POLYNOMIALLY GROWING HARMONIC FUNCTIONS ON CONNECTED GROUPS

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Abstract. We study the connection between the dimension of certain spaces of harmonic functions on a group and its geometric and algebraic properties.

Our main result shows that (for sufficiently “nice” random walk measures) a connected, compactly generated, locally compact group has polynomial volume growth if and only if the space of linear growth harmonic functions has finite dimension.

This characterization is interesting in light of the fact that Gromov’s theorem regarding finitely generated groups of polynomial growth does not have an analog in the connected case. That is, there are examples of connected groups of polynomial growth that are not nilpotent by compact. Also, the analogous result for the discrete case has only been established for solvable groups, and is still open for general finitely generated groups.

1. Introduction

1.1. Background. The study of harmonic functions on abstract groups has been quite fruitful in the past few decades. Bounded harmonic functions have a deep algebraic structure and have been used to study “boundaries” of groups, especially (but not only) in the discrete case. This topic was initiated by Furstenberg [Fur63, Fur73]. A search for “Poisson-Furstenberg boundary” will reveal an immense amount of literature, we refer to [KV83, Fur02] and references therein for the interested reader. As for unbounded harmonic functions, positive harmonic functions were studied in the Abelian case by Chouquet & Deny [CD60] (and further by Raugi [Rau04] for nilpotent groups). Yau [Yau75] studied positive harmonic functions on open manifolds of non-negative Ricci curvature. He conjectured that the space of harmonic functions that grow at most like

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some polynomial on such a manifold should have finite dimension. This was proved by Colding and Minicozzi [CM97]. Kleiner [Kle10] used Colding and Minicozzi’s approach for finitely generated groups of polynomial growth, to reprove Gromov’s theorem regarding such groups [Gro81].

These works bring to light a connection between algebraic properties (nilpotence), analytic properties (harmonic functions and random walks) and geometric properties (volume growth, curvature). They motivate the following meta-questions: Given a group $G$, and some space of harmonic functions on $G$, what can be said about the dimension of the space and its relation to the algebraic and geometric properties of the group? Is the dimension independent of the choice of specific random walk? Does the finiteness of the dimension depend only on the group’s algebraic properties? In general, one would like to understand the structure of representations of the group given by its canonical action on some specific space of harmonic functions; how do these representations vary as the underlying random walk measure is changed?

An example for a precise formulation of one such question is the following conjecture, which has been open for quite some time.

**Conjecture 1.1.** Let $G$ be a compactly generated locally compact group. Let $\mu, \nu$ be two symmetric, adapted probability measures on $G$, with an exponential tail. Then, $(G, \mu)$ is Liouville if and only if $(G, \nu)$ is Liouville.

Here $(G, \mu)$ is Liouville means that any bounded $\mu$-harmonic function is constant. It is well known that the space of bounded harmonic functions is either only the constant functions (i.e. Liouville) or has infinite dimension. (For finitely generated groups this is also an easy consequence of Theorem 1.5 or Theorem 1.7 below.) So an equivalent formulation of the above conjecture is that the dimension of the space of bounded harmonic functions does not depend on the specific choice of (nicely behaved) measure $\mu$.

As stated, this question regarding bounded harmonic functions has been open for a while. This is part of the motivation for the following conjecture, from [MY16].
**Conjecture 1.2.** Let $G$ be a compactly generated locally compact group. Let $\mu$ be a symmetric, adapted probability measure on $G$, with an exponential tail. Then, $G$ has polynomial growth if and only if the space of linearly growing $\mu$-harmonic functions on $G$ is finite dimensional.

Note that a group $G$ with measure $\mu$ may be Liouville but still have an infinite dimension of linearly growing harmonic functions (see e.g. [KV83, MY16] and below for examples).

In [MY16] this conjecture is verified for $G$ finitely generated and (virtually) solvable. In fact, it is known that for finitely generated $G$, the dimension of the space of linear growth harmonic functions itself is either infinite or some number independent of the choice of specific measure, see [MPTY17].

The main result of this paper is a proof of Conjecture 1.2 for connected topological groups. In order to precisely state the results we introduce some notation.

### 1.2. Notation and main results

Let $G$ be a compactly generated locally compact (CGLC) group, and fix $K$ a compact generating set. Assume it is symmetric (i.e. $K = K^{-1} = \{x^{-1} : x \in K\}$). Let $K^n = \{x_1x_2\cdots x_n : x_1,\ldots, x_n \in K\}$. $K$ induces a metric on $G$ by

$$d_K(x, y) = d_K(1, x^{-1}y) := \min\{n : x^{-1}y \in K^n\}$$

and we use the notation $|x| = |x|_K = d_K(1, x)$. Note that this metric is left invariant, that is $d_K(x, y) = d_K(zx, zy)$, and that for two choices of generating sets $K_1$ and $K_2$, the respective metrics are bi-Lipschitz, i.e. there exists a constant $c > 0$ such that $c^{-1}|x|_{K_1} \leq |x|_{K_2} \leq c|x|_{K_1}$ for all $x \in G$.

The **growth of** $G$ is the growth rate of the sequence $(m(K^n))_n$, where $m = m_K$ is the Haar measure on $G$ normalized to $m(K) = 1$. We are mainly interested in polynomial growth: $G$ is said to have polynomial growth if there exist constants $C > 0, k > 0$ such that for all $n \geq 1$ we have $m(K^n) \leq Cn^k$.

For functions $f : G \to \mathbb{R}$, define the following (perhaps infinite) quantity:

$$\|f\|_k = \lim_{r \to \infty} \sup_{r \leq x \leq r} \sup_{|x| \leq r} |f(x)|.$$
We say that $f : G \to \mathbb{R}$ has \textit{degree-$k$ polynomial growth} if $||f||_k < \infty$. In the case $k = 1$ we say that $f$ has linear growth. Note that $||f||_k < \infty$ is equivalent to the existence of a constant $c > 0$ such that $|f(x)| \leq c(1 + |x|)^k$ for all $x \in G$. The group $G$ acts naturally on $\mathbb{R}^G$ by $(\gamma.f)(x) = f(\gamma^{-1}x)$. By bi-Lipschitzness, the property $||f||_k < \infty$ is independent of the choice of specific generating set (although the specific value of $||f||_k$ does depend on the metric induced by $K$).

Throughout, we will consider a probability measure $\mu$ on $G$. We will always assume that

- It is adapted, i.e. there is no proper closed subgroup $H \leq G$ such that $\mu(H) = 1$.
- It is symmetric, i.e. $\mu(A) = \mu(A^{-1})$ for any measurable set $A$.
- It has a third moment, i.e. $\int_G |s|^3d\mu(s) < \infty$.

For short, we call a probability satisfying these three assumption \textit{courteous}. If $\mu$ satisfies $\int_G e^{\varepsilon |s|}d\mu(s) < \infty$ for some $\varepsilon > 0$, we say that $\mu$ has an exponential tail. Note that the property of having a third moment or of having an exponential tail is independent of specific choice of generating set, again because the different metrics are bi-Lipschitz.

For measurable functions $f : G \to \mathbb{R}$, we define the Laplace operator by

$$(\Delta_\mu f)(x) = f(x) - \int_G f(xs)d\mu(s),$$

and we say that a function $f : G \to \mathbb{R}$ is $\mu$-harmonic if $\Delta_\mu f \equiv 0$.

We can now define the space of $\mu$-harmonic functions with polynomial growth of degree at most $k$:

$$\text{HF}_k(G, \mu) := \{ f : G \to \mathbb{R} \mid \Delta_\mu f \equiv 0, ||f||_k < \infty, f \text{ is continuous } \}.$$

Note that since $G$ acts on the left and harmonicity is checked on the right, $\text{HF}_k(G, \mu)$ is a $G$-invariant subspace of $\mathbb{R}^G$. Our main result is:

**Theorem 1.3.** Let $G$ be any connected CGLC group, and let $\mu$ be a courteous probability measure. If $\dim \text{HF}_1(G, \mu) < \infty$ then $G$ has polynomial growth.

Here is a sketch of the main steps of the argument. The first step uses the result of [BBE97] stating that if $G$ is a closed subgroup of the $d$-dimensional affine group, $S_d$,
and if $G$ does not have polynomial growth, then there exists a non-constant positive continuous harmonic function $h$ on $G$. The second step is to show that the above function has linear growth. This was shown in [BB15] for the case $d = 1$, and we extend their result to general $d \geq 1$. The third step is a reduction from connected groups to the case of $S_d$, ultimately proving:

**Lemma 1.4.** Let $G$ be a connected CGLC group that does not admit polynomial growth. For any courteous measure $\mu$ on $G$, there exists a non-constant, positive, linearly growing $\mu$-harmonic function $f \in \text{HF}_1(G, \mu)$.

For the last step, we want to show that $\text{HF}_1$ has infinite dimension. We will actually prove that if $\text{HF}_1$ contains a non-constant, continuous positive function, then it must be infinite dimensional. This is articulated on the following theorem.

**Theorem 1.5.** Let $G$ be an amenable CGLC group. Let $\mu$ be a courteous measure on $G$.

If $\dim \text{HF}_1(G, \mu) < \infty$, then any $h \in \text{HF}_1(G, \mu)$ which is positive must be constant.

The above three steps prove that Theorem 1.3 holds for any connected subgroup of the affine group. The final step is a procedure from [BÉ95] which provides a reduction from general connected CGLC groups to the case of closed subgroups of the affine group $S_d$. This is done in Lemma 1.15.

1.3. **Convergence along random walks.** Let us stress here that Theorem 1.5 holds in the non-connected case as well, that is, for finitely generated groups. This may be of independent interest in other contexts. The main idea behind the proof of Theorem 1.5 is Theorem 1.7 which states that if a function converges a.s. along the random walk and has sub-exponential growth, then it must be constant. This is relevant to positive harmonic functions since a positive harmonic function evaluated on the corresponding random walk provides a positive martingale, which converges a.s. by the martingale convergence theorem (see [Dur10]).

**Definition 1.6.** A function $f : G \to \mathbb{R}$ converges along random walks if $(f(xX_t))_t$ converges a.s. for any $x \in G$, (where $(X_t)_t$ is the $\mu$-random walk).
For example, as mentioned above, any positive harmonic function converges along random walks. As do bounded harmonic functions. Hence, Theorem 1.5 follows from the following theorem, which may be of independent interest.

**Theorem 1.7.** Let $G$ be an amenable CGLC group and let $\mu$ be a courteous measure on $G$. Let $(X_t)_t$ denote the $\mu$-random walk. Let $f : G \to \mathbb{R}$ be a continuous function such that $f$ converges along random walks. Assume that $f$ has sub-exponential growth; that is

$$\limsup_{r \to \infty} \frac{1}{r} \sup_{|x| \leq r} \log |f(x)| = 0.$$ 

If $\dim \text{span}(G, f) < \infty$ then $f$ is constant.

The proof is carried out in Section 3.

The assumption of sub-exponential growth in Theorem 1.7 is technical, and most probably superfluous. In Section 3 we will show that in the discrete case, where $G$ is finitely generated, this assumption is not actually required. We conjecture that the assumption of sub-exponential growth can be removed in the general CGLC case, see Conjecture 1.12 below. In addition, although the proof heavily uses the amenability of $G$, it is not clear that this is a necessary condition for Theorem 1.7 to hold. Again see the open questions below.

1.4. **Characterization of polynomial growth.** As mentioned, Gromov’s theorem [Gro81] characterizes the geometric property of polynomial growth of a finitely generated group by the algebraic property of containing a finite index nilpotent subgroup. However, in the connected case, there is no such characterization. In fact, it is not true that any CGLC group of polynomial growth is nilpotent by compact. One can construct a connected 2-step solvable linear group of polynomial growth that is not nilpotent by compact, see [Bre14, Example 7.9].

We have the following theorem characterizing connected CGLC groups of polynomial growth using an analytic property, namely the finiteness of the dimension of $HF_1$.

Recall, for a probability measure $\mu$ on a CGLC group $G$, we say that $\mu$ has an exponential tail if there exists $\varepsilon > 0$ such that $\int_G e^{\varepsilon |s|} d\mu(s) < \infty$. Also recall that this
property of $\mu$ does not depend on the specific choice of compact symmetric generating set $K$ determining the metric.

**Theorem 1.8.** Let $G$ be a connected CGLC group. Let $\mu$ be a courteous measure with exponential tail. The following are equivalent:

1. $G$ has polynomial growth.
2. For any $k \geq 1$ we have $\dim \mathcal{HF}_k(G, \mu) < \infty$.
3. $\dim \mathcal{HF}_1(G, \mu) < \infty$.
4. The space $\mathcal{HF}_1(G, \mu)$ does not contain a non-constant positive function.

This is a solution of Conjecture 1.2 for the connected case.

**Proof.** In [Per18] it is shown that for any CGLC group of polynomial growth, $G$, and any courteous $\mu$ with exponential tail on $G$, the dimension of $\mathcal{HF}_k(G, \mu)$ is finite for all $k \geq 1$. (This is an extension of Kleiner’s work [Kle10] to non-compactly-supported measures, and to connected CGLC groups.) This gives the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) for connected CGLC groups is precisely the statement of Theorem 1.3.

(3) $\Rightarrow$ (4) follows from Theorem 1.5 (which is actually the contra-positive).

(4) $\Rightarrow$ (1) is exactly the contrapositive of Lemma 1.4. \qed

1.5. **Further questions.** Let us list some open questions motivated by the results mentioned above.

In [MPTY17] it is shown that for a finitely generated group $G$ and courteous measure $\mu$ with exponential tail, if $\dim \mathcal{HF}_k(G, \mu) < \infty$ then the space $\mathcal{HF}_k(G, \mu)$ is basically the space of **harmonic polynomials** on $G$ of degree at most $k$ (see [MPTY17] for precise definitions). This proves that $\dim \mathcal{HF}_k(G, \mu) \in \{\infty, d\}$ for some $d$ which depends only on the group $G$ and not on $\mu$.

**Conjecture 1.9.** Let $G$ be a CGLC group. Let $\mu, \nu$ be courteous measures on $G$ with an exponential tail. Then $\dim \mathcal{HF}_k(G, \mu) = \dim \mathcal{HF}_k(G, \nu)$ for any $k \geq 0$.

Note the we have also included the $k = 0$ case in the above conjecture, i.e. the space of bounded harmonic functions.
We have seen that the finiteness of the dimension of $\mathsf{HF}_1$ characterizes polynomial growth (at least for connected groups and for solvable groups). In the connected case, the same solvable linear non-nilpotent-by-compact example mentioned above from [Bre14] shows that one can no longer obtain results analogous to [MPTY17], since this group has finite dimensional $\mathsf{HF}_1$ but linear growth harmonic functions which are not polynomials. If the group $G$ is nilpotent however, one can show that functions in $\mathsf{HF}_k$ are polynomials even in the connected case.

**Question 1.10.** Let $G$ be a connected CGLC group. Let $\mu$ be a courteous measure on $G$ with an exponential tail. Let $P^k(G)$ denote the space of polynomials of degree at most $k$ on $G$ (see [MPTY17]). Fix $k \geq 1$. Is it true that $\mathsf{HF}_k(G,\mu) \subset P^k(G)$ if and only if $G$ is nilpotent?

The results of Choquet & Deny [CD60], Raugi [Rau04], and Yau [Yau75] motivate the question of the existence of non-constant positive harmonic functions on some group. Specifically:

**Question 1.11.** Let $G$ be a CGLC group of non-polynomial growth. Let $\mu$ be a courteous measure with an exponential tail on $G$.

Is it true that there exists a positive $\mu$-harmonic function that is non-constant?

Is it true that there exists such a function of linear growth?

Let us remark that our proof of Theorem 1.3 answers the above question affirmatively, in the connected case (see the characterization in Section 1.4). The results of [MY16] also provide an affirmative answer in the finitely generated solvable case.

For finitely generated groups in general we do not know the answer, even for some specific examples, e.g. the Grigorchuk groups.

Regarding Theorem 1.7, as mentioned above, the condition of sub-exponential growth seems to be superfluous.

**Conjecture 1.12.** Let $G$ be an amenable CGLC group and $\mu$ a courteous measure on $G$. Let $f : G \to \mathbb{R}$ be a continuous function that converges along random walks.

If $\dim \text{span}(G.f) < \infty$ then $f$ is constant.
Finally, we only know how to prove Theorem 1.7 for amenable groups. It would be quite surprising if this theorem does not hold in the non-amenable case. Precisely:

**Question 1.13.** If $G$ is a CGLC group, is it true that for any continuous function $f : G \to \mathbb{R}$ that converges along random walks, if $f$ is non-constant its orbit spans an infinite dimensional space?

1.6. **Proof of main result.** We now can summarize all the above results into a proof of Theorem 1.3.

**Lemma 1.14.** Let $G$ be a closed subgroup of $S_d$ and let $\mu$ be a courteous measure on $G$. Suppose $G$ does not have polynomial growth, Then, there exists a continuous, non-constant positive $\mu$-harmonic function on $G$, which admits linear growth. In other words, $\dim \text{HF}_1(G, \mu) > 1$.

The proof of this lemma is in Section 2.2.

**Lemma 1.15.** Let $G$ be a connected CGLC group and let $\mu$ be a courteous measure on $G$. Then, there exists a connected closed subgroup $G'$ of the $d$-dimensional affine group $S_d$, and a courteous measure $\mu'$ on $G'$ such that:

- $\dim \text{HF}_1(G', \mu') \leq \dim \text{HF}_1(G, \mu)$.
- If $G'$ has polynomial growth then also $G$ has polynomial growth.

This is shown in Section 2.3.

Lemma 1.4 follows directly from the combination of these two lemmas, as explained in Section 2.3.

**Proof of Theorem 1.3.** Let $G', \mu'$ be as guarantied by Lemma 1.15. If $G$ is not of polynomial growth then neither is $G'$. By Lemma 1.14, there exists a positive harmonic function $h \in \text{HF}_1(G', \mu')$. Since $h$ is a positive harmonic function, $(h(xX_t))_t$ converges a.s. as a positive martingale by the martingale convergence theorem, see e.g. [Dur10]. Since $G'$ is a subgroup of the affine group it is upper-triangular, and thus solvable. By Theorem 1.5,

$$\dim \text{HF}_1(G, \mu) \geq \dim \text{HF}_1(G', \mu') \geq \dim \text{span}(G'.h) = \infty.$$
2. Linear growth positive harmonic function

2.1. Stationary measure on $\mathcal{S}_d$. Denote by $\mathcal{S}_d$ the group of affine similarities on $\mathbb{R}^d$. An element of $\mathcal{S}_d$ is $g = (a, k, b)$ where $a \in (0, \infty), k \in O(d), b \in \mathbb{R}^d$. Here $O(d)$ is the group of $d \times d$ orthogonal real matrices. (Note that in the 1-dimensional case, $\mathcal{S}_1$, we may omit the k-coordinate). The group’s multiplication is defined by

$$g_1 \cdot g_2 = (a_1, k_1, b_1) \cdot (a_2, k_2, b_2) := (a_1a_2, k_1k_2, a_1k_1b_2 + b_1).$$

$\mathcal{S}_d$ acts from the left on $\mathbb{R}^d$ by $(a, k, b).x = akx + b$. For an element $g = (a, k, b)$, let $a(g) = a$.

Let $G$ be a closed subgroup of $\mathcal{S}_d$, and let $\mu$ be a courteous probability measure on $G$. Let $\nu$ be a measure on $\mathbb{R}^d$, and define

$$\mu \ast \nu(A) = \int_G \int_{\mathbb{R}^d} 1_A(g.x)d\nu(x)d\mu(g).$$

for any measurable set $A$.

A Radon measure $\nu$ on $\mathbb{R}^d$ is called $\mu$-stationary if $\mu \ast \nu = \nu$.

Remark 2.1. In [BBE97] and related texts, a measure satisfying $\mu \ast \nu = \nu$ is called $\mu$-invariant. One should be careful to distinguish between invariance of $\nu$ with respect to convolution with the measure $\mu$, and the different notion of a $G$-invariant measure, which means $g.\nu = \nu$ for all $g \in G$. In this text we only deal with the former. In order to avoid confusion, we prefer the terminology stationary.

The existence of a $\mu$-stationary Radon measure is shown in [BBE97]. Namely, we have:

Lemma 2.2 (Proposition 1.1 in [BBE97]). Let $G$ be a closed CGLC subgroup of $\mathcal{S}_d$, and $\mu$ a courteous measure on $G$. Then there exist a $\mu$-stationary Radon measure $\nu$ on $\mathbb{R}^d$. 

2.2. **Linear growth harmonic functions.** Let \( \nu \) be a \( \mu \)-stationary Radon measure on \( \mathbb{R}^d \). Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be a compactly supported function. Define

\[
    h(g) := \int_{\mathbb{R}^d} \phi(g^{-1}x) d\nu(x).
\]

It is straightforward to check that \( h \) is a \( \mu \)-harmonic function on \( G \). If \( \phi \geq 0 \) then \( h \geq 0 \).

If \( 1\{A_1\} \leq \phi \leq 1\{A_2\} \) for some measurable sets \( A_1 \subset A_2 \), then \( \nu(g.A_1) \leq h(g) \leq \nu(g.A_2) \) for all \( g \in G \). Also, if \( \phi \) is continuous, then \( h \) is continuous as well.

Let \( B = [-1, 1]^d \subset \mathbb{R}^d \). Fix a compactly supported continuous function \( \phi : \mathbb{R}^d \to \mathbb{R} \) such that \( 1\{(1/2)B\} \leq \phi \leq 1\{B\} \). Define \( h \) as in (1). By Lemma 2.10 in \([BÉ95]\) \( h \) is non-constant as soon as \( G \) does not have polynomial growth.

We want to show that \( h \) has linear growth, i.e. that there exists a constant \( c_h > 0 \) such that \( h(g) \leq c_h (1 + |g|) \) for all \( g \in G \). By compactness and the continuity of the action, there exists a constant \( M > 1 \) such that \( k.B \subset [-M, M]^d \) for all \( k \) in the compact symmetric generating set \( K \). By induction, this implies \( g.B \subset [-M|g|, M|g|]^d \) for all \( g \in G \). Hence, to show linear growth of \( h \), it will be enough to show that

\[
\nu([-z, z]^d) \leq C (1 + \log z) \quad \forall \ z > 1
\]

for some constant \( C > 0 \). In \([BB15]\), this is shown for the case of \( d = 1 \). Their proof relies on the total ordering of the real numbers, hence it does not generalize to \( \mathbb{R}^d \) in a straightforward manner. However, we now provide a reduction from the \( d \)-dimensional case to the 1-dimensional case.

Let \( \psi = (a, k, b) \) be a random element generated by the probability measure \( \mu \) on \( S_d \). We assume the following:

- **Recurrence:** \( \mathbb{E}[\log(a)] = 0 \) and \( \mathbb{P}[a = 1] \neq 1 \).
- **Non-degeneracy:** \( \mathbb{P}[\psi.x = x] < 1 \) for all \( x \in \mathbb{R}^d \).
- **Moment condition:** \( \mathbb{E}\left[ \left( | \log(a) | + \log(1 + ||b||) \right)^3 \right] < \infty \).

We note that by \([Éli82, Gui80]\), there exist constants \( C, D > 0 \) such that

\[
    C^{-1}|x| - D < |\log(a)| + \log(1 + ||b||) < C|x| + D.
\]

Hence, the above moment condition is equivalent to existence of a third moment of \( \mu \).
Next, for an element $\psi$ as above, we define

$$g_\psi := (a, \max\{||b||, 1\}) \in S_1.$$  

We denote by $(\psi_t)_{t \geq 0}$ a sequence of i.i.d. $\mu$ random elements, and abbreviate by $(g_t)_{t \geq 0}$ the induced sequence $(g_{\psi_t})_{t \geq 0}$. For an element $g = (a, b) \in S_1$, denote $a(g) = a, b(g) = b$.

Finally, we define $R_t := g_1 \cdots g_t$, and note that

$$b(R_{t+1}) = a(R_t) b(g_{t+1}) + b(R_t).$$

(4)

Lemma 2.3. Let $(\psi_t)_{t \geq 0}$ be a sequence of i.i.d. $\mu$ random variables. Let $U, V$ be two compact Borel subsets of $\mathbb{R}^d$. Let $T = T_{U,V} := \inf\{t \geq 0 : \psi_1 \cdots \psi_t(U) \subset V\}$. Then

$$\nu(V) \geq \mathbb{P}[T < \infty] \cdot \nu(U).$$

The proof relies on the fact that $\nu((\psi_1 \cdots \psi_t)^{-1}(V))$ is a martingale. We omit the proof since (while written for $d = 1$) it is essentially contained in [BB15]. The following inequality follows from (4) and is easy to verify:

$$||\psi_1 \cdots \psi_t(x)|| \leq R_t ||x|| \quad \forall x \in \mathbb{R}^d.$$  

(5)

Lemma 2.4. Fix a constant $k_0 > 1$. Define the following subsets of $S_1$:

$$V_0 = \{(a, b) : k_0^{-1} \leq a \leq k_0, |b| \leq k_0 \},$$

$$V_z = V_0 \cdot (z^{-1}, 0) = \{(a, b) : k_0^{-1} \cdot z^{-1} \leq a \leq k_0 \cdot z^{-1}, |b| \leq k_0 \}.$$  

We have

$$\nu([-2k_0, 2k_0]^d) \geq \mathbb{P}[T_{V_0} < \infty] \cdot \nu([-z, z]^d),$$

where $T_{V_0} = \inf\{t : R_t \in V_0\}$.

Proof. Note that if $R_t \in V_z$ and $x \in \mathbb{R}^d$ satisfies $||x|| \leq z$, then $R_t \cdot ||x|| \leq 2k_0$. Hence, by (5), $||\psi_1 \cdots \psi_t(x)|| \leq 2k_0$. Applying Lemma 2.3 with $V = [-2k_0, 2k_0]^d$ and $U = [-z, z]^d$, we get

$$\nu(V) \geq \mathbb{P}[T_{U,V} < \infty] \cdot \nu(U) \geq \mathbb{P}[T_{V_z} < \infty] \cdot \nu(U)$$

since $T_{V_z} \geq T_{U,V}$.  \qed
Thus, it is enough to show that there exists a set $V_0$ as in Lemma 2.4 and a constant $\delta > 0$ such that

$$\mathbb{P}[T_{V_0} < \infty] > \frac{\delta}{1 + \log z}.$$  

We stress the fact that the hitting time $T_{V_0}$ is with respect to the process $(R_t)_{t \geq 0}$ on $S_1$, allowing us to use [BB15]. Indeed, the content of Lemma 3.4(2) in [BB15] is exactly the above, i.e. there exists such a set $V_0$. It is straightforward to check that by our assumptions on $\mu$, and the fact that $b(g_t) \geq 1$ for all $n$, the walk $(R_t)_{t \geq 0}$ satisfies the assumptions of Lemma 3.4 in [BB15].

Putting the pieces of this subsection together, we conclude the proof of Lemma 1.14.

2.3. **From locally compact groups to $S_d$.** In this section we will overview the reduction from general CGLC connected groups to the case of closed subgroups of $S_d$, as carried out in [BÉ95].

Let $G$ be a connected CGLC group, and $\mu$ a courteous measure on $G$. Let $(X_t)_{t \geq 0}$ be a $\mu$-random walk on $G$, i.e. $X_0 = 1$ and the increments $X_t^{-1}X_{t+1}$ are independent $\mu$-random variables. Let $H \leq G$ be a finite-index subgroup, and define the return time to $H$ by $\tau_H = \inf \{ t : X_t \in H \}$. It is well known that since $H$ is of finite index, $\tau_H$ is almost surely finite. Define the hitting measure $\mu_H$ on $H$ by

$$\mu_H(A) = \mathbb{P}[X_{\tau_H} \in A].$$

In [BÉ95], it is shown that $\mu_H$ is a courteous measure on $H$. It is then shown, that if $f_H$ is a $\mu_H$-harmonic function on $H$, then $f(g) := \mathbb{E}[f_H(X_{\tau_H}) \mid X_0 = g]$ is a $\mu$-harmonic function on $G$. In fact, we have:

**Proposition 2.5** ([BÉ95] Lemma 3.4, [MY16] Proposition 3.4). Let $G$ be a CGLC group, $\mu$ a courteous measure, and $H$ a finite index subgroup. Then $\mu_H$ is a courteous measure on $H$. Moreover, for any $k$ the restriction map $f \mapsto f|_H$ is a linear isomorphism from $HF_k(G, \mu)$ to $HF_k(H, \mu_H)$.

Put simply, by passing to a finite index subgroup, the space of harmonic function of polynomial growth of degree at most $k$ is essentially the same. This proposition indicates that courteous measures provide a suitable framework to prove Conjecture 1.2.
One can also pass to a continuous image of the group. Let \( \pi : G \rightarrow Q \) be a continuous homomorphism. In [BÉ95], it is shown that if \( \mu \) is courteous on \( G \) then \( \mu_Q := \mu \circ \pi^{-1} \) is courteous on \( Q \). It is then straightforward to show that if \( f_Q \) is a \( \mu_Q \)-harmonic function with linear growth on \( Q \), then \( f = f_Q \circ \pi \) is a \( \mu \)-harmonic function with linear growth on \( G \).

By a series of passes to quotients and finite index subgroups, a reduction from a connected CGLC group to a closed solvable subgroup \( \tilde{G} \) of \( S_d \) is carried out in [BÉ95]. It is also shown that if \( G \) does not have polynomial growth, then the corresponding subgroup of \( S_d \) does not have polynomial growth as well.

This argument proves Lemma 1.15.

From this it is also very easy to get to Lemma 1.4: Combine the results of [BB15, BBE97, BÉ95] mentioned above, together with Lemma 1.14.

3. Infinite dimensional orbit

In this section we prove Theorem 1.7.

**Lemma 3.1.** Let \( G \) be a CGLC group and let \( \mu \) be a courteous measure on \( G \). Let \( H \leq G \) be a subgroup of finite index and let \( \mu_H \) be the hitting measure.

If \( f : G \rightarrow \mathbb{R} \) converges along random walks (with respect to \( \mu \)) then the restriction \( f|_H \) converges along random walks (with respect to \( \mu_H \)).

**Proof.** Let \((X_t)_t\) be a \( \mu \)-random walk started at \( X_0 = y \in H \). Let \( \tau_0 = 0 \) and let \( \tau_{n+1} = \inf\{t \geq \tau_n + 1 : X_t \in H\} \) be the successive return times to \( H \). So \((Y_n := X_{\tau_n})_n\) is a \( \mu_H \)-random walk started at \( Y_0 = y \).

Since \((f(X_t))_t\) converges a.s., also \((f(Y_n))_n\) converges a.s. as a sub-sequence. This holds for arbitrary \( y \in H \) completing the proof. \( \square \)

**Lemma 3.2.** Let \( G \) be a compact group and \( \mu \) a courteous measure on \( G \).

If \( f : G \rightarrow \mathbb{R} \) is a continuous function that converges along random walks then \( f \) is constant.

**Proof.** Assume for a contradiction that \( f \) is non-constant. Let \( x \in G \) be such that \( f(x) \neq f(1) \). \( f \) is continuous, so we may choose two open neighborhoods \( x \in U, 1 \in V \)
such that
\[
\sup_{z \in U} |f(x) - f(z)| < \frac{1}{2} |f(x) - f(1)| \quad \text{and} \quad \sup_{y \in V} |f(1) - f(y)| < \frac{1}{2} |f(x) - f(1)|.
\]
Specifically, $V \cap U = \emptyset$ and $U, V$ have positive Haar measure. Also, for any $z \in U, y \in V$ we have $f(z) \neq f(y)$.

Now, the ergodic theorem tells us that for any measurable subset $A \subset G$ we have that
\[
\frac{1}{t} 1_{\{A\}}(X_t) \to \lambda(A) \quad \text{a.s.}
\]
where $\lambda$ is the normalized Haar probability measure on $G$. Thus, a.s. the sequence $(f(X_t))_t$ contains an accumulation points in any open set of positive Haar measure, contradicting convergence along random walks.

\[\square\]

**Lemma 3.3.** Let $G$ be an amenable CGLC group and let $\mu$ be a courteous measure on $G$. Let $f : G \to \mathbb{R}$ be a continuous function that converges along random walks.

Assume that there exists a co-compact normal subgroup $H \triangleleft G$ such that $H$ acts trivially on $f$.

Then, $f$ is constant.

**Proof.** Since $H$ acts trivially on $f$, this induces a continuous function on the compact group $G/H$ via $\bar{f}(Hx) = f(x)$. If we consider the projected random walk (i.e. the process $(H X_t)_t$ on this compact group, then $\bar{f}$ converges along random walks (because $f$ does). Thus, by Lemma 3.2 $\bar{f}$ is constant. This implies that $f$ is constant as well. \[\square\]

We require the notion of a type $S$ action following [BÉ95].

**Definition 3.4.** Let $\rho : G \to GL(V)$ be an action of $G$ on a finite-dimensional real vector space $V$. We say that this action is of type $S$ if there exists a compact subgroup $K$ of $GL(V)$, a continuous homomorphism $k : G \to K$ and a continuous homomorphism $a : G \to (0, \infty)$ such that $\rho(g) = a(g)k(g)$ for all $g \in G$.

**Proof of Theorem 1.7.** We will denote $X_{t+1} = X_t U_{t+1}$ for $(U_t)_{t \geq 1}$ i.i.d.-$\mu$ random steps.

Let $V = \text{span}(G.f)$ and assume that $\dim V = d < \infty$. Note that $(h(x X_t))_t$ converges for all $h \in V$. Since functions in $V$ factor through the kernel of the $G$-action, we may assume that $G \leq GL(V)$. 


Under this assumption, \( G \) is now an amenable linear group. A result by Guivarc’h [Gui73] states that there exists a finite index normal subgroup \( G' \) of \( G \), for which there is a finite sequence \( \{0\} = V_0 \subset V_1 \subset ... V_n = V \cong \mathbb{R}^d \) of \( G' \)-invariant linear subspaces of \( V \) such that the action of \( G' \) on each \( V_{i+1}/V_i \) is of type \( S \). By Lemma 3.1, we may, without loss of generality, pass to the finite index subgroup, since we are only required to prove that \( G' \) acts trivially on \( f \), due to Lemma 3.3. So we assume that a sequence \( \{0\} = V_0 \subset V_1 \subset ... V_n = V \) exists with respect to \( G \). Specifically, by an appropriate choice of basis \( B \) we have

\[
[x]_B = \begin{pmatrix}
    a_1(x)k_1(x) & x_{12} & \cdots & x_{1n} \\
    0 & a_2(x)k_2(x) & \cdots & x_{2n} \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & a_n(x)k_n(x) &
\end{pmatrix}
\]

where \( k_i : G \to K_i \subset \text{GL}(\mathbb{R}^d) \) and \( a_i : G \to (0, \infty) \) are homomorphisms and \( K_i \) is a compact subgroup of \( \text{GL}(\mathbb{R}^d) \).

Let \( H \triangleleft G \) be the kernel of the homomorphism \( x \mapsto (k_1(x), \ldots, k_n(x)) \). So \( G/H \) is isomorphic to the compact group \( k_1(G) \times \cdots \times k_n(G) \). Also, for any \( x \in H \) we have that \( k_j(x) = I \).

**Step I.** First we show that \( V_1 \) is the space of constant functions (so \( G \) acts trivially on \( V_1 \)).

For any \( x \in G \) and \( h \in V_1 \) we have

\[
[x.h]_B = [x]_B[h]_B = a_1(x)k_1(x)[h]_B.
\]

This is a slight abuse of notation, since we regard \( a_1(x)k_1(x) \) as acting on the whole space \( \mathbb{R}^d \), by identifying

\[
a_j(x)k_j(x) = \begin{pmatrix}
    I & 0 & 0 & \cdots & 0 \\
    0 & \ddots & \cdots & \vdots & 0 \\
    \vdots & a_j(x)k_j(x) & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & I
\end{pmatrix}
\]
Consider $\langle \cdot, \cdot \rangle$, the inner product on $\mathbb{R}^d$. For any $y \in G$ the map $h \mapsto h(y)$ is a linear functional on $V$, so by the Riesz representation theorem there exists $v_y \in \mathbb{R}^d$ such that
\[
\langle [h]_B, v_y \rangle = h(y) \text{ for all } h \in V.
\]

Let $h \in V_1$ be any function. Then for any $x \in H$ and $y \in G$, since $k_1(x) = I$,
\[
h(x) = x^{-1} \cdot h(y) = a_1(x)^{-1} \cdot \langle [h]_B, v_y \rangle = a_1(x)^{-1} \cdot h(y).
\]

Thus, for any $x \in H$ we have $h(x^{-n}) = a_1(x)^n h(1)$. Because we assumed that $h$ grows sub-exponentially, this implies that $a_1(x) = 1$ for all $x \in H$. So $H$ acts trivially on any $h \in V_1$. By Lemma 3.3 this implies that $V_1$ is the space of constant functions.

**Step II.** We now show that $H$ acts trivially on $V_2$. (If $d = 1$ this step is redundant, since we have already shown that $V = V_1$ is the space of constant functions).

Let $h \in V_2$. Let $\delta_1 \in \mathbb{R}^d$ be the vector with 1 in the first coordinate and 0 elsewhere. Note that since $V_1$ is the space of constant functions, $\delta_1 \perp v_y - v_1$ for all $y \in G$.

For all $x \in G$ we have
\[
\langle x \cdot h \rangle_B = a_2(x) \cdot k_2(x) [h]_B + \langle (0, x_{12}, \ldots, x_{1n}), [h]_B \rangle \cdot \delta_1.
\]

The important observation here is that the coefficient of $\delta_1$ above depends only on $x$ and not on the specific point of evaluation of the function $\delta_1$. So if $x \in H$ then for any $y \in G$,
\[
h(xy) - h(x) = a_2(x)^{-1} \cdot \langle [h]_B, v_y - v_1 \rangle = a_2(x)^{-1} (h(y) - h(1)).
\]

This implies that for any $x \in H$,
\[
h(x^{-n}) - h(1) = \sum_{j=0}^{n-1} a_2(x)^j \cdot (h(x^{-1}) - h(1)) = \frac{a_2(x)^n - 1}{a_2(x) - 1} \cdot (h(x^{-1}) - h(1)).
\]

As before, since we assumed that $h$ has sub-exponential growth, this implies that $a_2(x) = 1$ for any $x \in H$, which is to say that $H$ acts trivially on any $h \in V_2$.

Thus, by Lemma 3.3 any $h \in V_2$ is constant, implying that $G$ acts trivially on $V_2$.

**Conclusion.** Since $V_2$ is the space of constant functions, it must be that actually $d = 1$ and $V_1 = V$ is the space of constant functions, and that originally in (6) the matrices were all the identity matrix. This shows that $G$ acts trivially on $G.f$ and specifically on $f$. □
Following the statement of Theorem 1.7 we remarked that in the case where $G$ is finitely generated this theorem holds without the sub-exponential growth assumption. Since the proof is almost identical we only sketch the proof of this observation.

**Sketch of proof.** As in the proof of Theorem 1.7 we arrive at a representation as in (6). Setting $H$ to be the co-compact subgroup which is the kernel of the map $x \mapsto (k_1(x), \ldots, k_n(x))$, we find that $H$ is of finite index (because the compact group $k_1(G) \times \cdots \times k_n(G)$ is actually finitely generated in this case, and thus finite).

Thus, by Lemma 3.1, we may pass to the subgroup $H$ instead of $G$. Then, the same reasoning as in Step I of the proof above gives that for any $h \in V_1$, we have $h(X_t) = a_1(X_t)^{-1} \cdot h(1)$. Thus, $\log \frac{h(X_t)}{h(1)}$ is a symmetric random walk on the additive group $\mathbb{R}$. Such a random walk can only converge if it is degenerate (see [Dur10]); that is, if $h(X_t) = h(1)$ a.s. for all $t$. Because $\mu$ is adapted this implies that $h$ is constant.

Once establishing that $V_1$ is the constant functions, as in Step II of the proof of Theorem 1.7, we arrive at

\begin{equation}
  h(X_{ty}) - h(X_t) = a_2(X_t)^{-1} \cdot (h(y) - h(1))
\end{equation}

for any $y$ and any $t$, and for any $h \in V_2$. Now, for a fixed $y$ there exist $n \in \mathbb{N}$ and $\alpha > 0$ such that $\mu^n(y) > \alpha$. Thus, as $t \to \infty$ for any $\varepsilon > 0$,

$$
\alpha \cdot \mathbb{P}[|h(X_{ty}) - h(X_t)| > \varepsilon] \leq \mathbb{P}[|h(X_{t+n}) - h(X_t)| > \varepsilon] \to 0.
$$

That is, the left hand side of (8) converges to 0 in probability. However, as before, $(\log a_2(X_t))_t$ is a symmetric random walk on $\mathbb{R}$, implying that it can only converge if it is degenerate. So it must be that $a_2 \equiv 1$ and we arrive at $h(xy) - h(x) = h(y) - h(1)$ for all $x, y \in H$. This implies that $h - h(1)$ is a homomorphism from $H$ into the additive group $\mathbb{R}$. Specifically, $(h(X_t) - h(1))_t$ forms a symmetric random walk on $\mathbb{R}$, and because this random walk must converge a.s., we obtain as before that $h$ is constant. \hfill \Box

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