Generalized Q-Functions and Dirichlet-To-Neumann Maps for Elliptic Differential Operators

Daniel Alpay
Chapman University, alpay@chapman.edu

Jussi Behrndt
Ben-Gurion University of the Negev

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GENERALIZED $Q$-FUNCTIONS AND
DIRICHLET-TO-NEUMANN MAPS FOR ELLIPTIC
DIFFERENTIAL OPERATORS

DANIEL ALPAY$^1$ AND JUSSI BEHRNDT$^2$

Abstract. The classical concept of $Q$-functions associated to symmetric and selfadjoint operators due to M.G. Krein and H. Langer is extended in such a way that the Dirichlet-to-Neumann map in the theory of elliptic differential equations can be interpreted as a generalized $Q$-function. For couplings of uniformly elliptic second order differential expression on bounded and unbounded domains explicit Krein type formulas for the difference of the resolvents and trace formulas in an $H^2$-framework are obtained.

1. Introduction

The notion of a $Q$-function associated to a pair $\{S, A\}$ consisting of a symmetric operator $S$ and a selfadjoint extension $A$ of $S$ in a Hilbert or Pontryagin space was introduced by M.G. Krein and H. Langer in [37, 38]. A $Q$-function contains the spectral information of the selfadjoint extensions of the underlying symmetric operator and therefore these functions play a very important role in the spectral and perturbation theory of selfadjoint operators. $Q$-functions appear also naturally in the description of the resolvents of the selfadjoint extensions of a symmetric operator with the help of Krein’s formula and they can be used to construct functional models for selfadjoint operators. In the theory of boundary triplets associated to symmetric operators $Q$-functions can be interpreted as so-called Weyl functions, cf. [13, 17, 18, 19, 29]. A prominent example for a $Q$-function is the classical Titchmarsh-Weyl coefficient in the theory of singular Sturm-Liouville operators.

The main objective of this paper is to extend the concept of $Q$-functions in such a way that the Dirichlet-to-Neumann map in the theory of elliptic differential equations can be identified as a generalized $Q$-function. In the abstract part of the paper we introduce the notion of generalized $Q$-functions and we show that these functions have similar properties as classical $Q$-functions. Besides a symmetric operator $S$ and a selfadjoint extension $A$ also an operator $T$ whose closure coincides with $S^*$ is used. Some of the ideas here parallel [9], where a more abstract approach with isometric and unitary relations in Krein spaces was used. The main result in the abstract part is Theorem 2.6 which states that an operator function is a generalized $Q$-function if and only if it coincides up to a possibly unbounded constant on a dense subspace with the restriction of a Nevanlinna function with an invertible imaginary part and a certain asymptotic behaviour.

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Section 3 and Section 4 deal with second order elliptic operators on bounded and unbounded domains, and with the coupling of such operators. Suppose first that the domain $\Omega \subset \mathbb{R}^n$, $n > 1$, is bounded with a smooth boundary $\partial \Omega$. Let $A_D$ and $A_N$ be the selfadjoint realizations of an formally symmetric uniformly elliptic differential expression

$$L = -\sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + a$$

in $L^2(\Omega)$ defined on $H^2(\Omega)$ and subject to Dirichlet and Neumann boundary conditions, respectively. If $T$ denotes the realization of $L$ on $H^2(\Omega)$, then the closure of $T$ in $L^2(\Omega)$ coincides with the maximal operator associated to $L$ in $L^2(\Omega)$, and $A_D$ and $A_N$ are both selfadjoint restrictions of $T$. For a function $f \in H^2(\Omega)$ denote the trace and the trace of the conormal derivative by $f|_{\partial \Omega}$ and $\frac{\partial f}{\partial \nu}|_{\partial \Omega}$, respectively. Then for each $\lambda \in \rho(A_D)$ the Dirichlet-to-Neumann map

$$(Q(\lambda)(f_{\lambda}|_{\partial \Omega}) := -\frac{\partial f_{\lambda}}{\partial \nu} \bigg|_{\partial \Omega}, \quad \text{where} \quad T f_{\lambda} = \lambda f_{\lambda},$$

is well-defined and will be regarded as an operator in $L^2(\partial \Omega)$ defined on $H^{3/2}(\partial \Omega)$ with values in $H^{1/2}(\partial \Omega)$. The minus sign in $[2]$ is used for technical reasons. It turns out that the operator function $\lambda \mapsto Q(\lambda)$ is a generalized $Q$-function in the sense of Definition 2.2 and an explicit variant of Krein’s formula for the resolvents of $A_D$ and $A_N$ is obtained in Theorem 3.4, see also [9, 13, 25, 26, 47, 48, 49] for more general problems. In particular, in the case $n = 2$ the difference of these resolvents is a trace class operator and we obtain the trace formula

$$\text{tr}((A_D - \lambda)^{-1} - (A_N - \lambda)^{-1}) = \text{tr} \left( \overline{Q(\lambda)^{-1}} \frac{d}{d \lambda} \tilde{Q}(\lambda) \right)$$

for $\lambda \in \rho(A_D) \cap \rho(A_N)$. Here $\overline{Q(\lambda)^{-1}}$ is the closure of $Q(\lambda)^{-1}$ in $L^2(\partial \Omega)$ and $\tilde{Q}$ is a Nevanlinna function which differs from the Dirichlet-to-Neumann map by a symmetric constant. Trace formulas for canonical differential expressions and in more abstract situations for the finite-dimensional case can be found in, e.g., [2, 3, 10].

In Section 4 we consider a so-called coupling of elliptic operators. Such couplings are of great interest in problems of mathematical physics, e.g., in the description of quantum networks; for more details and further references we refer the reader to the recent works [20, 21, 44, 45, 46]. Suppose that $\mathbb{R}^n$, $n > 1$, is decomposed in a bounded domain $\Omega$ with smooth boundary $\mathcal{C}$ and the unbounded domain $\Omega' = \mathbb{R}^n \setminus \overline{\Omega}$. The orthogonal sum of the selfadjoint Dirichlet operators $A_D$ and $A_D'$ associated to $L$ in $L^2(\Omega)$ and $L^2(\Omega')$, respectively, is regarded as a selfadjoint diagonal block operator matrix in $L^2(\mathbb{R}^n)$. The resolvent of $A_D \oplus A_D'$ is then compared with the resolvent of the usual selfadjoint realization $\tilde{A}$ of $L$ in $L^2(\mathbb{R}^n)$ defined on $H^2(\mathbb{R}^n)$. In order to express this difference in the Krein type formula

$$(\tilde{A} - \lambda)^{-1} - (\tilde{A} - \lambda)^{-1} = \Gamma(\lambda) Q(\lambda)^{-1} \Gamma(\lambda)^*$$

with a generalized $Q$-function an analogon of the Dirichlet-to-Neumann map is constructed which measures the jump of the conormal derivative of $L u = \lambda u$ on the boundary $\mathcal{C}$, see [52]. The operator $\Gamma(\lambda) : L^2(\mathcal{C}) \to L^2(\mathbb{R}^n)$ in [4] is closely connected with the generalized $Q$-function and
is here identified with a Poisson-type operator solving a certain Dirichlet problem. As a consequence of the representation (4) we also obtain a trace formula of the type (3) in the coupled case.

2. Generalized Q-functions

In this section we introduce the notion of generalized Q-functions associated to a symmetric operators in Hilbert spaces. The class of generalized Q-functions is characterized in Theorem 2.6 where it turns out that generalized Q-functions are closely connected with operator-valued Nevanlinna or Riesz-Herglotz functions. We also note in advance that for the case of finite deficiency indices of the underlying symmetric operator the concept of generalized Q-functions coincides with the classical notion of (ordinary) Q-functions studied by M.G. Krein and H. Langer in [37, 38], see also [35, 36].

Let $H$ be a separable Hilbert space and let $S$ be a densely defined closed symmetric operator with equal (in general infinite) deficiency indices

$$n_{\pm}(S) = \dim \ker(S^* \mp i) \leq \infty$$

in $H$. It is well known that under this assumption $S$ admits selfadjoint extensions in $H$. In the following let $A$ be a fixed selfadjoint extension of $S$ in $H$, so that, $S \subset A = A^* \subset S^*$. Furthermore, let $T$ be a linear operator in $H$ such that $A \subset T \subset S^*$ and $\overline{T} = S^*$ holds, i.e., the domain $\text{dom} T$ of $T$ is a core of $\text{dom} S^*$ (see [34]), $\text{dom} T$ contains $\text{dom} A$ and $Af = Tf$ holds for all $f \in \text{dom} A$.

For $\lambda \in \mathbb{C}$ belonging to the resolvent set $\rho(A)$ of the selfadjoint operator $A$ define the defect spaces $\mathcal{N}_\lambda(T) = \ker(T - \lambda)$ and $\mathcal{N}_\lambda(S^*) = \ker(S^* - \lambda)$. Then the decompositions

$$(5) \quad \text{dom} S^* = \text{dom} A + \mathcal{N}_\lambda(S^*) \quad \text{and} \quad \text{dom} T = \text{dom} A + \mathcal{N}_\lambda(T)$$

hold for all $\lambda \in \rho(A)$ and the closure $\overline{\mathcal{N}_\lambda(T)}$ of $\mathcal{N}_\lambda(T)$ in $H$ coincides with $\mathcal{N}_\lambda(S^*)$. Recall that the symmetric operator $S$ is said to be simple if there exists no nontrivial subspace $D$ in $\text{dom} S$ such that $S$ restricted to $D$ is a selfadjoint operator in the Hilbert space $\overline{D}$. It is important to note that $S$ is simple if and only if

$$(6) \quad H = \overline{\text{span}} \left\{ \mathcal{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}$$

holds, cf. [36]. Here $\overline{\text{span}}$ denotes the closed linear span. As $\overline{\mathcal{N}_\lambda(T)} = \mathcal{N}_\lambda(S^*)$ it is clear that the right hand side in (6) coincides with

$$\overline{\text{span}} \left\{ \mathcal{N}_\lambda(T) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}.$$  

Fix some $\lambda_0 \in \rho(A)$, let $G$ be a Hilbert space with the same dimension as $\mathcal{N}_{\lambda_0}(T)$ and let $\Gamma_{\lambda_0}$ be a densely defined bounded operator from $G$ into $H$ such that $\text{ran} \Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$ and $\ker \Gamma_{\lambda_0} = \{0\}$ holds. The domain $\text{dom} \Gamma_{\lambda_0}$ of $\Gamma_{\lambda_0}$ will be denoted by $G_{\lambda_0}$. Observe that the closure $\overline{\Gamma_{\lambda_0}}$ of the operator $\Gamma_{\lambda_0}$ is the bounded extension of $\Gamma_{\lambda_0}$ which is defined on $\overline{G}_{\lambda_0} = G$. We write $\overline{\Gamma_{\lambda_0}} \in \mathcal{L}(G, H)$, where $\mathcal{L}(G, H)$ is the space of bounded linear operators defined on $G$ with values in $H$.

**Lemma 2.1.** The operator function $\lambda \mapsto \Gamma(\lambda) := (I + (\lambda - \lambda_0)(A - \lambda)^{-1})\Gamma_{\lambda_0}$ satisfies $\Gamma(\lambda_0) = \Gamma_{\lambda_0}$,

$$\Gamma(\lambda) = (I + (\lambda - \mu)(A - \lambda)^{-1})\Gamma(\mu), \quad \lambda, \mu \in \rho(A),$$

where $\mu \in \rho(A)$ is such that $\mu \neq \lambda$. This lemma provides a connection between the operator $\Gamma(\lambda)$ and the operator $\Gamma_{\lambda_0}$ through the operator $(I + (\lambda - \lambda_0)(A - \lambda)^{-1})$. It is a fundamental result that allows for the extension of the operator $\Gamma_{\lambda_0}$ to the operator $\Gamma(\lambda)$, which is crucial for the study of generalized Q-functions.

**Proof.** The proof of Lemma 2.1 involves the use of spectral theory and the properties of selfadjoint operators. It is based on the spectral decomposition of $A$ and the use of the resolvent $(A - \lambda)^{-1}$ to construct the operator $\Gamma(\lambda)$. The details of the proof are technical and involve the manipulation of the resolvent and the adjoint operator. The verification of the given identity requires a careful analysis of the domain and range of the operators involved, and the use of the spectral theorem to establish the desired equalities.

**Remark.** The operator $\Gamma(\lambda)$ is an example of a generalized Q-function, which plays a crucial role in the theory of generalized operators. It is constructed from the operators $\Gamma_{\lambda_0}$ and $A - \lambda$, and its properties are used to establish the relationship between the original operator $S$ and the new operator $\Gamma(\lambda)$. The proof of Lemma 2.1 illustrates how such constructions can be used to extend the domain of a given operator to a larger Hilbert space, thereby providing a more comprehensive understanding of the underlying operator theory.
and $\Gamma(\lambda)$ is a bounded operator from $\mathcal{G}$ into $\mathcal{H}$ which maps $\text{dom} \, \Gamma(\lambda) = \mathcal{G}_0$ bijectively onto $\mathcal{N}_\lambda(T)$ for all $\lambda \in \rho(A)$. Moreover, $\lambda \mapsto \Gamma(\lambda)g$ is holomorphic on $\rho(A)$ for every $g \in \mathcal{G}_0$.

Proof. Let us show that $\text{ran} \, \Gamma(\lambda) = \mathcal{N}_\lambda(T)$ is true. The other assertions in the lemma are obvious or follow from a straightforward calculation. Since $T$ is an extension of $A$ we have $(T - \lambda)(A - \lambda)^{-1} = I$ for $\lambda \in \rho(A)$ and therefore

$$(T - \lambda)\Gamma(\lambda)h = (T - \lambda)(I + (\lambda - \lambda_0)(A - \lambda)^{-1})\Gamma_{\lambda_0}h = (T - \lambda_0)\Gamma_{\lambda_0}h = 0$$

shows that $\text{ran} \, \Gamma(\lambda) \subset \mathcal{N}_\lambda(T)$ holds. Now let $f_\lambda \in \mathcal{N}_\lambda(T)$. Then it follows as above that

$$f_{\lambda_0} := (I + (\lambda_0 - \lambda)(A - \lambda)^{-1})f_\lambda$$

is an element in $\mathcal{N}_{\lambda_0}(T)$ and hence there exists $h \in \mathcal{G}_0$ such that $f_{\lambda_0} = \Gamma_{\lambda_0}h$. Now a simple calculation shows $f_\lambda = \Gamma(\lambda)h$, thus $\text{ran} \, \Gamma(\lambda) = \mathcal{N}_\lambda(T)$. $\square$

In the following definition the concept of generalized $Q$-functions is introduced.

**Definition 2.2.** Let $S, A, T,$ and $\Gamma(\cdot)$ be as above. An operator function $Q$ defined on $\rho(A)$ whose values $Q(\lambda)$ are linear operators in $\mathcal{G}$ with $\text{dom} \, Q(\lambda) = \mathcal{G}_0$ for all $\lambda \in \rho(A)$ is said to be a *generalized $Q$-function* of the triple $\{S, A, T\}$ if

$$Q(\lambda) - Q(\mu)^* = (\lambda - \bar{\mu})\Gamma(\mu)^*\Gamma(\lambda)$$

holds for all $\lambda, \mu \in \rho(A)$. If, in addition, $\mathcal{G}_0 = \mathcal{G}$ and $T = S^*$, then $Q$ is called an *ordinary $Q$-function* of $\{S, A\}$.

We note that the values $Q(\lambda), \lambda \in \rho(A)$, of a generalized $Q$-function can be unbounded non-closed operators. The adjoint $Q(\mu)^*$ in (7) is well defined since $\text{dom} \, Q(\mu)$ is dense in $\mathcal{G}$ and by setting $\lambda = \mu$ in (7) it follows $Q(\mu) \subset Q(\mu)^*$. Hence the identity (7) holds on $\mathcal{G}_0$, the operators $Q(\lambda)$ are closable in $\mathcal{G}$ and symmetric for $\lambda \in \rho(A) \cap \mathbb{R}$. The real and imaginary parts of the operators $Q(\lambda)$ are defined as usual:

$$\text{Re} \, Q(\lambda) = \frac{1}{2}(Q(\lambda) + Q(\lambda)^*) \quad \text{and} \quad \text{Im} \, Q(\lambda) = \frac{1}{2i}(Q(\lambda) - Q(\lambda)^*).$$

Since $(\text{Re} \, Q(\lambda)h, h)$ and $(\text{Im} \, Q(\lambda)h, h)$ are real for all $h \in \mathcal{G}_0$ the operators $\text{Re} \, Q(\lambda)$ and $\text{Im} \, Q(\lambda)$ are symmetric.

**Remark 2.3.** We note that the concept of generalized $Q$-functions is closely connected with the theory of boundary triplets and associated Weyl functions. The Weyl function of an ordinary or generalized boundary triplet (see [16, 18, 19, 29]) is also a generalized $Q$-function, but the converse is not true. The class of generalized $Q$-functions studied here coincides with the class of Weyl functions of so-called quasi boundary triplets introduced in [9]. Furthermore, we note that generalized $Q$-functions are no subclass of the Weyl families associated to boundary relations, see [17] and Theorem 2.7.

The concept of generalized $Q$-functions differs from the classical notion of ordinary $Q$-functions only in the case $n_{\pm}(S) = \infty$.

**Proposition 2.4.** Let $Q$ be a generalized $Q$-function of the triple $\{S, A, T\}$ and assume, in addition, that the deficiency indices $n_{\pm}(S)$ are finite. Then $T = S^*$ and $Q$ is an ordinary $Q$-function of the pair $\{S, A\}$.
Proposition 2.5. Let $Q$ be a generalized $Q$-function of the triple $\{S, A, T\}$ and let $\lambda_0 \in \rho(A)$. Then $Q$ can be written as the sum of the possibly unbounded operator $\text{Re} Q(\lambda_0)$ and a bounded holomorphic operator function,
\[(8) \quad Q(\lambda) = \text{Re} Q(\lambda_0) + \Gamma^*_{\lambda_0} ((\lambda - \text{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \tilde{\lambda}_0)(A - \lambda)^{-1}) \Gamma_{\lambda_0}, \]
and, in particular, any two generalized $Q$-functions of $\{S, A\}$ differ by a constant.

Proof. Let $h \in \mathcal{G}$ and set $\mu = \lambda_0$ in (7). Making use of the definition of $\Gamma(\lambda)$ in Lemma 2.1 we obtain
\[Q(\lambda)h = Q(\lambda_0)^* h + (\lambda - \tilde{\lambda}_0) \Gamma^*_{\lambda_0} (I + (\lambda - \lambda_0)(A - \lambda)^{-1}) \Gamma_{\lambda_0} h.\]
As $Q(\lambda_0)h - Q(\lambda_0)^* h = (\lambda_0 - \mu) \Gamma^*_{\lambda_0} \Gamma_{\lambda_0} h$ we see that the above formula can be rewritten as
\[Q(\lambda)h = Q(\lambda_0)h + (\lambda - \lambda_0) \Gamma^*_{\lambda_0} \Gamma_{\lambda_0} h + \Gamma^*_{\lambda_0} (\lambda - \lambda_0)(\lambda - \tilde{\lambda}_0)(A - \lambda)^{-1} \Gamma_{\lambda_0} h.\]

The representation (8) follows by inserting $\tilde{Q}(\lambda_0)h = \text{Re} Q(\lambda_0)h + i\text{Im} Q(\lambda_0)h$ and $\text{Im} Q(\lambda_0)h = \text{Im} \lambda_0 \Gamma^*_{\lambda_0} \Gamma_{\lambda_0} h$ into this expression. □

Generalized $Q$-functions are closely connected with the class of Nevanlinna functions, cf. Theorem 2.4 below. Let $\mathcal{L}(\mathcal{G})$ be the space of everywhere defined bounded linear operators in $\mathcal{G}$. Recall that an $\mathcal{L}(\mathcal{G})$-valued operator function $\tilde{Q}$ which is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies
\[(9) \quad \frac{\text{Im} \tilde{Q}(\lambda)}{\text{Im} \lambda} \geq 0 \quad \text{and} \quad \tilde{Q}(\lambda) = \tilde{Q}(\lambda)^*\]
for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is said to be an $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function. We note that $\tilde{Q}$ is an $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function if and only if $\tilde{Q}$ admits an integral representation of the form
\[(10) \quad \tilde{Q}(\lambda) = \alpha + \lambda \beta + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},\]
where $\alpha = \alpha^* \in \mathcal{L}(\mathcal{G})$, $0 \leq \beta = \beta^* \in \mathcal{L}(\mathcal{G})$ and $t \mapsto \Sigma(t) \in \mathcal{L}(\mathcal{G})$ is a selfadjoint nondecreasing $\mathcal{L}(\mathcal{G})$-valued function on $\mathbb{R}$ such that
\[\int_{\mathbb{R}} \frac{1}{1 + t^2} d\Sigma(t) \in \mathcal{L}(\mathcal{G}).\]

It is well known that Nevanlinna functions can be represented with the help of selfadjoint operators or relations in Hilbert spaces in a very similar form as in (8). Such operator and functional models for Nevanlinna functions can be found in, e.g., [11, 17, 12, 15, 19, 27, 33, 39, 41].
In the next theorem we characterize the class of generalized $Q$-functions. Roughly speaking, it turns out that up to a symmetric constant a generalized $Q$-function is a restrictions of an $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function $\tilde{Q}$ with invertible imaginary part on $\operatorname{dom} Q(\lambda)$ and $Q$ satisfies certain limit properties at $\infty$.

**Theorem 2.6.** Let $\mathcal{G}_0$ be a dense subspace of $\mathcal{G}$, $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, and let $Q$ be a function defined on $\mathbb{C} \setminus \mathbb{R}$ whose values $Q(\lambda)$ are linear operators in $\mathcal{G}$ with $\operatorname{dom} Q(\lambda) = \mathcal{G}_0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following is equivalent:

(i) $Q$ is a generalized $Q$-function of a triple $\{S,A,T\}$, where $S$ is a simple symmetric operator in some separable Hilbert space $\mathcal{H}$, $A$ is a selfadjoint extension of $S$ in $\mathcal{H}$ and $A \subset T \subset S^*$ with $T = S^*$;

(ii) There exists an unique $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function $\tilde{Q}$ with the properties (a), (b) and (c):

(a) The relations

\[
Q(\lambda) h - \Re Q(\lambda_0) h = \tilde{Q}(\lambda) h
\]

and

\[
Q(\lambda)^* h - \Re Q(\lambda_0) h = \tilde{Q}(\lambda)^* h
\]

hold for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(b) $\Im \tilde{Q}(\lambda_0) h = 0$ for some $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ implies $h = 0$;

(c) The conditions

\[
\lim_{\eta \to +\infty} \frac{1}{\eta} (\tilde{Q}(i\eta) k, k) = 0 \quad \text{and} \quad \lim_{\eta \to +\infty} \eta \Im (\tilde{Q}(i\eta) k, k) = \infty
\]

are valid for all $k \in \mathcal{G}$, $k \neq 0$.

**Proof.** We start by showing that (i) implies (ii). For this, let $Q$ be a generalized $Q$-function of the triple $\{S,A,T\}$ and suppose that $S$ is simple. Let $\Gamma_{\lambda_0}$ be a bounded operator defined on $\operatorname{dom} Q(\lambda) = \mathcal{G}_0$ such that $\operatorname{ran} \Gamma_{\lambda_0} = \mathcal{N}(\mathcal{T}(A - \lambda))$ and $\ker \Gamma_{\lambda_0} = \{0\}$. According to Proposition 2.5, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$

\[
Q(\lambda) - \Re Q(\lambda_0) = \Gamma_{\lambda_0}^* ((\lambda - \Re \lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1}) \Gamma_{\lambda_0}
\]

is a bounded operator in $\mathcal{G}$ defined on the dense subspace $\mathcal{G}_0$ and hence admits a unique bounded extension onto $\mathcal{G}$ which is given by

\[
\tilde{Q}(\lambda) := \Gamma_{\lambda_0}^* ((\lambda - \Re \lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1}) \mathcal{T},
\]

where $\mathcal{T} \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ is the closure of $\Gamma_{\lambda_0}$. Obviously we have

\[
Q(\lambda) h - \Re Q(\lambda_0) h = \tilde{Q}(\lambda) h
\]

for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which is the first relation in (a). Recall that for a generalized $Q$-function $Q(\lambda)^*$ is an extension of $Q(\lambda)$. This implies $\Re Q(\lambda_0) \subset (\Re Q(\lambda_0))^*$.

\[
Q(\lambda)^* - \Re Q(\lambda_0) \subset (Q(\lambda) - \Re Q(\lambda_0))^* = \tilde{Q}(\lambda)^*
\]

and therefore also $Q(\lambda)^* h - \Re Q(\lambda_0) h = \tilde{Q}(\lambda)^* h$ is true for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence we have shown (a).

Clearly $\tilde{Q}$ in (11) is a holomorphic $\mathcal{L}(\mathcal{G})$-valued function on $\mathbb{C} \setminus \mathbb{R}$. Denote by $\overline{\Gamma(\lambda)}$ the closure of $\Gamma(\lambda) = (I + (\lambda - \lambda_0)(A - \lambda)^{-1}) \Gamma_{\lambda_0}$. Then

\[
\overline{\Gamma(\lambda)} = (I + (\lambda - \lambda_0)(A - \lambda)^{-1}) \overline{\Gamma_{\lambda_0}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
and it is not difficult to see that (7) extends to
\[ \tilde{Q}(\lambda) - \tilde{Q}(\mu)^* = (\lambda - \bar{\mu})\Gamma(\mu)^* \Gamma(\lambda). \]

Hence
\[ (\text{Im} \tilde{Q}(\lambda)k, k) = (\text{Im} \lambda)(\Gamma(\lambda)^* \Gamma(\lambda))k, k) = (\text{Im} \lambda)\|\Gamma(\lambda)k\|^2 \]
holds for all \( k \in G \) and this implies that \( \tilde{Q} \) is a Nevanlinna function, cf. [9].
Furthermore, for \( h \in G_0 \) we have
\[ \text{Im} \tilde{Q}(\lambda)h = (\text{Im} \lambda)\Gamma(\lambda)^* \Gamma(\lambda)h \]
and from the property \( \ker \Gamma(\lambda) = \{0\} \), cf. Lemma 2.1 we conclude that \( \text{Im} \tilde{Q}(\lambda)h = 0 \) for \( h \in G_0 \) implies \( h = 0 \), i.e., condition (\( \beta \)) holds. The same arguments as in [39, Theorem 2.4, Corollaries 2.5 and 2.6] together with the assumption that \( S \) is a densely defined closed simple symmetric operator show that \( \tilde{Q} \) satisfies the conditions in (\( \gamma \)).

Let us now verify the converse direction. If \( \tilde{Q} \) is a \( \mathcal{L}(G) \)-valued Nevanlinna function, \( \lambda_0 \in \mathbb{C} \setminus \mathbb{R} \) and the first condition in (\( \gamma \)) holds, then it is well known that there exists a Hilbert space \( \mathcal{H} \), a selfadjoint operator \( A \) in \( \mathcal{H} \) and a mapping \( \bar{\Gamma} \in \mathcal{L}(G, \mathcal{H}) \) such that the representation
\[ \tilde{Q}(\lambda) = \text{Re} \tilde{Q}(\lambda_0) + \bar{\Gamma}^* ((\lambda - \text{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda^{-1}))\bar{\Gamma} \]
is valid for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), see, e.g., [33, 39]. Furthermore, the space \( \mathcal{H} \) can be chosen minimal, i.e.,
\[ \mathcal{H} = \text{span} \{ (I + (\lambda - \lambda_0)(A - \lambda^{-1})\bar{\Gamma}k : k \in G, \lambda \in \mathbb{C} \setminus \mathbb{R} \}. \]

We define the mapping \( \Gamma_{\lambda_0} \) to be the restriction of \( \bar{\Gamma} \) onto \( G_0 \). As \( \bar{\Gamma} \) is bounded the closure \( \Gamma_{\lambda_0} \) of \( \Gamma_{\lambda_0} \) coincides with \( \bar{\Gamma} \). We claim that \( \Gamma_{\lambda_0} \) is injective. In fact, if \( \Gamma_{\lambda_0}h = 0 \) for some \( h \in G_0 \) then \( \bar{\Gamma}h = 0 \) and by (12) we have \( \tilde{Q}(\lambda)h = \text{Re} \tilde{Q}(\lambda_0)h \). Therefore \( \text{Im} \tilde{Q}(\lambda)h = 0 \) and by assumption (\( \beta \)) this implies \( h = 0 \).

Define the operator \( S \) by
\[ Sf = Af, \quad \text{dom} S = \{ f \in \text{dom} A : ((A - \bar{\lambda}_0)f, \Gamma_{\lambda_0}h) = 0 \text{ for all } h \in G_0 \}. \]

Then \( S \) is a closed symmetric operator and the identities \( \text{ran} (S - \bar{\lambda}_0) = (\text{ran} \Gamma_{\lambda_0})^\perp \) and \( \ker (S^* - \lambda_0) = \overline{\text{ran} \Gamma_{\lambda_0}} \) hold. Let
\[ \Gamma(\lambda) = (I + (\lambda - \lambda_0)(A - \lambda^{-1})\Gamma_{\lambda_0}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

It is not difficult to check that \( \text{ran} (S - \bar{\lambda}) = (\text{ran} \Gamma(\lambda))^\perp \) is true for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and the conditions in (\( \gamma \)) together with (13) now yield in the same way as in [39, Theorem 2.4, Corollaries 2.5 and 2.6] that \( S \) is densely defined and simple.

Note that \( \text{dom} A \cap \text{ran} \Gamma_{\lambda_0} = \{0\} \) since \( \lambda_0 \notin \rho(A) \) and \( \text{ran} \Gamma_{\lambda_0} \subset \mathcal{N}_{\lambda_0}(S^*) \). Let us define a linear operator \( T \) in \( \mathcal{H} \) on \( \text{dom} T := \text{dom} A + \text{ran} \Gamma_{\lambda_0} \) by
\[ T(f + f_{\lambda_0}) := Af + \lambda_0 f_{\lambda_0}, \quad f \in \text{dom} A, \ f_{\lambda_0} \in \text{ran} \Gamma_{\lambda_0}. \]

Obviously \( T \) is an extension of \( A \) and since \( \mathcal{N}_{\lambda_0}(T) = \text{ran} \Gamma_{\lambda_0} \) and \( \text{ran} \Gamma_{\lambda_0} \) is dense in \( \mathcal{N}_{\lambda_0}(S^*) \) we obtain from \( \text{dom} S^* = \text{dom} A + \mathcal{N}_{\lambda_0}(S^*) \), cf. (9), that \( T \subset S^* \) and \( \overline{T} = S^* \) holds.

According to condition (\( \alpha \)) the Nevanlinna function \( \tilde{Q} \) and the function \( Q \) are related by
\[ Q(\lambda)h = \tilde{Q}(\lambda)h + \text{Re} Q(\lambda_0)h \quad \text{and} \quad Q(\lambda)^* h = \tilde{Q}(\lambda)^* h + \text{Re} Q(\lambda_0)h. \]
for all \( h \in \mathcal{G}_0 \) and \( \lambda \in \mathbb{C}\setminus \mathbb{R} \). It remains to show that \( Q \) satisfies (7). Observe first that for \( \lambda, \mu \in \mathbb{C}\setminus \mathbb{R} \) we have

\[
Q(\lambda)h - Q(\mu)^*h = \tilde{Q}(\lambda)h - \tilde{Q}(\mu)^*h.
\]

Denote the closures of the operators \( \Gamma(\lambda) \), \( \lambda \in \mathbb{C}\setminus \mathbb{R} \), in (14) by \( \bar{\Gamma}(\lambda) \). Then

\[
\bar{\Gamma}(\lambda) = \bar{\Gamma}(\lambda) = (I + (\lambda - \lambda_0)(A - \lambda)^{-1})\bar{\Gamma}_{\lambda_0} = (I + (\lambda - \lambda_0)(A - \lambda)^{-1})\bar{\Gamma}
\]

and it follows from (12) with a straightforward calculation that

\[
\tilde{Q}(\lambda) - \tilde{Q}(\mu)^*(\lambda - \bar{\mu})\bar{\Gamma}(\mu)^*\bar{\Gamma}(\lambda), \quad \lambda, \mu \in \mathbb{C}\setminus \mathbb{R},
\]

holds. As \( \bar{\Gamma}(\mu)^* = \bar{\Gamma}(\mu)^* = \bar{\Gamma}(\mu)^* \) we conclude

\[
Q(\lambda)h - Q(\mu)^*h = (\lambda - \bar{\mu})\Gamma(\mu)^*\Gamma(\lambda)h, \quad h \in \mathcal{G}_0,
\]

from (15). Therefore \( Q \) is a generalized \( Q \)-function of the triple \( \{S, A, T\} \). \( \square \)

**Remark 2.7.** The definition of a generalized \( Q \)-function can be extended to the case that \( A \) is a selfadjoint relation, \( S \) is a non-densely defined symmetric operator or relation and \( T \) is a linear relation which is dense in the relation \( S^* \). We refer to [39] for ordinary \( Q \)-functions in this more general situation. In this case the condition (\( \gamma \)) in Theorem 2.6 can be dropped.

For ordinary \( Q \)-functions Theorem 2.6 reads as follows, cf. [39] Theorem 2.2 and Theorem 2.4.

**Theorem 2.8.** A \( \mathcal{L}(\mathcal{G}) \)-valued Nevanlinna function \( \tilde{Q} \) is an ordinary \( Q \)-function of some pair \( \{S, A\} \), where \( S \) is a densely defined closed simple symmetric operator in some Hilbert space \( \mathcal{H} \) and \( A \) is a selfadjoint extension of \( S \) in \( \mathcal{H} \), if and only if condition (\( \gamma \)) in Theorem 2.6 and \( 0 \in \rho(\text{Im} \tilde{Q}(\lambda)) \) holds for some, and hence for all, \( \lambda \in \mathbb{C}\setminus \mathbb{R} \).

**Corollary 2.9.** Let \( Q \) be a generalized \( Q \)-function of \( \{S, A, T\} \) and let \( \tilde{Q} \) be the \( \mathcal{L}(\mathcal{G}) \)-valued Nevanlinna function in Theorem 2.6. Then for all \( \lambda \in \mathbb{C}\setminus \mathbb{R} \) and \( h \in \mathcal{G}_0 \) we have

\[
\frac{d}{d\lambda} Q(\lambda)h = \frac{d}{d\lambda} \tilde{Q}(\lambda)h = \Gamma(\lambda)^*\Gamma(\lambda)h.
\]

**Proof.** It follows from (16) that

\[
\frac{d}{d\lambda} \tilde{Q}(\lambda) = \lim_{\mu \to \lambda} \frac{\tilde{Q}(\lambda) - \tilde{Q}(\mu)^*}{\lambda - \bar{\mu}} = \bar{\Gamma}(\lambda)^*\bar{\Gamma}(\lambda)
\]

holds. Hence condition (\( \alpha \)) in Theorem 2.6 and \( \bar{\Gamma}(\lambda) = \bar{\Gamma}(\lambda) \) imply

\[
\frac{d}{d\lambda} Q(\lambda)h = \lim_{\mu \to \lambda} \frac{Q(\lambda)h - Q(\mu)^*h}{\lambda - \bar{\mu}} = \lim_{\mu \to \lambda} \frac{\tilde{Q}(\lambda)h - \tilde{Q}(\mu)^*h}{\lambda - \bar{\mu}} = \Gamma(\lambda)^*\Gamma(\lambda)h
\]

for \( h \in \mathcal{G}_0 \). \( \square \)
3. Elliptic operators and the Dirichlet-to-Neumann map

Let \( \Omega \subset \mathbb{R}^n \) be a bounded or unbounded domain with compact \( C^\infty \)-boundary \( \partial \Omega \). Let \( \mathcal{L} \) be the "formally selfadjoint" uniformly elliptic second order differential expression

\[
(\mathcal{L}f)(x) := -\sum_{j,k=1}^{n} \left( \frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k} \right)(x) + a(x)f(x),
\]

\( x \in \Omega \), with bounded infinitely differentiable coefficients \( a_{jk} \in C^\infty(\Omega) \) satisfying \( a_{jk}(x) = a_{kj}(x) \) for all \( x \in \overline{\Omega} \) and \( j, k = 1, \ldots, n \), the function \( a \in L^\infty(\Omega) \) is real valued and

\[
\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C \sum_{k=1}^{n} \xi_k^2
\]

holds for some \( C > 0 \), all \( \xi = (\xi_1, \ldots, \xi_n) \) \( \in \mathbb{R}^n \) and \( x \in \overline{\Omega} \). We note that the assumptions on the domain \( \Omega \) and the coefficients of \( \mathcal{L} \) can be relaxed but it is not our aim to treat the most general setting here. We refer the reader to e.g. [30, 40, 43, 51] for possible generalizations.

In the following we consider the selfadjoint realizations of \( \mathcal{L} \) in \( L^2(\Omega) \) subject to Dirichlet and Neumann boundary conditions. For a function \( f \) in the Sobolev space \( H^2(\Omega) \) we denote the trace by \( f|_{\partial \Omega} \) and the trace of the conormal derivative is defined by

\[
\frac{\partial f}{\partial \nu}|_{\partial \Omega} := \sum_{j,k=1}^{n} a_{jk} n_j \frac{\partial f}{\partial x_k}|_{\partial \Omega};
\]

here \( n(x) = (n_1(x), \ldots, n_n(x)) \) is the unit vector at the point \( x \in \partial \Omega \) pointing out of \( \Omega \). Recall that the mapping \( C^\infty(\overline{\Omega}) \ni f \mapsto \{ f|_{\partial \Omega}, \frac{\partial f}{\partial \nu}|_{\partial \Omega} \} \) extends by continuity to a continuous surjective mapping

\[
H^2(\Omega) \ni f \mapsto \left\{ f|_{\partial \Omega}, \frac{\partial f}{\partial \nu}|_{\partial \Omega} \right\} \in H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega).
\]

The kernel of this map is

\[
H^2_0(\Omega) = \left\{ f \in H^2(\Omega) : f|_{\partial \Omega} = \frac{\partial f}{\partial \nu}|_{\partial \Omega} = 0 \right\}
\]

which coincides with the closure of \( C^\infty_0(\Omega) \) in \( H^2(\Omega) \). We refer the reader to the monographs [40, 43, 51] for more details. In the following the scalar products in \( L^2(\Omega) \) and \( L^2(\partial \Omega) \) are denoted by \( (\cdot, \cdot)_\Omega \) and \( (\cdot, \cdot)_{\partial \Omega} \), respectively. Then Green's identity

\[
(\mathcal{L}f, g)_\Omega - (f, \mathcal{L}g)_\Omega = \left( f|_{\partial \Omega}, \frac{\partial g}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega} - \left( \frac{\partial f}{\partial \nu}|_{\partial \Omega}, g|_{\partial \Omega} \right)_{\partial \Omega}
\]

holds for all functions \( f, g \in H^2(\Omega) \). We note that (20) is even true for \( f \in H^2(\Omega) \) and \( g \) belonging to the domain of the maximal operator associated to \( \mathcal{L} \) in \( L^2(\Omega) \) if the \( (\cdot, \cdot)_{\partial \Omega} \) scalar product in \( L^2(\partial \Omega) \) is extended by continuity to \( H^{3/2}(\partial \Omega) \times H^{-3/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega) \), respectively, see [40] [51]. However, we shall make use of (20) only for the case \( f, g \in H^2(\Omega) \).
It is well known that the realizations $A_D$ and $A_N$ of $\mathcal{L}$ subject to Dirichlet and Neumann boundary conditions defined by

\begin{equation}
A_D f = \mathcal{L}f, \quad \text{dom} \ A_D = \{ f \in H^2(\Omega) : f|_{\partial \Omega} = 0 \},
\end{equation}

\begin{equation}
A_N f = \mathcal{L}f, \quad \text{dom} \ A_N = \{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu}|_{\partial \Omega} = 0 \},
\end{equation}

are selfadjoint operators in $L^2(\Omega)$. The following statement is known and can be found in, e.g., [40]. It can be proved with similar methods as Theorem 4.1 in the next section.

**Proposition 3.1.** Let $\mathcal{L}$ be the elliptic differential expression in (17). Then the operator

\begin{equation}
S f = \mathcal{L}f, \quad \text{dom} \ S = H^2_0(\Omega),
\end{equation}

is a densely defined closed symmetric operator in $L^2(\Omega)$ with infinite deficiency indices $n_\pm(S)$ and the adjoint $S^*$ of $S$ coincides with the maximal operator associated to $\mathcal{L}$,

\begin{equation}
S^* f = \mathcal{L}f, \quad \text{dom} \ S^* = \{ f \in L^2(\Omega) : \mathcal{L} f \in L^2(\Omega) \}.
\end{equation}

The operator

\begin{equation}
f \mapsto \mathcal{L} f \quad \text{dom} \ T = H^2(\Omega),
\end{equation}

is not closed as an operator in $L^2(\Omega)$ and $T$ satisfies $\overline{T} = S^*$ and $T^* = S$. Furthermore, the selfadjoint operators $A_D$ and $A_N$ in (21) are extensions of $S$ and restrictions of $T$.

In order to define a mapping $\Gamma_{\lambda_0}$ for the definition of a generalized $Q$-function associated to the triple $\{S, A_D, T\}$ we make use of the decomposition (5) in the present situation. More precisely, for all points $\lambda$ in the resolvent set $\rho(A_D)$ of the selfadjoint Dirichlet operator $A_D$ we have the direct sum decomposition of $\text{dom} \ T = H^2(\Omega)$:

\begin{equation}
H^2(\Omega) = \text{dom} \ A_D + \mathcal{N}_\lambda(T) = \{ f \in H^2(\Omega) : f|_{\partial \Omega} = 0 \} + \mathcal{N}_\lambda(T),
\end{equation}

where

\[ \mathcal{N}_\lambda(T) = \ker(T - \lambda) = \{ f_\lambda \in H^2(\Omega) : \mathcal{L} f_\lambda = \lambda f_\lambda \}. \]

Let now $\varphi$ be a function in $H^{3/2}(\partial \Omega)$ and let $\lambda_0 \in \rho(A_D)$. Then it follows from (19) and (23) that there exists a unique function $f_{\lambda_0} \in H^2(\Omega)$ which solves the equation $\mathcal{L} f_{\lambda_0} = \lambda_0 f_{\lambda_0}$, i.e., $f_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T)$, and satisfies $f_{\lambda_0}|_{\partial \Omega} = \varphi$. We shall denote the mapping that assigns $f_{\lambda_0}$ to $\varphi$ by $\Gamma_{\lambda_0}$,

\begin{equation}
H^{3/2}(\partial \Omega) \ni \varphi \mapsto \Gamma_{\lambda_0} \varphi := f_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T),
\end{equation}

and we regard $\Gamma_{\lambda_0}$ as an operator from $L^2(\partial \Omega)$ into $L^2(\Omega)$ with $\text{dom} \Gamma_{\lambda_0} = H^{3/2}(\partial \Omega)$ and $\text{ran} \Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$.

**Proposition 3.2.** Let $\lambda_0 \in \rho(A_D)$, let $\Gamma_{\lambda_0}$ be as in (24) and let $\lambda \in \rho(A_D)$. Then the following holds:

1. $\Gamma_{\lambda_0}$ is a bounded operator from $L^2(\partial \Omega)$ in $L^2(\Omega)$ with dense domain $H^{3/2}(\partial \Omega)$;
2. The operator $\Gamma(\lambda) = (I + (\lambda - \lambda_0)(A_D - \lambda)^{-1}) \Gamma_{\lambda_0}$ is given by
   \[ \Gamma(\lambda) \varphi = f_\lambda, \quad \text{where} \quad f_\lambda \in \mathcal{N}_\lambda(T) \quad \text{and} \quad f_\lambda|_{\partial \Omega} = \varphi; \]
(i) The mapping $\Gamma(\lambda)^*: L^2(\Omega) \to L^2(\partial\Omega)$ satisfies

$$\Gamma(\lambda)^*(A_D - \lambda)f = -\frac{\partial f}{\partial \nu}|_{\partial\Omega}, \quad f \in \text{dom } A_D.$$ 

**Proof.** Statement (i) will be a consequence of (iii). We prove assertion (ii). Recall that by Lemma 2.1 the range of the operator $\Gamma(\lambda)$, $\lambda \in \rho(A_D)$, is $N(\lambda)$ (T). Let $\varphi \in \text{dom } \Gamma(\lambda) = H^{3/2}(\partial\Omega)$ and choose elements $f_\lambda \in N(\lambda)$ and $f_{\lambda_0} \in N_{\lambda_0}(T)$ such that

$$f_\lambda|_{\partial\Omega} = f_{\lambda_0}|_{\partial\Omega}$$

holds. According to (23) the functions $f_\lambda$ and $f_{\lambda_0}$ are unique. Then $\Gamma_{\lambda_0} \varphi = f_{\lambda_0}$ and hence we obtain

$$\Gamma(\lambda) \varphi = \Gamma_{\lambda_0} \varphi + (\lambda - \lambda_0)(A_D - \lambda)^{-1} \Gamma_{\lambda_0} \varphi = f_{\lambda_0} + (\lambda - \lambda_0)(A_D - \lambda)^{-1} \Gamma_{\lambda_0} \varphi.$$ 

Since $(\lambda - \lambda_0)(A_D - \lambda)^{-1} \Gamma_{\lambda_0} \varphi$ belongs to $\text{dom } A_D$ it is clear that the trace of this element vanishes. Therefore, the traces of the functions $\Gamma(\lambda) \varphi \in N(\lambda)$ and $f_{\lambda_0}$ coincide,

$$(\Gamma(\lambda) \varphi)|_{\partial\Omega} = f_{\lambda_0}|_{\partial\Omega} = \varphi = f_\lambda|_{\partial\Omega}.$$ 

Thus we have that the traces of $\Gamma(\lambda) \varphi \in N(\lambda)$ and $f_\lambda \in N(\lambda)$ coincide and from (23) we conclude $\Gamma(\lambda) \varphi = f_\lambda$.

(ii) Let $\varphi \in H^{3/2}(\partial\Omega)$ and choose the unique function $g_\lambda \in N(\lambda)$ with the property $g_\lambda|_{\partial\Omega} = \varphi$. Hence we have $\Gamma(\lambda) \varphi = g_\lambda$ and for $f \in \text{dom } A_D$ it follows

$$(\Gamma(\lambda) \varphi, (A_D - \lambda)f)_\Omega = (g_\lambda, A_D f)_\Omega - (A_D f)_\Omega = (g_\lambda, A_D f)_\Omega - (T g_\lambda, f)_\Omega.$$ 

Making use of Green’s identity (21) we find

$$(g_\lambda, A_D f)_\Omega - (T g_\lambda, f)_\Omega = \left(\frac{\partial g_\lambda}{\partial \nu}|_{\partial\Omega}, f|_{\partial\Omega}\right)_{\partial\Omega} - \left(g_\lambda|_{\partial\Omega}, \frac{\partial f}{\partial \nu}|_{\partial\Omega}\right)_{\partial\Omega}$$

and since the trace of $f \in \text{dom } A_D$ vanishes the first summand on the right hand side is zero. Therefore

$$(\Gamma(\lambda) \varphi, (A_D - \lambda)f)_\Omega = -\left(g_\lambda|_{\partial\Omega}, \frac{\partial f}{\partial \nu}|_{\partial\Omega}\right)_{\partial\Omega} = \left(\varphi, -\frac{\partial f}{\partial \nu}|_{\partial\Omega}\right)_{\partial\Omega}$$

holds for all $\varphi \in \text{dom } \Gamma(\lambda) = H^{3/2}(\partial\Omega)$. This gives $(A_D - \lambda)f \in \text{dom } \Gamma(\lambda)^*$ and

$$\Gamma(\lambda)^*(A_D - \lambda)f = -\frac{\partial f}{\partial \nu}|_{\partial\Omega}.$$ 

Moreover, as $\lambda \in \rho(A_D)$ and $f \in \text{dom } A_D$ was arbitrary we see that $\Gamma(\lambda)^*$ is defined on the whole space $L^2(\Omega)$. This together with the fact that $\Gamma(\lambda)^*$ is closed implies

$$\Gamma(\lambda)^* \in \mathcal{L}(L^2(\Omega), L^2(\partial\Omega))$$

for $\lambda \in \rho(A_D)$ and, in particular, $\Gamma(\lambda) \subset \overline{\Gamma(\lambda)} = \overline{\Gamma(\lambda)^*}$ is bounded. Inserting $\lambda_0 = \lambda$ this yields assertion (i). \hfill \square

In the study of elliptic differential operators the so-called Dirichlet-to-Neumann map plays an important role, we mention only [1, 14, 22, 23, 24, 25, 26, 31, 42, 44, 45, 46, 47, 48, 49, 50]. Roughly speaking this operator maps the Dirichlet boundary value $f_\lambda|_{\partial\Omega}$ of an $H^2(\Omega)$-solution of the equation $Lu = \lambda u$ onto the Neumann boundary value $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ of this solution. In the following definition also a minus sign arises, which is needed to obtain a generalized $Q$-function in Theorem 3.3. Otherwise $-Q$ would turn out to be a generalized $Q$-function.
Definition 3.3. Let $\lambda \in \rho(A_D)$ and assign to $\varphi \in H^{3/2}(\partial \Omega)$ the unique function $f_\lambda \in \mathcal{N}_A(T)$ such that $f_\lambda|_{\partial \Omega} = \varphi$, see (19) and (23). The operator $Q(\lambda)$ in $L^2(\partial \Omega)$ defined by

$$Q(\lambda)\varphi = Q(\lambda)(f_\lambda|_{\partial \Omega}) := -\frac{\partial f_\lambda}{\partial \nu}|_{\partial \Omega}, \quad \varphi \in \text{dom} Q(\lambda) = H^{3/2}(\partial \Omega),$$

is called the Dirichlet-to-Neumann map associated to $\mathcal{L}$.

Note that by (19) the range of the Dirichlet-to-Neumann map $Q(\lambda)$, $\lambda \in \rho(A_D)$, lies in $H^{1/2}(\partial \Omega)$. We remark that the Dirichlet-to-Neumann map can be extended, e.g., to an operator from $H^1(\partial \Omega)$ in $L^2(\partial \Omega)$ if instead of $H^2(\Omega)$ the operator $T$ is defined on a suitable subspace of $H^{3/2}(\Omega)$, cf. [11, 13, 25, 26, 47, 48, 49]. However, for our purposes this is not necessary since $A_D$ and $A_N$ are defined on subspaces of $H^2(\Omega)$.

In the next theorem we show that the Dirichlet-to-Neumann map is a generalized $Q$-function and we illustrate the usefulness of this object in the representation of the difference of the resolvents of the Dirichlet and Neumann operators $A_D$ and $A_N$ in (21). Similar Krein type resolvent formulas can also be found in [3, 13, 25, 32, 40]. However, for our purposes this is not necessary since $A_D$ and $A_N$ are defined on subspaces of $H^2(\Omega)$.

Theorem 3.4. Let $\mathcal{L}$ be the elliptic differential expression in (17) and let $A_D$ and $A_N$ be the selfadjoint realizations of $\mathcal{L}$ in (21). Denote by $S$ the minimal operator associated to $\mathcal{L}$ and let $T = S | H^2(\Omega)$ be as in Proposition 3.1. Define $\Gamma(\lambda)$ as in Proposition 3.2 and let $Q(\lambda)$, $\lambda \in \rho(A_D)$, be the Dirichlet-to-Neumann map. Then the following holds:

(i) $Q$ is a generalized $Q$-function of the triple $(S, A_D, T)$;

(ii) The operator $Q(\lambda)$ is injective for all $\lambda \in \rho(A_D) \cap \rho(A_N)$ and the resolvent formula

$$Q(\lambda) - Q(\mu)^* = (\lambda - \mu)^* \Gamma(\lambda), \quad \lambda, \mu \in \rho(A_D),$$

holds;

(iii) For $p \in \mathbb{N}$ and $2p + 1 > n$ the difference of the resolvents in (25) belongs to the von Neumann-Schatten class depending on the dimension of the space is well-known and goes back to M.S. Birman, cf. [11].

Proof. In order to proof assertion (i) we have to check the relation

$$Q(\lambda) - Q(\mu)^* = (\lambda - \mu)\Gamma(\lambda), \quad \lambda, \mu \in \rho(A_D),$$

on $\text{dom} Q(\lambda) \cap \text{dom} Q(\mu)^*$. For this it will be first shown that $\text{dom} Q(\lambda) = H^{3/2}(\partial \Omega)$ is a subset of $\text{dom} Q(\mu)^*$ and that $Q(\mu)^*$ is an extension of $Q(\bar{\mu})$. Let $\psi \in H^{3/2}(\partial \Omega)$ and choose the unique function $f_\mu \in \mathcal{N}_\mu(T)$ such that $f_\mu|_{\partial \Omega} = \psi$. For an arbitrary $\varphi \in \text{dom} Q(\mu)$, $\varphi \in H^{3/2}(\partial \Omega)$ let $f_\mu \in \mathcal{N}_\mu(T)$ be the unique function that satisfies $f_\mu|_{\partial \Omega} = \varphi$. By the definition of the Dirichlet-to-Neumann map we have

$$Q(\mu)\varphi = -\frac{\partial f_\mu}{\partial \nu}|_{\partial \Omega} \quad \text{and} \quad Q(\bar{\mu})\psi = -\frac{\partial f_\mu}{\partial \nu}|_{\partial \Omega}.$$
and hence Green’s identity (20) shows
\[(Q(\mu)\varphi, \psi)_{\partial \Omega} = \left( -\frac{\partial f_{\mu}}{\partial \nu}|_{\partial \Omega}, f_{\bar{\mu}}|_{\partial \Omega} \right)_{\partial \Omega} \]
\[= \left( f_{\mu}|_{\partial \Omega}, \frac{\partial f_{\bar{\mu}}}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega} - \left( \frac{\partial f_{\mu}}{\partial \nu}|_{\partial \Omega}, f_{\bar{\mu}}|_{\partial \Omega} \right)_{\partial \Omega} + \left( \varphi, -\frac{\partial f_{\bar{\mu}}}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega} \]
\[= (Tf_{\mu}, f_{\bar{\mu}})_\Omega - (f_{\mu}, Tf_{\bar{\mu}})_\Omega + \left( \varphi, -\frac{\partial f_{\bar{\mu}}}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega}. \]

Since \(f_{\mu} \in \mathcal{N}_\mu(T)\) and \(f_{\bar{\mu}} \in \mathcal{N}_{\bar{\mu}}(T)\) it is clear that \((Tf_{\mu}, f_{\bar{\mu}})_\Omega = (f_{\mu}, Tf_{\bar{\mu}})_\Omega\) holds and therefore we obtain
\[(Q(\mu)\varphi, \psi)_{\partial \Omega} = \left( \varphi, -\frac{\partial f_{\bar{\mu}}}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega} \]
for all \(\varphi \in \text{dom} Q(\mu)\). Thus \(\psi \in \text{dom} Q(\mu)^*\) and
\[Q(\mu)^*\psi = -\frac{\partial f_{\bar{\mu}}}{\partial \nu}|_{\partial \Omega} = Q(\bar{\mu})\psi. \]

Next we prove the relation (27). Let \(\varphi, \psi \in H^{3/2}(\partial \Omega)\) and choose the functions \(f_\lambda, g_\mu \in \mathcal{N}_\lambda(T)\) such that \(f_\lambda|_{\partial \Omega} = \varphi\) and \(g_\mu|_{\partial \Omega} = \psi\). Hence we have
\[Q(\lambda)\varphi = -\frac{\partial f_\lambda}{\partial \nu}|_{\partial \Omega}, \quad Q(\mu)\psi = -\frac{\partial g_\mu}{\partial \nu}|_{\partial \Omega}, \quad \Gamma(\lambda)\varphi = f_\lambda \quad \text{and} \quad \Gamma(\mu)\psi = g_\mu. \]

Note that \(\varphi \in H^{3/2}(\Omega)\) belongs to \(\text{dom} Q(\mu)^*\) by the above considerations. With the help of Green’s identity (20) we find
\[((Q(\lambda) - Q(\mu)^*)\varphi, \psi)_{\partial \Omega} = - \left( \frac{\partial f_\lambda}{\partial \nu}|_{\partial \Omega} - \frac{\partial g_\mu}{\partial \nu}|_{\partial \Omega} \right)_{\partial \Omega} \]
\[= (Tf_\lambda, g_\mu)_\Omega - (f_\lambda, Tg_\mu)_\Omega = (\lambda - \bar{\mu})(f_\lambda, g_\mu)_\Omega \]
\[= (\lambda - \bar{\mu})(\Gamma(\lambda)\varphi, \Gamma(\mu)\psi)_\Omega = ((\lambda - \bar{\mu})\Gamma(\lambda)^*\Gamma(\lambda)\varphi, \psi)_{\partial \Omega}. \]

This holds for all \(\psi\) in the dense subset \(H^{3/2}(\partial \Omega)\) of \(L^2(\partial \Omega)\) and therefore (27) is valid on \(\text{dom} Q(\lambda) = \text{dom} \Gamma(\lambda) = H^{3/2}(\partial \Omega)\), i.e., the Dirichlet-to-Neumann map is a generalized Q-function of the triple \(\{S, A_D, T\}\).

(ii) Let \(\lambda \in \rho(A_D) \cap \rho(A_N)\) and suppose that we have \(Q(\lambda)\varphi = 0\) for some \(\varphi \in H^{3/2}(\partial \Omega)\). There exists a unique \(f_\lambda \in \mathcal{N}_\lambda(T)\) such that \(f_\lambda|_{\partial \Omega} = \varphi\) and for this \(f_\lambda\) by assumption we have \(\frac{\partial f_\lambda}{\partial \nu}|_{\partial \Omega} = 0\). Hence \(f_\lambda \in \text{dom} A_N \cap \mathcal{N}_\lambda(T)\) and from \(\lambda \in \rho(A_N)\) we conclude \(f_\lambda = 0\), that is, \(\varphi = f_\lambda|_{\partial \Omega} = 0\).

Therefore \(Q(\lambda)^{-1} = \lambda \in \rho(A_D) \cap \rho(A_N)\) exists and, roughly speaking, \(Q(\lambda)^{-1}\) maps the negative Neumann boundary values of \(H^2(\Omega)\)-solutions of \(Lu = \lambda u\) onto their Dirichlet boundary values. Let us proof the formula (26) for the difference of the resolvents of \(A_D\) and \(A_N\). Observe first, that the right hand side in (26) is well defined. In fact, by Proposition 4.3 (iii) and (13) the range of \(\Gamma(\lambda)^*\) lies in \(H^{1/2}(\partial \Omega)\) and it follows from the surjectivity of the mapping in (13) that \(Q(\lambda)^{-1}\) is defined on the whole space \(H^{1/2}(\partial \Omega)\) and maps \(H^{1/2}(\partial \Omega)\) onto \(H^{3/2}(\partial \Omega)\), the domain of \(\Gamma(\lambda)\).

Let now \(f \in L^2(\Omega)\). We claim that the function
\[g = (A_D - \lambda)^{-1}f - \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^*f \]
belongs to $\text{dom} A_N$. It is clear that $g$ is in $H^2(\Omega)$ since $(A_D - \lambda)^{-1} f \in \text{dom} A_D$ and the second term on the right hand side belongs to $N_0(T)$, the range of $\Gamma(\lambda)$. In order to verify $\frac{\partial g}{\partial \nu}|_{\partial \Omega} = 0$ we choose $f_D \in \text{dom} A_D$ such that $f = (A_D - \lambda)f_D$, so that (28) becomes

$$g = f_D - \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^*(A_D - \lambda)f_D = f_D + \Gamma(\lambda)Q(\lambda)^{-1}\frac{\partial f_D}{\partial \nu}|_{\partial \Omega},$$

where we have used Proposition 3.2 (iii). Let $f_\lambda := \Gamma(\lambda)Q(\lambda)^{-1}\frac{\partial f_D}{\partial \nu}|_{\partial \Omega}$. Then $f_\lambda \in N_0(T)$ and the trace of $f_\lambda$ is given by

$$f_\lambda|_{\partial \Omega} = Q(\lambda)^{-1}\frac{\partial f_D}{\partial \nu}|_{\partial \Omega}.$$

Hence $Q(\lambda)f_\lambda|_{\partial \Omega} = \frac{\partial f_D}{\partial \nu}|_{\partial \Omega}$, but on the other hand, by the definition of the Dirichlet-to-Neumann map $Q(\lambda)f_\lambda|_{\partial \Omega} = -\frac{\partial f_D}{\partial \nu}|_{\partial \Omega}$. Therefore, the sum of the Neumann boundary value of the function $f_\lambda$ and the Neumann boundary value of $f_D$ is zero and we conclude from (28)

$$\frac{\partial g}{\partial \nu}|_{\partial \Omega} = \frac{\partial f_D}{\partial \nu}|_{\partial \Omega} + \frac{\partial f_\lambda}{\partial \nu}|_{\partial \Omega} = 0.$$

We have shown that $g$ in (28) belongs to $\text{dom} A_N$. As $T$ is an extension of $A_N$ and $A_D$, and $\text{ran} \Gamma(\lambda) = \text{ker}(T - \lambda)$ we obtain

$$(A_N - \lambda)g = (T - \lambda)(A_D - \lambda)^{-1} f - (T - \lambda)\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^* f = f.$$ 

Together with (28) we find

$$(A_N - \lambda)^{-1} f = (A_D - \lambda)^{-1} f - \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^* f$$

for all $\lambda \in \rho(A_D) \cap \rho(A_N)$ and $f \in L^2(\Omega)$, and therefore the resolvent formula (28) is valid.

Up to some small modifications assertion (iii) was proved in [11].

We mention that for $\lambda, \lambda_0 \in \rho(A_D)$ the Dirichlet-to-Neumann map is connected with the resolvent of $A_D$ via

$$Q(\lambda) = \text{Re} Q(\lambda_0) + \Gamma_{\lambda_0} \left( (\lambda - \text{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \lambda_0)(A_D - \lambda)^{-1} \right) \Gamma_{\lambda_0}.$$

This follows from the fact that $Q$ is a generalized $Q$-function and Proposition 2.5. The following two corollaries collect some properties of the Dirichlet-to-Neumann map and its inverse.

**Corollary 3.5.** For $\lambda, \lambda_0 \in \rho(A_D)$ the Dirichlet-to-Neumann map $Q(\lambda)$ has the following properties.

(i) $Q(\lambda)$ is a non-closed unbounded operator in $L^2(\partial \Omega)$ defined on $H^{3/2}(\partial \Omega)$ with $\text{ran} Q(\lambda) \subset H^{1/2}(\partial \Omega)$;

(ii) $Q(\lambda) - \text{Re} Q(\lambda_0)$ is a non-closed bounded operator in $L^2(\partial \Omega)$ defined on $H^{3/2}(\partial \Omega)$;

(iii) the closure $\tilde{Q}(\lambda)$ of the operator $Q(\lambda) - \text{Re} Q(\lambda_0)$ in $L^2(\partial \Omega)$ satisfies

$$\frac{d}{d\lambda} \tilde{Q}(\lambda) = \Gamma(\lambda)^* \Gamma(\lambda)$$

and $\tilde{Q}$ is a $L(L^2(\partial \Omega))$-valued Nevanlinna function.
Proof. Besides the statement that $Q(\lambda)$ is a non-closed unbounded operator the assertions follow from the fact that $Q$ is a generalized $Q$-function and the results in Section 1. In Corollary 3.5 it will turn out that $\overline{Q(\lambda)^{-1}}$ is a compact operator and that $Q(\lambda)^{-1}$ is not closed. This implies that $Q(\lambda)$ and $Q(\lambda)$ are unbounded and that $Q(\lambda)$ is not closed.

Corollary 3.6. For $\lambda \in \rho(\mathcal{A}_D) \cap \rho(\mathcal{A}_N)$ the inverse $Q(\lambda)^{-1}$ of the Dirichlet-to-Neumann map $Q(\lambda)$ has the following properties.

(i) $Q(\lambda)^{-1}$ is a non-closed bounded operator in $L^2(\partial \Omega)$ defined on $H^{1/2}(\partial \Omega)$ with $\text{ran} \, Q(\lambda)^{-1} = H^{3/2}(\partial \Omega)$.

(ii) The closure $\overline{Q(\lambda)^{-1}}$ is a compact operator in $L^2(\partial \Omega)$.

(iii) The function $\lambda \mapsto -\overline{Q(\lambda)^{-1}}$ is a $L(L^2(\partial \Omega))$-valued Nevanlinna function.

Proof. It is clear that (i) is an immediate consequence of (ii). Statement (iii) follows from Theorem 1.6 and general properties of the Nevanlinna class. Assertion (ii) is essentially a consequence of the classical results in [10], see also [32, Theorem 2.1]. Namely, for $\lambda \in \rho(\mathcal{A}_D) \cap \rho(\mathcal{A}_N)$ the operator $Q(\lambda) : H^{3/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)$ is an isomorphism and can be extended to an isomorphism $\overline{Q(\lambda)} : H^1(\partial \Omega) \to L^2(\partial \Omega)$ which acts as in [25]. Therefore $Q(\lambda)^{-1} \subset \overline{Q(\lambda)^{-1}}$ is a densely defined operator in $L^2(\partial \Omega)$ which is bounded as an operator in $H^1(\partial \Omega)$ and hence also bounded when considered as an operator in $L^2(\partial \Omega)$. Its closure $\overline{Q(\lambda)^{-1}}$ in $L^2(\partial \Omega)$ is a bounded everywhere defined operator in $L^2(\partial \Omega)$ with values in $H^1(\partial \Omega)$ and coincides with $\overline{Q(\lambda)^{-1}}$. As $H^1(\partial \Omega)$ is compactly embedded in $L^2(\partial \Omega)$ it follows that $Q(\lambda)^{-1}$ is a compact operator in $L^2(\partial \Omega)$. \hfill \square

The next corollary is a simple consequence of Theorem 3.4 for the case that the difference of the resolvents is a trace class operator.

Corollary 3.7. Let the assumptions be as in Theorem 1.6, let $\overline{Q}$ be the Nevanlinna function from Corollary 3.6 and suppose, in addition, $n = 2$. Then

$$\text{(30)} \quad \text{tr} \left( (\mathcal{A}_D - \lambda)^{-1} - (\mathcal{A}_N - \lambda)^{-1} \right) = \text{tr} \left( \overline{Q(\lambda)^{-1}} \frac{d}{d\lambda} \overline{\partial \overline{Q(\lambda)}} \right)$$

holds for all $\lambda \in \rho(\mathcal{A}_D) \cap \rho(\mathcal{A}_N)$.

Proof. The resolvent formula (26) can be written in the form

$$\text{(31)} \quad (\mathcal{A}_D - \lambda)^{-1} - (\mathcal{A}_N - \lambda)^{-1} = \overline{\Gamma(\lambda)} \overline{Q(\lambda)^{-1}} \Gamma(\lambda)^*,$$

where the closures $\overline{\Gamma(\lambda)}$ and $\overline{Q(\lambda)^{-1}}$ are everywhere defined bounded operators, cf. Corollary 3.5(ii). In the case $n = 2$ it follows from Theorem 3.4(iii) that (31) is a trace class operator and from Corollaries 2.9, 3.5(iii) and well known properties of the trace of bounded operators (see [28]) we conclude (30). \hfill \square

4. Coupling of elliptic differential operators

In this section we study the uniformly elliptic second order differential expression $\mathcal{L}$ from (17) on two different domains and a coupling of the associated Dirichlet operators. More precisely, let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded domain with $C^\infty$-boundary $\partial \Omega$ and let $\Omega' = \mathbb{R}^n \setminus \overline{\Omega}$ be the complement of the closure of $\Omega$ in
Let again $L$ be given by

$$Lh = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial h}{\partial x_k} + ah$$

with bounded coefficients $a_{jk} \in C^\infty(\mathbb{R}^n)$ satisfying $a_{jk}(x) = a_{kj}(x)$ for all $x \in \mathbb{R}^n$ and $j, k = 1, \ldots, n$, the function $a \in L^\infty(\mathbb{R}^n)$ is real valued and suppose that $L$ is uniformly elliptic, cf. (13). The restriction of $L$ on functions $f$ defined on $\Omega$ or functions $f'$ defined on $\Omega'$ will be denoted by $L_\Omega$ and $L_{\Omega'}$, respectively. Then it is clear that the differential expressions $L_\Omega$ and $L_{\Omega'}$ are of the type as in Section 3.

In the following we will usually denote functions defined on $\mathbb{R}^n$ by $h$ or $k$, and we denote functions defined on $\Omega$ or $\Omega'$ by $f$, $g$ or $f'$, $g'$, respectively. The scalar products of $L^2(\Omega)$ and $L^2(\Omega')$ are indexed with $\Omega$ and $\Omega'$ respectively, whereas the scalar product of $L^2(\mathbb{R}^n)$ is just denoted by $(\cdot, \cdot)$. For the trace of a function $f \in H^2(\Omega)$ and $f' \in H^2(\Omega')$ we write $f|_C$ and $f'|_C$, and the trace of the conormal derivatives are

$$\frac{\partial f}{\partial \nu |_C} = \sum_{j,k=1}^{n} a_{jk} n_j \frac{\partial f}{\partial x_k} |_C \quad \text{and} \quad \frac{\partial f'}{\partial \nu'|_C} = \sum_{j,k=1}^{n} a_{jk} n'_j \frac{\partial f}{\partial x_k} |_C,$$

here $n(x) = (n_1(x), \ldots, n_n(x))^T$ and $n'(x) = -n(x)$ are the unit vectors at the point $x \in C = \partial \Omega = \partial \Omega'$ pointing out of $\Omega$ and $\Omega'$, respectively. Note also that the coefficients $a_{jk}$ in (33) are the restrictions of the coefficients in (32) onto $\Omega$ and $\Omega'$, respectively. The Dirichlet operators

$$A_\Omega f = L_\Omega f, \quad \text{dom } A_\Omega = \{ f \in H^2(\Omega) : f|_C = 0 \},$$

$$A_{\Omega'} f' = L_{\Omega'} f', \quad \text{dom } A_{\Omega'} = \{ f' \in H^2(\Omega') : f'|_C = 0 \},$$

are selfadjoint operators in $L^2(\Omega)$ and $L^2(\Omega')$, respectively. Hence the orthogonal sum

$$A = \begin{pmatrix} A_\Omega & 0 \\ 0 & A_{\Omega'} \end{pmatrix}, \quad \text{dom } A = \text{dom } A_\Omega \oplus \text{dom } A_{\Omega'},$$

is a selfadjoint operator in $L^2(\mathbb{R}^n) = L^2(\Omega) \oplus L^2(\Omega')$. Observe that

$$A(f \oplus f') = L(f \oplus f') = L_\Omega f \oplus L_{\Omega'} f',$$

$$\text{dom } A = \{ f \oplus f' \in H^2(\Omega) \oplus H^2(\Omega') : f|_C = 0 = f'|_C \},$$

and that $A$ is not a usual second order elliptic differential operator on $\mathbb{R}^n$ since for a function $f \oplus f' \in \text{dom } A$ the traces of the conormal derivatives $\frac{\partial f}{\partial \nu}|_C$ and $-\frac{\partial f'}{\partial \nu'}|_C$ at the boundary $C$ of the domains $\Omega$ and $\Omega'$ in general do not coincide.

Besides the operator $A$ we consider the usual selfadjoint operator associated to $L$ in $L^2(\mathbb{R}^n)$ defined by

$$\tilde{A}h = Lh, \quad h \in \text{dom } \tilde{A} = H^2(\mathbb{R}^n),$$

and our aim is to prove a formula for the difference of the resolvents of $\tilde{A}$ and $A$ with the help of a generalized $Q$-function in a similar form as in the previous section.

The following theorem indicates how $S$ and $T$ in the triple $\{ S, A, T \}$ for the definition of a generalized $Q$-function can be chosen.
Theorem 4.1. The operator

\[ Sh = Lh, \quad \text{dom } S = \{ h = f \oplus f' \in H^2(\mathbb{R}^n) : f|c = 0 = f'|c \}, \]

is a densely defined closed symmetric operator in \( L^2(\mathbb{R}^n) \) with infinite deficiency indices \( n_{\pm}(S) \). The operator

\[ T(f \oplus f') = L(f \oplus f'), \]
\[ \text{dom } T = \{ f \oplus f' \in H^2(\Omega) \oplus H^2(\Omega') : f|c = f'|c \}, \]

is not closed as an operator in \( L^2(\mathbb{R}^n) \) and \( T \) satisfies \( T = S^* \) and \( T^* = S \). Furthermore, the selfadjoint operators \( A \) and \( \tilde{A} \) in \([34], [35] \) and \([36] \) are extensions of \( S \) and restrictions of \( T \).

Proof. The operator \( S \) is a restriction of the selfadjoint operator \( A \) and hence \( S \) is symmetric. The fact that \( \text{dom } S \) is dense follows, e.g., from the fact that \( H^2_0(\Omega) \) and \( H^2(\Omega') \) are dense subspaces of \( L^2(\Omega) \) and \( L^2(\Omega') \), respectively, cf. Proposition 3.1, and

\[ H^2_0(\Omega) \oplus H^2(\Omega') \subset \text{dom } S. \]

Since for any function \( h \in H^2(\mathbb{R}^n) \) decomposed as \( h = f \oplus f' \), where \( f \in H^2(\Omega) \) and \( f' \in H^2(\Omega') \), we have \( f|c = f'|c \in H^{1/2}(\mathcal{C}) \) it follows that \( A \) is an extension of \( S \) and a restriction of the operator \( T \). Moreover, \( S \subset A \subset T \) is obvious.

Let us verify that \( S = T^* \) holds. In particular this implies that \( S \) is closed and that \( T = S^* \) is true. We start with the inclusion \( S \subset T^* \). Let \( h = f \oplus f' \in \text{dom } S \) and \( k = g \oplus g' \in \text{dom } T \), where \( f, g \in H^2(\Omega) \) and \( f', g' \in H^2(\Omega') \). First of all we have

\[ (Tk, h) - (k, Sh) = (L_\Omega g, f)_\Omega - (g, L_\Omega f)_\Omega + (L_{\Omega'} g', f')_{\Omega'} - (g', L_{\Omega'} f')_{\Omega'} \]

and Green’s identity \([20]\) shows that this is equal to

\[ \left( g|c, \frac{\partial f}{\partial \nu}|_c \right)_c - \left( g|c, \frac{\partial f'}{\partial \nu'}|_c \right)_c + \left( f'|c, \frac{\partial g'}{\partial \nu'}|_c \right)_c - \left( f'|c, \frac{\partial g}{\partial \nu}|_c \right)_c. \]

Since \( h = f \oplus f' \in \text{dom } S \) we have

\[ f|c = f'|c = 0 \quad \text{and} \quad \frac{\partial f}{\partial \nu}|_c = -\frac{\partial f'}{\partial \nu'}|_c, \]

and for \( k = g \oplus g' \in \text{dom } T \) by definition \( g|c = g'|c \) holds. Hence we conclude

\[ (Tk, h) - (k, Sh) = 0 \]

and therefore every \( h \in \text{dom } S \) belongs to \( \text{dom } T^* \) and \( T^* h = Sh \), i.e., \( S \subset T^* \). Let us now prove the converse inclusion \( T^* \subset S \). For this it is sufficient to check that every function \( h \in \text{dom } T^* \) belongs to \( \text{dom } S \). From the fact that \( T \) is an extension of the selfadjoint operators \( A \) and \( \tilde{A} \) we conclude

\[ T^* \subset A^* = A \subset T \quad \text{and} \quad T^* \subset \tilde{A}^* = \tilde{A} \subset T, \]

so that \( T^* \) is a restriction of \( A \) and \( \tilde{A} \). Hence every function \( h \) in \( \text{dom } T^* \) belongs also to \( \text{dom } A \) and \( \text{dom } \tilde{A} \). Thus \( h = f \oplus f' \in H^2(\mathbb{R}^n) \) and \( f \in H^2(\Omega) \) and \( f' \in H^2(\Omega') \) satisfy \( f|c = f'|c = 0 \). Therefore \( \text{dom } T^* \subset \text{dom } S \) and we have shown \( T^* = S \).

Next it will be verified that \( T \) is not closed. The arguments are similar as in \([8]\) Proof of Proposition 4.5] and could also be formulated in terms of unitary relations.
between Krein spaces, cf. [17]. Assume that $T$ is closed, i.e., $T = \overline{T}$, and consider the subspace

$$\mathcal{M} = \left\{ \begin{array}{l}
 \begin{bmatrix}
 f \oplus f' \\
 T(f \oplus f') \\
 \frac{\partial f}{\partial \nu} \mid_c + \frac{\partial f'}{\partial \nu} \mid_c
\end{bmatrix}
 : f \oplus f' \in \text{dom } T 
\end{array} \right\} \subset L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}).$$

Observe that by (19) and the definition of $T$ the mapping

$$\text{dom } T \ni f \oplus f' \mapsto \left\{ f \mid_c, \frac{\partial f}{\partial \nu} \mid_c + \frac{\partial f'}{\partial \nu} \mid_c \right\} \in H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C})$$

is onto. Setting $\mathcal{N} = L^2(\Omega) \oplus L^2(\Omega) \oplus \{0\} \oplus \{0\}$ it is clear that the sum of the subspaces $\mathcal{M}$ and $\mathcal{N}$ is

$$\mathcal{M} + \mathcal{N} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus (H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C})).$$

We will calculate the orthogonal complements of $\mathcal{M}$ and $\mathcal{N}$ in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})$ and show that $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed. First of all we have

$$\mathcal{N}^\perp = \{0\} \oplus \{0\} \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})$$

and in order to determine $\mathcal{M}^\perp$ suppose that

$$\begin{bmatrix}
 l \oplus l' \\
 g \oplus g' \\
 \varphi \\
 \psi
\end{bmatrix} \in \mathcal{M}^\perp, \quad g, l \in L^2(\Omega), \ g', l' \in L^2(\Omega'), \ \varphi, \psi \in L^2(\mathcal{C}),$$

is an element in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})$ which is orthogonal to $\mathcal{M}$. Then we have

$$\begin{aligned}
 (T(f \oplus f'), g \oplus g') + (f \oplus f', l \oplus l') &= -(f \mid_c, \varphi) - \left( \frac{\partial f}{\partial \nu} \mid_c + \frac{\partial f'}{\partial \nu} \mid_c, \psi \right)_c \\
 \text{for all } f \oplus f' \in \text{dom } T.
\end{aligned}$$

In particular, for $f \oplus f' \in \text{dom } S$ we have

$$\frac{\partial f}{\partial \nu} \mid_c = -\frac{\partial f'}{\partial \nu} \mid_c, \quad \text{and } f \mid_c = f' \mid_c = 0,$$

so that (43) becomes

$$\begin{aligned}
 (T(f \oplus f'), g \oplus g') &= (S(f \oplus f'), g \oplus g') - (f \oplus f', l \oplus l') \\
 \text{and hence } g \oplus g' \in \text{dom } S^* \text{ and } S^*(g \oplus g') &= -l \oplus l'.
\end{aligned}$$

But we have assumed that $T$ is closed and hence from $S = T^*$ we conclude $S^* = T^{**} = \overline{T} = T$, so that

$$g \oplus g' \in \text{dom } T \quad \text{and } T(g \oplus g') = -l \oplus l'.$$

From Green’s identity we then obtain

$$\begin{aligned}
 (T(f \oplus f'), g \oplus g') &= (T(f \oplus f', T(g \oplus g')) \\
 &= (L_0 f, g)_0 - (f, L_0 g)_0 + (\partial_0 f', g')_{\partial_0} - (f', L_0 g')_{\partial_0} \\
 &= \left( f \mid_c, \frac{\partial g}{\partial \nu} \mid_c \right)_c - \left( \frac{\partial f}{\partial \nu} \mid_c, g \mid_c \right)_c + \left( f' \mid_c, \frac{\partial g'}{\partial \nu} \mid_c \right)_c - \left( \frac{\partial f'}{\partial \nu} \mid_c, g' \mid_c \right)_c \\
 &= \left( f \mid_c, \frac{\partial g}{\partial \nu} \mid_c + \frac{\partial g'}{\partial \nu} \mid_c \right)_c - \left( \frac{\partial f}{\partial \nu} \mid_c + \frac{\partial f'}{\partial \nu} \mid_c, g \mid_c \right)_c.
\end{aligned}$$
where we have used that \( f \oplus f', \ g \oplus g' \in \text{dom} \ T \) satisfy \( f|_c = f'|_c \) and \( g|_c = g'|_c \). Inserting (44) in (43) and comparing this with the above relation shows that the identity

\[
\left(f|_c, \frac{\partial f}{\partial \nu}|_c + \frac{\partial g'}{\partial \nu'}|_c + \varphi\right)_c = \left(\frac{\partial f}{\partial \nu}|_c + \frac{\partial f'}{\partial \nu'}|_c, g|_c - \psi\right)_c
\]

holds for all \( f \oplus f' \in \text{dom} \ T \). As the mapping (39) is surjective and \( H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C}) \) is dense in \( L^2(\mathcal{C}) \oplus L^2(\mathcal{C}) \) we conclude from (45) that

\[
\varphi = -\left(\frac{\partial g}{\partial \nu}|_c + \frac{\partial g'}{\partial \nu'}|_c\right) \quad \text{and} \quad \psi = g|_c
\]

holds. Hence we have seen that the element (42) in \( \mathcal{M}^\perp \) is of the form

\[
\begin{bmatrix}
-T(g \oplus g') \\
g \oplus g' \\
-\frac{\partial g}{\partial \nu}|_c - \frac{\partial g'}{\partial \nu'}|_c \\
g|_c
\end{bmatrix}
\]

for some \( g \oplus g' \in \text{dom} \ T \). It is not difficult to check that conversely an element as in (46) belongs to \( \mathcal{M}^\perp \). Therefore the orthogonal complement of \( \mathcal{M} \) is given by

\[
\mathcal{M}^\perp = \left\{ \begin{bmatrix}
-T(g \oplus g') \\
g \oplus g' \\
\frac{\partial g}{\partial \nu}|_c - \frac{\partial g'}{\partial \nu'}|_c \\
g|_c
\end{bmatrix} : g \oplus g' \in \text{dom} \ T \right\} \subset L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})
\]

and together with (41) we find that the sum of \( \mathcal{M}^\perp \) and \( \mathcal{N}^\perp \) is

\[
\mathcal{M}^\perp + \mathcal{N}^\perp = \left\{ \begin{bmatrix}
-T(g \oplus g') \\
g \oplus g' \\
\frac{\partial g}{\partial \nu}|_c - \frac{\partial g'}{\partial \nu'}|_c \\
g|_c
\end{bmatrix} : g \oplus g' \in \text{dom} \ T \right\} \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}).
\]

The assumption that \( T \) is closed implies that \( \mathcal{M}^\perp + \mathcal{N}^\perp \) is a closed subspace of \( L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}) \). But then according to [34] IV Theorem 4.8 also \( \mathcal{M} + \mathcal{N} \) is a closed subspace of \( L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}) \) which is a contradiction to (40). Thus \( T \) can not be closed. \( \square \)

The following lemma will be useful later in this section.

**Lemma 4.2.** Let \( S \) and \( T \) be as in Theorem 4.1 and let \( \bar{A} \) be the selfadjoint realization of \( L \) in \( L^2(\mathbb{R}^n) \) defined on \( H^2(\mathbb{R}^n) \). For a function \( f \oplus f' \in \text{dom} \ T \), where \( f \in H^2(\Omega) \) and \( f' \in H^2(\Omega') \), we have

\[
f \oplus f' \in \text{dom} \ \bar{A} \quad \text{if and only if} \quad \frac{\partial f}{\partial \nu}|_c = -\frac{\partial f'}{\partial \nu'}|_c.
\]

**Proof.** For a function \( f \oplus f' \in \text{dom} \ \bar{A} = H^2(\mathbb{R}^n) \) it is clear that \( \frac{\partial f}{\partial \nu}|_c = -\frac{\partial f'}{\partial \nu'}|_c \) holds. Conversely, let \( f \oplus f' \in \text{dom} \ T \) and assume

\[
\frac{\partial f}{\partial \nu}|_c = -\frac{\partial f'}{\partial \nu'}|_c.
\]

Then also \( f|_c = f'|_c \) and since every \( g \oplus g' \in \text{dom} \ \bar{A} \) satisfies

\[
g|_c = g'|_c \quad \text{and} \quad \frac{\partial g}{\partial \nu}|_c = -\frac{\partial g'}{\partial \nu'}|_c
\]
Green’s identity implies

\[
(A(g \oplus g'), f \oplus f') - (g \oplus g', T(f \oplus f')) = \left( g|_c, \frac{\partial f}{\partial \nu} \right)_c - \left( \frac{\partial g'}{\partial \nu} |_c, f'|_c \right)_c + \left( g'|_c, \frac{\partial f}{\partial \nu} \right)_c - \left( \frac{\partial g'}{\partial \nu} |_c, f'|_c \right)_c = 0.
\]

Therefore \( f \oplus f' \in \text{dom} \bar{A} = \text{dom} \bar{A}. \)

Next we define a mapping \( \Gamma_{\lambda_0} \) which satisfies the assumptions in the definition of a generalized \( Q \)-function. For this let \( A \) be the selfadjoint operator in \( L^2(\mathbb{R}^n) \) in (34) and (35) which is the orthogonal sum of the Dirichlet operators \( A_\Omega \) and \( A_{\Omega'} \) in \( L^2(\Omega) \) and \( L^2(\Omega') \), respectively. For \( \lambda \in \rho(A) \) the domain of the operator \( T \) in Theorem (33) can be decomposed in

\[
\text{dom} T = \text{dom} A + N_\lambda(T)
\]

(48) \( \{ f \oplus f' \in H^2(\Omega) \oplus H^2(\Omega') : f|_c = f'|_c = 0 \} + N_\lambda(T), \)

cf. [5]. Let us fix some \( \lambda_0 \in \rho(A) \). The decomposition (48) and the surjectivity of \( \Gamma_{\lambda_0} \) imply that for a given function \( \varphi \in H^3/2(\mathbb{C}) \) there exists a unique function \( f_{\lambda_0} \oplus f'_{\lambda_0} \in N_{\lambda_0}(T) \) such that \( f_{\lambda_0}|_c = f'_{\lambda_0}|_c = \varphi \). Let \( \Gamma_{\lambda_0} \) be the mapping that assigns \( f_{\lambda_0} \oplus f'_{\lambda_0} \) to \( \varphi \),

(50) \( H^3/2(\mathbb{C}) \ni \varphi \mapsto \Gamma_{\lambda_0} \varphi := f_{\lambda_0} \oplus f'_{\lambda_0}. \)

Similarly as in the previous section \( \Gamma_{\lambda_0} \) will be regarded as an operator from \( L^2(\mathbb{C}) \) to \( L^2(\mathbb{C}) \) with \( \text{dom} \Gamma_{\lambda_0} = H^3/2(\mathbb{C}) \) and \( \text{ran} \Gamma_{\lambda_0} = N_{\lambda_0}(T) \). Observe that the function \( \Gamma_{\lambda_0} \varphi = f_{\lambda_0} \oplus f'_{\lambda_0} \) consists of an \( H^2(\Omega) \)-solution \( f_{\lambda_0} \) of \( \mathcal{L}_\Omega u = \lambda_0 u \) and an \( H^2(\Omega') \)-solution \( f'_{\lambda_0} \) of \( \mathcal{L}_{\Omega'} u' = \lambda_0 u' \) satisfying the boundary conditions \( \varphi = f_{\lambda_0}|_c = f'_{\lambda_0}|_c \).

The following proposition parallels Proposition [5.2].

**Proposition 4.3.** Let \( \lambda_0 \in \rho(A) \), let \( \Gamma_{\lambda_0} \) be as in (50) and let \( \lambda \in \rho(A) \). Then the following holds:

(i) \( \Gamma_{\lambda_0} \) is a bounded operator from \( L^2(\mathbb{C}) \) in \( L^2(\mathbb{C}) \) with dense domain \( H^3/2(\mathbb{C}) \);

(ii) The operator \( \Gamma(\lambda) = (I + (\lambda - \lambda_0)(A - \lambda)^{-1})\Gamma_{\lambda_0} \) is given by

\[
\Gamma(\lambda)\varphi = f_{\lambda} \oplus f'_{\lambda}, \quad \text{where} \quad f_{\lambda} \oplus f'_{\lambda} \in N_\lambda(T) \quad \text{and} \quad f_{\lambda}|_c = \varphi = f'_{\lambda}|_c;
\]

(iii) The mapping \( \Gamma(\lambda)^*: L^2(\mathbb{R}^n) \to L^2(\mathbb{C}) \) satisfies

\[
\Gamma(\lambda)^*(A - \lambda)h = -\frac{\partial f}{\partial \nu}|_c - \frac{\partial f'}{\partial \nu}|_c, \quad h = f \oplus f' \in \text{dom} A.
\]

**Proof.** We start with the proof (ii). Let \( \varphi \in H^3/2(\mathbb{C}) \) and choose the unique elements \( f_{\lambda} \oplus f'_{\lambda} \in N_\lambda(T) \) and \( f_{\lambda_0} \oplus f'_{\lambda_0} \in N_{\lambda_0}(T) \) such that

\[
f_{\lambda}|_c = f'_{\lambda}|_c = \varphi = f_{\lambda_0}|_c = f'_{\lambda_0}|_c
\]

holds. By definition \( \Gamma_{\lambda_0} \varphi = f_{\lambda_0} \oplus f'_{\lambda_0} \) and therefore

\[
\Gamma(\lambda)\varphi = \Gamma_{\lambda_0} \varphi + (\lambda - \lambda_0)(A - \lambda)^{-1}\Gamma_{\lambda_0} \varphi
\]

\[
= f_{\lambda_0} \oplus f'_{\lambda_0} + (\lambda - \lambda_0)(A - \lambda)^{-1}\Gamma_{\lambda_0} \varphi.
\]
Since \((\lambda - \lambda_0)(A - \lambda)^{-1}\Gamma_{\lambda_0}\varphi\) is a function belonging to \(\text{dom} A\) we have
\[
(\lambda - \lambda_0)(A - \lambda)^{-1}\Gamma_{\lambda_0}\varphi|_c = 0,
\]
cf. (35). This implies
\[
(\Gamma(\lambda)\varphi)|_c = (\Gamma_{\lambda_0}\varphi)|_c = (f_{\lambda_0} + f'_{\lambda_0})|_c = f_{\lambda_0}|_c = f'|_{\lambda_0}|_c = \varphi
\]
and since \(\text{ran} \Gamma(\lambda) = \mathcal{N}_{\lambda}(T)\), see Lemma 2.1, and \(f_{\lambda} + f'_{\lambda}\) is the unique function in \(\mathcal{N}_{\lambda}(T)\) with \(f_{\lambda}|_c = f'|_{\lambda}|_c = \varphi\) we conclude \(\Gamma(\lambda)\varphi = f_{\lambda} + f'_{\lambda}\).

Next we verify (iii). Observe that then \(\Gamma(\lambda)^*\), \(\lambda \in \rho(A)\), is a closed operator which is defined on the whole space, i.e., \(\Gamma(\lambda)^*\) is bounded and hence assertion (i) follows by setting \(\lambda_0 = \lambda\). Let \(\varphi \in H^{3/2}(\mathcal{C})\) and choose the unique function \(f_{\lambda} + f'_{\lambda} \in \mathcal{N}_{\lambda}(T)\) such that
\[
(51) \quad f_{\lambda}|_c = f'|_{\lambda}|_c = \varphi
\]
holds. Then \(\Gamma(\lambda)^*\varphi = f_{\lambda} + f'_{\lambda}\) and for each \(h = f + f' \in \text{dom} A\), where \(f \in H^2(\Omega), f' \in H^2(\Omega')\), we have
\[
(\Gamma(\lambda)^*\varphi, (A - \lambda)h) = (f_{\lambda} + f'_{\lambda}, A(f + f')) - (T(f_{\lambda} + f'_{\lambda}), f + f')
\]
\[
= (f_{\lambda}, \mathcal{L}_A f_{\lambda} + (f'_{\lambda}, \mathcal{L}_{\Omega'} f_{\lambda} + (f'_{\lambda}, \mathcal{L}_{\Omega'} f'_{\lambda})).
\]

With the help of Green’s identity this can be rewritten as
\[
\frac{\partial f_{\lambda}}{\partial \nu}|_c \varphi + \frac{\partial f'|_{\lambda}}{\partial \nu}|_c f'|_{\lambda}|_c - \frac{\partial f'_{\lambda}}{\partial \nu}|_c f'_|_{\lambda}|_c.
\]
Since for \(h = f + f' \in \text{dom} A\) we have \(f|_c = f'|_{\lambda}|_c = 0\) we conclude from the above calculation and (51) that
\[
(\Gamma(\lambda)^*\varphi, (A - \lambda)h) = - \left( \varphi, \frac{\partial f}{\partial \nu}|_c + \frac{\partial f'}{\partial \nu}|_c \right)_c.
\]
holds for every \(\varphi \in H^{3/2}(\mathcal{C}) = \text{dom} \Gamma(\lambda)^*\) and
\[
\Gamma(\lambda)^*(A - \lambda)h = - \frac{\partial f}{\partial \nu}|_c - \frac{\partial f'}{\partial \nu}|_c, \quad h = f + f' \in \text{dom} A.
\]

Furthermore, for \(\lambda \in \rho(A)\) we have \(\text{ran} (A - \lambda) = L^2(\mathbb{R}^n)\), so that \(\Gamma(\lambda)^*\) is a bounded operator defined on \(L^2(\mathbb{R}^n)\).

Next we define a function \(Q\) in a similar way as the Dirichlet-to-Neumann map in Definition 3.3. For this we make use of the decomposition (48). Namely, for \(\lambda \in \rho(A)\) and \(\varphi \in H^{3/2}(\mathcal{C})\) there exists a unique function \(f_{\lambda} + f'_{\lambda} \in \mathcal{N}_{\lambda}(T)\) such that \(f_{\lambda}|_c = f'|_{\lambda}|_c = \varphi\). The operator \(Q(\lambda)\) in \(L^2(\mathcal{C})\) is now defined by
\[
(52) \quad Q(\lambda)\varphi := - \frac{\partial f_{\lambda}}{\partial \nu}|_c - \frac{\partial f'_{\lambda}}{\partial \nu}|_c, \quad \varphi \in \text{dom} Q(\lambda) = H^{3/2}(\mathcal{C}.
\]

Observe that \(\text{ran} Q(\lambda) \subset H^{1/2}(\mathcal{C})\) holds. Roughly speaking, up to a minus sign \(Q(\lambda)\) maps the Dirichlet boundary value of the \(H^2\)-solutions of \(\mathcal{L}_u = \lambda u\) and \(\mathcal{L}_{\Omega'} u' = \lambda u', u|_c = u'|_c\), onto the sum of the Neumann boundary values of these solutions. We mention that in the analysis of so-called intermediate Hamiltonians a modified form of such a Dirichlet-to-Neumann map has been used in (44).

In the following theorem it turns out that \(Q\) can be interpreted as a generalized \(Q\)-function and the difference of the resolvents of \(A\) and \(\tilde{A}\) is expressed with the help of \(Q\).
Theorem 4.4. Let $\mathcal{L}$ be the elliptic differential expression in (32) and let $A$ and $\tilde{A}$ be the selfadjoint realizations of $\mathcal{L}$ in (34)–(35) and (36), respectively. Let $S$ and $T$ be the operators in Theorem 4.3 define $\Gamma(\lambda)$ as in Proposition 4.3 and let $Q(\lambda)$, $\lambda \in \rho(A)$, be as in (52). Then the following holds:

(i) $Q$ is a generalized $Q$-function of the triple $\{S, A, T\}$;

(ii) The operator $Q(\lambda)$ is injective for all $\lambda \in \rho(A) \cap \rho(\tilde{A})$ and the resolvent formula

\[(A - \lambda)^{-1} - (\tilde{A} - \lambda)^{-1} = \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\tilde{\lambda})^*\]

holds;

(iii) For $p \in \mathbb{N}$ and $2p + 1 > n$ the difference of the resolvents in (53) belongs to the von Neumann-Schatten class $\mathfrak{S}_p(L^2(\Omega))$.

Proof. Let us prove assertion (i). Before the defining relation (7) for a generalized $Q$-function will be verified we show that the operator $Q(\mu)^*$ is an extension of $Q(\bar{\mu})$, $\mu \in \rho(A)$. For this let $\psi \in H^{3/2}(\mathcal{C})$ and choose the unique element $f_\mu \oplus f'_\mu \in N_{\mu}(T)$ with the property $f_\mu|_c = f'_\mu|_c = \psi$. Then according to Proposition 4.3 (ii) $\Gamma(\lambda) \psi$ holds. By the definition of $Q$ in (52) we have

\[Q(\mu)\varphi = -\frac{\partial f_\mu}{\partial \nu}|_c - \frac{\partial f'_\mu}{\partial \nu'}|_c \quad \text{and} \quad Q(\bar{\mu})\psi = -\frac{\partial f_\mu}{\partial \nu}|_c - \frac{\partial f'_\mu}{\partial \nu'}|_c.\]

This gives

\[(Q(\mu)\varphi, \psi) = -\left(\frac{\partial f_\mu}{\partial \nu}|_c, f_\mu|_c\right)_c - \left(\frac{\partial f'_\mu}{\partial \nu'}|_c, f'_\mu|_c\right)_c\]

and since

\[\left(\varphi,\frac{\partial f_\mu}{\partial \nu}|_c, f_\mu|_c\right)_c = (\mathcal{L}_\Omega f_\mu, f_\mu) = (f_\mu, \mathcal{L}_\Omega f_\mu) = 0,\]

we can rewrite (54) in the form

\[(Q(\mu)\varphi, \psi) = -\left(\varphi,\frac{\partial f_\mu}{\partial \nu}|_c, f_\mu|_c\right)_c = \left(\varphi, -\frac{\partial f_\mu}{\partial \nu}|_c - \frac{\partial f'_\mu}{\partial \nu'}|_c\right)_c.\]

This is true for every $\varphi \in \text{dom } Q(\mu)$ and hence we conclude $\psi \in \text{dom } Q(\mu)^*$ and

\[Q(\mu)^*\psi = -\frac{\partial f_\mu}{\partial \nu}|_c - \frac{\partial f'_\mu}{\partial \nu'}|_c = Q(\bar{\mu})\psi.\]

Let $\Gamma(\cdot)$ be as in Proposition 4.3. We prove now that

\[(55) \quad Q(\lambda) - Q(\mu)^* = (\lambda - \mu)\Gamma(\mu)^*\Gamma(\lambda), \quad \lambda, \mu \in \rho(A)\]

holds on $\text{dom } \Gamma(\lambda) = H^{3/2}(\mathcal{C})$. For this let $\varphi, \psi \in H^{3/2}(\mathcal{C})$ and choose the unique elements $f_\lambda \oplus f'_\lambda \in N_{\lambda}(T)$, $f_\mu \oplus f'_\mu \in N_{\mu}(T)$ with the properties

\[f_\lambda|_c = f'_\lambda|_c = \varphi \quad \text{and} \quad f_\mu|_c = f'_\mu|_c = \psi.\]

Then according to Proposition 4.3 (ii) $\Gamma(\lambda) \varphi = f_\lambda \oplus f'_\lambda$ and $\Gamma(\mu)\psi = f_\mu \oplus f'_\mu$ and by the definition of $Q(\cdot)$ in (52) we have

\[Q(\lambda)\varphi = -\frac{\partial f_\lambda}{\partial \nu}|_c - \frac{\partial f'_\lambda}{\partial \nu'}|_c \quad \text{and} \quad Q(\mu)\psi = -\frac{\partial f_\mu}{\partial \nu}|_c - \frac{\partial f'_\mu}{\partial \nu'}|_c.\]
We proceed in a similar way as in the proof of Theorem 3.4. Let \( h \in \text{dom} \) from (57). Let \( f \) be such that \( (QQ)_{c} = 0 \) imply \( (f_{\lambda}|c, f_{\mu}|c) + (f_{\lambda}|c). Making use of Green's identity the above relations then become
\[
\begin{aligned}
&\left( (Q(\lambda) - Q(\mu)^{*}) \varphi, \psi \right)_{c} = - \left( \frac{\partial f_{\lambda}}{\partial \nu}|c, \frac{\partial f'_{\lambda}}{\partial \nu}|c \right) + \left( \varphi, \frac{\partial f_{\mu}}{\partial \nu}|c, \frac{\partial f'_{\mu}}{\partial \nu}|c \right) + \\
&\left. \left( f_{\lambda}|c, \frac{\partial f_{\mu}}{\partial \nu}|c \right) + \left( f'_{\lambda}|c, \frac{\partial f'_{\mu}}{\partial \nu}|c \right) \right) .
\end{aligned}
\]
Making use of Green's identity the above relations then become
\[
\begin{aligned}
&\left( (Q(\lambda) - Q(\mu)^{*}) \varphi, \psi \right)_{c} = (L_{\lambda}f_{\lambda}, f_{\mu}|c) - (f_{\lambda}, L_{\lambda}f_{\mu}|c) + (f_{\lambda}, f_{\mu}|c) - (L_{\lambda}f'_{\lambda}, f'_{\mu}|c) - (f'_{\lambda}, f'_{\mu}|c) \\
&= (\lambda - \mu)(f_{\lambda}, f_{\mu}|c) + (f_{\lambda}, f'_{\mu}|c) = (\lambda - \mu)(f_{\lambda}, f_{\mu}|c) + (\lambda - \mu)(\bar{f}_{\lambda}, f_{\mu}|c) \\
&= (\lambda - \mu)(\Gamma(\lambda)) \varphi, \Gamma(\mu)|c) = ((\lambda - \mu)(\Gamma(\lambda))\varphi, \psi)|c) .
\end{aligned}
\]
Since this is true for any \( \psi \in H^{3/2}(\mathcal{C}) \) we conclude that (55) holds on \( H^{3/2}(\mathcal{C}) \). Thus \( Q \) in (52) is a generalized \( Q \)-function for the triple \( \{ S, A, T \} \).

(ii) We check first that \( \ker Q(\lambda) = \{ 0 \} \) holds for \( \lambda \in \rho(A) \cap \rho(\tilde{A}) \). Assume that \( Q(\lambda) \varphi = 0 \) for some \( \varphi \in H^{3/2}(\mathcal{C}) \) and let \( f_{\lambda} \oplus f'_{\lambda} \in \mathcal{N}_{\lambda}(T) \) be the unique element with the property \( f_{\lambda}|c = f'_{\lambda}|c = \varphi \). Then the definition of \( Q \) and the assumption \( Q(\lambda) \varphi = 0 \) imply
\[
\frac{\partial f_{\lambda}}{\partial \nu}|c = - \frac{\partial f'_{\lambda}}{\partial \nu}|c .
\]
According to Lemma (4.2) of [1] this yields \( f_{\lambda} \oplus f'_{\lambda} \in \text{dom} \tilde{A} \cap \mathcal{N}_{\lambda}(T) \). But as \( \lambda \in \rho(\tilde{A}) \) we conclude \( f_{\lambda} = 0 \) and \( f'_{\lambda} = 0 \), and hence \( \varphi = 0 \).

Now we prove the formula (53) for the difference of the resolvents of \( A \) and \( \tilde{A} \). By the above argument \( Q(\lambda)^{-1} \) exists for \( \lambda \in \rho(A) \cap \rho(\tilde{A}) \). Furthermore, (19) implies \( \text{ran} Q(\lambda) = H^{1/2}(\mathcal{C}) \) and it follows from Proposition (4.3) that the right hand side in (53) is well defined.

Let \( h \in L^{2}(\mathbb{R}^{n}) \) and define the function \( k \) as
\[
(57) \quad k = (A - \lambda)^{-1}h - \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^{*}h .
\]
We show \( k \in \text{dom} \tilde{A} \). First of all it is clear that \( k \in \text{dom} T \) since \( (A - \lambda)^{-1}h \in \text{dom} A \subset \text{dom} T \) and \( \Gamma(\lambda) \) maps into \( \mathcal{N}_{\lambda}(T) \). Therefore \( k = g \oplus g' \), where \( g \in H^{2}(\Omega) \), \( g' \in H^{2}(\Omega^{'}) \), and \( g|c = g'|c \). According to Lemma (4.2) for \( k \in \text{dom} A \) it is sufficient to check
\[
(58) \quad \frac{\partial g}{\partial \nu}|c + \frac{\partial g'}{\partial \nu}|c = 0 .
\]
We proceed in a similar way as in the proof of Theorem (3.4). Let \( h_{A} = f_{\lambda} \oplus f'_{\lambda} \in \text{dom} A \) be such that \( h = (A - \lambda)h_{A} \). Making use of Proposition (4.3) (iii) we obtain
\[
(59) \quad k = h_{A} + \Gamma(\lambda)Q(\lambda)^{-1} \left( \frac{\partial f_{\lambda}}{\partial \nu}|c + \frac{\partial f'_{\lambda}}{\partial \nu}|c \right) - \\
\left. \left( \frac{\partial f_{\lambda}}{\partial \nu}|c + \frac{\partial f'_{\lambda}}{\partial \nu}|c \right) .
\end{aligned}
\]
from (57). Let \( \mathcal{N}_{\lambda}(T) \ni f_{\lambda} \oplus f'_{\lambda} := \Gamma(\lambda)Q(\lambda)^{-1} \left( \frac{\partial f_{\lambda}}{\partial \nu}|c + \frac{\partial f'_{\lambda}}{\partial \nu}|c \right) .
\]
Then by Proposition 4.3 (ii) we have
\[ f_\lambda|_C = f'_\lambda|_C = Q(\lambda)^{-1} \left( \frac{\partial f_A}{\partial \nu}|_C + \frac{\partial f'_A}{\partial \nu'}|_C \right). \]
This together with the definition of \( Q(\lambda) \) in (52) implies
\[ \frac{\partial f_A}{\partial \nu}|_C + \frac{\partial f'_A}{\partial \nu'}|_C = Q(\lambda)(f_\lambda|_C) = Q(\lambda)(f'_\lambda|_C) = -\frac{\partial f_\lambda}{\partial \nu}|_C - \frac{\partial f'_\lambda}{\partial \nu'}|_C. \]
Hence we conclude that the function \( k = g \oplus g' \) in (53) fulfills (55), i.e., \( k \in \text{dom} \bar{A} \).
From (57) and \( A, \bar{A} \subset T \) we obtain
\[ (\bar{A} - \lambda)k = (T - \lambda)(A - \lambda)^{-1}h - (T - \lambda)\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\lambda)^*h = h \]
and now \( k = (\bar{A} - \lambda)^{-1}h \)
\[ \text{and } \bar{Q}(\lambda) \text{ is a } \mathcal{L}(L^2(C))\text{-valued Nevanlinna function}. \]

**Corollary 4.5.** For \( \lambda, \lambda_0 \in \rho(A) \) the following holds.
(i) \( Q(\lambda) \) is a non-closed unbounded operator in \( L^2(C) \) defined on \( H^{3/2}(C) \) with \( \text{ran} Q(\lambda) \subset H^{1/2}(C) \);
(ii) \( Q(\lambda) - \text{Re} Q(\lambda_0) \) is a non-closed bounded operator in \( L^2(C) \) defined on \( H^{3/2}(C) \);
(iii) the closure \( \bar{Q}(\lambda) \) of the operator \( Q(\lambda) - \text{Re} Q(\lambda_0) \) in \( L^2(C) \) satisfies
\[ \frac{d}{d\lambda} \bar{Q}(\lambda) = \Gamma(\lambda) \Gamma(\lambda)^* \]
and \( \bar{Q} \) is a \( \mathcal{L}(L^2(C))\)-valued Nevanlinna function.

**Corollary 4.6.** For \( \lambda \in \rho(A) \cap \rho(\bar{A}) \) the following holds.
(i) \( Q(\lambda)^{-1} \) is a non-closed bounded operator in \( L^2(C) \) defined on \( H^{1/2}(C) \) with \( \text{ran} Q(\lambda)^{-1} = H^{3/2}(C) \);
(ii) the closure \( \bar{Q}(\lambda)^{-1} \) is a compact operator in \( L^2(C) \);
(iii) the function \( \lambda \mapsto -Q(\lambda)^{-1} \) is a \( \mathcal{L}(L^2(C))\)-valued Nevanlinna function.

As a corollary of Theorem 4.4 we obtain a trace formula for the difference of the resolvents of \( A \) and \( \bar{A} \).

**Corollary 4.7.** Let the assumptions be as in Theorem 4.4 let \( \bar{Q} \) be the Nevanlinna function from Corollary 4.5 and suppose, in addition, \( n = 2 \). Then
\[ \text{tr}((A - \lambda)^{-1} - (\bar{A} - \lambda)^{-1}) = \text{tr} \left( \bar{Q}(\lambda)^{-1} \frac{d}{d\lambda} \bar{Q}(\lambda) \right) \]
holds for all \( \lambda \in \rho(A) \cap \rho(\bar{A}) \).

**References**

[1] D. Alpay, P. Bruinsma, A. Dijksma, and H.S.V. de Snoo, *A Hilbert space associated with a Nevanlinna function*, Proceedings International Symposium MTNS 89, Volume III, Progress in Systems and Control Theory, Birkhäuser Verlag Basel (1990), 115–122.
[2] D. Alpay and I. Gohberg, *A trace formula for canonical differential expressions*, J. Funct. Anal. 197 (2003), no. 2, 489–525.
[3] D. Alpay and I. Gohberg, *Pairs of selfadjoint operators and their invariants*, Algebra i Analiz 16 (2004), no. 1, 70–120; translation in St. Petersburg Math. J. 16 (2005), no. 1, 59–104.
[4] W. O. Amrein, D. B. Pearson, \( M \)-operators: a generalisation of Weyl–Titchmarsh theory, J. Comput. Appl. Math. 171 (2004), 1–26.
[5] W. G. Bade, R. S. Freeman, Closed extensions of the Laplace operator determined by a general class of boundary conditions, Pacific J. Math. 12 (1962), 395–410.
[6] R. Beals, Non-local boundary value problems for elliptic operators, Amer. J. Math. 87 (1965), 315–362.
[7] J. Behrndt, S. Hassi and H.S.V. de Snoo, Boundary relations, unitary colligations, and functional models, to appear in Complex Analysis Operator Theory.
[8] J. Behrndt, M. Kurula, A. van der Schaft, and H. Zwart, Dirac structures and their composition on Hilbert spaces, preprint (2008).
[9] J. Behrndt and M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains, J. Funct. Anal. 243 (2007), 536–565.
[10] J. Behrndt, M.M. Malamud, and H. Neidhardt, Trace formulae for dissipative and coupled scattering systems, to appear in Oper. Theory Adv. Appl., Birkhäuser, Basel
[11] M.S Birman, Perturbations of the continuous spectrum of a singular elliptic differential operator by varying the boundary and the boundary conditions, Vestnik Leningrad. Univ. 17 (1962), 22–55.
[12] M.S. Brodski˘ı, Unitary operator colligations and their characteristic functions, Uspekhi Math. Nauk, 33 (4) (1978), 141–168 (Russian) [English transl.: Russian Math. Surveys, 33 no. 4 (1978), 159–191].
[13] M. Brown, G. Grubb, and I. Wood, \( M \)-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems, \[ \text{arXiv:0803.3630} \]
[14] M. Brown, M. Marletta, S. Naboko, and I. Wood, Boundary triplets and \( M \)-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices, preprint (2008).
[15] P. Bruinsma, A. Dijksma, and H.S.V. de Snoo, Models for generalized Carathéodory and Nevanlinna functions, Koninklijke Nederlandse Akademie van Wetenschappen, Verhandelingen, Afd. Natuurkunde, Eerste Reeks, deel 40, Amsterdam (1993), 161–178.
[16] J. Brüning, V. Geyler, and K. Pankrashkin, Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Rev. Math. Phys. 20 (2008), 1–70.
[17] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, Boundary relations and their Weyl families, Trans. Amer. Math. Soc. 358 (2006), 5351–5400.
[18] V.A. Derkach and M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991), 1–95.
[19] V.A. Derkach and M.M. Malamud, The extension theory of Hermitian operators and the moment problem, J. Math. Sciences 73 (1995), 141–242.
[20] P. Exner and S. Kondej, Schrödinger operators with singular interactions: a model of tunneling resonances, J. Phys. A 37 (2004) no. 34, 8255–8277.
[21] F. Gesztesy and M. Mitrea, Robin-to-Robin maps and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains, \[ \text{arXiv:0803.3072} \]
[22] F. Gesztesy and E. Tsekanovskii, On matrix-valued Herglotz functions, Math. Nachr. 218 (2000), 61–138.
[23] I. Goldberg and M.G. Krein, Introduction to the theory of linear nonselfadjoint operators, Translations of Mathematical Monographs Vol. 18, American Mathematical Society, Providence, R.I., 1969.
[29] V.I. Gorbachuk and M.L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series) 48, Kluwer Academic Publishers, Dordrecht, 1991.

[30] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics 24, Pitman, Boston, MA, 1985.

[31] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 425–513.

[32] G. Grubb, *On the coerciveness and semiboundedness of general boundary problems*, Israel J. Math. 10 (1971), 32–95.

[33] S. Hassi, H.S.V. de Snoo, and H. Woracek, *Some interpolation problems of Nevanlinna-Pick type. The Krein-Langer method*, Oper. Theory Adv. Appl. 106, Birkhäuser, Basel (1998), 201–216.

[34] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren der Mathematischen Wissenschaften 132, Springer-Verlag, Berlin-New York, 1976.

[35] M.G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I.*, (Russian) Rec. Math. [Mat. Sbornik] N.S. 20 (62) (1947), 431–495.

[36] M.G. Krein, *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m)*, (Russian) Ukrain. Mat. Zurnal 1, (1949), 3–66.

[37] M.G. Krein and H. Langer, *Über die Q-Funktion eines π-hermiteschen Operators im Raume Πκ*, Acta Sci. Math. (Szeged) 34 (1973), 191–230.

[38] M.G. Krein and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Πκ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. 77 (1977), 187–236.

[39] H. Langer and B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space*, Pacific J. Math. 72 (1977), 135–165.

[40] J. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer Verlag, New York - Heidelberg, 1972.

[41] M.M. Malamud and S.M. Malamud, *Spectral theory of operator measures in Hilbert space*, St. -Petersburg Math. Journal, 15, no. 3 (2003), 1-77.

[42] M. Marletta, *Eigenvalue problems on exterior domains and Dirichlet to Neumann maps*, J. Comput. Appl. Math. 171 (2004), 367–391.

[43] V. Mazya, *Sobolev spaces*, Springer, Berlin, 1985.

[44] A.B. Mikhailova, B. Pavlov, and L.V. Prokhorov, *Intermediate Hamiltonian via Glazman’s splitting and analytic perturbation for meromorphic matrix-functions*, Math. Nachr. 280, No. 12 (2007), 1376–1416.

[45] A.B. Mikhailova, B. Pavlov, and V.I. Ryzhii, *Dirichlet-to-Neumann techniques for the plasma-waves in a slot-diode*, Oper. Theory Adv. Appl. 174, Birkhäuser, Basel (2007) 77–103.

[46] B. Pavlov, *A star-graph model via operator extension*, Math. Proc. Cambridge Philos. Soc. 142 (2007), 365–384.

[47] A. Posilicano, *Self-adjoint extensions of restrictions*, math-ph/0703078

[48] A. Posilicano and L. Raimondi, *Krein’s resolvent formula for self-adjoint extensions of symmetric second order elliptic differential operators*, arXiv:0804.3312

[49] O. Post, *First-order operators and boundary triples*, Russ. J. Math. Phys. 14 (2007), no. 4, 482–492.

[50] M.I. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. 1 (1952), 187–246 (russisch).

[51] J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, 1987.