Ising vectors and automorphism groups of commutant subalgebras related to root systems

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Abstract

In this article we study and obtain a classification of Ising vectors in vertex operator algebras associated to binary codes and \( \sqrt{2} \) times root lattices, where an Ising vector is a conformal vector with central charge 1/2 generating a simple Virasoro sub VOA. Then we apply our results to study certain commutant subalgebras related to root systems. We completely classify all Ising vectors in such commutant subalgebras and determine their full automorphism groups.

1 Introduction

Motivated by the problem of looking for maximal associative subalgebras of the monstrous Griess algebra [G1], a class of conformal vectors in the lattice vertex operator algebra \( V_{\sqrt{2}R} \) were studied and constructed in [DLMN], where \( R \) is a root lattice of type \( A, D \) or \( E \) of rank \( \ell \) and \( \sqrt{2}R \) denotes \( \sqrt{2} \) times an ordinary root lattice \( R \). We adopt the standard notation for lattice vertex operator algebras as in [FLM]. In [DLMN], Dong et al. constructed conformal vectors of \( V^+_{\sqrt{2}R} \) of the forms

\[
\begin{align*}
    s_R &= \frac{h}{h+2} \omega - \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \\
    \tilde{\omega}_R &= \frac{2}{h+2} \omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha},
\end{align*}
\]

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where \( \omega \) is the Virasoro element of \( V_{\sqrt{2}}^{+} \), \( h \) is the Coxeter number of \( R \) and \( \Phi(R) \) denotes the root system of \( R \). The central charges of \( s_{R} \) and \( \tilde{\omega}_{R} \) are respectively \( \ell h/(h+2) \) and \( 2\ell/(h+2) \). The Weyl group \( W(R) \) of the root system \( \Phi(R) \) induces a natural action on the lattice vertex operator algebra \( V_{\sqrt{2}}^{+} \) and its \( \mathbb{Z}_{2} \)-orbifold \( V_{\sqrt{2}}^{+} \). By the construction, both conformal vectors \( s_{R} \) and \( \tilde{\omega}_{R} \) are fixed by \( W(R) \) so that \( W(R) \) acts identically on the Virasoro vertex operator subalgebra \( \text{Vir}(\tilde{\omega}_{R}) \) generated by \( \tilde{\omega}_{R} \). Therefore, the commutant (or coset) subalgebra

\[
\text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) = \{ a \in V_{\sqrt{2}}^{+} | [Y(a, z), Y(u, z_{2})] = 0 \text{ for all } u \in \text{Vir}(\tilde{\omega}_{R}) \}
\]

affords an action of the Weyl group \( W(R) \).

In this article, we shall study the structure of the commutant vertex operator subalgebra \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) of \( V_{\sqrt{2}}^{+} \). As our main result, we shall determine all the conformal vectors of central charge 1/2 in \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) and show that the vertex operator algebra \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) is generated by its weight two subspace. We shall also determine the full automorphism group of \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) and show that it always contains a half of the Weyl group \( W(R) \).

Besides its own interest, the study of \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) also has a perspective on the Monster simple group. There is an attempt in [LYY1, LYY2] to elucidate McKay’s observation on the Monster simple group and the extended \( E_{8} \) diagram via vertex operator algebras. It is known (cf. [C, M1]) that a 2A-involution of the Monster is in one-to-one correspondence with a conformal vector with central charge 1/2 in the moonshine vertex operator algebra [FLM] via so-called Miyamoto involution. Taking notice of a fact that the conformal vector \( \tilde{\omega}_{E_{8}} \) of \( V_{\sqrt{2}}^{+} \) is of central charge 1/2, Lam et al. used \( V_{\sqrt{2}}^{+} \) to relate the \( E_{8} \) diagram with the Monster in [LYY1, LYY2]. To each node of the extended \( E_{8} \) diagram, in [LYY1] they constructed two conformal vectors of central charge 1/2 inside \( V_{\sqrt{2}}^{+} \) which shall induce two 2A-involutions of the Monster. Then they used commutant subalgebra \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(s_{R})) \) for a sublattice \( R \) of \( E_{8} \) to study the subalgebra generated by two such conformal vectors of \( V_{\sqrt{2}}^{+} \) in [LYY2]. Since \( s_{R} \) and \( \tilde{\omega}_{R} \) are mutually commutative conformal elements in \( V_{\sqrt{2}}^{+} \) and the sum \( s_{R} + \tilde{\omega}_{R} \) is the Virasoro vector of \( V_{\sqrt{2}}^{+} \), the subalgebra \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) is exactly equal to the commutant subalgebra of \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(s_{R})) \) in \( V_{\sqrt{2}}^{+} \) (cf. [FZ]). By this duality, we can relate the structure of \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) with the centralizer of two 2A-involutions of the Monster. In [GN], Glauberman and Norton suggested some interesting relations between Weyl groups and centralizers of two 2A-involutions of Monster. Since \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) naturally affords an action of the Weyl group \( W(R) \), the study of \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) may lead Glauberman-Norton’s observation to an appreciable settlement.

Let us denote \( \text{Com}_{V_{\sqrt{2}}^{+}}(\text{Vir}(\tilde{\omega}_{R})) \) by \( M_{R} \) for simplicity of notation. The structure of \( M_{R} \) is closely related to that of the root system \( \Phi(R) \) of \( R \), so the automorphism group
of $M_R$ has a similar structure to the Weyl group $W(R)$ of $R$. The Weyl group $W(R)$ is a 3-transposition group, and Miyamoto discovered in [M1] that the 3-transposition property of $W(R)$ acting on $V^+_{\sqrt{2}R}$ comes from the structure of conformal vectors with central charge 1/2. In [M1], Miyamoto introduce a way to define involutions of vertex operator algebras containing conformal vectors with central charge 1/2. He showed that in some cases these involutions generate a 3-transposition group, which is exactly the case for $V^+_{\sqrt{2}R}$. There are many results about the group generated by these involutions, see [KM, Ma, M1, La, LS, LYY1, LYY2, Y]. Recently, Matsuo classified all 3-transposition groups defined by conformal vectors with central charge 1/2 in [Ma]. According to his classification, the parameters of $M_R$ such as central charges or dimensions of weight two subspace coincide with those for 3-transposition groups in his list. In particular, if we take $R = E_6, E_7,$ or $E_8$, then the corresponding groups have nice symmetry. In fact, this is one of the main motivations of our work. We shall determine the automorphism group of $M_R$ by studying conformal vectors of central charge 1/2 inside $M_R$.

Let us explain our result more precisely. Since the conformal vector with central charge 1/2 plays a central role in our discussion, we will refer a conformal vector with central charge 1/2 to as an Ising vector if it generates a simple Virasoro vertex operator subalgebra. To study Ising vectors, we first study a class of vertex operator algebras, called code vertex operator algebras (cf. [M2]). In the case that a code vertex operator algebra has finitely many Ising vectors, we classify all the Ising vectors of a code vertex operator algebra. Moreover, we will present a formula to count the exact number of such conformal vectors in Corollary 3.9. Then for a (not necessarily indecomposable) root lattice $R$ we will classify all Ising vectors inside $V^+_{\sqrt{2}R}$ by giving an embedding of $V^+_{\sqrt{2}R}$ into a code vertex operator algebra. As a result, it is shown in Theorem 4.6 that for each Ising vector $e$ of $V^+_{\sqrt{2}R}$, there exists a sublattice $K$ of $\sqrt{2}R$ isometric to either $\sqrt{2}A_1$ or $\sqrt{2}E_6$ such that $e \in V^+_K \subset V^+_{\sqrt{2}R}$. We expect that this is true not only for a lattice vertex operator algebra associated to $\sqrt{2}R$ but also for any lattice without roots (cf. Remark 4.7). The classification of Ising vectors of $M_R$ immediately follows from that of $V^+_{\sqrt{2}R}$. We also show that $M_R$ is generated by its Ising vectors. This reduces the study of $\text{Aut}(M_R)$ to an analysis of the permutation group of the Ising vectors. Since we know the group generated by involutions associated to these Ising vectors by the results obtained in [KM, Ma], it is not difficult to determine the permutation group of the Ising vectors.

The organization of this paper is as follows. In Section 2 we prepare some basic notation and related facts about conformal vectors and commutant subalgebras. In Section 3 we study Ising vectors of code vertex operator algebras. We present a classification of Ising vectors of a code vertex operator algebras in Proposition 3.8. In Section 4 we study Ising vectors of lattice vertex operator algebras. We classify Ising vectors of $V^+_{\sqrt{2}R}$ and $M_R$ with $R$ an irreducible root lattice. In Section 5 we determine the shape of $\text{Aut}(M_R)$. We make use of the fact that $M_R$ is generated by its Ising vectors. In the case that
$R = E_6$ or $E_7$, the proof of this fact is rather technical and long. So we separately give it in Section 7. In Section 6 we study a relation between inductive structures of a 3-transposition group acting on a vertex operator algebra and its commutant subalgebras. As an example, we study a vertex operator algebra corresponding to an inductive structure $O_{10}^+(2)^{(2)} \cong O_5^-(2)$. Section 7 is devoted to the proof that $M_R$ with $R = E_6$ or $E_7$ is generated by its weight two subspace as a vertex operator algebra. We use some representation theory of the unitary Virasoro vertex operator algebras and $W$-algebras there. So we also review some facts about these vertex operator algebras.

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1.1 Notation

In this article, every vertex operator algebra (VOA for short) we considered is defined over $\mathbb{C}$ and supposed to have the grading $V = \oplus_{n \geq 0} V_n$ with $V_0 = \mathbb{C} \mathbb{1}$. For a VOA structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$ on $V$, the vector $\omega$ is called the Virasoro element or Virasoro vector of $V$. For simplicity, we often use $(V, \omega)$ to denote the structure $(V, Y(\cdot, z), \mathbb{1} ,\omega)$.

An element $u \in V$ is referred to as a conformal vector with central charge $c_u \in \mathbb{C}$ if $u \in V_2$ and it satisfies $u(1)u = 2u$ and $u(3)u = c_u \mathbb{1}$. It is well-known (cf. [M1, La]) that after setting $L^u(m) := u(m+1)$, $n \in \mathbb{Z}$, we obtain a representation of the Virasoro algebra on $V$:

$$\left[ L^u(m), L^u(n) \right] = (m - n)L^u(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_u.$$  

For $c, h \in \mathbb{C}$, we denote by $L(c, h)$ the irreducible highest weight module over the Virasoro algebra with central charge $c$ and highest weight $h$. It is well-known that $L(c, 0)$ has a simple VOA structure (cf. [FZ]).

For a positive definite even lattice $L$, we will denote the lattice VOA associated to $L$ by $V_L$ (cf. [FLM]). We adopt the standard notation for $V_L$ as in [FLM]. In particular, $V_L^+$ denotes the fixed point subalgebra of $V_L$ under the lift of $(-1)$-isometry on $L$.

Given an automorphism group $G$ of $V$, we denote by $V^G$ the $G$-fixed point subalgebra of $V$. The subalgebra $V^G$ is called the $G$-orbifold of $V$ in the literature.

We denote the ring $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$. Let $C \subset \mathbb{Z}_2^n$ be a linear code. For a codeword $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$, we define the support of $\alpha$ by $\text{supp}(\alpha) := \{ i \mid \alpha_i = 1 \}$. For a subset $A$ of $C$, we define $\text{supp}(A) := \cup_{\alpha \in A} \text{supp}(\alpha)$.
2 Commutant subalgebras and conformal vectors

We will present the notion of a commutant subalgebra and its description by a pair of mutually commutative conformal vectors.

2.1 Commutant subalgebras

Let \((V, \omega)\) be a VOA. A decomposition \(\omega = u^1 + \cdots + u^n\) is called orthogonal if \(u^i\) are conformal vectors and they are mutually commutative, i.e., \([Y(u^i, z_1), Y(u^j, z_2)] = 0\) if \(i \neq j\). The following lemma is well-known (cf. Theorem 5.1 of [FZ]).

**Lemma 2.1.** Assume that a VOA \((V, \omega)\) has a grading \(V = \bigoplus_{n \geq 0} V_n\) such that \(V_0 = C\mathbb{1}\) and \(V_1 = 0\). Then for any conformal vector \(u\) of \(V\), the decomposition \(\omega = u + (\omega - u)\) is orthogonal.

In this paper, a sub VOA of \(V\) is defined by means of a pair \((U, e)\) of a subalgebra \(U\) of \(V\) containing the vacuum element \(\mathbb{1}\) of \(V\) and a conformal vector \(e\) of \(V \cap U\) such that \(e\) is the Virasoro vector of \(U\) and the structure \((U, e)\) inherits the grading of \(V\), that is, \(U = \bigoplus_{n \geq 0} U_n\) with \(U_n = V_n \cap U\). In the case that the conformal vector of a sub VOA of \(V\) coincides with that of \(V\), we will refer such a sub VOA to as a full sub VOA. Every conformal vector \(u\) of \(V\) together with the vacuum \(\mathbb{1}\) generates a Virasoro sub VOA of \(V\) which we will denote by \(\text{Vir}(u)\).

Let \(S\) be a subset of \(V\). The following is easy to see.

**Lemma 2.2.** The subspace \(S^c := \{a \in V \mid a(\alpha)S = 0, \ i \geq 0\}\) forms a subalgebra and satisfies that \([Y(u, z_1), Y(v, z_2)] = 0\) for any \(u \in S, v \in S^c\).

By the lemma above, we define the commutant subalgebra of a subalgebra \(U\) of \(V\) by

\[
\text{Com}_V(U) := U^c := \{a \in V \mid a(\alpha)U = 0, \ i \geq 0\}.
\] (2.1)

If \(U\) has a conformal vector \(e\) such that \((U, e)\) forms a sub VOA of \(V\), then it is shown in Theorem 5.2 of [FZ] that \(\text{Com}_V(U) = \text{ker}_V e(0)\). Therefore, the commutant subalgebra is described in term of the conformal vector \(e\) of \(U\).

2.2 Conformal vectors associated to root systems

Let \(R\) be a root lattice with root system \(\Phi(R)\). Let \(\ell\) be the rank of \(R\) and \(h\) the Coxeter number of \(R\). We denote by \(\sqrt{2}R\) the lattice whose norm is twice of \(R\)'s. We consider the fixed point subalgebra \(V^+_{\sqrt{2}R}\) of the lattice VOA \(V_{\sqrt{2}R}\) under the lift of \((-1)\)-isometry on \(R\). It is clear that \(V^+_{\sqrt{2}R}\) has a grading \(V^+_{\sqrt{2}R} = \bigoplus_{n \geq 0} (V^+_{\sqrt{2}R})_n\) such that \((V^+_{\sqrt{2}R})_0 = C\mathbb{1}\) and \((V^+_{\sqrt{2}R})_1 = 0\). We can find conformal vectors of \(V^+_{\sqrt{2}R}\) defined as follows. Set

\[
s = s_R := \frac{h}{h + 2} \omega - \frac{1}{h + 2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2} \alpha} \in V^+_{\sqrt{2}R'}\] (2.2)
where $\omega$ is the Virasoro vector of $V^+_{\sqrt{2}R}$. Then it is shown in [DLMN] that $s$ defines a conformal vector with central charge $\ell h/(h+2)$. By Lemma 2.1,

$$\tilde{\omega} = \omega_R := \omega - s = \frac{2}{h+2} \omega + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \in V^+_{\sqrt{2}R}$$

(2.3)

is also a conformal vector with central charge $2\ell(h+2)$ and the decomposition $\omega = s + \tilde{\omega}$ is orthogonal. In this article, we will mainly consider the commutant subalgebra

$$M_R := \text{Com}_{V^+_{\sqrt{2}R}}(\text{Vir}(\tilde{\omega})) = \ker_{V^+_{\sqrt{2}R}} \tilde{\omega}(0).$$

(2.4)

It is clear from the expression in (2.3) that $\tilde{\omega}$ is invariant under the natural action of the Weyl group $W(R)$ associated to the root system $\Phi(R)$. Therefore, the commutant subalgebra $M_R$ naturally affords an action of the Weyl group $W(R)$.

### 3 Ising vectors and Ising frames

We will introduce the notion of an Ising vector by which one can define an involution of a vertex operator algebra. We will review basic facts about involutions associated to Ising vectors. Then we will study the classification of Ising vectors of a code vertex operator algebra. We will also study automorphisms of a code vertex operator algebra. A brief description of the automorphism group of a code vertex operator algebra will be presented.

#### 3.1 Miyamoto involution

We begin by the definition of an Ising vector.

**Definition 3.1.** A conformal vector $e$ of a VOA $V$ is called an *Ising vector* if the subalgebra $\text{Vir}(e)$ generated by $e$ is isomorphic to the simple Virasoro VOA $L(1/2,0)$ with central charge $1/2$. An orthogonal decomposition $\omega = e^1 + \cdots + e^n$ of the Virasoro vector $\omega$ is called an *Ising frame* if each $e^i$ is an Ising vector.

**Remark 3.2.** An Ising vector is often referred to as a rational conformal vector of central charge $1/2$ in the literature (cf. [M1, La]).

It is well-known that the Virasoro VOA $L(1/2,0)$ is rational and has three irreducible representations, $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$ (cf. [DMZ]).

Let $e$ be an Ising vector of a VOA $V$. Since $\text{Vir}(e)$ is rational, $V$ is a semisimple $\text{Vir}(e)$-module. For $h = 0, 1/2, 1/16$, denote by $V_e(h)$ the sum of all irreducible $\text{Vir}(e)$-submodules of $V$ isomorphic to $L(1/2, h)$. Then we have the isotypical decomposition:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16).$$

(3.1)
Define a linear automorphism $\tau_e$ on $V$ which acts on $V_e(0) \oplus V_e(1/2)$ by identity and on $V_e(1/16)$ by $-1$. Then it is shown in [M1] that $\tau_e \in \text{Aut}(V)$. On the $(\tau_e)$-fixed point subalgebra $V^{(\tau_e)} = V_e(0) \oplus V_e(1/2)$, define a linear automorphism $\sigma_e$ which acts on $V_e(0)$ by identity and on $V_e(1/2)$ by $-1$. Then it is also shown in [M1] that $\sigma_e \in \text{Aut}(V^{(\tau_e)})$. We will refer $\tau_e \in \text{Aut}(V)$ (resp. $\sigma_e \in \text{Aut}(V^{(\tau_e)})$) to as the Miyamoto involution of $\tau$-type (resp. $\sigma$-type). An Ising vector $e$ of $V$ is called of $\sigma$-type if $\tau_e$ defines identity on $V$, and we also refer an Ising frame $\omega = e^1 + \cdots + e^n$ of $V$ to as of $\sigma$-type if all $e^i$, $1 \leq i \leq n$, are of $\sigma$-type on $V$.

### 3.2 Code VOA

Let us review the construction of code VOAs in [M2] at least for what we need in this paper.

Let $\mathcal{A}$ be the algebra generated by $\{\psi_r \mid r \in \mathbb{Z} + 1/2\}$ subject to the defining relation $\psi_r \psi_s + \psi_s \psi_r = \delta_{r+s,0}$, $r, s \in \mathbb{Z} + 1/2$. Let $\mathcal{A}^+$ be the subalgebra of $\mathcal{A}$ generated by $\{\psi_r \mid r > 0\}$ and let $\mathbb{C}1$ be a trivial $\mathcal{A}$-module. Then set $X := \text{Ind}_{\mathcal{A}^+}^{\mathcal{A}} \mathbb{C}1$. Consider the generating function

$$
\psi(z) := \sum_{n \in \mathbb{Z}} \psi_{n+1/2} z^{-n-1}.
$$

(3.2)

It is well-known that the space $X$, with the standard $\mathbb{Z}_2$-grading, has a unique structure of a simple vertex operator superalgebra (SVOA for short) with the vacuum element $1$ such that $Y(\psi_{-1/2}1, z) = \psi(z)$. The vector $\omega = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}1$ is a Virasoro vector of $X$ with central charge $1/2$ and the quadruple $(X, Y(\cdot, z), 1, \omega)$ is isomorphic to $L(1/2, 0) \oplus L(1/2, 1/2)$ (cf. [KR]).

Set $X^0 := L(1/2, 0) \subset X$ and $X^1 := L(1/2, 1/2) \subset X$ under the isomorphism $X \simeq L(1/2, 0) \oplus L(1/2, 1/2)$. Then $X^0 \otimes \cdots \otimes X^0$ also forms an SVOA as a tensor product of SVOAs. For an even linear subcode $C$ of $\mathbb{Z}_2^n$, set

$$
V_C := \bigoplus_{\alpha = (\alpha_1, \ldots, \alpha_n) \in C} X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n},
$$

(3.3)

which is a subalgebra of $X^0 \otimes \cdots \otimes X^0$. This is a simple VOA called a code VOA associated to $C$ (cf. [M2]). A code VOA has the standard Ising frame of $\sigma$-type and the following theorem characterizes code VOAs via their standard Ising frames.

**Theorem 3.3.** ([M3]) Let $V$ be a simple VOA with an Ising frame $\omega = e^1 + \cdots + e^n$ of $\sigma$-type. Then there exists a unique even linear subcode $C$ of $\mathbb{Z}_2^n$ such that $V$ is isomorphic to a code VOA $V_C$ with respect to the Ising frame $\omega = e^1 + \cdots + e^n$.

We will use the following notation for code VOAs. Set $u^0 := 1 \in X^0$, $u^1 := \psi_{-1/2}1 \in X^1$, and for a codeword $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_2^n$, we set

$$
x^{\alpha} := u^{\alpha_1} \otimes \cdots \otimes u^{\alpha_n} \in X^0 \otimes \cdots \otimes X^0.
$$

(3.4)
Then $x^\alpha$ is a highest weight vector of $X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n}$ with norm $\langle x^\alpha, x^\alpha \rangle = \pm 1^1$, where the invariant bilinear form $\langle \cdot, \cdot \rangle$ on $X$ is normalized such as $\langle 1, 1 \rangle = 1$.

For a codeword $\beta \in \mathbb{Z}_2^n$, the subspace

$$V_{C+\beta} := \bigoplus_{\gamma = (\gamma_1, \ldots, \gamma_n) \in C+\beta} X^{\gamma_1} \otimes \cdots \otimes X^{\gamma_n}$$

uniquely forms an irreducible $V_C$-submodule of $X^\otimes n$ (cf. [M3]). We will call $V_{C+\beta}$ a coset type module over $V_C$.

### 3.3 Ising vectors of $\sigma$-type

In this subsection we consider a VOA $V$ with trivial weight one subspace, i.e., $V_1 = 0$. Then the weight two subspace $V_2$ equipped with the product $a \cdot b := a_{(1)} b$ for $a, b \in V_2$ forms a commutative algebra with an invariant bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle a, b \rangle 1 = a_{(3)} b$. This algebra is called the Griess algebra of $V$. Let $e, f$ be Ising vectors of $V$ of $\sigma$-type. Then $e/2$ and $f/2$ are idempotents of the Griess algebra $V_2$ with squared norm $1/16$. The following result is fundamental:

**Proposition 3.4.** ([MM][M1]) Let $e, f$ be distinct Ising vectors of $V$ of $\sigma$-type. Then one of the following holds:

1. $\langle e, f \rangle = 0$ and $e \cdot f = 0$. In this case $\sigma_e f = f$ and $\sigma_f e = e$.
2. $\langle e, f \rangle = 1/32$ and $e \neq \sigma_f e = \sigma_e f \neq f$. In this case $\sigma_e \sigma_f$ is of order 3 and the following equality holds in $V_2$:

   $$e \cdot f = \frac{1}{4} (e + f - \sigma_e f).$$

Therefore, $e$ and $f$ generates a three dimensional subalgebra $C e \oplus C f \oplus C \sigma_e f$ in the Griess algebra $V_2$ on which the symmetric group of degree three acts.

Let $H_8$ be the $[8,4,4]$-Hamming code:

$$H_8 := \text{Span}_{\mathbb{Z}_2}\{(11111111), (11110000), (11001100), (10101010)\} \subset \mathbb{Z}_2^8.$$ 

It is well-known that $H_8$ is the unique doubly even self-dual code of length 8. Let $V_{H_8}$ be the code VOA associated to $H_8$ and let $\omega = e^1 + \cdots + e^8$ be the standard Ising frame of $V_{H_8}$. Inside the Hamming code VOA $V_{H_8}$, we can find three Ising frames. For a codeword $\alpha \in \mathbb{Z}_2^8$, define

$$t^\alpha := \frac{1}{8} \sum_{i=1}^8 e^i + \frac{1}{8} \sum_{\beta \in H_8, \langle \alpha, \beta \rangle = 4} (-1)^{\langle \alpha, \beta \rangle} x^{\beta} \in V_{H_8},$$

where $x^{\beta}$ above are defined as in (3.4). Then it is shown in [M2] that $t^\alpha$ is an Ising vector of $\sigma$-type.

\[\text{1The signs depend on the choice of 2-cocycle which we have use to construct a tensor product } X^\otimes n.\]
**Proposition 3.5.** ([MM]/[M4]) Inside the Hamming code VOA $V_{H_8}$, there are exactly three Ising frames given as follows.

\[ I_0 := \{ e^i | 1 \leq i \leq 8 \}, \quad I_1 := \{ t^{\nu^j} | 1 \leq j \leq 8 \} \quad \text{and} \quad I_2 := \{ t^{\nu^1+\nu^k} | 1 \leq k \leq 8 \}, \quad (3.7) \]

where we have set $\nu^1 := (10000000)$, $\nu^2 := (01000000), \ldots, \nu^8 := (00000001) \in \mathbb{Z}_2^8$. Moreover, if $f \in I_a$, then $\sigma_f I_b = I_c$ if $\{a,b,c\} = \{0,1,2\}$ so that all the frames are mutually conjugate to each other.

Based on Propositions 3.4 and 3.5, the following result is established in [La].

**Proposition 3.6.** ([La]) Let $C$ be an even linear code whose minimum weight is greater than 2. If the code VOA $V_C$ contains an Ising vector $f$ of $\sigma$-type which is not a summand of the standard Ising frame of $V_C$, then $C$ contains a subcode $D$ isomorphic to $H_8$ such that $f \in V_D \subset V_C$ and $f$ is of the form (3.6) in $V_D$. In particular, there are exactly 24 Ising vectors inside $V_{H_8}$ and every Ising vector of $V_{H_8}$ is a summand of an Ising frame of $V_{H_8}$.

We give a generalization of the proposition above. The following lemma enables us to reduce a general case to the known case.

**Lemma 3.7.** Let $V$ be a VOA with $V_1 = 0$. Suppose that $V$ has two Ising vectors $e, f$ and $e$ is of $\sigma$-type. Then $e \in V^{(\tau_f)}$.

**Proof:** Consider the Griess algebra $V_2$ of $V$. Let $t$ be an Ising vector of $V$. Then it is shown in [M1] that the Griess algebra $V_2$ affords the orthogonal decomposition

\[ V_2 = \mathbb{C}t \perp (V_1(0) \cap V_2) \perp (V_1(1/2) \cap V_2) \perp (V_1(1/16) \cap V_2) \quad (3.8) \]

and $L^t(0) = t(1)$ acts on $V_t(h) \cap V_2$ by the scalar $h$. For convention of notation, we set $B_t(h) := V_t(h) \cap V_2$ for $h = 0, 1/2, 1/16$, and for $x \in V_2$, we will denote the corresponding decomposition by $x = \lambda_{x,t}t + x_1(0) + x_1(1/2) + x_1(1/16)$ with $\lambda_{x,t} \in \mathbb{C}$. Since $\langle x, t \rangle = \lambda_{x,t}(t,t) = \lambda_{x,t}/4$, the scalar $\lambda_{x,t}$ is given by $4 \langle x, t \rangle$. By the fusion rules of $L(1/2,0)$-modules (cf. [DMZ]), $V_2$ has the following structure:

\[ B_t(0) \cdot B_t(h) \subset B_t(h), \quad h = 0, 1/2, 1/16, \quad B_t(1/2) \cdot B_t(1/2) \subset \mathbb{C}t \oplus B_e(0), \]

\[ B_t(1/2) \cdot B_t(1/16) \subset B_t(1/16), \quad B_t(1/16) \cdot B_t(1/16) \subset \mathbb{C}t \oplus B_t(0) \oplus B_t(1/2). \quad (3.9) \]

Write $f = \lambda e + f_e(0) + f_e(1/2)$ and $e = \lambda f + e_f(0) + e_f(1/2) + e_f(1/16)$ with $\lambda = 4 \langle e, f \rangle$. We shall show that $e_f(1/16) = 0$. Since $f$ is an Ising vector, we have $f \cdot f = 2f$. Using (3.9), we compare the $V_2(1/2)$-parts of both sides of $f \cdot f = 2f$ and obtain

\[ f_e(0) \cdot f_e(1/2) = \frac{1}{2}(2 - \lambda)f_e(1/2). \quad (3.10) \]
Similarly, we compare the $V_f^{(1/16)}$-parts in the equality $e \cdot e = 2e$ and get

$$e_f(0) \cdot e_f^{(1/16)} + e_f^{(1/2)} \cdot e_f^{(1/16)} = \frac{1}{16} (16 - \lambda) e_f^{(1/16)}. \quad (3.11)$$

Since the Griess algebra $V_2$ is commutative, $e \cdot f = f \cdot e$ and we have

$$f_e^{(1/2)} = 4\lambda (1 - \lambda) f - 4\lambda e_f(0) + (1 - 4\lambda) e_f^{(1/2)} + \frac{1}{8} (1 - 32\lambda) e_f^{(1/16)}. \quad (3.12)$$

Using $f_e(0) = f - \lambda e - f_e^{(1/2)}$, we have

$$f_e(0) = (3\lambda - 1)(\lambda - 1) f + 3\lambda e_f(0) + (3\lambda - 1) e_f^{(1/2)} + \frac{1}{8} (24\lambda - 1) e_f^{(1/16)}. \quad (3.13)$$

From the $V_f^{(1/16)}$-part in the equality $e \cdot f_e(0) = 0$ we obtain

$$e_f^{(1/2)} \cdot e_f^{(1/16)} = \frac{1}{16} (92\lambda - 1) e_f^{(1/16)}. \quad (3.14)$$

Then by (3.11) we have

$$e_f(0) \cdot e_f^{(1/16)} = \frac{1}{16} (-93\lambda + 17) e_f^{(1/16)}. \quad (3.15)$$

By (3.14), (3.15) and the $V_f^{(1/16)}$-parts in the equality (3.10), we obtain

$$\frac{1}{128} (32\lambda - 1)(64\lambda + 13) e_f^{(1/16)} = 0. \quad (3.16)$$

Therefore, either $\lambda = 1/32$, $\lambda = -13/64$ or $e_f^{(1/16)} = 0$.

Now suppose that $e_f^{(1/16)} \neq 0$. Then $\lambda = 1/32$ or $-13/64$. By comparing the $V_f^{(1/2)}$-parts in the equality $e \cdot e = 2e$ we obtain

$$\left( e_f^{(1/16)} \cdot e_f^{(1/16)} \right) f_e^{(1/2)} = (2 - \lambda) e_f^{(1/2)} - 2 e_f(0) \cdot e_f^{(1/2)}. \quad (3.17)$$

Then from the $V_f^{(1/2)}$-part in the equality $e \cdot f_e(0) = 0$ together with (3.17) we get

$$e_f(0) \cdot e_f^{(1/2)} = \frac{1}{6} (29\lambda + 2) e_f^{(1/2)} \quad (3.18)$$

and from the $V_f^{(1/2)}$-parts in the equality (3.10) we also have

$$\frac{1}{32} (168\lambda + 1) e_f(0) \cdot e_f^{(1/2)} = \frac{1}{64} (600\lambda^2 - 17\lambda + 34) e_f^{(1/2)}. \quad (3.19)$$

Since $\lambda = 1/32$ or $-13/64$, the equalities (3.18) and (3.19) imply that $e_f^{(1/2)} = 0$. Then the equation (3.14) contradicts to our assumption. Thus $e_f^{(1/16)} = 0$ and the lemma follows.

By the lemma above, we can generalize Proposition 3.6 as follows.
Proposition 3.8. Let $C$ be an even linear code with minimum weight greater than 2, and let $f$ be an Ising vector of the associated code VOA $V_C$. If $f$ is not a summand of the standard Ising frame of $V_C$, then there is a subcode $D$ of $C$ isomorphic to $H_8$ such that $f \in V_D \subset V_C$ and $f$ is of the form (3.6) in $V_D$.

Proof: Let $\omega = e^1 + \cdots + e^n$ be the standard Ising frame of $V_C$. Then byLemma 3.7 all $e^i$ are contained in the $\tau_f$-fixed point subalgebra $V_{C'}^{(\tau_f)}$ of $V_C$. Therefore, there is a subcode $C'$ of $C$ with index at most two such that $V_{C'}^{(\tau_f)}$ is a code VOA $V_{C'}$ with respect to the Ising frame $\omega = e^1 + \cdots + e^n$. Since $f$ is of $\sigma$-type on $V_{C'}$, we can apply Proposition 3.6 to $V_{C'}$. This completes the proof.

Corollary 3.9. Let $C$ be an even linear code of length $n$ and assume that the minimum weight of $C$ is greater than 2.

(1) Every Ising vector of $V_C$ is a summand of an Ising frame of $V_C$.

(2) Let $N$ be the number of embeddings of $H_8$ into $C$. Then $V_C$ contains exactly $16N + n$ Ising vectors.

As an application of Proposition 3.6, we show the conjugacy property of Ising frames of a code VOA.

Lemma 3.10. Let $C$ be an even linear code of length $n$ with no weight two element. Suppose that there is a subcode $D$ of $C$ which is isomorphic to the Hamming code $H_8$. Then an Ising vector $t \in V_D \subset V_C$ of the form (3.6) is of $\sigma$-type if and only if $|\text{supp}(\alpha) \cap \text{supp}(D)|$ is even for all codeword $\alpha \in C$.

Proof: We may assume that $\text{supp}(D) = \{1, \ldots, 8\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a codeword of $C$. Then $V_C$ is a sum of a coset type $V_D$-submodules $V_{D+(\alpha_1, \ldots, \alpha_8)}$ as a $V_D$-module. It is shown in Theorem 2.2 of [M4] that $\tau_i$ is trivial on $V_{D+(\alpha_1, \ldots, \alpha_8)}$ if and only if $(\alpha_1, \ldots, \alpha_8)$ is an even codeword. So the assertion follows.

By Proposition 3.6 and the lemma above, we can in principal count the number of $\sigma$-type Ising frames of $V_C$. Denote by $G$ the subgroup of $\text{Aut}(V_C)$ generated by $\sigma$-type Miyamoto involutions. By Proposition 3.4, $G$ is a 3-transposition group (see Definition 6.1).

Proposition 3.11. Let $C$ be an even linear code of length $n$ whose minimum weight is greater than 2 and let $\omega = e^1 + \cdots + e^n$ and $\omega = f^1 + \cdots + f^n$ be $\sigma$-type Ising frames of the code VOA $V_C$. Then there is an element $\rho \in G$ such that $\{\rho e^1, \ldots, \rho e^n\} = \{f^1, \ldots, f^n\}$.

Proof: It is enough to show the assertion in the case that $\{e^1, \ldots, e^n\}$ is the standard Ising frame of $V_C$. Set $I = \{e^1, \ldots, e^n\}$ and $J = \{f^1, \ldots, f^n\}$. We shall prove the assertion inductively. Assume that there is an element $\rho_i \in G$ such that $\{f^1, \ldots, f^i\} \subset \rho_i I$, where in the case of $i = 0$ we set $\rho_0 = \text{id}$. Since $I$ and $\rho_i I$ are conjugate, the associated
binary code of \( V_C \) with respect to the frame \( \rho_i I \) is still isomorphic to \( C \). Therefore, by replacing \( I \) by \( \rho_i I \), we may assume that \( \{ f^1, \ldots, f^i \} \subset I \cap J \). If \( f^{i+1} \not\in I \), then there is a subcode \( D \) of \( C \) isomorphic to the Hamming code \( H_8 \) such that \( f^{i+1} \in V_D \subset V_C \) and \( f^{i+1} \) is of the form (3.6) in \( V_D \) by Proposition 3.6. Let \( \text{supp}(D) = \{ j_1, \ldots, j_8 \} \). Then \( \{ \sigma_{e_1} f^{i+1}, \ldots, \sigma_{e_{j_8}} f^{i+1} \} = \{ \sigma_{e_{j_1}} f^{j_1}, \ldots, \sigma_{e_{j_8}} f^{j_8} \} \) by Proposition 3.5. Therefore, \( f^{i+1} \in \sigma_{e_{j_1}} f^{j_1} I \). It follows from (3.6) that \( \langle f^{i+1}, e^j \rangle = 1/32 \) for \( j \in \text{supp}(D) \). Hence \( \{ f^1, \ldots, f^i \} \cap \{ e_j^1, \ldots, e_j^8 \} = \emptyset \) and \( \sigma_{e_{j_1}} f^{j_1} \{ f^1, \ldots, f^i \} = \{ f^1, \ldots, f^i \} \). Thus \( \rho_{i+1} = \sigma_{e_{j_1}} f^{j_1} \in G \) satisfies \( \{ f^1, \ldots, f^{i+1} \} \subset \rho_{i+1} I \). By this procedure, we will obtain \( \rho = \rho_n \in G \) such that \( \rho I = J \).

**Remark 3.12.** We can slightly generalize Proposition 3.11 as follows. Let \( \omega = e^1 + \cdots + e^n \) be a \( \sigma \)-type Ising frame of \( V_C \) and \( \omega = f^1 + \cdots + f^n \) any Ising frame of \( V_C \). Let \( H \) be the subgroup of \( \text{Aut}(V_C) \) generated by \( \tau \)-type Miyamoto involutions \{ \tau_{f^i} \mid 1 \leq i \leq n \}. Then \( f^i \) are \( \sigma \)-type Ising vectors on \( V_C^H \). By Lemma 3.7, \{ \omega, e^1, \ldots, e^n \} \) is contained in \( V_C^H \). Therefore, by Proposition 3.11, we can find \( \rho \in \text{Aut}(V_C^H) \) such that \( \rho \{ e^1, \ldots, e^n \} = \{ f^1, \ldots, f^n \} \). However, we cannot find such \( \rho \) inside \( \text{Aut}(V_C) \) unless \( H = 1 \). In fact, conjugating VOA structures by \( \rho \), we can perform a \( \mathbb{Z}_2 \)-twisted orbifold construction of a framed VOA (cf. [M5, Y]).

We have only considered code VOAs associated to codes without weight two elements. If an even linear code \( C \) contains a weight two element, then the code VOA \( V_C \) contains a subalgebra isomorphic to a lattice VOA \( V_{2\alpha} \) associated to a lattice \( \mathbb{Z}\alpha \) with \( \langle \alpha, \alpha \rangle = 4 \) (cf. [DMZ]). In this case, we can define continuous automorphisms on \( V_C \) by exponential and hence \( \text{Aut}(V_C) \) is always an infinite group. Conversely, if \( C \) contains no weight two element, then \( \text{Aut}(V_C) \) is finite. This result is established in [M2] in the case that \( V_C \) is considered over \( \mathbb{R} \). By Proposition 3.6, we know that the set of \( \sigma \)-type Ising vectors of \( V_C \) is finite so that \( \text{Aut}(V_C) \) is still finite even if \( V_C \) is defined over \( \mathbb{C} \). We give a brief description of \( \text{Aut}(V_C) \) as follows.

**Proposition 3.13.** Let \( C \) be an even linear code of length \( n \) without weight two codewords. Then the automorphism group \( \text{Aut}(V_C) \) of the code VOA \( V_C \) is a finite group generated by the lift of \( \text{Aut}(C) \) and \( \sigma \)-type Miyamoto involutions.

**Proof:** Let \( I = \{ e^1, \ldots, e^n \} \) be the standard Ising frame of \( V_C \) and let \( G \) be the 3-transposition subgroup of \( \text{Aut}(V_C) \) generated by \( \sigma \)-type Miyamoto involutions. Take any \( \phi \in \text{Aut}(V_C) \). By Proposition 3.11, there is an element \( \rho \in G \) such that \( \rho \phi I = I \). Then \( \rho \phi \) defines an automorphism of \( \text{Vir}(e^1) \otimes \cdots \otimes \text{Vir}(e^n) \). Therefore, \( \rho \phi \) preserves the set \( \{ \pm x^\alpha \subset V_C \mid \alpha \in C \} \) of normed highest weight vectors and thus there is a lift \( \tilde{g} \in \text{Aut}(V_C) \) of \( g \in \text{Aut}(C) \) such that \( \tilde{g} \rho \phi e^i = e^i \) for \( i = 1, \ldots, n \). Then by Schur’s lemma, \( \tilde{g} \rho \phi \) is written as a product of \( \sigma_{e_i} \), \( 1 \leq i \leq n \). Thus \( \text{Aut}(V_C) \) is generated by \( G \) and the lift of \( \text{Aut}(C) \) on \( V_C \). Since there are finitely many Ising frames inside \( V_C \), the argument above also shows that \( \text{Aut}(V_C) \) is finite.
Let $E$ be the set of $\sigma$-type Ising vectors of $V$. We have defined a map $\sigma : E \to \text{Aut}(V)$ by associating the $\sigma$-type Miyamoto involution $\sigma_e \in \text{Aut}(V)$ to each $e \in E$. The following injectivity is shown in Lemma 2.5.2 of [Ma].

**Lemma 3.14. ([Ma])** Assume that for each $e \in E$, there is an Ising vector $g \in V$ such that $\langle e, g \rangle = 1/32$. Then $\sigma : E \to \text{Aut}(V)$ is injective.

**Proof:** Suppose $\sigma_e = \sigma_f$ with $e, f \in E$. By the assumption, there exists an Ising vector $g \in V$ such that $\langle e, g \rangle = 1/32$. It follows from Lemma 3.7 that $e, f \in V^{(\tau_0)}$. Then by (2) of Proposition 3.4, $g \neq \sigma_e g = \sigma_f g$ and hence $g \neq \sigma_g e = \sigma_f g$. Thus $\langle f, g \rangle = 1/32$ and $\sigma_f g = \sigma_g f$ again by Proposition 3.4. Hence $\sigma_g e = \sigma_e g = \sigma_f g = \sigma_g f$ showing $e = f$.  

### 4 Ising vectors of lattice VOAs

We will classify Ising vectors of the lattice VOA $V$ and its code $C$. It is easy to see that all $4$-frame vectors $C$ are of the form $\omega^\pm := (0_1^0, \ldots, 0_n^0)$, where $\omega$ is the standard basis of $\mathbb{Z}^n$. By definition $L_C$ contains a 4-frame $2\mathbb{Z}x^1 \perp \cdots \perp 2\mathbb{Z}x^n$ so that $V_{L_C}$ is a lattice VOA associated with a root lattice $R$. This classification immediately leads to a classification of Ising vectors in $M_R$.

#### 4.1 A lattice VOA and its code

Let $\mathbb{Z}^n$ be the standard lattice and let $\rho : \mathbb{Z}^n \to \mathbb{Z}_2^n$ be the reduction mod 2, which is a group homomorphism. For an even linear code $C \subset \mathbb{Z}_2^n$, the preimage $L_C := \rho^{-1}(C) \subset \mathbb{Z}^n$ define a sublattice of $\mathbb{Z}^n$. Set $x^1 := (1, 0, \ldots, 0), x^2 := (0, 1, \ldots, 0), \ldots, x^n := (0, 0, \ldots, 1) \in \mathbb{Z}^n$. Then $\{x^1, \ldots , x^n\}$ is the standard basis of $\mathbb{Z}^n$. By definition $L_C$ contains a 4-frame $2\mathbb{Z}x^1 \perp \cdots \perp 2\mathbb{Z}x^n$ so that $V_{L_C}$ is a lattice VOA associated with a root lattice $R$. This classification immediately leads to a classification of Ising vectors in $M_R$.

Let $w^{i\pm}$ be the Virasoro vector of $V_{L_C}$. The lattice VOA $V_{L_C}$ contains an Ising frame

$$\omega = (w^{1-} + w^{1+}) + (w^{2-} + w^{2+}) + \cdots + (w^{n-} + w^{n+}).$$

Then $w^{i\pm}$ are mutually orthogonal Ising vectors. Let $\omega$ be the Virasoro vector of $V_{L_C}$. The lattice VOA $V_{L_C}$ contains an Ising frame

$$\omega = (w^{1-} + w^{1+}) + (w^{2-} + w^{2+}) + \cdots + (w^{n-} + w^{n+}).$$

It is easy to see that all $w^{i\pm}$ are of $\sigma$-type on $V_{L_C}$ so that $V_{L_C}$ is a code VOA with respect to the frame. It is obvious that $w^{i\pm}$ are contained in $V_{2\mathbb{Z}x_i}^\pm$, so that they are also contained in $V_{L_C}$. We define

$$D_0(C) := \{(u_1, u_2, u_3, \ldots, u_n, u_n) \in \mathbb{Z}_2^{2n} \mid (u_1, u_2, \ldots, u_n) \in C^\perp\} \subset \mathbb{Z}_2^{2n},$$

$$D_1(C) := D_0(C) \cup (D_0(C) + \gamma) \subset \mathbb{Z}_2^{2n}, \quad \gamma = (1010 \ldots 10) \in \mathbb{Z}_2^{2n}.$$ 

As $\text{Vir}(w^{i-}) \otimes \text{Vir}(w^{i+})$-modules, we have the following isomorphisms:

$$V_{2\mathbb{Z}x_i}^\pm \simeq L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0), \quad V_{2\mathbb{Z}x_i}^- \simeq L(\frac{1}{2}, 1/2) \otimes L(\frac{1}{2}, 1/2),$$

$$V_{2\mathbb{Z}x_i+x^i}^\pm \simeq L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 1/2), \quad V_{2\mathbb{Z}x_i+x^i}^- \simeq L(\frac{1}{2}, 1/2) \otimes L(\frac{1}{2}, 0).$$

13
By the isomorphisms above, the structure codes of $V_{L_C}$ and $V_{L_C}^+$ with respect to the frame (4.2) are described as follows.

**Proposition 4.1.** (1) The lattice VOA $V_{L_C}$ with the Ising frame (4.2) is isomorphic to the code VOA associated to $D_0(C)^\perp$.
(2) The VOA $V_{L_C}^+$ with the Ising frame (4.2) is isomorphic to the code VOA associated to $D_1(C)^\perp$.

If we take
\[ C_n := \text{Span}_{\mathbb{Z}_2}\{ (1^20^{2n-2}), (0^21^20^{2n-4}), \ldots, (0^{2n-2}1^2), (1010 \ldots 10) \} \subset \mathbb{Z}_2^{2n}, \]
then we obtain $L_{C_n} \simeq \sqrt{2}D_{2n}$. Therefore, the structure code of $V_{\sqrt{2}D_{2n}}$ as a code VOA is given by the dual of the following code:
\[ C_n := \text{Span}_{\mathbb{Z}_2}\{ (1^40^{4n-4}), (0^41^40^{4n-8}), \ldots, (0^{4n-4}1^4), (1^20^2 \ldots 1^20^2), \} \subset \mathbb{Z}_2^{4n}, \quad (4.3) \]
and that of $V_{\sqrt{2}E_{8}}^+$ is given by the dual code of the code generated by $C_n$ and $(1010 \ldots 10) \in \mathbb{Z}_2^{4n}$.

If we take the [8,4,4]-Hamming code $H_8$, then $H_8^\perp = H_8$ and we obtain $L_{H_8} \simeq \sqrt{2}E_8$. The structure code of $V_{\sqrt{2}E_{8}}$ as a code VOA is given by $D_1(H_8)^\perp$, where $D_1(H_8)$ has the following presentation:
\[ D_1(H_8) = \text{Span}_{\mathbb{Z}_2}\{ (1^{16}), (1^80^8), (1^40^41^40^4), (1^20^21^20^21^20^2), (\{10\}^8) \} \subset \mathbb{Z}_2^{16}. \]
We note that the code $D_1(H_8)$ is the same as the first order Reed-Muller code $RM(1,4)$ of length $2^4$ so that the code $D_1(H_8)^\perp$ is equal to the second order Reed-Muller code $RM(2,4)$ (cf. [CS]).

### 4.2 Ising vectors of $V_{\sqrt{2}R}^+$

Let $R$ be an indecomposable root lattice with root system $\Phi(R)$. We give a classification of Ising vectors of $V_{\sqrt{2}R}^+$. As in (4.1), for $\alpha \in \Phi(R)$ we set
\[ w^\pm(\alpha) := \frac{1}{8}\alpha_{(-1)}^21 \pm \frac{1}{4}\left(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}\right) \in V_{\sqrt{2}R}^+. \quad (4.4) \]
It is shown in [DMZ] [DLMN] that $w^\pm(\alpha), \alpha \in \Phi(R)$, are Ising vectors of $\sigma$-type.

**Proposition 4.2.** Let $R$ be an indecomposable root lattice whose root system is of type $A_n$ or $D_n$. Then the set $\{ w^\pm(\alpha) \mid \alpha \in \Phi(R) \}$ exhausts all the Ising vectors of $V_{\sqrt{2}R}^+$.

**Proof:** It suffices to show the assertion in the case of $R = D_{2n}$ since $A_k$ and $D_{2m+1}$, $1 \leq k, m \leq n - 1$, are sublattices of $D_{2n}$. Let $C$ be the dual code of the following code:
\[ \text{Span}_{\mathbb{Z}_2}\{ (1^40^{4n-4}), (0^41^40^{4n-8}), \ldots, (0^{4n-4}1^4), (1^20^2 \ldots 1^20^2), (1010 \ldots 1010) \} \subset \mathbb{Z}_2^{4n}. \]
As shown in Sec.4.1, $V^+_{\sqrt{2}D_{2n}}$ is isomorphic to a code VOA associated to $C$. By Corollary 3.9, it suffices to count the number $N$ of embeddings of $H_8$ into $C$ and one can easily show that $N = \binom{n}{4} = n(n-1)/2$. Therefore, there are $4n(2n-1)$ Ising vectors inside $V^+_{\sqrt{2}D_{2n}}$. Since there are $|\Phi(D_{2n})| = 4n(2n-1)$ Ising vectors of the form (4.4) in $V^+_{\sqrt{2}D_{2n}}$, they are all.

Next we consider the $E$-series. Since $E_6$ and $E_7$ are sublattices of $E_8$, we only need to consider $V^+_{\sqrt{2}E_8}$. Since the number of roots in $\Phi(E_8)$ is 240, we have 240 Ising vectors of $V^+_{\sqrt{2}E_8}$ of the form (4.4). We also have an Ising vector $\tilde{\omega}$ of $V^+_{\sqrt{2}E_8}$ defined as in (2.3). Since $\tau_{\tilde{\omega}} = \text{id}$ on $V^+_{\sqrt{2}E_8}$, $\tilde{\omega}$ is of $\sigma$-type. For $h \in \mathbb{C}E_8$, set

$$\varphi_h := \exp \left( \frac{\pi \sqrt{-1}}{2} (h(1) \mathbb{I})(0) \right) \in \text{Aut}(V^+_{\sqrt{2}E_8}).$$

Then $\varphi_h V^+_{\sqrt{2}E_8} = V^+_{\sqrt{2}E_8}$ if and only if $\varphi_{2h} = 1$. Therefore, we obtain a group homomorphism $\varphi : E_8 \ni x \mapsto \varphi_x \in \text{Aut}(V^+_{\sqrt{2}E_8})$ with $\ker \varphi = 2E_8$. So we have $2^8 = 256$ Ising vectors of $V^+_{\sqrt{2}E_8}$ of the form

$$\varphi_x \tilde{\omega}, \quad x \in E_8.$$  

(4.6)

It is shown in [CS] that $E_8/2E_8$ contains 1 class represented by 0, 120 classes represented by a pair of roots $\pm \alpha \in \Phi(E_8)$, and 135 classes represented by 16 vectors forming a 4-frame of $E_8$.

It is shown by Shimakura [S2] that these $240 + 256 = 496$ vectors exhaust all the Ising vectors of $V^+_{\sqrt{2}E_8}$. We give another proof here.

**Proposition 4.3. There are 496 Ising vectors inside $V^+_{\sqrt{2}E_8}$.**

**Proof:** As shown in Sec. 4.1, there is an Ising frame of $V^+_{\sqrt{2}E_8}$ such that $V^+_{\sqrt{2}E_8}$ is isomorphic to a code VOA associated to the Reed-Muller code $\text{RM}(2,4)$ with respect to the frame. The minimum weight of $\text{RM}(2,4)$ is 4 (cf. [CS]). So by Corollary 3.9, it suffices to count embeddings of the $[8,4,4]$-Hamming code $H_8$ into $\text{RM}(2,4)$. It is not difficult to see that for each embedding $H_8 \hookrightarrow \text{RM}(2,4)$, the support of $H_8$ belongs to $\text{RM}(2,4)^\perp = \text{RM}(1,4)$ so that there are exactly 30 embeddings of $H_8$ into $\text{RM}(2,4)$. Hence, there exists $30 \times 16 = 480$ Ising vectors of the form (3.6) inside $V^+_{\sqrt{2}E_8}$. Adding the 16 Ising vectors of summands of the standard Ising frame, there are $480 + 16 = 496$ Ising vectors which exhaust all the Ising vectors inside $V^+_{\sqrt{2}E_8}$. We also note that all the Ising vectors are of $\sigma$-type on $V^+_{\sqrt{2}E_8}$.

Combining Proposition 4.2 and 4.3, we obtain the following.

**Theorem 4.4. Let $R$ be an indecomposable root lattice with root system $\Phi(R)$.**

(1) If $R \neq E_8$, the set $\{w^+(\alpha) \mid \alpha \in \Phi(R)\}$ exhausts all the Ising vectors of $V^+_{\sqrt{2}R}$.

(2) There are 256 Ising vectors of $V^+_{\sqrt{2}E_8}$ other than $\{w^+(\alpha) \mid \alpha \in \Phi(E_8)\}$. The set $\{w^+(\alpha), \varphi_x \tilde{\omega} \mid \alpha \in \Phi(E_8), x \in E_8/2E_8\}$ exhausts all the Ising vectors of $V^+_{\sqrt{2}E_8}$. 

\[\text{15}\]
Corollary 4.5. There is no \(\tau\)-type Ising vector inside the \(\mathbb{Z}_2\)-orbifold subalgebra \(V^+_{\sqrt{2}R}\) of the lattice VOA \(V_{\sqrt{2}R}\) associated to an indecomposable root lattice \(R\).

We have treated only indecomposable root lattices. Now we classify all Ising vectors of the \(\mathbb{Z}_2\)-orbifold \(V^+_{\sqrt{2}L}\) of a lattice VOA \(V_{\sqrt{2}L}\) associated to any root lattice \(L\).

Theorem 4.6. Let \(L\) be a root lattice. If \(e\) is an Ising vector of \(V^+_{\sqrt{2}L}\), then there exists a sublattice \(K\) of \(L\) isometric to \(A_1\) or \(E_8\) such that \(e \in V^+_{\sqrt{2}K} \subset V^+_{\sqrt{2}L}\).

Proof: Since \(L\) is a root lattice, \(L\) is a direct sum of indecomposable root lattices of \(ADE\)-type. Let \(L^{(1)}\) be the sum of irreducible sublattices of \(L\) of \(AD\)-type and \(L^{(2)}\) the orthogonal complement of \(L^{(1)}\) in \(L\) which is the sum of irreducible sublattices of \(L\) of \(E\)-type. Let \(L^{(2)} = L^{(2,1)} \oplus \cdots \oplus L^{(2,n)}\) be the decomposition of \(L^{(2)}\) into irreducible components. Then there exists \(N \in \mathbb{N}\) such that we can embed \(L^{(1)}\) into \(D_{2N}\) and \(L^{(2,i)}\) into \(E_8\) for \(1 \leq i \leq n\). Set \(R^0 = D_{2N}\) and \(R^i = E_8\) for \(1 \leq i \leq n\). By using the embeddings above, we identify \(L\) as a sublattice of \(R^0 \oplus R^1 \oplus \cdots \oplus R^n \simeq D_{2N} \oplus E_8^{\oplus n}\) by which we regard \(V^+_{\sqrt{2}L}\) as a sub VOA of \(V^+_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}\). We shall show that if \(e\) is an Ising vector of \(V^+_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}\), then \(e\) is contained in \(V^+_{\sqrt{2}R^0} \otimes \cdots \otimes V^+_{\sqrt{2}R^n}\). Recall the code \(C_N\) of length \(4N\) defined by (4.3). Let \(C\) be the dual code of the code generated by \(C_N \oplus D_0(H_8)^{\oplus n}\) and \(\gamma = (1010 \ldots 10) \in \mathbb{Z}_2^{4N + 16n}\). As shown in Sec. 4.1, \(V^+_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}\) is isomorphic to a code VOA \(V_C\) associated to the code \(C\). Then by Proposition 3.8 there exists an embedding \(\phi : H_8 \hookrightarrow C\) such that \(e \in V_{\phi(H_8)} \subset V_C\). It is not difficult to verify that \(\text{supp}(\phi(H_8))\) is contained in one of the direct summands of \(C_N^+ \oplus (D_0(H_8))^{\oplus n}\). This implies that \(e\) must be in \(V^+_{\sqrt{2}R^0} \otimes \cdots \otimes V^+_{\sqrt{2}R^n}\) as we claimed. Then \(e \in V^+_{\sqrt{2}R_i}\) for some \(0 \leq i \leq n\). Therefore, by Theorem 4.4, either there is a root \(\alpha \in R^0 \oplus \cdots \oplus R^n\) such that \(e\) is of the form \(w^+(\alpha)\) as in (4.4), or \(e \in V^+_{\sqrt{2}R_i}\) for some \(i > 0\) and \(e\) is of the form (4.6) in \(V^+_{\sqrt{2}R_i}\). Anyway, there is a sublattice \(K\) of \(R^0 \oplus \cdots \oplus R^n\) isometric to \(A_1\) or \(E_8\) such that \(e \in V^+_{\sqrt{2}K} \subset V^+_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}\). If \(e\) is taken from \(V^+_{\sqrt{2}K}\), then \(K \subset L\) as we know the explicit form of \(e\) in \(V^+_{\sqrt{2}(R^0 \oplus \cdots \oplus R^n)}\). This completes the proof.

Remark 4.7. With reference to the theorem above, we believe in that the same is true for the \(\mathbb{Z}_2\)-orbifold \(V^+_L\) of a lattice VOA associated with any even lattice \(L\) without roots. For example, it is shown by Shimakura [S2] that this is true for the \(\mathbb{Z}_2\)-orbifold \(V^+_A\) of the Leech Lattice VOA \(V_A\). However, we do not know the answer at present.

### 4.3 Ising vectors of \(M_R\)

Let \(R\) be an indecomposable root lattice as before. We shall determine the Ising vectors of \(M_R\). Recall that

\[ w^+(\alpha) = \frac{1}{8} \alpha^2 (-1)^{1 \pm \frac{1}{4}} \left( e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha} \right), \quad \text{for} \ \alpha \in R, \]
and
\[
\hat{\omega} = \hat{\omega}_R = \frac{2}{h + 2} \omega + \frac{1}{h + 2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2} \alpha},
\]
where \( h \) denotes the Coxeter number of \( \Phi(R) \) as before. By definition, it is easy to show the following.

**Lemma 4.8.** Let \( R \) be a root lattice and consider \( V^+_{\sqrt{2} R} \). Let \( \alpha, \beta \in \Phi(R) \). Then

1. \( \langle w^+(\alpha), w^-(\beta) \rangle = 1/32 \) if \( \langle \alpha, \beta \rangle = \pm 1 \) and 0 otherwise.
2. \( \langle w^+(\alpha), w^-(\beta) \rangle = 1/4 \) if \( \alpha = \pm \beta \), 1/32 if \( \langle \alpha, \beta \rangle = \pm 1 \) and 0 if \( \langle \alpha, \beta \rangle = 0 \).
3. \( \langle w^-(\alpha), \omega \rangle = 0 \) and \( \langle w^+(\alpha), \omega \rangle = 1/(h + 2) \), where \( h \) is the Coxeter number of \( \Phi(R) \).
4. For \( R = E_8 \), let \( x \in E_8 \). Then we have \( \langle \varphi_x \hat{\omega}, \omega \rangle = 1/4 \) if \( x \in 2E_8 \), 1/32 if \( x \) is represented by a root in \( E_8/2E_8 \) and 0 otherwise.

**Proof:** We note that \( M_R = \ker_{V^+_{\sqrt{2} R}}(\hat{\omega}_{(1)}) \) since \( V^+_{\sqrt{2} R} \) is a completely reducible \( \text{Vir}(\hat{\omega}) \)-module. It is clear that \( \{w^-(\alpha) \mid \alpha \in \Phi(R)\} \) is a set of linearly independent vectors of the weight two subspace of \( M_R \), thanks to (3) of Lemma 4.8. If \( R = A_n \), it is shown in [DLMN] that the set \( \{w^+(\alpha) \mid \alpha \in \Phi(R)\} \) is a basis of the weight two subspace of \( V^+_{\sqrt{2} R} \). Therefore, by (3) of Lemma 4.8, the assertion holds in this case. It is shown in [DLY] that \( M_{D_n} \simeq V^+_{\sqrt{2} A_{n-1}} \). By Theorem 4.4 and (3) of Lemma 4.8, the assertion also holds if \( R = D_n \).

Consider the case that \( R = E_6 \) or \( E_7 \). The vacuum character of \( M_R \) is computed in Sec. 7 and one can verify that \( \dim(M_R)_{2} = |\Phi(R)|/2 \). So the assertion follows. If \( R = E_8 \), then \( M_{E_8} \) is a code VOA and it is easy to know the structure code from which we can compute the vacuum character. As a result, one also has \( \dim(M_{E_8})_{2} = |\Phi(E_8)|/2 \). This completes the proof.

### 5 Automorphism group of \( M_R \)

We will determine the automorphism group of the commutant subalgebra \( M_R \) where \( R \) is an indecomposable root lattice. It is shown that \( \text{Aut}(M_R) \) always contains a half of the Weyl group \( W(R) \) of \( R \).

#### 5.1 \( \text{Aut}(M_R) \): the case \( R \neq E_8 \)

Let \( R \) be an indecomposable root lattice. We suppose that \( R \) is not the \( E_8 \)-lattice. In this case, due to Theorem 4.4 and Lemma 4.8, the set of Ising vectors of \( M_R \) is given by \( E_R := \{w^-(\alpha) \mid \alpha \in \Phi(R)\} \). We have shown in Lemma 4.9 that the weight two subspace of \( M_R \) is spanned by \( E_R \). In fact, \( M_R \) is generated by \( E_R \) as a VOA.
Proposition 5.1. $M_R$ is generated by its weight two subspace.

Proof: The assertion is already shown in [LS] if $R = A_n$. Since $M_{D_n} \cong V_{\sqrt{2}A_{n-1}}^+$ by [DLY], the assertion is also true if $R = D_n$. So we only need to show the cases for $R = E_6$ and $R = E_7$. The proof is rather technical so that it will be given in Section 7.

Denote by $\Omega(E_R)$ the permutation group on $E_R$ and we define

$$\text{Aut}(E_R) := \{ \rho \in \Omega(E_R) \mid \langle pe, pf \rangle = \langle e, f \rangle \text{ for all } e, f \in E_R \}.$$ 

Since $M_R$ is generated by $E_R$ as a VOA, by restriction map we have an injection from $\text{Aut}(M_R)$ to $\text{Aut}(E_R)$. On the other hand, $g \in \text{Aut}(R)$ acts on $E_R$ by $gw^-(\alpha) := w^-(g\alpha)$. Hence we also have a group homomorphism $\phi : \text{Aut}(R) \to \text{Aut}(E_R)$.

Lemma 5.2. $\phi$ is surjective.

Proof: Let $\rho \in \text{Aut}(E_R)$. Take a simple system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\Phi(R)$, where $\ell$ is the rank of $R$. By Proposition 3.4 and Lemma 4.8, one can easily verify that $\{w^-(\alpha_1), \ldots, w^-(\alpha_\ell)\}$ is a set of generators of the Griess algebra of $M_R$. Write $\rho w^-(\alpha_i) = w^-(\beta_i)$ for $i = 1, \ldots, \ell$. By Proposition 3.4 and Lemma 4.8, there is a simple system $\Delta'$ of $\Phi(R)$ such that $\{\pm \beta_1, \ldots, \pm \beta_\ell\} = \Delta' \cup (-\Delta')$. Therefore, by choosing suitable signs, we can take the representatives $\{\beta_i \mid 1 \leq i \leq \ell\}$ to be a simple system of $\Phi(R)$. Then we can find $g \in \text{Aut}(R)$ such that $ga_i = \beta_i$ for $i = 1, \ldots, \ell$. This implies that $\rho = g$ since $\{w^-(\alpha_i) \mid 1 \leq i \leq \ell\}$ is a set of generators of $(M_R)_2 = \text{Span}_{\mathbb{C}}E_R$. This completes the proof.

By definition, it is clear that $\ker \phi = \langle \pm 1 \rangle$. Therefore, we have:

Proposition 5.3. $\text{Aut}(E_R) \cong \text{Aut}(R)/\langle \pm 1 \rangle$.

The Weyl group $W(R)$ also acts on $E_R$. Let us denote by $r_\alpha$ the reflection on $R$ defined by a root $\alpha \in \Phi(R)$. It follows from Proposition 3.4 and Lemma 4.8 that $\sigma_{w^-(\alpha)}w^-(\beta) = w^-(r_\alpha \beta)$ for $\alpha, \beta \in \Phi(R)$. Therefore, the group generated by $\{\sigma_e \mid e \in E_R\}$ realizes the action of the Weyl group on $E_R$. The shapes of the Weyl group $W(R)$ and the automorphism group $\text{Aut}(R)$ are as follows (cf. [CS]).

| $R$           | $W(R)$       | $\text{Aut}(R)$  |
|--------------|--------------|------------------|
| $A_n$        | $S_{n+1}$    | $S_{n+1} \times 2$ |
| $D_4$        | $2^3 : S_4$  | $(2^3 : S_4) : S_3$ |
| $D_n \ (n > 4)$ | $2^{n-1} : S_n$ | $2^{n-1} : S_n \times 2$ |
| $E_6$        | $U_4(2) : 2$ | $2.U_4(2) : 2$   |
| $E_7$        | $2 \times \text{Sp}_6(2)$ | $2 \times \text{Sp}_6(2)$ |
Theorem 5.4. Suppose $R \neq E_8$. Then $\text{Aut}(M_R)$ is as follows.

| $R$   | $A_1$ | $A_n$ $(n > 1)$ | $D_4$ | $D_n$ $(n > 4)$ | $E_6$ | $E_7$ |
|-------|-------|---------------|-------|---------------|-------|-------|
| $\text{Aut}(M_R)$ | 1     | $S_{n+1}$    | $(2^2 : S_4) : S_3$ | $2^{n-1} : S_n$ | $U_4(2) : 2 \simeq O_6^-(2)$ | $\text{Sp}_6(2)$ |

**Proof:** Let $R$ be either $A_n$, $n > 1$, $E_6$ or $E_7$. Under the injection, we can consider $\text{Aut}(M_R)$ as a subgroup of $\text{Aut}(E_R)$. By Proposition 5.3, we have the following relation.

$$\text{Aut}(M_R) \hookrightarrow \text{Aut}(E_R) \simeq \text{Aut}(R)/\langle \pm 1 \rangle.$$  

Since the action of the Weyl group on $E_R$ is realized by $\sigma$-type involutions $\{\sigma_e \mid e \in E_R\}$, the subgroup of $\text{Aut}(M_R)$ generated by $\{\sigma_e \mid e \in E_R\}$ coincides with $\text{Aut}(R)/\langle \pm 1 \rangle$. Hence $\text{Aut}(M_R) \simeq \text{Aut}(R)/\langle \pm 1 \rangle$. This establishes the cases (i), (iii) and (iv). The isomorphism in the case (ii) follows from the isomorphism $M_{D_n} \simeq V_{\sqrt{2}A_{n-1}}^+$ shown in [DLY] and the description of $\text{Aut}(V_{\sqrt{2}A_n}^+)$ obtained in [S1]. The case $R = A_1$ is trivial since $M_{A_1} \simeq L(1/2, 0)$. This completes the proof.

5.2 $\text{Aut}(M_{E_8}) \simeq \text{Sp}_8(2)$

By Theorem 4.4, the set of Ising vectors of $V_{\sqrt{2}E_8}^+$ is given by

$$\{w^+(\alpha) \mid \alpha \in \Phi(E_8)\} \cup \{\varphi_2 \tilde{\omega} \mid x \in E_8\}. \tag{5.1}$$

**Lemma 5.5.** Let $R$ be an indecomposable root lattice. Then all the Ising vectors of $V_{\sqrt{2}R}^+$ are conjugate under $\text{Aut}(V_{\sqrt{2}R}^+)$.  

**Proof:** If $R \neq E_8$, then the set of Ising vectors of $V_{\sqrt{2}R}^+$ is provided by $\{w^+(\alpha) \mid \alpha \in \Phi(R)\}$. It is clear that $V_{\sqrt{2}R}^+$ affords an action of the Weyl group $W(R)$ associated to the root system $\Phi(R)$ and $W(R)$ transitively acts on both $\{w^+(\alpha) \mid \alpha \in \Phi(R)\}$ and $\{w^-(\beta) \mid \beta \in \Phi(R)\}$ as $R$ is indecomposable. By (2) of Proposition 3.4 and (1) of Lemma 4.8, there is a pair $\{w^+(\alpha), w^-(\beta)\}$ of Ising vectors which are conjugate under $\text{Aut}(V_{\sqrt{2}R}^+)$. Therefore, all the Ising vectors are conjugate.

Now assume that $R = E_8$. We have seen that $\text{Aut}(V_{\sqrt{2}E_8}^+)$ acts on $\{w^+(\alpha) \mid \alpha \in \Phi(E_8)\}$ transitively. It is obvious that Ising vectors of the form (4.6) are mutually conjugate under $\text{Aut}(V_{\sqrt{2}E_8}^+)$. Then again by (2) of Proposition 3.4 and (3) of Lemma 4.8, all the Ising vectors of $V_{\sqrt{2}E_8}^+$ are conjugate.

As an immediate consequence, we have:

**Corollary 5.6.** For any Ising vector $e$ of $V_{\sqrt{2}E_8}^+$, $\text{Com}_{V_{\sqrt{2}E_8}^+}(\text{Vir}(e)) \simeq M_{E_8}$.

**Lemma 5.7.** $M_{E_8}$ is generated by its weight two subspace.
Proof: It is shown in Sec. 4.1 that $V^+_{\sqrt{2}E_8}$ is isomorphic to the code VOA $V_{RM(2,4)}$ associated to the Reed-Muller code $RM(2,4)$. By Corollary 5.6, we may assume that $\tilde{\omega}$ is the 1st summand of the standard Ising frame of $V_{RM(2,4)}$. Then $M_{Es}$ is also a code VOA and the associated code is obtained by collecting all the vectors of $RM(2,4)$ whose 1st entry is 0 and dropping their 1st entry 0. It is easy to see that this code is generated by weight four vectors. Therefore, $M_{Es}$ is generated by its weight two subspace. □

Lemma 5.8. (1) An Ising vector of $M_{Es}$ is equal to either $w^-(\alpha)$, $\alpha \in \Phi(E_8)$ or $\varphi_x \tilde{\omega}$, $x \in E_8$ and $x + 2E_8 \in E_8/2E_8$ is represented by a norm four vector of $E_8$. Hence, there are $120 + 135 = 255$ Ising vectors of $M_{Es}$.

(2) These 255 Ising vectors are mutually conjugate under $Aut(M_{Es})$.

Proof: (1) Since $V^+_{\sqrt{2}E_8}$ has a trivial weight one subspace, so does $M_{Es}$. By (1) of Proposition 3.4, an Ising vector $e$ of $V^+_{\sqrt{2}E_8}$ belongs to the commutant subalgebra $M_{Es} = \text{Com}_{V^+_{\sqrt{2}E_8}}(\text{Vir}(\tilde{\omega}))$ if and only if $\langle e, \tilde{\omega} \rangle = 0$. So the assertion follows from Lemma 4.8.

(2) It is clear that $w^-(\alpha)$, $\alpha \in \Phi(E_8)$, are mutually conjugate under $Aut(M_{Es})$ since $M_{Es}$ affords a natural action of the Weyl group $W(E_8)$. One can also verify that for each $\varphi_x \tilde{\omega} \in M_{Es}$, there is an Ising vector $w^-(\beta)$ in $M_{Es}$ such that $\langle \varphi_x \tilde{\omega}, w^-(\beta) \rangle = 1/32$. Therefore, all the Ising vectors are mutually conjugate by (2) of Proposition 3.4. □

Theorem 5.9. $Aut(M_{Es}) \simeq Sp_8(2)$.

Proof: Denote by $E$ the set of Ising vectors of $M_{Es}$ and set

$$\sigma(E) := \{ \sigma_e \in Aut(M_{Es}) \mid e \in E \}.$$ 

Let $G$ be the 3-transposition subgroup of $Aut(M_{Es})$ generated by $\sigma(E)$. It is shown in [KM, Ma] that $G$ is isomorphic to a simple group $Sp_8(2)$. Since $E$ is invariant under $Aut(M_{Es})$, $\sigma(E)$ is a normal set of $Aut(M_{Es})$ and hence $G$ is a normal subgroup of $Aut(M_{Es})$. It is shown in [ATLAS] that $Aut(G) = \text{Inn}(G) \simeq G$. So the kernel of the conjugate action of $Aut(M_{Es})$ on $G$ is $C_{\text{Aut}(M_{Es})}(G)$ and $Aut(M_{Es}) \simeq G \times C_{\text{Aut}(M_{Es})}(G)$. Let $g \in C_{\text{Aut}(M_{Es})}(G)$. Since $g$ commutes with $\sigma(E)$, $g$ acts on $E$ identically by Lemma 3.14. By Lemma 4.9, $g$ is also identical on the weight two subspace of $M_{Es}$. Since $M_{Es}$ is generated by $E$ as a VOA by Lemma 5.7, $g$ is trivial on $M_{Es}$. Thus $C_{\text{Aut}(M_{Es})}(G) = 1$ and $Aut(M_{Es}) = G \simeq Sp_8(2)$.

6 3-transposition group and inductive structure

We will study a relation between an inductive structure of a 3-transposition group acting on a vertex operator algebra and its commutant subalgebra. As an example, we study a commutant subalgebra of $V^+_{\sqrt{2}E_8}$ having $O_8^-(2)$ as its full automorphism group, which corresponds to an inductive structure $O_{10}^+(2)^{(2)} \simeq O_8^-(2)$. Note that $Aut(V^+_{\sqrt{2}E_8}) \simeq O_{10}^+(2)$ by [G2, S1].
### 6.1 3-transposition group

**Definition 6.1.** A (finite) 3-transposition group is a pair \((G, D)\) of a finite group \(G\) and a normal set \(D\) of involutions in \(G\) such that \(G\) is generated by \(D\) and if \(x, y \in D\) then the order of \(xy\) is either 1, 2 or 3.

A partial linear space consists of a set \(X\) called the set of points and a set \(L\) of subsets of \(X\) called the set of lines such that any two points lie on at most one line and any line has at least two points.

To a 3-transposition group \((G, D)\), we can associate a partial linear space called the Fischer space \((X, L)\) of \((G, D)\) as follows (cf. [Ma] and references therein). The set of points \(X\) is \(D\) and the set of lines \(L\) is such that a subset \(\ell \subset D\) is a line if and only if \(\ell\) consists of three points which generate a subgroup of \(G\) isomorphic to \(S_3\). For \(x, y \in D\), we write \(x \sim y\) if they lie on a line and \(x \perp y\) if not.

The collinearity graph of \((X, L)\) is a graph \(\Gamma\) whose vertex set is \(X\) and \(x, y \in X\) are adjacent if \(x \sim y\), i.e., they are incident to a common line.

A 3-transposition group \((G, D)\) is referred to as of symplectic type if the affine plane of order 3 does not occur in the associated Fischer space. It is called indecomposable if the associated Fischer space has the connected collinearity graph.

**Remark 6.2.** Let \(\ell_1\) and \(\ell_2\) be two distinct lines with a common point in a Fischer space. It is known that the subspace generated by \(\ell_1\) and \(\ell_2\) is either isomorphic to a dual affine plane of order 2 or an affine plane of order 3 (cf. [Asc]). Thus if \((G, D)\) is of symplectic type, only the dual affine plane of order 2 can occur in the associated Fischer space. In this case, the subgroup generated by the corresponding involutions in \(\ell_1\) and \(\ell_2\) will be isomorphic to the symmetry group \(S_4\).

Let \((G, D)\) be a 3-transposition group and \((X, L)\) the associated Fischer space. Define \(\sigma : X \to \text{Aut}(X)\) by \(\sigma_x(y) := y^x\) for \(x \in X\), and for \(x_1, x_2, \ldots \in X\), denote by

\[
X_{x_1, x_2, \ldots} := \{x \in X \mid x \perp x_i, \ i = 1, 2, \ldots\}.
\]

If \(G\) is indecomposable, \(\sigma(G)\) acts on \(X = D\) transitively. For, if \(x \sim y\) with \(x, y \in X\), then there exists \(z \in X\) such that \(x \sim z \sim y\), which implies \(\sigma_x(y) = \sigma_y(x) = z\). Thus \(x\) and \(y\) are conjugate under the action of \(\sigma(G)\). Repeating this, we can establish the transitivity.

Given an (indecomposable) 3-transposition group \((G, D)\), one can consider its inductive structure as follows. Let \(D^{(1)}\) be the set of elements of \(D\) which commute with a fixed element of \(D\) and \(D^{(2)}\) be that with two fixed non-commuting elements of \(D\). We set \(G^{(1)} = \langle D^{(1)} \rangle\) and \(G^{(2)} = \langle D^{(2)} \rangle\). The group structure \((G^{(i)}, D^{(i)})\) is called the inductive structure of \((G, D)\).
Proposition 6.3. Let \((G, D)\) be a centerfree indecomposable 3-transposition group of symplectic type and \((X, L)\) the associated Fischer space. Suppose that \(X_{x,y} \neq \emptyset\) for any distinct \(x, y \in X\). If \(C_{\text{Aut}(X)}(\sigma_x)|_{X_x} = C_G(\sigma_x)|_{X_x}\) for any \(x \in X\), then \(\text{Aut}(X) = G\).

**Proof:** Let \(g \in \text{Aut}(X)\). Take any \(x \in X = D\). By the transitivity, there exists \(\rho_1 \in G\) such that \(g(x) = \rho_1(x)\). Hence \(\rho_1^{-1} g|_{X_x} \in C_{\text{Aut}(X)}(\sigma_x)|_{X_x} = C_G(\sigma_x)|_{X_x}\). Then there exists \(\rho_2 \in G\) such that \(\rho_2^{-1} \rho_1^{-1} g = \text{id}\) on \(X_x \cup \{x\}\). Take \(y \in X\) such that \(y \sim x\). Set \(\rho = \rho_1 \rho_2\). We show that \(\rho^{-1} g\) stabilizes the set \(\{x, y, y^x\}\). Assume that there exists \(z \in X_{x,y} \setminus \{x, y, x^y\}\). We may assume that \(z \sim x\). Since \((X, L)\) is a Fischer space of symplectic type, the subspace \(\langle x, y, z \rangle\) generated by \(x, y\) and \(z\) is isomorphic to a dual affine plane of order 2 and the configuration in the Fischer space is as in Figure 1:

\[ \begin{array}{c}
  \text{Fig. 1} \\
  x \\
  y \quad z \\
 y^x = x^y \\
 (y^x)^z \\
 z^x = x^z \\
 \end{array} \]

Since \(X\) is indecomposable and \(X_{x,y} \neq \emptyset\), there exists \(w \in X_{x,y}\) such that \(w\) is collinear to a point \(p\) in \(\langle x, y, z \rangle\). By Fig. 1., \(p = x, x^z \) or \((y^x)^z\). Nevertheless, \(z \in X_{x,y}^\perp\). Hence we have \(p = x^z\) or \(p = (y^x)^z\). In either case, \(p \sim z\) and \(z^p = x\) or \(y^x\). Now consider the subspace generated by \(w, z\) and \(p\). Then we have the configuration as in Figure 2:

\[ \begin{array}{c}
  \text{Fig. 2} \\
  p \\
  w \\
  z \\
 w^p = p^w \\
 (w^p)^z \\
 z^p = p^z \\
 \end{array} \]

In this case, \(w \sim z^p \in \{x, y^x\}\) which contradicts the fact that \(w \in X_{x,y}\). Hence \(X_{x,y}^\perp = \{x, y, x^y\}\). Thus by replacing \(\rho\) by \(\rho \sigma_x\) if needed, we may assume that \(\rho^{-1} g(y) = y\). Then \(\rho^{-1} g\) acts as identity on \(X_x \cup \{x, y, y^x\}\). Now we claim that \(\rho^{-1} g = \text{id}\) on \(X\). Take any \(t \in X\) such that \(t \notin X_x \cup \{x, y, y^x\}\). Then \(\{x, y, x^y\}\) and \(\{x, t, x^t\}\) are two distinct lines. Since \((X, L)\) is of symplectic type, the Fischer subspace generated by \(\{x, y, t\}\) has the
configuration as in Fig. 3:

\[ y^x = x^y \]
\[ x^t = t^x \]

Fig. 3

Hence, there exists \( s \in X \cap \langle x, y, t \rangle \) such that \( \langle x, y, t \rangle = \langle x, y, s \rangle \simeq S_4 \) and \( t \in \langle x, y, s \rangle \). Since \( \rho^{-1} g \) acts trivially on \( \{ x, y, s \} \), it acts trivially on \( t \), also. Thus \( G = \operatorname{Aut}(X) \).

\[ \square \]

### 6.2 Automorphism group of a commutant subalgebra

We consider a correspondence between an inductive structure of a 3-transposition group and a commutant subalgebra structure of a vertex operator algebra on which the group acts. We study an inductive structure \( O_{10}^+(2) \simeq O_{8}^-(2) \). It is shown in [G2, S1] that \( \operatorname{Aut}(V_{\sqrt{2}E_8}^+) \simeq O_{10}^+(2) \). By this inductive structure, we can find a commutant subalgebra of \( V_{\sqrt{2}E_8}^+ \) on which \( O_{8}^-(2) \) acts. We will show that in this case the inductive structure completely determines the full automorphism group of the commutant subalgebra.

Take a root \( \alpha_0 \in \Phi(E_8) \) and set \( L_1 := \mathbb{Z}\alpha_0 \) and \( L_2 := (\mathbb{Z}\alpha_0)^\perp = \{ \beta \in E_8 \mid \langle \alpha_0, \beta \rangle = 0 \} \). Then \( L_1 \) and \( L_2 \) are sublattices of \( E_8 \) isometric to \( A_1 \) and \( E_7 \), respectively. Let \( U \) be the subalgebra of \( V_{\sqrt{2}E_8}^+ \) generated by \( \tilde{\omega} \) and \( \varphi_{\alpha_0}\tilde{\omega} \). It is shown in [M1, LY1] that \( U \) is isomorphic to a simple VOA of the form

\[
L\left(\frac{1}{2},0\right) \otimes L\left(\frac{7}{10},0\right) \oplus L\left(\frac{1}{2},\frac{1}{2}\right) \otimes L\left(\frac{7}{10},\frac{3}{2}\right).
\]

By a direct computation, one has

\[
\tilde{\omega}(1)\varphi_{\alpha_0}\tilde{\omega} = \frac{1}{4} \left( \tilde{\omega} + \varphi_{\alpha_0}\tilde{\omega} - w^+(\alpha_0) \right)
\]

so that \( \sigma_{\tilde{\omega}}\varphi_{\alpha_0}\tilde{\omega} = \sigma_{\varphi_{\alpha_0}\tilde{\omega}}\tilde{\omega} = w^+(\alpha_0) \) (cf. Proposition 3.4). Therefore, \( U \) contains exactly three Ising vectors \( \tilde{\omega}, \varphi_{\alpha_0}\tilde{\omega} \) and \( w^+(\alpha_0) \). Set

\[
U^c = \operatorname{Com}_{V_{\sqrt{2}E_8}}(U).
\]

We shall show that \( U^c \) is again generated by a set of Ising vectors and its full automorphism group \( \operatorname{Aut}(U^c) \) is isomorphic to \( O_{8}^-(2) \).

**Proposition 6.4.** There are exactly 136 Ising vectors in the commutant subalgebra \( U^c \) and they are mutually conjugate under the action of \( \operatorname{Aut}(U^c) \).
Proof. Since $U$ is generated by $\tilde{\omega}$ and $\varphi_{\alpha_0}\omega$, we have

$$U^c = \text{Com}_{\sqrt{2E_8}}(\text{Vir}(\omega)) \cap \text{Com}_{\sqrt{2E_8}}(\text{Vir}(\varphi_{\alpha_0}\tilde{\omega})).$$

By Proposition 4.3 and Lemma 4.8, if $e$ is an Ising vector in $U^c$, then either

1. $e = w^-(\beta)$ for some $\beta$ with $\langle \alpha_0, \beta \rangle = 0, \pm 2$ or
2. $e = \varphi_x(\tilde{\omega})$, $x + 2E_8$ is represented by a norm 4 vector and $\langle \alpha_0, x \rangle \equiv 1 \mod 2$.

Note that the lattice $L_2 = (\mathbb{Z}\alpha_0)^\perp$ is isomorphic to $E_7$, which has 63 positive roots. Moreover, $\langle \alpha_0, \beta \rangle = \pm 2$ if and only if $\beta = \pm \alpha_0$. Hence, there are exactly $1 + 63 = 64$ Ising vectors of the form $w^-(\beta)$ in the case (1).

For the case (2), there are exactly 128 cosets $x + 2E_8$ of $E_8/2E_8$ such that $\langle \alpha_0, x \rangle = 1 \mod 2$. Among them, $120 - 64 = 56$ classes are represented by roots. Hence there are 72 Ising vectors of the form $\varphi_x(\tilde{\omega})$ in $U^c$. Therefore, there are totally $64 + 72 = 136$ Ising vectors in $U^c$.

Next we shall show all Ising vectors are mutually conjugate in $U^c$. First we shall note that the Weyl group $W(L_2) \simeq W(E_7)$ is naturally a subgroup of $\text{Aut}(U^c)$ since both $\tilde{\omega}$ and $w^+(\alpha_0)$ are fixed by $W(L_2)$. Hence, it is clear that $w^-(\beta)$ and $w^-(\gamma)$ are conjugate if $\beta, \gamma \in L_2$. Let $e = \varphi_x(\tilde{\omega})$ be an Ising vector of $U^c$. Then $x + 2E_8$ is represented a norm 4 vector and $\langle \alpha_0, x \rangle \equiv 1 \mod 2$. Since $E_8$ is generated by its roots, we may set $x = \beta_1 + \beta_2$ such that $\beta_1$ and $\beta_2$ are roots in $E_8$ and $\langle \beta_1, \beta_2 \rangle = 0$, $\langle \alpha_0, \beta_1 \rangle = 1$ and $\langle \alpha_0, \beta_2 \rangle = 0$. In this case, $\beta_2 \in (\mathbb{Z}\alpha_0 + \mathbb{Z}\beta_1)^\perp \simeq E_6$ and hence there exists a root $\gamma \in (\mathbb{Z}\alpha_0 + \mathbb{Z}\beta_1)^\perp$ such that $\langle \beta_1, \gamma \rangle = \langle \alpha_0, \gamma \rangle = 0$ and $\langle \beta_2, \gamma \rangle = 1$. Therefore, we have $\langle \varphi_x(\tilde{\omega}), w^-(\gamma) \rangle = 1/32$ and thus $\varphi_x(\tilde{\omega})$ is conjugate to $w^-(\gamma)$ by (2) of Proposition 3.4.

Finally, for any $e = \varphi_x(\tilde{\omega}) \in U^c$, we have $\langle \alpha_0, x \rangle \equiv 1 \mod 2$. Hence, we have $\langle \varphi_x(\tilde{\omega}), w^-(\alpha_0) \rangle = 1/32$ and $\varphi_x(\tilde{\omega})$ is conjugate to $w^-(\alpha_0)$ also.

Lemma 6.5. $U^c$ is generated by its Ising vectors.

Proof. Denote

$$L = \sqrt{2}(L_1 \oplus L_2) \simeq \sqrt{2}A_1 \oplus \sqrt{2}E_7.$$

Since $|\sqrt{2}E_8/L| = 2$, there is $\gamma \in \sqrt{2}E_8$ such that $\sqrt{2}E_8 = L \cup (\gamma + L)$ and hence

$$V_{\sqrt{2}E_8} = V_L \oplus V_{\gamma + L} \quad \text{and} \quad V_{\sqrt{2}E_8}^+ = V_L^+ \oplus V_{\gamma + L}^+.$$

Note that the quotient group structure $\sqrt{2}E_8/L$ induces an automorphism $\rho \in \text{Aut}(V_{\sqrt{2}E_8}^+)$ such that $\rho|_{V_L^+} = 1$ and $\rho|_{V_{\gamma + L}^+} = -1$.

By definition (cf. (2.3)), it is easy to show

$$\tilde{\omega}_{L_2}(-\beta) = 4\alpha(\tilde{\omega} + \varphi_{\alpha_0}\tilde{\omega}) - \frac{1}{5}w^+(\alpha_0).$$

24
Therefore, \( \tilde{\omega}_{L_2} \in U \) and the Virasoro element of \( U \) is an orthogonal sum of \( w^+(\alpha_0) \) and \( \tilde{\omega}_{L_2} \). Hence the commutant subalgebra \( U_c \) can be defined as follows.

\[
U_c = \{ v \in V_{\sqrt{2}E_8}^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0 \}.
\]

Let

\[
M^0 = U_c \cap V_{L_2}^+ = \{ v \in V_{L_2}^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0 \},
\]

\[
M^1 = U_c \cap V_{\gamma + L_2}^+ = \{ v \in V_{\gamma + L_2}^+ \mid (\tilde{\omega}_{L_2})_{(1)}v = w^+(\alpha_0)_{(1)}v = 0 \}.
\]

Then we have \( U_c = M^0 \oplus M^1 \). Moreover, the automorphism \( \rho \) induces a natural action on \( U_c \) such that \( \rho|_{M^0} = 1 \) and \( \rho|_{M^1} = -1 \). Note that

\[
M^0 \simeq L(1/2,0) \otimes M_{E_7},
\]

and \( M^1 \) is an irreducible \( M^0 \)-module in this case (cf. [DM]).

Now by Lemma 4.8 and Proposition 5.1, we know that \( M_{E_7} \) is generated by the Ising vectors of the form \( w^-(\beta), \beta \in L_2 \simeq E_7 \). Hence, \( M^0 \simeq L(1/2,0) \otimes M_{E_7} \) is generated by \( w^-(\alpha_0) \) and \( \{ w^-(\beta) \mid \beta \in L_2 \simeq E_7 \} \).

Let \( W \) be the sub VOA generated by the set of all Ising vectors of \( U_c \). Then \( M^0 \subset W \). Since \( W \) also contains Ising vectors of the form \( \varphi_+(\tilde{\omega}) \) which is not contained in \( M^0 \) (cf. Lemma 4.8), we know that \( W \neq M^0 \). Hence, \( W = M^0 \oplus M^1 = U_c \) as desired.

**Theorem 6.6.** \( \text{Aut}(U^c) \simeq O_{8}^{-}(2) \).

**Proof:** Let \( E \) be the set of Ising vectors of \( U^c \) and \( G \) the 3-transposition subgroup of \( \text{Aut}(U^c) \) generated by involutions \( \{ \sigma_e \mid e \in E \} \). It is shown in [Ma] that \( G \simeq O_{8}^{-}(2) \). By the proof of the previous lemma, we have

\[
\tilde{\omega}_{L_2} = \frac{4}{5}(\tilde{\omega} + \varphi_{\alpha_0} \tilde{\omega}) - \frac{1}{5} w^+(\alpha_0) \in U
\]

and \( \text{Com}_{U^c}(\text{Vir}(w^-(\alpha_0))) \simeq M_{E_7} \). By Theorem 5.4, \( \text{Aut}(M_{E_7}) \) is isomorphic to \( \text{Sp}_6(2) \) which is generated by \( \sigma \)-type involutions associated to Ising vectors of \( M_{E_7} \). Since the 3-transposition subgroup of \( G \) generated by \( \sigma(E)_{\sigma(w^-(\alpha))} = \{ \sigma_{w^-(\beta)} \mid \beta \in L_2 \} \) is isomorphic to \( \text{Sp}_6(2) \), we can apply Proposition 6.3 to \( G \) and we conclude that \( \text{Aut}(U^c) = G \simeq O_{8}^{-}(2) \) as \( U^c \) is generated by \( E \).

**7 Decomposition of \( M_R \)**

Next, we will complete the proof of Proposition 5.1. Since it requires the notion of \( W \)-algebras, we will first review some basic facts about \( W \)-algebras.
7.1 Modules over $W$-algebra

Let $\Lambda_0$ and $\Lambda_1$ be the fundamental weights of the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$. For positive integers $\ell, j$ with $0 \leq j \leq \ell$, consider the irreducible highest weight module $\mathcal{L}(\ell, j)$ over $\hat{\mathfrak{sl}}_2(\mathbb{C})$ with highest weight $(\ell - j)\Lambda_0 + j\Lambda_1$. It is well-known that $\mathcal{L}(\ell, 0)$ forms a simple VOA and the integrable $\hat{\mathfrak{sl}}_2(\mathbb{C})$-modules $\mathcal{L}(\ell, j)$, $0 \leq j \leq \ell$, provide all the inequivalent irreducible $\mathcal{L}(\ell, 0)$-modules (cf. [FZ]).

Now let $A^{\oplus \ell}_1 = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_\ell$ be an even lattice with $(\epsilon_i, \epsilon_j) = 2\delta_{i,j}$ and $V_{A^{\oplus \ell}_1}$ the lattice VOA associated with $A^{\oplus \ell}_1$. Then $V_{A^{\oplus \ell}_1} \cong (V_{A_1})^{\otimes \ell} \cong \mathcal{L}(1, 0)^{\otimes \ell}$. Recall that the weight one subspace of $V_{A^{\oplus \ell}_1}$ forms a Lie algebra by the Lie bracket $[x, y] := x(0)y$ for $x, y \in (V_{A^{\oplus \ell}_1})_1$.

Set $H^{(\ell)} = (\epsilon_1 + \cdots + \epsilon_\ell)(-1)1$, $E^{(\ell)} = e^{\epsilon_1} + \cdots + e^{\epsilon_\ell}$ and $F^{(\ell)} = e^{-\epsilon_1} + \cdots + e^{-\epsilon_\ell}$. Then the subspace $\mathbb{C}H^{(\ell)} + \mathbb{C}E^{(\ell)} + \mathbb{C}F^{(\ell)}$ of the weight one subspace of $V_{A^{\oplus \ell}_1}$ forms a simple Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and the sub VOA generated by $\{H^{(\ell)}, E^{(\ell)}, F^{(\ell)}\}$ is isomorphic to level $\ell$ affine VOA $\mathcal{L}(\ell, 0)$ (cf. [DL]).

Let $\gamma = \epsilon_1 + \cdots + \epsilon_\ell \in A^{\oplus \ell}_1$. Then $\gamma(-1)1 = H^{(\ell)}$ and it is easy to verify that

$$e^\gamma = \frac{1}{\ell!}(E^{(\ell)}(-1))^{\ell-1}E^{(\ell)}.$$

Thus $\mathcal{L}(\ell, 0)$ contains a subalgebra isomorphic to the lattice VOA $V_{\mathbb{Z}\gamma}$. We consider the following commutant subalgebra:

$$W_\ell := \text{Com}_{\mathcal{L}(\ell, 0)}(V_{\mathbb{Z}\gamma}). \quad (7.1)$$

Note that the central of $W_\ell$ is equal to $3\ell/(\ell + 2) - 1 = 2(\ell - 1)/(\ell + 2)$. It is clear that $\mathcal{L}(\ell, 0)$ contains a full sub VOA isomorphic to $V_{\mathbb{Z}\gamma} \otimes W_\ell$ so that we can regard $\mathcal{L}(\ell, j)$ as a $V_{\mathbb{Z}\gamma} \otimes W_\ell$-module. Following [DL, Li], we introduce the following spaces:

$$W_\ell(j, k) := \text{Hom}_{V_{\mathbb{Z}\gamma}} \left( V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}, \mathcal{L}(\ell, j) \right), \quad (7.2)$$

where $0 \leq j \leq \ell$ and $0 \leq k < 2\ell$. Then $W_\ell(j, k)$ denotes the space of multiplicity of $V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}$ in $\mathcal{L}(\ell, j)$. Thus, viewing $\mathcal{L}(\ell, j)$ as a $V_{\mathbb{Z}\gamma} \otimes W_\ell$-module, we have the following decomposition:

$$\mathcal{L}(\ell, j) = \bigoplus_{k=0}^{2\ell-1} V_{(k/2\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, k).$$

It is shown in [DL] that $W_\ell(j, k) = 0$ if $j + k \equiv 1 \pmod{2}$, and so

$$\mathcal{L}(\ell, j) = \begin{cases} \bigoplus_{k=0}^{\ell-1} V_{(k/\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, 2k), & \text{if } j \text{ is even}, \\ \bigoplus_{k=0}^{\ell-1} V_{((2k+1)/2\ell)\gamma + \mathbb{Z}\gamma} \otimes W_\ell(j, 2k+1), & \text{if } j \text{ is odd}. \end{cases} \quad (7.3)$$

The following basic fact is due to [DL] (see also [Li]).
Proposition 7.1. ([DL, Li])

(1) All $W_\ell(j, k)$, $0 \leq j \leq \ell$, $0 \leq k \leq 2\ell - 1$, $j \equiv k \mod 2$, are irreducible $W_\ell$-modules.

(2) As $W_\ell$-modules, $W_\ell(j_1, k_1) \simeq W_\ell(j_2, k_2)$ if $j_1 + j_2 = \ell$ and $k_2 \equiv k_1 + \ell \mod 2\ell$.

We will use $W_\ell$-modules $W_\ell(j, k)$ to study $M_{E_\ell}$ and $M_{E_7}$. In [LYY2], they recursively computed the vacuum characters of these modules. In the below we shall use the vacuum characters of $W_\ell$-modules obtained in (loc. cit.) without any comments.

7.2 $V_{\sqrt{2}A_N}$ and $W_{N+1}$-algebra

Let us consider a lattice VOA $V_{\sqrt{2}A_N}$. Let $s_{A_N}$ and $\tilde{\omega}_{A_N}$ be conformal vectors of $V_{\sqrt{2}A_N}$ defined as in (2.2) and (2.3), respectively. Then one has an orthogonal decomposition $\omega = s_{A_N} + \tilde{\omega}_{A_N}$ of the Virasoro vector $\omega$ of $V_{\sqrt{2}A_N}$. We use the following result established in [LY2].

Lemma 7.2. ([LY2]) There is a VOA-isomorphism $\text{Com}_{V_{\sqrt{2}A_N}}(M_{A_N}) \simeq W_{N+1}$.

Due to the lemma above, we shall identify the commutant subalgebra $\text{Com}_{V_{\sqrt{2}A_N}}(M_{A_N})$ with $W_{N+1}$. By the orthogonality, $(W_{N+1}, \tilde{\omega}_{A_N})$ forms a sub VOA of $V_{\sqrt{2}A_N}$. Therefore, $V_{\sqrt{2}A_N}$ contains a full subalgebra isomorphic to $M_{A_N} \otimes W_{N+1}$. It is shown in [DLMN] that $M_{A_N}$ contains a full sub VOA isomorphic to $L(c_1, 0) \otimes \cdots \otimes L(c_N, 0)$, where $c_m$ denotes the central charge of the unitary series of the Virasoro algebra:

$$c_m := 1 - \frac{6}{(m + 2)(m + 3)}, \quad m = 1, 2, \ldots.$$  (7.4)

By [W], the irreducible modules over $L(c_m, 0)$ are given by the irreducible highest weight modules $L(c_m, h^m_{r,s})$ whose highest weights are parameterized as follows.

$$h^m_{r,s} := \frac{[r(m + 3) - s(m + 2)]^2 - 1}{4(m + 2)(m + 3)}, \quad 1 \leq r \leq m + 1, \quad 1 \leq s \leq m + 2. \quad (7.5)$$

In [LY2], the following decomposition of $V_{\sqrt{2}A_N}$ as an $L(c_1, 0) \otimes \cdots \otimes L(c_N, 0) \otimes W_{N+1}$-module is obtained.

$$V_{\sqrt{2}A_N} = \bigoplus_{0 \leq k_j \leq j + 1 \atop j = 0, \ldots, N \atop k_j \equiv 0 \mod 2} L(c_1, h_{k_0,1,k_1+1}) \otimes \cdots \otimes L(c_N, h_{k_{N-1},1,k_{N+1}}) \otimes W_{N+1}(k_N, 0). \quad (7.6)$$

To describe $V_{\sqrt{2}A_N}$ as an $M_{A_N} \otimes W_{N+1}$-module, we introduce the following notation.

$$M_{A_N}(2s) := \bigoplus_{0 \leq k_j \leq j + 1 \atop j = 0, \ldots, N - 1 \atop k_j \equiv 0 \mod 2} L(c_1, h_{k_0,1,k_1+1}) \otimes \cdots \otimes L(c_N, h_{k_{N-1},1,2s+1}). \quad (7.7)$$

By (7.6), we have

$$V_{\sqrt{2}A_N} = \bigoplus_{0 \leq 2s \leq N+1} M_{A_N}(2s) \otimes W_{N+1}(2s, 0) \quad (7.8)$$

and it is shown in [LS] that $M_{A_N}(2s)$, $0 \leq 2s \leq N + 1$, are inequivalent irreducible $M_{A_N}$-modules. By construction, the vacuum characters of $M_{A_N}(2s)$ are obvious.
7.3 Decomposition of $M_{E_7}$

Let us recall that the root lattice of type $E_7$ can be written as

$$E_7 = \left\{ (x_1, \ldots, x_8) \in \mathbb{Q}^8 \mid \text{all } x_i \text{ are in } \mathbb{Z} \text{ or all } x_i \text{ are in } \frac{1}{2} + \mathbb{Z}, \quad x_1 + x_2 + \cdots + x_7 + x_8 = 0 \right\}.$$ 

Let $\epsilon_i$ be the vector of $\mathbb{Q}^8$ such that the $i$-th entry is 1 and all the other entries are zero, and set

$$N := \text{Span}_{\mathbb{Z}}\{-\epsilon_1 + \epsilon_2, \ldots, -\epsilon_7 + \epsilon_8\}.$$ 

Then $N$ is a root lattice of type $A_7$. Let $\xi = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in \mathbb{Q}^8$. Then the root lattice of type $E_7$ can be written as $E_7 = N \cup (\xi + N)$.

Let $s_{A_7}$ and $\tilde{\omega}_{A_7} = \omega - s_{A_7}$ be conformal vectors defined as in (2.2) and (2.3), respectively. Then $W_8$ is isomorphic to the parafermion algebra of central charge $7/5$ (cf. [LY2, ZF]). As we have seen, $V_{\sqrt{2}N}$ contains a full sub VOA isomorphic to $M_{A_7} \otimes W_8$.

The following decomposition is obtained in [LY2, LS].

**Lemma 7.3.** ([LY2, LS]) As a module over $M_{A_7} \otimes W_8$,

$$V_{\sqrt{2}E_7} \cong \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes W_8(2s, 8).$$

Hence, we have

$$V_{\sqrt{2}E_7} = V_{\sqrt{2}N} \oplus V_{\sqrt{2}E_7} = \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes (W_8(2s, 0) \oplus W_8(2s, 8)).$$

Let $\tilde{\omega}'_{E_7} = \tilde{\omega}_{A_7} - \tilde{\omega}_{E_7}$. Then $\tilde{\omega}_{E_7}$ and $\tilde{\omega}'_{E_7}$ are mutually orthogonal conformal vectors with central charge $7/10$. Denote

$$U := \text{Com}_{V_{\sqrt{2}E_7}}(\text{Vir}(s_{A_7})) \cong W_8(0, 0) \oplus W_8(0, 8).$$

Then $U$ contains $\tilde{\omega}_{E_7}$ and $\tilde{\omega}'_{E_7}$ so that $\text{Vir}(\tilde{\omega}'_{E_7}) \otimes \text{Vir}(\tilde{\omega}_{E_7})$ is a full sub VOA of $U$ isomorphic to $L(7/10, 0) \otimes L(7/10, 0)$. By the vacuum characters, it is easy to show that

$$U \cong L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \oplus L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right)$$

as a $\text{Vir}(\tilde{\omega}'_{E_7}) \otimes \text{Vir}(\tilde{\omega}_{E_7})$-module. Moreover, we have

**Lemma 7.4.** The lattice VOA $V_{\sqrt{2}E_7}$ can be decomposed as follows:

$$V_{\sqrt{2}E_7} \cong \bigoplus_{0 \leq 2s \leq 8} M_{A_7}(2s) \otimes U(2s),$$

28
where

\[ U(0) \simeq U(8) \simeq L(\frac{7}{10}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{3}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}), \]
\[ U(2) \simeq U(6) \simeq L(\frac{7}{10}, \frac{3}{5}) \otimes L(\frac{7}{10}, \frac{3}{5}) \oplus L(\frac{7}{10}, \frac{1}{10}) \otimes L(\frac{7}{10}, \frac{1}{10}), \]
\[ U(4) \simeq L(\frac{7}{10}, 0) \otimes L(\frac{7}{10}, \frac{3}{5}) \oplus L(\frac{7}{10}, \frac{3}{2}) \otimes L(\frac{7}{10}, \frac{1}{10}) \]
\[ \oplus L(\frac{7}{10}, \frac{3}{5}) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{1}{10}) \otimes L(\frac{7}{10}, \frac{3}{2}). \]

Since \( \omega = s_{A_7} + \bar{\omega}_{E_7} + \bar{\omega}_{E_7} \) is an orthogonal sum, we have the following isomorphism as \( M_{A_7} \otimes \text{Vir}(\bar{\omega}_{E_7}) \)-modules:

\[ M_{E_7} \simeq M_{A_7}(0) \otimes L(\frac{7}{10}, 0) \oplus M_{A_7}(4) \otimes L(\frac{7}{10}, \frac{3}{5}) \oplus M_{A_7}(8) \otimes L(\frac{7}{10}, 0). \]

**Proposition 7.5.** \( M_{E_7} \) is generated by its weight 2 subspace.

**Proof:** It is shown in Proposition 5.6 of [LS] that \( M_{A_7} \) is generated by its weight two subspace as a VOA. Thus so is \( M_{A_7}(0) \otimes L(\frac{7}{10}, 0) \). Recall that \( M_{A_7}(2s) \simeq \bigoplus_{0 \leq k_1 \leq s+1, \atop k_j \equiv 0 \mod 2}^{j=0,\ldots,7} L(c_1, h_{k_0+1,k_1+1}^1) \otimes \cdots \otimes L(c_7, h_{k_6+1,2s+1}^7). \)

Take \((2k_0+1, 2k_1+1, \ldots, 2k_6+1, 2s+1) = (1, 3, 3, 5, 5, 7, 7, 9)\). Then we obtain a highest weight vector of weight

\[ (\frac{1}{2}, \frac{1}{10}, \frac{2}{5}, \frac{1}{7}, \frac{1}{14}, \frac{1}{6}, \frac{1}{3}) \]

in \( M_{A_7}(8) \). Similarly, by taking \((2k_0+1, 2k_1+1, \ldots, 2k_6+1, 2s+1) = (1, 1, 1, 1, 1, 3, 5, 5)\), \( M_{A_7}(4) \) contains a highest weight vector of weight

\[ (0, 0, 0, 0, \frac{3}{4}, \frac{7}{12}, \frac{1}{15}). \]

Therefore, both \( M_{A_7}(4) \otimes L(\frac{7}{10}, \frac{3}{5}) \) and \( M_{A_7}(8) \otimes L(\frac{7}{10}, 0) \) contain weight 2 elements. Since \( M_{A_7}(4) \otimes L(\frac{7}{10}, \frac{3}{5}) \) and \( M_{A_7}(8) \otimes L(\frac{7}{10}, 0) \) are irreducible \( M_{A_7} \otimes L(\frac{7}{10}, 0) \)-modules (cf. [LS]), \( M_{E_7} \) is generated by its weight 2 subspace.

**7.4 Decomposition of \( M_{E_6} \)**

Let us recall that

\[ E_6 = \left\{ (x_1, \ldots, x_8) \in \mathbb{Q}^8 \mid \text{all } x_i \text{ are in } \mathbb{Z} \text{ or all } x_i \text{ are in } \frac{1}{2} + \mathbb{Z}, \text{ and } x_1 + x_8 = x_2 + \cdots + x_7 = 0 \right\}. \]

Define

\[ L_1 = \left\{ (0; x_2, \ldots, x_7; 0) \in \mathbb{Z}^8 \mid x_2 + \cdots + x_7 = 0 \right\}, \]
\[ L_2 = \left\{ (x_1; 0, \ldots, 0; x_8) \in \mathbb{Z}^8 \mid x_1 + x_8 = 0 \right\}. \]
Then $L_1 \simeq A_5$, $L_2 \simeq A_1$ and it gives an embedding of $A_5 \oplus A_1$ into $E_6$. Set

$$\xi = \left(\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; -\frac{1}{2}; -\frac{1}{2}\right)$$

and $L = L_1 \oplus L_2$.

Then we have $E_6 = L \cup (\xi + L)$ and $V_{\sqrt{\mathcal{T}}E_6} = V_{\sqrt{\mathcal{T}}L} \oplus V_{\sqrt{\mathcal{T}}\xi + \sqrt{\mathcal{T}}L}$. Note that

$$V_{\sqrt{\mathcal{T}}L} \simeq V_{\sqrt{\mathcal{T}A_5}} \otimes V_{\sqrt{\mathcal{T}A_1}} \quad \text{and} \quad V_{\sqrt{\mathcal{T}\xi + \sqrt{\mathcal{T}}L}} \simeq V_{\sqrt{\mathcal{T}\xi_{1} + \sqrt{\mathcal{T}A_5}}} \otimes V_{\sqrt{\mathcal{T}\xi_{2} + \sqrt{\mathcal{T}A_1}}},$$

where we have set $\xi_1 = (0; 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 0)$ and $\xi_2 = (1/2; 0, 0, 0, 0, 0, 0, -1/2)$.

Define $s_{A_5}, \tilde{\omega}_{A_5} \in V_{L_1}$ and $w^+(2\xi_5) \in V_{L_2}$ as in (2.2), (2.3) and (4.4), respectively, and we identify $(M_{A_5}, s_{A_5})$ as a sub VOA of $V_{L_1} \simeq V_{\sqrt{\mathcal{T}A_5}}$. Set

$$\omega^1 := \tilde{\omega}_{A_5} + w^+(2\xi_2) - \tilde{\omega}_{E_6} \in V_{\sqrt{\mathcal{T}E_6}} \quad \text{and} \quad \omega^2 := w^-(2\xi_2) \in V_{L_2}. \tag{7.9}$$

Then $\omega^1$ and $\omega^2$ are conformal vectors of $V_{\sqrt{\mathcal{T}E_6}}$ with central charges $25/28$ and $1/2$, respectively. We also note that $\text{Vir}(\omega^1) \simeq L(25/28, 0)$ and $\text{Vir}(\omega^2) \simeq L(1/2, 0)$. One can directly check that $\omega = s_{A_5} + \omega^1 + \omega^2 + \tilde{\omega}_{E_6}$ is an orthogonal sum (cf. Lemma 2.1). Therefore, $M_{\sqrt{\mathcal{T}E_6}}$ contains a full sub VOA $M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \simeq M_{A_5} \otimes L(25/28, 0) \otimes L(1/2, 0)$. By a similar method as in [LS, LY2], one can establish the following.

**Lemma 7.6.** As a module over $M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$,

$$M_{E_6} \simeq M_{A_5}(0) \oplus M_{A_5}(2) \oplus M_{A_5}(4) \oplus M_{A_5}(6)$$

$$\oplus \left\{ L\left(\frac{25}{28}, 0\right) \otimes L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{25}{28}, \frac{15}{2}\right) \otimes L\left(\frac{1}{2}, \frac{1}{2}\right) \right\},$$

where

$$M_{A_5}(s) = \bigoplus_{m=0,2,4} M_{A_4}(m) \otimes L\left(\frac{25}{28}, \frac{b_m^5}{h_{m+1,s+1}}\right). \tag{7.10}$$

Denote the conformal vectors of $M_{A_4}$ and $\text{Com}_{M_{A_5}}(M_{A_4})$ by $u$ and $v$, respectively. Then $\tilde{\omega}_{A_5} = u + v$ is an orthogonal sum and we have the following sequence of the full sub VOAs of $V_{\sqrt{\mathcal{T}E_6}}$:

$$M_{A_4} \otimes \text{Vir}(v) \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \subset M_{A_5} \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \subset M_{E_6}.$$
Lemma 7.7. As a module over $\text{Vir}(v) \otimes \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$,

$$\text{Com}_{M_{E_6}}(M_{A_4}) \simeq L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{25}{28}, \frac{3}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{1}{2}, 0)$$

$$\oplus L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{3}{4}) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{25}{28}, \frac{15}{2}) \otimes L(\frac{25}{28}, \frac{15}{2}) \otimes L(\frac{1}{2}, 0)$$

$$\oplus L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, \frac{15}{2}) \otimes L(\frac{1}{2}, 2) \oplus L(\frac{25}{28}, \frac{3}{4}) \otimes L(\frac{25}{28}, \frac{3}{4}) \otimes L(\frac{1}{2}, 2)$$

$$\oplus L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{1}{2}, 2) \oplus L(\frac{25}{28}, \frac{15}{2}) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{1}{2}, 2).$$

The key observation is that $\text{Com}_{M_{E_6}}(M_{A_4})$ is isomorphic to a $\mathbb{Z}_2$-orbifold subalgebra of the $5A$-algebra considered in [LYY2]. We refer the proof of the following fact to [LYY2].

Lemma 7.8. ([LYY2]) $\text{Com}_{M_{E_6}}(M_{A_4})$ is generated by its weight two subspace.

Now we can show the following.

Proposition 7.9. $M_{E_6}$ is generated by its weight two subspace.

Proof: Let $U$ be the sub VOA of $M_{E_6}$ generated by the weight two subspace of $M_{E_6}$. Since $M_{A_4}$ is generated by its weight two subspace (cf. Proposition 5.6 of [LS]), $U$ contains $M_{A_5}$. It is also shown in [LS] that all $M_{A_5}(s), s = 0, 2, 4, 6$, are irreducible $M_{A_5}$-submodules of $M_{E_6}$. On the other hand, thanks to Lemma 7.8, $U$ also contains all the highest weight vectors for $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$ which appear in the decomposition in Lemma 7.6. Therefore, $U$ contains all the components given in Lemma 7.6 and hence $U = M_{E_6}$. □

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32
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