PROBABILISTIC PROOFS OF SOME GENERALIZED MERTENS’ FORMULAS VIA GENERALIZED DICKMAN DISTRIBUTIONS

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Abstract. The classical Mertens’ formula states that \( \prod_{p \leq N} (1 - \frac{1}{p})^{-1} \sim e^\gamma \log N \), where the product is over all primes \( p \) less than or equal to \( N \), and \( \gamma \) is the Euler-Mascheroni constant. By the Euler product formula, this is equivalent to either of the following statements:

1. \( \lim_{N \to \infty} \sum_{p \mid n, p \leq N} \frac{1}{n} = e^\gamma \)

2. \( \sum_{n: p \mid n, p \leq N} \frac{1}{n} \sim e^\gamma \log N \).

Via some random integer constructions and a criterion for weak convergence of distributions to so-called generalized Dickman distributions, we obtain some generalized Mertens’ formulas, some of which are new and some of which have been proved using number-theoretic tools. For example, in the spirit of (i), we show that if \( A \) is a subset of the primes which has natural density \( \theta \in (0, 1] \) with respect to the set of all primes, then

\[
\lim_{N \to \infty} \frac{\sum_{n: p \mid n, p \leq N \text{ and } p \in A} \frac{1}{n}}{\sum_{n \leq N: p \mid n \text{ and } p \in A} \frac{1}{n}} = e^{\gamma \theta} \Gamma(\theta + 1),
\]

and also, for any \( k \geq 2 \),

\[
\lim_{N \to \infty} \frac{\sum_{n: p \mid n, p \leq N \text{ and } p \in A} \frac{1}{n}}{\sum_{n \leq N: p \mid n \text{ and } p \in A} \frac{1}{n}} = e^{\gamma \theta} \Gamma(\theta + 1),
\]

where \( \sum^{(k)} \) denotes that the summation is restricted to \( k \)-free positive integers. In the spirit of (ii), we show for example that

\[
\sum_{n: p \mid n, p \leq N} \frac{1}{n_{((k-1)-free)} \phi^n_{((k-1)-power)}} \sim e^\gamma \log N,
\]

where \( \phi \) is the Euler totient function, and \( n_{((k-1)-free)} \) and \( n_{((k-1)-power)} \) are the \( (k-1) \)-free part and the \( (k-1) \)-power part of \( n \).

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1. Introduction and Statement of Results

The classical Mertens’ formula states that

\[ \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \sim e^\gamma \log N \approx 1.78 \log N, \text{ as } N \to \infty, \]

where the product is over all primes \( p \) less than or equal to \( N \), and \( \gamma \) is the Euler-Mascheroni constant. (Actually, the classical formula states a little more; namely, \( \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} = e^\gamma \log N + O(1) \).) By the Euler product formula, (1.1) is equivalent to either of the following statements, which are more in the spirit of the results we present in this paper:

(i) \[ \lim_{N \to \infty} \sum_{n \in \mathbb{N}} \frac{1}{n} \mathbb{1}_{n \leq N} \mathbb{1}_{p \nmid n} = e^\gamma; \]

(ii) \[ \sum_{n \in \mathbb{N}} \frac{1}{n} \mathbb{1}_{n \leq N} \mathbb{1}_{p \nmid n} \sim e^\gamma \log N, \text{ as } N \to \infty, \]

where as usual, \( [N] \) denotes the set \( \{1, \cdots, N\} \). In words, (1.2) states that \( e^\gamma \) is the limit of the ratio between the harmonic series restricted to the positive integers all of whose prime factors are no greater than \( N \) and the harmonic series restricted to the positive integers no greater than \( N \).

One of the results in [5] involved the construction of a sequence of random integers whose distributions were shown to converge weakly to the so-called Dickman distribution. It was noted in that paper that Mertens’ formula follows readily as a corollary of this result. In this paper, we make a number of random integer constructions in a similar vein, and use our recent paper [4] to show that their distributions converge weakly to so-called generalized Dickman distributions. From these results, we obtain a number of generalizations of Mertens’ formula, some of which are known via number theoretic methods, and some of which appear to be new.

We begin by introducing some notation and constructing the three sequences of random integers that will be used in this paper. Then we state our generalized Mertens’ formulas.

Let \( \mathbb{P} \) denote the set of prime numbers. Recall that for \( k \geq 2 \), an integer \( n \in \mathbb{N} \) is called \( k \)-free if \( p^k \nmid n \), for all primes \( p \). Let \( A \subset \mathbb{P} \) be an infinite set of primes. Denote the primes in \( A \) in increasing order by \( p_{1;A}, p_{2;A}, \cdots \). Let \( \{T_j\}_{j=1}^\infty \) be a sequence of independent random variables
with $T_j$ distributed according to the geometric distribution with parameter $\frac{1}{p_j;A}$ ($T_j \sim \text{Geom}(\frac{1}{p_j;A})$), $j = 1, \ldots$; that is

$$P(T_j = m) = (1 - \frac{1}{p_j;A})(\frac{1}{p_j;A})^m, \ m = 0, 1, \cdots. \tag{1.3}$$

For $N \in \mathbb{N}$, we define a random integer by

$$I_{N;A,1} = \prod_{j=1}^{N} p_j;A. \tag{1.4}$$

By construction, the support of $I_{N;A,1}$ is $\{n \in \mathbb{N} : p|n \Rightarrow p \leq N \text{ and } p \in A\}$. See [5] for a detailed study of the random integer sequence $\{I_{N;A,1}\}_{N=1}^{\infty}$ when $A = \mathbb{P}$.

Let $k \geq 2$. We define a second random integer sequence by replacing the random variables $\{T_j\}_{j=1}^{\infty}$ by a sequence $\{U_j\}_{j=1}^{\infty}$ of independent random variables, where $U_j$ is distributed as $T_j$ conditioned on being less than $k$ ($U_j \overset{\text{dist}}{=} T_j\{|T_j < k\}$); that is

$$P(U_j = m) = P(T_j = m|T_j < k) = \frac{1 - \frac{1}{p_j;A}}{1 - (\frac{1}{p_j;A})^k}(\frac{1}{p_j;A})^m, \ m = 0, 1, \cdots, k - 1. \tag{1.5}$$

For $N \in \mathbb{N}$, define a random integer by

$$I_{N;A,2} = \prod_{j=1}^{N} U_j. \tag{1.6}$$

By construction, the support of $I_{N;A,2}$ is the set of $k$-free integers in $\{n \in \mathbb{N} : p|n \Rightarrow p \leq N \text{ and } p \in A\}$.

Finally, for $k \geq 2$, we construct a third random integer sequence from a sequence $\{V_j\}_{j=1}^{\infty}$ of independent random variables, where $V_j$ is distributed as $T_j$ truncated at $k - 1$ ($V_j \overset{\text{dist}}{=} T_j \wedge (k - 1)$); that is

$$P(V_j = m) = (1 - \frac{1}{p_j;A})(\frac{1}{p_j;A})^m, \ m = 0, \cdots, k - 2; \tag{1.7}$$

$$P(V_j = k - 1) = (\frac{1}{p_j;A})^{k-1}.$$ 

For $N \in \mathbb{N}$, define a random integer by

$$I_{N;A,3} = \prod_{j=1}^{N} V_j. \tag{1.8}$$
By construction, the support of $I_{N;A,3}$ is the set of $k$-free integers in $\{n ∈ \mathbb{N} : p|n ⇒ p ≤ N \text{ and } p ∈ A\}$.

We now present our generalized Mertens’ formulas. For $A ⊂ \mathbb{P}$, denote by
$$D_{\text{nat-prime}}(A) := \lim_{N \to \infty} \frac{|A ∩ [N]|}{|\mathbb{P} ∩ [N]|},$$
the natural density of $A$ in $\mathbb{P}$, if it exists. For $B ⊂ \mathbb{N}$, let $\sum_{B}^{(k)}$ denote the summation restricted to the $k$-free powers in $B$. Let $\Gamma(x) := \int_{0}^{\infty} t^{x-1}e^{-t}dt$, $x > 0$, denote the Gamma function.

**Theorem 1.** Let $A ⊂ \mathbb{P}$ be a subset of primes whose natural density in $\mathbb{P}$ is $D_{\text{nat-prime}}(A) = \theta ∈ (0, 1]$. Then

i.
$$\lim_{N \to \infty} \frac{\sum_{n ∈ \mathbb{N} : p|n ⇒ p ≤ N \text{ and } p ∈ A} \frac{1}{n}}{\sum_{n ≤ N : p|n ⇒ p ∈ A} \frac{1}{n}} = e^{\gamma \theta} \Gamma(\theta + 1);$$

ii. for $k ≥ 2$,
$$\lim_{N \to \infty} \frac{\sum_{n ∈ \mathbb{N} : p|n ⇒ p ≤ N \text{ and } p ∈ A} \frac{1}{n}^{(k)}}{\sum_{n ≤ N : p|n ⇒ p ∈ A} \frac{1}{n}^{(k)}} = e^{\gamma \theta} \Gamma(\theta + 1).$$

**Remark 1.** Part (i) of Theorem 1 in a slightly different but equivalent form appears in [10], and a refined version appears in [6]. Part (ii) seems to be new.

**Remark 2.** The function $\theta \to e^{\gamma \theta} \Gamma(\theta + 1)$, $\theta ∈ (0, 1]$, is increasing, is equal to 1 at $\theta = 0^+$ and is equal to $e^\gamma$ at $\theta = 1$.

**Remark 3.** When $A = \mathbb{P}$, (i) reduces to the classical Mertens’ formula.

**Remark 4.** For $l ∈ \mathbb{N}$ and $j$ satisfying $1 ≤ j < l$ and $(j, l) = 1$, let $A_{l;j} = \{p ∈ \mathbb{P} : p = j \mod l\}$ denote the set of primes that are equal to $j$ modulo $l$. Dirichlet’s arithmetic progression theorem states that $D_{\text{nat-prime}}(A_{l;j}) = \frac{1}{\phi(l)}$, where $\phi$ is Euler’s totient function. In [9], it was proved that

\begin{equation}
(1.10) \prod_{p≤N,p=j \mod l} (1-\frac{1}{p})^{-1} = \sum_{n:p|n⇒p≤N \text{ and } p ∈ A_{l;j}} \frac{1}{n} \sim (e^{\gamma} \frac{\phi(l)}{l} C(l,j))^{\frac{1}{\phi(l)}} (\log N)^{\frac{1}{\phi(l)}},
\end{equation}
as $N \to \infty$, where $C(l,j)$ is a complicated expression involving Dirichlet characters modulo $l$. Thus, by part (i) we obtain
$$\sum_{n≤N:p|n⇒p∈A_{l;j}} \frac{1}{n} \sim \frac{1}{\Gamma(1+\frac{1}{\phi(l)})} (e^{\gamma} \frac{\phi(l)}{l} C(l,j))^{\frac{1}{\phi(l)}} (\log N)^{\frac{1}{\phi(l)}}, \text{ as } N \to \infty,$$
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as was noted in [9]. A much simpler looking form for $C(l, j)$ was obtained in [1]; namely,

$$\phi(l) C(l, j) = \prod_p \left(1 - \frac{1}{p}\right)^{-\alpha(p, l, j)}, \quad \text{where} \quad \alpha(p, l, j) = \begin{cases} \phi(l) - 1, & p = j \text{ mod } l; \\ -1, & \text{otherwise.} \end{cases}$$

See also [2] for more on this constant.

From part (ii), we obtain for $k \geq 2$,

$$\lim_{N \to \infty} \frac{\sum'_{n:p|n \Rightarrow p \leq N} \frac{1}{n}}{\sum'_{n \leq N: p|n \Rightarrow p \in A_{l, j}} \frac{1}{n}} = e^{\gamma(\frac{1}{\phi(l)} + 1)}.$$

Whereas Theorem 1 is an asymptotic result and depends on the above-mentioned convergence to generalized Dickman distributions, the next result is a non-asymptotic identity that holds for all $N$, and that only requires the random integer $I_{N; A, 3}$. For each $k \geq 2$, every $n \in \mathbb{N}$ can be written uniquely as

$$n = n_{\{k\text{-free}\}} n_{\{k\text{-power}\}},$$

where $n_{\{k\text{-free}\}}$ is $k$-free and $n_{\{k\text{-power}\}}$ is a $k$th power. We extend this to $k = 1$ by defining $n_{\{1\text{-free}\}} = 1$ and $n_{\{1\text{-power}\}} = n$.

**Theorem 2.** Let $N \geq 1$ and let $A_N = \{p_{1; A}, \ldots, p_{N; A}\} \subset \mathbb{P}$. Then for $k \geq 2$,

$$\sum'_{n:p|n \Rightarrow p \in A_N} \frac{1}{\phi(n)} = \sum_{n:p|n \Rightarrow p \leq N} \frac{1}{\phi(n)},$$

where $\phi$ is the Euler totient function.

**Remark.** Theorem 2 seems to be new. Note that when $k = 2$, the result is

$$\sum'_{n:p|n \Rightarrow p \in A_N} \frac{1}{\phi(n)} = \sum_{n:p|n \Rightarrow p \leq N} \frac{1}{\phi(n)}.$$

We have the following corollary.

**Corollary 1.** For $k \geq 2$,

$$\sum'_{n:p|n \Rightarrow p \leq N} \frac{1}{n_{\{k\text{-free}\}} \phi(n_{\{k\text{-free}\}}) n_{\{k\text{-power}\}}} \sim e^\gamma \log N, \quad \text{as } N \to \infty.$$

**Proof of Corollary.** By Theorem 2 with $A_N$ replaced by $\mathbb{P} \cap [N]$, we have

$$\sum'_{n:p|n \Rightarrow p \leq N} \frac{1}{n_{\{k\text{-free}\}} \phi(n_{\{k\text{-free}\}}) n_{\{k\text{-power}\}}} = \sum_{n:p|n \Rightarrow p \leq N} \frac{1}{n},$$

and by Mertens’ formula (1.2-ii) we have $\sum_{n:p|n \Rightarrow p \leq N} \frac{1}{n} \sim e^\gamma \log N$. 

Our final theorem combines some of the ingredients of Theorems 1 and 2.
Theorem 3. Let \( A \subset \mathbb{P} \) be a subset of primes whose density in \( \mathbb{P} \) is \( D_{\text{nat-prime}}(A) = \theta \in (0,1] \). Then for all \( k \geq 2 \),

\[
\lim_{N \to \infty} \frac{\sum_{n : p \mid n \land p \leq N \land p \in A} \frac{1}{n}}{\sum_{n \leq N : p \mid n \land p \in A} \frac{1}{\phi(n)}} = e^{\gamma} \theta \Gamma(\theta + 1).
\]

Remark. Theorem 3 seems to be new. Note that when \( k = 2 \), the result is

\[
\lim_{N \to \infty} \frac{\sum_{n : p \mid n \land p \leq N \land p \in A} \frac{1}{n}}{\sum_{n \leq N : p \mid n \land p \in A} \frac{1}{\phi(n)}} = e^{\gamma} \Gamma(1) = e^{\gamma}.
\]

From Theorem 3, we obtain the following corollary

**Corollary 2.** Let \( A \subset \mathbb{P} \) be a subset of primes whose density in \( \mathbb{P} \) is \( D_{\text{nat-prime}}(A) = \theta \in (0,1] \). Then for all \( k \geq 2 \),

\[
\sum_{n \leq N : p \mid n \land p \leq N \land p \in A} \frac{1}{\phi(n)} \sim \sum_{n \leq N : p \mid n \land p \in A} \frac{1}{n}, \quad \text{as } N \to \infty.
\]

**Proof of Corollary.** Compare (1.9) to (1.11). \( \square \)

Remark. When \( A = \mathbb{P} \), Corollary 2 reduces to

\[
\sum_{n \leq N} \frac{1}{\phi(n)} \sim \log N, \quad \text{as } N \to \infty,
\]

for all \( k \geq 2 \).

When \( k = 2 \), this reduces to

\[
\sum_{n \leq N} \frac{1}{\phi(n)} \sim \log N, \quad \text{as } N \to \infty,
\]

which is known (see [8] or [3, p. 43, problem 17]). The generalization to all \( k \geq 2 \) seems to be new. As an aside, we note that

\[
\sum_{n \leq N} \frac{1}{\phi(n)} \sim \frac{\zeta(2) \zeta(3)}{\zeta(6)} \log N \approx 1.94 \log N, \quad \text{as } N \to \infty.
\]

(see [3] p. 42, problem 13-(d)).

We prove Theorems 1-3 in sections 2-4 respectively.

2. **Proof of Theorem 1**

Fix a subset \( A \subset \mathbb{P} \) which satisfies \( D_{\text{nat-prime}}(A) = \theta \in (0,1] \). Denote the primes in \( A \) in increasing order by \( p_1;A, p_2;A, \cdots \), and let

\[
A_N = \{p_1;A, p_2;A, \cdots, p_N;A\}.
\]

**Proof of part (i).** Let the random integer \( I_{N;A,1} \) be as in (1.3), where \( \{T_j\}_{j=1}^\infty \) is a sequence of independent random variables with distributions given by
The support of $I_{N;A,1}$ is \( \{ n \in \mathbb{N} : p|n \Rightarrow p \in A_N \} \), and for arbitrary \( n = \prod_{j=1}^{N} p_{j;A}^{c_{j}} \) in the support,

\[
P(I_{N;A,1} = n) = P(T_j = c_j, j \in [N]) = \prod_{j=1}^{N} P(T_j = c_j)
\]

(2.1)

\[
= \prod_{j=1}^{N} (1 - \frac{1}{p_{j;A}})(\frac{1}{p_{j;A}})^{c_{j}} = \frac{1}{n} \prod_{j=1}^{N} (1 - \frac{1}{p_{j;A}}).
\]

We have

\[
\log I_{N;A,1} = \sum_{j=1}^{N} T_j \log p_{j;A}.
\]

Noting that the expected value of $T_j$ is given by

(2.2)

\[
ET_j = \frac{1}{p_{j;A} - 1},
\]

we have

\[
E \log I_{N;A,1} = \sum_{j=1}^{N} \frac{\log p_{j;A}}{p_{j;A} - 1}.
\]

It follows by the assumption on the density of $A$ and by the prime number theorem that

(2.3)

\[
p_{j;A} \sim \frac{j \log j}{\theta}, \text{ as } j \to \infty,
\]

and thus that

(2.4)

\[
E \log I_{N;A,1} \sim \theta \log N, \text{ as } N \to \infty.
\]

We will demonstrate below that the conditions of a theorem in [4] are satisfied, from which it follows that

(2.5)

\[
\lim_{N \to \infty} \frac{\log I_{N;A,1}}{E \log I_{N;A,1}} \overset{\text{dist}}{=} \frac{1}{\theta} D_{\theta},
\]

where $D_{\theta}$ is a random variable distributed according to the generalized Dickman distribution $GD(\theta)$ with parameter $\theta$. This distribution has density function $p_\theta = \frac{\theta - 1}{\Gamma(\theta)} \rho_\theta$, where $\rho_\theta$ satisfies the differential-delay equation

(2.6)

\[
\rho_\theta(x) = 0, \quad x \leq 0;
\]

\[
\rho_\theta(x) = x^{\theta-1}, \quad 0 < x \leq 1;
\]

\[
x \rho_\theta(x) + (1 - \theta) \rho_\theta(x) + \theta \rho_\theta(x - 1) = 0, \quad x > 1.
\]
(The function $\rho_1$ is known as the Dickman function; we call $\rho_\theta$ a generalized Dickman function.)

On the one hand, by the convergence in distribution in (2.5) and the fact that the limiting distribution is a continuous one, for any sequence $\{\theta_N\}_{N=1}^\infty$ satisfying $\lim_{N \to \infty} \theta_N = \theta$, we have

$$
\lim_{N \to \infty} P(\frac{\log I_{N;A,1}}{E \log I_{N;A,1}} \leq \frac{1}{\theta_N}) = P(\frac{1}{\theta} D_\theta \leq \frac{1}{\theta}) = \int_0^1 p_\theta(x)dx
$$

(2.7)

$$
= \frac{e^{-\gamma \theta}}{\Gamma(\theta)} \int_0^1 x^{\theta-1}dx = \frac{e^{-\gamma \theta}}{\theta \Gamma(\theta)} = \frac{e^{-\gamma \theta}}{\Gamma(\theta+1)}.
$$

On the other hand, let $\theta_N := \frac{E \log I_{N;A,1}}{\log p_{N,A}}$ and note from (2.3) and (2.4) that $\lim_{N \to \infty} \theta_N = \theta$. It follows from (2.1) that

$$
P(\frac{\log I_{N;A,1}}{E \log I_{N;A,1}} \leq \frac{1}{\theta_N}) = P(I_{N;A,1} \leq \exp(\frac{E \log I_{N;A,1}}{\theta_N}))
$$

(2.8)

$$
= P(I_{N;A,1} \leq p_{N,A}) = \left(\prod_{j=1}^{N} \left(1 - \frac{1}{p_j;A}\right)\right) \sum_{1 \leq n \leq p_{N,A}; p|n \Rightarrow p \in A} \frac{1}{n}
$$

$$
= \sum_{n \leq p_{N,A}; p|n \Rightarrow p \in A} \frac{1}{n}.
$$

From (2.7) and (2.8), we conclude that

$$
\lim_{N \to \infty} \frac{\sum_{n \leq p_{N,A}; p|n \Rightarrow p \in A} \frac{1}{n}}{\sum_{n \leq p_{N,A}; p|n \Rightarrow p \in A} \frac{1}{n}} = e^{-\gamma \theta} \Gamma(\theta + 1).
$$

(2.9)

Now (2.9) is equivalent to part (i) of Theorem 1. Indeed, for any $M \in \mathbb{N}$, let $N^+(M) = \max\{n : p_{n,A} \leq M\}$. Then

$$
\sum_{n,p|n \Rightarrow p \leq M} \frac{1}{n} = \sum_{n,p|n \Rightarrow p \leq p_{N^+(M),A}} \frac{1}{n},
$$

(2.10)

and

$$
\sum_{n \leq M,p|n \Rightarrow p \in A} \frac{1}{n} = \sum_{n \leq p_{N^+(M),A}; p|n \Rightarrow p \in A} \frac{1}{n} + H_M,
$$

(2.11)

where

$$
H_M := \sum_{n \in [p_{N^+(M),A}+1,M]; p|n \Rightarrow p \in A} \frac{1}{n} \leq \sum_{n=p_{N^+(M),A}+1} p_{N^+(M)+1;A} \frac{1}{n} \leq \log \frac{p_{N^+(M)+1;A}}{p_{N^+(M);A}} = o(1).
$$

(2.12)
Thus, \((2.9)-(2.12)\) give
\[
\lim_{M \to \infty} \frac{\sum_{n:p|n \Rightarrow p \leq M \text{ and } p \in A} \frac{1}{n}}{\sum_{n \leq M:p|n \Rightarrow p \in A} \frac{1}{n}} = e^{\gamma \theta} \Gamma(\theta + 1),
\]
which is part (i) of the theorem.

We now show that \((2.5)\) holds. Let \(\{B_j\}_{j=1}^\infty\) and \(\{X_j\}_{j=1}^\infty\) be mutually independent random variables with distributions as follows:

\(B_j \sim \text{Ber}(\frac{1}{p_j};A)\); that is,
\[
q_j := P(B_j = 1) = 1 - P(B_j = 0) = \frac{1}{p_j},
\]
\(X_j \sim \log p_j;A \cdot T_j\); that is
\[
P(X_j = m \log p_j;A) = P(T_j = m|T_j \geq 1) = (1 - \frac{1}{p_j}) (\frac{1}{p_j})^{m-1}, m = 1, 2, \cdots.
\]

Then
\[
\mu_j := E X_j = \frac{p_j;A}{p_j;A - 1} \log p_j;A,
\]
and it follows that
\[
\lim_{j \to \infty} \frac{X_j}{\mu_j} \overset{\text{dist}}{=} 1.
\]

By the construction of \(\{B_j\}_{j=1}^\infty\) and \(\{X_j\}_{j=1}^\infty\), we have
\[
\log I_{N;A,1} = \sum_{j=1}^N T_j \log p_j;A \overset{\text{dist}}{=} \sum_{j=1}^N B_j X_j.
\]

From \((2.3)\), \((2.13)\) and \((2.14)\),
\[
\mu_j \sim \log j; \quad q_j \sim \frac{\theta}{j \log j}, \text{ as } j \to \infty.
\]

Let \(W_N = \frac{\sum_{j=1}^N B_j X_j}{E \sum_{j=1}^N B_j X_j}\). Now Theorem 1.2 in \([4]\) applies to \(W_N\). Our notation here coincides with the notation in that theorem except that the summation there is over \(k\) while here it is over \(j\), and \(p_k\) there corresponds to \(q_j\) here. In light of \((2.16)\), we have \(J_\mu = J_p = 1, a_0 = 0, a_1 = b_0 = b_1 = c_\mu = 1, c_p = \theta\) in the notation of that theorem. For these values, the theorem indicates that \(W_N\) converges in distribution to \(\frac{1}{\theta}D_\theta\). By \((2.15)\), \(\frac{\log I_{N;A,1}}{E \log I_{N;A,1}} \overset{\text{dist}}{=} W_N\); thus \((2.5)\) holds.

Proof of part (ii). Fix \(k \geq 2\). The proof follows the proof of part (i), except that we replace the random integer \(I_{N;A,1}\) by the random integer \(I_{N;A,2}\) from
[1.6], where \( \{U_j\}_{j=1}^\infty \) is a sequence of random variables with distributions given by (1.5). The support of \( I_{N;A,2} \) is the set of \( k \)-free integers in \( \{n \in \mathbb{N} : p|n \Rightarrow p \in A_N \} \), and for arbitrary \( n = \prod_{j=1}^{N} p_j^{c_j} \) in the support,

\[
P(I_{N;A,2} = n) = \prod_{j=1}^{N} P(U_j = c_j) = \prod_{j=1}^{N} \frac{1 - \frac{1}{p_j;A}}{1 - \left( \frac{1}{p_j;A} \right)^k} \left( \frac{1}{p_j;A} \right)^{c_j}
\]

(2.17)

\[
= \frac{1}{n} \prod_{j=1}^{N} \left( 1 + \frac{1}{p_j;A} + \left( \frac{1}{p_j;A} \right)^2 + \cdots + \left( \frac{1}{p_j;A} \right)^{k-1} \right)^{-1}.
\]

We have

\[
\log I_{N;A,2} = \sum_{j=1}^{N} U_j \log p_j;A.
\]

Since

\[
EU_j = \frac{1 - \frac{1}{p_j;A}}{1 - \left( \frac{1}{p_j;A} \right)^k} \sum_{m=0}^{k-1} m \left( \frac{1}{p_j;A} \right)^m
\]

\[
= \frac{1 - \frac{1}{p_j;A}}{1 - \left( \frac{1}{p_j;A} \right)^k} \frac{1}{1 + (k - 1) \left( \frac{1}{p_j;A} \right)^k - k \left( \frac{1}{p_j;A} \right)^{k-1} - 1 - \left( \frac{1}{p_j;A} \right)^k}
\]

we have

(2.18)

\[
EU_j \sim \frac{1}{p_j;A}, \text{ as } j \to \infty.
\]

From (2.2) note that \( EU_j \) and \( ET_j \) have the same asymptotic behavior.

From (2.3) and (2.18), we have

(2.19)

\[
E \log I_{N;A,2} \sim \sum_{j=1}^{N} \frac{\log p_j;A}{p_j;A} \sim \theta \log N, \text{ as } N \to \infty.
\]

Note from (2.4) that \( E \log I_{N;A,2} \) and \( E \log I_{N;A,1} \) have the same asymptotic behavior.

We now give the appropriate redefinition of the mutually independent random variables \( \{B_j\}_{j=1}^{\infty} \) and \( \{X_j\}_{j=1}^{\infty} \) that were defined in the proof of part (i):

\[
B_j \sim \text{Ber} \left( \frac{1 - \left( \frac{1}{p_j;A} \right)^k}{1 - \left( \frac{1}{p_j;A} \right)^k} \right); \text{ that is,}
\]

(2.20)

\[
q_j := P(B_j = 1) = 1 - P(B_j = 0) = \frac{1 - \left( \frac{1}{p_j;A} \right)^k}{1 - \left( \frac{1}{p_j;A} \right)^k}.
\]
For any sequence (2.21) and the fact that the limiting distribution is a continuous one, for any part (i) of the theorem.

\[ \lim_{j \to \infty} \frac{X_j}{p_j} = 1. \]

By the construction of \( \mu \), as in part (i), we have

\[ \lim_{j \to \infty} \frac{X_j}{p_j} = \frac{1}{\theta}D. \]

On the one hand, just as in (2.7), by the convergence in distribution in (2.21) and the fact that the limiting distribution is a continuous one, for any sequence \( \{\theta_N\}_{N=1}^{\infty} \) satisfying \( \lim_{N \to \infty} \theta_N = \theta \), we have

\[ \lim_{N \to \infty} \frac{\log I_{N,A,2}}{E \log I_{N,A,2}} = \frac{1}{\theta}D. \]

On the other hand, let \( \theta_N := \frac{E \log I_{N,A,2}}{\log p_N/A} \) and note from (2.3) and (2.19) that \( \lim_{N \to \infty} \theta_N = \theta \). It follows from (2.17) that

\[ P\left( \frac{\log I_{N,A,2}}{E \log I_{N,A,2}} \leq \frac{1}{\theta_N} \right) = P\left( \frac{1}{\theta}D \leq \frac{1}{\theta} \right) = \frac{e^{-\gamma \theta}}{\Gamma(\theta + 1)}. \]

From this and (2.22), we conclude that

\[ \lim_{N \to \infty} \frac{1}{\theta_N} = e^{\gamma \theta} \Gamma(\theta + 1), \]

which is equivalent to part (ii) of Theorem 1 just as (2.9) was equivalent to part (i) of the theorem. \( \square \)
3. Proof of Theorem 2

Fix $N \geq 1$, $k \geq 2$ and $A_N = \{p_{1,A}, \ldots, p_{N,A}\}$ as in the statement of the theorem. Consider the random integer $I_{N;A;3}$ that was defined in (1.8), where \( \{V_j\}_{j=1}^N \) are independent random variables with distributions given by (1.7). The support of $I_{N;A;3}$ is the set of $k$-free integers in \( \{n \in \mathbb{N} : p|n \Rightarrow p \in A_N\} \).

Let $f, g$ be functions satisfying

\[
\begin{align*}
    f(j) &= 1, \quad j = 0, \ldots, k - 2; \quad f(k - 1) = 0; \\
    g(j) &= j + 1, \quad j = 0, \ldots, k - 2; \quad g(k - 1) = k - 1.
\end{align*}
\]

For definiteness, we take

\[
\begin{align*}
    f(x) &= 1 - \left(\frac{x}{k - 1}\right); \\
    g(x) &= (x + 1) - \left(\frac{x}{k - 1}\right).
\end{align*}
\]

Then for arbitrary $n = \prod_{j=1}^N p_{j;A}^{c_j}$ in the support,

\[
(3.1) \quad P(I_{N;A;3} = n) = \prod_{j=1}^N P(V_j = c_j) = \prod_{j=1}^N \frac{(p_{j;A} - 1)f(c_j)}{p_{j;A}^{g(c_j)}}.
\]

Noting that $g(x) - f(x) \equiv x$, we rewrite the right hand side of (3.1) as

\[
(3.2) \quad \prod_{j=1}^N \frac{(p_{j;A} - 1)f(c_j)}{p_{j;A}^{g(c_j)}} = \frac{\prod_{j=1}^N \left(p_{j;A} - 1\right)f(c_j)}{\prod_{j=1}^N p_{j;A}^{g(c_j) - f(c_j)}} \times \frac{1}{n}
\]

\[
= \left(\prod_{j=1}^N (1 - \frac{1}{p_{j;A}})\right) \times \frac{1}{n_{\{k-1\text{-free}\}}} \times \frac{1}{n_{\{k-1\text{-power}\}}} \times \frac{1}{\prod_{j: c_j = k-1} (1 - \frac{1}{p_{j;A}})}
\]

\[
= \left(\prod_{j=1}^N (1 - \frac{1}{p_{j;A}})\right) \times \frac{1}{n_{\{k-1\text{-free}\}}} \times \frac{1}{n_{\{k-1\text{-power}\}}} \times \frac{1}{\phi(n_{\{k-1\text{-power}\}})}
\]

From (3.1) and (3.2) we obtain

\[
(3.3) \quad P(I_{N;A;3} = n) = \left(\prod_{j=1}^N (1 - \frac{1}{p_{j;A}})\right) \frac{1}{n_{\{k-1\text{-free}\}}} \frac{1}{\phi(n_{\{k-1\text{-power}\}})}.
\]
The theorem now follows from (3.3) along with the fact that
\[ \prod_{j=1}^{N} \left(1 - \frac{1}{p_{j} / A}\right)^{-1} = \sum_{n : p | n \Rightarrow p \in A \cap N} \frac{1}{n} \]
and that
\[ \sum_{n : p | n \Rightarrow p \in A \cap N} P(I_{N ; A, 3} = n) = 1. \]

\[ \Box \]

4. Proof of Theorem 3

Fix a subset \( A \subset \mathbb{P} \) which satisfies \( D_{\text{nat-prime}}(A) = \theta \in (0, 1] \). Denote the primes in \( A \) in increasing order by \( p_{1} / A, p_{2} / A, \cdots \), and let
\[ A_{N} = \{p_{1} / A, p_{2} / A, \cdots , p_{N} / A\}. \]
Let \( I_{N ; A, 3} \) be as in (1.8), where \( \{V_{j}\}_{j=1}^{\infty} \) are independent random variables with distributions given by (1.7). We have
\[ \log I_{N ; A, 3} = \sum_{j=1}^{N} V_{j} \log p_{j} / A. \]
It is easy to check that as with \( ET_{j} \) and \( EU_{j} \), we have
\[ EV_{j} \sim \frac{1}{p_{j} / A}, \text{ as } j \to \infty. \]
From (2.3) and (4.1), we have
\[ E \log I_{N ; A, 3} \sim \sum_{j=1}^{N} \frac{\log p_{j} / A}{p_{j} / A} \sim \theta \log N, \text{ as } N \to \infty. \]

As in the proof of Theorem 1, we define mutually independent random variables \( \{B_{j}\}_{j=1}^{\infty} \) and \( \{X_{j}\}_{j=1}^{\infty} \):
\[ B_{j} \sim \text{Ber}\left(\frac{1}{p_{j} / A}\right); \text{ that is,} \]
\[ q_{j} := P(B_{j} = 1) = 1 - P(B_{j} = 0) = \frac{1}{p_{j} / A}. \]
\[ X_{j} \overset{\text{dist}}{=} \log p_{j} / A \cdot V_{j} | V_{j} \geq 1; \text{ that is} \]
\[ P(X_{j} = m \log p_{j} / A) = P(V_{j} = m | V_{j} \geq 1) = \begin{cases} (1 - \frac{1}{p_{j} / A})(\frac{1}{p_{j} / A})^{m-1}, & m = 1, 2, \cdots , k - 2, \\ (\frac{1}{p_{j} / A})^{k-2}, & m = k - 1. \end{cases} \]
As in the proof of Theorem 1, we have $µ_j := EX_j \sim \log p_j; A$ and $\lim_{j \to \infty} X_j \overset{\text{dist}}{=} 1$. By the construction of $\{B_j\}_{j=1}^\infty$ and $\{X_j\}_{j=1}^\infty$, we have

$$\log I_{N;A,3} = \sum_{j=1}^N V_j \log p_j; A \overset{\text{dist}}{=} \sum_{j=1}^N B_j X_j.$$

By the same considerations as in the proof of parts (i) and (ii) of Theorem 1, it follows that $W_N := \frac{\sum_{j=1}^N B_j X_j}{E \sum_{j=1}^N B_j X_j}$ converges in distribution to $\frac{1}{\theta} D_\theta$; thus,

$$\lim_{N \to \infty} \frac{\log I_{N;A,3}}{E \log I_{N;A,3}} \overset{\text{dist}}{=} \frac{1}{\theta} D_\theta.$$

On the one hand, just as in (2.7) and (2.22), by the convergence in distribution in (4.4) and the fact that the limiting distribution is a continuous one, for any sequence $\{\theta_N\}_{N=1}^\infty$ satisfying $\lim_{N \to \infty} \theta_N = \theta$, we have

$$\lim_{N \to \infty} P\left( \frac{\log I_{N;A,3}}{\theta_N} \leq \frac{1}{\theta} \right) = P \left( \frac{1}{\theta} D_\theta \leq \frac{1}{\theta} \right) = e^{-\gamma \theta} \Gamma(\theta + 1).$$

On the other hand, let $\theta_N := \frac{E \log I_{N;A,3}}{\log p_N; A}$ and note from (2.3) and (4.2) that $\lim_{N \to \infty} \theta_N = \theta$. It follows from (3.3) that

$$P\left( \frac{\log I_{N;A,3}}{\theta_N} \leq \frac{1}{\theta} \right) = P \left( I_{N;A,3} \leq \exp\left( \frac{E \log I_{N;A,3}}{\theta_N} \right) \right) = P \left( I_{N;A,3} \leq p_N; A \right)$$

$$= \prod_{j=1}^N \left( 1 - \frac{1}{p_j; A} \right) \sum_{n \leq p_N; A} \frac{1}{n} \frac{1}{\phi(n \{(k-1)\text{-free}\})^{\frac{1}{\theta}}} \sum_{n \mid p \leq p_N; A} \frac{1}{\phi(n \{(k-1)\text{-power}\})^{\frac{1}{\theta}}}.$$

From this and (4.5) it follows that

$$\lim_{N \to \infty} \sum_{n \leq p_N; A} \frac{1}{n} \frac{1}{\phi(n \{(k-1)\text{-free}\})^{\frac{1}{\theta}}} = e^{\gamma \theta} \Gamma(\theta + 1),$$

which is equivalent to (1.11) just as (2.9) and (2.23) were equivalent to parts (i) and (ii) respectively of Theorem 1.

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