Ultra-fast propagation of Schrödinger waves in absorbing media

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We identify the characteristic times of the evolution of a quantum wave generated by a point source with a sharp onset in an absorbing medium. The “traversal” or “Büttiker-Landauer” time (which grows linearly with the distance to the source) for the Hermitian, non-absorbing case is substituted by three different characteristic quantities. One of them describes the arrival of a maximum of the density calculated with respect to position, but the maximum with respect to time for a given position becomes independent of the distance to the source and is given by the particle’s “survival time” in the medium. This latter effect, unlike the Hartman effect, occurs for injection frequencies under or above the cut-off, and for arbitrarily large distances. A possible physical realization is proposed by illuminating a two-level atom with a detuned laser.

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As it was well understood by Brillouin and Sommerfeld long ago, certain wave features may travel at velocities exceeding c in systems described by relativistic equations, without violating Einstein’s causality [1]. In non-relativistic equations one may similarly find “ultrafast” phenomena which are subject to non-relativistic causality conditions [2]. In particular, the Hartman effect [3] has been studied thoroughly theoretically and experimentally both for non-relativistic and relativistic equations [2, 4, 5, 6, 7, 8, 9, 10, 11]. The peak of the transmitted packet in a collision of a particle with a square barrier is dimensionless unless stated otherwise),

where \( \omega_0 \) is the injection frequency. This is a non-Hermitian generalization of the source with a sharp onset studied by several authors before [12, 13, 14, 15, 16, 17]. The imaginary potential \(-iV_1\) models the passage from the incident channel to other channels which are not represented explicitly (a physical example is provided below). The dispersion relation corresponding to Eq. (11) is

\[ k = \sqrt{\omega - 1 + iV_1}, \quad \text{Im}(k) \geq 0, \] (2)

and the solution to Eq. (11) is given by

\[ \psi(x, t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{ikx-i\omega t}}{\omega - \omega_0 + i\delta}, \] (3)

or, in the \( k \)-complex plane, as

\[ \psi(x, t) = \frac{ie^{-i\omega t}}{\pi} \int_{\Gamma_k} dk \frac{e^{ikx-ik^2 t}}{k^2 - k_0^2}, \] (4)

where

\[ V = 1 - iV_1, \] (5)

\[ k_0 = k(\omega_0) = \sqrt{\omega_0^2 - 1 + iV_1}, \quad \text{Im}(k_0) \geq 0, \] (6)

and the countour \( \Gamma_k \) goes from \(-\infty\) to \( \infty \) passing above all singularities. Introducing a new integration variable

\[ u = \frac{1 + i}{\sqrt{2}} \sqrt{t} \left( k - \frac{x}{2t} \right), \] (7)

Eq. (11) takes the form

\[ \psi(x, t) = \frac{ie^{-iV_1 + x^2/4t}}{2\pi} \int_{\Gamma_u} du e^{-u^2} \left( \frac{1}{u - u_0} + \frac{1}{u - u'_0} \right), \] (8)

where

\[ u_0 = \frac{1 + i}{\sqrt{2}} \sqrt{t} \left( k_0 - \frac{x}{2t} \right), \]

\[ u'_0 = \frac{1 + i}{\sqrt{2}} \sqrt{t} \left( -k_0 - \frac{x}{2t} \right), \] (9)

Consider, for \( x \geq 0 \), the following Schrödinger equation with “source” boundary conditions (all quantities are dimensionless unless stated otherwise),

\[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + (1 - iV_1)\psi \] (1)

\[ \psi(x = 0, t) = e^{-i\omega_0 t}\Theta(t) \]

\[ \psi(x, t) = 0 \quad x > 0, t < 0. \]
and the contour $\Gamma_u$ goes from $-\infty$ to $\infty$ passing above the two simple poles. Using the integral definition of the $w-$functions \[ 16 \] \[ 18 \], Eq. \[ 8 \] is finally given by \[
abla (x) = \frac{e^{-iVt\,e^{ix^2/2t}}}{2\pi} \left[ w(-u_0) + w(-u_0') \right]. \tag{10} \]

To interpret this exact solution in simple terms it is useful to find approximations. In particular, the contour may be deformed along the steepest descent path from the saddle at $x/2t$ in the $k$-plane (the origin in the $u$-plane). The path crosses the pole at $k_0$ at a time \[
abla \equiv x/2[\text{Re}(k_0) + \text{Im}(k_0)]. \tag{11} \]

Eq. \[ 8 \] can be written as the sum of contributions from the saddle point and the pole, \[
abla (x) = \psi_p(x, t) + \psi_s(x, t) \tag{12} \]
\[
abla (x) = e^{-i\omega t+xk_0^2} \Theta(t-\nabla) \tag{13} \]
\[
abla (x) = e^{-iVt\,e^{ix^2/2t}} \left( \frac{1}{u_0} + \frac{1}{u_0'} \right) \tag{14} \]
\[
abla (x) = e^{-iVt\,e^{ix^2/2t} (t/\pi)^{1/2}} \left( 1 + i \right) k_0^2 (t^2 + t) \tag{15} \]
where \[
abla \equiv \frac{x}{-2ik_0}. \tag{16} \]

Alternatively, Eq. \[ 12 \] may be obtained from the asymptotic expansion of the $w$’s for large values of the modulus of their arguments $|u_0|$ and $|u_0'|$. They become large for large $|k_0|$, $x$ or $t$, and also for very small $t$. Both have minima at $t = |\nabla|$, with minimum values \[ x[|k_0| \pm \text{Re}(k_0)]^{1/2}. \] (A scale for the validity of the saddle plus pole approximation for all $t$ is thus $x > |[k_0| - \text{Re}(k_0)^{-1})]$. For injection frequencies below the cut-off the saddle contribution dominates up to exponentially large times as in the Hermitian case \[ 19 \], so that $\nabla$ is not of much significance whereas, above threshold, $\nabla$ is a good scale for the arrival of the main signal. The discussion hereafter refers to the case $\omega_0 < 1$ (injection below cut-off) unless stated otherwise.

In Figure 1 we can see the formation of the forerunner in a sequence of three snapshots of the density at three different instants and for a fixed value of $V_1$. (For increasing values of $V_1$ the spatial peak appears at larger $x$ and it also takes a longer time to be formed.) From $d|\psi(t, x)|^2/dx = 0$ we may obtain the position $x(t)$ of the “spatial” maximum at time $t$. This also defines (by inverting $x(t)$) a function $\tau_s(x)$, namely, the time when this spatial maximum arrives at $x$. One finds from Eq. \[ 19 \] that the spatial maximum in the large-$x$ region is given by \[
abla_s = |\nabla|, \tag{17} \]
a role played by the real $\nabla$ in the absence of absorption \[ 20 \].

A single real quantity $\nabla (V_1 = 0)$ in the Hermitian case has been substituted, for a non-zero $V_1$, by three different quantities: the time for pole-cutting $\nabla$, a complex $\nabla$, and its modulus $|\nabla|$, all of which tend to the “Buttiker-Landauer” traversal time $\nabla (V_1 = 0)$ without absorption \[ 20 \].

Similarly, we may fix $x$, calculate $d|\psi(x, t)|^2/dt = 0$, and solve for $t$ to obtain a “temporal” maximum, $\tau_t(x)$.

In Figure 2 we have plotted the density, $|\psi|^2$, versus time for a large $x$, so that, in the scale used, the exact solution and the saddle approximation $\psi_s$ are indistinguishable. One clear effect of the complex potential is the decrease of the amplitude; also, the peak arrives earlier when $V_1$ increases. At variance with the Hermitian
case, the time of arrival of the maximum is not proportional to $\tau$ ($\tau_T = \tau/\sqrt{3}$ when $V_1 = 0$). The equation $dv_1^2/dt = 0$ cannot be solved analytically for $V_1 \neq 0$ in a generic case, so there is no explicit formula for $\tau_T$. Nevertheless, if $x \gg 2^{1/2}|k_0|/V_1$, one finds

$$\tau_T \approx \frac{1}{2V_1}, \quad (18)$$

i.e., the temporal maximum coincides with the mean survival time of a particle immersed in the absorbing potential, and it is independent of $x$ and $\omega_0$, which is the most important result of this work. A time-frequency analysis [16, 21] shows that the maximum is not tunnelling but it is dominated by frequencies above the cut-off, in particular by the frequency $1 + (xV_1)^2$ corresponding to the “classical” velocity required to arrive at $x$ at time $\tau_T(x)$. In fact, this effect is also present for injection frequencies above the cut-off. It has, nevertheless, no purely classical explanation, in the sense that any classical ensemble of particles injected at a constant rate into the absorbing medium with an arbitrary momentum distribution from $t = 0$ on, would lead, for fixed $x$, to a monotonous increase of the density up to the asymptotic, stationary value.

The important difference between temporal and spatial maxima may be understood from the scaling satisfied by the saddle term,

$$|\psi_s(x, t)|^2 = \frac{e^{-2V_1(x-1)}}{s} |\psi_s(x, t)|^2. \quad (19)$$

Since the exponential depends on $t$, but not on $x$, the spatial maximum travels at constant velocity whereas, the temporal maximum does not, except for $V_1 = 0$.

Some recent works have examined the time of arrival of the maximum at a given position for the source problem without absorption also for small values of $x$, where the pole-saddle approximation is not valid [17, 22]. For small $x$, $\tau_T(x)$ presents in that case a basin with a minimum. In Fig. 3a, we have plotted this quantity versus $x$ for different values of the absorbing potential $V_1$. Note that the basin disappears by increasing the absorption. The most prominent feature of the curves though is their constant value $1/2V_1$ for large enough $x$, instead of the linear dependence found without absorption. Unlike the Hartman effect, the arrival of the temporal peak in an absorbing medium stays constant for arbitrarily large $x$. The effect is also present for other boundary conditions, in particular for the “Moshinsky shutter” boundary condition [22] corresponding to the initial, truncated-plane-wave state

$$\psi(x, t = 0) = \frac{1}{\sqrt{2\pi}} e^{ik_0x} \Theta(-x), \quad (20)$$

and the potential

$$V(x) = (1 - iV_1) \Theta(x). \quad (21)$$

Using the same techniques applied in [22], we may calculate $\psi(x, t)$. The corresponding density is shown in Fig. 3b, where the same characteristic time $1/2V_1$ which describes the arrival of the peak at large $x$ is found.

A close physical realization of the Moshinki shutter problem with absorption may be based on a fluorescence experiment where an atom is first prepared according to a truncated plane wave in an internal state [1], and then let evolve after a sudden shutter opening at $x = t = 0$. The “external” region, $x > 0$, is illuminated with a perpendicular laser. Let us assume a $\Lambda$-configuration for three relevant atomic levels such that the laser couples levels $\{1\}$ and $\{2\}$ whereas $\{2\}$ decays irreversibly by spontaneous photon emission to a ground state $\{0\}$. According to the quantum jump technique [23] the amplitudes for levels $\{1\}$ and $\{2\}$ obey an effective Schrödinger equation with Hamiltonian (all quantities are now dimensional)

$$H = \frac{\hbar^2}{2m} \frac{\hbar^2}{2} \left( \begin{array}{cc} 0 & \Omega \Theta(X) \\ \Omega \Theta(X) & -2\Delta - i\gamma \end{array} \right), \quad (22)$$

where $\Omega$ is the Rabi frequency, $\Delta$ the detuning between the laser frequency and the transition frequency $\omega_{12}$, $\gamma$ is the inverse life time of $\{2\}$, and $\hat{p}$ the momentum operator for the initial atomic direction $X$. For large detuning, $|2\Delta + i\gamma| >> \Omega$, a further reduction is possible to an even simpler effective Schrödinger equation for the amplitude of level 1 with potential [20, 27, 31]

$$V(X) = \frac{\hbar^2 \Omega^2 \Theta(X)}{4\Delta^2 + \gamma^2} - i \frac{\hbar \gamma \Omega^2 \Theta(X)}{4\Delta^2 + \gamma^2}, \quad (23)$$

which can be transformed by appropriate scaling to the form of Eq. (21). We have solved numerically the full
two-channel Schrödinger equation with the Hamiltonian of Eq. (22) and the initial state of Eq. (20) for the $|1\rangle$ component. We have found $\tau_T(x) = 1/2V_1$ for arbitrarily large distances to the shutter, in full agreement with the solution of the one-channel equation with absorbing potential, Eq. (1).

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