Simplification of control design for driftless nonholonomic systems based on rough path analysis

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Abstract: This paper provides a strict system formulation for a class of nonholonomic systems with Lie bracket motions via rough path analysis. The dynamics of the resulting systems are represented by rough differential equations as augmented versions of ordinary differential equations. The rough differential equations are allowed to have rough signals generated by unbounded-variation functions and derived by classifying the functions according to “orders” of the boundedness. This paper clarifies rough differential equations driven by third-order rough signals, and the validity for control design problems is confirmed by considering a fourth-order chained system, which is a typical form of driftless affine nonholonomic systems.

Key Words: nonlinear control, nonholonomic systems, Lie bracket motions, rough path analysis.

1. Introduction
Nonholonomic systems are widely known to have difficulty in control designing because almost all the systems are incapable of being stabilized by any smooth time-independent state-feedback controllers. The feature, confirmed by considering Brockett’s theorem \cite{Brockett1978}, requires time-varying or discontinuous properties in their state-feedback laws. Therefore, the systems are mainly studied in the field of nonlinear control theory \cite{Khalil2002,Slotine1983}.

Basic classes of nonholonomic systems are also in the class of driftless affine systems; for example, chained systems, first-order systems, and two-generator systems \cite{Kumano-go1992, Kawanabe1994,Umeda2000}. All the systems have multiple control inputs independent of each other but creating \textit{Lie bracket motions} mediated by the interaction of the inputs. The generation of the motions enables controlling the systems although the dimensions of the original control inputs are less than the ones of state variables.

The effects of Lie bracket motions into the dynamics of driftless affine systems are investigated in the approximation algorithms \cite{Khalil1996,Morimoto1997} and the transverse function approaches \cite{Umeda1999,Watanabe2000}. Both methods try generating “the hidden control inputs” independent from all the original control inputs by adding highly-oscillatory sinusoidal waves as parts of the original control inputs. The resulting systems actually have the hidden control inputs; however, they are also just approximate models when periodic
signals with infinite frequencies are added into the original control inputs because ordinary differential
equations cannot be defined when the signals included in dynamical systems.

Recently, the investigation of dynamical systems driven by periodic signals with infinite frequencies
is developed [11, 12]. The works are based on rough path analysis, which deal with dynamical systems
including thumping vibrations such as white noises [13, 14]. The analysis considers the basis of the
vibrations as unbounded-variation functions such as Wiener and Poisson processes, and then clarifies
the seamless connection between ordinary and stochastic differential equations despite the unbound-
edness of the processes. The first authors’ previous works [11, 12] focus attention on applying the
rough path analysis into non-probabilistic systems driven by highly-oscillatory signals (rough sig-
als) so that the approximate models concerned with Lie bracket motions become the true dynamics.
As a result, the models are identified as strict representations of rough differential equations, which
are extended notions of ordinary differential equations allowing the existence of unbounded-variation
functions.

In rough path analysis, the notion of the order is provided for rough signals. The previous works [11,
12] deal with just the second-order rough signals, which generate the simplest Lie bracket motions.
Furthermore, if the symbol (5) is used for a product of $A, B \in T^3(\mathbb{R}^n)$, the product is defined by

$$A \otimes B := \left( A^0 \otimes B^0, \frac{1}{2} A^1 \otimes B^{1-1}, \ldots, \sum_{i=0}^\lambda A^l \otimes B^{\lambda-l} \right).$$

For any function $y_t : [0, \infty) \to Y$ for any set $Y$, $y_{s,t} : \Delta_T \to Y$ is defined by $y_{s,t} = y_t - y_s$. The
$i$-th element of a $d$-dimensional vector $a$ is represented by $a[i]$, and the $i$-th row and the $j$-th column
element of a $d \times d$-dimensional matrix $A$ is denoted by $A[i, j]$. For $b, c \in \mathbb{R}$, $b^c$ means $b$ to the power
of $c$ while $B^c$ denotes the $c$-th element of $B$ if $B \in T^3(\mathbb{R}^n)$.

2. Motivation

Let us consider an input-affine driftless system

432
\[ \dot{x}_\tau = g(x)u^\circ, \quad x_\tau \in \mathbb{R}^n, \quad u^\circ_\tau \in \mathbb{R}^m \]  

be considered with \( g : \mathbb{R}^n \to \mathbb{R}^m \) being smooth. Designing \( u^\circ = \check{u} \) with \( u \in \mathbb{R}^m \) being differentiable, the above system is equivalent to an integral equation

\[ x_{s,t} = \int_s^t g(x_\tau)du_\tau. \]  

However, if \( u \) is a function having an unbounded variation, the integral of the right-hand side is now well defined as a Riemann-Stieltjes integral.

The previous works of controlling nonholonomic systems \([7–10]\) deal with highly-oscillatory periodic signals included in \( u \). In the procedure, the signals become unbounded-variation functions. This causes that the resulting systems are just approximation models because of the indefinable property of Riemann-Stieltjes integrals.

The recent development of rough path analysis enables considering dynamical systems including unbounded-variation functions. The motivation of this paper is to develop this direction of control theory by employing rough path analysis.

Here we consider the meaning of unbounded-variation functions. The total variation of a function \( y : [0, \infty) \to \mathbb{R}^q \) is defined by

\[ ||y||_{1,J} := \sup_{D \subset J} \left| \sum_{k=1}^N |y_{t_{k-1}, t_k}| \right|, \quad (8) \]

where \( J \subset [0, \infty) \) is a time interval. If (8) has a finite value, \( y \) is said to have a bounded variation (in \( J \)); otherwise, \( y \) is said to have an unbounded variation (in \( J \)). If \( y \) has an unbounded variation, any Riemann-Stieltjes integrals are incapable of being defined because the limitation values of Riemann sums are not uniquely defined.

The concrete unbounded-variation functions treated in this paper are as follows. Let

\[ \check{u}^D_1(\eta) = \begin{bmatrix} \check{u}^D_1(\eta)[1] \\ \check{u}^D_1(\eta)[2] \end{bmatrix} = \begin{bmatrix} \frac{\cos(\eta_1^2 \tau) - 1}{\eta_1} \\ \frac{\sin(\eta_1^2 \tau)}{\eta_1} \end{bmatrix}, \]  

\[ \check{u}^D_2(\eta) = \begin{bmatrix} \check{u}^D_2(\eta)[1] \\ \check{u}^D_2(\eta)[2] \end{bmatrix} = \begin{bmatrix} 1 - \cos(\eta_2^2 \tau) \\ \frac{\sin(2\eta_2^2 \tau)}{\eta_2} \end{bmatrix}, \]  

where \( \eta_1, \eta_2 \in \mathbb{N}_1^{\infty} \). All the above functions have bounded variations when \( \eta_1 \) and \( \eta_2 \) are both finite. However, as \( \eta_1, \eta_2 \to \infty \), they become all unbounded-variation functions. Therefore, we cannot define ordinary differential equations when \( u = \lim_{\eta \to \infty} u^D_1 \) or \( u = \lim_{\eta \to \infty} u^D_2 \) is applied in (7); however, we obtain “differential equations” via rough path analysis \([13, 14]\). The equation is said to be rough differential equations.

In this paper, we provide a concrete system formulation for dynamical systems when unbounded-variation functions such as (9)–(10) to develop the control analysis of input-affine driftless nonholonomic systems.

### 3. Rough path analysis

In this section, we briefly summarize rough path analysis.

**Definition 1** (\( p \)-variation \([14]\)) Let \( p \geq 1 \), a continuous path \( z : \Delta_T \to \mathbb{R}^q \), and a sequence of time sub-intervals \( D_T \) be considered. Then,

\[ ||z||_{p,\Delta_T} = \left[ \sup_{D_T \subset \Delta_T} \left( \sum_{k=0}^{N-1} |z_{t_{k-1}, t_k}|^p \right)^{1/p} \right], \]  

where \( N \) is the length of \( D_T \).
is said to be the \( p \)-variation of \( z \) (on the interval \( J \)). Furthermore, if \(|z|_{p,\Delta_T} < \infty\), then \( z \) is said to have finite \( p \)-variation (on the interval \( J \)). \( \square \)

In what follows, we consider the situation of \( p \in [1,4) \).

**Definition 2 (multiplicative functional [14])** Let \( Z : \Delta_T \to T^{|p|}(\mathbb{R}^q) \) be a continuous map satisfying

\[
Z_{s,t} = (Z_{s,t}^0, \ldots, Z_{s,t}^{|p|})
\]

for each \((s,t) \in \Delta_T\), where \( Z_{s,t}^0 \in \mathbb{R} \) and \( Z_{s,t}^\lambda \in (\mathbb{R}^q) \otimes \lambda \) with \( \lambda \in \mathbb{N}_1^{[p]} \). The function \( Z \) is called a multiplicative functional if \( Z_{s,t}^0 = 1 \) and \( Z_{s,t} \otimes Z_{\tau,t} = Z_{s,t} \) for all \( \tau \in [s,t] \in \Delta_T \). \( \square \)

**Definition 3 (rough path [14])** Let \( Z, \tilde{Z}(1), \ldots, \tilde{Z}(\eta) : \Delta_T \to T^{|p|}(\mathbb{R}^q) \) be continuous maps. Let also \( p \geq 1 \) be assumed. Then,

1. \( Z_{s,t} \) is said to be a \( p \)-rough path (in \( \mathbb{R}^q \)) if it is a multiplicative functional of degree \( |p| \) with finite \( p \)-variation;
2. \( \tilde{Z}(1)_{s,t}, \ldots, \tilde{Z}(\eta)_{s,t} \) are said to be \((p-)\)smooth rough paths if they are multiplicative functionals of degree \( |p| \) (in \( \mathbb{R}^q \)) with finite 1-variations; and
3. \( Z_{s,t} \) is said to be a geometric \( p \)-rough path if it satisfies \( d_p(\tilde{Z}(\eta)_{s,t}, Z_{s,t}) \to 0 \) as \( \eta \to \infty \), where

\[
d_p(Z,Y) := \max_{1 \leq \lambda \leq |p|} \sup_{D_T \subseteq \Delta_T} \left( \sum_{k=1}^N \|Z_{t_{k-1},t_k}^\lambda - Y_{t_{k-1},t_k}^\lambda\|^2 \right)^{\frac{1}{2}}.
\]

Furthermore, an element \( Z_{s,t}^j \) for \( j \in \mathbb{N}^{[p]}_0 \) is said to be the \( j \)-th level path of \( Z_{s,t} \). \( \square \)

Then, we define integrals along rough paths. For \( s = t_0 \leq \cdots \leq t_k = t \), \([s,t] \in \Delta_T\) and a smooth function \( h : \mathbb{R}^q \to \mathbb{R}^q \oplus \mathbb{R}^q \), we consider

\[
I_h(Z,D_T)_{s,t} := \sum_{\lambda=1}^{|p|} \sum_{k=1}^N H^\lambda(Z_{t_{k-1}})Z_{t_{k-1},t_k}^\lambda,
\]

where \( H^1 = h \),

\[
\left( H^2(Z_{t_{k-1}})Z_{t_{k-1},t_k}^2 \right)[i_1] = \sum_{i_1=1}^q \frac{\partial h(Z^1)[i_1]}{\partial Z^1[i_1]} Z_{t_{k-1},t_k}^1[i_1],
\]

\[
\left( H^3(Z_{t_{k-1}})Z_{t_{k-1},t_k}^3 \right)[i_1,i_2] = \sum_{i_1=1}^q \sum_{i_2=1}^q \frac{\partial^2 h(Z^1)[i_2]}{\partial Z^1[i_1] \partial Z^1[i_2]} Z_{t_{k-1},t_k}^2[i_1,i_2]
\]

for \( i_1, i_2 \in \mathbb{N}^p_0 \).

**Definition 4 (rough integral)** Let \( Z_{s,t} : \Delta_T \to T^3(\mathbb{R}^q) \) be a geometric rough path and \( h(Z^1) \) be smooth enough. Then,

\[
\int_s^t h(Z_s) dZ_s^1 := \lim_{|D_T| \to 0} I_h(Z,D_T)_{s,t}
\]

is said to be an integral of \( h \) along \( Z^1 \). \( \square \)

Let us consider a nonlinear system

\[
x_{s,t} = \lim_{|D_T| \to 0} \sum_{k=1}^N \sum_{\alpha=1}^m g_{\alpha}(x_{t_{k-1}}) u_{t_{k-1},t_k} [\alpha],
\]

434
where $x \in \mathbb{R}^n$ is a state vector, $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$ is an input vector and $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}^n$ are all smooth. If $u = \tilde{u}(\eta)$ becomes a geometric rough path as $\eta \to \infty$ with $p \in [1, 4)$, then, letting $z = (u^T, x^T)^T$, we obtain

$$Z^1_{s,t} = \int_s^t \begin{bmatrix} I_m \\
g(X^1_r) \\
0 \\
0 
\end{bmatrix} dZ^1_r. \quad (19)$$

Furthermore, extracting the dynamics of $X^1$ from (19) and abbreviating the integral sign $\int$, we obtain

$$dX^1_t = g(X^1_t)dU^1_t, \quad (20)$$

which is said to be a rough differential equation, where $Z$ is a rough path of $z$; that is, $Z^1 = z$ with

$$Z^1 = \begin{pmatrix} U^1 \\
X^1 \end{pmatrix}, \quad Z^2 = \begin{pmatrix} U^2 \\
U^1 \otimes X^1 \\
X^2 \otimes U^1 \end{pmatrix}, \quad Z^3 = \begin{pmatrix} U^3 \\
X^2 \otimes U^1 \\
X^3 \end{pmatrix}, \quad (21)$$

where $U$ and $X$ are rough paths of $u$ and $x$, respectively: $U^1 = u$ and $X^1 = x$. The universal limit theorem in [14] and the assumption of the smoothness of all coefficients imply that the solution to (20) uniquely exists in $\tau \in [s,t]$.

**Remark 1** This section describes results for rough path analysis explicitly used in what follows. There are at least two things abbreviated despite their importance for constructing rough paths and rough differential equations. To be exact, we should consider “$p$-variation topology” in Definition 3.12 of [14] to define $d_p$ in (13). However, we omit the explanation because it awakens high complexity in this paper. Furthermore, the strict definition of rough integrals requires $I_h(Z, D_{\tau})_{t, \lambda}$ for $\lambda = 2, 3$ as with (14) so that the results of rough integrations become also rough paths. However, their representations are so complicated despite no use in any calculations in what follows. The details of both abbreviated discussions are written in [13, 14], see also [12]. 

4. Rough differential equations driven by rough paths

Hereinafter, let us consider that $\tilde{u}(\eta)$ has finite 1-variations for all $\eta \in \mathbb{N}_1^\infty$. Further, let us consider the following:

(A1) $\tilde{U}(\eta) : \Delta_{\infty} \to T([p])(M_U)$ is a smooth rough path satisfying

$$\tilde{U}^1_{s,t}(\eta) = \tilde{u}_{s,t}(\eta), \quad \tilde{U}^\lambda_{s,t}(\eta) = \int_s^t \tilde{U}^{\lambda - 1}_{s,r}(\eta) \otimes d\tilde{u}(\eta)_r, \quad (22)$$

for all $\eta \in \mathbb{N}_1^\infty$ and $\lambda \in \mathbb{N}_1^3$.

(A2) $U_{s,t} : \Delta_{\infty} \to T([p])(M_U)$ is a geometric $p$-rough path with $p \in [1, 4)$ such that $d_p(U_{s,t}(\eta), U_{s,t}) \to 0$ as $\eta \to \infty$.

The concrete system representation along rough path $U$ in (A2) requires a further transformation of rough integrals. Reconsidering rough integrals (14) as

$$\int_s^t h(Z^1_r)dZ^1_r = \sum_{\lambda=1}^{[p]} \int_s^t \frac{1}{\lambda!} H^\lambda(Z^1_r)d\xi^\lambda_r, \quad (23)$$

where

$$\int_s^t \frac{1}{\lambda!} H^\lambda(Z^1_r)d\xi^\lambda_r := \lim_{|D_{\eta}| \to 0} \sum_{k=1}^N H^\lambda(Z_{t_{k-1}})Z^\lambda_{t_{k-1}, t_k}, \quad (24)$$

Then, the rough system (20) is calculated as follows:
\[ dx_r = \sum_{j=1}^{m} g_j(x_r) du_r^1[j] + \frac{1}{2!} \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial g_j}{\partial x[l]}(x_r) g[l, k](x_r) du_r^2[k, j] + \frac{1}{3!} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{p=1}^{m} \sum_{l=1}^{m} \frac{\partial}{\partial x[q]} \left[ \frac{\partial g_j}{\partial x[l]}(x_r) g[l, k](x_r) \right] g[q, p](x_r) du_r^3[p, k, j], \]

where \( x = X^1 \).

Based on the above discussions, we obtain the following results:

**Theorem 1** Consider (25) with \( m = 2 \). If \( U \) satisfies (A2) with \( p \in [2, 3] \) and \( \tilde{u}_r(\eta) = \tilde{u}_r^{D1}(\eta) \), then the resulting rough system is

\[ dx_r = \frac{1}{2} [g_1, g_2](x_r) d\tau. \]  

**Theorem 2** Consider (25) with \( m = 2 \). If \( U \) satisfies (A2) with \( p \in [3, 4] \) and \( \tilde{u}_r(\eta) = \tilde{u}_r^{D2}(\eta) \), then the resulting rough system is

\[ dx_r = \frac{1}{4} [g_1, [g_1, g_2]](x_r) d\tau. \]

The proofs are shown in Appendix sections.

**Remark 2** In the conference paper version [16], there is a mistake on the reference of (25). It is referred as in [11] despite the condition of \( p \in [1, 3] \); in contrast, this paper considers \( p \in [1, 4] \). Furthermore, the representation (23) provides more strict formulation of rough differential equations than [11] because it enables clarifying the relationship between rough integrals over \( dU^1 \) and their elements \( du^1, du^2 \) and \( du^3 \).

5. Case study for controlling nonholonomic systems by rough paths

In this section, we consider a stabilization problem of a fourth-order chained system

\[ \dot{x} = g_1(x) u[1] + g_2(x) u[2], \quad x \in \mathbb{R}^4, \quad u \in \mathbb{R}^2, \quad g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ x & 3 \\ 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]  

5.1 Control design

In this subsection, we design a stabilizing control law for (28) including second- and third-order rough signals:

\[ u_{s,t}[1] = \nu[1](x_s)(t - s) + \nu[3](x_s) \lim_{\eta_1 \to \infty} \tilde{u}_s^{D1}(\eta_1)[1] + \lim_{\eta_2 \to \infty} \tilde{u}_s^{D2}(\eta_2)[1], \]

\[ u_{s,t}[2] = \nu[2](x_s)(t - s) + \nu[4](x_s) \lim_{\eta_1 \to \infty} \tilde{u}_s^{D1}(\eta_1)[2] + \nu[4](x_s) \lim_{\eta_2 \to \infty} \tilde{u}_s^{D2}(\eta_2)[2], \]

where \( \nu[1], \nu[2], \nu[3], \nu[4] : \Delta_T \to \mathbb{R} \). Then, we obtain a rough system

\[ dx_r = \left\{ g_1(x_r) \nu[1](x_r) + g_2(x_r) \nu[2](x_r) + \frac{1}{2} \left[ g_1, g_2 \right](x_r) \nu[3](x_r) + \frac{1}{4} [g_1, [g_1, g_2]](x_r) \nu[4](x_r) - \frac{1}{2} g_1(x_r) (L_{g_2} \nu[3])(x_r) + \frac{1}{4} g_2(x_r) (L_{g_1} L_{g_1} \nu[4])(x_r) + \frac{1}{2} [g_1, g_2](x_r) (L_{g_1} \nu[4])(x_r) \right\} d\tau \]

via Theorems 1–2. Note that, \( \nu[3] \) and \( \nu[4] \) are “hidden control inputs” generated by the interactions of rough signals because
\[
G(x) = \begin{bmatrix} g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]] \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
x[2] & 0 & -1 & 0 \\
x[3] & 0 & 0 & 1 \\
\end{bmatrix},
\]
which is fullrank for any \(x \in \mathbb{R}^4\). Thus, further designing
\[
\nu[1] = \tilde{\nu}[1] + \frac{1}{2} L_{g_2} \nu[3],
\]
\[
\nu[2] = \tilde{\nu}[2] - \frac{1}{4} L_{g_1} L_{g_2} \nu[4],
\]
\[
\nu[3] = 2\tilde{\nu}[3] - L_{g_1} \nu[4],
\]
\[
\nu[4] = 4\tilde{\nu}[4],
\]
where \(\tilde{\nu} = (\tilde{\nu}[1], \tilde{\nu}[2], \tilde{\nu}[3], \tilde{\nu}[4])^T : \Delta_T \to \mathbb{R}^4\) is a new input vector, we obtain
\[
dx_\tau = -G(x_\tau)\tilde{\nu}_\tau(x_\tau)d\tau.
\]
Therefore, for example, designing
\[
\tilde{\nu}_\tau = -G^{-1}(x_\tau)Kx_\tau, \quad K = \begin{bmatrix} k[1] & 0 & 0 & 0 \\
0 & k[2] & 0 & 0 \\
0 & 0 & k[3] & 0 \\
0 & 0 & 0 & k[4] \end{bmatrix},
\]
results in a linear system
\[
dx_\tau = -Kx_\tau d\tau.
\]
Thus, choosing \(k[1], k[2], k[3], k[4] > 0\), the origin of (39) is asymptotically stable.

5.2 Numerical simulation
Here, we confirm the validity of the resulting system (39) via the numerical simulation of (25) with the coefficients in (28), control inputs in (29)–(30), (33)–(36) and (38), and \(\eta_1\) and \(\eta_2\) being finite values.

Setting the designing parameters by \(k[1] = k[2] = k[3] = k[4] = 1\) and the initial values as \(x[1] = -0.2, x[2] = 0.15, x[3] = -0.25\) and \(x[4] = 0.2\). Figures 1–3 show the numerical simulation results for the states with \(\eta_1 = \eta_2 = 10^1\), \(\eta_1 = \eta_2 = 10^2\), and \(\eta_1 = \eta_2 = 10^3\), respectively. Figure 4 shows the numerical simulation result for the states of an ordinary differential equation \(\dot{x} = -Kx\) as the comparison with our rough differential equation. They imply that, as \(\eta_1\) and \(\eta_2\) get larger, then the time responses of the states converge to the solution to \(\dot{x} = -Kx\), which is the same form as the resulting rough differential equation (39).

![Fig. 1. Time responses of the state variables with \(\eta = 10^1\).](image-url)
6. Concluding remarks

In this paper, we provided a concrete system formulation of rough differential equations driven by third-order rough paths to show that the approximate models for nonholonomic systems become strict system representations when using rough path analysis. The validity of the results was confirmed by designing a stabilizer including rough signals for a fourth-order chained system. In the procedure, we also showed that the resulting system becomes a stable linear system.
The claim of this paper is to simplify control design for nonholonomic systems by adding rough signals. The way enables deriving strict linearized systems keeping controllability even if the target systems do not satisfy necessary conditions in Brockett’s theorem [1]. Recently, the way is proven to be useful for further analysis of local controllability for a class of nonlinear systems in [17]. The novelty of this paper is to clarify that the way of considering rough path analysis is exactly connecting to the various previous works for controlling nonholonomic systems and controllability analysis for nonlinear control systems.

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Appendix

A. Proof for Theorem 1

The proof is started from considering Theorem 4 in [11]; that is, we obtain the second-order geometric rough path $U$ as

$$U_{s,t} = \left(1, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} (t-s) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) .$$

(A-1)

The combination of the above rough path and our representations for rough integrals (23) yields that the rules

$$du^1_\tau = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad du^2_\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d\tau .$$

(A-2)

Substituting the rules into the rough differential equation (25), we obtain

$$dx_\tau = \frac{1}{2} \sum_{l=1}^{n} \left\{ \frac{\partial g_2}{\partial x[l]}(x_\tau)g[l,1](x_\tau)du^2_\tau[1,2] + \frac{\partial g_1}{\partial x[l]}(x_\tau)g[l,2](x_\tau)du^2_\tau[2,1] \right\}$$

$$= \frac{1}{2} \sum_{l=1}^{n} \left\{ \frac{\partial g_2}{\partial x[l]}(x_\tau)g[l,1](x_\tau) - \frac{\partial g_1}{\partial x[l]}(x_\tau)g[l,2](x_\tau) \right\} d\tau$$

(A-3)

Thus, using the notation of a Lie bracket, we obtain (26).

B. Proof for Theorem 2

The proof firstly needs a calculation result for the third-order geometric rough path $U$. The elements of the first-level path results in

$$U^1_{s,t}[1] = \lim_{\eta_2 \to \infty} \int_{s}^{t} \left[ \frac{-\cos(\eta_2^3 \tau) + \cos(\eta_2^3 s)}{\eta_2} \right] d\tau = 0 ,$$

(B-1)

$$U^1_{s,t}[2] = \lim_{\eta_2 \to \infty} \int_{s}^{t} \left[ \frac{\sin(2\eta_2^3 \tau) - \sin(2\eta_2^3 s)}{\eta_2} \right] d\tau = 0 .$$

(B-2)

Because the elements of the second-level path are represented by the form

$$U^2_{s,t}[i, j] = \lim_{\eta_2 \to \infty} \int_{s}^{t} \int_{s}^{\tau} \left[ \frac{\cos(2\eta_2^3 \tau) - \cos(\eta_2^3 \tau)}{4} \right] d\tau = 0 ,$$

(B-3)

we obtain

$$U^2_{s,t}[1, 1] = \lim_{\eta_2 \to \infty} \int_{s}^{t} \left[ \frac{\cos(2\eta_2^3 \tau) - \cos(\eta_2^3 \tau)}{4} \right] d\tau = 0 .$$

(B-4)
\[ U_{s,t}[1,2] = \lim_{\eta_2 \to \infty} \frac{2}{\eta_2^3} \left[ -\sin(3\eta_2^3 \tau) + \sin(\eta_2^3 \tau) \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(2\eta_2^3 \tau)}{2} \right\}_s = 0, \quad (B-5) \]
\[ U_{s,t}[2,1] = \lim_{\eta_2 \to \infty} \frac{1}{\eta_2^2} \left[ \frac{\sin(\eta_2^3 \tau)}{2} - \frac{\sin(3\eta_2^3 \tau)}{6} + \frac{\sin(2\eta_2^3 s) \cos(\eta_2^3 \tau)}{2} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(2\eta_2^3 \tau)}{2} \right\}_s = 0, \quad (B-6) \]
\[ U_{s,t}[2,2] = \lim_{\eta_2 \to \infty} \frac{1}{\eta_2^2} \left[ -\frac{\cos(4\eta_2^3 \tau)}{4} - \frac{\sin(2\eta_2^3 \tau) \sin(2\eta_2^3 \tau)}{2} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(2\eta_2^3 \tau)}{2} \right\}_s = 0. \quad (B-7) \]

Furthermore, because elements of the third-level path are represented by the form
\[ U_{s,t}[i,j,k] = \lim_{\eta_2 \to \infty} \int_s^{\tau_1} \int_s^{\tau_1} \int_s^{\tau_1} d\tilde{u}_{s,t}[i] d\tilde{u}_{s,t}[j] d\tilde{u}_{s,t}[k], \quad i,j,k = 1,2, \quad (B-8) \]
we obtain
\[ U_{s,t}[1,1,1] = \lim_{\eta_2 \to \infty} \left[ \frac{\cos(3\eta_2^3 \tau)}{24\eta_2^3} + \frac{\cos(\eta_2^3 \tau)}{8\eta_2^3} + \frac{\cos(2\eta_2^3 s) \cos(\eta_2^3 \tau)}{4\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(2\eta_2^3 s) \sin(\eta_2^3 \tau)}{2} \right\}_s = 0, \quad (B-9) \]
\[ U_{s,t}[1,1,2] = \lim_{\eta_2 \to \infty} \frac{2}{\eta_2^3} \left[ \frac{\sin(3\eta_2^3 \tau)}{24\eta_2^3} + \frac{1}{8} + \frac{\sin(3\eta_2^3 \tau) \cos(\eta_2^3 \tau)}{6\eta_2^3} - \frac{\sin(\eta_2^3 \tau) \cos(2\eta_2^3 s)}{2\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(\eta_2^3 \tau)}{2} \right\}_s = \frac{1}{4} (t-s), \quad (B-10) \]
\[ U_{s,t}[1,2,1] = \lim_{\eta_2 \to \infty} \frac{2}{\eta_2^3} \left[ \frac{\sin(4\eta_2^3 \tau)}{48\eta_2^3} - \frac{\sin(2\eta_2^3 \tau)}{24\eta_2^3} - \frac{1}{12} - \frac{\sin(-2\eta_2^3 \tau)}{8\eta_2^3} + \frac{\sin(\eta_2^3 \tau) \cos(\eta_2^3 \tau)}{2\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(\eta_2^3 \tau)}{2} \right\}_s = -\frac{1}{2} (t-s), \quad (B-11) \]
\[ U_{s,t}[1,2,2] = \lim_{\eta_2 \to \infty} \frac{2}{\eta_2^3} \left[ \frac{\cos(5\eta_2^3 \tau)}{60\eta_2^3} + \frac{\cos(\eta_2^3 \tau)}{12\eta_2^3} - \frac{\cos(\eta_2^3 \tau)}{4\eta_2^3} - \frac{\cos(-3\eta_2^3 \tau)}{12\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(2\eta_2^3 \tau)}{2} \right\}_s = 0, \quad (B-12) \]
\[ U_{s,t}[2,1,1] = \lim_{\eta_2 \to \infty} \left[ \frac{\sin(-2\eta_2^3 \tau)}{8\eta_2^3} + \frac{1}{4} + \frac{\sin(2\eta_2^3 \tau)}{24\eta_2^3} + \frac{\sin(4\eta_2^3 \tau)}{48\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \cos(\eta_2^3 \tau)}{2\eta_2^3} \right\}_s = \frac{1}{4} (t-s), \quad (B-13) \]
\[ U_{s,t}[2,1,2] = \lim_{\eta_2 \to \infty} \frac{2}{\eta_2^3} \left[ \frac{\cos(\eta_2^3 \tau)}{4\eta_2^3} - \frac{\cos(-3\eta_2^3 \tau)}{12\eta_2^3} + \frac{\cos(3\eta_2^3 \tau)}{12\eta_2^3} + \frac{\cos(\eta_2^3 \tau)}{4\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(2\eta_2^3 \tau)}{2} \right\}_s = 0, \quad (B-14) \]
\[ U_{s,t}[2,2,1] = \lim_{\eta_2 \to \infty} \left[ \frac{\cos(5\eta_2^3 \tau)}{40\eta_2^3} + \frac{\cos(-3\eta_2^3 \tau)}{24\eta_2^3} + \frac{\sin(3\eta_2^3 \tau) \sin(2\eta_2^3 \tau)}{6\eta_2^3} - \frac{\cos(-\eta_2^3 \tau) \sin(2\eta_2^3 \tau)}{-2\eta_2^3} \right] \left\{ \frac{1}{2} \frac{\cos(\eta_2^3 s) \sin(\eta_2^3 \tau)}{2} \right\}_s = 0 \]
Thus, we obtain (27).

Here, using the formula of Proposition 1.2.2 (iii) in [15]:

\[
U_{s,t}^3[2,2,2] = \lim_{\eta_2 \to \infty} 2 \left[ -\frac{\cos(\eta_2^3 t)}{4\eta_2^3} - \frac{\cos(\eta_2^3 \tau) \sin(2\eta_2^3 s)}{2\eta_2^3} \right]_{s}^{t} = 0, \tag{B-15}
\]

\[
\frac{\sin(6\eta_2^3 \tau)}{48\eta_2^3} - \frac{\sin(2\eta_2^3 \tau)}{16\eta_2^3} + \frac{\cos(4\eta_2^3 \tau) \sin(2\eta_2^3 s)}{8\eta_2^3} + \frac{\sin(2\eta_2^3 \tau) \sin^2(2\eta_2^3 s)}{2\eta_2^3} = 0. \tag{B-16}
\]

Thus, we obtain

\[
U_{s,t} = \left( 1, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \frac{1}{4}(t-s) \right) \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{B-17}
\]

The combination of the above rough path and our representations for rough integrals (23) yields that the rules

\[
du_1 = 0, \quad du_2 = 0 \quad du_3 = 0 \quad du_4 = \begin{bmatrix} 0 & 3/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \quad dr. \tag{B-18}
\]

Substituting the rules into (25), we obtain

\[
dx_r = \frac{1}{3!} \sum_{l=1}^{n} \sum_{q=1}^{n} \left\{ \frac{3}{2} \frac{\partial}{\partial x[q]} \right\} \left[ \frac{\partial g_2}{\partial x[l]}(x_r) g[l,1](x_r) \right] g[q,1](x_r) du_3^1[l,1,2] \]

\[
- \frac{3}{2} \frac{\partial}{\partial x[q]} \left[ \frac{\partial g_1}{\partial x[l]}(x_r) g[l,2](x_r) \right] g[q,1](x_r) du_3^1[l,1,2] \]

\[
+ \frac{3}{2} \frac{\partial}{\partial x[q]} \left[ \frac{\partial g_1}{\partial x[l]}(x_r) g[l,1](x_r) \right] g[q,2](x_r) du_3^1[l,1,1] \}
\]

\[
= \frac{1}{4} \sum_{l=1}^{n} \sum_{q=1}^{n} \left\{ \frac{\partial}{\partial x[q]} \right\} \left[ \frac{\partial g_2}{\partial x[l]}(x_r) g[l,1](x_r) \right] g[q,1](x_r) \]

\[
- \frac{2}{2} \frac{\partial}{\partial x[q]} \left[ \frac{\partial g_1}{\partial x[l]}(x_r) g[l,2](x_r) \right] g[q,1](x_r) \]

\[
+ \frac{\partial}{\partial x[q]} \left[ \frac{\partial g_1}{\partial x[l]}(x_r) g[l,1](x_r) \right] g[q,2](x_r) \}
\]

\[
= \frac{1}{4} \left\{ (L_{g_1} L_{g_2}) g_2(x_r) - 2(L_{g_1} L_{g_2} g_1)(x_r) + (L_{g_2} L_{g_1}) g_1(x_r) \right\} \]

\[
d\tau = \frac{1}{4} \left\{ (L_{[g_1,g_2]} g_1)(x_r) - 2(L_{[g_1,g_2]} g_1)(x_r) + (L_{g_2} L_{g_1}) g_1(x_r) \right\} \}
\]

\[
= \frac{1}{4} \left\{ (L_{g_1} L_{g_2}) g_2(x_r) - 2(L_{g_1} L_{g_2} g_1)(x_r) + (L_{g_2} L_{g_1}) g_1(x_r) \right\} \}
\]

\[
= \frac{1}{4} \left\{ (L_{[g_1,g_2]} g_1)(x_r) - 2(L_{[g_1,g_2]} g_1)(x_r) + (L_{g_2} L_{g_1}) g_1(x_r) \right\} \}
\]

\[
\tag{B-19}
\]

Here, using the formula of Proposition 1.2.2 (iii) in [15]:

\[
L_{[g_1,g_2]} g_1 = L_{g_1} L_{g_2} g_1 - L_{g_2} L_{g_1} g_1, \tag{B-20}
\]

we obtain

\[
L_{g_1} L_{g_2} g_2 - 2L_{g_1} L_{g_2} g_1 + L_{g_2} L_{g_1} g_1 = L_{g_1} L_{g_2} g_2 - L_{g_1} L_{g_2} g_1 - L_{[g_1,g_2]} g_1
\]

\[
= [g_1, [g_1, g_2]]. \tag{B-21}
\]

Thus, we obtain (27).

References

[1] R.W. Brockett, “Asymptotic stability and feedback stabilization,” Differential Geometric Control Theory, pp. 181–191, Birkhauser, 1983.

[2] K. Fujimoto and T. Sugie, “Stabilization of Hamiltonian systems with nonholonomic constraints based on time-varying generalized canonical transformations,” Systems & Control Letters, vol. 44, pp. 309–319, 2001.
[3] O.J. Sordalen and O. Egeland, “Exponential stabilization of nonholonomic chained systems,” IEEE Transactions on Automatic Control, vol. 40, pp. 35–49, 1995.

[4] A. Bloch, Nonholonomic Mechanics and Control, 2nd Edition, Springer-Verlag New York, 2015.

[5] R.W. Brockett, “Control theory and singular Riemann geometry,” New Directions in Applied Mathematics, pp. 11–27, Springer-Verlag, 1982.

[6] K. Nishimoto, M. Ishikawa, Y. Sugimoto, and K. Osuka, “Feedback control for nonholonomic systems using neural oscillator network,” Nonlinear Theory and Its Applications, IEICE, vol. 4, no. 4, pp. 365–374, 2013.

[7] W. Liu, “An approximation algorithm for nonholonomic systems,” SIAM Journal on Control and Optimization, vol. 35, no. 4, pp. 1328–1365, 1997.

[8] H.J. Sussmann and W. Liu, “Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories,” Proc. 30th IEEE Conference on Decision and Control, 1991.

[9] P. Morin, J.-B. Pomet, and C. Samson, “Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of Lie brackets in closed loop,” SIAM Journal on Control and Optimization, vol. 38, no. 1, pp. 22–49, 1999.

[10] P. Morin and C. Samson, “Practical stabilization of driftless systems on Lie groups: The transverse function approach,” IEEE Transactions on Automatic Control, vol. 48, no. 9, pp. 1496–1508, 2003.

[11] Y. Nishimura, “Stabilization by unbounded-variation noises,” International Journal of Robust and Nonlinear Control, vol. 26, no. 18, pp. 4126–4147, 2016.

[12] Y. Nishimura, “Rough linearization by one-dimensional rough paths,” Proc. 2nd IFAC Conference on Modeling, Identification and Control of Nonlinear Systems (MICNON 2018), 2018.

[13] P.K. Friz and M. Hairer, A Course on Rough Paths, Springer International Publishing, Switzerland, 2014.

[14] T.J. Lyons, M.J. Caruana, and T. Lévy, Differential Equations Driven by Rough Paths, Springer-Verlag, Berlin Heidelberg, 2007.

[15] A. Isidori, Nonlinear Control Systems Third Edition, Springer-Verlag London, 1995.

[16] K. Takeuchi and Y. Nishimura, “Stabilization of fourth-order chained system by rough signals,” Proc. NOLTA’18, 2018.

[17] Y. Nishimura and D. Tsubakino, “Local controllability of single-input nonlinear systems based on deterministic Wiener processes,” IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2019.2912452, to be published in 2020.