Homology supported in Lagrangian submanifolds in mirror quintic threefolds

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Abstract. In this note, we study homology classes in the mirror quintic Calabi-Yau threefold which can be realized by special Lagrangian submanifolds. We have used Picard-Lefschetz theory to establish the monodromy action and to study the orbit of Lagrangian vanishing cycles. For many prime numbers $p$ we can compute the orbit modulo $p$. We conjecture that the orbit in homology with coefficients in $\mathbb{Z}$ can be determined by these orbits with coefficients in $\mathbb{Z}/p\mathbb{Z}$.

1 Introduction

Given a symplectic manifold $(X, \omega)$ of dimension $2n$ there are homology classes in $H_n(X, \mathbb{Q})$ which may be represented by Lagrangian cycles. In [17], the authors define Lagrangian cycles as cycles in a symplectic 4-manifold, whose two-simplices are given by $C^1$ Lagrangian maps and a Lagrangian homology class is a homology class which can be represented by a Lagrangian cycle. In that article, they show a characterization of the Lagrangian homology classes in terms of the minimizers of an area functional. Moreover, they show for a compact Kähler 4-manifold $(X, \omega, J)$ and a homology class $\alpha \in H_2(X, \mathbb{Z})$, that $\alpha$ is a Lagrangian homology class if and only if $[\omega](\alpha) = 0$, $c_1(X)(\alpha) = 0$, $\alpha^2 = -2$ and $\alpha$ is represented by a smooth sphere. For 4-manifolds, the dimension of the 2-cycles allows us to relate the property of being represented by Lagrangian cycles with the vanishing of the periods $\int_\alpha \omega$ and $\int_\alpha c_1(X)$. For higher dimension manifolds this pairing is not well-defined, hence we do not have a natural generalization of the previous results. Despite of this, it is possible to show that in any regular hypersurface of $\mathbb{P}^n$ with $n$ even, all $(n-1)$-cycles can be written as a linear combination of cycles supported in Lagrangian spheres, see Proposition 2.4.

A more interesting question for $n = 4$, is to ask not only which homology classes are generated by Lagrangian spheres but which ones are supported in Lagrangian spheres. In this article we consider a family $\tilde{X}_\varphi$ of mirror quintic Calabi-Yau threefolds and study some classes in $H_1(\tilde{X}_\varphi, \mathbb{Z})$ which are supported in Lagrangian 3-spheres and Lagrangian 3-tori. This family is constructed as follows. Consider the Dwork family $X_\varphi$ in $\mathbb{P}^4$ given...
by the locus of the polynomial
\[ p_\varphi := \varphi z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 - 5z_0z_1z_2z_3z_4 = 0, \]
with critical values in \( \varphi = 0, 1, \infty \). For every \( \varphi \neq 0, 1, \infty \), \( \bar{X}_\varphi \) is obtained as a desingularization of the quotient of \( X_\varphi \) by the action of a finite group, see §3 and [5, 6, 8]. The rank of the free group \( H_3(\bar{X}_\varphi, \mathbb{Z}) \) is four and hence it is isomorphic to \( \mathbb{Z}^4 \) after choosing a basis. In this basis the homology class \( \delta_2 = (0 \ 1 \ 0 \ 0) \) is represented by a torus associated to the singularity of \( X_\varphi \) when \( \varphi \to 0 \) and the class \( \delta_4 = (0 \ 0 \ 0 \ 1) \) is represented by a sphere \( S^3 \) associated to the singularity of \( X_\varphi \) when \( \varphi \to 1 \). As in [5] we give an explicit description of these two cycles in §4, and furthermore we show that these cycles are Lagrangian submanifolds of \( \bar{X}_\varphi \).

The monodromy action of the family is given by symplectomorphisms at each regular fiber. It is possible to determine two matrices \( M_0 \) and \( M_1 \) such that the monodromy action over \( H_3(\bar{X}_\varphi, \mathbb{Z}) \) corresponds (with respect to the basis mentioned above) to the free subgroup of \( Sp(4, \mathbb{Z}) \) generated by \( M_0 \) and \( M_1 \), see §3. Therefore, the orbit of \( \delta_2 \) and \( \delta_4 \) by the action of \( M_0 \ast M_1 \) are homology classes which can be represented by Lagrangian submanifolds. Our main result is about \( H_3(\bar{X}_\varphi, \mathbb{Z}_p) \), where \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) for some primes \( p \), and it is summarized in the following theorem.

**Theorem 1.1** For the mirror quintic Calabi-Yau threefold \( \bar{X} := \bar{X}_\varphi \) with \( \varphi \neq 0, 1, \infty \), the homology classes

\[
(0 \ 0 \ 1 \ 1), \ (0 \ 1 \ 0 \ 0), \ (0 \ 1 \ 0 \ 1), \ (1 \ 0 \ 0 \ 1), \ (1 \ 0 \ 1 \ 1) \in H_3(\bar{X}, \mathbb{Z}_2) \quad (1.1)
\]

\[
(0 \ 1 \ 0 \ 0), \ (0 \ 1 \ 0 \ 1), \ (0 \ 1 \ 0 \ 2), \ (0 \ 1 \ 0 \ 3), \ (0 \ 1 \ 0 \ 4),
(0 \ 1 \ 1 \ 0), \ (0 \ 1 \ 1 \ 1), \ (0 \ 1 \ 1 \ 2), \ (0 \ 1 \ 1 \ 3), \ (0 \ 1 \ 1 \ 4),
(0 \ 1 \ 2 \ 0), \ (0 \ 1 \ 2 \ 1), \ (0 \ 1 \ 2 \ 2), \ (0 \ 1 \ 2 \ 3), \ (0 \ 1 \ 2 \ 4),
(0 \ 1 \ 3 \ 0), \ (0 \ 1 \ 3 \ 1), \ (0 \ 1 \ 3 \ 2), \ (0 \ 1 \ 3 \ 3), \ (0 \ 1 \ 3 \ 4),
(0 \ 1 \ 4 \ 0), \ (0 \ 1 \ 4 \ 1), \ (0 \ 1 \ 4 \ 2), \ (0 \ 1 \ 4 \ 3), \ (0 \ 1 \ 4 \ 4) \in H_3(\bar{X}, \mathbb{Z}_5) \quad (1.2)
\]

are represented by Lagrangian 3-tori. The homology classes

\[
(0 \ 0 \ 0 \ 1), \ (0 \ 0 \ 1 \ 0), \ (0 \ 1 \ 1 \ 0), \ (0 \ 1 \ 1 \ 1), \ (1 \ 0 \ 0 \ 0),
(1 \ 0 \ 1 \ 0), \ (1 \ 1 \ 0 \ 0), \ (1 \ 1 \ 1 \ 0), \ (1 \ 1 \ 1 \ 1) \in H_3(\bar{X}, \mathbb{Z}_2) \quad (1.3)
\]

\[
(0 \ 0 \ 0 \ 1), \ (0 \ 0 \ 1 \ 1), \ (0 \ 0 \ 2 \ 1), \ (0 \ 0 \ 3 \ 1), \ (0 \ 0 \ 4 \ 1) \in H_3(\bar{X}, \mathbb{Z}_5) \quad (1.4)
\]

are represented by Lagrangian 3-spheres. For \( p = 3, 7, 11, 13, 17, 19, 23 \), any homology class in \( H_3(\bar{X}, \mathbb{Z}_p) \) different from \( (0 \ 0 \ 0 \ 0) \) can be represented by Lagrangian 3-tori and by Lagrangian 3-spheres.

In general for a manifold \( M \), a class \( \delta \in H_k(M, \mathbb{Z}) \) is called primitive if there is no \( m \in \mathbb{Z} \) and \( \delta' \in H_k(M, \mathbb{Z}) \) such that \( \delta = m\delta' \). We believe that for any prime different to 2 and 5, all classes in \( H_3(\bar{X}, \mathbb{Z}_p) \) different to \( (0 \ 0 \ 0 \ 0) \) can be represented by Lagrangian 3-tori and by a Lagrangian 3-spheres. This is a consequence of the following conjecture.
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Conjecture 1.2 Let $\delta$ be a primitive class in $H_3(\tilde{X}, \mathbb{Z})$. If $\text{mod}_2(\delta)$ is a homology class in the list (1.1) and $\text{mod}_5(\delta)$ is a homology class in the list (1.2), then $\delta$ is represented by a Lagrangian 3-torus. If $\text{mod}_2(\delta)$ is a homology class in the list (1.3) and $\text{mod}_5(\delta)$ is a homology class in the list (1.4), then $\delta$ is represented by a Lagrangian 3-sphere.

We have analogous results for other 14 examples of Calabi-Yau threefolds which appear in Table 1. However, in these cases we do not know if the vectors $\delta_2$ and $\delta_4$ have Lagrangian submanifolds associated as in the Dwork family case.

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2 Basics on Picard-Lefschetz theory

We recall some facts about Lefschetz fibration in symplectic geometry. These results are in the literature, see for example [1, 2, 18, 19]. We collect them here to set notations and for quick reference throughout the article.

Let $Y$ be a complex manifold. A Lefschetz fibration is a surjective analytic map $f : Y \rightarrow P^1$ with a finite number of critical points, such that for any critical point $p$, there is a chart with Morse coordinates. This means that there is a coordinate system around $p$ such that $f(z) = f(p) + z_1^2 + \cdots + z_n^2$ for $z$ in a neighborhood of $p$.

Every projective manifold $Y \hookrightarrow P^N$ has a natural symplectic form $\omega$ given by the pullback of the Fubini study form in $P^N$. Since the fibers of $f$ over regular values are complex submanifolds of $Y$, the restriction of $\omega$ to each regular fiber remains symplectic. Furthermore, the regular fibers of the Lefschetz fibration $f : Y \rightarrow P^1$ are symplectomorphic. This follows from the following symplectic version of the Ehresmann lemma.

Proposition 2.1 Let $(E, \omega)$ be a symplectic manifold and $B$ be a connected manifold. Consider $f : E \rightarrow B$ a proper surjective map with a finite set of critical values $C$, such that $\omega$ is symplectic at every regular fiber of $f$. Then the regular fibers are symplectomorphic.

Proof Using $\omega$ we can decompose the tangent bundle $TE$, over the set of regular values, as a direct sum of a vertical bundle $V E$ and a horizontal bundle $HE := \left(VE\right)^\omega$. Here, the vertical space $V_e E$ is the space of vectors tangent to the fibers of $f$ and the horizontal space $H_e E$ is its symplectic complement. This is well-defined since the restriction of $\omega$ to the fibers is symplectic.

Let $b \in B$ be a regular value and $U \subset B \setminus C$ be a neighborhood of $b$. We take a vector field $W$ defined on $U$, without singularities. Since $f$ is a submersion on $U$, the map $f_*$ is an isomorphism between $H_e E$ and $T_{f(e)}B$ for all $f(e) \in U$, thus we can take the vector field $V := f^*W$ on $E_U$. Because the fibers are compact, the flow $\theta$ of $V$ is defined in a neighborhood of $E_b$ for all $t$ in some interval $I$. Therefore $\varphi_t := \theta(-, t)$ is a diffeomorphism between $E_b$ and some other fiber in a neighborhood.
In order to show that \( \varphi_t \) preserves the symplectic form at the fibers is enough to show that \( \frac{d}{dt}\big|_{t=t^*} \varphi_t^* \omega_b = 0 \) for \( t \in I \), where \( \omega_b \) is the form \( \omega \) restricted to the fiber \( E_b \). This follows noting that

\[
\frac{d}{dt}\big|_{t=t^*} \varphi_t^* \omega_b = \varphi_t^* (L_V \omega_b) = \varphi_t^* (d_V \omega_b + \iota_V d \omega_b) = \varphi_t^* (d_V \omega_b),
\]

and that \( \iota_V \omega_b = 0 \) since \( V \) is in \( HE \).

Let \( \gamma : [0,1] \to B \setminus C \) be a simple path. We denote by \( P_\gamma : E_\gamma(0) \to E_\gamma(1) \) the symplectomorphism given by the lifting of \( \gamma \) as in the previous proposition.

**Corollary 2.2** Let \( Y \) be a projective manifold and \( f : Y \to \mathbb{P}^1 \) be a Lefschetz fibration with critical values \( C \). For a simple path \( \gamma : [0,1] \to \mathbb{P}^1 \setminus C \), the map \( P_\gamma \) is a symplectomorphism.

From now on, we will denote by \( X := Y_b \hookrightarrow Y \) a regular fiber of \( f \). Since these fibers are symplectomorphic we simply denote any symplectic fiber by \( (X, \omega_X) \). Thus, we have a map \( \pi_1(\mathbb{P}^1 \setminus C) \to \text{Symp}(X, \omega_X) \) which descends to homology, inducing the so-called monodromy action \( \pi_1(\mathbb{P}^1 \setminus C) \cong H_*(X, \mathbb{Z}) \) given by \( (\gamma, \delta) \mapsto (P_\gamma)_* \delta \).

Let \( \gamma : [0,1] \to \mathbb{P}^1 \) be a simple path such that \( \gamma(1) \in C \) and \( \gamma(t) \in \mathbb{P}^1 \setminus C \) for \( t \in [0,1) \). Let \( p \) be a critical point in \( f^{-1}(\gamma(1)) \). The set of points

\[
V_\gamma = \{ z \in f^{-1}(\text{Im}(\gamma)) \mid \lim_{t \to 1} P_\gamma(t)(z) = p \}
\]

is called **Lefschetz thimble** and the intersection of \( V_\gamma \) with the fiber \( f^{-1}(\gamma(0)) \) is called the **vanishing cycle** \( \delta_\gamma \).

**Proposition 2.3** The Lefschetz thimble \( V_\gamma \) is a Lagrangian submanifold of \( (Y, \omega) \) and the vanishing cycle \( \delta_\gamma \) is a Lagrangian sphere of \( (X, \omega_X) \).

**Proof** In a compact neighborhood \( U \) of \( p \) we can suppose that \( f(z) = f(p) + z_1^2 + \cdots + z_n^2 \) and that \( \gamma \) is a real curve in \( \mathbb{C} \) with \( \gamma(1) = 0 \), and \( \gamma(0) > 0 \). Let \( H : U \to \mathbb{R} \) be the map given by \( H(z) = \text{Re}(f(z)) \). The Hamiltonian vector field \( X_H \) is horizontal because \( H \) is constant in the fibers of \( f \) and \( \omega(X_H, V) = dH(V) = 0 \) for any vertical vector field \( V \). Since \( J V \) is also vertical then \( \nabla H \) is horizontal. On the other hand \( -\nabla H \) projects to \( \frac{\partial}{\partial z} \) and so \( V_\gamma \) is the unstable set of \( p \).

By a direct computation \( H \) is a Morse function with index \( n \). Using the unstable manifold theorem [3, Thm. 4.2] we conclude that \( V_\gamma \) is a \( n \)-ball inside \( Y \). To see that \( V_\gamma \) is isotropic, consider \( u, v \in T_z V_\gamma \) for any \( z \in V_\gamma \). Since the horizontal component of \( V_\gamma \) is one-dimensional, we have \( \omega_H(u, v) = \omega_X(z)(u_v, v_v) \), where \( u_v \) and \( v_v \) are the vertical components of \( u \) and \( v \). As the fibers over \( \gamma(t) \) with \( t \in [0,1) \) are symplectomorphic via \( \varphi_t \), we have that

\[
\omega_X(z)(u_v, v_v) = \omega_X(z(t))(u_v(t), v_v(t))
\]

where \( z(t) = \varphi_t(z), u_v(t) = (\varphi_t(z))_v u_v \) and \( v_v(t) = (\varphi_t(z))_v v_v \). In the limit the tangent space is a point, then by continuity we can conclude the result.
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Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ and let $b \in \mathbb{C}$ be some regular value of $f$. Suppose that the origin is an isolated critical point of the highest-grade homogeneous piece of $f$. The $(n-1)$-homology group of the fiber over $b$ is generated by the vanishing cycles (see [10], [14, §7.4]). As a consequence we can prove the next proposition.

**Proposition 2.4** Let $F \in \mathbb{C}[z_0, \ldots, z_n]$ be a homogeneous polynomial with $n$ even. Suppose that $F$ defines a smooth variety $X$ in $\mathbb{P}^n$. Then, any homology class $\delta \in H_{n-1}(X, \mathbb{Z})$ can be written as a finite sum $\delta = \sum_j \alpha_j \delta_j$, where $\alpha_j \in \mathbb{Z}$ and $\delta_j$ is supported in a Lagrangian $(n-1)$-sphere.

**Proof** Consider a hyperplane that intersects transversally $X$, and let $Z$ be its intersection. We can suppose that the hyperplane section is $Z = X \cap \{z_0 = 0\}$. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be the polynomial $F(1, z_1, \ldots, z_n)$ and we define the affine variety $U := X \setminus Z = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | f(z_1, \ldots, z_n) = 0\}$. The pair $(X, U)$ induces the exact sequence in homology
\[
\cdots \to H_n(X, U) \to H_{n-1}(U) \to H_{n-1}(X) \to H_{n-1}(X, U) \to \cdots,
\]
where the map $H_k(X) \to H_k(U)$ comes from the inclusion $U \subset X$. By Leray-Thom-Gysin isomorphism we have $H_k(X, U) \cong H_{k-2}(Z)$. By Lefschetz hyperplane section theorem we know that $H_k(Z) = H_k(\mathbb{P}^{n-2})$ if $k \neq n - 2$ (see [14, §5.4]). Since $n$ is even we have that $H_{n-3}(Z) = 0$, then the map
\[
H_{n-1}(U) \to H_{n-1}(X) \to H_{n-3}(Z) = 0
\]
is surjective. The vanishing cycles associated to the fibration $f : \mathbb{C}^n \to \mathbb{C}$ generate the homology group $H_{n-1}(U)$, and they are supported in Lagrangian spheres of $U$. ■

3 Monodromy action on mirror quintic threefolds

In this section we recall the definition of a mirror quintic Calabi-Yau threefold and its monodromy action coming from the Picard-Fuchs equations. We also list the monodromy action of other 14 examples of Calabi-Yau threefolds. For a more detailed description, the reader is referred to [5, 8, 15, 16].

The family of hypersurfaces in $\mathbb{P}^4$ given by a generic polynomial of degree 5 is denoted $\mathbb{P}^4[5]$. The elements of $\mathbb{P}^4[5]$ are quintic Calabi-Yau threefolds, with Hodge numbers $h^{1,1} = 1$ and $h^{2,1} = 101$. Let $\{X_\varphi\}_{\varphi}$ be the one-parameter family of hypersurfaces in $\mathbb{P}^4$ given by
\[
\varphi = \varphi z_0^5 + z_3^5 + z_4^5 = 5z_0z_1z_2z_3z_4, \quad \varphi \neq 0, 1. \quad (3.1)
\]
Consider the finite group
\[
G = \{(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{C}^5 | \xi_i^5 = 1, \xi_0\xi_1\xi_2\xi_3\xi_4 = 1\}
\]
acting on $\mathbb{P}^4$, as $(\xi_0, \xi_1, \ldots, \xi_4) \cdot [z_0 : z_1 : \cdots : z_4] = [\xi_0z_0 : \cdots : \xi_4z_4]$. It is known that the action of $G$ is free away from the curves $C_{ijk} := \{z_i^5 + z_j^5 + z_k^5 = 0, z_l = 0 \text{ for all } l \neq i, j, k \}$ for $0 \leq i < j < k \leq 4$ (see [13]). The mirror quintic Calabi-Yau threefold, mirror
quintic for short, is the variety $\tilde{X}_\varphi$, obtained after resolving the orbifolds singularities of the quotient $X_\varphi / G$. The manifold $\tilde{X}_\varphi$, has Hodge numbers $h^{1,1} = 101$ and $h^{2,1} = 1$ and Betti number $b_3 = 4$. In terms of the mirror symmetry $\mathbb{P}^4[5]$ is called the $A$-model and $\{\tilde{X}_\varphi\}$ the $B$-model (see for example [9]).

The variety $\tilde{X}_\varphi$ has a holomorphic 3 form $\eta$ that vanishes nowhere. Moreover, $H^{3,0}$ is spanned by $\eta$. The periods of $\eta$ are functions $\int_\Delta \eta$, where the homology class $\delta = [\Delta] \in H_3(\tilde{X}_\varphi, \mathbb{Z})$ is supported in the submanifold $\Delta$. The fourth-order linear differential equation

$$\left(\theta^4 - \varphi \left(\theta + \frac{1}{5}\right)\left(\theta + \frac{2}{5}\right)\left(\theta + \frac{3}{5}\right)\left(\theta + \frac{4}{5}\right)\right) y = 0, \quad \theta = \varphi \partial / \partial \varphi$$

is called Picard-Fuchs equations, and its solutions are the periods of $\eta$.

The Picard-Fuchs ODE has 3 regular singular points $\varphi = 0, 1, \infty$. The analytic continuation of this ODE, gives us the monodromy operators $M_0, M_1, M_\infty$. Since the monodromy is a representation $\rho: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \rightarrow Sp(4)$, we have the relation $M_0 M_1 M_\infty = \text{Id}$. There exits a basis such that the monodromy operators in this basis are written as

$$M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

see for example [4, 6, 8].

The matrices $M_0$ and $M_1$ are conjugated to the matrices

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

appearing in [6, 8], via the matrix $P = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 5 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Thus $P^{-1} T_i P = M_i, i = 0, 1$. In [5],

the matrices for the monodromy are

$$S_\infty = \begin{pmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

these matrices are associated to the equation

$$p_\psi = \zeta_0^5 + \zeta_1^5 + \zeta_2^5 + \zeta_3^5 + \zeta_4^5 - 5 \psi \zeta_0 \zeta_1 \zeta_2 \zeta_3 \zeta_4,$$

with singularities at $\psi^5 = 1, \infty$. The change of variable $\psi = \varphi^{1/5}$ gives us the family defined by the equation (3.1). Moreover, the matrix $M_0^5$ is conjugated to $S_\infty$. In fact with
the matrix

\[ M = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (3.2)

we obtain the equations \( M^{-1}S_1M = M_1 \) and \( M^{-1}S_\infty M = M_0^5 \).

More generally, it is known that the differential equation

\[(\theta^4 - \varphi(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B))y = 0, \quad \theta = \varphi \frac{\partial}{\partial \varphi} \] (3.3)

corresponds to the Picard-Fuchs equation of a mirror Calabi-Yau threefold for 14 values of \((A, B)\), and the singularities are in \(\varphi = 0, 1, \infty\). We have listed the A-model of these 14 examples in Table 1.

| \((d, k)\) | \(A\) | \(B\) | A-model of equation (3.3) |
|-----------|------|------|---------------------------|
| (5, 5)    | 1/5  | 2/5  | \(X(5) \subset \mathbb{P}^4\) |
| (2, 4)    | 1/8  | 3/8  | \(X(8) \subset \mathbb{P}^4(1, 1, 1, 1, 4)\) |
| (1, 4)    | 1/12 | 5/12 | \(X(2, 12) \subset \mathbb{P}^3(1, 1, 1, 1, 4, 6)\) |
| (16, 8)   | 1/2  | 1/2  | \(X(2, 2, 2, 2) \subset \mathbb{P}^7\) |
| (12, 7)   | 1/3  | 1/2  | \(X(2, 2, 3) \subset \mathbb{P}^{10}\) |
| (8, 6)    | 1/4  | 1/2  | \(X(2, 4) \subset \mathbb{P}^5\) |
| (4, 5)    | 1/6  | 1/2  | \(X(2, 6) \subset \mathbb{P}^5(1, 1, 1, 1, 3)\) |
| (2, 3)    | 1/4  | 1/3  | \(X(4, 6) \subset \mathbb{P}^5(1, 1, 1, 2, 2, 3)\) |
| (1, 2)    | 1/6  | 1/6  | \(X(6, 6) \subset \mathbb{P}^5(1, 1, 2, 2, 3, 3)\) |
| (6, 5)    | 1/6  | 1/4  | \(X(3, 4) \subset \mathbb{P}^5(1, 1, 1, 1, 1, 2)\) |
| (3, 4)    | 1/6  | 1/3  | \(X(6) \subset \mathbb{P}^5(1, 1, 1, 1, 2)\) |
| (1, 3)    | 1/10 | 3/10 | \(X(5) \subset \mathbb{P}^5(1, 1, 1, 2, 5)\) |
| (4, 4)    | 1/4  | 1/4  | \(X(4, 4) \subset \mathbb{P}^5(1, 1, 1, 2, 2)\) |
| (9, 6)    | 1/3  | 1/3  | \(X(3, 3) \subset \mathbb{P}^5\) |

Table 1: Fourteen values for equation 3.3 with the corresponding Calabi-Yau threefold.

The notation \(X\left(d_1, d_2, \ldots, d_l\right) \subset \mathbb{P}^n(w_1, w_2, \ldots, w_n)\) denotes a complete intersection of \(l\) hypersurfaces of degrees \(d_1, d_2, \ldots, d_l\) in the weighted projective space with weight \((w_1, w_2, \ldots, w_n)\), see for example [6]. For these cases the monodromy matrices correspond to the same \(M_1\) as before and

\[ M_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}. \]
4 Lagrangian sphere and Lagrangian torus in mirror quintic threefold

In the basis of homology used in [5] there are two homology classes which are supported on Lagrangian submanifolds. We observe that one class is realized by a Lagrangian 3-sphere and the other by a Lagrangian 3-torus.

Consider the mirror quintic Calabi-Yau threefold \( \bar{X}_\psi \) associated to the equation

\[
p_\psi = z_0^4 + z_1^4 + z_2^4 + z_3^4 + z_4^4 - 5\psi z_0z_1z_2z_3z_4,
\]

with singularities in \( \psi = 1, \infty \). Let \( \eta \) be the holomorphic form on \( \bar{X}_\psi \). The basis of the matrices \( \Sigma_\infty \) and \( \Sigma_0 \) are the periods \( \int_{\Delta_k} \eta \) with \( k = 1, 2, 3, 4 \). The cycle \( \Delta_2 \) is a torus associated with the degeneration of the manifold as \( \psi \) goes to \( \infty \), see [5, §3]. In coordinates it can be described as

\[
\Delta_2 = \{ [1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid |z_1| = |z_2| = |z_3| = r \text{ and } z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to \infty \} \quad (4.1)
\]

for \( r > 0 \) small enough, and \( z_4 \) is defined as the branch of the solution \( p_\psi(z) = 0 \) which tends to zero as \( \psi \to \infty \). The cycle \( \Delta_2 \) does not intersect the curves \( C_{ijk} \), and so its quotient by the group \( G \) is again a torus.

**Proposition 4.1** The cycle \( \Delta_2 \) is a Lagrangian submanifold of \( (X_\psi, \omega) \), where \( \omega \) is the symplectic form given by the pullback of the Fubini-Study form.

**Proof** Consider the Hamiltonian \( S^1 \)-space \((\mathbb{C}^5 \setminus \{0\}, \omega_{can}, S^1, \mu)\), where \( \mu(z) = \frac{-|z|^2 + 1}{2} \). By Marsden-Weinstein-Meyer theorem, there exist a symplectic form in the reduction \( \mu^{-1}(0)/S^1 = \mathbb{P}^4 \), and in this case it corresponds to the Fubini-Study form \( \omega_{FS} \), see for example [12, §5] or [7, §23]. Furthermore if we denote the reduction by

\[
\mu^{-1}(0) = S^9 \xrightarrow{\pi} \mathbb{C}^5 \setminus \{0\} \xleftarrow{pr} \mathbb{P}^4
\]

the reduced form satisfies \( \pi^* \omega_{FS} = t^* \omega_{can} \). The canonical form can be written as \( \omega_{can} = \frac{1}{2} \sum_j d|z_j|^2 \wedge d\theta_j \). Therefore, for \( \epsilon > 0 \) small enough, the set

\[
T := \{ (z_0, z_1, z_2, z_3, z_4) \in \mathbb{C}^5 \mid |z_0| = \epsilon, |z_1| = |z_2| = |z_3| = r, |z_4|^2 = 1 - \epsilon^2 - 3r^2 \} \subset S^9,
\]

is a Lagrangian submanifold of \((\mathbb{C}^5, \omega_{can})\). Besides, \( \Delta_2 \) is the intersection of \( X_\psi \) with the projection of \( T \) to \( \mathbb{P}^4 \). Consequently, the tangent space of \( \Delta_2 \) is contained in the tangent space of \( \pi(T) \). Since \( 0 = \pi^*(\omega_{FS})|_T = (\omega_{FS})|_{\pi(T)} \), we conclude that \( (\omega_{FS})|_{\Delta_2} = 0 \). \( \square \)

The cycle \( \Delta_4 \) is associated with the degeneration of the manifold when \( \psi \) goes to 1 [5, §3]. In coordinates can be described as

\[
\Delta_4 = \{ [1 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 \mid z_1, z_2, z_3 \text{ reals and } z_4 \text{ given by } p_\psi = 0 \text{ when } \psi \to 1 \} \quad (4.2)
\]
where $z_4$ is defined as the branch of of the solution of $p_\psi(z) = 0$ which is an $S^3$ when $\psi \to 1$. Follows from the next proposition that $\Delta_4$ is an Lagrangian sphere $S^3$.

**Proposition 4.2** The cycle $\Delta_4$ is a vanishing cycle.

**Proof** In the chart $z_0 = 1$ consider the function $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f(z_1, \ldots, z_4) = p_\psi(1, z_1, \ldots, z_4)$. The critical points of $f$, $(\xi^{k_1} \psi, \xi^{k_2} \psi, \xi^{k_3} \psi, \xi^{k_4} \psi)$ where $\xi = e^{\frac{2\pi i}{5}}$, $k_1 = 1, \ldots, 5$ and $5\sum_{j=1}^4 k_j$ are non degenerated. After doing the quotient by the finite group $G$, these critical points are identified with $(\psi, \ldots , \psi)$.

For real $\psi > 1$ close enough to 1 and by taking $z_j = x_j + i y_j$ we have that the map can be locally defined as

$$f(z_1, \ldots, z_4) = (1 - \psi^5) + \sum x_j^2 - \sum y_j^2 + 2i \prod x_j y_j,$$

and so the vanishing cycle $\delta_y$ in Proposition 2.3 is the sphere $(\psi^5 - 1) = \sum x_j^2$.

Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the basis on which the matrices $S_1$ and $S_\infty$ are written. Consider the isomorphism between $\text{span}\{\delta_i\}^4_{i=1}$ and $\mathbb{R}^4$ with the canonical basis, given by $\sum_{i=1}^4 n_i \delta_i \to (n_1, n_2, n_3, n_4)$. Thus, the monodromy acting on a vector $\delta = \sum_{i=1}^4 n_i \delta_i$ corresponds to

$$S_j(\delta) = (n_1 \ n_2 \ n_3 \ n_4) S_j \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} \quad \text{with } j = 1, \infty.$$

From [5] and Picard-Lefschetz formula, we know that the monodromy matrices satisfy $S_\infty \Delta_2 = \Delta_2$ and $S_1 \Delta_2 = \Delta_2 + \Delta_4$. Therefore $\Delta_2 = \delta_2 \equiv [0 \ 1 \ 0 \ 0]$ and $\Delta_4 = \delta_4 \equiv [0 \ 0 \ 0 \ 1]$.

**5 Orbits for $\delta_2$ and $\delta_4$**

Let $H$ be the subgroup of $Sp(4)$, generated by $M_0$ and $M_1$. Moreover, the vectors $\delta_2 = (0 \ 1 \ 0 \ 0)$ and $\delta_4 = (0 \ 0 \ 0 \ 1)$ are invariants by the change of basis $M$ defined in (3.2). In this section we compute the orbit of $\delta_2$ and $\delta_4$ by the action of $H$ in $\mathbb{Z}_p$, for some prime numbers $p$.

For the mirror quintic $\hat{X}$, any element in $H_2(\hat{X}, \mathbb{Z})$ which is in the orbit $H \cdot \delta_4$ is a homology class supported in a Lagrangian 3-sphere, and any element in the orbit $H \cdot \delta_2$ a homology class supported in a Lagrangian 3-torus. So far we have not computed the orbits in $\mathbb{Z}$. However, considering $H_2(\hat{X}, \mathbb{Z}_p)$ for some primes $p$, it is possible to compute the orbits. The next lemma helps us to reduce the possible words appearing in $H$ mod $p\mathbb{Z}$.

**Lemma 5.1**

$$\mod_p (M_0^p) = Id_4, \ p \neq 2, 3, \ mod_2(M_0^3) = Id_4, \ mod_3(M_0^3) = Id_4,$$

$$\mod_p (M_1^p) = Id_4 \quad \forall \text{ prime } p.$$
Proof  Computing the power of theses matrix, we have

\[
M_0^m = \begin{pmatrix}
1 & m & 0 & 0 \\
0 & 1 & 0 & 0 \\
dm & am & 1 & 0 \\
bm & cm & -m & 1 \\
\end{pmatrix} \quad \text{and} \quad M_1^m = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & m \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

where \(a_m = \frac{d}{2}m(m + 1), b_m = \frac{d}{2}m(1 - m)\) and \(c_m = \frac{d}{6}m(1 - m^2) - km\). Thus, it is enough to show that \(p|\alpha_p, \beta_p|, \gamma_p, p|c_p\). However, this is immediate because \(2|(p + 1), 2|(1 - p)\) and \(6|(1 - p^2)\), for \(p \neq 2, 3\) prime.

Let \(v\) be the vector \(\delta_2\) (or \(\delta_4\)) and we denote by \(\text{orb}_p\) the list of the vector in the orbit of \(v\) modulo \(p\). Firstly, we start with \(\text{orb}_p = \{\text{mod}_p(v)\}\) and we compute the vectors \(\text{mod}_p(wM_j^iM_0^j)\) for \(j = 0, \ldots, p, i = 0, \ldots p\) and \(w \in \text{orb}_p\). If these vectors are not in \(\text{orb}_p\), we add them to \(\text{orb}_p\). This step is repeated with the new \(\text{orb}_p\), if there are not new vectors then the orbit is complete. This process is finite because we have at most \(p^4\) different vectors in \((\mathbb{Z}/p\mathbb{Z})^4\). We summarize the algorithm \(^1\) to compute the orbit of \(v\) modulo \(p\) as follows.

```
Input: v, M_0, M_1, p  
Output: Orb_p  
Orb_p = mod_p(v)  
norm = 1  
while norm > 0 do  
  W = Orb_p, L = length(Orb_p)  
  l = i, c = 1  
  while l ≤ L do  
    j = 0  
    while j ≤ p do  
      if i ≤ p do  
        vaux = mod_p(W(l)∗(M_j^iM_0^j))  
        if vaux ∉ Orb_p then  
          Orb_p(L + c) = vaux  
          c = c + 1, i = i + 1  
        else  
          i = i + 1  
        end  
      else  
        j = j + 1  
      end  
    end  
    l = l + 1  
  end  
  norm = length(Orb_p) − length(W)  
end
```

Proof  Consider the free group \(H\) when \(d = k = 5\). Given \(p\), we denote the orbit of \(\delta_2\) and \(\delta_4\) modulo \(p\) as \(\text{orb}_p(\delta_2)\) and \(\text{orb}_p(\delta_4)\), respectively. By using the previous algorithm we have,

\[
\text{orb}_2(\delta_2) = \{(0 0 1 1), (0 1 0 0), (0 1 0 1), (1 0 0 1), (1 0 1 1)\}.
\]

\[
\text{orb}_2(\delta_4) = \{(0 0 0 1), (0 0 1 0), (0 1 1 0), (0 1 1 1), (1 0 0 0), (1 0 1 0), (1 1 0 0), (1 1 0 1), (1 1 1 0), (1 1 1 1)\}.
\]

\(^1\)We have written a MATLAB code for the computation. It is available in https://github.com/danfelmath/mirrorquintic.git
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\[ \text{orb}_2(\delta_2) = \{(0 \ 1 \ 0 \ 0), (0 \ 1 \ 0 \ 1), (0 \ 1 \ 0 \ 2), (0 \ 1 \ 0 \ 3), (0 \ 1 \ 0 \ 4), \]
\[ (0 \ 1 \ 1 \ 0), (0 \ 1 \ 1 \ 1), (0 \ 1 \ 1 \ 2), (0 \ 1 \ 1 \ 3), (0 \ 1 \ 1 \ 4), \]
\[ (0 \ 1 \ 2 \ 0), (0 \ 1 \ 2 \ 1), (0 \ 1 \ 2 \ 2), (0 \ 1 \ 2 \ 3), (0 \ 1 \ 2 \ 4), \]
\[ (0 \ 1 \ 3 \ 0), (0 \ 1 \ 3 \ 1), (0 \ 1 \ 3 \ 2), (0 \ 1 \ 3 \ 3), (0 \ 1 \ 3 \ 4), \]
\[ (0 \ 1 \ 4 \ 0), (0 \ 1 \ 4 \ 1), (0 \ 1 \ 4 \ 2), (0 \ 1 \ 4 \ 3), (0 \ 1 \ 4 \ 4) \}. \]

\[ \text{orb}_4(\delta_4) = \{(0 \ 0 \ 0 \ 1), (0 \ 0 \ 1 \ 1), (0 \ 0 \ 2 \ 1), (0 \ 0 \ 3 \ 1), (0 \ 0 \ 4 \ 1) \}. \]

\[ \text{orb}_p(\delta_2) = \text{orb}_p(\delta_4) = (\mathbb{Z}/p\mathbb{Z})^4 \setminus (0 \ 0 \ 0 \ 0), \text{ for } p = 3, 7, 11, 13, 17, 19, 23. \]

From the map \( H_3(\tilde{X}, \mathbb{Z}) \xrightarrow{\text{mod}_p} H_3(\tilde{X}, \mathbb{Z}_p) \) we have that if \( \delta \in H_3(\tilde{X}, \mathbb{Z}) \) is a primitive class and it is in the orbit of \( \delta_2 \) (or \( \delta_4 \)), then \( \text{mod}_p(\delta) \in \text{orb}_p(\delta_2) \) (or \( \text{mod}_p(\delta_4) \in \text{orb}_p(\delta_4) \)) for all \( p \). We think that the converse should be true; that is the Conjecture 1.2.

For the other examples of quintic threefolds appearing in Table 1, we have analogous results. However, in this case we do not know if the vectors \( \delta_2 = (0 \ 1 \ 0 \ 0) \) and \( \delta_4 = (0 \ 0 \ 0 \ 1) \) are really supported in a Lagrangian submanifold. In Table 2 we present the orbits of the vectors \( \delta_2 \) and \( \delta_4 \) modulo \( p \) for the fourteen cases of \((d, k)\). If the orbit is \((\mathbb{Z}/p\mathbb{Z})^4 \setminus (0 \ 0 \ 0 \ 0)\) we call it complete. The orbits for the vector \( \delta_2 \) are presented in Table 3 and the orbits for the vector \( \delta_4 \) are presented in Table 4.
| $(d,k)$ | Prime | Orbit |
|---------|-------|-------|
| (5,5)   | $p = 5$ | (0001), (0011), (0021), (0031), (0041), (0100), (0101), (0102), (0103), (0104), (0110), (0111), (0112), (0113), (0114), (0120), (0121), (0122), (0123), (0124), (0130), (0131), (0132), (0133), (0134), (0140), (0141), (0142), (0143), (0144) |
|         | $p = 2, 3, 7, 11, 13, 17, 19, 23$ | Complete |
| (2,4)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,4)   | $p = 2, 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (16,8)  | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (12,7)  | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3$ | (0001), (0002), (0021), (0022), (0100), (0101), (0102), (0110), (0111), (0112), (0211), (0212), (0220), (0221), (0222) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (8,6)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,5)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (2,3)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,2)   | $p = 2, 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (6,5)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3$ | (0001), (0002), (0011), (0012), (0100), (0101), (0102), (0120), (0121), (0122), (0210), (0211), (0212), (0220), (0221), (0222) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (3,4)   | $p = 3$ | (0001), (0002), (0021), (0022), (0100), (0101), (0102), (0110), (0111), (0112), (0210), (0211), (0212), (0220), (0221), (0222) |
|         | $p = 2, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,3)   | $p = 2, 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,4)   | $p = 2$ | (0001), (0011), (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (9,6)   | $p = 3$ | (0001), (0011), (0021), (0100), (0101), (0102), (0110), (0111), (0112), (0120), (0121), (0122) |
|         | $p = 2, 5, 7, 11, 13, 17, 19, 23$ | Complete |

Table 2: Orbit of vectors $\delta_2$ and $\delta_4$ by the monodromy action for the fourteen mirror Calabi-Yau threefolds.
Table 3: Orbit of vector $\delta_2$ by the monodromy action for the fourteen mirror Calabi-Yau threefolds.

| $(d,k)$ | Prime | Orbit |
|---------|--------|-------|
| (5,5)   | $p = 2$ | (0011), (0100), (0101), (1001), (1011) |
|         | $p = 2$ | (0100), (0101), (0102), (0103), (0104), (0110), (0111), (0112), (0113), (0114), (0120), (0121), (0122), (0123), (0124), (0130), (0131), (0132), (0133), (0134), (0140), (0141), (0142), (0143), (0144) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (2,4)   | $p = 2$ | (0100), (0101)(0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,4)   | $p = 2$ | (0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (16,8)  | $p = 2$ | (0010), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (12,7)  | $p = 2$ | (0011), (0100), (0101) |
|         | $p = 3$ | (0021), (0022), (0100), (0101), (0102), (0210), (0221), (0212) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (8,6)   | $p = 2$ | (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,5)   | $p = 2$ | (0011), (0100), (0101) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (2,5)   | $p = 2$ | (0011), (0100), (0101) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,2)   | $p = 2$ | (0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (6,5)   | $p = 2$ | (0011), (0100), (0101) |
|         | $p = 3$ | (0011), (0012), (0100), (0101), (0102), (0220), (0221), (0222) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (3,4)   | $p = 2$ | (0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111) |
|         | $p = 3$ | (0021), (0022), (0100), (0101), (0102), (0210), (0221), (0212) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,3)   | $p = 2$ | (0011), (0100), (0101), (1001), (1011) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,4)   | $p = 2$ | (0100), (0101), (0110), (0111) |
|         | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (9,6)   | $p = 2$ | (0010), (0100), (0101), (0110), (0111), (1001), (1010), (1110), (1111) |
|         | $p = 3$ | (0100), (0101), (0102), (0110), (0111), (0112), (0120), (0121), (0122) |
|         | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| $(d,k)$ | Prime | Orbit |
|--------|-------|-------|
| (5,5)  | $p = 2$ | $(0001), (0010), (0110), (0111), (1000), (1010), (1100), (1101), (1110), (1111)$ |
|        | $p = 5$ | $(0001), (0011), (0021), (0031), (0041)$ |
|        | Complete | |
| (2,4)  | $p = 2$ | $(0001), (0011)$ |
|        | $p = 3, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,4)  | $p = 2$ | $(0001), (0011), (1000), (1101), (1110), (1101)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (16,8) | $p = 2$ | $(0001), (0011)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (12,7) | $p = 2$ | $(0001), (0110), (0111)$ |
|        | $p = 3$ | $(0001), (0002), (0110), (0111), (0112), (0220), (0221), (0222)$ |
|        | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (8,6)  | $p = 2$ | $(0001), (0011)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,5)  | $p = 2$ | $(0001), (0110), (0111)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (2,3)  | $p = 2$ | $(0001), (0110), (0111)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,2)  | $p = 2$ | $(0001), (0011), (1000), (1011), (1100), (1101)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (6,5)  | $p = 2$ | $(0001), (0110), (0111)$ |
|        | $p = 3$ | $(0001), (0002), (0120), (0121), (0122), (0210), (0221), (0222)$ |
|        | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (3,4)  | $p = 2$ | $(0001), (0011), (1000), (1011), (1100), (1101)$ |
|        | $p = 3$ | $(0001), (0002), (0110), (0111), (0112), (0220), (0221), (0222)$ |
|        | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (1,3)  | $p = 2$ | $(0001), (0010), (0110), (0111), (1000), (1010), (1100), (1101), (1110), (1111)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (4,4)  | $p = 2$ | $(0001), (0011)$ |
|        | $p = 3, 5, 7, 11, 13, 17, 19, 23$ | Complete |
| (9,6)  | $p = 2$ | $(0001), (0011), (1000), (1011), (1100), (1101)$ |
|        | $p = 3$ | $(0001), (0011), (0021)$ |
|        | $p = 5, 7, 11, 13, 17, 19, 23$ | Complete |

Table 4: Orbit of vector $\delta_4$ by the monodromy action for the fourteen mirror Calabi-Yau threefolds.
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