Polarization of the Fulling–Rindler vacuum by a uniformly accelerated mirror

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Abstract
Positive-frequency Wightman function and vacuum expectation values of the energy–momentum tensor are computed for a massive scalar field with general curvature coupling parameter, satisfying the Robin boundary condition on a uniformly accelerated infinite plate. Both the regions of the right Rindler wedge, (i) on the right (RR region) and (ii) on the left (RL region) of the plate are investigated. For the case (ii) the electromagnetic field is considered as well. The mode summation method is used in combination with a variant of the generalized Abel–Plana formula. This allows us to present the expectation values in the form of a sum of the purely Rindler and boundary parts. Near the plate surface, the vacuum energy–momentum tensor is dominated by the boundary term. At large distances from the plate and near the Rindler horizon, the main contribution comes from the purely Rindler part. In the RL region, the vacuum energy density of the electromagnetic field is negative near the horizon and is positive near the plate.

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1. Introduction
The influence of boundaries on the vacuum state of a quantum field leads to interesting physical consequences. A well-known example is the Casimir effect (see [1–3], and references therein), when the modification of the zero-point fluctuations spectrum by the presence of boundaries induces vacuum forces acting on these boundaries. Another interesting effect is the creation of particles from the vacuum by moving boundaries [4]. In this paper, we will study the scalar vacuum polarization brought about by the presence of an infinite plane boundary moving with uniform acceleration through the Fulling–Rindler vacuum. This problem for the conformally coupled Dirichlet and Neumann massless scalar and electromagnetic fields in four-dimensional spacetime was considered by Candelas and Deutsch [5]. Here we will investigate
the vacuum expectation values of the energy–momentum tensor for the massive scalar field with general curvature coupling, satisfying the Robin boundary condition on the infinite plane in an arbitrary number of spacetime dimensions. The dimensional dependence of physical quantities is of considerable interest in the Casimir effect and is investigated for various geometries of boundaries and boundary conditions (see [3, 6–10], and references therein). The Robin boundary conditions are an extension of those imposed on perfectly conducting boundaries and may, in some geometries, be useful in depicting the finite penetration of the field into the boundary with the ‘skin-depth’ parameter related to the Robin coefficient [11, 12]. It is interesting to note that the quantum scalar field satisfying the Robin condition on the boundary of a cavity violates the Bekenstein entropy-to-energy bound near certain points in the space of the parameter defining the boundary condition [13]. This type of condition also appears in the considerations of the vacuum effects for a confined charged scalar field in external fields [14] and in quantum gravity [15–17]. Mixed boundary conditions naturally arise for scalar and fermion bulk fields in the Randall–Sundrum model [18]. In this model, the bulk geometry is a slice of anti-de Sitter space and the corresponding Robin coefficient is related to the curvature scale of this space. Unlike [5], where only the region of the right Rindler wedge to the right of the barrier (in the following we will denote this region as RR) is considered, we investigate vacuum energy density and stresses in both regions, including the one between the barrier and Rindler horizon (region RL in the following consideration). Our calculation is based on the summation formula derived in appendix A on the basis of the generalized Abel–Plana formula [19, 20] (for the application of the Abel–Plana formula to the Casimir effect calculations, see also [1, 21]). This allows us to extract explicitly the part due to the unbounded Rindler spacetime and present the boundary part in terms of strongly convergent integrals.

This paper is organized as follows. In the following section, we consider the vacuum expectation values for the energy–momentum tensor of the scalar field in the region to the right of the uniformly accelerated infinite plane. By using the summation formula derived in appendix A, these quantities are presented in the form of a sum containing two parts. The first one corresponds to the energy–momentum tensor for the Rindler wedge without boundaries, and is investigated in section 3. The second term presents the contribution brought about by the presence of the boundary and is considered in section 4. The corresponding boundary terms for the region between the barrier and Rindler horizon are evaluated in section 5 for scalar and electromagnetic fields. In appendix B, we show that the direct evaluation of the energy–momentum tensor in $D = 1$ gives the same result as the analytic continuation from higher dimensions. And finally, in appendix C, we prove the identity used in section 3 to see the equivalence of our expressions for the expectation values of the energy–momentum tensor in the case of the Fulling–Rindler vacuum without boundaries to the results previously investigated in the literature for the conformally and minimally coupled scalars.

2. Vacuum expectation values and the Wightman function

Consider a real scalar field $\varphi(x)$ with curvature coupling parameter $\zeta$ in a $D$-dimensional space satisfying the Robin boundary condition

$$(A + BN_i \nabla_i)\varphi(x) = 0 \quad (2.1)$$

on a plane boundary uniformly accelerated normal to itself. Here $A$ and $B$ are constants, $n^i$ is the unit normal to the plane and $\nabla_i$ is the covariant derivative operator. The Dirichlet and Neumann boundary conditions are obtained from (2.1) as special cases. Of course, all results in the following will depend on the ratio $A/B$ only. However, to keep the transition to the
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Dirichlet and Neumann cases transparent we will use the form (2.1). The field equation has the form
\[ (\nabla_i \nabla^i + m^2 + \zeta R)\phi(x) = 0, \]
where \( R \) is the scalar curvature for the background spacetime. The values of the curvature coupling \( \zeta = 0 \) and \( \zeta = \zeta_c \equiv D^{-1}4D \) correspond to the minimal and conformal couplings, respectively. In the case of a flat background spacetime, the corresponding metric energy–momentum tensor (EMT) is given by (see, e.g., [4])
\[ T_{ik} = (1 - 2\zeta)\partial_i\phi\partial_k\phi + \left(2\zeta - 1/2\right)g_{ik}\nabla^l\phi\nabla_l\phi + (1/2 - 2\zeta)m^2g_{ik}\phi^2. \]
By using the field equation it can be seen that expression (2.3) can also be presented in the form
\[ T_{ik} = \partial_i\phi\partial_k\phi + \left[(\zeta - 1/4)g_{ik}\nabla^l\nabla_l - \zeta \nabla_i\nabla_k\right]\phi^2, \]
and the corresponding trace is equal to
\[ T_i^i = D(\zeta - \zeta_c)\nabla^i\nabla_i\phi^2 + m^2\phi^2. \]
By virtue of equation (2.4) for the vacuum expectation values (VEVs) of the EMT we have
\[ \langle 0|T_{ik}(x)|0 \rangle = \lim_{x' \to x} \nabla_i\nabla_k G_{+}(x, x') + \left[(\zeta - 1/4)g_{ik}\nabla^l\nabla_l - \zeta \nabla_i\nabla_k\right]\langle 0|\phi^2(x)|0 \rangle, \]
where \( |0 \rangle \) is the amplitude for the vacuum state and we introduced the positive-frequency Wightman function
\[ G^+(x, x') = \langle 0|\phi(x)\phi(x')|0 \rangle. \]
In equation (2.6), instead of this function one can choose any other bilinear function of fields such as the Hadamard function, Feynman’s Green function, etc. The regularized vacuum EMT does not depend on the specific choice. The reason for our choice of the Wightman function is that this function also determines the response of the particle detectors in a given state of motion. By expanding the field operator over eigenfunctions and using the commutation relations, one can see that
\[ G^+(x, x') = \sum_{\alpha} \phi_\alpha(x)\phi^{\alpha}_*(x'), \]
where \( \alpha \) denotes a set of quantum numbers and \( \{\phi_\alpha(x)\} \) is a complete set of solutions to the field equation (2.2) satisfying boundary condition (2.1).
Below we will assume that the plate is situated at the right Rindler wedge, \( x_1 > |t| \).

First, we will consider the region to the right of the boundary, \( x \geq a \), region RR, where \( a^{-1} \) is the proper acceleration of the barrier. In Rindler coordinates, boundary condition (2.1) for a uniformly accelerated mirror takes the form
\[ \left(A + B\frac{\partial}{\partial \xi}\right)\phi = 0, \quad \xi = a. \]
The problem symmetry allows us to write the solutions to field equation (2.2) as
\[ \psi(x) = C \phi(\xi) e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t}. \] (2.12)
Substituting this into equation (2.2) we see that the function \( \phi(\xi) \) satisfies the equation
\[ \frac{\xi}{\xi} \frac{d}{d\xi} \left( \frac{\xi}{\xi} \frac{d\phi}{d\xi} \right) + (\omega^2 - \xi^2 \lambda^2) \phi(\xi) = 0, \quad \lambda = \sqrt{k^2 + m^2}. \] (2.13)
The corresponding linearly-independent solutions are the Bessel modified functions \( I_{\nu}(\lambda \xi) \) and \( K_{\nu}(\lambda \xi) \) with the imaginary order. It can be seen that for any two solutions to equation (2.13), \( \phi_{\omega}^{(1)}(\xi) \) and \( \phi_{\omega}^{(2)}(\xi) \), the following integration formula takes place:
\[ \int \frac{d\xi}{\xi} \phi_{\omega}^{(1)}(\xi) \phi_{\omega}^{(2)}(\xi) = \frac{\xi}{\omega^2 - \xi^2} \left[ \frac{d\phi_{\omega}^{(1)}(\xi)}{d\xi} \frac{d\phi_{\omega}^{(2)}(\xi)}{d\xi} - \phi_{\omega}^{(1)}(\xi) \frac{d\phi_{\omega}^{(2)}(\xi)}{d\xi} - \phi_{\omega}^{(2)}(\xi) \frac{d\phi_{\omega}^{(1)}(\xi)}{d\xi} \right]. \] (2.14)

As we will see below, the normalization integrals for the eigenfunctions will have this form.

For the region \( \xi > a \), a complete set of solutions that are of positive frequency with respect to \( \partial / \partial \tau \) and bounded as \( \xi \to \infty \) is
\[ \psi_{\omega}(x) = C K_{\omega}(\lambda \xi) e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t}, \quad \alpha = (\omega, \mathbf{k}). \] (2.15)

From boundary condition (2.11), we find that the possible values for \( \omega \) are the roots of the equation
\[ AK_{\omega}(\lambda a) + B_{\omega} K'_{\omega}(\lambda a) = 0, \] (2.16)
where the prime denotes differentiation with respect to the argument. This equation has an infinite number of zeros. In the following, we will assume the values of \( A/B \) for which all these zeros are real (they include the important special cases of Dirichlet and Neumann boundary conditions). We will denote them by \( \omega = \omega_n = \omega_n(k), n = 1, 2, \ldots \) arranged in ascending order: \( \omega_n < \omega_{n+1} \). The coefficient \( C \) in (2.15) is determined by the normalization condition
\[ (\psi_{\omega}, \psi_{\omega'}) = -i \int dx \int \frac{d\xi}{\xi} \psi_{\omega}^* \frac{\partial}{\partial \xi} \psi_{\omega'} = \delta_{\omega\omega'}, \] (2.17)
with respect to the standard Klein–Gordon inner product, and the \( \xi \)-integration goes over the region \( (a, \infty) \). Using integration formula (2.14) with \( \phi_{\omega}^{(1)} = \phi_{\omega}^{(2)} = K_{\omega}(\lambda \xi) \) and Wronskian \( W[K_{\nu}(\xi), I_{\nu}(\xi)] = 1/\xi \), from (2.17) one finds
\[ C^2 = \frac{\pi}{(2\pi)^{D-1}} \int_a^\infty \frac{d\xi}{\xi} K^2_{\omega}(\lambda \xi) \left[ \int_a^\infty \frac{d\xi}{\xi} K^2_{\omega}(\lambda \xi) \right]^{-1} \frac{1}{(2\pi)^{D-1}} \frac{\tilde{f}_{\omega}(\lambda a)}{\delta_{\omega\omega'}}, \] (2.18)
where for a given function \( f(\zeta) \) we use the notation
\[ \tilde{f}(\zeta) = Af(\zeta) + B f'(\zeta) = 0, \quad b = B/a. \] (2.19)

From the first equality in equation (2.18) it follows that \( C^2 \) is real and positive. Now substituting the eigenfunctions (2.15) into (2.8) for the Wightman function we obtain
\[ G^+(x, x') = \int \frac{dk}{(2\pi)^{D-1}} e^{i k(x - x')} \sum_{n=1}^\infty \frac{\tilde{f}_{\omega}(\lambda a)}{\delta_{\omega\omega'} \delta_{\omega n}(\lambda a) K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi') e^{-i \omega(t - t')}} |_{\omega n=\omega_n}. \] (2.20)

For the further evolution of VEVs (2.20) we can apply to the sum over \( n \) summation formula (A.5). As a function \( F(z) \) in this formula let us choose
\[ F(z) = K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi') e^{-i \xi(\tau - \tau')} \] (2.21)
Using the asymptotic formulae for the Bessel modified functions it can easily be seen that condition (A.3) is satisfied if \( a^2 e^{i \xi(\tau - \tau')} < |\xi|^2 |. \) Note that this condition is satisfied in the
coincidence limit $\tau = \tau'$ for the points in the region under consideration, $\xi, \xi' > a$. With $F(z)$ from (2.21), the contribution corresponding to the integral term on the left of formula (A.5) is the Wightman function for the Fulling–Rindler vacuum $|0_R\rangle$ without boundaries:

$$\langle \xi, \xi' \rangle = \langle 0_R | \psi(x) \psi(x') | 0_R \rangle = \frac{1}{\pi^2} \int \frac{dk}{(2\pi)^{D-1}} e^{ik(x-x')} \int_{0}^{\infty} d\omega \sinh(\pi \omega) e^{-i\omega(\tau - \tau')} K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi').$$  

(2.22)

To see this, note that for the right Rindler wedge, the eigenfunctions to the field equation (2.2) are of the form (see, for instance, [22–25])

$$\varphi_{\omega k} = \frac{\sqrt{\sinh \pi \omega}}{(2\pi)^{D-1/2} \pi} e^{ikx-i\omega \tau} K_{\omega}(\lambda \xi).$$  

(2.23)

Substituting these modes into formula (2.8) we obtain equation (2.22). Hence, taking into account this and applying summation formula (A.5) to equation (2.20), we obtain

$$G^*(x, x') = G^*_M(x, x') + \langle \psi(x) \psi(x') \rangle^{(b)},$$  

(2.24)

where the second term on the right is induced by the existence of the barrier

$$\langle \psi(x) \psi(x') \rangle^{(b)} = -\frac{1}{\pi} \int \frac{dk}{(2\pi)^{D-1}} e^{ik(x-x')} \int_{0}^{\infty} d\omega I_{\omega}(\lambda a) K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi') \cosh(\omega(\tau - \tau')).$$  

(2.25)

and is finite for $ \xi > a$. All divergences in the coincidence limit are contained in the first term corresponding to the Fulling–Rindler vacuum without boundaries. The VEVs for the EMT are obtained by substituting equation (2.24) into equation (2.6). Similar to equation (2.24), these VEVs are also in the form of the sum containing purely Rindler and boundary parts. First, we will concentrate on the first term.

### 3. Vacuum EMT in the Rindler wedge without boundaries

The Wightman function for the right Rindler wedge given by formula (2.22) is divergent in the coincidence limit $x' \to x$. This leads to the divergences in the VEV for the EMT. To extract these divergences we can subtract from equation (2.22) the corresponding Wightman function for the Minkowski vacuum $|0_M\rangle$. By using mode sum formula (2.8) and the standard Minkowskian eigenfunctions it can easily be seen that the latter may be presented in the form

$$G^*_M(x, x') = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(x-x')} K_0(\lambda \sqrt{\xi^2 + \xi'^2 - 2\xi \xi' \cosh(\tau - \tau'))},$$  

(3.1)

where $\xi^2 + \xi'^2 - 2\xi \xi' \cosh(\tau - \tau') = (x_1 - x_1')^2 - (t - t')^2$. Using the integration formula [26]

$$K_0(\lambda \sqrt{\xi^2 + \xi'^2 + 2\xi \xi' \cosh b}) = \frac{2}{\pi} \int_{0}^{\infty} d\omega \cosh(b \omega) K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi'),$$  

(3.2)

equation (3.1) can be presented in the form convenient for subtraction:

$$G^*_M(x, x') = \frac{1}{\pi^2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(x-x')} \int_{0}^{\infty} d\omega \cosh(\omega(\tau - \tau')) K_{\omega}(\lambda \xi) K_{\omega}(\lambda \xi').$$  

(3.3)
As a result, for the difference between the Rindler and Minkowskian Wightman functions one obtains
\[ G^+_R(x, x') - G^+_M(x, x') = -\frac{1}{\pi^2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(x-x')} \times \int_0^\infty d\omega e^{-\omega \xi} \cos[\omega(\tau - \tau')] |K_\omega(\lambda_\xi)K_\omega(\lambda_\xi^*)|. \] (3.4)

After integrating over directions of the vector \( \mathbf{k} \), in particular, for the difference between VEVs of the field square it follows from here
\[ \langle \psi^2(x) \rangle_{(R)} = \langle 0_R | \psi^2(x) | 0_R \rangle - \langle 0_M | \psi^2(x) | 0_M \rangle = -\frac{1}{2^{D-2} \pi^{(D+3)/2} \Gamma \left( \frac{D-1}{2} \right)} \int_0^\infty dk \ k^{D-2} \int_0^\infty d\omega e^{-\omega \xi} K_\omega^2(\lambda_\xi). \] (3.5)

Using formula (2.6), for the corresponding difference between the VEVs of the EMT we find
\[ \langle T^{(R)}_{\lambda} \rangle_{sub} = \langle 0_R | T^{(R)}_{\lambda} | 0_R \rangle - \langle 0_M | T^{(R)}_{\lambda} | 0_M \rangle = -\frac{\delta^0_{\lambda}}{2^{D-2} \pi^{(D+3)/2} \Gamma \left( \frac{D-1}{2} \right)} \int_0^\infty dk \ k^{D-2} \lambda^2 \int_0^\infty d\omega e^{-\omega \xi} f^{(i)}(\omega) |K_\omega(\lambda_\xi)|, \] (3.6)

where we have introduced the following notation:
\[ f^{(0)}(g(z)) = \left( \frac{1}{2} - 2\xi \right) \left[ \frac{\partial g(z)}{\partial z} \right]^2 + \left[ \frac{\xi}{2} \right] \left( \frac{\partial g(z)}{\partial z} \right)^2 + \left[ 1 - 2\xi + \left( \frac{1}{2} + 2\xi \right) \frac{\omega^2}{\xi^2} \right] |g(z)|^2, \]
\[ f^{(1)}(g(z)) = -\frac{1}{2} \left[ \frac{\partial g(z)}{\partial z} \right]^2 - \left[ \frac{\xi}{2} \right] \left( \frac{\partial g(z)}{\partial z} \right)^2 + \left[ 1 - \frac{\omega^2}{\xi^2} \right] |g(z)|^2, \]
\[ f^{(i)}(g(z)) = \left( \frac{1}{2} - 2\xi \right) \left[ \frac{\partial g(z)}{\partial z} \right]^2 + \left[ 1 - \frac{\omega^2}{\xi^2} \right] |g(z)|^2 \quad (D = 1)_{\lambda}^2 \] \[ \left( \frac{D-1}{2} \right)^2 |g(z)|^2, \] \[ i = 2, 3, \ldots, D, \]

and indices 0 and 1 correspond to the coordinates \( \tau \) and \( \xi \).

We consider first the massless limit. In this limit, interchanging the orders of integration, the integrals over \( k \) can be expressed in terms of the gamma function (see, for example, [26]). The standard formulae for the latter [27] allow us to write the purely Rindler parts (3.5) and (3.6) in the form
\[ \langle \psi^2 \rangle_{sub} = -\frac{\xi^{D-1}}{2^{D-1} \pi^{D/2} \Gamma(D/2)} \int_0^\infty \frac{d\omega}{e^{2\tau\omega} + (-1)^D} \prod_{l=1}^{l_m} \left[ \left( \frac{D - 1 - 2l}{2\omega} \right)^2 + 1 \right], \] (3.8)
\[ \langle T^{(R)}_{\lambda} \rangle_{sub} = -\frac{\delta^0_{\lambda} \xi^{D-1}}{2^{D-1} \pi^{D/2} \Gamma(D/2)} \int_0^\infty \frac{d\omega}{e^{2\tau\omega} + (-1)^D} f_0^{(i)}(\omega) \prod_{l=1}^{l_m} \left[ \left( \frac{D - 1 - 2l}{2\omega} \right)^2 + 1 \right], \] (3.9)

where \( l_m = D/2 - 1 \) for even \( D > 2 \) and \( l_m = (D - 1)/2 \) for odd \( D > 1 \), and the value for the product over \( l \) is equal to 1 for \( D = 1, 2, 3 \). In equation (3.9), we have introduced notation
\[ f_0^{(0)}(\omega) = -D f_0^{(1)}(\omega) = 1 + D(D - 1)(\xi_c - \xi)/\omega^2, \]
\[ f_0^{(i)}(\omega) = -1/D + (D - 1)^2(\xi_c - \xi)/\omega^2, \quad i = 2, 3, \ldots, D. \] (3.10)
Formulae (3.8) and (3.9) illustrate the thermal properties of the Minkowski vacuum relative to the Rindler observer [22–24]. As we see from equation (3.9), though thermal with temperature \( T = (2\pi\xi)^{-1} \), the spectrum of the vacuum EMT, in general, has a non-Planckian form: the density-of-states factor is not proportional to \( \omega^{D-1} \). It is interesting to note that for even values of the space dimension \( D \), this thermal distribution for the scalar field is of Fermi–Dirac type (see also [28–30]).

Note that for the case of a two-dimensional spacetime we cannot directly put \( D = 1 \) into the integral (3.9). Now, though the factor \( D - 1 \) in the second term on the right of expression (3.10) for \( J_0^{(0)}(\omega) \) is equal to zero, the corresponding \( \omega \)-integral diverges and this term gives finite contribution to the energy density. Keeping \( D > 1 \), the \( \omega \)-integral in equation (3.9) can be expressed in terms of the Riemann zeta function \( \zeta_R(x) \), and in the limit \( D \to 1 \) one obtains

\[
\langle T_0^{(R)} \rangle_{\text{sub}} = -\langle T_1^{(R)} \rangle_{\text{sub}} = -\frac{1}{\pi\xi^2} \left( \frac{1}{24} + \zeta \right) \Gamma(D)\zeta_R(D - 1) = \frac{1}{2\pi\xi^2} \left( \zeta - \frac{1}{12} \right),
\]

(3.11)

where \( \zeta_R(0) = -1/2 \).

The density-of-states factor is Planckian for the conformally coupled scalar and for \( D = 1, 2, 3 \):

\[
\langle T_0^{(R)} \rangle_{\text{sub}} = -\langle T_1^{(R)} \rangle_{\text{sub}} = \frac{\xi^{-D-1}}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty \frac{\omega^D d\omega}{e^{\omega\xi} + (-1)^D \text{diag} \left( 1, -\frac{1}{D}, \ldots, -\frac{1}{D} \right)},
\]

(3.12)

This result for the \( D = 3 \) case was obtained by Candelas and Deutsch [5]. Expression (3.12) corresponds to the absence of thermal distribution from the vacuum. This means that the Minkowski vacuum corresponds to a thermal state with respect to uniformly-accelerated observers [24].

We now treat the case of a massive scalar field. First of all, let us present the difference (3.4) between Rindler and Minkowski Wightman functions in another alternative form. For this, we will follow the procedure already used in [25] for the Green function. Substituting [31]

\[
K_\gamma(\lambda_\xi)K_\gamma(\lambda_\xi') = \frac{1}{2} \int_{-\infty}^{+\infty} dy e^{i\eta y} K_0(\lambda_\gamma),
\]

(3.13)

where \( \gamma^2 = \xi^2 + \xi'^2 + 2\xi\xi' \cosh \gamma \), into equation (3.4) and integrating over \( \omega \) and \( k \) on the basis of standard formulae [26], for the difference between Wightman functions one obtains

\[
G_R^0(x, x') - G_M^0(x, x') = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dy}{\pi^2 + y^2} \left( \frac{m}{2\pi\gamma} \right)^{(D-1)/2} K_{(D-1)/2}(m\gamma),
\]

(3.14)

where \( \gamma^2 = \xi^2 + \xi'^2 + 2\xi\xi' \cosh(y - \tau + \tau') + |x - x'|^2 \). A similar relation for Feynman’s Green function is considered in [25]. Noting that \( \gamma_1 \) is just the geodesic separation of the points \( x = (\tau, \xi, x) \) and \( x'' = (\tau' + y, -\xi', x) \) and taking into account the standard expression for the Minkowskian Wightman function, relation (3.14) can be written as

\[
G_R^+(x, x') = G_M^+(x, x') - \int_{-\infty}^{+\infty} \frac{dy}{\pi^2 + y^2} G_M^+(x, x'').
\]

(3.15)

Following [25], we can interpret the second term on the right of this formula as coming from an image charge density \( -1/(\pi^2 + y^2) \) distributed on a line \( (-\xi', x) \) in the left Rindler wedge.

In the coincidence limit, equation (3.14) yields

\[
\langle \psi^2(x) \rangle_{\text{sub}} = -\frac{2m^{D-1}}{(2\pi)^{(D+1)/2}} \int_0^{+\infty} \frac{dy}{\pi^2 + y^2} K_{(D-1)/2}(z) z^{(D-1)/2}, \quad z = 2m\xi \cosh(y/2).
\]

(3.16)
This quantity is a monotone increasing negative function on $\xi$. Substituting equations (3.14) and (3.16) into formula (2.6) for the difference between the corresponding VEVs of the EMT we obtain (no summation over $i$)

$$\left\langle T^{(R)}_{0i}\right\rangle_{\text{sub}} = -\frac{2mD+1}{(2\pi)^{1+D/2}} \int_0^{\infty} \frac{dy}{y^2 + \pi^2} \left[ \frac{K_{(D-1)/2}(z)}{\eta^{(D-1)/2}} + D \frac{K_{(D+1)/2}(z)}{\eta^{(D+1)/2}} \right] \left( 1 - 4\xi \cosh^2(y/2) \right),$$

(3.17)

$$\left\langle T^{(R)}_{1i}\right\rangle_{\text{sub}} = -\frac{2mD+1}{(2\pi)^{1+D/2}} \int_0^{\infty} \frac{dy}{y^2 + \pi^2} \left[ \frac{K_{(D+1)/2}(z)}{\eta^{(D+3)/2}} \right] \left( 1 - 4\xi \cosh^2(y/2) \right),$$

(3.18)

$$\left\langle R_{1i}\right\rangle_{\text{sub}} = -\frac{2mD+1}{(2\pi)^{1+D/2}} \int_0^{\infty} \frac{dy}{y^2 + \pi^2} \left[ \frac{K_{(D+1)/2}(z)}{\eta^{(D+3)/2}} \right] \left( 4\xi - 1 \right) \cosh^2(y/2), \quad i = 2, 3, \ldots, D.$$  

(3.19)

Note that near the horizon, in the limit $m\xi \ll 1$, the leading terms of the asymptotic expansion for the vacuum EMT components (3.17)–(3.19) do not depend on the mass and coincide with equation (3.9). Hence, near the horizon these components diverge as $\xi^{-D-1}$. The VEVs of the EMT for the real massive scalar fields with minimal and conformal couplings on a background of the Rindler spacetime were considered in [32] by using the covariant functional Schrödinger formalism, developed in [33]. In this paper, for the EMT the form (2.3) is used with $\xi = 0$ and $\xi = 0$, and the corresponding formulae (formulae (2.39) and (2.45) in [32]) are more complicated. The equivalence of these formulae to our expressions (3.17)–(3.19) with the special values $\xi = 0$, $\xi_{c}$, can be seen using the identity

$$\frac{4}{m^2 \xi^2} \int_0^{\infty} \frac{dy}{\pi^2 + y^2} \frac{K_{(D-1)/2}(z)}{\eta^{(D-1)/2}} = \int_0^{\infty} \frac{dy}{\pi^2 + y^2} \left( 1 - \cosh y \right) \frac{K_{(D-1)/2}(z)}{\eta^{(D-1)/2}}$$

$$+ \left( D - 1 \right) \cosh y \frac{K_{(D+1)/2}(z)}{\eta^{(D+1)/2}},$$

(3.20)

where $z$ is defined as in equation (3.16). We give the proof of this identity in appendix C.

From expressions (3.16)–(3.19) it follows that for the massive scalar field the energy spectrum is not strictly thermal and the quantities $\langle 0_M | \psi^2 | 0_M \rangle - \langle 0_R | \psi^2 | 0_R \rangle = -\langle \psi^2(x) \rangle_{\text{sub}}$ and $\langle 0_M | \psi^2 | 0_M \rangle - \langle 0_R | \psi^2 | 0_R \rangle = -\langle \psi^2(x) \rangle_{\text{sub}}$ do not coincide with the corresponding ones for the Minkowski thermal bath, $\langle \psi^2 \rangle_{\text{th}}^M$ and $\langle \psi^2 \rangle_{\text{th}}^M$, with temperature $(2\pi \theta)^{-1}$ (for this statement in a variety of situations, see [34–37] and references therein). To illustrate this for the case $D = 3$, we present in figure 1 the quantities $-\langle \psi^2 \rangle_{\text{sub}}$ (curve a) and $\langle \psi^2 \rangle_{\text{th}}^M$ (curve b) in dependence of $m\xi$.

The integrals in formulae (3.17)–(3.19) are strongly convergent and useful for numerical evaluation. In figure 2, the results of the corresponding numerical evaluation for the vacuum EMT components are presented in the Rindler case without boundaries for minimally and conformally coupled scalars.

In the limit $m\xi \gg 1$, using the asymptotic formula for the Macdonald function and taking into account that the main contribution to the integrals in equations (3.17)–(3.19) comes from
Figure 1. The $D = 3$ expectation values for the field square in the case of the Rindler vacuum without boundaries, $-\xi^2 (\langle \phi^2 \rangle)_{\text{sub}}$ (curve a) and for the Minkowski thermal bath, $\xi^2 (\langle \phi^2 \rangle)_{\text{th}}$ (curve b), given by formula (3.21), versus $m\xi$.

Figure 2. The expectation values for the diagonal components of the energy–momentum tensor, $\xi D_{i}^1 (\langle T_{i}^i \rangle)_{\text{sub}}$ (no summation over $i$), versus $m\xi$ in the case of the Fulling–Rindler vacuum without boundaries for minimally (left) and conformally (right) coupled scalar fields in $D = 3$. The curves a, b, c correspond to the values $i = 0, 1, 2$, respectively.

In the massless limit, $m = 0$, from equation (3.16) for the field product one obtains

$$\langle \phi^2 (x) \rangle_{\text{sub}} = -\frac{\Gamma \left( \frac{D-1}{2} \right)}{2\pi \xi (D+1)/2} \int_0^\infty \frac{dy}{\pi^2 + y^2} \frac{4\xi^2 (y/2) - 1}{\cosh^{D+1} (y/2)}.$$  

The corresponding formulae for the VEVs of the EMT are written as

$$\langle T_{0}^0 \rangle_{\text{sub}} = -D \langle T_{1}^1 \rangle_{\text{sub}} \approx \frac{D \Gamma \left( \frac{D-1}{2} \right)}{2\pi \xi (D+1)/2} \int_0^\infty \frac{dy}{\pi^2 + y^2} \frac{4\xi \cosh^2 (y/2) - 1}{\cosh^{D+1} (y/2)}.$$
\begin{align}
\langle T_i^{(R)} \rangle_{ab} &= \frac{\Gamma \left( \frac{D+1}{2} \right)}{(2\sqrt{\pi} \xi)^{D+1}} \int_0^\infty \frac{dy}{y^2 + \pi^2} \frac{1 - (1 - 4\zeta)(D - 1) \cosh^2(y/2)}{\cosh^{D+1}(y/2)}, \quad i = 2, \ldots, D.
\end{align}

(3.25)

Note that relation (3.24) between the energy density and the vacuum effective pressure also follows from the continuity equation for the EMT (see equation (4.5) below). Formulae (3.23)–(3.25) are alternative representations for the VEVs (3.8), (3.9).

4. VEVs in the RR region

In section 2, we have seen that the Wightman function in the case of the presence of a barrier can be presented in the form (2.24), where the first summand on the right corresponds to the Fulling–Rindler vacuum without boundaries and the second one is induced by the barrier and is given by expression (2.25). In the coincidence limit, from this formula for the boundary part of the field square we have

\begin{align}
\langle \varphi^2(x) \rangle^{(b)} &= \frac{-1}{2^{D-3}\pi^{(D+1)/2}} \frac{\Gamma \left( \frac{D+2}{2} \right)}{\Gamma \left( \frac{D+1}{2} \right)} \int_0^\infty dk \int_0^\infty d\omega \frac{T_k(\lambda_a)F_i(\lambda_a)}{K_\omega(\lambda_a)K_\omega(\lambda_b)}, \quad \xi > a.
\end{align}

(4.1)

This quantity is a monotone increasing negative function on $\xi$ for a Dirichlet scalar and the monotone decreasing positive function for the Neumann scalar. Substituting the Wightman function (2.24) into equation (2.6) and taking into account equations (2.25) and (4.1) for the VEVs of the EMT in the region $\xi > a$, one finds

\begin{align}
\langle 0| T^k_i(x)|0 \rangle = \langle 0| T^k_i(x)|0 \rangle^{(b)} + \langle T^k_i(x) \rangle^{(b)} ,
\end{align}

(4.2)

where the first term on the right is the VEV for the Fulling–Rindler vacuum without boundaries and has been investigated in the previous section. The second term is the contribution due to the presence of the barrier at $\xi = a$:

\begin{align}
\langle T^k_i \rangle^{(b)} &= \frac{-\delta_i^k}{2^{D-3}\pi^{(D+1)/2}} \frac{\Gamma \left( \frac{D+2}{2} \right)}{\Gamma \left( \frac{D+1}{2} \right)} \int_0^\infty dk \int_0^\infty d\omega \frac{T_k(\lambda_a)}{K_\omega(\lambda_a)} F_i^{(b)}[K_\omega(\lambda_b)],
\end{align}

(4.3)

where the functions $F_i^{(b)}[g(z)], i = 0, 1, 2$ are obtained from the functions $f_i^{(b)}[g(z)]$ in equations (3.7) by replacing $\omega \to i\omega$:

\begin{align}
F_i^{(b)}[g(z)] = f_i^{(b)}[g(z)], \quad \omega \to i\omega.
\end{align}

(4.4)

It can be easily checked that both the summands on the right of equation (4.2) satisfy the continuity equation $T^{i,k}_{i,k} = 0$, which for the geometry under consideration takes the form

\begin{align}
\frac{d}{d\xi} \langle \xi T^i_j \rangle - T^0_0 = 0.
\end{align}

(4.5)

By using the inequalities

\begin{align}
I_\omega'(z) &< \sqrt{1 + \omega^2/\zeta^2}I_\omega(z), \quad K_\omega'(z) > \sqrt{1 + \omega^2/\zeta^2}K_\omega(z)
\end{align}

(4.6)

it can be easily seen that $F_i^{(b)}[K_\omega(z)] > 0, F^{(1)}[K_\omega(z)] < 0$ for $\zeta < 0$ and $F^{(1)}[K_\omega(z)] < 0, i = 2, 3, \ldots, D$ for $\zeta \leq \frac{\omega}{4(D-1)}$. In the cases of Dirichlet and Neumann scalars for the boundary parts of the EMT components these lead to

\begin{align}
\left( T^0_0 \right)^{(b)}_{\text{Dirichlet}} < 0, \quad \left( T^1_1 \right)^{(b)}_{\text{Dirichlet}} > 0
\end{align}

(4.7)
for $\zeta \leq 0$, and (no summation over $i$)
\[
\begin{align*}
\left(T^0_i\right)_{\text{Neumann}} &< 0, &\left(T^1_i\right)_{\text{Neumann}} &> 0 \quad (4.8)
\end{align*}
\]
for $\zeta \leq \frac{D-1}{2D}$. If we want to realize a direct evaluation in $D = 1$, the prescription described in section 2 has to be modified for the massless case. In appendix B, we show that the corresponding evaluation leads to the same result.

Boundary part (4.3) is finite for all values $\xi > a$ and diverges at the plate surface $\xi = a$. These surface divergences are well known in quantum field theory with boundaries and are investigated for various types of boundary conditions and geometries. To extract the leading part of the boundary divergence note that near the boundary the main contribution to the $\omega$-integral comes from large values of $\omega$ and we can use the uniform asymptotic expansions for the modified Bessel functions [27]. Introducing a new integration variable $k \to \omega k$ and replacing the Bessel modified functions by their uniform asymptotic expansions in the leading order one obtains
\[
\begin{align*}
\langle T^0_i(\xi) \rangle &\sim \frac{D(2\delta_{B0} - 1)(\xi - \xi_c)}{2^{D/2}(D+1/2)(\xi - a)^{D+1}} \left(\frac{D + 1}{2}\right), &\xi &\to a \quad (4.13)
\end{align*}
\]
\[
\begin{align*}
\langle T^1_i(\xi) \rangle &\sim \frac{(1 - 2\delta_{B0})(\xi - \xi_c)}{2^{D/2}(D+1/2)(\xi - a)^{D+1}} \left(\frac{D + 1}{2}\right). \quad (4.14)
\end{align*}
\]
These terms do not depend on mass and Robin coefficients, and have opposite signs for the Dirichlet and Neumann boundary conditions. Note that the leading terms for $\langle T^0_i(\xi) \rangle$ and $\langle T^2_i(\xi) \rangle$ are the same as for the plate in Minkowski spacetime (see, for instance, [8]). This statement is also valid for the spherical and cylindrical boundaries on the background of the Minkowski spacetime [9, 10].

Now let us consider the asymptotic behaviour of the boundary part (4.3) for large $\xi$, $\xi \gg a$. Introducing a new integration variable $y = \lambda \xi$ in equation (4.3) and using the asymptotic formulae for the Bessel modified functions for small values of the argument, we see that the subintegrand is proportional to $(ya/2\xi)^{2\omega}$. It follows from here that the main contribution to the $\omega$-integral comes from the small values of $\omega$. Expanding with respect to $\omega$ in the leading order we obtain
\[
\begin{align*}
\langle T^k_i(\xi) \rangle &\sim -\frac{\delta_k^i(-1)^{\delta_{B0}}\xi^{-D-1}A_0^{(i)}(m\xi)}{2^{D/2}(D+1/2)(1 + \delta_{B0})^2}\ln^2(\xi/\alpha), \quad (4.15)
\end{align*}
\]
where

\[ A_0^{(i)}(m\xi) = \int_{m\xi}^{\infty} dy y^3 (y^2 - m^2 \xi^2)^{(D-3)/2} F^{(i)}(\xi/a) \big|_{\omega=0}. \] (4.16)

If, in addition, one has \( m\xi \gg 1 \), the integral in this formula can be evaluated replacing the Macdonald function by its asymptotic for large values of the argument. In the leading order this yields

\[ A_0^{(i)} \sim -2m\xi A_0^{(1)} \sim \pi(1/4 - \xi) \Gamma\left(\frac{D - 1}{2}\right) (m\xi)^{\frac{D+1}{2}} e^{-2m\xi}, \]

\[ m\xi \gg 1, \quad i = 2, 3, \ldots, D. \] (4.17)

For the massless case, the integral in equation (4.16) may be evaluated using standard formulae and one obtains

\[ A_0^{(i)} \sim -DA_1^{(i)} \sim \frac{D}{D-1} A_2^{(2)} \sim \frac{2^D D(\xi - \xi)}{(D - 1)^2 \Gamma(D)} \frac{\Gamma\left(\frac{D+1}{2}\right)}{2}, \quad m = 0. \] (4.18)

For the conformally coupled scalar the leading terms in the asymptotic expansions vanish and the VEVs are proportional to \( \xi^{-D-1} \ln^{-1}(2\xi/a) \) for \( \xi/a \gg 1 \). As follows from equations (3.9), (3.22) and (4.15), in both cases (4.17) and (4.18) and in the region far from the boundary the VEVs (4.2) are dominated by the purely Fulling–Rindler part (the first summand on the right): \( \{T_0^{(b)}\}_{\subscript{0ab}}^R \sim \ln^{-2}(2\xi/a) \). From equation (4.15) we see that for a given \( \xi \) the boundary part tends to zero as \( a \to 0 \) (the barrier coincides with the right Rindler horizon) and the corresponding VEVs of the EMT are the same as those for the Fulling–Rindler vacuum without boundaries. Hence, the barrier located at the Rindler horizon does not alter the vacuum EMT.

And finally, we turn to the asymptotic \( a, \xi \to \infty, \xi - a = \text{const} \). In this limit, \( \xi/a \to 1 \) and \( \omega \)-integrals in equation (4.3) are dominated by large \( \omega \). Replacing the modified Bessel functions by their uniform asymptotic expansions, keeping the leading terms only and introducing a new integration variable \( v = \omega/a \) for the energy density one obtains

\[ \langle T_0^{(b)} \rangle - \frac{1}{2^D-1} \pi^{D-1/2} \Gamma\left(\frac{D+1}{2}\right) \int_{0}^{\infty} dk k^{D-2} \int_{0}^{\infty} dv \frac{e^{-2(\xi - 1)\sqrt{v^2 + \lambda^2}}}{\sqrt{v^2 + \lambda^2}} \times \frac{A + B \sqrt{v^2 + \lambda^2}}{A - B \sqrt{v^2 + \lambda^2}} \left[-4\xi v^2 + \lambda^2 (1 - 4\xi)\right]. \] (4.19)

Converting to polar coordinates, \( v = r \cos \theta, k = r \sin \theta \), we can integrate over angles with the help of the formula

\[ \int_{0}^{\pi/2} d\theta \sin^n \theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(n/2 + 1\right)}. \] (4.20)

Introducing the new integration variable \( y = \sqrt{r^2 + m^2} \), we obtain

\[ \langle 0 | T_0^{(b)} | 0 \rangle \sim \frac{1}{2^D\pi^{D-1/2} \Gamma(D/2)} \int_{m}^{\infty} dy (y^2 - m^2)^{D/2} e^{-2(\xi - 1)\frac{A + By}{A - By}} \times \left[4(\xi - \xi) + \frac{m^2}{y^2} (4\xi - 1)\right], \quad a, \xi \to \infty, \quad \xi - a = \text{const}. \] (4.21)

Here we have taken into account that in the limit \( \xi \to \infty \) the purely Rindler part \( \langle 0 | T_0^{(b)} | 0 \rangle \) vanishes and in equation (4.2) only the boundary term remains. In the same order the evaluation of the components, corresponding to the vacuum effective pressures in the plane parallel to the barrier, leads (no summation over \( i \)) to

\[ \langle 0 | T_i^{(b)} | 0 \rangle \sim \langle 0 | T_0^{(b)} | 0 \rangle, \quad i = 2, 3, \ldots, D. \] (4.22)
The leading term for the component \( \langle 0 | T^1_1 | 0 \rangle \) vanishes,
\[
\langle 0 | T^1_1 | 0 \rangle = \mathcal{O}(1/a), \quad a \to \infty.
\] (4.23)

The term to the next order can be found using continuity equation (4.5) and expression (4.21). The expectation values (4.21)–(4.23) coincide with the corresponding VEVs induced by a single plate in the Minkowski spacetime with the Robin boundary conditions on it, and are investigated in [8] for the massless field and in [38] for the massive scalar field.

5. Vacuum stresses in the RL region

5.1. Scalar field

In the previous section, we have considered VEVs for the EMT in the region \( \xi > a \). Now we will turn to the case of the scalar field in the region between the barrier and the Rindler horizon (RL region), \( 0 < \xi < a \), satisfying boundary condition (2.11) on the boundary \( \xi = a \). For this region we have to take both the solutions to equation (2.13) and the eigenfunctions have the form (2.12), where now
\[
\phi(\xi) = Z_{\omega}(\lambda \xi, \lambda a) \equiv K_{\omega}(\lambda \xi) - \frac{\tilde{K}_{\omega}(\lambda a)}{I_{\omega}(\lambda a)} I_{\omega}(\lambda \xi),
\] (5.1)

and the coefficient is obtained from boundary condition (2.11). Here the quantities with overbars are defined in accordance with equation (2.19). As we see, unlike the previous case, the spectrum for \( \omega \) is continuous. The normalization coefficient is determined from condition (2.17), where integration over \( \xi \) goes between the limits \((0, a)\). Using integration formula (2.14) and the representation for the delta function
\[
\delta(x) = \lim_{\sigma \to \infty} \frac{\sin \sigma x}{\pi x},
\] (5.2)

it can be seen that
\[
C = \frac{\sqrt{\sinh \omega \pi}}{\pi (2\pi)^{D-1/2}}.
\] (5.3)

Substituting eigenfunctions (2.12) with the function (5.1) into the mode sum for the Wightman function one finds
\[
G^+(x, x') = \frac{1}{\pi} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(x-x')} \int_0^{\infty} d\omega \sinh(\pi \omega) e^{-i\omega(\tau-\tau')} Z_{\omega}(\lambda \xi, \lambda a) Z^{\ast}_{\omega}(\lambda \xi', \lambda a).
\] (5.4)

The part induced by the barrier can be obtained by subtracting from here the Wightman function for the right Rindler wedge without boundaries, given by expression (2.22). To transform the corresponding difference note that
\[
Z_{\omega}(\lambda \xi, \lambda a) Z^{\ast}_{\omega}(\lambda \xi', \lambda a) - K_{\omega}(\lambda \xi) K^{\ast}_{\omega}(\lambda \xi') = \frac{\pi \tilde{K}_{\omega}(\lambda a)}{2i \sinh \pi \omega} \sum_{\sigma = -1, 1} \frac{I_{\omega \sigma}(\lambda \xi) I_{\omega \sigma}(\lambda \xi')}{I_{\sigma \omega}(\lambda a)},
\] (5.5)

where we have used the standard relation
\[
K_\nu(z) = \frac{\pi}{2 \sin \nu \pi} [I_{\nu}(z) - I_\nu(z)].
\] (5.6)
This allows us to present the boundary part in the form

\[
\langle \varphi(x)\varphi(x') \rangle^{(b)} = G^+(x, x') - G^-(x, x')
\]

\[
= \frac{1}{\pi} \int \frac{d^{D-1}k}{(2\pi)^D} e^{ik(x-x')} \int_0^\infty d\omega e^{-i\omega(\tau-\tau')} \sum_{\sigma=-1,1} \bar{K}_{\omega}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma') \cosh[\omega(\tau-\tau')].
\]

This allows us to present the boundary part in the form

\[
\langle \varphi(x)\varphi(x') \rangle^{(b)} = \frac{1}{\pi} \int \frac{d^{D-1}k}{(2\pi)^D} e^{ik(x-x')} \int_0^\infty d\omega \frac{\bar{K}_{\omega}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma')}{I_{\omega,\sigma}(\lambda\sigma)}.
\]

Assuming that the function \( I_{\omega,\sigma}(\lambda\sigma) / I_{-\omega,\sigma}(\lambda\sigma) \) has no zeros for \(-\pi/2 \leq \text{arg } \omega < 0\) (\(0 < \text{arg } \omega < \pi/2\)), we can rotate the integration contour over \( \omega \) by angle \(-\pi/2\) for the term with \( \sigma = 1 \) and by angle \(\pi/2\) for the term with \( \sigma = -1 \). The integrals taken around the arcs of large radius tend to zero under the condition \( |\xi| \sim a^\gamma e^{i\pi}\bar{z}\) (note that, in particular, this is the case in the coincidence limit for the region under consideration). As a result for the difference of the Wightman functions one obtains

\[
\langle \varphi(x)\varphi(x') \rangle^{(b)} = \frac{1}{\pi} \int \frac{d^{D-1}k}{(2\pi)^D} e^{ik(x-x')} \int_0^\infty d\omega \frac{\bar{K}_{\omega}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma')}{I_{\omega,\sigma}(\lambda\sigma)} \cosh[\omega(\tau-\tau')].
\]

This quantity is a monotone decreasing negative function on \( \xi \) for the Dirichlet scalar and a monotonically increasing positive function for the Neumann scalar. From equation (2.6), the VEVs of the EMT are obtained in the form (4.2) with the boundary part

\[
\left\{ T^{(b)}_i \right\} = \left\{ \frac{\delta^i}{2D-2\pi(D+1)/2} \right\} \int_0^\infty d\omega \frac{K_{\omega}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma)}{I_{\omega,\sigma}(\lambda\sigma)} F^{(i)}[I_{\omega,\sigma}(\lambda\sigma)]
\]

where, as in equation (4.3), the functions \( F^{(i)}[g(z)] \) are obtained from expressions (3.7) by replacing \( \omega \rightarrow i\omega \) (see equation (4.4)). Comparing equations (5.9), (5.10), (4.1), and (4.3) we see that the boundary parts for the region \( 0 < \xi < a \) can be obtained from the corresponding ones for the region \( \xi > a \) by the replacements \( I_{\omega,\sigma} \rightarrow K_{\omega,\sigma} \). Note that here the situation is the same as that for the interior and exterior regions in the case of cylindrical and spherical surfaces on the background of the Minkowski spacetime (see [9, 10]). The reason for this analogy is that the uniformly accelerated observers produce worldlines in Minkowski spacetime which in Euclidean space would correspond to circles. This correspondence also extends to the field equation and its solutions, and modes (2.12) are the Minkowski spacetime analogue corresponding to cylinder harmonics.

For the \( D = 1 \) case, the integral over \( \lambda \) in equation (5.8) is absent and \( \lambda = m \). The corresponding formulae for the field square and vacuum EMT are obtained from equations (5.9) and (5.10) by replacing

\[
\frac{1}{2D-2\pi(D+1)/2} \int_0^\infty d\omega \frac{K_{\omega}(\lambda\sigma) I_{\omega,\sigma}(\lambda\sigma)}{I_{\omega,\sigma}(\lambda\sigma)} F^{(i)}[I_{\omega,\sigma}(\lambda\sigma)] \rightarrow \frac{1}{\pi}, \quad \lambda \rightarrow m.
\]

In particular, for the massless \( D = 1 \) case we have

\[
\left\{ T^{(b)}_i \right\} = \frac{\zeta \delta^i}{\pi \xi^2} \int_0^\infty d\omega \frac{A - B\omega/a}{A + B\omega/a} e^{-2\omega \ln(a/r)} F_0^{(i)}
\]
where
\[ F_0^{(0)} = 2\omega - 1, \quad F_0^{(1)} = 1. \] (5.13)

Expressions (5.10) are divergent on the plate surface, \( \xi = a \). The leading terms for the corresponding asymptotic expansions are derived in a way similar to that for equations (4.13) and (4.14), and can be obtained from these formulae by replacing \( \xi - a \) by \( a - \xi \) and changing the sign of \( \{T_i\}^{(b)} \).

Now let us consider the behaviour of the boundary part (5.10) for \( \xi \ll a \). Introducing a new integration variable \( y = \lambda a \) instead of \( k \) and using the formula for \( I_\mu(y\xi/a) \) in the case of small values of the argument, we see that the \( \omega \)-subintegrand is proportional to \( e^{-2\omega \ln(a/\xi)} \).

It follows from here that the main contribution comes from the small values of \( \omega \). Expanding over \( \omega \) we obtain the standard integrals of the form \( \int_0^{\infty} \omega^n e^{-2\omega \ln(a/\xi)} \, d\omega \). Performing these integrals one obtains in the leading order

\[
\{T_0\}^{(b)} = \frac{\xi B_0(ma)(1 + O(\ln^{-1}(2a/\xi)))}{2^{D-1}\pi^{(D+1)/2}\Gamma\left(\frac{D-1}{2}\right) \omega^{-1}\xi^2 \ln^2(2a/\xi)} \] (5.14)

\[
\{T_i\}^{(b)} = \frac{(4\xi - 1)B_0(ma)(1 + O(\ln^{-1}(2a/\xi)))}{2^{D-1}\pi^{(D+1)/2}\Gamma\left(\frac{D-1}{2}\right) \omega^{-1}\xi^2 \ln^3(2a/\xi)}, \quad i = 2, 3, \ldots, D, \] (5.15)

where
\[
B_0(ma) = \int_0^\infty dy \, y^{D-2} e^{-y} I_0(y). \] (5.16)

Note that the function \( B_0(ma) \) is positive for the Dirichlet boundary condition and is negative for the Neumann one. For the \( D = 1 \) case, one has

\[ \{T_0\}^{(b)} \sim \frac{\xi}{2\pi \xi^2 \ln^2(2/m\xi)} \frac{\tilde{K}_0(y)}{I_0(ma)}. \] (5.17)

As we see, the boundary part is divergent at the Rindler horizon. Recall that near the horizon the purely Rindler part behaves as \( \xi^{-D-1} \) and, therefore, in this limit the total vacuum EMT is dominated by this part.

5.2. Electromagnetic field

We now turn to the case of the electromagnetic field in the region \( 0 < \xi < a \) for the case \( D = 3 \). We will assume that the mirror is a perfect conductor with the standard boundary conditions of vanishing of the normal component of the magnetic field and the tangential components of the electric field, evaluated at the local inertial frame in which the conductor is instantaneously at rest. As has been shown in [5], the corresponding eigenfunctions for the vector potential \( A_\mu(x) \) may be resolved into transverse electric (TE) and transverse magnetic (TM) (with respect to \( \xi \)-direction) modes:

\[
A_\mu^{(\sigma)}(x) = \begin{cases} 
(0, 0, -ik3, ik_2)\phi^{(0)}(x), & \text{for } \sigma = 0, \\
(-\xi\partial/\partial\xi, io/\xi, 0, 0)\phi^{(1)}(x), & \text{for } \sigma = 1,
\end{cases}
\] (5.18)

where \( k = (k_2, k_3) \), \( \sigma = 0 \) and \( \sigma = 1 \) correspond to the TE and TM waves, respectively. From the perfect conductor boundary conditions on the vector potential we obtain the corresponding boundary conditions for the scalar fields \( \phi^{(\sigma)}(x) \):

\[
\phi^{(0)}(x) = 0, \quad \frac{\partial \phi^{(1)}(x)}{\partial \xi} = 0, \quad \xi = a. \] (5.19)
As a result, the TE/TM modes correspond to the Dirichlet/Neumann scalars. In the corresponding expressions for the eigenfunctions \( A^{(a)}_{\mu}(x) \), the normalization coefficient is determined from the orthonormality condition
\[
\int_0^\infty \int_0^{2\pi} d\xi \frac{d\chi}{\chi} A^{(a)\dagger}_{\mu}(x) A^{(a)\nu}(x) = -\frac{2\pi}{\omega} \delta_{\omega\omega} \delta_{\mu\nu}, \quad \alpha = (k, \omega). \tag{5.20}
\]
From here for the corresponding Dirichlet and Neumann modes we have
\[
\varphi^{(a)}_\sigma(x) = \frac{\sqrt{\sinh \omega \pi}}{\pi^{3/2} k} \left[ K_\omega(k\xi) - \frac{K^{(a)}_\omega(k\alpha)}{I^{(a)}_\omega(k\alpha)} I_\omega(k\xi) \right] e^{ikx - i\omega \tau}, \tag{5.21}
\]
where \( K^{(0)}_\omega(z) = K_\omega(z) \), \( K^{(1)}_\omega(z) = dK_\omega(z)/dz \) and the same notation is used for the function \( I_\omega(z) \). The VEVs for the EMT can be obtained substituting eigenfunctions (5.18) into the mode sum formula
\[
\langle 0 | T^k_{\alpha}(x) | 0 \rangle = \sum_{\sigma=0,1} \int_0^\infty d\omega \int_0^\infty d\omega' \left\{ A^{(a)\dagger}_{\mu}(x), A^{(a)\nu}(x) \right\}, \tag{5.22}
\]
with the standard bilinear form for the electromagnetic field EMT:
\[
T^k_{\alpha}(x, A_{\mu}(x)) = \frac{1}{4\pi} \left( -F_{ik} F^{ik} + \frac{1}{4} \delta_{ik} E^2 \right), \tag{5.23}
\]
where \( F_{ik} = \partial A_i / \partial x^k - \partial A_k / \partial x^i \) is the field tensor. Substituting eigenfunctions (5.18) into the mode sum (5.22) one finds
\[
\langle 0 | T^k_{\alpha}(x) | 0 \rangle = \frac{\delta^k_\alpha}{4\pi} \sum_{\sigma=0,1} \int_0^\infty d\omega \int_0^\infty d\omega' \sinh(\omega \pi) f^{(0)}_{\text{em}} \left[ K_\omega(k\xi) - \frac{K^{(a)}_\omega(k\alpha)}{I^{(a)}_\omega(k\alpha)} I_\omega(k\xi) \right]. \tag{5.24}
\]
Here for a given function \( g(z) \) the following notation is introduced:
\[
f^{(0)}_{\text{em}}[g(z)] = \left[ \frac{dg(z)}{dz} \right]^2 + \left( 1 + \frac{\omega^2}{z^2} \right) |g(z)|^2, \tag{5.25}
\]
\[
f^{(1)}_{\text{em}}[g(z)] = -\left[ \frac{dg(z)}{dz} \right]^2 + \left( 1 - \frac{\omega^2}{z^2} \right) |g(z)|^2, \tag{5.25}
\]
\[
f^{(2)}_{\text{em}}[g(z)] = f^{(3)}_{\text{em}}[g(z)] = -|g(z)|^2.
\]
It can be easily checked that components (5.24) satisfy the zero-trace condition and continuity equation (4.5). The way of subtracting from these quantities the parts due to the Fulling–Rindler vacuum without boundaries is the same as that for the scalar field given above in this section. This yields the following result:
\[
\langle 0 | T^k_{\alpha}(x) | 0 \rangle = \langle 0_R | T^k_{\alpha}(x) | 0_R \rangle = \frac{\delta^k_\alpha}{4\pi} \int_0^\infty d\omega \int_0^\infty d\omega' \left[ K_\omega(k\xi) + \frac{K^{(a)}_\omega(k\alpha)}{I^{(a)}_\omega(k\alpha)} I_\omega(k\xi) \right] F^{(0)}_{\text{em}}[I_\omega(k\xi)], \tag{5.26}
\]
where
\[
\langle 0_R | T^k_{\alpha}(x) | 0_R \rangle = \langle 0_M | T^k_{\alpha}(x) | 0_M \rangle = \frac{1}{\pi \xi^2} \int_0^\infty d\omega (\omega^3 + \omega) \frac{1}{e^{2\pi \omega} - 1} \text{diag} \left( 1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right) \tag{5.27}
\]
are the VEVs for the Fulling–Rindler vacuum without boundaries [5], and the functions \( F^{(0)}_{\text{em}}[g(z)] \) are obtained from equation (5.25) replacing \( \omega \rightarrow i\omega' \): \( F^{(1)}_{\text{em}}[g(z)] = f^{(0)}_{\text{em}}[g(z), \omega \rightarrow i\omega'] \). Note that the \( \omega' \)-integral in equation (5.27) is equal to \( 11/240 \). Comparing the boundary part in equation (5.26) with the corresponding expression for the RR region, \( \xi > a \), derived
in [5], we see that, as in the scalar case, these results are obtained from each other by the replacements $I_\omega \to K_\omega$, $K_\omega \to I_\omega$.

Using inequalities (4.6), we conclude that

$$\frac{K_\omega(z)}{I_\omega(z)} + \frac{K'_\omega(z)}{I'_\omega(z)} < 0. \quad (5.28)$$

This, together with the observation that $I_\omega(z)$ is a monotone increasing function, immediately shows that the components $\{T_i^{(b)}\}$, $i = 2, 3$ are negative and monotone decreasing on $\xi$. Further, from the first inequality (4.6) one has $F_{em}^{(i)}[I_\omega(k\xi)] > 0$ and it can be seen that this function is monotone increasing on $\xi$. Hence, the component $\{T_i^{(b)}\}$ is positive and monotone increasing. And, finally, by using equation (4.6) we see that $F_{em}^{(b)}[I_\omega(k\xi)] > 0$, and as a result one obtains that the boundary part of the energy density is positive, $\{T_0^{(b)}\} > 0$.

Now let us consider the asymptotics of the boundary part in equation (5.26) near the barrier, $\xi \to a$. In this limit the corresponding expressions are divergent. It follows from here that the main contributions to the VEVs are due to the large $\omega$. Rescaling the integration variable $k \to \omega k$ and replacing the Bessel modified functions by their uniform asymptotic expansions in the leading order one finds

$$\langle T_0^{(b)} \rangle \sim -2\langle T_1^{(b)} \rangle \sim \frac{1}{30\pi^2a(a-\xi)^2}, \quad i = 2, 3, \quad \xi \to a - 0. \quad (5.29)$$

Note that the corresponding asymptotics for the region $\xi > a$ are given by the same formulae [5].

Now we turn to the asymptotics in the limit $\xi \to 0$. Replacing the function $I_\omega(k\xi)$ by its asymptotic, and noting that the main contribution to the $\omega$-integrals comes from the region with small $\omega$, in the leading order we have (no summation over $i$)

$$\langle T_0^{(b)} \rangle \sim \langle T_1^{(b)} \rangle \sim -\frac{1}{8\pi^2a^4\ln(2a/\xi)} \int_0^\infty dy y^3 \frac{K_0(y)}{I_0(y)} - \frac{K_1(y)}{I_1(y)}. \quad (5.30)$$

In this formula the value for the integral is equal to $-1.326$. As we see, unlike the case for the scalar field, the boundary part of the electromagnetic vacuum EMT tends to zero as $\xi \to 0$.

And finally, let us consider the leading terms in the vacuum EMT in the limit $a, \xi \to \infty$, assuming $\xi - a = \text{constant}$. Introducing a new integration variable $ka$ in equation (5.26) we see that in this limit, the corresponding terms have the same form as those in the case $\xi \to a, a = \text{constant}$ and are given by expressions (5.29). Note that in this limit the total vacuum EMT is dominated by the boundary part.

In figure 3, we have plotted the electromagnetic vacuum energy density, $a^4 \langle T_0^{(b)} \rangle$ (left), and perpendicular pressure, $-a^4 \langle T_1^{(b)} \rangle$ (right), in dependence on $\xi/a$, generated by a single perfectly conducting plate uniformly accelerated normal to itself. For the plate $\xi = a$. The dashed curves represent the corresponding regularized quantities for the Fulling–Rindler vacuum without boundaries, given by the second summand on the right of formula (5.27). The full curves are for the boundary parts, the second term on the right of formula (5.26). In addition to the region $0 < \xi < a$, investigated in this section, we have also included the region $a < \xi < \infty$. The formulae for the boundary part in the latter case are derived in [5] and, as has been mentioned above, are obtained from the ones considered here by the replacements $I_\omega \to K_\omega$, $K_\omega \to I_\omega$. From figure 3, we see that the vacuum perpendicular pressure is always negative. The total energy density (as a sum of purely Rindler and boundary parts) is always
negative in the region $a < \xi < \infty$. For the region $0 < \xi < \infty$, the total energy density is negative near the horizon and is positive near the plate. As a result, for some intermediate value of $\xi$ the energy density vanishes.

6. Conclusions

Quantum field theory in accelerated systems contains many of the special features produced by a gravitational field without the complications due to a curved background spacetime. In this paper, we have considered the VEVs of the EMT induced by a uniformly-accelerated plane boundary in the Fulling–Rindler vacuum. This problem for the region to the right of the plane has been previously studied by Candelas and Deutsch in [5]. These authors consider a conformally coupled scalar field with Dirichlet or Neumann boundary conditions and electromagnetic field. Here we generalize the corresponding results for (i) an arbitrary number of spatial dimensions, (ii) for a scalar field with general curvature coupling parameter and (iii) non-zero mass, and (iv) for mixed boundary conditions of Robin type. In addition, (v) we consider the VEVs in the region between the plane and Rindler horizon for scalar and electromagnetic fields.

To obtain the expectation values for the energy–momentum tensor, we first construct the positive-frequency Wightman function (this function is also important in considerations of the response of a particle detector [4]). For the region to the right of the plate, $\xi > a$, the application of the generalized Abel–Plana formula to the mode sum over zeros of the function $K_{\omega \theta}$ allows us to extract the purely Rindler part without boundaries and to present the additional boundary part in terms of strongly convergent integrals (formulae (2.24), (2.25)). The expectation values for the EMT are obtained by applying a certain second-order differential operator and taking the coincidence limit. First, in section 3, we consider the vacuum EMT in the Rindler wedge without boundaries. After the subtraction of the Minkowskian part the corresponding components are presented in the form (3.6) with notation (3.7). For the massless scalar the VEVs of the EMT take the form (3.9). This expression corresponds to the absence of a thermal distribution with standard temperature $T = (2\pi \xi)^{-1}$ from the vacuum. In general, this distribution has a non-Planckian spectrum: the density-of-states factor is not
proportional to $\omega^{D-1} d\omega$. The spectrum takes the Planckian form for conformally coupled scalars in $D = 1, 2, 3$. It is interesting to note that for even values of spatial dimension the distribution is of Fermi–Dirac type. Further, on the basis of the procedure already used in [25], for the massive case we present the vacuum EMT in another alternative form, formulae (3.17)–(3.19). By using the identity proved in appendix C, we show that for the cases of conformally and minimally coupled scalars these formulae are equivalent to those previously obtained in [32]. For the massive scalar, the energy spectrum is not strictly thermal and the corresponding quantities do not coincide with those for the Minkowski thermal bath with temperature $(2\pi \xi)^{-1}$. This fact is illustrated in figure 1. In the limit $m\xi \gg 1$, the vacuum EMT components are exponentially decreasing functions on $m\xi$. In the massless limit, we obtain another equivalent representation, equations (3.24) and (3.25).

In section 4, we investigate the boundary part of the vacuum EMT induced by a single plate in the region $\xi > a$ (formula (4.3)). The corresponding result for the massless 1D field can be obtained by the analytic continuation and has the form (4.11). In appendix B, we show that this result coincides with the formula derived by direct calculation in $D = 1$. Further, we investigate the asymptotic properties of the boundary EMT. Near the plate surface the total VEVs are dominated by the boundary part and the corresponding components diverge at the boundary. For non-conformally coupled scalars the leading terms are given by formulae (4.13) and (4.14) and are the same as those for an infinite plane boundary in Minkowski spacetime with Dirichlet and Neumann boundary conditions. These terms do not depend on the mass or Robin coefficients, and have opposite signs for Dirichlet and Neumann cases. For large values of $\xi \gg a$, the leading term in the asymptotic expansion of the boundary-induced EMT does not depend on the Robin coefficient $B$ and has the form (4.15). In this limit $\langle T_k^i \rangle^{(b)} / \langle T_k^i \rangle^{(R)} \sim \ln^{-2}(2\xi / a)$, and the vacuum EMT is dominated by the purely Rindler part. For a given $\xi$, the boundary part of the EMT tends to zero as $a \to 0$, i.e., $\langle T_k^i \rangle^{(b)} \to 0$ in the limit when the barrier coincides with the Rindler horizon. As a result, the corresponding VEVs of the EMT are the same as those for the Fulling–Rindler vacuum without boundaries. Hence, the barrier located at the Rindler horizon does not alter the vacuum EMT.

The vacuum stresses in the region $0 < \xi < a$ are investigated in section 5 for scalar and electromagnetic fields. The corresponding boundary parts can be presented by equations (5.10) and (5.26), respectively. These formulae differ from those for the region $\xi > a$ by the replacements $I_{\omega} \to K_{\omega}, K_{\omega} \to I_{\omega}$. For the scalar field the boundary part diverges at the horizon with the leading behaviour (5.14), (5.15). The divergence of the purely Rindler part is stronger and this part dominates near the Rindler horizon. Unlike the scalar case, the boundary parts of the vacuum EMT components for the electromagnetic field vanish at the horizon (see asymptotic formulae (5.30)). In figure 3, we present the vacuum densities for the electromagnetic field, generated by a perfectly conducting plate in both RR and RL regions. As is seen, unlike the RR region, where the energy density is negative for all $a < \xi < \infty$, in the RL region the energy density is negative near the Rindler horizon, but is positive near the plate. As a result, for some intermediate value of $\xi$ we have zero energy density.

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Appendix A. Summation formula over zeros of $\bar{K}_\omega$

In section 2, we have shown that the VEVs for the EMT components in the RR region contain sums over zeros $\omega = \omega_n(\eta), n = 1, 2, \ldots$ of the function $\bar{K}_\omega(\eta) = AK_{in}(\eta) + b K'_{in}(\eta)$ for a given $\eta$. To obtain a summation formula over these zeros, we will use the generalized Abel–Plana formula derived in [19, 20]. In this formula, as functions $f(z)$ and $g(z)$ let us choose

$$f(z) = \frac{2i}{\pi} \sinh \pi z F(z), \quad g(z) = \frac{\bar{I}_{ic}(\eta)}{K_{ic}(\eta)} F(z),$$

with a function $F(z)$ analytic in the right half-plane $\text{Re} z \geq 0$, and quantities with overbars defined in accordance with equation (2.19). For the sum and difference of these functions one has

$$g(z) \pm f(z) = \frac{2}{\pi} F(z) \bar{I}_{\pm ic}(\eta) \bar{K}_{ic}(\eta).$$

(A.1)

By using the asymptotic formulae for the Bessel modified functions for large values of the index, the conditions for the generalized Abel–Plana formula can be written in terms of the function $F(z)$ as follows:

$$|F(z)| < \epsilon (|z|) e^{-\pi x (z \eta)}, \quad z = x + iy, \quad x > 0, \quad |z| \to \infty,$$

where $|z| \epsilon(|z|) \to 0$ when $|z| \to \infty$. Let $\omega = \omega_n$ be zeros for the function $\bar{K}_{\omega}(\eta)$ in the right half-plane. Here we will assume values of $A$ and $b$ for which all these zeros are real. It can be seen that they are simple. This directly follows from the relation

$$\frac{\partial}{\partial \omega} \bar{K}_{\omega}(\eta) \bigg|_{\omega = \omega_n} = \frac{2\omega_n b K_{in}(\eta) + b K'_{in}(\eta)}{\bar{K}_{ic}(\eta)} \int_0^\infty \frac{d\xi}{\xi} K_{ic}(\xi \eta),$$

(A.4)

which is a direct consequence of integral formula (2.14) with $\phi^{(1)}(\xi) = \phi^{(2)}(\xi) = K_{in}(\eta)$. Substituting functions (A.1) into generalized Abel–Plana formula (formula (2.11) in [20]) and taking into account that the points $z = \omega_n$ are simple poles for the function $g(z)$ and the relation $\bar{I}_{ic}(\eta) = \bar{I}_{-ic}(\eta)$, one obtains

$$\lim_{l \to \infty} \left\{ \sum_{n=1}^{n_l} \frac{\bar{I}_{ic}(\eta)}{\bar{K}_{ic}(\eta)} F(\omega_n) \frac{1}{\pi^2} \int_0^\infty \sinh \pi z F(z) \, dz \right\}
= -\frac{1}{2\pi} \int_0^\infty \frac{\bar{I}_{ic}(\eta)}{\bar{K}_{ic}(\eta)} [F(z e^{\pi i/2}) + F(z e^{-\pi i/2})] \, dz,$$

(A.5)

where $\omega_n < l < \omega_{n+1}$. Here we have assumed that the function $F(z)$ is analytic in the right half-plane. However, this formula can be easily generalized for the functions having poles in this region.

Appendix B. $D = 1$ massless case: direct evaluation

Here we show that the direct evaluation of the boundary VEVs in $D = 1$ gives the same result as the analytical continuation from the higher values of the space dimension. Now in equation (2.13) for the function $\phi(\xi)$ one has $\lambda = 0$. The corresponding linearly-independent solutions are $e^{\pm i \omega \ln(\xi/a)}$. The normalized eigenfunctions satisfying boundary condition (2.11) are of the form

$$\varphi_\omega(\xi, \tau) = \frac{e^{-i \omega \tau}}{\sqrt{\pi \omega}} \sin[\omega \ln(\xi/a) - \beta], \quad 0 < \omega < \infty,$$

(B.1)
where
\[ e^{i\beta} = \frac{A + i\omega B/a}{\sqrt{A^2 + (\omega B/a)^2}} \] (B.2)

Note that these eigenfunctions have the same form for the regions \( \xi > a \) and \( \xi < a \). Unlike the \( D > 1 \) cases, now both types of eigenfunction are bounded and the spectrum for \( \omega \) in the region \( \xi > a \) is continuous. Substituting eigenfunctions (B.1) into the mode sum formula (2.8) after some algebra one finds the Wightman function in the form (2.24), where the first summand on the right is the \( D = 1 \) Wightman function for the Rindler wedge without boundaries:
\[ G_R^+(x, x') = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{d\omega}{\omega} e^{-i\omega(x-x')} \cos \left[ \omega \ln(\xi/\xi') \right], \] (B.3)
and for the boundary part one has
\[ \langle \phi(x)\phi(x') \rangle^{(b)} = -\frac{1}{2\pi} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{A \pm i\omega B/a}{A \mp i\omega B/a} e^{-\omega(\ln(\xi'/\xi) - 2\beta)}. \] (B.4)

To transform this expression we write the \( \cos \) function in terms of exponentials and rotate the integration contour by angle \( \pm \pi/2 \) for the term with \( \exp[i\omega \ln(\xi'/a^2)] \) and by \( \mp \pi/2 \) for the term with \( \exp[-i\omega \ln(\xi'/a^2)] \). Here and below the upper and lower signs correspond to the cases \( \xi, \xi' > a \) and \( \xi, \xi' < a \), respectively. Under the condition \( |\tau - \tau'| < |\ln(\xi'/a^2)| \), the integrals over the arcs with large radius in the complex \( \omega \)-plane tend to zero and we obtain
\[ \langle \phi(x)\phi(x') \rangle^{(b)} = -\frac{1}{2\pi} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{A \pm i\omega B/a}{A \mp i\omega B/a} e^{-\omega(\ln(\xi'/\xi^2))} \cosh[\omega(\tau - \tau')]. \] (B.5)

The boundary contribution to the vacuum EMT can be found by substituting this expression into formula (2.6). It can easily be seen that as a result for the boundary part, we obtain formulae (4.11) and (5.12). Hence, we have shown that the direct evaluation gives the same result as the analytic continuation.

**Appendix C. Proof of identity (3.20)**

In this appendix, we will prove identity (3.20), which was used to see the equivalence of our formulae (3.17)–(3.19) for the VEVs of the EMT in the special cases of the conformally and minimally coupled massive scalars, to the results derived by Hill in [32]. Our starting point is the integral
\[ \mathcal{L}(\xi, \xi') = \int_{0}^{\infty} dk k^{D-2} \int_{0}^{\infty} d\omega \omega^2 e^{-\pi\omega} K_{\omega,\omega}(\lambda\xi) K_{\omega,\omega}(\lambda\xi') \] (C.1)

with \( \lambda \) defined as in equation (2.13). By using formulae (3.13) and [26]
\[ \int_{0}^{\infty} dk k^{D-2} K_{\omega,\omega}(\gamma \sqrt{k^2 + m^2}) = 2^{(D-3)/2} m^{D-1} \Gamma\left(\frac{D-1}{2}\right) K_{\omega,\omega}(\gamma/m') \] (D.1)

for this integral one obtains
\[ \mathcal{L}(\xi, \xi') = 2^{(D-1)/2} \pi m^{D-1} \Gamma\left(\frac{D-1}{2}\right) \int_{0}^{\infty} dy \frac{\pi^2 - 3y^2}{(\pi^2 + y^2)^3} K_{\omega,\omega}(\gamma/m') \] (C.2)

where \( \gamma \) is defined by equation (3.13). In particular, taking \( \xi' = \xi \) we receive
\[ \mathcal{L}(\xi, \xi) = 2^{(D-1)/2} \pi m^{D-1} \Gamma\left(\frac{D-1}{2}\right) \int_{0}^{\infty} dy \frac{\pi^2 - 3y^2}{(\pi^2 + y^2)^3} e^{(D-1)/2}. \] (C.3)
with $z$ defined in accordance with equation (3.16). Another form of $\mathcal{L}(\xi, \xi')$ can be obtained by using the relation

$$\omega^2 K_{\nu}(\lambda \xi) = \lambda^2 \xi^2 K_{\nu}(\lambda \xi) = -\xi^2 \frac{d^2 K_{\nu}(\lambda \xi)}{d\xi^2} - \xi \frac{d K_{\nu}(\lambda \xi)}{d\xi}. \quad (C.5)$$

The substitution into equation (C.1) yields

$$\mathcal{L}(\xi, \xi') = 2^{(D-3)/2} \pi m^{D+1} \Gamma \left( \frac{D-1}{2} \right) \int_0^\infty \frac{dy}{\pi^2 + y^2} \left\{ (D-1) \frac{K_{(D+1)/2}(m\gamma)}{(m\gamma)^{(D+1)/2}} 
+ \left( 1 - \frac{d^2}{dm^2} \frac{1}{m\xi} \frac{d(m\xi)}{d\xi} \right) \frac{K_{(D-1)/2}(m\gamma)}{(m\gamma)^{(D-1)/2}} \right\}. \quad (C.6)$$

For the coincidence case $\xi' = \xi$, from here one obtains

$$\mathcal{L}(\xi, \xi) = 2^{(D-5)/2} \pi m^{D+1} \xi^2 \Gamma \left( \frac{D-1}{2} \right) \int_0^\infty \frac{dy}{\pi^2 + y^2} \left\{ [D + 1 - (D - 1) \cosh y] \frac{K_{(D+1)/2}(z)}{z^{(D+1)/2}} + [1 - \cosh y] \frac{K_{(D-1)/2}(z)}{z^{(D-1)/2}} \right\}. \quad (C.7)$$

Comparing expressions (C.4) and (C.7), one obtains identity (3.20).

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