STRONG FELLER PROCESSES WITH MEASURE-VALUED DRIFTS

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Abstract. We construct a strong Feller process associated with $-\Delta + \sigma \cdot \nabla$, with drift $\sigma$ in a wide class of measures (weakly form-bounded measures, e.g. combining weak $L^d$ and Kato class measure singularities), by exploiting a quantitative dependence of the smoothness of the domain of an operator realization of $-\Delta + \sigma \cdot \nabla$ generating a holomorphic $C_0$-semigroup on $L^p(\mathbb{R}^d)$, $p > d - 1$, on the value of the form-bound of $\sigma$. Our method admits extension to other types of perturbations of $-\Delta$ or $(-\Delta)^{\frac{d}{2}}$, e.g. to yield new $L^p$-regularity results for Schrödinger operators with form-bounded measure potentials.

1. Let $L^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, L^d)$, $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^d, L^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d, L^d)$ the standard Lebesgue, weak Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of Hölder continuous functions ($0 < \gamma < 1$), $C^0_b = C^0_b(\mathbb{R}^d)$ the space of bounded continuous functions, endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $W^{s,p}$, $s > 0$, the Bessel space endowed with norm $\|u\|_{p,s} := \|g\|_p$, $u = (1 - \Delta)^{-\frac{s}{2}}g$, $g \in L^p$, $W^{-s,p}$ the dual of $W^{s,p}$, and $S = S(\mathbb{R}^d)$ the L. Schwartz space of test functions. We denote by $B(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \rightarrow Y$, endowed with operator norm $\| \cdot \|_{X \rightarrow Y}$; $B(X) := B(X,X)$. Set $\| \cdot \|_{p,q} := \| \cdot \|_{L^p \rightarrow L^q}$. We denote by $\overset{w}{\rightharpoonup}$ the weak convergence of $\mathbb{R}^d$- or $\mathbb{C}^d$-valued measures on $\mathbb{R}^d$, and the weak convergence in a given Banach space.

By $\langle u, v \rangle$ we denote the inner product in $L^2$,

$$\langle u, v \rangle = \langle u \bar{v} \rangle := \int_{\mathbb{R}^d} u \bar{v} L^d \quad (u, v \in L^2).$$

2. Let $d \geq 3$. The problem of constructing a Feller process having infinitesimal generator $-\Delta + b \cdot \nabla$, with singular drift $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, has been thoroughly studied in the literature (cf. [AKR] [KR] and references therein), motivated by applications, as well as the search for the maximal (general) class of vector fields $b$ such that the associated process exists. This search culminated in the following classes of critical drifts:

Definition 1. A vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to belong to $F_\delta$, the class of form-bounded vector fields, if $b$ is $L^d$-measurable and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{d}{2}}\|_{2 \rightarrow 2} \leq \sqrt{\delta}.$$
In particular, there exist above). The vector fields in $F$ of $F$ to multiplication by a constant), at isolated points or along hypersurfaces, respectively.

Earlier, the Kato class $K^{d+1}$, with $\delta > 0$ sufficiently small (but nevertheless allowed to be positive), has been recognized as ‘the right one’ for the existence of the Gaussian upper and lower bounds on the fundamental solution of $-\Delta + b \cdot \nabla$, see [S, ZH]; the Gaussian bounds yield an operator realization of $-\Delta + b \cdot \nabla$ generating a (contraction positivity preserving) $C_0$-semigroup in $C_{\infty}$ (moreover, in $C_b$), whose integral kernel is the transition probability function of a Feller process. In turn, $b \in F_\delta$, $\delta < 4$, ensures that $-\Delta + b \cdot \nabla$ is dissipative in $L^p$, $p > \frac{2}{2-\sqrt{\delta}}$ [KS]; then, if $\delta < \min\{1, \left(\frac{2}{\sqrt{\delta}-2}\right)^2\}$, the $L^p$-dissipativity allows to run a Moser-type iterative procedure of [KS], which takes $p \to \infty$ and

\[ b \in F_{\delta} \text{ (or $K^{d+1}_\delta$) } \iff \varepsilon b \in F_{\varepsilon \delta} \text{ (respectively, } K^{d+1}_{\varepsilon \delta}, \varepsilon > 0.\]

In particular, there exist $b \in F_\delta$ ($K^{d+1}_\delta$) such that $\varepsilon b \notin F_0$ ($K^{d+1}_0$) for any $\varepsilon > 0$ (cf. examples above). The vector fields in $F_\delta \setminus F_0$ and $K^{d+1}_\delta \setminus K^{d+1}_0$ have critical order singularities (i.e. sensitive to multiplication by a constant), at isolated points or along hypersurfaces, respectively.

We have:

1) $b(x) = \sqrt{\delta x} x|x|^{-2} \in F_\delta$ (Hardy inequality).

2) Also, if $|b(x)| \leq 1_{|x_1|<1}|x_1|^{s-1}$, where $0 < s < 1$, $x = (x_1, \ldots, x_d)$, $1_{|x_1|<1}$ is the characteristic function of $\{x : |x_1| < 1\}$, then $b \in K^{d+1}_\delta$. An example of a $b \in K^{d+1}_\delta \setminus K^{d+1}_0$ can be obtained e.g. by modifying [AS] p. 250, Example 1 [Examples 1), 2) demonstrate that $K^{d+1}_\delta \setminus F_{\delta_1} \neq \emptyset$, and $F_{\delta_1} \setminus K^{d+1}_\delta \neq \emptyset$.

It is clear that $\varepsilon > 0$. The value of the relative bound $\delta$ plays a crucial role in the theory of $-\Delta + b \cdot \nabla$, e.g. if $\delta > 4$, then the uniqueness of solution of Cauchy problem for $\partial_t - \Delta + \sqrt{\delta} \frac{2}{\sqrt{\delta}} x|x|^{-2} \cdot \nabla$ fails in $L^p$, see [KS] Example 7], see also comments below.
thus produces an operator realization of \(-\Delta + b \cdot \nabla\) generating a \(C_0\)-semigroup in \(C_\infty\), hence a Feller process.

The natural next step toward determining the general class of drifts \(b\) `responsible' for the existence of an associated Feller process is to consider \(b = b_1 + b_2\), with \(b_1 \in \mathcal{F}_{\delta_1}\), \(b_2 \in \mathcal{K}^{d+1}_{\delta_2}\). Although it is not clear how to reconcile the dissipativity in \(L^p\) and the Gaussian bounds, it turns out that neither of these properties is responsible for the existence of the process; in fact, the process exists for any \(b\) in the following class [Ki]:

**Definition 3.** A vector field \(b : \mathbb{R}^d \to \mathbb{R}^d\) is said to belong to \(\mathcal{F}^{\frac{1}{\delta}}_{\delta}\), the class of weakly form-bounded vector fields, if \(b\) is \(\mathcal{L}^d\)-measurable, and there exists \(\lambda = \lambda_\delta > 0\) such that

\[
|||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{2}}||_{2 \to 2} \leq \sqrt{\delta}.
\]

The class \(\mathcal{F}^{\frac{1}{\delta}}\) has been introduced in [S2, Theorem 5.1]. We have

\[
\mathcal{K}^{d+1} \subseteq \mathcal{F}^{\frac{1}{\delta}}_\delta, \quad \mathcal{F}^{d+2}_\delta \subseteq \mathcal{F}^{\frac{1}{\delta}}_\delta,
\]

\[b \in \mathcal{F}_\delta, \quad f \in \mathcal{K}^{d+1}_\delta \quad \implies \quad b + f \in \mathcal{F}^{\frac{1}{\delta}}_\delta, \quad \sqrt{\delta} = \sqrt{\delta_1} + \sqrt{\delta_2} \quad (1)
\]

(see [Ki]). In [Ki], the construction of the process goes as follows: the starting object is an operator-valued function \((b \in \mathcal{F}^{\frac{1}{\delta}}_\delta)\)

\[
\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1}\left\{(\zeta - \Delta)^{-\frac{1}{2}} \cdot \nabla (\zeta - \Delta)^{-\frac{1}{2}} \right\}^{\frac{1}{2}} b \cdot \nabla (\zeta - \Delta)^{-\frac{1}{2}} \cdot (\zeta - \Delta)^{-\frac{1}{2}} \cdot b
\]

where \(\text{Re} \zeta > \frac{d}{2} \lambda_\delta, \quad b^{\frac{1}{p}} := |b|^{\frac{1}{p}} b^{\frac{1}{p}-1}, \quad p\) is in a bounded open interval determined by the form-bound \(\delta\) (and expanding to \((1, \infty)\) as \(\delta \downarrow 0\)), and \(1 < r < p < q\). Then (see [Ki] for details)

\[
\Theta_p(\zeta, b) = (\zeta + \Lambda_p(b))^{-1},
\]

where \(\Lambda_p(b)\) is an operator realization of \(-\Delta + b \cdot \nabla\) generating a holomorphic \(C_0\)-semigroup \(e^{-t\Lambda_p(b)}\) on \(L^p\), and the very definition of \(\Theta_p(\zeta, b)\) implies that the domain of \(\Lambda_p(b)\)

\[
D(\Lambda_p(b)) \subseteq \mathcal{W}^{1+\frac{1}{q}, p}, \quad \text{for any } q > p.
\]

The information about smoothness of \(D(\Lambda_p(b))\) allows us to leap, by means of the Sobolev embedding theorem, from \(L^p\), \(p > d - 1\), to \(C_\infty\), while moving the burden of the proof of convergence in \(C_\infty\) (in the Trotter’s approximation theorem) to \(L^p\), a space having much weaker topology (locally). Then (see [Ki]) \(\Theta_p(\mu, b)|_S = (\mu + \Lambda_{C_\infty}(b))^{-1}|_S\), where \(\Lambda_{C_\infty}(b)\) is an operator realization of \(-\Delta + b \cdot \nabla\) generating a contraction positivity preserving \(C_0\)-semigroup on \(C_\infty\), hence a Feller process.

3. The primary goal of this note is to extend the method in [Ki] to weakly form-bounded measure drifts.

The study of measure perturbations of \(-\Delta\) has a long history, see e.g. [AM, SV], where the \(L^p\)-regularity theory of \(-\Delta\) (more generally, of a Dirichlet form) perturbed by a measure potential in the corresponding Kato class was developed, \(1 \leq p < \infty\) (cf. Corollary 1 below).
Recently, [BC] constructed a strong Feller process associated with $-\Delta + \sigma \cdot \nabla$ with a $\mathbb{R}^d$-valued measure $\sigma$ in the Kato class $K^{d+1}_\delta$ (see definition below), for $\delta = 0$, running perturbation-theoretic techniques in $C_0$, thus obtaining e.g. a Brownian motion drifting upward when penetrating certain fractal-like sets. We strengthen their result in Theorem 2 below.

**Definition 4.** A $\mathbb{C}^d$-valued Borel measure $\sigma$ on $\mathbb{R}^d$ is said to belong to $F^\frac{1}{2}_\delta$, the class of weakly form-bounded measures, if there exists $\lambda = \lambda_\delta > 0$ such that

$$\int_{\mathbb{R}^d} \left( (\lambda - \Delta)^{-\frac{1}{2}}(x,y)f(y)dy \right)^2 |\sigma|(dx) \leqslant \delta \|f\|_2^2, \quad f \in S,$$

where $|\sigma| := |\sigma_1| + \cdots + |\sigma_d|$ is the variation of $\sigma$. Clearly, $F^\frac{1}{2}_\delta \subset F^\frac{1}{2}_0$.

**Definition 5.** A $\mathbb{C}^d$-valued Borel measure $\sigma$ on $\mathbb{R}^d$ is said to belong to the Kato class $K^{d+1}_\delta$ if there exists $\lambda = \lambda_\delta > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\lambda - \Delta)^{-\frac{1}{2}}(x,y)|\sigma|(dy) \leqslant \delta.$$

See [BC] for examples of measures in $K^{d+1}_0$.

It is clear that $K^{d+1}_\delta \subset K^{d+1}_0$. By Lemma 1 below, $K^{d+1}_\delta \subset F^\frac{1}{2}_\delta$.

The operator-valued function $\Theta_p(\zeta, \sigma)$, $\text{Re} \zeta > \frac{d}{\alpha - 1} \lambda_\delta$ (see above), ‘a candidate’ for the resolvent of the desired operator realization of $-\Delta + \sigma \cdot \nabla$ generating a $C_0$-semigroup on $C_\infty$, is not well defined for $\sigma$ having non-zero singular part. We modify the method in [K]. Also, in contrast to the setup in [K], a general $\sigma$ doesn’t admit a monotone approximation by regular vector fields $v_k$ (i.e. by $v_kL^d$), which complicates the proof of convergence $\Theta_2(\zeta, v_kL^d) \mathop{\rightarrow}^{\ast} \Theta_2(\zeta, \sigma)$ in $L^2$, needed to carry out the method. We overcome this difficulty using an important variant of the Kato-Ponce inequality by [GO] (see also [BL]) (Proposition 5 below).

Our method depends on the fact that the operators $-\Delta, \nabla$ constituting $-\Delta + \sigma \cdot \nabla$ commute. In particular, our method admits a straightforward generalization to $(-\Delta)^\frac{\alpha}{2} + \sigma \cdot \nabla$, where $(-\Delta)^\frac{\alpha}{2}$ is the fractional Laplacian, $1 < \alpha < 2$, with measure $\sigma$ weakly form-bounded with respect to $\Delta^{\alpha - 1}$, i.e.

$$\int_{\mathbb{R}^d} \left( (\lambda - \Delta)^{-\frac{\alpha}{2}}(x,y)f(y)dy \right)^2 |\sigma|(dx) \leqslant \delta \|f\|_2^2, \quad f \in S$$

for some $\lambda = \lambda_\delta > 0$. (We note that the potential theory of operator $-\Delta^\frac{\alpha}{2}$ perturbed by a drift in the corresponding Kato class, as well as its associated process, attracted a lot of attention recently, see [BJ], [CKS], [KSo] and references therein.)

In Theorems 1, 2 (but not in Corollary 1) we assume that $\sigma$ admits an approximation by (weakly) form-bounded measures $\ll L^d$ having the same form-bound $\delta$ (in fact, $\delta + \varepsilon$, for an arbitrarily small $\varepsilon > 0$ independent of $k$). We verify this assumption for $\sigma = bL^d + \nu$,

$$bL^d \in F^\frac{1}{2}_\delta, \quad \nu \in K^{d+1}_\delta, \quad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2},$$

but do not address, in this note, the issue of constructing such an approximation for a general $\sigma$; we also do not address the issue (we believe, related) of constructing weakly form-bounded vector fields whose singularities are principally different from those of $F^\frac{1}{2}_\delta + K^{d+1}_\delta$ (cf. 1)).
The general classes of drifts studied in the literature in connection with operator $-\Delta + \sigma \cdot \nabla$.

Here we identify $b(x)$ with $b(x)L^d$. 

4. We proceed to precise formulations of our results.

**Notation.** Let

$$m_d := \inf_{\kappa > 0} \sup_{x \neq y, \ \Re \zeta > 0} \frac{|\nabla (\zeta - \Delta)^{-1}(x,y)|}{(\kappa^{-1} \Re \zeta - \Delta)^{-\frac{3}{2}}(x,y)} \quad \text{(2)}$$

(note that $m_d$ is bounded from above by $\frac{1}{2 \pi} (2e)^{-\frac{1}{2}} \sqrt{d} (d-1)^{\frac{1-d}{2}} < \infty$, see [Ki (A.1)],)

$$\mathcal{J} := \left(1 + \frac{1}{1 + \sqrt{1 - m_d \delta}}, 1 + \frac{1}{1 - \sqrt{1 - m_d \delta}}\right).$$

**Theorem 1** ($L^p$-theory of $-\Delta + \sigma \cdot \nabla$). Let $d \geq 3$. Assume that $\sigma$ is a $\mathbb{C}^d$-valued Borel measure in $\bar{F}_0^\frac{1}{2}$ such that $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{C}^d$, $b\mathcal{L}^d \in \bar{F}_0^\frac{1}{2}$, $\nu \in \bar{K}_{\delta_2}^{d+1}$, $\sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}$. 

$$L^p + L^\infty (p > d)$$
or, more generally (see Lemma 7 below), \( \sigma \in \Phi^1_\delta(\lambda) \) is such that there exist \( v_k \in C^\infty_0(\mathbb{R}^d, \mathbb{C}^d) \), \( v_k L^d \in \Phi^1_\delta(\lambda) \), \( v_k L^d \xrightarrow{w} \sigma \).

If \( m_\delta \delta < 1 \), then for every \( p \in J \):

(i) There exists a holomorphic \( C_0 \)-semigroup \( e^{-t\Lambda_p(\sigma)} \) in \( L^p \) such that, possibly after replacing \( v_k L^d \)'s with a sequence of their convex combinations (also weakly converging to measure \( \sigma \)), we have

\[
e^{-t\Lambda_p(v_k L^d)} \Rightarrow e^{-t\Lambda_p(\sigma)} \quad \text{in } L^p,
\]
as \( k \uparrow \infty \), where

\[
\Lambda_p(v_k L^d) := -\Delta + v_k \cdot \nabla, \quad D(\Lambda_p(v_k L^d)) = W^{2,p}.
\]

(ii) The resolvent set \( \rho(-\Lambda_p(\sigma)) \) contains a half-plane \( \mathcal{O} \subset \{ z \in \mathbb{C} : \text{Re} \ z > 0 \} \), and the resolvent \( (\zeta + \Lambda_p(\sigma))^{-1} \), \( \zeta \in \mathcal{O} \), admits an extension by continuity to a bounded linear operator in \( \mathcal{B}(W^{-\frac{1}{r},p}, W^{1+\frac{1}{r},p}) \), where \( 1 \leq r < \min\{2, p\} \), \( \max\{2, p\} < q \).

(iii) The domain of the generator \( D(\Lambda_p(\sigma)) \) is \( W^{1+\frac{1}{r},p} \) for every \( q > \max\{p, 2\} \).

**Remarks.** I. If \( \sigma \ll \mathcal{L}^d \), then the interval \( J \ni p \) in Theorem 1 can be extended, see [Ki] (in [Ki] we work directly in \( L^p \), while in the proof of Theorem 1 we have to first prove our convergence results in \( L^2 \), and then transfer them to \( L^p \) (Proposition 7), hence the more restrictive assumptions on \( p \)).

II. A straightforward modification of the proof of Theorem 1 yields:

**Corollary 1** (\( L^p \)-theory of \(-\Delta + \Psi\)). Let \( d \geq 3 \). Assume that \( \Psi \) is a \( \mathbb{C} \)-valued Borel measure such that

\[
\int_{\mathbb{R}^d} (\lambda - \Delta)^{-\frac{1}{2}}(x,y) f(y) dy \leq \delta \|f\|_2^2,
\]
for some \( \lambda = \lambda_\delta > 0 \). We write \( \Psi \in \Phi(\delta, \lambda) \). Set \( V_k := \rho_k e^{\varepsilon_k \Delta} \Psi, \varepsilon_k \downarrow 0 \), where \( \rho_k \in C^\infty_0 \), \( 0 \leq \rho_k \leq 1 \), \( \rho \equiv 1 \) in \( \{ |x| \leq k \} \), \( \rho \equiv 0 \) in \( \{ |x| \geq k + 1 \} \), so that

\[
V_k L^d \in \Phi(\delta, \lambda) \quad \text{for all } k, \quad V_k L^d \xrightarrow{w} \Psi \quad \text{as } k \uparrow \infty
\]
(see Lemma 2 below). If \( \delta < 1 \), then for every \( p \in (1 + \frac{1}{1 + \sqrt{1 - \delta}}, 1 + \frac{1}{1 - \sqrt{1 - \delta}}) \) there exists a holomorphic \( C_0 \)-semigroup \( e^{-t\Pi_p(\Psi)} \) in \( L^p \) such that

\[
e^{-t\Pi_p(V_k L^d)} \Rightarrow e^{-t\Pi_p(\Psi)} \quad \text{in } L^p,
\]
where \( \Pi_p(V_k L^d) := -\Delta + V_k, \ D(\Pi_p(V_k L^d)) = W^{2,p} \), possibly after replacing \( V_k L^d \)'s with a sequence of their convex combinations (also weakly converging to \( \Psi \)), and the domain of the generator \( D(\Pi_p(\Psi)) \subset W^{1+\frac{1}{r},p} \), for any \( q > \max\{2, p\} \).

Corollary 1 extends the results in [AM, SV] (applied to operator \(-\Delta + \Psi\)), where a real-valued \( \Psi \) is assumed to be in the Kato class \( K^d_\delta \) of measures (e.g. delta-function concentrated on a hypersurface). One disadvantage of Corollary 1 compared to [AM, SV], is that it requires \( |\Psi| \leq \delta(\lambda - \Delta) \) (in the sense of quadratic forms) rather than \( \Psi_- \leq \delta(\lambda - \Delta + \Psi_+) \), where \( \Psi = \Psi_+ - \Psi_- \), \( \Psi_+, \Psi_- \geq 0 \).

The purpose of Theorem 1 is to prove
Theorem 2 ($C_\infty$-theory of $-\Delta + \sigma \cdot \nabla$). Let $d \geq 3$. Assume that $\sigma$ is a $\mathbb{R}^d$-valued Borel measure in $\overline{F}_\delta^2$ such that $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{R}^d$,

$$b\mathcal{L}^d \in \overline{F}_\delta^2, \quad \nu \in \overline{K}_{\delta_2}^{d+1}, \quad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2},$$

or, more generally (see Lemma 2 below), $\sigma \in \overline{F}_\delta^2(\lambda)$ is such that there exist $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $v_k \mathcal{L}^d \in \overline{F}_\delta^2(\lambda)$, $v_k \mathcal{L}^d \xrightarrow{w} \sigma$.

If $m, \delta < \frac{2d-\delta}{(d^2-2p)}$, then:

(i) There exists a positivity preserving contraction $C_0$-semigroup $e^{-t\Lambda_{C_\infty}(\sigma)}$ on $C_\infty$ such that, possibly after replacing $v_k \mathcal{L}^d$’s with a sequence of their convex combinations (also weakly converging to measure $\sigma$) we have

$$e^{-t\Lambda_{C_\infty}(v_k \mathcal{L}^d)} \xrightarrow{s} e^{-t\Lambda_{C_\infty}(\sigma)} \text{ in } C_\infty, \quad t \geq 0,$$

as $k \uparrow \infty$, where

$$\Lambda_{C_\infty}(v_k \mathcal{L}^d) := -\Delta + v_k \cdot \nabla, \quad D(\Lambda_{C_\infty}(v_k \mathcal{L}^d)) = C^2 \cap C_\infty.$$

(ii) [Strong Feller property] $(\mu + \Lambda_{C_\infty}(\sigma))^{-1}|_{S}$ can be extended by continuity to a bounded linear operator in $\mathcal{B}(L^p, C^{0,\gamma})$, $\gamma < 1 - \frac{2}{d-1}$, for every $d - 1 < p < 1 + \frac{1}{1-\sqrt{1-\delta}}$.

(iii) The integral kernel $e^{-t\Lambda_{C_\infty}(\sigma)}(x,y)$ ($x, y \in \mathbb{R}^d$) of $e^{-t\Lambda_{C_\infty}(\sigma)}$ determines the (sub-Markov) transition probability function of a Feller process.

Remark. If $\sigma \ll \mathcal{L}^d$, then the constraint on $\delta$ in Theorem 2 can be relaxed, see [K], cf. Remark I above.

1. Approximating measures

1. In Theorems 1 and 2. Suppose $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{C}^d$, $b\mathcal{L}^d \in \overline{F}_\delta^2(\lambda)$, and $\nu \in \overline{K}_{\delta_2}^{d+1}(\lambda)$.

The following statement is a part of Theorems 1 and 2.

Lemma 1. There exist vector fields $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $k = 1, 2, \ldots$ such that

1. $v_k \mathcal{L}^d \in \overline{F}_\delta^2(\lambda)$, $\sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}$, for every $k$, and
2. $v_k \mathcal{L}^d \xrightarrow{w} \sigma$ as $k \uparrow \infty$.

Proof. We fix functions $\rho_k \in C_0^\infty$, $0 \leq \rho_k \leq 1$, $\rho \equiv 1$ in $\{|x| \leq k\}$, $\rho \equiv 0$ in $\{|x| \geq k + 1\}$, and define $v_k \mathcal{L}^d := b_k \mathcal{L}^d + \nu_k$, where, for some fixed $\varepsilon_k \downarrow 0$,

$$\nu_k := \rho_k e^{\varepsilon_k \Delta} \nu, \quad b_k := \rho_k e^{\varepsilon_k \Delta} b.$$

It is clear that $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $v_k \mathcal{L}^d \xrightarrow{w} \sigma$ as $k \uparrow \infty$. Let us show that $\nu_k \in \overline{K}_{\delta_2}^{d+1}(\lambda)$ for every $k$. Indeed, we have the following pointwise (a.e.) estimates on $\mathbb{R}^d$:

$$(\lambda - \Delta)^{-\frac{1}{2}}|\nu_k| \leq (\lambda - \Delta)^{-\frac{1}{2}} e^{\varepsilon_k \Delta} |\nu| \leq (\lambda - \Delta)^{-\frac{1}{2}} e^{\varepsilon_k \Delta} |\nu| = e^{\varepsilon_k \Delta} (\lambda - \Delta)^{-\frac{1}{2}} |\nu|.$$
Since \( \|e^{\varepsilon_k \Delta}(\lambda - \Delta)^{-\frac{1}{2}}\|_{\infty} \leq \|\lambda - \Delta\|^{-\frac{1}{2}} \|\|_{\infty} \) and, in turn, \( \|\lambda - \Delta\|^{-\frac{1}{2}} \|\|_{\infty} \leq \delta_2 \) \( \Leftrightarrow \nu \in \tilde{K}_{\delta_2}^{d+1}(\lambda) \), we have \( \nu_k \in \tilde{K}_{\delta_2}^{d+1}(\lambda) \). By interpolation, \( \nu_k \in \tilde{F}_{\delta_1}^{+}(\lambda) \). A similar argument yields \( b_k L^d \in \tilde{F}_{\delta_1}^{+}(\lambda) \). Thus, \( v_k L^d \in \tilde{F}_{\delta}^{+}(\lambda) \), for every \( k \).

\[ \square \]

2. **In Corollary** \( \mathbb{1} \) Suppose \( \Psi \in \tilde{F}_{\delta}(\Delta, \lambda) \). Select \( \rho_k \in C_0^\infty \), \( 0 \leq \rho_k \leq 1 \), \( \rho \equiv 1 \) in \( \{|x| \leq k\} \), \( \rho \equiv 0 \) in \( \{|x| \geq k + 1\} \). Fix some \( \varepsilon_k \downarrow 0 \).

**Lemma 2.** We have \( V_k := \rho_k e^{\varepsilon_k \Delta} \Psi \in C_0^\infty(\mathbb{R}^d) \), and

1. \( V_k L^d \in \tilde{F}_{\delta}(\Delta, \lambda) \) for every \( k \),
2. \( \lim_{k \to \infty} V_k L^d = \Psi \) as \( k \uparrow \infty \).

**Proof.** Assertion (2) is immediate. Let us prove (1). It is clear that \( V_k L^d \in \tilde{F}_{\delta}(\Delta, \lambda) \) if and only if

\[ \langle |V_k| \varphi, \varphi \rangle \leq \delta \langle (\lambda - \Delta)^{\frac{1}{2}} \varphi, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle, \quad \varphi \in \mathcal{S}. \]

We have \( |V_k| = \rho_k e^{\varepsilon_k \Delta} |\Psi| \leq e^{\varepsilon_k \Delta} |\Psi| \), so

\[ \langle |V_k| \varphi, \varphi \rangle \leq \langle e^{\varepsilon_k \Delta} |\Psi| \varphi, \varphi \rangle = \langle |\Psi|, e^{\varepsilon_k \Delta} (\varphi^2) \rangle \quad \text{(since } \Psi \in \tilde{F}_{\delta}(\Delta) \text{)} \]

\[ \leq \delta \left( \langle (\lambda - \Delta)^{\frac{1}{2}} [e^{\varepsilon_k \Delta} (\varphi^2)]^{\frac{1}{2}} \rangle^{2} \right) = \delta \langle (\lambda - \Delta)[e^{\varepsilon_k \Delta} (\varphi^2)]^{\frac{1}{2}}, [e^{\varepsilon_k \Delta} (\varphi^2)]^{\frac{1}{2}} \rangle \]

\[ = \delta (e^{\varepsilon_k \Delta} \varphi^2) + \delta \langle \nabla [e^{\varepsilon_k \Delta} (\varphi^2)]^{\frac{1}{2}}, \nabla [e^{\varepsilon_k \Delta} (\varphi^2)]^{\frac{1}{2}} \rangle \quad \text{(we are using } \langle e^{\varepsilon_k \Delta} \varphi^2 \rangle = \langle \varphi^2 \rangle \text{)} \]

\[ = \delta \langle \varphi^2 \rangle + \delta \langle (e^{\varepsilon_k \Delta} \varphi^2)^{-1} (e^{\varepsilon_k \Delta} \varphi \nabla \varphi)^2 \rangle \quad \text{(by Hölder inequality)} \]

\[ \leq \delta \langle \varphi^2 \rangle + \delta (e^{\varepsilon_k \Delta} (\nabla \varphi)^2) = \langle (\lambda - \Delta)^{\frac{1}{2}} \varphi, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle, \quad \text{as needed.} \]

\[ \square \]

**2. Proofs of Theorem** \( \mathbb{1} \)

**Preliminaries.** 1. By Lemma \( \mathbb{1} \), there exist vector fields \( v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d) \), \( k = 1, 2, \ldots \), such that \( v_k L^d \in \tilde{F}_{\delta}^{+}(\lambda) \), \( \sqrt{\sigma} := \sqrt{\sigma_1} + \sqrt{\sigma_2} \), and \( v_k L^d \to \sigma \) as \( k \uparrow \infty \).

2. Due to the strict inequality \( m_d \delta < 1 \), we may assume that the infimum \( m_d \) (cf. (2)) is attained, i.e. there is \( \kappa_d > 0 \)

\[ |\nabla (\zeta - \Delta)^{-1}(x, y)| \leq m_d \left( \kappa_d^{-1} \text{Re}\zeta - \Delta \right)^{-\frac{1}{2}}(x, y), \quad x, y \in \mathbb{R}^d, x \neq y, \text{Re}\zeta > 0. \]

3. Set \( \mathcal{O} := \{ \zeta \in \mathbb{C} : \text{Re}\zeta \geq \kappa_d \lambda \delta \} \).

**The method of proof.** We modify the method of \( \mathbb{K}_1 \). Fix some \( p \in \mathcal{J} \), and some \( r, q \) satisfying \( 1 \leq r < \min\{2, p\} \leq \max\{2, p\} < q \). Our starting object is an operator-valued function

\[ \Theta_p(\zeta, \sigma) := (\zeta - \Delta)^{-\frac{1}{2}} - \frac{1}{2} \Omega_p(\zeta, \sigma, r)(\zeta - \Delta)^{-\frac{1}{2}} \in \mathcal{B}(L^p), \quad \zeta \in \mathcal{O}, \]
which is ‘a candidate’ for the resolvent of the desired operator realization $\Lambda_p(\sigma)$ of $-\Delta + \sigma \cdot \nabla$ on $L^p$. Here

$$\Omega_p(\zeta, \sigma, q, r) := \left. \left( \Omega_2(\zeta, \sigma, q, r) \right|_{L^p \cap L^2} \right) \in B(L^p),$$

where, on $L^2$,

$$\Omega_2(\zeta, \sigma, q, r) := (\zeta - \Delta) - \frac{i}{4} \left( \frac{1}{2} - \frac{1}{4} \right) \left( 1 + Z_2(\zeta, \sigma) \right) (\zeta - \Delta) - \frac{i}{4} \left( \frac{1}{2} - \frac{1}{4} \right) \in B(L^2),$$

$$Z_2(\zeta, \sigma) h(x) := (\zeta - \Delta) - \frac{i}{4} \sigma \cdot \nabla (\zeta - \Delta) - \frac{i}{4} h(x)$$

$$= \int_{\mathbb{R}^d} (\zeta - \Delta) - \frac{i}{4} (x, y) \left( \int_{\mathbb{R}^d} \nabla (\zeta - \Delta) - \frac{i}{4} (y, z) dz \right) \cdot \sigma(y) dy, \quad x \in \mathbb{R}^d, \quad h \in \mathcal{S},$$

and $\|Z_2\|_{2 \to 2} < 1$, so $\Omega_2(\zeta, \sigma, q, r) \in B(L^2)$, see Proposition 1 below. We prove that $\Omega_p(\zeta, \sigma, q, r) \in B(L^p)$ in Proposition 6 below.

We show that $\Theta_p(\zeta, \sigma)$ is the resolvent of $\Lambda_p(\sigma)$ (assertion (i) of Theorem 1) by verifying conditions of the Trotter approximation theorem:

1) $\Theta_p(\zeta, v_k \mathcal{L}^d) = (\zeta + \Lambda_p(v_k \mathcal{L}^d))^{-1}$, $\zeta \in \mathcal{O}$, where $\Lambda_p(v_k \mathcal{L}^d) := -\Delta + v_k \cdot \nabla$, $D(\Lambda_p(v_k \mathcal{L}^d)) = W^{2, p}$.
2) $\sup_{n \geq 1} \|\Theta_p(\zeta, v_k \mathcal{L}^d)\|_{p \to p} \leq C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$.
3) $\mu \Theta_p(\zeta, v_k \mathcal{L}^d) \to 1$ in $L^p$ as $\mu \to \infty$ uniformly in $k$.
4) $\Theta_p(\zeta, v_k \mathcal{L}^d)$ converges weakly in $L^p$ for some $\zeta \in \mathcal{O}$ as $k \to \infty$ (possibly after replacing $v_k \mathcal{L}^d$ with a sequence of their convex combinations, also weakly converging to measure $\sigma$), see Propositions 2 - 7 below for details.

We note that a priori in 1) the set of $\zeta$’s for which $\Theta_p(\zeta, v_k \mathcal{L}^d) = (\zeta + \Lambda_p(v_k \mathcal{L}^d))^{-1}$ may depend on $k$; the fact that it actually does not is the content of Proposition 3.

The proofs of 2), 3), contained in Proposition 2 and 4, are based on an explicit representation of $\Omega_p(\zeta, v_k \mathcal{L}^d, q, r)$, $k = 1, 2, \ldots$, see formula (4) below. (The representation (4) doesn’t exist if $\sigma$ has a non-zero singular part; we have to take a detour via $L^2$, (cf. (3)), which requires us to put somewhat more restrictive assumptions on $\delta$ (compared to [K1], where the case of a $\sigma$ having zero singular part is treated).)

Next, 4) follows from $\Theta_2(\zeta, v_k \mathcal{L}^d) \xrightarrow{\delta} \Theta_2(\zeta, \sigma)$, combined with $\sup_n \|\Theta_p(\zeta, v_k \mathcal{L}^d)\|_{2(p-1) \to 2(p-1)} < \infty \Leftrightarrow 2$ and Hölder inequality, see Proposition 4. Our proof of $\Theta_2(\zeta, v_k \mathcal{L}^d) \xrightarrow{\delta} \Theta_2(\zeta, \sigma)$ (Proposition 5) uses the Kato-Ponce inequality by [GO].

Finally, we note that the very definition of the operator-valued function $\Theta_p(\zeta, \sigma)$ ensures smoothing properties $\Theta_p(\zeta, \sigma) \in B\left(W^{-\frac{1}{2}, p}, W^{1+\frac{1}{2}, p}\right) \Rightarrow$ assertion (ii). Assertion (iii) is immediate from (ii).

Now, we proceed to formulating and proving Propositions 1 - 7.

**Proposition 1.** We have for every $\zeta \in \mathcal{O}$

1) $\|Z_2(\zeta, v_k \mathcal{L}^d)\|_{2 \to 2} \leq \delta$ for all $k$.
2) $\|Z_2(\zeta, \sigma)f\|_2 \leq \delta \|f\|_2$, for all $f \in \mathcal{S}$, all $k$. 


Proof. (1) Define $H := |v_k|^\frac{1}{2} (\zeta - \Delta)^{-\frac{1}{2}}$, $S := v_k^\frac{1}{2} \nabla (\zeta - \Delta)^{-\frac{1}{2}}$ where $v_k^\frac{1}{2} := |v_k|^{-\frac{1}{2}} v_k$. Then $Z_2(\zeta, v_k L^d) = H^* S$, and we have

$$\|Z_2(\zeta, v_k L^d)\|_{2 \rightarrow 2} \leq \|H\|_{2 \rightarrow 2} \|S\|_{2 \rightarrow 2} \leq \|H\|_{2 \rightarrow 2} \|\nabla (\zeta - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} \leq \delta,$$

where $\|\nabla (\zeta - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} = 1$, and $\|H\|_{2 \rightarrow 2} \leq \delta$ (cf. Lemma [1](1)).

(2) We have, for every $f, g, \in S$,

$$\langle g, Z_2(\zeta, \sigma) f \rangle = \langle (\zeta - \Delta)^{-\frac{1}{2}} g, \sigma \cdot \nabla (\zeta - \Delta)^{-\frac{3}{2}} f \rangle$$

(here we are using $v_k L^d \overset{w}{\to} \sigma$)

$$= \lim_k \langle (\zeta - \Delta)^{-\frac{1}{2}} g, v_k \cdot \nabla (\zeta - \Delta)^{-\frac{3}{2}} f \rangle$$

(here we are using assertion (1))

$$\leq \delta \|g\|_2 \|f\|_2,$$

i.e. $\|Z_2(\zeta, \sigma) f\|_2 \leq \delta \|f\|_2$, as needed. \qed

The natural extension of $Z_2(\zeta, \sigma)|_S$ (by continuity) to $B(L^2)$ will be denoted again by $Z_2(\zeta, \sigma)$. Since $\|Z_2(\zeta, v_k L^d)\|_{2 \rightarrow 2}, \|Z_2(\zeta, \sigma)\|_{2 \rightarrow 2} \leq \delta < 1$, we have $\Omega_2(\zeta, v_k L^d(q, r), \Omega_2(\zeta, \sigma, q, r) \in B(L^2)$.

Set

$$\mathcal{I} := \left( \frac{2}{1 + \sqrt{1 - m_d \delta}}, \frac{2}{1 - \sqrt{1 - m_d \delta}} \right).$$

In the next few propositions, given a $p \in \mathcal{I}$, we assume $r, q$ satisfy $1 \leq r < \min \{2, p\} \leq \max \{2, p\} < q$.

The following proposition plays a principal role:

**Proposition 2.** Let $p \in \mathcal{I}$. There exist constants $C_p, C_{p, q, r} < \infty$ such that for every $\zeta \in \mathcal{O}$

(1) $\|\Omega_p(\zeta, v_k L^d(q, r))\|_{p \rightarrow p} \leq C_{p, q, r}$ for all $k$,

(2) $\|\Omega_p(\zeta, v_k L^d(\infty, 1))\|_{p \rightarrow p} \leq C_p |\zeta|^{-\frac{1}{2}}$ for all $k$.

Proof. Denote $v_k^\frac{1}{2} := |v_k|^{\frac{1}{2}-1} v_k$. Set:

$$\Theta_p(\zeta, v L^d(q, r)) := Q_p(q)(1 + T_p)^{-1} G_p(r), \quad \zeta \in \mathcal{O}, \quad (4)$$

where

$$Q_p(q) := (\zeta - \Delta)^{-\frac{1}{2}} \frac{1}{v_k}, \quad T_p := v_k^\frac{1}{2} \cdot \nabla (\zeta - \Delta)^{-\frac{1}{2}} |v_k|^{-\frac{1}{2}}, \quad G_p(r) := v_k^\frac{1}{2} \cdot \nabla (\zeta - \Delta)^{-\frac{3}{2}} - \frac{1}{r},$$

are uniformly (in $k$) bounded in $B(L^p)$, and, in particular, $\|T_p\|_{p \rightarrow p} \leq \frac{\pi}{2} m_d \delta$ (see the proof of [K3], Prop. 1(i)), and $\frac{\pi}{2} m_d \delta < 1$ since $p \in \mathcal{I}$. It follows that $C_{p, q, r} := \sup_k \|\Omega_p(\zeta, v L^d(q, r))\|_{p \rightarrow p} < \infty$.

Now, $\|\Theta_p(\zeta, v L^d(q, r))\|_{L^2 \cap L^p} = \|\Omega_p(\zeta, v L^d(q, r))\|_{L^2 \cap L^p}$ (by expanding $(1 + T_p)^{-1}, (1 + Z_2)^{-1}$ in the K. Neumann series in $L^p$ and in $L^2$, respectively). Therefore, $\Omega_p = \Theta_p \Rightarrow$ assertion (1). The proof of assertion (2) follows closely the proof of [K1], Prop. 1(ii)]. \qed

Clearly, $\Theta_p(\zeta, v_k L^d)$ does not depend on $q, r$. Taking $q = \infty, r = 1$, we obtain from Proposition [2]

$$\|\Theta_p(\zeta, v_k L^d)\|_{p \rightarrow p} \leq C_p |\zeta|^{-1}, \quad \zeta \in \mathcal{O}, \quad (5)$$
Proposition 3. Let $p \in I$. For every $k = 1, 2, \ldots \mathcal{O} \subset \rho(-\Lambda_p(v_k \mathcal{L}^d))$, the resolvent set of $-\Lambda_p(v_k \mathcal{L}^d)$, and

$$
\Theta_p(\zeta, v_k \mathcal{L}^d) = (\zeta + \Lambda_p(v_k \mathcal{L}^d))^{-1}, \quad \zeta \in \mathcal{O},
$$

where $\Lambda_p(v_k \mathcal{L}^d) := -\Delta + v_k \cdot \nabla$, $D(\Lambda_{C^\infty}(v_k \mathcal{L}^d)) = W^2 p$.

Proof. The proof repeats the proof of [K3 Prop. 4].

Proposition 4. For $p \in I$, $\mu \Theta_p(\mu, v_k \mathcal{L}^d) \overset{s}{\to} 1$ in $L^p$ as $\mu \uparrow \infty$ uniformly in $k$.

Proof. The proof repeats the proof of [K3 Prop. 3].

Proposition 5. There exists a sequence $\{\hat{v}_n\} \subset \text{conv}\{v_k\} \subset C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$
\hat{v}_n \mathcal{L}^d \xrightarrow{w} \sigma \text{ as } n \uparrow \infty,
$$

and

$$
\Omega_2(\zeta, \hat{v}_n \mathcal{L}^d, q, r) \overset{s}{\to} \Omega_2(\zeta, \sigma, q, r) \text{ in } L^2, \quad \zeta \in \mathcal{O}.
$$

Proof. To prove (1), it suffices to establish convergence $Z_2(\zeta, \hat{v}_n \mathcal{L}^d) \overset{s}{\to} Z_2(\zeta, \sigma)$ in $L^2$, $\zeta \in \mathcal{O}$.

Let $\eta_r \in C_0^\infty(0 \leq \eta_r \leq 1, \eta_r \equiv 1$ on $\{x \in \mathbb{R}^d : |x| \leq r\}$ and $\eta_r \equiv 0$ on $\{x \in \mathbb{R}^d : |x| \geq r + 1\}$.

Claim 1. We have

- (j) $\|(\zeta - \Delta)^{-\frac{1}{4}} v_k |(\zeta - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} \leq \delta$ for all $k$.
- (jj) $\|(\zeta - \Delta)^{-\frac{1}{4}}|\sigma|(\zeta - \Delta)^{-\frac{1}{4}} f\|_2 \leq \delta \|f\|_2$, for all $f \in \mathcal{S}$.

Proof. Define $H := |v_k|^{\frac{1}{2}}(|\zeta - \Delta)^{-\frac{1}{4}}$. We have $\|(\zeta - \Delta)^{-\frac{1}{4}} v_k |(\zeta - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} = \|H^* H\|_{2 \to 2} = \|H\|^2_{2 \to 2} \leq \delta$, where $\|H\|^2_{2 \to 2} \leq \delta (\leftrightarrow v_k \mathcal{L}^d \in \mathbb{P}^+(\lambda)$, cf. Lemma (1)), i.e. we have proved (j). An argument similar to the one in the proof of Proposition 4 but using assertion (j), yields (jj).

Claim 2. There exists a sequence $\{\hat{v}_n\} \subset \text{conv}\{v_k\}$ such that (1) holds, and for every $r > 1$

$$(\zeta - \Delta)^{-\frac{1}{4}} \eta_r (\hat{v}_n - \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{2}} \overset{s}{\to} 0 \text{ in } L^2, \quad \text{Re} \zeta \geq \lambda.$$

(Here and below we use shorthand $\hat{v}_n - \sigma := \hat{v}_n \mathcal{L}^d - \sigma$).

Proof of Claim 2 In view of Claim 1(j), (jj), it suffices to establish this convergence over $\mathcal{S}$. Let $c(x) = e^{-x^2}$, so that $c \in \mathcal{S}$, $|(\zeta - \Delta)^{-\frac{1}{4}} c| > 0$ on $\mathbb{R}^d$.

Step 1. Let $r = 1$, so $\eta_r = \eta_1$. Let us show that there exists a sequence $\{v_{\ell_1}^1\} \subset \text{conv}\{v_k\}$ such that

$$(\lambda - \Delta)^{-\frac{1}{4}} \eta_1(v_{\ell_1}^1 - \sigma) \cdot \nabla (\lambda - \Delta)^{-\frac{3}{2}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_1 \uparrow \infty.
$$

First, we show that

$$(\lambda - \Delta)^{-\frac{1}{4}} \eta_1(v_k - \sigma)(\lambda - \Delta)^{-\frac{1}{4}} c \overset{w}{\to} 0 \text{ in } L^2.
$$

Indeed, by Claim 1(j), (jj), $\|(\lambda - \Delta)^{-\frac{1}{4}} \eta_1(v_k - \sigma)(\lambda - \Delta)^{-\frac{1}{4}} c\|_2 \leq 2\delta\|c\|_2$ for all $k$. Hence, there exists a subsequence of $\{v_k\}$ (without loss of generality, it is $\{v_k\}$ itself) such that $(\lambda - \Delta)^{-\frac{1}{4}} \eta_1(v_k - \sigma)(\lambda - \Delta)^{-\frac{1}{4}} c \overset{s}{\to} 0 \text{ in } L^2$.
We may assume without loss of generality that each \(v\) i.e. \((f, h) = 0\). Since \(f \in \mathcal{S}\) was arbitrary, we have \(h = 0\), which yields (9).

Now, in view of (9), by Mazur’s Theorem, there exists a sequence \(\{v_{\ell_1}\} \subset \text{conv}\{v_k\}\) such that
\[
(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v_{\ell_1} - \sigma)(\lambda - \Delta)^{-\frac{1}{4}}c \overset{\text{w}}{\to} 0 \text{ in } L^2.
\]
(10)

We may assume without loss of generality that each \(v_{\ell_1} \in \text{conv}\{v_n\}_{n \geq \ell_1}\).

Next, set \(\ell := \ell_1\), \(\varphi_{\ell} := \eta_1(v_{\ell} - \sigma)\), \(\Phi := (\lambda - \Delta)^{-\frac{1}{4}}c\), fix some \(u \in \mathcal{S}\). We estimate:
\[
\|((\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u\|^2_2 = \langle \varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u, (\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u \rangle \\
\left(\text{since } \varphi_{\ell} \equiv 0 \text{ on } \{|x| \geq 2\}, \text{ in the left multiple } \varphi_{\ell} = \varphi_{\ell}\Phi\right) \\
= \langle \varphi_{\ell}\Phi\frac{\nabla(\lambda - \Delta)^{-\frac{1}{4}}u, (\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u \rangle \\
= \langle \varphi_{\ell}\Phi\frac{\nabla(\lambda - \Delta)^{-\frac{1}{4}}u, [\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{3}{4}}u] \rangle \\
\text{(here we are using in the left multiple that } \varphi_{\ell} = \lambda - \Delta)^{-\frac{1}{4}}(\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell}) \\
= \langle (\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell}\Phi, (\lambda - \Delta)^{-\frac{1}{4}}(fg_{\ell}) \rangle
\]
where we set \(f := \Phi\frac{\nabla(\lambda - \Delta)^{-\frac{1}{4}}u \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d), g_{\ell} := (\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u \in (\lambda - \Delta)^{-\frac{1}{4}}L^2 \text{ (in view of Claim (11)}, (j)).\) Thus, in view of the above estimates,
\[
\|((\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell} \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}u\|^2_2 \leq \|((\lambda - \Delta)^{-\frac{1}{4}}\varphi_{\ell}\Phi\|\|\lambda - \Delta)^{-\frac{1}{4}}(fg_{\ell})\|_2\).
\]
By the Kato-Ponce inequality of [GO] Theorem 1],
\[
\|((\lambda - \Delta)^{-\frac{1}{4}}(fg_{\ell})\|_2 \leq C\left(\|f\|_{\infty}\|\lambda - \Delta)^{-\frac{1}{4}}g_{\ell}\|_2 + \|((\lambda - \Delta)^{-\frac{1}{4}}f\|_{\infty}\|g_{\ell}\|_2\right),
\]
for some \(C = C(d) < \infty\). Clearly, \(\|f\|_{\infty}, \|((\lambda - \Delta)^{-\frac{1}{4}}f\|_{\infty} < \infty\), and \(\|((\lambda - \Delta)^{-\frac{1}{4}}g_{\ell}\|_2, \|g_{\ell}\|_2\) are uniformly (in \(\ell\)) bounded from above according to Claim (11), (j)). Thus, in view of (11), we obtain (S) (recalling that \(\ell_1 = \ell\), and \(\varphi_{\ell_1} = \eta_1(v_{\ell_1} - \sigma)\)).

Step 2. Now, we can repeat the argument of Step 1, but starting with sequence \(\{v_{\ell_1}\}\) in place of \(\{v_{\ell_2}\}\), thus obtaining a sequence \(\{v_{\ell_2}\} \subset \text{conv}\{v_{\ell_1}\}\) such that
\[
(\lambda - \Delta)^{-\frac{1}{4}}\eta_2(v_{\ell_2} - \sigma) \cdot \nabla(\lambda - \Delta)^{-\frac{1}{4}}c \overset{\text{w}}{\to} 0 \text{ in } L^2 \text{ as } \ell_2 \uparrow \infty.
\]
We may assume without loss of generality that each \(v_{\ell_2} \in \text{conv}\{v_{\ell_1}\}_{\ell_1 \geq \ell_2}\). Therefore, we also have
\[
(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v_{\ell_2} - \sigma) \cdot \nabla(\lambda - \Delta)^{-\frac{3}{4}}c \overset{\text{w}}{\to} 0 \text{ in } L^2 \text{ as } \ell_2 \uparrow \infty.
\]
Repeating this procedure \(n - 2\) times, we obtain a sequence \(\{v^n_{\ell_n}\} \subseteq \text{conv}\{v^{n-1}_{\ell_{n-1}}\} \subset \text{conv}\{v_k\}\) such that

\[
(\lambda - \Delta)^{-\frac{3}{4}} \eta_i (v^n_{\ell_n} - \sigma) \cdot \nabla (\lambda - \Delta)^{-\frac{3}{4}} \overset{\ell_n \uparrow \infty}{\to} 0 \text{ in } L^2, \quad 1 \leq i \leq n.
\]

**Step 3.** We set \(\hat{v}_n := v^n_{\ell_n}, n \geq 1\), so for every \(r \geq 1\)

\[
(\lambda - \Delta)^{-\frac{3}{4}} \eta_r (\hat{v}_n - \sigma) \cdot \nabla (\lambda - \Delta)^{-\frac{3}{4}} \overset{\ell_n \uparrow \infty}{\to} 0 \text{ in } L^2.
\]

Since \(v^n_{\ell_n} \in \text{conv}\{v^{n-1}_{\ell_{n-1}}\}_{\ell_{n-1} \geq \ell_n}, v^{n-1}_{\ell_{n-1}} \in \text{conv}\{v^{n-2}_{\ell_{n-2}}\}_{\ell_{n-2} \geq \ell_{n-1}}, \) etc, we obtain that \(v^n_{\ell_n} \in \text{conv}\{v_k\}_{k \geq \ell_n}, i.e.\) we also have \(\square\). Finally, \(\square\) combined with the resolvent identity yield

\[
(\zeta - \Delta)^{-\frac{3}{4}} \eta_r (\hat{v}_n - \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} \overset{\ell_n \uparrow \infty}{\to} 0 \text{ in } L^2, \quad \text{Re } \zeta \geq \lambda.
\]
i.e. we have proved Claim \(2\) \(\square\)

We are in a position to complete the proof of Proposition \(\square\). Let us show that, for every \(\zeta \in \mathcal{O}\)

\[
Z_2(\zeta, \hat{v}_n L^d)g - Z_2(\zeta, \sigma)g = (\zeta - \Delta)^{-\frac{3}{4}} (\hat{v}_n - \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} g \overset{\ell_n \uparrow \infty}{\to} 0 \text{ in } L^2, \quad g \in \mathcal{S}.
\]

Let us fix some \(g \in \mathcal{S}\). We have

\[
(\zeta - \Delta)^{-\frac{3}{4}} (\hat{v}_n - \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} g = (\zeta - \Delta)^{-\frac{3}{4}} (\hat{v}_n - \eta_r \hat{v}_n) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} g + (\zeta - \Delta)^{-\frac{3}{4}} (\eta_r \hat{v}_n - \eta_r \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} g + (\zeta - \Delta)^{-\frac{3}{4}} (\eta_r \sigma - \sigma) \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}} g =: I_{1,r,n} + I_{2,r,n} + I_{3,r,n}.
\]

**Claim 3.** Given any \(\varepsilon > 0\), there exists \(r\) such that \(\|I_{3,r,n}\|_2, \|I_{1,r,n}\|_2 < \varepsilon\), for all \(n, \zeta \in \mathcal{O}\).

**Proof of Claim \(3\)** It suffices to prove \(\|I_{1,r,n}\|_2 < \varepsilon\) for all \(n\). We will need the following elementary estimate: \(|\nabla (\zeta - \Delta)^{-\frac{3}{4}}(x,y)| \leq M_d(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}}(x,y), x, y \in \mathbb{R}^d, x \neq y\). We have

\[
\|I_{1,r,n}\|_2 = \|(\text{Re }\zeta - \Delta)^{-\frac{3}{4}} (1 - \eta_r) \hat{v}_n \cdot \nabla (\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2 \\
\leq c_d M_d \|(\text{Re }\zeta - \Delta)^{-\frac{3}{4}} (1 - \eta_r)|\hat{v}_n|(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2 \\
\leq c_d M_d \|(\text{Re }\zeta - \Delta)^{-\frac{3}{4}} |\hat{v}_n|^\frac{3}{4} \|_2 \|(1 - \eta_r)|\hat{v}_n|^\frac{1}{4}(|\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2.
\]

We have \(\|(\text{Re }\zeta - \Delta)^{-\frac{3}{4}} |\hat{v}_n|^\frac{1}{2} \|_{2 \to 2} \leq \delta\) in view of Lemma \(\square\)(1). In turn,

\[
|1 - \eta_r)|\hat{v}_n|^\frac{1}{4}(|\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2 = |\hat{v}_n|^\frac{1}{4}(|\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} (\kappa_d^{-1}\text{Re }\zeta - \Delta)^\frac{1}{4} (1 - \eta_r)(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g,
\]

so

\[
\|(1 - \eta_r)|\hat{v}_n|^\frac{1}{4}(|\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2 \leq \delta\|(\kappa_d^{-1}\text{Re }\zeta - \Delta)^\frac{1}{4} (1 - \eta_r)(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2,
\]

where \(\delta\|(\kappa_d^{-1}\text{Re }\zeta - \Delta)^\frac{1}{4} (1 - \eta_r)(\kappa_d^{-1}\text{Re }\zeta - \Delta)^{-\frac{3}{4}} g\|_2 \to 0\) as \(r \uparrow \infty\). The proof of Claim \(3\) is completed. \(\square\)
Claim 2, which yields convergence \( \|I_{2,r,n}\|_2 \to 0 \) as \( n \to \infty \) for every \( r \), and Claim 3 imply that

\[
Z_2(\zeta, \hat{v}_n L^d) - Z_2(\zeta, \sigma) \xrightarrow{\mathbb{D}} 0 \quad \text{in} \quad L^2, \quad g \in \mathcal{S}, \quad \zeta \in \mathcal{O},
\]

which, in view of Claim 1(j), (jj), yields \( Z_2(\zeta, \hat{v}_n L^d) - Z_2(\zeta, \sigma) \xrightarrow{\mathbb{D}} 0 \), \( \zeta \in \mathcal{O} \), in \( L^2 \) (\( \Rightarrow 17 \)). By Claim 2 we also have (6). This completes the proof of Proposition 5. \( \square \)

**Proposition 6.** Let \( p \in \mathcal{I} \). There exist constants \( C_p, C_{p,q,r} < \infty \) such that for every \( \zeta \in \mathcal{O} \)

\[
\begin{align*}
(1) & \|\Omega_p(\zeta, \sigma, q, r)\|_{p \to p} \leq C_{p,q,r} \quad \text{for all} \quad k, \\
(2) & \|\Omega_p(\zeta, \sigma, \infty, 1)\|_{p \to p} \leq C_p|\zeta|^\frac{p}{2}, \quad \text{for all} \quad k.
\end{align*}
\]

**Proof.** Immediate from Proposition 2 and Proposition 5. \( \square \)

Now, we assume that \( p \in \mathcal{J} \subset \mathcal{I} \).

**Proposition 7.** Let \( \{\hat{v}_n\} \) be the sequence in Proposition 5. For any \( p \in \mathcal{J} \),

\[
\Omega_p(\zeta, \hat{v}_n L^d, q, r) \xrightarrow{\mathbb{D}} \Omega_p(\zeta, \sigma, q, r) \quad \text{in} \quad L^p, \quad \zeta \in \mathcal{O}.
\]

**Proof.** Set \( \Omega_p := \Omega_p(\zeta, \sigma, q, r) \), \( \Omega_p^n := \Omega_p(\zeta, \hat{v}_n L^d, q, r) \). Recall that since \( p \in \mathcal{J} \), we have \( 2(p-1) \in \mathcal{I} \). Since \( \Omega_p, \Omega_p^n \in \mathcal{B}(L^p) \), it suffices to prove convergence on \( \mathcal{S} \). We have (\( f \in \mathcal{S} \)):

\[
\|\Omega_p f - \Omega_p^n f\|_p^p \leq \|\Omega_p f - \Omega_p^n f\|_{2(p-1)}^{p-1} \|\Omega_p f - \Omega_p^n f\|_2.
\]  \( (12) \)

Let us estimate the right-hand side in \( (12) \):

1) \( \Omega_p f - \Omega_p^n f = \Omega_2 f - \Omega_2^n f \xrightarrow{\mathbb{D}} 0 \) in \( L^2 \) as \( k \to \infty \) by Proposition 5.

2) \( \Omega_p f - \Omega_p^n f \xrightarrow{\mathbb{D}} \Omega_p f \) in \( L^p \), as needed. \( \square \)

This completes the proof of assertion \((i)\), and thus the proof of Theorem 1.

### 3. PROOF OF THEOREM 2

\((i)\) The approximating vector fields \( v_k \) were constructed in Section 1. The proof repeats the proof of [K1] Theorem 2. Namely, we verify conditions of the Trotter approximation theorem for \( \Lambda_{C_{\infty}}(v_k) := -\Delta + v_k \cdot \nabla, \ D(\Lambda_{C_{\infty}}(v_k)) = C^2 \cap C_{\infty} :\)

1) \( \sup_n \| (\mu + \Lambda_{C_{\infty}}(v_k))^{-1} \|_{\infty \to \infty} \leq \mu^{-1}, \quad \mu \geq \kappa_d \lambda \delta \).

2) \( \mu (\mu + \Lambda_{C_{\infty}}(v_k))^{-1} \to 1 \) in \( C_{\infty} \) as \( \mu \uparrow \infty \) uniformly in \( n \).

3) \( \Theta_p(\mu, \sigma) := (\mu - \Delta)^{-\frac{1}{2}} \Omega_p(\mu, \sigma, q, 1) \in \mathcal{B}(L^p), \quad \mu \geq \kappa_d \lambda \),

where \( \max\{2, p\} < q \), see the proof of Theorem 1. We will be using the properties of \( \Theta_p(\mu, \sigma) \) established there. Without loss of generality, we may assume that \( \{v_k\} \) is the sequence constructed in Proposition 7 that is, \( v_k \xrightarrow{\mathbb{D}} \sigma \), and \( \Omega_p(\mu, v_k L^d, q, 1) \xrightarrow{\mathbb{D}} \Omega_p(\mu, \sigma, q, 1) \) in \( L^p \) as \( k \to \infty \).
Given any $\gamma < 1 - \frac{d-1}{p}$ we can select $q$ sufficiently close to $p$ so that by the Sobolev embedding theorem,

$$(\mu - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} [L^p] \subset C^{0, \gamma} \cap L^p,$$

and $(\mu - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} \in B(L^p, C_{\infty})$.

Then Proposition 7 yields $\Theta_p(\mu, v_n L^d) f \overset{S}{\rightarrow} \Theta_p(\mu, \sigma) f$ in $C_{\infty}$, $f \in S$, as $n \uparrow \infty$. The latter, combined with the next proposition and 1°), verifies condition 3°):

**Proposition 8.** For every $k = 1, 2, \ldots$, $\Theta_p(\mu, v_k L^d) S \subset S$, and

$$(\mu + \Lambda_{C_{\infty}} (v_k L^d))^{-1} |S = \Theta_p(\mu, v_k L^d)|_S, \quad \mu \geq \kappa d \lambda.$$

**Proof.** The proof repeats the proof of [Ki] Prop. 6. \hfill \Box

**Proposition 9.** $\mu \Theta_p(\mu, v_k) \overset{S}{\rightarrow} 1$ in $C_{\infty}$ as $\mu \uparrow \infty$ uniformly in $k$.

**Proof.** The proof repeats the proof of [Ki] Prop. 8. \hfill \Box

The last two proposition yield 2°). This completes the proof of assertion (i).

(ii) follows from the equality $\Theta_p(\mu, \sigma) |_S = (\mu + \Lambda_{C_{\infty}} (C_{\infty}))^{-1} |_S$ (by construction), representation (13), and the Sobolev embedding theorem.

(iii) It follows from (i) that $e^{-t \Lambda_{C_{\infty}} (\sigma)}$ is positivity preserving. The latter, 1°) and the Riesz-Markov-Kakutani representation theorem imply (iii).

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