Photon blockade effect in a coupled cavity system

Fen Zou, Deng-Gao Lai, and Jie-Qiao Liao

Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China

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We study the photon blockade effect in a coupled cavity system, which is formed by a linear cavity coupled to a Kerr-type nonlinear cavity via a photon-hopping interaction. We explain the physical phenomenon from the viewpoint of the conventional and unconventional photon blockade effects. The corresponding physical mechanisms of these two photon blockade effects are based on the anharmonicity in eigenenergy spectrum and the destructive quantum interference between two different transition paths, respectively. We find that the quantum interference effect also exists in the conventional photon blockade regime. Our results are confirmed by analytically and numerically solving the quantum master equation and calculating the second-order correlation function of the cavity fields. This model is general and hence it can be implemented in various experimental setups such as coupled optical cavities, coupled photon-magnon systems, and coupled superconducting resonators. We present some discussions on the experimental implementation.

I. INTRODUCTION

The photon blockade effect describes a physical phenomenon that the occupation of one photon in a cavity will blockade the consequent injection of the second photon [1]. The basic idea of photon blockade is in analogy to the concept of the Coulomb blockade [2] of electrons in mesoscopic physics: the electrons inside an island will create a strong Coulomb repulsion preventing other electrons from flowing. Conventionally, photonic nonlinearity in the eigenenergy spectrum is considered as the physical origin of the occurrence of photon blockade [1]. Owing to the anharmonicity of the eigenenergy spectrum, a resonant physical transition between the low excitation states will lead to an off-resonance in the consequent transition to the upper energy levels, and hence the injection of the second photon is blocked. In contrast to the conventional photon blockade [1], a new kind of physical mechanism for creating photon blockade was recently proposed. This new physical mechanism (called as unconventional photon blockade) is based on the destructive quantum interference effect between different excitation pathways [3, 4]. As a result, the unconventional photon blockade mechanism usually requires that the physical systems involve several degrees of freedom so that multi-transition paths can be established.

So far, both of the above mentioned two physical mechanisms have been widely studied in various quantum systems. For example, the conventional photon blockade effect has been studied in quantum optical systems such as cavity-QED systems [5–15] and circuit-QED systems [16–20], the Kerr-type nonlinear cavity [1, 21–23], the optomechanical systems [24–35], and the coupled-atom system [36]. Meanwhile, the unconventional photon blockade effect has been studied in coupled Kerr-cavity systems [4, 37–40], the Jaynes-Cummings model [41, 42], and the double cavity optomechanical systems [43, 44], and other systems [45–48]. In these schemes, either the light-matter interactions or the moving boundary of the fields has been introduced to manipulate the optical mode so that the photon blockade effect can be realized.

In this paper, we study the photon blockade effect in a new kind of quantum optical model, which is formed by a linear cavity coupled to a Kerr-type nonlinear cavity, where the two cavities are coupled to each other through the photon-hopping interaction. Our study is inspired by the recent experimental advances in coupled optical cavity systems, coupled photon-magnon systems [49, 50], and coupled superconducting resonator systems [51]. These systems can be described by a common physical model which will be considered in this paper. For simplicity and without loss of generality, below we will use the terms of optical cavity and all the results work for other physical degrees of freedom. The photon blockade effects are evaluated by calculating the equal-time second-order correlation function of the linear cavity. We will analyze how does the nonlinear cavity mode modulate the linear cavity mode through two different physical mechanisms corresponding to conventional and unconventional photon blockade effects. In particular, we will analyze the anharmonicity of the eigenenergy spectrum and destructive quantum interference effect between two transition paths in this system. By exploring the relationship between the energy anharmonicity and the linewidth of the energy levels, we will study the inherent parameter condition under which the conventional photon blockade can be realized. We will also study the optimal driving frequency to meet the single-photon resonance and two-photon off-resonance conditions. The influence of the thermal noise on the photon blockade effect will also be considered. In addition, we will derive the parameter condition under which the unconventional
photon blockade occurs. The rest of this paper is organized as follows. In Sec. II, we introduce the physical model and present the Hamiltonian. In Sec. III, we study the conventional photon blockade effect by calculating the equal-time second-order correlation function and analyzing the dependence of the correlation function on the system parameters. We also prove that the photon blockade effect in the large Kerr parameter case can be reduced to the case corresponding to the Jaynes-Cummings model. In Sec. IV, we study the unconventional photon blockade effect by analyzing the quantum interference effect among the transition channels in this model. In Sec. V, we investigate the influence of the thermal noise on the photon blockade effects. In Sec. VI, we present some discussions on the experimental implementation of this scheme with various physical systems, such as the coupled cavity systems, the coupled photon-magnon system, and the coupled superconducting-resonator systems. Finally, we present a brief conclusion in Sec. VII. An appendix is presented to display the approximate steady-state solution of the quantum master equation.

II. MODEL AND HAMILTONIAN

We consider a coupled cavity system, which is formed by a linear cavity coupled to a Kerr-type nonlinear cavity [see Fig. 1(a)]. In order to manipulate the quantum state of this coupled cavity system, a monochromatic field is introduced to drive the linear cavity. The Hamiltonian of the total system reads ($\hbar = 1$)

$$
H_{\text{sys}} = \omega_a a^\dagger a + \omega_b b^\dagger b + K b^\dagger b b^\dagger b + J(a^\dagger b + ab^\dagger) + \Omega(a^\dagger e^{-i\omega_d t} + ae^{i\omega_d t}),
$$

(1)

where $a$ ($a^\dagger$) and $b$ ($b^\dagger$) are, respectively, the annihilation (creation) operators of the two cavity modes, with the corresponding resonance frequencies $\omega_a$ and $\omega_b$. The term $K b^\dagger b b^\dagger b$ represents the Kerr nonlinearity for the nonlinear cavity, with $K$ being the Kerr parameter. The parameter $J$ denotes the strength of the photon hopping interaction between the two cavity modes. The parameters $\Omega$ and $\omega_d$ are the drive amplitude and driving frequency of the linear cavity, respectively. In a rotating frame with respect to $H_0 = \omega_d (a^\dagger a + b^\dagger b)$, the Hamiltonian of the system becomes

$$
H_I = \Delta_a a^\dagger a + \Delta_b b^\dagger b + K b^\dagger b b^\dagger b + J(a^\dagger b + ab^\dagger) + \Omega(a^\dagger + a),
$$

(2)

where we introduce the driving detunings $\Delta_a = \omega_a - \omega_d$ and $\Delta_b = \omega_b - \omega_d$. In the absence of the cavity field driving, the Hamiltonian of the coupled cavity system reads $H_{\text{c}} = \Delta_a a^\dagger a + \Delta_b b^\dagger b + K b^\dagger b b^\dagger b + J(a^\dagger b + ab^\dagger)$, and the total excitation number operator $\hat{N} = a^\dagger a + b^\dagger b$ in this system is a conserved quantity because of the commutative relation $[H_{\text{c}}, \hat{N}] = 0$. For studying the photon blockade effect in this system, we consider the weak-driving regime. In this case, the photon number involved is small and hence we can restrict the system within the low-excitation subspaces. Below, we will consider the total excitation numbers as $N = 0$, 1, and 2 in our analytical studies. The subspaces corresponding to $N = 0$, 1, and 2 are spanned over the basis states $\{|0\rangle_{a,b}\}$, $\{|1\rangle_{a,b}\}$, and $\{|2\rangle_{a,b}\}$, respectively. Here $m$ photons in the linear cavity and $n$ photons in the nonlinear cavity. The eigensystems of the Hamiltonian $H_{\text{c}}$ in these three subspaces can be obtained as follows. In the zero-excitation subspace, the eigen-equation is $H_{\text{c}} |\varepsilon_{00}\rangle = E_{00} |\varepsilon_{00}\rangle$ with the eigenstate $|\varepsilon_{00}\rangle = |0\rangle_{a,b}$ and the eigenenergy $E_{00} = 0$. In the single-excitation subspace, the eigen-equation is $H_{\text{c}} |\varepsilon_{1\pm}\rangle = E_{1\pm} |\varepsilon_{1\pm}\rangle$, where the eigenstates and eigenvalues are defined by

$$
|\varepsilon_{1+}\rangle_{a,b} = C^{[1+]}_{0,1} |0\rangle_{a,b} + C^{[1+]}_{1,0} |1\rangle_{a,b},
$$

$$
|\varepsilon_{1-}\rangle_{a,b} = C^{[1-]}_{0,1} |0\rangle_{a,b} + C^{[1-]}_{1,0} |1\rangle_{a,b},
$$

(3)

and

$$
E_{1\pm} = \frac{\Delta_a + \Delta_b + K}{2} \pm \frac{\sqrt{(\Delta_b - \Delta_a + K)^2 + 4J^2}}{2}.
$$

(4)
The superposition coefficients in the eigenstates are defined by
\[ C_{0,1}^{[1]} = C_{1,0}^{[1]} = \cos \theta, \quad C_{1,0}^{[1]} = -C_{0,1}^{[1]} = \sin \theta, \quad (5) \]
where the mixing angle \( \theta \) is defined by the relation
\[ \tan(2\theta) = \frac{2J}{\Delta_b - \Delta_a + K}. \quad (6) \]

In the two-excitation subspace, the eigenstates and eigenvalues can be obtained by solving the eigensystem of the matrix
\[ H_{\text{ec}}^{[2]} = \begin{pmatrix} 2\Delta_b + 4K & \sqrt{2J} & 0 \\ \sqrt{2J} & \Delta_a + \Delta_b + K & \sqrt{2J} \\ 0 & \sqrt{2J} & 2\Delta_a \end{pmatrix}, \quad (7) \]
which is defined based on the basis states \(|0,2\rangle_{a,b} = (1,0,0)^T, |1,1\rangle_{a,b} = (0,0,1)^T, \) and \(|2,0\rangle_{a,b} = (0,1,0)^T\), where "\( T \)" denotes the matrix transpose. The eigensystem of the Hamiltonian in the two-excitation subspace is defined by \( H_{\text{ec}}^{[2]} |\varepsilon_{2s}\rangle = E_{2s} |\varepsilon_{2s}\rangle \) with \( s = \pm, 0 \). The eigenvalues are given by
\[ E_{2-} = -\frac{1}{3}[A + \sqrt{3D} \{ \cos(\alpha/3) + \sqrt{3} \sin(\alpha/3) \}], \]
\[ E_{20} = -\frac{1}{3}[A + \sqrt{3D} \{ \cos(\alpha/3) - \sqrt{3} \sin(\alpha/3) \}], \]
\[ E_{2+} = -\frac{1}{3}[A - 2\sqrt{3D} \cos(\alpha/3)], \quad (8) \]
with the corresponding eigenstates
\[ |\varepsilon_{2s}\rangle = C_{0,2}^{[2s]} |0,2\rangle_{a,b} + C_{1,1}^{[2s]} |1,1\rangle_{a,b} + C_{2,0}^{[2s]} |2,0\rangle_{a,b} \quad (9) \]
for \( s = \pm, 0 \). The superposition coefficients and the relating parameters used in the eigensystem are defined by
\[ C_{0,2}^{[2s]} = -\sqrt{2} \frac{1}{2} (E_{2s} - 2\Delta_a) N_{2s}^{-1/2}, \]
\[ C_{1,1}^{[2s]} = (E_{2s} - 2\Delta_a) (2\Delta_b + 4K - E_{2s}) N_{2s}^{-1/2}, \]
\[ C_{2,0}^{[2s]} = \sqrt{2} \frac{1}{2} (2\Delta_b + 4K - E_{2s}) N_{2s}^{-1/2}, \quad (10) \]
and
\[ A = -5K - 3\Delta_a - 3\Delta_b, \]
\[ B = -4J^2 + 4K^2 + 14K\Delta_a + 2\Delta_a^2 + 6K\Delta_b + 8\Delta_a\Delta_b + 2\Delta_b^2, \]
\[ C = 8J^2 K + 4J^2 \Delta_b - 8K^2 \Delta_a - 8K\Delta_a^2 + 4J^2 \Delta_b - 12K\Delta_a\Delta_b - 4\Delta_a^2 \Delta_b - 4\Delta_a \Delta_b^2, \]
\[ D = B - \frac{1}{3} A^2, \]
\[ E = C + \frac{2}{27} A^3 - \frac{1}{3} AB, \]
\[ \alpha = \arccos \left[ -3E \sqrt{3D} / (2D^2) \right], \]
\[ N_{2s} = (E_{2s} - 2\Delta_a)^2 [2J^2 + (2\Delta_b + 4K - E_{2s})^2] + 2J^2 (2\Delta_b + 4K - E_{2s})^2, \quad (11) \]
for \( s = \pm, 0 \).

### III. CONVENTIONAL PHOTON BLOCKADE

In this section, we study the conventional photon blockade effect by analytically and numerically calculating the equal-time second-order correlation function of the linear cavity and analyzing the dependence of the correlation function on the system parameters.

#### A. Quantum master equation

To treat the damping and noise in this system, we assume that the two cavity modes are coupled to two independent Markovian environments. Then the evolution of the system is governed by the quantum master equation
\[ \rho = i[\rho, H_I] + \frac{\kappa_a}{2} (\bar{n}_a + 1) \mathcal{L}_a[\rho] + \frac{\kappa_a}{2} \bar{n}_a \mathcal{L}_a^\dagger[\rho] \]
\[ + \frac{\kappa_b}{2} (\bar{n}_b + 1) \mathcal{L}_b[\rho] + \frac{\kappa_b}{2} \bar{n}_b \mathcal{L}_b^\dagger[\rho], \quad (12) \]
where the Hamiltonian \( H_I \) is given by Eq. (2) and
\[ \mathcal{L}_a[\rho] = (2\omega a^\dagger \omega^\dagger - a^\dagger a \omega^\dagger - \omega a \omega^\dagger - \omega^\dagger a) \quad (13) \]
denotes the Lindblad superoperator for an operator \( o \) [52]. The parameters \( \kappa_a \) and \( \kappa_b \) are, respectively, the dissipation rates of the two cavity fields, and \( \bar{n}_a = [\exp(\hbar \omega_a/k_B T_a) - 1]^{-1} \) and \( \bar{n}_b = [\exp(\hbar \omega_b/k_B T_b) - 1]^{-1} \) are the average thermal excitation numbers of the baths at temperatures \( T_a \) and \( T_b \), with \( k_B \) being the Boltzmann constant.

For the study of photon blockade, we assume that the driving of the cavity is very weak and then we can safely restrict the system within the low-excitation subspace in the case of zero-temperature baths. The master equation (12) can be solved with the bases of number states \( |m, n\rangle_{a,b} \), where \( m \) and \( n \) denote the photon numbers in the linear cavity and the nonlinear cavity, respectively. In the weak-driving limit \( \Omega/\kappa_a \ll 1 \), the Hilbert space of the total system can be truncated up to \( m + n = 2 \). The eigensystems of the undriving Hamiltonian, namely \( H_{\text{ec}} \), in the low-excitation subspace with \( m + n = 0, 1, \) and 2 have been given in Sec. II. In the weak-driving regime, we solve the equations of motion for the density matrix elements in the low-excitation subspace with the perturbation method and obtain the steady-state solution (see the Appendix). Meanwhile, we also solve the master equation numerically to obtain the exact solution of the steady state of the system, and then we evaluate the performance of the perturbation approximation by comparing the perturbative results with the exact results.

#### B. The equal-time second-order correlation function

The photon blockade effect in the linear cavity can be characterized by evaluating the equal-time second-order
correlation function

\[ g^{(2)}(0) \equiv \frac{\langle a^\dagger a^\dagger a^a \rangle_{ss}}{\langle a^a \rangle_{ss}^2}, \]

(14)

where the average value is taken over the steady state of the system. In the weak-driving case, the system is restricted into the low-excitation subspace up to \( m + n = 2 \), and the state bases in this subspace are denoted as \( |0, 0\rangle_{a,b} \rightarrow |S_1\rangle, |0, 1\rangle \rightarrow |S_2\rangle, |0, 2\rangle_{a,b} \rightarrow |S_3\rangle, |1, 0\rangle_{a,b} \rightarrow |S_4\rangle, |1, 1\rangle_{a,b} \rightarrow |S_5\rangle, \) and \( |2, 0\rangle_{a,b} \rightarrow |S_6\rangle \). Then the density matrix elements of the system can be expressed in this subspace as

\[ \rho = \sum_{m,n=1}^{6} \rho_{mn}|S_m\rangle\langle S_n|, \quad \rho_{mn} = \langle S_n|\rho|S_m\rangle. \]

(15)

In the weak-driving case, the density matrix elements can be divided into groups of different orders (denoted as the \( l \)th order) of the small number \( \Omega/\kappa_a \),

\[ l = 0, \{ \rho_{11} \}, \]
\[ l = 1, \{ \rho_{12}, \rho_{14}, \rho_{21}, \rho_{41} \}, \]
\[ l = 2, \{ \rho_{13}, \rho_{15}, \rho_{16}, \rho_{31}, \rho_{51}, \rho_{61}, \rho_{22}, \rho_{24}, \rho_{42}, \rho_{44} \}, \]
\[ l = 3, \{ \rho_{23}, \rho_{25}, \rho_{26}, \rho_{32}, \rho_{52}, \rho_{62}, \rho_{43}, \rho_{45}, \rho_{46}, \rho_{54}, \rho_{64} \}, \]
\[ l = 4, \{ \rho_{33}, \rho_{35}, \rho_{36}, \rho_{53}, \rho_{56}, \rho_{63}, \rho_{65} \}. \]

(16)

Then the second-order correlation function of the linear cavity can be approximately expressed as

\[ g^{(2)}(0) \approx \frac{2\rho_{66}}{(\rho_{44} + \rho_{55} + 2\rho_{66})^2} \approx \frac{2\rho_{66}}{\rho_{44}}. \]

(17)

Here we have used the relation \( \rho_{44} \gg \{ \rho_{55}, \rho_{66} \} \) because \( \rho_{55} \) and \( \rho_{66} \) are the fourth-order small quantity of \( \Omega/\kappa_a \), while \( \rho_{44} \) is the second-order small quantity of \( \Omega/\kappa_a \). Further, we solve the steady-state solution of these density matrix elements by using the perturbation method [53], i.e., by discarding the higher-order elements in the equations of the lower-order elements, and then the second-order correlation function of the steady state can be obtained.

To check the performance of the perturbation, we numerically [54] solve the quantum master equation and compare the exact numerical results with the approximate analytical results. We also solve the master equation to obtain the steady-state properties of the system, such as the photon number distribution and the second-order correlation function \( g^{(2)}(0) \). In the following, we illustrate the dependence of the \( g^{(2)}(0) \) on other parameters such as the driving detuning \( \Delta_a \), the Kerr parameter \( K \), and the photon-hopping interaction strength \( J \).

As we know, the conventional photon blockade can be explained from the viewpoint of the anharmonicity in the eigenenergy. Usually, the photon blockade effect is determined by the transitional resonance, and hence the driving detuning is an important parameter to control the occurrence of photon blockade. To see how does the driving detuning affect the photon number distributions, in Fig. 2(a) we plot the state occupations \( \rho_{11} \) for the bare states \( |S_{1=1,4,6}\rangle \) versus the scaled driving detuning \( \Delta/\kappa \). The equal-time second-order correlation function \( g^{(2)}(0) \) versus the driving detuning \( \Delta/\kappa \). Other parameters are \( \kappa_a = \kappa_b = \kappa \), \( \Omega/\kappa = 0.1 \), \( K/\kappa = 25 \), \( J/\kappa = 50 \), and \( n_a = n_b = 0 \).
\( s = \pm, 0 \). The locations of these three main peaks are determined by the resonance conditions \( E_{2s} - E_{00} = 0 \) for \( s = \pm, 0 \). The two-photon resonance means that the system absorbs two driving photons and then transits from the state |0, 0\) to one of the states |\( \varepsilon_{2s} \rangle \) for \( s = \pm, 0 \). In this physical process, the single-photon processes are not resonant, but the energy change in the two-photon processes are conserved, i.e., the energy difference between the states |\( \varepsilon_{2s} \rangle \) and the state |0, 0\)\( a,b \rangle \) equals to the energy of two driving photons. Note that the frequency of the driving photon is zero in the rotating frame with respect to the Hamiltonian \( H_0 = \omega_d (a^\dagger a + b^\dagger b) \). For the parameters used in our simulation, the locations of these three peaks in \( \rho_{66} \) are given by \( \Delta/J = -76.3368, -22.5197, \) and 36.3566. In addition, there are two peaks induced by the single photon resonance, and hence the locations of these two peaks are the same as those of the two peaks in \( \rho_{44} \).

Besides the peaks, we can also see some dips in the state occupations \( \rho_{44} \) and \( \rho_{66} \). Physically, these dips are caused by the quantum interference effect existing in the state transitions induced by the driving photon. To clarify this point, we study the steady state of the system in the eigenstate representation and analyze the relationship between the bare state |\( S_4 \rangle \) (|\( S_6 \rangle \)) and these eigenstates |\( \varepsilon_{1\pm} \rangle \) (|\( \varepsilon_{2s} \rangle \) for \( s = \pm, 0 \)). For clearly seeing the physical picture of quantum interference, we adopt the effective Hamiltonian method to simulate the evolution of the system. Here the evolution of the system is governed by the non-Hermitian Hamiltonian which is formed by adding phenomenologically the imaginary dissipation terms into the Hamiltonian as follows [55]

\[
H_{\text{int}} = \left( \Delta_a - i \frac{K_a}{2} \right) a^\dagger a + \left( \Delta_b - i \frac{K_h}{2} \right) b^\dagger b + K b^\dagger b a^\dagger a + J (a^\dagger b + ab^\dagger) + \Omega (a^\dagger + a),
\]

(18)

In terms of this non-Hermitian Hamiltonian and the state of the system expressed in the eigenstate representation

\[
|\psi(t)\rangle = D_{00}(t)|\varepsilon_{00}\rangle + D_{1+}(t)|\varepsilon_{1+}\rangle + D_{1-}(t)|\varepsilon_{1-}\rangle + D_{2+}(t)|\varepsilon_{2+}\rangle + D_{20}(t)|\varepsilon_{20}\rangle + D_{2-}(t)|\varepsilon_{2-}\rangle
\]

we can obtain the equations of motion for these probability amplitudes \( D_{00}(t) \), \( D_{1\pm}(t) \), and \( D_{2\pm,0}(t) \). Using the perturbative method, the steady-state solution of these probability amplitudes can be obtained. Based on the state in Eq. (19) and the steady-state solution of these probability amplitudes, the state occupations \( \rho_{44} \) and \( \rho_{66} \) can be obtained as follows

\[
\rho_{44} = |D_{1+} C_{1,0}^{[1+]}|^2 + |D_{1-} C_{1,0}^{[1-]}|^2 + 2 \mathrm{Re} \left[ D_{1+} C_{1,0}^{[1+]} D_{1-} C_{1,0}^{[1-]*} \right],
\]

(20)

and

\[
\rho_{66} = |D_{2+} C_{2,0}^{[2+]}|^2 + |D_{2-} C_{2,0}^{[2-]}|^2 + |D_{20} C_{2,0}^{[20]}|^2 + 2 \mathrm{Re} \left[ D_{2+} C_{2,0}^{[2+]} D_{2-} C_{2,0}^{[2-]*} + D_{2+} C_{2,0}^{[2+]} D_{20} C_{2,0}^{[20]*} + D_{2-} C_{2,0}^{[2-]} D_{20} C_{2,0}^{[20]*} \right],
\]

(21)

where “Re” gives the real part of the variable.

The terms in the first line of \( \rho_{44} \) and \( \rho_{66} \) in Eqs. (20) and (21) are, respectively, the non-quantum-interference contribution of the state occupations of |\( S_4 \rangle \) and |\( S_6 \rangle \), and the rest parts are induced by the quantum interference effect among the eigenstates in the same subspace of \( N = 0, 1, \) and 2. To see the quantum interference effect in the photon blockade effect, in Fig. 2(a) we also show the non-quantum-interference part (the dashed curves) of the state occupations \( \rho_{44} \) and \( \rho_{66} \) as a reference. Here we can confirm that for \( \rho_{44} \) there are two main peaks which match the exact numerical results. The dip confirmed by the exact result disappears in the non-quantum-interference result. This means that the dip in

![3D plot](image-url)

**Fig. 3.** (Color online) The state occupations (a) \( \log_{10} \rho_{44} \) and (b) \( \log_{10} \rho_{66} \), and the equal-time second-order correlation function (c) \( \log_{10} \rho^{(2)}(0) \) as a function of the photon hopping interaction strength \( J/K \) and the Kerr parameter \( \Omega/K \). The driving detunings are given by \( \Delta_a = \Delta_b = -(K + \sqrt{K^2 + 4 J^2})/2 \), which corresponds to the single-photon resonance transition |\( \varepsilon_{00} \rangle \rightarrow |\varepsilon_{1+} \rangle \). Other parameters are \( \Omega/K = 0.1 \) and \( \bar{n}_a = \bar{n}_b = 0 \).
$\rho_{44}$ should be caused by the quantum interference effect between two transition paths associated with the states $|e_{1+}\rangle$ and $|e_{1-}\rangle$. For the two-photon state occupation $\rho_{66}$, we can also see a dip, which is induced by the quantum interference effect. Note that the state occupations given by Eqs. (20) and (21) are the same as the numerical results obtained based on the master equation.

To clearly see the photon blockade effect in the linear cavity, in Fig. 2(b) we plot the equal-time second-order correlation function $g^{(2)}(0)$ as a function of the driving detuning $\Delta/J$. Here, the blue-dashed curves are plotted using the numerical solution of Eq. (12), while the red-solid curves are based on the analytical solution given in Eq. (17). To see the quantum interference effect in the explanation the conventional photon blockade, moreover, we plot the result (i.e., the first line terms in Eqs. (20) and (21) corresponding to the non-quantum-interference case using the grey-dashed curve. We can see that the analytical results can match well with the numerical results, and that the non-quantum-interference evaluation can predict the location of the optimal driving detuning, but it cannot give the exact value of the correlation function $g^{(2)}(0)$. The location of the dips ($d_1$ and $d_2$) of the correlation function $g^{(2)}(0)$ corresponds to the single-photon resonance, namely the two peaks of the state occupation $\rho_{44}$. The location of the peaks ($p_1$, $p_2$, and $p_2$) in the correlation function $g^{(2)}(0)$ correspond to the peaks of the state occupation $\rho_{66}$, and the dip in the state occupation $\rho_{44}$ also modify one of the peak in $g^{(2)}(0)$ slightly.

In Fig. 3, we plot the state occupations $\rho_{44}$ and $\rho_{66}$, and the equal-time second-order correlation function $g^{(2)}(0)$ as a function of the photon hopping interaction strength $J/\kappa$ and the Kerr parameter $K/\kappa$ under the single-photon resonance condition $\Delta_a = \Delta_b = -(K + \sqrt{K^2 + 4J^2})/2$. We can see that approximately along the line $J \approx 2K$, the correlation function $g^{(2)}(0)$ reach its minimum value. By comprising the correlation function and the state occupations, we can see that the state occupation $\rho_{66}$ has minimal values along the line $J \approx 2K$. This means that the photon blockade effect in this case is mainly caused by the suppressed transitions to the two-photon states.

**C. Going back to the Jaynes-Cummings model**

From the viewpoint of the coupled cavity-atom system, we can understand this system as a single-mode cavity field coupled to a Kerr-type multilevel atomic system. The eigenstates and eigenenergies of the multilevel system are given by $|m\rangle_b$ and $m\omega_a + m^2K$. Below, we consider an interesting special case in which the coupled cavity system can be reduced to a Jaynes-Cummings (JC) system which describes a single-mode cavity field coupled to a two-level system. The parameter condition for this special case is that the Kerr parameter $K$ should be much larger than the photon hopping strength $J$, and that the cavity field frequency should be near resonant with the lowest two states $|0\rangle_b$ and $|1\rangle_b$ of the nonlinear cavity, namely $\omega_a \sim \omega_b + K$. In this case, the high-excitation states of the Kerr multilevel system will not be populated and then the Hamiltonian of this reduced system can be written as

$$H_{JC} = \omega_a a^\dagger a + \frac{\omega_b + K}{2} \sigma_z + J(a^\dagger \sigma_- + \sigma_+ a) + \Omega(a^\dagger e^{-i\omega_d t} + ae^{i\omega_d t}),$$

(22)

where the Pauli operators are defined with the states $|0\rangle_b$ and $|1\rangle_b$ as $\sigma_z = |1\rangle_b \langle 1| - |0\rangle_b \langle 0|$, $\sigma_+ = |1\rangle_b \langle 0|$, and $\sigma_- = |0\rangle_b \langle 1|$. In a rotating frame with respect to $H_0 = \omega_a(a^\dagger a + \sigma_z/2)$, the Hamiltonian of the system becomes

$$H_{JC}' = \Delta_a a^\dagger a + \frac{\Delta_b + K}{2} \sigma_z + J(a^\dagger \sigma_- + \sigma_+ a) + \Omega(a^\dagger + a),$$

(23)

where $\Delta_a = \omega_a - \omega_d$ and $\Delta_b = \omega_b - \omega_d$ are the driving detunings.

To prove the above analyses, we compare the second-order correlation functions of the mode $a$ for the JC model and the coupled cavity model, which are described by the Hamiltonian (23) and the Hamiltonian (2), respectively. In Fig. 4, we plot the correlation function $g^{(2)}(0)$ as a function of the driving detuning $\Delta/J$ when the Kerr parameter takes various values $K/J = 2, 4, 20$. We can see that the correlation functions in these two cases match better for a larger value of the ratio $K/J$, as expected by our above analyses. This means that the coupled cavity model can be reduced to the JC model under the parameter condition $K/J \gg 1$. 

![FIG. 4. (Color online) The equal-time second-order correlation function $g^{(2)}(0)$ as a function of the driving detuning $\Delta/J$ at various values of $K/J$. The dashed curves and the solid (circled) curve are the results corresponding to the coupled cavity model and the JC model, respectively. Other parameters are given by $\kappa_a/J = 0.05$, $\kappa_b/J = 0.05$, $\Omega/J = 0.005$, and $\Delta_a = \Delta_b + K$.](image)
IV. UNCONVENTIONAL PHOTON BLOCKADE

In this section, we study the unconventional photon blockade effect in this coupled cavity system. For simplicity, we consider the case of zero-temperature environments, and then we can use the effective Hamiltonian method to describe the evolution of the system. Note that a coupled-resonator system has been considered mainly in a different parameter range to study the phonon antibunching effect at a finite temperature [56]. In the weak-driving case, we restrict the system within the low-excitation subspace up to $N = 2$, as explained in the above section. To solve the steady-state solution of the system, we expand the wave function of the system with the bare-state bases as

$$|\psi\rangle = C_{0,0}(t)|0,0\rangle_{a,b} + C_{1,0}(t)|1,0\rangle_{a,b} + C_{0,1}(t)|0,1\rangle_{a,b}$$

$$+ C_{2,0}(t)|2,0\rangle_{a,b} + C_{1,1}(t)|1,1\rangle_{a,b} + C_{0,2}(t)|0,2\rangle_{a,b},$$

(24)

where $C_{m,n}$ for $m, n = 0, 1, 2$ are the probability amplitudes corresponding to the basis state $|m,n\rangle_{a,b}$. Based on the Hamiltonian in Eq. (18) and the wave function in Eq. (24), we can obtain the equations of motion for these probability amplitudes $C_{m,n}$. In the weak-driving case, these probability amplitudes can also be classified into various groups of different orders of the small ratio $\Omega/\kappa_a$. The amplitude $C_{0,0}(t)$ is of the zero order of $\Omega/\kappa_a$. The coefficients $C_{1,0}(t)$ and $C_{1,1}(t)$ are of the first order of $\Omega/\kappa_a$, and $C_{2,0}(t)$, $C_{1,1}(t)$, and $C_{0,2}(t)$ are of the second order of $\Omega/\kappa_a$. In this case, we can solve the equations of motion for the probability amplitudes $C_{m,n}$ using the perturbative method [53], namely discarding the higher-order terms in the equations of motion for the lower-order variables. In particular, we focus on the steady-state solution of these probability amplitudes for studying the photon blockade effect. Below, we consider the case of $\Delta_a = \Delta_b = \Delta$ and $\kappa_a = \kappa_b = \kappa$, and then obtain the steady-state solution as

$$C_{1,0} = \frac{2(2K - i\kappa + 2\Delta)\Omega}{4J^2 + (\kappa + 2i\Delta)(2iK + \kappa + 2i\Delta)}$$

$$C_{0,1} = \frac{-4J\Omega}{4J^2 + (\kappa + 2i\Delta)(2iK + \kappa + 2i\Delta)},$$

$$C_{2,0} = 2\sqrt{2}(2K - i\kappa + 2\Delta)(4K - i\kappa + 2\Delta)$$

$$\times (2K - 2i\kappa + 4\Delta) + 8J^2K\Omega^2M^{-1},$$

$$C_{1,1} = -8J(4K - i\kappa + 2\Delta)$$

$$\times (2K - 2i\kappa + 4\Delta)\Omega^2M^{-1},$$

$$C_{0,2} = 8\sqrt{2}J^2(2K - 2i\kappa + 4\Delta)\Omega^2M^{-1},$$

(25)

where the variable $M$ is defined by

$$M = [4J^2 + (\kappa + 2i\Delta)(2iK + \kappa + 2i\Delta)]$$

$$\times [(\kappa + 2i\Delta)(4K - i\kappa + 2\Delta)(2iK + 2\kappa + 4i\Delta)]$$

$$+ 4J^2(4K - 2i\kappa + 4\Delta)].$$

(26)

Mathematically, for observing photon blockade effect in the linear cavity, it means that the two-photon probability in this cavity is suppressed. For the idea case, this corresponds to the probability amplitude for the state $|2,0\rangle_{a,b}$ is zero. Based on the solution (25), the parameter condition for $C_{2,0} = 0$ can be obtained as

$$R[J, K, \Delta, \kappa] = 4J^2K + 8K^3 + 28K^2\Delta + 28K\Delta^2$$

$$+ 8\Delta^3 - 7K\Delta^2 - 6\Delta\kappa^2 = 0,$$

$$I[J, K, \Delta, \kappa] = 14K^2 + 28K\Delta + 12\Delta^2 - \kappa^2 = 0.$$  

(27)

We can see that $R[J, K, \Delta, \kappa]$ is a function of $J^2$ and that there exists the symmetric relations

$$R[J, -K, -\Delta, \kappa] = -R[J, K, \Delta, \kappa],$$

$$I[J, -K, -\Delta, \kappa] = I[J, K, \Delta, \kappa].$$

(28)

Based on the symmetric relations and Eq. (27), we know that if $(K, \Delta)$ is a solution, then $(-K, -\Delta)$ is also a solution. As an example, we choose a moderate photon-hopping interaction strength $J/\kappa = 50$, then the solutions of Eq. (27) are: $K/\kappa \approx \pm 125.859, \Delta/\kappa \approx \pm 91.3359; K/\kappa \approx \pm 68.81i; \Delta/\kappa \approx \pm 110.618i; K/\kappa \approx \pm 1.54 \times 10^{-4}, \Delta/\kappa \approx \pm 0.288$.

In order to confirm the optimal condition, we inspect the second-order correlation function $g^{(2)}(0)$ when the system parameters take the values corresponding to the

FIG. 5. (a) The equal-time second-order correlation function $g^{(2)}(0)$ versus the driving detuning $\Delta/\kappa$ at various values $K/\kappa = (1, 1.54, 2) \times 10^{-4}$. (b) The correlation function $g^{(2)}(0)$ versus the Kerr parameter $K/\kappa$ at various values $\Delta/\kappa = (0.2, 0.288, 0.4)$. (c) The correlation function $\log_{10} g^{(2)}(0)$ versus the driving detuning $\Delta/\kappa$ and the Kerr parameter $K/\kappa$. Other parameters are $J/\kappa = 50$ and $\Omega/\kappa = 0.1$. 

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real solutions of Eq. (27). In Fig. 5(a), we plot the correlation function $g^{(2)}(0)$ as a function of the average thermal excitation number $\bar{n}_0$ in the conventional photon blockade case (a) $\Delta_a = \Delta_b = -(K + \sqrt{K^2 + 4J^2})/2$ and $K/\kappa = 25$, and in the unconventional photon blockade case (b) $\Delta_a = \Delta_b = 0.288/\kappa$ and $K/\kappa = 1.54 \times 10^{-4}$. Other parameters are $J/\kappa = 50$ and $\Omega/\kappa = 0.1$.

confirmed from the 3-D plot in Fig. 6(c) and Fig. 6(d). Here the locations of the two minimal value points are also determined by the symmetric relations.

Physically, the photon blockade effect corresponding to the above four real solutions $\{K/\kappa \approx \pm 1.54 \times 10^{-4}, \Delta/\kappa \approx \pm 0.288\}$ and $\{K/\kappa \approx \mp 125.859, \Delta/\kappa \approx \pm 91.3359\}$ can be explained using the destructive interference between two different paths of photon excitation. In Fig. 1(c), we show a sketch of the quantum interference effect responsible for this unconventional photon blockade. The quantum interference effect occurs between these two paths: (i) the direct excitation from $|1,0 \rangle \xrightarrow{\Omega} |2,0 \rangle$ (red arrow) and (ii) the indirect transition path $|1,0 \rangle \xrightarrow{\omega} |0,1 \rangle \xrightarrow{\Omega} |1,1 \rangle \xrightarrow{\omega} |2,0 \rangle$ (blue arrow). When the system parameters take the values corresponding to the real solutions of Eq. (27), the probability amplitude $C_{2,0} = 0$ is obtained, which indicates that the state $|2,0 \rangle_{a,b}$ will not be populated due to the destructive interference.
V. THE INFLUENCE OF THE THERMAL EXCITATION ON THE PHOTON BLOCKADE EFFECT

The above discussions focus on the zero-temperature environment case. For optical cavities, the thermal photon number is negligible, and hence our above assumption of the zero-temperature environments is reasonable. However, for other bosonic excitations such as phonon and magnon, it is needed to consider the influence of the thermal excitations on the photon blockade effects. Below, we will discuss the thermal bath case by considering a finite thermal excitations in the quantum master equation (12). In Fig. 7, we plot the equal-time second-order correlation function \(g^{(2)}(0)\) in the steady state as a function of the thermal excitation number \(\bar{n}_b\) in mode \(b\). The value of \(g^{(2)}(0)\) increases with the increase of the thermal photon number \(\bar{n}_b\) of mode \(b\). It is clear that the thermal photons have a significant effect on both the conventional and unconventional photon blockade effects, and that the latter is more fragile against the thermal noise than the former. We can conclude that the thermal noise is fatal to the photon blockade effect in this coupled cavity system and the photon blockade effect only exists in the zero temperature case.

VI. DISCUSSIONS ON THE EXPERIMENTAL IMPLEMENTATION

In this section, we present some discussions on the experimental implementation of this scheme with some quantum optical systems, such as coupled optical cavity systems, coupled photon-magnon systems, and coupled superconducting resonator systems. We also present some analyses on the parameter conditions of these systems. In this model, there are three components: the self-Kerr interaction of mode \(b\), the excitation hopping interaction between the two modes \(a\) and \(b\), the monochromatic driving of mode \(a\). To implement this scheme, the candidate physical systems should have these three physical processes.

For coupled optical cavity systems, the Kerr interaction can be implemented with a Kerr-type nonlinear cavity, and the Kerr parameter can enter the strong coupling regime. The photon-hopping interaction between the two cavities can also be realized in the coupled cavity systems. Based on the detailed systems, the optical cavity could be various semiconductor microcavities and the Fabry-Perot cavity. In these two cases, the driving on the cavity can be implemented.

For coupled photon-magnon systems, the two bosonic modes could be implemented with an electromagnetic mode in the superconducting resonator and a magnon mode of a YIG. Some recent experiments [49, 50] reported that the Kerr interaction in the magnon mode can be implemented and the excitations hopping between photons and magnons can be realized. The cavity field driving can also be realized in superconducting resonator by introducing a microwave driving. In this system, the resonance frequency of the linear cavity (photon) mode is \(\omega_a \approx 2\pi \times 10.1\) GHz, and the resonance frequency \(\omega_b\) of the nonlinear bosonic (magnon) mode could range from several hundreds of megahertz to 28 GHz. The coupling strength between the electromagnetic field and the magnonic mode is \(J \approx 2\pi \times 42\) MHz. The decay rates of the cavity mode and the magnon mode are given by \(\kappa_a \approx 2\pi \times 2.87\) MHz and \(\kappa_b \approx 2\pi \times 24.3\) MHz. However, the magnitude of the self-Kerr interaction of the magnon mode is very small and hence the coupled photon-magnon model might be a possible candidate of the unconventional photon blockade effect.

For coupled superconducting resonator systems, the Kerr interaction can be implemented with the Josephson nonlinearity, and the photon hopping interaction can be implemented with a capacitor or other superconducting elements. Recently, some experiments have been reported that similar systems have been realized in superconducting setups. In a recent reported experiment [51], the authors proposed a method to realize a highly coherent Kerr medium by coupling a superconducting vertical transmon qubit to two three dimensional waveguide cavities. In this system, the resonance frequency of the cavity is \(\omega_a \approx 2\pi \times 9.2747\) GHz, the decay rate of the cavity mode is \(\kappa_a \approx 2\pi \times 10\) kHz, and the Kerr parameters is \(K \approx 2\pi \times 325\) kHz. Here we can see that the ratio \(K/\kappa\) could be larger than 30, and hence the strong Kerr nonlinearity in coupled cavity system is accessible. The coupling between two superconducting resonators can be realized through a capacitor or a qubit, and the coupling strength between the two resonators could reach the order of megahertz. Therefore, the parameters used in the present paper is accessible with current experimental condition in superconducting quantum circuits. The monochromatic driving of the superconducting resonator can be realized through the microwave field, the driving frequency and driving amplitude can be controlled on demand. For current experimental condition, the resonance frequency of the superconducting resonators is of the order of \(2\pi \times 5 - 10\) GHz, and the working temperature is about 15 - 25 mK. Based on these two parameter, we can estimate the thermal occupation number in the superconducting resonators. As an example, for \(T \approx 25\) mK, we have \(n_{th} \approx 4.9 \times 10^{-9} - 6.7 \times 10^{-5}\) when \(\omega_b \approx 2\pi \times 5 - 10\) GHz. In this case, the conventional photon blockade effect is observable in the coupled superconducting resonator system.

VII. CONCLUSION

In conclusion, we have studied the conventional and unconventional photon blockade effects in a coupled cavity system, which is formed by a linear cavity coupled to a nonlinear cavity. Here the linear cavity is weakly driven by a monochromatic laser field. We have calculated the
equal-time second-order correlation function of the linear cavity field, and these results have been confirmed by analytically and numerically solving the quantum master equation. We have found that the photon blockade effects in this system can be explained based on both the conventional and unconventional photon blockade physical mechanisms. We have also clarified the quantum-interference-induced physical phenomenon existing in the conventional photon blockade regime. Some discussions on the experimental implementation of this model have been presented.

Appendix: Equations of motion for the density matrix elements

In this Appendix, we present the steady-state solution of these density matrix elements at zero temperature. In the weak-driving case, we truncate the Hilbert space of the system up to two excitations $m + n = 2$ and assume that the total system is initially in the vacuum state $|0, 0\rangle_{a,b}$. For below convenience, we introduce the notations: $|0, 0\rangle_{a,b} \rightarrow |S_1\rangle$, $|0, 1\rangle_{a,b} \rightarrow |S_2\rangle$, $|1, 0\rangle_{a,b} \rightarrow |S_3\rangle$, $|1, 1\rangle_{a,b} \rightarrow |S_4\rangle$, and $|2, 0\rangle_{a,b} \rightarrow |S_5\rangle$. Then the density matrix elements can be expressed as $\rho_{mn} = \langle S_m | \rho | S_n \rangle$. In the weak-driving case, these density matrix elements can be divided into different groups of the orders of the small quantity $\Omega / \kappa_a$ as $\rho_{00} \gg \rho_{10}, \rho_{01} \gg \rho_{20}, \rho_{11}, \rho_{02}$ [53]. The steady-state equations of these density matrix elements can accordingly be grouped based on the marked orders. The zero-order equation can be obtained as $\rho_{11} = 1$. The first-order equations are

\[
i(\Delta + K)\rho_{12} + iJ\rho_{14} - \kappa_b\rho_{12}/2 = 0, \]
\[
i\Delta\rho_{14} + iJ\rho_{12} + i\Omega\rho_{11} - \kappa_a\rho_{14}/2 = 0, \tag{A.1}\]

then the solution can be obtained as

\[
\rho_{12} = \rho_{12}^* = -\frac{4J\Omega}{4J^2 - (2K + i\kappa_b + 2\Delta)(2\Delta + i\kappa_a)}, \]
\[
\rho_{14} = \rho_{14}^* = -\frac{2(2K + i\kappa_b + 2\Delta)\Omega}{-4J^2 + (2K + i\kappa_b + 2\Delta)(2\Delta + i\kappa_a)}. \tag{A.2}\]

The second-order equations are

\[
i(2\Delta + 4K)\rho_{13} + i\sqrt{2}J\rho_{15} - \kappa_b\rho_{13} = 0, \]
\[
i(2\Delta + K)\rho_{15} + i\sqrt{2}J(\rho_{16} + \rho_{13}) + i\Omega\rho_{12} - (\kappa_a/2 + \kappa_b/2)\rho_{15} = 0, \]
\[
i2\Delta\rho_{16} + i\sqrt{2}J\rho_{15} + i\sqrt{2}\Omega\rho_{14} - \kappa_a\rho_{16} = 0, \]
\[
iJ(\rho_{24} - \rho_{42}) - \kappa_b\rho_{22} = 0, \]
\[
iJ(\rho_{42} - \rho_{24}) + i\Omega(\rho_{44} - \rho_{14}) - \kappa_a\rho_{44} = 0, \]
\[
-iK\rho_{24} + iJ(\rho_{22} - \rho_{44}) + i\Omega\rho_{21} - (\kappa_a/2 + \kappa_b/2)\rho_{24} = 0, \]
\[
iK\rho_{42} + iJ(\rho_{44} - \rho_{22}) - i\Omega\rho_{12} - (\kappa_a/2 + \kappa_b/2)\rho_{42} = 0, \tag{A.3}\]

which leads to the solutions

\[
\rho_{13} = \rho_{31}^* = 8\sqrt{2}J^2[2K + i(\kappa_b - 4\Delta + \kappa_a)]\Omega^2P^{-1}Q^{-1}, \]
\[
\rho_{15} = \rho_{51}^* = -8J[4K + i\kappa_b + 2\Delta][2K + i(\kappa_b - 4\Delta + \kappa_a)]\Omega^2P^{-1}Q^{-1}, \]
\[
\rho_{16} = \rho_{61}^* = 2\sqrt{2}[8J^2K + (2K + i\kappa_b + 2\Delta)(4K + i\kappa_b + 2\Delta)][2K + i(\kappa_b - 4\Delta + \kappa_a)]\Omega^2P^{-1}Q^{-1}, \tag{A.4}\]

and

\[
\rho_{22} = 16J^2\Omega^2\langle |P|^2 \rangle^{-1}, \]
\[
\rho_{44} = 4[\kappa_b^2 + 4(K + \Delta)^2]\Omega^2\langle |P|^2 \rangle^{-1}, \]
\[
\rho_{24} = \rho_{42}^* = 8J(2K + i\kappa_b + 2\Delta)\Omega^2\langle |P|^2 \rangle^{-1}, \tag{A.5}\]

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Finally, we obtain the fourth-order equations

\[
P = [4J^2 - (2K + i\kappa_b + 2\Delta)(2\Delta + i\kappa_a)],
\]
\[
Q = -(4K + i\kappa_b + 2\Delta)(2\Delta + i\kappa_a)[2K + i(\kappa_b - 4i\Delta + \kappa_a)] + 4J^2[4K + i(\kappa_b - 4i\Delta + \kappa_a)].
\]  
(A.6)

By solving the third-order equations

\[
i(\Delta + 3K)\rho_{23} + i\sqrt{2}J\rho_{25} - iJ\rho_{43} - 3\kappa_b\rho_{23}/2 = 0,
\]
\[
i\Delta\rho_{25} + iJ(\sqrt{2}\rho_{26} + \sqrt{2}\rho_{23} - \rho_{25}) + i\Omega\rho_{22} - (\kappa_a/2 + \kappa_b)\rho_{25} = 0,
\]
\[
i(\Delta - K)\rho_{25} + iJ(\sqrt{2}\rho_{26} - \rho_{25}) + i\sqrt{2}\Omega\rho_{24} - (\kappa_b/2 + \kappa_a)\rho_{26} = 0,
\]
\[
i(\Delta + 4K)\rho_{43} + iJ(\sqrt{2}\rho_{45} - \rho_{23}) - i\Omega\rho_{13} - (\kappa_a/2 + \kappa_b)\rho_{13} = 0,
\]
\[
i(\Delta + K)\rho_{45} + iJ(\sqrt{2}\rho_{46} + \sqrt{2}\rho_{43} - \rho_{25}) + i\Omega(\rho_{12} - \rho_{15} - (\kappa_b/2 + \kappa_a)\rho_{45} = 0,
\]
\[
i\Delta\rho_{46} + iJ(\sqrt{2}\rho_{45} - \rho_{26}) + i\Omega(\sqrt{2}\rho_{44} - \rho_{16}) - 3\kappa_a\rho_{46}/2 = 0,
\]  
(A.7)

we obtain the solution as

\[
\rho_{23} = \rho_{23}^{*2} = -32\sqrt{2}J^3[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1},
\]
\[
\rho_{25} = \rho_{25}^{*2} = 32J^2(4K + i\kappa_b + 2\Delta)[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1},
\]
\[
\rho_{26} = \rho_{26}^{*2} = -8\sqrt{2}J[8J^2K + (2K + i\kappa_b + 2\Delta)(4K + i\kappa_b + 2\Delta)[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1},
\]
\[
\rho_{43} = \rho_{43}^{*4} = 16\sqrt{2}J^2(2K - i\kappa_b + 2\Delta)[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1},
\]
\[
\rho_{45} = \rho_{45}^{*4} = -16J(2K - i\kappa_b + 2\Delta)(4K + i\kappa_b + 2\Delta)[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1},
\]
\[
\rho_{46} = \rho_{46}^{*4} = 4\sqrt{2}(2K - i\kappa_b + 2\Delta)[8J^2K + (2K + i\kappa_b + 2\Delta)(4K + i\kappa_b + 2\Delta)]
\times[2K + i(\kappa_b - 4i\Delta + \kappa_a)]\Omega^3(|P|^2Q)^{-1}.
\]  
(A.8)

Finally, we obtain the fourth-order equations

\[
i\sqrt{2}J\rho_{35} - i\sqrt{2}J\rho_{33} - 2\kappa_b\rho_{33} = 0,
\]
\[
-i3K\rho_{35} + i\sqrt{2}J(\rho_{36} + \rho_{33} - \rho_{55}) + i\Omega\rho_{32} - (\kappa_a/2 + 3\kappa_b/2)\rho_{35} = 0,
\]
\[
i3K\rho_{53} - i\sqrt{2}J(\rho_{53} + \rho_{33} - \rho_{55}) - i\Omega\rho_{23} - (\kappa_a/2 + 3\kappa_b/2)\rho_{53} = 0,
\]
\[
i4K\rho_{35} - i\sqrt{2}J(\rho_{35} - \rho_{55}) - i\Omega\rho_{34} - (\kappa_b + \kappa_a)\rho_{35} = 0,
\]
\[
i4K\rho_{53} - i\sqrt{2}J(\rho_{53} - \rho_{55}) - i\Omega\rho_{43} - (\kappa_b + \kappa_a)\rho_{53} = 0,
\]
\[
i\sqrt{2}J(\rho_{56} + \rho_{53} - \rho_{65} - \rho_{35}) + i\Omega(\rho_{52} - \rho_{53}) - (\kappa_b + \kappa_a)\rho_{56} = 0,
\]
\[
i4K\rho_{56} + i\sqrt{2}J(\rho_{55} - \rho_{65} - \rho_{35}) + i\Omega(\sqrt{2}\rho_{54} - \rho_{26}) - (\kappa_b/2 + 3\kappa_a/2)\rho_{56} = 0,
\]
\[
i\sqrt{2}J(\rho_{65} - \rho_{56}) + i\sqrt{2}\Omega(\rho_{64} - \rho_{46}) - 2\kappa_a\rho_{66} = 0,
\]  
(A.9)

and their solutions

\[
\rho_{33} = 128J^4[4(K + 2\Delta)^2 + (\kappa_b + \kappa_a)^2]\Omega^4(|P|^2Q)^{-1},
\]
\[
\rho_{35} = \rho_{35}^{*5} = -64\sqrt{2}J^3(4K + i\kappa_b + 2\Delta)[4(K + 2\Delta)^2 + (\kappa_b + \kappa_a)^2]\Omega^4(|P|^2Q)^{-1},
\]
\[
\rho_{36} = \rho_{36}^{*6} = 32J^2[2K - i(\kappa_b + 4i\Delta + \kappa_a)]\Omega^4(|P|^2Q)^{-1},
\]
\[
\times[2K + i(\kappa_b + 4i\Delta + \kappa_a)]\Omega^4(|P|^2Q)^{-1},
\]
\[
\rho_{55} = 64J^2[\kappa_b^2 + 4(2K + \Delta)^2]([4(K + \Delta)^2 + (\kappa_b + \kappa_a)^2]\Omega^4(|P|^2Q)^{-1},
\]
\[
\rho_{56} = \rho_{56}^{*6} = -16\sqrt{2}J(4K + i\kappa_b + 2\Delta)[2K - i(\kappa_b + 4i\Delta + \kappa_a)]\Omega^4(|P|^2Q)^{-1},
\]
\[
+ (2K + i\kappa_b + 2\Delta)(4K + i\kappa_b + 2\Delta)[2K + i(\kappa_b + 4\Delta + i\kappa_a)]\Omega^4(|P|^2Q)^{-1},
\]
\[
\rho_{66} = 8\{64J^4K^2 + 128J^2K(\kappa_b + 2\Delta)(-\kappa_b + 2K^2 + 5K\Delta + 2\Delta^2) - 32J^2K\kappa_b(3K + 2\Delta)\kappa_a
\]
\[
+ \kappa_b^2 + 4(2K + \Delta)^2]\kappa_a^2\Omega^4(|P|^2Q)^{-1}.
\]  
(A.10)

Based on these solutions of the state occupations, we can calculate the approximate analytical result of the equal-time second-order correlation function of the cavity field.
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