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Simplex Complex Forms for Two-Sided Quaternion Linear Canonical Transform

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Abstract. The two-sided quaternion linear canonical transform is a general form of the two-sided quaternion Fourier transform. Based on simplex complex forms of the two-sided quaternion Fourier transform we derive in detail the simplex complex forms of the two-sided quaternion linear canonical transform. Finally, a consequence of the generalized simplex complex forms is also presented.

1. Introduction
It is well known that the quaternion Fourier transform is general form of the traditional Fourier transform. There are many works devoted to the development of theories and applications of the quaternion Fourier transform [2, 3, 4, 6, 10, 11, 12, 18, 22]. According to the non-commutative property of quaternion multiplication, there are three kinds of the two-dimensional quaternion Fourier transform. They are so-called a left-sided quaternion Fourier transform, a right-sided quaternion Fourier transform, and a two-sided quaternion Fourier transform, respectively.

The two-sided quaternion linear canonical transform is generalization of the two-sided the quaternion Fourier transform. It also can be regarded as generalized form of the the traditional linear canonical transform [5, 13, 23]. Some results in the linear canonical transform domain have been extended in the quaternion linear canonical transform domain. For instance, in [21, 24], the authors established component-wise uncertainty principle. More results can be found in [7, 8, 9, 20, 22, 25] and the references therein. In the present paper, we establish simplex complex forms of two-sided quaternion linear canonical transform based on simplex complex forms the two-sided quaternion Fourier transform. We also present a consequence of the proposed simplex complex forms.

2. Preliminaries
In the following we provide definitions and some basic properties of the quaternions which will be needed later (for more details, see [1]).

2.1. Quaternions
We will be working with real quaternions. Let \( \mathbb{H} \) denotes of the set of real quaternions. Its elements can be written in the following form

\[
\mathbb{H} = \{ q = q_0 + i q_1 + j q_2 + k q_3 ; q_0, q_1, q_2, q_3 \in \mathbb{R} \},
\]
which the imaginary units $i, j$ and $k$ fulfill the following rules:

\[ \begin{align*}
  ij &= -ji = k, & jk &= -kj = i,
  ki &= -ik = j, & i^2 &= j^2 = k^2 = ijk = -1.
\end{align*} \]  

(1)

For a quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$, $q_0$ is simply called scalar part of $q$ denoted by $\text{Sc}(q)$ and $q = iq_1 + jq_2 + kq_3$ is called vector part of $q$ denoted $\text{Vec}(q)$.

Let $p, q \in \mathbb{H}$ and $p, q$ be their vector parts, respectively. Based on equation (1) we obtain the multiplication of two quaternions $qp$ as

\[ qp = q_0p_0 - q \cdot p + q_0p + p_0q + q \times p, \]  

(2)

where

\[ q \cdot p = q_1p_1 + q_2p_2 + q_3p_3, \]

\[ q \times p = i(q_2p_3 - q_3p_2) + j(q_3p_1 - q_1p_3) + k(q_1p_2 - q_2p_1). \]

Analogously to the complex case, a quaternionic conjugation $\overline{q}$ is given by

\[ \overline{q} = q_0 - iq_1 - jq_2 - kq_3, \]  

(3)

which leads to the anti-involution, that is,

\[ qp = \overline{p}\overline{q}. \]

With the help of (3) we get the norm or modulus of $q \in \mathbb{H}$ as

\[ |q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \]  

(4)

One can easily verify that

\[ |qp| = |q||p| \quad \text{and} \quad |q + p| \leq |q| + |p|, \quad \forall p, q \in \mathbb{H}. \]  

(5)

Like complex numbers, based on the conjugate (3) and the modulus of $q$, we get the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

\[ q^{-1} = \frac{\overline{q}}{|q|^2}. \]

In quaternionic notation, we may define an inner product for quaternion-valued functions $f, g : \mathbb{R}^2 \rightarrow \mathbb{H}$ as follows:

\[ (f, g) = \int_{\mathbb{R}^2} f(x)\overline{g(x)} \, dx, \]  

(6)

provided that the integral exists. Here $dx = dx_1dx_2$ and $x \in \mathbb{R}^2$. The symmetric real scalar part is defined by

\[ (f, g) = \frac{1}{2}[(f, g) + (g, f)] = \int_{\mathbb{R}^2} \text{Sc}(f(x)\overline{g(x)}) \, dx. \]  

(7)

In particular, for $f = g$, we obtain the $L^2(\mathbb{R}^2; \mathbb{H})$-norm

\[ \|f\| = \sqrt{(f, f)} = \left( \int_{\mathbb{R}^2} |f(x)|^2 \, dx \right)^{1/2}. \]  

(8)

A quaternion module $L^2(\mathbb{R}^2; \mathbb{H})$ is then defined as

\[ L^2(\mathbb{R}^2; \mathbb{H}) = \{ f : \mathbb{R}^2 \rightarrow \mathbb{H}, \|f\| < \infty \}. \]  

(9)
2.2. Split Quaternion and Properties
In this section we discuss the basic formulas of split quaternion (see [11]), which will be used to derive the useful results in the next section.

**Definition 2.1.** For two quaternion square roots \(\mu, \nu\) such that \(\mu^2 = \nu^2 = -1\), we may express a quaternion \(q\) as

\[
q = q_+ + q_- = \frac{1}{2}(q \pm \mu \nu).
\]

(10)

Especially, when \(\mu = \nu\), then any quaternion \(q\) may be split up into the commuting and anticommuting parts with respect with \(\mu\), i.e,

\[
\mu q_- = q_- \mu, \quad \mu q_+ = -q_+ \mu.
\]

(11)

It is easily proved the commuting and anticommuting parts satisfy the interesting properties:

\[
\mu^2 = \mu_+^2 + \mu_-^2 = -1, \quad \mu_+ \mu_- + \mu_- \mu_+ = 0.
\]

(12)

We derive from the above equation that

\[
q_\pm e^{i\mu \theta} = e^{\mp i\theta} q_\pm,
\]

(13)

where

\[
\cos \theta = \frac{q_0}{|q|}, \quad \sin \theta = \frac{\sqrt{q_1^2 + q_2^2 + q_3^2}}{|q|}.
\]

(14)

Especially, taking \(\mu = i\) and \(\nu = j\) equation (10) becomes

\[
q = q_+ + q_- = \frac{1}{2}(q \pm ij).
\]

(15)

Applying the above identity yields

\[
q_\pm = \{(q_0 \pm q_3) + i(q_1 \mp q_2)\} \frac{1 \pm k}{2} = \frac{1 \pm k}{2}\{(q_0 \pm q_3) + j(q_2 \mp q_1)\}.
\]

(16)

This leads to the following modulus identity

\[
|q|^2 = |q_-|^2 + |q_+|^2.
\]

(17)

Furthermore, one can obtain

\[
\text{Sc}(p_+ q_-) = 0.
\]

3. Simplex Complex Forms for Quaternion Linear Canonical Transform
In [14, 15, 16, 17, 19], Hitzer has been introduced the simplex complex forms for the two-sided quaternion Fourier transform (QFT). Following his idea, it is possible to extend the simplex complex forms in the two-sided quaternion linear canonical transform (QLCT) domains. For clarity we introduce the following QLCT definition.
**Definition 3.1 (QLCT definition).** Let be $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ be real matrix parameters satisfying $\det(A_1) = \det(A_2) = 1$. The two-sided QLCTs of a quaternion signal $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is given by

$$L_{A_1, A_2}^\mathbb{H}\{f\}(\omega) = \int_{\mathbb{R}^2} K_{A_1}(x_1, \omega_1)f(x_1, x_2, \omega_2) \, dx,$$

where $\omega \in \mathbb{R}^2$ and the kernels $K_{A_1}(x_1, \omega_1)$ and $K_{A_2}(x_2, \omega_2)$ are defined by

$$K_{A_1}(x_1, \omega_1) = \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{i}{2}x_1^2} e^{\frac{i}{2}x_1\omega_1 + \frac{i}{2}x_1\omega_1^3 - \frac{i}{2}} & \text{for } b_1 \neq 0 \\ \frac{1}{\sqrt{2\pi a_1}} e^{i(a_1\omega_1)} & \text{for } b_1 = 0, \end{cases}$$

and

$$K_{A_2}(x_2, \omega_2) = \begin{cases} \frac{1}{\sqrt{2\pi a_2}} e^{\frac{i}{2}x_2^2} e^{\frac{i}{2}x_2\omega_2 + \frac{i}{2}x_2\omega_2^3 - \frac{i}{2}} & \text{for } b_2 \neq 0 \\ \frac{1}{\sqrt{2\pi b_2}} e^{i(b_2\omega_2)} & \text{for } b_2 = 0. \end{cases}$$

From the definition mentioned above, it is easily seen that, when $b_1b_2 = 0$ and $b_1 = b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. Therefore, in this article we always suppose $b_1b_2 \neq 0$. More specifically, with $A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ for $i = 1, 2$, we get the following relation

$$L_{A_1, A_2}^\mathbb{H}\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x_1, x_2, \omega_2) e^{-i\omega_2 x_2} \frac{e^{-i\omega_1^{\frac{1}{2}}}}{\sqrt{2\pi}} \frac{e^{-i\omega_2^{\frac{1}{2}}}}{\sqrt{2\pi}} \, dx = \frac{e^{-i\omega_1^{\frac{1}{2}}}}{\sqrt{2\pi}} F_q\{f\}(\omega) e^{-i\omega_2^{\frac{1}{2}}},$$

where in this case $F_q\{f\}$ is the two-sided quaternion Fourier transform defined by (see, for example, [3, 6, 15])

$$F_q\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x_1, x_2) e^{-i\omega_2 x_2} \, dx.$$ 

As a consequence of above definition, we now obtain the following important theorem, which is the main result of the present section.

**Theorem 3.1.** If the quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$, then the QLCT of $f_\pm$ has the simplex complex forms

$$L_{A_1, A_2}^\mathbb{H}\{f_\pm\} = \frac{1}{\sqrt{2\pi b_2}} \int_{\mathbb{R}^2} f_\pm e^{\frac{1}{2}x_1^2} e^{\frac{1}{2}x_1\omega_1 + \frac{1}{2}x_1\omega_1^3 - \frac{i}{2}} \, dx,$$

and

$$L_{A_1, A_2}^\mathbb{H}\{f_\pm\} = \frac{1}{\sqrt{2\pi b_2}} \int_{\mathbb{R}^2} f_\pm e^{\frac{1}{2}x_1^2} e^{\frac{1}{2}x_1\omega_1 + \frac{1}{2}x_1\omega_1^3 - \frac{i}{2}} \, dx,$$
Proof. An easy computation gives

\[
L_{A_1,A_2}^B\{f_\pm\} = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i\frac{1}{2}\left(\frac{a_1}{b_1} x_1^2 - \frac{b_1}{a_1} x_1 \omega_1 + \frac{a_1}{b_1} \omega_1^2 - \frac{\pi}{2}\right)} f_\pm(x) \frac{1}{\sqrt{2\pi b_2}} e^{i\frac{1}{2}\left(\frac{a_2}{b_2} x_2^2 + \frac{a_2}{b_2} x_2 \omega_2 + \frac{a_2}{b_2} \omega_2^2 - \frac{\pi}{2}\right)} dx
\]

With the help of (16) we immediately obtain

\[
L_{A_1,A_2}^H\{f_\pm\} = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i\frac{1}{2}\left(\frac{a_1}{b_1} x_1^2 - \frac{b_1}{a_1} x_1 \omega_1 + \frac{a_1}{b_1} \omega_1^2 - \frac{\pi}{2}\right)} f_\pm(x) \frac{1}{\sqrt{2\pi b_2}} e^{i\frac{1}{2}\left(\frac{a_2}{b_2} x_2^2 - \frac{b_2}{a_2} x_2 \omega_2 + \frac{a_2}{b_2} \omega_2^2 - \frac{\pi}{2}\right)} dx
\]

On the other hand,

\[
L_{A_1,A_2}^H\{f_\pm\}(\omega) = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} \frac{1}{\sqrt{2\pi b_1}} \{(f_0 \pm f_3) + i(f_1 \mp f_2)\} \frac{1}{\sqrt{2\pi b_2}} e^{i\frac{1}{2}\left(\frac{a_1}{b_1} x_1^2 - \frac{b_1}{a_1} x_1 \omega_1 + \frac{a_1}{b_1} \omega_1^2 - \frac{\pi}{2}\right)} dx
\]

This gives the desired result. □

It is obvious that equation (23) can be rewritten in the form

\[
L_{A_1,A_2}^H\{f_-\}
\]
\[
= \frac{1}{2\pi b_0} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} f(x) e^{i \frac{1}{2} \left( \left( \frac{d_1}{b_2} x_1^2 - \frac{d_2}{a_2} x_2 x_0 + \frac{d_3}{a_3} x_3^2 - \frac{d_4}{a_4} \right) x_1 \omega_1 + \left( \frac{d_5}{b_8} x_1^2 - \frac{d_6}{a_6} x_2 x_0 + \frac{d_7}{a_7} x_3^2 - \frac{d_8}{a_8} \right) x_2 \omega_2 \right)} \, dx,
\]

and
\[
L_{A_1,A_2}^H \{ f_+ \} = \frac{1}{2\pi} \int_{\mathbb{R}^2} f_+ e^{i x_0 \omega_1} \, dx, \quad L_{A_1,A_2}^H \{ f_- \} = \frac{1}{2\pi} \int_{\mathbb{R}^2} f_- e^{i x_0 \omega_2} \, dx.
\]

In particular, when \( A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0) \) for \( i = 1, 2 \), equations (25) and (26) above will reduce to
\[
L_{A_1,A_2}^H \{ f_- \} = \frac{e^{-i \pi}}{2\pi} \int_{\mathbb{R}^2} f_- e^{i x_0 \omega} \, dx, \quad L_{A_1,A_2}^H \{ f_+ \} = \frac{1}{2\pi} \int_{\mathbb{R}^2} f_+ e^{i x_0 \omega} \, dx.
\]

An application of (17) to split \( f = f_- + f_+ \) we easily obtain the modulus identities for the QLCT, that is,
\[
|L_{A_1,A_2}^H \{ f \}(\omega)|^2 = |L_{A_1,A_2}^H \{ f_- \}(\omega)|^2 + |L_{A_1,A_2}^H \{ f_+ \}(\omega)|^2.
\]

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