On systems of interacting populations influenced by multiplicative white noise

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Abstract

We discuss a model of a system of interacting populations for the case when: (i) the growth rates and the coefficients of interaction among the populations depend on the populations densities; and (ii) the environment influences the growth rates and this influence can be modelled by a Gaussian white noise. The system of model equations for this case is a system of stochastic differential equations with: (i) deterministic part in the form of polynomial nonlinearities; and (ii) state-dependent stochastic part in the form of multiplicative Gaussian white noise. We discuss both the cases when the formal integration of the stochastic differential equations leads: (i) to integrals of Ito kind; or (ii) to integrals of Stratonovich kind. The systems of stochastic differential equations are reduced to the corresponding Fokker-Planck equations. For the Ito case and for the case of 1 population analytic results is obtained for the stationary PDF of the the population density. For the case of more than one population and for the both Ito case and Stratonovich case the detailed balance conditions are not satisfied and because of this exact analytic solutions of the corresponding Fokker-Planck equations for the stationary PDFs for the population densities are not known. We obtain approximate solutions for this case by the methodology of the adiabatic elimination.

1 Introduction

The research on the nonlinear dynamics of the complex systems increases steadily in the last two decades (for several examples see Appendix A). Many complex systems are influenced by random events. Because of this the theory of stochastic processes is much used in the modeling of the processes in the complex systems [1]-[5]. In this paper we discuss some mathematical aspects of the theory of interacting populations for the case when the growth rates are influenced by environmental fluctuations. For the case when the fluctuations can be modelled by Gaussian white noise we shall obtain as model equations a system of stochastic differential equations that contain multiplicative noise. For the case of single population the model equation will be of the kind

\[ \dot{\rho} = F(\rho) + \eta G(\rho) \]  

(1.1)
where \( F(\rho) \) and \( G(\rho) \) are polynomials of \( \rho \) and \( \eta \) is Gaussian white noise. For the case of system of interacting populations the corresponding model equations will be of the kind

\[
\dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) + \eta_i G_i(\rho_1, \ldots, \rho_n); \quad i = 1, \ldots, n \tag{1.2}
\]

where \( F \) and \( G \) are polynomials and \( \eta_i \) are Gaussian white noises.

The organization of the article is as follows. We discuss the model equations for the dynamics of interacting populations in the following section. The presence of Gaussian white noise in the growth rates of populations leads to a system of stochastic differential equations with multiplicative noise. The integration of these stochastic differential equations leads in principle to stochastic integrals of Itô kind or to stochastic integrals of Stratonovich kind. Section 3 is devoted to the theory for the case when the stochastic integrals are of Itô kind. Section 4 is devoted to theory for the case when the stochastic integrals are of Stratonovich kind. Several concluding remarks are summarized in Section 5. In addition four appendices supply the reader with information about the examples of research on complex systems, about the theory of stochastic differential equations containing multiplicative white noise, theory of stochastic differential equations of Itô and Stratonovich kind and their relation to the Fokker-Plank equation (known also as forward Kolmogorov equation).

## 2 Investigated equations and population dynamics

The classical model of interacting populations is based on a system of equations of Lotka-Volterra kind [6, 7]:

\[
\dot{\rho}_i = r_i \rho_i(t) \left( 1 - \sum_{j=1}^{n} \alpha_{ij} \rho_j(t) \right) \tag{2.1}
\]

where \( \rho_i \) are the densities of the population members, \( r_i \) are the birth rates, and \( \alpha_{ij} \) are coefficients of interaction between the populations \( i \) and \( j \). Let us now suppose that the birth rates and interaction coefficients depend on the density of the populations and in addition the birth rates fluctuate:

\[
r_i = r_i^0 \left( 1 + \sum_{j=1}^{n} r_{ij} \rho_j \right) + \eta_i; \quad \alpha_{ij} = \alpha_{ij}^0 \left( 1 + \sum_{j=1}^{n} \alpha_{ijk} \rho_k \right) \tag{2.2}
\]

in Eq. (2.2) \( r_{ij} \) and \( \alpha_{ijk} \) are parameters and \( \eta_i \) are Gaussian white noises. The system of equations (2.2) for \( \eta_i = 0 \) has been introduced and investigated in [8]-[14]. The presence of \( \eta_i \) however influences much the system dynamics [15, 16].

The substitution of Eq. (2.2) in Eq. (2.1) leads to a system of model equations.
of the kind

\[ \dot{\rho}_i = F_i(\rho_1, \ldots, \rho_n) + \eta_i G_i(\rho_1, \ldots, \rho_n); \]

\[ F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left\{ 1 - \sum_{j=1}^{n} (\alpha_{ij}^0 - r_{ij}) \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}^0 (\alpha_{ij} + r_{ij}) \rho_j \rho_k \right\} \]

\[ G_i(\rho_1, \ldots, \rho_n) = \rho_i \left( 1 - \sum_{j=1}^{n} \alpha_{ij}^0 \rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}^0 \alpha_{ijk} \rho_j \rho_k \right) \]  (2.3)

For the case of one population (we set \( r_i^0 = r; r_{11} = 0; \alpha_{11}^0 = \alpha; \alpha_{111} = 0 \)) the model equation is

\[ \dot{\rho} = F(\rho) + \eta G(\rho) \]

\[ F(\rho) = r \rho - \alpha \rho^2; \quad G(\rho) = \rho - \alpha \rho^2 \]  (2.4)

Below we shall discuss more general equation in comparison to Eq. (2.4). We shall discuss the case where \( F(\rho) \) and \( G(\rho) \) are polynomials of arbitrary orders \( p_1 \) and \( p_2 \), i.e.,

\[ F(\rho) = \sum_{i=1}^{p_1} \mu_i \rho^i; \quad G(\rho) = \sum_{i=1}^{p_2} \theta_i \rho^i \]  (2.5)

where \( \mu_i \) and \( \theta_i \) are parameters. In this case Eq. (2.4) becomes

\[ \dot{\rho} = \sum_{i=1}^{p_1} \mu_i \rho^i + \sum_{i=1}^{p_2} \theta_i \rho^i \]  (2.6)

The formal integration of Eq. (2.4) (see also Appendix B) leads to the equation

\[ \rho(t) = \rho(t = 0) + \int_0^t d\tau F[\rho(\tau)] + \int_0^t dW_\tau G[\rho(\tau)], \]  (2.7)

where \( W_\tau \) is a Wiener process. The integral \( \int_0^t dW_\tau G(\rho(\tau)) \) can be integral of Ito kind or integral of Stratonovich kind (for more discussion see Appendix B). In the next two sections we shall discuss these two cases.

3 Case of stochastic differential equations of Ito kind

For this case Eq. (2.6) can be written as

\[ d\rho_i = F(\rho_i)dt + G(\rho_i)dW_t, \]  (3.1)
where we denote the time dependence as subscript and in general $F$ and $G$ are
given by Eqs.(2.5). The Fokker-Planck equation that corresponds to Eq.(3.1) is
\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \left\{ p(x, t) \left[ \sum_{i=1}^{p_1} \mu_i x^i \right] \right\} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ p(x, t) \left[ \sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j} \right] \right\}
\]  
(3.2)

We can formulate the following

**Proposition 1.** Let $b_1$ and $b_2$ be natural boundary points ($-\infty \leq b_1 < b_2 \leq \infty$). Let in addition \( \sigma(x) = \sum_{i=1}^{p_2} \theta_i x^i > 0 \) in \((b_1, b_2)\). Then the diffusion process $X_t$ that is solution of the stochastic differential equation Eq.(3.1) has unique invariant distribution with p.d.f.

\[
p^0(x) = \frac{N}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \frac{2 \sum_{i=1}^{p_1} \mu_i y^i}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j y^{i+j}} \right), \quad \forall x \in (b_1, b_2) \]  
(3.3)

if the quantity

\[
N^{-1} = \int_{b_1}^{b_2} dx \frac{1}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \frac{2 \sum_{i=1}^{p_1} \mu_i y^i}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j y^{i+j}} \right), \quad b_1 < c < b_2 \]  
(3.4)

has finite value. In addition each time-dependent solution $p(x, t)$ of the Fokker-Planck equation (3.2) in \((b_1, b_2)\) satisfies

\[
\lim_{t \to \infty} p(x, t) = p^0(x) \]  
(3.5)

**Proof.** The proposition follows from the Observation 1 from the Appendix B for the case when

\[
f(x) = \sum_{i=1}^{p_1} \mu_i x^i; \quad \sigma(x) = \sum_{i=1}^{p_2} \theta_i x^i.\]

\[\square\]

Let us apply the Proposition 1 to the case of one population modelled by Eq.(2.4). In this case $\mu_1 = r; \mu_2 = -\alpha r; \theta_1 = 1; \theta_2 = -\alpha$. We note that Proposition 1 is valid when $\sigma > 0$. In our case this means that $\rho < 1/\alpha$ ($\rho \geq 0$).

For the quantity $N$ from Eq.(3.3) we obtain

\[
N^{-1} = \frac{(\alpha c - 1)^{2r}}{r(4r^2 - 1)c^{2r}} \left[ \frac{b_2^{2r-1}((r - \alpha b_2)(2r + 1) + \alpha^2 b_2^2)}{(1 - \alpha b_2)^{2r+1}} - \frac{b_1^{2r-1}((r - \alpha b_1)(2r + 1) + \alpha^2 b_1^2)}{(1 - \alpha b_1)^{2r+1}} \right]
\]  
(3.6)
and for $p^0(x)$ from Eq. (3.3) we obtain

$$p^0(x) = r(1 - 4r^2) \left[ - \frac{b_2^{2r-1}((r - \alpha b_2)(2r + 1) + \alpha^2 b_2^2)}{(1 - \alpha b_2)^{2r+1}} + \frac{b_1^{2r-1}((r - \alpha b_1)(2r + 1) + \alpha^2 b_1^2)}{(1 - \alpha b_1)^{2r+1}} \right]^{-1} \frac{1}{x^{2-2r}(1 - \alpha x)^{2r+2}} \quad (3.7)$$

Let us now discuss the case of more than one population. For this case we have to solve the system of stochastic differential equations

$$dX_i(t) = F_i[X_1(t), \ldots, X_n(t)] + G_i[X_1(t), \ldots, X_n(t)]dW_i(t), \ i = 1, \ldots, n \quad (3.8)$$

where $W_j(t)$ are independent Wiener processes and

$$F_i(\rho_1, \ldots, \rho_n) = r_i^0 \rho_i \left\{ 1 - \sum_{j=1}^{n} (\alpha_{ij} - r_j)\rho_j - \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_{ij}(\alpha_{ijl} + r_l)\rho_j\rho_l - \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{n} \alpha_{ij}\alpha_{ijkl}\rho_j\rho_l \rho_k \right\}$$

$$G_i(\rho_1, \ldots, \rho_n) = \rho_i \left( 1 - \sum_{j=1}^{n} \alpha_{ij}\rho_j - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}\alpha_{ijk}\rho_j\rho_k \right) \quad (3.9)$$

The corresponding Fokker-Planck equation is ($G_{ij} = G_i \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta-symbol)

$$\frac{\partial}{\partial t}p = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[pF_i(x_1, \ldots, x_n, t)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}[pG_{ij}(x_1, \ldots, x_n, t)G_{ji}(x_1, \ldots, x_n, t)] \quad (3.10)$$

We are interested in stationary solutions $p_s$ of Eq. (3.10). In the general case such solutions can be obtained numerically. A hope to obtain analytic solutions exists mainly when the conditions for detailed balance are satisfied [17]. For the case of Eq. (3.10) these conditions are

$$\epsilon_i F_i(\vec{c} \cdot \vec{x})p_s(\vec{x}) = -F_i(\vec{x})p_s(\vec{x}) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[G_{ij}(\vec{x})p_s(\vec{x})]$$

$$\epsilon_i^2 G_{ii}(\vec{c} \cdot \vec{x}) = G_{ii}^2(\vec{x}) \quad (3.11)$$

where $\epsilon_i = \pm 1$. One can easily show that that the structure of $G_i$ from (3.9) is such that the second condition for existence of detailed balance from (3.11) is not satisfied. Then one can hope to obtain approximate analytic solutions for particular cases of Eq. (3.10).

One possible way for obtaining approximate solutions of the Fokker-Planck equation for the case of more than one population is the connected to the method of adiabatic elimination [17]. In order to illustrate this method we consider the following particular case of the Eqs. (2.3). Let $\alpha_{ij}^0 = 0$ and $\alpha_{ijk} = 0$. In addition
let $\eta_2 = 0$. For the case of two populations we obtain the following system of equations

\begin{align*}
\frac{d\rho_1}{dt} &= r_0^1 \rho_1 (1 + r_{11} \rho_1 + r_{12} \rho_2) dt + \rho_1 dW_1 \\
\frac{d\rho_2}{dt} &= r_0^2 \rho_2 (1 + r_{21} \rho_1 + r_{22} \rho_2) dt
\end{align*}

(3.12)

Let $\rho_2$ be the fast relaxing variable, i.e., $d\rho/\,dt$ tends to 0 very fast in the time. Then in the second equation of Eqs. (3.12) one can set $d\rho/\,dt = 0$ and then the resulting equation has solutions $\rho_2 = 0$ (extinction of the second population) or

$$
\rho_2 = -\frac{1}{r_{22}} - \frac{r_{21}}{r_{22}} \rho_1
$$

(3.13)

which corresponds to a ”slaving” of the ”fast” variable $\rho_2$ by the ”slow” variable $\rho_1$. The substitution of Eq. (3.13) in the first equation of Eqs. (3.12) leads to the stochastic differential equation

$$
\frac{d\rho_1}{dt} = \left[ r_1^0 \left( r_{11} - \frac{r_{12} r_{21}}{r_{22}} \right) \rho_1^2 + r_1^0 \left( 1 - \frac{r_{12}}{r_{22}} \right) \rho_1 \right] dt + \rho_1 dW_1
$$

(3.14)

Eq. (3.14) can be treated by the methodology discussed above. In order to obtain an analytic result we have to assume

$$
r_1^0 = \frac{2}{1 - r_{12}/r_{22}}
$$

(3.15)

The application of the methodology connected to Proposition 1 leads to the distribution

$$
p^0(\rho_1) = A \rho_1^2 \exp(2\mu_2 \rho_1)
$$

(3.16)

where

$$
A = \frac{\mu_2^3}{\left( \frac{\mu_2^2 b_2^2}{2} - \frac{\mu_2 b_2}{2} + \frac{1}{4} \right) \exp(2\mu_2 b_2) - \left( \frac{\mu_2^2 b_1^2}{2} - \frac{\mu_2 b_1}{2} + \frac{1}{4} \right) \exp(2\mu_2 b_1)}
$$

and

$$
\mu_2 = \frac{2r_{11}(r_{22} - r_{11})}{r_{22} - r_{12}} < 0
$$

(3.17)

### 4 Case of stochastic differential equations of Stratonovich kind

For this case Eq. (2.6) can be written as

$$
\frac{d\rho_t}{dt} = \left[ F(\rho_t) + \frac{1}{2} G'(\rho_t)G(\rho_t) \right] dt + G(\rho_t) dW_t,
$$

(4.1)

where we again again denote the time dependence as subscript and in general $F$ and $G$ are given by Eqs. (2.5). According to Appendix B the Fokker-Planck
The equation that corresponds to Eq.(4.1) is

\[ \frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} \left\{ p(x,t) \left[ \left( \sum_{i=1}^{p_1} \mu_i \rho^i \right) + \frac{1}{2} \left( \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} i \theta_i \theta_j \rho^{i+j-1} \right) \right] \right\} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ p(x,t) \left[ \sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j \rho^{i+j} \right] \right\} \]

(4.2)

We can formulate the following

**Proposition 2.** Let \( b_1 \) and \( b_2 \) be natural boundary points \((-\infty \leq b_1 < b_2 \leq \infty)\). Let in addition \( \sigma(x) = \sum_{i=1}^{p_2} \theta_i x^i > 0 \) in \((b_1, b_2)\). Then the diffusion process \( X_t \) that is solution of the stochastic differential equation Eq.(4.1) has unique invariant distribution with p.d.f.

\[ p^0(x) = \frac{\mathcal{N}}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \frac{2 \left( \sum_{i=1}^{p_1} \mu_i y^i + \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} i \theta_i \theta_j y^{i+j-1} \right)}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j y^{i+j}} \right), \quad \forall x \in (b_1, b_2), \]

(4.3)

if the quantity

\[ \mathcal{N}^{-1} = \int_{b_1}^{b_2} dx \frac{1}{\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \theta_i \theta_j x^{i+j}} \exp \left( \int_c^x dy \frac{2 \left( \sum_{i=1}^{p_1} \mu_i y^i + \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} i \theta_i \theta_j y^{i+j-1} \right)}{\sum_{i=1}^{p_2} \sum_{j=1}^{p_2} \theta_i \theta_j y^{i+j}} \right), \]

(4.4)

has finite value. In addition each time-dependent solution \( p(x,t) \) of the Fokker-Planck equation (4.2) in \((b_1, b_2)\) satisfies

\[ \lim_{t \to \infty} p(x,t) = p^0(x) \]

(4.5)

**Proof.** The proposition follows from the Observation 2 from the Appendix C for the case when

\[ f(x) = \sum_{i=1}^{p_1} \mu_i x^i; \quad \sigma(x) = \sum_{i=1}^{p_2} \theta_i x^i. \]

\( \square \)

Let us apply the Proposition 2 to the case of one population modelled by Eq.(2.4). In this case \( \mu_1 = r; \ \mu_2 = -\alpha r; \ \theta_1 = 1; \ \theta_2 = -\alpha \). We note that
Proposition 2 is valid when $\sigma > 0$. In our case this means that $\rho < 1/\alpha$ ($\rho \geq 0$). For the quantity $N$ from Eq. (4.4) we obtain

$$N^{-1} = \int_{b_1}^{b_2} dx \frac{(\alpha c - 1)^{2r-3} x^{2r-1}}{e^{2r+1}(\alpha x - 1)^{2r-1}} \exp \left[ -2\alpha c + \frac{2\alpha x}{(\alpha x - 1)(\alpha c - 1)} \right]$$  \hspace{1cm} (4.6)

and for the distribution $p^0(x)$ we obtain

$$p^0(x) = \int_{b_1}^{b_2} dy \frac{y^{2r-1}}{(\alpha y - 1)^{2r-1}} \exp \left[ -2\alpha c + \frac{2\alpha y}{(\alpha y - 1)(\alpha c - 1)} \right]$$  \hspace{1cm} (4.7)

Let us now discuss the case of more than one population. For this case we have to solve the system of stochastic differential equations

$$dX_i(t) = \left\{ F_i[X_1(t), \ldots, X_n(t)] + \frac{1}{2} G_i(X_1(t), \ldots, X_n(t)) \frac{\partial}{\partial x_i} [G_i(X_1(t), \ldots, X_n(t))] \right\} + G_i[X_1(t), \ldots, X_n(t)]dW_i(t), \; i = 1, \ldots, n \hspace{1cm} (4.8)$$

where $W_j(t)$ are independent Wiener processes and $F_i(\rho_1, \ldots, \rho_n)$ and $G_i(\rho_1, \ldots, \rho_n)$ are the same as in Eq. (3.29). The corresponding Fokker-Planck equation is ($G_{ij} = G_i\delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta-symbol)

$$\frac{\partial}{\partial t} p = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ p \left[ F_i(x_1, \ldots, x_n, t) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} G_{ij}(x_1, \ldots, x_n, t) \frac{\partial}{\partial x_i} [G_{ik}(x_1, \ldots, x_n, t)] \right] \right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left[ p G_{ij}(x_1, \ldots, x_n, t) G_{ji}(x_1, \ldots, x_n, t) \right]$$  \hspace{1cm} (4.9)

We are interested in stationary solutions $p_s$ of Eq. (4.9). In the general case such solutions can be obtained only numerically. Analytic solutions can be obtained when the conditions for detailed balance are satisfied \cite{17}. For the case of Eq. (4.9) the second of these conditions is the same as the second condition from (3.11). $G_i$ for the case of Stratonovich is the same as $G_i$ from the case of Ito. Then the second condition for existence of detailed balance is not satisfied. Then one can hope to obtain approximate analytic solutions for particular cases of Eq. (4.9) by the method of the adiabatic elimination discussed in the previous section.

Let us consider the following particular case of the Eqs. (2.3). Let $\alpha_{ij} = 0$ and $\alpha_{ijk} = 0$. In addition let $\eta_2 = 0$. For the case of two populations we obtain the following system of equations

$$d\rho_1 = [\rho_1^0(1 + r_{11}\rho_1 + r_{12}\rho_2) + \rho_1/2]dt + \rho_1dW_1$$

$$d\rho_2 = r_2^0\rho_2(1 + r_{21}\rho_1 + r_{22}\rho_2)dt$$  \hspace{1cm} (4.10)
Let again $\rho_2$ be the fast relaxing variable, i.e., $d\rho/dt$ tends to 0 very fast in the time. Then in the second equation of Eqs. (4.10) one can set $d\rho/dt = 0$ and then the resulting equation has solutions $\rho_2 = 0$ (extinction of the second population) or

$$\rho_2 = -\frac{1}{r_{22}} - \frac{r_{21}}{r_{22}} \rho_1$$

which corresponds to a "slaving" of the "fast" variable $\rho_2$ by the "slow" variable $\rho_1$. The substitution of Eq. (4.11) in the first equation of Eqs. (4.10) leads to the stochastic differential equation

$$d\rho_1 = \left[r_1^0 \left(r_{11} - \frac{r_{12}r_{21}}{r_{22}}\right) \rho_1^2 + \left[r_1^0 \left(1 - \frac{r_{12}}{r_{22}}\right) + 1/2\right] \rho_1\right] dt + \rho_1 dW_1$$

Eq. (4.12) can be treated by the methodology discussed above. In order to obtain an analytic result we have to assume

$$r_1^0 = \frac{3/2}{1 - r_{12}/r_{22}}$$

The application of the methodology connected to Proposition 2 leads to the distribution

$$p^0(\rho_1) = A \rho_1^{b_1} \exp(2\mu_2 \rho_1)$$

where

$$A = \frac{\mu_2^3}{\left(\frac{\mu_2 b_2^2}{2} - \frac{\mu_2 b_2}{2} + \frac{1}{4}\right) \exp(2\mu_2 b_2) - \left(\frac{\mu_2 b_1^2}{2} - \frac{\mu_2 b_1}{2} + \frac{1}{4}\right) \exp(2\mu_2 b_1)}$$

and

$$\mu_2 = \frac{3r_{11}(r_{22} - r_{11})}{2(r_{22} - r_{12})} < 0$$

5 Concluding remarks

We note that the environment can influence not only the birth rates of the interacting populations. The environment can influence also the interaction coefficients. Thus the discussed above model is the simplest of the three categories models of interacting populations: (i) Models accounting for the influence of environment on the growth rates (one model of this class is discussed in this paper); (ii) Models accounting for the influence of the environment on the coefficients of the interaction among the populations; and (iii) Models accounting for the influence of the environment both on the growth rates and the coefficients of interactions among the populations. The equations for all classes of the models are discussed elsewhere [15].

One result of our study above is that analytic solutions of the Fokker-Planck equations connected to the dynamic of interacting populations can be obtained for the case of one population. For two or more populations one can obtain approximate solutions in some particular cases. In the general case one has to solve the model equations numerically with the help of computers.
A Nonlinear dynamics and interacting populations

The nonlinear characteristics of the complex systems are intensively studied in different areas of science \[18, 19\] such for an example as the optics \[20, 21\], fluid mechanics \[22\], biology \[23\] or population dynamics \[24\] - \[27\], etc. \[28\] - \[44\]. Various mathematical methods connected to nonlinear time series analysis \[45\] and nonlinear PDEs \[46\] - \[55\] are used in the study of these systems. In this paper we discuss a class of models of the the dynamics of interacting populations. These models consist of equations that contain only time dependence of the population densities. What we add to the previous version of the models \[8\] - \[12\] is an influence of the environment on the growth rates of the interacting populations. This (random) influence has the following effect: instead of equations for the trajectories of the populations in the phase space of the population densities we shall write and solve equations for the probability density functions of the population densities.

B Multiplicative white noise. Stochastic integrals of Ito and Stratonovich kind

Let us consider a system that is influenced by noise. The current state of the system is \(X(t)\) and the intensity of the noise depends on \(X(t)\). Let the evolution of the system state be described by the stochastic differential equation

\[
\dot{X}(t) = f[X(t)] + \sigma[X(t)]\zeta(t), \quad X(0) = X_0 \tag{B.1}
\]

If \(\sigma[X(t)] = 0\) then Eq.\,(B.1) is deterministic one. If \(\sigma[X(t)] = \text{const}\) and if \(\zeta(t)\) is Gaussian white noise then Eq.\,(B.1) is equation of Langevin kind. As a more general case \(\sigma[X(t)]\) is not a constant and if \(\zeta(t)\) is a Gaussian white noise then Eq.\,(B.1) describes the case of multiplicative Gaussian white noise. Below we shall discuss several features of the solution of Eq.\,(B.1) for the case of presence of the multiplicative Gaussian white noise.

The formal integration of eq.(B.1) leads to the integral equation \[56, 57\]

\[
X_t = X_0 + \int_0^t d\tau \ f(X_\tau) + \int_0^t dW_\tau \ \sigma(X_\tau), \tag{B.2}
\]

where \(W_\tau\) is a Wiener process \((dW_\tau = \zeta(\tau)d\tau)\). The second integral from Eq.\,(B.2), namely \(\int_0^t dW_\tau \ \sigma(X_\tau)\), is a stochastic integral. There are two interpretations of this integral: (a) as integral of Ito kind; and (b) as integral of Stratonovich kind. It depends on the characteristics of the modelled system which kind of integral has to be used.

B.1 Interpretation of the stochastic integral as an integral of Ito kind

Let us interpret the above stochastic integral as

\[
I_t = \int_0^t dW_\tau \sigma(X_\tau) = \lim_{\delta_n \downarrow 0} I_t^{(n)} \tag{B.3}
\]
In Eq. (B.3) qa lim,\( n \rightarrow 0 \) is a quadratic average limit. This limit has to be understood as tendency to 0 of the expectation \( E \mid I_t - I_t^{(n)} \mid^2 \):

\[
\lim_{\delta_n \downarrow 0} E \mid I_t - I_t^{(n)} \mid^2 = 0,
\]

(B.4)

where

\[
I_t^{(n)} = \sum_{i=0}^{n-1} \sigma(X_{t_i})(W_{t_{i+1}} - W_{t_i}),
\]

(B.5)

and \( 0 = t_0 < t_1 < \cdots < t_n \); \( \delta_n = \max_i(t_{i+1} - t_i) \). The integral of kind (B.3) is called integral of Ito kind. Then the equation (B.2) can be written in the following differential form

\[
dX_t = f(X_t)dt + \sigma(X_t)dW_t
\]

(B.6)

The initial condition is \( X_0 = X(0) \) and \( X_0 \) is a random variable which probability density function is independent on the Wiener process \( W_t \).

### B.2 Interpretation of the stochastic integral as an integral of Stratonovich kind

The model system can be of such kind that the stochastic integral in Eq. (B.2) is not an integral of Ito kind. This situation arises when for an example the noise process has a finite correlation time. Such processes are present frequently in the real systems and the corresponding stochastic integral is integral of Stratonovich kind. The interpretation of the stochastic integral from Eq. (B.2) for the last case is as follows:

\[
S_t = \int_0^t dW(\tau) \circ \sigma(X_\tau) = qalim_{\delta_n \downarrow 0} \sum_{i=0}^{n-1} \sigma\left[\frac{1}{2}(X_{t_{i+1}} + X_{t_i})\right](W_{t_{i+1}} - W_{t_i}).
\]

(B.7)

Then Eq. (B.2) can be written in the following differential form

\[
dX_t = f(X_t)dt + \sigma(X_t) \circ dW_t.
\]

(B.8)

Let \( \sigma \) be continuous differentiable function. Then a relationship exists between the integrals of Ito and Stratonovich kind. The relationship is as follows:

\[
\int_0^t dW_\tau \circ \sigma(X_\tau) = \int_0^t dW_\tau \sigma(X_\tau) + \frac{1}{2} \int_0^t d\tau \sigma'(X_\tau)\sigma(X_\tau)
\]

(B.9)

We note that for the case of additive white noise \( \sigma = \text{const} \). Then \( \sigma' = 0 \) and the Ito integral coincides with the Stratonovich integral.

We obtain on the basis of Eq. (B.9) that the Stratonovich differential equation (B.8) is equivalent to the following stochastic differential equation of Ito kind:

\[
dX_t = [f(X_t) + \frac{1}{2} \sigma'(X_t)\sigma(X_t)]dt + \sigma(X_t)dW_t
\]

(B.10)

In this paper we shall assume that \( \sigma \) is continuously differentiable.
C Probability density function for the case of multiplicative white noise and Ito kind of stochastic differential equation

In this case the solution $X_t$ of the Eq.(B.6) is a Markov process and the p.d.f. $p(x,t)$ for the values of $X$ (if the p.d.f. exists) is given by the Fokker-Planck equation [17]:

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}[p(x,t)f(x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2}[p(x,t)\sigma^2(x)],$$  \hspace{1cm} (C.1)

with initial condition $p(x,0) = p_0(x)$. Let us now discuss the behavior of the solution $p(x,t)$ of Eq.(C.1) at $t \to \infty$. We shall formulate an Observation and for this we need the notion of natural boundary point.

Let the interval of possible values of the diffusion process $X$ that is solution of Eq.(B.6) be within the interval $[b_1, b_2]$. If $f$ and $\sigma$ are continuously differentiable in this interval then the solution of Eq.(B.6) exists till the time point when one of the boundary points $b_{1,2}$ is reached. After that time the behavior of the system depends on the boundary conditions. When the boundary point $b_1$ can’t be reached for finite time it is called inaccessible (the same is the situation with the point $b_2$). The inaccessible boundary point $b_1$ is called natural when the solution $X(x_0)$ that starts from $x_0 \in (b_1,c), c < b_2$ accesses first the point $c$ with probability 1. This means that for $t \to \infty$ the point $b_1$ almost surely will be not accessed.

What follows is [58]

Observation 1:

Let $b_1$ and $b_2$ be natural boundary points ($-\infty \leq b_1 < b_2 \leq \infty$). Let in addition $\sigma(x) > 0$ in $(b_1, b_2)$. Then the diffusion process $X_i$ that is solution of the stochastic differential equation Eq.(B.6) has unique invariant distribution with p.d.f.

$$p^0(x) = \frac{N}{\sigma^2(x)} \exp \left( \int_c^x dy \frac{2f(y)}{\sigma^2(y)} \right), \hspace{1cm} \forall x \in (b_1, b_2) \hspace{1cm} (C.2)$$

if the quantity

$$N^{-1} = \int_{b_1}^{b_2} dx \frac{1}{\sigma^2(x)} \exp \left( \int_c^x dy \frac{2f(y)}{\sigma^2(y)} \right), \hspace{1cm} b_1 < c < b_2 \hspace{1cm} (C.3)$$

has finite value. In addition each time-dependent solution $p(x,t)$ of the Fokker-Planck equation (C.1) in $(b_1, b_2)$ satisfies

$$\lim_{t \to \infty} p(x,t) = p^0(x) \hspace{1cm} (C.4)$$

Let us now consider the system of coupled stochastic equations

$$\dot{X}_i(t) = f_i[X_1(t), \ldots, X_n(t)] + \sum_{j=1}^m g_{ij}[X_1(t), \ldots, X_n(t)]\zeta_j(t), \hspace{1cm} i = 1, \ldots, n \hspace{1cm} (C.5)$$
where \( \zeta_j(t) \) are independent white Gaussian noises. If the arising in the process of solution of Eq. (C.5) stochastic integrals are of Ito kind then one has to solve the system of coupled stochastic differential equations

\[
dX_i(t) = f_i[X_1(t), \ldots, X_n(t)] + \sum_{j=1}^{m} g_{ij}[X_1(t), \ldots, X_n(t)]dW_j(t), \quad i = 1, \ldots, n
\]

(C.6)

where \( W_j(t) \) are independent Wiener processes. For the conditional probability density \( p = p(x_1, \ldots, x_n, t \mid x_{01}, \ldots, x_{0n}, t) \) of \( \mathbf{X} = (X_1, \ldots, X_n) \) one obtains the Fokker-Planck equation

\[
\frac{\partial}{\partial t} p = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [pf_i(x_1, \ldots, x_n, t)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [pg_{ij}(x_1, \ldots, x_n, t)g_{ji}(x_1, \ldots, x_n, t)]
\]

(C.7)

D Probability density function for the case of multiplicative white noise and Stratonovich kind of stochastic differential equation

In this case the solution \( X_t \) of the Eq. (B.10) is a Markov process and the p.d.f. \( p(x, t) \) for the values of \( X \) (if the p.d.f. exists) is given by the Fokker-Planck equation \( [50, 57] \):

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} [p(x, t)f^*(x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [p(x, t)\sigma^2(x)],
\]

(D.1)

where

\[
f^*(X_t) = f(X_t) + \frac{1}{2} \sigma'(X_t)\sigma(X_t)
\]

(D.2)

with initial condition \( p(x, 0) = p_0(x) \). The behavior of the solution \( p(x, t) \) of Eq. (C.11) at \( t \to \infty \) is as follows. Let the interval of possible values of the diffusion process \( X \) that is solution of Eq. (B.6) be within the interval \([b_1, b_2] \). Let \( f^* \) and \( \sigma \) are continuously differentiable in this interval then the solution of Eq. (B.6) exists till the time point when one of the boundary points \( b_{1,2} \) is accessed. Then we can formulate \([58]\).

Observation 2:

Let \( b_1 \) and \( b_2 \) be natural boundary points \((-\infty \leq b_1 < b_2 \leq \infty)\). Let in addition \( \sigma(x) > 0 \) in \((b_1, b_2)\). Then the diffusion process \( X_t \) that is solution of the stochastic differential equation Eq. (B.6) has unique invariant distribution with p.d.f.

\[
p^0(x) = \frac{\mathcal{N}}{\sigma^2(x)} \exp \left( \int_c^x dy \frac{2f^*(y)}{\sigma^2(y)} \right), \quad \forall x \in (b_1, b_2)
\]

(D.3)

if the quantity

\[
\mathcal{N}^{-1} = \int_{b_1}^{b_2} dx \frac{1}{\sigma^2(x)} \exp \left( \int_c^x dy \frac{2f^*(y)}{\sigma^2(y)} \right), \quad b_1 < c < b_2
\]

(D.4)
has finite value. In addition each time-dependent solution \( p(x,t) \) of the Fokker-Planck equation (D.1) in \((b_1,b_2)\) satisfies

\[
\lim_{t \to \infty} p(x,t) = p^0(x)
\]  

Let us now consider the system of coupled stochastic equations

\[
\dot{X}_i(t) = f_i[X_1(t), \ldots, X_n(t)] + \sum_{j=1}^{m} g_{ij}[X_1(t), \ldots, X_n(t)]\zeta_j(t), \ i = 1, \ldots, n
\]  

where \(\zeta_j(t)\) are independent white Gaussian noises. If the arising in the process of solution of Eq.(C.5) stochastic integrals are of Stratonovich kind then one has to solve the system of coupled stochastic differential equations

\[
dX_i(t) = \left\{ f_i[X_1(t), \ldots, X_n(t)] + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} g_{kj}(X_1(t), \ldots, X_n(t)) \frac{\partial}{\partial x_i} [g_{jk}(X_1(t), \ldots, X_n(t))] \right\} + \sum_{j=1}^{m} g_{ij}[X_1(t), \ldots, X_n(t)]dW_j(t), \ i = 1, \ldots, n
\]  

where \(W_j(t)\) are independent Wiener processes. For the conditional probability density \( p = p(x_1, \ldots, x_n, t \mid x_{01}, \ldots, x_{0n}, t) \) of \(X = (X_1, \ldots, X_n)\) one has the Fokker-Planck equation

\[
\frac{\partial}{\partial t} p = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ p \left[ f_i(x_1, \ldots, x_n, t) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} g_{jk}(x_1, \ldots, x_n, t) \frac{\partial}{\partial x_j} [g_{jk}(x_1, \ldots, x_n, t)] \right] \right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} [p g_{ij}(x_1, \ldots, x_n, t) g_{ji}(x_1, \ldots, x_n, t)]
\]  

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