Generalized Stallings’ decomposition theorem for
pro-$p$ groups

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Abstract

The celebrated Stallings’ decomposition theorem states that the splitting of a finite index subgroup $H$ of a finitely generated group $G$ as an amalgamated free product or an HNN-extension over a finite group implies the same for $G$. We generalize the pro-$p$ version of it proved by Weigel and the second author in [25] to splittings over infinite pro-$p$ groups. This generalization does not have any abstract analogs. We also prove that generalized accessibility of finitely generated pro-$p$ groups is closed for commensurability.

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1 Introduction

In 1965, J-P. Serre showed that a torsion free virtually free pro-$p$ group must be free (cf. [19]). This motivated him to ask the question whether the same statement holds also in the discrete context. His question was answered positively some years later. In several papers (cf. [20], [22], [24]), J.R. Stallings and R.G. Swan showed that free groups are precisely the groups of cohomological dimension 1, and at the same time J-P. Serre himself showed that, in a torsion free group $G$, the cohomological dimension of a subgroup of finite index coincides with the cohomological dimension of $G$ (cf. [18]).

One of the major tools for obtaining this type of result - the theory of ends - provided deep results also in the presence of torsion. The first result to be mentioned is ‘Stallings’ decomposition theorem’ (cf. [23]). It generalizes the previously mentioned result as follows.

Theorem 1 (J.R. Stallings). Let $G$ be a finitely generated group containing a subgroup $H$ of finite index which splits as a non-trivial free amalgamated product or HNN-extension over a finite group. Then $G$ also splits either as a free product with amalgamation or as an HNN-extension over a finite group.

The pro-$p$ version of Theorem 1 was proved by Thomas Weigel and the second author in [25] generalizing the result of W.N. Herfort and the second author in [10], where it
was proved for virtually free pro-$p$ groups. The objective of this paper is to show that, in the category of pro-$p$ groups, splitting theorems hold beyond splitting over finite groups. More precisely, the result holds for a splitting over a general pro-$p$ group $K$ provided that the factors (resp. the base group) are indecomposable over any conjugate of any subgroup of $K$, i.e. do not split as a free amalgamated pro-$p$ product or pro-$p$ HNN-extension. Note that, in the pro-$p$ case, an amalgamated free pro-$p$ product or HNN-extension might be not proper (see Subsections 2.4 and 2.5), i.e. the free factors (resp. the base group) do not embed in general in the free amalgamated product (in the HNN-extension); the verb split will mean in this paper that these embeddings hold.

**Theorem 2.** Let $H = H_1 \ast_K H_2$ be a free amalgamated pro-$p$ product of finitely generated pro-$p$ groups $H_1, H_2$ that are indecomposable over any conjugate of any subgroup of $K$. Let $G$ be a pro-$p$ group having $H$ as an open normal subgroup. Then $G$ splits as a free amalgamated pro-$p$ product $G = G_1 \ast_K G_2$ such that $G_i \cap H$ are contained in some conjugate of $H_i$, $i = 1, 2$ and $L \cap H$ is contained in some conjugate of $K$.

Of course, if $H_1, H_2$ do not split as a free amalgamated pro-$p$ product or HNN-extension at all (such groups called FA-groups by Serre [17] and so we are going to use this term in the pro-$p$ context) then Theorem 2 holds independently of $K$.

The class of FA pro-$p$ groups is quite large and includes many important examples. All Fab pro-$p$ groups, i.e., pro-$p$ groups whose open subgroups have finite abelianization are FA pro-$p$ groups. Note that Fab pro-$p$ groups include all just-infinite pro-$p$ groups and play very important role in the class field theory (in particular have importance to the Fontaine-Mazur Conjecture, cf. [3]), p-adic representation theory [11] and include for example all open pro-$p$ subgroups of $SL_n(Z_p)$. The pro-$p$ completion of Grigorchuk, Gupta-Sidki groups and other branch groups are FA pro-$p$ groups as well as the Nottingham pro-$p$ group. Splittings as amalgamated free products of Fab analytic pro-$p$ groups occur naturally in the study of generalized RAAG pro-$p$ groups [13] Subsection 5.5 where it is also proved that an amalgamated free pro-$p$ product of uniformly powerful pro-$p$ groups is always proper. Thus Theorem 2 applies to these splittings of generalized RAAG pro-$p$ groups.

In fact, if $H_1, H_2$ are FA, we even do not need hypothesis of normality on $H$ for odd $p$.

**Corollary 3.** Let $p > 2$ and $H = H_1 \ast_K H_2$ be a free amalgamated pro-$p$ product of finitely generated FA pro-$p$ groups $H_1, H_2$. Let $G$ be a pro-$p$ group having $H$ as an open subgroup. Then $G$ splits as a free amalgamated pro-$p$ product $G = G_1 \ast_K G_2$ such that $G_i \cap H$ are contained in some conjugate of $H_i$, $i = 1, 2$ and $L \cap H$ is contained in some conjugate of $K$.

For an HNN-extension the corresponding statement admits two types of splittings.

**Theorem 4.** Let $H = HNN(H_1, K, t)$ be a pro-$p$ HNN-extension of a finitely generated pro-$p$ group $H_1$ that is indecomposable over any conjugate of any subgroup of $K$. Let $G$ be a pro-$p$ group having $H$ as an open normal subgroup. Then $G$ splits as a free amalgamated pro-$p$ product $G = G_1 \ast_K G_2$ or HNN-extension $G = (G_1, L, t)$ such that $G_i \cap H$, $i = 1, 2$ are contained in some conjugate of $H_1$, and $L \cap H$ is contained in some conjugate of $K$. 2
If $H_1$ is FA then for $p > 2$ we can drop the hypothesis of normality on $H$.

**Corollary 5.** Let $p > 2$ and $H = HNN(H_1, K, t)$ be a pro-$p$ HNN-extension of a finitely generated FA pro-$p$ group $H_1$. Let $G$ be a pro-$p$ group having $H$ as an open subgroup. Then $G$ splits as a pro-$p$ HNN-extension $G = (G_1, L, t)$ such that $G_1 \cap H$ is contained in some conjugate of $H_1$, and $L \cap H$ is contained in some conjugate of $K$.

Of course, in general, the factors of an amalgamated free pro-$p$ product $H = H_1 \ast_K H_2$ or the base group of a pro-$p$ HNN-extension $H = HNN(H_1, K, t)$ can split further, so to extend our results to a more general context we need to have some pro-$p$ version of JSJ-decomposition, i.e. $H$ should be the fundamental pro-$p$ group of a graph of pro-$p$ groups whose vertex groups do not split further over edge groups. Thus we need to exploit a pro-$p$ version of the Bass-Serre theory of groups acting on trees.

**Theorem 6.** Let $G$ be a finitely generated pro-$p$ group having an open normal subgroup $H$ acting on a pro-$p$ tree $T$. Suppose $\{H_v \mid v \in V(T)\}$ is $G$-invariant. Then $G$ is the fundamental group of a profinite graph of pro-$p$ groups such that each vertex group intersected with $H$ stabilizes a vertex of $T$. In particular $G$ splits as a non-trivial free amalgamated pro-$p$ product or a pro-$p$ HNN-extension.

If the stabilizers $H_v$ are FA, then the $G$-invariance $\{H_v \mid v \in V(T)\}$ is automatic; moreover, if the $H_v$ are Fab, then we can drop the normality assumption on $H$.

**Corollary 7.** Let $G$ be a finitely generated pro-$p$ group having an open subgroup $H$ acting on a pro-$p$ tree $T$ such that each stabilizer $H_v$ is Fab. Then $G$ is the fundamental group of a profinite graph of pro-$p$ groups such that each vertex group intersected with $H$ stabilizes a vertex of $T$. In particular $G$ splits as a non-trivial free amalgamated pro-$p$ product or a pro-$p$ HNN-extension.

Note also that Theorem 6 does not require necessarily existence of JSJ-decomposition or even accessibility (see Section 2 for definition); in other words, we do not require that $H \setminus T$ is finite. Indeed, G. Wilkes [26] constructed an example of a finitely generated inaccessible pro-$p$ group $G$ (that acts on a pro-$p$ tree with infinite $G \setminus T$), but our theorem holds for his example as well (see Section 6).

However, if we assume accessibility, we can tell more.

**Theorem 8.** Let $G$ be a finitely generated pro-$p$ group having an open normal subgroup $H$ that splits as the fundamental pro-$p$ group of a finite graph of finitely generated pro-$p$ groups $(\mathcal{H}, \Delta)$. Suppose conjugacy classes of vertex groups are $G$-invariant. Then $G$ is the fundamental group of a reduced finite graph of pro-$p$ $(\mathcal{G}, \Gamma)$ groups such that its vertex and edge groups intersected with $H$ are subgroups of vertex and edge groups of $H$ respectively. Moreover, $|E(\Gamma)| \leq |E(\Delta)|$.

Once more, if the vertex groups $\mathcal{H}(v)$ are Fab, then we can omit $G$-invariance and normality hypotheses.

**Corollary 9.** Let $G$ be a finitely generated pro-$p$ group having an open subgroup $H$ that splits as a finite graph of finitely generated pro-$p$ groups $(\mathcal{H}, \Delta)$. Suppose the vertex groups of $(\mathcal{H}, \Delta)$ are Fab. Then $G$ is the fundamental group of a reduced finite graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ such that its vertex and edge groups intersected with $H$ are subgroups of vertex and edge groups of $H$ respectively. Moreover, $|E(\Gamma)| \leq |E(\Delta)|$. 
Theorem 8 is a generalization of the pro-p version of Stallings’ decomposition theorem proved in [25], namely if in Theorem 8 we suppose that $H$ is a non-trivial free pro-p product, we obtain as a particular case the pro-p version of [25 Theorem 1.1]. Theorem 1.4 of [29] gives an example of a situation when Theorem 8 is applicable, namely if all vertex groups are Poincaré duality of dimension $n$ ($PD^n$ pro-p groups) and the edge groups have cohomological dimension $\leq n - 1$. Moreover, many 3-manifold groups admit a $p$-efficient JSJ-decomposition by [28 Theorem A] and if the vertex groups of them are arithmetic, then the pro-p version of Theorem [27 Proposition 6.23] combined with [9 Theorem 5.13] give the pro-p JSJ-decomposition of their pro-p completion that meets the hypothesis of Theorem 8.

The proofs of Theorem 8 and Corollaries 3, 5, and 9 are more subtle and require the following theorem that is of independent interest. Note that for an open subgroup $H$ of the fundamental pro-p group $G = \Pi_1(\mathcal{G}, \Gamma, v)$ of a finite graph of pro-p groups, the pro-p version of the Bass-Serre theorem for subgroups works, i.e. $H = \Pi_1(\mathcal{H}, \mathcal{H}\backslash S(G))$ in the standard manner (see Proposition 2.25).

**Theorem 10 (Limitation Theorem).** Let $G = \Pi_1(\mathcal{G}, \Gamma, v)$ be the fundamental pro-p group of a finite reduced graph of pro-p groups. Let $H$ be an open normal subgroup of $G$ and $H = \Pi_1(\mathcal{H}, \Delta, v')$ be a decomposition as the fundamental pro-p group of a reduced graph of pro-p groups $(\mathcal{H}, \Delta, v')$ obtained by a reduction process from $(\mathcal{H}, \mathcal{H}\backslash S(G))$. Then $|E(\Delta)| \geq |E(\Gamma)|$. Moreover, for $p > 2$ the inequality is strict unless $\Gamma = \Delta$.

Recall that two pro-p groups $G_1, G_2$ are *commensurable* if there exist $H_1$ open in $G_1$ and $H_2$ open in $G_2$ such that $H_1 \simeq H_2$. Theorem 10 allows us to prove that the accessibility of a pro-p group with respect to a family $\mathcal{F}$ of pro-p groups is preserved by commensurability. For accessible abstract groups such a result can be deduced from the Stallings splitting theorem; we are not aware of such a result for accessible groups with respect to a family of infinite groups in the abstract situation.

**Theorem 11.** Let $\mathcal{F}$ be a family of pro-p groups closed for commensurability. Let $G$ be a finitely generated pro-p group and $H$ an open subgroup of $G$. Then $G$ is $\mathcal{F}$-accessible if and only if $H$ is $\mathcal{F}$-accessible.

Note that the hypothesis of non-splitting in Theorems 2, 4 and 6 are essential. The pro-5 completion of the triangle group $G = \langle x, y \mid x^5, y^5, (xy)^5 \rangle$ for example contains the pro-5 completion $\hat{S}$ of a surface group $S$ as a subgroup of index 5. The group $\hat{S}$ is a free pro-5 product of free pro-5 groups with cyclic amalgamation, but $G$ does not split as a non-trivial amalgamated free pro-5 product. Indeed, if it does, i.e. if $G = G_1 \ast_H G_2$ then all torsion elements $x, y$ and $xy$ have to belong to some free factor up to conjugation, but then they belong to the normal closure of the same free factor, say $G_1^G$; it means that $G_1^G = G$ which is impossible, since $G/G_1^G \cong G_2/H^{G_2} \neq 1$.

This paper is organized as follows. In Section 2 we recall the elements of the pro-p version of the Bass-Serre theory of groups acting on trees that will be used through the text. Section 3 contains the proof of the Limitation Theorem. Section 4 starts with the proof of Theorem 6. Then with Limitation Theorem in hand, we prove Theorem 8. Theorems 2 and 4 then follow immediately, but their corollaries require some work. Section 5 deals with finitely generated pro-p accessible groups, where we prove Theorem...
In the last section we show that our Theorem 6 also works for Wilkes’ example of a finitely generated inaccessible pro-p group.

2 Main concepts of the pro-p version of the Bass-Serre theory

In this section we recall the necessary notions of the pro-p version of the Bass-Serre theory (see [14, 15, 16] for further details).

2.1 Pro-p trees

**Definition 2.1** (Profinite graph). A profinite graph is a profinite space \( \Gamma \) with a distinguished closed nonempty subset \( V(\Gamma) \) called the vertex set, \( E(\Gamma) = \Gamma - V(\Gamma) \) the edge set and two continuous maps \( d_0, d_1 : \Gamma \to V(\Gamma) \) whose restrictions to \( V(\Gamma) \) are the identity map \( id_{V(\Gamma)} \). We refer to \( d_0 \) and \( d_1 \) as the incidence maps of the profinite graph \( \Gamma \).

A morphism (\( q \)-morphism in the terminology of [15]) \( \alpha : \Gamma \to \Delta \) of profinite graphs is a continuous map with \( \alpha d_i = d_i \alpha \) for \( i = 0, 1 \). Note that this definition allows edges to be mapped to vertices. By [15, Proposition 2.1.4] every profinite graph \( \Gamma \) is an inverse limit of finite quotient graphs of \( \Gamma \).

A profinite graph is called connected if every finite quotient graph of it is connected. So a connected profinite graph is an inverse limit of finite connected graphs.

**Definition 2.2** ([1], Definition 3.4). If \( \Gamma \) is a connected finite graph, its pro-p fundamental group \( \pi_1(\Gamma, v) \) can be defined as the pro-p completion \( (\pi_{1}^\text{abs}(\Gamma, v))_\hat{p} \) of the abstract (usual) fundamental group \( \pi_{1}^\text{abs}(\Gamma, v) \). If \( \Gamma \) is a connected profinite graph and \( \Gamma = \varprojlim \Gamma_i \) its decomposition as inverse limit of finite graphs \( \Gamma_i \), then \( \pi_1(\Gamma, v) \) can be defined as the inverse limit \( \pi_1(\Gamma, v) = \varprojlim (\pi_1(\Gamma_i, v_i))_\hat{p} \), where \( v_i \) is the image of \( v \) in \( \Gamma_i \) (see [15, Proposition 3.3.2 (b)])). We say that \( \Gamma \) is a pro-p tree if \( \pi_1(\Gamma) = 1 \).

If \( v \) and \( w \) are elements of a pro-p tree \( T \), one denotes by \( [v, w] \) the smallest pro-p subtree of \( T \) containing \( v \) and \( w \).

If \( T \) is a pro-p tree, then we say that a pro-p group \( G \) acts on \( T \) if it acts continuously on \( T \) and the action commutes with \( d_0 \) and \( d_1 \). For \( t \in V(T) \cup E(T) \) we denote by \( G_t \) the stabilizer of \( t \) in \( G \). For a pro-p group \( G \) acting on a pro-p tree \( T \) let \( \tilde{G} \) denote the subgroup generated by all vertex stabilizers.

2.2 Fundamental pro-p group of a profinite graph of pro-p groups

**Definition 2.3** (Sheaf of pro-p groups). Let \( T \) be a profinite space. A sheaf of pro-p groups over \( T \) is a triple \( (\mathcal{G}, \pi, T) \), where \( \mathcal{G} \) is a profinite space and \( \pi : \mathcal{G} \to T \) is a continuous surjection satisfying the following conditions:

(a) For every \( t \in T \), the fiber \( \mathcal{G}(t) = \pi^{-1}(t) \) over \( t \) is a pro-p group (whose topology is induced by the topology of \( \mathcal{G} \) as the subspace topology);
(b) If we define 
\[ \mathcal{G}^2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \pi(g) = \pi(h)\}, \]
then the map \( \mu : \mathcal{G}^2 \to \mathcal{G} \) given by \( \mu(g, h) = gh^{-1} \) is continuous.

**Definition 2.4.** A morphism \( \alpha = (\alpha, \alpha') : (\mathcal{G}, \pi, T) \to (\mathcal{G}', \pi', T') \) of sheaves of pro-\( p \) groups consists of a pair of continuous maps \( \alpha : \mathcal{G} \to \mathcal{G}' \) and \( \alpha' : T \to T' \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\alpha} & \mathcal{G}' \\
\pi \downarrow & & \downarrow \pi' \\
T & \xrightarrow{\alpha'} & T'
\end{array}
\]

commutes and the restriction of \( \alpha \) to \( \mathcal{G}(t) \) is a homomorphism from \( \mathcal{G}(t) \) into \( \mathcal{G}'(\alpha'(t)) \), for each \( t \in T \).

**Definition 2.5** (Profinite graph of pro-\( p \) groups). Let \( \Gamma \) be a connected profinite graph with incidence maps \( d_0, d_1 : \Gamma \to V(\Gamma) \). A profinite graph of pro-\( p \) groups over \( \Gamma \) is a sheaf \( (\mathcal{G}, \pi, \Gamma) \) of pro-\( p \) groups over \( \Gamma \) together with two morphisms of sheaves \( (\hat{c}_i, d_i) : (\mathcal{G}, \pi, \Gamma) \to (\mathcal{G}_V, \pi, V(\Gamma)) \), where \( (\mathcal{G}_V, \pi, V(\Gamma)) \) is a restriction sheaf of \( (\mathcal{G}, \pi, \Gamma) \) and the restriction of \( \hat{c}_i \) to \( \mathcal{G}_V \) is the identity map \( id_{\mathcal{G}_V} \), \( i = 0, 1 \); in addition, we assume that the restriction of \( \hat{c}_i \) to each fiber \( \mathcal{G}(m) \) is an injection.

**Remark 2.6.** If \( \Gamma \) is finite then the notion of sheaf is not needed, since \( \mathcal{G} = \bigcup_{m \in \Gamma} \mathcal{G}(m) \) has the disjoint union topology. A finite graph of finite \( p \)-groups is just a usual graph of groups from the Bass-Serre theory.

**Definition 2.7.** A morphism of graphs of groups \( \varphi = (\nu, \nu') : (\mathcal{G}, \Gamma) \to (\mathcal{H}, \Delta) \) is a morphism of sheaves such that \( \nu \hat{c}_i = \hat{c}_i \nu \).

As was already mentioned in the introduction unlike the situation for abstract graphs of groups, the vertex groups of a profinite graph of pro-\( p \) groups \( (\mathcal{G}, \Gamma) \) do not always embed in its fundamental pro-\( p \) group \( \Pi_1(\mathcal{G}, \Gamma) \). This motivates the following definition:

**Definition 2.8** (Injective graph of pro-\( p \) groups, cf. Section 6.4 of [13]). We say that a graph of pro-\( p \) groups \( (\mathcal{G}, \Gamma) \) is injective if the restriction of \( \nu : \mathcal{G} \to \Pi_1(\mathcal{G}, \Gamma) \) to each fiber \( \mathcal{G}(m) \) \( (m \in \Gamma) \), is injective.

To achieve such embedding one has to replace the vertex and edge group with their images in \( \Pi_1(\mathcal{G}, \Gamma) \) (see [13] for details). Since we consider here splittings of pro-\( p \) groups as a graph of pro-\( p \) groups, we take a different approach developed in [1] to define \( \Pi_1(\mathcal{G}, \Gamma, \nu) \) with respect to a base point that gives the embedding above automatically.

Let \( I \) be a partially ordered set and \( \{(\mathcal{G}_i, \pi_i, \Gamma_i), \nu_{ij}\} \) an inverse system of finite graphs of finite \( p \)-groups. Then \( (\mathcal{G}, \pi, \Gamma) = \lim_{\to I} (\mathcal{G}_i, \pi_i, \Gamma_i) \) is a profinite graph of pro-\( p \) groups.

**Proposition 2.9** ([1], Proposition 2.15). Let \( (\mathcal{G}, \Gamma) \) be a profinite graph of pro-\( p \) groups. Then \( (\mathcal{G}, \Gamma) \) decomposes as an inverse limit \( (\mathcal{G}, \Gamma) = \lim_{\to I} (\mathcal{G}_i, \Gamma_i) \) of finite graphs of finite \( p \)-groups.
We need the following concepts to define the fundamental group of a graph of finite $p$-groups with a base point.

**Definition 2.10** (The group $F(\mathcal{G}, \Gamma)$ [17 Sect. I.5.1]). The path group $F(\mathcal{G}, \Gamma)$ is defined by $F(\mathcal{G}, \Gamma) = W_1/N$, where $W_1 = \langle \ast_{v \in V(\Gamma)} \mathcal{G}(v) \rangle \ast F(E(\Gamma))$, where $F(E(\Gamma))$ denotes the free group with basis $E(\Gamma)$ and $N$ is a normal subgroup of $W_1$ generated by the set \begin{equation*} \{c_0(x)^{-1} e_1(x) e^{-1} \mid x \in \mathcal{G}(e), e \in E(\Gamma)\}. \end{equation*}

**Definition 2.11** (Words of $F(\mathcal{G}, \Gamma)$ [17 Sect. I.5.1, Definition 9]). Let $c = v_0, e_0, \cdots, v_n, e_n$ be a path in $\Gamma$ with length $n = l(c)$ such that $v_j \in V(\Gamma), e_j \in E(\Gamma)$, $j = 0, \cdots, n$. A word of type $c$ in $F(\mathcal{G}, \Gamma)$ is a pair $(c, \mu)$ where $\mu = (g_0, \cdots, g_n)$ is a sequence of elements $g_j \in \mathcal{G}(v_j)$. The element $[c, \mu] = g_0, e_0, g_1, e_1, \cdots, e_n, g_n$ of $F(\mathcal{G}, \Gamma)$ is said to be associated with the word $(c, \mu)$.

**Definition 2.12** (The fundamental group of $(\mathcal{G}, \Gamma)$ [17 Sect. I.5.1, Definition 9(a)]). Let $v$ be a vertex of $\Gamma$. We define $\pi_1(\mathcal{G}, \Gamma, v)$ as the set of elements of $F(\mathcal{G}, \Gamma)$ of the form $[c, \mu]$, where $c$ is a path whose extremities both equal $v$. One sees immediately that $\pi_1(\mathcal{G}, \Gamma, v)$ is a subgroup of $F(\mathcal{G}, \Gamma)$, called the fundamental group of $(\mathcal{G}, \Gamma)$ at $v$. In particular, if $\mathcal{G}$ consists of trivial groups only then $\pi_1(\mathcal{G}, \Gamma, v)$ becomes a usual fundamental group of the graph $\Gamma$ and denoted by $\pi_1(\Gamma, v)$. It can be viewed of course as a subgroup that consists of set of elements of $F(\mathcal{G}, \Gamma)$ of the form $[c, \mu] = g_0, e_0, g_1, e_1, \cdots, e_n, g_n$, where $c$ is a path whose extremities both equal $v$ and $g_0 = 1 = g_1 = \cdots = g_n$. This way $\Gamma = \pi_1(\mathcal{G}, \Gamma, v)$ is a semidirect product $\pi_1(\mathcal{G}, \Gamma, v) = \langle \mathcal{G}(v) \mid v \in V(\Gamma) \rangle^{\mathcal{G}} \rtimes \pi_1(\Gamma, v)$.

**Proposition 2.13** ([1], Proposition 3.6). An inverse limit $(\mathcal{G}, \Gamma) = \varprojlim_i (\mathcal{G}_i, \Gamma_i)$ of finite abstract graphs of finite $p$-groups induces an inverse limit $\varprojlim_i (\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_p$ of the pro-$p$ completions of fundamental abstract groups $\pi_1(\mathcal{G}_i, \Gamma_i, v_i)$.

**Definition 2.14** ([1], Definition 3.7). Let $(\mathcal{G}, \Gamma)$ be a profinite graph of pro-$p$ groups and $(\mathcal{G}, \Gamma) = \varprojlim_i (\mathcal{G}_i, \Gamma_i)$ be the decomposition as the inverse limit of finite graphs of finite $p$-groups (see Proposition 2.29). Let $v$ be a vertex of $\Gamma$ and $v_i$ its image in $\Gamma_i$. The group $\varprojlim_i (\Pi^{ab}_1(\mathcal{G}_i, \Gamma_i, v_i))_p$ from Proposition 2.13 will be called the pro-$p$ fundamental group of the graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ at point $v$ and denoted by $\Pi_1(\mathcal{G}, \Gamma, v)$.

By [1] Theorem 3.9] our definition is equivalent to one in [15] assuming the vertex groups of $(\mathcal{G}, \Gamma)$ embed in $\Pi_1(\mathcal{G}, \Gamma)$, i.e., $(\mathcal{G}, \Gamma)$ is injective. We denote $\nu(\mathcal{G}(m))$ by $\Pi(m)$. Also note that, if $(\mathcal{G}, \Gamma)$ is a finite graph of finite $p$-groups, then $\Pi_1(\mathcal{G}, \Gamma, v) = (\Pi^{ab}_1(\mathcal{G}, \Gamma, v))_p$.

### 2.3 Reduced graph of pro-$p$ groups

**Definition 2.15** (Reduced graph of groups). A profinite graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ is said to be reduced if for every edge $e$, which is not a loop, neither $\tilde{e}_1 : \mathcal{G}(e) \to \mathcal{G}(d_1(e))$ nor $\tilde{e}_0 : \mathcal{G}(e) \to \mathcal{G}(d_0(e))$ is an isomorphism; we say that an edge $e$ is fictitious if it is not a loop and one of the edge maps $\tilde{e}_i$ is an isomorphism.

Any finite graph of groups can be transformed into a reduced finite graph of groups by collapsing fictitious edges using the following procedure. If $e$ is a fictitious edge, we can
remove \{e\} from the edge set of \( \Gamma \), and identify \( d_0(e) \) and \( d_1(e) \) to a new vertex \( y \). Let \( \Gamma' \) be the finite graph given by \( V(\Gamma') = y \cup V(\Gamma) \setminus \{d_0(e), d_1(e)\} \) and \( E(\Gamma') = E(\Gamma) \setminus \{e\} \), and let \((\mathcal{G}', \Gamma')\) denote the finite graph of groups based on \( \Gamma' \) given by \( \mathcal{G}'(y) = \mathcal{G}(d_1(e)) \) if \( e_0(e) \) is an isomorphism, and \( \mathcal{G}'(y) = \mathcal{G}(d_0(e)) \) if \( e_0(e) \) is not an isomorphism. This procedure can be continued until there are no fictitious edges. The resulting finite graph of groups \((\mathcal{G}, \Gamma)\) is reduced.

**Remark 2.16.** The reduction procedure described above does not change the fundamental group (as a group given by presentation), i.e. choosing a maximal subtree to contain the collapsing edge, the morphism \((\mathcal{G}, \Gamma) \to (\mathcal{G}', \Gamma')\) induces the identity map on the fundamental group with presentation given by eliminating redundant relations associated with fictitious edges that are just collapsed by reduction.

**Remark 2.17.** The reduction procedure can not be applied, however, if \( \Gamma \) is infinite, since removal of an edge results in a non-compact object. To obtain a reduced graph of pro-\( p \) groups in this case one has to reconstruct the profinite graph of pro-\( p \) groups following the procedure performed in the proof of Theorem 6.

The reduction procedure allows us to refine the main result of [10] as follows:

**Theorem 2.18.** Let \( G \) be a finitely generated pro-\( p \) group with a free open subgroup \( F \). Then \( G \) is the pro-\( p \) fundamental group of a reduced finite graph of finite pro-\( p \)-groups \((\mathcal{G}, \Gamma)\) with orders of vertex groups bounded by \([G : F]\). Moreover, if \( G = \Pi_1(\mathcal{G}', \Gamma') \) is another splitting as a reduced finite graph of finite pro-\( p \)-groups then \(|\Gamma| = |\Gamma'|, |V(\Gamma)| = |V(\Gamma')|, |E(\Gamma)| = |E(\Gamma')|\).

**Proof.** By [10] Theorem 1.1] \( G \) is the pro-\( p \) fundamental group of a finite graph of finite pro-\( p \)-groups \((\mathcal{G}, \Gamma)\) with orders of vertex groups bounded by \([G : F]\) and applying the reduction procedure we get the first statement. By [15] Theorem 1.1.2] maximal finite subgroups of \( G \) are exactly the vertex groups of \((\mathcal{G}, \Gamma)\) and \((\mathcal{G}', \Gamma')\) up to conjugation, so \(|V(\Gamma)| = |V(\Gamma')|\). Now by [4] Proposition 3.4\] \( G/\mathcal{G} = \pi_1(\Gamma) = \pi_1(\Gamma') \) is a free pro-\( p \) groups of rank \(|E(\Gamma) - |V(\Gamma)| + 1 = |E(\Gamma') - |V(\Gamma')| + 1 implying |E(\Gamma)| = |E(\Gamma')| \) and \(|\Gamma| = |\Gamma'|\). The proof is complete.

Two essential particular cases of the fundamental group of a finite graph of pro-\( p \) groups are amalgamated free pro-\( p \) products and pro-\( p \) HNN-extensions.

### 2.4 Free pro-\( p \) products with amalgamation

**Definition 2.19 ([14], Section 9.2).** Let \( G_1 \) and \( G_2 \) be pro-\( p \) groups and let \( f_i : H \to G_i \) \((i = 1, 2)\) be continuous monomorphisms of pro-\( p \) groups. An amalgamated free pro-\( p \) product of \( G_1 \) and \( G_2 \) with amalgamated subgroup \( H \) is defined to be a pushout of \( f_i \) \((i = 1, 2)\)

\[
\begin{array}{ccc}
H & \xrightarrow{f_1} & G_1 \\
\downarrow{f_2} & & \downarrow{\varphi_1} \\
G_2 & \xrightarrow{\varphi_2} & G
\end{array}
\]
in the category of pro-$p$ groups, i.e., a pro-$p$ group $G$ together with continuous homomorphisms $\varphi_i : G_i \to G$ ($i = 1, 2$) satisfying the following universal property: for any pair of continuous homomorphisms $\psi_i : G_i \to K$ ($i = 1, 2$) into a pro-$p$ group $K$ with $\psi_1 f_1 = \psi_2 f_2$, there exists a unique continuous homomorphism $\psi : G \to K$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
H & f_1 & G_1 \\
\downarrow f_2 & & \downarrow \varphi_1 \\
G_2 & \varphi_2 & G \\
& \psi & \downarrow \psi_1 \\
& & K \\
& \psi_2 & \\
\end{array}
\]

This amalgamated free pro-$p$ product, also referred to as free pro-$p$ product with amalgamation, is denoted by $G = G_1 \cup_H G_2$.

Following the abstract notion, we can consider $H$ as a common subgroup of $G_1$ and $G_2$ and think of $f_1$ and $f_2$ as inclusions. However, unlike the abstract case where the canonical homomorphisms $\varphi_i^{abs} : G_i \to G_1 \star_H G_2$ ($i = 1, 2$) are always monomorphisms (cf. Theorem I.1 in [17]), the corresponding maps in the category of pro-$p$ groups

$$\varphi_i : G_i \to G_1 \cup_H G_2$$

($i = 1, 2$) are not always injective. This motivates the next definition:

**Definition 2.20.** An amalgamated free pro-$p$ product $G = G_1 \cup_H G_2$ will be called proper if the canonical homomorphisms $\varphi_i$ ($i = 1, 2$) are monomorphisms. In that case we shall identify $G_1$, $G_2$ and $H$ with their images in $G$, when no possible confusion arises.

Throughout the paper all free pro-$p$ products with amalgamation will be proper.

The next example shows that an amalgamated free pro-$p$ product appears as a particular case of the fundamental pro-$p$ group of a profinite graph of pro-$p$ groups.

**Example 2.21** ([15], Example 6.2.3(d)). Let $G_1$, $G_2$ and $H$ be pro-$p$ groups and consider the following graph of groups:

\[
\begin{array}{ccc}
G_1 & & G_2 \\
& H & \\
\end{array}
\]

Then its fundamental pro-$p$ group will be $G = G_1 \cup_H G_2$. 
2.5 Pro-$p$ HNN-extensions

**Definition 2.22** ([14], Section 9.4). Let $H$ be a pro-$p$ group and let $f : A \to B$ be a continuous isomorphism between closed subgroups $A$, $B$ and $H$. A pro-$p$ HNN-extension of $H$ with associated groups $A$, $B$ consists of a pro-$p$ group $G = HNN(H, A, f)$, an element $t \in G$ called the stable letter, and a continuous homomorphism $\varphi : H \to G$ with $t(\varphi(a))^{-1} = \varphi f(a)$ and satisfying the following universal property: for any pro-$p$ group $K$, any $k \in K$ and any continuous homomorphism $\psi : H \to K$ satisfying $k(\psi(a))^{-1} = \psi f(a)$ for all $a \in A$, there is a continuous homomorphism $\omega : G \to K$ with $\omega(t) = k$ such that the diagram

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K$$

is commutative.

By construction, $G = HNN(H, A, f)$ arises as the pro-$p$ completion of the abstract HNN-extension $G^{\text{abs}} = HNN(H, A, f)$ (cf. [14] Proposition 9.4.1]).

In contrast with the abstract situation, the canonical homomorphism $\varphi : H \to G = HNN(H, A, f)$ is not always a monomorphism. When $\varphi$ is a monomorphism, we shall call $G = HNN(H, A, f)$ a proper pro-$p$ HNN-extension. Throughout the paper all pro-$p$ HNN-extensions will be proper.

The next example shows that a pro-$p$ HNN-extension appear as a very particular case of the pro-$p$ fundamental group of a profinite graph of pro-$p$ groups.

**Example 2.23** ([15], Example 6.2.3(e)). Let $G(v)$ and $G(e)$ be pro-$p$ groups and consider the following graph of groups:

Then its fundamental pro-$p$ group will be the pro-$p$ HNN-extension $G = HNN(G(v), \tilde{c}_0(G(e)), t, f)$ of $G(v)$, where $f : \tilde{c}_0(G(e)) \to \tilde{c}_1(G(e))$ is the isomorphism defined by $\tilde{c}_0(x) \mapsto \tilde{c}_1(x)$, for all $x \in G(e)$ and $t$ is the stable letter related to $e \in E(\Gamma)$.

2.6 Standard pro-$p$ tree

Next we shall describe that standard pro-$p$ tree on which $G = \Pi_1(G, \Gamma, v)$ acts. We shall assume that $\Gamma$ is finite, since we use it here only for this case in which the notation is much simpler; see [15] Section 6.3] for the general case.

**2.24. Standard (universal) pro-$p$ tree (cf. [15] Example 6.3.1).** Associated with the finite graph of pro-$p$ groups $(G, \Gamma)$ there is a corresponding standard pro-$p$ tree (or universal covering graph) $S = S(G) = \bigcup_{m \in \Gamma} G/\Pi(m)$ (cf. [30] Proposition 3.8). The vertices of $S$ are cosets of the form $g\Pi(v)$, with $v \in V(\Gamma)$ and $g \in G$; its edges are the
cosets of the form $g\Pi(e)$, with $e \in E(\Gamma)$; choosing a maximal subtree $D$ of $\Gamma$, the incidence maps of $S$ are given by the formulas:

$$d_0(g\Pi(e)) = g\Pi(d_0(e)); \quad d_1(g\Pi(e)) = gt_e\Pi(d_1(e)) \quad (e \in E(\Gamma), t_e = 1 \text{ if } e \in D).$$

There is a natural continuous action of $G$ on $S$ given by

$$g(g'\Pi(m)) = gg'\Pi(m),$$

where $g, g' \in G, m \in \Gamma$. Clearly $G \backslash S = \Gamma$. There is a standard connected transversal $s : \Gamma \rightarrow S$, given by $m \mapsto \Pi(m)$. Note that $s|_D$ is an isomorphism of graphs and the elements $t_e$ satisfy the equality $d_1(s(e)) = t_e s(d_1(e))$. Using the map $s$, we shall identify $\Pi(m)$ with the stabilizer $G_{s(m)}$ for $m \in \Gamma$:

$$\Pi(e) = G_{s(e)} = G_{d_0(s(e))} \cap G_{d_1(s(e))} = \Pi(d_0(e)) \cap t_e\Pi(d_1(e))t_e^{-1} \quad (1)$$

with $t_e = 1$ if $e \in D$. Remark also that since $\Gamma$ is finite, $E(S)$ is compact.

We shall often use the following result from [31] stating that for open subgroups of the fundamental pro-$p$ group of finite graph of pro-$p$ groups the subgroup theorem of the Bass-Serre theory works.

**Proposition 2.25.** ([31], Corollary 4.5 combined with 5.4) Let $G = \Pi_1(G, \Gamma)$ be the pro-$p$ fundamental group of a finite graph of pro-$p$ groups and $H$ an open subgroup of $G$. Let $s : H \backslash S(G) \rightarrow S(G)$ be a connected transversal. Then $H = \Pi_1(H, H \backslash S(G))$ with $H(m) = H_{s(m)}$ for each $m \in H \backslash S(G)$.

### 3 The proof of the Limitation Theorem for virtually free pro-$p$ groups

In this section we prove a special case of our main technical result, namely Theorem [10]. We will provide the additional elements to prove Theorem [10] in the next section.

**Theorem 3.1.** Let $G = \Pi_1(G, \Gamma, v)$ be the fundamental pro-$p$ group of a finite reduced graph of finite $p$-groups. Let $H$ be an open normal subgroup of $G$ and $H = \Pi_1(H, \Delta, v')$ be a decomposition as the fundamental pro-$p$ group of a reduced graph of finite $p$-groups. Then $|E(\Delta)| \geq |E(\Gamma)|$.

**Proof.** Using induction on the index $[G : H]$ we may assume that $[G : H] = p$. Consider the action of $G$ on its standard pro-$p$ tree $S(G)$ (see Section 2.6). Then $G/H$ acts naturally on the quotient graph $H \backslash S(G)$.

Denote by $V_1$ the set of fixed vertices by this action and by $V_2$ the moved ones. By Proposition 2.25, $H = \Pi_1(\mathcal{H}, H \backslash S(G))$ and $\mathcal{H}(w)$ is a conjugate of some vertex group $\mathcal{G}(v) \leq H$ for each $w \in V_2$. If $(\mathcal{H}, H \backslash S(G))$ is not reduced, we can apply the procedure described after Definition 2.15 to obtain the reduced graph of finite $p$-groups $(\mathcal{H}, \Delta)$. Since $G$ is virtually free pro-$p$ one can use then Theorem 2.18 to deduce that it suffices to prove the statement for $(\mathcal{H}, \Delta)$.
Identifying $V_1$ with its bijective image in $\Gamma$ we have that for each $v \in V_1$ the vertex group $\mathcal{H}(v) = \mathcal{G}(v) \cap H$ is of index $p$ in $\mathcal{G}(v)$. If $V_1 = \emptyset$ then all the edge and vertex groups of $(\mathcal{H}, H \setminus S(G))$ are conjugates of some edge and vertex groups of $(\mathcal{G}, \Gamma)$. It follows that $(\mathcal{H}, H \setminus S(G))$ is reduced, since $(\mathcal{G}, \Gamma)$ is by hypothesis. But $|H \setminus E(S(G))| = p|E(\Gamma)|$ and the result follows in this case.

Assume $V_1$ is non-empty. Denote by $\Gamma(V_i)$ the spanned graph of $V_i$, $i = 1, 2$ and $E_{12}$ the edges that connect vertices of $V_1$ to vertices of $V_2$. If $(\mathcal{H}, H \setminus S(G))$ is not reduced, then the fictitious edges can be only the moved ones that are in $E(\Gamma(V_1)) \cup E_{12}$. Moreover, only one such edge from its $G/H$-orbit can be collapsed. Indeed, after collapsing an edge $e \in E_{12}$ its vertex from $V_1$ disappears (and the rest of the vertices of $Ge$ are in $\Gamma(V_2)$); on the other hand, if $e \in E(\Gamma(V_1))$ then, after collapsing it, all the other edges from its orbit become loops. Here are the pictures for the case $p = 2$, where $g \in G/H$.

![Figure 1: Graph of groups $(\mathcal{H}, S(G)/H)$](image1)

![Figure 2: Reduced graph of groups $(\mathcal{H}, \Delta)$ assuming $e_1$ and $e_{12}$ are collapsed](image2)

Thus we can deduce that $E(\Delta) \geq |E(\Gamma(V_1))| + p|E(\Gamma(V_2))| + (p-1)|E_{12}| \geq |E(\Gamma(V_1))| + $
$|E(\Gamma(V_2))| + |E_{12}| \geq E(\Gamma)$. This finishes the induction and concludes the theorem. \qed

Remark 3.2. It follows from the first 3 lines of the proof of Theorem 3.1 that $|\Delta| \leq [G : H]|\Gamma|$, $|E(\Delta)| \leq [G : H]|E(\Gamma)|$ and $|V(\Delta)| \leq [G : H]|V(\Gamma)|$.

Remark 3.3. The proof also shows that for $p > 2$ one has $|E(\Gamma)| < |E(\Delta)|$ unless $\Gamma = \Delta$. This means that if $[G : H] > |E(\Delta)|$ then there exists intermediate subgroup $H \leq K < G$ such that $K = \Pi_1(K, \Gamma, v)$. For $p = 2$ the equality $|E(\Delta)| = |E(\Gamma)|$ can happen either in the case $\Delta = \Gamma$ or if for every edge $e \in E_{12}$ one has $[\mathcal{G}(d_0(e)) : \mathcal{G}(e)] = 2$.

In fact Remark 3.3 combined with [21, Lemma 3.2] gives a short proof of [12, Theorem B]. We refer to the latter paper for the definition of a pro-$p$ limit group.

Corollary 3.4. Let $G$ be a pro-$p$ limit group and $U$ be a proper open subgroup of $G$. Then the minimal number of generators $d(U)$ is strictly bigger than the minimal number of generators $d(G)$.

4 Finitely generated pro-$p$ groups virtually acting on trees

In this section we prove the main results stated in the introduction and deduce several consequences. The proof of Theorem 6 follows the proof of [4, Lemma 4.1] whose original idea appears in the proof of the main result of [25].

Proof of Theorem 6

Let $\mathcal{U}$ be the collection of open normal subgroups $U$ of $G$ contained in $H$. Denote by $\bar{U}$ the topological closure of $U$ generated by the $U$-stabilizers of the vertices of $T$, i.e.,

$$\bar{U} = cl(\langle U \cap H_v | v \in V(T) \rangle).$$

Then $\bar{U}$ is a closed normal subgroup of $G$. To see this it suffices to observe that $H^g_{\bar{v}} \leq \bar{H}$ for any $g \in G$ which is exactly our hypothesis.

Note that $\bar{U} \setminus T$ is a pro-$p$ tree and $H/\bar{U}$ acts on $\bar{U} \setminus T$ with $U/\bar{U}$ acting freely. Therefore $G/\bar{U}$ contains the open normal subgroup $U/\bar{U}$ which is finitely generated and free pro-$p$ (cf. [30, Theorem 2.6]). By Theorem 2.18 $G/\bar{U}$ is isomorphic to the pro-$p$ fundamental group $\Pi_1(\mathcal{G}_U, \Gamma_U, v_U)$ of a finite graph of finite $p$-groups.

Although neither the finite graph $\Gamma_U$ nor the finite graph of finite $p$-groups $\mathcal{G}_U$ are uniquely determined by $U$ (resp. $\bar{U}$), the index $U$ in the notation shall express that both these objects are depending on $U$. Using the procedure described after Definition 2.15 we have a morphism $\eta : (\mathcal{G}_U, \Gamma_U) \rightarrow (\mathcal{G}_U, \Gamma_U)$ to a reduced graph of groups.

For $V \subseteq U$ both open and normal in $G$ the decomposition $G/\bar{V} = \Pi_1(\mathcal{G}_V, \Gamma_V, \bar{v}_V)$ gives rise to a natural decomposition of $G/\bar{U}$ as the fundamental group $G/\bar{U} = \Pi_1(\mathcal{G}_{V,U}, \Gamma_V, \bar{v}_V)$ of a finite graph of finite $p$-groups $(\mathcal{G}_{V,U}, \Gamma_V)$, where the vertex and edge groups are $\mathcal{G}_{V,U}(x) = \mathcal{G}_V(x)\bar{U}/\bar{U}$, $x \in \Gamma_V$. Thus we have a morphism $\nu_{V,U} : (\mathcal{G}_V, \Gamma_V) \rightarrow (\mathcal{G}_{V,U}, \Gamma_V)$ of graphs of groups such that the induced homomorphism on the pro-$p$ fundamental groups
coincides with the canonical projection \( \varphi_{V, U} : G / \tilde{V} \to G / \tilde{U} \). Choose a reduction morphism \( \eta_U : (\mathcal{G}_{\tilde{V}, \tilde{U}}, \tilde{\Gamma}_V) \to (\mathcal{G}_{V, U}, \Gamma_U) \) to a reduced graph of groups \((\mathcal{G}_{V, U}, \Gamma_U)\) (it is not unique); it induces the identity map on the fundamental group \( G / \tilde{U} \) (see Remark 2.16) and so \( \eta_U \) induces the homomorphism \( \Pi_1(\mathcal{G}_{V, U}, \Gamma_U, \pi_U) \to \Pi_1(\mathcal{G}_{V, U}, \Gamma_U, \pi_U) \) on the pro-\( p \) fundamental groups that coincides with the canonical projection \( \varphi_{UV} : G / \tilde{V} \to G / \tilde{U} \).

Using the aforementioned reduction, we have that \( G / \tilde{U} = \Pi_1(\mathcal{G}_{U}, \Gamma_U, \pi_U) \). Then, by [23, Corollary 3.3], the number of isomorphism classes of finite reduced graphs of finite \( p \)-groups \((\mathcal{G}_{U}, \Gamma')\) which are based on \( \Gamma' \) satisfying \( G / \tilde{U} \cong \Pi_1(\mathcal{G}_{U}, \Gamma', v_0) \) is finite.

Suppose that \( \Omega_U \) is a set containing a copy of every such isomorphism class. Since \( G \) is finitely generated, we may choose \( V_i, i \in \mathbb{N} \), be a decreasing chain of open normal subgroups of \( G \) with \( V_0 = U \) and \( \bigcap_i V_i = \{1\} \). For \( X \subseteq \Omega_{V_i} \) define \( T(X) \) to be the set of all reduced graphs of groups in \( \Omega_{V_{i+1}} \) that can be obtained from graphs of groups of \( X \) by the procedure of reduction explained above (note that \( T \) is not a map). Define \( \Omega_1 = T(\Omega_{V_1}) \), \( \Omega_2 = T(\Omega_{V_2}) \), \ldots, \( \Omega_i = T^i(\Omega_{V_1}) \) and note that it is a non-empty subset of \( \Omega_U \) for every \( i \in \mathbb{N} \). Clearly \( \Omega_{i+1} \subseteq \Omega_i \) and since \( \Omega_U \) is finite there is an \( i_1 \in \mathbb{N} \) such that \( \Omega_j = \Omega_{i_1} \) for all \( j > i_1 \) and we denote this \( \Omega_0 \) by \( \Sigma_U \). Then \( T(\Sigma_{V_i}) = \Sigma_{V_{i+1}} \) and so we can construct an infinite sequence of graphs of groups \((\mathcal{G}_{V_j}, \Gamma_j) \in \Omega_{V_j}\) such that \( (\mathcal{G}_{V_{j+1}}, \Gamma_{j+1}) \in \Omega_{V_j} \) for all \( j \). This means that \((\mathcal{G}_{V_j}, \Gamma_j)\) can be reduced to \((\mathcal{G}_{V_{j-1}}, \Gamma_{j-1})\), i.e. this sequence \( \{(\mathcal{G}_{V_j}, \Gamma_j)\} \) is an inverse system of reduced graph of groups satisfying the required conditions. Therefore \( (\mathcal{G}, \Gamma) = \lim \mathcal{G}_{V_j}(\Gamma_j) \) is a reduced finite graph of finitely generated pro-\( p \) groups satisfying \( G \cong \Pi_1(\mathcal{G}, \Gamma, v) \).

Moreover, denoting by \( x_\Gamma \) the image of \( x \in \Gamma \) in \( \Gamma_V \) we have \( \mathcal{G}(x) = \lim \mathcal{G}_{V_j}(x_\Gamma) \) if \( x \) is either a vertex or an edge of \( \Gamma \). Since \( \mathcal{G}_{V_j}(x) \cap H / \tilde{V}_j \) fixes a vertex in \( \tilde{V}_j / T \) for each \( V_j \), and the set of fixed vertices of \( \mathcal{G}_{V_j}(x) \cap H / \tilde{V}_j \) is compact, the projective limit argument implies that \( \mathcal{G}(x) \cap H \) fixes a vertex of \( T \).

By [41, Theorem 4.2] a finitely generated pro-\( p \) group that acts on a pro-\( p \) tree splits as an amalgamated free pro-\( p \) product or pro-\( p \) HNN-extension over the stabilizer of an edge. Using the fact that the fundamental pro-\( p \) group of a graph of pro-\( p \) groups acts on its standard pro-\( p \) tree (see [15, Chapter 6]) we can deduce that \( G \) splits as non-trivial free amalgamated pro-\( p \) product or pro-\( p \) HNN-extension. This finishes the proof of the theorem.

\[ \square \]

Proof of Corollary 7

Let \( N = H_G \) be the normal core of \( H \) in \( G \). Since \( H_G \) is Fab, so is and \( N_e \) and therefore \( N^g_e \) must fix a vertex of \( T \). Hence hypotheses of Theorem 6 are satisfied for \( N \) and the result follows.

\[ \square \]

Corollary 4.1. \( |E(\Gamma)| \leq |E(H \setminus T)| \). Moreover, if \( p > 2 \) and \( \Gamma \neq H \setminus T \), then the inequality is strict.

Proof. It make sense to prove the statement assuming that \( H \setminus T \) is finite. By Theorem 3.3 combined with Theorem 2.18 \( |E(\Gamma_V)| \leq |E(H \setminus T)| \). Hence \( |E(\Gamma)| \leq |E(H \setminus T)| \) as

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required. Moreover, if \( \Gamma \neq H \setminus T \) then \(|E(\Gamma_{V_j})| < |E(H \setminus T)|\) by Remark 3.3 and hence \(|E(\Gamma)| < |E(H \setminus T)|\).

We are ready to prove Theorem 10. It will be crucial to deal with the generalized version of Stallings’ decomposition theorem and with accessibility in finitely generated pro-\( p \) groups.

**Proof of Theorem 10.** In this case we consider the action of \( H \) on the standard pro-\( p \) tree \( S(G) \) of \( G \) and so \( U \) from the proof of Theorem 6 is automatically normal in \( G \). Moreover, all \( \Gamma_j \) in the proof of Theorem 6 can be assumed to be equal to \( \Gamma \). Hence Corollary 4.1 valid for this case. This finishes the proof.

**Remark 4.2.** Corollary 4.1 shows that if \([G : H] > |E(H \setminus T)|\) then there exists an intermediate open subgroup \( H \leq K \leq G \) such that \( K \) splits as the fundamental pro-\( p \) group of a graph of pro-\( p \) groups with the same underlying graph. To illustrate this, let \( H = H_1 \sqcup L H_2 \) be a (proper) free amalgamated product of \( FA \) pro-\( p \) groups (for example open pro-\( p \) subgroups of \( SL_n(\mathbb{Z}_p) \)) and \( G \) be a finite extension of \( H \), i.e. \( G/H = P \) with \( P \) being finite \( p \)-group. Then \( G = G_1 \sqcup M G_2 \). Moreover, \( G/H = G_1/(G_1 \cap H) \sqcup M \cap H G_2/(G_2 \cap H) \) has to be fictitious, which means that, up to renumbering, \( P = G_1/(H \cap G_1) \) and \( G_2/(G_2 \cap H) = M/(M \cap H) \) since \( G/H \) is finite.

One of the obstacles to obtain the main structure result in the pro-\( p \) version of Bass-Serre theory is that a maximal subtree of a profinite graph \( \Gamma \) does not always exist. The next corollary shows that, for the finitely generated case, this difficulty can be surpassed.

**Corollary 4.3.** \( \Gamma \) possesses a closed maximal pro-\( p \) subtree.

**Proof.** By [17, Section 2.3, Corollary 2] the inverse image of a maximal subtree under a collapse is a maximal subtree. Hence we can choose maximal subtrees \( D_j \) of \( \Gamma_{V_j} \) from the proof of Theorem 6 such that they form the inverse subsystem. Then \( D = \varprojlim D_j \) is a pro-\( p \) tree with \( V(D) = V(\Gamma) \).

**Proof of Theorem 8.**

Consider the action of \( H \) on its standard pro-\( p \) tree \( S(H) \) (see Section 2.6) and apply Theorem 6 together with Corollary 4.1 to get a splitting of \( G \) as the fundamental group \( \Pi_1(G, \Gamma, v) \) of a finite graph of pro-\( p \) groups, where the vertex groups intersected with \( H \) are subgroups of vertex groups of \( H \). Now observe that Corollary 4.1 or Theorem 6 allows us to assume that all \( \Gamma_{V_j} = \Gamma \) in the proof of Theorem 6 and applying it we get the last statement of the theorem.

We are left with the statement about edge stabilizers. Assume w.l.o.g. that \((H, \Delta)\) is reduced. Then the vertex stabilizers of \( \tilde{V}_j \setminus S(H) \) in \( H/\tilde{V}_j \) are exactly maximal finite subgroups of \( H/\tilde{V}_j \). This implies, in particular, that, for \( V_{j+1} \leq V_j \), the maximal finite subgroups of \( H/\tilde{V}_{j+1} \) map onto maximal finite subgroups of \( H/\tilde{V}_j \). It induces the bijection of the conjugacy classes of the maximal finite subgroups of \( H/\tilde{V}_{j+1} \) and \( H/\tilde{V}_j \). Then, if \( e \) is an edge of \( \Gamma \), starting from some \( j \), one has \( G_{V_j}(e) = G_{V_j}^e(v) \cap G_{V_j}(w)^q \), where \( v, w \) are the extremity vertices of \( e \) and are maximal finite subgroups of \( G/\tilde{V}_j \).
Let $H_{V_j} = \Pi_1(\mathcal{H}_j, \Delta_j, \nu'_j)$ be the splitting of $H_{V_j}$ as an open subgroup of $\Pi_1(\mathcal{G}_V, \Gamma_j, \nu_j)$ (see Proposition 2.25). Then $H_{V_j} \cap \mathcal{G}_V(e) \leq H_\nu(e', h)$ for some $e' \in E(\Delta_j), \ h \in H_{V_j}$. It follows that $H_{V_j} \cap \mathcal{G}_V(e)$ is contained in the intersection of at least two distinct maximal finite subgroups of $H_{V_j}$ (some vertex stabilizers of $\tilde{H}_{V_j} \setminus S(H)$). Hence $\mathcal{G}(e) \cap H$ is contained in the intersection of at least two distinct vertex stabilizers of $\mathcal{T}$ and so fixes an edge of $\mathcal{T}$. This finishes the proof. 

□

Proof of Corollary 9
Let $N = H_G$ be the normal core of $H$ in $G$. Since $\mathcal{H}(v), \ v \in V(\Delta)$ is Fab, $N_v$ must fix a vertex of the standard pro-$p$ tree $S(G)$ (see Section 2.6) and so its conjugacy class is $G$-invariant. Hence the result follows from Theorem 8.

□

Proof of Theorem 2
By Theorem 8, $G$ is the fundamental group of a graph of pro-$p$ groups with one edge only (cf. Example 2.21). However, $H/\tilde{H}$ is trivial in this case and since $\tilde{H} \leq \tilde{G}, G/\tilde{G}$ is finite. Hence $G$ cannot be an HNN-extension.

□

Proof of Corollary 3 We use induction on $[G : H]$. The base of induction $[G : H] = p$ follows from Theorem 2 as $H$ is normal in $G$ in this case. Suppose $[G : H] > p$ and $H < N \leq G$ with $[N : H] = p$. Then, by Theorem 2, $N = N_1 \cup_{M_1} N_2$ and $N_1 \cap H, N_2 \cap H$ are conjugate into $H_1$ or $H_2$. To apply the induction step, we just need to show that $N_1$ and $N_2$ are FA.

Let $S(N)$ be a standard pro-$p$ tree on which $N$ acts. Since $H_1, H_2$ are FA, each of them fix a vertex in $S(N)$ and hence is conjugate into $(N_1 \cap H)$ or $(N_2 \cap H)$. Suppose w.l.o.g $H_1 \leq N_1 \cap H$. Since $H_1$ is not conjugate into $H_2$ we deduced that $N_1 \cap H$ is conjugate into $H_1$ and hence is equal to $H_1$. Then $H_1$ has at most index $p$ in $N_1$ and so $N_1$ can not split, because otherwise $H_1$ would split by Proposition 2.25 and this splitting is non-trivial by Theorem 10 so $N_1$ is FA by [1, Theorem 4.2].

If $N_2 \cap H$ is conjugate into $H_2$ then by the same argument one deduces that $N_2$ is FA.

We claim that $N_2 \cap H$ is not conjugate into $H_1$. Suppose it is, so $H$ is contained in the normal closure $G_1^G$ in $G$ and, by Proposition 2.25, $G_1^G$ splits as a fundamental graph of groups $(G_1, \Delta)$ that we may assume to be reduced (see Remark 2.16). Moreover, by Theorem 10, $E(\Delta) = 1$ only if $p = 2$. Since $p = 2$ case is excluded by the hypothesis, the proof is complete.

□

Theorem 4 follows by direct application of Theorem 5 and Corollary 4.1 (cf. Example 2.23).

Proof of Corollary 5 By Corollary 9 we just need to prove the last statement of the corollary.

We use induction on $[G : H]$. The base of induction $[G : H] = p$ follows from Theorem 4 as $H$ is normal in $G$ in this case. Suppose $[G : H] > p$ and $H < N \leq G$ with
\[N : H] = p.\] Then, by Theorem 2 either \(N = N_1 \sqcup N_2\) with \(N_1 \cap H, N_2 \cap H\) conjugate into \(H_1\) or \(N = H/N(N_1, M, t)\) with \(N_1 \cap H\) is conjugate into \(H_1\). But for \(p > 2\) the first case \(N = N_1 \sqcup N_2\) does not occur by Theorem 10. Thus to apply the induction step, we just need to show that \(N_1\) is FA.

Let \(S(N)\) be a standard pro-\(p\) tree on which \(N\) acts. Since \(H_1\) is FA, it fixes a vertex in \(S(N)\) and hence is conjugate into \((N_1 \cap H)\). Hence \(H_1\) and \(N_1 \cap H\) are conjugate so, w.l.o.g, we may assume that with \(N_1 \cap H = H_1\). Then \(H_1\) has index at most \(p\) in \(N_1\). Then \(N_1\) can not split, because then \(H_1\) would split non-trivially by Proposition 2.25 and Theorem 10 contradicting the hypothesis, so \(N_1\) is FA by [3, Theorem 4.2].

\[\square\]

5 Generalized accessible pro-\(p\) groups

Abstract accessibility was studied in a series of papers by M.J. Dunwoody (cf. [5], [6], [7], [8]), where he proved that every finitely presented group is accessible, but not every finitely generated group over an arbitrary family of groups. In fact, he presented an example of a finitely generated inaccessible group. Generalized accessible groups were studied by Bestvina and Feighn ([2]). The pro-\(p\) version of accessibility was introduced by G. Wilkes in [20], and Z. Chatzidakis and the second author generalized this definition as follows:

**Definition 5.1** (Generalized accessible pro-\(p\) group, cf. Definition 5.1 of [1]). Let \(\mathcal{F}\) be a family of pro-\(p\) groups. We say that a pro-\(p\) group \(H\) is \(\mathcal{F}\)-accessible if any splitting of \(H\) as the fundamental group of a reduced finite graph \((\mathcal{G}, \Gamma)\) of pro-\(p\) groups such that the edge groups are in \(\mathcal{F}\) has bound on \(\Gamma\).

Now we prove Theorem 11

**Proof of Theorem 11.** Using the obvious induction on \([G : H]\) we may assume that \([G : H] = p\) and so \(H\) is normal in \(G\).

Suppose \(H\) is \(\mathcal{F}\)-accessible and \(G\) is not. Then for any \(n \in \mathbb{N}\) there exists a finite reduced graph of pro-\(p\) groups \((\mathcal{G}, \Gamma)\) such that \(G = \Pi_{1}(G, \Gamma, v)\) with edge groups in \(\mathcal{F}\) and \(|E(\Gamma)| > n\). It follows from the proof of Theorem 6 that there exists an open normal subgroup \(U\) of \(G\) contained in \(H\) such that \(G/\tilde{U} = (G_U, \Gamma)\) is the fundamental group of a reduced quotient graph of finite \(p\)-groups of \((\mathcal{G}_U, \Gamma)\) over the same underlying graph \(\Gamma\). Then, by Theorem 10, \(H/\tilde{U} = \Pi_1(\mathcal{H}_{U}, \Delta_U, v_U)\) is the fundamental group of a finite reduced graph of pro-\(p\) groups with \(|E(\Delta_U)| > n\). Then \(|E(\Delta_U)| > n\) for each open normal \(V\) contained in \(U\). By the proof of Theorem 3, the set \(\{(\mathcal{H}_V, \Delta_V) \mid V \unlhd U\}\) contains a subset that form a surjective inverse system \(\{\mathcal{H}_{V_j}, \Delta_{V_j}\}\) with \((\mathcal{H}, \Delta) = \varprojlim(\mathcal{H}_{V_j}, \Delta_{V_j})\) being the reduced graph of pro-\(p\) groups such that and \(H = \Pi_1(\mathcal{H}, \Delta)\). Moreover, it is proved in Theorem 8 that edge groups of \((\mathcal{H}, \Delta)\) are virtually \(\mathcal{F}\). Therefore, \(|E(\Delta)| > n\) for an arbitrary chosen \(n \in \mathbb{N}\) contradicting \(\mathcal{F}\)-accessibility of \(H\).

Suppose now \(G\) is \(\mathcal{F}\)-accessible with accessibility number \(m\) and \(H\) is not. Then for any \(n \in \mathbb{N}\) there exists a finite reduced graph of pro-\(p\) groups \((\mathcal{H}, \Delta)\) such that \(H = \Pi_1(\mathcal{H}, \Delta)\) with edge groups in \(\mathcal{F}\) and \(|E(\Delta)| > n\). Again it follows from the proof of
Theorem 6 that there exists an open normal subgroup \( U \) of \( G \) contained in \( H \) such that \( H_U = (\mathcal{H}_U, \Delta, v) \) is the fundamental group of a reduced quotient graph of finite \( p \) groups with the same underlying graph \( \Delta \). On the other hand, the graph of groups \((\tilde{G}_U, \Gamma_U, \nu_U)\) with \( \tilde{G}_U = \Pi_1(\tilde{G}_U, \tilde{\Gamma}_U, \tilde{\nu}_U) \) constructed in the proof of Theorem 6 must have at most \( m \) edges and therefore by Theorem 2.18 and Remark 3.2 \( \Delta \) has at most \( m[G : H] \) edges. This contradiction completes the proof of the theorem.

6 Adaptation of Wilkes’ example

In this section we show that our Theorem 6 also works for the inaccessible finitely generated group presented by Wilkes in [26, Section 4.2].

**Example 6.1.** First define the map \( \mu_n : \{0, \ldots, p^{n+1} - 1\} \rightarrow \{0, \ldots, p^n - 1\} \) by sending an integer to its remainder modulo \( p^n \). Define \( H_n = \mathbb{F}_p[\{0, \ldots, p^n - 1\}] \) to be the \( \mathbb{F}_p \)-vector space with basis \( \{h_0, \ldots, h_{p^n-1}\} \). There are inclusions \( H_n \subseteq H_{n+1} \) given by inclusions of bases, and retractions \( \eta_n : H_{n+1} \rightarrow H_n \) defined by \( h_k \mapsto h_{\mu_n(k)} \). Note also that there is a natural action of \( \mathbb{Z}/p^n\mathbb{Z} \) on \( H_n \) given by cyclic permutation of the basis elements, and that these actions are compatible with the retractions \( \eta_n \). The inverse limit of the \( H_n \) along these retractions is the completed group ring \( H_\infty = \mathbb{F}_p[[\mathbb{Z}_p]] \) with multiplication ignored. The continuous action of \( \mathbb{Z}_p \) on the given basis of \( H_\infty \) allows to form a sort of a pro-\( p \) wreath product \( H_\omega = \mathbb{F}_p[[\mathbb{Z}_p]] \rtimes \mathbb{Z}_p = \varprojlim (H_n \rtimes \mathbb{Z}/p^n) \) which is a pro-\( p \) group into which \( H_\omega \) embeds.

Next set \( K_n = \mathbb{F}_p \times H_n = \langle k_n \rangle \times H_n \). Set \( G_1 = K_1 \times \mathbb{F}_p \). For \( n > 1 \), let \( G_n \) be a finite \( p \)-group with presentation \( G_n = \langle k_{n-1}, k_n, h_0, \ldots, h_{p^n-1} \mid k_i^p = h_i^p = 1, h_i \leftrightarrow h_j, k_{n-1} \leftrightarrow k_n \rangle \) for all \( i \neq p^{n-1}, k_n = [k_{n-1}, h_{p^n-1}] \) central\rangle where \( \leftrightarrow \) denotes the relation ‘commutes with’.

The choice of generator names describes maps \( H_n \rightarrow G_n, K_{n-1} \rightarrow G_n, \) and \( K_n \rightarrow G_n \). One may easily see that all three of these maps are injections. Define a retraction map

\[ \rho_n : G_n \rightarrow K_{n-1} \]

by killing \( k_n \) and by sending \( h_k \rightarrow h_{\mu_{n-1}(k)} \). Note that \( \rho_n \) is compatible with \( \eta_n : H_n \rightarrow H_{n-1} \) that is, there is a commuting diagram

\[
\begin{array}{ccc}
K_{n-1} & \xrightarrow{\rho_n} & G_n \\
\uparrow & & \uparrow \\
H_{n-1} & \xrightarrow{\eta_n} & H_n
\end{array}
\]

Define \( \Pi_1(G_m, \Gamma_m, \nu_m) \) to be the pro-\( p \) fundamental group of the following graph of groups:

\[
\begin{array}{cccccccc}
& G_1 & \xrightarrow{K_1} & G_2 & \cdots & G_{m-1} & \xrightarrow{K_{m-1}} & G_m \\
& \downarrow & & \uparrow & & \downarrow & & \uparrow
\end{array}
\]

Note that the retraction \( \rho_n : G_n \rightarrow K_{n-1} \) induces the retraction \( P_{m+1} \rightarrow P_m \) represented by the collapse the last right edge of the picture.
Then $P = \lim_{m \to \infty} \Pi_1(G_m, \Gamma_m, v_m)$ is the fundamental group of the following profinite graph of pro-$p$ groups

\[
\begin{align*}
&G_1 \quad K_1 \quad G_2 \quad K_2 \quad G_3 \quad \cdots \quad G_\infty
\end{align*}
\]

where the vertex at infinity is a one point compactification of the edge set of the graph and so does not have an incident edge to it; thus the edge set is not compact. The vertex group $G_\infty$ of the vertex at infinity is $G_\infty = K_\infty = \lim_{i \to \infty} K_i = H_\infty$. Let $J = P \cup H_\infty$. Then $J$ is the fundamental group of the following profinite graph of groups

\[
\begin{align*}
&G_1 \quad K_1 \quad G_2 \quad \cdots \quad H_\infty \quad H_\infty \quad H_\omega
\end{align*}
\]

By [26, Section 4.3], this graph of pro-$p$ groups is injective and by [26, Section 4.4] $J = \langle G_1, H_\omega \rangle$. Since $G_1$ is finite and $H_\omega$ is 2-generated, $J$ is finitely generated (in fact for $p = 2$ the group $J$ is 3-generated). Collapsing the right edge we shall get the reduced graph of pro-$p$ groups since no vertex group equals to an edge group of an incident edge. Note that the latter graph of groups has a unique vertex $\infty$ whose vertex group is infinite and isomorphic to $F_p \wr \mathbb{Z}_p$ which does not split over a finite $p$-group, so satisfies the hypotheses of Theorem 6.

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