Classification of irreducible modules for the vertex operator algebra $M(1)^+$

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Abstract

We classify the irreducible modules for the fixed point vertex operator subalgebra of the vertex operator algebra associated to the Heisenberg algebra with the central charge 1 under the $-1$ automorphism.

1 Introduction

Let $\mathfrak{h}$ be a finite dimensional complex vector space of dimension $d$ with a nondegenerate symmetric bilinear form and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ the corresponding affine algebra. Then the free bosonic Fock space $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ is a vertex operator algebra of central charge $d$ (cf. [FLM]). If $d = 1$ the automorphism group of $M(1)$ is $\mathbb{Z}_2$ generated by $\theta$ (see Subsection 2.3). Then $M(1)$ has only two proper subalgebras, namely, $M(1)^+$ and the vertex operator subalgebra generated by the Virasoro algebra [DG]. In this paper we determine the Zhu’s algebra $A(M(1)^+)$ and classify the irreducible modules for $M(1)^+$. The classification result says that any irreducible $M(1)^+$-module is isomorphic to either a submodule of a $M(1)$-module or a submodule of $\theta$-twisted $M(1)$-module.

The vertex operator algebra $M(1)^+$ is closely related to $W$-algebra. It was shown in [DG] that $M(1)^+$ is generated by the Virasoro element $\omega$ and a highest weight vector $J$ of weight 4 (see Section 2 below). Thus $M(1)^+$ can be regarded as the vertex operator algebra associated to the $W$-algebra $W(2, 4)$ with central charge 1 (cf. [BFKNRV]). So

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this paper also gives a classification of irreducible modules for the $W$-algebra $W(2,4)$ which can be lifted to modules for $M(1)^+$. The result in this paper is fundamental in the classification of irreducible modules for the vertex operator algebras $V_L^+$ [DN]. Let $L$ be a positive definite even lattice of rank 1. The corresponding vertex operator algebra $V_L$ is a tensor product of $M(1)$ with the group algebra $\mathbb{C}[L]$. The structure and representation theory of $V_L$ including the fusion rules are well understood (see [B], [FLM], [D], [DL], [DLM1]). Then $\theta$ can be extended to an automorphism of $V_L$ of order 2. Moreover the fixed point vertex operator subalgebra $V_L^+$ contains $M(1)^+$ as a subalgebra and $V_L^+$ is a completely reducible $M(1)^+$-module. The result of the present paper has been used in [DN] to determine the Zhu’s algebra $A(V_L^+)$ and to classify the irreducible modules for $V_L^+$.

It should be pointed out that conformal field theory associated to $M(1)^+$ is an orbifold theory (cf. [DVVV]) for the nonrational vertex operator algebra $M(1)$. Let $V$ be a rational vertex operator algebra and $G$ be a finite group of automorphisms of $V$. The orbifold theory conjectures that any irreducible module of the fixed point vertex operator subalgebra $V_G^+$ is isomorphic to a submodule of a $g$-twisted $V$-module for some $g \in G$. Our result in this paper suggests that it may be true even when $V$ is not rational.

One important tool in the representation theory of vertex operator algebra is the Zhu’s algebra [Z]. In [Z] it was shown that for any vertex operator algebra $V$ there is an associative algebra $A(V)$ associated to $V$ such that there is a one to one correspondence between the irreducible admissible $V$-modules and irreducible $A(V)$-modules (see Subsection 2.2 for more detail). The main idea in the present paper is to determine the Zhu’s algebra $A(M(1)^+)$ which turns out to be a commutative algebra over $\mathbb{C}$ with two variables.

We should mention an important role played by a generalized PBW type theorem in this paper. The classical PBW theorem gives a basis for the universal enveloping algebra of a Lie algebra and a nice spanning set of modules. For an arbitrary vertex operator algebra $V$ the component operators of the fixed generators of $V$ in general do not form a Lie algebra, so one cannot use the classical PBW theorem to get a good spanning set in terms of the component operators of the generators. As mentioned before $M(1)^+$ is generated by $\omega$ and $J$. Although the component operators of $\omega$ and $J$ do not form a Lie algebra as the commutators involves quadratic or higher products, we manage to obtain a kind of PBW type result which is good enough to give nice spanning
sets of $M(1)^+$ and $A(M(1)^+)$. The same idea and technique have been developed further in [DN] to yield nice spanning sets of $V_L^+$ and $A(V_L^+)$. A PBW-type generating property for general vertex operator algebras has been given in [KL] recently.

The structure of this paper is as follows. In Section 2 we recall the definition of admissible twisted modules for a vertex operator algebra, the notion of Zhu’s algebra and related results, and the construction of vertex operator algebra $M(1)^+$. In Section 3 we give the commutator relations for the components operators of $\omega$ and $J$, and we produce a kind of generalized PBW theorem. This enables us to get spanning sets of $M(1)^+$ and $A(M(1)^+)$. Section 4 shows how to evaluate the generators of $A(M(1)^+)$ on the top levels of the known irreducible modules for $M(1)^+$ to yield the relations which are good enough to determine the algebra structure of $A(M(1)^+)$. We then use $A(M(1)^+)$ to classify the irreducible modules for $M(1)^+$.

2 Preliminaries

This section is divided into three parts. In the first part we recall various notions of (twisted) modules for a vertex operator algebra $V$ (cf. [DLM2]). The Zhu’s algebra [Z] and related results are explained in second part. In the last part we review the vertex operator algebra $M(1)$ and its (twisted) modules (cf. [FLM]).

2.1 Modules

Let $V$ be a vertex operator algebra (cf. [B], [FLM]) and $g$ an automorphism of $V$ of finite order $T$. Denote the decomposition of $V$ into eigenspaces with respect to the action of $g$ as $V = \bigoplus_{r \in \mathbb{Z}/T \mathbb{Z}} V^r$ where $V^r = \{ v \in V | gv = e^{-2\pi i r/T} v \}$.

An admissible $g$-twisted $V$-module (cf. [DLM2], [Z])

$$M = \sum_{n=0}^{\infty} M(\frac{n}{T})$$

is an $\frac{1}{T} \mathbb{Z}$-graded vector space with the top level $M(0) \neq 0$ equipped with a linear map

$$V \longrightarrow (\text{End } M) \{ z \}$$

$$v \longrightarrow Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \text{End } M)$$
which satisfies the following conditions; for all $0 \leq r \leq T - 1$, $u \in V^r$, $v \in V$, $w \in M$,

$$Y_M(u, z) = \sum_{n \in \frac{r}{T} + Z} u_n z^{-n-1},$$

$$u_n w = 0 \quad \text{for} \quad n \gg 0,$$

$$Y_M(1, z) = 1,$$

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1)$$

$$= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(v, z_0) v, z_2), \quad (2.1)$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ (elementary properties of the $\delta$-function can be found in [FLM]) and all binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable;

$$u_m M(n) \subset M(\text{wt}(u) - m - 1 + n)$$

if $u$ is homogeneous. If $g = 1$, this reduces to the definition of an admissible $V$-module.

A $g$-twisted $V$-module is an admissible $g$-twisted $V$-module $M$ which carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$. That is, we have

$$M = \prod_{\lambda \in \mathbb{C}} M_{\lambda}$$

where $M_{\lambda} = \{ w \in M | L(0) w = \lambda w \}$. Moreover we require that $\dim M_{\lambda}$ is finite and for fixed $\lambda$, $M_{\frac{m}{T} + \lambda} = 0$ for all small enough integers $n$. Again if $g = 1$ we get an ordinary $V$-module.

### 2.2 Zhu’s algebra

Let us recall that a vertex algebra $V$ is $\mathbb{Z}$-graded:

$$V = \prod_{n \in \mathbb{Z}} V_n, \quad v \in V_n, \quad n = \text{wt}(v).$$

Each $v \in V_n$ is a homogeneous vector of weight $n$. In order to define Zhu’s algebra $A(V)$ we need two products $\ast$ and $\circ$ on $V$. For $u \in V$ homogeneous and $v \in V$

$$u \ast v = \text{Res}_z \left( \frac{(1 + z)^{\text{wt}(u)}}{z} Y(u, z) v \right) = \sum_{i=0}^{\infty} \binom{\text{wt}(u)}{i} u_{i-1} v, \quad (2.2)$$

$$u \circ v = \text{Res}_z \left( \frac{(1 + z)^{\text{wt}(u)}}{z^2} Y(u, z) v \right) = \sum_{i=0}^{\infty} \binom{\text{wt}(u)}{i} u_{i-2} v \quad (2.3)$$

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and extend both (2.2) and (2.3) to linear products on $V$. Define $O(V)$ to be the linear span of all $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. For $u \in V$ we denote $o(u)$ the weight zero component operator of $u$ on any admissible module. Then $o(u) = u_{\text{wt}(u) - 1}$ if $u$ is homogeneous. The following theorem is essentially due to Zhu [Z].

**Theorem 2.1.** (i) The product $*$ induces an associative algebra structure on $A(V)$ with the identity $1 + O(V)$. Moreover $\omega + O(V)$ is a central element of $A(V)$.

(ii) The map $u \mapsto o(u)$ gives a representation of $A(V)$ on $M(0)$ for any admissible $V$-module $M$. Moreover, if any admissible $V$-module is completely reducible, then $A(V)$ is a finite dimensional semisimple algebra.

(iii) The map $M \to M(0)$ gives a bijection between the set of equivalence classes of simple admissible $V$-modules and the set of equivalence classes of simple $A(V)$-modules.

For convenience we write $[u] = u + O(V) \in A(V)$. We define $u \sim v$ for $u, v \in V$ if $[u] = [v]$. This induces a relation on $\text{End} V$ such that for $f, g \in \text{End} V$, $f \sim g$ if and only if $fu \sim gu$ for all $u \in V$.

The following proposition is useful later (cf. [W], [Z]).

**Proposition 2.2.** (i) Assume that $u \in V$ homogeneous, $v \in V$ and $n \geq 0$. Then

$$\text{Res}_z \left( \frac{(1 + z)^{\text{wt}(u)}}{z^{2+n}} Y(u, z)v \right) = \sum_{i=1}^{\infty} \binom{\text{wt}(u)}{i} u_{i-n-2}v \in O(V).$$

(ii) If $u$ and $v$ are homogeneous elements of $V$, then

$$u * v \sim \text{Res}_z \left( \frac{(1 + z)^{\text{wt}(v)-1}}{z} Y(v, z)u \right).$$

(iii) For any $n \geq 1$,

$$L(-n) \sim (-1)^n \{(n-1)(L(-2) + L(-1)) + L(0)\} \quad (2.4)$$

where $L(n)$ are the Virasoro operators given by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

### 2.3 Vertex operator algebras $M(1)$ and $M(1)^+$

Finally we discuss the construction of vertex operator algebra $M(1)$ and its (twisted) modules (cf. [FLM]). We also define the vertex operator subalgebra $M(1)^+$. 

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Let \( \mathfrak{h} \) be a finite dimensional vector space with a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) and \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \mathcal{C} \) the corresponding affine Lie algebra. Let \( \lambda \in \mathfrak{h} \) and consider the induced \( \hat{\mathfrak{h}} \)-module

\[
M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t, \lambda])} \mathbb{C} \simeq S(\mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}]) \quad \text{(linearly)}
\]

where \( \mathfrak{h} \otimes t \mathbb{C}[t] \) acts trivially on \( \mathbb{C} \), \( h \) acts as \( \langle \alpha, \lambda \rangle \) for \( \alpha \in \mathfrak{h} \) and \( c \) acts as 1. For \( \alpha \in \mathfrak{h} \) and \( n \in \mathbb{Z} \), we write \( \alpha(n) \) for the operator \( \alpha \otimes t^n \) and put

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}.
\]

Among \( M(1, \lambda), \lambda \in \mathfrak{h} \), \( M(1) = M(1, 0) \) has the special interesting because it has a natural vertex operator algebra structure as explained below. For \( \alpha_1, ..., \alpha_k \in \mathfrak{h}, n_1, ..., n_k \in \mathbb{Z} (n_i > 0) \) and \( v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \in M(1) \), we define a vertex operator corresponding to \( v \) by

\[
Y(v, z) = \circ \underbrace{\partial_z^{(n_1-1)} \alpha_1(z) \cdots \partial_z^{(n_k-1)} \alpha_k(z)}_{n_i \geq 0},
\]

where

\[
\partial_z^{(n)} = \frac{1}{n!} \left( \frac{d}{dz} \right)^n
\]

and a normal ordering procedure indicated by open colons signifies that the expression above is to be reordered if necessary so that all the operators \( \alpha(n) \) (\( \alpha \in \mathfrak{h}, n < 0 \)) are to be placed to the left of all the operators \( \alpha(n) \) (\( n \geq 0 \)) before the expression is evaluated. We extend \( Y \) to all \( v \in V \) by linearity. Let \( \{ \beta_1, ..., \beta_d \} \) be an orthonomal basis of \( \mathfrak{h} \). Set \( 1 = 1 \) and \( \omega = \frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)^2 \) The following theorem is well known (cf. [FLM]):

**Theorem 2.3.** The space \( M(1) = (M(1), Y, 1, \omega) \) is a simple vertex operator algebra and \( M(1, \lambda) \) for \( \lambda \in \mathfrak{h} \) gives a complete list of inequivalent irreducible modules for \( M(1) \).

We define an automorphism \( \theta \) of \( M(1) \) by

\[
\theta(\alpha_1(n_1) \cdots \alpha_k(n_k)) = (-1)^k \alpha_1(n_1) \cdots \alpha_k(n_k).
\]

Then \( \theta \)-invariants \( M(1)^+ \) of \( M(1) \) form a simple vertex operator subalgebra and the \(-1\)-eigenspace \( M(1)^- \) is an irreducible \( M(1)^+ \)-module (see Theorem 2 of [DM2]). Clearly \( M(1) = M(1)^+ \oplus M(1)^- \).
Following [DM1] we define \( \theta \circ M(1, \lambda) = (\theta \circ M(1, \lambda), Y_\theta) \) where \( Y_\theta(v, z) = Y(\theta v, z) \). Then \( \theta \circ M(1, \lambda) \) is also an irreducible \( M(1) \)-module isomorphic to \( M(1, -\lambda) \). The following proposition is a direct consequence of Theorem 6.1 of [DM2]:

**Proposition 2.4.** If \( \lambda \neq 0 \) then \( M(1, \lambda) \) and \( M(1, -\lambda) \) are isomorphic and irreducible \( M(1)^+ \)-modules.

Next we turn our attention to the \( \theta \)-twisted \( M(1) \)-modules (cf. [FLM]). The twisted affine algebra is defined to be \( \hat{h}[-1] = \sum_{n \in \mathbb{Z}} h \otimes t^{1/2+n} \oplus Cc \) and its canonical irreducible module is

\[
M(1)(\theta) = U(\hat{h}[-1]) \otimes_{U(h \otimes t^{1/2} C[t] \oplus Cc)} C \simeq S(h \otimes t^{-1/2} C[t^{-1/2}])
\]

where \( h \otimes t^{1/2} C[t] \) acts trivially on \( C \) and \( c \) acts as 1. As before there is an action of \( \theta \) on \( M(1)(\theta) \) by \( \theta(\alpha_1(n_1) \cdots \alpha_k(n_k)) = (-1)^k \alpha_1(n_1) \cdots \alpha_k(n_k) \) where \( \alpha_i \in h \), \( n_i \in \frac{1}{2} + \mathbb{Z} \) and \( \alpha(n) = \alpha \otimes t^n \). We denote the \( \pm 1 \)-eigenspace of \( M(1)(\theta) \) under \( \theta \) by \( M(1)(\theta) \pm \).

Let \( v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \in M(1) \). We define

\[
W_\theta(v, z) = \circ \partial_z^{(n_1-1)} \alpha_1(z) \partial_z^{(n_2-1)} \alpha_2(z) \cdots \partial_z^{(n_k-1)} \alpha_k(z) \circ,
\]

where the right side is an operator on \( M(1)(\theta) \), namely,

\[
\alpha(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \alpha(n) z^{-n-1}
\]

and the normal ordering notation is obvious. Further we extend this to all \( v \in V_L \) by linearity. Define constants \( c_{mn} \in \mathbb{Q} \) for \( m, n \geq 0 \) by the formula

\[
\sum_{m,n \geq 0} c_{mn} x^m y^n = -\log \left( \frac{(1 + x)^{1/2} + (1 + y)^{1/2}}{2} \right).
\]

Set

\[
\Delta_z = \sum_{m,n \geq 0} \beta_1(m) \beta_1(n) z^{-m-n}.
\]

Now we define twisted vertex operators \( Y_\theta(v, z) \) for \( v \in M(1) \) as follows:

\[
Y_\theta(v, z) = W_\theta(e^{\Delta_z} v, z).
\]

Then we have
Theorem 2.5. (i) \( (M(1)(\theta), Y_\theta) \) is the irreducible \( \theta \)-twisted \( M(1) \)-module.

(ii) \( M(1)(\theta)^\pm \) are irreducible \( M(1)^+ \)-modules.

Part (i) was a result of Chapter 9 of [FLM] and part (ii) follows Theorem 5.5 of [DLi].

In this paper, we mainly consider the case that \( \mathfrak{h} \) is one dimensional. From now on we always assume that \( \mathfrak{h} = \mathbb{C} h \) with the normalized inner product \( \langle h, h \rangle = 1 \).

Remark 2.6. It is easy to see in this case that the automorphism group of \( M(1) \) is generated by \( \theta \). It was pointed out in [DG] that \( M(1)^+ \) is the only proper vertex operator subalgebra of \( M(1) \) which differs from the vertex operator subalgebra generated by \( \omega \).

For the later purpose we need to know the first few coefficients of \( z \) in \( \Delta_z \). Note that

\[
-\log \left( \frac{(1 + x)^{1/2} + (1 + y)^{1/2}}{2} \right) = -\frac{1}{4} x - \frac{1}{4} y + \frac{3}{32} x^2 + \frac{1}{16} x y + \frac{3}{32} y^2 - \frac{5}{96} x^3 - \frac{1}{32} x^2 y - \frac{1}{32} x y^2 - \frac{5}{96} y^3 + \frac{35}{1024} x^4 + \frac{5}{256} x^3 y + \frac{9}{512} x^2 y^2 + \frac{5}{256} x y^3 + \frac{35}{1024} y^4 + \cdots.
\]

Thus

\[
\Delta_z = -\frac{1}{2} h(0) h(1) z^{-1} + \left( \frac{3}{16} h(0) h(2) + \frac{1}{16} h(1)^2 \right) z^{-2} + \left( -\frac{5}{48} h(0) h(3) - \frac{1}{16} h(1) h(2) \right) z^{-3} + \left( \frac{35}{512} h(0) h(4) + \frac{5}{128} h(1) h(3) + \frac{9}{512} h(2)^2 \right) z^{-4} + \cdots \tag{2.5}
\]

3 A spanning set of \( A(M(1)^+) \)

In this section we use a result in [DG] to yield a spanning set of \( M(1)^+ \) and then use it to produce a spanning set of \( A(M(1)^+) \). We also list known irreducible modules for \( M(1)^+ \) and the actions of \( L(0) \) and \( o(J) \) on the top levels of these modules where \( J \) is a singular vector of \( M(1)^+ \) of weight 4 defined in Subsection 3.1.
3.1 Some commutator relations

Recall that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ where the component operators $L(n)$ together with 1 spanned a Virasoro algebra of central charge 1 on $M(1)$. It is well known that $M(1)$ is a unitary representation for the Virasoro algebra and $M(1)^+$, as the submodule for the Virasoro algebra, is a direct sum of irreducible modules

$$M(1)^+ = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L(1, 4m^2)$$

(3.1)

where $L(1, 4m^2)$ is an irreducible highest weight Virasoro module with highest weight $4m^2$ and central charge 1 (See [DG] Theorem 2.7 (1)).

Let

$$J = h(-1)^4 1 - 2h(-3)h(-1)1 + \frac{3}{2} h(-2)^2 1$$

(3.2)

which is a singular vector of weight 4 for the Virasoro algebra. Then the field

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$$

is a primary field. We have commutation relations

$$[L(m), J(z)] = z^m (z \partial_z + 4(m + 1)) J(z) \quad (m \in \mathbb{Z})$$

which follows from the Jacobi identity (2.1) and which is equivalent to

$$[L(m), J_n] = (3(m + 1) - n) J_{n+m} \quad (m, n \in \mathbb{Z})$$

(3.3)

where $J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}$.

Next we compute the commutator relation $[J_m, J_n]$ for $m, n \in \mathbb{Z}$. Again by the Jacobi identity (2.1) we know

$$[J_m, J_n] = \sum_{i=0}^{\infty} \binom{m}{i} (J_i J)_{m+n-i}.$$

Since the weight of $J$ is 4, we see that $\text{wt}(J_i J) = 7 - i \leq 7$. Then it follows from the decomposition (3.1) that for any $i \in \mathbb{Z}_{\geq 0}$, we have $J_i J \in L(1, 0) \bigoplus L(1, 4)$ and then all these are expressed as linear combinations of

$$L(-m_1) \cdots L(-m_s) 1, \quad L(-n_1) \cdots L(-n_t) J$$

where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$, $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$ and $s, t \leq 3$. Note that for any vertex operator algebra $V$, $u, v \in V$ and $m, n \in \mathbb{Z}$, $(u_m v)_n$ is a linear combination of operators $u_s v_t$ and $v_t u_s$ for $s, t \in \mathbb{Z}$. Using (3.3) we obtain the following lemma.
Lemma 3.1. For any $m, n \in \mathbb{Z}$, commutators $[J_m, J_n]$ are expressed as linear combinations of

$$L(p_1) \cdots L(p_s), \quad L(q_1) \cdots L(q_t)J_r$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t, r \in \mathbb{Z}$ and $s, t \leq 3$.

3.2 A spanning set for $M(1)^+$

We first note the following theorem.

Theorem 3.2. ([DG], Theorem 2.7 (2)) As a vertex operator algebra, $M(1)^+$ is generated by the Virasoro element $\omega$ and any singular vector of weight greater than 0. In particular, $M(1)^+$ is generated by $\omega$ and $J$.

From this theorem we see that $M(1)^+$ is spanned by

$$\{u_{m_1}^1 \cdots u_{m_k}^k \mid u^i = \omega, J, m_i \in \mathbb{Z}\}$$

which are not necessarily linearly independent. We say that an expression $u_{m_1}^1 \cdots u_{m_k}^k$ has length $t$ with respect to $J$, which we write $\ell_J(u_{m_1}^1 \cdots u_{m_k}^k) = t$, if $\{i \mid u^i = J\}$ has cardinality $t$. Note that $\omega_i = L(i - 1)$. An induction on $\ell_J(u_{m_1}^1 \cdots u_{m_k}^k)$ using (3.3) and Lemma 3.1 shows that $u_{m_1}^1 \cdots u_{m_k}^k$ is a linear combination of vectors of type

$$\{L(m_1)L(m_2) \cdots L(m_s)J_{n_1}J_{n_2} \cdots J_{n_t} \mid m_a, n_b \in \mathbb{Z}\}.$$ 

Thus $M(1)^+$ is spanned by those vectors.

Using the commutator relations (3.3) and fact that $L(m)1 = 0, m \geq -1$, we get the following lemma.

Lemma 3.3. Let $W$ be a subspace of $M(1)^+$ spanned by $J_{n_1} \cdots J_{n_t}1$ with $n_i \in \mathbb{Z}$. Then $W$ is invariant under the action of $L(m), m \geq -1$.

Proposition 3.4. The vertex operator algebra $M(1)^+$ is spanned by

$$L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t}1$$

where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2, n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$. 

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Proof. We have already known that $\mathcal{M}(1)^+$ is spanned by

$$L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t}1$$

where $m_a, n_b \in \mathbb{Z}$. Using the PBW theorem for the Virasoro algebra we can assume that $m_1 \geq \cdots \geq m_s$. By Lemma 3.3 we can further assume that $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$. We proceed by induction on the length with respect to $J$ that $v = L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t}1$ can be spanned by the indicated vectors in the proposition.

If the length is 0, it is clear. Suppose that it is true for all monomials $v$ such that $\ell_J(v) < t$. Since $J_k1 = 0$ for $k \geq 0$ we can assume $n_t \geq 1$. If $n_1 \geq \cdots \geq n_t$ we are done. Otherwise there exists $n_a$ such that $n_{a+1} \geq \cdots \geq n_t$ but $n_a < n_{a+1}$. There are two cases $n_a \leq 0$ and $n_a > 0$ which are dealt with separately. If $n_a \leq 0$ then $J_{-n_a}1 = 0$ and

$$L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t}1$$

$$= \sum_{j=a+1}^{t} L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots \overset{\nu}{J}_{-n_a} \cdots [J_{-n_a}, J_{-n_j}] \cdots J_{-n_t}1$$

where $\overset{\nu}{J}_{-n_a}$ means that we omit the term $J_{-n_a}$. However by Lemma 3.1, $[J_{-n_a}, J_{-n_j}]$ are linear combinations of operators of type

$$L(p_1) \cdots L(p_{s'}), \quad L(q_1) \cdots L(q_{q'})J_r.$$ 

By substituting these into the above and using commutation relation (3.3) again, the right hand side is a linear combination of monomials whose lengths with respect to $J$ are less than or equal to $t - 1$. Thus by induction hypothesis, this is expressed as linear combinations of expected monomials.

If $n_a > 0$ then either $n_a < n_t$ or there exists $b$ with $t > b > a$ so that $n_b > n_a \geq n_{b+1}$. Then we have either

$$L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t}1$$

$$= \sum_{j=a+1}^{t} L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots \overset{\nu}{J}_{-n_a} \cdots [J_{-n_a}, J_{-n_j}] \cdots J_{-n_t}1$$

$$+ L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots \overset{\nu}{J}_{-n_a} \cdots J_{-n_t}J_{-n_a}1$$

11
or

\[ L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t} \]

\[ = \sum_{j=a+1}^{b} \quad L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_{a_j}} \cdots J_{-n_t} \]

\[ + L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_{a_j}} J_{-n_1} J_{-n_1+1} \cdots J_{-n_t} \]

From the discussion of case \( n_a \leq 0 \) it is enough to show either

\[ L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_{a_j}} \cdots J_{-n_1} \]

or

\[ L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_{a_j}} \cdots J_{-n_{a_j}} J_{-n_1} J_{-n_1+1} \cdots J_{-n_t} \]

can be expressed as linear combinations of desired vectors. But this follows from an induction on \( a \). □

### 3.3 A spanning set for \( A(M(1)^+) \)

For short we set

\[ v^s = v^s \]

for \( v \in M(1)^+ \). Recalling \( [v] = v + O(M(1)^+) \) for \( v \in M(1)^+ \). We will also use a similar notation \([v]^s\). Then it is easy to see that \([v]^s = [v]^t\).

**Theorem 3.5.** The Zhu’s algebra \( A(M(1)^+) \) is spanned by \( S = \{ [\omega]^s \cdot [J]^t \mid s, t \geq 0 \} \).

**Proof.** By Proposition 3.4, it is enough to show that for any monomial

\[ v = L(-m_1) \cdots L(-m_s)J_{-n_1} \cdots J_{-n_t} \]

where \( m_1 \geq m_2 \geq \cdots \geq m_1 \geq 2, n_1 \geq n_2 \geq \cdots \geq n_t \geq 1 \), \([v]\) is a linear combination of \( S \). We prove by induction on \( \ell_J(v) \) that \([v]\) is spanned by vectors \([\omega]^s \cdot [J]^t \) in \( S \) such that \( q \leq t \) and weights of its homogeneous components are less than or equal to the weight of \( v \).

In case that \( \ell_J(v) = 0 \), then \( v = L(-m_1) \cdots L(-m_s)1 \), which is spanned by \( \{ [\omega]^s \mid s \geq 0 \} \) (cf. [FZ]). Now let \( t > 0 \) and assume that the statement is true for all \( v \) with \( \ell_J(v) < t \). We will prove by induction on weight of \( v \) that \([v]\) is a linear
combination of $S$. Clearly, the smallest weight is $t\,\text{wt}(J)$ and corresponding $v$ has the form

$$v = J_{-1} \cdots J_{-1}.$$  

Then by (2.2),

$$J^{st} - v = \sum_{n_i \in \{-1,0,1,2,3\} \setminus (n_i) \neq (-1,\ldots,-1)} a_{n_1n_2\ldots n_t} J_{n_1} J_{n_2} \cdots J_{n_t},$$

Since each term appeared in the right hand side involves $J_{n_i}$ for some nonnegative $n_i$, we can write the right hand side as a linear combination of spanning vectors in Proposition 3.4 whose lengths are strictly less than $t$. Thus by induction hypothesis, the image of right hand side in $A(M(1)^+)$ is spanned by $S$ and so is $[v]$.

Now consider general $v = L(-m_1) \cdots L(-m_s) J_{-n_1} J_{-n_2} \cdots J_{-n_t}$. Without loss of generality, we can assume that $m_1 = m_2 = \cdots = m_s = 2$, namely,

$$v = L(-2) \cdots L(-2) J_{-n_1} \cdots J_{-n_t},$$

since suppose there exits $m_i$ such that $m_i \geq 3$, then $m_1 \geq 3$ and by (2.4),

$$v \sim (-1)^{m_1} \{(m_1 - 1)(L(-2) + L(-1)) + L(0)\} L(-m_2) \cdots L(-m_s) J_{-n_1} \cdots J_{-n_t},$$

which is a sum of three homogeneous vectors of weight strictly less than $\text{wt}(v)$. Then we see

$$v = \omega^{s*} * (J_{-n_1} J_{-n_2} \cdots J_{-n_t}) + v'$$

where $\text{wt}(v') < \text{wt}(v)$. Then again by using induction hypothesis about weight, it is enough to show that the image of

$$v = \omega^{s*} * (J_{-n_1} J_{-n_2} \cdots J_{-n_t})$$

in $A(M(1)^+)$ is spanned by $S$. Since $\omega$ is a central element in $A(M(1)^+)$, we have

$$v = (J_{-n_1} J_{-n_2} \cdots J_{-n_t}) * \omega^{s*} * (J_{-n_2} \cdots J_{-n_t}) + v'$$

where $\text{wt}(v') < \text{wt}(v)$. If $n_1 > 1$, we can use the fact that $J_{-n_1} u$ is congruent to a sum of vectors whose lengths are less than or equal to $t$ and whose weights are smaller than
wt(v) (cf. (2.3)) to show that \([v]\) is spanned by \(S\). If \(n_1 = 1\), then \(n_2 = \cdots = n_t = 1\) and

\[
v = \omega^{**} J^{*t} + \text{lower weight terms.}
\]

Again it is done by the induction assumption. □

**Remark 3.6.** From the proof of Theorem 3.5, we see \(v\) is spanned by \(\omega^{**} J^{*t}\) with \(2s + 4t \leq \text{wt}(v)\).

### 3.4 List of irreducible modules

As mentioned in Subsection 2.3, \(M(1)^+\) has the following irreducible modules,

\(M(1)^+, M(1)^-, M(1, \lambda) (0 \neq \lambda \in \mathbb{C}), M(1)(\theta)^+, M(1)(\theta)^-\).

Recall that \(M(1, \lambda)\) and \(M(1)(\theta)\) are symmetric algebras on \(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]\) and \(\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]\) as vector spaces.

The following table gives the action of \(\omega\) and \(J\) on the top levels of these modules.

|         | \(M(1)^+\) | \(M(1)^-\) | \(M(1), \lambda \in \mathbb{C}^x\) | \(M(1)(\theta)^+\) | \(M(1)(\theta)^-\) |
|---------|-------------|-------------|---------------------------------|-----------------|-----------------|
| \(M(0)\) | \(\mathbb{C}1\) | \(\mathbb{C}h(-1)1\) | \(\mathbb{C}\) | \(\mathbb{C}\) | \(\mathbb{C}h(-1/2)\) |
| \(\omega\) | 0 | 1 | \(\lambda^2/2\) | \(1/16\) | \(9/16\) |
| \(J\) | 0 | -6 | \(\lambda^4 - \lambda^2/2\) | \(3/128\) | \(-45/128\) |

Here we give some explanations on how to get the table. The actions of \(\omega\) and \(J\) on these spaces except \(M(1)(\theta)^\pm\) are easily verified. From the definition of \(Y_{\theta}(u, z)\), we see

\[
Y_{\theta}(\omega, z) = \frac{1}{2} \circ h(z)^2 \circ + \frac{1}{16} z^{-2}.
\]

Recall the expression of \(J\) from (3.2). Then by using (2.5), we get

\[
e^{\Delta_t} J = J + \frac{3}{4} h(-1)^2 1 z^{-2} + \frac{3}{128} z^{-4}
\]

and thus

\[
Y_{\theta}(J, z) = J(z) + \frac{3}{4} \circ h(z)^2 \circ z^{-2} + \frac{3}{128} z^{-4}
\]

where \(h(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} h(n) z^{-n-1}\). The actions of \(\omega\) and \(J\) on the top levels of \(M(1)(\theta)^+\) and \(M(1)(\theta)^-\) are immediately derived.
4 Classification of irreducible modules for $M(1)^+$

In this section we explicitly determine the algebra structure of $A(M(1)^+)$ and use this result together to prove that the list of the irreducible modules in Subsection 3.4 is complete.

4.1 The structure of $A(M(1)^+)$

It has been proved in Subsection 3.3 that Zhu’s algebra $A(M(1)^+)$ as an associative algebra is generated by $[\omega]$ and $[J]$. Since $[\omega]$ is a central element, $A(M(1)^+)$ is a commutative associative algebra and must be isomorphic to a quotient of the polynomial algebra $\mathbb{C}[x, y]$ with variables $x$ and $y$ modulo an ideal $I$. We still need to determine the ideal explicitly. For this purpose we will find relations between $[\omega]$ and $[J]$ in $A(M(1)^+)$. For convenience we simply write $[u]$ by $u$ for $u \in M(1)^+$ and $u * v$ by $uv$.

**Proposition 4.1.** In $A(M(1)^+)$

$$J^2 = p(\omega) + q(\omega)J$$

where

\[
p(x) = \frac{1816}{35}x^4 - \frac{212}{5}x^3 + \frac{89}{10}x^2 - \frac{27}{70}x, \quad q(x) = -\frac{314}{35}x^2 + \frac{89}{14}x - \frac{27}{70}.
\]

Or equivalently,

\[(J + \omega - 4\omega^2)(70J + 908\omega^2 - 515\omega + 27) = 0.\]

**Proof.** Recall that as a module for the Virasoro algebra $M(1)^+$ has the decomposition $M(1)^+ = \bigoplus_{m \geq 0} L(1, 4m^2)$. Since $J$ is the singular vector with weight 4, we see

$$J^2 = \sum_{i \geq 0} \binom{4}{i} J_{i-1} J \in L(1, 0) \bigoplus L(1, 4).$$

Therefore from Remark 3.6, we get

$$J^2 = p(\omega) + q(\omega)J \quad (4.1)$$

where $p, q$ are polynomials of degrees less than or equal to 4 and 2 respectively. Let

\[
p(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon \quad \text{and} \quad q(x) = ax^2 + bx + c.
\]
In order to determine the coefficients of \( p(x) \) and \( q(x) \) we evaluate both sides of equation (4.1) on modules listed in Subsection 3.4.

Since \( \omega = J = 0 \) on the top level of \( M(1)^+ \) we have \( \epsilon = 0 \). On the top level of \( M(1)^- \), \( \omega = 1 \) and \( J = -6 \) give \( \alpha + \beta + \gamma + \delta - 6(a + b + c) = 36 \). Further on the top levels of \( M(1, \lambda) \) for \( \lambda \in \mathbb{C}^\times \) we know \( \omega = \lambda^2/2 \) and \( J = \lambda^4 - \lambda^2/2 \). Comparing the coefficients of \( \lambda^i \)'s tells us

\[
\alpha + 4a = 16, \quad \beta + 4b - a = -8, \quad \gamma + 4c - b = 1, \quad \delta - c = 0.
\]

Finally, we get two more equations by substituting \( \omega = 1/16, J = 3/128 \) on \( M(1)^+ \) and \( \omega = 9/16, J = -45/128 \) on \( M(1)^- \). Solving this linear system gives the desired result. □

**Proposition 4.2.** In \( A(M(1)^+) \)

\[
(\omega - 1)(\omega - \frac{1}{16})(\omega - \frac{9}{16})(J + \omega - 4\omega^4) = 0.
\]

As a vertex operator algebra, \( M(1)^+ \) has the weight space decomposition \( M(1)^+ = \bigoplus_{m \geq 0} M(1)^+_m \). The following is the list of \( \dim M(1)^+_m \) for \( m \) up to 10.

| \( m \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|---|
| \( \dim M(1)^+_m \) | 1 | 0 | 1 | 1 | 3 | 3 | 6 | 7 | 12 | 14 | 22 |

In order to produce the second relation we need the following lemma whose proof is given in the Appendix.

**Lemma 4.3.** The vectors

\[
L(-1)M(1)^+_0, \quad L(-3)M(1)^+_7, \quad h(-1)^4h(-1)^4\mathbf{1}, \quad L(-2)^5\mathbf{1}
\]

span \( M(1)^+_10 \).

Now we can prove Proposition 4.2. First note that any weight 10 vector is contained in \( L(1, 0) \oplus L(1, 4) \) and is a linear combination of vectors of type \( L(-n_1) \cdots L(-n_s), L(-m_1) \cdots L(-m_t)J \) where \( n_1 \geq \cdots \geq n_s \geq 2, m_1 \geq \cdots \geq m_t \geq 1, \sum n_a = 10 \) and \( \sum m_b + 4 = 10 \). From the proof of Theorem 5.3 the images of this kind of vectors in
A(M(1)^+) can be are expressed as linear combinations of \( \omega^i \) \( (i = 0, 1, \ldots, 5) \), \( \omega^i J \) \( (i = 0, 1, 2, 3) \).

By Proposition 2.2 (i) and (iii), \( L(-1)M(1)^+_0 \), \( L(-3)M(1)^+_7 \), \( h(-1)^4 h(-1)^4 1 \) are congruent to vectors whose homogeneous components have weights less than 10. Note that \( \omega^i = L(-2)^i 1 + \) lower weight terms. Then it follows from Remark 3.6 and Proposition 4.1 that \( L(-1)M(1)^+_0 \), \( L(-3)M(1)^+_7 \), \( h(-1)^4 h(-1)^4 1 \) are congruent to vectors spanned by \( \omega^i \) \( (i = 0, 1, \ldots, 4) \), \( \omega^i J \) \( (i = 0, 1, 2) \). As a result we have

\[
\omega^3 J = P(\omega) + Q(\omega) J
\]

where \( \deg P \leq 5 \), \( \deg Q \leq 2 \). Evaluating this equation on the top levels of the modules listed in Subsection 3.4 gives the desired result.

Now we can state our first main theorem.

**Theorem 4.4.** We have the following algebra isomorphism:

\[
\mathbb{C}[x, y]/\langle P, Q \rangle \cong A(M(1)^+)
\]

where

\[
P = (y + x - 4x^2)(70y + 908x^2 - 515x + 27), \quad Q = (x - 1)(x - \frac{1}{16})(x - \frac{9}{16})(y + x - 4x^2).
\]

**Proof.** By Theorem 3.5, we have a surjective algebra homomorphism

\[
\varphi : \mathbb{C}[x, y] \longrightarrow A(M(1)^+)
\]

\[
x \longmapsto \omega \\
y \longmapsto J.
\]

Let \( K(x, y) \in \text{Ker } \varphi \) and regard \( K(x, y) \) as a polynomial in variable \( y \). Note that \( P(x, y) \) has degree 2 in \( y \). Using the division algorithm we can write \( K(x, y) = A(x, y) P(x, y) + R(x, y) \) where \( A(x, y), R(x, y) \in \mathbb{C}[x, y] \) so that \( R(x, y) \) has degree 1 in \( y \). We can express \( R(x, y) \) as \( R(x, y) = B(x)(y + x - 4x^2) + C(x) \). By Proposition 4.1 \( P(x, y) \in \text{Ker } \varphi \). So we have

\[
B(\omega)(J + \omega - 4\omega^2) + C(\omega) = 0\quad (4.3)
\]

Evaluating the above equation on the top levels of modules \( M(1, \lambda) \) yields \( C(\lambda^2/2) = 0 \) since \( J + \omega - 4\omega^2 = 0 \) on the top level of \( M(1, \lambda) \) for all \( \lambda \in \mathbb{C}^\times \). Thus \( C(x) = 0 \) as a
polynomial. Further evaluating equation (4.3) on the top levels of $M(1)^-$, $M(1)(\theta)^\pm$ and noting that $J + \omega - 4\omega^2 \neq 0$, we get $B(1) = B(1/16) = B(9/16) = 0$. This implies $(x - 1)(x - 1/16)(x - 9/16)|B(x)$. Thus we reach to

$$K(x, y) = A(x, y)P(x, y) + D(x)Q(x, y)$$

for some polynomial $D(x)$. Since $Q(x, y)$ lies in Ker $\varphi$ already by Proposition 4.2, we conclude that Ker $\varphi = \langle P(x, y), Q(x, y) \rangle$. □

4.2 Classification of irreducible modules for $M(1)^+$

Finally we can use $A(M(1)^+)$ whose algebra structure was determined in the previous section to classify the irreducible modules for $M(1)^+$.

**Theorem 4.5.** The set

$$\{M(1)^\pm, M(1)(\theta)^\pm, M(1, \lambda) \cong M(1, -\lambda), \lambda \in \mathbb{C}^\times\}$$

gives a complete list of inequivalent irreducible $M(1)^+$ modules. Moreover, any irreducible admissible $M(1)^+$-module is an ordinary module.

**Proof.** Let $M = \oplus_{n \geq 0} M(n)$ be an irreducible admissible $M(1)^+$-module with $M(0) \neq 0$. Then $M(0)$ is an irreducible $A(M(1)^+)$-module. Since $A(M(1)^+)$ is commutative, $M(0)$ is one-dimensional. So both $\omega$ and $J$ act as scalars $\alpha$ and $\beta$ on $M(0)$. From Theorem 4.4 we have

$$(\beta + \alpha - 4\alpha^2)(70\beta + 908\alpha^2 - 515\alpha + 27) = 0$$

and

$$(\alpha - 1)(\alpha - \frac{1}{16})(\alpha - \frac{9}{16})(\beta + \alpha - 4\alpha^2) = 0.$$ 

If $\beta + \alpha - 4\alpha^2 = 0$ and $\alpha \neq 0$, then $M(0)$ is isomorphic to the top level of $M(1, \sqrt{2\alpha})$ and $M$ is isomorphic to $M(1, \sqrt{2\alpha})$. If $\beta + \alpha - 4\alpha^2 = 0$ and $\alpha = 0$ then $M$ is isomorphic to $M(1)^+$. Otherwise we have $(\alpha - 1)(\alpha - 1/16)(\alpha - 9/16) = 0$ and $70\beta + 908\alpha^2 - 515\alpha + 27 = 0$. One can easily verify that $M$ is isomorphic to $M(1)^-$, $M(1)(\theta)^+$ and $M(1)(\theta)^-$ when $\alpha = 1, 1/16$ and $9/16$. □
Appendix

Here we give the details of a proof of Lemma 4.3. First, we list bases of $M(1)_{7}^{+}, M(1)_{9}^{+}$ and $M(1)_{10}^{+}$ which have dimension 7, 14 and 22 respectively.

A basis of $M(1)_{7}^{+}$:

$$
e_1 = h(-6)h(-1)1, \quad e_2 = h(-5)h(-2)1, \quad e_3 = h(-4)h(-3)1 \quad e_4 = h(-4)h(-1)^31, \quad e_5 = h(-3)h(-2)h(-1)^21, \quad e_6 = h(-2)^3h(-1)1, \quad e_7 = h(-2)h(-1)^51.$$

A basis of $M(1)_{9}^{+}$:

$$f_1 = h(-8)h(-1)1, \quad f_2 = h(-7)h(-2)1, \quad f_3 = h(-6)h(-3)1, \quad f_4 = h(-6)h(-1)^31, \quad f_5 = h(-5)h(-4)1, \quad f_6 = h(-5)h(-2)h(-1)^21, \quad f_7 = h(-4)h(-3)h(-1)^21, \quad f_8 = h(-4)h(-2)^2h(-1)1, \quad f_9 = h(-4)h(-1)^51, \quad f_{10} = h(-3)^2h(-2)h(-1)1, \quad f_{11} = h(-3)h(-2)^31, \quad f_{12} = h(-3)h(-2)h(-1)^41, \quad f_{13} = h(-2)^3h(-1)^31, \quad f_{14} = h(-2)h(-1)^71.$$

A basis of $M(1)_{10}^{+}$:

$$g_1 = h(-9)h(-1)1, \quad g_2 = h(-8)h(-2)1, \quad g_3 = h(-7)h(-3)1, \quad g_4 = h(-7)h(-1)^31, \quad g_5 = h(-6)h(-4)1, \quad g_6 = h(-6)h(-2)h(-1)^21, \quad g_7 = h(-5)^21, \quad g_8 = h(-5)h(-3)h(-1)^21, \quad g_9 = h(-5)h(-2)^2h(-1)1, \quad g_{10} = h(-5)h(-1)^51, \quad g_{11} = h(-4)^2h(-1)^21, \quad g_{12} = h(-4)h(-3)h(-2)h(-1)1, \quad g_{13} = h(-4)h(-2)^31, \quad g_{14} = h(-4)h(-2)h(-1)^41, \quad g_{15} = h(-3)^3h(-1)1, \quad g_{16} = h(-3)^2h(-2)^21, \quad g_{17} = h(-3)^2h(-1)^41, \quad g_{18} = h(-3)^2h(-2)^2h(-1)^31, \quad g_{19} = h(-3)h(-1)^71, \quad g_{20} = h(-2)^4h(-1)^21, \quad g_{21} = h(-2)^2h(-1)^61, \quad g_{22} = h(-1)^{10}1.$$
It is easy to see that

\[ h(-1)^4 \frac{1}{2} h(-1)^4 \]

\[ = 96h(-9)h(-1)1 + 144h(-7)h(-1)^3 1 + 144h(-6)h(-2)h(-1)^2 1 \]

\[ + 144h(-5)h(-3)h(-1)^2 1 + 72h(-4)^2 h(-1)^2 + 48h(-5)h(-1)^5 1 \]

\[ + 96h(-4)h(-2)h(-1)^4 1 + 48h(-3)^2 h(-1)^4 1 + 48h(-3)h(-2)^2 h(-1)^3 1 \]

\[ + 4h(-3)h(-1)^7 1 + 6h(-2)^2 h(-1)^6 1. \]

The tables below give the precise linear combinations of certain vectors in terms of \( g_i \) for \( i = 1, \ldots, 22 \). For example, \( L(-1)f_1 = 8g_1 + g_2 \). We know from the tables that the vectors in (4.2) without \( L(-2)^5 1 \) span a 21-dimensional subspace of \( M(1)^{+}_{10} \) and none of these vectors involves the term \( h(-1)^{10} 1 \). On the other hand, \( L(-2)^5 1 \) involves the term \( h(-1)^{10} 1 \). Thus the vectors in (4.2) span \( M(1)^{+}_{10} \), as expected.
| L(−1)f_1 | L(−1)f_2 | L(−1)f_3 | L(−1)f_4 | L(−1)f_5 | L(−1)f_6 | L(−1)f_7 | L(−1)f_8 | L(−1)f_9 | L(−1)f_10 | L(−1)f_11 | L(−1)f_12 | L(−1)f_13 | L(−1)f_14 | L(−3)e_1 | L(−3)e_2 | L(−3)e_3 | L(−3)e_4 | L(−3)e_5 | L(−3)e_6 | L(−3)e_7 | h(−1)^2_3 h(−1)^4_1 |
|-----------|------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 8         | 1          | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 7          | 2         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 6         | 0         | 3         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 6         | 0         | 3         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 5         | 0         | 4         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 0         | 5         | 0         | 2         | 2         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 0         | 0          | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |

Table A1
|            | $g_{12}$ | $g_{13}$ | $g_{14}$ | $g_{15}$ | $g_{16}$ | $g_{17}$ | $g_{18}$ | $g_{19}$ | $g_{20}$ | $g_{21}$ | $g_{22}$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $L(-1)f_1$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_2$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_3$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_4$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_5$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_6$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_7$ | 2       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_8$ | 4       | 1       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_9$ | 0       | 0       | 5       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_{10}$ | 6       | 0       | 0       | 2       | 1       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_{11}$ | 0       | 3       | 0       | 0       | 6       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-1)f_{12}$ | 0       | 0       | 3       | 0       | 0       | 2       | 4       | 0       | 0       | 0       | 0       |
| $L(-1)f_{13}$ | 0       | 0       | 0       | 0       | 0       | 0       | 6       | 0       | 3       | 0       | 0       |
| $L(-1)f_{14}$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 2       | 0       | 7       | 0       |
| $L(-3)e_1$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-3)e_2$ | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-3)e_3$ | 1       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-3)e_4$ | 0       | 0       | 1       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
| $L(-3)e_5$ | 2       | 0       | 0       | 0       | 0       | 0       | 1       | 0       | 0       | 0       | 0       |
| $L(-3)e_6$ | 0       | 1       | 0       | 0       | 0       | 0       | 0       | 1       | 0       | 0       | 0       |
| $L(-3)e_7$ | 0       | 0       | 5       | 0       | 0       | 0       | 0       | 0       | 0       | 1       | 0       |
| $h(-1)^4g_{12}$ | 0       | 0       | 0       | 0       | 96      | 0       | 0       | 48      | 48      | 0       | 6       |

Table A2

References

[BFKNRV] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, W-algebras with two and three generators, *Nucl. Phys. B*361 (1991), 255-289.

[B] Borcherds, R.: Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* 83, 3068–3071(1986)
[DVVV] Dijkgraaf, R., Vafa, C., Verlinde, E., Verlinde, H.: The operator algebra of orbifold models, Commu. Math. Phys. 123, 485–526 (1989)

[D] C. Dong, Vertex algebras associated with even lattices, J. Algebra 160 (1993), 245-265.

[DG] Dong, C., Griess, R.L., Jr.: Rank one lattice type vertex operator algebras and their automorphism groups, J. Algebra, to appear, [q-alg/9710017]

[DL] Dong, C., Lepowsky, J.: Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. Vol. 112, Birkhäuser, Boston 1993.

[DLM] Dong, C., Li, H., Mason, G.: Regularity of rational vertex operator algebras, Advances. in Math. 132 (1997), 148-166

[DLM] Dong, C., Li, H., Mason, G.: Twisted representation of vertex operator algebras, Math. Ann. 310 (1998), 571-600.

[DLi] Dong, C., Lin, Z.: Induced modules for vertex operator algebras, Commu. Math. Phys. 179 (1996), 157-184.

[DM] Dong, C., Mason, G.: Nonabelian orbifolds and boson-fermion correspondence, Commu. Math. Phys. 163 (1994), 523-559.

[DM2] Dong, C., Mason, G.: On quantum Galois theory, Duke Math. J. 86 (1997), 305-321.

[DN] Dong, C., Nagatomo, K.: Representations of vertex operator algebra $V_L^+$ for rank one lattice $L$, preprint.

[FHL] Frenkel, I.B., Huang, Y., Lepowsky, J.: On axiomatic approach to vertex operator algebras and modules, Mem. Amer. Math. Soc. 104 No.494(1993)

[FLM] Frenkel, I.B., Lepowsky, J., Meurman, A.: Vertex operator algebras and the Monster, Academic Press, 1988

[FZ] Frenkel, I.B, Zhu, Y.: Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66, 123–168 (1992)
[KL] M. Karel and H. Li, Certain generating subspaces for vertex operator algebras, preprint.

[W] Wang, W.: Rationality of Virasoro vertex operator algebras, *Duke Math. J.* **71**, IMRN No. 7, 197–211(1993)

[Z] Zhu, Y.: Modular invariance of characters of vertex operator algebras, *J. AMS* **9**, 237–301(1996)