A comparison theorem for stochastic differential equations under a Novikov-type condition

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Abstract

We consider a system of stochastic differential equations driven by a standard $n$-dimensional Brownian motion where the drift coefficient satisfies a Novikov-type condition while the diffusion coefficient is the identity matrix. We define a vector $Z$ of square integrable stochastic processes with the following property: if the filtration of the translated Brownian motion obtained from the Girsanov transform coincides with the one of the driving noise then $Z$ coincides with the unique strong solution of the equation; otherwise the process $Z$ solves in the strong sense a related stochastic differential inequality. This fact together with an additional assumption will provide a comparison result similar to well known theorems obtained in the presence of strong solutions.

Keywords: stochastic differential equations, Girsanov theorem, Novikov condition, convex envelope, comparison theorems.

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1 Introduction

The present paper is devoted to the study of the system of stochastic differential equations

$$dX_t^x = b(X_t^x) + dB_t, \quad t \in [0, T]$$

$$X_0^x = x \in \mathbb{R}^n$$

where $b: \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function, $\{B_t\}_{0 \leq t \leq T}$ is a standard $n$-dimensional Brownian motion and $T \in \mathbb{R}$ is a fixed positive constant.

Equation (1.1) has attracted the attention of many authors since it represents one of the simplest (the noise appears additively) nontrivial stochastic perturbations of the ordinary differential equation

$$\frac{dX_t^x}{dt} = b(X_t^x), \quad t \in [0, T]$$

$$X_0^x = x \in \mathbb{R}^n.$$
One of the most fascinating features of equation (1.1) is that it possesses a (strong) solution under very weak assumptions on the drift function \( b \) (assumptions that are not sufficient for the existence of a solution to the deterministic equation (1.2)). It was in fact proved by Zvonkin [17] that in the one dimensional case it is sufficient to require the boundedness of \( b \) in order to have a unique strong solution for the equation (1.1). This result was then generalized to several dimensions by Veretennikov [16] who employed the so-called Yamada-Watanabe principle (see e.g. [8]) and techniques from the theory of partial differential equations. Equation (1.1) was also considered by Krylov and Röckner in [7]: in this paper they obtain strong solvability of the equation under an integrability condition of drift function \( b \).

In [10] Meyer-Brandis and Proske proved that in the one dimensional case, when the drift function is bounded, the unique strong solution of (1.1) is even differentiable in the sense of the Malliavin calculus (see e.g. [12]). Their approach is based on a representation formula for solutions of stochastic differential equations obtained in Lanconelli and Proske [9], white noise techniques and an approximation argument. We also mention the recent paper by Nilssen [11] where the above mentioned Malliavin differentiability is obtained for stochastic equations with a sub-linear drift.

We would like to mention the paper by Pilipenko [14] who proved the existence of a unique strong solution for the one dimensional version of (1.1) with a bounded drift \( b \) and with \( B_t \) replaced by a stable symmetric Levy process. In [2] Da Prato et al. obtained existence and uniqueness of a mild solution of an infinite dimensional version of (1.1) with bounded drift. Finally we remark that the above regularizing effect obtained with the introduction of a noise into a deterministic differential equation was also investigated for stochastic partial differential equations in Flandoli et al. [1] and Fedrizzi and Flandoli [3]. Here the noise is added in a multiplicative way through a Stratonovich integral.

In this paper we consider the stochastic differential equation (1.1) with a drift function \( b \) satisfying a Novikov-type condition (see (2.1) below). Under this assumption the Girsanov theorem guarantees the existence of a weak solution for that equation. This solution becomes the unique strong solution of (1.1) if the shift utilized in the Girsanov theorem is invertible (see the paper by Üstünel [15] for an elegant approach to stochastic equations via results on invertible shifts on the Wiener space) or equivalently if the filtration of the translated Brownian motion coincides with the filtration of the original Brownian motion. We define a \( n \)-dimensional vector of stochastic processes \( Z \), characterized through a duality relation (see (2.2) below) and we show that it coincides with the unique strong solution of (1.1) when the above mentioned filtrations agree. Observe that the process \( Z \) is well defined without the assumption of the invertibility of the shift or the equality of the filtrations. We then ask ourselves the following question: does the process \( Z \) with the sole assumption (2.1) solve in the strong sense any equation related to (1.1)? Our main theorem states that the process \( Z \) solves in the strong sense a stochastic differential inequality where the drift function is the convex envelope of \( b \), i.e. the best convex minorant of \( b \). This fact together with the local Lipschitz-continuity of convex functions will provide a comparison theorem for equation (1.1) where the role of the a priori non-existing strong solution is played by the process \( Z \).

(By comparison theorem we mean roughly speaking a result of the following type: if the drift function of a stochastic differential equation is greater than the drift function of another one then the strong solution of the first equation is greater than the strong
solution of the second one).

The paper is organized as follows: In Section 2 after a quick description of our framework we state the main result (Theorem 2.1) together with a corollary; Section 3 collects a number of properties and useful remark about the process $Z$ that will be utilized in Section 4 in proving the main results.

2 Statement of the main results

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the classical Wiener space associated with a standard $n$-dimensional Brownian motion over the time interval $[0, T]$; more precisely, $\Omega$ coincides with the space of continuous functions $\omega$ defined on the interval $[0, T]$ with values on $\mathbb{R}^n$ and with $\omega(0) = 0 \in \mathbb{R}^n$ ($T$ is a fixed positive number), $\mathcal{F}$ is the Borel $\sigma$-algebra associated to the topology induced by the sup-norm on $\Omega$ and $\mathcal{P}$ is the Wiener measure on $\Omega$.

We denote by $\{B_t\}_{0 \leq t \leq T}$ the coordinate process defined as $B_t(\cdot) : \omega \in \Omega \mapsto \omega(t) \in \mathbb{R}^n$, which turns out to be a standard $n$-dimensional Brownian motion under the measure $\mathcal{P}$.

We also denote by $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ its $\mathcal{P}$-augmented natural filtration.

Let $b : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function satisfying the Novikov-type condition

$$E\left[ \exp \left\{ \int_0^T \|b(B_t + x)\|^2 dt \right\} \right] < +\infty \quad (2.1)$$

for all $x \in \mathbb{R}^n$. Here $E$ denotes the expectation on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\| \cdot \|$ stands for the $n$-dimensional Euclidian norm. Observe that in the usual Novikov condition there is a factor $\frac{1}{2}$ in front of the integral of formula (2.1). Therefore our condition is stronger than the classical one. The need of this stronger condition will be evident in the proof of Proposition 3.4 below. Roughly speaking condition (2.1) guarantees that any process of the form $\{b(B_t + x) + f(t)\}_{0 \leq t \leq T}$, where $f$ is deterministic, fulfils the usual Novikov condition.

According to the Girsanov theorem (see e.g. [8] page 190) the process

$$B^b_t := B_t - \int_0^t b(B_s)ds, \quad t \in [0, T],$$

is a standard $n$-dimensional Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and the measure

$$d\mathcal{Q} := \exp \left\{ \int_0^T \langle b(B_s), dB_s \rangle - \frac{1}{2} \int_0^T \|b(B_s)\|^2 ds \right\} d\mathcal{P}.$$ 

(Here $\int_0^T \langle b(B_s), dB_s \rangle$ stands for $\sum_{i=1}^n \int_0^T b_i(B_s)dB^i_s$). We also denote by $\{\mathcal{F}^b_t\}_{0 \leq t \leq T}$ the $\mathcal{P}$-augmented natural filtration of the process $B^b$.

We recall that condition (2.1) guarantees that the process

$$t \in [0, T] \mapsto \exp \left\{ \int_0^t \langle b(B_s), dB_s \rangle - \frac{1}{2} \int_0^t \|b(B_s)\|^2 ds \right\}$$

is a continuous $\{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathcal{P}$-martingale belonging to $\mathcal{L}^q(\Omega, \mathcal{F}, \mathcal{P})$ for all $q \in [1, +\infty]$ (see [8] page 198).
In the sequel for an \( n \)-dimensional \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \)-adapted process \( \gamma = (\gamma^1, ..., \gamma^n) \) satisfying \( \mathcal{P}(\sum_{i=1}^n \int_0^T |\gamma_i|^2 \, ds < +\infty) = 1 \) the following notation will be adopted

\[
\mathcal{E}_t(\gamma) := \exp \left\{ \int_0^t \langle \gamma_s, dB_s \rangle - \frac{1}{2} \int_0^t \|\gamma_s\|^2 \, ds \right\} \quad t \in ]0, T[.
\]

We will simply write \( \mathcal{E}(\gamma) \) when \( t = T \) and \( \mathcal{E}(b) \) when \( \gamma_t = b(B_t) \) for any \( t \in [0, T] \).

We now introduce the translation (or shift) operator: for \( k \in \mathbb{N} \), \( \varphi \in C^\infty_0(\mathbb{R}^m) \) and \( t_1, ..., t_k \in [0, T] \) define

\[
\mathcal{T}_b(\varphi(B_{t_1}, ..., B_{t_k})) := \varphi(B_{t_1} - \int_0^{t_1} b(B_s) \, ds, ..., B_{t_k} - \int_0^{t_k} b(B_s) \, ds).
\]

This operator can be extended to \( \mathcal{L}^q(\Omega, \mathcal{F}, \mathcal{P}) \) for all \( q \in [1, +\infty[ \). It is easy to see that under the condition (2.1), \( \mathcal{T}_b \) maps \( \mathcal{L}^q(\Omega, \mathcal{F}, \mathcal{P}) \) into \( \cap_{r<q} \mathcal{L}^r(\Omega, \mathcal{F}, \mathcal{P}) \). Note that in this notation we have \( B^k_t = \mathcal{T}_{-b}B_t \).

If \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) is a measurable function we denote by \( \hat{h} \) the convex envelope of \( h \), i.e. \( \hat{h} \) is the greatest element of the set

\[
\{ g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ convex and } g(x) \leq h(x) \text{ for all } x \in \mathbb{R}^n \}.
\]

(Convex envelopes are well studied objects in optimization theory; see the interesting paper by Oberman [13] for a fully nonlinear partial differential equation solved in the sense of viscosity solutions by the convex envelope of a given function).

We conclude our framework saying that a function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is quasi-monotonously increasing if for any \( i \in \{1, ..., n\} \) the inequality

\[
f_i(x) \leq f_i(y)
\]

is satisfied for all \( x, y \in \mathbb{R}^n \) such that \( x_i = y_i \) and \( x_j \leq y_j \) if \( j \neq i \). (This condition is well known in the theory of systems of deterministic ordinary differential equations; see for instance the discussion in Assing and Manthey [1] and the references quoted there).

All the equalities and inequalities involving \( n \)-dimensional vectors are understood in a componentwise manner.

We are now ready to state our main result.

**Theorem 2.1**

Fix \( x \in \mathbb{R}^n \) and let \( b : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a measurable function fulfilling condition (2.1).

For \( i \in \{1, ..., n\} \) and \( t \in [0, T] \) define \( Z_{t,i}^{x,i} \) to be the unique element of \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \) verifying for any \( Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \) the following identity

\[
E[Z_{t,i}^{x,i}Y] = E[(B_t^i + x_i)\mathcal{E}(b)\mathcal{T}_bY].
\]

(2.2)

Then:

i) The process \( Z_t^x := (Z_t^{x,1}, ..., Z_t^{x,n}) \) is continuous and \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted.

ii) If \( \mathcal{F}_t^b = \mathcal{F}_t \) for all \( t \in [0, T] \), the process \( \{Z_t^x\}_{0 \leq t \leq T} \) is the unique strong solution of the system of stochastic differential equations

\[
\begin{align*}
dX_t^x &= b(X_t^x) \, dt + dB_t \quad t \in [0, T] \\
X_0^x &= x
\end{align*}
\]

(2.3)
iii) If $\mathcal{F}_t^b \subset \mathcal{F}_t$ for some $t \in [0, T]$, the process $\{Z_t^x\}_{0 \leq t \leq T}$ solves the following system of stochastic differential inequalities

$$
\begin{align*}
\frac{dZ_t^x}{dt} &\geq \hat{b}(Z_t^x)dt + dB_t, \quad t \in [0, T] \\
Z_0^x &= x.
\end{align*}
$$

where $\hat{b} := (\hat{b}_1, ..., \hat{b}_n)$ and $\hat{b}_i$ is the convex envelope of $b_i$ for each $i \in \{1, ..., n\}$. If in addition $\hat{b}$ is quasi-monotonously increasing one has with probability one that

$$
Z_t^x \geq Y_t^x \quad \text{for all} \quad t \in [0, \tau]
$$

where $\{Y_t^x\}_{0 \leq t \leq \tau}$ is the unique (possibly local) strong solution of the stochastic differential equation

$$
\begin{align*}
\frac{dY_t^x}{dt} &= \hat{b}(Y_t^x)dt + dB_t, \quad t \in [0, T] \\
Y_0^x &= x.
\end{align*}
$$

and $\tau$ its explosion time.

**Corollary 2.2**

Fix $x \in \mathbb{R}^n$ and let $b : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function fulfilling condition (2.1). Assume in addition that

$$
b(x) \geq Ax + b \quad \text{for all} \quad x \in \mathbb{R}^n
$$

where $A = (a_{ij})_{1 \leq i, j \leq n}$ is a $n \times n$-matrix with $a_{ij} \geq 0$ if $i \neq j$ and $b \in \mathbb{R}^n$. Then the process $\{Z_t^x\}_{0 \leq t \leq T}$ defined in (2.2) verifies

$$
Z_t^x \geq \Phi(t) \left( x + \int_0^t \Phi^{-1}(s) bds + \int_0^t \Phi^{-1}(s) dB_s \right) \quad \text{for all} \quad t \in [0, T]
$$

with probability one, where $t \mapsto \Phi(t)$ is the unique solution of the following matrix differential equation

$$
\frac{d}{dt} \Phi(t) = A \Phi(t), \quad \Phi(0) = I.
$$

($I$ denotes the $n \times n$ identity matrix).

### 3 Preliminary results

In this section we will only assume that $b : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function satisfying condition (2.1).

Let $q > 1$, $X \in L^q(\Omega, \mathcal{F}, \mathcal{P}) := \cup_{r > q} L^r(\Omega, \mathcal{F}, \mathcal{P})$ and consider the linear operator

$$
Y \mapsto E[X \mathcal{E}(b) \mathcal{L}_b Y].
$$

This map is continuous on $L^{q'}(\Omega, \mathcal{F}, \mathcal{P})$, where $q'$ denotes the conjugate exponent of $q$. In fact,

$$
\begin{align*}
\left| E[X \mathcal{E}(b) \mathcal{L}_b Y] \right| &\leq \left( E[|X|^{q} \mathcal{E}(b)] \right)^{\frac{1}{q'}} \left( E[|Y|^{q'} \mathcal{E}(b)] \right)^{\frac{1}{q'}} \\
&\leq \left( \|X\|_{L^q} \|E(b)\|_{L^{q'}} \right)^{\frac{1}{2}} \|Y\|_{L^{q'}} \\
&\leq \|X\|_{L^p} \|E(b)\|_{L^{p'}} \|Y\|_{L^{q'}} \\
&= C \|X\|_{L^p} \|Y\|_{L^{q'}}
\end{align*}
$$

(3.1)
where we used the Hölder inequality twice and the Girsanov theorem ($p$ and $p'$ are conjugate exponents with $p > 1$). By the Riesz representation theorem there exists a unique element $Z(X)$ (we stress only the dependence on $X$ and not on $b$ since this function is arbitrary but fixed) belonging to $L^q(\Omega, \mathcal{F}, \mathcal{P})$ such that

$$E[X\mathcal{E}(b)\mathcal{T}_bY] = E[Z(X)Y] \tag{3.2}$$

This equality will be crucial for the rest of the paper. The next proposition explains how $Z(X)$ depends on $X$.

**Proposition 3.1**

The operator

$$Z : \mathcal{L}^{q+}(\Omega, \mathcal{F}, \mathcal{P}) \rightarrow \mathcal{L}^q(\Omega, \mathcal{F}, \mathcal{P})$$

$$X \mapsto Z(X)$$

is linear and continuous.

**Proof.** The linearity follows immediately from (3.2). Let us prove the continuity:

$$\|Z(X)\|_q = \sup_{\|Y\|_{q'} \leq 1} |E[Z(X)Y]|$$

$$= \sup_{\|Y\|_{q'} \leq 1} |E[X\mathcal{E}(b)\mathcal{T}_bY]|$$

$$\leq \sup_{\|Y\|_{q'} \leq 1} C\|X\|_{q+}\|Y\|_{q'}$$

$$= C\|X\|_{q+}$$

where in the inequality we used the estimate obtained in (3.1).

**Proposition 3.2**

Let $X \in \mathcal{L}^{q+}(\Omega, \mathcal{F}, \mathcal{P})$. Then:

i) If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function such that $\varphi(X) \in \mathcal{L}^{q+}(\Omega, \mathcal{F}, \mathcal{P})$, we have

$$\varphi(Z(X)) \leq Z(\varphi(X)). \tag{3.3}$$

ii) If $U \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{P})$, the following identity holds true:

$$UZ(X) = Z(X\mathcal{T}_bU). \tag{3.4}$$

In particular for $X = 1$ one gets

$$U = Z(\mathcal{T}_bU) \quad (Z \text{ is the left-inverse of } \mathcal{T}_b). \tag{3.5}$$

iii) If $E_Q[|\mathcal{F}_T^b|]$ denotes the conditional expectation given $\mathcal{F}_T^b$ under the measure $Q$, we have

$$Z(X) = Z(E_Q[X|\mathcal{F}_T^b]). \tag{3.6}$$
Proof. We prove each part of the statement separately.
i) In order to prove (3.3) it is sufficient to show that $Z(1) = 1$ and that $Z$ preserves the positivity. These facts together with the representation of a convex as a supremum of affine functions will provide the desired inequality. Now using (3.2) and the Girsanov theorem we get

$$
E[Z(1)Y] = E[E(b)\mathcal{T}_b Y]
= E[Y].
$$

The previous identities hold for any $Y \in L^q(\Omega, \mathcal{F}, P)$ implying that $Z(1) = 1$. Now if $X \geq 0$ then for any $Y \in L^q(\Omega, \mathcal{F}, P)$ with $Y \geq 0$ we obtain

$$
E[Z(X)Y] = E[XE(b)\mathcal{T}_b Y] \geq 0.
$$

Hence $Z(X) \geq 0$.
i) The proof of (3.4) is obtained as follows

$$
E[UZ(X)Y] = E[XE(b)\mathcal{T}_b (UY)]
= E[XE(b)\mathcal{T}_b U\mathcal{T}_b Y]
= E[Z(X\mathcal{T}_b U)Y].
$$

Since $Y$ is arbitrary in $L^q(\Omega, \mathcal{F}, P)$ the proof is complete.
i) Let $Y \in L^q(\Omega, \mathcal{F}, P)$. Then

$$
E[Z(X)Y] = E[XE(b)\mathcal{T}_b Y]
= E_Q[X\mathcal{T}_b Y]
= E_Q[E_Q[XF^b_T]|F^b_T]\mathcal{T}_b Y]
= E[Z(E_Q[XF^b_T])Y].
$$

In the third equality we used the fact that $\mathcal{T}_b$ is $F^b_T$-measurable since $\mathcal{T}_b$ transforms functionals of $B$ into functionals of $B^b$.

Remark 3.3

Equality (3.5) implies a non trivial property of the operator $Z$:
If $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded function,

$$
\varphi(Z(\mathcal{T}_b U)) = \varphi(U)
= Z(\mathcal{T}_b \varphi(U))
= Z(\varphi(\mathcal{T}_b U))
$$

i.e.

$\varphi(Z(\mathcal{T}_b U)) = Z(\varphi(\mathcal{T}_b U))$.

This means that on the set of bounded functionals of the Brownian motion $B^b$ ($\mathcal{T}_b U$ is an element of this kind) the operator $Z$ commutes with any bounded nonlinear function. Outside this set we can only get (3.3).

Note that when $\mathcal{F} = \mathcal{F}^b$ the set of bounded functionals of the Brownian motion $B^b$ coincides with the whole $L^\infty(\Omega, \mathcal{F}, P)$. In this case equation (2.3) possess a unique strong solution (see Section 4 below).
Let \( \{X_t\}_{0 \leq t \leq T} \) be an \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted stochastic process such that \( X_t \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \) for all \( t \in [0, T] \). Then the process \( \{Z(X_t)\}_{0 \leq t \leq T} \) is also \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted.

**Proposition 3.4**

Let \( \{X_t\}_{0 \leq t \leq T} \) be an \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted stochastic process such that \( X_t \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \) for all \( t \in [0, T] \). Then the process \( \{Z(X_t)\}_{0 \leq t \leq T} \) is also \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted.

**Proof.** We recall that the set

\[
\{\mathcal{E}(f), f \in \mathcal{L}^2([0, T]; \mathbb{R}^n)\}
\]

is total in \( \mathcal{L}^q(\Omega, \mathcal{F}, \mathcal{P}) \) for all \( q \geq 1 \) and that for any \( Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \),

\[
E[Y\mathcal{E}(f)] = \sum_{k \geq 0} \langle h_k, f^\otimes 2 \rangle_{\mathcal{L}^2([0, T]^k; \mathbb{R}^k)}
\]

where the sequence \( \{h_k\}_{k \geq 0} \) represents the kernels of the Wiener-Itô chaos decomposition of \( Y \) with respect to the Brownian motion \( B \) (see e.g. [5]).

We have

\[
E[Z(X_t)\mathcal{E}(f)] = E[X_t\mathcal{E}(b)\mathcal{T}_b\mathcal{E}(f)] = E[X_t\mathcal{E}(f + b)].
\]

Observe that, as we already mentioned in Section 2, condition (2.1) ensures that the process \( \{\mathcal{E}_t(b + f)\}_{0 \leq t \leq T} \) is a continuous (\( \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathcal{P} \))-martingale. Therefore

\[
E[Z(X_t)\mathcal{E}(f)] = E[X_t\mathcal{E}(f + b)] = E[X_t\mathcal{E}(f + b)] = E[X_t\mathcal{E}(b)\mathcal{T}_b\mathcal{E}(f)] = E[X_t\mathcal{E}(b)\mathcal{T}_b\mathcal{E}(f)] = E[Q[X_t\mathcal{T}_b\mathcal{E}(f)]] = E[Q[Q[X_t|\mathcal{F}_t^b]\mathcal{T}_b\mathcal{E}(f)]] = E[Q[Q[X_t|\mathcal{F}_t^b]\mathcal{T}_b\mathcal{E}(f)]] = \sum_{k \geq 0} \langle h_k(\cdot, t), f^\otimes k \rangle_{\mathcal{L}^2([0, T]^k; \mathbb{R}^k)}
\]

where \( \{h_k(\cdot, t)\}_{k \geq 0} \) are the kernels of \( E[Q[X_t|\mathcal{F}_t^b]] \) with respect to the Brownian motion \( B^b \). Observe that in the seventh equality we utilized the martingale property of \( \{\mathcal{T}_b\mathcal{E}(f)\}_{0 \leq t \leq T} \) with respect to the filtration \( \{\mathcal{F}_t^b\}_{0 \leq t \leq T} \) and the measure \( \mathcal{Q} \) (the role of \( \mathcal{T}_b\mathcal{E}(f) \) in \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{Q}) \) is the same as the one of \( \mathcal{E}(f) \) in \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \)).

Comparing the first and the last terms of the above chain of equalities we deduce that \( \{h_k(\cdot, t)\}_{k \geq 0} \) are also the kernels of \( Z(X_t) \) with respect to the Brownian motion \( B \).

Since \( E[Q[X_t|\mathcal{F}_t^b]] \) is \( \{\mathcal{F}_t^b\}_{0 \leq t \leq T} \)-adapted we get (see [5] Lemma 2.5.3) that for each \( k \geq 0 \), \( h_k(t_1, ..., t_n, t) = 0 \) if \( t_i > t \) for some \( i \in \{1, ..., n\} \). This last condition (recall that the \( h_k \)'s are also the kernels of \( Z(X_t) \) with respect to the Brownian motion \( B \)) implies that \( Z(X_t) \) is \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted. \( \square \)
Remark 3.5
Looking through the proof of Proposition 3.4 one can understand how the operator $Z$ acts:
- Take $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$
- Compute the conditional expectation $E_Q[X|\mathcal{F}_T^b]$
- Write the Wiener-Itô chaos decomposition of $E_Q[X|\mathcal{F}_T^b]$ as $\sum_{k\geq 0} I^b_k(h_k)$ where $I^b_k$ are multiple Itô integrals with respect to the Brownian motion $B^b$
- Take the kernels from above and write the Wiener-Itô chaos decomposition $\sum_{k\geq 0} I_k(h_k)$ where $I_k$ are now multiple Itô integrals with respect to the Brownian motion $B$
- The result is $Z(X)$, i.e. $Z(X) = \sum_{k\geq 0} I_k(h_k)$.

4 Proofs of the main results

To ease the notation we will assume that $x = 0$ and denote the processes $X^0_t, Y^0_t$ and $Z^0_t$, appearing in the statement of the theorem, by $X_t, Y_t$ and $Z_t$, respectively. Observe that the one dimensional processes $Z^i_t$ as defined in (2.2) correspond in the notation of the previous section to $Z(B^i_t)$. Therefore all the properties that we proved in the previous section for $Z(X)$ are valid also for $Z^i_t$. We will however continue to use the symbol $Z(B^i_t)$ when we will use the results of the previous section.

Proof of Theorem 2.1

i) Adaptedness follows immediately from Proposition 3.4. Let us prove the continuity. By means of (3.3) we have for any $i \in \{1, ..., n\}$ and $s, t \in [0, T]$ that

$$E[|Z(B^i_t) - Z(B^i_s)|^3] = E[|Z(B^i_t - B^i_s)|^3] \leq E[|Z(B^i_t - B^i_s)|^3] = E[|I^i_t - I^i_s|^3 \mathcal{E}(b)] \leq E[|I^i_t - I^i_s|^6]^{1/2} E[\mathcal{E}(b)^2]^{1/2} = C(|t - s|)^{3/2}.$$  

From the Kolmogorov’s continuity theorem we deduce that $Z^i_t$ has a continuous modification which we will continue to denote by $Z^i_t$.

ii) Assume that $\mathcal{F}^b_t = \mathcal{F}_t$ for all $t \in [0, T]$. This means that any functional of the path of $B$ is also a functional of the path of $B^b$ and hence from Remark 3.3 we obtain the identity

$$\varphi(Z(U)) = Z(\varphi(U)) \quad (4.1)$$

for any measurable and bounded function $\varphi : \mathbb{R} \to \mathbb{R}$ and $U \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathcal{P})$. Actually these hypothesis can be relaxed: in fact, looking through the proof of identity (3.4) one can easily see that (4.1) holds for instance for all $U \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(U) \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$. In our case $U = B^i_t$ and $\varphi = b_i$. Since we are assuming condition (2.1) on the function $b$ a simple application of the inequality $e^x \geq 1 + x$ provides the condition $E[b^2_i(B_t)] < +\infty$ t.a.e. (The null set where the previous condition is not
fulfilled will play no role since property (4.1) will be used under the Lebesgue-integral sign).

Now fix a function $f \in L^2([0, T]; \mathbb{R}^n)$; an application of the Itô formula gives

$$E[Z_i \mathcal{E}(f)] = E[B^i \mathcal{E}(b) T_b \mathcal{E}(f)]$$

$$= E[B^i \mathcal{E}(f + b)]$$

$$= E\left[ B^i \left( 1 + \int_0^T \langle (f(s) + b(B_s)) \mathcal{E}_s(f + b), dB_s \rangle \right) \right]$$

$$= E\left[ \int_0^t (f_i(s) + b_i(B_s)) \mathcal{E}_s(f + b) ds \right]$$

$$= \int_0^t E[b_i(B_s) \mathcal{E}_s(f + b)] ds + \int_0^t f_i(s) ds$$

$$= \int_0^t E[b_i(B_s) \mathcal{E}(f + b)] ds + E[B^i \mathcal{E}(f)]$$

$$= \int_0^t E[b_i(B_s) \mathcal{E}(b) T_b \mathcal{E}(f)] ds + E[B^i \mathcal{E}(f)]$$

$$= \int_0^t E[Z(b_i(B_s)) \mathcal{E}(f)] ds + E[B^i \mathcal{E}(f)]$$

$$= \int_0^t E[b_i(Z(B_s)) \mathcal{E}(f)] ds + E[B^i \mathcal{E}(f)]$$

$$= E\left[ \left( \int_0^t b_i(Z(B_s)) ds + B^i_t \right) \mathcal{E}(f) \right].$$

Comparing the first and the last term of this chain of equalities we get for each $i \in \{1, ..., n\}$ that

$$Z^i_t = \int_0^t b_i(Z_s) ds + B^i_t.$$

This shows that the process $Z_t = (Z^1_t, ..., Z^n_t)$ solves (2.3) in the strong sense. To prove uniqueness we simply observe that if $\{V_t\}_{0 \leq t \leq T}$ is another square integrable, continuous, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$-adapted solution to (2.3) by applying the Girsanov theorem we get for every $f \in L^2([0, T]; \mathbb{R}^n)$ and $i \in \{1, ..., n\}$ that

$$E[V^i_t \mathcal{E}(f)] = E[B^i_t \mathcal{E}(f + b)];$$

since by definition

$$E[Z^i_t \mathcal{E}(f)] = E[B^i_t \mathcal{E}(b) T_b \mathcal{E}(f)]$$

$$= E[B^i_t \mathcal{E}(f + b)]$$

we conclude that

$$Z_t = V_t$$

for all $t \in [0, T]$ with probability one.
iii) Let us now assume that $F_b^t \subset F_t$ for some $t \in [0, T]$. We can repeat the reasoning that brought to the chain of equalities obtained above; however now we have to stop before the interchange between the action of the operator $Z$ and the nonlinear function $b_i$ (in the present case this commutation is not allowed since the Brownian motion $B$ can not be written as a functional of $B^b$). Therefore we can write

$$E[Z_i \mathcal{E}(f)] = \int_0^t E[Z(b_i(B_s))\mathcal{E}(f)]ds + E[B_i \mathcal{E}(f)]$$

$$= E\left[\left(\int_0^t Z(b_i(B_s))ds + B_i\right)\mathcal{E}(f)\right]$$

or equivalently

$$Z_i^t = \int_0^t Z(b_i(B_s))ds + B_i^t.$$

(4.2)

Recall that in proving inequality (3.3) we showed that $Z$ is a positivity preserving operator; due to the linearity of $Z$ this is equivalent to say that if $\mathcal{P}(X \leq Y) = 1$ then $\mathcal{P}(Z(X) \leq Z(Y)) = 1$. Therefore if we denote by $\hat{b}_i$ the convex envelope of $b_i$ we can continue (4.2) as

$$Z_i^t = \int_0^t Z(b_i(B_s))ds + B_i^t \geq \int_0^t \hat{b}_i(Z(B_s))ds + B_i^t$$

$$\geq \int_0^t \hat{b}_i(Z_s)ds + B_i^t$$

where in the second inequality we used (3.3). Hence we proved that $Z_t = (Z_t^1, ..., Z_t^n)$ is a solution to the system of stochastic differential inequalities

$$Z_t \geq \int_0^t \hat{b}(Z_s)ds + B_t.$$

Now, the function $\hat{b} = (\hat{b}_1, ..., \hat{b}_n)$ is a vector of convex functions; since convex functions are locally Lipschitz-continuous the stochastic differential equation

$$dY_t = \hat{b}(Y_t)dt + dB_t \quad t \in [0, T]$$

$$Y_0 = 0$$

possesses a unique strong solution up to the explosion time

$$\tau := \lim_{N \to +\infty} \inf\{t \in [0, T] : \|Y_t\| > N\}$$

(see e.g. [6]). If $\hat{b} : \mathbb{R}^n \to \mathbb{R}^n$ happens to be quasi-monotonously increasing then by Proposition 3.3 in [1] we conclude that

$$Z_t \geq Y_t \text{ for all } t \in [0, \tau]$$
with probability one.

**Proof of Corollary 2.2**

First observe that for all \( x \in \mathbb{R}^n \) we have \( b(x) \geq \hat{b}(x) \geq Ax + b \). Therefore following the line of reasoning in the proof of iii) above we can immediately say that

\[
dZ_t \geq (AZ_t + b)dt + dB_t.
\]

Since the function \( x \mapsto Ax + b \) is linear, the stochastic differential equation

\[
dY_t = (AY_t + b)dt + dB_t \quad Y_0 = 0
\]

possesses a unique global strong solution; this solution is explicitly represented in the right hand side of (2.7) (see [8] page 354). Moreover the assumption on \( A \) guarantees that the function \( x \mapsto Ax + b \) is quasi-monotonously increasing. These facts together with Proposition 3.3 in [1] implies the desired inequality (2.7).

**References**

[1] Assing, S. and Manthey, R.: The behaviour of solutions of stochastic differential inequalities, *Probab. Theory Relat. Fields* **103** (1995) 493-514

[2] Da Prato, G., Flandoli, F., Priola, E. and Röckner, M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, to appear on *Annals Probab.*

[3] Fedrizzi, E. and Flandoli, F.: Noise prevents singularities in linear transport equations, *J. Funct. Anal.* **264** (2013) 1329-1354

[4] Flandoli, F., Gubinelli, M. and Priola, E.: Well posedness of the transport equation by stochastic perturbation, *Invent. Math.* **180** (2010) 1-53

[5] Holden, H., Øksendal, B., Ubøe, J. and Zhang, T.-S.: *Stochastic Partial Differential Equations- A Modeling, White Noise Functional Approach*, Birkhäuser, Boston 1996

[6] Ikeda, N. and Watanabe, S.: *Stochastic differential equations and diffusion processes*, North Holland, Amsterdam 1981

[7] Krylov, N.V. and Röckner, M.: Strong solutions of stochastic equations with singular time dependent drift, *Probab. Theory Relat. Fields* **131** (2005) 154-196

[8] Karatzas, I. and Shreve, S.E.: *Brownian motion and stochastic calculus, II edition*, Springer, New York 1991

[9] Lanconelli, A. and Proske, F.N.: On explicit strong solutions of Itô-SDE’s and the Donsker delta function of a diffusion, *Inf. Dim. Anal. Quantum Probab. Relat. Topics* **7** (2004) 437-447
[10] Meyer-Brandis, T. and Proske, F.N.: Construction of strong solutions of SDE’s via Malliavin calculus, *J. Funct. Anal.* **258** (2010) 3922-3953

[11] Nilssen, T.K.: One-dimensional SDE’s with discontinuous unbounded drift and continuously differentiable solutions of the stochastic transport equation, Preprint University of Oslo (2012)

[12] Nualart, D.: *Malliavin calculus and Related Topics, II edition*, Springer, New York 2006

[13] Oberman, A.M.: The convex envelope is the solution of a nonlinear obstacle problem, *Proceedings AMS* **135** (2007) 1689-1694

[14] Pilipenko, A.Y.: On existence and properties of strong solutions of one-dimensional stochastic equations with an additive noise, *arXiv:1306.0212v1*

[15] Üstünel, A.S.: Entropy, invertibility and variational calculus of adapted shifts on Wiener space, *J. Funct. Anal.* **257** (2009) 3655-3689

[16] Veretennikov, A.Y.: On the strong solutions of stochastic differential equations, *Theory Probab. Appl.* **24** (1979) 354-366

[17] Zvonkin, A.K.: A Transformation of the state space of a diffusion process that removes the drift, *Math. USSR (Sbornik)* **22** (1974) 129-149