Representation of Analytic Functions by Exponential Series in Half-Plane

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(Submitted by A. B. Muravnik)

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Received December 23, 2021; revised March 22, 2022; accepted April 5, 2022

Abstract—In this paper we study representations of analytic in the half-plane \( \Pi_0 = \{ z = x + iy : x > 0 \} \) functions by the exponential series taking into consideration a given growth. A.F. Leontiev proved that for each bounded convex domain \( D \) there exists a sequence \( \{ \lambda_n \} \) of complex numbers depending only on the given domain such that each function \( F \) analytic in \( D \) can be expanded into an exponential series \( F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z} \) (the convergence of which is uniform on compact subsets of \( D \)). Later a similar results on expansions into exponential series, but taking into consideration the growth, was also obtained by A.F. Leontiev for the space of analytic functions of finite order in a convex polygon. He also showed that the series of absolute values \( \sum_{n=1}^{\infty} |a_n e^{\lambda_n z}| \) admits the same upper bound as the initial function \( F \). In 1982, this fact was extended to the half-plane \( \Pi_0^+ \) by A.M. Gaisin. However, the expansion in \( \Pi_0^+ \) contains an additional term — an entire function. In the present paper we study a similar case, when as a comparing function, some decreasing convex majorant serves and this majorant is unbounded in the neighborhood of zero. We have found out in which case the entire component from the expansion in the half-plane is bounded in the strip containing the imaginary axis.

DOI: 10.1134/S1995080222090086

Keywords and phrases: analytic function, exponential series, growth majorant, Legendre transform, bilogarithmic Levinson condition.

1. INTRODUCTION

This paper\(^{1}\) is devoted to the problem on expanding analytic in a half-plane functions into exponential series taking into account the growth determined by some convex majorant. As is known (see [1]), such an expansion contains a term representing an entire function. It would be interesting to find out when this entire function is bounded in a vertical strip adjacent to the imaginary axis.

Let’s talk about the representation of functions by exponential series in arbitrary convex domains.

Let \( D \) be a convex domain in the complex plane \( \mathbb{C} \) and \( A(D) \) be the space of analytic in \( D \) functions with the topology of uniform convergence on compact subsets in \( D \).

In the theory of exponential series one of the main results is the following one by Leontiev [1, Ch. V, Sect. 3, Subsect. 1]:

Let \( D \) be a bounded convex domain. Then there exists a sequence \( \{ \lambda_n \} \) depending only on the domain \( D \) such that each function \( F \) from \( A(D) \) can be expanded into an exponential series

\[
F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}
\]

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The introduction was compiled by A. M. Gaisin, and the stated in paragraph 1 and paragraph 2 belongs to G. A. Gaisina.
in $D$.

In this theorem the sequence of exponents $\lambda_n \, (n = 1, 2, \ldots)$ is chosen as simple zeroes of an entire function $L$ of exponential type and completely regular growth with appropriate estimates for $|L'(\lambda_n)|$ from below. Such entire function always exists, see [1, Ch. IV, Sect 6, Subsect. 2]. In view of this we recall that the problem on existence of entire functions with prescribed asymptotic properties in the most general form was solved in [2], while in [3] this result was specified both for estimates of the entire function and for the size of exceptional sets, see [4].

It was also shown in [1] that each entire function $\Phi$ can be represented by an exponential series

$$
\Phi(z) = \sum_{n=1}^{\infty} A_n e^{\nu_n z},
$$

in the entire plane, and the exponents $\nu_n \, (n = 1, 2, \ldots)$ which can be chosen on at least three rays, are the zeroes of an entire function $L$ of a proximate order $\rho(r), \lim_{r \to \infty} \rho(r) = 1$ [1, Ch. VIII, Sect. 1, Subsect. 3].

The issues on representation by exponential series in unbounded domains $D$, $D \neq \mathbb{C}$ are of a special interest. In [1], a case of such domains of special shape was considered. Later it was found out that each function $F \in A(D)$, where $D$ is an arbitrary unbounded domain, can be represented by a series

$$
F(z) = \sum_{n=1}^{\infty} c_n e^{\mu_n z}
$$

in $D$, see [5]. This is implied by the results of [3, 6] on approximating subharmonic functions by the logarithm of the absolute value of an entire function. Here we are interested in the case of the half-plane, which was considered separately in [1].

**Theorem 1.** Let $F$ be a function regular in the left half-plane $\Pi_0 = \{z = x + iy: \ x < 0\}$. Then there exists a sequence $\{\mu_n\}, \mu_n > 0, \lim_{n \to \infty} \frac{\mu_n}{\mu_n^2} = \tau, \ 0 < \tau < \infty \ (\rho > 1 \ is \ arbitrary)$ independent of $F$ such that

$$
F(z) = \sum_{n=1}^{\infty} B_n e^{\mu_n z} + \text{entire function}, \quad z \in \Pi_0.
$$

We mention that in this theorem the condition $\rho > 1$ is essential: the exponents $\mu_n \, (n = 1, 2, \ldots)$ can not be zeroes of entire functions of exponential type, see [1, Ch. VIII, Sect. 1, Subsect. 3].

Theorem 1 is implied by the following statement [1, Ch. VIII, Sect. 1, Subsect. 3]:

Let $F$ be a function regular in the half-plane $\Pi_0^\circ$. Then there exist a function $f$, regular in $\Pi_0^\circ$ and continuous in the closure $\overline{\Pi}_0^\circ$, and satisfying in $\overline{\Pi}_0^\circ$ the condition $f(z) = O \left( \frac{1}{z^\tau} \right)$ as $z \to \infty$, an entire function $M(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ with a growth at most of the right order of the minimal type and an entire function $\Phi$ such that

$$
F(z) = M(D) f(z) + \Phi(z), \quad z \in \Pi_0^\circ.
$$

We recall that a differential operator of an infinite order $M(D)$ can act on the function $f$ in the entire regularity domain, which is $\Pi_0^\circ$ in our case.

In the article [7], the following problem was considered.

Let the growth of the function $F$, $F \in A(\Pi_0^\circ)$, in the neighborhood of the imaginary axis be controlled in certain sense by some majorant $H: \ [-1, 0) \to (0, +\infty), \ H(x) \uparrow 0$ as $x \to 0^-$. Then find an expansion of form (1) such that the growth of the series of the absolute values $\sum_{n=1}^{\infty} |B_n e^{\mu_n z}|$ be also governed by the majorant $H$.

We mention that in terms of the growth order, this problem was first studied by Leontiev in [8] for convex polygons and later by Gaisin in [9] for a half-plane.

We will be interested in the following question: in which case the entire function in expansion (1) will be bounded in the vertical strip $\{z = x + iy: \ |x| < 1\}$? In the present article, we have obtained an answer to this question in one important case, in which the Levinson–Shoberg–Wolff theorem on normal families of analytic functions is applicable.
2. PRELIMINARY RESULTS AND REQUIRED INFORMATION

Let \( \Pi_0^- = \{z = x + iy: x < 0\} \), \( \Pi_0^+ = \{z = x + iy: x > 0\} \).

By \( K_0 \) we denote the class of functions \( F \) possessing the properties:

1. \( F \) is regular in \( \Pi_0^+ \);
2. \( F(z) \to 0 \) as \( z \to \infty \) in each half-plane \( \Pi_s^+ = \{z = x + iy: x \geq s > 0\} \) uniformly with respect to \( \arg z \);
3. for each \( s > 0 \)

\[
T_F(s) = \frac{1}{2\pi} \int_{\text{Re}z=s>0} |F(z)||dz| < \infty.
\]

Given \( F \in K_0 \), we let

\[
A(t) = \frac{1}{2\pi i} \int_{\text{Re}z=s>0} F(z)e^{zt}dz.
\]

Then the inversion formula holds [10, Ch. VI, Sect. 1, Subsect. 79]

\[
F(z) = \int_0^{+\infty} A(t)e^{-zt}dt, \quad z \in \Pi_0^+.
\]

We introduce one more class of functions. We shall say that \( F \in K_1 \) if and only the function \( F \) is regular in \( \Pi_0^+ \), continuous in \( \Pi_0^+ = \{x = x + iy: x \geq 0\} \) and in \( \Pi_0^+ \) it obeys the condition: as \( |z| \to \infty \)

\[
|F(z)| = O \left( \frac{1}{|z|^\rho} \right).
\]

In [7] is proved

**Theorem 2.** Let \( F \in K_0 \) and \( T_F(s) \leq A_FH(s) \), \( s > 0 \), where \( H: \mathbb{R}_+ \to \mathbb{R}_+ \), \( H \) is a decreasing function, \( H(s) \downarrow 0 \) as \( s \to +\infty \), \( H(s) \uparrow \infty \) as \( s \to 0^+ \), \( H(d) = e \). We also assume that

\[
\lim_{s \to 0} s^kH(s) = \infty \quad (k \text{ is arbitrary, } k \in \mathbb{N}),
\]

while the functions \( m(s) = \ln H(s) \) (\( s > 0 \)) and \( m(e^{-t}) \) (\( t \in \mathbb{R} \)) are convex. Then there exists an entire function

\[
M(\lambda) = \sum_{n=0}^{\infty} c_n\lambda^n, \quad \ln |M(\lambda)| \leq C_M\varphi(|\lambda|)
\]

and a function \( f \in K_1 \) such that

\[
F(z) = M(D)f(z) + \Phi(z), \quad z \in \Pi_0^+,
\]

where \( \Phi \) is some entire function, \( \varphi(r) \) (\( r = |\lambda| \)) is the lower Legendre transform\(^2\) of the function \( m(s) \), \( \varphi(r) = o(r) \) as \( r \to \infty \) and \( \varphi \) is logarithmically convex.

As a corollary, the prove a theorem on expansion into an exponential series.

**Theorem 3.** Let \( F \in K_0 \) and \( T_F(s) \leq A_FH(s) \), \( s > 0 \), where the majorant \( H \) satisfies the assumptions of theorem 2. Then there exists a sequence of exponents \( \{\lambda_n\}, \lambda_n > 0 \), \( \lim_{n \to \infty} \frac{n}{\lambda_n} = \tau, 0 < \tau < \infty \) (\( \rho > 1 \) is arbitrary), independent of \( F \) such that

\[
F(z) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n z} + \text{entire function}, \quad z \in \Pi_0^+,
\]

\(^2\)The lower transformation of the function \( m(s) \) is called \( (Lm)(x) = \inf_{0<s}[m(s) + sx] \) [11].
and for some $k \in \mathbb{N}$

$$
\sum_{n=1}^{\infty} |B_n e^{-\lambda_n z}| \leq B H^k \left( \frac{x}{k} \right), \quad z = x + iy \in \Pi_0^+.
$$

(3)

It is easy to see that Theorem 3 generalizes the corresponding result from [9] on the expansion in $\Pi_0^+$ taking into account the order of growth. Indeed, it suffices to consider the majorant $H(s) = \exp \left( \left( \frac{1}{s} \right)^\rho \right)$, where $\rho$ is the order of the function $T_F(s)$.

We note that if the majorant $H$ for the initial function $F$ obeys a bilogarithmic Levinson condition

$$
\int_0^d \ln \ln H(x) dx < \infty,
$$

then the majorant $H_k(x) = B H^k \left( \frac{x}{k} \right)$ for series (3) of the absolute values obeys condition (4). This aspect is essential in issues related with the normality of families of holomorphic functions, namely, in theorems of Levinson–Sjöberg–Wolf type, see, for instance, [11]. It turns out that if condition (4), holds, then in some cases one can answer the following question: under which conditions an entire function in expansion (2) is bounded in the vertical strip $\{z = x + iy: |x| < 1\}$?

3. ON AN ENTIRE FUNCTION FROM A REPRESENTATION IN A HALF-PLANE

In Theorem 3 it was shown that for any function $F \in K_0$ for which

$$
T_F(s) \leq A_F H(s), \quad s > 0,
$$

(5)

there is an entire function $\Psi$ such that

$$
F(z) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n z} + \Psi(z), \quad z \in \Pi_0^+,
$$

(6)

and for some $B > 0, k \in \mathbb{N}$

$$
\sum_{n=1}^{\infty} B_n e^{-\lambda_n z} \leq B H^k \left( \frac{x}{k} \right).
$$

(7)

Let us find an upper bound for $|F(z)|$. By the Cauchy integral formula

$$
F(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{F(t)}{t - z} dt, \quad z = x + iy.
$$

(8)

where $\Gamma_R$ is rectangle border $\{s = \sigma + it: \frac{3}{2} \leq \sigma \leq \frac{3}{2} x, \ |\tau| = R\}$. Since $F(z) = o(1)$ as $z \to \infty$ and $\text{Re} z \geq s > 0$, then from (8), obviously, we will have

$$
F(z) = -\frac{1}{2\pi i} \int_{\text{Re}=\frac{3}{2}} \frac{F(t)}{t - z} dt + \frac{1}{2\pi i} \int_{\text{Re}=-\frac{3}{2}} \frac{F(t)}{t - z} dt.
$$

Therefore, taking into account (5), we obtain

$$
|F(z)| \leq \frac{2}{x} \left( T \left( \frac{x}{2} \right) + T \left( \frac{3}{2} x \right) \right) \leq A_F \frac{4}{x} H \left( \frac{x}{2} \right).
$$

But then from (6), (7) we have

$$
|\Psi(z)| \leq |F(z)| + \sum_{n=1}^{\infty} |B_n e^{-\lambda_n z}| \leq A_F \frac{4}{x} H \left( \frac{x}{2} \right) + B H^k \left( \frac{x}{k} \right).
$$
But \( x^N H(x) \to \infty \) as \( x \to 0 \) (\( N > 0 \) is arbitrary), so
\[
|\Psi(z)| \leq CH^k \left( \frac{x}{k} \right), \quad k \geq 2.
\]  

Thus, in Theorem 3 on the expansion «in a series of exponentials», the entire term \( \Psi \) is also subject to an estimate of type (7).

The question naturally arises about the behavior of the entire function \( \Psi \) in a neighborhood of the straight line \( i\mathbb{R} \), for example, in a vertical strip \( \Pi = \{ z = x + iy: \ 0 < x < 1 \} \). We will give an answer to this question in one important special case. Let’s dwell on this in more detail.

Let \( I \) be a segment of the imaginary axis \([-ai, ai], \ a > 0 \). By \( K_0^I(H) \) we denote the class of functions \( F \) that are analytic outside \( I \) and satisfy the estimate
\[
|F(z)| \leq H(|x|), \quad z \in \mathbb{C} \setminus I,
\]
where \( H(t) \) is any function decreasing with respect to \( t > 0, H(t) \uparrow \infty \) as \( t \to 0^+ \).

It is known that if the majorant \( H \) satisfies the logarithmic condition 4, then the family of functions \( K_0^I(H) \) is normal, that is, for any \( \delta > 0 \)[12]
\[
M^*(\delta) = \sup \{|F(z)|: \ F \in K_0^I(H), \ \rho(z, I) \geq \delta \} < \infty.
\]
Here \( \rho(z, I) = \inf_{\xi \in I} |z - \xi| \). Thus \( M^* \) is the smallest function for which
\[
|F(z)| \leq M^*(\rho(z, I)), \quad z \in \mathbb{C} \setminus I,
\]
for all \( F \in K_0^I(H) \). It characterizes the behavior of all functions from \( K_0^I(H) \) near the set \( I \) of singularities. In [13] a solution was obtained to the problem posed in [12] about the "effective estimate" of the majorant \( M^* \), namely, the following statement was proved.

If the majorant \( H \) from the definition of the class \( K_0^I(H) \) satisfies condition (4), and the functions \( m(s) (s > 0) \) and \( m(e^{-t}) (t \in \mathbb{R}) \), \( m(s) = \ln H(s) \), are convex, then for \( s \to 0 \)
\[
\ln M^*(s) = (1 + o(1))P_\varphi(s),
\]
where
\[
P_\varphi(s) = \sup_{x > 0} \left[ \frac{2x}{\pi} \int_0^\infty \frac{\varphi(t)}{t^2 + x^2} dt - xs \right],
\]
and \( \varphi(t) = (Lm)(t) \) is the lower Legendre transform of the function \( m(s) \). Note that the problem from [12] is far from trivial and essentially relies on one of the versions (for half-strips \( \{ z = x + iy: \ |x| < 1, \ |y| > a \} \) the following theorem of N. Levinson (see, for example, in [12], [14]):

Let the majorant \( H \) from the definition of \( K_0^I(H) \) satisfy condition (4), and \( K_P(H) \) be the family of functions analytic in the rectangle
\[
P = \{ z + x + iy: \ |x| < 1, \ |y| < 1 \}
\]
and satisfying in \( P \) the estimate \( |F(z)| \leq H(|x|) \). Then for any \( \delta \in (0, 1) \) there exists a constant \( C = C_H(\delta) \), depending only on \( H \) and \( \delta \), such that for any function \( F \in K_P(H) \) in rectangle
\[
P_\delta = \{ z = x + iy: \ |x| < 1, \ |y| < 1 - \delta \}
\]
fair estimate \( |F(z)| \leq C_H(\delta) \).

Let \( F \) be an even function that is analytic outside \( I \) and belongs to the class \( K_0 \). Suppose that condition (5) is satisfied, where the majorant \( H \) satisfies the conditions of Theorem 2. Then, by what has been proved,
\[
F(z) = \begin{cases} 
\sum_{n=1}^{\infty} B_n e^{-\lambda_n z} + \Psi(z), & z \in \Pi_0^+; \\
\sum_{n=1}^{\infty} B_n e^{\lambda_n z} + \Psi(-z), & z \in \Pi_0^-.
\end{cases}
\]
Therefore, taking into account (9), the entire function \( \Psi \) in the vertical strip \( \Pi = \{ z = x + iy: |x| < 1 \} \) satisfies the estimate

\[
|\Psi(z)| \leq H_0(|x|), \quad H_0(|x|) = CH^k \left( \frac{|x|}{k} \right).
\]

If the function \( H \) satisfies condition (4), then \( H_0 \) also satisfies this condition. But then the entire function \( \Psi \) is bounded in the strip \( \Pi \): \( \sup_{\Pi} |\Psi(z)| < \infty \). This statement follows from the following version of a Levinson-type theorem (see [15]).

**Theorem 4** [T. Carleman, A. Beurling, N. Levinson, N. Sjöberg, F. Wolf, and Y. Domar]. Let \( H \) be the majorant from theorem 2 satisfying condition (4). Then any function \( f \), that is analytic in the strip \( \Pi \) and satisfies the estimate

\[
|f(z)| \leq H(|x|), \quad z \in \Pi,
\]

is bounded in \( \Pi \). If the integral (4) diverges, then there is a function \( f \), satisfying inequality (10), which will not be bounded in \( \Pi \).

We formulate the result obtained in the form of the following theorem.

**Theorem 5.** Let \( F \) be an even function that is analytic outside the segment \( I = [-ai, ai], a > 0 \), and belongs to the class \( K_0 \). Let’s pretend that

\[
T_F(s) \leq A_F H(s), \quad s > 0,
\]

where the majorant \( H \) satisfies all the conditions of Theorem 2. If \( H \) also satisfies the Levinson logarithmic condition (4), then the entire function \( \Psi \) from representation (6) is bounded in the strip \( \Pi = \{ z = x + iy: |x| \leq 1 \} \).

**FUNDING**

This work was supported by the Russian Science Foundation grant no. 21-11-00168.

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3) This statement easily follows from the above theorem of N. Levinson. Indeed, it suffices to consider the family of functions \( \{ f_n(z) \}, f_n(z) = f(z + im), z \in P, n = \pm 1, \pm 2, \ldots \).