A NONLINEAR GENERALIZATION OF
THE CAMASSA-HOLM EQUATION WITH PEAKON SOLUTIONS

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Abstract. A nonlinearly generalized Camassa-Holm equation, depending an arbitrary
nonlinearity power \( p \neq 0 \), is considered. This equation reduces to the Camassa-Holm
equation when \( p = 1 \) and shares one of the Hamiltonian structures of the Camassa-Holm
equation. Two main results are obtained. A classification of point symmetries is presented
and a peakon solution is derived, for all powers \( p \neq 0 \).

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1. Introduction

There has been much recent interest in nonlinear dispersive equations that model break-
ing waves. One of the first well-studied equations of this kind is the Camassa-Holm (CH)
equation \[5\]
\[
 u_t - u_{txx} = 3uu_x - 2u_x u_{xx} - uu_{xxx}
\] (1)
for \( u(t,x) \). This equation arises from the theory of shallow water waves \[5\ [4\] and provides
a model of wave breaking for a large class of solutions in which the wave slope blows up in
a finite time while the wave amplitude remains bounded \[8\ [9\ [10\ [11\]. A special class of
weak solutions of this equation describes peaked solitary waves, known as peakons \[1\ [5\ [6\],
whose wave slope is discontinuous at the wave peak. More remarkably, the CH equation is
an integrable system \[5\ [13\ [14\], possessing a Lax pair, a bi-Hamiltonian structure, and an
infinite hierarchy of symmetries and conservation laws.

Recently in the literature, some generalizations of the CH equation \[11\] that admit breaking
wave solutions have been studied. For example, the \( b \)-equation \[12\ [15\]
\[
 u_t - u_{txx} = -(b+1)uu_x + bu_x u_{xx} + uu_{xxx}
\] (2)
with parameter \( b \neq 0 \), and the 4-parameter family of equations \[2\]
\[
 u_t - u_{txx} = -au^p u_x + bu^{p-1} u_x u_{xx} + cu^p u_{xxx}
\] (3)
with parameters \( a,b,c \) (not all zero) and \( p \neq 0 \).

In this paper we discuss an interesting nonlinear generalization of the CH equation, given by
\[
 u_t - u_{txx} = \frac{1}{2}(p+1)(p+2)u^p u_x - \frac{1}{2}p(p-1)u^{p-2} u_x^3 - 2pu^{p-1} u_x u_{xx} - u^p u_{xxx}, \quad p \neq 0, \quad (4)
\]
where \( p \) is the nonlinearity parameter. We call this generalization the gCH equation. It is motivated by one of the Hamiltonian structures of the CH equation and has a close analogy to the relationship between the generalized Korteweg de Vries (gKdV) equation \( u_t = v^p v_x + v_{xxx} \), with \( p \neq 0 \), and the ordinary Korteweg de Vries (KdV) equation \( u_t = vv_x + v_{xxx} \). In particular, the generalized KdV equation reduces to the KdV equation when \( p = 1 \) and shares one of its two Hamiltonian structures. The same relationship holds between the gCH and CH equations.

In section 2, we review the Hamiltonian structures of the CH equation and use one of these structures to derive the gCH equation. We also discuss some conservation laws admitted by the gCH equation. In section 3, we show that point symmetries admitted by the gCH equation consist of translations in \( t \) and \( x \) and a scaling involving \( t \) and \( u \). We use these symmetries to reduce the gCH equation to ordinary differential equations that describe the corresponding group invariant solutions. In section 4, we consider weak solutions of the gCH equation and derive a peakon solution for all nonlinearity powers \( p \neq 0 \). We make some concluding remarks in section 5.

2. Derivation

The Hamiltonian structures of the CH equation are most naturally formulated by first introducing the variable

\[
m = u - u_{xx},
\]

Then the CH equation takes the form of an evolutionary equation

\[
m_t = 2u_x m + um_x = \left( \frac{1}{2} (u^2 - u_x^2) + um \right)_x
\]

for \( m(t, x) \), where \( u = \Delta^{-1} m \) is expressed in terms of the operator

\[
\Delta = 1 - D_x^2.
\]

This evolutionary equation has two Hamiltonian structures given by the Hamiltonian operators

\[
\mathcal{H} = m D_x + D_x m
\]

and

\[
\mathcal{E} = D_x - D_x^3 = \Delta D_x = D_x \Delta,
\]

where the Hamiltonians are

\[
H = \int_{-\infty}^{+\infty} \frac{1}{2} m u \, dx
\]

and

\[
E = \int_{-\infty}^{+\infty} \frac{1}{2} u^2 (u + m) \, dx.
\]

Their variational derivatives are computed by using the relation

\[
\frac{\delta}{\delta m} = \Delta^{-1} \frac{\delta}{\delta u},
\]

with the Hamiltonians expressed only in terms of \( u \) and its \( x \) derivatives,

\[
H = \int_{-\infty}^{+\infty} \frac{1}{2} (u^2 + u_x^2) \, dx
\]
and
\[ E = \int_{-\infty}^{+\infty} \frac{1}{2} u(u^2 + u_x^2) \, dx \]  
(15)

(after dropping total \( x \)-derivative terms in the integrals). The Hamiltonian operators (9) and (10) determine corresponding Poisson brackets defined by

\[ \{ \frac{\delta F_1}{\delta m}, \frac{\delta F_2}{\delta m} \}_D = \int_{-\infty}^{+\infty} \frac{\delta F_1}{\delta m} \frac{\delta F_2}{\delta m} \, dx \]  
(16)
in terms of \( D = \mathcal{H} \) and \( D = \mathcal{E} \), where \( F_1 \) and \( F_2 \) are arbitrary functionals in terms of \( x, u \) and \( x \)-derivatives of \( u \). The bracket (16) will be skew and satisfy the Jacobi identity if and only if the operator \( D \) is Hamiltonian [17]. One aspect of integrability of the CH equation (6) is that these two Poisson brackets given by \( D = \mathcal{H} \) and \( D = \mathcal{E} \) are compatible in the sense that any linear combination of them produces a Poisson bracket. Correspondingly, any linear combination of the Hamiltonian operators \( \mathcal{H} \) and \( \mathcal{E} \) is a Hamiltonian operator.

The two Hamiltonians (14) and (15) of the CH equation (6) are conserved integrals (under suitable asymptotic decay conditions on \( u \))

\[ \frac{dH}{dt} = 0, \quad \frac{dE}{dt} = 0 \]  
(17)
due to the antisymmetry of the Poisson brackets (16). Since the CH equation (6) itself is in the form of a conservation law, the integral

\[ P = \int_{-\infty}^{+\infty} m \, dx \]  
(18)
is also conserved,

\[ \frac{dP}{dt} = 0. \]  
(19)

There is a third Hamiltonian structure of the CH equation (6) for which this integral (18) is the Hamiltonian [16].

It is useful to observe that the second Hamiltonian structure given by the operator (10) can be equivalently expressed in a strictly local variational form in terms of \( u \) through the identity

\[ \mathcal{E}(\delta F/\delta m) = D_x(\delta F/\delta u) \]  
(20)
(which holds for any functional \( F \)). This formulation gives

\[ m_t = -D_x(\delta E/\delta u) \]  
(21)
where \( E \) is the Hamiltonian (15).

A natural nonlinear generalization of the variational formulation (21) consists of simply replacing the Hamiltonian (15) by

\[ E_{(p)} = \int_{-\infty}^{+\infty} \frac{1}{2} u^p(u^2 + u_x^2) \, dx, \quad p \neq 0, \]  
(22)
which yields the Hamiltonian evolutionary equation

\[ m_t = -D_x(\delta E_{(p)}/\delta u) = -\mathcal{E}(\delta E_{(p)}/\delta m), \]  
(23)
where \( p \) is an arbitrary nonlinearity power. In this formulation, the Hamiltonian (22) can be equivalently expressed in terms of \( u \) and \( m \), as given by

\[ E_{(p)} = \int_{-\infty}^{+\infty} \frac{1}{2}(p + 1)^{-1} u^{p+1}(pu + m) \, dx. \]  
(24)
(after dropping a total $x$-derivative term). The gCH equation (23) reduces to the CH equation (6) when $p = 1$. For $p \neq 1$, the gCH equation (23) is a nonlinear variant of the CH equation (6), analogous to how the gKdV equation nonlinearly generalizes the KdV equation.

Like the CH equation, the gCH equation (4) is in the form of a conservation law

$$u_t - u_{txx} = \left(\frac{1}{2}pu^{p-1}(u^2 - u_x^2) + u^p(u - u_{xx})\right)_x.$$  

(25)

Thus the integral (19) is conserved (under suitable asymptotic decay conditions on $u$). Another conserved integral is provided by the Hamiltonian (22). This integral gives rise to the conservation law

$$D_t T + D_x X = 0,$$  

(26)

with

$$T = \frac{1}{2}u^p(u^2 + u_x^2),$$

$$X = -u^p u_t u_x + \frac{1}{2}(\delta E/\delta m)^2 - \frac{1}{2}(D_x(\delta E/\delta m))^2,$$  

(27)

where

$$\delta E/\delta m = \Delta^{-1}(\delta E/\delta u) = \Delta^{-1}\left(\frac{1}{2}pu^{p-1}(u^2 - u_x^2) + u^p(u - u_{xx})\right).$$  

(28)

We will present a complete classification of conservation laws of the gCH equation (4) elsewhere.

3. Symmetry Analysis

We will now consider the gCH equation (4) written in the form of a system

$$m = u - u_{xx},$$

$$m_t = 2pu^{p-1}u_x u + u^p u_x + \frac{1}{2}p(p - 1)u^{p-2}(u^2 - u_x^2)u_x$$  

(29)

for $u(t, x), m(t, x)$.

A point symmetry [3, 17] of system (29) is a one-parameter Lie group of transformations on $(t, x, u, m)$ generated by a vector field of the form

$$X = \tau(t, x, u, m)\partial_t + \xi(t, x, u, m)\partial_x + \eta(t, x, u, m)\partial_u + \phi(t, x, u, m)\partial_m$$  

(30)

which is required to leave invariant the solution space of system (29). The condition of invariance is given by applying the prolongation of $X$ to each equation in the system (29). After prolongation, the resulting equations split with respect to the $t$ and $x$ derivatives of $u$ and $m$, yielding an overdetermined, linear system of 47 equations for $\tau(t, x, u, m), \xi(t, x, u, m), \eta(t, x, u, m), \phi(t, x, u, m)$, together with the parameter $p$. We derive and solve this linear system by using the Maple package GeM [7].

**Proposition 3.1.** The infinitesimal point symmetries admitted by the gCH system (29) for $p \neq 0$ are generated by

$$X_1 = \partial_x, \quad \text{translation in } x,$$  

(31)

$$X_2 = \partial_t, \quad \text{translation in } t,$$  

(32)

$$X_3 = m\partial_m + u\partial_u - pt\partial_t, \quad \text{scaling},$$  

(33)

(there are no extra symmetries admitted only for special values of $p \neq 0$). All of these symmetries (31)–(33) project to point symmetries of the gCH equation (4).
Each admitted point symmetry (30) can be used to reduce the gCH system (29) to a system of ordinary differential equations (ODEs) whose solutions correspond to invariant solutions \((u(t, x), m(t, x))\) of system (29) under the point symmetry. These invariant solutions are naturally expressed in terms of similarity variables

\[
(z(t, x), U(t, x, u, m), M(t, x, u, m)),
\]

which are found by solving the invariance conditions

\[
\begin{align*}
\eta(t, x, u, m) - \tau(t, x, u, m)u_t - \xi(t, x, u, m)u_x &= 0, \\
\phi(t, x, u, m) - \tau(t, x, u, m)m_t - \xi(t, x, u, m)m_x &= 0,
\end{align*}
\]

with \(U_u \neq 0\) and \(M_m \neq 0\).

3.1. Reduction under translations. Reductions under the separate translation symmetries (31) and (32) are not interesting. Instead we consider the combined space-time translation symmetry

\[
X = \partial_t - c\partial_x,
\]

where \(c\) is a non-zero constant. For this symmetry the invariance conditions (35) are given by

\[
\begin{align*}
u_t - cu_x &= 0, \\
m_t - cm_x &= 0,
\end{align*}
\]

which yields the similarity variables

\[
z = x + ct, \quad U = u, \quad M = m.
\]

The resulting form for invariant solutions of the gCH system (29) is a travelling wave given by

\[
u = U(z), \quad m = M(z),
\]

satisfying the ODE system

\[
M = U - U'',
\]

\[
cM' = \frac{1}{2}p(U^{p-1}(U^2 - U'^2))' + (MU^p)',
\]

This system (40) is equivalent to the nonlinear third order ODE

\[
0 = \left(\frac{1}{2}pU^{p-1}(U^2 - U'^2) + (-c + U^p)(U - U'')\right)'.
\]

We look for solutions \(U(z)\) that describe solitary waves, as characterized by the asymptotic boundary conditions

\[
U, U', U'' \to 0 \quad \text{for} \quad |z| \to \infty.
\]

The ODE (41) has the obvious first integral

\[
\frac{1}{2}pU^{p-1}(U^2 - U'^2) + (-c + U^p)(U - U'') = \alpha = \text{const.}
\]

Imposing the asymptotic conditions (42), we have

\[
\alpha = 0.
\]

The resulting second order ODE is

\[
\frac{1}{2}pU^{p-1}(U^2 - U'^2) + (-c + U^p)(U - U'') = 0.
\]

This ODE (43) has an integrating factor \(U'\), which yields the first integral

\[
(-c + U^p)(U^2 - U'^2) = \beta = \text{const.}
\]
Again from imposing the asymptotic conditions (42), we get
\[ \beta = 0. \] (47)
Thus, we obtain
\[ (-c + U^p)(U^2 - U'^2) = 0, \] (48)
which implies that \( U = \pm U' \) whenever \( U^p \neq c \). Clearly, no smooth function \( U(z) \) can satisfy both \( U' = \pm U \) and \( U^p \neq c \) such that \( U \to 0 \) as \( |z| \to \infty \). Thus there do not exist any smooth, asymptotically decaying solutions of the travelling wave ODE (41).

This analysis suggests that we look for weak solutions, called peakons,
\[ U(z) = c^{1/p} \exp(-|z|), \] (49)
obeying the asymptotic conditions (42). In the next section, we will show that this expression (49) does satisfy the weak form of the travelling wave ODE (41).

**Proposition 3.2.** The gCH equation (4) admits peaked travelling waves (which are weak solutions)
\[ u = c^{1/p} \exp(-|x + ct|), \quad p \neq 0, \] (50)
where \( c \) is an arbitrary constant.

When \( p = 1 \), this result coincides with the well-known peakon solution \( u = c \exp(-|x + ct|) \) of the CH equation (1). When \( p \neq 1 \) is an odd integer, or more generally when \( p \) is rational with an odd denominator, then the peakon solution (50) holds with no restriction on the sign of \( c \). In all other cases, \( c \) must be positive.

3.2. **Reduction under scaling.** For the scaling symmetry (33), the invariance conditions (35) are given by
\[ u + ptu_t = 0, \quad m + ptm_t = 0, \] (51)
which yields the similarity variables
\[ z = x, \quad U = t^{1/p}u, \quad M = t^{1/p}m. \] (52)
The resulting form for scaling-invariant solutions of the gCH system (29) is given by
\[ u = t^{-1/p}U(z), \quad m = t^{-1/p}M(z), \] (53)
satisfying the ODE system
\[ M = U - U'', \]
\[-p^{-1}M = (-\frac{1}{2}pU^{p-1}(U^2 - U'^2) - U^p M)' . \] (54)
This system (54) is equivalent to the nonlinear third order ODE
\[ 0 = p^{-1}(U - U'') - (\frac{1}{2}pU^{p-1}(U^2 - U'^2) + U^p(U - U''))' \] (55)
for \( U(z) \). Solutions of this ODE (55) yield similarity solutions
\[ u = t^{-1/p}U(x) \] (56)
of the gCH equation (4).

When \( p > 0 \), these similarity solutions (56) will exhibit decay \( u \to 0 \) as \( t \to \infty \) and a blow-up \( u \to \infty \) at \( t = 0 \). By applying a time translation, we will get a solution
\[ u = (t - t_0)^{-1/p}U(x), \quad t_0 = \text{const.} \] (57)
which still decays to 0 for large $t$ but has no blow-up for $t \geq 0$ when $t_0 < 0$. We would like to find solutions that have spatial decay $u \to 0$ for large $x$. Correspondingly, we want solutions $U(z)$ of the similarity ODE (55) that satisfy the asymptotic boundary conditions (42).

The similarity ODE (55) can be shown to admit no point symmetries other than translations in $z$, and no contact symmetries, as well as no integrating factors at most linear in $U''$. The $z$-translation symmetry yields only a trivial invariant solution, $U(z) = 0$. Thus, standard integration methods fail to yield any non-trivial solutions of ODE (55). By inspection, however, we see that $U(z) = a \exp(\pm z)$ is an exact solution of ODE (55), where $a$ is an arbitrary constant. These exponential solutions fail to satisfy the asymptotic conditions (42), but this might suggest looking for a peakon solution $U(z) = a \exp(-|z|)$. However, we will show in the next section that the weak form of the similarity ODE (55) does not admit such solutions.

3.3. Other reductions. If two symmetry generators (30) are related by conjugation with respect to some subgroup in the full group of point symmetry generated by $X_1, X_2, X_3$, then the action of this symmetry subgroup on solutions $(u(t, x), m(t, x))$ will map the group-invariant solutions determined by the two symmetry generators into each other. Consequently, for the purpose of finding all group-invariant solutions, it is sufficient to work with any maximal set of symmetry generators that are conjugacy inequivalent. From the point symmetry algebra $[X_1, X_2] = [X_1, X_3] = 0, [X_2, X_3] = -pX_2$, the equivalence classes under conjugation consist of $X_1 + aX_2, X_3 + bX_1$, where $a, b$ are arbitrary parameters. We consider the reductions with $a \neq 0$ and $b \neq 0$ elsewhere.

4. Peakon Solutions

To show that the peakon (49) is a weak solution of the travelling wave ODE (41), we start from an equivalent integral formulation obtained by multiplying the ODE with a test function $\psi$ (which is smooth and has compact support) and integrating over $-\infty < z < \infty$, leaving at most first derivatives of $\psi$ in the integral. This yields, after integration by parts,

\[ 0 = \int_{-\infty}^{+\infty} \left( -c(\phi U + \phi' U') + \frac{1}{2}p\phi U^{p-1}(U^2 - U'^2) + \phi U^{p+1} + U'((\phi U')^d) \right) dz, \tag{58} \]

where $\phi = \psi'$ is also a test function. Weak solutions of ODE (45) are functions $U(z)$ that belong to the Sobolev space $W^{1,p+1}_{\text{loc}}(\mathbb{R})$ and that satisfy the integral equation (58) for all smooth test functions $\phi(z)$ with compact support.

Now we substitute a peakon expression

\[ U = ae^{-|z|}, \quad U' = -a\sgn(z)e^{-|z|} \tag{59} \]

into equation (58) and split up the integral into the intervals $-\infty < z \leq 0$ and $0 \leq z < +\infty$. From the first term in equation (58), after integrating by parts, we obtain

\[ -\int_{-\infty}^{0} c(\phi U + \phi' U') dz - \int_{0}^{+\infty} c(\phi U + \phi' U') dz = -2ac\phi(0). \tag{60} \]

The second term in equation (58) is 0 since the peakon expression (59) satisfies $U^2 = U'^2$. Expanding the fourth term in equation (58), we have

\[ \int_{-\infty}^{+\infty} (\phi' U'' U^p + p\phi U^{p-1}U'^2) dz. \tag{61} \]
The first term in the integral \(61\) yields, after integration by parts,
\[
\int_{-\infty}^{0} \phi' U^p \, dz + \int_{0}^{+\infty} \phi' U^p \, dz = 2a^{p+1} \phi(0) - a^{p+1}(p + 1) \int_{-\infty}^{+\infty} \phi e^{-(p+1)|z|} \, dz. \tag{62}
\]

The second term in the integral \(61\) yields
\[
\int_{-\infty}^{+\infty} p\phi U^{p-1} U'^2 \, dz = pa^{p+1} \int_{-\infty}^{+\infty} \phi e^{-(p+1)|z|} \, dz. \tag{63}
\]

Similarly, the third term in the integral \(58\) yields
\[
\int_{-\infty}^{+\infty} \phi U^{p+1} \, dz = a^{p+1} \int_{-\infty}^{+\infty} \phi e^{-(p+1)|z|} \, dz. \tag{64}
\]

Combining the terms \(60\)–\(64\), we get
\[
2(-ac + a^{p+1})\phi(0) = 0. \tag{65}
\]

Since \(\phi(0)\) is arbitrary, this implies
\[
a = c^{1/p}, \tag{66}
\]
which establishes that the peakon \((49)\) satisfies equation \((58)\).

The integral formulation of the similarity ODE \((55)\) is given by
\[
0 = \int_{-\infty}^{+\infty} \phi \left( p^{-1}(U - U'') - \left( \frac{1}{2} p U^{p-1}(U'^2 - U''^2) + U^p(U - U'') \right)' \right) \, dz, \tag{67}
\]
where \(\phi(z)\) is a test function. After integrating by parts, we obtain
\[
0 = \int_{-\infty}^{+\infty} \left( p^{-1}(\phi U + \phi' U') + \frac{1}{2} p\phi' U^{p-1}(U'^2 - U''^2) + \phi U^{p+1} + (\phi' U^p)' U' \right) \, dz. \tag{68}
\]

We substitute a peakon expression \((59)\) into equation \((68)\) and split up the integral into the intervals \(-\infty < z \leq 0\) and \(0 \leq z < +\infty\). From the first term in equation \((68)\), after integrating by parts, we obtain
\[
\int_{-\infty}^{0} p^{-1}(\phi U + \phi' U') \, dz + \int_{0}^{+\infty} p^{-1}(\phi U + \phi' U') \, dz = 2ap^{-1}\phi(0). \tag{69}
\]

The second term in equation \((68)\) is 0 since the peakon expression \((59)\) satisfies \(U^2 = U'^2\). The third and fourth terms in equation \((68)\) yield, after integration by parts,
\[
\int_{-\infty}^{0} (\phi' U^{p+1} + (\phi' U^p)' U') \, dz + \int_{0}^{+\infty} (\phi' U^{p+1} + (\phi' U^p)' U') \, dz = 2a^{p+1}\phi'(0). \tag{70}
\]

Combining the terms \((69)\)–\(70\), we get
\[
2ap^{-1}\phi(0) + 2a^{p+1}\phi'(0) = 0. \tag{71}
\]

Since \(\phi\) is arbitrary, this implies
\[
a = 0. \tag{72}
\]

Thus, the weak form of the similarity ODE \((55)\) does not admit peakon solutions \((49)\).
5. Remarks

We have introduced a non-linearly generalized CH equation (4), depending on an arbitrary nonlinearity power \( p \neq 0 \). This equation reduces to the CH equation when \( p = 1 \) and shares one of the Hamiltonian structures of CH equation (1). For all \( p \neq 0 \), it admits a peakon solution (50).

The gCH equation is worth further study to understand how its nonlinearity affects properties of its solutions compared to the CH equation. In particular, the CH equation is an integrable system, admits multi-peakon weak solutions, and exhibits wave-breaking for a large class of classical solutions. Is the gCH equation integrable for some nonlinearity power \( p \neq 1 \)? Does it admit multi-peakon solutions for nonlinearity powers \( p \neq 1 \)? Is it well-posed for all \( p \neq 1 \)? Does it exhibit the same wave-breaking behavior for all \( p \neq 1 \)? Is there a critical power \( p \) for which a different kind of blow-up occurs (other than wave breaking)?

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