About the spectra of a real nonnegative matrix and its signings

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Abstract. For a complex matrix $M$, we denote by $\text{Sp}(M)$ the spectrum of $M$ and by $|M|$ its absolute value, that is the matrix obtained from $M$ by replacing each entry of $M$ by its absolute value. Let $A$ be a nonnegative real matrix, we call a signing of $A$ every real matrix $B$ such that $|B| = A$. In this paper, we characterize the set of all signings of $A$ such that $\text{Sp}(B) = \alpha \text{Sp}(A)$ where $\alpha$ is a complex unit number. Our motivation comes from some recent results about the relationship between the spectrum of a graph and the skew spectra of its orientations.

1. Introduction

Throughout this paper, all matrices are complex, unless otherwise noted. The identity matrix of order $n$ is denoted by $I_n$ and the transpose of a matrix $A$ by $A^T$. Let $\Sigma$ be a subgroup of $\mathbb{C}^*$, the group of nonzero complex numbers under multiplication. Two square matrices $A$ and $B$ are $\Sigma$-diagonally similar if $B = \Lambda^{-1} A \Lambda$ for some diagonal matrix $\Lambda$ with diagonal entries in $\Sigma$. A square matrix $A$ is reducible if there exists a permutation matrix $P$, so that $A$ can be reduced to the form $PAP^T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ where $X$ and $Z$ are square matrices. A square matrix which is not reducible is

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said to be irreducible. A real matrix $A$ is nonnegative, (we write $A \geq 0$), if all its entries are nonnegative.

Let $A$ be an $n \times n$ real or complex matrix. The multiset $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of $A$ is called the spectrum of $A$ and is denoted by $\text{Sp}(A)$. We usually assume that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. The spectral radius of $A$ is $\rho(A) := |\lambda_1|$. The relationship between the spectrum of a graph and the skew spectra of its orientations is studied in many papers (see for example [1,2,4–6,8,10]). Our work is closely related to the result of Shader and So [8]. To state this result, we need to introduce some definitions and notations.

Let $G$ be a finite simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of $G$ is the symmetric matrix $A(G) = (a_{ij})_{1 \leq i,j \leq n}$ where $a_{ij} = a_{ji} = 1$ if $\{v_i, v_j\}$ is an edge of $G$ and $a_{ij} = a_{ji} = 0$ otherwise. Since the matrix $A(G)$ is symmetric, its eigenvalues are real. The adjacency spectrum $\text{Sp}(G)$ of $G$ is defined as the spectrum of $A(G)$. Let $G^\sigma$ be an orientation of $G$, which assigns to each edge a direction so that the resultant graph $G^\sigma$ becomes an oriented graph. The skew-adjacency matrix of $G^\sigma$ is the real skew-symmetric matrix $S(G^\sigma) = (a^\sigma_{ij})_{1 \leq i,j \leq n}$ where $a^\sigma_{ij} = -a^\sigma_{ji} = 1$ if $(v_i, v_j)$ is an arc of $G^\sigma$ and $a^\sigma_{ij} = 0$ otherwise. The skew-spectrum $\text{Sp}(G^\sigma)$ of $G^\sigma$ is defined as the spectrum of $S(G^\sigma)$. Note that $\text{Sp}(G^\sigma)$ consists of only purely imaginary eigenvalues because $S(G^\sigma)$ is a real skew-symmetric matrix.

Let $G$ be a bipartite graph with bipartition $[I,J]$, the orientation $G^\varepsilon$ that assigns to each edge of $G$ a direction from $I$ to $J$ is called the canonical orientation. Shader and So [8] showed that $\text{Sp}(G^\varepsilon) = i \text{Sp}(G)$. Moreover, they proved that a graph $G$ is bipartite if and only if $\text{Sp}(G^\sigma) = i \text{Sp}(G)$ for some orientation $G^\sigma$ of $G$.

Consider now two orientations $G^\sigma$ and $G^\tau$ of $G$. We say that $G^\sigma$ and $G^\tau$ are switching-equivalent if there exists a subset $W$ of $V(G)$ such that $G^\sigma$ is obtained from $G^\tau$ by reversing the direction of all arcs between $W$ and $V(G)\setminus W$. Clearly, the skew-adjacency matrices of switching-equivalent orientations are $\{-1,1\}$-diagonally similar. Hence, they have the same spectrum. When $G$ is bipartite, Anuradha et al. [1] proved that $\text{Sp}(G^\tau) = i \text{Sp}(G)$ if and only if $G^\sigma$ is switching-equivalent to the canonical orientation.

These results can be stated in term of matrices as follows.

**Proposition 1.** Let $A$ be a $\{0,1\}$-symmetric matrix. Then the following statements are equivalent:
i) There exists a real skew-symmetric matrix $B$ such that $|B| = A$ and $\text{Sp}(B) = i\text{Sp}(A)$;

ii) There exists a permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$$

where the zero diagonal blocks are square.

Proposition 2. Let $A = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$ be a $\{0,1\}$-symmetric matrix and let $B$ be a skew-symmetric matrix such that $|B| = A$. Then, the following statements are equivalent:

i) $\text{Sp}(B) = i\text{Sp}(A)$;

ii) $B$ is $\{-1,1\}$-diagonally similar to $\tilde{A} = \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix}$.

For a $\{0,1\}$-symmetric matrix $A$, the propositions above characterize the set of all skew-symmetric signings $B$ of $A$, such that $\text{Sp}(B) = i\text{Sp}(A)$. In this paper, we consider the more general problem.

Problem 1. Let $A$ be a nonnegative real matrix and let $\alpha$ be a complex unit number. Characterize the set of all signings $B$ of $A$ such that $\text{Sp}(B) = \alpha\text{Sp}(A)$.

We solve this problem when $A$ is an irreducible matrix. To state our main result, we need some terminology. A digraph $D$ is a pair consisting of a finite set $V(D)$ of vertices and a subset $E(D)$ of ordered pairs of vertices called arcs. Let $v,v'$ be two vertices of $D$, a path $P$ from $v$ to $v'$ is a finite sequence $v_0 = v, \ldots, v_k = v'$ such that $(v_0, v_1), \ldots, (v_{k-1}, v_k)$ are arcs of $D$. The length of $P$ is the number $k$ of its arcs. If $v_0 = v_k$, we say that $P$ is a closed path. A digraph is said to be strongly connected if for any two vertices $v$ and $v'$, there is a path joining $v$ to $v'$. It is easy to see that a strongly connected digraph contains a closed path. The period of a strongly connected digraph is the greatest common divisor of the lengths of its closed paths.

With each $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, we associate a digraph $D_A$ on the vertex set $[n] = \{1, \ldots, n\}$ and with arc set $\{(i,j) : a_{ij} \neq 0\}$. It is easy to show that $A$ is irreducible if and only if $D_A$ is strongly connected. The period of an irreducible matrix is the period of its associate digraph. For example, if $A$ is the adjacency matrix of a connected graph $G$, then
About the spectra of a real nonnegative matrix

its period is either 1 or 2. Moreover, the period of $A$ is 2 if and only if $G$ is bipartite.

Let $A$ be an irreducible nonnegative real matrix of period $p$. For each complex unit number $\alpha$, we denote by $\mathcal{M}(\alpha, A)$ the set of all signings $B$ of $A$ such that $\text{Sp}(B) = \alpha \text{Sp}(A)$.

In Corollary 2, we prove that if $\mathcal{M}(\alpha, A)$ is nonempty, then $\alpha = e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$. Moreover, we prove in Proposition 6 that $e^{-\frac{i\pi}{p}} \text{sp}(A) = \text{sp}(A)$. This implies that,

$$\mathcal{M}(1, A) = \mathcal{M}(e^{\frac{2i\pi}{p}}, A) = \mathcal{M}(e^{\frac{2(p-1)i\pi}{p}}, A) = \cdots = \mathcal{M}(e^{\frac{(2p-1)i\pi}{p}}, A)$$

Therefore, it suffices to characterize $\mathcal{M}(1, A)$ and $\mathcal{M}(e^{\frac{i\pi}{p}}, A)$.

In the proof of Corollary 1, we give an explicit construction of a matrix $B_0 \in \mathcal{M}(e^{\frac{i\pi}{p}}, A)$ which is used in our main theorem below.

**Theorem 1.** Under the notation above, the following statements hold

i) $\mathcal{M}(1, A)$ is the set of matrices $\{-1, 1\}$-diagonally similar to $A$;

ii) $\mathcal{M}(e^{\frac{i\pi}{p}}, A)$ is the set of matrices $\{-1, 1\}$-diagonally similar to $B_0$.

**2. Some properties of $\mathcal{M}(\alpha, A)$**

Throughout, $A$ is an $n \times n$ irreducible nonnegative matrix, $p$ its period and $\alpha$ a unit complex number. We will use the following theorem due to Helmut Wielandt [9].

**Theorem 2.** Let $B$ be a complex $n \times n$ matrix such that $|B| \leq A$. Then $\rho(B) \leq \rho(A)$. Moreover, if equality holds (i.e., $\rho(A)e^{i\theta} \in \text{Sp}(B)$ for some real number $\theta$) then $B = e^{i\theta}LAL^{-1}$, where $L$ is a complex diagonal matrix such that $|L| = I_n$.

We will use Theorem 2 to prove the following.

**Proposition 3.** Let $B$ be a signing of $A$ such that $\rho(B) = \rho(A)$. If $\lambda$ is an eigenvalue of $B$ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)e^{\frac{i\pi k}{p}}$ for some $k \in \{0, \ldots, 2p-1\}$.

**Proof.** Let $A := (a_{ij})_{1 \leq i,j \leq n}, B := (b_{ij})_{1 \leq i,j \leq n}$ and $\lambda = \rho(A)e^{i\theta}$. By Theorem 2, we have $B = e^{i\theta}LAL^{-1}$ where $L$ is a complex diagonal matrix such that $|L| = I_n$. It follows that $b_{ij} = e^{i\theta}l_{ij}a_{ij}l_{ij}^{-1}$ for $i, j \in \{1, \ldots, n\}$,
where \( l_1, \ldots, l_n \) are the diagonal entries of \( L \). Consider now a closed path \( C = (i_1, i_2, \ldots, i_r, i_1) \) of \( D_A \). By the previous equality, we have
\[
\frac{b_{i_1i_2} \cdots b_{i_{r-1}i_r} b_{i_r i_1}}{a_{i_1 i_2} \cdots a_{i_{r-1} i_r} a_{i_r i_1}} = (e^{i\theta} l_{i_1} l_{i_2}^{-1}) \cdots (e^{i\theta} l_{i_{r-1}} l_{i_r}^{-1})(e^{i\theta} l_{i_r} l_{i_1}^{-1}) = (e^{i\theta})^r
\]
Then \( (e^{i\theta})^r \in \{1, -1\} \) because \( |B| = A \).

Since \( p \) is the greatest common divisor of the lengths of the closed paths in \( D_A \), we have \( (e^{i\theta})^p \in \{1, -1\} \) and then \( \lambda = \rho(A) e^{i\pi k} \) for some \( k \in \{0, \ldots, 2p - 1\} \).

**Remark 1.** Let \( \lambda \) be an eigenvalue of \( A \) such that \( |\lambda| = \rho(A) \). By applying Proposition 3 to \( B = A \), we have \( \lambda = \rho(A) e^{i\pi k} \) for some \( k \in \{0, \ldots, 2p - 1\} \).

The following result gives a necessary condition under which \( \mathcal{M}(\alpha, A) \) is nonempty.

**Corollary 1.** If \( \mathcal{M}(\alpha, A) \) is nonempty then \( \alpha = e^{i\pi k} \) for some \( k \in \{0, \ldots, 2p - 1\} \), or equivalently \( \alpha^p = \pm 1 \).

**Proof.** Let \( \lambda \) be an eigenvalue of \( A \) such that \( |\lambda| = \rho(A) \). By Remark 1, we have \( \lambda = \rho(A) e^{i\pi k} \) for some \( k \in \{0, \ldots, 2p - 1\} \). Let \( B \in \mathcal{M}(\alpha, A) \). Then \( \alpha \rho(A) e^{i\pi k} \in \text{Sp}(B) \) because \( \text{Sp}(B) = \alpha \text{Sp}(A) \). It follows from Proposition 3 that \( \alpha \rho(A) e^{i\pi k} = \rho(A) e^{i\pi h} \) for some \( h \in \{0, \ldots, 2p - 1\} \) and hence \( \alpha = e^{i\pi (h-k) p} \).

It is easy to see that if \( B \in \mathcal{M}(\alpha, A) \), then \( \Lambda^{-1} B \Lambda \in \mathcal{M}(\alpha, A) \) for every \( \{-1, 1\} \)-diagonal matrix \( \Lambda \). Conversely,

**Proposition 4.** The matrices in the set \( \mathcal{M}(\alpha, A) \) are all \( \{-1, 1\} \)-diagonally similar.

**Proof.** Let \( B_1, B_2 \in \mathcal{M}(\alpha, A) \). Then \( \text{Sp}(B_1) = \text{Sp}(B_2) = \alpha \text{Sp}(A) \). It follows that \( B_1 \) and \( B_2 \) have a common eigenvalue of the form \( \rho(A) e^{i\theta} \) for some real number \( \theta \). By Theorem 2, we have \( B_1 = e^{i\theta} L_1 A L_1^{-1} \) and \( B_2 = e^{i\theta} L_2 A L_2^{-1} \) where \( L_1, L_2 \) are complex diagonal matrices such that \( |L_1| = |L_2| = 1 \). It follows that \( B_1 = (L_2 L_1^{-1})^{-1} B_2 L_2 L_1^{-1} \). To conclude, it suffices to apply Lemma 1 below.

**Lemma 1.** Let \( B, B' \) be two signings of \( A \). If there exists a complex diagonal matrix \( \Gamma \) such that \( B' = \Gamma B \Gamma^{-1} \) and \( |\Gamma| = I_n \) then \( B \) and \( B' \) are \( \{-1, 1\} \)-diagonally similar.
Proof. Let \( A := (a_{ij})_{1 \leq i, j \leq n} \), \( B := (b_{ij})_{1 \leq i, j \leq n} \) and \( B' := (b'_{ij})_{1 \leq i, j \leq n} \). We denote by \( \gamma_1, \ldots, \gamma_n \) the diagonal entries of \( \Gamma \). Let \( \Delta := \gamma_1^{-1} \Gamma \). Clearly, we have \( \Delta B \Delta^{-1} = \Gamma B \Gamma^{-1} = B' \). Hence, to prove our lemma, it suffices to check that \( \Delta \) is a \( \{-1, 1\}\)-diagonal matrix. For this, let \( j \in \{2, \ldots, n\} \). As \( A \) is irreducible, the digraph \( D_A \) is strongly connected and then there is a path \( j = i_1, \ldots, i_r = 1 \) of \( D_A \) from \( j \) to \( 1 \). By definition of \( D_A \), we have \( a_{i_1 i_2} \neq 0, \ldots, a_{i_r-1 i_r} \neq 0 \). It follows that \( b_{i_1 i_2} \neq 0, \ldots, b_{i_r-1 i_r} \neq 0 \) and \( b'_{i_1 i_2} \neq 0, \ldots, b'_{i_r-1 i_r} \neq 0 \) because \( |B| = |B'| = A \). Moreover, from the equality \( B' = \Gamma B \Gamma^{-1} \) we have \( b'_{i_1 i_2} = \gamma_1 b_{i_1 i_2} \gamma_1^{-1} \), \( b'_{i_2 i_3} = \gamma_2 b_{i_2 i_3} \gamma_2^{-1} \), \ldots, \( b'_{i_{r-1} i_r} = \gamma_{i_{r-1}} b_{i_{r-1} i_r} \gamma_{i_{r-1}}^{-1} \). Then \( b'_{i_1 i_2} \cdots b'_{i_{r-1} i_r} = \gamma_1 \gamma_2^{-1} b_{i_1 i_2} \cdots b_{i_{r-1} i_r} \). But by hypothesis, \( B, B' \) are real matrices and \( |B| = |B'| \), then \( b'_{i_1 i_2} \cdots b'_{i_{r-1} i_r} = \pm b_{i_1 i_2} \cdots b_{i_{r-1} i_r} \) and hence \( \gamma_j \gamma_i^{-1} = \gamma_i \gamma_i^{-1} = \gamma_1 \gamma_1^{-1} \in \{-1, 1\} \), which completes the proof of the lemma. \( \Box \)

3. Proof of the main theorem

Assertion \( i \). (resp. assertion \( ii \). for \( p = 1 \)) follows from Proposition 4 and the fact that \( A \in \mathcal{M}(1, A) \) (resp. \( -A \in \mathcal{M}(-1, A) \)). To prove assertion \( ii \). for \( p > 1 \), we will use the cyclic form of irreducible matrices with period \( p \). To define \( k \)-cyclic matrices, let \( n \) be a positive integer and let \( \{r_1, \ldots, r_k\} \) be a partition of \( n \), that is \( r_1, \ldots, r_k \) are positive integers and \( r_1 + \cdots + r_k = n \). For \( i = 1, \ldots, k - 1 \), let \( A_i \) be a \( r_i \times r_{i+1} \) matrix and let \( A_k \) be a \( r_k \times r_1 \) matrix. The matrix

\[
\begin{pmatrix}
0 & A_1 & 0 & \cdots & 0 \\
0 & 0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & A_{k-1} \\
A_k & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

is denoted by \( \text{Cyc}(A_1, A_2, \ldots, A_k) \). Each matrix of this form is called \( k \)-cyclic.

The characterization of irreducible matrices with period \( p > 1 \) is given by the following result due to Frobenius.

**Proposition 5.** Let \( A \) be an irreducible nonnegative real matrix with period \( p > 1 \), then there exists a permutation matrix \( P \) such that \( PAP^T \) is \( p \)-cyclic.
Proposition 6. Let \( A = \text{Cyc}(A_1, A_2, \ldots, A_p) \) be a nonnegative \( p \)-cyclic matrix where \( A_i \) is a \( r_i \times r_{i+1} \) matrix for \( i = 1, \ldots, p-1 \) and \( A_p \) is a \( r_p \times r_1 \) matrix. Let \( \tilde{A} \) be the matrix obtained from \( A \) by replacing the block \( A_p \) by \(-A_p \). Let \( k \in \{0, \ldots, 2p-1\} \), then

i) if \( k \) is even, \( e^{i\frac{\pi k}{p}} A \) is diagonally similar to \( A \), in particular \( \text{Sp}(A) = e^{i\frac{\pi k}{p}} \text{Sp}(A) \);

ii) if \( k \) is odd, \( e^{i\frac{\pi k}{p}} A \) is diagonally similar to \( \tilde{A} \), in particular \( \text{Sp}(\tilde{A}) = e^{i\frac{\pi k}{p}} \text{Sp}(A) \).

Proof. Let

\[
L := \begin{pmatrix}
I_{r_1} & 0 & 0 & \cdots & 0 \\
0 & e^{i\frac{\pi k}{p}} I_{r_2} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & 0 & e^{i\frac{\pi k(p-1)}{p}} I_{r_p}
\end{pmatrix}.
\]

It easy to check that if \( k \) is even, \( e^{i\frac{\pi k}{p}} LAL^{-1} = A \) and if \( k \) is odd, \( e^{i\frac{\pi k}{p}} LAL^{-1} = \tilde{A} \). \qed

The next corollary is a consequence of the above proposition and Proposition 5.

Corollary 2. Let \( A \) be an irreducible nonnegative matrix with period \( p \). Then \( \mathcal{M}(e^{i\frac{\pi k}{p}}, A) \) is nonempty.

Proof. As \((-A) \in \mathcal{M}(-1, A)\), we can assume that \( p > 1 \). By Proposition 5, there exists a permutation matrix \( P \) such that \( PAP^T \) is \( p \)-cyclic. Let \( A' := PAP^T := \text{Cyc}(A'_1, A'_2, \ldots, A'_p) \) and let \( \tilde{A}' \) be the matrix obtained from \( A' \) by replacing the block \( A'_p \) by \(-A'_p \). It follows from Proposition 6 that \( \text{Sp}(\tilde{A}') = e^{i\frac{\pi k}{p}} \text{Sp}(A') \), and hence \( \text{Sp}(P^T \tilde{A}'P) = e^{i\frac{\pi k}{p}} \text{Sp}(P^T A'P) = e^{i\frac{\pi k}{p}} \text{Sp}(A) \). Let \( B_0 := P^T A'P \). Since \( |\tilde{A}'| = A' \), we have \(|B_0| = P^T A'P = A \) and then \( B_0 \in \mathcal{M}(e^{i\frac{\pi k}{p}}, A) \). \qed

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