Decreasing circumference for increasing radius in axially symmetric gravitating systems

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Abstract

Apart from the flat space with an angular deficit, Einstein general relativity possesses another cylindrically symmetric solution. Because this configuration displays circles whose “circumferences” tend to zero when their “radius” go to infinity, it has not received much attention in the past. We propose a geometric interpretation of this feature and find that it implies field boundary conditions different from the ones found in the literature if one considers a source consisting of the scalar and the vector fields of a $U(1)$ system. To obtain a non-increasing energy density the gauge symmetry must be unbroken. For the Higgs potential this is achieved only with a vanishing vacuum expectation value but then the solution has a null scalar field. A non-trivial scalar behaviour is exhibited for a potential of sixth order. The trajectories of test particles in this geometry are studied, its causal structure discussed. We find that this bosonic background can support a normalizable fermionic condensate but not such a current.

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I Introduction

Among the most important characteristics of cosmic strings is the existence of a symmetry axis and the concentration of energy around this axis [1]. One can ask if cosmic strings are the only configurations displaying this feature. Taking into account gravity, the existence of a symmetry axis implies cylindrical symmetry for the metric as well. All static cylindrically symmetric solutions of Einstein equations in vacuum are of the Kasner type. The vanishing of the energy-momentum tensor in the “asymptotic” region of any physically reasonable solution implies that the geometry should approach a Kasner line there. As the energy momentum tensor corresponding to these axial configurations implies the invariance of the metric under boosts, one is left with only two Kasner geometries: a flat space presenting a conical singularity and a Melvin like space [2]. The first case leads to the well known cosmic strings whose possible role in cosmology ranges from structure formation to baryogenesis. The second solution has received less attention because it seems to present a singular behaviour in the metrical sector: circles of increasing “radius” display decreasing circumferences. Recent investigations suggest that for an Abelian-Higgs system with a symmetry breaking scale much smaller than the Planck scale, asymptotically Melvin-like solution coexist with the habitual string configuration as long as the angular deficit of the latter does not exceed $2\pi$ [3].

The inertial and Tolman masses of the two solutions have been contrasted [4]. However, the unusual behaviour of the metric has not been addressed. The key point of this article is the interpretation of this unusual geometry. We shall argue that the coordinate which has been considered as the radius in the second special Kasner-line element (i.e. the Melvin-like geometry) is a rather complicated function of the “true” radius. To support our proposal, we shall invoke the coordinatization of the sphere by polar stereographic variables in the first section. This example, although elementary, strongly suggests that letting $r$ go to infinity in the second special Kasner line element, one enters an asymptotic region which shrinks to the symmetry axis. In the third section, an Abelian Lagrangian is coupled to the Einstein-Hilbert one. Looking for an axially symmetric configuration different of the string, the preceding section imposes the vanishing of the vector and the scalar field as the coordinate $r$, formerly interpreted as a radius, goes to infinity. This is at odds with previous treatments and results in a divergent inert mass per unit length if the v.e.v of the Higgs field does not vanish for example. In the fourth section
we find that the trajectories of massive particles are bounded from above in this space time. We also study massless particle trajectories and discuss the causal structure of the new solution. Due to the cylindrical symmetry, we propose a Penrose-like diagram, but with each point representing a two geometry conformal to a cylinder. In the fifth section, we study the wave function which describes a condensate or a current of a fermion charged under the Abelian field considered.

II The second special Kasner line element.

The metric
\[ ds^2 = c r^{4/3}(dt^2 - dz^2) - dr^2 - d r^{-2/3}d\theta^2 \] (2.1)
where \( c \) and \( d \) are positive constants is what we call the second Kasner line element. It has been shown to satisfy Einstein equations in the vacuum \( [2] \); it displays a cylindrical symmetry if \( \theta = 0 \) is identified with \( \theta = 2\pi \).

If one considers the circle parameterized by \( t = t_0, z = z_0, r = r_0 \) and \( \theta \in [0, 2\pi] \), its circumference \( 2\pi r_0^{-2/3} \) decreases as the “radius” \( r_0 \) increases. But, the circle given by \( r_0 = \infty \) being space-like everywhere and having a null length can not be anything else than a point.

This is already realized on the sphere. Consider a point \( Q(x, y, \sqrt{\sigma^2 - x^2 - y^2}) \) located on the sphere of radius \( \sigma \) centered at \( (0, 0, \sigma) \). The straight line joining \( Q \) to the north pole intersects the plane \( z = 0 \) at the point \( P(a, b, 0) \) specified by
\[ x = a(1 - \lambda) \quad y = b(1 - \lambda) \] (2.2)
where
\[ \lambda = \frac{2(a^2 + b^2) + \sigma^2 + \sigma \sqrt{4\sigma^2 + 3(a^2 + b^2)}}{2(a^2 + b^2 + \sigma^2)}. \] (2.3)
Introducing the polar stereographic coordinates \((r, \theta)\) by \( a = r \cos \theta \), \( b = r \sin \theta \) one obtains
\[ ds^2 = \frac{16\sigma^4}{(4\sigma^2 + r^2)^2} dr^2 + \frac{16\sigma^4 r^2}{(4\sigma^2 + r^2)^2} d\theta^2. \] (2.4)
The coefficient \( g_{\theta \theta} \) becomes a decreasing function for large values of the coordinate \( r \). If one does not know where this metric comes from and interpret it as a radius, a circle of infinite radius turns out to be of null length. This is not surprising since \( r = \infty \) corresponds to the point at infinity on the plane.
which is mapped into the north pole by the stereographic projection. Here $r = \infty$ is just the north pole. When the coordinate $r$ vanishes, one has another circle displaying a vanishing circumference: the south pole. The two are on the symmetry axis. Introducing the variable

$$r_* = 2\sigma \arctan(r/2\sigma)$$

(2.5)
The metric reads

$$ds^2 = dr_*^2 + \sigma^2 \sin^2 \left( \frac{r_*}{\sigma} \right) d\theta^2.$$  

(2.6)
The relation between $r_*$ and $r$ is bijective provided that $r_* \in [0, \pi\sigma]$. The points located on the symmetry axis once again are those for which the coefficient $g_{\theta\theta}$ vanish.

The section $t = c^{st}$, $z = c^{st}$ of the second Kasner line element

$$ds^2 = dr^2 + b r^{-2/3} d\theta^2$$

(2.7)
is a two dimensional surface; the preceding discussion strongly suggests that $r = \infty$ is a "point" on the symmetry axis. The difference with the sphere lies in the fact that the second special Kasner geometry is unbounded since the meridian $\theta = c^{st}$ is of infinite length; $r = \infty$ is a point rejected at infinity on the symmetry axis.

This interpretation is also supported by an embedding diagram. The previous two geometry can be rewritten as

$$ds^2 = 9b^{-3} \rho^{-8} d\rho^2 + \rho^2 d\theta^2.$$  

(2.8)
by the trivial change of coordinates $r = (b^{-1} \rho^2)^{-3/2}$. This line element can be realized in the Euclidean space as the surface of revolution

$$z(\rho) = \int_0^\rho d\xi (\rho^{8}_{cr} \xi^{-8} - 1)^{1/2}.$$  

(2.9)
which is real only for the values of $\rho$ satisfying the inequality

$$\rho \leq \rho_{cr} = (9b^{-3})^{1/8}.$$  

(2.10)
This can be translated for the geometry given in Eq. (2.8) by $r \geq (b \rho_{cr}^{-2})^{3/2}$. The coordinate $r$ measures the length along a generating line while $\rho$ is linked to the circumference by the usual formula $2\Pi \rho$. The two geometry looks like a bottle with an infinite neck which becomes thinner and thinner.
( as $r^- > \infty, \rho^- > 0$) and a basis which becomes larger and larger (as $r^- > 0, \rho^- > \infty$). The region $r \approx 0$ (where the components of the Riemann tensor associated to geodesic deviations become infinite) is excluded from the embedding and will be excluded when according the second Kasner line element to non vacuum solutions of Einstein equations. Our embedding is similar to the one developed in the case of the Melvin solution [5]. The new feature here is that this fact will be used with the example of the sphere to argue that new boundary conditions arise.

What is the causal structure of this second Kasner special geometry? As the relevant symmetry here is axial, we will factorize the geometry of a cylinder. Introducing a radial coordinate $\rho$ (different from the one appearing in Eq. (2.8)) by

$$r = b^{1/2}a^{-1/2}\rho^{-1}$$

one obtains

$$ds^2 = a^{-1/3}b^{2/3}\rho^{-4/3}[-dt^2 + a^{-4/3}b^{1/3}\rho^{-8/3}d\rho^2] + a^{-1/3}b^{2/3}\rho^{-4/3}[dz^2 + \rho^2 d\theta^2].$$

(2.12)

In a spherically symmetric geometry like the Schwarzschild solution, the causal structure is found by making the coordinates $\theta$ and $\phi$, which parameterizes a sphere, constants. The remaining geometry is written in terms of bounded ingoing and outgoing null coordinates [6]. In the Penrose diagram, each point represents a sphere. Following this procedure, we shall take $z$ and $\theta$ (which parameterizes the cylinder) to be constants in Eq.(2.12) : this gives the light-cone coordinates

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \pm t - 3a^{-2/3}b^{1/6}\rho^{-1/3}$$

(2.13)

in terms of which the metric reads

$$ds^2 = -\frac{1}{6}a^{2/3}b^{5/6}\frac{(\tan \bar{u} + \tan \bar{v})}{\cos^2 \bar{u}\cos^2 \bar{v}}d\bar{u}d\bar{v}.$$  

(2.14)

In a conformal diagram, the region $A$ specified by ($r = 0, t$ finite) would be a singular region (The Gauss scalar curvature diverges there) while the "region" $B$ given by ($r = \infty, t$ finite) would correspond simply to a point. It should be remembered that each point on such a diagram is a surface conformal to a cylinder. The difference with the spherical solutions lies in
the fact that here we probe the space-time structure using horizontal light rays which cross the symmetry axis at right angles while in the former one uses radial ones. The intermediary variable \( \rho \) was only illustrative since Eq. (2.13) and Eq. (2.11) give the link between the null coordinates and the set of variables \( t, r \). We shall omit this step in the coming section.

We shall be interested in the causal structure of a solution of the gravitating Abelian system which has the same asymptotic behavior than the Kasner geometry.

### III The gravitating \( U(1) \) system.

Let us consider a self gravitating Abelian-Higgs system minimally coupled to gravity. The classical field equations are derived from the action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} D_\mu \Phi D^\mu \Phi^* - \frac{\lambda}{4} (\Phi \Phi^* - v^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{R}{16\pi G} \right].
\] (3.1)

The \( U(1) \) charge \( e \) is embodied in the covariant derivative \( D_\mu \Phi = \partial_\mu \Phi - ieA_\mu \Phi \).

For a static cylindrically symmetric configuration, the ensätze can be given the form

\[
ds^2 = N^2(r) dt^2 - dr^2 - L^2(r) d\theta^2 - K^2(r) dz^2
\] (3.2)

\[
\Phi = f(r) e^{i\theta}; \quad A_\mu dx^\mu = e^{-1}(1 - P(r)) d\theta.
\] (3.3)

The field’s coordinate dependence leads to the equality \( T^0_0 = T^z_z \) in the entire space-time and so implies \( N(r) = K(r) \). Among all the Kasner geometries, this condition selects the cosmic string solution (a flat geometry with an angular deficit) and the second Kasner line element given in Eq. (2.1). The cosmic string solution has been extensively studied in the literature. In that configuration, the smoothness of the geometry on the symmetry axis is guaranteed by the initial conditions [2]

\[
L(0) = 0, \quad L'(0) = 1, \quad K(0) = 1, \quad K'(0) = 0
\] (3.4)

while the matter fields are non singular on the core provided that

\[
f(0) = 0, \quad P(0) = 1.
\] (3.5)

The finiteness of energy implies

\[
f(\infty) = v, \quad P(\infty) = 0.
\] (3.6)
What happens if one considers the second special Kasner line element of Eq.(2.1) as giving the asymptotic behavior of the metric i.e. when the coordinate $r$ goes to infinity? In the previous section, we argued that $r = \infty$ is the point at infinity on the symmetry axis. For a regular configuration, the Higgs and the vector fields must vanish there

$$f(\infty) = 0 \quad \text{and} \quad P(\infty) = 1.$$  \hspace{1cm} (3.7)

Extracting the expression of the inertial mass from Eq.(3.1) one has

$$E = \int dr d\theta dz L K^2 \left[ \frac{1}{2} g_{rr} |D_r \Phi|^2 + \frac{1}{2} g_{\theta\theta} |D_\theta \Phi|^2 + \frac{1}{4} F_{r\theta} F^{r\theta} + \frac{\lambda}{4} (\Phi^* \Phi - v^2)^2 \right].$$  \hspace{1cm} (3.8)

In the asymptotic region (i.e. $r \to \infty$) one has $L K^2 \sim r$ so that the volume element is not bounded. The first three terms decrease in the asymptotic region provided that $f(r)$ and $P(r)$ approach constants there; this is already satisfied by Eq.(3.7). The contribution of the Higgs potential in this part of the space is reduced to the integral of $r^\lambda v^4$. This has a chance to converge only when $v = 0$ i.e. when the system does not face a spontaneous symmetry breaking, (at least at tree level). The vanishing of the v.e.v makes it impossible to use the habitual parameterization $\Phi = v f(x) e^{i\theta}$ in terms of the dimensionless length $x = \sqrt{\lambda v^2} r$. Nevertheless, the dimensionfull Newton constant $G$ makes possible the parameterization $\Phi = (1/\sqrt{G\lambda}) f(x) e^{i\theta}$ in terms of the dimensionless quantity $x = \sqrt{Gr}$ measuring the length. In the same spirit, we can introduce other dimensionless functions $P(x), K(x), L(x)$ and write the field equations obtained by extremizing the action given in Eq.(3.1). In this case, we obtained a vanishing scalar field for any value of the parameters. Imposing the boundary conditions specified by Eq.(3.3) and Eq.(3.7), one obtains a function $f(x)$ whose values are many orders of magnitude smaller than the error fixed for numerical computations. This has to be discarded. Physically this can be understood as follows. Forcing the scalar field to go from zero to zero as $r$ goes from zero to infinity, one obtains that it vanishes identically since there is no source. Such a source would be for example a local maximum of the potential but as the vacuum expectation value vanishes, such a maximum does not exist. Taking an identically vanishing scalar field, we found a solution. In order to obtain a smooth embedding diagram, we took the coordinate $r$ to range from $-\infty$ to $\infty$. This is realized simply by taking $r$ to be the oriented length on a meridian of the two surface $t = c^s$, $z = c^s$: above the equator $r$ is taken positive and below
it, it becomes negative. To ensure a symmetric embedding, we chose the conditions

\[ f'(0) = P'(0) = L'(0) = K'(0) = 0 \]  \hspace{1cm} (3.9)

while at spatial infinity (\( r = \pm \infty \)), the boundary conditions displayed in Eq. (3.7) were imposed. To give a unified description with the case where a non vanishing scalar field is present, the equations of motion are given below (Eq.(3.11)- Eq.(3.14) ) in the case of an appropriate potential.

Is it possible to construct a solution with a non vanishing scalar field? To do this we need a potential which vanishes with the scalar field so that the minimum is attained at spatial infinity. We also need a local maximum which will correspond to a source. These conditions are for example satisfied by the gauge invariant potential

\[ V(\Phi) = \frac{\lambda}{v^2} \Phi \Phi^* ( (\Phi \Phi^*) - v^2 )^2 \]  \hspace{1cm} (3.10)

\( v \) being an inverse length scale. A local maximum is attained at \( |\phi| = v/\sqrt{3} \) while there are two minima, at \( |\phi| = 0, v \). The U(1) symmetry is broken spontaneously in the second vacuum and preserved in the first one. We disregard the renormalizability since we are interested only in classical solutions. Looking for a configuration whose geometry has an embedding similar to the one constructed with a vanishing scalar field, the radius \( r = 0 \) is particular because it corresponds to the greatest circumference as can be seen from Eq.(3.9). Here one uses the usual parameterization \( r = \sqrt{\lambda v^2 x} \), \( \Phi = v f(x) \exp i \theta, \cdots \) of \( \Phi \) and the dimensionless quantities \( \alpha = e^2/\lambda, \gamma = 8\pi G v^2 \). The Euler Lagrange equations read

\[ f''(x) + \frac{f'(x)L'(x)}{L(x)} + 2\frac{f'(x)K'(x)}{K(x)} - \frac{f(x)P(x)^2}{L(x)^2} - 6f(x)^5 + 8f(x)^3 - 2f(x) = 0 \]  \hspace{1cm} (3.11)

\[ P''(x) - \frac{L'(x)P(x)}{L(x)} + \frac{K'(x)P'(x)}{K(x)} - \alpha f(x)^2 P(x) = 0 \]  \hspace{1cm} (3.12)

\[ K''(x) = \gamma \frac{K(x)P'(x)^2}{2\alpha L(x)^2} + \frac{K'(x)L'(x)}{L(x)} + \frac{K'(x)^2}{K(x)} + \gamma f(x)^6 K(x) - 2\gamma f(x)^4 K(x) + \gamma f(x)^2 K(x) = 0 \]  \hspace{1cm} (3.13)
\[ L''(x) + \frac{P'(x)^2}{2\alpha L(x)} + 2\frac{K'(x)L(x)}{K(x)} + \frac{\gamma f(x)^2P(x)^2}{L(x)} + \gamma f(x)^6L(x) \\ - 2\gamma f(x)^4L(x) + \gamma f(x)^2L(x) = 0 \] (3.14)

The fields dependence of this solution are plotted in Fig.1.1 - Fig.1.4 for the parameters \( \alpha = 1, \gamma = 0.01 \). The scalar is the most rapidly varying field, followed by the vector field. The components of the metric change significantly on a scale approximately equal to ten times the one needed for the matter fields. The energy density per unit length along \( z \), noted \( \epsilon(x) \) and defined by \( E = 2\pi v^2 \int dx \epsilon(x) \) is plotted in Fig 1.5. It has significant contributions at \( x = \pm 1.39 \) where the scalar field attains the value corresponding to the maximum of the potential. An asymptotic analysis shows the total energy converges. For our example, \( E = 2\pi v^2 0.79 \).

### IV The classical trajectories.

Let us first consider the case of the massive particles. The metric specified in Eq.(3.2) has three cyclic coordinates; its geodesics are by way of consequence characterized by three constants of motion \( E, l, a \) such that \( N(r)\dot{t} = E, L(r)\dot{\phi} = l, K(r)\dot{z} = a \). The derivatives are performed with respect to the proper time of the observer. Introducing the proper time per unit mass \( \tau \), the energy-momentum relations reads

\[ \left( \frac{dr}{d\tau} \right)^2 = \frac{E^2 - a^2}{\mu^2} \frac{1}{K^2(r)} - \frac{l^2}{\mu^2 L^2(r)} - 1 \equiv A(r) \] (4.1)

\( \mu \) being the mass of the particle. A physical motion possesses a real velocity. A necessary condition for this to be possible is \( E^2 > a^2 \). The asymptotic behaviour of the metric displayed in Eq.(2.1) shows that the trajectory of a massive particle is always bounded from above: such a particle will never reach the point at infinity on the symmetry axis( \( r = \infty \) ).

The physically allowed region exhibits positive values of \( A(r) \). One of the signatures of this geometry is the existence of a maximum of the function \( L(r) \) which means that the set of slices \( t=\text{constant}, z=\text{constant} \) possess a maximal circumference. The angular velocity of an observer attains its lowest value there.
The causal Structure is found by analyzing particular light geodesics. The bounded null coordinates in the new background are given by $\bar{u} = c^{st}$ or $\bar{v} = c^{st}$ with

$$\begin{align*}
\bar{u} & = \arctan[t \mp \sigma(r)] \\
\bar{v} & = \arctan[t \mp \sigma(r)]
\end{align*}$$

with

$$\sigma(r) = \int_{0}^{r} d\xi \frac{1}{K(\xi)}$$

The 2 metric obtained by fixing $\theta$ and $z$ can now be written as

$$ds^2 = \frac{d\bar{u}d\bar{v}}{\cos^2 \bar{u} \cos^2 \bar{v}} K \left( \sigma^{-1} \left( \frac{\tan \bar{u} - \tan \bar{v}}{2} \right) \right)$$

For the embedding of this two geometry in the Euclidean space, let us consider the change of the coordinate variable $\rho = L(r)$ which is valid only above or below the radius $r = 0$ which maximizes $L(r)$. The two dimensional geometry

$$ds^2 = L^2(r) d\theta^2 + dr^2$$

can be realized in an Euclidean 3 space as the surface of revolution

$$z(\rho) = \int_{\rho_0}^{\rho} d\xi \left[ \frac{1}{L'(L^{-1}(\xi))^2} - 1 \right]^{1/2}.$$}

This is possible only in the regions in which the term under the square root is positive.

The embedding of the slice $t = c^{ct}, z = c^{ct}$ is generated by the rotation of the curve displayed in Fig.3 around the $y$ axis; it looks like a bottle with two necks. The associated causal structure is given in Fig.3.

V The coupling to a fermionic field.

In the background of many configurations defined by bosonic fields, it proves useful to look for fermionic solutions. For example, the sphaleron possesses a fermionic charge [7] and a cosmic string can support a fermionic current [8].

Let us consider a fermion charged under the gauge field. As the gauge symmetry is manifest, it has to be massless. We shall write its wave equation
in the bosonic background studied above. Our choice for the $\gamma$ matrices is the following

$$\gamma^t = K(r)^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \gamma^r = \begin{pmatrix} 0 & e^{-i\theta} & 0 & 0 \\ -e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\theta} \\ 0 & 0 & e^{i\theta} & 0 \end{pmatrix}$$

(5.1)

$$\gamma^0 = L(r)^{-1} \begin{pmatrix} 0 & -ie^{-i\theta} & 0 & 0 \\ -ie^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{-i\theta} \\ 0 & 0 & ie^{i\theta} & 0 \end{pmatrix} \quad \gamma^z = K(r)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

(5.2)

The en\"satze we shall use for the Dirac field is

$$\psi^t = (\psi_1(r) \exp(i(kz - \omega t + n_1\theta)), \psi_2(r) \exp(i(kz - \omega t + n_2\theta)), \psi_3(r) \exp(i(kz - \omega t + n_3\theta)), \psi_4(r) \exp(i(kz - \omega t + n_4\theta))).$$

(5.3)

Factorizing the dependence in $\theta$ one finds the relations

$$n_2 = n_1 + 1, n_3 = n_2 - 1, n_4 = 1 + n_1$$

(5.4)

while the reparameterization

$$\psi_1(r) = i\chi_1(r), \psi_2(r) = \chi_2(r), \psi_3(r) = i\chi_3(r), \psi_4(r) = \chi_4(r)$$

(5.5)

renders the equations purely real. We shall specialize to two possibilities which have been analyzed for the cosmic string [8].

V-A The condensate

Assuming $k = \omega = 0$, one is left with the two decoupled equations

$$\chi_1'(r) - \frac{(-1 + n_1 + P(r))}{L(r)}\chi_1(r) = 0$$

(5.6)

$$\chi_2'(r) + \frac{(n_1 + P(r))}{L(r)}\chi_2(r) = 0$$

(5.7)
supplemented by the equalities

\[ \chi_3(r) = \chi_1(r), \chi_4(r) = \chi_2(r). \] (5.8)

The solution of the second differential equation can be obtained using quadratures:

\[ \chi_{2P}(r) = \exp \left( - \int_{r_0}^{r} \frac{(n_1 + P(\xi))}{L(\xi)} d\xi \right) \] (5.9)

\( r_0 \) being an arbitrary constant.

As \( P(\xi) \sim 0 \) and \( L(\xi) \sim \xi^{-1/3} \) in the asymptotic region, one obtains

\[ \chi_{2P}(r) \sim c^{\text{st}} \exp \left( -\frac{3}{4}(n_1 + 1)r^{4/3} \right) \] (5.10)

which is normalizable only if \( n_1 > -1 \) since the volume element yields a factor \( r^{1/3} \).

Similar considerations give

\[ \chi_{1P}(r) \sim \exp \left( \frac{3}{4}n_1 r^{4/3} \right) \] (5.11)

which is normalizable only when \( n_1 < 0 \).

To summarize, the wave functions

\[ \psi^t = i\chi_{1P}(r)e^{in_1\theta}(1, 0, 1, 0) \quad \psi^t = i\chi_{2P}(r)e^{in_1\theta}(0, 1, 0, 1) \] (5.12)

with \( n_1 < 0 \) in the first case and \( n_1 < -1 \) in the second case are normalized and describe condensates.

**V-B The current.**

The dispersion relation \( k = \omega \) preserves the equalities \( \chi_4 = \chi_2, \chi_3 = \chi_1 \) but changes the field equations into

\[ \chi_1'(r) - \frac{(-1 + n_1 + P(r))}{L(r)}\chi_1(r) - \frac{2\omega}{K(r)}\chi_2(r) = 0 \] (5.13)

\[ \chi_2'(r) + \frac{(n_1 + P(r))}{L(r)}\chi_2(r) = 0. \] (5.14)

The last equation is exactly one of those found for the condensate so that we can take \( \chi_2(r) = \chi_{2P}(r) \) with \( n_1 > -1 \). The remaining equation is solved by
\[ \chi_1(r) = \chi_1 P(r) \int_{r_0}^{r} d\xi \xi^{-2/3} e^{-\frac{1}{4}(2n_1+1)\xi}. \]  
(5.15)

The integral present in this formula converges only for the integers \( n_1 \) satisfying \( n_1 > -\frac{1}{2} \). But for these values, the function \( \chi_1 P(r) \) goes to infinity so that the wave function is not normalizable.

\[ \chi_1'(r) - \frac{(-1 + n_1 + P(r))}{L(r)} \chi_1(r) = 0 \]  
(5.16)

\[ \chi_2'(r) + \frac{(n_1 + P(r))}{L(r)} \chi_2(r) + \frac{2\omega}{K(r)} \chi_1(r) = 0. \]  
(5.17)

The solution to the first equation has been seen to be well defined when \( n_1 < 0 \). Calling \( \chi_2 P(r) \) the solution of Eq. (5.16), one finds that

\[ \chi_2(r) = -\chi_2 P(r) \int_{r_0}^{r} \frac{2\omega}{K(\xi)} \frac{\chi_1(\xi)}{\chi_2 P(\xi)} d\xi \]  
(5.18)

solves Eq. (5.13). However, the behaviour of \( \chi_1(\xi) \) and \( \chi_2 P(\xi) \) in the asymptotic region makes this \( \chi_2(r) \) diverge.

**VI Conclusion**

In the asymptotic region, the geometry we studied is close to the one exhibited by the Melvin solution and the ones constructed recently \([3, 4]\), but the magnetic field looks quite different.

The mechanism for the formation of this configuration has not been addressed. In particular, one can ask if like monopoles or domain walls, it does not come to dominate the energy density of the universe. This may be used to constrain the parameters of the model.

A section \( t = c^t, z = c^z \) of the geometry of a cosmic string is a plane with an angular deficit. Putting such planes in parallel, with the edges one above the other, one obtains the representation of the 3 geometry corresponding to a fixed time. We could not do the same for the solutions we constructed. This is an important and open problem.

Hawking et al. have shown that a generalization of the Melvin solution (the C metric) plays a role in the creation of black hole pairs \([5]\). It may prove interesting to study to which extent their picture is affected by our interpretation.
A point of view different from the one adopted here would consist in taking the variable $\theta$ to be a non compact coordinate. The configuration displayed in Eq.(2.1), being a solution depending on a sole distance $r$ may then interpreted as a sort of domain wall. Such a possibility would require a specific treatment and has not been considered here.

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VII Figures captions

- Fig1.1 - Fig1.5 displays the fields dependence and the energy density of the solution for the parameters $\alpha = e^2/\lambda = 1, \gamma = 8\Pi Gv^2 = 0.01$.

- Fig.2 gives the generating line of the embedding of the slice $t = c^{te}, z = c^{te}$ of the solution as a surface of revolution in the Euclidean space.
The part which hugs the rotation axis has not been rendered with precision. The rotation is carried around the $y$ axis.

- Fig. 3 gives the causal structure of the solution. The line $r = 0$ is the diagonal, the thick line represents a line $r = r_o$, a the dotted line corresponds $t = t_o$. The time-like infinities are $I^+ = (\bar{\mu} = \bar{\nu} = \pi/2)$ and $I^- = (\bar{\mu} = \bar{\nu} = -\pi/2)$. As $r$ can change sign, there are two space infinity: $I^\sigma_\sigma = (-\pi/2, \pi/2)$ and $I^\sigma_\sigma = (\pi/2, -\pi/2)$. 
