Polyak-Ruppert Averaged Q-Leaning is Statistically Efficient

Xiang Li∗ Wenhao Yang† Zhihua Zhang‡ Michael I. Jordan§

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Abstract

We study synchronous Q-learning with Polyak-Ruppert averaging (a.k.a., averaged Q-leaning) in a γ-discounted MDP. We establish asymptotic normality for the averaged iteration $\bar{Q}_T$. Furthermore, we show that $\bar{Q}_T$ is actually a regular asymptotically linear (RAL) estimator for the optimal Q-value function $Q^*$ with the most efficient influence function. It implies the averaged Q-learning iteration has the smallest asymptotic variance among all RAL estimators. In addition, we present a non-asymptotic analysis for the $\ell_\infty$ error $\mathbb{E}\|\bar{Q}_T - Q^*\|_\infty$, showing it matches the instance-dependent lower bound as well as the optimal minimax complexity lower bound. As a byproduct, we find the Bellman noise has sub-Gaussian coordinates with variance $O((1-\gamma)^{-1})$ instead of the prevailing $O((1-\gamma)^{-2})$ under the standard bounded reward assumption. The sub-Gaussian result has potential to improve the sample complexity of many RL algorithms. In short, our theoretical analysis shows averaged Q-Leaning is statistically efficient.

1 Introduction

Q-learning [Watkins, 1989], as a model-free approach seeking the optimal Q-function $Q^*$ of a Markov decision process (MDP), is perhaps the most popular learning algorithm in reinforcement leaning (RL) [Sutton and Barto, 2018]. Analysis towards its sample efficiency moves from the asymptotic fashion [Singh, 1993, Tsitsiklis, 1994, Borkar and Meyn, 2000] to the non-asymptotic style [Szepesvári et al., 1998, Even-Dar et al., 2003, Beck and Srikant, 2012, Zhang et al., 2021, Chen et al., 2020]. However, it is until recent years that the exact sample efficiency of Q-learning is clearly depicted, which is on the order of $\tilde{O}(\frac{|S \times A|}{(1-\gamma)^4\varepsilon^2})$ tight up to some log factor [Li et al., 2021, 2020b]. Given the minimax lower bound $\Omega(\frac{|S \times A|}{(1-\gamma)^3\varepsilon^2})$ [Azar et al., 2013], it is natural to explore ways to fill this gap on the effective horizon $(1-\gamma)^{-1}$.

Though various algorithms successfully reach the goal [Lattimore and Hutter, 2014, Sidford et al., 2018a,b, Wainwright, 2019c], we prefer a simple extension of standard Q-learning. For one thing, many RL algorithms can be viewed through the lens of stochastic approximation (SA) which is an iterative method for solving root-finding problems [Robbins and Monro, 1951]. Indeed, Q-learning is
a special SA algorithm to solve the fixed point of the Bellman equation $TQ^* = Q^*$ where $T$ is the population Bellman operator (see (3) for definition). For the other thing, it is well known that the Polyak-Ruppert averaging procedure stabilizes and accelerates SA algorithms by taking an average over iterates [Polyak and Juditsky, 1992]. For example, the Polyak-Ruppert averaging accelerates policy evaluation [Mou et al., 2020a,b] and shows empirical superiority [Lillicrap et al., 2016, Anschel et al., 2017]. Hence, it is natural to ask whether Q-Learning with Polyak-Ruppert averaging (a.k.a., averaged Q-learning) could close the gap on the effective horizon $(1 - \gamma)^{-1}$.

In our work, we will give an affirmative answer to the question. Furthermore, we will give both asymptotic and non-asymptotic analysis of averaged Q-learning on a $\gamma$-discounted infinite-horizon MDP and in the synchronous setting where a generative model produces independent samples for all state-action pairs in every iteration [Kearns et al., 2002]. Unlike policy evaluation where the underlying structure is linear in nature and the goal is essentially to solve a linear system, Q-learning itself is non-linear, non-smooth and non-stationary$^1$ due to the Bellman operator $T$. In this more challenging problem, we cannot use classic SA theories directly.$^2$ Therefore, we develop a new analysis for averaged Q-learning to uncover asymptotic statistical properties and to provide non-asymptotic analysis with finite samples.

1.1 Our Contribution

In particular, we develop a ‘sandwich’ argument to decompose the error $\bar{Q}_T - Q^*$ into several terms, which either have a nice structure (sum of i.i.d. variables) or vanish in the $\ell_\infty$ norm with probability one. In this way, the non-asymptotic analysis reduces to careful revisits of these diminishing rates. The analysis method might be of independent interests. The theoretical findings of this paper are summarized as follows.

On the asymptotic side, we show averaged Q-learning sequence $\bar{Q}_T$ enjoys an asymptotic normality with $\text{Var}_Q$ (see (9)) the asymptotic variance. Our results accommodate the polynomial step size as well as a more generally decaying step size (see Assumption 3.3). Furthermore, we establish a semiparametric efficiency Cramer-Rao lower bound for any regular asymptotically linear (RAL) estimators (see Definition 3.1 for detail) of the optimal Q-value function $Q^*$. We further show that $\bar{Q}_T$ is the most efficient RAL estimator with the smallest asymptotic variance, showing its optimality in the asymptotic regime.

On the non-asymptotic side, we provide the first finite-sample error analysis of $\mathbb{E}\|\bar{Q}_T - Q^*\|_\infty$ in the $\ell_\infty$ norm for both linearly rescaled and polynomial step sizes. The error is dominated by $O\left(\sqrt{\text{diag}(\text{Var}_Q) \|_\infty} \sqrt{\frac{\ln |S \times A|}{T}}\right)$, which matches the instance-dependent lower bound established by Khamaru et al. [2021b]. We then perform a worst-case analysis showing $\|\text{diag}(\text{Var}_Q)\|_\infty = O((1 - \gamma)^{-3})$ for any $\gamma$-discounted MDPs. It, together with the finite-sample bound, implies averaged Q-learning achieves the optimal minimax sample complexity $\tilde{O} \left(\frac{|S \times A|}{(1 - \gamma)^{3/2}}\right)$ established by Azar et al. [2013]. As a byproduct, we find the Bellman noise (see definition in (7)) has sub-Gaussian coordinates whose variance is merely $O((1 - \gamma)^{-1})$ rather than the prevailing $O((1 - \gamma)^{-2})$ under

$^1$Here the non-stationarity means the maintained policy $\pi_t$ changes with iteration $t$. By contrast, $\pi_t \equiv \pi_b$ for all iteration $t$ in policy evaluation where $\pi_b$ is the target policy.

$^2$On the asymptotic side, Polyak and Juditsky [1992] provided a set of conditions under which a given SA algorithm, when combined with Polyak-Ruppert averaging, is guaranteed to have asymptotically optimal behavior. For the non-linear and non-smooth SA method Q-learning, it is unclear whether we can find a well-behaved Lyapunov function satisfying the conditions. On the non-symptotic side, Moulines and Bach [2011] provided a finite-sample analysis for SGD with Polyak-Ruppert averaging on smooth loss functions, which, as argued, is not suitable for Q-learning.
the standard bounded reward assumption. The sub-Gaussian result has potential to improve the sample complexity of many RL algorithms. For example, it improves the finite-sample analysis of Q-learning in Wainwright [2019b] directly to the optimal without using a sophisticated recursion argument as in Li et al. [2021].

1.2 Related Work

Due to the rapidly growing literature on Q-learning, we review only the theoretical results that are highly relevant to our work. Interested readers can check references therein for more information.

Asymptotic normality in RL. The establishment of asymptotic normality helps conduct statistical inference and quantify randomness. Yang et al. [2021] studies asymptotic behaviors of robust estimators in MDPs, while most efforts focus on the off-policy evaluation (OPE) where one aims to compute the value function of a given policy using pre-collected data. Analysis is simplified a lot due to the linear nature of OPE. With a Cramer–Rao lower bound established in Jiang and Li [2016], asymptotic efficiency of estimators using linear approximation has been discussed [Uehara et al., 2020, Hao et al., 2021, Yin and Wang, 2020, Mou et al., 2020a] as well as a semiparametric doubly robust estimator [Kallus and Uehara, 2020]. To estimate the optimal $Q^*$, multi-stage algorithms have been proposed and their asymptotic behaviors have been analyzed [Luckett et al., 2019, Shi et al., 2020]. We supplement these upper bound works with a semiparametric efficiency lower bound and show averaged Q-learning is the most efficient RAL estimator to achieve it.

Sample complexity for Q-learning. To obtain an $\varepsilon$-accurate estimate of the optimal Q-function $Q^*$ in a $\gamma$-discounted MDP, in the presence of a generative model, model-based Q-value-iteration has been shown to achieve optimal minimax sample complexity $\tilde{O} \left( \frac{D}{\varepsilon^2 (1-\gamma)^2} \right)$ [Azar et al., 2013, Agarwal et al., 2020, Li et al., 2020a]. In the model-free context, Wainwright [2019b] empirically found that the usual Q-learning suffers from at least worst-case fourth-order scaling in $(1-\gamma)^{-1}$ in sample complexity. A complexity $\tilde{O} \left( \frac{D}{\varepsilon^2 (1-\gamma)^2} \right)$ is provided [Wainwright, 2019b, Chen et al., 2020], which is far from the optimal though better than previous efforts [Even-Dar et al., 2003, Beck and Srikant, 2012]. Li et al. [2021] gave a sophisticated analysis showing the complexity of Q-learning is $\tilde{O} \left( \frac{D}{\varepsilon^2 (1-\gamma)^2} \right)$ and provided a matching lower bound to confirm its sharpness. By contrast, our sub-gaussian result produces a better variance bound on Bellman noises and thus directly improves the analysis of Wainwright [2019b] to the optimal without additional efforts. Wainwright [2019c], Khamaru et al. [2021b] introduced a variance-reduced [Gower et al., 2020] variant of Q-learning to achieve the optimal sample complexity and instance complexity. Our results show that a simple average over all history $Q_t$ is sufficient to guarantee the same optimality. The averaged method is fully online without additional samples and storage space.

Non-linear stochastic approximation. Some researchers study Q-learning through the lens of non-linear stochastic approximation. Under this bigger framework, Q-learning is treated as a special case. In this avenue, analysis on asymptotic convergence is provided in Tsitsiklis [1994], Borkar and Meyn [2000]. On the non-asymptotic side, Q-learning is studied either in the synchronous setting [Shah and Xie, 2018, Wainwright, 2019b, Chen et al., 2020] or the asynchronous setting where only one sample from current state-action pair is available at a time [Qu and Wierman, 2020, Li et al., 2020b, Chen et al., 2021]. As argued, their sample complexity is far from the optimal. Others
consider Q-learning with linear function approximation in the ℓ₂ norm [Melo et al., 2008, Chen et al., 2019]. Asymptotic convergence of averaged Q-learning is studied in Lee and He [2019a,b] via the ODE (ordinary differential equation) approach. Our results are complementary to theirs, including asymptotic statistical properties and finite-sample analysis in the ℓ₂ norm. Though peculiar to averaged Q-learning, we believe our analysis can be extended to non-linear SA problems.

2 Preliminaries and Notation

Discounted infinite-horizon MDPs. We consider an infinite-horizon MDP as represented by a tuple \(\mathcal{M} = (S, A, \gamma, P, R, T)\). Here \(S\) is the state space with \(S = |S|\) the cardinality and \(A\) is the action space with \(A = |A|\) the cardinality. \(\gamma \in (0, 1)\) is the discount factor. \(P: S \times A \to \Delta(S)\) represents the probability transition kernel, i.e., \(P(s'|s, a)\) is the probability of transitioning to \(s'\) from a given state-action pair \((s, a)\) \(\in S \times A\). \(R: S \times A \to [0, \infty)\) stands for the random reward, i.e., \(R(s, a)\) is the immediate reward collected in state \(s \in S\) when action \(a \in A\) is taken.

Value function and Q-function. A deterministic policy \(\pi\) maps each \(s \in S\) to a single action \(a \in A\). In a \(\gamma\)-discounted MDP, a common objective is to maximize the expected long-term rewards called value functions or Q-functions. For a given deterministic policy \(\pi: S \to A\), the associated value function and Q-function are defined respectively by

\[
V^\pi(s) = \mathbb{E}_{\tau \sim \pi} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s \right] \quad \text{and} \quad Q^\pi(s, a) = \mathbb{E}_{\tau \sim \pi} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s, a_0 = a \right]
\]

for any state-action pair \((s, a) \in S \times A\). Here \(\tau = \{(s_t, a_t)\}_{t \geq 0}\) is a trajectory of the MDP induced by the policy \(\pi\) and the expectation \(\mathbb{E}_{\tau \sim \pi}(\cdot)\) is taken with respect to the randomness of the trajectory \(\tau\). The optimal value function \(V^*\) and optimal Q-function \(Q^*\) are defined respectively as \(V^*(s) = \max_{\pi} V^\pi(s)\) and \(Q^*(s, a) = \max_{\pi} Q^\pi(s, a)\) for any state-action pair \((s, a) \in S \times A\).

Q Learning. In this work, we assume access to a generative model [Kearns and Singh, 1999, Sidford et al., 2018a, Li et al., 2021]. In particular, in each iteration \(t\), we collect independent samples of rewards \(r_t(s, a)\) and the next-state \(s_t(s, a) \sim P(\cdot|s, a)\) for every state-action pair \((s, a) \in S \times A\).\(^3\)

Note that \(r_t(s, a)\) is identically distributed as \(R(s, a)\) with the expectation \(r(s, a)\). The synchronous Q-learning algorithm maintains a Q-function estimate \(Q_t: S \times A \to \mathbb{R}\) for all \(t \geq 0\) and updates all entries of the Q-function estimate via the following update rule

\[
Q_t = (1 - \eta_t) Q_{t-1} + \eta_t \hat{T}_t(Q_{t-1}), \quad (1)
\]

where \(\eta_t \in (0, 1]\) is the step size in the \(t\)-th iteration and \(\hat{T}_t\) is the empirical Bellman operator constructed by samples collected in the \(t\)-th iteration, that is,

\[
\hat{T}_t(Q)(s, a) = r_t(s, a) + \gamma \max_{a' \in A} Q(s_t, a') \quad \text{where} \quad r_t(s, a) \sim R(s, a) \quad \text{and} \quad s_t = s_t(s, a) \sim P(\cdot|s, a) \quad (2)
\]

for each state-action pair \((s, a) \in S \times A\). Obviously, \(\hat{T}_t\) is an unbiased estimate of the celebrated Bellman operator \(T\) given by

\[
T(Q)(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a)} \max_{a' \in A} Q(s', a') \quad \text{for all} \quad (s, a) \in S \times A. \quad (3)
\]

\(^3\)This means \(s_t(s, a)\) (as well as \(r_t(s, a)\)) are independent over different \((s, a) \in S \times A\).
The optimal Q-function $Q^*$ is the unique fixed point of the Bellman operator $\mathcal{T}(Q^*) = Q^*$. Let $\pi_t$ be the greedy policy w.r.t. $Q_t$, i.e., $\pi_t(s) \in \arg \max_{a \in A} Q_t(s,a)$ for $s \in S$ and $\pi^*$ the optimal policy.

**Averaged Q-learning.** Ruppert [1988], Polyak and Juditsky [1992] show that averaging iterates generated by a stochastic approximation (SA) algorithm has favorable asymptotic statistical properties. In RL, many studies focus on the application of Polyak-Ruppert averaging on policy evaluation [Bhandari et al., 2018, Khamaru et al., 2021a, Mou et al., 2020a]. Q-learning is more difficult than policy evaluation because of non-stationarity (i.e., $\pi_t$ changes during time) and nonlinearity of $\mathcal{T}$. The averaged Q-learning iterate has the following form

$$Q_T = \frac{1}{T} \sum_{t=1}^{T} Q_t$$

with $\{Q_t\}$ updated as in (1). When we conduct inference, we use the average estimate $\hat{Q}_T$ rather than the last iterative value $Q_T$ for an iterative budget $T$. The application of Polyak-Ruppert averaging in deep RL has benefits of error reduction and stability [Lillicrap et al., 2016, Anschel et al., 2017].

**Matrix notation.** Given a matrix $A \in \mathbb{R}^{D \times D}$, we use $\|A\|_{\infty}$ to denote the infinity operator norm of $A$, i.e., $\|A\|_{\infty} = \max_{i,j} |A(i,j)|$. We use $\|A\|_{\max}$ to denote the max norm, i.e., $\|A\|_{\max} = \max_{i,j} |A(i,j)|$. We use $\text{diag}(A)$ to denote the diagonal matrix obtained by moving all off-diagonal entries in $A$.

For simplicity, we define $D = |S \times A| = SA$. We introduce the transition matrix $P \in \mathbb{R}^{D \times S}$ to represent the probability transition kernel $P$, whose $(s,a)$-th row $P_{s,a}$ is a probability vector representing $P(\cdot|s,a)$. The square probability transition matrix $P^\pi \in \mathbb{R}^{D \times D}$ (resp. $P^\pi \in \mathbb{R}^{S \times S}$) induced by deterministic policy $\pi$ over the state-action pairs (resp. states) as

$$P^\pi := P \Pi^\pi \quad \text{and} \quad P^\pi := \Pi^\pi P,$$

where $\Pi^\pi \in \{0,1\}^{S \times D}$ is a projection matrix associated with the deterministic policy $\pi$:

$$\Pi^\pi = \text{diag}\{e_{\pi(1)}^\top, e_{\pi(2)}^\top, \cdots, e_{\pi(S)}^\top\}$$

with $e_i$ the $i$-th standard basis vector. We use $r_t \in \mathbb{R}^D$ to represent the random reward $R$ generated at iteration $t$ such that for any $(s,a) \in S \times A$, the $(s,a)$-th entry of $r_t$ is given by $r_t(s,a) = R(s,a)$. $r = \mathbb{E}r_t \in \mathbb{R}^D$ denotes the expected reward vector. Similar $P_t \in \mathbb{R}^{D \times S}$ is the empirical transition matrix at iteration $t$ with each row only one non-zero entry and $\mathbb{E}P_t = P$. Analogously, we employ the vectors $V^\pi, V^* \in \mathbb{R}^S$ and $Q^\pi, Q^*, Q_t, \hat{Q}_t \in \mathbb{R}^D$ to denote the functions $V^\pi, V^*, Q^\pi, Q^*, Q_t, \hat{Q}_t$.

### 3 Main Results

We define $Z_t \in \mathbb{R}^D$ as the Bellman noise (vector) at the $t$-th iteration. The $(s,a)$-th entry of $Z_t$ is

$$Z_t(s,a) = \hat{T}_t(Q^*)(s,a) - T(Q^*)(s,a).$$

In matrix form, the Bellman noise at iteration $t$ can be equivalently presented as $Z_t = (r_t - r) + \gamma (P_t - P)V^*$. Bellman noise $Z_t$ reflects the noise present in the empirical Bellman operator (2) using samples collected at iteration $t$ as an estimate of the population Bellman operator (3).
In our synchronous setting, \( r_t \) and \( P_t \) are independent with each other and with all history data. Therefore, \( \{Z_t\} \) is an i.i.d. random vector sequence with each coordinate mean zero and mutually independent. When it is clear from the context, we would like to drop the dependence on \( t \) and use \( Z \) to denote an independent copy of \( Z_t \). We name \( Z \) as Bellman noise (vector). In our analysis, an important quantity is the covariance matrix of \( Z \), that is,

\[
\text{Var}(Z) = \mathbb{E}_{r_t,s_t} ZZ^\top \in \mathbb{R}^{D \times D},
\]

where the expectation \( \mathbb{E}_{r_t,s_t}(\cdot) \) is taken over the randomness of rewards \( r_t \) and next-states \( s_t \). Clearly, \( \text{Var}(Z) \) is a diagonal matrix with the \((s,a)\)-th diagonal entry given by \( \mathbb{E}Z^2(s,a) \).

### 3.1 Asymptotic Normality of Averaged Q-learning

We first turn to the asymptotic normality of the averaged Q-learning sequence \( \bar{Q}_T = \frac{1}{T} \sum_{t=1}^{T} Q_t \). For simplicity, we make two mild assumptions. One is the uniformly bounded and non-negative reward. The other is a positive optimality gap (defined in Assumption 3.2). The latter ensures that the optimal policy \( \pi^* \) is unique and thus the optimal variance is also unique as well as identifiable. All the detailed proof is provided in Appendix C.

**Assumption 3.1** (Uniformly bounded random reward). The random reward is non-negative and uniformly bounded, i.e., for all \((s,a)\) \( R(s,a) \leq 1 \) almost surely.

**Assumption 3.2** (Positive optimality gap). \( \text{gap} := \min_{s \in S} \min_{a_1 \neq a_2} |Q^*(s,a_1) - Q^*(s,a_2)| > 0 \).

**Theorem 3.1.** Under Assumptions 3.1 and 3.2, if we use a polynomial step size, that is, \( \eta_t = t^{-\alpha} \) with \( \alpha \in (0.5,1) \), then we have

\[
\sqrt{T}(Q_T - Q^*) \overset{d}{\to} N(0, \text{Var}_Q),
\]

where the asymptotic variance is given by

\[
\text{Var}_Q = (I - \gamma P^{\pi^*})^{-1} \text{Var}(Z)(I - \gamma P^{\pi^*})^{-\top} \in \mathbb{R}^{D \times D}.
\]

Here \( \text{Var}(Z) \) is the covariance matrix of the Bellman noise \( Z \) defined in (8).

**Remark 3.1.** Theorem 3.1 actually allows a more general choice of step sizes. In particular, any sequence of step sizes satisfying the following Assumption 3.3 leads to the same asymptotic normality. Similar conditions have been used in stochastic approximation literature [Polyak and Juditsky, 1992, Su and Zhu, 2018]. The proof is provided in Appendix C.5.

**Assumption 3.3.** Assume \( \{\eta_t\} \) satisfies (i) \( 0 \leq \sup_t \eta_t \leq 1, \eta_t \downarrow 0 \) and \( t \eta_t \uparrow \infty \) as \( t \to \infty \); (ii) \( \frac{\eta_{t+1} - \eta_t}{\eta_{t+1}} = o(\eta_{t+1}) \) and \((1 - (1 - \gamma) \eta_t) \eta_{t-1} \leq \eta_t \) for \( t \geq 1 \); and (iii) \( \frac{1}{\sqrt{T}} \sum_{t=0}^{T} \eta_t \to 0 \) as \( T \to \infty \).

**Asymptotic variance for \( Q^* \) estimation.** Theorem 3.1 implies the average of sequence \( Q_t \) has an asymptotic normal distribution with \( \text{Var}_Q \) the asymptotic variance. \( \text{Var}_Q \) is form of the covariance matrix \( \text{Var}(Z) \) of Bellman noise \( Z \) multiplied with a pre-factor \((I - \gamma P^{\pi^*})^{-1}\). By a von Neumann expansion, \((I - \gamma P^{\pi^*})^{-1}\) is equivalent to \( \sum_{t=0}^{\infty} (\gamma P^{\pi^*})^t \). As argued by Khamaru et al. [2021b], the sum of the powers of \( \gamma P^{\pi^*} \) accounts for the compounded effect of an initial perturbation when
following the MDP induced by \( \pi^* \). The Bellman noise \( Z \) reflects the noise present in the empirical Bellman operator (2) as an estimate of the population Bellman operator (3). By the way, it implies \( \| (I - \gamma P^*)^{-1} \| \leq \sum_{t=0}^{\infty} \gamma^t \| (P^*)^t \|_\infty = (1 - \gamma)^{-1} \). We find that \( \| \text{diag}(\text{Var}_Q) \|_\infty \) coincides with the instance-dependent functional proposed by Khamaru et al. [2021b] that controls the difficulty of estimating \( Q^* \) in the \( \ell_\infty \) norm. Hence, \( \| \text{diag}(\text{Var}_Q) \|_\infty \) quantifies the difficulty of \( Q^* \) estimation.

**Asymptotic normality for \( V^* \) estimation.** Sometimes one concerns the optimal value function \( V^* \) rather than the optimal Q-value function \( Q^* \). To use the asymptotic normality of \( \bar{Q}_T \), we define an estimator \( \bar{V}_T \in \mathbb{R}^S \) greedily from \( Q_T \in \mathbb{R}^D \): the \( s \)-th entry of \( \bar{V}_T \) is \( \bar{V}_T(s) \in \arg \max_{a \in A} Q_T(s,a) \).

As a corollary of Theorem 3.1, \( \bar{V}_T \) enjoys a similar asymptotic normality with the asymptotic variance defined by \( \text{Var}_V \). One can check that

\[
\text{Var}_V = \Pi^{\pi^*} \text{Var}_Q (\Pi^{\pi^*})^\top,
\]

where \( \Pi^{\pi^*} \in \{0,1\}^{S \times D} \) is the projection matrix associated with the deterministic optimal policy \( \pi^* \) (see the definition in (6)). Hence, \( \text{Var}_V \) is formed by selecting entries from \( \text{Var}_Q \). In particular, \( \text{Var}_V(s,s') = \text{Var}_Q((s,\pi^*(s)),(s',\pi^*(s'))) \) for any \( s,s' \in \mathcal{S} \). The proof is left in Appendix C.6.

**Corollary 3.1.** Let \( \bar{V}_T \in \mathbb{R}^S \) be the greedy value function computed from \( \bar{Q}_T \in \mathbb{R}^D \), i.e., \( \bar{V}_T(s) \in \arg \max_{a \in A} \bar{Q}_T(s,a) \). Then under Assumptions 3.1 and 3.2, and using the same choice of step sizes,

\[
\sqrt{T}(\bar{V}_T - V^*) \overset{d}{\to} N(0, \text{Var}_V),
\]

where the asymptotic variance is

\[
\text{Var}_V = (I - \gamma P^{\pi^*})^{-1} \text{Var}(\Pi^{\pi^*} Z)(I - \gamma P^{\pi^*})^{-\top} \in \mathbb{R}^{S \times S}
\]

and \( \text{Var}(\Pi^{\pi^*} Z) \) is the covariance matrix of the projected Bellman noise \( \Pi^{\pi^*} Z \).

**Insights on sample efficiency.** The asymptotic results shed light on the sample efficiency of averaged Q-learning. Noticing that \( Q_T \) is uniformly bounded from Assumption 3.1, the bounded convergence theorem yields that asymptotically

\[
\sqrt{T} \mathbb{E} \| Q_T - Q^* \|_\infty \to \mathbb{E} \| Z \|_\infty \approx \sqrt{\ln D} \sqrt{\| \text{diag}(\text{Var}_Q) \|_\infty} \quad \text{where} \quad Z \sim N(0, \text{Var}_Q).
\]

In this case, roughly speaking, to obtain an \( \varepsilon \)-accurate estimator of the optimal Q-value function \( Q^* \) (i.e., \( \mathbb{E} \| Q_T - Q^* \|_\infty \leq \varepsilon \)), it takes about \( T = O \left( \frac{\ln D}{\varepsilon^2} \| \text{diag}(\text{Var}_Q) \|_\infty \right) \) iterations or equivalently \( DT = O \left( \frac{D\ln D}{\varepsilon^2} \| \text{diag}(\text{Var}_Q) \|_\infty \right) \) samples. This explains why Khamaru et al. [2021b] regards \( \| \text{diag}(\text{Var}_Q) \|_\infty \) as the difficulty indicator because it affects the sample complexity directly.

### 3.2 Information Theoretical Lower Bound

Theorem 3.1 shows \( Q_T \) is a \( \sqrt{T} \)-consistent estimate for \( Q^* \) with asymptotic variance \( \text{Var}_Q \). It is of theoretical interest to investigate whether \( Q_T \) is asymptotically efficient or not. In parametric statistics [Lehmann and Casella, 2006], the Cramer-Rao lower bound (CRLB) is derived to evaluate the hardness of estimating the target parameter \( \beta(\theta) \) in a parametric model \( \mathcal{P}_\theta \) indexed by parameter \( \theta \). Any unbiased estimator whose variance achieves the CRLB is viewed as optimal and efficient.

Such a kind of CRLB can be extended both to those possibly biased but asymptotically unbiased estimators and to non-parametric statistical models where the dimension of parameter \( \theta \) is typically infinity [Van der Vaart, 2000, Tsiatis, 2006].
The semi-parametric model. In our case, the transition kernel \( \{P(\cdot|s,a)\}_{s,a} \) is specified by \( D \) parametric distributions on \( \Delta(S) \),\(^4\) while the random reward \( \{R(s,a)\}_{s,a} \) is totally non-parametric because the \( R(s,a) \) are not assumed to come from prescribed models. Hence, to figure out the corresponding CRLB for \( Q^* \) estimation, we enter the world of semi-parametric statistics. In particular, our MDP model \( \mathcal{M} = (S,A,\gamma,P,R,r) \) has parameter \( \theta = (P,R) \). The parameters \( P \) and \( R \) are variationally independent. Our interest parameter \( \beta(\theta) = Q^* \). At iteration \( t \), \( r_t \in \mathbb{R}^D \) collects all random rewards generated at each \( (s,a) \) and \( P_t \in \mathbb{R}^{D \times S} \) gathers all empirical transitions following the probability \( P_{s,a} := P(\cdot|s,a) \) starting from each \( (s,a) \). Specifically, the \( (s,a) \)-th entry of \( r_t \) is an independent copy of \( R(s,a) \), while the \( (s,a) \)-th row of \( P_t \) is a one-hot random vector with a single non-zero entry. The distribution of \( P_t \) is determined by its expectation \( P = \mathbb{E}P_t \) which belongs to

\[
P_P := \left\{ P \in \mathbb{R}^{D \times S} : P(s'|s,a) \geq 0, \forall (s,a,s') \in S \times A \times S \text{ and } \sum_{s' \in S} P(s'|s,a) = 1 \right\},
\]  

while \( R \) is non-parametric belonging to

\[
P_R = \{ R = \{R(s,a)\}_{s,a} : \mathbb{E}R(s,a) = r(s,a), \forall (s,a) \in S \times A \}.
\]

Each \( r_t \) and \( P_t \) are mutually independent and also independent with all history data. Let \( \mathcal{D} = \{(r_t,P_t)\}_{t \in [T]} \) contain the \( T \) samples generated as described above.

Semiparametric efficiency lower bound. Tsiatis [2006] argued it is reasonable to restrict ourselves to RAL estimators (see Page 27) with a great range of estimators falling into this class.

**Definition 3.1** (Regular asymptotically linear). Let \( \tilde{Q}_T \in \mathbb{R}^{D} \) be a measurable random function of \( \mathcal{D} \). We say \( \tilde{Q}_T \) to be regular asymptotically linear (RAL) for \( Q^* \) if it is regular and asymptotically linear with influence function (i.e., a measurable random function) \( \phi(r_t,P_t) \in \mathbb{R}^{D} \) such that

\[
\sqrt{T}(\tilde{Q}_T - Q^*) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi(r_t,P_t) + o_P(1),
\]

where \( \mathbb{E}\phi(r_t,P_t) = 0 \) and \( \mathbb{E}\phi(r_t,P_t)\phi(r_t,P_t)^\top \) is finite and non-singular.

**Remark 3.2.** Informally speaking, an estimator is called regular if its limiting distribution is unaffected by local changes in the data generating process. The regular assumption excludes super-efficient estimators, whose asymptotic variance can be smaller than the Cramer-Rao lower bound for some parameter values, but performs poorly in the neighbourhood of points of super-efficiency. We suggest interested readers refer to Section 3.1 in Tsiatis [2006] for a detailed introduction. We have an easy criterion on influence functions to check regularity for an asymptotically linear estimator (see Theorem 2.2 in Newey [1990] which we also use to prove our Theorem 3.3).

**Theorem 3.2.** Given the dataset \( \mathcal{D} = \{(r_t,P_t)\}_{t \in [T]} \), for any RAL estimator \( \tilde{Q}_T \) of \( Q^* \) computed from \( \mathcal{D} \), its variance satisfies

\[
\lim_{T \to \infty} T\mathbb{E}(\tilde{Q}_T - Q^*)(\tilde{Q}_T - Q^*)^\top \succeq \text{Var}_Q,
\]

where \( A \succeq B \) means \( A - B \) is semi-positive definite and \( \text{Var}_Q \) is given in (9).

---

\(^4\)To determine a distribution with a finite support (say, \( S \) discrete points), we only need to specify \( S - 1 \) parameters \( \{p_s\}_{s \in [S-1]} \) which satisfies \( 0 \leq p_s \leq 1 \) for all \( s \in [S-1] \) and \( 0 \leq \sum_{s \in [S-1]} p_s \leq 1 \).
By Definition 3.1, any influence function determines an asymptotic linear estimator for $Q^*$. The semi-parametric efficiency bound in Theorem 3.2 gives us a concrete target in the construction of the influence function. If we can find an influence function that achieves the bound, we know that it is the most efficient one among all RAL estimators. Fortunately, Theorem 3.3 implies that $\bar{Q}_T$ is the most efficient estimator among all RAL estimators with the efficient influence function $(I - \gamma P^*)^{-1}Z_t$. Theorem 3.3 is stronger than Theorem 3.1 because it not only implies Theorem 3.1, but also shows the regularity of $\bar{Q}_T$.

**Theorem 3.3.** Under Assumptions 3.1 and 3.2, the averaged Q-learning iterate $\bar{Q}_T$ is a RAL estimator for $Q^*$. In particular, we have the following decomposition

$$\sqrt{T}(\bar{Q}_T - Q^*) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (I - \gamma P^*)^{-1}Z_t + o_P(1),$$

where $Z_t = (r_t - r) + \gamma(P_t - P)V^*$ is the Bellman noise at iteration $t$.

### 3.3 Sub-Gaussian of Bellman Noises

We start to explore the non-asymptotic behavior of averaged Q-learning, i.e., how $\mathbb{E}\|\bar{Q}_T - Q^*\|_\infty$ depends on finite $T$ and $(1 - \gamma)^{-1}$. As discussed in Section 3.1, when $T$ is sufficiently large, the dominant term of $\mathbb{E}\|\bar{Q}_T - Q^*\|_\infty$ is approximately $\sqrt{\frac{\ln D}{T}} \sqrt{\|\text{diag}(\text{Var}Q)\|_\infty}$. Hence, the remaining issue is to determine the sharpest dependence of $\|\text{diag}(\text{Var}Q)\|_\infty$ on $(1 - \gamma)^{-1}$. Previous work in Azar et al. [2013], Li et al. [2020a] implies $\|\text{diag}(\text{Var}V)\|_\infty \leq \|\text{diag}(\text{Var}Q)\|_\infty = O((1 - \gamma)^{-3})$.\(^5\) Such a dependence is tight, since Khamaru et al. [2021b] provided an instance-dependent bound showing $\|\text{diag}(\text{Var}Q)\|_\infty = \Theta((1 - \gamma)^{-3+\lambda})$ in a family of MDPs parameterized by $\lambda \geq 0$.

However, to provide a finite analysis on the $\ell_\infty$ norm, we should also capture the tail distribution of each coordinate of $Z$. Additionally, sometimes $\|\text{Var}(Z)\|_\infty$ matters since it affects the sample complexity as $\|\text{diag}(\text{Var}Q)\|_\infty$ does. Indeed, using the inequality $\|\text{diag}(AA^\top)\|_\infty \leq \|V\|_\infty \|A\|_\infty^2$ (see Lemma C.2 for the proof), we have

$$\|\text{diag}(\text{Var}Q)\|_\infty \leq \|(I - \gamma P^*)^{-1}\|_\infty^2 \|\text{Var}(Z)\|_{\max} \leq \frac{1}{(1 - \gamma)^2} \|\text{Var}(Z)\|_\infty,$$

where the last inequality uses $\|(I - \gamma P^*)^{-1}\|_\infty \leq (1 - \gamma)^{-1}$ and the diagonal matrix $\text{Var}(Z)$ has equal $\|\cdot\|_\infty$ and $\|\cdot\|_{\max}$. Hence, any bound on $\|\text{Var}(Z)\|_\infty$ provides a concrete bound for $\|\text{diag}(\text{Var}Q)\|_\infty$. The currently known bound on $\|\text{Var}(Z)\|_\infty$ is given in Wainwright [2019b], who noticed that $\|Z\|_\infty \leq (1 - \gamma)^{-1}$ almost surely and thus gave a crude bound $\|\text{Var}(Z)\|_\infty = O((1 - \gamma)^{-2})$.

**Our results.** We show that actually $\|\text{Var}(Z)\|_\infty = O((1 - \gamma)^{-1})$ for any MDPs satisfying Assumption 3.1. The worst-case bound $O((1 - \gamma)^{-1})$ is tight as supported by the instance-dependent analysis in Khamaru et al. [2021b] when the parameter $\lambda$ is set as zero. Furthermore, we can prove the Bellman noise $Z$ has sub-Gaussian coordinates with parameter $O((1 - \gamma^{-2})^{-1})$ as Theorem 3.4 shows. Such a sub-Gaussian property not only gives us sharpest bounds for any powers of Bellman noise coordinates, but also provides tighter tail bounds for some functions of Bellman noises.

---

\(^5\)For example, Lemma 8 in Li et al. [2020a] implies that $\|(I - \gamma P^*)^{-1}\sqrt{\text{Var}(\Pi^{*\top}Z)}\|_\infty = O((1 - \gamma)^{-1.5})$. Using the inequality $\|\text{diag}(AA^\top)\|_\infty \leq \|A\|_\infty^2$ (see Lemma C.2), we have $\|\text{diag}(\text{Var}V)\|_\infty = O((1 - \gamma)^{-3})$. Lemma 7 in Azar et al. [2013] implies $\|\text{diag}(\text{Var}Q)\|_\infty = O((1 - \gamma)^{-3})$. Finally, the relation between $\text{Var}V$ and $\text{Var}Q$ shown in (10) yields $\|\text{diag}(\text{Var}V)\|_\infty \leq \|\text{diag}(\text{Var}Q)\|_\infty$. 

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Theorem 3.4. Let $e_i \in \mathbb{R}^D$ be the $i$-th standard basis with only the $i$-th coordinate non-zero and equal to 1. Then under Assumption 3.1, we have

$$\sup_{i \in [D]} \mathbb{E} \exp(\lambda e_i^\top Z) \leq \exp\left(\frac{1}{1 - \gamma^2} \frac{\lambda^2}{8}\right) \quad \text{for any } \lambda \in \mathbb{R}.$$ 

Corollary 3.2. Under Assumption 3.1, we have

$$\|\text{Var}(Z)\|_\infty \leq \frac{1}{4(1 - \gamma^2)} = \mathcal{O}\left(\frac{1}{1 - \gamma}\right).$$

Remark 3.3. Theorem 3.4 requires the random reward is uniformly bounded, i.e., $0 \leq R(s,a) \leq 1$ almost surely for all state-action pairs (see Assumption 3.1). It can be replaced by a much weaker assumption. More specifically, Theorem 3.4 still holds if we assume each $R(s,a)$ satisfies $\mathbb{E} \exp(\lambda (R(s,a) - r(s,a))) \leq \exp(\lambda^2/8)$ for any $\lambda \in \mathbb{R}$ and $0 \leq r(s,a) := \mathbb{E}R(s,a) \leq 1$ for any $(s,a) \in \mathcal{S} \times \mathcal{A}$. See Appendix A for the detail as well as all the proof.

Remark 3.4. Corollary 3.2 together with (14) implies $\|\text{diag}(\text{Var}(Q))\|_\infty \leq \frac{1}{4(1 - \gamma^2)}$, which coincides with previous results [Azar et al., 2013, Li et al., 2020a]. The constant is $4\ln 2$ in [Azar et al., 2013] and $8/\gamma^2$ in Li et al. [2020a]. Our proof is different from them in that we utilizes the sub-gaussian property of each coordinate of $Z$.

Impacts on prior works. As a measure of estimation difficulty, $\|\text{Var}(Z)\|_\infty$ (or $\|\text{diag}(\text{Var}(Q))\|_\infty$) will explicitly or implicitly appear at sample complexity (or iteration complexity) when one analyzes their target algorithms. However, one often uses a loose bound

$$\|\text{Var}(Z)\|_\infty = \mathcal{O}((1 - \gamma)^{-2}) \quad (15)$$

and thus often results in a sub-optimal sample complexity. For example, Wainwright [2019b] analyzed Q-learning with rescaled linear step size and showed that $\mathbb{E}\|Q_T - Q^*\|_\infty$ converges in the rate depicted in Proposition 3.1. At the worst case, $\|Q^*\|_{\text{span}} = \mathcal{O}((1 - \gamma)^{-1})$, which together with the coarse bound (15) yields a sub-optimal $\widetilde{O}\left(\frac{D}{(1 - \gamma)\gamma^2}\right)$ sample complexity. Li et al. [2021] tightens it to $\widetilde{O}\left(\frac{D}{(1 - \gamma^2)^{\gamma^2}}\right)$ using a sophisticated recursion analysis with a matching lower bound. However, with Corollary 3.2 and Proposition 3.1, the dominant term $\clubsuit$ becomes $\widetilde{O}\left(\sqrt{\frac{1}{(1 - \gamma)\gamma^2}}\right)$, by which we directly improve the sample complexity to the optimal $\widetilde{O}\left(\frac{D}{(1 - \gamma)^{\gamma^2}}\right)$ without any additional efforts. It reveals the importance of capturing the correct order of $\|\text{Var}(Z)\|_\infty$ on $(1 - \gamma)^{-1}$.

Proposition 3.1 (Corollary 3 in Wainwright [2019b]). If the step size is $\eta_t = \frac{1}{1 + (1 - \gamma)T}$, then

$$\mathbb{E}\|Q_T - Q^*\|_\infty \leq \frac{\|Q_0 - Q^*\|_\infty}{1 + (1 - \gamma)T} + \frac{c}{1 - \gamma} \sqrt{\ln(2D)\|\text{Var}(Z)\|_\infty} + \frac{c\|Q^*\|_{\text{span}} \ln(2eD(1 + (1 - \gamma)T))}{1 - \gamma}$$

where $c > 0$ is some universal constant and $\|Q^*\|_{\text{span}} := \sup_{s,a} Q^*(s,a) - \inf_{s',a'} Q^*(s',a')$. 
Additionally, the sub-Gaussian property of $Z$ helps further remove the $\|Q^*\|_{\text{span}}$ term in Proposition 3.1 and results the following Proposition 3.2. Here is an explanation of the improvement. The sum of ♣ and ♠ results from an autoregressive process of independent Bellman noises. The sub-Gaussian property allows us to replace the Bernstein inequality with a Hoeffding-style inequality when bounding the compound Bellman noises. In this way, only the variance-dependent term ♣ is left, while the offset-dependent term ♠ disappears. Hence, Theorem 3.4 helps tighten analysis on functions of Bellman noises.

Proposition 3.2. If the step size is $\eta_t = \frac{1}{1+(1-\gamma)t}$, using the sub-Gaussian property of $Z$, we have

$$\mathbb{E}\|Q_T - Q^*\|_\infty \leq \frac{\|Q_0 - Q^*\|_\infty}{1 + (1-\gamma)T} + c\sqrt{\frac{\ln(2eD)}{(1-\gamma)^3}} \frac{1}{\sqrt{1 + (1-\gamma)T}}.$$

Considering the prevalence of the crude bound (15), we believe the two results, Theorem 3.4 and Corollary 3.2, will help more RL researchers improve the analysis for sample complexity (or iteration complexity) of their target algorithms. For example, it is possible to provide a simpler analysis to obtain the same (or even better) results for Wang et al. [2021], Zhao et al. [2021].

3.4 Faster Non-asymptotic Convergence

We first provide a non-asymptotic convergence to validate (12).

Theorem 3.5. Under Assumptions 3.1 and 3.2, when $D$ is larger than a universal constant,

- If $\eta_t = \frac{1}{1+(1-\gamma)t}$, it follows that for all $T \geq 1$,

  $$\mathbb{E}\|Q_T - Q^*\|_\infty = \mathcal{O}\left(\sqrt{\|\text{diag}(\text{Var}Q)\|_\infty} \sqrt{\frac{\ln D}{T}} + \tilde{\mathcal{O}}\left(\frac{\gamma}{(1-\gamma)^5} \frac{1}{T}\right)\right).$$

- If $\eta_t = t^{-\alpha}$ with $\alpha \in (0.5, 1)$ for $t \geq 1$ and $\eta_0 = 1$, it follows that for all $T \geq 1$,

  $$\mathbb{E}\|Q_T - Q^*\|_\infty = \mathcal{O}\left(\sqrt{\|\text{diag}(\text{Var}Q)\|_\infty} \sqrt{\frac{\ln D}{T}} + \frac{\sqrt{\ln D}}{(1-\gamma)^2} \frac{1}{T^{1-\frac{1}{2}}} \right)$$

  $$+ \tilde{\mathcal{O}}\left(\frac{1}{(1-\gamma)^2+\frac{2}{1-\alpha}} \frac{1}{T} + \frac{\gamma}{(1-\gamma)^3+\frac{1}{1-\alpha}} \frac{1}{T^\alpha}\right),$$

where $\tilde{\mathcal{O}}(\cdot)$ hides polynomial dependence on $\alpha$ and logarithmic factors (namely $\ln D$ and $\ln T$).

To the best of our knowledge, Theorem 3.5 is the first finite-sample analysis of averaged Q-learning in the $\ell_\infty$ norm. On one hand, it shows the instance-dependent term $\mathcal{O}(\sqrt{\|\text{diag}(\text{Var}Q)\|_\infty} \sqrt{\frac{\ln D}{T}})$ dominates the $\ell_\infty$ error, which matches the instance-dependent lower bound established in Khamaru et al. [2021b]. On the other hand, when plugging in the worst-case bound $\|\text{diag}(\text{Var}Q)\|_\infty = \mathcal{O}((1-\gamma)^{-3})$, we find that for sufficiently small $\varepsilon$, averaged Q-learning already achieves the optimal

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Interested readers can compare our Lemma B.2 with Lemma 3 in Wainwright [2019b].
minimax sample complexity $\tilde{O}\left(\frac{D}{(1-\gamma)^2}\right)$ established in Azar et al. [2013]. To obtain the optimal complexity, prior works utilize variance reduction technique [Wainwright, 2019c, Khamaru et al., 2021b, Mou et al., 2020a], a procedure used to increase the precision of estimates [Gower et al., 2020]. The variance-reduced Q-learning [Wainwright, 2019c, Khamaru et al., 2021b] requires an additional collection of i.i.d. samples to perform an Monte Carlo approximation of population Bellman operator (3). However, our results show a simple average is sufficient to guarantee the optimality. Moreover, the computation of $Q_T$ is fully online with no additional samples needed.

**Confirming the theoretical predictions.** We provide numerical experiments to illustrate instance-adaptive as well as worst-case behaviors guaranteed by Theorem 3.5. We focus on the the sample complexity $T(\varepsilon, \gamma) = \inf\{T : \mathbb{E}\|Q_T - Q^*\|_{\infty} \leq \varepsilon\}$ for $\varepsilon = 10^{-4}$. We conduct $10^3$ independent trials in a random MDP to compute $T(\varepsilon, \gamma)$ under different values of $\gamma \in \Gamma$ and two step sizes. We then plot the least-squares fits through these points $\{(\log(1-\gamma)^{-1}, \log T(\varepsilon, \gamma))\}_{\gamma \in \Gamma}$ and $\{(\log \|\text{diag}(\text{Var}_Q)\|_{\infty}, \log T(\varepsilon, \gamma))\}_{\gamma \in \Gamma}$ and provide the slopes $k$ of these lines in the legend. More details are left in Appendix G. At a high level, we see that the averaged Q-learning with different step sizes produces sample complexity that is well predicted by our theory: all the slopes are no larger than the theoretical limit $k^*$.

**4 Our Proof Idea**

To prove Theorem 3.1, we first establish a ‘sandwich’ result which provides both lower and upper coordinate bounds on the error $\Delta_T := Q_T - Q^*$. In particular, we have $\Delta_T^2 \leq \Delta_T \leq \Delta_T^1$ for two iteratively defined auxiliary sequences $\{\Delta^i_t\} (i = 1, 2, t \geq 0)$ (see Lemma C.1). We then prove both $\sqrt{T}\Delta_T^2$ and $\sqrt{T}\Delta_T^1$ weakly converge to the same multivariate normal distribution $\mathcal{N}(0, \text{Var}_Q)$ and thus complete the proof. To prove the asymptotic normality of $\sqrt{T}\Delta_T^2$, we decompose $\sqrt{T}\Delta_T^2$ into five separate terms. One involves a sum of independent rescaled Bellman noises contributes to the normality (which is actually the influence function, see Lemma C.5), while the rest converge to zero in probability. As for the non-asymptotic result Theorem 3.5, we carefully analyze the five terms and provide finite sample error analysis in the $\ell_{\infty}$ norm. $\sqrt{T}\Delta_T^2$ can be analyzed in an almost
identical way. As a byproduct of our non-asymptotic attempt, the sub-gaussian result of Bellman noise Theorem 3.4 follows from the observation that any value function $V^\pi$ is the expectation of a discounted sum of random rewards following an MDP induced by $\pi$. Jensen’s inequality allows us to move the expectation outside $\exp(\cdot)$ when bounding $\mathbb{E}\exp(\lambda e_i^\top Z)$. In this way, the exponent is an infinite sum of martingale differences, by which we are motivated to use a similar argument in the proof of Azuma’s inequality to complete the proof. Finally, Theorems 3.2 and 3.3 borrow tools from semiparametric statistics to provide the asymptotic efficiency lower bound [Tsiatis, 2006] and to check the regularity of averaged Q-learning [Newey, 1990].

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Appendix

A Proof for Sub-Gaussian Bellman Error

A.1 Proof of Theorem 3.4

Theorem 3.4 requires uniformly bounded reward functions, i.e., \( 0 \leq R(s,a) \leq 1 \) almost surely for all state-action pairs (see Assumption 3.1). From a theoretical interest, one might concern about what is the minimum conditions we should pose on the random reward function \( R : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \) to guarantee the same result. Therefore, we provide a generalized version of Theorem 3.4 which uses a much weaker condition Assumption A.1 instead. In a nutshell, we assume each \( R(s,a) \) is sub-Gaussian with uniformly-bounded expectation rather than a almost surely uniformly-bounded random variable. As a immediate application of the Hoeffding’s lemma (see Lemma A.1), Assumption 3.1 implies Assumption A.1, while the verse is generally not true. Hence, Theorem A.1 implies Theorem 3.4.

Assumption A.1 (Sub-Gaussian random reward with uniformly bounded expectation). For any state-action pair \( (s,a) \in \mathcal{S} \times \mathcal{A} \) and \( \lambda \in \mathbb{R} \),

\[
\mathbb{E}\exp(\lambda (R(s,a) - r(s,a))) \leq \exp\left(\frac{\lambda^2}{8}\right)
\]

where \( r(s,a) = \mathbb{E}R(s,a) \) is the expected reward function of \( R(\cdot) \). What’s more, we assume the expected reward is non-negative and uniformly bounded, i.e., for all \( (s,a) \in \mathcal{S} \times \mathcal{A} \),

\[
0 \leq r(s,a) := \mathbb{E}R(s,a) \leq 1.
\]

Lemma A.1 (Hoeffding’s lemma). Let \( X \) be any real-valued random variable with zero mean and \( a \leq X \leq b \) almost surely. Then for any \( \lambda \in \mathbb{R} \),

\[
\mathbb{E}\exp(\lambda(X - \mathbb{E}X)) \leq \exp\left(\frac{\lambda^2(b - a)^2}{8}\right).
\]

Theorem A.1 (General version of Theorem 3.4). Recall \( Z_t = (r_t - r) + \gamma(P_t - P)V^* \in \mathbb{R}^D \). Let \( e_i \in \mathbb{R}^D \) be the \( i \)-th standard basis with only the \( i \)-th coordinate non-zero and equal to 1. Under Assumption A.1, we have

\[
\sup_{i \in [D]} \mathbb{E}\exp(\lambda e_i^\top Z_t) \leq \exp\left(\frac{1}{1 - \gamma^2} \frac{\lambda^2}{8}\right) \text{ for any } \lambda \in \mathbb{R}.
\]

Proof of Theorem A.1. Fixing an initial state-action pair \( (s,a) \in \mathcal{S} \times \mathcal{A} \) (which has the index \( i \)), we have \( e_i^\top Z_t = r_t(s,a) - r(s,a) + \gamma(P_t(s,a) - P(s,a))V^* \). Here \( r_t(s,a) \) is independent copy of \( R(s,a) \) at iteration \( t \) and we slightly abuse the notation using \( P(s,a) := P((s,a),\cdot) = P_{s,a} \) (resp. \( P_t(s,a) = P_t((s,a),\cdot) = (P_t)_{s,a} \)) to represent the \( (s,a) \)-th row vector of \( P \) (resp. \( P_t \)) for simplicity. Therefore,

\[
\mathbb{E}\exp(\lambda e_i^\top Z_t) \overset{(a)}{=} \mathbb{E}\exp(\lambda(r_t(s,a) - r(s,a)))\mathbb{E}\exp(\lambda\gamma(P_t(s,a) - P(s,a))V^*)
\]

\[
\overset{(b)}{=} \exp\left(\frac{\lambda^2}{8}\right)\mathbb{E}\exp(\lambda\gamma(P_t(s,a) - P(s,a))V^*)
\]

(16)
Here (a) uses the independence between the reward $r_t(s, a)$ and transition $P_t(s, a)$ and (b) uses Assumption A.1.

Without loss of generality, we assume the optimal policy $\pi^*$ is deterministic. Hence, we slightly abuse the notation and redefine $r(s_t) = r(s_t, \pi^*(s_t))$. It is clear that $r(s_t)$ is a deterministic function of $s_t$. We define a Markov process $\{s_t\}_{t \geq 0}$ as following, $s_0 \sim P(s, a)$ and $s_{t+1} \sim P(-|s_t, \pi^*(s_t))$. Recall $P_t(s, a)$ represents the one-hot vector standing for the next-state starting from the given state-action pair $(s, a)$. Without loss of generality, we assume $P_t(s, a)$ picks up $s_0$, i.e., $P_t(s, a)(s_0) = 1$ and $P_t(s, a)(s') = 0$ for any $s' \neq s_0$. We use $\tau_0 \sim \pi^*$ to mean the state trajectory after the initial state $s_0$ denoted by $\tau_0 = (s_1, s_2, \cdots)$ is generated according to the transition kernel $P_{\pi^*}$ which is the MDP induced by the policy $\pi^*$.

Let $\mathcal{F}_t = \sigma(\{s_t\}_{0 \leq t \leq t})$ be the $\sigma$-field that collect all sources of randomness before and including iteration $t$. By the generation of $\{s_t\}_{t \geq 0}$, we know that $r_t \in \mathcal{F}_t$. Hence,

$$
(P_t(s, a) - P(s, a))V^* = V^*(s_0) - \mathbb{E}_{s_0 \sim P(-|s, a)}V^*(s_0)
$$

$$
= \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t) \mid s_0 = s_0 \right] - \mathbb{E} \sum_{t=0}^{\infty} \gamma^t r(s_t)
$$

$$
= \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t) \mid s_0 = s_0 \right] - \mathbb{E} \sum_{t=0}^{\infty} \gamma^t \mathbb{E}[r(s_t) | \mathcal{F}_{t-1}]
$$

$$
= \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \tilde{r}_t \mid s_0 = s_0 \right] = \mathbb{E}_{\tau_0 \sim \pi^*} \sum_{t=0}^{\infty} \gamma^t \tilde{r}_t \tag{17}
$$

where the last line uses the following notation

$$
\tilde{r}_t := r(s_t) - \mathbb{E}[r(s_t) | \mathcal{F}_{t-1}] \tag{18}
$$

Clearly conditioning on $\mathcal{F}_{t-1}$, we have $\mathbb{E}[\tilde{r}_t | \mathcal{F}_{t-1}] = 0$ as well as $0 \leq r(s_t) \leq 1$.

For simplicity, we use $\mathbb{E}_{s_0}(\cdot)$ and $\mathbb{E}_{\tau_0}(\cdot)$ to denote $\mathbb{E}_{s_0 \sim P(s, a)}(\cdot)$ and $\mathbb{E}_{\tau_0 \sim \pi^*}(\cdot)$ respectively, while $\mathbb{E}(\cdot)$ means taking expectation with all randomness. Therefore, we have

$$
\mathbb{E}_{s_0 \sim P(s, a)} \exp(\lambda \gamma (P_t(s, a) - P(s, a))V^*) = \mathbb{E}_{s_0} \exp(\lambda \gamma (P_t(s, a) - P(s, a))V^*)
$$

$$
= \mathbb{E}_{s_0} \exp \left( \lambda \gamma \mathbb{E}_{\tau_0} \sum_{t=0}^{\infty} \gamma^t \tilde{r}_t \right)
$$

$$
(a) \leq \mathbb{E}_{s_0} \mathbb{E}_{\tau_0} \exp \left( \lambda \gamma \sum_{t=0}^{\infty} \gamma^t \tilde{r}_t \right)
$$

$$
(b) \mathbb{E}_{s_0} \mathbb{E}_{\tau_0} \lim_{T \to \infty} \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^t \tilde{r}_t \right)
$$

$$
(c) \lim_{T \to \infty} \mathbb{E}_{s_0} \mathbb{E}_{\tau_0} \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^t \tilde{r}_t \right)
$$

\footnote{If $\pi^*$ is a stochastic policy that doesn’t put all probability mass on a single action, then we can define $r(s_t) = \mathbb{E}_{a_t \sim \pi^*(s_t)} r(s_t, a_t)$ to serve the purpose. The change will not affect our following analysis.}
where (a) follows from the Jensen’s inequality for \( \exp(\cdot) \), (b) uses the fact that \( \sum_{t=0}^{T} \gamma^{t} \tilde{r}_{t} \) converges almost surely to \( \sum_{t=0}^{\infty} \gamma^{t} \tilde{r}_{t} \) as \( T \) goes to infinity, and (c) follows from dominated convergence theorem that uses the constant function \( \exp\left(\frac{\lambda \gamma}{8}\right) \) as the dominating function. In this way, we can safely exchange the order of expectation and limitation.

Therefore,

\[
\mathbb{E}_{s_0} \mathbb{E}_T \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^{t} \tilde{r}_{t} \right) = \mathbb{E} \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^{t} \tilde{r}_{t} \right) \\
= \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^{t} \tilde{r}_{t} \right) \mid \mathcal{F}_{T-1} \right] \right] \\
= \mathbb{E} \left[ \exp \left( \lambda \gamma \sum_{t=0}^{T-1} \gamma^{t} \tilde{r}_{t} \right) \mathbb{E} \left[ \exp \left( \lambda \gamma \sum_{t=0}^{T} \gamma^{t} \tilde{r}_{t} \right) \mid \mathcal{F}_{T-1} \right] \right] \\
= \left(\begin{array}{c}
(a) \\
(b)
\end{array}\right) \leq \mathbb{E} \left[ \exp \left( \lambda \gamma \sum_{t=0}^{T-1} \gamma^{t} \tilde{r}_{t} \right) \exp \left( \frac{\lambda^{2} \gamma^{2(T+1)}}{8} \right) \right] \\
\leq \exp \left( \sum_{t=1}^{T+1} \frac{\lambda^{2} \gamma^{2 t}}{8} \right)
\]

where (a) follows from the Hoeffding’s lemma using \( 0 \leq r(s_t) \leq 1 \) and (b) follows from repeatedly evoking (a) and induction.

Putting all pieces together, we have for all \( \lambda \in \mathbb{R} \),

\[
\mathbb{E} \exp(\lambda \gamma (P_t(s, a) - P(s, a))V^*) \leq \exp \left( \frac{\gamma^{2} \lambda^{2}}{1 - \gamma^{2}} \right)
\]

which together with (16) implies

\[
\mathbb{E} \exp(\lambda e_i^T Z_t) \leq \exp \left( \frac{1}{1 - \gamma^{2}} \frac{\lambda^{2}}{8} \right).
\]

\[\square\]

A.2 Proof of Corollary 3.2

Proof of Corollary 3.2. We provide two proofs for Corollary 3.2.

Way 1: use the sub-gaussian result Theorem A.1. Theorem A.1 implies

\[
\mathbb{E} \exp(\lambda e_i^T Z) \leq \exp \left( \frac{1}{1 - \gamma^{2}} \frac{\lambda^{2}}{8} \right) \text{ for any } \lambda \in \mathbb{R}.
\]

By Taylor expansion and dominated convergence theorem,

\[
1 + \lambda \mathbb{E} e_i^T Z + \frac{\lambda^{2}}{2} \mathbb{E}(e_i^T Z)^2 + o(\lambda^{2}) \leq 1 + \frac{1}{1 - \gamma^{2}} \frac{\lambda^{2}}{8} + o(\lambda^{2}).
\]
As $\mathbb{E}e_i^T Z = 0$, letting $\lambda \to 0$ gives

$$\mathbb{E}(e_i^T Z)^2 \leq \frac{1}{4(1 - \gamma^2)}.$$ 

By arbitrariness of $i \in [D]$, we have

$$\|\operatorname{Var}(Z)\|_\infty = \max_{i \in [D]} \mathbb{E}(e_i^T Z)^2 \leq \frac{1}{4(1 - \gamma^2)}.$$ 

**Way 2: decompose the variance by definition.** To distinguish the random variable and its expectation, we add the subscript $t$. Without loss of generality, we assume the $i$-th coordinate of $Z_t$ is the state-action pair $(s, a)$. By definition, $e_i^T Z_t = r_t(s, a) - r(s, a) + (P_t(s, a) - P(s, a))V^*$. It follows that

$$\mathbb{E}(e_i^T Z_t)^2 = \mathbb{E}[r_t(s, a) - r(s, a) + \gamma(P_t(s, a) - P(s, a))V^*)^2]$$

$$(a) = \mathbb{E}(r_t(s, a) - r(s, a))^2 + \gamma^2\mathbb{E}((P_t(s, a) - P(s, a))V^*)^2$$

$$(b) \leq \frac{1}{4} + \gamma^2\mathbb{E}((P_t(s, a) - P(s, a))V^*)^2$$

$$(c) \leq \frac{1}{4} + \gamma^2\mathbb{E}_{s_t \sim P(\cdot|s,a)}\left(\sum_{t=0}^{\infty} \gamma^t r_t\right)^2$$

$$(d) \leq \frac{1}{4} + \gamma^2\mathbb{E}_{s_t \sim P(\cdot|s,a)}\mathbb{E}_{\tau_0 \sim \pi^*}(\sum_{t=0}^{\infty} \gamma^t r_t)^2 = \frac{1}{4} + \gamma^2 \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t r_t\right)^2$$

$$(e) \leq \frac{1}{4} + \gamma^2 \sum_{t=0}^{\infty} \gamma^{2t} \mathbb{E}_{\tau_t}^2 \leq \frac{1}{4} \sum_{t=0}^{\infty} \gamma^{2t} = \frac{1}{4} \frac{1}{1 - \gamma^2}$$

where (a) uses the dependence between $r_t$ and $P_t$, (b) uses $\operatorname{Var}(r_t(s, a)) \leq 1/4$ since $0 \leq r_t(s, a) \leq 1$ almost surely, (c) uses (17) with with $\tilde{r}_t$ defined in (18), (d) uses Jensen’s inequality, and (e) follows from the Markov property of $\{\tilde{r}_t\}$ which implies $\mathbb{E}_{\tau_t} \tilde{r}_j = 0$ for any $t \neq j$, and (f) uses $\mathbb{E}_{\tau_t}^2 = \mathbb{E}\operatorname{Var}(r(s_t)|\mathcal{F}_{t-1}) \leq \operatorname{Var}(r(s_t)) \leq 1/4$ since $0 \leq r(s_t) \leq 1$ almost surely.\footnote{Here we use $\operatorname{Var}(X) = \operatorname{Var}(\mathbb{E}[X|\mathcal{F}]) + \mathbb{E}\operatorname{Var}(X|\mathcal{F})$ for any random variable $X$ and $\sigma$-field $\mathcal{F}$.}

### A.3 Proof of Proposition 3.2

**Proof of Proposition 3.2.** We will make use of two intermediate results in Appendix B. By Lemma B.1, we have

$$\mathbb{E}\|Q_T - Q^*\|_\infty \leq a_T + \mathbb{E}b_T + \mathbb{E}\|N_T\|_\infty$$

where the sequences $\{a_t\}, \{b_t\} \subseteq \mathbb{R}$ and $\{N_t\} \subseteq \mathbb{R}^D$ are defined therein. If we use $\eta_t = \frac{1}{1+(1-\gamma)t}$, then we have

$$a_T = \prod_{t=0}^{T}(1 - (1-\gamma)\eta_t) = \frac{\|Q_0 - Q^*\|_\infty}{1+(1-\gamma)T}$$
and
\[
\mathbb{E} b_T = \gamma \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - (1 - \gamma) \eta_j) \eta_t \mathbb{E} \| N_{t-1} \|_\infty = \gamma \eta_T \sum_{t=1}^{T} \mathbb{E} \| N_{t-1} \|_\infty \overset{(a)}{\leq} \gamma \eta_T \sum_{t=1}^{T} \sqrt{\frac{2 \ln(2eD)}{1 - \gamma^2}} \eta_{t-1}
\]
where (a) uses Lemma B.2. Hence,
\[
\mathbb{E} b_T + \mathbb{E} \| N_T \|_\infty \leq \eta_T \sum_{t=0}^{T-1} \sqrt{\frac{2 \ln(2eD)}{1 - \gamma^2}} \eta_t + \sqrt{\frac{2 \ln(2eD)}{1 - \gamma^2}} \eta_T = \eta_T \sqrt{\frac{2 \ln(2eD)}{1 - \gamma^2}} \left( \sum_{t=0}^{T-1} \sqrt{\eta_t} + \frac{1}{\sqrt{T}} \right).
\]
Since
\[
\sum_{t=0}^{T-1} \sqrt{\eta_t} = \sum_{t=0}^{T-1} \frac{1}{\sqrt{1 + (1 - \gamma)t}} \leq \int_{-1}^{T} \frac{1}{\sqrt{1 + (1 - \gamma)t}} dt \leq \frac{2 \sqrt{1 + (1 - \gamma)T}}{1 - \gamma},
\]
we then have for some \( c > 0 \),
\[
\sum_{t=0}^{T-1} \sqrt{\eta_t} \leq c \frac{\sqrt{1 + (1 - \gamma)T}}{1 - \gamma}.
\]
Putting all the pieces together, we then complete the proof.

\[\Box\]

**B A Convergence Result**

In this section, we aim to devise asymptotic rates for \( \mathbb{E} \| Q_t - Q^* \|_\infty^2 \) for all \( 0 \leq t \leq T \) under general choices of stepsizes. In the sequel, we denote by
\[
\Delta_t = Q_t - Q^*
\]
the error of the Q-function estimate in the \( t \)-th iteration. The main result is the following theorem.

**Theorem B.1.** Under Assumption 3.1, we have the following bounds for \( \mathbb{E} \| \Delta_T \|_\infty^2 \) for three choices of step sizes:

- If \( \eta_t = \frac{1}{1 + (1 - \gamma)t} \), it follows that for all \( T \geq 1 \),
  \[
  \mathbb{E} \| \Delta_T \|_\infty^2 \leq \frac{12 \| \Delta_0 \|_\infty^2}{(1 - \gamma)^2} \frac{1}{(T + 1)^2} + \frac{12 \ln(6D) \ln(3T)}{(1 - \gamma)^4} \frac{1}{T}.
  \]

- If \( \eta_t = t^{-\alpha} \) with \( \alpha \in (0, 1) \) for \( t \geq 1 \) and \( \eta_0 = 1 \), it follows that for all \( T \geq 1 \),
  \[
  \mathbb{E} \| \Delta_T \|_\infty^2 \leq \Delta_0 \exp \left( -\frac{1 - \gamma}{1 - \alpha} ((1 + T)^{1-\alpha} - 1) \right) + \frac{114 \ln(6D)}{(1 - \gamma)^3} \frac{1}{T^\alpha}
  \]
  where
  \[
  \Delta_0 = 3 \| \Delta_0 \|_\infty^2 + \frac{48 \gamma^2 \ln(6D)}{(1 - \gamma)^2} \left( \frac{2\alpha}{1 - \gamma} \right)^{\frac{1}{\alpha}}.
  \]
Theorem B.1 provides the convergence rates of $\mathbb{E}\|\Delta_T\|_\infty^2$ for all $T \geq 1$ and two typical step sizes, both of which are analyzed by Wainwright [2019b]. Though adopting the same induction analysis in Wainwright [2019b], we have a better dependence on $1 - \gamma$ thanks to Theorem 3.4. Previous state-of-the-art analysis in Li et al. [2021], though also tight on the dependence of $(1 - \gamma)^{-1}$, is more sophisticated and complicated. Moreover, it can neither cover all $0 \leq t \leq T$, nor provide convergence for polynomial-decaying step sizes. To prove Theorem B.1, a key lemma is to capture the dependence of $\mathbb{E}\|\Delta_T\|_\infty^2$ on the step size sequences $\{\eta_t\}$. We formulate this dependence in Lemma B.4. By plugging the specific form of $\{\eta_t\}$ and performing some algebra, we can prove Theorem B.1.

**Corollary B.1.** Under Assumption 3.1, there exist some positive constant $c > 0$ such that

- If $\eta_t = \frac{1}{1 + T(1 - \gamma)T}$, it follows that
  $$\frac{1}{T} \sum_{t=0}^{T} \mathbb{E}\|\Delta_t\|_\infty^2 \leq c \left[ \frac{\|\Delta_0\|_\infty^2}{T} + \frac{\ln(6D)\ln^2(3T)}{(1 - \gamma)^3 T} \right].$$

- If $\eta_t = t^{-\alpha}$ with $\alpha \in (0, 1)$ for $t \geq 1$ and $\eta_0 = 1$, it follows that
  $$\frac{1}{T} \sum_{t=0}^{T} \mathbb{E}\|\Delta_t\|_\infty^2 \leq c \left[ \frac{\Delta_0}{\sqrt{1 - \alpha(1 - \gamma)^{1 - \alpha}}} \frac{1}{T} + \frac{\ln(6D)}{(1 - \alpha)(1 - \gamma)^3} \frac{1}{T^\alpha} \right].$$

where $\Delta_0$ is defined in (19).

**Proof of Corollary B.1.** The result directly follows from Theorem B.1 with some further calculation. For the first item, we use $\sum_{t=1}^{\infty} t^{-2} = \frac{\pi^2}{6}$ and $\sum_{t=1}^{T} t^{-1} \leq 1 + \ln T \leq \ln(3T)$. For the second item, we use

$$\sum_{t=2}^{\infty} \exp \left( -\frac{1 - \gamma}{1 - \alpha} (t^{1-\alpha} - 1) \right) \leq \int_{1}^{\infty} \exp \left( -\frac{1 - \gamma}{1 - \alpha} (t^{1-\alpha} - 1) \right) \, dt$$

$$\leq \frac{\exp \left( \frac{1 - \gamma}{1 - \alpha} \right)}{1 - \gamma} \int_{0}^{\infty} e^{-x} \left( \frac{1 - \gamma}{1 - \alpha} x \right)^{1-\alpha} \, dx$$

$$\leq \frac{\exp \left( \frac{1 - \gamma}{1 - \alpha} \right) (1 - \alpha)^{\frac{\alpha}{1 - \alpha}} \Gamma \left( \frac{1}{1 - \alpha} \right)}{1 - \gamma} \leq \frac{\sqrt{2\pi} e}{\sqrt{1 - \alpha} (1 - \gamma)^{1 - \alpha}} \frac{1}{\sqrt{1 - \alpha}}$$

and $\sum_{t=1}^{T} t^{-\alpha} \leq \int_{0}^{T} t^{-\alpha} \, dt = \frac{T^{1-\alpha}}{1 - \alpha}$. Here (a) uses the change of variable $x = \frac{1 - \gamma}{1 - \alpha} t^{1-\alpha}$ and (b) uses the definition of gamma function $\Gamma(z) = \int_{0}^{\infty} e^{-x} x^{z-1} \, dx$. Finally (c) follows from a numeral inequality about gamma function. Since $\Gamma(1 + x) < \sqrt{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2}$ for any $x > 0$ (see Theorem 1.5 of Batir [2008]), then

$$\Gamma \left( \frac{1}{1 - \alpha} \right) \leq \sqrt{2\pi} \left( \frac{1 + \alpha}{2(1 - \alpha)} \right)^{\frac{1 + \alpha}{2(1 - \alpha)}} \exp \left( -\frac{1 + \alpha}{2(1 - \alpha)} \right)$$

which implies that

$$\exp \left( \frac{1 - \gamma}{1 - \alpha} \right) (1 - \alpha)^{\frac{\alpha}{1 - \alpha}} \Gamma \left( \frac{1}{1 - \alpha} \right) \leq \frac{\sqrt{2\pi} e}{\sqrt{1 - \alpha}}.$$

\(\square\)
B.1 A Sandwich $\ell_\infty$ Bound

Our proof is divided into three steps. The first is a upper bound for $\|\Delta_t\|_\infty$ that Lemma B.1 provides:

$$\|\Delta_t\|_\infty \leq a_t + b_t + \|N_t\|_\infty.$$  

As a result, $\|\Delta_t\|_\infty^2 \leq 3(a_t^2 + b_t^2 + \|N_t\|_\infty^2)$. We adopt a similar analysis as Theorem 1 in Wainwright [2019b] which views Q-learning as a cone-contractive operator and establishes a $\ell_\infty$-norm bound.

**Lemma B.1.** For any sequence of stepsizes $\{\eta_t\}$ in the interval $(0, 1)$, the iterates $\{\Delta_t\}$ satisfies the sandwich relation

$$-(a_t + b_t)1 + N_t \leq \Delta_t \leq (a_t + b_t)1 + N_t$$  

where $\{a_t\}, \{b_t\}$ are non-negative scalars and $\{N_t\}$ are random vectors collecting noises in empirical bellman operators. The three sequences are defined in a recursive way: they are initialized as $a_0 = \|\Delta_0\|_\infty, b_0 = 0$ and $N_0 = 0$ and satisfy the following recursion

$$a_t = (1 - \eta_t(1 - \gamma))a_{t-1}$$

$$b_t = (1 - \eta_t(1 - \gamma))b_{t-1} + \eta_t\gamma\|N_{t-1}\|_\infty$$

$$N_t = (1 - \eta_t)N_{t-1} + \eta_tZ_t$$

where $Z_t = (r_t - r) + \gamma(P_t - P)V^*$ is the empirical Bellman error at iteration $t$.

**Proof of Lemma B.1.** The synchronous Q-learning has the following update rule

$$Q_t = (1 - \eta_t)Q_{t-1} + \eta_t(r_t + \gamma P_t V_{t-1})$$  

in the $t$-th iteration. We start by decomposing the estimation error $\Delta_t$.

$$\Delta_t = Q_t - Q^* = (1 - \eta_t)Q_{t-1} + \eta_t(r_t + \gamma P_t V_{t-1} - Q^*)$$

$$= (1 - \eta_t)(Q_{t-1} - Q^*) + \eta_t(r_t + \gamma P_t V_{t-1} - Q^*)$$

$$\overset{(a)}{=} (1 - \eta_t)[r_t - r] + \gamma (P_t V_{t-1} - PV^*)]$$

$$= (1 - \eta_t)[r_t - r] + \gamma (P_t - P)V^* + \gamma P_t (V_{t-1} - V^*)]$$

$$\overset{(b)}{=} (1 - \eta_t)[Z_t + \gamma P_t (V_{t-1} - V^*)]$$

where (a) uses the equation $Q^* = r + \gamma PV^*$ and (b) uses the shorthand $Z_t = (r_t - r) + \gamma (P_t - P)V^*$. Further, the term $P_t (V_{t-1} - V^*)$ can be linked with $\Delta_{t-1}$ as following

$$\|P_t (V_{t-1} - V^*)\|_\infty \leq \|P_t\|_\infty \|V_{t-1} - V^*\|_\infty = \|V_{t-1} - V^*\|_\infty \leq \|Q_{t-1} - Q^*\|_\infty.$$  

Next we use mathematical induction to prove (21). For iteration $t = 0$, it follows that $-(a_0 + b_0)1 + N_0 = -\|\Delta_0\|_\infty 1 \leq \Delta_0 \leq \|\Delta_0\|_\infty 1 = (a_0 + b_0)1 + N_0$ due to the initialization $a_0 = \|\Delta_0\|_\infty, b_0 = 0$ and $N_0 = 0$.

We now assume that the claim holds at iteration $t - 1$, and show that it holds for iteration $t$. For the upper bound, by (23) and (24), we have

$$\Delta_t = (1 - \eta_t)[Z_t + \gamma P_t (V_{t-1} - V^*)]$$

$$\leq (1 - \eta_t)[Z_t + \gamma \|\Delta_{t-1}\|_\infty 1]$$

$$\leq (1 - \eta_t)[(a_{t-1} + b_{t-1})1 + N_{t-1}] + \eta_t[Z_t + \gamma (a_{t-1} + b_{t-1} + \|N_{t-1}\|_\infty)1]$$

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Similarly, for the lower bound, we have

\[
\Delta_t = (1 - \eta_t)\Delta_{t-1} + \eta_t [Z_t + \gamma P_t(V_{t-1} - V^*)] \\
\geq (1 - \eta_t)\Delta_{t-1} + \eta_t [Z_t - \gamma \|\Delta_{t-1}\|_{\infty}1] \\
\geq (1 - \eta_t) [- (a_{t-1} + b_{t-1})1 + N_{t-1}] + \eta_t [Z_t - \gamma(a_{t-1} + b_{t-1}) + \|N_{t-1}\|_{\infty}]1] \\
= -(1 - \eta_t(1 - \gamma))a_{t-1}1 - [(1 - \eta_t(1 - \gamma))b_{t-1} + \eta_t\|N_{t-1}\|_{\infty}]1 + ((1 - \eta_t)N_{t-1} + \eta_tZ_t) \\
= -(a_t + b_t)1 + N_t
\]

which completes the proof of the lower bound. \(\square\)

### B.2 Bound the Second Moment of A Sum of Bellman Noises

The second step is to bound \(E\|N_T\|_{\infty}^2\). The improvement comes from Theorem 3.4 that shows the Bellman error is sub-Gaussian with parameter \(O((1 - \gamma)^{-1})\).

**Lemma B.2.** Under Assumption 3.1 and \((1 - \eta_t)\eta_{t-1} \leq \eta_t\) for any \(t \geq 1\), we have

\[
E\|N_T\|_{\infty}^2 \leq 2\ln(2\pi D) \frac{2\ln(2\pi D)}{1 - \gamma^2} \eta_T.
\]

**Proof of Lemma B.2.** Recall that \(\{N_t\}\) is recursively defined in Lemma B.1: \(N_0 = 0\) and \(N_t = (1 - \eta_t)N_{t-1} + \eta_tZ_t\) with \(Z_t = (r_t - r) + \gamma(P_t - P)V^*\).

**Lemma B.3.** Define \(\sigma^2 = \frac{1}{1 - \gamma^2}\). If \((1 - \eta_t)\eta_{t-1} \leq \eta_t\) for any \(t \geq 1\), we have for any \(t \geq 0\),

\[
E\exp(\lambda\|N_t\|_{\infty}) \leq 2D\exp\left(\frac{\lambda^2\sigma^2\eta_t}{2}\right) \text{ for any } \lambda \in \mathbb{R}.
\]

Lemma B.3 implies \(\|N_T\|_{\infty}\) is essentially sub-Gaussian. Hence, we can compute any moment of \(\|N_T\|_{\infty}\) via a traditional argument in non-asymptotic statistics. Specifically, we first derive the tail distribution bound of \(\|N_T\|_{\infty}\). For any \(\tau \geq 0\),

\[
P(\|N_T\|_{\infty} \geq \tau) \overset{(a)}{\leq} \exp(-\lambda\tau)E\exp(\lambda\|N_T\|_{\infty}) \\
\overset{(b)}{\leq} 2D\exp\left(-\lambda\tau + \frac{\lambda^2\tau^2\eta_T}{2}\right) \\
\overset{(c)}{=} 2D\exp\left(-\frac{\tau^2}{2\sigma^2\eta_T}\right)
\]

where (a) follows from the Markov inequality, (b) follows from Lemma B.3, and (c) follows by setting \(\lambda = \frac{\tau}{\sigma^2\eta_T}\).

Let \(\tau_0 = 2\sigma^2\eta_T\ln(2D)\) such that \(2D\exp\left(-\frac{\tau_0}{2\sigma^2\eta_T}\right) = 1\). Then,

\[
E\|N_T\|_{\infty}^2 = \int_{0}^{\infty} P(\|N_T\|_{\infty}^2 \geq \tau)d\tau = \int_{0}^{\infty} P(\|N_T\|_{\infty} \geq \sqrt{\tau})d\tau
\]
\[ = \int_0^{\tau_0} \mathbb{P} (\| N_T \|_\infty \geq \sqrt{\tau}) d\tau + \int_{\tau_0}^\infty \mathbb{P} (\| N_T \|_\infty \geq \sqrt{\tau}) d\tau \]
\[ \leq \tau_0 + 2D \int_{\tau_0}^\infty \exp \left( -\frac{\tau}{2\sigma^2 \eta_T} \right) d\tau \]
\[ = \tau_0 + 4D\sigma^2 \eta_T \exp \left( -\frac{\tau_0}{2\sigma^2 \eta_T} \right) \leq 2\sigma^2 \eta_T \ln (2eD). \]

\[ \square \]

**Proof of Lemma B.3.** We first prove the following inequality that
\[ \sup_{i \in [D]} \mathbb{E} \exp \left( \lambda \epsilon_i^\top N_t \right) \leq \exp \left( \frac{\lambda^2 \sigma^2 \eta_t}{2} \right) \text{ for any } \lambda \in \mathbb{R} \text{ and } t \geq 0. \]

We will prove it by induction on \( t \). Fix any \( i \in [D] \). The statement is vacuous for \( t = 0 \) since \( N_0 = 0 \). We now assume that the claim holds at iteration \( t \), and then verify that it holds at iteration \( t + 1 \). We have
\[ \mathbb{E} \exp (\lambda \epsilon_i^\top N_{t+1}) = \mathbb{E} \exp (\lambda (1 - \eta_{t+1}) \epsilon_i^\top N_t + \lambda \gamma \eta_{t+1} \epsilon_i^\top Z_{t+1}) \]
\[ = \mathbb{E} \exp (\lambda (1 - \eta_{t+1}) \epsilon_i^\top N_t) \mathbb{E} \exp (\lambda \gamma \eta_{t+1} \epsilon_i^\top Z_{t+1}) \]
\[ \leq \mathbb{E} \exp (\lambda (1 - \eta_{t+1}) \epsilon_i^\top N_t) \cdot \exp \left( \frac{\lambda^2 \gamma^2 \eta_{t+1}^2}{2} \right) \]
\[ \leq \exp \left( \frac{\lambda^2 (1 - \eta_{t+1})^2 \sigma^2}{2} \right) \exp \left( \frac{\lambda^2 \gamma^2 \eta_{t+1}^2}{2} \right) \]
\[ \leq \exp \left( \frac{\lambda^2 \sigma^2 \eta_{t+1}}{2} \right). \]

where \((a)\) follows from Theorem 3.4, \((b)\) follows from the induction hypothesis, and \((c)\) follows from \((1 - \eta_{t+1}) \eta_t \leq \eta_{t+1} \).

Since \( \| N_T \|_\infty = \max_{i \in [D]} \{ \pm \epsilon_i^\top N_T \} \), we have
\[ \mathbb{E} \exp (\lambda \| N_T \|_\infty) \leq \sum_{i \in [D]} \left[ \mathbb{E} \exp (\lambda \epsilon_i^\top N_T) + \mathbb{E} \exp (\lambda \epsilon_i^\top N_T) \right] \leq 2D \exp \left( \frac{\lambda^2 \sigma^2 \eta_T}{2} \right). \]

\[ \square \]

### B.3 Capture the Step Size Dependence

The final step is to establish the dependence of \( \mathbb{E} \| \Delta_T \|_\infty^2 \) on \( \{ \eta_t \} \).

**Lemma B.4.** Let \( \tilde{\eta}_t = (1 - \gamma) \eta_t \) denote the rescaled step size. Under Assumption 3.1, if \((1 - \tilde{\eta}_t) \tilde{\eta}_{t-1} \leq \tilde{\eta}_t \) for any \( t \geq 1 \), then
\[ \mathbb{E} \| \Delta_T \|_\infty^2 \leq 3 \prod_{t=1}^T (1 - \tilde{\eta}_t)^2 \| \Delta_0 \|_\infty^2 + \frac{6 \gamma^2 \ln(6D)}{(1 - \gamma)^3} \sum_{t=1}^T \prod_{j=t+1}^T (1 - \tilde{\eta}_j) \tilde{\eta}_j \tilde{\eta}_{t-1} + \frac{6 \ln(6D)}{1 - \gamma} \eta_T. \] (25)
Proof of Lemma B.4. Wainwright [2019b] implies it is crucial to set $\eta_t$ proportion to $1/(1 - \gamma)$ to ensure the sample complexity is polynomial depends on $1/(1 - \gamma)$. We then set $\tilde{\eta}_t = (1 - \gamma)\eta_t$ as the rescaled step size. By the recursion of $\{a_t\}$ and $\{b_t\}$ in Lemma B.1, it follows that

$$a_T = \prod_{t=1}^{T} (1 - \tilde{\eta}_t)\|\Delta_0\|_\infty$$  
and  
$$b_T = \gamma \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\eta_t\|N_{t-1}\|_\infty.$$  

Hence, $a_T^2 = \prod_{t=1}^{T} (1 - \tilde{\eta}_t)^2\|\Delta_0\|_\infty^2$ and

$$E b_T^2 = \frac{\gamma^2}{(1 - \gamma)^2} E \left( \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\tilde{\eta}_t\|N_{t-1}\|_\infty \right)^2 \leq \frac{3\gamma^2}{(1 - \gamma)^2} \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\tilde{\eta}_t E\|N_{t-1}\|_\infty^2$$

where (a) uses $\sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\tilde{\eta}_t = 1 - \prod_{j=1}^{T} (1 - \tilde{\eta}_j) \leq 1$ and Jensen’s inequality. Therefore,

$$E\|\Delta_T\|_\infty^2 \leq 3(a_T^2 + E b_T^2 + E\|N_T\|_\infty^2) \leq 3 \prod_{t=1}^{T} (1 - \tilde{\eta}_t)^2\|\Delta_0\|_\infty^2 + \frac{3\gamma^2}{(1 - \gamma)^2} \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\tilde{\eta}_t E\|N_{t-1}\|_\infty^2 + 3E\|N_T\|_\infty^2.$$  

The remaining issue is to figure out the bound of $\sum_{t=1}^{T} E\|N_{t-1}\|_\infty^2$. If $\tilde{\eta}_t = (1 - \gamma)\eta_t$ satisfies $(1 - \tilde{\eta}_t)\tilde{\eta}_{t-1} \leq \tilde{\eta}_t$, we must have $(1 - \eta_t)\eta_{t-1} \leq \eta_t$, implying Lemma B.2 can be applied. By Lemma B.2, we have $E\|N_t\|_\infty^2 \leq \frac{2\eta_t \ln(6D)}{1 - \gamma^2} \leq \frac{2\eta_t \ln(6D)}{1 - \gamma}$. Plugging the bound complete the proof. \(\square\)

### B.4 Specific Rates Under Different Step Sizes

Proof of Theorem B.1. In the following, we derive specific rates by plugging the specific step sizes into the result of Lemma B.4.

**I) Linearly rescaled step size** If we use a linear rescaled step size, i.e., $\eta_t = \frac{1}{1 + (1 - \gamma) t}$ (equivalently $\tilde{\eta}_t = \frac{1 - \gamma}{1 + (1 - \gamma) t}$), then we have $1 - \eta_t \leq 1 - \tilde{\eta}_t = 1 + \frac{(1 - \gamma)(t-1)}{1 + (1 - \gamma) t} = \tilde{\eta}_t/\tilde{\eta}_{t-1} = \eta_t/\eta_{t-1}$ for $t \geq 1$. It implies Lemma B.4 is applicable and $\prod_{j=t+1}^{T} (1 - \tilde{\eta}_j)\tilde{\eta}_t \leq \tilde{\eta}_t$. Note that $\sum_{t=1}^{T} \tilde{\eta}_t \leq 1 + \sum_{t=1}^{T} \frac{1}{t} \leq 1 + \ln(T - 1) \leq \ln(3T)$ and

$$\ln \frac{(1 - \gamma)(T + 1)}{2} \leq \ln \frac{1 + (1 - \gamma)(T + 1)}{1 + (1 - \gamma)} = \int_1^{T+1} \frac{1 - \gamma}{1 + (1 - \gamma) t} dt \leq \sum_{t=1}^{T} \frac{1 - \gamma}{1 + (1 - \gamma) t} = \sum_{t=1}^{T} \tilde{\eta}_t.$$  

Hence,

$$E\|\Delta_T\|_\infty^2 \leq 3\|\Delta_0\|_\infty^2 \exp \left(-2\sum_{t=1}^{T} \tilde{\eta}_t\right) + \frac{6\gamma^2 \ln(6D)}{(1 - \gamma)^2} \tilde{\eta}_T \sum_{t=1}^{T} \tilde{\eta}_t + \frac{6 \ln(6D)}{(1 - \gamma)^2} \tilde{\eta}_T$$
At the end of this subsection, we will prove that

\[ \mathbb{E}\|\Delta_t\|^2 \leq 12\frac{\|\Delta_0\|^2}{(1-\gamma)^2} \frac{1}{(T+1)^2} + \frac{6\ln(6D)\ln(3T)}{(1-\gamma)^2} + \frac{6\ln(6D)}{(1-\gamma)^2} \]

which implies

\[ \mathbb{E}\|\Delta_t\|^2 \leq 12\frac{\|\Delta_0\|^2}{(1-\gamma)^2} \frac{1}{(T+1)^2} + 12\frac{\ln(6D)\ln(3T)}{(1-\gamma)^2}. \]  

(II) **Polynomial step size** If we choose a polynomial step size, i.e., \( \eta_t = t^{-\alpha} \) with \( \alpha \in (0, 1) \) for \( t \geq 1 \) and \( \eta_0 = 1 \), then we again have \( 1 - \eta_t = 1 - \frac{1}{t^\alpha} \leq \left( \frac{t+1}{t} \right)^\alpha = \eta_t / \eta_{t-1} \) for \( t \geq 1 \), which implies Lemma B.2 is applicable. Recalling that (25), we have

\[ \mathbb{E}\|\Delta_t\|^2 \leq 3 \prod_{t=1}^{T} (1-\tilde{\eta}_t)^2 \|\Delta_0\|^2 + \frac{6\gamma^2 \ln(6D)}{(1-\gamma)^2} \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1-\tilde{\eta}_j) \eta_{t-1} + \frac{6\ln(6D)}{1-\gamma} \eta_T. \]

Note that

\[ \frac{(T+1)^{1-\alpha} - (t+1)^{1-\alpha}}{1-\alpha} = \int_{t+1}^{T+1} j^{-\alpha} dj \leq \sum_{j=t+1}^{T} j^{-\alpha} \leq \int_{t}^{T} j^{-\alpha} dj = \frac{T^{1-\alpha} - t^{1-\alpha}}{1-\alpha}, \]  

which implies that \( \sum_{t=1}^{T} \eta_t \geq \sum_{t=1}^{T} t^{-\alpha} \geq \frac{1}{1-\alpha} ((T+1)^{1-\alpha} - 1) \) and \( (T+1)^{1-\alpha} \leq 1 + T^{1-\alpha} \). Hence,

\[ \prod_{t=1}^{T} (1-\tilde{\eta}_t)^2 \leq \exp \left( -2(1-\gamma) \sum_{t=1}^{T} \eta_t \right) \leq \exp \left( -2 \frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha} \right). \]

Additionally, using \( \eta_{t-1} \leq 2\eta_t \) for all \( t \geq 1 \) and (27), we have,

\[ \prod_{j=t+1}^{T} (1-\tilde{\eta}_j) \eta_{t-1} \leq 8 \prod_{j=t+1}^{T} (1-\tilde{\eta}_j) \eta_{t+1} \leq 8 \exp \left( - \sum_{j=t+1}^{T} \tilde{\eta}_j \right) \eta_{t+1}^2 \]

\[ \leq 8 \exp \left( - \frac{1-\gamma}{1-\alpha} (1+T)^{1-\alpha} \right) \frac{\exp \left( \frac{1-\gamma}{1-\alpha} (t+1)^{1-\alpha} \right)}{(t+1)^{2\alpha}}, \]

which implies

\[ \sum_{t=1}^{T} \prod_{j=t+1}^{T} (1-\tilde{\eta}_j) \eta_{t-1} \leq \sum_{t=1}^{T-1} \prod_{j=t+1}^{T} (1-\tilde{\eta}_j) \eta_{t-1} + 2\eta_T \]

\[ \leq 8 \sum_{t=2}^{T} \exp \left( - \frac{1-\gamma}{1-\alpha} (1+T)^{1-\alpha} \right) \frac{\exp \left( \frac{1-\gamma}{1-\alpha} (t+1)^{1-\alpha} \right)}{(t+1)^{2\alpha}} + \frac{2}{T^{2\alpha}}. \]

At the end of this subsection, we will prove that

**Lemma B.5.** For any \( \alpha \in (0, 1) \) and \( \beta > 0 \), it follows that

\[ \sum_{t=1}^{T} \frac{\exp \left( \frac{1-\gamma}{1-\alpha} (t+1)^{1-\alpha} \right)}{t^\beta} \leq \left( \frac{\beta}{1-\gamma} \right)^{\frac{1}{1-\alpha}} \exp \left( \frac{1-\gamma}{1-\alpha} \right) \frac{\exp \left( \frac{1-\gamma}{1-\alpha} (1+T)^{1-\alpha} \right)}{(1-\gamma)^{\alpha}}. \]

(28)
By setting $\beta = 2\alpha$, we have
\[
\sum_{t=1}^{T} \frac{\exp\left(\frac{1-\gamma}{1-\alpha} t^{-\alpha} - a\right)}{t^{2\alpha}} \leq \left(\frac{2\alpha}{1-\gamma}\right)^{\frac{1}{\gamma}} \exp\left(\frac{1-\gamma}{1-\alpha}\right) + \frac{2}{1-\gamma} \exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha}\right).
\]
Therefore,
\[
\sum_{t=1}^{T} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j) \eta_i t_{i-1} \leq 8 \left(\frac{2\alpha}{1-\gamma}\right)^{\frac{1}{\gamma}} \exp\left(\frac{1-\gamma}{1-\alpha}\right) \exp\left(-\frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha}\right) + \frac{16}{1-\gamma} \frac{1}{(1 + T)^{\alpha}} + \frac{2}{T^{2\alpha}}.
\]
Putting together the pieces, we can safely conclude that
\[
\mathbb{E}\|\Delta_T\|_2^2 \leq 3\|\Delta_0\|_2^2 \exp\left(-2\frac{1-\gamma}{1-\alpha} ((T + 1)^{1-\alpha} - 1)\right) + \frac{6 \ln(6D)}{1-\gamma} \frac{1}{T^{\alpha}} + \frac{96\gamma^2 \ln(6D)}{(1-\gamma)^3} \frac{1}{(1 + T)^{\alpha}}
\]
\[
+ \frac{12\gamma^2 \ln(6D)}{(1-\gamma)^2} \frac{1}{T^{2\alpha}} + \frac{48\gamma^2 \ln(6D)}{(1-\gamma)^2} \exp\left(-\frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha}\right) \left(\frac{2\alpha}{1-\gamma}\right)^{\frac{1}{\gamma}} \exp\left(\frac{1-\gamma}{1-\alpha}\right)
\]
\[
\leq \Delta_0 \exp\left(-\frac{1-\gamma}{1-\alpha} ((1 + T)^{1-\alpha} - 1)\right) + \frac{114 \ln(6D)}{(1-\gamma)^3} \frac{1}{T^{\alpha}}
\]
where
\[
\Delta_0 = 3\|\Delta_0\|_2^2 + \frac{48\gamma^2 \ln(6D)}{(1-\gamma)^2} \left(\frac{2\alpha}{1-\gamma}\right)^{\frac{1}{\gamma}}.
\]

Proof of Lemma B.5. We do this via a similar argument of Lemma 4 in Wainwright [2019b]. Let $f(t) = \frac{\exp\left(\frac{1-\gamma}{1-\alpha} t^{-\alpha} - a\right)}{t^\beta}$. By taking derivatives, we find that $f(t)$ is decreasing in $t$ on the interval $[0, t^*]$ and increasing for $[t^*, \infty)$, where $t^* = \left(\frac{\beta}{1-\gamma}\right)^{\frac{1}{\gamma}}$. Hence,
\[
\sum_{t=1}^{T} f(t) \leq \begin{cases} T f(1) & \text{if } T \leq [t^*], \\ [t^*] f(1) + \int_{t^*}^{T+1} f(t) dt & \text{if } T > [t^*]. \end{cases}
\]
Using integrating by parts, it follows that
\[
I^* := \int_{t^*}^{T+1} f(t) dt = \frac{\exp\left(\frac{1-\gamma}{1-\alpha} t^{-\alpha} - a\right)}{(1-\gamma)t^{\beta-\alpha}} \bigg|_{t^*}^{T+1} + \frac{\beta - \alpha}{1-\gamma} \int_{t^*}^{T+1} \frac{\exp\left(\frac{1-\gamma}{1-\alpha} t^{-\alpha} - a\right)}{t^{1+\beta-\alpha}} dt
\]
\[
\leq \frac{\exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{-\alpha} - a\right)}{(1-\gamma)(1 + T)^{\beta-\alpha}} + \frac{\beta - \alpha}{1-\gamma} \int_{t^*}^{T+1} \frac{f(t)}{t^{\beta-\alpha}} dt
\]
\[
\leq \frac{\exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{-\alpha} - a\right)}{(1-\gamma)(1 + T)^{\beta-\alpha}} + \frac{\beta - \alpha}{1-\gamma} \frac{1}{(t^*)^{1-\alpha}} \int_{t^*}^{T+1} f(t) dt
\]
\[
= \frac{\exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{-\alpha} - a\right)}{(1-\gamma)(1 + T)^{\beta-\alpha}} + \frac{\beta - \alpha}{\beta} I^*
\]

28
where the last equality uses definition of $t^*$ and $I^*$. Hence, we have

$$I^* = \int_{t^*}^{T+1} f(t) dt \leq \frac{\beta}{(1-\gamma)\alpha} \exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha}\right).$$

Putting together the pieces, we have shown that if $T > \lfloor t^* \rfloor$,

$$\sum_{t=1}^{T} f(t) \leq t^* f(1) + I^* = \left(\frac{\beta}{1-\gamma}\right) \frac{1}{\alpha} \exp\left(\frac{1-\gamma}{1-\alpha}\right) + \frac{\beta}{(1-\gamma)\alpha} \exp\left(\frac{1-\gamma}{1-\alpha} (1 + T)^{1-\alpha}\right).$$

If $T \leq \lfloor t^* \rfloor$, then

$$\sum_{t=1}^{T} f(t) \leq \lfloor t^* \rfloor f(1) \leq t^* f(1) = \left(\frac{\beta}{1-\gamma}\right) \frac{1}{\alpha} \exp\left(\frac{1-\gamma}{1-\alpha}\right).$$

Now we have proved the inequality is true for any choice of $T$. \hfill \Box

C Proof of Theorem 3.1

C.1 Main Idea

In this section, we provide a self-contained proof for our central limit theorem. The application of Polyak-Ruppert average [Polyak and Juditsky, 1992] gives an estimator for $Q^*$ with error given by

$$\bar{\Delta}_T := \frac{1}{T} \sum_{t=1}^{T} \Delta_t = \frac{1}{T} \sum_{t=1}^{T} (Q_t - Q^*).$$

(29)

To proceed the proof, we will use two auxiliary sequences $\{\Delta^1_t\}$ and $\{\Delta^2_t\}$ defined in Lemma C.1. The proof of Lemma C.1 can be found in Appendix C.4.1. The two sequences form a sandwich bound for $\Delta_t$, producing $\Delta^2_t \leq \Delta_t \leq \Delta^1_t$. We similarly define

$$\Delta^1_T := \frac{1}{T} \sum_{t=1}^{T} \Delta^1_t$$

and $\Delta^2_T := \frac{1}{T} \sum_{t=1}^{T} \Delta^2_t$.

Then, it is valid that

$$\sqrt{T} \Delta^2_T \leq \sqrt{T} \Delta_T \leq \sqrt{T} \Delta^1_T.$$ 

(30)

Lemma C.1. Denote $G = I - \gamma P^\pi$, $A_t = I - \eta_t G$ and $W_t = (r_t - r) + \gamma (P_t - P) V_{t-1}$ for short. The auxiliary sequences $\{\Delta^1_t\}$ and $\{\Delta^2_t\}$ are defined iteratively: $\Delta^1_0 = \Delta^2_0 = \Delta_0$ and for $t \geq 1$

$$\Delta^1_t = A_t \Delta^1_{t-1} + \eta_t \left[ W_t + \gamma (P^\pi_{t-1} - P^\pi) \Delta_{t-1}\right]$$

(31)

$$\Delta^2_t = A_t \Delta^2_{t-1} + \eta_t W_t.$$ 

(32)

As long as $\sup_t \eta_t \leq 1$, it follows that all $t \geq 0$,

$$\Delta^2_t \leq \Delta_t \leq \Delta^1_t.$$ 

(33)
Remark C.1. Some readers might wonder why we would bother to introduce two auxiliary sequences rather than focus on the raw sequence. This is because the auxiliary sequences are easier to analyze since they can be decomposed into several terms having nice structures. In this way, we can capture the whole effect of non-stationarity and show it is of high order (see $T_4$ in (38)).

In the following subsections, we will show that under Assumption 3.1 and 3.3, $\sqrt{T}\Delta^1_T$ and $\sqrt{T}\Delta^2_T$ weakly converge to the same normal distribution $\mathcal{N}(0, V)$ with $V$ defined in (9). By the sandwich inequality (30), we immediately conclude that $\sqrt{T}\Delta^T \overset{d}{\to} \mathcal{N}(0, V)$ by the definition of weakly convergence. Indeed, for any $x \in \mathbb{R}^D$, we have

$$\Phi_V(x) = \lim_{T \to \infty} \mathbb{P}(\sqrt{T}\Delta^1_T \leq x) \leq \lim inf_{T \to \infty} \mathbb{P}(\sqrt{T}\Delta_T \leq x) \leq \lim sup_{T \to \infty} \mathbb{P}(\sqrt{T}\Delta_T \leq x) \leq \Phi_V(x)$$

where $\Phi_V(x)$ is the cumulative distribution function of $\mathcal{N}(0, V)$. Hence, $\lim_{T \to \infty} \mathbb{P}(\sqrt{T}\Delta_T \leq x)$ exists and equal to $\Phi_V(x)$, which implies $\sqrt{T}\Delta^T \overset{d}{\to} \mathcal{N}(0, V)$.

The following is a useful inequality which would be used frequently in the subsequent proof.

**Lemma C.2.** For any matrices $A, V$ with the compatible order, we have

$$\|\text{diag}(AVA^\top)\|_\infty \leq \|V\|_{\text{max}} \|A\|_\infty^2 \quad \text{(34)}$$

where $\|V\|_{\text{max}} = \max_{i,k} |V(i,k)|$.

**Proof of Lemma C.2.** For any diagonal entry $i$, it follows that

$$|(AVA^\top)(i,i)| = \left| \sum_l (AV)(i,l)A(i,l) \right| = \left| \sum_l \sum_k A(i,k)V(k,l)A(i,l) \right|$$

$$\leq \sum_l \sum_k |A(i,k)| \cdot |V(k,l)| \cdot |A(i,l)|$$

$$\leq \|V\|_{\text{max}} \sum_k |A(i,k)| \cdot \sum_l |A(i,l)|$$

$$\leq \|V\|_{\text{max}} \|A\|_\infty^2.$$ 

\[\Box\]

### C.2 Asymptotic Normality of $\sqrt{T}\Delta^1_T$

We first show the asymptotic normality of $\sqrt{T}\Delta^1_T$. The asymptotic normality of $\sqrt{T}\Delta^2_T$ can be proved in an almost identical way. We start by rewriting (31) as

$$\Delta^1_t = A_t\Delta^1_{t-1} + \eta_t (Z_t + \gamma D^1_{t-1}). \quad \text{(35)}$$

where $A_t = I - \eta_t(I - \gamma P^\pi)$, $Z_t = (r_t - r) + \gamma(P_t - P)V^\pi$, and $D^1_{t-1} = (P_t - P)(V_{t-1} - V^\pi) + (P^\pi_{t-1} - P^\pi)V_{t-1}$. We comment that $\{Z_t\}$ collects the i.i.d. noises inherent in the empirical
bellman operator and \{D_{i-1}\} captures the closeness between the current Q-function estimator \(Q_{t-1}\) and the optimal one \(Q^*\). Recurring (35) gives

\[
\Delta_t^1 = \prod_{j=1}^{t} A_j \Delta_0 + \gamma \sum_{j=1}^{t} \prod_{i=j+1}^{t} A_i \eta_j (Z_j + \gamma D_{j-1}^1).
\]

Here we use the convention that \(\prod_{i=t+1}^{T} A_i = I\) for any \(t \geq 0\). Summing the last equality over \(t = 1, \ldots, T\) and scaling it properly, we have

\[
\sqrt{T} \Delta_T^1 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \Delta_t^1 = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \prod_{j=1}^{t} A_j \Delta_0 + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{i=j+1}^{t} A_i \eta_j (Z_j + \gamma D_{j-1}^1)
\]

\[
= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \prod_{j=1}^{t} A_j \Delta_0 + \frac{1}{\sqrt{T}} \sum_{t=J}^{T} \sum_{i=j+1}^{t} A_i \eta_j (Z_j + \gamma D_{j-1}^1)
\]

\[
= \frac{1}{\eta_0 \sqrt{T}} (A_T^T - \eta_0 I) \Delta_0 + \frac{1}{\sqrt{T}} \sum_{j=1}^{T} A_j^T (Z_j + \gamma D_{j-1}^1) \tag{36}
\]

where the last line uses the following notation

\[
A_j^T = \eta_j \prod_{t=j+1}^{T} A_i \text{ for any } T \geq j \geq 0. \tag{37}
\]

Define \(G = I - \gamma P^{\pi^*}\) with \(\gamma \in [0, 1)\), then \(A_i = I - \eta_i G\). Typically speaking, \(A_j^T\) approximates \(G\) uniformly well (see Lemma C.4). By the observation, we further expand (36) and decompose \(\sqrt{T} \Delta_T^1\) into five terms which will be analyzed respectively in the following:

\[
\sqrt{T} \Delta_T^1 = \frac{1}{ \eta_0 \sqrt{T}} (A_T^T - \eta_0 I) \Delta_0 + \frac{1}{\sqrt{T}} \sum_{j=1}^{T} G^{-1} Z_j + \frac{1}{\sqrt{T}} \sum_{j=1}^{T} (A_j^T - G^{-1}) Z_j
\]

\[
+ \gamma \frac{1}{\sqrt{T}} \sum_{j=1}^{T} A_j^T (P_j - P)(V_{j-1} - V^*) + \gamma \frac{1}{\sqrt{T}} \sum_{j=1}^{T} A_j^T (P^{\pi_{j-1}} - P^{\pi^*}) \Delta_{j-1}
\]

\[
:= \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4. \tag{38}
\]

**C.2.1 Properties of \{A_j^T\}_{0 \leq j \leq T}**

In this part, we introduce some properties of \(\{A_j^T\}_{0 \leq j \leq T}\) under different step sizes. Define \(\tilde{\eta}_t = (1 - \gamma) \eta_t\) as the rescaled step size for simplicity. We mainly consider two types of step sizes:

(S1) linear rescaled step size that uses \(\eta_t = \frac{1}{1+(1-\gamma)t}\) (equivalently \(\tilde{\eta}_t = \frac{1-\gamma}{1+(1-\gamma)t}\));

(S2) polynomial step size that uses \(\eta_t = t^{-\alpha}\) with \(\alpha \in (0, 1)\) for \(t \geq 1\) and \(\eta_0 = 1\).

**Lemma C.3 (Uniform boundedness).** It follows that for any \(T \geq j \geq 1\), there exists some \(c > 0\) such that

\[
\|A_j^T\|_\infty \leq C_0 := \begin{cases} \max(1+(1-\gamma)T), & (S1) \\ \frac{1}{\sqrt{1-\alpha}} \left( \frac{1}{1-\gamma} \right)^{1-\alpha}, & (S2) \end{cases}
\]
Prior work of Polyak and Juditsky [1992] shows that when the step size \( \eta_t \) is either sufficiently small or decreases at a slow rate, \( A_j^T \) is a good surrogate of \( G^{-1} := (I - \gamma P^{\pi^*})^{-1} \) in the asymptotic sense: \( \lim_{T \to \infty} \frac{1}{T} \sum_{j=1}^{T} ||A_j^T - G^{-1}||_2 = 0 \). For our purpose, we provide a non-asymptotic counterpart in the following lemma which captures the convergence rate of \( \frac{1}{T} \sum_{j=1}^{T} ||A_j^T G^{-1} I||_\infty^2 \) under the polynomial step size. We observe that as \( T \) grows, \( \frac{1}{T} \sum_{j=1}^{T} ||A_j^T G^{-1} I||_\infty^2 \) vanishes under (S2), but is only guaranteed to be bounded for (S1). The proof is deferred in Appendix C.4.3.

**Lemma C.4 (Uniform approximation).** It follows that for some constant \( c > 0 \),

\[
\sqrt{\frac{1}{T} \sum_{j=1}^{T} ||A_j^T - G^{-1}||_\infty^2} \leq \begin{cases} 
\frac{5}{1-\gamma} \frac{c_02\sqrt{T}}{\sqrt{T}} \left[ \frac{1}{(1-\alpha)^2} \frac{1}{1-\gamma} + \frac{1}{1-\gamma} \sqrt{\sum_{j=1}^{T} \frac{1}{j(1-\alpha)}} \right] + \frac{1}{(1-\gamma)} \sqrt{\frac{1}{T^{\eta_T}}} & (S1) \\
\end{cases}

(S2)

**C.2.2 Bound the Five Terms**

We are going to bound the the five terms of the right hand side of (38) in the following.

**Negligibility of \( \mathcal{T}_0 \).** Since \( \eta_0 \leq 1 \leq C_0 \), it is obvious that

\[
||\mathcal{T}_0||_\infty = \frac{1}{\eta_0 \sqrt{T}} ||(A_0^T - \eta_0 I)\Delta_0||_\infty \leq \frac{1}{\eta_0 \sqrt{T}} (||A_0^T||_\infty + \eta_0)||\Delta_0||_\infty \\
\leq \frac{2C_0}{\eta_0 T^{\eta_T}} \to 0 \text{ as } T \to \infty.
\]

**Asymptotic Normality of \( \mathcal{T}_1 \).** Recall that \( Z_j = (r_j - \tau) + \gamma (P_j - P)V^* \) is the noise inherent in the empirical Bellman operator at iteration \( j \). Since at each iteration the simulator generates rewards \( r_j \) and produces next states \( P_j \) in an i.i.d. fashion, \( \mathcal{T}_1 \) is the scaled sum of \( T \) independent copies of the random vector \( Z_j \) which has zero mean and finite variance denoted by \( \text{Var}(Z_j) = \text{Var}(r_j + \gamma P_j V^*) = \mathbb{E}Z_j Z_j^T \).

**Lemma C.5.** By multivariate central limit theorem, it follows that

\[
\mathcal{T}_1 = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} G^{-1} Z_j \overset{d}{\to} \mathcal{N}(0, \text{VarQ})
\]

where the variance is equivalent to

\[
\text{VarQ} = G^{-1} \text{Var}(Z_j) G^{-\top} = (I - \gamma P^{\pi^*})^{-1} \text{Var}(Z_j)(I - \gamma P^{\pi^*})^{-\top}.
\]

**Negligibility of \( \mathcal{T}_2 \).** We will prove \( \mathcal{T}_2 = o_p(1) \) by showing \( \mathbb{E}||\mathcal{T}_2||_\infty = o(1) \). To the end, we again make use of the fact that empirical Bellman error \( \{Z_i\} \) is sub-Gaussian and obtain the following lemma. Lemma C.4 implies that \( \frac{1}{T} \sum_{j=1}^{T} ||A_j^T - G^{-1}||_\infty^2 \to 0 \) under the polynomial step size. Hence, \( \mathbb{E}||\mathcal{T}_2||_\infty = o(1) \) and thus \( \mathcal{T}_2 = o_p(1) \). The proof is deferred in Appendix C.4.4.

**Lemma C.6.**

\[
\mathbb{E}||\mathcal{T}_2||_\infty \leq \sqrt{\frac{2 \ln(6D)}{1-\gamma^2} \frac{1}{T} \sum_{j=1}^{T} ||A_j^T - G^{-1}||_\infty^2}
\]

where \( A_j^T \) is defined in (37) and \( D = |S \times A| \).
Negligibility of \( \mathcal{T}_3 \). We will prove \( \mathcal{T}_3 = o_T(1) \) by showing \( \mathbb{E}\|\mathcal{T}_3\|_\infty = o(1) \). Lemma C.7 establishes a Bernstein-style expectation bound for \( \mathcal{T}_3 \). Corollary B.1 implies that \( \frac{\ln T}{T} \sum_{j=1}^{T} \|\Delta_j\|_\infty^2 \to 0 \) as \( T \to \infty \) under the polynomial step size, which directly leads to \( \mathbb{E}\|\mathcal{T}_3\|_\infty = o(1) \). The proof is deferred in Appendix C.4.5.

Lemma C.7.

\[
\mathbb{E}\|\mathcal{T}_3\|_\infty \leq 4\gamma C_0 \sqrt{\ln(2DT^2)} \cdot \sqrt{\frac{1}{T} \sum_{j=1}^{T} \mathbb{E}\|\Delta_{j-1}\|_\infty^2} + \frac{8\gamma C_0 \ln(3DT^2)}{3\sqrt{T}(1-\gamma)},
\]

where \( C_0 \) is the uniform bound given in Lemma C.3 and \( D = |S \times A| \).

Negligibility of \( \mathcal{T}_4 \). Similarly, we will prove \( \mathcal{T}_4 = o_T(1) \) by showing \( \mathbb{E}\|\mathcal{T}_4\|_\infty = o(1) \). From a retrospective view, \( \mathcal{T}_2 \) (or \( \mathcal{T}_3 \)) is a finite sum of mean-zero martingale differences whose cross terms are zero under expectation. It results the dominated term of \( \mathbb{E}\|\mathcal{T}_2\|_\infty \) (or \( \mathbb{E}\|\mathcal{T}_3\|_\infty \)) is its variance, when a Hoeffding-style or Bernstein-style concentration bound (see Lemma C.11 or F.1) is used. However, bounding \( \mathbb{E}\|\mathcal{T}_4\|_\infty \) should be treated differently since it is the sum of correlated random variables (which are even not mean-zero). To that end, we need a high-order residual condition (39), which is ensured by a positive optimality gap as Lemma C.8 shows. With such a Lipschitz condition, in Lemma C.9, we show \( \mathbb{E}\|\mathcal{T}_4\|_\infty \) is dominated by \( \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \mathbb{E}\|\Delta_{j-1}\|_\infty^2 \) which is \( o(1) \) for the polynomial step size with \( \alpha \in (0, 1) \) as suggested by Corollary B.1.

Remark C.2. It is worthy to mention that \( \mathcal{T}_4 \) is incurred by the non-stationary nature or Q-learning. If we consider a stationary update process, e.g., policy evaluation like \([\text{Mou et al., 2020a, b, Khamaru et al., 2021b}]\), \( \mathcal{T}_4 \) would disappear. It will make our analysis simpler and results nicer.

Lemma C.8. If \( \text{gap} = \min_{a_1 \neq a_2} |Q^*(s, a_1) - Q^*(s, a_2)| > 0 \), then the optimal policy \( \pi^* \) is unique. What’s more, for any Q-function estimator \( Q \), it follows that

\[
\| (P^{\pi_Q} - P^{\pi^*})(Q - Q^*) \|_\infty \leq L \| Q - Q^* \|_\infty \text{ with } L = \frac{4}{\text{gap}}
\]

where \( \pi_Q \) is the greedy policy with respect to \( Q \) defined by \( \pi_Q := \arg \max_{a \in A} Q(s, a) \). If \( \arg \max_{a \in A} Q(s, a) \) has more than one element, we break the tie by randomness.

Proof of Lemma C.8. For any \( Q \) satisfying \( \| Q - Q^* \|_\infty < \frac{\text{gap}}{2} \), we must have \( \| Q(s, \cdot) - Q^*(s, \cdot) \|_\infty < \frac{\text{gap}}{2} \) for any \( s \in S \). In this case, it must be true that \( \pi_Q(s) = \pi^*(s) \) for all \( s \in S \). Otherwise, there exists some \( s \in S \) such that \( \pi_Q(s) \neq \pi^*(s) \). Letting \( a = \pi_Q(s) \) and \( a^* = \pi^*(s) \) for simplicity, we then have

\[
Q(s, a) < Q^*(s, a) + \frac{\text{gap}}{2} \leq Q^*(s, a^*) - \frac{\text{gap}}{2} < Q(s, a^*)
\]

where \( (a) \) follows from the definition of the optimality gap. The result \( Q(s, a) < Q(s, a^*) \) contradicts with the fact that \( a = \pi_Q(s) \) is the greedy policy with respect to \( Q \) at state \( s \), which implies \( Q(s, a^*) \leq Q(s, a) \). This implies that the event \( \{ \pi_Q \neq \pi^* \} \subseteq \{ \| Q - Q^* \|_\infty \geq \frac{\text{gap}}{2} \} \) and thus \( 1_{\{ \pi_Q \neq \pi^* \}} \leq 1_{\{ \| Q - Q^* \|_\infty \geq \frac{\text{gap}}{2} \}} \). Hence,

\[
\| (P^{\pi_Q} - P^{\pi^*})(Q - Q^*) \|_\infty \leq \| P^{\pi_Q} - P^{\pi^*} \|_\infty \| Q - Q^* \|_\infty
\]

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We can repeat the above analysis for where the last line uses \(1_{\{\|Q - Q^*\|_\infty \geq \text{gap}\}} \leq \frac{2}{\text{gap}} \|Q - Q^*\|_\infty\).

**Lemma C.9.** With a positive optimality gap, define \(L = \frac{4}{\text{gap}}\). It follows that

\[
E\|\mathcal{T}_4\|_\infty \leq \gamma L C_0 \cdot \frac{1}{\sqrt{T}} \sum_{j=1}^{T} E\|\Delta_{j-1}\|^2_\infty.
\]

**Proof of Lemma C.9.** By Lemma C.8 and Lemma C.3, it follows that

\[
E\|\mathcal{T}_4\|_\infty = \frac{\gamma}{\sqrt{T}} E \left\| \sum_{j=1}^{T} A_j^T (P^\pi_{j-1} - P^{\pi*}) \Delta_{j-1} \right\|_\infty
\]

\[
\leq \frac{\gamma}{\sqrt{T}} E \sum_{j=1}^{T} \|A_j^T\|_\infty \left\| (P^\pi_{j-1} - P^{\pi*}) \Delta_{j-1} \right\|_\infty
\]

\[
\leq \gamma L C_0 \cdot \frac{1}{\sqrt{T}} E \sum_{j=1}^{T} \|\Delta_{j-1}\|^2_\infty.
\]

Putting pieces together. We have shown \(\sqrt{T} \Delta^1_T = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4\) with \(\mathcal{T}_0 = o_P(1), \mathcal{T}_1 \xrightarrow{d} \mathcal{N}(0, V), \mathcal{T}_2 = o_P(1), \mathcal{T}_3 = o_P(1), \mathcal{T}_4 = o_P(1)\). By Slutsky’s theorem, it follows that \(\sqrt{T} \Delta^1_T \xrightarrow{d} \mathcal{N}(0, V)\) with \(V\) defined in (9). We then complete the proof for the asymptotic normality of \(\sqrt{T} \Delta^1_T\).

### C.3 Asymptotic Normality of \(\sqrt{T} \Delta^2_T\)

We can repeat the above analysis for \(\sqrt{T} \Delta^2_T\). We rewrite (32) as

\[
\Delta^2_t = A_t \Delta^2_{t-1} + \eta_t (Z_t + \gamma D^1_{t-1}).
\]

where \(A_t = I - \eta_t (I - \gamma P^{\pi*})\) and \(Z_t = (r_t - r) + \gamma (P_t - P)V^*\) are the same as those defined in (35) except that \(D^1_{t-1}\) is replaced by \(D^2_{t-1} = (P_t - P)(V_{t-1} - V^*)\). Since \(D^1_{t-1}\) is much simpler than \(D^2_{t-1}\), the analysis for \(\sqrt{T} \Delta^2_T\) should be easier than \(\sqrt{T} \Delta^1_T\). We similarly decompose \(\sqrt{T} \Delta^2_T\) into four terms:

\[
\sqrt{T} \Delta^2_T = \frac{1}{\eta_0 \sqrt{T}} A_0^T \Delta_0 + \frac{1}{\sqrt{T}} \sum_{j=1}^{T} G^{-1} Z_j + \frac{1}{\sqrt{T}} \sum_{j=1}^{T} (A_j^T - G^{-1}) Z_j
\]

\[
+ \gamma \frac{1}{\sqrt{T}} \sum_{j=1}^{T} A_j^T (P_j - P)(V_{j-1} - V^*)
\]

\[
= \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.
\]

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Here \( \{T_i\}_{i=0}^{3} \) are exactly the same as in (38). Our previous analysis provides us a low-hanging fruit result: \( T_0 = o_p(1), T_1 \overset{d}{\rightarrow} \mathcal{N}(0, V), T_2 = o_p(1), T_3 = o_p(1) \). By Slutsky’s theorem, it follows that \( \sqrt{T} \Delta_{t}^{2} \overset{d}{\rightarrow} \mathcal{N}(0, V) \) where \( V \) is given in (9).

### C.4 Missing Proof of Lemmas

#### C.4.1 Proof of Lemma C.1

*Proof of Lemma C.1.* We use mathematical induction to prove the statement. When \( t = 0 \), the inequality (33) holds by initialization. Assume (33) holds at \( t - 1 \), i.e., \( \Delta_{t-1}^{2} \leq \Delta_{t-1} \leq \Delta_{t-1}^{1} \). Let us analyze the case of \( t \). By the Q-learning update rule (e.g., see (23)), it follows that

\[
\Delta_t = (1 - \eta_t) \Delta_{t-1} + \eta_t [(r_t - r) + \gamma (P_t V_{t-1} - P V^*)]
\]

\[(a) \quad (1 - \eta_t) \Delta_{t-1} + \eta_t [W_t + \gamma (P V_{t-1} - P V^*)]
\]

\[(b) \quad (1 - \eta_t) \Delta_{t-1} + \eta_t [W_t + \gamma (P \pi_{t-1} Q_{t-1} - P^* Q^*)]
\]

\[(c) \quad A_t \Delta_{t-1} + \eta_t [W_t + \gamma (P \pi_{t-1} - P^*) Q_{t-1}]
\]

(42)

where (a) uses \( W_t = (r_t - r) + \gamma (P_t - P) V_{t-1} \); (b) uses \( P V_{t-1} = P \pi_{t-1} Q_{t-1} \) and \( PV^* = P^* Q^* \), and (c) follows by arrangement and the shorthand \( A_t = I - \eta_t (I - \gamma P \pi^*) \). Since all the entries of \( A_t = I - \eta_t (I - \gamma P \pi^*) \) are non-negative (which results from the assumption \( \sup_t \eta_t \leq 1 \)), then \( A_t \Delta_{t-1}^{2} \leq A_t \Delta_{t-1} \leq A_t \Delta_{t-1}^{1} \).

For one hand, based on (42), we have

\[
\Delta_t^2 = A_t \Delta_{t-1}^2 + \eta_t W_t \leq A_t \Delta_{t-1} + \eta_t W_t
\]

\[
\leq A_t \Delta_{t-1} + \eta_t \left[ W_t + \gamma (P \pi_{t-1} - P^*) Q_{t-1} \right] = \Delta_t
\]

where the last inequality uses \( P \pi_{t-1} Q_{t-1} \geq P^* Q_{t-1} \) which results from the fact \( \pi_{t-1} \) is the greedy policy with respect to \( Q_{t-1} \). For the other hand, it follows that

\[
\Delta_t = A_t \Delta_{t-1} + \eta_t \left[ W_t + \gamma (P \pi_{t-1} - P^*) Q_{t-1} \right]
\]

\[
= A_t \Delta_{t-1}^1 + \eta_t \left[ W_t + \gamma (P \pi_{t-1} - P^*) \Delta_{t-1} + \gamma (P \pi_{t-1} - P^*) Q^* \right]
\]

\[
\leq A_t \Delta_{t-1}^1 + \eta_t \left[ W_t + \gamma (P \pi_{t-1} - P^*) \Delta_{t-1} \right] = \Delta_t^1
\]

where the last inequality uses \( P \pi_{t-1} Q^* \leq P^* Q^* \) which results from the fact \( \pi^* \) is the greedy policy with respect to \( Q^* \). Hence, we have proved \( \Delta_t^2 \leq \Delta_t \leq \Delta_t^1 \) holds at iteration \( t \).

#### C.4.2 Proof of Lemma C.3

*Proof of Lemma C.3.* By the definition of (37), we have \( \|A_j^T\|_\infty = \eta_j \sum_{t=j}^T \prod_{i=j+1}^t (1 - \eta_i) \). Plugging the specific form of \( \{\eta_t\} \), we have for \( (S1) \)

\[
\|A_j^T\|_\infty = \eta_j \sum_{t=j}^T \prod_{i=j+1}^t \frac{1 + (1 - \gamma)(i - 1)}{1 + (1 - \gamma)i} = \eta_j \sum_{t=j}^T \frac{1 + (1 - \gamma)i}{1 + (1 - \gamma)t}
\]

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\[
\begin{align*}
\|A_j^T\|_\infty &= \eta_j \sum_{t=j}^{T} \prod_{i=j+1}^{t} (1 - (1 - \gamma)i^{-\alpha}) \leq \eta_j \sum_{t=j}^{T} \exp \left( -(1 - \gamma) \sum_{i=j+1}^{t} i^{-\alpha} \right) \\
&\leq \eta_j \sum_{t=j+1}^{T+1} \exp \left( \frac{1 - \gamma}{1 - \alpha} \left( t^{1-\alpha} - j^{1-\alpha} \right) \right) \\
&\leq \eta_j \int_{j}^{\infty} \exp \left( \frac{1 - \gamma}{1 - \alpha} \left( t^{1-\alpha} - j^{1-\alpha} \right) \right) dt \\
&\leq \frac{\eta_j}{1 - \gamma} \int_{0}^{\infty} \left( \frac{1 - \alpha}{1 - \gamma} y + j^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} \exp (-y) dy \\
&\leq \frac{\eta_j}{1 - \gamma} \max \left\{ 2^{1-\alpha}, 1 \right\} \left[ \left( \frac{1 - \alpha}{1 - \gamma} \right)^{\frac{\alpha}{1-\alpha}} + j^\alpha \right] \exp (-j) \\
&\leq e \max \left\{ 2^{1-\alpha}, 1 \right\} \left[ \frac{\sqrt{2\pi e}}{\sqrt{1 - \alpha (1 - \gamma) \Gamma(\frac{1}{1-\alpha})}} + \frac{1}{1 - \gamma} \right] \\
&\leq e 2^{1-\alpha} \frac{1}{\sqrt{1 - \alpha (1 - \gamma) \Gamma(\frac{1}{1-\alpha})}}
\end{align*}
\]

where (a) uses \( \sum_{i=j}^{t} i^{-\alpha} \geq \frac{1}{1-\alpha} ((t + 1)^{1-\alpha} - j^{1-\alpha}) \) and \( \exp((1 - \gamma)j^{-\alpha}) \leq e \), (b) uses the change of variable \( y = \frac{1 - \gamma}{1 - \alpha} (t^{1-\alpha} - j^{1-\alpha}) \), (c) uses \( (a + b)^p \leq \max\{2^{p-1}, 1\}\alpha^p \) for any \( p > 0 \), and (d) uses \((1 - \alpha) \Gamma(\frac{1}{1-\alpha}) \leq \frac{\sqrt{2\pi e}}{\sqrt{\Gamma(\frac{1}{1-\alpha})}} \) from (20) and \( \max \left\{ 2^{1-\alpha}, 1 \right\} \leq 2^{1-\alpha} \). \hfill \Box

**C.4.3 Proof of Lemma C.4**

**Proof of Lemma C.4.** For (S1), we have

\[
\frac{1}{T} \sum_{j=1}^{T} \|A_j^T - G^{-1}\|_\infty^2 \leq \frac{2}{T} \sum_{j=1}^{T} (\|A_j^T\|_\infty^2 + \|G^{-1}\|_\infty^2)
\]

\[
\leq 2 + \frac{8}{(1 - \gamma)^2} \frac{1}{T} \sum_{j=1}^{T} \ln^2 \left( \frac{1 + (1 - \gamma)T}{1 + (1 - \gamma)(j - 1)} \right)
\]

\[
\leq 2 + \frac{8}{(1 - \gamma)^2} \left[ \frac{\ln^2(1 + (1 - \gamma)T)}{T} + \frac{1}{T} \sum_{j=1}^{T-1} \ln^2 \frac{T}{j} \right]
\]

\[
\leq 2 + \frac{7}{1 - \gamma} + \frac{16}{(1 - \gamma)^2} \leq \frac{25}{(1 - \gamma)^2}
\]

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where \( a \) uses \( \ln^2(1 + x)/x \leq \frac{7}{5} \) for all \( x \geq 0 \) and \( \int_0^1 \ln^2 x \, dx = \Gamma(3) = 2\Gamma(1) = 2 \).

For (S2), based on (37) and \( G = \eta_j^{-1}(I - (I - \eta_j G)) \), we have

\[
A_j^T - G^{-1} = (A_j^T G - I)G^{-1} = \sum_{t=j}^T \left( \prod_{i=j+1}^t (I - \eta_i G) - \prod_{i=j}^{t-1} (I - \eta_i G) \right) G^{-1} - G^{-1}
\]

\[
= \sum_{t=j+1}^T \left( \prod_{i=j+1}^t (I - \eta_i G) - \prod_{i=j}^{t-1} (I - \eta_i G) \right) G^{-1} - \prod_{t=j}^T (I - \eta_t G) G^{-1}
\]

\[
= \sum_{t=j+1}^T (\eta_j - \eta_t) \prod_{i=j+1}^{t-1} (I - \eta_i G) - \prod_{t=j}^T (I - \eta_t G) G^{-1}
\]

\[
:= M_{T,j}^{(1)} + M_{T,j}^{(2)}.
\]

For one thing,

\[
\|M_{T,j}^{(2)}\|_\infty \leq \|G^{-1}\|_\infty \prod_{t=j}^T \|I - \eta_t G\|_\infty \leq \prod_{t=j}^T \frac{(1 - \bar{\eta}_t)'}{1 - \gamma} \leq \frac{(1 - \bar{\eta}_T)'}{1 - \gamma}.
\]

For another thing,

\[
\|M_{T,j}^{(1)}\|_\infty = \left\| \sum_{t=j+1}^T (\eta_t - \eta_j) \prod_{i=j+1}^{t-1} (I - \eta_i G) \right\|_\infty
\]

\[
\leq \sum_{t=j+1}^T |\eta_t - \eta_j| \exp \left( - \sum_{i=j+1}^{t-1} \bar{\eta}_i \right)
\]

\[
\leq \sum_{t=j+1}^T \sum_{k=j}^{t-1} |\eta_{k+1} - \eta_k| \exp \left( - \sum_{i=j+1}^{t-1} \bar{\eta}_i \right)
\]

\[
\leq \sum_{t=j+1}^T \sum_{k=j}^{t-1} \alpha \eta_k \exp \left( - \sum_{i=j+1}^{t-1} \bar{\eta}_i \right)
\]

\[
\leq \frac{e\alpha}{j} \sum_{t=j+1}^T \tilde{m}_{j,t-1} \exp (-\tilde{m}_{j,t-1}) = \frac{e\alpha}{j} \sum_{t=j+1}^{T-1} \tilde{m}_{j,t} \exp (-\tilde{m}_{j,t})
\]

\[
\leq \frac{e\alpha}{j} \left[ \frac{2\Gamma(1 - \alpha)}{(1 - \alpha)^2 (1 - \gamma)^{1 - \alpha}} + \frac{2\Gamma(1 - \alpha)}{1 - \gamma} \right].
\]

where \( a \) uses the fact that for \( \eta_t = t^{-\alpha} \), we have

\[
\frac{\eta_t - \eta_{t+1}}{\eta_t} = 1 - \left( 1 - \frac{1}{t+1} \right)^\alpha \leq 1 - \exp(-\frac{\alpha}{t}) \leq \frac{\alpha}{t}.
\]

where we use \( \ln(1 + x) \geq x/(1 + x) \) in the first inequality and \( \ln(1 + x) \leq x \) in the second inequality.

\( b \) uses the notation \( \tilde{m}_{j,t} := \sum_{i=j}^t \bar{\eta}_i \) and \( \exp(\bar{\eta}_j) \leq \exp(1) = e \). \( c \) uses the following lemma.
Lemma C.10. Let \( \bar{m}_{j,t} := \sum_{i=j}^t \tilde{\eta}_i \) and recall \( \tilde{\eta}_i = (1 - \gamma)i^{-\alpha} \). Then \( T \geq j \geq 1 \), for some constant \( c > 1 \),

\[
\sum_{t=j}^T \bar{m}_{j,t} \exp(-\bar{m}_{j,t}) \leq c \left[ \frac{2^{\frac{1}{1-\alpha}}}{(1-\alpha)^{\frac{3}{2}}(1-\gamma)^{\frac{1}{1-\alpha}}} + \frac{2^{\frac{1}{1-\alpha}}}{1-\gamma}(j-1)^{\alpha} \right].
\]

Therefore,

\[
\frac{1}{T} \sum_{j=1}^T \| A_j^T - G^{-1} \|_\infty^2 \leq \frac{2}{T} \sum_{j=1}^T \left[ \| M_{T,j}^{(1)} \|_\infty^2 + \| M_{T,j}^{(2)} \|_\infty^2 \right]
\leq \frac{2c}{T} \sum_{j=1}^T \left[ \frac{\alpha^2}{j^2} \frac{2^{\frac{2}{1-\alpha}}}{(1-\alpha)^{\frac{3}{2}}(1-\gamma)^{\frac{1}{1-\alpha}}} + \frac{\alpha^2}{(1-\gamma)^2} \frac{2^{\frac{2}{1-\alpha}}}{j^{2(1-\alpha)}} + \frac{(1/\bar{\eta}_T)^{2(T-j+1)}}{(1-\gamma)^2} \right]
\leq \frac{c\alpha^2 2^{\frac{2}{1-\alpha}}}{T} \left[ \frac{1}{(1-\alpha)^{\frac{3}{2}}(1-\gamma)^{\frac{1}{1-\alpha}}} + \frac{1}{(1-\gamma)^2} \sum_{j=1}^T \frac{1}{j^{2(1-\alpha)}} \right] + \frac{1}{(1-\gamma)^2} \frac{1}{T\bar{\eta}_T}
\]

\( \square \)

Proof of Lemma C.10. Clearly we have

\[
\frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - j^{1-\alpha}) \leq \bar{m}_{j,t} = \sum_{i=j}^t \tilde{\eta}_i \leq \frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha})
\]

Then \( \bar{m}_{j,t} \leq \frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha}) \leq \bar{m}_{j-1,t-1} \). Hence,

\[
\sum_{t=j}^T \bar{m}_{j,t} \exp(-\bar{m}_{j,t}) = \sum_{t=j}^T \bar{m}_{j,t} \exp(-\bar{m}_{j-1,t-1} \exp(\bar{\eta}_{j-1} - \tilde{\eta}_k))
\]

\[
= e \sum_{t=j}^T \frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha}) \exp \left( -\frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha}) \right)
\leq 2e \int_{j-1}^\infty \frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha}) \exp \left( -\frac{1-\gamma}{1-\alpha} (t^{1-\alpha} - (j-1)^{1-\alpha}) \right) dt
\]

\[
= \frac{2e}{1-\gamma} \int_0^\infty y \exp(-y) \left( \frac{1-\alpha}{1-\gamma} y + (j-1)^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} dt
\]

\[
\leq \frac{e \max(2^{\frac{1-\alpha}{1-\gamma}}, 2)}{1-\gamma} \int_0^\infty y \exp(-y) \left[ \left( \frac{1-\alpha}{1-\gamma} y \right)^{\frac{\alpha}{1-\alpha}} + (j-1)^{\alpha} \right] dt
\]

\[
\leq \frac{e^{2^{\frac{1-\alpha}{1-\gamma}}}}{1-\gamma} \left[ \left( \frac{1-\alpha}{1-\gamma} \right)^{\frac{\alpha}{1-\alpha}} \Gamma \left( 1 + \frac{1}{1-\alpha} \right) + (j-1)^{\alpha} \right]
\]

\[
\leq \frac{\sqrt{2\pi} e^{2^{\frac{1-\alpha}{1-\gamma}}} \Gamma \left( \frac{1-\alpha}{1-\gamma} \right)}{(1-\alpha)^{\frac{1}{2}} (1-\gamma)^{\frac{1-\alpha}{2}}} + \frac{e^{2^{\frac{1-\alpha}{1-\gamma}} (j-1)^{\alpha}}}{1-\gamma}
\]

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where (a) uses the change of variable $y = \frac{1-\gamma}{1-\alpha} \left( t^{1-\alpha} - (j-1)^{1-\alpha} \right)$, (b) uses $(a+b)^p \leq \max\{2^{p-1}, 1\} (a^p + b^p)$ for any $p > 0$, (c) uses $\max\{2\alpha \gamma, 2\} \leq 2^{\frac{1}{\alpha}}$, (d) uses $\Gamma \left( 1 + \frac{1}{\alpha} \right) = \frac{1}{1-\alpha} \Gamma \left( \frac{1}{1-\alpha} \right)$ and $(1-\alpha)^{1-\alpha} \Gamma \left( \frac{1}{1-\alpha} \right) \leq \frac{2\pi}{\sqrt{1-\alpha}}$ from (20).

\[ \text{C.4.4 Proof of Lemma C.6} \]

Proof of Lemma C.6. Using Lemma C.11 and setting $B_j = \frac{1}{\sqrt{T}} \left( A_j^T - G^{-1} \right) \in \mathbb{R}^{D \times D}$, we complete the proof. The proof of Lemma C.11 is provided at the end of this subsubsection.

Lemma C.11. Recall that $Z_j = (r_j - r) + \gamma (P_j - P)V^*$ is the empirical Bellman noise at iteration $j$. For any sequence of deterministic matrices $\{B_j\} \subseteq \mathbb{R}^{D \times D}$, we have

\[
\mathbb{E} \left\| \sum_{j=1}^{T} B_j Z_j \right\|_{\infty} \leq \sqrt{\frac{2 \ln(6D)}{1-\gamma^2} \sum_{j=1}^{T} \|B_j\|_{\infty}^2}.
\]

Proof of Lemma C.11. By nature of the generator, each coordinate of $Z_j$ is mean-zero and independent. Moreover, Theorem 3.4 shows that each coordinate of $Z_j$ is sub-Gaussian with parameter $(1-\gamma^2)^{-1}$. Then for a given vector $b = (b_1, b_2, \ldots, b_D)^{\top} \in \mathbb{R}^D$, Theorem 3.4 yields that for any $\lambda \in \mathbb{R}$,

\[
\mathbb{E} \exp(\lambda b^{\top} Z_j) = \mathbb{E} \exp \left( \lambda \sum_{i=1}^{D} b_i e_i^{\top} Z_j \right) = \prod_{i=1}^{D} \mathbb{E} \exp \left( \lambda b_i e_i^{\top} Z_j \right)
\]

\[
\leq \prod_{i=1}^{D} \exp \left( \frac{1}{1-\gamma^2} \frac{\lambda^2 b_i^2}{2} \right) = \exp \left( \frac{1}{1-\gamma^2} \frac{\lambda^2 \|b\|^2}{2} \right).
\]

Hence, for sequence of vector $\{b_j\} \subseteq \mathbb{R}^D$, since each $\{Z_j\}$ are mutually independent, we have

\[
\mathbb{E} \exp \left( \frac{1}{\sum_{j=1}^{T} b_j^{\top} Z_j} \right) = \prod_{j=1}^{T} \mathbb{E} \exp \left( \lambda b_j^{\top} Z_j \right) \leq \exp \left( \frac{1}{1-\gamma^2} \frac{\lambda^2}{2} \sum_{j=1}^{T} \|b_j\|^2 \right).
\]

Then for any sequence of deterministic matrices $\{B_j\} \subseteq \mathbb{R}^{D \times D}$, we can show $\| \sum_{j=1}^{T} B_j Z_j \|_{\infty}$ is also sub-Gaussian. Let $B_j(i, \cdot)$ denote the $i$-th row of the matrix $B_j$. Then we already have that for any $\lambda \in \mathbb{R}$,

\[
\mathbb{E} \exp \left( \frac{1}{\sum_{j=1}^{T} B_j(i, \cdot) Z_j} \right) \leq \exp \left( \frac{1}{1-\gamma^2} \frac{\lambda^2}{2} \sum_{j=1}^{T} \|B_j(i, \cdot)\|^2 \right).
\]

Note that $\| \sum_{j=1}^{T} B_j Z_j \|_{\infty} = \max_{i \in D} \{ \pm \sum_{j=1}^{T} B_j(i, \cdot) Z_j \}$. We have

\[
\mathbb{E} \exp \left( \lambda \| \sum_{j=1}^{T} B_j Z_j \|_{\infty} \right) \leq \sum_{i \in D} \left[ \mathbb{E} \exp \left( \frac{1}{\sum_{j=1}^{T} B_j(i, \cdot) Z_j} \right) + \mathbb{E} \exp \left( -\lambda \sum_{j=1}^{T} B_j(i, \cdot) Z_j \right) \right]
\]
Hence, $E$ here the last inequality uses (34). To bound $U_j$ iteration $C.4.5$ Proof of Lemma C.7.

By a similar argument in the proof of Lemma B.2, we have

$$
\mathbb{E} \left\| \sum_{j=1}^{T} B_j Z_j \right\|_\infty^2 \leq \frac{2 \ln(6D)}{1 - \gamma^2} \max_{i \in D} \sum_{j=1}^{T} \left\| B_j(i, \cdot) \right\|^2
$$

Finally using the inequality $\mathbb{E} \left\| \sum_{j=1}^{T} B_j Z_j \right\|_\infty \leq \sqrt{\mathbb{E} \left\| \sum_{j=1}^{T} B_j Z_j \right\|_\infty^2}$ completes the proof.

C.4.5 Proof of Lemma C.7

Proof of Lemma C.7. Recall that $\mathcal{F}_j$ is the $\sigma$-field generated by all randomness before (and including) iteration $j$. We will apply Lemma F.1 to prove our lemma. Using the notation defined therein, we set $X_j = \frac{1}{\sqrt{T}}(P_j - P)(V_{j-1} - V^*)$ and $B_j = A_j^T$. Clearly, $\{X_j\}$ is a martingale difference sequence since $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = \frac{1}{\sqrt{T}}\mathbb{E}[P_j - P|\mathcal{F}_{j-1}](V_{j-1} - V^*) = 0$. As a result, $X = \frac{4\gamma}{\sqrt{T}(1 - \gamma)}, B = C_0, D = |S \times A|$ and $U_j = \text{Var}[X_j|\mathcal{F}_{j-1}]$.

Recall that $W_T = \text{diag}(\sum_{j=1}^{T} B_j U_j B_j^T)$. To upper bound $\mathbb{E}\|W_T\|_\infty$, we aim to find a upper bound for $\|W_T\|_\infty$. We first note that

$$
\|W_T\|_\infty = \left\| \text{diag} \left( \sum_{j=1}^{T} B_j U_j B_j^T \right) \right\|_\infty \leq \sum_{j=1}^{T} \left\| \text{diag} \left( B_j U_j B_j^T \right) \right\|_\infty \leq \sum_{j=1}^{T} \|B_j\|_\infty \|U_j\|_{\text{max}}
$$

Here the last inequality uses (34). To bound $\|U_j\|_{\text{max}}$, we find that for any $i \neq k$, $U_j(i, k) = \mathbb{E}[e_i^T X_j X_j^T e_k|\mathcal{F}_{j-1}] = 0$ due to each coordinate of $X_j$ are independent conditioning on $\mathcal{F}_{j-1}$. Hence,

$$
\|U_j\|_{\text{max}} = \max_{i, k} |U_j(i, k)| = \max_{i} |U_j(i, i)| = \|\mathbb{E}[\text{diag}(X_j X_j^T)|\mathcal{F}_{j-1}]\|_\infty
$$

$$
\leq \mathbb{E} \left\| \text{diag}(X_j X_j^T) \right\|_\infty |\mathcal{F}_{j-1}| \leq \mathbb{E}[\|X_j\|^2_\infty |\mathcal{F}_{j-1}] = \frac{\gamma^2}{T} \mathbb{E}[\|(P_j - P)(V_{j-1} - V^*)\|^2_\infty |\mathcal{F}_{j-1}] \leq \frac{\gamma^2}{T} \|V_{j-1} - V^*\|^2_\infty \mathbb{E}[\|P_j - P\|^2_\infty] \leq \frac{4\gamma^2}{T} \|V_{j-1} - V^*\|^2_\infty
$$

To distinguish $\text{Var}[X_j|\mathcal{F}_{j-1}]$ and the value function $V_j$, we use $U_j$ to denote the conditional variance.
where (a) again uses (34) and (b) uses \( \|P_j - P\|_\infty \leq \|P_j\|_\infty + \|P\|_\infty = 2 \).

Putting all the pieces together, we have

\[
\|W_T\|_\infty \leq \frac{4\gamma^2}{T} \sum_{j=1}^T \|B_j\|_\infty^2 \|V_{j-1} - V^*\|_\infty^2 \leq \frac{4\gamma^2 C_0^2}{T} \sum_{j=1}^T \|V_{j-1} - V^*\|_\infty^2
\]

where we use \( \sup_j \|B_j\|_\infty \leq B = \frac{C_0}{1 - \gamma} \). The rest follows from (60) in Lemma F.1 by plugging the corresponding \( B, X, D \) and \( \sigma^2 \) and the inequality \( \|V_{j-1} - V^*\|_\infty \leq \|Q_{j-1} - Q^*\|_\infty = \|\Delta_{j-1}\|_\infty \). □

C.5 Results for General Step Sizes

In this section, we instead use a more general rescaled step size \( \{\tilde{\eta}_t\} \) given in Assumption 3.2 which satisfies the following conditions

(C1) \( 0 \leq \sup_t \eta_t \leq 1, \eta_t \downarrow 0 \) and \( t\eta_t \uparrow \infty \) when \( t \to \infty \);

(C2) \( \frac{\eta_{t+1} - \eta_t}{\eta_t} = o(\eta_{t-1}) \) and \( (1 - \tilde{\eta}_t)\tilde{\eta}_{t-1} \leq \tilde{\eta}_t \) for all \( t \geq 1 \);

(C3) \( \frac{1}{\sqrt{T}} \sum_{t=0}^{T} \tilde{\eta}_t \to 0 \) when \( T \to \infty \);

to prove the same asymptotic normality given in Theorem 3.1. Based on our previous argument in Appendix C.2 and Appendix C.3, we only need the corresponding \( T_i(i = 0, 2, 3, 4) \) are all \( o_P(1) \).

First, prior work of Polyak and Juditsky [1992] shows that when the step size \( \eta_t \) satisfying (C1) and (C2), we have

- Uniform boundedness: \( \|A_j^T\|_2 < C_0 \) uniformly for all \( T \geq j \geq 1 \) for some constant \( C_0 > 0 \);

- Uniform approximation: \( \lim_{T \to \infty} \frac{1}{T} \sum_{j=1}^{T} \|A_j^T - G^{-1}\|_2 = 0 \).

By the equivalence between \( \|\cdot\|_2 \) and \( \|\cdot\|_\infty \), we already have a counterpart Lemma C.3 and C.4 for the general step size (which is likely to have a poor non-asymptotic convergence rate though an asymptotic convergence is guaranteed).

Such a uniform boundedness and uniform approximation together with Lemma C.6 yield that \( \mathbb{E}\|T_0\|_\infty \to 0 \) and \( \mathbb{E}\|T_2\|_\infty \to 0 \). From Lemma C.7 and C.9, to prove \( \mathbb{E}\|T_3\|_\infty \to 0 \) and \( \mathbb{E}\|T_4\|_\infty \to 0 \), we only need to prove

\[
\frac{1}{\sqrt{T}} \sum_{t=0}^{T} \mathbb{E}\|\Delta_t\|_\infty^2 \to 0.
\]

(45)

In the following, we are going to show the general step size ensures the establishment of (45). With (C1), Lemma B.4 is valid and yields that

\[
\mathbb{E}\|\Delta T\|_\infty^2 \leq 3 \prod_{t=1}^{T} (1 - \tilde{\eta}_t)^2 \|\Delta_0\|_\infty^2 + \frac{6 \gamma^2 \ln(6D)}{(1 - \gamma)^3} \sum_{t=1}^{T} \tilde{\eta}(t, T) \hat{\eta}_{t-1} + \frac{6 \ln(6D)}{1 - \gamma} \eta_T
\]

where we define for simplicity

\[
\tilde{\eta}(t, T) = \begin{cases} \prod_{j=1}^{T} (1 - \tilde{\eta}_j), & \text{if } t = 0 \\ \tilde{\eta} \prod_{j=t+1}^{T} (1 - \tilde{\eta}_j), & \text{if } 0 < t < T \\ \tilde{\eta}_T, & \text{if } t = T. \end{cases} \tag{46}
\]
Since $tn_t \uparrow \infty$, then we must have $\sum_{t=1}^{T} \tilde{\eta}_t - \frac{1}{4} \ln T \to +\infty$, which together with the Stolz–Cesaro theorem implies
\[
\sqrt{T} \prod_{t=1}^{T} (1 - \tilde{\eta}_t)^2 \leq \exp \left( \frac{1}{2} \ln T - 2 \sum_{t=1}^{T} \tilde{\eta}_t \right) \to 0 \quad \implies \quad \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \prod_{j=1}^{t} (1 - \tilde{\eta}_j)^2 \to 0.
\]

**Lemma C.12.** For any $T \geq t \geq 1$, $\sum_{t=1}^{T} \tilde{\eta}_{(t,l)} \leq c$. Here $\{\tilde{\eta}_{(t,l)}\}_{t \geq t}$ is defined in (46) and $\{\tilde{\eta}_t\}$ satisfies Assumption 3.3.

By (C3) and Lemma C.12, it follows that
\[
\frac{1}{\sqrt{T}} \sum_{l=1}^{T} \sum_{t=1}^{l} \tilde{\eta}_{(t,l)} \eta_{t-1} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \eta_{t-1} \cdot \sum_{t=1}^{T} \tilde{\eta}_{(t,l)} \leq \frac{c}{\sqrt{T}} \sum_{t=1}^{T} \eta_{t-1} \to 0.
\]

Putting all pieces together, we have established (45).

**Proof of Lemma C.12.** We define $\tilde{m}_{t,l} := \sum_{i=1}^{l} \tilde{\eta}_i$. Due to $t\tilde{\eta} \uparrow \infty$, we have $t\tilde{\eta}_i \leq \tilde{\eta}_i$ for all $i \geq t$ and thus
\[
\tilde{m}_{t,l} := \sum_{i=t}^{l} \tilde{\eta}_i \geq \tilde{\eta}_t \sum_{i=t}^{l} \frac{1}{i} \geq t\tilde{\eta}_l \left( \ln \frac{l}{t} - \frac{1}{2t} \right) = \frac{\tilde{\eta}_l}{2} + t\tilde{\eta}_l \ln \frac{l}{t}.
\]

Since $t\tilde{\eta} \uparrow \infty$, there exists some $t_0 > 0$ such that any $t \geq t_0$, we have $t\tilde{\eta}_t \geq 2$. Therefore, we have for all $l \geq t \geq t_0$,
\[
\frac{1}{\eta_l} \leq \frac{l}{t\tilde{\eta}_l} \leq \frac{1}{\eta_l} \exp \left( \frac{\tilde{m}_{t,l} + \frac{\eta_l}{2}}{t\tilde{\eta}_l} \right) \leq \sqrt{e} \exp \left( \frac{\tilde{m}_{t,l}}{2} \right).
\]

In the following, we will discuss three cases. If $T \geq t \geq t_0$, by definition, it follows that
\[
\sum_{t=1}^{T} \tilde{\eta}_{(t,l)} = \sum_{t=1}^{T} \tilde{\eta}_t \prod_{j=1}^{l} (1 - \tilde{\eta}_j) \leq \frac{\tilde{\eta}_t}{1 - \tilde{\eta}_t} \sum_{t=1}^{T} \exp \left( -\tilde{m}_{t,l} \right)
\]
\[
\leq \frac{\tilde{\eta}_t}{1 - \tilde{\eta}_t} \sum_{t=1}^{T} \tilde{\eta}_t \cdot \sqrt{e} \exp \left( -\frac{\tilde{m}_{t,l}}{2} \right)
\]
\[
\leq \sqrt{e} \sum_{t=1}^{T} \tilde{\eta}_t \exp \left( -\frac{\tilde{m}_{t,l}}{2} \right) = \frac{2\sqrt{e}}{\gamma}
\]

where (a) uses $1 - \tilde{\eta}_t \geq 1 - \tilde{\eta}_0 = \gamma$; and (b) uses $\sum_{t=1}^{T} \tilde{\eta}_t \exp \left( -\frac{\tilde{m}_{t,l}}{2} \right) \leq \int_{0}^{\infty} \exp(-x/2)dx = 2$ due to $\tilde{m}_{t,l} \uparrow \infty$ as $l \to \infty$.

If $T \geq t_0 \geq t$, by definition, $\tilde{\eta}_{(t,l)} = \tilde{\eta}_{(t,l_0)} / \tilde{\eta}_{l_0}$, then we have $\sum_{t=1}^{T} \tilde{\eta}_{(t,l)} \leq \sum_{t=1}^{l_0} \tilde{\eta}_{(t,l)} \leq \sum_{t=1}^{T} \tilde{\eta}_{(t,l_0)} / \tilde{\eta}_{l_0} \leq \sum_{t=1}^{T} \tilde{\eta}_{(t,l)} / \tilde{\eta}_{l_0} \leq t_0 + \frac{2\sqrt{e}}{\gamma\tilde{\eta}_0}$.

If $t_0 \geq T \geq t$, we have $\sum_{t=1}^{T} \tilde{\eta}_{(t,l)} \leq t_0$.

Putting the three cases together, we can set $c = t_0 + 2\sqrt{e} / (\gamma\tilde{\eta}_0)$ which ensures that $\sum_{t=1}^{T} \tilde{\eta}_{(t,l)} \leq c$ for any $T \geq t \geq 1$. \qed

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C.6 Proof of Corollary 3.1

Proof of Corollary 3.1. We first prove
\[ \text{Var}_V = \Pi^\pi \text{Var}_Q (\Pi^\pi)^\top. \] (47)
Recall the definition
\[ \text{Var}_Q = (I - \gamma P^\pi)^{-1}\text{Var}(Z)(I - \gamma P^\pi)^{-\top} \in \mathbb{R}^{D \times D} \]
\[ \text{Var}_V = (I - \gamma P^\pi)^{-1}\text{Var}(\Pi^\pi Z)(I - \gamma P^\pi)^{-\top} \in \mathbb{R}^{S \times S}. \]
For one thing, we have \( \text{Var}(\Pi^\pi Z) = \Pi^\pi \text{Var}(Z)(\Pi^\pi)^\top. \) For another thing, we have \( \Pi^\pi (I - \gamma P^\pi)^{-1} = (I - \gamma P^\pi)^{-1} \Pi^\pi. \) This is because \( (I - \gamma P^\pi) \Pi^\pi = \Pi^\pi - \gamma \Pi^\pi P \Pi^\pi = \Pi^\pi (I - \gamma P^\pi). \)
Putting these together, (47) follows from direct verification.

We then prove the asymptotic normality of \( \hat{V}_T. \) Let \( \bar{\pi}_t \) is the greedy policy with respect to \( \bar{Q}_t, \) i.e., \( \bar{\pi}_t(s) \in \arg\max_{a \in \mathcal{A}} \bar{Q}_t(s,a). \) From the definition of our estimator,
\[ \hat{V}_T = \Pi^\pi_t \bar{Q}_T \text{ and } V^* = \Pi^\pi Q^* \]
which implies
\[ \hat{V}_T - V^* = \left( \Pi^\pi_t \bar{Q}_T - \Pi^\pi_t Q_T \right) + \left( \Pi^\pi_t Q_T - \Pi^\pi Q^* \right). \]
On the other hand, it is easy to see that
\[ \sqrt{T} \left( \Pi^\pi Q_T - \Pi^\pi Q^* \right) \xrightarrow{d} N(0, \Pi^\pi \text{Var}(\Pi^\pi)^\top) = N(0, \text{Var}_V). \]
If we can prove
\[ \sqrt{T} \left( \Pi^\pi \bar{Q}_T - \Pi^\pi Q_T \right) = o_P(1), \] (48)
then the conclusion follows from the Slutsky’s theorem. We have that
\[ \sqrt{T} \text{E}[\Pi^\pi Q_T - \Pi^\pi Q_T] \leq \sqrt{T} \text{E}[\|\Pi^\pi - \Pi^\pi\|_\infty \|Q_T\|_\infty] \]
\[ \leq (a) \frac{\sqrt{T}}{1 - \gamma} \text{E}[\|\Pi^\pi - \Pi^\pi\|_\infty] \]
\[ \leq (b) \frac{2\sqrt{T}}{1 - \gamma} \text{P}(\bar{\pi}_T \neq \pi^*) \]
\[ \leq (c) \frac{2\sqrt{T}}{1 - \gamma} \text{P}\left(\|Q_T - Q^*\|_\infty \geq \frac{\text{gap}}{2}\right) \]
\[ \leq \frac{2\sqrt{T}}{1 - \gamma} \frac{4}{\text{gap}^2} \text{E}\|Q_T - Q^*\|_\infty^2 \]
\[ \leq (d) \frac{1}{1 - \gamma} \frac{8}{\text{gap}^2} \frac{1}{\sqrt{T}} \sum_{t=1}^T \text{E}\|Q_t - Q^*\|_\infty^2 \]
where (a) uses \( \|Q_T\|_\infty \leq (1 - \gamma)^{-1}, \) (b) uses the fact that both \( \bar{\pi}_T \) and \( \pi^* \) are deterministic policies and thus \( \|\Pi^\pi - \Pi^\pi\|_\infty = 2 \cdot 1_{\{\bar{\pi}_T \neq \pi^*\}} \), (c) uses the fact \( \{\bar{\pi}_t \neq \pi^*\} \subseteq \{Q_T - Q^*\|_\infty \geq \text{gap}/2\} \) which we derived in the proof of Lemma C.8, and finally (d) follows from the Jensen’s inequality.

From (45), we know \( \frac{1}{\sqrt{T}} \sum_{t=1}^T \text{E}\|Q_t - Q^*\|_\infty^2 \to 0 \) as \( T \to \infty. \) Therefore, we have that
\[ \sqrt{T} \text{E}[\Pi^\pi \bar{Q}_T - \Pi^\pi Q_T] = o(1) \] which implies (48) is true. \( \square \)
D  Proof of Theorem 3.5

Proof of Theorem 3.5. From Lemma C.1 and (30), we have $\sqrt{T} \Delta_f^2 \leq \sqrt{T} \Delta_T \leq \sqrt{T} \Delta_f^1$. For one thing, from (38), we have $\sqrt{T} \Delta_f^1 \leq \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4$ with $\{ \mathcal{T}_i \}_{i=0}^4$, defined therein. For another thing, we have $\sqrt{T} \Delta_f^2 \leq \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ from (41). These results implies

$$\mathbb{E}\|\bar{\Delta}_T\|_\infty \leq \mathbb{E}\max\{\|\bar{\Delta}_f^1\|_\infty, \|\bar{\Delta}_f^2\|_\infty\} \leq \frac{1}{\sqrt{T}} \sum_{i=0}^4 \mathbb{E}\|\mathcal{T}_i\|_\infty.$$

Bounds for $\mathbb{E}\|\mathcal{T}_i\|_\infty (i = 0, 2, 3, 4)$ have been provided in Appendix C.2.2.

For $\mathbb{E}\|\mathcal{T}_1\|_\infty$, the direct application of Lemma C.11 yields that

$$\mathbb{E}\|\mathcal{T}_1\|_\infty \leq \sqrt{\frac{2 \ln(6D)}{1 - \gamma^2}} \|G^{-1}\|_\infty \leq \frac{\sqrt{2 \ln(6D)}}{(1 - \gamma)^{\frac{3}{2}}}.$$

However, if we want to capture the dependence of $\mathbb{E}\|\mathcal{T}_1\|_\infty$ on $\|\text{diag}(\text{Var}_Q)\|_\infty$, we can use (59) in Lemma F.1,\(^{10}\) which yields

$$\mathbb{E}\|\mathcal{T}_1\|_\infty \leq 6 \sqrt{\ln(2D)} \sqrt{\|\text{diag}(\text{Var}_Q)\|_\infty} + \frac{4 \ln(6D)}{3 \sqrt{T}(1 - \gamma)^{\frac{3}{2}}}.$$

Clearly, when $T$ is sufficiently large, (50) is better than (49) due to $\|\text{diag}(\text{Var}_Q)\|_\infty \leq \frac{1}{4(1 - \gamma)^{\frac{3}{2}}}$ from Corollary 3.2. Hence, we will prefer (50) though it incurs an additional error term.

We then put the pieces together and have

$$\mathbb{E}\|\bar{\Delta}_T\|_\infty \leq \frac{2C_0}{\eta_0 T} + 6 \sqrt{\ln(2D)} \sqrt{\frac{\|\text{diag}(\text{Var}_Q)\|_\infty}{T}} + \frac{4 \ln(6D)}{3T(1 - \gamma)^{\frac{3}{2}}} + \frac{\sqrt{2 \ln(6D)}}{(1 - \gamma)^{\frac{3}{2}}} \frac{1}{\sqrt{T}} + \frac{2 \ln(6D)}{1 - \gamma^2} \frac{1}{T} \sum_{j=1}^{T} \|A_j^T - G^{-1}\|_\infty^2 + \frac{4 \gamma C_0 \sqrt{\ln(2DT^2)}}{T} \cdot \sqrt{\frac{1}{T} \sum_{j=1}^{T} \mathbb{E}\|\Delta_{j-1}\|_\infty^2} + \frac{8 \gamma C_0 \ln(3DT^2)}{3T(1 - \gamma)}$$

$$+ \gamma LC_0 \cdot \frac{1}{T} \sum_{j=1}^{T} \mathbb{E}\|\Delta_{j-1}\|_\infty^2.$$

(I) Linearly rescaled step size  If we use a linear rescaled step size, i.e., $\eta_t = \frac{1}{1 + (1 - \gamma)T}$ (equivalently $\bar{\eta}_t = \frac{1 - \gamma}{1 + (1 - \gamma)T}$), then Lemma C.3 and Lemma C.4 give

$$C_0 = \frac{2}{1 - \gamma} \ln(1 + (1 - \gamma)T) \text{ and } \frac{1}{T} \sum_{j=1}^{T} \|A_j^T - G^{-1}\|_\infty^2 \leq \frac{25}{(1 - \gamma)^2}.$$

\(^{10}\)We set $B_j \equiv G^{-1}$ and $X_j = Z_j$ which implies $B = X = \frac{1}{1 - \gamma}$ and $\|W_T\|_\infty \leq T \|\text{diag}(\text{Var}_Q)\|_\infty$.}

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Hiding constant factors in $c$, Corollary B.1 gives

$$\frac{1}{T} \sum_{t=0}^{T} \mathbb{E} \| \Delta_t \|_\infty^2 \leq c \left[ \frac{\| \Delta_0 \|_\infty^2}{(1-\gamma)^2 T} + \frac{\ln(6D)}{(1-\gamma)^4} \frac{\ln^2(3T)}{T} \right].$$

Hence, it follows that

$$\mathbb{E} \| \Delta_T \|_\infty = O \left( \frac{\ln T}{(1-\gamma)T} + \sqrt{\| \text{diag}(\text{Var}Q) \|_\infty} \sqrt{\frac{\ln D}{T}} + \frac{\ln(D)}{(1-\gamma)^2 T} + \frac{\gamma \sqrt{\ln(DT)}}{(1-\gamma)^3} \left( \frac{\ln T}{T} + \frac{\sqrt{\ln D}}{T} \right) \right)$$

$$\quad + \frac{\gamma}{(1-\gamma)^2} \frac{\ln T \ln DT}{T} + \frac{\gamma L \ln T}{1-\gamma} \left( \frac{1}{(1-\gamma)^4 T} + \frac{\ln(D)}{(1-\gamma)^4} \frac{\ln^2(T)}{T} \right)$$

$$= O \left( \sqrt{\| \text{diag}(\text{Var}Q) \|_\infty} \sqrt{\frac{\ln D}{T}} + \tilde{O} \left( \frac{\gamma}{(1-\gamma)^5} \frac{1}{T} \right) \right)$$

where $\tilde{O}(\cdot)$ hides polynomial dependence on logarithmic terms namely $\ln D$ and $\ln T$.

**II) Polynomial step size** If we choose a polynomial step size, i.e., $\eta_t = t^{-\alpha}$ with $\alpha \in (0.5, 1)$ for $t \geq 1$ and $\eta_0 = 1$, then hiding constant factors in $c$, Lemma C.3 and Lemma C.4 give

$$C_0 = O \left( \frac{1}{(1-\gamma)^{1+\frac{3}{1-\alpha}}} \right)$$

$$\sqrt{\frac{1}{T} \sum_{j=1}^{T} \| A_j^T - G^{-1} \|_\infty^2} = O \left( \frac{1}{(1-\gamma)^{\frac{1}{1-\alpha}}} \frac{1}{\sqrt{T}} + \frac{1}{(1-\gamma)^{\frac{2}{1-\alpha}}} \frac{1}{T^{\frac{\alpha}{2}}} \right).$$

where $O(\cdot)$ hides constant factors on $\alpha$. Corollary B.1 gives

$$\frac{1}{T} \sum_{t=0}^{T} \mathbb{E} \| \Delta_t \|_\infty^2 \leq O \left( \frac{\ln D}{(1-\gamma)^{2+\frac{1}{1-\alpha}}} \frac{1}{T} + \frac{\ln(D)}{(1-\gamma)^{3} T^\alpha} \right).$$

Hence, it follows that

$$\mathbb{E} \| \Delta_T \|_\infty = O \left( \frac{1}{(1-\gamma)^{1+\frac{3}{1-\alpha}} T} + \sqrt{\| \text{diag}(\text{Var}Q) \|_\infty} \sqrt{\frac{\ln D}{T}} + \frac{\ln(D)}{(1-\gamma)^2 T} \right)$$

$$\quad + \sqrt{\frac{\ln D}{(1-\gamma)T}} \left( \frac{1}{(1-\gamma)^{1+\frac{1}{1-\alpha}}} \frac{1}{\sqrt{T}} + \frac{1}{(1-\gamma)^{\frac{2}{1-\alpha}}} \frac{1}{T^{\frac{\alpha}{2}}} \right)$$

$$\quad + \frac{\gamma}{(1-\gamma)^{1+\frac{3}{1-\alpha}}} \frac{\ln DT}{T} + \frac{\gamma L}{1-\gamma} \left( \frac{\ln D}{(1-\gamma)^{2+\frac{1}{1-\alpha}}} \frac{1}{T} + \frac{\ln(D)}{(1-\gamma)^{3} T^\alpha} \right)$$

$$= O \left( \frac{\sqrt{\ln D}}{(1-\gamma)^{\frac{3}{2}} \sqrt{T}} \right) + O \left( \frac{\sqrt{\ln D}}{(1-\gamma)^2 T^{1-\frac{\alpha}{2}}} \right) + \tilde{O} \left( \frac{1}{(1-\gamma)^{2+\frac{2}{1-\alpha}}} \frac{1}{T} + \frac{\gamma}{(1-\gamma)^{3+\frac{1}{1-\alpha}}} \frac{1}{T^\alpha} \right)$$

where $\tilde{O}(\cdot)$ hides polynomial dependence on logarithmic terms namely $\ln D$ and $\ln T$. $\square$
E Proof for Information Theoretical Lower Bound

E.1 Proof of Theorem 3.2

The semiparametric model \( \mathcal{P} = \mathcal{P}_P \times \mathcal{P}_R \) described in Section 3.2 is described through an infinite-dimensional parameter \( \theta = (\mathcal{P}, R) \), which is partitioned into a finite-dimensional parameter \( \mathcal{P} \in \mathbb{R}^{D \times S} \) and an infinite-dimensional parameter \( R \). The reason why \( R \) is infinite-dimensional is because we don’t specify the probability model of each \( R(s, a) \), which is equivalent to consider the class of all p.d.f.’s on the interval \([0, 1]\) and thus has infinite dimensions. The parameter of interest is a smooth function of \( \theta \), denoted by \( \beta(\theta) = Q^* \in \mathbb{R}^D \). To compute the semiparametric Cramer-Rao lower bound (see Definition 4.7 of [Tsiatis, 2006]), we should figure out

\[
\sup_{\mathcal{P}_\gamma \subset \mathcal{P}} \Gamma(\gamma_0) I(\gamma_0)^{-1} \Gamma^\top(\gamma_0)
\]

where \( \mathcal{P}_\gamma \) is any parametric submodel containing the truth, i.e., \( \mathcal{P}_{\gamma_0} = \mathcal{P} \). Here \( \Gamma(\gamma_0) = \frac{\partial Q^*}{\partial \gamma}|_{\gamma = \gamma_0} \) and \( I(\gamma_0) \) is the Fisher information matrix. Due to the independence between \( \mathcal{P} \) and \( R \), (51) can be divided into two parts

\[
\Gamma(\mathcal{P}) I(\mathcal{P})^{-1} \Gamma^\top(\mathcal{P}) + \sup_{\mathcal{P}_\gamma(R) \subset \mathcal{P}_R} \Gamma(\gamma_0(R)) I(\gamma_0(R))^{-1} \Gamma^\top(\gamma_0(R))
\]

where \( \Gamma(\mathcal{P}) = \frac{\partial Q^*}{\partial \mathcal{P}} \) and \( I(\mathcal{P}) \) is the (constrained) information matrix. Here \( \mathcal{P}_\gamma(R) \) denotes the parametric submodel depending only on \( R \). \( \gamma_0(R) \) is the finite-dimensional part of \( \gamma_0 \) that relates with \( R \).

In the following, we will first handle the parametric part (i.e., the transition kernel \( P \)) by computing the (constrained) information matrix and then cope with the nonparametric part (i.e., the random reward \( R \)) by using semiparametric tools. Combining the two parts together, we find that the semiparametric efficiency bound is

\[
\frac{1}{T} \cdot (I - \gamma P^*)^{-1} \text{Var}(\gamma P_j V^*) (I - \gamma P^*)^{-\top} + \frac{1}{T} \cdot (I - \gamma P^*)^{-1} \text{Var}(r_j) (I - \gamma P^*)^{-\top}
\]

\[
= \frac{1}{T} \cdot (I - \gamma P^*)^{-1} \text{Var}(Z_j) (I - \gamma P^*)^{-\top}
\]

using the notation \( Z_j = r_j + \gamma P_j V^* \) and the independence of \( r_j \) and \( P_j \).

E.1.1 Parametric Part

We first investigate the Cramer-Rao lower bound of estimating \( Q^* \) using samples from \( \{P_t\}_{t \in [T]} \) whose distribution is determined by \( P \in \mathcal{P} \) with \( \mathcal{P} \) defined in (13). Note that \( P \in \mathcal{P} \) is linearly constrained, i.e.,

\[
h(P) = 0
\]

where \( h : \mathbb{R}^{D \times S} \rightarrow \mathbb{R}^D \) with its \((\bar{s}, \bar{a})\)-th coordinate of \( h \) given by

\[
h_{\bar{s}, \bar{a}}(P) = \sum_{s, a, s'} P(s'|s, a) 1\{(s, a) = (\bar{s}, \bar{a})\} - 1.
\]
Hence, we encounter the Cramer-Rao lower bound for constrained parameters. Let \( C_\pi(P) \) is the inverse fisher information matrix using \( T \) i.i.d. samples under the constraint \( h(P) = 0 \). Hence, \( C_\pi(P) = \frac{C_\pi(P)}{T} \) and the constrained Cramer-Rao lower bound [Moore Jr, 2010] is

\[
\Gamma(P) I(P)^{-1} \Gamma^T(P) = \left( \frac{\partial Q^*}{\partial P} \right)^T C_\pi(P) \frac{\partial Q^*}{\partial P} = \frac{1}{T} \cdot \left( \frac{\partial Q^*}{\partial P} \right)^T C_1(P) \frac{\partial Q^*}{\partial P} \tag{53}
\]

where \( \frac{\partial Q^*}{\partial P} \) is the partial derivatives computed ignoring the linear constraint \( h(P) = 0 \).

To give a precise formulation of the bound (53), we first compute \( \frac{\partial Q^*}{\partial P} \).

**Lemma E.1.** Under Assumption 3.2, \( Q^* \) is differentiable w.r.t. \( P \) with the partial derivatives given by

\[
\frac{\partial Q^*(s,a)}{\partial P(s'|\bar{s},\bar{a})} = \gamma V^*(s') \cdot (I - \gamma P^*)^{-1}((s,a), (\bar{s}, \bar{a})).
\]

We then compute \( C_1(P) \) via the following lemma.

**Lemma E.2.** The \((s,a)\)-th row of the random matrix \( P_t \) is given by \( P_t(s'|s,a) = 1_{s_t(s,a) = s'} \) where \( s_t(s,a) \) is the generated next-state from \((s,a)\) at iteration \( t \) with probability given as the \((s,a)\)-th row of \( P \). Hence \( P = EP_t \) and \( P \) belongs to the following parametric space

\[
P = \{ P \in \mathbb{R}^{D \times S} : P(s'|s,a) \geq 0 \text{ for all } (s,a,s') \text{ and } h(P) = 0 \},
\]

with \( h \) defined in (52). The constrained inverse Fisher information matrix \( C_1(P) \) is

\[
C_1(P) = \text{diag} \left( \left\{ \text{diag}(P(s'|s,a)) - P(s'|s,a)P(s'|s,a)^\top \right\}_{(s,a)} \right)
\]

By Lemma E.1 and E.2, we have

\[
\left( \frac{\partial Q^*}{\partial P} \right)^T C_1(P) \frac{\partial Q^*}{\partial P} = \sum_{(s,\bar{a})} \gamma^2 G^{-1}((s,a), (\bar{s}, \bar{a}))G^{-1}((\bar{s}, \bar{a}), (\bar{s}, \bar{a})) \cdot \left( \sum_{s'} V^*(s')^2 P(s'|\bar{s}, \bar{a}) - \sum_{s'} V^*(s') P(s'|\bar{s}, \bar{a})^2 \right).
\]

By verification, the Cramer-Rao lower bound is equal to

\[
\left( \frac{\partial Q^*}{\partial P} \right)^T C_\pi(P) \frac{\partial Q^*}{\partial P} = T \cdot (I - \gamma P^*)^{-1} \text{Var}(\gamma P_j V^*) (I - \gamma P^*)^{-\top}.
\]

At the end of this part, we provide the deferred proof for Lemma E.1 and E.2.

**Proof of Lemma E.1.** Notice that \( Q^* = R + \gamma PV^* \). Then by the chain rule, we have

\[
\frac{\partial Q^*(s,a)}{\partial P(s'|s,a)} = \gamma V^*(s') + \gamma \sum_{s_1} P(s_1|s,a) \frac{\partial V^*(s_1)}{\partial P(s'|s,a)}.
\]
\[
\frac{\partial Q^*(s,a)}{\partial P(s'|\tilde{s},\tilde{a})} = \gamma \sum_{s_1} P(s_1|\tilde{s},\tilde{a}) \frac{\partial V^*(s_1)}{\partial P(s'|\tilde{s},\tilde{a})} \text{ for any } (s,a) \neq (\tilde{s},\tilde{a}).
\]

Assumption 3.2 implies the optimal policy \(\pi^*\) is unique. Hence, using \(V^*(s_1) = \max_a Q^*(s_1,a) = Q^*(s_1,\pi^*(s_1))\), we have

\[
\frac{\partial V^*(s_1)}{\partial P(s'|s,a)} = \frac{\partial Q^*(s_1,\pi^*(s_1))}{\partial P(s'|s,a)}.
\]

Notice that \(P^*((s,a),(\tilde{s},\tilde{a})) = P(\tilde{s}|s,a)1_{\{\tilde{a} = \pi^*(\tilde{s})\}}\). Putting all the pieces together and solving \(\{\partial Q^*(s,a)\}_{s,a,s',\tilde{s},\tilde{a}}\) from the linear system, we have

\[
\frac{\partial Q^*(s,a)}{\partial P(s'|\tilde{s},\tilde{a})} = \gamma V^*(s') \cdot (I - \gamma P^*)^{-1}((s,a),(\tilde{s},\tilde{a})).
\]

\[\square\]

**Proof of Lemma E.2.** We write our the log-likelihood of sample \(P_t\) as

\[
\log f_P(P_t) = \sum_{s,a,s'} 1_{\{s_t=(s,a)=s'\}} \log P(s'|s,a)
\]

which implies \(\frac{\partial}{\partial P} \log f_P(P_t) \in \mathbb{R}^{S^2A}\) with the \((s,a,s')\)-th entry given by

\[
\frac{\partial \log f_P(P_t)}{\partial P(s'|s,a)} = \frac{1_{\{s_t=(s,a)=s'\}}}{P(s'|s,a)}.
\]

(54)

By definition of Fisher information matrix, we have

\[
I_1(P) = \mathbb{E} \left\{ \frac{\partial}{\partial P} \log f_P(P_t) \left[ \frac{\partial}{\partial P} \log f_P(P_t) \right]^\top \right\} \in \mathbb{R}^{S^2A \times S^2A}
\]

which implies

\[
I_1(P)((s,a,s'),(\tilde{s},\tilde{a},\tilde{s}') = \begin{cases} 
1_{\{s'=\tilde{s}'\}} P(s'|s,a) & \text{if } (s,a) = (\tilde{s},\tilde{a}), \\
1 & \text{if } (s,a) \neq (\tilde{s},\tilde{a}). 
\end{cases}
\]

By definition of \(h(P)\), we rearrange \(h(P)\) into a \(S^2A \times SA\) matrix given by

\[
H(P)((s,a,s'),(\tilde{s},\tilde{a})) := \frac{\partial h_{\tilde{s},\tilde{a}}(P)}{\partial P(s'|s,a)} = 1_{\{(\tilde{s},\tilde{a})=(s,a)\}}.
\]

Let \(U(P) \in \mathbb{R}^{S^2A \times (S^2A-SA)}\) be the orthogonal matrix whose column space is the orthogonal complement of the column space of \(H(P)\), which stands for \(H(P)^\top U(P) = 0\) and \(U(P)^\top U(P) = I\). Using results in Moore Jr [2010], the constrained CR lower bound is

\[
C_1(P) = U(P) \left( U(P)^\top I_1(P) U(P) \right)^{-1} U(P)^\top.
\]

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We define an auxiliary matrix $X \in \mathbb{R}^{S_A \times S^2_A}$ satisfying

$$X((s, a), (\bar{s}, \bar{a}, \bar{s}')) = -\frac{1}{2} \cdot 1\{ (s, a) \neq (\bar{s}, \bar{a}) \}.$$ 

By $H(P)^\top U(P) = 0$, we have

$$C_1(P) = U(P) \left( U(P)^\top (H(P)X + I_t(P) + X^\top U(P)^\top)U(P) \right)^{-1} U(P)^\top$$

$$:= U(P) \left( U(P)^\top D(P)U(P) \right)^{-1} U(P)^\top,$$

where $D(P)((s, a, s'), (s, a, s')) = 1/P(s'|s, a)$ and takes value 0 elsewhere. Now we reformulate $D(P)$ as a block diagonal matrix $D(P) = \text{diag}(\{D(s, a)\}_{(s, a)})$ where $D(s, a)$ is a diagonal matrix with $D(s, a)(s', s') = 1/P(s'|s, a)$. Similarly, we have $H(P) = \text{diag}(\{1_S\}_{(s, a)})$, where $1_S$ is an all-1 vector with dimension $S$, and $U(P) = \text{diag}(\{U(s, a)\}_{(s, a)})$, where $U(s, a) \in \mathbb{R}^{S^2 \times S - 1}$ satisfying $U_{(s, a)}^\top 1_S = 0$. In this way, $C_1(P)$ has an equivalent block diagonal formulation

$$C_1(P) = \text{diag} \left( \left\{ U(s, a) \left( U(s, a)^\top D(s, a)U(s, a) \right)^{-1} U(s, a) \right\}_{(s, a)} \right).$$

For each block $(s, a)$ of $C_1(P)$, the submatrix is exactly the constrained Cramer-Rao bound of a multinomial distribution $P_{s,a} = \{P(\cdot | s, a)\}$, which is equal to $\text{diag}(P_{s,a}) - P_{s,a} P_{s,a}^\top$. Therefore,

$$C_1(P) = \text{diag} \left( \left\{ \text{diag}(P(\cdot | s, a)) - P(\cdot | s, a)P(\cdot | s, a)^\top \right\}_{(s, a)} \right).$$

□

E.1.2 Nonparametric Part

Next, we move on discussing the efficiency on rewards. Unlike $P_t$ that is generated according to a parametric model, the generating mechanism of $r_t$ can be arbitrary. In other words, a finite dimensional parametric space is not enough to cover the possible distributions of $r_t$. Thus, semiparametric theory is needed here. Fortunately, our interest parameter $Q^* = (I - \gamma P_{\pi^*})^{-1}r$ is linear in $r := \mathbb{E}r_t$, implying only the expectation of $r_t$ matters. In semiparametric theory [Van der Vaart, 2000, Tsiatis, 2006], the efficient influence function for mean estimation is exactly the random variable minus its expectation. Lemma E.3 shows it is still true in our case.

**Lemma E.3.** Let Assumption 3.2 hold. Given a random sample $r_t$, the most efficient influence function for estimating $Q^*(s, a)$ for any $(s, a)$ is

$$\phi(s, a) = (I - \gamma P_{\pi^*})^{-1}(r_t - r)(s, a),$$

where $r = \mathbb{E}r_t$. Hence, the semiparametric efficiency bound of estimating $Q^*$ with $\{r_t\}_{t \in \mathbb{T}}$ is

$$\sup_{P_{\gamma(R)} \in \mathcal{P}_R} \Gamma(\gamma_0(R))I(\gamma_0(R))^{-1} \Gamma^\top(\gamma_0(R)) = \frac{1}{T} \cdot (I - \gamma P_{\pi^*})^{-1} \text{Var}(r_t)(I - \gamma P_{\pi^*})^{-1}. $$
Proof of Lemma E.3. As \( r_t(s, a) \) are independent with different \((s', a')\) pairs, we can only consider randomness of one pair \((s, a)\).

Firstly, we consider a submodel family \( \mathcal{P}_{R_e} \) of \( \mathcal{P}_R \) that is parameterized by \( \varepsilon \) such that when \( \varepsilon = 0 \), we recover the distribution of \( R(s, a) \). That is \( \mathcal{P}_{R_e} = \{ R_{\varepsilon} : \varepsilon \in [-\delta, \delta] \} \) and \( R(s, a) = R_{\varepsilon}(s, a)|_{\varepsilon=0} \). This can be achieved by manipulating density functions of each \( R(s, a) \). It is clear that \( \mathcal{P}_{R_e} \) is a parametric family on rewards and we can make use of results in parametric statistics for our purpose. By definition, we have for \((s, a)\),

\[
\frac{\partial Q^*(r, a)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \frac{\partial \mathbb{E}r_t(s, a)}{\partial \varepsilon} \bigg|_{\varepsilon=0} + \gamma \sum_{s'} P(s'|s, a) Q^*(s', \pi^*(s')) \left. \frac{\partial Q^*(s', \pi^*(s'))}{\partial \varepsilon} \right|_{\varepsilon=0}.
\]

For any \((\tilde{s}, \tilde{a}) \neq (s, a)\), we have

\[
\frac{\partial Q^*(\tilde{s}, \tilde{a})}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \gamma \sum_{s'} P(s'|\tilde{s}, \tilde{a}) \frac{\partial Q^*(s', \pi^*(s'))}{\partial \varepsilon} \bigg|_{\varepsilon=0}.
\]

Recursively expanding the above terms like what we have done in Lemma E.1, we have

\[
\frac{\partial Q^*(\tilde{s}, \tilde{a})}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \frac{\partial \mathbb{E}r_t(s, a)}{\partial \varepsilon} \bigg|_{\varepsilon=0} \cdot (I - \gamma P^*)^{-1}((\tilde{s}, \tilde{a}),(s, a)).
\]

Let \( F_{\varepsilon} \) denote the cumulative distribution function of \( R_{\varepsilon}(s, a) \). Then we have

\[
\frac{\partial \mathbb{E}r_t(s, a)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \int (r_t(s, a) - r(s, a)) \frac{\partial}{\partial \varepsilon} \log dF_{\varepsilon} \bigg|_{\varepsilon=0} dF_0,
\]

where \( r(s, a) = \mathbb{E}r_t(s, a) \) and \( \frac{\partial}{\partial \varepsilon} \log dF_{\varepsilon} \) is the score function. Therefore,

\[
\frac{\partial Q^*(\tilde{s}, \tilde{a})}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \int \phi(\tilde{s}, \tilde{a}) \frac{\partial}{\partial \varepsilon} \log dF_{\varepsilon} \bigg|_{\varepsilon=0} dF_0
\]

where

\[
\phi(\tilde{s}, \tilde{a}) = (r_t - r)(s, a) \cdot (I - \gamma P^*)^{-1}((\tilde{s}, \tilde{a}),(s, a)).
\]

Since the parametric submodel family \( \mathcal{R}_e \) is arbitrary, we conclude that the efficient influence function of \( Q^*(\tilde{s}, \tilde{a}) \) is \( \phi(\tilde{s}, \tilde{a}) \) by Theorem 2.2 in Newey [1990]. Finally, as \( r_t(s, a) \) is independent with each other \( r_t(s', a') \)'s, our final result is obtained by summing the above equation over all \((s, a)\).

\( \square \)

E.2 Proof of Theorem 3.3

Proof of Theorem 3.3. Recall that \( \Delta_T = \frac{1}{T} \sum_{t=1}^{T} (Q_T - Q^*) \). Combining (30), (38) and (41), we have

\[
T_0 + T_1 + T_2 + T_3 \leq \sqrt{T} \Delta_T^1 \leq \sqrt{T} \Delta_T \leq \sqrt{T} \Delta_T^2 \leq T_0 + T_1 + T_2 + T_3 + T_4.
\]

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where the inequality holds coordinately. As proved in Appendix C.2.2, \( T_t = o_P(1) \) for \( i = 0, 2, 3, 4 \). Hence,

\[
\Delta_T = T_t + o_P(1) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (I - \gamma P^{\pi^*})^{-1} Z_t + o_P(1) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi(r_t, P_t) + o_P(1)
\]

where \( Z_t = (r_t - r) + \gamma (P_t - P) V^* \) is the Bellman noise at iteration \( t \). It implies \( Q_T \) is asymptotically linear with influence function \( \phi(r_t, P_t) := (I - \gamma P^{\pi^*})^{-1} Z_t \).

The remaining issue is to prove it is regular. By definition, a RAL estimator for a semiparametric model \( P = P_P \times P_R \) if it is a RAL estimator for every parametric submodel \( P_\gamma = P_P \times P_{R_\gamma} \subset P \) where \( \gamma = (P, \varepsilon) \) is the finite-dimensional parameter controlling \( P_\gamma \). In a parametric submodel \( P_P \times P_{R_\gamma} \), by Theorem 2.2 in Newey [1990], for the asymptotically linear estimator \( Q_T \) of \( Q^* \) which has the influence function

\[
\phi(r_t, P_t) = (I - \gamma P^{\pi^*})^{-1} [(r_t - r) + \gamma (P_t - P) V^*],
\]

its regularity is equivalent to the equality

\[
\mathbb{E} \phi(r_t, P_t) S_\gamma^T(\gamma_0) = \frac{\partial Q^*}{\partial \gamma} \bigg|_{\gamma = \gamma_0}
\]

where \( S_\gamma(\cdot) \) is the score function, \( \gamma = (P', \varepsilon) \in P_P \times [-\delta, \delta] \) is the finite-dimensional parameter and \( \gamma_0 = (P, 0) \) is the true underlying parameter. Since \( P \) and \( \varepsilon \) are variational independence, \( S_\gamma(\gamma_0) = (SP(\gamma_0), S\varepsilon(\gamma_0)) \).

For the transition kernel \( P \). Since our parametric space \( P_P \) has a linear constraint, it is not easy to compute the constrained score function. Hence, for \( P = \{P(s'|s, a)\}_{s,a,s'} \), we regard \( \{P(s'|s, a)\}_{s,a,s' \neq s_0} \) as free parameters where \( s_0 \in S \) is any fixed state and use it as our new parameter. For a fixed \( s,a \), once \( P(s'|s, a) \) is determined for all \( s' \neq s_0 \), one can recover \( P(s_0|s, a) \) by \( P(s_0|s, a) = 1 - \sum_{s' \neq s_0} P(s'|s, a) \). In this way, each \( \{P(s'|s, a)\}_{s,a,s' \neq s_0} \) lies in an open regime. We still denote the set collecting all feasible \( \{P(s'|s, a)\}_{s,a,s' \neq s_0} \) as \( P \), but readers should remember that current \( P = \{P(s'|s, a)\}_{s,a,s' \neq s_0} \in \mathbb{R}^{|SA| \times (S - 1)} \). From (54) and under our new notation of \( P \), \( SP(\gamma_0) \in \mathbb{R}^{SA(S-1)} \) with entries given by

\[
SP(\gamma_0)(s, a, s') = \frac{1_{\{s, a, s' \neq s_0\}}}{P(s'|s, a)} - \frac{1_{\{s, a = s_0\}}}{P(s_0|s, a)} \text{ for any } s' \neq s_0.
\]

By Lemma E.1 and the chain rule, it follows that \( \frac{\partial Q^*}{\partial P} \in \mathbb{R}^{SA \times SA(S-1)} \) and its \((\tilde{s}, \tilde{a}, s')\)-th column is

\[
\gamma(I - \gamma P^{\pi^*})^{-1}(\cdot, (\tilde{s}, \tilde{a})) [V^*(s') - V^*(s_0)].
\]

Since \( (I - \gamma P^{\pi^*})^{-1} \) has a full rank (i.e., \( SA \)), it is easy to see that \( \frac{\partial Q^*}{\partial P} \) also has rank \( SA \) by varying \((\tilde{s}, \tilde{a})\) and fixing \( s', s_0 \) in (57). On the other hand, the \((\tilde{s}, \tilde{a}, s')\)-th column of \( \mathbb{E} \phi(r_t, P_t) SP(\theta_0) \) is

\[
\mathbb{E} \phi(r_t, P_t) SP(\gamma_0)^T(\cdot, (\tilde{s}, \tilde{a}, s')) = \mathbb{E} \phi(r_t, P_t) \left[ \frac{1_{\{s, a, s' \neq s_0\}}}{P(s'|s, a)} - \frac{1_{\{s, a = s_0\}}}{P(s_0|s, a)} \right]
= \gamma(I - \gamma P^{\pi^*})^{-1} \mathbb{E} (P_t - P) V^* \left[ \frac{1_{\{s, a, s' \neq s_0\}}}{P(s'|s, a)} - \frac{1_{\{s, a = s_0\}}}{P(s_0|s, a)} \right]
\]
\[
= \gamma (I - \gamma P^{\pi^*})^{-1} (\cdot, (\tilde{s}, \tilde{a})) [V^*(s') - V^*(s_0)]
\]

where the last equality uses the following result. By direct calculation, the \((s, a)\)-th entry of
\[\mathbb{E}(P_t - P)V^* \left[ \frac{1_{\{s_t(s,a) = s'\}}}{P(s'|s,a)} - \frac{1_{\{s_t(s,a) = s_0\}}}{P(s_0|s,a)} \right]\]
is 0 for all \((s, a) \neq (\tilde{s}, \tilde{a})\) (due to independence) and the
\((\tilde{s}, \tilde{a})\)-th entry is \(V^*(s') - V^*(s_0)\). Indeed, the \((\tilde{s}, \tilde{a})\)-th entry of the mentioned matrix is
\[
\mathbb{E} \sum_{i \in S} (1_{\{s_t(s,a) = i\}} - P(i|s,a))V^*(i) \left[ \frac{1_{\{s_t(s,a) = s'\}}}{P(s'|s,a)} - \frac{1_{\{s_t(s,a) = s_0\}}}{P(s_0|s,a)} \right]
\]
\[
= \left( V^*(s') - \sum_{i \neq s_0} P(i|s,a)V^*(i) \right) + \sum_{i \in S} P(i|s,a)V^*(i) = V^*(s') - V^*(s_0).
\]

Therefore, combining the results for all \((\tilde{s}, \tilde{a}, s')(s' \neq s_0)\), we have
\[
\mathbb{E}\phi(r_t, P_t)S_P(\gamma_0)^\top = \frac{\partial Q^*}{\partial P}
\]
which implies (56) holds for the \(P\) part.

**For the random reward \(R\).** Using the notation in the proof of Lemma E.3, \(S_\epsilon(\gamma_0) = \frac{\partial}{\partial \epsilon} \log dF_\epsilon|_{\epsilon=0}\).
By (55), we have
\[
\frac{\partial Q^*}{\partial \epsilon} \bigg|_{\epsilon=0} = \mathbb{E}(I - \gamma P^{\pi^*})^{-1} (r_t - r)S_\epsilon(\gamma_0) = \mathbb{E}\phi(r_t, P_t)S_\epsilon(\gamma_0)
\]
which implies (56) holds for the \(\epsilon\) part.

\(P_{R_\epsilon}\) can be arbitrary, so (56) holds for all parametric submodels. It means \(Q_T\) is regular for all
parametric submodels and thus is regular for our semiparametric model. \(\square\)

## F A Useful Concentration Inequality

We introduce a useful concentration inequality in this part. It captures the expectation and high
probability concentration of a martingale difference sum in terms of \(\| \cdot \|_\infty\). It uses a similar idea of
Theorem 4 in Li et al. [2021] and is built on the Freedman’s inequality [Freedman, 1975] and the
union bound.

**Lemma F.1.** Assume \(\{X_j\} \subseteq \mathbb{R}^d\) are martingale differences adapted to the filtration \(\{\mathcal{F}_j\}\) with zero
conditional mean \(\mathbb{E}[X_j|\mathcal{F}_{j-1}] = 0\) and finite conditional variance \(V_j = \mathbb{E}[X_jX_j^\top|\mathcal{F}_{j-1}]\). Besides,
we assume \(\{X_j\}\) is uniformly bounded, i.e., \(\sup_j \|X_j\|_\infty \leq X\). For any sequence of deterministic
matrices \(\{B_j\} \subseteq \mathbb{R}^{D \times d}\) satisfying \(\sup_j \|B_j\|_\infty \leq B\), we define the weighted sum as
\[
Y_T = \sum_{j=1}^T B_j X_j
\]
and \(W_T = \text{diag}(\sum_{j=1}^T B_j V_j (B_j)^\top)\) is a diagonal matrix that collects conditional quadratic variations.
Then, it follows that
\[
\mathbb{P} \left( \|Y_T\|_\infty \geq \frac{2BX}{3} \ln \frac{2D}{\delta} + \sqrt{2\sigma^2 \ln \frac{2D}{\delta}} \text{ and } \|W_T\|_\infty \leq \sigma^2 \right) \leq \delta
\]  
\[
(58)
\]

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Generally, we have
\[ \mathbb{E}\|Y_T\|_\infty 1_{\{\|W_T\|_\infty \leq \sigma^2\}} \leq 6\sigma \sqrt{\ln 2D} + \frac{4BX}{3} \ln 6D. \] (59)

Proof of Lemma F.1. Fixing any \( i \in [D] \), we denote the \( i \)-th row of \( B_j \) as \( b_j^\top \). For simplicity, we omit the dependence of \( b_j \) on \( i \). Then the \( i \)-th coordinate of \( Y_T \) is \( Y_T(i) = \sum_{j=1}^T b_j^\top X_j \) and \( W_T(i, i) = \sum_{j=1}^T b_j^\top V_j b_j \). Clearly \( \{ b_j^\top X_j \} \) is a scalar martingale difference with \( W_T(i, i) = \sum_{j=1}^T \mathbb{E}(b_j^\top X_j)^2 | F_{j-1} \) the quadratic variation and \( |b_j^\top X_j| \leq \|b_j\|_1 \|X_j\|_\infty \leq \|B_j\|_\infty \|X_j\|_\infty = BX \) the uniform upper bound. By Freedman’s inequality [Freedman, 1975], it follows that
\[ \mathbb{P}(\|Y_T(i)\| \geq \tau \) and \( \|W_T(i, i)\| \leq \sigma^2) \leq 2 \exp \left( -\frac{\tau^2/2}{\sigma^2 + BX \tau/3} \right). \]

Then by the union bound, we have
\[ \mathbb{P}(\|Y_T\|_\infty \geq \tau \) and \( \|W_T\|_\infty \leq \sigma^2) = \sum_{i \in [D]} \mathbb{P}(\|Y_T(i)\| \geq \tau \) and \( \max_{i \in [D]} \|W_T(i, i)\| \leq \sigma^2) \leq 2D \exp \left( -\frac{\tau^2/2}{\sigma^2 + BX \tau/3} \right). \] (61)

Solving for \( \tau \) such that the right hand side of (61) is equal to \( \delta \) gives
\[ \tau = \frac{BX}{3} \ln \frac{2D}{\delta} + \sqrt{\left( \frac{BX}{3} \ln \frac{2D}{\delta} \right)^2 + 2\sigma^2 \ln \frac{2D}{\delta}}. \]

Using \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) gives an upper bound on \( \tau \) and provides the high probability result.

The tail bound of \( \|Y_T\|_\infty 1_{\{\|W_T\|_\infty \leq \sigma^2\}} \) has already been derived in (61). For the expectation result, we refer to the conclusion of Exercise 2.8 (a) in Wainwright [2019a] which implies that
\[ \mathbb{E}\|Y_T\|_\infty 1_{\{\|W_T\|_\infty \leq \sigma^2\}} \leq 2\sigma(\sqrt{\pi} + \sqrt{\ln 2D}) + \frac{4BX}{3}(1 + \ln 2D) \leq 6\sigma \sqrt{\ln 2D} + \frac{4BX}{3} \ln 6D \]
where the last inequality uses \( \sqrt{a + b} \leq \sqrt{2(a + b)} \).

For the last result, we aim to bound \( \mathbb{E}\|Y_T\|_\infty \) without the condition \( \|W_T\|_\infty \leq \sigma^2 \) for some positive number \( \sigma \). We first assert that there exists a trivial upper bound for \( \|W_T\|_\infty \) which is \( \|W_T\|_\infty \leq TB^2 X^2 \). This is because
\[ \|W_T\|_\infty = \left\| \operatorname{diag} \left( \sum_{j=1}^T B_j V_j (B_j^\top) \right) \right\|_\infty \leq \sum_{j=1}^T \left\| \operatorname{diag} (B_j V_j (B_j^\top)) \right\|_\infty \leq \|V_j\|_{\max} \|B_j\|_2^2 \leq TB^2 X^2 \]
where \((a)\) uses Lemma C.2 and \((b)\) is due to \(\|V_j\|_{\text{max}} \leq X^2\) for all \(j \in [T]\). However, if we set \(\sigma^2 = TB^2X^2\) in \((59)\), the resulting expectation bound of \(\mathbb{E}\|Y_T\|_{\infty}\) has a poor dependence on \(T\).

To refine the dependence, we adapt and modify the argument of Theorem 4 in Li et al. [2021]. For any positive integer \(K\), we define

\[
\mathcal{H}_K = \left\{ \|Y_T\|_{\infty} \geq \frac{2BX}{3} \ln \frac{2DK}{\delta} + \sqrt{4 \max \left\{ \|W_T\|_{\infty}, \frac{TB^2X^2}{2K} \right\} \ln \frac{2DK}{\delta}} \right\}
\]

and claim that we have \(\mathbb{P}(\mathcal{H}_K) \leq \delta\). We observe that the event \(\mathcal{H}_K\) is contained within the union of the following \(K\) events: \(\mathcal{H}_K \subseteq \bigcup_{k \in [K]} \mathcal{B}_k\) where for \(0 \leq k < K\), \(\mathcal{B}_k\) is defined to be

\[
\mathcal{B}_k = \left\{ \|Y_T\|_{\infty} \geq \frac{2BX}{3} \ln \frac{2DK}{\delta} + \sqrt{2 \frac{TB^2X^2}{2k-1} \ln \frac{2DT}{\delta}} \text{ and } \|W_T\|_{\infty} \leq \frac{TB^2X^2}{2k-1} \right\}
\]

Invoking \((58)\) with a proper \(\sigma^2 = \frac{TB^2X^2}{2k-1}\) and \(\delta = \frac{\delta}{K}\), we have \(\mathbb{P}(\mathcal{B}_k) \leq \delta\) for all \(k \in [K]\). Taken this result together with the union bound gives \(\mathbb{P}(\mathcal{H}_K) \leq \sum_{k \in [K]} \mathbb{P}(\mathcal{B}_k) \leq \delta\). Then we have

\[
\mathbb{E}\|Y_T\|_{\infty} = \mathbb{E}\|Y_T\|_{\infty}1_{\mathcal{H}_K} + \mathbb{E}\|Y_T\|_{\infty}1_{\mathcal{H}_K^c}
\]

\[
(a) \leq TBX\mathbb{P}(\mathcal{H}_K) + \mathbb{E} \left[ \frac{2BX}{3} \ln \frac{2DK}{\delta} + \sqrt{4 \max \left\{ \|W_T\|_{\infty}, \frac{TB^2X^2}{2K} \right\} \ln \frac{2DK}{\delta}} \right]
\]

\[
(b) \leq BX + \frac{2BX}{3} \ln(2DT^2) + 2\mathbb{E} \sqrt{\max \left\{ \|W_T\|_{\infty}, B^2X^2 \right\} \ln(2DT^2)}
\]

\[
(c) \leq BX + \frac{8BX}{3} \ln(2DT^2) + 2\mathbb{E} \sqrt{\|W_T\|_{\infty} \ln(2DT^2)}
\]

\[
(d) \leq \frac{8BX}{3} \ln(3DT^2) + 2\sqrt{\mathbb{E}\|W_T\|_{\infty} \ln(2DT^2)}
\]

where \((a)\) uses \(\|Y_T\|_{\infty} \leq TBX\), \((b)\) follows by setting \(\delta = \frac{1}{T} \) and \(K = [\log_2 T] \leq T\), \((c)\) uses \(\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}\), and \((d)\) follows from Jensen’s inequality and \(\exp\left(\frac{3}{2}\right) \leq \frac{3}{2}\).

\[\square\]

\section{Experiment Detail}

According to Theorem 3.5, for sufficiently small error \(\varepsilon > 0\), we expect the sample complexity \(T(\varepsilon, \gamma)\) is always upper bounded by \(\|\text{diag(Var)}\|_{\infty}\|\text{diag(Cov)}\|_{\infty}\) and \(\frac{1}{(1-\gamma)^2T}\) at a worst case. To ensure Assumption 3.2, we consider a random MDP. In particular, for each \((s, a)\) pair, the random reward \(R(s, a) \sim \mathcal{U}(0, 1)\) is the uniformly sampled from \((0, 1)\) and the transition probability \(P(s'|s, a) = u(s')/\sum_s u(s)\), where \(u(s) \sim \mathcal{U}(0, 1)\). The size of the MDP we choose is \(|S| = 4, |A| = 3\). We consider 30 different values of \(\gamma\) equispaced between 0.6 and 0.9. For a given \(\gamma\), we run Q-learning algorithm for \(10^5\) steps (which already ensures convergence) and repeat the process independently for \(10^5\) times. Finally, we average
the $\ell_\infty$ error $\|\bar{Q}_T - Q^*\|_\infty$ of the $10^3$ independent trials as an approximation of $\mathbb{E}\|\bar{Q}_T - Q^*\|_\infty$ and compute $T(\varepsilon, \gamma)$ by definition. The polynomial step size $\eta_t = t^{-\alpha}$ uses $\alpha \in \{0.51, 0.55, 0.60\}$ and the rescaled linear step size is $\eta_t = (1 + (1 - \gamma)t)^{-1}$. In Figure 1, we choose $\varepsilon = e^{-4}$ and plot the results on a log-log scale. We have also plotted the least-squares fits through these points and the slopes of these lines are also provided in the legend. In particular, we run linear regression of pairs $(\log \frac{1}{1-\gamma_i}, \log T(\varepsilon, \gamma_i))$ and obtain the coefficient $k$ on $\log \frac{1}{1-\gamma}$ in the legend of Figure 1.