Regular graphs and the spectra of two-variable logic with counting

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Abstract
The spectrum of a first-order logic sentence is the set of natural numbers that are cardinalities of its finite models. In this paper we show that when restricted to using only two variables, but allowing counting quantifiers, the spectra of first-order logic sentences are semilinear and hence, closed under complement. At the heart of our proof are semilinear characterisations for the existence of regular and biregular graphs, the class of graphs in which there are a priori bounds on the degrees of the vertices. Our proof also provides a simple characterisation of models of two-variable logic with counting – that is, up to renaming and extending the relation names, they are simply a collection of regular and biregular graphs.

Keywords: two-variable logic with counting, first-order spectra, regular graphs, semi-linear, Presburger arithmetic.

1 Introduction

The spectrum of a first-order sentence \( \phi \), denoted by \( \text{Spec}(\phi) \), is the set of natural numbers that are cardinalities of finite models of \( \phi \). Or, more formally, \( \text{Spec}(\phi) = \{ n \mid \text{there is a model of } \phi \text{ of size } n \} \). A set is a spectrum, if it is the spectrum of a first-order sentence.

In this paper we consider the logic \( \text{C}^2 \), the class of first-order sentences using only two variables and allowing counting quantifiers \( \exists^k z \phi(z) \), where \( k \geq 1 \). Semantically \( \exists^k z \phi(z) \) means there exist at least \( k \) number of \( z \)'s such that \( \phi(z) \) holds. We prove that the spectra of \( \text{C}^2 \) are precisely semilinear sets. In fact, our proof also shows that the family of models of a \( \text{C}^2 \) formula can be viewed as a collection of regular graphs. This gives us another interesting insight why modal logic is so robustly decidable [11, 35]. In fact, it must also be noted that the satisfiability of a number of extensions of two-variable logic can be reduced to that of \( \text{C}^2 \).

Related works. Two-variable logic and its variant is an important class of logic used in many settings in computer science such as verification, specification, artificial intelligence, etc [11, 13, 30, 35]. The class \( \text{C}^2 \) itself was first introduced and studied in [14, 31]. In the context of specification and verification of concurrent systems, two-variable logic (without counting) \( \text{FO}^2 \) is especially an important class [13] due to its relation to modal logic commonly used in specification and artificial intelligence.

It was known that two-variable logic has small model property and hence decidable [29]. On the other hand, \( \text{C}^2 \) does not have finite model property, but still decidable [14, 31]. Recently in [24] it is shown that the satisfiability \( \text{FO}^2 \) extended with equivalence closure of two binary predicates, denoted by \( \text{EC}^2_2 \), is decidable in 2-NEXPTIME complete via reduction to Linear Integer Programming (LIP). However, this does not imply that the spectra of \( \text{EC}^2_2 \) or \( \text{FO}^2 \) are semilinear. The satisfiability proof only shows the existence of a model with size that satisfies the LIP, which of course, does not imply that the size of all models must satisfy the LIP too.

The notion of spectrum was introduced by Scholz in [33] where he also asked whether there exists a necessary and sufficient condition for a set to be a spectrum. Since its publication, Scholz’s

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question and many of its variants have been investigated by many researchers for the past 60 years. One of the arguably main open problems in this area is the one asked by in [1], known as Asser’s conjecture, whether the complement of a spectrum is also a spectrum.

The notion of spectrum has a deep connection with complexity theory as shown by Jones and Selman [20], as well as Fagin [4] independently that a set of integers is a spectrum if and only if its binary representation is in NE. Hence, Asser’s conjecture is equivalent to asking whether NE = co-NE. It also immediately implies that if Asser’s conjecture is false, i.e., there is a spectrum whose complement is not a spectrum, then NP ̸= co-NP, hence P ̸= NP. The converse implication is still open. An interesting result in [36] states that if spectra are precisely rudimentary sets, then NE = co-NE and NP ̸= co-NP.† There are a number of interesting connections between spectrum and various models of computation such as RAM as well as intrinsic computational behavior. See, for example, [8, 9, 10, 26, 32]. We refer the reader to [2] for a more comprehensive treatment on the spectra problem and its history.

The logic C2 is not the first logic known to have semilinear spectra. A well known Parikh theorem states the spectra of context-free languages are semilinear, and closed under complementation. Using the celebrated composition method, Gurevich and Shelah in [16] showed that the spectra of monadic second order logic with one unary function are semilinear. In [6] Fischer and Makowsky show that the many-sorted spectra of the monadic second-order logic with modulo counting over structures with bounded tree-width are semilinear. Intuitively, the many-sorted spectra of a formula are spectra which counts the cardinality of the unary predicates in the models of the formula, instead of just counting the sizes of the models.

On the other hand, structures expressible in C2 do not have bounded tree-width. An example is d-regular graphs for d ≥ 3. It should also be noted that in C2 one can express a few unary functions, hence our result does not follow from [16], and neither theirs from ours since we are restricted to using only two variables. As far as we know, C2 is the first logic known to have its spectra closed under complement without any restriction on the vocabulary nor in the interpretation.

The result closest to ours is the one by ´E. Grandjean in [10] where he considers the spectra of first-order sentences using only one variable. A similar result due to M. Grohe and stated in [2], says that for every Turing machine M, there exists a first-order sentence ϕM using only three variables such that Spec(ϕM) = \{t^2 \mid t \text{ is the length of an accepting run of } M\}.

**Sketch of our proof.** Consider the following instances of structures expressible in C2.†

(Ex.1) (c, d)-biregular graphs: the bipartite graphs on the vertices U ∪ V, where the degree of each vertex in U and V is c and d, respectively.

(Ex.2) (c, d)-regular digraphs: the directed graphs in which the in-degree and the out-degree of each vertex is c and d, respectively.

An observation from basic graph theory tells us that for “big enough” M and N,‡

(C1) there is a (c, d)-biregular graph in which M vertices are of degree c and N vertices of degree d if and only if Mc = Nd;

(C2) there is a (c, d)-regular digraph of N vertices if and only if Nc = Nd, and hence, c = d.

These characterisations immediately imply that the spectra of the formulas (Ex.1) and (Ex.2) above are linear sets. It is from these observations that we draw our inspiration to prove the semilinearity of the spectra of C2.

The main technical difficulty that we encounter is that we cannot rely on any available “pumping” argument, argument which is commonly used to prove that a set is semilinear. Hence, we proceed by constructing a Presburger formula that captures the spectrum of a given formula. Since

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†It should be noted that the class of rudimentary sets corresponds precisely to linear time hierarchy – the linear time analog of polynomial time hierarchy [37].

‡Though the result in this paper holds for arbitrary structures, it helps to assume that the structures of C2 are graphs in which the vertices and the edges are labelled with a finite number of colours.

§“Big enough” means M and N are greater than a constant K which depends only on c and d.
Presburger formulas are known to express precisely semilinear sets [7], our result follows immediately.

The crux of our construction is a generalisation of the characterisations (C1)–(C2) above to the following setting. Let $\mathcal{C}$ be a set of $\ell$-colors, denoted by $\text{col}_1, \text{col}_2, \ldots, \text{col}_\ell$, and let $C$ and $D$ be $(\ell \times m)$- and $(\ell \times n)$-matrices whose entries are all non-negative integers. We say that a bipartite graph $G = (U, V, E)$ is $(C, D)$-biregular, if we can color its edges with colors from $\mathcal{C}$ such that there is a partition $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ where

- for every integer $1 \leq i \leq m$, for every vertex $u \in U_i$, for every $1 \leq j \leq \ell$, the number of edges with color $\text{col}_j$ adjacent to $u$ is precisely $C_{j,i}$; and
- for every integer $1 \leq i \leq n$, for every vertex $v \in V_i$, for every $1 \leq j \leq \ell$, the number of edges with color $\text{col}_j$ adjacent to $v$ is precisely $D_{j,i}$.

Our setting also allows us to say that the number of edges with color $\text{col}_j$ adjacent to $v$ is at least $D_{j,i}$. In Theorem 5.1 we effectively construct a Presburger formula that characterises the set \{ $N$ | there is a $(C, D)$-biregular graph of $N$ vertices \}.

In a similar manner, we can define $(C, D)$-regular digraphs, where $C$ and $D$ control the number of incoming and outgoing edges of each vertex, respectively. Likewise, we obtain a similar Presburger formula that characterises the set \{ $N$ | there is a $(C, D)$-regular digraph of $N$ vertices \}. We then proceed to observe that the relations in every model of a $C^2$ formula can be partitioned in such a way that every part forms a $(C, D)$-regular digraph, and every two parts a $(C, D)$-biregular graph. In a sense this shows that the models of $C^2$ are simply a collection of regular graphs. Applying the Presburger formula that characterises the existence of these regular graphs, we obtain the semilinearity of the spectra of $C^2$ formulae.

For the converse direction, it is not that difficult to show that every semilinear set is also a spectrum of a $C^2$ sentence. Since semilinear sets are closed under complement, this establishes the fact that the spectra of $C^2$ are closed under complement. It can also be deduced immediately from our proof that the many-sorted spectra of $C^2$ are also semilinear. Moreover, our result extends trivially to the class $\exists$SOC$^2$, the class of sentences of the form: $\exists R_1 \cdots \exists R_m \phi$, where $R_1, \ldots, R_m$ are second-order variables and $\phi$ is a $C^2$ formula. We simply regard $R_1, \ldots, R_m$ as part of the signature.

Outline of the paper. This paper is organised as follows. We review the definitions of Presburger formulas in Section 2. Then in Section 3 we define the logic $C^2$ and state our main results. In Section 4 we present the modal logic for $C^2$, called quantified modal logic with counting (QMLC). For our purpose it is easier to work with QMLC than with $C^2$. In Sections 5 we introduce the notions of biregular graphs and regular directed graphs and present their Presburger characterisations (Theorems 5.2 and 5.4), which will then be used in Section 6 to show the semilinearity of the spectra of $C^2$. For the sake of readability, we postpone the proof of Theorems 5.2 and 5.4 until Sections 7 and 8. Finally we conclude with some concluding remarks in Section 9.

2 Presburger formula

Presburger formulas are first-order formulas with the relation symbols + and $\leq$ and constants 0 and 1 interpreted over the domain $\mathbb{N}$ in the natural way. For a vector of variables $X = (X_1, \ldots, X_k)$, we write $\phi(X)$ to denote that $X_1, \ldots, X_k$ are the free variables in $\phi$.

A set $\Gamma \subseteq \mathbb{N}^k$ is called a linear set, if there exist vectors $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n \in \mathbb{N}^k$ such that $\Gamma = \{ \vec{v}_0 + m_1 \vec{v}_1 + \cdots + m_n \vec{v}_n \mid m_1, \ldots, m_n \in \mathbb{N} \}$. A semilinear set is a finite union of linear sets.

Theorem 2.1 [7] For every Presburger formula $\phi(X)$, the set $\{ N \mid \phi(N) \text{ holds} \}$ is semilinear.
3 The logic $\mathcal{C}^2$

In this section we review the definition of $\mathcal{C}^2$ and mention the main result in this paper and its corollaries. Let $\mathcal{P} = \{P_1, P_2, \ldots\}$ be the set of predicate symbols of arity 1; while $\mathcal{R} = \{R_1, R_2, \ldots\}$ the set of predicate symbols of arity 2. Two-variable logic with counting, denoted by $\mathcal{C}^2$, is defined by the following syntax.

$$\phi ::= \neg \phi \mid R(z, z) \mid P(z) \mid \phi \land \phi \mid \exists^k z \phi,$$

where the variable $z$ ranges over $x, y$, and the symbols $R$ and $P$ over $\mathcal{R}$ and $\mathcal{P}$, respectively.

The quantifier $\exists^k z \phi$ means there are at least $k$ elements $z$ such that $\phi$ holds. Note that $\exists^1 z \phi$ is the standard $\exists z \phi$, and $\forall z \phi$ is equivalent to $\neg \exists^1 z \neg \phi$. By default, we assume that $\exists^0 z \phi$ always holds.

For simplicity, we are going to use the following notations.

$$\exists^{=k} z \phi ::= \exists^k z \phi \land \neg (\exists^{k+1} z \phi)$$

$$\exists^{\leq k} z \phi ::= \neg (\exists^{k+1} z \phi)$$

As usual, we write $\mathfrak{A} \models \phi$ to denote that the structure $\mathfrak{A}$ is a model of $\phi$.

Theorem 3.1 below is the main result in this paper. We present its proof in Section 6.

**Theorem 3.1** For every $\phi \in \mathcal{C}^2$, there exists a Presburger formula $\text{PREB}(n)$ such that the set $\{n \mid \text{PREB}(n) \text{ holds}\} = \text{Spec}(\phi)$. Moreover, the formula $\text{PREB}(n)$ can be constructed effectively.

We should remark that Theorem 3.1 also holds for arbitrary vocabulary. Since $\mathcal{C}^2$ uses only two variables, relations of greater arity such as $R(x, y, x, y)$ can be viewed simply as unary or binary relations; so we can create new binary and unary relations for each possible combination, and easily verify whether the result is consistent.

An immediate consequence of Theorem 3.1 is the spectra of $\mathcal{C}^2(\text{Sym})$ are semilinear.

**Corollary 3.2** For every sentence $\phi \in \mathcal{C}^2$, the spectrum $\text{Spec}(\phi)$ is semilinear.

On the other hand, it is not that difficult to show that every semilinear set is a spectrum of a $\mathcal{C}^2$ sentence, as formally stated below.

**Proposition 3.3** For every semilinear set $\Lambda \subseteq \mathbb{N}$, there exists a sentence $\phi \in \mathcal{C}^2$ such that $\text{Spec}(\phi) = \Lambda$.

Combining Corollary 3.2 and Proposition 3.3, we obtain the following corollary.

**Corollary 3.4** The spectra of $\mathcal{C}^2$ sentences are closed under complement within $\mathcal{C}^2$.

**Proof.** It follows from Corollary 3.2 and Proposition 3.3 and the fact that semilinear sets are closed under complement.

Theorem 3.1 can be further generalised as follows. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$, where $P_1, P_2, \ldots, P_l$ are unary predicates. Define the image of a structure $\mathfrak{A}$ as $\text{Image}_\mathcal{P}(\mathfrak{A}) = \langle |P_1^\mathfrak{A}|, \ldots, |P_l^\mathfrak{A}| \rangle$. We also define the image of a formula $\varphi$ with predicates from $\mathcal{P}$ as $\text{Image}_\mathcal{P}(\varphi) = \{\text{Image}_\mathcal{P}(\mathfrak{A}) \mid \mathfrak{A} \models \varphi\}$. It must be noted here that $P_{1}^\mathfrak{A}, \ldots, P_{l}^\mathfrak{A}$ are not necessarily disjoint, and that they may not cover the whole domain $\mathcal{A}$. For this reason, the notion of image is more general than the notion of many-sorted spectrum which requires the unary predicates partition the whole domain. With a slight adjustment in our proof in Section 6, we can obtain the following two corollaries.

**Corollary 3.5** Let $\phi \in \mathcal{C}^2$ and $\mathcal{P} = \{P_1, \ldots, P_l\}$, where $P_1, \ldots, P_l$ be a set of unary predicates in $\phi$. The set $\{\text{Image}_\mathcal{P}(\mathfrak{A}) \mid \mathfrak{A} \models \phi\}$ is semilinear.

**Corollary 3.6** Let $\mathcal{P} = \{P_1, \ldots, P_l\}$. The following problem is decidable. Given a $\mathcal{C}^2$ formula $\phi$ and a Presburger formula $\Psi(x_1, \ldots, x_l)$, determine whether there exists a structure $\mathfrak{A} \models \phi$ such that $\Psi(\text{Image}_\mathcal{P}(\mathfrak{A}))$ holds.
4 Quantified modal logic with counting

In this section we present quantified modal logic with counting (QMLC), which for our purpose, will be easier to work with. We are going to show that $C^2$ and $QMLC$ are equivalent in terms of spectra. In fact, our proof shows that $C^2$ and $QMLC$ are equivalent up to renaming/deleting/adding relational symbols. It is an adaptation of the proof in [27] which shows that similar equivalence holds between two-variable logic and modal logic.

Moreover, for ease of presentation, we make the following assumption. Let $\varphi \in C^2$ and let $\mathcal{R}$ and $\mathcal{P}$ be the set of binary and unary predicates appearing in $\varphi$, respectively.

The class $MLC$ of modal logic with counting is defined with the following syntax.

$$\phi ::= \neg \phi \mid \alpha \mid \phi \land \phi \mid \Diamond^k \phi$$

where $\alpha$ ranges over $\mathcal{P}$ and $\beta$ over $\mathcal{R}$.

The semantics of $MLC$ is as follows. Let $\mathfrak{A}$ be a structure of $\tau$ and $a \in A$ and $\phi$ be an $MLC$ formula. That $\mathfrak{A}$ satisfies $\phi$ from $a$, denoted by $\mathfrak{A}, a \models \phi$, is defined as follows.

- $\mathfrak{A}, a \models \neg \psi$, if $\mathfrak{A}, a \not\models \psi$.
- $\mathfrak{A}, a \models \phi_1 \land \phi_2$, if $\mathfrak{A}, a \models \phi_1$ and $\mathfrak{A}, a \models \phi_2$.
- $\mathfrak{A}, a \models \Diamond^k \phi$, if there exist at least $k$ elements $b_1, \ldots, b_k \in A$ such that $R(a, b_i)$ holds in $\mathfrak{A}$ and $\mathfrak{A}, b_i \models \phi$ for $i = 1, \ldots, k$.

We define the class of quantified modal logic with counting, denoted by $QMLC$ with the following syntax.

$$\psi ::= \neg \psi \mid \psi_1 \land \psi_2 \mid \exists \phi$$

where the formula $\phi \in MLC$. A $QMLC$ formula $\psi$ is called a basic $QMLC$, if it is of the form $\exists \phi$, where $\phi \in MLC$.

The semantics of $QMLC$ is as follows. Let $\mathfrak{A}$ be a structure of $\tau$ and $\psi \in QMLC$. That $\mathfrak{A}$ satisfies $\psi$, denoted by $\mathfrak{A} \models \psi$, is defined as follows.

- $\mathfrak{A} \models \neg \psi$, if it is not the case that $\mathfrak{A} \models \psi$.
- $\mathfrak{A} \models \psi_1 \land \psi_2$, if $\mathfrak{A} \models \psi_1$ and $\mathfrak{A} \models \psi_2$.
- $\mathfrak{A} \models \exists \phi$, if there exist at least $k$ elements $a_1, \ldots, a_k \in A$ such that $\mathfrak{A}, a_i \models \phi$ for $i = 1, \ldots, k$.

We also denote by $\text{Spec}(\psi)$, the spectrum of a $QMLC$ formula $\psi$.

Theorem 4.1 below states the spectral equivalence between $C^2$ and $QMLC$.

**Theorem 4.1** For every $\varphi \in C^2$, there is a $QMLC$ formula $\psi$ such that $\text{spec}(\varphi) = \text{spec}(\psi)$.

**Proof.** Let $\varphi \in C^2$ and let $\mathcal{R}$ and $\mathcal{P}$ be the set of binary and unary predicates appearing in $\varphi$, respectively. By extending $\mathcal{R}$ and $\mathcal{P}$ and by modifying the sentence $\varphi$, if necessary, we can obtain another $\varphi' \in C^2$ such that $\text{spec}(\varphi') = \text{spec}(\varphi)$ and every structure $\mathfrak{A} \models \varphi'$ satisfies the following properties.

(N1) $\mathfrak{A}$ is a clique over $A$. That is, for every $a, b \in A$, either $a = b$ or $R(a, b)$ for some $R \in \mathcal{R}$.

(N2) Every binary relation in $\mathcal{R}$ is not reflexive. That is, for every $R \in \mathcal{R}$, for every $a, b \in A$, if $R(a, b)$, then $a \neq b$.

(N3) $\mathcal{R}$ is closed under reversal. That is, for every $R \in \mathcal{R}$, there exists $\overline{R} \in \mathcal{R}$ such that $\overline{R} \neq R$ and for every $a, b \in A$, $R(a, b)$ if and only if $\overline{R}(b, a)$.

(N4) The binary predicates in $\mathcal{R}$ are pairwise disjoint.

This can be obtained by replacing each Boolean combination of relations in $\mathcal{R}$ with a new binary relation.
Note that all these properties can be written in $C^2$ formulas.

Now, given a $C^2$ formula $\varphi$ that satisfies properties (N1)–(N4) above, we are going to describe the construction of the desired QMLC formula $\varphi$ which consists of the following two steps.

1. Convert the sentence $\varphi$ into its “normal form” $\varphi'$.

2. Convert the sentence $\varphi'$ into a “quantified modal logic” (QMLC) sentence $\psi$ such that for all structure $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi'$ if and only if $\mathfrak{A} \models \psi$.

In the following paragraphs we are going to describe formally these two steps.

A sentence $\varphi \in C^2$ is in normal form, if all the quantifiers are either of form $\exists k \ y \ (R(x,y) \wedge \theta(y))$, or $\exists k \ x \ \theta(x)$, and all other applications of variables are of form $P(x)$, where $P \in P$.

We claim that every sentence $\varphi \in C^2$ can be converted into its equivalent sentence in normal form. It can be done as follows.

• First, we rewrite every subformula of the form $\exists k \ y \ (R(x,y) \wedge \theta(y))$ with one free variable $x$ into the following form:

$$\exists^k y \ (R(x,y) \wedge \theta(y)),$$

and all other applications of variables are of form $P(x)$, where $P \in P$.

After such rewriting, we can assume that every quantifier in $\varphi$ is of the form $\exists k \ y \ (R(x,y) \wedge \theta(y))$.

• Second, every quantification $\exists^k y \ ((x \neq y) \wedge \theta(x,y))$, in which $\theta(x,y)$ contains a subformula $\alpha(x)$ depending only on $x$, can be rewritten into the form:

$$\neg \alpha(x) \wedge \exists^k y \ ((x \neq y) \wedge \theta(x,y))$$

$$\alpha(x) \wedge \exists^k y \ ((x \neq y) \wedge \theta(x,y))$$

where $\theta_0(x,y)$ and $\theta_1(x,y)$ are obtained from $\theta$ by replacing $\alpha(x)$ with false and true, respectively. We can repeat this until $\theta(x,y)$ no longer has a subformula depending only on $x$.

After such rewriting we can assume that every quantifier in $\varphi$ is of the form

$$\exists^k y \ ((x \neq y) \wedge \theta(x,y)),$$

where $\theta(x,y)$ does not contain any subformula depending only on $x$.

• Third, every quantification $\exists^k y \ ((x \neq y) \wedge \theta(x,y))$ can be rewritten into the form:

$$\bigvee_{f \in \Delta_k^R} \bigwedge_{R \in \mathcal{R}} \exists^{f(R)} y \ (R(x,y) \wedge \theta_R(y))$$

where $\Delta_k^R$ is the set of all functions $f : \mathcal{R} \to \mathbb{N}$ such that $\sum_{R \in \mathcal{R}} f(R) = k$, and $\theta_R(y)$ is obtained from $\theta(x,y)$ by replacing each $R'(x,y)$ with true if $R = R'$, and false otherwise.

By performing these three steps, we get a $C^2$ formula in the normal form.

Now given a $C^2$ sentence $\varphi$ in its normal form, the construction of its QMLC sentence $\psi = F(\varphi)$ can be done inductively as follows. There are two cases.
By a straightforward induction, we can show that for every structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models F(\varphi)$. This concludes the conversion from the normal forms of $\mathcal{C}^\sharp$ to $\text{QMLC}$ formulae. ■

5 Regular graphs

In this section we are going to introduce two types of regular graphs: biregular graphs (bipartite regular graphs) and regular digraphs. The main results in this section are Theorems 5.2 and 5.4, which will be used in our proof of Theorem 3.1. For the sake of readability, we postpone their proofs until Sections 7 and 8.

5.1 Biregular graphs

An $\ell$-type bipartite graph is $G = (U, V, E_1, \ldots, E_\ell)$, where $E_1, \ldots, E_\ell$ are pairwise disjoint subsets of $U \times V$. Elements in $E_i$ are called $E_i$-edges. It helps to think of $G$ as a bipartite graph in which the edges are coloured with $\ell$ number of colours.

For a vertex $u \in U \cup V$, $\deg_{E_i}(u)$ denotes the number of $E_i$-edges adjacent to it, and $\deg(u) = \sum_{i=1}^{\ell} \deg_{E_i}(u)$. We write $\deg(G) = \max\{\deg(u) | u \text{ is a vertex in } G\}$. For an integer $d \in \mathbb{N}$, we write $\deg_{E_i}(u) = \mathbf{d}$, if $\deg_{E_i}(u) \geq d$.

Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbf{\mathbb{N}} = \{\mathbf{1}, \mathbf{2}, \mathbf{2}, \ldots\}$ and $\mathbb{B} = \mathbb{N} \cup \mathbf{\mathbb{N}}$. We write $\mathbb{B}^{\ell \times m}$ to denote the set of $\ell \times m$ matrices whose entries are from $\mathbb{B}$. The entry in row $i$ and column $j$ of a matrix $D \in \mathbb{B}^{\ell \times m}$ is denoted by $D_{i,j}$.

Let $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$. An $\ell$-type bipartite graph $G = (U, V, E_1, \ldots, E_\ell)$ is $(C, D)$-biregular, if there is a partition $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ such that the following holds.

- For every $i = 1, \ldots, \ell$, for each $j = 1, \ldots, m$, for each vertex $u \in U_j$, $\deg_{E_i}(u) = C_{i,j}$.
- For every $i = 1, \ldots, \ell$, for each $j = 1, \ldots, n$, for each vertex $v \in V_j$, $\deg_{E_i}(v) = D_{i,j}$.

The partitions $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ are called the partitions of the $(C, D)$-biregularity of $G$. We say that the $(C, D)$-biregular graph $G$ is of size $(M, N)$, if $M = \{U_1, \ldots, U_m\}$ and $N = \{V_1, \ldots, V_n\}$.

Theorem 5.1 For every two matrices $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$, there is a Presburger formula $\text{BiReg}_{C,D}(\bar{x}, \bar{y})$, where $\bar{x} = (x_1, \ldots, x_m)$ and $\bar{y} = (y_1, \ldots, y_n)$ such that the following holds. There exists an $\ell$-type $(C, D)$-biregular graph of size $(M, N)$ if and only if $\text{BiReg}_{C,D}(M, N)$ holds.

Theorem 5.1 is then generalised to the case of complete bipartite graphs. An $\ell$-type bipartite graph $G = (U, V, E_1, \ldots, E_\ell)$ is complete, if $U \times V = E_1 \cup \cdots \cup E_\ell$. If $G$ is also a $(C, D)$-biregular graph, then we call it a $(C, D)$-complete-biregular graph.

The following theorem is the main result in this subsection that will be used in the proof in Section 6.
Theorem 5.2 For every two matrices \( C \in \mathbb{B}^{\ell \times m} \) and \( D \in \mathbb{B}^{\ell \times n} \), there is a Presburger formula \( \text{COMP-BiREG}_{C,D}(\bar{X},\bar{Y}) \), where \( \bar{X} = (X_1, \ldots, X_m) \) and \( \bar{Y} = (Y_1, \ldots, Y_n) \) such that the following holds. There exists a \((C,D)\)-complete-biregular graph of size \((M,N)\) if and only if \( \text{COMP-BiREG}_{C,D}(M,N) \) holds.

5.2 Regular digraphs

An \( \ell \)-type directed graph (or, digraph for short) is a tuple \( G = (V,E_1,\ldots,E_\ell) \), where \( E_1,\ldots,E_\ell \) are pairwise disjoint irreflexive relations on \( V \) and for every \( u,v \in V \), if \((u,v) \in E_i \cup \cdots \cup E_\ell \), then the reverse direction \((v,u) \notin E_i \cup \cdots \cup E_\ell \). Similarly, edges in \( E_i \) are called \( E_i \)-edges.

We will write \( \text{in-deg}_{E_i}(u) \) to denote the number of incoming \( E_i \)-edges toward the vertex \( u \), and \( \text{out-deg}_{E_i}(u) \) the number of outgoing \( E_i \)-edges from the vertex \( u \). As before, for an integer \( d \in \mathbb{N} \), we write \( \text{in-deg}_{E_i}(u) = \ast d \) and \( \text{out-deg}_{E_i}(u) = \ast d \), to indicate that \( \text{in-deg}_{E_i}(u) \geq d \) and \( \text{in-deg}_{E_i}(u) \geq d \), respectively.

Let \( C,D \in \mathbb{B}^{\ell \times m} \). An \( \ell \)-type digraph \( G = (V,E_1,\ldots,E_\ell) \) is \((C,D)\)-regular-digraph, if there exists a partition \( V = V_1 \cup \cdots \cup V_m \) such that for each \( i = 1,\ldots,\ell \), for each \( j = 1,\ldots,m \), for each vertex \( v \in V_j \), \( \text{in-deg}_{E_i}(v) = C_{i,j} \) and \( \text{out-deg}_{E_i}(v) = D_{i,j} \). We say that \( V_1 \cup \cdots \cup V_m \) is a partition of \((C,D)\)-regularity of \( G \) and the graph \( G \) is of size \( N = (N_1,\ldots,N_m) \), if \((N_1,\ldots,N_m) = (|V_1|,\ldots,|V_m|)\).

Theorem 5.3 For every \( C,D \in \mathbb{B}^{\ell \times m} \), there exists a Presburger formula \( \text{REG}_{C,D}(\bar{X}) \), where \( \bar{X} = (X_1,\ldots,X_m) \) such that the following holds. There exists a \((C,D)\)-regular-digraph of size \( N \) if and only if \( \text{REG}_{C,D}(N) \) holds.

Similar to Section 5.1, Theorem 5.3 will be generalised to the case of complete regular digraph. An \( \ell \)-type graph \( G = (V,E_1,\ldots,E_\ell) \) is a complete digraph, if every two different vertices \( u,v \) or \((u,v) \) is in \( E_1 \cup \cdots \cup E_\ell \). By the standard graph theoretic term, a complete digraph is a tournament with coloured edges. If \( G \) is also a \((C,D)\)-regular, then we call \( G \) a \((C,D)\)-complete-regular digraph.

The following theorem is the main result in this subsection that will be used in the proof in Section 6.

Theorem 5.4 For every \( C,D \in \mathbb{B}^{\ell \times m} \), there exists a Presburger formula \( \text{COMP-REG}_{C,D}(\bar{X}) \), where \( \bar{X} = (X_1,\ldots,X_m) \) such that the following holds. There exists a \((C,D)\)-complete-regular digraph of size \( N \) if and only if \( \text{COMP-REG}_{C,D}(N) \) holds.

6 Proof of Theorem 3.1

In this section we are going to prove Theorem 3.1. We start with the case of basic QMLC and its negation in Subsection 6.1. The full proof of Theorem 3.1 is presented in Subsection 6.2. In Subsection 6.3 we show that the family of models of a QMLC formula can be viewed as a collection of bipartite graphs and regular digraphs.

6.1 Presburger formula for basic QMLC and its negation

Recall that a QMLC formula \( \phi \) is a basic QMLC, if it is of the form \( \exists^k \varphi \), where \( \varphi \in \text{MLC} \).

We are going to use the following notion of type in our proof. Let \( \phi \) be a QMLC sentence. Let \( \mathcal{M}_\phi \) be the set of all MLC subformulae of \( \phi \) and their negations. A type in \( \phi \) is a subset \( T \subseteq \mathcal{M}_\phi \) such that

- if \( \varphi_1 \land \varphi_2 \in T \), then both \( \varphi_1, \varphi_2 \in T \);
- \( \varphi \in T \) if and only if \( \neg \varphi \notin T \);
- if \( \neg (\varphi_1 \land \varphi_2) \in T \), then at least one of \( \neg \varphi_1, \neg \varphi_2 \in T \).
Proof. Let $\phi$ be a basic QMLC formula of the form: $\exists^k \varphi$, where $\varphi \in \text{MLC}$. Let $\mathcal{S}$ be the set of unary predicates used in $\phi$ and $\mathcal{R} = \{R_1, \ldots, R_\ell, \overline{R}_1, \ldots, \overline{R}_\ell\}$ the set of binary relations used in $\phi$, where $\overline{R}_i$ is the reversed relation of $R_i$. Let $K$ be the integer such that for all subformula $\mathcal{O}^l_{R}\varphi$ in $\phi$, we have $l \leq K$.

Recall that $\mathcal{T}_\phi$ denotes the set of all types in $\phi$. We say that a function $f : \mathcal{T}_\phi \times \mathcal{R} \times \mathcal{T}_\phi \to \{0, 1, \ldots, K\} \cup \{\ast K\}$ is consistent, if for every $T \in \mathcal{T}_\phi$ the following holds.

- If $\mathcal{O}^l_{R}\mu \in T$, then $\sum_{T' \text{ s.t. } T' \models \mu} f(T, R, T') \geq l$.
- If $\neg(\mathcal{O}^l_{R}\mu) \in T$, then $\sum_{T' \text{ s.t. } T' \models \mu} f(T, R, T') \leq l - 1$, and $f(T, R, T') \in \mathbb{N}$, for every $R \in \mathcal{R}$ and for every type $T' \models \mu$.

Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be the set of all consistent functions.

For a type $T \in \mathcal{T}_\phi$, we define two matrices $D_T, \overline{D}_T \in \mathbb{B}^{l \times m}$ as follows.

$$D_T := \begin{pmatrix}
  f_1(T, R_1, T) & f_2(T, R_1, T) & \cdots & f_m(T, R_1, T) \\
  f_1(T, R_2, T) & f_2(T, R_2, T) & \cdots & f_m(T, R_2, T) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(T, R_\ell, T) & f_2(T, R_\ell, T) & \cdots & f_m(T, R_\ell, T)
\end{pmatrix}$$

and

$$\overline{D}_T := \begin{pmatrix}
  f_1(T, \overline{R}_1, T) & f_2(T, \overline{R}_1, T) & \cdots & f_m(T, \overline{R}_1, T) \\
  f_1(T, \overline{R}_2, T) & f_2(T, \overline{R}_2, T) & \cdots & f_m(T, \overline{R}_2, T) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(T, \overline{R}_\ell, T) & f_2(T, \overline{R}_\ell, T) & \cdots & f_m(T, \overline{R}_\ell, T)
\end{pmatrix}$$

Intuitively, the matrix $D_T$ collects all the information of the degrees of the relations $R_1, \ldots, R_\ell$, while $\overline{D}_T$ the degrees of the reverse relations $\overline{R}_1, \ldots, \overline{R}_\ell$.

For two different types $S, T \in \mathcal{T}_\phi$, we define the matrix $D_{S \rightarrow T}, \overline{D}_{S \rightarrow T} \in \mathbb{B}^{l \times m}$ as follows.

$$D_{S \rightarrow T} := \begin{pmatrix}
  f_1(S, R_1, T) & f_2(S, R_1, T) & \cdots & f_m(S, R_1, T) \\
  f_1(S, R_2, T) & f_2(S, R_2, T) & \cdots & f_m(S, R_2, T) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(S, R_\ell, T) & f_2(S, R_\ell, T) & \cdots & f_m(S, R_\ell, T)
\end{pmatrix}$$
and

$$
\overline{\mathcal{D}}_{S \rightarrow T} := \begin{pmatrix}
 f_1(T, \overline{R}_1, S) & f_2(T, \overline{R}_1, S) & \cdots & f_m(T, \overline{R}_1, S) \\
 f_1(T, \overline{R}_2, S) & f_2(T, \overline{R}_2, S) & \cdots & f_m(T, \overline{R}_2, S) \\
 \vdots & \vdots & \ddots & \vdots \\
 f_1(T, \overline{R}_\ell, S) & f_2(T, \overline{R}_\ell, S) & \cdots & f_m(T, \overline{R}_\ell, S) \\
 f_1(T, R_1, S) & f_2(T, R_1, S) & \cdots & f_m(T, R_1, S) \\
 f_1(T, R_2, S) & f_2(T, R_2, S) & \cdots & f_m(T, R_2, S) \\
 \vdots & \vdots & \ddots & \vdots \\
 f_1(T, R_{\ell}, S) & f_2(T, R_{\ell}, S) & \cdots & f_m(T, R_{\ell}, S)
\end{pmatrix}
$$

Notice that in the matrix $D_{S \rightarrow T}$ the first $\ell$ rows contain the information on the degree of $R_1, \ldots, R_\ell$, and the last $\ell$ rows the information on the degree of $\overline{R}_1, \ldots, \overline{R}_\ell$ from the type $S$ to the type $T$; while in the matrix $\overline{\mathcal{D}}_{S \rightarrow T}$ it is the opposite and the direction is from the type $T$ to the type $S$.

Now we define the formula $\text{PREB}_\phi(x)$. Recall that $\phi$ is a basic QMLC formula of the form: $\exists^k \varphi$. First, we enumerate the set $\mathcal{T}_\phi = \{T_1, \ldots, T_n\}$ and the set $\mathcal{T}_\phi \times \mathcal{F} = \{(T_1, f_1), \ldots, (T_n, f_m)\}$. The formula $\text{PREB}_\phi(x)$ is defined as follows:

$$
\exists x(T_1, f_1) \cdots \exists x(T_n, f_m) \left( x = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} x_{T_i, f_j} \right) \land \text{PREB-Atom}_\phi(X) \land \text{CON}(X)
$$

where $X = (X(T_1, f_1), \ldots, X(T_n, f_m))$ the vector of all the variables $X_{(T, f)}$’s and

$$
\text{PREB-Atom}_\phi(X) := \sum_{(T, f) s.t. \varphi \in T} X_{(T, f)} \geq k
$$

and

$$
\text{CON}(X) := \bigwedge_{T \in \mathcal{T}} \text{COMP-REG}_{DT, DT}(\hat{X}_T) \land \bigwedge_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq i-1} \text{COMP-BiREG}_{DT, DT}(\hat{X}_{T_i}, \hat{X}_{T_j})
$$

where $\hat{X}_T = (X_{(T_1, f_1)}, \ldots, X_{(T_{\ell}, f_{\ell})})$ and $\hat{X}_S = (X_{(S_1, f_1)}, \ldots, X_{(S_{\ell}, f_{\ell})})$ are the vector of variables associated with the types $T$ and $S$, respectively.

Notice that in the formula $\text{COMP-BiREG}_{DT, DT}(\hat{X}_{T_i}, \hat{X}_{T_j})$ the direction is always from $T_i$ to $T_j$ whenever $i \geq j + 1$.

We claim that $\text{PREB}_\phi$ defines precisely the spectrum of $\phi$. We simply observe that the incoming $R_i$ edges to an element $v$ are precisely the outgoing $\overline{R}_i$ edges from $v$. Vice versa, the outgoing $R_i$ edges from an element $v$ are precisely the incoming $\overline{R}_i$ edges to $v$.

The formal proof is as follows. Abusing the notation, we let $\text{PREB}_\phi$ itself to denote the set $\{n \mid \text{PREB}_\phi(n) \text{ holds}\}$. We claim that $\text{PREB}_\phi = \text{SPEC}(\phi)$.

We first show the $\subseteq$ direction. Let $N \in \text{PREB}_\phi$. Let $\hat{M} = (M_{T_1, f_1}, \ldots, M_{T_{n}, f_{m}})$ be the witnesses to $X$ such that $\text{PREB}_\phi(N)$ holds. In the following we are going to write $\hat{M}_T$ to denote $(M_{T_1, f_1}, \ldots, M_{T_{\ell}, f_{\ell}})$ for every type $T \in \mathcal{T}_\phi$.

By definition $N = \sum_{T \in \mathcal{T}_\phi} \hat{M}_T$. We take a set $V$ of $N$ vertices and we partition $V = V_{(T_1, f_1)} \cup \cdots \cup V_{(T_{\ell}, f_{\ell})}$ such that $|V_{(T, f)}| = M_{(T, f)}$ for each $T \in \mathcal{T}_\phi$ and $f \in \mathcal{F}$. We denote by $V_T = V_{(T_1, f_1)} \cup \cdots \cup V_{(T_{\ell}, f_{\ell})}$ for each $T \in \mathcal{T}_\phi$.

Since $\text{CON}(M)$ holds, by Theorems 5.4, for each $T \in \mathcal{T}_\phi$, there exists a $(DT, \overline{DT})$-complete-regular digraph $G_T = (V_T, R_{T, 1}, \ldots, R_{T, \ell})$ of size $\hat{M}_T$, with $V_T = V_{(T_1, f_1)} \cup \cdots \cup V_{(T_{\ell}, f_{\ell})}$ be the partition of $(DT, \overline{DT})$-regularity. This means that for every vertex $v \in V_{T_1, f_1}$, for every $R \in \{R_1, \ldots, R_{\ell}\}$,

- out-deg$_R(v)$ in the graph $G_T$ is $f_i(T, R, T)$;
• in-deg$_R(v)$ in the graph $G_T$ is $f_i(T, \overline{R}, T)$.

Now let $G_T = (V_T, R_{T,1}, \ldots, R_{T,\ell}, \overline{R}_{T,1}, \ldots, \overline{R}_{T,\ell})$ be the graph obtained by taking $\overline{R}_i$ as the reverse of $R_i$. Then for each vertex $v \in V_T$,

• out-deg$_R(v) = \text{in-deg}_{\overline{R}}(v)$ in the graph $G_T$;

• in-deg$_R(v) = \text{out-deg}_{\overline{R}}(v)$ in the graph $G_T$.

Similarly, by Theorem 5.2 for each $T_i, T_j \in \mathcal{T}_\phi$, where $j \leq i-1$, there exists a $(D_{T_j \rightarrow T_i}, \overline{D}_{T_i \rightarrow T_j})$-biregular-complete graph

$$G_{T_i, T_j} = (V_{T_i}, V_{T_j}, R_{T_i, T_j, 1}, \ldots, R_{T_i, T_j, \ell}, \overline{R}_{T_i, T_j, 1}, \ldots, \overline{R}_{T_i, T_j, \ell})$$

of size $(M_S, M_T)$, with $V_{T_i} = V(T_j, f) \cup \cdots \cup V(T_m, f)$ and $V_{T_j} = V(T_j, f) \cup \cdots \cup V(T_m, f)$ be the partition of $(D_{T_j \rightarrow T_i}, \overline{D}_{T_i \rightarrow T_j})$-biregularity. This means that for every $R \in \{R_1, \ldots, R_{\ell}, \overline{R}_1, \ldots, \overline{R}_{\ell}\}$,

• for every vertex $v \in V_{T_i}$, out-deg$_R(v)$ in the graph $G_{T_i, T_j}$ is $f(T_i, R, T_j)$;

• for every vertex $v \in V_{T_j}$, out-deg$_R(v)$ in the graph $G_{T_i, T_j}$ is $f(T_i, \overline{R}, T_j)$.

We put the orientation in every the edges in the graph $G_{T_i, T_j}$ going from $V_{T_i}$ to $V_{T_j}$. Now let $\tilde{G}_{T_i, T_j}$ be the graph obtained by adding $(u, v)$ into $\overline{R}$ in the graph $G_{T_i, T_j}$ whenever $(v, u)$ is an $R$-edge in $G_{T_i, T_j}$.

Hence, we have for each vertex $v \in V_{T_i} \cup V_{T_j}$, for each $R \in \mathcal{R}$

• out-deg$_R(v) = \text{in-deg}_{\overline{R}}(v)$ in the graph $\tilde{G}_{T_i, T_j}$;

• in-deg$_R(v) = \text{out-deg}_{\overline{R}}(v)$ in the graph $\tilde{G}_{T_i, T_j}$.

Let $G = (V_1, \ldots, V_\ell, \overline{R}_1, \ldots, \overline{R}_\ell)$ be the combination of all the graphs $\tilde{G}_{T_i}$’s and $\tilde{G}_{T_i, T_j}$’s. Formally,

$$V = \bigcup_T V(\tilde{G}_T)$$

$$R = \bigcup_T R(\tilde{G}_T) \cup \bigcup_{T_i, T_j} R(\tilde{G}_{T_i, T_j}) \text{ for each } R \in \{R_1, \ldots, R_{\ell}, \overline{R}_1, \ldots, \overline{R}_{\ell}\}$$

Moreover, we also label each vertex $v \in V$ with a subset of $\mathcal{S}$ as follows. For each $T \in \mathcal{T}_\phi$, for each $v \in V_T$, we “declare” that $v$ is labeled with a unary predicate $P \in \mathcal{S}$ if and only if $P \in T$.

We claim that $G \models \phi$. For that it is sufficient to show that for each $T \in \mathcal{T}_\phi$, for each $v \in V_T$,

• For each unary predicate $P \in \mathcal{P}$, it is by our labelling of the vertices of $G$ that $P(v)$ holds in $G$ if and only if $P \in T$.

• For each $\phi^P_R \mu \in T$, where $R \in \mathcal{R}$ we have

$$\sum_{T' \text{ s.t. } T' \supseteq \mu} f(T, R, T') \geq l$$

number of outgoing $R$-edges from $v$. Since every function $f \in \mathcal{F}$ is consistent, $\phi^P_R \mu \in \text{type}(v)$.

• Similarly, for each $\phi^P_R \mu \notin T$, and hence $\neg(\phi^P_R \mu) \in T$, where $R \in \mathcal{R}$ we have

$$\sum_{T' \text{ s.t. } T' \supseteq \mu} f(T, R, T') \geq l$$

number of outgoing $R$-edges from $v$. Since every function $f \in \mathcal{F}$ is consistent, $\phi^P_R \mu \in \text{type}(v)$. 

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Therefore the graph \( G \models \phi \), and hence \( N \in \text{Spec}(\phi) \).

Now we prove the direction \( \supseteq \). Suppose \( \mathfrak{A} \models \phi \) is of size \( N \). Let \( M = (M(T_1,f_1), \ldots, M(T_n,f_m)) \) where \( M(T,f) \) be the number of elements of type \( T \) from which there exist \( f(T,R,S) \) number of outgoing \( R \)-edges towards the elements of type \( S \). Take each \( M(T,f) \) to be the witness for \( X(T,f) \) for each \( T \in \mathcal{T} \) and \( f \in \mathcal{F} \). It immediately follows from Theorems 5.2 and 5.4 that \( \text{CON}(N,M) \) holds. Moreover, \( \text{PREB-Atom}_\phi(M) \) holds, since \( \mathfrak{A} \models \phi \). This completes the proof of our proposition.

Proposition 6.2 below handles the case of negations of basic QMLC formulas.

**Proposition 6.2** Let \( \phi \in \text{QLMC} \) be a negation of a basic QMLC formula. Then there exists a Presburger formula \( \text{PREB}_\phi(x) \) such that \( \text{Spec}(\phi) = \{ n \mid \text{PREB}_\phi(n) \text{ holds} \} \).

**Proof.** The proof is almost the same as in the proof of Proposition 6.1. Let \( \phi \) be \( \neg \exists^k \varphi \), where \( \varphi \in \text{MLC} \). We simply replace the formula \( \text{PREB-Atom} \) with \( \sum_{(T,f) \text{ s.t.} \varphi \in T} X(T,f) \leq k - 1 \). This completes our proof of Proposition 6.2.

### 6.2 Proof of Theorem 3.1

Let \( \phi \in \text{QLMC} \). First, we push all the negations inside so that they are applied only to basic QMLC. Then we define \( \text{PREB-Atom}_\phi \) inductively as follows.

- If \( \phi := \exists^k \varphi \), then \( \text{PREB-Atom}_\phi := \sum_{(T,f)} X(T,f) \geq k \).
- If \( \phi := \neg \exists^k \varphi \), then \( \text{PREB-Atom}_\phi := \sum_{(T,f)} X(T,f) \leq k - 1 \).
- If \( \phi := \phi_1 \land \phi_2 \), then \( \text{PREB-Atom}_\phi := \text{PREB-Atom}_{\phi_1} \land \text{PREB-Atom}_{\phi_2} \).
- If \( \phi := \phi_1 \lor \phi_2 \), then \( \text{PREB-Atom}_\phi := \text{PREB-Atom}_{\phi_1} \lor \text{PREB-Atom}_{\phi_2} \).

Now the formula \( \text{PREB}_\phi(x) \) is defined exactly like above.

\[
\exists X(T_1,f_1) \cdots \exists X(T_n,f_m) \left( x = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \ell_{T_i} \right) \land \text{PREB-Atom}_\phi(\bar{X}) \land \text{CON}(\bar{X})
\]

where the formula \( \text{CON} \) is as in Proposition 6.1. The correctness of the formula \( \text{PREB}_\phi(x) \) can be verified by a straightforward induction. This completes our proof of Theorem 3.1.

### 6.3 Regular characterisations of models of QMLC

In this subsection we remark that the family of models of a QMLC formula can be viewed as a collection of binary graphs and regular digraphs.

Let \( \phi \) be a QMLC formula and \( \mathcal{R} = \{ R_1, \ldots, R_\ell, \bar{R}_1, \ldots, \bar{R}_\ell \} \) be the set of binary relations used in \( \phi \) and that \( \bar{R}_i \) is the reversed relation of \( R_i \). Let \( K \) be the integer such that for all subformulae \( \hat{\phi} \psi \in \phi \), we have \( l \leq K \).

Let \( \mathcal{T} \) be the set of all types in \( \phi \). For a model \( \mathfrak{A} \models \phi \), we partition \( A = \bigcup_{T \in \mathcal{T}} A_T \), where \( A_T = \{ a \in A \mid a \text{ is of type } T \} \). That the model \( \mathfrak{A} \) is a collection of regular graphs is in the following sense. Recall the matrices \( D_T, \bar{D}_T, D_{S \rightarrow T} \) and \( D_{T \rightarrow S} \) as defined in the proof of Proposition 6.1.

For a type \( T \), by restricting the relations \( R_1, \ldots, R_\ell \) on the elements in \( A_T \), we obtain a \((D_T, \bar{D}_T)\)-regular digraph \( G_T = (A_T, E_1, \ldots, E_\ell) \), where for each \( E_i \),

\[
E_i = R_i \cap (A_T \times A_T)
\]

For two different types \( S, T \), by restricting the relations on \( A_S \times A_T \), we obtain a \((D_{S \rightarrow T}, \bar{D}_{S \rightarrow T})\)-biregular graph \( G_{S,T} = (A_S, A_T, E_1, \ldots, E_\ell, E'_1, \ldots, E'_\ell) \), where each \( E_i, E'_i \) are

\[
E_i = R_i \cap (A_S \times A_T) \quad \text{and} \quad E'_i = \bar{R}_i \cap (A_S \times A_T)
\]
7 Proof of Theorem 5.2

The proof of Theorem 5.2 is rather long. However, one could say that it is based on the following simple observation stated as Proposition 7.1 below.

**Proposition 7.1** Let $c, d \geq 0$. For every $M, N \in \mathbb{N}$, the following holds.

(a) There exists a $(c, d)$-biregular graph of size $(M, N)$ if and only if $N \geq c$, $M \geq d$ and $M \cdot c = N \cdot d$.

(b) There exists a $(c, d)$-biregular graph of size $(M, N)$ if and only if $M \geq d$, $N \geq c$ and $Mc \geq Nd$.

(c) There exists a $(c, d)$-biregular graph of size $(M, N)$ if and only if $M \geq d$, $N \geq c$.

**Proof.** Let $c, d \geq 0$, and let $M, N \in \mathbb{N}$. We first prove part (a). The “only if” direction follows from the fact that in $(c, d)$-biregular graph the number of edges is precisely $Mc = Nd$. That $M \geq d$ and $N \geq c$ is straightforward.

The “if” direction is as follows. Let $M \geq d$, $N \geq c$. Let $K = Mc = Nd$. First, we construct the following graph.

![Diagram](http://example.com/diagram.png)

On the left side, we have $M$ vertices, and each has degree $c$. On the right side, we have $K = Nd$ vertices, and each has degree 1. We are going to merge every $d$ vertices on the right side into one vertex of degree $d$. The merging is as follows. We merge every $d$ vertices $v_i, v_i + N, \ldots, v_i + (d - 1)N$ into one node for every $i = 1, \ldots, N$. Since $K = Nd$, it is possible to do such merging. Moreover, $N \geq c$, hence we do not have multiple edges between two vertices. Thus, we obtain the desired $(c, d)$-biregular graph of size $(M, N)$.

Now we consider part (b). The “only if” direction follows from the fact that in $(c, d)$-biregular graph the number of edges is precisely $Mc$, which should be greater than $Nd$. That $M \geq d$ and $N \geq c$ is straightforward.

For the “if” direction, the proof is almost the same as above. Suppose $M \geq d$, $N \geq c$ and $Mc \geq Nd$. Let $K = Mc$. We first construct the bipartite graph, in which on the left side, we have $M$ vertices, each of which has degree $c$; and on the right side we have $K = Mc$ vertices, each of which has degree 1. For every $i \in \{1, \ldots, N\}$, we set the set $I_i \subseteq \{1, \ldots, K\}$ as follows.

$$I_i := \{i + kN \mid 1 \leq i + kN \leq K\}.$$

Since $K = Mc \geq Nd$, the cardinality $|I_i| \geq d$ for every $1 \leq i \leq N$. Now for every $1 \leq i \leq N$, we merge the vertices $\{v_j \mid j \in I_i\}$ into one vertex. Hence, we obtain the desired $(c, d)$-biregular graph of size $(M, N)$.

Now we prove part (c). The “only if” direction is straightforward. The “if” direction is as follows. Suppose $M \geq d$ and $N \geq c$. There are two cases: either $Mc \geq Nd$, or $Mc < Nd$. In the former case, we construct a $(c, d)$-biregular graph of size $(M, N)$, while in the latter case, we construct a $(c, d)$-biregular graph of size $(M, N)$. In either case, we obtain a $(c, d)$-biregular graph of size $(M, N)$. This completes our proof of Proposition 7.1.

The proof is divided into five successive steps presented in Subsections 7.1–7.5.
In Subsection 7.1 we consider the case where $C \in \mathbb{N}^{1 \times m}$ and $D \in \mathbb{N}^{1 \times n}$. (Theorem 7.3)

In Subsection 7.2 we consider the case where $C \in \mathbb{N}^{\ell \times m}$ and $D \in \mathbb{N}^{\ell \times n}$. (Theorem 7.4)

In Subsection 7.3 we consider the case where $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$, but restricted to the graphs where the number of vertices whose degrees specified with $\triangleright d$ is “big enough.” (Theorem 7.6)

In Subsection 7.4 we introduce the notion of partial graphs to handle the case of graphs where the number of vertices whose degree is specified with $\triangleleft d$ is “small.” Combining it with Theorem 7.6 in Subsection 7.3, we obtain our proof of Theorem 5.1.

Finally in Subsection 7.5 we present our proof of Theorem 5.2.

Before we start, we need a few notations. We write $\bar{1}$ to denote the vector $(1, \ldots, 1) \in \mathbb{N}^m$, for an appropriate $m \geq 1$. That is, $\bar{1}$ is a vector whose components are all one. For two vectors $\bar{c} = (c_1, \ldots, c_m) \in \mathbb{N}^m$ and $\bar{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, the dot product between $\bar{c}$ and $\bar{d}$ is $\bar{c} \cdot \bar{d} = (c_1 d_1, \ldots, c_m d_m)$.

### 7.1 When $C \in \mathbb{N}^{1 \times m}$ and $D \in \mathbb{N}^{1 \times n}$

In this subsection we consider the case when $C$ and $D$ consist of only one vector each. In this case, we are going to write $(\bar{c}, \bar{d})$-biregular graph, where $\bar{c}$ and $\bar{d}$ are the only vectors of $C$ and $D$, respectively.

#### Lemma 7.2

Let $\bar{c} \in \mathbb{N}^m$ and $\bar{d} \in \mathbb{N}^n$ and both do not contain zero entry. For each $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$ such that $\bar{M} \cdot \bar{1} + \bar{N} \cdot \bar{1} \geq 2(\bar{c} \cdot \bar{1})(\bar{d} \cdot \bar{1}) + 3$, the following holds. There exists a $(\bar{c}, \bar{d})$-biregular graph of size $(\bar{M}, \bar{N})$ if and only if $\bar{M} \cdot \bar{c} = \bar{N} \cdot \bar{d}$.

#### Proof.

Let $\bar{c} \in \mathbb{N}^m$ and $\bar{d} \in \mathbb{N}^n$ and both do not contain zero entry. Let $\bar{M} \in \mathbb{N}^m$, $\bar{N} \in \mathbb{N}^n$ such that $\bar{M} \cdot \bar{1} + \bar{N} \cdot \bar{1} \geq 2(\bar{c} \cdot \bar{1})(\bar{d} \cdot \bar{1}) + 3$.

The “only if” direction is straightforward. If $G$ is a $(\bar{c}, \bar{d})$-biregular graph of size $(\bar{M}, \bar{N})$, then the number of edges in $G$ is precisely $\bar{M} \cdot \bar{c} = \bar{N} \cdot \bar{d}$.

Now we prove the “if” part. Suppose $\bar{M} \in \mathbb{N}^m$, $\bar{N} \in \mathbb{N}^n$ such that $\bar{M} \cdot \bar{c} = \bar{N} \cdot \bar{d}$. Let $\bar{c} = (c_1, \ldots, c_m)$ and $\bar{d} = (d_1, \ldots, d_n)$, and $\bar{M} = (M_1, \ldots, M_m)$ and $\bar{N} = (N_1, \ldots, N_n)$.

We are going to construct a $(\bar{c}, \bar{d})$-biregular graph of size $(\bar{M}, \bar{N})$. We first construct the following bipartite graph $G$. 

That is, the left side has $\bar{M} \cdot 1$ vertices, and there are $M_1$ vertices of degree $c_1$, $M_2$ nodes of degree $c_2$, etc. The right side has $\bar{M} \cdot \bar{c}$ number of vertices, each of degree one. We are going to do some merging of the vertices on the right side so that there are exactly $N_1$ vertices of degree $d_1$, $N_2$ vertices of degree $d_2$, etc. We do the following.

We “group” the vertices on the right side into $V_1, \ldots, V_n$, where $V_1$ has $N_1d_1$ vertices, $V_2$ has $N_2d_2$ vertices, etc. Such grouping is possible because $M \cdot \bar{c} = N \cdot \bar{d}$.

For each $i \in \{1, \ldots, n\}$, we do the following. We merge $d_i$ vertices in $V_i$ into one vertex, so that each vertex in $V_i$ has degree $d_i$. Let $V_i = \{v_{i,1}, \ldots, v_{i,K_i}\}$ where $K_i = N_id_i$. We merge the vertices $v_{1,1}, v_{N_1+1}, v_{2N_1+1}, \ldots, v_{(d_i-1)N_i+1}$ into one vertex; the vertices $v_{2,1}, v_{N_2+2}, v_{2N_2+2}, \ldots, v_{(d_i-1)N_i+2}$ into one vertex; and so on.

After such merging, each vertex in $V_i$ has degree $d_i$. However, it is possible that after we do the merging, we have “parallel” edges, i.e. more than one edges between two vertices. (See the left side of the illustration below.) We are going to “remove” such parallel edges one by one until there are no more parallel edges.

Suppose we have parallel edges between the vertices $u$ and $v$. We pick an edge $(u', v')$ such that $u'$ is not adjacent to $v$ and $v'$ is not adjacent to $u$. (See the left side of the illustration below.)

Such an edge $(u', v')$ exists since the number of vertices reachable in distance 2 from the vertices $u$ and $v$ is $\leq 2(\bar{c} \cdot \bar{1})(d \cdot \bar{1}) + 2$ and the number of vertices is $\bar{M} \cdot \bar{1} + N \cdot \bar{1} \geq 2(\bar{c} \cdot \bar{1})(d \cdot \bar{1}) + 3$. 

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Now we delete the edges \((u', v')\) and one of the parallel edge \((u, v)\), replace it with the edges \((u, v')\) and \((u', v)\), as illustrated on the right side of the illustration above. We perform such operation until there are no more parallel edges. This completes the proof of Lemma 7.2.

The following theorem is a straightforward application of Lemma 7.2.

**Theorem 7.3** For every \(\vec{c} \in \mathbb{N}^m\) and \(\vec{d} \in \mathbb{N}^n\), there exists a Presburger formula \(\text{BiREG}_{\vec{c}, \vec{d}}(\vec{X}, \vec{Y})\), where \(\vec{X} = (X_1, \ldots, X_m)\) and \(\vec{Y} = (Y_1, \ldots, Y_n)\) such that the following holds. There exists a \((\vec{c}, \vec{d})\)-biregular graph of size \((\vec{M}, \vec{N})\) if and only if the sentence \(\text{BiREG}_{\vec{c}, \vec{d}}(\vec{M}, \vec{N})\) holds.

**Proof.** The proof is a direct application of Lemma 7.2. We assume that all the entries in \(\vec{c}\) and \(\vec{d}\) are not zero. Otherwise, we do the following. Suppose \(\vec{c} = (c_1, \ldots, c_m), \vec{d} = (d_1, \ldots, d_n)\) and let \(I = \{i \mid c_i = 0\}\) and \(J = \{j \mid d_j = 0\}\). We define

\[
\text{BiREG}_{\vec{c}, \vec{d}}(\vec{X}, \vec{Y}) := \bigwedge_{i \in I} X_i \geq 0 \land \bigwedge_{j \in J} Y_j \geq 0 \land \text{BiREG}_{\vec{c}', \vec{d}'}(\vec{X}', \vec{Y}'),
\]

where \(\vec{c}'\) and \(\vec{X}'\) are the vectors \(\vec{c}\) and \(\vec{X}\) without the entries in \(I\), respectively, and \(\vec{d}'\) and \(\vec{Y}'\) are the vectors \(\vec{d}\) and \(\vec{Y}\) without the entries in \(J\), respectively.

For \(\vec{c}, \vec{d} \in \mathbb{N}^m\) which do not contain zero entry, we define the following set \(H\).

\[
H := \left\{ (\vec{M}, \vec{N}) \mid \vec{M} \cdot \vec{1} + \vec{N} \cdot \vec{1} \leq 2(\vec{c} \cdot \vec{1})(\vec{d} \cdot \vec{1}) + 2 \text{ and } \text{there exists a } (\vec{c}, \vec{d})\text{-biregular graph of size } (\vec{M}, \vec{N}) \right\}
\]

Such set can be computed greedily since the number of \((\vec{M}, \vec{N})\) such that \(\vec{M} \cdot \vec{1} + \vec{N} \cdot \vec{1} \leq 2(\vec{c} \cdot \vec{1})(\vec{d} \cdot \vec{1}) + 2\) is bounded. (Here we make use of the fact that neither \(\vec{c}\) nor \(\vec{d}\) contain zero entry.)

Now we define the formula \(\text{BiREG}_{\vec{c}, \vec{d}}(\vec{X}, \vec{Y})\) as follows.

\[
\left(\vec{M} \cdot \vec{1} + \vec{N} \cdot \vec{1} \geq 2(\vec{c} \cdot \vec{1})(\vec{d} \cdot \vec{1}) + 3 \land (\vec{X} \cdot \vec{c} = \vec{Y} \cdot \vec{d})\right) \lor \bigvee_{(\vec{M}, \vec{N}) \in H} \vec{X} = \vec{M} \land \vec{Y} = \vec{N}
\]

The formula is a Presburger formula since \(\vec{c}\) and \(\vec{d}\) are constants. That \(\text{BiREG}_{\vec{c}, \vec{d}}(\vec{X}, \vec{Y})\) is the desired formula follows immediately from Lemma 7.2.

### 7.2 When \(C \in \mathbb{N}^{\ell \times m}\) and \(D \in \mathbb{N}^{\ell \times n}\)

Theorem 7.4 below is the generalisation of Theorem 7.3 to the case where \(\ell \geq 1\). Recall that for a matrix \(C \in \mathbb{N}^{\ell \times m}\), we write \(C \cdot \vec{1}\) to denote the sum of all the entries in \(C\).

**Theorem 7.4** For every \(C \in \mathbb{N}^{\ell \times m}\) and \(D \in \mathbb{N}^{\ell \times n}\), there exists a Presburger formula \(\text{BiREG}_{C, D}(\vec{X}, \vec{Y})\), where \(\vec{X} = (X_1, \ldots, X_m)\) and \(\vec{Y} = (Y_1, \ldots, Y_n)\) such that the following holds. There exists a \((C, D)\)-biregular \(\ell\)-type graph of size \((\vec{M}, \vec{N})\) if and only if the sentence \(\text{BiREG}_{C, D}(\vec{M}, \vec{N})\) holds.

**Proof.** Let \(C \in \mathbb{N}^{\ell \times m}\) and \(D \in \mathbb{N}^{\ell \times n}\) be the given matrices. For simplicity, we assume that both \(C\) and \(D\) do not contain zero column. If column \(i\) in matrix \(C\) (or, \(D\), respectively) is zero column, then we add the constraint \(X_i \geq 0\) (or, \(Y_i \geq 0\), respectively) and ignore that column.

Let \(c_1, \ldots, c_\ell\) and \(d_1, \ldots, d_\ell\) be the row vectors of \(C\) and \(D\), respectively. For a vector \(\vec{t} = (t_1, \ldots, t_m) \in \mathbb{N}^m\), we define the characteristic vector of \(\vec{t}\) as \(\chi(\vec{t}) := (b_1, \ldots, b_m) \in \{0, 1\}^m\) where \(b_i = 0\) if \(c_i = 0\), and \(b_i = 1\) if \(c_i \neq 0\).

We first define the following set.

\[
H_{C, D} := \left\{ (\vec{M}, \vec{N}) \mid \vec{M} \cdot \vec{1} + \vec{N} \cdot \vec{1} < 2\ell(C \cdot \vec{1})(D \cdot \vec{1}) + 3\ell \text{ and } \text{there exists a } (C, D)\text{-biregular graph of size } (\vec{M}, \vec{N}) \right\}
\]

Again, such set can be computed greedily since the number of \((\vec{M}, \vec{N})\) such that \(\vec{M} \cdot \vec{1} + \vec{N} \cdot \vec{1} < 2\ell(C \cdot \vec{1})(D \cdot \vec{1}) + 3\ell\) is bounded. (Here we make use of the fact that neither \(\vec{c}\) nor \(\vec{d}\) contain zero column.)
Then, the formula $\text{BiREG}_{C,D}(\bar{X}, \bar{Y})$ can be defined inductively as follows. When $\ell = 1$,

$$
\text{BiREG}_{C,D}(\bar{X}, \bar{Y}) := \text{BiREG}_{\bar{c}_1, \bar{d}_1}(\bar{X}, \bar{Y})
$$

When $\ell \geq 2$,

$$
\text{BiREG}_{C,D}(\bar{X}, \bar{Y}) := \bigvee_{(\bar{M}, \bar{N}) \in H_{C,D}} \bar{X} = \bar{M} \land \bar{Y} = \bar{N}
\land \bigvee_{1 \leq j \leq \ell} \left( \bar{X} \cdot \chi(\bar{c}_j) + \bar{Y} \cdot \chi(\bar{d}_j) \geq 2(C \cdot 1)(D \cdot 1) + 3 \land \text{BiREG}_{C-\bar{c}_j, D-\bar{d}_j}(\bar{X}, \bar{Y}) \land \text{BiREG}_{\bar{c}_j, \bar{d}_j}(\bar{X}, \bar{Y}) \right)
$$

where $C - \bar{c}_j$, $D - \bar{d}_j$ denote the matrices $C$ and $D$ without row $j$, respectively.

We are going to prove that there exists a $(C, D)$-biregular graph of size $(\bar{M}, \bar{N})$ if and only if the statement $\text{BiREG}_{C,D}((\bar{M}, \bar{N})$ holds. The proof is by induction on $\ell$. The basis $\ell = 1$ has been established in Theorem 7.3. For the induction step, we assume that it holds for the case of $\ell - 1$ and we are going to prove the case $\ell$.

We first prove the “only if” direction. Suppose $G = (U, V, E_1, \ldots, E_\ell)$ is $(C, D)$-biregular of size $(\bar{M}, \bar{N})$. If $(\bar{M}, \bar{N}) \in H_{C,D}$, then $\text{BiREG}_{C,D}((\bar{M}, \bar{N})$ holds. So suppose $(\bar{M}, \bar{N}) \notin H_{C,D}$ and $\bar{M} \cdot 1 + \bar{N} \cdot 1 \geq 2(C \cdot 1)(D \cdot 1) + 3$. Since $C$ and $D$ does not contain zero column, there exists $j \in \{1, \ldots, \ell\}$ such that $\bar{M} \cdot \chi(\bar{c}_j) + \bar{N} \cdot \chi(\bar{d}_j) \geq 2(C \cdot 1)(D \cdot 1) + 3$. Moreover, if $G = (U, V, E_1, \ldots, E_\ell)$ is $(C, D)$-biregular of size $(\bar{M}, \bar{N})$, then $G$ is also $(C - \bar{c}_j, D - \bar{d}_j)$-biregular and $(\bar{c}_j, \bar{d}_j)$-biregular.

By the induction hypothesis, both $\text{BiREG}_{C-\bar{c}_j, D-\bar{d}_j}((\bar{M}, \bar{N})$ and $\text{BiREG}_{\bar{c}_j, \bar{d}_j}((\bar{M}, \bar{N})$ hold.

We now prove the “if” direction. Suppose $\text{BiREG}_{C,D}((\bar{M}, \bar{N})$ holds. If $(\bar{M}, \bar{N}) \in H_{C,D}$, then there exists a $(C, D)$-biregular graph of size $(\bar{M}, \bar{N})$ and we are done. So suppose $(\bar{M}, \bar{N}) \notin H_{C,D}$. Hence there exists $j \in \{1, \ldots, \ell\}$ such that $\bar{M} \cdot \chi(\bar{c}_j) + \bar{N} \cdot \chi(\bar{d}_j) \geq 2(C \cdot 1)(D \cdot 1) + 3$.

For simplicity, we assume that $j = \ell$. By the induction hypothesis, there exists a $(C - \bar{c}_\ell, D - \bar{d}_\ell)$-biregular graph $G_1 = (U_1, V_1, E_1, \ldots, E_{\ell-1})$ of size $(\bar{M}, \bar{N})$, and by definition, $E_1, \ldots, E_{\ell-1}$ are all pairwise disjoint. By Theorem 7.3, there exists a $(\bar{c}_\ell, \bar{d}_\ell)$-biregular graph $G_2 = (U_2, V_2, E_{\ell})$ of size $(\bar{M}, \bar{N})$. We can assume that $U_1 = U_2 = U$ and $V_1 = V_2 = V$ since $G_1$ and $G_2$ are of the same size $(\bar{M}, \bar{N})$.

We are going to combine $G_1$ and $G_2$ into one graph to get an $\ell$-type $(C, D)$-biregular graph $G = (U, V, E_1, \ldots, E_\ell)$ of size $(\bar{M}, \bar{N})$. If $E_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1}) = \emptyset$, then the graph $G = (U, V, E_1, \ldots, E_\ell)$ is the desired $(C, D)$-biregular $\ell$-type graph of size $(\bar{M}, \bar{N})$, and we are done.

Now suppose $E_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1}) \neq \emptyset$. We are going to construct another graph $G' = (U, V, E'_\ell)$ such that $|E'_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1})| < |E_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1})|$. We do this repeatedly until at the end we obtain a graph $G'' = (V, E''_\ell)$ such that $E''_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1}) = \emptyset$.

Let $(u, v) \in E_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1})$. The number of vertices reachable in from $u$ and $v$ within distance 2 (by any of edges in $E_1, \ldots, E_\ell$) is bounded from above by $2(C \cdot 1)(D \cdot 1) + 2$. Since $\bar{M} \cdot \chi(\bar{c}_\ell) + \bar{N} \cdot \chi(\bar{d}_\ell) \geq 2(C \cdot 1)(D \cdot 1) + 3$, there exists $(u', v') \in E_\ell$ such that $(u, u'), (v, v') \notin E_1 \cup \cdots \cup E_{\ell-1}$. See the left side of the illustration below.
Now we define $E'_\ell$ by deleting the edges $(u, v), (u', v')$ from $E_\ell$, while adding the edges $(u, v'), (u', v)$ into $E_\ell$. Formally,

$$E'_\ell := (E_\ell - \{(u, v), (u', v')\}) \cup \{(u, v'), (u', v)\}$$

See the right side of the illustration above.

Now it is straightforward that $G' = (U, V, E'_\ell)$ is still a $d\bar{r}$-regular graph of size $\bar{N}$, while

$$|E'_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1})| < |E_\ell \cap (E_1 \cup \cdots \cup E_{\ell-1})|$$

We perform this operation until $E_{\ell+1} \cap (E_1 \cup \cdots \cup E_\ell) = \emptyset$. This completes the proof of Theorem 7.4.

\[
\begin{align*}
\sum_{1 \leq i \leq \ell} (M \cdot \chi(\bar{c}_i) + \bar{N} \cdot \chi(\bar{d}_i)) & \geq 2(C \cdot 1)(D \cdot 1) + 3 \\
\sum_{1 \leq i \leq \ell} \left( \sum_{j \text{ such that } D_{i,j} \in \mathbb{N}} N_j \right) & \geq \max_{1 \leq j \leq m} (C_{i,j}) \\
\sum_{1 \leq i \leq \ell} \left( \sum_{j \text{ such that } C_{i,j} \in \mathbb{N}} M_j \right) & \geq \max_{1 \leq j \leq m} (D_{i,j})
\end{align*}
\]

In the following for a positive integer $k$, $I_k$ denotes the $(k \times k)$ identity matrix. In Lemma 7.5 we consider the case where in the matrices $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$ on each row $i = 1, \ldots, \ell$, at least one of the row-i of $C$ and $D$ consists entirely of $\mathbb{N}$.

We need the following notations. For $d \in \mathbb{N}$, we write $\lceil d \rceil$ to denote the number $d$. By default, we set $\lceil d \rceil = d$. For $d \in \mathbb{B}^m$ and $D \in \mathbb{B}^{\ell \times m}$, we define $\lceil d \rceil$ and $|D|$, where $\lceil \cdot \rceil$ is applied on each entry in $d$ and $D$, respectively. For two vectors $t_1, t_2 \in \mathbb{B}^m$, we define the dot product $t_1 \cdot t_2$ as $\lceil t_1 \rceil \cdot \lceil t_2 \rceil$. For a matrix $D \in \mathbb{B}^{\ell \times m}$, we write $D \cdot 1$ to denote the sum $\sum_{i,j} |D_{i,j}|$.

The lemma below characterises the existence of $(C, D)$-biregular graph of size $(\bar{M}, \bar{N})$, where $\bar{M}, \bar{N}$ are big enough with respect to $(C, D)$ and that for every row $i$, either the row-i of $C$ or of $D$ contains only elements from $\mathbb{N}$.

**Lemma 7.5** Let $C = \begin{pmatrix} C^{(1)} \\ C^{(2)} \\ C^{(3)} \end{pmatrix} \in \mathbb{B}^{\ell \times m}$ and $D = \begin{pmatrix} D^{(1)} \\ D^{(2)} \\ D^{(3)} \end{pmatrix} \in \mathbb{B}^{\ell \times n}$ where $\ell = \ell_1 + \ell_2 + \ell_3$ and
Let $M \in \mathbb{N}^m$ and $N \in \mathbb{N}^n$ be big enough with respect to $C$ and $D$ Then the following holds. There is a $(C, D)$-biregular graph of size $(M, N)$ if and only if $\text{BiREG}_{C', D'}((M, K_1, \ldots, K_{\ell_3}), (N, L_1, \ldots, L_{\ell_3}))$ holds, where
\[ C' = \begin{pmatrix} C^{(1)} & 0 \\ C^{(2)} & 0 \\ C^{(3)} \end{pmatrix} \in \mathbb{B}^{\ell \times m} \quad \text{and} \quad D' = \begin{pmatrix} D^{(1)} & 0 \\ D^{(2)} & 0 \\ D^{(3)} \end{pmatrix} \in \mathbb{B}^{\ell \times n}, \]
- each $K_i = \bar{d}_{i+1} \cdot N - \bar{c}_{i+1} \cdot N$,
- each $L_i = \bar{c}_{i+1} \cdot \bar{M} - \bar{d}_{i+1} \cdot N$.

**Proof.** Let $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$ and $\ell_1, \ell_2, \ell_3, \bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$ be as in the premises. We also assume that $\bar{c}_1, \ldots, \bar{c}_\ell$ and $\bar{d}_1, \ldots, \bar{d}_\ell$ are the row vectors of $C$ and $D$, respectively.

Before we present our proof, we have to remark here that we do not need the condition that $\bar{M}$ and $\bar{N}$ are big enough to establish the “only if” direction. For the “if” direction, we only need Inequalities 2 and 3. Inequality 1 is needed only to established Theorem 7.6.

We start with the “only if” direction. Suppose $G = (U, V, E_1, \ldots, E_\ell)$ is a $(C, D)$-biregular graph of size $(M, N)$. This means there exists a partition $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ such that for each $i = 1, \ldots, \ell$,
- for each $j = 1, \ldots, m$, for each $u \in U_j$, $\deg_{E_i}(u) = C_{i,j}$;
- for each $j = 1, \ldots, n$, for each $v \in V_j$, $\deg_{E_i}(v) = D_{i,j}$.

Now, the following holds.

- For each $i = \ell_1 + 1, \ldots, \ell_1 + \ell_2$, the number of $E_i$-edges in $G$ is $\bar{c}_i \cdot \bar{M}$, which should be greater than $[\bar{d}_i] \cdot \bar{N}$. We set $L_i - \ell_3 = \bar{c}_i \cdot \bar{M} - [\bar{d}_i] \cdot \bar{N}$.
- For each $i = \ell_1 + \ell_2 + 1, \ldots, \ell_1 + \ell_2 + \ell_3$, the number of $E_i$-edges in $G$ is $\bar{d}_i \cdot \bar{N}$, which should be greater than $[\bar{c}_i] \cdot \bar{M}$. We set $K_i - \ell_3 - \ell_3 = \bar{d}_i \cdot \bar{N} - [\bar{c}_i] \cdot \bar{M}$.

Let $C', D'$ be as defined in the lemma, and $\bar{K} = (K_1, \ldots, K_{\ell_3})$ and $\bar{L} = (L_1, \ldots, L_{\ell_3})$. We construct a $(C', D')$-biregular graph of size $((M, \bar{K}), (\bar{N}, \bar{L}))$ as follows.

- For each $i = \ell_1 + 1, \ldots, \ell_1 + \ell_2$, for each $j = 1, \ldots, n$, for each vertex $v \in V_j$, if $\deg_{E_i}(v) > [D_{i,j}]$, then we “split” $v$ into $v_0, v_1, \ldots, v_k$ vertices, where
  \[ - k = \deg_{E_i}(v) - [D_{i,j}], \\
  - \deg_{E_i}(v_0) = [D_{i,j}], \text{ and for each } i' \neq i, \deg_{E_{i'}}(v_0) = \deg_{E_{i'}}(v), \\
  - \deg_{E_i}(v_1) = \deg_{E_i}(v_2) = \cdots = \deg_{E_i}(v_k) = 1, \text{ and for each } i' \neq i, \deg_{E_{i'}}(v_1) = \deg_{E_{i'}}(v_2) = \cdots = \deg_{E_{i'}}(v_k) = 0. \]

- Similarly, for each $i = \ell_1 + \ell_2 + 1, \ldots, \ell_1 + \ell_2 + \ell_3$, for each $j = 1, \ldots, m$, for each vertex $u \in U_j$, if $\deg_{E_i}(u) > [C_{i,j}]$, then we “split” $u$ into $u_0, u_1, \ldots, u_k$ vertices, where
  \[ - k = \deg_{E_i}(u) - [C_{i,j}], \\
  - \deg_{E_i}(u_0) = [C_{i,j}], \text{ and for each } i' \neq i, \deg_{E_{i'}}(u_0) = \deg_{E_{i'}}(u), \\
  - \deg_{E_i}(u_1) = \deg_{E_i}(u_2) = \cdots = \deg_{E_i}(u_k) = 1, \text{ and for each } i' \neq i, \deg_{E_{i'}}(u_1) = \deg_{E_{i'}}(u_2) = \cdots = \deg_{E_{i'}}(u_k) = 0. \]
It should be obvious that the resulting graph is a \((C', D')\)-biregular graph of size \((\bar{M}, \bar{K}), (\bar{N}, \bar{L})\).

Now we prove the “if” direction. Suppose \(\text{BiREG}_{C',D'}((\bar{M}, \bar{K}), (\bar{N}, \bar{L}))\) holds, where each \(L_i = \bar{e}_i \cdot \bar{M} - [d_i] \cdot \bar{N}\) and \(K_i = d_i \cdot \bar{N} - [e_i] \cdot \bar{M}\).

By Theorem 7.4, there exists a \((C', D')\)-biregular graph \(G\) of size \((\bar{M}, \bar{K}), (\bar{N}, \bar{L})\). Let \(G = (U \cup A, V \cup B, E_1, \ldots, E_\ell)\), where \(U \cup A = U_1 \cup \cdots \cup U_m \cup A_1 \cup \cdots \cup A_{\ell_3}\) and \(V \cup B = V_1 \cup \cdots \cup V_n \cup B_1 \cup \cdots \cup B_{\ell_2}\) are the \((C', D')\)-biregular partitions.

To construct a \((C, D)\)-biregular graph of size \((M, N)\), we do the following. For each vertex \(u \in U\) adjacent by \(E_i\)-edges to, say, \(s\) vertices in \(B\), we pick \(s\) vertices \(v_1, \ldots, v_s\) from the set \(\bigcup_{j \text{ such that } d_{i,j} \in N} V_j\).

Such \(s\) vertices exist since by Inequality 2, \(\sum_j\) such that \(d_{i,j} \in N\) \(N_j\) is \(\geq \max(\bar{e}_i) \geq \deg(u)\). We delete those \(s\) vertices in \(B\), and connect \(u\) to each of \(v_1, \ldots, v_s\) by \(E_i\)-edges. We do this until the set \(B\) is empty. Similarly, by Inequality 3, we can perform similar operations until the set \(A\) is empty. The resulting graph is a \((C, D)\)-biregular graph of size \((M, N)\). This completes the proof of Lemma 7.5.

Now for every pair \(C \in \mathbb{B}^{\ell \times m}\) and \(D \in \mathbb{B}^{\ell \times n}\) of the form in Lemma 7.5, we can define a Presburger formula \(\text{BiREG}_{C,D}(\bar{X}, \bar{Y})\) as follows.

\[
\text{BiREG}_{C,D}(\bar{X}, \bar{Y}) := \exists Z_1 \cdots \exists Z_{\ell_3} \exists Z'_1 \cdots \exists Z'_{\ell_2} \quad \text{(4)}
\]

\[
\text{BiREG}_{C,D'}(\bar{X}, Z_1, \ldots, Z_{\ell_3}, \bar{Y}, Z'_1, \ldots, Z'_{\ell_2}) \quad \text{(5)}
\]

where \(C' = \begin{pmatrix} C^{(1)} & 0 \\ C^{(2)} & 0 \\ C^{(3)} & I_{\ell_3} \end{pmatrix} \in \mathbb{N}^{\ell \times (m+\ell_3)}\) and \(D' = \begin{pmatrix} D^{(1)} & 0 \\ D^{(2)} & I_{\ell_2} \\ D^{(3)} & 0 \end{pmatrix} \in \mathbb{N}^{\ell \times (n+\ell_2)}\).

Now we can prove the following theorem.

**Theorem 7.6** For every \(C \in \mathbb{B}^{\ell \times m}\) and \(D \in \mathbb{B}^{\ell \times n}\), there exists a Presburger formula \(\text{BiREG}_{C,D}(\bar{X}, \bar{Y})\) such that for every \(\bar{M}, \bar{N}\) big enough with respect to \(C, D\), the following holds. There exists a \((C, D)\)-biregular graph of size \((\bar{M}, \bar{N})\) if and only if the statement \(\text{BiREG}_{C,D}(\bar{X}, \bar{Y})\) holds.

**Proof.** Let \(C \in \mathbb{B}^{\ell \times m}\) and \(D \in \mathbb{B}^{\ell \times n}\), where \(\bar{e}_1, \ldots, \bar{e}_\ell\) and \(\bar{d}_1, \ldots, \bar{d}_\ell\) are the row vectors of \(C\) and \(D\), respectively.

We need an additional notation. For a set \(I \subseteq \{1, \ldots, \ell\}\), we write \(C(I)\) be the matrix \(C'\), in which each row vector \(\bar{e}_i'\) is defined as \(\bar{e}_i' = [\bar{e}_i]_I\), if \(i \in I\), and \(\bar{e}_i' = \bar{e}_i\), otherwise. We can define \(D(I)\) similarly.

We define the formula \(\text{BiREG}_{C,D}(\bar{X}, \bar{Y})\) as follows.

\[
\text{BiREG}_{C,D}(\bar{X}, \bar{Y}) := \bigvee_{I_0, I_1, I_2} \left( \bigwedge_{i \in I_0} \bar{X} \cdot [\bar{e}_i] = \bar{Y} \cdot [\bar{d}_i] \land \bigwedge_{i \in I_1} \bar{X} \cdot [\bar{e}_i] > \bar{Y} \cdot [\bar{d}_i] \land \bigwedge_{i \in I_2} \bar{X} \cdot [\bar{e}_i] < \bar{Y} \cdot [\bar{d}_i] \land \text{BiREG}_{C(I_0 \cup I_2), D(I_0 \cup I_1)}(\bar{X}, \bar{Y}) \right)
\]

where each \(\text{BiREG}_{C(I_0 \cup I_2), D(I_0 \cup I_1)}(\bar{X}, \bar{Y})\) is as defined in Equation 4 and \(I_0, I_1, I_2\) range over the partition \(I_0 \cup I_1 \cup I_2 = \{1, 2, \ldots, \ell\}\). Obviously, for every partition \(I_0 \cup I_1 \cup I_2 = \{1, \ldots, \ell\}\), on each row \(i = 1, \ldots, \ell\), either row-\(i\) from \(C(I_0 \cup I_2)\), or row-\(i\) from \(D(I_0 \cup I_1)\), or row-\(i\) from both consists entirely of \(N\). (Our intention is the application of Lemma 7.5 later on.)

We are going to prove that the formula \(\text{BiREG}_{C,D}(\bar{X}, \bar{Y})\) is the desired formula. The “if” direction follows from Lemma 7.5 and that every \(C(I_0 \cup I_2), D(I_0 \cup I_1)\)-biregular graph is obviously also a \((C, D)\)-biregular graph.

Now we prove the “only if” direction. Suppose \(\bar{M}, \bar{N}\) are big enough for \(C, D\). Let \(G = (U, V, E_1, \ldots, E_\ell)\) be a \((C, D)\)-biregular graph of size \((\bar{M}, \bar{N})\), where \(U = U_1 \cup \cdots \cup U_m\) and \(V = V_1 \cup \cdots \cup V_n\) be the partition of \(C, D\)-biregularity.
We construct a graph. Stage 3 is as follows. For each \( s \) holds. This can be achieved by doing the following. Suppose there is an edge \((u, v)\) from \( C, D \). Stage 2.

We are going to convert the graph \( G \) into \((C(I_0 \cup I_1), D(I_0, I_2))\)-biregular graph in which \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) are also the partition of \((C(I_0 \cup I_1), D(I_0, I_2))\)-biregularity. This, together with Lemma 7.5, implies that \( \text{BiREG}_{C(I_0 \cup I_1), D(I_0, I_2)}(\bar{M}, \bar{N}) \) holds, and hence, our theorem.

If \( G \) is already a \((C(I_0 \cup I_1), D(I_0, I_2))\)-biregular graph, then we are done. Suppose that \( G \) is not. We do the following three stages.

Stage 1. We delete some edges in \( G \) such that for every edge \((u, v) \in E'_i\), either

\[
\text{deg}_{E'_i}(u) = [C_{i,j}] \quad \text{or} \quad \text{deg}_{E'_i}(v) = [D_{i,k}].
\]

This can be achieved by doing the following. Suppose there is an edge \((u, v) \in E'_i\) such that \( \text{deg}_{E'_i}(u) > [C_{i,j}] \) and \( \text{deg}_{E'_i}(v) > [D_{i,k}] \). Since \( G \) is \((C, D)\)-biregular, this means that \( C_{i,j}, D_{i,k} \in \mathbb{N}_1 \).

Deleting the edge \((u, v)\), we still have \( \text{deg}_{E'_i}(u) \geq [C_{i,j}] \) and \( \text{deg}_{E'_i}(v) \geq [D_{i,k}] \), and hence \( G \) is still \((C, D)\)-biregular with \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) be the partition of the \((C, D)\)-biregularity. We repeatedly do this until the graph \( G \) satisfies condition (6).

Stage 2. We construct a graph \( G' = (U \cup S, V \cup T, E'_1, \ldots, E'_\ell) \), where for every \( i = 1, \ldots, \ell \),

- for every \( j = 1, \ldots, m \), for each \( u \in U_j \), \( \text{deg}_{E'_j}(u) = [C_{i,j}] \);
- for every \( s \in S \), \( \text{deg}(s) = 1 \);
- for every \( k = 1, \ldots, n \), for each \( v \in V_j \), \( \text{deg}_{E'_j}(v) = [D_{i,j}] \);
- for every \( t \in T \), \( \text{deg}(t) = 1 \).

The graph \( G' \) can be obtained by doing the same trick as in the proof of Lemma 7.5. For every vertex \( u \in U_j \), if \( \text{deg}_{E'_j}(u) - [C_{i,j}] = z > 0 \), then we “split” \( u \) into \( z + 1 \) vertices \( u', s_1, \ldots, s_z \), where

- \( \text{deg}_{E'_j}(u') = [C_{i,j}] \), for all other \( h \neq i \), \( \text{deg}_{E'_j}(u') = \text{deg}_{E'_j}(u) \);
- \( \text{deg}_{E'_j}(s_1) = \cdots = \text{deg}_{E'_j}(s_z) = 1 \), and for all other \( h \neq i \), \( \text{deg}_{E'_j}(s_1) = \cdots = \text{deg}_{E'_j}(s_z) = 0 \).

We can do similar operation to the vertices in \( v \in V_k \). Since \( G \) satisfies condition 6, there is no edge between vertices in \( S \) and \( T \). We also further partition \( S = S_1 \cup \cdots \cup S_\ell \) and \( T = T_1 \cup \cdots \cup T_\ell \), where each \( S_i \) and \( T_i \) contains the vertices whose \( \text{deg}_{E'_j} = 1 \).

Stage 3. Stage 3 is as follows. For each \( i = 1, \ldots, \ell \), if there are an edge \((s, v) \in E_i \) and an edge \((u, t) \in E_i \), for some \( s \in S_i \), \( v \in V_k \), \( u \in U_j \), \( t \in T_\ell \), we do the following.

- We delete the two edges \((s, v)\) and \((u, t)\) from \( E_i \), as well as the vertices \( s \) and \( t \).
- We add an edge \((u, v) \in E_i \).
- If there is already an existing edge \((u, v) \in E_1 \cup \cdots \cup E_\ell \), adding another \((u, v)\) may result in “parallel” edges. However, since \( M, N \) is big enough with respect to \( C, D \), and in particular, Inequality 1 holds, we can apply the same trick as in the proof of Theorem 7.4 to get rid of the parallel edge, while preserving the degree of the vertices.

We repeatedly do this until for each \( i = 1, \ldots, \ell \) either \( S_i = \emptyset \), or \( T_i = \emptyset \). In particular, the following holds.

- If \( i \in I_0 \), then \( S_i = T_i = \emptyset \).

Recall that \( i \in I_0 \) means that \( M \cdot \lceil \hat{c}_i \rceil = N \cdot \lceil \hat{d}_i \rceil \), which implies that the initial sets \( S_i, T_i \) have the same cardinality. Since we always delete a pair of vertices \( s, t \) from \( S_i, T_i \), respectively, we have at the end \( S_i = T_i = \emptyset \).

\( 1 \)Recall that \( \text{deg}(s) = \text{deg}_{E'_i}(s) + \cdots + \text{deg}_{E'_j}(s) \). Hence, \( \text{deg}(s) = 1 \) means that there is only one edge adjacent to \( s \).
Likewise, if \( i \in I_1 \), then \( S_i = 0 \).

This is because \( M \cdot [c_i] > \bar{N} \cdot [d_i] \), implies that initially \( |T_i| > |S_i| \), which further implies that at the end \( S_i = \emptyset \).

By symmetrical reasoning, if \( i \in I_2 \), then \( T_i = 0 \).

From here, we will "merge" back the vertices in \( S \) with vertices in \( U \), and the vertices in \( T \) with vertices in \( V \). Again, this can be done in a similar manner.

For each vertex \( u \in U \) adjacent by \( E_t \)-edges to, say, \( z \) vertices in \( T \), we pick \( z \) vertices \( v_1, \ldots, v_z \) from the set

\[
\bigcup_{j \text{ such that } u_{i,j} \in \mathbb{N}} V_j
\]

Such \( z \) vertices by Inequality 2 which holds because \( \bar{M}, \bar{N} \) are big enough w.r.t. \( C, D \). We delete those \( z \) vertices in \( T \), and connect \( u \) to each of \( v_1, \ldots, v_z \) by \( E_t \)-edges. We do this until the set \( T \) is empty. Similarly, by Inequality 3, we can do the same for the set \( S \).

The resulting graph is \((C(I_0 \cup I_1), D(I_0, I_2))\)-biregular graph, which by Lemma 7.5 the formula \( \text{BiREG}_{C(I_0 \cup I_1), D(I_0, I_2)}(M, \bar{N}) \) holds. This completes our proof of Theorem 7.6.

\[ \square \]

### 7.4 The notion of partial bipartite graphs

In this subsection we are going to generalise Theorem 7.6 to the case where one of the inequalities 1 and 2 does not hold. For this, we introduce the notion of partial graph.

An \( \ell \)-type partial bipartite graph is a tuple \( \mathcal{P} = (C, D, S, T, f, g) \), where

- \( C \in \mathbb{B}^{\ell \times m} \) and \( D \in \mathbb{B}^{\ell \times n} \),
- \( S \) is a finite set of vertices (possibly empty),
- \( T \) is a finite set of vertices (possibly empty),
- \( f : S \times \{E_1, \ldots, E_\ell\} \rightarrow \mathbb{B} \),
- \( g : T \times \{E_1, \ldots, E_\ell\} \rightarrow \mathbb{B} \).

Obviously, if \( S \) or \( T \) is empty, then \( f \) or \( g \), respectively, is also an "empty" function. In the following the term partial graph always means partial bipartite graph.

A completion of the partial graph \( \mathcal{P} = (C, D, S, T, f, g) \) is a bipartite graph \( G = (U \cup S, V \cup T, E_1, \ldots, E_\ell) \), where there are partitions \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \), where

- for every \( u \in U_j \), \( \deg_{E_i}(u) = C_{i,j} \),
- for every \( v \in V_j \), \( \deg_{E_i}(v) = D_{i,j} \),
- for every \( s \in S \), \( \deg_{E_i}(s) = f(s, E_i) \),
- for every \( t \in T \), \( \deg_{E_i}(t) = g(t, E_i) \).

When it is clear from the context, we also call \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) the \((C, D)\)-biregular partitions. Note that when both \( S \) and \( T \) are empty, then the completions of the partial graph \( \mathcal{P} \) are simply \((C, D)\)-biregular graphs.

We need a few additional notations:

\[
(\blacktriangledown c) - d = \begin{cases} 
\blacktriangledown (c - d) & \text{if } c \geq d \\
\blacktriangledown 0 & \text{otherwise}
\end{cases}
\]

Let \( C \in \mathbb{B}^{\ell \times m} \). We define a matrix \( \xi(C) \in \mathbb{B}^{\ell \times (\ell + 1)m} \) as follows.

\[
\xi(C) := \begin{pmatrix} 
C & M_1 & \cdots & M_m
\end{pmatrix}
\]
where each $M_i$ is the matrix is obtained by repeating the $i$th column vector of $C$ for $\ell$ number of times, and subtracted by the identity matrix $I_\ell$. Formally,

$$M_i := \begin{pmatrix} C_{1,i} & C_{1,i} & \cdots & C_{1,i} \\ C_{2,i} & C_{2,i} & \cdots & C_{2,i} \\ \vdots & \vdots & \ddots & \vdots \\ C_{\ell,i} & C_{\ell,i} & \cdots & C_{\ell,i} \end{pmatrix} - I_\ell$$

Lemma 7.7 below essentially states that every partial graph can be reduced into a “smaller” partial graph with the addition of some linear equalities.

**Lemma 7.7** Let $\mathcal{P} = (C,D,S,T,f,g)$ be a partial graph, where $T \neq \emptyset$. Let $t \in T$. Then the following holds.

1. For every completion graph $G = (U \cup S, V \cup T, E_1, \ldots, E_\ell)$ of the partial graph $\mathcal{P}$ with $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be the partitions of the $(C,D)$-biregularity, there exists a completion graph $G' = (U \cup S, (V \cup T) \setminus \{t\}, E'_1, \ldots, E'_\ell)$ of the partial graph $\mathcal{P}' = (\xi(C),D,S,T \setminus \{t\},f,g')$, with the partitions of the $(\xi(C),D)$-biregularity be

\[
U = U'_1 \cup \cdots \cup U'_m \cup (U'_{1,1} \cup \cdots \cup U'_{1,i}) \cup (U'_{1,2} \cup \cdots \cup U'_{1,2}) \cup \cdots \cup (U'_{1,m} \cup \cdots \cup U'_{1,m})
\]

and

$$V = V_1 \cup \cdots \cup V_n$$

and

$$\sum_{1 \leq j \leq m} |U'_{1,j}| = g(t, E_i) \quad \text{for each } i = 1, \ldots, \ell.$$ 

2. Visa versa, for every completion graph $G' = (U \cup S, (V \cup T) \setminus \{t\}, E'_1, \ldots, E'_\ell)$ of the partial graph $\mathcal{P}' = (\xi(C),D,S,T \setminus \{t\},f,g')$, with the partitions of the $(\xi(C),D)$-biregularity be

\[
U = U'_1 \cup \cdots \cup U'_m \cup (U'_{1,1} \cup \cdots \cup U'_{1,i}) \cup (U'_{1,2} \cup \cdots \cup U'_{1,2}) \cup \cdots \cup (U'_{1,m} \cup \cdots \cup U'_{1,m})
\]

and for each $i = 1, \ldots, \ell$, $\sum_{1 \leq j \leq m} |U'_{1,j}| = g(t, E_i)$, there exists a completion graph $G = (U \cup S, V \cup T, E_1, \ldots, E_\ell)$ of the partial graph $\mathcal{P}$ with $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be the partitions of the $(C,D)$-biregularity, and

$$|U_j| = |U'_j| + |U'_{1,j}| + \cdots + |U'_{\ell,j}| \quad \text{for each } j = 1, \ldots, m.$$ 

**Proof.** Let $\mathcal{P} = (C,D,S,T,f,g)$ be a partial graph, where $T \neq \emptyset$ and $t \in T$. First, we prove part (1). Let $G = (U \cup S, V \cup T, E_1, \ldots, E_\ell)$ be a completion graph of $\mathcal{P}$ with $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be the partitions of the $(C,D)$-biregularity.

For each $j = 1, \ldots, m$, we partition $U_j$ into

$$U_j = U'_j \cup \ldots \cup U'_{\ell,j},$$

where

- $U'_j$ be the set of vertices in $U_j$ that are not adjacent to the vertex $t$,
- for each $i = 1, \ldots, \ell$, $U'_{i,j}$ is the set of vertices in $U_j$ adjacent to $t$ via $E_i$-edges.
Now deleting the vertex \( t \) and all its adjacent edges, we obtain the desired completion graph
\[ G' = (U \cup S, (V \cup T) \setminus \{ t \}, E_1', \ldots, E_\ell') \] of \( P' = (\xi(C)D, S, T \setminus \{ t \}, f, g') \).

Now we prove part (2). Let \( G' = (U \cup S, (V \cup T) \setminus \{ t \}, E_1', \ldots, E_\ell') \) be a completion of the partial graph \( P' = (\xi(C)D, S, T \setminus \{ t \}, f, g') \), with the partitions of the \((\xi(C), D)\)-biregularity be
\[
U = U_1' \cup \cdots \cup U_m' \cup (U_{1,1}' \cup \cdots \cup U_{1,2}' \cup \cdots \cup U_{1,m}' \cup \cdots \cup U_{\ell,1}' \cup \cdots \cup U_{\ell,2}' \cup \cdots \cup U_{\ell,m}')
\]
\[
V = V_1 \cup \cdots \cup V_n
\]
and for each \( i = 1, \ldots, \ell \), \( \sum_{1 \leq j \leq m} |U_{i,j}'| = g(t, E_i) \).

The desired completion graph \( G = (U \cup S, V \cup T, E_1, \ldots, E_\ell) \) of the partial graph \( P \) can be obtained as follows. We put the vertex \( t \) back inside \( T \). Then, for each \( i = 1, \ldots, \ell \) and for each \( j = 1, \ldots, m \), we connect \( t \) with every vertex \( u \in U_{i,j}' \) with \( E_i \)-edge. This way we obtain the completion graph \( G \) with \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) be the partitions of the \((C, D)\)-biregularity, and
\[
|U_i| = |U_i'| + |U_{i,1}'| + \cdots + |U_{i,m}'| \quad \text{for each } i = 1, \ldots, m.
\]

This completes our proof of Lemma 7.7. ■

Following Lemma 7.7 above, we show that every partial graph can be translated into a Presburger formula that captures any of its completion, as stated in the following theorem.

**Theorem 7.8** For every partial graph \( P = (C, D, S, T, f, g) \), we can construct a Presburger formula \( \Psi_P(X, Y) \) such that for every \( M \) and \( N \) big enough w.r.t. \( C, D \), the following holds. There exists a completion graph \( G = (U \cup S, V \cup T, E_1, \ldots, E_\ell) \), such that \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) and \( M = (|U_1|, \ldots, |U_m|) \) and \( N = (|V_1|, \ldots, |V_n|) \) if and only if \( \Psi_P(M, N) \) holds.

**Proof.** Let \( P = (C, D, S, T, f, g) \) be a partial graph. If the matrix \( C \) is empty, there are only finitely many completion of \( P \). In this case \( \Psi_P \) simply contains the enumeration the sizes of all possible completions of \( P \). We can define \( \Psi_P \) in a similar manner when \( D \) is empty.

Now suppose both the matrices \( C \) and \( D \) are not empty. The construction of \( \Psi_P \) is done inductively as follows. The base case is \( S \cup T = \emptyset \), in which case \( \Psi_P \) is defined as follows.
\[
\Psi_P(X, Y) := \text{BiREG}_{C, D}(X, Y)
\]

Towards the induction step, let \( S \cup T \neq \emptyset \). Suppose \( T \neq \emptyset \) and \( t \in T \). (The case when \( S \neq \emptyset \) can handled in a symmetrical manner.)

We define \( \Psi_P(X, Y) \) as follows.
\[
\Psi_P(X, Y) := \exists Z_1 \cdots \exists Z_m \exists Z_{1,1} \cdots \exists Z_{1,1} \exists Z_{1,2} \cdots \exists Z_{1,2} \cdots \exists Z_{1,m} \cdots \exists Z_{\ell,m}
\]
\[
\wedge \bigwedge_{1 \leq i \leq m} \bigwedge_{1 \leq i \leq \ell} \sum_{1 \leq j \leq m} Z_{i,j} = g(t, E_i)
\]
\[
\wedge \bigwedge_{1 \leq i \leq \ell} \Psi_P((Z_1, \ldots, Z_m, Z_{1,1}, \ldots, Z_{1,1}, Z_{1,2}, \ldots, Z_{1,2}, \ldots, Z_{1,m}, \ldots, Z_{\ell,m}), Y)
\]
where \( \mathcal{P}' = (\xi(C), D, S, T \setminus \{ t \}, f, g') \) and \( g' \) is the function \( g \) restricted to \( T \setminus \{ t \} \).

By Theorem 7.6 in the previous section, the correctness of the base case is established. The induction step follows from Lemma 7.7, and hence, shows that the formula \( \Psi_P \) is the desired formula. This completes our proof of Theorem 7.8. ■

Before we move on to prove Theorem 5.1, we need the following notions. Let \( C \in B^{t \times m} \) and \( D \in B^{t \times n} \). We say that a partial graph \( P = (C', D', S, T, f, g) \) is compatible with \((C, D)\) with respect to a subset \( I \subseteq \{1, \ldots, m\} \) and a subset \( J \subseteq \{1, \ldots, n\} \), and the partitions \( S = S_1 \cup \cdots \cup S_{m'} \) and \( T = T_1 \cup \cdots \cup T_{n'} \), if the following four conditions hold.

\[24\]
where
whenever
a variable vector $\bar{Y} = (X_1, \ldots, X_{\ell})$, we write $X_I$ to denote the variables obtained by deleting $X_i$ whenever $i \in I$. We can define $\bar{Y}_J$ similarly when $J \subseteq \{1, \ldots, n\}$ and $\bar{Y} = (Y_1, \ldots, Y_{n})$.

We define the formula $\text{BiREG}_{C,D}(\bar{X}, \bar{Y})$ as follows.

$$
\text{BiREG}_{C,D}(\bar{X}, \bar{Y}) := \bigvee_{P} \left( \Psi_P(\bar{X}_I, \bar{Y}_J) \land \varphi \land X_{i_1} = |S_1| \land X_{i_2} = |S_2| \land \cdots \land X_{i_{m'}} = |S_{m'}| \land Y_{j_1} = |T_1| \land Y_{j_2} = |T_2| \land \cdots \land Y_{j_{n'}} = |T_{n'}| \right)
$$

where

- the disjunction ranges over all partial graph $P = (C', D', S, T, f, g)$ compatible with $(C, D)$ w.r.t. $I = \{i_1, \ldots, i_{m'}\}$ and $J = \{j_1, \ldots, j_{n'}\}$, as well as the partitions $S = S_1 \cup \cdots \cup S_{m'}$ and $T = T_1 \cup \cdots \cup T_{n'}$,
- $\varphi$ states that $\bar{X}_I, \bar{Y}_J$ are big enough w.r.t. $(C', D')$.

The correctness of the formula $\text{BiREG}_{C,D}$ follows immediately from the correctness of the formula $\Psi_P$ in Theorem 7.8. This completes our proof of Theorem 5.1.

### 7.5 Proof of Theorem 5.2

We start with the following lemma which essentially states that if there exists a $(C, D)$-biregular-complete graph of “big enough” size, then the pair of matrices $(C, D)$ itself has certain property.

**Lemma 7.9** Let $G = (U, V, E_1, \ldots, E_\ell)$ be an $\ell$-type $(C, D)$-biregular graph, where $C = \begin{pmatrix} \bar{c}_1 \\ \bar{c}_\ell \end{pmatrix} \in \mathbb{B}^{\ell \times m}$, $D = \begin{pmatrix} \bar{d}_1 \\ \cdots \\ \bar{d}_\ell \end{pmatrix} \in \mathbb{B}^{\ell \times n}$ and $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ is the partition of the regularity. Suppose that for each $i, j$, we have

$$
|U_i|, |V_j| \geq |C| \cdot \bar{1} + |D| \cdot \bar{1} + 1.
$$

If $G$ is a complete bipartite graph, then for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, there exists $l \in \{1, \ldots, \ell\}$ such that both $C_{i,l}, D_{l,j} \in \mathbb{N}$. 

**Proof.** Let $C \in \mathbb{B}^{\ell \times m}$ and $D \in \mathbb{B}^{\ell \times n}$, and $G = (U, V, E_1, \ldots, E_\ell)$ be a $(C, D)$-biregular-complete graph, where $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ are the partition of the $(C, D)$-biregularity. Suppose each $|U_i|$ and $|V_j|$ satisfy the inequality above.
For the sake of contradiction, we assume that there exist \( i,j \in \{1, \ldots, m\} \) such that for all \( l \in \{1, \ldots, \ell\} \), either \( C_{l,i} \in \mathbb{N} \) or \( D_{l,j} \in \mathbb{N} \). This means that for each \( l \in \{1, \ldots, \ell\} \), the number of \( E_l \)-edges between \( U_i \) and \( V_j \) is \( |U_i|C_{l,i} \) if \( C_{l,i} \in \mathbb{N} \), or \( |V_j|D_{l,j} \) if \( D_{l,j} \in \mathbb{N} \). For each \( l = 1, \ldots, \ell \),

\[
K_l = \begin{cases} 
|U_i|C_{l,i} & \text{if } C_{l,i} \in \mathbb{N}, \\
|V_j|D_{l,j} & \text{if } D_{l,j} \in \mathbb{N}.
\end{cases}
\]

Now the total number of edges between \( U_i \) and \( V_j \) must be \( \sum_{1 \leq l \leq \ell} K_l \), which must be equal to \( |U_i| \times |V_j| \) since \( G \) is a complete bipartite graph.

However, from the inequality

\[ |U_i|, |V_j| \geq |C| \cdot \bar{1} + |D| \cdot \bar{1} + 1, \]

a straightforward calculation shows that \( K \) is strictly less than \( |U_i| \times |V_j| \), a contradiction. Therefore, for every \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \), there exists \( l \in \{1, \ldots, \ell\} \) such that both \( C_{l,i}, D_{l,j} \in \mathbb{N} \). This completes the proof of our lemma.

We say that a pair of matrices \( (C,D) \in \mathbb{B}^{\ell \times m} \times \mathbb{B}^{\ell \times n} \) is an easy pair of matrices, if for every \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \) there exists \( l \in \{1, \ldots, \ell\} \) such that both \( C_{l,i}, D_{l,j} \in \mathbb{N} \).

**Lemma 7.10** Let \((C, D)\) be an easy pair of matrices, where \( C \in \mathbb{B}^{\ell \times m} \) and \( D \in \mathbb{B}^{\ell \times n} \). Then the following holds. There exists a \((C, D)\)-biregular-complete graph of size \((M, N)\) if and only if \( \text{BiREG}_{C,D}(M, N) \) holds.

**Proof.** The “only if” direction follows directly from Theorem 5.1. Now we prove the “if” direction. Suppose \( \text{BiREG}_{C,D}(M, N) \) holds. By Theorem 5.1, there exists a \((C, D)\)-biregular graph \( G = (U,V,E_1, \ldots, E_\ell) \) of size \((M, N)\). This graph \( G \) is not necessarily complete. So suppose \( U = U_1 \cup \cdots \cup U_m \) and \( V = V_1 \cup \cdots \cup V_n \) be the partition of the biregularity. If \( G \) is not complete, then we perform the following. For every \( u \in U \) and \( v \in V \) such that \( (u, v) \notin E_1 \cup \cdots \cup E_\ell \), we do the following.

- Let \( u \in U_i \) and \( v \in V_j \).
- Pick an index \( l \in \{1, \ldots, k\} \) such that \( C_{l,i}, D_{l,j} \in \mathbb{N} \).
  (Such an index \( l \) exists since \((C, D)\) is an easy pair.)
- Connect \( u \) and \( v \) with an \( E_l \)-edge.

The resulting graph is now complete and still \( D \)-regular. This completes our proof of Lemma 7.10.

---

(Proof of Theorem 5.2.) Let \( C \) and \( D \) be two matrices. The formula \( \text{COMP-BiREG}_{C,D}(\bar{X}, \bar{Y}) \) is constructed as follows.

- If \((C, D)\) is an easy pair, then \( \text{COMP-BiREG}_{C,D}(\bar{X}, \bar{Y}) = \text{BiREG}_{C,D}(\bar{X}, \bar{Y}) \).
- If \((C, D)\) is not an easy pair, then by Lemma 7.9, the values in the entries (in \( \bar{X} \) and \( \bar{Y} \)) corresponding to the columns in \( C \) and \( D \) that make them not an easy pair must be bounded. These values can be encoded as partial graphs as described in the previous section. We omit the details here since we use the same arguments.

This completes our proof of Theorem 5.2.

---

8 Proof of Theorem 5.4

The proof is by observing that the existence of a \((C, D)\)-directed-regular graph of size \( \bar{N} \) is equivalent to the existence of a \((C, D)\)-biregular graph of size \((\bar{N}, \bar{N})\). We explain it more precisely below.
Suppose \( G = (V, E_1, \ldots, E_{\ell}) \) is a \((C, D)\)-directed-regular graph of size \( \bar{N} \).

Then, for every vertex \( v \in V \), we “split” it into two vertices \( u \) and \( w \) such that \( u \) is only adjacent to the incoming edges of \( v \) and \( w \) to the outgoing edges of \( v \). See the illustration below. The left-hand side shows the vertex \( v \) before the splitting, and the right-hand side shows the vertices \( u \) and \( w \) after the splitting.

Let \( U \) be the set of vertices \( u \)’s and \( W \) the set of vertices \( w \)’s after splitting all the vertices in \( V \). Ignoring the orientation of the edges, the resulting graph is a bipartite graph with vertices \( U \cup W \) and it is a \((C, D)\)-biregular graph of size \( (\bar{N}, \bar{N}) \).

Suppose \( G = (U, W, E_1, \ldots, E_{\ell}) \) is a \((C, D)\)-biregular graph of size \( (\bar{N}, \bar{N}) \). Let \( U = U_1 \cup \cdots \cup U_m \) and \( W = W_1 \cup \cdots \cup W_m \) be the partition of \((C, D)\)-biregularity. We denote by \( U_i = \{u_{i,1}, \ldots, u_{i,K_i}\} \) and \( W_i = \{w_{i,1}, \ldots, w_{i,K_i}\} \) for each \( i = 1, \ldots, m \).

Now we put the orientation on all the edges from \( U \) to \( W \). Then we merge every two vertices \( u_{i,j} \) and \( w_{i,j} \) into one vertex \( v_{i,j} \). This way, we obtain a \((C, D)\)-directed-regular graph \( G = (V, E_1, \ldots, E_{\ell}) \) with \( V = V_1 \cup \cdots \cup V_m \) be the partition of regularity where \( V_i = \{v_{i,1}, \ldots, v_{i,K_i}\} \) for each \( i = 1, \ldots, m \).

However, with such merging it is possible that there is a self-loop \((v, v)\) in \( G \) or a pair of edges \((v, v'), (v', v)\) \( \in E_1 \cup \cdots \cup E_{\ell} \). We can get rid of the self-loop \((v, v)\) without violating the \((C, D)\)-directed-regularity as follows. The trick is similar to the one used before. Assuming that the size of each \(|V_1|, \ldots, |V_m|\) is big enough and there are enough edges in each \( E_1, \ldots, E_{\ell} \), there is an edge \((v', v'')\) of the same type. Deleting the edge \((v, v)\) and \((v', v'')\), and adding the edges \((v', v)\) and \((v, v'')\), we obtain a \((C, D)\)-directed-regular graph with one less self-loop. We do this repeatedly until there is no more self-loop. See the illustration below.

Similarly, we can get rid of a pair of edges \((v, v'), (v', v)\) \( \in E_1 \cup \cdots \cup E_{\ell} \) without violating the \((C, D)\)-directed-regularity as follows. Again, the trick is similar to the one used before. Assuming that the size of each \(|V_1|, \ldots, |V_m|\) is big enough and there are enough edges in each \( E_1, \ldots, E_{\ell} \), there is an edge \((w, w')\) of the same type. Deleting the edge \((v, v')\) and \((w, w')\), and adding the edges \((v, w')\) and \((w, v')\), we obtain a \((C, D)\)-directed-regular graph with one
less parallel edges. We do this repeatedly until there are no more parallel edges. See the illustration below.

If some sets \( V_i \) or \( |E_i| \) are of a fixed size, those can be encoded in a partial graph in the same manner discussed in Subsection 7.4.

We omit the technical details since we essentially run through the same arguments used in the previous section.

9 Concluding remarks

By showing Presburger formulas for the existence of regular graphs, we have shown that the spectra of \( C^2 \) formulae are constructive semilinear sets. From our proof, it can be immediately deduced that the many-sorted spectra of \( C^2 \) are also semilinear. As far as our knowledge is concerned, the logic \( C^2 \) is the first logic whose spectra is closed under complement without any restriction on the vocabulary nor in the interpretation. The semilinearity of its spectra also gives us an interesting insight on the nature of two-variable logic – that each of its models is simply a collection of regular graphs.

Another interesting open question is, how can \( C^2 \) be extended while keeping decidability? Using three variables (\( FO^3 \)) one can easily encode a grid; therefore, the satisfiability problem is no longer decidable (and thus the image membership problem). However, we could extend \( C^2 \) by giving access to a relation having a property which is undecidable in \( C^2 \), such as transitivity. In particular, \( C^2(\lt) \), that is, the logic \( C^2 \) with access to a total order on the universe, seems powerful: Petri net reachability \([28, 22, 23]\) reduces to image membership for \( C^2(\lt) \) formulae. We do not know whether a reduction exists in the other direction. Another possible extension is to add an equivalence relation.

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