HAMILTON–JACOBI EQUATIONS AND DISTANCE FUNCTIONS ON RIEMANNIAN
MANIFOLDS

CARLO MANTEGAZZA AND ANDREA CARLO MENNUCCI

ABSTRACT. The paper is concerned with the properties of the distance function from a closed subset
of a Riemannian manifold, with particular attention to the set of singularities.

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1. INTRODUCTION

We are concerned with the properties of the singular set of the distance function from a closed
subset of a \(n\)-dimensional, smooth and connected Riemannian manifold \((M, g)\), with particular
attention to its rectifiability.

Definition 1.1. We say that a subset \(S\) of \((M, g)\) is \(C^r\)-rectifiable, with \(r \geq 1\), if it can be covered
by a countable family of embedded \(C^r\) submanifolds of dimension \((n - 1)\), with the exception of
a set of \(\mathcal{H}^{n-1}\) zero measure, where \(\mathcal{H}^{n-1}\) is the \((n - 1)\)-dimensional Hausdorff measure on \(M\).
We will simply say that a set is rectifiable when it is at least \(C^1\)-rectifiable.

See [18, 29] for a complete discussion of the notion of rectifiability.

The distance function from a closed, not empty subset \(K\) of \((M, g)\) is defined in the usual way,
\[
d_K(x) = \inf_{y \in K} d(x, y)
\]
where \(d\) is the distance on \(M\) induced by the metric tensor \(g\).

The singular set \(\text{Sing}\) of \(d_K : M \to \mathbb{R}\) is the set where this function fails to be differentiable.

Our study of the rectifiability of \(\text{Sing}\) relies on the theory of viscosity solutions of Hamilton–Jacobi
equations. Indeed, we show that the distance function \(d_K\) is a viscosity solution of the following
problem
\[
\begin{aligned}
|\nabla u| &= 1 & \text{in } M \setminus K, \\
u &= 0 & \text{on } \partial K
\end{aligned}
\]
and we use the property of semiconcavity shared by such solutions to obtain a rectifiability result
for \(\text{Sing}\).

Then, we investigate under which hypotheses also the closure of \(\text{Sing}\) is rectifiable. This problem
is strictly connected to the analysis of the geodesic flow on \((M, g)\) originating from \(K\), hence,
we adapt ideas coming from the study of the cut locus of a point in a Riemannian manifold, which is actually the very special case when $K$ is a single point of $M$.

Our results lead to the conclusion that, under some conditions on the regularity of the set $K$, the Hausdorff dimension of the closure of the singular set is at most $(n-1)$ and that the gradient of the distance function from $K$ is locally a vector field with special bounded variation (see [3, 4, 5]). Moreover, we also study when the singular set shares an higher regularity and we analyse in detail its topological structure if $M$ is a two–dimensional analytic surface and $K$ an analytic subset.

The study of the distance function and of the associate eikonal equation $|\nabla u| = 1$ is a special example of a connection which can be extended to a large class of stationary Hamilton–Jacobi equations. In the last section we discuss some problems about the structure of the singular set of more general viscosity solutions, suggested by some geometric results for the cut locus of a point.

2. Stationary Hamilton–Jacobi Equations on Manifolds

Let $M$ be a smooth and connected, $n$–dimensional differentiable manifold.

We consider the following Hamilton–Jacobi problem in $\Omega \subset M$,

\[
\begin{aligned}
H(x, du(x), u(x)) &= 0 \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \partial\Omega
\end{aligned}
\]

where $H : T^*\Omega \times \mathbb{R} \to \mathbb{R}$ and $T^*$ denotes the cotangent bundle.

**Definition 2.1.** Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the superdifferential of $u$ at $x$ is the subset of $T^*_x M$ defined by

\[
\partial^+ u(x) = \left\{ d\varphi(x) \mid \varphi \in C^1(M), \varphi(x) - u(x) = \min_{M} \varphi - u \right\}.
\]

Similarly, the set

\[
\partial^- u(x) = \left\{ d\psi(x) \mid \psi \in C^1(M), \psi(x) - u(x) = \max_{M} \psi - u \right\}
\]

is called the subdifferential of $u$ at $y$.

Notice that it is equivalent to replace the $\max$ (min) on all $M$ with the maximum (minimum) in an open neighborhood of $x$ in $M$.

It is easy to see that $\partial^+ u(x)$ and $\partial^- u(x)$ are both nonempty if and only if $u$ is differentiable at $x \in M$. In this case we have

\[
\partial^+ u(x) = \partial^- u(x) = \{du(x)\}.
\]

We list here without proof some of the standard properties of the sub and superdifferentials which will be needed later.

**Proposition 2.2.** If $\psi : N \to M$ is a map between the smooth manifolds $N$ and $M$ which is $C^1$ around $x \in N$, then

\[
\partial^+(u \circ \psi)(x) \supset \partial^+ u(\psi(x)) \circ d\psi(x) = \{v \circ d\psi(x) \mid v \in \partial^+ u(\psi(x))\}.
\]

If $\psi$ is a local diffeomorphism near $x$, the inclusion becomes an equality. An analogous statement holds for $\partial^-$.

**Proposition 2.3.** If $\theta : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function such that $\dot{\theta}(u(x)) \geq 0$, then

\[
\partial^+(\theta \circ u)(x) \supset d\theta(u(x)) \circ \partial^+ u(x) = \{d\theta(u(x)) \circ v \mid v \in \partial^+ u(x)\},
\]

similarly for $\partial^-$. If $\dot{\theta}(u(x)) > 0$ then the inclusion is an equality.

For a locally Lipschitz function $u$ on a Riemannian manifold $(M, g)$, $\partial^+ u(x)$ and $\partial^- u(x)$ are compact convex sets, almost everywhere coinciding with the differential of the function $u$, by Rademacher’s Theorem.

For a generic continuous function $u$ we prove in the next proposition that $\partial^+ u(x)$ and $\partial^- u(x)$ are not empty in a dense subset.
Proposition 2.4. Let \( u : \Omega \to \mathbb{R} \) be a continuous function on an open subset \( \Omega \) of \( M \). Then the superdifferential \( \partial^+ u(x) \) (the subdifferential \( \partial^- u(x) \)) is not empty for every \( x \) in a dense subset of \( \Omega \).

Proof. It is always possible to endow \( M \) with a Riemannian structure giving a metric \( d(\cdot, \cdot) \) on \( M \) which generates the same topology.

Consider a generic point \( y \in \Omega \) and a geodesic ball \( B \) contained in \( \Omega \) with center \( y \). If the ball \( B \) is small enough, the function \( x \mapsto d^2(x, y) \) is smooth in \( \overline{B} \). Taking a large positive constant \( A \), the function \( F_A(x) = u(x) + Ad^2(x, y) \) has a local minimum at a point \( x_A \) in the interior of \( B \). At \( x_A \) the subdifferential of the function \( F_A \) must contain the origin of \( T^*_{x_A} M \), hence, being \( d^2(x, y) \) differentiable in the ball \( B \), the differential of \( -d^2(x, y) \) at \( x_A \) belongs to \( \partial^- u(x_A) \). As the point \( y \) and the ball \( B \) were arbitrarily chosen, the set of points where the subdifferential of \( u \) is not empty is dense in \( \Omega \).

The same argument holds for the superdifferential of \( u \), considering the function \( -u \). \( \square \)

Now we introduce the notion of semiconcavity which will play a central role in the first part of the paper.

Definition 2.5. Given an open set \( \Omega \subset \mathbb{R}^n \), a continuous function \( u : \Omega \to \mathbb{R} \) is called locally semiconcave if, for any open convex set \( \Omega' \subset \Omega \) with compact closure in \( \Omega \), there exists a constant \( C \) such that one of the following three equivalent conditions is satisfied,

1. \( \forall x, h \) with \( x, x + h, x - h \in \Omega' \),
   \[ u(x + h) + u(x - h) - 2u(x) \leq 2C|h|^2 \, , \]

2. \( Du^2 \leq 2CId \) in \( \Omega' \), as distributions (\( Id \) is the \( n \times n \) identity matrix).

In order to give a meaning to the concept of semiconcavity when the ambient space is a differentiable manifold \( M \), we analyse the stability of this property under composition with \( C^2 \) maps.

Proposition 2.6. Let \( \Omega \) and \( \Omega' \) two open subsets of \( \mathbb{R}^n \). If \( u : \Omega \to \mathbb{R} \) is a Lipschitz function such that \( u(x) - C|x|^2 \) is concave and \( \psi : \Omega' \to \Omega \) is a \( C^2 \) function with bounded first and second derivatives, then \( u \circ \psi : \Omega' \to \mathbb{R} \) is a Lipschitz function and \( u \circ \psi(y) - C'|y|^2 \) is concave, for a suitable constant \( C' \).

The proof is straightforward.

Then, the following definition is well–posed.

Definition 2.7. A continuous function \( u : M \to \mathbb{R} \) is called locally semiconcave if, for any local chart \( \psi : \mathbb{R}^n \to \Omega \subset M \), the function \( u \circ \psi \) is locally semiconcave in \( \mathbb{R}^n \).

The importance of semiconcave functions in connection with the generalized differentials is expressed by the following proposition (see [12]).

Proposition 2.8. Let the function \( u : M \to \mathbb{R} \) be locally semiconcave, then the superdifferential \( \partial^+ u \) is not empty at each point, moreover, \( \partial^+ v \) is upper semicontinuous, namely

\[ x_k \to x, \quad v_k \to v, \quad v_k \in \partial^+ u(x_k) \quad \implies \quad v \in \partial^+ u(x) \, . \]

In particular, if the differential \( du \) exists at every point of \( \Omega \in M \), then \( u \in C^1(\Omega) \).

Now we introduce the definition of viscosity solution.

Let \( \Omega \) be an open subset of \( M \) and \( H \), called Hamiltonian function, a continuous real function on \( T^* \Omega \times \mathbb{R} \). We are interested in the following Hamilton–Jacobi problem

\begin{equation}
H(x, du(x), u(x)) = 0 \quad \text{in } \Omega \, .
\end{equation}

Definition 2.9. We say that a continuous function \( u \) is a viscosity solution of equation (2.1) if for every \( x \in \Omega \),

\begin{equation}
\begin{cases}
H(x, v, u(x)) \leq 0 & \forall v \in \partial^+ u(x) \, , \\
H(x, v, u(x)) \geq 0 & \forall v \in \partial^- u(x) \, .
\end{cases}
\end{equation}

If only the first condition is satisfied (resp. the second) \( u \) is called a viscosity subsolution (resp. a viscosity supersolution).
If $\Omega'$ is an open subset of another smooth differentiable manifold $N$ and $\psi : \Omega' \to \Omega$ is a $C^1$ local diffeomorphism, we define the pull-back of the Hamiltonian function $\psi^*H : T^*\Omega' \times \mathbb{R} \to \mathbb{R}$ by

$$\psi^*H(y, v, r) = H(\psi(y), v \circ d\psi(y)^{-1}, r).$$

Taking into account Proposition 2.2, the following statement is obvious.

**Proposition 2.10.** If $u$ is a viscosity solution of $H = 0$ in $\Omega \subset M$ and $\psi : \Omega' \to \Omega$ is a $C^1$ local diffeomorphism, then $u \circ \psi$ is a viscosity solution of $\psi^*H = 0$ in $\Omega' \subset N$.

### 3. The Distance Function from a Subset of a Manifold

From now on, $(M, g)$ will be a smooth, connected and complete Riemannian manifold without boundary, of dimension $n$.

We will study the distance function $d_K$ from a closed and not empty subset $K$ of $M$ (for technical reasons, sometimes we consider also its square $d_K^2(x) : M \to \mathbb{R}$).

The distance between two points $x$ and $y$ or from the point $x$ to the set $K$ is defined as the infimum of the lengths of the $C^1$ curves starting at $x$ and ending at $y$, or on $K$, respectively. As $M$ is complete, by the Theorem of Hopf–Rinow, such infimum is reached by at least one curve which will be a smooth geodesic.

The distance from the set $K$ is a continuous function on $M$ but in general it is not everywhere differentiable, for instance, if the manifold $M$ is compact, the distance function from any proper subset will be singular at the points of absolute maximum. This section deals precisely with the set where the gradient of $d_K$ does not exist.

In the following we will denote the distance between two points $x, y \in M$ with $d(x, y)$ and the exponential map of $(M, g)$ with $\exp : TM \times \mathbb{R} \to M$. For simplicity, we will write $|v|$ for the modulus of a vector $v \in TM$, defined as $\sqrt{g(v, v)}$.

Moreover, we will take often into account the identification between the differential $du$ and the gradient $\nabla u$ of a function, induced by the scalar product $g$.

**Theorem 3.1.** The distance function $d_K$ is the unique viscosity solution of the following Hamilton–Jacobi problem

$$\begin{cases}
|\nabla u| - 1 = 0 & \text{in } M \setminus K, \\
u = 0 & \text{on } K
\end{cases}$$

(3.1)

in the class of continuous functions bounded from below.

The function $d_K^2/2$ is the unique viscosity solution of

$$\begin{cases}
|\nabla u| - 2u = 0 & \text{in } M, \\
u = 0 & \text{on } K
\end{cases}$$

(3.2)

in the class of continuous functions on $M$ such that their zero set is $K$.

**Remark 3.2.** The restriction to lower bounded functions is necessary, $\|x\|$ and $-\|x\|$ are both viscosity solutions of Problem (3.1) with $M = \mathbb{R}^n$ and $K = \{0\}$. Moreover, the completeness of $M$ plays an important role here, if $M$ is the open unit ball of $\mathbb{R}^n$ the same example shows that the uniqueness does not hold.

Notice also that every function $d_H^2/2$ where $H$ is a closed subset of $M$ with $H \supset K$, is a viscosity solution of Problem (3.2), equal to zero on $K$.

**Proof.** Notice that $d_K(x)$ is the minimum time $t \geq 0$ for any curve $\gamma$ to reach a point $\gamma(t) \in K$, subject to the conditions $\gamma(0) = 0$ and $|\dot{\gamma}| \leq 1$; $d_K$ is then the value function of a “minimum time problem”; this proves that $d_K$ is also a viscosity solution of Problem (3.1), by well known results (see for example Proposition 2.3, Chapter IV in [13]). Then we show that the function $d_K^2/2$ is a solution of Problem (3.2).

First of all, notice that the distance function from $K$ is a 1–Lipschitz function, hence $d_K^2$ is locally Lipschitz.
As $d_K$ is 1–Lipschitz, at every point of $K$ the function $d_K^2$ is differentiable and its differential is zero. Hence, the definition of viscosity solution holds for points belonging to $K$. In order to prove the theorem, it is then sufficient to test conditions (2.2) on the generalized differentials at the points of the open set $M \setminus K$.

Since $d_K^2/2$ is positive in $M \setminus K$, applying Proposition 2.3 with the function $\theta(t) = \sqrt{2t}$, we see that the function $d_K^2/2$ is a viscosity solution of

$$g \left( \frac{\nabla u}{\sqrt{2u}}, \frac{\nabla u}{\sqrt{2u}} \right) - 1 = 0$$

in $M \setminus K$. Being there positive, it also solves

$$g(\nabla u, \nabla u) - 2u = 0$$

in $M \setminus K$. This fact together with the previous remark about the behavior of $d_K^2$ at the points of $K$ gives the claim.

Suppose now that $u$ is a viscosity solution of Problem (3.1) then, $u$ is also a solution of

$$\begin{cases} |\nabla u| - 1 = 0 & \text{in } M \setminus K, \\ u = 0 & \text{on } K. \end{cases}$$

As in the work of Kružhkov [23], we consider the function $v = -e^{-u}$ which, by Proposition 2.3, turns out to be a viscosity solution of

$$\begin{cases} |\nabla v| + v = 0 & \text{in } M \setminus K, \\ v = -1 & \text{on } K. \end{cases}$$

moreover, $|v| \leq e^{-\inf u}$.

We establish a uniqueness result for this last problem in the class of bounded functions $v$, which clearly implies the first uniqueness result. We remark that the proof is based on similar ones in [14, 15, 20].

We argue by contradiction, suppose that $u$ and $v$ are two bounded solutions of (3), $|u|, |v| \leq C$, and that at a point $\overline{x}$ we have $u(\overline{x}) \geq 2\varepsilon + v(\overline{x})$ with $\varepsilon > 0$.

Let $b(x, y) : M \times M \to \mathbb{R}$ be a smooth function satisfying

- $b \geq 0$
- $|\nabla_x b(x, y)|, |\nabla_y b(x, y)| \leq 2$
- $\sup_{M \times M} |d(x, y) - b(x, y)| < \infty$

such a function can be obtained smoothing the distance function in $M \times M$.

We fix a point $x_0$ in $K$ and we define the smooth function $B(x) = b(x, x_0)^2$. By the properties of $b$ and the boundedness of $u$ and $v$, the following function $\Psi : M \times M \to \mathbb{R}$

$$\Psi(x, y) = u(x) - v(y) - \lambda d(x, y)^2 - \delta B(x) - \delta B(y)$$

has a maximum at a point $\hat{x}, \hat{y}$ (dependent on the positive parameters $\delta$ and $\lambda$) and such maximum $\Psi(\hat{x}, \hat{y})$ is less than $2C$. Hence, the function

$$x \mapsto [v(\hat{y}) + \lambda d(x, \hat{y})^2 + \delta B(x) + \delta B(\hat{y})] - u(x)$$

has a minimum at $\hat{x}$ while

$$y \mapsto [u(\hat{x}) - \lambda d(\hat{x}, y)^2 - \delta B(\hat{x}) - \delta B(y)] - v(y)$$

has a maximum at $\hat{y}$.

If $2\delta \leq \varepsilon / B(\overline{x})$ then

$$\Psi(\hat{x}, \hat{y}) \geq \Psi(\overline{x}, \overline{y}) \geq 2\varepsilon - 2\delta B(\overline{x}) \geq \varepsilon$$

hence, we get

$$2\delta B(\overline{x}) + \delta B(\overline{y}) + \lambda d(\overline{x}, \overline{y})^2 + \varepsilon \leq u(\overline{x}) - v(\overline{y}) \leq 2C.$$

This shows that, for a fixed $\delta$, the pair $\overline{x}, \overline{y}$ is contained in a bounded set and, if $\lambda$ goes to $+\infty$ the distance between $\overline{x}$ and $\overline{y}$ goes to zero. Possibly passing to a subsequence for $\lambda$ going to infinity, $\overline{x}$ and $\overline{y}$ converge to a common limit point $z$ which cannot belong to $K$, otherwise we would get
Since the matrix coordinates.

\[ \psi \] solution of \( R_v \) define \( d \)

Finally, we have that \( |\hat{v}| + u(\hat{x}) \leq 0 \) and \( |\hat{w}| + v(\hat{y}) \geq 0 \), hence

\[ u(\hat{x}) - v(\hat{y}) + |\hat{v}| - |\hat{w}| \leq 0. \]

Moreover,

\[ |\hat{v}| - |\hat{w}| = \left| \delta \nabla B(\hat{\xi}) \nabla^2 \hat{d}^2(\hat{x}, \hat{\gamma}) \right| \leq \left\| \delta \nabla B(\hat{\xi}) \nabla^2 \hat{d}^2(\hat{x}, \hat{\gamma}) \right\| \]

which implies,

\[ u(\hat{x}) - v(\hat{y}) - |\delta \nabla B(\hat{\gamma})| - |\delta \nabla B(\hat{x})| \leq 0. \]

Finally, we have that

\[ \delta |\nabla B(\hat{x})| = 2\delta |\nabla \hat{d}(\hat{x}, x_0) \nabla \hat{d}(\hat{x}, x_0)| \leq 4\delta \sqrt{\hat{d}(\hat{x})} \]

and using the estimate \( \delta \hat{d}(\hat{x}) \leq 2C \) which follows from equation (3.5),

\[ \delta |\nabla B(\hat{x})| \leq 8\sqrt{2\delta C} \leq \varepsilon / 4 \]

if \( \delta \) was chosen small enough. Holding the same for \( \hat{y} \), we conclude that

\[ u(\hat{x}) - v(\hat{y}) - \varepsilon / 2 \leq 0 \]

which is in contradiction with the fact that \( u(\hat{x}) - v(\hat{y}) \geq \varepsilon \).

About the second uniqueness claim, if \( u \) is a continuous viscosity solution of Problem (3.2) then, by Proposition 2.4 the superdifferential of \( u \) is not empty in a dense subset of \( M \setminus K \), hence, directly by the equation and by continuity, \( u \) is non negative. By the hypothesis on its zero set we conclude that \( u \) is positive in all \( M \setminus K \). Composing \( u \) with the function \( t \mapsto \sqrt{2t} \), we see that \( \sqrt{2u} \) is a positive, continuous viscosity solution of Problem (3.1), then it must coincide with \( d_K \), by the previous result. It follows that \( u = d_K^2 / 2 \).

We now study the rectifiability of the singular set of \( d_K \),

\[ \text{Sing} = \{ x \in M \mid d_K^2 \text{ is not differentiable at } x \} . \]

Remark 3.3. In this definition we used the squared distance function instead of the distance in order to avoid to consider also the points of the boundary of \( K \), which are singular for \( d_K \). It is easy to see that outside \( K \) the distance and its square have the same regularity.

Proposition 3.4. The function \( d_K \) is locally semiconcave in \( M \setminus K \).

Proof. The distance function \( d_K \) is a viscosity solution of \( H = 0 \) in \( M \setminus K \), where the Hamiltonian function is given by \( H(x, v, t) = |v|^2 - 1 \). We choose a smooth local chart \( \psi : \mathbb{R}^n \to \Omega \subset M \) and we define \( v = d_K \circ \psi \), which is a locally Lipschitz function and, by Proposition 2.10, it is a viscosity solution of \( \psi^* H = 0 \).

The pull-back of the Hamiltonian function on \( \mathbb{R}^n \) takes the form

\[ \psi^* H(y, w, s) = g_{\psi(y)}(d\psi(w), d\psi(w)) - 1 = g_{ij}(y)w_i w_j - 1 \]

for \( (y, w, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) and where \( g_{ij}(y) \) are the components of the metric tensor of \( M \) in local coordinates.

Since the matrix \( g_{ij}(y) \) is positive definite \( \psi^* H(y, w, s) \) is locally uniformly convex in \( w \), hence,
The semiconcavity of \( d_K \) allows us to work with the superdifferentials when the gradient does not exist. Indeed, notice that the points of \( \text{Sing} \) are precisely those where the superdifferential is not a singleton, then, the following result is a straightforward consequence of Proposition 2.8.

**Proposition 3.5.** The function \( d_K^2 \) is of class \( C^1 \) in the open set \( M \setminus \overline{\text{Sing}} \) and \( d_K \) is \( C^1 \) in \( M \setminus (K \cup \overline{\text{Sing}}) \).

**Remark 3.6.** The semiconcavity property also gives information about the propagation of singularities and the relations between the structure of the superdifferential at a point \( x \) and the set of minimal geodesics from \( x \) to \( K \) (see [2, 6]).

The set \( \text{Ext}(\partial^+ d_K^2(x)/2) \) of extremal points of the (convex) superdifferential set of \( d_K^2 \) at \( x \) is in one–to–one correspondence with the family \( \mathcal{G}(x) \) of minimal geodesics from \( x \) to \( K \). Precisely, \( \mathcal{G}(x) \) is described by

\[
\mathcal{G}(x) = \{ \text{Exp}(x, -v, \cdot) : [0, 1] \to M \mid v \in \text{Ext}(\partial^+ d_K^2(x)/2) \}.
\]

The set of points of \( K \) at minimum distance from \( x \) are given by \( \text{Exp}(x, -v, 1) \) for all \( v \) in the set of extremal points of the superdifferential set of \( d_K^2/2 \) at \( x \). As a particular case we have that if the function \( d_K^2 \) is differentiable at \( x \), then the point of \( K \) closest to \( x \) is uniquely determined and given by \( \text{Exp}(x, -\nabla d_K^2(x)/2, 1) \).

Finally, notice that \( \text{Sing} \) is the set of points \( x \) such that the distance \( d(x, K) \) is realized by more than one minimal geodesic between \( x \) and \( K \).

The rectifiability of \( \text{Sing} \) now follows.

**Proposition 3.7.** The set \( \text{Sing} \) is \( C^2 \)-rectifiable.

**Proof.** By a result proved in [1], the singular set of a locally semiconcave function in an open set of \( \mathbb{R}^n \) is \( C^2 \)-rectifiable. We take a countable family of local charts \( \psi_i : \mathbb{R}^n \to \Omega_i \) and consider the functions \( d_K \circ \psi_i \). These functions are locally semiconcave in \( \mathbb{R}^n \) with singular sets \( \text{Sing}_i \), hence, by the relation

\[
\text{Sing} \subset \bigcup_{i=1}^{\infty} \psi_i(\text{Sing}_i)
\]

we get the thesis.

The same statement does not hold for the closure of \( \text{Sing} \), for a generic closed set. We describe now a counterexample showing that indeed the set \( \overline{\text{Sing}} \) is not rectifiable for a set \( K \) of class \( C^{1,1} \) only.

We look for a convex open set \( \Omega \) with a \( C^{1,1} \) boundary in \( \mathbb{R}^2 \) such that the closure of the singular set \( \overline{\text{Sing}} \) of the distance function from its boundary has nonzero Lebesgue measure, hence it is not rectifiable.

We start with a Cantor–like set \( C \subset S^1 \), closed with empty interior in \( S^1 \), with no isolated points and positive Hausdorff \( H^1 \) measure. Such a set can be constructed as follows

\[
C = S^1 \setminus \bigcup_{i=1}^{\infty} I_i
\]

where \( \{I_i\} \) is a countable family of open disjoint connected arcs on \( S^1 \), whose middle points are \( p_i \in S^1 \) and such that the sum of their lengths is less than \( 2\pi \).

We claim that every point of \( C \) is a limit point of the sequence \( \{p_i\} \). If \( p \in C \) there must be a sequence of arcs \( I_{i_j} \) arbitrarily close to \( p \), since the arcs are countable and the sum of their lengths is bounded by \( 2\pi \) we have that they shrink when \( j \) goes to infinity, hence \( p_{i_j} \to p \).

We define an open convex set \( \Omega' \) as the intersection of the open halfplanes, containing the origin of \( \mathbb{R}^2 \), determined by the tangent lines to \( S^1 \) at the points of \( C \), see the following figure.
Let us take an arc $I$ with middle point $p$, bounded by $P$, $Q \in C$ and consider the associate quadrilateral $OPVQ$. If the point $x$ is inside the open triangle $OPV$ it is clear that the point of $\partial \Omega'$ closest to $x$ belongs to the segment $PV$ and it is unique ($\tilde{x}$ in the figure). Hence, for such points the distance from the boundary of $\Omega'$ coincide with the distance from the segment $PV$.

Applying the same argument to the open triangle $OQV$, we see that the segment $OV$ consists of singular points of $d_{\partial \Omega'}$, moreover, the segment $OV$ intersects $S^1$ at the point $p$.

It follows that the union $S$ of the segments from the middle points $p_i$ to the origin coincides with $\text{Sing for } d_{\partial \Omega'}(x)$. Being $p_i$ dense in $C$, the closure of $S$ contains $\lambda C \subset \mathbb{R}^2$ for every $\lambda \in [0, 1]$. As $C$ has $\mathcal{H}^1$ positive measure, the Lebesgue measure of $S$ is positive, by Fubini’s Theorem.

Now let $\Omega$ be the set of points of $\mathbb{R}^2$ whose distance from the convex $\Omega'$ is less than 1. It is immediate to check that

$$d_{\partial \Omega}(x) = d_{\partial \Omega'}(x) + 1 \quad \forall x \in \Omega'$$

hence for every $x$ in the unit ball.

So the closure of $\text{Sing}$ for the distance function from the boundary of $\Omega$ (or from the complementary set of $\Omega$ in $\mathbb{R}^2$) has positive Lebesgue measure, moreover, by the properties of convex bodies, the boundary of $\Omega$ is of class at least $C^{1,1}$. Since the Lebesgue measure of $\text{Sing}$ is positive it cannot be rectifiable.

Remark 3.8. In the next section we will show that if the boundary of $K$ is of class at least $C^3$ then also the closure of $\text{Sing}$ is rectifiable. To our knowledge it is unknown even in $\mathbb{R}^2$ if the gap between such result and the previous counterexample can be filled, that is, if the $C^2$ (or maybe $C^{2,1}$) regularity of the boundary of $K$ is enough to get the rectifiability of the closure of the singular set.

4. Rectifiability of the Closure of the Singular Set

In this section we are going to show that an higher regularity of the set $K$ implies the rectifiability also of the closure of the singular set. Moreover, we determine a relation between the regularity of $K$ and of the hypersurfaces covering $\text{Sing}$.

In all this section $K$ is a $k$-dimensional embedded $C^r$ submanifold of $M$ without boundary, with $0 \leq k \leq n - 1$ (the case $k = n$ is trivial) and $r \geq 2$.

Let $UK$ be the unit normal bundle of $K$ in $M$, we denote with $F : UK \times [0, +\infty) \to M$ the restriction of the exponential map of $M$ to $UK \times [0, +\infty)$. Since $K$ is $C^r$, $UK$ is a manifold of
class \( C^{r-1} \) and being \( \text{Exp} \) a smooth map, \( F(q, v, t) \) and all its derivatives in the \( t \) variable are \( C^{r-1} \) functions.

Remark 4.1. The case when \( K \) is the closure of an open set of \( M \) with smooth boundary can be reduced to our case. Indeed, if \( x \in M \setminus K \), then the minimal geodesic from \( x \) to \( K \) ends in \( \partial K \) without touching the interior points, hence to study \( d_K \) we can simply consider the distance function from \( \partial K \) in every open connected component of \( M \setminus K \). However, in this case, some results like the higher smoothness of \( d_K^2 \) at the points of \( \partial K \) expressed by Proposition 4.2, could be lost.

The behavior of \( d_K \) near \( K \) is well known.

Proposition 4.2. There exist \( \varepsilon > 0 \) and an open neighborhood \( \Omega \) of \( K \) in \( M \) such that the map \( F|_{UK \times (0, \varepsilon)} : UK \times (0, \varepsilon) \to \Omega \setminus K \) is a \( C^{r-1} \) diffeomorphism.

Moreover,

- for every point in \( \Omega \) there is an unique point of minimum distance in \( K \) (the unique projection property holds for \( K \) in \( \Omega \)),
- the distance function \( d_K \) is \( C^r \) in \( \Omega \setminus K \),
- the squared distance function \( d_K^2 \) is \( C^r \) in \( \Omega \).

Remark 4.3. It can be proved that \( K \) has to be at least \( C^{1,1} \) in order to share the unique projection property in a neighborhood, in such case the squared distance function also turns out to be of class \( C^{1,1} \).

See [16, 17] for a detailed discussion of the relation between the regularity of \( K \) and of \( d_K \).

In order to study what happens far from \( K \) we have to analyse the sets of points where the unique projection property fails or \( F \) is not a local diffeomorphism. From a topological point of view, the problem is naturally connected with the study of the singularities of maps between Euclidean spaces. For instance, when \( K \) is a single point of \( M \) the singular sets were shown to be related to the classes of singularities considered by the Theory of Catastrophes, see [11].

Consider the geodesic curve \( t \mapsto F(q, v, t) \) for \( t \in [0, t_0] \) \( ((q, v) \in UK \) is fixed), for small values of \( t_0 \) it is the unique minimizer of the length functional between its end point \( p = F(q, v, t_0) \) and \( K \) but for large \( t_0 > 0 \) it could cease to be minimal. Hence, there exists a value \( \sigma \) (possibly \( +\infty \)) such that this geodesic is minimal between \( q \) and \( F(q, v, t) \) for every \( t < \sigma \), but not on any larger interval. If \( \sigma(q, v) < +\infty \), we say that the point \( F(q, v, \sigma(q, v)) \) is the cut point of the geodesic \( F(q, v, t) \) and we define the following set,

\[
V_K = \{(q, v, t) \in UK \times \mathbb{R}^+ \mid t < \sigma(q, v)\}.
\]

Notice that the set \( F(V_K) \) clearly contains \( \Omega \setminus K \), where \( \Omega \) is the open neighborhood of Proposition 4.2.

Definition 4.4. The set of points \( F(q, v, \sigma(q, v)) \) for \( (q, v) \in UK \) with \( \sigma(q, v) < +\infty \) is called the cut locus of \( K \), we denote it with \( \text{Cut}(K) \).

The reasons why a geodesic ceases to be minimal are explained by the following proposition (see [19, 28]).

Proposition 4.5. If for a geodesic \( F(q, v, t) \) we have \( \sigma(q, v) < +\infty \), at least one of the following two non exclusive conditions is satisfied:

1. at the point \( p = F(q, v, \sigma(q, v)) \) there arrives another minimal geodesic from \( K \),
2. the differential \( dF(q, v, \sigma(q, v)) \) is not invertible.

Conversely, if at least one of these conditions is satisfied the geodesic \( F(q, v, t) \) cannot be minimal on an interval larger that \( [0, \sigma(q, v)] \).

Notice that, by Remark 3.6, if condition 1 above is satisfied then the point \( p \) belongs to \( \text{Sing} \), while \( \text{Sing} \) is clearly included in \( \text{Cut}(K) \).

Then, we consider the following two subsets of \( \text{Cut}(K) \),
Proposition 4.6. The following statements hold,
1. \( \text{Cut}(K) = \text{Sing} \), that is, the cut locus of \( K \) is closed in \( M \) and \( \text{Sing} \) is a dense subset.
2. The set \( V_K \) is open in \( UK \times \mathbb{R}^+ \).
3. The map \( \sigma : UK \rightarrow \mathbb{R}^+ \) is continuous.
4. The map \( F \) is a \( C^{r-1} \) bijection between \( V_K \) and \( M \setminus (K \cup \text{Cut}(K)) \) with a \( C^{r-1} \) inverse.
5. The cut locus \( \text{Cut}(K) \) is equal to \( F(\partial V_K) \) where the boundary is considered in the ambient space \( UK \times \mathbb{R}^+ \).
6. The set \( M \setminus \text{Cut}(K) \) can be continuously retracted on \( K \), and, if \( \sigma(q,v) < +\infty \) for every \( (q,v) \in UK \), the set \( M \setminus K \) can be continuously retracted on the cut locus \( \text{Cut}(K) \).
7. The open set \( M \setminus \text{Cut}(K) \) has the unique projection property, moreover the squared distance function \( d^2_K \) is of class \( C^r \) in it. The distance \( d_K \) is \( C^r \) in \( M \setminus (\text{Cut}(K) \cup K) \).

By the point 1, the closure of \( \text{Sing} \) is precisely the cut locus of \( K \), which is the union of \( \text{Sing} \) and \( J \). We study then the rectifiability of these two sets separately, as the rectifiability property is clearly stable under countable union (see Definition 1.1).

We have seen in Proposition 3.7 that the set \( \text{Sing} \) is always \( C^2 \)-rectifiable. We partially improve this result when \( K \) is more regular.

Proposition 4.7. If \( K \) is of class \( C^r \) with \( r \geq 3 \) the set \( \text{Sing} \setminus J \) is a \( C^r \)-rectifiable subset of \( M \).

We need a preliminary lemma.

Lemma 4.8. If there are infinitely many minimal geodesics from \( p \) to \( K \), then \( p \) is an optimal focal point.

Proof. If \( F(q_i,v_i,\sigma(q_i,v_i)) = p \) for infinite distinct geodesics \( F(q_i,v_i,t_i) \), then \( \sigma(q_i,v_i) = d(q_i,p) = d_K(p) \) hence, by compactness, we may assume that \( (q_i,v_i) \rightarrow (q,v) \) for some \( (q,v) \in UK \). It follows (by the semicontinuity of the length functional) that \( F(q,v,t) \) is a minimal geodesic for \( p \) and that \( dF(q,v,t) \) is singular since \( F \) is not locally injective near \( (q,v,t) \).

Proof of Proposition 4.7. Let \( p \) be a point in \( \text{Sing} \setminus J \). We know that the number of minimal geodesics \( F(q_i,v_i,t) \) from \( p = F(q_i,v_i,\sigma(q_i,v_i)) \) to \( K \) is finite by the lemma above and greater than one, by the singularity at \( p \). Moreover, the differential \( dF(q_i,v_i,\sigma(q_i,v_i)) \) is invertible for every \( i \), then \( F \) is locally invertible in the neighborhood of every point \( (q_i,v_i,\sigma(q_i,v_i)) \in UK \times \mathbb{R}^+ \).

Let \( U_i \) be disjoint open neighborhoods of \( (q_i,v_i,\sigma(q_i,v_i)) \) such that \( F \) restricted to every \( U_i \) is a \( C^{r-1} \) diffeomorphism with its image. We can also suppose that \( F(U_i) = U \) where \( U \) is an open neighborhood of \( p \) in \( M \). We define the functions \( d_i : U \rightarrow \mathbb{R}^+ \) given by

\[
d_i(x) = \pi_{\mathbb{R}^+}(F^{-1}(x) \cap U_i)
\]

where \( \pi_{\mathbb{R}^+} : UK \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) denotes the projection on the second factor. Applying Proposition 4.2 to a small neighborhood of every \( q_i \), we can see that all the functions \( d_i \) are of class \( C^r \).

The singular set \( \text{Sing} \cap U \) is clearly contained in the union of the sets \( S_{ij} = \{ r \in U \mid d_i(x) = d_j(x) \} \) for \( i \neq j \). We now prove that such sets are locally \( C^r \) hypersurfaces. By the implicit function theorem, it is sufficient to show that \( \nabla d_i(x) - \nabla d_j(x) \neq 0 \) at the points of \( S_{ij} \). If \( r = F(s_i,w_i,t) = F(s_j,w_j,t) \), for \( (s_i,w_i,t) \in U_i \) and \( (s_j,w_j,t) \in U_j \) then

\[
\nabla d_i(x) = \nabla d_j(x) \implies \frac{dF}{dt}(s_i,w_i,t) = \frac{dF}{dt}(s_j,w_j,t).
\]
Hence, by the uniqueness of the geodesic from \( r \) with a certain initial velocity vector, we would get \((s_i, w_i) = (s_j, w_j)\), contradicting the hypothesis that \( U_i \) and \( U_j \) are disjoint. \( \square \)

The next proposition gives the rectifiability of the set \( J \) of optimal focal points which implies the rectifiability of the cut locus of \( K \), by the previous discussion.

**Proposition 4.9.** If \( K \) is of class \( C^r \) with \( r \geq 3 \), then the set \( J \) is \( C^{r-2} \)-rectifiable.

**Theorem 4.10.** If \( K \) is of class \( C^r \) with \( r \geq 3 \), the closure of \( \text{Sing} \), that is, the cut locus of \( K \), is \( C^{r-2} \)-rectifiable.

**Proof of Proposition 4.9.** We introduce the set \( \tilde{J} \) of the first focal points as follows. Let \( F(q, v, t) \) be a geodesic from \( K \) with \((q, v) \in UK \) and \( t \in \mathbb{R}^+ \), considering the first value \( t = c(q, v) \) such that \( dF(q, v, t) \) is not invertible or setting \( c(q, v) = +\infty \) if \( dF(q, v, t) \) is invertible for every \( t \in \mathbb{R}^+ \), we define the map \( c : UK \to \mathbb{R}^+ \cup \{+\infty\} \). If \( c(q, v) < +\infty \) we say that \( F(q, v, c(q, v)) \) is the first focal point along the geodesic \( F(q, v, t) \).

We consider the following set of points \( G \) in \( UK \times \mathbb{R}^+ \) where \( dF \) is not invertible

\[
G = \{(q, v, c(q, v)) \in UK \times \mathbb{R}^+ \mid c(q, v) < +\infty\}
\]

and we call \( \tilde{J} = F(G) \) locus of the first focal points of \( K \).

By Proposition 4.5, we have that \( J \supset \tilde{J} \), the set of optimal focal points, hence, it sufficient to show that the set \( \tilde{J} \) is rectifiable to prove the same for \( J \) and conclude the proof.

At the points of the set \( G \) the rank of \( dF \) is at most \((n - 1)\). We split \( G \) in two subsets,

\[
G_1 = \{(q, v, c(q, v)) \in G | \text{ Rank } dF(q, v, c(q, v)) = n - 1\}
\]

\[
G_2 = \{(q, v, c(q, v)) \in G | \text{ Rank } dF(q, v, c(q, v)) < n - 1\}
\]

The following version of Sard’s Theorem can be found in the book of Federer [18, Theorem 3.4.3].

**Lemma 4.11.** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a map of class \( C^l \) for some \( l \geq 1 \).

If we set for any \( k \in \{0, 1, \ldots, n - 1\} \)

\[
A_k = \{x \in \mathbb{R}^n | \text{ Rank } dF(x) \leq k\}
\]

then the Hausdorff measure \( \mathcal{H}^{k+\frac{n-k}{l}} \) of \( F(A_k) \) is zero.

Considering local charts of \( UK \times \mathbb{R}^+ \) and \( M \) and applying this lemma to our map \( F \) which is of class \( C^{r-1} \) we get that \( \mathcal{H}^{n-2+2/(r-1)}(F(G_2)) = 0 \), which implies \( \mathcal{H}^{n-1}(F(G_2)) = 0 \) as \( r \geq 3 \).

Now we show that the set \( G_1 \) is locally a \( C^{r-2} \) hypersurface in \( UK \times \mathbb{R}^+ \) using the implicit function theorem, this proves that \( F(G_1) \) is a \( C^{r-2} \)-rectifiable set in \( M \) and so \( \tilde{J} \).

Let \((q, v, c(q, v))\) be a point in \( G_1 \), by the lower semicontinuity of the rank, choosing a small neighborhood \( B \) of \((q, v, c(q, v))\) in \( UK \times \mathbb{R}^+ \) we can suppose that there are no points of \( G_2 \) in \( B \). The points of \( G_1 \) in \( B \) can be characterized as the zero set of the determinant \( \det dF(q, v, t) \) which is of class \( C^{r-2} \).

We claim that

\[
(4.1) \quad \frac{\partial \det dF}{\partial t}(q, v, c(q, v)) \neq 0
\]

at the points of \( G_1 \cap B \). By the implicit function theorem, this fact implies that \( G_1 \cap B \) is a regular \((n - 1)\)-dimensional submanifold of class \( C^{r-2} \) and we are done.

Let \( \nabla \) be the covariant derivative of \((M, g)\). Any vector in \( T_{(q, v)}UK \) can be represented by the velocity vector \((w, u) = \nabla_s(q(s), v(s))|_{s=0} \) at \( s = 0 \) of a \( C^{r-1} \) curve \((q(s), v(s)) \) in \( UK \), with \((q(0), v(0)) = (q, v)\). Such a curve is given by a \( C^r \) curve \( q(s) \) in \( K \) with a \( C^{r-1} \) unit vector field \( v(s) \) defined along \( q(s) \) and normal to \( K \). It is then clear that \( w = q'(0) \) belongs to the tangent space to \( K \) at \( q \)
We set \( u(s) = \nabla_s v(s) \) and \( u(0) = u \). Suppose that \( z(s) \) is an arbitrary vector field along \( q(s) \) tangent to \( K \) with \( z(0) = z \), then, by the orthogonality of \( v(s) \) and \( z(s) \), we have
\[
0 = \frac{d}{ds} [g(z(s), v(s))] = g(z(s), u(s)) + g(\nabla_s z(s), v(s))
\]
and at the point \( s = 0 \) we get
\[
0 = g(z, u) + g(\nabla_s z(s), v)|_{s=0}.
\]
Introducing the shape operator \( A_v : T_q K \to T_q K \) of \( K \) at \( q \), relative to the unit vector \( v \), we can rewrite this equation as
\[
g(z, u) + g(A_v z, w) = 0
\]
and, by the symmetry property of the shape operator,
\[
g(z, u) + g(A_v w, z) = 0.
\]
Hence, as \( z \) can be chosen arbitrarily in \( T_q K \), we obtain that \( u + A_v w \in N_q K \). Notice also that, since \( v \) is a unit vector, differentiating \( g(v(s), v(s)) = 1 \) we obtain \( g(u, v) = 0 \).

Resuming, the tangent space to \( UK \) at the point \( (q, v) \) is represented by the pairs of vectors \( (w, u) \in T_q M \times T_q M \) such that
\[
\tag{4.2}
w \in T_q K, \quad u + A_v w \in N_q K \quad \text{and} \quad g(u, v) = 0.
\]

Consider now a vector \( (w, u) \in T_{(q,v)} UK \), and the vector field
\[
X(t) = \partial_{UK} F(q, v, t)(w, u)
\]
along the normal geodesic \( \gamma(t) = F(q, v, t) \) with unit velocity vector
\[
\gamma'(t) = \partial_t F(q, v, t),
\]
where \( \partial_{UK} \) denotes the partial derivative with respect to the variable \( (q, v) \in UK \).

The field \( X \) is a Jacobi field along the geodesic \( \gamma(t) \), that is, it satisfies the following relations
\[
X(0) = w, \quad X'(0) = u,
\]
\[
X''(t) + R(X(t), \gamma'(t))\gamma'(t) = 0,
\]
where \( R \) is the Riemann curvature operator of \( M \) and we adopted the convention of denoting with \( T' \) the covariant derivative along the geodesic \( \gamma(t) \) of any vector or tensor field \( T \).

We take a basis \( \{(w_i, u_i)\} \), for \( i = 1, \ldots, n-1 \), of the tangent space \( T_{(q,v)} UK \) and we construct an \( n \)-vector \( \omega \) along \( \gamma \) as follows,
\[
\omega(t) = X_1(t) \wedge \cdots \wedge X_n(t)
\]
where the fields \( X_i(t) = \partial_{UK} F(q, v, t)(w_i, u_i) \) are Jacobi fields and \( X_n(t) = \gamma'(t) \). Notice that the relation
\[
X''_n(t) + R(X_n(t), \gamma'(t))\gamma'(t) = 0
\]
is satisfied by the field \( X_n \), since \( \gamma'' = 0 \). As \( \{(w_i, u_i)\} \) is a basis of \( T_{(q,v)} UK \), proving equation (4.1) is equivalent to show that \( \omega'(c(q, v)) \neq 0 \).

We argue by contradiction, by the Gauss Lemma (see [19, Chapter 3, Section E]) we have that \( X_n(t) \) is orthogonal to \( X_i(t) \) for every \( i = 1, \ldots, n-1 \), so we can suppose that the nonzero vector \( (w_1, u_1) \) is the generator of the kernel of \( \partial_{UK} F(q, v, c(q, v)) \). Hence, \( X_1(c(q, v)) = 0 \) and \( X_2(c(q, v)), \ldots, X_n(c(q, v)) \) generate a subspace of dimension \( (n-1) \), by the assumption on the rank of \( dF(q, v, c(q, v)) \).

Computing \( \omega'(c(q, v)) \) we get
\[
\omega'(c(q, v)) = X'_1(c(q, v)) \wedge X_2(c(q, v)) \wedge \cdots \wedge X_n(c(q, v))
\]
by linearity and the hypothesis that \( X_1(c(q, v)) = 0 \).

To conclude we need only to show that \( X'_1(c(q, v)) \) cannot belong to the \( (n-1) \)-dimensional subspace generated by \( X_2(c(q, v)), \ldots, X_n(c(q, v)) \).
Consider now the function \( f(t) \) given by
\[
f(t) = g(X'_1(t), X_i(t)) - g(X'_1(t), X'_i(t)).
\]
We have \( f(0) = g(u_1, w_i) - g(w_1, u_i) \) so, using the second relation in (4.2) and taking into account that \( w_i \in T_i K \) if \( i \leq n - 1 \), we obtain
\[
f(0) = g(w_1, A_v w_i) - g(A_v w_1, w_i)
\]
which is zero since the shape operator is symmetric. Moreover,
\[
f'(t) = g(X''_1(t), X_i(t)) - g(X'_1(t), X''_i(t)) = R(X'_1(t), \gamma'(t), \gamma'(t), X_i(t)) - R(X'_1(t), \gamma'(t), \gamma'(t), X'_i(t)) = 0
\]
by the properties of the curvature tensor.

Hence, the function \( f \) is identically zero and \( f(c(q, v)) = 0 \) gives
\[
g(X'_i(c(q, v)), X_i(c(q, v))) = 0 \quad \text{for } i = 2, \ldots, n - 1,
\]
so \( X'_i(c(q, v)) \) is orthogonal to every vector \( X_i(c(q, v)) \) and cannot belong to the subspace spanned by \( \{X_i(c(q, v))\} \) for \( i = 2, \ldots, n \). If \( X'_i(c(q, v)) \) would be zero, then by the differential relation (4.3), we would get \( X(t) = X'(t) = 0 \) for every \( t \) and in particular for \( t = 0 \), that is, \( (w_1, u_1) = (0, 0) \) contradicting the initial hypothesis.

By standard arguments of geometric measure theory (see [18]), the rectifiability of the cut locus has the following immediate consequence.

**Corollary 4.12.** The Hausdorff dimension of \( \text{Sing} \) (the cut locus of \( K \)) is at most \((n - 1)\).

To explain another consequence we need to introduce briefly the theory of functions with bounded variation, see [18, 29] for details. We say that a function \( u : \mathbb{R}^n \to \mathbb{R}^m \) is a function with locally bounded variation \( u \in BV_{loc} \), if its distributional derivative \( Du \) is a Radon measure. Such notion can be easily extended to maps between manifolds using smooth local charts.

A standard result says that the derivative of a locally semiconcave function stays in \( BV_{loc} \) in view of Proposition 3.4 this implies that the vector field \( \nabla d_K \) belongs to \( BV_{loc} \) in the open set \( M \setminus K \).

Now we define the subspace of \( BV_{loc} \) of functions (or vector fields, as before) with locally special bounded variation \( SBV_{loc} \) (see [3, 4, 5]).

The Radon measure representing the distributional derivative \( Du \) of a function \( u : \mathbb{R}^n \to \mathbb{R}^m \) with locally bounded variation can be always uniquely separated in three mutually singular measures
\[
Du = D^a u + Ju + Cu
\]
where the first term is the part absolutely continuous with respect to the Lebesgue measure \( L^n \), \( Ju \) is a measure concentrated on a \((n - 1)\)-rectifiable set and \( Cu \) (called the Cantor part) is a measure which does not charge the subsets of Hausdorff dimension \((n - 1)\).

The space \( SBV_{loc} \) is defined as the class of functions \( u \in BV_{loc} \) such that \( Cu = 0 \), that is, the Cantor part of the distributional derivative of \( u \) is zero.

**Corollary 4.13.** If \( K \) is of class \( C^r \) with \( r \geq 3 \), the vector field \( \nabla d_K \) belongs to the space \( SBV_{loc}(M \setminus K) \) of vector fields with locally special bounded variation.

**Proof.** Being the cut locus rectifiable, hence of Hausdorff dimension \((n - 1)\), the Cantor part of the distributional derivative of \( \nabla d_K \) cannot be concentrated on it, so it must be concentrated in the open set \( M \setminus (K \cup \text{Cut}(K)) \). By point 7 of Proposition 4.6, the field \( \nabla d_K \) belongs to \( C^{r - 1} \) in \( M \setminus (K \cup \text{Cut}(K)) \) then, by the hypotheses, it is at least \( C^2 \), hence its distributional derivative coincides with the product of the classical derivative with the Lebesgue measure, this shows that \( Cu(M \setminus (K \cup \text{Cut}(K))) = 0 \). These two facts together prove that \( \nabla d_K \) belongs to \( SBV_{loc}(M \setminus K) \).
A very important particular case is of our discussion is $K = \{p\}$. The cut locus of a point $p$ in $M$ arises naturally in various geometric problems, its definition is due to Poincaré [27] and its properties were studied by many authors, see for instance [10, 11, 21, 22, 25, 26, 30, 31]. A general discussion of its properties can be found in the books of Berger [9] and of Gallot, Hulin, Lafontaine [19].

Because of its importance, we resume here what we got in this special case.

**Proposition 4.14.** Let $p$ be a point of a smooth and connected Riemannian manifold $(M, g)$, of dimension $n$, then the squared distance function from $p$ is a locally semiconcave function on $M$ and its gradient is an SBV vector field on $M$. Moreover, $d_F^2$ is $C^\infty$ in $M \setminus \text{Cut}(p)$ which is an open neighborhood of $p$. The cut locus of $p$ is $C^\infty$ rectifiable with Hausdorff dimension at most $(n-1)$ and $\text{Cut}(p) \setminus J$ is locally a finite union of smooth hypersurfaces.

5. A Special Case

In this section we study a special example which is interesting for the discussion of the next section. We assume that $M$ is a two–dimensional analytic, connected and compact surface and $K$ is an one–dimensional embedded analytic submanifold of $M$ (a finite union of analytic curves). As before, our analysis also applies to closed sets $K$ with analytic boundary.

We look for topological results on the structure of the cut locus of $K$ generalizing some arguments introduced to study the special case $K = \{p\}$, see Myers [25, 26]. Our goal is to show that $\text{Cut}(K)$ is a finite graph and to connect its topological structure to the differential properties of the function $d_F^2$.

Clearly we have that $UK, F$ and $F^{-1}$, when it exists, are analytic. Notice that the fiber of $UK$ is $U_1K \cong \{-1, 1\}$.

The strong result given by analyticity is that the number of the optimal focal points is finite.

**Lemma 5.1.** The function $c : UK \to \mathbb{R}^+ \cup \{+\infty\}$ defined in the previous section is analytic in the open set where it is not $+\infty$.

**Proof.** With the same proof of Proposition 4.9, noticing that when $n = 2$ the set $G$ coincides with $G_1$, we can show that the set of points $(q, v, t) \in UK \times \mathbb{R}^+$ where $\det dF(q, v, t) = 0$ is a finite union of analytic curves. Hence, as this set is the graph of the map $c$, we have the thesis.

**Proposition 5.2.** The set $J$ of optimal focal points of $K$ is finite.

**Proof.** Being $M$ compact every geodesic cannot be minimal between its end points if it is longer than the diameter of $M$, hence a minimal geodesic joining an optimal focal point to $K$ has to be shorter than a fixed constant.

Consider an optimal focal point $p$ and let $F(q, v, t)$ be a minimal geodesic from $K$ to $p$ which has a non invertible differential $dF(q, v, \sigma(q, v))$ (notice that in this situation we have $c(q, v) = \sigma(q, v)$), we claim that $(q, v)$ is a critical point of the function $c$.

By the Gauss Lemma (see [19, Chapter 3, Section E]), the differential $dF(q, v, t)$ act on an element $(w, s) \in T(q, v, t)UK \times \mathbb{R}^+$ as follows,

\begin{equation}
\begin{aligned}
dF(q, v, t)(w, s) &= \partial_UK F(q, v, t)(w) + \partial_t F(q, v, t)(s) \\
&= X + sT
\end{aligned}
\end{equation}

where the two vectors $X, T \in T(q, v, t)M$ are mutually orthogonal and $T$ is the unit tangent vector to the geodesic $F(q, v, t)$. Taking into account that $UK$ is locally a curve, this shows that if $dF(q, v, t)$ is singular then $\partial_UK F(q, v, t) = 0$.

Consider now the pull–back $F^*g$ of the metric tensor $g$ on $T(q, v, t)UK \times \mathbb{R}^+$ via the map $F$. The set of points $(q, v, t)$ where this form is not positive definite covers the graph of $c$. Computing this form using equation (5.1) we have,

\begin{equation}
\begin{aligned}
(F^*g)_{(q, v, t)}((w, s), (w, s)) &= g_{F(q, v, t)}(dF(q, v, t)(w, s), dF(q, v, t)(w, s)) \\
&= s^2 + g_{F(q, v, t)}(\partial_UK F(q, v, t)(w), \partial_UK F(q, v, t)(w)) \\
&= s^2 + h(q, v, t)g_q(w, w)
\end{aligned}
\end{equation}
for a non-negative function $h : UK \times \mathbb{R}^+ \to \mathbb{R}$ and where $w \in T_{(q,v)}UK$ is considered as a vector in $T_qM$. Clearly, the set of points where the function $h : UK \times \mathbb{R}^+ \to \mathbb{R}$ is equal to zero contains the graph of $c$ by the previous discussion.

Suppose that $dc(q, v) \neq 0$, recalling that $UK$ is a curve there exists a small neighborhood $B$ of $(q, v)$ in $UK$ where the map $c$ is invertible and $c^{-1}$ is analytic in the open set $c(B) \subset \mathbb{R}^+$. Take a point $(r, z) \in B$ with $c(r, z) < c(q, v)$ and consider the curve $\gamma(s) : [0, c(q, v)] \to M$ defined by

\[
\begin{align*}
\gamma(0) &= r, \\
\gamma(s) &= F(r, z, s) \quad \text{for } s \in (0, c(r, z)], \\
\gamma(s) &= F(c^{-1}(s), s) \quad \text{for } s \in (c(r, z), c(q, v)],
\end{align*}
\]

that is, after a piece of geodesic, we follow the locus of first focal points. This is a piecewise analytic curve in $M$ starting from $K$ and ending at the point $p$. The first piece of $\gamma$ is a geodesic, hence its length is $c(r, z)$, the second piece follows the locus of first focal points. Using relation (5.2) we compute

\[
|\gamma(s)|^2 = g_{v(s)}(dF(c^{-1}(s), s)(w(s), 1), dF(c^{-1}(s), s)(w(s), 1)) \\
= 1 + h(c^{-1}(s), s)g(w(s), w(s)) \\
= 1
\]

for every $s \in (c(r, z), c(q, v)]$ and where $w(s) = \frac{d(c^{-1})}{ds}(s)$. In the last equality we used the fact that the point $(c^{-1}(s), s)$ belongs to the graph of $c$, where the function $h$ is zero.

Then the length of the second piece of $\gamma$ coincides with the variation in $s$, that is, $c(q, v) - c(r, z)$. Finally the total length of $\gamma$ is $c(q, v) = dK(p)$.

Such curve is $C^1$ since the tangent vectors of its two parts are equal at the point $F(r, z, c(r, z))$, but it is not a geodesic for $s \in (c(r, z), c(q, v)]$, otherwise (by uniqueness) $\gamma$ should coincide with $F(r, z, s)$ for every $s \in (0, c(q, v)]$ and this is impossible by construction.

This fact implies that there must exist a shorter curve joining $p$ with $r$ and this is in contradiction with the assumption that $F(q, v, t)$ is minimal, so the claim is proved.

Arguing by contradiction, if the set of optimal focal points would be infinite then in a connected component $C$ of $\{c(q, v) < +\infty\} \subset UK$ there would be infinite points $(q_i, v_i) \in C$ where $dc$ is zero and $F(q_i, v_i, c(q_i, v_i)) = p_i$ are distinct optimal focal points. By compactness and the initial argument on the length of a minimal geodesic from an optimal focal point, there must exists an accumulation point of the set $\{(q_i, v_i)\}$ in $C \subset \{c(q, v) < L\}$, for a suitable constant $L$. Then, by the analyticity of $c$, this would imply that $dc(q, v)$ is identically zero and $c(q, v)$ is constant in the component $C$.

Defining a function $H : UK \to M$ by $H(q, v) = F(q, v, c(q, v))$ we have that

\[
dH(q, v) = \partial_{UK}F(q, v, c(q, v)) + \partial_t F(q, v, c(q, v)) dc(q, v) = 0,
\]

as $\partial_{UK} F(q, v, c(q, v)) = 0$ and $dc(q, v) = 0$.

So the map $F(q, v, c(q, v))$ is constant in $C$. This implies that all the points $p_i = F(q_i, v_i, c(q_i, v_i))$ coincide contradicting the hypotheses.

As the map $\sigma : UK \to M$ is continuous, the cut locus $Cut(K)$ is given by a finite family of curves of kind $s \mapsto F(q(s), v(s), \sigma(q(s), v(s)))$ where $(q(s), v(s))$ is a curve describing a connected component of $UK$. We say that $p \in Cut(K)$ is an end point if at the point $p$ there arrives one and only one 1-cell of points of $Cut(K)$. We are going to prove that every end point is an optimal focal point.

First we exclude a very special case.

**Lemma 5.3.** If at a point $p \in Cut(K)$ there arrive an infinite number of minimal geodesics then all these geodesics start from a unique connected component of $K$ which is a geodesic circle around $p$, that is the set of points of $M$ at a certain distance $R$ from $p$.

Moreover, $p$ is an isolated point in $Cut(K)$, more precisely $Cut(K) \cap B_R(p) = \{p\}$. Conversely, if $p$ is an isolated point in $Cut(K)$ then there is a connected component of $K$ which is a geodesic circle around $p$. 

Remark 5.4. Notice that $p$ is an optimal focal point by Lemma 4.8.
Since the connected components of $K$ are finite, it follows that the isolated point of $\text{Cut}(K)$ are finite.

Proof. If $F(q_i, v_i, t)$ is the infinite family of minimal geodesics $F(q_i, v_i, t)$ of length $R$ ending at $p$, then all the distinct points $q_i$ belong to the geodesic circle of center $p$ and radius $R$ in $M$. The set of points $\{(q_i, v_i)\} \in UK$ clearly has an accumulation point, hence, by the analyticity of $UK$, the function $H(q, v) = F(q, v, R)$ is constantly equal to $p$ in the connected component of $UK$ containing such accumulation point. Again by the analyticity of the connected components of $UK$ and of the curves constituting $K$, we can conclude that the whole circle has to be a connected component of $K$. Hence, from every point of this circle there is a minimal geodesic ending at $p$ and there cannot be other points of $K$ inside the circle, otherwise their distance from $p$ would be less than the radius $R$.

Suppose now that $p$ is isolated in $\text{Cut}(K)$ and consider the open connected component $\Gamma$ of $M \setminus K$ which contains $p$. The boundary of $\Gamma$ is a subset $K'$ of $K$ and every minimal geodesic starting from $K'$ with an initial velocity vector pointing toward $\Gamma$, must necessarily cease to be minimal at $p$, as $\text{Cut}(K) \cap \Gamma = \{p\}$. This last assertion follows from the fact that $\Gamma$, by point 6 of Proposition 4.6, can be continuously retracted on $\text{Cut}(K) \cap \Gamma$, hence $\text{Cut}(K) \cap \Gamma$ is connected and then it coincides with $\{p\}$. This shows that there are infinite minimal geodesics from $K$ to $p$ and we can conclude as in the first part of the lemma.

Suppose now that $p \in \text{Cut}(K) \setminus J$, so the number $n > 1$ of minimal geodesics $F(q_i, v_i, t)$ ending at $p$ is finite. Consider a small ball $B$ around $p$ in $M$, then these $n$ minimal geodesics cut the ball $B$ in $n$ sectors that we call $S_i$. Any minimal geodesic starting in a sufficiently small neighborhood of $(q_i, v_i) \in UK$ has its cut point in the ball $B$ by continuity of the function $\sigma$, moreover this geodesic cannot cross one of the geodesics $F(q_i, v_i, t)$ before reaching its cut point, otherwise this latter ceases to be minimal. Hence, considering the continuous curve of the cut points of the geodesics starting at the points of $UK$ locally on the right side of $(q_i, v_i)$ (remember that $UK$ is one–dimensional), we have that it is all contained in one of the sectors $S_i$, more precisely, by continuity, in one of the two sectors adjacent to the geodesic $F(q_i, v_i, t)$. This curve gives a 1–cell of $\text{Cut}(K)$ approaching $p$.

With the same argument, considering the points locally on the left side of $(q_i, v_i)$ we obtain another 1–cell, in the other sector. Thus, we can conclude that the number of 1–cells of $\text{Cut}(K)$ arriving at a point $p$ is at least the number of the sectors $S_i$, hence at least the number of minimal geodesics from $p$ to $K$.

This implies that every end point of $\text{Cut}(K)$ where there arrives one and only one 1–cell, has a unique minimal geodesic to $K$ so it has to be an optimal focal point.

Putting together these facts and Lemma 5.3, by Proposition 5.2 the end points are finite. Following Myers [26], this result implies that the cut locus of $K$ is a linear graph and locally a tree, moreover the points where the order of the graph is greater than two are finite.

Now we introduce the map $\#G(p)$ from $M$ to $\mathbb{N} \cup \{\infty\}$ counting the number of minimal geodesics from $K$ to a point $p$.

Proposition 5.5. An arc in $\text{Cut}(K)$ containing no points of $J$ and no interior points $p$ with $\#G(p) > 2$ is a regular analytic arc.

Proof. Let $\gamma$ be such an arc in $\text{Cut}(K)$. Consider a point $p_0 \in \gamma$ with $F(q_1, v_1, \sigma(q_1, v_1)) = F(q_2, v_2, \sigma(q_2, v_2)) = p_0$, by the fact that $p_0$ is not an optimal focal point, applying the implicit function theorem, there is an open neighborhood $B$ of $p$ in $M \setminus K$ without optimal focal points and there exist analytic functions $z_1, z_2 : B \to UK$, $t_1, t_2 : B \to \mathbb{R}^+$ such that, $F(z_1(p), t_1(p)) = F(z_2(p), t_2(p)) = p$ for every $p \in B$ and $z_1(B) \cap z_2(B) = \emptyset$. If $B$ is small enough, for every point $p$ of $B$ we have $\#G(p) \leq 2$, then $t_1(p) = t_2(p) = d_K(p)$ if and only if $p \in \text{Cut}(K) \cap B = \gamma \cap B$.

The rest of the proof proceed as in Proposition 4.7.

Our last goal is to show that the order of a point $p \in \text{Cut}(K)$, as a graph, is equal to $\#G(p)$. The order of a point $p$ of $\text{Cut}(K)$ is defined as the number of distinct 1–cells of $\text{Cut}(K)$ arriving at $p$. 

We have already seen before that that the order of $p$ is always greater than the value $\#\mathcal{G}(p)$. We now prove the opposite inequality.

Notice that the optimal geodesics cannot cross $\text{Cut}(K)$, otherwise they cease to be minimal since they would intersect another minimal geodesic. We can take a small ball $B$ around a point $p \in \text{Cut}(K)$ so that the 1-cells divide it in $n$ sectors. Let $S$ be one of these sectors and consider a sequence of points $p_i \notin \text{Cut}(K)$ all contained in $S$ and converging to $p$ such that $F(q_i, v_i, t)$ are the minimal geodesics relative to $p_i$. By compactness, we can suppose that the points $(q_i, v_i) \in UK$ converge to a point $(q, v)$, hence the minimal geodesics $F(q_i, v_i, t)$ converge to a minimal geodesic $F(q, v, t)$ from $K$ to $p$. Being the points $p_i$ contained in $S$ the final part $F(q_i, v_i, t) \cap B$ of the respective minimal geodesics have to be contained in the sector $S$ and so also the final part $F(q, v, t) \cap B$ of the minimal geodesic for $p$.

Taking into account the fact that such minimal geodesic cannot intersect the cut locus, we conclude that there is at least a minimal geodesic for every sector $S$. Being the number of the sectors equal to the order of $\text{Cut}(K)$ as a graph, we proved the opposite inequality we claimed before. Hence, there are exactly $\#\mathcal{G}(p)$ 1-cells of the cut locus arriving at every point $p \in \text{Cut}(K)$.

We summarize all the discussion of this section in the following theorem.

**Theorem 5.6.** The set $\text{Cut}(K)$ is a disjoint finite union of isolated points and linear graphs, each one locally a tree.

The order of every point $p \in \text{Cut}(K)$ equals the function $\#\mathcal{G}(p)$, counting the number of minimal geodesics from $p$ to $K$. In particular, the set of points of $\text{Cut}(K)$ with only one or more than two minimal geodesics is finite.

The set of optimal focal points in $\text{Cut}(K)$ is finite.

All the isolated points and end points of $\text{Cut}(K)$ are optimal focal points.

Considering as vertices of the graph the optimal focal points and the points of order greater than two, the arcs connecting such vertices are regular analytic arcs.

**Remark 5.7.** The analysis of this section also applies with small modifications to the case when $K$ is a finite set of points, considering an auxiliary set $\tilde{K}$ consisting of a family of disjoint circles centered at the points of $K$ with a radius $R$ small enough.

### 6. Singularities of Solutions of Hamilton–Jacobi Equations

The ideas employed in the study of the distance from $K$ and of the cut locus can be extended to analyse also the set of singularities of viscosity solutions of general Hamilton–Jacobi problems

$$
\begin{aligned}
\{ & H(x, du(x), u(x)) = 0 & \text{in } \Omega \subset M, \\
& u = u_0 & \text{on } \partial \Omega. \\
\end{aligned}
$$

Moreover, geometric results on the cut locus suggest conjectures about the viscosity solutions of these equations. We give now an example.

Suppose that $A(x)$ is an analytic map from the closure of the unit ball $B$ of $\mathbb{R}^2$ to the space of positively defined $2 \times 2$-matrices.

We consider the following problem,

$$
\begin{aligned}
\{ & (A(x)\nabla u(x), \nabla u(x)) = 1 & \text{in } B, \\
& u = 0 & \text{on } \partial B. \\
\end{aligned}
$$

Using arguments similar to those of Section 5, it is possible to prove that the closure of the singular set of the viscosity solution is a finite graph.

If we look for the same result in the $C^\infty$ case, a counterexample can be found as follows. It is possible to endow the two-dimensional sphere with a $C^\infty$ metric tensor $g$ such that the cut locus of a certain point $p$ is very wild, that is, it is not triangulable, hence it is not a finite graph (see [21]). Cutting away from the sphere $S^2$ a small geodesic disc $D$ around $p$ whose intersection with the cut locus of $p$ is empty, and mapping stereographically from $p$ the set $S \setminus D$ on $\mathbb{R}^2$, we have that the closure of the singular set of the viscosity solution of Problem (6.2) in a ball of $\mathbb{R}^2$, where $A$
is given by the push–forward of the metric $g$ via the stereographic projection, coincides with a homeomorphic image of the cut locus of $p$, hence it is not a finite graph.

However, another results on the cut locus of a point says that for a generic $C^{\infty}$ metric (in the category sense, the cut locus of every point of a surface is triangulable and has no points of order higher than three (see [30]). Hence, changing a little our point of view, this discussion suggests the following conjecture.

**Conjecture 6.1.** For a generic $C^{\infty}$ function $A(x)$ from the closed unit ball $B$ of $\mathbb{R}^2$ to the space of positive definite $2 \times 2$–matrices, the closure of the singular set of the viscosity solution of problem

$$
\begin{cases}
  \langle A(x) \nabla u(x), \nabla u(x) \rangle = 1 & \text{in } B, \\
  u = 0 & \text{on } \partial B
\end{cases}
$$

is a finite graph.

More in general, the same question can be asked about Problem 6.1 for a generic function $H$, domain $\Omega$ or boundary data $u_0$.

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**References**

1. G. Alberti, *On the structure of singular sets of convex functions*, Calc. Var. **2** (1994), 17–27.
2. G. Alberti, L. Ambrosio, and P. Cannarsa, *On the singularities of convex functions*, Manuscripta Math. **76** (1992), 421–435.
3. L. Ambrosio, *A compactness theorem for a new class of functions of bounded variation*, Boll. Un. Mat. Ital. **3-B** (1989), 857–881.
4. ———, *Variational problems in $SBV$*, Acta Applicandae Mathematicae **17** (1989), 1–40.
5. ———, *Existence theory for a new class of variational problems*, Arch. Rat. Mech. Anal. **111** (1990), 291–322.
6. L. Ambrosio, P. Cannarsa, and H. M. Soner, *On the propagation of singularities of semi–convex functions*, Ann. Sc. Norm. Sup. Pisa **XX** (4) (1993), 597–616.
7. L. Ambrosio and C. Mantegazza, *Curvature and distance function from a manifold*, J. Geom. Anal. **8** (1998), no. 5, 719–744.
8. L. Ambrosio and H. M. Soner, *Level set approach to mean curvature flow in any codimension*, J. Diff. Geom. **43** (1996), no. 4, 693–737.
9. M. Berger, *Geometry*, Springer–Verlag, 1987.
10. A. L. Besse, *Manifolds all of whose Geodesics are Closed*, Springer–Verlag, 1978.
11. M. Buchner, *The structure of the cut–locus in dim $\leq 6$*, Compositio Math. **37** (1978), 103–119.
12. P. Cannarsa and H. M. Soner, *On the singularities of the viscosity solutions to Hamilton–Jacobi–Bellman equations*, Indiana Univ. Math. J. **36** (1987), 501–524.
13. M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.
14. M. G. Crandall, L. C. Evans, and P.-L. Lions, *Some properties of viscosity solutions of Hamilton–Jacobi equations*, Trans. Am. Math. Soc. **282** (1984), 487–502.
15. M. G. Crandall, H. Ishii, and P.-L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Am. Math. Soc. **27** (1991), 1–67.
16. M. Delfour and J.-P. Zolésio, *Shape analysis via oriented distance functions*, J. Functional Analysis **123** (1994), 129–201.
17. ———, *Shape analysis via distance functions: local theory, Boundaries, Interfaces and Transitions* (M. Delfour, ed.), CRM Proc. Lect. Notes Ser., AMS, 1998.
18. H. Federer, *Geometric Measure Theory*, Springer–Verlag, New York, 1969.
19. S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry*, Springer–Verlag, Berlin, 1990.
20. Y. Giga, S. Goto, H. Ishii, and M. H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J. **40** (1991), no. 2, 443–469.
21. H. Gluck and D. Singer, *The existence of non triangularizable Cut Loci*, Bull. Am. Math. Soc. **82** (1976), no. 4, 599–602.
22. S. Koyagush, *On Conjugate and Cut Loci in Global Geometry and Analysis*, Studies in Mathematics, vol. 4, The Mathematical Ass. of America, Washington D.C., 1976.
23. S. N. Kruzhkov, *Generalized solutions of the Hamilton–Jacobi equations of eikonal type*, I. Math. USSR Sb. **27** (1975), 406–446.
24. P. L. Lions, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, Boston, 1982.
25. S. B. Myers, *Connections between differential geometry and topology*, Duke Math J. **1** (1935), 376–391.
26. ———, *Connections between differential geometry and topology II*, Duke Math J. **2** (1936), 95–102.
27. H. Poincaré, *Sur les lignes géodésiques des surfaces convexes*, Trans. Am. Math. Soc. 6 (1905), 237–274.
28. T. Sakai, *Riemannian Geometry*, AMS, 1996.
29. L. Simon, *Lectures on Geometric Measure Theory*, Proc. Center for Mathematical Analysis, vol. 3, Australian National University, Canberra, 1983.
30. C. T. C. Wall, *Geometric properties of generic differentiable manifolds*, Geometry and Topology (J. Palis and M. do Carmo, eds.), Springer–Verlag, 1977.
31. A. Weinstein, *The cut–locus and conjugate points of a Riemannian manifold*, Ann. Math. 87 (1968), 29–41.

(Carlo Mantegazza) SCUOLA NORMALE SUPERIORE PISA, ITALY, 56126
E-mail address, C. Mantegazza: mantegaz@sns.it

(Andrea Mennucci) SCUOLA NORMALE SUPERIORE PISA, ITALY, 56126
E-mail address, A. C. Mennucci: mennucci@sns.it