STABILITY FOR MULTIVALUED MCKEAN-VLASOV STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. The work concerns multivalued McKean-Vlasov stochastic differential equations. First of all, we prove the existence and uniqueness of strong solutions for multivalued McKean-Vlasov stochastic differential equations with non-Lipschitz coefficients. Then, the classical Itô’s formula is extended to that for multivalued McKean-Vlasov stochastic differential equations. Finally, the asymptotic stability of second moments and the almost surely asymptotic stability for their solutions in terms of a Lyapunov function are shown.

1. Introduction

Given a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) and a \(m\)-dimensional standard Brownian motion \(W = (W^1, W^2, \ldots, W^m)\) defined on it. Consider the following multivalued McKean-Vlasov stochastic differential equation (SDE for short) on \(\mathbb{R}^d\):

\[
\begin{aligned}
dX_t &\in -A(X_t)dt + b(X_t, \mathcal{L}_X_t)dt + \sigma(X_t, \mathcal{L}_X_t)dW_t, \\
X_0 &= \xi,
\end{aligned}
\]

where \(\xi\) is a \(\mathcal{F}_0\)-measurable random variable with \(\mathbb{E}|\xi|^2 < \infty\), \(A : \mathbb{R}^d \mapsto 2^{\mathbb{R}^d}\) is a maximal monotone operator, \(\mathcal{L}_X_t\) is the probability distribution of \(X_t\) with respect to the probability measure \(\mathbb{P}\), and the coefficients \(b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^d\), \(\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}^d \times \mathbb{R}^m\) are Borel measurable (\(\mathcal{P}_2(\mathbb{R}^d)\) is defined in Subsection 2.1).

If \(A = 0\), Eq.(1) becomes McKean-Vlasov SDE. And the first work on McKean-Vlasov SDEs can be tracked back to McKean [12], who was inspired of Kac’s programme in Kinetic theory in [10]. From then on, a large number of results appear (c.f. [11]-[7]). Let us mention some related works. For Eq.(1), under non-Lipschitz coefficients, Ding and Qiao [5, 6] proved the well-posedness, exponential stability of the second moment, exponentially 2-ultimate boundedness and almost surely asymptotic stability of strong solutions to these equations. In [7], Hammersley, Siska and Szpruch also showed exponentially 2-ultimate boundedness of strong solutions for Eq.(1) under the local boundedness for \(b, \sigma\) and some extra properties for integrated Lyapunov functions.

If \(A \neq 0\), Eq.(1) is called a multivalued McKean-Vlasov SDE. Under global Lipschitz conditions, Chi [4] proved the existence and uniqueness of strong solutions for the following
multivalued McKean-Vlasov SDE on \( \mathbb{R}^d \):
\[
dX_t \in -A(X_t)dt + \bar{b}[X_t, \mathcal{L}X_t]dt + \bar{\sigma}[X_t, \mathcal{L}X_t]dW_t,
\]
where for any \( x \in \mathbb{R}^d \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \)
\[
\bar{b}[x, \mu] := \int_{\mathbb{R}^d} \bar{b}(x, y)\mu(dy), \quad \bar{\sigma}[x, \mu] := \int_{\mathbb{R}^d} \bar{\sigma}(x, y)\mu(dy),
\]
and \( \bar{b} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d, \bar{\sigma} : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^m \) are Borel measurable. When the operator \( A \) is the sub-differential of some convex function, Ren and Wang \cite{15} studied the following multivalued McKean-Vlasov SDE on \( \mathbb{R}^d \):
\[
dX_t \in -A(X_t)dt + \tilde{b}(X_t, \mathbb{E}[X_t])dt + \tilde{\sigma}(X_t, \mathbb{E}[X_t])dW_t,
\]
where \( \tilde{b}, \tilde{\sigma} \) satisfy global Lipschitz conditions, they showed well-posedness and a large deviation principle for Eq. (3). Note that Eq. (1) is more general than Eq. (2) and (3). However, as far as we know, Eq. (1) seems not to be studied in the literature.

In this paper, we first study the existence and uniqueness of strong solutions to Eq. (1) under non-Lipschitz conditions. Our result (Theorem 3.3) can cover \cite[Theorem 3.1]{4} and \cite[Theorem 3.3]{15} for the time homogeneous case. Then, the classical Itô formula for SDEs is extended to that for multivalued McKean-Vlasov SDEs. We emphasize that the Itô formula for multivalued McKean-Vlasov SDEs (Theorem 4.1) is different from the Itô formula for McKean-Vlasov SDEs (c.f. \cite[Proposition 3.1]{13}). After this, the stability for the strong solution to Eq. (1) is considered. We offer sufficient conditions to assure the asymptotic stability of the second moment in terms of Lyapunov functions. Finally, we prove the almost surely asymptotic stability for the strong solution to Eq. (1). Here we remind that the appearance of the maximal monotone operator causes a lot of trouble in the deduction.

The rest of the paper is organized as follows. In Section 2 we recall some basic notation and introduce maximal monotone operators and derivatives for functions on \( \mathcal{P}_2(\mathbb{R}^d) \) with respect to the measures. In Section 3 we prove that Eq. (1) has a unique strong solution under non-Lipschitz conditions. Next we extend the classical Itô formula to that for multivalued McKean-Vlasov SDEs in Section 4. Then in Section 5 we present the asymptotic stability of second moments and the almost surely asymptotic stability for the strong solution to Eq. (1). Finally, an example is given to explain our result in Section 6.

The following convention will be used throughout the paper: \( C \) with or without indices will denote different positive constants whose values may change from one place to another.

2. Preliminary

In the section, we introduce notations and concepts and recall some results used in the sequel.

2.1. Notations. In the subsection, we introduce some notations.

For convenience, we shall use \( | \cdot | \) and \( \| \cdot \| \) for norms of vectors and matrices, respectively. Furthermore, let \( \langle \cdot, \cdot \rangle \) denote the scalar product in \( \mathbb{R}^d \). Let \( B^* \) denote the transpose of the matrix \( B \).
Let $C(R^d)$ be the collection of continuous functions on $R^d$ and $C^2(R^d)$ be the space of continuous functions on $R^d$ which have continuous partial derivatives of order up to 2.

Let $\mathcal{B}(R^d)$ be the Borel $\sigma$-algebra on $R^d$ and $\mathcal{P}(R^d)$ be the space of all probability measures defined on $\mathcal{B}(R^d)$ carrying the usual topology of weak convergence. Let $\mathcal{P}_2(R^d)$ be the set of probability measures on $\mathcal{B}(R^d)$ with finite second order moments. That is,

$$\mathcal{P}_2(R^d) := \left\{ \mu \in \mathcal{P}(R^d) : \mu(|\cdot|^2) := \int_{R^d} |x|^2 \mu(dx) < \infty \right\}.$$ 

It is known that $\mathcal{P}_2(R^d)$ is a Polish space endowed with the $L^2$-Wasserstein distance defined by

$$W_2(\mu, \nu) := \inf_{\pi \in \Psi(\mu, \nu)} \left( \int_{R^d \times R^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_2(R^d),$$

where $\Psi(\mu, \nu)$ is the set of all couplings $\pi$ with marginal distributions $\mu$ and $\nu$. Moreover, if $\xi, \zeta$ are two random variables with distributions $\mathcal{L}_\xi, \mathcal{L}_\zeta$ under $\mathbb{P}$, respectively,

$$W_2(\mathcal{L}_\xi, \mathcal{L}_\zeta) \leq (\mathbb{E}|\xi - \zeta|^2)^{\frac{1}{2}},$$

where $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$.

### 2.2. Maximal monotone operators.

In the subsection, we introduce maximal monotone operators.

Fix a multivalued operator $A : R^d \mapsto 2^{R^d}$, where $2^{R^d}$ stands for all the subsets of $R^d$, and set

$$\mathcal{D}(A) := \{ x \in R^d : A(x) \neq \emptyset \}$$

and

$$Gr(A) := \{ (x, y) \in R^{2d} : x \in \mathcal{D}(A), \ y \in A(x) \}.$$ 

Then we say that $A$ is monotone if $(x_1 - x_2, y_1 - y_2) \geq 0$ for any $(x_1, y_1), (x_2, y_2) \in Gr(A)$, and $A$ is maximal monotone if

$$(x_1, y_1) \in Gr(A) \iff (x_1 - x_2, y_1 - y_2) \geq 0, \ \forall (x_2, y_2) \in Gr(A).$$

**Example 2.1.** Let $\mathcal{O} \subset R^d$ be a closed convex set with non-empty interior. The indicator function of $\mathcal{O}$ is defined by

$$I_{\mathcal{O}}(x) := \begin{cases} 0, & x \in \mathcal{O}, \\ +\infty, & x \notin \mathcal{O}. \end{cases}$$

Then one can justify that $I_{\mathcal{O}}$ is a lower semicontinuous convex function. The sub-differential of $I_{\mathcal{O}}$ is given by

$$\partial I_{\mathcal{O}}(x) := \begin{cases} \emptyset, & x \notin \mathcal{O}, \\ \{0\}, & x \in \text{Int}(\mathcal{O}), \\ \Pi_x, & x \in \partial \mathcal{O}, \end{cases}$$

where $\Pi_x$ is the cone of unit outward normal to $\mathcal{O}$ at $x$. It is well known that $\partial I_{\mathcal{O}}$ is a maximal monotone operator with $\text{Int}(\mathcal{D}(\partial I_{\mathcal{O}})) = \text{Int}(\mathcal{O}) \neq \emptyset$. 

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Given $T > 0$. Let $\mathcal{V}_0$ be the set of all continuous functions $K : [0, T] \mapsto \mathbb{R}^d$ with finite variations and $K_0 = 0$. For $K \in \mathcal{V}_0$ and $s \in [0, T]$, we shall use $|K|^s_0$ to denote the variation of $K$ on $[0, s]$. Set

$$\mathcal{A} := \{(X, K) : X \in C([0, T], D(A)), K \in \mathcal{V}_0,$$

and $\langle X_t - x, dK_t - ydt \rangle \geq 0$ for any $(x, y) \in Gr(A)\}.$

Then about $\mathcal{A}$ we recall three following results. (c.f. [2] [17])

**Lemma 2.2.** For $X \in C([0, T], D(A))$ and $K \in \mathcal{V}_0$, the following statements are equivalent:

(i) $(X, K) \in \mathcal{A}$,

(ii) For any $(x, y) \in C([0, T], \mathbb{R}^d)$ with $(x_t, y_t) \in Gr(A)$, it holds that

$$\langle X_t - x_t, dK_t - y_t dt \rangle \geq 0.$$

(iii) For any $(X', K') \in \mathcal{A}$, it holds that

$$\left\langle X_t - X'_t, dK_t - dK'_t \right\rangle \geq 0.$$

**Lemma 2.3.** Assume that $Int(D(A)) \neq \emptyset$, where $Int(D(A))$ denotes the interior of the set $D(A)$. For any $a \in Int(D(A))$, there exists constants $\gamma_1 > 0$, and $\gamma_2, \gamma_3 \geq 0$ such that for any $(X, K) \in \mathcal{A}$ and $0 \leq s < t \leq T$,

$$\int_s^t \langle X_r - a, dK_r \rangle \geq \gamma_1 |K|^s_t - \gamma_2 \int_s^t |X_r - a| dr - \gamma_3 (t - s).$$

**Lemma 2.4.** Assume that $\{K^n, n \in \mathbb{N}\} \subset \mathcal{V}_0$ converges to some $K$ in $C([0, T]; \mathbb{R}^d)$ and $\sup_n |K^n|^T_0 < \infty$. Then $K \in \mathcal{V}_0$, and

$$\lim_{n \to \infty} \int_0^T \langle X^n_s, dK^n_s \rangle = \int_0^T \langle X_s, dK_s \rangle,$$

where the sequence $\{X^n\} \subset C([0, T]; \mathbb{R}^d)$ converges to some $X$ in $C([0, T]; \mathbb{R}^d)$.

### 2.3. Derivatives for functions on $P_2(\mathbb{R}^d)$

In the subsection, we introduce derivatives for functions on $P_2(\mathbb{R}^d)$ (c.f. [11]).

A function $f : P_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is differential at $\mu \in P_2(\mathbb{R}^d)$, if for $\tilde{f}(\gamma) := f(\mathcal{L}_\gamma)$, $\gamma \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, there exists some $\zeta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ with $\mathcal{L}_\zeta = \mu$ such that $\tilde{f}$ is Fréchet differentiable at $\zeta$, that is, there exists a linear continuous mapping $D\tilde{f}(\zeta) : L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \mapsto \mathbb{R}$ such that for any $\eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$

$$\tilde{f}(\zeta + \eta) - \tilde{f}(\zeta) = D\tilde{f}(\zeta)(\eta) + o(|\eta|_{L^2}), \quad |\eta|_{L^2} \to 0.$$ 

Since $D\tilde{f}(\zeta) \in L(L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d), \mathbb{R})$, it follows from the Riesz representation theorem that there exists a $\mathbb{P}$-a.s. unique variable $\tilde{\eta} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that for all $\eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$

$$D\tilde{f}(\zeta)(\eta) = \langle \tilde{\eta}, \eta \rangle_{L^2} = \mathbb{E}[\tilde{\eta} \cdot \eta].$$
Definition 2.5. We say that \( f \in C^1(\mathcal{P}_2(\mathbb{R}^d)) \), if there exists for all \( \gamma \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) a \( \mathcal{L}_\gamma \)-modification of \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \), again denoted by \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \), such that \( \partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \mathbb{R}^d \) is continuous, and we identify this continuous function \( \partial_\mu f \) as the derivative of \( f \).

Definition 2.6. We say that \( f \in C^2(\mathcal{P}_2(\mathbb{R}^d)) \), if for any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \) is differentiable, and its derivative \( \partial_\gamma \partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^d \) is continuous, and for any \( y \in \mathbb{R}^d \), \( \partial_\mu f(\cdot)(y) \) is differentiable, and its derivative \( \partial_\mu^2 f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^d \) is continuous.

Definition 2.7. A function \( F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \) is said to be in \( C^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \), if (i) \( F \) is \( C^2 \) in \( x \in \mathbb{R}^d \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), respectively; (ii) these derivatives
\[
\partial_x F(x, \mu), \partial^2_x F(x, \mu), \partial_\mu F(x, \mu)(y), \partial_\gamma \partial_\mu F(x, \mu)(y), \partial_\mu^2 F(x, \mu)(y, y')
\]
are jointly continuous in the variable family \( (x, \mu), (x, \mu, y) \) and \( (x, \mu, y, y') \), respectively.

Definition 2.8. A function \( F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \) is said to be in \( \mathcal{C}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \), if \( F \in C^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \) and for any compact set \( \mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \),
\[
\sup_{(x, \mu) \in \mathcal{K}} \int_{\mathbb{R}^d} \left( \|\partial_y \partial_\mu F(x, \mu)(y)\|^2 + |\partial_\mu F(x, \mu)(y)|^2 \right) \mu(dy) < \infty.
\]

Definition 2.9. A function \( F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \) is said to be in \( C^{2,2}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \), if (i) \( F \in C^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \); (ii) \( F(x, \mu) \) and all its derivatives
\[
\partial_x F(x, \mu), \partial^2_x F(x, \mu), \partial_\mu F(x, \mu)(y), \partial_\gamma \partial_\mu F(x, \mu)(y), \partial_\mu^2 F(x, \mu)(y, y')
\]
are uniformly bounded in the variable family \( (x, \mu), (x, \mu, y) \) and \( (x, \mu, y, y') \), respectively.

Definition 2.10. The function \( F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \) is said to be in \( C^{2,2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \), if (i) \( F \in C^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \); (ii) \( F(x, \mu) \) and all its derivatives
\[
\partial_x F(x, \mu), \partial^2_x F(x, \mu), \partial_\mu F(x, \mu)(y), \partial_\gamma \partial_\mu F(x, \mu)(y), \partial_\mu^2 F(x, \mu)(y, y')
\]
are Lipschitz continuous in the variable family \( (x, \mu), (x, \mu, y) \) and \( (x, \mu, y, y') \), respectively. If \( F \in C^{2,2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \) and \( F \geq 0 \), we say that \( F \in C^{2,2,1,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \).

3. The existence and uniqueness of strong solutions

In this section, we study the existence and uniqueness of strong solution for Eq. (1).

First of all, we define strong solutions, weak solutions and the pathwise uniqueness of weak solutions for Eq. (1). Fix \( T > 0 \) and consider Eq. (1), i.e.
\[
\begin{cases}
dX_t \in -A(X_t)dt + b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t, & 0 \leq t \leq T, \\
X_0 = \xi.
\end{cases}
\]

Definition 3.1. (Strong solutions) We say that Eq. (1) admits a strong solution with the initial value \( \xi \) if there exists a pair of adapted processes \((X, K)\) on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\) such that
(i) \( \mathbb{P}(X_0 = \xi) = 1 \),
(ii) $X_t \in \mathcal{F}^W_t$, where $\{\mathcal{F}^W_t\}_{t \in [0,T]}$ stands for the $\sigma$-field filtration generated by $W$,

(iii) $(X(\omega), K(\omega)) \in \mathcal{A}$ a.s. $\mathbb{P}$,

(iv) it holds that

$$\mathbb{P}\left\{ \int_0^T (| b(X_s, \mathcal{L}_{X_s}) | + \| \sigma(X_s, \mathcal{L}_{X_s}) \|^2) \, ds < +\infty \right\} = 1,$$

and

$$X_t = \xi - K_t + \int_0^t b(X_s, \mathcal{L}_{X_s}) \, ds + \int_0^t \sigma(X_s, \mathcal{L}_{X_s}) \, dW_s, \quad 0 \leq t \leq T.$$

From the above definition, we know that $\mathcal{L}_{X_0} = \mathcal{L}_\xi$.

**Definition 3.2.** (Weak solutions) We say that Eq. (1) admits a weak solution with the initial law $\mathcal{L}_\xi \in \mathcal{P}(\mathbb{R}^d)$, if there exists a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P})}$, a $d$-dimensional standard $(\hat{\mathcal{F}}_t)$-Brownian motion $\hat{W}$ as well as a pair of $(\hat{\mathcal{F}}_t)$-adapted process $(\hat{X}, \hat{K})$ defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}})$ such that

(i) $\hat{\mathbb{P}} \circ \hat{X}_0^{-1} = \mathcal{L}_\xi$,

(ii) $(\hat{X}(\omega), \hat{K}(\omega)) \in \mathcal{A}$ a.s. $\hat{\mathbb{P}}$,

(iii) it holds that

$$\mathbb{P}\left\{ \int_0^T (| \hat{b}(\hat{X}_s, \mathcal{L}_{\hat{X}_s}) | + \| \hat{\sigma}(\hat{X}_s, \mathcal{L}_{\hat{X}_s}) \|^2) \, ds < +\infty \right\} = 1,$$

and

$$\hat{X}_t = \hat{X}_0 - \hat{K}_t + \int_0^t \hat{b}(\hat{X}_s, \mathcal{L}_{\hat{X}_s}) \, ds + \int_0^t \hat{\sigma}(\hat{X}_s, \mathcal{L}_{\hat{X}_s}) \, d\hat{W}_s, \quad 0 \leq t \leq T.$$

Such a solution will be denoted by $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}; \hat{W}, (\hat{X}, \hat{K}))$.

**Definition 3.3.** (Pathwise Uniqueness) Suppose $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}; \hat{W}, (\hat{X}_1^1, \hat{K}_1^1))$ and $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}; \hat{W}, (\hat{X}_2^2, \hat{K}_2^2))$ are two weak solutions for Eq. (1) with $\hat{X}_0^1 = \hat{X}_0^2$. If $\hat{\mathbb{P}}\{(\hat{X}_1^1, \hat{K}_1^1) = (\hat{X}_2^2, \hat{K}_2^2), t \in [0,T]\} = 1$, we say that the pathwise uniqueness holds for Eq. (1).

In the following, we give some conditions to assure the existence and pathwise uniqueness of weak solutions for Eq. (1). Assume:

(H$_{1.1}$) The function $b$ is continuous in $(x, \mu)$, and $b, \sigma$ satisfy for $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$$|b(x, \mu)|^2 + \|\sigma(x, \mu)\|^2 \leq L_1 (1 + |x|^2 + \mu(|\cdot|^2)),$$

where $L_1 > 0$ is a constant.

(H$_{1.2}$) The functions $b, \sigma$ satisfy for $(x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$$2(x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2)) \leq L_2 \left( |x_1 - x_2|^2 + \mathcal{W}_2^2(\mu_1, \mu_2) \right),$$

$$\| \sigma(x_1, \mu_1) - \sigma(x_2, \mu_2) \|^2 \leq L_2 \left( |x_1 - x_2|^2 + \mathcal{W}_2^2(\mu_1, \mu_2) \right),$$

where $L_2 > 0$ is a constant.
Next, we give a key lemma. Set
\[ \mathcal{C}_{\xi}^{\mathbb{R}^d} := \{ \mu \in C([0,T], \mathcal{P}_2(\mathbb{R}^d)), \mu_0 = \xi \}, \]
\[ \rho(\mu, \nu) := \sup_{t \in [0,T]} \mathbb{W}_2(\mu_t, \nu_t), \quad \mu, \nu \in \mathcal{C}_{\xi}^{\mathbb{R}^d}, \]
and the space \((\mathcal{C}_{\xi}^{\mathbb{R}^d}, \rho)\) is a complete metric space. For \(\mu \in \mathcal{C}_{\xi}^{\mathbb{R}^d}\), consider the following auxiliary multivalued SDE on \(\mathbb{R}^d:\)
\[
\begin{align*}
\left\{ 
\begin{array}{l}
   dX^\mu_t = -A(X^\mu_t)dt + b(X^\mu_t, \mu_t)dt + \sigma(X^\mu_t, \mu_t)dW_t, \\
   X^\mu_0 = \xi.
\end{array}
\right.
\end{align*}
\]

**Lemma 3.4.** Suppose that \(\text{Int}(\mathcal{D}(A)) \neq \emptyset\) and \(b, \sigma\) satisfy (H1.1)-(H1.2). Then for any \(\mathcal{F}_0\)-measurable random variable \(\xi\) with \(\mathbb{E}|\xi|^2 < \infty\), Eq. (4) has a unique strong solution \((X^\mu, K^\mu)\) and \(\mathcal{L}_{X^\mu} \in \mathcal{C}_{\xi}^{\mathbb{R}^d}\).

**Proof.** First of all, by the similar deduction to that in [14, Theorem 2.8], we obtain that Eq. (1) has a unique strong solution \((X^\mu, K^\mu)\). Then, we prove that \(\mathcal{L}_{X^\mu} \in \mathcal{C}_{\xi}^{\mathbb{R}^d}\).

Take any \(a \in \text{Int}(\mathcal{D}(A))\). By Itô’s formula, Lemma 2.3 and (H1.1), we have for \(0 < t \leq T\)
\[
|X^\mu_t - a|^2 = |\xi - a|^2 - 2 \int_0^t \langle X^\mu_s - a, dK^\mu_s \rangle + 2 \int_0^t \langle X^\mu_s - a, b(X^\mu_s, \mu_s) \rangle ds \\
+ 2 \int_0^t \langle X^\mu_s - a, \sigma(X^\mu_s, \mu_s) dW_s \rangle + \int_0^t \| \sigma(X^\mu_s, \mu_s) \|^2 ds \\
\leq |\xi - a|^2 - 2 \gamma_1 |K^\mu|_0^t + 2 \gamma_2 \int_0^t |X^\mu_s - a|^2 ds + 2 \gamma_3 t + \int_0^t |X^\mu_s - a|^2 ds \\
+ \int_0^t |b(X^\mu_s, \mu_s)|^2 ds + \int_0^t \| \sigma(X^\mu_s, \mu_s) \|^2 ds \\
+ 2 \int_0^t \langle X^\mu_s - a, \sigma(X^\mu_s, \mu_s) dW_s \rangle \\
\leq |\xi - a|^2 - 2 \gamma_1 |K^\mu|_0^t + 2 \gamma_2 \int_0^t (1 + |X^\mu_s - a|^2) ds + 2 \gamma_3 t + \int_0^t |X^\mu_s - a|^2 ds \\
+ L_1 \int_0^t \left( 1 + 2|a|^2 + 2|X^\mu_s - a|^2 + \mu_s(| \cdot |^2) \right) ds + 2 \int_0^t \langle X^\mu_s - a, \sigma(X^\mu_s, \mu_s) dW_s \rangle \\
\leq |\xi - a|^2 - 2 \gamma_1 |K^\mu|_0^t + \left( 2 \gamma_2 + 2 \gamma_3 + L_1(1 + 2|a|^2 + \sup_{s \in [0,T]} \mu_s(| \cdot |^2)) \right) T \\
+ (2 \gamma_2 + 1 + 2L_1) \int_0^t |X^\mu_s - a|^2 ds + 2 \int_0^t \langle X^\mu_s - a, \sigma(X^\mu_s, \mu_s) dW_s \rangle .
\] (5)

By the BDG inequality and the Hölder inequality, it holds that
\[
\mathbb{E} \left( \sup_{s \in [0,t]} |X^\mu_s - a|^2 \right) + 2 \gamma_1 \mathbb{E}|K^\mu|_0^t \\
\leq \mathbb{E} |\xi - a|^2 + \left( 2 \gamma_2 + 2 \gamma_3 + L_1(1 + 2|a|^2 + \sup_{s \in [0,T]} \mu_s(| \cdot |^2)) \right) T
\]
\[+(2\gamma_2 + 1 + 2L_1)E \int_0^t |X_s^\mu - a|^2 ds\]
\[+ 12E \left( \int_0^t |X_s^\mu - a|^2 \|\sigma (X_s^\mu, \mu_s)\|^2 ds \right)^{1/2}\]
\[\leq E |\xi - a|^2 + \left(2\gamma_2 + 2\gamma_3 + L_1 (1 + 2|a|^2 + \sup_{s[0,T]} \mu_s (|\cdot|^2))\right) T\]
\[+ (2\gamma_2 + 1 + 2L_1)E \int_0^t |X_s^\mu - a|^2 ds\]
\[+ \frac{1}{2} E \left( \sup_{s[0,t]} |X_s^\mu - a|^2 \right) + CE \int_0^t \|\sigma (X_s^\mu, \mu_s)\|^2 ds,\]

and furthermore
\[E \left( \sup_{s[0,t]} |X_s^\mu - a|^2 \right) + 4\gamma_1 E |K^\mu|_0^t\]
\[\leq 2E |\xi - a|^2 + 2\left(2\gamma_2 + 2\gamma_3 + CL_1 (1 + 2|a|^2 + \sup_{s[0,T]} \mu_s (|\cdot|^2))\right) T\]
\[+ 2(2\gamma_2 + 1 + 2CL_1) \int_0^t E \left( \sup_{r[0,s]} |X_r^\mu - a|^2 \right) ds.\]

Then by the Gronwall inequality, we know that
\[E \left( \sup_{s[0,t]} |X_s^\mu - a|^2 \right) \leq \left[2E |\xi - a|^2 + C_{\gamma_2,\gamma_3,L_1} T\right] e^{2(2\gamma_2 + 1 + 2CL_1) t},\]

where \(C_{\gamma_2,\gamma_3,L_1} := 2\left(2\gamma_2 + 2\gamma_3 + CL_1 (1 + 2|a|^2 + \sup_{s[0,T]} \mu_s (|\cdot|^2))\right),\) which yields that
\[E \left( \sup_{s[0,t]} |X_s^\mu|^2 \right) \leq 2E \left( \sup_{s[0,t]} |X_s^\mu - a|^2 \right) + 2|a|^2\]
\[\leq 2 \left[2E |\xi - a|^2 + C_{\gamma_2,\gamma_3,L_1} T\right] e^{2(2\gamma_2 + 1 + 2CL_1) t} + 2|a|^2.\]

Thus, it holds that \(\mathcal{L}_{X_t^\mu} \in P_2(\mathbb{R}^d).\)

Next, we estimate \(\mathcal{W}_2^2(\mathcal{L}_{X_s^\mu}, \mathcal{L}_{X_t^\mu})\) for \(s, t \in [0, T].\) From the definition of the metric \(\mathcal{W}_2,\) it follows that
\[\mathcal{W}_2^2(\mathcal{L}_{X_s^\mu}, \mathcal{L}_{X_t^\mu}) \leq E |X_s^\mu - X_t^\mu|^2.\]  

(6)

So, it is sufficient to compute \(E |X_s^\mu - X_t^\mu|^2.\)

Note that \(X_s^\mu, X_t^\mu\) satisfy the following equations
\[X_s^\mu = \xi - K_s^\mu + \int_0^s b(X_r^\mu, \mu_r) dr + \int_0^s \sigma(X_r^\mu, \mu_r) dW_r,\]
\[X_t^\mu = \xi - K_t^\mu + \int_0^t b(X_r^\mu, \mu_r) dr + \int_0^t \sigma(X_r^\mu, \mu_r) dW_r.\]
Assume \( s \leq t \) and take a stopping time sequence \( \{ \tau_N \} \) given by \( \tau_N := \inf\{ r \geq s, |X_r^\mu| \geq N \} \). Thus, by the Itô formula, it holds that

\[
|X_{t \wedge \tau_N}^\mu - X_s^\mu|^2 = -2 \int_{s}^{t \wedge \tau_N} < X_r^\mu - X_s^\mu, dK_r^\mu> + 2 \int_{s}^{t \wedge \tau_N} < X_r^\mu - X_s^\mu, b(X_r^\mu, \mu_r)> dr \\
+ 2 \int_{s}^{t \wedge \tau_N} < X_r^\mu - X_s^\mu, \sigma(X_r^\mu, \mu_r) dW_r> + \int_{s}^{t \wedge \tau_N} \| \sigma(X_r^\mu, \mu_r) \|^2 dr \\
\leq \int_{s}^{t \wedge \tau_N} |X_r^\mu - X_s^\mu|^2 dr + \int_{s}^{t \wedge \tau_N} |b(X_r^\mu, \mu_r)|^2 dr \\
+ 2 \int_{s}^{t \wedge \tau_N} < X_r^\mu - X_s^\mu, \sigma(X_r^\mu, \mu_r) dW_r> + \int_{s}^{t \wedge \tau_N} \| \sigma(X_r^\mu, \mu_r) \|^2 dr.
\]

Taking the expectation on two sides, by (H.1) we obtain that

\[
E|X_{t \wedge \tau_N}^\mu - X_s^\mu|^2 \leq E \int_{s}^{t \wedge \tau_N} |X_r^\mu - X_s^\mu|^2 dr + L_1E \int_{s}^{t \wedge \tau_N} (1 + |X_r^\mu|^2 + \mu_r(|\cdot|^2)) dr \\
\leq (1 + 2L_1)E \int_{s}^{t \wedge \tau_N} |X_r^\mu - X_s^\mu|^2 dr + L_1E \int_{s}^{t \wedge \tau_N} (1 + 2|X_s^\mu|^2 + \mu_r(|\cdot|^2)) dr \\
\leq (1 + 2L_1)E \int_{s}^{t} |X_r^\mu - X_s^\mu|^2 dr \\
+ L_1E \left( 1 + 2|X_s^\mu|^2 + \sup_{r \in [0,T]} \mu_r(|\cdot|^2) \right) (t - s).
\]

Using the Gronwall inequality ([9, Lemma 19.2, Page 172]) and the mean value theorem, one can get that

\[
E|X_{t \wedge \tau_N}^\mu - X_s^\mu|^2 \leq L_1E \left( 1 + 2|X_s^\mu|^2 + \sup_{r \in [0,T]} \mu_r(|\cdot|^2) \right) \frac{e^{(1+2L_1)(t-s)} - 1}{1 + 2L_1} \\
\leq L_1E \left( 1 + 2|X_s^\mu|^2 + \sup_{r \in [0,T]} \mu_r(|\cdot|^2) \right) (t - s) e^{(1+2L_1)t},
\]

and furthermore as \( N \to \infty \)

\[
E|X_s^\mu - X_t^\mu|^2 \leq L_1 \left( 1 + 2E|X_s^\mu|^2 + \sup_{r \in [0,T]} \mu_r(|\cdot|^2) \right) (t - s) e^{(1+2L_1)t}.
\] \hspace{1cm} (7)

Inserting (7) in (6), we have that

\[
\mathbb{W}_2(\mathcal{L}X_t^\mu, \mathcal{L}X_t^\mu) \leq L_1^{1/2} \left( 1 + 2E|X_s^\mu|^2 + \sup_{r \in [0,T]} \mu_r(|\cdot|^2) \right)^{1/2} (t - s)^{1/2} e^{(1+2L_1)t/2},
\]

which yields that

\[
\lim_{s \to t} \mathbb{W}_2(\mathcal{L}X_t^\mu, \mathcal{L}X_t^\mu) = 0.
\]

By the same deduction to above, it holds that for \( s \geq t \),

\[
\lim_{s \to t} \mathbb{W}_2(\mathcal{L}X_s^\mu, \mathcal{L}X_t^\mu) = 0.
\]

The proof is complete. \( \square \)

Now, it is the position to state and prove the existence and uniqueness of strong solutions for Eq. (II).
Theorem 3.5. Assume that \( \text{Int}(\mathcal{D}(A)) \neq \emptyset \) and the coefficients \( b, \sigma \) satisfy \( (H_{1.1})-(H_{1.2}) \). Then for any \( \mathcal{F}_0 \)-measurable random variable \( \xi \) with \( \mathbb{E}[|\xi|^2] < \infty \), there exists a unique strong solution \( (X, K) \) for Eq. (7), i.e. for any \( t \in [0, T] \)

\[
X_t = \xi - K_t + \int_0^t b(X_s, L_s)ds + \int_0^t \sigma(X_s, L_s)dW_s, \quad L_t = \mathbb{P} \circ X_t^{-1} \in \mathcal{P}_2(\mathbb{R}^d).
\]

Proof. Step 1. Define the mapping \( \Psi : \mathcal{C}_{0,t} \to \mathcal{C}_{0,t} \) by \( \Psi(\mu) = L_\mu \) for any \( \mu \in \mathcal{C}_{0,t} \), where \( 0 < t_0 < T \) is given later. Thus, by Lemma 3.4 we know that the mapping \( \Psi \) is well defined. Then, we prove that it is contractive.

First of all, by the definition of the metric on \( \mathcal{C}_{0,t} \), it holds that for \( \mu^1, \mu^2 \in \mathcal{C}_{0,t} \),

\[
\hat{\rho} \left( \Psi(\mu^1), \Psi(\mu^2) \right) = \sup_{t \in [0, t_0]} \mathbb{W}_2 \left( \Psi(\mu^1)_t, \Psi(\mu^2)_t \right) = \sup_{t \in [0, t_0]} \mathbb{W}_2 \left( L_{\mu^1}, L_{\mu^2} \right)
\leq \sup_{t \in [0, t_0]} \left( \mathbb{E}[|X_{t_0}^{\mu^1} - X_{t_0}^{\mu^2}|^2] \right)^{1/2} \leq \left( \mathbb{E} \sup_{t \in [0, t_0]} |X_{t_0}^{\mu^1} - X_{t_0}^{\mu^2}|^2 \right)^{1/2}. \tag{8}
\]

Next, we estimate \( \mathbb{E} \sup_{t \in [0, t_0]} |X_{t_0}^{\mu^1} - X_{t_0}^{\mu^2}|^2 \). By the Itô formula, Lemma 2.2 and \( (H_{1.2}) \), we have for \( 0 < t \leq t_0 \)

\[
|X_{t_0}^{\mu^1} - X_{t_0}^{\mu^2}|^2 = -2 \int_0^t \left( X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}, d \left( K_{s_0}^{\mu^1} - K_{s_0}^{\mu^2} \right) \right)
+ 2 \int_0^t \left( X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}, b \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - b \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right) ds
+ 2 \int_0^t \left( X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}, \left( \sigma \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - \sigma \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right) \right) dW_s
+ \int_0^t \left( \sigma \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - \sigma \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right)^2 ds
\leq 2L_2 \int_0^t \left( |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}| + \mathbb{W}^2_2 (\mu^1, \mu^2) \right) ds
+ 2 \int_0^t \left( X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}, \left( \sigma \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - \sigma \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right) \right) dW_s
\]

So, it follows from the BDG inequality and the Hölder inequality that

\[
\mathbb{E} \left( \sup_{s \in [0, t]} |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}|^2 \right) \leq 2L_2 \mathbb{E} \int_0^t \left( |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}| + \mathbb{W}^2_2 (\mu^1, \mu^2) \right) ds
+ 12 \mathbb{E} \left( \int_0^t |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}|^2 \left( \sigma \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - \sigma \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right)^2 ds \right)^{1/2}
\leq 2L_2 \mathbb{E} \int_0^t \left( |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}| + \mathbb{W}^2_2 (\mu^1, \mu^2) \right) ds
+ C \int_0^t \mathbb{E} \left( \left( \sigma \left( X_{s_0}^{\mu^1}, \mu^1_s \right) - \sigma \left( X_{s_0}^{\mu^2}, \mu^2_s \right) \right)^2 \right) ds
+ \frac{1}{2} \mathbb{E} \left( \sup_{s \in [0, t]} |X_{s_0}^{\mu^1} - X_{s_0}^{\mu^2}|^2 \right),
\]
and furthermore
\[
\mathbb{E}\left( \sup_{s \in [0,t]} \left| X^{\mu_1}_s - X^{\mu_2}_s \right|^2 \right) \leq 4L_2 \mathbb{E} \int_0^t \left( \left| X^{\mu_1}_s - X^{\mu_2}_s \right|^2 + \mathbb{W}_2^{(2)}(\mu_1^*, \mu_2^*) \right) ds + C \int_0^t \mathbb{E}\left( \left| \sigma\left(X^{\mu_1}_s, \mu_1^*\right) - \sigma\left(X^{\mu_2}_s, \mu_2^*\right) \right| \right)^2 ds \overset{(H_{1,2})}{\leq} C \mathbb{E} \int_0^t \left( \left| X^{\mu_1}_s - X^{\mu_2}_s \right|^2 + \mathbb{W}_2^{(2)}(\mu_1^*, \mu_2^*) \right) ds \leq C t_0 \sup_{s \in [0, t_0]} \mathbb{W}_2^{(2)}(\mu_1^*, \mu_2^*) + C \int_0^t \mathbb{E}\left( \sup_{s \in [0, s]} \left| X^{\mu_1}_s - X^{\mu_2}_s \right| \right)^2 ds.
\]

By the Gronwall inequality, we derive that
\[
\mathbb{E}\left( \sup_{s \in [0,t]} \left| X^{\mu_1}_s - X^{\mu_2}_s \right| \right) \leq C t_0 e^{C t_0} \sup_{s \in [0, t_0]} \mathbb{W}_2^{(2)}(\mu_1^*, \mu_2^*).
\]

Taking \( t_0 > 0 \) with \( C t_0 e^{C t_0} < \frac{1}{4} \), one can obtain that
\[
\left( \mathbb{E}\left( \sup_{s \in [0,t]} \left| X^{\mu_1}_s - X^{\mu_2}_s \right| \right) \right)^{1/2} \leq \frac{1}{2} \sup_{s \in [0, t_0]} \mathbb{W}_2^{(2)}(\mu_1^*, \mu_2^*) = \frac{1}{2} \hat{\rho}(\mu_1^*, \mu_2^*). \leq 0 \text{ for } t_0 \geq T.
\]

Combining (8) with (10), we have
\[
\hat{\rho}(\Psi(\mu^1), \Psi(\mu^2)) \leq \frac{1}{2} \hat{\rho}(\mu_1^*, \mu_2^*).
\]

**Step 2.** We prove that Eq. (11) has a unique strong solution.

By the conclusion in **Step 1.** and the fixed point theorem, it holds that there exists a unique fixed point \( \mu^* \in \mathcal{C}_{[0, t_0]} \) such that
\[
\Psi(\mu^*) = \mathcal{L}_{X^{\mu^*}} = \mu^*.
\]

Thus, \((X^{\mu^*}, K^{\mu^*})\) is a weak solution for Eq. (11) on \([0, t_0]\). If \(t_0 \geq T\), the proof of the existence for weak solutions to Eq. (11) is over; if \(t_0 < T\), by the same deduction as above, we obtain the existence for weak solutions to Eq. (11) on \([t_0, T]\).

Next, we prove the pathwise uniqueness for Eq. (11). Assume that \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}; W, X, K)\) and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}; W, \tilde{X}, \tilde{K})\) are two weak solutions of Eq. (11) with \(X_0 = \tilde{X}_0 = \xi\), i.e.
\[
X_t = \xi - K_t + \int_0^t b(X_s, \mathcal{L}_X) ds + \int_0^t \sigma(X_s, \mathcal{L}_X) dW_s,
\]
\[
\tilde{X}_t = \xi - \tilde{K}_t + \int_0^t b(\tilde{X}_s, \mathcal{L}_{\tilde{X}}) ds + \int_0^t \sigma(\tilde{X}_s, \mathcal{L}_{\tilde{X}}) dW_s.
\]

So, by the similar deduction to that in (9), we obtain that for any \(t \in [0, T]\)
\[
\mathbb{E}\left( \sup_{s \in [0,t]} \left| X_s - \tilde{X}_s \right|^2 \right) \leq C \mathbb{E} \int_0^t \left( \left| X_s - \tilde{X}_s \right|^2 + \mathbb{W}_2^{(2)}(\mathcal{L}_X, \mathcal{L}_{\tilde{X}}) \right) ds
\]
\[
\leq C \mathbb{E} \int_0^t \left( \left| X_s - \tilde{X}_s \right|^2 + \mathbb{E} \left| X_s - \tilde{X}_s \right|^2 \right) ds
\]
Corollary 3.6. Assume that \( K \) is complete. Then for any \( \xi \) it follows from the Gronwall inequality that
\[
\mathbb{E}\left( \sup_{s \in [0,t]} |X_s - \tilde{X}_s|^2 \right) = 0,
\]
which yields that
\[
X_t = \tilde{X}_t, \quad t \in [0, T] \text{ a.s.}. \mathbb{P}.
\]
Finally, note that for any \( t \in [0, T] \)
\[
K_t = \xi - X_t + \int_0^t b(X_s, \mathcal{L}_X) \, ds + \int_0^t \sigma(X_s, \mathcal{L}_X) \, dW_s
\]
\[
= \xi - \tilde{X}_t + \int_0^t b(\tilde{X}_s, \mathcal{L}_{\tilde{X}}) \, ds + \int_0^t \sigma(\tilde{X}_s, \mathcal{L}_{\tilde{X}}) \, dW_s
\]
\[
= \tilde{K}_t.
\]
Thus, the fact that \( K_t \) is continuous in \( t \) assures that \( K_t = \tilde{K}_t, t \in [0, T] \) a.s. \( \mathbb{P} \). The proof is complete. \( \square \)

By Theorem 3.3, we immediately have the following corollary.

Corollary 3.6. Assume that \( \text{Int}(\mathcal{D}(A)) \neq \emptyset \) and the coefficients \( b, \sigma \) satisfy (H1.1)-(H1.2). Then for any \( \mathcal{F}_0 \)-measurable random variable \( \xi \) with \( \mathbb{E}[|\xi|^2] < \infty \), the strong solution \((X, K)\) for Eq. (4) has the following moment property: for \( 0 \leq t \leq T \)
\[
\mathbb{E}\left( \sup_{s \in [0,t]} |X_s|^2 \right) \leq 2(2\mathbb{E}[|\xi - a|^2 + Ct)e^{Ct} + 2|a|^2), \quad \forall a \in \text{Int}(\mathcal{D}(A)). \tag{11}
\]

4. The generalized Itô formula for multivalued McKean-Vlasov SDEs

In the section, we will extend the classical Itô formula for SDEs to multivalued McKean-Vlasov SDEs. First of all, we strengthen the condition (H1.2) to the following assumption: (H1.3) The functions \( b, \sigma \) satisfy for \((x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)
\[
|b(x_1, \mu_1) - b(x_2, \mu_2)|^2 + \|\sigma(x_1, \mu_1) - \sigma(x_2, \mu_2)\|^2
\]
\[
\leq L_3(|x_1 - x_2|^2 + \mathbb{W}_2(\mu_1, \mu_2)),
\]
where \( L_3 > 0 \) is a constant.

Theorem 4.1. Suppose that the function \( \Phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \) belongs to \( C_b^{2,2}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \). Then, under \( \text{Int}(\mathcal{D}(A)) \neq \emptyset \) and (H1.1), (H1.3), for any \( \mathcal{F}_0 \)-measurable random variable \( \xi \) with \( \mathbb{E}[|\xi|^2] < \infty \), the following Itô formula holds: \( \forall 0 \leq s < t, \)
\[
\Phi(X_t, \mathcal{L}_X) - \Phi(X_s, \mathcal{L}_X)
\]
\[
= -\int_s^t (\partial_x \Phi(X_u, \mathcal{L}_X), dK_u) + \int_s^t (b^i \partial_{x_i} \Phi)(X_u, \mathcal{L}_X) \, du
\]
\[
+ \int_s^t (\sigma^{i j} \partial_{x_i} \Phi)(X_u, \mathcal{L}_X) \, dW^{j}_{u} + \frac{1}{2} \int_s^t \left( (\sigma \sigma^*)^{i j} \partial_{x_i} \partial_{x_j} \Phi)(X_u, \mathcal{L}_X) \right) \, du
\]

\[ + \int_s^t \int_{\mathbb{R}^d} b^i(y, \mathcal{L}_{X_u})(\partial_{\mu} \Phi)_i(X_u, \mathcal{L}_{X_u}(y)) \mathcal{L}_{X_u}(dy) \, du \]
\[ + \frac{1}{2} \int_s^t \int_{\mathbb{R}^d} (\sigma^*)^{ij}(y, \mathcal{L}_{X_u}) \partial_{y_j} (\partial_{\mu} \Phi)_j(X_u, \mathcal{L}_{X_u}(y)) \mathcal{L}_{X_u}(dy) \, du \]
\[ - \int_s^t \mathbb{E} \left\langle (\partial_{\mu} \Phi)(y, \mathcal{L}_{X_u})(X_u), dK_u \right\rangle |_{y=X_u}. \]  

**Proof. Step 1.** Suppose that \( b, \sigma \) are bounded. Fix \( x \in \mathbb{R}^d \) and define \( f(\mu) := \Phi(x, \mu) \). Now, we prove the Itô formula for \( f(\mathcal{L}_{X_t}) \).

For any positive integer \( N \), set
\[ x^1, x^2, \cdots, x^N \in \mathbb{R}^d, \quad f^N(x^1, x^2, \cdots, x^N) := f \left( \frac{1}{N} \sum_{l=1}^N \delta_{x^l} \right), \]  
and \( f^N(x^1, x^2, \cdots, x^N) \) is a function on \( \mathbb{R}^{d \times N} \). Moreover, by [3] Proposition 3.1, Page 15, it holds that \( f^N \) is \( C^2 \) on \( \mathbb{R}^{d \times N} \) and
\[ \partial_{x^i} f^N(x^1, x^2, \cdots, x^N) = \frac{1}{N} \partial_{\mu} f \left( \frac{1}{N} \sum_{l=1}^N \delta_{x^l} \right)(x^i), \]
\[ \partial_{x^i x^j} f^N(x^1, x^2, \cdots, x^N) = \frac{1}{N} \partial_{x^i} \partial_{\mu} f \left( \frac{1}{N} \sum_{l=1}^N \delta_{x^l} \right)(x^i) I_{i=j} + \frac{1}{N^2} \partial_{\mu}^2 f \left( \frac{1}{N} \sum_{l=1}^N \delta_{x^l} \right)(x^i, x^j), \]  
i, j = 1, 2, \cdots, N.  

Besides, we take \( N \) independent copies \( X^l_t, l = 1, 2, \cdots, N \) of \( X_t \). That is, consider these following equations
\[ \begin{cases} dX^l_t = -A(X^l_t)dt + b(X^l_t, \mathcal{L}_{X^l_t})dt + \sigma(X^l_t, \mathcal{L}_{X^l_t})dW^l_t, \\
X^l_0 = \xi, \quad l = 1, 2, \cdots, N, \end{cases} \]
where \( W^l_t, l = 1, 2, \cdots, N \) are mutually independent and identical distribution copies of \( W \). By Theorem [3.5] we know that for \( l = 1, 2, \cdots, N \), there exists \( (X^l_t, K^l_t) \in \mathcal{A} \) such that
\[ X^l_t - X^s_t = -(K^l_t - K^s_t) + \int_s^t b(X^l_r, \mathcal{L}_{X^l_r})dr + \int_s^t \sigma(X^l_r, \mathcal{L}_{X^l_r})dW^l_r, \quad 0 \leq s < t. \]
Then applying Itô’s formula to \( f^N(X^1_t, \cdots, X^N_t) \) and taking the expectation on both sides, we obtain that for \( 0 \leq t < t + \nu \)
\[ \mathbb{E} f^N(X^1_{t+\nu}, \cdots, X^N_{t+\nu}) \]
\[ = \mathbb{E} f^N(X^1_t, \cdots, X^N_t) + \sum_{i=1}^N \int_t^{t+\nu} \mathbb{E} \partial_{x^i} f^N(X^1_s, \cdots, X^N_s) b(X^i_s, \mathcal{L}_{X^i_s}) \, ds 
\]
\[ + \frac{1}{2} \sum_{i=1}^N \int_t^{t+\nu} \mathbb{E} \partial_{x^i x^j}^2 f^N(X^1_s, \cdots, X^N_s) \sigma \sigma^*(X^i_s, \mathcal{L}_{X^i_s}) \, ds 
\]
\[ - \sum_{i=1}^N \mathbb{E} \int_t^{t+\nu} \langle \partial_{x^i} f^N(X^1_s, \cdots, X^N_s), dK^i_s \rangle \]
\[ - \frac{1}{2} \sum_{i=1}^N \int_t^{t+\nu} \mathbb{E} \partial_{x^i x^j}^2 f^N(X^1_s, \cdots, X^N_s) \sigma \sigma^*(X^i_s, \mathcal{L}_{X^i_s}) \, ds. \]
where the property of the same distributions for \((X^l_t, K^l_t), l = 1, 2, \ldots, N\) is used in the second equality. Inserting (13) and (14) in the above equality, we get that

\[
\mathbb{E} f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_{t+\nu}} \right) = \mathbb{E} f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_t} \right) + \int_t^{t+\nu} \mathbb{E} \partial_\nu f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_s} \right) \left( X^1_s \right) b \left( X_s^1, \mathcal{L} X^1_s \right) ds
\]

\[
+ \frac{1}{2} \int_t^{t+\nu} \mathbb{E} \partial_y \partial_\nu f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_s} \right) \left( X^1_s \right) \sigma \sigma^* \left( X^1_s, \mathcal{L} X^1_s \right) ds
\]

\[
+ \frac{1}{2N} \int_t^{t+\nu} \mathbb{E} \partial^2_\nu f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_s} \right) \left( X^1_s, X^1_s \right) \sigma \sigma^* \left( X^1_s, \mathcal{L} X^1_s \right) ds
\]

\[- \mathbb{E} \int_t^{t+\nu} \left\langle \partial_\nu f \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_s} \right), dK^1_s \right\rangle.
\]

Next, we take the limit on both sides of the above equality. Note that

\[
\lim_{N \to \infty} \mathbb{E} \left[ \mathbb{W}^2 \left( \frac{1}{N} \sum_{l=1}^{N} \delta_{X^l_t}, \mathcal{L} X_t \right) \right] = 0,
\]

which comes from [8 Section 5]. And as \(N \to \infty\), by continuity and boundedness of \(f, \partial_\mu f, \partial_y \partial_\mu f\), and boundedness of \(\partial^2_\mu f, b, \sigma\), it follows from the dominated convergence theorem that

\[
f \left( \mathcal{L} X_{t+\nu} \right) = f \left( \mathcal{L} X_t \right) + \int_t^{t+\nu} \mathbb{E} \partial_\mu f \left( \mathcal{L} X_s \right) \left( X^1_s \right) b \left( X^1_s, \mathcal{L} X^1_s \right) ds
\]

\[
+ \frac{1}{2} \int_t^{t+\nu} \mathbb{E} \partial_y \partial_\mu f \left( \mathcal{L} X_s \right) \left( X^1_s \right) \sigma \sigma^* \left( X^1_s, \mathcal{L} X^1_s \right) ds
\]

\[- \mathbb{E} \int_t^{t+\nu} \left\langle \partial_\mu f \left( \mathcal{L} X_s \right), \left( X^1_s \right), dK^1_s \right\rangle
\]

\[
= f \left( \mathcal{L} X_t \right) + \int_t^{t+\nu} \int_{\mathbb{R}^d} \partial_\mu f \left( \mathcal{L} X_s \right) (y) b \left( y, \mathcal{L} X_s \right) \mathcal{L} X_s (dy) ds
\]

\[
+ \frac{1}{2} \int_t^{t+\nu} \int_{\mathbb{R}^d} \partial_y \partial_\mu f \left( \mathcal{L} X_s \right) (y) \sigma \sigma^* \left( y, \mathcal{L} X_s \right) \mathcal{L} X_s (dy) ds
\]

\[- \int_t^{t+\nu} \mathbb{E} \left\langle \partial_\mu f \left( \mathcal{L} X_s \right), \left( X^1_s \right), dK_s \right\rangle.
\]

\textbf{Step 2.} Assume that \((H_{1.1}) (H_{1.3})\) hold. Then we prove the Itô formula for \(f(\mathcal{L} X_t)\).
Let \( \phi_n : \mathbb{R}^d \to \mathbb{R}^d \) be a smooth function satisfying

\[
\begin{cases}
\phi_n(x) = x, & |x| \leq n, \\
\phi_n(x) = 0, & |x| > 2n,
\end{cases}
\]

such that

\[
|\phi_n(x)| \leq C, \quad ||\partial_x \phi_n(x)|| \leq C,
\]

where the positive constant \( C \) is independent of \( n \). Define

\[
b(n)(x,\mu) := b(\phi_n(x),\mu), \quad \sigma(n)(x,\mu) := \sigma(\phi_n(x),\mu),
\]

and by simple calculation it holds that \( b(n), \sigma(n) \) satisfy the assumption (H.1)-(H.2). Therefore, the following equation

\[
\begin{cases}
dX^{(n)}_t \in -A(X^{(n)}_t)dt + b(n)(X^{(n)}_t, \mathcal{L}_X^{(n)})dt + \sigma^{(n)}(X^{(n)}_t, \mathcal{L}_X^{(n)})dW_t, \\
X^{(n)}_0 = \xi,
\end{cases}
\]

has a unique solution \( (X^{(n)}, K^{(n)}) \in \mathcal{A} \). Besides, by (H.1), we know that \( b(n), \sigma(n) \) are bounded. Thus, by Step 1., it holds that for \( 0 \leq t < t + v, \)

\[
\begin{align*}
f(\mathcal{L}_X^{(n)}) &= f(\mathcal{L}_X^{(n)}) + \int_t^{t+v} \int_{\mathbb{R}^d} \partial_y f(\mathcal{L}_X^{(n)})(y)b(n)(y, \mathcal{L}_X^{(n)}) \mathcal{L}_X^{(n)}(dy)ds \\
&\quad + \frac{1}{2} \int_t^{t+v} \int_{\mathbb{R}^d} \partial_y \partial_y f(\mathcal{L}_X^{(n)})(y)\sigma(n)(\mathcal{L}_X^{(n)})(y, \mathcal{L}_X^{(n)}) \mathcal{L}_X^{(n)}(dy)ds \\
&\quad - \int_t^{t+v} \mathbb{E}\left\langle \partial_y f(\mathcal{L}_X^{(n)})(X^{(n)}_s), dK^{(n)}_s \right\rangle.
\end{align*}
\]

Next, we observe the limit of \( \mathcal{L}_X^{(n)} \) as \( n \to \infty \). Applying Itô’s formula to \( X^{(n)}_t - X_t \) and taking the expectation on both sides, we get that

\[
\begin{align*}
\mathbb{E}\left| X^{(n)}_t - X_t \right|^2 &= -2\mathbb{E} \int_0^t \left\langle X^{(n)}_s - X_s, dK^{(n)}_s \right\rangle ds \\
&\quad + 2\mathbb{E} \int_0^t \left\langle X^{(n)}_s - X_s, (b(n)(X^{(n)}_s, \mathcal{L}_X^{(n)}) - b(X_s, \mathcal{L}_X)) \right\rangle ds \\
&\quad + \mathbb{E} \int_0^t \left\| \sigma(n)(X^{(n)}_s, \mathcal{L}_X^{(n)}) - \sigma(X_s, \mathcal{L}_X) \right\|^2 ds \\
&\leq \mathbb{E} \int_0^t \left| X^{(n)}_s - X_s \right|^2 ds + \mathbb{E} \int_0^t \left| b(n)(X^{(n)}_s, \mathcal{L}_X^{(n)}) - b(X_s, \mathcal{L}_X) \right|^2 ds \\
&\quad + \mathbb{E} \int_0^t \left\| \sigma(n)(X^{(n)}_s, \mathcal{L}_X^{(n)}) - \sigma(X_s, \mathcal{L}_X) \right\|^2 ds \\
&\leq \mathbb{E} \int_0^t \left| X^{(n)}_s - X_s \right|^2 ds + L_3\mathbb{E} \int_0^t \left( |\phi_n(X^{(n)}_s) - X_s|^2 + \mathcal{W}_s^2 \right) \mathcal{L}_X^{(n)}(ds) \\
&\leq C\mathbb{E} \int_0^t \left| X^{(n)}_s - X_s \right|^2 ds + C\mathbb{E} \int_0^T |\phi_n(X_s) - X_s|^2 ds,
\end{align*}
\]
where we use (H\textsubscript{1,3}), (16) and the fact \( \mathbb{W}^2_t \left( L_{X_s^{(n)}}, L_{X_t} \right) \leq \mathbb{E} \left| X_s^{(n)} - X_t \right|^2 \). Therefore, by the Gronwall inequality, it holds that

\[
\mathbb{E} \left| X_t^{(n)} - X_t \right|^2 \leq C \mathbb{E} \int_0^T \left| \phi_n (X_s) - X_s \right|^2 ds.
\]

Combining \( \left| \phi_n (x) \right| \leq C \) for \( x \in \mathbb{R}^d \), with the estimate (11), by the dominated convergence theorem one can have that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \left| \phi_n (X_s) - X_s \right|^2 ds = 0,
\]

and

\[
\lim_{n \to \infty} \mathbb{E} \left| X_t^{(n)} - X_t \right|^2 = 0.
\]

Moreover, we obtain that

\[
\lim_{n \to \infty} \mathbb{W}^2_t \left( L_{X_t^{(n)}}, L_{X_t} \right) \leq \lim_{n \to \infty} \mathbb{E} \left| X_t^{(n)} - X_t \right|^2 = 0. \tag{19}
\]

Next, note that \( (X_s^{(n)}, K_s^{(n)}), (X_s, K_t) \) are the unique strong solution of Eq. (17) and (1), respectively, i.e.

\[
K_t^{(n)} = \xi - X_t^{(n)} + \int_0^t b(X_s^{(n)}, L_{X_s^{(n)}}) ds + \int_0^t \sigma(X_s^{(n)}, L_{X_s^{(n)}}) dW_s,
\]

\[
K_t = \xi - X_t + \int_0^t b(X_s, L_{X_s}) ds + \int_0^t \sigma(X_s, L_{X_s}) dW_s.
\]

Thus by (19) and (H\textsubscript{1,3}), we get

\[
\lim_{n \to \infty} \mathbb{E} \left| K_t^{(n)} - K_t \right|^2 = 0.
\]

Taking the limit on two sides of (18), by the dominated convergence theorem, one can conclude (15).

**Step 3.** We prove the Itô formula (12).

By the classical Itô’s formula and (13), it holds that

\[
\Phi(X_{t+v}, L_{X_{t+v}}) - \Phi(X_t, L_{X_t}) = - \int_{t}^{t+v} \langle \partial_x \Phi(X_s, L_{X_s}), dK_s \rangle + \int_{t}^{t+v} \langle \partial_x \Phi(X_s, L_{X_s}), b(X_s, L_{X_s}) \rangle ds
\]

\[
+ \int_{t}^{t+v} \langle \partial_x \Phi(X_s, L_{X_s}), \sigma(X_s, L_{X_s}) dW_s \rangle + \frac{1}{2} \int_{t}^{t+v} \mathrm{tr} \left( \sigma \sigma^* (X_s, L_{X_s}) \frac{\partial^2 \Phi(X_s, L_{X_s})}{\partial_x^2} \right) ds
\]

\[
+ \int_{t}^{t+v} \int_{\mathbb{R}^d} \langle b(y, L_{X_s}), \partial_y \Phi(X_s, L_{X_s}) (y) \rangle L_{X_s} (dy) ds
\]

\[
+ \frac{1}{2} \int_{t}^{t+v} \int_{\mathbb{R}^d} \mathrm{tr} \left( \sigma \sigma^* (y, L_{X_s}) \partial_y \partial_y \Phi(X_s, L_{X_s}) (y) \right) L_{X_s} (dy) ds
\]

\[
- \int_{t}^{t+v} \mathbb{E} \langle \partial_y \Phi(y, L_{X_s}) (X_s), dK_s \rangle |_{y=X_s}.
\]

The proof is complete.
Remark 4.2. Although we prove the Itô formula for any $\Phi \in C^{2,2}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, it also holds for $\Phi \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Please refer to [7, Proposition A.8] for details.

5. The stability of strong solutions

In the section, we require that $\xi = x_0$ is non-random and study the stability of strong solutions for Eq. (P) by the generalized Itô formula.

5.1. The asymptotic stability of the second moment for the strong solution. In the subsection, we consider the asymptotic stability of the second moment for the strong solution of Eq. (P). Assume:

(H2.1) There exists a function $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ satisfying

(i) $F \in C(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$,

(ii) $\int_{\mathbb{R}^d} \left( \mathcal{L}_\mu F(x, \mu) + \alpha F(x, \mu) \right) \mu(dx) \leq M_1$,

where $\mathcal{L}_\mu$ is defined by

$$\left( \mathcal{L}_\mu F \right)(x, \mu) := \left( b^i \partial_{x_i} F \right)(x, \mu) + \frac{1}{2} \left( (\sigma^*)^{ij} \partial^2_{x_i x_j} F \right)(x, \mu)$$

$$+ \int_{\mathbb{R}^d} b^i(y, \mu) \left( \partial_{\mu} F \right)_i(x, \mu)(y) \mu(dy)$$

$$+ \int_{\mathbb{R}^d} \frac{1}{2} (\sigma^*)^{ij}(y, \mu) \partial_{y_i} \left( \partial_{\mu} F \right)_j(x, \mu)(y) \mu(dy),$$

and $\alpha > 0, M_1 \geq 0$ are constants;

(iii) $a_1 \int_{\mathbb{R}^d} |x|^2 \mu(dx) - M_2 \leq \int_{\mathbb{R}^d} F(x, \mu) \mu(dx),$

where $a_1 > 0, M_2 \geq 0$ are constants.

In the following, we prepare an important lemma.

Lemma 5.1. Assume that $\text{Int}(\mathcal{D}(A)) \neq \emptyset$ and (H1.1) (H1.3) and (H2.1) hold. If the strong solution $(X_t, K_t)$ and the Lyapunov function $F$ satisfy for any $t > 0$

$$\langle \partial_t F(X_t, \mathcal{L}X_t), dK_t \rangle + \mathbb{E} \langle (\partial_{\mu} F)(x, \mathcal{L}X_t)(X_t), dK_t \rangle |_{x=X_t} \geq 0,$$

it holds that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}|X_s|^2 ds \leq \frac{\alpha M_2 + M_1}{\alpha a_1}.$$ 

Moreover, if $M_1 = M_2 = 0$,

$$\int_0^\infty \mathbb{E}|X_s|^2 ds < \infty.$$

Proof. By the Itô formula (12), it holds that

$$F(X_t, \mathcal{L}X_t) - F(x_0, \delta_{x_0})$$

$$= \int_0^t \left[ \left( b^i \partial_{x_i} F \right)(X_s, \mathcal{L}X_s) + \frac{1}{2} \left( (\sigma^*)^{ij} \partial^2_{x_i x_j} F \right)(X_s, \mathcal{L}X_s) \right]$$

$$+ \int_0^t \left[ b^i(y, \mu) \left( \partial_{\mu} F \right)_i(x, \mu)(y) \mu(dy) \right]$$

$$+ \int_0^t \frac{1}{2} (\sigma^*)^{ij}(y, \mu) \partial_{y_i} \left( \partial_{\mu} F \right)_j(x, \mu)(y) \mu(dy).$$

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which concludes that

Finally, based on \((iii)\) in \((H_{2.1})\), one can get that

\[
    a_1 E|X_t|^2 - M_2 \leq F(x_0, \delta_{x_0}) - a_1 \int_0^t E|X_s|^2 ds + (\alpha M_2 + M_1)t,
\]

which yields that

\[
    \frac{1}{t} \int_0^t E|X_s|^2 ds \leq \frac{F(x_0, \delta_{x_0}) + M_2}{\alpha a_1 t} + \frac{\alpha M_2 + M_1}{\alpha a_1},
\]

and furthermore

\[
    \limsup_{t \to \infty} \frac{1}{t} \int_0^t E|X_s|^2 ds \leq \frac{\alpha M_2 + M_1}{\alpha a_1}.
\]

The proof is complete.

Here we state and prove the main result in the subsection.

**Theorem 5.2.** Suppose that \(\text{Int}(\mathcal{D}(A)) \neq \emptyset\) and \((H_{1.1})\) \((H_{1.3})\) and \((H_{2.1})\) hold with \(M_1 = M_2 = 0\). If the strong solution \((X_t, K_t)\) and the Lyapunov function \(F\) satisfy for any \(t > 0\)

\[
    \langle \partial_x F(X_t, \mathcal{L}_{X_t}), dK_t \rangle + \mathbb{E} \langle (\partial_{\mu} F)(X_t) (X_t), dK_t \rangle \big|_{x=X_t} \geq 0,
\]
it holds that
\[
\lim_{t \to \infty} \mathbb{E}|X_t|^2 = 0.
\]

Proof. By (21) of Lemma 5.5 in order to obtain \(\lim_{t \to \infty} \mathbb{E}|X_t|^2 = 0\), we only need to prove that \(\mathbb{E}|X_t|^2\) is uniformly continuous in \(t\). Note that for any \(s, t\)
\[
|\mathbb{E}|X_t|^2 - \mathbb{E}|X_s|^2| \leq \mathbb{E}|X_t - X_s|^2 + 2(\mathbb{E}|X_t - X_s|^2)^{1/2}(\mathbb{E}|X_s|^2)^{1/2}.
\]
Therefore it is sufficient to show that \(\mathbb{E}|X_t - X_s|^2\) uniformly converges to 0 as \(s \to t\). For \(\mathbb{E}|X_t - X_s|^2\), by the same deduction to that of (7), we have that
\[
\mathbb{E}|X_t - X_s|^2 \leq L_1\left(1 + 3\mathbb{E}|X_s|^2\right)(t - s)e^{(1 + 2L_1)t},
\]
which together with (11) and the continuity of \(X_t\) in \(t\) implies that \(\mathbb{E}|X_t - X_s|^2\) uniformly converges to 0 as \(s \to t\). The proof is complete. \(\square\)

Remark 5.3. Here we remind that one can prove the above theorem without (21). In fact, when \(M_1 = M_2 = 0\), (22) becomes
\[
a_1\mathbb{E}|X_t|^2 \leq F(x_0, \delta_{x_0}) - \alpha a_1 \int_0^t \mathbb{E}|X_s|^2 ds.
\]
By the comparison theorem for integral equations, it holds that
\[
\mathbb{E}|X_t|^2 \leq \frac{F(x_0, \delta_{x_0})}{a_1} e^{-\alpha t},
\]
which yields that
\[
\lim_{t \to \infty} \mathbb{E}|X_t|^2 = 0.
\]

5.2. The almost surely asymptotic stability for the strong solution. In the subsection, we study the almost surely asymptotic stability of the strong solution for Eq.(1). We start with the concept of the almost surely asymptotic stability.

Definition 5.4. If for all \(x_0 \in \mathbb{R}^d\), it holds that
\[
\mathbb{P}\left\{\lim_{t \to \infty} |X_t| = 0\right\} = 1,
\]
we say that \(X\) is almost surely asymptotically stable.

Next we introduce a function class. Let \(\Sigma\) denote the family of functions \(\gamma : \mathbb{R}_+ \to \mathbb{R}_+\), which are continuous, strictly increasing, and \(\gamma(0) = 0\). And \(\Sigma_\infty\) means the family of functions \(\gamma \in \Sigma\) with \(\gamma(x) \to \infty\) as \(x \to \infty\). Then we present some assumptions.

(H1) The function \(b\) is continuous in \((x, \mu)\) and satisfies for \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)
\[
|b(x, \mu)|^2 \leq L_1'(1 + |x|^2 + \mu(\cdot)^2)),
\]
where \(L_1' > 0\) is a constant, and \(\sigma\) is bounded.

(H2) There exists a function \(F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) satisfying
\begin{enumerate}
  \item \(F \in C^{2,1}_{b+}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))\),
  \item \(\mathcal{L}_\mu F(x, \mu) + aF(x, \mu) \leq 0\),
  \item \(\gamma_1(|x|) \leq F(x, \mu) \leq \gamma_2(|x|),\) where \(\gamma_1, \gamma_2 \in \Sigma_\infty\).
\end{enumerate}
Theorem 5.5. Assume that Int(\(\mathcal{D}(A)\)) \(\neq \emptyset\) and (\(H_{\text{I.1}}\)) (\(H_{\text{I.3}}\)) and (\(H_{\text{II.2}}\)) hold. If the strong solution \((X, K)\) and the Lyapunov function \(F\) satisfy for any \(t > 0\)

\[
\langle \partial_x F(X_t, \mathcal{L}X_t), dK_t \rangle + \mathbb{E} \langle \partial_x F(x, \mathcal{L}X_t)(X_t), dK_t \rangle |_{x=X_t > 0},
\]

\(X\) is almost surely asymptotically stable, i.e.

\[
\mathbb{P}\left\{ \lim_{t \to \infty} |X_t| = 0 \right\} = 1.
\]

Proof. First of all, since under (\(H_{\text{I.1}}\)) (\(H_{\text{I.3}}\)) Eq.(\(1\)) has a unique strong solution \((X, K)\) with the initial value \((x_0, 0)\), the distribution family \(\{\mathcal{L}X_t\}_{t \geq 0}\) of \((X_t)_{t \geq 0}\) is known. Thus, we rewrite Eq.(\(1\)) as

\[
X_t = x_0 - K_t + \int_0^t \tilde{b}(u, X_u)du + \int_0^t \tilde{\sigma}(u, X_u)dW_u, \quad t \geq 0,
\]

where \(\tilde{b}(u, X_u) := b(X_u, \mathcal{L}X_u), \tilde{\sigma}(u, X_u) := \sigma(X_u, \mathcal{L}X_u)\). That is, Eq.(\(24\)) is a nonhomogeneous multivalued SDE. Set \(\tau_n := \inf\{t \geq 0, |X_t| > n\}\). Now applying Itô’s formula to \(|X_{s \wedge \tau_n} - x_0|^2\) for \(s \geq 0\), we get that

\[
|X_{s \wedge \tau_n} - x_0|^2 = -2 \int_0^{s \wedge \tau_n} < X_u - x_0, dK_u > + 2 \int_0^{s \wedge \tau_n} < X_u - x_0, \tilde{b}(u, X_u) > du + 2 \int_0^{s \wedge \tau_n} < X_u - x_0, \tilde{\sigma}(u, X_u)dW_u > + \int_0^{s \wedge \tau_n} \|\tilde{\sigma}(u, X_u)\|^2 du
\]

\[
\leq 2 \int_0^{s \wedge \tau_n} |X_u - x_0|^2 du + \int_0^{s \wedge \tau_n} \|\tilde{b}(u, X_u)\|^2 du + \int_0^{s \wedge \tau_n} \|\tilde{\sigma}(u, X_u)\|^2 du + 2 \int_0^{s \wedge \tau_n} < X_u - x_0, \tilde{\sigma}(u, X_u)dW_u > 
\]

Then by (\(H_{\text{I.1}}\)) and the Burkholder-Davis-Gundy inequality, it holds that

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + C\mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + n^2) du
\]

\[
+ C\mathbb{E} \left( \int_0^{t \wedge \tau_n} |X_u - x_0|^2 \|\tilde{\sigma}(u, X_u)\|^2 du \right)^{1/2}
\]

\[
\leq \mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + C\mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + n^2) du
\]

\[
+ \frac{1}{4} \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) + C\mathbb{E} \left( \int_0^{t \wedge \tau_n} \|\tilde{\sigma}(u, X_u)\|^2 du \right),
\]

which yields

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq C\mathbb{E} \int_0^{t \wedge \tau_n} |X_u - x_0|^2 du + C\mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_u|^2 + n^2) du.
\]
So, based on the boundedness of $X_t$ we conclude that
\[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0|^2 \right) \leq C \mathbb{E}(t \wedge \tau_n) \leq Ct, \]
where $C > 0$ depends on $L'_1$, $x_0$ and $n$. Then from the Chebyshev inequality, it follows that for any $\lambda > 0$,
\[ \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_{s \wedge \tau_n} - x_0| > \lambda \right\} \leq \frac{Ct}{\lambda^2}. \quad (25) \]

Next, we follow up the line in [3 Theorem 5.2] and apply (25) to obtain that
\[ \mathbb{P} \left\{ \lim_{t \to \infty} |X_t| = 0 \right\} = 1. \]
The proof is complete. \qed

6. An example

In this section, we present an example to explain our results.

**Example 6.1.** Let $\mathcal{O} := \{x \in \mathbb{R}^d; l(x) \geq 0\}$ be a convex closed domain with $\partial \mathcal{O} = \{x \in \mathbb{R}^d; l(x) = 0\}$, where $l$ belongs to $C^2_b(\mathbb{R}^d)$ with $\sum_{i=1}^d (\partial x_i l(x))^2 = 1$ and $\langle x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy), \partial_x l(x) \rangle \geq 0$ for $x \in \partial \mathcal{O}$ and $\mu \in \mathcal{P}_2(\partial \mathcal{O})$.

Next, we consider the following equation
\[ \begin{aligned}
\left\{ \begin{array}{l}
\text{d}X_t = -\partial I_{\mathcal{O}}(X_t)\text{d}t + \left( -X_t - \frac{1}{4} \int_{\mathbb{R}^d} y \mathcal{L}_{X_t}(dy) \right)\text{d}t + \text{d}W_t, \\
X_0 = x_0 \in \partial \mathcal{O},
\end{array} \right.
\end{aligned} \tag{26} \]

where
\[ A = \partial I_{\mathcal{O}}, \quad b(x, \mu) = -x - \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy), \quad \sigma(x, \mu) = I_d. \]

Moreover, by Example 2.7, it holds that
\[ \left\{ \begin{array}{l}
\text{d}X_t = \partial_x l(X_t)\text{d}|K|_0 + \left( -X_t - \frac{1}{4} \int_{\mathbb{R}^d} y \mathcal{L}_{X_t}(dy) \right)\text{d}t + \text{d}W_t, \\
X_0 = x_0 \in \partial \mathcal{O}, \quad \text{supp}(\text{d}|K|_0) \subset \{u \geq 0 : X_u \in \partial \mathcal{O}\}.
\end{array} \right. \]

So, one can justify that for $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$
\[ |b(x, \mu)|^2 = \left| x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy) \right|^2 \leq 2 \left( |x|^2 + \frac{1}{16} \int_{\mathbb{R}^d} y \mu(dy) \right)^2 \leq 2 \left( |x|^2 + \frac{1}{16} \int_{\mathbb{R}^d} y \mu(dy) \right)^2 \leq \frac{17}{8} \left( 1 + |x|^2 + \mu(| \cdot |)^2 \right), \]

and for $(x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,
\[ |b(x_1, \mu_1) - b(x_2, \mu_2)|^2 \leq 2 |x_1 - x_2|^2 + \frac{1}{16} \left| \int_{\mathbb{R}^d} y \mu_1(dy) - \int_{\mathbb{R}^d} y \mu_2(dy) \right|^2 \leq \frac{17}{8} \left( |x_1 - x_2|^2 + \mathbb{W}^2_2(\mu_1, \mu_2) \right). \]

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That is, $b, \sigma$ satisfy (H1.1) and (H1.3). By Theorem 3.3, we know that Eq. (20) has a unique strong solution $(X, K)$.

In the following, if we take the Lyapunov function

$$F(x, \mu) = \left| x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy) \right|^2,$$

$F$ satisfies (H2.1). Indeed, it is easily seen that $F$ satisfies (i). And by simple calculation, it holds that

$$\partial_x F(x, \mu) = 2(x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy)), \quad \partial_\mu F(x, \mu)(z) = \frac{1}{2}(x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy)),$$

$$\partial_{xx} F(x, \mu) = 2, \quad \partial_x \partial_\mu F(x, \mu) = 0.$$

Based on this, the following computations are reasonable:

$$(\mathcal{L}_\mu F)(x, \mu)$$

$$= -2 \left| x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy) \right|^2 + d - \frac{1}{2} \int_{\mathbb{R}^d} \left( z_i + \frac{1}{4} \int_{\mathbb{R}^d} y_i \mu(dy) \right) \left( x_i + \frac{1}{4} \int_{\mathbb{R}^d} y_i \mu(dy) \right) \mu(dz),$$

and

$$\int_{\mathbb{R}^d} \left( \mathcal{L}_\mu F(x, \mu) + 2F(x, \mu) \right) \mu(dx)$$

$$= -2 \int_{\mathbb{R}^d} \left| x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy) \right|^2 \mu(dx) + d - \sum_{i=1}^d \left( \int_{\mathbb{R}^d} \left( x_i + \frac{1}{4} \int_{\mathbb{R}^d} y_i \mu(dy) \right) \mu(dx) \right)^2$$

$$+ 2 \int_{\mathbb{R}^d} \left| x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy) \right|^2 \mu(dx)$$

$$\leq d.$$

That is, $F(x, \mu)$ satisfies (ii). Besides, note that

$$F(x, \mu) = \sum_{i=1}^d \left( x_i + \frac{1}{4} \int_{\mathbb{R}^d} y_i \mu(dy) \right)^2 \geq \sum_{i=1}^d \left( x_i^2 + \frac{x_i}{2} \int_{\mathbb{R}^d} y_i \mu(dy) + \frac{1}{16} \left( \int_{\mathbb{R}^d} y_i \mu(dy) \right)^2 \right)$$

$$\geq \sum_{i=1}^d \left( \frac{3x_i^2}{4} - \frac{3}{16} \left( \int_{\mathbb{R}^d} y_i \mu(dy) \right)^2 \right) \geq \sum_{i=1}^d \left( \frac{3x_i^2}{4} - \frac{3}{16} \int_{\mathbb{R}^d} y_i \mu(dy) \right).$$

Thus, we have

$$\int_{\mathbb{R}^d} F(x, \mu) \mu(dx) \geq \frac{9}{16} \int_{\mathbb{R}^d} |x|^2 \mu(dx),$$

which implies that $F(x, \mu)$ satisfies (iii).

Now, we verify that (21) holds. In fact, note that $\langle x + \frac{1}{4} \int_{\mathbb{R}^d} y \mu(dy), \partial_x l(x) \rangle \geq 0$ for $x \in \partial \mathcal{O}$ and $\mu \in P_2(\partial \mathcal{O})$. Thus, it holds that $\langle \partial_x F(x, \mu), \partial_x l(x) \rangle + \langle \partial_\mu F(x, \mu)(z), \partial_x l(x) \rangle \geq 0$ for $x \in \partial \mathcal{O}$ and $\mu \in P_2(\partial \mathcal{O})$. Therefore, (21) holds.

Finally, by the above deduction and Lemma 5.1, we know that the strong solution of Eq. (20) satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E} |X_s|^2 ds \leq \frac{8}{9} d.$$
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