On Picent for blocks with normal defect group

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Abstract

We prove that if \( b \) is a block of a finite group with normal abelian defect group and inertial quotient a direct product of elementary abelian groups, then Picent(\( b \)) is trivial. We also provide examples of blocks \( b \) of finite groups with non-trivial Picent(\( b \)). We even have examples with normal abelian defect group and abelian inertial quotient.

1 Introduction

Let \( \mathcal{O} \) be a complete discrete valuation ring with \( k := \mathcal{O}/J(\mathcal{O}) \) an algebraically closed field of prime characteristic \( p \). Let \( K \), the field of fractions of \( \mathcal{O} \), have characteristic zero. We take \( K \) large enough, meaning that it contains all \( |H|^{th} \) roots of unity, for all the groups \( H \) involved in the rest of the paper. By a block we always mean a block \( b \) of \( \mathcal{O}H \) for some finite group \( H \). We use \( \text{Irr}(H) \) to denote the set of irreducible characters of \( H \) and \( \text{Irr}(b) \) the subset of irreducible characters lying in the block \( b \).

Let \( b \) be a block of a finite group \( H \). The Picard group \( \text{Pic}(b) \) of \( b \) consists of isomorphism classes of \( b \)-\( b \)-bimodules which induce \( \mathcal{O} \)-linear Morita auto-equivalences of \( b \). For \( M, N \in \text{Pic}(b) \), the group multiplication is given by \( M \otimes b N \). We will often view \( M \) as an \( \mathcal{O}(H \times H) \)-module via \( (g, h).m = gmh^{-1} \), for all \( g, h \in H \) and \( m \in M \). This paper is concerned with Picent(\( b \)), the subgroup of \( \text{Pic}(b) \) consisting of Morita auto-equivalences that induce the trivial permutation of \( \text{Irr}(b) \).

In [2] it is proved that all 2-blocks with abelian defect group of rank at most three have trivial Picent. Our main result is that in certain situations Picent is always trivial (see Theorems 3.2 and 3.3).

**Theorem.** Let \( b \) be a block with normal abelian defect group and cyclic inertial quotient or inertial quotient a product of elementary abelian groups or a principal block with abelian inertial quotient. Then Picent(\( b \)) is trivial.

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Notwithstanding the above theorem, in this paper we show that blocks with non-trivial Picent do exist. Our focus is entirely on blocks with normal defect group. We construct three families of examples. The first, see Example 4.1, is simply any $p$-group with a non-inner, class-preserving automorphism. The second example, see Proposition 4.2, is given by a non-inner, class-preserving automorphism of a group $H$, with normal abelian Sylow $p$-subgroup, that induces a non-trivial element of Picent for the principal block of $OH$. Our final example, see Proposition 4.3, is concerned with a non-principal block with normal abelian defect group and abelian inertial quotient. An interesting point to note in this final case is that the relevant bimodule has vertex $\Delta D$, where $D$ is the defect group. This means the Morita auto-equivalence is simply given by tensoring with a linear character of the inertial quotient.

The following notation will hold for the remainder of the article. If $N \unlhd H$, for a finite group $H$, and $\chi \in \text{Irr}(N)$, then we denote by $\text{Irr}(\langle H \rangle | \chi)$ the set of irreducible characters of $H$ appearing as constituents of $\chi^{\langle H \rangle}$. Similarly, for a block $b$ of $H$, we define $\text{Irr}(b | \chi) := \text{Irr}(b) \cap \text{Irr}(\langle H \rangle | \chi)$. If $h \in H$, then we denote by $c_b \in \text{Aut}(H)$ the automorphism given by $g \mapsto hg^{-1}$. For any $ON$-module $M$, $hM$ will denote the $ON$-module equal to $M$ as an $O$-module but with the action of $N$ defined via $g.m = c_b^{-1}(g)m$, for all $g \in N$ and $m \in hM$. The character $h\chi \in \text{Irr}(N)$ is defined analogously. If $X \unlhd H$, then we set $I_X(M) := \{h \in X | hM \cong M\}$ and define $I_X(\chi)$ analogously. We write $O_H$ for the $O\ H$-module $O$ with the trivial action of $H$ and $1_H \in \text{Irr}(H)$ will denote the trivial character of $H$. We use $e_b \in O_H$ to signify the block idempotent of $b$ and $e_\eta \in KH$ the character idempotent associated to $\eta \in \text{Irr}(H)$.

If $\psi \in \text{Aut}(H)$, then we denote by $\Delta \psi := \{(h, \psi(h)) | h \in H\} \leq H \times H$. If $\psi = \text{Id}_H$, then we just denote $\Delta \psi$ by $\Delta H$. We will also view $\psi$ as an $O$-linear automorphism of $OH$, where appropriate. For an $O$-algebra $A$, $\text{Aut}_O(A)$ will stand for the set of $O$-algebra automorphisms of $A$. If $\alpha \in \text{Aut}_O(b)$, we denote by $\alpha b \in \text{Pic}(b)$ the equivalence induced by $\alpha$. In other words, $\alpha b = b$ as sets but with $x.m.y := \alpha(x)my$, for all $x, y \in b$ and $m \in \alpha b$.

The article is organised as follows. §2 contains an assortment of lemmas that will be needed in §3 and §4. §3 concerns Morita auto-equivalences induced by bimodules with trivial source and then goes on to prove our main theorems regarding families of blocks with trivial Picent. §4 gives our three classes of examples of blocks with non-trivial Picent.

## 2 Preliminaries

In this section we gather together various lemmas that will be used throughout the rest of the article.

**Lemma 2.1.** Let $V$ be an $n$-dimensional vector space over a field $\mathbb{F}$, for some $n \in \mathbb{N}$. In addition let $(V_i)_{i \in I}$ be a set of subspaces of $V$ of codimension one.
such that \( \bigcap_{i \in I} V_i = \{0\} \). Then there exists a subset \( \{i_1, \ldots, i_n\} \subseteq I \) such that \( \bigcap_{j=1}^n V_{i_j} = \{0\} \). Furthermore, setting \( U_l := \bigcap_{j \neq l} V_{i_j} \), for each \( 1 \leq l \leq n \), we have that each \( U_l \) has dimension one and \( V = \bigoplus_{l=1}^n U_l \).

**Proof.** Choose any \( i_1 \in I \). Then choose the remaining \( i_l \)'s iteratively by demanding that \( \bigcap_{j=1}^l V_{i_j} \) is strictly contained in \( \bigcap_{j=1}^{l-1} V_{i_j} \) for \( 2 \leq l \leq n \). Note that the next \( V_{i_l} \) always exists, until \( \bigcap_{j=1}^l V_{i_j} = \{0\} \), since \( \bigcap_{i \in I} V_i = \{0\} \). In fact \( \bigcap_{j=1}^l V_{i_j} = \{0\} \) precisely when \( l = n \), since \( \bigcap_{j=1}^{l-1} V_{i_j} \) has codimension one in \( \bigcap_{j=1}^l V_{i_j} \), for \( 1 \leq l \leq n \). In particular, each \( U_l \) has dimension one. Therefore, the final claim follows since

\[
\left( \sum_{l=1, l \neq m}^n U_l \right) \cap U_m \subseteq V_m \cap U_m = \bigcap_{j=1}^n V_{i_j} = \{0\},
\]

for all \( 1 \leq m \leq n \). \( \square \)

**Lemma 2.2.** Let \( H \) be a finite group, \( N \) a normal subgroup such that \( H/N \) is abelian, \( \ell \) a prime and \( N \leq L \leq H \) such that \( L/N \) is a cyclic \( \ell \)-group and \( \ell \nmid [H:L] \). If \( \chi \in \text{Irr}(N) \) is \( H \)-stable, then \( \chi \) extends to \( L \) and every extension of \( \chi \) is \( H \)-stable.

**Proof.** Since \( L/N \) is cyclic and \( \chi \) is \( L \)-stable, \( \chi \) certainly extends to \( L \). Now let \( h \in H \). Since \( H/N \) is abelian, \( L/N \) is a cyclic \( \ell \)-group and \( \ell \nmid [H:L] \), \( L/h)N/N \) is cyclic. Therefore, \( \chi \) extends to \( L/h) \) in \( [L/h) : N \) different ways. In particular, every extension of \( \chi \) to \( L \) extends to \( L/h) \) and so \( h \) stabilises \( \chi \). \( \square \)

For a finite \( \ell \)-group \( P \), we denote by \( \Phi(P) \) the Frattini subgroup of \( P \). For the following lemma we have in mind the semi-direct product \( P \times H \). We will borrow notation from this setup, for example \( C_P(H) \) will denote the set of fixed points in \( P \) under the action of \( H \).

**Lemma 2.3.** Let \( P \) be a finite abelian \( \ell \)-group and \( H \) an abelian \( \ell' \)-subgroup of \( \text{Aut}(P) \).

1. \( P = C_P(H) \times [H, P] \).

2. The natural homomorphism \( \phi : P \to \text{Aut}(P/\Phi(P)) \) is injective. Furthermore, \( H \) acts indecomposably on \( P \), that is there does not exist a non-trivial, \( H \)-invariant decomposition \( P = P_1 \times P_2 \), if and only if \( H \) acts indecomposably on \( P/\Phi(P) \).

3. The homomorphism \( \phi : \text{Aut}(\text{Irr}(P)) \to \text{Irr}(P) \) by \( h(\lambda)(x) = \lambda(h^{-1}(x)) \), for all \( h \in H \), \( x \in P \) and \( \lambda \in \text{Irr}(P) \), is injective and this action of \( H \) on \( \text{Irr}(P) \) is indecomposable if and only if the action of \( \phi(h) \) on \( P \) is.

4. If \( H \) does act indecomposably on \( P \), then \( H \) is cyclic and \( C_P(h) \) is trivial for all \( h \in H \setminus \{1\} \).
5. Let $\psi \in C_{\text{Aut}(P)}(H)$ such that, for all $x \in P$, $\psi(x) = h(x)$, for some $h \in H$. Then $\psi \in H$.

Proof.

1. This is [3, §5, Theorem 2.3].

2. The fact that the homomorphism is injective follows from [3, §5, Theorem 1.4]. The indecomposability statement is [8, Lemma 2.5].

3. We can identify $P$ with $\text{Irr} (\text{Irr}(P))$ as $H$-sets via $x \mapsto (\lambda \mapsto \lambda(x))$. Injectivity follows since, if some $h \in H$ fixes $\text{Irr}(P)$ pointwise, then $h$ also fixes $\text{Irr}(\text{Irr}(P))$ pointwise and hence also $P$. If $P = P_1 \times P_2$ is an $H$-invariant decomposition of $P$, then $\text{Irr}(P) = \text{Irr}(P, 1_{P_1}) \times \text{Irr}(P, 1_{P_2})$ is an $H$-invariant decomposition of $\text{Irr}(P)$. The reverse implication follows, once again, by identifying $P$ with $\text{Irr}(\text{Irr}(P))$.

4. [8, Lemma 2.6] gives that $H$ is cyclic. Next let $h \in H$. If $C_P(h)$ is non-trivial then, by part (1), $P = C_P(h) \times [h, P]$ is an $H$-invariant decomposition. Therefore, since $H$ acts indecomposably on $P$, $C_P(h) = P$ and $h = 1$.

5. Let’s first assume that $P$ is elementary abelian and $H$ acts indecomposably. By post-composing with a suitable element of $H$, we may assume that $\psi$ has a non-trivial fixed point $x \in P$. Now $\psi$ must fix all of $P$, since otherwise $C_P(\psi)$ is a non-trivial, proper $H$-invariant subgroup of $P$ which, by [3, §3, Theorem 3.2], contradicts the indecomposability of the action of $H$ on $P$.

We next drop the assumption that $H$ acts indecomposably. Decompose $P = P_1 \times \cdots \times P_n$ into indecomposable components. First note that the hypotheses of the lemma ensure that $\psi$ respects this decomposition. Now choose $x_i \in P_i \setminus \{1\}$, for $1 \leq i \leq n$. As above, we may assume that $\psi$ fixes $(x_1, \ldots, x_n) \in P$ and hence each $x_i$. Also as above, we must now have $\psi$ acting as the identity on each $P_i$ and therefore all of $P$.

For the general case, by the previous paragraph, we may assume that $\psi$ induces the trivial automorphism of $P/\Phi(P)$. Decompose $P = P_1 \times \cdots \times P_n$ into indecomposable components. Let $x_i \in P_i \setminus \Phi(P_i)$, for $1 \leq i \leq n$ and set $x := (x_1, \ldots, x_n) \in P$. Then

$$C_H(x \Phi(P)) = \bigcap_{i=1}^n C_H(x_i \Phi(P_i)) = \bigcap_{i=1}^n C_H(P_i/\Phi(P_i)) = \bigcap_{i=1}^n C_H(P_i) = \{1\},$$

where the second equality follows from part (4) and the third follows from part (2). Therefore, any $h \in H$ such that $\psi(x) = h(x)$ must be trivial. So $\psi$ is trivial on $(P_1 \setminus \Phi(P_1)) \times \cdots \times (P_n \setminus \Phi(P_n))$ and hence on all of $P$. 

\[ \Box \]
3 Blocks with trivial Picent

We set the following notation that will hold for the remainder of this section. Let $D$ be a finite $p$-group, $E$ a finite $p'$-group and $Z \leq E$ a central, cyclic subgroup such that we can identify $L := E/Z$ with a subgroup of $\text{Aut}(D)$. Through this identification we define $G := D \times E$ and $B := OG_{e_\varphi}$ for some fixed $\varphi \in \text{Irr}(Z)$. Since $D \triangleleft G$, any block idempotent of $OG$ is supported on $C_G(D) = Z(D) \times Z$. Therefore, $B$ is a block of $OG$ with defect group $D$. We set $C := D \times Z$. $T(B)$ will denote the subgroup of $\text{Pic}(B)$ consisting of bimodules that have trivial source when viewed as $OG \times G$-modules.

**Proposition 3.1.** Let $M \in T(B)$.

1. $M$ has vertex $\Delta \psi$ for some $\psi \in N_{\text{Aut}(D)}(L)$.

We denote by $\psi_L \in \text{Aut}(L)$ the automorphism of $L$ induced by conjugation by $\psi$.

2. $M \cong \alpha B$, where $\alpha \in \text{Aut}_G(B)$ satisfies $\alpha(xe_\varphi) = \psi(x)e_\varphi$, for all $x \in D$ and $\alpha(OgZe_\varphi) = O\psi_L(gZ)e_\varphi$, for all $g \in E$.

3. If $M$ has vertex $\Delta D$, then $M$ is just given by tensoring with some linear character $\lambda \in \text{Irr}(G|1_C)$. Moreover, different $\lambda$'s give non-isomorphic $M$'s.

**Proof.**

1. By [1, Theorem 1.1(i)], $M$ has vertex $\Delta \psi$ for some $\psi \in \text{Aut}(D,F)$, where $F$ is the fusion system on $D$ determined by $B$ and $\text{Aut}(D,F)$ is the subgroup of $\text{Aut}(D)$ which stabilises $F$. In particular, $\psi \in N_{\text{Aut}(D)}(L)$.

2. Since $M$ has vertex $\Delta \psi$ and $OG \uparrow_{D\psi}^{D \times D} \cong \psi(O\mathcal{D})$ is indecomposable, $M$ is a direct summand of $\psi(O\mathcal{D}) \uparrow_{D \psi}^{G \times G}$. Now let $(g,h) \in E \times E$. Then $(g,h)(\psi(O\mathcal{D}))$ has vertex

$$
(g,h)(\Delta \psi) = \{(gx,h \psi(x)|x \in D} = \{(x,h \psi(g^{-1}x)|x \in D
$$

$$
= \Delta(hZ \circ \psi \circ g^{-1}Z),
$$

where we are viewing $gZ, hZ \in \text{Aut}(D)$. Now if $hZ = \psi_L(gZ)$, then

$$
hZ \circ \psi \circ g^{-1}Z = \psi_L(gZ) \circ \psi \circ g^{-1}Z = \psi \circ gZ \circ \psi^{-1} \circ \psi \circ g^{-1}Z = \psi.
$$

Therefore, $(g,h)(\Delta \psi) = \Delta \psi$ and $S_\psi \leq I_{G \times G}(\psi(O\mathcal{D}))$, where $S_\psi \leq G \times G$ is generated by $D \times D$ and

$$
E_\psi := \{(g,h) \in E \times E| \psi_L(gZ) = hZ)
$$

If $I_{G \times G}(\psi(O\mathcal{D}))$ properly contains $S_\psi$, then there exists some $(g,1) \in I_{G \times G}(\psi(O\mathcal{D}))$, with $g \in E \setminus Z$. Since $p \nmid |L|$, $gZ$ has order prime to $p$ and so
cannot be an inner automorphism of $D$. Suppose $(x,y)(\Delta \psi) = \Delta(\psi \circ g^{-1}Z)$, for some $(x,y) \in D \times D$. Then, as in (\ref{lemma2}), $c_y \circ \psi \circ c_{x^{-1}} = \psi \circ g^{-1}Z$. Therefore,
\begin{equation}
\alpha \psi^{-1}(y) = g^{-1} \circ c_y \circ \psi = g^{-1}Z \circ c_x,
\end{equation}
a contradiction as $gZ$ is not inner, and so $S_\psi \equiv \I_G\psi(\psi(OD))$. Furthermore, $\psi(OD)$ must extend to an $\mathcal{OS}_\psi$-module since if it didn’t, then
\begin{equation}
\text{rk}_\mathcal{O}(M) > \text{rk}_\mathcal{O}(\psi(OD)), [G \times G : S_\psi] = |D||L| = \text{rk}_\mathcal{O}(B),
\end{equation}
contradicting \cite{lem2}. Therefore $M \cong N \uparrow_{S_\psi}^{G \times G}$, where $N \downarrow_{D \times D}^{S_\psi} \cong \psi(OD)$. Moreover, since $\psi M \psi = M$, $N \downarrow_{C \times C}^{S_\psi} \cong \psi(OD) \otimes \mathcal{O}Ze_\varphi$. Now
\begin{equation}
(\psi(KD) \otimes K Ze_\varphi) \uparrow_{C \times C}^{S_\psi} e_{1_D} \cong (\psi(KD)e_{1_D} \otimes K Z e_\varphi) \uparrow_{C \times C}^{S_\psi} \cong \bigoplus_{\chi \in \text{Irr}(E_\varphi | \varphi \otimes \varphi^-)} K \mathcal{S}_\psi e_\chi',
\end{equation}
where $\chi' := \text{Inf}_{E_\varphi}^{S_\psi}(\chi)$, the inflation of $\chi$ to $S_\psi$. Since $N$ is a direct summand of $(\psi(OD) \otimes \mathcal{O}Z e_\varphi) \uparrow_{C \times C}^{S_\psi}$ and $\dim_K((K \otimes \mathcal{O} N)e_{1_D}) = 1$, we have that $(K \otimes \mathcal{O} N)e_{1_D} \cong K \mathcal{S}_\psi e_\chi$ as $K \mathcal{S}_\psi$-modules, for some linear character $\chi \in \text{Irr}(E_\varphi | \varphi \otimes \varphi^-)$ and, for each such $\chi$, up to isomorphism, there is at most one extension $N_\chi$ of $\psi(OD) \otimes \mathcal{O}Z e_\varphi$ to an $\mathcal{OS}_\psi$-module with $(K \otimes \mathcal{O} N_\chi)e_{1_D} \cong K \mathcal{S}_\psi e_\chi$. Until further notice we fix the appropriate linear $\chi \in \text{Irr}(E_\varphi | \varphi \otimes \varphi^-)$ such that $N \cong N_\chi$ as $\mathcal{OS}_\psi$-modules.

Since $p \nmid |E_\varphi|$, $\chi(g,h) \in \mathcal{O}$, for all $(g,h) \in E_\varphi$. We define $\alpha \in \text{Aut}_\mathcal{O}(B)$ via $\alpha(xe_\varphi) = \psi(x)e_\varphi$, for all $x \in D$ and $\alpha(ge_\varphi) = \chi(g,h)he_\varphi$, for all $(g,h) \in E_\varphi$. One can readily check that $\alpha$ is a well-defined $\mathcal{O}$-algebra automorphism of $B$. We demonstrate the most difficult condition:
\begin{align*}
\alpha(ge_\varphi)\alpha(xe_\varphi)\alpha(g^{-1}e_\varphi) &= h(\psi(x)h^{-1}e_\varphi) = hZ(\psi(x))e_\varphi = \psi_L(gZ)(\psi(x))e_\varphi \\
&= (\psi \circ gZ \circ \varphi^{-1})(\psi(x))e_\varphi = \psi(gZ(x))e_\varphi = \alpha(gxg^{-1}e_\varphi),
\end{align*}
where we view $gZ, hZ \in \text{Aut}(D)$ and the third equality follows from the fact that $(g,h) \in E_\psi$. We claim that $N \uparrow_{S_\psi}^{G \times G} \equiv \alpha B$.

Set $G_\psi := \Delta \psi \rtimes E_\psi$ and $\mathcal{O}_\alpha$ the $\mathcal{O}G_\psi$-module $\mathcal{O}$ affording Inf_{E_\psi}^{G_\psi}(\chi)$. Since $(e_\psi)_B \subseteq \alpha B$ affords Inf_{E_\psi}^{G_\psi}(\chi), $\alpha B \cong \mathcal{O}_\alpha \uparrow_{G_\psi}^{G \times G}$. Next we consider the $\mathcal{OS}_\psi$-module $\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}$. Now
\begin{align*}
\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi} \cong \mathcal{O}_\alpha \uparrow_{G \times D}^{G \times G} \cong \mathcal{O}_\Delta \uparrow_{\Delta \psi}^{\Delta \psi} \cong \mathcal{O}_\Delta \uparrow_{\Delta \psi}^{\Delta \psi} \equiv \psi(OD).
\end{align*}
Therefore, $\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}$ is an extension of $\psi(OD) \otimes \mathcal{O}Z e_\varphi$ to $\mathcal{OS}_\psi$. In par-
In other words, or equivalently \( \chi \) defect group and inertial quotient do not change when we move from position 3.1. Note that in [6, Theorem 6.14.1], the isomorphism classes of the \( \alpha \) groups. Let \( \psi \) be as in Theorem 3.2. (3. What was proved in part (2) was essentially that any \( M \in T(B) \) is uniquely determined by its vertex, say \( \Delta \psi \), and the linear character \( \chi \in \text{Irr}(E\psi|\varphi \otimes \varphi^{-1}) \) afforded by \( (K \otimes N)_{1_D} \), where \( N \) is an extension of \( \psi(OD) \otimes_\mathcal{O} O Ze_\varphi \) to an \( OS_\varphi \)-module and \( M \cong N \uparrow_{S_\varphi}^{G \times G} \). If \( M \in T(B) \) induces the equivalence given by tensoring with \( \lambda \), a linear character in \( \text{Irr}(G|1_C) \), then \( \psi \) can be taken to be \( \text{Id}_D \) and \( \alpha \) to be given by \( \alpha(g)e_\varphi = \lambda(g)g e_\varphi \), for all \( g \in G \). The corresponding \( \chi \) is then given by \( \chi(g,h) = \lambda(g)\varphi(gh^{-1}) \), for all \( (g,h) \in E_{1_D} = (Z \times Z)(\Delta E) \). Since \( \chi \) is a linear character in \( \text{Irr}(E_{1_D}|\varphi \otimes \varphi^{-1}) \), in fact every \( \chi \) is of this form for some \( \lambda \in \text{Irr}(G|1_C) \). Therefore, every \( M \in T(B) \), with vertex \( \Delta D \), is given by tensoring with some linear character \( \lambda \in \text{Irr}(G|1_C) \).

Next suppose \( M \) is given by tensoring with some linear character \( \lambda \in \text{Irr}(G|1_C) \). As shown above \( M \) has vertex \( \Delta D \). Let \( N \) be the extension of \( OD \otimes_\mathcal{O} O Ze_\varphi \) to an \( OS_\varphi \)-module from the proof of part (2). Since \( S_{1_D} = I_{G \times G}(OD) \) and \( M \cong N \uparrow_{S_{1_D}}^{G \times G} \), \( N \) is in fact the unique summand of \( M \uparrow_{S_{1_D}}^{G \times G} \) that extends \( OD \otimes_\mathcal{O} O Ze_\varphi \). Therefore, once we’ve fixed \( \Delta D \) as a vertex, \( \chi \) is uniquely determined by \( M \). Finally, by the previous paragraph, \( \lambda \) is uniquely determined by \( \chi \).

\[ 3. \text{ What was proved in part (2) was essentially that any } M \in T(B) \text{ is uniquely determined by its vertex, say } \Delta \psi, \text{ and the linear character } \chi \in \text{Irr}(E\psi|\varphi \otimes \varphi^{-1}) \text{ afforded by } (K \otimes N)_{1_D}, \text{ where } N \text{ is an extension of } \psi(OD) \otimes_\mathcal{O} O Ze_\varphi \text{ to an } OS_\varphi \text{-module and } M \cong N \uparrow_{S_\varphi}^{G \times G}. \text{ If } M \in T(B) \text{ induces the equivalence given by tensoring with } \lambda, \text{ a linear character in } \text{Irr}(G|1_C), \text{ then } \psi \text{ can be taken to be } \text{Id}_D \text{ and } \alpha \text{ to be given by } \alpha(g)e_\varphi = \lambda(g)g e_\varphi, \text{ for all } g \in G. \text{ The corresponding } \chi \text{ is then given by } \chi(g,h) = \lambda(g)\varphi(gh^{-1}), \text{ for all } (g,h) \in E_{1_D} = (Z \times Z)(\Delta E). \text{ Since } \chi \text{ is a linear character in } \text{Irr}(E_{1_D}|\varphi \otimes \varphi^{-1}), \text{ in fact every } \chi \text{ is of this form for some } \lambda \in \text{Irr}(G|1_C). \text{ Therefore, every } M \in T(B), \text{ with vertex } \Delta D, \text{ is given by tensoring with some linear character } \lambda \in \text{Irr}(G|1_C). \text{ Next suppose } M \text{ is given by tensoring with some linear character } \lambda \in \text{Irr}(G|1_C). \text{ As shown above } M \text{ has vertex } \Delta D. \text{ Let } N \text{ be the extension of } OD \otimes_\mathcal{O} O Ze_\varphi \text{ to an } OS_\varphi \text{-module from the proof of part (2). Since } S_{1_D} = I_{G \times G}(OD) \text{ and } M \cong N \uparrow_{S_{1_D}}^{G \times G}, \text{ } N \text{ is in fact the unique summand of } M \uparrow_{S_{1_D}}^{G \times G} \text{ that extends } OD \otimes_\mathcal{O} O Ze_\varphi. \text{ Therefore, once we’ve fixed } \Delta D \text{ as a vertex, } \chi \text{ is uniquely determined by } M. \text{ Finally, by the previous paragraph, } \lambda \text{ is uniquely determined by } \chi. \]

\[ \square \]

**Theorem 3.2.** Let \( b \) be a block with normal abelian defect group and inertial quotient a product of elementary abelian groups. Then Picent(\( b \)) is trivial.

**Proof.** By [5, Theorem A] (see also [6, Theorem 6.14.1] for a more detailed description), we may assume \( b \) is of the form of \( B \) as described just before Proposition 3.1. Note that in [6, Theorem 6.14.1], the isomorphism classes of the defect group and inertial quotient do not change when we move from \( b \) to \( B \). In other words \( D \) is abelian and \( L = E/Z \) is a product of elementary abelian groups. Let \( M \in \text{Picent}(B) \). By [2, Propositions 4.3,4.4], \( M \in T(B) \) and so we let \( \psi \) and \( \alpha \) be as in Proposition 3.1.

We will use \( \psi^* \) to denote the self-bijection of \( \text{Irr}(D) \) given by \( \psi^*(\chi)(x) = \chi(\psi^{-1}(x)), \text{ for all } x \in D \text{ and } \chi \in \text{Irr}(D) \). For any \( C \leq H \leq G \), we denote by \( \alpha H \) the unique subgroup \( C \leq \alpha H \leq G \) such that \( O(\alpha H)e_\varphi = \alpha(OHe_\varphi) \) or equivalently \( \alpha H/C = \psi_L(H/C) \). For any such \( H \), we will use \( \alpha^* \) to denote
the bijection Irr\((H|\varphi) \rightarrow \text{Irr}^{a}(H|\varphi)\) given by \(\alpha^{*}(\chi)(x) = \chi(\alpha^{-1}(x))\), for all \(x \in aH\) and \(\chi \in \text{Irr}(H|\varphi)\). Since \(M \in \text{Picent}(B)\), when \(H = G\), \(\alpha^{*}\) is just the identity on \(\text{Irr}(B)\). Note that, since \(\alpha\) permutes the left cosets of \(C\) in \(G\), \(\alpha^{*}(\chi) \uparrow^{G}_{H} = \alpha^{*}(\chi \uparrow^{G}_{H})\), for any \(C \leq H \leq G\) and \(\chi \in \text{Irr}(H|\varphi)\).

Decompose \(D = D_{1} \times \cdots \times D_{n}\) into indecomposable components with respect to the action of \(L\). Let \(\theta_{i} \in \text{Irr}(D_{i})\setminus\{1_{D_{i}}\}\), for each \(1 \leq i \leq n\) and set \(\theta := \theta_{1} \otimes \cdots \otimes \theta_{n} \in \text{Irr}(D)\). Since \(\alpha^{*}(\theta \otimes \varphi) \uparrow^{G}_{C} = \alpha^{*}(\theta \otimes \varphi) \uparrow^{G}_{C} = (\theta \otimes \varphi) \uparrow^{G}_{C}\), \(\alpha^{*}(\theta \otimes \varphi) = \psi^{*}(\theta) \otimes \varphi\) must be conjugate to \(\theta \otimes \varphi\) via an element of \(E\). Therefore, by composing \(\psi\) and \(\alpha\) with an appropriately chosen \(c_{g}\), we may assume that \(\psi^{*}(\theta) = \theta\).

By an abuse of notation, we view each \(\theta_{i} \in \text{Irr}(D)\) by letting it act trivially on all the other \(D_{j}\)’s. An identical argument to the previous paragraph applied to each \(\theta_{i} \otimes \varphi \in \text{Irr}(C)\), for \(1 \leq i \leq n\), and the fact that \(\psi^{*}\) already fixes \(\theta\), gives that each \(\theta_{i}\) is also fixed by \(\psi^{*}\). In addition, by parts (3) and (4) of Lemma \ref{Lemma2.3} \(I_{L}(\theta_{i}) = C_{L}(D_{i})\) and \(I_{E}(\theta_{i}) = C_{E}(D_{i})\).

Until further notice we fix a prime \(\ell \mid |L|\). Since \(L\) is a product of elementary abelian groups, \(O_{\ell}(L) \cong (C_{\ell})^{t}\), for some \(t \in \mathbb{N}\). Therefore, by part (4) of Proposition \ref{Proposition2.3} for each \(1 \leq i \leq n\),

\[
O_{\ell}(L/I_{L}(\theta_{i})) = O_{\ell}(L/C_{L}(D_{i})) = \{1\} \text{ or } C_{\ell}.
\]

Also, since we’re identifying \(L\) with a subgroup of \(\text{Aut}(D)\),

\[
\bigcap_{i=1}^{n} O_{\ell}(I_{L}(\theta_{i})) = \bigcap_{i=1}^{n} O_{\ell}(C_{L}(D_{i})) = O_{\ell}(C_{L}(D)) = \{1\}.
\]

Therefore, viewing \(O_{\ell}(L)\) as an \(\mathbb{F}_{\ell}\)-vector space, by Lemma \ref{Lemma2.1} there exist \(1 \leq i_{1}, \ldots, i_{t} \leq n\) such that \(\bigcap_{j=1}^{t} O_{\ell}(I_{L}(\theta_{i_{j}})) = \{1\}\). In addition, setting

\[
\vartheta_{m} := \left( \bigotimes_{j=1}^{t} \theta_{i_{j}} \right) \otimes \left( \bigotimes_{j=1}^{n} 1_{D_{j}} \right) \otimes 1_{D_{m}} \in \text{Irr}(D),
\]

for each \(1 \leq m \leq t\), Lemma \ref{Lemma2.1} also gives that

\[
C_{m} := O_{\ell}(I_{L}(\vartheta_{m})) = \bigcap_{j=1}^{t} O_{\ell}(I_{L}(\theta_{i_{j}})) \cong C_{\ell},
\]

and that the \(C_{m}\)’s generate \(O_{\ell}(L)\).

For any subgroup \(H\) of \(L\) we denote by \(\hat{H}\) its preimage in \(E\). Until further notice we fix some \(1 \leq m \leq t\) and set \(I_{m} := I_{L}(\vartheta_{m} \otimes \varphi)\). Since \(\psi^{*}(\theta_{i}) = \theta_{i}\),
for all $1 \leq i \leq n$, $\psi^*(\vartheta_m) = \vartheta_m$. Therefore, $\psi_L(I_m) = I_m$ and $\psi_L(C_m) = C_m$, implying $\alpha(D \rtimes \hat{I}_m) = D \rtimes \hat{I}_m$ and $\alpha(D \rtimes \hat{C}_m) = D \rtimes \hat{C}_m$.

By [4 Theorem 6.11(b)], $\mathrm{Irr}(B|\vartheta_m \otimes \varphi) = \mathrm{Irr}(G|\vartheta_m \otimes \varphi)$ is in one-to-one correspondence, via induction, with $\mathrm{Irr}(I_G(\vartheta_m \otimes \varphi)|\vartheta_m \otimes \varphi) = \mathrm{Irr}(D \rtimes \hat{I}_m|\vartheta_m \otimes \varphi)$. Therefore, since $\alpha(D \rtimes \hat{I}_m) = D \rtimes \hat{I}_m$ and $\alpha^*$ respects induction, $\alpha^*$ must be the identity on $\mathrm{Irr}(D \rtimes \hat{I}_m|\vartheta_m \otimes \varphi)$. By Lemma 2.2, $\vartheta_m \otimes \varphi$ extends to $D \rtimes \hat{C}_m$ and every extension to $D \rtimes \hat{C}_m$ is stable in $D \rtimes \hat{I}_m$. In other words, every character in $\mathrm{Irr}(D \rtimes \hat{I}_m|\vartheta_m \otimes \varphi)$ lies above a unique character in $\mathrm{Irr}(D \rtimes \hat{C}_m|\vartheta_m \otimes \varphi)$. Therefore, again since $\alpha^*$ respects induction, $\alpha^*$ is also the identity on $\mathrm{Irr}(D \rtimes \hat{C}_m|\vartheta_m \otimes \varphi)$. Now every character in $\mathrm{Irr}(D \rtimes \hat{C}_m|\vartheta_m \otimes \varphi)$ is linear and is therefore determined by its restriction to $\hat{C}_m$. Hence, $\alpha$ is the identity on $\mathcal{Z}(\mathcal{O}\hat{C}_m e_\varphi) = \mathcal{O}\hat{C}_m e_\varphi$.

Since the $C_m$’s generate $O_L(L)$ and the $O_L(L)$’s generate $L$ as $\ell$ runs over all primes dividing $|L|$, $\alpha$ is the identity on $\mathcal{O}E e_\varphi$. In particular, $\psi_L$ is the identity. As noted earlier in this proof, for every $\chi \in \mathrm{Irr}(D)$, $\psi^*(\chi)$ is conjugate to $\chi$ via an element of $E$. Therefore, by part (5) of Lemma 2.3, $\psi^*$ is induced by an element of $E$ and so, by composing $\psi$ and $\alpha$ with an appropriately chosen $c_g$, we may assume that $\psi$ is the identity.

We now repeat the entire proof with $\psi$ being the identity until we’ve reproved that $\alpha$ is the identity on $\mathcal{O}E e_\varphi$. Therefore, $\alpha$ is the identity and $M$ is the trivial Morita auto-equivalence.

**Theorem 3.3.** Let $b$ be a block with normal abelian defect group and either cyclic inertial quotient or a principal block with abelian inertial quotient. Then $\text{Picent}(b)$ is trivial.

**Proof.** The proofs for both situations proceed similarly to that of Theorem 3.2. We first note that if $b$ is a principal block of a group $H$ with normal defect group $P$, then $P$ must be a Sylow $p$-subgroup of $H$. Therefore, by the Schur-Zassenhaus theorem, $H = P \rtimes F$, for some $p'$-subgroup $F \leq H$. Then, $C_F(P)$ must be in the kernel of $b$ and so, by factoring out by $C_F(P)$ and considering the relevant Morita equivalent block, we may assume that $b$ is already of the form of $B$ with $Z$ trivial. In other words, the block $B$ we reduce to is also principal. In particular, in both instances of the theorem, $E$ is abelian.

Now replace $\vartheta_m \in \mathrm{Irr}(D)$, in the proof of Theorem 3.2 with $1_D$. Proceeding as in that proof and noting that, since $E$ is abelian, every character in $\mathrm{Irr}(E|\varphi)$ is linear, we prove that $\alpha$ is the identity on $\mathcal{O}E e_\varphi$. We then continue to prove that $M$ is the trivial Morita auto-equivalence exactly as before.  


4 Examples of non-trivial Picent

Picard groups have not been calculated for any Morita equivalence class of blocks with non-abelian defect group, excluding nilpotent blocks but, as our first example shows, just taking the principal $p$-block of a suitably chosen $p$-group already yields an example of a block with non-trivial Picent.

We first set up some notation. A class-preserving automorphism of a group $H$ is an automorphism that leaves invariant every conjugacy class of $H$. Equivalently it leaves invariant every irreducible character of $H$. We define

$$\text{Aut}_c(H) = \{ \alpha \in \text{Aut}(H) | \alpha \text{ is class-preserving} \},$$

$$\text{Inn}(H) = \{ c_h \in \text{Aut}(H) | h \in G \},$$

$$\text{Out}(H) = \text{Aut}(H) / \text{Inn}(H) \text{ and } \text{Out}_c(H) = \text{Aut}_c(H) / \text{Inn}(H).$$

**Example 4.1.** Let $P$ be a finite $p$-group. We know, by [9], that $\text{Pic}(O_P) \cong \text{Hom}(P, O^\times) \rtimes \text{Out}(P)$. Since any element of Picent$(O_P)$ must fix the trivial character, Picent$(O_P) \cong \text{Out}_c(P)$ and finding a non-inner, class-preserving automorphism of $P$ immediately yields a non-trivial element of Picent$(O_P)$.

Many $p$-groups with such an automorphism have been constructed, [11] lists some of these. The smallest $p$-group $P$ with $\text{Out}_c(P) \neq \{1\}$ has order 32 and is described in [10].

For blocks with abelian defect group the situation is different. As $\text{Out}_c(P)$ is trivial, for any abelian $p$-group $P$, $p$-groups will no longer provide examples with non-trivial Picent. In addition, in all the cases the Picard group of a block with abelian defect group has been computed (namely [1] and [2]) Picent has been trivial. However, the families of blocks that we exhibit show explicitly that Picent of a block with abelian defect group is not trivial, in general. The first family we are going to describe yields a counterexample in the same spirit as Example 4.1, the non-trivial element of Picent$(B)$ is given by an outer automorphism of the relevant group.

Before stating the result we set up some notation. Let $t > 1$ be an integer coprime to $p$. For any $i \in \mathbb{N}$, we denote by $\omega_i$ a primitive $i^{th}$ root of unity and take $n$ to be the smallest positive integer such that $\omega_i \in F_{p^n}$. In other words, $n$ is the multiplicative order of $p$ mod $t^2$. Set $s := (p^{nt} - 1)/(t(p^n - 1))$ and $D := (C_p)^{nt}$ that we identify with $F_{p^{nt}}$. Certainly $s$ is coprime to $p$ and

$$(p^{nt} - 1)/(p^n - 1) = p^{n(t-1)} + \ldots + p^n + 1.$$  

Therefore, since each $p^{nt} \equiv 1$ mod $t^2$, $s \equiv 1$ mod $t$. In particular, $s$ and $t$ are coprime. For any $\lambda \in F_{p^{nt}}^\times$, we denote by $m_\lambda$ the group automorphism of $F_{p^{nt}}$ given by multiplication by $\lambda$ and we denote by $\Phi_{p^n}$ the automorphism of $F_{p^n}$ given by $(x \mapsto x^p)$.
We set $G := D \times E$, where $E := \langle (g) \times (h) \rangle$, $\langle g \rangle = \langle m_{\omega_i} \rangle \cong C_2$ and $\langle h \rangle = \langle \Phi_p^n \circ m_{\omega_j^2} \rangle \cong C_{t^2}$. Since $hg = gp^s$, $E$ is well-defined. What’s more, since $s$ and $t^2$ are coprime, $E/C_E(D)$ has order $st^2$. In other words, $E$ acts faithfully on $D$ and $OG$ is a block.

**Proposition 4.2.** With the above notation, $Out_c(G)$ is non-trivial. Moreover, the image of $Out_c(G)$ in $Picent(OG)$ is non-trivial.

**Proof.** We define $\psi = m_{\omega_j^2} \in Aut(D)$. This extends to an automorphism $\psi_G \in Aut(G)$ that acts trivially on $E$. We claim that $\psi_G$ gives a non-trivial element of $Out_c(G)$.

Note that $\psi \in Aut(D)$ is not already in $E$. Suppose it were, then $\psi^{-1} \circ h = \Phi_p^n \in E$ is an element of order $t$. By considering their images in $E/\langle g \rangle \cong C_{t^2}$, we see that all the elements of order $t$ in $E$ are in $\langle g \rangle \times \langle h^t \rangle$ and hence powers of $h^t = m_{\omega_j^2}$. However, $\Phi_p^n$ is not of this form as it has non-trivial fixed points on $D$.

We now show that $\psi$ preserves the $E$-conjugacy classes of $D$, that is for each $x \in D$, there exists $y \in E$ such that $\psi(x) = yx$.

For $x \in \mathbb{F}_{p^t}^\times$, $0 \leq i < s$ and $0 \leq j < t$,

$$c_g^i \circ c_h^{1+jt}(x) = \psi(x) \iff \omega_s^i \omega_{t^2}^{1+jt} x p^{n(1+jt)} = \omega_{t^2} x \iff x p^{t-1} = \omega_{t^2}^{1+jt} \omega_s^{-i}.$$

Thus, since $x p^{t-1}$ is a (not necessarily primitive) $(st)^{\text{th}}$ root of unity, there exist unique $i, j$ such that $c_g^i \circ c_h^{1+jt}(x) = \psi(x)$.

Next we prove that for all $x \in D \setminus \{1\}$, $C_E(x) = \{1\}$. This is crucial, since it will give us a description of the irreducible characters of the group $G$.

We first note that all the non-trivial elements of $\langle g \rangle \times \langle h^t \rangle = \langle m_{\omega_j^s} \rangle$ have no non-trivial fixed points on $D$. If $y \in E \setminus \langle g \rangle$, then, by considering its image in $E/\langle g \rangle \cong C_{t^2}$, there exists an integer $l$ such that $y^l \in (\langle g \rangle \times \langle h^t \rangle) \setminus \{1\}$. If $y$ has non-trivial fixed points on $D$ then $y^l$ would have non-trivial fixed points as well, thus $y$ can’t have any non-trivial fixed points. In other words, for all $x \in D \setminus \{1\}$, $C_E(x) = \{1\}$.

We now show that the analogous property holds, when we replace $D$ with $Irr(D)$. Let $\theta \in D \setminus \{1\}$ and decompose $D = D_1 \times \cdots \times D_m$ with respect to the action of $\langle g \rangle$. Let $\theta_i \in Irr(D_i)$, for each $1 \leq i \leq m$, and set $\theta := \theta_1 \otimes \cdots \otimes \theta_m \in Irr(D)$. If $\theta \neq 1_D$, say $\theta_j \neq 1_{D_j}$, for some $1 \leq j \leq m$, then

$$I_{\langle g \rangle}(\theta) \subseteq I_{\langle g \rangle}(\theta_j) = C_{\langle g \rangle}(D_j) = \{1\},$$

where the first equality follows from parts (3) and (4) of Lemma 2.3 and the final one from the previous paragraph. In other words, the action of $E$ on $Irr(D)$
is fixed-point-free as well. Thus all irreducible character of $G$ are either irreducible characters with $D$ in their kernel or irreducible characters induced from non-trivial characters of $D$. Characters of $G$ with $D$ in their kernel correspond to characters of $G/D$, that are obviously fixed by $\psi_G$, while the characters of $G$ induced from $D$ are characterised by their values on $G$-conjugacy classes of elements of $D$ but $\psi$ preserves the $E$-conjugacy classes of elements of $D$. Therefore, every character of $G$ is $\psi_G$-stable and so $\psi_G \in \text{Aut}_c(G)$.

Note that $\psi_G$ is a non-inner group automorphism of $G$, since we have shown that $\psi$ is not induced by an element of $E$. Finally, we note that, since $\psi_G(OG) \cong \mathcal{O}_{\Delta \psi_G} \Delta_G \times G$, $\psi_G(OG)$ has vertex $\Delta \psi$. Similarly $OG$ has vertex $\Delta D$. Since $\psi$ is not induced by an element of $E$, $\Delta \psi$ is not conjugate to $\Delta D$ in $G$. Therefore, $\psi_G(OG)$ is a non-trivial element of $\text{Picent}(OG)$.

We now exhibit another family of blocks with non-trivial Picent. The main difference with the previous family is that, as well as having abelian defect group, these blocks also have abelian inertial quotient and diagonal vertex. In fact, the non-trivial element of $\text{Picent}(B)$ we exhibit is given by multiplication by a linear character of the inertial quotient. Note that, by Theorem 3.3 these blocks are necessarily non-principal.

Let $\ell$ be a prime number different from $p$. Take $$E := ((z) \times \langle g \rangle) \rtimes \langle h \rangle \cong (C_\ell \times C_{\ell^2}) \rtimes C_\ell,$$
defined by $hz = zh$, $h^g = gz$ and set $F := \langle z \rangle \times \langle g^\ell \rangle \times \langle h \rangle \leq E$. Furthermore, let $n$ be the multiplicative order of $p \mod \ell^2$ and set $D := (C_p)^n \times (C_p)^n$ := $D_1 \times D_2$ that we identify with $\mathbb{F}_{p^n} \oplus \mathbb{F}_{p^n}$.

We define $G := D \rtimes E$, where the action of $E$ on $D_1$ has kernel $\langle z \rangle \times \langle h \rangle$, while on $D_2$ the kernel is $\langle z \rangle \times \langle g^\ell h \rangle$ and $g$ acts on both components by multiplication by $\omega \in \mathbb{F}_{p^n}^\times$, a primitive $\ell^2$-th root of of unity. Since $\omega \notin \mathbb{F}_{p^n}$ for any $m < n$, no proper $\mathbb{F}_p$-subspace of $\mathbb{F}_{p^n}$ has an $\mathbb{F}_p$-linear map with eigenvalue $\omega$. In particular, the actions of $E$ on both $D_1$ and $D_2$ are indecomposable. Of course, $C_E(D) = Z := \langle z \rangle$. Let $\varphi \in \text{Irr}(Z) \setminus \{1_Z\}$ and set $B := \mathcal{O}G_{\varphi}$.

**Proposition 4.3.** With the notation above, tensoring by any non-trivial $\lambda \in \text{Irr}(G|1_{D \times F})$ yields a non-trivial element of $\text{Picent}(B)$.

**Proof.** Let $\lambda \in \text{Irr}(G|1_{D \times F})$. Every character in $\text{Irr}(B|1_D)$ is of the form $\text{Inf}^G_E(\chi)$, for some $\chi \in \text{Irr}(E|\varphi)$. Then, by [R Lemma 3.1(2)], we have $\chi^{\otimes m} = \eta \uparrow_{Z(E)}^E$, for some $m \in \mathbb{N}$ and $\eta \in \text{Irr}(Z(E)|\varphi)$. Note that $Z(E) = Z \times \langle g^\ell \rangle \leq F$. Therefore,

$$\lambda \downarrow_{E}^G(\eta \uparrow_{Z(E)}^E) = (\lambda \downarrow_{Z(E)}^G \cdot \eta) \uparrow_{Z(E)}^E = \eta \uparrow_{Z(E)}^E,$$
giving that $(\lambda \downarrow_{E}^G) \chi = \chi$ and finally that $\lambda . \text{Inf}^G_E(\chi) = \text{Inf}^G_E(\chi)$.
Now take $\chi \in \text{Irr}(B|\theta)$, for some $\theta = \theta_1 \otimes \theta_2 \in \text{Irr}(D)$, where $\theta_i \in \text{Irr}(D_i) \backslash \{1_{D_i}\}$, for $i = 1, 2$. By part (3) and (4) of Lemma 2.3, $\mathcal{I}_E(\theta_i) = C_E(D_i)$, for $i = 1, 2$.

Therefore, $\mathcal{I}_E(\theta_1 \otimes \theta_2) = C_E(D_1) \cap C_E(D_2) = Z$ and so $\chi = (\theta \otimes \varphi)^G_D \times Z$.

Hence,

$$\lambda.\chi = \lambda.((\theta \otimes \varphi)^G_D \times Z) = ((\lambda \downarrow^G_D \times Z).(\theta \otimes \varphi))^G_D \times Z = (\theta \otimes \varphi)^G_D \times Z = \chi.$$ 

Finally, let $\chi \in \text{Irr}(B|\theta)$, where $\theta = \theta_1 \otimes 1_{D_2} \in \text{Irr}(D)$, for some $\theta_1 \in \text{Irr}(D_1) \backslash \{1_{D_1}\}$.

This time $\mathcal{I}_E(\theta) = \langle z \rangle \times \langle h \rangle \leq F$ and the argument concludes similarly to the previous paragraph. Of course, the case of $\theta = 1_{D_1} \otimes \theta_2$, for some $\theta_2 \in \text{Irr}(D_2) \backslash \{1_{D_2}\}$ is dealt with in an identical fashion.

We have proved that tensoring with $\lambda$ fixes $\text{Irr}(B)$ pointwise and thus defines an element of Picent$(B)$. The fact that non-trivial $\lambda$ induce non-trivial elements of Picent$(B)$ follows from part (3) of Proposition 3.1.

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