The Extended Diffusive Sachdev–Ye–Kitaev Model as a Sort of “Strange Metal”

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The linear dependence on temperature of the resistivity in metals undergoing high-temperature superconducting transitions is a puzzle, which is generally attributed to non-Fermi liquid behavior on a large range of temperature values. No particle and momentum conservation characterizes the $0 + 1 - d$ Sachdev–Ye–Kitaev model as a tractable non-Fermi liquid starting paradigm, when extended to higher space dimensions. Due to the conformal symmetry breaking in the incoherent phase, Goldstone bosons arise, which provide energy diffusion in the lattice. The linear response to energy diffusion in the model is discussed, which is in principle charge neutral and non-momentum conserving, and the temperature dependence of the thermal and electric conductivity in the incoherent prechaotic, intermediate temperature phase is obtained.

1. Introduction

After almost three decades, the strange high-temperature metal phase of the various families of materials which become superconducting at a high critical temperature $T_c$ and are denoted as high-Tc superconductors (HTS) is still a puzzle. The great number of families of materials which undergo superconductivity at high temperature, cuprates, iron-based material, and picnides, possibly nickelates, present rather general and basic surprising features, which require an advancement in the understanding of strongly correlated fermion systems. The most striking of these features is the linear increase with temperature of the electrical conductivity in the normal metal phase, up to $\approx 1000$ K, without any sign of saturation\cite{[1–3]} and the constancy of the thermal conductivity.\cite{[4]} The second puzzling feature is the mysterious mechanism for superconducting electron pairing at “high” temperature.

The phenomenon of superconductivity has always been an icebreaker vessel for the interpretation of the quantum nature of the metal realm. The Bardeen–Cooper–Schrieffer quantum model came out more than 40 years after the discovery of superconductivity in metals, most likely because the full-fledged quantum field theory was required to provide a convincing tool box. Something similar is happening today with the HTS because we do not have full understanding of strong correlation in many body systems. It was immediately realized by P.W. Anderson\cite{[5]} that the doped Mott insulator, which appears as a good starting point for interpreting the normal metal phase of HTS, breaks the frame of the Landau Fermi liquid (FL) theory\cite{[6,7]}, a theory which has dominated successfully up to the 1980s. A brand new approach to strong correlation in metals, the non-Fermi liquid (NFL) theory\cite{[8,9]} is required, which is still unsettled.

A burst of interest comes now from the appearance of a new exactly solvable model in $0 + 1 - d$ dimension, the Sachdev–Ye–Kitaev (SYK)\cite{[10–12]} model, which describes random all-to-all $J$ interaction between $N$ Majorana fermions. The SYK model has become highly popular as a holographic dual for gravity theories of black holes.\cite{[13,14]–[19]} Although it appears as the farthest possible from a realistic description of any real material, its extension to higher space dimensions could make it a useful prototype model for NFL.

Generalized SYK models have been proposed with extension to higher space dimensions\cite{[16–24]}, most of them having in mind applications to high critical temperature ($T_c$) superconducting materials. There are complex fermion versions of the SYK model\cite{[13,18,25,26]} Lattice anisotropy which characterizes the cuprates and their $2 - d$ CuO planes would motivate preference for two space dimensions. We are unable to bridge “bad” or “strange” metals and quantum spin liquids on one side\cite{[22–29]} with the SYK model on the other side,\cite{[14,18,30–34]} but we concentrate on the circumstance of approximate symmetry breaking in the infrared (IR) limit, when there is scaling to strong interaction $J \to \infty$, $N \to \infty$, but finite $\beta J / N$. Fermionic pseudo-Goldstone modes ($p$Gm) arise and additional bosonic excitations, when the hopping on a $2 - d$ space lattice also include the first UV corrections.\cite{[35]}

In fact, in the IR limit, hopping in the lattice can be dealt with within lowest perturbative order. The response of the fermionic excitations, in the conformal symmetry limit, to an external driving to be specified, gives rise to the celebrated linear temperature dependence of the resistivity and to the constancy in temperature of the thermal conductivity. However, the kinetic term added to
the model requires the addition of a complex $U(1)$ phase to the real fields, which couples with the fermionic $pGm$ beyond the IR limit and gives rise to bosonic collective and diffusive modes.[19,35,36] The original part of this work focuses on these modes, which we call Q-excitations. We set up a hydrodynamical description of these energy modes which, by transporting energy, are responsible for the thermalization of the system in a temperature window $T_{coh} \leq T \leq T_0$. The temperature $T_0$, with $\beta T_0 \sim O(\beta \gamma / N)$ (the notation $O(\ldots)$ means “order of”), is given by Equation (68) and is assumed to be the temperature threshold above which, on the basis of what found in the $0 + 1 - d$ SYK model, a “scrambled phase” leading to quantum chaos is expected in the SYK system.

On the other side, $T_{coh}$ denotes the temperature below which hopping in the lattice is coherent. If $W$ is the bandwidth which arises from the hopping in the lattice, $T_{coh} \approx W^2 / NJ$, as argued in Section 2. To guarantee that hopping is a marginal perturbation in the $\mathcal{J} N \rightarrow \infty$ limits is $\beta T_{coh} \sim O(\beta \mathcal{J} N)$, as well.

Section 2 collects the properties of the $0 + 1 - d$ SYK model perturbed by the lattice, in the conformal symmetry limit, including the hopping term in the action. Fermionic quasiparticles can be derived which smear the Fermi surface (FS), but the most relevant contribution to particle and thermal conductivities comes from the dominant $pGm$ which arise from the approximate symmetry breaking. These coherent excitations are the sources of space gradients of the chemical potential $\mu$, which is related to the temperature derivative of the phase $\partial_\phi$, introduced by the hopping term in the extended action. We characterize these phase modes and the related response of the system with the help of a hydrodynamical dissipative approach. The linearity of the $2 - d$ resistance $\sigma^{-1}$ with temperature and the constancy of the thermal conductivity $\kappa$ are obtained in this way, as reported in Section 2.3. They are $O(\beta / \mathcal{J} N)$ in the large $N$ strong $\mathcal{J}$ limit.

From now on we investigate the single particle and collective excitations of the model as well as their interactions to give an estimate of further contributions to the transport coefficients arising from these extra excitations beyond the IR limit. The bottom line of this analysis is that all extra contributions to $\kappa$ and $\sigma^{-1}$ coming from these extra excitations are subdominant with respect to the ones of Section 2.3 and can be ignored in the strong coupling limit. From Section 3 onward, corrections beyond the IR approximation are considered.

The phase $\phi$ arises in a reparametrization of the fermionic propagator in the IR limit. This allows to extract the first correction to the action $\approx O(N / \beta \mathcal{J})$, i.e., the one coming from UV corrections. This extra term entails a diffusive dynamics for the phase and its fluctuations. In the absence of coherence, hydrodynamics can help to extract the collective properties of the phase beyond the conformal limit, as discussed in Section 3. It is assumed that thermalization occurs at a Planck rate, with lifetime of the collective excitations $\tau_0 \approx h \beta$ (where $\beta = 1 / k_B T$). A temperature-independent diffusion coefficient $D_0$ can be derived in this frame, which turns out to be $O(\beta / \mathcal{J} N)$, and allows for the definition of a mean square diffusion length $\tilde{\alpha}_x \approx T_0 / T$. In this approximation, the gapless diffusive Q-excitation modes contribute to the thermal conductivity with a term $\alpha \propto \sqrt{T}$, which is subdominant, being $O(\beta / \mathcal{J} N^{1/2})$ (Section 3.2).

In Section 4, we consider an hypothetical $3 - d$ quantum liquid (QL), in which SYK sheets are embedded, which would be a FL of the same bandwidth $W$, when considered in isolation (see Figure 1). The complex fermionic quasiparticles of the QL, close to the Fermi energy, become ill defined when the local interaction with the $0 + 1 - d$ SYK systems is turned on. We show that the QL becomes a marginal Fermi liquid (MFL). Inserting the quasiparticle lifetime of the MFL in an approximate expression for the thermal conductance, Equation (78), provides again a subdominant contribution to $\kappa$, which is constant with temperature. In Section 4.2, the additional scattering rate induced by the presence of the Q-excitations is derived by means of the Fermi golden rule, and compared to the case of the usual electron–phonon interaction. Competing effects produce a big drop of the exponent in the $T$ power law, with respect to the electron–phonon case. On the one hand, incoherent scattering, in which thermalization is very effective, so that no bookkeeping for particle number and momentum is maintained, should imply higher scattering possibilities and higher scattering rates. On the other hand, the very local nature of the interaction reduces the energy matrix element and the scattering rate. The quasiparticle inverse lifetime acquires a FL-like contribution $\approx T^2$ to be summed to the MFL scattering rate in a sort of Matthiessen's

Figure 1. A caricature of the toy system modelized in this work. $2 - d$ sheets of strongly interacting Majorana fermions are described by a SYK model which includes a hopping term in the continuum limit approximation. The SYK action is given by Equation (8). In the temperature window considered here, the SYK system is incoherent and fully thermalized, but not chaotic. Linear response of the SYK system in the IR limit gives a thermal conductivity which is $T$ independent and a linear in $T$ resistivity. Breaking of the conformal symmetry which is present in the IR limit requires UV corrections for regularization. Bosonic collective diffusive Q-excitations arise, which contribute to the transport coefficients. A $3 - d$ simple single band FL of complex fermions (“electrons” represented as stars in the picture) becomes a marginal Fermi liquid (MFL), when perturbative interaction with the SYK sheets in the incoherent IR limit is turned on. Back interaction between the SYK sheets induced by the coupling with the MFL is not considered. The interaction of the Q-excitations with the MFL do not change the linear $T$ dependence of the particle current derived in the IR limit, but adds extra $T$ dependence to the thermal conductance.
rule. This $T^2$ inverse scattering rate implies a $1/T$ contribution to thermal conductivity which is, however, subdominant once more. Similarly happens for the resistivity contribution, which, though subdominant, turns out to maintain the experimentally measured linear $T$ dependence.\textsuperscript{[1]}

In Appendix, the derivation of the hydrodynamic relativistic equations used in the text is reported.

2. The SYK Model and Its Extension to Higher Space Dimensions

The Hamiltonian $\mathcal{H}_0$ for the uncoupled $0 + 1 - d$ SYK dots is

$$\mathcal{H}_0 = \sum_x \mathcal{H}_x = \frac{1}{4!} \sum_{klmn} J_{klmn} \chi_x[\varphi_{klmn}] \chi_x[\varphi_{klmn}]. \quad (1)$$

Lattice position is denoted by the subscript $x$ in the continuum limit. $\chi_x$ are Majorana fermion operators of $N$ fermionic flavors ($klmn \in \{1, \ldots, N\}$ on each site $x$.

In each uncoupled dot, the all-to-all interaction $J_{klmn}$ is averaged over. After this averaging, the incoherent dynamics produced by the random interaction and disorder is parametrized by an average interaction strength $J$. The large $J$ and $N$ limit (IR limit) is considered, which is exactly solvable. In the IR limit, the model develops an approximate conformal symmetry because it is invariant under a full reparametrization symmetry. Spontaneous breaking of this symmetry occurs, down to the $SL(2, \mathbb{R})$ group symmetry,\textsuperscript{[37]} with $pG_m$ which make the four point function divergent. Two point function, the time ordered, dot independent, single particle Green’s function, $G_x(\tau) = Tr \{ \chi_x(\tau) \chi_x(0) \}$ plays the role of an order parameter and is nonlocal in imaginary time $\tau$. However, symmetry breaking is induced anyhow, when the first ultraviolet correction is included to cure the divergency.\textsuperscript{[38,39]} An additional term of $O(N/\beta T)$ is added to the action, as a result of reparametrization of the “quasi-order-parameter” field. While the latter is bilocal in time, the added correction to the action is local in time and adds local dynamics which becomes extended in space when the model is extended to higher space dimensionality.

While the Green function and the self-energy of the original model are real, those of the extended model, $G_x(\theta_1, \theta_2)$, $\Sigma_x(\theta_1, \theta_2)$, have a space dependent modulus and phase, with small fluctuations around the saddle point values $G_x(\theta_{12})$, $\Sigma_x(\theta_{12})$ of the $0 + 1 - d$ SYK dot at each site $x$ (here $\theta_{12} \equiv \theta_1 - \theta_2$). We write

$$G_x(\theta_1, \theta_2) = |G_x(\theta_{12}) + \delta G(x, \theta_{12}, \theta_x)| e^{\phi_{\psi}(\theta_x)}$$

$$\Sigma_x(\theta_1, \theta_2) = |\Sigma_x(\theta_{12}) + \delta \Sigma(x, \theta_{12}, \theta_x)| e^{\phi_{\psi}(\theta_x)} \quad (2)$$

Here, $\theta = 2\pi \tau/\beta$ is a dimensionless time and $\theta_{12} = \theta_1 - \theta_2$, while $\theta_x = (\theta_1 + \theta_2)/2$ is a local time in the space evolution (however, we will most of the times denote the dimensionless time as $\tau$ unless differently specified).

The local correction $\delta G(x, \tau_1 - \tau_2, \tau_x)$ $e^{\phi_{\psi}(\tau_x)}$ arises from nearest neighbor hopping in a lattice of spacing $\bar{a}$. A simple argument for the determination of the relevance of the kinetic energy term in the scaling is reported below.

Let $\tau(\equiv \tau_x) \rightarrow \tau' = \tau/s$ with $s \rightarrow J/T$. Majorana fields scale as $\chi \approx \chi^{1/4}$ (scaling dimension is $\Delta = 1/4$). The kinetic energy due to the lattice is naively written as

$$t_0 \int \! dt \chi'_{x} \chi'_{x,x,m} \rightarrow t'_0 \int \! dt' \chi'_{x,x,x,x,m} \quad (3)$$

with $t'_0 \approx s^{1/2} t_0$. This is a relevant perturbation. Scaling up to $s_T = J/T$ implies

$$t_0 \rightarrow t_0(s_T) = t_0 \times \left( \frac{J}{T} \right)^{1/2} \quad (4)$$

which is a weak hopping, up to $t_0(s_T) \sim J$. This defines a threshold (coherence) temperature $T_{coh}$ when

$$J \sim t_0 \times \left( \frac{J}{T_{coh}} \right)^{1/2} \sim T_{coh} \sim \frac{t_0^{1/2}}{J} \quad (5)$$

For $T \gg T_{coh}$ the lattice can be dealt with as an irrelevant perturbation on the $0 + 1 - d$ SYK model, while for $T < T_{coh}$, the hopping is a marginal correction in the scaling limit.

We introduce now a more careful approach to the kinetic energy term to be added to Equation (1).

Electron quasiparticles hopping out of site $x$ to one of the neighboring sites require complex fermions. In a full paramagnetic system we drop out the spin, and the complex fermionic spinless operator $c_x^\dagger c_x$ for the particle can be represented in terms of two flavors of the neutral fermions on the same site

$$c_x^\dagger = \frac{1}{\sqrt{2}} (\chi_{x1} + i \chi_{x2})$$

$$c_x = \frac{1}{\sqrt{2}} (\chi_{x1} - i \chi_{x2}) \quad (6)$$

The hopping will change the interaction energy of all the Majorana Fermions of site $x$. As we ignore conservation of the particle number, we focus just on the site which is the origin of the jump. However, the local interference among neighboring sites involves the local phase factor $e^{i \phi_{\psi}(\tau_x)}$. It can be shown\textsuperscript{[35]} that the change of energy is, for one single realization of disorder and replica

$$\mathcal{H}_{x,\tau}(\tau) = i \frac{1}{3!} \sum_{klmn} J_{12lm} e^{i \phi_{\psi}(\tau_x)} \chi_{x,\tau}(\tau) \chi_{x,\tau}(\tau) + h.c. \quad (7)$$

The thermal average including annealed disorder (the average uses the replica trick in the standard SYK model) includes now an additional Gaussian average for $J_{12lm}$, which mixes terms from different sites, and gives rise to random interdot hopping of energy scale $t_0/\sqrt{N}$. As usual, we have assumed that the mixing of replicas can be ignored in the saddle point approximation and we have dropped the replica label everywhere. The temperature determines the relevance of the resulting kinetic term in the limit $J, N \rightarrow \infty$. To guarantee that the kinetic term is marginal also in the incoherent phase limit, it is required that $\beta T_{coh} \approx O(\bar{a}^2)$. $T_{coh}$ is defined as $\frac{t_0^{1/2}}{J}$, as suggested by Equation (5).

The next step is the integration over the Majorana fields $\chi_{x,\tau}(\tau)$, with the help of the Hubbard–Stratonovich fields of Equation (2). The saddle point action in terms of the complex bilocal auxiliary fields $G_x(\tau_1, \tau_2)$ and $\Sigma_x(\tau_1, \tau_2)$, characterized by the phase $\phi_{\psi}$, is
The last term of the action is the intersite kinetic term, which was also previously derived.\textsuperscript{119} The action has become complex due to the additional $U(1)$ minimal coupling in Equation (7). The real Green function in the saddle point approximation is

$$
\frac{1}{G(i\omega_n)} = -i\omega_n - \Sigma(i\omega_n)
$$

where $i\omega_n$ are fermionic frequencies.

The conformal limit in the IR corresponds to the dropping of $\partial_r$ in the determinant, or of $-i\omega_n$ in Equation (9). In this limit, at low temperature, assuming particle–hole (p–h) symmetry, $G_c(i\omega_n)$ is given by

$$
G_c(i\omega_n) = \frac{i \text{sign} (\omega_n)}{\sqrt{J} \sqrt{-i\omega_n}}
$$

From Equation (9), with $\Sigma(i\omega_n) \rightarrow \Sigma^c(i\omega_n) = \sqrt{J} \sqrt{-i\omega_n}$ in the same limit.

However, $\partial_r$ plays an important role in the extended model of Section 3.

### 2.1. Quasiparticle Energy Distribution of the SYK Extended Model Close to the FS at $T < T_{coh}$

To check the nature of the quasiparticles for energies close to the FS, when the SYK model is extended perturbatively to two space dimensions, we calculate the quasiparticle occupation number $n_k$ close to the Fermi energy $\varepsilon_F \approx \mu$.

The single particle Green’s function of the SYK model in the conformal symmetry limit, The Green function $G_c(i\omega_n)$ of Equation (10) is local in space. To include the $k$ dependence in the propagator, for $k$ vectors close to the FS, we add a single band energy $\varepsilon_k = \hbar^2 k$, in the continuum limit to the right-hand side of Equation (9), and expand to lowest order, obtaining\textsuperscript{24}

$$
G(\vec{k}, i\omega_n) \approx \left[ \frac{1}{\Sigma^c(i\omega_n)} + \frac{\varepsilon_k}{\Sigma^c(i\omega_n)^2} \right]
$$

The occupation number of the complex fermions can be rewritten in terms of the Majorana fermions by means of Equation (6) as

$$
\frac{I_{xx}}{N} = \sum_x \left[ -\frac{1}{2} \ln \det[\partial_r - \Sigma_x] + \int \text{d}r \text{d}r' \left\{ -\frac{J}{4} |G_x(r, r')|^2 \right. \right.
$$

$$
\left. + \Sigma_x(r, r') G_x(r, r') - \frac{\hbar^2}{N} \sum_{x \in n} G_x(r, r') G_x(r, r') \right\} \right] (8)
$$

We evaluate the Matsubara sum of $G(k, i\omega_n)$ in the complex frequency plane $i\omega_n \rightarrow z$

$$
\frac{1}{\hbar \beta} \sum_n e^{i\omega_n \Phi^0} G(k, i\omega_n) = \frac{1}{2\pi i} \int e^{i\omega} \frac{dz}{\sqrt{j^2 + 1}} G(k, z) (13)
$$

$$
G(k, z) \approx -\frac{1}{\sqrt{J}} \text{sign}[z] + \frac{\varepsilon_k - \mu}{J} (14)
$$

$G(k, z)$ has a cut and a pole at the origin. The cut can be taken on the real $z > 0$ axis and contributes to the integral of Equation (13) as follows

$$
\frac{1}{2\pi i} \int_{i\infty}^{0} \frac{dz}{\sqrt{j^2 + 1}} e^{i\omega} \left\{ e^{i\omega/2} - e^{-i\omega/2} \right\} = \frac{1}{\sqrt{\pi J}} \zeta(\frac{3}{2}) > 0 (15)
$$

where $\zeta(\frac{3}{2}) = -1.46035$ is the Riemann function. At zero temperature the contribution of the cut vanishes.

The second term in Equation (14) contains the pole at the origin. Defining $\xi = \frac{\varepsilon_k}{J}$, we get

$$
\frac{1}{2\pi i} \frac{\xi}{\eta - \xi} \int_{0}^{\infty} \frac{dz}{\sqrt{j^2 + 1}} e^{i\omega} \frac{1}{\eta - z} = -\xi (16)
$$

It follows that the final result for Equation (12), close to the Fermi energy, in the limit $\beta J \rightarrow \infty$ and low temperature, is

$$
n_k = \frac{1}{2} \left( 1 - \frac{\varepsilon_k - \mu}{J} \right) (17)
$$

The FS of the SYK model extended to higher space dimensionality is blurred in this perturbative approach and Fermionic quasiparticles are no longer well defined.

In Section 4.1 we will derive the lifetime of the quasiparticles of a $3 - d$ qL in perturbative interaction with the $0 + 1 - d$ SYK system and we find that the qL becomes itself a MFL.

### 2.2. Fluctuations around the IR Saddle Point Solution

Back to Equation (8), we can expand the action to second order and integrate out the $\delta \Sigma$ fluctuations. The resulting functional integral, in terms of the fluctuations of the $G$ field, $\delta g(\tau_2, \tau_1)$, is\textsuperscript{115}

$$
Z[\partial_r \varphi_p(\tau_n)] \propto \int D(\Pi \delta g^*_{\tau_2, \tau_3}) D(\Pi \delta g_{\tau_2, \tau_3}) e^{\frac{\delta g(\delta g^*_R K^{-1}_R - 1|\delta g) e^{-\frac{\delta g^2}{2}}}{}_{\text{det}} \left\{ \left( -i \delta \varphi_p(\tau_1) \delta \varphi_p(\tau_2) \right) \right\} e^{\frac{\delta g^2}{2}} \left\{ \left( -i \delta \varphi_p(\tau_1) \delta \varphi_p(\tau_2) \right) \right\} e^{\frac{\delta g^2}{2}} (18)
$$
\[ \langle \partial_i \varphi_p | R_c^{-1} \Lambda_c R_c^{-1} | \partial_i \varphi_p \rangle = \int d\tau_1 \int d\tau_2 [\delta G_{\varphi x}(\tau_1, \tau_2) \delta G_{\varphi x}^{*}(\tau_1, \tau_2)] \]  

(19)

where \( q = 4 \), as in the usual SYK notation and

\[
\begin{align*}
\delta G_{\varphi x}(\tau_1, \tau_2) & = [G_x(\tau_1, \tau_2) G_{\varphi x}^{*}(\tau_3, \tau_4) - G_x(\tau_1 - \tau_2) G_x(\tau_3 - \tau_4)] \\
& \approx \frac{1}{2} G_x(\tau_1) \left( e^{i \theta \varphi(\tau_1 - \varphi(\tau_2))} - 1 \right) G_x(\tau_4) + c.c.
\end{align*}
\]

(20)

Here, we keep only quadratic terms in the phase difference \( \varphi_\omega(\tau_2) - \varphi_\omega(\tau_4) \) because the first order drops out by adding the c.c. and we obtain

\[
\rightarrow - \frac{1}{2} G_x(\tau_1) (\vec{a} \cdot \nabla \varphi_\omega(\tau_2) - \varphi_\omega(\tau_4))^2 G_x(\tau_4).
\]

(21)

We now approximate \( \varphi_\omega(\tau_2) - \varphi_\omega(\tau_4) \approx (\tau_2 - \tau_4) \partial_\omega \varphi_\omega(\tau_2) \) in the kernel \( \Lambda_c \) of Equation (19). Owing to the self-averaging established for the SYK model at large \( N \), translational invariance allows space Fourier transform \((p, x) \) is a wavevector).

To highlight the physical meaning of the source term in the functional of Equation (18), we define the energy density current \( \vec{J} \) in a hydrodynamic relativistic approach\(^{40}\) (see Appendix)

\[
\vec{J} = \left( \frac{e + p}{n} \right) (\vec{J} - \vec{\nu}) = T_0^\nu \equiv \nu \vec{u}^\nu
\]

(22)

Here, \( \nu = e + p \) is the enthalpy with pressure \( \vec{p} \) and \( T_0^\nu \) is the energy-momentum tensor (italic indices \( i = 0, 1, 2, 3 \), while Greek indices \( \sigma = 1, 2, 3 \)). \( j \) is the average particle current density and we will adopt the linear response approach, so that the fluctuating current \( \vec{\nu} \) is a small contribution, \( \vec{\nu} \ll \vec{J} \). Here, \( \vec{u}^\nu \) are the components of the velocity and \( \nu^\nu = 1 \).

The starting point of Appendix is the conservation of the energy-momentum tensor \( T_i^\nu \), which includes dissipative processes. The viscosity and the thermal dissipation are accounted for by the stress tensor of Navier–Stokes origin, \( \tau^\nu_i \). Here, \( \nu^\nu = 0 \) because the dissipative components of the stress-energy tensor and current are orthogonal to \( \vec{u} \). In the absence of external forces and in the case of \( j = 0 \), it can be shown in Equation (A8) that the change of entropy (per unit volume) is given by

\[
T \left( \frac{D s}{D t} \right) = \mu \nabla \nu - \nu^\nu \frac{\partial u^\nu}{\partial x^\sigma}
\]

(23)

Here, \( \frac{D}{D t} = \nu \frac{\partial}{\partial x^\sigma} \) is the covariant derivative, which includes the convective derivative. The last two terms on the right-hand side are dissipative terms. When we integrate over space and time and we substitute \( \nu / n \rightarrow \vec{J} / k_B T \) from Equation (22), we get a source term to be added to the action of Equation (37).

\[
[\delta S]_{\text{source}} = \int dt \int d\nu_b \frac{1}{k_B T} \vec{J}^\nu \cdot \nabla \nu
\]

(24)

which reminds of the source term \( \alpha \int J \cdot E \) of electrodynamics.

T plays here the role of the background average constant temperature in linear response and \( \lambda_{b} \) is a small volume. As \( \mu(x, \tau_b) = h \partial_x \varphi_x \), it follows that fluid correlations contributing to energy density currents, \( \frac{1}{2} \left[ \hat{\nu} \cdot \hat{J} \right] \equiv \left\langle \mathcal{N}_p(\tau_b) \mathcal{N}_p(0) \right\rangle \), are generated by the functional of Equation (37). This allows us to discuss thermal transport in the extended SYK model within the conformal limit, what is done in the next subsection.

Meanwhile, it is found that the approximate conformal symmetry of the SYK model is spontaneously broken down to the \( SL(2,\mathbb{R}) \) group symmetry.\(^{37}\) The dominant contribution to the functional integration in Equation (18) comes from the pGm \( \delta g_{\tau_{\omega}} \). At this level of approximation, the \( \delta g \)’s do not depend on \( \tau \) and are eigenfunctions of the Kernel \( K_c \) with eigenvalue 1.

The signature of the symmetry breaking is the disappearance of the action of the \( \delta g \)’s \( \langle \delta g | K^{-1}_c - 1 | \delta g \rangle \) in Equation (18), what implies that the functional integral is actually divergent. Regularization requires UV corrections. In the next subsection we are going first to discuss transport in the conformal symmetric limit by considering the saddle point for the action only, which suffices for a linear response approximation. The divergence of the functional integration will be cured in Section 3.

### 2.3. Transport in the IR Conformal Symmetry Limit

In the conformal symmetry limit, we keep only the lowest order in the kinetic term by approximating \( G_x(\theta_1, \theta_2) \approx G_x(\theta_1, \theta_2) e^{\varphi(\tau_2 - \tau_1)} \) in Equation (2). The kinetic term operator appearing in Equation (19), \( \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} R_c^{-1} \Lambda_c R_c^{-1} \) is the space Fourier transform of

\[
\frac{1}{2} \left( \vec{a} \cdot \nabla \varphi \right) G_c(\tau_1)(\tau_2 - \tau_4)^2 G_c(\tau_4)(\vec{a} \cdot \nabla \varphi)
\]

(25)

derived from Equation (21). The function \( h \partial_\tau \varphi_\omega \) is canonically conjugated to the particle flux \( \mathcal{N}_p(\tau_1) = \frac{1}{h} \frac{\partial}{\partial \tau} \varphi_\omega \), where \( L_{ex}[\partial_\tau \varphi_\omega] \) originates from the action of Equation (8) and we have defined \( \tau_0 = \sqrt{(\tau_2 - \tau_1)^2} \) as an average timescale of the correlations in time, according to Equation (25).

The function \( \frac{2}{\Delta \tau} \left\langle \mathcal{N}_p(t) \right\rangle \) and the current-current correlator \( \left\langle \mathcal{N}_p(t) \mathcal{N}_p(0) \right\rangle \) can be derived from the second variation of the generating functional of Equation (18), \( \frac{2}{\Delta \tau} \ln Z \). Moving back to space and real time, the correlator in the hopping between sites, at saddle point, is

\[
\frac{1}{2} \int_{\tau_0}^{\tau_1} \left\langle \mathcal{N}_p(t) \mathcal{N}_p(0) \right\rangle \rightarrow \frac{1}{2} \int_{\tau_0}^{\tau_1} \left( i C_{\ell \tau}(t) - i C_{R \tau}(t) \right)
\]

(26)

where the labels L, R refer to left and right side in the hopping process and we have lumped in \( \tau_0 \) all prefactors. Here
where \( A_\pm \) are the fermionic spectral functions with \( A_\pm^0 (\omega) = e^{i\omega} A_\pm^0 (\omega) \). The \( A_\pm^0 \)'s are real and positive.

Let \( \langle N(t) \rangle \) be the currents to the left/right, which differ for a chemical potential \( \mu \) imbalance. From Equation (26), we have

\[
\frac{\delta}{\delta \phi} \langle N(t) \rangle = \frac{1}{4\hbar} \int d\omega \, [ A_+ A_- (\omega \mp i\epsilon) - A_- A_+ (\omega \pm i\epsilon)] e^{i\omega t}, \quad \text{for } \tau > 0
\]

The spectral functions are actually independent of the side: \( A_\pm^0 (\omega) = A_R A_L^0 (\omega) \).

According to Equation (24), the difference between left and right fluxes over \( \omega \), in the limit \( \omega \rightarrow 0 \), is connected to the response function to an external perturbation of the chemical equilibrium

\[
D_R^0 (\omega) = -i \int dt \theta(t) \left\langle \left[ N(t), N(0) \right] \right\rangle e^{i\omega t}
\]

with

\[
\left\langle \left[ N(t), N(0) \right] \right\rangle \approx \frac{\epsilon}{i\hbar} \left[ (i G^R (t)) (-i G^R (-t)) - (i G^G (t)) \right]
\]

Indeed, by comparing with Equation (28)

\[
\left. \left\langle \left[ \frac{\delta}{\delta \phi} N_+ - N_- \right] \right\rangle \right|_{\omega} = -\frac{\epsilon}{2\hbar} \int d\nu \, \text{Im} \{ D_R^0 (\nu) \} \frac{\delta}{\delta \phi} \text{Im} \{ G_R^0 (\nu) \}
\]

provides the real part of the conductivity, with the density of the incoming states \( \text{Im} \{ G_R^0 (\nu) \} / \pi \).

The thermal conduction arises from the response to the energy current in place of the particle current. It is enough to substitute \( A(\nu) \rightarrow \hbar A(\nu) \) in the final expression to obtain the thermal conduction response \( Re \{ \kappa \} \). On the other hand, at equilibrium, the imaginary part of the response function \( D_R^0 (\omega) \) and the retarded function \( G_R^0 (\omega) \) are simply related by

\[
\text{Im} \{ D_R^0 (\omega) \} = \frac{\tan \frac{\beta \omega}{2}}{\text{Im} \{ G_R^0 (\omega) \}}
\]

We obtain

\[
\kappa = \lim_{\omega \rightarrow 0} \frac{Re \{ \kappa(t) \}}{k_B^2} = \lim_{\omega \rightarrow 0} |\omega| \text{Re} \{ \kappa(\omega) \}
\]

\[
= N \frac{\epsilon^2}{2\hbar} \int d\nu \, \text{tanh} \frac{\beta \nu}{2} \left[ -\nu \frac{\partial}{\partial \nu} \text{Im} \{ G_R^0 (\nu) \} \right]^2
\]

The SYK Green function in the conformal limit, which was approximated in Equation (10), is

\[
G_R^0 (\omega) = -ib \beta \frac{1}{\sqrt{\pi}} \int \frac{d\nu}{2\pi} \left[ \frac{\Gamma \left( \frac{1}{2} - i \frac{\omega}{2\beta} \right)}{\Gamma \left( \frac{1}{2} - i \frac{\omega}{2\beta} \right)} \right] + \frac{\pi^{1/4}}{2\sqrt{\beta \Gamma}}
\]

and, integrating by parts, we get

\[
\lim_{\omega \rightarrow 0} \frac{\text{Re} \{ \kappa \}}{k_B^2} = \frac{1}{\pi^{1/2}} \frac{\epsilon^2 \hbar}{2\beta \Gamma} \int d\nu \, \text{sech}^2(\pi \nu) \left[ \text{Re} \{ \Gamma \left( \frac{1}{2} - i \nu \right) \} \right]^2
\]

\[
\propto N \frac{k_B T_{coh}}{\hbar}
\]

where we have introduced the definition of \( T_{coh} \).

By the same token, from Equation (30,31), the particle current can be derived following the same steps as before. Ignoring the fact that the current is neutral in this Majorana fermion system, a hypothetical electrical real conductivity \( \sigma \) would be:

\[
\lim_{\omega \rightarrow 0} \frac{\text{Re} \{ \sigma \}}{\epsilon} = \frac{\beta}{\pi^{1/2}} \frac{\epsilon^2 \hbar}{2\beta \Gamma} \int d\nu \, \text{sech}^2(\pi \nu) \left[ \text{Re} \{ \Gamma \left( \frac{1}{2} - i \nu \right) \} \right]^2
\]

\[
\propto N \frac{T_{coh}}{T}
\]

Equation (36) is the celebrated linear \( T \) dependence of the electrical resistivity offered by the SYK model. In the next section, we discuss the contribution of the collective Q-modes to the \( T \) dependence of the transport coefficients. These modes arise when the UV corrections to the action are included.

### 3. Collective Bosonic Excitations from UV Corrections

In the previous section, based on the spontaneous conformal symmetry breaking, we acted as including just the Goldstone modes in the functional integration and regarded Equation (18) as a generating functional for the Goldstone mode correlations, having \( \partial_\tau \varphi_p \) as a source field

\[
\mathcal{Z} = \int D\varphi_p \, e^{\frac{1}{2} \left\{ -i R^1 \varphi_p \partial_\varphi \varphi_p - \int d\nu \varphi_p (\nu) \right\}}
\]

\[
\times e^{\frac{1}{2} \int d\nu \varphi_p (\nu) \partial_\varphi \varphi_p + \frac{1}{2} \left( -i \partial_\nu \varphi_p (\nu) R^1 \varphi_p (\nu) \right)}
\]

We have swept under the carpet the fact that these massless Goldstone modes are responsible for the divergence of the functional integral of Equation (18). Moreover, \( \varphi_p (\tau, \nu) \) was given as an external source with no dynamics. However, symmetry breaking is only approximately spontaneous because the term \( \sim \partial_\tau \), in Equation (8) produces itself a symmetry breaking, which becomes relevant in the UV. The modes of the Gaussian action are, so to say, pGm and display a small mass. In the IR, a time reparametrization of the conformal Green’s function under the diffeomorphism \( e^{\partial_\tau} \rightarrow e^{\partial_{\tau} + \partial_\nu \varphi_p (\nu)} \) produces a local term to the action, which gives dynamics to the \( \varphi_p (\tau, \nu) \) fluctuations and generates diffusive collective bosonic excitations which we call
Q-exitations. As the \(0 + 1\)–SYK dots at the lattice sites are sinks of charge and momentum, the surviving dynamical variable is energy density and we interpret the Q-exitations as energy density currents.

The IR Green’s function under reparametrization reads

\[
\tilde{C}_{IR}(\theta_1, \theta_2) = \tilde{C}_e(\varphi(\theta_1), \varphi(\theta_2)) \varphi' (\theta_1) \Delta, \varphi' (\theta_2) \Delta
\]

which, when expanded to lowest order in \(\theta_{12}\), gives

\[
\approx \mathcal{C}_{\beta \rightarrow \infty}(\theta_1, \theta_2) \left[ 1 + \frac{\Delta}{6} (\theta_{12})^2 \text{Sch}(\varepsilon \varphi(\beta), \theta_{12}) \right]
\]

where \(\text{Sch}(\varepsilon \varphi(\beta), \theta_{12})\) is the Schwarzian.\(^{[11]} \)

The fluctuations \(\delta g_{\tau_{12}} \sim R_c (\tilde{C}_{IR} - \tilde{C}_{\beta \rightarrow \infty})\) can be substituted in the source term of Equation (37), giving rise to a local action of order \(1/\beta J\)

\[
\frac{\hbar}{N} \left[ \partial_t \varphi \right] = -2\pi \frac{\alpha_s}{\beta J} \int \frac{d\theta}{2\pi} \frac{1}{2} \left( (\varphi')^2 - (\varphi')^2 \right)
\]

which adds dynamics to the field \(\varphi_\tau (\tau_\tau)\). The constant term in Equation (40) gives a contribution \(\alpha J\), which is just a correction to the energy and will appear in Equation (66). The added local action of Equation (40) \(= O(\beta J^{-1})\) and a correction of the same order should be included to the action,\(^{[35,38]} \) coming from the \((\delta g)^2 / K_0^{-1} - 1\) \((\delta g)\) term appearing in Equation (18). We remind that \(\partial_t \varphi (\tau_\tau)\) corresponds to changes in the chemical potential \(\mu\), according to Equation (24).

The UV correction of the \(0 + 1\)–d SYK model is local in time. To extend the model to higher space dimensions, we use the macroscopic continuity equation

\[
\rho T \frac{D\mu}{Dt} = -\nabla \cdot J'
\]

where \(J'\) is the energy current density of Equation (22). This will help us in characterizing the collective bosonic Q-exitations. In the next subsection, we derive the response function to \(J' - J^e\) fluctuations, starting from the equation of motion, which is assumed to be diffusive

\[
J' = -\kappa \nabla T + g
\]

where \(g\) is an external stochastic source. Diffusivity, besides being physically justified in the incoherent regime, can also be justified starting from the full action \(I[\partial_t \varphi_\tau]\) in the large \(J\) limit.\(^{[33]} \)

3.1. Hydrodynamic Response to Energy Current Density Fluctuations

According to Equation (24), the gradients of the chemical potential can be obtained as a response to energy current density fluctuations in the \(2 + 1\)–d SYK system. This is of importance for us, as the perturbation of the local chemical potential in the \(3 - d\) QL, induced by the energy current density fluctuations of the \(2 + 1\)–d SYK system, is our final goal. While in Section 2.3 we have presented the response of the energy current density \(J^e\) to chemical potential gradients \(\nabla \mu\), in this subsection we study the response of the chemical potential to the \(J'\) fluctuations, starting from temperature gradients and from the continuity equation, Equation (41).

Let us define the canonical conjugate variables \(x_a, X_a\), appearing in the continuity equation, Equation (41)

\[
S = -x_a \frac{\partial S}{\partial x_a}, \quad \text{with } x_a = -\frac{\partial S}{\partial x_a}
\]

in terms of which the usual linearized response equation takes the form

\[
x_a = \sum_b (\gamma_{ab} X_b + g_b)
\]

(here \(g_b\) is the external stochastic source and \(a, b\) denote elementary space cells of volume \(v_a, v_b\).)

Space integration of Equation (41), together with Equation (42), gives

\[
\frac{S}{k_B} = \int dv_k \frac{1}{k_B T} \nabla \cdot (\kappa \nabla T) = -\int dv_k \left\{ \frac{J^e}{k_B} \right\} \left\{ \frac{\nabla T}{T} \right\}_b
\]

It follows from Equation (43–45) that

\[
x \approx g = \frac{f}{k_B}, \quad X = \nabla T, \quad \gamma_{ab} = -\frac{\kappa T^2}{k_B}
\]

The hydrodynamical approach developed in Appendix provides Equation (A8), which reads

\[
\frac{D x_b}{Dt} = \frac{\mu}{T} \nabla \cdot T + \nabla \left( \frac{\mu}{T} - \nabla \mu \right) \frac{\partial \omega}{\partial \mu}
\]

with \(\nu\) defined in Equation (22). When the particle current vanishes, the right-hand side contributes with dissipative terms which can be cumulatively expressed as \(-\frac{\partial \omega}{\partial \mu}\). The term \(\frac{\omega}{T} \nabla \omega\) provides the diffusive contribution to the change in time of the entropy. Equation (A12) shows that \(-\frac{\omega}{T} \approx \nabla^2 u\). Introducing the diffusive constant \(d = \kappa/nC\), where \(C\) is the specific heat, and posing \(CV^2 T \approx V^2 u\), we obtain

\[
k_B v_x X_b = -\frac{D n_B}{Dt} = \frac{n}{w T} \left( \frac{\nabla^2 u}{T} - \frac{\partial \omega}{\partial \mu} \right) \frac{\partial \omega}{\partial \mu}
\]

Fourier transforming in space and inverting this equation, we get the response equation that we are looking after \((\mu \rightarrow \mu(\omega \approx 0))\)

\[
\mu_{\omega, \omega} = k_B T \frac{X_{\omega, \omega}}{\frac{d^2}{\omega^2} - \frac{i\omega}{\omega} X_{\omega, -\omega}}
\]

The source \(X_{\omega, \omega}\) in this equation can be interpreted as an elementary energy current density source: \(J'_{-q, -\omega} \approx \frac{k_B}{2} [\nabla^2 T]_q\).

In fact \(k_0 = \frac{\gamma_{1, 1}}{2}\) has the dimension of a thermal conductance\(^{[41]} \) [C/time]. We obtain
\( \mu_{q,n} \omega = \frac{\hbar}{(k_B T)^2} f_{q,-q} J_{q,n} \frac{-i \omega}{D q^2 - i \omega} \)  

(50)

In the following we consider the thermal average of \( \int \frac{d \omega}{\pi} f_{q,-q, \omega} J_{q,n} \omega \), as inducing a response on the space and time-dependent chemical potential energy density \( \mu \). We will use this result when discussing the interaction between the low energy qL and the bosonic Q-excitation. In Section 3.2, the thermal average \( \int \frac{d \omega}{\pi} \langle \{ J_{\omega} J_{q,n} \} \rangle \) will be denoted as

\[ \langle \{ J_{\omega} J_{q,n} \} \rangle \]

Classically, the symmetrized correlation derived from Equation (44) is \( \frac{1}{2} \langle \{ \delta \theta(t), \delta \theta(t') \} \rangle = \langle \delta \theta(t) \delta \theta(t') \rangle \) and, with the consequence of Equation (46), it becomes in the quantum case

\[ \frac{1}{2} \int d^2 r \int dt \int dt' \langle \{ J_{\omega}(r, t), J_{\omega}(r, t') \} \rangle e^{i \omega(t-t')} = -i \hbar \omega \text{coth} \left( \frac{\hbar \omega}{2 k_B T} \right) \]

(51)

where \( \kappa \approx \lim_{\omega \to 0} \text{Re} \{ \kappa \} = \lim_{\omega \to 0} \text{Re} \{ \kappa(\omega) \} \), so that, substituting the \( J' - J' \) fluctuations with the imaginary part of the response via the fluctuation dissipation theorem, we have

\[ \omega \text{Re} \{ \kappa(\omega) \} = \frac{1}{2} \text{Im} \left\{ D_{J', J'}^\omega (\omega) \right\} \frac{1}{\hbar \omega T} \text{tanh} \left( \frac{\hbar \omega}{2 k_B T} \right) \]

(52)

where

\[ D_{J', J'}^\omega (\omega) = -i \int dt \int dt' \theta(t-t') \int d^2 k \langle \{ J_{\omega}(t), J_{\omega}(t') \} \rangle e^{i \omega(t-t')} \]

(53)

The response function of Equation (53) will be evaluated in the next subsection for damped Q-excitation within a restricted \( 2 - d \) volume in which correlation is vanishing. The space volume integration is explicit in Equation (50) for the \( J' - J' \) fluctuations and implicit in the \( k \)- integral of Equation (53). We will denote this restricted \( 2 - d \) volume as \( \hat{a}^2 \) in Section 3.2.

3.2. Lagrangian Approach to Gapless Diffusive Excitation Modes

Equation (42) is a classical diffusion equation of a nonconserving system. We now construct a Lagrangian of excitation modes which is conserving but, assuming a relaxation time \( \tau_0 \) for these modes, it reproduces diffusive motion. We will quantize this Hamiltonian and derive the fluctuations of these modes from the response function by means of the fluctuation-dissipation theorem. The canonical conjugate variables and the corresponding Lagrangian (in \( 2 - d \)) are

\[ \theta = \left( \frac{\kappa C}{\hbar T} \right)^{1/2} \frac{f_0}{k_B} \]

\[ \nabla \theta = \left( \frac{\hbar C}{\kappa T} \right)^{1/2} \frac{\partial}{\partial V T} \]

(54)

\[ \mathcal{L} = \frac{1}{2} \int d^2 x \left[ \frac{k_B}{T} \left( \frac{f_0}{k_B} \right)^2 + \frac{h}{\kappa C} \left( \frac{\nabla V T}{T} \right)^2 \right] \]

\[ - \frac{1}{2} \int d^2 x \left[ A \theta^2 + B (\nabla \theta)^2 \right] \]

(55)

\[ A = \frac{\hbar k_B}{\kappa C}, \quad B = T \]

Note that the dimension of the two terms in square brackets is \( [\epsilon/\epsilon^2] \).

From the equation of motion

\[ \frac{d}{dt} \left( \frac{\partial \theta}{\partial \theta} \right) - \frac{\partial L}{\partial \nabla \theta} = A \theta + B \nabla \theta = 0 \]

(56)

the classical motion equation of Equation (42) can be recovered if we put \( \tau_0 f_0 \approx f_c \).

Although \( \hbar \) is already in the Lagrangian, we proceed to quantize the theory. Fourier transforming the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \sum_k \left[ A \theta_k \theta_k - B \theta_k^2 \theta_k \right] \]

(57)

the canonical momentum for \( \theta_k \) is \( \pi_k = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} = A \theta_k \), and \( \theta_k, \pi_k = i \delta_{kk'} \). It follows that

\[ \pi_k = -i T \left( \frac{2 \eta k}{\kappa C} \right)^{1/2} \left| \theta_k - \theta_k^\dagger \right| \]

\[ \theta_k = T^{-1/2} \left( \frac{2 \eta k}{\kappa C} \right)^{1/2} \left( \theta_k + \theta_k^\dagger \right) \]

(58)

\[ \mathcal{H} = \sum_k \eta k \theta_k^\dagger \theta_k + \text{cnst}, \quad \eta_k = \frac{\left( \frac{\kappa C}{\hbar} \right)^{1/2}}{\left| k \right|} \equiv \left| \nu \right| \]

what defines the velocity \( \nu \) in terms of a lattice parameter \( \hat{a} \). The approach is similar to the one for phonons, but \( \pi(x) \) plays the role of the lattice displacement \( d(x, t) \), while \( \nabla \theta \) plays the role of the corresponding momentum \( \pi_\theta(x, t) \). The atomic mass \( M \) of the phonon problem corresponds to \( 1/T \) here. With the definitions given in Equation (54,58)

\[ \frac{1}{2} \langle \{ J_{\omega} \} \rangle_{\omega} = \frac{1}{\tau_0} \frac{\kappa C}{2 \hbar} \left( \langle \pi_{\omega}, \pi_\omega \rangle \right) \]

(59)

The space integral of the response function, \( D_{\text{ex}}(\omega) \), corresponding to Equation (53), but referring to the symmetrized correlation \( \langle \{ \pi_{\omega}, \pi_{\omega} \} \rangle \), can be expressed in terms of the convolution in \( k \) space. In turn, the latter can be transformed into an integral over the mode energy \( \eta_k \) defined in Equation (58). To account for thermalization, we introduce the inverse lifetime of the collective mode appearing in Equation (54), as a damping of the mode, \( \Gamma = \hbar/\tau_0 \) in the response function

\[ D_{\text{ex}}(\omega) = \frac{\pi T}{\nu^2 \hbar} \hat{a}^2 \int_0^\infty d\eta \left( \frac{1}{\omega + \eta + i \Gamma} - \frac{1}{\omega - \eta + i \Gamma} \right) e^{-\eta \omega} \]

(60)

so that
\[ \Im(D_{qL}(\omega)) \rightarrow -\frac{2\pi T}{\nu^2} \hat{a}_f^2 \omega \frac{\Gamma}{\hbar} \text{ for } \Gamma/\omega \ll 1 \]  
\hspace{1cm} (61)

Here, we have assumed that the correlations range a space area \( \hat{a}_f^2 \gg \hat{a} \). Using Equation (59), we get

\[ \frac{1}{2} \hat{a}_f^2 \int dk \langle \langle \hat{J}_f^+ \hat{J}_f^- \rangle \rangle_{qL} = -\hbar \omega T \coth \left( \frac{\hbar \omega}{2k_B T} \right) \times \left[ \frac{\hat{a}_f^2}{\hbar^2} \right] \left( \frac{2\pi T}{\nu^2} \right) \]  
\hspace{1cm} (62)

Comparing Equation (51) with Equation (62) we see that the square bracket provides an expression for \( \kappa \)

\[ \kappa = \frac{k_B \Gamma}{\hbar} \left( \frac{\pi \alpha_c}{\hat{a}_f \sqrt{a}} \right) \]  
\hspace{1cm} (63)

where the velocity given by Equation (58) has been inserted.

The diffusivity \( D_Q \) defined in Equation (89) is connected to the thermal conductivity \( \kappa \) by the Einstein relation

\[ D_Q = \frac{\kappa}{\rho_0 C} \]  
\hspace{1cm} (64)

The “particle density” \( \rho_0 \) appearing here is associated with the Q-excitations and can be defined consistently, on the basis of the parameters used in the hydrodynamical model of this section, as

\[ \rho_0 = 1/\nu^2 \hat{a}_f^2 . \]

Inserting Equation (63) in Equation (64) and using \( \rho_0 \) with the velocity given in Equation (58) gives

\[ D_Q = \hat{a}_f^2 \pi \Gamma \frac{k_B T}{\hbar} \frac{\hat{a}_f^2}{\sqrt{a}} \frac{1}{CT} \]  
\hspace{1cm} (65)

As the energy diffusion is mainly due to heat transport in a highly thermalizable environment, an estimate of \( CT \hat{a}^2/\hat{a}_f^2 \), appearing in Equation (63,65), can be derived from the saddle point contribution to the energy of the SYK model\(^{[38]} \)

\[ \ln Z = -\beta E_0 + S_0 + \frac{c}{2\beta} + \ldots + \frac{c}{2} = \frac{2\pi^2 \alpha_c N}{\beta J} \]  
\hspace{1cm} (66)

By posing \( \frac{\hbar}{\nu^2} CT \sim \partial_\beta \ln Z = c/2\beta^2 \), we get

\[ D_Q = \frac{\beta J}{\pi \alpha_c N} \hat{a}_f^2 \]  
\hspace{1cm} (67)

Equation (67) allows to define \( \hat{a}_f^2 \) as the mean squared diffusive length, such that \( D_Q \rho_0 \approx \hat{a}_f^2 \). In turn, a temperature \( T_0 \) can be defined

\[ \hat{a}_f^2 = \left( \frac{\beta J}{\pi \alpha_c N} \right), \quad T_0 \approx \frac{\Gamma}{k_B} \left( \frac{\beta J}{\pi \alpha_c N} \right) \]  
\hspace{1cm} (68)

\( T_0 \) is the threshold temperature for thermalization to be efficient and is independent of temperature if \( \tau_0 \sim \hbar \Gamma^{-1} \approx \hbar \beta \). Indeed, a “Planck” relaxation time scale is expected when the full incoherent SYK regime sets in.\(^{[35,44]} \) In this case, both \( D_Q \) of Equation (67) and \( T_0 \) of Equation (68) are temperature independent.

![Sketch of the temperature dependence of the thermal conductivity \( \kappa \) and of the resistivity \( \rho \) in the incoherent phase of the strongly interacting fermionic model presented here, which is a sort of extension to higher space dimensions of the \( 0+1-d \) SYK model in its prechaotic phase.](image)

Figure 2. Sketch of the temperature dependence of the thermal conductivity \( \kappa \) and of the resistivity \( \rho \) in the incoherent phase of the strongly interacting fermionic model presented here, which is a sort of extension to higher space dimensions of the \( 0+1-d \) SYK model in its prechaotic phase.

Equation (68) confirms that \( T_0 \) is \( O(\beta J/\pi \alpha_c N) \), if just the zero order for \( \Gamma \) is retained. If we make the same substitution for \( \beta J/\pi \alpha_c N \) in Equation (63) that we did in Equation (67), we get

\[ \kappa = \frac{k_B \Gamma}{\hbar} \left( \frac{\beta J}{\pi \alpha_c N} \right)^{1/2} \]  
\hspace{1cm} (69)

This contribution to the thermal conductance is \( \approx O\left( \left( \frac{\beta J}{\pi \alpha_c N} \right)^{1/2} \right) \) and is reported in the sketch of Figure 2 as \( \sim \sqrt{T} \). The definition of \( \hat{a}_f^2 \sim \hat{a}^2 T_0/T \), in Equation (67), emerged in a previous study,\(^{[35]} \) by including scaling to finite \( \beta J/\pi \alpha_c \). Both \( T_0 \) and \( T_0 \) are \( O(\beta J/\pi \alpha_c \) because they are parameter scales of marginal perturbations in the large \( N \), large \( J \) limit. It follows that the contribution to \( \kappa \) given by Equation (69) is subdominant with respect to the constant one coming from the IR limit given by Equation (35).

### 4.3 - \( 3-d \) Quantum Liquid in Interaction with the SYK System in the Temperature Range \((T_0, T_0)\)

In this section, we consider a \( 3-d \) qL, which would be a single band FL if in isolation, and we consider its interaction with our extended \( 2+1-d \) SYK model. In Section 2 and 3 we have recognized that there is a temperature threshold \( T_0 \) given by Equation (5), below which a \( 3-d \) qL in interaction with the SYK system acquires coherence and a threshold temperature \( T_0 \) given by Equation (68), at which the Q-excitations due to the UV corrections become dominant. In this section we discuss how the fermionic SYK excitations and the collective bosonic Q-excitations influence the coherence of the qL. Our aim is to derive changes in the lifetime of the quasiparticles of the qL due to the interaction. We start with the correction to the self-energy of the FL induced by the energy distribution of the fermionic SYK excitations in the conformal limit (for \( T \leq T_0 \)). We will find that the qL becomes a MFL (Section 4.1). Next, we derive the lifetime reduction induced by the interaction with the bosonic Q-excitations in the temperature range \( T \leq T_0 \) (Section 4.2).
4.1. MFL for $T \gtrsim T_{\text{coh}}$

In a perturbative approach, let us assume a local interaction strength $J$ in the FL. The quasiparticles of the low energy FL have a residue $Z$, such that $Z k_F \sim J^{-1}$. The single particle energy is $\varepsilon_k = \varepsilon_0 k^2$, in the continuum limit, in the vicinity of the FS, with a renormalized physical velocity $v_F = Z k_F$. The isotropic self-energy arising from the interaction, for $k$ on the Fermi surface, is

$$\Sigma(k, \omega) = J \sum_q \frac{d\Omega}{2\pi} G(\varepsilon_{k+q} - \varepsilon_k, i\omega + i\Omega) \Pi(q, i\Omega)$$

$$\approx J \sum_q \frac{d\Omega}{2\pi} \frac{1}{iZ^{-1}(\omega + \Omega) - \varepsilon_0 \cos \theta} \Pi(q, i\Omega),$$

(70)

where $\Pi(q, i\Omega)$ is the polarization function

$$\Pi(q, i\Omega) = \sum_p \sum_{\alpha_n} G(\varepsilon_p, i\omega_n) G(\varepsilon_{p+q}, i\omega_n + i\Omega)$$

(71)

In Equation (69), $\theta$ is the angle between $q$ and $k = \varepsilon_0 k$. We have approximated $\varepsilon_{k+q} - \varepsilon_k \approx \varepsilon_0 \cos \theta$.

In the range of frequencies $\Omega \sim T_{\text{coh}} = W^2/J$, where $W$ is the bandwidth ($k_F = 1$ here), the contribution to the polarization, coming from the virtual FL quasiparticles, gives the well-known result

$$\Im \{\Sigma^{(1)}(k, \omega)\} \approx a Z^{-2} \text{sign} |\omega| \omega^2$$

(72)

($a$ is a parameter of order one). The contribution coming from interaction with the extended SYK system, $\Pi^2(i\Omega)$, is evaluated from the single particle Green’s function of the SYK model, in the conformally symmetric limit, which is $q$ independent, and reported in Equation (10)

$$\Pi^2(i\Omega) = -\frac{1}{J} \left[ \int_{-J}^{J} \frac{d\omega}{\Omega} \frac{\text{sign}(\omega + \Omega) \text{sign}(\omega)}{\sqrt{\omega} \sqrt{\omega + \Omega}} \right]$$

$$\approx -\frac{4}{J} \ln \left( \frac{J}{\max(\Omega, T_{\text{coh}}, 1)} \right) \approx -\frac{8}{J} \ln \left( \frac{W}{T_{\text{coh}}} \right).$$

(73)

The final result is

$$\Im \{\Sigma^{(2)}(\omega)\} \approx -\frac{e_F}{2T_{\text{coh}}} |\omega| \ln \left( \frac{J}{T_{\text{coh}}} \right) \text{sign}(\omega)$$

(74)

For $T \gg T_{\text{coh}}$, we should put $\ln J / T_{\text{coh}} \rightarrow \ln(J/T)$ in Equation (73). $\sum^{(1,2)}(k, \omega)$ changes sign at $\omega = 0$ when the quasiparticle becomes a quasihole. Putting both terms together, the quasiparticle relaxation rate is

$$\frac{1}{\tau} \approx -Z |\Im \Sigma(k, \omega)|$$

$$= |\omega| \frac{e_F}{k_B T_{\text{coh}}} \left( \frac{J}{T_{\text{coh}}} \right) + \frac{a}{Z} \nu_0 |\omega|^2,$$

(75)

which shows that, to the lowest approximation, the perturbed FL is a MFL. The lowest lying collective excitations of the MFL can also be derived in the present perturbative frame. A hydrodynamic collective excitation, the would-be acoustic plasmon, is rather well defined. The acoustic plasmon is on the verge to emerge as a bound state at low energies, split off the p–h continuum. Its dispersion tends to the boundary of the p–h continuum, where the imaginary part vanishes.

In Section 2.3, we have derived a thermal conductance $\kappa$ in the conformal limit, which is independent of $T$. This $T$ independence appears to be consistent with the present result. Semiclassically, the left-hand side of Equation (51) can be approximated as ($\omega \approx 0$)

$$\frac{1}{\tau} \int d\omega \int dt \int d\omega' \langle [J'(r, t), J'(r', t')] \rangle \approx \eta_{\text{MFL}} \frac{L}{2\pi} \frac{v_{\text{F}}^2}{\Omega} (|\delta E|^2) \tau_{\text{MFL}}$$

(76)

The saddle point contribution to energy of the SYK model of Equation (66) provides us with the energy fluctuation, given by

$$\langle |\delta E|^2 \rangle = \frac{\alpha}{\rho^2} \ln \Omega = \frac{e}{\rho^2}$$

(77)

By plugging Equation (76) in Equation (51) with Equation (77), we get, in the limit $\hbar \omega / k_B T \ll 1$

$$\kappa \approx k_B \frac{L}{2\pi} \eta_{\text{MFL}} \frac{v_F^2}{(k_B T)^2} \frac{4\pi a_\Omega^2 N}{\rho^2 J} \tau_{\text{MFL}}$$

(78)

The MFL lifetime of Equation (75) is $\tau_{\text{MFL}} \approx T^{-1}$, so that $T$ power counting gives the $\kappa$ contribution of Equation (78) independent of $T$, at this perturbative level. However, this contribution to the thermal conductivity is again subdominant as $\mathcal{O}(|\beta J/N|^3)$, while the contribution coming from the IR limit of Equation (36) is $\mathcal{O}(\beta J/N)$.

4.2. Influence of the Q-Excitations on the Lifetime of the MFL ($T \gtrsim T_{\text{coh}}$)

The Q-excitations introduced in Section 3.2 are an additional scattering mechanism which contributes to the reduction of the lifetime of the quasiparticles of the QL.

According to the source term of Equation (25), $J' - J''$ fluctuations couple to change of the chemical potential, as can be read off Equation (50). Changes in the chemical potential perturb the QL. Our aim in this subsection is to derive the effect on the lifetime of the quasiparticles of the QL due to this kind of interaction. The thermal average of Equation (50) can be newly defined as

$$G_{+}^{\text{LO}}(q, i\omega) = \frac{\hbar}{\rho_0} \int \frac{dk'}{(k_B T)^2} \frac{d\omega'}{2\pi} J_{+}(k', \omega', -i\omega)$$

(79)

In real time, $G_{+}^{\text{LO}}$ is a response function $G_{+}(t) = \Theta(t)G^{(1)}$ and has the dimensions of $[e]$. Here, we face a problem in the construction of the interaction matrix element between the SYK sheet system and the $3 - d$ QL. The transferred momentum $q$ is a $3 - d$ vector in the liquid, but it is only defined in two dimensions in the $J' - J''$ term. We overcame the same difficulty in Equation (51,53), by considering the convolution (with $k$ integration) as in Equation (62) of...
Section 3.2, which corresponds to averaging in space, but by assuming that the fluctuation correlations are localized within a volume $\sim a_\parallel^2$. We eventually get

$$
G_{\nu}^{\perp}(q, i\omega) = 2\pi \frac{\hbar}{\sigma_{\nu}^2 r_0^2} \frac{a_\parallel^2}{H} \frac{\Gamma}{-i\omega + \hbar D q^2} \quad (80)
$$

Assuming in the T power counting $\rho_\nu^2 r_0^2 \approx O([T]^{0})$, with $\hbar/r_0 \approx \Gamma \approx T$, the prefactor of the energy $G_{\nu}^{\perp}$ provides a squared coupling prefactor $|g|^2 \approx T^0$ while, in the case of the e–ph interaction in a metal,[44] $|g|^2 \approx T$.

Equation (80) can play the role of an interaction energy matrix element for the scattering between Q-excitations and a MFL quasiparticle. We plug it in the Fermi golden rule and derive the thermalization rate[38] ($\omega$ has dimension $[e]$ from now on)

$$
\frac{1}{\tau_\nu} = \frac{2\pi}{\hbar} \frac{L}{(2\pi)} \int_0^\infty \nu_\nu \omega d\omega \int_{\omega/\hbar v_F}^{2k_F} q^{d-1} dq \left[ G_{\nu}^{\perp}(q, i\omega) \right]^2 \quad (81)
$$

$\nu_\nu$ is the density of states at $E_F$ for a $3-d$ volume. The integration over the direction of the momentum has already been performed and gives the factor of $(\hbar v_F)^2$ in the denominator, reflecting the increased time available for small deflections. As a minimum momentum must be transferred to give a change in energy of $\omega$, $\omega$ appears also in the lower limit of the momentum integral. The integral over $\omega$ is simply the number of possible hole excitations that can be created in the FL close to the FS. If $G_{\nu}^{\perp}(q, i\omega)$ is independent of $q$ and $\omega$, the integral is not sensitive to the value of the lower limit and is independent of $\omega$. The subsequent integration over $\omega$ recovers the usual $3-d$ result, $\tau_\nu^{-1} \sim \epsilon^2$, which is expected for a FL.

Inserting $G_{\nu}^{\perp}$ of Equation (80) in Equation (81), we have

$$
\frac{1}{\tau_\nu} = \frac{2\pi}{\hbar} \frac{L}{(2\pi)} \int_0^\infty \nu_\nu \omega d\omega \int_{\nu_\nu/\hbar v_F}^{2k_F} q^{d-1} dq \left[ \frac{\hbar}{\rho_\nu^2 r_0^2} \Gamma \right]^2 \quad (82)
$$

which gives

$$
\frac{1}{\tau_\nu} = \frac{2\pi}{\hbar} \frac{L}{(2\pi)} \int_0^\infty \nu_\nu \omega d\omega \int_{\nu_\nu/\hbar v_F}^{2k_F} q^{d-1} dq \frac{1}{\hbar D q^2 - i\omega} \quad (83)
$$

In the evaluating the integral, we have excluded the upper limit because it implies very large transferred $q$ values, which we neglect as it gives a contribution $O(\epsilon^2)$ to the result. To proceed to $T$ power counting, we assume that the volume is factorized into an in-plane contribution $L^2 \sim a_\parallel^2$ where the diffusion dynamics takes place, times an out of plane one, represented by the term $\nu_\nu \hbar v_F L/a_\parallel^2$. Assuming again $\rho_\nu^2 r_0^2 \approx T^0$, we get

$$
\frac{1}{\tau_\nu} \propto T^2 \left( \frac{T_0}{T} \right)^3 \epsilon^3 \approx T^2 \quad (84)
$$

This $T$ dependence cannot be distinguished from the FL one, although it arises from a very different origin, that is from the scattering of MFL single particle excitations by the Q-excitations. Just for comparison the electron–photon scattering rate, is[38]:

$$
\tau_{e-\gamma}^{-1} \approx T^4 \quad (85)
$$

If we use Equation (78) as an approximate expression for the thermal conductance, in which we insert the result of Equation (83), we get $\kappa \approx 1/T$ as in the FL case. In fact, in the FL case, $\kappa = C v_F e = C v_F^2 r_0$. As $r_0 \approx T^{-1}$ and $C \approx T^2$, then $\kappa \approx T^{-1}$. However, as $a_\parallel^2$ appears in Equation (83), this $\kappa$ contribution $\approx O([N/\beta J]^3)$ and it is irrelevant for large $\beta J$.

On the other hand, resistivity is ruled by a relaxation time in which the forward scattering is subtracted from the response appearing in Equation (80). For very small $\omega$, both integrals are convergent and we have

$$
\frac{L}{2\pi} \int_{\omega/\hbar v_F}^{2k_F} \frac{d\omega}{(\hbar v_F)^2} \left[ \frac{-i\omega}{\hbar D q^2 - i\omega} - 1 \right] = \frac{1}{2\pi} \frac{2k_F L}{(\hbar v_F)^2} \quad (85)
$$

where we have excluded the bottom limit of integration. Hence

$$
\frac{1}{\tau_\nu} = \frac{1}{\hbar} \frac{a_\parallel^2 L}{(\hbar v_F)^2} \int_0^{2k_F} q^{d-1} dq \left[ \frac{1}{\rho_\nu^2 r_0^2} \right] \frac{1}{\hbar D q^2 - i\omega} \quad (86)
$$

Here, the $T$ power counting provides $\frac{1}{\tau_\nu} \propto T^2 \left( \frac{T_0}{T} \right)^3 \epsilon^2 \approx T$ which points to the unbound dependence $\sim T$ of the resistivity, once more. This result should be compared with the fifth $T$ power arising from the electron–photon scattering and shows that, although the constraints on momentum conservation has been relaxed, the localized nature of the Q-excitations does not increase resistance with increasing temperature any further. This can also be seen from the absence of $a_\parallel^2$ in Equation (86) which implies that a contribution to the resistivity would drop at increasing temperature as $O([N/\beta J]^3)$, again, and soon become irrelevant.

5. Conclusions

The SYK model, borrowed from gravity theories, is solvable at large flavor number $N$ and is intrinsically NFL. Nonvanishing zero temperature entropy is the signature of a crucial role of quantum fluctuations in the build-up of the ground state of the system. The model is originally in $0+1-d$ dimensions and its fermionic fields are Majorana neutral fermions. This feature, if taken seriously, already implies that the charge and spin degrees of freedom have been dropped out of the model[10] when describing the electronic carriers. Indeed, Majorana Fermions are neutral and interaction $J$ grows to infinity but is fully local and momentum is not conserved. As transport on the c-axis could be attributed to tunnelling, we propose in this work an extension of the SYK model to two space dimensions and include its interaction with a 3D qL.

Our focus is on the symmetry breaking of an approximate conformal symmetry of the SYK model in the IR, strong coupling limit. For temperature in the neighborhood of $T_{coh}$, the symmetry breaking generates a “quasi-order parameter” $G_{\nu}(r)$ and “single” particle neutral fermion excitations which are responsible for transforming the qL into a MFL. The linear dependence of the resistivity on $T$ and the independence of $T$ of the thermal conductivity, derived with the linear response, follow. However, the spontaneous breaking of the conformal symmetry is in reality
explicitly produced by UV corrections. The UV corrections are local in time and they are found to be extended, but rather localized in space, when the SYK model is extended to higher space dimensions. They give rise to bosonic collective modes which we nickname as Q-excitations, in interaction with the qL. The Q-excitations are diffusive modes in the lattice. We assume that they have lifetime \( \tau_0 \approx \hbar / \beta \) and we infer by scaling\(^{[35]} \) that they have mean square space extension \( \delta \sim T_0 / T \). In this work, we have derived the contribution of these excitations to the lifetime of the qL quasiparticles. The trends in the temperature dependence of the resistivity \( \rho \) and of the thermal conductivity \( \kappa \) derived in our approach are reported in Figure 2 for the temperature window \( T_{\text{coh}} \leq T \leq T_0 \), in which there is a crossover to the incoherent regime. For \( T > T_0 \), the SYK model saturates a bound on chaos\(^{[45]} \) and no perturbative approach is any more feasible.

To sum up, we have studied the effect of interactions between the SYK model extended to \( 2 - d \) and the quasiparticles of a \( 3 - d \) qL in which the SYK system is embedded. The \( 2 - d \) SYK model can be thought of as a collection of disordered dots in a continuum limit, in which strong disorder restores space invariance due to self-averaging. In Section 2, we have introduced the Hamiltonian of the SYK model extended to the lattice. The temperature threshold, \( T_{\text{coh}} \), below which the hopping in the lattice establishes coherence in the system has been defined in Section 2.1 and the complex fermion occupation number \( n_f \) close to the Fermi energy has been derived in the conformal symmetry limit, by including the lattice perturbation. We show that these complex fermion single particle excitations which remind of FL quasiparticles are ill defined at the FS, confirming the NFL nature of the extended version of the SYK model. This fact justifies our choice of concentrating on the neutral excitations of the model, which are the fermionic excitations originating from the IR saddle point strong coupling limit and the collective diffusive Q-excitations arising from UV corrections to the IR theory. The generating functional of the soft mode correlations, derived in the Gaussian approximation,\(^{[35]} \) is reported in Section 2.2. A hydrodynamics of energy, instead of particles, is introduced in this section. It is crucial to keep in mind that the underlying physics of the incoherent SYK system implies that the \( 2 - d \) system acts as a sink for particles and momentum, so that the usual particle hydrodynamics is not justified. On the contrary, the energy current density has a diffusive dynamics which does not require momentum and particle conservation. The source term to be added to the action is derived in this way and is used in Section 2.3, to obtain the thermal conductivity \( \kappa \) and the resistivity \( \rho \) of the model in the conformal symmetry limit.\(^{[19]} \) These transport coefficients are proportional the temperature scale \( T_{\text{coh}} \) which is \( \sim O(\beta \sqrt{\mathcal{N}} / N) \) in the large \( N \), large \( \mathcal{J} \) limit.

The role of the UV corrections in giving dynamics to the collective bosonic excitations, the Q-excitations, is presented in Section 3. The Q-excitations can be shown to be diffusive in \( 2 - d \)\(^{[38]} \) and to produce energy current density fluctuations which become dominant for \( T \sim T_0 \). The response of the chemical potential to these fluctuations is derived in Section 3.1. A Lagrangian approach to the dynamics of the Q-excitations, which includes diffusivity in an approximate way, and their quantization is presented in Section 3.2. The \( T \) dependence of their contribution to transport for \( T \approx T_0 \) follows from the choice of their lifetime \( \tau_0 \approx \hbar / \beta \), which implies full thermalization of the SYK system. It turns out that this contribution to thermal conductivity is subdominant (\( \sim O(\beta \sqrt{\mathcal{J} / N} / N^{1/2}) \)) with respect to the one of Section 2.3.

Section 4 is dedicated to the \( 3 - d \) qL and to its interaction both with the single particle fermionic excitations\(^{[24]} \) and with the Q-excitations introduced in Section 3. The lifetime of the quasiparticles of the qL is derived in Section 4.1. The interaction with the conformally symmetric SYK system is dealt with perturbatively, and makes the liquid a MFL. The contribution of the MFL to the thermal conductivity is derived in a semiclassical approximation and is found to be consistent with the one derived in Section 2.3. In Section 4.2 the influence of the Q-excitations on the lifetime of the MFL is derived by using the Fermi golden rule. These scattering rates allow to estimate the resistivity and thermal conductivity for \( T \approx T_0 \). The estimates of the temperature dependence of \( \kappa \) and \( \rho \) are visualized in the sketch of Figure 2.

The model confirms, also beyond the IR limit, the temperature dependence of the resistivity that is expected in the measurements, if the model does indeed entail features of the real “strange” NFL metals. All additional contributions to the thermal conductivity estimated here are subdominant with respect to the IR limit contribution (Equation (36)), which is of \( O(\beta \sqrt{\mathcal{J} / N}) \) in the large \( N \), strong interaction \( \mathcal{J} \) limit. The pivotal quantity of our hydrodynamic approach to the collective Q-excitations is their mean square diffusion area \( \delta^2 \approx T_0 / T \). In the scaling to the strong interaction limit, both \( T_0 \) and the diffusion coefficient \( D_0 \) are temperature independent. The incipient localization with increasing temperature locates our result straightforwardly out of the Mott–Ioffe–Regel resistivity limit.\(^{[46]} \) However, for \( T > T_0 \) the \( 0 + 1 - d \) SYK model enters into a scrambled phase which is fully chaotic. The analysis of this phase in the extended model is out of scope of this work.

Appendix: Relativistic Approach to Transport

In a particle system at equilibrium, the free energy per particle is \( F = \frac{U}{N} - \frac{T S}{N} \), where \( U / N = \epsilon \) is the internal energy per particle and \( s = S / N \) is the entropy per particle. Adding and subtracting \( p / n \) where \( p \) is the pressure and \( n \) is the particle density, equilibrium implies

\[
dF = 0 \rightarrow \frac{dU}{n} = T \frac{dS}{n} + \frac{dp}{n} \quad \text{(A1)}
\]

Here, \( w = \epsilon + p \) is the enthalpy and small letters are thermodynamic quantities per unit volume.

Besides, in terms of the Gibbs potential, \( G = \mu N = F + p V \) we write

\[
\frac{w}{n} - \frac{T S}{n} = \mu \quad \text{(A2)}
\]

where \( \mu \) is the chemical potential.

Here, we consider heat flow, even in the absence of matter convection, starting from \( T^k \), the energy-momentum tensor

\[
T^k_{\mu} = w u_{\mu} v^k + p g^k_{\mu} + \tau^k_{\mu} \quad \text{(A3)}
\]
where \( g_{ij} \) is the metric and \( \tau_i^k \) is the stress tensor. In the proper frame, the velocity \( u^i = 0 \) and \( u^0 = 1 \) (italic indices \( i = 0, 1, 2, 3 \), while Greek indices \( \alpha = 1, 2, 3 \).) Separating the center of mass flux from a small fluctuating contribution, we have, as in Equation (23)

\[
\vec{J} = n\vec{u} + \vec{v} \rightarrow \vec{T}_0^a
\]

\[
\vec{J}^* = \left( \frac{\epsilon + p}{n} \right) (\vec{J} - \vec{v}) = \vec{T}_0^a \equiv w u^a
\]  

(A4)

where \( \vec{J} \) is the particle current density while \( \vec{J}^* \) is the energy current density. In the proper frame, the vector flux density \( n u^i + \nu^i \) implies \( \nu^i u_i = 0 \) because its 0 component must be equal to the particle number density \( -n : n u^i u_k + \nu^i u_k = -n \), by definition. The continuity equation reads

\[
\frac{\partial}{\partial x^k} \left[ n u^k + \nu^k \right] = 0
\]  

(A5)

\( \vec{T}_i^k \) includes dissipative processes (viscosity and thermal diffusion). However, in the absence of external forces, conservation requires \( \frac{\partial}{\partial x^i} \vec{T}_i^k = 0 \), or\(^{[40]}\)

\[
0 = u^i \left( \frac{\partial}{\partial x^{\nu}} \vec{T}_i^k + \frac{\partial}{\partial x^k} g_i^k + \frac{\partial}{\partial x^i} n u^k + \frac{\partial w}{\partial x^k} \right)
+ u^i u_k^l + \frac{\partial w}{\partial x^k} u_i^l + \frac{\partial}{\partial x^k} \tau_i^k
\]  

(A6)

Here, \( u^i u_k = -1 \), so that \( \frac{\partial w}{\partial x^k} = 0 \) and the fourth term vanishes. In the third term, we substitute \( \frac{\partial u^k}{\partial x^i} \rightarrow -\frac{\partial u^k}{\partial x^i} \), according to Equation (A4). Next, from Equation (A1), we have

\[
u_i = -\kappa \left( \frac{nT}{w} \right)^2 \left( \frac{\partial}{\partial x^i} \left( \frac{\mu}{T} \right) + u_i u^k \frac{\partial}{\partial x^k} \left( \frac{\mu}{T} \right) \right)
\]  

(11)

\( \kappa \) is the heat current response to \( -\nabla T \) in the absence of a driving particle current. The choice of the prefactor is in agreement with the Wiedemann–Franz law \( \sigma_\alpha = \frac{1}{2} \left( \frac{k_b}{2} \right)^2 \). Thermal conduction with no particle flux implies vanishing of the second term. We have

\[
T_\alpha = w u^\alpha u_\alpha = -w \frac{\mu_x}{n} \frac{\partial}{\partial x^i} \left( \frac{nT}{w} \right) \frac{\partial}{\partial x^i} T
\]  

(A12)

The last equality arises from the following manipulations

\[
d \left( \frac{nT}{w} \right) = -\frac{1}{T^2} \frac{1}{n} dT - \frac{1}{n} \frac{1}{T} d \left[ \frac{T^2}{T^2} \right]
\]  

(A13)

where Equation (A1) has been used.

Equation (A12) is the energy flux, which also includes a \( V \rho \) term that is absent in the nonrelativistic result.\(^{[40]}\)

From linear response, the energy density current \( \vec{j} \) can be written as\(^{[47]}\)

\[
\vec{J}^* = w \vec{J} + \kappa V \vec{T} + \frac{\sigma_{\epsilon \nu}}{n} \left[ -\nabla \vec{p} \right] + \frac{\sigma_{\epsilon \nu}}{n} \frac{\vec{e} \times \vec{J}}{c^2} \frac{1}{\epsilon} \int d \vec{r} \cdot \vec{j}
\]  

(14)

where \( \sigma_{\epsilon \nu} \) is the conductivity tensor with \( \epsilon, \nu \) unitary vector in real space. We have added the last term, which is the Magnus force. The Magnus force is present in case there is a current circulation \( \oint d \vec{r} \cdot \vec{j} \neq 0 \) in the flow. In the case of an electromagnetic current it would take the form \( we \vec{J} \times \frac{\partial}{\partial x^i} \vec{J} \), when induced by a magnetic field \( B \), with \( \sigma_{\epsilon \nu} \) being the Hall conductance.\(^{[48]}\)

The dimension of \( [J^*] \) is \( \left[ \frac{\eta}{\sqrt{T}} \right] \).

We can define the compressibility \( \chi \) and the pair of diffusion constants for the Q-excitations \( D_Q \) and for the particles \( \tilde{D} \), respectively, according to

\[
\chi = -\frac{1}{V} \frac{\partial V}{\partial \rho \vec{p}}, \quad \tilde{D}_Q = \frac{\kappa}{\rho_0 C}, \quad \tilde{D}_e = \frac{\sigma}{\tilde{\chi}}
\]  

(A15)

(space isotropy is assumed here for simplicity). The diffusivity \( D_Q \) is related to the thermal capacitance \( C \) and \( \rho_0 \) is a particle density associated to the Q-excitations which does not coincide with the particle density \( n_i \) of the carriers and is defined after Equation (64). A generalized diffusion constant \( \tilde{D} \) can be defined, according to

\[
\frac{1}{n} \vec{J} = \tilde{D}_Q CVT + \frac{\sigma}{nT \tilde{\chi}} \left[ \frac{1}{V} \frac{\partial V}{\partial \rho \vec{p}} + \frac{n^2}{m^2} \epsilon \right] \nabla \vec{T} \equiv \tilde{D} C VT
\]  

(A16)
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Conflict of Interest
The author declares no conflict of interest.

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