A sensitive test for models of Bose-Einstein correlations

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Abstract

Accurate and sensitive measurements of higher order cumulants open up new approaches to pion interferometry. It is now possible to test whether a given theoretical prediction can consistently match cumulants of both second and third order. Our consistency test utilizes a new technique combining theoretically predicted functions with experimentally determined weights in a quasi-Monte Carlo approach. Testing a general quantum-statistics-based framework of Bose-Einstein correlations with this technique, we find that predictions for third order cumulants differ significantly from UA1 data. This discrepancy may point the way to more detailed dynamical information.

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Pion interferometry has been a vital part of multiparticle physics for several decades [1]. While traditionally experimental effort in this field has centered around second order correlations, much progress has been made recently in accurately quantifying higher order correlations by means of so-called correlation integrals [2] to the point where these now yield statistically significant conclusions not only for moments but also for higher order cumulants. Because cumulants are so sensitive to details of the dynamics, they represent a stringent testing ground for proposed theoretical models.

A number of theoretical predictions for higher orders exist [3,4]. In particular, Andreev, Plümer and Weiner (APW) [6] have suggested a very general quantum-statistical framework, based on the classical source current formalism applied successfully in quantum optics. Its basic assumptions are (1) a Gaussian density functional for the classical random currents and (2) isotropy in isospin space. These two assumptions determine all higher order correlation functions in terms of the basic correlator, independent of the structure of the sources. All further assumptions concern only a more detailed specification of the space-time evolution of the sources. Thus, the APW framework includes as special cases more specific models of Bose-Einstein correlations such as the GKW model [7] and the approach of Biyajima et al. [8]. Because the APW model is so important, we test a simple version of its predictions below. It will also serve as an example to show how our approach works.

While higher order cumulant measurements are valuable in their own right, they can be used to even greater effect in consistency checks: once an assumed parameterization is found to fit the second order data, the same set of parameter values ought to fit all predicted higher order correlations as well. Departing from tried and tested ways, we therefore concentrate not so much on numerical values of source parameters, but rather on utilizing their required constancy over cumulants of different orders to test for consistency and ultimately falsifiability of a given theoretical prediction.

Pion interferometry measures correlations in terms of pair variables such as 3- or 4-momentum differences of two particles. In the correlation integral formalism [2], this means that the original variables in the $r$th factorial moment density $\rho_r(p_1,\ldots,p_r)$ must be converted to $r(r-1)/2$ (not necessarily independent) pair variables $(q_{12}, q_{13}, \ldots, q_{r-1,r})$. Here, we use the specific definition of a “distance” $|p_i - p_j| = \sqrt{(p_i - p_j)^2 - (E_i - E_j)^2}$ between two phase-space points $p_i$ and $p_j$, but other pair variables and distance definitions are also possible. The differential moment densities as well as the corresponding cumulant densities can be written in terms of delta functions involving pair variables, $\delta_{12} \equiv \delta(q_{12} - |p_1 - p_2|)$. For example, in third order we have

$$\rho_3(q_{12}, q_{23}, q_{31}) = \int dp_1 dp_2 dp_3 \rho_3(p_1, p_2, p_3) \delta_{12} \delta_{23} \delta_{31} .$$

The pair-variable moments are measured by direct and exact sampling prescriptions [3] in terms of the measured four-momentum differences between track pairs, $Q_{ij}$, and event averages $\langle \cdots \rangle$,

$$\rho_2 = \langle \sum_{i \neq j} \delta(q_{12} - Q_{ij}) \rangle .$$

(1)
\[\rho_3 = \langle \sum_{i\neq j \neq k} \delta(q_{12} - Q_{ij})\delta(q_{23} - Q_{jk})\delta(q_{31} - Q_{ki}) \rangle,\]  

etc. Pair-variable cumulants

\[C_2(q_{12}) = \rho_2(q_{12}) - \rho_1 \otimes \rho_1(q_{12}),\]

\[C_3(q_{12}, q_{23}, q_{31}) = \rho_3 - \sum_{(3)} \rho_2 \otimes \rho_1 + 2 \rho_1 \otimes \rho_1 \otimes \rho_1,\]

are similarly measured in exact prescriptions. Mixed terms such as \(\rho_1 \otimes \rho_1\) require multiple event averages in terms of distances \(Q_{ij}^a = |p_i^a - p_j^a|\) between particles \(p_i^a\) and \(p_j^a\) taken from different events \(a\) and \(b\),

\[\rho_1 \otimes \rho_1(q_{12}) = \langle \langle \sum_{i,j} \delta(q_{12} - Q_{ij}^a) \rangle \rangle\]

and so on (see Refs. [2] for details).

These experimentally measured cumulants are, after normalization, to be compared to the corresponding theoretical predictions. The APW normalized cumulant predictions are built up from normalized correlators \(d_{ij}\), the on-shell Fourier transforms of the space-time classical current correlators. The specific parameterizations of the correlator we shall be testing are, in terms of the 4-momentum difference \(q_{ij}\),

- Gaussian: \(d_{ij} = \exp(-r^2 q_{ij}^2)\),
- exponential: \(d_{ij} = \exp(-r q_{ij})\),
- power law: \(d_{ij} = q_{ij}^{-\alpha}\).

For constant chaoticity \(\lambda\) and real-valued currents, APW predict the second and third-order normalized cumulants \(k_{2}^{th}(q_{12})\) and \(k_{3}^{th}(q_{12}, q_{23}, q_{31})\) to have the form [6]

\[k_{2}^{th} = \frac{C_2}{\rho_1 \otimes \rho_1} = 2\lambda(1-\lambda) d_{12} + \lambda^2 d_{12}^2,\]

\[k_{3}^{th} = \frac{C_3}{\rho_1 \otimes \rho_1 \otimes \rho_1} = 2\lambda^2(1-\lambda)[d_{12} d_{23} + d_{23} d_{31} + d_{31} d_{12}] + 2\lambda^3 d_{12} d_{23} d_{31},\]

with any of the above \(d_{ij}\) parameterizations.

A comparison of sample cumulants [3]–[4] with the theoretical APW cumulants in [3]–[4] encounters two basic difficulties: projection and normalization. In dealing with these, we are led to the “Monte Karli” prescription outlined below.

Statistical sampling limitations render a fully multidimensional measurement of cumulants such as \(C_3(q_{12}, q_{23}, q_{31})\) impossible, necessitating projections onto a single variable. Such projections must of course then be applied to theoretical predictions such as [3] also. However, the complicated combinatorics of \(d_{ij}\) entering the cumulants make it generally impossible to give analytical formulae in terms of any one combined variable (such as \(S = q_{12} + q_{23} + q_{31}\)) without making further approximations. While two such approximations, leading to simple third order formulae, have been used previously by us and other groups [3]–[4], conclusions based on them are weak because the effects of making such approximations are seldom quantifiable.

The difficulty in normalization arises because experimental and theoretical procedures differ. Theoretical cumulants are normalized fully differentially with \(\rho_1 \otimes \rho_1(q_{12})\) and \(\rho_1 \otimes \rho_1 \otimes \rho_1(q_{12}, q_{23}, q_{31})\) respectively. Such normalizations assume essentially perfect measurement accuracy for the \(q_{ij}\)’s and infinite sample size. Experimentally, however, one always measures unnormalized cumulants over some phase-space region \(\Omega\) and then divides by an uncorrelated reference sample integrated over the same \(\Omega\). For example, in second order one actually measures

\[\Delta K_{2}^{ex}(\Omega) = \frac{\int_{\Omega} C_{2}^{ex}(q) dq}{\int_{\Omega} \rho_1 \otimes \rho_1^{ex}(q) dq},\]

which approaches the differential \(k_{2}^{th}(q)\) or the bin-integrated normalized theoretical cumulant

\[\Delta K_{2}^{th}(\Omega) = \frac{1}{\Omega} \int_{\Omega} k_{2}^{th}(q) dq\]

only in the limit \(\Omega \to 0\) and may significantly differ from \(\Delta K_{2}^{ex}\) for larger \(\Omega\). Non-stationary single-particle distributions can significantly distort results, especially when they vary strongly within a given bin, as they do in the case of momentum differences. The problem is exacerbated in higher orders.

“Monte Karli” (MK) integration provides a solution to both these problems [8]. It is conceptually quite simple: Since \(C_2(q) = k_2(q) \rho_1 \otimes \rho_1(q)\) we can, in analogy to \(\Delta K_{2}^{ex}\), define an MK normalized cumulant

\[\Delta K_{2}^{MK}(\Omega) = \frac{\int_{\Omega} k_{2}^{th}(q) \rho_1 \otimes \rho_1^{ex}(q) dq}{\int_{\Omega} \rho_1 \otimes \rho_1^{ex}(q) dq} = \frac{\int_{\Omega} C_{2}^{MK}(q) dq}{\int_{\Omega} \rho_1 \otimes \rho_1^{ex}(q) dq},\]

where \(k_{2}^{th}\) is a function supplied by theory while \(\rho_1 \otimes \rho_1^{ex}\) is taken directly from experiment. Thus \(\Delta K_{2}^{MK}\) by construction satisfies both the theoretical function \(k_{2}^{th}\) and the experimental one-particle distributions. Drawing \(k_{2}^{th}\) into the event averages of the correlation integral prescription [8], the unnormalized MK cumulant becomes

\[C_{2}^{MK}(q) = \langle \langle \sum_{i,j} \delta(q - Q_{ij}^a) k_{2}^{th}(Q_{ij}^a) \rangle \rangle,\]

so that the argument of the theoretical \(k_{2}^{th}\) is now an experimental particle pair distance, obtained by event mixing. Prescriptions such as [3] and [4] below amount to Monte Carlo-style sampling of the theoretical function weighted by the actual experimental particle distributions, thereby earning Monte Karli its name. Integrated over the bin domain \(\Omega\), the normalized MK cumulant reads

\[\Delta K_{2}^{MK}(\Omega) = \frac{\langle \langle \sum_{i,j} \int_{\Omega} dq \delta(q - Q_{ij}^a) k_{2}^{th}(Q_{ij}^a) \rangle \rangle}{\langle \langle \sum_{i,j} \int_{\Omega} dq \delta(q - Q_{ij}^a) \rangle \rangle},\]
where binning of track pairs is represented by \( \int dq \delta(q - Q_{ij}^{ab}) \); whenever a pair’s \( Q_{ij}^{ab} \) falls within \( \Omega \), the numerator counts \( k_2^{th}(Q_{ij}^{ab}) \) while the denominator counts 1.

Similarly, the unnormalized third-order Monte Karli cumulant \( C_3^{MK}(q) \) can be written in terms of \( k_3^{th}(Q_{ij}^{ab}) \) times \( \rho_1 \otimes p_1 \otimes \rho_2 \). For the projection onto the single variable \( q \), different “topologies” exist which quantify the size of a given cluster of \( r \) particles in phase space \( \Omega \). The “GHP max” prescription used here bins cluster sizes according to the maximum 4-momentum difference. Inserting this into the correlation integral prescription, one finds

\[
C_3^{MK}(q) = \langle \sum_{i,j,k} \delta[q - \max(Q_{ij}, Q_{jk}, Q_{ki})] \rangle, \\
k_3^{th}(Q_{ij}^{ab}, Q_{jk}^{bc}, Q_{ki}^{ca}),
\]

and the normalized binned MK cumulant becomes

\[
\Delta K_3^{MK}(\Omega) = \frac{\langle \sum_{i,j,k} \int dq \delta[q - \max(\cdots)] k_3^{th}(Q_{ij}^{ab}, Q_{jk}^{bc}, Q_{ki}^{ca}) \rangle}{\langle \sum_{i,j,k} \int dq \delta[q - \max(Q_{ij}^{ab}, Q_{jk}^{bc}, Q_{ki}^{ca})] \rangle}.
\]

Multiple event averages are handled efficiently within a “reduced event mixing” algorithm using unbiased estimators, cf. Eggers and Lipa in [2]. Because the identical projection is applied to both experimental and theoretical cumulants, their comparison is not subject to any approximation or limitation. Clearly, the MK procedure includes, besides the free parameters \( \lambda \) and \( r \) (or \( \alpha \)), an overall additive constant \( A \) as free parameter. Such additive constants are necessary for our choice of normalization because of the non-Poissonian nature of UA1 data, which leads to nonvanishing cumulants at large \( q \). Best-fit parameter values obtained were \( \lambda = 0.505 \pm 0.005 \), \( \alpha = 0.64 \pm 0.03 \), \( A = 0.269 \pm 0.012 \) for the APW power law, and \( \lambda = 0.59 \pm 0.04 \), \( r = 1.16 \pm 0.02 \) for the APW exponential. The Gaussian parameters are \( \lambda = 0.23 \pm 0.01 \), \( r = 0.81 \pm 0.02 \) for the APW power, exponential and Gaussian respectively.

We have measured second- and third-order normalized cumulants using a sample of around 160,000 non-single-diffractive events at 630 GeV/c recorded in the UA1 central detector. Only vertex-associated charged tracks with transverse momentum \( p_\perp \geq 0.15 \) GeV/c, pseudorapidity \( |\eta| \leq 3 \) and good measurement quality were used. We restricted our analysis to the azimuthal angle region \( 45^\circ \leq |\phi| \leq 135^\circ \) because there acceptance corrections are small enough to be safely neglected. To calculate the four momentum difference \( q \), all particles were assumed to be pions. The resolution \( \Delta q \approx 8 \) MeV, estimated from the errors of track fits, is approximately constant over the whole range \( q \leq 1 \) GeV. For further details of the detector and pair selection criteria, see Ref. [11].

In the following, we check the APW formulae (6)–(10) for consistency with the data. The procedure is to find numerical values for \( r \) (or \( \alpha \)) and \( \lambda \) by fitting the model’s second-order prediction (11) for \( k_2 \) to the experimentally measured differential cumulant. These best-fit values are then compared to third-order data using (13).

In Figure 1, we show the second-order cumulant of like-sign particles \( \Delta K_2 = \langle f \rho_2/\int p_1 \otimes p_1 \rangle - 1 \), where numerator and denominator are integrals over bins spaced logarithmically between \( q = 1 \) GeV and 30 MeV. Fits to the data were performed using the three parameterizations (11)–(13) in the APW form (11). All fits shown include, besides the free parameters \( \lambda \) and \( r \) (or \( \alpha \)), an overall additive constant \( A \) as free parameter. Such additive constants are necessary for our choice of normalization. Fits shown include, besides the free parameters \( \lambda \) and \( r \) (or \( \alpha \)), an overall additive constant \( A \) as free parameter. Such additive constants are necessary for our choice of normalization because of the non-Poissonian nature of UA1 data, which leads to nonvanishing cumulants at large \( q \). Best-fit parameter values obtained were \( \lambda = 0.505 \pm 0.005 \), \( \alpha = 0.64 \pm 0.03 \), \( A = 0.269 \pm 0.012 \) for the APW power law, and \( \lambda = 0.59 \pm 0.04 \), \( r = 1.16 \pm 0.02 \) for the APW exponential. The Gaussian parameters are \( \lambda = 0.23 \pm 0.01 \), \( r = 0.81 \pm 0.02 \) for the APW power, exponential and Gaussian respectively.

For \( \Delta K_3 \), we have performed three separate consistency checks, two based on approximations (shown in [8]), the third involving the Monte Karli method (shown below) which is exact. In Ref. [8], the APW predictions for \( \Delta K_3 \) based on the two approximations were found to differ substantially from the data.

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1 It should be remarked that previous work has shown the utility of using logarithmic rather than linear binning: much of what is interesting in correlations happens at small \( q \), and this region is probed better by using logarithmic bins.
In Figure 2, the results of implementing the MK prescription are shown, using the GHP max topology and the parameterizations (7) and (8). Parameter values used for the respective power law and exponential MK points were taken from the fit to $\Delta K^2$ of Figure 1. (The effect of finite binning was checked by inserting these parameter values back into the MK formulae for $\Delta K^2$ and finding good agreement between UA1 data and MK output.) Again, all MK points shown are determined only up to an additive constant, so that they may be shifted up and down in unison. It is clear, though, that the shape of the measured third-order cumulant differs significantly from that predicted by the APW formulae and $\Delta K^2$ parameter values. Similar discrepancies were found when using other topologies such as the GHP sum variable.

This result does not contradict an earlier UA1 paper whose higher order moment data was claimed to be in agreement with the APW formalism. The reason is that moments are numerically dominated by a combinatoric background of lower orders, so that comparisons based on moments are not very sensitive. Higher-order cumulants are a much more sensitive test of the theory, as we have just shown.

We have considered the following sources of errors:

- No corrections for Coulomb repulsion were included. However, $\Delta K^2$ data rise more strongly than theoretical predictions even for large $q$ (several hundred MeV) where Coulomb repulsion is not expected to be important.
- We have checked through additional small-$q$ cuts that possible residual track mismatching (multiple counting of tracks) does not explain the discrepancy between predicted and observed $\Delta K_3$.
- Our sample consists of 15% unidentified kaons and protons which were treated as pions in our analysis. This impurity is expected to weaken the correlations. We have repeated the analysis with a low-momentum ($|p| \leq 0.6 \text{ GeV/c}$) sample of particles by using the information on $dE/dx$ to get an almost pure sample of pions. No reduction of the discrepancy between the predicted and observed $\Delta K_3$ was observed.
- We have checked whether the restriction to the azimuthal angle region $45^\circ \leq |\phi| \leq 135^\circ$ distorts $\Delta K^2$ or $\Delta K_3$. In the region $q < 1 \text{ GeV}$, no distortion is observed.

Nevertheless, we do not regard our result as a failure of the APW formalism per se. One caveat relates to the structure of the UA1 multiplicity distribution which is non-Poissonian (i.e. the cumulants are nonzero at large $q$). The use of an additive constant in the cumulants is only an approximate remedy which can be improved substantially.

Furthermore, we assumed the chaoticity $\lambda$ to be momentum-independent and the source currents real-valued, as has been done implicitly in all previous experimental work known to us. More general versions of the APW theory have yet to be tested. Indeed, we hope that with our sensitive new testing methods the role of such hitherto inaccessible dynamical information can be investigated. Beyond such extensions there may lurk a non-Gaussian source current.

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