Dwell-Time Based Stability Analysis and $L_2$ Control of LPV Systems with Piecewise Constant Parameters and Delay

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Abstract

Dwell-time based stability conditions for a class of LPV systems with piecewise constant parameters under time-varying delay are derived using clock-dependent Lyapunov-Krasovskii functional. Sufficient synthesis conditions for clock-dependent gain-scheduled state-feedback controllers ensuring $L_2$-performance are also provided. Several numerical and practical examples, to illustrate the efficacy of the results, are given.

Key words: LPV systems, time delay, $L_2$-performance, dwell-time, clock-dependent L-K functional.

1 Introduction

The framework of LPV systems has proven to be a systematic way to model nonlinear real-world phenomena and synthesize gain-scheduled controllers for nonlinear systems; see Briat (2015). Mohammadpour and Scherer (2012), and Toli (2010). The applications of LPV systems include the automotive industry (Sename et al. (2013)), turbofan engines (Gilbert et al. (2010)), robotics (Kajiwara et al. (1999)), and aerospace systems (Shin et al. (2000)). Apart from nonlinearity, real-world applications are often affected by time delays that can degrade the performance of the dynamical systems, or in the worst case, they can cause instability; see Niculescu (2001). Time delays frequently appear in communication networks, mechanical systems, PVTOL aircrafts, robotized teleoperation, and many other domains; see Chiasson and Loiseau (2007) and Ahmed et al. (2018). Since time delays can also adversely affect the stability of the LPV systems (Briat (2015b); Zakwan and Ahmed (2020)), it is quite natural to consider LPV systems with time delays.

The point of view usually considered in LPV control is worst-case analysis, i.e., parameters are assumed to behave in an extreme way almost all the time by (i) either considering them to vary arbitrarily fast/discontinuously, or (ii) by assuming that they have bounded derivatives. Both of them are quite extreme cases, and there is a room in the parameter space in-between parameters varying arbitrarily fast/discontinuously and parameters having bounded derivatives. To fill this gap, we consider the class of LPV systems with piecewise constant parameters as introduced in Briat (2015). The rationale of LPV systems with piecewise constant parameters lies in reduced conservatism with improved performance. The main idea is to utilize the prior knowledge of the parameters’ trajectory for stability analysis rather than performing worst-case analysis. LPV systems with piecewise constant parameters arise naturally in the context of sampled-data control of LPV systems (Joo and Kim (2015)) and control of buck converters with piecewise constant loads (Tan et al. (2002)). LPV systems with piecewise constant parameters can be considered as switched systems with an uncountable number of modes in a bounded compact set, Zakwan (2020). LPV systems with piecewise constant parameters subject to spontaneous Poissonian jumps are also discussed in Briat (2018) and Zakwan (2020).

The main aim of this paper is to study the dwell-time based stability properties and control of LPV systems with piecewise constant parameters under a time-varying delay. Stability analysis and control of LPV systems with piecewise constant parameters is also discussed in Briat (2015). However, there are two main differences between our work and Briat (2015). First, no delay is present in Briat (2015). However, there are two main differences between our work and Briat (2015). First, no delay is present in Briat (2015). Here we extend the results of Briat (2015) to the difficult case when there is a time-varying delay in the dynamics of LPV systems with piecewise constant parameters. Second, our work provides $L_2$-performance for controller synthesis, which was not considered in Briat (2015).

At first glance, establishing quadratic stability for the class of LPV systems with piecewise constant parameters seems to be a natural choice, since the parameters belong to the class of arbitrarily fast varying parameters. However, by doing so, we will fail to capture the fact that the parameters are constant between the consecutive jumps, hence, leading to conservative results. To reduce this conservatism, we
employ clock-dependent Lyapunov-Krasovskii functionals, introduced in [Briat (2013)], for stability analysis. These functionals inherit a clock that measures the time elapsed since the last jump in the parameters’ trajectory yielding clock-dependent stability conditions. These conditions result in infinite-dimensional semi-definite programs that are intractable. Several techniques such as gridding methods [Briat (2015)], Appendix C] and sum-of-squares (SOS) polynomials [Wu and Prajna, 2003; Scherer and Hol, 2006] are available to approximate semi-infinite constraint LMI by a finite number of LMIs. After obtaining the dwell-time-based stability conditions, we also use them to derive synthesis conditions for clock-dependent gain-scheduled state-feedback controller ensuring $L_2$-performance.

The paper unfolds as follows. In Section 2, we provide some preliminary results followed by dwell-time-based stability conditions for LPV systems with piecewise constant parameters under a time-varying delay whereas synthesis conditions for clock-dependent gain-scheduled controllers with guaranteed $L_2$ performance for these systems are provided in Section 3. Section 4 provides numerical and practical examples to illustrate our main results. Finally, some concluding remarks and future research directions are briefly discussed in Section 5.

We employ standard notation throughout the paper. The sets of positive integers and whole numbers are denoted by $\mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, respectively. The identity and null matrices of dimension $n$ are denoted by $I_n$ and $O_n$, respectively. We write $M > 0$ (resp. $M \leq 0$) to indicate that $M$ is a symmetric positive definite (resp. negative semi-definite) matrix. The cone of symmetric positive definite (resp. positive semi-definite) matrices is denoted by $\mathbb{S}^n_+$ (resp. $\mathbb{S}^n_-$).

For some square matrix $A$, $A + A^T$ will be denoted by $\text{Sym}[A]$. The Banach space of continuous functions from a set $X$ to a set $Y$ is denoted by $C(X,Y)$. The asterisk symbol $(\ast)$ denotes the complex conjugate transpose of a matrix and $x(\theta)$ is the shorthand notation for the translation operator acting on the trajectory such that $x(\theta) = x(t + \theta)$ for some non-zero interval $\theta \in [-h, 0]$.

2 Stability analysis of LPV systems with piecewise constant parameters and delay

2.1 Preliminaries

We consider in this paper LPV systems with piecewise constant parameters and time-varying delay that can be described as

$\Sigma_s : \begin{cases} \dot{x}(t) = A(\rho)x(t) + A_d(\rho)x(t - d(t)) \\
+ B(\rho)u(t) + E(\rho)w(t) \\
\dot{z}(t) = C(\rho)x(t) + C_d(\rho)x(t - d(t)) \\
+ D(\rho)u(t) + F(\rho)w(t) \\
x(\theta) = \phi(\theta), \forall \theta \in [-h, 0], \end{cases}$

where $x \in \mathbb{R}^n$ is the system state, $w \in \mathbb{R}^m$ is the exogenous input, $u \in \mathbb{R}^p$ is the control input, $z \in \mathbb{R}^r$ is the controlled output, and $\phi \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ is the functional initial condition. The time-varying delay $d(t)$ is assumed to belong to the set $D := \{d : \mathbb{R}_0 \rightarrow [0, h], \ \dot{d} \leq \mu < 1\}$ with $h < +\infty$. The parameter vector trajectory $\rho : \mathbb{R}_0 \rightarrow \mathcal{P} \subset \mathbb{R}^s$, $\mathcal{P}$ compact and connected, is assumed to be piecewise constant and measurable, and that the matrix-valued functions $A(\cdot), A_d(\cdot), B(\cdot), C(\cdot), C_d(\cdot), D(\cdot), E(\cdot)$, and $F(\cdot)$ are bounded and continuous on $\mathcal{P}$. We define the sequence $\{t_k\}_{k \in \mathbb{N}_0}, t_0 = 0$, of time instants where the parameters change values. We assume that there exists an $\epsilon > 0$ such that $T_k := t_{k+1} - t_k \geq \epsilon$ for all $k \in \mathbb{N}_0$, and $T_D = T_k$ is referred to as minimum dwell-time.

We now provide the following integral inequality based on Jensen’s inequality to be used in the proof of our stability theorem to bound the derivative of the Lyapunov-Krasovskii functional. This inequality plays an important role in the stability problem of time-delay systems, [Gu et al. (2003)].

**Proposition 1 (Gu et al. (2003))** For any matrix-valued function $R : \mathcal{P} \mapsto \mathbb{S}^n_+$, scalar $h > 0$ such that the integrations concerned are well defined, it holds that

$$\left(\int_{t-h}^{t} \dot{x}(s)ds\right)^T R \left(\int_{t-h}^{t} \dot{x}(s)ds\right) \leq h \int_{t-h}^{t} \dot{x}(s)^T \dot{R} \dot{x}(s)ds .$$

2.2 Dwell-time based stability results

In this section, we derive dwell-time-based stability conditions for the the system $\Sigma_2$ by employing clock-dependent Lyapunov-Krasovskii functional. To this aim, we define the set

$$\mathcal{P}_{\geq T_D} = \left\{ \rho : \mathbb{R}_0 \rightarrow \mathcal{P} : \rho(t) = \alpha_k \in \mathcal{P}, \ \forall t \in [t_k, t_{k+1}), t_{k+1} \geq t_k + T_D, k \in \mathbb{N}_0 \right\} ,$$

which contains all the possible parameter trajectories.

**Theorem 1** For given constants $h > 0, \mu \in [0, 1), \kappa > 0$, and $T_D > 0$, if there exist matrix-valued functions $P : [0, T_D] \times \mathcal{P} \mapsto \mathbb{S}^n_+$, $Q : [0, T_D] \times \mathcal{P} \mapsto \mathbb{S}^n_-$, and $R : [0, T_D] \times \mathcal{P} \mapsto \mathbb{S}^n_-$ such that the LMIs

$$\begin{bmatrix} \Gamma_{11}(\tau, \rho) & \Gamma_{12}(\tau, \rho) & hA^T(\rho)R(\tau, \rho) \\
* & \Gamma_{22}(\tau, \rho) & hA^T(\rho)R(\tau, \rho) \\
* & * & -R(\tau, \rho) \end{bmatrix} < 0 \quad (1)$$

$$\Gamma_{33}(T_D, \rho) < 0 \quad (2)$$

$$P(T_D, \rho) - P(0, \eta) \succeq 0 \quad (3)$$

$$Q(T_D, \rho) - Q(0, \eta) \succeq 0 \quad (4)$$

$$R(T_D, \rho) - R(0, \eta) \succeq 0 \quad (5)$$

$$\kappa Q(T_D, \rho) - Q(0, \eta) \succeq 0 \quad (6)$$

$$\kappa R(T_D, \rho) - R(0, \eta) \succeq 0 \quad (7)$$
Employing Proposition 1, we deduce from (8) that
\[ \dot{V}(t,x_t) \leq \xi^T(t) \begin{bmatrix} \Gamma_{11}(\tau,\rho) & \Gamma_{12}(\tau,\rho) \\ * & \Gamma_{22}(\tau,\rho) \end{bmatrix} \xi(t) + h^2 \dot{x}^T(t)R(\tau,\rho)\dot{x}(t), \]

where
\[ \xi(t) = \begin{bmatrix} x^T(t) \\ x^T(t - d(t)) \end{bmatrix}^T. \]

Taking Schur compliment of (9) yields the LMI (1). This condition will ensure that Lyapunov-Krasovskii function is decreasing between two consecutive jumps of the parameter vector \( \rho \). Moreover, the change in Lyapunov-Krasovskii functional at the jumping instant \( t_k \) of the parameters’ trajectory is given as
\[ V(t_k^-, x_t, \rho) - V(t_k^+, x_t, \rho) = x^T(t)[P(T_D, \rho) - P(0, \eta)]x(t) \]
\[ + \int_{t_k^+}^{t_k^-} e^{-\kappa(s-t_k)}x^T(s)Q(T_D, \rho) - Q(0, \eta)x(s)ds \]
\[ + h \int_{t_k^-}^{t_k^+} \int_{t_k^-}^{t_k^+} e^{-\kappa(s-t_k)}x^T(s)R(T_D, \rho) - R(0, \eta)x(s)dsd\theta. \]

Since (3), (4), and (5) hold, the Lyapunov-Krasovskii functional cannot increase at the time instant \( t_k \) as
\[ V(t_k^-, x_t, \rho) - V(t_k^+, x_t, \theta) \geq 0. \]

Therefore, \( \Sigma_\eta \) is uniformly asymptotically stable. This concludes the proof. \( \square \)

**Remark 1** The Lyapunov-Krasovskii functional is parameter and clock-dependent during the holding time \( t \in [t_k, t_k + T_D] \). For \( t > t_k + T_D \), the matrix-valued functions \( P(\tau, \rho), Q(\tau, \rho), R(\tau, \rho) \) are chosen to be only parameter-dependent such that \( P(\tau, \rho) = P(T_D, \rho) \).

### 3 Stabilization with guaranteed \( \mathcal{L}_2 \)-performance by state-feedback

In this section, we aim at obtaining synthesis conditions for the clock-dependent gain-scheduled state-feedback controllers of the form
\[ \Sigma_c : \quad u(t) = \begin{cases} K(t - t_k, \rho(t_k))x(t), & t \in [t_k, t_k + T_D) \\ K(T_D, \rho(t_k))x(t), & t \in [t_k + T_D, t_{k+1}) \end{cases} , \]

where \( \rho \in \mathcal{P}_{\geq T_D}, \tau \in [0, T_D], \tau = \min\{t - t_k, T_D\} \), \( \kappa > 0 \), and \( \mu > 0 \). If there exist matrix-valued functions \( \hat{P} : [0, T_D] \times \mathcal{P} \rightarrow \mathbb{S}_{n \times n}, \hat{Q} : [0, T_D] \times \mathcal{P} \rightarrow \mathbb{S}_{n \times n}, \hat{R} : [0, T_D] \times \mathcal{P} \rightarrow \mathbb{S}_{n \times n}, \hat{U} : [0, T_D] \times \mathcal{P} \rightarrow \mathbb{R}^{u \times n}, \) and \( \bar{X} : \mathcal{P} \rightarrow \mathbb{R}^{n \times n} \) such that the LMIs (10)-(16) are feasible:
\[ \hat{P}(T_D, \rho) < 0 \]
\[ \hat{P}(T_D, \rho) - \hat{P}(0, \eta) \succeq 0 \]
\[ \hat{Q}(T_D, \rho) - \hat{Q}(0, \eta) \succeq 0 \]
\[ \hat{R}(T_D, \rho) - \hat{R}(0, \eta) \succeq 0 \]

Then
\[ \begin{cases} \hat{P}(t_k, \rho(t_k))x(t), & t \in [t_k, t_k + T_D) \\ \hat{K}(T_D, \rho(t_k))x(t), & t \in [t_k + T_D, t_{k+1}) \end{cases} \]
\[ \kappa \dot{Q}(T_D, \rho) - \ddot{Q}(0, \eta) \geq 0 \quad (15) \]
\[ \kappa \dot{R}(T_D, \rho) - \ddot{R}(0, \eta) \geq 0 \quad (16) \]

for all \( \tau \in [0, T_D] \) and all \( \rho, \eta \in \mathcal{P} \), where
\[ \dot{\Gamma}_{12}(\tau, \rho) = \dot{P}(\tau, \rho) + A(\rho)X(\rho) + B(\rho)\dot{U}(\tau, \rho) \]
\[ \dot{\Gamma}_{25}(\tau, \rho) = (C(\rho)X(\rho) + D(\rho)\dot{U}(\tau, \rho))^T \]
\[ \dot{\lambda}_{31}(\tau, \rho) = -(1 - \mu)Q(\tau, \rho)e^{-\kappa h} - e^{-\kappa h}\dot{R}(\tau, \rho) \]
\[ \dot{\gamma}(\tau, \rho) = \tilde{\gamma}(\tau, \rho) + \tilde{Q}(\tau, \rho) - e^{-\kappa h}\dot{R}(\tau, \rho) - \tilde{\gamma}(\tau, \rho) \]

then the closed-loop system \((\Sigma_s, \Sigma_r)\) with \( \rho \in \mathcal{P}_{\geq T_D} \) is uniformly asymptotically stable in the absence of disturbance \( w \) and the \( L_2 \)-gain of the map \( w \mapsto z \) is at most \( \gamma \).

Proof: From Theorem 1, it follows that
\[ \dot{V}(t, x_1, \rho) \leq \xi^T(t)\left[ \begin{array}{cc} \Gamma_{11}(\tau, \rho) & \Gamma_{12}(\tau, \rho) \\ * & \Gamma_{22}(\tau, \rho) \end{array} \right] \xi(t) + h^2 \dot{x}^T(t)R(\tau, \rho) \dot{x}(t), \]
where \( \Gamma_{11}(\tau, \rho), \Gamma_{12}(\tau, \rho), \) and \( \Gamma_{22}(\tau, \rho) \) are given in Theorem 1. To ensure the prescribed \( L_2 \) performance level of \( \gamma \), we further require
\[ \dot{V}(t, x_1, \rho) - \gamma^2 w^T(t)w(t) + z^2(t)z(t) < 0. \quad (17) \]

Substituting \( z(t) \) from \( \Sigma_s \) in (17) yields
\[ \zeta^T(t)\Psi(\tau, \rho)\zeta(t) + h^2 \dot{x}^T(t)R(\tau, \rho) \dot{x}(t) < 0, \quad (18) \]
where
\[ \zeta = [x(t) \ x(t - d(t)) \ w(t)] \]

and
\[ \Psi(\tau, \rho) = \left[ \begin{array}{ccc} \Psi_{11}(\tau, \rho) & \Psi_{12}(\tau, \rho) & P(\tau, \rho)E(\rho) + C^T(\rho)D(\rho) \\ * & \Psi_{22}(\tau, \rho) & C^T(\rho)D(\rho) \\ * & * & -\gamma^2 I + F^T(\rho)F(\rho) \end{array} \right] \]

with
\[ \Psi_{11}(\tau, \rho) = \text{Sym}[P(\tau, \rho)A(\rho)] + \dot{P}(\tau, \rho) + Q(\tau, \rho) - e^{-\kappa h}R(\tau, \rho) + C^T(\rho)C(\rho) \]
\[ \Psi_{12}(\tau, \rho) = P(\tau, \rho)A_d(\rho) + e^{-\kappa h}R(\tau, \rho) + C^T(\rho)C_d(\rho) \]
\[ \Psi_{22}(\tau, \rho) = -(1 - \mu)e^{-\kappa h}Q(\tau, \rho) - e^{-\kappa h}R(\tau, \rho) + C^T(\rho)C_d(\rho) \]

for all \( \rho \in \mathcal{P}_{\geq T_D} \) and all \( \tau \in [0, T_D] \).

Applying Schur complement twice on the LMI (18), we obtain
\[ \Lambda(\tau, \rho) = \left[ \begin{array}{ccc} \Lambda_{11}(\tau, \rho) & \Lambda_{12}(\tau, \rho) & P(\tau, \rho)E(\rho) + C^T(\rho)hA^T(\rho)R(\tau, \rho) \\ * & \Lambda_{22}(\tau, \rho) & 0 \\ * & * & -\gamma^2 I + F^T(\rho)hE^T(\rho)R(\tau, \rho) \end{array} \right] < 0, \quad (19) \]

where
\[ \Lambda_{11}(\tau, \rho) = \text{Sym}[P(\tau, \rho)A(\rho)] + \dot{P}(\tau, \rho) + Q(\tau, \rho) - e^{-\kappa h}R(\tau, \rho) \]
\[ \Lambda_{12}(\tau, \rho) = P(\tau, \rho)A_d(\rho) + e^{-\kappa h}R(\tau, \rho) \]
\[ \Lambda_{22}(\tau, \rho) = -(1 - \mu)e^{-\kappa h}Q(\tau, \rho) - e^{-\kappa h}R(\tau, \rho). \]

The structure of (19) is not adapted to the controller design due to the existence of the multiple product terms \( A(\rho)\dot{P}(\tau, \rho) \) and \( A(\rho)R(\tau, \rho) \) that prevent finding a linearizing change of variable even after congruence transformations. A relaxation approach based on the idea [Briat et al. 2010] is applied to remove these multiple product terms as follows. We will first prove that feasibility of (23) guarantees the feasibility of (19). To this aim, we let (23) be called \( \tilde{\Gamma}(\tau, \rho) \) with \( \gamma(\tau, \rho) = \dot{\gamma}(\tau, \rho) + Q(\tau, \rho) - e^{-\kappa h}R(\tau, \rho) - P(\tau, \rho) \), and decompose it as follows:
\[ \tilde{\Gamma}(\tau, \rho) = \tilde{\gamma}(\tau, \rho) |_{X=0} + U^TXV + V^TX^TU, \quad (20) \]
where \( U = [-I_n A(\rho) A_d(\rho) E(\rho) \mathcal{O}_n I_n I_n] \) and \( V = [I_n \mathcal{O}_{n \times 6n}] \). Then invoking the projection lemma (Gahinet and Apkarian 1994), the feasibility of \( \tilde{\Gamma}(\tau, \rho) \prec 0 \) implies the feasibility of the LMIs
\[ \mathcal{N}^{-1}_U \tilde{\gamma}(\tau, \rho) |_{X=0} < 0, \quad (21a) \]
\[ \mathcal{N}^{-1}_V \tilde{\Gamma}(\tau, \rho) |_{X=0} < 0, \quad (21b) \]
where \( \mathcal{N}_U \) and \( \mathcal{N}_V \) are basis of the null space of \( U \) and \( V \) and given as
\[ \mathcal{N}_U = \left[ \begin{array}{ccc} A(\rho) & A_d(\rho) & E(\rho) & I & I \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{array} \right], \quad \mathcal{N}_V = \left[ \begin{array}{ccc} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]. \quad (22) \]

Subsequently, the projection lemma yields two inequalities, where the first inequality (21a) yields (19) and (21b) yields \(-R(\tau, \rho) < 0 \) for all \( \rho \in \mathcal{P}_{\geq T_D} \) and all \( \tau \in [0, T_D] \). Note that this inequality is a relaxed form of the right bottom \( 1 \times 1 \) block of the inequality (19) and is always satisfied. Hence, the feasibility of (23) implies the feasibility of (19).

Finally, for the controller synthesis, substituting the closed-
4.1 Example 1: Illustration of Theorem 1

The application of our results to the consensus problem of Theorem 2. The third example demonstrates ample is to illustrate Theorem 1 whereas the second one.

We now provide three examples. The purpose of first ex-

4 Illustrations

into the inequality (23), and then performing a congru-

loop matrices

\[ A(\rho) \leftarrow A_d(\tau, \rho) := A(\rho) + B(\rho) K(\tau, \rho) \]

\[ C(\rho) \leftarrow C_d(\tau, \rho) := C(\rho) + D(\rho) K(\tau, \rho) \]

into the inequality (23), and then performing a congru-

cence transformation with respect to matrix diag\((X^{-1}(\rho), X^{-1}(\rho), X^{-1}(\rho), X^{-1}(\rho))\) along with the linear-

yield the LMI (10). This concludes the proof. \(\square\)

4 Illustrations

We now provide three examples. The purpose of first ex-

ample is to illustrate Theorem 1 whereas the second one illustrates Theorem 2. The third example demonstrates the application of our results to the consensus problem of multi-agent systems.

4.1 Example 1: Illustration of Theorem 1

Let us consider the following LPV system with time delay considered in [Pang and Zhang (2015)]:

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 \\
-2 - \rho & -1
\end{bmatrix} x(t) + \begin{bmatrix}
-1 & 0 \\
-1 - \rho & -1
\end{bmatrix} x(t - d(t)),
\]

(25)

where \(d(t) < 0.5\) and \(\dot{d}(t) \leq \mu < 0.5\). We solve the LMIs in Theorem 1 via gridding approach with fifty points in YALMIP, [Löfberg (2004)]. Since the LMIs in Theorem 1 yield intractable infinite-dimensional semi-definite programs, we relax them by using parameter-dependent polynomials of order 1. Choosing \(T_D = 1 \times 10^{-4}\) and \(\kappa = 0.005\), and applying Theorem 1, one can corroborate that the system (25) is stable for \(0 < \rho(t) \leq 0.76\).

4.2 Example 2: Illustration of Theorem 2

We now consider the system \(\Sigma_a\) with

\[
A(\rho) = \begin{bmatrix}
2 - \rho & -0.5 - 0.5\rho \\
-1 & -2 + 0.1\rho
\end{bmatrix}, \quad A_d(\rho) = \begin{bmatrix}
-1 & 0 \\
0.05 - 0.45\rho & -1
\end{bmatrix},
\]

\[
B(\rho) = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad E(\rho) = \begin{bmatrix}
0.01 \\
0.01
\end{bmatrix}, \quad C(\rho) = C_d(\rho) = \begin{bmatrix}
0 & 1
\end{bmatrix},
\]

\[
D(\rho) = F(\rho) = 0, \quad \mathcal{P} = [0, \rho].
\]

(26)

Choosing \(\bar{\rho} = 1, \bar{h} = 0.2, \mu = 0.9, \kappa = 1 \times 10^{-8}\), and \(T_d = 0.01\), and solving the LMIs in Theorem 2 yields the following controller gain

\[
K(\tau, \rho) = \frac{1}{\text{den}(\tau, \rho)} \begin{bmatrix}
K_1(\tau, \rho) & K_2(\tau, \rho)
\end{bmatrix}
\]

where

\[K_1(\tau, \rho) = 2031.1\rho + 2364.7\tau + 6.8217\rho \tau - 46431.0\]

\[K_2(\tau, \rho) = 10.565\rho \tau - 2181.9\tau - 2588.6\rho + 26748.0\]

\[\text{den}(\rho) = 47.187\rho + 1639.5.\]

We simulate both the open-loop system and the closed-loop system under a unit-step disturbance. It can be observed in Fig. 1 (top) that the open-loop system is unstable. At the bottom of the same figure, a random parameter trajectory is shown, where the time between two successive jumps
The upper bounds on delay and its derivative are chosen to be $h = 0.2$ and $\mu = 0.9$, respectively. For the system (27), we consider a switching topology represented by the following time-varying Laplacian matrix

$$\mathcal{L}(t) = \sigma(t)\mathcal{L}_1 + (1 - \sigma(t))\mathcal{L}_2,$$

where $\sigma(t)$ is a piecewise constant switching signal that takes value in $[0,1]$ with $T_D = 0.1$, and $\mathcal{L}_1, \mathcal{L}_2$ is a pair of symmetric commutative Laplacian matrices given by

$$\mathcal{L}_1 = \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix},$$

$$\mathcal{L}_2 = \begin{bmatrix} 1 & -0.25 & -0.25 & 0 & -0.25 & -0.25 \\ -0.25 & 1 & -0.25 & -0.25 & 0 & -0.25 \\ -0.25 & -0.25 & 1 & -0.25 & -0.25 & 0 \\ 0 & -0.25 & -0.25 & 1 & -0.25 & -0.25 \\ -0.25 & 0 & -0.25 & -0.25 & 1 & -0.25 \\ -0.25 & -0.25 & 0 & -0.25 & -0.25 & 1 \end{bmatrix}. \tag{31}$$

For the matrices $\mathcal{L}_1$ and $\mathcal{L}_2$, the maximum and minimum eigenvalues are $\min\{\lambda_1, \lambda_2\} = 0$, $\max\{\lambda_1, \lambda_2\} = 2$, respectively. By defining a new piecewise constant parameter $\rho \in \mathbb{P}_{\mathcal{S}_D}$ with $\mathbb{P} = [0, \bar{\rho}]$ where $\bar{\rho} = 2$ and using an approach similar to Corollary 2 of [Zakwan and Ahmed (2020)], the distributed system (27) can be modeled as the following LPV system with piecewise constant parameter:

$$\dot{x}(t) = A_\rho x(t) + A_d(\rho)x(t - d(t)) + B(\rho)u(t) + E(\rho)w(t) \tag{32}$$

$$z(t) = C_\rho x(t) + C_d(\rho)x(t - d(t)) + D(\rho)u(t) + F(\rho)w(t)$$

where $A_\rho = A$, $A_d(\rho) = \rho A_d$, $B(\rho) = B$, $E(\rho) = E$, $C_\rho = C$, $C_d(\rho) = \rho C_d$, $D(\rho) = D$, and $F(\rho) = F$. Our goal is to design a clock-dependent distributed controller $K(\tau, t) = I_N \otimes K_{\alpha}(\tau) + \mathcal{L}(t) \otimes K_{\alpha}(\tau)$. Using an approach similar to Corollary 2 of [Zakwan and Ahmed (2020)], the distributed controller $K(t)$ can be modeled as the clock-dependent gain-scheduled controller $\Sigma_c$ with

$$\dot{x}(t) = (I_N \otimes A)\bar{x}(t) + (\mathcal{L}(t) \otimes A_d)\bar{x}(t - d(t)) + (I_N \otimes B)\bar{u}(t) + (I_N \otimes \mathcal{E})\bar{w}(t)$$

$$\dot{\bar{z}}(t) = (I_N \otimes C)\bar{x}(t) + (\mathcal{L}(t) \otimes C_d)\bar{x}(t - d(t)) + (I_N \otimes D)\bar{u}(t) + (I_N \otimes \mathcal{F})\bar{w}(t)$$

where $\bar{x}(\theta) = \phi(\theta)$ for all $\theta \in [-h, 0]$, $\bar{x} \in \mathbb{R}^{Nn}$, $\bar{u} \in \mathbb{R}^{Nq}$, $\bar{w} \in \mathbb{R}^{Nm}$, $\bar{z} \in \mathbb{R}^{Nr}$, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad \mathcal{E} = 0.05 \times B, \quad C = C_d = I_2, \quad D = O_2, \quad F = 0.1I_2.$$
gain $K(\tau, \rho) = K_a(\tau) + \rho K_b(\tau)$. We Choose $\kappa = 0.01$ and then solve the LMIs in Theorem 2 using gridding approach with fifty points in YALMIP, Löfberg [2004]. Note that to inherit same parametrization for controller as of the plant, we used $\tilde{U}(\tau, \rho) = \tau \tilde{U}_a + \rho \tilde{U}_b$. Once LMIs are feasible, we compute the matrix-valued function $\tilde{U}(\tau, \rho)$ which results in the following controller gains

$$
K_a(\tau) = \begin{bmatrix} -11.102\tau & -11.573\tau \\ 17.813\tau & -14.79\tau \end{bmatrix} \\
K_b(\tau) = \begin{bmatrix} -0.5078 & -0.63061 \\ 0.87181 & -0.78384 \end{bmatrix}
$$

The distributed controller gain $K(t)$ can be constructed from $K_a(\tau) = K_a(\tau)$ and $K_b(\tau) = K_b(\tau)$.

We simulate the closed-loop system with a time-varying delay $\tau(t) = 0.09\sin(0.9t) + 0.1$ under a unit-step disturbance. Fig. 3 shows the evolution of the state trajectories reaching a consensus subject to a typical switching signal shown in Fig. 4. For the random parameter trajectory shown in Fig. 4, the time between two successive jumps is taken equal to $T_D = 0.1$ and the next value for the parameter is simply drawn from $U(0, 1)$. The consensus of multi-agent system depicted in Fig. 3 under a switching topology and time-varying delay reflects the efficacy of the approach.

5 Concluding Remarks

Dwell-time based stability conditions and synthesis conditions for gain-scheduled $L_2$ state-feedback controllers are derived for a class of LPV systems with piecewise parameters under time-varying delay. One of the main advantages of our approach is reduced conservatism with improved performance as compared to more generalized methods such as quadratic stability.

Several extensions of this work are possible; for instance, dynamic $L_2$ output feedback control, improving the bound on the rate of change of time delay, considering stochastic time-delays and stochastic piecewise constant parameter trajectories.

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