Periodic solutions of wave equations for asymptotically full measure sets of frequencies

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1 Introduction

The aim of this Note is to prove existence and multiplicity of small amplitude periodic solutions of the completely resonant wave equation

\[
\begin{align*}
\Box u + f(x, u) &= 0 \\
u(t, 0) &= u(t, \pi) = 0
\end{align*}
\]

where \(\Box := \partial_{tt} - \partial_{xx}\) is the D’Alambertian operator and

\[
f(x, u) = a_2 u^2 + a_3(x)u^3 + O(u^4) \quad \text{or} \quad f(x, u) = a_4 u^4 + O(u^5)
\]

for a Cantor-like set of frequencies \(\omega\) of asymptotically full measure at \(\omega = 1\).

Equation (1) is called completely resonant because any solution \(v = \sum_{j \geq 1} a_j \cos(j t + \vartheta_j) \sin(j x)\) of the linearized equation at \(u = 0\)

\[
\begin{align*}
u_{tt} - u_{xx} &= 0 \\
u(t, 0) &= u(t, \pi) = 0
\end{align*}
\]

is \(2\pi\)-periodic in time.

Existence and multiplicity of periodic solutions of completely resonant wave equations had been proved for a zero measure, uncountable Cantor set of frequencies in [4] for \(f(u) = u^3 + O(u^5)\) and in [5]-[6] for any nonlinearity \(f(u) = a_p u^p + O(u^{p+1})\), \(p \geq 2\).

Existence of periodic solutions for a Cantor-like set of frequencies of asymptotically full measure has been recently proved in [7] where, due to the well known “small divisor difficulty”, the “0th order bifurcation equation” is required to possess non-degenerate periodic solutions. Such property was verified in [7] for nonlinearities like \(f = a_2 u^2 + O(u^4)\), \(f = a_3(x)u^3 + O(u^4)\). See also [11] for \(f = u^3 + O(u^5)\).

In this Note we shall prove that, for quadratic, cubic and quartic nonlinearities \(f(x, u)\) like in (2), the corresponding 0th order bifurcation equation possesses non-degenerate periodic solutions – Propositions 1 and 2 –, implying, by the results of [7], Theorem 1 and Corollary 1 below.

We remark that our proof is purely analytic (it does not use numerical calculations) being based on the analysis of the variational equation and exploiting properties of the Jacobi elliptic functions.

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1.1 Main results

Normalizing the period to $2\pi$, we look for solutions of
\[
\begin{cases}
\omega^2 u_{tt} - u_{xx} + f(x, u) = 0 \\
u(t, 0) = u(t, \pi) = 0
\end{cases}
\]
in the Hilbert algebra (for $s > 1/2$, $\sigma > 0$)
\[
X_{\sigma, s} := \left\{ u(t, x) = \sum_{l \geq 0} \cos(lt) \, u_l(x) \mid u_l \in H^1_0((0, \pi), \mathbb{R}) \, \forall l \in \mathbb{N} \text{ and } \|u\|_{2, s, \sigma}^2 := \sum_{l \geq 0} \exp(2\sigma l)(l^2 + 1)\|u_l\|_{H^1_0}^2 < +\infty \right\}.
\]

It is natural to look for solutions which are even in time because equation (1) is reversible. We look as well for solutions of (1) in the subalgebras
\[
X_{\sigma, s, n} := \left\{ u \in X_{\sigma, s} \mid u \text{ is } \frac{2\pi}{n}\text{-periodic} \right\} \subset X_{\sigma, s}, \quad n \in \mathbb{N}
\]
they are particular $2\pi$-periodic solutions).

The space of the solutions of the linear equation (3) that belong to $H^1_0(\mathbb{T} \times (0, \pi), \mathbb{R})$ and are even in time is
\[
V := \left\{ v(t, x) = \sum_{l \geq 1} \cos(\bar{l}t) \sin(lx) \mid u_l \in \mathbb{R}, \sum_{l \geq 1} l^2|u_l|^2 < +\infty \right\}
\]
\[
= \left\{ v(t, x) = \eta(t + x) - \eta(t - x) \mid \eta \in H^1(\mathbb{T}, \mathbb{R}) \text{ with } \eta \text{ odd} \right\}.
\]

**Theorem 1.** Let
\[
f(x, u) = a_2u^2 + a_3(x)u^3 + \sum_{k \geq 4} a_k(x)u^k
\]
where $(a_2, \langle a_3 \rangle) \neq (0, 0)$, $\langle a_3 \rangle := \pi^{-1}\int_0^\pi a_3(x)dx$ or
\[
f(x, u) = a_4u^4 + \sum_{k \geq 5} a_k(x)u^k
\]
where $a_4 \neq 0$, $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$. Assume moreover $a_k(x) \in H^1((0, \pi), \mathbb{R})$ with $\sum_k \|a_k\|_{H^1}\rho^k < +\infty$ for some $\rho > 0$.

Then there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there is $\delta_0 > 0$, $\bar{\delta} > 0$ and a $C^\infty$-curve $[0, \delta_0) \ni \delta \mapsto u_\delta \in X_{\sigma, s, n}$ with the following properties:

- (i) $\|u_\delta - \delta \bar{v}_n\|_{\sigma/2, s, n} = O(\delta^2)$ for some $\bar{v}_n \in V \cap X_{\sigma, s, n} \setminus \{0\}$ with minimal period $2\pi/n$;
- (ii) there exists a Cantor set $C_n \subset [0, \delta_0)$ of asymptotically full measure, i.e. satisfying
\[
\lim_{\varepsilon \to 0^+} \frac{\text{meas}(C_n \cap (0, \varepsilon))}{\varepsilon} = 1,
\]
such that, $\forall \delta \in C_n$, $u_\delta(\omega(\delta)t, x)$ is a $2\pi/(\omega(\delta)n)$-periodic, classical solution of (1) with $\omega(\delta) = \begin{cases} \sqrt{1 - 2s^2\delta^2} & \text{if } f \text{ is like in } (\mathbb{A}) \\ \sqrt{1 - 2\delta^2} & \text{if } f \text{ is like in } (\mathbb{B}) \end{cases}$ and
\[
s^* = \begin{cases} -1 & \text{if } \langle a_3 \rangle \geq \pi^2a_3^2/12 \\ \pm 1 & \text{if } 0 < \langle a_3 \rangle < \pi^2a_3^2/12 \\ 1 & \text{if } \langle a_3 \rangle \leq 0. \end{cases}
\]

\footnote{Note how the interaction between the second and the third order terms $a_2u^2$, $a_3(x)u^3$ changes the bifurcation diagram, i.e. existence of periodic solutions for frequencies $\omega$ less or/and greater of $\omega = 1$.}
By (6) also each Cantor-like set of frequencies \( W_n := \{ \omega(\delta) \mid \delta \in C_n \} \) has asymptotically full measure at \( \omega = 1 \).

**Corollary 1. (Multiplicity)** There exists a Cantor-like set \( W \) of asymptotically full measure at \( \omega = 1 \), such that \( \forall \omega \in C \), equation (11) possesses geometrically distinct periodic solutions

\[
u_{n_0}, \ldots, u_n, \ldots u_{N_\omega}, \quad N_\omega \in \mathbb{N}
\]

with the same period \( 2\pi/\omega \). Their number increases arbitrarily as \( \omega \) tends to 1:

\[
\lim_{\omega \to 1} N_\omega = +\infty.
\]

**Proof.** The proof is like in [7] and we report it for completeness. If \( \delta \) belongs to the asymptotically full measure set (by (6))

\[
D_n := C_{n_0} \cap \ldots \cap C_n, \quad n \geq n_0
\]

there exist \( (n - n_0 + 1) \) geometrically distinct periodic solutions of (11) with the same period \( 2\pi/\omega(\delta) \) (each \( u_n \) has minimal period \( 2\pi/\omega(\delta) \)).

There exists a decreasing sequence of positive \( \varepsilon_n \to 0 \) such that

\[
\text{meas}(D_n \cap (0, \varepsilon_n)) \leq \varepsilon_n 2^{-n}.
\]

Let define the set \( C \equiv D_n \) on each \([\varepsilon_{n+1}, \varepsilon_n]\). \( C \) has asymptotically full measure at \( \delta = 0 \) and for each \( \delta \in C \) there exist \( N(\delta) := \max\{n \in \mathbb{N} : \delta < \varepsilon_n \} \) geometrically distinct periodic solutions of (11) with the same period \( 2\pi/\omega(\delta) \). \( N(\delta) \to +\infty \) as \( \delta \to 0 \).

**Remark 1.** Corollary 1 is an analogue for equation (11) of the well known multiplicity results of Weinstein-Moser [15]-[13] and Fadell-Rabinowitz [10] which hold in finite dimension. The solutions form a sequence of functions with increasing norms and decreasing minimal periods. Multiplicity of solutions was also obtained in [7] (with the "optimal" number \( N_\omega \approx C/\sqrt{\vert \omega - 1 \vert} \)) but only for a zero measure set of frequencies.

The main point for proving Theorem 1 relies in showing the existence of non-degenerate solutions of the 0th order bifurcation equation for \( f \) like in (2). In these cases the 0th order bifurcation equation involves higher order terms of the nonlinearity, and, for \( n \) large, can be reduced to an integro-differential equation (which physically describes an averaged effect of the nonlinearity with Dirichlet boundary conditions).

**Case** \( f(x,u) = a_4u^4 + O(u^5) \). Performing the rescaling

\[
u \to \delta u, \quad \delta > 0
\]

we look for \( 2\pi/\nu \)-periodic solutions in \( X_{\sigma,s,n} \) of

\[
\begin{cases}
\omega^2 u_{tt} - u_{xx} + \delta^3 g(\delta, x, u) = 0 \\
u(t, 0) = u(t, \pi) = 0
\end{cases}
\]

(7)

where

\[
g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^4} = a_4u^4 + \delta a_5(x)u^5 + \delta^2 a_6(x)u^6 + \ldots.
\]

To find solutions of (11) we implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

\[
X_{\sigma,s,n} = (V_n \cap X_{\sigma,s,n}) \oplus (W \cap X_{\sigma,s,n})
\]

where

\[
V_n := \left\{ v(t, x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(T, \mathbb{R}) \text{ with } \eta \text{ odd} \right\}
\]
and
\[ W := \left\{ w = \sum_{l \geq 0} \cos(l \pi) w_l(x) \in X_{0,s} \mid \int_0^\pi w_l(x) \sin(l x) \, dx = 0, \forall l \geq 0 \right\}. \]

Looking for solutions \( u = v + w \) with \( v \in V_n \cap X_{\sigma,s,n}, \ w \in W \cap X_{\sigma,s,n} \), and imposing the frequency-amplitude relation
\[ \frac{(\omega^2 - 1)}{2} = -\delta^6 \]
we are led to solve the bifurcation equation and the range equation
\[
\begin{align*}
\Delta v &= \delta^{-3} \Pi_{V_n} g(\delta, x, v + w) \\
L_\omega w &= \delta^3 \Pi_{W_n} g(\delta, x, v + w)
\end{align*}
\]
where
\[ \Delta v := v_{xx} + v_{tt}, \quad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx} \]
and \( \Pi_{V_n} : X_{\sigma,s,n} \to V_n \cap X_{\sigma,s,n}, \Pi_{W_n} : X_{\sigma,s,n} \to W \cap X_{\sigma,s,n} \) denote the projectors.

With the further rescaling
\[ w \to \delta^3 w \]
and since \( v^4 \in W_n \) (Lemma 3.4 of \([9]\)), \( a_5(x)v^5, a_6(x)v^6, a_7(x)v^7 \in W_n \) because \( a_5(\pi - x) = -a_5(x), \ a_6(\pi - x) = a_6(x), \ a_7(\pi - x) = -a_7(x) \) (Lemma 7.1 of \([7]\)), system \( \mathcal{S} \) is equivalent to
\[
\begin{align*}
\Delta v &= \Pi_{V_n} \left( 4a_4v^3w + \delta r(\delta, x, v, w) \right) \\
L_\omega w &= a_4v^4 + \delta \Pi_{W_n} \tilde{r}(\delta, x, v, w)
\end{align*}
\]
where \( r(\delta, x, v, w) = a_5(\delta)v^8 + 5a_5(\delta)v^4w + O(\delta) \) and \( \tilde{r}(\delta, x, v, w) = a_5(\delta)v^5 + O(\delta) \).

For \( \delta = 0 \) system \( \mathcal{S} \) reduces to \( w = -a_4 \square^{-1} v^4 \) and to the 0th order bifurcation equation
\[ \Delta v + 4a_4^2 \Pi_{V_n} (v^3 \square^{-1} v^4) = 0 \]
which is the Euler-Lagrange equation of the functional \( \Phi_0 : V_n \to \mathbb{R} \)
\[ \Phi_0(v) = \frac{||v||^2_{H^1}}{2} - \frac{a_4^2}{2} \int_\Omega v^4 \square^{-1} v^4 \]
where \( \Omega := \mathbb{T} \times (0, \pi) \).

**Proposition 1.** Let \( a_4 \neq 0 \). \( \exists n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0 \) the 0th order bifurcation equation \( \mathcal{S} \) has a solution \( \tilde{v}_n \in V_n \) which is non-degenerate in \( V_n \) (i.e. \( \text{Ker}D^2 \Phi_0 = \{0\} \)), with minimal period \( 2\pi/n \).

**Case** \( f(x, u) = a_2u^2 + a_3(u)^3 + O(u^4) \). Performing the rescaling \( u \to \delta u \) we look for \( 2\pi/n \)-periodic solutions of
\[
\begin{align*}
\omega^2 u_{tt} - u_{xx} + \delta g(\delta, x, u) = 0 \\
u(t, 0) = u(t, \pi) = 0
\end{align*}
\]
where
\[ g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^2} = a_2u^2 + \delta a_3(u)u^3 + \delta^2 u_4(x)u^4 + \ldots. \]

With the frequency-amplitude relation
\[ \frac{\omega^2 - 1}{2} = -s^* \delta^2 \]
where \( s^* = \pm 1 \), we have to solve
\[
\begin{align*}
-\Delta v &= -s^* \delta^{-1} \Pi_{V_n} g(\delta, x, v + w) \\
L_\omega w &= \delta \Pi_{W_n} g(\delta, x, v + w).
\end{align*}
\]
With the further rescaling $w \to \delta w$ and since $\nu^2 \in W_n$, system (12) is equivalent to

\[
\begin{align*}
-\Delta v &= s^* \Pi_{V_n} \left( -2a_2 \nu^2 - a_2 \delta v^2 - a_4(x)(v + \delta \nu)^3 - \delta r(\delta, x, v + \delta \nu) \right) \\
L_\omega w &= a_2 \nu^2 + \delta \Pi_{V_n} \left( 2a_2 \nu^2 + 2a_3(x)(v + \delta \nu)^3 + 8(\nu^2) dr(\delta, x, v + \delta \nu) \right)
\end{align*}
\]

(13)

where $r(\delta, x, u) := \delta^{-4}[f(x, \delta u) - a_2 \delta^2 u^2 - \delta^3 a_3(x) u^3] = a_4(x) u^4 + \ldots$. For $\delta = 0$ system (13) reduces to $w = -a_2 \nu^{-1} v^2$ and the 0th order bifurcation equation

\[
-s^* \Delta v = 2a_2^2 \Pi_{V_n} (v^{\circ -1} v^2) - \Pi_{V_n} (a_3(x) v^3)
\]

(14)

which is the Euler-Lagrange equation of $\Phi_0 : V_n \to \mathbb{R}$

\[
\Phi_0(v) := s^* \frac{\|v\|_{H^1}^2}{2} - a_2^2 \frac{\Omega v^2}{2} + \frac{1}{4} \int_\Omega a_3(x) v^4.
\]

(15)

\textbf{Proposition 2.} Let $(a_2, (a_3)) \neq 0$. \exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation \textbf{(14)} has a solution $\bar{v}_n \in V_n$ which is non-degenerate in $V_n$, with minimal period $2\pi/n$.

\section{Case $f(x, u) = a_4 u^4 + O(u^5)$}

We have to prove the existence of \textit{non-degenerate} critical points of the functional

\[
\Phi_n : V \to \mathbb{R}, \quad \Phi_n(v) := \Phi_0 (\mathcal{H}_n v)
\]

where $\Phi_0$ is defined in (11). Let $\mathcal{H}_n : V \to V$ be the linear isomorphism defined, for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$, by

\[
a_8(x) v^8 (\mathcal{H}_n v)(t, x) := \eta(n(t + x)) - \eta(n(t - x))
\]

so that $V_n \equiv \mathcal{H}_n V$.

\textbf{Lemma 1.} See [6]. $\Phi_n$ has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$

\[
\Phi_n (\beta n^{1/3} \nu) = 4\pi^2 n^{8/3} \left[ \Psi(\eta) + o \frac{\mathcal{R}(\eta)}{n^2} \right],
\]

(16)

where $\beta := (3/(\pi^2 a_4^2))^{1/6}$, $\alpha := a_2^2/(8\pi)$,

\[
\Psi(\eta) := \frac{1}{2} \int_T \eta^2(t) \, dt - \frac{2\pi}{8} \left( \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)^2,
\]

(17)

$\langle \cdot \rangle$ denotes the average on $\mathbb{T}$, and

\[
\mathcal{R}(\eta) := -\int_\Omega v^4 \nabla^{-1} v^4 \, dt = \frac{\pi^4}{6} \left( \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)^2.
\]

(18)

\textbf{Proof.} Firstly the quadratic term writes

\[
\frac{1}{2} \|\mathcal{H}_n v\|_{H^1}^2 = \frac{n^2}{2} \|v\|_{H^1}^2 = n^2 \pi \int_T \eta^2(t) \, dt.
\]

(19)

By Lemma 4.8 in [6] the non-quadartic term can be developed as

\[
\int_\Omega (\mathcal{H}_n v)^4 \nabla^{-1} (\mathcal{H}_n v)^4 = \frac{\pi^4}{6} \langle m \rangle^2 - \frac{\mathcal{R}(\eta)}{n^2}
\]

(20)

where $m : \mathbb{T}^2 \to \mathbb{R}$ is $m(s_1, s_2) := (\eta(s_1) - \eta(s_2))^4$, $\langle m \rangle := (2\pi)^{-2} \int_{\mathbb{T}^2} m(s_1, s_2) \, ds_1 \, ds_2$ denotes its average, and

\[
\mathcal{R}(\eta) := -\int_\Omega v^4 \nabla^{-1} v^4 + \frac{\pi^4}{6} \langle m \rangle^2.
\]

(21)
is homogeneous of degree 8. Since $\eta$ is odd we find
\[
\langle m \rangle = 2\left(\langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2\right)
\]  
(22)
where $\langle \cdot \rangle$ denotes the average on $\mathbb{T}$.

Collecting (14), (20), (21) and (22) we find out
\[
\Phi_n(\eta) = 2\pi n^2 \int_\mathbb{T} \eta^2(t) \, dt - \frac{\pi^4}{3} A^2 \left(\langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2\right) + \frac{\alpha_1^2}{2n^2} \mathcal{R}(\eta).
\]

Via the rescaling $\eta \to \beta n^{1/3} \eta$ we get the expressions (17) and (18). $\blacksquare$

By (10), in order to find for $n$ large enough a non-degenerate critical point of $\Phi_n$, it is sufficient to find a non-degenerate critical point of $\Psi(\eta)$ defined on
\[
E := \{ \eta \in H^1(\mathbb{T}), \eta \text{ odd}\},
\]
namely non-degenerate solutions in $E$ of
\[
\dddot{\eta} + A(\eta) \left(3\langle \eta^2 \rangle \eta + \eta^3\right) = 0 \quad A(\eta) := \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2.
\]  
(23)

**Proposition 3.** There exists an odd, analytic, $2\pi$-periodic solution $g(t)$ of (23) which is non-degenerate in $E$. $g(t) = V \mathrm{sn}(\Omega t, m)$ where $\mathrm{sn}$ is the Jacobi elliptic sine and $V > 0$, $\Omega > 0$, $m \in (-1, 0)$ are suitable constants (therefore $g(t)$ has minimal period $2\pi$).

We will construct the solution $g$ of (23) by means of the Jacobi elliptic sine in Lemma 6. The existence of a solution $g$ follows also directly applying to $\Psi: E \to \mathbb{R}$ the Mountain-Pass Theorem [2]. Furthermore such solution is an analytic function arguing as in Lemma 2.1 of [1].

### 2.1 Non-degeneracy of $g$

We now want to prove that $g$ is non-degenerate. The linearized equation of (23) at $g$ is
\[
\dddot{h} + 3A(g) \left(g^2 h + g^2 h\right) + 6A(g)g(gh) + A'(g)[h] \left(3\langle g^2 \rangle g + g^3\right) = \ddot{h} + 3A(g) \left(g^2 + g^2\right) h + 6g(gh) \left(\langle g^4 \rangle + 3\langle g^2 \rangle^2\right) + 4g \left(\langle g^3 h \rangle + 3\langle g^2 \rangle (gh)\right) \left(3\langle g^2 \rangle + g^2\right) = 0
\]
that we write as
\[
\dddot{h} + 3A(g) \left(g^2 + g^2\right) h = -\langle gh \rangle I_1 - \langle g^3 h \rangle I_2
\]  
(24)
where
\[
\begin{align*}
I_1 & := 6\left(9\langle g^2 \rangle^2 + \langle g^4 \rangle\right) g + 12\langle g^2 \rangle g^3 \\
I_2 & := 12g\langle g^2 \rangle + 4g^3.
\end{align*}
\]  
(25)
For $f \in E$, let $H := L(f)$ be the unique solution belonging to $E$ of the non-homogeneous linear system
\[
\dddot{h} + 3A(g) \left(g^2 + g^2\right) h = f;
\]  
(26)
an integral representation of the Green operator $L$ is given in Lemma 1. Thus (24) becomes
\[
h = -\langle gh \rangle L(I_1) - \langle g^3 h \rangle L(I_2).
\]  
(27)

Multiplying (24) by $g$ and taking averages we get
\[
\langle gh \rangle \left[1 + \langle gL(I_1)\rangle\right] = -\langle g^3 h \rangle \langle gL(I_2)\rangle,
\]  
(28)
while multiplying (27) by $g^3$ and taking averages
\[
\langle g^3 h \rangle \left[1 + \langle g^3 L(I_2)\rangle\right] = -\langle gh \rangle \langle g^3 L(I_1)\rangle.
\]  
(29)
Since $g$ solves (23) we have the following identities.
Lemma 2. There holds

\[ 2A(g)\langle g^3 L(g) \rangle = \langle g^2 \rangle \quad (30) \]
\[ 2A(g)\langle g^3 L(g^3) \rangle = \langle g^4 \rangle . \quad (31) \]

Proof. \[ (30) \] is obtained by the identity for \( L(g) \)

\[ \frac{d^2}{dt^2}(L(g)) + 3A(g) \left( \langle g^2 \rangle + g^2 \right) L(g) = g \]

multiplying by \( g \), taking averages, integrating by parts,

\[ \langle \tilde{g}L(g) \rangle + 3A(g) \left( \langle g^2 \rangle L(g) + \langle g^3 L(g) \rangle \right) = \langle g^2 \rangle \]

and using that \( g \) solves \[ (23) \].

Analogously, \[ (31) \] is obtained by the identity for \( L(g^3) \)

\[ \frac{d^2}{dt^2}(L(g^3)) + 3A(g) \left( \langle g^2 \rangle + g^2 \right) L(g^3) = g^3 \]

multiplying by \( g \), taking averages, integrating by parts, and using that \( g \) solves \[ (23) \]. \[ \Box \]

Since \( L \) is a symmetric operator we can compute the following averages using \[ (25), (30), (31) \):

\[
\begin{cases}
\langle gL(I_1) \rangle = 6 \left( \langle g^4 \rangle + 9 \langle g^2 \rangle^2 \right) \langle gL(g) \rangle + 6 A(g)^{-1} \langle g^2 \rangle^2 \\
\langle gL(I_2) \rangle = 12\langle g^2 \rangle \langle gL(g) \rangle + 2 A(g)^{-1} \langle g^2 \rangle \\
\langle g^3 L(I_1) \rangle = 9\langle g^2 \rangle \\
\langle g^3 L(I_2) \rangle = 2 .
\end{cases}
\]

(32)

Thanks to the identities \[ (32) \], equations \[ (28), (29) \] simplify to

\[
\begin{cases}
\langle gh \rangle \left[ A(g) + 6 \langle g^2 \rangle^2 \right] B(g) = -2 \langle g^2 \rangle B(g) \langle g^3 h \rangle \\
\langle g^3 h \rangle = -3 \langle g^2 \rangle \langle gh \rangle 
\end{cases}
\]

(33)

where

\[ B(g) := 1 + 6A(g)\langle gL(g) \rangle . \]

(34)

Solving \[ (33) \] we get

\[ B(g)\langle gh \rangle = 0 . \]

We will prove in Lemma \[ 5 \] that \( B(g) \neq 0 \), so \( \langle gh \rangle = 0 \). Hence by \[ (33) \] also \( \langle g^3 h \rangle = 0 \) and therefore, by \[ (27) \], \( h = 0 \). This concludes the proof of the non-degeneracy of the solution \( g \) of \[ (23) \].

It remains to prove that \( B(g) \neq 0 \). The key is to express the function \( L(g) \) by means of the variation of constants formula.

We first look for a fundamental set of solutions of the homogeneous equation

\[ \ddot{h} + 3A(g) \left( \langle g^2 \rangle + g^2 \right) h = 0 . \]

(HOM)

Lemma 3. There exist two linearly independent solutions of \( (HOM) \), \( \tilde{u} := \dot{g}/\dot{g}(0) \) and \( \tilde{v} \), such that

\[
\begin{cases}
\tilde{u} \text{ is even, } 2\pi \text{ periodic} \\
\tilde{u}(0) = 1 , \; \dot{\tilde{u}}(0) = 0 \\
\tilde{v} \text{ is odd, not periodic} \\
\tilde{v}(0) = 0 , \; \dot{\tilde{v}}(0) = 1
\end{cases}
\]

and

\[ \tilde{v}(t + 2\pi) - \tilde{v}(t) = \rho \tilde{u}(t) \quad \text{for some } \rho > 0 . \]

(35)
PROOF. Since \( \varphi \) is autonomous, \( \dot{\varphi}(t) \) is a solution of the linearized equation (HOM). \( \dot{\varphi}(t) \) is even and 2\( \pi \)-periodic.

We can construct another solution of (HOM) in the following way. The super-quadratic Hamiltonian system (with constant coefficients)

\[
\ddot{y} + 3A(y)(y^2) y + A(y) y^3 = 0
\]  

(36)

possesses a one-parameter family of odd, \( T(E) \)-periodic solutions \( y(E, t) \), close to \( g \), parametrized by the energy \( E \). Let \( \bar{E} \) denote the energy level of \( g \), i.e. \( g = y(\bar{E}, t) \) and \( T(\bar{E}) = 2\pi \).

Therefore \( l(t) := (\partial_{E}y(E, t))_{|E=\bar{E}} \) is an odd solution of (HOM).

Deriving the identity \( y(E, t + T(E)) = y(E, t) \) with respect to \( E \) we obtain at \( E = \bar{E} \)

\[
l(t + 2\pi) - l(t) = -(\partial_{E}T(E))_{|E=\bar{E}} \dot{\varphi}(t)
\]

and, normalizing \( \tilde{v}(t) := l(t)/l(0) \), we get (37) with

\[
\rho := -(\partial_{E}T(E))_{|E=\bar{E}} \left( \frac{\dot{\varphi}(0)}{l(0)} \right).
\]

(37)

Since \( y(E, 0) = 0 \) \( \forall E \), the energy identity gives \( E = \frac{1}{2}(\dot{\varphi}(E, 0))^2 \). Deriving w.r.t \( E \) at \( E = \bar{E} \), yields \( 1 = \dot{\varphi}(0)l(0) \) which, inserted in (37), gives

\[
\rho = -(\partial_{E}T(E))_{|E=\bar{E}} (\dot{\varphi}(0))^2.
\]

(38)

\( \rho > 0 \) because \((\partial_{E}T(E))_{|E=\bar{E}} < 0 \) by the superquadraticity of the potential of (HOM). It can be checked also by a computation, see Remark after Lemma 4.

Now we write an integral formula for the Green operator \( L \).

**Lemma 4.** For every \( f \in \mathcal{E} \) there exists a unique solution \( H = L(f) \) of (26) which can be written as

\[
L(f) = \left( \int_{0}^{t} f(s) \bar{u}(s) \, ds \right) + \frac{1}{\rho} \int_{0}^{2\pi} f \bar{v}(s) \, ds \bar{v}(t) - \left( \int_{0}^{t} f(s) \bar{v}(s) \, ds \right) \bar{u}(t) \in \mathcal{E}.
\]

(39)

**Proof.** The non-homogeneous equation (26) possesses the particular solution

\[
\bar{H}(t) = \left( \int_{0}^{t} f(s) \bar{u}(s) \, ds \right) \bar{v}(t) - \left( \int_{0}^{t} f(s) \bar{v}(s) \, ds \right) \bar{u}(t)
\]

as can be verified noting that the Wronskian \( \bar{u}(t) \dot{\bar{v}}(t) - \dot{\bar{u}}(t) \bar{v}(t) \equiv 1, \forall t \). Notice that \( \bar{H} \) is odd.

Any solution \( \bar{H}(t) \) of (26) can be written as

\[
H(t) = \bar{H}(t) + a \bar{u} + b \bar{v}, \quad a, b \in \mathbb{R}.
\]

Since \( \bar{H} \) is odd, \( \bar{u} \) is even and \( \bar{v} \) is odd, requiring \( H \) to be odd, implies \( a = 0 \). Imposing now the 2\( \pi \)-periodicity yields

\[
0 = \left( \int_{0}^{t} f \bar{u} \right) \bar{v}(t + 2\pi) - \left( \int_{0}^{t} f \bar{v} \right) \bar{u}(t + 2\pi) - \left( \int_{0}^{t} f \bar{u} \right) \bar{v}(t) + \left( \int_{0}^{t} f \bar{v} \right) \bar{u}(t) + b \left( \bar{v}(t + 2\pi) - \bar{v}(t) \right)
\]

\[
= (b + \int_{0}^{t} f \bar{u}) \left( \bar{v}(t + 2\pi) - \bar{v}(t) \right) - \bar{u}(t) \left( \int_{0}^{t} f \bar{v} \right)
\]

(40)

using that \( \bar{u} \) and \( f \bar{u} \) are 2\( \pi \)-periodic and \( \langle f \bar{u} \rangle = 0 \). By (28) and (36) we get

\[
\rho \left( b + \int_{0}^{t} f \bar{u} \right) - \int_{0}^{t+2\pi} f \bar{v} = 0.
\]

(41)
The left hand side in (41) is constant in time because, deriving w.r.t. $t$,

$$\rho f(t)\bar{u}(t) - f(t)(\bar{v}(t + 2\pi) - \bar{v}(t)) = 0$$

again by (39). Hence evaluating (41) for $t = 0$ yields $b = \rho^{-1} \int_0^{2\pi} f\bar{v}$. So there exists a unique solution $H = L(f)$ of (26) belonging to $E$ and (39) follows. □

Finally

**Lemma 5.** There holds

$$\langle gL(g) \rangle = \frac{\rho}{4\pi A(g)} + \frac{1}{2\pi\rho} \left( \int_0^{2\pi} g\bar{v} \right)^2 > 0$$

because $A(g), \rho > 0$.

**Proof**. Using (39) we can compute

$$\langle gL(g) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t g\bar{u} \right) \bar{v}(t)g(t) dt + \frac{1}{2\pi\rho} \left( \int_0^{2\pi} g\bar{v} \right)^2 - \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t g\bar{v} \right) \bar{u}(t)g(t) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t g\bar{u} \right) \bar{v}(t)g(t) dt + \frac{1}{2\pi\rho} \left( \int_0^{2\pi} g\bar{v} \right)^2$$

(42)

because, by $\int_0^{2\pi} g\bar{u} = 0$, we have

$$0 = \int_0^{2\pi} \frac{d}{dt} \left[ \left( \int_0^t g\bar{v} \right) \left( \int_0^t g\bar{u} \right) \right] dt = \int_0^{2\pi} \left[ \left( \int_0^t g\bar{v} \right) \bar{u}(t)g(t) + \left( \int_0^t g\bar{u} \right) \bar{v}(t)g(t) \right] dt.$$

Now, since $\bar{u}(t) = \dot{g}(t)/\dot{g}(0)$ and $g(0) = 0$,

$$\frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t g\bar{u} \right) \bar{v}(t)g(t) dt = \frac{1}{2\pi\dot{g}(0)} \int_0^{2\pi} \left( \int_0^t g\bar{v} \right) \bar{u}(t)g(t) + \left( \int_0^t g\bar{u} \right) \bar{v}(t)g(t) dt = \frac{1}{4\pi\dot{g}(0)} \int_0^{2\pi} g^3\bar{v}. \quad (43)$$

We claim that

$$\int_0^{2\pi} g^3\bar{v} = \frac{\rho\dot{g}(0)}{2A(g)}. \quad (44)$$

By (22), (38), (44) we have the thesys.

Let us prove (44). Since $g$ solves (26) multiplying by $\bar{v}$ and integrating

$$\int_0^{2\pi} \bar{v}(t)\dot{g}(t) + 3A(g)(g^2)\bar{v}(t)g(t) + A(g)g^3(t)\bar{v}(t) dt = 0 \quad (45)$$

Next, since $\bar{v}$ solves (HOM), multiplying by $g$ and integrating

$$\int_0^{2\pi} g(t)\bar{v}(t) + 3A(g)(g^2)\bar{v}(t)g(t) + 3A(g)g^3(t)\bar{v}(t) dt = 0. \quad (46)$$

Subtracting (45) and (46), gives

$$\int_0^{2\pi} \bar{v}(t)\dot{g}(t) - g(t)\bar{v}(t) = 2A(g) \int_0^{2\pi} g^3\bar{v}. \quad (47)$$

Integrating by parts the left hand side, since $g(0) = g(2\pi) = 0$, $\bar{u}(0) = 1$ and (15), gives

$$\int_0^{2\pi} \bar{v}(t)\dot{g}(t) - g(t)\bar{v}(t) = \dot{g}(0)[\bar{v}(2\pi) - \bar{v}(0)] = \rho\dot{g}(0). \quad (48)$$

(47) and (48) give (44). □
2.2 Explicit computations

We now give the explicit construction of $g$ by means of the Jacobi elliptic sine defined as follows. Let $\text{am}(\cdot, m) : \mathbb{R} \to \mathbb{R}$ be the inverse function of the Jacobi elliptic integral of the first kind

$$\varphi \mapsto F(\varphi, m) := \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}.$$ 

The Jacobi elliptic sine is defined by

$$\text{sn}(t, m) := \sin(\text{am}(t, m)).$$

$\text{sn}(t, m)$ is $4K(m)$-periodic, where $K(m)$ is the complete elliptic integral of the first kind

$$K(m) := F\left(\frac{\pi}{2}, m\right) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}$$

and admits an analytic extension with a pole in $iK(1/(1-m))/\sqrt{1-m}$ for $m < 0$. Moreover, since

$$\partial_t \text{am}(t, m) = \sqrt{1 - m \text{sn}^2(t, m)},$$

the elliptic sine satisfies

$$(\dot{\text{sn}})^2 = (1 - \text{sn}^2)(1 - m \text{sn}^2).$$

(49)

Lemma 6. There exist $V > 0$, $\Omega > 0$, $m \in (-1, 0)$ such that $g(t) := V\text{sn}(\Omega t, m)$ is an odd, analytic, $2\pi$-periodic solution of (23) with pole in

$$\frac{iK(1/(1-m))/\sqrt{1-m}}{\Omega}. $$

Proof. Deriving (49) we have $\ddot{\text{sn}} + (1 + m)\text{sn} - 2m\text{sn}^3 = 0$. Therefore $g_{(V, \Omega, m)}(t) := V\text{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of

$$\ddot{g} + \Omega^2(1 + m)g - 2m\Omega^2 g^3 = 0.$$

(50)

The function $g_{(V, \Omega, m)}$ will be a solution of (23) if $(V, \Omega, m)$ verify

$$\begin{cases}
\Omega^2(1 + m) = 3A(g_{(V, \Omega, m)}) \langle g_{(V, \Omega, m)}^2 \rangle \\
-2m\Omega^2 = V^2 A(g_{(V, \Omega, m)}) \\
2K(m) = \Omega\pi.
\end{cases}$$

(51)

Dividing the first equation of (51) by the second one

$$-\frac{1 + m}{6m} = \langle \text{sn}^2(\cdot, m) \rangle.$$ 

(52)

The right hand side can be expressed as

$$\langle \text{sn}^2(\cdot, m) \rangle = \frac{K(m) - E(m)}{mK(m)}$$

(53)

where $E(m)$ is the complete elliptic integral of the second kind

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m \sin^2 \vartheta} d\vartheta = \int_0^{K(m)} 1 - m \sin^2(\xi, m) d\xi$$

(in the last passage we make the change of variable $\vartheta = \text{am}(\xi, m)$).

Now, we show that system (51) has a unique solution. By (52) and (53)

$$(7 + m)K(m) - 6E(m) = 0.$$ 

(54)
By the definitions of $E(m)$ and $K(m)$ we have

$$\psi(m) := (7 + m)K(m) - 6E(m) = \int_0^{\pi/2} \frac{1 + m(1 + 6 \sin^2 \vartheta)}{(1 - m \sin^2 \vartheta)^{1/2}} \, d\vartheta. \quad (55)$$

For $m = 0$ it holds $\psi(0) = \pi/2 > 0$ and, for $m = -1$, $\psi(-1) = -\int_0^{\pi/2} 6 \sin^2 \vartheta (1 + \sin^2 \vartheta)^{-1/2} \, d\vartheta < 0$. Since $\psi$ is continuous there exists a solution $\bar{m} \in (-1, 0)$ of \(51\). Next the third equation in \(51\) fix $\bar{\Omega}$ and finally we find $\bar{V}$. Hence $g(t) = \bar{V} \text{sn}(\bar{\Omega}t, \bar{m})$ solves \(56\).

Analyticity and poles follow from \[1\], 16.2, 16.10.2, pp.570,573.

At last, $\bar{m}$ is unique because $\psi'(m) > 0$ for $m \in (-1, 0)$ as can be verified by \(50\). One can also compute that $\bar{m} \in (-0.30, -0.28)$. \(\blacksquare\)

**Remark.** We can compute explicitly the sign of $dT/dE$ and $\rho$ of \(58\) in the following way.

The functions $g(v, \Omega, m)$ are solutions of the Hamiltonian system \(54\) imposing

$$\begin{cases}
\Omega^2(1 + m) = \alpha \\
-2m\Omega^2 = V^2\beta
\end{cases} \quad (56)$$

where $\alpha := 3A(g) < g^2)$, $\beta := A(g)$ and $g$ is the solution constructed in Lemma \(49\).

We solve \(51\) w.r.t $m$ finding the one-parameter family $(y_m)$ of odd periodic solutions $y_m(t) := V(m) \text{sn}(\Omega(t), m)$, close to $g$, with energy and period

$$E(m) = \frac{1}{2}V^2(m)\Omega^2(m) = -\frac{1}{\beta} m \Omega^4(m), \quad T(m) = \frac{4K(m)}{\Omega(m)}.$$

It holds

$$\frac{dT(m)}{dm} = \frac{4K'(m)\Omega(m) - 4K(m)\Omega'(m)}{\Omega^2(m)} > 0$$

because $K'(m) > 0$ and from \(50\) $\Omega'(m) = -\Omega(m)(2(1 + m))^{-1} < 0$. Then

$$\frac{dE(m)}{dm} = -\frac{1}{\beta} \Omega^4(m) - \frac{1}{\beta} m 4\Omega^3(m)\Omega'(m) < 0,$$

so

$$\frac{dT}{dE} = \frac{dT(m)}{dm} \left(\frac{dE(m)}{dm}\right)^{-1} < 0$$

as stated by general arguments in the proof of Lemma \(3\).

We can also write an explicit formula for $\rho$,

$$\rho = \frac{m}{m - 1} \left[2\pi + (1 + m) \int_0^{2\pi} \frac{\text{sn}^2(\Omega t, m)}{\text{dn}^2(\Omega t, m)} \, dt\right]. \quad (57)$$

From \(54\) it follows that $\rho > 0$ because $-1 < m < 0$.

### 3 Case $f(x, u) = a_2u^2 + a_3(x)u^3 + O(u^4)$

We have to prove the existence of non-degenerate critical points of the functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ where $\Phi_0$ is defined in \(15\).

**Lemma 7.** See \(1\). $\Phi_n$ has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$,

$$\Phi_n(\beta v) = 4\pi \beta^2 n^4 \left[\Psi(\eta) + \frac{\beta^2}{4\pi} \left(R_2(\eta) n^2 + R_3(\eta)\right)\right] \quad (58)$$
where
\[ \Psi(\eta) := \frac{s^4}{2} \int_T \dot{\eta}^2 + \frac{\beta^2}{4\pi} \left[ \alpha \left( \int_T \eta^2 \right)^2 + \gamma \int_T \eta^4 \right] \]
\[ R_2(\eta) := -\frac{a^2 \beta}{2} \left[ \int_T v^2 \phi v^2 - \frac{\pi^2}{6} \left( \int_T \eta^2 \right)^2 \right], \quad R_3(\eta) := \frac{1}{4} \int_T (a^3(x) - \langle a^3 \rangle)(H_n v)^4, \quad (59) \]
\[ \alpha := (9(a_3) - \pi^2 a_2^2)/12, \quad \gamma := \pi(a_3)/2, \text{ and} \]
\[ \beta = \begin{cases} (2|\alpha|)^{-1/2} & \text{if } \alpha \neq 0, \\ (\pi/\gamma)^{1/2} & \text{if } \alpha = 0. \end{cases} \]

**Proof.** By Lemma 4.8 in [8] with \( m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^2 \), for \( v(t, x) = \eta(t + x) - \eta(t - x) \) the operator \( \Phi_n \) admits the development
\[ \Phi_n(v) = 2\pi s^4 n^2 \int_T \dot{\eta}^2(t) dt - \frac{\pi^2 a^2}{12} \left( \int_T \eta^2(t) dt \right)^2 - \frac{a^2}{2n^2} \left( \int_T v^2 \phi v^2 - \frac{\pi^2}{6} \left( \int_T \eta^2(t) dt \right)^2 \right) + \frac{1}{4} \int_T v^4 + \frac{1}{4} \int_T (a^3(x) - \langle a^3 \rangle)(H_n v)^4. \]
Since
\[ \int_T v^4 = 2\pi \int_T \eta^4 + 3 \left( \int_T \eta^2 \right)^2, \]
we write
\[ \Phi_n(v) = 2\pi s^4 n^2 \int_T \dot{\eta}^2 + \frac{\pi^2 a^2}{12} \left( \int_T \eta^2 \right)^2 + \frac{1}{4} \langle a^3 \rangle \left[ 2\pi \int_T \eta^4 + 3 \left( \int_T \eta^2 \right)^2 \right] + \frac{R_2(\eta)}{n^2} + R_3(\eta), \]
where \( R_2, R_3 \) defined in (59) are both homogenous of degree 4. So
\[ \Phi_n(v) = 2\pi s^4 n^2 \int_T \dot{\eta}^2 + \alpha \left( \int_T \eta^2 \right)^2 + \gamma \int_T \eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta) \]
where \( \alpha, \gamma \) are defined above. With the rescaling \( \eta \to \eta \beta n \) we get decomposition (58). 

In order to find for \( n \) large a non-degenerate critical point of \( \Phi_n \), by (58) it is sufficient to find critical points of \( \Psi \) on \( E = \{ \eta \in H^1(\mathbb{T}), \eta \text{ odd} \} \) (like in Lemma 6.2 of [7] also the term \( R_3(\eta) \) tends to 0 with its derivatives).

If \( \langle a^3 \rangle \in (-\infty, 0) \cup (\pi^2 a_2^2/9, +\infty) \), then \( \alpha \neq 0 \) and we must choose \( s^4 = -\text{sign}(\alpha) \), so that the functional becomes
\[ \Psi(\eta) = \text{sign}(\alpha) \left( -\frac{1}{2} \int_T \dot{\eta}^2 + \frac{1}{8\pi} \left( \left( \int_T \eta^2 \right)^2 + \frac{\gamma}{\alpha} \int_T \eta^4 \right) \right). \]
Since in this case \( \gamma/\alpha > 0 \), the functional \( \Psi \) clearly has a mountain pass critical point, solution of
\[ \ddot{\eta} + \langle \eta^2 \rangle \eta + \lambda \eta^3 = 0, \quad \lambda = \frac{\gamma}{2\pi \alpha} > 0. \quad (60) \]
The proof of the non-degeneracy of the solution of (60) is very simple using the analytical arguments of the previous section (since \( \lambda > 0 \) it is sufficient a positivity argument).

If \( \langle a^3 \rangle = 0 \), then the equation becomes \( \ddot{\eta} + \langle \eta^2 \rangle \eta = 0 \), so we find again what proved in [7] for \( a^3(x) \equiv 0 \).

If \( \langle a^3 \rangle = \pi^2 a_2^2/9 \), then \( \alpha = 0 \). We must choose \( s^4 = -1 \), so that we obtain
\[ \Psi(\eta) = -\frac{1}{2} \int_T \dot{\eta}^2 + \frac{1}{4} \int_T \eta^4, \quad \ddot{\eta} + \eta^3 = 0. \]
This equation has periodic solutions which are non-degenerate because of non-isocronicity, see Proposition 2 in [8].
Finally, if \( (a_3) \in (0, \pi^2a_2^2/9) \), then \( \alpha < 0 \) and there are both solutions for \( s^* = \pm 1 \). The functional

\[
\Psi(\eta) = \frac{s^*}{2} \int_T \dot{\eta}^2 + \frac{1}{8\pi} \left[ -\left( \int_T \eta^2 \right)^2 + \frac{\gamma}{|\alpha|} \int_T \eta^4 \right] = \frac{s^*}{2} \int_T \dot{\eta}^2 + \frac{1}{4} \int_T \eta^4 \lambda - Q(\eta)
\]

where

\[
\lambda := \frac{\gamma}{2\pi|\alpha|} > 0, \quad Q(\eta) := \left( \frac{\int_T \eta^2}{2\pi \int_T \eta^4} \right)^2
\]

possesses Mountain pass critical points for any \( \lambda > 0 \) because (like in Lemma 3.14 of [6])

\[
\inf_{\eta \in E \setminus \{0\}} Q(\eta) = 0, \quad \sup_{\eta \in E \setminus \{0\}} Q(\eta) = 1
\]

(for \( \lambda \geq 1 \) if \( s^* = -1 \), and for \( 0 < \lambda < 1 \) for both \( s^* = \pm 1 \)).

Such critical points satisfy the Euler Lagrange equation

\[
-s^* \ddot{\eta} - \langle \eta^3 \rangle \dot{\eta} + \lambda \eta^3 = 0 \tag{61}
\]

but their non-degeneracy is not obvious. For this, it is convenient to express this solutions in terms of the Jacobi elliptic sine.

**Proposition 4.** (i) Let \( s^* = -1 \). Then for every \( \lambda \in (0, +\infty) \) there exists an odd, analytic, \( 2\pi \)-periodic solution \( g(t) \) of \( [61] \) which is non-degenerate in \( E \). \( g(t) = V \text{sn}(\Omega t, m) \) for \( V > 0 \), \( \Omega > 0 \), \( m \in (-\infty, -1) \) suitable constants.

(ii) Let \( s^* = 1 \). Then for every \( \lambda \in (0, 1) \) there exists an odd, analytic, \( 2\pi \)-periodic solution \( g(t) \) of \( [61] \) which is non-degenerate in \( E \). \( g(t) = V \text{sn}(\Omega t, m) \) for \( V > 0 \), \( \Omega > 0 \), \( m \in (0, 1) \) suitable constants.

We prove Proposition 4 in several steps. First we construct the solution \( g \) like in Lemma 6

**Lemma 8.** (i) Let \( s^* = -1 \). Then for every \( \lambda \in (0, +\infty) \) there exist \( V > 0 \), \( \Omega > 0 \), \( m \in (-\infty, -1) \) such that \( g(t) = V \text{sn}(\Omega t, m) \) is an odd, analytic, \( 2\pi \)-periodic solution of \( [61] \) with a pole in \( \Omega \sqrt{1-m} K \left( \frac{1}{1-m} \right) \).

(ii) Let \( s^* = 1 \). Then for every \( \lambda \in (0, 1) \) there exist \( V > 0 \), \( \Omega > 0 \), \( m \in (0, 1) \) such that \( g(t) = V \text{sn}(\Omega t, m) \) is an odd, analytic, \( 2\pi \)-periodic solution of \( [61] \) with a pole in \( iK(1-m)/\Omega \).

**Proof.** We know that \( g(t) := V \text{sn}(\Omega t, m) \) is an odd, \( (4K(m)/\Omega) \)-periodic solution of \( [50] \), see Lemma 6. So it is a solution of \( [61] \) if \( (V, \Omega, m) \) verify

\[
\begin{align*}
\Omega^2(1+m) &= s^*V^2 \langle \text{sn}^2 (\cdot, m) \rangle \\
2m\Omega^2 &= s^*V^2 \lambda \\
2K(m) &= \Omega \pi \cdot
\end{align*}
\]

Conditions [62] give the connection between \( \lambda \) and \( m \):

\[
\lambda = \frac{2m}{1+m} \langle \text{sn}^2 (\cdot, m) \rangle. \tag{63}
\]

Moreover system [62] imposes

\[
\begin{cases}
m \in (-\infty, -1) & \text{if } s^* = -1 \\
m \in (0, 1) & \text{if } s^* = 1.
\end{cases}
\]

We know that \( m \mapsto \langle \text{sn}^2 (\cdot, m) \rangle \) is continuous, strictly increasing on \((-\infty, 1)\), it tends to 0 for \( m \to -\infty \) and to 1 for \( m \to 1 \), see Lemma 12. So the right-hand side of [63] covers \((0, +\infty)\) for \( m \in (-\infty, 0) \), and
it covers \((0,1)\) for \(m \in (0,1)\). For this reason for every \(\lambda > 0\) there exists a unique \(\bar{m} < -1\) satisfying (63), and for every \(\lambda \in (0,1)\) there exists a unique \(\bar{m} \in (0,1)\) satisfying (63).

The value \(\bar{m}\) and system (62) determine uniquely the values \(\bar{V}, \Omega\).

Analyticity and poles follow from (1), 16.2, 16.10.2, pp.570,573.

Now we have to prove the non-degeneracy of \(g\). The linearized equation of (70) at \(g\) is

\[
\dot{h} + s^*(g^2 - 3\lambda g^2) h = -2s^*(gh)g.
\]  

(64)

Let \(L\) be the Green operator, i.e. for \(f \in E\), let \(H := L(f)\) be the unique solution belonging to \(E\) of the non-homogeneous linear system

\[
\dot{H} + s^*(g^2 - 3\lambda g^2) H = f.
\]  

We can write (64) as

\[
h = -2s^*(gh)L(g).
\]  

(65)

Multiplying by \(g\) and integrating we get

\[
\langle gh \rangle [1 + 2s^*(gL(g))] = 0.
\]

If \(A_0 := 1 + 2s^*(gL(g)) \neq 0\), then \(\langle gh \rangle = 0\), so by (65) \(h = 0\) and the non-degeneracy is proved.

It remains to show that \(A_0 \neq 0\). As before, the key is to express \(L(g)\) in a suitable way. We first look for a fundamental set of solutions of the homogeneous equation

\[
\dot{h} + s^*(g^2 - 3\lambda g^2) h = 0.
\]  

(66)

**Lemma 9.** There exist two linearly independent solutions of (66), \(\bar{u}\) even, \(2\pi\)-periodic and \(\bar{v}\) odd, not periodic, such that \(\bar{u}(0) = 1\), \(\bar{u}(0) = 0\), \(\bar{v}(0) = 0\), \(\dot{\bar{v}}(0) = 1\), and

\[
\bar{v}(t + 2\pi) - \bar{v}(t) = \rho \bar{u}(t) \quad \forall t
\]  

(67)

for some \(\rho \neq 0\). Moreover there hold the following expressions for \(\bar{u}, \bar{v}\):

\[
\bar{u}(t) = \frac{\dot{\bar{g}}(t)}{\bar{g}(0)} = \sin(\bar{\Omega} t, \bar{m})
\]  

(68)

\[
\bar{v}(t) = \frac{1}{\bar{\Omega}(1 - \bar{m})} \sin(\bar{\Omega} t) + \frac{\bar{m}}{\bar{m} - 1} \bar{\sin}(\bar{\Omega} t) \left[ t + \frac{1 + \bar{m}}{\bar{\Omega}} \int_{0}^{\bar{\Omega} t} \frac{\sin^2(\xi, \bar{m})}{\sin^2(\xi, \bar{m})} d\xi \right].
\]  

(69)

**Proof.** \(g\) solves (64) so \(\dot{g}\) solves (66); normalizing we get (68).

By (66), the function \(y(t) = V\sin(\Omega t, m)\) solves

\[
\dot{y} + s^*(g^2)y - s^*\lambda y^3 = 0
\]  

(70)

if \((V, \Omega, m)\) satisfy

\[
\begin{cases}
\Omega^2(1 + m) = s^*(g^2) \\
2m\Omega^2 = s^*V^2\lambda.
\end{cases}
\]  

(71)

We solve (64) w.r.t. \(m\) finding the one-parameter family \((y_m)\) of odd periodic solutions of (64), \(y_m(t) = V(m)\sin(\Omega m t, m)\). So \(l(t) := (\partial_m y_m)_{m=\bar{m}}\) solves (66). We normalize \(\bar{v}(t) := l(t)/l(0)\) and we compute the coefficients differentiating (64) w.r.t. \(m\). From the definitions of the Jacobi elliptic functions it holds

\[
\partial_m \sin(x, m) = -\frac{1}{2} \int_{0}^{x} \frac{\sin^2(\xi, m)}{\sin^2(\xi, m)} d\xi;
\]

thanks to this formula we obtain (69).
Since $2\pi \Omega = 4K(\bar{m})$ is the period of the Jacobi functions $sn$ and $dn$, by (38), (39) we obtain (72) with
\[
\rho = \bar{m} - 1 - 2\pi \left( 1 + (1 + \bar{m}) \langle \frac{\text{sn}^2}{\text{dn}^2} \rangle \right).
\]
If $s^* = 1$, then $\bar{m} \in (0, 1)$ and directly we can see that $\rho < 0$. If $s^* = -1$, then $\bar{m} < -1$. From the equality $\langle \text{sn}^2/\text{dn}^2 \rangle = (1 - m)^{-1} \left( 1 - \langle \text{sn}^2 \rangle \right)$ (see [3], Lemma 3, (L.2)), it results $\rho > 0$. ■

We can note that the integral representation (39) of the Green operator $L$ holds again in the present case. The proof is just like in Lemma 4.

**Lemma 10.** We can write $A_0 := 1 + 2s^* \langle gL(g) \rangle$ as function of $\lambda$, $\bar{m}$,
\[
A_0 = \frac{\lambda(1 - \bar{m})^2 q - (1 - \lambda)^2 (1 + \bar{m})^2 + \bar{m} q^2}{\lambda(1 - \bar{m})^2 q}, \quad q = q(\lambda, \bar{m}) := 2 - \lambda \frac{(1 + \bar{m})^2}{2\bar{m}} > 0.
\]

**Proof.** First, we calculate $\langle gL(g) \rangle$ with the integral formula (39) of $L$. The equalities (42), (43) still hold, while similar calculations give
\[
\int_0^{2\pi} g^3 \bar{v} = -s^* \frac{\rho}{4\pi} + \frac{1}{2\pi \rho} \left( \int_0^{2\pi} g \bar{v} \right)^2
\]
instead of (44). So
\[
\langle gL(g) \rangle = -s^* \frac{\rho}{4\pi} + \frac{1}{2\pi \rho} \left( \int_0^{2\pi} g \bar{v} \right)^2
\]
and the sign of $A_0$ is not obvious. We calculate $\int_0^{2\pi} g \bar{v}$ recalling that $g(t) = \bar{V} \text{sn}(\Omega t, \bar{m})$, using formula (63) for $\bar{v}$ and integrating by parts
\[
\int_0^{2\pi} \text{sn}(\Omega t) \text{sn}(\Omega t) \mu(t) dt = -\frac{1}{2\Omega} \int_0^{2\pi} \text{sn}(\Omega t) \dot{\mu}(t) dt
\]
where $\mu(t) := t + (1 + \bar{m}) \Omega^{-1} \int_0^\Omega \text{sn}^2(\xi)/\text{dn}^2(\xi) d\xi$. From [3], (L.2), (L.3) in Lemma 3, we obtain the formula
\[
\langle \frac{\text{sn}^4}{\text{dn}^2} \rangle = \frac{1 + (m - 2) \langle \text{sn}^2 \rangle}{m(1 - m)}
\]
and consequently
\[
\int_0^{2\pi} g \bar{v} = \frac{-\pi \bar{V}}{2\Omega(1 - \bar{m})^2} \left( 1 + \bar{m} - 2\bar{m} \langle \text{sn}^2 \rangle \right).
\]
By the second equality of (42) and (43) we get
\[
A_0 = 1 + \frac{2\lambda}{\lambda} \left[ -\frac{\rho}{4\pi} + \frac{\pi \bar{m}}{\rho(1 - \bar{m})^2} \left( 1 + \bar{m} - 2\bar{m} \langle \text{sn}^2 \rangle \right)^2 \right]
\]
both for $s^* = \pm 1$. From the proof of Lemma 4 we have $\rho = -2\pi \bar{m} q (1 - \bar{m})^{-2}$, where $q$ is defined in (72); inserting this expression of $\rho$ in (75) we obtain (44).

Finally, for $\bar{m} < -1$ we have immediately $q > 0$, while for $\bar{m} \in (0, 1)$ we get $q = 2 - (1 + \bar{m}) \langle \text{sn}^2 \rangle$ by (63). Since $\langle \text{sn}^2 \rangle < 1$, it results $q > 0$. ■

**Lemma 11.** $A_0 \neq 0$. More precisely, $\text{sign}(A_0) = -s^*$. 
PROOF. From \((\ref{53})\), \(A_0 > 0\) iff \(\lambda(1 - \bar{m})^2 q - (1 - \lambda)^2(1 + \bar{m})^2 + \bar{m} q^2 > 0\). This expression is equal to \(- (1 - \bar{m})^2 p\), where
\[
p = p(\lambda, \bar{m}) = \frac{(1 + \bar{m})^2}{4\bar{m}} \lambda^2 - 2\lambda + 1,
\]
so \(A_0 > 0\) iff \(p < 0\). The polynomial \(p(\lambda)\) has degree 2 and its determinant is \(\Delta = -(1 - \bar{m})^2/\bar{m}\). So, if \(s^* = 1\), then \(\bar{m} \in (0, 1), \Delta < 0\) and \(p > 0\), so that \(A_0 < 0\).

It remains the case \(s^* = -1\). For \(\lambda > 0\), we have \(p(\lambda) < 0\) iff \(\lambda > x^*\), where \(x^*\) is the positive root of \(p\), \(x^* := 2R(1 + R)^{-2}, R := |\bar{m}|^{1/2}\). By \((\ref{53})\), \(\lambda > x^*\) iff
\[
\langle \text{sn}^2(\cdot, \bar{m}) \rangle > \frac{R - 1}{(R + 1) R}.
\]
By formula \((\ref{58})\) and by definition of complete elliptic integrals \(K\) and \(E\) we can write \((\ref{55})\) as
\[
\int_0^{\pi/2} \left( \frac{R - 1}{(R + 1)R} - \sin^2 \vartheta \right) \frac{d\vartheta}{\sqrt{1 + R^2 \sin^2 \vartheta}} < 0.
\]
We put \(\sigma := R - 1/(R + 1)R\) and note that \(\sigma < 1/2\) for every \(R > 0\).

\(\sigma - \sin^2 \vartheta > 0\) iff \(\vartheta \in (0, \vartheta^*)\), where \(\vartheta^* := \arcsin(\sqrt{\sigma})\), i.e. \(\sin^2 \vartheta^* = \sigma\). Moreover \(1 < 1 + R^2 \sin^2 \vartheta < 1 + R^2\) for every \(\vartheta \in (0, \pi/2)\). So
\[
\int_0^{\pi/2} \frac{\sigma - \sin^2 \vartheta}{\sqrt{1 + R^2 \sin^2 \vartheta}} d\vartheta < \int_0^{\vartheta^*} (\sigma - \sin^2 \vartheta) d\vartheta + \int_{\vartheta^*}^{\pi/2} \frac{\sigma - \sin^2 \vartheta}{\sqrt{1 + R^2 \sin^2 \vartheta}} d\vartheta.
\]
Thanks to the formula
\[
\int_a^b \sin^2 \vartheta d\vartheta = \frac{b - a}{2} - \frac{\sin(2b) - \sin(2a)}{4}
\]
the right-hand side term of \((\ref{58})\) is equal to
\[
\frac{\sin(2\vartheta^*)}{4} \left[ (2\sigma - 1) \left( \frac{2\vartheta^*}{\sin(2\vartheta^*)} + \frac{1}{\sqrt{1 + R^2}} \right) - 2\vartheta^* + \left(1 - \frac{1}{\sqrt{1 + R^2}}\right) \right].
\]
Since \(2\sigma - 1 < 0\) and \(\alpha > \sin \alpha\) for every \(\alpha > 0\), this quantity is less than
\[
\frac{\sin(2\vartheta^*)}{4} \left[ (2\sigma - 1) \left(1 + \frac{1}{\sqrt{1 + R^2}}\right) - 2\vartheta^* + \left(1 - \frac{1}{\sqrt{1 + R^2}}\right) \right].
\]
By definition of \(\sigma\), the last quantity is negative for every \(R > 0\), so \((\ref{58})\) is true. Consequently \(\lambda > x^*\), \(p < 0\) and \(A_0 > 0\). \(\blacksquare\)

As Appendix, we show the properties of the function \(m \mapsto \langle \text{sn}^2(\cdot, m) \rangle\) used in the proof of Lemma \(S\n\)

**Lemma 12.** The function \(\varphi : (-\infty, 1) \rightarrow \mathbb{R}, m \mapsto \langle \text{sn}^2(\cdot, m) \rangle\) is continuous, differentiable, strictly increasing, and \(\lim_{m \to -\infty} \varphi(m) = 0, \lim_{m \to 1} \varphi(m) = 1\).

**Proof.** By \((\ref{55})\) and by definition of complete elliptic integrals \(K\) and \(E\),
\[
\varphi(m) = \frac{K(m) - E(m)}{m K(m)} = \int_0^{\pi/2} \frac{\sin^2 \vartheta d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}} \left( \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}} \right)^{-1},
\]
so the continuity of \(\varphi\) is evident.

Using the equality \(\sin^2 + \cos^2 = 1\) and the change of variable \(\vartheta \to \pi/2 - \vartheta\) in the integrals which define \(K\) and \(E\), we obtain the formulae
\[
K(m) = \frac{1}{\sqrt{1 - m}} K\left(\frac{m}{m - 1}\right), \quad E(m) = \sqrt{1 - m} E\left(\frac{m}{m - 1}\right) \quad \forall m < 1.
\]
We put $\mu := m/(m-1)$, so it results

$$\varphi(m) = 1 - \frac{1}{\mu} + \frac{E(\mu)}{\mu K(\mu)}, \quad (80)$$

Since $\mu$ tends to 1 as $m \to -\infty$, $E(1) = 1$ and $\lim_{\mu \to 1} K(\mu) = +\infty$, (79), (80) give $\lim_{m \to -\infty} \varphi(m) = 0$.

Since $E(m)/K(m)$ tends to 0 as $m \to 1$, (53) gives $\lim_{m \to 1} \varphi(m) = 1$.

Differentiating the integrals which define $K$ and $E$ w.r.t. $m$ we obtain the formulae

$$E'(m) = \frac{E(m) - K(m)}{2m}, \quad K'(m) = \frac{1}{2m} \left( \int_0^{\pi/2} \frac{d\vartheta}{(1 - m \sin^2 \vartheta)^{3/2}} - K(m) \right),$$

so the derivative is

$$\varphi'(m) = \frac{1}{2m^2 K^2(m)} \left[ E(m) \int_0^{\pi/2} \frac{d\vartheta}{(1 - m \sin^2 \vartheta)^{3/2}} - K^2(m) \right].$$

The term in the square brackets is positive by strict Hölder inequality for $(1 - m \sin^2 \vartheta)^{-3/4}$ and $(1 - m \sin^2 \vartheta)^{1/4}$.

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