LOWEST-DEGREE POLYNOMIAL DE RHAM COMPLEX ON GENERAL QUADRILATERAL GRIDS

QIMENGQUAN, XIA JI, SHUO ZHANG

Abstract. This paper devotes to the construction of finite elements on grids that consist of general quadrilaterals not limited in parallelograms. Two finite elements are established for the \( H^1 \) and \( H(\text{rot}) \) elliptic problems, respectively. The two finite element spaces on general quadrilateral grids, together with the space of piecewise constant functions, formulate a discretised de Rham complex. First order convergence rate can be proved for both of them under the asymptotic-parallelogram assumption on the grids.

The local shape functions of the two spaces are piecewise polynomials, and the global finite element functions are nonconforming. A rigorous analysis is given in this paper that it is impossible to construct a practically useful finite element defined as Ciarlet’s triple whose shape functions are always piecewise polynomials and which can form conforming subspaces on a grid that consists of arbitrary quadrilaterals.

1. Introduction

There has been a long history on the study of finite element methods on general quadrilateral grids. Many conforming and nonconforming finite elements are established for various model problems. The classical strategy for constructing quadrilateral elements is to utilize isoparametric technique (cf., e.g., [5]). With this strategy, one begins with a given shape function space on a reference square and a class of bilinear transforms (or Piola transforms, etc.), the finite element on any convex quadrilateral cell can be constructed correspondingly. Great success has been achieved via this approach, particularly in constructing conforming finite element spaces; we refer to [2, 3, 7] for more details. On the other hand, a solid difficulty of these methods, as discussed in, e.g., Zhang [14], is that one will encounter the problem of rational function integration in practical numerical computation due to non-constant Jacobian determinants and inverse Jacobian matrices. Besides, when we take the discretized differential complexes, which is one of the most fundamental structural feature of the finite element schemes and has been a central topic of the finite element methods during the passed decades, into consideration, we

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have to pay extra attention to the different but relevant Jacobian matrices for the spaces, respectively. We are thus motivated to study finite element schemes with piecewise polynomials on every cell that possess clearer structure and more friendly implementation. In this paper, we present three finite element spaces on grids that consist of general quadrilaterals, and they form a discrete analogue of the de Rham complex which reads in 2D

\[
\mathbb{R} \xrightarrow{\text{inclusion}} H^1 \xrightarrow{\nabla} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \xrightarrow{\text{integration}} \mathbb{R}.
\]

There have been various finite element schemes on general quadrilateral grids with piecewise polynomials; we refer to [8,9,11,15] for example. Also, the first finite element complex with piecewise polynomials on general quadrilateral grids can be found in [15]. It is worthy of attention that all these aforementioned finite element spaces are nonconforming, except those for the space \(L^2\). Indeed, as will be proved in Section 2, it is impossible to construct a practically useful finite element whose shape functions are always piecewise polynomials and which can form conforming subspaces on a grid that consists of arbitrary quadrilaterals rather than parallelograms only. We thus do not seek to construct conforming elements; in view of (1), we impose the continuity of evaluation on vertices for the approximation of \(H^1\) functions and the continuity of the average of the tangential component along the edges for the approximation of \(H(\text{rot})\) functions, and these spaces can be called quasi-conforming ones. For low-degree polynomials, the continuity can guarantee the conformity of the respective finite element spaces on parallelograms, while on general quadrilaterals, the continuity does not even guarantee the functions to pass the patch test. In the analysis and the implementation, we will then adopt the general \(O(h^2)\) asymptotic-parallelogram regularity assumption for the convergence. This assumption is normally used for the analysis of nonconforming finite element schemes. Further, due to the lack of full elliptic regularity, for the nonconforming finite element scheme of the \(H(\text{curl})\) problem, higher than general regularity assumption may in general be needed for the same convergence rate. In contrast, by the aid of this assumption, we can prove the optimal convergence rate for the proposed elements with the proper regularity assumptions of the exact solutions. Though is crucial for our lowest-degree case, the assumption can be weakened for higher degree schemes, such as [1,6].

The remaining of the paper is organized as follows. In the remaining of this section, we introduce some necessary notations. In Section 2, we introduce some geometrical features of the grids. We will particularly prove that, shortly speaking, no practically useful conforming element with piecewise polynomials can be constructed for general quadrilateral grids. In Section 3, three finite elements are introduced with a commutative diagram. The approximation error is analyzed by the technique of combining the classical Taylor expansion procedure and a specific commutative property. In Section 4, the modulus of the continuity of the finite element functions are given. Then in Section 5, the performance of the finite elements are studied for the
$H^1$ and $H(\text{rot})$ problems; both theoretical analysis and numerical verifications are given. Some conclusions and comments are given in Section 6.

**Notations.** In this paper, conventional notations for the Sobolev spaces and grid-related quantities will be used. Let $\Omega \subset \mathbb{R}^2$ be a simple connected Lipschitz domain and $\Gamma = \partial \Omega$ be the piecewise boundary with $\eta$ the outward unit normal vector and $t$ the counterclockwise unit tangential vector. “$\tilde{\cdot}$” representing the vector valued quantities. Denote by $H^m(\Omega)$ and $H^m_0(\Omega)$ the standard Sobolev spaces equipped with the norm $\| \cdot \|_{m, \Omega}$ and seminorm $| \cdot |_{m, \Omega}$ as usual, and $L^2_0(\Omega) := \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \}$. We also denote by $L^2(\Omega) = (L^2(\Omega))^2$, $H^m(\Omega) = (H^m(\Omega))^2$ and $H^m_0(\Omega) = (H^m_0(\Omega))^2$. The inner product of $L^2$ and $L^2$ is denoted by $(\cdot, \cdot)$ on the domain $\Omega$. We define two forms of rotation operator in two-dimensional case by

$$\text{Given a vector } \sigma(x_1, x_2) = (\sigma_1, \sigma_2)^T \quad \text{rot} \sigma = \frac{\partial \sigma_2}{\partial x_1} - \frac{\partial \sigma_1}{\partial x_2}$$

$$\text{Given a scalar function } \sigma = \sigma(x_1, x_2) \quad \text{curl} \sigma = \left( \frac{\partial \sigma}{\partial x_2}, -\frac{\partial \sigma}{\partial x_1} \right)^T.$$

Superscript $T$ indicates transposition of vector or matrix as usual. We also use these notations to denote Sobolev spaces $H(\text{rot}, \Omega) = \{ \sigma | \sigma \in L^2(\Omega), \text{rot} \sigma \in L^2(\Omega) \}$ and $H_0(\text{rot}, \Omega) = \{ \sigma | \sigma \in H(\text{rot}, \Omega), \sigma \cdot t = 0 \text{ on } \Gamma \}$ equipped with standard norm $\| \cdot \|_{\text{rot}, \Omega} = \| \cdot \|_{0, \Omega}^2 + | \cdot |_{0, \Omega}^2$. Specially, a new notation is used for the space $H^1(\text{rot}, \Omega) \doteq \{ \sigma | \sigma \in H^1(\Omega), \text{rot} \sigma \in H^1(\Omega) \}$.

Let $\mathcal{J}_h$ be a regular subdivision of domain $\Omega$, with the elements being convex quadrilaterals, i.e., $\Omega = \bigcup_{K \in \mathcal{J}_h} K$. And any two distinct quadrilaterals $K_1$ and $K_2$ in $\mathcal{J}_h$ with $\bar{K}_1 \cap \bar{K}_2 \neq \emptyset$, share exactly one vertex or have one edge in common. Denote a finite element $(K, P_K, D_K)$ by Ciarlet’s triple [5], the subscription $K$ implying the dependence of the quadrilateral $K$. Let $\mathcal{N}_h$ denote the set of all the vertexes, $\mathcal{N}_h = \mathcal{N}_h^i \cup \mathcal{N}_h^b$, with $\mathcal{N}_h^i$ and $\mathcal{N}_h^b$ consisting of the interior vertexes and the boundary vertexes, respectively. Similarly, let $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ denote the set of all the edges, with $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ consisting of the interior edges and the boundary edges, respectively. The subscript $h$ in various notations implies the dependence of the subdivision. Denote by $h_K$ the diameter of each quadrilateral $K$ and the grid size $h \doteq \max_{K \in \mathcal{J}_h} h_K$. On the edge $e$, we use $\llbracket \cdot \rrbracket_e$ for the jump across $e$.

Throughout the paper we denote by $C$ a positive constant not necessarily the same at each occurrence but always independent of the diameter $h_K$ or the grid size $h$. Denote $\lambda_{F,G}$ by the
generalized eigenvalue of matrix pair \((F, G)\), i.e., \(Fx = \lambda_F G x\). We use notations \(P_e\) and \(P_K\) to denote the average of the integral on the edge \(e\) and quadrilateral \(K\), respectively.

2. Geometry of the quadrilaterals

2.1. Quadrilateral and functions. Let \(K\) be a convex quadrilateral with \(A_i\) the vertices and \(e_i\) the edges, \(i = 1 : 4\), see Figure 1. Let \(m_i\) be the mid-point of \(e_i\), then the quadrilateral \(\square m_1m_2m_3m_4\) is a parallelogram. The cross point of \(m_1m_3\) and \(m_2m_4\), which is labelled as \(O\), is the midpoint of both \(m_1m_3\) and \(m_2m_4\). Denote \(r = \overrightarrow{Om_4}\) and \(s = \overrightarrow{Om_1}\). Then, the coordinates of the vertices in the coordinate system \(rOs\) are \(A_1(1 + \alpha, 1 + \beta)\), \(A_2(-1 + \alpha, 1 - \beta)\), \(A_3(-1 + \alpha, -1 + \beta)\), \(A_4(1 - \alpha, -1 - \beta)\) and for some \(\alpha, \beta\). Since \(K\) is convex, \(|\alpha| + |\beta| < 1\). Without loss of generality, we assume \(\alpha > 0, \beta > 0\) and \(r \times s > 0\). Here and after, we call \(\alpha, \beta\) local shape parameters.

Define the shape regularity indicator of the quadrilateral \(K\) by \(R_K = \max\{|r|, |s|, |r \times s|\}\). Evidently \(R_K \geq 1\), and \(R_K = 1\) if and only if \(K\) is a square. A given family of quadrilateral subdivisions \(\{\mathcal{S}_h\}\) of \(\Omega\) is called regular, if all the shape regularity indicators of the quadrilaterals of all the subdivisions are uniformly bounded.

Define two linear functions \(\xi\) and \(\eta\) by \(\xi(ar + bs) = a\) and \(\eta(ar + bs) = b\). The two functions play the same role on quadrilaterals as that played by barycentric coordinates on triangles. Additionally we also define two functions \(\hat{\xi}\) and \(\hat{\eta}\) by \(\hat{\xi} = \xi - \int_K \xi \, dx, \hat{\eta} = \eta - \int_K \eta \, dx\) for convenience of calculation in Section 4. Technically, we construct two tables about the evaluation of some functions which will be useful in theoretical analysis and numerical computation.
Table 1. boundary integral evaluation of some functions

| Function | $\xi^2$ | $\xi \eta$ | $\eta^2$ |
|----------|---------|-----------|---------|
| $\int_{e_1}$ | $(1+\alpha^2) |e_1|/3$ | $(1+\alpha) |e_1|/3$ | $(3+\beta^2)|e_1|/3$ |
| $\int_{e_2}$ | $(3+\alpha^2) |e_2|/3$ | $\alpha |e_2|/3$ | $(1-\beta^2)|e_2|/3$ |
| $\int_{e_3}$ | $(1-\alpha^2) |e_3|/3$ | $(-1+\alpha) |e_3|/3$ | $(3+\beta^2)|e_3|/3$ |
| $\int_{e_4}$ | $(3+\alpha^2) |e_4|/3$ | $(1+\alpha) |e_4|/3$ | $(1+\beta^2)|e_4|/3$ |

Table 2. integral evaluation of some functions in domain K

| Function | $\int_K$ | $1$ | $\xi$ | $\eta$ | $\xi^2$ | $\xi \eta$ | $\eta^2$ |
|----------|----------|-----|-------|-------|-------|-------|-------|
| $\int_{e_1}$ | $4r \times s$ | $\frac{2\alpha}{3}r \times s$ | $\frac{\alpha}{3}r \times s$ | $\frac{4}{3}(1+\alpha^2)r \times s$ | $\frac{4}{3}(1+\alpha^2)r \times s$ | $\frac{4}{3}(1+\beta^2)r \times s$ |
| $\int_{e_2}$ | $\frac{4}{3}(1+\alpha^2)r \times s$ | $\frac{8}{3} \alpha \beta r \times s$ | $\frac{8}{3}(3+\beta^2) \alpha \beta r \times s$ |

2.2. Grid refinement. Denote by $d_K$ the distance between the midpoints of the diagonals of the quadrilateral $K$, then we introduce a lemma.

Lemma 1. All refined quadrilaterals produced by a bisection scheme of grid subdivisions have the property $d_K = O(h_K^2)$.

Proof. The proof can be found in [13, 16]. □

Here and after, we call the grid generated by the bisection scheme as an asymptotically parallelogram grid. We notice that the quantity $\max_{K \in J_h}\{\alpha_K, \beta_K\}$ is of order $O(h)$ uniformly for asymptotically regular parallelogram grid by Lemma 1. This proposition will be used frequently in the Section 4.

2.3. On the construction of conforming element with piecewise polynomials. Hereby, we prove that, as rigorously presented in the below theorem, for general quadrilateral grids, no practically useful conforming elements can be constructed with piecewise polynomials.

Theorem 2. Let $\text{FEM}_{pq} = (K, P_K, N_K)$ be a finite element defined by Ciarlet’s triple, with $K$ being any quadrilateral, and $P_K$ being a space of polynomials on $K$. If the finite element space generated by $\text{FEM}_{pq}$ by the continuity of nodal parameters is an $H^1$ subspace on any grid that consists of arbitrary quadrilaterals, then $P_K$ only contains polynomials that vanish on the boundary of $K$.

Remark 3. Shortly speaking, if a finite element $\text{FEM}_{pq}$, the subscripts “p” for polynomial and “q” for quadrilateral, can formulate continuous piecewise polynomial space on general quadrilateral grids, then the shape function space of the finite element consists of bubble functions on
This theorem shows the non-existence of practically useful conforming finite element defined by Ciarlet’s triple on general quadrilateral grids. However, we emphasize that it does not exclude the possibility that, on a given quadrilateral grid, a subspace of $H^1_0(\Omega)$ that consists of piecewise $k$-th degree polynomials can contain more than cell bubbles.

**Proof of Theorem 2.** By the assumption that FEM\(_{pq}\) can formulate an $H^1$ subspace on a grid that consists of arbitrary quadrilaterals, given a quadrilateral $K_1$ with $D$ being one of its vertices, without loss of generality, we assume there is a shape function (polynomial) $q$ defined on $K_1$, such that $q = 0$ along the two edges of which $D$ is **not** an endpoint and $q$ is nontrivial along one edge $e \ni D$ of $K_1$. By the continuity of the finite element space generated from FEM\(_{pq}\) by the continuity of nodal parameters and by the arbitrariness of choosing the evaluation of nodal parameters, for any groups of quadrilaterals (including $K_1$) which can form a patch $\omega_D$ centered at $D$ (see Figure 2 for a reference), there exists a finite element function $r \in H^1_0(\omega_D)$, such that $r|_{K_1} = q$. Again, we emphasize that the evaluation on these common nodal parameters do not inflect the evaluation of the finite element function on the edges which do not intersect with $K_1$, and this is why $r$ can be chosen in $H^1_0(\omega_D)$.

Now, since $r|_f = 0$, we can rewrite $r|_{K_2} = r_{-1} \cdot l_f$, where $l_f$ is a first degree polynomial which vanishes on $f$ and $r_{-1}$ is a polynomial with one degree lower than $r$. Without loss of generality, we assume $f$ is not parallel to $e$. By the continuity of $r$ on $e$, we can rewrite $q|_e = q_{-1}l_f |_e = q_{-1}(t - \theta)$, where $t$ is the length parameter of $e$, $q_{-1}$ is a polynomial on $e$ with one degree lower than $q$ and $\theta$ varies as the angle between $e$ and $f$ varies. Recall that $K_1$ and $q_{K_1}$ are fixed, but $f$ can be arbitrary. Change $f$ to another direction $f'$, by elementary calculation, it follows that $q|_e = q_{-2}(t - \theta)(t - \theta')$. This way, by repeating the procedure, we can see that $q|_e$ contains a polynomial factor with growing degree and thus can not be a nontrivial polynomial. This leads to a contradiction to the assumption that $q|_e \neq 0$ and completes the proof. \[\square\]

**Remark 4.** Similarly, conforming finite elements can not be defined for $H(\text{rot})$ with piecewise polynomials for general quadrilateral grids. Indeed, the assertion can be generalized to general Sobolev spaces.
3. A sequence of lowest-degree finite elements

3.1. Definitions of the finite elements. In the subsection we introduce three types of finite elements.

The quadrilateral finite element presented below is similar to the bilinear element on rectangle, and we call it the quadrilateral bilinear (QBL) element.

The QBL element is defined by \((K, P^\text{QBL}_K, D^\text{QBL}_K)\) with

1. \(K\) is a convex quadrilateral with vertexes \(A_i, i = 1 : 4\)
2. \(P^\text{QBL}_K \widehat{=} \text{span}\{1, \xi, \eta, \xi\eta\}\)
3. \(D^\text{QBL}_K \widehat{=} \{u(A_i), i = 1 : 4\}\) for any \(u \in H^2(K)\)

QBL element defined above is unisolvent. Indeed, define

\[
\begin{align*}
\phi_1 &= \frac{\alpha + \beta - 1}{4(\alpha^2 + \beta^2 - 1)}\xi\eta + \frac{(\beta - 1)(-\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\xi + \frac{(\alpha - 1)(\alpha - \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\eta + \frac{-(\alpha - 1)(\beta - 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)} \\
\phi_2 &= \frac{-\alpha + \beta + 1}{4(\alpha^2 + \beta^2 - 1)}\xi\eta + \frac{-(\beta + 1)(\alpha + \beta - 1)}{4(\alpha^2 + \beta^2 - 1)}\xi + \frac{(\alpha - 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\eta + \frac{(\alpha - 1)(\beta + 1)(\alpha - \beta + 1)}{4(\alpha^2 + \beta^2 - 1)} \\
\phi_3 &= \frac{-(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\xi\eta + \frac{-(\beta + 1)(\alpha - \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\xi + \frac{(\alpha + 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\eta + \frac{(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)} \\
\phi_4 &= \frac{\alpha - \beta + 1}{4(\alpha^2 + \beta^2 - 1)}\xi\eta + \frac{-(\beta - 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\xi + \frac{-(\alpha + 1)(\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\eta + \frac{(\alpha + 1)(\beta - 1)(-\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}
\end{align*}
\]

then we can verify directly \(\phi_i(A_j) = \delta_{ij}, i, j = 1 : 4\).

Here and after, the functions \(\{\phi_i\}_{i=1:4}\) are called local basis of \(P^\text{QBL}_K\). We use the notation \(\phi_i^{(j)}\) to denote \(j\)-th coefficient of \(i\)-th basis, for example, \(\phi_1^{(2)} = \frac{(\beta - 1)(-\alpha + \beta + 1)}{4(\alpha^2 + \beta^2 - 1)}\).

Given a QBL element \((K, P^\text{QBL}_K, D^\text{QBL}_K)\), define the local interpolation operator \(J_K\) by

\[
J_Ku = \sum_{i=1}^{4} u(A_i)\phi_i, \quad \forall u \in H^2(K).
\]

Furthermore, given a family of QBL elements \((K_i, P^\text{QBL}_{K_i}, D^\text{QBL}_{K_i})\) in a subdivision \(\mathcal{J}_h\), define the global interpolation operator \(J_h\) by

\[
J_hu|_{K_i} = J_{K_i}u, \quad \forall K_i \in \mathcal{J}_h.
\]

The quadrilateral finite element presented below is similar to the Raviart-Thomas element on rectangle, and we call it the quadrilateral Raviart-Thomas (QRT) element.
The QRT element is defined by \((K, P^{\text{QRT}}_K, D^{\text{QRT}}_K)\) with

1. \(K\) is a convex quadrilateral with edges \(e_i, i = 1:4\)
2. \(P^{\text{QRT}}_K \cong \text{span}[\nabla \xi, \nabla \eta, \xi \nabla \eta, \eta \nabla \xi]\)
3. \(D^{\text{QRT}}_K \cong \{ \int_{e_i} \sigma \cdot t_j \, ds, i = 1:4 \} \) for any \(\sigma \in H^1(K)\)

Here \(t_j\) is the unit tangential vector of \(e_i\) respectively and the positive direction is counterclockwise along \(\partial K\).

The QRT element as above is unisolvent. Indeed, define

\[
\begin{align*}
\phi_1 &= \frac{(1 - \alpha)(1 - \beta^2)|e_1|}{4(\alpha^2 + \beta^2 - 1)} \nabla \xi + \frac{\alpha(1 - \alpha)\beta|e_1|}{4(\alpha^2 + \beta^2 - 1)} \nabla \eta + \frac{-\alpha(1 - \alpha)|e_1|}{4(\alpha^2 + \beta^2 - 1)} \xi \nabla \eta + \frac{(1 - \alpha - \beta^2)|e_1|}{4(\alpha^2 + \beta^2 - 1)} \eta \nabla \xi \\
\phi_2 &= \frac{\alpha \beta(1 + \beta)|e_2|}{4(\alpha^2 + \beta^2 - 1)} \nabla \xi + \frac{(1 - \alpha^2)(1 + \beta)|e_2|}{4(\alpha^2 + \beta^2 - 1)} \nabla \eta + \frac{-\alpha(1 + \beta)|e_2|}{4(\alpha^2 + \beta^2 - 1)} \xi \nabla \eta + \frac{\beta(1 + \beta)|e_2|}{4(\alpha^2 + \beta^2 - 1)} \eta \nabla \xi \\
\phi_3 &= \frac{-\alpha(1 + \alpha)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \nabla \xi + \frac{(1 - \alpha^2)(1 + \beta)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \nabla \eta + \frac{-\beta(1 + \beta)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \xi \nabla \eta + \frac{2\beta(1 - \beta)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \eta \nabla \xi \\
\phi_4 &= \frac{(1 - \alpha)(1 - \beta^2)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \nabla \xi + \frac{(1 - \alpha)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \nabla \eta + \frac{\beta(1 - \beta)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \xi \nabla \eta + \frac{(1 - \alpha - \beta^2)|e_4|}{4(\alpha^2 + \beta^2 - 1)} \eta \nabla \xi.
\end{align*}
\]

Denote \(\{D^{\text{QRT}}_i\}_{i=1:4}\) by the components of \(D^{\text{QRT}}_K\), then we can verify directly \(D^{\text{QRT}}_i(\phi_j) = \delta_{ij}\), \(i, j = 1:4\).

Here and after, the functions \(\{\phi_i\}_{i=1:4}\) are called local basis of \(P^{\text{QRT}}_K\).

Given the QRT element \((K, P^{\text{QRT}}_K, D^{\text{QRT}}_K)\), define the local interpolation operator \(\nabla_K\) by

\[
\nabla_K \sigma = \sum_{i=1}^{4} D^{\text{QRT}}_i(\sigma) \phi_i \quad \forall \sigma \in H^1(K).
\]

Furthermore, given a family of QRT elements \((K_i, P^{\text{QRT}}_{K_i}, D^{\text{QRT}}_{K_i})\) in a subdivision \(\mathcal{J}_h\), define the global interpolation operator \(\nabla_h\) by

\[
\nabla_h \sigma|_{K_i} = \nabla_K \sigma \quad \forall K_i \in \mathcal{J}_h.
\]

Finally, for any \(q \in L^2(\Omega)\) define the interpolation operator \(P_h\) by \(P_h q|_{K_i} = P_{K_i} q, \forall K_i \in \mathcal{J}_h\).

3.1.1. Exact sequences on a quadrilateral.
Theorem 5. The commutative diagram holds as below:

\[
\begin{array}{cccccc}
\mathbb{R} & \rightarrow & H^2(K) & \triangledown & H^1(K) & \text{rot} & L^2(K) & \text{rot} & P_K & \rightarrow & \mathbb{R} \\
\downarrow J_K & & \downarrow \cap_K & & \downarrow P_K & & \downarrow \cap_K & & \downarrow P_K & & \downarrow P_K & & \rightarrow & \mathbb{R} \\
\mathbb{R} & \rightarrow & P^\text{QBL}_K & \triangledown & P^\text{QRT}_K & \rightarrow & \mathbb{R} & \rightarrow & \mathbb{R}.
\end{array}
\]

Proof. We first prove the discretized de Rham complex. Evidently \( \ker(\triangledown) = \mathbb{R} \) and \( \nabla P^\text{QBL}_K \subset P^\text{QRT}_K \). On the other hand, \( \text{rot} P^\text{QRT}_K = \mathbb{R} \). It remains to prove that \( \ker(\text{rot}) = \nabla P^\text{QBL}_K \). Given a \( \tau \in P^\text{QRT}_K \), such that \( \text{rot} \tau = 0 \). Since \( \tau = d_1 \nabla \xi + d_2 \nabla \eta + d_3 \xi \nabla \eta + d_4 \eta \nabla \xi \), then we have \( d_3 = d_4 \) and \( \tau \in \nabla P^\text{QBL}_K \).

Then we are going to show that \( \nabla J_K = \cap_K \nabla \) on \( H^2(K) \), and \( \text{rot} \cap_K = P_K \text{rot} \) on \( H^1(K) \). We first prove the former. Given a \( \sigma \in H^1(K) \), let \( \cap_K \sigma = g_1 \nabla \xi + g_2 \nabla \eta + g_3 \xi \nabla \eta + g_4 \eta \nabla \xi \). By definition, we have

\[
\frac{(1 - \alpha)(1 - \beta^2)\xi_1 \cdot P_{e_1}(\sigma) + \alpha \beta(1 + \beta)\xi_2 \cdot P_{e_2}(\sigma) - (1 + \alpha)(1 - \beta^2)\xi_2 \cdot P_{e_1}(\sigma) - \alpha \beta(1 - \beta)\xi_4 \cdot P_{e_4}(\sigma)}{4(\alpha^2 + \beta^2 - 1)},
\]

\[
\frac{\alpha(1 - \alpha)\xi_1 \cdot P_{e_1}(\sigma) + (1 - \alpha^2)(1 + \beta)\xi_2 \cdot P_{e_2}(\sigma) - \alpha(1 + \alpha)\xi_3 \cdot P_{e_3}(\sigma) - (1 - \alpha^2)(1 - \beta)\xi_4 \cdot P_{e_4}(\sigma)}{4(\alpha^2 + \beta^2 - 1)},
\]

\[
\frac{-\alpha(1 - \alpha)\xi_1 \cdot P_{e_1}(\sigma) - (1 - \alpha^2 + \beta)\xi_2 \cdot P_{e_2}(\sigma) + \alpha(1 + \alpha)\xi_3 \cdot P_{e_3}(\sigma) - (1 - \alpha^2 - \beta)\xi_4 \cdot P_{e_4}(\sigma)}{4(\alpha^2 + \beta^2 - 1)},
\]

\[
\frac{(1 - \alpha - \beta^2)\xi_1 \cdot P_{e_1}(\sigma) - \beta(1 + \beta)\xi_2 \cdot P_{e_2}(\sigma) + (1 + \alpha - \beta^2)\xi_3 \cdot P_{e_3}(\sigma) + \beta(1 - \beta)\xi_4 \cdot P_{e_4}(\sigma)}{4(\alpha^2 + \beta^2 - 1)}.
\]

Now given a \( u \in H^2(K) \), we take \( \sigma = \nabla u \in H^1(K) \) and the former follows by simple calculation.

It remains to prove the latter. Since \( \nabla \xi = \frac{\partial}{\partial x} \xi \times \frac{\partial}{\partial y} \xi \), \( \nabla \eta = \frac{\partial}{\partial x} \eta \times \frac{\partial}{\partial y} \eta \), then for \( \sigma \in H^1(K) \) it holds

\[
\text{rot}(\cap_K \sigma) = (g_3 - g_4) \nabla \xi \times \nabla \eta = \frac{g_3 - g_4}{\xi \times \eta} = \frac{1}{4\xi \times \eta} \int_{\partial K} \sigma \cdot t \, ds = P_K(\text{rot} \sigma).
\]

The proof is completed. \( \square \)

3.2. Interpolation error estimation.
3.2.1. Interpolation error estimations in $L^2$ norm.

**Theorem 6.** Let $K$ be a convex quadrilateral, then it holds

$$\|u - J_K u\|_{0,K} \leq C h_K^2 |u|_{2,K} \quad \forall u \in H^2(K).$$

**Proof.** By density, it suffices to consider $u \in C^2(\bar{K})$. Let $A$ be any point in the quadrilateral $K$ with vertexes $\{A_i\}_{i=1,4}$. Using Taylor expansion with integral remainder, we have

$$u(A) = u(A) + \nabla u(A) \cdot (A_i - A) + R_i(A), \quad R_i(A) = \int_0^1 (1 - t) \frac{d^2 u}{dt^2} (\xi_i, \eta_i) \, dt.$$

Here $\xi_i = tx_i + (1 - t)x, \eta_i = ty_i + (1 - t)y$.

Since $J_K$ preserves linear polynomial

$$J_K u(A) = \sum_{i=1}^4 u(A_i) \phi_i(A) = u(A) + \sum_{i=1}^4 R_i(A) \phi_i(A).$$

Then we obtain

$$u(A) - J_K u(A) = - \sum_{i=1}^4 R_i(A) \phi_i(A).$$

Evidently

$$|R_i(A)|^2 = \left| \int_0^1 (1 - t)^2 \frac{\partial^2 u}{\partial \xi_i \partial \eta_i} (x_i - x)^2 + 2 \frac{\partial^2 u}{\partial \xi_i \partial \eta_i} (x_i - x)(y_i - y) + \frac{\partial^2 u}{\partial \eta_i^2} (y_i - y)^2 \right| dt^2$$

$$\leq 4 h_K^4 \int_0^1 (1 - t)^2 \sum_{|m|\geq 2} |\partial^m u(\xi_i, \eta_i)|^2 \, dt$$

$$\|u - J_K u\|_{0,K}^2 \leq C h_K^4 \sum_{i=1}^4 \sum_{|m|\geq 2} \int_0^1 (1 - t)^2 \int_\mathcal{K} |\partial^m u(\xi_i, \eta_i)|^2 \, dx \, dy \, dt.$$

Take integral variable substitution: $d\xi_i = (1 - t)dx, d\eta_i = (1 - t)dy$, then we have

$$\|u - J_K u\|_{0,K}^2 \leq C h_K^4 \sum_{i=1}^4 \sum_{|m|\geq 2} \int_0^1 \int_\mathcal{K} |\partial^m u(\xi_i, \eta_i)|^2 \, d\xi_i \, d\eta_i \, dt = C h_K^4 |u|_{2,K}^2.$$

The proof is completed. \qed

**Theorem 7.** Let $K$ be a convex quadrilateral, then it holds

$$\|\sigma - \cap_K \sigma\|_{0,K} \leq C h_K |\sigma|_{1,K} \quad \forall \sigma \in H^1(K).$$
We postpone the proof of the theorem after a technical lemma. Define a new interpolation operator \( \nabla Q : H^1(K) \to \text{span}[1, \xi, \eta, \xi^2 - \eta^2] \) by \( \int_K \varsigma \, \text{ds} = \int_{K_i} \nabla Q \varsigma \, \text{ds} , i = 1 : 4 \). Evidently \( \nabla Q \) is well-defined.

**Lemma 8.** The local interpolation operator \( \nabla K \) is \( H^1 \) stable, namely

\[
| \nabla K \varsigma |_{1,K} \leq C | \nabla \varsigma |_{1,K} \quad \forall \varsigma \in H^1(K).
\]

**Proof.** Let \( \nabla Q \varsigma = d_1 \cdot 1 + d_2 \cdot \xi + d_3 \cdot \eta + d_4 \cdot (\xi^2 - \eta^2) \), then by definition we have

\[
d_1 = \frac{\alpha^2 - \beta^2 + 2}{8} P_1(\sigma) + \frac{-\alpha^2 + \beta^2 + 2}{8} P_2(\sigma) + \frac{\alpha^2 - \beta^2 + 2}{8} P_3(\sigma) + \frac{-\alpha^2 + \beta^2 + 2}{8} P_4(\sigma)
\]

\[
d_2 = -\beta \frac{4}{P_1(\sigma)} + \beta \frac{2}{4} P_2(\sigma) - \beta \frac{4}{P_3(\sigma)} + \frac{\beta + 2}{4} P_4(\sigma)
\]

\[
d_3 = \frac{\alpha}{4} P_1(\sigma) - \frac{\alpha}{4} P_2(\sigma) + \frac{\alpha - 2}{4} P_3(\sigma) + \frac{\alpha}{4} P_4(\sigma)
\]

\[
d_4 = -\frac{3}{8} P_1(\sigma) + \frac{3}{8} P_2(\sigma) - \frac{3}{8} P_3(\sigma) + \frac{3}{8} P_4(\sigma).
\]

Denote by row vector \( \sigma = (\sigma_1, \sigma_2), p_1 = (P_1(\sigma_i))_{i=1:4}, p_2 = (P_1(\sigma_i))_{i=1:4}, \) and column vector \( p = (p_1, p_2)^T \). Denote \( \zeta \) by \( = \) up to a constant, independent of diameter \( h_K \), then

\[
\int_K |\nabla (\nabla Q \varsigma)|^2 \, \text{dx} = h_K^2 \int_K |\partial_\xi (\nabla Q \varsigma)|^2 + |\partial_\eta (\nabla Q \varsigma)|^2 \, \text{dx} = h_K^2 \int_K |d_2 + 2d_4 \xi^2 + 2d_3 - 2d_4 \eta^2| \, \text{dx}
\]

\[
\leq |d_2|^2 + |d_3|^2 + |d_4|^2 = p_1 B_1^T B_1 p_1 + p_2 B_1^T B_1 p_2 = p^T B p.
\]

Here \( p \in \mathbb{R}^8 \) and

\[
B_1 = \begin{bmatrix}
-\beta & -\beta & -\beta & -\beta & -\beta & -\beta & -\beta & -\beta \\
\frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} & \frac{\beta}{4} \\
\frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} \\
\frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1^T B_1 & 0 \\
0 & B_1^T B_1
\end{bmatrix}.
\]

Similarly, let \( \nabla K \varsigma = g_1 \cdot \nabla \xi + g_2 \cdot \nabla \eta + g_3 \cdot \xi \nabla \eta + g_4 \cdot \eta \nabla \xi \), recalling \{g_i\}_{i=1:4} \) in Theorem 5, then there exists a semi-positive matrix \( D \) with \( \mathcal{O}(1) \) elements such that \( | \nabla K \varsigma |_{1,K}^2 = p^T D p \).

Since \( \nabla Q \) is \( H^1 \) stable (see [15]), it remains to show that \( | \nabla Q \varsigma |_{1,K} \leq C | \nabla Q \varsigma |_{1,K} \).
We first show that $\ker B \subset \ker D$. Given a $p \in \ker B$, then $|Q_\sigma|^2_{1,K} = 0$ and $\nabla Q_\sigma$ is a constant vector. Since $\nabla K_\sigma = \nabla K(\nabla Q_\sigma)$, thus $\nabla K_\sigma$ is also a constant vector and $|\nabla K_\sigma|^2_{1,K} = 0$, i.e. $p \in \ker D$.

Subsequently, we calculate all the eigenvalues of $B_1^T B_1$, (below we use $\lambda_i = \lambda_i(B_1^T B_1)$, $i = 1 : 4$)

$$
\begin{align*}
\lambda_1 &= 0 \\
\lambda_2 &= \frac{1}{2} \\
\lambda_3 &= \frac{17}{32} + \frac{a^2 + \beta^2}{8} - \frac{\sqrt{16(a^4 + \beta^4) + 32a^2\beta^2 + 136(a^2 + \beta^2)} + 1}{32} \\
\lambda_4 &= \frac{17}{32} + \frac{a^2 + \beta^2}{8} + \frac{\sqrt{16(a^4 + \beta^4) + 32a^2\beta^2 + 136(a^2 + \beta^2)} + 1}{32}.
\end{align*}
$$

The eigenvalues $\lambda(B) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of matrix $B$ are all double eigenvalues. Let $\{\nu_1, \nu_2\}$ be two eigenvectors subordinating to the eigenvalue $\lambda_1 = 0$. Then we can decompose $\mathbb{R}^8 = \text{span}\{\nu_1, \nu_2\} \oplus (\text{span}\{\nu_1, \nu_2\})^\perp$. Suppose $p = \psi_1 + \psi_2$, $\psi_1 \in \text{span}\{\nu_1, \nu_2\}$, $\psi_2 \in (\text{span}\{\nu_1, \nu_2\})^\perp$.

Rayleigh quotient theorem reads

$$
|\nabla K_\sigma|^2_{1,K} = p^T D p = \psi_2^T D \psi_2 \leq \lambda_{\max}(D)\|\psi_2\|^2,
$$

$$
|Q_\sigma|^2_{1,K} = p^T B p = \psi_2^T B \psi_2 \geq \min(\lambda_2, \lambda_3, \lambda_4)\|\psi_2\|^2.
$$

This finishes the proof. □

**Proof of Theorem 7.** Dividing a quadrilateral $K$ along the diameter into two triangles, we have the following estimation from the similar argument referred to Theorem 5.1 in [4].

$$
\|\sigma - \nabla K_\sigma\|_{0,K} \leq C h_K \|\nabla(\sigma - \nabla K_\sigma)\|_{0,K}.
$$

This proves (8) by Lemma 8.

**Theorem 9.** Let $K$ be a convex quadrilateral, then it holds

$$
\|q - P_K q\|_{0,K} \leq C h_K \|q\|_{1,K} \quad \forall q \in H^1(K).
$$

**Proof.** The proof can be found in [10]. □

**3.2.2. Interpolation error estimations in energy norms.**

**Theorem 10.** Let $K$ be a convex quadrilateral, then it holds

$$
|u - J_K u|_{1,K} \leq C h_K |u|_{2,K} \quad \forall u \in H^2(K),
$$

(9)
and

\begin{equation}
|\sigma - \nabla K \nabla \sigma|_{\text{rot}, K} \leq C h_K |\text{rot} \sigma|_{1, K} \quad \forall \sigma \in H^1(\text{rot}, K).
\end{equation}

\textbf{Proof.} We prove the estimations by the commutative diagrams. Since \( \nabla (J_K) = \nabla K \nabla \) on \( H^2(K) \), we have

\[ |u - J_K u|_{1, K} = \| \nabla u - \nabla K \nabla u \|_{0, K} \leq C h_K |u|_{2, K}. \]

Similarly, since \( \text{rot} \nabla K \nabla = P_K \text{rot} \) on \( H^1(K) \), we have

\[ \| \text{rot} \sigma - \text{rot}(\nabla K \nabla) \|_{0, K} = \| \text{rot} \sigma - P_K (\text{rot} \sigma) \|_{0, K} \leq C h_K |\text{rot} \sigma|_{1, K}. \]

The proof is completed. \( \square \)

3.3. Finite element spaces on a grid \( \mathcal{J}_h \).

\textbf{Definition 11.} Associated with the QBL element, define the finite element spaces \( V_{h}^{\text{QBL}} \) and \( V_{h0}^{\text{QBL}} \) by

\[ V_{h}^{\text{QBL}} \triangleq \{ v_h \in L^2(\Omega) : v_h|_K \in P_{K}^{\text{QBL}}, \text{v}_h \text{ is continuous at two endpoints of edge } e \in \mathcal{E}_h \}, \]

and \( V_{h0}^{\text{QBL}} \triangleq \{ v_h \in V_{h}^{\text{QBL}} : v_h = 0 \text{ at two endpoints of edge } e \in \mathcal{E}_h \}. \)

\textbf{Definition 12.} Associated with the QRT element, define the finite element spaces \( V_{h}^{\text{QRT}} \) and \( V_{h0}^{\text{QRT}} \) by

\[ V_{h}^{\text{QRT}} \triangleq \{ \tau_h \in L^2(\Omega) : \tau_h|_K \in P_{K}^{\text{QRT}}, \int_e \tau_h \cdot t_e \, ds \text{ is continuous at the edge } e \in \mathcal{E}_h \}, \]

and \( V_{h0}^{\text{QRT}} \triangleq \{ \tau_h \in V_{h}^{\text{QRT}} : \int_e \tau_h \cdot t_e \, ds = 0 \text{ at the edge } e \in \mathcal{E}_h \}. \)

\textbf{Definition 13.} Define the piecewise constant finite element spaces \( W_h \) and \( W_{h0} \) by

\[ W_h \triangleq \{ q_h \in L^2(\Omega) : q_h|_K = P_K q, q \in L^2(\Omega) \}, \quad \text{and} \quad W_{h0} \triangleq \{ q_h \in W_h : \int_{\Omega} q_h \, dx = 0 \}. \]

The properties on a single cell can be generalized on a grid. We firstly adopt a lemma below.

\textbf{Lemma 14.} It holds on the subdivision that

\begin{equation}
\sup_{\tau_h \in V_{h}^{\text{rot}}} (\text{rot} \tau_h, q_h) \geq C \| \tau_h \|_{\text{rot}, h} \| q_h \|_{0, \Omega}, \quad \text{for any } q_h \in W_h,
\end{equation}
and
\[
\sup_{\tau_h \in V_{h0}^{QRT}} (\text{rot}_h \tau_h, q_h) \geq C \|\tau_h\|_{\text{rot}_h} \|q_h\|_{0,\Omega}, \quad \text{for any } q_h \in W_{h0}.
\]

**Proof.** Given \( q_h \in W_h \), there exists a \( \tau \in H^1(\Omega) \), such that \( \text{rot}\tau = q_h \), and \( \|\tau\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega} \).

Set \( \tau_h := \cap_h \tau \), then \( \text{rot}_h \tau_h = q_h \), and \( \|\tau_h\|_{\text{rot}_h} \leq C \|\tau\|_{1,\Omega} \). This proves (11). Similarly is (12) proved.

**Theorem 15.** The commutative diagrams hold as below:

\[
\begin{array}{cccccc}
\mathbb{R} & \rightarrow & H^2(\Omega) & \xrightarrow{\nabla} & H^1(\Omega) & \xrightarrow{\text{rot}} & L^2(\Omega) & \xrightarrow{\text{h}} & \mathbb{R} \\
\downarrow J_h & & \downarrow \cap_h & & \downarrow P_h & & & & \\
\mathbb{R} & \rightarrow & V_h^{QBL} & \xrightarrow{\nabla} & V_h^{QRT} & \xrightarrow{\text{rot}_h} & W_h & \xrightarrow{\text{h}} & \mathbb{R},
\end{array}
\]

and
\[
\begin{array}{cccccc}
\{0\} & \rightarrow & H^2(\Omega) \cap H^1_0(\Omega) & \xrightarrow{\nabla} & H^1(\Omega) \cap H_0(\text{rot}, \Omega) & \xrightarrow{\text{rot}} & L^2_0(\Omega) & \xrightarrow{\text{h}} & \{0\} \\
\downarrow J_h & & \downarrow \cap_h & & \downarrow P_h & & & & \\
\{0\} & \rightarrow & V_{h0}^{QBL} & \xrightarrow{\nabla} & V_{h0}^{QRT} & \xrightarrow{\text{rot}_h} & W_{h0} & \xrightarrow{\text{h}} & \{0\}.
\end{array}
\]

**Proof.** We first consider (13). The commutativity is trivial by Theorem 5 and it remains us to verify the discretized de Rham complex by the standard dimension counting technique.

Evidently \( \ker(\nabla_h) = \mathbb{R} \) and \( \nabla_h V_{h0}^{QBL} \subset V_{h0}^{QRT} \). On the other hand, by (11), \( \text{rot}_h V_{h}^{QRT} = W_h \).

This way, (13) follows by noting that \( \dim(V_{h}^{QBL}) = \#(N_h) \) and \( \dim(V_{h}^{QRT}) = \#(\mathcal{E}_h) \), and that \( \dim(V_{h}^{QRT}) = \dim(V_{h}^{QBL}) + \dim(W_h) - 1 \) by the Euler formula.

Similarly is (14) proved. The proof is completed.

The error estimation of the global interpolator is the same as that of the respective local ones.

**Theorem 16.** There exists a constant \( C \) depending on the shape regularity of \( \mathcal{F}_h \) only, such that

1. \( \|u - J_h u\|_{0,\Omega} + h\|u - J_h u\|_{1,\Omega} \leq Ch^2\|u\|_{2,\Omega} \quad \forall u \in H^2(\Omega); \)
2. \( \|\sigma - \cap_h \sigma\|_{\text{rot}_h} \leq Ch(|\sigma|_{1,\Omega} + |\text{rot}\sigma|_{1,\Omega}) \quad \forall \sigma \in H^1(\text{rot}, \Omega); \)
3. \( \|q - P_h q\|_{0,\Omega} \leq Ch|q|_{1,\Omega} \quad \forall q \in H^1(\Omega). \)

4. Nonconforming finite element spaces and their modulus of continuity

In this section, we show that on a grid that consists of arbitrary quadrilaterals and satisfies the condition that the cells are asymptotically parallelograms, the spaces \( V_{h}^{QBL} \) and \( V_{h}^{QRT} \), though
not subspaces of $H^1$ and $H(\text{rot})$ respectively, the consistency can be controlled well. We begin with an analysis that $V^{\text{QBL}}_h$ is in general not continuous.

4.1. **Continuity and non-continuity of $V^{\text{QBL}}_{h0}$**. Let $\mathcal{G}_D$ be a patch with the center $D$ and four cells $K_1, K_2, K_3, K_4$, see Figure 3. Let $V^{\text{QBL}}_{h0}(\mathcal{G}_D)$ be QBL finite element space defined on $\mathcal{G}_D$ with zero boundary condition. Denote by local shape parameters $\alpha_i, \beta_i$ of $K_i$, $i = 1 : 4$. Here $\alpha_i = \beta_i = 0$ for $i = 2 : 4$.

![Figure 3. Illustration of a patch $\mathcal{G}_D$](image)

**Theorem 17.** For $\alpha_1, \beta_1 \neq 0$ and $v_h \neq 0 \in V^{\text{QBL}}_{h0}(\mathcal{G}_D)$, there exists a function $\varphi \in C^\infty(\mathcal{G}_D)$ such that

$$(\nabla \varphi, \nabla_h v_h) + (\Delta \varphi, v_h) \neq 0$$

**Proof.** Since $\alpha_i, \beta_i = 0$ for $i = 2 : 4$, then

$$(\nabla \varphi, \nabla_h v_h) + (\Delta \varphi, v_h) = \int_{\partial K_i} \frac{\partial \varphi}{\partial n}(v_h - q) \, ds.$$  

Here $q$ is the linear interpolation of $v_h$ on $e$ with respect to endpoints. Without loss of generality, we assume $\alpha_1 \neq 0$. Noticing that $\dim V^{\text{QBL}}_{h0}(\mathcal{G}_D) = 1$, we denote by $\phi_D$ the basis of $V^{\text{QBL}}_{h0}(\mathcal{G}_D)$ and $\phi_D|_{K_1} = \phi_1$ (see (2)), then $v_h|_{K_1} = v_h(D)\phi_1$. Let $\varphi$ be a linear polynomial whose gradient is $\xi_{K_1}/\xi_{K_1} \times \xi_{K_1}$, defined on the patch $\mathcal{G}_D$. Then the proof is completed by simple calculation. □

4.2. **Modulus of continuity of $V^{\text{QBL}}_{h0}$**. Define consistency functional $E(\zeta, v_h)$ by

$$E(\zeta, v_h) = (\zeta, \nabla_h v_h) + (\text{div} \zeta, v_h) \quad \text{for} \quad \zeta \in H^1(\Omega), v_h \in V^{\text{QBL}}_{h0}$$

$$E(\tilde{\zeta}, v_h) = (\tilde{\zeta}, \nabla_h v_h) + (\text{div} \tilde{\zeta}, v_h) \quad \text{for} \quad \tilde{\zeta} \in H^1_0(\Omega), v_h \in V^{\text{QBL}}_h \cap L^2_0(\Omega).$$
Let \{\phi_i\}_{i=1}^{4} be local basis of \mathcal{P}_k^{Q_{BL}}, then define by

\[ t_1 = \sum_{i=1}^{4} \phi_i^{(1)} v_h(A_i), \quad t_2 = \sum_{i=1}^{4} \phi_i^{(2)} v_h(A_i), \quad t_3 = \sum_{i=1}^{4} \phi_i^{(3)} v_h(A_i). \]

**Theorem 18.** For \( \zeta \in H^1(\Omega), v_h \in V_h^{Q_{BL}} \) or \( \zeta \in H^1_0(\Omega), v_h \in V_h^{Q_{BL}} \), it holds

\[
E(\zeta, v_h) \leq C h |\zeta|_{1, \Omega} |v_h|_{1, h}.
\]

**Proof.** Evidently the consistency functional can be decomposed into

\[
E(\zeta, v_h) = \sum_{e \in \delta \Omega_h} \int_e \zeta \cdot n_e [v_h - q] \, ds = \sum_{K \in \delta \Omega_h} \sum_{e \in \partial K} \int_e \zeta \cdot n_e (v_h - q) \, ds.
\]

Here \( q \) is the linear interpolation of \( v_h \) on \( e \) with respect to endpoints. Then by direct calculation, we have

\[
\int_{\partial K} \zeta \cdot n(v_h - q) \, ds = \beta_k t_1 \int_{e_1} \zeta \cdot n_1 \left( \frac{\xi^2}{1 + \alpha_k} - (1 + \alpha_k) \right) \, ds + \alpha_k t_1 \int_{e_2} \zeta \cdot n_2 \left( \frac{\eta^2}{1 + \beta_k} - (1 + \beta_k) \right) \, ds
\]

\[
+ \beta_k t_1 \int_{e_3} \zeta \cdot n_3 \left( \frac{\xi^2}{1 + \alpha_k} - (1 + \alpha_k) \right) \, ds + \alpha_k t_1 \int_{e_4} \zeta \cdot n_4 \left( \frac{\eta^2}{1 + \beta_k} - (1 + \beta_k) \right) \, ds
\]

\[
\leq C h t_1 |\zeta|_{1, K}.
\]

On the other hand, setting \( \ell = (t_1, t_2, t_3)^T \) and \( |v_h|_{1, K}^2 = \ell^T G \ell \), then

\[
G = \begin{bmatrix}
4(1 + \beta_k^2) S_k \cdot S_k - 8 \alpha_k \beta_k \ell \cdot S_k + 4(1 + \alpha_k^2) \ell \cdot \ell \\
3 \ell \times S_k & 4 \alpha_k S_k \cdot S_k - 4 \beta_k \ell \cdot S_k \\
4 \alpha_k S_k \cdot S_k - 4 \beta_k \ell \cdot S_k & 3 \ell \times S_k \\
3 \ell \times S_k & 3 \ell \times S_k
\end{bmatrix}
\]

By the generalized Rayleigh quotient theorem, then for \( F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and all \( \ell \in \mathbb{R}^3 \)

\[
\ell^T F \ell \leq \lambda_{F,G} \ell^T G \ell, \quad \lambda_{F,G} = \frac{9 \ell \cdot \ell}{4((3 - \alpha_k^2 + 3 \beta_k^2) S_k \cdot S_k - 4 \alpha_k \beta_k \ell \cdot S_k + (3 + 3 \alpha_k^2 - \beta_k^2) \ell \cdot \ell)}.
\]
In summary, we have
\[ E(\zeta, v_h) \leq C h \sum_{K \in \mathcal{T}_h} \| \zeta \|_{1,K} \| v_h \|_{1,K} \leq C h \| \zeta \|_{1,\Omega} \| v_h \|_{h}. \]

The proof is completed. \qed

4.3. **Modulus of continuity of \( V_{h_0}^{\text{QRT}} \).** Define consistency functional \( E(w, \tau_h) \) by

\[
(17) \quad E(w, \tau_h) = (w, \rot \tau_h) - (\text{curl } w, \tau_h) \quad \text{for } w \in H^1(\Omega), \forall \tau_h \in V_{h_0}^{\text{QRT}}
\]

\[
(18) \quad E(w, \tau_h) = (w, \rot \tau_h) - (\text{curl } w, \tau_h) \quad \text{for } w \in H_0^1(\Omega), \forall \tau_h \in V_h^{\text{QRT}}.
\]

Evidently the consistency functional can be decomposed into

\[
(19) \quad E(w, \tau_h) = \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \int_e (w - c_K) (\tau_h \cdot t_e - P_e(\tau_h \cdot t_e)) \, ds.
\]

Here \( c_K \) is an arbitrary constant.

**Theorem 19.** For \( w \in H^1(\Omega), \tau_h \in V_{h_0}^{\text{QRT}} \) or \( w \in H_0^1(\Omega), \tau_h \in V_h^{\text{QRT}}, \) it holds

\[
(20) \quad E(w, \tau_h) \leq C h \| w \|_{1,\Omega} \| \tau_h \|_{\text{rot},h}.
\]

**Proof.** Evidently, we have \( \inf_{c_K \in \mathbb{R}} \| w - c_K \|_{0,\partial K} \leq C h_K \| w \|_{1,K} \). Let \( \tau_h = \gamma_1 \nabla \xi + \gamma_2 \nabla \eta + \gamma_3 \hat{\xi} \nabla \eta + \gamma_4 \hat{\eta} \nabla \xi \) and \( \| \tau_h \|_{1,K}^2 = \gamma^T V \gamma \), with \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T \) and

\[
(21) \quad V = \begin{bmatrix}
4\xi_x \xi_x & -4\xi_x \xi_k & 0 & 0 \\
-4\xi_k \xi_x & 4\xi_k \xi_k & 0 & 0 \\
-4\xi_k \xi_k & 4\xi_k \xi_k & 0 & 0 \\
0 & 0 & 4(3+3\alpha_k^2-\beta_k)\xi_k \cdot \xi_k+36 & -8\alpha_k \beta_k \xi_k \cdot \xi_k-36 \\
0 & 0 & -8\alpha_k \beta_k \xi_k \cdot \xi_k-36 & 4(3+3\beta_k^2-\alpha_k^2)\xi_k \cdot \xi_k+36 \\
0 & 0 & 4(3+3\beta_k^2-\alpha_k^2)\xi_k \cdot \xi_k+36 & 9\xi_k \cdot \xi_k
\end{bmatrix}.
\]

On the other hand

\[
\| \tau_h \cdot t_e - P_e(\tau_h \cdot t_e) \|_{0,e}^2 = \int_e (\tau_h \cdot t_e - P_e(\tau_h \cdot t_e))^2 \, ds = \int_e \tau_h \cdot t_e (\tau_h \cdot t_e - P_e(\tau_h \cdot t_e)) \, ds,
\]
then by simple calculation

\[
\int_{e_1} \tau_h \cdot \xi_1 (\tau_h \cdot \xi_1 - P_e (\tau_h \cdot \xi_1)) \, ds
\]

\[
= \int_{e_1} (\gamma_3 \nabla \eta \cdot \xi_1 \xi_1 + \gamma_4 \nabla \xi \cdot \xi_1 \eta_1) \cdot (\gamma_3 \nabla \eta \cdot \xi_1 (\xi_1 - P_e (\xi_1)) + \gamma_4 \nabla \xi \cdot \xi_1 (\eta_1 - P_e (\eta_1))) \, ds
\]

\[
= \int_{e_1} (\gamma_3 \nabla \eta \cdot \xi_1)^2 \xi_1^2 + 2(\gamma_3 \nabla \eta \cdot \xi_1)(\gamma_4 \nabla \xi \cdot \xi_1)\xi_1 \eta_1 + (\gamma_4 \nabla \xi \cdot \xi_1)^2 \eta_1 (\eta_1 - 1) \, ds = \frac{4(1 + \alpha_K)^2 \beta_K^2}{3|e_1|} \chi^T U \chi.
\]

Here \( U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \). Similarly

\[
\| \tau_h \cdot \xi_2 - P_e(\tau_h \cdot \xi_2) \|^2_{0,e_2} = \frac{4\alpha_K^2 (-1 + \beta_K)^2}{3|e_2|} \gamma^T U \gamma
\]

\[
\| \tau_h \cdot \xi_3 - P_e(\tau_h \cdot \xi_3) \|^2_{0,e_3} = \frac{4(1 + \alpha_K)^2 \beta_K^2}{3|e_3|} \gamma^T U \gamma
\]

\[
\| \tau_h \cdot \xi_4 - P_e(\tau_h \cdot \xi_4) \|^2_{0,e_4} = \frac{4\alpha_K^2 (1 + \beta_K)^2}{3|e_4|} \gamma^T U \gamma.
\]

By the generalized Rayleigh quotient theorem, then for all \( \gamma \in \mathbb{R}^4 \)

\[
\| \tau_h \cdot \xi_e - P_e(\tau_h \cdot \xi_e) \|^2_{0,K} \leq C \lambda_{U,V} h_K \| \tau_h \|^2_{0,K}.
\]

Here \( \lambda_{U,V} = p(\alpha_K, \beta_K, \xi_K, \xi_K) / q(\alpha_K, \beta_K, \xi_K, \xi_K) \) with

\[
p(\alpha_K, \beta_K, \xi_K, \xi_K) = 9((3 + 3\alpha_K^2 - \beta_K^2)\xi_K \cdot \xi_K + 4\alpha_K\beta_K \xi_K \cdot \xi_K + (3 + 3\beta_K^2 - \alpha_K^2)\xi_K \cdot \xi_K + 36)\xi_K \times \xi_K
\]

and

\[
q(\alpha_K, \beta_K, \xi_K, \xi_K) = 4((3 + 3\alpha_K^2 - \beta_K^2)(3 + 3\beta_K^2 - \alpha_K^2)(\xi_K \cdot \xi_K)(\xi_K \cdot \xi_K) - 4\alpha_K^2 \beta_K^2 (\xi_K \cdot \xi_K)^2
\]

\[
+ (27 - 9\alpha_K^2 + 27\beta_K^2) \xi_K \cdot \xi_K - 36\alpha_K\beta_K(\xi_K \cdot \xi_K + (27 + 27\alpha_K^2 - 9\beta_K^2)\xi_K \cdot \xi_K).
\]

(20) follows from the Cauchy-Schwarz inequality, the proof is completed.

5. **Finite element schemes for respective model problems**
5.1. **A finite element scheme for the Poisson equation.** We consider the Poisson problem with homogeneous boundary condition

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \Gamma
\end{aligned}
\]

The variational formulation is to find \( u \in H^1_0(\Omega) \), such that

\[
(22) \quad \int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall \ v \in H^1_0(\Omega).
\]

The finite element problem is to find \( u_h \in V_{QBL} \), such that

\[
(23) \quad \sum_{K \in J_h} \int_K \nabla u_h \nabla v_h \, dx = \int_\Omega f v_h \, dx, \quad \forall \ v_h \in V_{QBL}^{h_0}.
\]

**Theorem 20.** Let \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( u_h \) be the solutions of (22), and (23), respectively. Then

\[
(24) \quad |u - u_h|_{1,h} \leq C h \|u\|_{2,\Omega}.
\]

**Proof.** The theorem is proved by the standard technique. \( \square \)

5.1.1. **Numerical verification.** We choose the computational domain to be the quadrilateral with vertexes \((0,0), (1,0), (2,2), (-1,1)\). The data \( f \) is chosen such that the exact solution is the polynomial \( u = y(x+y)(x-3y+4)(2x-y-2) \). We subdivide the domain with quadrilateral grids and triangle grids, respectively, and numerical solutions are computed on both grids. To generate quadrilateral grids, we use bisection strategy. To generate triangle grids, we firstly subdivide the domain with quadrilateral grids, then bisect all of them each to two triangles, see Figure 4. We first test the performance of the QBL element on the quadrilateral grids, then we test Courant element on the triangle grids as a comparison. The results are recorded in Table 3.

![Figure 4](image-url)  
**Figure 4.** Two different sequences of grids
Table 3. Numerical results for Poisson problem

| Size($\mathcal{J}_h$) | On quadrilateral grids | On triangle grids |
|------------------------|------------------------|------------------|
|                        | $|u - u_h|_{1,h}$   | $||u - u_h||_{0,\Omega}$ | $|u - u_h|_{1,h}$   | $||u - u_h||_{0,\Omega}$ |
| 8 × 8                  | 1.67E0                 | 1.39E-1          | 3.59E0 | 2.57E-1          |
| 16 × 16                | 8.35E-1                | 3.52E-2          | 1.83E0 | 6.73E-2          |
| 32 × 32                | 4.18E-1                | 9.69E-3          | 9.23E-1 | 1.70E-2          |
| 64 × 64                | 2.09E-1                | 2.42E-3          | 4.62E-1 | 4.57E-3          |

Convergence order 1 2 1 2

Figure 5 reports on approximation results of QBL and Courant elements for Poisson equation. The $x$-axis and the $y$-axis represent the logarithm of grid size $h$ and of the error, respectively. The dashed line and the solid line represent the error associated with the norm $| \cdot |_{1,h}$ and $|| \cdot ||_{0,\Omega}$, respectively. The results confirm our conclusion: a clear first-order of convergence is observed with $| \cdot |_{1,h}$.

5.2. Application on Laplace eigenvalue equation. We consider the Laplace eigenvalue problem with homogeneous boundary condition

\[
\begin{aligned}
-\Delta u &= \lambda u & \text{in } \Omega \\
u &= 0 & \text{on } \Gamma.
\end{aligned}
\]

The variational formulation is to find $(\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)$, such that

\[
\int_{\Omega} \nabla u \nabla v \, dx = \lambda \int_{\Omega} u v \, dx, \quad \forall v \in H^1_0(\Omega).
\]
The finite element problem is to find \((\lambda_h, u_h) \in \mathbb{R} \times V_{h,h}^{QBL}\), such that

\[
\sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \nabla v_h \, dx = \lambda_h \int_\Omega u_h v_h \, dx, \quad \forall v_h \in V_{h,h}^{QBL}.
\]

**Theorem 21.** Let the eigenvalues of the problem (26) and (27) be sorted from small to big. Let \((\lambda, u)\) and \((\lambda_h, u_h)\) be the k-th eigenpair of (26) and (27), respectively. Then for \(h\) small enough,

\[
|\lambda - \lambda_h| = O(h^2) \quad \text{and} \quad |u - u_h|_{1,h} = O(h).
\]

**Proof.** The theorem is proved by the standard technique. \(\square\)

5.2.1. *Numerical verification.* We choose the computational domain to be the unit square \(\Omega = (0,1) \times (0,1)\). The eigenvalue \(\lambda\) is chosen such that the exact solution is the function \(u = \sin(\pi x) \sin(\pi y)\). We first divide the computational domain into four trapezoids, then use the same strategy as Subsection 5.1 to generate the grids, see Figure 6, and repeat numerical test by same elements as before. The results are recorded in Table 4.

![Figure 6. Two different sequences of grids](image)

**Table 4.** Numerical results for Laplace eigenvalue problem

| Size(\(\mathcal{F}_h\)) | On quadrilateral grids | On triangle grids |
|-------------------------|------------------------|------------------|
|                         | \(\lambda_h\) | \(|\lambda - \lambda_h|\) | \(\lambda_h\) | \(|\lambda - \lambda_h|\) | \(\lambda\) |
| 8 \times 8              | 2.035E1 | 6.090E-1 | 2.074E1 | 9.995E-1 | 2\pi^2 |
| 16 \times 16            | 1.988E1 | 1.359E-1 | 1.998E1 | 2.424E-1 | 2\pi^2 |
| 32 \times 32            | 1.977E1 | 3.210E-2 | 1.980E1 | 6.120E-2 | 2\pi^2 |
| 64 \times 64            | 1.9747E1 | 7.800E-3 | 1.9754E-1 | 1.480E-2 | 2\pi^2 |
| Convergence order       | 2          | 2          |

Figure 7 reports on approximation results of QBL and Courant elements for Laplace eigenvalue equation. The x-axis and the y-axis represent the logarithm of grid size \(h\) and of the error, respectively. In this numerical experiment, a clear second-order of convergence is observed and the numerical performance of QBL element is better than that of Courant element.
5.3. Application on $H(\text{rot})$ equation. We consider the problem with homogeneous boundary condition

$$
\left\{
\begin{array}{ll}
\text{curl}(\text{rot}\tilde{\sigma}) + \tilde{\sigma} = f & \text{in } \Omega \\
\tilde{\sigma} \tau = 0 & \text{on } \Gamma.
\end{array}
\right.
$$

The variational formulation is to find $\sigma \in H_0(\text{rot}, \Omega)$, such that

$$
\int_{\Omega} \text{rot}\tilde{\sigma} \text{rot}\tau + \sigma \tau \, dx = \int_{\Omega} f \tau \, dx, \quad \forall \, \tau \in H_0(\text{rot}, \Omega).
$$

The finite element problem is to find $\sigma_h \in V_{h0}^{\text{QRT}}$, such that

$$
\sum_{K \in T_h} \int_{K} \text{rot}\sigma_h \text{rot}\tau_h + \sigma_h \tau_h \, dx = \int_{\Omega} f \tau_h \, dx, \quad \forall \, \tau_h \in V_{h0}^{\text{QRT}}.
$$

**Theorem 22.** Let $\sigma \in H^1(\text{rot}, \Omega) \cap H_0(\text{rot}, \Omega)$ and $\sigma_h$ be the solutions of \(29\), and \(30\), respectively. Then

$$
\|\sigma - \sigma_h\|_{\text{rot}, h} \leq Ch(|\sigma|_{1, \Omega} + |\text{rot}\sigma|_{1, \Omega}).
$$

**Proof.** The theorem is proved by the standard technique. \qed
5.3.1. **Numerical verification.** We choose the computational domain to be the unit square $\Omega = (0, 1) \times (0, 1)$. The source term $f_\tilde{\sigma}$ is chosen such that the exact solution is given by $\sigma = (xy^2 - xy, x^2y - xy)^T$. Then we test the performance of the QRT element on the quadrilateral grids as Subsection 5.2 and the results are recorded in Table 5.

**Table 5. Numerical results for $H(\text{rot})$ problem**

| Size($\mathcal{F}_h$) | $\|\sigma - \sigma_h\|_{0,\Omega}$ | $\|\sigma - \sigma_{\text{rot},h}\|_{\text{rot},h}$ | $\|\sigma - \sigma_{\text{rot},h}\|_{\text{rot},h}$ |
|-------------------------|------------------------------------|---------------------------------|---------------------------------|
| $8 \times 8$            | 1.13E-1                            | 5.95E-2                         | 1.28E-1                         |
| $16 \times 16$          | 5.53E-2                            | 2.98E-2                         | 6.28E-2                         |
| $32 \times 32$          | 2.78E-2                            | 1.49E-2                         | 3.15E-2                         |
| $64 \times 64$          | 1.40E-2                            | 7.45E-3                         | 1.58E-2                         |
| Convergence order       | 1                                  | 1                               | 1                               |

Figure 8 reports on approximation results of QRT element for $H(\text{rot})$ equation. The $x$-axis and the $y$-axis represent the logarithm of grid size $h$ and of the error, respectively. The error associated with $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{\text{rot},h}$ and $\|\cdot\|_{\text{rot},h}$ are plotted by dotted line, dashed line and solid line, respectively. The results confirm our conclusion: a clear first-order of convergence is observed.

![Figure 8](image.png)

**Figure 8.** The log-log plot of the error of QRT for $H(\text{rot})$ equation

6. **Concluding remarks**

In this paper, we present a polynomial de Rham complex on a grid that consists of arbitrary quadrilaterals by constructing two nonconforming finite elements for the $H^1$ and $H(\text{rot})$ problems, respectively. As is proved in the present paper, cf. Theorem 2 and Remark 4 we can not
theoretically expect the finite element be conforming anyway; however, the two spaces are both quasi-conforming and are consistent to the requirement of the differential complex. Moreover, with respect to the $O(h^2)$ asymptotic parallelogram assumption, the scheme for $H(\text{rot})$ problem is $O(h)$ convergent for $H^1(\text{rot})$ exact solution; namely, this element does not suffer from the extra requirement on the regularity for general nonconforming $H(\text{curl})$ element (cf. [12]). For the Poisson equation, numerical experiments show that the QBL element plays superior to the triangular linear element with the same amount of unknowns for both source and eigenvalue problems, which confirms the need of quadrilateral elements.

It follows immediately that a piecewise polynomial complex can be constructed by rotation according to

\begin{equation}
\mathbb{R} \xrightarrow{\text{inclusion}} H^1 \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \xrightarrow{\text{integration}} \mathbb{R}
\end{equation}

by simply a rotation. Further, the methodology which use the same shape functions as that on a framework parallelogram and the nodal parameters on a physical cells can be generalized to more complicated cases, such as higher order schemes and higher dimension problems. These would be discussed in future.

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