Projection Pressure and Bowen’s Equation for a Class of Self-similar Fractals with Overlap Structure

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Abstract

Let \( \{S_i\}_{i=1}^l \) be an iterated function system (IFS) on \( \mathbb{R}^d \) with attractor \( K \). Let \( \pi \) be the canonical projection. In this paper we define a new concept called “projection pressure” \( P_{\pi}(\varphi) \) for \( \varphi \in C(\mathbb{R}^d) \) under certain affine IFS, and show the variational principle about the projection pressure. Furthermore we check that the unique zero root of “projection pressure” still satisfies Bowen’s equation when each \( S_i \) is the similar map with the same compression ratio. Using the root of Bowen’s equation, we can get the Hausdorff dimension of the attractor \( K \).

Key words: projection entropy, projection pressure, Hausdorff dimension, variational principle, Bowen’s equation

1. Introduction

Let \( \{S_i : X \to X\}_{i=1}^l \) be a family of contractive maps on a nonempty closed set \( X \subset \mathbb{R}^d \). Following Barnsley [1], we say that \( \{S_i\}_{i=1}^l \) is an \textit{iterated function system} (IFS) on \( X \). Hutchinson [10] showed that there is a unique nonempty compact set \( K \subset X \), called the \textit{attractor} of \( \{S_i\}_{i=1}^l \), such that \( K = \bigcup_{i=1}^l S_i(K) \).

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There are many references to compute the Hausdorff dimension of $K$ or the Hausdorff dimension of multifractal spectrum, such as [14], [15] and [12]. Thermodynamic formalism played a significant role when we try to compute the Hausdorff dimension of $K$, especially the Bowen’s equation. Usually we call $P_J(t\psi) = 0$ the Bowen’s equation, where $P_J$ is the topological pressure of the map $f : J \to J$, and $\psi$ is the geometric potential $\psi(z) = \log |f'(z)|$. The root of Bowen’s equation always approaches the Hausdorff dimension of some sets. In [5], Bowen first discovered this equation while studying the Hausdorff dimension of quasi-circles. Later Ruelle [20], Gatzouras and Peres [9] showed that Bowen’s equation gives the Hausdorff dimension of $J$ whenever $f$ is a $C^1$ conformal map on a Riemannian manifold and $J$ is a repeller. According to the method for calculating Hausdorff dimension of cookie-cutters presented by Bedford [3], Keller discussed the relation between classical pressure and dimension for IFS [11]. He concluded that if $\{S_i\}_{i=1}^l$ is a conformal IFS satisfying the disjointness condition with local energy function $\psi$, then the pressure function has a unique zero root $t_0 = \dim_H K$. In 2000, using the definition of Carathéodory dimension characteristics, Barreira and Schmeling [2] introduced the notion of the $u$-dimension $\dim_u Z$ for positive functions $u$, showing that $\dim_u Z$ is the unique number $t$ such that $P_Z(-tu) = 0$.

On the progress of calculating $\dim_H K$, references [14], [15] and [12] depend on the open set condition and separable condition. In fact, there are a lot of examples don’t satisfy this disjointness condition. Hui Rao and Zhiying Wen once discussed a kind of self-similar fractal with overlap structure called $\lambda$-Cantor set [17].

In order to study the Hausdorff dimension of an invariant measure $\mu$ for conformal and affine IFS with overlaps, Dejun Feng and Huyi Hu introduce a notion projection entropy (see[8]), which plays the similar role as the classical entropy of IFS satisfying the open set condition, and it becomes the classical entropy if the projection is finite to one.

Bedford pointed out that the Bowen’s equation works if three elements are present:(i) conformal contractions, (ii) open set conditions, and (iii) subshift of finite type (Markov) structure. Chen [7] proved that subshift of finite type (Markov) structure can be replaced by any subshift structure. The motivation of this paper is to find a new pressure function satisfying the Bowen’s equation without the open set conditions. We will show the variational principle for the new pressure under certain affine IFS, and check that the zero of projection pressure is equal to $\dim_H K$ when each $S_i$ has the same compression ratio. Thus, using our results we can easily get the dimension of $\lambda$
2. The projection pressure for certain affine IFS: Definition and variational principle

Let \( \{S_i\}_{i=1}^l \) be an IFS on a closed set \( X \subset \mathbb{R}^d \). Denote by \( K \) its attractor. Let \( \sum = \{1, \cdots, l\}^\mathbb{N} \) associated with the left shift \( \sigma \). Let \( M_\sigma(\Sigma) \) denote the space of \( \sigma \)-invariant measure on \( \Sigma \), endowed with the weak-star topology, \( C(X) \) the space of real-valued continuous functions of \( X \) and \( \pi : \Sigma \to K \) be the canonical projection defined by

\[
\{\pi(x)\} = \bigcap_{n=1}^{\infty} S_{x_1} \circ S_{x_2} \circ \cdots \circ S_{x_n}(K), \quad \text{where} \quad x = \{x_i\}_{i=1}^{\infty}. \tag{1}
\]

A measure \( \mu \) on \( K \) is called invariant (resp. ergodic) for the IFS if there is an invariant (resp. ergodic) measure \( \nu \) on \( \Sigma \) such that \( \mu = \nu \circ \pi^{-1} \).

Let \( (\Omega, \mathcal{F}, \nu) \) be a probability space. For a sub-\( \sigma \)-algebra \( \mathcal{A} \) of \( \mathcal{F} \) and \( f \in L^1(\Omega; \mathcal{F}; \nu) \), we denote by \( E_\nu(f|\mathcal{A}) \) the conditional expectation of \( f \) given \( \mathcal{A} \). For countable \( \mathcal{F} \)-measurable partition \( \xi \) of \( \Omega \). We denote by \( I_\nu(\xi|\mathcal{A}) \) the conditional information of \( \xi \) given \( \mathcal{A} \), which is given by the formula

\[
I_\nu(\xi|\mathcal{A}) = -\sum_{A \in \xi} \lambda_A \log E_\nu(\lambda_A|\mathcal{A}), \tag{2}
\]

where \( \lambda_A \) denotes the characteristic function on \( \mathcal{A} \).

The conditional entropy of \( \xi \) given \( \mathcal{A} \), written \( H_\nu(\xi|\mathcal{A}) \) is defined by the formula \( H_\nu(\xi|\mathcal{A}) = \int I_\nu(\xi|\mathcal{A}) d\nu \).

The above information and entropy are unconditional when \( \mathcal{A} = \mathcal{N} \), the trivial \( \sigma \)-algebra consisting of sets of measure zero and one, and in this case we write

\[
I_\nu(\xi|\mathcal{N}) = I_\nu(\xi) \quad \text{and} \quad H_\nu(\xi|\mathcal{N}) = H_\nu(\xi). \tag{3}
\]

Now we consider the space \((\Sigma, \mathcal{B}(\Sigma), m)\), where \( \mathcal{B}(\Sigma) \) is the Borel \( \sigma \)-algebra on \( \Sigma \) and \( m \in M_\sigma(\Sigma) \). Let \( \mathcal{P} \) denote the Borel partition

\[
\mathcal{P} = \{[j] : 1 \leq j \leq l\}
\]

of \( \Sigma \), where \( [j] = \{(x_i)_{i=1}^{\infty} \in \Sigma, \ x_1 = j\} \). Let \( \mathcal{I} \) denote the \( \sigma \)-algebra

\[
\mathcal{I} = \{B \in \mathcal{B}(\Sigma) : \sigma^{-1}B = B\}.
\]
For convenience, we use $\gamma$ to denote the Borel $\sigma$-algebra $B(\mathbb{R}^d)$ of $\mathbb{R}^d$. For $f \in C(X)$, denote $\|f\| = \sup_{x \in X} f(x)$ and $S_n f(x) = \sum_{i=0}^{n-1} f(\sigma^n x)$ $x \in X$.

Let $\Sigma = \{[b] : [b] = (x_1, x_2, \cdots, x_n), x_i \in \Sigma, \ i = 1, \cdots, n\}$.

**Definition 1.** For any $m \in M_\sigma(\Sigma)$, we call

$$h_\pi(\sigma, m) = H_m(\mathcal{P}|\sigma^{-1} \pi^{-1} \gamma) - H_m(\mathcal{P}|\pi^{-1} \gamma)$$

the projection entropy of $m$ under $\pi$ w.r.t. $\{S_i\}_{i=1}^l$, and we call

$$h_\pi(\sigma, m, x) = E_m(f|\mathcal{I})(x)$$

the local projection entropy of $m$ at $x$ under $\pi$ w.r.t $\{S_i\}_{i=1}^l$, where $f$ denotes the function $I_m(\mathcal{P}|\sigma^{-1} \pi^{-1} \gamma) - I_m(\mathcal{P}|\pi^{-1} \gamma)$.

It is clear that $h_\pi(\sigma, m) = \int h_\pi(\sigma, m, x) dm(x)$.

**Definition 2.** Let $k \in \mathbb{N}$ and $\nu \in M_{\sigma^k}(\Sigma)$. Define

$$h_\pi(\sigma^k, \nu) = H_\nu(\mathcal{P}|\sigma^{-k} \pi^{-1} \gamma) - H_\nu(\mathcal{P}|\pi^{-1} \gamma).$$

The term $h_\pi(\sigma^k, \nu)$ can be viewed as the projection measure-theoretic entropy of $\nu$ w.r.t. the IFS $\{S_i \circ \cdots \circ S_{ij} : 1 \leq i_j \leq l \text{ for } 1 \leq j \leq k\}$. The following lemma exploits the connection between $h_\pi(\sigma^k, \nu)$ and $h_\pi(\sigma, \nu)$, where $m = \frac{1}{k} \sum_{i=0}^{k-1} \nu \circ \sigma^{-i}$.

**Lemma 3.** Let $k \in \mathbb{N}$ and $\nu \in M_{\sigma^k}(\Sigma)$. Set $m = \frac{1}{k} \sum_{i=0}^{k-1} \nu \circ \sigma^{-i}$. Then $m$ is $\sigma$-invariant, and $h_\pi(\sigma, \nu) = \frac{1}{k} h_\pi(\sigma^k, \nu) = \frac{1}{k} h_\pi(\sigma^k, m)$.

**Proof:** See Proposition 4.3 in [8].

The following two Lemmas will be used in our results.

**Lemma 4.** Let $a_1, a_2, \cdots, a_k$ be given real numbers. If $p_i \geq 0$ and $\sum_{i=0}^{k} p_i = 1$, then

$$\sum_{i=0}^{k} p_i (a_i - \log p_i) \leq \log (\sum_{i=0}^{k} e^{a_i})$$

and equality holds iff $p_i = \frac{e^{a_i}}{\sum_{j=1}^{k} e^{a_j}}$.

**Proof:** See Lemma 9.9 in [22].
Lemma 5. Assume that $\Omega$ is a subset of $\{1, \cdots, l\}$ such that $S_i(K) \cap S_j(K) = \emptyset$ for all $i, j \in \Omega$ with $i \neq j$. Suppose that $\nu$ is an invariant measure on $\Sigma$ supposed on $\Omega^\mathbb{N}$, i.e., $\nu([j]) = 0$ for all $j \in \{1, \cdots, l\} \setminus \Omega$. Then $h_\pi(\sigma, \nu) = h(\sigma, \nu)$.

**Proof:** See Lemma 4.19 in [8].

Let $S_i(x) = Ax + c_i$ $(i = 1, \cdots, l)$ be an IFS on $\mathbb{R}^d$, where $A$ is a $d \times d$ non-singular matrix with $\|A\| < 1$ and $c_i \in \mathbb{R}^d$. Let $K$ denote its attractor and $\pi : \Sigma \to K$ be the canonical projection. Let $Q$ denote the partition $\{[0,1)^d + \alpha : \alpha \in \mathbb{Z}^d\}$ of $\mathbb{Z}^d$. We set $Q_n = \{A^nQ : Q \in Q\}$ for $n = 0, 1, \cdots$.

Lemma 6. Let $m \in M_\sigma(\Sigma)$, then $h_\pi(\sigma, m) = \lim_{n \to \infty} \frac{H_m(\pi^{-1}Q_n)}{n}$.

**Proof:** See Proposition 4.18 (i) in [8].

Theorem 7. If an IFS $\{S_i\}_{i=1}^l$ has the form as above and $f \in C(K)$. Then

$$\lim_{n \to \infty} \frac{1}{n} (\log \sum_{Q \in Q_n} \sup_{Q \cap K \neq \emptyset} e^{S_nf(\alpha)}) = \sup \{h_\pi(\sigma, m) + \int f \circ \pi dm : m \in M_\sigma(\Sigma)\}.$$  

**Proof:** We assume $K \subset [0,1)^d$, without loss of generality. We divided the proof into two steps.

**step 1.**

$$\liminf_{n \to \infty} \frac{1}{n} (\log \sum_{Q \in Q_n} \sup_{Q \cap K \neq \emptyset} e^{S_nf(\alpha)}) \geq \sup \{h_\pi(\sigma, m) + \int f \circ \pi dm : m \in M_\sigma(\Sigma)\}.$$  

For arbitrary $n \in \mathbb{N}, Q \in Q_n, m \in M_\sigma(\Sigma)$, let $g_n(Q) = \sup_{\alpha \in \pi^{-1}(Q \cap K)} S_nf(\alpha), P(Q) = m(\pi^{-1}Q)$. By lemma 4, we have

$$\log \sum_{Q \in Q_n} \sup_{Q \cap K \neq \emptyset} e^{S_nf(\alpha)} \geq \sum_{Q \in Q_n} P(Q)(g_n(Q) - \log P(Q))$$

$$= H_m(\pi^{-1}Q_n) + \sum_{Q \in Q_n} P(Q)g_n(Q)$$
\[
\geq H_m(\pi^{-1}Q_n) + \int S_nf\pi(\alpha)dm \\
= H_m(\pi^{-1}Q_n) + n \int f\pi dm.
\]

Using Lemma 6 yields

\[
\liminf_{n \to \infty} \frac{1}{n} (\log \sum_{Q \in Q_n} \sup_{Q \cap K \neq \emptyset} e^{S_n f\pi(\alpha)}) \geq \lim_{n \to \infty} \frac{H_m(\pi^{-1}Q_n)}{n} + \int f\pi dm \\
= h_\pi(\sigma, m) + \int f\pi dm.
\]

By the arbitraryness of \( m \), we have step1.

**step2.**

\[
\sup \{ h_\pi(\sigma, m) + \int f\circ\pi dm : m \in M_\sigma(\Sigma) \} \geq \limsup_{n \to \infty} \frac{1}{n} (\log \sum_{Q \in Q_n} \sup_{Q \cap K \neq \emptyset} e^{S_n f\pi(\alpha)}).
\]

By the continuity of \( f\pi \), for arbitrary \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for arbitrary \( a_N \in \Sigma_N \), and any \( x, y \in a_N \) we have

\[
|f\pi(x) - f\pi(y)| < \epsilon.
\]

Now, for any \( n \in \mathbb{N} \) and \( a_{n+N} \in \Sigma_{n+N} \)

\[
|S_{n+N} f\pi(x) - S_{n+N} f\pi(y)| \leq n\epsilon + 2N\|f\pi\| \quad \forall x, y \in a_{n+N}.
\]

For \( x \in K \) and \( n \in \mathbb{N} \), we denote

\[
g_n(x) = \sup_{\alpha \in \pi^{-1}x} e^{S_n f\pi(\alpha)}.
\]

For \( X \subset \mathbb{R}^d \) and \( X \cap K \neq \emptyset \), we denote

\[
g_n(X) = \sup_{x \in X \cap K} g_n(x).
\]

For any \( Q \in Q_n \), we denote

\[
2Q = \bigcup_{P \subseteq Q_n} P \quad \text{if} \ P \cap Q \neq \emptyset
\]
and

$$3Q = \bigcup_{P \in Q, \, T \cap 2Q \neq \emptyset} P.$$  

Claim: There exists a subset $\Gamma$ of $\{Q \in Q_{n+N} : Q \cap K \neq \emptyset\}$ such that

(i) $\sum_{Q \in \Gamma} g_{n+N}(Q) \geq \frac{\sum_{Q \in Q_{n+N}} g_{n+N}(Q)}{7^d};$

(ii) $2Q \cap 2\tilde{Q} = \emptyset$, $Q, \tilde{Q} \in \Gamma$, $Q \neq \tilde{Q}.$

Since $K$ is compact, we can find $Q_1 \in Q_{n+N}$ and $x_1 \in Q_1 \cap K$ such that

$$g_{n+N}(x_1) = g_{n+N}(Q \cap K) = g_{n+N}(K).$$

If $K \setminus 3Q = \emptyset$, then $\Gamma = \{Q_1\}$. Otherwise, let $K_2 = K \setminus 3Q_1$. We can find $Q_2 \in Q_{n+N}$ and $x_2 \in Q_2 \cap K$ such that

$$g_{n+N}(x_2) = g_{n+N}(Q \cap K) = g_{n+N}(K).$$

If $K_2 \setminus 3Q_2 = \emptyset$, then $\Gamma = \{Q_1, Q_2\}$. Repeat above steps and we can finish it in finite steps. Clearly, there exists $M = M(n, N)$ and $\Gamma = \{Q_1, Q_2, \ldots, Q_M\}$ such that (i), (ii) are satisfied. Let $X = \{x_1, x_2, \ldots, x_M\}$, according to our claim, we have

$$\sum_{x \in X} g_{n+N}(x) \geq \frac{\sum_{Q \in Q_{n+N}} g_{n+N}(Q)}{7^d}. \tag{4}$$

For each $x \in X$, since $x \in K$, we can pick a word $[u]_x \in \Sigma_{n+N}$ such that $x \in S_{[u]_x} K$ and

$$\sup_{\alpha \in [u]_x} e^{(S_{n+N}f\pi)(\alpha)} \geq \sup_{\alpha \in \pi^{-1}x} e^{(S_{n+N}f\pi)(\alpha)}. \tag{5}$$

Consider the collection $W = \{[u]_x, x \in X\}$. The separation condition for elements in $X$ guarantees that $S_{[u]_x} K \cap S_{[u]_y} K = \emptyset$ for all $x, y \in X$ with $x \neq y$. Define a Bernoulli measure $\nu$ on $W^\mathbb{N}$ by

$$\nu([u]_x) = \frac{\sup_{\alpha \in [u]_x} e^{(S_{n+N}f\pi)(\alpha)}}{\sum_{x \in X} \sup_{\beta \in [u]_x} e^{(S_{n+N}f\pi)(\beta)}},$$
\[ \nu([w_1, \cdots, w_k]) = \prod_{i=1}^{k} \nu([w_i]), \quad w_i \in W, \ k \in \mathbb{N}. \]

Then \( \nu \) can be viewed as a \( \sigma^{n+N} \)-invariant measure on \( \Sigma \) (by viewing \( W^{\mathbb{N}} \) as a subset of \( \Sigma \)). By Lemma 5, we have \( h(\sigma^{n+N}, \nu) = h(\sigma^{n+N}, \nu). \) Define

\[ \mu = \frac{1}{n+N} \sum_{i=0}^{n+N-1} \nu \circ \sigma^{-i} \in M_\sigma(\Sigma) \]

and \( \xi = \{[u]_x, x \in X\} \cup \{\Sigma \setminus \bigcup_{x \in X} [u]_x\}. \)

According to lemma 3, we have

\[ h(\sigma, \mu) + \int f \pi d\mu = \frac{h(\sigma^{n+N}, \nu))}{n+N} + \frac{\int S_{n+N} f \pi d\nu}{n+N} \]

\[ = \frac{1}{n+N} (h(\sigma^{n+N}, \nu) + \int S_{n+N} f \pi d\nu) \]

\[ = \frac{1}{n+N} (H_v(\xi) + \int S_{n+N} f \pi d\nu) \]

\[ \geq \frac{1}{n+N} \left( \sum_{x \in X} \left( -\nu([u]_x) \log \nu([u]_x) + \nu([u]_x) \inf_{\alpha \in [u]_x} S_{n+N} f \pi(\alpha) \right) \right) \]

\[ \geq \frac{1}{n+N} \left( \sum_{x \in X} \left( -\nu([u]_x) \log \nu([u]_x) + \nu([u]_x) \sup_{\alpha \in [u]_x} S_{n+N} f \pi(\alpha) \right) \right) \]

\[ \geq \frac{1}{n+N} \left( \sum_{x \in X} \left( -\nu([u]_x) \log \nu([u]_x) + \nu([u]_x) \sup_{\alpha \in [u]_x} S_{n+N} f \pi(\alpha) \right) \right) \]

\[ -n \epsilon - 2N \| f \pi \| \]

\[ = \frac{1}{n+N} \left( \sum_{x \in X} \left( -\nu([u]_x) \log \nu([u]_x) + \nu([u]_x) \sup_{\alpha \in [u]_x} S_{n+N} f \pi(\alpha) \right) \right) \]

\[ -n \epsilon + 2N \| f \pi \| \]

\[ = \frac{1}{n+N} \log \sum_{x \in X} \sup_{\alpha \in [u]_x} e^{(S_{n+N} f \pi)(\alpha)} - \frac{n \epsilon + 2N \| f \pi \|}{n+N} \]

\[ \geq \frac{1}{n+N} \log \sum_{x \in X} \sup_{\alpha \in \pi^{-1} x} e^{(S_{n+N} f \pi)(\alpha)} - \frac{n \epsilon + 2N \| f \pi \|}{n+N} \]

\[ \geq \frac{1}{n+N} \log \left( \frac{\sum_{Q \in \mathcal{Q}} g_{n+N}(Q)}{7d} \right) - \frac{n \epsilon + 2N \| f \pi \|}{n+N}. \]
Let $k = n + N$ and let $n \to \infty$, then $k \to \infty$. We have
\[
\sup \{h_\pi(\sigma, m) + \int f \circ \pi dm, m \in M_\sigma(\Sigma)\} \geq \limsup_{n \to \infty} \frac{1}{n} \left( \log \sum_{\substack{Q \in \mathcal{Q}_n \setminus \emptyset \atop Q \cap K \neq \emptyset}} \sup_{\alpha \in \pi^{-1}(Q \cap K)} e^{S_n f \pi(\alpha)} \right) - \epsilon.
\]

Since $\epsilon$ is arbitrary, we finish the proof of step 2.

**Definition 8.** If $f \in C(\mathbb{R}^d, \mathbb{R})$, we call
\[
P_\pi(f) = \lim_{n \to \infty} \frac{1}{n} \left( \log \sum_{\substack{Q \in \mathcal{Q}_n \setminus \emptyset \atop Q \cap K \neq \emptyset}} \sup_{\alpha \in \pi^{-1}(Q \cap K)} e^{S_n f \pi(\alpha)} \right)
\]

the projection pressure of $f$ under $\pi$ w.r.t. $\{S_i\}_{i=1}^l$, where $\{S_i\}_{i=1}^l$ as in Theorem 2.1.

It is clearly that if $f = 0$ we have the same result of Proposition 4.18 (ii) in [8].

**Corollary 9.** \[
\lim_{n \to \infty} \frac{\log \# \{Q \in \mathcal{Q}_n : \mathcal{Q}^n \cap K \neq \emptyset \}}{n} = \sup \{h_\pi(\sigma, m) : m \in M_\sigma(\Sigma)\}.
\]

3. **Bowen’s equation for certain self-similar set**

**Definition 10.** The IFS $\{S_i\}_{i=1}^l$ is conformal if for every $i \in \{1, 2, \ldots, l\}$,
(1) $S_i : U \to S_i(U)$ is $C^1$, (2) $\|S_i'(x)\| \neq 0$ for all $x \in U$, (3) $|S_i'(x)| = \|S_i'(x)\| |y|$ for all $x \in U, y \in \mathbb{R}^d$.

In the following, we assume that $S_i(x) = Ax + c_i$ ($i = 1, \ldots, l$) be an IFS and $A$ be a $d \times d$ compressed orthogonal matrix, which means $A$ satisfies $AA^T = cE$, $c_i \in \mathbb{R}^d$ and $0 < c < 1$. Clearly, such a IFS is conformal. Let $\|S\|$ denote the spectral norm of $S$.

**Lemma 11.** Let $K$ be the attractor of a conformal IFS $S_i(x) = Ax + c_i$ ($i = 1, \ldots, l$). Then we have
\[
\dim_H K \leq \frac{\sup \{h_\pi(\sigma, m) : m \in M_\sigma(\Sigma)\}}{-\log \|A\|}.
\]

**Proof:** See Theorem 2.13 in [8].
Theorem 12. Let \( S_i(x) = Ax + c_i \quad (i = 1, \cdots, l) \) be an IFS and \( A \) be a \( d \times d \) compressed orthogonal matrix. Let \( \pi : \Sigma \to K \) be the canonical projection. Then \( \dim_H K \) is the unique root of \( P_\pi(\log \| A \| \cdot t) = 0 \).

Proof: According to Theorem 2.1 and Lemma 11, we have

\[
P_\pi(\log \| A \| \cdot t) = \sup \{ h_\pi(\sigma, m) : m \in M_\sigma(\Sigma) \} + \log \| A \| \cdot t.
\]

Hence \( \dim_H K \) is the unique root of \( P_\pi(\log \| A \| \cdot t) = 0 \).

Thus we verify the projection pressure function satisfies the Bowen’s equation without any disjoint property.

Example 13. (\( \lambda \)-Cantor set) Let \( \lambda \in [0, 1] \) be a real number and let \( S_1(x) = x/3, S_2(x) = x/3 + \lambda/3, S_3(x) = x/3 + 2/3 \) be three similarities on \( \mathbb{R} \). Then the self-similar set generated by these three similarities, denoted by \( F_\lambda \), will be called a \( \lambda \)-Cantor set.

If \( \lambda = 0 \), then \( S_1 = S_2 \) and \( F_\lambda \) is exactly the classical middle third Cantor set. In this case \( \dim_H(K) = \log 2 / \log 3 \).

If \( \lambda = 1 \), then \( F_\lambda = [0, 1] \) and \( \dim_H(K) = 1 \).

If \( 0 < \lambda < 1 \), the structure of \( F_\lambda \) is quite complicated.

However, according to our result, the Hausdorff dimension of \( F_\lambda \) should be the unique root \( t_0 \) of \( P_\pi(-\log 3 \cdot t) = 0 \).

Example 14. (Overlapping Sierpiński triangle) Consider a certain IFS as:

\[
S_1 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right), \quad S_2 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right), \quad \text{where } 0 \leq a_1 \leq 1/2 \quad \text{and} \quad 0 \leq a_2 \leq \sqrt{3}/4,
\]

\[
S_3 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} 1/4 \\ \sqrt{3}/4 \end{array} \right) \text{ on } \{(x, y) : y \leq \sqrt{3}x, y \geq 0 \text{ and } y \geq -\sqrt{3}x + \sqrt{3}\} (\text{See Figure 1}).
\]

Let \( K_{a_1, a_2} \) be its attractor.

If \( a_1 = a_2 = 0 \) or \( a_1 = 1/4 \) and \( a_2 = \sqrt{3}/4 \), then \( \dim_H K_{a_1, a_2} = 1 \).

If \( a_1 = 1/2 \) and \( a_2 = 0 \), then \( K_{a_1, a_2} \) is classical Sierpiński triangle and its Hausdorff dimension is \( \log 3 / \log 2 \).

If \( 0 < a_1 < 1/2 \) and \( 0 < a_2 < \sqrt{3}/4 \), the Hausdorff dimension of \( K_{a_1, a_2} \) should be the unique root \( t_0 \) of \( P_\pi(-\log 2 \cdot t) = 0 \).
Figure 1:

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