the properties of the set \( S_- \) defined in eq. (B.6) arising when either \( p \) or \( q \) is 0 (but not both)

\[
4^M^0 = 1, 4^M^0(k) = 0 \quad \text{for } (k) > 0
\]  

With the case \( n = 0 \) given in eq. (C.7) we can use for \( n \geq 1 \) as generating function for the mod 4 sums of binomial coefficients the generating polynomial

\[
G(n; z) = (1 + z)^n \quad ; \quad n \geq 1
\]

\[
= \sum_{k=0}^{n} a^n_k z^k ; \quad a^n_k = \binom{n}{k}
\]  

The base functions \( G(\tau) \) in eq. (C.5) thus become

\[
G_{(0)}(n; z) = (1 + z)^n
\]

\[
G_{(1)}(n; z) = (1 + iz)^n
\]

\[
G_{(2)}(n; z) = (1 - z)^n
\]

\[
G_{(3)}(n; z) = (1 - iz)^n
\]

\[
4^M^n(l) = \sum_{(\tau)} y(l)(\tau) Y^n_{(\tau)} ; \quad Y^n_{(\tau)} = G_{(\tau)}(n; z = 1)
\]  

With \( y \) determined in eq. (C.2) it remains to calculate the constants \( Y \) defined in eq. (C.9)

\[
Y^n_{(0)} = 2^n \quad , \quad Y^n_{(1)} = (1 + i)^n
\]

\[
Y^n_{(2)} = 0 \quad , \quad Y^n_{(3)} = (1 - i)^n = \left( Y^n_{(1)} \right)^* \]  

It is the powers \((1 \pm i)^n\) for odd \((\tau)\) within the mod 4 logic, which bring about the mod 8 dependence inherent to Majorana representations, for even \( d = p + q = 2\nu \).
We thus rewrite the quantities

\[ n = 8u + \{n\} \quad \{n\} = 0, 1 \cdots 7, \quad u = 0, 1 \cdots \]

\[ (1 + i)^n = 2^{4u} \ (1 + i)^{\{n\}} \]

| \{n\} | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| \(\frac{1}{\sqrt{2}} (1 + i)^{\{n\}}\) | 1 | \(i\) | -1 | -\(i\) |
| \{n\} | 1 | 3 | 5 | 7 |
| \(\frac{1}{\sqrt{2}} (1 + i)^{\{n\}}\) | \(\frac{1}{\sqrt{2}} (1 + i)\) | \(\frac{1}{\sqrt{2}} (-1 + i)\) | \(\frac{1}{\sqrt{2}} (-1 - i)\) | \(\frac{1}{\sqrt{2}} (1 - i)\) |

(\ref{eq:C11})

**Appendix D. mod 4 sums of binomials and powers of \(2^\frac{1}{2}\) and \(i^\frac{1}{2} \equiv \exp \left(\frac{2\pi}{8}i\right)\)**

It is through the generating polynomials that half-integer powers of 2 (and i) enter. We rewrite eq. (\ref{eq:C10}) and use y in the form given in eq. (\ref{eq:C4})

\[ Y_{(0)}^n = 2^n, \quad Y_{(1)}^n = \left(2^\frac{1}{2}\right) \left(i^\frac{1}{2}\right) \]

\[ Y_{(2)}^n = 0, \quad Y_{(3)}^n = \left(2^\frac{1}{2}\right) \left(i^{-\frac{n}{2}}\right) = \left(Y_{(1)}^n\right)^* \]

\[ i^\frac{n}{2} = i^{\{n\}} ; \quad \{n\} = n \mod 8 ; \quad y_{(1)}(\tau) = \frac{1}{4} (i)^{-(\ell)(\tau)} \]

This yields the characteristic sums

\[ Y_{(\ell)}^n = \frac{1}{4} \sum_{(\tau)} (i)^{-(\ell)(\tau)} Y_{(\tau)}^n = 2^{n-2} + \frac{1}{4} \Delta_{n(\ell)} \]

\[ \Delta_{n(\ell)} = 2^\frac{1}{2} \left( (i)^{-3(\ell)} (i)^{\frac{n}{2}} + (i)^{-(\ell)} (i)^{-\frac{n}{2}} \right) \]

\[ = 2^\frac{1}{2} \left( (i)^{-3(\ell)} (i)^{\frac{n}{2}} + (i)^{(\ell)} (i)^{-\frac{n}{2}} \right) \]

\[ (i)^{-3(\ell)} = (i)^{(\ell)} \]

Equation (\ref{eq:D2}) yields

\[ \Delta_{n(\ell)} = (2)^{\left(\frac{\ell}{2}\right)+1} \Re \left( (i)^{\frac{\ell}{2} -(\ell)} \right) \]

\[ (i)^{\frac{\ell}{2} -(\ell)} = \left( (i)^{\left[\frac{n}{2}\right] -(\ell)} \right) \left\{ \begin{array}{ll} 1 & \text{for } n \text{ even} \\ i^{\frac{n}{2}} & \text{for } n \text{ odd} \end{array} \right. \]

\[ \left[\frac{n}{2}\right] = \left\{ \begin{array}{ll} \frac{n}{2} & \text{for } n \text{ even} \\ \frac{n}{2} - \frac{1}{2} & \text{for } n \text{ odd} \end{array} \right. \]

(\ref{eq:D3})

We thus rewrite the quantities \(Y_{(\ell)}^n\) in eq. (\ref{eq:D2})

\[ 4M_{(\ell)}^n = Y_{(\ell)}^n = 2^{n-2} + 2^{\left[\frac{\ell}{2}\right]}^{-1} F(n - 2(\ell)) \quad ; \quad n \geq 1 \]

\[ F(n - 2(\ell)) = \Re \left( \exp \left( i \frac{\pi}{4} (n - 2(\ell)) \right) \right) \times \left\{ \begin{array}{ll} 1 & \text{for } n \text{ even} \\ \sqrt{2} & \text{for } n \text{ odd} \end{array} \right. \]

\[ F \rightarrow F(j) ; \quad j = 0, \pm 1, \pm 2 \cdots \text{ with } j \rightarrow n - 2(\ell) \]

\[ \text{(\ref{eq:D4})} \]

\[ \text{(\ref{eq:D4})} \]
The function \( F(j) \) defined in eq. (D.4) over the signed integers \( j \) takes only integer values \( \{F\} = \{0, \pm 1\} \)

\[
F(j) = F(-j) = F(j + 8)
\]

\[
\begin{array}{ccccccccccc}
  & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
F & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1
\end{array}
\]  
(D.5)

\( F \) can be visualized as projection on the real axis of a side-centered quadrangle in the complex plane, with the side centers forming an inscribed quadrangle rotated by 45 degrees, as shown in figure D1 below.

![Figure D1](image)

**Figure D1.** The side-centered quadrangle(s) associated with the function \( F(j) \).

We collect the formulae determining the mod 4 sums of binomial coefficients (eqs. C.7, (D.4))

\[
4M^0_{(0)} = 1, 4M^0_{(l)} = 0 \text{ for } (l) > 0
\]

\[
4M^l = Y^{n(l)} = 2^{n-2} + 2^{[\frac{n}{2}]}F(n - 2(l)) ; \quad n \geq 1
\]

with \( F(j) \) defined in eq. (D.5)

Care must be taken if using eq. (D.6) for \( n = 1 \) and 2 whenever \((l) > n\).

**Appendix E. The spin (10) product representations** \((16 \oplus \overline{16}) \otimes (16 \oplus \overline{16})\)

We follow the spin (10) decomposition discussed in section 3 (eq. (17) repeated below)

\[
\text{spin}(10) \rightarrow \text{SU5} \times U1J_5
\]

Further let us denote representations of spin (10) as opposed to those pertaining to SU5 and associated \( J_5 \) quantum number by

\[
\text{spin} (10) : \{\text{dim}\} \quad \text{SU5} \times U1J_5 : \{\text{dim}\}_J \]

Thus eq. (21) translates to

\[
[16] = \{1\}_+ + \{10\}_+ + \{5\}_- \\
[\overline{16}] = \{1\}_- + \{10\}_- + \{5\}_+
\]

(E.3)

In turn SU5 representations shall be decomposed along the standard model gauge group \(\text{SU3}_c \otimes \text{SU2}_L \otimes \text{U1}_Y\), where \( Y \) denotes the electroweak hypercharge (with a factor \( \frac{1}{2} \) included)

\[
Y = Q_{e.m.}/e - I_{3L}
\]

(E.4)
The dimensions $\{\text{dim}\} \rightarrow \sum (\text{dim SU}_3, \text{dim SU}_2_L)_Y$ \[ (E.5) \]

The brackets on the right hand side of eq. (E.5) are reversed in order not to confuse spin(10) and standard model representations. Then the base 16 (16) decompose to

\[
\begin{align*}
\{1\}_5 + & \rightarrow \{1, 1\}_0 \bigg[ \begin{array}{c} \{3, 2\}_+ \frac{1}{6} \bigg] + \\
\{10\}_{+1} & \rightarrow \bigg[ \begin{array}{c} \{3, 1\}_- \frac{2}{3} \\
\{1, 1\}_+ \bigg] \bigg[ \begin{array}{c} \{3, 1\}_+ \frac{1}{3} \bigg] + \\
\{5\}_{-3} & \rightarrow \bigg[ \begin{array}{c} \{3, 1\}_+ \frac{1}{3} \bigg] + \\
\end{align*}
\]
\[ (E.6) \]

The product representations $\{16 \oplus \overline{16}\} \otimes \{16 \oplus \overline{16}\}$ generate all SO(10) antisymmetric tensor ones, of which we encountered the fivefold antisymmetric in section 3 (eq. (26)).

To elaborate we specify the $n$-fold antisymmetric tensors obtained from the 10-representation of SO(10)

\[
\begin{align*}
[t_0] & \sim 1 \\
[t_1]^A & \sim z^A ; \ A = 1, 2, \cdots, 10 \leftrightarrow [t_1] = \{5\}_2 \oplus \{5\}_{-2} \\
[t_2]^{[A_1A_2]} & \sim \frac{1}{2} \left( z_1^{A_1} z_2^{A_2} - z_2^{A_1} z_1^{A_2} \right) \\
\cdots \\
[t_n]^{[A_1A_2\cdots A_n]} & \sim \frac{1}{n!} \sum \text{sgn} \left( \begin{array}{c} 1 \\ \pi_1 \\ \vdots \\ \pi_n \end{array} \right) z_1^{A_{\pi_1}} z_2^{A_{\pi_2}} \cdots z_n^{A_{\pi_n}} \\
n & \leq 10
\end{align*}
\[ (E.7) \]

The quantities $[t_n]$ defined in eq. (E.7) form irreducible real representations of SO(10) except for $n = 5$, which is composed of the relatively complex irreducible representations 126 and 126 (eq. (26)). The tenfold antisymmetric invariant corresponds to $[t_{n=10}]$. The product of two full Clifford algebras pertaining to spin (10) contains all $[t_n]$; $n = 0 \cdots 10$ representations exactly once. Treating the $n = 5$ tensor as one representation – it is reducible only over $\mathbb{C}$ – the dimensions of the $[t_n]$ representations follow Pascal’s triangle (figure C1) of binomial coefficients for $N = 10$, whereby $n$ even and odd shall be distinguished

\[
\begin{array}{ccccccc}
[t_0] & [t_1] & [t_2] & [t_3] & [t_4] & [t_5] & [t_6] \ & [t_7] & [t_8] & [t_9] & [t_{10}] \\
1 & 45 & 210 & 210 & 45 & 1 & & & & & \\
10 & 120 & 252 & 120 & 10 & & & & & & \\
\end{array}
\]
\[ (E.8) \]

This corresponds to the following products of $16 + \overline{16}$

\[
\begin{array}{c|c|c}
\{16\} & \{10\} + \{126\} & \{1\} + \{45\} + \{210\} \\
\{16\} & s: & a: \{120\} \\
\{16\} & \{1\} + \{45\} + \{210\} & s: & a: \{120\} \\
\end{array}
\]
\[ (E.9) \]
The correspondence of product representations of the $16 + \overline{16} = 32$ associative Clifford algebra with the sum of antisymmetric tensor ones follows from the completeness of all products of $\gamma$ matrices forming the spin(10) algebra i.e. are of dimension

$$ (32)^2 = (2^5)^2 = 2^{10} \quad \text{(E.10)} $$

We proceed to reduce the $[16] \otimes [16]$ product with respect to $J_5$, SU5 and SU3c × SU2L × U1Y. The individual products are $\left( a_s : (a)\text{symmetric} \right)$

$$ \begin{array}{c|c|c|c}
{\{ 1 \}}_5 & {\{ 10 \}}_1 & {\{ 5 \}}_{-3} \\
\hline
{\{ 10 \}}_1 & {\{ 10 \}}_6 & {\{ 5 \}}_2 \\
{\{ 5 \}}_{-3} & {\{ 5 \}}_2 & \left( \begin{array}{c} \{ 5 \}_2^+ \\ \{ 45 \}_2^+ \end{array} \right) \\
\end{array} \quad \text{(E.11)} $$

We proceed to decompose the diagonal $\{ SU5 \} J_5$ representations (eq. (E.6))

$$ \{10\}_1 \otimes \{10\}_1 = \{\overline{5}\}_2 + \{\overline{50}\}_2 \quad \downarrow $$

$$ \begin{array}{c|c|c|c|c}
 & (3, 2)_{\pm} & (\overline{3}, 1)^{\mp} & (1, 1)^{\pm} \\
\hline
(3, 2)_{\pm}^+ & \left( (6, 3)_{\pm}^+ + (8, 2)_{\pm}^+ \right) & \left( (1, 2)_{\pm}^+ \right) & \left( (3, 2)_{\pm}^+ \right) \\
(\overline{3}, 1)^{\mp} & \left( (\overline{6}, 1)^{\pm} \right) & \left( (\overline{3}, 1)^{\pm} \right) & \left( (1, 1)^{\pm} \right) \\
(1, 1)^{\pm} & \left( (1, 1)^{\pm} \right) & \left( (1, 1)^{\pm} \right) & \left( (1, 1)^{\pm} \right) \\
\end{array} \quad \text{(E.12)} $$

$$ \left( \{\overline{5}\}_{-3} \otimes \{\overline{5}\}_{-3} \right)_s = \{\overline{15}\}_{-6} \quad \downarrow $$

$$ \begin{array}{c|c|c|c|c}
 & (\overline{3}, 1)^{\mp} & (1, 2)^{\pm} & (1, 3)^{\pm} \\
\hline
(\overline{3}, 1)^{\mp} & \left( (\overline{6}, 1)^{\pm} \right) & \left( (\overline{3}, 2)^{\pm} \right) & \left( (1, 1)^{\pm} \right) \\
(1, 2)^{\pm} & \left( (1, 3)^{\pm} \right) & \left( (1, 3)^{\pm} \right) & \left( (1, 3)^{\pm} \right) \\
\end{array} \quad \text{(E.13)} $$

complex e.w. triplet coupling to

$$ \frac{1}{\sqrt{2}} \left( \nu_F^\alpha \right)^{\alpha} \left( \nu_G^\alpha \right)^{\alpha} $$

Next we assemble the (anti)symmetric products $\left( [16] \otimes [16] \right)_s = [10] \oplus [126]$ and $\left( [16] \otimes [16] \right)_a =$
with respect to SU5 ⊗ U1,t using eq. (E.11)

\[
([16] \otimes [16])_s = [10] \oplus [126] \\
\]

\[
= \begin{cases}
\{5\}_{-2} + \{5\}_{2} \\
\{1\}_{10} + \{\overline{5}\}_{2} + \{10\}_{6} + \{\overline{15}\}_{-6} + \{45\}_{-2} + \{\overline{50}\}_{2}
\end{cases}
\]

\[
([16] \otimes [16])_a = [120] \\
\]

\[
= \begin{cases}
\{5\}_{-2} + \{\overline{5}\}_{2} \\
\{10\}_{6} + \{10\}_{-6}
\end{cases}
\]

The roman indices \( I, \overline{I} \) in eq. (E.14) indicate that appropriate linear combinations of the two \( \{\overline{5}\}_{2} \) representations form parts of \([10]\) and \([126]\) respectively.

It remains to decompose the SU5 ⊗ U1,t representations in eq. (E.14) with respect to SU3c × SU2L × U1y. We do this associating according to the product representations as they appear in eq. (E.14)

\[
\begin{array}{ccc}
[10] [120] & \{5\}_{-2} & (3, 1)_{-\frac{1}{3}} \left[ \begin{array}{c} + \\ l_{+3}
\end{array} \right] + (1, 2)_{+\frac{1}{3}} \left[ \begin{array}{c} + \\ l_{+3}
\end{array} \right] \\
[10] [126] [120] & \{\overline{5}\}_{+2} & (\overline{3}, 1)_{\frac{1}{3}} \left[ \begin{array}{c} - \\ l_{-3}
\end{array} \right] + (1, 2)_{-\frac{1}{3}} \left[ \begin{array}{c} + \\ l_{+3}
\end{array} \right] \\
[126] & \{1\}_{+10} & (1, 1)_{0} \left[ \begin{array}{c} + \\ l_{+10}
\end{array} \right]
\end{array}
\]

\[
\begin{array}{ccc}
[126] [120] & \{10\}_{+6} & (3, 2)_{-\frac{1}{3}} \left[ \begin{array}{c} + \\ l_{+6}
\end{array} \right] + (\overline{3}, 1)_{\frac{1}{3}} \left[ \begin{array}{c} - \\ l_{+6}
\end{array} \right] + (1, 1)_{+1} \left[ \begin{array}{c} + \\ l_{+6}
\end{array} \right]
\end{array}
\]

\[
\begin{array}{ccc}
[120] & \{\overline{10}\}_{-6} & (\overline{3}, 2)_{+\frac{1}{3}} \left[ \begin{array}{c} - \\ l_{-6}
\end{array} \right] + (3, 1)_{-\frac{1}{3}} \left[ \begin{array}{c} + \\ l_{-6}
\end{array} \right] + (1, 1)_{-1} \left[ \begin{array}{c} + \\ l_{-6}
\end{array} \right]
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{\overline{15}\}_{-6} & (\overline{5}, 1)_{+\frac{2}{3}} \left[ \begin{array}{c} - \\ l_{-6}
\end{array} \right] + (\overline{3}, 2)_{-\frac{1}{3}} \left[ \begin{array}{c} - \\ l_{-6}
\end{array} \right] + (1, 3)_{-1} \left[ \begin{array}{c} + \\ l_{-6}
\end{array} \right]
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{45\}_{-2} & c.c.
\end{array}
\]

\[
\begin{array}{ccc}
[120] & \{\overline{15}\}_{+2} & \\
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{\overline{50}\}_{+2} & \\
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{\overline{50}\}_{+2} & \\
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{\overline{50}\}_{+2} & \\
\end{array}
\]

\[
\begin{array}{ccc}
[126] & \{\overline{50}\}_{+2} & \\
\end{array}
\]
\[
\{10\}_1 \otimes \{10\} = \{45\}
\]

\[
\begin{array}{c|ccc}
\{3, 2\} & \{3, 2\}_1 & \{3, 1\} & \{1, 1\} \\
\hline
\{5\} & 5_2 & 5_+ \\
\{10\} & 10_2 & 10_+ & 10_+ \\
\end{array}
\]

(E.17)

I end the collection of representation decompositions with the adjoint \([45]\) representation of \(SO(10)\)

\[
\{10\} \otimes \{10\} = \{45\}
\]

(E.18)

It should be noted that despite coinciding dimensions the following entities are most distinct

\[
\{10\} \neq \{10\}_4, \{10\}_6
\]

\[
\{45\} \neq \{45\}_2 : \cdots
\]

(E.19)

References.

[1] Konetschny W and Kummer W 1977 Nonconservation of total Lepton number with scalar bosons Phys. Lett. B 70 433

Cheng T P and Li L F 1980 Neutrino masses, mixings and oscillations in SU(2)xU(1) models of electroweak interactions Phys. Rev. D 22 2860

Lazarides G, Shafi Q and Wetterich C 1981 Proton lifetime and fermion masses in an SO(10) model Nucl. Phys. B 181 287

Schechter J and Valle J W F 1980 Neutrino masses in SU(2)xU(1) theories Phys. Rev. D 22 2227

Mohapatra R N and Senjanović G 1981 Neutrino masses and mixings in gauge models with spontaneous parity violation Phys. Rev. D 23 165

for a review of extended lepton multiplets see also: Fileviez Perez P, Han T, Huang G, Li T and Wang K 2008 Neutrino masses and the CERN LHC: testing type II seesaw Phys. Rev. D 78 015018 (Preprint arXiv:0805.3536 [hep-ph])

[2] Cartan É 1926 Sur une classe remarquable d’espaces de Riemann Bull. Soc. Math. de France 54 214; 1927 55 114

[3] Sohnius M F 1976 The Conformal Group in Superspace MPI Report MPI-PAE/PTh 32/76, 11p (Proc. 2nd Conf. on Structure of Space and Time (Tutzting, West Germany, 1976) (1977 Proc. Quantum Theory and the Structure Of Time and Space [In Memoriam Werner Heisenberg] (Peldafing 1976) vol 2 pp 241–252)

Sohnius M F and West P C 1981 Conformal invariance in N=4 supersymmetric Yang-Mills theory Phys. Lett. 100B 245

[4] Brauer R and Weyl H 1935 Spinors in n dimensions Amer. J. Math. 57 425–449

[5] Coquereaux R 1982 Modulo 8 Periodicity Of Real Clifford Algebras And Particle Physics Phys. Lett. B 115 389
[6] Minkowski P 2001 On amplified and tiny signatures of neutrino oscillations Proc. NO-VE Int. Workshop on Neutrino Oscillations (Venice, Italy 24–26 July 2001) p 291

There is an error in the definition of the $\hat{A}$ genus in ref. [6], which is corrected in eq. (3) here.

[7] Hirzebruch F 1966 Topological methods in algebraic geometry Die Grundlehren der mathematischen Wissenschaften vol 131 (Berlin: Springer Verlag)

[8] For a more complete account of left- and right-chiral bases see also: Minkowski P Neutrino flavors light and heavy – how heavy? http://www.mink.itp.unibe.ch/data/nus-albufeira2007.pdf and references therein

[9] Fritzsch H and Minkowski P 1975 Unified interactions of leptons and hadrons Annals Phys. 93 193

Georgi H 1975 The state of the art – gauge theories AIP Conf. Proc. 23 575

[10] Fritzsch H, Gell-Mann M and Minkowski P 1975 Vector-like weak currents and new elementary fermions Phys. Lett. B 59 256

[11] Fritzsch H and Minkowski P 1976 Vector-like weak currents, massive neutrinos and neutrino beam oscillations Phys. Lett. B 62 72

[12] Fukugita A and Yanagida T 1994 Physics of neutrinos Physics and Astrophysics of Neutrinos ed A Fukugita and A. Suzuki (Springer-Verlag)

[13] Pati J and Salam A 1974 Lepton number as the fourth color Phys. Rev. D 10 275; erratum: 1975 Phys. Rev. D 11 703

[14] Minkowski P 1977 $\mu \rightarrow e\gamma$ at a rate of one out of 1-billion muon decays? Phys. Lett. B 67 421

Heusch C and Minkowski P 1994 Lepton flavor violation induced by heavy Majorana neutrinos Nucl. Phys. B 416 3

[15] Gell-Mann M, Ramond P and Slansky R 1979 Complex spinors and unified theories Supergravity ed P van Nieuwenhuizen and D Z Freedman (North Holland Publ. Co.) (1979 Proc. Stony Brook Workshop 0315 (QC178:S8:1979))

Yanagida T 1979 Horizontal symmetry and masses of neutrinos Proc. Workshop on the Baryon Number of the Universe and Unified Theories (Tsukuba, Japan, 13–14 February 1979) ed O Sawada and A Sugamoto (QCD161:W69:1979)

Glashow S 1980 Quarks and leptons Proc. of the Cargèse Lectures ed M Lévy (New York: Plenum Press)

Mohapatra R and Senjanović G 1980 Neutrino mass and spontaneous parity violation Phys. Rev. Lett. 44 912

[16] Georgi H and Glashow S L 1974 Unity of all elementary particle forces Phys. Rev. Lett.32 438

[17] Minkowski P 1980 Fermionic oscillators and the embedding $\text{SO}(2n) \supset \text{SU}(n)$ seminar at the Enrico Fermi Summer School of Physics, Varenna, Italy, July 21–August 2, 1980 (unpublished)

[18] Gursey F, Ramond P and Sikivie P 1976 A universal gauge theory model based on $E_6$ Phys. Lett.B 60 177 (1975 Yale Report YALE-3075-118)