Nonlinear Jaynes-Cummings model of atom-field interaction

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August 21, 2018

PACS Nos:42.50 Md, 42.50.Dv, 05.30.-d

Abstract

Interaction of a two-level atom with a single mode of electromagnetic field including Kerr nonlinearity for the field and intensity-dependent atom-field coupling is discussed. The Hamiltonian for the atom-field system is written in terms of the elements of a closed algebra, which has SU(1,1) and Heisenberg-Weyl algebras as limiting cases. Eigenstates and eigenvalues of the Hamiltonian are constructed. With the field being in a coherent state initially, the dynamical behaviour of atomic-inversion, field-statistics and uncertainties in the field quadratures are studied. The appearance of nonclassical features during the evolution of the field is shown. Further, we explore the overlap of initial and time-evolved field states.

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1 Introduction

Interaction of a single mode of electromagnetic field with a two-level atom is the simplest problem in matter-radiation coupling. A model for the interaction, introduced by Jaynes and Cummings [1], treats the atom as a dipole placed in an external field. The Jaynes-Cummings (JC) model has provided a lot of impetus for theoretical explorations and experimental verifications [2, 3, 4, 5, 6]. The Hamiltonian for the model is

\[ H_{JC} = H_0 + g(\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+). \] (1)

Here \( H_0 \), the Hamiltonian for the atom and field in the absence of interaction, is \( \omega \hat{a}^\dagger \hat{a} + \frac{1}{2} \nu \sigma_z \). The atomic transition frequency is \( \nu \), the field frequency is \( \omega \) and \( g \) is the coupling constant. We denote the creation operator by \( \hat{a}^\dagger \), annihilation operator by \( \hat{a} \), and their action on the basis states of the harmonic oscillator are

\[ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \] (2)
\[ \hat{a}|0\rangle = 0, \] (3)
\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle. \] (4)

The atom has two-levels, \( |g\rangle \) and \( |e\rangle \), the ground and excited states. The operators \( \sigma_z, \sigma_+ \) and \( \sigma_- \) act on the states as given below:

\[ \sigma_z|g\rangle = -|g\rangle \] (5)
\[ \sigma_z|e\rangle = |e\rangle \] (6)
\[ \sigma_\pm|g\rangle = \frac{1 \pm \frac{1}{2}}{2}|e\rangle \] (7)
\[ \sigma_\pm|e\rangle = \frac{1 \mp \frac{1}{2}}{2}|g\rangle \] (8)

Atomic inversion, defined as the difference in probabilities for the atom to be in the excited and ground states, as predicted by JC model is a sum of quasiperiodic functions with incommensurate frequencies. The model predicts collapses, revivals and ringing revivals in the time-development of atomic inversion[7, 8, 9, 10, 11] The revival phenomenon is entirely quantal, and hence the model is very important for testing predictions of quantum theory. The model has been generalized in very many ways. We list the respective Hamiltonian for some of the models:

1) Buck-Sukumar model[12]

\[ H_{JC} = H_0 + g(\sqrt{\hat{a}^\dagger \hat{a}} \sigma_+ + \hat{a}^\dagger \sqrt{\hat{a}^\dagger \hat{a}} \sigma_-) \] (10)

With this particular form of intensity-dependent coupling the atomic-inversion is a sum of periodic functions. This model is very interesting as it can be written as a combination of generators of SU(1,1) algebra[13]. Generalization to
two-photon case, where two-photons are absorbed or emitted in atom-field interactions, has been done. Once again the SU(1,1) algebraic structure in the model is used to solve the problem \[14\].

2) Kerr-nonlinearity\[15, 16, 17\]

\[
H_{Kerr} = H_{JC} + \chi \hat{a}^2 \hat{a}^2
\] (11)

This is an effective Hamiltonian for a system in which the electromagnetic field mode is excited in a Kerr medium. The medium is modelled as an anharmonic oscillator\[18, 19\].

3) Dicke-Tavis-Cummings model\[20, 21\]

In this model, the interaction between field and a group of two-level atoms is considered and the Hamiltonian is

\[
H_{DT C} = \omega \hat{a}^\dagger \hat{a} + \frac{\nu}{2} \sum_i \sigma_{i,z} + \text{Interaction part}
\] (12)

4) Nonlinear Jaynes-Cummings model

By including the motion of atom in the external field, the coupling is made position dependent. This offers enormous possibilities to tailor the form of atom-field interaction \[22, 23\]. The general form for the Hamiltonian is

\[
H_{NL} = H_0 + g(f(\hat{a}^\dagger \hat{a})\hat{a}^m + \text{adjoint})
\] (13)

Here \( f(\hat{a}^\dagger \hat{a}) \) is an operator-valued function of the number operator \( \hat{a}^\dagger \hat{a} \) and \( m \) is an integer.

The objective of the present paper is to study the dynamics of a two-level atom interacting with a single mode of electromagnetic field. The interaction is governed by the Hamiltonian

\[
H = \omega \left[ \hat{a}^\dagger \hat{a} + \frac{r}{2} \sigma_z \right] + \chi \hat{a}^2 \hat{a}^2 + g\omega (\sqrt{1 + k\hat{a}^\dagger \hat{a} \sigma_+ + \hat{a}^\dagger \sqrt{1 + k\hat{a}^\dagger \hat{a} \sigma^-}}) (14)
\]

This Hamiltonian is an example for nonlinear JC model including Kerr term. Note that we have scaled the coupling constant \( g \) by \( \omega \). The parameter \( r \) is \( \frac{\nu}{2} \) and the coupling constant for the Kerr term is \( \chi = k\omega \). This will simplify the expressions we derive in sequel. In particular, we set \( \omega = 1 \) which amounts to studying a new Hamiltonian \( \frac{H}{\omega} \). The speciality of the Hamiltonian \( H \) is that it becomes \( H_{JC} \) when the parameter \( k \) is set equal to zero. Further, the usual Holstein-Primikoff realization is obtained when \( k = 1 \) so that the interaction is approximately given by \( g(\sqrt{\hat{a}^\dagger \hat{a} \sigma_+ + \text{adjoint}}) \), which is same as the interaction studied in Ref.\[13\]. Yet another form of interaction occurs when \( k \ll 1 \). If the
photon number distribution is such that $kn \ll 1$ for all $n$ under the peak of the distribution, then Eq. 14 leads to

$$H \rightarrow \omega \left[ \hat{a}^\dagger \hat{a} + k\hat{a}^\dagger \hat{a}^2 + g(1 + \hat{a}^\dagger \hat{a}/2)\hat{a}\sigma_z + \text{adjoint} \right]. \quad (15)$$

Further, when $k \ll 1$ so that we can neglect $kn$ in comparison to unity but retain $kn^2$, we arrive at $H_{JC}$ with an additional Kerr term. This system has been studied in Ref.[24]. Many well studied systems are thus special cases of the Hamiltonian considered for discussion in the present paper. The organization of the paper is as follows. In section II we study the algebraic aspects of the Hamiltonian and construct the eigenvectors and eigenvalues. Section III is devoted to study the atomic inversion and approximate expressions are obtained for first collapse and revival periods when the field is in a coherent state. The time-development of field statistics is discussed in section IV and the results are summarized in Section V.

2 Model Hamiltonian and its properties

In this section we study the algebraic aspects of the generalized Hamiltonian given in Eq. (14). We introduce a set of operators which are closed under commutation. This algebraic structure is exploited to determine the time-evolution operator to evolve the initial state of the atom-field system. The eigenvalues and the corresponding eigenvectors are constructed.

2.1 Eigenvalues and eigenvectors

The Hamiltonian given in Eq.14 is written as

$$H = \omega K_+ K_- + \frac{\nu}{2} \sigma_z + g(K_+ \sigma_- + K_- \sigma_+), \quad (16)$$

wherein we have set $K_- = \sqrt{1 + k\hat{a}^\dagger \hat{a}}$ and $K_+ = \hat{a}^\dagger \sqrt{1 + k\hat{a}^\dagger \hat{a}}$. Further, we assume that $k$ is nonnegative and restricted to take values less than or equal to unity. Formally, the Hamiltonian $H$ has the same structure as $H_{JC}$ with $\hat{a}$ and $\hat{a}^\dagger$ replaced by $K_-$ and $K_+$. However, the former corresponds to Kerr-type medium with intensity-dependent coupling for atom-field interaction. The difference is very clear in the commutation relations among the operators. From the realisation of $K_-$ and $K_+$ in terms of $\hat{a}^\dagger$ and $\hat{a}$, we arrive at

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm kK_\pm. \quad (17)$$

The operator $K_0$ is $k\hat{a}^\dagger \hat{a} + \frac{1}{2}$. Thus, the operators $K_-$, $K_+$ and $K_0$ form a closed algebra. It worth noting that the commutation relations define the SU(1,1) algebra when $k = 1$. On the other hand, to get the well-known Heisenberg-Weyl
algebra generated by \( \hat{a}^\dagger, \hat{a} \) and the identity \( I \) we set \( k = 0 \). Two different algebras are realized depending on the value of \( k \) and hence the algebra of \( K_-, K_+ \) and \( K_0 \) is said to be an “interpolating algebra”. An invariant operator, which commutes with \( K_\pm \) and \( K_0 \), for the algebra is given by \( K_0^2 - (k/2)(K_- K_+ + K_+ K_-) \). The coherent states corresponding to this algebra and their Hilbert space properties are known [25].

The atom-field evolution is studied in the space of \( |e,n\rangle \) and \( |g,n\rangle \), where \( n = 0, 1, 2, \cdots \). The state \( |e,n\rangle \) means that the atom is in the excited state \( |e\rangle \) and the field in the \( n \)th excited state \( |n\rangle \). The states \( |e,n\rangle \) and \( |g,n\rangle \) are eigenstates of \( \omega K_+ K_- + \frac{\nu}{2} \sigma_z \) and the respective eigenvalues are \( \mathcal{E}_{e,n} = (n + kn^2 - kn)\omega + \frac{\nu}{2} \) and \( \mathcal{E}_{g,n} = (n + kn^2 - kn)\omega - \frac{\nu}{2} \). The Hamiltonian \( H \) admits a constant of motion \( N \) such that the commutator \([N, H]\) vanishes. Explicitly, \( N = K_+ K_- + 2K_0 \sigma_+ \sigma_- \). Note that when \( k = 0 \), \( N \) becomes \( \hat{a}^\dagger \hat{a} + \sigma_+ \sigma_- \), the constant of motion for \( H_{JC} \).

The interaction part of \( H \) is such that the state \( |e,n\rangle \) is taken to \( |g,n + 1\rangle \) and vice versa, during the evolution of the atom-field system. Thus, the entire Hilbert space is split into subspaces spanned by \( |e,n\rangle \) and \( |g,n + 1\rangle \) and the dynamics confined to individual subspaces. In one such subspace, specified by the value of \( n \), the Hamiltonian matrix is

\[
H = \begin{pmatrix}
(n + \frac{1}{2} + kn^2 - kn)\omega + \Delta/2 & g\sqrt{(1 + kn)(1 + n)} \\
g\sqrt{(1 + kn)(1 + n)} & (n + \frac{1}{2} + kn^2 + kn)\omega - \Delta/2
\end{pmatrix}
\]  
(18)

The detuning parameter \( \Delta \) is \( (r - 1)\omega \). The eigenvalues of the Hamiltonian are

\[
E_{\pm,n} = (kn^2 + n + \frac{1}{2})\omega \pm \frac{1}{2}\sqrt{(\Delta - 2k\omega n)^2 + 4g^2\omega^2(1 + kn)(1 + n)}
\]  
(19)

and the corresponding eigenvectors are

\[
|+,n\rangle = \cos \theta_n |e,n\rangle + \sin \theta_n |g,n + 1\rangle,
\]  
(20)

\[
|-,n\rangle = \sin \theta_n |e,n\rangle - \cos \theta_n |g,n + 1\rangle.
\]  
(21)

The expansion coefficients are

\[
\cos \theta_n = \frac{2g\omega \sqrt{(1 + n)(1 + kn)}}{\sqrt{(\Omega_n - \Delta_n)^2 + 4g^2\omega^2(1 + n)(1 + kn)}}
\]  
(22)

\[
\sin \theta_n = \frac{\Omega_n - \Delta_n}{\sqrt{(\Omega_n - \Delta_n)^2 + 4g^2\omega^2(1 + n)(1 + kn)}}.
\]  
(23)

in which we have set \( \Delta_n = \Delta - 2k\omega n \) and \( \Omega_n = \sqrt{\Delta_n^2 + 4g^2\omega^2(1 + n)(1 + kn)} \).

The energy difference between the levels \( E_{+,n} \) and \( E_{-,n} \) is \( \sqrt{\Delta_n^2 + 4g^2\omega^2(1 + kn)(1 + n)} \). The minimum of the separation occurs when \( \Delta \) equals \( 2k\omega n \) and the corresponding difference is \( 2g\omega \sqrt{(1 + kn)(1 + n)} \). In Fig. 1 we have plotted the energy eigenvalues \( E_+ \) and \( E_- \) as a function of \( \Delta \). The dashed lines represent the eigenvalues when \( g = 0 \), i.e., \( \mathcal{E}_{e(g),n} \). In this case the eigenvalues cross each
other as $\Delta$ increases from negative to positive values. The continuous lines represent the energy eigenvalues for $g = 10^{-3}$. The diverging eigenvalue separation beyond the minimum separation indicates “level repulsion” in the eigenvalues of the dressed atom. The effect of nonzero $k$ is to shift the value of $\Delta$ at which the minimum separation or the crossing occurs. If $k = 0$, the minimum as well as the crossing occur at $\Delta = 0$.

2.2 Evolution of atom-field state

To understand the dynamics of the atom-field system, we solve for the state of the system in interaction picture, where the evolution equation is

$$i \frac{\partial \vert \psi \rangle}{\partial t} = \tilde{V} \vert \psi \rangle.$$  \hspace{1cm} (24)

Here $\tilde{V}$ is the transformed interaction given by

$$\tilde{V} = g \exp[it(\omega K_+ K_- + \frac{\nu}{2} \sigma_z)](\sigma_+ K_+ + \sigma_- K_-) \exp[-it(\omega K_+ K_- + \frac{\nu}{2} \sigma_z)].$$  \hspace{1cm} (25)

The effect of the transformation on the interaction term is obtained from the following results:

$$\exp(it\omega K_+ K_-)K_+ \exp(-it\omega K_+ K_-) = K_+ \exp(it\omega K_0),$$  \hspace{1cm} (26)

$$\exp(i\nu \sigma_z)\sigma_- \exp(-i\nu \sigma_z) = \exp(-i\nu)\sigma_-, $$  \hspace{1cm} (27)

and their adjoints. Note that the ordering of operators should be maintained in the rhs of the first of the results. Using these relations, the interaction picture Hamiltonian is written as

$$\tilde{V} = g(\sigma_- \exp[it(2\omega K_0 - \nu)]) + \text{adjoint}.$$  \hspace{1cm} (28)

At any time $t$, let the state of the atom-field system be represented as

$$\vert \psi(t) \rangle = \sum_{n=0}^{\infty} C_{e,n}(t) \vert e, n \rangle + C_{g,n}(t) \vert g, n \rangle.$$  \hspace{1cm} (29)

The coefficients $C_{e,n}(t)$ and $C_{g,n}(t)$, determined in terms of their initial values by the evolution equation Eq.24, are

$$\exp\left(-i\frac{\Delta_n t}{2}\right) C_{e,n}(t) = \left[ \cos\left(\frac{\Omega_n t}{2}\right) - i\frac{\Delta_n}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] C_{e,n}(0) - \frac{2ig\sqrt{(n+1)(1+kn)}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) C_{g,n+1}(0),$$  \hspace{1cm} (30)

$$\exp\left(i\frac{\Delta_n t}{2}\right) C_{g,n+1}(t) = \left[ \cos\left(\frac{\Omega_n t}{2}\right) + i\frac{\Delta_n}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] C_{g,n+1}(0) - \frac{2ig\sqrt{(n+1)(1+kn)}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) C_{e,n}(0).$$  \hspace{1cm} (31)
The Rabi frequency $\Omega_n$ is $\sqrt{\Delta_n^2 + 4g^2\omega^2(1 + kn)(1 + n)}$. The dependence of $\Omega_n$ on $n$ such is that there is a minimum value for the Rabi frequency when $n$ satisfies $\Delta = 2k\omega + g^2\omega^2(1 + k + 2kn)(k\omega)^{-1}$, provided $k \neq 0$. The variation of $\Omega_n$ with respect to $n$ is shown in Fig. 2. In the case of $H_{JC}$, the Rabi frequency varies linearly with $n$ and hence there is no minimum. The existence of a minimum Rabi frequency has important consequences for the dynamics of atomic-inversion, squeezing, photon-statistics, etc. and they are discussed in the following sections.

Let the Rabi frequency attain its minimum for some specific value of $n$, denoted by $\bar{n}$. Therefore, we have

$$\Delta = \Delta_e = 2k\omega\bar{n} + \frac{g^2(1 + k + \bar{n})}{k\omega}. \quad (32)$$

The $n$ dependence of Rabi frequency, for values of $n$ close to $\bar{n}$, is obtained by Taylor expanding $\Omega_n$ around $\bar{n}$, up to second order. The resultant expression is

$$\Omega_n = \Omega_{\bar{n}} + \frac{2(n - \bar{n})^2(k^2\omega^2 + kg^2)}{\Omega_{\bar{n}}} \quad (33)$$

In Fig. 2 the values predicted by the approximate expression for $\Omega_n$ are compared with those of the exact expression. It is clear that for the chosen values of $g$, $k$ and $\bar{n}$, the values match very well for those values of $n$ under the peak of the photon number distribution. Quantitatively, the fractional difference is less than 3%.

### 3 Evolution of atomic inversion

In the previous section we constructed the complete state of the atom-field system in the dressed atom basis in interaction picture. The state $|\phi(t)\rangle$ in the Schrödinger picture is easily obtained by premultiplying the interaction picture state function $|\psi(t)\rangle$ by $\exp(i(K_+K_- + \nu/2)t)$. In this section, we study the temporal evolution of atomic inversion.

The time-dependent state vector $|\psi(t)\rangle$ of the system is determined completely once the coefficients $C_{e,n}(t)$ and $C_{g,n}(t)$ are known, which, in turn, are specified by their initial values. For instance, if the atom is initially in the excited state, we have $C_{g,n}(0) = 0$ and $C_{e,n}(0) = \langle n|\psi(0)\rangle$. The probability that the atom is in the excited state, irrespective of the state of the field, is $\sum_{n=0}^{\infty} P_n|C_{e,n}(t)|^2$ and to be in the ground state is $\sum_{n=0}^{\infty} P_n|C_{g,n}(t)|^2$. The photon number distribution of the field is $P_n$. The difference of these two probabilities is atomic inversion. For the specified initial condition, namely, the atom
is initially in the excited state, the atomic inversion \( W(t) \) is

\[
W(t) = 1 + \sum_{n=0}^{\infty} P_n \left[ \frac{4g^2\omega^2(1+n)(1+kn)}{\Omega_n^2} \right] (\cos(\Omega_n t) - 1),
\]

(34)

and time-dependent part of \( W(t) \) is

\[
W_T(t) = 4g^2\omega^2 \sum_{n=0}^{\infty} P_n \left[ \frac{(1+n)(1+kn)}{\Omega_n^2} \right] \cos(\Omega_n t).
\]

(35)

The quantity \( W_T \) exhibits rich structure in its evolution. It exhibits collapse and revivals when the initial photon distribution is taken to be a Poissonian distribution of mean photon number \( \bar{n} \), which corresponds to the field being in a coherent state \( |\sqrt{\bar{n}}\rangle \). The photon number distribution for the state is \( P_n = \exp(-\bar{n})\bar{n}^n/n! \). This distribution has a single peak and the standard deviation is \( \bar{n} \). Hence, the major contribution to the sum in Eq.35 comes from a few terms with \( n \) around the peak. With this choice of \( P_n \), we have plotted in Fig. 3a the evolution of \( W_T \) as a function of time. The time required for the first collapse and the following revival, denoted by \( T_C \) and \( T_R \) respectively, can be estimated approximately. For the revival to occur, the terms corresponding to those \( n \) around the peak, should be in phase. Thus, we require \( T_R(\Omega_{\bar{n}+1} - \Omega_{\bar{n}}) = 2\pi \). The difference of the nearest-neighbour Rabi frequencies is

\[
\Omega_{\bar{n}+1} - \Omega_{\bar{n}} = \frac{2A\bar{n} + A + B}{2\Omega_{\bar{n}}},
\]

(36)

The constants \( A \) and \( B \) are \( 4(k^2\omega^2 + kg^2) \) and \( 4(g^2 + kg^2 - \Delta k\omega) \) respectively. The revival time \( T_R \) is

\[
\frac{4\pi\Omega_{\bar{n}}}{2A\bar{n} + A + B}.
\]

For the inversion to collapse, the terms in the sum on the rhs of Eq.35 should be uncorrelated. Since the width of a Poissonian distribution is where the probability \( P_n \) is appreciable, the condition for collapse is written as \( T_C(\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Omega_{\bar{n}-\sqrt{\bar{n}}}) = 1 \). If \( \bar{n} \) is large, the expression for \( T_C \) is

\[
\frac{T_C}{2\pi\sqrt{\bar{n}}}.
\]

The function \( W_T \) exhibits rich features when detuning is close to \( \Delta_c \). In Fig. 3, we have shown the behaviour of \( W_T \) for three different values of \( \Delta \). The values for the parameters are \( g = 10^{-3}, k = 10^{-4}, \omega = 1, \bar{n} = 30 \) and the corresponding \( \Delta_c \) is 0.01606. The values are so chosen that the Rabi frequency attains its minimum when \( n \) is near \( \bar{n} \), the average photon number. The evolution of \( W_T \) with \( \Delta = 0.01 < \Delta_c \) is shown as the top most figure, marked (a) in Fig. 3. The figure (b) corresponds to \( \Delta = \Delta_c \) and the one marked (c) is for \( \Delta = 0.22 > \Delta_c \). The envelope of \( W_T \) when detuning equals \( \Delta_c \), is distinct with structures repeating without much distortion. This should be compared with the top and bottom figures, which correspond to \( \Delta \neq \Delta_c \), which exhibit random oscillations and do not have neat envelope.
As noted in Section II, SU(1,1) algebra is realized in terms of $K_{\pm}$ and $K_0$ when $k = 1$. If we consider resonant interaction ($r = 1$), then $W_T$ can be estimated approximately for the SU(1,1) case. The field is taken to be in a coherent state $|\alpha\rangle$ such that $|\alpha| = \sqrt{n} \gg 1$. We use $(1 + n)(1 + kn) \approx n^2$ and $g^2 \ll 1$ to arrive at $\Omega_n = 2n\omega$. With these approximations, we arrive at

$$W_T = g^2 \exp(\bar{n}(\cos(\omega t) - 1)) \cos(\bar{n}\sin(\omega t)), \quad (37)$$

in which we have used $(1 + n)(1 + kn) \approx \Omega_n^2$. The magnitude of $W_T$ is negligible in this case and so there is no perceptible collapse or revival. This is due to the presence of Kerr term and the fact that it dominates over $\hat{a}^\dagger \hat{a}$ in the Hamiltonian. However, we point out that collapses and revivals are present in $W_T$ if $\chi \ll \omega$.

4 Dynamics of field properties

In the previous section, we studied the dynamics of the two-level atom, in particular, the atomic inversion. In the present section, we explore the temporal behaviour of field statistics and field amplitudes. As in the previous section, the field is initially in a coherent state of complex amplitude $\alpha$ and the atom is taken to be in its excited state. With these initial conditions, the probability distribution of photons at time $t$ is

$$P(n, t) = |C_{e,n}(t)|^2 + |C_{g,n}(t)|^2 = \frac{P_n}{2} \left[ 1 + \frac{\Delta_n^2}{\Omega_n^2} + \frac{4g^2\omega^2(1 + n)(1 + kn)}{\Omega_n^2} \cos(\Omega_n t) \right] \quad (38)$$

Coherent states are the wavepackets whose behaviour is closest to that of a classical particle and hence are called classical states. Nevertheless, during their evolution in time the states may not be classical. The photon statistics of coherent states is Poissonian. Any deviation from this behaviour is characterized by Mandel’s Q parameter defined as $\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle$. Using the time-dependent probability distribution $P(n, t)$, the expectation values in the expression for Q parameter are

$$Q = \frac{\sum_{n=0}^{\infty} n^2 P(n, t) - [\sum_{n=0}^{\infty} P(n, t)n]^2}{\sum_{n=0}^{\infty} P(n, t)n} \quad (39)$$

For coherent states, $Q$ is unity. Any value of $Q$ less than unity is nonclassical. In Fig. 4 the time evolution of $Q$ parameter for an initially coherent state and $\Delta = 0.01 < \Delta_c$ is given. The emergence of nonclassical behaviour ($Q < 1$) is seen. Though not shown in figure, we point out that for $\Delta = \Delta_c = 0.016061$, the statistics does not become sub-Poissonian. When $k = 1$ and $\Delta = 0$, the time-dependent part of $P(n, t)$ is of negligible magnitude and so $Q$ does not evolve in time.
4.1 Squeezing

We define the field amplitudes to be

\[ X(t) = \frac{\hat{a}^\dagger(t) + \hat{a}(t)}{\sqrt{2}}, \]  
\[ Y(t) = \frac{i\hat{a}^\dagger(t) - \hat{a}(t)}{\sqrt{2}}. \]

These amplitudes satisfy the commutation relation \([X, Y] = i\) and hence they satisfy \((\delta X)(\delta Y) = 1/2\). The symbol \((\delta X)\) stands for the expression \(\sqrt{\langle X^2 \rangle - \langle X \rangle^2}\), the variance in \(X\) for a given field state. For coherent states of any amplitude \(\alpha\), variances in \(X\) and \(Y\) are the same and equal to \(1/2\). A state is nonclassical if \((\delta X)\) is less than \(1/2\), the coherent state value. Using the time-dependent state function given in Eq. (30), the variances in the field amplitudes are given by

\[(\delta X)^2 = \frac{1}{2} \left[ 1 + \langle 2\hat{a}^\dagger\hat{a} + \hat{a}^4 \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^4 \rangle - 2\langle \hat{a}^2 \rangle \langle \hat{a} \rangle \right], \tag{43}\]

and

\[(\delta Y)^2 = \frac{1}{2} \left[ 1 + \langle 2\hat{a}^\dagger\hat{a} - \hat{a}^4 \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^4 \rangle - 2\langle \hat{a}^2 \rangle \langle \hat{a} \rangle \right]. \tag{44}\]

The expectation values of various operators in these expressions are

\[ \langle \hat{a} \rangle = \sum_{n=0}^{\infty} \sqrt{n+1} \left[ C^*_{e,n} C_{e,n+1} + C^*_{g,n} C_{g,n+1} \right], \tag{45}\]

\[ \langle \hat{a}^2 \rangle = \sum_{n=0}^{\infty} \sqrt{(n+1)(1+kn)} \left[ C^*_{e,n} C_{e,n+2} + C^*_{g,n} C_{g,n+2} \right], \tag{46}\]

and

\[ \langle \hat{a}^4 \rangle = \sum_{n=0}^{\infty} n \left[ C^*_{e,n} C_{e,n} + C^*_{g,n} C_{g,n} \right]. \tag{47}\]

The expectation values \(\langle \hat{a}^3 \rangle\) and \(\langle \hat{a}^4 \rangle\) are the complex conjugates of \(\langle \hat{a} \rangle\) and \(\langle \hat{a}^2 \rangle\) respectively.

The evolution of \(\delta X\) is shown in Fig. 5. As the field evolves in time, the variance in \(X\) falls below 0.5 indicating that the quadrature exhibits squeezing. As a consequence of uncertainty relation, the \(Y\) quadrature does not show squeezing. However, when we set \(\alpha = i\sqrt{30}\), the situation is reversed. In this case, squeezing is possible in \(Y\) and not in \(X\). An interesting feature is that the uncertainty in \(Y\) with \(\alpha = i\sqrt{30}\) is same as in \(X\) when \(\alpha = \sqrt{30}\). The reason for this is that the field states are of the form \(\sum_{n=0}^{\infty} B_n \alpha^n |n\rangle\) wherein the coefficients \(B_n\) are real for all \(n\). For such superpositions, the uncertainty in \(X\) for a given \(\alpha\) is same as that in \(Y\) with \(\alpha\) replaced by \(i\alpha\).
4.2 Overlap of initial and time-evolved states

A quantity of interest is the overlap of the state of the atom-field at time \( t \) and that at \( t = 0 \). With the same initial conditions for the atom-field state as in the previous section, the overlap is

\[
|\langle \psi(0) | \psi(t) \rangle|^2 = \exp(-|\alpha|^2) \left| \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \left[ \cos(\frac{\Omega_n t}{2}) + i \frac{\Delta_n}{\Omega_n} \sin(\frac{\Omega_n t}{2}) \right]^2 \right|.
\] (48)

The numerical value of the overlap lies between zero and unity. It is seen from Fig. 6 that the overlap becomes zero at longer times. In other words, the time-evolved state is almost orthogonal to initial state. For short durations, an approximate expression for the decay of overlap is derived by replacing expression \( \exp(-\frac{\bar{n}}{n}) \) with the Gaussian distribution \( \exp(-\frac{(n-\bar{n})^2}{2\bar{n}}) \) and the sum over \( n \) by integration. Further, we set \( \Omega_n = \Omega_N + (N-n)\Omega'_N + \Omega''_N (N-n)^2 \), with the assumption that \( N \) is close to but not less than \( \bar{n} \). Here, prime denotes taking derivative with respect to \( n \) and the suffix represents the value of \( n \) where the derivative is evaluated. With these approximations we get, after neglecting oscillatory terms,

\[
\sum_{n=0}^{\infty} \exp(-\frac{(n-\bar{n})^2}{2\bar{n}}) \cos(\frac{\Omega_n t}{2}) = \text{Re} \int_0^{\infty} \exp(-\frac{(n-\bar{n})^2}{2\bar{n}} + i \frac{\Omega_n t}{2}) dn \\
\propto \left[ 1 + N^2 \frac{\partial^2 \Omega_n}{\partial^2 n} \right]^{-\frac{1}{2}}
\]

\[
\times \exp \left[ -N^2 \frac{\partial \Omega_n}{\partial n} \right]_{n=N} \frac{t^2}{4} \] (49)

Similar expression can be derived for summation with \( \sin(\frac{\Omega_n t}{2}) \). The above expression indeed predicts that the overlap function decays with time. When \( N = \bar{n} \), the first derivative of \( \Omega_n \) vanishes and the exponential term in the envelope is absent. Consequently the decay is slower. However, if the photon number distribution of the field is very broad, it is incorrect to truncate the Taylor series and the expression in Eq. (49) is invalid.

5 Summary

The Hamiltonian that contains the usual and intensity-dependent (including Kerr term) JC models as limiting cases has been constructed. The nondissipative dynamics dictated by the generalized Hamiltonian \( H \) is completely solvable. The eigenvalues of the Hamiltonian, with Kerr term and intensity-dependent coupling, exhibits level-repulsion. In the case of nonvanishing Kerr term, the Rabi frequency, considered as a function of the photon number \( n \), attains a
minimum. The dynamical behaviour of atomic-inversion, when detuning is so chosen to have the minimum Rabi frequency, exhibits superstructures, which is absent in the usual JC model. The overlap of the initial coherent state and the time-evolved state decays with time. In the language of inner product, the initial coherent state is almost orthogonal to the time-evolved state. The results, except the occurrence of superstructures in atomic inversion, a consequence of nonvanishing Kerr term, go over to those of the JC model, in the limit of \( k \to 0 \).

We note that the formal equivalence between \( H_{JC} \) and \( H \) is obtained by identifying \( K_+ \) with \( \hat{a}^\dagger \) and \( K_- \) with \( \hat{a} \). This, in conjunction with the fact both the sets of operators \( \{ \hat{a}^\dagger, \hat{a}, I \} \) and \( \{ K_\pm, K_0 \} \) are closed under commutation, implies that the expansion coefficients \( C_{e/g,n}(t) \) for the evolution governed by \( H \) are obtained from the corresponding expressions for the usual JC model by replacing \( \Delta \) by \( \Delta_n \) and \( g \) by \( g\sqrt{1 + kn} \). Hence, all those physical quantities, like the atomic inversion, quadrature fluctuations, etc., computed in terms of the expansion coefficients are derivable from those of the JC model.

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List of Figures

Fig. 1 Dependence of eigenvalues $E_+$ and $E_-$ on detuning $\Delta$. The continuous curve corresponds to $g = 0.1$ and the dashed curve corresponds to $g = 0$. Here $k = .1$ and $\bar{n} = 30$. The dashed curves with positive and negative slopes correspond respectively to $E_{e,n}$ and $E_{g,n}$. Lower part of the figure is for $n = 1$ and upper part for $n = 2$.

Fig. 2 Variation of $\Omega_n$ with $n$. We have set $g = 10^{-3}$, $k = 10^{-4}$, $\bar{n} = 30$ and $\Delta = 0.016061$. The approximate and actual Rabi frequencies are compared in the upper figure. Dotted line corresponds to the approximate expression in Eq.33 and continuous curve corresponds to exact expression. Values of $k$, $g$, $\bar{n}$ and $\Delta$ are $10^{-4}$, $10^{-3}$, $30$ and $0.016061$ respectively. The bottom curve shows the photon number distribution for the coherent state $|\alpha = \sqrt{30}\rangle$.

Fig. 3 Time-dependent part of atomic inversion. Case (a) corresponds to $\Delta = 0.01 < \Delta_c$. Case (b) is for $\Delta = \Delta_c = 0.016061$ and Case (c) refers to $\Delta = 0.022 > \Delta_c$. Here Case (b) shows $1 + W_T$ and Case (c) shows $2 + W_T$.

Fig. 4 Time variation of $(\delta X)^2$, as the atom-field system evolves. The evolution is shown for three values of detuning, 0.01 (continuous), 0.016061 (dotted) and 0.02 (dashed). Instants of $(\delta X)^2$ less than 0.5 correspond to squeezing in $X$ quadrature.

Fig. 5 Mandel’s $Q$ parameter as a function of time. Instants of $Q < 1$ correspond to sub-Poissonian statistics. Values of $g$, $k$ and $\bar{n}$ are same as in Fig. 2 and $\Delta = 0.01$.

Fig. 6 Overlap of initial and time-evolved field states. Y-axis corresponds to $|\langle \psi(0) | \psi(t) \rangle|^2$. The envelope of the overlap function decays with time implying that the initial state is almost orthogonal to the evolved state. The values of the parameters are same as in Fig. 2 and the detuning $\Delta$ is equal to 0.016061.
\[ \mathcal{E}_+^{(n=2)} \]

\[ \mathcal{E}_-^{(n=2)} \]

\[ \mathcal{E}_+^{(n=1)} \]

\[ \mathcal{E}_-^{(n=1)} \]
\[ (\delta X)^2 \]
