RESTRICTED SET OF PATTERNS, CONTINUED FRACTIONS, AND CHEBYSHEV POLYNOMIALS

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Abstract
We study generating functions for the number of permutations in $S_n$ subject to set of restrictions. One of the restrictions belongs to $S_3$, while the others to $S_k$. It turns out that in a large variety of cases the answer can be expressed via continued fractions, and Chebyshev polynomials of the second kind.

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1. Introduction
Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. An occurrence of $\tau$ in $\pi$ is a subsequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $(\pi_{i_1}, \ldots, \pi_{i_k})$ is order-isomorphic to $\tau$; in such a context $\tau$ is usually called a pattern. We say that $\pi$ avoids $\tau$, or is $\tau$-avoiding, if there is no occurrence of $\tau$ in $\pi$. The set of all $\tau$-avoiding permutations in $S_n$ is denoted $S_n(\tau)$.

Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [Kn], [Ta] to theory of Kazhdan-Lusztig polynomials [Fo], and singularities of Schubert varieties [El], [Sc]. A natural generalization of single pattern avoidance is subset avoidance; that is, we say that $\pi \in S_n$ avoids a subset $T \subset S_k$ if $\pi$ avoids any $\tau \in T$. A complete study of subset avoidance for the case $k = 3$ is carried out in [SS] (see also [W,M1,M2]).

Several recent papers [CW,RWZ,MV1,Kr,JR,MV2,MV3,MV4] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs $\tau_1, \tau_2$. Another natural question is to study permutations avoiding $\tau_1$ and containing $\tau_2$ exactly $t$ times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [Kr], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [CW,RWZ,MV1,KJ,JR,MV2,MV3,MV4]. The tools involved in these papers include continued fractions, Chebyshev polynomials, Dyck paths, and ordered trees.
Definition 1. A finite continued fraction with \( n \) steps define as the following expression

\[
\frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n + a_{n+1}}}}.
\]

There are many faces for applications of theory of continued fractions as an examples: Theory of functions, Approximation theory, Numerical analysis, and Restricted pattern. As an application in restricted pattern is appear the continued fraction

\[
\frac{1}{1 - \frac{x}{1 - \frac{x}{\cdots}}}
\]

in [RWZ, CW], and later than in [MV1, Kr, JR, MV2, MV3, MV4]. Now we generalize this continued fraction by the following.

Definition 2. Let us denote the continued fraction with \( k \) steps

\[
\frac{1}{1 - \frac{x}{1 - \frac{x}{\cdots}}}
\]

by \( R_{k;E}(x) \) for any \( k \geq 1 \), and for \( k = 0 \) we define \( R_{0;E}(x) = E \). Also for simplicity we denote \( R_{k;0}(x) \) by \( R_k(x) \).

Properties for \( R_{k;E} \) is given by the following proposition.

Proposition 1. Let \( E \) any expression. Then

(i) For all \( k \geq 1 \);

\[
R_{k;E}(x) = \frac{U_{k-1} \left( \frac{1}{2\sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-2} \left( \frac{1}{2\sqrt{x}} \right)}{\sqrt{x} \left( U_k \left( \frac{1}{2\sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2\sqrt{x}} \right) \right)},
\]

where \( U_k \) if the \( k \)th Chebyshev polynomials of the second kind;

(ii) For all \( k \geq 1 \);

\[
\prod_{j=1}^{k} R_{j;E}(x) = \frac{1}{x^{\frac{1}{2}} \left[ U_k \left( \frac{1}{2\sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2\sqrt{x}} \right) \right]},
\]

where \( U_k \) is the \( k \)th Chebyshev polynomial of the second kind;
(iii) \[
\lim_{k \to \infty} R_{k;E}(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

Proof. (i). For \(k = 1\) the proposition is trivial. By definitions
\[
R_{k+1;E}(x) = \frac{1}{1 - xR_k;E(x)},
\]
and by us induction we yields
\[
R_{k+1;E}(x) = \frac{\sqrt{x} \left( U_k \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) \right)}{\sqrt{x} \left( U_k \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) \right) - x \left( U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-2} \left( \frac{1}{2 \sqrt{x}} \right) \right)},
\]
which means that
\[
R_{k+1;E}(x) = \frac{U_k \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right)}{U_k \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} \left( U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} E \cdot U_{k-2} \left( \frac{1}{2 \sqrt{x}} \right) \right)}.
\]
On the other hand, by definition of Chebyshev polynomials of the second kind have the following property
\[
\sqrt{x} U_k \left( \frac{1}{2 \sqrt{x}} \right) = U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) - \sqrt{x} U_{k-2} \left( \frac{1}{2 \sqrt{x}} \right).
\]
Hence the Proposition holds for \(k + 1\).

Again by us induction it is easy to see the second property, and the third property its yield immediately from \[MV1, Lemma 3.1\].

Example 1. By Proposition \[R_{k;0} = R_k(x), R_{k,1}(x) = R_{k+1}(x), \]
\[
R_{k+1;x}(x) = \frac{U_k \left( \frac{1}{2 \sqrt{x}} \right) - x U_{k-2} \left( \frac{1}{2 \sqrt{x}} \right)}{\sqrt{x} \left[ U_{k+1} \left( \frac{1}{2 \sqrt{x}} \right) - x U_{k-1} \left( \frac{1}{2 \sqrt{x}} \right) \right]}.
\]

Now, for any three set of patterns \(A, B, \) and \(C\) let us define \(F^C_{A;B}(n)\) be the number of all \(\alpha \in S_n(A)\) such that \(\alpha\) containing every pattern in \(B\) exactly once and containing every pattern in \(C\) at least once. The corresponding generating function we denote by \(F^C_{A;B}(x)\). For simplicity, we write \(F_A(x), F_{A;B}(x)\) and \(G_B(x)\) when \(B = C = \emptyset, C = \emptyset,\) and \(A = C = \emptyset\) respectively.

The paper is organized as the following. In section 2, we find a recurrence in terms of generating functions for \(F^C_{A;B}(x)\), and we prove \(F^C_{A;B}(x)\) for all \(A, B, C\) such that \(A \cup B \neq \emptyset\) is a rational function. In section 3 we present an examples of the main results, which are present the relation between the restricted patterns and continued fractions.
2. Main results

Consider an arbitrary pattern \( \tau = (\tau_1, \ldots, \tau_k) \in S_k(132) \). Recall that \( \tau_i \) is said to be a right-to-left maximum if \( \tau_i > \tau_j \) for any \( j > i \). Let \( m_0 = k, m_1, \ldots, m_r \) be the right-to-left maxima of \( \tau \) written from left to right. Then \( \tau \) can be represented as

\[
\tau = (\tau^0, m_0, \tau^1, m_1, \ldots, \tau^r, m_r),
\]

where each of \( \tau^i \) may be possibly empty, and all the entries of \( \tau^i \) are greater than \( m_{i+1} \) and all the entries of \( \tau^{i+1} \). This representation is called the canonical decomposition of \( \tau \). Given the canonical decomposition, we define the ith prefix of \( \tau \) by \( \pi^i(\tau) = (\tau^0, m_0, \ldots, \tau^i, m_i) \) for \( 1 \leq i \leq eq \) and \( \pi^0(\tau) = \tau^0, \pi^{-1}(\tau) = \emptyset \).

Besides, the ith suffix of \( \tau \) is defined by \( \sigma^i(\tau) = (\tau^i, m_i, \ldots, \tau^r, m_r) \) for \( 0 \leq i \leq r \) and \( \sigma^{r+1}(\tau) = \emptyset \). Strictly speaking, prefixes and suffixes themselves are not patterns, since they are not permutations (except for \( \pi^r(\tau) = \sigma^0(\tau) = \tau \)). However, any prefix or suffix is order-isomorphic to a unique permutation, and in what follows we do not distinguish between a prefix (or suffix) and the corresponding permutation. Now, let us find \( f_A^\tau(n) \) in terms of \( f_T(n) \) by the following lemma.

**Lemma 1.** Let \( A \) set of patterns, and let \( \{\beta^{(i)}\}_{i=1}^m \) be sequence of patterns. Then

\[
f_A^{\beta^{(1)}, \ldots, \beta^{(m)}}(n) = \sum_{j=0}^{m} \left( -1 \right)^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq m} f_{A,\beta^{(i_1)}, \ldots, \beta^{(i_j)}}(n).
\]

**Proof.** By definitions

\[
f_A^{\beta^{(1)}}(n) + f_{A,\beta^{(1)}}(n) = f_A(n),
\]

which means this statement holds for \( m = 1 \). So more generally,

\[
f_A^{\beta^{(1)}, \ldots, \beta^{(m+1)}}(n) = \sum_{j=0}^{m} \left( -1 \right)^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq m} f_{A,\beta^{(i_1)}, \ldots, \beta^{(i_j)}}(n),
\]

by use induction and the same argument in the case \( m = 1 \), the theorem holds.

Immediately, we can represent this result (Lemma 1) by another way, as the following.

**Theorem 1.** Let \( \{\alpha^{(i)}\}_{i=1}^m \), \( \{\beta^{(i)}\}_{i=1}^m \) be two sequences of patterns such that \( \alpha^{(i)} \) contains \( \beta^{(i)} \) for all \( i = 1, 2, \ldots, m \), and let \( A \) set of patterns. Then

\[
f_{A,\alpha^{(i_1)}, \ldots, \alpha^{(i_j)}}^{\beta^{(1)}, \ldots, \beta^{(m)}}(n) = \sum_{j=0}^{m} \left( -1 \right)^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq m} f_{A_{i_1}, \ldots, i_j}(n),
\]

where \( A_{i_1}, \ldots, i_j = A \bigcup_{d=1}^{m} \{\alpha^{(d)}\} \setminus \{\alpha^{(i_{d-1})}, \ldots, \alpha^{(i_j)}\} \bigcup_{d=1}^{j} \{\beta^{(i_d)}\} \).


Theorem 3. By induction, by uses Theorem 1, Theorem 2, and us result in \[MV3, \text{Th 3.1}\].

In the current subsection we find a recurrence to calculate the generating function \( F \). The generating function is a rational function satisfying the relation

\[
F_{\tau[1], \ldots, \tau[p]}(x) = 1 + x \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_p=0}^{m_p} \prod_{j=p}^{(j_1-1)} F_{\tau[1], \ldots, \tau[j]}^{(j_1-1), \ldots, (j_p-1)}(x) F_{\tau[1], \ldots, \tau[p]}^{(j_1), \ldots, (j_p)}(x),
\]

where

\[
F_{\tau[1], \ldots, \tau[p]}^{(j_1-1), \ldots, (j_p-1)}(x) = \sum_{j=0}^{p} (-1)^j \sum_{1 \leq t_1 < \cdots < t_j \leq p} F_{A_{t_1}, \ldots, t_j}^{(j_1), \ldots, (j_p)}(x),
\]

such that \( A_{t_1}, \ldots, t_j = \bigcup_q \{ \tau_{q[t]} \} \bigcup \bigcup_q = \{ \tau_{q[t]} \} \).

The generating function \( F_{A,B}(x) \).

Here, we find a recurrence to calculate \( F_{A,B}(x) \). This calculation immediately by induction, by uses Theorem 2, Theorem 3 and us result in \[MV3, \text{Th 2.1}\].

Theorem 2. Let \( \tau[i] = (\tau^{i,0}, d_i, 0, \tau^{i,1}, d_i, 1, \ldots, \tau^{i,m_i}, d_i, m_i) \in S_{d_i,0}(132) \) for \( i = 1, 2, \ldots, p \) such no there two patterns one contain the another. Then

\[
F_{\tau[1], \ldots, \tau[p]} (x) = 1 + x \sum_{a_i=0}^{m_i} \sum_{b_j=0}^{1+r_{\gamma_j}} \prod_{a_i=0}^{m_i} F_{A_{1;B_1}}^{(a_i)} (x) F_{A_{2;B_2}}^{(b_j)} (x), \quad B \neq \emptyset
\]

where \( A_i = \{ \tau_i \} \) and \( B = \{ \gamma_j \} \), and

\[
C = \{ \pi^{a_i-1} (\tau_i) | i = 1, 2, \ldots, a \}, \quad A_1 = \{ \pi^{a_i} (\tau_i) | i = 1, 2, \ldots, a \} \cup \{ \pi^{b_j} (\gamma_j) | j = 1, 2, \ldots, b \},
\]

\[
B_1 = \{ \pi^{b_j-1} (\gamma_j) | j = 1, 2, \ldots, b \}, \quad A_2 = \{ \sigma^{a_i} (\tau_i) | i = 1, 2, \ldots, a \} \cup \{ \sigma^{b_j-1} (\gamma_j) | j = 1, 2, \ldots, b \},
\]

\[
B_2 = \{ \sigma^{b_j} (\gamma_j) | j = 1, 2, \ldots, b \}.
\]
(ii) for two sequences of patterns \(\{\alpha^{(i)}\}_{i=1}^{m}, \{\beta^{(i)}\}_{i=1}^{m}\) such that \(\alpha^{(i)}\) contains \(\beta^{(i)}\) for all \(i = 1, 2, \ldots, m\), and for any set of patterns \(T\),

\[
F_{T,\alpha^{(i)},\ldots,\alpha^{(m)},B}(x) = \sum_{j=0}^{m} (-1)^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq m} F_{A_{i_1},...,i_j:B}(x),
\]

where \(A_{i_1,...,i_j} = T \bigcup_{d=1}^{m} \{\alpha^{(d)}\} \setminus \{\alpha^{(i_1)},...,\alpha^{(i_j)}\} \bigcup_{d=1}^{j} \{\beta^{(i_d)}\}\).

3. Examples and continued fractions

Though elementary, Theorem 2 enables us to derive easily various known and new results for a fixed set of patterns.

Example 2. (see [3]) An a numerical case, by Theorem 2 we yields

\[
F_{\{2341,3241\}}(x) = 1 + x F_{\{23,32\}}(x) F_{\{2341,3241\}}(x) + x(F_{\{23,32\}}(x) - F_{\{23,32\}}(x)) F_{\{2341,1\}}(x) + x(F_{\{2341,32\}}(x) - F_{\{23,32\}}(x)) F_{\{1,3241\}}(x) + x(F_{\{2341,3241\}}(x) - F_{\{23,3241\}}(x) - F_{\{23,32\}}(x) + F_{\{23,32\}}(x)) F_{\{1,1\}}(x).
\]

On the other hand, by definition it is easy to see \(F_{\{23,32\}}(x) = 1 + x, F_{\{\tau,1\}}(x) = 1\), and \(F_{\{23,3241\}}(x) = F_{\{2341,32\}} = \frac{1}{1-x}\). Hence, it is easy to get \(F_{\{2341,3241\}}(x) = \frac{1-x-x^2}{1-2x-x^2}\).

As a corollary of Theorem 2 we obtain the following.

Corollary 1. Let \(T\) set of pattern, and let \(T = \{(\tau_1,\ldots,\tau_{k-1},k)|(\tau_1,\ldots,\tau_{k-1}) \in T'\}\). Then

\[
F_T(x) = \frac{1}{1-xF_T(x)}.
\]

Example 3. (see [5], Pr. 15., and [3], Sec 4.1) An another numerical example, by Corollary 2 we yields

\[
F_{\{123,213\}}(x) = \frac{1}{1-xF_{\{21,12\}}(x)},
\]

and by definitions we get the result [5], Pr. 15., which is

\[
F_{\{123,213\}}(x) = \frac{1}{1-x-x^2}.
\]

In the same way, by use Corollary 2 twice we yields

\[
F_{\{1234,2134\}}(x) = \frac{1-x-x^2}{1-2x-x^2},
\]

which is result [3, Sec. 4.1].
Now let us generalize the above example. First let define a special set of patterns.

**Definition 3.** For any \( k \geq l \geq 1 \), let \( U^k_l \) be the set of all permutations \( \tau \in S_k \) such that \((\tau_{l+1}, \tau_{l+2}, \ldots, \tau_k) = (l+1, l+2, \ldots, k)\). Clearly \( |U^k_l| = l! \).

By \([Kn]\) and definitions

\[
F_{U^k_l}(x) = \sum_{j=0}^{l-1} c_j x^j,
\]

where \( c_j \) is the \( j \)th Catalan number. Hence, consequentially to Example 3 we yields similarly the following.

**Corollary 2.** Let \( k \geq l \geq 1 \); then

\[
F_{U^k_l}(x) = R_{k-l,E(x)}(x),
\]

where \( E(x) = \sum_{j=0}^{l-1} c_j x^j \), and \( c_j \) is the \( j \)th Catalan number.

**Example 4.** (see \([CW, MV1, Kr]\)) For \( l = 1 \), by Corollary 3 we yields

\[
F_{U^k_1}(x) = F_{12\ldots k}(x) = R_{k,0}(x) = R_k(x).
\]

Now we present another direction to use continued fractions.

**Corollary 3.** Let \( k > l \geq 1 \). For any \( \tau \in U^k_l \),

\[
F_{U^k_l \setminus \{\tau\};\tau}(x) = \frac{x^{l^{\downarrow}}}{\left(U_k^{-l}(\frac{1}{2 \sqrt{x}}) - \sqrt{x E(x) U_{k-1-l}(\frac{1}{2 \sqrt{x}})}\right)^2},
\]

where \( E(x) = \sum_{j=0}^{l-1} c_j x^j \), and \( c_j \) is the \( j \)th Catalan number.

**Proof.** Let us fix \( \tau \in U^k_l \) such that \((\tau = \tau', k)\), and let us denote \( U^k_l \setminus \{\tau\} \) by \( M^k_l \). Immediately by Theorem 3 we yields

\[
F_{M^k_l;\tau}(x) = x F_{M^k_l-1,\tau'}(x) F_{U^k_l}(x) + x F_{U^k_l-1}(x) F_{M^k_l;\tau}(x),
\]

which means by Corollary 2

\[
F_{M^k_l;\tau}(x) = \frac{x R_{k-l,E(x)}(x) F_{M^k_l-1,\tau'}(x)}{1 - x R_{k-l-1,E(x)}} F_{M^k_l-1,\tau'}(x),
\]

where \( E = \sum_{j=0}^{l-1} c_j x^j \), and \( c_j \) is the \( j \)th Catalan number. Since \( \frac{1}{1-x R_{m-1,E(x)}} = R_{m,E(x)} \) we obtain

\[
F_{M^k_l;\tau}(x) = x R_{k-l,E(x)}(x) F_{M^k_l-1,\tau'}(x),
\]
By use induction we yields

\[ F_{M^k;\tau}(x) = x^{k-l}F_{M;\beta}(x) \prod_{j=1}^{k-l} R^2_{j;E}(x), \]

where \( \tau = (\beta, l+1, l+2, \ldots, k) \). On the other hand, by definitions \( F_{M;\beta}(x) = x^l \), so

\[ F_{M^k;\tau}(x) = x^k \prod_{j=1}^{k-l} R^2_{j;E}(x). \]

Hence by Proposition 1 the corollary holds.

**Example 5.** (see [MV1, Kr]) For either \( l = 1, E = 1 \); or \( l = 0, E = 0 \), we yields from Corollary 3 for all \( k \geq 1 \)

\[ G_{(12\ldots k)}(x) = F_{\emptyset;12\ldots k}(x) = \frac{1}{U^2_k \left( \frac{1}{2\sqrt{x}} \right)}. \]

For \( l = 2, E = 1 + x \) we obtain for all \( k \geq 3 \)

\[ F_{123\ldots k;213\ldots k}(x) = F_{213\ldots k;123\ldots k}(x) = \frac{x}{U_{k-1} \left( \frac{1}{2\sqrt{x}} \right) - xU_{k-3} \left( \frac{1}{2\sqrt{x}} \right)^2}. \]

**Example 6.** Now we complete all the calculation for either containing exactly once, or avoiding a two patterns from \( U^k_2 = \{123\ldots k, 213\ldots k\} \). By Corollary 2 and Example 5 its left to find \( G_{U^k_2}(x) \). Let \( k \geq 3 \); by Theorem 3

\[ G_{U^k_2}(x) = xF_{U^k_2-1}(x)G_{U^k_2}(x) + xF_{123\ldots k-1,213\ldots k-1}(x)F_{213\ldots k-1,213\ldots k}(x) + xG_{U^k_2-1}(x)F_{U^k_2}(x), \]

so by Corollary 2 Example 5 and by use Proposition 1 we yields

\[ G_{U^k_2}(x) = \frac{2x^2\sqrt{x}}{W_{k;2}(x)W_{k;1}^2(x)} + xR^2_{k-2;1+x}(x)G_{U^k_2-1}(x), \]

where \( W_{k;j}(x) = U_{k-j} \left( \frac{1}{2\sqrt{x}} \right) - xU_{k-2-j} \left( \frac{1}{2\sqrt{x}} \right) \). Besides \( G_{U^k_2}(x) = 0 \) for all \( k = 0, 1, 2, 3 \), hence

\[ G_{U^k_2}(x) = \frac{2x^2\sqrt{x}}{W_{k;2}^2(x)} \sum_{j=3}^{k-2} \frac{1}{W_{k;j-1}(x)W_{k;j}(x)}. \]
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