PRIESTLEY-STYLE DUALITY FOR FILTER-DISTRIBUTIVE CONGRUENTIAL LOGICS

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ABSTRACT. We first present a Priestley-style duality for the classes of algebras that are the algebraic counterpart of some congruential, finitary and filter-distributive logic with theorems. Then we analyze which properties of the dual spaces correspond to properties that the logic might enjoy, like the deduction theorem or the existence of a disjunction.

1. INTRODUCTION

The classes of algebras that correspond to many well-known logics have a distributive lattice reduct. Among them we find Boolean algebras, Heyting algebras, modal algebras, positive modal algebras, De Morgan algebras and MV-algebras. These classes of algebras are also the algebraic counterpart of some congruential logic equal or closely related to the logic from which they originally arise. This fact can be taken to explain from a logical perspective why topological Priestley dualities exist for many of them: the prime filters of the algebras are in fact the irreducible logical filters of the congruential logic. A crucial property of these congruential logics is that in any of their algebras the lattice of logical filters is distributive. In abstract algebraic logic the logics with this property are known as filter-distributive.

Besides the logics whose algebras have a distributive lattice reduct, there are logics which are congruential and filter-distributive but whose algebras have only a meet-semilattice or a join-semilattice reduct or even no semilattice reduct at all; for example Hilbert algebras, which are the algebras that constitute the algebraic counterpart of the implication fragment of intuitionistic logic, a fragment which is a congruential and filter-distributive logic.

Our aim in this paper is to develop a framework to obtain Priestley-style dualities for the classes of algebras that are the algebraic counterparts of the congruential, finitary and filter-distributive logics. The point of view we take is that of logic. The starting point is any logic $S$ with the mentioned properties, its algebraic counterpart, denoted by $\text{Alg}S$, and the lattices of the logical filters of the algebras in $\text{Alg}S$. In any of these lattices we have its irreducible elements, which are also prime because the lattice is distributive. In general, these irreducible logical filters are not enough to be the points of a dual space if we are interested in a Priestley-style duality. We need a less restrictive notion encompassing the irreducible logical filters. To introduce it, we consider the notion of strong logical ideal and define the
optimal logical filters as the logical filters whose complement is a strong logical ideal. The optimal filters will be the points of the dual space.

A way to understand the optimal logical filters of an algebra \( A \in \text{Alg}S \) is to consider the \( S \)-semilattice of \( A \), a notion introduced in [10]. It is the dual of the meet-semilattice of the finitely generated logical filters of \( A \). The lattice of the filters of this meet-semilattice turns out to be isomorphic to the lattice of the logical filters of \( A \) and therefore inherits the distributivity from this later one. Hence, we have that the \( S \)-semilattice of \( A \) is a distributive meet-semilattice and we can use and take inspiration from the duality for distributive meet-semilattices developed in [1]

to obtain the dualities we are after.

For every finitary, congruential and filter distributive logic \( S \) with theorems we present a duality between the category of the algebras in \( \text{Alg}S \) together with the algebraic homomorphisms between them and a category of Priestly-style spaces augmented with an algebra of clopen up-sets; such structures will be called \( S \)-Priestley spaces. Then we characterize the properties of the the category of \( S \)-Priestley spaces that correspond to basic logical properties that the logic \( S \) might enjoy. We do it for the property of having a binary formula that behaves like a conjunction, the property of having a set of binary formulas that collectively behaves as a disjunction, the property of having a binary formula that behaves like an implication that satisfies the modus ponens rule and the deduction theorem, and finally, the property of having an inconsistent formula, i.e., a formula that implies every formula.

The paper is structured as follows. In Section 2 we present the preliminaries we need on posets, distributive meet-semilattices, and congruential logics. We also review the duality given in [1]. Section 3 is devoted to the representation theorems for \( S \)-algebras that we obtain using the optimal filters. In Section 4 we study the \( S \)-semilattice of an \( S \)-algebra. In Section 5 we introduce the dual objects of the \( S \)-algebras and in Section 6 the duals of the homomorphisms between \( S \)-algebras. Section 7 shows the categorical duality. Finally, in Section 8 we present the results on the dual properties of the logical properties we mentioned above.

2. Preliminaries

2.1. Notation. Let \( X, Y \) be sets. For any \( B \subseteq X \), we denote by \( B^c \) the relative complement of \( B \) w.r.t. \( X \) when no confusion can arise, i.e., \( B^c = \{ a \in X : a \notin B \} \).

For any binary relation \( R \subseteq X \times Y \) and every \( x \in X \) we let \( R(x) := \{ y \in Y : xRy \} \), and we define the function \( \Box_R : \mathcal{P}(Y) \to \mathcal{P}(X) \) by setting for every \( U \subseteq Y \)

\[
\Box_R(U) := \{ x \in X : R(x) \subseteq U \}.
\]

We denote the powerset of a set \( X \) by \( \mathcal{P}(X) \). We indicate that \( B \) is a finite subset of \( X \) by \( B \subseteq^\omega X \).

For algebras \( A \) and \( B \) of the same type, we denote by \( \text{Hom}(A, B) \) the set of all homomorphisms from \( A \) to \( B \).

If \( \langle X, \tau, \leq \rangle \) is an ordered topological space, \( \text{CIUp}(X) \) denotes the set of all its clopen up-sets.

2.2. Posets and distributive meet-semilattices. Let \( P = \langle P, \leq \rangle \) be a partial order. For every \( a \in P \), we let \( \uparrow a := \{ b \in P : a \leq b \} \) and \( \downarrow a := \{ b \in P : b \leq a \} \).

For every \( U \subseteq P \), we define \( \uparrow U := \bigcup \{ \uparrow a : a \in U \} \) and \( \downarrow U := \{ \downarrow a : a \in U \} \). Moreover, we say that \( U \subseteq P \) is an up-set of \( P \) (resp. a down-set) when \( \uparrow U = U \) (resp.
\[ U = \{ 1 \} \]. By \( \mathcal{P}^\uparrow(P) \) we denote the collection of all up-sets of \( P \), and for \( U \subseteq P \), we denote by \( \max(U) \) the collection of maximal elements of \( U \). A set \( U \subseteq P \) is up-directed (resp. down-directed) when for every \( a, b \in U \) there exists \( c \in U \) such that \( a, b \leq c \) (resp. \( c \leq a, b \)).

An order filter of \( P \) is any non-empty up-set that is down-directed and an order ideal is any non-empty down-set that is up-directed. By \( \text{Id}(P) \) we denote the collection of the order ideals of \( P \) and by \( \text{Fi}(P) \) the collection of its order filters. These collections may not be closure systems. A Frink ideal of \( P \) is a set \( I \subseteq M \) such that for every finite \( I' \subseteq I \) and every \( b \in M \), \( \bigcap \{ \uparrow a : a \in I' \} \subseteq \uparrow b \) implies \( b \in I \); in other words: if every lower bound of the set of upper bounds of \( I' \) belongs to \( I \). We denote by \( \text{Id}_P(P) \) the collection of all Frink-ideals of \( P \), which is a closure system, and by \( \llbracket \mathcal{I} \rrbracket \) the closure operator associated, i.e., for any \( B \subseteq M \), \( \llbracket B \rrbracket \) is the least Frink ideal containing \( B \). This Frink ideal can be described as follows

\[ a \in \llbracket B \rrbracket \text{ iff there exists a finite } B' \subseteq B \text{ s.t. } \bigcap_{b \in B'} \uparrow b \subseteq \uparrow a. \tag{2.1} \]

Notice that according to the definition the emptyset may be a Frink ideal, but this happens if and only if there is no bottom element in \( P \).

An algebra \( M = (M, \wedge, 1) \) of type \((2,0)\) is a meet-semilattice with top element (in short, with top) when the binary operation \( \wedge \) is idempotent, commutative, associative, and \( a \wedge 1 = 1 \) for all \( a \in M \). The (meet) partial order of \( M \) is the relation \( \leq \) such that for every \( a, b \in M \), \( a \leq b \) if and only if \( a \wedge b = a \). Then for every \( a, b \in M \), \( a \wedge b \) is the meet of \( a, b \) w.r.t. \( \leq \) and 1 is its greatest upper bound. The meet-semilattices with top element that also have a least lower bound are called bounded meet-semilattices; the least lower bound is then denoted by 0.

Let \( M \) be a meet-semilattice with top. As a poset, we have the order ideals and the order filters of \( M \). These last ones turn out to be the non-empty up-sets closed under \( \wedge \) and they are usually called meet filters, or simply filters. Now the collection \( \text{Fi}(M) \) is a closure system, and therefore it is a complete lattice where the infimum of a subset of \( \text{Fi}(M) \) is given by the intersection. We denote the closure operator associated by \( \llbracket \mathcal{I} \rrbracket \). It assigns to each \( B \subseteq M \) the filter generated by \( B \), i.e. the least filter containing \( B \), which can be characterized as follows. For every \( a \in M \):

\[ a \in \llbracket B \rrbracket \text{ iff } a = 1 \text{ or } (\exists n \in \omega)(\exists b_0, \ldots, b_n \in B) b_0 \wedge \cdots \wedge b_n \leq a. \]

Since \( \text{Fi}(M) \) is a lattice, we have its meet-irreducible elements. We denote the set of these filters by \( \text{Irr}_{\wedge}(M) \) and call them the irreducible filters of \( M \).

A meet-semilattice \( M \) with top is distributive (cf. \([11, \text{Sec. II.5}]\)) when for each \( a, b_1, b_2 \in M \) with \( b_1 \wedge b_2 \leq a \), there exist \( c_1, c_2 \in M \) such that \( b_1 \leq c_1 \), \( b_2 \leq c_2 \) and \( a = c_1 \wedge c_2 \). It is well known that \( M \) is distributive if and only if the lattice of the filters of \( M \) is distributive. Another characterization of distributive meet-semilattices is given in \([3, \text{Thm. 10}]\): A meet-semilattice with top \( M \) is distributive if and only if for all \( F \in \text{Fi}(M) \), \( F \in \text{Irr}_{\wedge}(M) \) if and only if \( F^c \in \text{Id}(M) \).

In \([1]\) it is presented a Priestley-style duality for two categories both with objects the bounded distributive meet-semilattices and one with morphisms the usual algebraic homomorphisms and the other with morphisms the algebraic homomorphisms that in addition preserve the existing finite joins (including the lower bounds when they exist); these morphisms are called there sup-homomorphisms. The duality in \([1]\) can be slightly modified to obtain a duality for distributive meet-semilattices
with top (but not necessarily a least lower bound) as explained in a sketchy manner in [1, Sec. 9]. It should be noticed that the duality sketched there works only if we modify the definitions of Frink ideal and optimal filter of [1] so that they may include the empty set and the total set respectively, when a lower bound does not exist. A careful presentation of this modified duality can be found in [7]. We expound it briefly here since we will make use of it later in the paper. A useful way to define sup-homomorphism is the following. Let $M_1$ and $M_2$ be meet-semilattices with top, a homomorphism $h : M_1 \to M_2$ is a sup-homomorphism if for every $b \in M_2$ and every finite $A \subseteq M_1$

$$\text{if } \bigcap_{a \in A} \uparrow a \subseteq \uparrow b, \text{ then } \bigcap_{a \in A} \uparrow h(a) \subseteq \uparrow h(b).$$

(2.2)

Note that taking $A$ empty, $\bigcap_{a \in A} \uparrow a \subseteq \uparrow b$ holds if and only if $b$ is a lower bound of $M_1$; henceforth, if $h : M_1 \to M_2$ is a sup-homomorphism and $M_1$ has a lower bound, then $M_2$ should have one and $h$ should preserve the lower bounds.

Regarding objects, the strategy followed in [1] to obtain the dual of a distributive meet-semilattice with top can be described as follows. First every distributive meet-semilattice $M$ with top is embedded into a distributive lattice $L(M)$ with top by a meet-semilattice embedding $e$ that is also a sup-homomorphism and such that $e[M]$ is join dense in $L(M)$. The pair $(L(M), e)$ is called the distributive envelope of $M$. This lattice is in category-theoretic terms the free distributive lattice extension of $M$ w.r.t. the forgetful functor from the category of distributive lattices with top together with their algebraic homomorphism to the category of distributive meet-semilattices with top and the sup-homomorphisms. Thus the distributive envelope of $M$ is (up to isomorphism) the only distributive lattice $L$ with top such that there exists a meet-semilattice embedding $e : M \to L$ which is a sup-homomorphism and $e[M]$ is join-dense in $L$. It holds that $M$ has a lower bound if and only if $L(M)$ has a lower bound.

The Priestley dual of $M$ is essentially taken to be the Priestley dual space of $L(M)$ together with a dense set that allows to recover $M$ inside the lattice of the clopen up-sets. We can take as points of the dual space the inverse images of the prime filters of $L(M)$ by the embedding $e : M \to L(M)$, together with $M$ when $M$ has no lower bound. Under the identification of $M$ with $e[M]$, the points of the space, which are called optimal filters, are then the intersection with $M$ of the prime filters of $L(M)$, with the addition of $M$ if $M$ has no lower bound.

The notion of optimal filter can be defined for every meet-semilattice $M$ using the notion of Frink ideal. Moreover, the optimal filters can be used to give one of the particular constructions of the distributive envelope of a distributive meet-semilattice. Let $M$ be a meet-semilattice with top. A filter $F \in \text{Fi}(M)$ is optimal when there is $I \in \text{Id}_F(M)$ such that $F$ is a maximal element of $\{G \in \text{Fi}(M) : G \cap I = \emptyset\}$ and $I$ is a maximal element of $\{J \in \text{Id}_F(M) : F \cap J = \emptyset\}$. We denote by $\text{Op}(M)$ the set of all optimal filters of $M$. It is easy to check that the irreducible filters are optimal. Moreover, $M$ is an optimal filter if and only if there is no bottom element in $M$. If $M$ is distributive, then $F \in \text{Op}(M)$ if and only if $F^c \in \text{Id}_F(M)$. This shows that the definition of optimal filter given here and the definition in [1] are coextensive. For distributive meet-semilattices with top the irreducible filters and the optimal filters are characterized in the next proposition using the concept of $\wedge$-prime set. A set $X \subseteq M$ of a meet-semilattice $M$ with top is said to be $\wedge$-prime (or simply prime) if it is proper and for all non-empty finite $U \subseteq M$, if
Let $M$ be a distributive meet-semilattice with top. For every $F \subseteq M$,

1. $F \in \text{Irr}_{\wedge}(M)$ if and only if $F^c$ is a $\wedge$-prime order ideal,
2. $F \in \text{Op}(M)$ if and only if $F^c$ is a $\wedge$-prime Frink ideal.

For a given distributive meet-semilattice $M = \langle M, \wedge, 1 \rangle$ with top, the representation map $\sigma : M \rightarrow \mathcal{P}(\text{Op}(M))$ is defined by setting for every $a \in M$

$$\sigma(a) := \{ P \in \text{Op}(M) : a \in P \}.$$ 

The set $\sigma[M] := \{ \sigma(a) : a \in M \}$ is closed under the binary operation of intersection of sets and $\sigma(1) = \text{Op}(M)$. Therefore, $\sigma[M] := \langle \sigma[M], \cap, \sigma(1) \rangle$ is a meet-semilattice with top and $\sigma$ is an isomorphism between $M$ and $\sigma[M]$. Note that $M$ has a bottom element if and only if $\emptyset \in \sigma[M]$.

A particular construction of the distributive envelope of $M$ is given by the distributive lattice $L(M)$ that we obtain by closing $\sigma[M]$ under the binary operation of set-theoretic union. The embedding from $M$ to $L(M)$ that shows that this lattice is the distributive envelope of $M$ is $\sigma$. A very useful property of $\sigma$ that follows easily from the fact that $\sigma$ is a sup-homomorphism is the following. For every $a, b_0, \ldots, b_n \in M$,

$$\bigcap_{i \leq n} \uparrow b_i \subseteq \uparrow a \iff \sigma(a) \subseteq \bigcup_{i \leq n} \sigma(b_i). \quad (2.3)$$

In this particular construction of the distributive envelope, the relation between the filters of $M$ and the filters of $L(M)$ is given by the map $[\sigma[,]]$. When it is applied to $\text{Fi}(M)$ establishes an isomorphism between $\text{Fi}(M)$ and the lattice of the filters of $L(M)$ and the inverse is easily se to be given by the map $\sigma^{-1}[,]$. Moreover, the lattice of the Frink ideals of $M$ and the lattice of the ideals of $L(M)$ together with the empty set, when $M$ has no lower bound, are also isomorphic. The isomorphism is given by the map that sends a Frink filter $I$ of $M$ to the ideal generated by $\sigma[I]$, when $I$ is non-empty, and to the emptyset otherwise. The inverse of this isomorphism is given by the map $\sigma^{-1}[,]$. It follows that $[\sigma[,]]$ establishes an isomorphism between $\text{Op}(M) \setminus \{ M \}$ and the set $\text{Pr}(L(M))$ of the prime filters of $L(M)$, when $M$ has no bottom element, and between $\text{Op}(M)$ and $\text{Pr}(L(M))$, when it has.

The dual objects of distributive meet-semilattices with top are called $\ast$-generalized Priestley spaces in [1]. We delete the star in this paper.

A generalized Priestley space is a tuple $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$ such that

1. $\langle X, \tau, \leq \rangle$ is a Priestley space,
2. $X_B$ is a dense subset of $X$,
3. $X_B = \{ x \in X : \{ U \in X^* : x \notin U \} \text{ is non-empty and up-directed} \}$,
4. for all $x, y \in X$, $x \leq y$ if and only if $y \in U$ for all $U \in X^*$ such that $x \in U$, where $X^* := \{ U \in \text{ClUp}(X) : \max(U^c) \subseteq X_B \}$. The elements of $X^*$ are called the $X_B$-admissible clopen up-sets of $\mathfrak{X}$. The collection $X^*$ is closed under the binary operation of intersection, and the structure $X^* := \langle X^*, \cap, X \rangle$ is a distributive meet-semilattice with top. It follows from the denseness of $X_B$ that $X^*$ has a lower bound if and only if $\emptyset \in X^*$. The meet-semilattice $X^*$ is the dual of $\mathfrak{X}$.
Proposition 2.2. Let $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$ be a generalized Priestley space.

1. The closure $(X^*)^\mu$ of $X^*$ under the binary operation of set-theoretic union is the set of all non-empty clopen up-sets, if $\emptyset \notin X^*$, and it is the set of all clopen up-sets, if $\emptyset \in X^*$.
2. The distributive envelope of $X^*$ is (up to isomorphism) the lattice $L(X^*) = ((X^*)^\mu, \cap, \cup, X)$, with embedding the identity map.

Let $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$ be a generalized Priestley space. Then for every proper optimal filter $F$ of $X^*$, the filter generated by $F$ in $L(X^*)$ is a prime filter and for every prime filter $G$ of $L(X^*)$, $G \cap X^*$ is an optimal filter of $X^*$.

The dual of a distributive meet-semilattice $M = \langle M, \land, 1 \rangle$ with top is defined as follows. Let $\tau_M$ be the topology on $\text{Op}(M)$ determined by the subbasis $\{ \sigma(a) : a \in M \} \cup \{ \sigma(b)^c : b \in M \}$. Then $(\text{Op}(M), \tau_M, \subseteq)$ is a Priestley space. And the structure $\mathfrak{P}_\lambda(M) := \langle \text{Op}(M), \tau_M, \subseteq, \text{Irr}_\lambda(M) \rangle$ turns out to be a generalized Priestley space such that $\text{Op}(M)^* = \sigma[M]$. This implies that $M$ is isomorphic to $(\mathfrak{P}_\lambda(M))^*$ by means of the map $\sigma$. Furthermore, the following facts hold (for a proof see [1, Sec. 5]):

Proposition 2.3. Let $M = \langle M, \land, 1 \rangle$ be a distributive meet-semilattice with top.

1. Every non-empty clopen-upset of $(\text{Op}(M), \tau_M, \subseteq)$ is the union of a finite and non-empty subset of $\sigma[M]$.
2. For every clopen up-set $U \in \text{ClUp}(\text{Op}(M))$, $U$ is an $\text{Irr}_\lambda(M)$-admissible clopen up-set if and only if there exists $a \in M$ such that $U = \sigma(a)$.

Any generalized Priestley space $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$ is order homeomorphic to $\langle \text{Op}(X^*), \tau_{X^*}, \subseteq, \text{Irr}_\lambda(X^*) \rangle$ by means of the map $\varepsilon : X \to \text{Op}(X^*)$, given by:

$$\varepsilon(x) := \{ U \in X^* : x \in U \}.$$ 

Moreover, $\varepsilon[X_B] = \text{Irr}_\lambda(X^*)$.

Regarding morphisms, we just need to recall how the dual of an algebraic homomorphism, which is a relation, is defined in [1].

For generalized Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is a generalized Priestley morphism ([1, Def. 6.2]) when:

(DSR1) $\Box_R(U) \in X_1^*$ for all $U \in X_2^*$,
(DSR2) if $(x, y) \notin R$, then there exists $U \in X_2^*$ such that $y \notin U$ and $R(x) \subseteq U$.

A generalized Priestley morphism $R$ is functional when in addition:

(DSR3) for every $x \in X_1$ there exists $y \in X_2$ such that $R(x) = \uparrow y$.

For a given generalized Priestley morphism $R \subseteq X_1 \times X_2$, the map $\Box_R : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ is an algebraic homomorphism between the distributive meet-semilattices with top $X_2^*$ and $X_1^*$. If $R$ is functional, then $\Box_R$ is a sup-homomorphism. Moreover, for all $x \in X_1$ and all $y \in X_2$, $(x, y) \in R$ if and only if $(\varepsilon(x), \varepsilon(y)) \in R_{\Box_R}$. For every homomorphism $h : M_1 \to M_2$ between distributive meet-semilattices with top, the relation $R_h \subseteq \text{Op}(M_2) \times \text{Op}(M_1)$, defined by:

$$(P, Q) \in R_h \iff h^{-1}[P] \subseteq Q,$$

is a generalized Priestley morphism between the dual generalized Priestley spaces $\mathfrak{P}_\lambda(M_2)$ and $\mathfrak{P}_\lambda(M_1)$; $R_h$ is functional when $h$ is a sup-homomorphism. Moreover, $\sigma_\lambda(h(a)) = \Box_{R_h}(\sigma_\lambda(a))$ for all $a \in M_1$.

The notion of generalized Priestley morphism has a slight drawback: the usual composition of relations does not always produce a generalized Priestley morphism.
when applied to two of them; hence it can not be taken as a category-theoretic operation of composition. Instead, we have that for any generalized Priestley spaces $X_1$, $X_2$ and $X_3$ and any generalized Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, the operation of composition of $R$ and $S$ that we need in order to obtain a category is the relation $S \star R \subseteq X_1 \times X_3$ defined by:

$$(x, z) \in (S \star R) \iff \forall U \in X_3^* (x \in (\sqcup_R \circ \sqcup_S)(U) \Rightarrow z \in U).$$

2.3. Congruential logics. Congruential logic is a concept studied in abstract algebraic logic. To be able present it we need to go throughout some of the basic concepts of this field. We follow the survey [9]. Given a logical language $\mathcal{L}$ (i.e., a set of connectives, a.k.a. function symbols, possibly of arity 0), a logic $L$ is a pair $\mathcal{S} := \langle \text{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}} \rangle$, where $\text{Fm}_{\mathcal{L}}$ is the algebra of formulas of $\mathcal{L}$ (i.e., the absolutely free algebra of type $\mathcal{L}$ over a countably infinite set of generators: the variables) and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\text{Fm}_{\mathcal{L}}) \times \text{Fm}_{\mathcal{L}}$ is a substitution-invariant consequence relation on the set of formulas $\text{Fm}_{\mathcal{L}}$.

A logic $\mathcal{S}$ is finitary when for all $\Gamma \cup \{\delta\} \subseteq \text{Fm}_{\mathcal{L}}$, if $\Gamma \vdash_{\mathcal{S}} \delta$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathcal{S}} \delta$. A logic $\mathcal{S}$ has theorems when there is at least one formula $\delta \in \text{Fm}_{\mathcal{S}}$ such that $\emptyset \vdash_{\mathcal{S}} \delta$.

From now on let $\mathcal{S}$ be a logic in the language $\mathcal{L}$. We say that an algebra $A$ has the same type as $\mathcal{S}$ when the algebraic language of $A$ is $\mathcal{L}$. In what follows, when we pick an arbitrary algebra $A$ we assume that it is an algebra of the same type as $\mathcal{S}$, if not stated otherwise.

**Definition 2.4.** Let $A$ be an algebra. A set $F \subseteq A$ is an $\mathcal{S}$-filter of $A$ when for every $h \in \text{Hom}(\text{Fm}_{\mathcal{L}}, A)$ and for every set of formulas $\Gamma \cup \{\delta\} \subseteq \text{Fm}_{\mathcal{L}}$:

if $\Gamma \vdash_{\mathcal{S}} \delta$ and $h(\gamma) \in F$ for all $\gamma \in \Gamma$, then $h(\delta) \in F$.

We denote by $\text{Fi}_{\mathcal{S}}(A)$ the collection of all $\mathcal{S}$-filters of $A$. This collection is always a closure system and therefore a complete lattice under the order of inclusion. And if $\mathcal{S}$ is a finitary logic, it is a finitary closure system. We call the meet-irreducible elements of the lattice of $\mathcal{S}$-filters irreducible $\mathcal{S}$-filters, and we denote by $\text{Ir}_{\mathcal{S}}(A)$ the collection of all of them. Let us denote by $Fg_{\mathcal{S}}^A$ the closure operator associated with $\text{Fi}_{\mathcal{S}}(A)$. Thus, for any subset $U \subseteq A$, $Fg_{\mathcal{S}}^A(U)$ denotes the least $\mathcal{S}$-filter of $A$ that contains $U$. We abbreviate as usual $Fg_{\mathcal{S}}^A(\{a\})$ by $Fg_{\mathcal{S}}^A(a)$. Recall that when $\text{Fi}_{\mathcal{S}}(A)$ is finitary, then $Fg_{\mathcal{S}}^A$ is a finitary closure operator.

A basic property of $\mathcal{S}$-filters whose use is ubiquitous in abstract algebraic logic is the following. Let $A_1, A_2$ be algebras. For every $h \in \text{Hom}(A_1, A_2)$ and every $\mathcal{S}$-filter $F$ of $A_2$, $h^{-1}[F]$ is an $\mathcal{S}$-filter of $A_1$.

A logic $\mathcal{S}$ is filter-distributive when for every algebra $A$, the lattice of $\mathcal{S}$-filters of $A$ is distributive.

The closure operator $Fg_{\mathcal{S}}^A$ allows us to define the specialization quasiorder $\preceq_{\mathcal{S}}^A$ on $A$ by saying that for all $a, b \in A$:

$$a \preceq_{\mathcal{S}}^A b \iff Fg_{\mathcal{S}}^A(b) \subseteq Fg_{\mathcal{S}}^A(a).$$

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1This means that the following conditions are satisfied:

(C1) if $\gamma \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \gamma$,

(C2) if $\Delta \vdash_{\mathcal{S}} \gamma$ for all $\gamma \in \Gamma$ and $\Gamma \vdash_{\mathcal{S}} \delta$, then $\Delta \vdash_{\mathcal{S}} \delta$,

(C3) if $\Gamma \vdash_{\mathcal{S}} \delta$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\delta)$ for all substitutions $\sigma \in \text{Hom}(\text{Fm}_{\mathcal{L}}, \text{Fm}_{\mathcal{L}})$. 

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We denote by $\equiv_A$ the equivalence relation associated with $\leq_A$, i.e., $\equiv_A := \leq_A \cap \geq_A$. We use this relation to introduce the following concepts:

**Definition 2.5.** A logic $\mathcal{S}$ is congruential\(^2\) when for every algebra $A$, the relation $\equiv_A$ is a congruence of $A$.

This definition is equivalent to the more usual one of the concept as shown in [8, Prop. 2.42]. The next definition is also equivalent to the more usual one given in for example [8].

**Definition 2.6.** An algebra $A$ is an $\mathcal{S}$-algebra when for every congruence $\theta$ of $A$, if $\theta \subseteq \equiv_A$, then $\theta$ is the identity. We denote by Alg$\mathcal{S}$ the collection of all $\mathcal{S}$-algebras. This class of algebras is the algebraic counterpart of $\mathcal{S}$.

Note that the trivial algebras, namely the algebras with a single element, are $\mathcal{S}$-algebras.

Many well-known logics, including classical and intuitionistic propositional logics, are congruential. The next theorem provides a useful characterization of congruentiality.

**Theorem 2.7** ([10, Theorem 2.2]). A logic $\mathcal{S}$ is congruential if and only if for any algebra $A$, $A \in$ Alg$\mathcal{S}$ if and only if $(A, \leq_A^S)$ is a poset.

Notice that when $\mathcal{S}$ is congruential, for every $\mathcal{S}$-algebra $A$ we have the collection $\mathrm{Fi}(A)$ of the order filters of the poset $(A, \leq_A^S)$ and the collection $\mathrm{Id}(A)$ of its order ideals. All $\mathcal{S}$-filters of $A$ are up-sets with respect to $\leq_A^S$, but not necessarily order filters of $(A, \leq_A^S)$ because they may not be down-directed. Note also that for every $a \in A$, $Fg^A_S(a) = \uparrow_{\leq_A^S} a$. When the context is clear, we drop the subscripts of $\uparrow_{\leq_A^S}$ and $\downarrow_{\leq_A^S}$. When a congruential logic $\mathcal{S}$ has theorems, all $\mathcal{S}$-filters of $A$ are non-empty, and the poset $(A, \leq_A^S)$ has a top element, that we denote by $1_A$. Note that then $\{1_A\}$ is the least $\mathcal{S}$-filter of $A$. Furthermore, from the previous theorem we infer that for any congruential logic $\mathcal{S}$,

$$\text{Alg}\mathcal{S} = \{A : \equiv_A^S \text{ is the identity}\}.$$

We recall now the definition of $\mathcal{S}$-ideal given in [10].

**Definition 2.8.** A subset $I \subseteq A$ is an $\mathcal{S}$-ideal of an algebra $A$ provided that for any finite $I' \subseteq I$ and any $a \in A$, if $\bigcap\{Fg^A_S(b) : b \in I'\} \subseteq Fg^A_S(a)$, then $a \in I$.

We denote by $\mathrm{Id}_S(A)$ the collection of all $\mathcal{S}$-ideals of $A$. Notice that when $A$ is an $\mathcal{S}$-algebra of a congruential logic, then the $\mathcal{S}$-ideals of $A$ are exactly the Frink ideals of the poset $(A, \leq_A^S)$. Then we have that $\emptyset \in \mathrm{Id}_S(A)$ if and only if the poset $(A, \leq_A^S)$ has no bottom element. We are interested in a certain type of $\mathcal{S}$-ideals; they will help to define the notion of optimal $\mathcal{S}$-filter we need.

**Definition 2.9.** An $\mathcal{S}$-ideal $I$ of $A$ is strong when for any finite $I' \subseteq I$ and any non-empty and finite $B \subseteq A$, if $\bigcap\{Fg^A_S(b) : b \in I'\} \subseteq Fg^A_S(B)$, then $Fg^A_S(B) \cap I \neq \emptyset$.

Note that the definition implies that $A$ is a strong $\mathcal{S}$-ideal.

\(^2\)We follow here the terminology used in [10]. Congruential logics were previously called strongly selfextensional [8] and fully selfextensional [12]. The last terminology is currently widely used in abstract algebraic logic.
**Lemma 2.10.** Let $S$ be a congruential logic and $A$ an $S$-algebra. A set $I \subseteq A$ is a strong $S$-ideal if and only if it is a down-set w.r.t. $\leq^A_S$ and satisfies the condition of Definition 2.9.

**Proof.** We only need to proof the implication from right to left. Assume that $I$ is a down-set w.r.t. $\leq^A_S$ and satisfies the condition of Definition 2.9. We have to see that it is an $S$-ideal. Suppose that $I' \subseteq I$ is finite and $\bigcap\{Fg^A_S(b) : b \in I'\} \subseteq Fg^A_S(a)$ for $a \in A$. By the condition on Definition 2.9, $I \cap Fg^A_S(a) \neq \emptyset$. Thus there is $c \in I$ such that $a \leq^A_S c$. Hence $a \in I$. \hfill $\square$

**Remark 2.11.** If $S$ has theorems, then in the definition of strong $S$-ideal we can delete the condition that $B$ is non-empty and we obtain an equivalent definition. If we take $B$ possibly empty in the case that $S$ does not have theorems, then $A$ may not be a strong $S$-ideal in case that there are $a, b \in A$ such that $Fg^A_S(a) \cap Fg^A_S(b) = \emptyset$.

We denote by $\text{Id}_{sS}(A)$ the collection of all strong $S$-ideals. It is easy to check that when $A$ is an $S$-algebra all order ideals of $(A, \leq^A_S)$ are strong $S$-ideals, in particular for every $a \in A$, $\downarrow a$ is a strong $S$-ideal. The next notion helps to characterize when the emptyset is a strong $S$-ideal.

**Definition 2.12.** Let $S$ be a congruential logic and $A$ an $S$-algebra. A non-empty finite set $U \subseteq A$ of incomparable elements with respect to $\leq^A_S$ is a bottom-family of $A$ when $Fg^A_S(U) = A$.

Note that if $\leq^A_S$ has a bottom element, then its singleton is a bottom-family. In particular, in the trivial $S$-algebras the domain of the algebra is a bottom-family.

**Lemma 2.13.** Let $S$ be a congruential logic and $A$ an $S$-algebra. Then $A$ has a bottom-family if and only if there exists a finite $U \subseteq A$ such that $Fg^A_S(U) = A$.

**Proof.** The right implication is an immediate consequence of the definition. Let $U \subseteq A$ be finite and such that $Fg^A_S(U) = A$. If $U = \emptyset$, then for every $a \in A$, $Fg^A_S(a) = A$. Therefore $A$ is trivial and hence has a bottom-family. If $U \neq \emptyset$, then being finite it is easy to see that there is a non-empty $U' \subseteq U$ which is a set of incomparable elements w.r.t. $\leq^A_S$ and $Fg^A_S(U') = A$. Hence $A$ has a bottom-family. \hfill $\square$

A straightforward argument shows that $\emptyset \in \text{Id}_{sS}(A)$ if and only if $A$ has no bottom-family.

We are now in a position to define the notion of optimal $S$-filter. Note the similarity with the definition of optimal filter of a meet-semilattice.

**Definition 2.14.** An $S$-filter $F \in \text{Fg}_S(A)$ is optimal when there is a strong $S$-ideal $I \in \text{Id}_{sS}(A)$ such that $F$ is a maximal element of the collection $\{G \in \text{Fg}_S(A) : G \cap I = \emptyset\}$ and $I$ is a maximal element of the collection $\{J \in \text{Id}_{sS}(A) : F \cap J = \emptyset\}$.

We denote by $\text{Op}_S(A)$ the collection of all optimal $S$-filters of $A$.

**Remark 2.15.** From the definition it follows that $\emptyset \in \text{Id}_{sS}(A)$ if and only if $A \in \text{Op}_S(A)$. Therefore, $A \in \text{Op}_S(A)$ if and only if $A$ has no bottom-family. Hence, in the trivial $S$-algebras there is no non-empty optimal $S$-filter.

**Remark 2.16.** If $S$ does not have theorems, then since $\emptyset$ is an $S$-filter and $A$ a strong $S$-ideal, $\emptyset$ is an optimal $S$-filter.
For any finitary congruential logic, we have the following two separation lemmas that we gather in one proposition. They rely on Zorn’s lemma and the fact that for any logic $\mathcal{S}$ both $Id_{\mathcal{S}}(\mathbf{A})$ and $Id(\mathbf{A})$ are closed under unions of non-empty chains.

**Proposition 2.17** (Optimal and irreducible $\mathcal{S}$-filter lemmas). Let $\mathcal{S}$ be a finitary congruential logic, $\mathbf{A}$ an $\mathcal{S}$-algebra and $F \in \text{Fi}_{\mathcal{S}}(\mathbf{A})$,

1. If $I \in \text{Id}_{\mathcal{S}}(\mathbf{A})$ is such that $F \cap I = \emptyset$, then there exists $Q \in \text{Op}_{\mathcal{S}}(\mathbf{A})$ such that $F \subseteq Q$ and $Q \cap I = \emptyset$.
2. If $I \in \text{Id}(\mathbf{A})$ is such that $F \cap I = \emptyset$, then there exists $Q \in \text{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $F \subseteq Q$ and $Q \cap I = \emptyset$.

**Proof.** (1) First of all note that $Id_{\mathcal{S}}(\mathbf{A})$ is closed under unions of non-empty chains. Let $F \in \text{Fi}_{\mathcal{S}}(\mathbf{A})$ and $I \in \text{Id}_{\mathcal{S}}(\mathbf{A})$ be such that $F \cap I = \emptyset$ and consider the set $\mathcal{F} := \{ G \in \text{Fi}_{\mathcal{S}}(\mathbf{A}) : F \subseteq G \land G \cap I = \emptyset \}$, which is non-empty and closed under unions of non-empty chains because the closure operator of $\mathcal{S}$-filter generation is finitary. By Zorn’s lemma there exists a maximal element $Q$ of $\mathcal{F}$. Consider now the set $\mathcal{I} := \{ H \in \text{Id}_{\mathcal{S}}(\mathbf{A}) : I \subseteq H \land H \cap Q = \emptyset \}$. Then $I \in \mathcal{I}$ and $\mathcal{I}$ is closed under unions of non-empty chains. Let, by Zorn’s lemma, $H \in \mathcal{I}$ be maximal. Then $H$ is $Q$-maximal and it is easy to see that $Q$ is $H$-maximal. Therefore, $Q$ is optimal.

(2) Let $F \in \text{Fi}_{\mathcal{S}}(\mathbf{A})$ and $I \in \text{Id}(\mathbf{A})$ be such that $F \cap I = \emptyset$. Consider the set $\mathcal{F}' := \{ G \in \text{Fi}_{\mathcal{S}}(\mathbf{A}) : F \subseteq G \land G \cap I = \emptyset \}$. Then $F \in \mathcal{F}'$ and $\mathcal{F}'$ is closed under unions of non-empty chains. By Zorn’s lemma we take a maximal element $Q'$ of $\mathcal{F}'$. Clearly, since $I \neq \emptyset$, then $Q'$ is proper. To show that $Q'$ is irreducible suppose that $F_1, F_2 \in \text{Fi}_{\mathcal{S}}(\mathbf{A})$ are such that $F_1 \cap F_2 = Q$ and in search of a contradiction that $F_1 \neq Q$ and $F_2 \neq Q$. Let then $a \in F_1 \setminus Q$ and $b \in F_2 \setminus Q$. By the maximality of $Q$ in $\mathcal{F}'$, there are $a' \in \text{Fg}_{\mathcal{S}}(Q \cup \{ a \}) \cap I$ and $b' \in \text{Fg}_{\mathcal{S}}(Q \cup \{ b \}) \cap I$. Since $I$ is up-directed let $c \in I$ such that $a' \leq_{\mathcal{S}} c$. Then $c \in \text{Fg}_{\mathcal{S}}(Q \cup \{ a \}) \cap \text{Fg}_{\mathcal{S}}(Q \cup \{ b \}) \subseteq F_1 \cap F_2 = Q$, contradicting the fact that $Q \cap I = \emptyset$. \hfill $\square$

When we restrict ourselves to filter-distributive logics, we have good characterizations of the optimal and of the irreducible $\mathcal{S}$-filters.

**Theorem 2.18.** Let $\mathcal{S}$ be a filter-distributive, finitary, and congruential logic and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. Then for every $F \in \text{Fi}_{\mathcal{S}}(\mathbf{A})$,

1. $F \in \text{Op}_{\mathcal{S}}(\mathbf{A})$ if and only if $F^{c} \in \text{Id}_{\mathcal{S}}(\mathbf{A})$,
2. $F \in \text{Irr}_{\mathcal{S}}(\mathbf{A})$ if and only if $F^{c} \in \text{Id}(\mathbf{A})$.

**Proof.** (1) If $F^{c} \in \text{Id}_{\mathcal{S}}(\mathbf{A})$, then obviously $F$ is $F^{c}$-maximal and $F^{c}$ is $F$-maximal; hence $F$ is optimal. To prove the converse, suppose that $F$ is optimal. Let then $I \in \text{Id}_{\mathcal{S}}(\mathbf{A})$ be such that $F$ is $I$-maximal and $I$ is $F$-maximal. We prove that $F^{c} \in \text{Id}_{\mathcal{S}}(\mathbf{A})$. We reason by cases. If $F = A$, then $F^{c} = \emptyset$ and therefore $I = \emptyset = F^{c}$; hence $F^{c} \in \text{Id}_{\mathcal{S}}(\mathbf{A})$. Suppose then that $F \neq A$. Let $B$ be a finite and non-empty subset of $A$ and $H$ a finite subset of $F^{c}$. Suppose that $\bigcap \{ \text{Fg}_{\mathcal{S}}(b) : b \in H \} \subseteq \text{Fg}_{\mathcal{S}}(B)$. If $H = \emptyset$, then $\text{Fg}_{\mathcal{S}}(B) = A$, and since $F^{c} \neq \emptyset$, $\text{Fg}_{\mathcal{S}}(B) \cap F^{c} \neq \emptyset$. If $H \neq \emptyset$, since $F$ is $I$-maximal, for every $b \in H$, let $a_{b} \in \text{Fg}_{\mathcal{S}}(F, b) \cap I$, and by the finitariness of the closure operator $\text{Fg}_{\mathcal{S}}$, let $F_{b} \subseteq F$ be a finite and such that $a_{b} \in \text{Fg}_{\mathcal{S}}(F_{b}, b)$. Consider then the finite set $G = \bigcup_{b \in H} F_{b}$. Then for every $b \in H$,
From the fact that \( I \) is a strong \( S \)-ideal that includes \( \{a_b : b \in H\} \), it follows that 
\( \text{Fg}_S^A(G \cup B) \cap I \neq \emptyset \). Suppose towards a contradiction that \( \text{Fg}_S^A(B) \cap F^c = \emptyset \). Then 
\( \text{Fg}_S^A(B) \subseteq F \) and therefore \( \text{Fg}_S^A(G \cup B) \subseteq F \). Since \( G \subseteq F \) we get that \( F \cap I \neq \emptyset \), a contradiction.

(2) Suppose that \( F \in \text{Irr}_S(A) \). By assumption \( F \) is a proper up-set w.r.t. \( \leq^A \), and hence \( F^c \) is a non-empty down-set. To show that it is up-directed, let \( a, b \in F^c \), so that 
\( \text{Fg}_S^A(a), \text{Fg}_S^A(b) \nsubseteq F \). Since \( \text{Fg}_S^A(A) \) is distributive and \( F \) is meet-irreducible, \( F \) is meet-prime. Therefore, 
\( \text{Fg}_S^A(a) \cap \text{Fg}_S^A(b) \nsubseteq F \). Let \( c \in \text{Fg}_S^A(a) \cap \text{Fg}_S^A(b) \) be such that \( c \notin F \). Then \( a, b \leq^A c \) and \( c \in F^c \). Thus \( F^c \) is up-directed. To prove the converse, assume that \( F^c \in \text{Id}(A) \). We show that \( F \) is a meet-prime element of \( \text{Fg}_S(A) \).

Since \( F^c \) is non-empty, \( F \) is proper. Suppose that \( F_1, F_2 \in \text{Fg}_S(A) \) are such that \( F_1 \cap F_2 \subseteq F \), \( F_1 \nsubseteq F \) and \( F_2 \nsubseteq F \). Let then \( b_1 \in F_1 \setminus F \) and \( b_2 \in F_2 \setminus F \). Since \( F^c \) is up-directed, let \( c \in F^c \) be such that \( b_1, b_2 \leq^A c \). Then \( c \in \text{Fg}_S^A(b_1) \cap \text{Fg}_S^A(b_2) \subseteq F_1 \cap F_2 \) and hence \( c \in F \), a contradiction. \( \square \)

As a consequence we have that when \( S \) is a filter-distributive, finitary, and congruential logic all the optimal \( S \)-filters are proper if and only if \( A \) has a bottom-family.

Remark 2.19. We can separate the proof if we like in two cases depending on whether \( S \) has theorems or not. In the second case, condition (1) holds for \( \emptyset \) even if \( S \) is not filter-distributive, so the real proof handles the non-empty case. But even in this case, to carry on the proof we need the given definition of strong \( S \)-ideal (with \( B \) non-empty).

We conclude this section by introducing a concept of primness that will be used later on.

Definition 2.20. A subset \( X \subseteq A \) is \( S \)-prime when it is a proper subset such that for all nonempty and finite \( B \subseteq A \), if \( \text{Fg}_S^A(B) \cap X \neq \emptyset \), then \( B \cap X \neq \emptyset \).

Lemma 2.21 ([7, Lemma 4.4.11]). Let \( S \) be a finitary logic, \( A \) an \( S \)-algebra and \( X \subseteq A \) non-empty. Then \( X \in \text{Fg}_S(A) \) if and only if \( X^c \) is \( S \)-prime.

Remark 2.22. The requirement that \( X \) is non-empty is important when \( S \) has no theorems. Otherwise, the lemma would fail for \( \emptyset \). The definition of \( S \)-prime implies that \( A \) is not \( S \)-prime.

3. Representation theorem for \( S \)-algebras

From now on we focus on congruential logics. Let us fix an arbitrary congruential logic \( S \) and an \( S \)-algebra \( A \). We are interested in closure bases for the closure system \( \text{Fg}_S(A) \). Recall that \( F \subseteq \text{Fg}_S(A) \) is a closure base for \( \text{Fg}_S(A) \) provided that any
Theorem 3.3. Let \( S \) be a congruential logic, \( A \) an \( S \)-algebra and \( \mathcal{F} \) a closure base for \( \text{Fi}_S(A) \). Then

1. \( \{ \varphi_F[P] : P \in \mathcal{F} \} \) is a closure base for \( \text{Fi}_S(\varphi_F[A]) \),
2. for any \( a \in A \) and any \( B \subseteq A \):
   \[
a \in \text{F}_S^\mathcal{F}(B) \iff \varphi_F(B) \subseteq \varphi_F(a) \iff \varphi_F(a) \in \text{F}_S^\mathcal{F}(\varphi_F[B]).
   \]
that converse assume that $M(\cdot)$ and there is no optimal $\emptyset$. 

**Proof.** Let Lemma 4.2.

Lemma concerning the bottom element and bottom-families handles the situation.

element, which is $\emptyset$. Hence, if $\emptyset \notin \text{Id}_S(A)$, then $\emptyset \notin \text{Id}_F(M(A))$.

**Proposition 3.4.** Let $S$ be a finitary, filter-distributive and congruential logic, $A$ an $S$-algebra and $\mathcal{F}$ a closure base for $\text{Fi}_S(A)$. Then, for every $F \in \text{Op}_S(A)$, $F \in \text{Irr}_S(A)$ if and only if $\varphi_F[F^c] = \{ \varphi_F(a) : a \notin F \}$ is non-empty and up-directed in $\langle \varphi_F[A], \subseteq \rangle$.

**Proof.** Let $F \in \text{Op}_S(A)$. Note that from Theorem 3.2 we get that $F^c$ is an order ideal of $\langle A, \leq_A \rangle$ if and only if $\{ \varphi_F(a) : a \notin F \}$ is non-empty and up-directed in $\langle \varphi_F[A], \subseteq \rangle$. From (2) of Theorem 2.18 we have that $F \in \text{Irr}_S(A)$ if and only if $F^c$ is an $\leq_A$-order ideal of $A$. Thus we obtain the proposition. \qed

4. THE $S$-SEMILATTICE OF AN $S$-ALGEBRA $A$

The $S$-semilattice of an $S$-algebra $A$ of a finitary congruential logic $S$ was introduced in [10]. It can be described as the dual of the join-semilattice of the finitely generated $S$-filters of $A$. Alternatively, to obtain a specific definition of it we can work with certain closure bases $\mathcal{F}$ for $\text{Fi}_S(A)$, and obtain the $S$-semilattice of $A$ as the closure under finite intersections of the image of $A$ under the representation map $\varphi_\mathcal{F}$.

Let $S$ be a congruential logic, $A$ an $S$-algebras and $\mathcal{F}$ a closure base for the closure system $\text{Fi}_S(A)$. We denote by $M_\mathcal{F}(A)$ the closure of $\varphi_\mathcal{F}[A]$ under intersections of finite subsets. In other words, $M_\mathcal{F}(A) = \{ \varphi_\mathcal{F}(B) : B \subseteq A \}$. Note that $\mathcal{F} \in M_\mathcal{F}(A)$ because $\mathcal{F} = \varphi_\mathcal{F}(\emptyset)$. By convenience, we dispense with the subscript $\mathcal{F}$ in $\varphi_\mathcal{F}$ and $M_\mathcal{F}(A)$, and we use $\varphi$ and $M(A)$ instead. Notice that we are justified to do so because from Theorem 3.3 it follows that for closure bases $\mathcal{F}$ and $\mathcal{F}'$ for $A$, $\langle M_\mathcal{F}(A), \cap, \mathcal{F} \rangle$ and $\langle M_\mathcal{F'}(A), \cap, \mathcal{F}' \rangle$ are isomorphic semilattices.

**Definition 4.1.** For any congruential logic $S$ and any $S$-algebra $A$, the algebra $M(A) := \langle M(A), \cap, \mathcal{F} \rangle$ is called the $S$-semilattice of $A$.

By definition, $M_\mathcal{F}(A)$ is a meet-semilattice with top element $\mathcal{F}$. If $S$ has theorems, then $\varphi(1^A) = \mathcal{F}$. In this case we can describe $M_\mathcal{F}(A)$ as the set $\{ \varphi_\mathcal{F}(B) : B \subseteq A \}$.

When $S$ is finitary, the optimal and the irreducible $S$-filter lemmas (Proposition 2.17) imply that for every $S$-algebra $A$ the sets $\text{Op}_S(A)$ and $\text{Irr}_S(A)$ are closure bases for $\text{Fi}_S(A)$. The closure bases $\mathcal{F} \subseteq \text{Op}_S(A)$ for $\text{Fi}_S(A)$ of an $S$-algebra $A$ will be called optimal $S$-bases.

We should be careful when dealing with the bottom element. The following lemma concerning the bottom element and bottom-families handles the situation.

**Lemma 4.2.** Let $S$ be a finitary congruential logic, $A$ an $S$-algebra and $\mathcal{F}$ an optimal $S$-base. Then $A$ has a bottom-family if and only if $M(A)$ has a bottom element, which is $\emptyset$. Hence, if $\emptyset \notin \text{Id}_S(A)$, then $\emptyset \notin \text{Id}_F(M(A))$.

**Proof.** If $A$ has a bottom-family $B$, then $\emptyset \notin \text{Id}_S(A)$. Therefore $A \notin \text{Op}_S(A)$, and there is no optimal $S$-filter containing $B$. So $\varphi(B) = \emptyset \in M(A)$. To prove the converse assume that $M(A)$ has a bottom element. Let $B \subseteq A$ be a finite set such that $\varphi(B)$ is the bottom element of $M(A)$. Then $\varphi(B) \subseteq \varphi(a)$ for every $a \in A$. 

(3) for all $B \subseteq A$, $F_{gS}(\varphi_\mathcal{F}[B]) = \varphi_\mathcal{F}[F_{gS}(B)]$.

(4) for all $B, B' \subseteq A$, $F_{gS}(B) = F_{gS}(B')$ iff $\varphi_\mathcal{F}(B) = \varphi_\mathcal{F}(B')$. 

In the case of finitary, filter-distributive and congruential logics we have the following characterization of the irreducible $S$-filters inside the optimal ones.
Hence, by Theorem 3.3 it follows that $Fg^A(B) = A$. If $B = \emptyset$, then $A$ should be a trivial algebra. Hence it has a bottom-family. If $B \neq \emptyset$, we also have that $A$ has a bottom-family. By the first part of the proof this implies that $\emptyset \in M(A)$ and hence the bottom element of $M(A)$ is the emptyset.

We will now study the relations between the different families of filters and ideals in $A$ and $M(A)$ we have considered so far. Let us begin with the $S$-filters of $A$ and the filters of $M(A)$. The operation of meet filter generation $[\cdot]$ is taken in $M(A)$.

**Lemma 4.3.** Let $S$ be a finitary congruential logic, $A$ an $S$-algebra and $F$ an optimal $S$-base. For every finite $B \subseteq A$ and every finite family $\{B_i : i \in K\}$ of finite subsets of $A$:

$$\bigcap_{i \in K} Fg^A(B_i) \subseteq Fg^A(B) \iff \bigcap_{i \in K} \llbracket \bar{\varphi}(B_i) \rrbracket \subseteq \llbracket \bar{\varphi}(B) \rrbracket.$$

**Proof.** We distinguish two cases depending on whether $K$ is empty or not. Assume first that $K \neq \emptyset$. Suppose that $\bigcap \{Fg^A(B_i) : i \in K\} \subseteq Fg^A(B)$ and let $D \subseteq A$ be finite and such that $\bar{\varphi}(D) \in \bigcap \{\llbracket \bar{\varphi}(B_i) \rrbracket : i \leq n\}$, i.e., $\bar{\varphi}(B_i) \subseteq \bar{\varphi}(D)$ for all $i \leq n$. Then, by Theorem 3.3, $D \subseteq Fg^A(B_i)$ for all $i \leq n$. Thus from the hypothesis $D \subseteq Fg^A(B)$ and by Theorem 3.3 again we get $\bar{\varphi}(B) \subseteq \bar{\varphi}(D)$; hence $\bar{\varphi}(D) \in \llbracket \bar{\varphi}(B) \rrbracket$. For the converse, assume that $\bigcap \{\llbracket \bar{\varphi}(B_i) \rrbracket : i \in K\} \subseteq \llbracket \bar{\varphi}(B) \rrbracket$ and let $a \in \bigcap \{Fg^A(B_i) : i \in K\}$. Then for each $i \leq n$, $a \in Fg^A(B_i)$, and so, by Theorem 3.3, $\bar{\varphi}(B_i) \subseteq \varphi(a)$. This implies that $\varphi(a) \in \bigcap \{\llbracket \bar{\varphi}(B_i) \rrbracket : i \in K\}$, and so, by hypothesis $\varphi(a) \in \llbracket \bar{\varphi}(B) \rrbracket$, i.e., $\bar{\varphi}(B) \subseteq \varphi(a)$. Then by Theorem 3.3 again, we get $a \in Fg^A(B)$.

Now we assume that $K = \emptyset$. If $\bigcap \{Fg^A(B_i) : i \in K\} \subseteq Fg^A(B)$, then $Fg^A(B) = A$ and by Lemma 2.13 we obtain that $A$ has a bottom-family. Therefore, $A$ is not an optimal $S$-filter. Thus since the only $S$-filter that includes $B$ is $A$, $\bar{\varphi}(B) = \emptyset$. Moreover, Lemma 4.2 implies that $\emptyset \in M(A)$. Hence, $\llbracket \bar{\varphi}(B) \rrbracket = M(A)$. Conversely if $\bigcap_{i \in K} \llbracket \bar{\varphi}(B_i) \rrbracket \subseteq \llbracket \bar{\varphi}(B) \rrbracket$, then $\llbracket \bar{\varphi}(B) \rrbracket = M(A)$; thus $\bar{\varphi}(B)$ is the bottom element of $M(A)$ and by Lemma 4.2 we have that $\bar{\varphi}(B) = \emptyset$. Using Theorem 3.3 it follows that $Fg^A(B) = A$. □

The next proposition (cf. [7, Lemmas 4.5 and 4.8]) shows the relation between the $S$-filters of $A$ and the filters of $M(A)$ provided $S$ is finitary.

**Proposition 4.4.** Let $S$ be a finitary congruential logic, $A$ an $S$-algebra and $F$ an optimal $S$-base.

1. If $F \in Fis(A)$, then $\llbracket \varphi[F] \rrbracket \in Fi(M(A))$ and $\varphi^{-1} \llbracket \varphi[F] \rrbracket = F$.
2. If $F \in Fi(M(A))$, then $\varphi^{-1}[F] \in Fis(A)$ and $\llbracket F \cap \varphi[A] \rrbracket = F$.

**Proof.** (1) Let $F \in Fis(A)$. By definition $\llbracket \varphi[F] \rrbracket \in Fi(M(A))$ and it is clear that $F \subseteq \varphi^{-1} \llbracket \varphi[F] \rrbracket$. To prove the other inclusion let $a \in \varphi^{-1} \llbracket \varphi[F] \rrbracket$, i.e., $\varphi(a) \in \llbracket \varphi[F] \rrbracket$. Then there is a finite $B \subseteq F$ such that $\bar{\varphi}(B) \subseteq \varphi(a)$. Hence, by Theorem 3.3 we have $a \in Fg^A(B) \subseteq F$.

(2) Let $F \in Fi(M(A))$. By definition $\varphi^{-1}[F] \subseteq Fg^A(\varphi^{-1}[F])$. Let now $a \in Fg^A(\varphi^{-1}[F])$. By finitarity let $B \subseteq \varphi^{-1}[F]$ be finite with $a \in Fg^A(B)$. Then by Theorem 3.3 we have $\bar{\varphi}(B) \subseteq \varphi(a)$. Since $\varphi[B] \subseteq F$, $\bar{\varphi}(B) = \bigcap \varphi[B] \in F$, and since $F$ is an up-set, $\varphi(a) \in F$. Hence $a \in \varphi^{-1}[F]$. This proves the other inclusion.

For the remaining statement, note that $\llbracket \varphi^{-1}[F] \rrbracket = \llbracket F \cap \varphi[A] \rrbracket$, having then $\llbracket F \cap \varphi[A] \rrbracket \subseteq F$. For the other inclusion let $B \subseteq A$ be such that $\bar{\varphi}(B) \in F$. If
\( B = \emptyset, \) then \( \widehat{\varphi}(B) = \mathcal{F} \in [F \cap \varphi[A]] \). If \( B \neq \emptyset \), then \( \varphi[B] \subseteq F \cap \varphi[A] \) and hence \( \widehat{\varphi}(B) = \bigcap \varphi[B] \in [F \cap \varphi[A]] \).

**Corollary 4.5.** For any finitary congruential logic \( S \) and any \( S \)-algebra \( A \), there is an isomorphism between the lattice \( F \pi S(A) \) and the lattice \( Fi(M(A)) \) given by the following maps, each one inverse of the other:

\[
\langle \varphi[I] \rangle : F \pi S(A) \cong Fi(M(A)) : \varphi^{-1}[I].
\]

As a consequence, we obtain that for any finitary congruential logic \( S \) and any \( S \)-algebra \( A \), the lattice \( F \pi S(A) \) is distributive if and only if the lattice \( Fi(M(A)) \) is distributive. Therefore if \( S \) is a filter-distributive logic, then \( M(A) \) is a distributive meet-semilattice. Moreover, the previous isomorphism maps irreducible \( S \)-filters of \( A \) to irreducible filters of \( M(A) \). But even if this happens, it may not be an isomorphism between the optimal \( S \)-filters of \( A \) and the optimal filters of \( M(A) \). However, under the assumption of filter-distributivity of the logic, it is an isomorphism. In order to show it, since the behaviour of optimal \( S \)-filters and optimal filters depends on the behaviour of the strong \( S \)-ideals of \( A \) and the Frink ideals of \( M(A) \), we need to make a detour and study first the relation between strong \( S \)-ideals of \( A \) and Frink ideals of \( M(A) \). Throughout the next proofs we use \( \downarrow \) instead of \( \downarrow_{M(A)} \). The operation \( \downarrow \) of Frink ideal generation is taken in \( M(A) \).

**Lemma 4.6.** Let \( S \) be a finitary congruential logic, \( A \) an \( S \)-algebra and \( F \) an optimal \( S \)-base. For every strong \( S \)-ideal \( I \) of \( A \), \( \langle \varphi[I] \rangle = \downarrow_{M(A)} \varphi[I] \).

**Proof.** Let \( I \in Id_{S}(A) \). If \( I = \emptyset \), then \( A \) has no bottom-family and hence \( M(A) \) has no bottom element, which implies that \( \emptyset \) is a Frink ideal of \( M(A) \). Thus \( \langle \varphi[I] \rangle = \emptyset \) and we are done. If \( I \) is non-empty, it is enough to show that \( \downarrow \varphi[I] \) is a Frink ideal. So let \( K \) be a finite set of indexes and \( \{B_{i} : i \in K \} \) a family of finite subsets of \( A \) such that \( \widehat{\varphi}(B_{i}) \subseteq \varphi(I) \) for all \( i \in K \). Then for every \( i \in K \) there exists \( a_{i} \in I \) such that \( \widehat{\varphi}(B_{i}) \subseteq \varphi(a_{i}) \). Moreover, let \( B \subseteq A \) be finite and such that \( \bigcap \{ \widehat{\varphi}(B_{i}) : i \in K \} \subseteq \langle \varphi(I) \rangle \). If \( K = \emptyset \), then \( \langle \varphi(I) \rangle = M(A) \), and so \( \widehat{\varphi}(B) \) is the bottom element of \( M(A) \) that belongs to \( \varphi[I] \) since \( I \) is non-empty. If \( K \neq \emptyset \), then \( \bigcap \{ \varphi(a_{i}) : i \in K \} \subseteq \langle \varphi(B) \rangle \). From Lemma 4.3 it follows \( \bigcap \{ \text{Frink}_S(a_{i}) : i \in K \} \subseteq \text{Frink}_S(B) \), and then since \( \{a_{i} : i \in K \} \subseteq I \) and \( I \) is a strong \( S \)-ideal, there exists \( c \in \text{Frink}_S(B) \cap I \neq \emptyset \). Then by Theorem 3.3 \( \varphi[B] \subseteq \varphi(c) \subseteq \varphi[I] \), as required.

**Proposition 4.7.** Let \( S \) be a finitary congruential logic, \( A \) an \( S \)-algebra and \( F \) an optimal \( S \)-base:

1. If \( I \in Id_{S}(A) \), then \( \langle \varphi[I] \rangle \in Id_{F}(M(A)) \), \( \varphi^{-1}[\langle \varphi[I] \rangle] = I \) and if \( I \) is \( \wedge \)-prime, then \( \langle \varphi[I] \rangle \) is \( \wedge \)-prime.
2. If \( I \in Id_{F}(M(A)) \) is \( \wedge \)-prime, then \( \varphi^{-1}[I] \in Id_{S}(A) \), \( \langle \varphi[I] \rangle = I \) and it is \( S \)-prime.

**Proof.** (1) Let \( I \in Id_{S}(A) \). By definition \( \langle \varphi[I] \rangle \in Id_{F}(M(A)) \), and clearly \( I \subseteq \varphi^{-1}[\langle \varphi[I] \rangle] \). Let us prove the other inclusion. If \( I = \emptyset \) we get as in the previous proof that \( \langle \varphi[I] \rangle = \emptyset \) and we are done. Assume \( I \neq \emptyset \), and let \( a \in \varphi^{-1}[\langle \varphi[I] \rangle] \), i.e., \( \varphi(a) \in \langle \varphi[I] \rangle \). By the characterization of the generated Frink ideal, there is a finite \( I' \subseteq I \) such that \( \bigcap \{ \varphi(b) : b \in I' \} \subseteq \varphi(a) \). As \( I \neq \emptyset \), we can assume, without loss of generality, that \( I' \neq \emptyset \). Then by Lemma 4.3, \( \bigcap \{ \text{Frink}_S(b) : b \in I' \} \subseteq \text{Frink}_S(a) \). And since \( I \) is an \( S \)-ideal, we get \( a \in I \), as required.
Assume now that $I$ is $S$-prime. Thus $I$ is proper. Using that $\varphi^{-1}[\langle\langle I]\rangle] = I$, \langle\langle I]\rangle should also be proper. To prove that \langle\langle I]\rangle is $\land$-prime, let $B_1, B_2 \subseteq A$ be finite and such that \(\varphi(B_1) \cap \varphi(B_2) \in \langle\langle I]\rangle\). Using that \(\varphi(B_1) \cap \varphi(B_2) = \varphi(B_1 \cup B_2)\) and \(\langle\langle I]\rangle = \downarrow \varphi[I]\) (Lemma 4.6), we get that there is $c \in I$ such that $\varphi(B_1 \cup B_2) \subseteq \varphi(c)$. This implies that $B_1 \cup B_2$ is non-empty, otherwise $c$ is a bottom element and $I$ is not proper. Then by Theorem 3.3, $c \in \text{Fg}_S^\land(B_1 \cup B_2)$, so $\text{Fg}_S^\land(B_1 \cup B_2) \cap I \neq \emptyset$. Moreover, since $I$ is $S$-prime, we get $(B_1 \cup B_2) \cap I \neq \emptyset$, so $B_1 \cap I \neq \emptyset$ or $B_2 \cap I \neq \emptyset$. This implies, by Theorem 3.3 again, that either $\varphi(B_1) \in \downarrow \varphi[I]$ or $\varphi(B_2) \in \downarrow \varphi[I]$. We conclude that \langle\langle I]\rangle is $\land$-prime.

(2) Let now $I \in \text{Id}_F(M(A))$ be $\land$-prime. Thus $I$ is proper. We show that the condition in the definition of strong $S$-filter holds for $\varphi^{-1}[I]$. By Lemma 2.10 this implies that $\varphi^{-1}[I]$ is an $S$-ideal because $\varphi^{-1}[I]$ is easily seen to be a downset. Let $I' \subseteq \varphi^{-1}[I]$ be finite and let $C \subseteq A$ be finite, non-empty, and such that $\bigcap(Fg_S^\land(b) : b \in I') \subseteq Fg_S^\land(C)$. By Lemma 4.3 we have $\bigcap(Fg_S^\land(b) : b \in I') \subseteq \langle\langle C]\rangle$. This implies that $\varphi(C) \in I$. Now, by $\land$-primeness of $I$ we get that $\varphi(c) \in I$ for some $c \in C$ and so $Fg_S^\land(C) \cap \varphi^{-1}[I] \neq \emptyset$.

We proceed to show that \langle\langle [\varphi^{-1}[I]]\rangle = I$. Clearly the inclusion from left to right holds, so we just have to show the other inclusion. Since $I$ is proper, $\varphi(\emptyset) \not\in I$. Let $B \subseteq A$ be non-empty, finite and such that $\varphi(B) \in I$. Then, as $I$ is $\land$-prime, there is $b \in B$, such that $\varphi(b) \in I$. So $\varphi(b) \in \varphi^{-1}[I]$ and as $\varphi(B) \subseteq \varphi(b)$ and Frink ideals are down-sets, we have $\varphi(B) \in \langle\langle [\varphi^{-1}[I]]\rangle$.

Now we prove that $\varphi^{-1}[I]$ is proper. Suppose that it is not. Then $\varphi^{-1}[I] = \varphi[A]$ and since, as we already proved, \langle\langle [\varphi^{-1}[I]]\rangle = I$, we have, $\varphi[A] = I$. We show that $\langle\langle \varphi[A]\rangle = M(A)$. Let $B$ be a finite subset of $A$. Observe that
\[
\bigcap_{a \in B} \uparrow_{M(A)} \varphi(a) \subseteq \uparrow_{M(A)} \varphi(B).
\]
Since $\{\varphi(a) : a \in B\}$ is a finite subset of $\varphi[A]$ it follows that $\varphi(B) \in \langle\langle \varphi[A]\rangle$. Hence, $\langle\langle \varphi[A]\rangle = M(A)$ and therefore $I$ is not proper. To conclude we show that $\varphi^{-1}[I]$ is $S$-prime. Let $B \subseteq A$ be non-empty, finite and such that $\text{Fg}_S^\land(B) \cap \varphi^{-1}[I] \neq \emptyset$, and let $c \in \text{Fg}_S^\land(B)$, then by Theorem 3.3 $\varphi(B) \subseteq \varphi(c)$. Moreover, since $\varphi(c) \in I$, and $I$ is a down-set, we get $\varphi(B) \in I$. Now, as $I$ is $\land$-prime, there is $b \in B$ such that $\varphi(b) \in I$, so $B \cap \varphi^{-1}[I] \neq \emptyset$, as required.

**Corollary 4.8.** For any finitary congruential logic $S$ and any $S$-algebra $A$, the map $\langle\langle .\rangle\rangle$ suitably restricted establishes an isomorphism between the poset of the $S$-prime and strong $S$-ideals of $A$ and the poset of the $\land$-prime Frink ideals of $M(A)$; and the inverse map is given by $\varphi^{-1}$.

**Proof.** It follows from Proposition 4.7.

Up to this point, all the results in the present section are valid in general for any finitary congruential logic. If we assume besides that the logic is filter-distributive, then we get further results.

**Lemma 4.9.** Let $S$ be a filter-distributive finitary congruential logic and $A$ an $S$-algebra. For any finite $B, B_1, \ldots, B_n \subseteq A$:
\[
\bigcap_{i \leq n} \text{Fg}_S^\land(B_i) \subseteq \text{Fg}_S^\land(B) \iff \varphi(B) \subseteq \bigcup_{i \leq n} \varphi(B_i).
\]
Proof. Assume \( \bigcap \{ F_{S} \mathcal{A}(B_i) : i \leq n \} \subseteq F_{S} \mathcal{A}(B) \) and let \( G \in \widehat{\mathcal{G}}(B) \). Then we have \( B \subseteq G \), and so \( F_{S} \mathcal{A}(B) \subseteq G \). Suppose, towards a contradiction, that \( G \notin \bigcup \{ \widehat{\mathcal{G}}(B_i) : i \leq n \} \). Then for each \( i \leq n \) there is \( b_i \in B_i \) such that \( b_i \notin G \). Notice that then \( \bigcap \{ F_{S} \mathcal{A}(B_i) : i \leq n \} \subseteq \bigcap \{ F_{S} \mathcal{A}(B_i) : i \leq n \} \subseteq F_{S} \mathcal{A}(B) \). As \( G \) is an optimal \( S \)-filter, by Theorem 2.18 we know that \( G^c \) is a strong \( S \)-ideal such that \( \{ b_i : i \leq n \} \subseteq G^c \), and thus we obtain \( F_{S} \mathcal{A}(B) \cap G^c \neq \emptyset \), a contradiction. For the converse, assume \( \widehat{\mathcal{G}}(B) \subseteq \bigcup \{ \widehat{\mathcal{G}}(B_i) : i \leq n \} \) and let \( a \in \bigcap \{ F_{S} \mathcal{A}(B_i) : i \leq n \} \). Thus \( \widehat{\mathcal{G}}(B_i) \subseteq \varphi(a) \) for all \( i \leq n \), and then by the assumption and Theorem 3.3 we obtain \( a \in F_{S} \mathcal{A}(B) \). □

Proposition 4.10. Let \( S \) be a filter-distributive and finitary congruential logic, \( \mathcal{A} \) an \( S \)-algebra, and \( F \) an optimal \( S \)-base:

1. If \( F \in \text{Op}_{S}(\mathcal{A}) \) is non-empty, then \( \varphi[F] \in \text{Op}(\mathcal{A}) \).
2. If \( F \in \text{Op}(\mathcal{A}) \), then \( \varphi^{-1}[F] \in \text{Op}_{S}(\mathcal{A}) \).

Proof. (1) Let \( F \in \text{Fis}_{S}(\mathcal{A}) \) be optimal and non-empty. Then by Theorem 2.18 and Lemma 2.21, \( F^c \) is an \( S \)-prime strong \( S \)-ideal of \( \mathcal{A} \), and so by Proposition 4.7 \( \langle \varphi[F] \rangle \) is a \( \wedge \)-prime Frink ideal of \( \mathcal{A} \), hence proper, and moreover by Proposition 2.1 \( \langle \varphi[F] \rangle^c \) is an optimal filter of \( \mathcal{A} \). Therefore, it is enough to show that \( \langle \varphi[F] \rangle \equiv \langle \varphi[F^c] \rangle \). To this end, we prove first the inclusion from right to left. Let \( B \subseteq A \) be finite and such that \( \widehat{\mathcal{G}}(B) \in \langle \varphi[F^c] \rangle^c \). If \( B = \emptyset \), then \( \widehat{\mathcal{G}}(B) = \mathcal{F} \) and hence \( \widehat{\mathcal{G}}(B) \in \langle \varphi[F] \rangle \). If \( B \) is non-empty, then for all \( b \in B \), \( b \notin \varphi(B) \), and thus by Proposition 4.7 we get that \( b \notin \varphi^{-1}[\varphi[F^c]] \). Therefore, \( \varphi^{-1}[\varphi[F^c]] \) for all \( b \in B \), and thus \( \bigcap \{ \varphi(b) : b \in B \} = \varphi(B) \in \langle \varphi[F] \rangle \). For the other inclusion, let \( B \subseteq A \) be finite and such that \( \widehat{\mathcal{G}}(B) \in \langle \varphi[F^c] \rangle \). Then either \( \widehat{\mathcal{G}}(B) = \mathcal{F} \) or there is a non-empty and finite \( B' \subseteq F \) such that \( \varphi(B') \subseteq \varphi(B) \). In the first case, since \( \langle \varphi[F^c] \rangle \) is proper, \( \varphi(B) \notin \langle \varphi[F^c] \rangle \), and we are done. In the second case, by Lemma 4.9, \( F_{S} \mathcal{A}(B) \subseteq F_{S} \mathcal{A}(B') \subseteq F \), so \( B \subseteq F \). Therefore, for all \( b \in B \), \( b \notin F^c = \varphi^{-1}[\varphi[F^c]] \). Hence \( \varphi(b) \notin \langle \varphi[F^c] \rangle \) for all \( b \in B \). Moreover, since \( \langle \varphi[F^c] \rangle \) is a \( \wedge \)-prime Frink ideal, \( \varphi(B) \notin \langle \varphi[F^c] \rangle \), i.e., \( \varphi(B) \in \langle \varphi[F^c] \rangle^c \), as required.

(2) For \( F \in \text{Op}(\mathcal{A}) \), by Corollary 2.1 \( F^c \) is a \( \wedge \)-prime Frink ideal of \( \mathcal{A} \), and so by Proposition 4.7, \( \varphi^{-1}[F^c] \) is an \( S \)-prime strong \( S \)-ideal of \( \mathcal{A} \), and moreover by Lemma 2.21 and Theorem 2.18, \( \varphi^{-1}[F^c] \) is an optimal \( S \)-filter of \( \mathcal{A} \). Notice that \( \langle \varphi^{-1}[F^c] \rangle \in \langle \varphi^{-1}[F] \rangle \). Therefore \( \varphi^{-1}[F] \) is an optimal \( S \)-filter of \( \mathcal{A} \). □

Corollary 4.11. For any filter-distributive and finitary congruential logic \( S \) and any \( S \)-algebra \( \mathcal{A} \), there is an order isomorphism between the poset \( \langle \text{Op}_{S}(\mathcal{A}) \setminus \{ \emptyset \}, \subseteq \rangle \) and the poset \( \langle \text{Op}(\mathcal{A}), \subseteq \rangle \) given by the following maps, one inverse to the other:

\[
\langle \varphi[\cdot] \rangle : \langle \text{Op}_{S}(\mathcal{A}) \setminus \{ \emptyset \}, \subseteq \rangle \cong \langle \text{Op}(\mathcal{A}), \subseteq \rangle : \varphi^{-1}[\cdot].
\]

Note that if \( S \) has theorems, then for every algebra \( \mathcal{A} \) all the \( S \)-filters are non-empty, in particular so are the optimal filters of the \( S \)-algebras. Hence in this case the corollary establishes an isomorphism between \( \langle \text{Op}_{S}(\mathcal{A}), \subseteq \rangle \) and \( \langle \text{Op}(\mathcal{A}), \subseteq \rangle \).

Also note that the previous isomorphism provides a new characterization of the non-empty optimal \( S \)-filters of \( \mathcal{A} \) as the images of optimal filters of \( \mathcal{A} \) by the map \( \varphi^{-1}[\cdot] \). This is the keystone why we build the dual Priestley space of \( \mathcal{A} \) from the dual Priestley space of \( \mathcal{A} \), as it is explained in next section.
5. Duality for objects

In this section we present the correspondence between \( S \)-algebras and a certain class of Priestley-style spaces for the logics \( S \) that are finitary, congruential, filter-distributive and with theorems.

In order to characterize such spaces, we use the concept of referential algebra, that goes back to Wójcicki [13] (see also [12]).

Given a logical language \( \mathcal{L} \), an \( \mathcal{L} \)-referential algebra is a structure \( \mathcal{X} = \langle X, \mathcal{B} \rangle \) where \( X \) is a set and \( \mathcal{B} \) is an \( \mathcal{L} \)-algebra whose elements are subsets of \( X \).

For any \( \mathcal{L} \)-referential algebra \( \mathcal{X} = \langle X, \mathcal{B} \rangle \), the relation \( \preceq_\mathcal{X} \subseteq X \times X \) defined by setting for every \( x, y \in X \):

\[
x \preceq_\mathcal{X} y \text{ iff } (\forall U \in \mathcal{B})(x \in U \Rightarrow y \in U)
\]

is a quasiorder on \( X \). Whenever \( \preceq_\mathcal{X} \) is a partial order, the \( \mathcal{L} \)-referential algebra \( \mathcal{X} \) is said to be reduced. In this case, we denote \( \preceq_\mathcal{X} \) by \( \leq_\mathcal{X} \), or even by \( \leq \) when the context is clear.

Referential algebras can be used to define logics in the following way. For any \( \mathcal{L} \)-referential algebra \( \mathcal{X} = \langle X, \mathcal{B} \rangle \) we define the relation \( \vdash_\mathcal{X} \subseteq \mathcal{P}(\text{Fm}_\mathcal{L}) \times \text{Fm}_\mathcal{L} \) such that for all \( \Gamma \cup \{ \delta \} \subseteq \text{Fm}_\mathcal{L} \):

\[
\Gamma \vdash_\mathcal{X} \delta \text{ iff } (\forall h \in \text{Hom}(\text{Fm}_\mathcal{L}, \mathcal{B})) \bigcap_{\gamma \in \Gamma} h(\gamma) \subseteq h(\delta).
\]

This relation is such that \( \langle \text{Fm}_\mathcal{L}, \vdash_\mathcal{X} \rangle \) is a logic.

Given a logic \( S \) in the language \( \mathcal{L} \) and an \( \mathcal{L} \)-referential algebra \( \mathcal{X} \), we say that \( \mathcal{X} \) is an \( S \)-referential algebra provided that \( \vdash_S \subseteq \vdash_\mathcal{X} \). Moreover, we say that \( S \) admits a (complete local) referential semantics if there is a class of referential algebras \( \mathcal{X} \) such that \( \vdash_S = \bigcap \{ \vdash_\mathcal{X} : \mathcal{X} \in \mathcal{X} \} \).

Remark 5.1. It is easy to see that if \( \mathcal{X} = \langle X, \mathcal{B} \rangle \) is an \( S \)-referential algebra, then for every \( x \in X \), the set \( \{ U \in \mathcal{B} : x \in U \} \) is an \( S \)-filter.

Remark 5.2. If \( \langle X, \mathcal{B} \rangle \) is a reduced \( S \)-referential algebra, then \( \mathcal{B} \in \text{Alg}_S \) (see [12, Remark 5.2]).

We return now to consider \( S \)-algebras and closure bases for their closure systems of \( S \)-filters, as they can be seen as reduced \( S \)-referential algebras when \( S \) is congruential.

Theorem 5.3. Let \( S \) be a congruential logic, \( \mathcal{A} \) an \( S \)-algebra, and \( \mathcal{F} \) a closure base for \( \text{Fi}_S(\mathcal{A}) \). Then \( \langle \mathcal{F}, \mathcal{F}[\mathcal{A}] \rangle \) is a reduced \( S \)-referential algebra and the associated order is given by the inclusion relation.

Proof. By definition, \( \langle \mathcal{F}, \mathcal{F}[\mathcal{A}] \rangle \) is a referential algebra. We show first that it is reduced. Consider the quasiorder \( \preceq \subseteq \mathcal{F} \times \mathcal{F} \) of this referential algebra, and note that for every \( P, Q \in \mathcal{F} \), \( P \preceq Q \) if and only if for every \( a \in \mathcal{A} \) such that \( P \in \mathcal{F}[\mathcal{A}] \), \( a \in \mathcal{F}[\mathcal{A}] \). It follows that \( \preceq \) is the inclusion relation on \( \mathcal{F} \). Therefore the referential algebra is reduced. Let us show now that \( \langle \mathcal{F}, \mathcal{F}[\mathcal{A}] \rangle \) is an \( S \)-referential algebra. Let \( \Gamma \cup \{ \delta \} \subseteq \text{Fm}_\mathcal{L} \) be such that \( \Gamma \vdash_S \delta \), and let \( h \in \text{Hom}(\text{Fm}_\mathcal{L}, \mathcal{F}[\mathcal{A}]) \). Since

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3 We admit \( X \) to be empty to cover the case of the trivial algebras. In this case the domain of \( \mathcal{B} \) is \( \{ \emptyset \} \) and \( \mathcal{B} \) is trivial.

4 If \( X \) is empty, we consider the empty relation as a partial order and the referential algebra \( \langle X, \mathcal{B} \rangle \) is reduced.
\[ \varphi_x \in \text{Hom}(A, \varphi_x[A]) \] is an isomorphism, there is \( h' \in \text{Hom}(\mathbf{Fm}, A) \) such that 
\( \varphi_x \circ h' = h \). To show that \( \bigcap \{ h(\gamma) : \gamma \in \Gamma \} \subseteq h(\delta) \), suppose that \( P \in \mathcal{F} \) is such that \( P \in \bigcap \{ h(\gamma) : \gamma \in \Gamma \} = \bigcap \{ \varphi_x(h'(\gamma)) : \gamma \in \Gamma \} \). Then \( h'\Gamma \subseteq P \). And since \( P \in \text{FIS}(A) \), \( h' \in \text{Hom}(\mathbf{Fm}, A) \) and \( \Gamma \vdash \delta \) we obtain \( h'(\delta) \in P \), so \( P \in \varphi_x(h'(\delta)) = h(\delta) \), as required. \( \square \)

In [13, Section 5.6.7] the referential algebra \( \langle \mathcal{F}, \varphi_x[A] \rangle \) is called the canonical referential algebra for \( \text{FIS}^A \) determined by \( \mathcal{F} \). Notice that Theorem 5.3, Remark 5.2, and Theorem 3.3 together imply that for any congruential logic \( \mathcal{S} \), there is a correspondence between the reduced \( \mathcal{S} \)-referential algebras and the structures of the form \( \langle A, \mathcal{F} \rangle \), where \( A \) is an \( \mathcal{S} \)-algebra and \( \mathcal{F} \) is a closure base for \( \text{FIS}(A) \). This correspondence between objects, first addressed by Czelakowski in [6], was formulated as a full-fledged duality in [12], for the case when the collection \( \text{FIS}(A) \) is taken itself as the closure base. But this closure base is not the closure base that properly generalizes the representation theorems on which the Stone/Priestley dualities that we find in the literature are based. For instance, the algebraic counterpart of intuitionistic logic is the variety of Heyting algebras and the intuitionistic logical filters of the Heyting algebras are the lattice filters. Yet the representation theorem on which the Esakia duality for Heyting algebras is based relies on the prime filters and not on all the lattice filters. Therefore, we should not for our purposes work with the whole collection of \( \mathcal{S} \)-filters, but rather we should identify a closure base that provides us with a direct generalization of the mentioned representation theorems. The base we need is the collection of all optimal \( \mathcal{S} \)-filters.

From now on, we let \( \mathcal{S} \) to be a filter-distributive finitary congruential logic with theorems and \( A \) an \( \mathcal{S} \)-algebra, if we do not say the contrary. Let us further assume that \( \text{Ops}(A) \) is the optimal \( \mathcal{S} \)-base from which the representation map \( \varphi \) is defined, i.e., for any \( a \in A \), \( \varphi(a) = \{ P \in \text{Ops}(A) : a \in P \} \). Since \( \mathcal{S} \) has theorems, \( A \) has a top element that we denote by \( 1^A \), having that \( \varphi(1^A) = \text{Ops}(A) \). We define on \( \text{Ops}(A) \) the topology \( \tau_A \) obtained by taking as subbasis the collection:
\[
\{ \varphi(a) : a \in A \} \cup \{ \varphi(b)^c : b \in A \}.
\]

**Proposition 5.4.** The isomorphism \( \llbracket \varphi[\cdot] \rrbracket : \text{FIS}(A) \to \text{FI}(M(A)) \) suitably restricted establishes an order homeomorphism between the ordered topological spaces \( \langle \text{Ops}(A), \tau_A, \subseteq \rangle \) and \( \langle \text{Op}(M(A)), \tau_{M(A)}, \subseteq \rangle \), whose inverse is \( \varphi^{-1}[:]. \)

**Proof.** By Corollary 4.11 we already know that \( \llbracket \varphi[\cdot] \rrbracket \) establishes an order isomorphism between \( \langle \text{Ops}(A), \subseteq \rangle \) and \( \langle \text{Op}(M(A)), \subseteq \rangle \), whose inverse is \( \varphi^{-1}[:]. \) Therefore, to prove the proposition, using that inverse maps preserve intersections, we just need to show that \( \varphi^{-1}[: \} \) sends subbasic opens of the space \( \langle \text{Op}(M(A)), \tau_{M(A)}, \subseteq \rangle \) to opens of \( \langle \text{Ops}(A), \tau_A, \subseteq \rangle \). Recall that \( \{ \sigma(U) : U \in M(A) \} \cup \{ \sigma(V)^c : V \in M(A) \} \) is a subbasis for \( \tau_{M(A)} \). Using Definition 4.1 this subbasis can be described as the union of \( \{ \sigma(\varphi(B)) : B \subseteq A \) finite \} and \( \{ \sigma(\varphi(B))^c : B \subseteq A \) finite \}. So let \( B \subseteq A \) be finite. If \( B = \emptyset \), then \( \varphi(B) = \text{Ops}(A) \) and so \( \sigma(\varphi(B)) = \text{Op}(M(A)) \). Therefore, \( \varphi^{-1}[\sigma(\varphi(B))] = \text{Ops}(A) \), which is open, and \( \varphi^{-1}[\sigma(\varphi(B))^c] = \emptyset \), which is also open. In the case that \( B \) is non-empty, we prove that \( \varphi^{-1}[\sigma(\varphi(B))] = \bigcup \{ \varphi(b) : b \in B \} \), which implies that \( \varphi^{-1}[\sigma(\varphi(B))] \) is a basic open subset of \( \langle \text{Ops}(A), \tau_A, \subseteq \rangle \) and \( \varphi^{-1}[\sigma(\varphi(B))^c] = \bigcup \{ \varphi(b)^c : b \in B \} \) an open subset of \( \langle \text{Ops}(A), \tau_A, \subseteq \rangle \). First note that if \( F \in \text{Ops}(A) \), then, since \( F = \varphi^{-1}[\llbracket \varphi[F] \rrbracket] \), \( F \in \varphi^{-1}[\sigma(\varphi(B))] \) if and only if \( \llbracket \varphi[F] \rrbracket \in \sigma(\varphi(B)) \). But, \( \llbracket \varphi[F] \rrbracket \in \sigma(\varphi(B)) \) if and only if \( \varphi(B) \in \llbracket \varphi[F] \rrbracket \) and this is
equivalent to say that $\varphi(b) \in \langle \varphi[F] \rangle$ for every $b \in B$, which in turn is easily seen to be equivalent to say that $F \in \bigcap \{ \varphi(b) : b \in B \}$.

\textbf{Corollary 5.5.} Let $A$ be an $S$-algebra. Then

1. the space $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$ is a Priestley space,
2. the set $\text{Irr}_S(A)$ is dense in $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$,
3. $\text{Irr}_S(A) = \{ P \in \text{Op}_S(A) : \{ \varphi(a) : a \not\in P \} \text{ is non-empty and up-directed} \}$.

\textbf{Proof.} (1) follows from the fact that $\langle \text{Op}(M(A)), \tau_{M(A)}, \subseteq \rangle$ is a Priestley space, (2) follows from the fact that the set $\text{Irr}_A(M(A))$ is dense in that space and that $\text{Irr}_S(A) = \{ \varphi^{-1}[F] : F \in \text{Irr}_A(M(A)) \}$, and (3) is a restatement of Proposition 3.4. \hfill $\square$

\textbf{Remark 5.6.} Note that for every $P \in \text{Op}_S(A)$, $\{ \varphi(a) : a \not\in P \}$ is non-empty and up-directed if and only if $\{ \varphi(B) \in M(A) : P \not\in \varphi(B) \}$ is non-empty and up-directed.

\textbf{Remark 5.7.} In the case of a trivial $S$-algebra $A$, since $S$ has theorems the set $\text{Op}_S(A)$ is empty. This forces us to consider the Priestley space with an empty set of points.

We say that a clopen-upset $U$ of $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$ is $\text{Irr}_S(A)$-admissible whenever $\max(U^c) \subseteq \text{Irr}_S(A)$. The next proposition shows that the set of $\text{Irr}_S(A)$-admissible clopen-upsets of $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$ is the closure of $\varphi[A]$ under the binary operation of intersection.

\textbf{Proposition 5.8.} Let $A$ be an $S$-algebra and $U$ be a clopen up-set of the Priestley space $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$. Then $U = \varphi(C)$ for some finite $C \subseteq A$ if and only if $\max(U^c) \subseteq \text{Irr}_S(A)$.

\textbf{Proof.} First note that if $U = \text{Op}_S(A)$, then $\varphi(\emptyset) = U$ and $U^c = \emptyset$ and therefore the statement holds. Let $U \neq \text{Op}_S(A)$. Suppose first that $U = \varphi(C)$ for some finite $C \subseteq A$. Then $C \neq \emptyset$ and there is $P \in \max(\varphi(C)^c)$, because $U^c$ is clopen and non-empty. Hence, $C \not\subseteq P$. Therefore there is $b \in C \setminus P$. Then by the irreducible $S$-filter lemma, there is $Q \in \text{Irr}_S(A)$ such that $b \notin Q$ and $P \subseteq Q$. This implies $C \not\subseteq Q$, so $\varphi(C)^c \not\subseteq Q$ and by maximality of $P$ we conclude $P = Q$, i.e., $P$ is an irreducible $S$-filter. For the converse, let $U$ be a clopen up-set such that $\max(U^c) \subseteq \text{Irr}_S(A)$. Notice that

$$\langle \varphi[P] \rangle = \max(\{ \langle \varphi[F] \rangle : F \in U \}) \text{ iff } P \in \max(U^c).$$

This follows from the isomorphism between $\langle \text{Op}_S(A), \subseteq \rangle$ and $\langle \text{Op}(M(A)), \subseteq \rangle$ given in Proposition 4.10. Therefore, using the homeomorphism given in Proposition 5.4, from $U$ being an $\text{Irr}_S(A)$-admissible clopen up-set of $\text{Op}_S(A)$ we obtain that $\{ \langle \varphi[F] \rangle : F \in U \}$ is an $\text{Irr}_A(M(A))$-admissible clopen up-set of $\text{Op}(M(A))$. And then by Proposition 2.3, there is a finite subset $C \subseteq A$ such that $\sigma_{M(A)}(\varphi_A(C)) = \{ \langle \varphi[F] \rangle : F \in U \}$, and then we obtain that $U = \varphi_A(C)$, as required. \hfill $\square$

We are ready to introduce the definition of the Priestley-dual objects of the $S$-algebras.

\textbf{Definition 5.9.} A structure $\mathfrak{X} = \langle X, \tau, B \rangle$ is an $S$-Priestley space when:

1. $(X, B)$ is a reduced $S$-referential algebra, whose associated order is denoted by $\leq$,
Proof. Let $\langle X, \tau, \mathcal{B} \rangle$ be an $\mathcal{S}$-Priestley space and let $U \subseteq X$. 

(1) $U$ is a non-empty open up-set of $\langle X, \tau, \leq \rangle$ if and only if $U$ is the union of a non-empty set of non-empty sets which are intersections of non-empty finite subsets of $\mathcal{B}$.

(2) $U$ is a non-empty clopen up-set of $\langle X, \tau, \leq \rangle$ if and only if $U$ is the union of a non-empty finite set of non-empty sets which are intersections of non-empty finite subsets of $\mathcal{B}$.

Proof. Let $U$ be a non-empty open up-set of $\langle X, \tau, \leq \rangle$. When $U = X$ there is nothing to prove, so assume that $U \neq X$ and that $x \in U$. Because $U$ is an up-set, we have that for all $y \notin U$, $x \not\leq y$. Then by (Pr1), since the $\mathcal{S}$-referential algebra $\langle X, \mathcal{B} \rangle$ is reduced, for all $y \notin U$ there is $V^x_y \in \mathcal{B}$ such that $x \in V^x_y$ and $y \notin V^x_y$. Then we have a closed set $U^c$ and open sets $\{ (V^x_y)^c : y \notin U \}$ such that $U^c \subseteq \bigcup \{ (V^x_y)^c : y \notin U \}$. Now by the compactness of the space given by (Pr3), there are $y_0, \ldots, y_n \notin U$ such that $U^c \subseteq (V^x_{y_0})^c \cup \cdots \cup (V^x_{y_n})^c$. Hence $V^x_{y_0} \cap \cdots \cap V^x_{y_n} \subseteq U$. Notice that $x \in V^x_{y_0} \cap \cdots \cap V^x_{y_n}$. Therefore $U \subseteq \bigcup_{x \in U} (V^x_{y_0} \cap \cdots \cap V^x_{y_n}) \subseteq U$. Thus, as $U \neq \emptyset$, $U$ is the union of a non-empty set of non-empty sets which are intersections of finite and non-empty subsets of $\mathcal{B}$, and (1) has been proven, since the other direction is clear. (2) follows easily from the compactness of the space. □
Let $X = \langle X, \tau, B \rangle$ be an $S$-Priestley space. The map $\xi : X \to \mathcal{P}(B)$ is defined as follows:

$$\xi(x) := \{ U \in B : x \in U \}.$$ 

We will show later that $\xi$ establishes a homeomorphism between the $S$-Priestley space $X$ and the dual space of its dual algebra. In this way we will have the natural transformation we need to establish the categorical duality.

**Lemma 5.14.** Then map $\xi : X \to \mathcal{P}(B)$ is injective and for every $x \in X$, $\xi(x) \in \text{Figs}(B)$.

**Proof.** The injectivity of $\xi$ follows easily from (Pr1) because in reduced referential algebras the elements of the algebra separate points. Moreover, since $\langle X, B \rangle$ is a reduced $S$-referential algebra, for every $x \in X$ the set $\{ U \in B : x \in U \}$ is an $S$-filter.

**Remark 5.15.** Notice that Lemma 5.14 implies that the converse of condition (Pr2) holds for any $S$-Priestley space $\langle X, \tau, B \rangle$, i.e., for all non-empty and finite $V \subseteq B$ and all $U \in B$, if $U \in \text{Fgs}\{B\}(V)$, then $\bigcap V \subseteq U$. Assume $U \in \text{Fgs}\{B\}(V)$ and let $x \in \bigcap V$, so $V \subseteq \xi(x)$. Since $\xi(x)$ is an $S$-filter, $\text{Fgs}\{B\}(V) \subseteq \xi(x)$, and therefore, by the assumption, $U \in \xi(x)$, i.e., $x \in U$.

**Remark 5.16.** From Remark 5.15 and condition (Pr2) we obtain that for any finite $U, V \subseteq B$, if $U$ is non-empty, then

$$\bigcap U \subseteq \bigcap V \iff \text{Fgs}\{B\}(V) \subseteq \text{Fgs}\{B\}(U).$$

In particular, for any $U, V \subseteq B$:

$$V \subseteq U \iff U \in \text{Fgs}\{B\}(V).$$

Therefore, the specialization order $\subseteq_S$ on $B$ coincides with the inclusion relation on $B$. We will repeatedly use this fact as well as the next generalization.

**Lemma 5.17.** Let $\langle X, \tau, B \rangle$ be an $S$-Priestley space. For any non-empty and finite $U_0, \ldots, U_n, V \subseteq B$:

$$\bigcap V \subseteq \bigcup_{i \leq n} \bigcap U_i \iff \bigcap_{i \leq n} \text{Fgs}\{B\}(U_i) \subseteq \text{Fgs}\{B\}(V) \iff \bar{\varphi}_B(V) \subseteq \bigcup_{i \leq n} \bar{\varphi}_B(U_i).$$

**Proof.** We prove the first equivalence. The second one is an application of Lemma 4.9. Assume first that $\bigcap V \subseteq \bigcup_{i \leq n} \bigcap U_i$ and assume that $U \in \bigcap_{i \leq n} \text{Fgs}\{B\}(U_i)$. By Remark 5.15 we get $\bigcup_{i \leq n} \bigcap U_i \subseteq U$; therefore $\bigcap V \subseteq U$. It follows from (Pr2) that $U \in \text{Fgs}\{B\}(V)$. Assume now that $\bigcap_{i \leq n} \text{Fgs}\{B\}(U_i) \subseteq \text{Fgs}\{B\}(V)$. We show that $\bigcap V \cap X_B \subseteq \bigcup_{i \leq n} \bigcap U_i$, and then the claim follows from the denseness of $X_B$ and from $\bigcup_{i \leq n} \bigcap U_i$ being clopen. Let $x \in \bigcap V \cap X_B$ and suppose, towards a contradiction, that $x \notin \bigcup_{i \leq n} \bigcap U_i$. Then for every $i \leq n$ there exists $U_i \in U_i$ such that $x \notin U_i$. Then using condition (Pr5), there is $U \in B$ such that $U_i \subseteq U$ and $x \notin U$. Thus for every $i \leq n$, $\bigcap U_i \subseteq U_i \subseteq U$. Therefore, by (Pr2), $U \in \bigcap_{i \leq n} \text{Fgs}\{B\}(U_i)$. Hence $U \in \text{Fgs}\{B\}(V)$ and this by Remark 5.15 implies $\bigcap V \subseteq U$. As $x \in \bigcap V$, we get $x \in U$, a contradiction.

We can give characterizations of when $B$ has a bottom element and of when it has a bottom-family.

**Lemma 5.18.** Let $\langle X, \tau, B \rangle$ be an $S$-Priestley space.
(1) \(B\) has a bottom element if and only if \(\emptyset \in B\).

(2) \(B\) has a bottom-family if and only if there is a finite \(D \subseteq B\) such that \(\bigcap D = \emptyset\).

Proof. (1) If \(\emptyset \in B\), then \(\emptyset\) is the bottom element of \(B\). For the converse, assume that \(U\) is the bottom element of \(B\) and suppose, towards a contradiction, that there is \(x \in U \cap X_B\). Then by condition (Pr5), there is \(V \in B\) such that \(x \notin V\), but since \(U\) is the bottom element, then \(U \subseteq V\). This implies \(x \in V\), a contradiction. Hence, \(U \cap X_B = \emptyset\), and then from the denseness of \(X_B\), \(U = \emptyset\).

For (2), note first that if \(X = \emptyset\), then \(B = \{\emptyset\}\) and so \(B\) has a bottom-family and moreover \(\bigcup \{\emptyset\} = \emptyset\). Assume then that \(X\) is non-empty. Suppose first that \(D \subseteq B\) is finite and \(\bigcap D = \emptyset\). Note that \(D\) must be non-empty, otherwise \(\bigcap D = X\). Then being \(D\) finite we can take a non-empty family \(D' \subseteq D\) of incomparable elements such that \(\bigcap D' = \emptyset\). Then for any \(U \in B\), \(\bigcap D' = \emptyset \subseteq U\), and by (Pr2) it follows \(U \in \text{Fg}_B(D')\), so \(\text{Fg}_B(D') = B\) and \(D'\) is a bottom-family. For the converse, assume that \(B\) has a bottom-family \(D \subseteq B\), and suppose, towards a contradiction, that there is \(x \in \bigcap D\). Since \(D\) is finite, by denseness of \(X_B\) we can assume, without loss of generality, that \(x \in X_B\). Then by condition (Pr5) there is \(V \in B\) such that \(x \notin V\). But being \(D\) a bottom-family, \(V \in \text{Fg}_B(D)\). Remark 5.15 then implies \(\bigcap D \subseteq V\); hence, \(x \in V\), a contradiction. \(\square\)

Remark 5.19. Note that from the proof of Lemma 5.18 it follows that if \(D\) is a bottom-family of \(B\), then \(\bigcap D = \emptyset\).

Proposition 5.20. Let \((X, \tau, B)\) be an \(S\)-Priestley space. Then for any \(x \in X\), \(\xi(x) \in \text{Op}_S(B)\). Moreover if \(x \in X_B\), then \(\xi(x) \in \text{Irr}_S(B)\).

Proof. Notice that if \(\xi(x) = B\), then \(B\) has no bottom-family. On the contrary, using Lemma 5.18, there would be \(D \subseteq B\) finite such that \(\bigcap D = \emptyset\), but this contradicts the assumption, which implies \(x \in \bigcap D\). Thus if \(\xi(x) = B\), then \(\xi(x)\) is an optimal \(S\)-filter of \(B\).

Suppose now that \(\xi(x) \neq B\). To prove that \(\xi(x)^c = \{U \in B : x \notin U\}\) is a strong \(S\)-ideal we use Lemma 2.10. Since \(\leq_B^\text{c}\) is the inclusion relation, it is clear that \(\xi(x)^c\) is a downset w.r.t. \(\leq_B^\text{c}\). Now we show that the condition on the definition of strong \(S\)-ideal is satisfied. Let \(V \subseteq B\) be finite and let \(I \subseteq \xi(x)^c\) be finite and such that \(\bigcap \{\text{Fg}_S(B)(U) : U \in I\} \subseteq \text{Fg}_S(B)(V)\). If \(I = \emptyset\), the hypothesis turns into \(\text{Fg}_S(B)(\emptyset) = B\). This implies that there is \(V' \subseteq V\) that is a bottom-family for \(B\). Thus, reasoning as in the proof of Lemma 5.18, \(\bigcap V' = \emptyset\), and then there is \(V \in V\) such that \(x \notin V\), i.e., \(V \in \xi(x)^c\). If \(I \neq \emptyset\) and \(V = \emptyset\), then since \(S\) is assumed to have theorems, \(\text{Fg}_S(B)(V) = \text{Fg}_S(B)(\{X\}) = \{X\}\), because \(1_B = X\). Hence, by Lemma 5.17 we get \(X \subseteq \bigcup \{U : U \in I\}\), and therefore \(x \in U\) for some \(U \in I\), a contradiction. Thus, \(V\) is non-empty. By Lemma 5.17 we get \(\bigcap V \subseteq \bigcup \{U : U \in I\}\), and then from \(x \notin U\) for all \(U \in I\) we get that there is \(V \in V\) such that \(x \notin V\), i.e., \(V \in \xi(x)^c\). From either case we get that \(V \cap \xi(x)^c \neq \emptyset\), and so \(\text{Fg}_S(B)(V) \cap \xi(x)^c \neq \emptyset\). Thus, we have shown that \(\xi(x)^c\) is a strong \(S\)-ideal, and by Theorem 2.18 we conclude that \(\xi(x)\) is an optimal \(S\)-filter.

Finally, if \(x \in X_B\), we know from (Pr5) that \(\xi(x)^c\) is non-empty and up-directed, i.e., an order ideal of \(B\). Hence, by Theorem 2.18, \(\xi(x)\) is an irreducible \(S\)-filter. \(\square\)

We aim to show that the map \(\xi\) is onto \(\text{Op}_S(B)\). To this end we note that \(B^\cap := \langle B^\cap, \cap, X\rangle\) is a meet-semilattice isomorphic to the \(S\)-semilattice \(M(B)\) of \(B\).
by the map \( h \) given by:
\[
\bigcap \mathcal{U} \mapsto \bigcap \{ \varphi_B(U) : U \in \mathcal{U} \} = \hat{\varphi}_B(\mathcal{U}),
\]
where \( \mathcal{U} \) is a finite subset of \( B \). From the condition \((\text{Pr}2)\) and Lemma 4.9 it follows that the map is well defined and one-to-one. It is obviously onto. Finally, it is easy to see that it is a homomorphism preserving the top element. From the distributivity of \( M(B) \), which follows from the fact that the logic \( \mathcal{S} \) is distributive, we obtain that \( B^\cap := \langle B^\cap, \cap, X \rangle \) is a distributive meet-semilattice.

Using the isomorphism \( h : B^\cap \cong M(B) \) and the isomorphisms
\[
\left\langle \{ \varphi[\cdot] \} : \text{Fi}_S(B) \cong \text{Fi}(M(B)) : \varphi^{-1}[\cdot] \right\rangle.
\]
that restrict to the optimals as
\[
\left\langle \{ \varphi[\cdot] \} : \text{Op}_S(B), \subseteq \cong \text{Op}(M(B)), \subseteq : \varphi^{-1}[\cdot] \right\rangle
\]
it is easy to see that for every \( S \)-filter \( F \) of \( B \), \( F \in \text{Op}_S(B) \) if and only if \( \{ E \in B^\cap : \exists U_0, \ldots, U_n \in F \text{ s.t. } U_0 \cap \ldots \cap U_n \subseteq E \} \in \text{Op}(B^\cap) \).

Let \( B^{\cap \cup} \) be the closure of \( B^\cap \) under the binary operation of union, so that \( \emptyset \in B^\cap \) if and only if \( \emptyset \in B^{\cap \cup} \). We prove that the identity embedding from \( B^\cap \) to the distributive lattice with top \( B^{\cap \cup} := \langle B^{\cap \cup}, \cap, \cup, X \rangle \) is a sup-homomorphism. Then, since the set \( B^\cap \) is obviously join-dense in \( B^{\cap \cup} \), it follows that \( B^{\cap \cup} \) is the distributive envelope of \( B^\cap \).

**Proposition 5.21.** The identity map from \( B^\cap \) to \( B^{\cap \cup} \) is a sup-homomorphism, that is, for all non-empty and finite subsets \( \mathcal{U}_0, \ldots, \mathcal{U}_n, \mathcal{V} \) of \( B \)

\[
\text{if } \bigcap_{i \leq n} \uparrow_B \cap \bigcap_{i \leq n} \mathcal{U}_i \subseteq \uparrow_B \cap \bigcap \mathcal{V}, \text{ then } \bigcap_{i \leq n} \uparrow_B \cap \bigcap_{i \leq n} \mathcal{U}_i \subseteq \uparrow_B \cap \bigcap_{i \leq n} \mathcal{V}
\]

**Proof.** Suppose that \( \bigcap_{i \leq n} \uparrow_B \cap \bigcap_{i \leq n} \mathcal{U}_i \subseteq \uparrow_B \cap \bigcap \mathcal{V} \) and let \( C = \bigcup_{j \leq m} \bigcap \mathcal{W}_j \) for some non-empty finite subsets \( \mathcal{W}_0, \ldots, \mathcal{W}_m \) of \( B \). We prove that \( \bigcap \mathcal{V} \cap X_B \subseteq C \). Then from denseness it will follow that \( \bigcap \mathcal{V} \subseteq C \). Suppose that \( x \in \bigcap \mathcal{V} \cap X_B \) and \( x \notin C \). Then \( x \notin \bigcup_{j \leq m} \bigcap \mathcal{W}_j \). Let for every \( j \leq m \), \( W_j \in \mathcal{W}_j \) such that \( x \notin W_j \). Being \( x \in X_B \), the set \( \{ U \in B : x \notin U \} \) is non-empty and up-directed. Thus there exists \( W \in B \) such that \( W_0 \cup \ldots \cup W_m \subseteq W \) and \( x \notin W \). It follows that \( \bigcup_{j \leq m} \bigcap \mathcal{W}_j \subseteq W \), i.e., \( C \subseteq W \); therefore \( W \in \bigcap_{i \leq n} \uparrow_B \cap \bigcap_{i \leq n} \mathcal{U}_i \). The assumption implies that \( \bigcap \mathcal{V} \subseteq W \). As \( x \in \bigcap \mathcal{V} \), \( x \in W \), a contradiction. \( \square \)

From the fact that \( B^\cap \) and \( M(B) \) are isomorphic it follows that \( B^{\cap \cup} \) is (isomorphic to) the distributive envelope \( L(M(B)) \) of \( M(B) \). We describe the isomorphism in the next proposition.

**Proposition 5.22.** The map \( g : B^{\cap \cup} \to L(M(B)) \) defined by
\[
g(\bigcup_{i \leq n} \bigcap \mathcal{U}_i) := \bigcup_{i \leq n} \{ \sigma_M(B)(\hat{\varphi}_B(\mathcal{U}_i)) : i \leq n \},
\]
where the \( \mathcal{U}_i \)'s are non-empty and finite subsets of \( B \), is a lattice isomorphism between \( B^{\cap \cup} := \langle B^{\cap \cup}, \cap, \cup, X \rangle \) and the distributive envelope \( L(M(B)) \) of \( M(B) \).

**Proof.** First of all we need to see that \( g \) is well defined. To this end we prove that if \( \{ \mathcal{U}_i : i \leq n \} \) and \( \{ \mathcal{V}_j : j \leq m \} \) are finite families of non-empty finite subsets of \( B \) such that \( \bigcup_{i \leq n} \bigcap \mathcal{U}_i \subseteq \bigcup_{j \leq m} \bigcap \mathcal{V}_j \), then \( \bigcup_{i \leq n} \sigma_M(B)(\hat{\varphi}_B(\mathcal{U}_i)) \subseteq \bigcup_{j \leq m} \sigma_M(B)(\hat{\varphi}_B(\mathcal{V}_j)) \)
\[ \bigcup_{j \leq m} \sigma_{M(B)}(\mathcal{F}_B(V_j)). \] From this fact also follows that the map is order-preserving. Suppose that \( \bigcup_{i \leq n} \mathcal{U}_i \subseteq \bigcup_{j \leq m} \mathcal{V}_j. \) From Lemma 5.17 it follows that \( \bigcup_{i \leq n} \mathcal{F}_B(\mathcal{U}_i) \subseteq \bigcup_{j \leq m} \mathcal{F}_B(V_j). \) Then \( \bigcap_{j \leq m} \mathcal{F}_B(V_j) \subseteq \mathcal{F}_B(\mathcal{U}_i) \) for every \( i \leq n. \) Using condition (2.3) we obtain that \( \bigcup_{i \leq n} \sigma_{M(B)}(\mathcal{F}_B(\mathcal{U}_i)) \subseteq \bigcup_{j \leq m} \sigma_{M(B)}(\mathcal{F}_B(V_j)). \)

The next move is to show that \( g \) is order reflecting. Suppose that \( \{\mathcal{U}_i : i \leq n\} \) and \( \{\mathcal{V}_j : j \leq m\} \) are finite families of non-empty finite subsets of \( B \) such that \( \bigcup_{i \leq n} \sigma_{M(B)}(\mathcal{F}_B(\mathcal{U}_i)) \subseteq \bigcup_{j \leq m} \sigma_{M(B)}(\mathcal{F}_B(V_j)). \) Condition (2.3) implies that \( \bigcup_{i \leq n} \mathcal{F}_B(\mathcal{U}_i) \subseteq \bigcup_{j \leq m} \mathcal{F}_B(V_j). \) Applying Lemma 5.17 we obtain that \( \bigcup_{i \leq n} \mathcal{U}_i \subseteq \bigcup_{j \leq m} \mathcal{V}_j. \)

Notice that the emptyset \( \emptyset \in CIUP(X) \) can be trivially described as an (empty) finite union of non-empty finite intersections of elements of \( B. \) Therefore, the previous proposition implies that for any \( S \)-Priestley space \( \langle X, \tau, B \rangle, \) the lattice of clopen up-sets \( CIUP(X) \) (which is the dual distributive lattice of the Priestley space \( \langle X, \tau, \leq \rangle \)) is isomorphic to \( L(M(B)) \cup \{\emptyset\} \), which is the distributive envelope of \( M(B) \) augmented with a bottom element whenever \( M(B) \) has no bottom element. In particular, this implies that the optimal filters of \( L(M(B)) \) are in one-to-one correspondence with the prime filters of \( CIUP(X). \) This fact will be used in the proof of the following proposition.

**Proposition 5.23.** Let \( \langle X, \tau, B \rangle \) be an \( S \)-Priestley space. Then:

1. For every \( P \in Op_S(B) \) there is \( x \in X \) such that \( \xi(x) = P. \)
2. For every \( Q \in Irr_S(B) \) there is \( x \in X_B \) such that \( \xi(x) = Q. \)

**Proof.** (1) Let \( P \) be an optimal \( S \)-filter of \( B. \) Then we know by Proposition 4.10 that \( \|\varphi[P]\| \) is an optimal filter of \( M(B). \) Therefore, \( \|\sigma(\|\varphi[P]\|)\| \) is a prime filter of \( L(M(B)), \) if \( \|\varphi[P]\| \neq M(B) \) or is \( L(M(B)) \) in case \( \|\varphi[P]\| = M(B). \) By the isomorphism \( g \) in Proposition 5.22, \( g^{-1}(\|\sigma(\|\varphi[P]\|)\|) \) is a prime filter of \( B^{\tau_0}. \) And in any case it is a prime filter of \( CIUP(X). \) By Priestley duality there exists \( x \in X \) such that \( g^{-1}(\|\sigma(\|\varphi[P]\|)\|) = \{C \in CIUP(X) : x \in C\}. \) Now it is easy to see that \( B \cap g^{-1}(\|\sigma(\|\varphi[P]\|)\|) = P. \) It follows that \( P = \{U \in B : x \in U\}. \) Hence, \( P = \xi(x). \)

(2) Let \( Q \in Irr_S(B). \) By (1) we know that there is \( x \in X \) such that \( Q = \{U \in B : x \notin U\} \) is order ideal, so it is non-empty and up-directed. Hence by (Pr5) we conclude that \( x \in X_B. \) \hfill \( \square \)

**Corollary 5.24.** Let \( \langle X, \tau, B \rangle \) be an \( S \)-Priestley space. The map \( \xi \) is an order homeomorphism between the ordered topological spaces \( \langle X, \tau, \leq \rangle \) and \( \langle Op_S(B), \tau_B, \subseteq \rangle \) such that for every \( U \in B, \) \( \xi^{-1}(\varphi_B(U)) = U \) and \( \xi(U) = \varphi_B(U). \)

**Proof.** Notice that for all \( x \in X \) and all \( U \in B \) we have: \( x \in U \) if and only if \( U \in \xi(x) \) if and only if \( \xi(x) \in \varphi_B(U). \) Thus, \( \xi^{-1}(\varphi_B(U)) = U \) and moreover:

\[ x \in \xi^{-1}(\varphi_B(U)) \] \iff \[ \xi(x) \in \varphi_B(U) \] \iff \[ U \notin \xi(x) \] \iff \[ x \in U^c. \]

Therefore \( \xi^{-1}(\varphi_B(U^c)) = U^c. \) Since inverse maps preserve intersections, this implies that the inverse of \( \xi \) sends basic opens to basic opens. From condition (Pr1) it follows that \( \xi \) is order preserving. As \( \xi \) is one-to-one (Lemma 5.14), onto (Proposition 5.23), and its inverse sends basic opens of \( \langle Op_S(B), \tau_B \rangle \) to basic opens of \( \langle X, \tau \rangle, \) we conclude that \( \xi \) is an homeomorphism, as required. \hfill \( \square \)
Corollary 5.25. Let $X = \langle X, \tau, B \rangle$ be an $S$-Priestley space. Then the structure $\langle \text{Op}_S(B), \tau_B, \varphi_B[B] \rangle$ is an $S$-Priestley space such that $\langle X, \tau \rangle$ and $\langle \text{Op}_S(B), \tau_B \rangle$ are homeomorphic topological spaces by means of the map $\xi : X \rightarrow \text{Op}_S(B)$, that moreover is an order isomorphism between $\langle X, \leq \rangle$ and $\langle \text{Op}_S(B), \subseteq \rangle$. Furthermore $B$ and $\varphi_B[B]$ are isomorphic $S$-algebras by means of the map $\varphi_B : B \rightarrow \varphi_B[B]$ whose inverse if the map $\xi^{-1} : \varphi_B[B] \rightarrow B$.

The previous corollary establishes that $S$-algebras and $S$-Priestley spaces are equivalent objects by means of the maps $\varphi$ and $\xi$. Before dealing with morphisms, let us investigate some other properties of $S$-Priestley spaces. Notice that from Proposition 5.13 we know that for any $S$-algebra, the collection of clopen up-sets of the Priestley space $\langle \text{Op}_S(A), \tau_A, \subseteq \rangle$ is $\varphi[A]^{\omega} \cup \{\emptyset\}$. We show in the following proposition that within all clopen up-sets of $\text{Op}_S(A)$, those that are $\text{Irr}_S(A)$-admissible admit a simpler description as finite intersections of elements of $\varphi[A]$, i.e., as elements of $M(A)$.

Proposition 5.26. Let $\langle X, \tau, B \rangle$ be an $S$-Priestley space. Then the collection of $X_B$-admissible clopen up-sets of $X$ coincides with $B^n$.

Proof. We claim that for any finite subsets $\{V_i \subseteq B : i \leq n\}$ for some $n \in \omega$, we have

$$x \in \max((\bigcup_{i \leq n} V_i)^c) \iff \xi(x) \in \max((\bigcup_{i \leq n} \varphi_B(V_i))^c).$$

This follows easily from propositions 5.20 and 5.23 using that for any $V \in B$, $x \in V$ if and only if $\xi(x) \in \varphi_B(V)$.

Let first assume that $U \subseteq B^n$, i.e., $U = \bigcap V$ for some non-empty and finite $V \subseteq B$. By Proposition 5.8 we know that $\varphi_B(V)$ is an $\text{Irr}_S(B)$-admissible clopen up-set of $\langle \text{Op}_S(B), \tau_B, \subseteq \rangle$. Then from the claim and using that $X_B = \xi^{-1}[\text{Irr}_S(B)]$, we obtain that $\bigcap V$ is an $X_B$-admissible clopen up-set of $X$.

Suppose now that $U$ is an $X_B$-admissible clopen up-set of $X$. If $U = \emptyset$, then from the claim we get that $\emptyset$ is an $\text{Irr}_S(B)$-admissible clopen up-set of $\text{Op}_S(B)$. Then by Proposition 5.8, there is a finite $W \subseteq B$ such that $\emptyset = \varphi_B(W)$, and this implies $U = \emptyset = \bigcap W$. Assume now that $U \neq \emptyset$. Then by Proposition 5.13 we know that $U = \bigcup_{i \leq n} V_i$ for finite $V_i \subseteq B$ for all $i \leq n$ for some $n \in \omega$. Then from the assumption and the claim, using that $\xi[X_B] = \text{Irr}_S(B)$, we obtain that $\bigcup_{i \leq n} \varphi_B(V_i)$ is an $\text{Irr}_S(B)$-admissible clopen up-set of $\text{Op}_S(B)$. By Proposition 5.8, there is a finite $W \subseteq B$ such that $\bigcup_{i \leq n} \varphi_B(V_i) = \varphi_B(W)$, and this implies $\bigcup_{i \leq n} \bigcap V_i = \bigcap W$, as required.

Proposition 5.27. Let $\langle X, \tau, B \rangle$ be an $S$-Priestley space. Then the tuple $\langle X, \tau, \leq , X_B \rangle$ is a generalized Priestley space and its dual meet-semilattice, i.e., the meet-semilattice of all $X_B$-admissible clopen up-sets, is isomorphic to $M(B)$.

Proof. The condition (DS1) in the definition of generalized Priestley space, namely that $\langle X, \tau, \leq \rangle$ is a Priestley space, follows from (Pr1), (Pr3) and (Pr4). The condition (DS2) follows from from (Pr5). Recall that $X^*$ denotes the collection of $X_B$-admissible clopen up-sets of $\langle X, \tau, \leq , X_B \rangle$. By 5.26 $X^*$ coincides with $B^n$. Then from (Pr5) and Remark 5.10 it follows that $X_B = \{x \in X : \{U \in B^n : x \notin U\}$ is non-empty and up-directed$, so condition (DS3) also holds. Moreover, from (Pr1) it follows that for all $x, y \in X$, $x \leq y$ if and only if for all $U \in B^n$, if $x \in U$ then $y \in U$, so condition (DS4) also holds.
Corollary 5.28. Let $A$ be an $S$-algebra. Then $\langle \text{OP}_S(A), \tau_A, \preceq, \text{Irr}_S(A) \rangle$ is a generalized Priestley space, whose dual meet-semilattice is isomorphic to $M(A)$.

Since for any Priestley space $X$ the collection $\{U \setminus V : U, V \in \text{ClUp}(X)\}$ is a basis for the space, from Proposition 5.13 we obtain that for any $S$-Priestley space $X$ the collection $B \cup \{U^c : U \in B\}$ is a subbasis of the space $X$. The next proposition highlights that this issue is strongly connected with the fact that the $S$-referential algebra $\langle X, B \rangle$ is reduced, and leads us to an alternative definition of $S$-Priestley space.

Proposition 5.29. For any $S$-referential algebra $\langle X, B \rangle$ augmented with a topology $\tau$ and an order $\preceq$ on $X$, if $\langle X, \tau, \preceq \rangle$ is a Priestley space, $X \in B$ and $\text{ClUp}(X) = B^{\cup \cup} \cup \{\emptyset\}$, then $\langle X, B \rangle$ is reduced.

Proof. Let $\langle X, \tau, \preceq \rangle$ be a Priestley space satisfying the conditions above mentioned. We show that $\langle X, B \rangle$ is reduced by showing that $\preceq$ is the quasiorder $\preceq$ associated with the referential algebra.

Let first $x, y \in X$ be such that $x \preceq y$. As the elements of $B$ are up-sets, it follows that for all $V \in B$, if $x \in V$ then $y \in V$. Let now $x, y \in X$ be such that $x \not\preceq y$. Then by totally order disconnectedness of the space, there is $U$ a clopen up-set such that $x \in U$ and $y \not\in U$. Clearly $U \neq \emptyset$, so by assumption $U \in B^{\cup \cup}$. Then there are non-empty and finite subsets $U_i \subseteq B$, with $i \leq n$, for some $n \in \omega$, such that $x \in \bigcup\{\bigcap U_i : i \leq n\} = U$ and $y \notin \bigcup\{\bigcap U_i : i \leq n\}$. So there is $i \leq n$ such that $x \in \bigcap U_i$ and $y \notin \bigcap U_i$. And then there is $U_i \subseteq U \subseteq B$ such that $x \in U_i$ and $y \notin U_i$.

We conclude that for all $x, y \in X$, $x \preceq y$ if and only for all $V \in B$, if $x \in V$ then $y \in V$. Hence $\preceq = \preceq$. And since $\preceq$ is a partial order, it follows that the referential algebra $\langle X, B \rangle$ is reduced.

Corollary 5.30. A structure $\mathfrak{X} = \langle X, \tau, B \rangle$ is an $S$-Priestley space if and only if the following conditions are satisfied:

1. (Pr1) $\langle X, B \rangle$ is an $S$-referential algebra, whose associated quasiorder is denoted by $\preceq$.
2. (Pr2) for any finite $V \subseteq B$ and any $U \in B$, if $\bigcap V \subseteq U$, then $U \notin Fg^B(V)$.
3. (Pr3) $\langle X, \tau, \preceq \rangle$ is a Priestley space, and $B \cup \{U^c : U \in B\}$ is a subbasis for it.
4. (Pr4) $X \in B$ and $\text{ClUp}(X) = B^{\cup \cup} \cup \{\emptyset\}$.
5. (Pr5) the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed} is dense in $\langle X, \tau \rangle$.

6. Duality for morphisms

The approach for this section is similar to that of [1]. From now on, let $S$ be a filter-distributive and finitary congruential logic with theorems.

Let $A_1$ and $A_2$ be $S$-algebras and let $h \in \text{Hom}(A_1, A_2)$. The dual relation of $h$ is the relation $R_h \subseteq \text{OP}_S(A_2) \times \text{OP}_S(A_1)$ defined by:

$(P, Q) \in R_h \iff h^{-1}[P] \subseteq Q$.

Proposition 6.1. Let $A_1$, $A_2$ be $S$-algebras and $h \in \text{Hom}(A_1, A_2)$. For all $a \in A_1$:

1. $R_h^{-1}(\varphi_1(a)^c) = \varphi_2(h(a))^c$.
2. $\Box R_h(\varphi_1(a)) = \varphi_2(b(a))$.
3. $\Box R_h \in \text{Hom}(\varphi_1[A_1], \varphi_2[A_2])$. 

Proof. (1) First we show that \( R_h^{-1}(\varphi_1(a)^c) \subseteq \varphi_2(h(a))^c \). Let \( P \in \text{Op}_S(A_2) \) such that \( P \in R_h^{-1}(\varphi_1(a)^c) \), i.e., \( h^{-1}[P] \subseteq Q \) for some \( Q \notin \varphi_1(a) \). Then from \( a \notin Q \) we get \( a \notin h^{-1}[P] \), hence \( h(a) \notin P \) and so \( P \in \varphi_2(h(a))^c \). For the converse, let \( P \in \varphi_2(h(a))^c \), i.e., \( a \notin h^{-1}[P] \). As \( P \) is an \( S \)-filter of \( A_2 \), \( h^{-1}[P] \) is an \( S \)-filter of \( A_1 \). By the optimal \( S \)-filter lemma, since \( \downarrow a \) is a strong \( S \)-ideal and \( \downarrow a \cap h^{-1}[P] = \emptyset \), there is \( Q \in \text{Op}_S(A_1) \) such that \( a \notin Q \supseteq h^{-1}[P] \). So, \( Q \in \varphi_1(a)^c \) and \( Q \in R_h(P) \). Hence \( P \in R_h^{-1}(\varphi_1(a)^c) \).

(2) First we show that \( \Box_{R_h}(\varphi_1(a)) \subseteq \varphi_2(h(a)) \), so let \( P \in \Box_{R_h}(\varphi_1(a)) \), i.e., \( R_h(P) \subseteq \varphi_1(a) \). Suppose, towards a contradiction, that \( P \notin \varphi_2(h(a)) \). Then by item (1) we have \( P \in R_h^{-1}(\varphi_1(a)^c) \), so there is \( Q \in R_h(P) \) such that \( Q \notin \varphi_1(a) \), a contradiction. For the converse, let \( P \in \varphi_2(h(a)) \), so \( a \in h^{-1}[P] \). Then for any \( Q \in R_h(P) \), from \( h^{-1}[P] \subseteq Q \) we get \( a \in Q \), i.e., \( Q \in \varphi_1(a) \). This implies that \( R_h(P) \subseteq \varphi_1(a) \), i.e., \( P \in \Box_{R_h}(\varphi_1(a)) \), as required.

(3) Let \( f \) be an \( n \)-ary connective of the language and let \( a_i \in A_1 \) for each \( i \leq n \). Using the definition of \( \varphi_1[A_1] \) and \( \varphi_2[A_2] \), item (2), and the fact that \( h \in \text{Hom}(A_1,A_2) \), we get:

\[
\Box_{R_h}(f^\varphi_1[A_1](\varphi_1(a_1), \ldots, \varphi_1(a_n))) = \Box_{R_h}(\varphi_2(f^{A_1}(a_1, \ldots, a_n))) \\
= \varphi_2(h(f^{A_1}(a_1, \ldots, a_n))) \\
= \varphi_2(f^{A_2}(h(a_1), \ldots, h(a_n))) \\
= f^\varphi_2[A_2](\varphi_2(h(a_1)), \ldots, \varphi_2(h(a_n))) \\
= f^\varphi_2[A_2](\Box_{R_h}(\varphi_1(a_1)), \ldots, \Box_{R_h}(\varphi_1(a_n))).
\]

\( \Box \)

**Proposition 6.2.** Let \( A_1, A_2 \) be \( S \)-algebras and \( h \in \text{Hom}(A_1,A_2) \). For any \( P \in \text{Op}_S(A_2) \) and \( Q \in \text{Op}_S(A_1) \) such that \( (P,Q) \notin R_h \), there is \( a \in A_1 \) such that \( Q \notin \varphi(a) \) and \( R_h \subseteq \varphi(a) \).

**Proof.** From \( (P,Q) \notin R_h \) we get \( h^{-1}[P] \notin Q \), so there is \( a \in A \) such that \( a \in h^{-1}[P] \) and \( a \notin Q \). This implies that \( Q \notin \varphi(a) \) and for all \( Q' \in \text{Op}_S(A_1) \) such that \( (P,Q') \in R_h \), \( a \in Q' \). Therefore \( R_h(P) \subseteq \varphi(a) \) and we are done. \( \Box \)

Notice that the previous propositions hold in general for any finitary congruential logic with theorems, not necessarily a filter-distributive one. They lead us to the definition of the Priestley-dual morphisms between \( S \)-algebras.

**Definition 6.3.** Let \( \mathcal{X}_1 = (X_1,\tau_1,B_1) \) and \( \mathcal{X}_2 = (X_2,\tau_2,B_2) \) be two \( S \)-Priestley spaces. A relation \( R \subseteq X_1 \times X_2 \) is an \( S \)-Priestley morphism when:

(PrR1) \( \Box_R \in \text{Hom}(B_2,B_1) \),

(PrR2) if \( (x,y) \notin R \), then there is \( U \in B_2 \) such that \( y \notin U \) and \( R(x) \subseteq U \).

**Proposition 6.4.** Let \( A_1, A_2 \) be \( S \)-algebras and \( h \in \text{Hom}(A_1,A_2) \). Then \( R_h \) is an \( S \)-Priestley morphism between \( S \)-Priestley spaces \( \text{Op}_S(A_2) \) and \( \text{Op}_S(A_1) \).

**Proof.** (PrR1) follows from Proposition 6.1 and (PrR2) follows from Proposition 6.2. \( \Box \)

Recall that in Proposition 5.27 we proved that for any \( S \)-Priestley space \( (X,\tau,B) \), the structure \( (X,\tau,\leq,X_B) \) is a generalized Priestley space. Analogously, in the next theorem we show how \( S \)-Priestley morphisms and generalized Priestley morphisms are related.
**Theorem 6.5.** Let $R \subseteq X_1 \times X_2$ be an $S$-Priestley morphism between $S$-Priestley spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. Then $R$ is a generalized Priestley morphism between generalized Priestley spaces $\langle X_1, \tau_1, \leq_1, X_{B_1} \rangle$ and $\langle X_2, \tau_2, \leq_2, X_{B_2} \rangle$.

**Proof.** We just need to check that for any $X_{B_2}$-admissible clopen up-set of $X_2$, we have that $\Box_R(U)$ is an $X_{B_1}$-admissible clopen up-set of $X_1$. So let $U \in \text{ClUp}(X_2)$ be such that $\max(U^c) \subseteq X_{B_2}$. By Proposition 5.26 there are $U_0, \ldots, U_n \in B_2$ such that $U = U_0 \cap \cdots \cap U_n$. Then we have that $\Box_R(U) = \{ x \in X : R(x) \subseteq U_0 \cap \cdots \cap U_n \} = \Box_R(U_0) \cap \cdots \cap \Box_R(U_n)$. And then by (PrR1) and Proposition 5.26 again, $\max((\Box_R(U))^c) \subseteq X_{B_1}$, as required. \qed

The order associated with the $S$-referential algebra plays a prominent role in the duality. The next propositions show that it is an $S$-Priestley morphism and that its relational composition with any $S$-Priestley morphism $R$ is included in $R$.

**Proposition 6.6.** For any $S$-Priestley space $\mathcal{X} = \langle X, \tau, B \rangle$, the order associated with the $S$-referential algebra $\langle X, B \rangle$ is an $S$-Priestley morphism.

**Proof.** Recall that we denote the order associated with the $S$-referential algebra $\langle X, B \rangle$ by $\leq$. As the referential algebra is reduced, for any $x, y \in X$ such that $x \nleq y$, there is $U \in B$ such that $x \in U$ and $y \notin U$. Moreover, as $B$ is a family of clopen up-sets, for every $z \in \uparrow x$ we get $z \in U$. Therefore $\uparrow x \subseteq U$, hence condition (PrR2) is satisfied by $\leq$. Notice also that $\Box_<(Y) = \{ x \in X : \uparrow x \subseteq Y \}$. Since the elements of $B$ are up-sets with respect to $\leq$, for all $U \in B$ we have $\Box_<(U) = U$. Therefore $\Box_<$ is the identity map from $B$ to $B$, and so $\Box_\leq \in \text{Hom}(B, B)$ and condition (PrR1) is also satisfied by $\leq$. Hence the relation $\leq \subseteq X \times X$ is an $S$-Priestley morphism. \qed

**Proposition 6.7.** Let $\mathcal{X}_1 = \langle X_1, \tau_1, B_1 \rangle$ and $\mathcal{X}_2 = \langle X_2, \tau_2, B_2 \rangle$ be two $S$-Priestley spaces and $R \subseteq X_1 \times X_2$ an $S$-Priestley morphism. Then $\leq_1 \circ R \subseteq R$ and $R \circ \leq_2 \subseteq R$.

**Proof.** If $x \leq_1 y$, $(y, z) \in R$ and $(x, z) \notin R$, let $U \in B_2$ such that $z \notin U$ and $R(x) \subseteq U$. Thus, $x \in \Box_R(U)$ and since $\Box_R(U) \in B_1$ it is an up-set; therefore $y \in \Box_R(U)$ and $R(y) \subseteq U$. Hence $z \in U$, a contradiction. This proves that $\leq_1 \circ R \subseteq R$. A similar reasoning gives that $R \circ \leq_2 \subseteq R$. \qed

### 7. Categorical duality

We conclude the presentation of the duality for the algebraic counterpart $\text{Alg}S$ of a filter-distributive, finitary, and congruential logic $S$ with theorems by showing the functors and the natural transformations involved in it. Clearly $S$-algebras and homomorphisms between them form a category, that we denote by $\text{Alg}S$. Before proving the categorical duality for $\text{Alg}S$, we need to show that $S$-Priestley spaces and $S$-Priestley morphisms form a category as well.

Similarly to the case of distributive meet-semilattices, the set-theoretic relational composition of two composable $S$-Priestley morphisms may not be an $S$-Priestley morphisms. Hence we cannot use this operation to obtain a category. The operation that works is, as for distributive meet-semilattices, the following one. If $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ are $S$-Priestley spaces and $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ are $S$-Priestley morphisms.
morphisms, the composition \((S \ast R) \subseteq X_1 \times X_3\) is the relation defined by:

\[
(x, z) \in (S \ast R) \iff \forall U \in B_3(x \in \Box_R \circ \Box_S(U) \Rightarrow z \in U) \\
\iff \forall U \in B_3((S \circ R)(x) \subseteq U \Rightarrow z \in U).
\]

**Theorem 7.1.** Let \((X_1, \tau_1, \mathcal{B}_1), (X_2, \tau_2, \mathcal{B}_2)\) and \((X_3, \tau_3, \mathcal{B}_3)\) be \(S\)-Priestley spaces and let \(R \subseteq X_1 \times X_2\) and \(S \subseteq X_2 \times X_3\) be \(S\)-Priestley morphisms. Then:

1. The \(S\)-Priestley morphism \(\leq_2 \subseteq X_2 \times X_2\) satisfies:
   
   (a) \(\leq_2 \circ R = R\) and \(S \circ \leq_2 = S\),
   
   (b) \(\leq_2 \circ R = \leq_2 \ast R\) and \(S \circ \leq_2 = S \ast \leq_2\).

2. \((S \ast R) \subseteq X_1 \times X_3\) is an \(S\)-Priestley morphism.

**Proof.** To prove (2) note that Conditions \((\text{PrR1})\) and \((\text{PrR2})\) follow easily from the definition of \(\ast\). We proceed to prove (1.a). First we show that \(\leq_2 \circ R = R\). Let \(y \in R(x)\) and \(y \leq_2 z\), and suppose, towards a contradiction, that \(z \notin R(x)\). By \((\text{PrR2})\) there is \(U \in B_2\) such that \(R(x) \subseteq U\) and \(z \notin U\). Then by assumption \(y \in U\), and since \(U\) is an up-set, we get \(z \notin U\), a contradiction. Hence we have \(\leq_2 \circ R \subseteq R\).

The other inclusion is immediate. Now we show that \(S \circ \leq_2 = S\). Let \(x \leq_2 y\) and \(z \in S(y)\), and suppose, towards a contradiction, that \(z \notin S(x)\). By \((\text{PrR2})\) again, there is \(U \in B_3\) such that \(S(x) \subseteq U\) and \(z \notin U\). Then we have \(x \in \Box_S(U)\) and by \((\text{PrR1})\) we get \(\Box_S(U) \in B_2\). In particular \(\Box_S(U)\) is an up-set, thus \(y \in \Box_S(U)\).

Then \(S(y) \subseteq U\), and therefore \(z \in U\), a contradiction. Hence we have \(S \circ \leq_2 = S\).

The other inclusion is immediate. Finally we prove (1.b) The inclusion from left to right follows by definition. For the other inclusion, let \((x, z) \in (\leq_2 \ast R)\) and suppose, towards a contradiction, that \((x, z) \notin \leq_2 \circ R\). By item (1) we know that \(\leq_2 \circ R = R\), and then from the hypothesis and \((\text{Pr2})\), there is \(U \in B_2\) such that \(R(x) \subseteq U\) and \(z \notin U\). But since \((\leq_2 \circ R)(x) = R(x)\), we conclude \((x, z) \notin (\leq_2 \ast R)\), a contradiction. A similar proof shows that \(S \circ \leq_2 = S \ast \leq_2\). \(\square\)

Proposition 6.6 and Theorem 7.1 imply the next corollary

**Corollary 7.2.** The \(S\)-Priestley spaces and the \(S\)-Priestley morphisms with composition \(\ast\) form a category.

Let us denote by \(\text{PrS}\) the category of \(S\)-Priestley spaces and \(S\)-Priestley morphisms. On the one hand, we consider the functor \(\mathcal{D}_{\text{PrS}} : \text{Alg}_S \rightarrow \text{PrS}\) such that for any \(S\)-algebras \(A, A_1, A_2\) and any homomorphism \(h \in \text{Hom}(A_1, A_2)\):

\[
\mathcal{D}_{\text{PrS}}(A) := (O_{\text{PrS}}(A, \tau_A, \varphi[A]),) \\
\mathcal{D}_{\text{PrS}}(h) := R_h \subseteq O_{\text{PrS}}(A_2) \times O_{\text{PrS}}(A_1).
\]

Clearly, for the identity morphism \(\text{id}_A\) for \(A \in \text{Alg}_S\), we get \(R_{\text{id}_A} = \subseteq\), and this is the identity morphism for \(\mathcal{D}_{\text{PrS}}(A)\) in \(\text{PrS}\). Moreover, it easily follows from the definition of the dual relation of a homomorphism between \(S\)-algebras that for \(S\)-algebras \(A_1, A_2\) and \(A_3\) and homomorphisms \(f \in \text{Hom}(A_1, A_2)\) and \(g \in \text{Hom}(A_2, A_3)\), \(R_{gf} = R_f \ast R_g\). Therefore, using propositions 5.12 and 6.4, we conclude that the functor \(\mathcal{D}_{\text{PrS}}\) is well defined.

On the other hand, we consider the functor \((\cdot)\ast : \text{PrS} \rightarrow \text{Alg}_S\) such that for any \(S\)-Priestley spaces \(X, X_1, X_2\) and any \(S\)-Priestley morphism \(R \subseteq X_1 \times X_2\):

\[
X \ast := B, \\
R \ast := \Box_R : B_2 \rightarrow B_1.
\]
In order to complete the duality, we need to define two natural isomorphisms, \( \Phi_S \) and \( \Xi \). Consider first the family of morphisms in \( \mathbf{AlgS} \):

\[
\Phi_S: (\varphi_A: A \rightarrow \varphi_A[A])_{A \in \mathbf{AlgS}}
\]

**Theorem 7.3.** \( \Phi_S \) is a natural isomorphism between the identity functor on \( \mathbf{AlgS} \) and \( (\mathbb{D}_S(\cdot))^* \).

**Proof.** Let \( A_1, A_2 \in \mathbf{AlgS} \) and let \( h \in \text{Hom}(A_1, A_2) \). It is enough to show that

\[
\Box_R \circ \varphi_1 = \varphi_2 \circ h.
\]

For \( a \in A_1 \) and \( P \in \Box_R(\varphi_1(a)) \), we have \( R_h(P) \subseteq \varphi_1(a) \). It follows that \( h(a) \in P \), so \( P \in \varphi_2(h(a)) \). For \( P' \in \varphi_2(h(a)) \), we have \( h(a) \in P' \). It follows that \( R_h(P') \subseteq \varphi_1(a) \), so \( P' \in \Box_R(\varphi_1(a)) \).

From this we have get that \( \Phi_S \) is a natural transformation, and since by Theorem 3.2 we know that \( \varphi_1 \) is an isomorphism from \( A_1 \) to \( \varphi_1[A_1] \), we conclude that \( \Phi_S \) is a natural isomorphism. \( \square \)

Before defining the other natural isomorphism, we need to do some work. Recall that for any \( S \)-Priestley space \( \mathbf{X} = \langle X, \tau, \mathbf{B} \rangle \), the map \( \xi_X: X \rightarrow \mathbb{O}_S(\mathbf{B}) \) defined on Section 5 is a homeomorphism between the topological spaces \( \langle X, \tau \rangle \) and \( \langle \mathbb{O}_S(\mathbf{B}), \tau_B \rangle \). This map encodes the natural isomorphism we are looking for, but since morphisms in \( \mathbf{PrS} \) are relations, we need to give a relation associated with this map. We define the relation \( \mathcal{T}_\mathbf{X} \subseteq X \times \mathbb{O}_S(\mathbf{B}) \) by:

\[
(x, P) \in \mathcal{T}_\mathbf{X} \iff \xi_X(x) \subseteq P.
\]

**Proposition 7.4.** \( \mathcal{T}_\mathbf{X} \) is an \( S \)-Priestley morphism.

**Proof.** We have to show that \( \Box_{\mathcal{T}_\mathbf{X}} \in \text{Hom}(\varphi_B[\mathbf{B}], \mathbf{B}) \). Notice that for all \( \varphi_B(b) \in \varphi_B[B] \), we have:

\[
\Box_{\mathcal{T}_\mathbf{X}}(\varphi_B(b)) = \{ x \in X : \forall y \in X (\langle x, \xi_X(y) \rangle \in \mathcal{T}_\mathbf{X} ) \Rightarrow \xi_X(y) \in \varphi_B(b) ) \}
\]

\[
= \{ x \in X : \forall y \in X (\xi_X(x) \subseteq \xi_X(y) ) \Rightarrow b \in \xi_X(y) ) \}
\]

\[
= \{ x \in X : b \in \xi_X(x) ) = b. \}
\]

Therefore \( \Box_{\mathcal{T}_\mathbf{X}} = \varphi_B^{-1} \). And since \( \mathbf{B} \) and \( \varphi_B[\mathbf{B}] \) are isomorphic \( S \)-algebras by means of the map \( \varphi_B \), it follows that \( \Box_{\mathcal{T}_\mathbf{X}} \in \text{Hom}(\varphi_B[\mathbf{B}], \mathbf{B}) \). This proves that condition \( (\text{PrR1}) \) is satisfied by \( \mathcal{T}_\mathbf{X} \).

We prove now that condition \( (\text{PrR2}) \) is also satisfied by \( \mathcal{T}_\mathbf{X} \). Notice that for each \( x \in X \), we have \( \mathcal{T}_\mathbf{X}(x) = \uparrow \xi_X(x) \). Let \( x \in X \) and \( P \in \mathbb{O}_S(\mathbf{B}) \) be such that \( (x, P) \notin \mathcal{T}_\mathbf{X} \). We have to show that there is \( U \in B \) such that \( P \notin \varphi_B(U) \) and \( \mathcal{T}_\mathbf{X}(x) \subseteq \varphi_B(U) \). By definition of \( \mathcal{T}_\mathbf{X} \), we have that \( \xi_X(x) \nsubseteq P \), so there is \( U \in B \) such that \( U \in \xi_X(x) \) and \( U \notin P \). Hence \( P \notin \varphi_B(U) \) and \( \xi_X(x) \in \varphi_B(U) \). Now since \( \mathcal{T}_\mathbf{X}(x) = \uparrow \xi_X(x) \), we obtain that \( \mathcal{T}_\mathbf{X}(x) \subseteq \varphi_B(U) \), as required. Finally, by previous argument we conclude that \( \mathcal{T}_\mathbf{X} \) is an isomorphism in \( \mathbf{PrS} \). \( \square \)

Consider now the family of morphisms in \( \mathbf{PrS} \):

\[
\Xi = (\mathcal{T}_\mathbf{X} \subseteq X \times \mathbb{O}_S(\mathbf{B}))_{\mathbf{X} \in \mathbf{PrS}}
\]

**Theorem 7.5.** \( \Xi \) is a natural isomorphism between the identity functor on \( \mathbf{PrS} \) and \( (\mathbb{D}_S(\cdot))^* \).
Proof. Let $x_1 = \langle X_1, \tau_1, B_1 \rangle$ and $x_2 = \langle X_2, \tau_2, B_2 \rangle$ be two $S$-Priestley spaces and let $R \subseteq X_1 \times X_2$ be an $S$-Priestley morphism. First we show that:

$$(x, y) \in R \iff (\xi_1(x), \xi_2(y)) \in R_{\bigtriangleup}.$$ 

Let $x \in X_1$ and $y \in X_2$ be such that $(x, y) \in R$, and let $U \subseteq B_2$. Notice that we have:

$$U \in \square^{-1}[\xi_1(x)] \iff \square_R(U) \subseteq \xi_1(x) \iff x \in \square_R(U) \iff R(x) \subseteq U.$$ 

Thus if $U \in \square^{-1}[\xi_1(x)]$, then $R(x) \subseteq U$, and since $(x, y) \in R$, we obtain $y \in U$, i.e., $U \in \xi_2(y)$, and therefore $(\xi_1(x), \xi_2(y)) \in R_{\bigtriangleup}$. For the converse, let $x \in X_1$, $y \in X_2$ be such $(\xi_1(x), \xi_2(y)) \in R_{\bigtriangleup}$ and suppose, towards a contradiction, that $y \notin R(x)$. Since $R$ is an $S$-Priestley morphism, by (PrR1), there is $U \in B_2$ such that $y \notin U$ and $R(x) \subseteq U$. From previous equivalences we obtain $U \in \square^{-1}[\xi_1(x)]$. But then from the hypothesis $U \subseteq \xi_2(y)$, so $y \in U$, a contradiction.

The equivalence that we just proved implies that $R_{\bigtriangleup} \ast T_{x_1} = T_{x_2} \ast R$. Thus $\Xi_S$ is a natural equivalence. Moreover, as $T_X$ is an isomorphism for each $S$-Priestley space $X$, then $\Xi_S$ is a natural isomorphism. \hfill $\square$

**Theorem 7.6.** The categories $\text{Alg}S$ and $\text{Pr}S$ are dually equivalent by means of the contravariant functors $\square_{\text{Pr}S}$ and $(\cdot)^*$ and the natural equivalences $\Phi_S$ and $\Xi_S$.

8. **Dual correspondence of some logical properties**

In this final section we examine how the correspondences between objects of the categories we are considering can be refined depending on the properties of the logic under consideration. For information on the abstract properties of logics we consider in the sequel we refer the reader to [8]. Given the abstract character of our general approach, we carry out this study modularly, treating each property independently, obtaining in this way results that can be combined afterwards.

8.1. **The Property of Conjunction.** A logic $S$ has the property of conjunction (PC) for a binary term $p \land q$ if :

$$p \land q \vdash_S p, \quad p \land q \vdash_S q, \quad p, q \vdash_S p \land q.$$ 

By the substitution-invariance of $\vdash_S$ we can replace $p$ and $q$ by arbitrary formulas. The property of conjunction transfers to every algebra in the sense that if $S$ has (PC) for $p \land q$, then for every algebra $A$ and every $a, b \in A$

$$\text{Fg}_S^A(a \land^A b) = \text{Fg}_S(a, b),$$

see page 50 in [8]. For the remaining part of the subsection, let $S$ be a filter-distributive and finitary congruential logic with theorems.

**Lemma 8.1.** If $S$ satisfies (PC), then for every $S$-algebra $A$ and all $a, b \in A$,

$$\varphi(a) \cap \varphi(b) = \varphi(a \land^A b).$$

**Proof.** If follows easily from the fact that (PC) transfers to every algebra. \hfill $\square$

Notice that by the associativity of intersection, the previous lemma implies that for any non-empty and finite $B \subseteq A$, $\bigcap\{\varphi(b) : b \in B\} = \varphi(\land^A B)$. Recall that we defined the $S$-semilattice of $A$ as the closure of $\varphi[A]$ under finite intersections. Therefore, if $S$ satisfies (PC), then $\langle A, \land^A, 1^A \rangle$ and $M(A)$ are isomorphic.
**Proposition 8.2.** If $S$ satisfies (PC), then for every $S$-algebra $A$ and all $U \subseteq O_{P_S}(A)$, if $U$ is an $\text{Irr}_S(A)$-admissible clopen up-set of the space $(O_{P_S}(A), \tau_A, \subseteq)$, then there is an $A \cap a$ such that $U = \varphi(a)$.

**Proof.** Let $U \subseteq O_{P_S}(A)$ be a clopen up-set of $(O_{P_S}(A), \tau_A, \subseteq)$ such that $\max(U^c) \subseteq \text{Irr}_S(A)$. Then by Proposition 5.8, there is a non-empty finite $B \subseteq A$ such that $U = \hat{\varphi}(B) = \bigcap\{\varphi(b) : b \in B\}$. Lemma 8.1 implies that $U = \varphi(A^B B)$, as required. \hfill $\Box$

**Corollary 8.3.** If $S$ satisfies (PC), then for every $S$-algebra $A$, $\varphi[A]$ is the collection of $\text{Irr}_S(A)$-admissible clopen up-sets of $(O_{P_S}(A), \tau_A, \subseteq)$.

From the previous corollary we can conjecture that the property that corresponds on any $S$-Priestley space $(X, \tau, B)$ to (PC) is “$B$ is the collection of $X_B$-admissible clopen up-sets”. We prove that this condition is indeed enough for recovering the conjunction. To prove this we recall a general fact from the theory of congruential logics (cf. [8]):

**Proposition 8.4.** Let $S$ be a congruential logic. For every algebra $A$, the quotient homomorphism $\pi_A : A \rightarrow A/\equiv_S$ induces an isomorphism between the lattices $F_S(A)$ and $F_S(A/\equiv_S)$ given by $F \mapsto \pi_A[F]$ and whose inverse is given by $G \mapsto \pi_A^{-1}[G]$.

**Proposition 8.5.** Let $(X, \tau, B)$ be an $S$-Priestley space such that $B$ is the collection of the $X_B$-admissible clopen up-sets of $X$. Then for all $U, V \in B$, $F_{g_B}^{B}(U, V) = F_{\mathcal{B}}^{B}(U \cap V).

**Proof.** First notice that the hypothesis implies that $B$ is closed under finite intersections. Now let $U, V \in B$. By (Pr2) we get $\bigcap\{U, V\} \subseteq U \cap V$ if and only if $U \cap V \in F_{g_B}^{B}(U, V)$. Therefore, $U \cap V \in F_{g_B}^{B}(U, V)$. Then, considering Remark 5.16, it is easy to see that $F_{g_B}^{B}(U, V) = F_{g_B}^{B}(U \cap V)$. \hfill $\Box$

**Proposition 8.6.** If $S$ is such that for every $S$-Priestley space $(X, \tau, B)$ the set $B$ is the collection of the $X_B$-admissible clopen up-sets of $X$, then $S$ satisfies (PC).

**Proof.** Recall that the Lindenbaum-Tarski algebra $Fm/\equiv_S$ is an $S$-algebra. We abbreviate $\equiv_S$ by $\equiv$. Let $p, q \in Var$. The assumption implies that there is a formula $\varphi \in Fm_\mathcal{B}$ such that $\varphi(p/\equiv) \cap \varphi(q/\equiv) = \varphi(p/\equiv)$. Moreover, by Proposition 8.5 we have

$$F_{\mathcal{B}}[Fm/\equiv](\varphi(p/\equiv), \varphi(q/\equiv)) = F_{\mathcal{B}}[Fm/\equiv](\varphi(p/\equiv) \cap \varphi(q/\equiv)).$$

Then by Theorem 3.3 we obtain $F_{\mathcal{B}}[Fm/\equiv](p/\equiv, q/\equiv) = F_{\mathcal{B}}[Fm/\equiv](p/\equiv)$. And using Proposition 8.4 we get $C_{n_S}(p, q) = C_{n_S}(\rho)$. By the substitution-invariance of $\vdash_S$, it follows that there is a formula $\rho(p, q)$ in at most the variables $p$ and $q$ such that $C_{n_S}(p, q) = C_{n_S}(\rho(p, q))$. Hence $S$ satisfies (PC) for $\rho(p, q)$. \hfill $\Box$

Corollary 8.3 and Proposition 8.6 imply the desired theorem.

**Theorem 8.7.** Let $S$ be a filter-distributive and finitary congruential logic with theorems. Then $S$ satisfies (PC) if and only if for every $S$-Priestley space $(X, \tau, B)$, $B$ is the collection of the $X_B$-admissible clopen up-sets of $X$. 


8.2. The Property of Disjunction. A logic $S$ satisfies the property of weak disjunction (PWDI) for a set of formulas in two variables $\nabla(p, q)$ if for all formulas $\delta, \gamma, \mu \in Fm_S$:

(a) $\delta \vdash_S \nabla(\delta, \gamma)$,  
(b) $\delta \vdash_S \nabla(\gamma, \delta)$,  
(c) if $\delta \vdash_S \mu$ & $\gamma \vdash_S \mu$, then $\nabla(\delta, \gamma) \vdash_S \mu$.

A logic $S$ satisfies the property of disjunction (PDI) for a set of formulas in two variables $\nabla(p, q)$ if for all formulas $\{\delta, \gamma, \mu\} \cup \Gamma \subseteq Fm_S$ besides the conditions (a) and (b) above we have:

(d) if $\Gamma, \delta \vdash_S \mu$ & $\Gamma, \gamma \vdash_S \mu$, then $\Gamma, \nabla(\delta, \gamma) \vdash_S \mu$.

If $S$ satisfies (PDI) for $\nabla(p, q)$, then this property transfers to every algebra (cf. Corollary 2.5.4 in [6] or Theorem 2.52 in [8]), that is, for every algebra $A$ and every $\{a, b\} \cup X \subseteq A$:

$$\text{F}_{S}(A)(X, \nabla^A(a, b)) = \text{F}_{S}(A)(X, a) \cap \text{F}_{S}(A)(X, b).$$

Moreover, it is known that if a logic satisfies (PDI) then it is filter-distributive (cf. [5]). It is also known that for any filter-distributive logic $S$, $S$ satisfies (PWDI) for a set of formulas $\nabla(p, q)$ if and only if it satisfies (PDI) for the same set $\nabla(p, q)$ (cf. [4]). Next lemma (see Proposition 2.5.7 in [6]) will be used in the sequel.

**Lemma 8.8.** If $S$ satisfies (PDI) for a set of formulas $\nabla(p, q)$, then for every $S$-algebra $A$, an $S$-filter $F \in \text{Fi}_S(A)$ is irreducible if and only if for every $a, b \in A$, if $\nabla^A(a, b) \subseteq F$, then $a \in F$ or $b \in F$.

Another important fact on the property of disjunction is the following result, that follows from Theorem 2.5.9 in [6], as observed in [4] (taking into account that in [6] the irreducible $S$-filters are called prime):

**Theorem 8.9.** A filter-distributive and finitary logic $S$ satisfies (PDI) for some set of formulas $\nabla(p, q)$ if and only if for all algebras $A, B$, every homomorphism $h \in \text{Hom}(A, B)$ and every $G \in \text{Irr}_S(B)$, the $S$-filter $h^{-1}[G]$ of $A$ is irreducible.

In our setting of congruential, filter-distributive and finitary logics, we can restrict the class of algebras in Theorem 8.9 to $\text{Alg}_S$:

**Proposition 8.10.** Let $S$ be a congruential logic. If for all algebras $A, B \in \text{Alg}_S$, every homomorphism $h \in \text{Hom}(A, B)$ and every $G \in \text{Irr}_S(B)$ the $S$-filter $h^{-1}[G]$ of $A$ is irreducible, then the same holds for arbitrary algebras.

**Proof.** Let $A, B$ be arbitrary algebras and let $h : A \to B$ be a homomorphism. Consider the quotient algebras $A/\equiv_A^S$ and $B/\equiv_B^S$, which belong to $\text{Alg}_S$. Let $\pi_A : A \to A/\equiv_A^S$ and $\pi_B : B \to B/\equiv_B^S$ the quotient homomorphisms. We define the map $h^* : A/\equiv_A^S \to B/\equiv_B^S$ by setting for every $a \in A$, $h(a)/\equiv_B^S = h(a)/\equiv_B^S$. This map is well defined, as if $a \equiv_A^S b$ and $F \in \text{Fi}_S(B)$ is such that $h(a) \in F$, then $a \in h^{-1}[F] \in \text{Fi}_S(A)$ and therefore, since $a \equiv_S b$, we obtain $b \in h^{-1}[F]$ and so $h(b) \in F$. Moreover, $h'$ is a homomorphism and $h' \circ \pi_A = \pi_B \circ h$. By Proposition 8.4, $\pi_A[\cdot]$ establishes a lattice isomorphism between $\text{Fi}_S(A)$ and $\text{Fi}_S(A/\equiv_A^S)$ and $\pi_B[\cdot]$ a lattice isomorphism between $\text{Fi}_S(B)$ and $\text{Fi}_S(B/\equiv_B^S)$. Thus if $G$ is an irreducible $S$-filter of $B$, then $\pi_B[G]$ is an irreducible $S$-filter of $B/\equiv_B^S$. Hence from the assumption follows that $h'^{-1}[\pi_B[G]]$ is an irreducible $S$-filter of $A/\equiv_A^S$. Therefore, $\pi_A^{-1}[h'^{-1}[\pi_B[G]]]$ is an irreducible $S$-filter of $A$. But $\pi_A^{-1}[h'^{-1}[\pi_B[G]]] = \pi_A^{-1}h^{-1}[\pi_B[G]]$. 


The following conditions are equivalent:

**Proposition 8.12.** Proposition 8.12.

finitary logic spaces. From now on in this subsection we fix a congruential, filter-distributive and irreducible. □

We obtain the next corollary:

**Corollary 8.11.** A congruential, filter-distributive and finitary logic $S$ satisfies (PDI) for some set of formulas $\nabla(p, q)$ if and only if for all algebras $A, B \in \text{Alg}_S$, every homomorphism $h \in \text{Hom}(A, B)$ and every $G \in \text{Irr}_S(B)$, the $S$-filter $h^{-1}[G]$ of $A$ is irreducible.

We look now for a translation of this property on the inverse images of irreducible $S$-filters under homomorphisms into a property of morphisms of the dual $S$-Priestley spaces. From now on in this subsection we fix a congruential, filter-distributive and finitary logic $S$.

**Proposition 8.12.** For all algebras $A, B \in \text{Alg}_S$ and every $h \in \text{Hom}(A, B)$ the following conditions are equivalent:

1. for every $G \in \text{Irr}_S(B)$, $h^{-1}[G] \in \text{Irr}_S(A)$,
2. for every $G \in \text{Irr}_S(B)$ there exists $F \in \text{Irr}_S(A)$ such that for all $a \in A$, $F \in \varphi(a)$ if and only if $R_h(G) \subseteq \varphi(a)$.

**Proof.** Note that condition (2) is equivalent to saying that $\bigcap R_h(G) \in \text{Irr}_S(A)$, for every $G \in \text{Irr}_S(B)$. Also note that from Proposition 2.17 it follows that for any $G \in \text{Fg}_S(B)$, $h^{-1}[G]$ equals the intersection of all the irreducible $S$-filters of $A$ that include $h^{-1}[G]$. Moreover, since every irreducible $S$-filter is optimal, every irreducible $S$-filter that includes $h^{-1}[G]$ belongs to $R_h(G)$. Therefore, since $h^{-1}[G] \subseteq H$ for every $H \in R_h(G)$ it follows that $h^{-1}[G] = \bigcap R_h(G)$. We proceed to prove that (1) implies (2). Suppose now (2). Let $G \in \text{Irr}_S(B)$. Then, by (2), $\bigcap R_h(G) \in \text{Irr}_S(A)$. Since $h^{-1}[G] = \bigcap R_h(G)$ we conclude that $h^{-1}[G] \in \text{Irr}_S(A)$.

We obtain the following characterization of having (PDI).

**Theorem 8.13.** The logic $S$ satisfies (PDI) if and only if for every $S$-Priestley morphism $R \subseteq X_1 \times X_2$ from an $S$-Priestley space $(X_1, \tau_1, B_1)$ to an $S$-Priestley space $(X_2, \tau_2, B_2)$ the following condition holds:

$$\forall x \in X_1 \exists y \in X_2 \forall U \in B_2 (y \in U \iff R(x) \subseteq U). \quad (E2)$$

**Proof.** If $S$ satisfies (PDI), then using the duality we have developed and Proposition 8.12 we obtain condition (E2). Conversely, if (E2) holds for every $S$-Priestley morphism $R \subseteq X_1 \times X_2$ from an $S$-Priestley space $(X_1, \tau_1, B_1)$ to an $S$-Priestley space $(X_2, \tau_2, B_2)$, then the duality we have developed, Proposition 8.12 and Corollary 8.11 allow us to conclude that $S$ satisfies (PDI).

**Proposition 8.14.** If $S$ satisfies (PDI) for a finite set of formulas $\nabla(p, q)$, then for every $S$-algebra $A$, $\text{Op}_S(A) = \text{Irr}_S(A)$.

**Proof.** We just need to show that every optimal $S$-filter of $A$ is irreducible, so let $P$ be an optimal $S$-filter of $A$ and let $a, b \in A$ be such that $\nabla^A(a, b) \subseteq P$. Assume, towards a contradiction, that $a, b \notin P$. Since $F_gS^A(a) \cap F_gS^A(b) = F_gS^A(\nabla^A(a, b))$ and $P^c$ is a strong $S$-ideal it follows that $F_gS^A(\nabla^A(a, b)) \cap P^c \neq \emptyset$, a contradiction. Hence if $\nabla^A(a, b) \subseteq P$, then $a \in P$ or $b \in P$. Thus, Lemma 8.8 implies that $P$ is irreducible. □
Note that taking into account Proposition 8.14 we also have the following result:

**Proposition 8.15.** If the logic \( S \) satisfies (PDI) for some finite set of formulas \( \nabla(p, q) \), then for every \( S \)-Priestley space \( \langle X, \tau, B \rangle \) we have \( X = X_B \) and for every \( S \)-Priestley morphism \( R \subseteq X_1 \times X_2 \) from an \( S \)-Priestley space \( \langle X_1, \tau_1, B_1 \rangle \) to an \( S \)-Priestley space \( \langle X_2, \tau_2, B_2 \rangle \) it holds that

\[
(\forall x \in X_1)(\exists y \in X_2)R(x) = \uparrow y. \tag{E3}
\]

**Proof.** If \( S \) satisfies (PDI) for a finite set of formulas \( \nabla(p, q) \), then the duality we have developed and Proposition 8.14 show that \( X = X_B \) for every \( S \)-Priestley space \( \langle X, \tau, B \rangle \). Then it is easily seen that condition (E2) in Theorem 8.13 implies (E3).

We say that an \( S \)-Priestley morphism \( R \subseteq X_1 \times X_2 \) from an \( S \)-Priestley space \( \langle X_1, \tau_1, B_1 \rangle \) to an \( S \)-Priestley space \( \langle X_2, \tau_2, B_2 \rangle \) is *functional* if condition E3 holds. The name is due to the fact that then from \( R \) we can define a function \( f_R : X_1 \to X_2 \) by setting \( f_R(x) \) as the only \( y \) such that \( R(x) = \uparrow y \).

Note that the consequent of the statement of Proposition 8.15 implies that \( S \) has (PDI), because it implies condition (E2) in Theorem 8.13.

Now we concentrate in the case where (PDI) holds for a one-element set \( \nabla = \{p \lor q\} \), where \( p \lor q \) is a formula in two variables. In this case we say \( S \) satisfies (PDI) for \( p \lor q \).

**Lemma 8.16.** If \( S \) satisfies (PDI) for \( p \lor q \), then for every \( S \)-algebra \( A \) and all \( a, b \in A \), \( \varphi(a) \lor \varphi(b) = \varphi(a \lor^A b) \).

**Proof.** Notice that since (PDI) transfers to every algebra, for all \( a, b \in A \) we have \( Fg_S^A(a \lor^A b) = Fg_S^A(a) \cap Fg_S^A(b) \). This implies that \( a, b \leq^A a \lor^A b \) and therefore for any \( P \in Op_S(A) \), we have that if \( a \in P \) or \( b \in P \), then \( a \lor^A b \in P \), because \( P \) is an up-set. This proves that \( \varphi(a) \lor \varphi(b) \subseteq \varphi(a \lor^A b) \). To prove the other inclusion assume that \( P \in \varphi(a \lor^A b) \). Since, by Proposition 8.14, \( P \) is irreducible, by Lemma 8.8 follows that \( a \in P \) or \( b \in P \); hence \( P \in \varphi(a) \lor \varphi(b) \). \( \square \)

**Corollary 8.17.** If \( S \) satisfies (PDI) for a single formula \( p \lor q \), then for every \( S \)-algebra \( A \), \( \varphi[A] \) is closed under the binary operation of union.

We recall that for every logic \( S \) and every algebra \( A \), every \( S \)-filter \( F \) of \( A \) is equal to the intersection of all the irreducible \( S \)-filters of \( A \) that include \( F \). In particular this holds for the theories of \( S \).

**Proposition 8.18.** If \( S \) satisfies (PDI) and for every \( S \)-algebra \( A \) the set \( \varphi[A] \) is closed under the binary operation of union, then \( S \) satisfies (PDI) for a single formula.

**Proof.** Let us consider the relation \( \equiv^F_{\mathsf{fm}} \) that we abbreviate all along the proof by \( \equiv \). Then the quotient algebra \( \mathsf{fm}/\equiv \in \mathsf{Alg}S \). Let \( p, q \) be variables and consider the equivalence classes \( p/\equiv, q/\equiv \). By assumption \( \varphi(p/\equiv) \lor \varphi(q/\equiv) = \varphi(\delta/\equiv) \) for some formula \( \delta \).

We first prove that \( Cn_S(p) \cap Cn_S(q) = Cn_S(\delta) \). Let \( T \) be an irreducible \( S \)-theory such that \( Cn_S(p) \cap Cn_S(q) \subseteq T \). Then \( T = (Cn_S(p) \cap Cn_S(q)) \cup T \). Since \( S \) is filter-distributive, \( T = (Cn_S(p) \cup T) \cap (Cn_S(q) \cup T) \). Now being \( T \) irreducible, it follows that \( T = Cn_S(p) \cup T \) or \( T = Cn_S(q) \cup T \). Hence \( p \in T \) or \( q \in T \). Therefore, \( p/\equiv \in \pi[T] \) or \( q/\equiv \in \pi[T] \), where \( \pi \) is the quotient homomorphism
from \( Fm \) onto \( Fm/\equiv \). By Proposition 8.4, the fact that \( T \) is irreducible implies that \( \pi[T] \) is an irreducible \( S \)-filter of \( Fm/\equiv \); therefore \( \pi[T] \in \varphi(p/\equiv) \cup \varphi(q/\equiv) \).

Hence, \( \delta/\equiv \in \pi[T] \). This implies, since by the definition of \( \equiv \), \( \delta \equiv \delta' \) if and only if \( Cn_S(\delta) = Cn_S(\delta') \), that \( \delta \in T \). We conclude that \( Cn_S(\delta) \subseteq Cn_S(p) \cap Cn_S(q) \). To prove the other inclusion, let \( T \) be an irreducible \( S \) theory such that \( \delta \in T \). Then \( \delta/\equiv \in \pi[T] \) and \( \pi[T] \) is an irreducible \( S \)-filter of \( Fm/\equiv \). Therefore, \( \pi[T] \equiv \varphi(\delta/\equiv) \).

Hence \( p/\equiv \in \pi[T] \) or \( q/\equiv \in \pi[T] \). This, by a similar reasoning as before, implies that \( p \in T \) or \( q \in T \). In both cases \( Cn_S(p) \cap Cn_S(q) \subseteq T \). We conclude that \( Cn_S(p) \cap Cn_S(q) \subseteq Cn_S(\delta) \).

Now, since \( S \) has (PDI), let us assume that \( S \) has (PDI) for \( \{ \varphi(p) \} \). Then \( Cn_S(p) \cap Cn_S(q) = Cn_S(\varphi(p, q)) \). Therefore, \( Cn_S(\varphi(p, q)) = Cn_S(\delta) \). Let \( \sigma \) be the substitution that maps \( p \) to \( p \) and all the remaining variables to \( q \). Then \( \sigma[\varphi(p, q)] = \varphi(p, q) \) and therefore \( Cn_S(\varphi(p, q)) = Cn_S(\sigma(\delta)) \) using invariance under substitutions. It follows that \( Cn_S(p) \cap Cn_S(q) = Cn_S(\sigma(\delta)) \) and the variables in \( \sigma(\delta) \) are at most \( p, q \). Thus \( S \) has (WPDI) for \( \sigma(\delta) \) and being filter-distributive it also has (PDI) for \( \sigma(\delta) \).

Combining Proposition 8.15 and Proposition 8.18 we easily obtain that the property of \( S \)-Priestley spaces that corresponds to (PDI) for a single formula is “\( X = X_B \), \( B \) is closed under the binary operation of union and the \( S \)-Priestley morphisms are functional”.

We find another equivalent condition, this time on morphisms. To this end we consider the next proposition.

**Proposition 8.19.** If \( S \) satisfies (PDI) for a single formula \( p \lor q \), then for every \( A, B \in \text{Alg} S \) and every \( h \in \text{Hom}(A, B) \), the relation \( R_h \subseteq \text{Op}_S(B) \times \text{Op}_S(A) \) satisfies that for every \( a, b \in A \) and every \( P \in \text{Op}_S(B) \), if \( R_h(P) \subseteq \varphi^A(a) \lor \varphi^A(b) \), then \( R_h(P) \subseteq \varphi^A(a) \lor \varphi^A(b) \).

**Proof.** Assume that \( R_h(P) \subseteq \varphi^A(a) \lor \varphi^A(b) \). Then \( R_h(P) \subseteq \varphi^A(a \lor^A b) \). Suppose that \( R_h(P) \not\subseteq \varphi^A(a) \lor \varphi^A(b) \). Let \( Q, Q' \subseteq R_h(P) \) such that \( a \notin Q \) and \( b \notin Q' \). Then \( h^{-1}[P] \subseteq Q \) and \( h^{-1}[P] \subseteq Q' \). Hence, \( h(a), h(b) \notin P \) and therefore, \( h(a \lor^A b) \notin P \).

**Lemma 8.20.** Let \( (X, \tau, B) \) be an \( S \)-Priestley space such that \( B \) is closed under the binary operation of union. Then for all \( \{ U, V \} \subseteq B \),

\[
F_{gS}^B(W, U) \cap F_{gS}^B(W, V) = F_{gS}^B(W, U \lor V).
\]

**Proof.** We first prove that for all \( U, V \subseteq B \), \( F_{gS}^B(U) \cap F_{gS}^B(V) = F_{gS}^B(U \lor V) \). By hypothesis we have for any \( W \subseteq B \) that \( W \in F_{gS}^B(U) \cap F_{gS}^B(V) \) if and only if \( U \subseteq W \) and \( V \subseteq W \) if and only if \( W \in F_{gS}^B(U \lor V) \), as required. Now using the filter distributivity of the logic we have:

\[
F_{gS}^B(W, U) \cap F_{gS}^B(W, V) = (F_{gS}^B(W) \cup F_{gS}^B(U)) \cap (F_{gS}^B(W) \cup F_{gS}^B(V)) = F_{gS}^B(W) \cup (F_{gS}^B(U) \cap F_{gS}^B(V)) = F_{gS}^B(W) \cup F_{gS}^B(U \lor V) = F_{gS}^B(W, U \lor V).
\]

Lemma 8.20 implies the following characterization of the irreducible \( S \)-filters.
Lemma 8.21. Let \((X, \tau, B)\) be an \(S\)-Priestley space such that \(B\) is closed under the binary operation of union and let \(F\) be an \(S\)-filter of \(B\). Then \(F\) is irreducible if and only if for all \(U, V \in B\), if \(U \cup V \in F\), then \(U \in F\) or \(V \in F\).

Proof. Suppose that \(F\) is an irreducible \(S\)-filter of \(B\) and \(U \cup V \in F\) with \(U, V \in B\).

By Lemma 8.20, \(Fg^B_S(F, U) \cap Fg^B_S(F, V) = Fg^B_S(F, U \cup V)\). Since \(U \cup V \in F\), it follows that \(Fg^B_S(F, U) \cap Fg^B_S(F, V) = F\). This implies, being \(F\) irreducible, that \(Fg^B_S(F, U) = F\) or \(Fg^B_S(F, V) = F\); hence \(U \in F\) or \(V \in F\). Conversely, suppose that for all \(U, V \in B\), if \(U \cup V \in F\), then \(U \in F\) or \(V \in F\). Assume that \(F = H \cap G\) with \(F, G \in Fg^B_S(B)\) and that \(F \neq H\) and \(F \neq G\). Let then \(U \in H \setminus F\) and \(V \in G \setminus F\). Since \(B\) is closed under finite unions \(U \cup V \in B\). Hence, since \(H\) and \(G\) are upsets, \(U \cup V \in H \cap G = F\). From the hypothesis follows that \(U \in F\) of \(V \in F\), a contradiction. \(\square\)

Lemma 8.22. Let \(R \subseteq X_1 \times X_2\) be an \(S\)-Priestley morphism from an \(S\)-Priestley space \(\langle X_1, \tau_1, B_1 \rangle\) to an \(S\)-Priestley space \(\langle X_2, \tau_2, B_2 \rangle\) such that \(B_1\) and \(B_2\) are closed under the binary operation of union and moreover for every \(U, V \in B_2\) and every \(x \in X_1\), if \(R(x) \subseteq U \cup V\), then \(R(x) \subseteq U\) or \(R(x) \subseteq V\). Then condition \((E2)\) holds.

Proof. Assume the antecedent. Let \(x \in X_{B_1}\). Then \(\varepsilon(x)\) is an irreducible \(S\)-filter of \(B_1\). Consider the homomorphism \(\Box_R : B_2 \rightarrow B_1\). We prove that \(\Box_R^{-1}[\varepsilon(x)]\) is an irreducible \(S\)-filter of \(B_2\). Let \(U, V \in B_2\) be such that \(U \cup V \in \Box_R^{-1}[\varepsilon(x)]\). Then \(\Box_R(U \cup V) \in \varepsilon(x)\). Therefore, \(x \in \Box_R(U \cup V)\). Thus \(R(x) \subseteq U \cup V\). Therefore, by the assumption, \(R(x) \subseteq U\) or \(R(x) \subseteq V\). Hence \(x \in \Box_R(U)\) of \(x \in \Box_R(V)\). This implies that \(\Box_R(U) \in \varepsilon(x)\) or \(\Box_R(V) \in \varepsilon(x)\); hence \(U \in \Box_R^{-1}[\varepsilon(x)]\) or \(V \in \Box_R^{-1}[\varepsilon(x)]\). By Lemma 8.21 we obtain that \(\Box_R^{-1}[\varepsilon(x)]\) an irreducible \(S\)-filter of \(B_2\).

The next theorem follows using the duality we have developed, Proposition 8.19

and the last lemma.

**Theorem 8.23.** The logic \(S\) has (PDI) for a single formula if and only if for every \(S\)-Priestley space \(\langle X, \tau, B \rangle\) the set \(B\) is closed under the binary operation of union and for every \(S\)-Priestley morphism \(R \subseteq X_1 \times X_2\) from an \(S\)-Priestley space \(\langle X_1, \tau_1, B_1 \rangle\) to an \(S\)-Priestley space \(\langle X_2, \tau_2, B_2 \rangle\) it holds that for every \(x \in X_1\) and every \(U, V \in B_2\), if \(R(x) \subseteq U \cup V\), then \(R(x) \subseteq U\) or \(R(x) \subseteq V\).

To conclude, let us consider the case where \(S\) satisfies both (PC) and (PDI) for a single formula. Then it is well known that all \(S\)-algebras have a distributive lattice reduct (see Proposition 2.8 in [10]) and the \(S\)-filters are the same as the order filters of the specialization order of the algebras in \(\text{Alg}S\). In this case, by corollaries 8.3 and 8.17 and Proposition 5.13 we know the following: if \(A\) has a bottom element, then \(\varphi[A]\) is the collection of clopen up-sets of \(\langle \text{Op}_S(A), \tau_A, \leq \rangle\). Since in this case the optimal \(S\)-filters coincide with the prime filters, we obtain exactly what Priestley duality for bounded distributive lattices gives us. Notice that if no bottom element is assumed, we still need to deal with \(\text{Irr}_S(A)\)-admissible clopen up-sets for recovering the algebra from the space. This collection coincides with all clopen up-sets when the algebra has a bottom element, but excludes the emptyset when the algebra has no bottom element.
8.3. Deduction-Detachment Theorem. A logic $S$ has the deduction-detachment property (DDT) for a non-empty set of formulas in two variables $\Delta(p, q)$ if for all
\[
\{\delta, \gamma\} \cup \Gamma \subseteq \text{Fm}_S:
\]
\[
\Gamma, \delta \vdash \gamma \iff \Gamma \vdash_S \Delta(\delta, \gamma).
\]
If $S$ has (DDT) for $\Delta$, then this property transfers to every algebra in the sense that for every algebra $A$, and every $\{a, b\} \subseteq X \subseteq A$
\[
b \in \text{Fg}_S^A(X, a) \iff \Delta^A(a, b) \subseteq \text{Fg}_S^A(X),
\]
see, for example, Theorem 2.48 in [8]. A logic $S$ satisfies (uDDT) for a term $p \rightarrow q$ if it satisfies (DDT) for the set $\{p \rightarrow q\}$. It is well known that (DDT) implies filter-distributivity of the logic (see [5]).

Again we fix for the remaining part of the subsection a filter-distributive finitary congruential logic $S$ with theorems.

**Lemma 8.24.** If $S$ satisfies (uDDT), then for every $S$-algebra $A$ and all $a, b \in A$,
\[
(\downarrow (\varphi(a) \cap \varphi(b)^c))^c = \varphi(a \rightarrow^A b).
\]

**Proof.** Since (uDDT) transfers to every algebra, for any $\{a, b\} \cup X \subseteq A$ we have $b \in \text{Fg}_S^A(X, a)$ if and only if $a \rightarrow^A b \in \text{Fg}_S^A(X)$. Let first $P \in \varphi(a \rightarrow^A b)$, and suppose, towards a contradiction, that $P \notin (\downarrow (\varphi(a) \cap \varphi(b)^c))^c$. Then it follows that $P \in (\downarrow (\varphi(a) \cap \varphi(b)^c))^c$, and so there is $Q \in \text{Op}_S(A)$ such that $P \subseteq Q$, $Q \in \varphi(a)$ and $P \notin \varphi(b)$. By assumption, from $P \subseteq Q$ we get $a \rightarrow^A b \in Q$, and then by (uDDT) we obtain $b \in \text{Fg}_S^A(Q, a)$. Since $a \in Q$, then $b \in \text{Fg}_S^A(Q, a) = \text{Fg}_S^A(Q) = Q$, a contradiction. We conclude that $P \in (\downarrow (\varphi(a) \cap \varphi(b)^c))^c$, as required.

Let now $P \in \text{Op}_S(A)$ be such that $P \notin \varphi(a \rightarrow^A b)$, i.e., $a \rightarrow^A b \notin P$. By (uDDT) we get that $b \notin \text{Fg}_S^A(P, a)$. Then by the optimal $S$-filter lemma, there is $Q \in \text{Op}_S(A)$ such that $b \notin Q$ and $\text{Fg}_S^A(P, a) \subseteq Q$. So, we have $a \in Q$, $P \subseteq Q$ and $b \notin Q$, i.e., $Q \in \varphi(a) \cap \varphi(b)^c$, and so $P \notin (\downarrow (\varphi(a) \cap \varphi(b)^c))^c$. Therefore $P \notin (\downarrow (\varphi(a) \cap \varphi(b)^c))^c$, as required. $\Box$

From the previous corollary we conjecture that for any $S$-Priestley space $\langle X, \tau, B \rangle$, the Priestley-dual property of (uDDT) is the property “$B$ is closed under $(\downarrow (\cdot))^c$”. Let us check now that this condition is enough for recovering the implication.

**Proposition 8.25.** Let $\langle X, \tau, B \rangle$ be an $S$-Priestley space such that for all $U, V \in B$, $(\downarrow (U \cap V)^c)^c \in B$. Then for all $\{U, V\} \cup W \subseteq B$:
\[
V \in \text{Fg}_S^B(W, U) \iff (\downarrow (U \cap V^c))^c \in \text{Fg}_S^B(W).
\]

**Proof.** Assume first that $(\downarrow (U \cap V^c))^c \in \text{Fg}_S^B(W)$. Then as the logic $S$ is finitary, there is a finite $W' \subseteq W$ such that $(\downarrow (U \cap V^c))^c \in \text{Fg}_S^B(W')$. Thus by (Pr2), $\bigcap W' \subseteq (\downarrow (U \cap V^c))^c$. We show that $U \cap \bigcap W' \subseteq V$, so assume that $x \in U \cap \bigcap W'$ and suppose, towards a contradiction, that $x \notin V$. On the one hand $x \in U$. Moreover $x \in \bigcap W' \subseteq (\downarrow (U \cap V^c))^c$, i.e., $x \notin (U \cap V^c)$. But from $x \notin V$ and $x \in U$ we get $x \in U \cap V^c \subseteq (\downarrow (U \cap V^c))$, a contradiction. We conclude that $U \cap \bigcap W' \subseteq V$, and thus by (Pr2), $V \in \text{Fg}_S^B(W') \subseteq \text{Fg}_S^B(W)$.

Assume now that $V \in \text{Fg}_S^B(W, U)$. Then by finitariness again, there is $W' \subseteq W$ a finite subset such that $V \in \text{Fg}_S^B(W', U)$. We show that $\bigcap W' \subseteq (\downarrow (U \cap V^c))^c$. Suppose that $x \in \bigcap W'$ and, towards a contradiction, that $x \notin (\downarrow (U \cap V^c))^c$. Then there is $y \in U \cap V^c$ such that $x \leq y$. Let $\xi(y) = \{W \in B : y \in W\}$. This
set is an optimal $\mathcal{S}$-filter of $\mathcal{B}$ by Proposition 5.20. Suppose that $W \in \mathcal{W}'$. By assumption $x \in W$ and since $W$ is an up-set, $y \in W$, i.e., $W \in \xi(y)$. Therefore $\mathcal{W}' \subseteq \xi(y)$ and moreover, since $y \in U$, $U \in \xi(y)$. Furthermore, as $\xi(y)$ is an $\mathcal{S}$-filter $F_{\mathcal{S}}^\mathcal{B}(\mathcal{W}',U) \subseteq \xi(y)$, so by hypothesis $V \in \xi(y)$, i.e., $y \in V$, a contradiction. Thus we conclude that $\bigcap \mathcal{W}' \subseteq (\downarrow(U \land V))^c$, and then by (Pr2), $(\downarrow(U \land V))^c \in F_{\mathcal{S}}^\mathcal{B}(\mathcal{W}') \subseteq F_{\mathcal{S}}^\mathcal{B}(\mathcal{W})$, as required. \hfill $\square$

Assuming that $\mathcal{S}$ is protoalgebraic, a property implied by (DDT), we can find conditions over the dual space that imply that the logic has (uDDT).\footnote{Recall that a logic $\mathcal{S}$ is protoalgebraic, following the definition of Block and Pigozzi [2], when for any $\mathcal{Cn}_\mathcal{S}$-closed set of formulas $\Gamma \subseteq Fm_{\mathcal{S}}$ and any formulas $\delta, \mu \in Fm_{\mathcal{S}}$, if $(\delta, \mu) \in \Omega_{Fm}^\mathcal{S}(\Gamma)$, then $\Gamma, \delta \vdash_{\mathcal{S}} \mu$ and $\Gamma, \mu \vdash_{\mathcal{S}} \delta$. Remind that $\Omega_{Fm}^\mathcal{S}(\Gamma)$ is the Leibniz congruence of $\Gamma$ relative to $Fm$.} This result is supported in the following theorem due to Czelakowski.

**Theorem 8.26** (Theorem 2.6.8 in [6]). Let $\mathcal{S}$ be a protoalgebraic logic. Then $\mathcal{S}$ satisfies (DDT) if and only if for every $\mathcal{S}$-algebra $\mathcal{A}$, the lattice of $\mathcal{S}$-filters $F_{\mathcal{S}}(\mathcal{A})$ is infinitely meet-distributive over its compact elements, i.e., for any finite $B \subseteq A$ and any $\{G_i : i \in I\} \subseteq F_{\mathcal{S}}(\mathcal{A})$:

$$F_{\mathcal{S}}^\mathcal{A}(B) \cup \bigcap_{i \in I} G_i = \bigcap_{i \in I} (F_{\mathcal{S}}^\mathcal{A}(B) \cup G_i).$$

**Theorem 8.27.** Let $\mathcal{S}$ be a protoalgebraic logic. If for every $\mathcal{S}$-Priestley space $(X, \tau, \mathcal{B})$, $(\downarrow(U \land V))^c \in B$ for all $U, V \in B$, then $\mathcal{S}$ satisfies (uDDT).

**Proof.** Let $\mathcal{S}$ be a protoalgebraic logic satisfying the assumption. First we prove that $\mathcal{S}$ has (DDT) and then we will see that it satisfies (uDDT). Let $\mathcal{A}$ be an $\mathcal{S}$-algebra. By Theorem 8.26, it is enough to show that $F_{\mathcal{S}}(\mathcal{A})$ is infinitely meet-distributive over its compact elements. By the representation theorem for $\mathcal{S}$-algebras, and Corollary 5.12, we know that for any $\mathcal{S}$-algebra $\mathcal{A}$ there is an $\mathcal{S}$-Priestley space $(X, \tau, \mathcal{B})$ such that $\mathcal{A}$ is isomorphic to $\mathcal{B}$. Therefore, it is enough to show that for any $\mathcal{S}$-Priestley space $(X, \tau, \mathcal{B})$, $F_{\mathcal{S}}(\mathcal{B})$ is infinitely meet-distributive over its compact elements.

So let $(X, \tau, \mathcal{B})$ be an $\mathcal{S}$-Priestley space, let $\{G_i : i \in I\} \subseteq F_{\mathcal{S}}(\mathcal{B})$ and let $U_1, \ldots, U_n \subseteq B$ be finite sets. We show that $F_{\mathcal{S}}(\{U_1, \ldots, U_n\}) \cup \bigcap\{G_i : i \in I\} = \bigcap\{F_{\mathcal{S}}(\{U_1, \ldots, U_n\} \cup G_i) : i \in I\}$.

Notice that the inclusion from left to right is immediate by finitarity of the logic, so we just have to show the other inclusion. Let $G := \bigcap\{G_i : i \in I\}$ and suppose that $V \in \bigcap\{F_{\mathcal{S}}(\{U_1, \ldots, U_n\} \cup G_i) : i \in I\}$. Then for each $i \in I$ we have that $V \in F_{\mathcal{S}}(\{U_1, \ldots, U_n\} \cup G_i)$. For any $W_1, W_2 \in B$, let us denote $(\downarrow(W_1 \land W_2))^c$ by $W_1 \Rightarrow W_2$. Then for each $i \in I$, by assumption and Proposition 8.25 we get $U_1 \Rightarrow (\ldots(U_n \Rightarrow V)\ldots) \subseteq G_i$. Thus $U_1 \Rightarrow (\ldots(U_n \Rightarrow V)\ldots) \in G$. Recall that $\bigcap\{G_i : i \in I\} = F_{\mathcal{S}}(\bigcap\{G_i : i \in I\})$ is an $\mathcal{S}$-filter of $\mathcal{B}$. So by assumption and Proposition 8.25 again we conclude

$$V \in F_{\mathcal{S}}(\{U_1, \ldots, U_n\} \cup G) = F_{\mathcal{S}}(F_{\mathcal{S}}(\{U_1, \ldots, U_n\} \cup G)) = F_{\mathcal{S}}(\{U_1, \ldots, U_n\}) \cup \bigcap\{G_i : i \in I\}.$$
Let us consider now the congruence relation \( \equiv_{S} \) on \( Fm \), that we abbreviate all along the proof by \( \equiv \). Let \( \pi \) be the quotient homomorphism from \( Fm/\equiv \). Recall that \( Fm/\equiv \in \text{Alg}S \). Let \( p, q \) be two variables and consider the equivalence classes \( p/\equiv \) and \( q/\equiv \). By the assumption we have \((\downarrow(\varphi(p/\equiv) \cap \varphi(q/\equiv)^c) = \varphi(\delta/\equiv)\) for some formula \( \delta \). We prove that for every \( \Gamma \subseteq Fm \)
\[ q \in Cn_{S}(\Gamma, p) \quad \text{iff} \quad \delta \in Cn_{S}(\Gamma). \] (E4)

Suppose that \( q \in Cn_{S}(\Gamma, p) \) and \( \delta \notin Cn_{S}(\Gamma) \). Then there is an irreducible \( S \)-theory \( T \) such that \( Cn_{S}(\Gamma) \subseteq T \) and \( \delta \notin T \). Then \( \pi[T] \) is irreducible in \( Fm/\equiv \) and \( \pi[T] \notin \varphi(\delta/\equiv) \). Thus, \( \pi[T] \in (\varphi(p/\equiv) \cap \varphi(q/\equiv)^c) \). Let \( Q \in (\varphi(p/\equiv) \cap \varphi(q/\equiv)^c) \) such that \( \pi[T] \subseteq Q \). Then \( \pi^{-1}[Q] \) is an \( S \)-theory such that \( \Gamma \cup \{ p \} \subseteq \pi^{-1}[Q] \). Therefore \( q \in \pi^{-1}[Q] \) and this implies that \( Q \in \varphi(q/\equiv) \), a contradiction. To prove the converse, assume that \( \delta \in Cn_{S}(\Gamma) \) and \( q \notin Cn_{S}(\Gamma, p) \). Let \( T \) be an irreducible \( S \)-theory such that \( Cn_{S}(\Gamma, p) \subseteq T \) and \( q \notin T \). Then \( \pi[T] \in \varphi(p/\equiv) \cap \varphi(q/\equiv)^c \). Therefore, \( \pi[T] \notin \varphi(\delta/\equiv) \); hence \( \delta \notin T \). Since \( \Gamma \subseteq T \) and \( \delta \in Cn_{S}(\Gamma) \) we have a contradiction.

By the first part of the proof let \( \Delta(p, q) \) be a (DDT) set for \( S \). We show that \( Cn_{S}(\delta) = Cn_{S}(\Delta) \). This easily follows from (E4) and the assumption that \( \Delta(p, q) \) is a (DDT) set. Indeed, \( q \in Cn_{S}(\Delta, p) \) holds. Then by (E4) we have \( \delta \in Cn_{S}(\Delta) \). On the other hand, since \( \delta \in Cn_{S}(\delta) \), (E4) gives \( q \in Cn_{S}(\delta, p) \). Therefore, \( \Delta \subseteq Cn_{S}(\delta) \).

Now let \( \sigma \) be the substitution that maps \( p \) to \( p \) and all the remaining variables to \( q \). The by invariance under substitutions follows that \( Cn_{S}(\Delta) = Cn_{S}(\sigma(\delta)) \). Then it easily follows that \( S \) has (DDT) for the formula \( \delta'(p, q) = \sigma(\delta) \).

**Corollary 8.28.** \( S \) has (uDDT) if and only if \( S \) is protoalgebraic and for every \( S \)-Priestley space \((X, \tau, B)\), \((\downarrow(U \cap V^c))^c \in B \) for all \( U, V \in B \).

### 8.4. The Property of an Inconsistent element.

A logic \( S \) satisfies the property of an inconsistent element (PIE) for a formula \( \psi \) if for every formula \( \delta \in Fm_{\varphi} \):
\[ \psi \vdash_{S} \delta. \]

Such a formula is known as an inconsistent formula. It is immediate that (PIE) transfers to every algebra in the following sense. If \( S \) has (PIE) for \( \psi \), then for every algebra \( A \) and any homomorphism \( h : Fm \to A \),
\[ a \in Fg^{A}_{S}(h(\psi)) \]
for every \( a \in A \). If \( S \) is congruential, then it easily follows that if \( S \) satisfies (PIE) for a formula \( \psi \), then for every \( S \)-algebra \( A \) and all \( h, h' \in \text{Hom}(Fm, A) \), \( h(\psi) = h'(\psi) \), that is, \( \psi \) is a constant term on \( S \)-algebras. Moreover, it also holds that if \( S \) satisfies (PIE) for two inconsistent formulas \( \psi \) and \( \psi' \), then their interpretations on every \( S \)-algebra are the same. Thus if \( S \) satisfies (PIE), then in every \( S \)-algebra \( A \) there is a unique element that is the unique possible interpretation of all the inconsistent formulas and this element is the bottom element of \( A \) (w.r.t. \( \preceq_{A} \)). We denote this element by \( \perp_{A} \) or \( 0_{A} \) and refer to it as the inconsistent element of \( A \).

For the remaining part of the subsection let \( S \) be a filter-distributive finitary congruential logic with theorems.

**Lemma 8.29.** If \( S \) satisfies (PIE), then for every \( S \)-algebra \( A \) and all \( a \in A \),
\[ \varphi(0_{A}) = \emptyset \subseteq \varphi(a). \]
Proof. Notice that, since (PIE) transfers to every algebra, we have that $A$ has a bottom element $0^A$. Then we have that for any $P \in \text{Op}_S(A)$:

$$P \in \varphi(0^A) \iff 0^A \in P \iff A \subseteq P.$$ 

Recall that when $A$ has a bottom element, then $\emptyset \notin \text{Id}_S(A)$, so optimal $S$-filters are proper, and we get $\varphi(0^A) = \emptyset$, so it follows trivially that $\varphi(0^A) = \emptyset \subseteq \varphi(a)$ for all $a \in A$. □

Theorem 8.30. If for any $S$-Priestley space $\langle X, \tau, B \rangle$, $\emptyset \in B$, then $S$ satisfies (PIE).

Proof. Recall that the Lindenbaum-Tarski algebra $\text{Fm}/\equiv^S_\text{Fm}$ is an $S$-algebra. Let us abbreviate $\equiv^S_\text{Fm}$ by $\equiv$. By assumption $\emptyset \in \varphi[\text{Fm}/\equiv^S_\text{Fm}]$. Therefore there is $\psi \in \text{Fm}_S$ such that $\emptyset = \varphi(\psi/\equiv)$. Let $\delta \in \text{Fm}_S$. If $\psi \nvdash_S \delta$, there is an irreducible $S$-theory $T$ such that $\psi \in T$ and $\mu \notin T$. Then $\psi/\equiv \in \pi[T]$, which implies that $\pi[T] \in \varphi(\psi/\equiv)$, a contradiction. Therefore, $\psi$ is an inconsistent formula. □

Corollary 8.31. Let $S$ be a logic. Then $S$ satisfies (PIE) if and only if for any $S$-Priestley space $\langle X, \tau, B \rangle$ it holds that $\emptyset \in B$.

Observe that when the logic $S$ satisfies (PIE), we have that $\emptyset$ is the inconsistent element in $B$, i.e., the inconsistent element in the referential algebra $B$ is represented by the emptyset.

When $S$ satisfies both (PC) and (PIE) we know that for any $S$-Priestley space $\langle X, \tau, B \rangle$, the emptyset is an $X_B$-admissible clopen up-set of $X$, so $\max(X) \subseteq X_B$ or, in other words, $\downarrow X_B = X$. This property corresponds, in the general case, to the property that the $S$-algebras have a bottom-family.

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