Convergence Rates in Strong Ergodicity by
Hitting Times and $L^2$-exponential
Convergence Rates*

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Abstract

The computable convergence rates in strong ergodicity for Markov processes are obtained by using uniformly bounded moments of hitting times and the convergence rates in $L^2$-exponential ergodicity. We reveal a phenomenon for a class of Markov processes which are strongly ergodic that the two convergence rates in both strong ergodicity and exponential ergodicity are identical. These processes vary from Markov chains, diffusions to SDEs driven by symmetric stable processes.

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1 Introduction and general results

Strong ergodicity (or uniform ergodicity) is an important topic in ergodic theory for Markov processes. In this paper, we are interested in the convergence rate in strong ergodicity.

It is well known that the criteria for a Markov process to be strongly ergodic, is to use the uniformly bounded moment of the first return time related to any petite set (or equivalently, a bounded Lyapunov function), especially for the Markov chains. See [1, 13, 21].

To get the (exponential) convergence rates for discrete-time Markov chain, several types of classical methods are used, such as minorization condition ([22]), Foster-Lyapunov criteria ([2]) and Dobrushin’s ergodicity coefficients ([23]) which can be used conceptually to continuous-time Markov processes, as in [1, Chapter 6].

Coupling methods can be generally used to estimate the convergence rate via the moments of the so-called coupling time (see [6], [7]). This was done in [17] for the

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convergence rate in strong ergodicity of Markov chain and diffusion process, then was improved by [18]. However, to apply the coupling method, the stochastic monotone property is needed to estimate the moment of the coupling time.

Historically, the study of the convergence rate in strong ergodicity was much later than that in exponential ergodicity, although both the convergence rates are exponential. One reason for this may lie on the fact that $L^\infty$-norm for strong ergodicity is less smooth than the $L^2$-norm for exponentially ergodicity, especially for the reversible Markov processes. Even for the reversible Markov processes, no functional inequality can be adopted directly for the convergence rates in strong ergodicity. For the reversible Markov processes, the spectral gap given by the classical Poincaré inequality is identical to the optimal convergence rate in exponential ergodicity. See, for example, [6], [7] or [27].

A “mixed” method appeared in [19] where the moment of hitting time and spectral gap for reversible Markov chains are used to estimate the convergence rate in strong ergodicity. The advantage of the “mixed” method is two-fold. On the one hand, in many cases, the uniform moment of the hitting time can also afford the lower bounds for the spectral gap, so that we can get the explicit bound given by using the moment of the hitting time in many concrete models. On the other hand, if it happens that the upper bound can be given to the spectral gap, then we find a phenomenon that the optimal convergence rate in strong ergodicity $\kappa$ equals to the spectral gap $\lambda_1$ whenever the process is strongly ergodic. This is an interesting phenomenon which was first proved in [20] for the birth-death process. In general, if the Markov semigroup $P_t$ is ultra-bounded, i.e. $\|P_t\|_{2\rightarrow\infty} < \infty$ for some $t > 0$, then $\kappa = \lambda_1$, see [18] for an argument. However, ultra-boundedness is a much stronger property to be satisfied. As we will see soon, we actually find an extensive class of Markov processes, from Markov chains, diffusion processes to Lévy type processes, satisfying $\kappa = \lambda_1$.

Let $(X_t)_{t \geq 0}$ be a Markov process on state space $(E, \mathcal{B})$ with transition function $P_t(x, \cdot)$ which admits a stationary probability measure $\pi$.

**Definition 1.1.** The (exponential) convergence rate in strong ergodicity is defined by

$$\kappa = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}}.$$ 

If $X$ is strongly ergodic, then $\sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}} \to 0$ as $t \to \infty$. This convergence must be exponential, since by Markov property:

$$\sup_{x \in E} \|P_{t+s}(x, \cdot) - \pi\|_{\text{Var}} \leq \sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}} \times \sup_{x \in E} \|P_s(x, \cdot) - \pi\|_{\text{Var}}.$$ 

So $\exists C < \infty$ and $\epsilon > 0$ such that $\sup_{x \in E} \|P_t(x, \cdot) - \pi\|_{\text{Var}} \leq Ce^{-\epsilon t}$. Then $\kappa$ is the maximal $\epsilon$ in the previous estimate.

Two basic concepts are related to our study on the convergence rate in strong ergodicity.

The first concept is the convergence rate in exponential ergodicity: there exist $\epsilon > 0$ and a non-negative function $C(x) < \infty$ such that for any $x \in E$,

$$\|P_t(x, \cdot) - \pi(\cdot)\|_{\text{Var}} \leq C(x)e^{-\epsilon t}.$$  

(1.1)

Denote by $\lambda$ the maximal $\epsilon$ in the above inequality, which is called the convergence rate in exponential ergodicity. Obviously, $\lambda \geq \kappa$. A closed quantity to $\lambda$ is the $L^2$-exponential
convergence rate $\lambda_1$:

$$\lambda_1 := -\lim_{t \to \infty} \frac{1}{t} \log \|P_t - \pi\|_{L^2(\pi) \to L^2(\pi)},$$

where $L^2(\pi)$ is the usual $L^2$-space with respective to $\pi$. For the reversible Markov processes, $\lambda_1$ is just the $L^2$-spectral gap

$$\lambda_1 = \inf \{ D(f, f) : f \in \mathcal{D}, \pi(f) = 0, \pi(f^2) = 1 \}$$

where $(D, \mathcal{D})$ is the Dirichlet form of $X$. For the general Markov process, $\lambda$ and $\lambda_1$ may not be equal, but usually $\lambda \geq \lambda_1$. See Corollary 1.3 below.

The second concept related to $\kappa$ is the uniform moment of hitting time:

$$M_H := \sup_{x \in E} \mathbb{E}_x \tau_H,$$

where $\tau_H = \inf \{ t \geq 0 : X_t \in H \}$ is the hitting time to a subset $H$. It is well-known that under some regular condition, $X$ is strongly ergodic if and only if $M_H < \infty$ for some "petite" set $H$. Cf. [1, 21] and reference therein.

In this paper, we will use exponential ergodicity convergence rate $\lambda$ or $\lambda_1$ and the moment $M_H$ to derive the convergence rate $\kappa$ in strong ergodicity. For this, unless otherwise stated, we always make the following assumptions:

(A1) The state space $(E, \mathcal{B})$ is a locally compact Polish space with metric $\rho$, $X = (X_t)_{t \geq 0}$ is a progressive measurable right continuous strong Markov process on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration;

(A2) $X$ is non-explosive, admits a stationary probability measure $\pi$.

Under Assumption (A1), $\tau_H$ is a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$ and $X_{\tau_H} \in H$ for non-empty closed set $H$. Denote by $M_H = \sup_{x \in E} \mathbb{E}_x \tau_H$ and

$$\mathcal{H} = \{ H \in \mathcal{B} : H is a bounded closed set, such that M_H = \sup_{x \in E} \mathbb{E}_x \tau_H < \infty \}.$$

Now we can claim our general result, giving the relationship among $\kappa$ and $\lambda, M_H$.

**Theorem 1.2.** Let $\lambda$ be the convergence rate in exponential ergodicity. Assume that for any $\epsilon < \lambda$, (1.1) holds with $\sup_{x \in H} C(x) < \infty$ for some $H \in \mathcal{H}$. Then

$$\kappa \geq \min \left\{ \lambda, \frac{1}{M_H} \right\} > 0. \quad (1.2)$$

Consequently,

(R1) if there exists $H \in \mathcal{H}$ such that $\lambda \leq 1/M_H$, then $\kappa = \lambda$;

(R2) if there exists $H \in \mathcal{H}$ such that $\lambda \geq 1/M_H$, then $\kappa \geq 1/M_H$.

To apply Theorem 1.2, we need to prove the local boundedness of $C(x)$ on some $H \in \mathcal{H}$ in the exponential ergodicity (1.1). For Markov chain, we can consider $B$ as a single point and represent $C(x)$ explicitly by stationary distribution $\pi$ (such as Example 1.5). By using heat kernel $p_t(\cdot, \cdot)$ to represent $C(x)$, we can replace the exponential convergence rate $\lambda$ by the $L^2$-exponential convergence rate $\lambda_1$. 

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Corollary 1.3. Assume that \( P_t(x, dy) = p_t(x, y)\pi(dy), x, y \in E \). If there is \( s > 0 \) such that \( \phi(x) := \|p_s(x, \cdot)\|_{L^2(x)} < \infty \) (in the case of reversible processes, we have \( \phi(x) = p_{2s}(x, x) \)), then \( \lambda \geq \lambda_1 \). If further \( \sup_{x \in H} \phi(x) < \infty \) for some \( H \in \mathcal{H} \), then

\[
\kappa \geq \min \{\lambda_1, 1/M_H\}.
\]

Remark 1.4. (1) According to [18], if \((X_t)_{t \geq 0}\) is strongly ergodic, then

\[
\mathcal{G} := \{B \in \mathcal{B} : B \text{ is a bounded closed set, and } \pi(B) > 0\} \subset \mathcal{H}.
\]

In fact, there may exist some set \( H \in \mathcal{H} \setminus \mathcal{G} \). For example, we can take \( H \) a singleton, say \( \{0\} \), for the one-dimensional \( \alpha \)-stable process with \( \alpha \in (1, 2) \). Although \( \pi(\{0\}) = 0 \) as \( \pi \) has density with respect to the Lebesgue measure, \( M_{\{0\}} \) can be represented explicitly for the ergodic time-changed \( \alpha \)-stable process (see Theorem 4.3 below).

(2) When \( \lambda_1 \leq 1/M_H \), it is interesting to get that \( \kappa = \lambda \) (or \( \lambda_1 \) for the reversible process), that is the convergence rates in strong ergodicity and exponential ergodicity are identical. The prototype for this phenomenon is the birth-death process, see Example 1.5 below.

To end this section, we would like to give two examples to illustrate that the situations (R1) and (R2) in Theorem 1.2 respectively.

Example 1.5. [20, Theorem 1.1] Let \((X_t)_{t \geq 0}\) be a birth-death process on \( \mathbb{Z}_+ \) with birth rates \( b_i > 0 (i \geq 0) \) and death rates \( a_i > 0 (i \geq 1) \). Assume the process has the \( \infty \)-entrance boundary in Feller’s sense:

\[
\sum_{i=0}^{\infty} \mu_i \sum_{j=i}^{\infty} \frac{1}{\mu_j b_j} = \infty \quad \text{and} \quad S := \sum_{i=0}^{\infty} \frac{1}{\mu_i b_i} \sum_{j=i+1}^{\infty} \mu_j < \infty,
\]

where \( \mu_0 = 1, \mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i (i \geq 1) \).

The process is reversible with the stationary distribution \((\pi_i)_{i \geq 0} : \pi_i = \mu_i/\sum_{j=0}^{\infty} \mu_j \). Moreover, the process is stochastically monotone. By using the coupling method and the stochastically monotone property, the estimate \( \kappa \geq 1/(eS) \) was firstly given in [17] and then was improved to \( \kappa \geq 1/S \) in [18]. Now this estimate is improved further in two ways by applying Theorem 1.2. First, we see that [11] holds with \( \epsilon = \lambda_1 \) and \( C(x) = \sqrt{\pi^{-1}_x} - 1 \) for \( x \in \mathbb{Z}_+ \). By putting \( H_n = \{0, 1, \cdots, n\} \), we have \( \lim_{n \rightarrow \infty} M_{H_n} = 0 \), so that \( \lambda_1 \leq 1/M_{H_n} \) for \( n \) large enough. Hence \( \kappa = \lambda_1 > 0 \). From [5], we have

\[
\delta^{-1} \geq \kappa = \lambda_1 \geq (4\delta)^{-1}.
\]

where \( \delta = \sup_{n \geq 0} \sum_{i=n}^{\infty} \mu_i \sum_{i=0}^{n-1} \frac{1}{\mu_i b_i} \). Moreover, the approximation procedure in [5] can be applied to \( \kappa \). Second, from Theorem 4.2 in Section 4 below, we have

\[
\kappa = \lambda_1 \geq \sup_{i \geq 0} \left( \max \left\{ S_i, \overline{S}_i \right\} \right)^{-1} \geq 1/S,
\]

where \( S_i = \sum_{k=i}^{\infty} \frac{1}{\mu_k b_k} \sum_{j=k+1}^{\infty} \mu_j \) and \( \overline{S}_i = \sum_{k=0}^{i-1} \frac{1}{\mu_k b_k} \sum_{j=0}^{k} \mu_j \).

We also remark that the strong ergodicity can not imply the ultra-contraction. In [29], the examples of the strongly ergodic birth-death processes were given to exclude the hyper-contraction, let alone ultra-contraction.

The argument in Theorem 1.2 can be also applied to the discrete-time Markov chains, as shown in the following example.
Example 1.6. [Theorem 1.4] Let \((X_n)_{n \geq 0}\) be a reversible Markov chain on a discrete state space, with nonnegative definite transition matrix \(P\) and stationary distribution \(\pi\). Let \(H = \{0\}\) and \(M_0 := \sup_{x \in E} \E \tau_0 < \infty\). According to [24, Lemmas 3.11-3.12], the spectral gap \(\lambda_1 \geq 1/M_0\), therefore \(\kappa \geq 1/M_0\) by Theorem [12] (R2).

The paper is organized as follows. In Section 2, we prove our main results, which establish the relation among \(\kappa, \lambda_1\) and \(M_H\) for general Markov processes and obtain a new estimate of lower bound for \(\kappa\). In Section 3 and 4, we study two typical situations which made \(\kappa = \lambda_1\) and \(\kappa \geq 1/M_0\) respectively. The processes include single death process, birth-death process on tree, diffusion process on manifold, and SDE driven by stable process.

## 2 Proof of main results

The following lemma is the start point of our method, which can be seen as a mixture of hitting time and exponential ergodicity.

**Lemma 2.1.** For \(H \in \mathcal{H}\), let \(F_{x,H}(t) = \P_x[\tau_H \leq t]\) be the distribution of \(\tau_H\) and \(f(x,t) = \|P_t(x,\cdot) - \pi\|_{\text{Var}}\). Then

\[
f(x,t) \leq \P_x[\tau_H > t] + \int_0^t \sup_{y \in H} f(y,t-s) dF_{x,H}(s), \quad x \notin H.
\]

**Proof.** For \(x \notin H\) and \(A \in \mathcal{B}\), we have

\[
|P_t(x,A) - \pi(A)| = \left|\P_x[X_t \in A, \tau_H > t] + \P_x[X_t \in A, \tau_H \leq t] - \pi(A)\right|
\leq \left|\P_x[X_t \in A, \tau_H > t] - \pi(A)\P_x[\tau_H > t]\right|
\leq \left|\P_x[X_t \in A, \tau_H \leq t] - \pi(A)\P_x[\tau_H \leq t]\right|
\leq \P_x[\tau_H > t] + \left|\P_x[X_t \in A, \tau_H \leq t] - \pi(A)\P_x[\tau_H \leq t]\right|,
\]

where in the second inequality we use the fact \(|a - b| \leq c\) for \(0 \leq a, b \leq c\).

Note that by strong Markov property, for any \(A \in \mathcal{B}\), on \(\{\tau_H \leq t\}\),

\[
\P_x[X_t \in A|\mathcal{F}_{\tau_H}] = \P_x[X_{t-\tau_H} \circ \theta_{\tau_H} \in A|\mathcal{F}_{\tau_H}] = \P_{X_{\tau_H}}[X_t \in A],
\]

where \(\theta_s\) is the usual shift operator such that \(X_{s+t} = X_t \circ \theta_s\) for \(s,t \geq 0\). Using the conditional expectation with respect to the stopping \(\sigma\)-algebra \(\mathcal{F}_{\tau_H}\), it follows from \([22]\) that

\[
P_x[X_t \in A, \tau_H \leq t] = \E_x[\P_x[X_t \in A, \tau_H \leq t|\mathcal{F}_{\tau_H}]] = \E_x[1_{\{\tau_H \leq t\}}\P_x[X_t \in A|\mathcal{F}_{\tau_H}]]
\leq \int_E \int_0^t P_{t-s}(y,A)\P_x[\tau_H \in ds, X_{\tau_H} \in dy].
\]

Since \(X_{\tau_H} \in H\), we have

\[
\left|\P_x[X_t \in A, \tau_H \leq t] - \pi(A)\P_x[\tau_H \leq t]\right| = \left|\int_E \int_0^t (P_{t-s}(y,A) - \pi(A))\P_x[\tau_H \in ds, X_{\tau_H} \in dy]\right|
\leq \int_0^t \sup_{y \in H} |P_{t-s}(y,A) - \pi(A)| dF_{x,H}(s).
\]
By combining (2.1) and (2.4), the desired result is obtained.

Now, we use the above lemma to prove Theorem 1.2.

**Proof of Theorem 1.2**

(a) Thanks exponential ergodicity (1.1), the integral by parts gives for $x \notin H$,

\[
\int_0^t \sup_{y \in H} f(y, t-s) dF_x(s) \leq C_H \int_0^t e^{-\epsilon(t-s)} d(-\mathbb{P}_x[\tau_H > s])
\]

\[
= C_H e^{-\epsilon t} \left( 1 - e^{-\epsilon t} \mathbb{P}_x[\tau_H > t] + \int_0^t \mathbb{P}_x[\tau_H > s] \lambda e^{\epsilon t} ds \right)
\]

\[
\leq C_H e^{-\epsilon t} \left( 1 + \int_0^t \mathbb{P}_x[\tau_H > s] \epsilon e^{\epsilon t} ds \right)
\]

(2.5)

where $C_H := \sup_{x \in H} C(x)$.

(b) By [11, Lemma 3.7],

\[
\sup_{x \in E} \mathbb{E}_x[\tau_H^n] \leq n! M_H^n, \quad \text{for } n = 0, 1, 2, \ldots
\]

so that

\[
\mathbb{E}_x[e^{\beta \tau_H}] = \sum_{n=0}^{\infty} \frac{\mathbb{E}_x[\tau_H^n]}{n!} \leq \frac{1}{1 - \beta M_H}, \quad \text{for } 0 < \beta < 1/M_H.
\]

Thus

\[
\mathbb{P}_x[\tau_H > t] \leq \mathbb{E}_x[e^{\beta \tau_H}] e^{-\beta t} \leq \frac{1}{1 - \beta M_H} e^{-\beta t}.
\]

(2.6)

By (a), we have for $\epsilon \neq \beta$,

\[
\int_0^t \mathbb{P}_x[\tau_H > s] (\epsilon e^{\epsilon t}) ds \leq \frac{1}{1 - \beta M_H} \int_0^t \mathbb{P}_x[\tau_H > s] e^{-\beta s} e^{\epsilon t} ds = \frac{e^{(\epsilon - \beta) t} - 1}{(\epsilon - \beta)(1 - \beta M_H)},
\]

(2.7)

where in the case of $\beta = \epsilon$, the last term is understood as the limit of $\beta \to \epsilon$.

(c) From (a) and (b), it follows that for $x \notin H$,

\[
f(x, t) \leq \frac{2e^{-\beta t}}{1 - \beta M_H} + C_H e^{-\epsilon t} \left( 1 + \frac{e^{(\epsilon - \beta) t} - 1}{(\epsilon - \beta)(1 - \beta M_H)} \right),
\]

(2.8)

while obviously for $x \in H$,

\[
f(x, t) \leq C_H e^{-\epsilon t}.
\]

Therefore we have

\[
\kappa = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in E} f(x, t) \geq \min \{ \epsilon, \beta \}
\]

for any $\beta < 1/M_B$ and $\epsilon < \lambda$, so that

\[
\kappa \geq \min \left\{ \lambda, \frac{1}{M_H} \right\}.
\]

□

**Proof of Corollary 1.3**
(a) In the reversible case, we have \( p_t(x, y) = p_t(y, x) \), \( \pi \times \pi \) a.s. \((x, y)\), hence \( \phi(x) = \|p_s(x, \cdot)\|_{L^2(\pi)} = p_{2s}(x, x) \). Since \( \lambda_1 \) is equal to the spectral gap, \( \lambda \geq \lambda_1 \) by [7, Theorem 8.8].

(b) For the general case, by definition of \( \lambda_1 \), for any \( \epsilon < \lambda_1 \), there is \( C_1 < \infty \) such that

\[
\|P_t - \pi\|_{L^2(\pi) \to L^2(\pi)} \leq C_1 e^{-t\epsilon}, \quad t \geq 0.
\]

Let \( P^*_t \) be the dual semigroup of \( P_t \) with respect to \( \pi \). Then

\[
\|P^*_t - \pi\|_{L^2(\pi) \to L^2(\pi)} = \|P_t - \pi\|_{L^2(\pi) \to L^2(\pi)} \leq C_1 e^{-t\epsilon}, \quad t \geq 0.
\]

(c) For \( t \geq s \), we have

\[
P_t f(x) = P_s P_{t-s} f(x) = \int_E p_s(x, y) P_{t-s} f(y) \pi(dy) = \int_E [P^*_t (p_s(x, \cdot))(y)] f(y) \pi(dy).
\]

So by Cauchy-Schwartz inequality and (b),

\[
\|P_t(x, \cdot) - \pi\|_{\text{Var}} = \sup_{|f| \leq 1} |P_t f(x) - \pi(f)| = \sup_{|f| \leq 1} \left| \int_E [P^*_t (p_s(x, \cdot))(y)] f(y) \pi(dy) - \int_E p_s(x, \cdot) f(y) \pi(dy) \right|
\]

\[
\leq \|P^*_t (p_s(x, \cdot) - 1)\|_{L^2(\pi)} \leq \|p_s(x, \cdot) - 1\|_{L^2(\pi)} C_1 e^{-s\epsilon(t-s)}.
\]

Since \( \|P_t(x, \cdot) - \pi\|_{\text{Var}} \leq 2 \), \([1.1]\) holds for all \( t \geq 0 \) by choosing \( C = \max\{2, \|p_s(x, \cdot) - 1\|_{L^2(\pi)} C_1 e^{\epsilon s}\} \). Thus \( \lambda \geq \epsilon \) for any \( \epsilon < \lambda_1 \), so that \( \lambda \geq \lambda_1 \). Then the desired result follows from Theorem \([1.2]\). \(\square\)

Next, Sections \([3]\) and \([4]\) will discuss two classes of models which respectively satisfy the assumptions (R1) and (R2) in Theorem \([1.2]\).

## 3 Estimate of \( \kappa \) by \( \lambda \) or \( \lambda_1 \)

In this section, we will seek the situation that the moment of hitting time can be used as the upper bound for \( \lambda_1 \) or \( \lambda \). More technically, we give the conditions for which \( M_H \) goes to zero when \( H \) becomes bigger and bigger. That is, if there exists an \( H \in \mathcal{H} \), such that

\[
\lambda \leq \frac{1}{M_H}, \quad \text{or,} \quad \lambda_1 \leq \frac{1}{M_H},
\]

then it holds that

\[
\kappa = \lambda \quad \text{or} \quad \kappa = \lambda_1
\]

respectively.

As have done in Example \([1.7]\) for the birth-death process, we will do this by seeking a sequence \( \{H_n\} \subset \mathcal{H} \) such that \( H_n \uparrow E \) and

\[
\limsup_{n \to \infty} \mathbb{E}_x \tau_{H_n} = 0,
\]

(3.1)
So there exists \( H_n \in \mathcal{H} \) such that \( \lambda_1 \leq 1/M_{H_n} \) or \( \lambda \leq 1/M_{H_n} \).

In the following subsections, to study this situation, we present a class of models including Markov processes with \( \infty \) instantaneous entrance boundary, Markov chains, Diffusion processes and SDEs driven by stable processes.

### 3.1 Markov processes with \( \infty \) instantaneous entrance boundary

Let \( E = [0, \infty) \) and \( X \) be a non-explosive Markov process on \( E \). We say \( \infty \) is an instantaneous entrance boundary, if for any \( t \geq 0 \),

\[
\lim_{b \to \infty} \limsup_{x \to \infty} \mathbb{P}_x(\tau_{[0,b]} > t) = 0.
\]

(Cf. [14].) It is proved in [14, Definition 1.1] that for Feller process with non-negative jump, (3.2) is equivalent to

\[
\lim_{b \to \infty} \limsup_{x \to \infty} \mathbb{E}_x \tau_{[0,b]} = 0,
\]

This ensures that \( \kappa = \lambda \) by Theorem 1.2 if \( X \) is strongly ergodic.

Now, we turn to \( \mathbb{R}^d \) by extending the \( \infty \)-instantaneous entrance as follows.

**Definition 3.1.** We say \( \infty \) is an instantaneous entrance boundary, if the process is not explosive and there exist a sequence of bounded closed sets \( \{B_n\}_{n=1}^\infty \) such that \( B_n \uparrow \mathbb{R}^d \) and for any \( t \geq 0 \),

\[
\lim_{n \to \infty} \limsup_{x \to \infty} \mathbb{P}_x(\tau_{B_n} > t) = 0,
\]

where “\( \limsup_{x \to \infty} \)” means that \( \lim_{n \to \infty} \sup_{x \in E_n} \). Here \( \{E_n\}_{n=1}^\infty \) is a sequence of bounded sets such that \( E_n \uparrow \mathbb{R}^n \); \( \{E_n\}_{n=1}^\infty \) may be not identical with \( \{B_n\}_{n=1}^\infty \).

**Lemma 3.2.** Let \( \mathbb{R}^d \) be equipped with the usual partial order ‘\( \leq \)’. Assume that \( X \) is stochastically monotone, i.e. for any bounded monotone increasing function \( f \),

\[
P_t f(x) \leq P_t f(y) \quad \text{for} \quad x \leq y \quad \text{and} \quad t \geq 0.
\]

Then \( \infty \)-instantaneous entrance boundary is equivalent to (3.1) by choosing \( B_n := \{x : \forall 1 \leq i \leq d, x_i \leq n\} \).

**Proof.** Let \( \delta_B(t) = \sup_x \mathbb{P}_x(\tau_B > t) \). Given \( t_0 > 0 \), there exists \( N \), such that for \( n \geq N \), \( \delta_{B_n}(t_0) < 1 \). By using Markov property, we have

\[
\delta_B(kt) = \sup_x \mathbb{P}_x(\tau_B > kt) \leq \delta_B(t)^k.
\]

(Cf. [14, Lemma 2.1]). So for \( n \geq N \), by a similar argument to [17, Lemma 2.1],

\[
\mathbb{E}_x \tau_{B_n} = \int_{0}^{\infty} \mathbb{P}_x(\tau_{B_n} > t)dt = \sum_{k=0}^{\infty} \int_{k t_0}^{(k+1)t_0} \mathbb{P}_x(\tau_{B_n} > t)dt \\
\leq t_0 \sum_{k=0}^{\infty} \mathbb{P}_x(\tau_{B_n} > k t_0) \leq \frac{t_0}{1 - \delta_{B_n}(t_0)}.
\]

Since \( X \) is stochastically monotone,

\[
\mathbb{P}_x[X_s \notin B_n] \geq \mathbb{P}_y[X_s \notin B_n] \quad \text{for} \quad y \leq x,
\]

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so that
\[ P_x[\tau_{B_n} > t] = P_x[X_s \notin B_n, \text{ for all } 0 \leq s \leq t] \geq P_y[\tau_{B_n} > t]. \] (3.5)
Thus
\[ E_x \tau_{B_n} = \int_0^\infty P_x[\tau_{B_n} > t]dt \geq E_y \tau_{B_n} \text{ for } y \leq x, \]
which gives
\[ \sup_x E_x \tau_{B_n} = \limsup_{x \to \infty} E_x \tau_{B_n}. \]

Now assume that $\infty$ is an instantaneous entrance boundary. It follows from (3.5) that for any $t \geq 0$,
\[ \lim_{n \to \infty} \delta_{B_n}(t) = \limsup_{n \to \infty} P_x(\tau_{B_n} > t) = \lim_{n \to \infty} \limsup_{x \to \infty} P_x(\tau_{B_n} > t) = 0. \]

It follows from (3.4) that for any $t_0 > 0$,
\[ \lim_{n \to \infty} \sup_x E_x \tau_{B_n} \leq t_0. \]

Now by letting $t_0 \to 0$, we get the necessity.

The sufficiency can be obtained directly by
\[ P_x(\tau_{B_r} > t) \leq \frac{1}{t} E_x[\tau_{B_r}] \]
and monotonicity of $E_x \tau_{B_r}$ in $x$. \qed

By using Lemma 3.2, the following corollary is direct.

**Corollary 3.3** (Stochastically monotone Markov processes). Assume that $X$ is a stochastically monotone Markov process on $\mathbb{R}^d$ with $\infty$-instantaneous entrance boundary. If (1.1) holds for any $\epsilon < \lambda$ with $C(x)$ locally bounded, then $\kappa \geq \lambda$. Furthermore, if $X$ is reversible, then $\kappa = \lambda_1$.

### 3.2 Single death processes

As a counterpart of Markov process on $[0, \infty)$ with no negative jump, we consider the so-called single death process (or downwardly skip free process) on $\mathbb{Z}_+$. The $Q$-matrix $Q = (q_{ij})_{i,j \in \mathbb{Z}_+}$ is called a single death $Q$-matrix, if $q_{i,i-1} > 0$ for all $i \geq 1$, and $q_{i,i-j} = 0$ for $i \geq j \geq 2$. Assume that $Q$ is regular and irreducible. Let
\[ q^{(k)}_n = \sum_{j=k}^{\infty} q_{nj}, \quad k > n \geq 0, \]
and define inductively
\[ G_n^{(n)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q^{(k)}_n G_k^{(i)}, \quad 1 \leq n < i. \]
It is proved in [31, Lemma 2.7] that the single death process is strongly ergodic if and only if

\[ S := \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_{l,l-1}} < \infty. \]

Furthermore, for \( i > n \),

\[ E_i \tau_n \leq \sum_{k=n+1}^{i} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_{l,l-1}}. \]

By choosing \( H_n = \{0, 1, \ldots, n\} \), we have \( E_i \tau_{H_n} = E_i \tau_n \) by skip free property, so that

\[ \sup_{i > n} E_i \tau_{H_n} \leq \sum_{k=n+1}^{\infty} \sum_{l=k}^{\infty} \frac{G_k^{(l)}}{q_{l,l-1}} \to 0, \quad \text{as} \quad n \to \infty, \]

provided \( S < \infty \). Then \( \kappa = \lambda > 0 \) by applying Theorem 1.2.

### 3.3 Birth-death processes on trees

Let \( T \) be a connected tree with root \( o \). For any \( i \in T \setminus \{o\} \), there is a unique shortest path \( \mathcal{P}(i) \) from \( i \) to root \( o \). Let \( |i| = \# \mathcal{P}(i) - 1 \) be the distance from \( i \) to \( o \) and \( i^* \) be the unique point connected to \( i \) with \( |i^*| = |i| - 1 \), called the father of \( i \). Assume the tree has the finite degree property, that is, the set \( \{j : |j| = |i| = 1\} \) is finite for any \( i \in T \). Let \( Q = (q_{ij})_{i,j \in T} \) with \( q_{ii^*} > 0 \) and \( q_{i^*i} > 0 \). Assume that \( Q \) is regular. Let

\[ \mu_0 = 1, \quad \mu_i = \prod_{j \in \mathcal{P}(i)} q_{j^*j}, \quad i \neq 0. \]

Then \( \mu = (\mu_i)_{i \in T} \) is the symmetric measure for the \( Q \)-process. Set \( m_j = (\mu_j q_{jj^*})^{-1} \sum_{i \in T_j} \mu_i \) for \( j \in T \).

It is proved in [32, Corollary 2.6] that the \( Q \)-process is strongly ergodic if and only if

\[ \sup_{i \in T \setminus \{o\}} \sum_{j \in \mathcal{P}(i)} m_j < \infty. \]

Let \( \Gamma \) be collection of paths approaching to infinity. Then

\[ S := \sup_{\gamma \in \Gamma} \sup_{i \in \mathcal{P}_i} \sum_{j \in \mathcal{P}(i)} m_j < \infty, \]

so that \( \forall n \geq 1 \), for any \( \gamma \in \Gamma \) there exists \( i_\gamma \in \gamma \) such that

\[ \sum_{j \in \mathcal{P}(i_\gamma)} m_j \leq \frac{1}{n}. \]

Let \( H_n = \bigcup_{\gamma \in \Gamma} \mathcal{P}(i_\gamma) \), which is obviously finite by the finite degree property. Thus

\[ \sup_{k \in T} E_k \tau_{H_n} \leq \sup_{j \notin H_n} E_j \tau_{H_n} = \sup_{j \in \Gamma} \sum_{j \in \mathcal{P}(i_\gamma)} m_j \leq \frac{1}{n}. \]

By applying Theorem 1.2, we have \( \kappa = \lambda_1 \). Actually, by Theorem 4.2 in Section 4, we get the estimate \( \kappa = \lambda_1 \geq 1/S \).
3.4 Diffusions processes

First we consider the one-dimensional diffusion process which is both stochastically monotone Markov process and Feller process with non-negative jump.

**Corollary 3.4 (One-dimensional diffusions).** Let $L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$ with $a(x) > 0$, and $a, b$ be continuous. Define $c(x) = \int_1^x \frac{h(y)}{a(y)} dy$, and $\pi(dz) = a(z)^{-1} e^{c(z)} dz$. Denote by $X$ the diffusion process on $[0, \infty)$ with generator $L$ and reflecting boundary at 0. Assume that the curvature condition is satisfied, i.e. there exists a constant $\kappa > 0$ such that Ric($\cdot$) $\geq \kappa |\cdot|^2$. Assume $X$ is also stochastically monotone and $\rho$-entrance Markov process and Feller process with non-negative jump. Assume that the curvature condition is satisfied, i.e. there exists a constant $\kappa > 0$ such that Ric($\cdot$) $\geq \kappa |\cdot|^2$.

Then $\kappa = \lambda_1 = \inf \{ \pi(af') : \pi(f) = 0, \pi(f^2) = 1 \}$.

**Proof.** By [16] Section 4.11, the heat kernel $p_t(x, x)$ can be chosen to be continuous in $x \in \mathbb{R}_+$, so by Corollary 1.3 $C(x)$ is locally bounded. Notice that

$$M_r := \sup_{x > r} P_x \tau_{[0, r]} = \int_r^\infty e^{-c(y)} \left( \int_y^\infty \frac{e^{c(z)}}{a(z)} dz \right) dy < \infty.$$

Then $\lim_{r \to \infty} M_r = 0$, so by Theorem 1.2 we have $\kappa = \lambda_1$.

The above result provides a way by using the spectral gap $\lambda_1$ to estimate $\kappa$ for the one-dimensional diffusion process with entrance boundary. For examples, the following estimate in [1] can be served as the estimate for $\kappa$:

$$\delta^{-1} \leq \kappa = (4\delta)^{-1},$$

where

$$\delta = \sup_x \int_x^\infty e^{-c(y)} dy \int_x^\infty \frac{e^{c(z)}}{a(z)} dz < \infty.$$

This estimate improves the estimate $\kappa \geq 1/M_0$ in [18] by using the coupling method. Moreover, in [1], the approximation procedure now can also be applied to estimate $\kappa$.

Next we turn to diffusion processes on manifolds. Let $M$ be a connected, complete Riemannian manifold with empty boundary or convex boundary, and $(X_t)_{t \geq 0}$ be a non-explosive diffusion process on $M$ generated by $L = \Delta + Z$ with invariant probability measure $\pi$ (cf. see [3] Theorem 3.1 for the sufficient condition for the existence of invariant measure). Assume that the curvature condition is satisfied, i.e. there exists a constant $K$ such that $\text{Ric}(X, X) - \langle \nabla_X Z, X \rangle \geq -K \|X\|^2$.

Under these assumptions, the dimensional free Harnack inequality holds (see [28] Theorem 2.3.3), thus by [26] Corollary 3.1(2), there exists density $p_t(x, y)$ with respect to $\pi$.

Let $\rho$ be the Riemannian metric. Fix a point $o \in M$, set $\rho(x) = \rho(o, x)$ and $D = \sup_{x \in M} \rho(x)$. Fix $r_0 > 0$, let

$$\overline{C}(r) = \int_{r_0}^r \overline{\beta}(s) ds \text{ and } \overline{\beta}(r) \geq \sup_{\rho(x) = r} L \rho(x) \text{ for } r > r_0.$$

We say the process has $\rho$-entrance boundary at infinity with respect to Riemannian metric $\rho$, if

$$\overline{\delta}(\rho) := \int_0^D e^{-\overline{C}(y)} \left( \int_y^D e^{\overline{C}(z)} dz \right) dy < \infty. \quad (3.6)$$
Theorem 3.5. If \((X_t)_{t \geq 0}\) is non-explosive and has \(\rho\)–entrance boundary, then the convergence rate

\[ \kappa \geq \lambda_1. \]

Specially in the reversible case, if \(Z = \nabla V\) for some \(V \in C^2(M)\), then

\[ \kappa = \lambda_1 = \inf \{ \pi(|\nabla f|^2) : \pi(f) = 0, \pi(f^2) = 1 \}, \]

where \(\pi(dx) = e^{V(x)}dx/\int_M e^{V(x)}dx\).

Before starting the proof of Theorem 3.5, we need the following lemma whose proof is similar to that of [28, Theorem 2.4.4].

Lemma 3.6. Let \(p_t(x, y)\) is the transition density. Then for any \(s, r > 0\), and \(x \in M\),

\[ \|p_s(x, \cdot)\|_{L^2(\pi)} \leq \frac{1}{\pi(B(x, r))} e^{U_s(r)}, \]  

where \(B(x, r) = \{y \in M : \rho(x, y) \leq r\}\) and \(U_s(r) = Kr^2/(e^{2Ks} - 1)\).

Proof. Let \(p = 2\) in dimension-free Harnack inequality (see [28, Theorem 2.3.3]), we have for any positive bounded function \(f\),

\[ (P_s f)^2(x) \leq P_s f^2(y) e^{U_s(\rho(x, y))}. \]

Hence

\[ \pi(f^2) = \pi P_s f^2 \geq (P_s f)^2(x) \int_M e^{-U_s(\rho(x, y))} \pi(dy) \geq (P_s f)^2(x) e^{-U_s(r)} \pi(B(x, r)). \]

By choosing \(f(y) = n \wedge p_s(x, y)\), we obtain that

\[ \left( \int_M (n \wedge p_s(x, y))p_s(x, y)\pi(dy) \right)^2 \leq \frac{1}{\pi(B(x, r))} e^{U_s(r)} \pi((n \wedge p_s(x, y))^2). \]

Since

\[ \int_M (n \wedge p_s(x, y))p_s(x, y)\pi(dy) \geq \pi((n \wedge p_s(x, y))^2); \]

we have

\[ \pi((n \wedge p_s(x, y))^2) \leq \frac{1}{\pi(B(x, r))} e^{U_s(r)}; \]

By letting \(n \to \infty\), we get (3.7). \(\square\)

Proof of Theorem 3.5

Let

\[ u_p(r) = \int_p^r e^{-\bar{\gamma}(y)} \left( \int_y^D e^{\bar{\gamma}(z)} dz \right) dy \]

and \(\bar{\beta}_p(\rho) = \lim_{r \to D} u_p(r)\). Then \(\bar{\beta}_p(\rho) < \infty\) and \(u_p\) satisfies that \(u''_p(r) + \bar{\beta}(r)u'_p(r) = -1\). Hence for \(x \in M\) with \(\rho(x) = r\),

\[ Lu_p \circ \rho(x) = u''_p[\rho(x)] + L\rho(x)u'_p[\rho(x)] \leq -1. \]  

(3.8)
Taking \( f_p(x) = u_p \circ \rho(x) \), and \( B_p = \{ x \in M : \rho(x) \leq p \} \), by the well-posedness of martingale problem, we have

\[
\mathbb{E}_x[f_p(X_{t \wedge \tau_{B_p}})] - f_p(x) = \mathbb{E}_x \left[ \int_0^{t \wedge \tau_{B_p}} Lf_p(X_s)ds \right] \leq -\mathbb{E}_x[t \wedge \tau_{B_p}],
\]

(3.9)

Note that \( \mathbb{E}_x[f_p(X_{\tau_{B_p}})] = 0 \) and \( \sup_{x \notin B_p} f_p(x) = \delta_p(\rho) \). By letting \( t \to \infty \), we have that

\[
M_p := \sup_{x \notin B_p} \mathbb{E}_x[\tau_{B_p}] \leq \delta_p(\rho) < \infty.
\]

Hence \( \lim_{p \to D} M_p = 0 \). According to Lemma 3.6, \( \|p_s(x, \cdot)\|_{L^2(\pi)} \) is locally bounded, consequently \( \kappa \geq \lambda_1 \) by Corollary 1.3. Specially, if \( Z = \nabla V \) for some \( V \in C^2(M) \), then the process is reversible with respect to \( \pi \), so \( \kappa = \lambda_1 = \inf\{ \pi(|\nabla f|^2) : \pi(f) = 0, \pi(f^2) = 1 \} \).

\[\square\]

Theorem 3.5 can improve the estimates in [18] for the diffusion processes on \( M \) by using coupling method. Here we show an example:

**Example 3.7.** [8, Example 1.9] Let \( (X_t)_{t \geq 0} \) be a diffusion process on \( \mathbb{R}^n \) with generator \( L = \Delta + \nabla V \), \( V(x) = -|x|^4 \). By [18, Theorem 3.2], we get a lower bound of \( \kappa \):

\[
\kappa \geq \frac{1}{4\delta} = \left( 4 \int_0^{\infty} e^{(y/2)^4} dy \int_y^{\infty} e^{-(z/2)^4} dz \right)^{-1}.
\]

By Gautschi’s estimate

\[
e^{xp} \int_x^{\infty} e^{-yp} dy \leq C_p \left( x^p + \frac{1}{C_p} \right)^{1/p}, \quad (x \geq 0, \ C_p := \Gamma(1 + \frac{1}{p})^{p-1}),
\]

we have

\[
4\delta \leq C_4 \int_0^{1} \left[ \left( x^4 + \frac{1}{C_4} \right)^{1/4} - x \right] dx + \int_1^{\infty} e^{y^4} \left( \int_1^{\infty} e^{-z^4} dz \right) dy \leq \int_0^{1} \frac{1}{\left[ \left( x^4 + 1/C_4 \right)^{1/4} + x \right]} \left( x^4 + 1/C_4 \right)^{1/2} + x^2 \right] dx + \int_1^{\infty} \frac{1}{4y^3} dy \quad (3.10)
\]

\[
\leq \int_0^{1} \frac{1}{\left( 1/C_4 \right)^{1/4}} \left( 1/\left( 1/C_4 \right)^{1/2} + x^2 \right] dx + 1/8 \approx 0.7203 + 1/8.
\]

then \( \kappa \geq \frac{1}{4\delta} \approx 1.1831 \).

But on the other hand, it is obvious that \( (X_t)_{t \geq 0} \) satisfies the condition of Theorem 3.5, hence \( \kappa = \lambda_1 \). In [8, Example 4.11], apply \( I \)-operator to \( f(x) = \log(1 + x) \) to derive

\[
\kappa = \lambda_1 \geq 2.4395.
\]
3.5 SDEs driven by symmetric stable processes

Let \((Z_t)_{t \geq 0}\) be a \(d\)-dimensional symmetric \(\alpha\)-stable process with generator \(\Delta^{\alpha/2} := -(\Delta)^{\alpha/2}\), which has the following expression:

\[-(\Delta)^{\alpha/2} f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{C_{d,\alpha}}{|z|^{d+\alpha}} dz,
\]

where

\[C_{d,\alpha} = \frac{\alpha^{\alpha-1} \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)}\]

is the normalizing constant.

Consider the following stochastic differential equation (SDE) driven by \(\alpha\)-stable process on \(\mathbb{R}^d\):

\[dX_t = dZ_t + b(X_t) dt, \quad X_0 = x,
\]

where \(b: \mathbb{R}^d \to \mathbb{R}^d\) is a continuous function satisfying there exists a constant \(K > 0\) such that for all \(x, y \in \mathbb{R}^d\),

\[\langle b(x) - b(y), x - y \rangle \leq K|x - y|^2.
\]

Note the SDE has the unique strong solution \((X_t)_{t \geq 0}\) which is strong Feller and Lebesgue irreducible, see, e.g. [30]. For the generator \(L\) of \(X\), we set

\[D_w(L) := \left\{ f : (E, \mathcal{B}) \to (\mathbb{R}, \mathcal{B}) \text{ is measurable such that } f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds \text{ is a local martingale} \right\}.
\]

For any \(f \in D_w(L)\),

\[Lf(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{C_{d,\alpha}}{|z|^{d+\alpha}} dz + \langle b(x), \nabla f(x) \rangle,
\]

and

\[\left\{ f \in C^2(\mathbb{R}^d) : \int_{\{|z| > 1\}} \left| f(x + z) - f(x) \right| \frac{1}{|z|^{d+\alpha}} dz < \infty, \text{ for } x \in \mathbb{R}^d \right\} \subset D_w(L).
\]

Define

\[g(r) = \inf_{|x| > r} \left\{ -\frac{\langle x, b(x) \rangle}{|x|^2} \vee 0 \right\} \quad \text{and} \quad \tilde{g}(r) = \frac{1}{r} \int_1^r g(s) ds, \quad r \geq 1.
\]

Obviously, \(g(r)\) is a non-decreasing function, so that \(\tilde{g}(r) \leq g(r)\).

**Theorem 3.8.** If for any \(r > 0\),

\[\delta_r := \int_r^\infty \frac{1}{s\tilde{g}(s)} ds < \infty,
\]

then \((X_t)_{t \geq 0}\) is strongly ergodic; moreover, for \(\alpha \in (1, 2)\), the convergence rate \(\kappa \geq \lambda_1\).
Proof. For any $r > 1$, we take nonnegative function $u_r(x) \in C_b^d(\mathbb{R}^d)$ such that $1 \leq u_r \leq \delta_r$ and for $|x| \geq r$, $u_r(x) = \int_{|x|}^{\infty} (sg^\alpha(s))^{-1} ds$. Therefore,

$$\int_{\{|z| > 1\}} \left[ u_r(x + z) - u_r(x) \right] \frac{1}{|z|^{d+\alpha}} dz \leq 2\delta_r \Gamma_d \int_1^\infty \frac{1}{r^{1+\alpha}} dr = \frac{2\delta_r \Gamma_d}{\alpha} < \infty, \quad (3.14)$$

where $\Gamma_d = 2\pi^{d/2}/\Gamma(d/2)$, so by (3.12), $u_r(x) \in D_w(L)$. A direct computation shows

$$\langle b(x), \nabla u_r(x) \rangle = \frac{\langle x, b(x) \rangle}{|x|^2} \frac{1}{\bar{g}(|x|)} \leq -\frac{g(|x|)}{\bar{g}(|x|)} \leq -1, \quad (3.15)$$

and for $|z| \leq 1$,

$$u_r(x + z) - u_r(x) - \langle z, \nabla u_r(x) \rangle = \frac{1}{2} \langle z, D^2 u_r(\xi) z \rangle = \frac{|z|^2}{|\xi|^2 \bar{g}(|\xi|)} - \frac{\langle z, \xi \rangle^2}{|\xi|^3 \bar{g}(|\xi|)} - \frac{\langle z, \xi \rangle^2 g(|\xi|)}{|\xi|^3 \bar{g}(|\xi|)} \leq \frac{1}{(1 - 1)^2 \bar{g}(|x| - 1)} |z|^2,$$

where $\xi = x + \theta z$, $\theta \in (0, 1)$. Thus

$$\int_{\{|z| \leq 1\}} \left[ u_r(x + z) - u_r(x) - \langle z, \nabla u_r(x) \rangle \right] \frac{dz}{|z|^{d+\alpha}} \leq \frac{1}{(1 - 1)^{2} \bar{g}(|x| - 1)} \int_{\{|z| \leq 1\}} |z|^2 dz \leq \frac{C_{d,\alpha} \Gamma_d}{(2 - \alpha)(1 - 1^2) \bar{g}(|x| - 1)^2}. \quad (3.16)$$

Combining (3.14), (3.15) and (3.16), we get that for $|x| > r$ and $r > 2$,

$$Lu_r(x) \leq -1 + \frac{2\delta_r \Gamma_d}{\alpha} + \frac{C_{d,\alpha} \Gamma_d}{(2 - \alpha)(r - 1)^2 \bar{g}(r - 1)},$$

so that $Lu_r(x) \leq -\frac{1}{2}$ for $|x| \geq r$ with $r$ large enough. By the definition of $D_w(L)$,

$$\mathbb{E}_x[u_r(X_{t \wedge \tau_r})] - u_r(x) = \mathbb{E}_x \left[ \int_0^{t \wedge \tau_r} Lu_r(X_s) ds \right] \leq -\frac{1}{2} \mathbb{E}_x[t \wedge \tau_r], \quad (3.17)$$

where $\tau_r := \inf\{t \geq 0 : |X_t| \leq r\}$. By letting $t \to \infty$, we obtain

$$M_r := \sup_x \mathbb{E}_x[\tau_r] \leq \sup_x u_r(x) = \int_r^\infty \frac{1}{s \overset{\cdot}{\bar{g}}(s)} ds < \infty,$$

hence $\lim_{r \to \infty} M_r = 0$ and $X$ is strongly ergodic.

Furthermore, if $1 < \alpha < 2$, then the dimensional free Harnack inequality holds (see [25 Corollary 2.2(3)]), so by [26 Corollary 3.1(2)], the density $p_r(x, y)$ with respect to $\pi$ exists. By Lemma 3.9 below, $\|p_r(x, \cdot)\|_{L^2(\pi)}$ is locally bounded, thus $\kappa \geq \lambda_1$ by Corollary 1.3.
Lemma 3.9. Assume that $1 < \alpha < 2$ and $p_t(x,y)$ is the transition density, then for any $s > 0$ and $x \in \mathbb{R}$, there exists a constant $C > 0$ such that for any $r > 0$,

\[
\|p_s(x, \cdot)\|_{L^2(x)} \leq \frac{1}{\pi(B(x,r))} e^{V_s(r)},
\]

where

\[
V_s(r) = \frac{2Cr^2}{(s \wedge 1)^{\frac{\alpha}{2}}} + \frac{C(2r^2)^{\frac{\alpha}{4}}}{(s \wedge 1)^{\frac{\alpha}{2}}},
\]

Proof. According to Harnack inequality (see [25, Theorem 2.1]), for any $s > 0$, $x, y \in \mathbb{R}^d$ and positive $f \in \mathcal{B}_b(\mathbb{R}^d)$,

\[
(P_s f(y))^2 \leq (P_s f^2(x)) e^{V_s(|x-y|)}.
\]

Now by choosing $f(y) = n \wedge p_s(x,y)$, the desired result follows from a similar argument to the proof of Lemma 3.7.

Remark 3.10. Note that if for some $\delta > 1$,

\[
\langle x, b(x) \rangle \leq -K|x|^{1+\delta}
\]

holds, then the condition (3.13) can be replaced by above drift condition. Specially, let $b(x) = -x|x|^{\eta}$, we obtain that $X$ is strongly ergodic if and only if $\eta > 0$, and for $\alpha \in (1, 2)$, $\kappa \geq \lambda_1$.

4 Estimate of $\kappa$ by hitting time

Now we are going to another direction, for seeking the lower bound of $\kappa$ by the uniform moment of hitting time to some bound set. In this case, we will first obtain the lower bound of $\lambda$ (or $\lambda_1$) by using the hitting time.

This strategy was done well for the reversible Markov chain on countable state space, see for example [19].

Let $X_t$ be a continuous-time Markov chain on a denumerable state space $E$. The transition function $P_t = (p_{ij}(t))_{i,j \in E}$ is reversible with respect to the stationary distribution $\pi = (\pi_i)_{i \in E}$:

\[
\pi_i p_{ij}(t) = \pi_j p_{ji}(t), \quad i, j \in E, t \geq 0.
\]

Let $\tau_x = \inf \{t \geq 0 : X_t = x\}$ be the hitting time to state $x \in E$. The following lemma can be found in [5, Proposition 3.2].

Lemma 4.1. Let $P^x(t) = (p^x_{ij}(t))_{i,j \neq x}$ be the killed process upon $x \in E$:

\[
p^x_{ij}(t) = \mathbb{P}_i[X_t = j, t < \tau_x].
\]

Then $\lambda_1 \geq \lambda^x$, where $\lambda^x$ is the Dirichlet spectral gap of killed process upon $x$:

\[
\lambda^x = -\lim_{t \to \infty} \frac{1}{t} \log \|P^x_t\|_{2 \to 2}
\]
Theorem 4.2. Under the above assumptions, it holds that

\[ \kappa \geq \sup_{x \in E} \left( \sup_{i \neq x} \mathbb{E}_i \tau_x \right)^{-1}. \]

Proof. To apply Theorem 1.2 by Lemma 4.1 we need only prove that

\[ \lambda^x \geq \left( \sup_{i \neq x} \mathbb{E}_i \tau_x \right)^{-1}. \]

Assume \( M_x := \sup_{i \in E} \mathbb{E}_i \tau_0 < \infty \). Then similar to the part b) of proof of Theorem 1.2 for any \( \beta < 1/M_x \),

\[ \sup_{i \in E} e^{\beta \tau_x} \leq \frac{1}{1 - \beta M_x}. \]

So

\[ \| P^x_t \|_{\infty \to \infty} = \sup_{i \in E} \sum_{j \in E} p^x_{ij}(t) = \sup_{i \in E} \mathbb{P}_i [t < \tau_x] \leq \frac{1}{1 - \beta M_0} e^{-\beta t}. \] (4.1)

By the symmetry \( \| P^x_t \|_{1 \to 1} = \| P^x_t \|_{\infty \to \infty} \), the interpolation theorem implies

\[ \| P^x_t \|_{2 \to 2} \leq \frac{1}{1 - \beta M_x} e^{-\beta t} \quad \text{for } 0 < \beta < 1/M_x. \] (4.2)

Hence

\[ \lambda^x \geq \frac{1}{M_x}. \]

Consequently, we have \( \kappa \geq \sup_{x \in E} 1/M_x. \) \( \square \)

Theorem 4.2 can be applied to the previous Birth-death processes on tree in Section 3.3, to give the estimate \( \kappa \geq 1/S \).

We can also use this strategy to one-dimensional reversible Markov processes. specially, as an example, we consider time-changed symmetric \( \alpha \)-stable process with \( \alpha \in (1, 2) \).

Let \( X \) be the symmetric \( \alpha \)-stable processes on \( \mathbb{R} \) with generator \( \Delta^{\alpha/2}, \alpha \in (1, 2) \).

Note that this process is recurrent but not ergodic.

Let \( a(x) \) be a strictly positive continuous function on \( \mathbb{R} \). Define

\[ T_t = \inf \left\{ s \geq 0 : \int_0^s a(X_u)^{-1} du > t \right\} \quad \text{and} \quad Y_t := X_{T_t}. \]

We say \( Y \) is a time-changed \( \alpha \)-stable process, which remains recurrent. By [10, Section 1.2], \( Y \) is a symmetric strong Markov process with the measure \( \pi(dx) = a(x)^{-1}dx \); so that \( Y \) is ergodic whenever \( \pi \) is finite. To distinguish \( X \) from \( Y \), we write \( \square X \) (resp. \( \square Y \)) as the quantity \( \square \) of \( X \) (resp. \( Y \)).

Theorem 4.3. For the time-changed process \( Y \), if \( I := \int \mathbb{R} a(x)^{-1} |x|^{\alpha-1} dx < \infty \), then the process is strongly ergodic and

\[ \kappa \geq \frac{1}{\omega_\alpha I} > 0, \]

where

\[ \omega_\alpha := \frac{2}{\cos(\pi \alpha/2) \Gamma(\alpha)} > 0. \]
Proof. Define the transition semigroup $P^0_t$ of killed process $X^0$ as

$$P^0_t(x, A) = \mathbb{P}_x[X_t \in A, t < \tau_0] \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}),$$

and the Green function $G^X_0(\cdot, \cdot)$ for $X$ killed upon 0 by

$$G^X_0(x, dy) = \int_0^\infty P^0_t(x, dy)dt = G^X_0(x, y)dy,$$

which by [15, Page 152],

$$G^X_0(x, y) = -\frac{1}{2\Gamma(\alpha) \cos \left(\frac{\pi\alpha}{2}\right)} \left(|y|^{\alpha-1} + |x|^{\alpha-1} - |y - x|^{\alpha-1}\right).$$

By [12, (4.25)], we can represent the Green function $G^Y_0(\cdot, \cdot)$ for $Y$ killed upon 0 as

$$G^Y_0(x, A) = \int_A G^X_0(x, y)a(y)^{-1}dy.$$

Therefore,

$$\mathbb{E}^X_0 \tau_0 = \int_0^\infty \mathbb{E}_x I_{\mathbb{R}}(X^0_t)dt = \int_\mathbb{R} G^Y_0(x, dy) = \int_\mathbb{R} G^X_0(x, y)a(y)^{-1}dy.$$

According to Lemma 4.4 below, we have

$$\sup_x \mathbb{E}_x^Y \tau_0^Y \leq -\frac{1}{\Gamma(\alpha) \cos \left(\frac{\pi\alpha}{2}\right)} \int_\mathbb{R} |y|^{\alpha-1}a(y)^{-1}dy = \omega \alpha I^{\sigma, \alpha},$$

hence we only need to prove that

$$\lambda_0 \geq \left(\sup_x \mathbb{E}_x^Y \tau_0^Y\right)^{-1}. \quad (4.3)$$

In fact, it is well known that $\lambda_1 \geq \lambda_0$ (see [5, Proposition 3.2]), where

$$\lambda_0 := \lambda_0(\{0\}^c) = \inf\{D(f, f) : \pi(f^2) = 1, f(0) = 0\},$$

which is equivalent to the spectral gap of killed process $X^0$:

$$\lambda_0 = -\lim_{t \to \infty} \frac{1}{t} \log \|P^0_t\|_{2 \to 2}.$$  

Then similar to the proof of Theorem 4.2, we see (4.3) holds, then our result follows by Theorem 1.2. \square

Lemma 4.4. For any $x, y \in \mathbb{R}$ and $\alpha \in (1, 2),$

$$|y|^{\alpha-1} + |x|^{\alpha-1} - |y - x|^{\alpha-1} \leq 2(|x| \wedge |y|)^{\alpha-1},$$
Proof. Let $a = |x \land y|, b = |x \lor y|$. Then $|x| \land |y| = a \land b$.

(1) When $xy = 0$, it is trivial; when $xy < 0$,

$$|y|^{\alpha - 1} + |x|^{\alpha - 1} - |y - x|^{\alpha - 1} = a^{\alpha - 1} + b^{\alpha - 1} - (b + a)^{\alpha - 1} \leq (a \land b)^{\alpha - 1}.$$ 

(2) When $xy > 0$, we only need to consider $x, y > 0$. In this case,

$$(x + y)^{\alpha - 1} \leq x^{\alpha - 1} + y^{\alpha - 1},$$

so $b^{\alpha - 1} \leq (b - a)^{\alpha - 1} + a^{\alpha - 1}$, or, $a^{\alpha - 1} + b^{\alpha - 1} - (b - a)^{\alpha - 1} \leq 2a^{\alpha - 1}$. Then

$$y^{\alpha - 1} + x^{\alpha - 1} - |y - x|^{\alpha - 1} \leq 2(x \land y)^{\alpha - 1}.$$ 

□

Remark 4.5. Theorem 4.3 is an extension of the following sufficient condition for strong ergodicity by using Lyapunov function:

$$\liminf_{|x| \to \infty} \frac{a(x)^{1/\alpha}}{|x|^\gamma} > 0 \text{ for some } \gamma > 1.$$ 

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