Functional integration on Regge geometries

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We adopt the standard definition of diffeomorphism for Regge gravity in \( D = 2 \) and give an exact expression of the Liouville action in the discretized case. We also give the exact form of the integration measure for the conformal factor. In \( D > 2 \) we extend the approach to any family of geometries described by a finite number of parameters. The ensuing measure is a geometric invariant and it is also invariant in form under an arbitrary change of parameters.

1. Introduction

In the following we shall consider the Regge manifold as a differential manifold equipped with a singular metric which is everywhere flat except on \( D - 2 \) dimensional simplices. A singular metric does not conflict with the differential structure. In fact the concept of differential manifold precedes that of Riemannian manifold [1] which means that the charts and transition functions are to be given before equipping the manifold with a metric; once the differential manifold is given one can consider also singular metrics on it [2].

In defining the functional integral the choice of the local fundamental variables plays a key role. The analog of the connection \( A_\mu \) of Yang-Mills theory is played in gravity by the metric tensor \( g_{\mu\nu} \) and the gauge transformations are replaced by the diffeomorphisms. Similarly to what happens in the finite dimensional case, the integration measure is induced by a distance in the space of the field configurations. Such a distance must be invariant under the relevant symmetry group of the theory and ultralocal; this last property is implied by the request that the integration measure should play a kinematical and not a dynamical role. In the case of gravity the most general distance which is invariant under diffeomorphisms and ultralocal has been given by De Witt [3]

\[
(\delta g, \delta g) = \int \sqrt{g} d^D x \delta g_{\mu\nu} G^{\mu\nu\mu'\nu'} \delta g_{\mu'\nu'} \quad (1)
\]

with

\[
G^{\mu\nu\mu'\nu'} = \delta^{\mu\nu} g_{\mu'\nu'} + \delta^{\mu'\nu} g_{\mu\nu'} + C g_{\mu\nu} g_{\mu'\nu'} \quad (2)
\]

The main job is that to extract from the integration measure induced by (1) the infinite volume of the diffeomorphisms. We shall examine first the simpler case of \( D = 2 \) and then go over to \( D > 2 \).

2. \( D = 2 \)

We recall that the De Witt metric (1) is the starting point of the treatment of 2 dimensional quantum gravity on the continuum [4] and this should be kept in mind when comparing the results of the continuum theory with the outcome of numerical simulations. On the continuum a complete reduction of the functional integral has been achieved in the conformal gauge [4] giving rise to the well known Liouville action

\[
S_L[\sigma, \hat{g}(\tau_i)] = -\frac{26}{24\pi} \int d^2 x \sqrt{|\hat{g}|} \left[ \hat{g}_{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + R_\delta \sigma \right] \quad (3)
\]

which is the log of the determinant of the Lichnerowicz operator \( P_1 P \) computed on the surface described by the conformal factor \( e^{2\sigma} \).

The Regge scheme where the curvature is localized at isolated points represents a natural discretization of the functional integral; it is the analog of restricting in ordinary quantum mechanics the functional integral, to piecewise linear paths, and let at the end the number of segments go to infinity. The conformal factor describing a Regge geometry with the topology of the sphere is given
by \[
\lambda' = \lambda_0 + \sum_{i=1}^{N}(\alpha_i - 1) \log |\omega_i c + d|
\]
with \(0 < \alpha_i\) and \(\sum_{i=1}^{N}(1 - \alpha_i) = 2\). As it happens on the continuum such a conformal factor is unique \[\Box\] up to the 6 parameter \(SL(2C)\) transformations

\[
\omega_i' = \frac{a \omega_i + b}{c \omega_i + d}, \quad \alpha_i' = \alpha_i
\]

with the complex parameters satisfying \(ad - bc = 1\). Such a description is equivalent to usual one in terms of link lengths. In fact the number of parameters in \(\Box\) are \(3N\) from which we have to subtract the dimension of the invariance group obtaining \(3N - 6\). This equals the number of links \(H\) given by the Euler relation \(H = F + N - 2\) with \(N\) the number of vertices and \(F = 2H/3\) the number of faces. On the other hand it is mathematically more advantageous because the links represent arcs of a very special set of geodesics connecting the vertices, among which one has to impose a large number of triangular inequalities, while the parameters \(\omega_i\) vary without constraints. It is possible to give an exact expression of the determinant of the Lichnerowicz operator on a Regge surface to obtain for the action \[\Box\]

\[
S_L = \frac{26}{12} \left\{ \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{\alpha_i} \log |w_i - w_j| + \lambda_0 \sum_i (\alpha_i - 1 \alpha_i) - \sum_i F(\alpha_i) \right\}
\]

where \(F(\alpha)\) is given by an integral representation. Action \(\Box\) is invariant under the \(SL(2C)\) group and in the continuum limit goes over to eq.(\ref{eq:3}).

The discrete counterpart of the functional measure \(\mathcal{D}[\sigma]\) is given by

\[
\mathcal{D}[\sigma] = \prod_{k=1}^{N} d^2 \omega_k \prod_{i=1}^{N-1} d \alpha_i d \lambda_0 \sqrt{\det \mathcal{J}}
\]

where \(\mathcal{J}\) is the determinant of the \(3N \times 3N\) matrix

\[
\mathcal{J}_{ij} = \int d^2 \omega e^{2\sigma} \frac{\partial \sigma}{\partial p_i} \frac{\partial \sigma}{\partial p_j},
\]

being \(p_i\) the parameters \(\omega_1 \ldots \omega_{N_y}\), \(\lambda_0, \alpha_1 \ldots \alpha_{N-1}\). Also such integration measure is invariant under \(SL(2C)\). Both the action and the measure can be written explicitly also for the torus topology \[\Box\]. In this case the product of the exponential of the action and the measure is invariant under translations and under the modular transformations.

Action \(\Box\) is not local but very simple; less simple, even though explicitly known, is the determinant of \(J_{ij}\).

On the numerical front, accurate simulations have been given of two dimensional gravity, both pure and coupled with Ising spins by adopting the measure \(\prod_i \frac{\lambda}{\pi} \). The results are consistent with the Onsager exponents and in definite disagreement with the KPZ exponents \[\Box\] while the situation for the string susceptibility is still unclear \[\Box\]. That measures of type \(\prod_i d l_i f(l_i)\) fail to reproduce the Liouville action can be understood by the following argument \[\Box\]: on the continuum for geometries which deviate slightly from the flat space one can compute approximately the Liouville action by means of a one loop calculation. If one tries to repeat a similar calculation for the Regge model with the measure \(\prod_i d l_i f(l_i)\) one realizes that being the Einstein action in two dimensions a constant, the only dynamical content of the theory is played by the triangular inequalities. But at the perturbative level triangular inequalities do not play any role and thus one is left with a factorized product of independent differentials which bear no dynamics and thus no Liouville action. The problem is that the Liouville action is not the result of integrating on fluctuations of the geometry but of integrating on the diffeomorphisms, while keeping the geometry exactly fixed.

3. \(D > 2\)

While in \(D = 2\) the geometry is completely described by a finite number of parameters plus a conformal factor, in \(D > 2\) this is no longer true. We consider a general scheme in which the \(D\)-dimensional geometries are described by a class of metrics \(\bar{g}_{\mu\nu}(x, l)\) depending on a finite number of parameters \(l_i\). In the Regge model one
can think of the $l_i$ as the link lengths, but any other parameterization or class of geometries is equally good. We want to treat exactly the diffeomorphisms $f$ and thus the functional integral will be performed on the class of metrics $g_{\mu\nu}(x,\tau, f) = f^*\delta_{\mu\nu}(x,\tau)$ and again the job will be that to extract from the De Witt measure the finite volume of the diffeomorphisms. To this purpose the variation of the metric will be decomposed in two orthogonal parts

$$\delta g_{\mu\nu} = (F\xi)_{\mu\nu} + (1-F(F^\dagger F)^{-1}F^\dagger)\frac{\partial g_{\mu\nu}}{\partial l_i} \delta l_i$$

(9)

being $(F\xi)_{\mu\nu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu}$ the action of an infinitesimal diffeomorphism. The adjoint of $F$ according to metric (1) is given by

$$(F^\dagger h)_{\nu} = -4\nabla^\mu h_{\mu\nu} - 2(C - \frac{2}{D})\nabla_{\nu} h^{\mu}_{\mu}$$

(10)

and the inverse of $F^\dagger F$ is well defined from $\text{Im}(F^\dagger)$ onto itself. Factoring the volume of the diffeomorphisms one reaches for the integration measure the expression

$$\prod_k d\ell_k (\det(t^i, t^j)) \text{Det}(F^\dagger F)^{\frac{1}{2}}$$

(11)

where

$$t^i_{\mu\nu} = (1-F(F^\dagger F)^{-1}F^\dagger)\frac{\partial g_{\mu\nu}}{\partial l_i}$$

(12)

The main feature of eq. (11) is to be a geometric invariant i.e. it does not change under diffeomorphisms which may also depend on the parameters $l$ and thus under a larger class of transformations than the De Witt metric. On the other hand it depends through $F^\dagger$ on the parameter $C$.

One expects the dependence on $C$ to disappear when the number of parameters $l$ becomes large (continuum limit). In fact if one enlarges the scheme by integrating on a class of metrics $g_{\mu\nu} = f^*e^{2\lambda}\delta_{\mu\nu}(x,\tau)$, where the finite number of parameters $\tau$ describe deformations transverse both to diffeomorphism and to the Weyl group, one reaches the $C$-independent measure

$$\prod_k d\tau_k D[\sigma] \left[\text{Det}(P^\dagger P)\right]$$

(13)

$$\cdot \det\left(k^i, (1-P(P^\dagger P)^{-1}P^\dagger)k^j)\right)^{\frac{1}{2}}$$

where $(P\xi)_{\mu\nu} = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} - \frac{4}{D}g_{\mu\nu}\nabla \cdot \xi$ and

$$k^i_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \tau_i} - g_{\mu\nu} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau_i}$$

(14)

The determinants appearing in eqs. (11, 13) are both well defined, being $F^\dagger F$ and $P^\dagger P$ elliptic operators, but contrary to what happens in the 2 dimensional case the dependence on $\sigma$ of $\det(P^\dagger P)$ cannot be reduced to the computation of a local variation of $\sigma$, due to the non ellipticity of $PP^\dagger$.

In conclusion in $D = 2$ an explicit form for the discretized path integral has been given, satisfying the correct invariance properties. The scheme can be extended to $D > 2$, but the explicit calculation of the determinants is not straightforward.

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