A variational representation for $G$-Brownian functionals

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Abstract

The purpose of this paper is to establish a variational representation

$$\log \mathbb{E} \left[ e^{f(B)} \right] = \sup_h \mathbb{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right]$$

for functionals of the $d$-dimensional $G$-Brownian motion $B$. Here $\mathbb{E}$ is a sublinear expectation called $G$-expectation, $f$ is any bounded function in the domain of $\mathbb{E}$ mapping $C([0, 1]; \mathbb{R}^d)$ to $\mathbb{R}$, the integrals are taken with respect to the quadratic variation of $B$, and the supremum runs over all $h$'s for which these integrals are well-defined. As an application, we give another proof of the results obtained by Gao-Jiang (2010), large deviations for $G$-Brownian motion.

1 Introduction

This paper is concerned with $G$-Brownian motion introduced by S. Peng. $G$-Brownian motion can be regarded as a Brownian motion with an uncertain variance process. One of its features is that, while the classical Brownian motion is defined on a probability space, $G$-Brownian motion is defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Here $\Omega$ is a given set and $\mathcal{H}$ is a vector lattice of real-valued functions on $\Omega$ containing 1, which is the domain of a sublinear expectation $\mathbb{E}$. Peng [7, 8] constructed a sublinear expectation space on which the canonical process of the space $\Omega = C([0, 1]; \mathbb{R}^d)$ of continuous paths starting from 0 becomes a $G$-Brownian motion. The sublinear expectation in this space is called $G$-expectation. Also defined in [7, 8] were the quadratic variation process of $G$-Brownian motion, and stochastic integrals with respect to $G$-Brownian motion and its quadratic variation for a certain class of stochastic processes. It is known that any sublinear expectation can be represented as a supremum of linear expectations, referred

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to as an upper expectation. Recently, L. Denis, M. Hu and S. Peng gave a concrete upper expectation representation for $G$-expectation in [3]. Through the upper expectation, a related capacity is defined, and it plays a similar role to a probability measure in the classical stochastic analysis. For instance, Gao-Jiang [5] formulated and proved large deviation principles for $G$-Brownian motion under this capacity.

In this paper, we establish a variational representation for functionals of $G$-Brownian motion:

$$\log \mathbb{E} \left[ e^{f(B)} \right] = \sup_{h \in (M^2_G(0,1))^d} \mathbb{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right]. \quad (1.1)$$

Here $\mathbb{E}$ is $G$-expectation, $B$ is the $d$-dimensional $G$-Brownian motion, $f$ is any bounded function in the domain of $G$-expectation that maps $C([0,1]; \mathbb{R}^d)$ to $\mathbb{R}$, the integrals are taken with respect to the quadratic variation $\langle B \rangle$ of $G$-Brownian motion, and the supremum runs over all $\mathbb{R}^d$-valued processes $h$ for which these integrals are well-defined. Precise definitions will be seen in Section 2.

One of our motivations for this representation comes from large deviation principles for $G$-Brownian motion. It is well known that, on a probability space, the large deviation principle for a given family of random variables is equivalent to its Laplace principle. In [2], M. Boué and P. Dupuis established a variational representation for functionals of Brownian motion and showed its usefulness in the derivation of Laplace principles when the family of concern consists of functionals of Brownian motion. Our variational representation (1.1) has the same application in the framework of $G$-expectation space; indeed, it is also true that the Laplace principle formulated under $G$-expectation is equivalent to the large deviation principle formulated under the capacity, and the representation (1.1) can be used to derive Laplace principles for families of random variables given as functionals of $G$-Brownian motion. As an illustration, we prove the Laplace principles for the families $\{\sqrt{\varepsilon}B; \varepsilon > 0\}$ and $\{(\sqrt{\varepsilon}B, \langle B \rangle); \varepsilon > 0\}$. Large deviations for these families were originally obtained by Gao-Jiang [5]; they employed a discretization technique. Our variational representation gives another proof.

The proof of the representation (1.1) is split into the derivations of the lower and upper bounds. By virtue of approximating method we employ, proofs of these bounds are reduced to showing their validity for a particular class of functions $f$, namely the class of bounded Lipschitz cylinder functions. To obtain the lower bound, Girsanov’s formula for $G$-Brownian motion in [6] allows us to use a similar argument to that in Boué-Dupuis [2]. The proof of the upper bound is in the same spirit as Zhang [10], which extended the representation of Boué-Dupuis to the framework of an abstract Wiener space as simplifying the proof of the upper bound by using the Clark-Ocone formula; we use a type of the Clark-Ocone formula under $G$-expectation (Lemma 3.8) to prove the upper bound. Prior to the proof of the representation (1.1), the well-definedness of the right-hand side has also to be verified, that is, it is needed to show that for any bounded function $f$ in the domain of $G$-expectation, functionals of the form $f(B + \int_0^1 d\langle B \rangle_s h_s)$ with $h$ as described above are again in the domain. A key is to establish an absolute continuity between $B$ and $B + \int_0^1 d\langle B \rangle_s h_s$ under the capacity (Proposition 5.1), which is done by using relative entropy estimates as given in Boué-Dupuis [2] and also by using Girsanov’s formula for $G$-Brownian motion.
We give an outline of the paper. In Section 2, we introduce necessary notions and related results as preliminaries: the construction of $G$-expectation, stochastic integrals for $G$-Brownian motion, the upper expectation for $G$-expectation due to Denis-Hu-Peng \cite{3}, and Girsanov’s formula for $G$-Brownian motion obtained by \cite{6}. Main results of this paper are stated and proved in Section 3; we verify the well-definedness of the right-hand side of (1.1) in Subsection 3.1, and prove the representation (1.1) in Subsections 3.2 and 3.3. In Section 4, we derive large deviation principles for $G$-Brownian motion as an application of our representation. In Section 5, we show an absolute continuity relationship between $B$ and $B + \int_0^\cdot d\langle B \rangle_s h_s$ under the capacity.

Throughout this paper, for a probability measure $P$, $E_P$ denotes the expectation with respect to $P$. For a real-valued function $f$ on any metric space $(X,d)$, we denote by $\text{Lip}(f)$ the Lipschitz constant of $f$:

$$\text{Lip}(f) := \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$  

Other notation will be introduced as needed.

2 $G$-Brownian motion and related stochastic analysis

In this section, we briefly recall from \cite{7} \cite{8} \cite{3} some notions and related results about $G$-Brownian motion and $G$-expectation space. As preparing some necessities such as the notions of $G$-stochastic integrals and $G$-martingales, we then introduce Girsanov’s formula for multidimensional $G$-Brownian motion established in \cite{6}.

2.1 $G$-expectation space and the related capacity

Let $\Omega$ be the set of $\mathbb{R}^d$-valued continuous functions $\omega : [0, 1] \to \mathbb{R}^d$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sup_{0 \leq t \leq 1} |\omega^1_t - \omega^2_t|, \quad \omega^1, \omega^2 \in \Omega. \quad (2.1)$$

For each $t \in [0, 1]$, we also set $\Omega_t := \{\omega_{\cdot, t} : \omega \in \Omega\}$. We denote by $B(\Omega)$ (resp. by $B(\Omega_t)$) the associated Borel $\sigma$-algebra of $\Omega$ (resp. of $\Omega_t$). In the sequel we denote by $B = \{B_t ; 0 \leq t \leq 1\}$ the canonical process in $\Omega$: $B_t(\omega) := \omega_t$, $0 \leq t \leq 1, \omega \in \Omega$. For each $t \in [0, 1]$, let $C_{b,Lip}(\Omega_t)$ be the set of bounded Lipschitzian cylinder functionals on $\Omega_t$:

$$C_{b,Lip}(\Omega_t) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \in \mathbb{N}, \ t_1, \ldots, t_n \in [0, t], \ \varphi \in C_{b,Lip}(\mathbb{R}^{dn})\};$$

when $t = 1$, we simply write $C_{b,Lip}(\Omega)$. Here and below, $C_{b,Lip}(\mathbb{R}^m)$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^m$. Let $\mathbb{R}^{d \times d}$ be the set of $d \times d$ matrices and $\Theta$ a non-empty, bounded and closed subset of $\mathbb{R}^{d \times d}$, the set $\Theta$ is a collection of parameters.
that represents the variance uncertainty of \( G \)-Brownian motion. We associate \( \Theta \) with two constants \( \sigma_1, \sigma_0 \geq 0 \) via
\[
\sigma_0^2 = \inf_{\gamma \in \Theta} \inf_{x \in \mathbb{R}^d} x \cdot \gamma^* x, \quad \sigma_1^2 = \sup_{\gamma \in \Theta} \sup_{x \in \mathbb{R}^d} x \cdot \gamma^* x.
\] (2.2)

For a \( d \times d \) symmetric matrix \( A \), define
\[
G(A) := \frac{1}{2} \sup_{\gamma \in \Theta} \text{tr} [A \gamma^* \gamma].
\]

For a given \( \varphi \in C_{b,Lip}(\mathbb{R}^d) \), we denote by \( u_{\varphi} \) the unique viscosity solution to the following nonlinear partial differential equation called the \( G \)-heat equation:
\[
\begin{cases}
\frac{\partial u}{\partial t} - G(D^2 u) = 0 & \text{in } (0, 1) \times \mathbb{R}^d, \\
u|_{t=0} = \varphi & \text{in } \mathbb{R}^d,
\end{cases}
\] (2.3)

where \( D^2 u = \left( \frac{\partial^2 u}{\partial x^i \partial x^j} \right)_{i,j=1}^d \) is the Hessian matrix of \( u \).

**Remark 2.1.** For the existence and uniqueness of the viscosity solution to (2.3), we refer to [9, Section C.3]. If \( \sigma_0 > 0 \), then the solution to (2.3) becomes a \( C^{1,2} \)-solution.

It is shown in [7, 8] that there exists a unique sublinear expectation functional \( \mathbb{E} : C_{b,Lip}(\Omega) \to \mathbb{R} \) that possesses the following two properties:

(i) for all \( 0 \leq s < t \leq 1 \) and \( \varphi \in C_{b,Lip}(\mathbb{R}^d) \),
\[
\mathbb{E}[\varphi(B_t - B_s)] = \mathbb{E}[\varphi(B_{t-s})] = u_{\varphi}(t-s,0);
\]

(ii) for all \( n \in \mathbb{N}, 0 \leq t_1 \leq \ldots \leq t_n \leq 1 \) and \( \psi \in C_{b,Lip}(\mathbb{R}^d)^n \),
\[
\mathbb{E}[\psi(B_{t_1}, \ldots, B_{t_n})] = \mathbb{E}[\psi_1(B_{t_1}, \ldots, B_{t_{n-1}})],
\] (2.4)

where \( \psi_1 : (\mathbb{R}^d)^{n-1} \to \mathbb{R} \) is defined by
\[
\psi_1(x_1, \ldots, x_{n-1}) = \mathbb{E}[\psi(x_1, \ldots, x_{n-1}, B_{t_{n-1}} - x_{n-1})]
\]
with \( B_{t_{n-1}} = B_t - B_s \) for \( 0 \leq s \leq t \leq 1 \).

For \( t_{k-1} \leq t < t_k \), the related conditional expectation of \( \varphi(B_{t_1}, \ldots, B_{t_n}) \) on \( C_{b,Lip}(\Omega_t) \) is defined by
\[
\mathbb{E}_t[\varphi(B_{t_1}, \ldots, B_{t_n})] := \varphi_{n-k}(B_{t_1}, \ldots, B_{t_{k-1}}, B_t),
\] (2.5)

where \( \varphi_{n-k}(x_1, \ldots, x_{k-1}, x_k) = \mathbb{E}[\varphi(x_1, \ldots, x_{k-1}, B_{t_{k-1}}^t + x_k, \ldots, B_t^t + x_k)] \). The completion of \( C_{b,Lip}(\Omega_t) \) with respect to the norm \( \mathbb{E}[\cdot] \) is denoted by \( \mathcal{L}^1_G(\Omega_t) \); we simply write \( \mathcal{L}^1_G(\Omega) \) for \( \mathcal{L}^1_G(\Omega_1) \). The functional \( \mathbb{E} \) (resp. \( \mathbb{E}_t \)) is then uniquely extended to a sublinear expectation (resp. a conditional sublinear expectation) on \( \mathcal{L}^1_G(\Omega) \). This extension is
called $G$-expectation (resp. conditional $G$-expectation) and will still be denoted by $\mathbb{E}$ (resp. by $\mathbb{E}_t$) in the sequel. The triplet $(\Omega, L^1_G(\Omega), \mathbb{E})$ is called $G$-expectation space, on which the canonical process $B$ is a $d$-dimensional $G$-Brownian motion; for more details, we send the reader to [9] and references therein.

Let $W = \{W_t = (W^1_t, \ldots, W^d_t) ; t \geq 0\}$, together with a probability measure $P$ defined on a suitable measurable space, be a $d$-dimensional Brownian motion starting from the origin: $P(W_0 = 0) = 1$. We denote by $\{\mathcal{F}_t\}_{t \geq 0}$ its augmented filtration:

$$\mathcal{F}_t := \sigma(W_s, 0 \leq s \leq t) \vee \mathcal{N}, \quad t \geq 0,$$

where $\mathcal{N}$ is the collection of $P$-null events. We denote by $\mathcal{A}^\Theta_{0,1}$ the set of $\Theta$-valued $\{\mathcal{F}_t\}$-progressively measurable processes over the interval $[0,1]$. For each $\theta \in \mathcal{A}^\Theta_{0,1}$, we denote by $P_\theta$ the law of the process

$$\int_0^t \theta_s dW_s, \quad 0 \leq t \leq 1,$$

induced on $\Omega$. Now we define the capacity $c : \mathcal{B}(\Omega) \to [0,1]$ by

$$c(A) := \sup_{\theta \in \mathcal{A}^\Theta_{0,1}} P_\theta(A) \quad \text{for } A \in \mathcal{B}(\Omega). \quad (2.6)$$

We list some capacity-related terms: (i) A set $A \in \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$; (ii) a property is said to hold quasi-surely (q.s.) if it holds outside a polar set; (iii) a mapping $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous (q.c.) if for all $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_O$ is continuous; (iv) we say that $X : \Omega \to \mathbb{R}$ has a q.c. version if there exists a q.c. function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s. For each $t \in [0,1]$, we denote by $L^0(\Omega_t)$ the set of $\mathcal{B}(\Omega_t)$-measurable real-valued functions, and write $L^0(\Omega)$ for $L^0(\Omega_1)$. For each $X \in L^0(\Omega)$ such that $E_{P_\theta}[X]$ exists for all $\theta \in \mathcal{A}^\Theta_{0,1}$, set

$$\mathbb{E}[X] := \sup_{\theta \in \mathcal{A}^\Theta_{0,1}} E_{P_\theta}[X].$$

The following characterization of $G$-expectation space is given by [3, Theorem 54]:

$$\mathcal{L}^1_G(\Omega_t) = \left\{ X \in L^0(\Omega_t) : \text{X has a q.c. version, } \lim_{n \to \infty} E[|X| 1_{|X| > n}] = 0 \right\}, \quad (2.7)$$

$$E[X] = \mathbb{E}[X] \quad \text{for all } X \in \mathcal{L}^1_G(\Omega). \quad (2.8)$$

We refer to the latter identity as (2.8) as the upper expectation representation for $G$-expectation. If we denote by $\mathcal{P}$ the closure of the family $\{P_\theta : \theta \in \mathcal{A}^\Theta_{0,1}\}$ with respect to the topology of weak convergence, then the same conclusion as (2.8) holds for the upper expectation relative to $\mathcal{P}$ [3, Theorem 52]: for each $X \in L^0(\Omega)$ such that $E_P[X]$ exists for all $P \in \mathcal{P}$, set $\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X]$. Then

$$\mathbb{E}[X] = \hat{\mathbb{E}}[X] \quad \text{for all } X \in \mathcal{L}^1_G(\Omega). \quad (2.9)$$
Let us consider another capacity
\[ \hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega), \tag{2.10} \]
associated to the upper expectation representation (2.9). If \( N \in \mathcal{B}(\Omega) \) is polar under the capacity \( c \), then \( N \) is also polar under the capacity \( \hat{c} \). Indeed, since the indicator function \( 1_N \) is equal to 0 q.s. under the capacity \( c \), we have in particular that \( 1_N(B) \in L^1_G(\Omega) \) by (2.7). Then by (2.9), we have \( \hat{c}(N) = \hat{E}[1_N] = 0 \). This shows that the quasi-sureness under \( \hat{c} \) is equivalent to that under \( c \). Thus we do not need to distinguish these two and simply write q.s.

Remark 2.2. If variational representations hold under the laws of continuous martingales, then the representation (1.1) is immediate from those and (2.8); however, as far as we know, such representations have not been obtained in any existing literature.

2.2 Girsanov’s formula for \( G \)-Brownian motion

In order to introduce the statement of Girsanov’s formula for \( G \)-Brownian motion from [6], we recall some notions first.

For each \( p \geq 1 \) and \( t \in [0, 1] \), we denote by \( \mathcal{L}^p_G(\Omega_t) \) the completion of \( C_{b,\text{Lip}}(\Omega_t) \) with respect to the norm \( \mathbb{E}[|\cdot|^p]^{1/p} \). When \( t = 1 \), we drop it from notation. Let
\[ M^p,0_G(0,1) := \left\{ \sum_{k=0}^{n-1} \xi_k 1_{[t_k,t_{k+1})} : n \in \mathbb{N}, \ 0 = t_0 < t_1 < \cdots < t_n = 1, \xi_k \in \mathcal{L}^p_G(\Omega_{t_k}), \ k = 0, \ldots, n-1 \right\}. \tag{2.11} \]
We denote by \( M^p_G(0,1) \) the completion of \( M^p,0_G(0,1) \) under the norm
\[ \|\eta\|_{M^p_G(0,1)} := \left\{ \int_0^1 \mathbb{E}[|\eta|^p] \, dt \right\}^{1/p}. \]
To \( \eta = (\eta^i)_{i=1}^d \in (M^p_G(0,1))^d \), we assign the norm \( \|\eta\|_{M^p_G(0,1;\mathbb{R}^d)} \) by
\[ \|\eta\|_{M^p_G(0,1;\mathbb{R}^d)} := \|\eta\|_{M^p_G(0,1)}. \]
For every \( h \in (M^p_G(0,1))^d \) and \( t \in [0, 1] \), the Itô integral for \( G \)-Brownian motion
\[ \int_0^t h_s \cdot dB_s := \sum_{i=1}^d \int_0^t h^i_s \, dB^i_s \]
is defined as an element of \( \mathcal{L}^2_G(\Omega_t) \). Here \( B^i \) denotes the \( i \)-th coordinate of \( B \). For \( i, j = 1, \ldots, d \), the mutual variation of \( B^i \) and \( B^j \)
\[ \langle B^i, B^j \rangle_t := B^i_t B^j_t - \int_0^t B^i_s dB^j_s - \int_0^t B^j_s dB^i_s \]
is also defined since $B^i, B^j$ belong to $M_G^2(0, 1)$. We denote the quadratic variation of $B$ by $\langle B \rangle_t := (\langle B^i, B^j \rangle_t)_{i,j=1}^d, 0 \leq t \leq 1$. In addition, for each $\eta \in (M_G^1(0, 1))^d$, we define
\[
\int_0^t d\langle B \rangle_s \eta_s := \left( \sum_{i=1}^d \int_0^t \eta_s^i d\langle B^i, B^i \rangle_s, \ldots, \sum_{i=1}^d \int_0^t \eta_s^i d\langle B^i, B^j \rangle_s \right)^*
\]
as an element of $(L_G^1(\Omega))^d$. Note that if $\eta^1, \eta^2 \in M_G^2(0, 1)$, then $\eta^1 \eta^2 \in M_G^2(0, 1)$. For each $h \in (M_G^2(0, 1))^d$, we write
\[
\int_0^t h_s \cdot (d\langle B \rangle_s h_s) := \sum_{i,j=1}^d \int_0^t h_s^i h_s^j d\langle B^i, B^j \rangle_s.
\]

**Remark 2.3.** From the upper expectation representation (2.8) and the definition of $P_\theta$, $\theta \in A_{0,1}^\theta$, it is seen that for every $\theta \in A_{0,1}^\theta$,
\[
\frac{d\langle B \rangle_s}{ds} \in \{ \gamma \gamma^* : \gamma \in \Theta \} \quad \text{for a.e. } s \in [0, 1] \text{ } P_\theta\text{-a.s.} \tag{2.12}
\]

An $\mathbb{R}$-valued process $\eta = \{ \eta_t; 0 \leq t \leq 1 \}$ defined on $(\Omega, L_G^1(\Omega), \mathbb{E})$ is called a $G$-martingale if $\eta_t \in L_G^1(\Omega_t)$ for every $0 \leq t \leq 1$ and its conditional $G$-expectations satisfy
\[
\mathbb{E}_s[\eta_t] = \eta_s \quad \text{in } L_G^1(\Omega_s)
\]
for all $0 \leq s \leq t$; $\eta$ is called a symmetric $G$-martingale if both $\eta$ and $-\eta$ are $G$-martingales.

With these notions, we now introduce Girsanov’s formula for $G$-Brownian motion. Let $h \in (M_G^2(0, 1))^d$. We define, for $0 \leq t \leq 1$,
\[
D_t^{(h)} := \exp \left( \int_0^t h_s \cdot dB_s - \frac{1}{2} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right),
\]
\[
\bar{B}_t := B_t - \int_0^t d\langle B \rangle_s h_s,
\]
and we set
\[
C_{b,Lip}^{(h)}(\Omega) := \{ \phi(\bar{B}_{t_1}, \ldots, \bar{B}_{t_n}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, 1], \phi \in C_{b,Lip}(\mathbb{R}^d)^n \}.
\]

**Theorem 2.4** ([6], Theorem 5.3). Assume that $\sigma_0$ defined by (2.2) is strictly positive and that $D^{(h)}$ is a symmetric $G$-martingale on $(\Omega, L_G^1(\Omega), \mathbb{E})$. Define a sublinear expectation $\mathbb{E}^h$ by
\[
\mathbb{E}^h[X] := \mathbb{E}[XD_1^{(h)}] \quad \text{for } X \in C_{b,Lip}^{(h)}(\Omega).
\]
Let $L_G^{1,(h)}(\Omega)$ be the completion of $C_{b,Lip}^{(h)}(\Omega)$ under the norm $\mathbb{E}^h[\cdot]$, and extend $\mathbb{E}^h$ to a unique sublinear expectation on $L_G^{1,(h)}(\Omega)$. Then the process $\{ \bar{B}_t; 0 \leq t \leq 1 \}$ is a $G$-Brownian motion on the sublinear expectation space $(\Omega, L_G^{1,(h)}(\Omega), \mathbb{E}^h)$. 
Remark 2.5. Suppose that the assumptions of Theorem 2.4 are fulfilled. If a functional $F \equiv F(B)$ on $\Omega$ belongs to $L^1_G(\Omega)$, then we see that by its construction, $L^1_G(\Omega)$ contains $F(B)$ and then by Theorem 2.4,

$$\mathbb{E}[F(B)] = \mathbb{E}^h[F(B)]; \quad (2.13)$$

this transformation of $G$-expectation will be seen in Sections 3 and 5.

A sufficient condition for $D^h$ to be a symmetric $G$-martingale, referred to as $G$-Novikov’s condition, is also given in [6]: there exists $\varepsilon > 0$ such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}(1 + \varepsilon) \int_0^1 h_s \cdot (d\langle B \rangle_s h_s)\right)\right] < \infty.$$

**Proposition 2.6** ([6], Proposition 5.9). If $h \in (M^2_G(0, 1))^d$ satisfies $G$-Novikov’s condition, then the process $D^h$ is a symmetric $G$-martingale.

In the sequel we denote by $\|\cdot\|_\infty$ the supremum norm under the capacity $c$:

$$\|X\|_\infty := \inf\{M \geq 0 : c(|X| > M) = 0\} \quad \text{for } X \in L^0(\Omega).$$

We say that $h \in (M^2_G(0, 1))^d$ is bounded if

$$\sup_{0 \leq t \leq 1} \|h_t\|_\infty < \infty.$$

$G$-Novikov’s condition implies that $D^h$ is a symmetric $G$-martingale if $h$ is bounded. We also write $\mathbb{E}^h[X]$ for $X \in L^0(\Omega)$ to denote $\mathbb{E}[XD^h]$ whenever $XD_1^h \in L^1_G(\Omega)$; we recall from [6, Remark 5.8] that under the assumptions of Theorem 2.4, we have $XD_1^h \in L^1_G(\Omega)$ if $X \in L^p_G(\Omega)$. We close this section with a lemma that will also be referred to in Sections 3 and 5.

**Lemma 2.7.** Let $h \in (M^2_G(0, 1))^d$ be bounded.

(i) Let $X \in L^0(\Omega)$ be such that $X \in L^p_G(\Omega)$ for some $p > 1$. Then $\mathbb{E}^h[X]$ is well-defined, that is, $XD_1^h \in L^1_G(\Omega)$.

(ii) Denote $\bar{B} = B - \int_0^1 d\langle B \rangle_s h_s$ as above. Then it holds that

$$\left(\int_0^1 h_s \cdot d\bar{B}_s\right) D_1^h \in L^1_G(\Omega)$$

and

$$\mathbb{E}^h\left[\int_0^1 h_s \cdot d\bar{B}_s\right] = -\mathbb{E}^h\left[-\int_0^1 h_s \cdot d\bar{B}_s\right] = 0.$$
Proof. (i) Let us check $XD_1^{(h)} \in L^1_G(\Omega)$ via the characterization \((2.7)\) of $L^1_G(\Omega)$. Since $X$ and $D_1^{(h)}$ are in $L^1_G(\Omega)$, it is clear that $XD_1^{(h)}$ has a q.c. version. Note that by the boundedness of $h$ and \((2.12)\),

$$\int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \leq \sigma_1^2 \sup_{0 \leq t \leq 1} ||h_t||^2_{\infty} \; \text{q.s.} \quad (2.14)$$

with the constant $\sigma_1$ given in \((2.2)\). Then for any $q \geq 1$,

$$\mathbb{E} \left[ (D_1^{(h)})^q \right] = \mathbb{E} \left[ \exp \left( \frac{q^2 - q}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right) D_1^{(qh)} \right] \leq \exp \left( \frac{q^2 - q}{2} \sigma_1^2 \sup_{0 \leq t \leq 1} ||h_t||^2_{\infty} \right) \mathbb{E} \left[ D_1^{(qh)} \right] = \exp \left( \frac{q^2 - q}{2} \sigma_1^2 \sup_{0 \leq t \leq 1} ||h_t||^2_{\infty} \right) < \infty, \quad (2.15)$$

where for the last equality, we used the fact that $D^{(qh)}$ is also a symmetric $G$-martingale due to the boundedness of $h$. Then by Hölder’s inequality,

$$\mathbb{E} \left[ |XD_1^{(h)}|^\frac{p+1}{p} \right] \leq \mathbb{E} \left[ |X|^{\frac{p+1}{p} \cdot \frac{2p}{p-1}} \mathbb{E} \left[ (D_1^{(h)})^{\frac{p+1}{p} \cdot \frac{2p}{p-1}} \right] \right] \leq \mathbb{E} \left[ |X|^p \right]^{\frac{2}{p} \cdot \frac{p+1}{p} \cdot \frac{2}{p-1}} \mathbb{E} \left[ (D_1^{(h)})^{\frac{p+1}{p} \cdot \frac{2p}{p-1}} \right]^{\frac{p-1}{2p}} < \infty.$$

Therefore we obtain

$$\lim_{n \to \infty} \mathbb{E} \left[ |XD_1^{(h)}| \chi_{\{|XD_1^{(h)}| \geq n\}} \right] = 0,$$

and hence $XD_1^{(h)} \in L^1_G(\Omega)$.

(ii) Set

$$Y_t := \int_0^t h_s \cdot dB_s = \int_0^t h_s \cdot dB_s - \int_0^t h_s \cdot (d\langle B \rangle_s h_s)$$

for $0 \leq t \leq 1$. It follows from \((2.14)\) that $Y_1 \in L^2_G(\Omega)$, and hence $Y_1D_1^{(h)} \in L^1_G(\Omega)$ by (i). Since $YD_1^{(h)}$ is a $P_\theta$-martingale for any $\theta \in A^{0,1}_0$, we have $E_{P_\theta}[Y_1D_1^{(h)}] = -E_{P_\theta}[-Y_1D_1^{(h)}] = 0$. As $\theta \in A^{0,1}$ is arbitrary, we obtain

$$E^h[Y_1] = -E^h[-Y_1] = 0$$

by the upper expectation representation \((2.8)\), and end the proof.

\[\square\]

### 3 The variational representation for $G$-Brownian functionals

In this section, we state and prove the main result of this paper, the variational representation \((1.1)\) for functionals on $G$-expectation space. In the rest of this paper, we
assume $\sigma_0 > 0$. This assumption allows us to apply Girsanov’s formula for $G$-Brownian motion (Theorem 2.4), which plays a central role throughout this section and Section 5.

When $f \in \mathcal{L}^1_G(\Omega)$ is q.s. bounded, we simply call it bounded.

**Theorem 3.1.** For any bounded elements $f$ of $\mathcal{L}^1_G(\Omega)$, it holds that

$$\log \mathbb{E} \left[ e^{f(B)} \right] = \sup_{h \in (M^2_G(0,1))^d} \mathbb{E} \left[ f \left( B + \int_0^t d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right]. \quad (3.1)$$

The well-definedness of the right-hand side of (3.1) will be seen in Proposition 3.3.

**Remark 3.2.** For any $M > 0$, the mappings

$$\mathbb{R} \ni x \mapsto \log (e^{-M} \vee (x \wedge e^M)) \quad \text{and} \quad \mathbb{R} \ni x \mapsto \exp \{ (-M) \vee (x \wedge M) \}$$

are both Lipschitz functions with Lipschitz constant $e^M$. Here and in what follows we write $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ for $x, y \in \mathbb{R}$.

### 3.1 A preliminary result

In this subsection, we see that $G$-expectation in the right-hand side of (1.1) is well-defined, that is, we prove the following:

**Proposition 3.3.** Let $f$ be a bounded element of $\mathcal{L}^1_G(\Omega)$. Then, for any $h \in (M^2_G(0,1))^d$, we have

$$f \left( B + \int_0^t d\langle B \rangle_s h_s \right) \in \mathcal{L}^1_G(\Omega).$$

A key step to the proof of this proposition is an absolute continuity stated in Proposition 5.1. In what follows, we denote

$$T^h(B)_t = B_t + \int_0^t d\langle B \rangle_s h_s, \quad 0 \leq t \leq 1,$$

for $h \in (M^2_G(0,1))^d$.

**Proof of Proposition 3.3.** Let $\{f_n\}_{n \in \mathbb{N}} \subset C_{b,\text{Lip}}(\Omega)$ be such that

$$\lim_{n \to \infty} \mathbb{E}[|f_n(B) - f(B)|] = 0.$$

Truncating $f_n$ if necessary, we may assume that $\|f_n\|_\infty \leq \|f\|_\infty$ for all $n \in \mathbb{N}$. Since every $f_n$ is in $C_{b,\text{Lip}}(\Omega)$ and $\int_0^t d\langle B \rangle_s h_s$ has a q.c. version for each $t \in [0,1]$, it is clear that the functional $f_n(T^h(B))$ also has a q.c. version, hence belongs to $\mathcal{L}^1_G(\Omega)$ due to (2.1). Therefore, in order to prove the proposition, it is sufficient to show

$$\mathbb{E} \left[ |f_n(T^h(B)) - f(T^h(B))| \right] \longrightarrow 0 \quad (3.2)$$
because of (2.8). To this end, fix $\varepsilon > 0$ arbitrarily. By the sublinearity of $\mathbb{E}$, the left-hand side of (3.2) is bounded from above by the sum

$$\mathbb{E} \left[ |f_n(T^h(B)) - f(T^h(B))| ; T^h(B) \in A_n^c \right] + \mathbb{E} \left[ |f_n(T^h(B)) - f(T^h(B))| ; T^h(B) \in A_n \right],$$

where $A_n = \{ \omega : |f_n(\omega) - f(\omega)| > \varepsilon \}$. The first term is less than or equal to $\varepsilon$ from the definition of $A_n$. On the other hand, since (5.2) in Proposition 5.1 yields the bounds $|f(T^h(B))| \leq \|f\|_{\infty}$ q.s. for all $n \in \mathbb{N}$, the second term of (3.3) is less than or equal to $2\|f\|_{\infty} c \left( T^h(B) \in A_n \right)$.

By Chebyshev’s inequality and (2.9),

$$c(A_n) \leq \varepsilon^{-1} \mathbb{E} \left[ |f_n(B) - f(B)| \right];$$

the right-hand converges to 0 by letting $n \to \infty$. Therefore Proposition 5.1 implies $c \left( T^h(B) \in A_n \right) < \varepsilon$ for sufficiently large $n$. Combining these estimates, we can bound (3.3) from above by

$$(1 + 2\|f\|_{\infty}) \varepsilon$$

for sufficiently large $n$, and hence obtain (3.2). \qed

### 3.2 Proof of the lower bound

In this subsection, we prove the lower bound in Theorem 3.1 for any bounded elements $f$ of $\mathcal{L}_G^1(\Omega)$,

$$\log \mathbb{E} \left[ e^{f(B)} \right] \geq \sup_{h \in (M_G^2(0,1))^d} \mathbb{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right].$$

**Lemma 3.4.** Let $h \in (M^2_G(0,1))^d$ be bounded. Then for any bounded $f \in \mathcal{L}_G^1(\Omega)$, we have

$$\log \mathbb{E} \left[ e^{f(B)} \right] \geq \mathbb{E}^h \left[ f(B) - \log D_1^{(h)} \right].$$

**Proof.** First note that $f(B) - \log D_1^{(h)} \in \mathcal{L}_G^1(\Omega)$ by the assumption. Hence the right-hand side of (3.5) is well-defined by (i) of Lemma 2.7 and is equal to

$$\mathbb{E}^h \left[ \log \mathbb{E} \left[ e^{f(B)} \right] - \log \mathbb{E} \left[ e^{f(B)} \right] + \log e^{f(B)} - \log D_1^{(h)} \right]$$

$$= \log \mathbb{E} \left[ e^{f(B)} \right] + \mathbb{E}^h \left[ - \log \left( \frac{\mathbb{E} \left[ e^{f(B)} \right]}{e^{f(B)}} D_1^{(h)} \right) \right].$$  \hspace{1cm} (3.6)
We rewrite the second term of (3.4) to obtain the bound
\[
\mathbb{E} \left[ \frac{e^{f(B)}}{e^{f(B)} - D_1^{(h)}} \log \left( \frac{\mathbb{E} \left[ e^{f(B)} \right]}{\mathbb{E} \left[ e^{f(B)} \right] - D_1^{(h)}} \right) \times \frac{e^{f(B)}}{\mathbb{E} \left[ e^{f(B)} \right]} \right] 
\leq \mathbb{E} \left[ \left( 1 - \frac{\mathbb{E} \left[ e^{f(B)} \right]}{\mathbb{E} \left[ e^{f(B)} \right] - D_1^{(h)}} \right) \times \frac{e^{f(B)}}{\mathbb{E} \left[ e^{f(B)} \right]} \right] 
= \mathbb{E} \left[ \frac{e^{f(B)}}{\mathbb{E} \left[ e^{f(B)} \right]} - D_1^{(h)} \right],
\]
where the inequality follows from that \(-x \log x \leq 1 - x\) for \(x > 0\). Since \(D^{(h)}\) is a symmetric \(G\)-martingale by the boundedness of \(h\) and \(G\)-Novikov’s condition (Proposition 2.6),
\[
\mathbb{E} \left[ \frac{e^{f(B)}}{\mathbb{E} \left[ e^{f(B)} \right]} - D_1^{(h)} \right] = \mathbb{E} \left[ \frac{e^{f(B)}}{\mathbb{E} \left[ e^{f(B)} \right]} \right] + \mathbb{E} \left[ -D_1^{(h)} \right] = 1 - 1 = 0.
\]
Therefore the lemma follows. \(\square\)

We denote by \(S_{b,Lip}\) the subset of \(M_G^{2,0}(0,1)\) consisting of elements with each \(\xi_k\) in the definition (2.11) of \(M_{G}^{2,0}(0,1)\) belonging to \(C_{b,Lip}(\Omega_{t_k})\). Since \(C_{b,Lip}(\Omega_{t_k})\) is dense in \(L_G^1(\Omega_{t_k})\), we may deduce that \(S_{b,Lip}\) is also dense in \(M_G^2(0,1)\). Let \(h \equiv h.(B) \in (S_{b,Lip})^d\) be written as
\[
h_t = \sum_{k=0}^{n-1} \xi_k \mathbf{1}_{(t_k, t_{k+1})}(t), \quad 0 \leq t \leq 1, \tag{3.7}
\]
for some \(n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1, \xi_0 \in \mathbb{R}^d, \) and \(\xi_k \equiv \xi_k(B) \in (C_{b,Lip}(\Omega_{t_k}))^d, k = 1, \ldots, n-1\). We associate \(h\) with a simple process \(\overline{h}\) defined as follows:
\[
\overline{\xi}_0 := \xi_0, \quad \overline{\xi}_1 := \xi_1 \left( B_t - \int_0^t d \langle B \rangle_s \overline{T}_s, t \leq t_1 \right), \quad \overline{\xi}_t := \overline{\xi}_1, \quad t_1 < t < t_2,
\]
\[
\vdots
\]
\[
\overline{\xi}_{n-1} := \xi_{n-1} \left( B_t - \int_0^t d \langle B \rangle_s \overline{T}_s, t \leq t_{n-1} \right), \quad \overline{\xi}_t := \overline{\xi}_{n-1}, \quad t_{n-1} < t < t_n.
\]
By this construction, it is clear that each \(\overline{\xi}_k\) belongs to \(L_G^1(\Omega)\) and is bounded, and hence \(\overline{h}\) is a bounded element of \((M_G^{2,0}(0,1))^d\); moreover,
\[
\overline{h}_t = h_t \left( T^{-\overline{T}}(B) \right) \quad \text{for all} \ 0 \leq t \leq 1. \tag{3.8}
\]

**Lemma 3.5.** (3.4) **holds for** \(f \in C_{b,Lip}(\Omega)\).
Proof. It is sufficient to show

\[ \log \mathbb{E} \left[ e^{f(B)} \right] \geq \mathbb{E} \left[ f \left( T^h(B) \right) - \frac{1}{2} \int_0^1 h_s \cdot \left( d\langle B \rangle_s \right) h_s \right] \]

(3.9)

for all \( h \in \mathbb{M}_G^{2}(0,1)^d \).

First we let \( h \in \mathcal{S}_{b,Lip}^d \). Let \( \overline{h} \) be the associated element of \((\mathbb{M}_G^{2}(0,1))^{d}\) as constructed above so that (3.8) holds. By the boundedness of \( \overline{h} \), Lemma 3.4 implies

\[ \log \mathbb{E} \left[ e^{f(B)} \right] \geq \mathbb{E} \left[ f(B) - \frac{1}{2} \int_0^1 \overline{h}_s \cdot \left( d\langle B \rangle_s \overline{h}_s \right) - \int_0^1 \overline{h}_s \cdot d\overline{B}_s \right] \]

\[ = \mathbb{E} \left[ f(B) - \frac{1}{2} \int_0^1 \overline{h}_s \cdot \left( d\langle B \rangle_s \overline{h}_s \right) \right], \]

where \( \overline{B} := T^{-\overline{h}}(B) \) and the equality follows from (ii) of Lemma 2.7. By (3.8) and the obvious identity \( \langle B \rangle = \langle \overline{B} \rangle \), we can rewrite the right-hand side as

\[ \mathbb{E} \left[ f \left( \overline{B} + \int_0^1 d\langle B \rangle_s h_s (\overline{B}) \right) - \frac{1}{2} \int_0^1 h_s (\overline{B}) \cdot \left( d\langle B \rangle_s h_s (\overline{B}) \right) \right]. \]

(3.10)

By the boundedness of \( \overline{h} \), Proposition 2.6 implies that \( D(\overline{h}) \) is a symmetric \( G \)-martingale, and hence by Girsanov’s formula for \( G \)-Brownian motion (Theorem 2.4), \( \overline{B} \) is a \( G \)-Brownian motion on \((\Omega, \mathcal{L}_G^1(\Omega), \mathbb{P})\). Since \( f \left( B + \int_0^1 d\langle B \rangle_s h_s (B) \right) \) is in \( \mathcal{L}_G^1(\Omega) \) by the assumption \( f \in C_{b,Lip}(\Omega) \) and \( \int_0^1 h_s (B) \cdot \left( d\langle B \rangle_s h_s (B) \right) \) is also in \( \mathcal{L}_G^1(\Omega) \), we see from (2.13) in Remark 2.5 that (3.10) is equal to

\[ \mathbb{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot \left( d\langle B \rangle_s h_s \right) \right]. \]

Therefore (3.9) holds for \( h \in \mathcal{S}_{b,Lip}^d \).

Now we let \( h \in \mathbb{M}_G^{2}(0,1)^d \) and take a sequence \( \{h^n\}_{n \in \mathbb{N}} \subset \mathcal{S}_{b,Lip}^d \) such that

\[ \| h - h^n \|_{\mathbb{M}_G^{2}(0,1)^d} \to 0 \quad \text{as} \quad n \to \infty. \]

As seen above, (3.9) holds for every \( h^n \), from which it follows that

\[ \log \mathbb{E} \left[ e^{f(B)} \right] \geq \mathbb{E} \left[ f \left( T^h(B) \right) - \frac{1}{2} \int_0^1 h_s \cdot \left( d\langle B \rangle_s h_s \right) \right] - \mathbb{E} \left[ f \left( T^{h^n}(B) \right) \right] - \frac{1}{2} \mathbb{E} \left[ \int_0^1 h^n_s \cdot \left( d\langle B \rangle_s h^n_s \right) - \int_0^1 h_s \cdot \left( d\langle B \rangle_s h_s \right) \right]. \]

(3.11)

Therefore it suffices to show that the second and third terms in the right-hand side of (3.11) tend to 0 as \( n \to \infty \). For the second term, since \( f \) is Lipschitz,

\[ |\mathbb{E} \left[ f \left( T^h(B) \right) - f \left( T^{h^n}(B) \right) \right]| \leq \text{Lip}(f)\mathbb{E} \left[ \sup_{0 \leq t \leq 1} \left| \int_0^t d\langle B \rangle_s (h_s - h^n_s) \right| \right] \]
where in the last line, we use (2.12) and the Cauchy-Schwarz inequality. On the other hand, for the third term, we also have

\[
\left| \mathbb{E} \left[ \int_0^1 h_s^n \cdot (d\langle B \rangle_s h_s^n) - \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right] \right|
\leq \mathbb{E} \left[ \left| \int_0^1 (h_s^n + h_s) \cdot (d\langle B \rangle_s (h_s^n - h_s)) \right| \right]
\leq \sigma^2 \|h^n + h\|_{M_2^\infty(0,1; \mathbb{R}^d)} \|h^n - h\|_{M_2^\infty(0,1; \mathbb{R}^d)}.
\] (3.13)

Since \( \{h^n\} \) is an approximate sequence of \( h \), (3.12) and (3.13) tend to 0 as \( n \to \infty \). Therefore (3.9) is valid for \( h \in (M_2^\infty(0,1))^d \) and we complete the proof. \( \square \)

We are ready to prove the lower bound in Theorem 3.1.

**Proposition 3.6.** (3.14) holds for all bounded elements \( f \) of \( \mathcal{L}^1_G(\Omega) \).

**Proof.** Fix \( h \in (M_2^\infty(0,1))^d \) arbitrarily and let \( \{f_n\}_{n \in \mathbb{N}} \) be as in the proof of Proposition 3.3. Lemma 3.5 implies that

\[
\log \mathbb{E} \left[ e^{f(B)} \right] - \mathbb{E} \left[ f(T^h(B)) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right]
\geq \log \mathbb{E} \left[ e^{f(B)} \right] - \log \mathbb{E} \left[ e^{f_n(B)} \right] - \mathbb{E} \left[ \|f(T^h(B)) - f_n(T^h(B))\| \right].
\]

As seen in the proof of Proposition 3.3, the third term in the right-hand side converges to 0 as \( n \to \infty \). By taking \( M = \|f\|_{\infty} \) in Remark 3.2, we also have

\[
|\log \mathbb{E} \left[ e^{f(B)} \right] - \log \mathbb{E} \left[ e^{f_n(B)} \right]| \leq e^{2\|f\|_{\infty}} \mathbb{E} \|f(B) - f_n(B)\| \to 0
\] (3.14)
as \( n \to \infty \), and hence obtain the proposition. \( \square \)

### 3.3 Proof of the upper bound

In this subsection, we prove the upper bound in Theorem 3.1 for any bounded elements \( f \) of \( \mathcal{L}^1_G(\Omega) \),

\[
\log \mathbb{E} \left[ e^{f(B)} \right] \leq \sup_{h \in (M_2^\infty(0,1))^d} \mathbb{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right].
\] (3.15)

Proofs of the next two lemmas proceed as those of Lemma 2.1 and Theorem 2.2 in Chapter IV of [9].

**Lemma 3.7.** Let \( 0 \leq t_1 \leq 1 \). For every \( \varphi \in C_{b,Lip}(\mathbb{R}^d) \), there exist a bounded \( h \in (M_2^\infty(0,1))^d \) and an \( A \in \mathcal{L}^2_G(\Omega) \) with \( A \geq 0 \), q.s. such that

\[
\varphi(B_{t_1}) = \log \mathbb{E}_{t_1} \left[ e^{\varphi(B_{t_1})} \right] + \int_{t_1}^1 h_s \cdot dB_s - \frac{1}{2} \int_{t_1}^1 h_s \cdot (d\langle B \rangle_s h_s) - A
\]
in \( \mathcal{L}^2_G(\Omega) \), where \( \mathbb{E}_{t_1} \) is the conditional G-expectation defined by (2.5).
Proof. If we let
\[ u(t, x) := E \left[ e^{\varphi(B_t - B_t + x)} \right] \]
for \((t, x) \in [0, 1] \times \mathbb{R}^d\), then by the assumption \(\sigma_0 > 0\), \(u\) is the \(C^{1,2}((0, 1) \times \mathbb{R}^d)\)-solution of
\[
\begin{cases}
\frac{\partial u}{\partial t} + G(D^2u) = 0 & \text{in } (0, 1) \times \mathbb{R}^d, \\
u(1, x) = e^{\varphi(x)} & \text{for } x \in \mathbb{R}^d;
\end{cases}
\tag{3.16}
\]
see Remark 2.1. Observe that by letting \(|\varphi| := \sup_{x \in \mathbb{R}^d} |\varphi(x)|\),
\[ 0 < e^{-|\varphi|} \leq u(t, x) \leq e^{|\varphi|} < \infty \tag{3.17} \]
for any \((t, x)\), and that by the Lipschitz continuity of \(\varphi\),
\[ \sup_{(t,x) \in (0,1) \times \mathbb{R}^d} |\nabla u(t,x)| < \infty, \tag{3.18} \]
where \(\nabla := \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}\right)^{\ast}\). Additionally, we note that by Theorem 4.5 in Appendix C in [9], there exists an \(\alpha \in (0, 1)\) such that
\[ \|u\|_{C^{1,\frac{1}{2}+\alpha}((0,1-\varepsilon) \times \mathbb{R}^d)} < \infty \quad \text{for every } \varepsilon \in (0, 1). \tag{3.19} \]
Here
\[
\|u\|_{C^{1,\frac{1}{2}+\alpha}((0,1-\varepsilon) \times \mathbb{R}^d)} := \sup_{s \in (0,1-\varepsilon), s \neq t} \frac{|u(s, x) - u(t, y)|}{|s-t|^{\alpha/2} + |x-y|^\alpha}.
\]
Set \(U(t, x) := \log u(t, x)\). Then \(U\) is also a member of \(C^{1,2}((0, 1) \times \mathbb{R}^d)\) and
\[
\begin{cases}
\frac{\partial U}{\partial t}(t, x) = \frac{1}{u(t, x)} \frac{\partial u}{\partial t}(t, x), \\
\nabla U(t, x) = \frac{1}{u(t, x)} \nabla u(t, x), \\
D^2 U(t, x) = -\nabla U(t, x) \nabla U(t, x)^{\ast} + \frac{1}{u(t, x)} D^2 u(t, x).
\end{cases}
\tag{3.20}
\]
Therefore we have by (3.17) and (3.19),
\[ \|U\|_{C^{1,\frac{1}{2}+\alpha}((0,1-\varepsilon) \times \mathbb{R}^d)} < \infty \quad \text{for any } \varepsilon \in (0, 1), \]
which allows us to apply G-Itô’s formula [11, Theorem 6.5 in Chapter III] to $U(t, B_t)$ on $[t_1, 1 - \varepsilon]$ for $0 < \varepsilon < 1 - t_1$. Then we have

$$U(1 - \varepsilon, B_{1 - \varepsilon}) = U(t_1, B_{t_1}) + \int_{t_1}^{1 - \varepsilon} \frac{\partial U}{\partial t}(s, B_s) \, ds + \int_{t_1}^{1 - \varepsilon} \nabla U(s, B_s) \, dB_s$$

$$+ \frac{1}{2} \int_{t_1}^{1 - \varepsilon} \text{tr} \left[ d(B)_s D^2 U(s, B_s) \right] \in L^2_G(\Omega).$$

By (3.20), together with (3.16), we obtain

$$U(1 - \varepsilon, B_{1 - \varepsilon}) = U(t_1, B_{t_1}) + \int_{t_1}^{1 - \varepsilon} h_s \, dB_s - \frac{1}{2} \int_{t_1}^{1 - \varepsilon} h_s \cdot (d(B)_s h_s) - A_{1 - \varepsilon}, \quad (3.21)$$

where

$$h_t := \nabla U(t, B_t), \quad 0 \leq t < 1,$$

$$A_t := \int_{t_1}^{t} G(D^2 u(s, B_s)) \frac{ds}{u(s, B_s)} - \int_{t_1}^{t} \frac{1}{2} \text{tr} \left[ d(B)_s D^2 u(s, B_s) \right], \quad t_1 \leq t \leq 1.\]$$

Note that $\{A_t; t_1 \leq t \leq 1\}$ is a nondecreasing process with $A_{t_1} = 0$ and each term of (3.21) is an element of $L^2_G(\Omega)$. For the left-hand side of (3.21), we first note that by Remark 3.2 and by the above estimates,

$$|u(s, x) - u(t, x)| \leq e^{\|\phi\|_{Lip}} \mathbb{E} \left[ |B_s - B_t| \right],$$

$$|u(t, x) - u(t, y)| \leq e^{\|\phi\|_{Lip}} |x - y|$$

for all $0 \leq s, t \leq 1$ and $x, y \in \mathbb{R}^d$. Again by Remark 3.2 and by the above estimates,

$$|U(1, B_1) - U(1 - \varepsilon, B_{1 - \varepsilon})|$$

$$\leq e^{\|\phi\|_{Lip}} |u(1, B_1) - u(1 - \varepsilon, B_{1 - \varepsilon})|$$

$$\leq e^{2\|\phi\|_{Lip}} \mathbb{E} \left[ |B_1 - B_{1 - \varepsilon}| \right] + |B_1 - B_{1 - \varepsilon}|$$

$$\to 0 \quad \text{as} \ \varepsilon \to 0 \quad \text{in} \ L^2_G(\Omega). \quad (3.22)$$

For the second and third terms in the right-hand side of (3.21), since $\nabla U$ is bounded because of (3.17), (3.18) and (5.20), we have

$$\int_{1 - \varepsilon}^{1} h_s \, dB_s \to 0 \quad \text{and} \quad \int_{1 - \varepsilon}^{1} h_s \cdot (d(B)_s h_s) \to 0 \quad (3.23)$$

as $\varepsilon \to 0$ in $L^2_G(\Omega)$. For the convergence of $A_{1 - \varepsilon}$, it is clear that $A_{1 - \varepsilon} \to A := A_1$ as $\varepsilon \to 0$ P$\theta$-a.s. for every $\theta \in A^\Theta_{0,1}$. Moreover, by (3.22) and (3.23),

$$A_{1 - \varepsilon} \to -U(1, B_1) + U(t_1, B_{t_1}) + \int_{t_1}^{1} h_s \cdot dB_s - \frac{1}{2} \int_{t_1}^{1} h_s \cdot (d(B)_s h_s)$$

as $\varepsilon \to 0$. Therefore, by the convergence of $A_{1 - \varepsilon}$, we have

$$\int_{1 - \varepsilon}^{1} h_s \cdot dB_s \to 0$$

and

$$\int_{1 - \varepsilon}^{1} h_s \cdot (d(B)_s h_s) \to 0 \quad (3.24)$$

as $\varepsilon \to 0$. For the convergence of $A_{1 - \varepsilon}$, it is clear that $A_{1 - \varepsilon} \to A := A_1$ as $\varepsilon \to 0$ P$\theta$-a.s. for every $\theta \in A^\Theta_{0,1}$. Moreover, by (3.22) and (3.23),

$$A_{1 - \varepsilon} \to -U(1, B_1) + U(t_1, B_{t_1}) + \int_{t_1}^{1} h_s \cdot dB_s - \frac{1}{2} \int_{t_1}^{1} h_s \cdot (d(B)_s h_s)$$

as $\varepsilon \to 0$. Therefore, by the convergence of $A_{1 - \varepsilon}$, we have
as \( \varepsilon \to 0 \) in \( \mathcal{L}_G^2(\Omega) \). For every \( \theta \in \mathcal{A}_{0,1}^{\Theta} \), taking a subsequence if necessary, we see that this convergence also holds \( P_\theta \)-a.s. Therefore we have

\[
A = -U(1, B_1) + U(t_1, B_{t_1}) + \int_{t_1}^1 h_s \cdot dB_s - \frac{1}{2} \int_{t_1}^1 h_s \cdot (d\langle B \rangle_s h_s) \tag{3.24}
\]

\( P_\theta \)-a.s. for all \( \theta \in \mathcal{A}_{0,1}^{\Theta} \). Since all terms in the right-hand side of (3.24) are in \( \mathcal{L}_G^2(\Omega) \), \( A \) is also in \( \mathcal{L}_G^2(\Omega) \). As a consequence, the equality (3.24) holds in \( \mathcal{L}_G^2(\Omega) \). By noting that

\[
U(1, B_1) = \varphi(B_1),
\]

\[
U(t_1, B_{t_1}) = \log \mathbb{E} \left[ e^{\varphi(B_1 - B_{t_1} + x)} \right]_{x = B_{t_1}} = \log \mathbb{E}_{t_1} \left[ e^{\varphi(B_1)} \right],
\]

we finally obtain

\[
\varphi(B_1) = \log \mathbb{E}_{t_1} \left[ e^{\varphi(B_1)} \right] + \int_{t_1}^1 h_s \cdot dB_s - \frac{1}{2} \int_{t_1}^1 h_s \cdot (d\langle B \rangle_s h_s) - A
\]
in \( \mathcal{L}_G^2(\Omega) \). \( \square \)

The following lemma is a type of the Clark-Ocone formula in the framework of \( G \)-expectation space.

**Lemma 3.8.** For every \( f \in C_{b,\text{Lip}}(\Omega) \), there exist a bounded \( h \in (M_G^2(0,1))^d \) and an \( A \in \mathcal{L}_G^2(\Omega) \) with \( A \geq 0 \) q.s. such that

\[
f(B) = \log \mathbb{E} \left[ e^{f(B)} \right] + \int_0^1 h_s \cdot dB_s - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) - A \quad \text{in} \quad \mathcal{L}_G^2(\Omega). \tag{3.25}
\]

**Proof.** Let \( 0 \leq t_1 \leq 1 \) and \( \varphi \in C_{b,\text{Lip}}(\mathbb{R}^d) \). It suffices to show the lemma holds when \( f(B) = \varphi(B_{t_1} - B_1, B_1) \). Set

\[
u(t,x,y) := \mathbb{E} \left[ e^{\varphi(x,B_1 - B_{t_1} + y)} \right], \quad U(t,x,y) := \log u(t,x,y)
\]

for \((t,x,y) \in [0,1] \times \mathbb{R}^d \times \mathbb{R}^d\). By Lemma 3.7, we have for every \( x \in \mathbb{R}^d \),

\[
U(1,x,B_1) = \log \mathbb{E} \left[ e^{\varphi(x,B_1 - B_{t_1} + y)} \right]_{y=B_{t_1}} + \int_{t_1}^1 \nabla_y U(s,x,B_s) \cdot dB_s
\]

\[
- \frac{1}{2} \int_{t_1}^1 \nabla_y U(s,x,B_s) \cdot (d\langle B \rangle_s \nabla_y U(s,x,B_s))
\]

\[
- \left( \int_{t_1}^1 G \left( D^2_y u(s,x,B_s) \right) \frac{ds}{u(s,x,B_s)} \right) - \int_{t_1}^1 \frac{1}{2} \text{tr} \left[ \frac{d\langle B \rangle_s D^2_y u(s,x,B_s)}{u(s,x,B_s)} \right] \int_{t_1}^1 \frac{d}{\text{tr}} D^2_y \frac{d}{\text{tr}} \frac{d}{\text{tr}} u(s,x,B_s)
\]

in \( \mathcal{L}_G^2(\Omega) \), where \( \nabla_y := \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_d} \right)^* \) and \( D^2_y := \left( \frac{\partial^2}{\partial y_i \partial y_j} \right)_{i,j=1}^d \). By the construction of the integrations with respect to \( dB_s \) and \( d\langle B \rangle_s \) (see, e.g., [8]), this identity still holds with \( x \) replaced by \( B_{t_1} \). Hence, by letting

\[
\varphi_1(x) := \log \mathbb{E} \left[ e^{\varphi(x,B_1 - B_{t_1} + x)} \right],
\]
\[ h_s := \nabla_y U(s, B_t, B_s), \ t_1 \leq s < 1, \]
\[ A^{(1)} := \int_{t_1}^1 G \left( D_y^2 u(s, B_t, B_s) \right) ds - \int_{t_1}^1 \frac{1}{2} \text{tr} \left[ D_y^2 u(s, B_t, B_s) \right], \]
we get
\[ \varphi(B_{t_1}, B_1) = U(1, B_{t_1}, B_1) = \varphi(1) + \int_0^1 h_s \cdot dB_s - \frac{1}{2} \int_0^1 \left( h_s \cdot (d(B)_s h_s) - A^{(1)} \right). \]
Since \( \varphi_1 \in C_{b,\text{Lip}}(\mathbb{R}^d) \), we may apply Lemma 3.7 to \( \varphi(B_{t_1}) \) to obtain \( \hat{h} \in (M^2_\lambda(0, 1))^d \) and \( A^{(2)} \in \mathcal{L}^2_G(\Omega) \) with \( A^{(2)} \geq 0 \) q.s. such that
\[ \varphi(B_{t_1}, B_1) = \log \mathbb{E} \left[ e^{\varphi_1(B_{t_1})} \right] + \int_0^1 h_s \cdot dB_s - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) - (A^{(1)} + A^{(2)}) \]
in \( \mathcal{L}^2_G(\Omega) \). Noting (2.4), we have
\[ \mathbb{E} \left[ e^{\varphi_1(B_{t_1})} \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{\varphi(x, B_{t_1})+x} \right] | x = B_{t_1} \right] = \mathbb{E} \left[ e^{\varphi(B_{t_1}, B_1)} \right]. \]
Therefore the lemma follows. \( \square \)

Recall that \( \mathcal{P} \) is the weak closure of \( \{P_\theta : \theta \in \mathcal{A}_{0,1}^\theta\} \); by the tightness of \( \{P_\theta : \theta \in \mathcal{A}_{0,1}^\theta\} \) (\( \mathcal{A}\) Proposition 50), \( \mathcal{P} \) is weakly compact.

**Lemma 3.9.** For \( f \in C_{b,\text{Lip}}(\Omega) \), let \( A \in \mathcal{L}^2_G(\Omega) \) be given in Lemma 3.8. Then there exists \( P \in \mathcal{P} \) such that
\[ A = 0 \quad P\text{-a.s.} \]

**Proof.** Rewriting (3.23) in Lemma 3.8 we have
\[ e^{f(B)+A} = D_{\hat{h}}^2 \mathbb{E} \left[ e^{f(B)} \right]. \]
By the boundedness of \( h \) and Proposition 2.6, \( D_{\hat{h}} \) is a symmetric \( G \)-martingale and hence the right-hand side is symmetric. Therefore the left-hand side is also symmetric, which yields
\[ \mathbb{E} \left[ e^{f(B)+A} \right] = -\mathbb{E} \left[ -e^{f(B)+A} \right] = \mathbb{E} \left[ e^{f(B)} \right]. \] (3.26)
Since every integrand in (3.26) is an element of \( \mathcal{L}^1_G(\Omega) \), we see from (2.9) that (3.26) still holds if \( \mathbb{E} \) is replaced by \( \hat{\mathbb{E}} \), from which it follows that
\[ E_P \left[ e^{f(B)+A} \right] = \hat{\mathbb{E}} \left[ e^{f(B)} \right] \quad \text{for all } P \in \mathcal{P}. \] (3.27)
Also observe that, since \( f \) is bounded and continuous, the mapping \( \mathcal{P} \ni P \mapsto E_P[e^{f(B)}] \) is continuous by the definition of weak convergence. Then by the compactness of \( \mathcal{P} \), there exists \( P' \in \mathcal{P} \) which attains the supremum of \( E_P[e^{f(B)}] \) over \( P \in \mathcal{P} \), namely
\[ E_{P'} \left[ e^{f(B)} \right] = \hat{\mathbb{E}} \left[ e^{f(B)} \right]. \] (3.28)
Combining (3.27) and (3.28) leads to \( P'(A = 0) = 1 \) since \( A \) is nonnegative q.s. \( \square \)
In the case of a classical Brownian motion, the following lemma is a consequence of Scheffé’s lemma, the equivalence between $L^1$-convergence and the convergence of $L^1$-norms for an a.s. convergent sequence of random variables. Although the setting is restricted, this lemma may be regarded as a sublinear counterpart to Scheffé’s lemma.

**Lemma 3.10.** For a bounded $h \in (M^2_G(0,1))^d$, let $\{h^n\}_{n \in \mathbb{N}} \subset (S_{b,Lip})^d$ be such that

$$\sup_{0 \leq t \leq 1} \|h^n_t\|_{\infty} \leq \sup_{0 \leq t \leq 1} \|h_t\|_{\infty}$$

and

$$\lim_{n \to \infty} \|h - h^n\|_{M^2_G(0,1;\mathbb{R}^d)} = 0.$$ 

Then we have

$$\lim_{n \to \infty} \mathbb{E} \left[ \left| D_1^{(h)} - D_1^{(h^n)} \right| \right] = 0.$$ 

In particular, it holds that for any bounded elements $f$ of $L^1_G(\Omega)$,

$$\mathbb{E}^h \left[ f(B) - \frac{1}{2} \int_0^1 h_s \cdot (dB)_s h_s \right] = \lim_{n \to \infty} \mathbb{E}^{h^n} \left[ f(B) - \frac{1}{2} \int_0^1 h^n_s \cdot (dB)_s h^n_s \right].$$ \hspace{1cm} (3.29)

**Proof.** Since $D_1^{(h)}$ and $D_1^{(h^n)}$ are $P_\theta$-martingales for any $\theta \in A_{0,1}^\Theta$, their expectations under $P_\theta$ are equal to 1, and hence

$$E_{P_\theta} \left[ D_1^{(h)} \right] = E_{P_\theta} \left[ D_1^{(h^n)} \right].$$ \hspace{1cm} (3.30)

The left-hand side and the right-hand side of (3.30) are equal to

$$E_{P_\theta} \left[ (D_1^{(h)} - D_1^{(h^n)})^+ \right] + E_{P_\theta} \left[ D_1^{(h)} \wedge D_1^{(h^n)} \right]$$

and

$$E_{P_\theta} \left[ (D_1^{(h^n)} - D_1^{(h)})^+ \right] + E_{P_\theta} \left[ D_1^{(h^n)} \wedge D_1^{(h)} \right],$$

respectively. Here $x^+ = x \vee 0$ for $x \in \mathbb{R}$. Combining these with (3.30), we have the relation $E_{P_\theta}[(D_1^{(h)} - D_1^{(h^n)})^+] = E_{P_\theta}[(D_1^{(h^n)} - D_1^{(h)})^+]$, and hence

$$E_{P_\theta} \left[ \left| D_1^{(h)} - D_1^{(h^n)} \right| \right] = 2E_{P_\theta} \left[ \left| (D_1^{(h)} - D_1^{(h^n)})^+ \right| \right].$$

As $\theta \in A_{0,1}^\Theta$ is arbitrary, it follows that

$$\mathbb{E} \left[ \left| D_1^{(h)} - D_1^{(h^n)} \right| \right] = 2\mathbb{E} \left[ \left| (D_1^{(h)} - D_1^{(h^n)})^+ \right| \right].$$ \hspace{1cm} (3.31)
By letting \( X_n := \log D_1^{(h^n)} - \log D_1^{(h)} \), the right-hand side of (3.31) is rewritten as

\[
2\mathbb{E} \left[ (1 - e^{X_n})^+ D_1^{(h)} \right],
\]

which is bounded from above by

\[
2\mathbb{E} \left[ |X_n| D_1^{(h)} \right] \leq 2\mathbb{E} \left[ |X_n|^2 \right]^{1/2} \mathbb{E} \left[ (D_1^{(h)})^2 \right]^{1/2}.
\]

To obtain the lemma, it is enough to show that \( \mathbb{E}[|X_n|^2] \) tends to 0 as \( n \to \infty \) since \( \mathbb{E}[(D_1^{(h)})^2] \) is finite by (2.15). Note that

\[
X_n = \int_0^1 (h^n_s - h_s) \cdot dB_s - \frac{1}{2} \int_0^1 (h^n_s + h_s) \cdot (dB)_s (h^n_s - h_s).
\]

Since it holds that

\[
\mathbb{E} \left[ \left| \int_0^1 (h^n_s - h_s) \cdot dB_s \right|^2 \right] \leq \sigma_1^2 \|h^n - h\|^2_{M^2_G(0,1;\mathbb{R}^d)}
\]

and

\[
\mathbb{E} \left[ \left| \int_0^1 (h^n_s + h_s) \cdot (dB)_s (h^n_s - h_s) \right|^2 \right] \leq \sigma_1^4 \mathbb{E} \left[ \int_0^1 |h^n_s + h_s|^2 \, ds \right] \mathbb{E} \left[ \int_0^1 |h^n_s - h_s|^2 \, ds \right] \\
\leq 4\sigma_1^4 \sup_{0 \leq s \leq 1} \|h_t\|_\infty^2 \|h^n - h\|^2_{M^2_G(0,1;\mathbb{R}^d)},
\]

we get \( \lim_{n \to \infty} \mathbb{E}[|X_n|^2] = 0 \), and complete the proof.

Let \( f \in \mathcal{L}^1_G(\Omega) \) be bounded and \( \{f_n\}_{n \in \mathbb{N}} \subset C_{b,Lip}(\Omega) \) such that

\[
\|f_n\|_\infty \leq \|f\|_\infty \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}[|f_n(B) - f(B)|] = 0. \tag{3.32}
\]

For each \( f_n \), let \( h^n \in (M^2_G(0,1))^d \) and \( A^n \in \mathcal{L}^2_G(\Omega) \) be as given by Lemma 3.8 and \( P^n \in \mathcal{P} \) as given by Lemma 3.9.

**Lemma 3.11.** For each \( n \in \mathbb{N} \), we have

\[
\log \mathbb{E} \left[ e^{f_n(B)} \right] = E_{Q^n} \left[ f_n(B) - \frac{1}{2} \int_0^1 h^n_s \cdot (dB)_s h^n_s \right], \tag{3.33}
\]

where \( Q^n := D_1^{(h^n)} P^n \).
Proof. Setting \( \bar{B} := T^{-h^n}(B) \), we have by Lemma 3.8
\[
\log \mathbb{E} [e^{f_n(B)}] = f_n(B) - \int_0^1 h^n_s \cdot d\bar{B}_s - \frac{1}{2} \int_0^1 h^n_s \cdot (d\langle B \rangle_s h^n_s) + A^n.
\]
Since \( h^n \) is bounded, we have by (ii) of Lemma 2.7,
\[
\mathbb{E} \left[ \left( \int_0^1 h^n_s \cdot d\bar{B}_s \right) D_1^{(h^n)} \right] = -\mathbb{E} \left[ -\left( \int_0^1 h^n_s \cdot d\bar{B}_s \right) D_1^{(h^n)} \right] = 0.
\]
By (2.9), this relation holds with \( \mathbb{E} \) replaced by \( \hat{\mathbb{E}} \), which results in
\[
E_P \left[ \left( \int_0^1 h^n_s \cdot d\bar{B}_s \right) D_1^{(h^n)} \right] = 0 \quad \text{for all } P \in \mathcal{P}.
\]
Combining this with \( P^n(A^n = 0) = 1 \) leads to (3.33).

Now we are in a position to prove the upper bound in Theorem 3.1.

**Proposition 3.12.** \( (3.15) \) holds for any bounded elements \( f \) of \( L^1_G(\Omega) \).

**Proof.** For a bounded \( f \in L^1_G(\Omega) \), let \( \{f_n\}_{n \in \mathbb{N}} \subset C_b,\text{Lip}(\Omega) \) satisfy (3.32), and let \( h^n, A^n \) and \( P^n \) be as above. Then for each \( n \in \mathbb{N} \), we have by Lemma 3.11
\[
\log \mathbb{E} [e^{f(B)}] = \log \mathbb{E} [e^{f(B)}] - \log \mathbb{E} [e^{f_n(B)}] + E_{Q^n}[f_n(B) - f(B)]
+ E_{Q^n} \left[ f(B) - \frac{1}{2} \int_0^1 h^n_s \cdot (d\langle B \rangle_s h^n_s) \right]. \tag{3.34}
\]

For the difference of the first two terms in the right-hand side of (3.34), as seen in (3.14), we have
\[
|\log \mathbb{E} [e^{f(B)}] - \log \mathbb{E} [e^{f_n(B)}]| \to 0 \quad \text{as } n \to \infty.
\]
As to the third term in the right-hand side of (3.34), we see from Lemmas 3.8 and 3.9 that
\[
D_1^{(h^n)} = \frac{e^{f_n(B)}}{\mathbb{E} [e^{f_n(B)}]} \leq e^{2\|f\|_\infty} P^n\text{-a.s.},
\]
and hence
\[
E_{Q^n}[f_n(B) - f(B)] \leq e^{2\|f\|_\infty} E_{P^n}[|f_n(B) - f(B)|]
\leq e^{2\|f\|_\infty} \hat{\mathbb{E}}[|f_n(B) - f(B)|],
\]
which converges to 0 as \( n \to \infty \) by (2.9) and (3.32).

Therefore it remains to estimate the last term in the right-hand side of (3.34). For this purpose, we first observe the bound
\[
E_{Q^n} \left[ f(B) - \frac{1}{2} \int_0^1 h^n_s \cdot (d\langle B \rangle_s h^n_s) \right] \leq \hat{\mathbb{E}} \left[ \left( f(B) - \frac{1}{2} \int_0^1 h^n_s \cdot (d\langle B \rangle_s h^n_s) \right) D_1^{(h^n)} \right]
\]
\begin{equation}
E^h \left[ f(B) - \frac{1}{2} \int_0^1 h_s^+ \cdot (d\langle B \rangle_s h_s^+) \right],
\end{equation}
(3.35)

where the equality follows from (2.9). Fix \( \varepsilon > 0 \) arbitrarily. From (3.29) in Lemma 3.10 we see that, for every \( n \in \mathbb{N} \), there exists an \( h \equiv h^{(n, \varepsilon)} \in (S_{b, Lip})^d \) such that

\begin{equation}
E^h \left[ f(B) - \frac{1}{2} \int_0^1 h_s^+ \cdot (d\langle B \rangle_s h_s^+) \right] \leq E^h \left[ f(B) - \frac{1}{2} \int_0^1 h_s^+ \cdot (d\langle B \rangle_s h_s^+) \right] + \varepsilon.
\end{equation}
(3.36)

Let \( h \in (S_{b, Lip})^d \) be written as (3.7) and define a simple process \( \hat{h} \) as follows:

\begin{align*}
\hat{\xi}_0 &= \xi_0, \\
\hat{\xi}_1 &= \xi_1 \left( B_t + \int_0^t d\langle B \rangle_s \hat{h}_s, \ t \leq t_1 \right), \\
&\vdots \\
\hat{\xi}_{m-1} &= \xi_{m-1} \left( B_t + \int_0^t d\langle B \rangle_s \hat{h}_s, \ t \leq t_{m-1} \right), \\
\hat{h}_t &= \hat{\xi}_m, \ t_{m-1} \leq t < t_m.
\end{align*}

By this construction, it is obvious that \( \hat{h} \) is in \((M^2_G(0, 1))^d\) and bounded. Furthermore, it can be shown inductively that

\[ h_t = \hat{h}_t(T^{-h}(B)), \quad t_k \leq t < t_{k+1}. \]

for all \( k = 0, 1, \ldots, m - 1 \). Put \( \hat{B} := T^{-h}(B) \). By Theorem 2.4 we have

\begin{align*}
E^h \left[ f(B) - \frac{1}{2} \int_0^1 h_s^+ \cdot (d\langle B \rangle_s h_s^+) \right] \\
= E^h \left[ f \left( \hat{B} + \int_0^t d\langle B \rangle_s \hat{h}_s(B) \right) - \frac{1}{2} \int_0^1 \hat{h}_s(\hat{B}) \cdot \left( d\langle \hat{B} \rangle_s \hat{h}_s(\hat{B}) \right) \right] \\
= E \left[ f \left( B + \int_0^t d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot \left( d\langle B \rangle_s h_s \right) \right];
\end{align*}

for the validity of the second equality, see Remark 2.5. Combining this with (3.35) and (3.36), we obtain

\begin{align*}
E_Q^n \left[ f(B) - \frac{1}{2} \int_0^1 h_s^+ \cdot (d\langle B \rangle_s h_s^+) \right] \\
\leq \sup_{h \in (M^2_G(0, 1))^d} E \left[ f \left( B + \int_0^t d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot \left( d\langle B \rangle_s h_s \right) \right] + \varepsilon
\end{align*}

for all \( n \in \mathbb{N} \), and complete the proof. \( \square \)
4 An application to large deviations

In this section, we apply Theorem 3.1 to the derivation of the Laplace principles for \( \{ (\sqrt{\varepsilon} B, \langle B \rangle); \varepsilon > 0 \} \) and \( \{ \sqrt{\varepsilon} B; \varepsilon > 0 \} \). Similarly to the classical case, the Laplace principle implies the large deviation principle, and hence we recover the large deviation principles for these two families, which are originally proved by Gao-Jiang [5] through discretization technique.

First we formulate the Laplace principle under \( G \)-expectation as follows: Let \( \{ X^\varepsilon; \varepsilon > 0 \} \) be a family of random variables taking values in a Polish space \( \mathcal{X} \). We let \( I \) be a rate function on \( \mathcal{X} \), that is, a mapping from \( \mathcal{X} \) into \( [0, \infty) \) such that for each \( M > 0 \) the revel set \( \{ x \in \mathcal{X} : I(x) \leq M \} \) is a compact subset of \( \mathcal{X} \). We say that \( \{ X^\varepsilon; \varepsilon > 0 \} \) satisfies the Laplace principle on \( \mathcal{X} \) with rate function \( I \) if for all bounded and continuous functions \( \Phi : \mathcal{X} \rightarrow \mathbb{R} \), it holds that

\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Phi(X^\varepsilon)}{\varepsilon} \right) \right] = \sup_{x \in \mathcal{X}} \{ \Phi(x) - I(x) \}.
\]

(4.1)

The following proposition can be proved through the same argument as in the classical case (see, e.g., the proof of [4, Theorem 1.2.3]). We remark that the validity of the converse assertion is ensured by Lemma A.3 in [5].

**Proposition 4.1.** The Laplace principle implies the large deviation principle with the same rate function. More precisely, if the Laplace limit (4.1) holds for all bounded continuous functions \( \Phi : \mathcal{X} \rightarrow \mathbb{R} \), then \( \{ X^\varepsilon; \varepsilon > 0 \} \) satisfies the large deviation principle on \( \mathcal{X} \) with rate function \( I \).

**Remark 4.2.** As in the classical case, it is enough to check that the Laplace limit (4.1) is valid for all bounded Lipschitz continuous functions \( \Phi \) in order to see \( \{ X^\varepsilon; \varepsilon > 0 \} \) satisfies the Laplace principle.

Denote by \( C([0, 1]; \mathbb{R}^{d \times d}) \) (resp. by \( C([0, 1]; \mathbb{R}^d) \)) the space of all \( \mathbb{R}^{d \times d} \)-valued (resp. \( \mathbb{R}^d \)-valued) continuous functions on \([0, 1] \) vanishing at 0. We equip \( C([0, 1]; \mathbb{R}^{d \times d}) \) with the distance
dist\( (y^1, y^2) := \sup_{0 \leq t \leq 1} \| y^1(t) - y^2(t) \|, \quad y^1, y^2 \in C([0, 1]; \mathbb{R}^{d \times d}), \)

where \( \| A \| = \sqrt{\text{tr}[AA^*]} \) for \( A \in \mathbb{R}^{d \times d} \), and equip \( C([0, 1]; \mathbb{R}^d) \) with the distance \( \rho \) defined by (2.1). Set

\[
\mathbb{H} := \left\{ x \in C([0, 1]; \mathbb{R}^d) : x \text{ is absolutely continuous and } \int_0^1 |\dot{x}(t)|^2 \, dt < \infty \right\},
\]

\[
\mathbb{A} := \left\{ y \in C([0, 1]; \mathbb{R}^{d \times d}) : y \text{ is absolutely continuous and } \dot{y}(t) \in \{ \gamma \gamma^* : \gamma \in \Theta \} \text{ for a.e. } t \in [0, 1] \right\}.
\]

Here \( \dot{x} \) and \( \dot{y} \) denote the derivatives \( dx/dt \) and \( dy/dt \), respectively, provided that they exist. We define rate functions \( J : C([0, 1]; \mathbb{R}^d) \times C([0, 1]; \mathbb{R}^{d \times d}) \rightarrow [0, \infty] \) and \( I :
\( C([0, 1]; \mathbb{R}^d) \rightarrow [0, \infty] \) by

\[
J(x, y) = \begin{cases} 
\frac{1}{2} \int_0^1 \dot{x}(t) \cdot (\dot{y}(t)^{-1} \dot{x}(t)) \, dt & \text{if } (x, y) \in \mathbb{H} \times A, \\
+\infty & \text{otherwise},
\end{cases}
\]

\[
I(x) = \begin{cases} 
\frac{1}{2} \int_0^1 \inf_{\gamma \in \Theta} |\gamma^{-1} \dot{x}(t)|^2 \, dt & \text{if } x \in \mathbb{H}, \\
+\infty & \text{otherwise}.
\end{cases}
\]

In the following, we abbreviate the notation as sup \( x \) (resp. sup \( (x, y) \)) when we take the supremum over \( x \in C([0, 1]; \mathbb{R}^d) \) (resp. \( (x, y) \in C([0, 1]; \mathbb{R}^d) \times C([0, 1]; \mathbb{R}^{d \times d}) \)).

**Proposition 4.3.** (i) For any bounded Lipschitz continuous function \( \Psi : C([0, 1]; \mathbb{R}^d) \times C([0, 1]; \mathbb{R}^{d \times d}) \rightarrow \mathbb{R} \),

\[
\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Psi(\sqrt{\varepsilon} B, \langle B \rangle)}{\varepsilon} \right) \right] = \sup_{(x, y)} \{ \Psi(x, y) - J(x, y) \}.
\]

(ii) For any bounded Lipschitz continuous function \( \Phi : C([0, 1]; \mathbb{R}^d) \rightarrow \mathbb{R} \),

\[
\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Phi(\sqrt{\varepsilon} B)}{\varepsilon} \right) \right] = \sup_x \{ \Phi(x) - I(x) \}.
\]

From Proposition 4.1 and Remark 4.2, we see that Proposition 4.3 implies that the family \( \{ (\sqrt{\varepsilon} B, \langle B \rangle) ; \varepsilon > 0 \} \) (resp. \( \{ \sqrt{\varepsilon} B ; \varepsilon > 0 \} \)) satisfies the large deviation principle on \( C([0, 1]; \mathbb{R}^d) \times C([0, 1]; \mathbb{R}^{d \times d}) \) (resp. \( C([0, 1]; \mathbb{R}^d) \)) with rate function \( J \) (resp. \( I \)).

We begin with a lemma which is an application of Theorem 3.1.

**Lemma 4.4.** For every \( \varepsilon > 0 \), we have the following.

(i) For any bounded Lipschitz continuous function \( \Psi : C([0, 1]; \mathbb{R}^d) \times C([0, 1]; \mathbb{R}^{d \times d}) \rightarrow \mathbb{R} \),

\[
\varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Psi(\sqrt{\varepsilon} B, \langle B \rangle)}{\varepsilon} \right) \right] = \sup_{h \in (M^2_{d}(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon} B + \int_0^1 d(B)_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right].
\]

(ii) For any bounded and Lipschitz continuous function \( \Phi : C([0, 1]; \mathbb{R}^d) \rightarrow \mathbb{R} \),

\[
\varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Phi(\sqrt{\varepsilon} B)}{\varepsilon} \right) \right] = \sup_{h \in (M^2_{d}(0, 1))^d} \mathbb{E} \left[ \Phi \left( \sqrt{\varepsilon} B + \int_0^1 d(B)_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right].
\]
Proof. The proofs of (i) and (ii) are similar, so we only show (i).

We first check that the functional $\Psi(\sqrt{\varepsilon}B, \langle B \rangle)$ is a bounded element of $L^1_G(\Omega)$. For $n \in \mathbb{N}$, let $\Delta_n = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be the partition of $[0, 1]$ such that $t_k - t_{k-1} = 1/n$ for all $k = 1, \ldots, n$. For every $y \in C([0, 1]; \mathbb{R}^{d \times d})$ we denote by $(y)^{\Delta_n}$ the polygonal approximation of $y$ such that $(y)^{\Delta_n}(t_k) = y(t_k)$, $k = 1, \ldots, n$. Since $\langle B \rangle_t \in (L^1_G(\Omega))^{d \times d}$ for each $t \in [0, 1]$, the mapping
\[
\Omega \ni \omega \mapsto \Psi_n(\omega) := \Psi \left( \sqrt{\varepsilon} \omega, (\langle B \rangle)^{\Delta_n}(\omega) \right)
\]
has a q.c. version. $\Psi_n$ is also bounded, and hence is in $L^1_G(\Omega)$. By the Lipschitz continuity of $\Psi$, we have
\[
\mathbb{E} \left[ |\Psi(\sqrt{\varepsilon}B, \langle B \rangle) - \Psi_n(B)| \right] \leq \text{Lip}(\Psi) \mathbb{E} \left[ \text{dist}(\langle B \rangle, (\langle B \rangle)^{\Delta_n}) \right]. \tag{4.3}
\]

Note that
\[
\text{dist}(\langle B \rangle, (\langle B \rangle)^{\Delta_n}) = \max_{1 \leq k \leq n} \sup_{t_k - 1 \leq t \leq t_k} \| (\langle B \rangle)_t - (\langle B \rangle)_{t_k - 1} - n(t - t_k)(\langle B \rangle)_{t_k} - (\langle B \rangle)_{t_k - 1} \| \leq 2 \max_{1 \leq k \leq n} \sup_{t_k - 1 \leq t \leq t_k} \| (\langle B \rangle)_t - (\langle B \rangle)_{t_k - 1} \|, \tag{4.4}
\]
and that $\| (\langle B \rangle)_t - (\langle B \rangle)_s \| \leq d\sigma^2|t - s|$ for all $0 \leq s, t \leq 1$ q.s. Then the right-hand side of (4.3) is estimated from above by
\[
\frac{2d\text{Lip}(\Psi)\sigma^2}{n},
\]
which tends to 0 as $n \to \infty$, and hence $\Psi(\sqrt{\varepsilon}B, \langle B \rangle) \in L^1_G(\Omega)$.

Now let us verify (4.2). By Theorem 3.1 we have
\[
\varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Psi(\sqrt{\varepsilon}B, \langle B \rangle)}{\varepsilon} \right) \right] = \sup_{h \in (M^2_2(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon}T^h(B), \langle T^h(B) \rangle \right) - \frac{\varepsilon}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right]
\]
\[
= \sup_{h \in (M^2_2(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon}B + \int_0^1 d\langle B \rangle_s \sqrt{\varepsilon}h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 \sqrt{\varepsilon}h_s \cdot (d\langle B \rangle_s \sqrt{\varepsilon}h_s) \right]
\]
\[
= \sup_{h \in (M^2_2(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon}B + \int_0^1 d\langle B \rangle_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right],
\]
which is (4.2). \qed

By using Lemma 4.4 we prove Proposition 4.3.
Proof of Proposition 4.4 (i) By the Lipschitz continuity of $\Psi$, we have
\[
\sup_{h \in (M^2_0(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon} B + \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^t h_s \cdot \langle d\langle B \rangle_s h_s \rangle \right]
- \sup_{h \in (M^2_0(0, 1))^d} \mathbb{E} \left[ \Psi \left( \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^t h_s \cdot \langle d\langle B \rangle_s h_s \rangle \right]
\leq \sup_{h \in (M^2_0(0, 1))^d} \mathbb{E} \left[ \Psi \left( \sqrt{\varepsilon} B + \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) - \Psi \left( \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) \right]
\leq C \sqrt{\varepsilon}.
\] (4.5)

Here $C := \text{Lip}(\Psi) \mathbb{E} \left[ \sup_{0 \leq t \leq 1} |B_t| \right]$, whose finiteness follows from the Cauchy-Schwarz inequality and Doob's inequality:
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq 1} |B_t| \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq 1} |B_t|^2 \right]^{1/2}
= \sup_{\theta \in A_{0,1}^\Theta} \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \int_0^t \theta_s dW_s \right]^{1/2}
\leq 2 \sup_{\theta \in A_{0,1}^\Theta} \mathbb{E} \left[ \int_0^1 \text{tr} [\theta_s^2] ds \right]^{1/2} \leq 2 \sqrt{d \sigma_1};
\] (4.6)
for the equality, recall that, in the definition of the upper expectation $\mathbb{E}$, the supremum is taken over the laws of $\int_0^t \theta_s dW_s$, $\theta \in A_{0,1}^\Theta$, under the probability measure $P$. Combining Lemma 4.4 with (4.5), we see that
\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Psi(\sqrt{\varepsilon} B, \langle B \rangle)}{\varepsilon} \right) \right]
= \sup_{h \in (M^2_0(0, 1))^d} \mathbb{E} \left[ \Psi \left( \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^t h_s \cdot \langle d\langle B \rangle_s h_s \rangle \right].
\]
Hence what to show is that
\[
\sup_{h \in (M^2_0(0, 1))^d} \mathbb{E} \left[ \Psi \left( \int_0^t d\langle B \rangle_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^t h_s \cdot \langle d\langle B \rangle_s h_s \rangle \right]
= \sup_{(x, y)} \{ \Psi(x, y) - J(x, y) \}.
\] (4.7)

First we prove the upper bound. For $\theta \in A_{0,1}^\Theta$ and $h \in (M^2_0(0, 1))^d$, we set
\[
X_t \equiv X_t^{(\theta, h)} := \int_0^t \theta_s h_s^\theta ds, \quad Y_t \equiv Y_t^{(\theta)} := \int_0^t \theta_s ds \quad \text{for} \quad 0 \leq t \leq 1.
\]
Here $h^{(\theta)}$ is defined by
\[
h_t^{(\theta)} = h_t \left( \int_0^t \theta_s dW_s \right) \quad \text{for} \quad 0 \leq t \leq 1.
\]
Then $X \in \mathbb{H}$ and $Y \in \mathbb{A}$ $P$-a.s., and hence

$$E_P \left[ \Psi \left( \int_0^t d(B)_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right] = E_P \left[ \Psi(X, Y) - J(X, Y) \right]$$

$$\leq \sup \left\{ \Psi(x, y) - J(x, y) \right\}.$$ 

Since $\theta \in \mathcal{A}^{\Theta}_{0,1}$ and $h \in (M^2_G(0, 1))^d$ are arbitrary, we get the upper bound in (4.7).

Next we prove the lower bound in (4.7). If $x \not\in \mathbb{H}$ or $y \not\in \mathbb{A}$, the right-hand side of (4.7) is $-\infty$, so we take an arbitrary $(x, y) \in \mathbb{H} \times \mathbb{A}$. Let $y$ be written as

$$y(t) = \int_0^t g(s)g(s)^* \, ds, \quad 0 \leq t \leq 1,$$

for some deterministic measurable function $g : [0, 1] \to \Theta$. We denote by $P_y$ the law of $\int_0^t g(s) \, dW_s$ and define a deterministic function $\eta$ by

$$\eta_t = \dot{y}(t)^{-1} \dot{x}(t) \quad \text{for a.e. } t \in [0, 1].$$

Then $\eta$ is in $(M^2_G(0, 1))^d$, whence

$$\sup_{h \in (M^2_G(0, 1))^d} \mathbb{E} \left[ \Psi \left( \int_0^t d(B)_s h_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right]$$

$$\geq E_P \left[ \Psi \left( \int_0^t d(B)_s \eta_s, \langle B \rangle \right) - \frac{1}{2} \int_0^1 \eta_s \cdot (d(B)_s \eta_s) \right]$$

$$= E_P \left[ \Psi \left( \int_0^t \dot{y}(s) \eta_s \, ds, y \right) - \frac{1}{2} \int_0^1 \eta_s \cdot (\dot{y}(s) \eta_s) \, ds \right]$$

$$= \Psi(x, y) - J(x, y).$$

Taking the supremum of the right-hand side, we obtain the lower bound.

(ii) Similarly to (i), we have by Lemma 4.4

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \left[ \exp \left( \frac{\Phi(\sqrt{\varepsilon} B)}{\varepsilon} \right) \right]$$

$$= \sup_{h \in (M^2_G(0, 1))^d} \mathbb{E} \left[ \Phi \left( \int_0^t d(B)_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right].$$

Therefore it is sufficient to show

$$\sup_{h \in (M^2_G(0, 1))^d} \mathbb{E} \left[ \Phi \left( \int_0^t d(B)_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right] = \sup_x \{ \Phi(x) - I(x) \}. \quad (4.8)$$

For the upper bound, take $\theta \in \mathcal{A}^{\Theta}_{0,1}$ and $h \in (M^2_G(0, 1))^d$ arbitrarily and let $X \equiv X^{(\theta, h)}$ be as in the proof of (i). Then we have

$$E_P \left[ \Phi \left( \int_0^t d(B)_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d(B)_s h_s) \right] = E_P \left[ \Phi(X) - \frac{1}{2} \int_0^1 \theta^{-1} \dot{X}_s^2 \, ds \right]$$
\[ \leq E_P [\Phi(X) - I(X)] \]
\[ \leq \sup_x \{\Phi(x) - I(x)\}. \]

For the lower bound in (4.8), we fix any \( x \in \mathbb{H} \). By the measurable selection (see, e.g., [2, Lemma A.1]), there exists a measurable map \( \Gamma : \mathbb{R}^d \to \Theta \) such that
\[ |\Gamma(\xi)^{-1}\xi| = \inf_{\gamma \in \Theta} |\gamma^{-1}\xi| \quad \text{for all } \xi \in \mathbb{R}^d. \]

For such \( \Gamma \), define \( g(s) := \Gamma(\dot{x}(s)) \) for a.e. \( s \in [0, 1] \), and note that
\[ I(x) = \frac{1}{2} \int_0^1 |g(s)^{-1}\dot{x}(s)|^2 ds. \]

Denote by \( P_g \) the law of \( \int_0^1 g(s) dW_s \) and set
\[ z(t) := \int_0^t (g(s)g(s)^*)^{-1} \dot{x}(s) ds. \]

Since \( \dot{z} \in (M^2_{G}(0, 1))^d \), it follows that
\[ \sup_{h \in (M^2_{G}(0, 1))^d} \mathbb{E} \left[ \Phi \left( \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right] \]
\[ \geq E_{P_g} \left[ \Phi \left( \int_0^1 d\langle B \rangle_s \dot{z}(s) \right) - \frac{1}{2} \int_0^1 \dot{z}(s) \cdot (d\langle B \rangle_s \dot{z}(s)) \right] \]
\[ = \Phi(x) - \frac{1}{2} \int_0^1 |g(s)^{-1}\dot{x}(s)|^2 ds \]
\[ = \Phi(x) - I(x), \]

and hence by taking the supremum of the rightmost side, we obtain the lower bound in (4.8).

\[ \square \]

Remark 4.5. Let \( \mathcal{E}[X] := -\mathbb{E}[-X] \) for \( X \in \mathcal{L}_G^1(\Omega) \). By similar arguments to those in Section 3, we can also derive a variational representation
\[ \log \mathcal{E}[e^{f(B)}] = \sup_{h \in (M^2_{G}(0, 1))^d} \mathcal{E} \left[ f \left( B + \int_0^1 d\langle B \rangle_s h_s \right) - \frac{1}{2} \int_0^1 h_s \cdot (d\langle B \rangle_s h_s) \right] (4.9) \]

for any bounded elements \( f \) of \( \mathcal{L}_G^1(\Omega) \). Note that by (2.8), \( \mathcal{E}[X] = \inf_{\theta \in \mathcal{A}_{0, 1}^G} E_{P_\theta}[X] \), to which we may associate the “lower” capacity \( \underline{\mathcal{C}} \) via
\[ \underline{\mathcal{C}}(A) = \inf_{\theta \in \mathcal{A}_{0, 1}^G} P_\theta(A) \quad \text{for } A \in \mathcal{B}(\Omega). \]

We expect that similar applications to those given in this section are also possible under \( \mathcal{E} \) and \( \underline{\mathcal{C}} \). A key to the validity of (4.9) is the identity (3.29) in Lemma 3.10.
5 An absolute continuity result

We conclude this paper with the proof of an absolute continuity relationship between $B$ and $B + \int_0^t d(B)_s h_s$ under the capacity, which plays a key role in Subsection 3.1 and is of independent interest itself. Let $c$ and $\widehat{c}$ be the capacities given in (2.6) and (2.10), respectively, and $T^h(B)$ for $h \in (M_G^2(0,1))^d$ as in Section 3.

**Proposition 5.1.** Let $h \in (M_G^2(0,1))^d$ be arbitrary. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$c(T^h(B) \in A) < \varepsilon \quad \text{for all } A \in \mathcal{B}(\Omega) \text{ with } \widehat{c}(A) < \delta. \quad (5.1)$$

In particular,

$$c(T^h(B) \in N) = 0 \quad \text{for all } N \in \mathcal{B}(\Omega) \text{ with } c(N) = 0. \quad (5.2)$$

**Remark 5.2.** Note that, since the reverse Fatou lemma does not hold in full generality under $c$, we cannot conclude from (5.2) the stronger result than (5.1):

$$c(T^h(B) \in A) < \varepsilon \quad \text{for all } A \in \mathcal{B}(\Omega) \text{ with } c(A) < \delta.$$

We start with the following lemma.

**Lemma 5.3.** Let $h \in (S_{b,Lip})^d$. It holds that

$$E^\widehat{\Pi} \left[ \log D_{1}^{(\widehat{\Pi})} \right] \leq \frac{1}{2} \sigma_1^2 \| h \|^2_{M_G^2(0,1;\mathbb{R}^d)}.$$

Here $\widehat{\Pi}$ is the associated bounded element of $(M_G^2(0,1))^d$ defined so that the relation (3.8) holds.

**Proof.** First observe that by the boundedness of $h$, $(\log D_{1}^{(\widehat{\Pi})})D_{1}^{(\widehat{\Pi})} \in \mathcal{L}_G^1(\Omega)$, which may be deduced from the proof of (ii) of Lemma 2.7. Denoting $\widehat{B} = T^{-\widehat{\Pi}}(B)$, we have

$$\log D_{1}^{(\widehat{\Pi})} = \int_0^1 \widehat{h}_s \cdot d\widehat{B}_s + \frac{1}{2} \int_0^1 \widehat{h}_s \cdot (d\langle B \rangle_s \widehat{h}_s). \quad (5.3)$$

Taking the sublinear expectation of both sides of (5.3) under $E^\widehat{\Pi}$, we have by (ii) of Lemma 2.7

$$E^\widehat{\Pi} \left[ \log D_{1}^{(\widehat{\Pi})} \right] = \frac{1}{2} E^\widehat{\Pi} \left[ \int_0^1 \widehat{h}_s \cdot (d\langle B \rangle_s \widehat{h}_s) \right]$$

$$= \frac{1}{2} E^\widehat{\Pi} \left[ \int_0^1 h_s(\widehat{B}) \cdot (d\langle B \rangle_s h_s(\widehat{B})) \right],$$

where the second equality follows from (3.8) and the obvious identity $\langle B \rangle = \langle \widehat{B} \rangle$. Then by Theorem 2.4 and Remark 2.5

$$\frac{1}{2} E^\widehat{\Pi} \left[ \int_0^1 h_s(\widehat{B}) \cdot (d\langle B \rangle_s h_s(\widehat{B})) \right] = \frac{1}{2} E \left[ \int_0^1 h_s(B) \cdot (d\langle B \rangle_s h_s(B)) \right]$$

$$\leq \frac{1}{2} \sigma_1^2 E \left[ \int_0^1 |h_s(B)|^2 ds \right],$$

and therefore the assertion is proved. \qed
Lemma 5.4. Let \( f : \Omega \to \mathbb{R} \) be bounded and Lipschitz continuous. Then, for every \( h \in (M^2_G(0,1))^d \), we have \( f\left(T^h(B)\right) \in \mathcal{L}^1_G(\Omega) \).

Proof. For each \( n \in \mathbb{N} \), let the partition \( \Delta_n \) of \([0,1]\) and the polygonal approximation \((\omega)^{\Delta_n}\) of \( \omega \in \Omega \) as in the proof of Lemma 3.4. Since \( f \) is continuous and \( \int_0^t d(B)_s h_s \) has a q.c. version for each \( t \in [0,1] \), it is clear that the functional \( f\left((T^h(B))^{\Delta_n}\right) \) also has a q.c. version, hence belongs to \( \mathcal{L}^1_G(\Omega) \) due to (2.7). Therefore, in order to prove the lemma, it is sufficient to show

\[
\mathbb{E} \left[ f\left(T^h(B)\right) - f\left((T^h(B))^{\Delta_n}\right) \right] \xrightarrow{n \to \infty} 0. \tag{5.4}
\]

To this end, note that, similarly to (4.4), we have

\[
\rho\left(\omega, (\omega)^{\Delta_n}\right) \leq 2 \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} |\omega_t - \omega_{t_k-1}| \quad \text{for } \omega \in \Omega.
\]

Combining this estimate with Lipschitz continuity of \( f \), we see that the left-hand side of (5.4) is bounded from above by

\[
2\text{Lip}(f) \left( I^1_n + I^2_n \right)
\]

with

\[
I^1_n = \mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} |B_t - B_{t_k-1}| \right],
\]

\[
I^2_n = \mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} \left| \int_{t_{k-1}}^t d(B)_s h_s \right| \right].
\]

As to \( I^2_n \), we see that by (2.12) and the Cauchy-Schwarz inequality,

\[
I^2_n \leq \sigma^2 \mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} \int_{t_{k-1}}^t |h_s| \, ds \right] \leq \sigma^2 \mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} \sqrt{t - t_{k-1}} \left( \int_0^1 |h_s|^2 \, ds \right)^{1/2} \right] \leq \sigma^2 n^{-1/2} \|h\|_{M^2_G(\mathbb{R}^d)}.
\]

Therefore \( I^2_n \) tends to 0 as \( n \to \infty \). In order to see the convergence of \( I^1_n \) to 0, fix \( \varepsilon > 0 \) arbitrarily. By the tightness of the family \( \{P_{\theta} : \theta \in \mathbb{A}^\Theta_{0,1}\} \), there exists a compact \( K_\varepsilon \subset \Omega \) such that \( c(K_\varepsilon) \leq \varepsilon \). We bound \( I^1_n \) from above by the sum

\[
\mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} |B_t - B_{t_k-1}| ; K_\varepsilon \right] + \mathbb{E} \left[ \max_{1 \leq k \leq n} \sup_{t_k-1 \leq t \leq t_k} |B_t - B_{t_k-1}| ; K^c_\varepsilon \right]. \tag{5.5}
\]

Due to the uniform equicontinuity of \( K_\varepsilon \) by the Arzelà-Ascoli theorem (see [1, Theorem 7.2]), the first term of (5.5) converges to 0 as \( n \to \infty \). On the other hand, the second term of (5.5) is dominated by

\[
2 \mathbb{E} \left[ \sup_{0 \leq t \leq 1} |B_t| ; K^c_\varepsilon \right] \leq 2 \mathbb{E} \left[ \sup_{0 \leq t \leq 1} |B_t|^2 \right]^{1/2} c(K^c_\varepsilon)^{1/2}
\]
by the Cauchy-Schwarz inequality. Combining these with the estimate given in (4.6) yields
\[
\limsup_{n \to \infty} I^1_n \leq 4 \sqrt{d} \sigma_1 \sqrt{\varepsilon},
\]
which leads to (5.4) as \( \varepsilon \) is arbitrary.

Using Lemmas 5.3 and 5.4, we prove Proposition 5.1.

Proof of Proposition 5.1. Let \( \delta > 0 \) and \( A \in \mathcal{B}(\Omega) \) be such that \( \hat{c}(A) < \delta \). Let \( h \in M^2_G(0,1)^d \) and fix a compact subset \( K \subset \subset A \) arbitrarily. We take a sequence \( \{ h^n \}_{n \in \mathbb{N}} \subset (\mathcal{S}_{b \text{Lip}})^d \) so that
\[
\lim_{n \to \infty} \| h - h^n \|_{M^2_G(0,1;\mathbb{R}^d)} = 0.
\]
(5.6)

For each \( m \in \mathbb{N} \), set the function \( g_m \) by \( g_m(a) = 1 - ma \) for \( 0 \leq a \leq 1/m \) and \( g_m(a) = 0 \) for \( a > 1/m \). Then the mapping \( \Omega \ni \omega \mapsto g_m(\rho(\omega,K)) \) is bounded and Lipschitz continuous, and hence by Lemma 5.3
\[
g_m(\rho(T^h(B),K)), g_m(\rho(T^{h^n}(B),K)) \in \mathcal{L}^1_G(\Omega), \quad m = 1,2,\ldots
\]
(5.7)

Fix \( \alpha > 1 \). We start with the proof of
\[
\mathbb{E}_{\Omega} \left[ g_m(\rho(T^h(B),K)) \right] \leq \alpha \mathbb{E}_{\Omega} \left[ g_m(\rho(B,K)) \right] + \frac{1}{\log \alpha} C_h,
\]
(5.8)
\[
C_h := \frac{1}{e} + \frac{1}{2} \sigma_1^2 \sup_{n \in \mathbb{N}} \| h^n \|_{M^2_G(0,1;\mathbb{R}^d)}^2;
\]

note that \( C_h < \infty \) by (5.6). For each \( n \in \mathbb{N} \), we bound the left-hand side of (5.8) from above by
\[
\mathbb{E}_{\Omega} \left[ | g_m(\rho(T^h(B),K)) - g_m(\rho(T^{h^n}(B),K)) | \right] + \mathbb{E}_{\Omega} \left[ g_m(\rho(T^{h^n}(B),K)) \right].
\]
The first term is dominated by
\[
\text{Lip}(g_m) \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \int_0^t d\langle B \rangle_s (h_s - h^n_s) \right] \leq \text{Lip}(g_m) \sigma_1^2 \| h - h^n \|_{M^2_G(0,1;\mathbb{R}^d)},
\]
(5.9)
which vanishes by letting \( n \to \infty \). As to the second term of (5.9), we let \( \bar{h}^n \) be a bounded element of \( M^2_G(0,1) \) that satisfies (3.8) with \( h \) replaced by \( h^n \). Noting that (3.8) yields the relation \( T^{h^n} \circ T^{-\bar{h}^n}(B) = B \), we have, by (5.7) and Theorem 2.4 (see Remark 2.5),
\[
\mathbb{E}_{\Omega} \left[ g_m(\rho(T^{h^n}(B),K)) \right] = \mathbb{E}_{\Omega}^{\bar{h}^n} \left[ g_m(\rho(B,K)) \right].
\]
(5.10)

Following an argument in the proof of [2] Lemma 2.8, we estimate (5.10) from above by
\[
\mathbb{E} \left[ g_m(\rho(B,K)) D_1^{(\bar{h}^n)}; D_1^{(\bar{h}^n)} \leq \alpha \right] + \mathbb{E} \left[ g_m(\rho(B,K)) D_1^{(\bar{h}^n)}; D_1^{(\bar{h}^n)} > \alpha \right]
\]
\[ \leq \alpha \mathbb{E} \left[ g_m (\rho (B, K)) \right] + \frac{1}{\log \alpha} \mathbb{E} \left[ g_m (\rho (B, K)) D_1^{(\ell_1)} \log D_1^{(\ell_1)} ; D_1^{(\ell_1)} > \alpha \right] \]
\[ \leq \alpha \mathbb{E} \left[ g_m (\rho (B, K)) \right] + \frac{1}{\log \alpha} \left( \frac{1}{e} + \mathbb{E} \left[ D_1^{(\ell_1)} \log D_1^{(\ell_1)} \right] \right). \] (5.11)

Here for the last line, we used the inequality
\[ \mathbb{I}_{(\alpha, \infty)}(x) x \log x \leq \frac{1}{e} + x \log x \text{ for all } x > 0. \]

Using Lemma 5.3 we see that (5.11) is dominated from above by
\[ \alpha \mathbb{E} \left[ g_m (\rho (B, K)) \right] + \frac{1}{\log \alpha} C_h. \]

Combining these leads to (5.8).

For the the left-hand side of (5.8), we have
\[ c \left( T^h (B) \in K \right) \leq \mathbb{E} \left[ g_m (\rho (T^h (B), K)) \right] \] (5.12)
since \( \mathbb{I}_K(\omega) \leq g_m (\rho (\omega, K)) \) for all \( \omega \in \Omega \). On the other hand, when we let \( m \to \infty \), \( g_m (\rho (\omega, K)) \) converges to \( \mathbb{I}_K(\omega) \) for all \( \omega \in \Omega \) by the closedness of \( K \), and this convergence is decreasingly monotone. Therefore by [3, Theorem 31] and (2.9),
\[ \lim_{m \to \infty} \mathbb{E} \left[ g_m (\rho (B, K)) \right] = \mathbb{E} \left[ \mathbb{I}_K(B) \right] \leq \hat{c}(A), \] (5.13)
where the inequality follows from the inclusion \( K \subset A \). Then by (5.8), (5.12) and (5.13) with \( \alpha = 1/\sqrt{\delta} \), we have for any compact subset \( K \subset A \),
\[ c \left( T^h (B) \in K \right) \leq \frac{1}{\sqrt{\delta}} \hat{c}(A) + \frac{2}{\log(1/\delta)} C_h \]
\[ < \sqrt{\delta} + \frac{2}{\log(1/\delta)} C_h \]
as \( \hat{c}(A) < \delta \). Therefore
\[ c \left( T^h (B) \in A \right) = \sup_{\theta \in \mathcal{A}_0^{\Theta}} P_\theta \circ (T^h (B))^{-1} (A) \]
\[ = \sup_{\theta \in \mathcal{A}_0^{\Theta}} \sup_{K \subset A, K \text{ compact}} P_\theta \circ (T^h (B))^{-1} (K) \]
\[ = \sup_{K \subset A, K \text{ compact}} c \left( T^h (B) \in K \right) \]
\[ \leq \sqrt{\delta} + \frac{2}{\log(1/\delta)} C_h, \] (5.14)
where the second line follows from the fact that \( P_\theta \circ (T^h (B))^{-1} \) is a regular measure for each \( \theta \in \mathcal{A}_{0,1}^{\Theta} \) as \( \Omega \) is a complete separable metric space. As the rightmost side of (5.14) can be arbitrarily small by letting \( \delta \downarrow 0 \), we conclude (5.1). Moreover, as mentioned just before Remark 2.2 \( c(N) = 0 \) implies \( \hat{c}(N) = 0 \), from which we have (5.2). \( \square \)
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