THIN SEQUENCES AND THE GRAM MATRIX

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Abstract. We provide a new proof of Volberg’s Theorem characterizing thin interpolating sequences as those for which the Gram matrix associated to the normalized reproducing kernels is a compact perturbation of the identity. In the same paper, Volberg characterized sequences for which the Gram matrix is a compact perturbation of a unitary as well as those for which the Gram matrix is a Schatten-2 class perturbation of a unitary operator. We extend this characterization from 2 to \( p \), where \( 2 \leq p \leq \infty \).

1. Introduction

Let \( \mathbb{D} \) denote the open unit disk and \( \mathbb{T} \) the unit circle. Given \( \{\alpha_j\} \), a Blaschke sequence of points in \( \mathbb{D} \), we let \( B \) denote the corresponding Blaschke product and \( B_n \) denote the Blaschke product with the zero \( \alpha_n \) removed. Further, we let \( \delta_j = |B_j(\alpha_j)| \), \( k_j = \frac{1}{1 - \bar{\alpha}_j z} \) denote the Szegö kernel (the reproducing kernel for \( H^2 \)) at \( \alpha_j \), \( g_j = k_j/\|k_j\| \) the \( H^2 \)-normalized kernel, and \( G \) the Gram matrix with entries \( G_{ij} = \langle g_j, g_i \rangle \). In the second part of [10, Theorem 2], Volberg’s goal was to develop a condition ensuring that \( \{g_n\} \) is near an orthogonal basis; by this, one means that there exist \( U \) unitary and \( K \) compact such that \( g_n = (U + K)e_n \), where \( \{e_n\} \) is the standard orthogonal basis for \( \ell^2 \). By [7, Section 3] or [4, Proposition 3.2], this is equivalent to the Gram matrix defining a bounded operator of the form \( I + K \) with \( K \) compact. Following Volberg and anticipating the connection to the Schatten-\( p \) classes, we call such bases \( U + S_\infty \) bases. Volberg showed that \( \{g_n\} \) is a \( U + S_\infty \) basis if and only if \( \lim_n \delta_n = 1 \); in other words, if and only if \( \{\alpha_n\} \) is a thin sequence. Assuming \( \{g_n\} \) is a \( U + S_\infty \) basis, it is not difficult to show that the sequence \( \{\alpha_n\} \) must be thin. But Volberg’s proof of the converse is more difficult and depends on the main lemma of a paper of Axler, Chang and Sarason [2, Lemma 5], estimating the norm of a certain product of Hankel operators as well as a factorization theorem for Blaschke products. The lemma in [2] uses maximal functions and a certain distribution function inequality. A more direct proof of Volberg’s result is desirable, and we provide a simpler proof of this result in Theorem 3.5 of this paper.

In a second theorem, letting \( S_2 \) denote the class of Hilbert-Schmidt operators, Volberg showed (see [10, Theorem 3]) that \( \{g_n\} \) is a \( U + S_2 \) basis if and only if \( \prod_{n=1}^{\infty} \delta_n \) converges. We are interested in estimates for the “in-between” cases. We provide a new proof of Volberg’s theorem for \( p = \infty \) and prove the following theorem.

Theorem 1.1. For \( 2 \leq p < \infty \), the operator \( G - I \in S_p \) if and only if \( \sum_n (1 - \delta_n^2)^{p/2} < \infty \).

Volberg’s theorem covered the cases \( p = 2 \) and \( p = \infty \), but our proofs differ in the following way: Instead of using the results of [2] and theorems about Hankel operators, we use the...
relationship between growth estimates of functions that do interpolation on thin sequences (see [5], [6]) and the norm of the Gram matrix. This simplifies previous proofs and provides the best estimates available.

2. Preliminaries and notation

Let \{\alpha_j\} be a sequence in \(\mathbb{D}\) with corresponding Blaschke product \(B\), and \(B_j\) be the Blaschke product with zeroes at every point in the sequence except \(\alpha_j\) and \(\delta_j = |B_j(\alpha_j)|\). The separation constant \(\delta\) is defined to be \(\delta := \inf_j \delta_j\). Carleson’s interpolation theorem says that the sequence \{\alpha_j\} is interpolating if and only if \(\delta > 0\), [3]. The sequence \{\alpha_j\} is said to be thin if \(\lim_{j \to \infty} \delta_j = 1\). Given a thin sequence we may arrange the \(\delta_j\) in increasing order and rearrange the zeros of the Blaschke product accordingly.

Recall that if \(T\) is an operator on a Hilbert space \(\mathcal{H}\) and \(\lambda_n\) is the \(n\)th singular value of \(T\), then given \(p\) with \(1 \leq p < \infty\) the Schatten-\(p\) class, \(S_p\), is defined to be the space of all compact operators with corresponding singular sequence in \(\ell^p\), the space of \(p\)-summable sequences. Then \(S_p\) is a Banach space with norm

\[ \|T\|_p = \left( \sum |\lambda_n|^p \right)^{1/p}. \]

For \(p = \infty\), we let \(S_\infty\) denote the space of compact operators.

Recall that \(k_j\) denotes the Szegö kernel, \(g_j = k_j/\|k_j\|\), and \(G\) the Gram matrix with entries \(G_{ij} = \langle g_j, g_i \rangle\). (The Gram matrix depends of course on the sequence \{\alpha_j\}, but we suppress this in the notation). For \{\alpha_j\} interpolating, we let \(D\) be the diagonal matrix with entries \(1/B_j(\alpha_j)\). It is known (see, for example, formula (26) of [8]) that

\[ G^{-1} = D^* G^t D. \]

For a given sequence \{\alpha_j\}, the interpolation constant is the infimum of those \(M\) such that for any sequence \{a_j\} in \(\ell^\infty\), one can find a function \(f\) in \(H^\infty\) with \(f(\alpha_j) = a_j\) and \(\|f\|_\infty \leq M\|a\|_\infty\). We shall let \(M(\delta)\) denote the supremum of the interpolation constants over all sequences \{\alpha_j\} with separation constant \(\delta\).

The following result is due essentially to A. Shields and H. Shapiro [9]. See [1, Proposition 9.5] for a proof of this version.

**Proposition 2.1.** Let \{\alpha_j\} be an interpolating sequence in \(\mathbb{D}\).

(i) If the interpolation constant is \(M\), then both \(\|G\|\) and \(\|G^{-1}\|\) are bounded by \(M^2\).

(ii) If \(\|G\| = C_1\) and \(\|G^{-1}\| = C_2\), then the interpolation constant is bounded by \(\sqrt{C_1C_2}\).

We shall use the following estimate of J.P. Earl (see [5] or [6]) to obtain our results.

**Theorem 2.2** (Earl’s Theorem). The interpolation constant \(M(\delta)\) satisfies

\[ M(\delta) \leq \left( \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^2. \]

3. Schatten-\(p\) classes

In this section we provide estimates on the Schatten-\(p\) norm of \(G - I\). We will need the theorem and lemma below.
Theorem 3.1. (see e.g. [11, Theorem 1.33]) Let $T$ be an operator on a separable Hilbert space, $\mathcal{H}$.

If $0 < p \leq 2$ then
\[
\|T\|_{\mathcal{S}^p}^p = \inf \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}
\]
and if $2 \leq p < \infty$
\[
\|T\|_{\mathcal{S}^p}^p = \sup \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}.
\]

We say that two sequences $\{x_n\}$ and $\{y_n\}$ of positive numbers are equivalent if there exist constants $c$ and $C$, independent of $n$, such that $cy_n \leq x_n \leq Cy_n$ for all $n$. We write $x_n \asymp y_n$.

We will also write $A \asymp B$ to indicate that there exists a constant $C$ such that $A \leq CB$.

Lemma 3.2. Let $\{e_n\}$ denote the standard orthonormal basis for $\ell^2$ and $\{\alpha_j\}$ be an interpolating sequence in $D$ with corresponding $\delta_j$. Then
\[
\|(G - I)e_n\| \asymp \sqrt{1 - \delta_n^2}.
\]

Proof. We have
\[
\|(G - I)e_n\|^2 = \langle (G^* - I)(G - I)e_n, e_n \rangle
\]
\[
= \left\langle \begin{pmatrix} \langle g_n, g_1 \rangle \\ \vdots \\ \langle g_n, g_{n-1} \rangle \\ 0 \\ \langle g_n, g_{n+1} \rangle \\ \vdots \\ \langle g_n, g_j \rangle \\ \vdots \\ \langle g_n, g_j \rangle \\ \langle g_n, g_j \rangle \\ \end{pmatrix}, \begin{pmatrix} \langle g_n, g_1 \rangle \\ \vdots \\ \langle g_n, g_{n-1} \rangle \\ 0 \\ \langle g_n, g_{n+1} \rangle \\ \vdots \\ \langle g_n, g_j \rangle \\ \vdots \\ \langle g_n, g_j \rangle \\ \langle g_n, g_j \rangle \\ \end{pmatrix} \right\rangle
\]
\[
= \sum_{j \neq n} \langle g_n, g_j \rangle \langle g_n, g_j \rangle
\]
\[
= \sum_{j \neq n} \left| \frac{\sqrt{1 - |\alpha_j|^2} \sqrt{1 - |\alpha_n|^2}}{1 - \bar{\alpha}_n \alpha_j} \right|^2
\]
\[
= \sum_{j \neq n} \left| \frac{1 - |\alpha_j - \alpha_n|^2}{1 - \bar{\alpha}_n \alpha_j} \right|^2.
\]

But $-\log x \geq 1 - x$ for $x > 0$ and $-\log x < c(1 - x)$ for $x < 1$ bounded away from 0 and some constant $c$ independent of $x$, so $-\log x \asymp 1 - x$ for $x$ bounded away from 0. Consequently,
\[
\|(G - I)e_n\|^2 \asymp \sum_{j \neq n} -\log \left| \frac{\alpha_j - \alpha_n}{1 - \bar{\alpha}_n \alpha_j} \right|^2
\]
\[
= -\log \prod_{j \neq n} \left| \frac{\alpha_j - \alpha_n}{1 - \bar{\alpha}_n \alpha_j} \right|^2
\]
\[
= -\log \delta_n^2
\]
\[
\asymp 1 - \delta_n^2.
\]
Note that the constants involved do not depend on \(n\).

Combining the lemma with Theorem 3.1, we obtain the following theorem.

**Theorem 3.3.** The following estimates hold:

- If \(2 \leq p < \infty\) then
  \[
  \sum_n (1 - \delta_n)^\frac{p}{4} \preceq \|G - I\|_{\mathcal{S}_p}^p;
  \]

- If \(0 < p \leq 2\) then
  \[
  \|G - I\|_{\mathcal{S}_p}^p \preceq \sum_n (1 - \delta_n)^\frac{p}{4};
  \]

- If \(p = 2\) then
  \[
  \sum_n (1 - \delta_n) \asymp \|G - I\|_{\mathcal{S}_2}^2.
  \]

**Lemma 3.4.** Let \(\{\alpha_j\}\) be an interpolating sequence and \(G\) the corresponding Gram matrix. Let \(C = \|G^{-1}\|\). Then \(\|G - I\| \leq C - 1\).

**Proof.** By (2.1), we have \(G \leq CI\), and as \(G\) is a positive operator, we have \(G \geq (1/C)I\). Therefore
\[
\left(\frac{1}{C} - 1\right) I \leq G - I \leq (C - 1) I,
\]
and as \(C > 1\), we get \(\|G - I\| \leq C - 1\).

In what follows, for a positive integer \(N\), we let \(G_N\) denote the lower right-hand corner of the Gram matrix obtained by deleting the first \(N\) rows and columns of \(G\). Thus,
\[
G_N = \begin{pmatrix}
1 & \langle g_{N+1}, g_N \rangle & \cdots & \langle g_{N+j}, g_N \rangle & \cdots \\
\langle g_N, g_{N+1} \rangle & 1 & \cdots & \langle g_{N+j}, g_{N+1} \rangle & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle g_{N+j}, g_{N+j} \rangle & \langle g_{N+j}, g_{N+j+1} \rangle & \cdots & 1 & \cdots \\
\end{pmatrix}
\]
and \(\lambda_n \geq 0\) denotes the \(n\)-th singular value of \(G - I\), where the singular values are arranged in decreasing order.

We are now ready to provide our simpler proof of Volberg’s result [10, Theorem 2, p. 215].

**Theorem 3.5.** The sequence \(\{\alpha_n\}\) is a thin sequence if and only if the Gram matrix \(G\) is the identity plus a compact operator.

**Proof.** \((\Rightarrow)\) Suppose \(\{\alpha_n\}\) is thin. By discarding finitely many points in the sequence \((\alpha_n)\), we can assume that the sequence has a positive separation constant, and hence is interpolating.

Let \(G_N\) be the Gram matrix of \(\{g_j\}\) for \(j \geq N\). We shall let \(\delta_{N,j}\) denote the \(\delta_j\) defined for \(G_N\) (that is, corresponding to the Blaschke sequence \(\{\alpha_j\}_{j \geq N}\)). Note that \(\delta_{N,j} \geq \delta_{N+j}\) for \(j = 0, 1, 2, \ldots\), and so we have that \(\delta(N) := \inf_j \delta_{N,j} \geq \inf_j \delta_{N+j} = \delta_N\). By Theorem 2.2 and Proposition 2.1,
\[
\|G_N^{-1}\| \leq (M(\delta(N)))^2 \leq \frac{(1 + \sqrt{1 - \delta(N)})^4}{\delta(N)^4} \leq \frac{(1 + \sqrt{1 - \delta_N^4})^4}{\delta_N^4}.
\]
Applying Lemma 3.4
\[ \|G_N - I_N\| \leq \left( \frac{(1 + \sqrt{1 - \delta_N^2})^4}{\delta_N^4} - 1 \right) \leq C\sqrt{1 - \delta_N}, \]
where \( C \) is a constant independent of \( N \). Since \( \sqrt{1 - \delta_N} \to 0 \) as \( N \to \infty \), we conclude that \( G - I \) is compact.

(\( \Leftarrow \)) From (2.1), we have

\[ G^{-1} - I = D^*(G^t - I)D + [D^*D - I]. \]

If \( G - I \) is compact, then so are \( G^t - I \) and \( G^{-1} - I = G^{-1}(I - G) \). Therefore from (3.1), we have \( D^*D - I \) is compact, which means \( \lim_{j \to \infty} \delta_j^2 = 1 \). Consequently, the sequence is thin.

**Theorem 3.6.** For \( 2 \leq p < \infty \), the operator \( G - I \) \( \in \mathcal{S}_p \) if and only if \( \sum_n (1 - \delta_n^2)^{p/2} < \infty \).

**Proof.** By Theorem 3.3, if \( G - I \) \( \in \mathcal{S}_p \), then the sum is finite.

Now suppose the sum is finite. Using Lemma 3.4 as in Theorem 3.5, we have

\[ \|G_N - I_N\| \leq C\sqrt{1 - \delta_N}, \]

where \( C \) is independent of \( N \).

By [11, Theorem 1.4.11],

\[ |\lambda_{N+1}| \leq \inf \{ \| (G - I) - F \| : F \in \mathcal{F}_N \}, \]

where \( \mathcal{F}_N \) is the set of all operators of rank less than or equal to \( N \). Therefore, taking \( F \) to be the matrix with the same first \( N \) rows and columns as \( G - I \), which is of rank at most \( 2N \), we have

\[ |\lambda_{2N+1}| \leq \|G_{N+1} - I_{N+1}\| \leq C\sqrt{1 - \delta_{N+1}}, \]

by our computation above. Therefore

\[ |\lambda_{2N+1}|^p \leq C^p(1 - \delta_{N+1})^{p/2}. \]

Since the singular values are arranged in decreasing order, \( |\lambda_{2n+1}| > |\lambda_{2n}| \) for each \( n \). Thus, if \( \sum_N (1 - \delta_N)^{p/2} < \infty \), then \( \sum_n |\lambda_{2n}|^p \leq 2 \sum_n |\lambda_{2n+1}|^p < \infty \) and we conclude that \( G - I \) \( \in \mathcal{S}_p \).

We conclude by remarking that it is possible to trace through the proofs above to determine constants \( c \) and \( C \), which depend only on \( \delta = \inf_n \delta_n \), such that for \( 2 \leq p \leq \infty \),

\[ c\|\sqrt{1 - \delta_n}\|_{\ell^p} \leq \|G - I\|_{\mathcal{S}_p} \leq C\|\sqrt{1 - \delta_n}\|_{\ell^p}. \]

In particular, by choosing \( \delta \) close enough to 1, one can choose \( c \) and \( C \) in (3.2) arbitrarily close to \( \sqrt{2} \) and \( 4\sqrt{2}(2^{1/p}) \), respectively.

**Question 3.7.** Is Theorem 3.6 true for \( p < 2 \)?

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