Mutex Graphs and Multicliques: Reducing Grounding Size for Planning

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We present an approach to representing large sets of mutual exclusions, also known as mutexes or mutex constraints. These are the types of constraints that specify the exclusion of some properties, events, processes, and so on. They are ubiquitous in many areas of applications. The size of these constraints for a given problem can be overwhelming enough to present a bottleneck for the solving efficiency of the underlying solver. In this paper, we propose a novel graph-theoretic technique based on multicliques for a compact representation of mutex constraints and apply it to domain-independent planning in ASP. As computing a minimum multiclique covering from a mutex graph is NP-hard, we propose an efficient approximation algorithm for multiclique covering and show experimentally that it generates substantially smaller grounding size for mutex constraints in ASP than the previously known work in SAT.

1 Introduction

Mutual exclusion (mutex) can be traced back to concurrency control, which refers to the condition that prevents simultaneous accesses to a shared resource. In knowledge representation, they specify the constraints that some properties cannot hold at the same time. For example, an object cannot be at different locations at the same time. These constraints frequently occur in applications from model-checking problems in computer-aided verification [2], computer vision [12, 17], graph algorithms [11], and AI planning [14].

The goal of this paper is to develop a graph-theoretic technique for compactly encoding large sets of mutex constraints and apply it to planning in ASP. We do this by focusing on domain-independent AI planning as started out by SATPlan [10]. That is, we will first obtain an ASP planner by a straightforward translation from SATPlan and then study how to encode mutex constraints compactly for the planner.

In SAT/ASP planning, mutex constraints are specified by formulas/rules that, for any state (which involves a time step, also called a layer in this paper), the actions with conflicting preconditions or effects, and the fluents that are inferred to be conflicting, are mutually exclusive. A naive encoding of these constraints can certainly generate enough rules to overwhelm the underlying solver for large planning instances. For example, in SAT planning these constraints can be expressed by 2-literal clauses (a 2-literal clause is of the form \( l_1 \lor l_2 \) where \( l_1 \) and \( l_2 \) are literals), which, according to [14], constitute about 50-95% of the formulas, and sometimes they used so much memory that they could not fit in a 32-bit address space.

As shown in [14], significant space-savings can be gained by considering the way in which we encode mutex constraints. We may view the set of mutex constraints on fluents as an undirected graph, called a mutex graph, where each fluent is a vertex and each constraint is an edge. When a solver selects one fluent to be true at a given layer, it infers by unit-propagation that each fluent joined directly by an edge
with the selection must be false. Thus, the set of fluents which are true at a given layer constitutes an independent set on the mutex graph.

Rintanen [14] shows that there exist other smaller encodings besides the naive approach of listing out every individual binary constraint and implies that since these encodings are smaller, they must be superior. In their experiments, they use instances of the AIRPORTS domain from an IPC planning competition. This domain is notable because of the vast number of mutex constraints it generates. The larger instances of this problem emit complex mutex graphs which can overwhelm the underlying SAT solver if encoded naively (in a one-constraint-per-edge fashion).

Rintanen further shows that the mutex graphs in these planning problems (even in benchmark AIRPORTS) tend to be highly structured and that in SAT it is possible to cover the mutex graph (somewhat more compactly) with cliques (complete subgraphs) or with bicliques (complete bipartite subgraphs). A biclique can be expressed in SAT using only one auxiliary variable and one binary clause per assignment. Rintanen demonstrates that cliques can be expressed using only a logarithmic set of bicliques. He concludes that the best way to express a mutex graph in SAT is with a biclique edge-covering.

In this paper, we show that for ASP, cardinality constraints give us more power than is available in SAT and indeed we can directly encode a mutex graph by its clique covering (without the extra cost of a logarithmic factor), but further we can eliminate the choice of whether to use cliques or bicliques entirely and instead cover the graph with multicliques (complete multi-partite subgraphs) which is a generalization of both. Indeed, we find that with multicliques, the number of clauses (namely ASP rules) and literals required to encode mutex constraints can be further reduced over Rintanen’s results.

The next section provides an ASP planner as the context of dealing with mutex constraints. We also review the definitions of cliques/bicliques and comment on the complexity and representation issues. Section 3 then presents an approximation algorithm for multiclique covering and Section 4 shows how to construct action mutex constraints simultaneously. In Section 5 we present experimental results. Section 6 comments on related work and Section 7 concludes the paper with final remarks.

The ASP encodings in this paper are constructed to run on CLINGO and follow the ASP-Core-2 Standard [4] except that (i) we will use ; to separate rule body atoms since the more conventional comma sign , is overloaded and has a different meaning in more complex rules (CLINGO supports both), and (ii) the disjunctive head of a rule may be written by a conditional literal. The work reported here has been used in a recent construction of a cost-optimal planner in ASP [18].

2 Preliminaries

2.1 STRIPS Planning in ASP

We adopt a direct translation of 5 rules of SATPlan [10] into ASP and call the resulting planner ASPPlan.

| Rule          | ASP Rule                                      |
|---------------|----------------------------------------------|
| 1.            | `holds(F,K) :- goal(F); finalStep(K).`       |
| 2.            | `happens(A,K-1) : add(A,F), validAct(A,K-1) :- holds(F,K); K > 0.` |
| 3.            | `holds(F,K) :- pre(A,F); happens(A,K); validFluent(F,K).` |
| 4.            | `:- mutexAct(A,B); happens(A,K); happens(B,K).` |
| 5.            | `:- mutex(F,G); holds(F,K); holds(G,K).`     |

An independent set on a graph is a set of vertices where no two vertices in the set share an edge [16]; equivalently this is a clique in the complement graph.
where \( \text{validAct}(A,K) \) means that action \( A \) can occur at time \( K \) and \( \text{validFluent}(F,K) \) means fluent \( F \) can be true at time \( K \). Time steps used in constructing a plan are also called layers.

Rule 1 says that goals hold at the final layer. In rule 2, if a fluent holds at layer \( K \), the disjunction of actions that have that fluent as an effect hold at layer \( K - 1 \). The next rule says that actions at each layer imply their preconditions. The last two rules are mutex constraints: in rule 4, actions with (directly) conflicting preconditions or effects are mutually exclusive, and in rule 5, the fluents that are inferred to be mutually exclusive are encoded as constraints.

Following SATPlan, we add to our plan “preserving” actions for each fluent. The goal is to simulate the frame axioms by using the existing machinery for having an action add a fluent that gets used some steps later. These preserving actions can be specified as:

\[
\begin{align*}
\text{action}(\text{preserve}(F)) & : - \text{fluent}(F). \\
\text{pre}(\text{preserve}(F),F) & : - \text{fluent}(F). \\
\text{add}(\text{preserve}(F),F) & : - \text{fluent}(F).
\end{align*}
\]

where each fluent \( F \) has a corresponding preserving action denoted by term \( \text{preserve}(F) \). Preserving actions can be easily distinguished from regular actions. Now that an action occurs at time \( K \) indicates that its add-effect \( F \) will hold at time \( K + 1 \).

Note that the reason why rule 5 of ASPPlan prevents fluents from being deleted before they’re used is a bit subtle. In order for a fluent to hold, it must occur in conjunction with a preserving action at each time step it’s held for. A preserving action has that fluent as a precondition and so would be mutex with any action that has it as a delete effect. This means that deleting actions cannot occur as long as that fluent is held (by rule 4).

Like SATPlan, we run this planner by solving at some initial makespan \( K \), where \( K \) is the first layer at which \( \text{validFluent}(F,K) \) holds for all \( \text{goal}(F) \), and if it is UNSAT, we increment \( \text{finalStep} \) by 1 until we find a plan.

This is a straightforward and unsurprising encoding in every respect, but has a somewhat surprising consequence as compared to SATPlan. Because ASP models are stable, for any fluent \( F \), \( \text{holds}(F,K) \) can only be true if there exists some action which requires its truth as per rule 3. Similarly for actions as per rule 2. Furthermore, since rule 2 is disjunctive at every step, the set of actions which occurs is a minimal set required to support the fluents at the subsequent step. This conforms exactly to the approach to planning by Blum and Furst [3]: First build the planning graph, then start from the goal-state planning backwards, at each step selecting a minimal set of actions necessary to add all the preconditions for the current set of actions. That is, in this ASP translation, the neededness-analysis as carried out in [15] is accomplished automatically during grounding or during the search for stable models.

**Smart Encoding of Action Mutexes:** Let us first consider action mutex constraints as expressed by Rule 4 of ASPPlan, which can blow up in size when grounded because nearly any two actions acting on the same fluent can be considered directly conflicting. For example, assume a planning problem in which there is a crane which we must use to load boxes onto freighters and there are many boxes and many freighters available but only one crane. Then we will have one such constraint for every two actions of

\[2\]Blum and Furst [3] give a handy way to identify for each action and each fluent, what is the first layer at which this action/fluent might occur by building the planning graph. Note that validAct/2 and validFluent/2 as well as predicates mutexAct/2 and mutex/2 are all extracted from the planning graph.

\[3\]As a further note, when PDDL (planning domain definition language) without any extensions is defined, goals can only be positive and actions can only have positive preconditions. There is a :negative-preconditions extension to PDDL, but we didn’t use it. Any problem which uses :negative-preconditions can be trivially adapted to avoid using it by adding a fluent :not-F for every fluent :F and then adding a corresponding add-effect wherever there’s a delete-effect and vice versa.
the form, load(Crate, Freighter), for any crate and any freighter. As there is already a quadratic number of actions in the problem description size (crates × freighters), the number of mutex constraints over pairs of actions is quartic in the initial (non-ground) problem description size.

We would like to avoid such an explosion by introducing new predicates to keep the problem size down. We will only consider two actions to be mutex if one deletes the other’s precondition. But we will take extra steps to ensure that no add-effect is later used if the same fluent is also deleted at that step. Here is the revised encoding of rule 4.

\[
\text{used}_{\text{preserved}}(F,K) :- \text{happens}(A,K); \text{pre}(A,F); \text{not del}(A,F).
\]

\[
\text{deleted}_{\text{unused}}(F,K) :- \text{happens}(A,K); \text{del}(A,F); \text{not pre}(A,F).
\]

\[
:\text{happens}(A,K): \text{pre}(A,F), \text{del}(A,F)} > 1; \text{valid}_{\text{at}}(F,K).
\]

\[
\text{deleted}(F,K) :- \text{happens}(A,K); \text{del}(A,F).
\]

\[
:\text{holds}(F,K); \text{deleted}(F,K-1).
\]

Effectively, we are splitting the ways in which we care that an action A can relate to a fluent F into three different cases: (i) A has F as a precondition, but not a delete-effect; (ii) A has F as a delete-effect, but not a precondition; and (iii) A has F as both a precondition and a delete-effect.

By explicitly creating two new predicates for properties (i) and (ii), we have packed this restriction into one big cardinality constraint. Further, we must account for conflicting effects, so we define one more predicate (deleted/2) which encapsulates the union of all actions from properties 2 and 3 (those that delete F) and assert that F cannot hold at this step if any of those actions occurred in the previous one.

### 2.2 Cliques and Bicliques

We review the definitions of cliques and bicliques and comment on their possible encodings in SAT.

Let \( G = (V,E) \) be an undirected graph. A clique is a subgraph \( (C,E') \) of \( G \) such that \( C \subseteq V \) and \( E' = \{(v,u) \in E \mid v,u \in C, u \neq v\} \). A biclique is a subgraph \( (C,C',E') \) of \( G \) such that \( C,C' \subseteq V \), \( C \cap C' = \emptyset \), and \( E' = \{(u,v) \in E \mid u \in C, v \in C'\} \).

That is, cliques are complete subgraphs of a graph and bicliques are complete bipartite subgraphs of a graph. Deciding if a graph has a clique of size \( n \) is known to be NP-complete [6, 9]. This is also the case for bicliques under several size measures [6, 13, 19]. There are approximation algorithms for the computation of cliques and bicliques, with approximation guarantees [7], or without [14].

In SAT, given \( n \) fluents, besides the naive \( O(n^2) \) size representation, cliques can be represented in size \( O(n) \) using \( O(n) \) many auxiliary variables, or in size \( O(n \log n) \) using only \( O(\log n) \) many auxiliary variables. Bicliques enjoy a more compact representation: if \( C \) and \( C' \) form a biclique, then \(|C| \times |C'|\) many binary constraints can be represented by \(|C| + |C'|\) many 2-literal clauses using only one auxiliary variable [14]. The idea is that for any literals \( l \in C \) and \( l' \in C' \), mutex constraints of the form \( l \lor l' \) can all be represented using one new variable, say \( x \), by \( \neg l \rightarrow x \) and \( x \rightarrow l' \).

\[4\]There is a minor difference between the definition of mutex as given in [3], which appears to be overly restrictive, and the definition we’re using. Whereas graphplan treats any two actions as mutex if they have conflicting effects (one adds a fluent which the other deletes), we only consider them to be mutex if they have conflicting effects and the add-effect is used at that layer. So we allow actions to occur simultaneously with conflicting effects as long as the relevant fluent doesn’t hold afterwards.
3 An Approximation Algorithm for Multiclique Covering

In this section, we formulate a polynomial-time, approximation algorithm for multiclique covering. First, let us have a formal definition of multiclique.

**Definition 3.1** Let \( G = (V, E) \) be an undirected graph. A multiclique of \( G \) is a subgraph \((C_1, \ldots, C_k, E')\) of \( G \), such that \( C_1 \cup \cdots \cup C_k \subseteq V \), \( C_i \cap C_j = \emptyset \) for all \( 1 \leq i, j \leq k \) where \( i \neq j \), and \( E' = \{(u, v) \in E \mid u \in C_i, v \in C_j, i \neq j\}\).

We call each \( C_i \) \((1 \leq i \leq k)\) above a partition.

**Proposition 3.1** A multiclique is a graph whose complement is a cluster graph, i.e., a set of disjoint cliques.

The claim is easy to verify. Consider any graph \( G \) which is a multiclique by definition. In the complement graph \( G^c \), every partition is a clique. Further, since any two vertices \( u \) and \( v \) must have an edge if they belong to separate partitions in \( G \), it follows that there are no edges between partitions in \( G^c \), therefore, the only edges in \( G^c \) belong to cliques. Similarly, if \( G^c \) is a cluster-graph, then the connected components form the partitions in \( G \) as a multiclique.

Given a mutex graph, a naive encoding of mutex constraints in ASP is to list each edge between two vertices by a 2-literal constraint. With a multiclique covering, mutex constraints in a mutex graph can be encoded compactly.

Given a graph \( G = (V, E) \), the goal of multiclique covering is to produce a sequence of multicliques \( \Pi = (S_1, \ldots, S_n) \) for some \( n \), where each \( S_i \) is a multiclique subgraph of \( G \), for all \( j > i \), \( S_j \) contains at least one edge not in \( S_i \), and the union of edges in \( S_k \) \((1 \leq k \leq n)\) is \( E \). In the multiclique covering \( \Pi \), \( S_i \) and \( S_j \) may share some vertices. In general, to cover all mutex constraints in a mutex graph, the edges covered in different multicliques in \( \Pi \) need not be non-overlapping. In our algorithm, we do allow overlapping if it leads to more compact representation. In summary, as the edges in a mutex graph represent constraints, multiclique covering is to cover the edges of the mutex graph where the edges are spread out in multicliques that are constructed.

For each multiclique constructed, we can encode a constraint graph in ASP as:

\[
\begin{align*}
\% \text{Covering is given by } \text{inPartition}(F,P) & \text{ if fluent } F \text{ belongs to partition } P, \\
\% \text{ and } \text{inMulticlique}(P,M) & \text{ if partition } P \text{ belongs to multiclique } M. \\
\% p(P,K) & : \text{P is a partition at layer } K.
\end{align*}
\]

\[
\text{partitionHolds}(P,K) \leftarrow holds(F,K); \text{inPartition}(F,P). \\
:- \{p(P,K): \text{partitionHolds}(P,K), \text{inMulticlique}(P,M)\} > 1; \\
\text{multiclique}(M); \text{layer}(K).
\]

Here we have a cardinality constraint expressing the rule that among all partitions \( P \) of multiclique \( M \), at most one holds at layer \( K \). Furthermore, if any fluent \( F \) holds at layer \( K \), so does its partition \( P \).

Additionally, we can avoid some unnecessary rules by handling singleton partitions specially. A singleton partition can be packed directly into the cardinality constraint rather than introduced through an auxiliary atom:

\[
\begin{align*}
:- \{\text{partitionHolds}(P,K): \text{inMulticlique}(P,M); \\
& \quad holds(F,K) : \text{singletonPartitionOf}(F,M)\} > 1; \\
& \text{multiclique}(M); \text{layer}(K).
\end{align*}
\]

Now, our ASPPlan given in Section [2.1](#) is updated by replacing Rule 5 therein with the above rules.
Algorithm 1 Multiclique Covering

1: procedure FIND_COVER(g : Graph) → Set MultiClique
2: var uncovered ← g.edges :: Set Edge
3: var multicliques ← {} :: Set MultiClique
4: while uncovered.nonempty do
5:   new_multiclique ← NEXT_MULTICLIQUE()
6:   multicliques ← multicliques ∪ {new_multiclique}
7:   uncovered ← uncovered \ EDGES_COVERED_BY(new_multiclique)
8: end while
9: return multicliques
10: end procedure

Hence, given a planning instance, if we can construct a multiclique covering from its mutex graph, we can use ASP to encode these constraints compactly. Now let us find an algorithm for this task.

In general, finding a minimum multiclique covering (using as few multicliques as possible) is NP-hard. To see this, consider the problem of finding a minimum multiclique covering on a bipartite graph. It’s easy to see that a multiclique on a bipartite graph is a biclique. Thus the minimum multiclique covering of a bipartite graph is the minimum biclique covering. The size of the minimum biclique covering of a bipartite graph is also known as its bipartite dimension. Finding the bipartite dimension of a graph is known to be NP-hard [1]. Thus, finding a minimum multiclique cover is also NP-hard.

Nonetheless, we can still use approximation algorithms similar to those used in [7]. One critical observation is that under the restriction that a multiclique must use exactly a particular set of vertices, there is always only one optimal way to partition those vertices into a multiclique to cover a maximal set of edges: If there is a path between two vertices $v$ and $w$ in the complement of the induced graph, then they must belong to the same partition. If there is no path, then we might as well put them in separate partitions. Therefore, the best partition is the one which makes a partition for each connected component in the complement of the induced graph.

Let us use an example to illustrate. Consider the mutex graph $G = (V, E)$ on the left of the figure below and its complement graph $G^c$ on the right. The connected components of $G^c$ give us a multiclique $\{\{a, b, d\}, \{c\}, \{e\}\}$, which covers almost all edges in $E$ except edge $(a, b)$. So edge $(a, b) \in E$ will have to be captured in another multiclique.

![Figure 1: An example mutex graph and its complement.](image)

Our algorithm is given in Algorithm 1, with supporting functions given in Algorithm 2. The algorithm is greedy, simple, and polynomial-time. We track the set of uncovered edges and tack multicliques on one at a time, greedily building each multiclique in such a way so as to maximize the difference $\phi_1 - \phi_2$, where $\phi_1$ is the number of literals in the naive encoding and $\phi_2$ is the number of literals in our ASP encoding of the corresponding multiclique. A difference indicates an encoding reduction.
Algorithm 2 Multiclique Covering Helper Functions

1: type MCPartition = Set Vertex
2: type MultiClique = Set MCPartition
3: function MAKE_MULTICLIQUE(vs :: Set Vertex) → MultiClique
   4:     return g.induced_subgraph(vs).complement().connected_components()
5: end function
6: function EDGES_COVERED_BY(mc :: MultiClique) → Edge
   7:     return \{(x, y) | p ∈ mc, q ∈ mc, p ≠ q, x ∈ p, y ∈ q\}
8: end function
9: function COUNT_UNCOVERED_INCIDENT_EDGES(x :: Vertex) → N
   10:     return |\{(g.incident_edges(x) ∩ uncovered)\}|
11: end function
12: procedure DEFAULTS_FOR(vs :: Set Vertex) → MCPartition
   13:     candidates ← \{g.neighbors(v) | v ∈ vs\} :: Set Vertex
   14:     return \{c | c ∈ candidates, |g.incident_edges(c) ∩ uncovered| ≥ 2\}
15: end procedure
16: procedure SCORE(vs :: Set Vertex) → Z
   17:     multiclique :: MultiClique
   18:     multiclique ← MAKE_MULTICLIQUE(vs) ∪ DEFAULTS_FOR(vs)
   19:     newly_covered :: Set Edge
   20:     newly_covered ← EDGES_COVERED_BY(multiclique) ∩ uncovered
   21:     complexity_cost :: Z
   22:     complexity_cost ← \[\sum_{p \in \text{multiclique}} \begin{cases} 1 & \text{if } |p| = 1 \\ 2 \times |p| + 1 & \text{if } |p| > 1 \end{cases}\]
   23:     return 2 * |newly_covered| − complexity_cost
24: end procedure
25: procedure NEXT_MULTICLIQUE → MultiClique
   26:     first_vertex :: Vertex
   27:     first_vertex ← argmax_{w \in \text{vertices}} (\lambda w. COUNT_UNCOVERED_INCIDENT_EDGES(w))
   28:     var vertex_set ← \{first_vertex\} :: Set Vertex
   29:     repeat
   30:         next :: Vertex
   31:         next ← argmax_{w \in \text{vertices}} (\lambda w. SCORE(vertex_set ∪ \{w\}))
   32:         improved ← SCORE(vertex_set ∪ \{next\}) > SCORE(vertex_set)
   33:         if improved then
   34:             vertex_set ← vertex_set ∪ \{next\}
   35:         end if
   36:     until improved
   37:     return MAKE_MULTICLIQUE(vertex_set ∪ DEFAULTS_FOR(vertex_set))
38: end procedure

For more details, in Algorithm 1, the variable multicliques is empty to start with. Then it iteratively adds one new multiclique at a time until all edges are covered.
In the helper function NEXT_MULTICLIQUE in Algorithm 2, we select the first vertex by finding the one incident to the most uncovered edges. This is accomplished at Line 27 (line numbers below all refer to Algorithm 2), where we use a lambda function which is applied to each vertex for the parameter w. We then repeatedly select each subsequent vertex to greedily maximize the size difference mentioned above under the assumption that we will finish by adding on a “default partition” of vertices, until no improvement can be generated (lines 29-36 of Algorithm 2). The default partition consists of all vertices which have an edge to every vertex we have selected so far including at least two edges not yet covered (lines 12-15).

Given a set of vertices vs, the function MAKE_MULTICLIQUE(vs) generates a multiclique, where the partitions are obtained by finding the connected components of the complement graph induced from vs, along with the covered edges (lines 3-5).

Note that, instead of removing edges from the graph once they’ve been assigned to a multiclique, we keep a separate record of “uncovered” edges which still remain to be assigned. In this way the same edge may be covered twice by different multicliques if that helps minimize the encoding (cf. line 27).

**Theorem 3.1** Given a mutex graph $G = (V, E)$, the algorithm FIND_COVER terminates after a number of execution steps in polynomial time in the size of $G$, and after termination, a sequence of multicliques $\{(V_1, E_1), \ldots, (V_n, E_n)\}$ is generated such that $V_1 \cup \cdots \cup V_n \subseteq V$ and $E_1 \cup \cdots \cup E_n = E$.

**Proof:** First, we verify that each $(V_i, E_i) (1 \leq i \leq n)$ is a multiclique. $V_i$ is returned as a set of vertices by NEXT_MULTICLIQUE and partitioned by MAKE_MULTICLIQUE into $C_1, \ldots, C_k$ satisfying the following statement: for any $i \neq j$, $v \in C_i$ and $v' \in C_j$ iff there is no path between $v$ and $v'$ in the complement of the graph induced from $V_i$ iff there is an edge between $v$ and $v'$ in the given mutex graph. Hence, each vertex in any partition is connected to every vertex in a different partition. Then, to obtain a multiclique, we only need to let $E_i$ be the set of edges that connect vertices of different partitions.

The algorithm terminates since each $E_j$ covers at least one of the uncovered edges. The first vertex is selected such that it maximizes the number of uncovered edges to which it’s incident, so as long as there are uncovered edges, we’re guaranteed to select a first vertex which is incident to at least one of them. Trivially we can extend this to a multiclique which covers an uncovered edge by selecting the vertex on the other side of any one of them for a score of at least zero. Since the score of the multiclique is only allowed to improve from there and the score measures the number of uncovered edges we’ve covered, it must be the case that every multiclique will cover at least one new uncovered edge (otherwise its score would be negative).

The claim on polynomial time holds because the number of multicliques is bounded by $|E|$ and there are at most $|E|$ calls to NEXT_MULTICLIQUE; further, it can be easily checked that the computation of each such call takes polynomial time. □

Let’s take a look at how this behaves on an example graph. We’ll start with a mutex graph for a ferry crossing problem in which we have three islands, a ferry and a car. The ferry can be at any of the three islands and it can have just moved or be in the process of loading. The car can be on the ferry or at one of the three islands. If loading then the car is not currently on the ferry. Figure 2 shows what the mutex graph for the problem looks like.

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5 If there is only one, there will be no savings in encoding size, as it would require the same number of literals/rules to include a vertex in a partition.
Now let's run our multiclique cover algorithm on it. We get:

```prolog
% Multiclique 0 has all singleton parts
:- {holds(just Moved(ferry, island_a), T); 
    holds(just Moved(ferry, island_b), T); 
    holds(just Moved(ferry, island_c), T); 
    holds(loading(ferry), T) 
} > 1; step(T).

% Multiclique 1 has all singleton parts
:- {holds(car At(island_a), T); 
    holds(car At(island_b), T); 
    holds(car At(island_c), T); 
    holds(on_ferry(car), T) 
} > 1; step(T).

% Multiclique 2 has three non-singleton partitions
partitionHolds(part(2,0), T) :- holds(ferry_at(island_a), T).
partitionHolds(part(2,0), T) :- holds(just_moved(ferry, island_a), T).
partitionHolds(part(2,1), T) :- holds(ferry_at(island_b), T).
partitionHolds(part(2,1), T) :- holds(just_moved(ferry, island_b), T).
partitionHolds(part(2,2), T) :- holds(ferry_at(island_c), T).
partitionHolds(part(2,2), T) :- holds(just_moved(ferry, island_c), T).
```

Figure 2: Mutex graph for the ferry problem.
:- \{\text{partitionHolds(part(2,0),T); partitionHolds(part(2,1),T); partitionHolds(part(2,2),T)} \} > 1; \text{step(T)}.

\% Multiclique 3 has two singleton parts and so is just a normal \% mutex constraint.
:- \text{holds(loading(ferry),T); holds(on_ferry(car),T)}.

In total we have (per-layer) a grounded 10 rules with 25 literals. Had we used the naive encoding it would have been 22 rules with 44 literals so we can see this encoding is quite a bit more compact.

To give a better picture, in Figure 3 we color each edge with the multiclique to which it belongs. Note that three of the edges ended up in two distinct multicliques and so are duplicated in the image:

![Image](image.png)

Figure 3: Colored multiclique covering for the ferry problem.

### 4 Eventual Fluent Mutex Constraints

In Section 2.1 we found a way for the ASP solver to avoid explicitly dealing with action mutex constraints and so were able to save on grounded encoding space. But we still have a problem because the algorithm presented by Blum and Furst [3] for generating fluent mutex constraints in the first place requires simultaneously constructing action mutex constraints.

Indeed, Rintanen [14] reports being unable to run experiments on the largest AIRPORTS instances from IPC-2004 because the action mutex constraints used so much memory they wouldn’t fit in a 32-bit address space.
In this section, we find a way to circumvent this problem and were able to generate mutex constraints on the very largest (AIRPORTS-50) instance while using only about a gigabyte of memory.

Mutex constraints as defined in [3] are “per-layer”. You determine the set of mutex constraints at each layer by looking at what actions, fluents and mutex constraints were in the previous layer. Two actions are mutex if they are directly mutex or have any mutex preconditions. Two fluents are mutex if all respective pairs of causing actions are mutex. However, suppose we only care to discover and encode which fluents are always mutex in the sense that for every layer up to an arbitrarily large makespan they cannot both be true.

One way to obtain this set is to build the planning graph outward until the set of mutex constraints stabilizes. That is, we can stop once we find two consecutive layers at which the set of mutex constraints doesn’t change. But this would still require tracking action mutex constraints for all pairs of actions.

The key insight is that fluents which are always mutex will be so in sequential planning (where exactly one action happens at each layer) as well as in parallel planning. A parallel plan is just a way of compressing a sequential plan into fewer steps so the set of pairs of things which can be true at some point will be the same regardless of how we express it.

Since a sequential plan can be expressed as a parallel plan where at most one non-preserving action happens at each layer, we can run the mutex generation algorithm under the assumption that all non-preserving actions are mutex with each other. Then we only need to explicitly keep track of which actions are mutex with each of the preserving actions. There are generally significantly fewer preserving actions than total actions. When the set of mutex fluent-pairs stabilizes, it should come out the same as if we had obtained these pairs by building the planning graph normally and waiting for the mutex fluents to stabilize.

5 Experiments

We implemented the multiclique generation algorithm in Haskell, representing a fluent or action as an Int and a collection of mutex constraints as an IntMapIntSet. Both IntMap and IntSet come from the containers package. A partition of a multiclique was represented as an IntSet, a multiclique as a list of partitions, and a multiclique covering as a list of multicliques.

We ran this algorithm on the same instances as Rintanen (as well as on the AIRPORTS-50 instance, the largest problem in the set) and found a significant improvement over his results. Note that these edge-counts do not take into account neededness. That is, they cover many fluents and actions which are irrelevant to the goal of the problem and are guaranteed not to be explored by the solver. When we accounted for neededness we found the graphs got much smaller (approximately 5-fold). But we chose not to utilize this so that our results would be better comparable to Rintanen’s.

In Table 5 “Edges” is the number of edges in the mutex graph for each instance. “CL” is the number of grounded clauses (rules) we used to encode this graph. These clauses are a mix of binary constraints and “at most 1” cardinality constraints. Because not all the clauses are binary, we are compelled to give the sum number of literals among all the constraints. This is the “Lit” column.

During our experiments, after a look at a couple of example instances, it became immediately clear to us that the majority of edges belong to the first few multicliques found. After that the number of edges covered per clause drops off rapidly. Thus, if we are willing to forget a small percentage of the edges, we can reduce the number of clauses necessary to encode the graph much further. For each instance, we reran the multiclique generation algorithm terminating it as soon as it had covered 90% of the total number of
Table 1: Multiclique Reduction for AIRPORTS (Abbreviated AP)

| Instance | Edges | CL | Lit | Edges* | CL* | Lit* | R-Lit |
|----------|-------|----|-----|--------|-----|------|-------|
| AP-21    | 181884| 7531| 16437| 166229 | 2336| 4783 | 26382 |
| AP-22    | 275515| 11310| 25014| 249173 | 3464| 7104 | 42776 |
| AP-23    | 371062| 14969| 33100| 336209 | 4806| 9929 | 63552 |
| AP-24    | 373188| 15353| 33894| 337385 | 4907| 10103| 60814 |
| AP-25    | 467653| 18834| 41821| 421181 | 6208| 12816| 107442 |
| AP-26    | 566948| 22507| 50252| 511401 | 8025| 16625| 100494 |
| AP-27    | 571298| 22777| 50801| 514978 | 8155| 16890| 107442 |
| AP-28    | 669336| 26488| 59201| 602737 | 9941| 20616| 132120 |
| AP-29    | 653096| 20486| 45150| 599396 | 6351| 12956| 89294 |
| AP-30    | 261373| 7531 | 16437| 2336   | 4783 | 26382 |

The resulting numbers of edges covered, clauses, and literals required are given respectively by the columns “Edges*”, “CL*”, and “Lit*”. “R-Lit” gives the number of literals required for Rintanen’s biclique encoding. It’s twice the number of constraints he reports [14] since all his constraints are binary clauses (having exactly two literals).

It is worth mentioning that our implementation of multiclique covering has been employed in a cost-optimal planner in ASP [18]. That is, all the experiment results reported in [18] for that planner used this implementation for the representation of mutex constraints, where every plan produced by the planner was validated by the Strathclyde Planning Group plan verifier VAL [8].

6 Related Work

In [14], an algorithm called IDENTIFY-BICLIQUE is presented. Given a graph $G = (V, E)$, the algorithm starts with the trivial biclique $\emptyset, V$, and repeatedly adds nodes to the first part. Nodes from the second part are removed if there is no edge between them and the new node in the first part. The nodes are chosen to maximize the size reduction. The algorithm terminates when the size reduction is no longer possible.

Our algorithm on multiclique covering is a natural extension of the IDENTIFY-BICLIQUE algorithm with some key differences.

- We’re generating multicliques rather than bicliques so there can be more than two partitions. In contrast with Rintanen’s explicit construction of two partitions, which is possible and convenient because of the limit on two, we generate partitions for a multiclique based on a graph-theoretic property.
- As commented earlier, instead of removing edges from the graph once they’ve been assigned to
An Approximation Algorithm for Multiclique Covering

In this paper, we address this problem by proposing an algorithm for a multiclique covering from a given mutex graph. As computing a minimum multiclique covering from a mutex graph is NP-hard, we propose an intuitive, approximation algorithm and show experimentally that it generates substantially smaller grounding size for mutex constraints in ASP than the previously known work in SAT.

Like [14], our approximation algorithm does not provide any approximation guarantees. A question of interest is whether such a guarantee can be formulated and proved.

7 When dealing with strictly binary clauses (as in Rintanen’s case), these behave identically since latter metric is just the former multiplied by two.

7

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