Stability and Evolution of Color Skyrmions in the Quark-Gluon Plasma

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Abstract

We show the existence of unstable color skyrmions in a class of nonabelian fluid models. Oscillating and expanding solutions are found in the time-dependent case.

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1 Introduction

Hydrodynamics is the natural framework for the nuclear matter at finite temperature and at long distance scales. A novel nuclear matter state has been discovered in the Relativistic Heavy Ion Collision (RHIC) experiment [1] (for review see for example [2]). It is widely believed that a deconfinement phase transition happens and the corresponding deconfined state is the quark gluon plasma (QGP) [3] (for early work prior to QCD, see [4]). It has been estimated that the critical density and critical temperature are $\mu_C \sim 1 GeV/fm^3$, ten times the density of the normal nuclear matter, and $T_C \sim 170 MeV$ at which $\alpha_s \sim 1$. So the reliability of the perturbative description of QGP is highly suspicious. To apply hydrodynamics in this case requires the consideration of the propagation of the color charges. Therefore the corresponding liquid is “colored”. More interestingly, experiment shows that the shear viscosity of the deconfined nuclear matter is close to the lower bound set by theory based on AdS/CFT [5]. This implies that a perfect fluid is a good approximation to the real QGP. The framework for a perfect color liquid has been developed in [6] (for review see [7]).

In using the nonabelian fluid mechanics to explore the physics of QGP, the natural first step is to discuss the classical solution space. So we ask whether there exists any topological nontrivial solution. In fact within the framework of the nonabelian fluid, the configuration of fluid is described by a field $g$ which takes values in the color group, say $SU(n)$. Because QGP only exists in finite space region, a natural boundary condition is that $g$ should go to a constant at spacial infinity. Therefore, the fluid configurations are classified by $\Pi_3(SU(n)) = Z$. Moreover, in each topological class, we hope to check whether there exists a configuration minimizing the total energy. Such solutions are referred to as skyrmions related to color group or simply color skyrmions. In our previous paper [8], the existence of color skyrmion is shown for a particular choice of the hydrodynamic Hamiltonian. In this paper, we continue the discussion of the existence of color skyrmions in both time-independent and time-dependent cases.

We will consider the nonabelian fluid system whose equation of state (EOS) follows the so-called $\gamma$-law, namely the pressure density is proportional to the energy density:

$$\varphi = (\gamma - 1)\varepsilon. \quad (1)$$

From the relativistic fluid mechanics, we know that the case $\gamma = 4/3$ corresponds to radiation ($\varphi = \varepsilon/3$) while the case $\gamma = 1$ corresponds to dust ($\varphi = 0$). The up-to-date results on EOS for the isentropic expansion of QGP come from lattice simulation [9]. In the temperature region relevant to RHIC, for example, a phenomenological parametrization of the resulting equation of state is given by

$$\frac{\varphi}{\varepsilon} = \frac{1}{3} \left( 1 - \frac{1.2}{1 + 0.5\varepsilon \text{ fm}^3/\text{GeV}} \right). \quad (2)$$
We see it is a smooth transition from radiation to dust. In addition, we will take $\gamma = 2$ as a heuristic example. Consequently, we will restrict $1 \leq \gamma \leq 2$ in this paper. Our major results can be summarized as follows. For the time-independent case,

- $\gamma = 2$: we showed the existence of color skyrmion for this case in a particular ansatz for configuration in previous work. In this paper, the existence is shown for more general cases.
- $6/5 \leq \gamma \leq 5/3$: the topological configuration are in general unstable. The total energy gets minimized by infinite dilution in the fluid and the final value of energy is zero.
- $\gamma = 1$: the energy is minimized to a finite value when the fluid is infinitely diluted.
- $1 < \gamma < 6/5$ and $5/3 < \gamma < 2$: unstable topological configurations exist; however, the existence of any metastable soliton cannot be ruled out.

For time-dependent case, both oscillating and expanding solutions are found.

The organization of this paper is the following. In Sect. 2, preliminaries materials on nonabelian fluid mechanics are reviewed. In Sect. 3 and 4, static color skyrmions are thoroughly considered for different $\gamma$. In Sect. 5, time-dependent solutions are given. In Sect. 6, conclusions are given and some open issues are addressed.

## 2 Preliminaries

We describe the internal degrees of freedom for a single colored particle by a group element $g$ in the color group $SU(n)$. And the corresponding phase space structure is determined by the symplectic potential:

$$\Theta = -i \rho_s Tr(T_s g^{-1} dg)$$

from which a key Poisson bracket is determined

$$\{\rho_s, g\} = -ig T_s.$$  (4)

$T_s$ is a generator in Cartan subalgebra and $\rho_s$ is the conjugate momentum. The index $s$ runs over all the generators $T_s$. The dynamics of color degrees of freedom is therefore determined by the Lagrangian

$$L = -i \rho_s Tr(T_s g^{-1} \dot{g}) - H(\rho_s, g)$$

where $H$ is the Hamiltonian.
The generalization to many-particle system gives the nonabelian fluid Lagrangian density:

\[ \mathcal{L} = \sum_s j^\mu_s \omega_{s\mu} - F(\{n_s\}) \]  

(6)

where \( j^\mu_s \) is the color current, \( n_s^2 = j^\mu_s j_{s\mu} \) and \( F \) an invariant potential function. And

\[ \omega_{s\mu} = -i Tr(T_s g^{-1} \partial_\mu g) \]  

(7)

is the generalized velocity field. The action in a general metric background is given by

\[ S = \int d^4x \sqrt{g} \mathcal{L}. \]

In this paper, we will only consider rank-one group \( SU(2) \); therefore, the index \( s \) will be omitted hereafter.

Recall that the energy-momentum tensor of an ideal fluid is of the standard form

\[ T^{\mu\nu} = (\varepsilon + \varphi) u^\mu u^\nu - \varphi g^{\mu\nu} \]  

(8)

where \( u^\mu \) is the unit velocity field such that \( u^\mu u_\mu = 1 \). Now we identify \( u^\mu \) in the following way:

\[ j^\mu = n u^\mu. \]  

(9)

Note an on-shell condition is

\[ n \omega_\mu = F' j_\mu. \]  

(10)

With the identity \( \delta g = g g^{\mu\nu} \delta g_{\mu\nu} \) and the definition of the energy-momentum tensor,

\[ T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}, \]  

(11)

it can be verified that \( T^{\mu\nu} \) for the nonabelian fluid system in Eq. (6) is in accordance with the form for an ideal fluid in Eq. (8) and that

\[ \varepsilon = F, \quad \varphi = n F' - F. \]  

(12)

So we claim that the Lagrangian in Eq. (6) does describe an ideal fluid system. Eq. (12) shows that for the \( \gamma \)-law of Eq. (1), we solve

\[ F(n) = \frac{\alpha}{\gamma} n^\gamma, \]  

(13)

with \( \alpha \) a dimensional constant.

Remember the physical boundary condition for \( g \) as a field defined in space \( \mathbb{R}^3 \) in the previous introductory section. In accordance with the boundary condition, the configuration space is classified by the mapping class \( g : S^3 \to SU(n) \). We do not need to specify the topological number in this paper. The assumption that the configuration carries certain topological charges is enough in practice below.
3 Variational Analysis of Time-Independent Configurations

We discuss the existence of the static soliton solution with defined topological charge in the nonabelian fluid in this section.

3.1 General Setting

To do so, it is convenient to use Hamiltonian. In fact, the Lagrangian in Eq. (6) can be reexpressed as \( \mathcal{L} = \rho \omega_0 - \mathcal{H} \) where \( \rho = j^0 \). The Hamiltonian density is given by

\[
\mathcal{H} = j \cdot \tilde{\omega} + F
\]

and there is an on-shell constraint:

\[
\frac{\delta \mathcal{H}}{\delta j} = \tilde{\omega} + \tilde{\omega} = \omega - \frac{F'}{n} \frac{j}{n} = 0.
\]

With some simple algebra, we get a dynamical system

\[
\mathcal{H} = \frac{n}{F''} \omega^2 + F, \quad \rho^2 = n^2 + \frac{n^2}{F''} \omega^2
\]

where \( \omega = |\tilde{\omega}| \). In principle, the invariant density \( n \) can be eliminated from Hamiltonian by solving the second equation and the Hamiltonian becomes a function of \( \rho \) and \( \omega \). However, the constraint is in general unsolvable algebraically. This is the hard part of this issue.

Now we introduce the major physical assumptions of this paper, under which the existence of color skyrmions are considered.

1. We only consider the fluid configurations with a single characteristic scale, which is written as \( R \).
2. \( \rho \) scales like \( 1/R^3 \).
3. \( \omega \) scales like \( 1/R \).
4. \( F \) follows the \( \gamma \)-law.

We will simply call \( R \) as the size of fluid configuration. Remarkably, these assumptions are general enough regardless of the spin and the color of the fluid configuration. Accordingly, we introduce the dimensionless quantities in the following:

\[
\vec{X} = \vec{x}/R, \quad P = R^3 \rho, \quad \Omega = R \omega.
\]
Moreover, we parameterize the dimensional constant in the definition of $\gamma$-law as

$$\mu = \alpha^{1/(4-3\gamma)}.$$  \hspace{1cm} (18)

So we have two scales in the problem: $\mu$ which is fixed, and $R$ which can be varied to minimize the energy. It is natural to make the size of the configuration dimensionless by defining

$$r = \mu R.$$  \hspace{1cm} (19)

Instead of using the invariant density $n$, it is more convenient to use the ratio

$$Z = \frac{\rho}{\nu}.$$  \hspace{1cm} (20)

With these technical preparations and by putting the $\gamma$-law in, we can recast the physical quantities in a more compact form:

\[
\begin{align*}
H &= T + P, \\
T &= \frac{\mu}{\mu^{\nu-3\gamma}} \int d^3 X \frac{A}{Z^{2-\gamma}}, \\
P &= \frac{\mu}{\nu^{3(\gamma-1)}} \int d^3 X \frac{B}{Z^{\gamma}}, \\
Z^2 &= 1 + cZ^{2(\gamma-1)}. \hspace{1cm} (21)
\end{align*}
\]

where we introduce the following definitions

$$c = \frac{C}{r^{2(4-3\gamma)}}.$$  \hspace{1cm} (23)

and

$$A = \Omega^2 P^{2-\gamma}, \quad B = \frac{P^{\gamma}}{\gamma}, \quad C = \frac{\Omega^2}{P^{2(\gamma-1)}}.$$  \hspace{1cm} (24)

We will refer $T$ as tension term and $P$ as potential term. We can now define the basic problem of the variational procedure: Given $\gamma$, whether the total energy in Eq. (21) is able to be minimized by the variation of $r$ with the constraint given in Eq. (22)? The solution to this question gives rise to a stable configuration in nonabelian fluid within the variational method.

To solve this question, we first consider some general implications of the constraint equation (22) and the solution $Z$ as a function in $c$. Eq. (22) cannot be given algebraically for the generic value of $\gamma$. In other words, Eq. (22) is transcendental in general; even for the simple rational powers like $\gamma = 7/6, 6/5, 7/5, 8/5, 9/5, 11/6$, the roots of this equation cannot be written down. Now we rewrite Eq. (22) in another form

$$c = \frac{Z^2 - 1}{Z^{2(\gamma-1)}}.$$  \hspace{1cm} (25)
we see immediately that $Z > 1$ for all $c > 0$ since $Z$ as a ratio of two densities is always nonnegative. 1 The next step is to consider the monotonicity of $Z$ in $c$. We take derivatives on both sides of Eq. (25) to get

$$\frac{dc}{dZ} = \frac{2(2 - \gamma)}{Z^{2\gamma - 3}} + \frac{2(\gamma - 1)}{Z^{2\gamma - 1}}.$$  

(26)

So as long as $1 < \gamma < 2$, $dc/dZ > 0$ and $Z$ is strictly monotonic in $c$. Henceforth, we will assume

$$1 \leq \gamma \leq 2.$$  

(27)

Furthermore, we claim a very useful inequality:

$$Z > (\gamma - 1)cZ^{2\gamma - 3}.$$  

(28)

In fact, take derivatives on the both side of Eq. (22) and one can derive the following relation:

$$Z - (\gamma - 1)cZ^{2\gamma - 3} = \frac{Z^{2\gamma - 2}}{2Z^\gamma}.$$  

(29)

The claimed statement follows the fact that the righthand side of above equation is positive.

So much for our general consideration of $Z$.

3.2 $\gamma = 4/3$

In the following we will discuss the existence of variational solution for different $\gamma$. Recall the definition $c = Cr^{2(3\gamma - 4)}$. An important observation is that $\gamma = 4/3$ is a critical value for which the density ratio $Z$ is independent of the configuration size $R$. So even though the constraint in Eq. (22) can be solved algebraically in this case, the detail of the solution is irrelevant to the existence of stable configuration. In fact, the total energy is of the form

$$E = \text{const.} \frac{R}{R}$$  

(30)

where the unspecified constant comes from the dimensionless integral. Physically, this simplicity follows directly from the fact that the case $\gamma = 4/3$ describes the massless particle system. The immediate inference is that no stable fluid configuration with fixed topological charge exists at finite $R$. For the physics of QGP, this is not bad news because we expect to see the expansion process of the fireball of nuclear matter. So we refer this type of unstable configuration as the unstable color skyrmion. We can imagine that once an unstable configuration is generated, it can be stabilized through a process in which the size $R$ is enlarged and the energy is dissipated. Because the process is continuous, the topological charge is not changed during this process. However, the final state is a null state in which the configuration is completely diluted and the energy is completely dissipated.

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1Technically, $\rho$ can be zero at some points. However, such points have no contribution to energy; therefore in the integral they can be excluded.
3.3 Asymptotic Analysis

We discuss the asymptotic behaviors of the total energy in the limits $r \to 0$ and $\infty$ for $1 < \gamma < 4/3$ and $4/3 < \gamma < 2$. To do so, we need to solve Eq. (22) approximately in these two limits. In fact if $c = 0$ the physical solution to Eq. (22) is $Z = 1$. So, for small $c$, the equation can be solved approximately by

$$Z \approx 1 + \frac{c}{2}.$$  

(31)

For large $c$, we recast Eq. (22) in the form

$$\frac{Z^2}{c} = \frac{1}{c} + Z^{2(\gamma - 1)}.$$  

(32)

Let $Z \sim c^\lambda$ with a positive power $\lambda$ such that the first term on the right hand side of (32) can be dropped. The approximative solution is

$$Z = c^{1/2(2-\gamma)}.  

(33)

In the region $4/3 < \gamma < 2$ for small $r$, the contribution of $Z$ can be omitted and the potential term is the dominant part of the energy in Eq. (21),

$$H \sim \frac{\mu}{r^{3(\gamma - 1)}}.$$  

(34)

In the region $4/3 < \gamma < 2$ for large $r$,

$$Z \sim r^{(3\gamma - 4)/(2-\gamma)}, \quad T \sim \frac{\mu}{r}, \quad P \sim \frac{\mu}{r^{(5\gamma - 6)/(2-\gamma)}},  

(35)

therefore, the tension term dominates. In the region $1 < \gamma < 4/3$ for large $r$, the contribution of $Z$ can be omitted and again the potential term is dominant. In the region $1 < \gamma < 4/3$ for small $r$ from the same equation (35), the tension term dominates. The common feature of both regions of $\gamma$ is that the energy blows up in the $r \to 0$ limit and vanishes as $r \to \infty$. So the most immediate conclusion from above asymptotic analysis is that there always exists unstable color skyrmion in each region for $\gamma$.

Nevertheless, there is a subtle sub-region for both $1 < \gamma < 4/3$ and $4/3 < \gamma < 2$ from the behavior of energy in the small $r$ limit. In the sub-region $5/3 < \gamma < 2$ for small $r$, the tension term behaves like

$$T \sim \mu r^{3\gamma - 5} \to 0.$$  

(36)

Combining the second formula in Eq. (35) and Eq. (36), we conclude the profile of tension term has maximum for some finite $r$ in the region $5/3 < \gamma < 2$. In the sub-region $1 < \gamma < 6/5$ for small $r$, the potential term behaves like

$$P \sim \mu r^{(6-5\gamma)/(2-\gamma)},  

(37)
which implies that the profile of potential term has maximum for some finite \( r \) in the region \( 1 < \gamma < 6/5 \). These results reveal the possibility that there may exist metastable color skyrmions in these two sub-regions. By a metastable soliton, we mean that the energy has a local minimum. The implication for physics is the following. At the classical level, if the configuration is generated with a large enough size \( R \) then it is unstable and it can only be stabilized by expansion and dissipation until the configuration is completely diluted. For small sized configuration however, there is a locally stable point with finite energy.

### 3.4 \( 6/5 < \gamma < 5/3 \)

Beyond the asymptotic analysis, we conduct a monotonicity analysis to scrutinize a claim that there does not exist metastable color skyrmion in \( 6/5 < \gamma < 5/3 \). This conclusion is certainly true for \( \gamma = 4/3 \) from our previous discussion. For \( 4/3 < \gamma < 5/3 \), the conclusion is also easy to make. In fact, from the monotonicity of \( Z \) in \( c \) and the dependency of \( c \) on \( r \), it is easy to conclude that the total energy is strictly decreasing in \( r \). And the two limiting cases \( r \to 0, \infty \) have been considered in the asymptotic analysis.

The case for \( 6/5 < \gamma < 4/3 \) is more subtle. The monotonicity of total energy is determined by the two combinations,

\[
W_1 = \frac{Z}{c^{4\gamma-(1-3\gamma)}} \quad W_2 = \frac{Z}{c^{2(4-3\gamma)(2-\gamma)}},
\]

which are implicit in the expressions of potential term and tension term in total energy. By Eq. (22), we know \( W_2 \) satisfies the relation:

\[
W_2^2 = c^{-\frac{2\gamma-5}{(3\gamma-4)(2-\gamma)}} + c^{\frac{1}{3\gamma-4}} W_2^{2\gamma-1}.
\]

Taking derivatives on both sides,

\[
2 \left(W_2 - \frac{\gamma-1}{c^{4-3\gamma}} W_2^{2\gamma-3}\right) \frac{dW_2}{dc} = (-) \frac{Z^{2(\gamma-1)} + \frac{5-3\gamma}{(2-\gamma)c}}{(4 - 3\gamma)c^{(4-3\gamma)(2-\gamma)}},
\]

where we have expressed the right hand side back in terms of \( Z \). The bracketed factor on the left hand side can be also expressed in terms of \( Z \) as

\[
W_2 - \frac{\gamma-1}{c^{4-3\gamma}} W_2^{2\gamma-3} = \frac{Z - (\gamma-1)cZ^{2\gamma-3}}{c^{2(4-3\gamma)(2-\gamma)}} > 0.
\]

The last inequality is because of (28) so we conclude that

\[
\frac{dW_2}{dc} < 0,
\]

(42)
namely $W_2$ is strictly decreasing in $c$, hence increasing in $r$. Due to Eq. (22), $W_1$ satisfies
\begin{equation}
W_1^2 = c^{\frac{3(\gamma - 1)}{\gamma(4-3\gamma)}} + c^{\frac{6-5\gamma}{\gamma(4-3\gamma)}} W_1^{2\gamma - 2}.
\end{equation}
Taking derivatives on both sides of Eq. (38),
\begin{equation}
2(W_1 - \frac{\gamma - 1}{c^{\frac{5\gamma - 6}{\gamma(4-3\gamma)}}} W_1^{2\gamma - 3}) \frac{dW_1}{dc} = (-)(5\gamma - 6)Z^{2(\gamma - 1)} + \frac{3(\gamma - 1)}{\gamma(4 - 3\gamma)c^{\frac{4(\gamma - 1)}{\gamma(4-3\gamma)}}}.
\end{equation}
Again, the sign of $dW_1/dc$ is determined by the righthand side because the inequality (28) implies the left hand side is positive. So Eq. (44) implies
\begin{equation}
\frac{dW_1}{dc} < 0
\end{equation}
namely $W_1$ is also strictly decreasing in $c$. \footnote{Certain uniformity conditions for the profile functions $\Omega$ and $P$ are expected to guarantee that the conclusion does not change after the integral over the space coordinates $\int d^3X$.} To conclude this part of the monotonicity analysis, we see the total energy is strictly decreasing in $r$ in the region $6/5 < \gamma < 5/3$. In addition, the monotonicity cannot be established from the righthand sides of Eqs. (40,44) if $\gamma > 5/3$ or $\gamma < 6/5$. We leave the issue for possible metastable configurations to further work.

\subsection*{3.5 $\gamma = 2$}
Now we consider the boundary value of $\gamma = 2$. This is the case which has been worked out with a more specific ansatz in our previous paper [8]. The same result can be established on more general grounds here. For $\gamma = 2$, Eq. (22) can be solved by
\begin{equation}
n = \sqrt{\rho^2 - \frac{\omega^2}{\alpha^2}}.
\end{equation}
The Hamiltonian density in Eq. (14) is then
\begin{equation}
\mathcal{H} = \frac{\omega^2}{2\alpha} + \frac{\alpha\rho^2}{2}.
\end{equation}
The total energy is given by
\begin{equation}
H = \frac{\mu}{2} \int d^3X \left( r\Omega^2 + \frac{P^2}{r^2} \right).
\end{equation}
The existence of color skyrmion that minimizes the energy at certain finite $R$ is straightforward.
3.6 \( \gamma = 1 \)

In this case, Eq. (22) is solved by

\[
n = \frac{\rho}{\sqrt{1 + \frac{\omega^2}{\alpha^2}}}
\]

and the energy is

\[
E = \mu \int d^3 X \ P \sqrt{1 + \frac{\Omega^2}{r^2}}.
\]

The unstable color skyrmion has a finite energy in the limit \( R \to \infty \) and the final state energy is

\[
E \rightarrow \mu \int d^3 X \ P.
\]

To this point, we establish the results of the time-independent color skyrmions listed in the introductory section.

4 \( \gamma \)-Dependence of Total Energy

In previous section, we solve the variational problem for the nonabelian fluid system with fixed \( \gamma \). However, in reality the power \( \gamma \) is a changing quantity during the process of expansion of QGP. So in this section, we consider the dependence of the total energy on the power \( \gamma \). To do so, we will calculate \( \partial H/\partial \gamma \). We assume the profiles of \( \Omega, \ P \) and the parameters \( \mu, \ R \) are independent to \( \gamma \). We claim that \( \partial H/\partial \gamma < 0 \) in the “physical parameter region” \( 1 < \gamma < 4/3 \) provided the soliton size \( R \) is large enough. In fact,

\[
\frac{1}{\mu} \frac{\partial H}{\partial \gamma} = \int d^3 X \left( \frac{\Omega^2 P^{2-\gamma}}{r^{5-3\gamma} Z^{2-\gamma}} - \frac{P^\gamma}{\gamma P^{3(\gamma-1)Z}} \ln \frac{r^3 Z}{P} \right.
\]

\[
- \left[ (2 - \gamma) \frac{\Omega^2 P^{2-\gamma}}{r^{5-3\gamma} Z^{3-\gamma}} + \frac{P^\gamma}{r^{3(\gamma-1)Z^{\gamma+1}}} \right] \frac{\partial Z}{\partial \gamma}
\]

\[
- \frac{P^\gamma}{\gamma^2 r^{3(\gamma-1)Z^{\gamma}}} \right).
\]

We can impose the following sufficient conditions to guarantee \( \partial H/\partial \gamma < 0 \):

\[
\frac{Z^{2(\gamma-1)}}{r^{2(4-3\gamma)}} < \frac{P^{2(\gamma-1)}}{\gamma \Omega^2} ;
\]

\[
\frac{\partial Z}{\partial \gamma} > 0;
\]

\[
r^3 Z > P.
\]
By using Eq. (22) and the definition of $C$ in Eq. (24), one can show that (53) is equivalent to

$$Z^2 < \frac{\gamma + 1}{\gamma}.$$  

(56)

By the fact that $dZ/dc > 0$ and the relation in Eq. (23), we conclude that for large enough $r$, the inequality (56), hence (53), is satisfied if $1 < \gamma < 4/3$. For (54), we take derivatives with respect to $\gamma$ on both sides of (22),

$$\left(Z - (\gamma - 1)cZ^{2\gamma - 3}\right)\frac{\partial Z}{\partial \gamma} = cZ^{2\gamma - 2}\ln \frac{r^3 Z}{P}. \tag{57}$$

Recall the relation in (28). So (54) is satisfied if (55) is satisfied. To show (55), we need to check the monotonicity of the combination

$$W_3 \equiv c^{3/(6\gamma - 8)} Z. \tag{58}$$

We claim that

$$\frac{dW_3}{dc} < 0. \tag{59}$$

In fact, $W_3$ satisfies the equation

$$W_3^2 = c^{3/3\gamma - 4} + c^{2/3\gamma - 4} W_3^{2(\gamma - 1)}. \tag{60}$$

Take derivatives in $c$:

$$2(W_3 - (\gamma - 1)c^{2/3\gamma - 4} W_3^{2\gamma - 3})\frac{dW_3}{dc} = -\frac{1}{4 - 3\gamma}(3c^{7 - 3\gamma/3\gamma - 4} + 2c^{6 - 3\gamma/3\gamma - 4} W_3^{2\gamma - 2}). \tag{61}$$

It is easy to show

$$W_3 - (\gamma - 1)c^{2/3\gamma - 4} W_3^{2\gamma - 3} = c^{3/6\gamma - 8}(Z - (\gamma - 1)cZ^{2\gamma - 3}). \tag{62}$$

Therefore, this factor is positive. So from (61), it is obvious that $dW_3/dc < 0$ for $\gamma < 4/3$. We see that the conditions (54) and (55) are universally true provided $\gamma$ is in the physical region. But (53) requires a large configuration size.

Now we want to estimate the lower bound for $r$. From (22) and (55), we derive an interesting relation:

$$Z^2 > 1 + \frac{\Omega^2}{r^2}. \tag{63}$$

Combined with (56), we have

$$1 + \frac{\Omega^2}{r^2} < Z^2 < \frac{1 + \gamma}{\gamma} \Rightarrow r^2 > \gamma \Omega^2. \tag{64}$$

So given an ansatz profile function $\Omega$, we know how to estimate the configuration size such that the energy is strictly decreasing with $\gamma$. 

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The physical meaning of the condition $\partial H/\partial \gamma < 0$ is the following. It is easy to understand from the previous section that with expanding soliton size $r$, the energy decreases. On the other hand, the equation of state in Eq. (2) implies with the decreasing energy density, $\gamma$ becomes smaller as well. So there is a trajectory in the $(\gamma, r)$-plane along which the total energy can be expected to be unchanged during the expansion process.

5 Time Evolution of Color Skyrmions

So far we have dealt with time-independent fluid configurations. In this section, we consider the time-dependent configurations. To do so, we need to take account of the presence of $\omega_0$ which plays no role in the time-independent case. Accordingly, we need to eliminate $j^0$ as well as $\vec{j}$ by the equation of motion,

$$\delta \mathcal{L} = \omega_{\mu} - \alpha n^{\gamma-2} j_{\mu} = 0 \quad (65)$$

where we have used the $\gamma$-law. Eliminating $j^\mu$ by $j^\mu \omega_{\mu} = \alpha n^{\gamma}, \omega^\mu \omega_{\mu} = \alpha n^{\gamma-2} j_{\mu} \omega^\mu$, we get

$$\omega^\mu \omega_{\mu} = \alpha^2 n^{2\gamma-2}. \quad (66)$$

The Lagrangian density is therefore expressed in terms of $\omega_{\mu}$ as

$$\mathcal{L} = \frac{\gamma - 1}{\gamma} \alpha n^{\gamma} \gamma \frac{\gamma - 1}{\gamma} \alpha^{\gamma - 1} (\omega^\mu \omega_{\mu})^{\frac{\gamma}{2(\gamma-1)}}. \quad (67)$$

Since in this paper we will concentrate on classical configurations, the pre-factor $(\gamma-1) / \gamma \alpha^{1/\gamma-1}$ does not matter to us at this level. So we will deal with the following Lagrangian

$$L = \int d^3 x \left( \omega_0^2 - \vec{\omega} \cdot \vec{\omega} \right)^{\frac{\gamma}{2(\gamma-1)}}. \quad (68)$$

As in the case for the time-independent configurations, we make the following scaling assumptions:

$$\omega_0^2 = f_1 \frac{\dot{R}^2}{R^2}, \quad \vec{\omega} \cdot \vec{\omega} = f_2 \frac{\dot{R}^2}{R^2} \quad (69)$$

where $f_{1,2}$ are two dimensionless functions depending only on $\vec{X} = \vec{x}/R$. Then the Lagrangian in Eq. (68) is transformed to be

$$L(R, \dot{R}) = \dot{R}^{\frac{2\gamma-3}{\gamma-1}} \int d^3 X \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{\gamma}{2(\gamma-1)}}. \quad (70)$$

Next we will derive the Euler-Lagrangian equation for the Lagrangian in Eq. (70). First of all,

$$\frac{\partial L}{\partial \dot{R}} = \frac{\gamma}{\gamma - 1} \dot{R}^{\frac{2\gamma-3}{\gamma-1}} \int d^3 X f_1 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{\gamma-1}{2(\gamma-1)}}, \quad (71)$$
from which
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{R}} = \frac{\gamma}{\gamma - 1} \left( \frac{2 - \gamma}{\gamma - 1} R^{2\gamma - 3} \dot{R}^2 \int d^3 X f_1^2 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{4 - 3\gamma}{2(\gamma - 1)}} \right) + R^{2\gamma - 3} \int d^3 X f_1 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{2 - \gamma}{2(\gamma - 1)}} + \frac{2\gamma - 3}{\gamma - 1} R^{2\gamma - 2} \int d^3 X f_1 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{2 - \gamma}{2(\gamma - 1)}} \right).
\] (72)

Further
\[
\frac{\partial L}{\partial R} = \frac{2\gamma - 3}{\gamma - 1} R^{\frac{2\gamma - 2}{2(\gamma - 1)}} \int d^3 X \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{2}{2(\gamma - 1)}} \]. (73)

Thus the Euler-Lagrangian equation is given by
\[
\frac{2 - \gamma}{\gamma - 1} R \ddot{R}^2 \dot{R} \int d^3 X f_1^2 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{4 - 3\gamma}{2(\gamma - 1)}} + R \dot{R} \int d^3 X f_1 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{2 - \gamma}{2(\gamma - 1)}} + \frac{2\gamma - 3}{\gamma - 1} \dot{R} \int d^3 X f_1 \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{2 - \gamma}{2(\gamma - 1)}} - \frac{2\gamma - 3}{\gamma} \int d^3 X \left( f_1 \dot{R}^2 - f_2 \right)^{\frac{\gamma}{2(\gamma - 1)}} = 0. \] (74)

In the following we will investigate into two particular cases for $\gamma = 2$ and $\gamma = 4/3$.

5.1 $\gamma = 2$

In this case, stable time-independent color skyrmion exists. So we expect the solution to the time-evolution equation to be of the form of oscillations around the stable point. In fact, the Euler-Lagrangian equation (74) reduces to
\[
R \ddot{R} + \frac{\dot{R}^2}{2} + \mu = 0 \] (75)

where
\[
\mu = \frac{\int d^3 X f_2}{\frac{3}{2} \int d^3 X f_1}. \] (76)

To solve Eq. (75), we change its form to
\[
\frac{d^2}{dt^2} R^{3/2} + \frac{3\mu}{2 R^{1/2}} = 0. \] (77)

Let $q = R^{3/2}$ and $p = dq/dt$ then
\[
\frac{p}{\mu} \frac{dp}{dq} + \frac{3}{2q^{1/3}} = 0 \] (78)
which can be integrated as
\[ \frac{p^2}{2\mu} + \frac{9}{4}q^{2/3} = \mathcal{E} \]  
(79)
where \( \mathcal{E} \) is an integral constant. The solution in Eq. (79) means the color skyrmion for \( \gamma = 2 \) forms a one-dimensional Hamiltonian system with the potential of the form \( q^{2/3} \) and the motion is always bounded and oscillating!

5.2 \( \gamma = 4/3 \)

We know that only unstable color skyrmions exist in the time-independent case. Therefore, for the time-dependent case, we expect the solution to describe the expansion of the color skyrmion with the size going from some initial value to infinity. In fact, the Euler-Lagrangian equation (74) becomes
\[ \ddot{R}^2 \left( R^2 - \frac{\dot{R}^2}{4} \right) - \beta_2 \left( R^2 - \frac{\dot{R}^2}{2} \right) + \beta_4 = 0 \]  
(80)
where
\[ \beta_2 = \frac{\int d^3X f_1 f_2}{3 \int d^3X f_1^2}, \quad \beta_4 = \frac{\int d^3X f_2^4}{12 \int d^3X f_1^2}. \]  
(81)
We will solve Eq. (80) approximately. To do this, we first rewrite this equation by the form
\[ \frac{R\dot{R}}{R^2} - \frac{1}{4} = \frac{\beta_2}{R^2} \left( \frac{R\dot{R}}{R^2} - \frac{1}{2} \right) - \frac{\beta_4}{R^4} \]  
(82)
and make the following scaling assumptions:
\[ |\dot{R}| \gg 1, \quad |R\dot{R}/\dot{R}^2| \sim 1, \quad \beta_{2,4} \sim 1. \]  
(83)
To the zero order \( R \approx R^{(0)} \), Eq. (82) can be approximated by
\[ \frac{R^{(0)} \dot{R}^{(0)}}{R^{(0)}^2} - \frac{1}{4} = 0, \]  
(84)
which is solved by
\[ R^{(0)} = R_0^\left( \frac{t}{\tau} + 1 \right)^{4/3} \]  
(85)
with \( R_0, \tau \) two integral constants. The scaling rule in Eq. (83) works if the ratio
\[ \epsilon = \frac{\tau}{R_0} \ll 1. \]  
(86)
It is very interesting to see that \( R^{3/4} \) is linear in time. Eq. (85) describes the expansion of the color skyrmion for \( \tau > 0 \) and the contraction or collapse for \( \tau < 0 \). Actually, we will
only take the former case and consider the latter as unphysical. We will refer the motion in Eq. (85) as linear expansion.

To check the stability of the above-mentioned expansion motion, we do the perturbation to the order of $\epsilon^2$ by writing

$$ R = R^{(0)} + \delta R $$

(87)

and introducing the following quantities:

$$ \delta R = R_0 \epsilon^2 y, \quad x = \frac{t}{\tau} + 1, \quad f' = \frac{df}{dx}. $$

(88)

The perturbation satisfies the following equation

$$ y'' - \frac{2y'}{3x} + \frac{4y}{9x^2} = -\frac{\beta_2}{4x^{4/3}}. $$

(89)

Eq. (89) has the particular solution

$$ y = \frac{9\beta_2}{8} x^{2/3}. $$

(90)

It is easy to see that the linear independent solutions to the homogenous equation in (89) are $x^{4/3}$, $x^{1/3}$. So the general solution is given by

$$ y = \frac{9\beta_2}{8} x^{2/3} + Ax^{4/3} + Bx^{1/3} $$

(91)

where $A$, $B$ are two integration constants. By examining the initial conditions, we can fix $A$, $B$. Actually we let

$$ R|_{t=0} = R^{(0)}|_{t=0}, \quad \dot{R}|_{t=0} = \dot{R}^{(0)}|_{t=0} $$

(92)

then

$$ y|_{x=1}, \quad y'|_{x=1} = 0, $$

(93)

from which we get

$$ A = -\frac{3}{8} \beta_2, \quad B = \frac{3}{4} \beta_2. $$

(94)

The solution to Eq. (80), to $\epsilon^2$ order, is then given by

$$ R = R_0 \left( (1 - \frac{3}{8} \beta_2 \epsilon^2) x^{4/3} + \frac{9}{8} \beta_2 \epsilon^2 x^{2/3} - \frac{3}{4} \beta_2 \epsilon^2 x^{1/3} \right) $$

(95)

where $x = 1 + t/\tau$. To see the stability of the expansion, we consider the long time behavior. It is easy to see that the leading order contribution in the limit $t \to \infty$ is given by the $x^{4/3}$ term, which is the linear expansion as in Eq. (85).
6 Conclusion and Discussion

With the assumption of $\gamma$-law for the equation of state for a nonabelian fluid, we show in this paper the value of $\gamma$ is crucial to the existence and the properties of color skyrmion for time-independent configurations. For $\gamma$ between $6/5$ and $5/3$, there is only an unstable color skyrmion. For $\gamma = 2$, the skyrmion is stable. The case $\gamma = 1$ is special for the unstable skyrmion, since it has finite energy even after it is diluted infinitely. For $1 < \gamma < 6/5$ and $5/3 < \gamma < 2$, we cannot rule out the possibility of the existence of metastable skyrmions besides the unstable ones. For two particular values $\gamma = 4/3, 2$, we furthermore consider the time-dependent configurations. And for the latter we find the oscillating evolution and for the former we find linear expansion.

We now turn to the question of how this relates to QGP. As mentioned in the introduction, if we try to model the lattice estimate of the equation of state as a $\gamma$-law, the value of $\gamma$ varies between $4/3$ and $1$, as the energy density decreases. Therefore, our analysis of the expanding soliton for $\gamma = 4/3$ takes on a special significance. In the creation of the QGP by nuclear collision, we are starting at high energy densities, or at values of $\gamma$ close to $4/3$. The collision process also favors the creation of small solitons during the phase transition process since establishing coherent fields or color densities over large scales is generally more difficult, less probable. Based on these two observations and in the light of our general analysis, an expanding soliton would seem to be the generic case we can expect for the QGP. There are, of course, some caveats to our analysis. We have neglected thermal gradients as well as dissipative effects such as viscosity. (The latter quantity, although small, is not zero.) It would be interesting to incorporate some of these effects. The soliton by virtue of its nontrivial topology is a generic feature of nonabelian fluid dynamics. We may expect many qualitative aspects to hold true even with thermal gradients, viscosity, etc., although there can be significant changes in details.

We note that the possibility of color skyrmions arises also in the recent effective action description of Wilson lines [10, 11]. It is clear that skyrmions in the QGP merit further analysis. Other hydrodynamics approach towards QGP can be found for example in [12].

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