Existence and Exponential Growth of Global Classical Solutions to the Compressible Navier-Stokes Equations with Slip Boundary Conditions in 3D Bounded Domains

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Abstract

We investigate the barotropic compressible Navier-Stokes equations with slip boundary conditions in a three-dimensional (3D) bounded domain, whose smooth boundary has a finite number of two-dimensional connected components. For any adiabatic exponent bigger than one, after discovering some new estimates on boundary integrals related to the slip boundary condition, we prove that both the weak and classical solutions to the initial-boundary-value problem of this system exist globally in time provided the initial energy is suitably small. Moreover, the density has large oscillations and contains vacuum states. Finally, it is also shown that for the classical solutions, the oscillation of the density will grow unboundedly in the long run with an exponential rate provided vacuum appears (even at a point) initially. This is the first result concerning the global existence of classical solutions to the compressible Navier-Stokes equations with density containing vacuum states initially for general 3D bounded smooth domains.

Keywords: compressible Navier-Stokes equations; global existence; slip boundary condition; vacuum.

1 Introduction

The viscous barotropic compressible Navier-Stokes equations for isentropic flows express the principles of conservation of mass and momentum in the absence of exterior forces:

\begin{align}
\begin{cases}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla\text{div} u + \nabla P(\rho) &= 0,
\end{cases}
\end{align}

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where \((x, t) \in \Omega \times (0, T]\), \(\Omega\) is a domain in \(\mathbb{R}^N\), \(t \geq 0\) is time, \(x\) is the spatial coordinate, and \(\rho \geq 0\), \(u = (u^1, \cdots, u^N)\) are the unknown fluid density, velocity respectively. The constants \(\mu\) and \(\lambda\) are the shear and bulk viscosity coefficients respectively satisfying the following physical restrictions:

\[
\mu > 0, \quad 2\mu + N\lambda \geq 0.
\]

(1.2)

We consider the barotropic case, that is, the pressure \(P(\rho)\) satisfies

\[
P(\rho) = a\rho^\gamma
\]

(1.3)

with constants \(a > 0\) and \(\gamma > 1\) which is the adiabatic exponent.

In this paper, we assume that \(\Omega\) is a bounded domain in \(\mathbb{R}^3\), its boundary \(\partial \Omega\) is of class \(C^\infty\) and only has a finite number of 2-dimensional connected components. In addition, the system is studied subject to the given initial data

\[
\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = \rho_0 u_0(x), \quad x \in \Omega,
\]

(1.4)

and slip boundary condition

\[
u \cdot n = 0, \quad \text{curl} \, u \times n = -A \, u \text{ on } \partial \Omega,
\]

(1.5)

where \(A = A(x)\) is a 3 \times 3 symmetric matrix defined on \(\partial \Omega\).

The mathematical study of compressible Navier-Stokes equations dates back to the late 1950s. For density away from vacuum, Serrin [36] and Nash [29] first considered the mathematical questions of compressible viscous fluid dynamics. An intensive treatment of compressible Navier-Stokes equations started with pioneering works by Itaya [20], Matsumura-Nishida [27], Kazhikhov-Solonnikov [23], and Hoff [12] on the local theory for nonstationary problems, and by Beirão da Veiga [5,6], Padula [35], and Novotný-Padula [32,33] on the theory of stationary problems for small data. For the case that density contains vacuum, Lions [26] proved the global existence of so called finite-energy weak solutions when the adiabatic exponent \(\gamma\) is suitably large, for example, \(\gamma \geq 9/5\) for 3D case. These results were further improved by Feireisl-Novotný-Petzeltová [9] to \(\gamma > 3/2\) for three-dimensional case. Moreover, Hoff [11,15] considered a new type of global weak solutions for any \(\gamma > 1\) with small energy that have extra regularity information compared with Lions-Feireisl’s large weak ones. However, the regularity and uniqueness of those weak solutions [9,13,15,26] are completely open. Recently, Huang-Li-Xin [19] and Li-Xin [25] established the global well-posedness of classical solutions to the Cauchy problem for the 3D and 2D barotropic compressible Navier-Stokes equations in whole space with smooth initial data that are of small energy but possibly large oscillations, in particular, the initial density is allowed to vanish. However, when the domains are bounded, the global existence of strong and/or classical solutions with vacuum to the compressible Navier-Stokes equations (1.1) remains open.

For bounded domains, the usual Navier-type slip condition can be stated as follows:

\[
u \cdot n = 0, \quad (2D(u) n + \vartheta u)_{\text{tan}} = 0 \text{ on } \partial \Omega,
\]

(1.6)

where \(D(u) = (\nabla u + (\nabla u)^T)/2\) is the shear stress, \(\vartheta\) is a scalar friction function which measures the tendency of the fluid to slip on the boundary, and the symbol \(v_{\text{tan}}\) represents the projection of tangent plane of the vector \(v\) on \(\partial \Omega\). Indeed, introduced originally by Navier [30] and later independently by Maxwell [28], the Navier-type slip
condition (1.6), which shows that there is a stagnant layer of fluid close to the wall allowing a fluid to slip with the slip velocity being proportional to the shear stress, is frequently used in numerical studies and analysis for various fluid mechanical problems (see for instance [8, 21, 37] and their references therein). Especially to deserve to be mentioned, the restriction that \( \vartheta \) is non-negative is usual, in order to ensure the conservation of energy. But mathematically, we can take into account the negative values of \( \vartheta \) as well. For compressible Navier-Stokes equations (1.1) with Navier-type slip boundary condition (1.6), Novotný-Straškraba [34] obtained the global existence of weak solutions for \( \gamma > 3/2 \) and \( \vartheta = 0 \) in a non-axisymmetric domain when \( \mu > 0 \) and \( 2\mu + 3\lambda > 0 \). Hoff [13] studied the global existence of weak solutions on the half space in \( \mathbb{R}^3 \) provided the initial energy is suitably small. However, the boundary of \( \Omega \) in [13] is flat. Therefore, it remains completely open even for the existence of global weak solutions for any \( \gamma \in (1, 3/2] \) in general bounded domains. Hence, it is interesting to study the global existence of weak and classical solutions to the initial-boundary-value problem (1.1)–(1.5) with any \( \gamma > 1 \) for general bounded smooth domains \( \Omega \) with density containing vacuum initially.

Before stating the main results, we explain the notations and conventions used throughout this paper. We first give the definition of simply connected domains.

**Definition 1.1** Let \( \Omega \) be a domain in \( \mathbb{R}^3 \). If the first Betti number of \( \Omega \) vanishes, namely, any simple closed curve in \( \Omega \) can be contracted to a point, we say that \( \Omega \) is simply connected. If the second Betti number of \( \Omega \) is zero, we say that \( \Omega \) has no holes.

Then we define weak and classical solutions as follows.

**Definition 1.2** Let \( T > 0 \) be a finite constant. A solution \((\rho, u)\) to (1.1) is called a weak solution if it satisfies (1.1) in the sense of distribution. Moreover, when all the derivatives involved in (1.1) are continuous functions, and (1.1) holds everywhere in \( \Omega \times (0, T) \), we call the solution a classical one.

Next, we set

\[
\int f \, dx \triangleq \int_{\Omega} f \, dx,
\]

and

\[
\overline{f} \triangleq \frac{1}{|\Omega|} \int_{\Omega} f \, dx,
\]

which is the average of a function \( f \) over \( \Omega \). For integer \( k \) and \( 1 \leq q < +\infty \), \( W^{k,q}(\Omega) \) is the standard Sobolev spaces, \( H^k(\Omega) \triangleq W^{k,2}(\Omega) \), and

\[
H^1_\omega(\Omega) \triangleq \{ f \in H^1 : f \cdot n = 0, \ \text{curl} f \times n = -Af \ \text{on} \ \partial \Omega \}.
\]

For some \( s \in (0, 1) \), the fractional Sobolev space \( H^s(\Omega) \) is defined by

\[
H^s(\Omega) \triangleq \left\{ u \in L^2(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy < +\infty \right\},
\]

which is a Banach space with the norm:

\[
\|u\|_{H^s(\Omega)} \triangleq \|u\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2}.
\]
For simplicity, we denote \( L^q(\Omega), W^{k,q}(\Omega), H^k(\Omega), \) and \( H^s(\Omega) \) by \( L^q, W^{k,q}, H^k, \) and \( H^s \) respectively.

For two \( n \times n \) matrices \( A = \{a_{ij}\}, \ B = \{b_{ij}\}, \) the symbol \( A: B \) represents the trace of \( AB, \) that is,

\[
A: B \triangleq \text{tr}(AB) = \sum_{i,j=1}^n a_{ij}b_{ji}.
\]

Finally, we denote the initial total energy of (1.1) as

\[
C_0 \triangleq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) dx,
\]

with

\[
G(\rho) \triangleq \rho \int_0^\rho \frac{P(s) - P(\bar{\rho})}{s^2} ds.
\]

Then one of the main purposes of this paper is to establish the following global existence of classical solutions of (1.1)-(1.5) in a general smooth bounded domain \( \Omega \subset \mathbb{R}^3. \)

**Theorem 1.1** Let \( \Omega \) be a simply connected bounded domain in \( \mathbb{R}^3 \) and its smooth boundary \( \partial \Omega \) is a finite number of 2-dimensional connected components. For given positive constants \( M, \tilde{\rho}, \) suppose that the \( 3 \times 3 \) symmetric matrix \( A \) in (1.5) is smooth and positive semi-definite, and the initial data \( (\rho_0, u_0) \) satisfy for some \( q \in (3, 6) \) and \( s \in (1/2, 1], \)

\[
(\rho_0, P(\rho_0)) \in W^{2,q}, \quad u_0 \in H^2 \cap H^1_\omega(\Omega), \quad 0 \leq \rho_0 \leq \tilde{\rho}, \quad \|u_0\|_{H^s} \leq M,
\]

and the compatibility condition

\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g,
\]

for some \( g \in L^2. \) Then there exists a positive constant \( \varepsilon \) depending only on \( \mu, \lambda, \gamma, a, s, \tilde{\rho}, M, \Omega, \) and the matrix \( A \) such that if

\[
C_0 \leq \varepsilon
\]

with initial energy \( C_0 \) as in (1.7), the initial-boundary-value problem (1.1)-(1.5) has a unique classical solution \((\rho, u)\) in \( \Omega \times (0, \infty) \) satisfying that for any \( 0 < \tau < T < \infty, \)

\[
\begin{aligned}
(\rho, P) &\in C([0, T]; W^{2,q}), \\
\nabla u &\in C([0, T]; H^1) \cap L^\infty(\tau, T; W^{2,q}), \\
u_t &\in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1),
\end{aligned}
\]

and that for any \( 0 < T < \infty, \)

\[
\mathcal{C}(T) \inf_{x \in \Omega} \rho_0(x) \leq \rho(x,t) \leq 2\tilde{\rho}, \quad (x,t) \in \Omega \times [0, T],
\]

for some positive constant \( \mathcal{C}(T) \) depending only on \( T, \mu, \lambda, \gamma, a, s, \tilde{\rho}, M, \Omega, \) and the matrix \( A. \) Moreover, for any \( r \in [1, \infty) \) and \( p \in [1, 6], \) there exist positive constants \( C \) and \( \eta_0 \) depending only on \( \mu, \lambda, \gamma, a, s, \tilde{\rho}, M, \rho_0, \Omega, r, p, \) and the matrix \( A \) such that for any \( t \geq 1, \)

\[
\|\rho - \rho_0\|_{L^r} + \|u\|_{W^{1,p}} + \|\sqrt{\rho} u_t\|_{L^2}^2 \leq Ce^{-\eta_0 t}.
\]
Then, with the exponential decay rate \((1.15)\) at hand, motivated by the proof of \([24, \text{Theorem 1.2}]\), we will establish the following large-time behavior of the spatial gradient of the density when vacuum appears initially.

**Theorem 1.2** Under the conditions of Theorem 1.1, assume further that there exists some point \(x_0 \in \Omega\) such that \(\rho_0(x_0) = 0\). Then the unique global classical solution \((\rho, u)\) to the problem \((1.1)-(1.3)\) obtained in Theorem 1.1 satisfies that for any \(r_1 > 3\), there exist positive constants \(\hat{C}_1\) and \(\hat{C}_2\) depending only on \(\mu, \gamma, a, \hat{p}, \hat{\rho}_0, M, \Omega, r_1\) and the matrix \(A\) such that for any \(t \geq 1\),

\[
\|\nabla \rho(\cdot, t)\|_{L^{r_1}} \geq \hat{C}_1 e^{\hat{C}_2 t}.
\]

(1.16)

The third result concerns the global existence of weak solutions.

**Theorem 1.3** Under the conditions of Theorem 1.1 except \((1.1)\), where the condition \((1.3)\) is replaced by

\[
\rho_0 \in L^\infty(\Omega), \quad u_0 \in H^1_\omega(\Omega),
\]

(1.17)

assume further that the initial energy \(C_0\) as in \((1.1)\) satisfies \((1.12)\) with \(\varepsilon\) as in Theorem 1.1. Then there exists at least one weak solution \((\rho, u)\) of the problem \((1.1)-(1.5)\) in \(\Omega \times (0, \infty)\) satisfying \((1.14), (1.15)\) and for any \(0 < \tau \leq T < \infty\) and \(q \in [1, \infty)\),

\[
\begin{cases}
\rho \in L^\infty(0, T; L^\infty) \cap C([0, T]; L^6), \\
u \in L^\infty(0, T; H^1), \ u_t \in L^2(\tau, T; L^6), \ \nabla u \in L^\infty(\tau, T; L^6), \\
\text{curl}\ u, \ (2\mu + \lambda)\text{div} u - P \in L^\infty(\tau, T; H^1) \cap L^2(\tau, T; W^{1,6}).
\end{cases}
\]

(1.18)

A few remarks are in order:

**Remark 1.1** It should be mentioned here that the Navier-type slip condition \((1.3)\) is in fact a particular case of the slip boundary one \((1.5)\). Indeed, since \(u \cdot n = 0\) on \(\partial\Omega\), we have, for any unit tangential vector \(\nu\),

\[
0 = \frac{\partial}{\partial \nu}(u \cdot n) = (D(u) \cdot n) \cdot \nu = \frac{1}{2} \text{curl} \times n \cdot \nu + \nu \cdot \nabla n \cdot u = (D(u) \cdot n) \cdot \nu - \frac{1}{2} \text{curl} \times n \cdot \nu + u \cdot D(n) \cdot \nu,
\]

(1.19)

where in the last equality we have used the fact \(\text{curl} n = 0\). Consequently, \((1.3)\) is equivalent to

\[
\text{curl} \times n = (2D(n) - \vartheta I)u
\]

(1.20)

where \(I\) is \(3 \times 3\) identity matrix and \(\vartheta I - 2D(n)\) is indeed a \(3 \times 3\) symmetric matrix.

**Remark 1.2** Since \(q > 3\), it follows from Sobolev’s inequality and \((1.13)_1\) that

\[
\rho, \nabla \rho \in C(\overline{\Omega} \times [0, T]).
\]

(1.21)

Moreover, it also follows from \((1.13)_2\) and \((1.13)_3\) that

\[
u, \nabla u, \nabla^2 u, u_t \in C(\overline{\Omega} \times [\tau, T]),
\]

(1.22)

due to the following simple fact that

\[
L^2(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^2).
\]
Finally, by (1.1), we have

\[ \rho_t = -u \cdot \nabla \rho - \rho \text{div} u \in C(\bar{\Omega} \times [\tau, T]), \]

which together with (1.21) and (1.22) shows that the solution obtained by Theorem 1.1 is a classical one.

**Remark 1.3** Theorem 1.2 implies that the oscillation of the density will grow unboundedly in the long run with an exponential rate provided vacuum (even a point) appears initially. This new phenomena is somewhat surprisingly compared with the Cauchy problem (19, 25) where there is not any result concerning the growth rate of the gradient of the density.

**Remark 1.4** For the sake of simplicity, we assume that the matrix \( A \) is smooth and positive semi-definite. However, these conditions can be relaxed. Indeed, we only use the assumption that the matrix \( A \) is positive semi-definite in the proof of (3.7) and (3.10) (see (3.10) and (3.11)). Thus, let \( \lambda_i(x) (i = 1, 2, 3) \) be the eigenvalues of \( A \) whose negative parts are denoted by \( \lambda_1^- (x), \lambda_2^- (x), \) and \( \lambda_3^- (x) \) respectively. Then one can deduce that (3.7) and (3.10) both still hold provided \( \lambda_1 (x), \lambda_2 (x), \lambda_3 (x) \) are bounded by some suitably small positive constant depending only on \( \lambda, \mu, \) the constants of Poincaré’s inequality and the constant \( C_1 \) in (3.1) for \( p = 2 \). The other restriction on \( A \) comes from a priori estimates related to \( A \), in which (4.11) plays a decisive role. In fact, by Lemma 2.25, we have

\[
\| \nabla^3 u \|_{L^p} \leq C(\| \text{div} u \|_{W^{2,p}} + \| \text{curl} u \|_{W^{2,p}}) \\
\leq C(\| \rho u \|_{W^{1,p}} + \| \nabla (Au) \|_{W^{1,p}} + \| \nabla u \|_{L^2}) + C\| \nabla P \|_{W^{1,p}} + C\| P - P \|_{L^p}.
\]

Therefore, compared with (4.11), it is sufficient to assume that \( A \in W^{2,6} \) rather than smooth. Moreover, for the lower order priori estimates in Section 3, it is enough to suppose that \( A \in W^{1,6} \) and then the dependence of \( \varepsilon \) on \( A \) in Theorem 1.7 can be determined by \( \| A \|_{W^{1,6}} \).

The following result removes the condition that the region is simply connected provided \( 2\mu + 3\lambda > 0 \).

**Theorem 1.4** Assume that \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \), not necessarily simply connected, and its smooth boundary \( \partial \Omega \) has a finite number of 2-dimensional connected components. Let \( A \in W^{2,6} (\Omega) \) satisfy that \( A + 2D(n) \) is a positive semi-definite \( 3 \times 3 \) symmetric matrix. Moreover, \( A + 2D(n) \) is positive on some \( \Sigma \subset \partial \Omega \) with \( |\Sigma| > 0 \) when \( \Omega \) is axially symmetric. Then, the conclusions of Theorems 1.1, 1.3 are still valid provided \( 2\mu + 3\lambda > 0 \).

By virtue of (1.20), \( A = \partial I - 2D(n) \) corresponds to Navier-type slip boundary condition (1.6). As a direct consequence of Theorems 1.2, 1.4 (see their remarks also), for compressible Navier-Stokes equations (1.1) with Navier-type slip boundary condition (1.6), we have the following conclusion on the global existence and large-time behavior of classical or weak solutions.
Corollary 1.5  Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^3 \) and its smooth boundary \( \partial \Omega \) has a finite number of 2-dimensional connected components. Then, for \( \vartheta \in W^{2,6}(\Omega) \), the conclusions of Theorems 1.1-1.3 still hold where (1.5) is replaced by (1.6) provided one of the following two conditions holds:

1. \( \Omega \) is simply connected, \( A = \vartheta I - 2D(n) \) satisfies the assumption given by Remark 1.4;
2. \( 2\mu + 3\lambda > 0, \vartheta \geq 0 \) and in addition, \( \vartheta > 0 \) on some \( \Sigma \subset \partial \Omega \) with \( |\Sigma| > 0 \) if \( \Omega \) is axially symmetric.

Remark 1.5  Compared with [34, Theorem 7.69] where they obtain the global weak solutions to the compressible Navier-Stokes equations with Navier-slip boundary conditions (\( \vartheta = 0 \)) for \( \gamma > 3/2 \) in a non-axisymmetric domain, our results establish the global existence of both weak and classical solutions (which may contain vacuum states) for any \( \gamma > 1 \) and more relaxed conditions for \( \vartheta \) provided that the initial energy is suitably small. Moreover, (1.14) indicates that vacuum states will not exhibit in any finite time even for weak solutions provided that no vacuum states are present initially.

We now comment on the analysis of this paper. Indeed, compared with the previous results ([19, 25]) where they treated the Cauchy problem, the slip boundary condition (1.5) causes additional difficulties in developing a priori estimates for solutions of the compressible Navier-Stokes equations. To overcome the difficulties, our research bases on three observations. First, it is important to find an equivalence of norms in \( H^1 \). Thanks to [41], for simply connect bounded domains, we have

\[
\| \nabla u \|_{L^2} \leq C (\| \text{div} u \|_{L^2} + \| \text{curl} u \|_{L^2}).
\]

And for more general bounded domains, we have (6.1). Next, for \( v = (v^1, v^2, v^3) \), denoting the material derivative of \( v \) by

\[
\dot{v} \triangleq v_t + u \cdot \nabla v,
\]

we rewrite (1.1) in the form

\[
\rho \dot{u} = \nabla F - \mu \nabla \times \text{curl} u,
\]

with

\[
\text{curl} u \triangleq \nabla \times u, \quad F \triangleq (\lambda + 2\mu) \text{div} u - (P - \bar{P}),
\]

where the vorticity \( \text{curl} u \) and the so-called the effective viscous flux \( F \) both play an important role in our following analysis. Since \( u \cdot n = 0 \) on \( \partial \Omega \), we check that

\[
u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u,
\]

which implies (see (2.32))

\[
(\dot{u} + (u \cdot \nabla) \times u^\perp) \cdot n = 0 \text{ on } \partial \Omega,
\]

with \( u^\perp \triangleq -u \times n \) on \( \partial \Omega \). As a direct consequence of this observation, we have (see (2.29))

\[
\| \dot{u} \|_{L^6} \leq C (\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}^2).
\]
Similarly, one can get \((\text{curl} u + (Au)^\perp) \times n = 0\) on \(\partial \Omega\) by the other boundary condition \(\text{curl} u \times n = -Au\). Combining this with \((1.25)\) implies that one can treat \((1.1)\) as a Helmholtz-Wyle decomposition of \(\rho\dot{u}\) which makes it possible to estimate \(\nabla F\) and \(\nabla \text{curl} u\) (see \((2.24)\)). Finally, since \(u \cdot n = 0\) on \(\partial \Omega\), we have

\[ u = u^\perp \times n \text{ on } \partial \Omega, \]

which, combined with the simple fact that \(\text{div}(\nabla u^\perp \times u^\perp) = -\nabla u^\perp \cdot \nabla \times u^\perp\), implies that we can bound the following key boundary integrals concerning the effective viscous flux \(F\), the vorticity \(\text{curl} u\), and \(\nabla u\) (see \((3.37)\) and \((3.41)\))

\[ \int_{\partial \Omega} F(u \cdot \nabla)u \cdot \nabla n \cdot ud\sigma, \quad \int_{\partial \Omega} \text{curl} u \times n \cdot ud\sigma. \]

All these treatments are the key to estimating the crucial integrals on the boundary \(\partial \Omega\).

The rest of the paper is organized as follows. First, some notations, known facts and elementary inequalities needed in later analysis are collected in Section 2. Sections 3 and 4 are devoted to deriving the necessary a priori estimates on classical solutions which can guarantee the extension of the local classical solution to be a global one. Finally, the main results, Theorems \(1.1-1.4\) will be proved in Sections 5 and 6.

## 2 Preliminaries

In this section, we recall some known facts and elementary inequalities which will be used later.

First, similar to the proof of \([16, \text{Theorem 1.4}]\), we have the local existence of strong and classical solutions.

**Lemma 2.1** Let \(\Omega\) be as in Theorem \(1.1\), assume that \((\rho_0, u_0)\) satisfies \((1.9)\) and \((1.11)\). Then there exist a small time \(T > 0\) and a unique strong solution \((\rho, u)\) to the problem \((1.1)-(1.5)\) on \(\Omega \times (0, T)\) satisfying for any \(\tau \in (0, T), \quad \)

\[
\begin{aligned}
& (\rho, P) \in C([0, T]; W^{2,q}), \\
& \nabla u \in C([0, T]; H^1) \cap L^\infty(\tau, T; W^{2,q}), \\
& u_t \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1), \\
& \sqrt{\rho} u_t \in L^\infty(0, T; L^2).
\end{aligned}
\]

Next, the well-known Gagliardo-Nirenberg’s inequality (see \([31]\)) will be used frequently later.

**Lemma 2.2 (Gagliardo-Nirenberg)** Assume that \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^2\). For \(p \in [2, 6]\), \(q \in (1, \infty)\), and \(r \in (3, \infty)\), there exist generic constants \(C, C_1, C_2\) which depend only on \(p, q, r, \) and \(\Omega\) such that for any \(f \in H^1(\Omega)\) and \(g \in L^q(\Omega) \cap D^{1,r}(\Omega), \quad \)

\[
\begin{aligned}
& \|f\|_{L^p(\Omega)} \leq C\|f\|_{L^2}^{\frac{6-p}{2}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2}} + C_1\|f\|_{L^2}, \\
& \|g\|_{C(\Omega)} \leq C\|g\|_{L^q}^{\frac{(r-3)/(3r+q(r-3))}{3r/(3r+q(r-3))}} \|\nabla g\|_{L^r}^{\frac{3r/(3r+q(r-3))}{3r/(3r+q(r-3))}} + C_2\|g\|_{L^2}. \quad (2.1)
\end{aligned}
\]

Moreover, if either \(f \cdot n|_{\partial \Omega} = 0\) or \(\bar{f} = 0\), we can choose \(C_1 = 0\). Similarly, the constant \(C_2 = 0\) provided \(g \cdot n|_{\partial \Omega} = 0\) or \(\bar{g} = 0\).
In order to get the uniform (in time) upper bound of the density $\rho$, we need the following Zlotnik’s inequality.

**Lemma 2.3** ([42]) Suppose the function $y$ satisfies

$$y'(t) = g(y) + b'(t) \text{ on } [0, T], \quad y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \tag{2.3}$$

for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then

$$y(t) \leq \max \{y^0, \zeta_0\} + N_0 < \infty \text{ on } [0, T],$$

where $\zeta_0$ is a constant such that

$$g(\zeta) \leq -N_1 \quad \text{for} \quad \zeta \geq \zeta_0. \tag{2.4}$$

Next, consider the Lamé’s system

$$
\begin{aligned}
-\mu \Delta u - (\lambda + \mu)\nabla \text{div} u &= f, \quad x \in \Omega, \\
u \cdot n = 0, \text{curl} u \times n &= -Au, \quad x \in \partial\Omega,
\end{aligned} \tag{2.5}
$$

where $u = (u^1, u^2, u^3)$, $f = (f^1, f^2, f^3)$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^3$, and $\mu, \lambda$ satisfy the condition ([12]). It follows from [39] that the Lamé’s system is of Petrovsky type. In Petrovsky’s systems, roughly speaking, different equations and unknowns have the same “differentiability order”, see [38]. We also recall that Petrovsky’s systems belong to an important subclass of Agmon-Douglis-Nirenberg (ADN) elliptic systems (see [24]), which has the same good properties of self-adjoint ADN systems. Thus, we have the following standard estimates.

**Lemma 2.4** ([24]) Let $u$ be a smooth solution of the Lamé’s equation (2.5). Then for $q \in (1, \infty)$, $k \geq 0$, there exists a positive constant $C$ depending only on $\lambda, \mu, q, k, \Omega$ and the matrix $A$ such that

1. If $f \in W^{k,q}$, then

$$
\|u\|_{W^{k+2,q}} \leq C(\|f\|_{W^{k,q}} + \|u\|_{L^q}),
$$

2. If $f = \nabla g$ and $g \in W^{k,q}$, then

$$
\|u\|_{W^{k+1,q}} \leq C(\|g\|_{W^{k,q}} + \|u\|_{L^q}).
$$

Next, the following two lemmas can be found in [41, Theorem 3.2] and [3, Propositions 2.6-2.9].

**Lemma 2.5** Let $k \geq 0$ be an integer and $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^{k+1,1}$ boundary $\partial \Omega$, $1 < q < +\infty$. Then for $v \in W^{k+1,q}$ with $v \cdot n = 0$ on $\partial \Omega$, there exists a constant $C = C(q, k, \Omega)$ such that

$$
\|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}} + \|v\|_{L^q}). \tag{2.6}
$$

Furthermore, assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ with $C^{k+1,1}$ boundary $\partial \Omega$, we have

$$
\|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}}). \tag{2.7}
$$
Lemma 2.6 Let $k \geq 0$ be an integer, $1 < q < +\infty$. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^3$ and its $C^{k+1,1}$ boundary $\partial \Omega$ only has a finite number of 2-dimensional connected components. Then for $v \in W^{k+1,q}$ with $v \times n = 0$ on $\partial \Omega$, there exists a constant $C = C(q,k,\Omega)$ such that

$$\|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}} + \|v\|_{L^q}).$$

In particular, if $\Omega$ has no holes, then

$$\|v\|_{W^{k+1,\Omega}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}}).$$

Next, to estimate the $L^1(0,T;L^\infty(\Omega))$-norm of $\nabla u$, we need the following Beale-Kato-Majda type inequality with respect to the slip boundary condition (1.5), which was first proved in [4,22] in the whole 3D spatial space when $\text{div} u \equiv 0$.

Lemma 2.7 Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. For $3 < q < \infty$, assume that $u \cdot n = 0$ and $\text{curl} u \times n = -Au$ on $\partial \Omega$, $u \in W^{2,q}$, then there is a constant $C = C(q,\Omega,A)$ such that the following estimate holds

$$\|\nabla u\|_{L^\infty} \leq C(\|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C. \quad (2.8)$$

Proof. We borrow some ideas of [17,18] and make some slight modifications. It follows from [38,39] that $u$ can be represented in the form

$$u^i = \int G_{i,.(x,y)} \cdot (\mu \Delta_y u + (\lambda + \mu)\nabla_y\text{div}_y u) dy$$

$$= \int G_{ij}(x,y)(\mu \Delta_y u^j + (\lambda + \mu)\partial_y^j\text{div}_y u)(y) dy,$$

where, $G = \{G_{ij}\}$ with $G_{ij} = G_{ij}(x,y) \in C^\infty(\Omega \times \Omega \setminus D)$, $D \equiv \{(x,y) \in \Omega \times \Omega : x = y\}$, is Green matrix of the Lamé’s system (2.5) and satisfies that for every multi-indexes $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ and $\beta = (\beta^1, \beta^2, \beta^3)$, there is a constant $C_{\alpha,\beta}$ such that for all $(x,y) \in \Omega \times \Omega \setminus D$, and $i,j = 1,2,3$,

$$|\partial^\alpha_x \partial^\beta_y G_{ij}(x,y)| \leq C_{\alpha,\beta}|x-y|^{-1-|\alpha|-|\beta|},$$

here $|\alpha| = \alpha^1 + \alpha^2 + \alpha^3$ and $|\beta| = \beta^1 + \beta^2 + \beta^3$.

Notice that according to the definition of $A$ in (1.1), $Au$ is still a tangential vector on $\partial \Omega$, and then we set

$$(Au)^\perp \triangleq -(Au) \times n, \quad (2.9)$$

so $Au = (Au)^\perp \times n$. Therefore,

$$u^i(x) = (\lambda + 2\mu) \int G_{i,.(x,y)} \cdot \nabla_y\text{div}_y u(y) dy$$

$$- \mu \int G_{i,.(x,y)} \cdot \nabla_y \times \text{curl} u(y) dy$$

$$= (\lambda + 2\mu) \int G_{i,.(x,y)} \cdot \nabla_y \text{div}_y u(y) dy$$

$$- \mu \int G_{i,.(x,y)} \cdot \nabla_y \times (\text{curl} u + (Au)^\perp) dy$$

$$+ \mu \int \nabla_y \times (Au(y))^\perp \cdot G_{i,.(x,y)} dy \triangleq \sum_{j=1}^{3} U^i_j. \quad (2.10)$$
It suffices to estimate the three terms \( U_j^i, \ j = 1, 2, 3 \). Let \( \delta \in (0, 1] \) be a constant to be chosen and introduce a cut-off function \( \eta_\delta(x) \) satisfying \( \eta_\delta(x) = 1 \) for \( |x| < \delta \); \( \eta_\delta(x) = 0 \) for \( |x| > 2\delta \), and \( |\nabla \eta_\delta(x)| < C\delta^{-1} \). Notice that \( G_i, (x,y) \cdot n = 0 \) on \( \partial \Omega \), \( \nabla U_1 \) can be written as

\[
\nabla U_1^i = (\lambda + 2\mu) \int \eta_\delta(|x-y|) \nabla_x G_i, (x,y) \nabla y \text{div} u(y) dy \\
+ (\lambda + 2\mu) \int \nabla_y \eta_\delta(|x-y|) \cdot \nabla_x G_i, (x,y) \text{div} u(y) dy \\
- (\lambda + 2\mu) \int (1 - \eta_\delta(|x-y|)) \nabla_x \text{div} y G_i, (x,y) \text{div} u(y) dy
\]

(2.11)

\( \Delta \) (\( \lambda + 2\mu \sum_{k=1}^3 \tilde{I}_k \)).

Now we estimate \( \tilde{I}_k, \ k = 1, 2, 3 \).

\[
|\tilde{I}_1| \leq C \| \eta_\delta(|x-y|) \nabla_x G_i, (x,y) \|_{L^{\delta/(q-1)}} \| \nabla^2 u \|_{L^q} \\
\leq C \left( \int_0^{2\delta} r^{-2q/(q-1)}r^2 dr \right)^{(q-1)/q} \| \nabla^2 u \|_{L^q}
\]

(2.12)

\[
|\tilde{I}_2| = \left| \int \nabla_y \eta_\delta(|x-y|) \cdot \nabla_x G_i, (x,y) \text{div} u(y) dy \right| \\
\leq C \int |\nabla_y \eta_\delta(y) \cdot \nabla_x G_i, (x,y)| dy \| \text{div} u \|_{L^\infty}
\]

(2.13)

\[
|\tilde{I}_3| = \left| \int (1 - \eta_\delta(|x-y|)) \nabla_x \text{div} y G_i, (x,y) \text{div} u(y) dy \right| \\
\leq C \left( \int_{|x-y| \leq 1} + \int_{|x-y| > 1} \right) |\nabla_x \text{div} y G_i, (x,y)| \| \text{div} u(y) \| dy \\
\leq C \int_{\delta}^{1} r^{-3}r^2 dr \| \text{div} u \|_{L^\infty} + C \left( \int_{1}^{\infty} r^{-6}r^2 dr \right)^{1/2} \| \text{div} u \|_{L^2}
\]

(2.14)

It follows from (2.11), (2.12), (2.13) that

\[
\| \nabla U_1 \|_{L^\infty} \leq C \left( \delta^{(q-3)/q} \| \nabla^2 u \|_{L^q} + (1 - \ln \delta) \| \text{div} u \|_{L^\infty} + \| \nabla u \|_{L^2} \right).
\]

(2.15)

Since by (1.5), \( (\text{curl} u + (Au)^{\perp}) \times n = 0 \) on \( \partial \Omega \), we rewrite \( \nabla U_2^i \) as

\[
\nabla U_2^i = -\mu \int \eta_\delta(|x-y|) \nabla_x G_i, (x,y) \cdot \nabla y \times (\text{curl} u + (Au)^{\perp}) dy \\
+ \mu \int \nabla_y \eta_\delta(|x-y|) \times \nabla_x G_i, (x,y) \cdot (\text{curl} u + (Au)^{\perp}) dy \\
- \mu \int (1 - \eta_\delta(|x-y|)) \nabla_y \times \nabla_x G_i, (x,y) \cdot (\text{curl} u + (Au)^{\perp}) dy.
\]
A discussion similar to the previous term gives
\[
\|\nabla U_2\|_{L^\infty} \leq C\delta^{(q-3)/q}\|\nabla^2 u\|_{L^q} + C(1 - \ln \delta)(\|\text{curl} u\|_{L^\infty} + \|u\|_{L^\infty}) + C\|\nabla u\|_{L^2}. \tag{2.16}
\]

Finally, it is clear that
\[
\|\nabla U_3\|_{L^\infty} \leq C\left(\delta^{(q-3)/q}\|\nabla^2 u\|_{L^q} + (1 - \ln \delta)(\|\text{curl} u\|_{L^\infty} + \|u\|_{L^\infty})\right). \tag{2.17}
\]

Combining (2.10) with (2.15)-(2.17), and utilizing (2.6) leads to Lemma 2.7.

Next, for the problem
\[
\begin{align*}
\text{div} v &= f, \quad x \in \Omega, \\
v &= 0, \quad x \in \partial \Omega,
\end{align*}
\tag{2.18}
\]
one has the following conclusion.

Lemma 2.8 [11, Theorem III.3.1] There exists a linear operator \( B = [B_1, B_2, B_3] \) enjoying the properties:

1) The operator \( B : \{ f \in L^p(\Omega) : \bar{f} = 0 \} \mapsto (W_0^{1,p}(\Omega))^3 \) is a bounded linear one, that is,
\[
\|B[f]\|_{W_0^{1,p}(\Omega)} \leq C(p)\|f\|_{L^p(\Omega)}, \text{ for any } p \in (1, \infty).
\]

2) The function \( v = B[f] \) solves the problem (2.11).

3) If, moreover, for \( f = \text{div} u \) with a certain \( g \in L^r(\Omega) \), \( g \cdot n|_{\partial \Omega} = 0 \), then for any \( r \in (1, \infty) \),
\[
\|B[f]\|_{L^r(\Omega)} \leq C(r)\|g\|_{L^r(\Omega)}.
\]

Now, for \( F, \text{curl} u \) as in (1.23), we have the following key a priori estimates which will be used frequently.

Lemma 2.9 Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) and its smooth boundary \( \partial \Omega \) only has a finite number of 2-dimensional connected components. Let \( (\rho, u) \) be a smooth solution of (1.1) with slip boundary condition (1.5). Then for any \( p \in [2, 6] \), there exists a positive constant \( C \) depending only on \( p, \mu, \lambda \) and \( A \) such that
\[
\|F\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2})^{(6-p)/(2p)}
+ C(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2}), \tag{2.19}
\]
\[
\|\text{curl} u\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)}\|\nabla u\|_{L^2}^{(6-p)/(2p)} + C\|\nabla u\|_{L^2}, \tag{2.20}
\]
\[
\|F\|_{L^p} + \|\text{curl} u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}), \tag{2.21}
\]
\[
\|\nabla u\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2})^{(6-p)/(2p)}
+ C(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^p}), \tag{2.22}
\]
\[
\|\nabla F\|_{L^p} + \|\nabla \text{curl} u\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^p}). \tag{2.23}
\]

Proof. For \((Au)\) as in (2.29) and \((\text{curl} + (Au)\) \(\times n = 0\) on \(\partial \Omega\), we have, for any \(\eta \in C^\infty(\mathbb{R}^3)\),

\[
\int \nabla \times \text{curl} \cdot \nabla \eta \, dx = \int \nabla \times (\text{curl} + (Au)) \cdot \nabla \eta \, dx - \int \nabla \times (Au) \cdot \nabla \eta \, dx
\]

which together with (1.1) yields that the viscous flux \(F\) satisfies that for any \(\eta \in C^\infty(\mathbb{R}^3)\),

\[
\int \nabla F \cdot \nabla \eta \, dx = \int \left( \rho \dot{u} - \nabla \times (Au) \right) \cdot \nabla \eta \, dx,
\]

that is,

\[
\begin{aligned}
\Delta F &= \text{div}(\rho \dot{u} - \nabla \times (Au)), \quad x \in \Omega, \\
\frac{\partial F}{\partial n} &= (\rho \dot{u} - \nabla \times (Au)) \cdot n, \quad x \in \partial \Omega.
\end{aligned}
\]

It follows from [34, Lemma 4.27] that

\[
\|\nabla F\|_{L^q} \leq C \left( \|\rho \dot{u}\|_{L^q} + \|\nabla \times (Au)\|_{L^q} \right)
\leq C \left( \|\rho \dot{u}\|_{L^q} + \|\nabla u\|_{L^q} \right),
\tag{2.24}
\]

and that for any integer \(k \geq 0\),

\[
\|\nabla F\|_{W^{k+1, q}} \leq C \left( \|\rho \dot{u}\|_{W^{k+1, q}} + \|\nabla \times (Au)\|_{W^{k+1, q}} \right).
\tag{2.25}
\]

Furthermore, since \(\bar{F} = 0\), one deduces from (2.1) and (2.24) that for \(p \in [2, 6]\),

\[
\|F\|_{L^p} \leq C \|\nabla F\|_{L^2} \leq C (\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}),
\tag{2.26}
\]

and

\[
\|F\|_{L^p} \leq C \|F\|_{L^2}^{(6-p)/(2p)} \|\nabla F\|_{L^2}^{(3p-6)/(2p)}
\leq C (\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)} (\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2})^{(6-p)/(2p)}
+ C \|\nabla u\|_{L^2} + C \|P - \bar{P}\|_{L^2},
\]

which is (2.19).

One rewrites (1.1) as \(\mu \nabla \times \text{curl} = \nabla F - \rho \dot{u}\). Since \((\text{curl} + (Au) \times n = 0\) on \(\partial \Omega\) and \(\text{div} (\nabla \times \text{curl}) = 0\), by Lemma 2.6 we get

\[
\|\nabla \text{curl} u\|_{L^q} \leq C (\|\nabla \times \text{curl} u\|_{L^q} + \|\nabla u\|_{L^q})
\leq C (\|\rho \dot{u}\|_{L^q} + \|\nabla u\|_{L^q}),
\tag{2.27}
\]

and for any integer \(k \geq 0\),

\[
\|\nabla \text{curl} u\|_{W^{k+1, q}} \leq C (\|\nabla \times \text{curl} u\|_{W^{k+1, q}} + \|\text{curl} u\|_{L^q} + \|Au\|_{W^{k+2, q}})
\leq C (\|\rho \dot{u}\|_{W^{k+1, q}} + \|\nabla (Au)\|_{W^{k+1, q}} + \|\nabla u\|_{L^q}),
\tag{2.28}
\]

where we have taken advantage of (2.25). Thus, by Gagliardo-Nirenberg’s inequality and (2.27), we get (2.20). Combining (2.20) with (2.26) shows (2.21).
Next, by virtue of (2.6), (2.19) and (2.20), it indicates that
\[ \| \nabla u \|_{L^p} \leq C(\| \text{div} u \|_{L^p} + \| \text{curl} u \|_{L^p} + \| u \|_{L^p}) \]
\[ \leq C(\| F \|_{L^p} + \| \text{curl} u \|_{L^p} + \| P - \bar{P} \|_{L^p} + \| \nabla u \|_{L^2}) \]
\[ \leq C(\| \rho u \|_{L^2}^{(3p-6)/(2p)}(\| \nabla u \|_{L^2} + \| P - \bar{P} \|_{L^2})^{(6-p)/(2p)}) + C(\| \nabla u \|_{L^2} + \| P - \bar{P} \|_{L^p}). \]
which together with (2.24) and (2.27) gives (2.22).

The inequality (2.23) is a direct consequence of (2.24) and (2.27) and we complete the proof of Lemma 2.9.

Finally, using the boundary condition (1.5), we have the following estimates on the material derivative of \( u \).

**Lemma 2.10** Under the assumption of Lemma 2.9, there exists a positive constant \( \Lambda \) depending only on \( \Omega \) such that
\[ \| \dot{u} \|_{L^6} \leq \Lambda(\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}^2), \]
\[ \| \nabla \dot{u} \|_{L^2} \leq \Lambda(\| \text{div} \dot{u} \|_{L^2} + \| \text{curl} \dot{u} \|_{L^2} + \| \nabla u \|_{L^2}^2). \]

**Proof.** First, setting \( u^\perp \triangleq -u \times n \), we have by (1.5)
\[ \dot{u} \cdot n = u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u = -(u \cdot \nabla n) \times u^\perp \cdot n \text{ on } \partial \Omega, \]
due to the following simple fact
\[ v \times (u \times n) = (v \cdot n)u - (v \cdot u)n, \]
with \( v = u \cdot \nabla n \). It thus follows for (2.31) that
\[ (\dot{u} + (u \cdot \nabla n) \times u^\perp) \cdot n = 0 \text{ on } \partial \Omega, \]
which together with Poincaré's inequality gives
\[ \| \dot{u} + (u \cdot \nabla n) \times u^\perp \|_{L^\frac{3}{2}} \leq C\| \nabla (\dot{u} + (u \cdot \nabla n) \times u^\perp) \|_{L^\frac{3}{2}}. \]
Thus, we obtain
\[ \| \dot{u} \|_{L^\frac{3}{2}} \leq C(\| \nabla \dot{u} \|_{L^\frac{3}{2}} + \| \nabla u \|_{L^2}^2), \]
which together with Sobolev's embedding theorem yields (2.29).

Finally, taking \( v = \dot{u} + (u \cdot \nabla n) \times u^\perp \) in (2.6) proves (2.30) due to (2.32).

**3 A priori estimates (I): lower order estimates**

Let \( T > 0 \) be a fixed time and \( (\rho, u) \) be a smooth solution to (1.1)-(1.5) on \( \Omega \times (0, T) \) with smooth initial data \( (\rho_0, u_0) \) satisfying (1.9) and (1.10). In this section, we always assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^3 \) and its boundary \( \partial \Omega \) is of class \( C^\infty \) and only has a finite number of 2-dimensional connected components. Since
\( u \cdot n = 0 \) on \( \partial \Omega \), by (2.7), for any \( 1 < q < +\infty \), there exists a positive constant \( C_1 \) depending only on \( q, \mu, \lambda \) and \( \Omega \) such that
\[
\| \nabla u \|_{L^q} \leq C_1 (\| \text{div} u \|_{L^q} + \| \text{curl} u \|_{L^q}). \tag{3.1}
\]

We will establish some necessary a priori bounds for smooth solutions to the problem (1.1)-(1.5) to extend the local classical solutions guaranteed by Lemma 2.1.

For \( \sigma = \sigma(t) \triangleq \min\{1, t\} \) and \( \dot{u} \) as in (1.24), we define
\[
A_1(T) \triangleq \sup_{0 \leq t \leq T} \left( \sigma \| \nabla u \|_{L^2}^2 + \int_0^T \int \sigma \rho |\dot{u}|^2 dx dt \right), \tag{3.2}
\]
\[
A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^3 \int \rho |\dot{u}|^2 dx + \int_0^T \int \sigma^3 |\nabla u|^2 dx dt, \tag{3.3}
\]
and
\[
A_3(T) \triangleq \sup_{0 \leq t \leq T} \int \rho |u|^3 dx. \tag{3.4}
\]

Now we will give the main result in this section, which guarantees the existence of a global classical solution of (1.1)–(1.5).

**Proposition 3.1** Under the conditions of Theorem 1.1, for \( \delta_0 \triangleq \frac{2s-1}{4s} \in (0, \frac{1}{4}] \), there exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M \) and the matrix \( A \) such that if \( (\rho, u) \) is a smooth solution of (1.1)–(1.5) on \( \Omega \times (0, T] \) satisfying
\[
\sup_{\Omega \times [0, T]} \rho \leq 2\hat{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{1/3}, \quad A_3(\sigma(T)) \leq 2C_0^{\delta_0}, \tag{3.5}
\]
then the following estimates hold
\[
\sup_{\Omega \times [0, T]} \rho \leq 7\hat{\rho}/4, \quad A_1(T) + A_2(T) \leq C_0^{1/3}, \quad A_3(\sigma(T)) \leq C_0^{\delta_0}, \tag{3.6}
\]
provided \( C_0 \leq \varepsilon \).

**Proof.** Proposition 3.1 is a consequence of the following Lemmas 3.6–3.8.

One can extend the function \( n \) to \( \Omega \) such that \( n \in C^3(\Omega) \), and in the following discussion we still denote the extended function by \( n \).

In the following, we will use the convention that \( C \) denotes a generic positive constant depending on \( \mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M \) and the matrix \( A \), and use \( C(\alpha) \) to emphasize that \( C \) depends on \( \alpha \).

We begin with the following standard energy estimate for \( (\rho, u) \).

**Lemma 3.2** Let \( (\rho, u) \) be a smooth solution of (1.1)–(1.5) on \( \Omega \times (0, T] \). Then there is a positive constant \( C \) depending only on \( \mu, \lambda \) and \( \Omega \) such that for \( G(\rho) \) as in (1.8),
\[
\sup_{0 \leq t \leq T} \int \left( \rho |u|^2 + G(\rho) \right) dx + \int_0^T \| \nabla u \|_{L^2}^2 dt \leq CC_0. \tag{3.7}
\]
Proof. First, integrating (1.1) over $\Omega \times (0, T)$ and using (1.5), one has
\[ \bar{\rho} = \frac{1}{|\Omega|} \int \rho(x, t)dx \equiv \frac{1}{|\Omega|} \int \rho_0 dx. \quad (3.8) \]

Next, since
\[ -\Delta u = -\nabla \text{div} u + \nabla \times \text{curl} u, \]
we rewrite (1.2) as
\[ \rho \dot{u} - (\lambda + 2\mu)\nabla \text{div} u + \mu \nabla \times \text{curl} u + \nabla P = 0. \quad (3.9) \]

Multiplying (3.9) by $u$ and integrating the resulting equality over $\Omega$, along with (1.5), we arrive at
\[ \frac{1}{2} \left( \int \rho|u|^2 dx \right)_t + (\lambda + 2\mu) \int (\text{div} u)^2 dx + \mu \int |\text{curl} u|^2 dx \\
+ \mu \int_{\partial\Omega} u \cdot A \cdot u ds = \int P \text{div} u dx. \quad (3.10) \]

Finally, by (1.1), one can check that
\[ (G(\rho))_t + \text{div}(G(\rho)u) + (P - P(\bar{\rho}))\text{div} u = 0, \]
which together with (3.10) gives
\[ \left( \int \frac{1}{2} \rho|u|^2 + G(\rho) dx \right)_t + \phi(t) = 0, \quad (3.11) \]
with
\[ \phi(t) \triangleq (\lambda + 2\mu)\|\text{div} u\|_{L_2}^2 + \mu\|\text{curl} u\|_{L_2}^2 + \mu \int_{\partial\Omega} u \cdot A \cdot u ds. \]

The energy estimate (3.7) thus follows from the positive semi-definiteness of $A$, (3.10) and (3.1). \qed

A direct consequence of Lemma 3.2 is the following estimates on the pressure.

Lemma 3.3 Let $(\rho, u)$ be a smooth solution of (1.1)-(1.5) on $\Omega \times (0, T]$ satisfying (3.5). Then there is a positive constant $C$ depending only on $\mu, \lambda, \gamma, \hat{\rho}, \Omega$ and the matrix $A$ such that
\[ \sigma \int (P - \bar{P})^2 dx \leq CC_0^{\frac{1}{2}}, \quad \int_0^T \sigma \int (P - \bar{P})^2 dx dt \leq CC_0^{\frac{3}{2}}. \quad (3.12) \]

Proof. First, it follows from (1.1) that
\[ P_t + \text{div}(Pu) + (\gamma - 1)P\text{div} u = 0, \quad (3.13) \]
which gives
\[ \bar{P}_t + (\gamma - 1)\bar{P}\text{div} u = 0. \quad (3.14) \]
Next, multiplying (1.1) by $B[P - \bar{P}]$ and integrating the resulting equality over $\Omega$, we get
\[
\int (P - \bar{P})^2 \, dx = \left( \int \rho u \cdot B[H \, df] \, \biggl|_t \right) - \left( \int \rho u \cdot \nabla B[H \, df] \cdot u \, dx \right)
\]
\[
- \left( \int \rho u \cdot B[P_t - \bar{P}_t] \, dx \right) + \mu \left( \int \nabla u \cdot \nabla B[H \, df] \, dx \right)
\]
\[
+ (\lambda + \mu) \left( \int (P - \bar{P}) \text{div} u \, dx \right)
\]
\[
\leq \left( \int \rho u \cdot B[H \, df] \, \biggl|_t \right) + C \|u\|^{2}_{L^2} \|P - \bar{P}\|_{L^{3/2}}
\]
\[
+ C \|u\|_{L^2} \|\nabla u\|_{L^2} + C \|P - \bar{P}\|_{L^2} \|\nabla u\|_{L^2}
\]
\[
\leq \left( \int \rho u \cdot B[H \, df] \, \biggl|_t \right) + \delta \|P - \bar{P}\|^{2}_{L^2} + C(\delta) \|\nabla u\|^{2}_{L^2},
\]
where in the first inequality we have used
\[
\|B[H \, df] \|_{L^{2}} = \|B[\text{div}(Pu)] + (\gamma - 1)B[P \text{div} - \bar{P} \text{div}]\|_{L^{2}}
\]
\[
\leq C \|\nabla u\|_{L^2}.
\]
Combining (3.15), (3.7), and Lemma 2.8 shows
\[
\int_0^T \int (P - \bar{P})^2 \, dx \, dt \leq CC^\frac{1}{2}.
\]
Next, using (1.1), we have
\[
P_t + u \cdot \nabla P + \gamma P \text{div} u = 0,
\]
which together with (3.14) gives
\[
(P - \bar{P})_t + u \cdot \nabla (P - \bar{P}) + \gamma P \text{div} u - (\gamma - 1)P \text{div} u = 0.
\]
Multiplying (3.18) by $2\sigma (P - \bar{P})$ and integrating the result over $\Omega$, one checks that
\[
\sigma \left( \int (P - \bar{P})^2 \, dx \right) \leq \sigma(\sigma + \sigma') \int (P - \bar{P})^2 \, dx + C \int \|\nabla u\|^2 \, dx,
\]
which together with (3.7) and (3.16) leads to
\[
\sigma \int (P - \bar{P})^2 \, dx \leq CC^\frac{1}{2}.
\]
Finally, combining (3.19) with (3.7) implies
\[
\sigma \left| \int \rho u \cdot B[H \, df] \right| \leq C \left( \int \rho |u|^2 \, dx \right)^{1/2} \left( \sigma \int (P - \bar{P})^2 \, dx \right)^{1/2}
\]
\[
\leq CC^\frac{3}{4},
\]
which together with (3.7), (3.15) and (3.16) yields
\[
\int_0^T \sigma \int (P - \bar{P})^2 \, dx \, dt \leq CC^\frac{3}{4}.
\]
Combining this with (3.19) proves Lemma 3.3.
The following conclusion concerns preliminary estimates on the $L^2$-norm of $\nabla u$ and $ho^{1/2} \dot{u}$.

**Lemma 3.4** Let $(\rho, u)$ be a smooth solution of (1.1) - (1.5) on $\Omega \times (0, T]$ satisfying (3.5). Then there is a positive constant $C$ depending only on $\mu, \lambda, a, \gamma, \hat{\rho}, \Omega$ and the matrix $A$ such that

$$A_1(T) \leq CC_0^2 + C \int_0^T \sigma \int |\nabla u|^3 dx dt,$$

and

$$A_2(T) \leq CC_0 + CA_1(T) + C \int_0^T \sigma^3 \int |\nabla u|^4 dx dt.$$

**Proof.** Motivated by Hoff [11], multiplying (1.1) by $\sigma m \dot{u}$ with $m \geq 0$ and then integrating the resulting equality over $\Omega$, one gets

$$\int \sigma^m \rho |\dot{u}|^2 dx = - \int \sigma^m \dot{u} \cdot \nabla P dx + (\lambda + 2\mu) \int \sigma^m \nabla \div u \cdot \dot{u} dx$$

$$- \mu \int \sigma^m \nabla \times \curl u \cdot \dot{u} dx$$

$$\triangleq I_1 + I_2 + I_3.$$  

We will estimate $I_1$, $I_2$ and $I_3$ one by one. First, it follows from (3.13) that

$$I_1 = - \int \sigma^m \dot{u} \cdot \nabla P dx$$

$$= \int \sigma^m P \div u_t dx - \int \sigma^m u \cdot \nabla \cdot \nabla P dx$$

$$= \left( \int \sigma^m P \div u dx \right)_t - \sigma^{m-1} \sigma' \int P \div u dx + \int \sigma^m P \nabla u : \nabla u dx$$

$$+ (\gamma - 1) \int \sigma^m P (\div u)^2 dx - \int_{\partial \Omega} \sigma^m P u \cdot \nabla n ds$$

$$\leq \left( \int \sigma^m P \div u dx \right)_t + C \|\nabla u\|^2_{L^2} + C m \sigma^{m-1} \sigma'\|P - \bar{P}\|_{L^2}^2,$$

where in the last inequality we have used

$$- \int_{\partial \Omega} \sigma^m P u \cdot \nabla u \cdot n ds = \int_{\partial \Omega} \sigma^m P u \cdot \nabla n \cdot u ds$$

$$\leq C \int_{\partial \Omega} \sigma^m |u|^2 ds \leq C \sigma^m \|\nabla u\|^2_{L^2},$$

due to (1.27). Hence,

$$I_1 \leq \left( \int \sigma^m P \div u dx \right)_t + C \|\nabla u\|^2_{L^2} + C m \sigma^{m-1} \sigma'\|P - \bar{P}\|_{L^2}^2.$$

(3.24)
Similarly, by (1.27), it indicates that
\[
\frac{I_2}{\lambda + 2\mu} = \int \sigma^m \nabla \text{div} \cdot \dot{u} dx
\]
\[
= \int_{\partial \Omega} \sigma^m \text{div}(\dot{u} \cdot n) ds - \int \sigma^m \text{div} \text{div} \dot{u} dx
\]
\[
= \int_{\partial \Omega} \sigma^m \text{div}(u \cdot \nabla u \cdot n) ds - \frac{1}{2} \left( \int \sigma^m (\text{div}u)^2 dx \right)_t
\]
\[
- \int \sigma^m \text{div} \text{div}(u \cdot \nabla u) dx + \frac{m}{2} \sigma^{-1} \sigma^' \left( \int (\text{div}u)^2 dx \right)_t
\]
\[
\leq - \int_{\partial \Omega} \sigma^m \text{div}(u \cdot \nabla n \cdot u) ds - \frac{1}{2} \left( \int \sigma^m (\text{div}u)^2 dx \right)_t
\]
\[
+ C \int \sigma^m |\nabla u|^3 dx + \frac{m}{2} \sigma^{-1} \sigma^' \left( \int (\text{div}u)^2 dx \right)_t.
\]

For the first term on the righthand side of (3.25), we have
\[
\left| (\lambda + 2\mu) \int_{\partial \Omega} \text{div} (u \cdot \nabla n \cdot u) ds \right|
\]
\[
= \left| \int_{\partial \Omega} (F + P - \bar{P})(u \cdot \nabla n \cdot u) ds \right|
\]
\[
\leq C(\|F\|_{H^1} + 1)\|u\|_{H^1}^2
\]
\[
\leq \frac{1}{4} \|\rho^2 \dot{u}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) ,
\]
where in the last inequality, we have used
\[
\|F\|_{H^1} + \|\text{curl} u\|_{H^1} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2})
\]

Due to Lemma 2.9 and (2.29). Therefore,
\[
I_2 \leq - \lambda + 2\mu \left( \int \sigma^m (\text{div}u)^2 dx \right)_t + C\sigma^m \|\nabla u\|_{L^3}^3
\]
\[
+ \frac{1}{4} \sigma^m \|\rho^2 \dot{u}\|_{L^2}^2 + C\sigma^m \|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2.
\]

Finally, by (1.5) and (2.23), a straightforward computation shows that
\[
I_3 = -\mu \int \sigma^m \nabla \times \text{curl} u \cdot \dot{u} dx
\]
\[
= -\mu \int \sigma^m \text{curl} u \cdot \text{curl} \dot{u} dx + \mu \int_{\partial \Omega} \sigma^m \text{curl} u \times n \cdot \dot{u} ds
\]
\[
= -\frac{\mu}{2} \left( \int \sigma^m |\text{curl} u|^2 dx + \int_{\partial \Omega} \sigma^m u \cdot A \cdot u ds \right)_t
\]
\[
+ \frac{\mu m}{2} \sigma^{-1} \sigma' \left( \int |\text{curl} u|^2 dx + \frac{\mu m}{2} \sigma^{-1} \sigma' \left( \int_{\partial \Omega} u \cdot A \cdot u ds \right)_t
\]
\[
- \mu \int \sigma^m \text{curl} u \cdot \nabla u ds + \mu \int_{\partial \Omega} \sigma^m (Au)^\perp \times (u \cdot \nabla u) \cdot n ds
\]
\[
\leq -\frac{\mu}{2} \left( \int \sigma^m |\text{curl} u|^2 dx + \int_{\partial \Omega} \sigma^m u \cdot A \cdot u ds \right)_t + C\|\nabla u\|_{L^2}^2
\]
\[
+ C\sigma^m \|\nabla u\|_{L^3}^3 + C\sigma^m \|\nabla u\|_{L^2}^4 + \frac{1}{4} \sigma^m \|\rho^2 \dot{u}\|_{L^2}^2,
\]
where in the last inequality, we have used
\[
\mu \int_{\partial \Omega} (Au)^\perp \times (u \cdot \nabla u) \cdot nds
\]
\[
= \mu \int \text{curl}((Au)^\perp) \cdot (u \cdot \nabla u) dx - \mu \int (\nabla u \times \nabla_i u) \cdot (Au)^\perp dx
\]
\[
- \mu \int (u \cdot \nabla \text{curl} u) \cdot (Au)^\perp dx
\]
\[
\leq C\|\nabla u\|^3_{L^3} + C(\eta)\|\nabla u\|^4_{L^2} + \eta\|\rho \dot{u}\|^2_{L^2},
\]
due to (3.26). It thus follows from (3.22) and (3.24)-(3.28) that
\[
\left(\sigma_m \int (2\mu + \lambda)(\text{div} u)^2 + \mu|\text{curl} u|^2) dx + \mu \sigma_m \int_{\partial \Omega} u \cdot A \cdot nds\right)_t
\]
\[
+ \int \sigma_m \rho |\dot{u}|^2 dx
\]
\[
\leq \left(2 \int \sigma_m (P - \bar{P}) \text{div} u dx\right)_t + C\sigma_m \sigma^{-1}\|P - \bar{P}\|^2_{L^2}
\]
\[
+ C\sigma_m \|\nabla u\|^4_{L^2} + C\|\nabla u\|^2_{L^2} + C\sigma_m \|\nabla u\|^3_{L^3}.
\]
Integrating (3.29) over (0, T), by (3.1), (3.7), (3.15), (3.16) and Young’s inequality, we conclude that for any \(m \geq 1\),
\[
\sigma^m \|\nabla u\|^2_{L^2} + \int_0^T \int \sigma^m \rho |\dot{u}|^2 dx dt
\]
\[
\leq CC_0^2 + C \int_0^T \sigma^m \|\nabla u\|^4_{L^2} dt + C \int_0^T \sigma^m \|\nabla u\|^3_{L^3} dt,
\]
which, after choosing \(m = 1\), together with (3.5) and (3.7) gives (3.20).

Now we will prove (3.21). First, rewrite (1.1) as
\[
\rho \dot{u} = \nabla F - \mu \nabla \times \text{curl} u.
\]
Operating \(\sigma^m \dot{u}^j [\partial / \partial t + \text{div}(u \cdot \cdot \cdot)] \) to (3.31), summing with respect to \(j\), and integrating it over \(\Omega\), one has
\[
\left(\frac{\sigma^m}{2} \int_0^T \rho |\dot{u}|^2 dx\right)_t - \frac{m}{2} \sigma^{m-1} \sigma' \int_0^T \rho |\dot{u}|^2 dx
\]
\[
= \int \sigma^m (\dot{u} \cdot \nabla F + \dot{u} \cdot \text{div}(u \partial_j F)) dx
\]
\[
+ \mu \int \sigma^m (\nabla \times \text{curl} u_i - \dot{u} \text{div}(\nabla \times \text{curl} u)^j u) dx
\]
\[
\triangleq J_1 + \mu J_2.
\]
For \(J_1\), it follows from (3.14) that
\[
F_i = (2\mu + \lambda) \text{div} u_i - P_i + \bar{P}_i
\]
\[
= (2\mu + \lambda) \text{div} \dot{u} - (2\mu + \lambda) \nabla u : \nabla u - u \cdot \nabla F + \gamma P \text{div} u - (\gamma - 1) \bar{P} \text{div} u,
\]
which together with (3.33), (3.26), (3.35) and (3.37) yields

\[
J_1 = \int_{\partial \Omega} \sigma^m \hat{u} \cdot \nabla \hat{F} \, ds + \int \sigma^m \hat{u} \cdot \nabla (u \hat{\nabla} F) \, dx \\
= \int_{\partial \Omega} \sigma^m \hat{F} \hat{u} \cdot nds - \int \sigma^m \hat{F} \hat{u} \, dx - \int \sigma^m \hat{u} \cdot \nabla \hat{\nabla} \hat{\nabla} F \, dx \\
= \int_{\partial \Omega} \sigma^m \hat{F} \hat{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\nabla \hat{u})^2 \, dx \\
+ (2\mu + \lambda) \int \sigma^m \nabla \hat{u} \cdot \nabla u \, dx + \int \sigma^m \nabla \hat{u} \cdot \nabla F \, dx \\
- \gamma \int \sigma^m P \nabla \hat{u} \cdot \nabla u \, dx - \int \sigma^m \hat{u} \cdot \nabla \hat{\nabla} F \, dx \\
- (\gamma - 1) \int \sigma^m \nabla \hat{u} \cdot \nabla \hat{u} \, dx
\]

(3.33)

\[
\leq \int_{\partial \Omega} \sigma^m \hat{F} \hat{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\nabla \hat{u})^2 \, dx \\
+ C \left( \| \nabla u \|_{L^4}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \| \nabla F \|_{L^6}^{\frac{1}{2}} \| \nabla F \|_{L^6}^{\frac{1}{2}} \right) \| \nabla \hat{u} \|_{L^2} \\
\leq \int_{\partial \Omega} \sigma^m \hat{F} \hat{u} \cdot nds - (2\mu + \lambda) \int \sigma^m (\nabla \hat{u})^2 \, dx + \delta \sigma^m \| \nabla \hat{u} \|_{L^2}^2 \\
+ C(\delta) \sigma^m \left( \| \nabla u \|_{L^4}^4 \| \hat{\nabla} \hat{u} \|_{L^2}^2 + \| \nabla u \|_{L^4}^2 + \| \nabla u \|_{L^2}^2 \right),
\]

where in the last equality we have used (3.26) and

\[
\| \nabla F \|_{L^6} + \| \nabla \nabla u \|_{L^6} \\
\leq C \| \hat{u} \|_{H^1} + C \| \nabla u \|_{L^2} + C \| P - \hat{P} \|_{L^6} \\
\leq C(\| \nabla \hat{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2}^2 + \| P - \hat{P} \|_{L^6}),
\]

(3.34)
due to Lemma 2.9 and (2.29).

For the first term on the righthand side of (3.33), we have

\[
\int_{\partial \Omega} \sigma^m F \hat{u} \cdot nds \\
= - \int_{\partial \Omega} \sigma^m F_t (u \cdot \nabla \nabla u) \, ds \\
= - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla \nabla u) \, F \, ds \right)_t + m \sigma^{m-1} \sigma' \int_{\partial \Omega} (u \cdot \nabla \nabla u) \, F \, ds \\
+ \sigma^m \int_{\partial \Omega} F \hat{u} \cdot \nabla \nabla \hat{u} \, ds + \sigma^m \int_{\partial \Omega} F \hat{u} \cdot \nabla \nabla \hat{u} \, ds \\
- \sigma^m \int_{\partial \Omega} F (u \cdot \nabla) \hat{u} \cdot \nabla \nabla u \, ds - \sigma^m \int_{\partial \Omega} F (u \cdot \nabla) \hat{u} \cdot \nabla \nabla u \, ds \\
\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla \nabla u) \, F \, ds \right)_t + C m \sigma' \sigma^{m-1} \| \nabla u \|_{L^2}^2 \| F \|_{H^1} \\
+ \delta \sigma^m \| \hat{u} \|_{H^1}^2 + C(\delta) \sigma^m \| \nabla u \|_{L^2}^2 \| F \|_{H^1}^2 \\
- \sigma^m \int_{\partial \Omega} F (u \cdot \nabla) \hat{u} \cdot \nabla \nabla u \, ds - \sigma^m \int_{\partial \Omega} F (u \cdot \nabla) \hat{u} \cdot \nabla \nabla u \, ds,
\]

(3.35)
where in the last inequality we have used
\[
\left| \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \right| \leq C \| \nabla u \|_{L^2}^2 \| F \|_{H^1}. \tag{3.36}
\]

Now we are in a position to estimate the last term on the right-hand side of (3.35), which indeed plays a crucial role in our analysis. Since \( u \cdot n \big|_{\partial \Omega} = 0 \), we observe that

\[
u = -(u \times n) \times n = u^\perp \times n \quad \text{on } \partial \Omega,
\]

which yields that

\[
- \int_{\partial \Omega} F(u \cdot \nabla) u \cdot \nabla n \cdot u ds
= - \int_{\partial \Omega} u^\perp \times n \cdot \nabla n \cdot u F ds
= - \int_{\partial \Omega} n \cdot (\nabla u^i \times u^\perp) \nabla n \cdot u F ds
= - \int_{\Omega} \text{div}((\nabla u^i \times u^\perp) \nabla n \cdot u F) dx
= - \int_{\Omega} \nabla (\nabla n \cdot u F) \cdot (\nabla u^i \times u^\perp) dx + \int_{\Omega} \nabla u^i \cdot \nabla \times u^\perp \nabla n \cdot u F dx \tag{3.37}
\]

\[
\leq C \int_{\Omega} \| \nabla F \| \| \nabla u \| \| u \|^2 dx + C \int_{\Omega} \| F \| (\| \nabla u \| \| u \| + \| \nabla u \| \| u \|^2) dx
\leq C \| \nabla u \|_{L^6} \| \nabla u \|_{L^2} \| u \|_{L^6}^2 + C \| F \|_{L^{12/5}} \| \nabla u \|_{L^4} \| u \|_{L^6}^2
+ C \| F \|_{L^\infty} \| \nabla u \|_{L^4} \| u \|_{L^6}^2
\leq C \| \nabla u \|_{L^6}^2 + C \| \nabla u \|_{L^6} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^4}^2 + C \| \nabla u \|_{L^4}^4
+ C \| \rho^{\frac{1}{2}} \nabla u \|_{L^2}^2 \left( \| \nabla u \|_{L^2}^2 + 1 \right),
\]

where in the fourth equality we have used the following standard fact:

\[
\text{div}(\nabla u^i \times u^\perp) = -\nabla u^i \cdot \nabla \times u^\perp.
\]

Similarly, we get

\[
- \int_{\partial \Omega} F u \cdot \nabla n \cdot (u \cdot \nabla) u ds
\leq \delta \| \nabla \hat{u} \|_{L^2}^2 + C(\delta) \| \nabla u \|_{L^2}^6 + C(\| \nabla u \|_{L^2} + \| \nabla u \|_{L^4}^4) \tag{3.38}
+ C \| \rho^{\frac{1}{2}} \nabla \hat{u} \|_{L^2}^2 \left( \| \nabla u \|_{L^2}^2 + 1 \right).
\]

Substituting (3.33), (3.37), and (3.38) into (3.35), we obtain after using (2.29), (3.7), (3.29), and (3.32) that

\[
J_1 \leq C m \sigma^{m-1} \sigma' \left( \| \rho^{\frac{1}{2}} \nabla \hat{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right)
- \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C \delta \sigma^m \| \nabla \hat{u} \|_{L^2}^2
+ C(\delta) \sigma^m \| \rho^{\frac{1}{2}} \nabla \hat{u} \|_{L^2}^2 \left( \| \nabla u \|_{L^2}^4 + 1 \right) - (\lambda + 2\mu) \int \sigma^m (\text{div} \hat{u})^2 dx
+ C(\delta) \sigma^m \left( \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^4}^6 + \| \nabla u \|_{L^4}^4 \right). \tag{3.39}
\]
For $J_2$, since $\text{curl} u_t = \text{curl} \dot{u} - u \cdot \nabla \text{curl} u - \nabla u^i \times \nabla_i u$, 

$$
J_2 = -\int \sigma^m |\text{curl} \dot{u}|^2 dx + \int \sigma^m \text{curl} \dot{u} : (\nabla u^i \times \nabla_i u) dx 
+ \int_{\partial \Omega} \sigma^m \text{curl} u_t \times n \cdot \dot{u} ds + \int \sigma^m u \cdot \nabla \text{curl} u \cdot \text{curl} \dot{u} dx 
+ \int \sigma^m u \cdot \nabla \dot{u} \cdot (\nabla \times \text{curl} u) dx 
\leq -\int \sigma^m |\text{curl} \dot{u}|^2 dx + \delta \sigma^m \|\nabla \dot{u}\|^2_{L^2} + C(\delta) (\|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^6}) 
+ C(\delta) \sigma^m \|\nabla u\|^4_{L^4} + C(\delta) \sigma^m \|\nabla \nabla u\|^2_{L^2} \|\nabla \text{curl} u\|^2_{L^2},
$$

(3.40)

where in the last inequality we have utilized (3.26), (3.31) and the fact

$$
\int_{\partial \Omega} \text{curl} u_t \times n \cdot \dot{u} ds = -\int_{\partial \Omega} u_t \cdot A \cdot \dot{u} ds 
= -\int_{\partial \Omega} \dot{u} \cdot A \cdot \dot{u} ds + \int_{\partial \Omega} (u \cdot \nabla u) \cdot A \cdot \dot{u} ds 
= -\int_{\partial \Omega} \dot{u} \cdot A \cdot \dot{u} ds + \int_{\partial \Omega} u^i \times n \cdot \nabla u^i (A_i \cdot \dot{u}) ds 
= -\int_{\partial \Omega} \dot{u} \cdot A \cdot \dot{u} ds + \int_{\partial \Omega} n \cdot (\nabla u^i \times u^i) (A_i \cdot \dot{u}) ds 
= -\int_{\partial \Omega} \dot{u} \cdot A \cdot \dot{u} ds + \int_{\partial \Omega} \nabla (A_i \cdot \dot{u}) \cdot (\nabla u^i \times u^i) ds 
- \int_{\partial \Omega} \nabla u^i \cdot \nabla \times u^i (A_i \cdot \dot{u}) dx 
(3.41)
$$

here the symbol $A_i$ denotes the $i$-th row of the matrix $A$.

Putting (3.39) and (3.40) into (3.32) yields

$$
\left( \frac{\sigma^m}{2} \rho^2 \dot{u}^2_t \right)_{L^2_t} + (\lambda + 2\mu) \sigma^m \|\nabla \dot{u}\|^2_{L^2} + \mu \sigma^m \|\text{curl} \dot{u}\|^2_{L^2} + \int_{\partial \Omega} \dot{u} \cdot A \cdot \dot{u} ds 
\leq C m \sigma^{m-1} \sigma' (\rho^2 \dot{u}^2_{L^2_t} + \|\nabla u\|^2_{L^2} + \|\nabla u\|^4_{L^2}) - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t 
+ \delta \sigma^m \|\nabla \dot{u}\|^2_{L^2} + C(\delta) \sigma^m \|\rho^2 \dot{u}\|^2_{L^2} (\|\nabla u\|^4_{L^2} + 1) 
+ C(\delta) \sigma^m (\|\nabla u\|^2_{L^2} + \|\nabla u\|^6_{L^2} + \|\nabla u\|^4_{L^1}),
$$

(3.42)

which together with (2.30), after choosing $\delta$ suitably small, implies

$$
\left( \frac{\sigma^m}{2} \rho^2 \dot{u}^2_t \right)_{L^2_t} + \mu \Lambda^{-1} \sigma^m \|\nabla \dot{u}\|^2_{L^2} 
\leq C m \sigma^{m-1} \sigma' (\rho^2 \dot{u}^2_{L^2_t} + \|\nabla u\|^2_{L^2} + \|\nabla u\|^4_{L^2}) 
- \left( 2 \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + C \sigma^m \|\rho^2 \dot{u}\|^2_{L^2} (\|\nabla u\|^4_{L^2} + 1) 
+ C \sigma^m (\|\nabla u\|^2_{L^2} + \|\nabla u\|^6_{L^2} + \|\nabla u\|^4_{L^1}).
$$

(3.43)

Finally, integrating (3.43) with $m = 3$ over $(0, T)$, we get (3.21) from (3.36) and (3.5), which completes the proof of Lemma 3.4.
Lemma 3.5 Assume that \((\rho, u)\) is a smooth solution of (1.1)-(1.5) satisfying (3.5). Then there exist positive constants \(C\) and \(\varepsilon_1\) depending only on \(\mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M\) and the matrix \(A\) such that

\[
\sup_{0 \leq t \leq \sigma(T)} t^{1-s} \|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-s} \int_0^\Omega \rho|\dot{u}|^2dxdt \leq C(\hat{\rho}, M),
\]

(3.44)

\[
\sup_{0 \leq t \leq \sigma(T)} t^{2-s} \int_\Omega \rho|\dot{u}|^2dx + \int_0^{\sigma(T)} t^{2-s} \int_0^\Omega |\nabla u|^2dxdt \leq C(\hat{\rho}, M),
\]

(3.45)

provide that \(C_0 \leq \varepsilon_1\).

Proof. First, for \(L f \triangleq \rho \dot{f} - \mu \Delta f - (\lambda + \mu) \nabla \text{div} f\), suppose that \(w_1(x, t)\) and \(w_2(x, t)\) solve the following problems respectively

\[
\begin{align*}
Lw_1 &= 0, & x \in \Omega, \\
w_1(x, 0) &= w_{10}(x), & x \in \Omega, \\
w_1 \cdot n &= 0, \nabla w_1 \times n &= -Aw_1 & x \in \partial \Omega,
\end{align*}
\]

(3.46)

and

\[
\begin{align*}
Lw_2 &= -\nabla (P - \bar{P}), & x \in \Omega, \\
w_2(x, 0) &= 0, & x \in \Omega, \\
w_2 \cdot n &= 0, \nabla w_2 \times n &= -Aw_2 & x \in \partial \Omega.
\end{align*}
\]

(3.47)

Then, just as what we have done in the proof of Lemma 2.9 by Lemma 2.4 and Sobolev’s inequality, for any \(p \in [2, 6]\), we have

\[
\|\nabla^2 w_1\|_{L^2} \leq C(\|\rho \dot{w}_1\|_{L^2} + \|\nabla w_1\|_{L^2}),
\]

\[
\|\nabla w_1\|_{L^p} \leq C\|w_1\|_{W^{2,2}} \leq C(\|\rho \dot{w}_1\|_{L^2} + \|\nabla w_1\|_{L^2}),
\]

(3.48)

\[
\|\nabla F_2\|_{L^p} \leq C(\|\rho \dot{w}_2\|_{L^p} + \|\nabla w_2\|_{L^2} + \|P - \bar{P}\|_{L^p}),
\]

(3.49)

\[
\|F_2\|_{L^p} \leq C\|\nabla F_2\|_{L^2} \leq C(\|\rho \dot{w}_2\|_{L^2} + \|\nabla w_2\|_{L^2}),
\]

(3.50)

\[
\|\nabla w_2\|_{L^p} \leq C(\rho \dot{2} \dot{w}_2)_{L^p}^{\frac{3p-6}{2p}} \left(\|\nabla w_2\|_{L^2} + \|P - \bar{P}\|_{L^2}\right)^{\frac{6-p}{2p}} + C(\|\nabla w_2\|_{L^2} + \|P - \bar{P}\|_{L^p}),
\]

(3.51)

where we denote the viscous effective flux of \(w_2\) as

\[
F_2 \triangleq (\lambda + 2\mu) \text{div} w_2 - (P - \bar{P}).
\]

Next, a similar way as for the proof of (3.7) shows that

\[
\sup_{0 \leq t \leq \sigma(T)} \int_\Omega \rho|w_1|^2dx + \int_0^{\sigma(T)} \int_\Omega |\nabla w_1|^2dxdt \leq C \int_\Omega |w_{10}|^2dx,
\]

(3.52)

and

\[
\sup_{0 \leq t \leq \sigma(T)} \int_\Omega \rho|w_2|^2dx + \int_0^{\sigma(T)} \int_\Omega |\nabla w_2|^2dxdt \leq CC_0.
\]

(3.53)
Then, multiplying (3.46) by \( w_{1t} \) and integrating over \( \Omega \), by (3.48), (3.5), and Young’s inequality, we obtain
\[
\left( \frac{\lambda + 2\mu}{2} \right) \int (\text{div} w_1)^2 dx + \frac{\mu}{2} \int |\text{curl} w_1|^2 dx + \frac{\mu}{2} \int_{\partial \Omega} w_1 \cdot A \cdot w_1 ds \right)_t + \int \rho|\dot{w}_1|^2 dx \\
= \int \rho \dot{w}_1 \cdot (u \cdot \nabla w_1) dx \\
\leq C \| \rho^{\frac{1}{2}} \dot{w}_1 \|_{L^2} \| \rho^{\frac{1}{2}} u \|_{L^2} \| \nabla w_1 \|_{L^6} \\
\leq C \| \rho^{\frac{1}{2}} \dot{w}_1 \|_{L^2}^2 + \| \nabla w_1 \|_{L^2}^2, \tag{3.54}
\]
which together with (3.52), Gronwall’s inequality and Lemma 2.5 yields
\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_1|^2 dx dt \leq C \| \nabla w_{10} \|_{L^2}^2, \tag{3.55}
\]
and
\[
\sup_{0 \leq t \leq \sigma(T)} t |\nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} t \int \rho |\dot{w}_1|^2 dx dt \leq C \| w_{10} \|_{L^2}^2, \tag{3.56}
\]
provided \( C_0 \leq \tilde{\varepsilon} \stackrel{\Delta}{=} (2C)^{-\frac{3}{6}} \).

Next, since the solution operator \( w_{10} \mapsto w_1(\cdot, t) \) is linear, one can deduce from Calderón’s interpolation theorem [10], Lemmas 22.3 and 36.1, (3.55) and (3.56) that for any \( \theta \in [0, 1] \),
\[
\begin{align*}
\sup_{0 \leq t \leq \sigma(T)} & t^{1-\theta} |\nabla w_1 \|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |\dot{w}_1|^2 dx dt \\
& \leq C \| w_{10} \|_{H^\theta}^2, \tag{3.57}
\end{align*}
\]
with a uniform constant \( C \) independent of \( \theta \).

Next, multiplying (3.47) by \( w_{2t} \) and integrating over \( \Omega \), we give
\[
\begin{align*}
\left( \frac{\lambda + 2\mu}{2} \right) \int (\text{div} w_2)^2 dx + \frac{\mu}{2} \int |\text{curl} w_2|^2 dx - \int P \text{div} w_2 dx \\
+ \left( \frac{\mu}{2} \int_{\partial \Omega} w_2 \cdot A \cdot w_2 ds \right)_t + \int \rho |\dot{w}_2|^2 dx \\
= \int \rho \dot{w}_2 \cdot (u \cdot \nabla w_2) dx - \int P \text{div} w_2 dx \\
= \int \rho \dot{w}_2 \cdot (u \cdot \nabla w_2) dx - \frac{1}{\lambda + 2\mu} \int P (F_2 \text{div} u + \nabla F_2 \cdot u) dx \\
- \frac{1}{2(\lambda + 2\mu)} \int (P - \tilde{P}) \text{div} u dx + \gamma \int P \text{div} u \text{div} w_2 dx \\
\leq C \| |\rho^{\frac{1}{2}} \dot{w}_2 \|_{L^2} \| \rho^{\frac{1}{2}} u \|_{L^2} \| \nabla w_2 \|_{L^6} + \| \nabla u \|_{L^2} \| F_2 \|_{L^2} + \| \nabla F_2 \|_{L^2} \| u \|_{L^2} \\
+ C \| (P - \tilde{P}) \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2} \| \nabla w_2 \|_{L^2} \\
\leq C \| \rho^{\frac{1}{2}} \dot{w}_2 \|_{L^2}^2 (\| \rho^{\frac{1}{2}} \dot{w}_2 \|_{L^2} + \| \nabla w_2 \|_{L^2} + \| P - \tilde{P} \|_{L^6}) \\
+ C \| \nabla u \|_{L^2} \| \rho^{\frac{1}{2}} \dot{w}_2 \|_{L^2} + C \| (P - \tilde{P}) \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla u \|_{L^2} \| \nabla w_2 \|_{L^2} \\
\leq C \| \rho^{\frac{1}{2}} \dot{w}_2 \|_{L^2}^2 + \| \nabla w_2 \|_{L^2}^2 + \| P - \tilde{P} \|_{L^6}, \tag{3.58}
\end{align*}
\]
where we have utilized \( (3.35) \), \( (3.17) \), \( (3.49) \), \( (3.51) \), Hölder’s, Poincaré’s and Young’s inequalities. After choosing \( C_0 \leq \varepsilon_2 \triangleq (4C)^{-\frac{1}{\sigma_0}} \), combining this with Gronwall’s inequality, \( (3.53) \), and Lemmas 2.5, 3.2 and 3.3 we get

\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_2 \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_2|^2 dx dt \leq C. \tag{3.59}
\]

Now letting \( w_{10} = u_0 \), we have \( w_1 + w_2 = u \), which combined with \( (3.57) \) and \( (3.59) \) directly proves \( (3.44) \) provided \( C_0 \leq \varepsilon_1 \triangleq \min\{\varepsilon_1, \varepsilon_2\} \).

Finally, it remains to prove \( (3.45) \). Taking \( m = 2 - s \) in \( (3.43) \), and integrating over \((0, \sigma(T))\), we obtain by \( (2.30) \),

\[
\sup_{0 \leq t \leq \sigma(T)} t^{2-s} \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 + \int_0^{\sigma(T)} t^{2-s} \| \nabla \dot{u} \|_{L^2}^2 dt \\
\leq C \int_0^{\sigma(T)} t^{1-s} \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 dt + C \int_0^{\sigma(T)} t^{2-s} \| \nabla \dot{u} \|_{L^2}^2 \| \nabla u \|_{L^2}^2 + 1 dt \\
\quad + C \int_0^{\sigma(T)} t^{2-s} (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^4) dt + C \int_0^{\sigma(T)} t^{2-s} \| \nabla u \|_{L^4}^2 dt \tag{3.60}
\]

\[
\leq C \int_0^{\sigma(T)} t^{2-s} \| \nabla u \|_{L^4}^4 dt + C(\hat{\rho}, M),
\]

where we have taken advantage of \( (3.36) \) and \( (3.44) \).

By \( (2.22) \) and \( (3.44) \), we have

\[
\int_0^{\sigma(T)} t^{2-s} \| \nabla u \|_{L^4}^4 dt \\
\leq C \int_0^{\sigma(T)} t^{2-s} \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 (\| \nabla u \|_{L^2} + \| P - \bar{P} \|_{L^2}) dt \\
\quad + C \int_0^{\sigma(T)} t^{2-s} (\| \nabla u \|_{L^2}^4 + \| P - \bar{P} \|_{L^2}^4) dt \\
\leq C \int_0^{\sigma(T)} t^{2-s} \| \nabla u \|_{L^2}^2 \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 + C(\hat{\rho}, M) \left( \sup_{0 \leq t \leq \sigma(T)} t^{2-s} \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 \right)^{\frac{1}{2}} + C,
\]

which together with \( (3.60) \) gives \( (3.45) \). \( \square \)

**Lemma 3.6** If \((\rho, u)\) is a smooth solution of \((1.1)-(1.5)\) satisfying \( (3.5) \) and the initial data condition \( \| u_0 \|_{H^s} \leq M \) in \( (1.10) \), then there exists a positive constant \( \varepsilon_2 \) depending only on \( \mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M \) and the matrix \( A \) such that

\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 dx \leq C_0^5, \tag{3.61}
\]

provided \( C_0 \leq \varepsilon_2 \).
Proof. First, multiplying (1.1) by 3|u|u and integrating the resulting equality over \( \Omega \), we find that
\[
\left( \int \rho |u|^3 \, dx \right)_t + 3(\lambda + 2\mu) \int \text{div} \text{div}(|u|u) \, dx + 3\mu \int \text{curl} \cdot \text{curl}(|u|u) \, dx \\
+ 3\mu \int_{\partial \Omega} |u|u \cdot A \cdot ds - 3 \int (P - \bar{P}) \text{div}(|u|u) \, dx = 0,
\]
which together with (3.44) and (3.45) yields
\[
\int \rho |u|^3 \, dx \leq C \int |u| |\nabla u|^2 \, dx + C \int u^3 \, ds + C \int |P - \bar{P}| |\nabla u| \, dx \leq C \| \nabla u \|_{L^6} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| P - \bar{P} \|_{L^3} \| \nabla u \|_{L^6} + C \| \nabla u \|_{L^2}^2
\]
(3.62)

where in the third inequality we have used
\[
\| \nabla u \|_{L^6} \leq C \| \rho u \|_{L^2} + C \| P - \bar{P} \|_{L^2} + C \| P - \bar{P} \|_{L^6} + C \| \nabla u \|_{L^2}
\]
due to (2.22).

Thus, combining (3.62) and (3.7) implies
\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 \, dx \\
\leq C(\rho, M) \left( \int_0^{\sigma(T)} \rho \sigma^{-\frac{(6 - 8\delta_0)(1-s)+1}{4(1 - 2\delta_0)}} \, dt \right)^{1 - 2\delta_0} \left( \int_0^{\sigma(T)} \| \nabla u \|_{L^2}^2 \, dt \right)^{2\delta_0}
\]

provided \( C_0 \leq \epsilon_1 \), where in the last inequality we have used both \( \frac{(6 - 8\delta_0)(1-s)+1}{4(1 - 2\delta_0)} \) < 1 due to \( \delta_0 = \frac{2s - 1}{4s} \in (0, \frac{1}{4}] \) and \( s \in (\frac{1}{2}, 1] \) and the following simple fact (see [7, Theorem 1])
\[
\int \rho_0 |u_0|^3 \, dx \leq C \| \rho_0 \|_{L^2}^{3(2s - 1)/2s} \| u_0 \|_{H^s}^{3/2s} \leq C(\rho, M) C_0^{2\delta_0}.
\]
(3.63)

Finally, we obtain (3.61) by setting \( \epsilon_2 \triangleq \min \{ \epsilon_1, C(\rho, M) \}^{\frac{1}{3\delta_0}} \}. \) The proof of Lemma 3.6 is finished. \( \square \)

**Lemma 3.7** Let \((\rho, u)\) be a smooth solution of (1.1)-(1.5) on \( \Omega \times (0, T) \) satisfying (3.5) and the initial data condition \( \| u_0 \|_{H^s} \leq M \) in (1.10). Then there exists a positive constant \( \epsilon_3 \) depending only on \( \mu, \lambda, \gamma, a, s, \rho, M, \Omega \) and the matrix \( A \) such that
\[
A_1(T) + A_2(T) \leq C_0^\frac{1}{\epsilon}.
\]
(3.64)
provided \( C_0 \leq \epsilon_3. \)
Proof. First, we will prove (3.64). By (2.22) and (3.35), one can check that

\[
\sigma^3\|\nabla u\|^4_{L^4} \leq C\sigma^3\|\rho^\frac{3}{2} \tilde{u}\|^3_{L^2}(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2}) + C\sigma^3(\|\nabla u\|^4_{L^4} + \|P - \bar{P}\|^4_{L^4}) \leq C\rho_0^{1/2} \sigma\|\rho^\frac{3}{2} \tilde{u}\|^2_{L^2} + C(\|\nabla u\|^2_{L^2} + \|P - \bar{P}\|^2_{L^2}),
\]

which together with (3.5), (3.7) and (3.12) leads to

\[
\int_0^T \sigma^3\|\nabla u\|^4_{L^4} dt \leq CC_0^{\frac{1}{2}}. \tag{3.66}
\]

Next, it follows from (2.22), (3.5), (3.7), (3.41) and (3.12) that

\[
\int_0^{\sigma(T)} \sigma\|\nabla u\|^3_{L^3} dt \leq C \int_0^{\sigma(T)} \sigma\|\rho^\frac{3}{2} \tilde{u}\|^2_{L^2}(\|\nabla u\|^2_{L^2} + \|P - \bar{P}\|^2_{L^2}) dt + C \int_0^{\sigma(T)} \sigma(\|\nabla u\|^3_{L^3} + \|P - \bar{P}\|^3_{L^3}) dt 
\]

\[
\leq C \int_0^{\sigma(T)} (\sigma\|\nabla u\|^2_{L^2})(\|\nabla u\|^2_{L^2} + \|P - \bar{P}\|^2_{L^2}) dt + C \int_0^{\sigma(T)} \sigma\|\nabla u\|^2_{L^2} dt 
\]

\[
\leq C(M) \left( \int_0^{\sigma(T)} \|\nabla u\|^2_{L^2} dt \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} \sigma\|\rho^\frac{1}{2}\|^2_{L^2} dt \right)^{\frac{1}{4}} + CC_0^{\frac{1}{2}} 
\]

\[
\leq C(\bar{\rho}, M)(A_1(T))^\frac{3}{4} + CC_0^{\frac{1}{2}} 
\]

\[
\leq C(\bar{\rho}, M)C_0^{\frac{3}{4}},
\]

provided \(C_0 \leq \varepsilon_2\).

On the other hand, by (3.66) and (3.7),

\[
\int_{\sigma(T)}^T \sigma\|\nabla u\|^3_{L^3} dt \leq \int_{\sigma(T)}^T \sigma\|\nabla u\|^4_{L^4} dt + \int_{\sigma(T)}^T \sigma\|\nabla u\|^2_{L^2} dt
\]

\[
\leq CC_0^{\frac{1}{2}}. \tag{3.68}
\]

Finally, by (3.20), (3.21) and (3.66)-(3.68), we have

\[
A_1(T) + A_2(T) \leq C(\bar{\rho}, M)C_0^{\frac{3}{4}},
\]

which gives (3.64) provided \(C_0 \leq \varepsilon_3\) with \(\varepsilon_3 \triangleq \min\{\varepsilon_2, (C(\bar{\rho}, M))^{-24}\}\). \(\square\)

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtaining all the higher order estimates and thus to extending the classical solution globally in time. We will use an approach motivated by the works [19][21].
Lemma 3.8 There exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M, \) and the matrix \( A \) such that, if \((\rho, u)\) is a smooth solution of \((1.1) - (1.5)\) on \( \Omega \times (0, T) \) satisfying \((3.5)\) and the initial data condition \( \|u_0\|_{H^s} \leq M \) in \((1.10)\), then for \((x, t) \in \Omega \times (0, T) \)

\[
\sup_{0 \leq t \leq T} \| \rho(t) \|_{L^\infty} \leq \frac{7\hat{\rho}}{4},
\]

provided \( C_0 \leq \varepsilon \). Moreover, if \( C_0 \leq \varepsilon \), there exists some positive constant \( \tilde{C}(T) \) depending only on \( T, \mu, \lambda, \gamma, a, \hat{\rho}, s, \Omega, M, \) and the matrix \( A \) such that for \((x, t) \in \Omega \times (0, T) \)

\[
\rho(x, t) \geq \tilde{C}(T) \inf_{x \in \Omega} \rho_0(x).
\]

**Proof.** First, the equation of mass conservation \((1.1)\) can be rewritten in the form

\[
D_t \rho = g(\rho) + b'(t),
\]

where

\[
D_t \rho \triangleq \rho_t + u \cdot \nabla \rho, \quad g(\rho) \triangleq -\frac{\rho(P - \bar{P})}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho F dt.
\]

Then, it follows from \((3.34), (3.5), (3.44),\) and \((3.45)\) that for \( \delta_0 \) as in Proposition \(3.1\) and \( t \in [0, \sigma(T)] \),

\[
\| \nabla F \|_{L^6} \leq C \| \nabla \hat{u} \|_{L^2} + C \sigma^{(1-s)} \| \frac{1}{2} \hat{u} \|_{L^2} \leq C \sigma^{\frac{(2-s)(1-\delta_1) + 3\delta_0}{2}} \| \nabla \hat{u} \|_{L^2} \quad C_0^{\frac{\delta_0}{6}},
\]

provide \( C_0 \leq \varepsilon_1 \). Combining this, \((2.22), (2.21),\) and \((3.5)\) yields

\[
\| F(\cdot, t) \|_{L^\infty} \leq C \| F \|_{L^6} \| \nabla F \|_{L^6} \leq C \left( \| \frac{1}{2} \hat{u} \|_{L^2} + \| \nabla \hat{u} \|_{L^2} \right)^\frac{1}{2} \left( \| \nabla \hat{u} \|_{L^2} + \sigma^{(1-s)} \right)^\frac{1}{2} \leq C \sigma^{\frac{(2-s)(1-\delta_0) + 3\delta_0}{4}} \sigma^{\frac{\delta_0}{6}} \| \nabla \hat{u} \|_{L^2} + \sigma^{s},
\]

which together with \((3.45)\) and Holder’s inequality thus implies that for all \( 0 \leq t_1 \leq t_2 \leq \sigma(T) \),

\[
|b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^\infty} dt \leq C \sup_{t \in [0, \sigma(T)]} \| \rho \|_{L^\infty} \leq \hat{\rho} + C(\hat{\rho}, M) C_0^{\frac{\delta_0}{6}} \leq \frac{3\hat{\rho}}{2},
\]

due to \((2-s)(2-\delta_0) + 3\delta_0 < 3\). Thus, choosing \( N_1 = 0, N_0 = C(\hat{\rho}, M) C_0^{\frac{\delta_0}{6}} \), and \( \zeta_0 = \hat{\rho} \) in Lemma \(2.3\) we use \((3.72), (3.71)\), and Lemma \(2.25\) to get

\[
\sup_{t \in [0, \sigma(T)]} \| \rho \|_{L^\infty} \leq \hat{\rho} + C(\hat{\rho}, M) C_0^{\frac{\delta_0}{6}} \leq \frac{3\hat{\rho}}{2},
\]

provided \( C_0 \leq \varepsilon_4 \triangleq \min \left\{ \varepsilon_3, \left( \frac{\hat{\rho}}{2C(\hat{\rho}, M)} \right)^{\frac{6}{\delta_0}} \right\} \).

On the other hand, for \( \sigma(T) \leq t_1 \leq t_2 \leq T \), we have

\[
|b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} \| F \|_{L^\infty} dt \leq \frac{a_0 \gamma + 1}{2(\lambda + 2\mu)} (t_2 - t_1) + C \int_{\sigma(T)}^{T} \| F \|_{L^\infty}^{4} dt \leq \frac{a_0 \gamma + 1}{2(\lambda + 2\mu)} (t_2 - t_1) + CC_0^{\frac{\delta_0}{6}},
\]
where in the last inequality we have used
\[
\begin{aligned}
\int_T^{\sigma(T)} \| F \|_{L^\infty}^4 dt &\le C \int_T^{\sigma(T)} \| F \|_{L^6}^2 \| \nabla F \|_{L^6}^2 dt \\
&\le CC_0^\frac{1}{3} \int_T^{\sigma(T)} \| \nabla \dot{u} \|_{L^2}^2 + CC_0^\frac{1}{2} \\
&\le CC_0^\frac{1}{3},
\end{aligned}
\]  
(3.75)
due to (2.23), (2.21), (2.29), (3.5), (3.7) and (3.12).

Now choosing \( N_0 = CC_0^\frac{1}{3}, N_1 = \frac{a\hat{\rho}^{\gamma+1}}{2(\lambda+2\mu)} \) in (2.3) and setting \( \zeta_0 = \frac{3\hat{\rho}}{2} \) in (2.4), we have for all \( \zeta \ge \zeta_0 = \frac{3\hat{\rho}}{2} \),
\[
g(\zeta) = -\frac{\zeta(a\zeta - \bar{P})}{\lambda + 2\mu} \le -\frac{a\hat{\rho}^{\gamma+1}}{2(\lambda + 2\mu)} = -N_1,
\]
which together with Lemma 2.3, (3.73), and (3.74) leads to
\[
\sup_{t \in [\sigma(T), T]} \| \rho \|_{L^\infty} \le \frac{3\hat{\rho}}{2} + CC_0^\frac{1}{2} \le \frac{7\hat{\rho}}{4},
\]
(3.76)
provided
\[
C_0 \le \varepsilon \triangleq \min \left\{ \varepsilon_{4\lambda} \left( \frac{\hat{\rho}}{4C} \right)^2 \right\}.
\]
(3.77)
Combining (2.4) and (3.76) thus gives (3.69).

Finally, it remains to prove (3.70). Indeed, without loss of generality, assume that
\[
\inf_{x \in \Omega} \rho_0(x) > 0.
\]
We have by (3.71),
\[
(2\mu + \lambda)D_t\rho^{-1} - \rho^{-1}(P - \bar{P}) - \rho^{-1}F = 0,
\]
which in particular shows that
\[
D_t\rho^{-1} \le C \rho^{-1} (\| F \|_{L^\infty} + 1).
\]
Combining this with Gronwall’s inequality, (3.72) and (3.75) gives (3.70) and finishes the proof of Lemma 3.8.

With Proposition 3.1 at hand, we are now in a position to prove the following result concerning the exponential decay rate of both weak and classical solutions. It should be noted here that both the rate \( \eta_0 \) and the constant \( C \) depend on \( \bar{\rho}_0 \) also which is different from the constants \( C \) in the proof of Proposition 3.1 where they are independent of \( \bar{\rho}_0 \).

**Proposition 3.9** For any \( r \in [1, \infty) \) and \( p \in [1, 6] \), there exist positive constants \( C \) and \( \eta_0 \) depending only on \( \mu, \lambda, \gamma, a, s, \hat{\rho}, \bar{\rho}_0, M, \Omega, r, p, \) and the matrix \( A \) such that (1.15) holds for \( t \ge 1 \).

**Proof.** First, by (1.8) and (3.8), there exists a positive constant \( \tilde{C} < 1 \) depending only on \( \gamma, \bar{\rho}_0, \) and \( \hat{\rho} \) such that for any \( \rho \in [0, 2\hat{\rho}] \),
\[
\tilde{C}^2(\rho - \hat{\rho})^2 \le \tilde{C}G(\rho) \le (\rho^\gamma - \bar{\rho}^\gamma)(\rho - \bar{\rho}),
\]
(3.78)
and
\[ \| P - \bar{P} \|^2_{L^2} \leq C\| P - P(\bar{\rho}) \|^2_{L^2} \leq C \int G(\rho) dx. \] (3.79)

Then, multiplying (1.1) by \( B[\rho - \bar{\rho}] \), one has
\[
\int (P - P(\bar{\rho}))(\rho - \bar{\rho}) dx \\
= \left( \int \rho u \cdot B[\rho - \bar{\rho}] dx \right)_t - \int \rho u \cdot \nabla B[\rho - \bar{\rho}] \cdot u dx - \int \rho u \cdot B[\rho] dx \\
+ \mu \int \nabla u \cdot \nabla B[\rho - \bar{\rho}] dx + (\lambda + \mu) \int (\rho - \bar{\rho}) \text{div} u dx \\
\leq \left( \int \rho u \cdot B[\rho - \bar{\rho}] dx \right)_t + C\| \rho^2 u \|_{L^4}^2 \| \rho - \bar{\rho} \|_{L^2} + C\| \rho u \|_{L^2}^2 \\
+ C\| \rho - \bar{\rho} \|_{L^2} \| \nabla u \|_{L^2} \\
\leq \left( \int \rho u \cdot B[\rho - \bar{\rho}] dx \right)_t + \delta \| \rho - \bar{\rho} \|_{L^2}^2 + C(\delta) \| \nabla u \|_{L^2}^2,
\]

which, along with (3.78) and (3.1), leads to
\[
a\tilde{C} \int G(\rho) dx \leq a \int (\rho^\gamma - \rho^\gamma)(\rho - \bar{\rho}) dx \\
\leq 2 \left( \int \rho u \cdot B[\rho - \bar{\rho}] dx \right)_t + \tilde{C}_1 \phi(t). \] (3.80)

Moreover, it follows from (3.78) that
\[
\left\| \int \rho u \cdot B[\rho - \bar{\rho}] dx \right\| \leq \tilde{C}_2 \left( \frac{1}{2} \| \sqrt{\rho} u \|_{L^2}^2 + \int G(\rho) dx \right),
\]

which gives
\[
\frac{1}{2} \left( \frac{1}{2} \| \sqrt{\rho} u \|_{L^2}^2 + \int G(\rho) dx \right) \leq W(t) \leq 2 \left( \frac{1}{2} \| \sqrt{\rho} u \|_{L^2}^2 + \int G(\rho) dx \right), \] (3.81)

where
\[
W(t) = \int \left( \frac{1}{2} \rho |u|^2 + G(\rho) \right) dx - \delta_1 \int \rho u \cdot B[\rho - \bar{\rho}] dx,
\]

with \( \delta_1 = \min\{ (2\tilde{C}_1)^{-1}, (2\tilde{C}_2)^{-1} \} \).

Adding (3.80) multiplied by \( \delta_1 \) to (3.11) and utilizing
\[
\int \rho |u|^2 dx \leq C \| \nabla u \|_{L^2}^2 \leq C_3 \phi(t),
\]

we obtain for \( \eta_0 = \min\{ a\tilde{C} \frac{1}{4}, \frac{1}{4C_3} \} \),
\[
W'(t) + 2\eta_0 W(t) \leq 0.
\]

Combining this with (3.81) yields that for any \( t > 0 \),
\[
\int \left( \frac{1}{2} \rho |u|^2 + G(\rho) \right) dx \leq 4C_0 e^{-2\eta t}, \] (3.82)
which together with (3.11) shows
\[ \int_0^T \phi(t)e^{\eta_0 t} dt \leq C. \] (3.83)

Choosing \( m = 0 \) in (3.29) along with (3.1), (2.22), (3.5) and (3.79) leads to
\[ \left( \phi(t) - 2 \int (P - P(\bar{\rho})) \text{div}u dx \right)_t + \frac{1}{2} \|\sqrt{\bar{\rho}}u\|_{L^2}^2 \leq C\phi(t) + CG(\rho). \] (3.84)

Noticing that
\[ \int (P - P(\bar{\rho})) \text{div}u dx \leq CG(\rho) + \frac{1}{4}\phi(t), \]
we obtain after multiplying (3.84) by \( e^{\eta_0 t} \) and using (3.82) and (3.83) that for any \( T \geq 1 \),
\[ \sup_{0 \leq t \leq T} \left( e^{\eta_0 t} \|\nabla u\|_{L^2}^2 \right) + \int_0^T e^{\eta_0 t} \|\sqrt{\bar{\rho}}u\|_{L^2}^2 dt \leq C. \] (3.85)

Finally, a similar analysis based on (3.83) where \( m = 3 \), (3.65), (3.36), (3.79) and (3.85) indicates that for any \( t \geq 1 \),
\[ \|\sqrt{\bar{\rho}}u\|_{L^2}^2 \leq Ce^{-\eta_0 t}, \]
which together with (3.82), (3.85), and (2.22) gives (1.15) and finishes the proof of Proposition 3.9.

4 A priori estimates (II): higher order estimates

Let \((\rho, u)\) be a smooth solution of (1.1)-(1.5). The purpose of this section is to derive some necessary higher order estimates, which make sure that one can extend the classical solution globally in time. Here we adopt the method of the article [19,25], and follow their work with a few modifications. We sketch it here for completeness.

In this section, we always assume that the initial energy \( C_0 \) satisfies (3.77), and the positive constant \( C \) may depend on \( T, \|g\|_{L^2}, \|\nabla u_0\|_{L^1}, \|\rho_0\|_{W^{2,q}}, \|P(\rho_0)\|_{W^{2,q}}, \) for \( q \in (3, 6) \) besides \( \mu, \lambda, a, \gamma, \hat{\rho}, s, \Omega, M \) and the matrix \( A \), where \( g \in L^2(\Omega) \) is given by (1.11).

**Lemma 4.1** There exists a positive constant \( C \) such that
\[ \sup_{0 \leq t \leq T} \|\rho \frac{4}{7} \dot{u}\|_{L^2} + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 dt \leq C, \] (4.1)
\[ \sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^6} + \|u\|_{H^2}) + \int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^6}) dt \leq C. \] (4.2)
Proof. First, Taking $s = 1$ in (3.43) along with (3.63) gives

$$
\sup_{t \in [0,T]} \| \nabla u \|_{L^2}^2 + \int_0^T \int_\Omega \rho |\dot{u}|^2 dx dt \leq C. \tag{4.3}
$$

Choosing $m = 0$ in (3.43), by (2.22) and (2.31), we have

$$
\left( \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 \right)_t + \mu \Lambda^{-1} \| \nabla \dot{u} \|_{L^2}^2 \\
\leq - \left( 2 \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \right)_t + C \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 (\| \nabla u \|_{L^2}^4 + 1) \\
+ C (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^6 + \| \nabla u \|_{L^2}^4) \tag{4.4}
$$

$$
\leq - \left( 2 \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds \right)_t + C \| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 (\| \rho^{\frac{1}{2}} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^4 + 1) \\
+ C (\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^6 + \| P - \bar{P} \|_{L^2}^4).
$$

By Gronwall’s inequality and the compatibility condition (1.11), we deduce (4.1) from (4.4), (4.3) and (3.36).

Next, we will follow the proof of [18, Lemma 5] to show (4.2). For $2 \leq p \leq 6$, $|\nabla \rho|^p$ satisfies

$$
|\nabla \rho|^p_t + \text{div} (|\nabla \rho|^p u) + (p - 1) |\nabla \rho|^p \text{div} u \\
+ p |\nabla \rho|^{p-2} (\nabla \rho)^{tr} \nabla u (\nabla \rho) + p p |\nabla \rho|^{p-2} \nabla \rho \cdot \text{div} \nabla u = 0,
$$

where $(\nabla \rho)^{tr}$ is the transpose of $\nabla \rho$.

Thus, taking $p = 6$, by (2.23), (2.29) and (4.3),

$$
(\| \nabla \rho \|_{L^6})_t \leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^6} + C \| \nabla F \|_{L^6} \\
\leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^6} + C \| \rho \dot{u} \|_{L^6} \tag{4.5}
$$

$$
\leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^6} + C (\| \rho \dot{u} \|_{L^2} + 1).
$$

Then, it follows from the Gagliardo-Nirenberg inequality, (2.29), (2.23), and (4.3) that

$$
\| \text{div} u \|_{L^\infty} + \| \text{curl} u \|_{L^\infty} \\
\leq C (\| F \|_{L^\infty} + \| P - \bar{P} \|_{L^\infty}) + \| \text{curl} u \|_{L^\infty} \\
\leq C (\| F \|_{L^2} + \| \nabla F \|_{L^6} + \| \nabla \text{curl} u \|_{L^6} + \| P - \bar{P} \|_{L^\infty}) \\
\leq C (\| \nabla u \|_{L^2} + \| P - \bar{P} \|_{L^2} + \| \rho \dot{u} \|_{L^6} + \| P - \bar{P} \|_{L^\infty}) \\
\leq C (\| \nabla \dot{u} \|_{L^2} + 1),
$$

which together with Lemma 2.7 and (2.29) yields

$$
\| \nabla u \|_{L^\infty} \leq C (\| \text{div} u \|_{L^\infty} + \| \text{curl} u \|_{L^\infty} \ln (e + \| \nabla^2 u \|_{L^6}) + C \| \nabla u \|_{L^2} + C \\
\leq C (1 + \| \nabla \dot{u} \|_{L^2}) \ln (e + \| \nabla \dot{u} \|_{L^2} + \| \nabla \rho \|_{L^6}) \tag{4.6}
$$

$$
\leq C (1 + \| \nabla \dot{u} \|_{L^2}^2 + C (1 + \| \nabla \dot{u} \|_{L^2}) \ln (e + \| \nabla \rho \|_{L^6}),
$$

where in the second inequality, we have used the fact that for any $p \in [2,6],

$$
\| \nabla^2 u \|_{L^p} \leq C (\| \rho \dot{u} \|_{L^p} + \| \nabla P \|_{L^p} + \| \nabla u \|_{L^2} + \| P - \bar{P} \|_{L^p}), \tag{4.7}
$$
which can be obtained by applying Lemma 2.4 to the following system

\[
\begin{aligned}
-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= -\rho \dot{u} - \nabla (P - \bar{P}), \\
u \cdot n &= 0, \quad \text{curl} \, u \times n = -Au,
\end{aligned}
\]

\[(4.8)\]

Next, it follows from (4.6) and (4.5) that

\[
(e + \|\nabla \rho\|_{L^6})_t \\
\leq C \left(1 + \|\nabla \dot{u}\|_{L^2}^2 + (1 + \|\nabla \dot{u}\|_{L^2}) \ln(e + \|\nabla \rho\|_{L^6})\right) (e + \|\nabla \rho\|_{L^6}),
\]

which together with Gronwall’s inequality and (4.1) shows that

\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} \leq C.
\]

Combining this, (4.6), (4.7), (2.29), (4.1), and (4.3) gives (4.2) and finishes the proof of Lemma 4.1.

**Lemma 4.2** There exists a positive constant $C$ such that

\[
\sup_{0 \leq t \leq T} \|\rho^\frac{1}{2} u_t\|_{L^2}^2 + \int_0^T \int |\nabla u_t|^2 \, dx \, dt \leq C, \quad \text{(4.9)}
\]

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^2} + \|P\|_{H^2}) \leq C. \quad \text{(4.10)}
\]

**Proof.** By Lemma 4.1, a straightforward calculation yields that

\[
\|\rho^\frac{1}{2} u_t\|_{L^2}^2 \leq \|\rho^\frac{1}{2} \dot{u}\|_{L^2}^2 + \|\rho^\frac{1}{2} u \cdot \nabla u\|_{L^2}^2 \\
\leq C + C\|u\|_{L^1}^2 \|\nabla u\|_{L^4}^2 \\
\leq C + C\|\nabla u\|_{L^2}^2 \|u\|_{H^2}^2 \\
\leq C,
\]

and

\[
\int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \leq \int_0^T \|\nabla \dot{u}\|_{L^2}^2 \, dt + \int_0^T \|\nabla (u \cdot \nabla u)\|_{L^2}^2 \, dt \\
\leq C + \int_0^T \|\nabla u\|_{L^4}^4 + \|u\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2}^2 \, dt \\
\leq C + C \int_0^T (\|\nabla^2 u\|_{L^2}^4 + \|\nabla u\|_{H^1}^2 \|\nabla^2 u\|_{L^2}^2) \, dt \\
\leq C.
\]

It remains to prove (4.10). We deduce from (4.8) and Lemma 2.4 that for any $p \in [2, 6],$

\[
\|\nabla^3 u\|_{L^p} \leq C(\|\rho \dot{u}\|_{W^{1,p}} + \|\nabla P\|_{W^{1,p}} + \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^p}), \quad \text{(4.11)}
\]

which together with (3.17), (1.11), (4.7), and Lemma 4.1 gives

\[
\frac{d}{dt}(\|\nabla^2 P\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) \\
\leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla^2 P\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + C\|\nabla \dot{u}\|_{L^2}^2 + C.
\]
Consequently, combining this, Gronwall’s inequality, and Lemma 4.1 leads to
\[ \sup_{0 \leq t \leq T} (\| \nabla^2 P \|_{L^2}^2 + \| \nabla^2 \rho \|_{L^2}^2) \leq C. \]

Thus the proof of Lemma 4.2 is finished. \( \square \)

**Lemma 4.3** There exists a positive constant \( C \) such that
\[ \sup_{0 \leq t \leq T} (\| \rho_t \|_{H^1} + \| P_t \|_{H^1}) + \int_0^T (\| \rho u \|_{L^2}^2 + \| P_{tt} \|_{L^2}^2) \, dt \leq C, \quad (4.12) \]
\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{L^2}^2 + \int_0^T \sigma \| \rho_{tt} u_{tt} \|_{L^2}^2 \, dt \leq C. \quad (4.13) \]

**Proof.** First, it follows from (3.17) and Lemma 4.1 that
\[ \| P_t \|_{L^2} \leq C \| u \|_{L^\infty} \| \nabla P \|_{L^2} + C \| \nabla u \|_{L^2} \leq C. \quad (4.14) \]

Next, applying \( \nabla \) to (3.17), one gets
\[ \nabla P_t + u \cdot \nabla \nabla P + \nabla u \cdot \nabla P + \gamma \nabla P \text{div} u + \gamma P \nabla \text{div} u = 0, \]
which together with Lemmas 4.1 and 4.2 gives
\[ \| \nabla P_t \|_{L^2} \leq C \| u \|_{L^\infty} \| \nabla^2 P \|_{L^2} + C \| \nabla u \|_{L^3} \| \nabla P \|_{L^6} + C \| \nabla^2 u \|_{L^2} \leq C. \]
Combining this with (4.14), one has
\[ \sup_{0 \leq t \leq T} \| P_t \|_{H^1} \leq C. \quad (4.15) \]

Next, it follows from (3.17) that \( P_{tt} \) satisfies
\[ P_{tt} + \gamma P_t \text{div} u + \gamma P \text{div} u_t + u_t \cdot \nabla P + u \cdot \nabla P_t = 0. \quad (4.16) \]
Multiplying (4.16) by \( P_{tt} \) and integrating over \( \Omega \times [0,T] \), we deduce from (4.15), Lemmas 4.1 and 4.2 that
\[ \int_0^T \| P_{tt} \|_{L^2}^2 \, dt = -\int_0^T \int \gamma P_{tt} P_t \text{div}u \, dx \, dt - \int_0^T \int \gamma P_{tt} P \text{div}u_t \, dx \, dt \]
\[ -\int_0^T \int P_{tt} u_t \cdot \nabla P \, dx \, dt - \int_0^T \int P_{tt} u_t \cdot \nabla P \, dx \, dt \]
\[ \leq C \int_0^T \| P_{tt} \|_{L^2} (\| P_t \|_{L^6} \| \nabla u_t \|_{L^2} + \| \nabla u_t \|_{L^2}) \, dt \]
\[ + C \int_0^T \| P_t \|_{L^2} (\| u_t \|_{L^3} \| \nabla P \|_{L^6} + \| u \|_{L^\infty} \| \nabla P_t \|_{L^2}) \, dt \]
\[ \leq C \int_0^T \| P_{tt} \|_{L^2} (1 + \| \nabla u_t \|_{L^2}) \, dt \]
\[ \leq \frac{1}{2} \int_0^T \| P_{tt} \|_{L^2}^2 \, dt + C \int_0^T \| \nabla u_t \|_{L^2}^2 \, dt + C \]
\[ \leq \frac{1}{2} \int_0^T \| P_{tt} \|_{L^2}^2 \, dt + C, \]
where we have utilized Sobolev’s inequality. Therefore, it holds
\[
\int_0^T \|P_{tt}\|_{L^2}^2 dt \leq C.
\]

One can deal with \(\rho_t\) and \(\rho_{tt}\) similarly. Thus, \((4.12)\) is proved.

It remains to prove \((4.13)\). Since \(u_t \cdot n = 0\) on \(\partial \Omega\), by Lemma \(2.5\) we have
\[
\|\nabla u_t\|_{L^2}^2 \leq CH(t),
\]
with \(H(t) \triangleq (\lambda + 2\mu) \int (\text{div}u_t)^2 dx + \mu \int |\text{curl}u_t|^2 dx\).

Differentiating \((11)\) with respect to \(t\), then multiplying by \(u_{tt}\), we obtain
\[
\frac{d}{dt} \left( H(t) + \mu \int_{\partial \Omega} u_t \cdot A \cdot u_t ds \right) + 2 \int \rho |u_{tt}|^2 dx = \frac{d}{dt} \tilde{I}_0 + \int \rho_t |u_t|^2 dx + 2 \int (\rho_t u \cdot \nabla u)_{tt} \cdot u_t dx
\]
\[
- 2 \int \rho u_t \cdot \nabla u \cdot u_{tt} dx - 2 \int \rho u_t \cdot \nabla u_t \cdot u_t dx - 2 \int P_{tt} \text{div}u_t dx \tag{4.18}
\]

where
\[
\tilde{I}_0 \triangleq - \int \rho_t |u_t|^2 dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t dx + 2 \int P_t \text{div}u_t dx
\]
\[
\leq \left| \int \text{div}(\rho u) |u_t|^2 dx \right| + C ||\rho_t||_{L^1} ||u||_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6}
\]
\[
+ C ||P_t||_{L^2} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \int |u||\rho u_t| \|\nabla u_t\| dx + C \|\nabla u_t\|_{L^2} \tag{4.19}
\]
\[
\leq C ||u||_{L^6} ||\rho^{1/2} u_t||_{L^2}^{1/2} ||u_t||_{L^6}^{1/2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|\nabla u\|_{L^2} ||\rho^{1/2} u_t||_{L^2}^{1/2} ||u_t||_{L^2}^{3/2} + C \|\nabla u_t\|_{L^2}
\]
\[
\leq \frac{1}{2} H(t) + C,
\]
de to \((1.1)\), \((2.20)\), \((4.1)\), \((4.2)\), \((4.3)\), \((4.9)\), \((4.12)\), \((4.17)\) and Sobolev’s and Poincaré’s inequalities.

Then, standard arguments yield that
\[
|\tilde{I}_1| = \left| \int \rho u_t |u_t|^2 dx \right|
\]
\[
= \left| \int \text{div}(\rho u)_t |u_t|^2 dx \right|
\]
\[
= 2 \left| \int (\rho_t u + \rho u_t) \cdot \nabla u_t \cdot u_t dx \right|
\]
\[
\leq C \left( ||\rho_t||_{H^1} ||u||_{H^2} + ||\rho^{1/2} u_t||_{L^2}^{1/2} ||u_t||^{1/2}_{L^2} \right) \|\nabla u_t\|_{L^2}^2
\]
\[
\leq C \|\nabla u_t\|_{L^2}^4 + C \|\nabla u_t\|_{L^2}^2 + C
\]
\[
\leq C \|\nabla u_t\|_{L^2}^2 H(t) + C \|\nabla u_t\|_{L^2}^2 + C,
\]
\[
36
\]
\[ |I_2| = 2 \left| \int (\rho_t u \cdot \nabla u)_t \cdot u_t dx \right| \]

\[ = 2 \left| \int (\rho_t u \cdot \nabla u \cdot u_t + \rho_t u_t \cdot \nabla u \cdot u_t + \rho_t u \cdot \nabla u_t \cdot u_t) dx \right| \]

\[ \leq \|\rho_t\|_{L^3} \|u \cdot \nabla u\|_{L^3} \|u_t\|_{L^6} + \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \]

\[ + \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \]

\[ \leq C \|\rho_t\|^2_{L^2} + C \|\nabla u_t\|^2_{L^2} \]

\[ |I_3| + |I_4| = 2 \left| \int \rho u_t \cdot \nabla u \cdot u_t dx \right| + 2 \left| \int \rho u_t \cdot \nabla u_t \cdot u_t dx \right| \]

\[ \leq C \|\rho^{1/2} u_t\|_{L^2} \left( \|u_t\|_{L^2} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \right) \]

\[ \leq \|\rho^{1/2} u_t\|^2_{L^2} + C \|\nabla u_t\|^2_{L^2} , \]

and

\[ |I_5| = 2 \left| \int P_t \text{div} u_t dx \right| \]

\[ \leq C \|P_t\|_{L^2} \|\text{div} u_t\|_{L^2} \]

\[ \leq C \|P_t\|^2_{L^2} + C \|\nabla u_t\|^2_{L^2} . \]

Finally, putting (4.20), (4.23) into (4.18) gives

\[ \frac{d}{dt} (\sigma H(t) + \mu \sigma \int_{\partial \Omega} u_t \cdot A \cdot u_t ds) + \sigma \int \rho |u_t|^2 dx \]

\[ \leq C(1 + \|\nabla u_t\|^2_{L^2}) \sigma H(t) + C(1 + \|\nabla u_t\|^2_{L^2} + \|\rho_t\|^2_{L^2} + \|P_t\|^2_{L^2}) , \]

which together with Gronwall’s inequality, (4.9), (4.12) and (4.19) leads to

\[ \sup_{0 \leq t \leq T} \left( \sigma H(t) + \mu \sigma \int_{\partial \Omega} u_t \cdot A \cdot u_t ds \right) + \int_0^T \sigma \|\rho^{1/2} u_t\|^2_{L^2} dt \leq C. \]

As a result, by (4.17),

\[ \sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|^2_{L^2} + \int_0^T \sigma \|\rho^{1/2} u_t\|^2_{L^2} dt \leq C , \]

which finishes the proof of Lemma 4.3 \( \blacksquare \)

**Lemma 4.4** For \( q \in (3,6) \), there exists a positive constant \( C \) such that

\[ \sup_{t \in [0,T]} \sigma \|\nabla u\|^2_{H^2} + \int_0^T \left( \|\nabla u\|^2_{H^2} + \|\nabla^2 u\|^2_{W^{1,q}} + \sigma \|\nabla u_t\|^2_{H^1} \right) dt \leq C , \]

\[ \sup_{t \in [0,T]} \left( \|\rho\|^2_{W^{2,q}} + \|P\|^2_{W^{2,q}} \right) \leq C , \]

where \( p_0 = \frac{9q-6}{10q-12} \in (1, \frac{7}{5}) \).

**Proof.** First, by Lemma 4.1 and Poincaré’s and Sobolev’s inequalities, one can check that

\[ \|\nabla (\rho u_t)\|_{L^2} \leq \|\nabla \rho\|_{L^2} + \|\rho \nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^2} \|u\|_{L^6} \]

\[ + \|\nabla \rho\|_{L^3} \|u_t\|_{L^6} + C \|\nabla u_t\|_{L^2} + C \|\nabla \rho\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^6} \]

\[ + C \|\nabla \rho\|_{L^3} \|\nabla u\|_{L^6} + C \|u\|_{L^\infty} \|\nabla^2 u\|_{L^2} \]

\[ \leq C + C \|\nabla u_t\|_{L^2} . \]
which together with (4.10) and Lemma 4.1 yields
\[ \| \nabla^2 u \|_{H^1} \leq C(\| \rho \hat{u} \|_{H^1} + \| P - \bar{P} \|_{H^2} + \| u \|_{L^2}) \]
\[ \leq C + C \| \nabla u_t \|_{L^2}. \]
Combining this, (4.12), (4.9) and (4.13) leads to
\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{H^1}^2 + \int_0^T \| \nabla u_t \|_{H^2}^2 dt \leq C. \quad (4.28) \]

Next, it follows from (4.1) and (3.5) that \( u_t \) satisfies
\[ \begin{cases} 
\mu \Delta u_t + (\lambda + \mu) \nabla \text{div} u_t = (\rho \hat{u})_t + \nabla P_t & \text{in } \Omega, \\
u_t \cdot n = 0, \ curl u_t \times n = -Au_t & \text{on } \partial \Omega, 
\end{cases} \quad (4.29) \]
which together with Lemmas 2.4 and 4.1, 4.10 and 4.13 shows
\[ \| \nabla^2 u_t \|_{L^2} \leq C(\| (\rho \hat{u})_t \|_{L^2} + \| P_t \|_{H^1} + \| u_t \|_{L^2}) \]
\[ \leq C(\| \rho u_t \| + \rho u_t \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t) + C \]
\[ \leq C(\| \rho u_t \|_{L^2} + \| \rho u \|_{L^3} \| u_t \|_{L^3} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2}) \]
\[ + C(\| u_t \|_{L^6} \| \nabla u_t \|_{L^2} + \| u_t \|_{L^6} \| \nabla u_t \|_{L^2} + \| \nabla P_t \|_{L^2} + \| u_t \|_{L^2}) \]
\[ \leq C\| \rho \hat{u} \|_{L^2} + C \| \nabla u_t \|_{L^2} + C, \]
Combining this, (4.30) and (4.13) yields
\[ \int_0^T \sigma \| \nabla u_t \|_{H^1}^2 dt \leq C. \quad (4.31) \]

Next, by Sobolev’s inequality, (2.29), (4.2), (4.10) and (4.13), we get for any \( q \in (3, 6), \)
\[ \| \nabla(\rho \hat{u}) \|_{L^q} \leq C \| \nabla \rho \|_{L^q} (\| \nabla \hat{u} \|_{L^6} + \| \nabla \hat{u} \|_{L^2} + \| \nabla \hat{u} \|_{L^2}^2) + C \| \nabla \hat{u} \|_{L^6} \]
\[ \leq C(\| \nabla \hat{u} \|_{L^2} + \| \nabla \hat{u} \|_{L^2}^2) + C(\| \nabla u_t \|_{L^q} + \| (u \cdot \nabla u) \|_{L^q}) \]
\[ \leq C(\| \nabla u_t \|_{L^2} + 1) + C \| \nabla u_t \|_{L^\infty} \| \nabla u_t \|_{L^\infty}^{\frac{6-q}{2q}} \| \nabla u_t \|_{L^\infty}^{\frac{3(q-2)}{2q}} \]
\[ + C(\| u \|_{L^\infty} \| \nabla^2 u \|_{L^q} + \| u \|_{L^\infty} \| \nabla^2 u \|_{L^q}) \]
\[ \leq C \sigma^{-\frac{1}{2}} + C \| \nabla u \|_{H^2} + C \sigma^{-\frac{1}{2}} (\| \nabla u_t \|_{H^1}^2) \quad (4.32) \]
Integrating this over \((0, T), \) by (4.11) and (4.31), we have
\[ \int_0^T \| \nabla(\rho \hat{u}) \|_{L^q}^2 dt \leq C. \quad (4.33) \]

On the other hand, the combination of (3.17) with (4.10) gives
\[ \begin{align*}
(\| \nabla^2 P \|_{L^q})_t & \leq C \| \nabla u \|_{L^\infty} \| \nabla^2 P \|_{L^q} + C \| \nabla^2 u \|_{W^{1,q}} \\
& \leq C(1 + \| \nabla u \|_{L^\infty}) \| \nabla^2 P \|_{L^q} + C(1 + \| \nabla u_t \|_{L^2}) \\
& \quad + C \| \nabla(\rho \hat{u}) \|_{L^q},
\end{align*} \quad (4.34) \]
where in the last inequality we have used the following simple fact that
\[
\|\nabla^2 u\|_{W^{1,q}} \leq C(\|\rho u\|_{L^2} + \|\nabla (\rho u)\|_{L^q} + \|\nabla^2 P\|_{L^q} + \|\nabla P\|_{L^q} + \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|P - \bar{P}\|_{L^q})
\]
(4.35)
due to (4.7), (4.11), (4.1) and (4.10).

Hence, applying Gronwall’s inequality in (4.34), we deduce from (4.2), (4.9) and (4.33) that
\[
\sup_{t \in [0,T]} \|\nabla^2 P\|_{L^q} \leq C,
\]
which along with (4.9), (4.10), (4.35) and (4.33) gives
\[
\sup_{t \in [0,T]} \|P\|_{W^{2,q}} + \int_0^T \|\nabla^2 u\|^p_{W^{1,q}} dt \leq C.
\]
(4.36)
Similarly, one has
\[
\sup_{0 \leq t \leq T} \|\rho\|_{W^{2,q}} \leq C,
\]
which together with (4.36) gives (4.27). The proof of Lemma 4.4 is finished.

**Lemma 4.5** For \( q \in (3,6) \), there exists a positive constant \( C \) such that
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla u_t\|_{H^1} + \|\nabla u\|_{W^{2,q}} \right) + \int_0^T \sigma^2 \|\nabla u_{tt}\|_{L^2}^2 dt \leq C.
\]
(4.37)

**Proof.** First, differentiating (1.1) with respect to \( t \) twice implies
\[
\rho_{ttu} + \rho u \cdot \nabla u_{tt} - (\lambda + 2\mu)\nabla \text{div} u_{tt} + \mu \nabla \times \text{curl} u_{tt} = 2\text{div}(\rho u_{tt}) + \text{div}(\rho u) u_t - 2(\rho u_t) \cdot \nabla u_t - (\rho u_t + 2\rho u_t) \cdot \nabla u - \rho u_{tt} \cdot \nabla u - \nabla P_{tt}.
\]
(4.38)

Then, multiplying (4.38) by \( 2u_{tt} \) and integrating over \( \Omega \), we get
\[
\frac{d}{dt} \int \rho |u_{tt}|^2 dx + 2(\lambda + 2\mu) \int (\text{div} u_{tt})^2 dx + 2\mu \int |\text{curl} u_{tt}|^2 dx
\]
\[
= -8 \int \rho u_{tt} u \cdot \nabla u_{tt} dx - 2 \int (\rho u_{tt}) \cdot [\nabla (u_t \cdot u_{tt}) + 2\nabla u_t \cdot u_{tt}] dx
\]
\[
- 2 \int (\rho u_t + 2\rho u_t) \cdot \nabla u \cdot u_{tt} dx - 2 \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx
\]
\[
+ 2 \int P_{tt} \text{div} u_{tt} dx \triangleq \sum_{i=1}^5 K_i.
\]
(4.39)

Let us estimate each \( K_i (i = 1, \ldots, 5) \) as follows. Hölder’s inequality and (4.2) give
\[
|K_1| \leq C \|\rho^{1/2} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \|u\|_{L^\infty}
\]
\[
\leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \|\rho^{1/2} u_{tt}\|_{L^2}^2.
\]
(4.40)
By (4.1), (4.9), (4.12) and (4.13), we conclude that
\[ |K_2| \leq C (\| \rho u_t \|_{L^3} + \| \rho_t u \|_{L^3}) (\| u_t \|_{L^6} \| \nabla u_t \|_{L^2} + \| \nabla u_t \|_{L^2} \| u_t \|_{L^6}) \]
\[ \leq C \left( \| \rho^{1/2} u_t \|_{L^2}^{1/2} \| u_t \|_{L^6}^{1/2} + \| \rho_t \|_{L^6} \| u_t \|_{L^6} \right) \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2} \]
\[ \leq \delta \| \nabla u_t \|_{L^2}^2 + C(\delta) \sigma^{-3/2}, \]
\[ |K_3| \leq C \left( \| \rho u_t \|_{L^2} \| u \|_{L^\infty} \| \nabla u \|_{L^3} + \| \rho_t \|_{L^6} \| u_t \|_{L^6} \| \nabla u \|_{L^2} \right) \| u_t \|_{L^6} \]
\[ \leq \delta \| \nabla u_t \|_{L^2}^2 + C(\delta) \| \rho_t \|_{L^2}^2 + C(\delta) \sigma^{-1}, \]
and
\[ |K_4| + |K_5| \leq C (\| \rho u_t \|_{L^2} \| u \|_{L^\infty} \| \nabla u \|_{L^3} + \| \rho_t \|_{L^6} \| u_t \|_{L^6} \| \nabla u_t \|_{L^2}^2 + C \| P_t \|_{L^2} \| \nabla u_t \|_{L^2}^2 \]
\[ \leq \delta \| \nabla u_t \|_{L^2}^2 + C(\delta) \| \rho^{1/2} u_t \|_{L^2}^2 + C(\delta) \| P_t \|_{L^2}^2. \]

Since \( u_t \cdot n = 0 \) on \( \partial \Omega \), Lemma 2.5 implies that there exists some positive constant \( \tilde{\mu} \) depending only on \( \Omega \) such that
\[ \tilde{\mu} \| \nabla u_t \|_{L^2} \leq \| \text{div} u_t \|_{L^2} + \| \text{curl} u_t \|_{L^2}. \]

Thus, substituting (4.40)–(4.43) into (4.39) and choosing \( \delta \) small enough, one has
\[ \frac{d}{dt} \| \rho^{1/2} u_t \|_{L^2}^2 + \mu \tilde{\mu} \| \nabla u_t \|_{L^2}^2 \leq C(\| \rho^{1/2} u_t \|_{L^2}^2 + \| \rho_t \|_{L^2}^2 + \| P_t \|_{L^2}^2) + C \sigma^{-3/2}, \]
which together with (4.12), (4.13), and Gronwall’s inequality yields that
\[ \sup_{0 \leq t \leq T} \sigma \| \rho^{1/2} u_t \|_{L^2}^2 + \int_0^T \sigma^2 \| \nabla u_t \|_{L^2}^2 dt \leq C. \]

Furthermore, it follows from (4.39) and (4.13) that
\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u_t \|_{H^1}^2 \leq C. \]

Finally, we deduce from (4.35), (4.32), (4.13), (4.27), (4.26), (4.45) and (4.46) that
\[ \sigma \| \nabla^2 u \|_{W^{1,1}} \leq C(\sigma + \sigma \| \nabla u_t \|_{L^2} + \sigma \| \nabla (\rho u_t) \|_{L^6} + \sigma \| \nabla^2 P \|_{L^3}) \]
\[ \leq C(\sigma + \sigma^{1/2} + \sigma \| \nabla u \|_{H^2}^2 + \sigma^{1/2} \sigma \| \nabla u_t \|_{H^1}^2)^{\frac{3(q-2)}{4q}} \]
\[ \leq C \sigma^{1/2} + C \sigma^{1/2} (\sigma^{-1})^{\frac{3(q-2)}{4q}} \]
\[ \leq C, \]
which together with (4.45) and (4.46) leads to (4.37) and completes the proof of Lemma 4.5.

5 Proofs of Theorems 1.1-1.3

With all the a priori estimates in Sections 3 and 4 at hand, we are going to prove the main results, Theorems 1.1-1.3.
5.1 Proof of Theorem 1.1

By Lemma 2.1, there exists a $T^*>0$ such that the system (1.1)-(1.5) has a unique classical solution $(\rho, u)$ on $\Omega \times (0, T^*)$. One may use the a priori estimates, Proposition 3.1 and Lemmas 4.3-4.5 to extend the classical solution $(\rho, u)$ globally in time.

First, by the definition of $A_1(T)$, $A_2(T)$ (see (3.2), (3.3)), the assumption of the initial data (1.10) and (3.63), one immediately checks that

$$A_1(0) + A_2(0) = 0, \quad 0 \leq \rho_0 \leq \hat{\rho}, \quad A_3(0) \leq C_0^{\delta_0},$$

(5.1)

Therefore, there exists a $T_1 \in (0, T^*)$ such that

$$0 \leq \rho_0 \leq 2\hat{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{\frac{1}{2}}, \quad A_3(\sigma(T)) \leq 2C_0^{\delta_0}$$

hold for $T = T_1$.

Next, we set

$$T^* \triangleq \sup \{ T | (5.1) \text{ holds} \}.$$ 

Then $T^* > T_1 > 0$. Hence, for any $0 < \tau < T \leq T^*$ with $T$ finite, it follows from Lemmas 4.3-4.5 that

$$\rho \in C([0, T]; W^{2,q}), \quad \nabla u_t \in C([\tau, T]; L^q), \quad \nabla u, \nabla^2 u \in C([\tau, T]; C(\Omega)),$$

where one has taken advantage of the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q), \quad \text{for any } q \in [1, 6).$$

This in particular yields

$$\rho^{1/2} u_t, \quad \rho^{1/2} \dot{u} \in C([\tau, T]; L^2).$$

Finally, we claim that

$$T^* = \infty.$$

Otherwise, $T^* < \infty$. Then by Proposition 3.1 it holds that

$$0 \leq \rho \leq \frac{7}{4} \hat{\rho}, \quad A_1(T^*) + A_2(T^*) \leq C_0^{\frac{1}{2}}, \quad A_3(\sigma(T^*)) \leq C_0^{\delta_0}.$$

It follows from Lemmas 4.3-4.5 and (5.2) that $(\rho(x, T^*), u(x, T^*))$ satisfy the initial data condition (1.9)-(1.11) except $u(\cdot, T^*) \in H^s$, where $g(x) \triangleq \rho^{1/2} u(x, T^*)$, $x \in \Omega$. Thus, Lemma 2.1 implies that there exists some $T^{**} > T^*$ such that (5.1) holds for $T = T^{**}$, which contradicts the definition of $T^*$.

By Lemmas 2.1 and 4.3-4.5 it indicates that $(\rho, u)$ is in fact the unique classical solution defined on $\Omega \times (0, T^*)$ for any $0 < T < T^* = \infty$. Moreover, Proposition 3.9 gives (1.15) and we finish the proof of Theorem 1.1. $\square$
5.2 Proof of Theorem [1.2]

For $T > 0$, we introduce the Lagrangian coordinates

$$\begin{align*}
\frac{\partial}{\partial \tau} X(\tau; t, x) &= u(X(\tau; t, x), \tau), \quad 0 \leq \tau \leq T \\
X(t; t, x) &= x, \quad 0 \leq t \leq T, \ x \in \tilde{\Omega}.
\end{align*}$$  \hspace{1cm} (5.3)

By virtue of (1.13), the transformation (5.3) is well-defined. Therefore, by (1.14), we get

$$\rho(x, t) = \rho_0(X(0; t, x)) \exp \left\{ - \int_0^t \text{div}(X(\tau; t, x), \tau) d\tau \right\}. $$  \hspace{1cm} (5.4)

Since $\rho_0(x_0) = 0$ for some point $x_0 \in \Omega$, for any $t > 0$, there is a point $x_0(t) \in \tilde{\Omega}$ such that $X(0; t, x_0(t)) = x_0$. Hence, by (5.4), $\rho(x_0(t), t) \equiv 0$ for any $t \geq 0$. As a result of Gagliardo-Nirenberg’s inequality [22], we get that for $r_1 \in (3, \infty)$,

$$\rho_0 \equiv \rho \leq \|\rho - \bar{\rho}\|_{C(\mathcal{B})} \leq C\|\rho - \bar{\rho}\|_{L^2}^{1-\theta_1}\|\nabla \rho\|_{L^{r_1}}^{\theta_1}$$

where $\theta_1 = 2(r_1 - 3)/(5r_1 - 6)$. Combining this with (1.15) gives (1.16) and completes the proof of Theorem 1.2.

5.3 Proof of Theorem [1.3]

Let $(\rho_0, u_0)$ be the initial data as in Theorem 1.3. We construct an approximation initial value $(\rho_0^\delta, u_0^\delta)$ satisfying (1.17) and for any $p \geq 1$,

$$\lim_{\delta \to 0} \left( \|\rho_0^\delta - \rho_0\|_{L^p} + \|u_0^\delta - u_0\|_{H^1} \right) = 0,$$

$$\rho_0^\delta \to \rho_0 \text{ in } W^* \text{ topology of } L^\infty \text{ as } \delta \to 0.$$

Following the proofs in Section 3, one can check that

$$\sup_{0 \leq t \leq T} \left( \|\rho^\delta\|_{L^\infty} + \|\nabla u^\delta\|_{L^2}^2 + \sigma^3 \|\rho^\delta u_t^\delta\|_{L^2}^2 \right)$$

$$+ \int_0^T \sigma \|\rho^\delta u_t^\delta\|_{L^2}^2 dt + \int_0^T \sigma^3 \|\nabla u_t^\delta\|_{L^2}^2 dt \leq C,$$

and then, by Lemma 2.9

$$\sup_{0 \leq t \leq T} \left( \sigma^3 \|\nabla u^\delta\|_{L^6}^2 \right) + \int_0^T \sigma^3 \|u_t^\delta\|_{L^6}^2 dt \leq C,$$

$$\sup_{0 \leq t \leq T} \left( \sigma^3 ||\text{curl} u^\delta||_{H^1} + \sigma^3 ||F^\delta||_{H^1} \right)$$

$$+ \int_0^T \sigma^3 (||\text{curl} u^\delta||_{W^{1,6}}^2 + ||F^\delta||_{W^{1,6}}^2) dt \leq C,$$

where $F^\delta \triangleq (\lambda + 2\mu) \text{div} u^\delta - P(\rho^\delta) + P(\bar{\rho})$. By virtue of Aubin-Lions Lemma, there exists a subsequence that

$$u^\delta \to u \text{ weakly } * \text{ in } L^\infty(0, T; H^1),$$

$$u^\delta \to u \text{ in } C([\tau, T]; L^6),$$

$$F^\delta \to F \text{ weakly } * \text{ in } L^2(0, T; W^{1,6}),$$

$$\omega^\delta \to \omega, \ F^\delta \to F \text{ in } L^2(\tau, T; L^6),$$


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for any \( \tau \in (0, T) \). By standard arguments (see [13] or [26]), one can deduce the strong convergence of \( \rho^\delta \), that is,

\[
\rho^\delta \to \rho \quad \text{in} \quad C([0, T]; L^q(\Omega)),
\]

for any \( q \in [1, \infty) \). Moreover, (1.18) is established directly. Therefore, we conclude that \((\rho, u)\) is a weak solution as in Theorem 1.3 which finishes our proof of Theorem 1.3.

\[\Box\]

### 6 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Compared with the previous theorems, since the domain is not necessarily simply connected, the inequality (2.7) with \( k = 0 \) and \( p = 2 \) is no longer valid. Therefore, the difficulty of this proof is that we need an alternative inequality. Thanks to [1, Proposition 3.7], the following lemma gives equivalent norms of \( H^1 \).

**Lemma 6.1** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Then for \( v \in H^1 \) with \( v \cdot n = 0 \) on \( \partial \Omega \), we have the following equivalence of norms:

\[
\|v\|_{H^1} \simeq \begin{cases} \\
\|D(v)\|_{L^2}, & \Omega \text{ is not axially symmetric}, \\
\|D(v)\|_{L^2} + \int_{\partial \Omega} u \cdot B \cdot vds, & \Omega \text{ is axially symmetric},
\end{cases}
\]

(6.1)

where \( \simeq \) denotes the equivalence of two norms and \( D(v) = (\nabla v + (\nabla v)^{tr})/2 \) and \( B \in W^{2,6}(\Omega) \) is a positive semi-definite \( 3 \times 3 \) symmetric matrix satisfying \( B > 0 \) on some \( \Sigma \subset \partial \Omega \) with \( |\Sigma| > 0 \).

**Remark 6.1** Similar to what have done in [1], when \( \Omega \) is axially symmetric with respect to a constant vector \( b \in \mathbb{R}^3 \), we have to add the term \( \int_{\partial \Omega} u \cdot B \cdot vds \) in order to exclude such a special case that \( v = C b \times x \).

The following lemma enables us to replace the inequality (2.7) (\( k = 0 \) and \( p = 2 \)) with (6.1) so that the previous proofs are still available.

**Lemma 6.2** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^3 \). Then for \( v \in H^2 \) with \( v \cdot n = 0 \) on \( \partial \Omega \), it holds that

\[
2 \int D(v) \cdot D(v)dx = 2 \int (\text{div} v)^2dx + \int |\text{curl} v|^2dx - 2 \int_{\partial \Omega} v \cdot D(n) \cdot vds.
\]

(6.2)

**Proof.** Observe that

\[
\Delta v = \nabla \text{div} v - \nabla \times \text{curl} v = 2\text{div}(D(v)) - \nabla \text{div} v,
\]

which together with (1.19) gives (6.2).

**Proof of Theorem 1.4** First, it is sufficient to find out where the inequality (2.7) with \( k = 0 \) and \( p = 2 \) is used, that is, (3.11), (3.29), (3.42), (3.54), (3.58), the proof of Proposition 3.9 and (4.24).

Then, we will take advantage of the above two lemmas to deduce the similar results step by step.
Indeed, setting \( B = A + 2D(n) \), by Lemma 6.1 and the extra assumptions of Theorem 1.4, for any \( v \in H^1 \) with \( v \cdot n = 0 \) on \( \partial \Omega \), we have

\[
\| \nabla v \|^2_{L^2} \leq C \left( 2\mu \| D(v) \|^2_{L^2} + \lambda \| \text{div} v \|^2_{L^2} + \int_{\partial \Omega} v \cdot B \cdot v ds \right). \tag{6.3}
\]

For (3.11), since \( A = B - 2D(n) \), by Lemma 6.2, \( \phi \) can be rewritten as

\[
\phi = 2\mu \| D(u) \|^2_{L^2} + \lambda \| \text{div} u \|^2_{L^2} + \int_{\partial \Omega} u \cdot B \cdot u ds, \tag{6.4}
\]

which, together with (6.3) gives (3.7).

Next, (3.29), (3.54), (3.58) and (4.24) can be similarly dealt with to get the results of their next step respectively, and the proof of Proposition 3.9 remains valid if we use (6.3) instead of (3.1).

Finally, it remains to handle (3.42). Setting \( v = \dot{u} + (u \cdot \nabla n) \times u^\bot \) in (6.2), by (2.32) and Young’s inequality, we deduce from (3.43) that

\[
\left( \frac{\sigma^m}{2} \| \rho^{1/2} \dot{u} \|^2_{L^2} \right)_t + 2\mu \sigma^m \| D(v) \|^2_{L^2} + \lambda \sigma^m \| \text{div} v \|^2_{L^2} + \mu \sigma^m \int_{\partial \Omega} v \cdot B \cdot v ds
\]

\[
\leq C m \sigma^{m-1} \sigma' \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| \nabla u \|^4_{L^2} \right)
\]

\[
+ 2\delta \sigma^m \| \nabla \dot{u} \|^2_{L^2} + C \sigma^m \| \rho^{1/2} \dot{u} \|^2_{L^2} (\| \nabla u \|^4_{L^2} + 1)
\]

\[
+ C(\delta) \sigma^m (\| \nabla u \|^2_{L^2} + \| \nabla u \|^6_{L^2} + \| \nabla u \|^4_{L^4}). \tag{6.5}
\]

On the other hand, by (6.3), we find that

\[
\| \nabla \dot{u} \|^2_{L^2} \leq C \left( 2\mu \| D(v) \|^2_{L^2} + \lambda \| \text{div} v \|^2_{L^2} + \int_{\partial \Omega} v \cdot B \cdot v ds \right)
\]

\[
+ C(\| \nabla u \|^2_{L^2} + \| \nabla u \|^4_{L^4}).
\]

Combining this with (6.5), we get (3.43) by letting \( \delta \) is suitably small and thus finish the proof of Theorem 1.4.

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