INSTABILITY OF AN EQUILIBRIUM OF A PARTIAL
DIFFERENTIAL EQUATION

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Abstract. A nonlinear parabolic differential equation is presented which has
at least one equilibrium. This equilibrium is shown to have a negative definite
linearization, but a spectrum which includes zero. An elementary construction
shows that the equilibrium is not stable.

1. Introduction

This article demonstrates that in infinite-dimensional settings, negative definiteness
of an equilibrium of a dynamical system is not sufficient to ensure that the
equilibrium is stable. This is in stark contrast to the situation in finite-dimensional
settings, where negative definiteness implies stability of the equilibrium. (See [1],
for instance.) A crucial point is that this system has a spectrum which includes
zero, although zero is not an eigenvalue. As a result, stability is possible (as in the
unforced heat equation), though not guaranteed.

The particular problem we study is the Cauchy problem

\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - 2f(x)u(t,x) - u^2(t,x) \\
& \quad \text{for } t > 0, x \in \mathbb{R}^n \text{ for } n \geq 1,
\end{aligned}
\tag{1}
\]

where \( f \in C_0^\infty(\mathbb{R}^n) \) is a positive function with two bounded derivatives. Since
the linear portion of the right side of (1) is a sectorial operator, we can use (1) to
define a nonlinear semigroup. [6] [12] This turns (1) into a dynamical system, the
behavior of which is largely controlled by its equilibria. This problem evidently has
an equilibrium: \( u(t,x) = 0 \) for all \( t, x \). Depending on the exact choice of \( f \), there
may be other equilibria, however they will not concern us here. We show that the
spectrum of the equilibrium \( u \equiv 0 \) includes zero by an elementary construction akin
to that of [13]. More interestingly, a technique pioneered by Fujita [5] shows that
this equilibrium is not stable in \( L^p(\mathbb{R}^n) \) for \( n = 1 \) and any \( 1 \leq p \leq \infty \).

2. Motivation

The problem (1) arises as a transformation of a related problem, namely

\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - 2f(x)u(t,x) + \phi(x) \\
u(0,x) &= u_0(x) \in C^\infty(\mathbb{R}^n) \\
t > 0, x \in \mathbb{R}^n \text{ for } n \geq 1,
\end{aligned}
\tag{2}
\]

with \( \phi \in C_0^\infty(\mathbb{R}^n) \). This equation describes a reaction-diffusion equation [11], or a
diffusive logistic population model with a spatially-varying carrying capacity. The
spatial inhomogeneity of $\phi$ makes the analysis of (2) much more complicated than that of typical reaction-diffusion equations. The existence of the equilibria for (2) is a fairly difficult problem, which depends delicately on $\phi$. We will not treat the existence of equilibria for (2) here, but assume that $f$ is a positive equilibrium for (2). (See [2] for a proof of existence for such a positive equilibrium.) Then we can look at the behavior of perturbations near $f$, for instance

$$\begin{align*}
\frac{\partial (f + u)}{\partial t} &= \Delta (f + u) - (f + u)^2 + \phi \\
\frac{\partial u}{\partial t} &= \Delta f + \Delta u - f^2 - 2fu - u^2 + \phi \\
\frac{\partial u}{\partial t} &= \Delta u - 2fu - u^2,
\end{align*}$$

which is (1). Notice that this transforms the equilibrium $f$ of (2) to the zero function in (1). The situation of (1) is considerably easier to examine.

3. Properties of the spectrum

We need to linearize (1) in order to examine the spectrum of the equilibrium and to begin to analyze its stability. In doing so, we roughly follow the outline given in [6]. Recall the following definition of the derivative map in a Banach space:

**Definition 1.** Suppose $R : B_1 \rightarrow B_2$ is a map from one Banach space to another. The derivative map of $R$ at $u \in B_1$ is the unique linear map $D : B_1 \rightarrow B_2$ such that for each sequence $\{h_n\}_{n=1}^\infty$ with $\|h_n\| \rightarrow 0$,

$$\lim_{n \rightarrow \infty} \frac{D(h_n) - R(u + h_n) + R(u)}{\|h_n\|} = 0.$$

Of course, such a map may not exist. If it does exist, we say $R$ is differentiable at $u$. The linearization $L$ of $R$ is the affine map given by the formula $L(h) = R(u) + D(h)$.

For this section, we shall work in the Hilbert space $L^2(\mathbb{R}^n)$ with the usual norm (using the fact that $\Delta$ is densely defined wherever necessary). The linearization of (1) at $u \equiv 0$ is easily computed to be

$$(3) \quad \frac{\partial h(t, x)}{\partial t} = \Delta h(t, x) - 2f(x)h(t, x).$$

Suppose $h(x, t) = X(x)T(t)$, then we can separate variables in (3), obtaining

$$T'(t) - \lambda T(t) = 0$$

$$\Delta X(x) - (\lambda + 2f(x))X(x) = 0.$$

The separation constant $\lambda$ can be determined by examining the eigenvalue problem

$$(4) \quad (\Delta - 2f(x))X(x) = \lambda X(x),$$

which is essentially the computation of the energy levels of a Schrödinger equation. The operator $(\Delta - 2f)$ is a Schrödinger operator with potential $-2f$. Due to its importance in quantum mechanics, much is known about Schrödinger operators (see [12] for a summary).

If $\Re(\lambda) < 0$ over all of the eigenvalues $\lambda$ in (4), we would normally conclude that $h \rightarrow 0$ as $t \rightarrow \infty$, that $u \equiv 0$ is a stable equilibrium. However, as we shall see in Section 4, this is false. The cause of the instability is that although $\Re(\lambda) < 0$ for all eigenvalues, $\lambda = 0$ is in the spectrum of the operator $(\Delta - 2f)$. 
Lemma 2. The spectrum of a self-adjoint, negative definite operator $T$ has spectrum which is confined to the closed left half-plane \( \{ \lambda \in \mathbb{C} | \Re(\lambda) \leq 0 \} \).

Proof. This is a standard argument (for instance, see [10]), which we sketch briefly. First, suppose \( \lambda \) is an eigenvalue of \( T \) with an eigenfunction \( \psi \). Then
\[
\lambda = \frac{\langle \psi, T\psi \rangle}{\langle \psi, \psi \rangle} = \bar{\lambda} \leq 0.
\]
On the other hand, the Fredholm alternative (see [9]) implies that \( T - \lambda \) is surjective for \( \lambda > 0 \).

Finally, we note that for \( \Re(\lambda) > 0 \), \( (T - \lambda)^{-1} \) is bounded:
\[
\langle (T - \lambda)^{-1} \psi, (T - \lambda)^{-1} \psi \rangle = \langle T\psi, T\psi \rangle - 2\Re(\lambda) \langle \psi, T\psi \rangle + |\lambda|^2 \langle \psi, \psi \rangle \geq |\lambda|^2 \langle \psi, \psi \rangle,
\]
by the negative definiteness of \( T \). Hence, for \( \Re(\lambda) > 0 \), \( (T - \lambda) \) has a bounded inverse. \( \square \)

Lemma 3. The self-adjoint operator \( (\Delta - 2f(x)) \) is negative definite if and only if \( f > 0 \) almost everywhere. (See [13] for a generalization.)

Proof. It is well-known and easily shown that \( (\Delta - 2f) \) is self-adjoint. See [8], for example. The self-adjointness of \( (\Delta - 2f) \) follows immediately from that of \( \Delta \). It is also well-known that \( \Delta \) is negative definite: with zero boundary conditions, the divergence theorem gives
\[
\langle u, \Delta u \rangle = \int \bar{u} \Delta u \, dx = -\int \nabla \bar{u} \cdot \nabla u \, dx < 0.
\]
So the only thing that will spoil the negative definiteness is \( f \). Suppose \( f > 0 \) almost everywhere, and \( u \in L^2 \). Then
\[
\langle u, -2fu \rangle = -2 \int \bar{u} f u \, dx = -2 \int f |u|^2 \, dx < 0.
\]
On the other hand, suppose \( A = \{ x \in \mathbb{R}^n | f(x) \leq 0 \} \) has positive measure. Then let \( u = 1_A \) and compute
\[
\langle u, (\Delta - 2f)u \rangle = \langle u, -2fu \rangle = -2 \int \bar{u} f u \, dx = -2 \int f |u|^2 \, dx \geq 0.
\]
So we have that \( (\Delta - 2f) \) is not negative definite in that case. \( \square \)

Lemma 4. Suppose \( f \) is a positive continuous function on \( \mathbb{R}^n \). Then \( (\Delta - 2f) \) is injective on \( C^2_0(\mathbb{R}^n) \).

Proof. Let \( u \in C^2_0(\mathbb{R}^n) \) satisfy \( (\Delta - 2f)u = 0 \). Let \( y = \sup_{x \in \mathbb{R}^n} u(x) \). We claim that \( y = 0 \). Suppose the contrary, that \( y > 0 \). Since \( u \in C^2_0(\mathbb{R}^n) \), there is an \( R > 0 \) such that for all \( ||x|| > R \), \( u(x) < y \). Thus \( M = u^{-1}(\{y\}) \) is compact. By the maximum principle, there exists an \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood of \( M \),
\[
M_\epsilon = \{ x \in \mathbb{R}^n | \inf_{z \in M} ||z - x|| < \epsilon \}
\]
has \( \Delta u(M_e - M) < 0 \). On the other hand, \( N = M_e \cap u^{-1}((0, y)) \) is a nonempty open set on which \( u|N > 0 \) and \( \Delta u|N < 0 \). But since \( f \) is positive and \( \Delta u = 2fu \), this is a contradiction. Similar reasoning leads to \( \inf_{x \in \mathbb{R}^n} u(x) = 0 \), so in fact \( u \equiv 0 \). \( \square \)

Since \( C_0^\infty(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), this implies that \( \lambda = 0 \) is not an eigenvalue of \( (\Delta - 2f) \) over \( L^2(\mathbb{R}^n) \).

**Lemma 5.** The spectrum of \((\Delta - 2f)\) includes zero when \( f \in C_0^\infty(\mathbb{R}^n) \) is a positive function. (See [13] for the most general result of this kind.)

**Proof.** By Lemma 4 and the Fredholm alternative, \((\Delta - 2f)^{-1}\) exists. We show that \((\Delta - 2f)^{-1}\) is not bounded, by constructing a sequence \( \{\psi_m\} \) such that

\[
\lim_{m \to \infty} \frac{\langle (\Delta - 2f)\psi_m, (\Delta - 2f)\psi_m \rangle}{\langle \psi_m, \psi_m \rangle} = 0.
\]

Let \( \psi_m \) be the function

\[
\psi_m(x) = \left( \frac{1}{2A_m \sqrt{\pi}} \right)^{n/2} e^{-\frac{x^2 - 2B_m x}{4A_m^2}},
\]

where \( A_m \in \mathbb{R} \) and \( B_m \in \mathbb{R}^n \) are constructed as follows. Choose \( A_m \) so that

\[
\langle \Delta \psi_m, \Delta \psi_m \rangle < \frac{1}{2m}
\]

(that this is possible follows from an easy computation). Then select \( B_m \) so that

\[
\langle f \psi_m, f \psi_m \rangle < \frac{1}{2m},
\]

which is possible since \( f \in C_0(\mathbb{R}^n) \). Notice that \( \langle \Delta \psi_m, \Delta \psi_m \rangle \) is independent of \( B_m \), so the second choice does not interfere with the first. Evidently

\[
\lim_{m \to \infty} \langle (\Delta - 2f)\psi_m, (\Delta - 2f)\psi_m \rangle = 0,
\]

by the Schwarz inequality. On the other hand, \( \langle \psi_m, \psi_m \rangle = 1 \) for all \( m \). As a result, (5) holds. \( \square \)

As a result of Lemmas 4 and 5 we have three things: (1) that the spectrum is contained in the closed left half plane, (2) the spectrum includes zero, and (3) zero is not an eigenvalue.

### 4. Instability of the Equilibrium

Now we construct an initial condition \( h_\epsilon \in C^\infty \cap L^p(\mathbb{R}) \) for the problem (1) for \( u (\epsilon > 0 \text{ and } 1 \leq p \leq \infty \text{ arbitrary}) \), with \( \|h_\epsilon\|_p < \epsilon \), for which \( \|u(t)\|_p \to \infty \). In particular, this implies that \( u \equiv 0 \) is not a stable equilibrium of (1). We employ a technique of Fujita, which appears in the first part of [5]. (Additionally, [3] contains a more elementary discussion with a similar construction.) The case for \( p = \infty \) is somewhat more delicate than the case where \( 1 \leq p < \infty \), so we must treat it separately.
4.1. The technique of Fujita. The technique of Fujita examines the blow-up behavior of nonlinear parabolic equations by viewing them as ordinary differential equations on a Hilbert space. Suppose $u(t)$ solves

$$\frac{\partial u(t)}{\partial t} = Lu(t) + N(u(t)),$$

where $L$ is a linear operator not involving $t$, and $N$ may be nonlinear and may depend on $t$. Suppose that $v(t)$ solves

$$\frac{\partial v(t)}{\partial t} = -L^*v(t),$$

where $L^*$ is the adjoint of $L$. Let $J(t) = \langle v(t), u(t) \rangle$. We observe that if $|J(t)| \to \infty$ then either $\|v(t)\|$ or $\|u(t)\|$ also does. So if $v(t)$ does not blow up, then we can show that $\|u(t)\|$ blows up, and perhaps more is true. The behavior of $J(t)$ is often easy to understand, because

$$\frac{d}{dt} J(t) = \frac{d}{dt} \langle v(t), u(t) \rangle$$

$$= \left\langle \frac{dv}{dt}, u(t) \right\rangle + \left\langle v(t), \frac{du}{dt} \right\rangle$$

$$= \langle -L^*v(t), u(t) \rangle + \langle v(t), Lu(t) + N(u(t)) \rangle$$

$$= \langle v(t), N(u(t)) \rangle$$

where there is typically a technical justification required for the second equality. It is often possible to find a bound for $\langle v(t), N(u(t)) \rangle$ in terms of $J(t)$. So then the method provides a fence (in the sense of [7]) for $J(t)$, which we can solve to give an bound on $|J(t)|$. As a result, the blow-up behavior of $u(t)$ is controlled by the solution of an ordinary differential equation (for $J(t)$) and a linear parabolic equation (for $v(t)$), both of which are much easier to examine than the original nonlinear parabolic equation.

4.2. Instability in $L^p$ for $1 \leq p < \infty$. We begin our application of the method of Fujita to the $L^p$ case by working with $L = \frac{d^2}{dx^2}$ and $N(u) = -2fu - u^2$ in [6]. Since [7] is then the backwards heat equation, which is not well-posed for all $t$, we must be a little more careful than the method initially suggests. As a result, we consider a family of solutions $v_\epsilon$ to [7] that have slightly extended domains of definition.

**Definition 6.** Let $H(t, x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|x|^2}{4t} \right)$, which is the heat kernel. Let $v_\epsilon(s, x) = H(t-s+\epsilon, x)$ for fixed $t$ and $s < t + \epsilon$.

**Lemma 7.** Suppose $u(t, x) \leq 0$ satisfies [1], and $u(t) \in L^p(\mathbb{R})$ for each $t$. Define

$$J_\epsilon(s) = \int v_\epsilon(s, x)u(s, x)dx.$$ 

Then $\frac{dJ_\epsilon(s)}{ds} \leq -(J_\epsilon(s))^2 - 2\|f\|_\infty J_\epsilon(s)$. 

**Proof.** First of all, we observe that since $u \in L^p(\mathbb{R})$, $v_\epsilon(s, \cdot)u(s, \cdot)$ is in $L^1(\mathbb{R})$ for each $s < t$.

Now suppose we have a sequence $\{m_n\}$ of compactly supported smooth functions with the following properties: [11]

- $m_n \in C^\infty(\mathbb{R})$,
- $m_n(x) \geq 0$ for all $x$. 

• supp($m_n$) is contained in the interval $(-n - 1, n + 1)$, and
• $m_n(x) = 1$ for $|x| \leq n$.

Then it follows that

$$J_\epsilon(s) = \lim_{n \to \infty} \int v_\epsilon(s, x) u(s, x) m_n(x) dx.$$

Now

$$\frac{d}{ds} J_\epsilon(s) = \frac{d}{ds} \lim_{n \to \infty} \int v_\epsilon(s, x) u(s, x) m_n(x) dx$$

$$= \lim_{h \to 0} \lim_{n \to \infty} \frac{1}{h} \int (v_\epsilon(s + h, x) u(s + h, x) - v_\epsilon(s, x) u(s, x)) m_n(x) dx.$$

We’d like to exchange limits using uniform convergence. To do this we show that

$$\lim_{n \to \infty} \lim_{h \to 0} \frac{1}{h} \int (v_\epsilon(s + h, x) u(s + h, x) - v_\epsilon(s, x) u(s, x)) m_n(x) dx$$

exists and the inner limit is uniform. We show both together by a little computation, using uniform convergence and LDCT:

$$\lim_{n \to \infty} \lim_{h \to 0} \frac{1}{h} \int (v_\epsilon(s + h, x) u(s + h, x) - v_\epsilon(s, x) u(s, x)) m_n(x) dx$$

$$= \lim_{n \to \infty} \int \left( \frac{d}{ds} v_\epsilon(s, x) u(s, x) + v_\epsilon(s, x) \frac{d}{ds} u(s, x) \right) m_n(x) dx$$

$$= \lim_{n \to \infty} \int (-\Delta v_\epsilon(s, x) u(s, x) + v_\epsilon(s, x)(\Delta u(s, x) - u^2(s, x) - 2f(x)u(x))) m_n(x) dx$$

$$= \lim_{n \to \infty} \int (-v_\epsilon(s, x) u^2(s, x) - 2v_\epsilon(s, x) f(x)u(s, x)) m_n(x) dx.$$

Minkowski’s inequality has that

$$\int v_\epsilon u m_n dx \leq \left( \int v_\epsilon m_n dx \right)^{1/2} \left( \int v_\epsilon u^2 m_n dx \right)^{1/2},$$

since $v_\epsilon, m_n \geq 0$. This gives that

$$\int (-v_\epsilon(s, x) u^2(s, x) - 2v_\epsilon(s, x) f(x)u(s, x)) m_n(x) dx$$

$$\leq -\frac{\left( \int v_\epsilon u m_n dx \right)^2}{\int v_\epsilon m_n dx} - 2\|f\|_\infty \int v_\epsilon u m_n dx$$

$$\leq -\frac{\left( \int v_\epsilon u dx \right)^2}{\int v_\epsilon m_1 dx} - 2\|f\|_\infty J_\epsilon(s) < \infty,$$

hence the inner limit of (9) is uniform. On the other hand,
\[
\lim_{n \to \infty} \int (-v_e(s, x)u^2(s, x) - 2v_e(s, x)f(x)u(s, x))m_n(x)dx
\leq \lim_{n \to \infty} \left( -\frac{\int v_um_n dx^2}{\int v_m dx} - 2\|f\|_\infty \int v_um_n dx \right)
\leq -(J_e(s))^2 - 2\|f\|_\infty J_e(s) < \infty,
\]
so the double limit of \((11)\) exists.

**Lemma 8.** Suppose \(u(t, x) \leq 0\) satisfies \((1)\), \(u(t) \in L^p(\mathbb{R})\) for each \(t\), and that for some \(t_0 > 0\),
\[
G(t_0) = \int H(t_0, x)u(0, x)dx < -2\|f\|_\infty.
\]
Then \(\|u(t)\|_p \to \infty\) for \(1 \leq p \leq \infty\).

**Proof.** First, observe that
\[
G(t) = \int H(t, x)u(0, x)dx
= \int \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} u(0, x)dx
\leq \int_{|x| < \delta} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} u(0, x)dx
\]
\[
(10)
\]
for some \(K < 0\) and \(\delta > 0\). (We have used the assumption that \(u(t, x) \leq 0\).) Additionally, by hypothesis,
\[
J_e(0) = \int v_e(0, x)u(0, x)dx
= \int H(t + \epsilon, x)u(0, x)dx
< -2\|f\|_\infty,
\]
since we may choose \(\epsilon > 0\) and \(t\) such that \(t + \epsilon = t_0\). Lemma 7 gives a bound for \(J_e(s)\) by solving \(\frac{d}{ds}J_e(t) \leq -J_e^2 - 2\|f\|_\infty J_e\) (see \([1]\), for instance). The above bound on \(J_e(0)\) selects the following branch of the solution:
\[
J_e(s) \leq \|f\|_\infty \frac{J_e(0) - \|f\|_\infty + (J_e(0) + \|f\|_\infty) e^{2\|f\|_\infty s}}{J_e(0) + \|f\|_\infty - (J_e(0) - \|f\|_\infty) e^{2\|f\|_\infty s}}.
\]
Since \(v_{1/n}\) is a \(\delta\)-sequence as \(n \to \infty\), we have that \(J_e(t) \to u(t, 0)\) and \(J_e(0) \to G(t)\) as \(\epsilon \to 0\). (Note that \(J_e(t)\) is well-defined.) This implies that \((11)\) becomes
\[
(12)\quad u(t, 0) = \lim_{\epsilon \to 0} J_e(t) \leq \|f\|_\infty \frac{G(t) - \|f\|_\infty + (G(t) + \|f\|_\infty) e^{2\|f\|_\infty t}}{G(t) + \|f\|_\infty - (G(t) - \|f\|_\infty) e^{2\|f\|_\infty t}}.
\]
We claim that this implies that \(J_e(s) \to -\infty\). For small \(t\), the denominator in \((12)\) is positive, so unless there is an asymptote in the right side of \((12)\),
\[
(13)
\]
\[
G(t) + \|f\|_\infty - (G(t) - \|f\|_\infty) e^{2\|f\|_\infty t} \geq 0.
\]
On the other hand, solving for $G(t)$ in (13) and applying (10), we have

$$\frac{K}{\sqrt{4\pi t}} \geq G(t) \geq -\|f\|_\infty + e^{2\|f\|_\infty} - e^{2\|f\|_\infty}t.$$  

The right side of (14) becomes positive for large $t$ while the left side remains negative for all $t$. This implies that there is a $T > 0$ such that $\lim_{t \to T^-} u(t, 0) = -\infty$. This implies that $J_\epsilon(t) \to -\infty$ as well for sufficiently small $\epsilon$. On the other hand,

$$|J_\epsilon(t)| \leq \int |v_\epsilon(t, x)||u(t, x)| dx \leq \frac{1}{\sqrt{4\pi \epsilon}} \|u(t)\|_1,$$

so we have that $\|u(t)\|_1$ and $\|u(t)\|_\infty$ both blow up. Finally,

$$\int |v_\epsilon(t, x)||u(t, x)| dx \leq \int |v_\epsilon||u|^p dx \leq \frac{1}{\|u\|_\infty^{-1} \sqrt{4\pi \epsilon}} \|u\|_p^p \leq \frac{1}{\sqrt{4\pi \epsilon}} \|u\|_p^p$$

since $\|u(t)\|_\infty \to \infty$. Hence $\|u(t)\|_p \to \infty$. \qed

Finally, we show that $u \equiv 0$ is unstable. Let $\epsilon > 0$ be given and $1 \leq p < \infty$. Take $h_\epsilon(0) \leq -4\|f\|_\infty$ to be arbitrary. We can construct $h_\epsilon \in L^p \cap L^\infty \cap C^\infty_0(\mathbb{R})$ such that additionally $\|h_\epsilon\|_p < \epsilon$, using the smooth Urysohn lemma. Then for sufficiently small $t > 0$,

$$\int H(t, x)h_\epsilon(x) dx < -2\|f\|_\infty$$

by the fact that $\{H(1/n, \cdot)\}$ is a $\delta$-sequence as $n \to \infty$. Hence by Lemma 8, if $u$ solves (14) with $h_\epsilon$ as its initial condition, then $\|u\|_p \to \infty$.

4.3. Instability for $L^\infty$. Note that the construction above fails for $p = \infty$, since we cannot ensure that both $h_\epsilon(0) \leq -4\|f\|_\infty$ and $\|h_\epsilon\|_\infty < \epsilon$. We resolve this difficulty by a different application of the method of Fujita. We take $L = \frac{\partial^2}{\partial x^2} - 2f$ and $N(u) = -u^2$ in (6) instead. It will also be important, for technical reasons, to enforce the assumption that the first and second derivatives of $f$ be bounded. As before, the resulting situation in (7) is not well-posed, so we make the following definitions.

Definition 9. Suppose $w = w(t, x)$ solves

$$\begin{cases}
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - 2f(x)w(t, x) \\
w(0, x) = w_0(x) \geq 0.
\end{cases}$$

Define $v_\epsilon(s, x) = w(t - s + \epsilon, x)$ for fixed $t > 0$, by analogy with Definition 6. Notice that by the comparison principle, $v_\epsilon(s, x) \geq 0$.

Lemma 10. Suppose that $w$ solves (15). Then $w, \frac{\partial w}{\partial x} \in C_0(\mathbb{R})$. 

Proof. The standard existence and regularity theorems for linear parabolic equations (see [14], for example) give that $w, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2} \in L^2(\mathbb{R})$ and that $w \in C^2(\mathbb{R})$. The maximum principle, applied to $\frac{\partial w}{\partial t}$ and $\frac{\partial^2 w}{\partial x^2}$ gives that the first and second derivatives of $w$ are bounded for each fixed $t$. (This uses our assumption that $f$ has two bounded derivatives.)

The lemma follows from a more general result: if $g \in C^1 \cap L^p(\mathbb{R})$ for $1 \leq p < \infty$ and $g' \in L^\infty(\mathbb{R})$, then $g \in C_0(\mathbb{R})$. To show this, we suppose the contrary, that $\lim_{x \to \infty} g(x) \neq 0$ (and possibly doesn’t exist). The definition of this implies that there is an $\epsilon > 0$ such that for all $x > 0$, there is a $y$ satisfying $y > x$ and $|g(y)| > \epsilon$. Let $S = \{y \mid |g(y)| > \epsilon\}$, which is a union of open intervals, is of finite measure, and has $\sup S = \infty$. Let $T = \{y \mid |g(y)| > \epsilon/2\}$. Note that $T$ contains $S$, but since $g'$ is bounded, for each $x \in S$, there is a neighborhood of $x$ contained in $T$ of measure at least $\epsilon/\|g'\|_\infty$. Hence, since $\sup T = \sup S = \infty$, $T$ cannot be of finite measure, which contradicts the fact that $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$. □

**Lemma 11.** Suppose $u(t, x) \leq 0$ satisfies (1), and $u(t) \in L^\infty(\mathbb{R})$ for each $t$. Then

\begin{equation}
- \int w(t, x)u(0, x)dx \leq \left( \int_0^t \frac{1}{\|w(s)\|_1} ds \right)^{-1},
\end{equation}

where $w$ is defined as in Definition [2].

Proof. Define

\begin{equation}
J_\epsilon(s) = \int v_\epsilon(s, x)u(s, x)dx.
\end{equation}

Then

\begin{equation}
\frac{dJ_\epsilon(s)}{ds} \leq -\frac{(J_\epsilon(s))^2}{\|v_\epsilon(s)\|_1},
\end{equation}

follows according to the method of Fujita, with technical justifications as in Lemma 7 mutatis mutandis. We solve the fence (18) to obtain (note $J_\epsilon \leq 0$)

\begin{equation}
\int_0^t \frac{1}{\|v_\epsilon(s)\|_1} ds \leq -\frac{dJ_\epsilon(s)}{ds} \frac{1}{(J_\epsilon(s))^2} \leq \frac{1}{J_\epsilon(t)} - \frac{1}{J_\epsilon(0)}
\end{equation}

Taking the limit as $\epsilon \to 0$ of both sides of the inequality yields

\begin{equation}
- \int w(t, x)u(0, x)dx \leq \left( \int_0^t \frac{1}{\|w(t-s)\|_1} ds \right)^{-1} = \left( \int_0^t \frac{1}{\|w(s)\|_1} ds \right)^{-1},
\end{equation}

as desired. □

Since we are interested in proving the instability of the zero function in (1), let $u(0, x) = h_\epsilon(x) = -\epsilon$ for $\epsilon > 0$. Then (19) takes on the simple form

\begin{equation}
\epsilon \int_0^t \frac{\|w(t)\|_1}{\|w(s)\|_1} ds \leq 1.
\end{equation}
So in particular, \(\|u(t)\|_\infty\) blows up if there exists a \(T > 0\) such that \(\epsilon \int_0^T \frac{\|u(t)\|_1}{\|w(s)\|_1} \, ds > 1\).

**Remark 12.** The \(L^\infty\) stability of the zero function in (1) depends on the stability of the zero function in (15) – the linearized problem. If the zero function in the linearized problem is very strongly attractive, say \(\|w(t)\|_1 \sim e^{-t}\), then

\[
\int_0^t e^{-t} e^{-s} \, ds = (1 - e^{-t}) < 1,
\]

and so a very small \(\epsilon > 0\) does not cause blow-up via a violation of (19). On the other hand, blow-up occurs if it is less attractive, say \(\|w(t)\|_1 \sim t^{-\alpha}\) for \(\alpha \geq 0\).

Because then

\[
\int_0^t s^{\alpha} t^{\alpha+1} \, ds = t^{\alpha+1},
\]

whence blow-up occurs before \(t = \frac{\alpha+1}{\alpha+1}\).

In the particular case of \(f(x) = 0\) for all \(x\), we note that \(w\) is simply a solution to the heat equation, which has \(\|w(t)\|_1 = \|w_0\|_1\) for all \(t\) (by direct computation using the fundamental solution, say), so blow up occurs. Thus we can recover a special case of the original blow-up result of Fujita in [5].

**Lemma 13.** Suppose \(\epsilon > 0\) is given. Then for a certain choice of \(w_0\), the solution to (15) violates (19) for a certain \(t > 0\).

**Proof.** Begin by making several definitions.

- Choose \(\gamma\) small enough so that
  \[
  \frac{\epsilon}{27\gamma^2} > 1.
  \]
- Since \(0 \leq f \in C_0^\infty(\mathbb{R})\), we can choose an \(x_1\) such that
  \[
  f(x) \leq \gamma \ \text{when} \ x < x_1.
  \]
- Next, we choose \(x_0 < x_1\) small enough so that
  \[
  \sqrt{t} \|f\|_\infty \left(1 - \operatorname{erf} \left(\frac{x_1 - x_0}{2\sqrt{t}}\right)\right) < \gamma
  \]
  for all \(0 < t < \frac{1}{\gamma^2}\). Notice that a smaller \(x_0\) will also work.
- Finally, let \(w_0(y) = \delta(y - x_0)\) (the Dirac \(\delta\)-distribution), and suppose that \(w\) solves (15). By this, we note that \(w\) will be a fundamental solution to (15). Note the the maximum principle ensures both that \(w(t,x) \geq 0\) for all \(t > 0\) and \(x \in \mathbb{R}\) and that \(\|w(t)\|_1 \leq \|w(0)\|_1 = 1\) for all \(t > 0\).

Now we estimate \(\|w(t)\|_1\). Notice that

\[
\frac{d}{dt} \|w(t)\|_1 = \frac{d}{dt} \int |w(t,x)| \, dx = \frac{d}{dt} \int w(t,x) \, dx
\]

\[
= \int \frac{\partial}{\partial t} w(t,x) \, dx = \int \frac{\partial^2 w}{\partial x^2} - 2f(x)w(t,x) \, dx
\]

\[
= -2 \langle f, w(t) \rangle \leq 0,
\]
which recovers the fact that \( \|w(t)\|_1 \leq 1 \). Note that we have used Lemma \([10]\) to eliminate the boundary terms. Now suppose \( z \) solves the heat equation with the same initial condition as \( w \), namely

\[
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} \\
z(0, x) &= w_0(x) = \delta(x - x_0).
\end{aligned}
\]

The maximum principle establishes that \( z(t, x) \geq w(t, x) \) for all \( t > 0 \) and \( x \in \mathbb{R} \), since \( f, w \geq 0 \). As a result, we have that

\[
\frac{d}{dt} \|w(t)\|_1 = -2 \langle f, w(t) \rangle \geq -2 \langle f, z(t) \rangle,
\]

which is an integrable equation for \( \|w(t)\|_1 \). As a result,

\[
\|w(t)\|_1 \geq 1 - 2 \int_0^t \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-x_0)^2}{4s}} w_0(y) dy \, dx \, ds.
\]

On the other hand using our choice for \( w_0 \),

\[
\int_0^t \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-x_0)^2}{4s}} w_0(y) dy \, dx \, ds = \int_0^t \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-x_0)^2}{4s}} dx \, ds
\]

\[
\leq \int_0^t \int_0^{x_1} e^{-\frac{(x-x_0)^2}{4s}} dx + \|f\|_{\infty} \int_{x_1}^{\infty} e^{-\frac{(x-x_0)^2}{4s}} dx \, ds
\]

\[
\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} \|f\|_{\infty} \int_0^t 1 - \text{erf} \left( \frac{x_1 - x_0}{2\sqrt{s}} \right) \, ds
\]

\[
\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} \|f\|_{\infty} \int_0^t 1 - \text{erf} \left( \frac{x_1 - x_0}{2\sqrt{t}} \right) \, ds
\]

\[
\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} \|f\|_{\infty} \left( 1 - \text{erf} \left( \frac{x_1 - x_0}{2\sqrt{t}} \right) \right)
\]

\[
\leq \frac{3\gamma\sqrt{t}}{4} \leq \gamma\sqrt{t},
\]

we have used \([21] \), \([22]\), and assumed that \( 0 < t < \frac{1}{\gamma^2} \). Then \([24]\) becomes

\[
\|w(t)\|_1 \geq 1 - 2\gamma\sqrt{t},
\]

whence

\[
\epsilon \int_0^t \frac{\|w(t)\|_1}{\|w(s)\|_1} ds \geq \epsilon \int_0^t \frac{\|w(t)\|_1}{\|w(0)\|_1} ds \geq \epsilon t - 2\epsilon \gamma t \sqrt{t}.
\]

Now \( \epsilon t - 2\epsilon \gamma t \sqrt{t} \) has a maximum which occurs at \( t_0 = \frac{1}{4\gamma^2} \), so

\[
\epsilon \int_0^{t_0} \frac{\|w(t)\|_1}{\|w(s)\|_1} ds \geq \epsilon t_0 - 2\epsilon \gamma t_0 \sqrt{t_0} = \frac{\epsilon}{27\gamma^2} > 1,
\]

by \([20]\). Hence \([19]\) is violated when \( t = t_0 \). \( \Box \)

As a result of Lemma \([13]\) we can conclude that for each \( \epsilon > 0 \), choosing \( h_\epsilon(x) = -\epsilon \) for all \( x \) results in a solution to \([1]\) which blows up in finite \( t \).
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