Schrödinger functional at $N_f = -2$

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We study the Schrödinger functional coupling for lattice Yang-Mills theory coupled to an improved bosonic spinor field, which corresponds to QCD with minus two light flavors. This theory serves as a less costly testcase than QCD for the scaling of the coupling.

1. Introduction

The non-perturbative computation of the strong coupling constant $\alpha_s$ from hadronic scales to perturbative high energy scales is one of the main interests of the ALPHA collaboration. For this computation, a finite volume renormalization scheme is used in which the coupling runs with the space-time volume [1]. The scale evolution of the coupling is computed recursively with the help of the step scaling function, which is an integrated variant of the beta function for finite changes of the scale. Furthermore, Schrödinger functional boundary conditions and $O(\alpha)$ improvement are employed. An important property of the step scaling function is that it can be calculated by simulating pairs of lattices with different sizes $L$ for the same $a$ and taking the continuum limit.

After this programme had been successfully used in the quenched approximation [2], first results for full QCD with two flavors are now available [3]. Unfortunately, full QCD is notoriously costly to simulate, and thus the data for the step scaling function do not reach very close to the continuum limit. In order to learn more about the extrapolation to the continuum and about the effect of $O(\alpha)$ improvement, we have studied the step scaling function in a Yang-Mills theory coupled to a bosonic spinor field, which has a local interaction and is therefore much cheaper to simulate. It corresponds to setting the number of flavors $N_f = -2$ in the partition function, and is therefore closely related to full QCD in perturbation theory. In the literature, this model is known as bermion model [4]. Our focus here however is not on extrapolating results from negative to positive flavor number.

2. Bermion model

For a detailed discussion of the lattice setup, boundary conditions and the algorithm we have used, we refer to [5] and references therein.

The Schrödinger functional, as the partition function of the system, is an integral over all gauge and quark fields which fulfill the given boundary conditions. After integrating out the quark fields, it is

$$Z = e^{-\Gamma} = \int D[U] D[\bar{\psi}] D[\psi] e^{-S[U, \bar{\psi}, \psi]} = \int D[U] e^{-S_g} \text{det}(M^\dagger M)^{N_f/2}$$

with a gauge action $S_g$. For $N_f = -2$, the determinant can be written as

$$Z = \int D[U] D[\phi^+] D[\phi] e^{-S_g - S_h},$$

with a now local bosonic action

$$S_h[U, \phi] = a^4 \sum_x |(M \phi)(x)|^2.$$

The fields $\phi(x)$ carry color and Dirac indices.

We have chosen the bulk improvement coefficient $c_{sw}$ by extrapolating the non-perturbative data for $N_f = 0$ [6] and $N_f = 2$ [7] in $N_f$. An explicit calculation of $c_{sw}$ along the lines of these references for the most critical parameters in our simulations has proven a good accuracy of this extrapolation. The boundary improvement coefficients $c_t$ and $\tilde{c}_t$ have been set to their perturbative values.
As the improved bosonic action depends quadratically on each link $U_{x\mu}$ through the clover term, it is difficult to employ finite step size algorithms such as the combination of heatbath and overrelaxation steps used in [6]. Because of the large additional cost of hybrid Monte Carlo algorithms, we have decided to use global heatbath steps for the bosonic fields and local overrelaxation steps with respect to the unimproved action for the gauge fields. The clover term is taken into account by an acceptance step. The acceptance rate has turned out to be high enough (about 70\%) for such an approach. Nevertheless, there is a significant overhead due to the calculation of the action difference necessary for the acceptance step, resulting in a cost factor of about 12 compared to unimproved Wilson bermions. On the other hand, comparing with data from [3], improved bermions are about a factor of 10 cheaper than dynamical fermions, with a better scaling in $a$. In figure 1 we see a scaling of the cost with $a^{-2.5}$.

3. Results

We have computed the step scaling function $\Sigma(u, a/L)$ for the couplings $u = \bar{g}^2 = 0.9793$ and $u = 1.5145$ and lattice sizes $L = 4, 5, 6, 8$. Most simulations were done on APE100 parallel computers with up to 256 nodes with 50 MFlops each. For the simulation of the largest lattice $L/a = 16$ at $u = 1.5145$ we have also used a crate of APEnile.

Figure 2 shows the results for the step scaling function plotted against $(a/L)^2$. Within the error bars, no linear dependence an $a/L$ is visible, and therefore an extrapolation linear in $(a/L)^2$ is justified. The extrapolated values are consistent with perturbation theory, and their error bars are of the same size as the perturbative 3-loop contribution.
Figure 3. Results for the step scaling function at \( u = 0.9793 \), together with a quadratic fit under the constraint of universality.

In [6], the step scaling function was already computed in the unimproved bermion theory for \( u = 0.9793 \). Figure 3 shows these results together with the data after implementing improvement. The linear cutoff effects for this observable are of the order of a few percent. In the plot, both data sets are fitted under the constraint of universality, i.e. that their continuum limit agrees. This fit is linear plus quadratic in \( a/L \) for the Wilson bermion data and quadratic in \( a/L \) in the improved data. Although the additional input from the Wilson data is included, the joint continuum limit \( \sigma_{\text{combined}}(0.9793) = 1.1059(43) \) is almost the same as the value \( \sigma_{\text{improved}}(0.9793) = 1.1063(46) \). A linear plus quadratic fit in \( a/L \) of the unimproved data alone would have given the continuum result \( \sigma_{\text{unimproved}}(0.9793) = 1.103(12) \).

This indicates a success of the improvement programme. The main contribution to the cost for the calculation of \( \sigma(u) \) comes from the largest lattice. When using an improved action, the lattice size needed for a reliable extrapolation to the continuum is smaller than without improvement. Even in the model used here, where the algorithmic implementation implies a large overhead for the inclusion of the clover term, this leads to the improved case being more cost effective.

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REFERENCES

1. Martin Lüscher, Peter Weisz and Ulli Wolff, *Nucl. Phys.*, B359:221–243, 1991.
2. Martin Lüscher, Rainer Sommer, Peter Weisz and Ulli Wolff, *Nucl. Phys.*, B413:481–502, 1994.
3. Roberto Frezzotti, Martin Hasenbusch, Ulli Wolff, Jochen Heitger and Karl Jansen, *Comput. Phys. Commun.*, 136:1–13, 2001.
4. Giulia de Divitiis et al., *Nucl. Phys.*, B437:447–470, 1995.
5. Bernd Gehrmann, Stefan Kurth, Juri Rolf and Ulli Wolff, *Nucl. Phys.*, B612:3–24, 2001
6. Juri Rolf and Ulli Wolff, *Nucl. Phys. Proc. Suppl.*, 83:899–901, 2000.
7. Martin Lüscher, Stefan Sint, Rainer Sommer, Peter Weisz, Ulli Wolff, *Nucl. Phys.*, B530:185–203, 1998.