Pairing Instability on a Luttinger Surface: A Non-Fermi Liquid to Superconductor Transition and its Gravity Dual

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We show that fluctuation thermodynamics on a model Luttinger surface – a contour of zeros of the many-body Green function – mimics black hole thermodynamics in the strong coupling limit. At zero temperature ($\beta \rightarrow \infty$) and a critical interaction strength ($u_{c0}$) characterized by the self-energy pole, we find that the pair susceptibility diverges leading to a superconducting instability. We evaluate the pair fluctuation partition function and find that the spectral density in the normal state has an interaction-driven, power-law $1/\omega^2$ type, van-Hove singularity (vHS) indicating non-Fermi liquid (NFL) physics. Crucially, in the strong coupling limit ($\beta u \gg 1$), the free energy in the normal state of this NFL-SC transition resembles well studied class of models with gravity duals and takes the form $-\beta F = \beta u_{c0} - \gamma \ln(\beta u_{c0})$ where $\gamma$ is a constant equal to $\frac{1}{2}$. Weak impurity scattering ($\tau \gg \beta^{-1}$) leaves the low-energy spectral density unaffected, but leads to an interaction-driven enhancement of superconductivity. Our results shed ligth on the role played by order-parameter fluctuations in providing the key missing link between Mott physics and strongly coupled toy-models exhibiting gravity duals such as Sachdev-Ye-Kitaev (SYK)-type models.

INTRODUCTION

A central notion that captures the failure of single-particle physics in quantum matter is the Luttinger surface (LS) – a contour in momentum space where the many-body Green function, $G(p, \omega)$, vanishes [1]. This lies in contrast to the normal Fermi Liquid (FL) where particle excitations are characterized by poles in the single-particle propagator. The LS has been invoked to reconcile several key experimental observations [2, 7] and its apparent violation [11–22], pseudo-gap and Fermi arcs [15, 17, 23–26], spectral weight transfer [18, 23], as well as features in the self-energy, $\Sigma(p, \omega)$ [27].

A salient property of the LS which gives rise to the aforementioned observations is a divergent $\Sigma(p, \omega)$ [13, 15, 17, 27, 28]. The breakdown of the LSR – a rule which relates the density of electrons at fixed chemical potential to the number of excitations in the FL and whose generalizations were shown to hold in broader contexts [10, 12, 13, 29, 32] – serves as an illustrative example to highlight the consequences of a singular self-energy. While the total particle density equals the area enclosed by the surface of propagator-poles when $\Sigma(p, \omega)$ is regular, there is an anomalous contribution to the density, proportional to $I = \int G_{qq} \partial_{\omega} G_{qq} / \partial_{\omega}$, that averages to zero in a FL [1, 8, 9]. The integral $I$ counts the excess density in addition to the volume contained inside contours where $G(p, \omega)$ changes sign [1, 11, 12] and can, however, be non-vanishing when $\Sigma(p, \omega)$ diverges [17, 20, 33]. These many-body properties follow entirely from explicit electron-electron interactions in the problem.

Nevertheless, the normal state of a superconductor can exhibit anomalies that deviate from a FL even in the absence of explicit electron correlations. This class of phenomena originates from Cooper-pair fluctuations [31, 32] and lead to precursor effects wherein certain characteristics of the SC are retained for temperatures $T > T_c$, and in some cases, can even persist for $T > T_c$. With knowledge of the fluctuation propagator $L(q, \omega)$ – the fundamental object in the theory of pair fluctuations constructed from the ground state of the system for $T > T_c$ – various measurable quantities can be evaluated systematically and compared with experiment [31, 35]. Several observations such as paraconductivity, rounding of transverse resistance peak, excess tunneling current, pseudo-gapid behavior etc. [36, 37, 38, 39], for example, as well as [31, 35] (for a more detailed review) have been successfully understood via fluctuation physics derived from a free electron Green function. More generic models describing the thermodynamics of fluctuations in multi-band systems have also been examined in the context of MgB$_2$ [40].

In this work, we introduce interactions explicitly by analyzing pair fluctuations for a system with a LS formed by a pole in $\Sigma$. We find a quantum phase transition into the superconducting state at a critical interaction strength ($u_{c0}$) where the pair susceptibility diverges. By calculating the pair fluctuation propagator $L(q, \omega)$ and partition-function, we determine the spectral density in the normal state. We find an interaction-driven, power-law $1/\omega^2$ type, van-Hove singularity (vHS) at low energies that signals NFL physics. Hence pair fluctuations combined with LS physics describe a NFL-SC transition at $T = 0$. Crucially, in the strong coupling limit ($\beta u \gg 1$), the free energy resembles well studied models with gravity duals and takes the form $-\beta F = \beta u_{c0} - \gamma \ln(\beta u_{c0})$ where $\gamma = \frac{1}{2}$. Here $u$ is the interaction parameter and is equal to square-root of the residue of the self-energy pole. Moreover, we do not require random couplings or explicit long-range interactions for our conclusions to hold. In
the presence of weak impurity scattering, the low-energy spectral density is unaffected in the strong coupling limit and gives rise to an interaction-driven enhancement of superconductivity. Our results demonstrate that orderparameter fluctuations provide the key link between Mott physics and strongly coupled toy-models exhibiting gravity duals. Hence we conclude that fluctuation thermodynamics on a Luttinger surface mimics black hole thermodynamics in the strong coupling limit 41,42.

**MODEL**

LSs have been obtained in numerous models in manybody literature, both at a phenomenological level 15 as well as microscopic Hubbard-type 13,14,17–19,20,22,27,28,43 and holographic 44,45 models. Other models study emergent gauge fields in a FL that nevertheless violate the LSR 46–49. The simplest Green function that vanishes along contours in the Brillouin zone has a simple pole in the self-energy and is given by

\[ G(p, \epsilon_n) = \frac{i\epsilon_n - \xi(p)}{\Sigma(p, \epsilon_n) + \epsilon(p) - \mu} \]

where \( \xi(p) = \epsilon(p) - \mu \) is the bare dispersion with chemical potential \( \mu \) and \( \epsilon_n \) is the fermionic Matsubara frequency. We choose a self-energy ansatz motivated by the well-studied Yang-Rice-Zhang model (YRZ) 15 with a pole structure given by

\[ \Sigma(p, \epsilon_n) = V + \frac{u^2}{i\epsilon_n + \xi(p)}. \]

Here \( V \) is a constant potential and plays the role of the Hartree-Fock potential if the case of interest is a density-wave order 43. As evident from the choice of \( \Sigma \), the LS and the bare electron FS occur for the same momenta set by \( \mu \) at zero energy. This need not be the case in more generic systems where the self-energy can acquire multiple poles each with distinct residues. In the presence of impurities, a finite life-time \( \tau \) is introduced in the Green function.

**Strong coupling (\( \beta u \gg 1 \)) in clean limit (\( \tau \to \infty \)):** The fluctuation propagator can be evaluated from Bethe-Salpeter-type equations in the particle-particle channel (see Fig. 1) for momentum \( q \) and frequency \( \Omega \)

\[ L^{-1}(q, \Omega) = -g^{-1} + \Pi(q, \Omega) \]

where \(-g\) is a constant bare (attractive) interaction vertex and \( \Pi(q, \Omega) \) is the pair susceptibility. The latter is defined in \( d \)-dimensions as

\[ \Pi(q, \Omega) = \frac{1}{\beta(2\pi)^d} \sum_{n} \int d^d p \ G(p, q + \epsilon_n+k)G(-p, -\epsilon_n) \]

The latter is defined in \( d \)-dimensions as

\[ \Pi(q, \Omega) = \frac{1}{\beta(2\pi)^d} \sum_{n} \int d^d p \ G(p, q + \epsilon_n+k)G(-p, -\epsilon_n) \]

where we make the replacements \( \epsilon_1 \equiv \epsilon_{n} \to \epsilon_{n+k} \)

and \( \epsilon_2 \equiv \epsilon_{2n} \to -\epsilon_n \), and introduce primed notation \( \epsilon'_{jn} = \sqrt{\epsilon_j^2 + u^2} \). The angular brackets \( \langle \ldots \rangle \) denote angular average, and \( m \) and \( \phi \) are the bare electron mass and azimuth angle respectively. We also introduce the ratio \( r = \frac{\mu}{\phi} \ll u, \epsilon_f \), where \( \epsilon_f \) is the Fermi energy. To recover well-known expressions of the pair-susceptibility for a FL, one only needs to take the limit of \( u \to 0 \) (see Appendix B). Performing an expansion in the parameter \( r \) and taking the static limit, the inverse fluctuation propagator is

\[ L^{-1}(q, \Omega \to 0) \simeq -g^{-1} + \Pi^{(0)}(0,0) + \Pi^{(2)}(q,0), \]

where

\[ \Pi^{(0)}(0,0) = \frac{m}{4 \beta} (2S_3 - u^2 S_5), \quad \Pi^{(2)}(q,0) = -\frac{m^2}{32 \beta^2} (2S_3 - u^2 S_5) \]

and

\[ S_\nu = \frac{1}{\beta} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p_\nu} \frac{1}{\beta} \equiv \frac{1}{\beta} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p_\nu} \]

where \( S_\nu = \frac{1}{\beta} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p_\nu} \)

and

\[ S_\nu = \frac{1}{\beta} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p_\nu} \]

for \( \nu = 1 \)

for \( \nu = 3, 5, ... \)

Here \( \Lambda \) is the ultraviolet cut-off of the divergent Matsubara sum for \( \nu = 1 \) (plays the role of the Debye frequency...
Finally, one can evaluate the spectral density
\[ \rho(\omega) = \frac{1}{\Omega(\omega)} \]
for the inverse fluctuation propagator, and expanding the resulting expression in powers of \( e^{-\kappa} \) and its polynomial products, we obtain the final expression for \( L^{-1}(q, 0) \) in the clean limit
\[
L^{-1}(q, \Omega \to 0) \simeq -g^{-1} + N_0 \left[ \ln \frac{\Lambda}{u} + \frac{\pi K}{2} e^{-\kappa} \right] - \frac{N_0 \tau^2}{12 u^2} \left[ 1 + \frac{\pi K^4}{8} e^{-\kappa} \right]. \tag{6}
\]
Here \( N_0 \) is the density of states at the Fermi level in two dimensions. Note that the above expression cannot be adiabatically connected to the FL result \([54]\) any longer as it is valid only in the strong coupling limit. There are several conclusions that can be drawn from the structure of the fluctuation propagator above. First, a divergence of the zero frequency, long-wavelength limit of the propagator signals a superconducting instability. At \( \beta = \infty \) and constant \( \Lambda \), this condition is achieved at the quantum critical point \( u = u_{\infty} = \Lambda e^{-\frac{1}{\kappa \tau}} \), a form analogous to the thermal BCS-type transition. Hence, interactions can destroy superconductivity even at zero temperature if \( u > u_{\infty} \). Second, the conformal structure of the theory is highlighted by setting \( u = u_{\infty} \) where the static, long-wavelength propagator takes a familiar form
\[
L^{-1}(q \to 0, \Omega = 0)_{u = u_{\infty}} = N_0 \sqrt{\frac{\pi u_{\infty}}{2\tau}} e^{-\frac{u_{\infty}}{\Lambda}}. \tag{7}
\]
From this expression, it is illuminating to evaluate the fluctuation contribution to the free energy to zeroth order in \( q \) above the critical point. Following the procedures described in \([34, 40]\) for the case of a single band model and using Eq. 7 we obtain the pair fluctuation free energy
\[
-\beta F = \beta u_{\infty} - \gamma \ln(\beta u_{\infty}). \tag{8}
\]
where \( \gamma = \frac{1}{2} \). This result must be compared with other quantum critical models having gravity duals such as the Sachdev-Ye-Kitaev (SYK) model and its variants \([41, 50-63]\) where sub-leading contributions to the free energy acquire a form similar to Eq. 3 but with \( \gamma = \frac{3}{2} \) \([41, 42]\]. Finally, one can evaluate the spectral density \( \rho(\omega) \) by taking the inverse Laplace transform of the partition function and the resulting integral can be solved by the saddle point method \([41]\). While \( \rho(\omega) \) is a constant independent of \( \omega \) at low energies in the SYK-type models \([41]\), our model yields a vHS \( \rho(\omega) \sim \frac{1}{\omega^2} \) at low energy leading to NFL transport \([53]\). This contrast is entirely due to the difference in the coefficient \( \gamma \) of the log term in Eq. 8. The conclusions drawn above are summarized in Fig. 2. The \( u-T \) phase diagram in Fig. 2 (left panel) plots the strong coupling phase boundary (solid light blue line) separating the SC and NFL phases for a constant \( \Lambda \). The dashed lines are extrapolations of the phase boundary where approximations made above fail. Fig. 2 (right panel, solid curve) plots the \( \kappa-(N_0 g)^{-1} \) phase diagram and shows the same phase boundary for close to zero temperatures and constant \( \beta \Lambda \). The intensity of fluctuations is indicated by the color scale and is largest in magnitude right above the phase boundary.

**Weak coupling (\( \kappa = \beta u \ll 1 \) in clean limit (\( \tau \to \infty \)):** That the \( T = 0 \) pair instability is only a feature at strong coupling can be confirmed by calculating \( L^{-1}(q, 0) \) in the opposite (weak coupling) limit \( \beta u \ll 1 \). We begin with Eq. 4 and expand \( \Pi(0)(0, 0) \) and \( \Pi(2)(q, 0) \) to quadratic power in \( \kappa = \beta u \) to obtain
\[
\Pi(0)(0, 0) \simeq \frac{m}{4\beta} \sum_{\epsilon_n} \left[ \frac{2}{|\epsilon_n|} - \frac{2u^2}{|\epsilon_n|^3} \right] \ln(\beta u_{\infty}). \tag{9}
\]
\[
\Pi(2)(q, 0) \simeq -\frac{m\tau^2}{32\beta} \sum_{\epsilon_n} \frac{2}{|\epsilon_n|^3} - \frac{4u^2}{|\epsilon_n|^5}. \tag{10}
\]
The sums above can be performed and substituted back into the static limit of the propagator (see Appendix D) and we find,
\[
L^{-1}(q, \Omega \to 0) = -\frac{1}{g} + N_0 \left[ \ln \frac{\Lambda}{2\pi T} - \psi \left( \frac{1}{2} \right) - \frac{u^2C_2}{8\pi^2T^2} \right] - \frac{N_0 \tau^2}{128\pi^2T^2} \left[ 2C_2 - \frac{u^2C_4}{12\pi^2T^2} \right]. \tag{11}
\]
where \( C_2 = |\psi'' \left( \frac{1}{2} \right) |, C_4 = |\psi^{(4)} \left( \frac{1}{2} \right) | \) are numerical constants equal to the second and fourth derivatives of the digamma function \( \psi(x) \) respectively. Setting \( q \to 0 \), this form of the fluctuation propagator resembles its thermal BCS counterpart plus the correction term proportional to

**FIG. 2.** (Left) Schematic plot of the \( u-T \) phase diagram in the clean limit. The red solid line denotes a Fermi liquid (\( u = 0 \)) while the light (dark) blue contours define the phase boundary in the strong coupling \( \beta u \gg 1 \) (weak coupling \( \beta u \ll 1 \)) limit. The strong coupling normal state is a NFL with a power-law divergence of the spectral density \( \rho(\omega) \). We have defined \( T_{cd} \equiv T_{c}(u = 0) \) and \( u_{\infty} \equiv u_{c}(\beta \to \infty) \). (Right) Strong coupling, weak impurity scattering (\( \tau \equiv \theta \gg 1 \)) limit of the \( \kappa-(N_0 g)^{-1} \) phase diagram. On the solid (dotted-dashed) curve, the pair fluctuations diverge in the absence (presence) of impurity scattering. Inset shows the weak enhancement of SC phase due to impurities.
It is hence clear that there is no sensible way to obtain a zero temperature transition into the superconducting state (since \( \beta u \ll 1 \)). Moreover, as the correction term is negative, its effect on BCS result is to reduce the thermal transition temperature \( T_c \) for a given interaction strength \( g \) and energy cut-off \( \Lambda \). This is shown in Fig. 2 (left panel) where we have defined \( T_{c0} \equiv T_c(u = 0) \) and the dashed lines are extrapolations of the phase boundary where approximations made above fail.

**Strong coupling (\( \beta u > 1 \)) and dilute impurities** \(( \theta = T \tau \gg 1 \)): The fluctuation propagator in the presence of impurities is shown in Fig. 3 - the solid lines are now impurity Green functions that acquire zeros and the shaded disk denotes vertex corrections due to impurities. The pair susceptibility bubble then becomes \( \Im \) \((d = 2)\)

\[
\Pi(q, \Omega_k) = \frac{1}{\beta} \sum_{\epsilon_n} \frac{P(q, \epsilon_{n-k}, -\epsilon_n)}{1 - \frac{\epsilon_n}{2\pi N_0 \tau}},
\]

where \( P(q, \epsilon_1, \epsilon_2) = \frac{1}{(2\pi)^2} \int d^2 p \ G(p + q, \epsilon_1)G(-p, \epsilon_2) \), \( \epsilon_n = \epsilon_n + \frac{\text{sgn}(\epsilon_n)}{2} \), and \( \text{sgn}(x) \) is the sign function. For \( \Omega_k = 0 \), one can perform an expansion in \( r \) similar to the clean case and write

\[
P(q, \epsilon_n, -\epsilon_n) \approx P^{(0)}(q = 0, \epsilon_n, -\epsilon_n) + P^{(2)}(q, \epsilon_n, -\epsilon_n),
\]

\[
P^{(0)}(q, 0) = \frac{\beta m^2}{4} \left( 2\tilde{S}_1 - u^2 \tilde{S}_3 \right),
\]

\[
P^{(2)}(q, 0) = \frac{\beta m^2}{32} \left( 2\tilde{S}_3 - u^2 \tilde{S}_3 \right)
\]

and \( \tilde{S}_\nu = \frac{1}{\beta} \sum_{\epsilon_n} (\epsilon_n^2 + u^2)^{-\nu/2} \). In the limit \( \beta u \gg 1 \) and \( \theta \equiv T \tau \gg 1 \), the denominator in Eq. \( 12 \) can be approximated by unity. This is equivalent to ignoring vertex corrections due to impurity scattering and hence \( \Pi(q, \Omega_k = 0) \approx \frac{1}{\beta} \sum_{\epsilon_n} P(q, \epsilon_n, -\epsilon_n) \) \((\text{Appendix G})\) gives additional numerical justification for this approximation). The Matsubara sums can be performed exactly for \( u > \tau^{-1} \) \((\text{Appendix F})\) and the final expression for the fluctuation propagator is only slightly modified from the clean limit and given by

\[
L^{-1}(q, \Omega \to 0) \approx -g^{-1} + N_0 \left[ \ln \frac{\Lambda}{u} + \frac{\sqrt{\pi \kappa}}{2} e^{-\epsilon + \frac{1}{2} u} \right]
\]

\[
- \frac{N_0 u^2}{12 u^2} \left[ 1 + \frac{\sqrt{\pi \kappa^3}}{8} e^{-\epsilon + \frac{1}{2} u} \right]. \quad (15)
\]

Hence its conformal structure at the quantum critical point, \( L^{-1}(q \to 0, \Omega = 0) = N_0 \sqrt{\frac{\pi \kappa u}{2}} e^{-\epsilon + \frac{1}{2} u} \), as well as the free energy contribution and vHS in \( \rho(\omega) \) are left essentially unchanged. On the other hand, as shown in Fig. 2 (right panel), there is a weak enhancement of the superconducting phase in the strong coupling phase diagram.

**Weak coupling (\( \beta u \ll 1 \)) and dilute impurities (\( T \tau \gg 1 \))**: The final case we consider is the weak coupling limit in the presence of dilute impurities. In this limit, vertex corrections become more important than in the strong coupling case \((\text{Appendix G})\) and the static long-wavelength limit of the pair susceptibility is

\[
\Pi(q \to 0, \Omega = 0) = \frac{1}{\beta} \sum_{n} \frac{2\pi N_0 \tau \hat{A}(\epsilon_n, u)}{2\pi N_0 \tau - A(\epsilon_n, u)} \quad \quad (16)
\]

\[
\hat{A}(\epsilon_n, u) = \frac{2\pi N_0}{4} \left[ \frac{2}{\epsilon_n^2} - \frac{u^2}{\epsilon_n^3} \right]. \quad (17)
\]

Like in the case of the clean limit, we can perform an expansion in \( \beta u \) and the Matsubara summations have been performed in Appendix E. The final result for the fluctuation propagator in this limit takes the form

\[
L^{-1}(q \to 0, \Omega = 0) = -g^{-1} + N_0 \left[ \ln \left( \frac{\Lambda}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) + 4u^2 \tau^2 \left[ \psi \left( \frac{1}{2} + \frac{1}{4\pi \theta} \right) - \frac{1}{4\pi \theta} \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) \right] \right]. \quad (18)
\]

This expression for the propagator looks similar to that obtained in the limit of low \( q \) and \( \tau < \infty \), but with \( u^2 \) replacing the energy scale arising from the squared momentum factor \( \Im \). In this limit, as anticipated from the clean case, the conformal structure of the propagator is lost and there is only a thermal transition into the superconducting state.

**Discussion**

The conformal structure of the fluctuation propagator and free energy in Eqs. 7 and 8 obtained from the YRZ Green function, and the associated power-law divergence of the spectral density is reminiscent of the “\( q = 4 \) SYK” model discussed for a \( q \)-body interaction. The Green function in this model is local and given by \( G(\omega_n) = -i\omega_n - \Sigma(\omega_n) \). At low temperatures, a Fourier transform gives \( G(\tau) \sim \frac{1}{\tau \Delta} \) where \( \Delta = q^{-1} \) and, therefore, the spectral density scales as \( \rho(\omega) \sim \omega^{\frac{1}{q} - 1} \). For \( q = 4 \), this reduces to the spectral density described in our model with a YRZ-type LS. The key difference, however, is the absence of any disorder \( \Im \) or explicit long-range interactions needed in our calculations; instead, we require a momentum-dependent (non-local) self-energy.
to obtain the same physical content. In addition, a large-N parameter, typically used in SYK-type models, is absent. We also emphasize that the equivalence between Eqs. [7] and [8] and the corresponding quantities in gravity-type models holds even in the absence of vertex corrections from Coulomb interactions. Therefore, these terms are expected to be irrelevant to establish this equivalence. Furthermore, the weak enhancement of the superconducting phase induced by the interplay of electron correlations and dilute impurities is also consistent with previous studies [55–57]. It would be of considerable interest to examine the consequences of fluctuation-driven vHVs on properties such as the entanglement entropy and energy-level spacing near the quantum critical point for a model with a LS.

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APPENDIX A

In this Appendix, we derive expression for the pair susceptibility \(\Pi(q, \Omega_k)\) appearing in Eq. 3 of the main text. For the clean limit in \(d = 2\) we begin with the definition (\(\epsilon_1 \equiv \epsilon_{1n}, \epsilon_2 \equiv \epsilon_{2n}\))

\[
I(q, \epsilon_1, \epsilon_2) = \int d^2p \ G(p + q, \epsilon_1)G(-p, \epsilon_2)
\]

where

\[
G(p, \epsilon_n) = \frac{1}{i\epsilon_n - p^2/2m + \mu - \Sigma(p, \epsilon_n)},
\]

\[
\Sigma(p, \epsilon_n) = V + \frac{u^2}{i\epsilon_n + \xi(p)},
\]

\(\xi(p) = \epsilon(q) - \mu,\) and \(V\) is a constant potential. \(u^2\) is the residue of the self-energy pole and its square-root plays the role of an interaction strength which other quantities can be compared. Substituting \(\Sigma(p, \epsilon_n)\) back into \(I(q, \epsilon_1, \epsilon_2)\) and taking the limits \(|q| \ll p_f, \omega \equiv \epsilon_1 - \epsilon_2 \sim p_f^2/m \ll p_f^2\) we get

\[
I(q, \epsilon_1, \epsilon_2) = m \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\phi \frac{(i\epsilon_1 + x + r \cos \phi)(i\epsilon_2 + x)}{((i\epsilon_1)^2 - x^2 - u^2)((i\epsilon_2)^2 - (x + r \cos \phi)^2 - u^2)}.
\]

To obtain the above, we have made the replacements \(x = \frac{p^2}{2m} - \mu,\) \(\int d^2p = m \int d\left(\frac{p^2}{2m}\right) d\phi,\) and absorbed \(V\) into the definition of the chemical potential which is set to be large. The poles of the integrand in \(I(q, \epsilon_1, \epsilon_2)\) are located at \(x = \pm i\sqrt{\epsilon_1^2 + u^2} \equiv \pm i\epsilon_1'\) and \(\pm i\sqrt{\epsilon_2^2 + u^2} - r \cos \phi \equiv \pm i\epsilon_2' - r \cos \phi.\) Along side these definitions, we can write the pair susceptibility as

\[
\Pi(q, \Omega_k) = \frac{1}{\beta(2\pi)^2} \sum_{\epsilon_n} I(q, \epsilon_1, \epsilon_2)
\]

\[
= \left\langle \frac{m}{2\pi \beta} \sum_{\epsilon_n} \int_{-\infty}^{\infty} dx \left[ \frac{(i\epsilon_1 + x + r \cos \phi)(i\epsilon_2 + x)}{((i\epsilon_1)^2 - x^2 - u^2)((i\epsilon_2)^2 - (x + r \cos \phi)^2 - u^2)} \right] \right\rangle,
\]

with the replacements \(\epsilon_1 \rightarrow \epsilon_{n+k}\) and \(\epsilon_2 \rightarrow -\epsilon_n\) and the angular average \(\langle ... \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi.\) The integral over the variable \(x\) can be performed exactly by the method of residues. Using poles of the \(x\) integrand and summing over residues in the upper-half plane, we obtain the pair susceptibility as

\[
\Pi(q, \Omega_k) = \left\langle \frac{m}{2\pi \beta} \sum_{\epsilon_n} \begin{array}{c}
\frac{\pi (\epsilon_1 + \epsilon_1') (\epsilon_2 + \epsilon_1 + i r \cos \phi)}{\epsilon_1' (\epsilon_1' - \epsilon_2 - r^2 \cos \phi + 2i r \epsilon_1' \cos \phi)} + \frac{\pi (\epsilon_2 + \epsilon_2') (\epsilon_1 + \epsilon_2 - i r \cos \phi)}{\epsilon_2' (\epsilon_2' - \epsilon_1 - r^2 \cos \phi - 2i r \epsilon_2' \cos \phi)}
\end{array} \right\rangle\]

\[
= \left\langle \frac{m}{2\pi \beta} \sum_{\epsilon_n} \begin{array}{c}
\frac{\pi (\epsilon_1 + \epsilon_1') (\epsilon_2 + \epsilon_1 + i r \cos \phi)}{\epsilon_1' (\epsilon_1' - \epsilon_2 - r^2 \cos \phi + 2i r \epsilon_1' \cos \phi)} + \frac{\pi (\epsilon_2 + \epsilon_2') (\epsilon_1 + \epsilon_2 - i r \cos \phi)}{\epsilon_2' (\epsilon_2' - \epsilon_1 - r^2 \cos \phi - 2i r \epsilon_2' \cos \phi)} + c.c(1 \leftrightarrow 2)
\end{array} \right\rangle.
\]

This is the expression that appears in Eq. 3 of the main text.

APPENDIX B

In this Appendix, we show that Eq. 3 of the main text indeed reduces to the correct FL result in the limit \(u \to 0.\) We begin with the expression for \(\Pi(q, \Omega_k)\) (we make the replacements \(\epsilon_1 \equiv \epsilon_{1n} \rightarrow \epsilon_{n+k}\) and \(\epsilon_2 \equiv \epsilon_{2n} \rightarrow -\epsilon_n\) to recover
the $\Omega_k$ dependence)

\[ \Pi(q, \Omega_k) = \left\langle \frac{m}{2\pi\beta} \sum_{\epsilon_n} \left[ \frac{\pi (\epsilon_1 + \epsilon_1') (\epsilon_2 + \epsilon_1' + ir\cos\phi)}{\epsilon_1'(\epsilon_1'^2 - \epsilon_2'^2 + r^2\cos^2\phi + 2ir\epsilon_1'\cos\phi)} + \frac{\pi (\epsilon_2 + \epsilon_2') (\epsilon_1 + \epsilon_2' - ir\cos\phi)}{\epsilon_2'(\epsilon_2'^2 - \epsilon_1'^2 - r^2\cos^2\phi - 2ir\epsilon_2'\cos\phi)} \right] \right\rangle. \]

Noting that all square-roots appearing above are positive, we have $\epsilon_i' \to |\epsilon_i|$ as $u \to 0$. Hence, in this limit we have

\[ \Pi(q, \Omega_k)_{u \to 0} = \left\langle \frac{m}{2\pi\beta} \sum_{\epsilon_n} \left[ \frac{\pi (\epsilon_1 + |\epsilon_1|) (\epsilon_2 + |\epsilon_1| + ir\cos\phi)}{|\epsilon_1|(\epsilon_1^2 - r^2\cos^2\phi + 2ir|\epsilon_1|\cos\phi)} + \frac{\pi (\epsilon_2 + |\epsilon_2|) (\epsilon_1 + |\epsilon_2| - ir\cos\phi)}{|\epsilon_2|(\epsilon_2^2 - r^2\cos^2\phi - 2ir|\epsilon_2|\cos\phi)} \right] \right\rangle. \]

Case 1, $\epsilon_1 < 0; \epsilon_2 < 0$: The numerators of both the terms vanish since $\epsilon_1 + |\epsilon_1| = -|\epsilon_1| + |\epsilon_1| = 0$, hence $\Pi(q, \Omega_k)_{u \to 0} = 0$ for this case.

Case 2, $\epsilon_1 > 0; \epsilon_2 > 0$: In this case, both the numerators are non-zero but the two terms cancel, i.e.,

\[ \Pi(q, \Omega_k)_{u \to 0} = 2\pi \frac{|\epsilon_2 + |\epsilon_1| + ir\cos\phi|}{(|\epsilon_1| + ir\cos\phi)^2 - \epsilon_1^2} = 0. \]  

(27)

Case 3, $\epsilon_1 < 0; \epsilon_2 > 0$: Here, the first term equals zero but the second remains non-zero and we have

\[ \Pi(q, \Omega_k)_{u \to 0} = 2\pi \frac{|\epsilon_1| - |\epsilon_2| + ir\cos\phi}{\epsilon_1^2 - (|\epsilon_2| - ir\cos\phi)^2} = \frac{2\pi}{|\epsilon_1| + |\epsilon_2| - ir\cos\phi}. \]  

(28)

Case 4, $\epsilon_1 > 0; \epsilon_2 < 0$: Similar to the case above, we have a non-zero contribution from the first term to give

\[ \Pi(q, \Omega_k)_{u \to 0} = 2\pi \frac{|\epsilon_1| - |\epsilon_2| + ir\cos\phi}{-\epsilon_2^2 + (|\epsilon_1| + ir\cos\phi)^2} = \frac{2\pi}{|\epsilon_1| + |\epsilon_2| + ir\cos\phi}. \]  

(29)

We can combine all the cases above to write

\[ \Pi(q, \Omega_k)_{u \to 0} = \frac{2\pi \Theta(-\epsilon_1\epsilon_2)}{|\epsilon_1 - \epsilon_2| + sgn(\epsilon_1 - \epsilon_2)ir\cos\phi}, \]

(30)

which is the same as the expression derived for the FL case $\text{Re}$.

**APPENDIX C**

In this Appendix, we evaluate the fractional Matsubara sums appearing in the main text. We recall that an expansion of the inverse fluctuation propagator in the parameter $r$ in the static limit gives

\[ L^{-1}(q, \Omega \to 0) \simeq -g^{-1} + \Pi^{(0)}(0, 0) + \Pi^{(2)}(q, 0), \]

(31)

where $\Pi^{(0)}(0, 0) = \frac{m^2}{\pi} (2S_1 - u^2 S_3)$, $\Pi^{(2)}(q, 0) = -\frac{m^2}{3\pi} (2S_3 - u^2 S_5)$ and $S_\nu = \frac{1}{\beta} \sum_{\epsilon_n} (\epsilon_n^2 + u^2)^{-\nu/2}$. We now wish to evaluate $S_\nu$ for odd $\nu = 1, 3, 5, \ldots$

Case 1, $\nu = 1$: We want to evaluate the divergent sum $S_1 = \frac{1}{\beta} \sum_{\epsilon_n} (\epsilon_n^2 + u^2)^{-1/2}$. To this end, consider an integral over the contour $C$ in the complex plane (shown in Fig. [3] (left)) with branch points at $\pm u$ and a branch cut extending out to $\pm \infty$ from their respective branch points. Using Cauchy’s theorem, we can relate this integral to the sum $S_1$ using the formula

\[ \oint_{C = C_1 + C_2 + C_3} \frac{g_F(z)dz}{(-z^2 + u^2)^{1/2}} = \frac{2\pi i}{\beta} \sum_{\epsilon_n} \frac{1}{(\epsilon_n^2 + u^2)^{1/2}}, \]

(32)

where $g_F(x) = \frac{1}{2} \tanh \left( \frac{\beta x}{2} \right)$, and the right hand side is simply a sum of residues of the poles at the fermionic Matsubara frequencies. To determine this integral, we divide the total contour into three parts, $C_{1,2,3}$, and evaluate each individually. We begin with the circular contour $C_3$ with a radius ($\epsilon$) that has a zero limiting value. This is given as

\[ \frac{1}{2\pi i} \oint_{C_3} \frac{g_F(z)dz}{(-z^2 + u^2)^{1/2}} = -\frac{1}{2\pi} \oint_{C_3} \frac{g_F(z)dz}{(z + u)^{1/2}(z - u)^{1/2}}. \]

(33)
We can parameterize the variable near the $z = u$ branch point as $z = u + \epsilon e^{i\phi}$ where $0 \leq \phi \leq 2\pi$ and $dz = i\epsilon e^{i\phi} d\phi$. With this substitution we obtain

$$\frac{1}{2\pi i} \int_{C_1} \frac{g_F(z)dz}{(-z^2 + u^2)^{1/2}} = -\frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi (iRe^{i\phi})g_F(Re^{i\phi})}{(R^2e^{2i\phi} - u^2)^{1/2}},$$

leading to a vanishing contribution as $\sqrt{\epsilon}$ as $\epsilon \to 0$. We can similarly parameterize the variable near the $z = -u$ branch point as $z = -u + \epsilon e^{i\phi}$ where $-\pi \leq \phi \leq \pi$ and $dz = i\epsilon e^{i\phi} d\phi$. This contribution to the total integral also vanishes as $\sqrt{\epsilon}$ as $\epsilon \to 0$. We now consider the contour integral over the large circle $C_1$ with radius $R$. As the circle is centered around $z = 0$, we can use the parameterization $z = Re^{i\phi}$ where $0 \leq \phi \leq 2\pi$ and $dz = iRe^{i\phi} d\phi$. With this substitution, the $C_1$ contribution is

$$\frac{1}{2\pi i} \int_{C_1} \frac{g_F(z)dz}{(-z^2 + u^2)^{1/2}} \to -\frac{1}{2\pi} \int_0^{2\pi} d\phi \ g_F(Re^{i\phi}) = \#,$$

Taking the limit $R \to \infty$, we have

$$-I_{C_2} = \frac{1}{2\pi} \int_{u+}^{\infty} \frac{g_F(z+)dz_+}{(z_+ + u)^{1/2}(z_+ - u)^{1/2}} + \frac{1}{2\pi} \int_{-\infty}^{u-} \frac{g_F(z-)dz_-}{(z_- + u)^{1/2}(z_- - u)^{1/2}}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{-u-} \frac{g_F(z-)dz_-}{(z_- + u)^{1/2}(z_- - u)^{1/2}} + \frac{1}{2\pi} \int_{-\infty}^{-u+} \frac{g_F(z+)dz_+}{(z_+ + u)^{1/2}(z_+ - u)^{1/2}}.$$
Since \((z \pm u)^{1/2}\) and \(g_P(z)\) are analytic across the branch points \(z = \pm u\) respectively, we can rewrite \(I_C_2\) as

\[
-I_{C_2} = \frac{1}{2\pi} \int_{u+\epsilon}^{\infty} \frac{g_P(z)dz}{(z+u)^{1/2}} - \frac{1}{(z-u)^{1/2}} - \frac{1}{(z+u)^{1/2}}.
\]

The quantities in the brackets above can be evaluated using the relations

\[
\frac{1}{(z-u)^{1-\alpha}} - \frac{1}{(z-u)^{1-\alpha}} = \frac{2i\sin\pi\alpha e^{-i\pi(1-\alpha)}}{|z-u|^{1-\alpha}},
\]

\[
\frac{1}{(z+u)^{1-\alpha}} - \frac{1}{(z+u)^{1-\alpha}} = -\frac{2i\sin\pi\alpha}{|z+u|^{1-\alpha}}.
\]

Using these relations by setting \(\alpha \to 1/2\) we can simplify \(I_{C_2}\) to write

\[
I_{C_2} = \lim_{\epsilon \to 0} -\frac{1}{\pi} \int_{u+\epsilon}^{\infty} \frac{(g_P(-z) - g_P(z))dz}{\sqrt{z^2 - u^2}} = \text{P.V.} \left[ \frac{1}{\pi} \int_u^{\infty} \frac{\tanh(\frac{u}{2})}{\sqrt{z^2 - u^2}} dz \right],
\]

where \(\text{P.V.}\) denotes principal value. As is evident from the form above, \(I_{C_2}\) (and consequently \(S_1\)) is UV divergent; hence, we set a cut-off energy parameter \(\Lambda\) to isolate the divergence. Changing variables \(z = z^\prime\), we can evaluate the integral in the strong coupling limit \(\beta u \gg 1\) where we can approximate \(\tanh x \simeq 1 - 2e^{-2x}\). Taking the limit \(\Lambda/u \gg 1\) and substituting \(I_{C_2}\) back into \(S_1\) we have

\[
S_1 \simeq \frac{1}{\pi} \ln \left( \frac{\Lambda}{u} \right) - \frac{2}{\pi} K(0, \beta u) + \#,
\]

where \(K(x, y)\) is the modified Bessel function of the second kind.

**Case 2, \(\nu = 3, 5, \ldots\):** We will now evaluate the convergent sums \(S_\nu = \frac{1}{\beta} \sum_n (\epsilon_n^2 + u^2)^{-\nu/2}\) where \(\nu = 3, 5, \ldots\) Similar to the case of \(\nu = 1\), we can break up the sums into three individual pieces of integration around \(C_{1,2,3}\) shown in Fig. 3 (left). Hence we write

\[
\left[ \oint_{C_1} + \oint_{C_2} + \oint_{C_3} \right] \frac{g_P(z)dz}{(-z^2 + u^2)^{\nu/2}} = \frac{2\pi i}{\beta} \sum_n \frac{1}{(\epsilon_n^2 + u^2)^{\nu/2}}.
\]

We begin evaluating the large contour \(C_1\) by replacing \(z = Re^{i\phi}\) and \(dz = iRe^{i\phi}d\phi\). With this substitution we have

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{g_P(z)dz}{(-z^2 + u^2)^{\nu/2}} = \frac{e^{-i\pi\nu/2}}{2\pi i} \int_0^{2\pi} \frac{g_P(Re^{i\phi})(iRe^{i\phi})d\phi}{(R^2e^{2i\phi} - u^2)^{\nu/2}}.
\]

Taking the limit \(R \to \infty\) the integral becomes

\[
\frac{1}{2\pi i} \oint_{C_1} \frac{g_P(z)dz}{(-z^2 + u^2)^{\nu/2}} = \frac{e^{-i\pi\nu/2}}{2\pi i} \int_0^{2\pi} \frac{g_P(Re^{i\phi})(iRe^{i\phi})d\phi}{R^{\nu}e^{\nu i\phi}} \sim \frac{1}{R^{\nu-1}} \to 0 \quad \text{for} \quad \nu > 1.
\]

Hence the contour \(C_1\) does not contribute to \(S_\nu\).

We will now show that the IR divergent contribution from contour \(C_3\) is cancelled with that of \(C_2\) yielding \(S_\nu\) that is finite as must be anticipated for \(\nu = 3, 5, \ldots\). We begin with the \(C_3\) contribution from the \(z = u\) branch point. Like before, we make the substitution \(z = u + \epsilon e^{i\phi}\) where \(0 \leq \phi \leq 2\pi\) and we obtain

\[
\frac{1}{2\pi i} \oint_{C_{3, z=0}} \frac{g_P(z)dz}{(-z^2 + u^2)^{\nu/2}} = \frac{e^{-i\pi\nu/2}}{2\pi i} \int_0^{2\pi} \frac{g_P(u + \epsilon e^{i\phi})d\phi}{(2u + \epsilon e^{i\phi})^{\nu/2}(\epsilon e^{i\phi})^{\nu/2}}, \quad \text{for} \quad z = u.
\]

Taking the limit of \(\epsilon \to 0\) and solving the \(\phi\) integral we have the IR divergent term from \(z = u\)

\[
\frac{1}{2\pi i} \oint_{C_{3, z=0}} \frac{g_P(z)dz}{(-z^2 + u^2)^{\nu/2}} = -\frac{e^{-i\pi\nu/2}}{(2u)^{\nu/2}} \frac{g_P(u)}{2\pi i} \left[ \frac{2}{2 - \nu} \right] \quad \text{for} \quad z = u, \quad \nu = 3, 5, \ldots
\]
Similarly the contribution from the $z = -u$ branch point can be obtained by the substitution $z = -u + \epsilon e^{i\phi}$ where $-\pi \leq \phi \leq \pi$. The result is equal to that obtained for the $z = u$ case discussed above and thus gives a total contribution from the $C_3$ contour

$$
\frac{1}{2\pi i} \int_{C_3} \frac{g_F(z)dz}{(z^2 + u^2)^{\nu/2}} = -\frac{2e^{-i\pi \nu/2}}{(2\nu)^{\nu/2}} \frac{g_F(u)}{\epsilon^{\nu/2}} \left[ \frac{2}{2-\nu} \right] \quad \nu = 3, 5, \ldots
$$

This term is IR divergent as $\sim \frac{1}{\nu}$. We now evaluate the contribution from the $C_2$ contour by following a similar procedure as the $\nu = 1$ case. We have

$$
\frac{1}{2\pi i} \int_{C_2} \frac{e^{-i\pi \nu/2}g_F(z)dz}{(z^2 - u^2)^{\nu/2}} = \frac{e^{-i\pi \nu/2}}{2\pi i} \left\{ \int_{u+\epsilon}^{\Lambda} g_F(z)dz \left[ \frac{2i \sin \pi (1 - \frac{\nu}{2}) e^{-i\pi \nu/2}}{z - u} \right] \\
+ \int_{-\Lambda}^{-u-\epsilon} g_F(z)dz \left[ \frac{2i \sin \pi (1 - \frac{\nu}{2})}{z + u} \right] \right\}
$$

We can now utilize Eqs. [10] and [11] to substitute for quantities appearing in the square brackets above. We make the replacement $\alpha \rightarrow 1 - \frac{\nu}{2}$ and after simplifications we are left with

$$
\frac{1}{2\pi i} \int_{C_2} \frac{e^{-i\pi \nu/2}g_F(z)dz}{(z^2 - u^2)^{\nu/2}} = -\frac{e^{-i\pi \nu/2}}{2\pi i} \left\{ \int_{u+\epsilon}^{\Lambda} g_F(z)dz \left[ \frac{2i \sin \pi (1 - \frac{\nu}{2})}{z - u} \right] \\
+ \int_{-\Lambda}^{-u-\epsilon} g_F(z)dz \left[ \frac{2i \sin \pi (1 - \frac{\nu}{2})}{z + u} \right] \right\}
$$

where in the last step we changed variables $z = uz'$. To be able to solve the integrals above and extract the IR divergence, we perform the strong coupling expansion $\tanh x \approx 1 - 2e^{-2x}$. The integral of the first term in the expansion gives for $\Lambda/u \rightarrow \infty$

$$
\int_{1+\epsilon}^{\infty} \frac{dz'}{|z'^2 - 1|^{\nu/2}} = \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{\nu-1}{2} \right) \sin \left( \frac{\pi \nu}{2} \right)}{2 \Gamma (\nu/2)} \right] \quad \nu = 3, 5, \ldots
$$

where the second term diverges as $\sim \frac{1}{\nu}$ and cancels the IR divergence in Eq. [49] for $\beta u \rightarrow \infty$. Therefore, we only need to keep the principal value of the integral over the contour $C_2$, i.e.,

$$
\frac{1}{2\pi i} \int_{C_2} \frac{e^{-i\pi \nu/2}g_F(z)dz}{(z^2 - u^2)^{\nu/2}} \approx \frac{e^{-i\pi \nu/2}}{2\pi i} \frac{2i \sin \pi (1 - \nu/2)}{u^{\nu-1}} \text{P.V.} \left\{ \int_{1}^{\infty} dz' \left[ \frac{1 - 2e^{-\beta uz'}}{|z'^2 - 1|^{\nu/2}} \right] \right\}
$$

The principal value integral can be solved exactly and can be combined with the $\nu = 1$ case to give the sum $S_{\nu}$ as

$$
S_{\nu} = \begin{cases} 
\frac{1}{\pi} \ln \frac{\Delta}{u} - \frac{2}{\pi} K(0, \kappa) & \text{for } \nu = 1 \\
\frac{e^{-i\pi \Gamma(1-\frac{\nu}{2})} \sin \pi (1 - \nu/2)}{u^{\nu-1} 2^{\nu/2}} \left[ 2^{\nu/2} K \left( \frac{\nu-1}{2}, \kappa \right) \right] - \Gamma \left( \frac{\nu+1}{2} \right) & \text{for } \nu = 3, 5, \ldots
\end{cases}
$$

where $\kappa \equiv \beta u$ and $K(x, y)$ is the modified Bessel function of the second kind. This is Eq. [5] in the main text.

**APPENDIX D**

In this section, we will evaluate relevant Matsubara sums to arrive at the expression for the fluctuation propagator in the weak coupling ($\kappa \ll 1$) clean limit ($\tau \rightarrow \infty$). We begin with the small $u$ expansions of the pair susceptibilities appearing in the main text (the powers (0) and (2) on top of the pair susceptibility components denote powers of the
Noting that
\[ \Pi^{(0)}(0, 0) \simeq \frac{m}{4\beta} \sum_{\epsilon_n} \left[ \frac{2}{|\epsilon_n|} - \frac{2u^2}{|\epsilon_n|^3} \right], \]
\[ \Pi^{(2)}(q, 0) \simeq -\frac{m\nu^2}{32\beta} \sum_{\epsilon_n} \left[ \frac{2}{|\epsilon_n|^3} - \frac{4u^2}{|\epsilon_n|^5} \right]. \]

Consider the sum
\[ \sum_{n=-\infty}^{n=\infty} \frac{1}{|n + \frac{1}{2} + x|^p} = \left( \sum_{n=0}^{\infty} + \sum_{n=-\infty}^{1} \right) \frac{1}{|n + \frac{1}{2} + x|^p} = \sum_{n=0}^{\infty} \left( \frac{1}{|n + \frac{1}{2} + x|^p} + \frac{1}{|n + \frac{1}{2} - x|^p} \right). \]

Using this relation we can write \( \Pi^{(0)}(0, 0) \) as (for \( x = 0 \))
\[ \Pi^{(0)}(0, 0) \simeq \frac{m}{4\beta} \sum_{n=0}^{\infty} \left[ \frac{4}{2\pi T(n + \frac{1}{2})^3} - \frac{4u^2}{(2\pi T)^3(n + \frac{1}{2})^3} \right]. \]

Noting that \( \sum_{n=0}^{n=A/2\pi T} (n + \frac{1}{2})^{-1} \simeq \ln \left( \frac{L}{2\pi T} \right) - \psi(1/2) \) and \( \sum_{n=0}^{\infty} (n + \frac{1}{2})^{-3} = -\frac{1}{2} \psi''(1/2) \), we arrive at
\[ \Pi^{(0)}(0, 0) \simeq N_0 \left[ \ln \left( \frac{\Lambda}{2\pi T} \right) - \psi(1/2) + \frac{u^2}{2(2\pi T)^2} \psi''(1/2) \right]. \]

Similarly we can write
\[ \Pi^{(2)}(q, 0) \simeq -\frac{m\nu^2}{32\beta} \sum_{n=0}^{\infty} \left[ \frac{4}{(2\pi T)^3(n + \frac{1}{2})^3} - \frac{8u^2}{(2\pi T)^5(n + \frac{1}{2})^5} \right]. \]

Noting again that \( \sum_{n=0}^{\infty} (n + \frac{1}{2})^{-5} = -\frac{1}{2^7} \psi^{(4)}(1/2) \), we arrive at
\[ \Pi^{(2)}(q, 0) \simeq -\frac{N_0\nu^2}{32} \left[ \frac{2!\psi''(1/2)}{(2\pi T)^2} - \frac{u^2|\psi^{(4)}(1/2)|}{3(2\pi T)^4} \right]. \]

Combining \( \Pi^{(0)}(0, 0) \) and \( \Pi^{(2)}(q, 0) \) we obtain the fluctuation propagator in the weak coupling, clean limit as
\[ L^{-1}(q, \Omega \to 0) = -\frac{1}{g} + N_0 \left[ \ln \frac{\Lambda}{2\pi T} - \psi \left( \frac{1}{2} \right) - \frac{u^2C_2}{8\pi^2T^2} \right] - \frac{N_0\nu^2}{128\pi^2T^2} \left[ 2C_2 - \frac{u^2C_4}{12\pi^2T^2} \right]. \]

**APPENDIX E**

In this Appendix we derive the fluctuation propagator in the weak coupling limit \( \kappa \ll 1 \) with dilute impurities \( \theta \equiv T \tau \gg 1 \). We recall the static long-wavelength pair susceptibility for weak impurity scattering from the main text
\[ \Pi(q \to 0, \Omega = 0) = \frac{1}{\beta} \sum_{\epsilon_n} \frac{2\pi N_0 \tau \tilde{A}(\epsilon_n, u)}{2\pi N_0 \tau - \tilde{A}(\epsilon_n, u)} \]
\[ \tilde{A}(\epsilon_n, u) = \frac{2\pi N_0}{4} \left[ \frac{2}{\epsilon_n^2} - \frac{u^2}{\epsilon_n^2} \right]. \]
Keeping only terms quadratic in \( u \), we obtain

\[
\Pi(q \to 0, \Omega = 0) \simeq \frac{2\pi N_0}{\beta} \sum_{n=-\infty}^{\infty} \left[ \frac{\tau}{2|\epsilon_n + \frac{\text{sgn}(\epsilon_n)}{2\tau}|} - \frac{2u^2\tau^2}{(2|\epsilon_n + \frac{\text{sgn}(\epsilon_n)}{2\tau}|)^2} \right] \tag{67}
\]

\[
= \frac{4\pi N_0}{\beta} \sum_{n=0}^{\infty} \left[ \frac{1}{2\epsilon_n} - \frac{u^2}{2(\epsilon_n + \frac{1}{2\tau})} \right] \tag{68}
\]

\[
= N_0 \sum_{n=0}^{\infty} \left( \frac{1}{(n + \frac{1}{2})^2} - 4u^2\tau^2 \left\{ (n + \frac{1}{2} + \frac{1}{4\pi\tau}) + \frac{1}{4\pi\tau} \left( n + \frac{1}{2} \right)^2 - \frac{1}{(n + \frac{1}{2})^2} \right\} \right). \tag{69}
\]

In the second line we changed the summation to positive integers by inverting sign of the summation variable. Using definitions and properties of Gamma functions we can write the final expression for the pair susceptibility (\( \theta \equiv T\tau \))

\[
\Pi(q \to 0, \Omega = 0) = N_0 \left[ \ln \left( \frac{\Lambda}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) + 4u^2\tau^2 \left\{ \psi \left( \frac{1}{2} + \frac{1}{4\pi\theta} \right) - \frac{1}{4\pi\theta} \psi' \left( \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) \right\} \right], \tag{70}
\]

which appears in the final expression of the fluctuation propagator in the main text.

**APPENDIX F**

In this Appendix, we evaluate Matsubara sums appearing in the strong coupling (\( \kappa \gg 1 \)), dilute impurity (\( \theta \gg 1 \)) limit. The derivation follows along similar lines as the case of the clean limit but with branch points shifted by \( \tau^{-1} \). See Fig. 3 (right) for a sketch of the integration contour chosen. We begin by recalling the pair susceptibility in the limit \( \kappa \gg 1, \theta \gg 1 \) where vertex corrections can be ignored,

\[
\Pi(q, \Omega_k) \simeq \frac{1}{\beta} \sum_n P(q, \tilde{\epsilon}_{n+k}, -\tilde{\epsilon}_n). \tag{71}
\]

For \( \Omega_k = 0 \), one can perform an expansion in \( \nu \) similar to the clean case and write

\[
P(q, \tilde{\epsilon}_n, -\tilde{\epsilon}_n) \simeq P^{(0)}(q = 0, \tilde{\epsilon}_n, -\tilde{\epsilon}_n) + P^{(2)}(q, \tilde{\epsilon}_n, -\tilde{\epsilon}_n),
\]

\[
P^{(0)}(0, 0) = \frac{\beta m}{4} \left( 2\tilde{S}_1 - u^2\tilde{S}_3 \right),
\]

\[
P^{(2)}(q, 0) = -\frac{\beta m r^2}{32} \left( 2\tilde{S}_3 - u^2\tilde{S}_5 \right)
\]

and \( \tilde{S}_\nu = \frac{1}{\beta} \sum_{\epsilon_n} (\tilde{\epsilon}_n^2 + u^2)^{-\nu/2} \). To evaluate \( \tilde{S}_\nu \), consider the integral over the contour \( C \) in Fig. 3 (right)

\[
\tilde{I}_\nu = \oint_C \frac{g_F(z)dz}{u^2 - (z + i \text{sgn}(z/\nu)/2\tau)^2(1 + \frac{1}{2\tau|z|})^{1/2}} \tag{74}
\]

where we used the definition of the complex signum function \( \text{sgn}(z) = z/|z| \) to obtain the right hand side. For \( u > \frac{1}{2\tau} \), the branch points can be solved as \( z = \pm (u - \frac{1}{2\tau}) \) with the branch cuts originating from these points to \( \pm \infty \) (see Fig. 3 (right)). Using Cauchy’s theorem, we can easily see that the sum \( \tilde{I}_\nu = \frac{1}{2\pi i} \tilde{I}_\nu \). Like the clean case, in the limit \( \kappa \gg 1, \theta \gg 1 \) and \( u > \frac{1}{2\tau} \), the non-trivial contribution to the summation comes from the \( C_2 \) part of the contour. This is true for both the \( \nu = 1 \) and \( \nu = 3, 5, ... \) cases. Extending the results for the clean case, we obtain for \( \tau < \infty \) and \( \theta \gg 1 \)

\[
\tilde{S}_\nu = e^{-i\pi\nu \sin \pi \left( 1 - \frac{\nu}{2} \right)} \int_{u-\frac{1}{2\tau}+\nu}^{u} \frac{(g_F(z) - g_F(-z))}{-u^2 + z^2 \left( 1 + \frac{1}{2\tau|z|} \right)^2} dz \tag{75}
\]

where \( \Lambda \) can be extended to infinity for the cases \( \nu = 3, 5, ... \) As the integration is now over real variables, we can make the substitution \( z = xu \) to yield

\[
\tilde{S}_\nu = e^{-i\pi\nu \sin \pi \left( 1 - \frac{\nu}{2} \right)} \int_{1-\frac{1}{2\tau}+\nu}^{\Lambda/u} \frac{d\tau}{\text{tanh} \left( \frac{\beta u \tau}{2} \right)} \tag{76}
\]

\[
\left( x + \frac{1}{2\tau} \right)^{-\nu} \frac{d\tau}{\text{tanh} \left( \frac{\beta u \tau}{2} \right)}.
\]

\[
\left( x + \frac{1}{2\tau} \right)^{-\nu} \frac{d\tau}{\text{tanh} \left( \frac{\beta u \tau}{2} \right)}.
\]
Setting \( x + \frac{1}{2\pi} = t \) and expanding the numerator hyperbolic function

\[
\tilde{S}_\nu = \frac{e^{-i\pi\nu}}{\pi u^{\nu-1}} \int_{1+\frac{1}{2\pi}}^{\frac{1}{2}+\frac{1}{2\pi}} dt \frac{\left[ \tanh \left( \frac{\beta ut}{2} \right) - \tanh \left( \frac{1}{4\pi T} \right) \right]}{(t^2 - 1)^{\nu/2}} \left[ 1 - \frac{\beta u}{2} \tanh \left( \frac{1}{4\pi T} \right) \right].
\]

(77)

Since \( \beta u \gg 1, T\tau \gg 1 \) we can rewrite the sum as

\[
\tilde{S}_\nu \simeq \frac{e^{-i\pi\nu}}{\pi u^{\nu-1}} P.V. \left[ \int_{1}^{\Lambda} dt \frac{1 - 2e^{-\beta ut}e^{\frac{t^2}{2\pi T}}}{(t^2 - 1)^{\nu/2}} \right].
\]

(78)

The above integral looks similar to the one that is obtained in the clean limit except for the additional factor \( e^{\frac{t^2}{2\pi T}} \). Hence, the expression for the fluctuation propagator in the strong coupling limit with dilute impurities can be readily generalized as

\[
L^{-1}(q, \Omega \rightarrow 0) \simeq -g^{-1} + N_0 \left[ \ln \frac{\Lambda}{u} + \sqrt{\frac{\pi \kappa}{2}} e^{-\kappa} \right] - \frac{N_0 \tau^2}{12u^2} \left[ 1 + \sqrt{\frac{\pi \kappa^3}{8}} e^{-\kappa} \right].
\]

(79)

This is the final expression that appears in the main text.

**APPENDIX G**

**FIG. 4.** Numerical plots of the pair susceptibility as a function of \( \kappa = \beta u \) (left column) and \( \theta = T\tau \) (right column) for \( \kappa \gg 1 \) and \( \theta \gg 1 \). The top (bottom) row are calculations with (without) vertex corrections due to impurities. The relatively flat behavior of the pair susceptibility as a function \( \theta \) for \( \kappa \gg 1 \) is due to the weak exponential dependence \( \sim e^{\frac{t^2}{2\pi T}} \). These results demonstrate that in the limit \( \kappa \gg 1 \) and \( \theta \gg 1 \), impurity vertex corrections have a negligible effect on the pair susceptibility.
FIG. 5. Same caption as Fig. 4 but for $\kappa \ll 1$ and $\theta \gg 1$. These results demonstrate that in the limit $\kappa \ll 1$ and $\theta \gg 1$, impurity vertex corrections have a discernible yet small effect on the pair susceptibility.

FIG. 6. Same caption as Fig. 4 but for $\theta \ll 1$ over a range of $\kappa$. These results demonstrate that in the limit $\theta \ll 1$, impurity vertex corrections completely alter the pair susceptibility leading to an eventual breakdown of perturbation theory.