Algorithmic complexity of quantum capacity

Samad Khabbazi Oskouei · Stefano Mancini

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Abstract We analyze the notion of quantum capacity from the perspective of algorithmic (descriptive) complexity. To this end, we resort to the concept of semi-computability in order to describe quantum states and quantum channel maps. We introduce algorithmic entropies (like algorithmic quantum coherent information) and derive relevant properties for them. Then we show that quantum capacity based on semi-computable concept equals the entropy rate of algorithmic coherent information, which in turn equals the standard quantum capacity. Thanks to this, we finally prove that the quantum capacity, for a given semi-computable channel, is limit computable.

Keywords Algorithmic complexity · Quantum entropies · Quantum channels · Quantum capacity

1 Introduction

Quantum channels are maps on the set of quantum states (density operators) that are linear, completely positive, and trace preserving [1]. They generalize the notion of classical channels, that is stochastic maps acting on probability distributions. As such they permit to transfer quantum information (besides classical one), namely quantum correlations also known as entanglement. Their quantum information transmission
ability would be captured by a capacity notion (quantum capacity), similar to what happens in the classical information theoretical framework with the Shannon capacity of classical channels. A remarkable difference is that the quantum capacity requires a regularization formula [1], that is, the computation of an entropic rate over infinite many quantum channel uses. This is due to the possible effects arising by employing entangled input across different channel uses. Due to that, quantum capacity evaluation results in a daunting task.

One way to deal with this problem can be to write down a computer program in a classical computer or even in a quantum one and try to evaluate the quantity expressed by a capacity formula.

This motivates us to study the quantum capacity when restricting to the usage of operators (and super-operators) that can be accepted by a computer. As matter of fact, in simulating physical phenomena by computer programs, we use algebraic and computable numbers (like \(e, \pi\)). Therefore, the entries of matrix representations of operators (and super-operators) should be taken as algebraic numbers, or limit of computable sequence of them. To this end, we invoke the notion of semi-computability introduced in [2] for a given function \(f : \mathbb{N} \rightarrow \mathbb{R}\) and then extended to density matrices and operators in separable Hilbert spaces [3–5]. Of course, this notion can be considered as well for super-operators, and hence, quantum channels. The main features of semi-computability is the universality concept, that is the existence of a semi-measure \(\mu\) which dominates all other semi-computable semi-measures up to a positive multiplicative constant [2]. A universal semi-measure is related to the concept of algorithmic complexity in that \(-\log \mu(i_1 \ldots i_n) \leq K(i_1 \ldots i_n) + c\) and \(K(i_1 \ldots i_n) \leq -\log \mu(i_1 \ldots i_n) + c'\) where \(i_1 \ldots i_n\) is a sequence of symbols from a finite alphabet, \(c, c' > 0\) are constant independent of \(i_1 \ldots i_n\), and \(K\) is the Kolmogorov complexity. The latter notion was developed by [6], Kolmogorov [7], and Chaitin [8]. In a nutshell, the complexity of a target object is measured by the difficulty to describe it; in the case of targets describable by binary strings, they are algorithmically complex when their shortest binary descriptions are essentially of the same length in terms of necessary bits, the descriptions being binary programs such that any universal Turing machine that runs them outputs the target string. Taking this approach, the quantum capacity should be characterized in terms of algorithmic (Kolmogorov) complexity. There are several ways this complexity can be extended to the quantum realm [3,9–11]. Here we follow the Gacs approach [3]. This is based on the notion of universal semi-density matrix,\(^1\) that is, on the existence of a density matrix \(\hat{\mu}\) on separable Hilbert space that dominates any other semi-computable density matrix up to a multiple positive constant number. The algorithmic complexity for a semi-computable density matrix \(\rho\) is given by \(-Tr(\rho \log \hat{\mu})\).

Along this way, we algorithmically define the coherent information in quantum systems. The nice feature is that this quantity is linear with respect to its argument, a property that does not hold true for the standard entropies. Nonetheless, we will prove that their rates equal those of the standard entropies. Then we show that quantum capacity based on semi-computable concept, namely the entropy rate of algorithmic

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\(^1\) Semi-density matrix is a broader notion with respect to the density matrix, in that it conceives positive matrices with trace not necessarily equal to 1, although finite.
coherent information, equals the standard quantum capacity. Thanks to this, we finally prove that quantum capacity is limit computable, thus shading light on the issue of computability or non-computability of quantum channel capacity [12,13]. Actually, given the hierarchy levels of non-computability, this result represents a step forward since the simple non-computability claimed in Ref. [13].

Although we cannot state computability of quantum capacity, at the end, we present a method to compute it on a restricted subset of density matrices which might be useful for computer programmers.

The organization of the paper foresees an initial Sect. 2 where we recall some basic notions and set the notation. Then Sect. 3 revisits relevant entropic quantities from an algorithmic point of view and contains the derivation of some of their relevant properties, including the fact that the algorithmic quantum capacity coincides with the standard one. In Sect. 4, one can find the proof that this is limit computable. Furthermore, an algorithmic method to compute it on a restricted subset of density matrices is presented there. Section 5 is for conclusions and outlook.

2 Preliminaries

A quantum channel is a completely positive and trace-preserving linear map \( \Phi : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B) \), where \( \mathcal{B}(\mathcal{H}) \) is the algebra of bounded linear operators defined on the Hilbert space \( \mathcal{H} \). Actually, states at input or output of a quantum channel are bounded operators of unit trace (density operators) constituting a proper subset \( \mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \).

A relevant entropic quantity for a quantum channel is the quantum coherent information

\[
I_c(\rho, \Phi) := S(\Phi(\rho)) - S(\rho, \Phi).
\]

(2.1)

Here \( S(\Phi(\rho)) \) is the output state von Neumann entropy, being

\[
S(\rho) := -Tr(\rho \log \rho).
\]

Additionally \( S(\rho, \Phi) \) is the so-called exchange entropy defined by

\[
S(\rho, \Phi) := S\left(\tilde{\Phi}(\rho)\right),
\]

where \( \tilde{\Phi} \) is the complementary channel. That is, let the isometry \( V : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B) \otimes \mathcal{B}(\mathcal{H}_E) \) be the Stinespring dilation of the channel \( \Phi \) [14], where \( E \) labels the environment, then

\[
\Phi(\rho) = Tr_E(V\rho V^\dagger),
\]

\[
\tilde{\Phi}(\rho) := Tr_B(V\rho V^\dagger).
\]

\footnote{Throughout, the paper the \log is intended on base 2.}
The entropy rate corresponding to (2.1) is

\[ Q_c (\Phi) := \lim_{n \to \infty} \frac{1}{n} \max_{\rho_n} I_c (\rho_n, \Phi^\otimes n), \] (2.2)

where \( \rho_n \in S(\mathbb{H}_A^\otimes n) \). In Refs. [15,16], it is shown that the quantum capacity for a quantum channel \( \Phi \) is given by \( Q_c (\Phi) \).

Furthermore, for a channel \( \Phi \), it is known that [17]

\[ S (\Phi (\rho), \Phi (\sigma)) \leq S (\rho, \sigma), \] (2.3)

where

\[ S (\rho, \sigma) := \text{Tr} (\rho (\log \rho - \log \sigma)). \] (2.4)

is the relative entropy between two density operators \( \rho, \sigma \in S(\mathbb{H}_A) \).

The aim of this paper is to revisit the characterization (2.2) of a quantum channel \( \Phi \) by an algorithmic approach. Let us first recall few basic notion about computability and algorithmic complexity.

A function from \( \mathbb{N}^k \) to \( \mathbb{N} \) is called partially computable if it is computed by a Turing machine [18] (here Turing machine, program, and algorithm are used interchangeably).

A real number \( x \in \mathbb{R} \) is called computable if there exists a computable function \( f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) such that \( |x - f (n)| < 2^{-n} \). For example, \( e \) and \( \pi \) are computable real numbers. A real number \( x \in \mathbb{R} \) is called limit computable if there exists a computable function \( f : \mathbb{N} \to \mathbb{Q} \) such that \( x = \lim_{n \to \infty} f (n) \). This means that we can approximate \( x \) by rationals, since \( \mathbb{Q} \) is isomorphic with \( \mathbb{N} \times \mathbb{N} \). However, a limit computable real number may not be computable. In fact for a limit computable number, we cannot always find the modulus of convergence, i.e., the number of steps we do need to get close to it by a chosen distance \( \epsilon \).

The set of limit computable numbers, which contains all computable numbers, is countable and so there are uncountable non-limit computable numbers.3

Turning to the quantum world, we will denote by \( \mathbb{H} \) the infinite-dimensional separable Hilbert space obtained by the closure of the union of the nested infinite sequence \( \mathbb{H}_{[0,n]} \subset \mathbb{H}_{[0,n+1]} \) with respect to the norm coinciding with the usual Hilbert norm on each \( \mathbb{H}_{[0,n]} \). Here \( \mathbb{H}_{[0,n]} = \mathbb{C}^{n+1} \).4 The corresponding orthogonal projections from \( \mathbb{H} \) onto \( \mathbb{H}_{[0,n]} \) will be denoted by \( P_n \), and the canonical injected subspace \( \mathbb{H}_{[0,n]} \) into \( \mathbb{H} \) will be identified by \( \mathbb{H}_{[0,n]} \). The concept of semi-computable semi-density matrices on infinite-dimensional separable Hilbert spaces are introduced in [4]. In such a context, we consider a fixed orthonormal basis \( \{|i\rangle \}_{i \in \Omega^*_2} \), where \( \Omega^*_2 \) is the set of all finite length binary strings. Indeed, for any \( i, j \in \Omega^*_2 \), \( \langle i|j \rangle = 1 \), if \( i = j \) and 0 otherwise.

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3 Example: Let us consider the subset \( A \) of all limit computable numbers between 0 and 1 which are represented as binary numbers. We can enumerate them by a Turing machine as \( \psi_1, \psi_2, \ldots, \psi_k, \ldots \). We denote by \( \varepsilon_k = 0, \psi_1 (1), \psi_2 (2), \ldots, \psi_k (n) \ldots \) the \( k \)th element of \( A \) being \( \psi_k (n) \) the \( n \)th digit of \( \varepsilon_k \). Now, consider a number \( \lambda \in [0, 1] \) whose \( n \)th digit is equal to \( \psi_n (n) + 1 \) modulo 2. It is obvious that \( \lambda \) is different of any \( \varepsilon_k \), and so \( \lambda \) is non-limit computable. The set of such \( \lambda \)'s is clearly uncountable.

4 Following [4], the embedding is obtained by turning the last bit of each canonical basis element to 0.
A linear operator $T : \mathbb{H}_{[0,n-1]} \rightarrow \mathbb{H}_{[0,n-1]}$, will be called elementary if the real and imaginary parts of all of its matrix entries are algebraic numbers. The linear operator $T : \mathbb{H} \rightarrow \mathbb{H}$, is a semi-density matrix if $T$ is positive and $0 \leq Tr(T) \leq 1$.

Let $n_1, n_2 \in \mathbb{N}$ and $n_1 \leq n_2$. Let $T_j : \mathbb{H}_{[0,n_j-1]} \rightarrow \mathbb{H}_{[0,n_j-1]}$, $j = 1, 2$, be two linear operators: $T_2$ will be said to be quasi-greater than $T_1$, written as $T_1 \leq_q T_2$, if $P_{n_1} T_2 P_{n_1} - T_1 \geq 0$, where $P_{n_1}$ is the canonical projector from $\mathbb{H}_{n_2}$ to $\mathbb{H}_{n_1}$. A sequence of linear operators $T_n : \mathbb{H}_{[0,n-1]} \rightarrow \mathbb{H}_{[0,n-1]}$ will be called quasi-increasing if for all $n \geq 1$, $T_{n+1} \geq_q T_n$.

A linear operator $T$ on $\mathbb{H}$ is a semi-computable semi-density matrix, if there exists a computable quasi-increasing sequence of elementary semi-density matrices $T_n \in \mathbb{B}(\mathbb{H}_{[0,n-1]}) \subseteq \mathbb{B}(\mathbb{H})$ such that $\lim_{n \to \infty} \| T - T_n \|_1 = 0$.

It has been shown in [4] that there exists a universal semi-computable semi-density matrix $\hat{\mu}$ in the following sense:

For any semi-computable semi-density matrix $\rho$, there exists a constant number $c_\rho > 0$ such that $c_\rho \rho \leq \hat{\mu}$.

**Definition 2.1** The upper Gacs complexity for any semi-computable semi-density matrix $\rho \in \mathcal{S}(\mathbb{H})$ is defined as follows:

$$G(\rho) := -Tr(\rho \log \hat{\mu}).$$ (2.5)

Let $A B$ be a composite quantum system with two subsystems $A$ and $B$. The associated Hilbert space $\mathbb{H}_{AB}$ can be extended to infinite-dimensional separable Hilbert space $\mathbb{H}$ containing all Hilbert subspaces $\mathbb{H}^n_{AB}, n \in \mathbb{N}$.

Then, $\hat{\mu}_{AB} := P_{\dim \mathbb{H}_{AB}} \hat{\mu} P_{\dim \mathbb{H}_{AB}}$ (with $P_{\dim \mathbb{H}_{AB}} : \mathbb{H} \rightarrow \mathbb{H}_{AB}$ the canonical projector onto finite-dimensional Hilbert space $\mathbb{H}_{AB}$), $\hat{\mu}_A := Tr_B(\hat{\mu}_{AB})$ and $\hat{\mu}_B := Tr_A(\hat{\mu}_{AB})$, are universal semi-density matrices on $\mathbb{H}_{AB}, \mathbb{H}_A$ and $\mathbb{H}_B$, respectively. In addition, we can establish universal semi-density matrices on Hilbert spaces $\mathbb{H}^n_{AB}, \mathbb{H}^n_A$ and $\mathbb{H}^n_B$, respectively, as follows:

$$\hat{\mu}^n_{AB} := P_{\dim \mathbb{H}^n_{AB}} \hat{\mu} P_{\dim \mathbb{H}^n_{AB}}, \quad \hat{\mu}^n_B := Tr_A(\hat{\mu}^n_{AB}), \quad \hat{\mu}^n_A := Tr_B(\hat{\mu}^n_{AB}).$$

**3 Algorithmic quantum information**

**Definition 3.1** A quantum channel $\Phi : \mathcal{B}(\mathbb{H}_A) \rightarrow \mathcal{B}(\mathbb{H}_B)$ is defined semi-computable if for any semi-computable semi-density matrix $\rho \in \mathcal{S}(\mathbb{H}_A), \Phi(\rho)$ is a semi-computable semi-density matrix on $\mathcal{S}(\mathbb{H}_B)$.

For example, quantum channels with algebraic entries in the Choi matrix representation [19] are semi-computable. As another example, let us consider the depolarizing quantum channel $\Phi(\rho) := p \rho + \frac{1-p}{d} I$, where $d$ is the dimension of Hilbert space. If $p$ is a limit computable number, then $\Phi$ is semi-computable channel. On the contrary, if $p$ is not limit computable, then the channel is not semi-computable.

We are now going to describe a fundamental property of relative entropy under the action of a semi-computable quantum channel.
Theorem 3.1 Let $\mathcal{H}_A, \mathcal{H}_B$ be finite-dimensional Hilbert spaces and $\Phi : \mathbb{B}(\mathcal{H}_A) \to \mathbb{B}(\mathcal{H}_B)$ be a semi-computable quantum channel. Then,

$$S(\Phi^\otimes(n)(\rho_n), \hat{\mu}_B^n) \leq S(\rho_n, \hat{\mu}_A^n) + \alpha(n),$$

where $\hat{\mu}_A^n$ and $\hat{\mu}_B^n$, for each $n$, are universal semi-density matrices on $\mathcal{H}_A^\otimes(n)$ and $\mathcal{H}_B^\otimes(n)$, respectively, and $\lim_{n \to \infty} \frac{\alpha(n)}{n} = 0$.

Proof Since $\Phi$ is semi-computable then $\Phi^\otimes(n)(\hat{\mu}_A^n)$, for any $n$, is also semi-computable semi-density matrix. Now, let us consider the semi-computable semi-density matrix $\sum_{n=1}^{\infty} \delta(n) \Phi^\otimes(n)(\hat{\mu}_A^n)$ where

$$\delta(n) = \frac{1}{n \log^2 n}. \quad (3.1)$$

There exists a constant number $c_\Phi$ such that

$$c_\Phi \delta(n) \Phi^\otimes(n)(\hat{\mu}_A^n) \leq c_\Phi \sum_{n=1}^{\infty} \delta(n) \Phi^\otimes(n)(\hat{\mu}_A^n) \leq \hat{\mu}. \quad (3.1)$$

Then,

$$c_\Phi \delta(n) \Phi^\otimes(n)(\hat{\mu}_A^n) = c_\Phi \delta(n) Tr_A(P_{\dim \mathcal{H}_A^\otimes(n)} \Phi^\otimes(n)(\hat{\mu}_A^n) P_{\dim \mathcal{H}_A^\otimes(n)})$$

$$\leq Tr_A(P_{\dim \mathcal{H}_A^\otimes(n)} \hat{\mu} P_{\dim \mathcal{H}_A^\otimes(n)}) = \hat{\mu}_B^n.$$}

Furthermore,

$$-Tr (\Phi^\otimes(n)(\rho_n) \log \hat{\mu}_B^n) \leq -Tr (\Phi^\otimes(n)(\rho_n) \log \Phi^\otimes(n)(\hat{\mu}_A^n) \log c_\Phi - \log \delta(n).$$

By adding $Tr (\Phi^\otimes(n)(\rho_n) \log \Phi^\otimes(n)(\rho_n))$ to both sides of inequality, we obtain

$$S(\Phi^\otimes(n)(\rho_n), \hat{\mu}_B^n) \leq S(\Phi^\otimes(n)(\rho_n), \Phi^\otimes(n)(\hat{\mu}_A^n)) - \log c_\Phi - \log \delta(n).$$

By the relation 2.3, the proof is complete. \(\square\)

Definition 3.2 The algorithmic coherent information for a given quantum channel $\Phi : \mathbb{B}(\mathcal{H}_A) \to \mathbb{B}(\mathcal{H}_B)$ is defined as follows:

$$IG_c(\rho, \Phi) := G(\Phi(\rho)) - G(\rho, \Phi), \quad (3.2)$$

where following Definition 2.1

$$G(\rho) := -Tr(\rho \log \hat{\mu}_A), \quad G(\rho, \Phi) := -Tr \left( \Phi(\rho) \log \hat{\mu}_E \right).$$
Definition 3.3 For a given quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, the algorithmic coherent information entropy rate is defined as follows:

$$QG_c(\Phi) := \lim_{n \to \infty} \frac{1}{n} \max_{\rho_n} IG_c(\rho_n, \Phi \otimes^n),$$

(3.3)

where $\rho_n \in S(\mathcal{H}_A \otimes^n)$ are semi-computable density matrices.

The following theorem shows that the quantum capacity based on semi-computability equals the standard one. This is remarkable because the former is computed by linear algorithmic coherent information while the latter by nonlinear coherent information.

Theorem 3.2 For a semi-computable quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, we have

$$Q_c(\Phi) = QG_c(\Phi),$$

where the maximum in the information entropy rate is taken over all semi-computable density matrices.

Proof First we show that

$$QG_c(\Phi) \leq Q_c(\Phi).$$

Let $\rho_n \in S(\mathcal{H}_A \otimes^n)$ be a computable sequence of semi-computable semi-density matrices. Defining $\rho := \sum_{n=1}^{\infty} \delta(n)\Phi \otimes^n(\rho_n)$, where $\delta(n)$ is like in Eq. (3.1), it is clear that $\rho$ is semi-computable semi-density matrix, and hence there exists a constant number $c_B > 0$, such that

$$c_B \rho \leq \hat{\mu} \Rightarrow c_B \delta(n)\rho_n \leq \hat{\mu} \Rightarrow -\log \hat{\mu} \leq -\log c_B - \log \delta(n) - \log \tilde{\Phi} \otimes^n(\rho_n).$$

Now, let $P_{AB} : \mathcal{H} \to \mathcal{H}_{AB} \otimes^n$ be the canonical projector. Then, we have

$$-\log \hat{\mu}_B^n = -Tr_A(P_{AB} \log \hat{\mu} P_{AB}) \leq -\log c_B - \log \delta(n) - Tr_B(P_{AB} \log \tilde{\Phi} \otimes^n(\rho_n) P_{AB}) = -\log c_A - \log \delta(n) - \log \tilde{\Phi} \otimes^n(\rho_n).$$

Therefore,

$$IG_c(\rho_n, \Phi \otimes^n) = G(\Phi \otimes^n(\rho_n)) - G(\tilde{\Phi} \otimes^n(\rho_n)) \leq S(\Phi_E^n(\rho_n)) - \log \delta(n) \log c_B - G(\tilde{\Phi} \otimes^n(\rho_n)) \leq S(\Phi_E^n(\rho_n)) - \log \delta(n) \log c_B - S(\Phi \otimes^n(\rho_n)).$$

The first inequality comes from the definition of Gacs complexity (Definition 2.1) and the fact that $\Phi \otimes^n(\rho_n)$ is semi-computable semi-density matrix. For the second
inequality, we used the fact that $S(\rho) \leq G(\rho)$. By taking the limit on both sides of the inequality, we get

$$
\lim_{n \to \infty} \frac{1}{n} I_G (\rho_n, \Phi^\otimes n) \leq \lim_{n \to \infty} \frac{1}{n} I_c (\rho_n, \Phi^\otimes n) \leq \lim_{n \to \infty} \frac{1}{n} \max_\rho I_c (\rho, \Phi^\otimes n) \leq Q_c (\Phi).
$$

Therefore,

$$Q_G c (\Phi) \leq Q_c (\Phi).$$

To prove the inverse relation, namely

$$Q_G c (\Phi) \geq Q_c (\Phi),$$

we may notice from Theorem 3.1 that

$$S(\Phi^\otimes n (\rho_n), \hat{\mu}_B) \leq S(\rho_n, \hat{\mu}_A) + \alpha(n),$$

which using relative entropy (Eq. 2.4) and Gacs complexity (Definition 2.1) yields

$$S(\Phi^\otimes n (\rho_n)) - G(\Phi^\otimes n (\rho_n)) \leq S(\rho_n, \Phi^\otimes n) - G(\rho_n, \Phi^\otimes n) + \alpha(n).$$

Thus rearranging l.h.s. and r.h.s. terms

$$I_c (\rho_n, \Phi^\otimes n) \leq I_G c (\rho_n, \Phi^\otimes n) + \alpha(n).$$

Finally, taking the limit on both sides of inequality, having in mind that \(\lim_{n \to \infty} \frac{\alpha(n)}{n} = 0\), gives the desired result. \(\blacksquare\)

4 Algorithmic quantum capacity

Here we show that the standard quantum capacity can be approximated by a quantity linear in the channel’s input.

**Theorem 4.1** Let \(\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)\) be a semi-computable quantum channel. Then its quantum capacity \(Q_c (\Phi)\) is a limit computable number.

**Proof** Let \(\rho_n \in \mathcal{S}(H^\otimes n)\) be an arbitrary semi-computable density matrix with the following decomposition

$$\rho_n = \sum_{i_1 \ldots i_n, j_1 \ldots j_n} \lambda_{i_1 \ldots i_n, j_1 \ldots j_n} |i_1 \ldots i_n \rangle \langle j_1 \ldots j_n|.$$
Let also $V_n$ be the Stinespring dilation of $\Phi^\otimes n$ [14]. Since $\rho_n$ and $V_n$ are linear, we have

$$IG_c(\rho_n) = -Tr\left(\Phi^\otimes n(\rho_n) \log \hat{\mu}_B^n\right) + Tr\left(\Phi^\otimes n(\rho_n) \log \hat{\mu}_E^n\right)$$

$$= \sum_{i_1...i_n} \lambda_{i_1...i_n} t_{i_1...i_n} \lambda_{j_1...j_n},$$

where we defined

$$t_{i_1...i_n} := -Tr(\Phi^\otimes n(|i_1...i_n><j_1...j_n|) \log \hat{\mu}_B^n)$$

$$+ Tr(\Phi^\otimes n(|i_1...i_n><j_1...j_n|) \log \hat{\mu}_E^n).$$

These can be intended as entries of a matrix $T$. If we now write $\rho$ and $T$ as vectors $v_\rho$ and $v_T$, respectively, then it follows:

$$|IG_c(\rho_n)|^2 = |\langle v_\rho | v_T \rangle|^2 \leq ||v_\rho|| \cdot ||v_T|| \leq ||v_T||.$$ 

Next, let us consider $\lambda$ as the largest eigenvalue of $|T|$ with eigenvector $|\Lambda\rangle$ and define $\rho = |\Lambda\rangle \langle \Lambda|$. It is clear that $||\rho|| = 1$ and hence

$$\max_\rho |\langle v | w \rangle| = ||v_T|| = |\lambda|.$$ 

Since $\hat{\mu}_B^n$ and $\hat{\mu}_E^n$ are semi-computable semi-density matrix, then there exist quasi-increasing computable sequences of elementary matrices that convergence to them. In turn it is know that the entries of elementary matrices are computable numbers, hence $t_{i_1...i_n} j_1...j_n$’s are limit computable numbers. This means that we can find a computable sequence of eigenvalues whose corresponding rate converges to the quantum capacity. Therefore, the maximum of $IG_c(\rho_n)$ can be derived in an algorithmic way, and hence, invoking the results of Theorem 3.2, we may conclude that the quantum channel capacity is approximated by this algorithmic method.

It is worth noticing that by means of Theorem 4.1, we remove the maximization of algorithmic coherent information at each level $n$ in proving the limit computable of $QG_c(\Phi)$. Furthermore $QG_c(\Phi)$, hence $Q_c(\Phi)$, results approximated by a quantity $(IG_c(\Phi))$ that is linear in the channel’s input.

**Remark 4.1** Theorem 4.1 does not tell us that if the quantum capacity is limit computable, then it will be semi-computable. For example, it is known that the quantum capacity of the qubit depolarizing channel for $p \geq 1/3$ is 0. But there are uncountable non-limit computable numbers satisfying this condition.

**Remark 4.2** Although Theorem 4.1 does not answer the question of whether quantum capacity is computable or non-computable, it sheds light on computational aspects of quantum capacity. In [12,13], it is shown that for a given $n$ there exists a quantum
channel whose capacity is zero after \( n \) uses, but nonzero after \( n + 1 \). This is interpreted as an indication that quantum capacity is a non-computable quantity. Since non-computable numbers have a hierarchy levels, it is important to characterize the non-computability of semi-computable quantum channels. Using Theorem 4.1, this reduces to the problem of finding a semi-computable quantum channel \( \Phi \) and a computable function \( f : \mathbb{N} \rightarrow \mathbb{Q} \) such that \( |QG_c(\Phi) - f(n)| \) cannot be dominates by any computable function from \( \mathbb{N} \) to \( \mathbb{Q} \).

We henceforth show a case where the algorithmic coherent information can be exactly (with any desired degree of precision) computed.

Quite generally, to find the quantum channel capacity by means of a computer program, we need a finite set of computable density matrices at each level \( n \), the dimension of the Hilbert space. Clearly, the quantum channel capacity evaluated in such a way will depend on the chosen set of density matrices. We assume that the cardinality of such set obeys the condition \( \lim_{n \rightarrow \infty} \log \frac{f(n)}{n} = 0 \), with \( f(n) \) a computable function denoting the number of chosen density matrices at level \( n \). For example, computable polynomial functions have this property. Then, the set can be represented as follows:

\[
S := \{ \rho_1, \ldots, \rho_{f(1)}, \rho_1 + f(1), \rho_2 + f(1), \ldots, \rho_{f(2)} + f(1), \ldots, \rho_1 + f(n-1) + \ldots + f(2) + f(1), \ldots, \rho_n + f(n-1) + \ldots + f(2) + f(1), \ldots \}
\]

Define

\[
\tilde{\rho} = \sum_{i=1}^{\infty} \delta(i) \rho_i,
\]

with \( \delta(i) \) as in Eq. (3.1). We emphasize that \( \tilde{\rho} \) is not universal semi-computable likewise \( \hat{\mu} \). Moreover, since we are working with computable density matrices, we should henceforth consider computable quantum channels.

**Theorem 4.2** Let \( \Phi : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B) \) be a computable quantum channel. Then the quantum capacity \( Q_c(\Phi) \) restricted to the set \( S \) turns out to be

\[
Q_c(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\rho_{i_n} \in S} \left\{ -Tr \left[ \Phi \otimes n (\rho_{i_n}) \log \Phi \otimes n (\tilde{\rho}) \right] + Tr \left[ \tilde{\Phi} \otimes n (\rho_{i_n}) \log \tilde{\Phi} \otimes n (\tilde{\rho}) \right] \right\}.
\]

**Proof** Let us assume that \( \{\rho_{i_1}, \rho_{i_2}, \ldots \} \) is a set of density matrices taken from \( S \). By considering the position of each element in this set, one can define the following density matrix

\[
\sigma = \sum_{n=1}^{\infty} \delta(i_n) \rho_{i_n}.
\]
It is trivial to show that $\sigma \leq \tilde{\rho}$. Then, we have

$$-Tr(\rho_{i_n} \log \tilde{\rho}) \leq -Tr(\rho_{i_n} \log \rho_{i_n}) - \log(\delta(i_n)). \quad (4.1)$$

On the other hand, thanks to the positivity of relative entropy, we have the following relation

$$-Tr(\rho_{i_n} \log \rho_{i_n}) \leq -Tr(\rho_{i_n} \log \tilde{\rho}). \quad (4.2)$$

Now, it is

$$I_c(\rho_{i_n}, \Phi^{\otimes n}) \leq -Tr[\Phi^{\otimes n}(\rho_{i_n}) \log \Phi^{\otimes n}(\tilde{\rho})] + Tr[\tilde{\Phi}^{\otimes n}(\rho_{i_n}) \log \tilde{\Phi}^{\otimes n}(\tilde{\rho})] - \log(\delta(i_n)).$$

Therefore, the maximum of coherent information over all density matrices from the set $S$ is less than the maximum of the r.h.s. over $S$. Taking into account that $i_n \leq f(1) + f(2) + \cdots + f(n) \leq nf(n)$, hence

$$-\log(\delta(i_n)) \leq -\log(\delta(nf(n)) \Rightarrow \lim_{n \to \infty} \frac{\log(\delta(i_n))}{n} = 0,$$

we get

$$Q_c(\Phi) \leq \lim_{n} \max_{\rho_{i_n}} \left\{ -Tr[\Phi^{\otimes n}(\rho_{i_n}) \log \Phi^{\otimes n}(\tilde{\rho})] + Tr[\tilde{\Phi}^{\otimes n}(\rho_{i_n}) \log \tilde{\Phi}^{\otimes n}(\tilde{\rho})] \right\}. \quad (4.3)$$

However, using the relations (4.1) and (4.2), we have

$$-Tr[\Phi^{\otimes n}(\rho_{i_n}) \log \Phi^{\otimes n}(\tilde{\rho})] + Tr[\tilde{\Phi}^{\otimes n}(\rho_{i_n}) \log \tilde{\Phi}^{\otimes n}(\tilde{\rho})] - \log(\delta(i_n)) \leq I_c(\rho_{i_n}, \Phi^{\otimes n}),$$

from which it follows

$$\lim_{n} \max_{\rho_{i_n}} \left\{ -Tr[\Phi^{\otimes n}(\rho_{i_n}) \log \Phi^{\otimes n}(\tilde{\rho})] + Tr[\tilde{\Phi}^{\otimes n}(\rho_{i_n}) \log \tilde{\Phi}^{\otimes n}(\tilde{\rho})] \right\} \leq Q_c(\Phi).$$

□

Notice that the quantity inside the brackets in Theorem 4.2 (as well as its maximum) can be computed with arbitrary degree of precision. Then, as consequence of Theorem 4.1, finding the limit computable of $Q_c$ restricted to the set of density matrices $S$ results is possible.

Furthermore, from an algorithmic point of view, we can borrow from Theorem 4.2 a lower bound on the quantum capacity.
5 Conclusion

We have investigated the quantum channel capacity based on the computability concept. In this process, the von Neumann entropy is replaced by the Gacs entropy [3] which is defined based on the universal semi-measure. The algorithmic coherent information is rewritten in terms of this entropy, which results linear with respect to density matrix. Nonetheless, we have proven that quantum capacity based on semi-computable concept, namely the entropy rate of algorithmic coherent information, equals the standard quantum capacity. Thanks to this, we have shown that quantum channel capacity for a given quantum semi-computable semi-measure is limit computable (Theorem 4.1). This constitutes a step forward since the negative claim about computability of quantum channel capacity [12,13]. In fact the set of limit computable numbers is a proper subset of the real numbers, which contains the set of all computable numbers and a subset of non-computable ones. Exactly because non-computable numbers have a hierarchy levels, it is important to characterize the non-computability of semi-computable quantum channel capacity. Using Theorem 4.1, this reduces to the problem of finding a semi-computable quantum channel such that the absolute value difference between its capacity and a computable function cannot be dominated by $1/2^n$. In other words, we should find a semi-computable quantum channel $\Phi$ and a limit computable number $x$ which is not computable, such that $|Q(n, \Phi) - x_n|$ goes to 0, where $Q(n, \Phi) \xrightarrow{n \to \infty} Q_c(\Phi)$ and $x_n \xrightarrow{n \to \infty} x$, respectively. Thus, to this end, we can get rid of non-limit computable numbers $x$.

Since the universal semi-measure is not computable, then the algorithmic coherent information may not be computable and so we have subsequently introduced a restriction to compute it with any degree of precision (as a computable number) (Theorem 4.2). We believe it will be useful for computer programmers to know that quantum channel capacity can be limit computable by removing the maximization of coherent information at each channel’s usage level. As well as the fact that restricting to the set of computable density matrices, a lower bound on the quantum capacity can be computed.

Finally, given that the quantum capacity is related to entanglement distillability, it would be worth extending the pursued algorithmic approach to the subject of entanglement manipulation (distillation and dilution). This is left for future investigations.

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