Realizability in tropical geometry and unobstructedness of Lagrangian submanifolds

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Abstract

We say that a tropical subvariety $V \subset \mathbb{R}^n$ is $B$-realizable if it can be lifted to an analytic subset of $(\Lambda^*)^n$. When $V$ is a smooth curve or hypersurface, there always exists a Lagrangian submanifold lift $L_V \subset (\mathbb{C}^*)^n$. We prove that whenever $L_V$ has well-defined Floer cohomology, we can find for each point of $V$ a Lagrangian torus brane whose Lagrangian intersection Floer cohomology with $L_V$ is non-vanishing. As a consequence, whenever $L_V$ is a Lagrangian submanifold that can be made unobstructed by a bounding cochain, the tropical subvariety $V$ is $B$-realizable by applying mirror symmetry.

As an application, we show that the Lagrangian lift of a genus zero tropical curve is unobstructed, thereby giving a purely symplectic argument for Nishinou and Siebert’s proof that genus-zero tropical curves are $B$-realizable. We also prove that tropical curves inside tropical abelian surfaces are $B$-realizable.

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1 Introduction

Mirror symmetry is a collection of equivalences between symplectic geometry ($A$-model) and algebraic geometry ($B$-model) on a pair of mirror spaces. A general proposal for constructing mirror pairs of a symplectic space $X_A$ and algebraic space $X_B$ comes from Strominger, Yau, and Zaslow, who conjectured that mirror pairs can be presented as dual torus fibrations over an integral affine manifold $Q$. One relation between these spaces arises in the form of Kontsevich’s homological mirror symmetry (HMS) conjecture, which predicts an equivalence between the Fukaya category of $X_A$ and the category of coherent sheaves on a mirror manifold $X_B$. Roughly, the objects of the
Fukaya category of $X_A$ are Lagrangian submanifolds $L \subset X_A$. A blueprint for mirror symmetry is that Lagrangian submanifolds of $X_A$ relate to sheaves supported on a subvariety of $X_B$ via mutual comparison to tropical subvarieties on the base $Q$.

We consider the relatively well-understood example of $X_A = T^*\mathbb{R}^n/\mathbb{T}^{*}\mathbb{Z}^n \mathbb{R}^n$. $X_B = (\Lambda^*)^n$, and $Q = \mathbb{R}^n$. On the $A$ side, it will be convenient for us to identify $X_A$ with $(\mathbb{C}^*)^n$, which has holomorphic coordinates $x_i = e^{q_i + i\theta_i}$ and standard symplectic form $\sum_{i=1}^{n} dq_i \land d\theta_i$. Note that $X_A$ does naturally come with a complex structure. On the $B$-side, we take $\Lambda$ to be the Novikov field

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^\lambda_i \mid a_i \in \mathbb{C}, \lambda \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

whose valuation map $\text{val} : \Lambda \to \mathbb{R} \cup \{\infty\}$ is the smallest exponent appearing the expansion of $\sum_{i=0}^{\infty} a_i T^\lambda_i$. On the $A$-side, the torus fibration is given by

$$\pi_A : X_A \to Q$$

$$(x_1, \ldots, x_n) \mapsto (\log |x_1|, \ldots, \log |x_n|) = (q_1, \ldots, q_n).$$

whose fibers are Lagrangian tori. The dual fibration $\text{TropB} : X_B \to Q$ is given by taking coordinate-wise valuation

$$\text{TropB} : X_B \to Q$$

$$(z_1, \ldots, z_n) \mapsto (\text{val}(z_1), \ldots, \text{val}(z_n)).$$

Instead of using tropical geometry as an intuition for HMS, this paper uses HMS and our understanding of the tropical to $A$-correspondence to study the tropical to $B$ correspondence. We now review these correspondences before stating our results.

**Tropical to $B$ correspondence**

The tropical-to-complex correspondence and its applications to enumerative geometry have been a particularly rich field of study since the pioneering work of [Mik05] which related counts of tropical curves in $\mathbb{R}^2$ to counts of curves in the complex algebraic torus (as opposed to the $\Lambda$ analytic torus we study). This relation consists of two parts: tropicalization, which associates to a holomorphic curve in $(\mathbb{C}^*)^2$ a tropical curve in $\mathbb{R}^2$; and realization, which lifts every tropical curve $V \subset \mathbb{R}^2$ to a holomorphic curve in $(\mathbb{C}^*)^2$. Both of these constructions have been extended to greater generality; we provide a coarse overview of the constructions here:

- **$B$-Tropicalization**: The tropicalization map associates to a closed analytic subset $Y \subset X_B$ its tropicalization $\text{TropB}(Y) \subset Q$. The expectation (which holds for algebraic subvarieties, [GBS4]) is that the tropicalization is a tropical subvariety (definition 2.2.1).

- **$B$-Realization**: Starting with $V \subset Q$ a tropical subvariety we say that $V$ is $B$-realizable if there exists closed analytic subset $Y \subset X_B$ with $\text{TropB}(Y) = V$.

One goal of tropical geometry is to determine which tropical subvarieties $V \subset Q$ are $B$-realizable. Examples such as [Mik04, Example 5.12] show that there exist tropical curves $V \subset \mathbb{R}^n, n > 2$ that are non-realizable. In some cases, there are criteria determining if a tropical subvariety is $B$-realizable.
For example, if \( V \subset Q \) is a tropical hypersurface, then there exists a tropical polynomial (piecewise integral affine convex function) \( f : Q \to \mathbb{R} \) so that \( V \) is the locus of points where \( f \) is non-affine. The function \( f \) is called a tropical polynomial as it can be written using the tropical sum and product operations:

\[
\oplus : (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\}) \to (\mathbb{R} \cup \{\infty\}) \\
q_1 \oplus q_2 = \min(q_1, q_2)
\]

\[
\odot : (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\}) \to (\mathbb{R} \cup \{\infty\}) \\
q_1 \odot q_2 = q_1 + q_2
\]

Let \( f = \oplus_{\alpha \in \mathbb{N}^n} a_{\alpha} \odot q^{\odot \alpha} \) be a tropical polynomial whose non-affine locus is \( V \). Let \( \Lambda \) be a complete non-Archimedean valued field, and let \( X_B = (\Lambda^*)^n \) be the algebraic torus. For each \( a_{\alpha} \), select a coefficient \( c_{\alpha} \in \Lambda \) whose valuation is \( \text{val}(c_{\alpha}) = a_{\alpha} \). Then the zero set of the polynomial \( \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} z^{\alpha} \) defines a subvariety of \( X_B \) which is the \( B \)-realization of \( V \).

The other examples where we have \( B \)-realization criteria are tropical curves. In [NS+06], it was shown that if \( V \subset Q \) is a smooth tropical curve of genus 0, then \( V \) is realizable. This was extended to all balanced maps from trees in [Ran17]. In higher genus, it is possible that the space of deformations of a tropical curve shows up in higher dimension than the expected dimension of a possible \( B \)-realization. In this case, we say that the tropical curve is superabundant (Mik05).

We expect that a generically chosen superabundant curve is not \( B \)-realizable. It is known that all 3-valent non-superabundant curves are realizable [Che+16]. In the superabundant setting, [Spe14, Theorem 3.4] established that if \( V \subset Q \) is a tropical curve of genus 1 and satisfies a condition called well-spacedness, then \( V \) is realizable.

### Tropical to \( A \) correspondence

The tropical-to-Lagrangian correspondence is a more recent construction, independently arrived at in a series of papers [Mat18, Mik19, Hic19a, MR20]. Each of the papers associates to a (certain type of) tropical subvariety \( V \subset Q \) a Lagrangian submanifold \( L_V^A \subset X_A \) whose projection to the base of the Lagrangian torus fibration \( \pi_A(L_V^A) \) is contained within an \( \epsilon \)-neighborhood of the tropical subvariety \( V \). We call this a geometric Lagrangian lift of \( V \). When \( V \) is a hypersurface, [Hic19a] proves that under homological mirror symmetry \( L_V^A \subset (\mathbb{C}^*)^n \) is identified with a sheaf whose support is a hypersurface \( Y \subset (\Lambda^*)^n \).

In contrast to \( B \)-realization, the constructions in [Mat18, Mik19, Hic19a, MR20] can construct a geometric Lagrangian lift \( L_V \) of any smooth tropical curve \( V \subset Q \). This difference occurs because the map \( \pi_A : X_A \to Q \) does not provide a good tropicalization map. For example, for any subset \( U \subset Q \) and \( \epsilon > 0 \) there exists a Lagrangian submanifold \( L \subset X_A \) with the property that the Hausdorff distance between \( \pi_A(L) \) and \( U \) is less than \( \epsilon \). Additionally, it would be desirable to have a tropicalization map that only depends on the Hamiltonian isotopy class of the Lagrangian submanifold — and \( \pi_A(L) \) can change substantially when we apply a Hamiltonian isotopy to \( L \).

To obtain a correspondence from Lagrangian submanifolds to tropical subsets of \( Q \), and justify why the Lagrangian \( L_V \) is the “correct” \( A \)-model realization of a tropical curve \( V \), one needs to employ techniques from Floer theory. Not all Lagrangian submanifolds are amenable to such analysis. We call a Lagrangian submanifold unobstructed if its filtered \( A_{\infty} \) algebra \( CF^*(L) \) admits a bounding cochain. The pair \((L, b)\) of Lagrangian submanifold equipped with a bounding cochain is called a Lagrangian brane. Examples of unobstructed Lagrangian submanifolds include those which bound no pseudoholomorphic disks for a given choice of almost complex structure. In particular, if \( L \) is exact, it is unobstructed.
If \((L_1, b_1), (L_2, b_2)\) are two Lagrangian branes then there exists a cochain complex \(CF^\bullet((L_1, b_1), (L_2, b_2))\) which is generated by the intersections of \(L_1\) and \(L_2\), and whose cohomology groups \(HF^\bullet((L_1, b_1), (L_2, b_2))\) are invariant under Hamiltonian isotopies of either \(L_1\) or \(L_2\). We can use this to define \(A\)-tropicalization and \(A\)-realization.

- **\(A\)-Tropicalization:** Starting with the fibration \(X_A \to Q\) and a Lagrangian brane \((L, b) \subset X_A\), we define the \(A\)-tropicalization

  \[
  \text{Trop}_A(L, b) := \{ q \in Q : \exists (F_q, \nabla) \text{ with } HF^\bullet((L, b), (F_q, \nabla)) \neq 0 \}
  \]

  where \(F_q = \pi_A^{-1}(q)\) is equipped with a unitary local system \(\nabla\), and \(HF^\bullet((L, b), (F_q, \nabla))\) is the Lagrangian intersection Floer cohomology of \((L, b)\) with \(F_q\) deformed by the local system \(\nabla\). An advantage of \(\text{Trop}_A(L, b)\) over \(\pi_A(L)\) is that the former depends only on the Hamiltonian isotopy class of \(L\).

- **\(A\)-Realizability:** In light of the definition of \(A\)-tropicalization, we say that \(V \subset Q\) is \(A\)-realizable if there exists a Lagrangian brane \((L, b) \subset X_A\) with \(\text{Trop}_A(L, b) = V\).

The Lagrangian submanifold \(L^\epsilon_V\) associated to \(V\) provides a geometric candidate for an \(A\)-realization of \(V\). However, to verify \(A\)-realizability, one still needs to check that \(L^\epsilon_V\) is unobstructed with bounding cochain \(b\) and that \(\text{Trop}_A(L^\epsilon_V, b) = V\). We call this last condition *faithfulness*.

### 1.1 Results

The three components (geometric realizability, unobstructedness, and faithfulness) of the \(A\)-realizability problem and its implications for the \(B\)-realizability problem in \(Q = \mathbb{R}^n\) are summarized in the following diagram.

The correspondences given by solid black lines always exist. Geometric \(A\)-realizability (the solid red arrow) is only known to exist for certain examples of subvarieties of \(Q\). For the applications we
consider (smooth tropical hypersurfaces and curves) we always have geometric \(A\)-realizability. We conjecture that every tropical subvariety of \(Q\) is geometrically \(A\)-realizable. For any given tropical subvariety \(V \subset Q\), there is no reason for either of the dashed arrows to hold. However, the following conjecture seems natural.

**Conjecture 1.1.1.** Let \(V \subset \mathbb{R}^n\) be a tropical curve. Then a geometric Lagrangian lift \(L_V\) is unobstructed if and only if \(V\) is \(B\)-realizable.

The main step in proving the forward direction of the conjecture is to establish the faithfulness of the Lagrangian brane lift, that is showing that \(\text{TropA}((L_V, b)) = V\). The primary result of this article is to prove faithfulness (for all tropical subvarieties admitting unobstructed Lagrangian lifts).

**Theorem A** (Restatement of Lemma \[5.2.3\]). Let \(V \subset Q\) be a tropical subvariety. Let \((L_V^x, b)\) be a Lagrangian brane lift of \(V\). Then \(\text{TropA}(L_V^x, b) = V\).

By applying homological mirror symmetry, we obtain the forward direction of conjecture 1.1.1. Depending on the affine manifold \(Q\) and Lagrangian \(L_V\), we may require assumption \[6.1.2\] which states that the family Floer construction of \[Abo17a\] extends to the non-compact and unobstructed setting.

**Theorem B** (Restatement of corollary \[6.2.1\]). Suppose assumption \[6.1.2\]. Let \(V \subset \mathbb{R}^n\) be a tropical subvariety. Suppose there exists \((L_V, b) \subset (\mathbb{C}^*)^n\) a Lagrangian brane lift of \(V\). Then \(V\) is \(B\)-realizable.

The second goal of this paper is to show that this can be used to produce realizability criteria. We first recover a theorem of \[NS+06\].

**Corollary C** (Restatement of corollary \[4.3.3\]). Suppose assumption \[6.1.2\]. Every smooth genus zero tropical curve \(V \subset \mathbb{R}^n\) has a Lagrangian brane lift \((L_V, b)\), and is therefore \(B\)-realizable.

The results of \[Nis20\] give necessary and sufficient conditions for when a tropical curve can be realized by a family of algebraic curves in a degenerating family of complex tori. In contrast to those results, our results show that every 3-valent tropical curve can be realized by a closed analytic subset. The following result does not assume assumption \[6.1.2\].

**Corollary D** (Restatement of corollary \[6.2.4\]). Let \(Q = T^2\) be a tropical abelian surface. Let \(V \subset Q\) be a 3-valent tropical curve. \(V\) has a Lagrangian brane lift \((L_V, 0)\), and is, therefore, \(B\)-realizable.

In summary we can recover \(B\)-realizability results using unobstructedness for the first 5 cases in table 1. We also provide some insight into the existence of holomorphic curves with boundary on tropical Lagrangian submanifolds.

**Example E** (Restatement of example \[6.3.2\]). Let \(V_c \subset \mathbb{R}^3\) be a generic tropical line. The Lagrangian \(L_{V_c}\) is unobstructed, but not tautologically unobstructed.

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1The realization result of \[NS+06\] considers \(B\)-tropicalizations coming from degenerating families of abelian surfaces, so that a tropical curve is realized by a parameterized algebraic curve. The \(B\)-realization we take is by closed analytic subsets. In the setting of genus zero stable tropical curves in toric varieties these tropicalizations can be related by \[Ran17\].
| $V$ and $Q$                     | $A$-model (Unobstructedness) | $B$-model (Realizability) | HMS Status |
|-------------------------------|-----------------------------|--------------------------|------------|
| Curves in abelian surfaces    | Corollary 6.2.4             | Nis20                    | ✓          |
| Curves in $\mathbb{R}^2$      | Hic20                       | Mik05                    | (*)        |
| Hypersurfaces of $\mathbb{R}^n$ | Hic20                       | Folklore                 | (*)+(**)   |
| Hypersurfaces in abelian varieties | Corollary 6.2.3             | —                        | (**)       |
| Genus 0 curves in $\mathbb{R}^n$ | Theorem C                   | NS+06                    | (*)+(**)   |
| Compact genus 0 curves in dim($Q$) = 3 | MR20                       | NS+06                    | —          |
| Well-Spaced Genus 1 curves    | Spec. in section 6.4        | Spe14                    | (*)+(**)   |

Table 1: Relating $A$-unobstructedness to $B$-realizability. (*) and (**) refer to the needed extensions of family Floer cohomology (assumption 6.1.2) to the non-compact and non-tautologically unobstructed settings.

### 1.2 Outline

In section 2, we give a toy computation that explores the entire road map above for a simple example, $V_{pants} \subset \mathbb{R}^2$, the tropical pair of pants. In addition to providing context for the remainder of the paper, the computation reviews some background for tropical geometry and symplectic geometry. We also use this section to fix notation for the paper. It is our hope that this section will be accessible to both tropical and symplectic geometers.

Section 3 discusses the geometric lifting problem. Definition 3.0.2 defines when a family of Lagrangian submanifolds $L_\varepsilon^V$ is a geometric Lagrangian lift of a tropical subvariety $V$. We show that definition 3.0.2 distinguishes tropical subvarieties among all polyhedral complexes as the ones which permit geometric Lagrangian lifts. Definition 3.0.2 requires that geometric Lagrangian lifts are monomially admissible, graded, and spin. In sections 3.1 to 3.3, we show that known constructions of geometric Lagrangian lifts of tropical subvarieties from [Mat18; Mik19; Hic19a] satisfy these conditions. We also prove lemma 3.3.1 which shows that for smooth genus zero tropical curves the map $H^2(L_V) \to H^2(\partial L_V)$ is an injection.

Section 4 investigates Lagrangian submanifolds which can be unobstructed by a bounding cochain (Lagrangian branes). We provide a brief overview of the pearly model for Lagrangian submanifolds in section 4.1. This is followed by examples of unobstructed geometric Lagrangian lifts (Lagrangian brane lifts) of tropical subvarieties in section 4.2 (summarized in table 1). Section 4.3 gives a new method for checking unobstructedness of Lagrangian submanifolds inside non-compact symplectic spaces which have a potential function $W : X_A \to \mathbb{C}$ (see definition A.0.1).

**Theorem F** (Restatement of theorem 4.3.1). Let $W : X_A \to \mathbb{C}$ be a symplectic fibration outside of a compact set of $\mathbb{C}$. Let $L \in X_A$ be a $W$-admissible Lagrangian submanifold with boundary $M \subset W^{-1}(t)$, $t \in \mathbb{R}_{\geq 0}$. Suppose $M$ is a tautologically unobstructed Lagrangian submanifold of $W^{-1}(t)$, and the connecting map $H^1(M) \to H^2(L, M)$ is surjective. Then there exists a bounding cochain $b$ so that $(L, b)$ is a Lagrangian brane.

The proof uses a lemma on filtered $A_\infty$ algebras (lemma B.2.3). Since we have previously proven
in lemma 3.3.1 that the geometric Lagrangian lifts $L_V$ of smooth genus zero tropical curves satisfy the criterion of theorem 4.3.1, we obtain that such $L_V$ are unobstructed (corollary 4.3.3).

In section 5, we prove faithfulness (lemma 5.2.3) which shows that the $A$-tropicalization (Floer theoretic support) of a Lagrangian brane lift $L_V$ is $V$. The proof uses that the Lagrangian intersection Floer cohomology between $(L_V, b)$ and $F_q$ is a deformation of the cohomology of a subtorus of $F_q$. An application of lemma 3.3.1 shows that this can be “undeformed” by a bounding cochain so that $HF^0((L, \nabla_0, b_0), (F_q, \nabla, b)) = \Lambda$.

Section 6 applies the previous constructions to address questions of realizability for tropical subvarieties. [Abo17b, Remark 1.1] states that we expect that the family Floer functor can be adapted to include unobstructed Lagrangians. We instead use assumption 6.1.2 — the weaker assumption that the family Floer construction of [Abo17a] can be employed for unobstructed Lagrangian submanifolds in the Lagrangian torus fibration $(\mathbb{C}^*)^n \to \mathbb{R}^n$ to construct a sheaf on the mirror space. We give a brief outline of the modifications to [Abo17a] which would be required to prove assumption 6.1.2. With this assumption, we prove the forward direction of conjecture 1.1.1 in corollary 6.2.1. We also discuss the first 5 cases in table 1.

Finally, we discuss evidence towards the reverse direction of conjecture 1.1.1. This requires us to understand some of the holomorphic disks which appear on Lagrangian lifts of tropical subvarieties. In example 6.3.2, we show that the lift of the tropical line in $\mathbb{R}^3$ bounds a holomorphic disk whose symplectic area is dictated by the internal edge length on the tropical line. We also discuss applications of $B$ non-realizability to obstructedness in section 6.4. Consider the superabundant tropical elliptic curve $V \subset \mathbb{R}^3$ from [Mik04, Example 5.12]. We provide a sketch for how Speyer’s well-spacedness criterion might be recovered from holomorphic disk counts on $L_V$. Section 6.5 looks at how to relate tropical line bundles on tropical curves to Lagrangian isotopies of their geometric Lagrangian lifts. We conjecture a relation between superabundance of a tropical curve $V$, and the relative ranks of $HF(L_V, b)$ and $H(L_V)$ (wide versus non-wide).

We provide some auxiliary results in the appendices. Appendix A discusses how to adapt the pearly model of Floer cohomology in [CW19] to the setting of non-compact spaces equipped with a potential $W : X \to \mathbb{C}$. In appendix B we prove the results on filtered $A_\infty$ algebras used in this paper.

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2 A guided calculation to the support of the Lagrangian pair of pants

This section contains an expository computation that is designed to frame the main ideas of the paper, provide background, and fix notation. The exposition here is not intended to be comprehensive, although we hope that through explicit examples, direct computations, and additional references, we've made this section accessible to both the tropical and symplectic geometry communities. As a result, the materials outside of section 2.5 are expository. As we will frequently use notation from examples 2.4.3 and 2.4.4, we suggest that the readers take a look at these computations of Lagrangian intersection Floer cohomology for conormal bundles in the cotangent bundle of the torus.

2.1 A-model, B-model, and Lagrangian torus fibrations

We provide a high-level overview of the SYZ96, Gro01 viewpoint on mirror symmetry. Let $Q$ be an integral affine manifold, that is a manifold equipped with a choice of full-rank lattice $T_Z Q \subset T_{\mathbb{Z}} Q$. This identifies a dual lattice $T^\ast_{\mathbb{Z}} Q \subset T^\ast_{\mathbb{R}} Q$, and also a flat connection on $T_{\mathbb{Z}} Q$. There are three kinds of geometries that we may associate with $Q$: symplectic geometry, complex geometry, and tropical geometry.

A-model

A symplectic manifold is a $2n$-manifold $X_A$ with a choice of two form $\omega \in \Omega^2(X_A)$ which is closed ($d\omega = 0$) and nondegenerate ($\omega^n \neq 0$). The submanifolds of interest for us in $X_A$ are Lagrangian submanifolds $L \subset X_A$, which are $n$-dimensional submanifolds on which the symplectic form vanishes ($\omega|_L = 0$). For any manifold $Q$, the cotangent bundle $T^\ast Q$ (whose local coordinates are $(q, p)$) carries a canonical symplectic form $\sum_{i=1}^n dq_i \wedge dp_i$. This descends to a symplectic form on the quotient $X_A := T^\ast Q / T^\ast_{\mathbb{Z}} Q$.

Given an integral affine submanifold $V \subset Q$ so that $T_Z V \subset T_Z Q$, the periodized conormal bundle $L_V := N^\ast V / N^\ast_{\mathbb{Z}} V \subset X_A$ is an example of a Lagrangian submanifold. The simplest example is when we pick a point $q \in Q$ so that

$$L_q = N^\ast q / N^\ast_{\mathbb{Z}} q = T^\ast q Q / T^\ast_{\mathbb{Z}} q Q$$

is a Lagrangian torus of $X_A$. We will call this Lagrangian torus $F_q \subset X_A$. For this reason we call the projection $\pi_A : X_A \to Q$ a Lagrangian torus fibration.

B-model

We can also build an almost-complex manifold from the data of $Q$. An almost complex structure on $X_B^C$ is an endomorphism $J : TX_B^C \to TX_B^C$ which squares to $-\text{id}$. The submanifolds of interest in the B-model are the \textit{almost-complex submanifolds} $Y^C \subset X_B^C$ whose tangent spaces are fixed under the almost complex structure so that $J(T_y Y^C) = T_y Y^C$.

As $Q$ is integral affine, there exists a connection on $T Q$ whose flat sections are locally constant sections of $T_Z Q$. This provides a splitting $T(T Q) = T_q Q \oplus \ker(\pi)$. We define an almost complex structure on $T Q$ which interchanges the components of this splitting with a sign:

$$J := \begin{pmatrix} 0 & -\text{id} \\ \text{id} & 0 \end{pmatrix}.$$
The almost complex structure on $TQ$ descends to an almost complex structure on $X^C_B := T^*Q/T^*_ZQ$; the fibers of $\pi_B : X^C_B \to Q$ are real tori.

Given an integral affine submanifold $\mathcal{V} \subset Q$, the periodized tangent bundle

$$Y^C_{\mathcal{V}} := T\mathcal{V}/T\mathcal{V} \subset X^C_B$$

is an example of an almost-complex submanifold. If we start with $q \subset Q$ a point, we see that $Y_q \subset X^C_B$ is a point of $X^C_B$.

**Mirror symmetry from Lagrangian torus fibrations**

We now describe in more detail the relationship between the Lagrangian tori of $X_A$ and the points of $X^C_B$. First, we note that for fixed $q \in Q$, there is a torus worth of points $z$ in $X^C_B$ with the property that $\pi_B(z) = q$.

In contrast to the complex lift, there is only one Lagrangian torus $F_q \subset X_A$ with $\pi_A(F_q) = \{q\}$. To get a matching family of Lagrangian lifts to our complex lift, we consider Lagrangian tori equipped with the additional data of a local systems. Let $(F_q, \nabla)$ be a pair consisting of a Lagrangian torus $F_q$ and a choice of $U(1)$ local system on $F_q$. Then there is a bijection between pairs $(F_q, \nabla) \subset X_A$ and points $z \in X^C_B$. A similar story holds for the Lagrangian and complex lifts of integral affine subspace $\mathcal{V} \subset Q$.

To generalize beyond the submanifolds $\mathcal{V}, L_{\mathcal{V}}$ and $Y_{\mathcal{V}}$ discussed above, we need to look at tropical geometry.

**Notation 2.1.1.** Unless otherwise stated, we will only consider $Q = \mathbb{R}^n$, so that $X_A = (\mathbb{C}^*)^n$ and $X^C_B = (\mathbb{C}^*)^n$.

### 2.2 A quick introduction to tropical geometry and $B$-tropicalization

A convex polyhedral domain is the intersection of finitely many closed half-spaces in $\mathbb{R}^n$,

$$\mathcal{V} = \{q \in Q : \langle q, \bar{v}_i \rangle \geq \lambda_i\},$$

where $\bar{v}_i$ is a collection of vectors in $\mathbb{R}^n$, and $\lambda_i$ is some set of constants in $\mathbb{R}$. We say that this is a rational convex domain if each of the $\bar{v}_i \in \mathbb{Z}^n$, equivalently if there is a full lattice $T_Z \mathcal{V} \subset T\mathcal{V}$ which is a sublattice of $T_Z Q$. A tropical subvariety is built out of these pieces.

**Definition 2.2.1.** A $k$-dimensional tropical subvariety $V \subset Q$ is a collection of $k$-dimensional disjoint rational convex polyhedral domains $\{\mathcal{V}_s \subset Q\}$ and weights $\{w_s \in \mathbb{N}\}$ which is required to satisfy the following conditions:

- **Polyhedral Complex condition:** At each pair of rational convex polyhedral domains, the intersection $\mathcal{V}_s \cap \mathcal{V}_t$ is either empty, or a boundary facet of both $\mathcal{V}_s$ and $\mathcal{V}_t$.

- **Balancing condition:** At top-dimensional boundary $\mathcal{W} \subset \mathcal{V}$, let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be the rational polyhedral domains containing $\mathcal{W}$. Consider lattices $T_Z \mathcal{V}_i$, each of which is a sublattice of $T_Z \mathcal{V}_i$ for each $i \in \{1, \ldots, k\}$. Select for each $i$ a vector $\bar{v}_i \in T_Z \mathcal{V}_i$, so that $T_Z \mathcal{V}_i = T_Z \mathcal{W} \oplus \langle \bar{v}_i \rangle$ as oriented lattices. We require that

$$\sum_i w_i \bar{v}_i \equiv 0 \in T_Z Q/T_Z \mathcal{W}.$$
Example 2.2.2. Consider the polyhedral domains in $Q = \mathbb{R}^2$

- $V_1 = \{(-t, 0) : t \in \mathbb{R}_{\geq 0}\}$
- $V_2 = \{(0, -t) : t \in \mathbb{R}_{\geq 0}\}$
- $V_3 = \{(t, t) : t \in \mathbb{R}_{\geq 0}\}$

As the directions $(-1, 0) + (0, -1) + (1, 1)$ sum to zero this is balanced and gives us a tropical curve. The collection of these three polyhedral domains is called the standard tropical pair of pants. The curve $V_{\text{pants}} \subset \mathbb{R}^2$ is drawn in fig. 1a.

We say that a tropical curve $V \subset \mathbb{R}^n$ is smooth if every 0-dimensional stratum is locally modelled after the pair of pants.

Notation 2.2.3. Given $V \subset Q$ a tropical subvariety, we will use $V^{(0)}$ to denote the union of the interiors of the $V$; we use $V^{(1)}$ to denote the union of the interiors of the boundaries of the $V$; more generally we will use $V^{(k)}$ to denote the codimension $k$ linearity strata of $V$. For any $V_s \subset V^{(k)}$, let $\text{star}(V_s)$ be the set of all stratum which contain $V_s$. If $V$ is a tropical curve, we will usually call the strata vertices and edges, and use $v, w$ for vertices and $e, f$ for edges.

2.3 $B$-tropicalization

$B$-tropicalization is the process of taking a subvariety of $X_B^C$ and obtaining a tropical subvariety of $Q$. The first approach one considers is the image of $Y^C \subset X_B^C$ under the $B$-torus fibration

$$\pi_B : X_B^C \to Q.$$ 

Under good conditions, $\pi_B(Y^C) \subset Q$ approximates a tropical subvariety of $Y$; see for instance [Mik04]. The image $\pi_B(Y^C)$ is called the amoeba of $Y^C$, which computationally can be checked to approach the tropical curve (see fig. 1b).

To obtain a theory where the tropicalization of a subvariety is a tropical subvariety, we look to non-Archimedean geometry. Let $\Lambda$ be the Novikov field. Given $M$ a rank $n$ lattice, denote by $X_B$ the torus $\text{Spec} \Lambda[M]$. The points of $X_B$ can be identified with $n$-tuples of invertible elements of $\Lambda$ so we will frequently write

$$X_B = \{(z_1, \ldots, z_n) : z_i \in \Lambda^*\}.$$
We build a tropicalization map by taking the valuation coordinate-wise:

\[ \text{TropA} : X_B \rightarrow M \otimes \mathbb{R} = Q \]
\[ (z_1, \ldots, z_n) \mapsto (\text{val}(z_1), \ldots, \text{val}(z_n)) \]

Given a \( Y \subset X_B \) a closed analytic subset, we call the image \( \text{TropA}(Y) \subset Q \) its tropicalization.

**Example 2.3.1.** Consider \( M = \mathbb{R}^2 \), and the closed analytic subset \( Y \subset (\Lambda^*)^2 \) given by

\[ Y = \left\{ (z_1, z_2) : 1 + z_1 + z_2 = 0 \right\} \]

We compute the valuation of such a point \((z_1, z_2) \in Y\). Since \( \text{val}(1+z_1+z_2) \geq \min(\text{val}(1), \text{val}(z_1), \text{val}(z_2)) \), with equality holding whenever the valuations differ, we obtain that for all \((z_1, z_2) \in Y\) at least one of the following equalities hold:

\[ \text{val}(z_1) = \text{val}(z_2) \quad \text{val}(z_1) = \text{val}(1) \quad \text{val}(z_2) = \text{val}(1) \]

This means that the image of \( \text{TropB}(Y) \) agrees with \( V_{\text{pants}} \subset \mathbb{R}^2 \) from example 2.2.2. It follows that \( V \) is \( B \)-realizable.

This phenomenon holds much more broadly.

**Theorem 2.3.2** ([GB84; Gub07]). Let \( Y \subset X_B \) be an irreducible \( k \)-dimensional analytic subset. Then \( \text{TropB}(Y) \) is a \( k \)-dimensional polyhedral complex.

It is expected that when \( Y \) is an irreducible \( k \)-dimensional analytic subset, \( \text{TropB}(Y) \) is a \( k \)-dimensional tropical subvariety. To our knowledge this result has not appeared in the literature. A discussion on the current status of tropicalization for analytic subsets is included in [SS20, Section 5.3].

### 2.4 Floer cohomology and \( A \)-tropicalization

The definition of the \( A \)-tropicalization of a Lagrangian submanifold requires a little more exposition because we wish to do some computations of the \( A \)-tropicalization. Our goal is to replace the Lagrangian torus fibration map \( \pi_A : X_A \rightarrow Q \) with a correspondence of subsets

\[ \text{TropA} : \{\text{Lagrangian branes}\} \rightarrow \{\text{subsets of } Q\} \]

which only depends on the Hamiltonian isotopy class of the Lagrangian brane.

#### 2.4.1 Lagrangian Intersection Floer Cohomology

The main computational tool that we will use in this paper is Lagrangian intersection Floer cohomology. We first equip a symplectic manifold \((X, \omega)\) with an \( \omega \)-compatible choice of almost complex structure \( J \).

**Definition 2.4.1** ([Flo88]). Given a pair of transversely intersecting Lagrangian submanifolds \( L_0, L_1 \subset X \) and choice of almost complex structure \( J \) satisfying the following conditions:

(i) \( X, L_1, L_2 \) are compact,
(ii) The symplectic area of all disks with boundary on \( L_i \) vanish \( \omega(\pi_2(X, L_i)) = 0 \),

(iii) The Lagrangians \( L_i \) are equipped with spin structures

(iv) The Lagrangians \( L_i \) are graded (in the sense of \([\text{Sei}00]\))

(v) The moduli spaces of \( J \)-holomorphic strips in eq. (2) are regular.

the Lagrangian intersection Floer cohomology is a chain complex

- Whose generators are the points of intersection between \( L_0 \) and \( L_1 \), so that as a vector space
  \[
  CF^*(L_0, L_1) := \bigoplus_{x \in L_0 \cap L_1} \Lambda_x,
  \]
  where \( \Lambda \) is the Novikov field. The grading \( \deg(x) \) of an intersection point \( x \in L_0 \cap L_1 \) is determined by the Maslov index.

- The differential on this complex is defined by a count of holomorphic strips with boundary on \( L_0 \cup L_1 \) and ends limiting to the intersection points. Namely, let \( x_\pm \in L_0 \cap L_1 \) be two intersection points, and \( \beta \in H^2(X, L_0 \cup L_1) \). Let
  \[
  \mathcal{M}_\beta(L_0, L_1, x_+, x_-) := \left\{ u : \mathbb{R} \times [0, 1] \to X \middle| \begin{array}{l}
  u(s, 0) \in L_0, u(s, 1) \in L_1 \\
  \lim_{s \to \pm \infty} u(s, t) = x_\pm \\
  \partial_s u = 0 \\
  [u] = \beta \in H_2(X, L_0 \cup L_1) \end{array} \right\} / (s \mapsto s + c) \tag{2}
  \]
  denote the moduli space of holomorphic strips with ends limiting to \( x_\pm \) in the relative homology class \( \beta \), up to reparameterization of the strip along the \( s \)-coordinate. Using the grading data on \( L_0, L_1 \), one can compute that
  \[
  \dim(\mathcal{M}_\beta(L_0, L_1, x_+, x_-)) = \deg(x_-) - \deg(x_+) - 1.
  \]
  The spin structures on \( L_0, L_1 \) provide orientations for the spaces \( \mathcal{M}_\beta(L_0, L_1, x_+, x_-) \); in particular when \( \deg(x_+) + 1 = \deg(x_-) \), then \( \dim(\mathcal{M}_\beta(L_0, L_1, x_+, x_-)) = 0 \) and we can count the points in this moduli space with sign. The structure coefficients of the differential \( d : CF^*(L_0, L_1) \to CF^*(L_0, L_1) \) are obtained by counting the elements in \( \mathcal{M}_\beta(L_0, L_1, x_+, x_-) \),
  \[
  \langle d(x_+), x_- \rangle = \sum_{\beta \in H^2(X, L_0 \cup L_1)} T^{\omega(\beta)} \# \mathcal{M}_\beta(L_0, L_1, x_+, x_-)
  \]
  where \# is the signed count of points with orientation and \( T^{\omega(\beta)} \) records the symplectic area of the strip \( u \) whose homology class is \( \beta \).

The proof that this is a chain complex proceeds in a similar method to Morse theory. Because of item (1), one can use Gromov compactness to prove that the 1-dimensional moduli spaces of strips have compactifications whose boundaries are given by products of the 0-dimensional moduli spaces of strips
  \[
  \partial \mathcal{M}_\beta(L_0, L_1, x_+, x_-) = \bigsqcup_{x_0 \in L_0 \cap L_1} \mathcal{M}_\beta(L_0, L_1, x_+, x_0) \times \mathcal{M}_\beta(L_0, L_1, x_0, x_-))
  \]
To ensure that the only broken configurations which show up in the compactification are given by strips breaking (as opposed to disk bubbling), we use item [iii] which states that there are no holomorphic disks with boundary on either \( L_0 \) or \( L_1 \). The compactification is compatible with the orientations given to the moduli spaces of holomorphic strips. Since the signed count of boundary components of a 1-dimensional manifold is zero, \( \langle d^2(x_+), x_- \rangle = 0 \). Unless otherwise stated, all Lagrangians we consider will be \( \mathbb{Z} \)-graded and spin. A major feature of Lagrangian intersection Floer cohomology is its invariance under Hamiltonian isotopy.

**Theorem 2.4.2** (Flo88). Let \( L_0, L_1 \) be Lagrangian submanifolds of \((X, \omega)\) satisfying conditions items [i] to [v]. Let \( \phi : X \to X \) be a Hamiltonian isotopy. Suppose that \( L_0, L_1 \) intersect transversely and we’ve picked \( \phi \) in such a way that \( \phi(L_0), L_1 \) intersect transversely. Then \( HF^*(L_0, L_1) = HF^*(\phi(L_0), L_1) \).

For this reason, whenever \( L_0, L_1 \) do not intersect transversely, we can compute their Floer cohomology by taking a Hamiltonian perturbation which makes their intersection transverse; the resulting cohomology groups are independent of the choice of perturbation taken. One can similarly show that it does not depend on the choice of an almost complex structure.

The conditions of items [ii] and [iii] can be weakened. For example, item [i] — which is required to prove that the moduli space of strips admit compactifications — can be replaced with the weaker condition of monomial admissibility (definition 3.1.1). Later we will look at weakening the condition item [ii] to *unobstructedness* (section 4). We now drop item [i] and compute the Lagrangian intersection Floer cohomology between two Lagrangians in a cotangent bundle. The computation we give is a direct generalization of [Sm15, Example 3.1].

**Example 2.4.3** (Running Example). Let \( F_0 = T^n \) be the \( n \)-dimensional torus. Let \( T^{n-k} \subset F_0 \) be the subtorus spanning the first \( n-k \) coordinates on \( T^n \). Then \( T^*F_0 \) is an example of an exact symplectic manifold. The zero section \( F_0 \) and the conormal bundle \( N^*T^{n-k} \) are examples of exact Lagrangian submanifolds. Lagrangian intersection Floer cohomology requires that our Lagrangians intersect transversely, so we will apply a Hamiltonian perturbation to one of the Lagrangians to achieve transverse intersections. Pick \( \lambda_0 \in \mathbb{R}_{>0} \). Consider the Hamiltonian function

\[
H = \sum_{i=1}^{n-k} \lambda_0 \cos(\theta_i)
\]

on \( T^*F_0 \). Let \( \phi : T^*F_0 \to T^*F_0 \) be the time-one Hamiltonian flow of \( H \). The resulting intersections of \( \phi(N^*T^{n-k}) \) with \( F_0 \) are the points

\[
\phi(N^*T^{n-k}) \cap F_0 = \{(a_1 \pi, \ldots, a_{n-k} \pi, 0, \ldots, 0) : a_i \in \{0, 1\}\}
\]

and the index of each intersection point \( x \) is given by \( \text{deg}(x) = \sum_{i=1}^{n-k} a_i \). We will call the corresponding generators of Floer cohomology \( x_I \), where \( a_i = 1 \) whenever \( i \in I \subset \{1, \ldots, n-k\} \). Write \( I \lessdot J \) if \( I = J \cup \{x_i\} \) for some \( i \). As a vector space \( CF^*(\phi(N^*T^{n-k}), F_0) \) matches \( CM^*(T^{n-k}) \) for the Morse function \( H \).

The differential on \( CF^*(\phi(N^*T^{n-k}), F_0) \) is related to the Morse differential. Let \( |I| + 1 = |J| \), so that \( \text{deg}(x_I) \) and \( \text{deg}(x_J) \) differ by one. If \( I \) and \( J \) differ at more than two elements, then \( \mathcal{M}\phi(\phi(N^*T^{n-k}), F_0, x_I, x_J) \) has non-zero dimension. If \( I \lessdot J \) differ at a single element \( j \), then there are exactly two holomorphic strips travelling between \( x_I \) and \( x_J \),

\[
\mathcal{M}(\phi(N^*T^{n-k}), F_0, x_I, x_J) = \{u_{I<j}^+, u_{I<j}^-\}
\]
which as points receive opposite orientations. By our choice of perturbation, the symplectic areas of the strips \( u_\perp \) and \( u_\perp \) agree (and is exactly \( \lambda_0 \)). Therefore,

\[
\langle d(x_I), x_J \rangle = \begin{cases} 
T^\omega(u_\perp) - T^\omega(u_\perp) = 0 & \text{if } I \neq J \\
0 & \text{otherwise}
\end{cases}
\]

and we conclude that

\[
HF^*(\phi(N^*T^{n-k}), F_q) = \Lambda \langle x_I \rangle = \bigwedge_{i \in \{1,\ldots,n-k\}} \Lambda \langle x_i \rangle.
\]

The example relates to the discussion of tropicalization as \( T^*F_0 \) can be identified with \( X_A = (\mathbb{C}^*)^n = T^*Q/T_0^*Q \). If \( T^{n-k} \) is a linear subtorus of \( F_q \), it corresponds to a \( n-k \)-dimensional subspace of \( T_0^{n-k} \subset T_0^*Q \); let \( V \subset T_0Q \) correspond to the set of vectors which are annihilated by \( T^{n-k} \). By abuse of notation, we use \( V \) to denote the integral affine subspace of \( Q \) with prescribed tangent space at 0. Under this identification \( N^*T^{n-k} \subset T^*F_0 \) is \( L_V \subset X_A \). Using that Lagrangian intersection Floer cohomology is invariant under symplectomorphisms, and noting that if \( q \not\in V \) then \( F_q \cap L_V = \emptyset \), we have computed

\[
HF(L_V, F_q) = \begin{cases} 
\bigwedge_{i \in \{1,\ldots,n-k\}} \Lambda \langle x_i \rangle & \text{if } q \in V \\
0 & \text{if } q \not\in V
\end{cases}
\]

### 2.4.2 Local Systems

Recall that the points of \( X_B \) are in bijection with pairs \((F_q, \nabla)\) of Lagrangian torus fibers equipped with local systems. We now discuss how to incorporate this data into Lagrangian intersection Floer cohomology. The unitary Novikov elements

\[
U_\Lambda := \left\{ a_0 + \sum_{i=1}^{\infty} a_i \tau^{\lambda_i} : \lim_{i \to \infty} \lambda_i = \infty, \lambda_i > 0, a_0 \in \mathbb{C}^*, a_i \in \mathbb{C} \right\}
\]

are those elements whose non-zero lowest order term is a constant. We now consider \( (L_i, \nabla_i) \), which are Lagrangian submanifolds with the additional choice of a trivial \( \Lambda \)-line bundles \( E_i \) and a \( U_\Lambda \) local system \( \nabla_i \). Given \( L_0, L_1 \) which intersect transversely satisfying items [i] to [v] we define \( CF^*(\{L_0, \nabla_0\}, (L_1, \nabla_1)) \) to be the chain complex

- whose underlying vector space is \( \bigoplus_{x \in L_0 \cap L_1} \hom((E_0)_x, (E_1)_x) \) and;
- whose differential is given by taking a \( \nabla_i \)-weighted count of the holomorphic strips with boundary in \( L_0 \cup L_1 \). More precisely: denote by \( \partial_u \) be the boundary of \( u \) contained in \( L_i \), and let \( P_{\gamma, i} : (E_i)_{\gamma(0)} \to (E_i)_{\gamma(1)} \) be the parallel transport induced by the local system along a path \( \gamma : [0,1] \to L_i \).

As in the definition of Lagrangian intersection Floer cohomology without local systems, let \( x_+, x_- \in L_0 \cap L_1 \) be intersection points with \( \deg(x_+) + 1 = \deg(x_-) \). Given \( \phi_x \in \hom((E_0)_{x_+}, (E_1)_{x_+}) \) and a holomorphic strip \( u \in M_\beta(L_0, L_1, x_+, x_-) \) we obtain a map between the fibers above \( x_0 \),

\[
P_{\gamma, i} \circ \phi_x \circ P_{\gamma, i}^{-1} \in \hom((E_0)_{x_-}, (E_1)_{x_-}).
\]
The differential on $\text{CF}^\bullet(L_0, L_1)$ is defined by taking the contributions $u \cdot \phi_{x_+}$ over all holomorphic strips between $x_+$ and $x_-$, weighted by the symplectic area.

\[
d_{\nabla_0, \nabla_1}(\phi_{x_+}) := \sum_{x_-, \text{ deg}(x_-) = \text{deg}(x_)+1} \sum_{u \in M_\beta(L_0, L_1, x_+, x_-)} \pm T^\omega(\beta) P_{(\partial^1 u)^{-1}} \circ \phi_{x_+} \circ P_{\nabla_0}^v - 1.
\]

where the sign is determined by the orientation of the moduli space.

When $\nabla_i$ are the trivial local systems, this recovers $\text{CF}^\bullet(L_0, L_1)$.

**Example 2.4.4** (Running Example, continued). We now return to example 2.4.3. Fix coordinates on $F_q$ and let $\{c_1, \ldots, c_n\}$ be generators of $H^1(F_q, \mathbb{Z})$ associated to the coordinate directions. A local system on $F_q$ is determined completely by its monodromy on the $c_i$. Given a $\Lambda$-unitary local system $\nabla_1$ on $F_q$, we write $z_i = P_{c_i}^{\nabla_1}$. Let $\nabla_0$ be the trivial local system on $L_Y$. We now compute the differential on $\text{CF}^\bullet((L_Y, \nabla_0), (F_q, \nabla_1))$. Given $\phi_{x_I} \in \text{hom}((E_0)_{x_I}, (E_1)_{x_I})$, and $I \subset J$ an index which differs at one spot $j$, we have

\[
d_{\nabla_0, \nabla_1}(\phi_I) = \left( T^{\omega(u^+_{I,J})} P_{(\partial^1 u^+_{I,J})} P^{\nabla_1} \circ \phi_I \circ P_{(\partial^1 u^+_{I,J})^{-1}} \right)
- \left( T^{\omega(u^-_{I,J})} P_{(\partial^1 u^-_{I,J})} P^{\nabla_1} \circ \phi_I \circ P_{(\partial^1 u^-_{I,J})^{-1}} \right)
\]

Recall that all of the holomorphic strips between intersection points differing in index by 1 have the same area $\lambda_0 = \omega(u^+_{I,J}) = \omega(u^-_{I,J})$. Using that $\nabla_0$ is trivial local system

\[
= T^{\lambda_0} P_{c_j} P_{(\partial^1 u^-_{I,J})} (\text{id} - P_{c_j}) \circ \phi_I \circ P_{(\partial^1 u^+_{I,J})^{-1}}
\]

This vanishes if and only if $P_{c_j} = z_j = 1$ for all $1 \leq j \leq n - k$. We conclude that

\[
\text{HF}^\bullet((L_Y, \text{id}), (F_q, \nabla_1)) = \begin{cases}
H^\bullet(T^{n-k}) & z_j = 1 \text{ for all } 1 \leq j \leq n - k \\
0 & \text{otherwise}
\end{cases}
\]

**Notation 2.4.5**. Given two Lagrangians $L_0, L_1$ which intersect transversely, we will pick at each intersection point $x \in L_0 \cap L_1$ an isomorphism in $\text{hom}((E_0)_{x}, (E_1)_{x})$; by abuse of notation, we will denote this isomorphism also by $x \in \text{hom}((E_0)_{x}, (E_1)_{x})$. We can in this way write

\[
\text{CF}^\bullet(L_0, L_1) = \Lambda \langle x \rangle
\]

and the differential on this complex will be given by the structure coefficients

\[
\langle d(x), y \rangle = \sum_{u \in M_\beta(L_0, L_1, x, y)} T^{\omega(\beta)} P_{(\partial^1 u) \circ x \circ (\partial^1 u)^{-1}}.
\]

where $P_{(\partial^1 u) \circ x \circ (\partial^1 u)^{-1}} \in U_\Lambda$ is a unitary element determined by $P_{(\partial^1 u) \circ x \circ (\partial^1 u)^{-1}}$. This allows us to use the simpler (and more commonly employed) notation from definition 2.4.4.
2.4.3 $A$-tropicalization

We are now ready to define the $A$-tropicalization. When considering a complex space $X_B^C$ on the $B$ side, we used the projection $\pi_B : X_B^C \to Q$ to obtain from each subvariety of $X_B^C$ an amoeba which approximated the tropical subvariety. Just as with the $B$-tropicalization, given a Lagrangian submanifold $L \subset Q$ we could consider the Lagrangian torus fibration image of a Lagrangian submanifold $\pi_A(L) \subset Q$. However, since even Hamiltonian isotopic Lagrangian submanifolds can have very different image in the base of the Lagrangian torus fibration, this does not provide a very good definition of $A$-tropicalization. Instead, we use Lagrangian intersection Floer theory to define the $A$-tropicalization.

**Definition 2.4.6 (Preliminary).** Let $L \subset X_A$ be a Lagrangian submanifold satisfying the conditions of definition 2.4.1. We define the $A$-tropicalization or Floer theoretic support of $L$ to be the set

$$\text{Trop}_A(L) := \left\{ q \in Q \mid \text{There exists a Lagrangian brane } (F_q, \nabla) \text{ with } HF^\bullet(L, (F_q, \nabla)) \neq 0 \right\}.$$

The $A$-tropicalization is a decategorification of a much more powerful invariant captured by family Floer theory due to [Fuk02; Abo17]. From this viewpoint, the chain complexes $CF^\bullet(L, (F_q, \nabla))$ should be considered as the stalks of a sheaf which appropriately bundled together into a sheaf on $X_B$. This viewpoint on tropicalization is also employed in [SS20]. The $A$-tropicalization is a refinement of projection to the base of the Lagrangian torus fibration in the following sense:

**Proposition 2.4.7.** Let $L \subset X_A$ be a Lagrangian brane. Then $\text{Trop}_A(L) \subset \pi_A(L)$.

**Proof.** Suppose that $q \notin \pi_A(L)$. Then $F_q = \pi_A^{-1}(q)$ is disjoint from $L$. As the Floer intersection is complex is generated on the intersection points, $CF^\bullet(L, (F_q, \nabla)) = 0$. \hfill $\square$

While $\text{Trop}_A(L) \subset \pi_A(L)$ always holds, it will rarely be the case that $\pi_A(L) \subset \text{Trop}_A(L)$. By invariance of $\text{Trop}_A(L)$ under Hamiltonian isotopies, we obtain that

$$\text{Trop}_A(L) \subset \bigcap_{\phi \in \text{Ham}(X_A)} (\pi_A(\phi(L)))$$

where $\text{Ham}(X_A)$ is the set of Hamiltonian isotopies of $X_A$. However, there is no reason to expect even this to be an equality. Section 5 proves that when $L$ is a Lagrangian constructed from the data of a tropical subvariety of $Q$ the above inclusion becomes an equality. We see a toy version of this statement below.

**Example 2.4.8 (Continuation of Running Example).** We now are able to compute the $A$-tropicalization of a Lagrangian submanifold. Let $\overline{V} \subset Q$ be an integral affine k-subspace, so that $L_{\overline{V}} \subset X_A$ is a $T^{n-k} \times \mathbb{R}^k$ Lagrangian submanifold. We now compute the $A$-tropicalization of $q$. Since $\pi_A(L_{\overline{V}}) = \overline{V}$, by proposition 2.4.7 we have that $\text{Trop}_A(L_{\overline{V}}) \subset \overline{V}$. By example 2.4.4 whenever $q \in \overline{V}$ there exists a local system so that $HF^\bullet(L_{\overline{V}}, (F_q, \nabla_1)) \neq 0$. Therefore $\text{Trop}_A(L_{\overline{V}}) = \overline{V}$.

In this example, we see there are three steps of the $A$-realizability problem.

1. First, we constructed a geometric lift $L_{\overline{V}}$ of $\overline{V}$.
2. The second step is to show that we have well defined Floer cohomology groups. In the example above, this follows from \( \pi_2(X_A, L_V) = 0 \), but more generally amounts to showing that the Lagrangian \( L_V \) is unobstructed.

3. Finally, the computation of support from example 2.4.4 proves that this is a faithful lift of \( V \).

In the example of the lift of \( V \), we can do slightly more than compute the tropicalization of \( L_V \); we can compute the \( A \)-support, which is the set of pairs \((F_q, \nabla)\) which have non-trivial pairing with \( L_V \).

\[
\text{Supp}_A(L_V) = \left\{ (F_q, \nabla) : q \in \mathcal{V}, P_c^\nabla = 0 \text{ for } c \cdot \mathcal{V} = 0 \right\}
\] (4)

Here, we identify \( H_1(\mathbb{C}^*)/Z \) with \( T_\mathbb{C}(Q) \). At each point \( q \in \mathcal{V} \), there is a \((U^*_\Lambda)^k\) choice of local systems satisfying the above criteria. The support can be identified with the set \( \text{Supp}_A(L_V) = \mathcal{V} \times (U^*_\Lambda)^k \subset X_B \).

2.5 \( A \)-tropicalization for the pair of pants

In this subsection, we carry out the entire \( A \)-realizability process with the tropical curve \( V_{pants} \) from example 2.2.2. This computation first appeared in unpublished work from [Hic19a, Section 4.3], and stems from a discussion with Diego Matessi. We use this example computation to outline the remainder of the paper.

Geometric Realizability: section 3

We first need to discuss the process of building a Lagrangian submanifold which geometrically is a lift of \( V \) in the sense that \( \pi_2(\mathbb{C}^*)/Z \) approximates \( V_{pants} \). In dimension 2, one can obtain Lagrangian submanifolds in \((\mathbb{C}^*)^2\) by hyperKähler rotation of complex curves. We therefore can build a Lagrangian lift of \( V_{pants} \) by starting with the holomorphic lift \( \{(z_1, z_2) : 1 + z_1 + z_2 = 0\} \subset (\mathbb{C}^*)^2 \) and applying hyperKähler rotation. For every \( \epsilon > 0 \), we can find a Lagrangian submanifold \( L_{V_{pants}}^\epsilon \subset X_A \) Hamiltonian isotopic to our hyperKähler rotation with the following properties:

- When restricted to the complement of a neighborhood of \( 0 \in Q \), we have

\[
L_{V_{pants}}^\epsilon \setminus \pi^{-1}_A(B_\epsilon(0)) = L_{V_{1}} \cup L_{V_{2}} \cup L_{V_{3}} \setminus \pi^{-1}_A(B_\epsilon(0)).
\]

This is one of the properties which characterizes a Lagrangian lift of a tropical curve.

- Furthermore, we can construct this Lagrangian so that it is symmetric under the permutation of coordinates \((z_1, z_2)\) on \( X_A \).

Unobstructedness: section 4

The next step to the \( A \)-realization process is to show that the Lagrangian submanifold one builds can be analyzed with Floer theory. In this example, \( L_{V_{pants}}^\epsilon \) is exact and so \( \omega(\pi_2(X_A, L_{V_{pants}}^\epsilon)) \) vanishes. It follows that \( HF^*(L_{V_{pants}}^\epsilon, F_q) \) will be well defined.

\[^{2}\text{We only use this construction for the ease by which it builds a Lagrangian pair of pants in dimension 2; we emphasize at this juncture that hyperKähler rotation is not mirror symmetry.}\]
(a) The intersection of the blue holomorphic cylinder and the tropical Lagrangian pair of pants is clean, and gives a holomorphic strip with boundary on $L_{V_{\text{pants}}}$ and $F_q$.

(b) The argument projection of $L_{V_{\text{pants}}}$ to $F_q$. The intersection points are labelled. The three holomorphic strips are denoted by the arrows, with $u^{qv}$ drawn in blue.

Figure 2

Faithfulness: section 5

We now compute TropA($L_{V_{\text{pants}}}$). Consider the Lagrangian pair of pants $L_{V_{\text{pants}}}$, the Lagrangian fiber $F_q$ and the holomorphic cylinder $z_1 = z_2$ as drawn in fig. 2a. We take Hamiltonian perturbations so that the Lagrangian submanifolds intersect transversely. Nearby the point $q$, the Lagrangian $L_{V_{\text{pants}}}$ agrees with $L_{V_{\text{pants}}}$; therefore $F_q \cap L_{V_{\text{pants}}} = F_q \cap L_{V_{\text{pants}}}$. Following the notation from example 2.4.4, we call the degree 0 intersection point $x_0$, and the degree 1 intersection point $x_1$.

In addition to agreement of intersection points, there are two “small strips” contributing to the differential on $CF^\bullet(L_{V_{\text{pants}}}, F_q)$ which match the strips in the differential of $CF^\bullet(L_{V_{\text{pants}}}, F_q)$. We call these holomorphic strips $u^+_{x_0 < x_1}$, $u^-_{x_0 < x_1}$.

From the symmetry of our setup, the Lagrangian $L_{V_{\text{pants}}}$ intersects the complex plane $z_1 = z_2$ cleanly along a curve. Furthermore, the holomorphic cylinder $z_1 = z_2$ intersects $F_q$ along a circle; therefore the portion of $z_1 = z_2$ bounded by $L_{V_{\text{pants}}}$ and $F_q$ gives an example of a holomorphic strip with boundary on $L_{V_{\text{pants}}}$ and $F_q$. The ends of this holomorphic strip limit toward $x_0$ and $x_1$. The valuation projection of this strip is a line segment connecting the point $q$ with the vertex of the tropical pair of pants. For this reason, we will call this holomorphic strip $u^{qv}$. The area of this strip is the length of the line segment corresponding to $\pi_A(u^{qv})$. The three holomorphic strips are more readily seen by considering the argument projection of $L_{V_{\text{pants}}}$ to $F_q$ as in fig. 2b.

We will think of $u^{qv}$ as being a “big strip” as we can choose $\lambda_0$ small enough so that $\lambda_0 = \omega(u^+_{x_0 < x_1}) = \omega(u^-_{x_0 < x_1}) \ll \omega(u^{qv})$. If no local systems are used, the differential on $CF^\bullet(L_{V_{\text{pants}}}, F_q)$ is

$$d(x_0) = T^{\omega(u^+_{x_0 < x_1})} - T^{\omega(u^-_{x_0 < x_1})} + T^{\omega(u^{qv})} \cdot x.$$ 

This does not vanish, so $HF^\bullet(L_{V_{\text{pants}}}, F_q) = 0$.

However, to compute the $A$ support we must compute Lagrangian intersection Floer cohomology where we equip $F_q$ with a local system. We characterize the local system $\nabla$ on $F_q$ in terms of its holonomy along the arg($z_1$) and arg($z_2$) loops of $F_q$, giving us quantities $(\exp(b_1), \exp(b_2)) \in (U_\Lambda)^2$. We’ll denote such a local system by $b_1, b_2$. We’ll denote this non-unitary local system by $\nabla_{b_1, b_2}$. 

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Given a point \( q = (-a, -a) \in V_p \), we compute the quantities

\[
\omega(u_{x^g < x_1}^+) = \lambda_0 \quad \omega(u_{x^0 < x_1}^-) = \lambda_0 \quad \omega(u_{0}) = -a + \lambda_0
\]

\[
P_{\partial u_{x^g < x_1}} = \exp \left( \frac{1}{2} (b_1 - b_2) \right) \quad P_{\partial u_{x^0 < x_1}} = \exp \left( \frac{1}{2} (b_2 - b_1) \right) \quad P_{\partial u_{0}} = \exp \left( \frac{1}{2} (b_1 + b_2) \right).
\]

The weights given by the local system are determined by the paths drawn in fig. 2b, from which we obtain the differential on the \( CF^\bullet(L_{V_{pants}}, (F_q, \nabla b_1, b_2)) \):

\[
\langle d\nabla_{b_1, b_2} (x^g), x_1 \rangle = \left( \left( P_{\partial u_{x^g < x_1}} - P_{\partial u_{x^0 < x_1}} - P_{\partial u_{0}} \right) - \left( T^{2 \frac{1}{2} (b_1 - b_2)} - \exp \left( \frac{1}{2} (b_2 - b_1) \right) + \exp \left( \frac{1}{2} (b_1 + b_2) \right) \cdot T^{-a} \right) \right)
\]

This always has a \( U_A \)-worth of solutions obtained by setting \( b_1 = \log(T^{-a} \exp(b_2) - 1) \). Therefore \( (-a, -a) \in \text{TropA}(L_{V_{pants}}) \). From this we conclude that \( \text{TropA}(L_{V_{pants}}) = V_{pants} \).

This is one of the rare situations where we can compute the Floer theoretic support explicitly: under the substitution \( z_1 = T^{a_1} \exp(b_1), z_2 = T^{a_2} \exp(b_2) \), the Lagrangian tori \((F_{a_1, a_2}, \nabla b_1, b_2)\) belong to the support of \( L_{V_{pants}} \) if and only if \( z_1 - z_2 + 1 = 0 \). This should be compared with the computation of the \( B \)-realization of \( V_{pants} \) from example 2.2.2.

**B-realizability: section 6**

The matching of the supports of the \( A \)- and \( B \)-realizations of \( V_{pants} \) can be captured in the language of homological mirror symmetry. This requires a description of the *Fukaya category* of a symplectic manifold. The *Fukaya pre-category* of a compact symplectic manifold \((X, \omega)\) is given by:

- Objects are given by mutually transverse Lagrangian submanifolds \( L \subset X \) which are graded, spin, and tautologically unobstructed (section 4).
- For \( L_0 \neq L_1 \), the morphisms \( \text{hom}(L_0, L_1) \) are given by Lagrangian intersection Floer cochains \( CF^\bullet(L_0, L_1) \).
- \( k \)-compositions of morphisms

\[
m^k : \bigotimes_{i=0}^{k-1} \text{hom}^{g_i}(L_i, L_{i+1}) = \text{hom}^{2-k + \sum g_i}(L_0, L_k),
\]

are given by counts of holomorphic polygons with boundary on the \( L_k \).

This is an \( A_\infty \) *pre-category*, meaning that for every collection of objects \( L_0, \ldots, L_k \), the filtered \( A_\infty \) relations hold:

\[
\sum_{j_1 + j_2 = k} (-1)^\bullet m^{j_1 + j_2} (\text{id}^\otimes j_1 \otimes m^j \otimes \text{id}^\otimes j_2) = 0.
\]

Here \( \bullet = j_1 + \sum g_i \), and \( k \geq 1 \).
The precategory can be appropriately completed to give a triangulated $A_\infty$ category, the Fukaya category $\text{Fuk}(X_A)$. Some of the hypotheses of the construction can be dropped or modified: for example, if $X_A$ is a cotangent bundle (and not compact) there is a version of the Fukaya category (the wrapped Fukaya category, $\text{W}(X_A)$) which can be defined with appropriate Lagrangian submanifolds. $X_A = (\mathbb{C}^*)^n = T^*F_0$ is one of these cases.

The homological mirror symmetry conjecture predicts that on mirror spaces the Fukaya category and derived category of coherent sheaves are derived equivalent.

**Theorem.** Let $X_A = (\mathbb{C}^*)^n$, and $X_B = (\mathbb{C}^*)^n$. There is an equivalence of derived categories:

$$\mathcal{F} : \text{W}(X_A) \rightarrow D^b_{dg}\text{Coh}(X_B^\circ).$$

between the wrapped Fukaya category of exact admissible Lagrangian submanifolds of $X_A$ and the bounded derived category of coherent sheaves on $X_B^\circ$.

The proof of the theorem first shows that the zero section $L(0) \subset X_A$ is a Lagrangian submanifold which generates $\text{Fuk}(X_A)$. Then, $HF^\bullet(L(0), L(0))$ is shown to be the algebra $\mathbb{C}[[\mathbb{Z}^n]] = \text{hom}(\mathcal{O}(\mathbb{C}^n), \mathcal{O}(\mathbb{C}^n))$. Since this generates $D^b_{dg}\text{Coh}(X_B^\circ)$, we know that these two categories are equivalent. However, this proof is non-constructive: given an arbitrary exact Lagrangian submanifold $L \subset X_A$, there is no immediate way of determining the corresponding mirror sheaf in $D^b_{dg}\text{Coh}(X_B^\circ)$. There are a few objects which we can match up under this functor. Let $(F_q, \nabla)$ be an exact fiber of the Lagrangian torus fibration. Then $\mathcal{F}(F_q, \nabla) \simeq \mathcal{O}_z$ for some $z \in X_B$. From here, we obtain the following toy-result, whose extension to the general $V$ is the objective of the remainder of this paper.

**Theorem 2.5.1.** $V_{\text{pants}} \subset \mathbb{R}^2$ is $B$-realizable.

**Proof.** From section 2.5, we proved that $V_{\text{pants}}$ is $A$-realizable by a Lagrangian $L_{V_{\text{pants}}}$. The support of the mirror sheaf $\mathcal{F}(L_{V_{\text{pants}}})$ is a $B$-realization of $V_{\text{pants}}$. $\square$

### 3 Geometric realization

The flexibility of Lagrangian submanifolds both complicates and simplifies the construction of a Lagrangian lift of a tropical subvariety. The additional flexibility means that we have a lot of wiggle room to construct a potential lift; however, identifying a Lagrangian submanifold as “the” lift of a tropical subvariety becomes impossible. For example, given any candidate lift $L_V$ of a tropical subvariety $V$, one could apply a Hamiltonian isotopy to $V$ to obtain a new Lagrangian submanifold.

More generally, each potential Lagrangian lift $L_V$ of $V$ is supposed to represent the data of a sheaf on $X_B$ whose support has tropicalization $V$; there are many such sheaves!

Despite all of this flexibility, we already have a good idea of what the Lagrangian lift $L_V$ of $V$ should look like from eq. (1). Recall that $V^{(0)}$ is the union of the interiors of the top dimensional polyhedral domains $\underline{V}$ defining $V$. At each component we can take the conormal torus construction to obtain a Lagrangian chain:

$$L_{V^{(0)}} := \bigcup_{\underline{V} \subset V^{(0)}} L_{\underline{V}}.$$  

Intuitively, a geometric Lagrangian lift of $V$ should approximate the chain $L_{V^{(0)}}$. 

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Remark 3.0.1. Fix an orientation on $F_q$, a fiber of the SYZ fibration. Then $L_V$ inherits an orientation (which in local coordinates comes from $dq_1 \wedge \cdots \wedge dq_k \wedge dp_{k+1} \wedge \cdots \wedge dp_n$). We will assume that we have fixed an orientation on $F_q$ in advance so that $L_V$ are equipped with a standard orientation.

We propose the following definition for a geometric Lagrangian lift of a tropical subvariety (which is similar to that proposed in [Mik19, Definition 2.1]).

Definition 3.0.2. A family of oriented Lagrangian submanifolds $L_\varepsilon V$ for $\varepsilon > 0$ is a geometric Lagrangian lift of an weight-1 polyhedral complex $V \subset Q$ if the following conditions hold:

(i) The Lagrangians $L_\varepsilon V$ are all Hamiltonian isotopic,

(ii) Let $V^{(i)}$ be the collection of codimension $i$ strata of $V$. We require that away from the codimension 1 strata,

$$L_\varepsilon V \setminus \pi_A^{-1}(B_\varepsilon(V^{(1)})) = L_{\varepsilon V}^{(0)} \setminus \pi_A^{-1}(B_\varepsilon(V^{(1)}))$$

as oriented submanifolds.

(iii) The Lagrangians $L_\varepsilon V$ are embedded, graded, spin, and admissible (in the sense of definition 3.1.1).

Remark 3.0.3. Definition 3.0.2 has two simplifying requirements; one is included due to current technical limitations in the definition of Floer cohomology, and the second is for convenience.

The requirement that $L_\varepsilon V$ is embedded is a technically needed assumption; we believe that this condition can be dropped without modifying the main results of this paper. Our reason for restricting ourselves to the embedded setting is that the Charest-Woodward pearly model as written does not include a description of Floer cohomology for immersed Lagrangian submanifolds.

While definition 3.0.2 looks only at weight-1 polyhedral complexes, one can extend the story to weighted polyhedral complexes by asking that at each top dimensional stratum $\mathcal{V} \subset \mathcal{V}^{(0)}$ with weight $m$, the realization $L_{\varepsilon V}$ is $m$-disjoint copies of $N^*_V/N^*_V$. All results in this paper can be extended to the weighted setting.

The constructions from [Mat18, Mik19, MR20, Hic19a] all satisfy (definition 3.0.2 items (i) and (ii)). To prove that the previous definitions give examples of geometric Lagrangian lifts, we need to additionally show that they are admissible, graded and spin. We prove these properties for certain examples of Lagrangian lifts in sections 3.1 to 3.3.

While definition 3.0.2 only asks that we take the lift of an weight-1 polyhedral complex, the only polyhedral complexes which admit such lifts are tropical ones.

Proposition 3.0.4. Let $V$ be a weight-1 rational polyhedral complex, and suppose that it has a Lagrangian lift $L_\varepsilon V$ satisfying (definition 3.0.2 items (i) and (ii)). Then $V$ is a tropical subvariety.

Proof. Select an interior point $r \in \mathcal{W} \subset \mathcal{V}^{(1)}$ of the codimension 1 stratum of $V$. Pick $U_r \subset T_r Q$ a rational subspace so that $U_r \oplus T_r W = T_r Q$. Let $R \subset Q$ be a small polyhedral domain passing through $q$ with tangent space $U_q$. Then $V|_R$ is a weight-1 rational polyhedral curve. By taking $R$ small enough, $V|_R$ has a single vertex and edges pointing in directions $v_1, \ldots, v_k$ corresponding to facets $F_1, \ldots, F_k$ containing $W$. We need to prove that $\sum_{i=1}^k v_i = 0$. See fig. 21.
Consider the symplectic manifold $Y_A := T^*R/TZ^*R \subset T^*Q/TZ^*Q$, with the Lagrangian torus fibration $\pi_{Y_A}: Y_A \to R$. Let $i: R \to Q$ be the inclusion. Select $\varepsilon$ small enough so that $B^\varepsilon(W) \cap R$ is an interior set of $R$. Given a Lagrangian submanifold $L \subset T^*Q/TZ^*Q$, we can take a Hamiltonian perturbation of $L$ so that $L_i^* \circ L := \{(r, i^*(p)) : (r, p) \in L, r \in R\}$ is a Lagrangian submanifold of $Y$. See [HH22, Section 5.2] for a more general discussion of this construction from the perspective of Lagrangian correspondences. By definition $\pi_{Y_A}(L_i^* \circ L) = \pi_{Y_A}(L) \cap R$, so $L_{V|_R}^\varepsilon := L_i^* \circ L_{V|_R}^\varepsilon$ is a geometric realization of $V|_R \subset R$. We, therefore, have reduced to the setting which is the lift of a tropical curve with a single vertex.

Given a tropical curve $V|_R \subset R$ with a single vertex $v$, the Lagrangian $L_{V|_R}^\varepsilon$ is a manifold with boundary. Consider the projection $\text{arg}_R: Y_A \to F_r = T^*_rR/TZ^*_rR$. Considering $\text{arg}_R(L_{V|_R}^\varepsilon)$ as a $(\dim(F_r) - 1)$ chain, we obtain the relation in homology

$$0 = [\text{arg}_R(\partial(L_{V|_R}^\varepsilon))] \in H_{\dim F_r - 1}(F_r).$$

There is an identification (as vector spaces) which sends an integral basis $e_1, \ldots, e_n$ to the class of the perpendicular subtorus

$$T_q R \to H_{\dim F_r - 1}(F_r),
\quad e_i \mapsto \{\eta \in T^*_rR : \eta(e_i) = 0\}.$$ 

The boundary lies of $L_{V|_R}^\varepsilon$ in the region where eq. [5] holds and we have an agreement of oriented submanifolds, we can therefore compute:

$$[\text{arg}_R(\partial(L_{V|_R}^\varepsilon))] = \sum_{i=1}^k \{\eta \in T^*_rR : \eta(e_i) = 0\}$$

proving that $\sum e_i = 0$. 

**Notation 3.0.5.** From here on, we will drop the $\varepsilon$ in $L_{V|_R}^\varepsilon$ and simply write $L_V$ for a Lagrangian which belongs to such a family.
3.1 Geometric Lagrangian lift: admissibility

When Lagrangian submanifolds are non-compact, we need to place taming conditions on them so that they are Floer-theoretically well behaved.

**Definition 3.1.1** ([Han19]). Let $W_\Sigma : X_A \to \mathbb{C}$ be a Laurent polynomial whose monomials are indexed $A$, the set of rays of a fan $\Sigma$. A monomial division $\Delta_\Sigma$ for $W_\Sigma = \sum_{\alpha \in A} c_\alpha z^\alpha$ is an assignment of a closed set $U_\alpha \subset \mathbb{Q}$ to each monomial $\alpha \in A$ so that the following conditions hold:

- The sets $U_\alpha$ cover the complement of a compact subset of $Q = \mathbb{R}^n$;
- There exist constants $k_\alpha \in \mathbb{R}_{>0}$ so that for all $z$ with $\text{val}(z) \in U_\alpha$ the expression
  \[ \max_{\alpha \in A} (|c_\alpha z^\alpha|^{k_\alpha}) \]
  is always achieved by $|c_\alpha z^\alpha|^{k_\alpha}$; and
- $U_\alpha$ is a subset of the open star of the ray $\alpha$ in the fan $\Sigma$.

A Lagrangian $L \subset X_A$ is $\Delta_\Sigma$-monomially admissible if over $\text{val}^{-1}(U_\alpha)$ the argument of $c_\alpha z^\alpha$ restricted to $L$ is zero outside of a compact set.

We will always assume that the $\text{arg}(c_\alpha) = 0$. An advantage of using the monomial admissibility condition for Lagrangian submanifolds is that it is a relatively simple check to see if a Lagrangian submanifold satisfies the condition.

**Theorem** (Theorem 3.1.7 of [Hic20]). Suppose that $V$ is the tropicalization of a hypersurface whose Newton polytope has dual fan $\Sigma$. Then the construction of $L_V$ from [Hic19a] is $\Delta_\Sigma$-monomially admissible.

Let $V \subset Q$ be a tropical curve. We say that $V$ is adapted to $\Sigma$ if each semi-infinite edge of $V$ points in the direction of a ray of $\Sigma$.

**Claim 3.1.2.** Suppose that $V \subset \mathbb{R}^n$ is an weight-1 tropical curve adapted to $\Sigma$. Any Lagrangian lift $L_V$ is $\Delta_\Sigma$-monomially admissible.

*Proof.* Let $V^{(0)}_\infty = \{ e_i \}_{i=1}^k$ denote the semi-infinite edges of $V$. We note that there exists a compact set $K \subset Q$ so that $L_C \setminus \pi^{-1}_A(K) = \bigsqcup_{e \in V^{(0)}_\infty} L_e \setminus \pi^{-1}_A(K)$. Furthermore $K$ can be chosen so that $e \setminus K \subset U_\alpha$ if and only if $e$ points in the direction $\alpha \in \Sigma$. Over this region, we observe that $\text{arg}(z^\alpha)|_{N^*e/N^*_e} = 0$. \(\square\)

**Remark 3.1.3.** In the event that some of the semi-infinite edges of $V$ are weighted, we must replace the last condition in monomially admissible with “there exists a discrete set of values $\{\theta_i\}$ such that the argument of $\text{arg}(c_\alpha z^\alpha)|_{L \cap C} \subset \{\theta_i\}$”. The Floer theoretic arguments in [Han19] can be applied to this setting as well (simply by letting $\theta_i$ be $k$-roots of unity, and replacing $\alpha$ with $k\alpha$).
3.2 Geometric Lagrangian lifts: homologically minimal and graded

The additional amount of flexibility that symplectic geometry affords us means that there are many geometric Lagrangian lifts of a single tropical subvariety. Some of these lifts differ for unimportant reasons: for instance, we could have included some extra topology in our Lagrangian by attaching a Lagrangian with vanishing Floer cohomology to a previously constructed lift. The following condition is imposed to weed out some of these worst offenders.

**Definition 3.2.1.** Let \( j : L_{V(0)} \setminus \pi_A^{-1}(B'(V^{(1)})) \hookrightarrow L_V \) be the inclusion of the top dimensional components of \( V \) into \( V \). We say that a lifting is homologically minimal if there exists a section \( i : V \to L_V \subset X_A \) so that \( H_1(L_V) \) is generated by the images of \( (i)_* : H_1(V) \to H_1(L_V) \) and \( (j)_* : H_1(L_{V(0)} \setminus \pi_A^{-1}B(V^{(1)})) \to H_1(L_V) \).

Let \( i_{L_V} : L_V \to X_A \) be the inclusion of our Lagrangian submanifold. We say that \( L_V \) is an untwisted realization of \( V \) if the composition

\[
V \xrightarrow{(i_{L_V})} X_A \xrightarrow{\arg} F_q
\]

is null-homologous.

**Remark 3.2.2.** For a fixed tropical subvariety \( V \), there can be several geometric Lagrangian lifts of \( V \) which are meaningfully different. We expand on how these map to choices of lifts correspond to tropical line bundles of \( V \) in section 6.3.

The homologically minimal condition places some constraints on our Lagrangian submanifolds.

**Lemma 3.2.3.** If \( L_V \) is homologically minimal and untwisted, then \( L_V \) is graded.

**Proof.** We recall the definition of graded from [Sei00, Example 2.9]. Since \( c_1(X_A) = 0 \), we can take a section \( \bigwedge_{i=1}^n (dq_i + id\theta_i)^{\otimes 2} \) of \( \bigwedge^n (TX_A, J)^{\otimes 2} \). This determines a map

\[
\det^2 \circ s_L : L \to S^1
\]

\[
x \mapsto \left( \bigwedge (dq_i + id\theta_i) (T_x L) \right)^2
\]

A Lagrangian is \( \mathbb{Z} \)-graded if this map can be lifted to \( \mathbb{R} \).

Consider a homologically minimal Lagrangian and untwisted Lagrangian \( L_V \). There exist generators \( \{[\alpha_k], [\beta_l] \} \) for \( H_1(L_V) \) so that \( \alpha_k \) is in the image of \( i \) and \( \beta_l \) are in the image of \( j \). Since the compositions

\[
\det^2 \circ s_{L_V} \circ i : V \to S^1
\]

\[
\det^2 \circ s_{L_V} \circ j : (L_{V(0)}) \to S^1
\]

are constantly 0, it follows that there is no obstruction to lifting \( \det^2 \circ s_{L_V} : L_V \to S^1 \) to \( \mathbb{R} \).

**Proposition 3.2.4.** Suppose that \( V \subset \mathbb{R}^n \) is either a smooth tropical curve or a smooth tropical hypersurface. Then the construction of \( L_V \) given by [Hic19a; Mik19; Mat18] produce homologically minimal Lagrangian lifts \( L_V \). The lifts are therefore graded.
Figure 4: Covering our tropical curve $V$ with two charts: $W$ and a pair of pants $\text{star}(v)$ centered at $v$.

Proof. In the cases of tropical curves, this follows from computing the homology of $L_V$ from a cover given by $L_{\text{star}(v)}$. For hypersurfaces, this is proven in [Hic20, Proposition 3.18].

Unless otherwise specified, the lift of a smooth tropical curve or hypersurface will always be the one given by [Hic19a; Mik19; Mat18].

3.3 Geometric Lagrangian lifts: spin

We start with a lemma on the topology of lifts of smooth genus zero tropical curves.

Lemma 3.3.1. Let $V \subset \mathbb{R}^n$ be a smooth genus 0 tropical curve.

(i) For any semi-infinite edge $f \in V^{(0)}_{\infty}$, the restriction map $\text{res}_f^V : H^1(L_V) \to H^1(L_f)$ is a surjection.

(ii) For any semi-infinite edge $f$, the restriction map $\text{res}_{V\setminus f}^V : H^2(L_V) \to \bigoplus_{g \neq f} H^2(L_g)$ is an injection.

Proof. We prove items [(i)] and [(ii)] by induction on the number of vertices in $V$.

Base Case: Suppose that $V$ has one vertex. Then $V$ is planar, and there exists a splitting of $(\mathbb{C}^*)^n = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-2}$ so that $L_V = L_{\text{pants}} \times T^{n-2}$, where $L_{\text{pants}} \subset (\mathbb{C}^*)^2$ is the standard pair of pants. The boundary of the pair of pants is $S_{e_1} \cup S_{e_2} \cup S_{e_3}$, where $e_1, e_2, e_3$ label the three edges of the pair of pants. A direct computation shows that $H^0(L_{\text{pants}}) \to H^0(S_{e_1})$ surjects, and that $H^1(L_{\text{pants}}) \to H^1(S_{e_1})$ inject. An application of Künneth formula gives items [(i)] and [(ii)] for $L_V$.

Inductive Step: Let $f \in V^{(0)}_{\infty}$ be any semi-infinite edge and let $v$ be the vertex of $V$ belonging to that edge. Let $W$ be the tropical curve given by vertices not equal to $v$, so that $L_{\text{star}(v)}, L_W$ cover $L_V$ with intersection $L_{\text{star}(v)} \cap L_W = L_e = T^{n-1} \times e$, as in fig. 4. This can be done because $V$ is a tree.
We first prove item (i). We use $L_{\text{star}(V)}, L_W$ to compute the first cohomology of $L_V$ using Mayer-Vietoris, and show that the red arrow in the diagram below is a surjection.

\[
\begin{array}{ccc}
H^1(L_V) \oplus res^V_{\text{star}(V)} & \xrightarrow{\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W} & H^1(L_{\text{star}(V)}) \oplus H^1(L_W) \\
\downarrow & & \downarrow \\
H^1(L_f) \oplus H^1(L_f) & \xrightarrow{\text{res}_f^V \oplus 0} & H^1(L_f) \\
\end{array}
\]

From the base case: given $\alpha \in H^1(L_f)$, there exists $\alpha' \in H^1(L_{\text{star}(V)})$ with $\text{res}^V_{\text{star}(V)}(\alpha') = \alpha$. From the induction hypothesis, there exists $\beta' \in H^1(L_W)$ with $\text{res}_e^W(\beta') = \text{res}^V_{\text{star}(V)}(\alpha')$. Therefore $(\alpha', \beta') \in \ker(\text{res}_e^V - \text{res}_e^W)$, and by exactness of the rows is in the image of $\text{res}_e^V \oplus \text{res}^V_W$. Let $\alpha''$ be in the preimage of $(\alpha', \beta')$. By commutativity of the below diagram, we conclude $\text{res}^V_e(\alpha'') = \alpha$.

We now prove item (ii). We compute $H^2(L_V)$ using Mayer-Vietoris, and show that the blue arrow below is injective. By item (i) the leftmost arrow is surjective. Using exactness of the sequence, we conclude that $\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W$ is injective on the second cohomology groups.

\[
\begin{array}{ccc}
H^1(L_{\text{star}(V)}) \oplus H^1(L_W) & \xrightarrow{\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W} & H^1(L_V) - 0 \\
\oplus g \neq f \text{ res}^V_g & & \oplus g \in W_0 \bigoplus g \neq e \text{ res}^V_g \\
\downarrow & & \downarrow \\
H^2(L_f) & \oplus g \in V_0^{(0)} \bigoplus g \neq f & H^2(L_f) + H^2(L_g) \\
\end{array}
\]

Let $C = \bigoplus_{g \in \text{star}(V)_0} \text{res}^V_g$ and $D = \bigoplus_{g \in W_0} \text{res}^V_g$. Consider now a class $\alpha \in H^2(L_V)$. Suppose that $\bigoplus_{g \neq f} \text{res}^V_g(\alpha) = 0$. We will show that $\alpha = 0$. By commutativity of the diagram, $(C \oplus D) \circ (\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W) = 0$. Because $D$ is injective, and the $(\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W)$ is injective, we obtain that $C \circ \text{res}^V_{\text{star}(V)}(\alpha) = 0$, and $\text{res}^V_W(\alpha) = 0$. We now break into two cases:

- **Case I:** $\text{res}^V_{\text{star}(V)}(\alpha) = 0$. This implies $(\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W)(\alpha) = 0$, which by injectivity of $\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W$ tells us that $\alpha = 0$.

- **Case II:** $\text{res}^V_{\text{star}(V)}(\alpha) \neq 0$. Observe then that $(C \oplus \text{res}^V_{\text{star}(V)} \circ \text{res}^V_{\text{star}(V)})$ is injective from the base case, so $\text{res}^V_{\text{star}(V)} \circ \text{res}^V_{\text{star}(V)}(\alpha) \neq 0$. Since $\text{res}^V_W(\alpha) = 0$, we obtain that

\[
(\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W) \circ (\text{res}^V_{\text{star}(V)} \oplus \text{res}^V_W)(\alpha) \neq 0.
\]

This violates exactness of the top row, so case II cannot occur.
Proposition 3.3.2. In the setting where $V \subset \mathbb{R}^n$ has genus 0, the constructions of [Mik19; Mat18; Hic20] give homologically minimal untwisted geometric Lagrangian lifts $L_V$ of $V$.

Proof. We prove that this Lagrangian submanifold is homologically minimal because the homology of the pair of pants is generated by the homology of the legs. If $n = 2$, then $L_V$ is a surface, and therefore spin.

To prove that the $n \geq 3$ cases are spin, we induct on the number of vertices in $V$. For the 1-vertex case, $L_{\text{star}(v)} \cong L_{\text{pants}} \times T^{n-2}$. The manifolds $L_{\text{pants},v} \times T^{n-2}$ have trivializations given by embedding $L_{\text{pants},v}$ into $\mathbb{R}^2$, and is therefore spin.

As in the proof of lemma 3.3.1, write $V = L_W \cup L_{\text{star}(v)}$, where $e$ is the common edge $L_W \cap L_{\text{star}(v)}$. By the induction hypothesis a spin structure on $L_W$. By pullback, this gives a spin structure over $L_e$. Since $H^1(L_{\text{star}(v)}, \mathbb{Z}/2\mathbb{Z}) \to H^1(L_e)$ surjects, there is no obstruction to picking a spin structure on $L_{\text{star}(v)}$ agree with the prescribed spin structure on $L_e$.

This method of proof can be extended to a slightly larger set of examples. We say that a smooth tropical curve $V$ has planar genus if there exists cycles $c_1, \ldots c_k \subset V$ so that $\{[c_1], \ldots, [c_k]\}$ generate $H_1(V)$, and there exist 2-dimensional planes $V_k \subset \mathbb{R}^n$ so that $c_i \subset V_k$.

Corollary 3.3.3. If $V \subset \mathbb{R}^n$ is a smooth tropical curve of genus $V$ with planar genus, then $L_V$ is spin.

The other setting where tropical Lagrangian lifts have been studied is the setting of hypersurfaces.

Lemma 3.3.4. If $V \subset \mathbb{R}^n$ is a smooth tropical hypersurface, the construction of [Mat18, Hic20] of $L_V$ is spin.

Proof. We break into several cases.

- If $n = 2$, then $L_V$ is a punctured surface (and therefore spin).
- If $n = 3$, then $L_V$ is an orientable 3-manifold (and therefore spin).
- If $n \geq 4$, then by [Hic20] the Lagrangian $L_V$ is the connect sum of two copies of $\mathbb{R}^n$ at several contractible regions $U_{\alpha}$ indexed by $\Delta$, the Newton polytope of the defining tropical polynomial for $V$. Assume that $\dim(\Delta) = n$ (as otherwise we may reduce to one of the previous cases). Following [Hic20, Proposition 3.18], we take two charts $L_r, L_s \simeq \mathbb{R}^n \setminus \bigcup_{\alpha \in \Delta} U_{\alpha}$ so that $L_V = L_r \cup L_s$. The $L_r, L_s$ are homotopic to $V$. The overlap $L_r \cap L_s \simeq \bigcup_{\alpha \in \Delta} \partial U_{\alpha}$, where each $\partial U_{\alpha}$ is homotopic to either $S^{n-1}$ or $D^{n-1}$. By Mayer-Vietoris, we compute

$$\bigoplus_{\alpha \in \Delta} H^1(\partial U_{\alpha}) \to H^2(L_V, \mathbb{Z}/2\mathbb{Z}) \to H^2(L_r, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(L_s, \mathbb{Z}/2\mathbb{Z}).$$

The left and right term are zero when $n \geq 4$, so $H^2(L_V, \mathbb{Z}/2\mathbb{Z}) = 0$ and our Lagrangian is spin.

□
4 Unobstructed Lagrangian lifts of tropical subvarieties

Since the geometric Lagrangian lifts $L_V$ we construct will not be exact, to obtain a Lagrangian Floer cohomology theory we need to show that these Lagrangian submanifolds have $\Lambda$-filtered $A_\infty$ algebra which can be unobstructed.

4.1 Pearly model for Floer cohomology

We will adopt the model employed in [CW19] to define $CF^\bullet(L)$.

**Theorem (CW19).** Let $L \subset X$ be a compact relative spin and graded Lagrangian submanifold inside a rational compact symplectic manifold $X$. Pick $h : L \to \mathbb{R}$ a Morse function, and $D \subset X \setminus L$ a stabilizing divisor. There exists a choice of perturbation datum $P$ which defines a filtered $A_\infty$ algebra $CF^\bullet(L, h, P, D)$ whose

- Chains are given by the Morse cochains of $L$, so that $CF^\bullet(L, h, P, D) = \Lambda(\text{Crit}(h))$.
- Product structures come from counting configurations of treed disks. More precisely, given a collection of critical points $\underline{x} = (x_1, \ldots, x_k)$, we define the structure coefficients

$$\langle m^k(x_1 \otimes \cdots \otimes x_k), x_0 \rangle = \sum_{\beta \in H_2(X, L)} (-1)^\triangledown (\sigma(u)!)^{-1} T^\omega(\beta) \cdot \#\mathcal{M}_P(X, L, D, \underline{x}, \beta)$$

which determine the $A_\infty$ product structure. Here, $\#\mathcal{M}_P(X, L, D, \underline{x}, \beta)$ is the count of points in the moduli space of $P$-perturbed pseudoholomorphic treed disks, $\sigma(u)$ denotes the number of stabilizing points on each of these treed disks, and $\triangledown = \sum_{i=1}^k i |x_i|$.

The $\Lambda$-filtered $A_\infty$ homotopy class does not depend on the choices of perturbation, divisor, and Morse function taken in the construction.

When the choice of $h$, $P$ and $D$ are unimportant, we will write $CF^\bullet(L)$ instead of $CF^\bullet(L, h, P, D)$. The most visible difference between the tautologically unobstructed setting and this more general definition is that there now exists a curvature term $m_0 : \Lambda \to CF^\bullet(L)$, which obstructs the squaring of the differential to zero. We say that $L$ is unobstructed if $CF^\bullet(L)$ has a bounding cochain $b \in CF^\bullet(L)$ (appendix B.1). When $L$ is unobstructed, the deformed $A_\infty$ structure on $CF^\bullet(L, b)$ is a chain complex.

In this section, we discuss whether a geometric Lagrangian lift $L_V$ of a tropical subvariety is an unobstructed Lagrangian submanifold. We give an example computation in the pearly disk model to fix notation.

**Example 4.1.1 (Running Example, Continued).** We return to example 2.4.4. First, we examine $CF^\bullet(F_q, \nabla_1)$. Since $F_q$ is exact, it bounds no holomorphic disks, so this is simply the Morse-tree algebra of $F_q$. Give $F_q$ the torus the Morse function

$$f = \sum_{i=1}^n \cos(\theta_i)$$

(6)

We label the generators of

$$CF^\bullet(F_q, \nabla_1) = \Lambda(y_1)$$

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where the \( I \subset \{1, \ldots, n\} \). The differential is given by \( m^1(y_I^1) = 0 \), and for a particular set of perturbations the product structure is

\[
m^2(y_I^1 \otimes y_J^1) = \begin{cases} 
\pm y_{I \cup J}^1 & \text{if } I \cap J = \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

where the sign is determined by the number of transposition required to reorder \( I \cup J \).

**Remark 4.1.2.** To our knowledge, it is unknown if there exists a perturbation scheme for Morse flow trees so that all higher products \( m^k : CM^\bullet(S^1)^{\otimes k} \to CM^\bullet(S^1)[2-k] \) vanish.

To work in the setting where \( X_A \) is non-compact, we need to place restrictions on the non-compact behavior of the Lagrangian \( L \) to ensure that the moduli spaces of pseudoholomorphic treed-disks considered by [CW19] remain compact. A natural condition to impose is that \( L \subset X_A \) is admissible (definition A.0.1) with respect to a potential function \( W : X_A \to \mathbb{C} \), so that the projection \( W(L) \) fibers over the real axis \( \mathbb{R}_{>0} \) outside of a compact set. Denote by \( Y_A = W^{-1}(t) \) for \( t \in \mathbb{R}_{>0} \). Choices of different sufficiently large \( t \) will yields fibers which are symplectomorphic. The restriction of \( L \) to a large fiber will be called \( M := L|_{Y_A} \); this is a Lagrangian submanifold of \( Y_A \). By theorem A.0.2 there exists a treed-disk model for Lagrangian Floer cohomology \( CF^\bullet(L) \) for \( W \)-admissible Lagrangians \( L \). Furthermore, there exist compatible choices of perturbation datum so the standard projection

\[
CF^\bullet(L) \to CF^\bullet(M)
\]

is a \( \Lambda \)-filtered map of \( A_\infty \) algebras.

A useful lemma of [Han19] states that when we have a monomially admissible Lagrangian \( L \), there exists a potential function \( W \) so that \( L \) is \( W \)-admissible. From the data of a fan and \( t \in \mathbb{R} \), [Abo09] constructs **tropicalized potential**, which is a symplectic fibration \( W^t : (\mathbb{C}^*)^n \to \mathbb{C} \) outside of a compact set.

**Lemma 4.1.3** (Section 4.4 of [Han19], Remark 2.10 [HH22]). Suppose that \( L \) is Lagrangian submanifold which is monomially admissible with respect to a monomial division adapted to \( \Sigma \) (in the sense of definition 3.1.1). Then \( L \) can be made admissible for the tropicalized potential.

### 4.2 Geometric Lagrangians versus Lagrangian branes

**Definition 4.2.1.** We say that an unobstructed Lagrangian submanifold \( (L_V, b) \) is a Lagrangian brane lift of \( V \) if \( L_V \) is a geometric Lagrangian lift of \( V \).

Before developing constructions of bounding cochains for geometric Lagrangian lifts, we give some examples of geometric Lagrangian lifts which are known to be unobstructed (or tautologically unobstructed) Lagrangian submanifolds.

**Example 4.2.2** (Lagrangian pair of pants). In [Mat18], it was shown that the tropical pair of pants centered at the origin was an exact Lagrangian submanifold; a similar proof was given in [Hic21b] which showed that all tropical Lagrangian submanifolds constructed from the data of a dimer are exact.

**Claim 4.2.3.** Let \( V \subset \mathbb{R}^n \) be a tropical variety so that 0 \( \in V \), for all facets \( V \subset V \). Let \( L_V^1 \) be a homologically minimal lift of \( V \). Then \( V \) is exact.
Proof. Let $\eta = pdq$ be the primitive for $\omega$ on $X_A = T^*T^n$. We need to show that $\eta$ is exact on $L^\epsilon_V$; equivalently show that $\eta(\gamma) = 0$ for all $[\gamma] \in H_1(L^\epsilon_V)$. Observe that $L^\epsilon_V \simeq \pi^{-1}B(0)$, then for every loop $\gamma \in H_1(L^\epsilon_V)$, there exists $\gamma'$ which is homotopic to $\gamma$ and lives within in $\pi^{-1}B(0)$; by letting $\epsilon \to 0$ we obtain $[\gamma'] = [\gamma]$ and $\gamma' \in F_0$. As $F_0$ is exact, $\eta(\gamma') = 0$.

Since these Lagrangians are exact, they are tautologically unobstructed and we can conclude that $L^\epsilon_V$ is a tropical Lagrangian brane.

In some cases, one obtains tautological unobstructedness (or unobstructedness) of the Lagrangian submanifold for free.

**Example 4.2.4.** Curves $V \subset \mathbb{R}^2$ provide an example of where we can obtain tautological unobstructedness. Since $L_V$ is a graded Lagrangian submanifold, the only holomorphic curves which might cause us difficulty are Maslov index 0 curves. However, the expected dimension of Maslov index 0 disks with boundary on a 2-dimensional Lagrangian is negative, therefore for a generic choice of almost complex structure these disks disappear and $L^\epsilon_V$ is tautologically unobstructed.

It is possible for non-regular Maslov index 0 disks to appear with boundary on $L^\epsilon_V$, even in simple examples (see example 4.2.11). More generally, [Hic21b] shows that there exists a “wall-crossing” phenomenon which occurs for isotopies between tropical Lagrangian submanifolds, and that the count of these Maslov index 0 holomorphic disks play a crucial role in understanding coordinates on the moduli space of tropical Lagrangian submanifolds.

**Example 4.2.5.** We now examine a setting outside of the mirrors to toric varieties. Let $Q$ be any tropical abelian surface; then $X_A := T^*Q/T^*_Q$ is a symplectic 4-torus. Given any tropical curve $V \subset Q$, there is a Lagrangian surface $L^\epsilon_V \subset X_A$. By the same reasoning as above, $L^\epsilon_V$ is tautologically unobstructed for generic choice of almost complex structure.

**Example 4.2.6.** Another example where we know unobstructedness for geometric Lagrangian lifts is [MR20]. In that setting, the base of the Lagrangian torus fibration has non-trivial discriminant locus, and the tropical Lagrangians constructed are lifts of compact genus-0 tropical curves in the base. Mak and Ruddat show that the associated tropical Lagrangians are homology spheres and therefore are always unobstructed by a choice of bounding cochain ([Fuk+10]).

In general, other techniques are required to prove that a geometric Lagrangian lift of a tropical subvariety is unobstructed.

**Example 4.2.7.** Given any smooth tropical hypersurface $V \subset \mathbb{R}^n$, [Hic19a] shows that the tropical Lagrangian lift can be equipped with a bounding cochain so that $(L_V, b)$ is an unobstructed Lagrangian submanifold of $(\mathbb{C}^*)^n$. The proof uses that $L_V$ can be constructed as a mapping cone of two Lagrangian sections in the Fukaya category; as these sections bound no holomorphic strips or disks, one expects that their Lagrangian connect sum can be equipped with a bounding cochain. In practice, the process of constructing the bounding cochain is delicate.

We furthermore expect that similar methods should show that given $V = V_1 \cap \cdots \cap V_k$ a transverse intersection of tropical hypersurfaces $V_i$, there exists $L_V$ an unobstructed Lagrangian lift of $V$. The Lagrangian $L_V$ is constructed by using the fiberwise sum of the lifts [Sub10, HH22], so that $L_V = L_{V_1} + q \cdots + q L_{V_k}$.
While the resulting Lagrangian submanifold $L_V$ may be immersed, over the top dimensional stratum of $V$ the Lagrangian submanifold $L_V$ satisfies definition 3.0.2. This provides the geometric realization. To obtain unobstructedness, we can also write $L_V$ as the geometric composition of unobstructed Lagrangian correspondences (each giving the fiberwise sum with $L_{V_i}$). It is expected (from the work [WW10; Fuk17]) that the geometric composition of unobstructed Lagrangian correspondences is unobstructed in this setting, from which it follows that $L_V$ is unobstructed by the pushforward bounding cochain.

Example 4.2.8. Given a smooth tropical hypersurface $V$ of a tropical abelian variety $Q = \mathbb{R}^n/M\mathbb{Z}$, [Hic19a, Example 5.2.0.7] constructs an unobstructed Lagrangian lift $(L_V, b)$ inside the symplectic torus $T^*Q/T^*_ZQ$. The proof of unobstructedness is easier than the hypersurface setting (as one does not need to worry about issues of compactness).

The next two examples were suggested by Dhruv Ranganathan.

Example 4.2.9. Suppose that $L_1 \subset X_1$, $L_2 \subset X_2$ be tautologically unobstructed Lagrangian submanifolds. Then $L_1 \times L_2 \subset X_1 \times X_2$ is again a tautologically unobstructed Lagrangian submanifold. Furthermore, if the methods in [Amo17] can be adapted to the Charest-Woodward model of Floer cohomology that we use, then the product of unobstructed Lagrangians is unobstructed. It follows that when $V_i \subset Q_i$ have Lagrangian brane lifts, then so does $V_1 \times V_2 \subset Q_1 \times Q_2$.

Example 4.2.10. Suppose for $t \in [0, 1]$ we have geometric Lagrangian lifts $L_{V_t}$ of a family of tropical subvarieties $V_t$. Furthermore, suppose that for $t \in [0, 1)$ the lift is a Lagrangian brane lift. Then $L_{V_1}$ is a Lagrangian brane lift of $V_1$. The proof uses Fukaya’s trick to choose perturbation data so that for $t$ close to 1, the $L_{V_t}$ all have the same moduli spaces of pseudoholomorphic disks. Then there exists a subsequence of bounding cochains for the $L_t$ which converge to a bounding cochain on $L_1$.

There are few general criteria for determining if a Lagrangian submanifold is unobstructed. To highlight some of the subtlety of the problems, we exhibit a tropical Lagrangian which bounds a non-regular Maslov-index zero disk.

Example 4.2.11. Consider the projection to the base of the Lagrangian torus fibration of the tropical Lagrangian submanifold $L_{V_4}$ drawn in fig. [3]. Let $\ell$ be the dashed red line. Take the standard metric on $\mathbb{R}^2$ so we may identify $T\mathbb{R}^2$ with $T^*\mathbb{R}^2$. Provided that one takes a symmetric construction of the Lagrangian pairs of pants (for example, using the construction of [Mat18]), the holomorphic cylinder $T\ell/T\mathbb{Z}\ell$ tropicalizing to the line $\ell$ intersects the Lagrangian $L_{V_4}$ cleanly along an $S^1$. This yields an isolated holomorphic disk with boundary on $L_{V_4}$. This is not a regular holomorphic disk.

Further examples of non-regular Maslov-index zero disks are given in [Hic21a]. In section 6.3 we give examples of geometric Lagrangians lifts which are unobstructed, but not tautologically unobstructed for any choice of admissible almost complex structure on $(\mathbb{C}^*)^n$. In section 6.4 we show that there exists $V$ such that $L_V$ is obstructed.

4.3 Unobstructedness at boundary

We now give a method for constructing a bounding cochain for a Lagrangian which is $W$-admissible.
Theorem 4.3.1. Let $W : X \to \mathbb{C}$ be a potential function, and suppose that $L$ is a $W$-admissible Lagrangian submanifold whose restriction to a large fiber is $M = L \cap (W^{-1}(t))$ (where $t \in \mathbb{R}_{>0}$). Suppose that there exists $M_0 \subset M$ a union of connected components of $M$ with the property that

(i) the Lagrangian $M_0$ bounds no holomorphic disks; and

(ii) the map $H^1(M_0) \to H^2(L, M_0)$ is surjective.

Then $L$ is unobstructed.

The idea of proof is to construct the bounding cochain for $L$ by lifting the curvature term of $L$ to the boundary $M_0".$ The condition that $H^1(M_0) \to H^2(L, M_0)$ shows that curvature term (which takes values in the subcomplex $H^2(L, M)$) is the coboundary of something coming from the boundary $M_0$ of $L.$ The algebraic content of this statement is lemma \[B.2.8\].

Proof. We show that the $A_\infty$ algebras $A = CF^*(L, M_0), B = CF^*(L), C = CF^*(M_0)$ satisfy the conditions items (i) to (iii) of lemma \[B.2.8\]. From theorem \[A.0.2\] the sequence $A \to B \to C$ is exact, and $A$ is an $A_\infty$ ideal. Since $M_0$ bounds no holomorphic disks, $C$ is tautologically unobstructed and $A$ is a strong ideal, giving us (lemma \[B.2.8\] item (i)). Because $M_0$ bounds no holomorphic disks, $CF^*(M_0) = CM^*(M_0),$ which is quasi-isomorphic to $\Omega^*(M).$ Thus we have (lemma \[B.2.8\] item (ii)). Finally, the hypothesis that $H^1(M_0) \to H^2(L, M_0)$ surjects is exactly (lemma \[B.2.8\] item (iii)). □

We give an example which relates to the discussion in [Ekh13, Section 5.2].

Example 4.3.2 (Aganagic-Vafa Brane). Let $A \in \mathbb{R}_{>0}$ be some constant. The Aganagic-Vafa (AV) brane is a Lagrangian submanifold $L_A \subset \mathbb{C}^3$ parameterized by

$$D^2 \times S^1 \to \mathbb{C}^3$$

$$(r_1, \theta_1, \theta_2) \mapsto \left(\sqrt{A^2 + r^2} e^{-i(\theta_1 + \theta_2)}, r e^{i\theta_1}, r e^{i\theta_2}\right)$$

The Lagrangian $L_A$ is admissible for the potential function $W(z_1, z_2, z_3) = z_1 z_2 z_3.$ The restriction to the fiber $M_A \subset W^{-1}(s) = (\mathbb{C}^*) \times (\mathbb{C}^*)$ is a product-type torus, so it bounds no holomorphic disks, and we may apply theorem 4.3.1 to conclude that this Lagrangian is unobstructed by a bounding cochain.

The bounding cochain provides a correction to this Lagrangian submanifold so that it agrees with predictions from mirror symmetry. By application of the open mapping theorem, the only
Figure 6: The projection of the AV Lagrangian $L_A$ to the base $(\mathbb{R}_{\geq 0})^3$ of the Lagrangian torus fibration of the Aganagic-Vafa Lagrangian $L_A \subset \mathbb{C}^3$. We also draw the projection of the single simple holomorphic disk which contributes to a bounding cochain for $L_A$.

Holomorphic disks with boundary on $L_A$ for the standard complex structure must lie in the fiber $W^{-1}(0)$; in fact, the only simple holomorphic disk with boundary on $L_A$ is parameterized by

$$u : (D^2, \partial D^2) \rightarrow (\mathbb{C}^2, L_A)$$

$$z \mapsto (Az, 0, 0)$$

A computation shows that the partial Maslov indices of this disk are $(2, -1, -1)$, and therefore this is a regular Maslov index zero disk by [Oh95]. This shows that the bounding cochain constructed by theorem 4.3.1 is nontrivial.

Under an additional assumption [Hic19, Assumption 5.2.3] one can compute the $m^0$-term, which counts the multiple covers of the disk $u$ with an appropriate weight. The bounding cochain is

$$\sum_{k=1}^{\infty} \frac{1}{k} T^{kw(u)} x, \text{ where } x \in CM^\bullet(M_A)$$

is a meridional class of the torus.

We remark that the Lagrangian $L_A$ is an example of a tropical Lagrangian submanifold considered in [Mik19], and the projection of $L_A$ under the moment map $\mathbb{C}^3 \rightarrow Q = \mathbb{R}_{\geq 0}^3$ is the ray $(|A|^2, 0, 0) + t(1, 1, 1)$.

Corollary 4.3.3. Let $V \subset Q$ be a genus zero smooth tropical curve. Let $L_V$ be a homologically minimal geometric Lagrangian lift of $V$. Then $L_V$ is unobstructed, so there exists $(L_V, b)$ a Lagrangian brane lift of $V$.

Proof. We show that the Lagrangian $L_V$ satisfies the criteria of theorem 4.3.1. Let $V^{(0)} \subset V$ be the set of semi-infinite edges of $V$. The boundary of this tropical Lagrangian realization $M \subset Y_A$ is contained within the lift of the semi-infinite edges $\bigcup_{e \in V^{(0)}} L_e = T_{e}^{-1} \times e$. Therefore, $M$ is the disjoint union of tori indexed by the semi-infinite edges of $V$, $\bigcup_{e \in V^{(0)}} T_{e}^{-1}$. At each edge, we see that $\pi_2(X_A, L_e) = 0$. It follows that $M \subset Y_A$ bounds no holomorphic disks, so we satisfy (theorem 4.3.1 item (i)). Select $f \in V^{(0)}$ any edge, and let $M_0 = \bigcup_{e \in V\setminus f} T_{e}^{-1}$. It remains to prove (theorem 4.3.1 item (ii)) that the image of $H^1(M_0)$ generates $H^2(L_V, M_0)$. From lemma 3.3.1 for any semi-infinite edge $f$ of $V$, $\text{res}_{V\setminus f} : H^2(L_V) \rightarrow \bigoplus_{g \in V\setminus f} H^2(L_g)$ is an injection. From the long exact sequence for relative cohomology,

$$\bigoplus_{g \in V^{(0)}\setminus f} H^1(L_g) \rightarrow H^2(L_V, M_0) \rightarrow 0 \rightarrow H^2(L_V) \rightarrow \bigoplus_{g \in V^{(0)}\setminus f} H^2(L_g)$$

the leftmost arrow surjects.

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5 Faithfulness: unobstructed lifts as $A$-realizations

Given a Lagrangian torus fibration $X_A \to Q$, the $A$-tropicalization of a tautologically unobstructed Lagrangian submanifold $L$ is the set of points $q \in Q$ so that $HF^\bullet(L, (F_q, \nabla)) \neq 0$ for some choice of local system $\nabla$ on $F_q$ (eq. (4)). We now describe the $A$-tropicalization when $L$ is unobstructed by bounding cochain. Because we again work in the scenario where the space $X$ is non-compact, we must apply a tampering condition at infinity to study Floer cohomology. By the same arguments for theorem A.0.2, whenever $L_0$ is admissible and $L_1$ is compact for a potential $W : X_A \to \mathbb{C}$, there exists a well defined $CF^\bullet((L_0, \nabla_0), (L_1, \nabla_1))$ bimodule $CF^\bullet((L_0, \nabla_0), (L_1, \nabla_1))$ given by $\text{[CW19]}$. As in the setting of definition 2.4.1, $CF^\bullet((L_0, \nabla_0), (L_1, \nabla_1))$ is generated on the transverse intersections between $L_0$ and $L_1$. The $A_\infty$ bimodule structure comes from counting pseudoholomorphic treed strips. Observe that we require $L_1$ to be compact to avoid issues of determining how to apply wrapping Hamiltonians in the definition.

Example 5.0.1 (Running Example, Continued). We return to example 4.1.1. Now we bring in the second Lagrangian $(L_V, \nabla_0)$ which we give the trivial local system. The bimodule $CF^\bullet((L_V, \nabla_0), (F_q, \nabla_1))$ has the same generators as example 2.4.4, following notation 2.4.5. We call these generators $x_{I}^{01}$, where $J \subset \{0, \ldots, n-k\}$. Since neither $F_q$ nor $L_V$ bound holomorphic disks, the differential agrees with example 2.4.4

$$m^1(x_I^{01}) = \sum_{I \subset J} \pm T^{\lambda_0}(\text{id} - P_{\nabla_1})x_J.$$ 

where $\lambda_0$ is the area of the small holomorphic strips.

We now describe the module product structure. This is given by counts of configurations of a Morse flow-line on $F_q$ which are incident to a strip with boundary on $F_q \cup L_V$. Recall that the Hamiltonian push off for $L_V$ is given by eq. (3) while the Morse function for $F_q$ is given by eq. (6). As before, we use $\{y_I\}_{I \subset \{1, \ldots, n\}}$ to label the critical points of $f : F_q \to \mathbb{R}$. The moduli space of strips from $x_I^{01}$ and $x_J$ is non-empty when $I < J$; the boundary of the strips sweep out the subtorus spanned by the indices of $J \setminus I$. The downward flow space of $y_K$ is the subtorus spanned by indices $\{1, \ldots, n\} \setminus K$. These two subtori intersect transversely only when $(J \setminus I) \cup \{1, \ldots n\} \setminus K) = \{1, \ldots, n\}$, which can be rephrased as

$$J = K \cup I \quad K \cap I = \emptyset.$$ 

See fig. 7 for a treed strip which contributes to the product. From this, it follows that the module product structure is given by

$$m^2(x_I^{01} \otimes y_J^{01}) = \begin{cases} P_{\nabla_1}^\bullet T^{|J| \cdot \lambda_0} x_I^{01} & \text{if } I \cap J = \emptyset \text{ and } I \cup J \subset \{0, \ldots, n-k\} \\ 0 & \text{otherwise} \end{cases}$$

Here, $|J| \cdot \lambda_0$ is the area of a holomorphic strip from $x_I^{01}$ to $x_{I \cup J}^{01}$, and $P_{\nabla_1}$ is the holonomy of the local system along the $F_q$ boundary of the strip. We remark that when $J = \emptyset$, the same formula holds (simply that $u^+$ is regarded as the constant strip at $x_I^{01}$). The map $m^2(x_0^{01}, -) : HF^1((F_q, \nabla_0)) \to T^{\lambda_0}HF^1((L_V, \nabla_0), (F_q, \nabla_1))$ surjects whenever the local system $\nabla_1$ has holonomy of the form $\text{id} + T^{\lambda_1}A$ along all the $F_q$ boundary of all strips.
5.1 Definition of support

When Lagrangians \((L_0, \nabla_0)\) and \((L_1, \nabla_1)\) are unobstructed by bounding cochains \(b_0, b_1\), we can deform the Lagrangian intersection Floer cohomology \(CF^*((L_0, \nabla_0), (L_1, \nabla_1))\) by these bounding cochains to obtain \(CF^*((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1))\), a \(CF^*(L_0, \nabla_0, b_0) - CF^*(L_1, \nabla_1, b_1)\) bimodule. Since \(CF^*(L_j, \nabla_0, b_i)\) have no curvature, the differential \(m^1 : CF^*((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1)) \rightarrow CF^*((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1))\) squares to zero, giving us cohomology groups which we can study.

**Definition 5.1.1.** Let \((L, \nabla, b) \subset X_A\) be an admissible Lagrangian brane. The \(A\)-tropicalization of \((L, \nabla, b)\) is the set

\[
\text{Trop}_A(L, \nabla, b) := \{ q : \exists (F_q, \nabla') \text{ such that }, HF^*((L, \nabla, b), (F_q, \nabla')) \neq 0 \}.
\]

**Remark 5.1.2.** Suppose that there is a bounding cochain \(b'\) for \(F_q\) so that

\[
HF^*((L, \nabla, b), (F_q, \nabla', b')) \neq 0.
\]

As \(F_q\) is tautologically unobstructed, a general principle of Lagrangian Floer cohomology (the divisor axiom) states that there exists a local system called \(\nabla''\) so that

\[
HF^*((L, \nabla, b), (F_q, \nabla'', b')) = HF^*((L, \nabla, b), (F_q, \nabla', b'))
\]  \(\text{(7)}\)

The local system \(\nabla''\) is usually denoted as \(\exp(b')\). To our knowledge, the divisor axiom has not been proven for the Charest-Woodward model of Lagrangian intersection Floer cohomology. In [Aur08] a proof of the divisor axiom was given for the de Rham version of open Gromov-Witten invariants. The central idea of the proof is that the coefficients in the exponential function make an appearance through the application of the “forgetting boundary points” relation between moduli spaces of holomorphic disks. The coefficients \(m^1\) in the expansion of the exponential function show up via the number of ways one can forget boundary marked points. Under the assumptions that Auroux uses, the forgetful axiom for pseudoholomorphic disks holds. In the Charest-Woodward model for \(CF^*(L_0, L_1)\) we do not expect that perturbations for Morse theory admit a “forgetting marked point” axiom. In our setting (where \(\omega(\pi_2(X_A, F_q)) = 0\)) the arguments used in lemma 5.2.3 show that for all \((F_q, \nabla', b')\) there exists \((F_q, \nabla'')\) so that the identity on \((F_q, \nabla', b')\) factors through \((F_q, \nabla'')\) and vice-versa. Provided that a Charest-Woodward model of the Fukaya category with homotopy unit exists, this would prove eq. (7) (although not give the closed-form expression for \(\nabla'\) as the exponential of the bounding cochain as in
the de Rham version). From the divisor axiom, it follows that the $A$-tropicalization can be rewritten as:

$$\text{Trop}_A(L, \nabla, b) = \{ q : \exists (F_q, \nabla', b') \text{ with } HF^*((L, \nabla, b), (F_q, \nabla', b')) \neq 0 \}$$

(8)

There remain some subtle differences between bounding cochains and local systems in general. It is clear that we can only expect to replace bounding cochains with local systems in the setting where $L$ is tautologically unobstructed. Furthermore, we do not expect that when $L$ is tautologically unobstructed that we can replace $(L, \nabla)$ with $(L, b)$. This is because $\text{val}(b) > 0$, so it can only be expected to represent local systems whose holonomy is of the form $\text{id} + T^A$ where $A \in U_A, \lambda > 0$. In the specialization to Lagrangian tori in a Lagrangian torus fibration, we believe that the requirement that $\text{val}(b) > 0$ may be loosened to $\text{val}(b) \geq 0$ by application of the reverse isoperimetric inequality (in the same fashion way that the reverse isoperimetric inequality is used to prove that the family Floer sheaf has structure coefficients defined over an affinoid algebra).

5.2 $A$-tropicalization of tropical lagrangian lifts

In general, it is difficult to compute $\text{Trop}_A(L, b)$, as it requires having a very good understanding of the differential on $CF^*((L, b), (F_q, b'))$. In section 2.5 we performed this computation for the pair of pants $V \subset \mathbb{R}^2$. Computation of the $A$-tropicalization is more tractable when the Lagrangian $L_V$ is a geometric lift of a tropical subvariety because we have good control of leading order contributions to the differential.

The main tool that we use to compute the $A$-tropicalization is the following lemma.

**Lemma 5.2.1.** Let $L_V$ be a Lagrangian lift of an affine subspace. Let $C$ be chosen so that $L_V$ has a neighborhood of radius $C$. There exists a function $A_{L_V, C} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, depending only on $L_V, C$ which provides the following property/bound on holomorphic strips.

- Given any Lagrangian submanifold $L \subset X$, and subset $U = \pi^{-1}_A(Q_U)$ such that $L|_U = L_V|_U$ and
- given any point $q \in Q_U$ whose distance (as measured in $Q$ with the standard metric) to $Q \setminus Q_U$ is $R$

every holomorphic strip $u$ with boundary on $L \cup F_q$ either:

- has image contained within $U$, and therefore describes a holomorphic strip with boundary on $L_V \cup F_q$ or
- has symplectic energy greater than $A_{L_V, C}(R)$.

Furthermore,

$$\lim_{C \to \infty} A_{L_V, C}(R) = 2R$$

Additionally, we may replace $F_q$ with a small Hamiltonian push off of $F_q$ while preserving the bound.

**Proof.** The lemma is an application of the reverse isoperimetric inequality from [GS14]. We use the proof for holomorphic strips which is employed by [Abo17b] following [Duv16]. Recall that the
reverse isoperimetric inequality states that given Lagrangians $L, F_0$, there exists a constant $A$ so that for strip $u$ with boundary on $L, F_0$ we can lower-bound the energy of the strip by:

$$A \cdot \ell(\partial u) \leq \int_u \omega.$$  \hfill (9)

Suppose that $\partial u \subset U$; then we can show that $u$ gives a strip with boundary on $L \cup V \cup F_0$; we know that all such strips are contained within $U$. Therefore $\partial u \not\subset U$, and we can conclude that

$$2A \cdot R < \int_u \omega$$

providing us a bound for the symplectic area.

The content of the lemma then lies in computing $A$ in terms of the set $Q_U$ and $q$. To simplify the argument, we will

- $F_q = F_0 = \{(q, p) : q_1 = \ldots = q_n = 0\}$, and that $L_V = \{(q, p) : p_1 = \ldots = p_k = q_{k+1} = \ldots = q_n = 0\}$.

- The neighborhood $U$ is of the form $q_1^2 + \cdots q_k^2 < R, p_k^2 + \cdots + p_n^2 < C$.

We sketch the construction of the bound $A_{V, C}$ from eq. (9). Let $g_\chi$ be the metric on $X$ induced by $\omega$ and the standard almost complex structure $J$. Following [Abo17b], take $d \ll C, R$ and let $K = B_d(X \setminus U) \cup B_d F_q$, and let $\ell_K(\partial u)$ be the length of $\partial u \cap (X \setminus K)$. Take $K' = B_{d/2} F_q \cup \{(q, p) : q_1^2 + \cdots + q_k^2 \geq R - \epsilon/2\}$. Note that in order to obtain regularity for “large” pseudoholomorphic strips we may choose perturbations of almost complex structure which match the standard $J$ on the complement of $K$. Let $\rho(q, p) = \sum_{i=1}^k p_i^2 + \sum_{i=k+1}^n q_i^2$.

Take a tubular neighborhood $B_{\rho(q)}$ of $L$, whose restriction to $U$ is given by $\{(q, p) \in U : \rho(p, q) \leq C\}$. Let $r_k(q) = \sqrt{\sum_{i=1}^k q_i^2}$.

The non-negative weakly $J$-plurisubharmonic function $\hat{\rho} : B_{\rho} \to \mathbb{R}$ considered in [Abo17b, Lemma A.1] can be taken so that over $L \setminus K$ it is given in coordinates by

$$\hat{\rho}(q, p) = \chi(r_k) \rho$$

where $\chi$ is a cutoff function drawn in fig. [8] chosen so that it satisfies the bounds

$$|\chi''| \leq \frac{10}{d^2}, \quad |\chi'| \leq \frac{10}{d^2 r_k}, \quad \chi|_{d \leq t \leq R - d} = 1 \quad \chi|_{t < d/2} = \chi|_{t > R} = 0.$$  

For this choice of $\hat{\rho}$, the pseudometric $g_{\hat{\rho}}$ induced by $\hat{\rho}$ and $J$ is given by the matrix of mixed partials

$$\frac{\partial^2 \hat{\rho}}{\partial z_i \partial \bar{z}_j} = \rho \cdot (\chi'' \partial_i r_k \partial_j r_k + \chi' \partial_i \partial_j r_k) + \chi' \partial_i r_k \partial_j \rho + \partial_i \rho \partial_j r_k + \chi \delta_{ij}$$
From the bounds
\[ \rho \leq C \quad \left| \partial_i r_k \right| \leq 1 \quad \left| \partial_i \partial_j r_k \right| \leq \frac{1}{r_k} \quad \rho < C \quad \left| \partial_i \rho \right| \leq 2\sqrt{C} \]
we obtain
\[ \left| \frac{\partial^2 \hat{\rho}}{\partial z_i \partial z_j} - \chi \delta_{ij} \right| \leq E_{C,d} := \frac{100\sqrt{C} + \sqrt{d^2}}{d^2} \quad \left| \nabla \hat{\rho} \right| \leq \left( 1 + \frac{50n}{d^2} \right) \sqrt{\rho}. \]

The metric is constructed so that it satisfies the following properties:

1. \( g_{\hat{\rho}} \) and \( g_X \) agree over the region \( B^*_C(L) \setminus V \).
2. \( g_{\hat{\rho}} \) is dominated by \( (1 + n^2 E_{C,d})g_X \) everywhere.

From the first property, we can strengthen [Abo17a, A.10] to
\[ \ell_K(\partial u) \leq \ell_{\rho}(\partial u). \]

From the second property, we see that
\[ \int u d\omega \geq \frac{C}{(1 + n^2 E_{C,d})} \lim_{s \to 0} a(s) \geq \ell_{\rho}(\partial u). \]

The inequality is proven using Fubini’s theorem, and the constant \( F_{n,d} \) is required to satisfy the bound
\[ \left| \nabla \hat{\rho} \right| \leq (1 + F_{n,d}) \cdot s \text{ whenever } \hat{\rho} < s^2 \]
From our estimates, we can choose \( F_{n,d} < \frac{10m}{d^2} \). We conclude that
\[ \int u \omega \geq \frac{Ca(c)}{(1 + n^2 E_{C,d})} \geq \lim_{s \to 0} \frac{Ca(s)}{(1 + n^2 E_{C,d})} \geq \frac{C\ell_{\rho}(\partial u)}{(1 + n^2 E_{C,d})(1 + F_{n,d})} \geq \frac{C(R - 2d)}{(1 + n^2 E_{C,d})(1 + F_{n,d})} \]
We note that as \( d \to \infty \), \( E_{C,d}, F_{n,d} \to 0 \). Therefore, if we let \( d = \sqrt{R} \), we obtain a function \( A_{V,C}(R) \) only dependent on \( L_V \) so that
\[ \int u \omega < A_{V,C}(R) \sim 2C \cdot R. \]

The bound holds if we replace \( F_q \) with any Lagrangian contained within \( B_{d/2}F_q \); in particular we can use a small Hamiltonian push off of \( F_q \). □
Remark 5.2.2. The constant $C$ is related to $\nabla$. In the 2-dimensional setting, we obtain the following nice relation. At a top dimensional stratum (edge) $e$ with integral primitive direction $\vec{v}$, the constant $C$ is in lemma 5.2.1 is $\frac{1}{2|v|}$. The bound for the holomorphic energy of the strips becomes

$$2C \cdot R = \frac{R}{|\vec{v}|}$$

which is the affine radius of the neighborhood around the point $q$. This can be observed section 2.5 and example 6.3.2 where the affine lengths of edges in tropical curves govern the areas of holomorphic disks and strips which appear those computations.

Lemma 5.2.3. Let $U \subset X_A$ be a neighborhood of $F_q$, a fiber of the Lagrangian torus fibration. Suppose that $(L, \nabla, b_0) \subset X_A$ is a Lagrangian brane, whose restriction to a neighborhood of $q$ is

$$L|_U = L_{\nabla,m}|_U$$

where $\nabla \subset U$ is a $k$-dimensional linear subspace, and $m$ the multiplicity. Then there exists a choice of bounding cochain and local system on $F_q$ so that

$$HF^0((L, \nabla, b_0), (F_q, \nabla, b)) = \Lambda.$$ 

Proof. To reduce notation in the proof, we will take the same simplifying assumptions as in lemma 5.2.1. Additionally, we assume that the local system $\nabla_0$ the local system $\nabla_0$ and bounding cochain $b_0$ on $L$ are trivial.

We see that $L|_U \cap F_q$ cleanly intersect along a $T^{n-k} \subset F_0$, morally we now apply spectral sequence of [Po94, Sch16] to compute the Floer cohomology of $CF^\bullet(L, F_0)$ as a deformation of $C^\bullet(T^{n-k})$. Because $L|_U = L_{\nabla}|_U$, we can apply lemma 5.2.1. Following example 5.0.1 apply a Hamiltonian isotopy to $L$ so that $L, F_q$ intersect transversely. Take the perturbation small enough so that the area of the holomorphic strips $\lambda_0$ is less than the bound $\lambda_1 := A_{\nabla,C}(r)$ provided by lemma 5.2.1. By lemma 5.2.1, the map $m^2 : CF^\bullet(L_\nabla, F_q) \otimes CF^\bullet(F_q) \to CF^\bullet(L_\nabla, F_q)$ agrees with example 5.0.1 at valuation less than $\lambda_1$.

$$m^2(x_0^{I \cap J}, x_j) \equiv \begin{cases} T^{\lambda_0}x_0^{I \cap J} & \text{if } I \cap J = \emptyset \text{ and } I \cup J \subset \{0, \ldots, n-k\} \\ 0 & \text{otherwise} \end{cases} \mod T^{\lambda_1} \tag{10}$$

Let $CF^\bullet(F_q, \Lambda_{\geq 0})$ and $CF^\bullet(L_\nabla, F_q, \Lambda_{\geq 0})$ be the filtered $A_{\infty}$ algebra and bimodule where we use $\Lambda_{\geq 0}$ rather than $\Lambda$-coefficients. It follows that the map on chains

$$m^2 : (x_0^{I \cap J}, x_j) \otimes CF^1(F_q, \Lambda_{\geq 0}) \to CF^1(L_\nabla, F_q, \Lambda_{\geq \lambda_0})/CF^1(L_\nabla, F_q, \Lambda_{\geq \lambda_1})$$

surjects. Therefore $CF^\bullet(L_\nabla, F_q, \Lambda_{\geq 0})$ as a right $CF^\bullet(F_q, \Lambda_{\geq 0})$ module satisfies the criterion of lemma B.3.1 and there exists $b \in CF^\bullet(F_q)$ so that $HF^0(L_\nabla, (F_q, b)) \neq 0$.

To extend to the setting where $L$ has a local system $\nabla_0$, we simply require that $F_q$ be equipped with a local system $\nabla_1$ which agrees with $\nabla_0$ on torus spanned by the classes $\{c_1, \ldots, c_{n-k}\}$. \hfill $\Box$

Remark 5.2.4. The constant $\lambda_0$ can be taken to zero provided that one works with a model of $CF^\bullet(L_\nabla, F_q)$ which allows for clean intersections between $L_\nabla$ and $F_q$; the proof of lemma B.3.1 becomes slightly simpler in that setting. The pearly model developed by [CW14] allows for such configurations of Lagrangian submanifolds.
Corollary 5.2.5. Let \((L, \nabla_0, b_0)\) and \((F_q, \nabla, b)\) be as above. Then
\[
\text{HF}^*((L, \nabla_0, b_0), (F_q, \nabla, b)) = \bigwedge_{i \in \{1, \ldots, n-k\}} \Lambda \langle x_i \rangle
\]

Proof. Again for expositional purposes, we assume that \(\nabla_0\) and \(\nabla\) are trivial local systems, assume that the multiplicity of the local model \(V\) is one, and suppress the bounding cochain on \(L\). On chains, the action of \(\text{CF}^*(F_q, b)\) on \(\text{CF}^*(L, (F_q, b))\) is a deformation of the action of \(\text{CF}^*(F_q, b)\) on \(\text{CF}^*(L, (F_q, b))\). By using an argument on the filtration similar to above, the map
\[
m^2(x_0, -) : \text{CF}^*((F_q, b)) \to \text{CF}^*(L, (F_q, b))
\]
is a surjection. As every class in \(\text{CF}^*((F_q, b))\) is closed and we’ve proven that \(x_0\) is closed, every element in \(\text{CF}^*(L, (F_q, b))\) is closed. This proves that \(m^1_{\text{CF}^*(L, (F_q, b))}(x_0) = 0\), and that \(\text{HF}^*(L, (F_q, b)) = \bigwedge_{i \in \{1, \ldots, n-k\}} \Lambda \langle x_i \rangle\).

Corollary 5.2.6. Let \((L, b)\) be an unobstructed geometric Lagrangian lift of \(V\). Then \(V^{(0)} \setminus V^{(1)} \subset \text{TropA}(L, b)\).

If we assume eq. (8), this immediately follows.

Proof. The proof of lemma 5.2.3 can be modified to replace the bounding systems everywhere with local systems. The needed observation is that the map from the space of local systems
\[
H^1(F_q, U_{\Lambda}) \to \text{CF}^1(L, F_q, \Lambda_{\geq \lambda_0})/\text{CF}^1(L, F_q, \Lambda_{\geq \lambda_1})
\]
is surjective. The same argument as in lemma B.3.1 can be used to construct a local system term by term so that \(m^1_{(L, (F_q, b))}(x_0) = 0\). See also [SS20, Proposition 5.13] which proves a similar statement for tropical curves using the implicit function theorem [Abh01, Section 10.8].

6 \(B\)-realizability and unobstructedness

6.1 HMS for \((\mathbb{C}^*)^n\)

6.1.1 Construction of the mirror space

Given \(\pi_A : X_A \to Q\) a Lagrangian torus fibration, there is a rigid analytic space \(X_B\) with a tropicalization map \(\text{TropB} : X_B \to Q\). As a set, \(X_B\) is the set of Lagrangian torus fibers equipped with \(U_{\Lambda}\) local system,
\[
X_B := \{(F_q, \nabla)\}
\]
which comes with a map \(\pi_B : X_B \to Q\) sending \((F_q, \nabla) \mapsto q\). When \(Q = \mathbb{R}^n\), the points of \(X_B\) are in bijection with \((\Lambda^*)^n\). We now describe, following [Abo14; EKL06], how this can be realized as the set of points of a rigid analytic space. We also recommend the discussion in [SS20, Section 5.1].

The Tate algebra in \(n\)-variables over \(\Lambda\) is the set of formal power series
\[
T_n := \left\{ \sum_{A \in \mathbb{Z}^n} f_A z^A : f_A \in \Lambda, \text{val}(f_A) \to \infty \text{ as } |A| \to \infty \right\},
\]
which is equipped with the sup-norm

\[ \left\| \sum_{A \in \mathbb{Z}^n} f_A z^A \right\| := \max_A |f_A| \geq 0 \]

We note that the maximal ideals of \( T_n \) are \( \{ (f_1, \ldots, f_n) : \text{val}(f_i) \leq 1 \} \).

To build our spaces we will glue together affinoid algebras, which are quotients of the Tate algebra. The affinoid algebras we will look at are the polytope algebras. Given a bounded rational polytope \( P \subset \mathbb{R}^n \), define

\[ O_P := \left\{ \sum_{A \in \mathbb{Z}^n} f_A z^A : \text{val}(f_A) + A \cdot p \to \infty \text{ as } \| A \| \to \infty \text{ for all } p \in P \right\}. \]

This is the affinoid algebra. The elements of this affinoid algebra have the property that they converge when evaluated on \( z \in (\Lambda^*)^n \) with \( \text{val}(z) \in P \). Furthermore, the points of \( O_P \) are seen to be in bijection with the points of \( \pi_B^{-1}(P) \). When \( Q \) is compact \( X_B \) can be covered by finitely many sets \( \pi_B^{-1}(P) \) giving \( X_B \) the structure of a rigid analytic space.

### 6.1.2 From Lagrangians to Coherent Sheaves

Due to the limitations on currently existing constructions for Fukaya categories, we do not have homological mirror symmetry for a category of non-exact Lagrangian submanifolds in \((\mathbb{C}^*)^n\). However, different aspects of this homological mirror symmetry statement exist in the literature with strengthened hypotheses.

- The family Floer functor associates to a compact Lagrangian torus fibration \( \pi_A : X_A \to Q \), a rigid analytic space \( X_B \to Q \) whose points are in bijection with Lagrangian tori \( F_q \subset X_A \) equipped with \( U_A \) local system. Furthermore, \([\text{Abo17b}, \text{Theorem 2.10}]\) constructs a faithful \( \mathbb{A}_\infty \)-functor \( \mathcal{F} : \text{Fuk}^{\text{taut}}(X_A) \to \text{Perf}(X_B) \). Here \( \text{Fuk}^{\text{taut}}(X_A) \) is the Fukaya category of tautologically unobstructed Lagrangian submanifolds.

- In the exact setting, we have a complete proof of homological mirror symmetry for \((\mathbb{C}^*)^n\). The proof comes from recasting a section \( L(0) \) of the fibration \( \pi_A : (\mathbb{C}^*)^n \to Q \) as a cotangent fiber in \( T^\ast T^n \), which is known to generate the exact Fukaya category. A computation shows that the \( \mathbb{A}_\infty \) algebra \( \text{CF}(L(0), L(0)) \) is homotopy equivalent to \( \text{hom}(\mathcal{O}_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) \).

For this paper, we will only compute \( \text{CF}((L_V, b), (F_q, \nabla)) \), which means that we need substantially less than a HMS functor of \([\text{Abo17b}]\).

**Theorem 6.1.1** ([\text{Abo14}]). Consider the Lagrangian torus fibration \( X_A = (\mathbb{C}^*)^n \to Q \), with \( Q \) compact. From this data we can construct a rigid analytic mirror space \( X_B \) whose points \( z \) are in bijection with pairs \( (F_q, \nabla) \). For any tautologically unobstructed Lagrangian brane \( L \subset X_A \), there exists a coherent sheaf \( \mathcal{F}(L) \) on \( X_B \) so that

\[ \text{hom}(\mathcal{F}(L), \mathcal{O}_z) = HF^0(L, (F_q, \nabla)). \]

**Assumption 6.1.2.** Theorem 6.1.1 still holds under the following weakened assumptions:
The base is allowed to be $Q = \mathbb{R}^n$, and we additionally require that the Lagrangian $L$ be monomially admissible.

The Lagrangian $L$ is allowed to be unobstructed by bounding cochain, in which case there exists a coherent sheaf $\mathcal{F}(L, b)$ on $X_B$ so that
\[
\text{hom}(\mathcal{F}(L, b), \mathcal{O}_z) = HF^0((L, b), (F_q, \nabla)).
\]

We now discuss the difficulties, expectations, and progress of proving the assumption. The primary difficulties arise from non-compactness and unobstructedness.

Non-compactness presents three immediate issues. The first is Gromov compactness. We expect that after one places appropriate taming conditions on our Lagrangian submanifolds (as in appendix A) the moduli spaces needed to construct the family Floer functor can be given appropriate compactifications.

The second more difficult issue regards the role that wrapping plays in computing the Floer cohomology between two non-compact Lagrangians. In the exact setting, the morphism space between two Lagrangians is computed as the limit of $CF^\bullet(\phi^i(L_0), L^i)$, where $\phi^i$ is a wrapping Hamiltonian, and the limit is taken over continuation maps. In the non-compact setting, these continuation maps have a non-zero valuation, and only have inverses defined over the Novikov field (with possibly negative valuation). To our knowledge, this version of the Fukaya category has not been constructed. However, since for our application we only need to compute Floer cohomology against Lagrangian torus fibers (which are compact), we can ignore the issues of the wrapping Hamiltonian.

Finally, there is the issue of coherence of $\mathcal{F}(L, b)$. Here, we use the monomial admissibility condition. We recall the proof of coherence when $Q$ is compact. The sheaf $\mathcal{F}(L, b)$ is constructed by defining it over affinoid domains on the mirror, which correspond to convex domains $U \subset Q$. The convex domain $U$ is “small enough” if there exists a Hamiltonian isotopy of $L$ so that it intersects all Lagrangian torus fibers $F_q$ with $q \in U$ transversely. Over each small enough $U$, the sheaf is computed by $CF^\bullet((L, b), (F_q, \nabla)) \otimes \mathcal{O}_U$, where $\mathcal{O}_U$ is the affine ring of the affinoid domain $X_{U,B}$ associated to the convex domain $U$. Since $CF^\bullet((L, b), (F_q, \nabla))$ is finitely generated, and (in the compact setting) we can cover $Q$ with finitely many such $U$, we obtain that the mirror sheaf is coherent. If we drop the condition of $Q$ being compact, and impose the condition that $L$ is monomially admissible, we can still cover $Q$ with a finite set of convex (possibly non-compact) small enough domains $U \subset Q$ by using invariance of the Lagrangian submanifold under symplectic flow in the direction of the monomial ray over each monomial region.

We now remark upon the difficulty of unobstructedness. [Abo17, Remark 1.1] states that the “tautologically unobstructed” hypothesis for construction of the family Floer functor is technical in nature, and it is expected that the family Floer functor should carry through using unobstructed Lagrangian submanifolds. As we do not require functoriality, such an adaptation of family Floer cohomology to the Charest-Woodward model would not require studying moduli spaces beyond those already studied in [CW19]. We believe the main items left to prove for this construction are the following:

- Showing that “Fukaya’s trick” for pulling back perturbation datum between Lagrangian fibers over sufficiently small convex domains can be worked out in the more technically challenging setting of domain-dependent perturbations. This does not appear to present a problem when working with the set up of [CW19].
Showing that “homotopies of continuation maps” exist in the version of Lagrangian intersection Floer cohomology one is working with. In [CW19], continuation maps are constructed using holomorphic quilts. There is also an additional challenge of showing that one can construct homotopies of continuation maps corresponding to changes in choice of stabilizing divisor.

Finally, we note that work in progress of Abouzaid, Gromann, and Varolgunes generalizing [Gro18; Var21] to the Fukaya category will prove homological mirror symmetry for unobstructed Lagrangian submanifolds of \((\mathbb{C}^*)^n\), giving us assumption 6.1.2.

6.2 Unobstructed Lagrangian lift implies B-realizability

By employing [Abo17a] (with the possible extensions stated in assumption 6.1.2) we can associate to each Lagrangian brane \((L_V, b)\) a closed analytic subset of \(X_B\):

\[
Y(L_V, b) := \text{Supp}(H^0(\mathcal{F}(L_V, b))).
\]

**Corollary 6.2.1.** Consider the Lagrangian torus fibration \(\pi_A : (\mathbb{C}^*)^n = X_A \to Q\) and a tropical subvariety \(V \subset Q\). Suppose that \((L_V, b)\) is a Lagrangian brane lift of \(V\). Then:

- \((L_V, b)\) is an \(A\)-realization of \(V\) in the sense that \(\text{Trop}_A(L_V, b) = V\)
- \(V\) is \(B\)-realizable.

**Proof.** By assumption [6.1.2] \(\text{Trop}_A(L_V, b) = \text{Trop}_B(Y(L_V, b))\). In corollary 5.2.6 we proved that \(V^{(0)} \subset \text{Trop}_A(L_V, b) \subset V\). Since \(Y(L_V, b)\) is a closed analytic subset, \(\text{Trop}_B(Y(L_V, b))\) is the union of closed rational polyhedra in \(N_{\mathbb{R}}\) [Gub07, Proposition 5.2]. As a result, \(\text{Trop}_B\) is closed and contains \(\overline{V^{(0)}} = V\). It follows that \(Y(L_V, b)\) is a closed analytic subset of \(X_B\) which realizes \(V\). \(\square\)

**Corollary 6.2.2.** Assuming assumption [6.1.2 (*)], (**) let \(V\) be a smooth hypersurface or a smooth genus zero tropical curve in \(\mathbb{R}^n\). Then \(V\) is \(B\)-realizable.

**Corollary 6.2.3.** Assuming assumption [6.1.2 (**)], let \(V\) be a smooth tropical hypersurface of a tropical abelian variety \(Q = \mathbb{R}^n/M_{\mathbb{Z}}\). Then \(V\) is \(B\)-realizable.

**Corollary 6.2.4.** Without assuming any portion of assumption 6.1.2 let \(V\) be a 3-valent tropical curve in a tropical abelian surface \(Q\). Then \(V\) is \(B\)-realizable.

**Proof.** The condition of 3-valency comes from using

- [Hol22] to build an affine dimer model associated to each 3-valent vertex and
- [Hic21b] to build tropical Lagrangian lifts from a dimer model.

We now address why assumption 6.1.2 may be dropped. Since \(Q\) is a tropical abelian surface (and is therefore compact), the symplectic manifold \(X_A\) is compact. Since the Lagrangian lift \(L_V\) is graded of dimension 2, it is tautologically unobstructed for a generic choice of almost complex structure (as Maslov index 0 disks appear in expected dimension -1). \(\square\)
6.3 Nonplanar tropical curves do not have tautologically unobstructed lifts

Even in the setting where \( V \) is a genus zero tropical curve, it is rare for the Lagrangian lift \( L_V \) to be a tautologically unobstructed Lagrangian submanifold.

Before constructing an example, we observe that the valuations of the “big-strips” in lemma 5.2.3 are dictated by the radius of the neighborhood \( U_q \) that we can construct around the point \( q \) which is disjoint from \( V^{(1)} \). In particular, this can be applied to [SS20, Proposition 5.10] to show that tautologically unobstructed Lagrangian lifts of tropical curves have supports that extend to an appropriate toric compactification of the mirror algebraic torus.

**Proposition 6.3.1.** Let \( \Sigma \) be a fan. Suppose that \( V \) is a tropical curve with semi-infinite edges in the directions of the rays of \( \Sigma \). Suppose the fan of \( \Sigma \) has the additional property that \( \langle \alpha, \beta \rangle < 0 \) for all 1-dimensional cones \( \alpha \neq \beta \). Then \( Y((L_V, b), 0) \) compactifies to a rigid analytic space inside \( X_B(\Sigma) \), the rigid analytic toric variety with fan \( \Sigma \).

**Proof.** We first describe the rigid analytic structure on \( X_B(\Sigma) \) given by [Rab12]. From [Pav09], the space \( X_B(\Sigma) \) comes with a fibration \( \text{TropB} : X_B(\Sigma) \to Q(\Sigma) \), which is a partial compactification of \( Q \) (see [Rab12, Definition 3.6]). Rabinoff then covers \( X_B(\Sigma) \) with charts given by the max-spec of affinoid algebras.

Let \( P_{\sigma} \subset Q \) denote a convex set which can be written as the form \( P' + \sigma \) for \( \sigma \in \Sigma \) and some convex compact polytope \( P' \subset Q \). Associated to \( P_{\sigma} \) is a subset \( \overline{P_{\sigma}} \subset Q(\Sigma) \), and an affinoid algebra

\[
\mathcal{O}_{P_{\sigma}} := \left\{ \sum_{A \in (\sigma') \cap \mathbb{Z}^n} f_A z^A : \text{val}(f_a) + a \cdot p \to \infty \quad \text{as} \quad \| A \| \to \infty \quad \text{for all} \quad p \in P_{\sigma} \right\}.
\]

We can cover \( X_B(\Sigma) \) with charts given by the max-spec of \( \mathcal{O}_{P_{\sigma}} \) (which covers \( \text{TropB}^{-1}(\overline{P_{\sigma}}) \)).

We now unpack what it means for a Lagrangian submanifold \( (L, b) \) constructed via family Floer theory to give a coherent sheaf \( \mathcal{F}((L, b)) \) on the rigid analytic space \( X_B(\Sigma) \). In the family Floer construction, for a sufficiently small convex polytope \( P \) in the base of \( Q \), one takes a Hamiltonian perturbation \( L_P \) of \( L \) so that \( L_P|_{\pi_A^{-1}(P)} \) is a disjoint set of flat sections of \( \pi_A^{-1}(P) \to P \), and that the bounding cochain is similarly parallel to the flat section. As a result, we may identify the chains \( CF^*((L_P, b), (F_q, \nabla)) \) for all \( q \in P \). Additionally, for \( q \in P \), one can appropriately choose almost complex structures (using Fukaya’s trick) so that the moduli spaces of strips contributing to the differential on \( CF^*((L, b), (F_q, \nabla)) \) does not depend on \( q \). Because the bounding cochain on \( L \) is parallel to the flat section, the contribution of \( b \) to the differential on \( CF^*((L, b), (F_q, \nabla)) \) does not depend on \( q \in P \). As a consequence, the dependence of the structure coefficients \( \langle m^1_{(L, b), (F_q, \nabla)}(x), y \rangle \) on \( (F_q, \nabla) \) factors through the flux homomorphism. Pick a base point \( x_0 \) on \( F_q \) and for each \( x \in L \cap F_q \) a path \( \gamma_x \) from \( x_0 \) to \( x \). By identifying \( (F_q, \nabla) \) with a point \( z \in \text{TropB}^{-1}(P) \), we obtain that

\[
\langle m^1(x), y \rangle = \sum_{a \in H_1(F_q)} c_a z^a
\]

where \( c_a \) is the area and local-system weighted count of pseudoholomorphic strips \( u \) so that \( [\gamma_x \cdot \partial_{F_q} u \cdot y^{-1}] = a \). The Lagrangian \( L \) defines a complex of sheaves \( \mathcal{F}(L) \) over \( (X_B, p) \) if these structure coefficients belong to \( \mathcal{O}_P \). The restriction maps compose up to homotopy of the chain complex. This is proven using the reverse isoperimetric inequality to bound the area of holomorphic strips \( u \) (which govern convergence) below by the winding of the \( F_q \) boundary component of \( U \) (which
governs the exponent appearing in $z^a$. To obtain a coherent sheaf of complexes on $X_B$, one must be able to cover $Q$ with finitely many sufficiently small sets $P$. When $Q$ is compact, this is always possible. In the setting we study, we must take some of the sets $P$ to be of the form $P_\sigma$ in order to construct a finite cover.

We now perform this construction for $L_V$ our tautologically unobstructed Lagrangian lift of a tropical curve $V$. Let $e$ be a semi-infinite edge of $V$ pointing in the $\alpha$ direction, where $\alpha \in \Sigma$ is a 1-dimensional cone. Then there exists a $P_\alpha$ so that $V|_{P_\alpha}$ is a 1-dimensional ray. Since $\langle \alpha, \beta \rangle < 0$ for all 1-dimensional rays $\beta \neq \alpha$, the projection $\chi^\alpha : X_A \to \mathbb{C}$ given by the $\alpha$-monomial has the property that the $\chi^\alpha|_{L_V} : L_V \to \mathbb{C}$ fibers over a real ray outside of a compact set. This is the main input needed in [SS20, Proposition 5.10] to show that the differential on $CF^*(L_V, (F_q, \nabla))$ is of the form $\sum_{a \in H_1(F_q), \langle a, \alpha \rangle \geq 0} c_a z^a$, and that $val(c_a) + a \cdot p \to \infty$ as $\|A\| \to \infty$ for all $p \in P_\alpha$. It follows that $\langle m^1_\chi(x), y \rangle \in O_{P_\alpha}$.

We can choose a finite cover of $Q$ by sets of the form $P_\sigma$ so that $\pi_A(L_V) \subset P_\sigma$ if and only if $|\sigma| \leq 1$. It follows that $F(L_V)$ define a sheaf on $X_B(\Sigma)$.

In the setting above (where $L_V$ is unobstructed and equipped with the tautological local system), the above computation not only shows that $F(L_V)$ extends to $X_B(\Sigma)$ but also shows that we can compute the points in the compactifying locus. For a semi-infinite edge $e$, let $P_\alpha = P + \langle \alpha \rangle$ a convex polytope whose only intersection with $V$ is along the edge $e$. Without loss of generality, we will assume that the edge $e$ is of the form $(t,0,\ldots,0) \subset Q = \mathbb{R}^n$, with $t$ tending to $\infty$. We can write the max-spec of $P_\alpha$ as
\[
\{(z_1,\ldots,z_n) \in \Lambda \times (\Lambda^*)^{n-1} : val(z_1,\ldots,z_n) \in \overline{P_\alpha} \subset (\mathbb{R} \cup \infty) \times \mathbb{R}^{n-1}\}.
\]
We prove that the point $(0,1,\ldots,1) \in \text{Supp}(F(L_V))$. The $\langle a, \alpha \rangle = 0$ terms of $\langle m^1_\chi(x_0), x_I \rangle$ agree with holomorphic strips for the differential on $CF^*(L_{\Sigma}, (F_q, \nabla))$, so we can write
\[
\langle m^1_\chi(x_0), x_I \rangle = (1 - z^{(I,\alpha)}) + \sum_{a \in H_1(F_q), \langle a, \alpha \rangle > 0} c_a z^a,
\]
When we have a sequence of points $\{z^k\}_{k \in \mathbb{N}}$ with the property that $m^1_{z^k}(x_0) = 0$ (i.e. $z^k \in \text{Supp}(F(L_V))$) and $\lim_{k \to \infty} val(z^k)^k \to \infty$ (so that the limit belongs to the compactifying toric divisor), the above equation states that $\lim_{k \to \infty} val(z^k)^k = 1$ for all $i \neq 1$. We conclude that the closure of $\text{Supp}(F(L_V))$ inside of $X_B(\Sigma)$ contains the point $(0,1,\ldots,1)$.

We now construct an example of a Lagrangian brane lift of a tropical curve which is unobstructed, but not tautologically unobstructed.

Figure 9: Monomial admissibility forces strips with $\langle \partial u, \alpha \rangle > 0$ to have large symplectic area.
Example 6.3.2. Consider the tropical line \( V_c \in \mathbb{R}^3 \) drawn in fig. 10. The tropical line \( V_c \) has two pants centered at the points \((0,0,0)\) and \((-c,-c,0)\), and whose legs at \( (0,0,0) \) point in the directions

\[
    e_1 = (1,0,0) \quad \quad e_2 = (0,1,0) \quad \quad e_c = (-1,-1,0)
\]

and whose legs at \((-c,-c,0)\) point in the directions

\[
    e_3 = (0,0,1) \quad \quad e_4 = (-1,-1,-1) \quad \quad -e_c = (1,1,0).
\]

We prove that \( L_{V_c} \) bounds a holomorphic disk for all but at most 1 value of \( c \).

Assume for contradiction that for all values of \( c \) the Lagrangian submanifold \( L_{V_c} \) is tautologically unobstructed, and requires no bounding cochain. Then the Lagrangians \( L_{V_c} \) satisfy the conditions of proposition 6.3.1 so each \( Y_{V_c} := \text{Supp}(\mathcal{F}(L_{V_c})) \) compactifies to give a curve inside of \( \mathbb{P}^3 \). Since this curve intersects each of the toric divisors at a single point, we conclude that every \( Y_{V_c} \) is a line in \( \mathbb{P}^3 \). Furthermore, every one of these lines contain the points \((1 : 0 : 0 : 0)\) and \((0 : 1 : 0 : 0)\) in \( \mathbb{P}^3 \). Since a line in \( \mathbb{P}^3 \) is determined by two points, this implies that \( Y_{V_c} = Y_{V_c'} \). However, as \( V_c \neq V_c' \), they cannot be realized by the same subvariety, a contradiction.

This doesn’t contradict the realizability of \( V_c \). Indeed, by corollary 4.3.3 the bounding cochain on \( L_{V_c} \) need only be supported on three of the four legs of \( L_{V_c} \). However, the above argument shows that one cannot construct a bounding cochain for \( L_{V_c} \) which restricts to zero on the two semi-infinite edges which share a vertex (which implies that the bounding cochain cannot be zero).

Using mirror symmetry, we can “back solve” for the valuation of the holomorphic disk which necessitates the use of a bounding cochain on \( L_{V_c} \). We may assume that the bounding cochain has trivial restriction to the \( e_1 \) edge. It follows that the tropical line \( Y_{V_c} \) may intersect toric divisors at the points \((0,1,1), (1 + \exp(b_1), 0, 1 + \exp(b_3)), (z^{-c} + \exp(c_1), z^{-c} + \exp(c_2), 0)\). Since these have to satisfy the equation of a line, there exists \( t \) so that

\[
    (1 - t)(0,1,1) + t(\exp(b_1), 0, \exp(b_2)) = (z^{-c}(\exp(c_1)), z^{-c}(\exp(c_2)), 0).
\]
From examining the third term, \( t = (1 - \exp(b_2))^{-1} \), we already see that \( b_2 \neq 0 \). From examining the third term

\[
(1 - \exp(b_2))^{-1} \exp(b_1) = z^c(\exp(c_1))
\]

from which we see that \( \text{val}(b_2) = c \). From this we conclude that there exists a pseudoholomorphic disk of energy \( c \) on \( L_{V_c} \).

### 6.4 Speculation on Speyer’s well-spacedness criterion

Corollary 6.2.1 proves the forward direction of conjecture 1.1.1. To investigate the reverse direction, we look at an example of a non-realizable tropical curve. In [Mik04] it was observed that every cubic curve in \( \mathbb{C}P^3 \) is planar (fig. 11a). Consequently, the example drawn in fig. 11b — a tropical cubic which is not contained within any tropical plane — cannot arise as the tropicalization of any curve in \( \mathbb{C}P^3 \).

**Corollary 6.4.1.** Let \( V \) be the tropical curve from [Mik04, Example 5.12]. Then the standard lift of \( L_V \) is an obstructed Lagrangian.

A general criteria understanding this phenomenon was stated in [Spe14].

**Theorem** (Speyer’s Well-Spacedness). Let \( V \) be a genus-one tropical curve whose cycle is contained within a linear subspace \( H \). Let \( d_1, \ldots, d_k \) be the affine lengths of paths along the edges of \( V \) to the boundary of \( V \cap H \). If the minimal distance occurs at least twice, the curve \( V \) is realizable.

We now speculate on how Speyer’s well-spacedness criteria can be understood in terms of holomorphic disks with boundary on \( L_V \). For \( L_V \) to be unobstructed, it is necessary for the lowest energy terms in \( m^0 \) to be null-homologous. In particular: the set

\[
\left\{ u : \omega(u) \leq \min_{0 \neq |\partial u'| \in H^2(L,M)} \omega(u') \right\}
\]

of minimal area non-null-homologous disks must contain at least two elements. This matches the “two minimal distance” criterion of Speyer’s Well-Spacedness theorem.
In [Hic21a], we saw that tropical cycles on $W \subset \mathbb{R}^2$ are related to non-regular Maslov-index zero disks with boundaries on the Lagrangian lifts $L_V$; it was speculated that these Maslov-index zero disks could appear regularly if they were glued onto a regular holomorphic disk or strip. In example 6.3.2 we saw that the Lagrangian brane lift of a small neighborhood of the green segment in fig. 11b must have a regular disk with energy given by the affine length of the edge.

In the example given by fig. 11b, we conjecture that there are regular holomorphic disks with boundaries on $L_V$ whose projections under the moment map are:

- The union of the blue hexagon (a non-regular disk) and green path (a regular disk); call this speculative disk $u_1$.
- The union of the blue hexagon (a non-regular disk) and red path (a regular disk); call this speculative disk $u_2$.

Using that the area of homology classes of disks with boundary on $L_V$ correspond to affine length, the disks $u_1$ and $u_2$ have the matching symplectic area if the affine lengths of the green and blue path match. In this case, the homology class of $[\partial u_1] - [\partial u_2]$ doesn’t wrap around the portion of the homology of $L_V$ which arises from $V$, and by a similar argument used in corollary 4.3.3 we see that $[\partial u_1] - [\partial u_2] \subset H_1(L_V(0))$. We could then apply the methods used in the proof of corollary 4.3.3 to conclude that $L_V$ is unobstructed.

In the event that $\omega(u_2)$ is uniquely minimal, the boundary of $\partial(u_2)$ is a non-trivial homology class in $H_1(L_V)$, suggesting that the contribution to $m^0 \in CF^0(L_V)$ is a non-removable obstruction.

### 6.5 Deformations, superabundance, and not-wide

#### 6.5.1 Geometric Deformations of $L$ and $(V, L)$

Given $V \subset \mathbb{R}^n$ a tropical subvariety, a Lagrangian $L_V$ should correspond to a lift of $V$ equipped with a line bundle. In this section, we examine how the deformations of $L_V$ up to Hamiltonian isotopy match deformations of a tropical curve equipped with a line bundle $(V, \mathcal{L})$.

Given a fixed tropical line bundle $\mathcal{L} \to V$ we can identify deformations of $\mathcal{L}$ with $H^1(V, \mathbb{R})$: this is because deformations of invertible locally integral affine functions from $U$ to $\mathbb{R}$ correspond to constant differences. Similarly, the deformations of $V \subset \mathbb{R}^n$ as a smooth tropical subvariety can be computed sheaf-theoretically. We choose a cover conducive to this computation. To each $v \in V$, let $\text{star}(v)$ be the union of the edges which contain $v$. We allow $v$ to be a leaf (at the end of a semi-infinite edge). Then the $\text{star}(v)$ form a cover of $V$, with $\text{star}(v) \cap \text{star}(w) = \overline{vw}$ whenever $vw$ is an edge. There are two types of vertices $v$ which we must consider:

- If $v$ is an internal vertex, then the deformations of $\text{star}(v)$ are identified with the integral affine space $Nv = T_v \mathbb{R}^n = \mathbb{R}^n$.
- If $v_\infty$ is a boundary vertex incident to edge $e$, then the deformations of $\text{star}(v_\infty)$ are identified with the integral affine space $\mathbb{R}^{n-1}$ perpendicular to the semi-infinite edge attached to $v_\infty$.

Over each edge $e$, the deformations of the tropical curve are given by the normal bundle to $e$. In summary, let $\text{Def}_V$ be the sheaf of deformations of the tropical embedding of $V$, and let $\text{Def}_{\mathcal{L}}$ be the deformations of a fixed line bundle $\mathcal{L}$ over $V$. We have:

$$\text{Def}_V(\text{star}(v)) = \mathbb{R}^n \quad \text{Def}_V(\text{star}(v_\infty)) = e_\perp^v \quad \text{Def}_V(\text{star}(e)) = e_\perp^e$$
For compact Lagrangian $L$, the infinitesimal deformations of $L$ up to Hamiltonian isotopy are described by classes in $H^1(L, \mathbb{R})$. Since $L_V$ is non-compact, we only consider the \textit{admissible} deformations of non-compact $L_V$ which preserve the condition in definition 3.1.1. Let $\Omega^1_{\text{admis}}(L_V, \mathbb{R})$ be the 1-forms on $L_V$ with the property that

- For each monomial region $U_\alpha$, the 1-form $\eta|_{L_V \cap U_\alpha}$ is invariant under the flow in the $\alpha$-direction.
- Furthermore, $\eta(\alpha) = 0$.

We let $\Omega^0_{\text{admis}}(L_V, \mathbb{R})$ be those functions which, outside of a compact set, are invariant under the flow in the $\alpha$ direction of the corresponding monomial region from definition 3.1.1.

We can similarly decompose $L_V$ into sets $L_{\text{star}}(v)$, which we will take to be:

- The standard Lagrangian pair of pants when $v$ is in an interior vertex so that $L_{\text{star}}(v) \cap L_{\text{star}}(w) = L_{\text{Vvw}}$. In this case $\Omega^i_{\text{admis}}(L_V, \mathbb{R}) = \Omega^i(L_V, \mathbb{R})$.
- A non-compact cylinder extending to the boundary whenever $w$ is a vertex at a non-compact edge.

We then compute $H^1(\Omega^\bullet_{\text{admis}}(L_V))$. The cohomology is the same as the first cohomology of the total complex; the first page in the spectral sequence is

$$
\begin{array}{c}
\bigoplus_{v \in V} H^0(\Omega^\bullet(L_v)) \\
\downarrow \\
\bigoplus_{e \in E} H^0(\Omega^\bullet(L_e)) \\
\downarrow \\
0
\end{array} \quad \quad \quad
\begin{array}{c}
\bigoplus_{v \in V} H^1(\Omega^\bullet(L_v)) \\
\downarrow \\
\bigoplus_{e \in E} H^1(\Omega^\bullet(L_e)) \\
\downarrow \\
0
\end{array} \quad \quad \quad \cdots
$$

We now start to identify these with deformations of tropical curves.

$$
\text{Def}_V(\text{star}(v)) = H^1(\Omega^\bullet(L_v)) \quad \quad \quad \text{Def}_V(\text{star}(e)) = H^1(\Omega^\bullet(L_e))
$$

$$
\mathbb{R} = H^0(\Omega^\bullet(L_v)) \quad \quad \quad \mathbb{R} = H^0(\Omega^\bullet(L_e))
$$

turning the first page of the spectral sequence into

$$
\begin{array}{c}
\bigoplus_{v \in V} \mathbb{R} \\
\downarrow \\
\bigoplus_{e \in E} \text{Def}_V(e) \\
\downarrow \\
0
\end{array} \quad \quad \quad
\begin{array}{c}
\bigoplus_{v \in V} \text{Def}_V(\text{star}(e)) \\
\downarrow \\
\bigoplus_{e \in E} \text{Def}_V \\
\downarrow \\
0
\end{array} \quad \quad \quad \cdots
$$

The spectral sequence for $H^1(\Omega^\bullet(L))$ converges at the second page for this covering, so

$$
H^1(\Omega^\bullet(L_V)) = H^0(V, \text{Def}_V) \oplus H^1(V, \mathbb{R}) = H^0(V, \text{Def}_V) \oplus H^0(V, \text{Def}_L).
$$
In general, understanding the moduli space of Lagrangian submanifolds isotopic to $L_V$ modulo Hamiltonian isotopy is a difficult question. In the setting of Lagrangian torus fibrations, there is a smaller class of Hamiltonian isotopies that we can hope to understand. We say that a Lagrangian isotopy $i_t : L_V \to X_A$ is a fiberwise isotopy if $\pi_A(i_t(q))$ is constant for all $q \in L_V$.

**Claim 6.5.1.** Let $L_V$ be a homologically minimal Lagrangian lift of a tropical curve $V$. Then the subspace of $H^1(L_V, \mathbb{R})$ arising from the flux classes of fiberwise Lagrangian isotopies is identified with $H^0(V, \text{Def}_L)$. Additionally,

$$\{\text{Fiberwise isotopies}\}/\{\text{Fiberwise Hamiltonian isotopies}\} \simeq H^1(V, \text{Aff}_V),$$

the tropical Jacobian.

Mirror symmetry, therefore, identifies fiberwise isotopies of a tropical Lagrangian $L_V$ with Modifications of the line bundle on the mirror curve $Y_V$.

### 6.5.2 Not-Wide and Superabundance

A tropical curve is called *superabundant* if the space of deformations $\text{Def}_V$ has a higher dimension than the expected dimension of deformations of the $B$-realization. Superabundance is a computable criterion that indicates that a curve may not be realizable. For example, the tropical curve examined in section 6.4 is a superabundant curve. It is known in certain cases [Che+16] that non-superabundant implies realizable.

In symplectic geometry, there are two ways to make sense of deformations of Lagrangian submanifolds. The first kind of deformations are the deformations of geometric Lagrangian submanifolds up to Hamiltonian isotopy. The infinitesimal deformations of Lagrangian submanifolds modulo Hamiltonian isotopy are given by $H^1(L, \mathbb{R})$. The second deformation space which we can consider is the component of the moduli space of objects at $L$ [TV07]. The tangent space to this moduli space is $HF^1(L)$. We note that as $CF^*(L)$ is a deformation of $C^*(L)$, we have that $\dim HF^1(L) \leq HF^1(L)$.

If $\dim HF(L) = \dim H(L)$, then the Lagrangian $L$ is called *wide*.

As the previous section identifies infinitesimal deformations of the pair $(V, L)$ with $H^1(L_V)$, we are led to conjecture:

**Conjecture 6.5.2.** Let $V$ be a smooth tropical curve, and let $L_V$ be its Lagrangian lift. Then $V$ is superabundant if and only if $L_V$ is not wide.

### A Pearly model in symplectic fibrations

Given a compact, spin, and graded Lagrangian $L$ inside of a rational compact symplectic manifold $X$, [CW19] constructs a filtered $A_\infty$ algebra $CF^*(L, h, \mathcal{P}, D)$. In [CW19] it is assumed that the space $X$ is compact. In this appendix, we outline how to extend [CW19] to the setting where $X$ is non-compact and is equipped with a potential function $W : X \to \mathbb{C}$; and $L$ is a Lagrangian submanifold which is admissible with respect to $W$.

**Definition A.0.1.** Let $X$ be a symplectic manifold, and let $W : X \to \mathbb{C}$ be a function. We say that $W$ is a potential if there exists a compact subset $U \subset \mathbb{C}$ so that

- $W^{-1}(U)$ is compact, and
the restriction $W : X \setminus W^{-1}(U) \to \mathbb{C} \setminus U$ is a symplectic fibration with compact fibers.

We say that a Lagrangian $L$ is $W$-admissible if there exists $R \in \mathbb{R}$ so that $W(L) \cap \{z : |z| > R\} \subset \mathbb{R}_{>R}$.

Given a $W$-admissible $L$, we say that a Morse function $h : L \to \mathbb{R}$ is admissible if there exists $R' > R$ so that
\[ W(\text{Crit}(h)) \cap \{z : |z| > R\} \subset \{R'\}. \]

and $\text{grad } h$ points outwards from $R'$ under the projection $W$.

Let $Y = W^{-1}(R')$. Given a $W$-admissible Lagrangian submanifold $L$, the restriction to the fiber $M := L \cap Y$ is a Lagrangian submanifold of $Y$. Because $h$ points outwards along the collar $M \times \mathbb{R}_{>R'} \subset L$, the Morse complex $CM^*(L, h)$ is well defined. The compatibility of Morse function with the potential function means that $h^+ := h|_M$ is a Morse function for $M$ and that we have a map of $A_\infty$ algebras
\[ \pi : CM^*(L, h) \longrightarrow CM^*(M, h^+). \]

This should be interpreted as the pullback map of the inclusion of the boundary.

We show that [CW19] extends to the setting of $W$-admissible Lagrangian submanifolds.

**Theorem A.0.2.** Let $W : X \to \mathbb{C}$ be a potential function. Let $L$ be a $W$-admissible Lagrangian submanifold, whose restriction to a large fiber is $M \subset Y = W^{-1}(t)$. Let $h : L \to \mathbb{R}$ and $h^+ := h|_M : M \to \mathbb{R}$ be admissible Morse functions. There exist:

- stabilizing symplectic divisors $D_X \subset X, D_Y \subset Y$; and
- regular choices for perturbation systems $\mathcal{P}_L, \mathcal{P}_M$ for $L$ and $M$;

so that the construction of [CW19] can be applied to give a well defined $A_\infty$ algebra $CF^*(L, h, \mathcal{P}_L, D_X)$. Furthermore, the choices of perturbations and divisors can be taken so that the projection on chains
\[ \pi : CF^*(L, h, \mathcal{P}_L, D_X) \to CF^*(M, h^+, \mathcal{P}_M, D_Y) \]

is a $\Lambda$-filtered $A_\infty$ algebra homomorphism.

The theorem consists of two statements: construction of a pearly model of stabilized treed disks in the setting of potential functions, and the compatibility between the pearly model of total space of the fibration and the pearly model of the fiber. These are analogous to [Hic19a, Corollary C.4.2 and Theorem C.5.1] which handle the setting where $X = Y \times \mathbb{C}$ and $W : X \to \mathbb{C}$ is projection to the second factor. In this appendix, we prove that $CF^*(L, h, \mathcal{P}_L, D_X)$ is well defined; the existence of the projection $\pi : CF^*(L, h, \mathcal{P}_L, D_X) \to CF^*(M, h^+, \mathcal{P}_M, D_Y)$ is the same as the proof of [Hic19a, Theorem C.5.1].

To construct $CF^*(L, h, \mathcal{P}_L, D_X)$ one needs to

1. Construct a stabilizing divisor for $X$ which is suitably compatible with the potential $W : X \to \mathbb{C}$;

2. Show that we can pick perturbations for almost complex structure for which the map $W : X \to \mathbb{C}$ is holomorphic outside of a compact set; and

3. Prove that for such choices of perturbations the moduli spaces have appropriate Gromov compactifications.
Item 1: Constructing a stabilizing divisor

Pick \( R \) sufficiently large so that outside of \( U = B_R(0) \) the Lagrangian submanifold \( L \) fibers over the positive real ray, and the map \( X \setminus W^{-1}(U) \to \mathbb{C} \setminus U \) is a symplectic fibration. For \( \theta \in [0, 2\pi], r \geq R \) we take a path

\[
\gamma_{\theta, r}(t) = \begin{cases} 
Re^{i\theta(2t)} & t \in [0, 1/2) \\
(R + (2t - 1)(r - R))e^{i\theta} & t \in [1/2, 1]
\end{cases}
\]

which travels first in the angular, then radial direction from \( R \) to \( re^{i\theta} \). To every path \( \gamma(t) : I \to \mathbb{C}_{|z|>R} \) we have a symplectic parallel transport map \( P_{\gamma} : Y_{\gamma(0)} \to Y_{\gamma(1)} \). Consider the monodromy \( P_{\gamma_{2\pi, r}} : Y_{\gamma} \to Y_{\gamma} \) given by parallel transport around the loop \( Re^{i\theta} \) in the positive direction. Pick a path of \( \omega_Y \)-tamed almost complex structures \( J_{Y_{R, \theta}} : [0, 2\pi] \to J_{\gamma}(Y_{R}, \omega_Y) \) such that \( P_{\gamma_{2\pi, r}}^{*}J_{2\pi} = J_{0} \).

This gives us endomorphism of the subbundle of the tangent spaces to the fibers

\[
J_{re^{i\theta}} : TY_{re^{i\theta}} \to TY_{re^{i\theta}} \\
J_{re^{i\theta}} = P_{\gamma_{0, r}}^{*}J_{Y_{R, \theta}}
\]

Since over every point with \( |z| > R \) we have a splitting \( T_{y,z}X = T_{y}Y \oplus T_{z}\mathbb{C} \), we can give \( T(X \setminus W^{-1}(U)) \) the same almost complex structure locally defined by \( J_{re^{i\theta}} \oplus J_{\mathbb{C}} \).

**Definition A.0.3.** We say that a \( \omega \)-tame almost complex structure on \( X \) is \( W \)-admissible if, when restricted to \( W^{-1}(U) \) it can be written as \( J_{re^{i\theta}} \oplus J_{\mathbb{C}} \) for some path of almost complex structures \( J_{Y_{R, \theta}} \in J_{\gamma}(Y_{R}, \omega) \). We denote the space of such almost complex structures \( J_{\gamma_{W,R}}(X, \omega_X) \).

The goal will be to construct a stabilizing divisor \( D_X \subset X \) in such a way that \( D_X \) is transverse to all \( Y_{re^{i\theta}} \) with \( r \geq R \), and subsequently show that there exists an open dense set of almost complex structures belonging to \( J_{\gamma_{W,R}}(X, \omega_X) \) which are \( E \)-stabilized by \( D_X \) ([CW19, Definition 4.24]). In the setting of Lagrangian cobordisms, the comparable statements are proven in [Hic19], Appendix C.3 and Lemma C.1.3.

We first construct the divisor \( D_X \). Take \( E_X \rightarrow X \) a vector bundle whose first Chern class is \( 1/2|\omega_X| \), so that the pullback \( E_Y \rightarrow Y \) a vector bundle whose first Chern class is \( 1/2|\omega_Y| \). Pick a family of Hermitian structures on \( E_{Y, \theta} \rightarrow Y_{re^{i\theta}} \) depending on \( \theta \) so that the curvature is \( -i\omega_Y \) and so that \( P_{\gamma_{2\pi, r}}^{*}E_{Y_{2\pi}} = E_{Y_0} \) as Hermitian line bundles. Let \( i_{\theta_0} : Y_{re^{i\theta_0}} \rightarrow X \) be the inclusion of the fiber over \( Re^{i\theta_0} \). Take a Hermitian structure on \( E_X \rightarrow X \) with curvature \( -i\omega_X \) and the property that \( \tau_{\theta_0}P_{\gamma_{0, r}}^{*}E_X = E_{Y_{\theta_0}} \) as Hermitian line bundles.

We will construct the stabilizing divisor \( D_X \) as the zero locus of an asymptotically holomorphic sections \( s_{k,X} : X \rightarrow E_X \). First, using [AGM01] we can pick asymptotically holomorphic sections \( s_{k,Y} : Y \rightarrow E_Y \) with the property that \( s_{k,Y}^{-1}(0) \) is disjoint from \( M \). We obtain a second asymptotically holomorphic section by pullback \( P_{\gamma_{2\pi, r}}^{*}s_{k,Y} \). By [Aur97] we can find a family \( s_{k,Y, \theta} \) of such sections so that \( s_{k,Y, 0} = s_{k,Y} \) and \( s_{k,Y, 2\pi} = P_{\gamma_{2\pi, r}}^{*}s_{k,Y} \).

Using this family of sections, we create an asymptotically holomorphic section \( s_{k,X, out} : X \rightarrow E_X \) which is given by

\[
s_{k,X, out}(z) := \rho_{k,R+1}(|z|) \cdot P_{\gamma_{0, r}}^{*}s_{k,Y, \theta}
\]

where \( \rho_{k,R+1}(|z|) : \mathbb{C} \rightarrow \mathbb{R} \) is a function which is concentrated (in the sense of [AGM01, Definition 2]) at the circle of radius \( R + 1 \). The zero set \( s_{k,X, out}^{-1}(0) \) enjoys the properties that

- \( s_{k,X, out}^{-1}(0) \) is disjoint from \( L \);
• for $k$ sufficiently large, $s_{k,X,out}^{-1}(0)$ is a symplectic divisor in $W^{-1}(\{z : |z| > R\})$; and

• $s_{k,X,out}^{-1}(0)$ intersects $Y_{re\theta}$ transversely for all $r > R$.

This constructs the sections taking the place of $s_{k,X \subset out}$ in [Hic19b, Appendix C.3.2]. The remainder of the construction of $D_X$ involves subsequently perturbing this section over the region $W^{-1}(U)$ which exactly follows [Hic19b, Appendix C.3.2].

**Item 2: Finding perturbations**

The construction of an open dense set of $E$-stabilized almost complex structures proceeds in the same fashion as [Hic19b, Section C.3.3] (itself based on the argument of [CW19, Section 4.5]). The main tool needed for the argument to run is to show that the space of almost complex structures regularizing holomorphic disks of energy up to $E$ is dense in $J_{\tau,W,R}(X, \omega_X)$. By application of the open mapping principle to $W$, every pseudoholomorphic disk in consideration must either:

• pass through $W^{-1}(U)$, where they can be made regular through perturbations confined to the region $W^{-1}(U)$ by application of [CM07, Lemma 5.6]; or

• be confined to a fiber $W^{-1}(t)$ with $t \in U$, in which case they can be made regular through perturbations constrained in the fiberwise direction. Since the fiber is compact, the set of such perturbations is open and dense.

**Item 3: Compactness of moduli spaces**

The proof that the moduli spaces of pseudoholomorphic treed disks considered are compact uses that we may apply open-mapping principle type arguments for perturbations chosen from $J_{\tau,W,R}(X, \omega_X)$, and that the Morse flow line components of treed disks point outwards at the boundary. ([Hic19b, Proposition C.4.1])

**Remark A.0.4.** In the examples we consider (potentials coming from tropicalized superpotentials associated to a monomial admissibility data) the fibers of the potential will in general not be compact. However, the monomially admissible condition ensures that the restriction of monomially admissible $L \subset X$ to $M \subset Y$ will be compact. As a result, all pseudoholomorphic disks contributing to treed disks will have boundary contained within a compact subset of $X$; we conclude that the moduli space of treed disks has compactification given by broken treed disks.

**B Auxiliary results for filtered $A_\infty$ algebras and modules**

In this section, we give some background for filtered $A_\infty$ algebras and bimodules, as well as provide some methods for constructing bounding cochains using the filtration on the $A_\infty$ algebra.

**B.1 A short review of bounding cochains**

The *Novikov ring* with $\mathbb{C}$-coefficients is the ring of formal power series

$$A_{\geq 0} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in \mathbb{C}, \lambda \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$
The field of fractions is the Novikov field Λ. A filtered \( A_\infty \) algebra \((A, m^k_A)\) is a free graded \(\Lambda \geq 0\)-module \(A^\bullet\) equipped with \(\Lambda \geq 0\)-linear products

\[
m^k : (A^\bullet)^{\otimes k} \to (A^\bullet)^{2-k}
\]

for each \(k \geq 0\). These are required to satisfy the axioms of [Fuk+10, Definition 3.2.20]. Among these axioms are:

- The quadratic filtered \( A_\infty \) relationship

\[
0 = \sum_{k_1 + k' + k_2 = k} (-1)^{\bullet(x,k_1)} m^{k_1+1+k_2}_A \circ (\text{id}^{\otimes k_1} \otimes m^{k'}_A \otimes \text{id}^{\otimes k_2}) (x_1, \cdots, x_k)
\]

The sign is determined by \(\bullet(x,k_1) := k_1 + \sum_{j=1}^{k_1} \deg(x_j)\).

- Each \( A^i \) have a filtration \( F^\Lambda A^i \) respecting the filtration on \( \Lambda \geq 0\), and a basis belonging to \( F^0(A^i) \setminus \bigcup_{\lambda > 0} F^\lambda A^i \).

Given a filtered-\( A_\infty \) algebra, we can also consider the \( \Lambda \)-linear products on \( A \otimes_{\Lambda \geq 0} \Lambda \). We call \( A \otimes_{\Lambda \geq 0} \Lambda \) a \( \Lambda \)-filtered \( A_\infty \) algebra.

Let \((A, m^k_A)\) and \((B, m^k_B)\) be \( A_\infty \) algebras. A filtered \( A_\infty \) homomorphism from \( A \) to \( B \) is a sequence of filtered graded maps

\[
f^k : A^{\otimes k} \to B
\]

satisfying the quadratic \( A_\infty \) homomorphism relations:

\[
\sum_{k_1 + k' + k_2 = k} (-1)^{\bullet(x,k_1)} f^{k_1+1+k_2}_B \circ (\text{id}^{\otimes k_1} \otimes m'^{k'}_B \otimes \text{id}^{\otimes k_2}) = \sum_{i_1 + \cdots + i_j = k} m^j_B \circ (f^{i_1} \otimes \cdots \otimes f^{i_j}).
\]

There similarly exists a notion of a homotopy between filtered \( A_\infty \) homomorphisms.

The main difficulty with filtered \( A_\infty \) algebras is that they do not have cohomology groups, as

\[
(m^1_A)^2 = m^2_A (m^0_A \otimes \text{id}) \pm m^2_A (\text{id} \otimes m^0_A).
\]

When \( m^0_A = 0 \), the right-hand side of the relation is zero and we say that \( A \) is a tautologically unobstructed unobstructed \( A_\infty \) algebra.

It is desirable to work with tautologically unobstructed \( A_\infty \) algebras as they can be studied with the standard tools employed for cochain complexes. Therefore, one might restrict one’s study to tautologically unobstructed filtered \( A_\infty \)-algebras. Problematically, tautologically unobstructed filtered \( A_\infty \) algebras are not closed under the relation of filtered \( A_\infty \) homotopy equivalence. This can be remedied by considering filtered \( A_\infty \) algebras equipped with bounding cochains.

Let \( A \) be a filtered \( A_\infty \) algebra. A deforming cochain is an element \( d \in A^1 \) with \( \val(d) > 0 \). The \( d \)-deformation of \( A \) is the filtered \( A_\infty \) algebra \((A, d)\) whose

- underlying chains groups agree with \( A \) and,
- composition maps are given by the \( d \)-deformed \( A_\infty \) products,

\[
m^k_{(A,d)} = \sum_{l=0}^{\infty} \sum_{j_0 + \cdots + j_k = l} m^{k+l}_{A} (d^{\otimes j_0} \otimes \text{id} \otimes d^{\otimes j_1} \otimes \cdots \otimes \text{id} \otimes d^{\otimes j_k}).
\]
Definition B.1.1. When \( m^0_{(A,b)} = 0 \), we say that \( b \) is a bounding cochain, and we say that the algebra \( A \) is unobstructed.

Given \( f : A \to B \) a filtered \( A_\infty \) homomorphism and \( b \in A \) a bounding cochain, there is a pushforward bounding cochain \( f_*(b) \in B \) so that \((B, f_*(b))\) is unobstructed. When \( f^k = 0 \) for \( k \neq 0 \) then \( f_*(b) = f(b) \). The existence of a pushforward bounding cochain shows that unobstructedness is a property of filtered \( A_\infty \) algebras which is preserved under the equivalence relation of filtered \( A_\infty \) homotopy equivalence.

In applications, we use \( \Lambda \)-filtered \( A_\infty \) algebras as opposed to filtered \( A_\infty \) algebras\(^3\). However, the homological algebra of filtered \( A_\infty \) algebras is notationally easier to describe (as there exist elements living in a minimal filtration level). A computation allows us to understand deformations and bounding cochains for the former (defined using the eq. (11)) in terms of the latter.

Claim B.1.2. Suppose that \( A \) is a filtered \( A_\infty \) algebra and \( b \) a bounding cochain for \( A \). Then \( b \otimes 1 \in A \otimes A_{\geq 0} \Lambda \) is a bounding cochain for the \( \Lambda \)-filtered \( A_\infty \) algebra \( A \otimes \Lambda_{\geq 0} \Lambda \).

B.2 Extending an unobstructed ideal

Following ideas from \([Fuk+10]\), we will provide a method for constructing bounding cochains by inducting on the valuation. In order to do this, we need a slight refinement of a filtered \( A_\infty \) algebra which states that the valuation of the structure coefficients is ordered by a monoid. A gapped \( A_\infty \) algebra is a filtered \( A_\infty \) algebra for which there exists a finitely generated monoid \( G \) and a monoid homomorphism \( \omega : G \to \mathbb{R}_{\geq 0} \) so that \( \omega(\beta) = 0 \) implies that \( \beta = 0 \), and so that we have the decomposition

\[
m^k = \sum_{\beta \in G} T^{\omega(\beta)} m^{k,\beta}
\]

where \( m^{k,\beta} \) are graded with respect to the filtration. We say that it satisfies the gapped \( A_\infty \) relations if for all \( \beta \in G \)

\[
\sum_{\beta_1 + \beta_2 = \beta} \sum_{j_1 + j_2 = k} (-1)^{j_1} m^{j_1 + 1, j_2, \beta_1} (\text{id} \otimes m^{j_2} \otimes \text{id} \otimes j_2) = 0
\]

Given \( b = \sum_{\beta \in G \setminus \{0\}} b_\beta \), we can deform the product structure by

\[
m^{k,\beta}_{(B,b)} = \sum_{\beta_0 + \cdots + \beta_k = \beta} m^{k+1}_{B} (\beta_0, 0 \otimes \cdots \otimes \beta_0, l_0 \otimes \text{id} \otimes \cdots \otimes \text{id} \otimes \beta_{k,0} \otimes \cdots \otimes \beta_{k,l_k}).
\]

so that \( m^k_{(B,b)} := \sum_{\beta \in G} T^{\omega(\beta)} m^{k,\beta}_B \) gives a \( G \)-gapped \( A_\infty \) algebra satisfying the gapped \( A_\infty \) relations. There similarly exists \( G \)-gapped filtered \( A_\infty \) homomorphisms, which also contains the data of a morphism of monoids \( \phi : G_A \to G_B \).

We will also need some basic statements about ideals in filtered \( A_\infty \) algebras.

Definition B.2.1. A subspace \( A \subset B \) is a weak \( A_\infty \) ideal if for all \( k = k_1 + 1 + k_2 > 0 \), the map

\[
m^k : B^{\otimes k_1} \otimes A \otimes B^{\otimes k_2} \to B
\]

\(^3\)This is because the continuation maps in Lagrangian intersection Floer cohomology are usually only weakly-filtered.
has image contained in \( A \).

Notably, we do not require that the curvature term \( m^0_A \) be an element of \( A \). As a result, it is not necessarily the case that \( A \) is itself a filtered \( A_\infty \) algebra. We say that \( A \) is a strong \( A_\infty \) ideal if additionally \( m^0_B \in A \).

**Claim B.2.2.** Let \( A \subset B \) be an \( A_\infty \) ideal. The quotient \( C = A/B \) inherits a filtered \( A_\infty \) structure. \( A \) is a strong \( A_\infty \) ideal if and only if \( C \) is tautologically unobstructed.

**Proof.** The filtered \( A_\infty \) structure is the natural one,

\[
m^k_C([x_1] \otimes \cdots \otimes [x_k]) := [m^k_B(x_1 \otimes \cdots \otimes x_k)]
\]

Because the \( m^k_B \) are multilinear, we see that if \( [x_i] = [x_i'] \), that \( m^k_C([x_1] \otimes \cdots \otimes [x_i] \otimes \cdots \otimes [x_k]) = m^k_C([x_1] \otimes \cdots \otimes [x_i'] \otimes \cdots \otimes [x_k]). \) \( A \) is a strong \( A_\infty \) ideal if and only if \( m^0_B = m^0_B = [0]. \)

**Example B.2.3.** Given a formal filtered \( A_\infty \) morphism \( f : B \to C \) (so that \( f^k = 0 \) for all \( k \neq 1 \)) the kernel of \( f \) is a weak \( A_\infty \) ideal.

**Example B.2.4.** Given a filtered \( A_\infty \) algebra \( A \), the set \( A_{>0} \) of positively filtered elements is an example of a strong \( A_\infty \) ideal. The quotient \( A := A_{>0} \) is an example of a tautologically unobstructed \( A_\infty \) algebra. A relevant example comes from Lagrangian Floer cohomology, where \( C \mathcal{F}^*(L) = CM^*(L) \).

**Claim B.2.5.** Suppose that \( A \subset B \) is an \( A_\infty \) ideal, and \( d \in B \) is a deforming cochain. Then \( A \) is an \( A_\infty \) ideal of \( (B, d) \). If \( A \subset B \) is a strong \( A_\infty \) ideal, and \( m^0_B \in A \), then \( A \) is a strong \( A_\infty \) ideal of \( B \). In particular, if \( d \in A \) then \( A \) is a strong \( A_\infty \) ideal of \( (B, d) \).

**Proof.** Suppose that \( a \in A \) is some element. Then

\[
m^k_{(B, d)}(x_1 \otimes \cdots \otimes a \otimes \cdots \otimes x_k)
\]

\[
= \sum_{l=0}^\infty \sum_{j_0+\cdots+j_k=l} m^{k+l}(d^{j_1} \otimes id \otimes d^{j_1} \otimes \cdots \otimes a \otimes \cdots \otimes id \otimes d^{j_k}) \in A.
\]

proving that \( A \) is a \( A_\infty \) ideal of \( (B, d) \). \( \square \)

The vector space \( H^1(A) \) is a lowest order approximation to the space of bounding cochains. When \( \overline{C} \) is an anti-commutative differential graded algebra elements of \( H^1(\overline{C}) \) are bounding cochains.

**Claim B.2.6.** Suppose that \( C \) is tautologically unobstructed. Suppose that \( f : C \to \overline{C} \) is an \( A_\infty \) map with gapped \( A_\infty \) homotopy inverse \( g : \overline{C} \to C \). Assume that \( \overline{C} \) is an anti-commutative differential graded algebra. Then for every class \( [c] \in H^1(C) \) with \( \mathrm{val}(c) > 0 \), there exists a bounding cochain \( c' \in C \) and \( \lambda > \mathrm{val}(c') \) with \( [c'] = [c] \in H^1(C/T^\lambda C) \).

**Proof.** Since \( C, \overline{C} \) are gapped, we can select \( \lambda > \mathrm{val}(c) \) so that \( \omega(\beta) < \lambda \) implies \( \omega(\beta) \leq \mathrm{val}(c) \). We observe that \( f(c) \in \overline{C} \) is closed, and therefore provides a bounding cochain for \( \overline{C} \), as

\[
m^0_{(\overline{C}, f(c))} = m^1_\overline{C}(f(c)) + m^2_\overline{C}(f(c), f(c)) = 0.
\]

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We then take \( c' \) to be the pushforward bounding cochain

\[
g_*(f(c)) = \sum_{k=1}^{\infty} g^k((f(c))^\otimes k)
\]

Since \( c' = (g \circ f)(c) \mod T^\lambda \), we obtain that \([c] = [c'] \in H^1(C/T^\lambda C)\). \( \square \)

**Claim B.2.7** (Claim [A.4.8 Hic19b].) Suppose that \( A' = (A, a) \). Given a deforming cochain \( a' \in A' \), the chain \( a'' = a + a' \in A \) is a deforming cochain so that \((A', a') = (A, a'')\).

We now come to the main lemma of this appendix. Suppose that we have an exact sequence (on the chain level) \( A \to B \to C \). If \( A \) is a strong \( A_\infty \) ideal containing the curvature of \( B, A \to C \) is zero, then we prove that there is no obstruction to finding a bounding cochain for \( B \). The argument is in the style of [Fuk+10, Theorem 3.6.18].

**Lemma B.2.8.** Consider a \( G \)-gapped \( A_\infty \) algebra \( B \) satisfying the gapped \( A_\infty \) relations. Suppose that:

(i) \( A \) is a strong \( A_\infty \) ideal of \( B \), and \( C = B/A \) giving us an exact sequence \( A \xrightarrow{i} B \xrightarrow{\pi} C \) of gapped \( A_\infty \) algebras,

(ii) There exists \( \overline{C} \) which is \( A_\infty \) homotopic to \( C \) and is an anti-commutative DGA,

(iii) Additionally, suppose that the connecting map \( \delta : H^1(\overline{C}) \to H^2(A) \) surjects.

Then for every \( \lambda > 0 \) there exists a deforming cochain \( b = \sum_{\beta \in G \setminus \{0\}} b_\beta \) for \( B \) so that for all \( \beta \) with \( \omega(\beta) \leq \lambda \), \( m_{0,\beta}^{(B,b)} = 0 \).

**Proof.** Because \( A, B, C \) are gapped \( A_\infty \) algebras, there exists \( \{\lambda_i\}_{i=1}^{n} \) an ordering of the image \( \omega(G) \in [0, \lambda] \).

We prove the statement by induction on \( \lambda_i \). Suppose that \( A' = (A, a_{i-1}), B' = (B, b_{i-1}), C' = (C, c_{i-1}) \) are \( G \)-gapped \( A_\infty \) algebras satisfying items (i) to (iii) and additionally

(iv) The curvature has large valuation, \( \text{val}(m_{0,B}^0) > \lambda_{i-1} \)

The inductive step will construct deforming cochains \( a', b', c' \) so that the algebras \((A', a'), (B', b'), (C', c')\) satisfy items (i) to (iv) where \( \lambda_{i-1} \) is replaced with \( \lambda_i \). By claim B.2.7 we can then construct the \( A_\infty \) algebras \((A, a_i), (B, b_i)\) and \((C, c_i)\).

Write \( m_{0,B}' = \sum_{j=1}^{\infty} \sum_{\omega(\beta) = \lambda_j} b_{j,\beta} T^{\lambda_j} \), where the \( b_{j,\beta} \) are elements of \( B' = B \) of degree 2. Because \( A \) is a strong \( A_\infty \) ideal, we can find \( a_{i,\beta} \in A \) with \( i(a_{i,\beta}) = b_{i,\beta} \).

We examine the lowest order terms of the \( A_\infty \) relation \( m_{1,A}^1 \circ m_{0,A}^0 = 0 \), and obtain

\[
m_{1,A}^1(a_{i,\beta}) = 0
\]

Since \( [a_{i,\beta}] \in H^2(A) \), by item (iii) \([b_{i,\beta}] = 0\). Therefore there exists \( b_{i,\beta} \) so that \( m_{1,B'}(b_{i,\beta}) = b_{i,\beta} \). The class \( c_{i,\beta} := \pi(b_{i,\beta}) \) is closed. Using claim B.2.6 we can find \( c'_j \) with \( j \geq i \) so that \( c' = \sum_{j=i}^{\infty} \sum_{\omega(\beta) = \lambda_j} c_{j,\beta} T^{\lambda_j} \) is a bounding cochain for \( C' \) with the property that \([c'_{i,\beta}] = [c_{i,\beta}] \in H^1(C'/T^{\lambda_{i+1}}C') \).

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Because $\pi : B \to C$ surjects, we can find for all $j \geq i$ cochains $b'_{j,\beta} \in B$ with $\pi(b'_{j,\beta}) = c''_{j,\beta}$. Let
\[
b' = -\sum_{j=i}^{\infty} \sum_{\beta : \omega(\beta) = \lambda} b'_{j,\beta} T^{\lambda_i}.
\]
This constructed $b'$ satisfies the property
\[
m^1_{B'}(b') \equiv -m^0_{B'} \mod T^{\lambda_{i+1}}.
\]
Since $\pi$ is a filtered $A_{\infty}$ homomorphism without higher terms, the pushforward $\pi \ast (b') = c'$ and
\[
\pi \circ m^0_{(B,b')} = m^0_{(C,c')} = 0
\]
Therefore $m^0_{(B',b')}$ is contained in $A'$, and we write $a'$ for the corresponding element in $A'$. Claim [B.2.5] states that $(A',a')$ is a strong $A_{\infty}$ ideal of $(B',b')$, whose quotient is $(C',c')$.

This gives us the $G$-gapped $A_{\infty}$ algebras $(A',b'), (B',b')$ and $(C',c')$, which we’ve shown satisfy item [i]. We now show these algebras satisfy items [ii] to [iv]. Item [ii] follows from observing that deformations by Maurer-Cartan classes preserves having anti-commutative model. Since the deformation occurs at valuation greater than 0, the map $H^1(C) \to H^2(A)$ continues to surject (item [iii]).

To check item [iv],
\[
\text{val}(m^0_{(B',b')}) = \text{val}\left(\sum_{k=0}^{\infty} m^k_{B'}((b')^{\otimes k})\right)
\geq \min\left(\text{val}(m^0_{B'} + m^1_{B'}(b')), \sum_{k=2}^{\infty} m^k_{B'}((b')^{\otimes k})\right)
\]
Given that $m^0_{B'} \equiv m^1_{B'}(b') \mod T^{\lambda_i}$
\[
\geq \lambda_{i+1}.
\]

\[\square\]

**Corollary B.2.9.** Let $A, B, C$ be $A_{\infty}$ algebras as in lemma [B.2.8]. Then there exists a bounding cochain for $B$.

**Proof.** The deforming cochains constructed in the above proof satisfy the condition that
\[
b_i \equiv b_{i+1} \mod T^{\lambda_i}.
\]
It follows that if we use the inductive procedure to build a sequence of deforming cochains $\{b_i\}_{i=0}^{\infty}$ so that $\text{val}(m^0_{(B,b_i)}) > \lambda_i$, the limit $\lim_{i \to \infty} b_i$ is a bounding cochain. \[\square\]

### B.3 $A_{\infty}$-bimodules and bounding cochains

Let $A, B$ be $A_{\infty}$ algebras. An $(A, B)$-bimodule is a filtered graded $\Lambda_{\geq 0}$-module $M$, along with a set of maps for all $k_1, k_2 \geq 0$.
\[
m^{k_1|k_2}_{A|M|B} : A^{\otimes k_1} \otimes M \otimes B^{\otimes k_2} \to M
\]
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Lemma B.3.1. Let \( A \) be an \( A_\infty \) bimodule and that \( b \) is a \( \lambda \)-filtered \( A \)-bimodule product on \( A \). Then there exists a choice of bounding cochain \( G \) such that \( \sum_{k_1 \geq 0} m_{A|B}^{k_1}(A, a) \) is a filtered \( \lambda \)-gapped structure and induct on valuations. For simplicity of exposition, \( 0 \) is trivial, and item (iv) follows from the gapped structure. Item (v) has no content.

Proof. We again use the gapped structure and induct on valuations. For simplicity of exposition, \( 0 \) is trivial, and item (iv) follows from the gapped structure. Item (v) has no content.

Lemma B.3.1. Let \( M \) be a \( G \)-gapped \((A, B)\)-bimodule. Suppose that \( A, B \) are tautologically unobstructed and that \( A \) has an anti-commutative DGA model \( \overline{A} \) as in claim \( B.2.6 \). Suppose that there exists \( \lambda_0 < \lambda_1 \in \mathbb{R} \) with the following properties:

(i) The maps \( m_{A|M|B}^{k_1|0} : A^\otimes k \otimes M \to M \) all have image contained within \( T^{\lambda_0} M \) and

(ii) There exists \( e \in H^1(M) \) an element so that the map

\[
H^1(A) \to H^1(T^{\lambda_0} M / T^{\lambda_1} M)
\]

is surjective.

Then there exists a choice of bounding cochain \( a \in A^1 \) and element \( e \in M^0 \) so that \( m_{(A,a)|M|B}^{0|1|0}(a \otimes e) = 0 \).

Proof. We again use the gapped structure and induct on valuations. For simplicity of exposition, we will assume that the monoid \( G \) is \( \mathbb{N} \), so that \( \omega(G) = \{ n \lambda : n \in \mathbb{N} \} \subset \mathbb{R} \). We will construct a sequence of bounding cochains \( a_i \) and elements \( e_i \in M^0 \) with the property that

(iii) \( a_i \) are bounding cochains;

(iv) \( m_{(A,a_i)|M|B}^{0|1|0}(e_i) \in T^{\lambda_0 + i \lambda_1} M \) and

(v) For \( i > 1 \), \( a_i - a_{i-1} \in T^{\lambda_0 + i \lambda_1} A \) and \( e_i - e_{i-1} \in T^{\lambda_0 + i \lambda_1} M \).

Base case: Let \( a_0 = 0 \), and \( e_0 = e \). Items (i) and (ii) are given by the hypothesis. Item (iii) is trivial, and item (iv) follows from the gapped structure. Item (v) has no content.

Inductive Step: Suppose we have constructed \( a_i, e_i \) satisfying the induction hypothesis. By item (iv), we can write \( m_{(A,a_i)|M|B}^{0|1|0}(e_i) \equiv c_i \mod T^{\lambda_0 + (i+1) \lambda_1} \), where \( c_i \in T^{\lambda_0 + i \lambda_1} M \). At order \( T^{\lambda_0 + (i+1) \lambda_1} \),

\[
\sum_{k_1 \geq 0} m_{A|M|B}^{k_1}(A, a) \equiv \sum_{k_1 \geq 0} m_{(A,a)|M|B}^{k_1|0}(A, a) \equiv \sum_{k_1 \geq 0} m_{(A,a)|M|B}^{0|1|0} \circ m_{(A,a)|M|B}^{0|1|0}(e_i) = 0 \mod T^{\lambda_0 + (i+1) \lambda_1}
\]
We therefore obtain a class \([c_i] \in T^{\lambda_0+i\lambda_1}H^1(M)\). Using item (ii), we have a homology class \(a \in T^{\lambda_0+\lambda_1}A\) with

\[
[m_{A|M|B}^{1}|0}(a) \equiv [c_i] \mod T^{\lambda_0+(i+1)\lambda_1}.
\]

By claim B.2.6, there exists a bounding cochain \(a' \in T^{\lambda_1}A\) for the product structures \(m^k_{(A,a_i)}\) satisfying

\[
a' \equiv a \mod T^{\lambda_0+(i+1)\lambda_1}
\]

\[
[m_{A|M|B}^{1}|0}(a' \otimes e) = [c_i] \text{ in } H^1(T^{\lambda_0+i\lambda_1}M/T^{\lambda_0+(i+1)\lambda_1}M).
\]

Write \(m_{A|M|B}^{1}|0}(a' \otimes e) = c_i + m_{A|M|B}^{0}|0}(e')\), where \(e' \in T^{\lambda_0+i\lambda_1}\). Then let \(e_{i+1} = e_i + e'\) and let \(a_{i+1} = a_i - a'\). By construction, we satisfy item (v). By claim B.2.7, \(a_{i+1}\) is a bounding cochain for \(A\), and we therefore obtain item (iii). Conditions items (i) and (ii) are unchanged by deformations.

It remains to prove item (iv): \(m_{A|M|B}^{1}|0}(a_{i+1}) = m_{(A,a_i)}^{0}|0}(e_i)

\[
-m_{A|M|B}^{1}|0}(a', e) + m_{A|M|B}^{1|0}(e') - m_{A|M|B}^{1|0}(a', e') \equiv 0 \mod T^{\lambda_0+(i+1)\lambda_1}.
\]

To complete the proof of the lemma, we can take the bounding cochain \(a\) and element \(e\) to be

\[
a = \lim_{i \to \infty} a_i \quad \quad \quad e = \lim_{i \to \infty} e_i.
\]

\[
\]

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