Generalized Bäcklund–Darboux transformations for Coxeter–Toda flows from a cluster algebra perspective

by

Michael Gekhtman
University of Notre Dame
Notre Dame, IN, U.S.A.

Michael Shapiro
Michigan State University
East Lansing, MI, U.S.A.

Alek Vainshtein
University of Haifa
Haifa, Mount Carmel, Israel

1. Introduction

This is the third in the series of papers in which we investigate Poisson geometry of directed networks. In [21] and [22], we studied Poisson structures associated with weighted directed networks in a disk and in an annulus. The study was motivated in part by Poisson properties of cluster algebras. In fact, it was shown in [21] that if a universal Poisson bracket on the space of edge weights of a directed network in a disk satisfy an analogue of the Poisson–Lie property with respect to concatenation, then the Poisson structure induced by this bracket on the corresponding Grassmannian is compatible with the cluster algebra structure in the homogeneous coordinate ring of the Grassmannian. In this paper we deal with an example that ties together objects and concepts from the theory of cluster algebras and directed networks with the theory of integrable systems.

Integrable systems in question are the Toda flows on $\text{GL}_n$. These are commuting Hamiltonian flows generated by conjugation-invariant functions on $\text{GL}_n$ with respect to the standard Poisson–Lie structure. Toda flows (also known as characteristic Hamiltonian systems [30]) are defined for an arbitrary standard semisimple Poisson–Lie group, but we will concentrate on the $\text{GL}_n$ case, where as a maximal algebraically independent family of conjugation-invariant functions one can choose $F_k: \text{GL}_n \ni X \mapsto (1/k) \text{tr} X^k$, $k=1,...,n-1$. The equation of motion generated by $F_k$ has the Lax form

$$\frac{d}{dt}X = \left[ X, - \frac{1}{2} (\pi_+ (X^k) - \pi_- (X^k)) \right], \quad (1.1)$$
where $\pi_+(A)$ and $\pi_-(A)$ denote strictly upper and lower parts of a matrix $A$.

Any double Bruhat cell $G^{u,v}$, $u,v \in S_n$, is a regular Poisson submanifold in $GL_n$ invariant under the right and left multiplication by elements of the maximal torus (the subgroup of diagonal matrices) $H \subset GL_n$. In particular, $G^{u,v}$ is invariant under the conjugation by elements of $H$. The standard Poisson–Lie structure is also invariant under the conjugation action of $H$ on $GL_n$. This means that Toda flows defined by (1.1) induce commuting Hamiltonian flows on $G^{u,v}/H$, where $H$ acts on $G^{u,v}$ by conjugation.

In the case when $v = u^{-1} = (n \ 1 \ 2 \ ... \ n-1)$, $G^{u,v}$ consists of tridiagonal matrices with non-zero off-diagonal entries, and $G^{u,v}/H$ can be conveniently described as the set Jac of Jacobi matrices of the form

$$L = \begin{pmatrix} b_1 & 1 & 0 & ... & 0 \\ a_1 & b_2 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & a_{n-2} & b_{n-1} & 1 \\ 0 & ... & 0 & a_{n-1} & b_n \end{pmatrix}, \quad a_1 ... a_{n-1} \neq 0, \quad \det L \neq 0. \quad (1.2)$$

The Lax equations (1.1) then become the equations of the finite non-periodic Toda hierarchy

$$\frac{d}{dt} L = [L, \pi_-(L^k)],$$

the first of which, corresponding to $k=1$, is the celebrated Toda lattice

$$\frac{d}{dt} a_j = a_j(b_{j+1} - b_j), \quad j = 1, ..., n-1,$$

$$\frac{d}{dt} b_j = (a_j - a_{j-1}), \quad j = 1, ..., n,$$

with the boundary conditions $a_0 = a_n = 0$. Recall that $\det L$ is a Casimir function for the standard Poisson–Lie bracket. The level sets of the function $\det L$ foliate Jac into $2(n-1)$-dimensional symplectic manifolds, and the Toda hierarchy defines a completely integrable system on every symplectic leaf. Note that although Toda flows on an arbitrary double Bruhat cell $G^{u,v}$ can be exactly solved via the so-called factorization method (see, e.g. [31]), in most cases the dimension of symplectic leaves in $G^{u,v}/H$ exceeds $2(n-1)$, which means that conjugation-invariant functions do not form a Poisson commuting family rich enough to ensure Liouville complete integrability.

An important role in the study of Toda flows is played by the Weyl function

$$m(\lambda) = m(\lambda; X) = ((\lambda 1 - X)^{-1} e_1, e_1) = \frac{q(\lambda)}{p(\lambda)}, \quad (1.3)$$