Note on “Electromagnetism and Gravitation”

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Abstract

We obtain Hamilton equations for the gravitational field and demonstrate the conservation of total energy. We derive the Poisson bracket equation for a general dynamical variable.
1. Lagrange Equations

The gravitational Lagrangian is given by [1]

\[ L_G = \frac{c^4}{8\pi G} g^{\alpha\beta} Q_{[\alpha]}^\rho Q_{[\beta]}^\eta \]

\[ = \frac{c^4}{8\pi G} \left\{ g^{00} Q^0_m Q^m_0 + g^{lm} Q^0_0 Q^0_m \right\} \]

\[ = \frac{c^4}{32\pi G} \left\{ g^{00} g^{la} g^{mb} \frac{\partial g_{am}}{\partial x^0} \frac{\partial g_{bl}}{\partial x^0} + g^{lm} g^{00} \frac{\partial g_{00}}{\partial x^l} \frac{\partial g_{00}}{\partial x^m} \right\} \] (1)

\( Q_{[\nu\lambda]}^\mu \) is the gravitational field tensor, with non-zero components

\[ Q^{ij}_{[j0]} = Q^i_{j0} = \frac{1}{2} g^{ja} \frac{\partial g_{aj}}{\partial x^0} \] (2)

\[ Q^0_{[0i]} = Q^0_{0i} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^i} \] (3)

Field equations were derived in [1] under the assumption that the seven potentials \( g_{\mu\nu} = (g_{00}, g_{ij}) \) are independent. However, \( L_G \) does not contain the time derivative of \( g_{00} \) and, therefore, it cannot be a true dynamical variable. In this note, we will eliminate \( g_{00} \) from the Lagrangian, in order to establish Hamilton equations of motion. This is accomplished via the principle of space-time reciprocity.

According to Einstein, an observer at rest in a gravitational field is equivalent to an accelerated observer in free space. Moreover, the difference in gravitational potential between two points \( P \) and \( P' \) is equivalent to a relative velocity between observers at \( P \) and \( P' \) [2]. It follows that:

(a) time intervals measured at \( P \) and \( P' \) are related by

\[ \Delta t = \Delta t' / \sqrt{1 - v^2/c^2} \] (time dilatation);

(b) distance intervals measured at \( P \) and \( P' \) are related by

\[ \Delta l = \Delta l' / \sqrt{1 - v^2/c^2} \] (length contraction).

The reciprocity in space and time gives way to the equality \( \Delta t \Delta l = \Delta t' \Delta l' \). We state the more general principle as follows: the space-time volume element \( \sqrt{-g} d^4 x \) is not affected by the presence of a gravitational field.

The array of potentials \( g_{\mu\nu} \) always takes the form
\[
g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\
0 & 0 & g_{ij} \\
0 & g_{ij} & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix} \quad (4)
\]

Setting \( \det g_{\mu\nu} = -g \) and \( \det g_{ij} = -h \), we have

\[
\sqrt{-g} = \sqrt{g_{00}} \sqrt{-h} \quad (5)
\]

However, by space-time reciprocity, the density \( \sqrt{-g} \) is equal to the associated flat-space density \( \sqrt{-h_0} \). For example, in rectangular coordinates, \( \sqrt{-g} = 1 \); in spherical coordinates, \( \sqrt{-g} = r^2 \sin \theta \); etc. Thus,

\[
\sqrt{-g} = \sqrt{g_{00}} \sqrt{-h} = \sqrt{-h_0} \quad (6)
\]
or

\[
g_{00} = \frac{h_0}{h} \quad (7)
\]

This constraint serves to eliminate \( g_{00} \) from the Lagrangian.

We now derive the corresponding field equations. Since \( \sqrt{-g} = \sqrt{-h_0} \) does not depend upon the gravitational field, its variation is zero

\[
\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} = 0 \quad (8)
\]
or

\[
g_{00} \delta g^{00} = -g_{ij} \delta g^{ij} \quad (9)
\]

This shows that, at any point, \( \delta g^{00} \) is determined by the \( \delta g^{ij} \). Variation and integration by parts yields

\[
\delta \int L_G \sqrt{-g} d^4x =
\int \frac{c^4}{8\pi G} \left\{ \frac{\partial}{\partial x_0} \left( \sqrt{-g} g^{00} Q^i_0 \right) - \delta^i_j \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{lm} Q^0_{0m} \right) \right\} g_{il} \delta g^{lj} d^4x
\]
\[
+ \frac{1}{2} \int \sqrt{g^{00}} (T^i_j - \delta^i_j T^0_0) g_{il} \delta g^{lj} \sqrt{-g} d^4x \quad (10)
\]

The gravitational stress-energy-momentum tensor is

\[
T_G^{\mu\nu} = \frac{c^4}{4\pi G} \sqrt{g_{00}} \left\{ g^{\mu\rho} g^{\nu\beta} Q^\rho_{[\rho]} Q^\beta_{[\beta]} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} Q^\rho_{[\rho]} Q^\eta_{[\eta]} \right\} \quad (11)
\]
and it is understood that \( g_{00} = h_0/h \). The contributions of matter and electromagnetism are expressed by

\[
\delta \int L_M \sqrt{-g} \, d^4x = \frac{1}{2} \int \sqrt{g^{00}} (T_{iM}^i - \delta^i_j T_{0 M}^0) g_{ii} \delta g^{ij} \sqrt{-g} \, d^4x
\]

where

\[
T_{\mu\nu}^M = \sqrt{g_{00}} \left\{ \rho c^2 u^\mu u^\nu + F_{\alpha}^\mu F_{\alpha\nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right\}
\]

(The factor of \( \sqrt{g_{00}} \) in (11) and (13) is discussed in section 3.) Combining (10) and (12), then setting coefficients of \( \delta g^{ij} \) equal to zero, we arrive at the six field equations

\[
c^4 \frac{1}{4\pi G} \left\{ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} (\sqrt{-g} g^{00} Q^i_j) - \delta^i_j \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} (\sqrt{-g} g^{im} Q^0_{0m}) \right\} \\
+ \sqrt{-g} (T_i^i - \delta^i_j T_0^0) = 0
\]

(14)

\( T^\mu_\nu \) is the total energy tensor

\[
T^\mu_\nu = T^\mu_\nu_G + T^\mu_\nu_M
\]

Newton’s law of gravitation is to be found, as a first approximation, in all three diagonal equations.

Before proceeding to the Hamilton equations, we re-express (14) in Lagrangian form. Setting \( \mathcal{L} = \sqrt{-g} L \), where

\[
L = L_G + L_M
\]

\[
\delta \int \mathcal{L} \, d^4x = \\
= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial g_{00}} \delta g_{00} + \frac{\partial \mathcal{L}}{\partial (\partial_k g_{00})} \delta (\partial_k g_{00}) + \frac{\partial \mathcal{L}}{\partial g_{ij}} \delta g_{ij} + \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} \delta (\partial_0 g_{ij}) \right\} \\
= \int d^4x \left\{ \left( \frac{\partial \mathcal{L}}{\partial g_{00}} - \partial_k \frac{\partial \mathcal{L}}{\partial (\partial_k g_{00})} \right) \delta g_{00} + \left( \frac{\partial \mathcal{L}}{\partial g_{ij}} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} \right) \delta g_{ij} \right\}
\]

(17)

However, \( \delta g_{00} = -g_{00} g^{ij} \delta g_{ij} \), therefore
\[ \delta \int \mathcal{L} d^4 x = \]
\[ = \int d^4 x \left\{ -g_{00} g^{ij} \left( \frac{\partial \mathcal{L}}{\partial g_{00}} - \partial_k \frac{\partial \mathcal{L}}{\partial (\partial_k g_{00})} \right) + \frac{\partial \mathcal{L}}{\partial g_{ij}} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} \right\} \delta g_{ij} \] (18)

In order to satisfy \( \delta \int \mathcal{L} d^4 x = 0 \), the coefficients of \( \delta g_{ij} \) must be zero

\[- g_{00} g^{ij} \left( \frac{\partial \mathcal{L}}{\partial g_{00}} - \partial_k \frac{\partial \mathcal{L}}{\partial (\partial_k g_{00})} \right) + \frac{\partial \mathcal{L}}{\partial g_{ij}} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} = 0 \] (19)

These are identical to field equations (14).

2. Hamilton Equations

The six independent dynamical variables \( g_{ij} \) possess conjugate momenta

\[ \pi^{ij} = \frac{\partial \sqrt{-g} \mathcal{L}_G}{\partial (\partial_0 g_{ij})} = \frac{c^4}{16\pi G} g^{00} g_{ia} g_{jb} \partial_0 g_{ab} \sqrt{-g} \] (20)

Solving for \( \partial g_{ij}/\partial x^0 \), the Hamiltonian density is

\[ \mathcal{H}_G = \sqrt{-g} H_G = \pi^{ij} \partial_0 g_{ij} - \mathcal{L}_G \]
\[ = \frac{8\pi G}{c^4} g_{00} g_{ma} g_{mb} \pi^{ia} \pi^{mb} \frac{1}{\sqrt{-g}} - \frac{c^4}{32\pi G} g^{lm} g^{00} \partial_l g_{00} \partial_m g_{00} \sqrt{-g} \] (21)

For simplicity, we now represent matter by the real scalar field

\[ L_M = \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right) \] (22)

with field equations

\[ \frac{\partial \mathcal{L}_M}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_M}{\partial (\partial_\mu \phi)} = 0 \] (23)

or

\[ \frac{1}{\sqrt{-g} g^{\mu\nu}} \left( \sqrt{-g} g^{\mu\nu} \partial_\phi \partial_\nu \right) + m^2 \phi = 0 \] (24)

The conjugate momentum is
\[\pi = \frac{\partial \sqrt{-g} L_M}{\partial (\partial_0 \phi)} = g^{00} \partial_0 \phi \sqrt{-g}\]  

and Hamiltonian density

\[
H_M = \sqrt{-g} H_M = \pi \partial_0 \phi - \mathcal{L}_M = \frac{1}{2} g_{00} \pi \frac{1}{\sqrt{-g}} - \frac{1}{2} (g^{lm} \partial_l \phi \partial_m \phi - m^2 \phi^2) \sqrt{-g}
\]  

Consider the variation of the spatial integral of \(H = H_G + H_M\):

\[
\delta \int \mathcal{H} \, d^3x = \delta \int d^3x \left\{ \frac{\partial \mathcal{H}}{\partial g_{00}} \delta g_{00} + \frac{\partial \mathcal{H}}{\partial (\partial_k g_{00})} \delta (\partial_k g_{00}) + \frac{\partial \mathcal{H}}{\partial g_{ij}} \delta g_{ij} + \frac{\partial \mathcal{H}}{\partial \pi} \delta \pi + \frac{\partial \mathcal{H}}{\partial \phi} \delta \phi \right\}
\]

\[
= \int \left\{ \left[ -g_{00} g^{ij} \left( \frac{\partial \mathcal{H}}{\partial g_{00}} - \frac{\partial \mathcal{H}}{\partial (\partial_k g_{00})} \right) + \frac{\partial \mathcal{H}}{\partial g_{ij}} \right] \delta g_{ij} + \frac{\partial \mathcal{H}}{\partial \pi} \delta \pi \delta \phi \right\}
\]

Setting (27) aside for the moment, the definition of \(\mathcal{H}\) provides the variation

\[
\delta \int \mathcal{H} \, d^3x = \delta \int d^3x \left\{ \pi^{ij} \partial_0 g_{ij} + \pi \partial_0 \phi - \mathcal{L} \right\}
\]

\[
= \int \left\{ \delta \pi^{ij} \partial_0 g_{ij} + \pi^{ij} \delta (\partial_0 g_{ij}) + \delta \pi \partial_0 \phi + \pi \delta (\partial_0 \phi) \right\}
\]

\[
- \frac{\partial \mathcal{L}}{\partial g_{00}} \delta g_{00} - \frac{\partial \mathcal{L}}{\partial (\partial_k g_{00})} \delta (\partial_k g_{00}) - \frac{\partial \mathcal{L}}{\partial g_{ij}} \delta g_{ij} - \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} \delta (\partial_0 g_{ij})
\]

\[
- \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial (\partial_k \phi)} \delta (\partial_k \phi) - \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \delta (\partial_0 \phi)
\]  

(28)

Cancel terms in (28), then integrate by parts, to find

\[
\delta \int \mathcal{H} \, d^3x =
\]
\[ \int d^3x \left\{ \left[ g_{00}g^{ij} \left( \frac{\partial L}{\partial g_{00}} - \partial_k \frac{\partial L}{\partial \left( \partial_k g_{00} \right)} \right) - \frac{\partial L}{\partial g_{ij}} \right] \delta g_{ij} + \partial_0 g_{ij} \delta \pi^{ij} \right\} \]

Finally, substitute the field equations (19) and (23)

\[ \delta \int \mathcal{H} d^3x = \int d^3x \left\{ -\partial_0 \pi^{ij} \delta g_{ij} + \partial_0 g_{ij} \delta \pi^{ij} - \partial_0 \pi \delta \phi + \partial_0 \phi \delta \pi \right\} \]

The Hamilton equations follow by equating coefficients in (27) and (30):

\[ -\frac{\partial \pi^{ij}}{\partial x^0} = -g_{00}g^{ij} \left( \frac{\partial H}{\partial g_{00}} - \partial_k \frac{\partial H}{\partial \left( \partial_k g_{00} \right)} \right) + \frac{\partial H}{\partial g_{ij}} \]  

\[ \frac{\partial g_{ij}}{\partial x^0} = \frac{\partial H}{\partial \pi^{ij}} \]  

\[ \frac{\partial \pi}{\partial x^i} = \frac{\partial H}{\partial \phi} - \partial_k \frac{\partial H}{\partial \left( \partial_k \phi \right)} \]  

\[ \frac{\partial \phi}{\partial x^0} = \frac{\partial H}{\partial \pi} \]

3. Conservation of Energy

Let us calculate the rate of change of the time-dependent quantity

\[ H(x^0) = \int \mathcal{H} d^3x \]  

where \( \mathcal{H} \) is the total Hamiltonian density:

\[ \frac{dH(x^0)}{dx^0} = \frac{d}{dx^0} \int \mathcal{H} d^3x \]

\[ = \int d^3x \left\{ \frac{\partial H}{\partial g_{00}} \partial_0 g_{00} + \frac{\partial H}{\partial \left( \partial_k g_{00} \right)} \partial_0 \left( \partial_k g_{00} \right) + \frac{\partial H}{\partial g_{ij}} \partial_0 g_{ij} + \frac{\partial H}{\partial \pi^{ij}} \partial_0 \pi^{ij} \right\} \]

\[ = \int d^3x \left\{ \left[ -g_{00}g^{ij} \left( \frac{\partial H}{\partial g_{00}} - \partial_k \frac{\partial H}{\partial \left( \partial_k g_{00} \right)} \right) + \frac{\partial H}{\partial g_{ij}} \right] \partial_0 g_{ij} + \frac{\partial H}{\partial \pi^{ij}} \partial_0 \pi^{ij} \right\} \]
We have made use of

\[
\frac{\partial g_{00}}{\partial x^0} = -g_{00} g^{ij} \frac{\partial g_{ij}}{\partial x^0}
\]  

(37)

Here, the volume must be large enough that surface integrals may be neglected. Substitute the Hamilton equations, in order to obtain

\[
\frac{dH(x^0)}{dx^0} = \int d^3 x \left\{ -\partial_0 \pi^{ij} \partial_0 g_{ij} + \partial_0 g_{ij} \partial_0 \pi^{ij} - \partial_0 \pi \partial_0 \phi + \partial_0 \phi \partial_0 \pi \right\} = 0
\]  

(38)

Therefore, the integral quantity \( H(x^0) \) is conserved, if the field equations are satisfied.

The differential law of energy-momentum conservation is [1]

\[
\text{div} T^\nu_{\mu} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^\nu_{\mu}}{\partial x^\nu} - Q^\nu_{\mu \lambda} T^\lambda = 0
\]  

(39)

\( Q^\mu_{\nu \lambda} \) are the connection coefficients of the theory

\[
\nabla_{\nu} e_{\mu} = e_{\lambda} Q^\lambda_{\mu \nu}
\]  

(40)

(The gravitational field is \( Q^\mu_{[\nu \lambda]} = Q^\mu_{\nu \lambda} - Q^\mu_{\lambda \nu} \).) Energy conservation is given by

\[
\text{div} T^0_{\nu} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^0_{\nu}}{\partial x^\nu} - Q^\nu_{0 \lambda} T^\lambda = 0
\]

\[
= \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^0_0}{\partial x^0} + \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^0_k}{\partial x^k} - Q^0_{00} T^0_0 - Q^0_{0k} T^0_k = 0
\]  

(41)

where

\[
Q^0_{00} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^0} = -\frac{1}{\sqrt{g^{00}}} \frac{\partial \sqrt{g^{00}}}{\partial x^0}
\]  

(42)

\[
Q^0_{0k} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^k} = -\frac{1}{\sqrt{g^{00}}} \frac{\partial \sqrt{g^{00}}}{\partial x^k}
\]  

(43)

Coefficients \( Q^0_{00} \) and \( Q^0_{0k} \) are identically zero. It follows that
\[
\text{div} T_0^\nu = \frac{1}{\sqrt{-h}} \frac{\partial \sqrt{-h} T_0^0}{\partial x^0} + \frac{1}{\sqrt{-h}} \frac{\partial \sqrt{-h} T_0^k}{\partial x^k} = 0 \quad (44)
\]

or
\[
\frac{\partial \sqrt{-h} T_0^0}{\partial x^0} = - \frac{\partial \sqrt{-h} T_0^k}{\partial x^k} \quad (45)
\]

On the other hand, the quantity of energy in an infinitesimal region, \(dV_0 = d^3x\), is given by the first term in the expansion
\[
\epsilon_\mu T_{\mu\nu} \sqrt{-g} dV_\nu = \epsilon_0 T^{00} \sqrt{-g} dV_0 + \epsilon_0 T^{0k} \sqrt{-g} dV_k + \epsilon_1 T^{i0} \sqrt{-g} dV_0 + \epsilon_1 T^{ik} \sqrt{-g} dV_k \quad (46)
\]

The scalar basis is a function, \(e_0 = \sqrt{g_{00}}\), and this is crucial. It allows us to consider the rate of change of the energy integral
\[
\frac{d}{dx^0} \int e_0 T^{00} \sqrt{-g} dV_0 = \frac{d}{dx^0} \int \sqrt{g_{00}} g^{00} T_0^0 \sqrt{-g} d^3x = \int \frac{\partial \sqrt{-h} T_0^0}{\partial x^0} d^3x \quad (47)
\]

This gives zero upon integration of (45) over a sufficiently large volume. It follows that if (38) is to represent conservation of total energy, then the integrands in (35) and (47) must be identical
\[
\mathcal{H} = \sqrt{-g} H = \sqrt{-h} T_0^0 \quad (48)
\]

or
\[
T_0^0 = \sqrt{g_{00}} H \quad (49)
\]

The gravitational Hamiltonian (21) may be evaluated in terms of the field \(Q_{\mu\nu}^{[\lambda]}\)
\[
T_0^0_G = \sqrt{g_{00}} H_G = \frac{c^4}{8\pi G} \sqrt{g_{00}} \left\{ g^{00} Q_{m0}^l Q_{l0}^m - g^{lm} Q_{00}^0 Q_{0m}^0 \right\} \quad (50)
\]

which implies
\[
T_G^{\mu\nu} = \frac{c^4}{4\pi G} \sqrt{g_{00}} \left\{ g^{\mu\alpha} g^{\nu\beta} Q_{[\alpha\eta\beta]} Q_{[\rho\beta]} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} Q_{[\rho\alpha]} Q_{[\rho\beta]} \right\} \quad (51)
\]

In similar fashion, the Hamiltonian (26) gives rise to a factor of \(\sqrt{g_{00}}\) in the matter tensor \(T_M^{\mu\nu}\).
4. The Poisson Bracket Equation

This section is of a purely formal nature, in which we consider general dynamical variables, \( U = \sqrt{-g}U \), that are functionals of the fields, their spatial derivatives, the momenta, their spatial derivatives, and the time:

\[
\frac{dU(x^0)}{dx^0} = \frac{d}{dx^0} \int U d^3x
\]

\[
= \int d^3x \left\{ \frac{\partial U}{\partial g_{00}} \partial_0 g_{00} + \frac{\partial U}{\partial (\partial_k g_{00})} \partial_0 (\partial_k g_{00}) + \frac{\partial U}{\partial g_{ij}} \partial_0 g_{ij} \\
+ \frac{\partial U}{\partial (\partial_k g_{ij})} \partial_0 (\partial_k g_{ij}) + \frac{\partial U}{\partial \pi} \partial_0 \pi + \frac{\partial U}{\partial (\partial_k \pi)} \partial_0 (\partial_k \pi) + \frac{\partial U}{\partial \pi} \right\} (52)
\]

Define the functional derivatives

\[
\frac{\delta U}{\delta g_{ij}} = -g_{00} g^{ij} \left( \frac{\partial U}{\partial g_{00}} - \partial_k \frac{\partial U}{\partial (\partial_k g_{00})} \right) + \left( \frac{\partial U}{\partial g_{ij}} - \partial_k \frac{\partial U}{\partial (\partial_k g_{ij})} \right) (53)
\]

\[
\frac{\delta U}{\delta \pi^{ij}} = \frac{\partial U}{\partial \pi^{ij}} - \partial_k \frac{\partial U}{\partial (\partial_k \pi^{ij})} (54)
\]

\[
\frac{\delta U}{\delta \phi} = \frac{\partial U}{\partial \phi} - \partial_k \frac{\partial U}{\partial (\partial_k \phi)} (55)
\]

\[
\frac{\delta U}{\delta \pi} = \frac{\partial U}{\partial \pi} - \partial_k \frac{\partial U}{\partial (\partial_k \pi)} (56)
\]

In terms of these derivatives, the Hamilton equations (31 – 34) become

\[
- \frac{\partial \pi^{ij}}{\partial x^0} = \frac{\delta H}{\delta g_{ij}} (57)
\]

\[
\frac{\partial g_{ij}}{\partial x^0} = \frac{\delta H}{\delta \pi^{ij}} (58)
\]

\[
\frac{\partial \pi}{\partial x^0} = \frac{\delta H}{\delta \phi} (59)
\]

\[
\frac{\partial \phi}{\partial x^0} = \frac{\delta H}{\delta \pi} (60)
\]
Integrate (52) by parts and neglect surface terms, in order to obtain the Poisson bracket equation

\[ \frac{dU(x^0)}{dx^0} = \int d^3 x \left\{ \left( \frac{\delta U}{\delta g_{ij}} \frac{\delta \mathcal{H}}{\delta \pi^{ij}} - \frac{\delta U}{\delta \pi^{ij}} \frac{\delta \mathcal{H}}{\delta g_{ij}} \right) + \left( \frac{\delta U}{\delta \phi} \frac{\delta \mathcal{H}}{\delta \pi} - \frac{\delta U}{\delta \pi} \frac{\delta \mathcal{H}}{\delta \phi} \right) + \frac{\partial U}{\partial x^0} \right\} \]  

(61)

Energy conservation (38) is a special case of (61), in which \( \mathcal{H} \) is a functional of the fields and momenta but is not an explicit function of the time.

References

1. K. Dalton, “Electromagnetism and Gravitation,” Hadronic Journal 17 (1994) 483; also, [http://xxx.lanl.gov/gr-qc/9512027](http://xxx.lanl.gov/gr-qc/9512027).

2. A. Einstein, “On the Influence of Gravitation on the Propagation of Light,” in *The Principle of Relativity* (Dover, New York, 1952).