On reductions of families of crystalline Galois representations
Part II.

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Abstract
We compute the semisimplified modulo $p$ reductions of the families of crystalline representations of the absolute Galois group of $\mathbb{Q}_{p^f}$ constructed in [3].

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1 Introduction
Let $p$ be a fixed prime number, $K = \mathbb{Q}_{p^f}$ the finite unramified extension of $\mathbb{Q}_p$ of degree $f$, and $E$ a finite large enough extension of $K$ with maximal ideal $m_E$ and residue field $k_E$. In [3] we constructed families of Wach modules, and compute the semisimplified modulo $p$ reductions of some of the corresponding crystalline representations $V_{\vec{w},\vec{i}}$, where $\vec{\alpha} \in m_E^f$, $\vec{w}$ is a parameter indicating the labeled Hodge-Tate weights and $\vec{i}$ a vector specifying the family in question. The method used was to first show that $V_{\vec{w},\vec{i}} \simeq V_{\vec{w},\vec{i}}^{\bar{\vec{i}}}$ and then compute the reduction $\bar{V}_{\vec{w},\vec{i}}^{\bar{\vec{i}}}$. This was done in complete generality when the representation $V_{\vec{w},\vec{i}}^{\bar{\vec{i}}}$ was a principal series, and subject to some divisibility condition involving labeled Hodge-Tate weights which ensured that the modulo $p$ reduction was reducible, when it was supercuspidal. Here, we remove these conditions and get explicit formulas in all cases. This is achieved by displaying irreducible constituents of the restrictions of the characteristic zero representations to $G_{\mathbb{Q}_{p^f}}$ and using Frobenius reciprocity.

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Let $K_n = K(\mu_{p^n})$ where $\mu_{p^n}$ is a primitive $p^n$-th root of unity inside $\bar{\mathbb{Q}}_p$ and $K_\infty = \cup_{n \geq 1} K_n$. Let $\chi : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character. We denote $H_K = \ker \chi = \text{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$ and $\Gamma_K = G_K/H_K = \text{Gal}(K_\infty/K)$. Let $\mathbb{A}_K$ be the ring defined by

$$\mathbb{A}_K = \{ \sum_{n=-\infty}^{\infty} \alpha_n \pi^n : \alpha_n \in \mathcal{O}_K \text{ and } \lim_{n \to -\infty} \alpha_n = 0 \},$$

where $\pi$ is a formal variable. $\mathbb{A}_K$ is equipped with a Frobenius endomorphism $\varphi$ which extends the absolute Frobenius of $\mathcal{O}_K$ and is such that $\varphi(\pi) = (1 + \pi)^p - 1$. It is also equipped with a commuting with the Frobenius $\Gamma_K$-action which is $\mathcal{O}_K$-linear and is such that $\gamma \varphi(\pi) = (1 + \pi)^{\chi(\gamma)} - 1$ for all $\gamma \in \Gamma_K$. The ring $\mathbb{A}_K$ is local with maximal ideal $(p)$, residue field $\mathbb{B}_K = k_K((\pi))$, where $k_K$ is the residue field of $K$, and fraction field $\mathbb{B}_K = \mathbb{A}_K[[\frac{1}{p}]]$. A $(\varphi, \Gamma)$-module over $\mathbb{A}_K$ (respectively $\mathbb{B}_K$) is an $\mathbb{A}_K$-module of finite type (respectively finite dimensional $\mathbb{B}_K$-vector space) with continuous commuting semilinear actions of $\varphi$ and $\Gamma_K$. A $(\varphi, \Gamma)$-module $M$ over $\mathbb{A}_K$ is called étale if $\varphi^*(M) = M$, where $\varphi^*(M)$ is the $\mathbb{A}_K$-module generated by the set $\varphi(M)$.

A $(\varphi, \Gamma)$-module $M$ over $\mathbb{B}_K$ called étale if it contains an $\mathbb{A}_K$-lattice which is étale over $\mathbb{A}_K$. Fontaine proved that there is an equivalence of categories between $p$-adic (respectively $\mathbb{Z}_p$-adic) representations of $G_K$ and étale $(\varphi, \Gamma)$-modules over $\mathbb{B}_K$ (respectively $\mathbb{A}_K$) given by the functor $V \mapsto D(V) = (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^K$ (respectively $T \mapsto D(T) = (\mathbb{A} \otimes_{\mathbb{Z}_p} T)^K$), where $\mathbb{A}$ and $\mathbb{B}$ are period rings constructed by Fontaine with the properties that $\mathbb{A}_K = \mathbb{A}^H_K$ and $\mathbb{B}_K = \mathbb{B}^H_K$. Let $\mathbb{A}_K = \mathcal{O}_K[[\pi]] \subset \mathbb{A}_K$ and $\mathbb{B}_K = \mathbb{A}_K[[\frac{1}{p}]] \subset \mathbb{B}_K$. When studying the category of $E$ (respectively $\mathcal{O}_E$)-linear representations of $G_K$ we replace $\mathbb{B}_K$ by $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K$ (respectively $\mathbb{A}_K$ by $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$). In this case the $\varphi$ and $\Gamma_K$-actions are $E$ (respectively $\mathcal{O}_E$)-linear. A natural question is to determine the types of étale $(\varphi, \Gamma_K)$-modules which correspond to crystalline representations of $G_K$ via Fontaine’s functor. This is the content of the following.

**Theorem 1.1 (Berger)** Let $V$ be an $E$-linear representation of $G_K$. The representation $V$ is crystalline with Hodge-Tate weights in $[-k, 0]$ for some non-negative integer $k$, if and only if there exists a unique $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^\perp$-module $N(V)$ contained in $D(V)$ such that:

1. $N(V)$ is free of rank $d = \dim_E(V)$ over $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^\perp$;
2. The $\Gamma_K$-action preserves $N(V)$ and is trivial on $N(V)/\pi N(V)$;
3. $\varphi(N(V)) \subset N(V)$ and $N(V)/\varphi^*(N(V))$ is killed by $q^k$, where $q = \varphi(\pi)/\pi$. The module $N(V)$ is endowed with the filtration $\text{Fil}^j (N(V)) = \{ x \in N(V) : \varphi(x) \in q^j N(V) \}$ for $j \geq 0$ and $N(V)/\pi N(V)$ is endowed with the induced filtration. Then

$$D_{\text{cris}}(V) \simeq (N(V)/\pi N(V))$$

as filtered $\varphi$-modules over $E[1]$. Moreover, if $T$ is a $G_K$-stable, $\mathcal{O}_E$-lattice in $V$, then $N(T) = D(T) \cap N(V)$ is a $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$-lattice in $N(V)$ and the functor $T \mapsto N(T)$ gives a bijection between the $G_K$-stable, $\mathcal{O}_E$-lattices in $V$ and the $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$-lattices in $N(V)$ satisfying the following conditions:
1. $N(T)$ is free of rank $d = \dim_E(V)$ over $\mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{A}_K$;

2. The $\Gamma_K$-action preserves $N(T)$ and is trivial on $N(T)/\pi N(T)$;

3. $\varphi(N(T)) \subset N(T)$ and $N(T)/\varphi^*(N(T))$ is killed by $q^\ell$. \square

The modules $N(V)$ and $N(T)$ are called Wach modules over $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+$ (respectively $\mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{A}_K^+$). The étale $(\varphi, \Gamma_K)$-module $D(V)$ (respectively $D(T)$) is obtained from $N(V)$ (respectively $N(T)$) by extension of scalars.

### 1.1 Wach modules of restricted representations

Let $L$ be any finite unramified extension of $K$ of degree $n$. We relate the Wach module of the effective crystalline representation $V$ of $G_K$ to the Wach module of $V$ restricted to $G_L$. The notations $V_K$ and $V_L$ have the obvious meaning.

**Proposition 1.2** Let $N(V_K)$ be the Wach module of the effective crystalline representation $V_K$. The Wach module of $V_L$ is

$$N(V_L) = \left( E \otimes_{\mathbb{Q}_p} \mathbb{B}_L^+ \right) \bigotimes_{E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+} N(V_K).$$

**Proof.** Follows immediately from Theorem 1.1. \square

Fix once and for all an embedding $\varepsilon : L \hookrightarrow E$ and let $\tau_j^L = \varepsilon \circ \sigma_j^L$ for $j = 0, 1, \ldots, nf - 1$, where $\sigma_L$ is the absolute Frobenius of $L$. We fix the $nf$-tuple of embeddings $| \tau_j^L | := (\tau_0^L, \tau_1^L, \ldots, \tau_{nf-1}^L)$. Let $\sigma_K$ be the absolute Frobenius of $K$ and $\varepsilon_K = \varepsilon|_L$. We fix the $f$-tuple of embeddings $| \tau_j^K | := (\varepsilon_K \circ \sigma_0^K, \varepsilon_K \circ \sigma_1^K, \ldots, \varepsilon_K \circ \sigma_{nf-1}^K)$. There is a ring isomorphism $\xi_K : \mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{A}_K^+ \longrightarrow \prod_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ given by $\xi_K(a \otimes b) = \left( a\tau_0^K(b), a\tau_1^K(b), \ldots, a\tau_{nf-1}^K(b) \right)$, where the embeddings are ordered as above and

$$\tau_j^K \left( \sum_{n=0}^{\infty} \beta_n \pi_n \right) = \sum_{n=0}^{\infty} \beta_n \pi_n \text{ for all } b = \sum_{n=0}^{\infty} \beta_n \pi_n \in \mathbb{A}_K^+.\text{ The ring } \mathcal{O}_E[[\pi]]|_{\tau_K} := \prod_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]] \text{ is equipped with } \mathcal{O}_E\text{-linear actions of } \varphi \text{ and } \Gamma_K \text{ given by}

$$\varphi(a_0(\pi), a_1(\pi), \ldots, a_{nf-1}(\pi)) = (a_1(\varphi(\pi)), \ldots, a_{nf-1}(\varphi(\pi)), a_0(\varphi(\pi)))$$

and $\gamma(a_0(\pi), a_1(\pi), \ldots, a_{nf-1}(\pi)) = (a_0(\gamma\pi), a_1(\gamma\pi), \ldots, a_{nf-1}(\gamma\pi))$ for all $\gamma \in \Gamma_K$. Since the restriction of $\sigma_L$ to $K$ is $\sigma_K$, we get the following commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{A}_K^+ & \xrightarrow{\xi_K} & \mathcal{O}_E[[\pi]]|_{\tau_K} \\
1_{\mathcal{O}_E} \otimes i & \downarrow & \theta \\
\mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{A}_L^+ & \xrightarrow{\xi_L} & \mathcal{O}_E[[\pi]]|_{\tau_L}
\end{array}$$

where $i$ is the natural inclusion of $\mathbb{A}_K^+$ to $\mathbb{A}_L^+$ and $\theta$ the ring homomorphism defined by
\[\theta(\alpha_0, \alpha_1, \ldots, \alpha_{f-1}) = (\alpha_0, \alpha_1, \ldots, \alpha_{f-1}, \alpha_0, \alpha_1, \ldots, \alpha_{f-1}) = (\alpha_0, \alpha_1, \ldots, \alpha_{f-1})^{\otimes n}.\]

For any matrix \( A \in M_d \left( \mathcal{O}_E^{|\tau \kappa|}[[\pi]] \right) \), we denote \( A^{\otimes n} \) the matrix gotten by replacing each entry \( \alpha \) by \( \alpha^{\otimes n} \). The following Proposition follows immediately from the discussion above and Prop. 1.2.

**Proposition 1.3** Let \( V_K \) and \( V_L \) be as in Prop. 1.2. If the Wach module of \( N(V_K) \) is defined by the actions of \( \varphi \) and \( \Gamma_K \) given by \((\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_d)) = \xi \cdot \Pi_K \) and \((\gamma(e_1), \gamma(e_2), \ldots, \gamma(e_d)) = \xi \cdot G_K^\gamma \) for all \( \gamma \in \Gamma_K \) for some ordered base \( \xi = (e_1, e_2, \ldots, e_d) \), then the Wach module \( N(V_L) \) of \( V_L \) is defined by \((\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_d)) = \xi \cdot \Pi_L \) and \((\gamma(e_1), \gamma(e_2), \ldots, \gamma(e_d)) = \xi \cdot G_L^\gamma \) for all \( \gamma \in \Gamma_L \), where \( \Pi_L = (\Pi_K)^{\otimes n} \) and \( G_L^\gamma = (G_K^\gamma)^{\otimes n} \) for all \( \gamma \in \Gamma_L \).

**Remark 1.4** The analogue of Proposition 1.3 for filtered \( \varphi \)-modules and for an arbitrary base is generally false.

## 2 Computing reductions

For each \( i \in \{0, 1, 2, \ldots, 2f-1\} \) let \( \chi_i \) be the \( E^\times \)-valued Lubin-Tate character of \( G_{Q_p,f} \) constructed in [3 & 2] with \( k_j = 0 \) for \( j \in \{0, 1, 2, \ldots, 2f-1\} \) with \( j \neq i-1 \) (we use the convention \( k_0 = k_{2f} \)) and \( k_{i+1} = 1 \). For any \( C \in \mathcal{O}_E^\times \), let \( \eta_C : G_{Q_p,f} \to \mathcal{O}_E^\times \) be the unramified character which sends \( \sigma_{Q_p,f} \) to \( C \). Recall the notations of [3 & 1.2] which will be used throughout. We will need the following elementary fact.

**Lemma 2.1** Let \( F \) be any field, \( G \) any group and \( H \) any finite index subgroup. Let \( V \) be an irreducible finite-dimensional \( FG \)-module whose restriction to \( H \) contains some \( FH \)-submodule \( W \) with \( \dim_F V = [G : H] \dim_F W \). Then \( V \simeq \text{Ind}_H^G(W) \).

**Proof.** By Frobenius reciprocity there exists some nonzero \( \alpha \in \text{Hom}_{FG}(\text{Ind}_H^G(W), V) \). It is an isomorphism because \( V \) is irreducible and \( \text{Ind}_H^G(W) \) and \( V \) have the same dimension over \( E \).

**Example 2.2** Let \( k_i, i = 0, 1 \) be positive integers and \( \bar{a} \in m_2^E \). Consider the crystalline two-dimensional \( E \)-representation \( V_{w,\bar{a}}^{(2,5)} \) of \( G_{Q_p} \), with labeled Hodge-Tate weights \( \{(0, -k_0), (0, -k_1)\} \) corresponding to the weakly admissible filtered \( \varphi \)-modules \( (B_{w,\bar{a}}^{(2,5)}, \varphi) \) with

\[(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \left( \begin{array}{cc} (\alpha_1, p^{\kappa_0}) & (-1, 0) \\ (p^{\kappa_1}, \alpha_0) & (0, 1) \end{array} \right),\]

where \( \alpha_i = a_ip^{\kappa_0} \) and

\[Fw(D_{w,\bar{a}}^{(2,5)}) = \begin{cases} (E \times E)\eta_1 \oplus (E \times E)\eta_2 & \text{if } j \leq 0, \\ (E \times E)\eta_1 \oplus (E \times E)\eta_2 & \text{if } 1 \leq j \leq w_0, \\ (E \times E)\eta_1 \oplus (E \times E)\eta_2 & \text{if } 1 + w_0 \leq j \leq w_1, \\ 0 & \text{if } j \geq 1 + w_1, \end{cases}\]

where \( \bar{x} = (1, 1) \) and \( \bar{y} = (-\alpha_0, \alpha_1) \). By [3 & 6.1], \( V_{w,\bar{a}}^{(2,5)} \simeq V_{w,\bar{a}}^{(2,5)} \) for all \( \bar{a} \in m_2^E \).
Proposition 2.3 \( V^{(2,5)} \approx \text{Ind}_{G_{Q_{p,4}}}^{G_{Q_{p,4}}^1} (\eta_{\sqrt{-1}}, \chi_{k_2}, \chi_{k_3}) \approx \text{Ind}_{G_{Q_{p,4}}}^{G_{Q_{p,4}}^1} (\eta_{\sqrt{-1}}, \chi_{k_0}, \chi_{k_1}) \).

Proof. The Wach module of \( V^{(2,5)} \) has been constructed in \([3 \& 6.1]\) and the Wach module of the restriction to \( G_{Q_{p,4}} \) is given by Prop. \([3 \& 6.1]\). The filtered \( \varphi \)-module \((D, \varphi)\) corresponding to the restriction of \( V^{(2,5)} \) to \( G_{Q_{p,4}} \) (by a computation as in \([3 \& 6.1]\)) is

\[
(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \begin{pmatrix}
(\alpha_1, \rho^{k_0}, \alpha_1, \rho^{k_0}) & (-1, 0, -1, 0) \\
(p^{k_1}, \alpha_0, p^{k_1}, \alpha_0) & (0, 1, 0, 1)
\end{pmatrix},
\]

where \( \alpha_i = a_i \rho^{m} \) and

\[
\text{Fil}^j(D) = \begin{cases}
(E \times E \times E \times E)\eta_1 \otimes (E \times E \times E \times E)\eta_2 & \text{if } j \leq 0, \\
(E \times E \times E \times E) (f_0)_{\otimes 2} \eta_1 & \text{if } 1 \leq j \leq w_0, \\
(E \times E \times E \times E) (f_1)_{\otimes 2} \eta_1 & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1.
\end{cases}
\]

with \( f_i \) as in \([3 \& 6.1]\). Notice that the restriction of \( V^{(2,5)} \) to \( G_{Q_{p,4}} \) has labeled Hodge-Tate weights \( \{0, -k_0\}, \{0, -k_1\}, \{0, -k_0\}, \{0, -k_1\} \). We change the base to have the matrix of Frobenius in the standard form of \([3 \& 5.2.1]\). We write \([\varphi]_2 = P_1 \times P_2 \times P_3 \times P_0 \) and look for a change of base matrix \( Q = Q_1 \times Q_2 \times Q_3 \times Q_0 \) such that \([\varphi]_2 := Q[\varphi]_2 \varphi(Q)^{-1} \) is diagonal. Since \([\varphi]_2 = \text{diag}(\rho^{k_0+k_1}, \rho^{k_0+k_1}, \rho^{k_0+k_1}) \) we proceed as in the proof of \([2]\) Proof of Claim in Lemma 2.3.

Let \( Q_1 = Q_2 = I d, Q_0 = Q_3 = \begin{pmatrix} 0 & \sqrt{-1}p^{\ell_2} \\ \sqrt{-1}p^{\ell_2} & 0 \end{pmatrix} \), then

\[
Q = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \sqrt{-1}p^{\ell_2} & 0, 0, \sqrt{-1}p^{\ell_2} & 0, 1, 1, 0 \end{pmatrix},
\]

\((e_1, e_2) = (\eta_1, \eta_2) Q, [\varphi]_2 = \text{diag}(\rho^{k_0}, \rho^{k_0}, \rho^{k_0}) \)
and

\[
\text{Fil}^j(D) = \begin{cases}
(E \times E \times E \times E) e_1 \oplus (E \times E \times E \times E) e_2 & \text{if } j \leq 0, \\
(E \times E \times E \times E) (f_0)_{\otimes 2} (x e_1 + \tilde{y} e_2) & \text{if } 1 \leq j \leq w_0, \\
(E \times E \times E \times E) (f_1)_{\otimes 2} (x e_1 + \tilde{y} e_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1.
\end{cases}
\]

where \( x = (0, 1, 1, 0) \) and \( \tilde{y} = \sqrt{-1}p^{\ell_2}, 0, 0, \sqrt{-1}p^{\ell_2} \). By \([3, 5.1]\) \((D, \varphi)\) is decomposable and \( D_2 := (E \times E \times E \times E)_{\otimes 2} \) with \( \text{Fil}^j(D_2) = D_2 \otimes \text{Fil}^j(D) \) is a weakly admissible summand. By a direct computation (or by the proof of Prop. 3.7 in \([2]\)),

\[
\text{Fil}^j(D_2) = \begin{cases}
(E \times E \times E \times E) e_2 & \text{if } j \leq 0, \\
(E \times E \times E \times E) (f_0)_{\otimes 2} (1, 0, 0, 1) e_2 & \text{if } 1 \leq j \leq w_0, \\
(E \times E \times E \times E) (f_1)_{\otimes 2} (1, 0, 0, 1) e_2 & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1.
\end{cases}
\]

and by \([3, \text{Prop 2.5}]\) the latter is isomorphic to the weakly admissible filtered \( \varphi \)-module with \( \varphi(e_2) = (\sqrt{-1}, \sqrt{-1}, \sqrt{-1}p^{k_1}, \sqrt{-1}p^{k_0}) \) and filtration as above. The character \( \eta_{\sqrt{-1}} \) is the
restriction to $\mathbb{Q}_{p^3}$ of the unramified character of $G_{\mathbb{Q}_p}$ which maps $\sigma_{\mathbb{Q}_p}$ to $\sqrt{-1}$, and by Prop. 1.2 the filtered $\varphi$-module corresponding to $\eta_{\sqrt{-1}}$ is $(D(\eta_{\sqrt{-1}}), \varphi)$ with trivial filtration and $\varphi(e) = (\sqrt{-1}, \sqrt{-1}, \sqrt{-1}, \sqrt{-1})$. By Prop. 2.4, the crystalline representation corresponding to $(D_2, \varphi)$ is $\chi = \eta_{\sqrt{-1}} \cdot \chi^{\kappa_1}_{\ell_1} \cdot \chi^{\kappa_0}_{\ell_1}$. Since the representation $V_{(2,5)}$ is irreducible (c.f. 3 & 6.1), the first isomorphism of the Proposition follows by Lemma 2.1. The other isomorphism is shown similarly.

Remark 2.4 By Prop. 1.2 and Prop. 2.4, one easily sees that the character $\eta_{\sqrt{-1}} \cdot \chi^{\kappa_1}_{\ell_1} \cdot \chi^{\kappa_0}_{\ell_1}$ (and therefore $\chi^{\kappa_1}_{\ell_1} \cdot \chi^{\kappa_0}_{\ell_1}$) does not extend to $G_{\mathbb{Q}_p^3}$. This is also clear by Frobenius reciprocity, since $V_{(2,5)}$ is irreducible and $\chi$ is an irreducible constituent of its restriction to $G_{\mathbb{Q}_p^3}$.

In the followin Corollary and for the rest of the paper the characters $\omega_{k, \tau}$, are as in 3 & 5.1 with $K = \mathbb{Q}_{p'}$.

Corollary 2.5 For any $\vec{a} \in m_E^2$,

\[
\left( \left( V_{(2,5)} \right)^{ss} \right)_{I_{\mathbb{Q}_p^2}} \simeq \left( \omega_{4, \tau_1}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0} \right) \bigoplus \left( \omega_{4, \tau_1}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0} \right)^{p^2}.
\]

Moreover, for any $\vec{a} \in m_E^2$, the representation $\tilde{V}_{(2,5)}$ is irreducible if and only if $(1 + p^2)(1 + p) \mid k_1 + p k_0$.

Proof. By Cor. 5.5, \[\det \tilde{V}_{(2,5)} = \omega_{4, \tau_0}^{-k_0} \cdot \omega_{2, \tau_1}^{-k_1} = \left( \omega_{4, \tau_0}^{-k_0} \cdot \omega_{4, \tau_1}^{-k_1} \right)^{1 + p^2} \] . By the previous Proposition and 3 & 5.1, \[
\left( \left( V_{(2,5)} \right)^{ss} \right)_{I_{\mathbb{Q}_p^2}} = \left( \omega_{4, \tau_0}^{-k_0} \cdot \omega_{4, \tau_1}^{-k_1} \right) \bigoplus \left( \omega_{4, \tau_1}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0} \right).
\]

The Corollary follows from the formulas in 3 & 5.1. If $(1 + p^2)(1 + p) \mid k_1 + p k_0$, the reduction \[
\left( \left( V_{(2,5)} \right)^{ss} \right)_{I_{\mathbb{Q}_p^2}}
\]
was computed in 3 by showing that $\tilde{V}_{(2,5)}$ is reducible. Conversely, suppose that $\tilde{V}_{(2,5)}$ is reducible, then by Prop. 2.7, there exist integers $m_1, m_2$ such that $\omega_2^{m_1} \oplus \omega_2^{m_2} \simeq \omega_{4, \tau_1}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0} \oplus \omega_{4, \tau_0}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0}$, where $\omega_f$ is as in 3.2.1. This implies that $\omega_4^{(1 + p^2)m_1} \oplus \omega_4^{(1 + p^2)m_2} \simeq \omega_{4, \tau_1}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0} \oplus \omega_{4, \tau_0}^{-k_1} \cdot \omega_{4, \tau_2}^{-k_0}$, therefore $(1 + p^2)m_1 \equiv (-k_1 - p k_0) \mod(p^4 - 1)$ and $(1 + p^2)m_2 \equiv (-k_1 - p k_0) \mod(p^4 - 1)$, or $(1 + p^2)m_1 \equiv (-k_1 - p k_0) \mod(p^4 - 1)$. In any case $m_1 \equiv m_2 \mod(p^2 - 1)$ and we may assume that $m_1 = m_2$. Then $(-k_1 - p k_0) \mod(p^4 - 1)$ and $(1 + p^2)(1 + p) \mid k_1 + p k_0$.

Example 2.6 Let $k_i, i = 0, 1$ be positive integers and $\vec{a} \in m_E^2$. Consider the crystalline two-dimensional $E$-representation $V_{(2,8)}^{(2,5)}$ of $G_{\mathbb{Q}_p}$ with labeled Hodge-Tate corresponding to the weakly admissible filtered $\varphi$-modules $(D_{(2,8)}^{(2,5)}, \varphi)$ defined by

\[
(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \left( \begin{array}{cc} \alpha_1 & 1 \\ p k_1, \alpha_0 & -1, 0 \end{array} \right),
\]

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where $\alpha_i = a_i p^m$ and

$$\text{Fil}^i (D_{\vec{x}, \vec{y}}^{(2, 8)}) = \begin{cases} 
(E \times E) \eta_1 \oplus (E \times E) \eta_2 & \text{if } j \leq 0, \\
(E \times E) f_{\eta} (\vec{x} \eta_1 + \vec{y} \eta_2) & \text{if } 1 \leq j \leq w_0, \\
(E \times E) f_{\eta} (\vec{x} \eta_1 + \vec{y} \eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1,
\end{cases}$$

with $\vec{x} = (0, 1)$ and $\vec{y} = (1, \alpha_1)$. In [8 & 6.2] we proved that $\tilde{V}_{\vec{x}, \vec{y}}^{(2, 8)} \simeq \tilde{V}_{\vec{x}, \vec{y}}^{(2, 8)}$ for all $\vec{a} \in m_E^2$.

**Proposition 2.7** $V_{\vec{x}, \vec{y}}^{(2, 8)} \simeq \text{Ind}_{G_{q, 2}}^{G_{q, 2}^{\eta}} (\eta \sqrt{-1} \cdot \chi_{e_1}^{k_0} \cdot \chi_{e_2}^{k_1}) \simeq \text{Ind}_{G_{q, 2}}^{G_{q, 2}^{\eta}} (\eta \sqrt{-1} \cdot \chi_{e_0} \cdot \chi_{e_3}^{k_2}).$

**Proof.** We proceed as in the previous Example, preserving the notation. Here,

$$Q = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \sqrt{-1} p^{\frac{k_1}{2}} & 0 & 0 & \sqrt{-1} p^{\frac{k_1}{2}} \end{pmatrix},$$

$$(e_1, e_2) = (\eta_1, \eta_2) Q, [\varphi]_\mathbb{L} = \text{diag} (\sqrt{-1} p^{\frac{k_1}{2}}, 1, \sqrt{-1} p^{\frac{k_1}{2}}, p^{k_0}), (\sqrt{-1} p^{\frac{k_1}{2}}, p^{k_0}, \sqrt{-1} p^{\frac{k_1}{2}}, 1)$$

and

$$\text{Fil}^i (D) = \begin{cases} 
(E \times E \times E \times E) e_1 \oplus (E \times E \times E) e_2 & \text{if } j \leq 0, \\
(E \times E \times E \times E) (f_{\eta}) \otimes (\vec{x} \eta_1 + \vec{y} \eta_2) & \text{if } 1 \leq j \leq w_0, \\
(E \times E \times E \times E) (f_{\eta}) \otimes (\vec{x} \eta_1 + \vec{y} \eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1,
\end{cases}$$

with $\vec{x} = (\sqrt{-1} p^{\frac{k_1}{2}}, 1, 0, 0)$ and $\vec{y} = (0, 0, 1, \sqrt{-1} p^{\frac{k_1}{2}})$. By Prop. 5.1 in [3], $(D, \varphi)$ is decomposable and $D_2 := (E \times E \times E \times E) \eta_2$ with $\varphi (e_2) = (1, \sqrt{-1} p^{\frac{k_1}{2}}, p^{k_0}, \sqrt{-1} p^{\frac{k_1}{2}}, 1) e_2$ and $\text{Fil}^i (D_2) = D_2 \cap \text{Fil}^i (D)$ is a weakly admissible summand. By a direct computation (or by the proof of Prop. 3.7 in [2]),

$$\text{Fil}^i (D_2) = \begin{cases} 
(E \times E \times E \times E) e_2 & \text{if } j \leq 0, \\
(E \times E \times E \times E) (f_{\eta}) \otimes (1, 0, 0, 1) e_2 & \text{if } 1 \leq j \leq w_0, \\
(E \times E \times E \times E) (f_{\eta}) \otimes (1, 0, 0, 1) e_2 & \text{if } 1 + w_0 \leq j \leq w_1, \\
0 & \text{if } j \geq 1 + w_1,
\end{cases}$$

The corresponding crystalline representation is $\eta \sqrt{-1} \cdot \chi_{e_1}^{k_0} \cdot \chi_{e_2}^{k_1}$. Since the representation $V_{\vec{x}, \vec{y}}^{(2, 8)}$ is irreducible (c.f. [3 & 6.2]), the Proposition follows by Lemma 2.1. Notice that the character $\chi_{e_1}^{k_0} \cdot \chi_{e_2}^{k_1}$ does not extend to $G_{q, 2}^{\eta}$.

**Corollary 2.8** For any $\vec{a} \in m_E^2$,

$$\left( \left( \tilde{V}_{\vec{x}, \vec{y}}^{(2, 8)} \right)_{G_{q, 2}}^{s_\mathbb{L}} \right)_{G_{q, 2}} \simeq \left( \omega_{\frac{k_0}{4}, \frac{k_1}{4}, \frac{k_1}{4}, \frac{k_1}{4}} \right) \bigoplus \left( \omega_{\frac{k_0}{4}, \frac{k_1}{4}, \frac{k_1}{4}, \frac{k_1}{4}} \right) p^2.$$

Moreover, for any $\vec{a} \in m_E^2$ the representation $\tilde{V}_{\vec{x}, \vec{y}}^{(2, 8)}$ is irreducible if and only if $(1 + p^2)(1 + p) \nmid k_0 + pk_1$. 

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Proof. Identical to that of Cor. 2.5. Notice that in [3] Prop. 6.4 there is a sign error; one should replace $-pk_0 + p^2k_1$ by $-pk_0 - p^2k_1$. 

Example 2.9 Let $K = \mathbb{Q}_p(t)$ and let $k_0, k_1, ..., k_{f-1}$ be positive integers with $k = \max\{k_0, k_1, ..., k_{f-1}\}$. Consider the following four types of matrices $P_i(X_i)$

\[
(I): \left( \begin{array}{cc}
0 & -1 \\
p^{k_i} & X_i p^{m_i}
\end{array} \right) \quad (II): \left( \begin{array}{cc}
X_i p^m & -1 \\
p^{k_i} & 0
\end{array} \right) \quad (III): \left( \begin{array}{cc}
X_i p^m & p^{k_i} \\
-1 & 0
\end{array} \right) \quad (IV): \left( \begin{array}{cc}
0 & p^{k_i} \\
-1 & X_i p^m
\end{array} \right),
\]

where $X_i$ are polynomial variables and $m = \lfloor \frac{k-1}{p} \rfloor$. Let $\mathbf{a} = (a_1, ..., a_{f-1}, a_0) \in m_f$. Consider the families of matrices of $GL_2(\prod_{\tau:K \to E} E)$ obtained from the matrices $P_i(X_i)$ for all $i \in I_0$. The filtered $\varphi$-modules $\tilde{D}_{\mathbf{a}, \mathbf{d}}$ are vectors in $\{1, 2, 3, 4\}^f$ with $i_j$ being the type of the matrix $P_j(X_j)$ for all $j \in I_0$. The filtered $\varphi$-modules $\tilde{D}_{\mathbf{a}, \mathbf{d}} = \left( \prod_{\tau:K \to E} E \right) \eta_1 \oplus \left( \prod_{\tau:K \to E} E \right) \eta_2$ with Frobenius endomorphisms defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\tilde{P}(\mathbf{a})$ and filtrations

\[
\text{Fil}^j(D_{\mathbf{a}, \mathbf{d}}) = \left\{
\begin{array}{ll}
\prod_{\tau:K \to E} E \eta_1 \oplus \prod_{\tau:K \to E} E \eta_2 & \text{if } j \leq 0, \\
\prod_{\tau:K \to E} E (\bar{x}\eta_1 + \bar{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\
\prod_{\tau:K \to E} E f_{1_i}(\bar{x}\eta_1 + \bar{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\
\prod_{\tau:K \to E} E f_{1_{i-1}}(\bar{x}\eta_1 + \bar{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\
0 & \text{if } j \geq 1 + w_{t-1},
\end{array}
\right.
\]

with $\bar{x} = (x_0, x_1, ..., x_{f-1})$ and $\bar{y} = (y_0, y_1, ..., y_{f-1})$ and

\[
(x_i, y_i) = \left\{\begin{array}{ll}
(1, 0) & \text{if } i_1 = 1, \\
(\alpha p^m, 1) & \text{if } i_1 = 2, \\
(0, 1) & \text{if } i_1 = 3, \\
(1, \alpha p^m) & \text{if } i_1 = 4
\end{array}\right.
\]

are weakly admissible (c.f. [3] & 4.2), and correspond to two-dimensional $E$-linear crystalline representations $\tilde{V}_{\mathbf{a}, \mathbf{d}}$ of $G_K$. For $[K: \mathbb{Q}_p]$ even the reductions $\tilde{V}_{\mathbf{a}, \mathbf{d}}$ have been computed in [3] Cor. 5.10, Prop 5.15 and Thm 5.6. There it was also shown that $\tilde{V}_{\mathbf{a}, \mathbf{d}} = \tilde{V}_{\mathbf{a}, \mathbf{d}}^\mathbf{t}$ for any $\mathbf{t} \in \{1, 2, 3, 4\}^f$ and $\mathbf{d} \in m_f$. We compute $\tilde{V}_{\mathbf{a}, \mathbf{d}}^\mathbf{t}$ for $[K: \mathbb{Q}_p]$ odd. We write $P_j(X_j) = \left( \begin{array}{cc} 0 & \beta_j \\ \gamma_j & 0 \end{array} \right)$ with $\{\beta_j, \gamma_j\} = \{-1, p^{k_j}\}$ for all $j$. Let

\[
\ell_j = \left\{\begin{array}{ll}
0 & \text{if } j \text{ is odd and } \beta_j = -1, \\
k_i & \text{if } j \text{ is odd and } \beta_j = p^{k_i}, \\
o & \text{if } j \text{ is even and } \gamma_j = -1, \\
k_i & \text{if } j \text{ is even and } \gamma_j = p^{k_i},
\end{array}\right.
\]

$t = \#\{j : \ell_j = 0\}$ and $s_j = k_j - \ell_j$. In the following Proposition $\chi_{s_j}$ are the $E^\times$-valued Lubin-Tate characters of $G_{\bar{Q}_p^{s_j}}$ defined in the beginning of the Section.
Proposition 2.10 If $f$ is odd, $V^\tau_{\vec{a},\overline{0}} \simeq \text{Ind}_{G^f_{\overline{0},\overline{2}f}}^{G^f_{\overline{0},\overline{f}}}(\eta_{2\sqrt{-1}} \cdot \chi_{\ell_1} \cdot \chi_{\ell_2} \cdots \cdot \chi_{\ell_{f-1}} \cdot \chi_{\ell_0} ) \simeq \text{Ind}_{G^f_{\overline{0},\overline{2}f}}^{G^f_{\overline{0},\overline{f}}}(\eta_{2\sqrt{-1}} \cdot \chi_{\ell_1} \cdot \chi_{\ell_2} \cdots \cdot \chi_{\ell_{f-1}} \cdot \chi_{\ell_0} ).$

Proof. By the construction of the Wach modules of the representations $V^\tau_{\vec{a},\overline{0}}$ (c.f. [3] & 4.1), there exists a $G^f_{\overline{0},\overline{f}}$-stable $\mathcal{O}_E$-lattice $T^f_{\overline{0},\overline{2}f}$ in $V^\tau_{\vec{a},\overline{0}}$, with Wach module whose Wach module is defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \Pi$, with $\Pi = \left( \begin{array}{cccccc} 0 & 0 & \cdots & 0 \\ b_1 & b_2 & \cdots & b_{f-1} & b_0 \end{array} \right)$, where $b_i = \begin{cases} 1 & \text{if } \beta_i = -1, \\ q^{k_i} & \text{if } \beta_i = p^{k_i}, \end{cases}$ and $c_i = \begin{cases} 0 & \text{if } \gamma_i = -1, \\ q^{k_i} & \text{if } \gamma_i = p^{k_i}. \end{cases}$ The lattice $T^f_{\overline{0},\overline{2}f}$ has Wach module whose Wach module is defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \Pi^{\otimes 2}$. We base change by the matrix $Q = Q_0 \times Q_1 \times \cdots \times Q_{f-1}$, where $Q_i = \text{Id}$ for even $i$ and $Q_i = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ for odd $i$ and the matrix of $\varphi$ with respect to the base $(e_1, e_2) := (\eta_1, \eta_2) Q$ is $[\varphi]_Q^E = \text{diag}((c_1, c_2, b_3, \cdots, b_{f-1}, c_0), (c_1, b_2, c_3, \cdots, c_{f-1}, b_0))$ and see that the rank-one Wach module with Frobenius endomorphism defined by $\varphi(\cdot) = (b_1, c_2, b_3, \cdots, b_{f-1}, c_0) e$ is its Wach submodule. The corresponding crystalline representation $\chi = \eta_{2\sqrt{-1}} \cdot \chi_{\ell_1} \cdot \chi_{\ell_2} \cdots \cdot \chi_{\ell_{f-1}} \cdot \chi_{\ell_0}$ is a subrepresentation of the restriction of $V^\tau_{\vec{a},\overline{0}}$ to $G^f_{\overline{0},\overline{f}}$. The Proposition follows by Lemma 2.11 since $V^\tau_{\vec{a},\overline{0}}$ is irreducible (c.f. [3] Prop. 5.12). □

Corollary 2.11 If $f$ is odd, 

\[
\left( \left( V^\tau_{\vec{a},\overline{0}} \right)^{ss} \right)_{I_{G^f_{\overline{0},\overline{f}}}} \simeq \left( \prod_{i=1}^{2f} \omega_{f_i, r_i}^{-1} \right) \oplus \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right) \cdot \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right)^{1+p'},
\]

for any $\vec{a} \in m^f_E$ and any $\vec{r}$. Moreover, for any $\vec{a} \in m^f_E$ the representation $V^\tau_{\vec{a}, \vec{r}}$ is irreducible if and only if $\ell_{f-1} + p\ell_0 + p^2\ell_1 + \cdots + p^{2f}\ell_{f-1} \neq 0 \mod(p^f + 1)$.

Proof. As in the proof of Prop. 2.5

\[
\left( \left( V^\tau_{\vec{a},\overline{r}} \right)^{ss} \right)_{I_{G^f_{\overline{0},\overline{f}}}} = \left( \prod_{i=1}^{2f} \omega_{f_i, r_i}^{-1} \right) \oplus \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right) \cdot \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right)^{1+p'},
\]

with indices viewed modulo $2f$. Since $(\omega_{f_i, r_i})^{1+p'} = \omega_{f_i, r_i}$ for all $i = 0, 1, \cdots, f-1$,

\[
\left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right) \cdot \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right)^{1+p'} = \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right) \cdot \left( \prod_{i=1}^{2f} \omega_{f_i, r_i} \right)^{1+p'}.
\]

By the definition of the $\ell_i$, and given that $\{\beta_i, \gamma_i\} = \{-1, p^{k_i}\}$ for all $i$,

\[
\prod_{i=0}^{2f-1} \omega_{f_i, r_i}^{k_i}, \quad \prod_{i=0}^{2f-1} \omega_{f_i, r_i}^{k_i} = \prod_{i=0}^{2f-1} \omega_{f_i, r_i}^{k_i},
\]

and $\beta_i = p^{k_i}$ and $\gamma_i = p^{k_i}$. 9
and the product \( \star \) equals

\[
\left( \prod_{i=0}^{2f-1} \omega_{2f,i}^{-k_i} \right)^{p^f} = \left( \prod_{i=0}^{2f-1} \omega_{2f,i}^{-k_i} \right)^{p^f} = \left( \prod_{i=0}^{2f-1} \omega_{2f,i}^{-k_i} \right)^{p^f}.
\]

The irreducibility statement is proved exactly as in Cor. 2.5 where the reducibility of \( \bar{V}_{\vec{w},\vec{a}} \) when

\[
\ell_{2f-1} + p\ell_0 + p^2\ell_1 + \cdots + p^{2f}\ell_{2f-1} \equiv 0 \mod(p^f + 1)
\]

is known by the proof of [3, Thm. 5.13].

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