Structures of not-finitely graded Lie algebras related to generalized Heisenberg-Virasoro algebras

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Abstract: In this paper, we study the structure theory of a class of not-finitely graded Lie algebras related to generalized Heisenberg-Virasoro algebras. In particular, the derivation algebras, the automorphism groups and the second cohomology groups of these Lie algebras are determined.

Key words: not-finitely graded Lie algebras, generalized Heisenberg-Virasoro algebras, derivations, automorphisms, 2-cocycles.

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1 Introduction

The Heisenberg-Virasoro algebra contains the classical Heisenberg algebra and Virasoro algebra as subalgebras. As the universal central extension of the Lie algebra of differential operators on a circle of order at most one, the Heisenberg-Virasoro algebra is an important object in mathematics and physics, whose theory has been widely studied in the mathematical and physical literatures. For example, the twisted Heisenberg-Virasoro algebra has been first studied by E. Arbarello et al. in [1]. Various generalizations of the Heisenberg-Virasoro algebra have been extensively studied by several authors (e.g., [2, 8–10]). However, it seems to us that little has been known on not-finitely graded aspect of generalized Heisenberg-Virasoro algebras.

In recent years, some researches on Lie algebras concerning their derivation algebras, automorphisms, second cohomology groups have been undertaken by many authors (see, e.g., [3–5, 11–17]). It is well known that central extensions, which are determined by second cohomology groups, are closely related to the structures of Lie algebras (see, e.g., [6, 7]). The computation of the second cohomology groups seems to be important and interesting. Not-finitely graded Lie algebras are important objects in Lie theory, whose structure and representation theories are subjects of studies with more challenge than that of finitely graded Lie algebras. In [3], the authors have studied the structure theory of a class of not-finitely graded Lie algebras related to generalized Virasoro algebras.

In this paper, we consider the following Lie algebras, which are referred to as generalized Heisenberg-Virasoro algebra $HV(\Gamma)$: Let $\Gamma$ be any nontrivial additive subgroup of $\mathbb{C}$, and $\mathbb{C} [\Gamma \times \mathbb{Z}_+]$ the semigroup algebra of $\Gamma \times \mathbb{Z}_+$ with basis $\{x^{\alpha,i} := x^{\alpha ti} | \alpha \in \Gamma, i \in \mathbb{Z}_+\}$ and product $x^{\alpha,i} x^{\beta,j} = x^{\alpha+\beta,i+j}$. Let $\partial_x, \partial_t$ be the derivations of $\mathbb{C}[\Gamma \times \mathbb{Z}_+]$ defined by $\partial_x(x^{\alpha,i}) = \alpha x^{\alpha,i}$, $\partial_t(x^{\alpha,i}) = ix^{\alpha,i-1}$ for $\alpha \in \Gamma, i \in \mathbb{Z}_+$. Denote $\partial = \partial_x + \partial_t$. Then $HV(\Gamma)$ is

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the Lie algebra with the underlining space $\mathbb{C}[\Gamma \times \mathbb{Z}_+]\partial \oplus \mathbb{C}[\Gamma \times \mathbb{Z}_+]$ and basis $\{L_{\alpha,i} := x^{\alpha,i} \partial, H_{\beta,j} := x^{\beta,j} | \alpha, \beta \in \Gamma, i, j \in \mathbb{Z}_+\}$ and relations

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta - \alpha)L_{\alpha+\beta,i+j} + (j - i)L_{\alpha+\beta,i+j-1},$$  \hspace{1cm} (1.1)  

$$[L_{\alpha,i}, H_{\beta,j}] = \beta H_{\alpha+\beta,i+j} + jH_{\alpha+\beta,i+j-1},$$  \hspace{1cm} (1.2)  

$$[H_{\alpha,i}, H_{\beta,j}] = 0.$$  \hspace{1cm} (1.3)  

Let $W$ be $\mathbb{C}[\Gamma \times \mathbb{Z}_+]\partial$ with basis $\{L_{\alpha,i} := x^{\alpha,i} \partial | \alpha \in \Gamma, i \in \mathbb{Z}_+\}$ and relation (1.1), and $H$ be $\mathbb{C}[\Gamma \times \mathbb{Z}_+]$ with basis $\{H_{\beta,j} := x^{\beta,j} | \beta \in \Gamma, j \in \mathbb{Z}_+\}$ and relation (1.3). We simply denote $HV = HV(\Gamma)$, then $W$ and $H$ are subalgebras of $HV$. The algebra $W$ is the generalized Witt algebra $W = W(0,1,0;\Gamma)$ of Witt type studied in [13]. Similar to Lie algebras studied in [3], the Lie algebra $HV$ has the following significant features:

1. It has a \textit{finitely graded filtration} in the sense that there exists a filtration $0 \subset HV^{(0)} \subset HV^{(1)} \subset \cdots$, satisfying $[HV^{(i)}, HV^{(j)}] \subseteq HV^{(i+j)}$ for all $i, j \in \mathbb{Z}_+$ and each $HV^{(i)}$ is finitely graded, i.e., there exists some abelian group $G$ which is independent of $i$ (one can simply choose $G = \Gamma$ in this case) such that $HV^{(i)} = \oplus_{g \in G} HV^{(i)}_g$, $[HV^{(i)}_g, HV^{(j)}_h] \subseteq HV^{(i+j)}_{g+h}$, $\dim HV^{(i)}_g < \infty$ for all $g, h \in G, i, j \in \mathbb{Z}_+$ (cf. (2.1)) and $HV^{(0)}_0 \neq 0$.

2. It has the set $\mathfrak{F}$ of ad-locally finite elements, where

$$\mathfrak{F} = \{a L_{\alpha,0} + \sum_{\alpha,j} b_{\alpha,j} H_{\alpha,j} | a, b_{\alpha,j} \in \mathbb{C}, \alpha \in \Gamma, j \in \mathbb{Z}_+\}.$$  \hspace{1cm} (1.4)  

3. $HV$ has exactly two ideals: $\mathbb{C}H_{0,0}$ and $H$. In particular, $H$ is the unique maximal ideal of $HV$ (cf. Theorem 2.1).

In this paper we shall mainly study the structure theory of $HV$ (namely, derivations, automorphisms, 2-cocycles). The Lie algebra $HV$ is $\Gamma$-graded

$$HV = \oplus_{\alpha \in \Gamma} HV_{\alpha}, \quad HV_{\alpha} = \text{span}\{L_{\alpha,i}, H_{\alpha,i} | i \in \mathbb{Z}_+\} \text{ for } \alpha \in \Gamma.$$  \hspace{1cm} (1.5)  

However, it is not finitely graded. Nevertheless, as stated in [3], due to the fact that $\Gamma$ may not be finitely generated (as a group), and so $HV$ may not be finitely generated as a Lie algebra, the classical techniques (such as those in [5]) cannot be directly applied to our situation here. One must employ some new techniques in order to tackle problems associated with not-finitely graded and not-finitely generated Lie algebras (this is also one of our motivations to present our results here). For instance, one of our strategies used in the present paper is to study the actions on $L_{\alpha,0}, L_{0,i}$ respectively so that the determination of derivation algebra and automorphism group can be done much more efficiently. The main results of the present paper are summarized in Theorems 2.1, 3.1, 4.1, 4.2 and 5.1.

Throughout the paper, we denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}, \mathbb{Z}_+, \Gamma^*$ the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers, nonzero elements of $\Gamma$ respectively.
2 Some properties of HV

We first study some properties of the Lie algebra HV, which will be summarized in Theorem 2.1. Now we recall some concepts.

A Lie algebra \( L \) is finitely graded if there exists an abelian group \( G \) such that \( L = \bigoplus_{a \in G} L_a \) is \( G \)-graded satisfying

\[
[L_a, L_b] \subset L_{a+b} \quad \text{and} \quad \dim L_a < \infty \quad \text{for} \quad a, b \in G.
\]  

(2.1)

An element \( x \in L \) is ad-locally finite if for any \( y \in L \) the subspace span\{ad\_x (y)| i \in \mathbb{Z}_+ \} is finite-dimensional, where \( \text{ad}_x : y \mapsto [x, y] \quad (y \in L) \) is the adjoint operator of \( x \).

Theorem 2.1. (1) The Lie algebra HV is not finitely graded.

(2) The Lie algebra HV has the set \( \mathcal{F} \) of ad-locally finite elements defined in (1.4).

(3) HV has exactly two ideals: \( \mathbb{C}H_{0,0} \) and \( H \). In particular, \( H \) is the unique maximal ideal of HV.

Proof. (1) It can be obtained that by [3].

(2) It can be obtained easily.

(3) Let \( H' \) be a proper ideal of HV which satisfies \( \mathbb{C}H_{0,0} \neq H' \), then there must exist a nonzero element \( \sum_{\gamma,s} a_{\gamma,s} H_{\gamma,s} + \sum_{\beta,j} c_{\beta,j} L_{\beta,j} \) in \( H' \). Let any \( L_{\alpha,i} \) and \( H_{\alpha,i} \) act on it respectively, we could get

\[
[L_{\alpha,i}, \sum_{\gamma,s} a_{\gamma,s} H_{\gamma,s} + \sum_{\beta,j} c_{\beta,j} L_{\beta,j}] = \sum_{\gamma,s} a_{\gamma,s}(\gamma H_{\alpha+i,s} + sH_{\alpha+i+s-1})
\]

\[
+ \sum_{\beta,j} c_{\beta,j}((\beta - \alpha)L_{\alpha+i+j} + (j-i)L_{\alpha+i+j-1})
\]  

(2.2)

and

\[
[H_{\alpha,i}, \sum_{\gamma,s} a_{\gamma,s} H_{\gamma,s} + \sum_{\beta,j} c_{\beta,j} L_{\beta,j}] = -\sum_{\beta,j} c_{\beta,j}(\alpha H_{\alpha+i+j} + iH_{\alpha+i+j-1}).
\]  

(2.3)

(i) If all the \( c_{\beta,j} \) are equal to zero, there must exist some \( a_{\gamma,s} \neq 0 \) with \( (\gamma, s) \neq (0,0) \). From (2.2), we have \( H = H' \).

(ii) If all the \( a_{\gamma,s} \) are equal to zero, there must exist some \( c_{\beta,j} \neq 0 \). Then from (2.2), we have \( W \subseteq H' \). From (2.3), we have \( H \subseteq H' \). Thus we get \( HV = H' \). It is a contradiction.

(iii) If there exist some \( a_{\gamma,s}, c_{\beta,j} \neq 0 \), from (2.2) and (2.3) we can get \( H' = HV \). It is also a contradiction.

From the above discussion, HV has only two ideals: \( \mathbb{C}H_{0,0} \) and \( H \). Due to \( \mathbb{C}H_{0,0} \not\subseteq H \), we can get that \( H \) is the unique maximal ideal of HV. \( \square \)
3 Derivation algebra

Recall that a linear map $D : HV \to HV$ is a derivation of $HV$ if $D([x,y]) = [D(x),y] + [x,D(y)]$ for any $x,y \in HV$. For any $z \in HV$, the adjoint operator $\text{ad}_z : HV \to HV$ is a derivation, called an inner derivation. Denote by $\text{Der} HV$ and $\text{ad} HV$ the vector spaces of all derivations and inner derivations respectively. Then the first cohomology group $H^1(HV, HV) \cong \text{Der} HV / \text{ad} HV$. We say that a derivation $D \in \text{Der} HV$ is of degree $\gamma$, if it satisfies that $D(HV \alpha) \subset HV \alpha_{\gamma}$. Denote by $(\text{Der} HV)_\gamma$ the space of all derivations of degree $\gamma$.

Let $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$ denote the space of group homomorphisms from $\Gamma$ to (the additive group) $\mathbb{C}$ (for each $\phi \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$, the scalar multiplication $\phi$ by $c \in \mathbb{C}$ is defined by $(c\phi)(\gamma) = c\phi(\gamma)$, thus $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$ is a vector space). For each $\phi \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$, we can define a derivation $D_\phi$ as follows,

$$D_\phi(L_{\alpha,i}) = \phi(\alpha)L_{\alpha,i}, \quad D_\phi(H_{\alpha,i}) = \phi(\alpha)H_{\alpha,i} \quad \text{for} \quad \alpha \in \Gamma, \ i \in \mathbb{Z}_+.$$  \hspace{1cm} (3.1)

We still use $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$ to denote the corresponding subspace of $\text{Der} HV$. In particular, since $\phi_0 : \alpha \mapsto \alpha$ is in $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$, we have the derivation

$$D_0 = D_{\phi_0} : L_{\alpha,i} \mapsto \alpha L_{\alpha,i}, \ H_{\alpha,i} \mapsto \alpha H_{\alpha,i} \quad \text{for} \quad \alpha \in \Gamma, \ i \in \mathbb{Z}_+.$$  \hspace{1cm} (3.2)

Now we define three derivations of degree 0, which are obviously not inner derivations of $HV$.

$$\begin{align*}
D_1(L_{\alpha,i}) &= iH_{\alpha,i-1}, \quad D_1(H_{\alpha,i}) = 0, \\
D_2(L_{\alpha,i}) &= \alpha H_{\alpha,i-1}, \quad D_2(H_{\alpha,i}) = 0, \\
D_3(L_{\alpha,i}) &= 0, \quad D_3(H_{\alpha,i}) = H_{\alpha,i} \quad \text{for any} \quad \alpha \in \Gamma, \ i \in \mathbb{Z}_+.
\end{align*}$$  \hspace{1cm} (3.3)

**Theorem 3.1.** The derivation space of $HV$ can be written as

$$\text{Der} HV = \bigoplus_{\alpha \in \Gamma} (\text{Der} HV)_\alpha = \text{ad} HV \oplus \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) \oplus CD_1 \oplus CD_2 \oplus CD_3,$$  \hspace{1cm} (3.4)

where $(\text{Der} HV)_\gamma \subset \text{ad} HV$, if $\gamma \neq 0$, and $(\text{Der} HV)_0 = (\text{ad} HV)_0 \oplus \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) \oplus CD_1 \oplus CD_2 \oplus CD_3$.

**Proof.** It can be obtained by the following claims 1, 2, 3 and lemmas 3.1, 3.2 immediately.

**Claim 1.** For any $D \in \text{Der} HV$, replacing $D - \text{ad}_u$ by $D$ for some $u \in HV$, we can suppose

$$D(L_{0,0}) = 0.$$  \hspace{1cm} (3.5)

**Proof.** Suppose $D(L_{0,0}) = \sum_{a,j}(a_{\alpha,j} L_{\alpha,j} + c_{\alpha,j} H_{\alpha,j}) \in HV$ for some $a_{\alpha,j}$, $c_{\alpha,j} \in \mathbb{C}$. For any $\alpha \in \Gamma$, we define $b_{\alpha,j}$, $d_{\alpha,j} \in \mathbb{C}$ inductively on $j \geq 0$ by

$$b_{\alpha,j} = \begin{cases} j^{-1}(-a_{\alpha,j-1} - ab_{\alpha,j-1}) & \text{if} \ j \neq 0, \\ 0 & \text{if} \ j = 0. \end{cases}$$  \hspace{1cm} (3.6)
and
\[d_{\alpha,j} = \begin{cases} 
  j^{-1}(-c_{\alpha,j-1} - \alpha d_{\alpha,j-1}) & \text{if } j \neq 0, \\
  0 & \text{if } j = 0. 
\end{cases} \tag{3.7}\]

Take \(u = \sum_{\alpha,j} (b_{\alpha,j} L_{\alpha,j} + d_{\alpha,j} H_{\alpha,j})\). Note that \(u \in HV\), by (3.6) and (3.7), it gives
\[
D(L_{0,0}) - \text{ad}_u(L_{0,0}) = \sum_{\alpha,j} (a_{\alpha,j} L_{\alpha,j} + c_{\alpha,j} H_{\alpha,j}) - \sum_{\alpha,j} (-\alpha b_{\alpha,j} - (j + 1)b_{\alpha,j+1}) L_{\alpha,j} \\
- \sum_{\alpha,j} (-\alpha c_{\alpha,j} - (j + 1)c_{\alpha,j+1}) H_{\alpha,j} \\
= 0.
\]

\(\square\)

For any \(D \in \text{Der HV}\) and \(\gamma \in \Gamma\), define the homogeneous operator \(D_\gamma\) of degree \(\gamma\) in the following way
\[
D(\sum_{\alpha} u_{\alpha}) = \sum_{\alpha} \pi_{\alpha+\gamma} D_\gamma(u_{\alpha}),
\]
where \(u_{\alpha} \in HV_\alpha\) and \(\pi_{\alpha} : HV \rightarrow HV_\alpha\) is the natural projection. Then \(D = \sum_{\gamma \in \Gamma} D_\gamma\)(may be infinite) and \(D_\gamma \in \text{Der HV}\).

**Lemma 3.2.** Every derivation \(D \in HV\) can be written as
\[
D = \sum_{\gamma \in \Gamma} D_\gamma, \quad D_\gamma \in (\text{Der HV})_\gamma \tag{3.8}
\]
such that only finitely many \(D_\gamma(x) \neq 0\) for every \(x \in HV\) (such a sum in (3.8) is called summable).

**Proof.** For a derivation \(D \in \text{Der HV}\) and an element \(x_{\alpha} \in HV_\alpha\), assume that
\[
D(x_{\alpha}) = \sum_{\beta \in \Gamma} y_{\beta}.
\]
we define \(D_\gamma(x_{\alpha}) = y_{\alpha+\gamma}\). Then a direct computation shows that \(D_\gamma\) is a derivation. \(\square\)

**Claim 2.** If \(0 \neq \gamma \in \Gamma\), \(D \in (\text{Der HV})_\gamma\) and \(D(L_{0,0}) = 0\), then
\[
D = 0. \tag{3.9}
\]

**Proof.** In order to prove this claim, we need to prove the following two facts:

(i) \(D(L_{\alpha,0}) = 0, D(L_{0,1}) = 0\) for any \(\alpha \in \Gamma\).
(ii) \(D(H_{\alpha,0}) = 0\), for any \(\alpha \in \Gamma\).
For (i), applying $D$ to $[L_{0,0}, L_{a,0}] = \alpha L_{a,0}$ with $\alpha \neq 0$, we have

$$[L_{0,0}, D(L_{a,0})] = \alpha D(L_{a,0}). \tag{3.10}$$

Assume $D(L_{a,0}) = \sum_{j \in \mathbb{Z}^+} (b_j L_{\alpha+\gamma,j} + d_j H_{\alpha+\gamma,j})$, and from (3.10) we have

$$\begin{cases} 
\gamma b_j = -(j + 1)b_{j+1} \quad \text{for } j \geq 0, \\
\gamma d_j = -(j + 1)d_{j+1} \quad \text{for } j \geq 0.
\end{cases} \tag{3.11}$$

As $b_j = 0, j \gg 0$, all $b_j = 0$ for all $j \geq 0$. Similarly, all $d_j = 0$ for all $j \geq 0$. Thus we have $D(L_{a,0}) = 0$, for all $\alpha \in \Gamma$.

Applying $D$ to $[L_{0,0}, L_{0,1}] = L_{0,0}$, we have

$$[L_{0,0}, D(L_{0,1})] = D(L_{0,0}). \tag{3.12}$$

Assume $D(L_{0,j}) = \sum_k f(j,k)L_{\gamma,k} + g(j,k)H_{\gamma,k}$, for $f(j,k), g(j,k) \in \mathbb{C}$, and note that $f(0,k) = g(0,k) = 0$. From (3.12), when $j = 1$, we have

$$\begin{cases} 
\gamma f(1,k) = -(k + 1)f(1,k+1) \quad \text{for } k \geq 0, \\
\gamma g(1,k) = -(k + 1)g(1,k+1) \quad \text{for } k \geq 0.
\end{cases} \tag{3.13}$$

Then we can get $f(1,k) = g(1,k) = 0$ for all the $k \in \mathbb{Z}^+$, and we have $D(L_{0,1}) = 0$.

For (ii), applying $D$ to $[L_{0,0}, H_{a,0}] = \alpha H_{a,0}$, we have

$$[L_{0,0}, D(H_{a,0})] = \alpha D(H_{a,0}). \tag{3.14}$$

Assume $D(H_{a,0}) = \sum_{j \in \mathbb{Z}^+} (e_j L_{\alpha+\gamma,j} + f_j H_{\alpha+\gamma,j})$, and from (3.14) we have

$$\begin{cases} 
\gamma e_j = -(j + 1)e_{j+1} \quad \text{for } j \geq 0, \\
\gamma f_j = -(j + 1)f_{j+1} \quad \text{for } j \geq 0.
\end{cases} \tag{3.15}$$

As $e_j = 0, j \gg 0$, all $e_j = 0$ for all $j \geq 0$. Similarly, all $f_j = 0$ for all $j \geq 0$. Thus we have $D(H_{a,0}) = 0$, for all $\alpha \in \Gamma$.

Since $HV$ is generated by $\{L_{a,0}, L_{0,1}, H_{a,0} \mid \alpha \in \Gamma\}$ from the relation (1.1), we can get $D = 0$ for any $D \in (\text{Der } HV)_{\gamma}, \gamma \in \Gamma, \gamma \neq 0$ eventually. \hfill \Box

**Claim 3.** Assume $D \in (\text{Der } HV)_0$ and $D(L_{0,0}) = 0$, then we have two main results:

(i) $D(L_{a,0}) = b_\alpha L_{a,0} + d_\alpha H_{a,0}$, for any $\alpha \in \Gamma$, with $b_\alpha \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$, $d_\alpha = \alpha d_1$, $d_1 \in \mathbb{C}$.

(ii) $D(H_{a,0}) = f_\alpha H_{a,0}$, for any $\alpha \in \Gamma$, with $f_\alpha = b_\alpha + f_0$, $f_0 \in \mathbb{C}$. 

6
Proof. For (i), assume $D(L_{a,0}) = \sum_{\alpha,j}(b_{\alpha,j}L_{\alpha,j} + d_{\alpha,j}H_{\alpha,j})$, and from (3.10) we have

\[
\begin{cases}
  b_{\alpha,j} = 0 & \text{for } j \geq 1, \\
  d_{\alpha,j} = 0 & \text{for } j \geq 1.
\end{cases}
\] (3.16)

Simply denote $b_{\alpha,0}, d_{\alpha,0}$ by $b_{\alpha}, d_{\alpha}$ respectively, we have $D(L_{a,0}) = b_{\alpha}L_{a,0} + d_{\alpha}H_{a,0}$, for all $\alpha \in \Gamma$. Applying $D$ to $[L_{a,0}, L_{\beta,0}] = (\beta - \alpha)L_{a+\beta,0}$, we can get that

\[
\begin{cases}
  b_{\alpha} + \beta = b_{\alpha+\beta} & \text{for any } \alpha \neq \beta, \\
  \beta d_{\alpha} = \alpha d_{\alpha} = (\beta - \alpha)d_{\alpha+\beta} & \text{for any } \alpha, \beta \in \Gamma.
\end{cases}
\] (3.17)

As $D(L_{0,0}) = b_{0}L_{0,0} = 0$, we have $b_{0} = 0$. According to (3.17), we also have $-b_{\alpha} = b_{-\alpha}$. Set $\alpha = \beta$ in (3.17), we can obtain that $b_{2\alpha} = b_{(\alpha+\eta)+(\alpha-\eta)} = b_{\alpha+\eta} + b_{\alpha-\eta} = b_{\alpha} + b_{\eta} + b_{\eta} = 2b_{\alpha}$, for any $\eta \in \Gamma \setminus \{\pm \alpha, 0\}$. Thus (3.17) holds for all $\alpha, \beta \in \Gamma$, which shows that the map $\Phi : \alpha \mapsto b_{\alpha}$ is an element in Hom$_{Z}(\Gamma, \mathbb{C})$. Hence we can get that $b_{\alpha} \in$ Hom$_{Z}(\Gamma, \mathbb{C})$.

Suppose that $\Gamma' = c\Gamma$ for some $c \in \mathbb{C}^{*}$, we have $HV(\Gamma) \cong HV(\Gamma')$. Thus without loss of generality, we can always suppose $1 \in \Gamma$. Then we can obtain that $d_{\alpha} = (d_{1} - d_{0})\alpha + d_{0}$, for $d_{1} \in \mathbb{C}$. Due to $d_{0} = 0$, we have $d_{\alpha} = \alpha d_{1}$, for $d_{1} \in \mathbb{C}$.

For (ii), assume $D(H_{0,0}) = \sum_{j \in \mathbb{Z}_{+}}(e_{j}L_{0,j} + f_{j}H_{0,j})$, and apply $D$ to $[L_{0,0}, H_{0,0}] = 0$. Then we have $e_{j} = f_{j} = 0$, for any $j \geq 1$, and we get $D(H_{0,0}) = e_{0}L_{0,0} + f_{0}H_{0,0}$. Applying $D$ to $[L_{0,1}, H_{0,0}] = 0$, then we have $e_{0} = 0$. Thus we have $D(H_{0,0}) = f_{0}H_{0,0}$. Assume $D(H_{a,0}) = \sum_{\alpha \in \Gamma, j \in \mathbb{Z}_{+}}(e_{\alpha,j}L_{\alpha,j} + f_{\alpha,j}H_{\alpha,j})$, and from (3.14), we have

\[
\begin{cases}
  e_{\alpha,j} = 0 & \text{for } j \geq 1, \\
  f_{\alpha,j} = 0 & \text{for } j \geq 1.
\end{cases}
\] (3.18)

So we have $D(H_{a,0}) = e_{a,0}L_{0,0} + f_{a,0}H_{0,0} := e_{a}L_{a,0} + f_{a}H_{a,0}$. Applying $D$ to $[L_{-\alpha,0}, H_{a,0}] = \alpha H_{0,0}$, then we have $e_{\alpha} = 0$. Thus we can assume $D(H_{a,0}) = f_{\alpha}H_{a,0}$. Now applying $D$ to $[L_{a,0}, H_{\beta,0}] = \beta H_{a+\beta,0}$, we have $b_{\alpha} + \beta = f_{a+\beta}$. So we have $b_{\alpha} + f_{0} = f_{\alpha}$, with $f_{0} \in \mathbb{C}$ for any $\alpha \in \Gamma$.

If we replace $D'$ by $D - D_{0}(\text{cf. (3.1)})$(note that this replacement does not affect (3.5)), we can suppose $b_{a} = 0$ for all $\alpha \in \Gamma$. Then we have $D'(L_{a,0}) = \alpha d_{1}H_{a,0}$ and $D'(H_{a,0}) = f_{0}H_{a,0}$.

Assume $D'(L_{a,1}) = \sum_{j}(f(1, j)L_{a,j} + g(1, j)H_{a,j})$, for $f(1, j), g(1, j) \in \mathbb{C}$. Applying $D'$ to $[L_{0,0}, L_{a,1}] = \alpha L_{a,1} + L_{a,0}$, we get $[L_{0,0}, D'(L_{a,1})] = \alpha D'(L_{a,1}) + \alpha d_{1}H_{a,0}$. By computation, we can get $f(1, j) = g(1, j) = 0$ for $j \geq 1$. So we have $D'(L_{a,1}) = f(1, 0)L_{a,0} + g(1, 0)H_{a,0} + \alpha d_{1}H_{a,1}$. By induction on $i$, we have $D'(L_{a,i}) = if(1, 0)L_{a,i-1} + ig(1, 0)H_{a,i-1} + \alpha d_{1}H_{a,i}$.

Applying $D'$ to $[L_{-\alpha,i}, H_{a,0}] = \alpha H_{a,i}$ for $\alpha \neq 0$, we have $D'(H_{0,i}) = if(1, 0)H_{0,i-1} + f_{0}H_{0,i}$.

Applying $D'$ to $[L_{0,i}, H_{a,0}] = \alpha H_{a,i}$ for $\alpha \neq 0$, we obtain $D'(H_{a,i}) = if(1, 0)H_{a,i-1} + f_{0}H_{a,i}$.
with $\alpha \neq 0$. So we can get $D'(H_{\alpha,i}) = if(1,0)H_{\alpha,i-1} + f_0H_{\alpha,i}$ for $\alpha \in \Gamma$, $i \in \mathbb{Z}_+$. And it follows that

$$
\begin{cases}
D'(L_{\alpha,i}) = if(1,0)L_{\alpha,i-1} + ig(1,0)H_{\alpha,i-1} + \alpha d_1H_{\alpha,i}, \\
D'(H_{\alpha,i}) = if(1,0)H_{\alpha,i-1} + f_0H_{\alpha,i},
\end{cases} 
(3.19)
$$

for any $\alpha \in \Gamma$, $i \in \mathbb{Z}_+$. Subtract $D'$ by a derivation in ad $HV$ and a derivation in $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$, and still denote it by $D'$, then we can get

$$
\begin{cases}
D'(L_{\alpha,i}) = ig(1,0)H_{\alpha,i-1} + \alpha d_1H_{\alpha,i}, \\
D'(H_{\alpha,i}) = f_0H_{\alpha,i},
\end{cases} 
(3.20)
$$

with $g(1,0)$, $f_0$, $d_1 \in \mathbb{C}$, for any $\alpha \in \Gamma$, $i \in \mathbb{Z}_+$.

**Lemma 3.3.** Under the above notations, we have a decomposition for $(\text{Der} HV)_0$:

$$(\text{Der} HV)_0 = (\text{ad} HV)_0 \oplus \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2 \oplus \mathbb{C}D_3,$$  
(3.21)

where $D_1(L_{\alpha,i}) = iH_{\alpha,i-1}$, $D_1(H_{\alpha,i}) = 0$, $D_2(L_{\alpha,i}) = \alpha H_{\alpha,i}$, $D_2(H_{\alpha,i}) = 0$, $D_3(L_{\alpha,i}) = 0$, $D_3(H_{\alpha,i}) = H_{\alpha,i}$, for any $\alpha \in \Gamma$, $i \in \mathbb{Z}_+$.

### 4 Automorphism group of HV

Firstly, we study the automorphism group of $W$. Denote by $\text{Aut} W$ the automorphism group of $W$. Let $\chi(\Gamma)$ be the set of characters of $\Gamma$, i.e., the set of group homomorphisms $\tau : \Gamma \to \mathbb{C}^*$. Set $\Gamma^{\mathbb{C}^*} = \{c \in \mathbb{C}^* | c\Gamma = \Gamma\}$. We define a group structure on $\chi(\Gamma) \times \Gamma^{\mathbb{C}^*}$ by

$$(\tau_1, c_1) \cdot (\tau_2, c_2) = (\tau, c_1c_2), \quad \text{where} \quad \tau : \alpha \mapsto \tau_1(c_2\alpha)\tau_2(\alpha) \quad \text{for} \quad \alpha \in \Gamma. \quad (4.1)$$

It turns out that the group $\chi(\Gamma) \times \Gamma^{\mathbb{C}^*}$ is just the semidirect product $\chi(\Gamma) \rtimes \Gamma^{\mathbb{C}^*}$ under the action given by $(c\tau)(\alpha) = \tau(c\alpha)$ for all $c \in \Gamma^*$, $\tau \in \chi(\Gamma)$, $\alpha \in \Gamma$. We define a group homomorphism $\phi : (\tau, c) \mapsto \phi_{\tau,c}$ from $\chi(\Gamma) \times \Gamma^{\mathbb{C}^*}$ to $\text{Aut} W$ such that $\phi_{\tau,c}$ is the automorphism of $W$ defined by

$$
\phi_{\tau,c} : L_{\alpha,i} \mapsto \tau(\alpha)c^{i-1}L_{\alpha,i} \quad \text{for} \quad \alpha \in \Gamma, i \in \mathbb{Z}_+.
(4.2)
$$

One can easily verify that $\phi_{\tau,c}$ is indeed an automorphism of $W$.

**Theorem 4.1.** We have $\phi : \text{Aut} W \cong \chi(\Gamma) \rtimes \Gamma^{\mathbb{C}^*}$.

**Proof.** The theorem can be obtained by proving the following three claims. As $L_{0,0}$ is the unique locally finite element of $W$, for any $\sigma \in \text{Aut} W$, we can suppose $\sigma(L_{0,0}) = aL_{0,0}$, $a \neq 0$.

**Claim 1.** For any $\alpha \in \Gamma$, we have $\sigma(L_{\alpha,0}) = c^{-1}\tau(\alpha)L_{\alpha,0}$, with $c = a^{-1}$, $\tau \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}^*)$.  

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Assume that \( \sigma(L_{a,0}) = \sum_{\beta \in \Gamma, j \in \mathbb{Z}_+} b_{\beta,j}^\beta L_{\beta,j} \) (finite sum) for some \( b_{\beta,j}^\beta \in \mathbb{C} \). Applying \( \sigma \) to \([L_{0,0}, L_{a,0}] = \alpha L_{a,0}, \) we have \([L_{0,0}, \sum_{\beta \in \Gamma, j \in \mathbb{Z}_+} b_{\beta,j}^\alpha L_{\beta,j}] = \alpha a^{-1} \sum_{\beta \in \Gamma, j \in \mathbb{Z}_+} b_{\beta,j}^\alpha L_{\beta,j} \). By (1.1), we have

\[
\sum_{\beta \in \Gamma, j \in \mathbb{Z}_+} b_{\beta,j}^\beta (\beta L_{\beta,j} + j L_{\beta,j-1}) = \alpha a^{-1} \sum_{\beta \in \Gamma, j \in \mathbb{Z}_+} b_{\beta,j}^\beta L_{\beta,j}.
\]

Comparing the coefficients of \( L_{\beta,j} \) on the both sides of the above formula, it follows that \( \beta b_{\beta,j}^\beta + (j + 1) b_{\beta,j+1}^\alpha = \alpha a^{-1} b_{\beta,j}^\beta \). Then it gives that \( \beta = \alpha a^{-1}, b_{\beta,j}^\alpha = 0, j \geq 1 \). So \( \sigma(L_{a,0}) = \varphi(\alpha)L_{a,0}^{-1}, c^{-1} \tau(\alpha)L_{a,0}^{-1}, \) where \( \varphi \) is a function from \( \Gamma \) to \( \mathbb{C}^* \) and \( \varphi(0) = a. \)

Let \( \tau(\alpha) = a^{-1} \varphi(\alpha) \) and we can verify that \( \tau \in \text{Hom}_Z(\Gamma, \mathbb{C}^*) \).

**Claim 2.** For any \( i \in \mathbb{Z}_+ \), we have \( \sigma(L_{0,i}) = c^{-i} L_{0,0} \).

Let \( N = \text{span}\{L_{0,i} \mid i \in \mathbb{Z}_+\} \), \( \text{Witt}_+ = \text{span}\{d_i \mid i \in \mathbb{Z} \text{ and } i \geq -1\} \). Then we have \( N \cong \text{Witt}_+ \). The automorphism here is given by \( \sigma_1 : L_{0,i} \mapsto d_{i-1} \). Now we show that \( \sigma(N) = N \). Firstly, we have \( \sigma(L_{0,0}) = c^{-1} L_{0,0} \), with \( c \in \mathbb{C}^* \). Applying \( \sigma \) to \([L_{0,0}, L_{0,1}] = L_{0,0} \), we have \([L_{0,0}, \sigma(L_{0,1})] = L_{0,0} \). Then we can get \( \sigma(L_{0,1}) = L_{0,1} + b L_{0,0} \), with \( b \in \mathbb{C} \). Applying \( \sigma \) to \([L_{0,0}, L_{0,2}] = 2 L_{0,1}, \) we have \([c^{-1} L_{0,0}, \sigma(L_{0,2})] = 2 \sigma(L_{0,1}) = 2 L_{0,1} + 2 b L_{0,0} \). Then we can get \( \sigma(L_{0,2}) = c L_{0,2} + 2 b c L_{0,1} + d L_{0,0}, \) with \( d \in \mathbb{C} \). By induction on \( i \), we have \( \sigma(N) = N \).

So \( \sigma|_N \in \text{Aut}(N) \cong \text{Aut}(\text{Witt}_+) \cong \mathbb{C}^* \). And we already known \( \sigma(d_i) = c^i d_i \). From the automorphism between \( N \) and \( \text{Witt}_+ \), we can get \( \sigma(L_{0,i+1}) = c^i L_{0,i+1} \). Hence, we get \( \sigma(L_{0,i}) = c^{-i} L_{0,0} \).

**Claim 3.** For any \( \alpha \in \Gamma, \ i \in \mathbb{Z}_+, \) we have \( \sigma(L_{\alpha,i}) = \tau(\alpha)c^i L_{ca,i} \).

If \( i = 0 \), the claim is already right. Now suppose \( n = i - 1 \), the claim holds. Then applying \( \sigma \) to \([L_{\alpha,0}, L_{0,i}] = -\alpha L_{\alpha,i} + i L_{\alpha,i+1} \), we have \([\sigma(L_{\alpha,0}), \sigma(L_{0,i})] = -\alpha \sigma(L_{\alpha,i}) + i \sigma(L_{\alpha,i+1}) \). Using this and \( \sigma(L_{\alpha,i-1}) = \tau(\alpha)c^{i-2} L_{ca,i-1} \), we immediately obtain \( \sigma(L_{\alpha,i}) = \tau(\alpha)c^{i-1} L_{ca,i} \).

Thus we complete the proof of the theorem. \( \Box \)

Now we begin to study the automorphism group of \( H \).

**Claim 4.** For any \( \alpha \in \Gamma, \) we have \( \sigma(H_{a,0}) = \tau(\alpha)e H_{ca,0}, \) where \( e \in \mathbb{C}^* \), \( \tau \in \text{Hom}_Z(\Gamma, \mathbb{C}) \).

Applying \( \sigma \) to \([L_{0,0}, H_{a,0}] = \alpha H_{a,0}, \) since \( H \) is an abelian ideal, we get \([L_{0,0}, \sigma(H_{a,0})] = \alpha c \sigma(H_{a,0}) \). That is to say, \( \sigma(H_{a,0}) \) is an eigenvector of \( \text{ad}_{L_{0,0}} \) with eigenvector \( \alpha c \). Hence we obtain that \( \sigma(H_{a,0}) = f(\alpha) H_{ca,0} \). Now applying \( \sigma \) to \([L_{\alpha,0}, H_{\beta,0}] = \beta H_{\alpha+\beta,0} \), we have \( \tau(\alpha)c^{-1}[L_{\alpha,0}, f(\beta) H_{\beta,0}] = \beta f(\alpha + \beta) H_{c(\alpha+\beta),0}, \) i.e., \( f(\beta)\tau(\alpha) = f(\alpha + \beta) \). As \( f(0) = e \), we get \( f(\alpha) = \tau(\alpha)e \) and \( \tau \in \text{Hom}_Z(\Gamma, \mathbb{C}) \).

**Claim 5.** For any \( j \in \mathbb{Z}_+ \), we have \( \sigma(H_{0,j}) = ec^{j} H_{0,j} \).
As $H$ is an abelian ideal, applying $\sigma$ to $[L_{0,1}, H_{0,j}] = jH_{0,j}$, we get $[L_{0,1}, \sigma(H_{0,j})] = j\sigma(H_{0,j})$. Assume that $\sigma(H_{0,j}) = \sum_{\beta,k} x_{\beta,k} H_{\beta,k}$, then from the above equation, we have $\sum_{\beta,k} x_{\beta,k} (\beta H_{\beta,k+1} + k H_{\beta,k}) = j \sum_{\beta,j} x_{\beta,k} H_{\beta,k}$. Comparing the coefficients of $H_{\beta,k}$ on the both sides of this formula, we have $\beta x_{\beta,k-1} = (j - k) x_{\beta,k}$. Thus we have $\beta = 0$. We can assume that $\sigma(H_{0,j}) = e_j H_{0,j}$, $e_j \in \mathbb{C}^*$. Applying $\sigma$ to $[L_{0,0}, H_{0,j}] = jH_{0,j-1}$, we get $e_j = e e_{j-1}$, $j > 0$. As $e_0 = e$, we get $e_j = e e^j$. Then the claim holds. $\square$

**Claim 6.** For any $\alpha \in \Gamma$, $i \in \mathbb{Z}_+$, we have $\sigma(H_{\alpha,i}) = (\tau(\alpha)) e^j H_{\alpha,j}$.

If $\alpha = 0$, the claim holds. If $\alpha \neq 0$, by applying $\sigma$ to $[L_{0,j}, H_{\alpha,0}] = \alpha H_{\alpha,j}$, we have $[e^{-1}_{\alpha} L_{0,j}, \tau(\alpha) e H_{e,0}] = \alpha e \tau(\alpha) H_{\alpha,j} = \alpha \sigma(H_{\alpha,j})$. Thus we get $\sigma(H_{\alpha,j}) = (\tau(\alpha)) e^j H_{\alpha,j}$. $\square$

Based on the above conclusions about the automorphism group of $W$ and $H$, now we can study the automorphism group of $HV$. Denote by $\text{Aut} HV$ the automorphism group of $HV$. Let $\sigma \in \text{Aut} HV$, so we have $\sigma(H) = H$, $\sigma(H_{0,0}) = e H_{0,0}$, $e \in \mathbb{C}^*$. Let $\tilde{\sigma} \in \text{Aut} HV/H(i.e., \text{Aut} W)$, we have $\tilde{\sigma}(L_{0,i}) = \tau(\alpha) e^{-i-1}(L_{0,i})$, i.e., $\sigma(L_{0,i}) \equiv \tau(\alpha) e^{-i-1} L_{\alpha,i}$ mod $H$.

Since $\sigma(L_{0,0}) \equiv e^{-1}_{\alpha} L_{0,0}$ mod $H$, we can assume $\sigma(L_{0,0}) = c^{-1}_{\alpha} L_{0,0} + \sum a_{\alpha,i} H_{\alpha,i}$ for some $a_{\alpha,i} \in \mathbb{C}$. Define

$$b_{\alpha,i} = \begin{cases} -i^{-1} a_{\alpha,i} - 1 & \text{for } i \geq 0, \\ 0 & \text{for } i = 0. \end{cases} (4.3)$$

Denote $\tau = e^{\text{ad} \sum a_{\alpha,i} b_{\alpha,i} H_{\alpha,i}} \in \text{Inn} H\mathbb{V}$, so $\tau(e^{-1}_{\alpha} L_{0,0}) = c^{-1}_{\alpha} L_{0,0} + \sum a_{\alpha,i} b_{\alpha,i} H_{\alpha,i}$. Let $\pi = \tau^{-1} \sigma$, and we can obtain $\pi(L_{0,0}) = \tau^{-1} \sigma(L_{0,0}) = c^{-1}_{\alpha} L_{0,0}$. Then from claims 4, 5 and 6, we have $\pi(L_{0,i}) = \tau(\alpha) e^{-i} L_{\alpha,i}$. Thus we complete the proof of the following theorem.

**Theorem 4.2.** We have $\pi : \text{Aut} HV \cong \text{Inn} H\mathbb{V} \rtimes \left( (\chi(\Gamma) \rtimes \Gamma \mathbb{C}^*) \rtimes \mathbb{C}^* \right)$.

## 5 Second cohomology group

Recall that a bilinear form $\psi : HV \times HV \to \mathbb{C}$ is called a 2-cocycle on $HV$ if the following conditions are satisfied:

$$\psi(x, y) = -\psi(y, x), \quad \psi(x, [y, z]) + \psi(y, [z, x]) + \psi(z, [x, y]) = 0,$$

for $x, y, z \in HV$. Denote by $C^2(HV, \mathbb{C})$ the vector space of 2-cocycles on $HV$. For any linear function $f : HV \to \mathbb{C}$, one can define a 2-cocycle $\psi_f$ by $\psi_f(x, y) = f([x, y])$ for $x, y \in HV$. Such a 2-cocycle is called a trivial 2-cocycle or a 2-coboundary on $HV$. Denote by $B^2(HV, \mathbb{C})$ the vector space of 2-coboundaries on $L$. The quotient space

$$H^2(HV, \mathbb{C}) = C^2(HV, \mathbb{C}) / B^2(HV, \mathbb{C})$$
is called the 2-cohomology group of $HV$. There exists a one-to-one correspondence between the set of equivalence classes of one-dimensional central extensions of $HV$ by $\mathbb{C}$ and the 2-cohomology group of $HV$.

**Theorem 5.1.** We have $H^2(HV, \mathbb{C}) = \mathbb{C}\overline{\phi}$, where $\overline{\phi}$ is the equivalence class of the 2-cocycle $\phi$, which is defined by

$$\phi(L_{\alpha,i}, L_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0} \frac{\alpha^3 - \alpha}{12}, \quad (5.1)$$

$$\phi(L_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0} (\alpha^2 - \alpha), \quad (5.2)$$

$$\phi(H_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0} \alpha. \quad (5.3)$$

**Proof.** $(5.1)$ can be easily obtained. $(5.2)$ and $(5.3)$ can be proved by the following Theorem 5.2 and Theorem 5.3 respectively.

**Theorem 5.2.** We have

$$\phi(L_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0} (\alpha^2 - \alpha), \quad (5.4)$$

for any $\alpha, \beta \in \Gamma$, $i, j \in \mathbb{Z}_+$. This theorem is obtained by induction on $i$ as follows

$$f(H_{\alpha,i}) = \begin{cases} 
\psi(L_{0,0}, H_{0,1}) & \text{if } \alpha = i = 0, \\
\frac{1}{\alpha} \psi(L_{0,1}, H_{\alpha,i}) & \text{if } \alpha = 0, \ i \neq 0, \\
\frac{1}{\alpha} (\psi(L_{0,0}, H_{\alpha,i}) - if(H_{\alpha,i-1})) & \text{if } \alpha \neq 0.
\end{cases} \quad (5.5)$$

Set $\phi = \psi - \psi f$, then we have

$$\phi(L_{0,0}, H_{\beta,j}) = \psi(L_{0,0}, H_{\beta,j}) - f([L_{0,0}, H_{\beta,j}]) = 0 \quad \text{for } \beta \neq 0. \quad (5.6)$$

Then we only need to consider $\phi(L_{0,0}, H_{0,j})$. Firstly, we can easily verify that $\phi(L_{0,0}, H_{0,0}) = \phi(L_{0,0}, H_{0,1}) = 0$. For $j \neq 1$, we can obtain $\phi(L_{0,0}, H_{0,j}) = \psi(L_{0,0}, H_{0,j}) - j f(H_{0,j-1}) = \psi(L_{0,0}, H_{0,j}) - \frac{j}{j-1} \phi(L_{0,1}, H_{0,j-1}) = \psi(L_{0,0}, H_{0,j}) - \frac{j}{j-1} \phi(L_{0,1}, [L_{0,0}, H_{0,j}]) = 0$. From the above discussion, we have

$$\phi(L_{0,0}, H_{\beta,j}) = 0, \quad (5.7)$$

for any $j \in \mathbb{Z}_+$, $\beta \in \Gamma$.

Now we want to prove $\phi(L_{\alpha,i}, H_{0,0}) = 0$, for any $\alpha \in \Gamma, i \in \mathbb{Z}_+$. If $\alpha \neq 0$, we use induction on $i$. It holds for $i = 0$. Suppose it also holds for $i - 1$, then we can get $\phi(L_{\alpha,i}, H_{0,0}) = \frac{1}{\alpha} \psi([L_{0,0}, L_{\alpha,i}] - iL_{\alpha,i-1}, H_{0,0}) = 0$. So we have $\phi(L_{\alpha,i}, H_{0,0}) = 0$ for $\alpha \neq 0$. We can also
verify $\phi(L_{0,i}, H_{0,0}) = \psi(\frac{1}{\Gamma}[L_{0,1}, L_{0,i}], H_{0,0}) = 0$, for $i \neq 0$. From the above discussion, we can get that
\[
\phi(L_{\alpha,i}, H_{0,0}) = 0,
\] for any $\alpha \in \Gamma, i \in \mathbb{Z}_+$.

Next we can also have $\phi(L_{\alpha,0}, H_{\beta,0}) = \psi(L_{\alpha,0}, H_{\beta,0}) - \beta f(H_{\alpha+\beta,0}) = \psi(L_{\alpha,0}, H_{\beta,0}) - \frac{\beta}{\alpha+\beta} \psi(L_{0,0}, H_{\alpha+\beta,0}) = \psi(L_{\alpha,0}, H_{\beta,0}) - \frac{1}{\alpha+\beta} \psi(L_{0,0}, [L_{0,0}, H_{\beta,0}]) = 0$, for $\alpha + \beta \neq 0$.

Similarly, if $i + j = 1$, we have $\phi(L_{0,0}, H_{0,1}) = \phi(L_{0,1}, H_{0,0}) = 0$. For $i + j \neq 1$, we have $\phi(L_{0,i}, H_{0,j}) = \psi(L_{0,i}, H_{0,j}) - j f(H_{0,i+j-1}) = \psi(L_{0,i}, H_{0,j}) - \frac{j}{i+j-1} \psi(L_{0,1}, [L_{0,i}, H_{0,j}]) = 0$. From the above discussion, we can get that
\[
\phi(L_{0,i}, H_{0,j}) = 0,
\] for any $i, j \in \mathbb{Z}_+$.

By induction on $i+j$, we have $\phi(L_{\alpha,i}, H_{\beta,j}) = \psi(L_{\alpha,i}, H_{\beta,j}) - \beta f(H_{\alpha+\beta,i+j}) - j f(H_{\alpha+\beta,i+j-1}) = \psi(L_{\alpha,i}, H_{\beta,j}) - \frac{\beta}{\alpha+\beta} \psi(L_{0,0}, H_{\alpha+\beta,i+j}) - (i+j)f(H_{\alpha+\beta,i+j-1}) - \frac{1}{\alpha+\beta} \psi(L_{0,0}, H_{\alpha+\beta,i+j-1}) - (i+j-1)f(H_{\alpha+\beta,i+j-2}) = -\frac{1}{\alpha+\beta}(j\phi(L_{\alpha,j}, H_{\beta,j-1}) + i\phi(L_{\alpha,j-1}, H_{\beta,j})) = 0$, for $\alpha + \beta \neq 0$. So we can obtain $\phi(L_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \phi(L_{\alpha,i}, H_{-\alpha,j})$.

Next let us consider $\phi(L_{\alpha,0}, H_{-\alpha,0})$. Firstly, we have $\phi(L_{\alpha,0}, H_{-\alpha,0}) = \psi(L_{\alpha,0}, H_{-\alpha,0}) - f([L_{\alpha,0}, H_{-\alpha,0}]) = \psi(L_{\alpha,0}, H_{-\alpha,0}) + \alpha f(H_{0,0}) = \psi(L_{\alpha,0}, H_{-\alpha,0}) + \alpha \psi(L_{0,0}, H_{0,1}) = c(\alpha)$. From the equation $\phi([L_{\alpha,0}, L_{\beta,0}], H_{-\alpha-\beta,0}) + \phi([L_{\beta,0}, H_{-\alpha-\beta,0}], L_{\alpha,0}) + \phi([H_{-\alpha-\beta,0}, L_{\alpha,0}], L_{\beta,0}) = 0$, we can get
\[
(\beta - \alpha)c(\alpha + \beta) + (\beta + \alpha)c(\alpha) = (\beta + \alpha)c(\beta),
\] for any $\alpha, \beta \in \Gamma$.

From (5.10), replacing $\alpha$ by $\alpha - 1$ and taking $\beta = 1, 2$ respectively, we can get
\[
\begin{aligned}
\left\{ \begin{array}{l}
\alpha c(1) = 0, \\
(3 - \alpha)c(\alpha + 1) + (\alpha + 1)c(\alpha - 1) = (\alpha + 1)c(2).
\end{array} \right.
\end{aligned}
\] (5.11)

If $\alpha \neq 0$, then we can obtain
\[
c(\alpha) = \frac{c(2)}{2} (\alpha^2 - \alpha).
\] (5.12)

As $\phi(L_{0,0}, H_{0,0}) = 0 = c(0)$ also satisfies (5.12), we can conclude that (5.12) holds for any $\alpha \in \Gamma$, with $c(2) \in \mathbb{C}$. So we have $\phi(H_{\alpha,0}, H_{\beta,0}) = \delta_{\alpha+\beta,0}(\alpha^2 - \alpha)$. Thus we can get that $\phi(L_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0}(\alpha^2 - \alpha)$.

\[
\textbf{Theorem 5.3.} \text{ We have } H^2(H, \mathbb{C}) = \mathbb{C}\phi, \text{ where } \phi \text{ is the equivalence class of the 2-cocycle } \phi, \text{ which is defined by }
\]
\[
\phi(H_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i+j,0}(\alpha^2 - \alpha).
\] (5.13)
Proof. First without loss of generality, we can always suppose $1 \in \Gamma$. Let $\psi \in C^2(H, \mathbb{C})$, we define a linear function $f : H \to \mathbb{C}$ and set $\phi = \psi - \psi_f$. Then we can obtain

$$\phi(H_{0,0}, H_{0,i}) = \psi(H_{0,0}, H_{0,i}) - f([H_{0,0}, H_{0,i}]) = 0. \quad (5.14)$$

Similarly, we have

$$\phi(H_{0,0}, H_{\beta,0}) = \psi(H_{0,0}, H_{\beta,0}) - f([H_{0,0}, H_{\beta,0}]) = 0 \text{ for } \beta \neq 0. \quad (5.15)$$

By the induction on $i$, we have

$$\phi(H_{0,0}, H_{\beta,i}) = 0. \quad (5.16)$$

for any $i \in \mathbb{Z}_+, \beta \in \Gamma$.

Furthermore, we can have

$$\phi(H_{\alpha,0}, H_{\beta,0}) = \psi(H_{\alpha,0}, H_{\beta,0}) - f([H_{\alpha,0}, H_{\beta,0}]) = -\frac{\beta}{\alpha} \phi(H_{\alpha,0}, H_{\beta,0}) \text{ for } \alpha \neq 0. \quad (5.17)$$

Comparing the two sides of the equation, we can get $\phi(H_{\alpha,0}, H_{\beta,0}) = 0$, for $\beta \neq -\alpha$.

Similarly, we can have $\phi(H_{0,i}, H_{0,j}) = 0$ for $j \neq 0, -i$.

Due to $\phi(H_{0,0}, H_{0,0}) = 0$ and $i \in \mathbb{Z}_+, j \in \mathbb{Z}_+$, we can obtain that

$$\phi(H_{0,i}, H_{0,j}) = 0 \text{ for } i, j \in \mathbb{Z}_+. \quad (5.18)$$

Furthermore, we can have

$$\phi(H_{0,i}, H_{\beta,0}) = \psi(H_{0,i}, H_{\beta,0}) - f([H_{0,i}, H_{\beta,0}]) = -\frac{(-1)^{|i|}}{\beta^i} \phi(H_{0,0}, H_{\beta,0}) = 0 \text{ for } \beta \neq 0. \quad (5.19)$$

By induction on $j$, we can get $\phi(H_{0,i}, H_{\beta,j}) = 0$, for any $\beta \in \Gamma$.

Then we can prove $\phi(H_{\alpha,i}, H_{\beta,j}) = 0$ for $\alpha \neq 0$, by using induction on $i + j$. Firstly, it holds for $i = 0, j = 0; i = 0, j = 1$ and $i = 1, j = 0$. Now assume $\phi(H_{\alpha,i-1}, H_{\beta,j}) = \phi(H_{\alpha,i}, H_{\beta,j-1}) = 0$.

Considering $\phi(H_{\alpha,i}, H_{\beta,j}) = \psi(H_{\alpha,i}, H_{\beta,j}) - f([H_{\alpha,i}, H_{\beta,j}]) = -\frac{\beta}{\alpha} \phi(H_{\alpha,i}, H_{\beta,j}) - \frac{2}{\alpha} \phi(H_{\alpha,i}, H_{\beta,j-1}) - \frac{i}{\alpha} \phi(H_{\alpha,i-1}, H_{\beta,j})$ for $\alpha \neq 0$ and from the assumption, we can get $\phi(H_{\alpha,i}, H_{\beta,j}) = -\frac{\beta}{\alpha} \phi(H_{\alpha,i}, H_{\beta,j})$. Thus we have

$$\phi(H_{\alpha,i}, H_{\beta,j}) = 0 \text{ for } \beta \neq -\alpha, \alpha \neq 0. \quad (5.20)$$

So we can obtain

$$\phi(H_{\alpha,i}, H_{\beta,j}) = \delta_{\alpha + \beta,0} \phi(H_{\alpha,i}, H_{-\alpha,j}) \text{ for } \alpha \neq 0. \quad (5.21)$$

Firstly, we have $\phi(H_{\alpha,i}, H_{-\alpha,j}) = 0$ for $\alpha = 0$. 

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Then we have \( \phi(H_{\alpha,i}, H_{-\alpha,j}) = -\frac{\alpha}{i+1} \phi(H_{0,j+1}, H_{-\alpha,i}) + \frac{j}{i-1} \phi(H_{0,j}, H_{-\alpha,i}) = 0 \), for any \( \alpha \neq 0, \ i \geq 2 \). Because \( \phi(H_{\alpha,0}, H_{-\alpha,1}) = \psi(H_{\alpha,0}, H_{-\alpha,1}) - f([H_{\alpha,0}, H_{-\alpha,1}]) = 0 \) and \( \phi(H_{\alpha,1}, H_{-\alpha,0}) = \psi(H_{\alpha,1}, H_{-\alpha,1}) - f([H_{\alpha,0}, H_{-\alpha,1}]) = 0 \), we can get \( \phi(H_{\alpha,i}, H_{-\alpha,j}) = 0 \) for \( \alpha \neq 0, \ i \geq 2 \). So we have \( \phi(H_{\alpha,i}, H_{-\alpha,j}) = \delta_{i+j,0} \phi(H_{\alpha,0}, H_{-\alpha,0}) \) for \( i+j \geq 1 \). Thus we can obtain

\[
\phi(H_{\alpha,i}, H_{-\alpha,j}) = \delta_{i+j,0} \phi(H_{\alpha,0}, H_{-\alpha,0}).
\] (5.22)

Suppose \( \phi(H_{\alpha,0}, H_{-\alpha,0}) = g(\alpha) \) and assume \( 1 \in \Gamma \), we can get \( \phi(H_{1,0}, H_{-1,0}) = g(1) \in \mathbb{C} \). And we also have \( \phi(H_{\alpha,0}, H_{-\alpha,0}) = \psi([L_{\alpha-1,0}, H_{1,0}], H_{-\alpha,0}) = \alpha \psi(H_{1,0}, H_{-1,0}) = \alpha g(1) \), for \( g(1) \in \mathbb{C} \). Finally, we can obtain

\[
\phi(H_{\alpha,i}, H_{-\alpha,j}) = \delta_{i+j,0} \alpha.
\] (5.23)

\[\square\]

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