Representations with $\text{Sp}(1)^k$-reductions and quaternion-Kähler symmetric spaces

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Abstract We classify non-polar irreducible representations of connected compact Lie groups whose orbit space is isometric to that of a representation of a finite extension of $\text{Sp}(1)^k$ for some $k > 0$. It follows that they are obtained from isotropy representations of certain quaternion-Kähler symmetric spaces by restricting to the “non-$\text{Sp}(1)$-factor”.

Keywords Orbit spaces · Orthogonal representations · Reductions · Quaternion-Kähler symmetric spaces

Mathematics Subject Classification 57S15 · 22E46 · 53C35

1 Introduction

The aim of this paper is to contribute to the program initiated in [9], namely, hierarchize the representations of compact Lie groups in terms of the complexity of their orbit spaces, viewed as metric spaces. We say that two representations are quotient-equivalent if they have isometric orbit spaces. Given a representation, if there is a quotient-equivalent representation of a lower-dimensional group, then we say that the former representation reduces to the latter one and that the latter representation is a reduction of the former one. A minimal reduction of a representation is a reduction with smallest possible dimension of the underlying group.

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At the basis of the hierarchy lie the polar representations, namely, those representations that reduce to finite group actions, and which turn out to be related to symmetric spaces (indeed every polar representation of a connected compact Lie group has the same orbits as the isotropy representation of a symmetric space [5]). In [10], there were studied and classified those irreducible representations of connected groups that reduce to actions of groups with identity component a torus $S^1 \times \cdots \times S^1$. It was shown that, mostly, those are close relatives of Hermitian symmetric spaces. Herein we study the quaternionic version, namely, those irreducible representations of connected groups that reduce to an action of a group whose identity component is a “quaternionic torus” $S^3 \times \cdots \times S^3$. Interestingly enough, these are related to quaternion-Kähler symmetric spaces, in a stricter sense than in the Hermitian case.

We state our main result as follows (where $\text{A}_n$ stands for the Coxeter group).

**Theorem** Let $\tau : H \to \text{O}(W)$ be a non-polar irreducible representation of a connected compact Lie group. Assume that $\tau$ is quotient-equivalent to a representation $\rho : G \to \text{O}(V)$ where $G^0 \equiv \text{Sp}(1)^k$ for some $k > 0$ and $\dim G < \dim H$. Then $k = 3$, $G$ is disconnected and $V = \otimes^3 \mathbb{C}^2$; moreover, the cohomogeneity of $\tau$ is 7 and it is obtained by restricting the isotropy representation of a certain quaternion-Kähler symmetric space to the “non-$\text{Sp}(1)$-factor”. More precisely, $\tau$ is one of:

| $\tau$ | qK symmetric space | $G/G^0$ | Condition |
|--------|---------------------|---------|-----------|
| $(\text{SO}(n) \times \text{Sp}(1), \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{H})$ | $\text{SO}(n+4)/(\text{SO}(n) \times \text{SO}(4))$ | $\text{A}_1$ | $n \geq 5$ |
| $(\text{Sp}(3), A^3 \mathbb{C}^6 \ominus \mathbb{C}^6)$ | $F_4/(\text{Sp}(3)\text{Sp}(1))$ | $\text{A}_2$ | $-$ |
| $(\text{SU}(6), A^3 \mathbb{C}^6)$ | $E_6/(\text{SU}(6)\text{SU}(2))$ | $\text{A}_2$ | $-$ |
| $(\text{Spin}(12), \mathbb{C}^{32})$ | $E_7/(\text{Spin}(12)\text{SU}(2))$ | $\text{A}_2$ | $-$ |
| $(\text{E}_7, \mathbb{C}^{56})$ | $E_8/(\text{E}_7\text{SU}(2))$ | $\text{A}_2$ | $-$ |

Finally, $\rho$ is a minimal reduction of $\tau$.

Note that there are three additional families of quaternion-Kähler symmetric spaces absent from the table, namely, the restrictions of their isotropy representations do not have the kind of reduction as in the Theorem. The Theorem says in particular that $k$ must be odd, so $\rho$ is a representation of quaternionic type. A posteriori we find that the $\text{Sp}(1)$-group of isometries of $W/H = V/G$ induced by the normalizer of $G$ in $\text{O}(V)$ can be lifted to a $\text{Sp}(1)$-subgroup of $\text{O}(W)$. In this connection, we note that the general problem of lifting isometries of orbit spaces to isometries deserves further attention ([2, Question 1.6] and [9, Question 1.13]).

The structure of the proof of the Theorem goes as follows. The situation of a non-trivial reduction from $\tau$ to $\rho$ entails the presence of non-empty boundary for the orbit space of $\rho$, according to results in [9, §5]. A careful analysis of the existence of boundary points then forces $k = 3$. Now $\rho$ is of quaternionic type, so its orbit space admits an $\text{Sp}(1)$-group of isometries. An application of a theorem of Thorbergsson, as in [10], shows that $\tau$ must be the restriction of a polar representation of Coxeter type $\text{B}_4$ or $\text{F}_4$. The argument is finished by invoking Dynkin’s classification of maximal connected closed subgroups of compact Lie groups and ruling out two additional cases by metric arguments.

We follow notation and terminology from [9]. See also [11] for some background material on stratification of orbit spaces and their metric structures.

## 2 Structure of the examples

In this section we show that the representations $\tau$ listed in the table of the Theorem indeed admit reductions to a group whose identity component is $\text{Sp}(1)^3$. In all cases, the reduction
in hand is the Luna–Richardson–Straume reduction to the normalizer of a principal isotropy group acting on the fixed point set of this principal isotropy group \[11, \S 2.6\]. It will follow from the discussion in Sect. 3.3 that those reductions are minimal.

2.1 The real Grassmannian

The isotropy representation of the rank 4 real Grassmannian manifold is \(SO(n) \times SO(4)\) acting on \(\mathbb{R}^n \otimes \mathbb{R}^4\) and \(S(\mathbb{Z}_2^n) \times SO(n - 4)\) is its principal isotropy group. Note that the diagonal \(S(\mathbb{Z}_2^n)\)-subgroup of \(SO(4)\) corresponds in \(Sp(1)\) to the diagonally embedded quaternion group \(Q = \{ \pm 1, \pm i, \pm j, \pm k \}\). In this case \(\tau\) is given by \(H = SO(n) \times SU(2)\) acting on \(\mathbb{R}^4 \otimes \mathbb{C}^2\), and it follows that it has principal isotropy group \(H_{princ} = SO(n - 4)\), so the effective normalizer \(NH(H_{princ})/H_{princ} = SO(4) \times SU(2)/H_{princ} = O(4) \times SU(2) = \mathbb{Z}_2 \cdot Sp(1)^3\) has the desired form.

2.2 The exceptional cases

For each one of the rank 4 exceptional quaternion-Kähler symmetric spaces, the isotropy representation \(\hat{\tau}\) is given by \(\hat{H} = H \cdot Sp(1)\) acting on \(\hat{W} = W \otimes \mathbb{H} \cong W\) and \(\tau = \hat{\tau}|_H : H = O(W)\) is a representation of quaternionic type. The principal isotropy group of \(\hat{\tau}\) has the form \(\hat{H}_{princ} = H_{princ} \cdot Q\) where \(Q\) is the quaternion group diagonally embedded in \(H \cdot Sp(1)\).

Since \(\hat{\tau}\) is asystatic (see \[21, pp. 11–12\] or \[8, \S 2.2\]), it is polar and the fixed point set \(W_{\hat{H}_{princ}}\) is a section; this is a totally real subspace of \(W\) of dimension 4. Now \(W_{H_{princ}}\) is a quaternionic subspace of \(W\), and it must be the quaternionic span of \(W_{\hat{H}_{princ}}\), of real dimension 16. Since the cohomogeneity of \(\tau\) is \(c(\tau) = c(\hat{\tau}) + \dim Sp(1) = 4 + 3 = 7\) (compare \[13, Table A\]), we deduce that \(\dim N_H(H_{princ})/H_{princ} = \dim W_{H_{princ}} - c(\tau) = 16 - 7 = 9\). The group \(N_H(H_{princ})/H_{princ}\) acts irreducibly on \(W_{H_{princ}}\) \[9, \S 5.2\], and so does its identity component (\[9, Thm. 1.7\] and \[10, Thm. 1.1\]), implying that its center is at most one-dimensional. A quick enumeration of the possible groups reveals that \([N_H(H_{princ})/H_{princ}]^0\) is locally isomorphic to \(Sp(1)^3\) or \(U(3)\), but \(W_{H_{princ}}\) is a representation of quaternionic type, and the latter group admits none. Note that the only 16-dimensional irreducible representation of \(Sp(1)^3\) is \(\otimes^3 \mathbb{C}^2\).

3 Proof of the Theorem

Throughout this section we let \(\tau : H \to O(W)\) and \(\rho : G \to O(V)\) be as in the statement of the Theorem. For ease of reference, we first gather all polar irreducible representations of \(Sp(1)^k\) using \[5,7\] and \[23, Tables 8.11.2 and 8.11.5\].

**Lemma 1** The faithful polar irreducible representations of \(Sp(1)^k\) together with their principal isotropy groups are listed as follows:
and then check via Lemma 1 that these admit an invariant real form of $\text{SL}_6$, which then must have a torus as identity component, owing to [9, Thm. 1.5]. However, the Luna–Richardson–Straume procedure gives a reduction to a group of dimension less than no invariant complex structure, then $V_c$ of $G$ is trivial. In the range $3 \leq k \leq 6$, we deduce that $k \leq 5$. In case $k = 5$, if $U$ is of quaternionic type then $\dim \mathbb{C}V^c = 2 \dim \mathbb{C}U \geq 2 \dim (\mathbb{C} \otimes \mathbb{C}) = 64 > 8 \cdot 5$, and if $U$ is of real type then $\dim \mathbb{C}V^c = \dim \mathbb{C}U \geq \dim (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) = 48 > 8 \cdot 5$, so in both cases the sgp is trivial. In the range $3 \leq k \leq 4$ the formula similarly implies that the sgp is trivial for most representations. In fact, after discarding two polar representations, the remaining real forms left are exactly $(\text{SO}(4) \times \text{SO}(1), \mathbb{R}^4 \otimes \mathbb{R}^2)$, $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes \mathbb{R}^8)$, $(\text{SO}(4) \times \text{SO}(3), \mathbb{R}^4 \otimes \mathbb{R}^5)$ and $(\text{SO}(3) \times \text{SO}(4), \mathbb{R}^3 \otimes \mathbb{R}^8)$ (here $(\text{SO}(3), \mathbb{R}^3 = S^2(\mathbb{R}^3))$ and $(\text{SO}(4) = \text{Sp}(1)\text{Sp}(1), \mathbb{R}^8 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$)) which are directly seen to have trivial pig by taking appropriate slice representations.

### 3.1 Triviality of the principal isotropy group

We observe that the principal isotropy group of $\rho^0 := \rho|_{G^0}$ is trivial. Indeed we have the following general result.

**Lemma 2** A non-polar irreducible representation of $\text{Sp}(1)^k$ has trivial principal isotropy group.

**Proof** We shall collect the irreducible representations with possibly non-trivial principal isotropy group and note that they are all polar. The complexification of the principal isotropy group (pig) of a representation of $\text{Sp}(1)^k$ on $V$ is the stabilizer in general position (sgp) of the corresponding complex representation of $\text{SL}(2, \mathbb{C})^k$ on $V^c = V \otimes \mathbb{C}$ [20, §5]. If $V$ has no invariant complex structure, then $V^c$ is irreducible; otherwise, $V^c$ consists of two copies of $V$. Now it suffices to start listing irreducible representations $U$ of $\text{SL}(2, \mathbb{C})^k$ such that:

1. $U$ has non-trivial sgp, if $U$ is a representation of real type;
2. $U \oplus U$ has non-trivial sgp, if $U$ is a representation of quaternionic type;

and then check via Lemma 1 that these admit an invariant real form $V$ which is a polar representation of $\text{Sp}(1)^k$.

At this point, one can apply [4, Theorem] to reduce the analysis to the case of finite sgp and then refer to the classifications of representations with finite non-trivial sgp in [17, Table 1] and [18, Table 1] to finish the job.

Alternatively, we can avoid using those classification results as follows. The cases $k = 1$ and $k = 2$ are easy. Indeed we deal directly with the real representations and note that if a non-polar irreducible representation of $\text{Sp}(1)$ or $\text{Sp}(1)^2$ has a non-trivial pig then the Luna–Richardson–Straume procedure gives a reduction to a group of dimension less than six, which must then have a torus as identity component, owing to [9, Thm. 1.5]. However, according to [10, Thm. 1.1], this is never the case for an irreducible representation of $\text{Sp}(1)$ or $\text{Sp}(1)^2$.

In general, the simple estimate in [3, Thm. 2] (see also [19, Thm. 7.10]) immediately yields that a necessary condition for non-triviality of sgp of $V^c$ is that $\dim V = \dim \mathbb{C}V^c \leq 8k$. Since $\dim \mathbb{C}U \geq 2^k$, we deduce that $k \leq 5$. In case $k = 5$, if $U$ is of quaternionic type then $\dim \mathbb{C}V^c = 2 \dim \mathbb{C}U \geq 2 \dim (\mathbb{C}^2 \otimes \mathbb{C}^2) = 64 > 8 \cdot 5$, and if $U$ is of real type then $\dim \mathbb{C}V^c = \dim \mathbb{C}U \geq \dim (\mathbb{C}^2 \otimes \mathbb{C}^3) = 48 > 8 \cdot 5$, so in both cases the sgp is trivial. In the range $3 \leq k \leq 4$ the formula similarly implies that the sgp is trivial for most representations. In fact, after discarding two polar representations, the remaining real forms left are exactly $(\text{SO}(4) \times \text{SO}(1), \mathbb{R}^4 \otimes \mathbb{R}^2)$, $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes \mathbb{R}^8)$, $(\text{SO}(4) \times \text{SO}(3), \mathbb{R}^4 \otimes \mathbb{R}^5)$ and $(\text{SO}(3) \times \text{SO}(4), \mathbb{R}^3 \otimes \mathbb{R}^8)$ (here $(\text{SO}(3), \mathbb{R}^3 = S^2(\mathbb{R}^3))$ and $(\text{SO}(4) = \text{Sp}(1)\text{Sp}(1), \mathbb{R}^8 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$)).
3.2 General setting

Note that $V$ equals $\mathbb{C}^{r_1} \otimes \mathbb{C} \cdots \otimes \mathbb{C}^{r_k}$ or a real form $[\mathbb{C}^{r_1} \otimes \mathbb{C} \cdots \otimes \mathbb{C}^{r_k}]_\mathbb{R}$ according to whether the dimension $r_i$ is even for an odd, resp. even, number of indices $i$. We have that $V$ is of quaternionic type in the first case, and of real type in the second one. We may assume $2 \leq r_1 \leq \cdots \leq r_k$.

Due to [9, §5], $X = V/G = W/H$ has non-empty boundary as an Alexandrov space. We recall that the boundary is given by the closure of the codimension one strata of $X$. Since the principal isotropy is trivial, a point $p \in V$ projects to such a stratum—called a $G$-important point in [9]—if and only if its isotropy group $G_p$ is a sphere $S^\ell$ of dimension $\ell$ equal to 0, 1 or 3 and

$$\dim V - 1 - \ell = 3k - m + f,$$

(1)

where $m$ is the dimension of the normalizer $N_G(G_p)$ and $f$ is the dimension of the fixed point set $V^{G_p}$ of $G_p$ in $V$, see e.g. [9, Lem. 4.1].

In the following two subsections, we will analyze the boundary of $V/G$ and show that it can be non-empty only if $G$ is disconnected, $k = 3$ and $V = \otimes^3 \mathbb{C}^2$. The cases $k = 1$ and $k = 2$ were discussed in [9, §10], so we may assume $k \geq 3$.

3.3 Connected case

We first suppose $G$ is connected and show that this assumption is incompatible with $V/G$ having non-empty boundary. So suppose that $p \in V$ is a $G$-important point. Since $G$ is connected, no $G$-important point may lie in an exceptional orbit [15], so $G_p$ is not discrete. Moreover, any SU(2)-subgroup of $G$ contains a unique involution that is central in $G$. Since such an involution cannot have fixed points by irreducibility, we cannot have $G_p \cong \text{SU}(2)$. We deduce that $G_p \cong S^1$.

The dimension formula (1) yields

$$\theta \cdot r_1 \cdots r_k - 2 = 3k - m + f,$$

where $\theta = 1$ or 2 in case $V$ is of real or quaternionic type, respectively. It is moreover clear that $m \geq k$. Since $V^{G_p}$, resp. its complexification, is the sum of weight spaces whose weights lie in a hyperplane of the dual Lie algebra of the maximal torus of $G$, it is not hard to see that $f \leq \theta \cdot r_2 \cdots r_k$. We deduce that

$$\theta(r_1 - 1)r_2 \cdots r_k \leq 2k + 2.$$

In particular, $\theta \cdot 2^{k-2} \leq k + 1$ implying $k = 3$ or 4. If $k = 3$, we obtain $(\text{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$ and $(\text{SO}(4) \times \text{SO}(3), \mathbb{R}^4 \otimes_\mathbb{R} \mathbb{R}^3)$; the latter representation is polar, which cannot be. If $k = 4$, we obtain $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes_\mathbb{R} \mathbb{R}^4)$ which is also polar and again is excluded.

It remains to analyze the case of $(\text{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$. Here the dimension formula (1) says that $5 + m = f$, and $m \geq 3$ implying $f \geq 8$. Recall that the weights of $\mathbb{C}^2$ are $\pm \epsilon$, where $2\epsilon$ is the positive root of $\text{Sp}(1)$. It is apparent that $V^{G_p}$ can contain at most four weight spaces so $f = 8$. However, in this case $G_p$ is a circle diagonally embedded in two factors of $\text{Sp}(1)^3$ which gives $m = 5$ and contradicts the dimension formula. We deduce that $G$ cannot be connected, as desired.

In particular, we have shown that the orbit space of any non-polar irreducible representation of $\text{Sp}(1)^k$ has empty boundary. Together with [9, Prop. 5.2], this implies:

Every irreducible representation of an extension of $\text{Sp}(1)^k$ by a finite group is either polar or reduced (i.e. it cannot be further reduced).
3.4 Disconnected case

We now suppose \( G \) is disconnected and prove that \( V/G \) can have non-empty boundary only if \( k = 3 \) and \( V = \otimes^3 \mathbb{C}^2 \). Since \( H \) is connected, there is an involution \( w \in G \backslash G^0 \), called a *nice involution*, that acts as a reflection on \( V/G^0 \) (see Proposition 3.2 and §4.3 in [9]). The dimension formula reads

\[
\dim V - 1 = \dim G - \dim C(w) + \dim V^w
\]

where \( C(w) \) is the centralizer of \( w \) in \( G \) and \( V^w \) is the fixed point set of \( w \) in \( V \). The element \( w \) acts on \( G^0 \) by conjugation.

3.4.1 Inner automorphism

We first consider the case in which \( w \) acts on \( G^0 \) as an inner automorphism and show that this case gives nothing. Write \( w = q j \) where \( q \) centralizes \( G^0 \) and \( j \in G^0 \).

Consider \( V \) as a representation \( \rho^0 \) of \( G^0 \) and reorder the factors to write

\[
V = \mathbb{R}^{m_1} \otimes \cdots \otimes \mathbb{R}^{m_a} \otimes \mathbb{H}^{n_1/2} \otimes \cdots \otimes \mathbb{H}^{n_b/2}
\]

where \( m_i \geq 3 \) is odd and \( n_j \geq 2 \) is even. Suppose \( V \) is of real type (\( b \) is even). Then \( q = \pm 1 \).

Since \( q \) does not lie in \( G^0 \), we must have \( q = -1 \) and \( b = 0 \), namely, all factors of \( G^0 \) are isomorphic to \( \text{SO}(3) \) and \( w = -j \), where \( j \in G^0 \). Write \( j = j_1 \cdots j_a \) where \( j_i \) is the component of \( j \) in the \( i \)th factor of \( G^0 \), and assume \( j_i \neq 1 \) precisely for \( 1 \leq i \leq a' \) for some \( 0 \leq a' \leq a \). On one hand, the dimension formula (2) gives

\[
\dim V^j = \dim V - \dim V^w = \dim G - \dim C(w) + 1 = 3a - (a' + 3(a - a')) + 1 = 2a' + 1.
\]

On the other hand,

\[
\dim V^j = m_{a'+1} \cdots m_a \dim (\mathbb{R}^{m_1} \otimes \cdots \otimes \mathbb{R}^{m_a'})^{j_1 \cdots j_a} = m_{a'+1} \cdots m_a \sum g_1 \cdots g_{a'}
\]

where \( g_i = e_i \) or \( f_i \) and the sum runs through all possibilities with an even number of \( f_i \)’s; here

\[
e_i = \dim (\mathbb{R}^{m_i})^j = \begin{cases} p_i & \text{if } p_i \text{ is odd,} \\ p_i + 1 & \text{if } p_i \text{ is even} \end{cases}
\]

and

\[
f_i = \dim (\mathbb{R}^{m_i})^{-j} = \begin{cases} p_i + 1 & \text{if } p_i \text{ is odd,} \\ p_i & \text{if } p_i \text{ is even} \end{cases}
\]

and \( m_i = 2p_i + 1 \geq 3 \) for all \( i \).
Since $e_i \geq 1$ and $f_i \geq 2$, we estimate
\[
\sum g_1 \cdots g_{a'} = e_1 \cdots e_{a'} + f_1 f_2 e_3 \cdots e_{a'} + \cdots
\]
\[
\geq \sum_{\ell=0}^{[\frac{a'}{2}]} \left( \frac{a'}{2} \right)^{2\ell}
\]
\[
= \frac{1}{2} [3^{a'} + (-1)^{a'}].
\]
We deduce that
\[
2a' + 1 \geq m_{a'+1} \cdots m_a \frac{1}{2} [3^{a'} + (-1)^{a'}] \quad \text{and} \quad 2a' + 1 \geq \frac{1}{2} [3^{a'} + (-1)^{a'}].
\]
The second inequality can be satisfied only if $a' \leq 2$; in this case, using $a \geq 3$, we see that the first inequality is never satisfied. Hence $V$ cannot be of real type.

We finally take up the case in which $V$ is of quaternionic type ($b$ is odd). Then $q^2 = j^{-2}$ lies in the center of $G^0$, which is isomorphic to $(\mathbb{Z}_2)^b$, and $q$ is a unit quaternion multiplying on the right. It follows that $q^2 = \pm 1$. If $q^2 = 1$, then irreducibility of $G^0$ implies that one of the $\pm 1$-eigenspaces of $q$ is trivial, namely, $q = \pm 1$, but then $q \in G^0$ as $b$ is odd, a contradiction. If $q^2 = -1$, then $q$ defines a complex structure with respect to which $w$ is a complex involution. In particular, the fixed point set $V^w$ has even dimension. Note that $m := \dim C(w) \equiv a + b \mod 2$. The dimension formula yields
\[
\dim V - 1 = 3(a + b) - m + \dim V^w
\]
leading to a contradiction as $\dim V$ is even, too.

\subsection{3.4.2 Outer automorphism}

We next discuss the case in which $w$ acts on $G^0$ by an outer automorphism. Since $w$ is not inner, its action on $G^0$ induces a non-trivial involutive permutation $\sigma$ of the factors, and its action on $V$ induces a corresponding involutive permutation of the factors of $V$. Consider $w_0 \in O(V)$ given by (on complex tensors)
\[
w_0(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \tag{3}
\]
Then $w_0^{-1}w$ induces an inner automorphism of $G^0$, so we can write $w = w_0 h z$ where $h = (h_1, \ldots, h_k) \in G^0$ and $z \in O(V)$ centralizes $G^0$.

Consider $V$ as a representation $\rho^0$ of $G^0$ and reorder the indices to write
\[
V = V_1 \otimes_R \cdots \otimes_R V_a \otimes_R \mathbb{R}^{m_1} \otimes_R \cdots \otimes_R \mathbb{R}^{m_b} \otimes_R \mathbb{H}^{p_1/2} \otimes_R \cdots \otimes_F \mathbb{H}^{p_c/2}
\]
where $n_i \geq 3$ is odd, $p_j \geq 2$ is even, $F = \mathbb{H}$ or $\mathbb{R}$ according to whether $c$ is even or odd; moreover, $V_i = \mathbb{R}^{m_i} \otimes_R \mathbb{R}^{m_i}$ and $w_0$ interchanges the coordinates in case $m_i$ is odd, or $V_i = \mathbb{H}^{m_i/2} \otimes_\mathbb{H} \mathbb{H}^{m_i/2}$ and $-w_0$ interchanges the coordinates in case $m_i$ is even; finally, $w_0$ fixes the other factors of $V$.

If $c$ is even, then $V$ is of real type, so $z = \pm 1$; if $c$ is odd, then $V$ is of quaternionic type, so $z$ is right multiplication on $\mathbb{H}^{p_c/2}$ by some element of $Sp(1)$ (we view quaternionic vector spaces as right $\mathbb{H}$-modules); in the latter case, $w_0$ fixes the factor $\mathbb{H}^{p_c/2}$, so in any case $w_0$ commutes with $z$. Now $w^2 = 1$ gives that
\[
(h_{\sigma(1)}h_1, \ldots, h_{\sigma(k)}h_k) = z^{-2}
\]
lies in the center of $G^0$. We deduce that $h_{2i} = \pm h_{2i-1}^{-1}$ for $i = 1, \ldots, a$. Now we can take
\[
\tilde{h} = (1, h_1^{-1}, \ldots, 1, h_{\ell}^{-1}, 1, \ldots, 1) \in G^0,
\]
where $\ell = 2a - 1$, to replace $w$ by the conjugate element $\tilde{w} = \tilde{h} w \tilde{h}^{-1}$ in somehow simpler form, namely,
\[
\tilde{w} = w_0(1, \pm 1, \ldots, 1, \pm 1, h_{2a+1}, \ldots, h_k) z,
\]
or yet
\[
\tilde{w} = w_0(1, \ldots, 1, h_{2a+1}, \ldots, h_k) z,
\]
by replacing $z$ by $-z$, if necessary, taking into account the kernel of $\rho$.

We have
\[
\dim V = m_1^2 \cdots m_a^2 \cdot n_1 \cdots n_b \cdot p_1 \cdots p_c \cdot \theta,
\]
where $\theta = 1$ or 2 whether $c$ is even or odd,
\[
\dim G = 3(2a + b + c) \quad \text{and} \quad \dim C(w) = \dim C(\tilde{w}) \geq 3a + b + c.
\]
The $\pm 1$-eigenspaces of $w_0$ on $V_i$ have dimension $\frac{m_i(m_i \pm 1)}{2}$. We deduce that
\[
\dim V^w = \dim V^{\tilde{w}} \leq \max\{\dim V^{w_0}, \dim V^{-w_0}\}
= M^{\text{even}} \cdot m_1 \cdots m_a \cdot n_1 \cdots n_b \cdot p_1 \cdots p_c \cdot \theta
\]
where
\[
M^{\text{even}} = \sum m_i^\pm \cdots m_a^\pm,
\]
and the sum runs through all combinations with an even number of negative signs. The dimension formula (2) now gives
\[
M^{\text{odd}} \cdot m_1 \cdots m_a \cdot n_1 \cdots n_b \cdot p_1 \cdots p_c \cdot \theta \leq 3a + 2b + 2c + 1,
\]
where $M^{\text{odd}} = \sum m_i^\pm \cdots m_a^\pm$ with the combinations being now taken with an odd number of negative signs. Since $m_i^\pm \geq \frac{3}{2}$ and $m_i^- \geq \frac{1}{2}$, we estimate
\[
M^{\text{odd}} \geq \sum_{\ell=0}^{[a-1]} \left( \begin{array}{c} a \\ 2\ell + 1 \end{array} \right) \frac{2a - (2\ell + 1)}{2^a}
= \left( \frac{3}{2} \right)^a \frac{1}{2} \left[ \left( 1 + \frac{1}{3} \right)^a - \left( 1 - \frac{1}{3} \right)^a \right]
= \frac{1}{2} (2^a - 1)
\geq 2^{a-2}.
\]
We deduce that
\[
2^{a-2} 3^b 2^c \cdot \theta \leq 3a + 2b + 2c + 1.
\]
It immediately follows from (6) that $a \leq 2$, and a quick run through the possibilities using (5) and (6), excluding polar representations, yields that $\rho^0$ must be $(\text{SO}(4) \times \text{SO}(3), \mathbb{R}^4 \otimes \mathbb{R}^8)$ for the values $a = b = 1, c = 0$, $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes \mathbb{R}^8)$ for $a = 1, b = 0, c = 2$ or

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are equivariant isomorphisms in the end of the proof of Lemma 2). The group $G$ is isomorphic to the subgroup of reflections $\mathbb{O}(5)$, and we get its image is a group generated by reflections $\mathbb{O}_+(5)$, but this implies $m = 6$ which contradicts the dimension formula. The second representation is also discarded by a completely analogous reasoning.

For the only remaining case $\rho^0 = (\text{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$, we check that $\tilde{w} = w_0$. Indeed we have $V = \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ and $\tilde{w} = w_0(1, 1, h_3)\tilde{z}$ where $w_0$ permutes the first two factors of $V$ and $h_3, \tilde{z} \in \text{Sp}(1)$ act on the third factor from the left and from right, respectively; in other words, $V = (\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \otimes \mathbb{R}^4$ and $\tilde{w} = w_0q$ where $q \in \text{SO}(4)$. Since $\tilde{w}$ and $w_0$ are involutions, and $w_0$ and $q$ commute, also $q$ is an involution. Since dim $V = 16$ and $m = 6$, the dimension formula (2) gives $m + 6 = f$ where $m = \dim C(w)$ and $f = \dim V^w$. Clearly $m \geq 4$ and thus $f \geq 10$. On the other hand,

$$f = \dim(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H})^{w_0} \dim(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H})^{-w_0} \dim(\mathbb{H} \otimes \mathbb{H})^{-q} = 3 \dim(\mathbb{H} \otimes \mathbb{H})^q + \dim(\mathbb{H} \otimes \mathbb{H})^{-q}$$

where $\dim(\mathbb{H} \otimes \mathbb{H})^{w_0}$ is even. It follows that $\dim(\mathbb{H} \otimes \mathbb{H})^q = 4$, that is, $q = 1$ and $\tilde{w} = w_0$.

Remark 1 Since we use both descriptions of $V$ in terms of complex tensor products and quaternionic-real tensor products, we explain some details in the quaternionic case. Recall that $\mathbb{H}^{m/2} \otimes \mathbb{H}^{m/2}$ can be interpreted as the module tensor product $\mathbb{H}^{m/2} \otimes \mathbb{H}^{m/2}$. There are equivalence isomorphisms

$$\mathbb{C}^m \otimes \mathbb{C}^m \rightarrow \mathbb{C}^m \otimes \epsilon \mathbb{C}^m \cong \text{End}_\mathbb{C}(\mathbb{C}^m) \quad u \otimes v \mapsto u \otimes (\epsilon v)^* \mapsto (\epsilon v)^*(\cdot)u$$

where $\epsilon$ denotes the quaternionic structure (conjugate-linear endomorphism with square equal to minus identity) yielding the identification $\mathbb{C}^m \cong \mathbb{H}^{m/2}$ and the conjugate-linear map $v \in \mathbb{C}^m \mapsto v^* \in \mathbb{C}^{m*}$ is via the invariant Hermitian inner product. The real form of $\mathbb{C}^m \otimes \mathbb{C}^m$ with respect to $\epsilon \otimes \epsilon$, that we call $\mathbb{H}^{m/2} \otimes \mathbb{H}^{m/2}$, corresponds to $\text{End}_\mathbb{H}(\mathbb{H}^{m/2})$ and is invariant under $w_0$. Finally, the action of $w_0$ on $\mathbb{H}^{m/2} \otimes \mathbb{H}^{m/2}$ corresponds to taking minus transpose quaternion-conjugate in $\text{End}_\mathbb{H}(\mathbb{H}^{m/2})$.

3.5 Discrete part

It follows from the discussion in the previous two subsections that $G$ is a finite extension of $G^0 = \text{Sp}(1)^3$ and $V = \otimes^3 \mathbb{C}^2$. The next step is to determine the exact nature of $G$. Since $G$ acts with trivial principal isotropy groups, $G/G^0 \rightarrow \text{Isom}(V/G^0)$ is injective; moreover, its image is a group generated by reflections [9, Prop. 3.2]. We have seen that we can select a generating set induced by nice involutions $w \in G$ that act on $G^0$ by an outer automorphism and act on $\otimes^3 \mathbb{C}^2$ as in (3), where $\sigma$ is a transposition.

Lemma 3 The group $G$ is $\Gamma \ltimes G^0$ where $\Gamma = A_1$ or $A_2$.

Proof The selected nice involutions generate a group $\Gamma$ of permutations of the factors of $V = \otimes^3 \mathbb{C}^2$ and thus $G = \Gamma \ltimes G^0$, where $\Gamma$ is a subgroup of $A_2$. Note that $\Gamma$ can only be $A_1$ or $A_2$, since it is non-trivial and contains an element of order 2. □
Both cases of $\Gamma$ described in Lemma 3 do occur: the first one for the rank 4 real Grassmannian and the second one for the rank 4 exceptional quaternion-Kähler symmetric spaces. We will see below that no further examples exist.

### 3.6 Symmetries of $X$

The key point is that $\rho : G \to O(V)$ is a representation of quaternionic type. Indeed $\Gamma$ clearly acts by complex transformations on $V = \otimes^3 \mathbb{C}^2$, but also preserves the quaternionic structure on $V$ as this is given by $\epsilon \otimes \epsilon \otimes \epsilon$ where $\epsilon$ is the quaternionic structure on $\mathbb{C}^2$. In particular, the centralizer $\text{Sp}(1)'$ of $G^0$ in $O(V)$ also centralizes $G$ and acts thus on $X = V / G = W / H$:

$$
\begin{align*}
W \\
\downarrow \\
X = W / H = \otimes^3 \mathbb{C}^2 / \Gamma \cdot \text{Sp}(1)^3 \sim \text{Sp}(1)'
\end{align*}
\downarrow
\begin{align*}
Y = \otimes^2 \mathbb{R}^4 / \Gamma \cdot \text{SO}(4)^2
\end{align*}
$$

Denote the composite map $W \to Y$ by $\pi$. Since $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes \mathbb{R}^4)$ is a polar representation (indeed the isotropy representation of the rank 4 real Grassmannian manifold), $Y$ is a flat Riemannian orbifold (of dimension 4) and hence the components of the level sets of $\pi$ yield an isoparametric foliation $\mathcal{F}$ by full irreducible submanifolds (compare [10, §2]). The codimension of $\mathcal{F}$ is 4, so by a theorem of Thorbergsson [22], $\mathcal{F}$ is homogeneous, namely, the maximal connected subgroup $\hat{H}$ of $O(W)$ that preserves the leaves of $\mathcal{F}$ acts transitively on them. By definition, $\hat{H}$ is closed, acts polarily, and contains $H$. In particular, it acts irreducibly on $W$. It follows that $\hat{\tau} : \hat{H} \to O(W)$ is the isotropy representation of an irreducible symmetric space, of rank 4.

The geometry of $Y$ can be understood. The Coxeter group of the polar representation $(\text{SO}(4) \times \text{SO}(4), \mathbb{R}^4 \otimes \mathbb{R}^4)$ is $D_4$, so $Y = \mathbb{R}^4 / \Gamma'$ where $\Gamma'$ is a finite extension of $D_4$ by $\Gamma$, of order $2\#(D_4)$ or $3\#(D_4)$, that acts irreducibly on $\mathbb{R}^4$. Recall that $\text{Aut}(D_4) \subset \text{Ad}(F_4)$, that is, every automorphism of $D_4$ becomes inner in $F_4$ (see e.g. [1, Thm. 14.2]). Since the representation of $\Gamma'$ on $\mathbb{R}^4$ is irreducible of real type, we deduce that every element of $\Gamma'$ differs from an element of $F_4$ by $\pm 1$, but $-1 \notin D_4$. This proves that $\Gamma'$ is a subgroup of $F_4$. Since $F_4 = D_4 \times A_2$ and $B_4 = D_4 \times A_1$, we deduce that $\Gamma' = B_4$ or $F_4$ according to whether $\Gamma = A_1$ or $A_2$.

We refer to the classification of isotropy representations of symmetric spaces [23, ch. 8] to list the possibilities for $\hat{\tau} : \hat{H} \to O(W)$:

| Case | $\hat{\tau}$ | Condition $\Gamma'$ |
|------|---------------|----------------------|
| 1    | $(\text{SO}(n) \times \text{SO}(4), \mathbb{R}^n \otimes \mathbb{R}^4)$ | $n \geq 5$ | $B_4$ |
| 2    | $(\text{SU}(n) \times \text{U}(4), \mathbb{C}^n \otimes \mathbb{C} \mathbb{C}^4)$ | $n \geq 4$ | $B_4$ |
| 3    | $(\text{Sp}(n) \times \text{Sp}(4), \mathbb{H}^n \otimes \mathbb{H} \mathbb{H}^4)$ | $n \geq 4$ | $B_4$ |
| 4    | $(\text{U}(8), \mathbb{L}^2 \mathbb{C}^6)$ | – | $B_4$ |
| 5    | $(\text{U}(9), \mathbb{L}^2 \mathbb{C}^9)$ | – | $B_4$ |
| 6    | $(\text{U}(4), \mathbb{L}^2 \mathbb{C}^4)$ | – | $B_4$ |
| 7    | $(\text{Sp}(3) \text{Sp}(1), (\mathbb{L}^3 \mathbb{C}^6 \otimes \mathbb{C}^6) \otimes \mathbb{H} \mathbb{C}^2)$ | – | $F_4$ |
| 8    | $(\text{SU}(6) \text{SU}(2), \mathbb{L}^3 \mathbb{C}^6 \otimes \mathbb{H} \mathbb{C}^2)$ | – | $F_4$ |
| 9    | $(\text{Spin}(12) \text{SU}(2), \mathbb{C}^{32} \otimes \mathbb{H} \mathbb{C}^2)$ | – | $F_4$ |
| 10   | $(\text{E}_7 \text{SU}(2), \mathbb{C}^{56} \otimes \mathbb{H} \mathbb{C}^2)$ | – | $F_4$ |
In each case, we need to find all closed subgroups $H$ of $\hat{H}$ that act irreducibly with cohomogeneity 7 on $W$. It suffices to test for maximal connected closed subgroups of $\hat{H}$ as compiled by Dynkin (6); see also [12, Thm. 2.3]) that act with cohomogeneity bounded by 7 and to continue iteratively as needed.

**Claim** The procedure above retrieves the representations listed in the Theorem plus $(\text{Spin}(7) \times \text{U}(2), \mathbb{R}^8 \otimes \mathbb{C}^2)$ and $(\text{SU}(4) \times \text{Sp}(2), \mathbb{C}^4 \otimes \mathbb{C}^4)$.

We defer the verification of this claim to the appendix.

### 3.7 End of the proof

To finish the proof, we use metric arguments to show that the representations $(\text{Spin}(7) \times \text{U}(2), \mathbb{R}^8 \otimes \mathbb{C}^2)$ and $(\text{SU}(4) \times \text{Sp}(2), \mathbb{C}^4 \otimes \mathbb{C}^4)$ do not admit the type of reduction as in the Theorem. Since both of them correspond to the case $I'' = B_4$, we need only show they cannot be quotient-equivalent to the model $(\mathbb{Z}_2 \rtimes \text{Spin}(1)^3, \otimes^3 \mathbb{C}^2) = (\text{O}(4) \times \text{Sp}(1), \mathbb{R}^4 \otimes \mathbb{R}^4)$. We accomplish this by comparing the boundaries of the orbit spaces.

Given a representation of a compact Lie group $G$ on an Euclidean space $V$ with orbit space $X$, for an isotropy group $K$ at a point of $V$, we denote the associated isotropy stratum of $X$ by $X_K$. Thus $X_K$ is the projection of the set of points of $V$ whose isotropy groups are conjugate to $K$.

The following notion turns out to be useful. We say the representation of $G$ on $V$ has *orbit space with linear boundary* if there is a unique isotropy stratum in the orbit space $X = V/G$ of codimension one. In this case, if $p \in V$ is a $G$-important point and $K = G_p$, then the image of the natural map $I_K : V^K/N_G(K) \to V/G$ is $\partial X$ [9, Lem. 10]. Whereas it is not clear whether this condition is preserved under quotient-equivalence (namely, we do not know whether a representation quotient-equivalent to one with this property must also have this property), in our applications we will replace this property by a stronger one which is clearly preserved under quotient-equivalence.

We first prove a lemma of general interest.

**Lemma 4** Let $\rho : G \to \text{O}(V)$ be any representation and consider the isotropy group $K$ of an arbitrary point in $V$. Denote the normalizer of $K$ in $G$ by $N$, the fixed point set of $K$ in $V$ by $W$, and the orbit space $W/N$ by $Y$. Then, for the natural map $I = I_K : Y \to X$ as above, we have:

(i) the image of any isotropy stratum $Y_L$ under $I$ is a finite union of isotropy strata of $X$;

(ii) $I(Y_{L_1}) \cap I(Y_{L_2}) = \emptyset$ if $(L_1) \neq (L_2)$;

(iii) if $X_M \subset I(Y_L)$, then $I^{-1}(X_M)$ is a submanifold of dimension $\dim X_M$ of $Y_L$.

**Proof** Take $q \in W$ projecting to $y \in Y_L$ such that $N_q = L$ and note that $N_q$ is the normalizer of $K$ in $G_q$. It follows that $I(y) \in X_{(G_q)}$ and $I(Y_L)$ contains $X_{(G_q)}$. Now (i) follows and (ii) is an easy consequence.

In order to prove (iii), denote the inclusion of $W$ in $V$ by $\iota$. We take $q \in W$ projecting to $y \in Z := I^{-1}(X_M)$ and construct a local chart of $Y_L$ around $y$ adapted to $Z$, where $L = N_q$ and $M = G_{\iota(q)} = G_q$. In fact, denote by $\pi_X : V \to X$ and $\pi_Y : W \to Y$ the natural projections, and let $S$ be a normal slice at $p$ for $(G, V)$. Then $S \cap W$ is a normal slice at $p$ for $(N, W)$ (compare [11, Lem. 10]), and $\pi_Y : (S \cap W)^L \to Y_L$ and $\pi_X : S^M \to X_M$ are local parametrizations around $y$ and $x := I(y)$, respectively.

Since $M$ contains $K$ and $L$, and $\pi_X^{-1}(X_M) \cap S = S^M$, it is straightforward to check that $I((S \cap W)^L \cap \pi_Y^{-1}(Z)) = S^M$. Therefore $\pi_Y : (S \cap W)^L \to Y_L$ restricts to a local parametrization $S_0 \to Z$, where $S_0 := (S \cap W)^L \cap \pi_Y^{-1}(Z)$. \qed
Proposition 1 Suppose \( \rho : G \to \mathcal{O}(V) \) and \( \rho' : G' \to \mathcal{O}(V') \) are quotient-equivalent and have orbit spaces \( X \) and \( X' \) with non-empty boundary. If \( X \) has no strata of codimension 2 contained in \( \partial X \), then \( X \) and \( X' \) have linear boundary and the corresponding representations \( (N := N_G(K), V^K) \) and \( (N' := N_{G'}(K'), V'^{K'}) \) are quotient-equivalent.

**Proof** By the Frankel–Petrunin Theorem [16, Cor. 3.3], any two codimension one strata of \( X \) must have a non-empty intersection, so the assumption on the codimension 2 strata of \( X \) implies that \( X \) and also \( X' \) has linear boundary. By Lemma 4, there are no codimension 1 strata in \( Y \), so \( \partial Y \) is empty and \( Y_0 := I^{-1}(X_{(K)}) \) is obtained from \( Y_{reg} \) by removing finitely many submanifolds of codimension bigger than 1. It follows that \( Y_0 \) is path-connected; moreover, in general the restriction of the metric of \( Y \) to \( Y_{reg} \) is intrinsic by convexity of the principal stratum, and in this case it remains intrinsic when restricted to \( Y_0 \). Similar remarks apply to \( Y_0' \).

Now the isometry \( X \to X' \) restricts to an isometry \( X_{(K)} \to X_{(K')} \). Since \( I : Y_0 \to X_{(K)} \) and \( I : Y_0' \to X_{(K')} \) are bijective local isometries [11, Lem. 10] and the metrics on \( Y_0 \) and \( Y_0' \) are intrinsic, we get an isometry \( f_0 : Y_0 \to Y_0' \). By a density-completeness argument, \( f_0 \) can be extended to an isometry \( f : Y \to Y' \), as wished.

Owing to the discussion in Sect. 3.4, the boundary of the orbit space of \( (\mathcal{O}(4) \times \text{Sp}(1), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4) \) originates from the fixed point set of a nice involution \( w \in \mathcal{O}(4) \setminus \text{SO}(4) \), which can be taken as \( w = \text{diag}(-1, 1, 1, 1) \). Namely, the representation has orbit space \( X \) with linear boundary and \( N = \mathcal{O}(3) \times \text{Sp}(1), V^K = \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4 \). In this case it follows from [9, §11.4] that \( X \) has no strata of codimension 2 contained in \( \partial X \), so Proposition 1 can be applied.

Proposition 2 Let \( \rho' \) be one of the representations \( (\text{SU}(4) \times \text{Sp}(2), \mathbb{C}^4 \otimes_{\mathbb{C}} \mathbb{C}^4) \) or \( (\text{Spin}(7) \times \text{U}(2), \mathbb{R}^8 \otimes_{\mathbb{R}} \mathbb{C}^2) \). Then \( \rho' \) and \( \rho = (\mathcal{O}(4) \times \text{Sp}(1), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4) \) are not quotient-equivalent.

**Proof** We argue by contradiction and apply Proposition 1. If \( \rho \) and \( \rho' \) are quotient-equivalent, then the former proposition says that \( (N'/K', V^{K'}) \) and \( (\mathcal{O}(3) \times \text{Sp}(1), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4) \) are quotient-equivalent. It follows from [9, Prop. 3.1] that \( ((N'/K')^0, V^{K'}) \) is quotient-equivalent to \( (\mathcal{O}(3) \times \text{Sp}(1), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4) \) or its restriction to the identity component, but it was proved in the same paper that the latter representations cannot be a reduction of an irreducible representation of a connected group of bigger dimension. We get \( \dim N'/K' = \dim \mathcal{O}(3) \times \text{Sp}(1) = 6 \); as argued in Sect. 2.2, the center of \( (N'/K')^0 \) is at most one-dimensional, so \( (N'/K')^0 \) is locally isomorphic to \( \text{Sp}(1)^2 \).

The representation \( \rho' \) has trivial pig. Since \( G' \) is connected, \( X' \) has no \( \mathbb{Z}_2 \)-boundary components. If \( K' = S^1 \) then \( \text{rank}(N') = \text{rank}(G') = 5 \). But this contradicts the fact that \( (N'/K')^0 \) is locally isomorphic to a group of rank 2.

Therefore it must be \( K' = S^3 \). In case \( G' = \text{SU}(4) \times \text{Sp}(2) \), this is not possible, for the result from [20, §13] implies that \( X' \) admits no \( S^3 \)-boundary components. In case \( G' = \text{Spin}(7) \times \text{U}(2) \), we argue that \( (N')^0 \) is locally isomorphic to \( \text{Sp}(1)^3 \) and hence semisimple, which contradicts the fact that \( G' \) has a 1-dimensional center.

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Appendix

Here we check the claim stated in Sect. 3.6. We run through the cases listed in the table therein.

In case 4, dim \( W = 56 \), so we need \( \dim H \geq 56 - 7 = 49 \). The only closed subgroups of \( \text{U}(8) \) in this dimension range are \( \text{SU}(8) \) and \( \text{U}(7) \times \text{U}(1) \); however the first group acts with cohomogeneity 5 and the second one does not act irreducibly on \( W \).

In case 5, we need \( \dim H \geq 65 \). The only possibilities are \( \text{SU}(9) \) which acts with cohomogeneity 4 (it is orbit-equivalent to \( \text{U}(9) \)), and \( \text{U}(8) \times \text{U}(1) \) which acts reducibly.

In case 6, \( \hat{H}_{\text{princ}} \) is finite so \( H \) would have to have dimension \( \dim W - 7 = 13 \), but \( \text{U}(4) \) admit no closed subgroups of dimension 13.

**Lemma** Let \( \xi : K \to \text{O}(U) \) be an representation of a compact Lie group \( K \) with \( \dim K > 0 \). Then \( c(2\xi) \geq 2c(\xi) + 1 \).

**Proof** It follows from the fact that the action of \( K_{\text{princ}} \) on \( U \) preserves the decomposition of \( U \) into the tangent and normal spaces to a principal orbit.

Consider now case 1. Take a maximal subgroup of \( \hat{H} \) of the form \( H_1 \times \text{SO}(4) \). Note that \( U(\tfrac{n}{2}) \times \text{SO}(4) \) with \( n \) even has cohomogeneity at least 9 if \( n \geq 6 \) and 6 if \( n = 4 \). Therefore, in case \( W_1 \) admits an \( H_1 \)-invariant complex structure, the only possibility is \( n = 4 \) and \( H = \text{SU}(2) \times \text{SO}(4) \), which is already reduced (see end of Sect. 3.3). Assume next \( W_1 \) does not admit an \( H_1 \)-invariant complex structure. Using the lemma above, for \( c(H_1) \geq 3 \) we have

\[
 c(H_1 \otimes \text{SO}(4)) \geq c(4H_1) - 6 \geq 15 - 6 = 9,
\]

which is too big. If \( c(H_1) = 2 \), we can follow ideas from [9, Lem. 14.1] to see that \( (H_1)_{\text{princ}} \) acts on the tangent space \( U \) to a principal orbit in \( W_1 \) with at least 3 invariant subspaces, so taking a slice representation at a pure tensor \( w_1 \otimes w_2 \) with the \( w_i \) regular points, we have

\[
 c(H_1 \otimes \text{SO}(4)) = 2 + c((H_1)_{\text{princ}} \times \text{SO}(3), (U \otimes_\mathbb{R} \mathbb{R}^3) \oplus \mathbb{R}^3) \\
= 3 + c((H_1)_{\text{princ}} \times \text{SO}(2), U \otimes_\mathbb{R} (\mathbb{R}^2 \oplus \mathbb{R})) \\
\geq 3 + c((H_1)_{\text{princ}}, 3U) - 1 \\
\geq 2 + 3c((H_1)_{\text{princ}}, U) \\
= 11,
\]

which is too big. If \( c(H_1) = 1 \), we run through the possible representations of real type (see e.g. [7]) to find that the condition \( c(H_1 \otimes \text{SO}(4)) \leq 7 \) implies \( H_1 = \text{Spin}(7) \). Now \( \text{Spin}(7) \otimes \text{SO}(4) \) is a representation with trivial pig and cohomogeneity 5, so we need a subgroup of codimension 2 of \( \text{SO}(4) \). We finally get the first candidate \( H = \text{Spin}(7) \times U(2) \hookrightarrow \text{SO}(8) \times \text{SO}(4) \).

Continuing with case 1, consider a maximal subgroup \( \text{SO}(n) \times H_2 \) which acts irreducibly on \( W \) with cohomogeneity 7 where \( H_2 \subset \text{SO}(4) \). The form of the pig of \( \text{SO}(n) \otimes \text{SO}(4) \) shows that \( H_2 \) has codimension 3 in \( \text{SO}(4) \); further it acts irreducibly on \( W_2 \) so \( H_2 \) must be one of the factors of \( \text{SO}(4) = \text{SU}(2)\text{SU}(2) \), as desired.

Consider next case 2. Take the maximal subgroup \( \text{Sp}(\tfrac{n}{2}) \times U(4) \) of \( \hat{H} \) for \( n \) even. We have \( c(\text{Sp}(\tfrac{n}{2}) \times U(4)) \geq 11 \) if \( n \geq 6 \) and \( c(\text{Sp}(2) \otimes U(4)) = 6 \). Since \( \text{Sp}(2) \times U(4) \) acts with trivial pig, we need a codimension one subgroup. We get our second candidate \( H = \text{Sp}(2) \times \text{SU}(4) \subset \text{S}(U(4) \times U(4)) \).
Still in case 2, consider the maximal subgroup $SU(p) \times SU(q) \times U(4)$ of $\hat{H}$, where $pq = n$, $p \geq q$, $p \geq 3$, $q \geq 2$. By the Monotonicity Lemma [9, Lem. 12.1], its cohomogeneity is bounded below by $c(U(2) \otimes SU(2) \otimes SU(4)) = c(SO(4) \otimes U(4)) = 10$, which is too big. The same estimate rules out the maximal subgroup $SO(n) \times U(4)$.

Continuing with case 2, we need also to consider maximal subgroups of $\hat{H}$ of the form $\tau_1(H_1) \times U(4)$ where $H_1$ is simple and $\tau_1$ is an irreducible representation of complex type. Since

$$7 \geq c(\tau_1(H_1) \otimes U(4)) \geq c(\tau_1(H_1) \otimes U(2)) \geq 2c(\tau_1(H_1)) - 4,$$

we deduce $2 \leq c(\tau_1(H_1)) \leq 5$. In this situation the underlying groups have dimension too small to attain the desired cohomogeneity [9, §12.8]. Next, a maximal subgroup of $\hat{H}$ of the form $U(n) \times H_2$ where $H_2 \subset Sp(4)$ has

$$c(U(n) \otimes H_2) \geq c(U(4) \otimes H_2) = c(H_2 \otimes U(4)) > 7$$

by the arguments above, unless $H_2 = Sp(2)$. Now we have $c(U(n) \otimes Sp(2)) = c(SU(n) \otimes Sp(2)) = 6$ for $n \geq 5$. Finally, in case $n = 4$, the diagonal $U(4)$-subgroup of $U(1)(SU(4) \times SU(4))$ acts reducibly on $W$ and this finishes case 2.

We next move to case 3. Consider maximal subgroups of $\hat{H}$ of the form $\tau_1(H_1) \times Sp(4)$ where $H_1$ is simple and $\tau_1$ is an irreducible representation of quaternionic type. Since

$$7 \geq c(\tau_1(H_1) \otimes Sp(4)) \geq c(\tau_1(H_1) \otimes Sp(2)) \geq 2c(\tau_1(H_1)) - 10,$$

we deduce $2 \leq c(\tau_1(H_1)) \leq 8$. Under this condition, all cases (see [9, §12.8]) give $c(\tau_1(H_1) \otimes Sp(4)) > 7$ simply by counting dimensions, but $\tau_1 = (Sp(1), \mathbb{H}^2)$ which gives $c(Sp(1) \times Sp(4), \mathbb{H}^2 \otimes \mathbb{H}^4) = 3$, and there are no more groups to be considered. Continuing, using the Monotonicity Lemma in the estimates

$$c(U(n) \otimes Sp(4)) \geq c(U(4) \otimes Sp(4)) = 12$$

and

$$c(SO(p) \otimes Sp(q) \otimes Sp(4)) \geq c(SO(3) \otimes Sp(1) \otimes Sp(4)) = 9$$

with $pq = n$, $p \geq 3$, $q \geq 1$, we further discard two classes of maximal subgroups of $\hat{H}$. We finish this case by excluding maximal subgroups of type $Sp(n) \times H_2$ and the diagonal subgroup of $Sp(4) \times Sp(4)$ in a similar manner to case 2.

The remaining cases (7 to 10) are associated to quaternion-Kahler symmetric spaces and give examples as described in Sect. 2. Note also that replacing the $Sp(1)$-factor by $U(1)$ gives a representation of cohomogeneity 6. In the following we show that subgroups of $\hat{H}$ of the form $H_1Sp(1)$, where $H_1$ is a closed subgroup of the “non-$Sp(1)$-factor” give no examples.

In case 7, dim $W = 28$ and $\hat{H}$ has finite pig, so we need a subgroup of $Sp(3)$ of dimension $28 - \text{dim } SU(2) - 7 = 18$, but it does not exist.

In case 8, dim $W = 40$ and we need closed subgroups of $SU(6)$ of dimension at least 30, however they do not exist.

In case 9, dim $W = 64$ and we need closed subgroups of $Spin(12)$ of dimension at least 54. The only closed maximal subgroup in this dimension range is $Spin(11)$, however it acts reducibly on $W$.

In case 10, dim $W = 112$, so we need closed subgroups $H_1$ of $E_7$ of dimension at least 102. According to [14, §8], there are no such subgroups.
References

1. Adams, J.F.: Lectures on exceptional Lie groups. In: Mahmud, Z., Mimura, M. (eds.) Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1996)
2. Alexandrino, M.M., Lytchak, A.: On smoothness of isometries between orbit spaces. In: Proceedings RIGA 2011, pp. 17–28. Ed. Univ. Bucureşti, Bucharest (2011)
3. Andreev, E.M., Popov, V.L.: Stationary subgroups of points of general position in the representation space of a semisimple lie group. (Russian). Funktsional. Anal. i Prilozhen. 5(4), 1–8 (1971) [English transl., Functional Analysis and Its Applications 5(4), 265–271 (1971)]
4. Andreev, E.M., Vinberg, E.B., Elashvili, A.G.: Orbits of greatest dimension in semi-simple linear Lie groups. (Russian). Funktsional. Anal. i Prilozhen. 1(4), 3–7 (1967) [English transl., Functional Analysis and Its Applications 1(4), 257–261 (1967)]
5. Dadok, J.: Polar coordinates induced by actions of compact Lie groups. Trans. Am. Math. Soc. 288, 125–137 (1985)
6. Dynkin, E.B.: The maximal subgroups of the classical groups. Am. Math. Soc. Trans. 6, 245–378 (1952)
7. Eschenburg, J., Heintze, E.: On the classification of polar representations. Math. Z. 232, 391–398 (1999)
8. Gorodski, C., Kollross, A.: Some remarks on polar actions. Ann. Glob. Anal. Geom. 49, 43–58 (2016)
9. Gorodski, C., Lytchak, A.: On orbit spaces of representations of compact Lie groups. J. Reine Angew. Math. 691, 61–100 (2014)
10. Gorodski, C., Lytchak, A.: Representations whose minimal reduction has a toric identity component. Proc. Am. Math. Soc. 143, 379–386 (2015)
11. Gorodski, C., Lytchak, A.: Isometric actions on spheres with an orbifold quotient. Math. Ann. 365(3–4), 1041–1067 (2016)
12. Gorodski, C., Podestà, F.: Homogeneity rank of real representations of compact Lie groups. J. Lie Theory 15(1), 63–77 (2005)
13. Hsiang, W.C., Hsiang, W.Y.: Differentiable actions of compact connected classical groups, II. Ann. Math. (2) 92(2), 189–223 (1970)
14. Kollross, A.: Low cohomogeneity and polar actions on exceptional compact Lie groups. Transform. Groups 14(2), 387–415 (2009)
15. Lytchak, A.: Geometric resolution of singular Riemannian foliations. Geom. Dedicata 149, 379–395 (2010)
16. Petrunin, A.: Parallel transportation for Alexandrov space with curvature bounded below. Geom. Funct. Anal. 8, 123–138 (1998)
17. Popov, A.M.: Irreducible simple linear Lie groups with finite standard subgroups of general position. (Russian). Funktsional. Anal. i Prilozhen. 9(4), 346–347 (1975) [English transl., Functional Analysis and Its Applications 9(4), 81–82 (1975)]
18. Popov, A.M.: Irreducible semisimple linear Lie groups with finite stationary subgroups of general position. (Russian). Funktsional. Anal. i Prilozhen. 12(2), 91–92 (1978) [English transl., Functional Analysis and Its Applications 12(2), 154–155 (1978)]
19. Popov V.L., Vinberg E.B.: Invariant theory. In: Parshin A.N., Shafarevich I.R. (eds.) Algebraic geometry IV. Encyclopaedia of Mathematical Sciences, vol 55, Ch. II, pp. 123–278. Springer, Berlin, Heidelberg
20. Schwarz, G.W.: Lifting smooth homotopies of orbit spaces. IHES Publ. Math. 51, 37–135 (1980)
21. Straume, E.: On the invariant theory and geometry of compact linear groups of cohomogeneity ≤ 3. Differ. Geom. Appl. 4, 1–23 (1994)
22. Thorbergsson, G.: Isoparametric foliations and their buildings. Ann. Math. 2(133), 429–446 (1991)
23. Wolf, J.: Spaces of Constant Curvature, 5th edn. Publish or Perish, Houston (1984)