Research Article

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The strong nil-cleanness of semigroup rings

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Abstract: In this paper, we study the strong nil-cleanness of certain classes of semigroup rings. For a completely 0-simple semigroup \( M = M^{(G;I, \Lambda;P)} \), we show that the contracted semigroup ring \( R_0[M] \) is strongly nil-clean if and only if either \( |I| = 1 \) or \( |\Lambda| = 1 \), and \( R[G] \) is strongly nil-clean; as a corollary, we characterize the strong nil-cleanness of locally inverse semigroup rings. Moreover, let \( S = [Y;S_0, q_{\alpha}^\beta] \) be a strong semilattice of semigroups, then we prove that \( R[S] \) is strongly nil-clean if and only if \( R[S_\alpha] \) is strongly nil-clean for each \( \alpha \in Y \).

Keywords: strong nil-cleanness, semigroup rings, maximal subgroups, completely 0-simple semigroups, supplementary semilattice sum

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1 Introduction

Diesl [1] introduced the concept of nil-clean and strongly nil-clean rings and asked the question when a matrix ring is (strongly) nil-clean. As we have observed, it is difficult to characterize the arbitrary ring \( R \) such that \( \mathbb{M}_n(R) \) is nil-clean [2]. It is also known that the ring of all \( 2 \times 2 \) matrices over any commutative local ring is not strongly nil-clean [3]. On the other hand, the strong nil-cleanness of some generalizations of matrix rings has been considered. By [1], if \( R \) is a commutative ring with identity, then the formal block matrix ring \( \begin{pmatrix} A & R \\ 0 & B \end{pmatrix} \) is strongly nil-clean if and only if \( A, B \) are strongly nil-clean. Moreover, the strong nil-cleanness of Morita contexts, formal matrix rings and generalized matrix rings has also been studied [2].

Let \( M = M^{(G;I, \Lambda;P)} \) be a completely 0-simple semigroup. Then the contracted semigroup ring \( R_0[M] \equiv \lambda(R[G];I, \Lambda;P) \) is a Munn ring, which is isomorphic to the matrix ring \( \mathbb{M}_n(R[G]) \) if such \( M \) is a Brandt semigroup. Thus, general contracted completely 0-simple semigroup rings can be viewed as a generalization of semigroup rings. Motivated by this, we want to characterize the strong nil-cleanness of the contracted semigroup rings \( R_0[M] \) in terms of the structure of \( M \) and the strong nil-cleanness of the group ring \( R[G] \). Noting that semigroup rings are also an extension of group rings, the strong nil-cleanness of certain kinds of group rings has been studied in [2,4,5].

However, for a finite completely 0-simple semigroup \( M = M^{(G;I, \Lambda;P)} \), the contracted semigroup ring \( R_0[M] \) contains an identity if and only if \( |I| = |\Lambda| \) and \( P \) is an invertible matrix over \( G \), and if and only if \( R_0[M] \) is isomorphic to the matrix ring \( \mathbb{M}_n(R[G]) \) [6]. Note that the term “non-unital ring” or “general ring” refers to a ring that does not necessarily have an identity. It is known that Nicholson extended many of the results on the (strong) cleanness of rings with identity to non-unital rings [7], and that in [1,2], the (strong) nil-cleanness of non-unital rings has been studied.

The paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, we show that the strong nil-cleanness of the contracted semigroup ring \( R_0[M] \) of a completely 0-simple semigroup

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\[M = M^0(G; I, \Lambda; P)\] can be characterized by \(|I|, |\Lambda|\) and the strong nil-cleanness of \(R[G]\). As an application, we determine when the contracted semigroup ring of a locally inverse semigroup with idempotents set locally finite is strongly nil-clean. In Section 4, we consider the strong nil-cleanness of certain classes of supplementary semilattice sums of rings, and the relationship between the strong nil-cleanness of contracted semigroup rings and semigroup rings.

Throughout this paper, a ring always means an associate non-unital (or general) ring, and we always assume that \(R\) is a ring with identity (not necessarily commutative).

## 2 Preliminaries

In this section, the notations and definitions on semigroups and semigroup rings which will be used in the sequel are provided, see [6,8,9].

Unless otherwise stated, a semigroup \(S\) is always assumed to have a zero element (denoted by \(\theta\) or \(\theta_0\)). Denote by \(S^1\) the semigroup obtained from \(S\) by adding an identity if \(S\) has no identity, otherwise, let \(S^1 = S\). Green’s equivalence relations play an important role in the theory of semigroups, which were introduced by Green (1951): for \(a, b \in S\),

\[
\begin{align*}
\begin{array}{l}
\mathcal{L} \ a \ b \iff S^1a = S^1b, \\
\mathcal{R} \ a \ b \iff aS^1 = bS^1, \\
\mathcal{J} \ a \ b \iff S^1aS^1 = S^1bS^1
\end{array}
\end{align*}
\]

and \(\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \mathcal{D} = \mathcal{L} \cup \mathcal{R}\).

A semigroup is said to be regular if every \(\mathcal{L}\)-class and every \(\mathcal{R}\)-class of it contains an idempotent. An inverse semigroup is a regular semigroup with commutative idempotents. Let \(E(S) = \{e \in S \mid e^2 = e\}\) be the set of idempotents of \(S\). We call a regular semigroup \(S\) a locally inverse semigroup if \(eSe\) is an inverse subsemigroup of \(S\) for each \(e \in E(S)\). It is clear that an inverse semigroup is locally inverse. We say the idempotent set \(E(S)\) of a locally inverse semigroup \(S\) is locally finite if \(E(eSe)\) is finite for each \(e \in E(S)\).

We call a semigroup \(S\) a completely 0-simple semigroup if \(S\) is 0-simple and all its non-zero idempotents are primitive. Let \(G\) be a group, \(I\) and \(\Lambda\) be two non-empty sets and let \(P = (p_{ij})\) be a \(\Lambda \times I\)-matrix with entries in \(G^0 (= G \cup \{0\})\), and suppose that \(P\) is regular, in the sense that no row or column of \(P\) consists entirely of zeros. Let \(M = (I \times G \times \Lambda) \cup \{0\}\) and define a multiplication on \(M\) by

\[
(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, xp_{ij}y, \mu), & \text{if } p_{ij} \neq 0, \\ (0, 0), & \text{otherwise}, \end{cases}
\]

and

\[ (i, x, \lambda)0 = 0(i, x, \lambda) = 0. \]

Then \(M\) is a completely 0-simple semigroup, denoted by \(M^0(G; I, \Lambda; P)\). Note that the zero element of a completely 0-simple semigroup is denoted by \(0\) instead of \(\theta\). Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.

In particular, if \(P\) is with entries in \(G\), then the set \((I \times G \times \Lambda)\) forms a subsemigroup of \(M^0(G; I, \Lambda; P)\), we denote this subsemigroup by \(M(G; I, \Lambda; P)\), and call it a completely simple semigroup.

**Lemma 2.1.** [8, Theorem 3.4.1] Let \(M_1 = M^0(G; I, \Lambda; P)\) and \(M_2 = M^0(G; I, \Lambda; P)\) be two completely 0-simple semigroups. Then \(M_1 \equiv M_2\) if and only if there exist a \(\Lambda \times \Lambda\) diagonal matrix \(X\) over \(G^0\) and an \(I \times I\) diagonal matrix \(Y\) over \(G^0\) such that \(P = XQY\).

By Lemma 2.1, for a completely 0-simple semigroup \(M = M^0(G; I, \Lambda; P)\), we can always assume the sandwich matrix \(P\) satisfies \(p_{11} = e\), where \(1 \in I \cap \Lambda\), and where \(e\) is the identity of \(G\).
We introduce the concept of semigroup rings. Let $S$ be a semigroup. The semigroup ring $R[S]$ of $S$ is defined to be the ring consisting of all finite formal sums $\sum r_is_i$, where $r_i \in R$ and $s_i \in S$, with the obvious definition of addition and with multiplication induced by the given multiplication in $R$ and $S$ according to the rule

$$\sum r_is_i \sum l_js_j = \sum (r_il_j)s_is_j,$$

where $r_i, l_j \in R$ and $s_i, t_j \in S$. Define $R(\sum r_is_i)$ to be $\sum (r_i)s_i$, where $r_i \in R$ and $s_i \in S$, and if $I_R$ is the identity of $R$, then $I_RS = S$ for all $s \in S$. If $I$ is a subset of $S$, let $R[I]$ denote the set of all finite $R$-linear combinations of elements of $I$. We call the factor ring $R[S]/RI\theta$ the contracted semigroup ring of $S$ over $R$, denoted by $R_0[S]$. Note that if $S^0$ is the semigroup resulting from the adjunction of a zero element $\theta$ to $S$ (whether or not $S$ has a zero to begin with), then $R_0[S^0] \cong R[S]$. Thus, any semigroup ring can be regarded as a contracted semigroup ring.

Let $S$ be a semigroup and $\sum r_is_i \in R_0[S]$, where $r_i \in R$ and $s_i \in S$. Define the support set of $\sum r_is_i$ to be the subset $\text{supp}(\sum r_is_i) = \{s_i \in S \mid r_i \neq 0\}$ of $S$.

Recall the definition of a Munn ring. Let $T$ be a ring, $I, \Lambda$ be two non-empty sets. It will be necessary to describe a $I \times \Lambda$-matrix over $T$ in terms of its entries. In this case, we use such notations as

$$X = [a_{i\alpha}], \quad X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad X = [a_1, a_2, \ldots, a_n, \ldots].$$

We use $[a_{i\alpha}]_I$ to represent an $I \times \Lambda$-matrix with only one non-zero entry $a$ in the position $(i, \lambda)$. Let $X = [a_{i\alpha}]_I$ be an $I \times \Lambda$-matrix over $T$. We say that the matrix $X$ is bounded if $a_{i\alpha} = 0$ for all but finitely many $(i, \lambda)$. Let $P$ be a fixed $\Lambda \times I$ matrix over $T$ (possibly not bounded) with the property that every row and every column of $P$ contain an unit of $T$, and let $\mathcal{M}(T; I, \Lambda; P)$ be the set of all bounded $I \times \Lambda$ matrices over $T$. Then $\mathcal{M}(T; I, \Lambda; P)$ becomes a ring over $T$ with the usual matrix addition and the multiplication $\circ$ defined by: $X \circ Y = XYP$, which is the usual matrix multiplication. We call $\mathcal{M}(T; I, \Lambda; P)$ a Munn ring over $T$ with the sandwich matrix $P$, or briefly a Munn ring [9,10]. Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a completely $0$-simple semigroup over a group $G$. By [6, Lemma 5.17], $R_0[S] = \mathcal{M}(R[G]; I, \Lambda; P)$ is a Munn ring.

An element $a$ in a ring $U$ is said to be left quasi-regular if there exists $u \in U$ such that $u + a + ua = 0$. A left ideal $I$ of $U$ is said to be left quasi-regular if every element of $I$ is left quasi-regular. The Jacobson radical $\mathcal{J}(U)$ of $U$ is defined to be the left quasi-regular left ideal which contains every left quasi-regular left ideal of $U$ [11]. We provide a method for determining whether an element of $U$ belongs to $\mathcal{J}(U)$ or not.

**Lemma 2.2.** [11, Lemma 2.15(ii)] Let $U$ be a ring and $a \in U$. Then $a \in \mathcal{J}(U)$ if and only if $ua$ is a left quasi-regular left ideal of $U$.

Note that the aforementioned definitions and results on Jacobson radicals are valid with “left” replaced by “right.”

## 3 Strong nil-cleanness of completely $0$-simple semigroup rings

The first part of this section presents some elementary definitions and results on strongly nil-cleang rings. The remainder of this section is concerned with the strong nil-cleanness of the contracted semigroup rings of both completely $0$-simple semigroups and locally inverse semigroups.

Let $U$ be a ring. Following Diesl [1], an element $u \in U$ is said to be nil-clean if there is an idempotent $e \in U$ and nilpotent $b \in U$ such that $u = e + b$. The element $u$ is further said to be strongly nil-clean if such an idempotent and a nilpotent can be chosen to satisfy the equality $be = eb$. The ring $U$ is called a nil-clean (respectively, strongly nil-clean) ring if each element in it is nil-clean (respectively, strongly nil-clean).

We establish a basic result which will be used later.
Lemma 3.1. [1, Proposition 2.9] A left (resp., right) ideal of a strongly nil-clean ring is also strongly nil-clean.

The following lemma provides a very useful characterization of strongly nil-clean rings, in terms of Jacobson radicals. A ring is said to be boolean if every element of the ring is an idempotent.

Lemma 3.2. [2] A ring $U$ is strongly nil-clean iff $U/J(U)$ is boolean and $J(U)$ is nil.

By Lemma 3.2, it will be helpful to know the Jacobson radicals of contracted completely 0-simple semigroup rings.

Lemma 3.3. [12] Let $M = M(0; G; I, A; P)$ be a completely 0-simple semigroup. Then
\[ J(R_0[M]) = \{ X \in R_0[M] \mid PXP \in (J(R[G]))_{A \cup A} \}. \]

Noting that for a completely 0-simple semigroup $M = M(0; G; I, A; P)$, if we denote the identity of $G$ by $e$, then $eMe \cong G$, and thus $eR_0[M]e \cong R[G]$, whence $R[G]$ is a corner ring of $R_0[M]$. Diesl proved the inheritance of the strong nil-cleanliness by corner subrings of a strong nil-cleanly ring with identity from the ring with identity. This remains true for non-unital rings by applying similar argument in the proof of [1, Proposition 3.25].

Lemma 3.4. Let $U$ be a ring and $f$ be any idempotent in $U$. If $U$ is strongly nil-clean, then the ring $fUf$ is also strongly nil-clean.

Proof. Since $Uf$ is a left ideal of $U$ and $fUf$ is a right ideal of $Uf$, we apply Lemma 3.1 to obtain first that $Uf$ is a strongly nil-clean ring, and then that $fUf$ is a strongly nil-clean ring. The lemma is proved. \( \square \)

Later, Lemma 3.4 is used to give a necessary condition for a contracted completely 0-simple semigroup ring to be strongly nil-clean.

Lemma 3.5. Let $M$ be a completely 0-simple semigroup with exactly two $R$-classes and exactly two $L$-classes. Then its contracted semigroup ring $R_0[M]$ is not strongly nil-clean.

Proof. By hypothesis, the completely 0-simple semigroup $M$ is of the form $M(0; G; 2, 2; P)$, where $G$ is a group, $2 = \{1, 2\}$ and $P$ is a $2 \times 2$-matrix over $G^0$. Since $M$ is a regular semigroup, by Lemma 2.1, the sandwich matrix $P$ must be one of the matrices in the set $\{\Delta, N_1, N_1', N_2, N_2', N_3\}$, up to isomorphic, where
\[
\Delta = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, \quad N_1 = \begin{bmatrix} e & e \\ e & 0 \end{bmatrix}, \quad N_1' = \begin{bmatrix} e & 0 \\ e & e \end{bmatrix},
\]
\[
N_2 = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}, \quad N_2' = \begin{bmatrix} 0 & e \\ e & e \end{bmatrix}, \quad N_3 = \begin{bmatrix} e & e \\ e & g \end{bmatrix},
\]
and where $e$ is the identity of the group $G$ and $g \in G$. We claim that the contracted semigroup ring $R_0[M]$ is not strongly nil-clean, no matter which of the above normalized matrix $P$ is chosen. There are five cases to be considered. For convenience, let $\mathcal{F}$ denote the set consisting of all the fields of characteristic 2.

Case 1. $P = \Delta$. Then $R_0[M] \cong \mathcal{A}(R(G); 2, 2, P) \cong M_2(R[G])$, and thus $R_0[M]$ is not strongly nil-clean, see [3].

Case 2. $P = N_1$, $P = N_1'$, $P = N_2$ or $P = N_2'$. It is clear that $P$ is an invertible matrix, and thus the map $R_0[M] \rightarrow M_2(R[G])$ defined by $A \mapsto AP$ is an isomorphism of rings. Then, by Case 1, $R_0[M]$ is not strongly nil-clean.

Case 3. $P = N_3$ with $g \neq e$, and $R \not\in \mathcal{F}$. Noting that each element in the Munn ring $R_0[M] = \mathcal{A}(R(G); 2, 2; P)$ is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in R[G]$. By Lemma 3.3, it is easy to verify that
\[
J(R_0[M]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R_0[M] \mid a + b + c + d, a + c + (b + d)g \in J(R[G]) \right\}, \tag{3.1}
\]
By contrary, suppose that \( R_0[M] \) is strongly nil-clean. This, together with Lemma 3.2, yields that \( R_0[M]/J(R_0[M]) \) is boolean. Let \( c \) be an arbitrary element of \( R[G] \), define \( N = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \in R_0[M] \). Then \( N \circ N + J(R_0[M]) = (N + J(R_0[M]))^2 = N + J(R_0[M]) \), which is equivalent to say that \( N \circ N - N \in J(R_0[M]) \). Note that \( N \circ N = NN_jN = \begin{bmatrix} 0 & 0 \\ 0 & cgc - c \end{bmatrix} \), hence \( N \circ N - N = \begin{bmatrix} 0 & 0 \\ 0 & cgc - c \end{bmatrix} \). It follows that \( (cgc - c)g \in J(R[G]) \) by (3.1). Next, we show that \( J(R[G]) \) is nil. For this, note that the group ring \( R[G] = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} R_0[M] \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \) is a corner subring of \( R_0[M] \). Since \( R_0[M] \) is strongly nil-clean by assumption, \( R[G] \) is strongly nil-clean by Lemma 3.4. Then it follows from Lemma 3.2 that \( J(R[G]) \) is nil. Since \( g \) is an invertible element in \( G \), we deduce that \((cgc - c)g \) is a nilpotent element. Since \( c \) is arbitrary, we can take \( c = -g^{-1} \), it follows that \(-2g^{-1} \in R[G] \) is nilpotent. This, together with the fact \( R \not\in \mathcal{F} \), implies that the element \( g^{-1} \in G \) is nilpotent, that is, \((g^{-1})^m = 0 \) for some positive integer \( m \), which is a contradiction. Consequently, \( R_0[M] \) is not strongly nil-clean, as required.

**Case 4.** \( P = N_1 \) and \( g = e \), and \( R \not\in \mathcal{F} \). By Case 3, if \( J(R[G]) \) is not nil, then \( R_0[M] \) is not strong nil-clean. Suppose now that \( J(R[G]) \) is nil. We claim that \( R_0[M] \) is not strongly nil-clean. By Lemma 3.2, it is sufficient to prove that \( R_0[M]/J(R_0[M]) \) is not boolean. Noting that, by Lemma 3.3, we have

\[
J(R_0[M]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R_0[M] \mid a + b + c + d \in J(R[G]) \right\},
\]

(3.2)

Let \( c, d \) be two arbitrary elements of \( R[G] \). Define \( N = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \in R_0[M] \). Then

\[
N \circ N = NN_jN = \begin{bmatrix} 0 & 0 \\ (c + d)c & (c + d)d \end{bmatrix},
\]

thus

\[
N \circ N - N = \begin{bmatrix} 0 & 0 \\ (c + d)c - c & (c + d)d - d \end{bmatrix}.
\]

By contrary, assume that \( N \circ N - N \in J(R_0[M]) \). It follows that \((c + d)(c + d - e) \in J(R[G]) \). If we take \( c + d = -e \), then \( 2e = (c + d)(c + d - e) \in J(R[G]) \). Since \( J(R[G]) \) is nil, there exists a positive integer \( n \) such that \( 2^n e = 0 \), which is a contradiction, because \( R \not\in \mathcal{F} \). Therefore, \( R_0[M] \) is not strongly nil-clean.

**Case 5.** \( P = N_1 \) and \( R \in \mathcal{F} \). By contrary, suppose that \( R_0[M] \) is a strongly nil-clean ring. By similar arguments to the above, we infer that \( J(R[G]) \) is nil, and \( R_0[M]/J(R_0[M]) \) is boolean. It is easy to check that

\[
N_2 \circ N_2 - N_2 = \begin{bmatrix} 0 & e + g \\ e & e + g \end{bmatrix}.
\]

Since \( \text{char} R = 2 \), we have that \( e + g + (e + g) + e = e \), which does not belong to \( J(R[G]) \). This, together with (3.2), implies that \( N_2 + J(R_0[M]) \) is not an idempotent in \( R_0[M]/J(R_0[M]) \). This is a contradiction, because \( R_0[M]/J(R_0[M]) \) is boolean. Therefore, \( R_0[M] \) is not strongly nil-clean.

Consequently, in either case, the contracted semigroup ring \( R_0[M] \) is not strongly nil-clean, as required.

\[\square\]

If \( M = M^0(G; 2, 2; \Delta) \) is a Brandt semigroup, where \( \Delta \) is the identity matrix over \( G^0 \), by Lemma 3.5, \( R_0[M] \equiv \mathfrak{H}_2(R[G]) \) is not strongly nil-clean. This may be thought of as an extension of the result in [3].

By applying Lemmas 3.1 and 3.5, a necessary condition for a contracted completely \( 0 \)-simple semigroup ring to be strongly nil-clean is obtained.

**Proposition 3.6.** Let \( M = M^0(G; I, \Lambda; P) \) be a completely \( 0 \)-simple semigroup. If \( R_0[M] \) is strongly nil-clean, then either \(|I| = 1 \) or \(|\Lambda| = 1 \).

**Proof.** Suppose that \( R_0[M] \) is strongly nil-clean. Noting that \(|I| \geq 1 \) and \(|\Lambda| \geq 1 \). Then in order to show either \(|I| = 1 \) or \(|\Lambda| = 1 \), assume on the contrary that \(|I| \geq 2 \) and \(|\Lambda| \geq 2 \). It would then follow that the \( \Lambda \times I \)-matrix \( P \) may be written in the block form \( \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \), where \( P_1 \) is a \( 2 \times 2 \)-matrix over \( G^0 \). Define a subset
A = \{(a_{ij}) \in R_0[M] \mid a_{ij} = 0 \text{ for all } i \neq 1, 2\} \text{ of } R_0[M]. \text{ It is clear that } A \text{ is closed under multiplication. Moreover, let } (a_{ij}) \in A \text{ and } (b_{ij}) \in R_0[M], \text{ then the entry in the } (i, j)-\text{position of the product } (a_{ij}) \cdot (b_{ij}) \text{ is } \sum_{k=1}^{\infty} a_{ik}b_{kj}, \text{ which yields that } (a_{ij}) \cdot (b_{ij}) \in A, \text{ whence } A \text{ is a right ideal of } R_0[M]. \text{ Since } R_0[M] \text{ is strongly nil-clean, } A \text{ is strongly nil-clean by Lemma 3.1. Moreover, } A \text{ is isomorphic to the Munn ring } \mathcal{M}(R[G]; 2, \Lambda; \left[ \begin{array}{c} P_1 \\ P_2 \end{array} \right]). \text{ Then a similar argument shows that the Munn ring } U = \mathcal{M}(R[G]; 2, 2; P_1) \text{ is isomorphic to a left ideal of } A, \text{ and hence } U \text{ is strongly nil-clean by Lemma 3.1 again. This is a contradiction, because } U = R_0[M^0(G; 2, 2; P_1)] \text{ is not strongly nil-clean by Lemma 3.5. Therefore, we must have either } |I| = 1 \text{ or } |\Lambda| = 1. \hfill \Box

The interesting case occurs when a completely 0-simple semigroup } M = M^0(G; I, \Lambda; P) \text{ contains a unique } \mathcal{L}-\text{class} \text{ (or, a unique } \mathcal{R}-\text{class).}

**Lemma 3.7.** Let } M = M^0(G; I, \Lambda; P) \text{ be a completely 0-simple semigroup. If } |I| = 1 \text{ or } |\Lambda| = 1, \text{ then } R_0[M] \text{ is strongly nil-clean if and only if } R[G] \text{ is strongly nil-clean.}

**Proof.** It is sufficient to consider the case of } |\Lambda| = 1, \text{ since the case of } |I| = 1 \text{ is dual. Assume that } |\Lambda| = 1. \text{ Before moving on, we explore the structure of } M \text{ and determine } J(R_0[M]). \text{ For this, since } M \text{ is a completely 0-simple semigroup, } S \text{ is regular, whence every } \mathcal{R}\text{-class of } M \text{ contains at least one idempotent. Since } M \text{ has only one } \mathcal{L}\text{-class, it follows that each } \mathcal{R}\text{-class of } M \text{ is a maximal group in } M, \text{ and hence the } 1 \times 1 \text{-matrix } P \text{ must be of the form } [e, \ e, \ ..., \ e, \ ...] \text{ whose entries are all equal to } e, \text{ where } e \text{ is the identity of the group } G. \text{ Denote } \mathcal{N} \text{ by the set } \{1, 2, ..., n\} \text{ consisting of all positive integers. Let } x = [x_1, x_2, ..., x_n, 0, ..., 0]^T \in R_0[M] \text{ with } x_1, x_2, ..., x_n \in R[G] \text{ and whose other entries are all equal to } 0, \text{ where } n \in \mathcal{N}. \text{ Then it is easy to check that } Px = (\sum_{i=1}^{n} x_i) [e, \ e, \ ..., \ e, \ ...]. \text{ Thus, by Lemma 3.3,}

\[ J(R_0[M]) = \left\{ x = [x_1, x_2, ..., x_n, 0, ..., 0]^T \in R_0[M] \mid \sum_{i=1}^{n} x_i \in R[G] \right\}. \]  

(3.3)

Now we suppose that } R[G] \text{ is strongly nil-clean. By Lemma 3.2, } R[G]/J(R[G]) \text{ is boolean and } J(R[G]) \text{ is nil. We want to show that } R_0[M] \text{ is strongly nil-clean. It suffices to prove that } J(R_0[M]) \text{ is nil and } R_0[M]/J(R_0[M]) \text{ is boolean, by Lemma 3.2 again. For this, let } x = [x_1, x_2, ..., x_n, 0, ..., 0]^T \in J(R_0[M]), \text{ where } x_1, x_2, ..., x_n \in R[G] \text{ and } n \in \mathcal{N}. \text{ By (3.3), } \sum_{i=1}^{n} x_i \in J(R[G]), \text{ and hence } J(R[G]) \text{ is nil, it follows that } \sum_{i=1}^{n} x_i \text{ is a nilpotent element.}

Note that, for any element } y = [y_1, y_2, ..., y_t, 0, ..., 0]^T \in R_0[M], \text{ where } y_1, y_2, ..., y_t \in R[G] \text{ and } t \in \mathcal{N}, \text{ we have } y \cdot y = yPy = y(\sum_{i=1}^{t} y_i), \text{ and thus for any integer } k \geq 2, \text{ } y^k = y^{k-2} \cdot y \cdot y = y^{k-2} \cdot y(\sum_{i=1}^{t} y_i) = ... = y^{k-2} \cdot y(\sum_{i=1}^{t} y_i)^{k-1}. \text{ Therefore, } x \text{ is a nilpotent element when } x \in J(R_0[M]), \text{ which yields that } J(R_0[M]) \text{ is nil.}

On the other hand, for any } z = [z_1, z_2, ..., z_m, 0, ..., 0]^T \in R_0[M] \setminus J(R_0[M]), \text{ where } z_1, z_2, ..., z_m \in R[G] \text{ and } m \in \mathcal{N}, \text{ we have } \sum_{j=1}^{m} z_j \notin J(R[G]), \text{ whence } \sum_{j=1}^{m} z_j \in R[G] \setminus J(R[G]). \text{ Note that}

\[ z^2 - z = \left[ z_1 \sum_{j=1}^{m} z_j - z_1, ..., z_m \sum_{j=1}^{m} z_j - z_m, 0, ..., 0 \right]^T \]

\[ = \left[ z_1 \left( \sum_{j=1}^{m} z_j - e \right), ..., z_m \left( \sum_{j=1}^{m} z_j - e \right), 0, ..., 0 \right]^T \]

\[ = [z_1, ..., z_m, 0, ..., 0]^T \left( \sum_{j=1}^{m} z_j - e \right), \]

where } e \text{ is the identity of } G. \text{ Note also that } \sum_{j=1}^{m} z_j \left( \sum_{j=1}^{m} z_j - e \right) = \left( \sum_{j=1}^{m} z_j \right)^2 - \sum_{j=1}^{m} z_j \in J(R[G]), \text{ because } \sum_{j=1}^{m} z_j \in R[G] \setminus J(R[G]) \text{ and } R[G]/J(R[G]) \text{ is boolean. Then, by (3.3), } z^2 - z \in J(R_0[M]). \text{ It follows that } R_0[M]/J(R_0[M]) \text{ is boolean.
Conversely, suppose that $R_0[M]$ is strongly nil-clean. We claim that $R[G]$ is strongly nil-clean. Indeed, let $f$ denote the element $[e, 0, \ldots, 0, \ldots]^T$ in $R_0[M]$, where $e$ is the identity of $G$. Then $f$ is an idempotent, and moreover $f \circ R_0[M] \circ f \equiv R[G]$. This, together with Lemma 3.4, implies that $R[G]$ is strongly nil-clean. □

We have obtained all the preliminaries needed to prove the following principal theorem in this section.

**Theorem 3.8.** Let $M = M^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then $R_0[M]$ is strongly nil-clean if and only if $R[G]$ is strongly nil-clean, and either $|I| = 1$ or $|\Lambda| = 1$.

**Proof.** Note that if $R_0[M]$ is strongly nil-clean, then $|I| = 1$ or $|\Lambda| = 1$ by Lemma 3.6. Then the result follows immediately from Lemma 3.7. □

Note that for a completely 0-simple semigroup $M = M^0(G; I, \Lambda; P)$, if $R_0[M]$ is strongly nil-clean, it follows from the proof of Theorem 3.8 that all entries of $P$ must be non-zero, and hence $M$ must be of the form $M(G; I, \Lambda; P) \cup \{0\}$.

In the following, we provide two corollaries of Theorem 3.8.

**Corollary 3.9.** Let $M = M(G; I, \Lambda; P)$ be a completely simple semigroup. Then $R[M]$ is strongly nil-clean if and only if $R[G]$ is strongly nil-clean, and either $|I| = 1$ or $|\Lambda| = 1$.

**Proof.** Note that the set $M^0 = M(G; I, \Lambda; P) \cup \{0\}$ forms a completely 0-simple semigroup with $R_0[M^0] \equiv R[M]$. The result then follows from Theorem 3.8. □

**Corollary 3.10.** Let $S$ be a locally inverse semigroup with idempotent set locally finite and finitely many $D$-classes. Then $R_0[S]$ is strongly nil-clean if and only if the following two conditions hold.

(i) Each $D$-class $D$ of $S$ has either a unique $R$-class or a unique $L$-class;

(ii) For each maximal subgroup $G$ of $S$, $R[G]$ is strongly nil-clean.

**Proof.** By [13, Theorem 5.1], $R_0[S]$ is a finite direct product of the contracted completely 0-simple semigroup rings $R_0[\bar{D}_\alpha^0]$, where $\alpha$ runs over the set $(S/D)^\star$ consisting all non-zero elements in $S/D$. Then by [1, Proposition 3.13], $R_0[S]$ is strongly nil-clean if and only if $R_0[\bar{D}_\alpha^0]$ is strongly nil-clean for each $\alpha \in (S/D)^\star$. Furthermore, for each $\alpha \in (S/D)^\star$, the maximal subgroup of $\bar{D}_\alpha^0$ is isomorphic to the maximal subgroup in $D_\alpha$, and the number of $L$-classes (resp., $R$-classes) in $\bar{D}_\alpha^0$ is equal to the number of $L$-classes (resp., $R$-classes) in $D_\alpha$, see [13]. The result then follows from Theorem 3.8. □

4 Strong nil-cleanness and strong semilattice of semigroups

We recall the definition of strong semiflattice of semigroups. Let $Y$ be a semiflattice with natural partial order $\leq$ and $S_Y$ be a family of semigroups indexed by $Y$. We say a semigroup $S$ is a strong semiflattice of semigroups $S_Y (a \in Y)$ if $S$ is the disjoint union of the semigroups $S_a$, and for all $a > b$ in $Y$ there exists a homomorphism $q_{a,b} : S_a \rightarrow S_b$ satisfying the following conditions:

(i) $q_{a,a}$ is the identity map of $S_a$;

(ii) If $\alpha > \beta > \gamma$, then $q_{\alpha,\beta}q_{\alpha,\gamma} = q_{\alpha,\gamma}$;

(iii) If $a \in S_a$ and $b \in S_b$, then the multiplication in $S$ is given by $ab = q_{\alpha,\beta}q_{a,b}$.

If this is the case, denote the semigroup $S$ by $S = [Y; S_Y, q_{a,b}]$. Note that each $q_{a,b}$ can be $R$-linearly extended into a semigroup ring homomorphism from $R[S_a]$ to $R[S_b]$. Note also that $R[S]$ is always a supplementary semiflattice sum of subrings $R[S_a] (a \in Y)$. 

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For the rest of this section, we always assume that $S$ is a strong semilattice $Y$ of semigroups $S_\alpha$ ($\alpha \in Y$), where $Y$ is finite. The main aim of this section is to construct a relationship between the strong nil-cleanness of $R_0[S]$ and the strong nil-cleanness of $R[R_0[S]]$ ($\alpha \in Y$).

Let $\beta$ be a fixed maximal element in $Y$, and then denote the set $S\setminus \{S_\beta\}$ by $T$. Thus, $T$ is an ideal of $S$, whence $R_0[T]$ is an ideal of $R_0[S]$. We introduce some preliminary results and definitions which are used to prove the main result of this section later.

Lemma 4.1. Every idempotent of $R_0[S]/R_0[T]$ can be lifted to an idempotent of $R_0[S]$.

Proof. Suppose that $x_1 + x_2 + R_0[T] \in R_0[S]/R_0[T]$ is an idempotent, where $x_1 \in R[S_\beta]$ and $x_2 \in R_0[T]$. Then $(x_1^2 - x_1 + x_2) + (x_1x_2 + x_2x_1 + x_2^2 - x_2) \in R_0[T]$. Since $\beta$ is maximal in $Y$ and $R[S_\beta]$ is a subring of $R_0[S]$, we infer that $\supp(x_1^2 - x_1) \cap \supp(x_1x_2 + x_2x_1 + x_2^2 - x_2) = \emptyset$, which yields that $x_1 = x_1^2 \in R[S_\beta]$ is an idempotent. Since $x_1 + x_2 + R_0[T] = x_1 + R_0[T]$, the result follows. \(\Box\)

Lemma 4.2. Let $x = x_1 + x_2 \in R_0[S]$, where $x_1 \in R[S_\beta]$ and $x_2 \in R_0[T]$. If $x \in J(R_0[S])$, then $x_1 \in J(R[S_\beta])$.

Proof. Suppose that $0 \neq x = x_1 + x_2 \in J(R_0[S])$. We claim that $x_1 \in J(R[S_\beta])$. For this, let $z$ be an arbitrary element in $R[S_\beta]$, then there exists an element $y \in R_0[S]$ such that $zx + y + y(zx) = 0$. Write $y = y_1 + y_2$, where $y_1 \in R[S_\beta]$ and $y_2 \in R_0[T]$. It follows from the maximality of $\beta$ that $zx_1 + y_1 + y_1(zx_1) = 0$, whence $R[S_\beta]x_1$ is a left quasi-regular left ideal of $R[S_\beta]$. By Lemma 2.2, $x_1 \in J(R[S_\beta])$, as required. \(\Box\)

Let $E$ be a semilattice and $\alpha, \gamma \in E$. Then $\gamma$ is said to be maximal under $\alpha$ [14] if $\alpha > \gamma$ and there is no $\nu \in E$ such that $\alpha > \nu > \gamma$. Denote by $\hat{\alpha}$ the set $\{\gamma \in E: \gamma$ is maximal under $\alpha\}$. Let $\alpha$ be a non-zero element in the finite semilattice $Y$. For each $\alpha \in R[S_\alpha]$, by [14], define an element

$$\alpha^* = \alpha + \sum_{|\alpha_1, \ldots, \alpha_l| < \hat{\alpha}} (-1)^{|\alpha_1, \ldots, \alpha_l|} q_{\alpha_1, \alpha_2, \ldots, \alpha_l}(\alpha) \in R_0[S],$$

(4.1)

where $|\alpha_1, \ldots, \alpha_l|$ runs over all non-empty subsets of $\hat{\alpha}$. Noting that, in (4.1), each element $\alpha_1, \ldots, \alpha_l \in Y$ is strictly smaller than $\alpha$, and thus the element $q_{\alpha_1, \alpha_2, \ldots, \alpha_l}(\alpha) \in R_0[S]$ actually belongs to $\sum_{\gamma \in R[S_\gamma]} R_0[S]$. \(\Box\)

Lemma 4.3. [14] Let $\alpha$ be a non-zero element in $Y$. If $A$ is an ideal of $R[S_\alpha]$, then $A^* = \{\alpha^* : \alpha \in A\}$ is an ideal of $R_0[S]$. Moreover, the map $A \to A^* : \alpha \to \alpha^*$ is a ring isomorphism from $A$ to $A^*$.

Note that Lemma 4.3 is true for the more general case when $Y$ is taken to be a pseudofinite semilattice [14].

Lemma 4.4. $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$.

Proof. If $J(R_0[S]) = 0$, then $J(R[S_\beta]) = 0$ if for each $\alpha \in Y$, because $J(R_0[S]) = 0$ if and only if $J(R[S_\alpha]) = 0$ for each $\alpha \in Y$ by [14, Theorem 2]. Hence, in this case, $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$ holds true. If $J(R_0[S]) \neq 0$, then we claim that $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$. Indeed, for any element $0 \neq x \in J(R[S_\beta])$, the element $x^* \in R_0[S]$ defined by (4.1) is non-zero, and furthermore belongs to $J(R[S_\beta])^*$. Since $J(R[S_\beta])^*$ is an ideal of $R_0[S]$ and is isomorphic to $J(R[S_\beta])$ by Lemma 4.3, we infer that $J(R[S_\beta])^*$ is a left quasi-regular ideal of $R_0[S]$. Because $J(R_0[S])$ contains any left quasi-regular left ideal of $R_0[S]$, $J(R[S_\beta])^* \subseteq J(R_0[S])$. It follows that $x^* \in J(R_0[S])$. Therefore, by the definition of $x^*$ again, $x + R_0[T] = x^* + R_0[T] \in J(R_0[S]) + R_0[T]$. Consequently, we have $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$. \(\Box\)

Lemma 4.5. $J(R_0[S]/R_0[T]) = (J(R_0[S]) + R_0[T])/R_0[T]$.

Proof. First, we show that $J(R_0[S]/R_0[T]) \subseteq (J(R_0[S]) + R_0[T])/R_0[T]$. For this, let $x + R_0[T] \in J(R_0[S]/R_0[T])$ with $x \in R_0[S]$. Then for each $y \in R_0[S], yx + R_0[T]$ is a left quasi-regular element of $R_0[S]/R_0[T]$. This, together
with the maximality of \( \beta \), implies that \( R[S_\beta]x \) is a left quasi-regular left ideal of \( R[S_\beta] \), whence \( x \in J(R[S_\beta]) \). By Lemma 4.4, we infer that \( x \in J(R_0[S]) + R_0[T] \), and hence \( x + R_0[T] \in (J(R_0[S]) + R_0[T])/R_0[T] \), as required. Next, we prove that \( (J(R_0[S]) + R_0[T])/R_0[T] \subseteq f(R_0[S]/R_0[T]). \) Assume that \( x + R_0[T] \in J(R_0[S]) + R_0[T]/R_0[T] \), where \( x \in J(R_0[S]) \). Then \( x \) is a left quasi-regular element in \( R_0[S] \), which is equivalent to say that \( R_0[S]x \) is a left quasi-regular left ideal of \( R_0[S] \), whence \((R_0[S]/R_0[T])(x + R_0[T])\) is also a left quasi-regular left ideal of \( R_0[S]/R_0[T] \). This implies that \( x + R_0[T] \in J(R_0[S]/R_0[T]) \), as required.

The following result provides us an equivalent characterization for a non-unital ring to be strongly nil-clean, which plays an important role in the below proof.

**Lemma 4.6.** [2] Let \( A \) be a non-unital ring and \( U \) be an ideal of \( A \). Then \( A \) is strongly nil-clean if and only if the following three conditions hold:

(i) \( U \) and \( A/U \) are strongly nil-clean;

(ii) Every idempotent of \( A/U \) can be lifted to an idempotent of \( A \);

(iii) \( J(A/U) = (U + J(A))/U \).

We are now ready to prove the main result of this section concerning the strong nil-cleanness of the contracted semigroup ring \( R_0[S] \).

**Theorem 4.7.** Let \( S \) be a strong semilattice \( Y \) of semigroups \( S_\alpha (\alpha \in Y) \), where \( Y \) is finite. Then \( R_0[S] \) is strongly nil-clean if and only if \( R_0[S\alpha] \) is strongly nil-clean and \( R[S_\beta] \) is strongly nil-clean for each \( \theta \in \alpha \in Y \).

**Proof.** We prove the result by induction on the cardinality of the set \( Y \). If \( Y = \{\theta_1\} \), this is trivial. If \( |Y| \geq 2 \), then let \( \beta \) be a maximal element in \( Y \) and define \( Y' \) be the subset \( Y\setminus\beta \) of \( Y \). By induction, the result holds for \( T = \cup_{a \in Y'}[\beta]_{S_a} \), which means that \( R_0[T] \) is strongly nil-clean if and only if \( R[S_a] \) is strongly nil-clean for each \( a \notin \{\theta_1, \beta\} \) and \( R_0[S_{\beta_1}] \) is strongly nil-clean. Thus, in order to show that the result holds for \( S \), it is sufficient to prove that \( R_0[S] \) is strongly nil-clean if and only if both \( R[S_{\beta_1}] = R_0[S]/R_0[T] \) and \( R_0[T] \) are strongly nil-clean. Note that Lemma 4.6(iii) holds true if we replace \( A \) by \( R_0[S_{\beta_1}] \) and \( R_0[T] \) respectively, by Lemmas 4.1 and 4.5. Then the result is obtained immediately by Lemma 4.6.

A similar argument in the proof of Theorem 4.7 provides us a characterization of the strong nil-cleanness of the semigroup ring \( R[S] \).

**Corollary 4.8.** Let \( S \) be a strong semilattice \( Y \) of semigroups \( S_\alpha (\alpha \in Y) \), where \( Y \) is finite. Then \( R[S] \) is strongly nil-clean if and only if \( R[S_\alpha] \) is strongly nil-clean for each \( \alpha \in Y \).

We end this paper with a brief discussion of the relationship between the strong nil-cleaness of semigroup rings and the strong nil-cleanness of contracted semigroup rings. Let \( N \) be a semigroup. Since \( R_0[N] = R[N]/R_0[\theta] \), we get that \( R_0[N] \) is strongly nil-clean whenever \( R[N] \) is strongly nil-clean by Lemma 4.6. However, if \( R_0[N] \) is strongly nil-clean, it does not follow that \( R[N] \) is strongly nil-clean. The following example illustrates this fact.

**Example 4.1.** Let \( K \) be a field and \( N = \{a, \theta\} \) be a semigroup with multiplication given by \( a^2 = a\theta = \theta a = \theta^2 = \theta \).

(i) Note that each non-zero element of \( R_0[N] \) of the form \( ka \), where \( 0 \neq k \in K \). Then in \( R_0[N] \), \((ka)^2 = k^2a^2 = 0 \). Hence, \( R_0[N] \) is nil, whence \( R_0[N] \) is strongly nil-clean.

(ii) It is clear that the set \( \{ka - k\theta \mid k \in K \} \) (resp. \( \{\theta\} \)) collects all nilpotent elements (resp., idempotents) in \( R[N] \). We claim that \( R[N] \) is not strongly nil-clean. For this, it suffices to show that the element \( 2a \in R[N] \) is not strongly nil-clean. By contrary, suppose that \( 2a = \theta + (ka - k\theta) \) for some \( k \in K \), then \( k \) must be equal to \( 1_k \), whence \( 2a = a \), which is a contradiction.
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