ALTERNATING SIGN MATRICES WITH REFLECTIVE SYMMETRY
AND PLANE PARTITIONS: $n + 3$ PAIRS OF EQUIVALENT STATISTICS

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Abstract. Vertically symmetric alternating sign matrices (VSASMs) of order $2n + 1$ are known to be equinumerous with lozenge tilings of a hexagon with side lengths $2n + 2, 2n, 2n + 2, 2n, 2n + 2, 2n$ and a central triangular hole of size 2 that exhibit a cyclical as well as a vertical symmetry, but no bijection between these two classes of objects has been constructed so far. In order to make progress towards finding such a bijection, we generalize this result by introducing certain natural extensions for both objects along with $n + 3$ parameters and show that the multivariate generating functions with respect to these parameters coincide. The equinumeracy of VSASMs and the lozenge tilings is then an easy consequence of this result, which is obtained by specializing the generating functions to signed enumerations for both types of objects. In fact, we present several versions of such results (one of which was independently conjectured by Florian Aigner) but in all cases certain natural extensions of the original objects are necessary and that may hint at why it is so hard to come up with an explicit bijection for the original objects.

1. Introduction

An alternating sign matrix (ASM) is a square matrix with entries in $\{0, \pm 1\}$ such that, in each row and column, the non-zero entries alternate and add up to 1. An example is displayed next.

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

It is well-known, see [And79, Zei96, Kra06], that there is the same number of $n \times n$ ASMs as there is of cyclically symmetric lozenge tilings of a hexagon with side lengths $n + 1, n - 1, n + 1, n - 1, n + 1, n - 1$ with a central triangular hole of size 2, but no explicit bijection has been constructed so far. These lozenge tilings are in easy bijective correspondence with descending plane partitions (DPPs) with parts no greater than $n$. In [AFb], a certain refinement of this result was provided that involves $n + 3$ statistics on both sides. Prior to this result, four statistics were known on ASMs and on DPPs that have the same joint distribution. The refinement in [AFb] made it necessary to work with extended objects, which evolved naturally through the $n + 3$ pairs of statistics that were included, and it was shown that the joint distributions of the two sets of $n + 3$ statistics coincide. A general introduction into this fascinating area of missing bijections is provided in [Bre99] and a recent update is given in [FK20].

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Also the purpose of the current paper is to shed new light on the mysterious relation between ASMs and plane partitions. More specifically, it is known that, in perfect analogy, vertically symmetric \((2n+1) \times (2n+1)\) alternating sign matrices (VSASMs) are equinumerous with cyclically symmetric lozenge tilings of a hexagon with side lengths \(2n + 2, 2n, 2n + 2, 2n, 2n + 2, 2n\) and with a central triangular hole of size 2 that exhibit an additional axial symmetry, see [MRR87, Kup94]. Such lozenge tilings are illustrated in Figure 1. In the present paper, we provide \((n+3)\)-parameter refinements (several of them) of the relation between VSASMs and certain families of non-intersecting lattice paths or plane partition objects (which can also be translated into lozenge tilings). Most of these results extend the relation between VSASMs and the lozenge tilings mentioned above.

Before we give a detailed description of our main results in the next section, we explain briefly how we obtain them. We choose to work with monotone triangles instead of ASMs, but there is an easy bijective correspondence between \(n \times n\) ASMs and monotone triangles with bottom row \(1, 2, \ldots, n\) as indicated at the beginning of the next section. In [AFb], an \((n+3)\)-parameter refinement of the operator formula [Fis06] for the number of monotone triangles with prescribed bottom row has been established. In view of the various objects that are known to be equinumerous with ASMs (and thus monotone triangles with bottom row \(1, 2, \ldots, n\)), an obvious question is whether we can carry over the \(n+3\) parameters to these other objects. In this paper, we study this problem for VSASMs and their symmetric counterparts related to DPPs. The above mentioned \((n+3)\)-parameter refinement of the monotone triangle count can be expressed in terms of an antisymmetrizer formula, see (3.1). In a sense, this generating function can be seen as a variation of Schur polynomials when Schur polynomials are thought of as the generating function of Gelfand-Tsetlin patterns and the variation concerns certain decorated Gelfand-Tsetlin patterns.

This expression still exists for arbitrary (but fixed) bottom row of monotone triangles. The case of VSASMs corresponds to having \(0, 2, \ldots, 2n - 2\) as bottom row. In this particular case, we can use Lemma 4.1 to transform the antisymmetrizer formula into a bialternant-type formula (in the spirit of the bialternant formula for the Schur polynomial, but more complicated), which is a quotient of a determinant and a Vandermonde-type product. The same lemma was also applicable in the non-symmetric case, however, in the symmetric case, a key observation was the necessity to multiply with the product in (3.2). Comparing again with the Schur polynomial case, we transform the bialternant-type formula into a Jacobi-Trudi-type formula in the next step. Here, Lemma 5.1 is an important tool. There are several possibilities to accomplish this, which lead to our different results. The various versions of Jacobi-Trudi-type determinants are then interpreted combinatorially using the Lindström-Gessel-Viennot lemma (Lemma 5.3) as families of lattice paths. Such interpretations are still signed enumerations in our cases, but we use further algebraic manipulations or combinatorial reasons (we have two different proofs) to obtain a signless version, which we express in terms of pairs of plane partitions. Combinatorial reasoning is also used to relate one of our \((n+3)\)-parameter refinements to the holey cyclically and vertically symmetric lozenge tilings that are known to be equinumerous with \((2n+1) \times (2n+1)\) VSASMs.
2. Main results

A monotone triangle is a triangular array of integers of the following form

\[
\begin{array}{cccc}
  m_{1,1} \\
m_{2,1} & m_{2,2} \\
\ldots & \ldots & \ldots \\
m_{n-1,1} & m_{n-1,2} & \ldots & m_{n-1,n-1} \\
m_{n,1} & m_{n,2} & \ldots & m_{n,n}
\end{array}
\]

with weak increase along $\nearrow$- and $\searrow$-diagonals, and strict increase along rows, i.e., $m_{i,j} \leq m_{i+1,j} \leq m_{i+1,j+1}$ and $m_{i,j} < m_{i,j+1}$. There is the following simple bijection between monotone triangles with bottom row 1,2,\ldots,n and $n \times n$ ASMs: Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ ASM and consider the matrix obtained by adding to each entry all the entries that are in the same column above, i.e., $S = \left( \sum_{i=1}^{n} a'_{i,j} \right)_{1 \leq i,j \leq n}$. It is not hard to see that this always results in a $\{0,1\}$-matrix with $i$ occurrences of 1’s in the $i$-th row. The corresponding monotone triangle is obtained by recording row by row the columns of the 1’s in $S$ in $n$ centered rows.

An observation that will be useful later is the following: By rotating and considering only the top $n$ rows of the monotone triangles (which take care of the “fundamental domain”), $(2n+1) \times (2n+1)$ VSASMs correspond to monotone triangles with bottom row $2,4,6,\ldots,2n$, or, equivalently, when subtracting 2 from each entry, with bottom row $0,2,\ldots,2n-2$. In order to see this, it is crucial to observe that the central column of a VSASM is always $(1,-1,1,-1,\ldots,-1,1)^T$.

The following decorated versions of monotone triangles have first appeared in [AFb].

**Definition 2.1.** An arrowed monotone triangle (AMT) is a monotone triangle where each entry $e$ carries a decoration from $\{\searrow, \nearrow, \swarrow\}$ such that the following two conditions are satisfied:

- If $e$ has a $\searrow$-neighbor and is equal to it, then $e$ must carry $\nearrow$.
- If $e$ has a $\nearrow$-neighbor and is equal to it, then $e$ must carry $\swarrow$.

In summary, an arrow indicates that the entry is different from the next entry in the specified direction.

We assign the following weight to an arrowed monotone triangle with $n$ rows

\[
u^\#_{\nearrow}v^\#_{\searrow}w^\#_{\swarrow}\prod_{i=1}^{n}X_i^{(\text{sum of entries in row } i) - (\text{sum of entries in row } i-1) + (#_{\nearrow} \text{ in row } i) - (#_{\swarrow} \text{ in row } i)},
\]

where the sum of entries in row 0 is defined to be 0.
In our examples, we write \(^*\) for entry decorated with \(\blacklozenge\), \(\blacktriangleleft\), \(\blacktriangleleft\) respectively. Such an example is provided next.

\[
\begin{array}{cccccccccc}
\blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge \\
4 & 5 & 6' & 7' & 8' & 9' & 10' & \\
\blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \\
1 & 2' & 3' & 4' & 5' & 6' & 8' & \\
\blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & 10' & \\
0 & 2 & 4' & 6' & 8' & \\
\blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \\
\end{array}
\]

Its weight is

\[u^7v^8w^6X_1^4X_2^3X_3^5X_6^4X_8^5.\]

In the case \(n = 2\), there are three undecorated monotone triangles with bottom row \((0, 2)\), which amount to a total of 45 such monotone triangles with decorations. We list these arrowed monotone triangles, where \(^*\) means that the entry can be decorated with any of the arrows \(\blacklozenge\), \(\blacktriangleleft\), \(\blacktriangleleft\).

\[
\begin{array}{cccccc}
0^* & 2^* & 0^* & 1^* & 2^* & 0^* & 2^* \\
\end{array}
\]

The generating functions is

\[
(2.1) \quad vX_2(uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w) + X_1X_2(uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w)^2 \\
+ uX_2^2X_2(uX_1 + vX_1^{-1} + w)(uX_2 + vX_2^{-1} + w).
\]

The significance of arrowed monotone triangles originally stems from the following specialization: Setting \(u = v = 1, w = -1\) and \(X_i = 1\), for \(i = 1, \ldots, n\), the generating function of arrowed monotone triangles with bottom row \(k_1, \ldots, k_n\) is the number of monotone triangles with bottom row \(k_1, \ldots, k_n\). Indeed, if we want to decorate a given monotone triangle with elements from \(\{\blacklozenge, \blacktriangleleft, \blacktriangleleft\}\) in an eligible way, the arrows are actually prescribed for all entries except for those that are different from their \(\blacklozenge\)- and \(\blacktriangleleft\)-neighbours (if they exist). In this case, we are free to choose any decoration. We say that such an entry is free. Thus, suppose \(f\) is the number of free entries for a given monotone triangle, then there are \(3^f\) associated arrowed monotone triangles. Setting \(X_i = 1, i = 1, \ldots, n\), in the generating function, the contribution of these \(3^f\) arrowed monotone triangles in the generating function is up to a monomial in \(u\) and \(v\) (coming from the entries that are not free) equal to \((u + v + w)^f\), which evaluates to 1 if \(u = v = 1\) and \(w = -1\).

Our main results concern several non-intersecting lattice paths or plane partition objects that have the same generating function as arrowed monotone triangles with bottom row \(0, 2, \ldots, 2n - 2\). Before we state these results, recall that also cyclically and vertically symmetric lozenge tilings of a hexagon with side lengths \(2n + 2, 2n, 2n + 2, 2n, 2n + 2, 2n\) and a central triangular hole of size 2 are in easy bijective correspondence with families of non-intersecting lattice paths. An example of such a lozenge tiling is provided in Figure 1 (left).

Due to the symmetries of the tilings, the holey hexagon can be decomposed into six equal parts, up to rotation and reflection. Figure 2 shows such a fundamental area \(H_n\). The tilings of the area \(H_n\) correspond to families of non-intersecting lattice paths starting in \(A_i = (2i - 1, i - 1)\) and ending in \(E_i = (i - 1, 2i - 2)\) for \(1 \leq i \leq n\) with the step set \(\{(-1, 0), (0, 1)\}\). For our example, that is indicated in Figure 1 (right).
The first result involves lattice paths in $\mathbb{Z}^2$, see Figure 3 for an example. We divide the lattice points of $\mathbb{Z}^2$ into even and odd points depending on whether the sum of the coordinates is even or odd, respectively.

**Theorem 2.2.** For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row $0, 2, \ldots, 2n - 2$ is equal to the signed generating function of $n$ lattice paths with starting points $(-1, 1), (-2, 2), \ldots, (-n, n)$ and end points $(0, 1), (1, 0), \ldots, (n - 1, -n + 2)$ and the following properties:

- In the region $\{(x, y) | x \leq 0\}$, the step set is $\{(1, 1), (1, 0)\}$, and steps of type $(1, 0)$ are equipped with the weight $w$.
- In the region $\{(x, y) | x \geq 0, y \geq 1\}$, the step set is $\{(1, -1), (0, -2)\}$, and steps of type $(0, -2)$ are equipped the weight $-wv$.
- In the region $\{(x, y) | x \geq 0, y \leq 1\}$, the step set is $\{(-1, 0), (0, -1)\}$. Horizontal steps with distance $d$ from the line $y = 2$ are equipped with the weight $uX_d + vX_d^{-1}$.

The paths are non-intersecting in the first and in the third region. In the second region, we have two types of paths, even and odd, depending on whether they contain only even lattice points or only odd lattice points. Lattice paths of the same type are not intersecting each other, but an odd path may have an intersection with an even path.

The weight of a family of lattice paths is $\prod_{i=1}^{n} X_i^{n-1}$ multiplied by the product of the weights of all its steps where the weight of a step is 1 if it has not been specified. Let $\sigma$ be the
Figure 3. Example of the first interpretation for $n = 6$. The associated permutation is $\sigma = (1\ 2\ 3\ 6\ 4\ 5)$ and the weight is $(-uv)^5w^9X_1^5X_2^5X_3^5X_4^5X_5^5(uX_3 + vX_3^{-1})(uX_6 + vX_6^{-1})$. In the second region, we draw the even and odd paths in different colors.

permutation so that the $i$-th starting point is connected to the $\sigma(i)$-th end point, then the sign of the family is $\text{sgn}\sigma$.

Next we list all families of lattice paths from Theorem 2.2 for the case $n = 2$ along with their weights up to the overall factor of $X_1X_2$. 

\begin{align*}
\text{w}^3 & \quad \text{w}(uX_1 + vX_1^{-1}) \quad \text{w}(uX_2 + vX_2^{-1}) \quad \text{w}(uX_1 + vX_1^{-1}) \quad \text{w}(uX_2 + vX_2^{-1}) \\
\text{w}(-uv) & \quad \text{w}(-uv) \quad \text{w}(uX_1 + vX_1^{-1})^2 \quad \text{w}(uX_1 + vX_1^{-1})(uX_2 + vX_2^{-1}) \quad \text{w}(uX_2 + vX_2^{-1})^2
\end{align*}
It can be checked that the weights indeed add to the generating function in (2.1).

In Sections 5.2 and 5.3, we show, in a purely combinatorial manner, that, when setting $u = v = 1$, $w = -1$ and $X_i = 1$ for $i = 1, \ldots, n$, the number of families of lattice paths as described in Theorem 2.2 are equinumerous with the families of non-intersecting lattice paths that correspond to cyclically and vertically symmetric lozenge tilings of a hexagon with side lengths $2n+2, 2n, 2n+2, 2n, 2n+2, 2n$ and a central triangular hole of size 2. Interestingly, in order to establish this connection, we have to leave the “regime” of non-intersecting lattice paths and have to work with lattice paths that are possibly intersecting. This was not the case in [AFb] and it is an open problem whether this is also possible here.

Also the second interpretation is most conveniently expressed in terms of non-intersecting lattice paths.

**Theorem 2.3.** For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row $0, 2, \ldots, 2n - 2$ is equal to the signed generating function of $n$ lattice paths with starting points $(1, 1), (2, 2), \ldots, (n, n)$ and end points $(2, 1), (2, 0), \ldots, (2, n + 2)$ as follows:

- In the region $\{(x, y)|y \geq 1\}$, the step set is $\{(1, 0), (0, -1)\}$ and horizontal steps with distance $d$ from the $x$-axis are equipped with the weight $uX_d + w + vX_d^{-1}$.
- In the region $\{(x, y)|y \leq 1\}$, the step set is $\{(-1, -1), (-2, -2), (-2, -1)\}$. Steps of type $(-1, -1)$ are equipped with the weight $-w$, while steps of type $(-2, -2)$ are equipped with the weight $-w$.

The paths are non-intersecting in the region $\{(x, y)|y \geq 1\}$. In the region $\{(x, y)|y \leq 1\}$, there may be intersections, however, no two steps of different paths may have a common end point.

The weight of a family is $\prod_{i=1}^{n+1} X_i^{n-1}$ multiplied by the product of the weights of all its steps where the weight of a step is 1 if it has not been specified. Let $\sigma$ be the permutation so that the $i$-th starting point is connected to the $\sigma(i)$-th end point, then the sign of the family is $\mathrm{sgn} \sigma$.

An example is provided in Figure 4. Its weight is

$$
\mathrm{sgn} \sigma(-uv)^3(-w)^5X_1^5X_2^5X_3^5X_4^5X_5^5X_6^0(uX_1 + w + vX_1^{-1})(uX_2 + w + vX_2^{-1})^2 \\
\times (uX_3 + w + vX_3^{-1})^2(uX_4 + w + vX_4^{-1})(uX_5 + w + vX_5^{-1})^2(uX_6 + vX_6^{-1})^2
$$

with the associated permutation $\sigma = (1 \ 2 \ 4 \ 3 \ 5 \ 6)$. 
Figure 4. Example of the second interpretation for $n = 6$. The paths are drawn in alternating colors for a better distinction.

Again we list all families of paths from Theorem 2.3 for the case $n = 2$ along with their weights up to the overall factor $X_1X_2$.

Finally, we present a third interpretation in terms of families of lattice paths.

**Theorem 2.4.** For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row 0, 2, \ldots, 2n−2 is equal to the generating function of families of $n$ non-intersecting lattice paths with starting points in $A_i = \{(n+i, 2n), (n+i−1, 2n)\}$, $i = 1, 2, \ldots, n$, to $E_j = (j, −j+1)$, $j = 1, 2, \ldots, n$ with the following properties.

- In the region $\{(x, y) | y \geq 1\}$, the step set is $\{(1, 0), (0, −1)\}$. Horizontal steps at height 1, 2, 3, 4, \ldots above the $x$-axis have weight $uX_1, vX_1^{-1}, uX_2, vX_2^{-1}, \ldots$, respectively.
- In the region $\{(x, y) | y \leq 1\}$, the step set is $\{(-1, -1), (0, -1)\}$, where steps of type $(0, -1)$ are equipped with the weight $w$.

Let $1 < i_1 < i_2 < \ldots < i_k \leq n$ be precisely the indices for which we choose $(n+i_k−1, 2n) \in A_{i_k}$ as starting points (and not $(n−i_k+1, 2n) \in A_{i_k}$). The weight of a family is

$$\prod_{i=1}^{k}(-uv)^{i−1} \prod_{i=1}^{n} X_i^{n−1}$$

multiplied by the product of the weights of all its steps where the weight of a step is 1 if it has not been specified.

Note that $A_1 = \{(n, 2n)\}$ consists of a single point. Note as well that in this interpretation, the sign has been incorporated into the weight. An example for the case $n = 4$ is provided in Figure 5. For this interpretation, we omit the case $n = 2$, since it already involves 57 configurations.
The weight of this family of non-intersecting lattice paths is $(-uv)^{3-n}w^2(uX_1)(uX_2)(vX_3^{-1})X_1^3X_2^3X_3^3X_4^3$, which is equal to $u^5v^3w^2X_1^4X_2^4X_3^3X_4^3$.

Finally, we present a signless version that is in terms of pairs of plane partitions of the same shape, where one of them is a column-strict plane partition (CSPP) and the other one is a row-strict plane partition (RSPP). Recall that a CSPP is a filling of a Young diagram with positive integers that weakly decrease along rows and strictly decrease down columns, while an RSPP is a filling of a Young diagram with positive integers that weakly decrease down columns and strictly along rows. Next we display a CSPP (left) and an RSPP (right).

![Figure 5](image_url)

**Theorem 2.5.** For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row $0, 2, \ldots, 2n - 2$ is equal to the generating function of pairs $(P, Q)$ of plane partitions of the same shape with $n$ rows (allowing also rows of length zero) such that $P$ is a CSPP and $Q$ is an RSPP, and the entries of $P$ in the $i$-th row from the bottom are no greater than $2i$, while the entries of $Q$ in the $i$-th row from the bottom are no greater than $i$, and the weight of such a pair is given by the following monomial.

$$w_{\binom{n+1}{2} - \text{# of entries in } Q} \prod_{i=1}^{n} X_i^{n-1} (uX_i)^{\#2 - 1} \text{ in } P (vX_i^{-1})^{\#2i} \text{ in } P$$
Letting \( n = 6 \), the example above is a pair that has the properties described in Theorem 2.5. The weight is

\[
\begin{align*}
& w^{(2)}_{16} X_1^5 X_2^5 X_3^5 X_4^5 X_5^5 X_6^5 (uX_1)^0 (uX_2)^2 (uX_3)^2 (uX_4)^1 (uX_5)^0 \\
& \quad \times (vX_1^{-1})^2 (vX_2^{-1})^3 (vX_3^{-1})^3 (vX_4^{-1}) (vX_5^{-1})^0 (vX_6^{-1})^0,
\end{align*}
\]

which is equal to \( w^5 u^7 v^9 X_1^4 X_2^4 X_3^6 X_4^6 X_5^5 X_6^5 \). Next we list all pairs for the case \( n = 2 \) along with their weights, which indeed add up to (2.1).

\[
\begin{array}{cccccccc}
(\emptyset, \emptyset) & (\emptyset, 1) & (\emptyset, 2) & (1, \emptyset) & (1, 1) & (1, 2) & (2, 1) & (2, 2) \\
\hline
w^3 X_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 \\
(2, 1, 1) & (3, 1, 1) & (4, 1, 1) & (2, 2, 1) & (3, 2, 1) & (4, 2, 1) & (3, 3, 1) & (3, 2, 1) \\
\hline
w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 \\
(4, 1, 1, 1) & (3, 2, 1, 1) & (4, 2, 1, 1) & (3, 3, 1, 1) & (3, 2, 1, 1) & (4, 2, 1, 1) & (3, 3, 1, 1) & (3, 2, 1, 1) \\
\hline
w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 \\
(3, 1, 1, 1, 1) & (3, 2, 1, 1, 1) & (4, 2, 1, 1, 1) & (3, 3, 1, 1, 1) & (3, 2, 1, 1, 1) & (4, 2, 1, 1, 1) & (3, 3, 1, 1, 1) & (3, 2, 1, 1, 1) \\
\hline
w^2 uX_1 X_2 & w^2 vX_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 \\
(2, 1, 1, 1, 1, 1) & (3, 1, 1, 1, 1, 1) & (4, 1, 1, 1, 1, 1) & (3, 2, 1, 1, 1, 1) & (4, 2, 1, 1, 1, 1) & (3, 3, 1, 1, 1, 1) & (3, 2, 1, 1, 1, 1) & (4, 2, 1, 1, 1, 1) \\
\hline
w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 \\
(2, 1, 1, 1, 1, 1, 1) & (3, 1, 1, 1, 1, 1, 1) & (4, 1, 1, 1, 1, 1, 1) & (3, 2, 1, 1, 1, 1, 1) & (4, 2, 1, 1, 1, 1, 1) & (3, 3, 1, 1, 1, 1, 1) & (3, 2, 1, 1, 1, 1, 1) & (4, 2, 1, 1, 1, 1, 1) \\
\hline
w^2 uX_1 X_2 & w^2 vX_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 uX_1 X_2 & w^2 vX_2 & w^2 uX_1 X_2 & w^2 vX_2 \\
\end{array}
\]

Remark 2.6. RSPPs with at most \( n \) rows (allowing also rows of length zero) such that the entries in the \( i \)-th row from the bottom are no greater than \( i \) are in easy bijective correspondence with \((2n+2) \times (2n+2) \times (2n+2)\) totally symmetric self-complementary plane partitions: After replacing each part \( p \) of the RSPP by \( n + 1 - p \) and conjugating, we obtain a semistandard Young tableau with entries in \( \{1, 2, \ldots, n\} \) such that the entries in column \( i \) (counted from the left) are no smaller than \( i \). Phrased differently, entries \( i \) can only occur in the first \( i \) columns. For the example preceding Theorem 2.5, we obtain

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 5 & 6 & 6 \\
3 & 4 & 6 \\
5 & 6 \\
\end{array}
\]

when choosing \( n = 6 \).

Now we use the standard procedure to transform semistandard Young tableaux with entries in \( \{1, 2, \ldots, n\} \) into Gelfand-Tsetlin patterns with \( n \) rows: To obtain the \( i \)-th row of the Gelfand-Tsetlin pattern, consider the shape in the semistandard Young tableaux constituted by the entries less than or equal to \( i \), add 0’s to the corresponding partition so that it is of length \( i \) and write it in reverse order. In our running example, we obtain the following
Gelfand-Tsetlin pattern.

\[
\begin{array}{cccc}
1 \\
1 & 2 \\
1 & 2 & 3 \\
0 & 2 & 2 & 4 \\
0 & 1 & 2 & 3 & 5 \\
0 & 0 & 2 & 3 & 5 & 6
\end{array}
\]

As the entries less than or equal to \(i\) are confined to the first \(i\) columns it follows that all parts in the partition that constitute the \(i\)-th row of the Gelfand-Tsetlin pattern are less than or equal to \(i\). We obtain a Gelfand-Tsetlin pattern with \(n\) rows, parts in \(\{0, 1, \ldots, n-1\}\) such that the entries in \(i\)-th \(\nearrow\)-diagonal are no greater than \(i\). Adding 1 to each entry and prepending a \(\nearrow\)-diagonal of 1's on the left results in a Magog triangle of order \(n+1\) as defined in [Zei96, Fis18]. In our example, we obtain

\[
\begin{array}{cccc}
1 \\
1 & 2 \\
1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
1 & 1 & 3 & 3 & 5 \\
1 & 1 & 2 & 3 & 4 & 6 \\
1 & 1 & 1 & 3 & 4 & 6 & 7
\end{array}
\]

Such Magog triangles are known to be in bijective correspondence with \((2n+2) \times (2n+2) \times (2n+2)\) totally symmetry self-complementary plane partitions [MRR86].

On the other hand, the CSPPs with the constraints as they appear in the theorem are in easy bijective correspondence with symplectic tableaux as defined in [KT90, Section 4]. Thus, Theorem 2.5 can also be seen as to provide the expansion of the generating function of arrowed monotone triangles with bottom row \(0, 2, \ldots, 2n-2\) into symplectic characters. This was conjectured by Florian Aigner [Aig] independently.

**Outline of the paper.** The paper is organized as follows. In Section 3, we introduce and manipulate a formula for the generating function of arrowed monotone triangles. In Section 4, we observe that for our particular bottom row, we can express the generating function with the help of a determinant. In Section 5, we provide the first combinatorial interpretation in terms of families of lattice paths (Theorem 2.2). We also relate this interpretation to the holey symmetric lozenge tilings that are known to be equinumerous with VSASMs. In Section 6, we derive the second combinatorial interpretation (Theorem 2.3), and, in Section 7, we establish the third combinatorial interpretation (Theorem 2.4). In this section, we also relate the first and the third combinatorial interpretation to the signless interpretation in terms of plane partitions, that is, we provide two different proofs of Theorem 2.5.

### 3. The Generating Function of Arrowed Monotone Triangles

We point out that there is a close analogy between arrowed monotone triangles and its generating function with respect to a fixed bottom row on one side and semistandard Young tableaux and Schur polynomials on the other side: Recall that Gelfand-Tsetlin patterns are defined as monotone triangles with the condition of the strict increase along rows dropped. As demonstrated in Remark 2.6, semistandard Young tableaux are in bijective correspondence
with Gelfand-Tsetlin patterns. Schur polynomials are multivariate generating functions of semistandard Young tableaux with respect to the weight
\[X_1^\#\] of 1's \(X_2^\#\) of 2's \(\ldots\) \(X_n^\#\) of \(n\)'s.

However, in terms of Gelfand-Tsetlin patterns, this weight can obviously be expressed as
\[
\prod_{i=1}^{n} X_i^{(\text{sum of entries in row } i) - (\text{sum of entries in row } i - 1)},
\]
which is analogous to the weight we use for arrowed monotone triangles. This analogy extends also to the following formula for the generating function of arrowed monotone triangles with given bottom row, which can be expressed in terms of (extended) Schur polynomials. We define
\[
s_{(k_n, \ldots, k_1)}(X_1, \ldots, X_n) = \frac{\det_{1 \leq i, j \leq n}(X_i^{k_j + j - 1})}{\prod_{1 \leq i < j \leq n}(X_j - X_i)}.
\]

Note that, for \(0 \leq k_1 \leq k_2 \leq \ldots \leq k_n\), \(s_{(k_n, \ldots, k_1)}(X_1, \ldots, X_n)\) is the Schur polynomial of the partition \((k_n, k_{n-1}, \ldots, k_1)\). In [APb, Theorem 2.2], the following theorem has been established, which is one of our main ingredients for the present paper.

**Theorem 3.1.** The generating function of arrowed monotone triangles with bottom row \(k_1 < k_2 < \ldots < k_n\) is
\[
\prod_{i=1}^{n} \left(uX_i + vX_i^{-1} + w\right) \prod_{1 \leq i < j \leq n} \left(uE_{k_i} + vE_{k_j}^{-1} + wE_{k_i}E_{k_j}^{-1}\right) s_{(k_n, \ldots, k_1)}(X_1, \ldots, X_n),
\]
where \(E_x\) denotes the shift operator, defined as \(E_x p(x) = p(x + 1)\).

Using the antisymmetrizer, defined as
\[
\text{ASym}_{X_1, \ldots, X_n}[f(X_1, \ldots, X_n)] = \sum_{\sigma \in S_n} \text{sgn} \sigma f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}),
\]
we rewrite the expression from Theorem 3.1 and obtain
\[
\prod_{i=1}^{n} \left(uX_i + vX_i^{-1} + w\right) \prod_{1 \leq i < j \leq n} \left(uE_{k_i} + vE_{k_j}^{-1} + wE_{k_i}E_{k_j}^{-1}\right) \text{ASym}_{X_1, \ldots, X_n}\left[\prod_{i=1}^{n} X_i^{k_i + i - 1}\right] \prod_{1 \leq i < j \leq n}(X_j - X_i)
\]
\[
= \prod_{i=1}^{n} \left(uX_i + vX_i^{-1} + w\right) \text{ASym}_{X_1, \ldots, X_n}\left[\prod_{1 \leq i < j \leq n} \left(uE_{k_i} + vE_{k_j}^{-1} + wE_{k_i}E_{k_j}^{-1}\right) \prod_{i=1}^{n} X_i^{k_i + i - 1}\right] \prod_{1 \leq i < j \leq n}(X_j - X_i)
\]
\[
= \prod_{i=1}^{n} \left(uX_i + vX_i^{-1} + w\right) \text{ASym}_{X_1, \ldots, X_n}\left[\prod_{1 \leq i < j \leq n} \left(uX_i + vX_i^{-1} + wX_jX_i^{-1}\right) \prod_{i=1}^{n} X_i^{k_i + i - 1}\right] \prod_{1 \leq i < j \leq n}(X_j - X_i)
\]
\[
= \text{ASym}_{X_1, \ldots, X_n}\left[\prod_{1 \leq i < j \leq n} \left(uX_j + vX_i^{-1} + w\right) \prod_{i=1}^{n} X_i^{k_i + i - 1}\right] \prod_{1 \leq i < j \leq n}(X_j - X_i)
\]

We set \(k_i = 2i - 2\) to restrict to VSASMs. We also multiply with
\[
\prod_{1 \leq i < j \leq n} \left(u - vX_i^{-1}X_j^{-1}\right),
\]
which is a symmetric function and can therefore be put inside the antisymmetrizer. Thus, we obtain

\[
\text{ASym}_{x_1, \ldots, x_n} \left[ \prod_{1 \leq i, j \leq n} (uX_j + vX_i^{-1} + w)(u - vX_i^{-1}X_j^{-1}) \prod_{i=1}^n X_i^{i+n-2} \right] / \prod_{1 \leq i, j \leq n} (X_j - X_i) = \prod_{i=1}^n X_i^{n-2} \text{ASym}_{x_1, \ldots, x_n} \left[ \prod_{1 \leq i, j \leq n} (uX_j + vX_i^{-1} + w)(uX_j - vX_i^{-1}) \right] / \prod_{1 \leq i, j \leq n} (X_j - X_i).
\]

4. Transition to plane partitions

The following lemma is another important ingredient. It first appeared in [AFK+20] (with a different proof), while in [APa] a combinatorial proof is provided. The proof given here is probably the shortest.

**Lemma 4.1.** Let \( n \geq 1 \), and \( Y = (Y_1, \ldots, Y_n), Z = (Z_1, \ldots, Z_n) \) be two sets of indeterminants. Then

\[
\det_{1 \leq i, j \leq n} (Y_i^j - Z_i^j) = \text{ASym} \left[ \prod_{1 \leq i, j \leq n} (Y_j - Z_i) \right].
\]

with

\[
\text{ASym} \left[ f(Y; Z) \right] = \sum_{\sigma \in S_n} \text{sgn} \sigma f(Y_{\sigma(1)}, \ldots, Y_{\sigma(n)}; Z_{\sigma(1)}, \ldots, Z_{\sigma(n)}).
\]

*Proof.* The proof is by induction with respect to \( n \). The result is obvious for \( n = 1 \). Let \( A_n(Y; Z) \) denote the right-hand side of (4.1), and observe that we have

\[
A_n(Y; Z) = \sum_{i=1}^n (-1)^{i+1} \left( \prod_{k=1}^n (Y_k - Z_i) \right) A_{n-1}(Y_1, \ldots, \tilde{Y}_i, \ldots, Y_n; Z_1, \ldots, \tilde{Z}_i, \ldots, Z_n),
\]

where \( \tilde{Y}_i \) and \( \tilde{Z}_i \) means that \( Y_i \) and \( Z_i \) are omitted, respectively. Using the induction hypothesis, it follows that \( A_n(Y; Z) \) is

\[
(-1)^{n-1} \det_{1 \leq i, j \leq n} \left( \begin{array}{c} Y_i^j - Z_i^j, \\ \prod_{k=1}^n (Y_k - Z_i), \end{array} \begin{array}{c} j < n, \\ j = n. \end{array} \right)
\]

For \( j \in \{1, 2, \ldots, n-1\} \), we multiply the \( j \)-th column with \( (-1)^{j} e_{n-j}(Y_1, \ldots, Y_n) \) and add it to the last column, where \( e_{n-j} \) denotes the elementary symmetric polynomial of degree \( n - j \). The entry in the \( i \)-th row and \( n \)-th column is then

\[
\sum_{j=1}^{n-1} (-1)^{j}(Y_i^j - Z_i^j)e_{n-j}(Y_1, \ldots, Y_n) + \prod_{k=1}^n (Y_k - Z_i)
\]

\[
= \sum_{j=1}^{n-1} (-1)^{j}(Y_i^j - Z_i^j)e_{n-j}(Y_1, \ldots, Y_n) + \sum_{j=0}^{n} (-1)^{j} Z_i^j e_{n-j}(Y_1, \ldots, Y_n)
\]

\[
= \sum_{j=1}^{n-1} (-1)^{j} Y_i^j e_{n-j}(Y_1, \ldots, Y_n) + e_{n}(Y_1, \ldots, Y_n) + (-1)^{n} Z_i^n
\]

\[
= \prod_{k=1}^n (Y_k - Y_i) - e_{n}(Y_1, \ldots, Y_n) - (-1)^{n} Y_i^n + e_{n}(Y_1, \ldots, Y_n) + (-1)^{n} Z_i^n
\]

\[
= (-1)^{n} (Z_i^n - Y_i^n),
\]

and this proves the assertion. \( \square \)
The crucial observation is that
\[(uX_j + vX_i^{-1} + w)(uX_j - vX_i^{-1}) = u^2X_j^2 + uwX_j - v^2X_i^{-2} - vwX_i^{-1},\]
and so the lemma is applicable (note that for this it was important to multiply with (3.2)): Setting \(Y_j = u^2X_j^2 + uwX_j\) and \(Z_i = v^2X_i^{-2} + vwX_i^{-1}\) in the lemma, we obtain
\[
\prod_{i=1}^{n} X_i^{n-2} \frac{\det_{1 \leq i, j \leq n} \left( (u^2X_i^2 + uwX_i)^j - (v^2X_i^{-2} + vwX_i^{-1})^j \right)}{\prod_{1 \leq i < j \leq n} (X_j - X_i)}.
\]

In order to obtain the generating function of arrowed monotone triangles with bottom row \(0, 2, \ldots, 2n - 2\), we need to divide by (3.2). The result is
\[
(4.2) \quad \prod_{i=1}^{n} X_i^{n-2} (u - vX_i^{-2})^{-1} \frac{\det_{1 \leq i, j \leq n} \left( (u^2X_i^2 + uwX_i)^j - (v^2X_i^{-2} + vwX_i^{-1})^j \right)}{\prod_{1 \leq i < j \leq n} (X_j - X_i) (u - vX_i^{-1})(X_j^{-1} - X_i^{-1})}.
\]

5. First combinatorial interpretation

5.1. Proof of Theorem 2.2. The following lemma is an important tool to transform bialternate formulas into Jacobi-Trudi type formulas; the proof can be found in [AF, Lemma 7.2]. The lemma involves complete symmetry functions also of negative degree, which are defined as follows.

\[h_k(X_1, \ldots, X_n) = (-1)^{n+1} \sum_{l_1 + \ldots + l_n = k, l_i \geq 0} X_1^{l_1} X_2^{l_2} \ldots X_n^{l_n}\]

Note that

\[h_{-1}(X_1, \ldots, X_n) = h_{-2}(X_1, \ldots, X_n) = \ldots = h_{-n+1}(X_1, \ldots, X_n) = 0.\]

For \(k \leq -n\), they can also be expressed in terms of those with positive degree as
\[
(5.1) \quad h_k(X_1, \ldots, X_n) = (-1)^{n+1}(X_1 \ldots X_n)^{-1} \sum_{l_1 + \ldots + l_n = k - n, l_i \geq 0} X_1^{-l_1} X_2^{-l_2} \ldots X_n^{-l_n} = (-1)^{n+1}(X_1 \ldots X_n)^{-1} h_{k-n}(X_1^{-1}, \ldots, X_n^{-1}).
\]

In fact, it is not hard to see that the relation
\[h_k(X_1, \ldots, X_n) = (-1)^{n+1}(X_1 \ldots X_n)^{-1} h_{k-n}(X_1^{-1}, \ldots, X_n^{-1})\]
is true for any \(k\).

**Lemma 5.1.** Let \(f_j(Y)\) be formal Laurent series for \(1 \leq j \leq n\), and define
\[f_j[Y_1, \ldots, Y_i] = \sum_{k \in \mathbb{Z}} (Y^k) f_j(Y) \cdot h_{k-i+1}(Y_1, \ldots, Y_i),\]

where \((Y^k)f_j(Y)\) denotes the coefficient of \(Y^k\) in \(f_j(Y)\) and \(h_{k-i+1}\) denotes the complete homogeneous symmetric polynomial of degree \(k - i + 1\). Then
\[
\frac{\det_{1 \leq i, j \leq n} (f_j(Y_i))}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)} = \frac{\det_{1 \leq i, j \leq n} (f_j[Y_1, \ldots, Y_i])}{\prod_{1 \leq i < j \leq n} (Y_j - Y_i)}.
\]

We rewrite (4.2) as follows.
\[
(5.2) \quad \prod_{i=1}^{n} X_i^{n-1} \frac{\det_{1 \leq i, j \leq n} \left( (u^2X_i^2 + uwX_i)^j - (v^2X_i^{-2} + vwX_i^{-1})^j \right)}{\prod_{1 \leq i < j \leq n} (uX_j + vX_i^{-1})(uX_i + vX_j^{-1})}.
\]
We aim at applying Lemma 5.1 to this bialternant with $Y_i = uX_i + vX_i^{-1}$. For this purpose, we need to express the entries in the $i$-th row of the matrix that underlies the determinant in (5.2) as formal Laurent series in $Y_i$, which is possible since these entries are invariant under $X_i \rightarrow u^{-1}vX_i^{-1}$. We rewrite the entries as follows.

\begin{equation}
(5.3) \quad \left( \frac{u^2X_i^2 + uwX_i}{uX_i - vX_i^{-1}} \right)^j = \frac{(v^2X_i^{-2} + vwX_i^{-1})^j}{uX_i - vX_i^{-1}} = \sum_{k=0}^j \binom{j}{k} w^{j-k} \frac{u^{j+k}X_i^{j+k} - v^{j+k}X_i^{-(j+k)}}{uX_i - vX_i^{-1}}.
\end{equation}

The next lemma is useful to transform this further.

**Lemma 5.2.** For $l \geq 0$, we have

\begin{equation}
\frac{u^lX^l - v^lX^{-l}}{uX - vX^{-1}} = \frac{(-uv)^r}{r!} r(l - 1) \binom{l - 2r - 1}{r} (uX + vX^{-1})^{l - 2r - 1}.
\end{equation}

**Proof.** Using the geometric series expansion, the left-hand side can be written as $\sum_{k=0}^{l-1} u^k v^{-k-l+1} X^{2k+l+1}$. The right-hand side is equal to

\begin{equation}
\sum_{r=0}^{(l-1)/2} \sum_{s \geq 0} (-1)^{r+s} \binom{l - r - 1}{r} \binom{l - 2r - 1}{s} u^r v^{-r-s+l-1} X^{2r+2s-l+1} = \sum_{k \leq l - 1} u^k v^{-k-l+1} X^{2k+l+1} \sum_{r=0}^{(l-1)/2} (-1)^{r} \binom{l - r - 1}{l - 2r - 1} \binom{r}{k - r}.
\end{equation}

Now, using the Chu-Vandermonde in the third step, the inner sum simplifies as follows and the proof is complete.

\begin{equation}
\sum_{r=0}^{l-k-1} (-1)^{r} \binom{l - r - 1}{r} = \sum_{r=0}^{l-k-1} (-1)^{r} \binom{l - r - 1}{k - r} = \sum_{r=0}^{l-k-1} (-1)^{l-1-k} \binom{l - r - 1}{k - r} = \begin{cases} 1, & k \leq l - 1, \\ 0, & k \geq l \end{cases}
\end{equation}

We use the lemma to rewrite (5.3) as follows.

\begin{equation}
\sum_{k=0}^j \binom{j}{k} w^{j-k} (-uv)^r \binom{j+k-r-1}{r} (uX_i + vX_i^{-1})^{j+k-2r-1} = \sum_{k \geq 0} (-uv)^r w^{2j-k-2r-1} \binom{j}{k} h_{k+1} (uX_i + vX_i^{-1} - 1)^k
\end{equation}

Applying Lemma 5.1 to (5.2) with $Y_i = uX_i + vX_i^{-1}$, we have

\begin{equation}
f_j[uX_i + vX_i^{-1}, \ldots, uX_i + vX_i^{-1}] = \sum_{k \geq 0} (-uv)^r w^{2j-k-2r-1} \binom{j+k}{r} \binom{j}{2j-k-2r-1} h_{k+1} (uX_i + vX_i^{-1}, \ldots, uX_i + vX_i^{-1})
\end{equation}

\begin{equation}
= \sum_{k \geq 0} (-uv)^r w^k \binom{2j-k-r-1}{r} \binom{j}{k} h_{2j-k-2r-1} (uX_i + vX_i^{-1}, \ldots, uX_i + vX_i^{-1}).
\end{equation}
We set $p = 2j - k$ and eliminate $k$. Then we let $q = p - 2r$ and eliminate $r$. Also exchange the role of $i$ and $j$, which is possible since we are interested in the determinant. We obtain
\begin{equation}
(5.4) \sum_{p \geq 1} w^{2i-p} \binom{i}{2i-p} \sum_{q \geq 1, 2(p-q)} (-uv)^{(p-q)/2} \left(\frac{(p+q)/2 - 1}{(p-q)/2}\right) h_{q-j}(uX_1 + vX_1^{-1}, \ldots, uX_j + vX_j^{-1}),
\end{equation}
which we denote by $a_{i,j}$.

Our combinatorial interpretations are based on the Lindström-Gessel-Viennot lemma [Lin73, GV85, GV89]. We state a version that is useful for us.

**Lemma 5.3.** Suppose $G$ is a finite directed acyclic edge-weighted graph with weight function $w$. For $i \in \{1, 2, \ldots, n\}$, let $A_i, E_i$ be sets of vertices of $G$ and assume that weights are also assigned to the vertices in these sets, also indicated by $w$. Further, let $P(A_i, E_j)$ be the generating function of paths from $A_i$ to $E_j$, i.e.,
\[ P(A_i, E_j) = \sum_{(x,y) \in A_i \times E_j} w(x)w(y) \prod_{e \in P} w(e). \]

Then $\det_{1 \leq i, j \leq n}(P(A_i, E_j))$ is the “signed” generating function of families of $n$ pairwise non-intersecting paths $(P_1, \ldots, P_n)$ with the property that there exists a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ (not necessarily the same for all $n$-tuples) such that $P_i$ is a directed path from a vertex $x_i$ of $A_i$ to a vertex $y_i$ of $E_{\sigma(i)}$. The “signed” weight is
\[ \text{sgn} \sigma \prod_{i=1}^{n} w(x_i)w(y_i) \prod_{e \in P_i} w(e). \]

We interpret (5.4) as a certain generating function of lattice paths from $(-i, i)$ to $(j - 1, -j + 2)$ as follows, see Figure 6.

- In the region $\{(x,y)|x \leq 0\}$, the step set is $\{(1,1), (1,0)\}$. Assuming that $(0,p)$ is the first lattice point on $x = 0$, there are $p-i$ steps of type $(1,1)$ and $2i-p$ steps of type $(1,0)$, so that there are $\binom{i}{2i-p}$ such paths. Steps of type $(1,0)$ are equipped with the weight $w$.
- In the region $\{(x,y)|x \geq 0, y \geq 1\}$, the step set is $\{(1,-1), (0,-2)\}$, and we aim at reaching the line $y = 1$. The steps of type $(0,-2)$ are equipped weight $-uv$. Assuming that $(q-1,1)$ is the first lattice point on $y = 1$, there are $q-1$ steps of type $(1,-1)$ and $(p-q)/2$ steps of type $(0,-2)$, so that there are $\binom{(p+q)/2 - 1}{(p-q)/2}$ such paths. Note that the special form of the step set implies $2|(p-q)$.
- In the region $\{(x,y)|x \geq 0, y \leq 1\}$, the step set is $\{(-1,0), (0,-1)\}$. Horizontal steps with distance $h$ from the line $y = 2$ have weight $uX_h + vX_h^{-1}$. The generating function is $h_{q-j}(uX_1 + vX_1^{-1}, \ldots, uX_j + vX_j^{-1})$ as the number of $(-1,0)$ steps is $q-j$, while the number of $(0,-1)$ steps is $j-1$.
- Paths that start in the $i$-th starting point are equipped with the additional weight $X_i^{n-1}$.

Applying Lemma 5.3 shows that $\prod_{i=1}^{n} X_i^{n-1} \det_{1 \leq i, j \leq n}(a_{i,j})$ is the generating function of such lattice paths that start in $(-1,1), (-2,2), \ldots, (-n,n)$ and end in $(0,1), (1,0), \ldots, (n-1, -n+2)$, and are non-intersection in the first and in the third region. In the second region, paths can be thought of the being either even or odd in the following sense: a path is said to be even if it only reaches lattice points whose sums of coordinates are even, and otherwise they are said to be odd (in which case it only reaches lattice points whose sums of coordinates are
implies that the family of even paths is non-intersecting as well as the family of odd paths. This concludes the proof of Theorem 2.2.

Our next goal is to relate this combinatorially to cyclically and vertically symmetric lozenge tilings of a hexagon with side lengths $2n + 2, 2n, 2n + 2, 2n, 2n + 2, 2n$ and with a central triangular hole of size 2, which are known to be equinumerous with $(2n+1) \times (2n+1)$ VSASMs, see [MRR87, Kup94]. Recall that such tilings correspond to families of non-intersecting lattice paths starting in $A_i = (2i-1, i-1)$ and ending in $E_i = (i-1, 2i-2)$, with $1 \leq i \leq n$ and the step set $\{(-1,0), (0,1)\}$. The total number of families of non-intersecting lattice paths between the two sets $(A_i)_{1 \leq i \leq n}$ and $(E_i)_{1 \leq i \leq n}$ can be calculated by means of the Lindström-Gessel-Viennot lemma 5.3:

\begin{equation}
\det_{1 \leq i, j \leq n} \mathcal{P}(A_i \rightarrow E_j) = \det_{1 \leq i, j \leq n} \left( \begin{pmatrix} i + j - 1 \\ 2j - i - 1 \end{pmatrix} \right).
\end{equation}

We see that $\mathcal{P}(A_i \rightarrow E_j) = 0$ if $j > 2i$.

Noting that

\begin{equation}
h_k(X_1, \ldots, X_n)\big|_{(X_1, \ldots, X_n) = X} = X^k \binom{n+k-1}{k},
\end{equation}

we observe that, when setting $u = v = 1, w = -1$ and $X_i = 1$, for $i = 1, 2, \ldots, n$, then (5.4) simplifies to

\begin{equation}
\sum_{p \geq 1} (-1)^{2i-p} \binom{i}{2i-p} \sum_{q \geq 1,\geq 2(p-q)} 2^{q-j}(-1)^{(p-q)/2} \binom{(p+q)/2-1}{(p-q)/2} \binom{q-1}{j-1}.
\end{equation}

We will give a combinatorial proof for

\begin{equation}
\sum_{q \geq 1,\geq 2(p-q)} 2^{q-j}(-1)^{(p-q)/2} \binom{(p+q)/2-1}{(p-q)/2} \binom{q-1}{j-1} = \binom{p+j-1}{p-j}.
\end{equation}
within our model, showing that (5.7) simplifies to
\[ \sum_{p \geq 1} (-1)^{2i-p} \binom{i}{2i-p} \binom{p+j-1}{p-j}, \]
and then we show combinatorially, also within our model, that
\[ (5.9) \sum_{p \geq 1} (-1)^{2i-p} \binom{i}{2i-p} \binom{p+j-1}{p-j} = \binom{i+j-1}{2j-i+1}, \]
which is equal to the entries of the matrix underlying the determinant in (5.5).

5.2. Combinatorial proof of (5.8) within our combinatorial model. Recall that in our combinatorial model, the left-hand side of (5.8) is interpreted as

- lattice paths from \((0, p)\) to \((j - 1, -j + 2)\),
- with step set \(\{(1, -1), (0, -2)\}\) until we reach the line \(y = 1\) (at \((q - 1, 1)\)),
- while on and below the line \(y = 1\), the step set is \(\{(-1, 0), (0, -1)\}\).

Steps of type \((0, -2)\) are equipped with the weight \(-1\), while steps of type \((-1, 0)\) are equipped with the weight \(2\). We take the latter weight into account by coloring such steps either in blue or in red. All steps different from these steps are colored in green to distinguish them from the uncolored paths that will provide an interpretation for the right-hand side of (5.8). The set of these colored paths is denoted by \(A(j, p)\), and we refer to them as the \(A\)-paths.

The combinatorial model for the right-hand side of (5.8) are paths from \((0, p)\) to \((2j - 1, -j + 1)\) with step set \(\{(1, -1), (0, -1)\}\). The set of these uncolored paths is denoted by \(B(j, p)\), and we refer to them as the \(B\)-paths.

**From \(B(j, p)\) to \(A(j, p)\):** We explain how to transform paths from \(B(j, p)\) into paths from \(A(j, p)\). It is an inductive process. In each step, we will have a lattice path that starts in \((0, p)\). The initial section will be colored in green (using the step set \(\{(1, -1), (0, -2)\}\) of the first region of \(A\)-paths), while the ending section will be colored in green, red and blue (using the step set \(\{(0, -1), (-1, 0)\}\) of the second region of \(A\)-paths). For the middle section, we use the uncolored step set \(\{(1, -1), (0, -1)\}\) of \(B\)-paths.

When travelling along the path starting at \((0, p)\), we look for the first pair of uncolored steps \((s_1, s_2)\) and perform the following transformations, see also Figure 7.

1. If \((s_1, s_2) = ((1, -1), (1, -1))\), then we color \(s_1\) green, contract \(s_2\) and add a green \((0, -1)\)-step at the end.
2. If \((s_1, s_2) = ((1, -1), (0, -1))\), then we color \(s_1\) green, replace \(s_2\) by an uncolored \((1, -1)\)-step and add a blue \((-1, 0)\)-step at the end.
3. If \((s_1, s_2) = ((0, -1), (1, -1))\), then we replace \(s_1\) by a green \((1, -1)\)-step and add a red \((-1, 0)\)-step at the end.
4. If \((s_1, s_2) = ((0, -1), (0, -1))\) we make three copies, two of them with the same sign and a third with the opposite sign. In the first two, we replace \(s_1\) by a green \((1, -1)\)-step. In both cases we add a \((-1, 0)\)-step at the end — in one case it is blue and in the other case it is red. In the copy with the opposite sign, we replace \((s_1, s_2)\) by a \((0, -2)\)-step. The sign is taken into account, since \((0, -2)\)-steps have sign \(-1\).

If there is only one uncolored step left, delete it. This step is always a step of type \((1, -1)\) as can be seen as follows: Paths in \(B(j, p)\) have \(2j - 1\) steps of this type, and only Rule (1) transforms such steps. Among these \(2j - 1\) steps, there are \(j - 1\) pairs of such steps that are transformed using Rule (1), so that there is one such step left in the end. This also shows that we are always in the situation that there is one uncolored step left in the end.
We create (blue or red) steps of type \((0, -2)\)-steps are either transformed into green \((1, -1)\)-steps or pairs of them are transformed into green \((0, -2)\)-steps. Suppose that there are \(2t\) of the second type so that there are \(p - j - 2t\) of the first type. It follows that there are \(p - 2t - 1\) green steps of type \((1, -1)\) and \(t\) green steps of type \((0, -2)\). With these steps we reach indeed the line \(y = 1\), since

\[(0, p) + (p - 2t - 1) \cdot (1, -1) + t \cdot (0, -2) = (p - 2t - 1, 1)\]

Note that we set \(t\) such that \(p - 2t\) corresponds to \(q\) in (5.8).

Since none of the rules changes the height of the endpoint of our path except for the deletion of the final uncolored \((1, -1)\)-step, the endpoint of our path lies on the line \(y = -j + 2\).

We create (blue or red) steps of type \((-1, 0)\) precisely when uncolored \((0, -1)\) steps are transformed, which happens in Rules (2), (3), and (4) using either of the first two options. There are \(p - j - 2t\) such applications so that the end point lies on the line \(x = p - 2t - 1 - (p - j - 2t) = j - 1\). Therefore, the end point is \((j - 1, -j + 2)\). An example is given in Figure 8.

**From \(A(j, p)\) to \(B(j, p)\):** Conversely, paths from \(A(j, p)\) are transformed into paths from \(B(j, p)\) as follows. After the first lattice point on the line \(y = 1\) when coming from \((0, p)\), we add an uncolored step of type \((1, -1)\), so that the new end point is \((j, -j + 1)\). From now on, it is again an inductive procedure. In each step, we have a path starting in \((0, p)\). The starting section uses the step set of \(A(j, p)\) for above the line \(y = 1\), while the ending section uses the step set of \(A(j, p)\) for below the line \(y = 1\). The middle part of the path uses the step set of \(B(j, p)\). The rules are as follows (note that they reverse the procedure of how to transform a \(B\)-path into an \(A\)-path).

1. If the last step of the beginning section is a \((0, -2)\)-step, we replace it by two uncolored \((0, -1)\)-steps.
2. If the last step of the beginning section is a \((1, -1)\)-step and the last step of the path is a green \((0, -1)\)-step, we delete this last step and replace the \((1, -1)\)-step by two uncolored \((1, -1)\)-steps.
3. If the last step of the beginning section is a \((1, -1)\)-step and the last step of the path is a \((-1, 0)\) step, we do the following:
   a. If the \((1, -1)\)-step is followed by an uncolored \((0, -1)\) step, we replace the \((1, -1)\)-step by an uncolored \((0, -1)\) step and delete the last step of the path.
   b. If the \((1, -1)\)-step is followed by an uncolored \((1, -1)\)-step, we distinguish according to the color of the last step (which is by assumption a \((-1, 0)\) step): (i) If
Figure 8. The transformation of a $B$-path into a family of $A$-paths for $p = 5$ and $j = 2$.

the step is red, we replace the green $(1,-1)$-step by an uncolored step of type $(0,-1)$, and delete also the red step.  
(ii) If the step is blue, we replace the uncolored $(1,-1)$-step by an uncolored $(0,-1)$-step, uncolor the green $(1,-1)$-step, and delete the blue step.

An example is provided in Figure 9.

The procedure is well-defined: It remains to show the following.

- If the last step of the beginning section is of type $(1,-1)$ then the ending section is not empty.
- The end point of the final path is $(2j - 1, -j + 1)$.

To show the first assertion, note that in each step the number of green $(1,-1)$-steps is just the number of blue or red $(-1,0)$-step plus the number of green $(0,-1)$-steps. This is obviously true for paths in $A(j,p)$ and remains to be true in every step by the nature of the
Figure 9. The transformation of an $A$-path into a $B$-path for $p = 7$ and $j = 2$.

rules: Whenever a green $(1, -1)$-step is transformed, we delete precisely one step of the other type.

Concerning the change of the end point, note that the end point does not change, except when we are in case (2), in which case the end point is pushed horizontally by one unit to the right. We have such a case (2) for each green $(0, -1)$-step in the original path in $A(j, p)$. There are $j - 1$ such steps, so that the end point is $(j, -j + 1) + (j - 1) \cdot (1, 0) = (2j - 1, -j + 1)$ as required.

5.3. Combinatorial proof of (5.9) within our combinatorial model. Recall that we interpret the left-hand side of (5.9) as paths from $(2j - 1, -j + 1)$ to $(-i, i)$. For $x \geq 0$, we allow steps of type $(-1, 1)$ and $(0, 1)$, and, for $x < 0$, we allow steps of type $(-1, -1)$ and $(-1, 0)$. Steps of type $(-1, 0)$ are equipped with the weight $-1$. Such a path must contain precisely one lattice point, say $(0, p)$, on the line $x = 0$. We reflect the portion of the path that is to the left of the $y$-axis along this axis.

This results in paths from $(2j - 1, -j + 1)$ to $(i, i)$ that touch the line $x = 0$ precisely once, namely in $(0, p)$. From $(2j - 1, -j + 1)$ to $(0, p)$, we allow steps of type $(-1, 1)$ and $(0, 1)$, and from $(0, p)$ to $(i, i)$, we allow steps of type $(1, -1)$ and $(1, 0)$. Every step of type $(1, 0)$ is equipped with the weight $-1$. An example is provided in Figure 10.

Next, we construct a sign-reversing involution on a subset of these paths. For this purpose, observe that the point $(0, p)$ divides the path in two subpaths, an upper path, that ends in $(i, i)$, and a lower path, that starts in $(2j - 1, -j + 1)$. Seen as lattice paths, these two paths might coincide in some diagonal steps (in the initial section of the upper path and the ending section of the lower path): Let $k \in \{0, \ldots, i\}$ be the largest integer such that the vertices $(0, p), (1, p - 1), \ldots, (k, p - k)$ lie both on the upper and the lower path. $P = (k, p - k)$ is then the rightmost intersection point of these two paths. If $k < i$, we perform the following transformation on the paths which we depict as follows:
Figure 10. Example of the lattice paths interpretation of the left-hand side of (5.9) for $i = 5$, $j = 3$ and $p = 6$. The weight of the path is $(-1)^4 = 1$.

Depending on whether the upper path has a diagonal or a horizontal step after it has reached $P$, we delete the vertical step of the lower path incident to $P$ and replace the diagonal step of the upper path by the horizontal step or insert a vertical step to the lower path and replace the horizontal step of the upper path by a diagonal step, respectively. This transformation changes the height of the intersection point $P$ and therefore alters the point where the path touches the line $x = 0$. The upper path either gains or loses a horizontal step which changes the weight of the path.

We leave the path unchanged only if the rightmost intersection point $P$ is $(i, i)$. In this case, the path touches the line $x = 0$ in the point $(0, 2i)$ and the upper path as well as the lower path coincide in the $i$ diagonal steps from $(0, 2i)$ to $(i, i)$. We delete those diagonal steps, which leaves us paths from $(2j - 1, -j + 1)$ to $(i, i)$ with steps of type $(-1, 1)$ and $(0, 1)$. These paths are enumerated by $\binom{i+j-1}{j}$.

Figure 11 illustrates one instance of this sign-reversing involution for $i = 4$ and $j = 3$.

Figure 11. Sign-reversing involution for $i = 4$ and $j = 3$. 
6. Second combinatorial interpretation

In the determinant in (4.2), we add \( t \)-times the \((n-j)\)-th column to the \((n-j+1)\)-st column, for \( j = 1, \ldots, n-1 \), in this order. We repeat this for \( j = 1, \ldots, n-2 \), then for \( j = 1, \ldots, n-3 \), etc. We obtain

\[
\prod_{i=1}^{n} X_i^{n-2}(u - vX_i^{-2})^{-1} \det_{1 \leq i, j \leq n} \left( (uX_i + vX_i^{-1})^j \right) \prod_{1 \leq i < j \leq n}(X_j - X_i) \left( u - vX_i^{-1}X_j^{-1} \right)
\]

Now, for \( j = 1 \), we can replace \((uX_i + vX_i^{-1})^j\) by \((uX_i + vX_i^{-1} + t)^j\) and \((vX_i^{-2} + vX_i^{-1} + t)\), respectively, as the \( t \) cancels in this case. Then we add \( t \)-times the \( j \)-th column to the \((j+1)\)-st column, for \( j = 1, 2, \ldots, n-1 \), in this order. The result is

\[
\prod_{i=1}^{n} X_i^{n-1} \det_{1 \leq i, j \leq n} \left( (uX_i + w + vX_i^{-1})^j \right) \prod_{1 \leq i < j \leq n}(X_j - X_i) \left( u - vX_i^{-1}X_j^{-1} \right)
\]

Setting \( t = uv \), we obtain after some further modifications

\[
\prod_{i=1}^{n} X_i^{n-1} \det_{1 \leq i, j \leq n} \left( (uX_i + w + vX_i^{-1})^j \right) \prod_{1 \leq i < j \leq n}(X_j - X_i) \left( u - vX_i^{-1}X_j^{-1} \right)
\]

We will apply Lemma 5.1 with \( Y_i = uX_i + w + vX_i^{-1} \). We need to express the entries in the \( i \)-th row in terms of formal Laurent series in \( Y_i \). By Lemma 5.2, this entry is

\[
(uX_i + w + vX_i^{-1})^j \sum_{r=0}^{(j-1)/2} (-uv)^r \binom{j - r - 1}{r} (uX_i + vX_i^{-1})^{j-2r-1}
\]

\[
= (uX_i + w + vX_i^{-1})^j \sum_{r=0}^{(j-1)/2} (-uv)^r \binom{j - r - 1}{r} (uX_i + vX_i^{-1} + w - w)^{j-2r-1}
\]

\[
= \sum_{s=0}^{(j-1)/2} \sum_{r=0}^{s} (-uv)^r (-w)^{j-2r-1-s} \binom{j - r - 1}{r} (uX_i + w + vX_i^{-1})^{j+s}
\]

Using Lemma 5.1 and setting \( s = t - j - 1 \), the expression in (4.2) is equal to

\[
\prod_{i=1}^{n} X_i^{n-1} \det_{1 \leq i, j \leq n} \left( \sum_{t \in \mathbb{Z}} c_{t,j}(u,v,w) h_{t-i}(uX_1 + w + vX_1^{-1}, \ldots, uX_i + w + vX_i^{-1}) \right)
\]

with \( c_{t,j}(u,v,w) \) defined as

\[
\sum_{r=0}^{(j-1)/2} (-uv)^r (-w)^{2j-2r-1} \binom{j - r - 1}{t - j - 1} = \sum_{r=0}^{(j-1)/2} (-uv)^r (-w)^{2j-2r-1} \binom{j - r - 1}{r, 2j - 2r - t, t - j - 1}
\]

We claim that \( c_{t,j}(u,v,w) \) is the generating function of lattice paths from \((0,0)\) to \((t-2, j-1)\) with step set \(\{(1,1), (2,2), (2,1)\}\) with respect to the weight \((-uv)^{\#(2,2)\text{-steps}} (-w)^{\#(1,1)\text{-steps}}\).
Indeed, suppose that there are $r$ steps of type $(2, 2)$, $x$ steps of type $(1, 1)$ and $y$ steps of type $(2, 1)$. Then

$$r \cdot (2, 2) + x \cdot (1, 1) + y \cdot (2, 1) = (t - 2, j - 1),$$

which implies $x = 2j - 2r - t$ and $y = t - j - 1$. Therefore, the number of such paths is indeed $\binom{j - r - 1}{r, 2j - 2r - t, t - j - 1}$. From Lemma 5.3, it follows that the matrix underlying the determinant in (6.1) has the following combinatorial interpretation, see Figure 4: We consider lattice paths from $(1, 1), (2, 2), \ldots, (n, n)$ to $(2, 1), (2, 0), \ldots, (2, -n + 2)$ as follows:

- In the region $\{(x, y)|y \geq 1\}$, the step set is $\{(1, 0), (0, -1)\}$, and horizontal steps with distance $d$ from the $x$-axis are equipped with the weight $uX_d + w + vX_d^{-1}$. Suppose $(i, i)$ is the starting point and $(t, 1)$ is last lattice point in this region, then

$$h_{t-i}(uX_1 + w + vX_1^{-1}, \ldots, uX_i + w + vX_i^{-1})$$

is the generating function of such paths.

- In the region $\{(x, y)|y \leq 1\}$, the step set is $\{(-1, -1), (-2, -2), (-2, -1)\}$. Steps of type $(-1, -1)$ are equipped with the weight $-w$, while steps of type $(-2, -2)$ are equipped with the weight $-uv$. Suppose a path goes from $(t, 1)$ to $(2, -j+2)$, then the generating function of such paths is $c_{t,j}(u, v, w)$.

- Paths that start in the $i$-th starting point are equipped with the additional weight $X_i^{n-1}$.

The paths are non-intersecting in the region $\{(x, y)|y \geq 1\}$. In the region $\{(x, y)|y \leq 1\}$, there may be intersections, however, no two steps of different paths may have a common end point. This concludes the proof of Theorem 2.3.

### 7. Third Combinatorial Interpretation

Now we use a totally different procedure to obtain another Jacobi-Trudi type expression for (4.2). It will eventually be used to derive a third interpretation in terms of non-intersecting lattice paths (Theorem 2.4). We will then see how to transform this into a signless plane partition interpretation (Theorem 2.5). At the end of the section, we will also see that the first interpretation (Theorem 2.2) can be transformed into the same signless interpretation, in which case the reasoning is totally combinatorial. This provides a second proof of Theorem 2.5.

#### 7.1. Another Jacobi-Trudi type expression.

**Lemma 7.1.** In the following identities, the argument of all complete homogeneous symmetric functions $h$ is

$$(uX_1, \ldots, uX_n, vX_1^{-1}, \ldots, vX_n^{-1}).$$
(1) For $n \geq 1$, the following identity holds.

$$
\det_{1 \leq i,j \leq n} \left( (u^2 X_i^2 + uw X_i)^j + (v^2 X_i^{-2} + vw X_i^{-1})^j \right) \det_{1 \leq i,j \leq n} \left( (u^2 X_i^2 + uw X_i)^j - (v^2 X_i^{-2} + vw X_i^{-1})^j \right)
\prod_{1 \leq i,j \leq n} (X_j - X_i) (u^{i-1}v X_j^{-1} - u^{-i+1}v X_j^{-1}) \prod_{1 \leq i,j \leq n}^n (u^{i-1}v X_j^{-1} - X_i)
\frac{(-1)^n \nu^{n^2}(\nu)-w}{2} \sum_{k=j}^{2j} \left( \sum_{k=j}^{2j} \delta_{k-j} \right) u^{2j-k} \left( h_{k-i+1} - u^{-i+1+n} v^{i+1+n} h_{k-i-2n-1} \right)
\times \det_{1 \leq i,j \leq n} \left( \sum_{k=j}^{2j} \delta_{k-j} \right) w^{2j-k} \left( h_{k-i-1} + u^{i-1}v^{i-1} h_{k-i-1+n} \right)
$$

(2) For $n \geq 1$, the following identity holds.

$$
\det_{1 \leq i,j \leq n} \left( (u^2 X_i^2 + uw X_i)^j - (v^2 X_i^{-2} + vw X_i^{-1})^j \right) \det_{1 \leq i,j \leq n} \left( (u^2 X_i^2 + uw X_i)^j + (v^2 X_i^{-2} + vw X_i^{-1})^j \right)
\prod_{1 \leq i,j \leq n} (X_j - X_i) (u^{i-1}v X_j^{-1} - u^{-i+1}v X_j^{-1}) \prod_{1 \leq i,j \leq n}^n (u^{i-1}v X_j^{-1} - X_i)
\frac{(-1)^n \nu^{n^2}(\nu)-w}{2} \sum_{k=j}^{2j} \left( \sum_{k=j}^{2j} \delta_{k-j} \right) u^{2j-k} \left( h_{k-i} - u^{-i+n} v^{i+n} h_{k+i-2n} \right)
\times \det_{1 \leq i,j \leq n} \left( \sum_{k=j}^{2j} \delta_{k-j} \right) w^{2j-k} \left( h_{k+i-1} + u^{i-1}v^{i-1} h_{k+i-1+n} \right)
$$

Note that in the case $n = 1$ we set the evaluation of the “empty” determinant to be equal to 1.

**Proof.** Re (1). We consider the product

$$
\det \left( (u^2 X_i^2 + uw X_i)^j - (v^2 X_i^{-2} + vw X_i^{-1})^j \right) \det \left( (u^2 X_i^2 + uw X_i)^j + (v^2 X_i^{-2} + vw X_i^{-1})^j \right).
$$

For square matrices $A, B$ of the same size, we have

$$
\det(A - B) \det(A + B) = \det \begin{pmatrix} A - B & B \\ 0 & A + B \end{pmatrix} = \det \begin{pmatrix} A - B & B \\ B - A & A \end{pmatrix} = \det \begin{pmatrix} A & B \\ B & A \end{pmatrix},
$$

and this implies that our product is equal to

$$
\det \left( \left( (u^2 X_i^2 + uw X_i)^j \right)_{1 \leq i,j \leq n} \right) \left( \left( (v^2 X_i^{-2} + vw X_i^{-1})^j \right)_{1 \leq i,j \leq n} \right).
$$

Setting $X_{n+i} = u^{i-1}v X_i^{-1}$, we can also write this as

$$
\det \left( \left( (u^2 X_i^2 + uw X_i)^j \right)_{1 \leq i,j \leq n} \right) \left( \left( (v^2 X_i^{-2} + vw X_i^{-1})^j \right)_{1 \leq i,j \leq n} \right).
$$

We apply Lemma 5.1 with $Y_i = X_i$ to

$$
\left( (u^2 X_i^2 + uw X_i)^j \right)_{1 \leq i,j \leq n} \prod_{1 \leq i,j \leq n} (X_j - X_i)
$$

and obtain

$$
\det \left( \sum_{k \geq 0} \left( \frac{j}{k-j} \right) u^k w^{2j-k} h_{k-i+1}(X_1, \ldots, X_i) \right)_{1 \leq i,j \leq n} \left( \sum_{k \geq 0} \left( \frac{j}{k-j} \right) u^k w^{2j+k} h_{k-i+1}(X_1, \ldots, X_i) \right)_{1 \leq i,j \leq n}.$$
We multiply from the left with the following matrix

\[ (h_{j-i}(X_j, X_{j+1}, \ldots, X_{2n}))_{1 \leq i, j \leq 2n} \]

with determinant 1. For this purpose, note that

\[ \sum_{l=1}^{2n} h_{l-i}(X_l, X_{l+1}, \ldots, X_{2n}) h_{k-l+1}(X_1, \ldots, X_l) = h_{k-i+1}(X_1, \ldots, X_{2n}), \]

and, therefore, the multiplication results in

\[ \det \left( \sum_{k \in \mathbb{Z}} \left( \sum_{j} \frac{j}{k-j} u^k w^{2j-k} h_{k-i+1}(X_1, \ldots, X_{2n}) \right) \right). \]

Using (5.1) and specializing \( X_{n+i} = u^{-1} v X_i^{-1} \) for \( i = 1, 2, \ldots, n \) gives

\[ (-1)^n \det \left( \sum_{k \in \mathbb{Z}, j} \left( \sum_{j} \frac{j}{k-j} u^{-1+n} w^{2j-k} h_{k-i+1-n} \right) \right). \]

Omitting the arguments of the complete homogeneous symmetric functions \( h \) and some manipulations give

\[ (-1)^n \det \left( \sum_{k \in \mathbb{Z}, j} \left( \sum_{j} \frac{j}{k-j} u^{-1+n} w^{2j-k} h_{k-i+1-n} \right) \right). \]

For \( j = 1, 2, \ldots, n \), we subtract the \((n+j)\)-th column multiplied by \( u^{2n} \) from the \(j\)-th column. Then, for \( i = n + 2, n + 3, \ldots, 2n \), multiply the \((2n+2-i)\)-th row with \( u^{i-n-1} v^{i+n+1} \) and add it to the \(i\)-th row. After this, the entries in the first \( n \) columns that are in row \( n+1 \) or below vanish. This implies that the determinant is the product of the determinants of the upper left block and the lower right block, i.e.

\[ (-1)^n \det \left( \sum_{k \in \mathbb{Z}, j} \left( \sum_{j} \frac{j}{k-j} w^{2j-k} \left( u^{i-1+n} h_{k-i+1-n} - u^{2n} v^{-i+1+n} h_{k+i-2n-1} \right) \right) \right) \]

\[ \times \det \left( \sum_{k \in \mathbb{Z}, j} \left( \sum_{j} \frac{j}{k-j} w^{2j-k} \left( u^{i-1+n} h_{k+i-1-n} + [i \neq 1] u^{-1} h_{k-i-n+1} \right) \right) \right), \]

where we make use of the Iverson bracket: For a logical statement \( S \), we set \([S] = 1\) if \( S\) is true and \([S] = 0\) otherwise.

The assertion then follows by taking the numerator of (7.1) as well as the specialization and some further manipulations into account.

Re (2). We consider now the following product.

\[ \det \left( \left( u^2 X_i + u w X_i \right)^{j-1} + \left( v^2 X_i^{-2} + v w X_i^{-1} \right)^{j-1} \right) \]

\[ \det \left( \left( u^2 X_i^2 + u w X_i \right)^{j} - \left( v^2 X_i^{-2} + v w X_i^{-1} \right)^{j} \right). \]
Let $A, B$ be $n \times (n-1)$ matrices and $c, d, e$ be column-vectors of length $n$, then we will use
\[
\det(c|A + B) \det(A - B|e - d) = \det \begin{pmatrix} c & A + B & B \\ 0 & A - B & e - d \end{pmatrix} = \det \begin{pmatrix} c & A + B & B \\ c & A + B & A \end{pmatrix} = \det \begin{pmatrix} c & A + B & B \\ c & A & e \end{pmatrix},
\]
and this implies that our product is equal to
\[
2 \cdot \det \left( \frac{((u^2 X_i^2 + uw X_i)^{j-1})_{1 \leq i, j \leq n}}{((v^2 X_i^{-2} + vw X_i^{-1})^{j-1})_{1 \leq i, j \leq n}} \right),
\]
where the 2 compensates for $c = 2$, since the column is now 1. Setting $X_{n+1} = u^{-1}v X_i^{-1}$, we can also write this as
\[
2 \cdot \det \left( \frac{((u^2 X_i^2 + uw X_i)^{j-1})_{1 \leq i, j \leq n}}{((v^2 X_i^{-2} + vw X_i^{-1})^{j-1})_{1 \leq i, j \leq n}} \right) \prod_{1 \leq i < j \leq n} (X_j - X_i).
\]
In a similar manner as above, we obtain, after specializing again and omitting the arguments of the $h$’s, the following.
\[
(-1)^n \det \left( \frac{2j-2}{k-j} \sum_{k=j}^{j-1} u^{i-1+n} w^{2j-2-k} h_{k-i+1} \right) \prod_{1 \leq i \leq j} \left( 2j \sum_{k=j}^{j} \left( \frac{u^{-i+1+n} w^{2j-k} h_{k+i-2n-1}}{v^{-i+1+n} w^{2j-k} h_{k+i-2n-1}} \right) \right)_{1 \leq k \leq n}
\]
\[
= (-u)^n \det \left( \frac{2j}{k-j} \sum_{k=j}^{j} u^{i+n} w^{2j-k} h_{k-i} \right) \prod_{1 \leq i \leq j} \left( 2j \sum_{k=j}^{j} \left( \frac{u^{-i+n} w^{2j-k} h_{k+i-2n}}{v^{-i+n} w^{2j-k} h_{k+i-2n}} \right) \right)_{1 \leq k \leq n}
\]
For $j = 1, 2, \ldots, n - 1$, we subtract the $(n - 1 + j)$-th column multiplied by $u^{2n}$ from the $j$-th column. Then, for $i = n + 1, n + 2, \ldots, 2n - 1$, multiply the $(2n - i)$-th row with $u^{-n} v^{-i+n}$ and add it to the $i$-th row. After this, the entries in the first $n - 1$ columns that are in row $n$ or below vanish. This implies that the determinant is the product of the determinants of the upper left block and the lower right block, i.e.
\[
(-u)^n \det \left( \sum_{k=j}^{j} \left( \frac{u^{i+n} h_{k-i} - u^{2n} v^{-i+n} h_{k+i-2n}}{v^{-i+1+n} h_{k+i-2n-1} + u^{-i} h_{k-i+n-1}} \right) \right) \times \det \left( \sum_{k=j}^{j} \left( \frac{u^{i+n} h_{k-i} - u^{2n} v^{-i+n} h_{k+i-2n}}{v^{-i+1+n} h_{k+i-2n-1} + u^{-i} h_{k-i+n-1}} \right) \right)
\]
Taking the numerator of (7.2) as well as the specialization into account and some further manipulations, the assertion follows. \qed

The following lemma is of use in the next corollary.
Lemma 7.2. Let \( n \geq 1 \) and \( 0 \leq m \leq n - 1 \). Then we have
\[
\sum_{l=1}^{n} \frac{(-uX_l - vX_l^{-1})^m}{\prod_{j \neq l}(X_j - X_l)(u - vX_l^{-1}X_j^{-1})} = \begin{cases} 1, & m = n - 1, \\ 0, & m < n - 1. \end{cases}
\]

Proof. As \((X_j - X_l)(u - vX_l^{-1}X_j^{-1}) = (uX_j + vX_j^{-1}) - (uX_l + vX_l^{-1})\), it suffices to show
\[
\sum_{l=1}^{n} \frac{(-Y_l)^m}{\prod_{j \neq l}(Y_j - Y_l)} = \begin{cases} 1, & m = n - 1, \\ 0, & m < n - 1. \end{cases}
\]

Multiplying with \( \prod_{1 \leq i < j \leq n}(Y_j - Y_i) \), this is equivalent to
\[
\sum_{l=1}^{n} (-1)^{l+1}(-Y_l)^m \prod_{i,j \neq l}(Y_j - Y_i) = \begin{cases} \prod_{1 \leq i < j \leq n}(Y_j - Y_i), & m = n - 1, \\ 0, & m < n - 1. \end{cases}
\]

Now, by the Vandermonde determinant evaluation, the left-hand side of this equation is
\[
\det_{1 \leq i,j \leq n} \begin{pmatrix} Y_i^{j-1}, & j < n, \\ (-1)^{n-1}(-Y_i)^m, & j = n, \end{pmatrix}
\]
and the assertion follows. \( \square \)

Corollary 7.3. In the following identities, the argument of all complete homogeneous symmetric functions \( h \) is
\[
(uX_1, \ldots, uX_n, vX_1^{-1}, \ldots, vX_n^{-1}).
\]

(1) For \( n \geq 1 \), the following identity holds.
\[
\frac{\det_{1 \leq i,j \leq n} \left( (uX_i^2 + uwX_i)^j - (vX_i^{-2} + vwX_i^{-1})^j \right)}{\prod_{1 \leq i < j \leq n}(X_j - X_i)(vX_i^{-1}X_j^{-1} - u)\prod_{i=1}^{n}(uX_i - vX_i^{-1})} = \frac{1}{2} \det_{1 \leq i,j \leq n} \left( \sum_{k,j}^2 \binom{j}{k} w^{2j-k} (h_{k+i-n-1} + u^{-1}v^{-1}h_{k-i+n+1}) \right)
\]

(2) For \( n \geq 1 \), the following identity holds.
\[
\frac{\det_{1 \leq i,j \leq n} \left( (vX_i^2 + uwX_i)^j + (vX_i^{-2} + vwX_i^{-1})^j \right)}{\prod_{1 \leq i < j \leq n}(X_j - X_i)(u - vX_i^{-1}X_j^{-1})} = \det_{1 \leq i,j \leq n} \left( \sum_{k,j}^2 \binom{j}{k} w^{2j-k} (h_{k-i+1} - u^{-1}+n v^{-1}+n h_{k+i-2n-1}) \right)
\]

(3) For \( n \geq 1 \), the following identity holds.
\[
\frac{1}{2} \det_{1 \leq i,j \leq n} \left( (uX_i^2 + uwX_i)^{j-1} + (vX_i^{-2} + vwX_i^{-1})^{j-1} \right) \prod_{1 \leq i < j \leq n}(X_j - X_i)(u - vX_i^{-1}X_j^{-1}) = \det_{1 \leq i,j \leq n-1} \left( \sum_{k,j}^2 \binom{j}{k} w^{2j-k} (h_{k-i} - u^{-1}+n v^{-1}+n h_{k+i-2n}) \right)
\]
Proof. The proof is by induction with respect to \( n \). For small \( n \), the identities can be shown by direct computations. We assume that all identities are proved for \( n - 1 \). We start by proving the third identity. Now observe that, by expanding the determinant with respect to the first column and the induction hypothesis applied to the case \( n - 1 \) in (2), we obtain

\[
\frac{1}{2} \sum_{i=1}^{n} (-1)^{i+1} \prod_{j \neq i} (X_j - X_i) \det_{1 \leq i, j \leq n} \left( \left( uX_i^2 + uwX_i \right)^{i-1} + \left( v^2X_i^{j-2} + vwX_i^{-1}\right)^{i-1} \right)
\]

for \( i = 1, \ldots, n \).

where the argument of the complete symmetric functions in the \( l \)-th summand is

\[
\left( uX_1, \ldots, uX_l, \ldots, uX_n, vX_1^{-1}, \ldots, vX_l^{-1}, \ldots, vX_n^{-1} \right),
\]

which is indicated by \( h^{(l)} \). We will use

\[
h_k(X_1, \ldots, \bar{X}_p, \ldots, \bar{X}_q, \ldots, X_n) = h_k(X_1, \ldots, \bar{X}_p, \ldots, X_q, \ldots, X_n) - X_p h_{k-1}(X_1, \ldots, \bar{X}_p, \ldots, X_n)
\]

\[
= h_k(X_1, \ldots, X_n) - (X_p + X_q) h_{k-1}(X_1, \ldots, X_n) + X_p X_q h_{k-2}(X_1, \ldots, X_n).
\]

Applied to our argument in (7.4), we see that

\[
h_k^{(l)} = h_k - (uX_l + vX_l^{-1}) h_{k-1} + uv h_{k-2},
\]

and, therefore, (7.3) is further equal to

\[
\frac{1}{2} \sum_{i=1}^{n} \prod_{j \neq i} (X_j - X_i) \det_{1 \leq i, j \leq n-1} \left( \left( uX_i^2 + uwX_i \right)^{i-1} + \left( v^2X_i^{j-2} + vwX_i^{-1}\right)^{i-1} \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \prod_{j \neq i} (X_j - X_i) \det_{1 \leq i, j \leq n-1} \left( \left( uX_i^2 + uwX_i \right)^{i-1} + \left( v^2X_i^{j-2} + vwX_i^{-1}\right)^{i-1} \right)
\]

\[
\times \left( h_{k+i+1} - u^{-i+n+1} v^{-i+n+1} h_{k+i-2n-1} - (uX_l + vX_l^{-1}) (h_{k-i} - u^{-i+n} v^{-i+n} h_{k+i-2n+1}) + uv (h_{k+i-1} - u^{-i+n-1} v^{-i+n-1} h_{k+i-2n+2}) \right).
\]

Setting

\[
a_{i,j} = \sum_{k=j}^{2j} \binom{j}{k-j} w^{2j-k} (h_{k-i} - u^{-i+n} v^{-i+n} h_{k+i-2n}),
\]

the determinant is equal to

\[
\det_{1 \leq i, j \leq n-1} \left( a_{i-1,j} - (uX_l + vX_l^{-1}) a_{i,j} + uv a_{i+1,j} \right).
\]

By the linearity in the rows, we can write each of the \( n \) determinants in (7.5) as a sum of \( 3^{n-1} \) determinants, by choosing in a particular row \( i \) either of the three terms, namely \( a_{i-1,j} \), \( -(uX_l + vX_l^{-1}) a_{i,j} \) or \( uv a_{i+1,j} \). We pull out \( uX_l + vX_l^{-1} \) whenever we choose the second term, so that the determinant is independent of \( l \). Across the different choices of \( l \), we combine the determinants where in each row we have made precisely the same choices among the three terms (all these determinants are of course equal since we pulled out the appropriate power of \( uX_l + vX_l^{-1} \)). If we have chosen \( m \) times the second term, then we have a prefactor that is equal to the left-hand side of the identity in Lemma 7.2. By the lemma, the prefactor is only non-zero if \( m = n - 1 \). This establishes (3) for \( n \).
Now (1) follows from (3) and Lemma 7.1 (2), whereas (2) finally follows from (1) and Lemma 7.1 (1).

**Remark 7.4.** We will only use the identity in Corollary 7.3 (1). Presumably, it can also be shown by induction as we indicate in the following, although this does not give much hint on how to find the identity. More specifically, there is an obvious recursion (with respect to \( n \)) underlying the left-hand side of the determinant, which is obtained by expanding the determinant with respect to the last column. Proving the identity then amounts to showing that the right-hand side also satisfies the same recursion.

We illustrate this on an example which essentially shows the equivalence of the bialternant formula for Schur polynomials to the Jacobi-Trudi formula. More specifically, we show

\[
(7.6) \quad \frac{\det_{1 \leq i,j \leq n} (X_i^{k_j})}{\prod_{1 \leq i,j \leq n} (X_i - X_j)} = \det_{1 \leq i,j \leq n} (h_{k_i+j-n}(X_1, \ldots, X_n)).
\]

By induction, we have

\[
(7.7) \quad \frac{\det_{1 \leq i,j \leq n} (X_i^{k_j})}{\prod_{1 \leq i,j \leq n} (X_i - X_j)} = \sum_{p=1}^{\infty} \frac{X_p^{k_p}}{\prod_{q \neq p} (X_q - X_p)} \det_{1 \leq i,j \leq n} (h_{k_i+j-n+1}(X_1, \ldots, X_n)).
\]

We need to show that the right-hand side of (7.6) is equal to the right-hand side above. Now we use

\[ h_k(X_1, \ldots, X_n) = h_k(X_1, \ldots, \overline{X}_p, \ldots, X_n) + X_p h_{k-1}(X_1, \ldots, X_n) \]

to see that the right-hand side of (7.7) is equal to

\[
\sum_{p=1}^{\infty} \frac{X_p^{k_p}}{\prod_{q \neq p} (X_q - X_p)} \det_{1 \leq i,j \leq n} (h_{k_i+j-n+1}(X_1, \ldots, X_n) - X_p h_{k_i+j-n}(X_1, \ldots, X_n))
\]

\[ = \sum_{r=0}^{n-1} (-1)^{r+n-1} h_{k+r-n+1}(X_1, \ldots, X_n) \det_{1 \leq i,j \leq n} (h_{k_i+j-n+j-r}(X_1, \ldots, X_n))
\]

\[ = \det_{1 \leq i,j \leq n} (h_{k_i+j-n}(X_1, \ldots, X_n)).
\]

**7.2. Proof of Theorem 2.4.** Corollary 7.3 (1) implies that the generating function in (4.2) is equal to

\[
(7.8) \quad \frac{(-1)^{\binom{n}{2}}}{2} \prod_{i=1}^{n} X_i^{n-1} \det_{1 \leq i,j \leq n} \left( \sum_{k=j}^{j} \binom{j}{k-j} u^{2j-k}(h_{k+i-n-1} + u^{i-1}v^{i-1}h_{k-i+n-1}) \right),
\]

where the arguments \((uX_1, \ldots, uX_n, vX_1^{-1}, \ldots, vX_n^{-1})\) of the complete homogeneous symmetric functions are omitted.

We consider families of \( n \) non-intersecting lattice paths from \( A_i = \{(n - i + 1, 2n), (n + i - 1, 2n)\} \), \( i = 1, 2, \ldots, n \), to \( E_j = (j, -j + 1) \), \( j = 1, 2, \ldots, n \), see Figure 5. The step set as well as the edge weights depend on whether or not we are below the line \( y = 1 \).

- Above and on the line \( y = 1 \), the step set is \( \{(1,0),(0,-1)\} \). Horizontal steps at height \( d \) above the \( x \)-axis have weight \( uX_d \) if \( d \leq n \) and the weight \( vX_d^{-1} \) if \( d > n \). Assuming that \((k,1)\) is the last lattice point on the line \( y = 1 \), the generating function of such lattice paths from \((n-i+2n)\) to \((k,1)\)

\[
h_{k+i-n-1}(uX_1, \ldots, uX_n, vX_1^{-1}, \ldots, vX_n^{-1}),
\]
The result is an immediate consequence of the following two identities, which we show below.

**Proof.**

Still need to take the sign into account: Suppose we have the starting points $n$ precisely for $n_{precisely} = n$. Then the only permutation for which the non-intersecting property does not force that $p_i = n$ is connected to $E_i$ by a path, so we still need to take the sign into account: Suppose we have the starting points $(n - i + 1, 2n)$ precisely for $n \geq i_p > i_{p-1} > \ldots > i_1 = 1$ and starting points $(n + i - 1, 2n)$ for $1 < j_1 < j_2 < \ldots < j_{n-p} \leq n$. Then the only permutation for which the $n$-tuple of paths can be non-intersecting is

$$(i_p, i_{p-1}, \ldots, i_1, j_1, \ldots, j_{n-p})^{-1},$$

the sign of which is $(-1)^{(i_1-1)+(i_2-1)+\ldots+(i_{p-1})}$. Thus, taking the factor $(-1)^{(i_2)}$ into account, each path with $(n + i - 1, 2n)$ as starting point contributes $(-1)^i$ to the sign.

### 7.3. First proof of Theorem 2.5.

We will multiply the matrix underlying the determinant in (7.8) with the following matrix of determinant $\frac{1}{2}$.

$$\left( (-1)^{i+j} \frac{1}{1 + [j = 1]} e_{i-j} \left( uX_{n-i+2}, \ldots, uX_n, vX_{n-i+2}, \ldots, vX_n \right) \right)_{1 \leq i, j \leq n}$$

For this purpose, we need the following lemma.

**Lemma 7.5.** For $n \geq 1$, $m + n \geq 0$ and $1 \leq i \leq n$, we have

$$\sum_{l=1}^{n} (-1)^{i+l} \frac{1}{1 + [l = 1]} e_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n)$$

$$\times \left[ h_{m+l-1}(uX_1, vX_1^{-1}, \ldots, uX_n, vX_n^{-1}) + (uv)^{l-1} h_{m-l+1}(uX_1, vX_1^{-1}, \ldots, uX_n, vX_n^{-1}) \right]$$

$$= h_{m-i+1}(uX_1, vX_1^{-1}, \ldots, uX_{n-i+1}, vX_{n-i+1}).$$

**Proof.** The left-hand side can be written as

$$\sum_{l=0}^{i-1} (-1)^{i+l} e_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n) h_{m+l}(uX_1, vX_1^{-1}, \ldots, uX_n, vX_n^{-1})$$

$$+ \sum_{l=1}^{i-1} (-1)^{i+l-1} (uv)^l e_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n) h_{m-l}(uX_1, vX_1^{-1}, \ldots, uX_n, vX_n^{-1}).$$

The result is an immediate consequence of the following two identities, which we show below.

$$\sum_{l=i+1}^{i-1} (-1)^{i+l} e_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n) h_{m+l}(uX_1, vX_1^{-1}, \ldots, uX_n, vX_n^{-1})$$

$$= h_{m-i+1}(uX_1, vX_1^{-1}, \ldots, uX_{n-i+1}, vX_{n-i+1})$$
and

\begin{equation}
(7.11) \quad \sum_{l=-i+1}^{-1} (-1)^{i+l}e_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n)h_{m+l}(uX_1, vX_1, \ldots, uX_n, vX_n)
\end{equation}

\begin{equation}
+ \sum_{l=1}^{i-1} (-1)^{i+l}(uv)j_{i-l}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n)h_{m-l}(uX_1, vX_1, \ldots, uX_n, vX_n) = 0.
\end{equation}

In order to show the first identity, observe that the left-hand side of (7.10) can be written as

\begin{equation}
\sum_{l=-i+1}^{-1} (t+l-1)\left[ \prod_{p=0}^{n-2} (1-tuX_p)(1-tvX_p^{-1}) \right]^{(t+m)} \left[ \prod_{p=1}^{n} (1-tuX_p)^{-1}(1-tvX_p^{-1}) \right]^{-1}
\end{equation}

\begin{equation}
= \langle t+m \rangle \left[ \prod_{p=1}^{n-1} (1-tuX_p)^{-1}(1-tvX_p^{-1}) \right] = h_{i+m-1}(uX_1, \ldots, uX_{n-i+1}, vX_1, \ldots, vX_{n-i+1}).
\end{equation}

The second identity follows after showing that the \(j\)-th summand in the first sum in (7.11) is the negative of the \((i-j)\)-th summand in the second sum, i.e.,

\begin{equation}
(-1)^{j}e_{2i-j-1}(uX_{n-i+2}, vX_{n-i+2}, \ldots, uX_n, vX_n)h_{m+j}(uX_1, vX_1, \ldots, uX_n, vX_n) =
\end{equation}

\begin{equation}
-\left[ (-1)^{j+1}(uv)^{-j}e_{j-1}(uX_{n-i+2}, uX_{n-i+2}, \ldots, uX_n, vX_n) h_{m+j}(uX_1, vX_1, \ldots, uX_n, vX_n) \right],
\end{equation}

for \(j = 1, 2, \ldots, i-1\), which is equivalent to

\begin{equation}
e_{2i-j-1}(uY_1, vY_1^{-1}, \ldots, uY_{i-1}, vY_{i-1}^{-1}) = (uv)^{-j}e_{j-1}(uY_1, vY_1^{-1}, \ldots, uY_{i-1}, vY_{i-1}^{-1})
\end{equation}

by setting \(Y_k = X_{k+n-i+1} \text{ for } k = 1, 2, \ldots, i-1\).

The identity follows as in each monomial of the expression on the left hand side, we choose at least \(2i-j-1-(i-1) = i-j\) pairs of variables \(uY_k, vX_k^{-1}\) and each such pair contributes \(uv\).

The previous lemma implies that (7.8) multiplied with (7.9) is equal to

\begin{equation}
(7.12) \quad (-1)^{(2)}(n) \prod_{i=1}^{n} X_{i}^{n-1} \det_{i \leq j \leq n} \left[ \sum_{k=j}^{2j} \binom{j}{k-j} u^{2j-k} h_{k+i-n-1}(uX_1, vX_1^{-1}, \ldots, uX_{n-i-1}, vX_{n-i-1}) \right]
\end{equation}

\begin{equation}
= \prod_{i=1}^{n} X_{i}^{n-1} \det_{i \leq j \leq n} \left[ \sum_{k=j}^{2j} \binom{j}{k-j} u^{2j-k} h_{k-i}(uX_1, vX_1^{-1}, \ldots, uX_i, vX_i^{-1}) \right].
\end{equation}

We consider families of \(n\) non-intersecting lattice paths from \(A_i = (i, 2i)\), \(i = 1, 2, \ldots, n\), to \(E_j = (j, -j+1)\), \(j = 1, 2, \ldots, n\), see Figure 12. The step set as well as the edge weights depend on whether or not we are below the line \(y = 1\).

- Above and on the line \(y = 1\), the step set is \(\{(1, 0), (0, -1)\}\). Horizontal steps at height \(1, 2, 3, 4, \ldots, 2n\) have weight \(uX_1, vX_1^{-1}, uX_2, vX_2^{-1}, \ldots, uX_n, vX_n^{-1}\), respectively. Assuming that \((k, 1)\) is the last lattice point on the line \(y = 1\), the generating function of such lattice paths from \((i, 2i)\) to \((k, 1)\) is

\begin{equation}
h_{k-i}(uX_1, vX_1^{-1}, \ldots, uX_i, vX_i^{-1}),
\end{equation}

- Below the line \(y = 1\), the step set is \(\{(-1, -1), (0, -1)\}\), and since we want to reach \((j, -j+1)\), there are \(k-j\) steps of type \((-1, -1)\) and \(2j-k\) steps of type \((0, -1)\), which gives in total \(\binom{j}{k-j}\) choices. The latter steps carry the weight \(w\). Equivalently, we can
also choose that the \((-1,-1)\)-steps are equipped with the weight \(w^{-1}\) and that there is an overall factor of \(w^{(\ell_{2j-1})}\).

- Again we have to multiply with the overall factor \(\prod_{i=1}^{n} X_i^{n-1}\).

\(\text{Figure} 12\). Example of the lattice path interpretation of (7.12) for \(n = 6\). The weight of these paths is \(u^7 v^9 w^5 X_1^2 X_2^{-1} X_3^{-1} X_4 X_5\). Note that this family of non-intersecting lattice paths corresponds to the pair of CSPP and RSPP in Section 2.

Such paths correspond to pairs \((P, Q')\) of column-strict plane partitions of the same shape with at most \(n\) rows with the following properties.

- The entries of \(P\) in the \(i\)-th row from the bottom are no greater than \(2i\).
- The entries of \(Q'\) in the \(i\)-th row from the bottom are no greater than \(2i - 1\) and no less than \(\ell + i - 1\) if \(\ell\) is the length of the \(i\)-row.

The entries of \(P\) correspond to the heights of the horizontal steps of the paths on and above the line \(y = 1\). Each odd entry \(2i - 1\) contributes \(uX_i\) multiplicatively to the weight, while each even entry \(2i\) contributes \(vX_i^{-1}\). The entries of \(Q'\) correspond to the diagonal steps, more precisely their distance from the line \(y = x\), and each of them contributes \(w\) to the weight.

Now we subtract \(i - 1\) from all entries in the \(i\)-th row from the bottom of \(Q'\) and then \(j - 1\) from all entries in the \(j\)-th column of \(Q'\). We obtain a row-strict plane partition \(Q\) with positive entries such that the entries in the \(i\)-th row from the bottom are no greater than \(i\). The pair \((P, Q)\) of the non-intersecting lattice paths from Figure 12 is the one given in Section 2.

This establishes the proof of Theorem 2.5.

Note that when setting \(u = v = 1\), \(w = -1\) and \(X_i = 1\) in (7.12), we obtain \(\binom{i+j-1}{2j-i} = \binom{i+j-1}{2i-j-1}\) by (5.6) as entries of the determinant’s underlying matrix and thus the unrefined count (5.5) of cyclically and vertically symmetric lozenge tilings of a hexagon with a central triangular hole of size 2.
Remark 7.6. Ideas from [FK97] are applicable to provide a bijective proof of the equivalence of Theorems 2.4 and 2.5.

7.4. Theorem 2.2 implies Theorem 2.5. A drawback of Theorem 2.2 is that the lattice paths interpretation also involves signs although the generating function has only non-negative coefficients, since it is also the generating function of arrowed monotone triangles. In this section, we will define a sign-reversing involution on the families of lattice paths from Theorem 2.2 to see directly that also their generating function has no negative coefficients. In fact, we will arrive again at the signless interpretation in Theorem 2.5. To be more precise, we will first have to “move back” to possibly intersecting paths in the second and third region, and apply the sign-reversing involution on such paths. This sign-reversing involution will also involve a “Gessel-Viennot”-type sign-reversing involution in the third region.

For this purpose, we will first show that (5.4) has only non-negative coefficients. In fact, our combinatorial proof of (5.8) implies a sign-reversing involution that shows that the inner sum of (5.4) has no negative coefficients. To see this, we introduce essentially the weights from Theorem 2.2 in the combinatorial interpretation of the left-hand side of (5.8) to obtain a combinatorial interpretation of the inner sum of (5.4): Recall that we consider lattice paths from (0, p) to \((j - 1, -j + 2)\), with step set \(\{(1, -1), (0, -2)\}\) until we reach the line \(y = 1\) at \((q - 1, 1)\) such that \(q \leq p\), while on and below the line \(y = 1\) the step set is \(\{(-1, 0), (0, -1)\}\). We then equip steps of type \((0, -2)\) with the weight \(-w\), while steps of type \((-1, 0)\) are colored in blue or red, and blue and red steps are equipped with the weight \(uX_d\) and \(vX_d^{-1}\), respectively, where \(d\) is the distance from the line \(y = 2\). All other steps have weight 1.

Lemma 7.7. Let \(p, j\) be positive integers, then

\[
\sum_{q \geq 1, 2|(p-q)} (-uv)(p-q)/2 \left(\frac{(p+q)/2 - 1}{(p-q)/2} \right) h_{q-j}(uX_1 + vX_1^{-1}, \ldots, uX_j + vX_j^{-1})
\]

is the generating function of lattice paths from \((0, p)\) to \((j - 1, -j + 2)\) as given above, but without steps of type \((0, -2)\) and without consecutive pairs of horizontal steps with the first step being blue and the second step being red.

Clearly, forbidding steps of type \((0, -2)\) implies that above the line \(y = 1\) we go straight from \((0, p)\) to \((p - 1, 1)\), using only steps of type \((1, -1)\).

Proof. We define the following sign-reversing involution on the lattice paths from \((0, p)\) to \((j - 1, -j + 2)\) that do not fall into the class as described in the lemma: Let \(r\) be the number of \((1, -1)\)-steps at the beginning of the lattice path (so that the next step is either \((0, -2)\) or we have already reached the line \(y = 1\). Since there are in total \(q - 1\) diagonal steps, we know that \(r \in \{0, 1, \ldots, q - 1\}\). We also consider the first \(r\) steps of the path after the first point on the line \(y = 1\). Such \(r\) steps exist, since there are precisely \(q - 1\) steps in the region \(\{(x, y) | y \leq 1\}\). They are all in the step set \(\{(-1, 0), (0, -1)\}\).

We distinguish between two cases. If among the latter \(r\) steps, there are two consecutive horizontal steps such that the first is blue and the second is red (when traversing the path from \((p, 0)\) to \((j - 1, -j + 2)\)), we choose the first pair of such horizontals steps. We delete both of them, and, assuming they were the steps at position \(s\) and \(s + 1\) now counted from the first point on the line \(y = 1\), we replace the diagonal steps at position \(s\) and \(s + 1\), counted from the start, by a single \((0, -2)\)-step. We adjust the path so that it starts in \((p, 0)\) and ends in \((j - 1, -j + 2)\). Note that, by this replacement, the first intersection point of the path with the line \(y = 1\) is shifted from \((q - 1, 1)\) to \((q - 3, 1)\).
If there is no such pair, then the path does not fall into the class as described in the lemma if and only if \( r < q - 1 \), and, consequently, there is a \((0, -2)\)-step right after the \( r \)-th diagonal steps. In this case, we replace the \((0, -2)\)-step by two steps of type \((1, -1)\) and insert a blue horizontal step at position \( r + 1 \) and a red horizontal step at position \( r + 2 \), counted from the first point on the line \( y = 1 \). In this case, the first intersection point with the line \( y = 1 \) is moved from \((q - 1, 1)\) to \((q + 1, 1)\).

This mapping is an involution and clearly sign-reversing: Two consecutive horizontal steps such that a blue step is followed by a red step have in total the weight \( uX_dvX_d^{-1} = uv \), and, in a sense, they are replaced by a \((0, -2)\)-step, which has weight \(-uv\), or vice versa. \( \square \)

This involution can be related to the combinatorial proof of (5.8) from Section 5.2 as follows: Two \( A \)-paths that are paired off under this involution are mapped to the same \( B \)-path, but have opposite signs as \( A \)-paths.

Using Lemma 7.7 , we are left with families of paths with step sets as described in Theorem 2.2 that

- are non-intersecting in the region \( \{(x, y) | x \leq 0 \} \),
- have only diagonal steps \((1, -1)\) in the region \( \{(x, y) | x \geq 0, y \geq 1 \} \) (which implies automatically that the paths are non-intersecting in this region), and
- consecutive horizontals steps in the region \( \{(x, y) | x \geq 0, y \leq 1 \} \) are divided into an initial section of red steps (which may be empty) followed by an ending section of blue steps (which may be empty as well). The paths may still be intersecting in this region at this point.

The lattice point of a maximal section of consecutive horizontal steps that divides the section into two portions of red and blue steps, respectively, is said to be the center of the section. Note that the center can also be the left or right end point of these maximal sections if there are no blue or no red steps, respectively. Now we say that two paths in the region \( \{(x, y) | x \geq 0, y \leq 1 \} \) touch if they intersect, but their intersections are fully contained in maximal sections of consecutive horizontal steps and none of these intersections contain centers of either paths. Two paths that are intersecting but not touching are said to intersect strongly.

We make the following important observation: Suppose \( P_1, P_2 \) are two touching paths, and let \((-i_1, i_1), (-i_2, i_2)\) be their starting points, respectively, and \((j_1 - 1, -j_1 + 2), (j_2 - 1, -j_2 + 2)\) their end points. Then \( i_1 \leq i_2 \) if and only if \( j_1 \leq j_2 \). This implies that for families where no two paths intersect strongly, the path starting in \((-i, i)\) ends in \((i - 1, -i + 2)\), so that the underlying permutation is the identity, which does not contribute to the sign.

Now we define a sign-reversing involution on such families of paths with at least one pair of paths that is intersecting strongly. Among those pairs we chose a canonical one as follows: The starting point of one path \( P_1 \) should be as rightmost as possible, and among all paths that are intersecting strongly with this path, choose again the path \( P_2 \) whose starting point is as rightmost as possible.

We distinguish several cases: If \( P_1 \cap P_2 \) contains a vertical step, then we select the last such vertical step in \( P_1 \) and interchange the endportions of the two paths after this step. Thus we can assume that \( P_1 \cap P_2 \) contains only horizontal steps. Then the involution is provided by Figure 13, which is applied to the last connected component of the intersection. In all cases, the sign of the family of paths changes by a factor of \(-1\).

An example of families of paths that do not cancel under this involution is given in Figure 14.
Figure 13

Figure 14. This family of lattice paths corresponds to the pair of CSPP and RSPP in Section 2.

To conclude this proof of Theorem 2.5, we argue somewhat similar as in the previous subsection where we derived the plane partition representation. The horizontal steps on and below the line $y = 1$ correspond to the entries in the plane partition $P$, however, here red steps at distance $d$ from the line $y = 2$ correspond to the entries $2d - 1$, while blue steps at distance $d$ from the line $y = 2$ correspond to the entries $2d$. The entries of $Q$ correspond to the diagonal steps left of the $y$-axis as follows: As before, their distances from the line $y = x$ yield a column-strict shifted plane partition $Q'$. By the same manipulations as in the end of
Section 7.3, that is, first subtracting \(i - 1\) from the entries in the \(i\)-th row from the bottom of \(Q'\) and then subtracting \(j - 1\) from all entries in the \(j\)-th column of \(Q'\), we finally obtain \(Q\).

7.5. **Towards a signless version of Theorem 2.3?** It seems less clear how to show that the generating function of lattice paths from Theorem 2.3 has no negative coefficients. It is tempting to proceed as in the case of the first interpretation since Lemma 7.7 does have the following counterpart (Lemma 7.8). However, it is unclear what replaces the Gessel-Viennot involution from the previous section.

**Lemma 7.8.** Let \(i, j\) be positive integers, then

\[
\sum_{t \in \mathbb{Z}} c_{t,j}(u,v,w) h_{t-i}(uX_1 + w + vX_1^{-1}, \ldots, uX_i + w + vX_i^{-1})
\]

is the generating function of lattice paths from \((i,i)\) to \((2j,1)\) using the step set \{(1,0), (0,-1)\}, with three types of \((1,0)\)-steps, colored in red, blue and green, and with weights \(uX_d, vX_d^{-1}\) and \(w\), respectively, where \(d\) is the distance from the \(x\)-axis and \(c_{t,j}(u,v,w)\) is given as in (6.2), and right of the line \(y = x - j\) there is no green horizontal step and also no pair of consecutive steps where the first is blue and the second is red.

**Proof.** Recall that the sum in the lemma is the generating function of lattice paths from \((i,i)\) to \((2,-j+2)\) using the step set and weights as given in the lemma in the region \{\((x,y)\mid y \geq 1\}\), while below \(y = 1\), the step set is \{\((-1,-1), (-2,-2), (-2,-1)\}\), where the steps of type \((-1,-1)\) and \((-2,-2)\) are equipped with the weight \(-w\) and \(-uv\), respectively.

We will construct a sign-reversing involution on such paths where one of the following is satisfied:

- The path contains a step of type \((-1,-1)\).
- The path contains a step of type \((-2,-2)\).
- Right of the line \(y = x - j\), the path contains a green horizontal step.
- Right of the line \(y = x - j\), the path contains a pair of consecutive horizontal steps where the first is blue and the second is red.

A path that does not fall into that class has only steps of type \((-2,-1)\) below the line \(y = 1\), and, since its end point is \((2,-j+2)\), the last point on the line \(y = 1\) is \((2j,1)\). Consequently, we obtain a path as described in the lemma after deleting the (fixed) portion below the line \(y = 1\).

In order to construct the sign-reversing involution we observe the following. Assuming that \((t,1)\) is the last point on the line \(y = 1\), the number of \((-2,-1)\)-steps is \(t - j - 1\), while the total number of steps from \((i,i)\) to \((t,1)\) is \(t - 1\). Let \(r\) be the maximal number of consecutive \((-2,-1)\)-steps right after \((t,1)\). Now consider the \(r\) steps before \((t,1)\) is reached. Since \(r \leq t - j - 1 < t - 1\), these steps exist.

Among these, we consider occurrences of green horizontal steps and occurrences of pairs of consecutive horizontals steps where the first is blue and the second is red, and, if there is such an occurrence, we consider the one that is closest to \((t,1)\). Suppose it is a green horizontal step at position \(s\) before \((t,1)\), then we delete this step and replace the \(s\)-th \((-2,-1)\)-step after \((t,1)\) by a \((-1,-1)\)-step. Clearly, this shifts the last point on the line \(y = 1\) from \((t,1)\) to \((t - 1,1)\). If we have a pair of consecutive horizontal steps where the first is blue and the second is red, and they are in position \(s, s + 1\) before \((t,1)\), then we delete the blue and the red step and replace the two \((-2,-1)\)-steps at positions \(s\) and \(s + 1\) after \((t,1)\) by \((-2,-2)\). In this case, the last point on the line \(y = 1\) is shifted by 2 to the left.
If there is neither a green horizontal step nor a pair of consecutive horizontal steps where the first is blue and the second is red, then there has to be either a \((-1, -1)\)-step or a \((-2, -2)\)-step after the \(r\)-th \((-2, -1)\)-step. In the first case, we replace it by a \((-2, -1)\)-step and insert a green horizontal step in position \(r + 1\) before \((t, 1)\). This shifts the last point on the line \(y = 1\) from \((t, 1)\) to \((t + 1, 1)\). In the second case, we replace the \((-2, -2)\)-step by two \((-2, -1)\)-steps and insert a pair of consecutive horizontal steps where the first is blue and the second is red in positions \(r + 2\) and \(r + 1\) before \((t, 1)\), which shifts the last point on the line \(y = 1\) from \((t, 1)\) to \((t + 2, 1)\).

This involution is clearly sign-reversing. \(\square\)

8. ACKNOWLEDGMENT

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