On the Approximate Nearest Neighbor Queries among Curves under the Fréchet Distance

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Abstract

Approximate nearest neighbor search (ANNS) is a long-studied problem in computational geometry that has received considerable attentions by researchers in the community. In this paper we revisit the problem in the presence of curves under the Fréchet distance. Given a set \( P \) of \( n \) curves of size at most \( m \) each in \( \mathbb{R}^d \) and a real \( \delta > 0 \), we aim to preprocess \( P \) into a data structure so that for any given query curve \( Q \) of size \( k \), report all curves in \( P \) whose Fréchet distances to \( Q \) are at most \( \delta \). In case that \( k \) is known in the preprocessing stage we propose a fully deterministic data structure whose space is \( O(n(32d^{1/2}/\varepsilon^3)^d(k+1)) \) and can answer the \((1 + \varepsilon)\delta\)-ANNS queries in \( O(kd) \) query time. Considering \( k \) as part of the query slightly changes the space to \( O(n(64d^{1/2}/\varepsilon^3)^md) \) with \( O(kd) \) query time within \( 5(1 + \varepsilon) \) approximation factor. We also show that our data structure could give an alternative treatment of the approximate subtrajectory range counting (ASRC) problem studied by de Berg et al. [12].

1 Introduction

The nearest neighbor search (NNS) is a classical problem with a long history of consideration in computer science that has many applications in different areas such as pattern recognition, computer vision, DNA sequencing, databases, GIS, etc [7, 22, 25, 32, 34]. In the elementary version of this problem, a set of points in \( \mathbb{R}^d \) is given and for any query point the aim is to find the nearest point, in the point set, to the given query point. Classical data structures, for instance point location using voronoi diagram, can solve the problem in \( O(n^{|d/2|} \log n) \) preprocessing time that cause the curse of dimensionality issue as \( d \) increases. While a linear search serves to solve the problem readily, numerous methods are employed to design data structures overcoming the running time bottleneck using approximation [5, 11, 27, 29, 30]. There are several practical algorithms that make use of conventional data structures including KD-tree, R-tree, X-tree, SS-tree, SR-tree, VP-tree, metric-trees for when the dimension \( d \) is low [6, 35, 37].

There is relatively much less known about the approximate nearest neighbor search (ANNS) among curves. Broadly speaking, curve data can be driven from a sequence of time...
stamped locations so called trajectories. Trajectories can be formulated to piecewise linear functions since part of the movement tracked by the GPS between every two consecutive locations can be linearly interpolated. This can also be seen as a polygonal curve in the literature of computational geometry. This steers the research to the point where the idea of developing data structures for curves as more complicated geometric data compared to points arises. Recently, the nearest neighbor search among curves has received noticeable amount of attentions. The very first inspiring steps towards stating the nearest neighbor problem between trajectories were initiated by Langran [31] and Lorentzos [33] that are more focussed on indexing spatiotemporal data in large databases. Not surprisingly research had mainly focussed on indexing databases so that basic queries concerning the data can be answered efficiently. At the time, the most common queries considered in the literature were different variants of nearest neighbour queries and range searching queries. Considering the nearest neighbor problem in the presence of trajectories demands meaningful distances so that the proximity between them gets captured and taken into account while searching for the nearest object. One of the most popular distances between polygonal curves that have interested many researchers for the past two decades is the Fréchet distance.

Fréchet distance is a very well-known metric with a lifetime longer than a century that is proposed by Maurice Fréchet [21]. Fréchet distance has applications in many areas, e.g., in morphing [18], movement analysis [22], handwriting recognition [36] and protein structure alignment [28]. It is intuitively the minimum length of the leash that connects a man and dog walking across the curves without going backward. Alt and Godau [2] were the first who computed the Fréchet distance between curves with total complexity $n^2 \log n$ and their algorithm runs in $O(n^2 \log n)$ time. In 2014, Bringmann [8] showed that, assuming the Strong Exponential Time Hypothesis, the Fréchet distance cannot be computed in strongly subquadratic time, i.e., in time $O(n^{2-\epsilon})$ for any $\epsilon > 0$. For the discrete Fréchet distance, which considers only distances between the vertices, Agarwal et al. [1] gave an algorithm with a (mildly) subquadratic running time of $O(n^2 \log \log n / \log n)$. Buchin et al. [10] showed that the continuous Fréchet distance can be computed in $O(n^2 \sqrt{\log n (\log \log n)^{3/2}})$ expected time. Bringmann and Mulzer [9] gave an $O(n^2 / \phi + n \log n)$-time algorithm to compute a $\phi$-approximation of the discrete Fréchet distance for any integer $1 \leq \phi \leq n$. For the continuous Fréchet distance, there are also a few subquadratic algorithms known for restricted classes of curves such as $\kappa$-bounded, protein backbone chains, $c$-packed and long-edged curves [3, 4, 15, 23].

There are different complementary variants of the ANNS problem among trajectory data under the Fréchet distance that are closely related in essence and they might be as much difficult as ANNS problem is such as approximate subcurve range searching ASRS, approximate subcurve range counting ASRC, and approximate subcurve range deciding ASRD. While such problems are studied in the community of computational geometry there might be different titles devoted to them in the literature. We will illuminate the backgrounds and existing results on the aforementioned problems later on. The related decision problems that are commonly considered under the Fréchet distance are as follows:

**Problem 1 ($\delta$-ANNS).** Given a set $P = \{P_1, \cdots, P_n\}$ of polygonal curves in $\mathbb{R}^d$ of size at most $m$ each and real $\delta > 0$, preprocess $P$ into a data structure so that for any query curve $Q$ of size $k$, the data structure returns all $P \in P$ if their Fréchet distances to $Q$ are at most $\delta$.

**Problem 2 ($\delta$-ASRS).** Given a single polygonal curve $P$ of size $n$ in $\mathbb{R}^d$ and a range param-
eter δ > 0, preprocess P into a data structure so that for any query curve Q of size k, report all subcurves P′ ⊆ P whose Fréchet distances to Q are at most δ.

Problem 3 (δ-ASRC). Given a polygonal curve P of size n in \( \mathbb{R}^d \) and range parameter δ > 0, preprocess them into a data structure so that for any query curve Q of size k, count all subcurves P′ ⊆ P whose Fréchet distances to Q are at most δ.

Problem 4 (δ-ASRD). Given a polygonal curve P of size n in \( \mathbb{R}^d \) and a range parameter δ > 0, preprocess them into a data structure so that for any query curve Q of size k, ‘decide’ whether there exists a subcurve P′ ⊆ P whose Fréchet distance to Q is at most δ.

In this paper, we intend to consider the approximate version of some of the problems mentioned above and we believe that they are all somewhat similar to each other, hence solving any of them can open space for solving the other ones. We particularly are interested to study the (1 + ε)δ-ANNS problem where if a subset of curves in \( \mathcal{P} \) are reported, then their Fréchet distances to Q are at most (1 + ε)δ and greater than δ, otherwise, for any ε > 0.

1.1 Related work

The first known result on ANNS among sequences of points (curves) under the discrete Fréchet distance has been obtained by Indyk [26]. For any parameter t > 1, the data structure in [26] uses \( O(m^2|X|^t/n^{1/t}) \) space that answers queries under the discrete Fréchet distance in \( O(m + \log n)^{O(t)} \) query time within the approximation factor of \( O((\log m + \log \log n)^{t-1}) \) where \( X \) is the metric set in which the curves are defined. This data structure for \( t = 1 + o(1) \), provides a constant factor approximation. Until recently in 2017, there was not much consideration of the problem under the Fréchet distance. Then Driemel and Silverstri [17] showed a locality-sensitive hash under the Fréchet distance where the Euclidean metric was considered. Their data structure is randomized and of size \( O(n \log n + nm) \) that answers queries in \( O(kd \log n) \) time within \( O(k) \) approximation factor. If they increase the space usage and query time to \( O(2^{km}n) \) and \( O(2^{km} \log n) \), respectively, then they can obtain an \( O(d^{1.5}) \) approximation factor. Subsequently, Emirris and Psarros [19] proposed a randomized data structure with approximation factor of \((1 + \epsilon)\).

Very recently, Driemel et al. [16] and Filtser et al. [20] concurrently obtained randomized data structures that answer queries in \( O(kd) \) query time within \((1 + \epsilon)\) approximation factor. While their space and construction times are quite different their ideas in handling queries are somewhat similar as they are using rounding idea to speed up the query time. The data structure by Driemel et al. [16] is randomized and has \( O(kd) \) query time and can be derandomized (deterministic) if the query time significantly goes up to \( O(d^{5/2}k^2\epsilon^{-1}(\log n + kd\log(kd/\epsilon))) \). The only requirement of their data structure is to have k known, the size of the query, at the preprocessing stage. Filtser et al. [16] show that one can construct a data structure, under the same assumption, in \( n \cdot O(1/k^d) \) and \( n \cdot O(d \log m + O(1/k^d)) \) expected time. In their terminology this variant of the setting, when k is known beforehand, is called asymmetric. In the symmetric setting, when k is assumed to be part of the query algorithm, they achieve a data structure of size \( n \cdot O(1/k^d) \) and \( n \cdot O(1/k^d) \) expected construction time with \( O(md) \) query time. However, in the symmetric case it is not quite clear that whether they have any prior knowledge about the query size beforehand or not. What is supposed
An attempt to summarize the existing and our results on the ANNS can be found in Table 1.

| Result | Approx. | Query time | Space | Assumptions |
|--------|---------|------------|-------|-------------|
| det. [26] | $O(1)$ | $O(m + \log n)^O(1)$ | $O(m^2 |X|^{m^{1-o(1)}} \cdot O(n^{2-o(1)})$ | dF, symmetric |
| rand. [17] | $O(k)$ | $O(kd \log n)$ | $O(n(md + \log n))$ | dF, symmetric |
| rand. [17] | $O(d^{1.5})$ | $O(2^{4md \log n})$ | $O(2^{4mdn})$ | dF, symmetric |
| rand. [19] | $1 + \varepsilon$ | $O(dm^{(1+\frac{1}{\varepsilon})} 2^{4m \log n})$ | $O(n) \cdot (2 + \frac{d}{\log m})^\lambda$ | dF, symmetric |
| rand. [20] | $1 + \varepsilon$ | $O(kd)$ | $n \cdot O(\frac{1}{\varepsilon})^{kd}$ | dF, asymmetric |
| det. [20] | $1 + \varepsilon$ | $O(md)$ | $n \cdot O(\frac{1}{\varepsilon})^{md}$ | dF, symmetric |
| det. [20] | $1 + \varepsilon$ | $O(kd \log(\frac{knd}{\varepsilon}))$ | $n \cdot O(\frac{1}{\varepsilon})^{kd}$ | dF, asymmetric |
| rand. [16] | $1 + \varepsilon$ | $O(kd)$ | $n \cdot O(\frac{k^{d/2}}{\varepsilon})^{kd}$ | dF, asymmetric |
| det. [16] | $1 + \varepsilon$ | $O\left(\frac{\log n + kd \log(kd/\varepsilon)}{\varepsilon d^{5/2} k^{2/2}}\right)$ | $nk \cdot O\left(\frac{k^{d/2}}{\varepsilon}\right)^{kd}$ | dF, asymmetric |
| det. Thm 8 | $1 + \varepsilon$ | $O(kd)$ | $n \cdot O\left(\frac{32d^{1/2}}{\varepsilon}\right)^{d(k+1)}$ | F-dF, asymmetric |
| det. Thm 14 | $5(1 + \varepsilon)$ | $O(kd)$ | $n \cdot O\left(\frac{6kd^{1/2}}{\varepsilon}\right)^{md}$ | dF, symmetric |

Table 1: Results on ANNS under the (discrete) Fréchet distance. Here, $\hat{O}$ hides factors polynomial in $1/\varepsilon$, $|X|$ is the cardinality of the metric set where curves are defined and $\lambda = O(m^{(1+\frac{1}{\varepsilon})} \cdot d \log(1/\varepsilon))$. (d)F briefly denotes the (discrete) Fréchet distance, rand. and det. stand for randomized and deterministic data structures, respectively.

Research on ANNS under the Fréchet distance is closely tied to developing data structures for a single curve. Designing data structures for one curve only has been initiated by a few important works [12, 14, 24] where the type of queries varies based on the application. Compared to the ANNS, data structures for single curve variant might seem relatively basic, however it might help us to solve the ANNS among multiple curves more efficiently.

In 2013, Driemel and Har-Peled [14] considered the approximate Fréchet distance queries (AFD) and presented a near-linear size data structure for an input curve $P$ such that for any given query curve $Q$ with $k$ vertices, and a subcurve $P' \subseteq P$ a $O(1)$-approximation of the Fréchet distance between $Q$ and $P'$ can be computed in $O(k^2 \log n \log(k \log n))$ time. Concurrently de Berg et al. [12] considered $\delta$-ASRC, a slightly different setting than [14] and they gave a data structure of quadratic size that can quickly count the number of subpaths of $P$, in polylogarithmic query time, whose Fréchet distance to a given query segment $Q$ is at
most $\delta$. Their data structure ‘approximately’ counts all subpaths whose Fréchet distances are at most $\delta$, but this count may also include subpaths of $P$ whose Fréchet distance is up to $(2 + 3\sqrt{2})\delta$. Subsequently, Gudmundsson and Smid [24] considered the $\delta$-ASRS problem for a special case where the input curve is $c$-packed. A curve is $c$-packed if for any ball of radius $r > 0$, the length of the curve contained inside of the ball is at most $2cr$. They showed for any constant $\varepsilon > 0$, a data structure of size $O((c + (1/\varepsilon^2d))^2 \log^2(1/\varepsilon)n)$ can be built in $O(((1/\varepsilon^2d)) \log^2(1/\varepsilon)n \log^2 n + 4n \log n)$ time, such that for any polygonal query path $Q$ of size $k$ the query algorithm returns ‘Yes’ or ‘No’ in $O((c^2/\varepsilon^4)k \log n)$ time. If the output is ‘Yes’ then the algorithm also reports all the subpaths $P'$ (the start point and the end point) on $P$ such that the Fréchet distance between $P'$ and $Q$ is at most $3(1 + \varepsilon)\delta$. If the output is ‘No’, then the Fréchet distance between them is greater than $\delta$. The main advantages of their approach is that it works for any constant dimension, and it can also be extended to the case when $P$ is a tree, however it underperforms when $c = \Omega(n)$. A while after, de Berg et al. [13] and Gudmundsson et al. [23] considered the Fréchet distance (FD) and $\delta$-Fréchet distance decision ($\delta$-FD) queries, respectively, and achieved $O(\log^2 n)$ query time under some specific assumptions. A summary of the existing results and our result can be found in Table 2.

| Data structure   | Approx. | Query time                                      | Space                                      | Assumptions   |
|------------------|---------|------------------------------------------------|--------------------------------------------|---------------|
| $\delta$-ASRC [12] | $2+3\sqrt{2}$ | $O(\frac{nm}{\varepsilon^2} \log^{O(1)} n)$ | $O(s \log^{O(1)} n)$                      | $Q$: long segment, $\mathbb{R}^2$ |
| AFD [14]        | $O(|X|^4)$ | $O(k^2 \log n \log(k \log n))$              | $O(n \log n)$                             | $\mathbb{R}^d$ |
| $\delta$-ASRS [24] | $3(1 + \varepsilon)$ | $O(\frac{c^2}{\varepsilon^2} \log n)$ | $O\left(n \left(c + \frac{\log^{s+1}(\frac{1}{\varepsilon^4})}{\varepsilon^4} \right)\right)$ | $P$: c-packed, $\mathbb{R}^d$ |
| FD [13]         | $1$     | $O(\log^2 n)$                                 | $O(n^2 \log^2 n)$                         | $Q$: segment, $\mathbb{R}^2$ |
| $\delta$-FD [23] | $1$ | $O(k \log^2 n)$                                | $O(n \log n)$                             | $Q$: long, $\mathbb{R}^2$ |
| $\delta$-ASRS,Thm 16 | $1 + \varepsilon$ | $O(k)$                                          | $O(n \lambda^{k+1}(\frac{1}{\varepsilon^2})^{d(k+1)})$ | $\mathbb{R}^d$, $k$ is given |

Table 2: Known results on data structures for $\delta$-ASRS, $\delta$-ASRC, $\delta$-FD, AFD, FD. Here $s$ is a parameter where $n \leq s \leq n^2$, $\lambda$ is a constant and $|X|$ is interpreted as the diameter of the metric space where the curve is defined.

1.2 Our contribution

In this paper we present a generic data structure that can handle different variants such as $(1+\varepsilon)\delta$-ANNS in both symmetric and asymmetric ways and $(1+\varepsilon)\delta$-ASRS problems. Our data structure is fully deterministic, meaning that both our preprocessing and query algorithms run deterministically unlike the one in [20] whose construction is randomized. In the asymmetric case, we present a data structure of size $O(n \left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)})$ and construction time $O(nmk^2d\left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)})$ such that for any polygonal query curve $Q$ of size at most $k$ it answers the $(1 + \varepsilon)\delta$-ANNS under the both discrete and continuous Fréchet distances in $O(kd)$ query time. In the symmetric case, when $k$ is part of the query, then we modify the data structure to the one of size $O\left(n \left(\frac{64d^{1/2}}{\varepsilon^3}\right)^{dm}\right)$ and construction time $O\left(nm\left(\frac{64d^{1/2}}{\varepsilon^3}\right)^{dm}\right)$.

1In the proof of Lemma 6.8 in [14] it is shown that the approximation factor is $O(r)$ where $r$ is the Fréchet distance between $Q$ and $P$. Clearly $r$ could be $\Omega(n)$, however one can bound the domain $X$ of the space, where the curve lives, so that $r \leq |X| = O(1)$.
such that for any polygonal query curve $Q$ of size $k$ answers the $5(1 + \varepsilon)\delta$-ANNS under the discrete Fréchet distance in $O(kd)$ query time. We assume that the diameter of the Euclidean space where curves are defined is constant. This seems to be a fair assumption in practice since it meets the requirements in GIS data where the diameter of map is less dependant on the number of trajectories and their granularity on the map. What yields us a deterministic data structure is that, similar to [16, 20], we use a grid as our main structural ingredient to speed up the search, however our grid is somehow bounded and of no need for randomly shifting [16]. We transpose the complexity of the input curves to a limited number of grid points as candidates and we store them into a hashtable if the curve and grid path are close. We use hashtable since the transposed complexity is independant on the input curve and it also provides us the opportunity of quickly retrieving the desired information associated to each grid path when it comes to the query part.

However, the difference in the way of using the grid becomes exposed when we cautiously construct it. The idea behind the construction is to restrict the possible placement of the query curve into the space by discarding the placements that are far enough from the input curves. This idea assists us to develop a deterministic data structure since we can immediately decide which placements are not close enough to any input curves.

The main strength of our results compared to [16, 20] is that our data structure is simpler and faster in query time while it is deterministic. We can assert that our data structure is relatively simpler and easier to implement since it does not exploit any randomized methods for example randomly shifted grid in [16, 17]. In the asymmetric case, it also works under the both continuous and discrete Fréchet distances. It is controversial to state that our data structure outperforms in all other aspects (space and/or construction time) since each data structure works better only under some specific circumstances of input parameters.

Our data structure, in the asymmetric case, can be extended to handle queries for $(1 + \varepsilon)\delta$-ASRS problem. This data structure has size $O\left(n\lambda^{k+1}\left(\frac{1}{\varepsilon^3}\right)^{(k+1)d}\right)$ and construction time $O\left(nk^2\lambda^{k+1}\left(\frac{1}{\varepsilon^3}\right)^{(k+1)d}\right)$ that can answers the mentioned queries in $O(k)$ time within $(1 + \varepsilon)$ approximation factor where $\lambda$ and $d$ are constants. Clearly, when $Q$ is a single line segment our data structure with size $O(n/\varepsilon^{6d})$ and $O(1)$ query time, improves on the data structure proposed by de Berg et al. [12].

2 Preliminaries

We consider the ANNS problem among trajectory data that occurs in many applications. A trajectory is a sequence of time-stamped locations in $\mathbb{R}^d$ where every point comes with a $d + 1$ coordinates. A trajectory can be interpreted as a piecewise linear function since the portion of it between the time-stamped points can be represented by a linear interpolation resulting in a straight-line segment. In the computational geometry literature a piecewise linear function is viewed as a polygonal curve: let $P = \langle p_1, p_2, \ldots, p_m \rangle$ be an input polygonal curve with $m$ vertices. We treat $P$ as a continuous map $P : [1, m] \rightarrow \mathbb{R}^d$, where $P(i) = p_i$ for integer $i$, and the $i$-th edge is linearly parametrized as $P(i + \lambda) = (1 - \lambda)p_i + \lambda p_{i+1}$. We write $P[s, t]$ for the subcurve between $P(s)$ and $P(t)$ and denote the segment, i.e., the straight line connecting them, by $\langle P(s)P(t) \rangle$. The Fréchet distance between two polygonal curves $P$ and
While the complexity of the query (which is at most some positive integer value \( k \)) is \( \Theta(n) \), the main idea of our construction is to use a grid of limited side length and store the near neighbors curves with respect to all possible ways of placing a query segment \( Q \) onto the grid cells. Once \( Q \) is given all we need is to search for a sequence of grid cells where the vertices of \( Q \) lie within them. We then report the prprocessed curves that are near neighbors to the cells determined close enough to the vertices of \( Q \). The main ingredient of our approach is somewhat similar to [16] and [20], although the main challenge is to preprocess the curves and using a bounded grid to achieve a deterministic data structure of faster query time.

We believe that if one carefully builds the data structure and make use of parameters properly then is able to achieve a query time independent of \( n \) and \( m \) which are size of the input curves. We begin our approach by describing the construction of our data structure. While the complexity of the query (which is at most some positive integer value \( k \)) must be known in this stage, the key idea is to compute all paths of size \( L \in \{1, \ldots, k\} \) whose vertices are grid points and distances are at most \((1 + \varepsilon/2)\delta\) to each input curve. Those input curves that are validated to have the Fréchet distance at most \((1 + \varepsilon/2)\delta\) to such grid paths, are stored into a hashtable by their indices. Note that every grid point acquires an unique id. Each bucket of the hashtable is associated with one unique grid path whose id is simply the concatenations of grid points’ ids so that we could retrieve the

\[
\delta_F(P,Q) = \inf_{(\sigma,\theta)} \max_{t \in [1, .., L]} \|P(\sigma(t)) - Q(\theta(t))\|
\]

where \( \sigma \) and \( \theta \) are continuous non-decreasing functions from \([0, 1]\) to \([1, m]\) and \([1, k]\), respectively. Finally, the discrete Fréchet distance (\( \delta_{DF}(P,Q) \)) is a variant where \( \sigma \) and \( \theta \) are discrete functions from \([1, \ldots, l]\) to \([1, m]\) and \([1, \ldots, k]\) with the property that \(|\sigma(i) - \sigma(i+1)| \leq 1 \) and \(|\theta(i) - \theta(i+1)| \leq 1 \) for any \( i \in \{1, \ldots, l-1\} \).

Let \( \mathcal{P} = \{P_1, \ldots, P_n\} \) be a set of \( n \) polygonal curves. As mentioned, each polygonal curve \( P \in \mathcal{P} \) can be represented as \( P = (p_1, \ldots, p_m) \) consisting of \( m \) vertices in \( \mathbb{R}^d \). Let \( V(P) \) denote the vertex set of \( P \in \mathcal{P} \). Then \( V(\mathcal{P}) \) is the set of vertices over all curves belonging to \( \mathcal{P} \), i.e., \( V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V(P) \). The diameter \( D \) of \( \mathcal{P} \) is then induced by the farthest pair of points in \( V(\mathcal{P}) \). Let \( \text{Prt}(\mathbb{R}^d, \ell) \) be a partitioning of \( \mathbb{R}^d \) into a set of disjoint cells (hypercubes) of side length \( \ell \) that is induced by axis parallel hyperplanes placed consecutively at distance \( \ell \). This induces a grid region denoted by \( \mathcal{G} \). If the hyperplanes above cut only a specific range of each axis by amount \( \mathcal{L} \) then we denote the side length of \( \mathcal{G} \) by \( \mathcal{L} \) whose grid cells have side length \( \ell \). Clearly \( \mathcal{N} = (\mathcal{L}/\ell)^d \) is the number of cells in \( \mathcal{G} \). Let \( X \subseteq \mathbb{R}^d \), for any point \( x \in X \) and real \( r > 0 \), the ball centered at \( x \) of radius \( r \) is denoted by \( B(x, r) \).

## 3 Data structure for the ANNS under the Fréchet distance

In this section we consider the decision version of the ANNS problem under the Fréchet distance among curves which also works under the discrete Fréchet distance. Our data structure is designed to be deterministic and simple. We believe that if one carefully understands the dependantcy between the input parameters once making assumption on them, they can achieve an efficient data structure in terms of space, construction and query times.

### 3.1 The main sketch

The main idea of our construction is to use grid of limited side length and store the near neighbors curves with respect to all possible way of placing a query segment \( Q \) onto the grid cells. Once \( Q \) is given all we need is to search for a sequence of grid cells where the vertices of \( Q \) lie within them. We then report the prprocessed curves that are near neighbors to the cells determined close enough to the vertices of \( Q \). The main ingredient of our approach is to preprocess the curves and using a bounded grid to achieve a deterministic data structure of faster query time.

We believe that if one carefully builds the data structure and make use of parameters properly then is able to achieve a query time independent of \( n \) and \( m \) which are size of the input curves. We begin our approach by describing the construction of our data structure. While the complexity of the query (which is at most some positive integer value \( k \)) must be known in this stage, the key idea is to compute all paths of size \( L \in \{1, \ldots, k\} \) whose vertices are grid points and distances are at most \((1 + \varepsilon/2)\delta\) to every input curve. Those input curves that are validated to have the Fréchet distance at most \((1 + \varepsilon/2)\delta\) to such grid paths, are stored into a hashtable by their indices. Note that every grid point acquires an unique id. Each bucket of the hashtable is associated with one unique grid path whose id is simply the concatenations of grid points’ ids so that we could retrieve the
bucket of interest whose grid path is closest to the query curve. We will describe this in our query algorithm in the dedicated section. We begin with the preprocessing algorithm below and then we show some of the technical details that help us to analyze the data structure.

### 3.2 Preprocessing algorithm

The input to this procedure is $\mathcal{P}, k, \delta$ and $\varepsilon$. The steps are mentioned as follows in Algorithm 1:

**Algorithm 1: The Preprocessing Algorithm**

1. **PreprocessingAlgorithm**($\mathcal{P}, k, \delta, \varepsilon$):
2. Approximate $\mathcal{D}$ greedily: start from an arbitrary vertex in $V(\mathcal{P})$ and find the farthest point to it. Call this value $\mathcal{D}'$.
3. if $\mathcal{D}' \leq \delta$ then Set $\mathcal{L} = 4\delta$ and $\ell = \varepsilon \delta / 2 \sqrt{d}$.
4. else if $\mathcal{D}' > \delta$ then Set $\mathcal{L} = 4 \delta \mathcal{D}' / \varepsilon$ and $\ell = \varepsilon^2 \delta \mathcal{D}' / 2 \sqrt{d}$.
5. Build $\mathcal{G}$ of side length $\mathcal{L}' = 2\mathcal{L}$ and grid cells of side length $\ell$ centered at an arbitrary vertex in $V(\mathcal{P})$.
6. forall $\mathcal{I} \in \{1, \cdots, k\}$ do
7.     forall sequences of grid points $\mathcal{C} = \langle c_1, \cdots, c_\mathcal{I} \rangle$ in $\mathcal{G}$ do
8.         forall $P \in \mathcal{P}$ do
9.             if $\delta_\mathcal{F}(P, \mathcal{C}) \leq (1 + \varepsilon / 2)\delta$ then Store the index of $P$ into the bucket of id associated with $\mathcal{C}$.

In line 2 we approximate the diameter $\mathcal{D}$. The classic 2-approximation algorithm for finding the diameter of $N$ points in $\mathbb{R}^d$, with running time $O(Nd)$, is to choose an arbitrary point and then return the maximum distance to another point. The diameter is no smaller than this value and no larger than twice this value (Lemma 4). In line 3,4 we check that whether the approximate diameter $\mathcal{D}'$ is larger than $\delta$ or not and we set the appropriate values to $\mathcal{L}$ and $\ell$ in each case that helps us to approximate the solution later. Suppose $\mathcal{L}$ is the minimum side length under which if $\mathcal{G}$ is centered appropriately, can contain the entire $V(\mathcal{P})$. In line 5 we double the side length $\mathcal{L}$ in order to fit the whole input curves in $\mathcal{P}$ within the grid space $\mathcal{G}$ of side length $\mathcal{L}' = 2\mathcal{L}$. This is because the center of the grid of side length $\mathcal{L}$ might be a vertex in $V(\mathcal{P})$ that is a maxima point with respect to some axis in $\mathbb{R}^d$ and hence the grid of side length $\mathcal{L}$ may not be able to cover the entire $\mathcal{P}$ along the same axis. Doubling the length up would fix the problem. Further details on this are mentioned in the approximation section. Finally in line 6-7 we consider all grid paths $\mathcal{C}$ of number of grid points $\mathcal{I} \in \{1, \cdots, k\}$ and in line 8-10 we store the indices of those curves $P \in \mathcal{P}$ that are approximately close to $\mathcal{C}$ (line 9) into an appropriate bucket of the hashtable (line 10). This can be done by associating distinct integers as ids to the grid points in $\mathcal{G}$. Once a grid path $\mathcal{C}$ (a sequence of grid points) is selected (line 7) the id associated with $\mathcal{C} = \langle c_1, \cdots, c_\mathcal{I} \rangle$ is simply the concatenation of id associated with each $c_i$ for $1 \leq i \leq \mathcal{I}$. Now we are ready to analyze our data structure below:

**Preprocessing time:** In line 2 of Algorithm 1 we first approximate the diameter $\mathcal{D}$ of $nm$ points. As described earlier we can compute $\mathcal{D}'$ in $O(nmd)$ time. Line 3-4 together take
constant time. Constructing $G$ in line 5 takes $O(N)$ time where:

$$N = \left(\frac{L'}{\ell}\right)^d = \left(\frac{2L}{\ell}\right)^d \leq \max \left\{ \left(\frac{16d^{1/2}}{\varepsilon}\right)^d, \left(\frac{16d^{1/2}}{\varepsilon^3}\right)^d \right\} = \left(\frac{16d^{1/2}}{\varepsilon^3}\right)^d,$$

for all $\varepsilon \leq 1$.

The expensive part of Algorithm 1 is between lines 6-10. There are:

$$\sum_{I=1}^{k} O(2^{dI} \cdot N^I),$$

grid paths in $G$ (line 6-7). Deciding the Fréchet distance between two curves of size $I$ and $m$ takes $O(dIm)$ [2], therefore given $n$ curves in $P$ (line 9) of size at most $m$ we have:

$$\sum_{I=1}^{k} O(2^{dI} \cdot N^I \cdot nImd) = \sum_{I=1}^{k} O(2^{dI} \cdot N^I \cdot nmd) = O(nmk^2d2^{d(k+1)}N^{k+1}).$$

Finally line 10 only takes $O(1)$. Plugging $N$ into the upper bound above yields us the following lemma:

**Lemma 1.** For any $0 < \varepsilon \leq 1$, the construction time of the data structure is:

$$O\left(nmk^2d\left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)}\right).$$

**Space:** The space required is only for storing the indices of curves in $P$ whose Fréchet distances to all grid paths $C$ are at most $(1+\varepsilon/2)\delta$ (line 10). Therefore the space is proportional to the number of grid paths times the number of indices stored into each bucket associated with each grid path id:

$$\sum_{I=1}^{k} O(2^{dI} \cdot N^I \cdot n) = O(n2^{d(k+1)}N^{k+1}).$$

Plugging $N$ into the upper bound yields the following lemma:

**Lemma 2.** For any $0 < \varepsilon \leq 1$, the space required for the data structure is

$$O\left(n\left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)}\right).$$

### 3.3 Query algorithm

In this section we present our query algorithm. Our query algorithm is very simple and make use of the property of grid we constructed together with the hashtable that consists of the near-neighbors of certain grid paths. The query algorithm is described as follows:

Step (1): Given $Q = \langle q_1, \cdots, q_{k'} \rangle$ and $k$ as the input parameters to the algorithm, where $k' \leq k$, check that if any vertex of $Q$ is outside of $G$. If so, terminate the algorithm.

Step (2): Find a sequence of grid cells $A = \langle a_1, \cdots, a_{k'} \rangle$ in $G$ where every vertex $q_i$ of
Q lie within $a_i$.

Step (3): Find an arbitrary corner $c_i$ of $a_i$ for all $i \in \{1, \cdots, k'\}$ and make a sequence $C = \langle c_1, \cdots, c_{k'} \rangle$ as a grid path in $\mathcal{G}$.

Step (4): Return the indices of curves that are stored into the bucket of id associated with the id of $C$.

Since it seems clear that what the query algorithm is performing, we set out to analyzing the performance of our query algorithm and its quality of approximation.

**Query time:** Step (1) checks that whether the query curve is close enough to any of the curves in $\mathcal{P}$ or not. This can be done in $O(kd)$ to check if all vertices of $Q$ lie within $\mathcal{G}$ or not. Step (2) computes all cells in $\mathcal{G}$ that contain the vertices of $Q$. This can be done by perfoming a binary search per vertex of $Q$ over grid cells while comparing every vertex’s coordinates with halfplanes passing through the coordinate and then splitting the grid space into $2^d$ subgrid spaces and recurse into the appropriate subgrid that contains the vertex until we reach the single cell. This takes $O(kd \log N) = O(kd \log \left(\frac{32d^{d/2}}{\varepsilon^2}\right)^{d(k+1)}) = O(k^2 d^2 (\log d + \log(1/\varepsilon))).$

Step (3) only takes $O(d)$ per vertex, hence $O(kd)$ time overall and Step (4) takes constant time to retrieve a string of indices of preprocessed curves from a bucket of the hashtable associated with $C$. Therefore, the total runtime is dominated by Step (2) which is $O(k^2 d^2 (\log d + \log(1/\varepsilon))).$

Note that our deterministic data structure in query stage as it is now still runs faster than the deterministic one proposed by Driemel and Psarros in [16], nevertheless we can use the ‘rounding’ idea similar to [16, 20] in order to improve the query time to $O(kd)$ which is what both of the papers obtain in their randomized data structures.

**Improved query time:** The idea is to instead of performing a binary search and finding a sequence of cells containing the vertices of $Q$, we use the coordinates of each vertex of $Q$ only to decide what grid point is of the rounded coordinates of the vertex’s coordinates. This can be done by scaling $\mathcal{G}$ properly in the preprocessing stage such that every cell attains unit side length, therefore every grid point would be of integer coordinates. We remember this scale and apply it to the coordinates of every vertex of $Q$ in the query stage later. Once a query curve is given, we scale the coordinates first and then round the coordinates of $Q$ to obtain a sequence of grid points. We then use the id of the obtained grid path to find the bucket in which the curves in $\mathcal{P}$ are stored. Clearly, the rounding takes $O(d)$ per vertex in $Q$, so the running time for Step (2) is $O(kd)$. We have the following lemma:

**Lemma 3.** Given a query curve $Q$ of size at most $k$, the query algorithm runs in $O(kd)$ time.

**Approximation:** In this part we examine the quality of approximation and show that both the preprocessing and query algorithms are cooperating to solve the $(1 + \varepsilon)\delta$-ANNS. First we need to show some properties in the following technical lemmas:

**Lemma 4.** Let $\mathcal{D}'$ be the approximate diameter described in Algorithm 1. Then $\mathcal{D}/2 \leq \mathcal{D}' \leq \mathcal{D}$.

**Proof.** The upper bound is obvious. For the lower bound assume that the diameter $\mathcal{D}$ is attained between two vertices $x, y \in V(\mathcal{P})$ and let $\mathcal{D}'$ be attained between a pair of vertices $p, q \in V(\mathcal{P})$. By triangle inequality we have $\|x - y\| \leq \|p - x\| + \|p - y\|$. Observe that $\|p - x\|
and \( \|p - y\| \) are smaller than \( \|p - q\| \) otherwise either of them would be \( D' \). Therefore:
\[
D = \|x - y\| \leq \|p - x\| + \|p - y\| \leq \|p - q\| + \|p - q\| = 2\|p - q\| = 2D',
\]
which completes the proof.\[\square\]

**Lemma 5.** Let \( 0 < \varepsilon \leq \delta \) and let \( G \) be a grid space of side length \( L' = 2L \) centered at an arbitrary vertex in \( V(\mathcal{P}) \) where \( L = \min\{4\delta, 4\delta D'/\varepsilon\} \). If there exists a vertex \( q \in Q \) that lies outside of \( G \), then \( \delta_F (P, Q) > \delta \) for all curves \( P \in \mathcal{P} \).

*Proof.* We prove this by showing that \( \|q - p\| > \delta \) for all points \( p \in P \) and \( P \in \mathcal{P} \). Let \( U \) be the set of all points (not necessarily vertices) in \( \bigcup_{i=1}^{n} P_i \). Let \( x \) and \( y \) be two points in \( U \) under which \( D \) is attained. Note that by a simple argument one can show that \( x \) and \( y \) are necessarily two vertices in \( U \), i.e., \( x, y \in V(\mathcal{P}) \). Now let \( C \) be the smallest enclosing hypercube of \( U \). We define \( C_\delta \) to be the hypercube whose center is identical to the center of \( C \) and its sides are enlarged additively by amount of \( \delta \) along all axes in \( \mathbb{R}^d \). Clearly, for any point \( q' \in \mathbb{R}^d \) outside of \( C_\delta \) it holds that \( \|q' - p\| > \delta \) for all points \( p \in U \). Let \( L \) be the side length of \( C_\delta \). We thus have \( L \leq D + 2\delta \) because of the enlarging argument mentioned earlier (see Figure 1). There are two cases to consider as follows:

1. \( D' \leq \delta \), hence \( L \leq D + 2\delta \leq 2D' + 2\delta \leq 4\delta \), since \( D \leq 2D' \) by Lemma 4.

2. \( D' > \delta \), hence \( L \leq D + 2\delta \leq 2D' + 2\delta \leq 4D' \leq 4D'/\varepsilon \), since \( D \leq 2D' \) following Lemma 4 and \( \delta \geq \varepsilon \).

Putting the two cases above together we have \( L = \min\{4\delta, 4\delta D'/\varepsilon\} \). Now setting \( L = L \) we get a grid space \( G \) that itself and its center are identical to \( C_\delta \) and the center of \( C_\delta \), respectively. Therefore \( G = C_\delta \). If the center of \( G \) changes over the vertices in \( V(\mathcal{P}) \), then \( G \) may not cover the entire \( U \). Therefore doubling its side length will yield a grid space \( G \) that covers the entire points in \( U \). Therefore \( G \) has side length \( L' = 2L \) whose center can be an arbitrary vertex in \( U \). Since \( C_\delta = G \subseteq G \) then for any point \( q' \in \mathbb{R}^d \) outside of \( G \) it holds that \( \|q' - p\| > \delta \) for all points \( p \in U \). This implies that for any vertex \( q \in Q \) outside of \( G \) we have \( \|q - p\| > \delta \) as well which results in \( \delta_F (P, Q) > \delta \).\[\square\]

**Lemma 6.** Let \( Q = \langle q_1, \cdots, q_k \rangle \) be a query curve and let \( A = \langle a_1, \cdots, a_k \rangle \) be the sequence of cells of side length \( \ell = \max \{\varepsilon \delta / 2\sqrt{d}, \varepsilon^2 \delta D'/2\sqrt{d}\} \) where each \( a_i \) contains \( q_i \) for all \( 1 \leq i \leq k \). For any \( 0 < \varepsilon \leq \frac{1}{D'} \) and for any grid path \( C = \langle c_1, \cdots, c_k \rangle \) where every \( c_i \) is an arbitrary corner of \( a_i \) it holds that \( \delta_F (Q, C) \leq \varepsilon \delta / 2 \).

*Proof.* According to Algorithm 1 there are two following cases to consider:

1. \( \ell_1 = \varepsilon \delta / 2\sqrt{d} \), if \( D' \leq \delta \). Let \( D_1 \) denote the diameter of each grid cell \( a_i \) in this case. Then \( D_1 = \ell_1 \cdot \sqrt{d} = (\varepsilon \delta / 2\sqrt{d}) \cdot \sqrt{d} = \varepsilon \delta / 2 \).

2. \( \ell_2 = \varepsilon^2 \delta D'/2\sqrt{d} \), if \( D' > \delta \). Let \( D_2 \) denote the diameter of each grid cell \( a_i \) in this case. Then we have:
\[
D_2 = \ell_2 \cdot \sqrt{d} = (\varepsilon^2 \delta D'/2\sqrt{d}) \cdot \sqrt{d} = \varepsilon^2 \delta D'/2 \leq \varepsilon^2 \delta D'/2 = \varepsilon^2 \delta / 2 \leq 1/D' \leq 1/D' \text{ by Lemma 4,}
\]
\[
\leq \varepsilon^2 \delta / 2 \cdot 1/\varepsilon = \varepsilon \delta / 2
\]
Figure 1: Any point $q$ outside the enlarged minimum enclosing cube of $P$ cannot have close distance to any point inside of the cube.

Setting $D = \min\{D_2, D_2\} \leq \varepsilon \delta / 2$ implies that $\|q_i - c_i\| \leq D \leq \varepsilon \delta / 2$ for every arbitrary corner $c_i \in a_i$. A simple Fréchet matching of width at most $\varepsilon \delta / 2$ (linearly) matches every $q_i$ to the corresponding $c_i$ and therefore $\delta_F(Q, C) \leq \varepsilon \delta / 2$. 

**Lemma 7** (Approximation). Let $0 < \varepsilon < 1$ and $P \in \mathcal{P}$ be a polygonal curve. If $P$ is returned by the query algorithm then $\delta_F(P, Q) \leq (1 + \varepsilon)\delta$. If $P$ is not returned then $\delta_F(P, Q) > \delta$.

**Proof.** Let $A = \langle a_1, \ldots, a_k \rangle$ be the sequence of cells containing the vertices of $Q$ and $C = \langle c_1, \ldots, c_k \rangle$ be a grid path where every $c_i$ is an arbitrary corner of $a_i$. Following Lemma 6 $\delta_F(Q, C) \leq \varepsilon \delta / 2$. Since $P$ is returned, it is already stored in the hashtable in line 10 of Algorithm 1 and $\delta_F(P, C) \leq (1 + \varepsilon / 2)\delta$. Applying a triangle inequality between $P$, $C$ and $Q$ yields:

$$\delta_F(P, Q) \leq \delta_F(Q, C) + \delta_F(P, C) \leq \varepsilon \delta / 2 + (1 + \varepsilon / 2)\delta = (1 + \varepsilon)\delta.$$

Now suppose $P$ is not returned by the query algorithm. Since $P$ is not returned it is not stored in the bucket of $C$’s id in line 10 of Algorithm 1 hence $\delta_F(P, C) > (1 + \varepsilon / 2)\delta$. Applying the other side of the triangle inequality yields:

$$\delta_F(P, Q) \geq |\delta_F(P, C) - \delta_F(Q, C)| > (1 + \varepsilon / 2)\delta - \varepsilon \delta / 2 = \delta.$$

This completes the proof. 

We summarize this section with the following theorem:

**Theorem 8.** Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of $n$ polygonal curves of constant diameter $D$ in $\mathbb{R}^d$ each of size at most $m$, $\delta$ be a constant and $k$ be the size of query given beforehand. For any $0 < \varepsilon \leq \min\{\frac{1}{D}, \delta, 1\}$ one can construct a deterministic data structure of size $O\left(n \left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)}\right)$ and construction time $O\left(n mk^2d \left(\frac{32d^{1/2}}{\varepsilon^3}\right)^{d(k+1)}\right)$ such that for any polygonal query curve $Q$ of size at most $k$ it computes the $(1 + \varepsilon)\delta$-ANNS under the both continuous and discrete Fréchet distances in $O(kd)$ query time.
Proof. The construction time, space and query time follow from Lemmas 1, 2 and 3, respectively. In these lemmas it holds that $0 < \varepsilon \leq 1$. In Step (1) of the query algorithm if any vertex of $Q$ is outside of $G$ of side length $2 \max\{4\delta, 4\delta D'/\varepsilon\}$ where $D/2 \leq D' \leq D$, following Lemma 5 $\delta_F(P, Q) > \delta$. Hence the algorithm terminates as desired because there is no $P$ close to $Q$. In this lemma it is assumed that $\delta \geq \varepsilon$ and $\varepsilon \leq 1/D$, thus overall $0 < \varepsilon \leq \min\{1/\delta, 1\}$. 

Now in Step (2) and Step (3) the algorithm finds a grid path $C$ close enough to $Q$ of distance at most $\varepsilon \delta/2$ (Lemma 6) and Lemma 7 implies that if any curve like $P \in P$ is returned by the query algorithm from the bucket of id associated with $C$ then $\delta_F(P, Q) \leq (1 + \varepsilon)\delta$ and $\delta_F(P, Q) > \delta$ otherwise which is either way a solution to the $(1 + \varepsilon)\delta$-ANNS decision problem. Note that all Lemmas 5, 6 and 7 work under the discrete Fréchet distance as well, therefore they result in solving the $(1 + \varepsilon)\delta$-ANNS under the discrete Fréchet distance.

\end{proof}

4 Symmetric ANNS data structure

In this section we set out to extending the data structure for the case where $k$ in not part of the preprocessing but the query algorithm. Since $k$ is not given in the preprocessing stage, the prior method may not be efficient for computing all grid path of size at most $k$ that causes exponential growth in space and construction time complexity. Beside we use the discrete Fréchet distance since the present method does not proceed under the continuous Fréchet distance. What Filtser et al. \cite{20} consider is to assume that both the query and input curves’ complexities are equal to $m$. They differentiate between this setting and the one where the size of query may not be equal to the input curve’s but given in the preprocessing, however, it seems that this may not really affect the nature of the problem since in either cases they are assuming that the size of the query curve is bounded by some known amount (either $k$ or $m$). What we assume is more general where we have no prior knowledge about the size of the query curve beforehand.

We describe our preprocessing algorithm below. As the beginning part of our present algorithm is similar to Algorithm 1 lines 2-5 we skip the similar parts and start from where the differences appear for the rest of it: we intersect the grid $G$ with the balls of radii $(1 + \varepsilon/2)\delta$ centered at vertices of curves in $P$ which gives us a subset of grid points lying within the balls in $\mathbb{R}^d$ and we mark the confined grid points as candidates for placing the vertices of the rounded query grid path. Next, we compute the grid points induced by $G \cap B(p_j, 3(1 + \varepsilon/2)\delta)$ where $p_j \in P$ and we denote this set by $g_{i,j}$. Now for each $P_i \in P$ we construct a graph $G_i$ whose vertices are $g_{i,j}$ for all $1 \leq j \leq |V(P_i)|$ and edges $\langle c_j, c_{j+1} \rangle$ are a stright line segment where $c_j \in g_{i,j}$ and $c_{j+1} \in g_{i,j+1}$. For each grid path $C_i \in G_i$ starting from a grid point in $g_{i,1}$ and ending to $g_{i,|P_i|}$ where $l = |V(P_i)| \leq m$, associate a bucket with a unique id induced by the concatenation of its grid points’ ids, and store the index of $P_i$ (say $i$) into the bucket.

**Construction time:** Approximating $D'$ and constructing $G$ together take $O(nmd + N)$ time where $N = \left(\frac{32d^2}{\varepsilon^3}\right)^d$ with $0 < \varepsilon \leq 1$. The number of vertices in $G_i$ is $O(mN^2d^2)$ and the number of edges is $O(mN^22^d)$ since we connect an edge between any pair of grid points lying within balls around every two consecutive vertices along $P_i$. Therefore constructing $G = \cup_{i=1}^r G_i$ takes $O(nmN^22^d)$ time. Associating buckets with all different paths in $G_i$ starting from grid points in $g_{i,1}$ and ending to grid points in $g_{i,|P_i|}$ where $l \leq m$ takes $O(n(N^2d)^m)$. Therefore, the overall construction time would be:

$$O\left(nmN^24^d + n(N^2d)^m\right) = O\left(nm(N^2d)^m\right).$$
We have the following lemma:

**Lemma 9.** For any $0 < \varepsilon \leq 1$, the construction time of the data structure is:

$$O\left(\frac{64d^{1/2}}{\varepsilon^3} \cdot d^m\right).$$

**Space:** The number of paths in $G_i$ of edge set size $O(mN^{2m}4^d)$ is $O\left((N^d)^m\right)$ and the number of buckets is the same as the number of paths in $G_i$. For each bucket we may store at most $n$ curves $P \in \mathcal{P}$, therefore we have: $O\left(n(N^d)^m\right)$.

**Lemma 10.** For any $0 < \varepsilon \leq 1$, the space required for the data structure is:

$$O\left(\frac{64d^{1/2}}{\varepsilon^3} \cdot d^m\right).$$

### 4.1 Query algorithm

Now we present the query algorithm which is similar to what we presented for the previous data structure. We first check if all vertices of $Q$ lie entirely inside of $\mathcal{G}$. Next we round $Q$ onto a grid path $C_Q$ whose vertices are selected from the marked grid points in the preprocessing stage. We then simplify the rounded query grid path using $\mu$-simplification by [15]. This results a simplified query grid path $C'_Q$ whose edges are longer than $\mu$. We choose $\mu = 2(1 + \varepsilon/2)\delta$ as the threshold parameter to this simplification. In the end we use the id of $C'_Q$ to retrieve all stored curves from the hashtable. The $\mu$-simplification and rounding take linear time each so the query algorithm takes $O(kd)$ overall. We have the following lemmas:

**Lemma 11** (Algorithm 2.1 in [15]). There is a linear time $\mu$-simplification of $P$ that gives $P'$ fulfilling $\delta_{dF}(P, P') \leq \mu$ where every edge of $P'$ has length strictly greater than $\mu$ except possibly the last one.

**Proof.** We describe the algorithm for the sake of a comprehensive presentation. The simplification is as follows: set the first vertex of $P$ as the current vertex $u$. Start from $u$ and find the first vertex $v$ along $P$ that lies outside of $B(u, \mu)$. Draw a line segment between $u$ and $v$ as a simplified link and set $v$ the current vertex. Repeat this process until the last vertex is reached.

Note that for every edge $\langle uv \rangle \in P'$ all points in $P$ before reaching $v \in P'$ are inside $B(u, \mu)$ thus they all can be matched to $u$ and therefore $\delta_{dF}(P, P') \leq \mu$. Also since $v$ is outside the ball $B(u, \mu)$ the length of $\langle uv \rangle$ is greater than $\mu$. Clearly every vertex is processed only once so it all takes linear time.

**Lemma 12.** For a query curve $Q$ of size $k$, the query algorithm takes $O(kd)$.

**Proof.** Following Lemma 11 and the fact that rounding takes linear time, the query algorithm takes $O(kd)$.

**Lemma 13** (Correctness). Let $0 < \varepsilon \leq 1$ and $P_i \in \mathcal{P}$ be a polygonal curve for some $1 \leq i \leq n$. If $P_i$ is returned by the query algorithm then $\delta_{dF}(P_i, Q) \leq 5(1 + \varepsilon)\delta$. If $P_i$ is not returned then $\delta_{dF}(P_i, Q) > \delta$. 


Proof. Suppose $P_i$ is stored into the hashtable already and returned by the query algorithm. Note that $\delta_{DF}(P_i,C_i) \leq 3(1 + \varepsilon/2)\delta$, for all $C_i \in G_i$. Since $P_i$ is returned then there exists a $C_i' \in G_i$ such that $C_i' = C_Q'$ because there is a bucket of id associated with $C_i'$ that $C_Q'$ caught it through the query algorithm. Thus $\delta_{DF}(P_i,C_Q') \leq 3(1 + \varepsilon/2)\delta$. Note that $\delta_{DF}(Q,C_Q) \leq \varepsilon\delta/2$ because the distance between every vertex and the rounded one is at most the diameter of the grid cells (Lemma 6). On the other hand $\delta_{DF}(C_Q',C_Q) \leq 2(1 + \varepsilon/2)\delta$ because of the $2(1 + \varepsilon/2)\delta$-simplification between $C_Q$ and $C_Q'$ (Lemma 11). Now applying a triangle inequality twice, once between $P_i$, $Q$ and $C_Q'$ and once again between $C_Q'$, $C_Q$ and $Q$, yields us:

$$
\delta_{DF}(P_i,Q) \leq \delta_{DF}(P_i,C_Q') + \delta_{DF}(C_Q',Q)
\leq \delta_{DF}(P_i,C_Q') + \delta_{DF}(C_Q',C_Q) + \delta_{DF}(C_Q,Q)
\leq 3(1 + \varepsilon/2)\delta + 2(1 + \varepsilon/2)\delta + \varepsilon\delta/2 = (5 + 3\varepsilon)\delta < 5(1 + \varepsilon)\delta,
$$

as desired. Now suppose $P_i$ is not returned. We show that $\delta_{DF}(P_i,Q) > \delta$. There are two cases occuring if $P_i$ is not returned:

1. This might be that all vertices of $C_Q$ are not entirely rounded onto the marked grid points. Clearly this results in having $\delta_{DF}(P_i,C_Q) > (1 + \varepsilon/2)\delta$ because some vertex of $C_Q$ lies outside of $B(p_j, (1 + \varepsilon/2)\delta)$ for all $1 \leq j \leq |V(P_i)|$. Now this time applying the triangle inequality yields:

$$
\delta_{DF}(P_i,Q) > |\delta_{DF}(P_i,C_Q) - \delta_{DF}(Q,C_Q)| > (1 + \varepsilon/2)\delta - \varepsilon\delta/2 = \delta.
$$

2. Case (1) does not occur but $C_Q' \notin G_i$. For the sake of a contradiction assume that $\delta_{DF}(P,Q) \leq \delta$. Since Case (1) is not occuring it implies that all vertices of $C_Q$ are rounded onto the marked grid points hence they are a subset of $V(G_i)$. Following Lemma 11 any $2(1 + \varepsilon/2)\delta$-simplification results in having simplified curve of edges longer than $2(1 + \varepsilon/2)\delta$ thus $C_Q'$ has at most one vertex within every $B(p_j, (1 + \varepsilon/2)\delta)$ for all $1 \leq j \leq |V(P_i)|$. Now if it has exactly one vertex inside of each such ball then clearly $C_Q' \in G_i$. If there is a ball $B(p_j, (1 + \varepsilon/2)\delta)$ for some $1 \leq j \leq |V(P_i)|$ such that no vertex of $C_Q'$ falls into it, then a segment of $C_Q'$ intersects with the ball. In other words there is at least a vertex $c_j \in C_Q$ where $c_j \in B(p_j, (1 + \varepsilon/2)\delta)$ but it is removed due to the $2(1 + \varepsilon/2)\delta$-simplification. Therefore $c_j \in B(p_{j'}, 2(1 + \varepsilon)\delta)$ for some $j' < j$. This implies that there is some $c_{j'} \in C_Q$ where $c_{j'} \in B(p_{j'}, (1 + \varepsilon/2)\delta)$. Therfore:

$$
||p_j - c_{j'}|| \leq ||p_j - c_j|| + ||c_j - c_{j'}|| \leq (1 + \varepsilon/2)\delta + 2(1 + \varepsilon/2)\delta = 3(1 + \varepsilon/2)\delta,
$$

as $||c_j - c_{j'}|| \leq 2(1 + \varepsilon/2)\delta$ because of the fact that $c_j$ is simplified since it is within $2(1 + \varepsilon/2)\delta$ distance from $c_{j'}$.

Since $||p_j - c_{j'}|| \leq 3(1 + \varepsilon/2)\delta$, it follows that $c_{j'} \in g_{i,j}$ and correspondingly $C_Q' \in G_i$ since $C_Q'$ has now exactly one vertex in $B(p, 3(1 + \varepsilon/2)\delta)$ for all $p \in V(P_i)$. This implies that $P_i$ is stored in the bucket of $C_Q'$’s id and $P_i$ is returned already. We have a contradiction.

Therefore $\delta_{DF}(P,Q) > \delta$ in both cases and this completes the proof. \qed

We now summarize the section with the following theorem: 
Theorem 14. Let $\mathcal{P} = \{P_1, \cdots, P_n\}$ be a set of $n$ polygonal curves in $\mathbb{R}^d$ each of size at most $m$, $\delta > 0$ and $D$ be constants. For any $0 < \varepsilon \leq \min\{\frac{1}{D}, \delta, 1\}$ one can construct a deterministic data structure of size $O\left(n\left(\frac{64d^{l/2}}{\varepsilon^3}\right)^{dm}\right)$ and construction time $O\left(nm\left(\frac{64d^{l/2}}{\varepsilon^3}\right)^{dm}\right)$ such that for any polygonal query curve $Q$ of size $k$ it computes the $5(1 + \varepsilon)\delta$-ANN under the discrete Fréchet distance in $O(kd)$ query time.

Proof. The construction time, size and query time follow from Lemmas 9, 10 and 12 with $0 < \varepsilon \leq 1$. As we refer to Lemma 6 to set the property of having grid cells of diameter at most $\varepsilon\delta/2$ in Lemma 13 we assume that $\varepsilon \leq \frac{1}{D}$. We also use Lemma 5 in our query algorithm at the beginning thus $\varepsilon \leq \delta$. Putting them all together we get $\varepsilon \leq \min\{\frac{1}{D}, \delta, 1\}$. The approximation correctness directly follows from Lemma 13. \hfill \Box

5 Approximate subtrajectory range searching queries

Another application of our data structure is the ability of solving the approximate subtrajectory range searching ($\delta$-ASRS) problem. In particular Theorem 8 indicates that for a query segment we only need linear storage in terms of the complexity of the curve to range search its subtrajectories to a single query segment. Recall that the $(1 + \varepsilon)\delta$-ASRS problem asks for reporting all subcurves of a curve $P$ whose Fréchet distances to a given query curve $Q$ are at most $\delta$ but it may report curves whose distance are $(1 + \varepsilon)\delta$ as well. De Berg et al. [12] gave a preliminary treatment of the ASRC problem for the case that $Q$ is only a single line segment and $P$ is given in the plane ($\mathbb{R}^2$). While they make some other assumptions on the length of the query segment to be large enough and on the length of the subcurves in the solution, their data structure of quadratic size approximately counts the number of subtrajectory in polylogarithmic query time up to the distances $(2 + 3\sqrt{2})\delta$. In their range counting they consider only a certain type of subcurves called ‘inclusion-minimal subcurves’. There might be a lot of (infinitely many) subcurves whose Fréchet distances to $Q$ is small but since they are along the same subcurve and possibly overlapped, therefore only one representative of them is sufficient for the counting purposes. This representative is the ‘inclusion-minimal subcurve’ that is the smallest subcurves in length that have Fréchet distance at most $\delta$ to $Q$. We will simply use this notion to handle this type of queries using our generic data structure. Gudmundsson and Smid [24] considered the ASRD and used the same notion of inclusion-minimal subcurves to handle the queries. They considered $c$-packed curves in $\mathbb{R}^d$ as input ($d$ is assumed constant) and they obtained a near linear size data structure of polylogarithmic query time. The space, construction time of their data structure significantly increase once one assumes that $c = \Omega(n)$. Especially the query time significantly becomes slower under this assumption as it gets near quadratic. However it outperforms compared to the data structure in [12] when $c = O(\text{polylog } n)$.

As a beneficial conclusion of our result, especially in Theorem 8, one can construct a linear size data structure with constant query time. Moreover, our the data structure can report the subtrajectories within approximation factor $(1 + \varepsilon)$ in any dimension. While we have no constraints on the length of $Q$ our only restriction is $\delta$ and the diameter that have to be constant and this can occur in practice where the trajectory data are realistic.

To summarize our data structure’s features compared to the ones in [12] and [24], has the following advantageous:

1. Faster query time $O(k)$ for any arbitrary dimension $d = O(1)$
2. $(1 + \varepsilon)$-approximation factor unlike $2 + 3\sqrt{2}$ in [12] and $3(1 + \varepsilon)$ in [24].

3. Linear space in $n$ and for when the query is low complexity ($k$ is small).

Below we recall a lemma from de Berg et al. [12] that it also works for a polygonal curve $Q$ of arbitrary edge length, and not necessarily for a single line segment.

Lemma 15 (Lemmas 2 and 3 in [12]). The following statements are true:

1. If there exists a subcurve $P' \subseteq P$ with $\delta_F(P', Q) \leq \delta$ then there exists an inclusion-minimal subcurve $P'' \subseteq P'$ such that $\delta_F(P'', Q) \leq \delta$.

2. All inclusion-minimal subcurves of $P$ are pairwise disjoint.

The Algorithm: The way we exploit the data structure in Theorem 8 to handle the $\delta$-ASRS queries is as follows: suppose a curve $P$ of size $n$ is given. For every grid path $C = \{c_1, \ldots, c_I\}$ of all lengths $I = \{1, 2, \ldots, k\}$ we compute the free space diagram between $P$ and $C$. First realize that if there is a subcurve whose Fréchet distance is small to $Q$ then there exists an inclusion-minimal subcurve as well Lemma 15 (if $Q$ is short then the inclusion-minimal subcurve would be a single point which is consistent with what we aim on our end). Without loss of generality assume that $C$ is aligned along the horizontal axis of the free space. We compute the inclusion-minimal subcurves along $P$ with respect to $C$ by staying on $c_1$ in free space and going upward until we hit a block space (point). Then we use the conventional Alt and Godaus dynamic programming algorithm [2] to reach $c_I$ of the lowest free point. This can be done by propagating the rightmost reachable cells of the free space. Since by Lemma 15 all inclusion-minimal subcurves are disjoint, this entire process take $O(nI)$ time by repeatedly starting the process from $c_1$ to find another one. We store the subcurves into a bucket of id associated with $C$’s id. The index we use for each subcurve to store in the hashtable is simply the concatenation of its parametrized starting and ending points. In the query algorithm, we only need to retrieve the inclusion-minimal subcurves stored in the hastable that are close enough to the rounded query grid path.

Given that the number of cella is $N = (\frac{16d^{1/2}}{\varepsilon^3})^d = O(\lambda \cdot (\frac{1}{\varepsilon^3})^d)$ where $\lambda = (16d^{1/2})^d$ and $d$ are constants, and there are $O(2^d N^I)$ grid path of length $I$ and $O(n)$ inclusion-minimal subcurves to store per grid path $C$ (due to their disjointness property), the space required is:

$$\sum_{I=1}^{k} O(n2^d N^I) = O(n2^{d(k+1)} \lambda^{k+1}) = O\left(n\lambda^{k+1} \left(\frac{1}{\varepsilon^3}\right)^{d(k+1)}\right).$$

The preprocessing is only computing the free space between $P$ and $C$ in $O(nI)$ time and then the bottom up traversal of the free space along $P$ axis of it, therefore it takes:

$$\sum_{I=1}^{k} O(2^d N^I \cdot nI) = O\left(nk^2 \lambda^{k+1} \left(\frac{1}{\varepsilon^3}\right)^{d(k+1)}\right).$$

We have the following theorem:

Theorem 16. Let $P$ be a curve with $n$ vertices in $\mathbb{R}^d$, $\delta > 0$ and $D$ be constants. For any $0 < \varepsilon \leq \min\{\frac{1}{D}, \delta, 1\}$ one can construct a data structure of size $O\left(n\lambda^{k+1} \left(\frac{1}{\varepsilon^3}\right)^{d(k+1)}\right)$ and
construction time $O\left( n k^2 \lambda^{k+1} \left( \frac{1}{2^k} \right)^{d(k+1)} \right)$ such that for any polygonal query curve $Q$ of size at most $k$ it computes the $(1 + \varepsilon) \delta$-ASRS under the Fréchet distance in $O(k)$ query time, where $\lambda$ and $d$ are constants.

6 Concluding remarks

In this paper, we considered the $\delta$-ANNS among curves under the Fréchet distance and proved several results solving the problem when the size of the query is known and unknown in preprocessing stage. Our data structures have size exponential in terms of the size of the input and query curves but linear in terms of the number of input curves. In particular our data structures are more efficient compared to the existing deterministic data structures in most of the aspects. We also showed that one could modify our data structure to handle the $\delta$-ASRS queries more efficiently. The only drawback of our data structure is that it requires the diameter of the metric space and the threshold parameter to be constants. An interesting problem to consider, for the future studies, would be to compute the optimization version of the problem.

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