Strongly coupled QFT dynamics via TQFT coupling

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Motivation
The idea of adiabatic continuity in a picture...

- Phase transition(s)
- QGP
- high $- T$
- low $- T$

\[ \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1} \times S^1_{\beta} \rightarrow \mathbb{R}^d \]

Thermal: Rapid crossover/phase transition at strong scale

**We wanted continuity** and a weak coupling calculable domain continuously connected to strong coupling

\[ \mathbb{R}^{d-1} \times S^1_L \]

MU2007
w/ Yaffe 2008
w/ Shifman 2008
Still active research field, many remarkable results.
Many interesting results on small $R_3 \times S^1$ over the last 14 years

1) Mechanism of mass gap generation in deformed YM, QCD(adj), and deformed QCD in any rep. ferm.

Saddles with fractional top. charge and action $1/N$, and topological charge zero and action $2/N$ (bions) play prominent roles in YM and QCD(adj) in all NP phenomena.

2) Absence of mass gap in chiral limit of QCD, derivation of chiral Lag.

3) Confinement in YM and QCD with fermions in rep R.

4) Mechanism of both discrete and continuous chiral symmetry breaking in QCD-like theories

5) Correct theta angle dependence, multi-branch structure, topological susceptibility

6) Understanding of semi-classical approach more deeply eventually lead to “Resurgence in QFT and QM program”, and this is still ongoing. (More than 15 topical conferences all around the world since 2014, currently a long term program at Cambridge, INI, today: Sergey Gukov, The role of resurgence in QFT and in string theory.)
But I am quite disturbed by the following:

If weak coupling EFT on the calculable regime adiabatically connected to strong coupling regime knows so much about the strong coupling domain:

1) Is there anything from the semi-classical weak coupling regime that survives in the strong coupling regime?

2) Could there be a way to study strongly coupled dynamics on $R_4$ or large $T_4$ directly?

Ambitious goals, but it is a voyage...let us see where it takes us.
TQFT construction in simple QM
QM of particle on a circle with N-minima: $T_N$ vs $(T_N/Z_N)_p$ models

\[ V(q) = -\cos(Nq), \quad q \sim q + 2\pi \]

\[ \mathbb{Z}_N : q \mapsto q + \frac{2\pi}{N} \]

\[ E_k(\theta) = -2Ke^{-S/N} \cos \frac{\theta + 2\pi k}{N} \]

In Born-Oppenheimer approximation, we can work with tight-binding Hamiltonian, and deal with only lowest $N$ state, which are split by NP effects.
Euclidean path integral

$$Z(\beta) = \text{tr}[e^{-\beta H}] = \int_{q(\beta) = q(0)} Dq \exp(-S[q]),$$

Partition function in Euclidean path integral: Sum over periodic paths, with integer topological charge.

$$q(\tau) : S^1_\beta \to S^1, \quad \pi_1(S^1) = \mathbb{Z} \quad \Rightarrow \quad W = \frac{1}{2\pi} \int dq \in \mathbb{Z}.$$
Reverse engineering instanton sum

\[
Z(\beta, \theta) = \sum_{k=0}^{N-1} e^{2\beta Ke^{-\frac{S}{N}} \cos \frac{\theta + 2\pi k}{N}} = \sum_{k=0}^{N-1} \sum_{n=0}^{\infty} \sum_{n-\bar{n} = 0 \text{ mod } N} \frac{1}{n!} 1 \left( \beta Ke^{-\frac{S}{N}+i\frac{\theta + 2\pi k}{N}} \right)^n \left( \beta Ke^{-\frac{S}{N}-i\frac{\theta + 2\pi k}{N}} \right)^{\bar{n}}
\]

\[
= N \sum_{W \in \mathbb{Z}} \sum_{n=0}^{\infty} \sum_{\bar{n}=0}^{\infty} \frac{1}{n!} \bar{n}! \left( \beta Ke^{-\frac{S}{N}+i\frac{\theta}{N}} \right)^n \left( \beta Ke^{-\frac{S}{N}-i\frac{\theta}{N}} \right)^{\bar{n}} \delta_{n-\bar{n}-WN,0}
\]

\[
n - \bar{n} - WN = 0, \quad \text{i.e., } n - \bar{n} = 0 \text{ mod } N
\]

More precisely:
\[
n_1 - \bar{n}_1 = n_2 - \bar{n}_2 = \ldots = n_N - \bar{n}_N = W
\]

Contributing terms in the sum:

\[
e^{-\frac{SI}{N} (n+\bar{n})} e^{i \frac{\theta}{N} (n-\bar{n})} = e^{-\left(W + \frac{2\bar{n}}{N}\right) SI} e^{iW \theta}
\]

Integer topological charge, fractional action!
Below, I will describe how to couple a TQFT to QM. This will describe an abstract formalism for something simple in QM. At the end of next few pages, you may even think why we did this at all.

What I will do is: In $T_N$ model with $Z_N$ symmetry, I will describe steps to turn on a classical background for $Z_N$ or gauge $Z_N$ completely.

The point is: The abstract formalism will carry over verbatim to Yang-Mills theory, QCD(adj), and with slight changes to QCD(F) (any flavor), as well as many other interesting QFTs. And will reveal insights which are otherwise not obvious to see.
$Z_N$ TQFT

$$Z_{\text{top},p} = \int \mathcal{D}A^{(1)} \mathcal{D}A^{(0)} \mathcal{D}F^{(0)} \ e^{i \int F^{(0)} \wedge (NA^{(1)} - dA^{(0)}) + ip \int A^{(1)}}$$

A sophisticated way of writing $\delta_{p,0} \mod N$.

$(A^{(1)}, A^{(0)})$ pair describe a $Z_N$ gauge field that can be turned on in quantum mechanical $T_N$ model to probe saddles, in particular, to probe the fractional instantons.

$$A^{(1)} \mapsto A^{(1)} + d\lambda^{(0)}, \quad A^{(0)} \mapsto A^{(0)} + N\lambda^{(0)}, \quad F^{(0)} \mapsto F^{(0)}$$

To couple a classical $Z_N$ background field to the $q$-field: $q \mapsto q - \lambda^{(0)}$,

Gauge Inv. combos: $Nq + A^{(0)}$, $dq + A^{(1)} = (\dot{q} + A_\tau) d\tau$

Kapustin, Seiberg, 2014,

A little bit abstract, I will make it explicit.
QM coupled to TQFT background

\[
Z[(A^{(1)}, A^{(0)}), p] = \int \mathcal{D}F^{(0)} \int_{q(\beta) = q(0)} \mathcal{D}q \; e^{i \int F^{(0)} \wedge N A^{(1)} - d A^{(0)} + i p \int A^{(1)}} \\
\times \exp \left( -\frac{1}{g} \int d\tau \left( \frac{1}{2} (\dot{q} + A_\tau)^2 - \cos(Nq + A^{(0)}) \right) + \frac{i \theta}{2\pi} \int (dq + A^{(1)}) \right)
\]

Simple question: What does it calculate?
Twisted BC = TQFT background

\[ Z_\ell = \text{tr}[e^{-\beta H} U_\ell] = \sum_{j=1}^{N} \langle j + \ell | e^{-\beta H} | j \rangle = \int_{y(\beta) = y(0) + \frac{2\pi}{N} \ell} D y \exp(-S[y]), \]

U: Translation operator, \( \ell = 0, 1, N - 1 \) fixed

One can trade TBC with \( Z_N \) background field. Use field redef.

\[ q(\tau) = y(\tau) - \frac{2\pi \ell}{N\beta} \tau, \quad \text{hence} \quad q(\beta) = q(0) \mod 2\pi. \]

\[ S[q, \ell] = \frac{1}{g} \int d\tau \left[ \frac{1}{2} \left( \frac{2\pi \ell}{N\beta} \right)^2 - \cos \left( Nq + \frac{2\pi \ell}{\beta} \tau \right) \right] + \frac{i\theta}{2\pi} \int \left( dq + \frac{2\pi \ell}{N\beta} \right) d\tau \]

which is nothing but \( Z_N \) TQFT coupled to QM.
The fractional instanton data (non-trivial topological charge) is transmuted to data about $Z_N$ background gauge field.

TBC: $y(\beta) = y(0) + \frac{2\pi \ell}{N}$  \quad q(\tau) = y(\tau) - \frac{2\pi \ell}{N \beta} \tau$,

PBC: $q(\beta) = q(0) + Z_N$ background gauge field
Gauging $Z_N$ and $(T_N/Z_N)_p$ model

Gauging $Z_N$ is equivalent to identifying adjacent sites.

It dilutes Hilbert space by a factor of $N$.

$N$ dimensional Hilbert space reduce to a 1 dimensional one.

$$Z_{(T_N/Z_N)_p} = \int \mathcal{D}A^{(1)} \mathcal{D}A^{(0)} Z[(A^{(1)}, A^{(0)}), p] \delta(NA^{(1)} - dA^{(0)})$$

$$\equiv \frac{1}{N} \sum_{\ell=0}^{N-1} e^{-i\frac{2\pi \ell p}{N}} Z_{\ell}$$

$$= e^{\xi \cos \frac{\theta + 2\pi p}{N}}$$

Compare with $Z_{T_N} = \sum_{k=0}^{N-1} e^{\xi \cos \frac{\theta + 2\pi k}{N}}$

Discrete theta angle $\theta_p = \text{level } p$ Chern-Simons = picking Bloch state with momentum $p$
Summary of QM

Bundle topologies for $T_N$ vs. $T_N/Z_N$

$$\sum_{W \in \mathbb{Z}} \int_{W} dq \quad \text{vs.} \quad \sum_{\ell \in \mathbb{Z}_N} \sum_{W \in \mathbb{Z}} \int_{W,\ell} dq$$

$$Z_{T_N} = \sum_{W \in \mathbb{Z}} e^{i\theta W} Z_W$$

$$Z_{T_N}(\ell) = \sum_{W \in \mathbb{Z}} e^{i\theta(W + \frac{\ell}{N})} Z_W(\ell)$$

$$Z_{(T_N/Z_N)_p} = \sum_{W \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}_N} e^{i\frac{2\pi}{N} p \ell} e^{i\theta(W + \frac{\ell}{N})} Z_W(\ell)$$

All of this stuff was to convince you that we did nothing that changes local dynamics even a tiny bit..... Perturbation theory, saddles, tunneling amplitudes, Borel plane, everything remains the same.

It will also our strategy to change nothing in QFT and in Yang-Mills theory.
Quick reminder of Dynamics of (Deformed) Yang-Mills on $\mathbb{R}^3 \times S^1$
Yang – Mills on $\mathbb{R}^3 \times S^1$ circumference $L$ or $\beta$

Phase transition

QGP

$\mathbb{R}^{d-1}$

$\mathbb{R}^{d-1} \times S^1_{\beta}$

$\mathbb{R}^d$

- $Z_N$ zero-form part of center symmetry, order parameter = Wilson line $\Omega$
- $L > L_c$: unbroken center symmetry
- $L < L_c$: broken center symmetry
Gauge holonomy potential

\[ V[\Omega] = -\frac{2}{\pi^2 \beta^4} \sum_{n=1}^{[N/2]} \frac{1}{n^4} |\text{tr} (\Omega^n)|^2 \]

Minimum at center-broken configuration. The value at min is the Stefan-Boltzmann law for gluons.

\[ F = -\frac{\pi^2}{45} T^4 (N^2 - 1) \]

At high-temperature YM theory, this is inevitable and there is no room for negotiation. This is also true in any QCD-like theory, and there is no hope here.

All calculations except susy ones between 1980-2007 always gave a minus sign in the potential.
Evading the stumbling block

In 2006, I realized that the analog of the effective potential calculation in a supersymmetric gauge theory with one adjoint fermion gave zero. At the heart of the cancelation was following identity:

\[-1 + 1 = 0\]

More precisely,

\[-1 \times (\text{stuff}) + 1 \times (\text{same stuff}) = 0\]

The crucial point: +1 appears due to the boundary conditions for adjoint fermions and not supersymmetry!

Immediately, we deduce:

\[-1 + N_f > 0 \quad \text{for } N_f > 1\]

Our simple calculation was the first positive sign in such a calculation. All earlier calculations were done for a specific (thermal) boundary condition.
QCD(adj) on $\mathbb{R}^3 \times S^1$

$N_f \geq 1$ massless adjoint rep. fermions

periodic boundary conditions $\Rightarrow$ stabilized center symmetry

$\tilde{Z}(L) = \text{tr}[e^{-LH}(-1)^F]$

$Z = Z_B + Z_F$

$Z = Z_B - Z_F$

Susy-theory: Supersymmetric Witten Index, useful.

Non-susy theory: Twisted partition function, probably more useful!

$V_{1\text{-loop}}[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} (-1 + N_f) \left| \text{tr} \Omega^n \right|^2$

$m_n^2 < 0$ instability, “all calculations except susy between 1980-2007”

$m_n^2 = 0$ Supersymmetric case, $N_f = 1$, marginal,

$m_n^2 > 0$ QCD(adj), $N_f > 1$, stability Kovtun, Unsal, Yaffe,07

This sign flip probably gave birth to one of the most promising windows to non-perturbative QCD.
The idea of adiabatic continuity

Thermal: Rapid crossover/phase transition at strong scale

Adiabatic continuity
Can we achieve center-stability in YM in small-L?

\[
S_{YM}^* = S_{YM} + \int_{R^3 \times S^1} P[\Omega(x)]
\]

\[
P[\Omega] = A \frac{2}{\pi^2 L^4} \sum_{n=1}^{[N/2]} \frac{1}{n^4} |\text{tr}(\Omega^n)|^2
\]

- Motivated by QCD(adj), Yaffe and I proposed a double-trace deformation that prevents center-breaking. (Yaffe, MU, 2008).

\* We can now do reliable semi-classics here, and it is continuously connected to YM on R4.

The double-trace deformation is something extremely interesting and has some very deep aspects especially in the context of large-N volume independence, but it is not my goal to discuss it in this talk.
Abelianization and abelian duality

\[ SU(N) \rightarrow U(1)^{N-1} \]

Similar to Polyakov model in 3d (1974) and Seiberg-Witten in 4d (1994), dynamics abelianize, but via a compact group valued field

Three types of holonomy

(a) center broken weak coupling
(b) center-stable weak coupling
(c) center-stable strong coupling

\[ L = \frac{1}{4} F_{\mu\nu}^2 \leftrightarrow \frac{1}{2} (\partial_\mu \sigma)^2 \]

Gapless to all orders in perturbation theory. How about NP-effects?
Topological configurations: Monopole-instantons

1-defects, Monopole-instantons: Associated with the $N$-nodes of the affine Dynkin diagram of $SU(N)$ algebra. The $N$th type corresponds to the affine root and is present only because the theory is locally 4d!

$$M_k \sim e^{-S_k} e^{-\alpha_k \cdot b + i\alpha_k \sigma + i\theta / N}, \quad k = 1, \ldots, N$$

$$S_k = \frac{8\pi^2}{g^2 N} = \frac{S_l}{N}$$

Action $1/N$ of the 4d instanton, keep this in mind!

Proliferation of monopole-instantons generates a non-perturbative mass gap for gauge fluctuations, similar to 3d Polyakov model (Polyakov, 74). It is first generalization thereof to local 4d theory!
Long-distance 3d dual theory

\[ S^{\text{dual}} = \int_{\mathbb{R}^3} \left[ \frac{1}{2L} \left( \frac{g}{2\pi} \right)^2 (\nabla \sigma)^2 - \zeta \sum_{i=1}^{N} \cos(\alpha_i \cdot \sigma) \right] . \]

**Abelian duality**

\[ F^{(j)}_{\mu\nu} = \frac{g^2}{2\pi L} \epsilon_{\mu\nu\rho} \partial_{\rho} \sigma^j \]

Maxwell term

Monopole Operator

Monopole due to compactness of Higgs scalar

Lee, Yi, Kraan, van Baal, 97, 98

Monopole charges $\Delta^0_{\text{aff}} \equiv \{ \alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N \}$. usual N-1 monopoles
Deformed YM, Euclidean vacuum

Dilute gas of monopole instantons and correlated bion events
What are we summing over in dYM on $R_3 \times S_1$?

Configurations that contribute to the partition function possess integer topological charge, but fractional action!

$$S = \frac{S_I}{N}(2m_1 + \ldots + 2m_N) + S_I|W| \in S_I\left(\frac{2}{N}|k| + |W|\right), \quad k, W \in \mathbb{Z}$$

$$Q = W \in \mathbb{Z}$$

The sum is still over integer topological charge just like the BPST instanton on $R_4$, but there is something intriguing going on about action. It does satisfy BPS bound, but exhibits a far more refined structure!
Two remarkable result from lattice simulations of deformed theory at small $S^1 \times R^3$: Topological susceptibility and mass gap

Topological susceptibility and mass gap in SU(4) dYM on small $S^1 \times R^3$ ($500\text{MeV}$) vs pure YM on the confined phase approximating $R^4$.

The deformation parameter: $h$.

Green curve is roughly the sharp drop associated with the deconfinement phase transition.

The simulation results strongly suggest us that we should carefully think about deformed YM. Clearly, it knows something deep about YM on $R^4$!
TQFT coupling in Yang-Mills
Coupling $\mathbb{Z}_N$ TQFT to YM-formally

To turn on a classical background gauge field for the $\mathbb{Z}_N^{[1]}$ 1-form symmetry, introduce pair of $U(1)$ 2-form and 1-form gauge fields $(B^{(2)}, B^{(1)})$ satisfying

$$NB^{(2)} = dB^{(1)}, \quad N \int B^{(2)} = \int dB^{(1)} = 2\pi \mathbb{Z}$$

Promote $SU(N)$ gauge field to a $U(N)$:

$$\tilde{a} = a + \frac{1}{N} B^{(1)}$$

1-form gauge trans. and coupling TQFT:

\[
B^{(2)} \mapsto B^{(2)} + d\Lambda^{(1)}, \quad B^{(1)} \mapsto B^{(1)} + N\Lambda^{(1)} \\
\tilde{a} \mapsto \tilde{a} + \Lambda^{(1)}, \quad \tilde{F} \mapsto \tilde{F} + d\Lambda^{(1)}
\]

\[
S[B^{(2)}, B^{(1)}, \tilde{a}] = \frac{1}{2g_{YM}^2} \int \text{tr}[(\tilde{F} - B^{(2)}) \wedge * (\tilde{F} - B^{(2)})] + i \frac{\theta_{YM}}{8\pi^2} \int \text{tr}[(\tilde{F} - B^{(2)}) \wedge (\tilde{F} - B^{(2)})]
\]

Kapustin, Seiberg, 2014, Komargodski et.al. 2017
Modified instanton equation: \((\tilde{F} - B^{(2)}) = \mp \star (\tilde{F} - B^{(2)})\)

Action: 
\[
S = \mp \frac{8\pi^2}{g^2} \frac{1}{8\pi^2} \int \text{tr}[(\tilde{F} - B^{(2)}) \wedge (\tilde{F} - B^{(2)})] = \frac{S_I}{N}
\]

because 
\[
\frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} \in \frac{1}{N} \mathbb{Z}
\]

In SU(N) theory coupled to \(Z_N\) background gauge field, the configurations which satisfy BPS bound have action \(S_I/N\), just like our monopole-instantons on \(R_3 \times S_1\). Is this an accident? Are they related?

I will make above very formal stuff first a bit more concrete (twisted BC a la ’t Hooft), and then, even more concrete, describe in Hamiltonian formalism, in my own terms.
’t Hooft (1981) found constant topological charge $1/N$ and action $1/N$ configurations for certain aspect-ratio of $T_4$. (He mentions that the reason for writing the article about a constant solution was the difficulty in finding them.) Historically, however, it was not easy to determine the time or space-time dependent non-trivial solutions.

Tony Gonzalez-Arroyo, Margarita Garcia Perez et.al. (1990s-) found by numerical lattice simulations on latticized $T_3 \times \mathbb{R}$ that time-dependent fractional instanton solutions with action $1/N$ exist in the presence of ’t Hooft flux. (These works did not receive the attention they deserve. To my mind, quite important works... Current TQFT literature seems to be unaware of them, but these are exactly lattice studies of TQFT coupled to YM. Also Twisted Eguchi-Kawai model is YM theory on a TQFT background!)

I would like to argue that monopole-instantons are non-trivial configurations in the PSU($N$) bundle! These are called ’t Hooft-Polyakov monopoles, but ’t Hooft did not realize their role in PSU($N$) bundle despite the fact that he searched for non-trivial configurations in PSU($N$) bundle. Understanding this requires some things that happened after 2008 papers I wrote with Yaffe and Shifman, and interesting work by Cherman and Poppitz 2016 (and easy to figure out only in retrospect)
Reminder: Hamiltonian interpretation of monopole-instanton in zero ’t Hooft flux background

Consider compactifying $\mathbb{R}^3 \times S^1_L$ to $T^2 \times \mathbb{R} \times S^1_L$

First, let me provide a Hamiltonian interpretation of ’t Hooft-Polyakov monopole instanton in the absence of ’t Hooft flux background. (e.g. Bank’s book, page 226). The story I will tell you later will be crucially different from this standard (but not sufficiently well-known) discussion.

A monopole-instanton in the case of Polyakov model always changes the energy of vacuum state at finite $\text{Area}(T^2)$. If $\Phi = \int_{T^2} B = \frac{2\pi}{g} \alpha_a n_a$ is magnetic flux, then the change in energy between the zero-magnetic flux state and $\Phi$ flux state is:

$$
\Delta E = \int_{T^2} \frac{1}{2} B^2 = \frac{1}{2} \left( \frac{2\pi}{g} \right)^2 \frac{n_a^2}{\text{Area}(T^2)} > 0
$$

$$
\lim_{\text{Area}(T^2) \to \infty} \Delta E = 0
$$

These states become degenerate with the zero magnetic flux state.
Turn on ’t Hooft flux background in 3-direction

Multiple ways to think about it:

1) $\mathbb{Z}_N$ TQFT background.

2) Non-dynamical center-vortex.

3) There can be decorations of center-vortex by dynamical monopoles associated with root lattice which do not change ’t Hooft flux, but change magnetic flux through T2. (See Greensite’s review, but there vortex is dynamical.)

4) One can think of 1-unit of ’t Hooft flux as if it is sourced by fundamental monopole who’s charge is in weight lattice. But center-vortex can exist on its own right without any source, and fundamental monopole does not exist in SU($N$) theory. If you wish to think in this dangerous description, the center-vortex can be viewed as if a snake eating its own tail. (Ouroboros, from ancient Egypt)
Classification of tunnelings in ’t Hooft flux background

But now, there is something more interesting. Consider the following magnetic flux configurations (all of which have the same ’t Hooft flux), which can be connected by monopoles in root lattice.

\[ \Phi = \int_{T^2} B = \frac{2\pi}{g} \nu_a, \quad a = 1, \ldots, N. \]

which are exactly degenerate.

\[ E_a = \frac{1}{2} \int_{T^2} B_a^2 = \frac{1}{2A} \left( \frac{2\pi}{g} \right)^2 \nu_a^2 = \frac{1}{2} \frac{2\pi}{g} \left( 1 - \frac{1}{N} \right), \quad a = 1, \ldots, N. \quad E_a - E_b = 0 \]

But the rest of all other magnetic flux configurations have higher energy at finite Area(T2) and become only degenerate in the infinite Area(T2) limit.

On finite T2 x R x S_L, there are two types of tunnelings.

1) Between states that becomes degenerate in Area(T2) tends to infinity limit. Eg. Polyakov, dYM both without TQFT background.

2) Between states that are already degenerate at finite Area(T2). This one is new, in the presence of TQFT background.
Born-Oppenheimer and $T_N$ model

In the small-T2 limit, and within Born-Oppenheimer approximation, YM with center-symmetric holonomy along $S_L$ reduces to quantum mechanics on $(N-1)$-simplex

$$Z_{\ell_{12}} = \text{tr}[e^{-\beta H_{\ell_{12}}}] = \int_{pbc} \mathcal{D}a \: e^{-S(a,B^{(2)}_{12})}$$

As far as I know, this is the most faithful representation of YM at the level of QM. Instanton and fractional instanton have the same action as in full QFT. There are $N$-induced classical minima due to classical $Z_N$ background! In fact, these are the QM origins of $N$-metastable vacua in YM theory!
Hamiltonian description of $\frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} = \frac{\ell_{12}\ell_{34}}{N}$

$\ell_{12}$ induces a classical potential with $N$ minima.
Sum of transition amplitudes between minima which are $\ell_{34}$ units apart.

$$Z_{\ell_{12}\ell_{34}} = \text{tr}[e^{-\beta H_{\ell_{12}}(U_c)\ell_{34}}]$$

$$= \sum_{a=1}^{N} \langle \nu_a | e^{-\beta H_{\ell_{12}}(U_c)\ell_{34}} | \nu_a \rangle$$

$$= \sum_{a=1}^{N} \langle \nu_a | e^{-\beta H_{\ell_{12}}} | \nu_{a+\ell_{34}} \rangle$$

$$= \int_{pbc} \mathcal{D}a \ e^{-S(a,B^{(2)}_{12},B^{(2)}_{34})}$$

$$= \int \Phi(\beta) = \Phi(0) + \frac{2\pi}{g} \alpha_{a,a+\ell_{34}}$$
Here comes the heart of the matter.

Why topological charge and action $1/N$? There seems to be 2 unrelated answers!

In center - symmetric background,

$$S_a = \frac{4\pi}{g^2} \left( \alpha_a \cdot \phi_\star \right) = \frac{8\pi^2}{g^2 N}, \quad Q = \frac{1}{2\pi} \left( \alpha_a \cdot \phi_\star \right) = \frac{1}{N}$$

Diag($U_3$) = $e^{i\phi_\star}$ = $(1, \omega, \omega^2, \ldots, \omega^{N-1})$,

In $\mathbb{Z}_N$ TQFT background:

$$S = \frac{8\pi^2}{g^2} \frac{1}{8\pi^2} \int B^{(2)} \wedge B^{(2)} = \frac{8\pi^2}{g^2 N}, \quad Q = \frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} = \frac{\ell_{12} \ell_{34}}{N} = \frac{1}{N}$$

Are they really unrelated?
TBC (conventionally): \( U_3(\beta = L_4) = e^{\frac{2\pi i}{N} \ell_3} U_3(0) \).

According to Cherman and Poppitz (2016), the gauge invariant rewriting of \( U(i)^N \) photon is

\[
F_{\mu\nu,k} = \frac{1}{N} \sum_{p=0}^{N-1} e^{-i \frac{2\pi kp}{N}} \text{tr}(U_3^p F_{\mu\nu})
\]

and they transform cyclically under zero-form center transformation, and so does dual photons and monopole operators.

Hence, the zero form center transformation changes the magnetic flux through \( T^2 \) by a magnetic charge, valued in root lattice. This is our dynamical monopole instanton.

\[
\Delta \int_{T^2} B = \frac{2\pi}{g} (\nu_a - \nu_{a+1})
= -\frac{2\pi}{g} \alpha_a
\]

The crucial point here \( U_3 \) being a center symmetric background! Because of that, center transformation ends up cyclically shifting magnetic flux!
Arbitrary large $T_3 \times \text{large } S_L$

May be, the results that we obtained in deformed YM on $R_3 \times S_1$ in 2007 were not some weak coupling, small circle artifacts. May be, they were trying to tell us something deeper about the theory on $R_4$ limit. We thought our construction was not powerful enough, it did not extend to the strong coupling regime and failed us.

Perhaps, it was other way around. Our theory was much smarter than us, and was trying to guide us towards truth. It was us failing it.

$$S_a = \frac{4\pi}{g^2} (\alpha_a \cdot \phi_* \phi) = \frac{8\pi^2}{g^2 N}$$  \hspace{1cm}  $$Q = \frac{1}{2\pi} (\alpha_a \cdot \phi_*) = \frac{1}{N}$$

**True on arbitrarily large $T_4$**  emulating $R_4$.

$$S = \frac{8\pi^2}{g^2} \frac{1}{8\pi^2} \int B^{(2)} \wedge B^{(2)} = \frac{8\pi^2}{g^2 N}$$  \hspace{1cm}  $$Q = \frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} = \frac{\ell_1 \ell_3}{N} \frac{\ell_4}{N} = \frac{1}{N}$$
We reached to one of our goals. NP expansion on $R_4$ is controlled by $S_I/N$, but not $S_I$. This is not only for YM, but all QCD-like theories and true regardless of representations of fermions.

**Bundle topologies for $SU(N)$ vs. $PSU(N)$**

$$
\sum_{W \in \mathbb{Z}} \int da \quad \text{vs.} \quad \sum_{w_2 \in H^2(M_4, \mathbb{Z}_N)} \sum_{W \in \mathbb{Z}} \int da
$$

$$
Z_{SU(N)} = \sum_{W \in \mathbb{Z}} e^{i\theta W} Z_W
$$

$$
Z_{SU(N)}(\ell, m) = \sum_{W \in \mathbb{Z}} e^{i\theta (W + (\ell \cdot m)/N)} Z_W(\ell, m)
$$

$$
Z_{PSU(N)} = \sum_{W \in \mathbb{Z}} e^{i \frac{2\pi}{N} \rho (\ell \cdot m)} e^{i\theta (W + (\ell \cdot m)/N)} Z_W(\ell, m)
$$

**Refined classification**

- $Q_{top} = W \in \mathbb{Z}$
- $S = |W| S_I$

**Standard classification**

- Theory $T$
- $Q_{top} = W \in \mathbb{Z}$
- $S = |W| S_I$?
Conclusions

- NP expansion parameters in CP(N-1), YM, QCD-like and chiral theories is $\exp[-S_1/N]$, not BPST instanton factor $\exp[-S_1]$.

- ’t Hooft TBC [coupling to $Z_N$ TQFT] is possible even in 1-flavor QCD with fundamental matter. (’t Hooft thought otherwise.)

- Part of our history and attachment to BPST instanton seems to be a red herring now. (with all the tremendous respect to each one of the BPST). It is important, but not for the dynamics.

- There are no large-$N$ vs. instanton puzzles.