MEASURE PRESERVING HOLOMORPHIC VECTOR FIELDS, INARIANT ANTI-CANONICAL DIVISORS AND GIBBS STABILITY

ROBERT J. Berman

Dedicated to Laszlo Lempert on the occasion of his 70th anniversary

Abstract. Let $X$ be a compact complex manifold whose anti-canonical line bundle $-K_X$ is big. We show that $X$ admits no non-trivial holomorphic vector fields if it is Gibbs stable (at any level). The proof is based on a vanishing result for measure preserving holomorphic vector fields on $X$ of independent interest. As an application it shown that, in general, if $-K_X$ is big, there are no holomorphic vector fields on $X$ that are tangent to a non-singular irreducible anti-canonical divisor $S$ on $X$. More generally, the result holds for varieties with log terminal singularities and log pairs. Relations to a result of Berndtsson about generalized Hamiltonians and coercivity of the quantized Ding functional are also pointed out.

1. Introduction

Let $X$ be a compact complex manifold and denote by $K_X$ its canonical line bundle, i.e. the top exterior power of the holomorphic cotangent bundle. We will use additive notation for tensor products. For a given positive integer $k$ set

$$N_k := \dim_{\mathbb{C}} H^0(X, -kK_X)$$

and assume that $N_k > 0$. The $N_k$-fold products $X^{N_k}$ of $X$ come with a natural effective anti-canonical divisor $Q$-divisor $D_{N_k}$, whose support is defined by

$$\{(x_1, \ldots, x_{N_k}) \in X^{N_k} : \exists s_k \in H^0(X, -kK_X) : s_k(x_i) = 0, \; \forall i, \; s_k \not\equiv 0\}$$

In classical terminology the support of $D_{N_k}$ thus consists of all configurations $(x_1, \ldots, x_{N_k})$ of points on $X^{N_k}$ which are in “bad position” with respect to $H^0(X, -kK_X)$. Equivalently, the divisor $D_k$ in $X^{N_k}$ may be defined as $k^{-1}$ times the zero-locus (including multiplicities) of the holomorphic section $\det S^{(k)}$ of $-kK_{X^{N_k}} \to X^{N_k}$ defined as the following Slater determinant

$$(\det S^{(k)})(x_1, x_2, \ldots, x_{N_k}) := \det \left(s_i^{(k)}(x_j)\right),$$

in terms of a given basis $s_1^{(k)}, \ldots, s_{N_k}^{(k)}$ in $H^0(X, -kK_X)$ (see Lemma 2.4).

Following [5], we will say that $X$ is Gibbs stable at level $k$ if $N_k > 0$ and the $Q$-divisor $D_{N_k}$ on $X^{N_k}$ is klt (Kawamata Log Terminal). This means, loosely speaking, that the divisor $D_{N_k}$ has mild singularities in the sense of the Minimal Model Program in birational algebraic geometry [21]. Furthermore, $X$ is called Gibbs stable if it is Gibbs stable for any sufficiently large level $k$. The definition of Gibbs stability is motivated
by the probabilistic approach to the construction of Kähler-Einstein metrics on Fano manifolds introduced in [5], as briefly recalled in Section 1.1.

**Theorem 1.1.** Assume that $-K_X$ is big and that $X$ is Gibbs stable at some level $k$. Then $X$ admits no non-trivial holomorphic vector fields.

The special case when $X$ is Fano, i.e. $-K_X$ is ample and $k$ is taken sufficiently large appears in [5, Prop 6.5], but, unfortunately, the proof was inaccurately formulated. Here the proof will be corrected and - as a bonus - two more proofs of Theorem 1.1 will be provided (of statements that are a bit weaker). More generally, we will show that Theorem 1.1 holds more generally when $X$ is a normal variety. We recall that the class of varieties for which $-K_X$ is big contains the vast class of Fano type varieties (such as toric varieties). However, in general, the anti-canonical ring of a variety $X$ with big $-K_X$ need not be finitely generated, even if $X$ is non-singular [13, Ex 3.7].

The starting point of the proofs is the standard analytic characterization of the klt condition of a divisor, which in the present setup means that the measure on $X^{N_k}$ corresponding to the anti-canonical divisor $D_k$, symbolically denoted by $|\det S^{(k)}|^{-2/k}$, is in $L^p_{loc}$ for some $p > 1$. In particular, the corresponding probability measure $\mu^{(N_k)}$ on $X^{N_k}$ is then well-defined:

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} |\det S^{(k)}|^{-2/k}, \quad Z_{N_k} := \int_{X^{N_k}} |\det S^{(k)}|^{-2/k}. \quad (1.3)$$

This measure is canonically attached to $X$ and thus invariant under the diagonal action on $X^{N_k}$ by any biholomorphism of $X$ (see Lemma 2.4). In particular, $\mu^{(N_k)}$ is invariant under the diagonal action on $X^{N_k}$ of the flow of any given holomorphic vector field on $X$. Hence, theorem 1.1 follows from the following result of independent interest (applied to $X^{N_k}$):

**Theorem 1.2.** Let $X$ be a compact manifold such that $-K_X$ is big and $V^{1,0}$ a holomorphic vector field on $X$. If the flow of $V^{1,0}$ preserves a measure $\mu$ on $X$ with a density in $L^p_{loc}$ for some $p > 1$, then $V^{1,0}$ vanishes identically.

In the case when the local density of $\mu$ is of the form $e^{-\phi}$ for a local function $\phi$ of the form

$$\phi = \sum_{i=1}^m \lambda_i \log |f_i| + \psi \quad (1.4)$$

with $f_i$ holomorphic, $\lambda_i \in \mathbb{R}$ and $\psi$ plurisubharmonic it is enough to assume that the density of $\mu$ is in $L^1_{loc}$, i.e. that $\mu$ has finite total mass. Indeed, by the resolution of Demailly’s strong openness conjecture [19] this implies $L^p_{loc}$-integrability for some $p > 1$ (this is a strengthening of Demailly-Kollar’s openness conjecture, concerning the case when $\lambda_i \geq 0$, previously established in [11]). The assumption that $-K_X$ be big in the previous proposition can not be dispensed with, as illustrated by the case when
$X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mu$ is the measure on $X$ induced by Lebesgue measure on $\mathbb{C}$ (which is invariant under holomorphic translations). This example also shows, by taking products, that it is not enough to assume that $K_X$ is “next to big” in the sense that it has next to maximal Kodaira dimension. On the other hand, in the opposite case that $K_X$ is big it is well-known that $X$ admits no non-trivial holomorphic vector fields at all (see Remark 2.7).

In the second proof of Theorem 1.1 we observe that Theorem 1.2 can be deduced from a result of Berndtsson [11] about generalized Hamiltonians in the case that $-K_X$ is big and nef and the current defined by the Ricci curvature of $\mu$ is positive (i.e. when $\phi$ is psh). This approach, however, does not apply in the more general setting of log pairs $(X, \Delta)$ when $X$ is singular or the divisor $\Delta$ is non-effective. Conversely, we show that Theorem 1.2 implies a partial generalization of Berndtsson’s result.

The third proof of Theorem 1.1 exploits the connection between Gibbs stability at level $k$ and coercivity of the quantized Ding functional established in [27, 7] and applies when $k$ is taken sufficiently large to ensure that the corresponding Kodaira map is a bimeromorphism.

In the final section the concept of (log) Gibbs stability is applied to prove the following

**Theorem 1.3.** Let $X$ be a compact complex manifold such that $-K_X$ is big and $s$ a non-trivial holomorphic section of $-K_X$, whose zero-locus $S$ is a non-singular connected subvariety of $X$. Then $S$ is not preserved by the flow of any non-trivial holomorphic vector field on $X$. More generally, the result holds when the varieties $X$ and $S$ have log terminal singularities and $S$, viewed as an anti-canonical divisor on $X$, is irreducible.

In this case the divisor $S$ is not klt on $X$, i.e. the corresponding measure $|s|^{-2/k}$ is not in $L^1_{\text{loc}}$, which prevents a direct application of Theorem 1.2. But for any sufficiently small positive number $\gamma$ the anti-canonical divisor

$$\gamma D_{N_k} + (1 - \gamma) \left( \pi_1^* S + \cdots + \pi_{N_k}^* S \right),$$

on $X^{N_k}$ - where $\pi_i$ denotes the projection from $X^{N_k}$ to the $i$th factor $X$ - is klt for any fixed $k$ such that $N_k > 1$ (this, essentially, amounts to Gibbs stability of the log pair $(X, (1 - \gamma)S)$). Since, $D_{N_k}$ is invariant under the diagonal flow of any holomorphic vector field on $X$ Theorem 1.3 thus follows from Theorem 1.2 applied to $X^{N_k}$, endowed with the measure induced by the anti-canonical divisor in formula (1.5).

In the case when $-K_X$ is ample (and $S$ is assumed non-singular, which implies connectedness when $-K_X$ is ample and $\dim X > 1$) Theorem 1.3 was shown in [4], confirming a conjecture of Donaldson [17]. The proof in [4] uses the coercivity of the corresponding log Mabuchi functional (which, loosely speaking, can be viewed as the large $k$ limit of the present proof). A different proof was then given in [29, Thm 2.8], including a generalization to the case when $S$ is the non-singular zero-locus of a holomorphic section $s_k$ of $-kK_X$ for $k > 1$. Note, however, that for $k > 1$ one can directly apply Theorem 1.2 to the measure $|s_k|^{-2/k}$ on $X$ when $-K_X$ is big (without assuming connectedness of $S$). The proof in [29] does not seem to generalize to the case when $-K_X$ is only assumed big (or either $X$ or $S$ is singular). Indeed, it relies, in particular,
on the Kodaira vanishing $H^{\dim X-1}(X,\mathcal{O}_X) = 0$, which does not hold when $-K_X$ is merely big (see Example [3,4]).

Note that, by adjunction, the variety $S$ appearing in Theorem 1.3 is Calabi-Yau, i.e. $K_S$ is trivial. This construction of Calabi-Yau varieties $S$ and the corresponding log Calabi-Yau variety $X - S$ play a prominent role in mirror symmetry (see [2] for the case when $X$ is a Gorenstein toric Fano variety and [1] where $-K_X$ is only assumed to be effective).

1.1. **Relations to the Yau-Tian-Donaldson conjecture and K-stability.** We conclude the introduction with a discussion of some relations to the Yau-Tian-Donaldson conjecture, that motivate some of the considerations in the present work. According to the Yau-Tian-Donaldson conjecture a Fano variety $X$ admits a Kähler-Einstein metric if and only if $(X, -K_X)$ is K-polystable. The cases when $X$ is non-singular was established in [14] and, very recently, the singular case was settled in [23]. As for the notion of Gibbs stability it arose in the probabilistic construction of Kähler-Einstein metrics on Fano manifolds proposed in [5]. Relations to the Yau-Tian-Donaldson conjecture - in particular the relation between Gibbs stability and K-stability - are discussed in the survey [6]. Here we will just highlight the role of holomorphic vector fields, as a motivation for Theorem 1.1. First of all, the assumption that $X$ is Gibbs stable at level $k$ is precisely what ensures that $\mu^{(N_k)}$ in formula 1.3 is a well-defined probability measure on $X^{N_k}$. Assuming that this is the case for $k$ sufficiently large, i.e that $X$ is Gibbs stable, it is conjectured in [5] that the measure on $X$ defined as the push-forward of $\mu^{(N_k)}$ to $X$ converges weakly, as $k \to \infty$, to the normalized volume form $dV_{KE}$ of a Kähler-Einstein metric on $X$. Since $\mu^{(N_k)}$ is biholomorphically invariant so is $dV_{KE}$. As a consequence, $X$ can not admit any non-trivial holomorphic vector field $V_1$. Indeed, otherwise its flow $F_\tau$ would act non-trivially on the volume form $dV_{KE}$ (as follows, for example, from the argument in Remark 3.3). The idea of the first proof of Theorem 1.1 is to show that the flow in fact also acts non-trivially on the measures $\mu^{(N_k)}$, even though $\mu^{(N_k)}$ is singular. Thus, in Theorem 1.1 one only needs to assume that $X$ is Gibbs stable at *some* level $k$. Could this, a priori weaker assumption, imply that $X$ is Gibbs stable? After all, it seems natural to expect that the divisor $D_k$ becomes less singular at $k$ is increased. More precisely, the answer would be affirmative if the log canonical threshold lct($D_k$) of $D_k$ were increasing with respect to $k$, since $X$ is Gibbs stable at level $k$ iff lct($D_k$) $> 1$ (the definition of the log canonical threshold of a divisor is recalled in Remark 2.2). For example, this is the case when $X = \mathbb{P}^1$, or, more generally, for any Fano orbifold curve [8]. Unfortunately, the behavior of lct($D_k$) in higher dimensions appears, however, to be rather elusive.

Part of the aforementioned conjecture in [5] says that any Gibbs stable Fano manifold admits a unique Kähler-Einstein metric (leaving the convergence issue aside). A slightly weaker version of this conjecture was established in [18], where it was shown that if $X$ is uniformly Gibbs stable (in the sense that there exists $\epsilon > 0$ such that lct($D_k$) $\geq 1 + \epsilon$ for any sufficiently large $k$), then $X$ is uniformly K-stable (and thus admits a unique Kähler-Einstein metrics, by the solution of the Yau-Tian-Donaldson conjecture). In particular, $X$ is K-stable, which, in turn, directly implies that $X$ admit no holomorphic
vector fields (using that a holomorphic vector field induces a test configuration for $X$). This argument does not, however, apply if $X$ is merely assumed to be Gibbs stable (since the problem whether Gibbs stability implies K-stability is open). But in Section 4 a “quantized” analog of this line of argument is pursued, showing the Gibbs stability at a sufficiently high level $k$ implies that $X$ admits no holomorphic vector fields.

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2. The first proof of Theorem 1.1 and vector fields preserving singular metrics

2.1. Setup.

2.1.1. Vector fields and their flows $F_\tau$. Let $X$ be a complex manifold and $V^{1,0}$ a holomorphic vector field on $X$ of type $(1,0)$, i.e. if $z \in \mathbb{C}^n$ are local holomorphic coordinates on $X$, then $V^{1,0} = \sum_{i=1}^n v_i(z) \partial/\partial z_i$ for holomorphic functions $v_i(z)$. We will denote by $F_\tau$ the corresponding locally defined flow. When $X$ is compact this means that $F_\tau$ is the globally defined holomorphic map from $\mathbb{C} \times X$ to $X$ uniquely determined by

$$\left. \frac{\partial}{\partial \tau} \right|_{\tau = 0} F_\tau(x) = V^{1,0} \big|_x, \quad F_0 = I.$$  

In case $X$ is non-compact $F_\tau(x)$ is only defined in a neighborhood of a given $(0,x) \in \mathbb{C} \times X$. Given a holomorphic line bundle $L \to X$ a vector field $V^{1,0}$ is said to lift holomorphically to $L$ if there exists a holomorphic vector field on the total space of $L$ whose local flow is $\mathbb{C}^*-$equivariant and covers the local flow $F_\tau$ of $V^{1,0}$ on $X$. Abusing notation slightly we will use the same notation $F_\tau$ for the lifted flow on the total space of $L$.

Given a complex vector field $V$ we will denote by $\delta_V$ the corresponding contraction operator (interior product) acting on forms: if $\alpha$ is a $p-$form, then $\delta_V \alpha$ is the $p-1-$form obtained by plugging in $V$ into the first argument of $\alpha$.

2.1.2. Metrics on line bundles. We will use additive notation for metrics. This means that we identify an Hermitian metric $\|\cdot\|$ on $L$ with a collection of local functions $\phi_U$ associated to a given covering of $X$ by open subsets $U$ and trivializing holomorphic sections $e_U$ of $L \to U$:

$$\phi_U := -\log(\|e_U\|^2),$$

which defines a function on $U$. Of course, the functions $\phi_U$ on $U$ do not glue to define a global function on $X$, but the two-form $\partial \bar{\partial} \phi_U$ is globally well-defined and coincides with the curvature form of $\|\|$. Accordingly, as is customary, we will symbolically denote by $\phi$ a given Hermitian metric on $L$ and by $i\partial \bar{\partial} \phi$ its curvature form. More generally, we will allow the metric to be singular in the sense that we only demand that $\phi_U \in L^1_{\text{loc}}$. This ensures that $i\partial \bar{\partial} \phi$ is a well-defined current on $X$. In other words, fixing a smooth metric $\phi_0$ of $L$ we will assume that that $\phi - \phi_0 \in L^1(X)$ (but we do not assume that $i\partial \bar{\partial} \phi$ is
a positive current). Given a (possibly singular) metric on $L$ and a global holomorphic section $s$ of $L$, i.e. $s \in H^0(X, L)$ we have

$$\|s\|^2 = |s_U|^2 e^{-\phi_U},$$

where $s_U$ is the local holomorphic function corresponding to $s$, $s_U := s/e_U$. Abusing notation slightly, the previous expression shall simply be denoted by $|s|^2 e^{-\phi}$.

**Example 2.1.** A global holomorphic section $s$ on $L$ induces a singular metric $\phi$ on $L$ locally defined by $\phi_U := \log(|s/e_U|^2)$. Accordingly, the singular metric on $L$ shall be denoted by $\log(|s|^2)$, abusing notation slightly. By the Poincaré-Lelong formula its curvature current is the current of integration along the regular part of the zero-locus of $\Delta_s$ of $s$ in $X$ (including multiplicities) and thus defines a positive current. More generally, a meromorphic section of $L$ also induces a singular metric on $L$, whose curvature is the current defined by $\Delta_s$ (including multiplicities, defined to be negative for poles of $s$) which is positive iff $s$ is holomorphic.

If $F$ is a $\mathbb{C}^*-$equivariant holomorphic self-map of $L$ then $F$ acts on metrics by pull-back, which in the present additive notation means that

$$(F^*\phi)_U(x) := -\log \left( \| (F s_x) \|^2 (Fx) \right)$$

It should be stressed that, in general, $(F^*\phi)_U$ does not coincide with $F^*(\phi_U)$ i.e. the pull-back of the local function $\phi_U$. However, this is the case if the trivializing section $s$ is invariant under $F$, i.e. if $(Fs_x)(Fx) = s_{F(x)}$.

2.1.3. Metrics on $-K_X$, the corresponding measures on $X$ and anti-canonical klt divisors. Given a smooth metric $\phi$ on $-K_X$ we will use the symbolic notation $e^{-\phi}$ for the corresponding measure on $X$, defined as follows. Given local holomorphic coordinates $z$ on $U \subset X$ denote by $e_U$ the corresponding trivialization of $-K_X$, i.e. $e_U = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n$. The metric $\phi$ on $-K_X$ induces, as in the previous section, a function $\phi_U$ on $U$ and the measure in question is defined by

$$e^{-\phi_U} n^2 \, dz \wedge d\bar{z}, \quad dz := dz_1 \wedge \cdots \wedge dz_n,$$

on $U$, which glue to define a global measure on $X$. In the case when $\phi$ is a singular metric on $-K_X$ the corresponding measure $e^{-\phi}$ is a globally well-defined measure on $X$ as long as $e^{-\phi_U} \in L^1_{\text{loc}}$. In particular, if $s_k$ is a holomorphic section of $-K_X$ then one obtains a measure on $X$, symbolically denoted by $|s_k|^{-2/k}$, if $|s_k|^{-2/k}$ is in $L^1_{\text{loc}}$. If this is the case then the corresponding anti-canonical divisor $\Delta$ on $X$ (i.e. $k^{-1}$ times the zero-locus of $s_k$) is said to be klt. By the well-known algebraic characterization of klt divisors this is, in fact, equivalent to the condition that the measure $|s_k|^{-2/k}$ is in $L^p_{\text{loc}}$ for some $p > 1$ [21].

**Remark 2.2.** In general, if $\Delta$ is an effective divisor on a complex manifold $X$, cut out by a holomorphic section $s$ of a line bundle $L \to X$, then its log canonical threshold lct ($\Delta$) may be analytically defined as the sup over all $t \in [0, \infty]$ such that, locally, $|s|^{-2t} \in L^1$. The definition of lct ($\Delta$) is extended to $\mathbb{Q}$–divisor by imposing linearity and a $\mathbb{Q}$–divisor $\Delta$ is said to be klt if $\text{lct}(\Delta) > 1$ [21].
2.1.4. Lifts to \(\pm K_X\) and tensor products. Any holomorphic vector field on \(X\) admits a canonical lift to \(K_X\) using that the flow \(F_x\) on \(X\) naturally acts by pull-back on forms on \(X\). By duality, it also lifts to \(-K_X\). If \(L_i\) is a collection of \(r\) holomorphic line bundles over \(X\) and \(F^{(i)}\) are \(C^*\)–equivariant holomorphic maps from \(L_i\) to \(L_i\) (thus covering one and the same holomorphic self-map \(F\) of \(X\)) then one naturally obtains a \(C^*\)–equivariant holomorphic \(C^*\)–equivariant self-map on the line bundle over \(X\) defined as the tensor product \(L_1 + \cdots + L_r\), by locally demanding that \(F\) preserves tensor products.

2.2. Gibbs stability and invariant anti-canonical divisors. Let \(X\) be a compact complex manifold and set

\[ N_k := \dim \mathbb{C} H^0(X, -kK_X). \]

Let \(S^{(k)}\) be the holomorphic section of \(-kK_X^{N_k} \to X^{N_k}\) defined by

\[ (\det S^{(k)})(x_1, x_2, ..., x_{N_k}) := \det(s^{(k)}_i(x_j)), \]

in terms of a given basis \(s^{(k)}_1, ..., s^{(k)}_{N_k}\) in \(H^0(X, -kK_X)\). We will denote by \(D_k\) the anti-canonical divisor \(\mathbb{Q}\)–divisor on \(X^{N_k}\) cut out by the \(k\) th root of \(\det S^{(k)}\), i.e. \(D_k\) is defined as \(k^{-1}\) times the zero-locus of \(\det S^{(k)}\), including multiplicities.

**Definition 2.3.** A complex manifold \(X\) is said to be *Gibbs stable at level \(k\)* if the anti-canonical divisor \(D_k\) on \(X^{N_k}\) is klt.

We will adopt the analytic characterization of the klt condition, recalled in Section 2.1. It amounts to the condition that the measure on \(X^{N_k}\) induced by \(\det S^{(k)}\), symbolically denoted by \(|\det S^{(k)}|^{-2/k}\), is in \(L^p_{\text{loc}}\) for some \(p > 1\).

It should be stressed that the divisor \(D_k\) is canonically attached to \(X\), as made precise by the following

**Lemma 2.4.** The zero-locus of \((\det S^{(k)})\) in \(X^{N_k}\) is independent of the choice of basis in \(H^0(X, -kK_X)\) and, as a consequence, it is preserved by any biholomorphism \(F\) of \(X\) acting diagonally on \(X^{N_k}\).

**Proof.** Changing the basis in \(H^0(X, -kK_X)\) has the effect of multiplying the corresponding determinant section \((\det S^{(k)})\) of \(-K_X^{N_k} \to X^{N_k}\), by a non-zero complex number, namely the determinant of the change of basis matrix. Thus the zero-locus of \((\det S^{(k)})\) is not altered. Next, if \(F\) is a biholomorphism of \(X\) we also denote by \(F\) its canonical lift to \(-kK_X\). It induces an action on \(H^0(X, -kK_X)\) and a biholomorphism of \(X^{N_k}\), defined by

\[ (F \cdot s)(x) := F^{-1}s(Fx), \quad F(x_1, ..., x_{N_k}) = (F(x_1), ..., F(x_{N_k})), \]

respectively. Accordingly, \(F \cdot (\det S^{(k)})\) coincides with the determinant section defined with respect to the basis \(F \cdot s^{(k)}_1, ..., F \cdot s^{(k)}_{N_k}\) of \(H^0(X, -kK_X)\). By the previous step this shows that the zero-locus of \(\det S^{(k)}\) is preserved by the diagonal action on \(X^{N_k}\) of \(F\).
alternative geometric proof of the lemma may be obtained by noting that the following geometric description of the zero-locus of \( \det S^{(k)}(x_1, \ldots, x_{N_k}) \) holds:

\[
\det S^{(k)}(x_1, \ldots, x_{N_k}) = 0 \iff \exists s_k \in H(X, -kK_X) : s_k(x_i) = 0, s_k \neq 0.
\]

Indeed, the condition \( \det S^{(k)}(x_1, \ldots, x_{N_k}) = 0 \) is equivalent to the non-injectivity of the linear map from \( H^0(X, -kK_X) \) to \( L_{x_1} \oplus \cdots \oplus L_{x_{N_k}} \) defined by \( s \mapsto (s(x_1), \ldots, s(x_N)) \).

In order to prove Theorem 1.1 (i.e. that Gibbs stability of \( X \) with \( -K_X \) big, at some level \( k \), implies that \( X \) admits no non-trivial holomorphic vector fields) it will be enough to prove the following proposition:

**Proposition 2.5.** Let \( X \) be a compact complex manifold such that \( -K_X \) is big and \( V^{1,0} \) a non-trivial holomorphic vector field on \( X \) whose flow \( F_\tau \) preserves an analytic subvariety of \( X \) of codimension one. Then the analytic subvariety is not the support of any anti-canonical klt divisor \( D \). Equivalently, if \( F_\tau \) preserves an anti-canonical divisor \( D \), then \( D \) is not klt.

Indeed, if \( V^{1,0} \) is a holomorphic vector field on \( X \), then we can apply the previous proposition to the induced diagonal flow on \( X^{N_k} \), which, by the previous lemma, preserves the divisor \( D_k \) on \( X^{N_k} \). In turn, the previous proposition follows from Theorem 1.2 stated in the introduction, applied to the probability measure

\[
\mu = \frac{|s_k|^{-2/k}}{\int_X |s_k|^{-2/k}},
\]

where we have represented the \( \mathbb{Q} \)-divisor \( D \) as the \( k^{-1} \) times the zero-locus (including multiplicities) of a holomorphic section \( s_k \) of some power \( k \) th tensor power \( -kK_X \) and \( \mu \) denotes the corresponding measure (see Section 2.1). Indeed, since the zero-locus \( s_k \) is invariant under \( F_\tau \), we have \( F_\tau \cdot s_k = \rho(\tau)s_k \) for some non-vanishing function \( \rho(\tau) \) on \( \mathbb{C} \) and hence \( F_\tau^* \mu = \mu \). The proof of Theorem 1.2 is given in the following section.

2.3. **A vanishing result for holomorphic vector fields preserving singular metrics.** Recall that a holomorphic line bundle \( L \) over a compact complex manifold \( X \) is said to be big if the space \( H^0(X, kL) \) has maximal asymptotic growth:

\[
\dim H^0(X, kL) \geq C k^{\dim X}
\]

for some positive constant \( C \) (the converse inequality always holds). In analytic terms this equivalently means that \( L \) admits a (possibly singular) metric whose curvature current is strictly positive (i.e. it is bounded from below by a smooth Hermitian metric on \( X \)). The following result contains, in particular, Theorem 1.2 (by letting \( \phi \) be the metric on \( -K_X \) induced by the measure \( \mu \)).

**Theorem 2.6.** Let \( X \) be a compact complex manifold \( X \) and \( V^{1,0} \) a holomorphic vector field on \( X \). Assume that the flow of \( V^{1,0} \) preserves some (possibly singular) metric \( \phi \) on \( -K_X \), i.e. \( F_\tau^* \phi = \phi \). If \( e^{-p\phi} \) is locally integrable for some \( p > 1 \), then the flow of \( V^{1,0} \) acts trivially on the complex vector space \( H^0(X, -kK_X) \) for any positive integer \( k \). In particular, if \( -K_X \) is big, then \( V^{1,0} \) is trivial, i.e. vanishes identically on \( X \).
Proof. Given $\epsilon > 0$ denote by $N_\epsilon$ the function on $H^0(X, -K_X)$ defined by

$$N_\epsilon(s_k) := \left( \int_X (|s_k|^2 e^{-k\phi})^{\epsilon/k} e^{-\phi} \right)^{k/2\epsilon},$$

which, by assumption, defines a finite function on $H^0(X, -K_X)$ when $\epsilon$ is sufficiently small (recall that $e^{-\phi}$ denotes the measure on $X$ naturally attached to the metric $\phi$ on $-K_X$). By the invariance assumption $N_\epsilon$ is invariant under the flow $F_\tau$ of $V^{1,0}$, i.e. $F_\tau^* N_\epsilon = N_\epsilon$. Thus $\Omega := \{ N_\epsilon < 1 \}$ is a domain in the finite dimensional complex vector space $H^0(X, -K_X)$, which is invariant under the flow $F_\tau$ of $V^{1,0}$ and bounded (since $N_\epsilon$ is positively one-homogeneous of degree one). But then $F_\tau$ must be the identity map for any $\tau$. Indeed, for any given $w \in \Omega$ the orbit $\tau \rightarrow F_\tau(w)$ defines a holomorphic map from $\mathbb{C}$ to $\mathbb{C}^{N_k}$ which is bounded (since $\Omega$ is bounded) and thus constant, i.e. equal to $F_\tau(w) (= w)$.

Next, assume that $-K_X$ is big and fix a bases $s_1^{(k)}(x) : \cdots : s_{N_k}^{(k)}(x)$ in $H^0(X, -K_X)$ and consider the corresponding Kodaira map:

$$\Phi_k : X \longrightarrow \mathbb{P}^{N_k-1}, \quad x \mapsto [s_1^{(k)}(x) : \cdots : s_{N_k}^{(k)}(x)],$$

which is a holomorphic map on the complement in $X$ of the joint zero-locus of the bases elements $s_i^{(k)}$. If $F$ is a biholomorphism of $-K_X$, commuting with the $\mathbb{C}^*$-action on $-K_X$, then it induces an invertible endomorphism of $H^0(X, -K_X)$ which descends to a biholomorphism $[F]$ of $\mathbb{P}^{N_k-1}$ which intertwines the Kodaira map $\Phi_k$, i.e.

$$[F] \circ \Phi_k = \Phi_k \circ F.$$  \hfill (2.3)

In particular, if $F$ is taken as the flow $F_\tau$ of a holomorphic vector field $V^{1,0}$ satisfying the assumptions in the theorem then, by the first part of the theorem $[F_\tau]$ is the identity. But then it follows from the assumption that $-K_X$ is big that $F_\tau$ is the identity (i.e. $V^{1,0}$ is trivial). Indeed, by Siegel’s lemma \cite[Lemma 2.2.6]{24}, for $k$ sufficiently large, $\Phi_k$ has maximal rank, which (by the implicit function theorem) means that there exists an open subset $U$ of $X$ such that $\Phi_k$ is a biholomorphism between $U$ and $\Phi_k(U) \subset Y$. Since $[F_\tau]$ is the identity the intertwining property \ref{2.3} thus implies that $F_\tau$ is the identity on $U$ and hence everywhere on $X$, as desired. \hfill \Box

Remark 2.7. The proof of the previous theorem can be viewed as a variant of the proof in \cite{20} that any complex manifold of general type (i.e. such that $K_X$ is big) admits no non-trivial holomorphic vector fields. Indeed, replacing $-K_X$ with $K_X$ and $e^{-\phi}$ with $e^\phi$ (which defines a measure on $X$ if $\phi$ is a metric on $K_X$) the corresponding function $N_\epsilon$ is, for $\epsilon = 1$, independent of $\phi$ and thus preserved by any holomorphic vector field.

In the case when the metric $\phi$ on $-K_X$ is locally of the form

$$\phi = \lambda_i \sum_{i=1}^m \log |f_i| + \psi$$  \hfill (2.4)

with $f_i$ holomorphic, $\lambda_i \in \mathbb{R}$ and $\psi$ plurisubharmonic (i.e. $i\partial \bar{\partial} \psi \geq 0$) it is enough to assume $\phi$ is in $L^1_{loc}$ in the previous theorem. Indeed, by the resolution of the strong
openness conjecture [19] this implies the $L^p_{\text{loc}}$-condition for some $p > 1$. To see the necessity of the integrability assumption $e^{-\phi_U} \in L^1_{\text{loc}}$ when $i\partial\bar{\partial}\phi \geq 0$ assume that $X$ is toric (i.e. $X$ is an equivariant compactification of the complex torus $\mathbb{C}^n$). Any toric variety is of Fano type and thus $-K_X$ is big. Let $\phi$ be the singular metric on $-K_X$ induced by the standard invariant section $s$ in $H^0(X, -K_X)$, defined as the exterior product of the generators of the action of $\mathbb{C}^n$. This means that on the dense subset $\mathbb{C}^n \subset X$

\begin{equation}
\tag{2.5}
{s = z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n}, \quad e^{-\phi} = |s|^2 = |z_1|^2 \cdots |z_n|^2}
\end{equation}

The metric $\phi$ is preserved by the vector field $V^{1,0}$ on $X$ induced by any element in the Lie algebra of $\mathbb{C}^n$, but $e^{-\phi_U}$ is not locally integrable. On the other hand, $e^{-\phi_U}$ is “next to integrable” in the sense that $e^{-\phi_U}$ is in $L^p_{\text{loc}}$ for any $p < 1$.

In the proof of Theorem 2.6 the fractional power $\epsilon/k$ is needed unless $e^{-\phi}$ is in $L^p_{\text{loc}}$ for $p$ large and thus the proof corrects the argument in the proof of [5 Prop 6.5] where $\epsilon/k$ was taken to be one. On the other hand the argument in [5] yields the following result, which applies to any big line bundle $L$:

**Proposition 2.8.** Let $L$ be a holomorphic line bundle over a compact complex manifold $X$ and $V^{1,0}$ a holomorphic vector field on $X$ that admits a holomorphic lift to $L$. Assume that the flow of $V^{1,0}$ preserves a (possibly singular) metric $\phi$ on $L$, i.e. $F^*_\tau \phi = \phi$. If $e^{-k\phi_U}$ is locally integrable, then the flow of $V^{1,0}$ acts trivially on the complex vector space $H^0(X, kL + K_X)$. In particular, if $L$ is big, then there exists some integer $k$ with the following property: if $e^{-k\phi_U}$ is integrable, then $V^{1,0}$ is trivial, i.e. vanishes identically on $X$. More precisely, $k$ can be taken as the minimal integer with the property that the corresponding meromorphic Kodaira map from $X$ to $H^0(X, kL + K_X)$ has maximal rank (i.e. equals the dimension of $X$).

**Proof.** Let $N_k$ be the positively one-homogeneous function on $H^0(X, kL + K_X)$ defined by

$$N_k(s_k) := \left( \int_X |s_k|^2 e^{-k\phi} \right)^{1/2},$$

which, by the integrability assumption, defines a finite function on $H^0(X, kL + K_X)$. Here $|s_k|^2 e^{-k\phi}$ denotes the measure on $X$ naturally attached to the section $s_k$ of $kL + K_X$ and the metric $\phi$ on $L$ using that $s_k$ may be identified with a holomorphic top form with values in $kL$. By the invariance assumption $N_k$ is invariant under the flow $F^*_\tau$ of $V^{1,0}$, i.e. $F^*_\tau N_k = N_k$. One can then conclude precisely as before. Alternatively, in this case, exploiting that $N_k$ is an $L^2$-norm, one can also conclude using the following trivial fact, applied to the generator of the flow on $H^0(X, kL + K_X)$ of the real part of $V^{1,0}$: if a matrix is both Hermitian and anti-Hermitian, then it vanishes identically.

Next, assume that $L$ is big. Then there exists a positive constant $c$ such that

\begin{equation}
\dim H^0(X, kL + K_X) \geq c^{k \dim X}
\end{equation}

Indeed, $L$ is big iff it admits a metric $\phi$ with analytic singularities whose curvature defines a Kähler current [24 Thm 2.3.30] (i.e. it is strictly positive). The lower bound (2.6) then
follows from Bonavero’s singular holomorphic Morse inequalities [24, Cor 2.3.26] (which apply to \(kL + E\) for any given holomorphic line bundle \(E\)). It then follows, exactly, as in the proof of the previous theorem that there exists \(k_0\) such that the corresponding Kodaira map \(\Phi_{k_0}\) has maximal rank. This implies, just as before, that the flow \(F_\tau\) is trivial.

The previous proposition applies, in particular, when \(\phi\) is psh and has vanishing Lelong numbers, which is equivalent to \(e^{-\phi} \in L^p_{\text{loc}}\) for any \(p > 1\). It should be stressed that for a general big line bundle \(L\) the weaker integrability condition in Theorem 2.6 is, however, not sufficient. This is illustrated by the case when \(X\) is the simplest toric Fano manifold, \(X = \mathbb{P}^1\), and \(L = \mathcal{O}(1)\), identified with \(-K_X/2\), endowed with the metric \(\phi\) induced by the \(\mathbb{C}^*-\)invariant section \(s\) in \(-K_X\) (formula 2.5). Indeed, then \(e^{-\phi} \in L^p_{\text{loc}}\) for any \(p < 2\) and yet the metric \(\phi\) on \(L\) is invariant under the generator \(V^{1,0}\) of the \(\mathbb{C}^*-\) action on \(X\) (since \(s\) is).

3. The second proof and generalized Hamiltonians

As explained in Section 2.2 Theorem 1.1 follows from Theorem 2.6. We next show that in the case that \(-K_X\) is big and nef on a projective manifold \(X\) and \(\phi\) is assumed psh the latter theorem can be deduced from the following result [11, Prop 8.2]:

**Proposition 3.1.** (Berndtsson) [11]. Let \(L\) be a holomorphic line bundle over a compact Kähler manifold \(X\) such that \(H^0(X, L + K_X)\) is non-trivial and \(H^1(X, L + K_X)\) is trivial. If \(V^{1,0}\) is a holomorphic vector field on \(X\) such that

\[ \delta V^{1,0} \partial \bar{\partial} \phi = 0, \]

for a (possibly singular) metric \(\phi\) on \(L\) with positive curvature current \(\partial \bar{\partial} \phi\) (i.e. \(\phi\) is locally psh) such that \(e^{-\phi} \in L^1_{\text{loc}}\), then \(V^{1,0}\) vanishes identically.

The proof of the previous proposition is based on delicate integration by parts. The relation to Theorem 2.6 stems from the following completely local result:

**Lemma 3.2.** Let \(X\) be a (possibly non-compact) complex manifold endowed with a holomorphic line bundle \(L\) and \(\phi\) a (possibly singular) metric on \(L\). If \(\phi\) is preserved by a holomorphic \((1, 0)\)-vector field \(V^{1,0}\), then

\[ \delta V^{1,0} \partial \bar{\partial} \phi = 0 \]

in the sense of currents.

**Proof.** Consider the global complex-valued generalized function (i.e. current of degree zero) on \(X\) defined by

\[ h = \left. \frac{\partial}{\partial \tau} \right|_{\tau = 0} (F_\tau)^* \phi, \]

(this is indeed a globally defined generalized function since the difference between any two metrics is a globally well-defined function in \(L^1(X)\)). Then we can express

\[ h = \left. \frac{\partial}{\partial \tau} \right|_{\tau = 0} (F_\tau)^* \phi, \]
\( h = \delta_{V^{1,0}} \partial \phi + f \)

where \( \bar{\partial} f = 0 \). Indeed, let \( g(z, \tau) \) be the local non-vanishing holomorphic function on \( X \times \mathbb{C} \) defined by

\[
(F_\tau s_z)(F_\tau z) = gs(F_\tau z).
\]

Then

\[
\| (F_\tau s_z)(F_\tau z) \| = |g(z, \tau)| \| s(F_\tau z) \|
\]

and hence

\[
((F_\tau)^* \phi)_U(z) = ((F_\tau)^* \phi_U)(z) - \log g(z, \tau) - \overline{\log g(z, \tau)},
\]

giving

\[
\frac{\partial}{\partial \tau} ((F_\tau)^* \phi)_U(z) = \frac{\partial}{\partial \tau} (F_\tau)^* \phi_U - \frac{\partial}{\partial \tau} \log g(z, \tau).
\]

Evaluating this at \( \tau = 0 \) proves formula (3.2) with \( f(z) := -\frac{\partial}{\partial \tau} \log g(z, \tau) \) evaluated at \( \tau = 0 \). Hence, applying \( \bar{\partial} \) gives

\[
\bar{\partial} h = \delta_{V^{1,0}} \bar{\partial} \partial \phi + 0,
\]

(3.3)

(using that \( \bar{\partial} \) commutes with \( \delta_{V^{1,0}} \) for a holomorphic vector field), as desired. \( \Box \)

Remark 3.3. Of course, the assumption in the previous lemma do not, in general, imply that \( V^{1,0} \equiv 0 \) (as illustrated by the case when \( (L, \phi) \) is trivial). However, if one adds the global assumption that \( L \) be ample (positive) line bundle over a compact manifold and \( \phi \) is smooth then there exist some point \( x_0 \) where \( \partial \bar{\partial} \phi|_{x_0} > 0 \) (by the maximum principle) and thus \( i\partial \bar{\partial} \phi > 0 \) on a whole neighborhood \( U \). The previous proposition then forces \( V^{1,0} \equiv 0 \) on \( U \) and thus on all of \( X \), since \( V^{1,0} \) is holomorphic. However, the point of Theorem 2.6 and Prop 2.8 is that they apply to singular situations.

If \( L = -K_X \) the space \( H^0(X, L + K_X) \) is automatically non-trivial. Moreover, \( H^1(X, L + K_X) \) is, in general, trivial if \( L \) is big and nef, by the Kawamata–Viehweg vanishing theorem [22, Thm 9.1.18]. Hence, applying the previous proposition and lemma to \( L = -K_X \) yields an alternative proof of Theorem 2.6 in the case when \( -K_X \) is big and nef on a projective manifold \( X \) and \( \phi \) is psh. Note, however, that when \( -K_X \) is merely assumed to be big the space \( H^1(X, -K_X + K_X) \) can be non-trivial. In other words, it can happen that \( h^{0,1}(X) \neq 0 \), as illustrated by the following example.

Example 3.4. For a projective complex surface \( X \) with \( -K_X \) big it follows from the Castelnuovo’s rationality criterion that \( h^{0,1}(X) = 0 \) iff \( X \) is rational, which, in general need not be the case. For example, when \( X \) is a ruled surface (i.e. a \( \mathbb{P}^1 \)-bundle) over a curve \( C \) of strictly positive genus \( g \) the anti-canonical line bundle \( -K_X \) is big if \( -C \cdot C > 2g - 2 \) [28, 26]. But \( h^{0,1}(X) = h^{0,1}(C) = g > 0 \), since \( X \) is birationally equivalent to \( \mathbb{P}^1 \times C \) and \( h^{0,1}(X) \) is a birational invariant.
3.1. An extension of Berndtsson’s vanishing result and generalized Hamiltonians. We next observe that the converse to Lemma 3.2 holds if $X$ is assumed compact:

**Lemma 3.5.** Let $X$ be a compact complex manifold endowed with a holomorphic line bundle $L$ and $\phi$ a (possibly singular) metric on $L$. If

$$\delta_{V^{1,0}} \overline{\partial} \phi = 0,$$

then $V^{1,0}$ admits a holomorphic lift to $L$ preserving $\phi$.

**Proof.** We first show that $V^{1,0}$ admits a $C^\ast$-equivariant holomorphic lift to $L$. To this end let $\phi_0$ be a fixed smooth metric on $L$. Then there exists a smooth function $h_0$ such that

$$\delta_{V^{1,0}} dd^c \phi_0 = \overline{\partial} h_0.$$

Indeed, defining $u := \phi_0 - \phi \in L^1(X)$ gives

$$\delta_{V^{1,0}} dd^c \phi_0 = 0 + \delta_{V^{1,0}} \overline{\partial} u = \overline{\partial} (\delta_{V^{1,0}} \overline{\partial} u),$$

using that $\overline{\partial} V^{1,0} = 0$. Thus $\delta_{V^{1,0}} \overline{\partial} \phi_0$ is a smooth $(0,1)$-form which is $\overline{\partial}$-exact in the complex of currents on $X$ and hence, by standard regularization results, also $\overline{\partial}$-exact with respect to smooth forms. This proves the existence of a smooth function $h_0$ as in formula 3.4. But, as is well-known this implies that $V^{1,0}$ admits a $C^\ast$-equivariant holomorphic lift to $L$ (see [3, Lemma 13]). Next, as explained in the proof of Lemma 3.2 the function $h$ associated to $(V^{1,0}, \phi)$, defined by formula 3.1 satisfies $\overline{\partial} h = \delta_{V^{1,0}} \overline{\partial} \phi$, which, by assumption, vanishes identically. Hence, $h$ is holomorphic and since $X$ is compact it follows that $h$ is constant, $h = c$. By the very definition of $h$ this means that

$$\frac{\partial}{\partial \tau} |_{\tau = 0} ((F_\tau)^* \phi - \Re(c \tau)) = 0,$$

where $\Re(\cdot)$ denotes the real part. Hence, if we define a new holomorphic lift of $V^{1,0}$ to $L$ by adding a term proportional to the generator of the standard $C^\ast$-action on $L$ and denote its flow by $G_\tau$, then $(G_\tau)^* \phi = \phi$, as desired. \qed

Combining this lemma with Theorem 2.6 and Prop 2.X thus yields the following partial generalization of Prop 3.1, where, in particular, $\phi$ is not assumed to be psh:

**Proposition 3.6.** Let $L$ be a big holomorphic line bundle over a compact complex manifold $X$ and $V^{1,0}$ a holomorphic vector field on $X$ and assume that

$$\delta_{V^{1,0}} \overline{\partial} \phi = 0,$$

for a (possibly singular) metric $\phi$ on $L$ with positive curvature current $\partial \overline{\partial} \phi$ (i.e. $\phi$ is locally psh). Then $V^{1,0}$ vanishes identically if $L = -K_X$ and $e^{-\phi} \in L^p_{\operatorname{loc}}$ for some $p > 1$. In general, $V^{1,0}$ vanishes identical if $e^{-\phi} \in L^k_{\operatorname{loc}}$ for the minimal integer $k$ such that the meromorphic Kodaira map from $X$ to $H^0(X, kL + K_X)$ has maximal rank (i.e. equals the dimension of $X$).
This result generalizes Prop 3.1 in the case that \( L = -K_X \). However, in the case of a general line bundle \( L \), assumed big and nef, the assumption that \( H^0(X, kL + K_X) \) be non-trivial is weaker than demanding that the rank of the Kodaira map be maximal. This is illustrated by the example following Prop 2.8 where \( H^0(X, kL + K_X) \) is non-trivial for \( k \geq 2 \), while the maximal rank property holds for \( k \geq 3 \).

3.1.1. Generalized Hamiltonians. Let \( L \to X \) be a big holomorphic line bundle over a compact complex manifold, \( V^{1,0} \) a holomorphic vector field on \( X \) that lifts holomorphically to \( L \) and denote by \( F_\gamma \) the \( \mathbb{C}^* \)-equivariant flow on \( L \) covering the flow on \( X \). For example, if \( L = -K_X \) any holomorphic vector field \( V^{1,0} \) admits a canonical lift (see Section 2.1). Given a (possibly singular) metric \( \phi \) on \( L \) we can then associate a generalized (complex-valued) function \( h \) to \( V^{1,0} \), defined by formula 3.1. Equation 3.3 says that \( h \) is a complex Hamiltonian for the vector field \( V^{1,0} \), in a generalized sense. Moreover, Theorem 2.6 and Prop 2.8 show that the association \( V^{1,0} \mapsto h \) is injective under the corresponding integrability conditions on \( e^{-\phi_U} \).

Next, denote by \( V \) the real vector field on \( X \) defined as

\[
V := \left(V^{1,0} - \overline{V^{1,0}}\right)
\]

(i.e. the imaginary part of \( V^{1,0} \)). Then \( h_{V^{1,0}} \) is real-valued iff the flow of \( V \) preserves the metric \( \phi \) (and then \( h \) may be expressed as the derivative of \( \phi \) along the flow of \( JV/2 \), where \( J \) denotes the complex structure on \( TX \)). Moreover, equation 3.3 then translates to

\[
dh = \delta_V i\partial \bar{\partial} \phi
\]

showing that \( h \) is a Hamiltonian for the vector field \( V \), in a generalized sense. While Hamiltonians are usually associated to vector fields \( V \) that lift to \( L \) and whose flow preserve a given symplectic form, in the present setup \( i\partial \bar{\partial} \phi \) is neither assumed to be smooth nor non-degenerate. Still, the results above reveal that the association \( V \mapsto h \) is injective under the appropriate integrability assumption on \( e^{-\phi_U} \).

4. The third proof and coercivity of the quantized Ding functional

In this section yet another proof of Theorem 1.1 is provided, under the condition that \( k \) is sufficiently large to ensure that the corresponding Kodaira map is a biholomorphism onto its image. But we start with a more general setup where \( X \) is assumed to be a compact complex manifold and \( k \) is any given positive integer such that the dimension \( N_k \) of \( H^0(X, -kK_X) \) is strictly positive. To simplify the notation we will abbreviate \( N_k = N \). We fix once and for all a smooth metric \( \phi_0 \) on \( -K_X \), a bases \( s_1^{(k)}, \ldots, s_N^{(k)} \) in \( H^0(X, -kK_X) \) and an Hermitian metric \( H_0 \) on \( H^0(X, -kK_X) \) making \( s_1^{(k)}, \ldots, s_N^{(k)} \) orthonormal. Then, for any positive number \( \gamma \) we set

\[
Z_{N, -\gamma} = \int_{X^N} \left\| \det S^{(k)} \right\|_{\phi_0}^{-2\gamma/k} \left(e^{-\phi_0}\right)^{\otimes N},
\]

where \( \left\| \det S^{(k)} \right\|_{\phi_0} \) denotes the point-wise norm of the section \( \det S^{(k)} \) of \( -kK_{X^N} \) (formula 1.2) with respect to the metric on \( -kK_{X^N} \) induced by the fixed metric \( \phi_0 \) on \( -K_X \).
As before, $e^{-\phi_0}$ denotes the corresponding volume form on $X$. The definition is made so that $Z_{N,-1}$ coincides with the normalization constant $Z_N$ appearing in formula 1.3 (since the contributions from the norm and the volume form cancel when $\gamma = 1$). Thus $X$ is Gibbs stable at level $k$ if there exists $\gamma > 1$ (depending on $k$) such that $Z_{N,\gamma} < \infty$. We will apply the following result in 1.3, expressed in terms of the twisted quantized Ding functional $D_{k,\gamma}$ on the space $H_k$ of all Hermitian metrics $H_k$ on the $N-$complex vector space $H^0(X,-kK_X)$:

**Theorem 4.1.** [7, Thm 3.3] The following inequality holds:

$$-\frac{1}{\gamma N} \log Z_{N_k}(-\gamma) \leq \inf_{H_k} D_{k,-\gamma} + \frac{1}{kN} \log N.$$ 

We recall that the functional $D_{k,-\gamma}$ on $H_k$ is defined by

$$D_{k,-\gamma}(H) := \frac{1}{kN} \log \det_{ij \leq N} (H(s_i^{(k)}, s_j^{(k)})) - \frac{1}{\gamma} \log \int_X e^{-(\gamma FS(H_k) + (1-\gamma)\phi_0)},$$

where $FS(H)$ is the metric on $-K_X$ induced from the Fubini-Study metric on $O(1) \rightarrow \mathbb{P}^{N-1}$ corresponding to $H$:

$$FS(H) := k^{-1} \log \left( \frac{1}{N} \sum_{i=1}^{N} |s_i^H|^2 \right)$$

where $(s_i^H)$ is any basis in $H^0(X,-kK_X)$ which is orthonormal wrt $H$ (the metric $FS(H)$ is smooth on the complement of the base locus of $-kK_X$). The normalization by $N$ used here is non-standard.

**Remark 4.2.** The quantized Ding functional $D_{k,-1}$ was introduced in [9] (using a different notation), building on [16]. The case of a general $\gamma$ is studied in [27], where $D_{k,\gamma}$ corresponds to the functional denoted by $F_{m}^{f,\delta}$ with $m = k$ and $\delta = \gamma$. In these works it is usually assumed that $L$ is ample and $kL$ is base point free, but all the properties that we shall use extend to the case when $N_k > 0$ (with the same proofs).

The functional $D_{k,-\gamma}$ is invariant under scaling by positive numbers:

$$D_{k,-\gamma}(cH) = D_{k,-\gamma}(H) \quad \forall c \in \mathbb{R},$$

i.e. $D_{k,-\gamma}$ descends to the quotient space $H_k/\mathbb{R}$.

Now assume that $X$ is Gibbs stable at level $k$. Then it follows from the inequality in the previous theorem that there exists $\gamma > 1$ such that $D_{k,-\gamma}$ is bounded from below on $H_k$, i.e. there exists a constant $C$ such that

$$D_{k,-\gamma} \geq -C.$$ 

This implies that the quantized Ding functional $D_{k,-1}$ is coercive on $H_k/\mathbb{R}$ in the following sense: there exists $\epsilon > 0$ such that

$$D_{k,-1} \geq \epsilon J_k - C.$$
(after perhaps increasing $C$), where $J_k$ is the function on the space $H_k/\mathbb{R}$ defined by

$$J_k(H) := \frac{1}{kN} \log \det_{ij \leq N} \left( H(s_i^{(k)}, s_j^{(k)}) \right) + \sup_{X} (FS(H) - \phi_0).$$

Indeed, this follows from Hölder’s inequality, precisely as in the proof of [27, Prop 4.9]. Thus Theorem 1.1 (in its weaker form discussed above) follows from the following:

**Proposition 4.3.** Assume that the quantized Ding functional $D_{k,-1}$ is coercive on $H_k$, i.e. that inequality (4.4) holds. Then the flow of $V^{1,0}$ acts trivially on the complex vector space $H^0(X, -kK_X)$. In particular, if $-K_X$ is big and $k$ is sufficiently large to ensure that the corresponding Kodaira map is a bimeromorphism onto its image, then $X$ admits no non-trivial holomorphic vector fields.

**Proof.** Given a holomorphic vector field $V^{1,0}$ denote by $F_\tau$ the corresponding $\mathbb{C}^*$-equivariant canonical flow on $-K_X \to X$ and by $[F_\tau]$ the corresponding one-parameter subgroup of the automorphism group $GL(H^0(X, -kK_X))$ of the vector space $H^0(X, -kK_X)$. We shall identify $[F_\tau]$ with a one-parameter subgroup of $GL(N, \mathbb{C})$ (depending on the fixed bases in $H^0(X, -kK_X)$).

*Step 1:* $D_{k,-1}(F_\tau^* H_0)$ is independent of $\tau$.

To see this first note that, in general, $\tau \mapsto D_{k,-1}(F_\tau^* H_0)$ is harmonic on $\mathbb{C}$. Indeed, for any $\tau$

$$\int_X e^{-FS(F_\tau^* H_0)} = \int_X F_\tau^* \left( e^{-FS(H_0)} \right) = \int_X e^{-FS(H_0)}$$

and

$$\det_{ij \leq N} \left( (F_\tau^* H_0)(s_i^{(k)}, s_j^{(k)}) \right) = \left| \det_{ij \leq N} [F_\tau] \right|^2.$$

Thus

$$D_{k,-1}(F_\tau^* H_0) - D_{k,-1}(H_0) = \frac{1}{kN} \log \left( \left| \det_{ij \leq N} [F_\tau] \right|^2 \right),$$

which is harmonic since $\tau \mapsto \det_{ij \leq N} [F_\tau]$ is a non-zero holomorphic function. But, in the present situation $D_{k,-1}$ is bounded from below on $H_k$ and thus the harmonicity implies that $D_{k,-1}(F_\tau^* H_0)$ is constant, as desired.

*Step 2:* If $\tau \mapsto J_k(F_\tau^* H_0)$ is bounded from above, then $[F_\tau] = e^{a\tau} I$ for some $a \in \mathbb{C}$, where $I$ denotes the identity in $GL(N, \mathbb{C})$.

First observe that $J_k$ defines an exhaustion function on $H_k/\mathbb{R}$. Indeed, for any volume form $\nu$ on $X$ there exists a constant $c$ such

$$\sup_X u \geq \int_X \nu - c \forall u \in PSH(X, \omega_0),$$

where $\omega_0$ denotes the curvature form of $\phi_0$ and $PSH(X, \omega_0)$ denotes the space of all functions $u$ in $L^1(X)$ such that $i\partial \bar{\partial} u + \omega_0 \geq 0$ in the sense of currents (this is standard and follows, for example, from the local submean property of plurisubharmonic functions.
using a partition of unity). Thus by the proof of [16, Prop 3] (see also [9, Lemma 7.6]) $J_k$ is an exhaustion function on $\mathcal{H}_k/\mathbb{R}$. Next, consider the fibration

$$GL(N, \mathbb{C}) \to \mathcal{H}_k, \ A \mapsto A^*H_0(= A^*A),$$

whose fibers are isomorphic to $U(N)$ and thus compact. Hence, $J_k$ induces an exhaustion function of $GL(N, \mathbb{C})/\mathbb{C}^*$ that we denote by $j_k$. Now, consider the function $\tau \mapsto \det[F_\tau]$. Since this is a one-parameter subgroup of $\mathbb{C}^*$ it can be expressed as $e^{-a/N\tau}$ for some $a \in \mathbb{C}$. Thus

$$[\tilde{F}_\tau] := e^{a\tau}[F_\tau],$$

has unit-determinant, i.e. it defines a subgroup of $SL(N, \mathbb{C})$. Since $D_{k,-1}$ is invariant under scaling the previous step implies that $\tau \mapsto D_{k,-1}(\tilde{F}_\tau H_0)$ is constant. Hence, by the assumed coercivity, $j_k([\tilde{F}_\tau])$ is bounded from above. Since $j_k$ is an exhaustion function of $GL(N, \mathbb{C})/\mathbb{C}^*$ and $\tilde{F}_\tau$ has unit determinant, it follows that $\tilde{F}_\tau$ takes values in a bounded subset of $GL(N, \mathbb{C})$. Hence, $\tilde{F}_\tau$ is independent of $\tau$, i.e. $\tilde{F}_\tau = I$, which concludes the proof of Step 2. Finally, the latter step implies, in fact, that $a = 0$, using that $F_\tau$ is the canonical lift of $V^{1,0}$ to $-K_X$. Indeed, taking $\tau$ to be real and applying formula [4.5] reveals that the real part of $a$ vanishes. Likewise, taking $\tau$ to be imaginary forces the imaginary part of $a$ to vanish. Thus $V^{1,0}$ acts trivially on the complex vector space $H^0(X, -kK_X)$. The last statement then follows precisely as in the end of the proof of Theorem 2.6.

When $-K_X$ is ample the application of the inequality in Theorem 4.1 above may, alternatively, be replaced by the following argument. First, by [18, Thm 2.5],

$$\text{lct}(X^{N_k}, D_k) \leq \delta_k(X),$$

where $\delta_k(X)$ an invariant introduced in [18]. The assumed Gibbs stability of $X$ at level $k$ means that $\text{lct}(X^{N_k}, D_k) > 1$ and thus it implies that $\delta_k(X) > 1$. In turn, this entails, by [27], that the quantized Ding functional $D_{k,-1}$ is coercive on $\mathcal{H}_k$. Presumably, this argument also applies when $-K_X$ is big. Anyhow, one virtue of the inequality in Theorem 4.1 is that its proof is, essentially, elementary.

5. Extension to the setting of log pairs

5.1. Setup. We briefly recall the general setup of log pairs [21, 22, 10]. By definition, a log pair $(X, \Delta)$ is a normal variety $X$ together with a $\mathbb{Q}$–divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$–Cartier, i.e. defines as a $\mathbb{Q}$–line bundle (called the log canonical line bundle). A log pair $(X, \Delta)$ is said to be klt if the following property holds for some (or equivalently any) log resolution $\pi$ of $(X, \Delta)$ i.e. a holomorphic bimeromorphism from a non-singular variety $X'$ to $X$ such that $\pi^*\Delta + E$ has simple normal crossings, where $E$ denotes the sum of the exceptional divisors of $\pi$. Let $\Delta'$ be the divisor on $X'$ defined by

$$\pi^*(K_X + \Delta) = K_{X'} + \Delta'.$$

By assumption $\Delta'$ has simple normal crossing and $(X, \Delta)$ is said to be klt if all the coefficients of $\Delta'$ are strictly smaller than 1. A variety $X$ is said to have log terminal singularities if the log pair $(X, 0)$ is klt.
Here we shall adopt the analytic characterization of the klt condition of a log pair $(X, \Delta)$ which goes as follows (see [10] for further background). First assume that $X$ is non-singular. The $\mathbb{Q}$–divisor $\Delta$ induces a $\mathbb{Q}$–line bundle on $X$, denoted by the same symbol $\Delta$, and a singular metric on the $\mathbb{Q}$–line bundle $\Delta$, denoted by $\phi_\Delta$ (see Section 2.1). Let now $\phi$ be a locally bounded metric on $-(K_X + \Delta)$. Then $\phi + \phi_\Delta$ defines a singular metric on $-K_X$ and thus also a measure on $X$, denoted by $e^{-(\phi + \phi_\Delta)}$. The log pair $(X, \Delta)$ is klt iff the measure $e^{-(\phi + \phi_\Delta)}$ on $X$ has finite mass. If $X$ is singular the measure $e^{-(\phi + \phi_\Delta)}$ is first defined as before on the regular locus $X_{\text{reg}}$ of $X$ and then extended by zero to all of $X$ (so that it puts no mass on the singular locus $X - X_{\text{reg}}$). Then the previous discussion applies. Note that the construction of the measure $e^{-(\phi + \phi_\Delta)}$ is compatible with log resolutions, i.e. if $\pi$ is a log resolution of $(X, \Delta)$, then

$$
\pi_* \left( e^{-(\pi^* \phi + \phi_\Delta)} \right) = e^{-(\phi + \phi_\Delta)}
$$

as measures. More generally, a divisor $D$ is said to be klt wrt the log pair $(X, \Delta)$ if $(X, D + \Delta)$ is a log pair which is klt.

5.1.1. Vector fields and log pairs. By definition, a holomorphic vector field $V^{1,0}$ on a normal variety $X$ is holomorphic vector field on the regular locus $X_{\text{reg}}$ of $X$ such that there exists a holomorphic map $F_\tau$ from $\mathbb{C} \times X$ called the flow of $V^{1,0}$, satisfying formula 2.1 on $X_{\text{reg}}$. If $(X, \Delta)$ is a log pair a holomorphic vector field $V^{1,0}$ on $X$ is said to be tangent to $\Delta$ if it is tangent to $\Delta$ along the regular locus of $\Delta$. This equivalently means that the flow $F_\tau$ of $V^{1,0}$ preserves the support of $\Delta$. Next assume that $\Delta$ has positive integer coefficients. Equivalently, there exists a line bundle over $X$, also denoted by $\Delta$, with a holomorphic section $s$ cutting out $\Delta$. If $F_\tau$ preserves the support of $\Delta$ then there exists a canonical holomorphic $\mathbb{C}^*$–equivariant lift of $F_\tau$ to the corresponding line bundle such that $s$ is invariant under $F_\tau$. Indeed, since $X - \Delta$ is preserved by $F_\tau$ the lift over $X - \Delta$ may be defined by

$$
F_\tau(ws_x) := ws_{F_\tau(x)}
$$

for any $w \in \mathbb{C}$, using that $s$ defines a global trivialization of the line bundle over $X - \Delta$. This definition extends holomorphically over the support of $\Delta$ (as can be seen by expressing $s$ in terms of a fixed local trivialization of the line bundle over a neighborhood of a given point in $\Delta$). Finally, in the case that $\Delta$ has some negative integer coefficients the lifting of $F_\tau$ is achieved by first lifting $F_\tau$ to the dual of the corresponding line bundle.

5.2. Gibbs stability of log pairs. Given a log pair $(X, \Delta)$ we fix a sufficiently divisible positive integer $k$, assuring that $k(K_X + \Delta)$ is a bona fide line bundle. Then all the definitions in Section 2.2 can be generalized by replacing the canonical line bundle $K_X$ of $X$ with the log canonical line bundle $K_X + \Delta$ of $(X, \Delta)$. For example,

$$
N_k := \dim H^0(X, -k(K_X + \Delta))
$$
and \((\det S^{(k)})\) denotes the holomorphic section of \(-\left(K_{X,k} + \pi_1^*\Delta + ... + \pi_{N_k}^*\Delta\right)\) defined as in formula \([2.2]\) but now with \(s^{(1)}, ..., s^{(k)}\) denoting a fixed basis in \(H^0(X, -k(K_X + \Delta))\). Accordingly, now \(D_k\) denotes the log anti-canonical divisor on \(X^{N_k}\) cut-out by \((\det S^{(k)})^{1/k}\).

**Definition 5.1.** A log pair \((X, \Delta)\) is said to be *Gibbs stable at level \(k\)* if the anti-canonical divisor \(D_k\) on \(X^{N_k}\) is klt with respect to the log pair \((X^{N_k}, \pi_1^*\Delta + ... + \pi_{N_k}^*\Delta)\).

The following result is a generalization of Theorem \([1.1]\) to the setting of log pairs:

**Theorem 5.2.** Assume that \(-(K_X + \Delta)\) is big and that \((X, \Delta)\) is Gibbs stable at some level \(k\). Then \(X\) admits no non-trivial holomorphic vector fields which are tangent to \(\Delta\).

**Remark 5.3.** In particular, taking \(\Delta = 0\) the previous theorem extends Theorem \([1.1]\) to normal varieties \(X\). Note that the Gibbs stability of \(X\) directly implies that \((X, 0)\) is klt, i.e. that \(X\) has log terminal singularities. This is analogous to the result \([25, \text{Thm 1.3}]\) saying that a K-semistable Fano variety has log terminal singularities.

To prove Theorem \([5.2]\) denote by \(\phi\) the singular metric on \(-\left(K_{X,k} + \pi_1^*\Delta + ... + \pi_{N_k}^*\Delta\right)\) \(\rightarrow X^{N_k}\) corresponding to \(\det S^{(k)} \in H^0(X, -k(K_X + \Delta))\). By assumption, the diagonal flow \(F_\varepsilon\) on \(X^{N_k}\) induced by a given holomorphic vector field \(V^{1.0}\) preserves the line bundle \(-k(K_{X,k} + \pi_1^*\Delta + ... + \pi_{N_k}^*\Delta)\). Hence, it follows precisely as in the proof of Lemma \([2.4]\) that the singular metric \(\phi\) is invariant under \(F_\varepsilon\). Theorem \([5.2]\) thus follows from the following generalization of Theorem \([2.6]\) applied to \((X^{N_k}, \pi_1^*\Delta + ... + \pi_{N_k}^*\Delta)\).

To simplify the statement of the integrability assumption we assume that \(\phi\) is, locally, of the form \([1.4]\) and the measure \(e^{-(\phi + \phi_\Delta)}\) on \(X\) has finite total mass, then the flow of \(V^{1.0}\) acts trivially on the complex vector space \(H^0(X, -k(K_X + \Delta))\) for any positive integer \(k\). In particular, if \(-(K_X + \Delta)\) is big, then \(V^{1.0}\) is trivial, i.e. vanishes identically.

**Proof.** Given \(\varepsilon > 0\) let \(N_\varepsilon\) the positively one-homogeneous function on \(H^0(X, -k(K_X + \Delta))\) defined by

\[
N_\varepsilon(s_k) := \left(\int_X \left|s_k\right|^2 e^{-k\phi} e^{-\varepsilon k\phi_\Delta}\right)^{k/2}\varepsilon^{k/2},
\]

defines a finite function on \(H^0(X, -k(K_X + \Delta))\) when \(\varepsilon\) is sufficiently small, by the assumption on \(\phi\) and the resolution of the strong openness conjecture \([19]\) (applied to a log resolution of \((X, \Delta))\). By the invariance assumption \(N_\varepsilon\) is invariant under the flow \(F_\varepsilon\) of \(V^{1.0}\), i.e. \(F_\varepsilon N_\varepsilon = N_\varepsilon\). Hence, we can proceed exactly as in the proof of Theorem \([2.6]\) \(\square\)

**5.3. Examples: the case of log Fano curves.** It may be illuminating to consider Theorem \([5.2]\) in the simplest case where \(X\) is a complex curve, i.e. \(\dim X = 1\). Then

\[
\Delta := \sum_{i=1}^{m} x_i w_i
\]
for given points $x_1,\ldots,x_m$ on $X$ and rational coefficients $w_i$. First consider the case when $X$ has genus zero, $X = \mathbb{P}^1$. Then $-(K_X + \Delta)$ is ample iff

$$2 - \sum w_i > 0.$$  \hfill (5.1)

Moreover, as shown in [8] $(X, \Delta)$ is Gibbs stable at a sufficiently large level $k$ iff the following weight condition holds:

$$w_i < \sum_{i \neq j} w_j, \ \forall i$$  \hfill (5.2)

(this condition first appeared in the existence result for conical Kähler-Einstein metrics on $\mathbb{P}^1$ established in [30]). In particular, given three points there always exists weights $w_1, w_2$ and $w_3$ satisfying the inequalities (5.1) and (5.2) and thus the corresponding log pair $(X, \Delta)$ is Gibbs stable at some level $k$. Hence, Theorem 5.2 implies the basic fact that any holomorphic vector field on $\mathbb{P}^1$ with three zeros vanishes identically (which follows, for example from the fact that the space $H^0(X, -K_X)$ of holomorphic vector fields on $X$ is 3-dimensional).

Next, consider the case when $X$ has genus one (for genus strictly larger than one there are no non-trivial holomorphic vector fields at all). Given a point $x$ on $X$ let $\Delta$ be the divisor

$$\Delta = -k^{-1}x$$

for a fixed positive integer $k$. Then $-(K_X + \Delta)$ has positive degree (since $-K_X$ is trivial) and is thus ample. Moreover, $(X, \Delta)$ is Gibbs stable at level $k$. Indeed, by construction, $-k(K_X + \Delta) = -k\Delta$ coincides with the line bundle of degree one over $X$ defined by the point $x$. It thus follows from the Riemann-Roch theorem that $N_k = 1$. In other words, $-k(K_X + \Delta)$ admits a non-trivial holomorphic section $s$, uniquely determined up to multiplication by $\mathbb{C}^*$. Hence,

$$|\det S^{(k)}|^{-2/k} |s_\Delta|^{-2} = |s|^{-2/k} |s|^{2/k}$$

which is trivially in $L^1(X)$, i.e. $(X, \Delta)$ is Gibbs stable at level $k$. Theorem 5.2 thus implies the basic fact that there are no non-trivial holomorphic vector fields on $X$ preserving a given point $x$ on (which follows, for example, from the fact that the space of holomorphic vector fields $H^0(X, -K_X)$ on $X$ is one-dimensional and thus generated by the vector field induced from the vector field $\partial / \partial z$ on $\mathbb{C}$ under the standard isomorphism $X \simeq \mathbb{C}/\Lambda$).

5.4. Log stability with respect to a parameter. The Gibbs stability of a log pair $(X, \Delta)$ can, in fact, be defined for some non-integer levels $k$. Indeed, all that is needed is that $-k(K_X + \Delta)$ defines a line bundle (i.e. a Cartier divisor). For example, if

$$\Delta = (1 - \gamma)S, \ \gamma \in \mathbb{C}/0,\infty$$

for an anti-canonical divisor $S$, then $-k(K_X + \Delta)$ is a well-defined line bundle for any $k \in \gamma^{-1}\mathbb{Z}$. Indeed, if $k = \gamma^{-1}p$ for $p \in \mathbb{Z}$, then

$$-k(K_X + \Delta) = -pK_X.$$  

In this case it is convenient to make the following
Definition 5.5. Let $S$ be anti-canonical divisor on a variety $X$ with log terminal singularities and $\gamma \in [0, \infty]$. Then $(X, S)$ is said to be Gibbs stable at level $k$ with respect to the parameter $\gamma$ if $(X, (1 - \gamma)S)$ is Gibbs stable at level $\gamma^{-1}k$. Equivalently, this means that the $\mathbb{Q}$-divisor $\gamma D_{N_k} + (1 - \gamma) \left( \pi_1^* S + \cdots + \pi_k^* S \right)$ is klt on $X^{N_k}$, where $D_{N_k}$ is the anti-canonical divisor on $X^{N_k}$ defined in Section 2.2 and $\pi_i$ denotes the projection $X^{N_k} \to X$ from $X^{N_k}$ onto the $i$th factor of $X^{N_k}$.

One advantage of introducing the parameter $\gamma$ is that it can be used as a deformation parameter (which from the point of view of statistical mechanics plays the role of the negative of the inverse temperature; see the discussion in [8, Section 2.4]). In the following section we will provide a geometric application of log Gibbs stability of $(X, S)$ with respect to a small parameter $\gamma$.

Remark 5.6. Formally, when $k = \infty$ the parameter $\gamma$ corresponds to Donaldson’s time-parameter in his singular generalization of Aubin’s method of continuity [17] for Kähler metrics $\omega_\gamma$:

$$\text{Ric} \omega_\gamma = \gamma \omega_\beta \beta + (1 - \gamma) [S],$$

where $\text{Ric} \omega_\gamma$ denotes the Ricci curvature current of the singular Kähler metric $\omega_\gamma$ and $[S]$ denotes the current of integration along the divisor $S$. As explained in [8], this formal analogy can be made rigorous under a zero-free assumption on the zeta type function appearing in formula 6.1 below (viewed as a meromorphic function of $\gamma$).

6. A vanishing theorem for holomorphic vector fields preserving anti-canonical divisors that are not klt

In this final section we apply the concept of log Gibbs stability to prove Theorem 1.3, stated in the introduction:

Theorem 6.1. Let $X$ be a compact complex manifold such that $-K_X$ is big and $s$ a non-trivial holomorphic section of $-K_X$, whose zero-locus $S$ is a non-singular connected subvariety of $X$. If $S$ is preserved by the flow a holomorphic vector field $V^{1,0}$ on $X$, then $V^{1,0}$ vanishes identically. More generally, the result holds when the varieties $X$ and $S$ have log terminal singularities and the divisor on $X$ defined by $s$ is irreducible.

In contrast to Prop 2.8 the anti-canonical divisor on $X$ defined by $s$ is not klt (but $(S, 0)$ is klt, since $S$ is assumed to have log terminal singularities). Without the regularity and irreducibility assumption the result thus fails, in general. This is illustrated by the toric example following Theorem 2.6, where $s$ is the standard invariant anti-canonical divisor on $X$, $S = X - \mathbb{C}^{\text{en}}$. In particular, when $X = \mathbb{P}^1$ the support $S$ of this $\mathbb{C}^*$-invariant divisor consists of the two points 0 and $\infty$ in $\mathbb{P}^1$ and is thus non-singular, but not connected/irreducible.

6.1. Proof of Theorem 6.1. Fix a positive integer $k$ such that $N_k > 1$. Denote by $\pi_i$ the projection $X^{N_k} \to X$ from $X^{N_k}$ onto the $i$th factor of $X^{N_k}$. The section $s$ induces, by taking tensor products, a holomorphic section $s^{\otimes N_k}$ of $-K_{X^{N_k}}$. By Theorem 5.2 it is enough to show that $(X, (1 - \gamma)S)$ is Gibbs stable at level $\gamma^{-1}k$ (i.e. that $(X, S)$ is
Gibbs stable at level $k$ with respect to the parameter $\gamma$). In other words, it is enough to show that

\begin{equation}
\int_{X^N_k} \left| \det S^{(k)} \right|^{-2\gamma/k} \left| s^\otimes N_k \right|^{-2(1-\gamma)} < \infty.
\end{equation}

To this end we factorize

$$\det S^{(k)}(x_1, \ldots, x_{N_k}) = s(x_1)^{\otimes l_1} s(x_2)^{\otimes l_2} \cdots q(x_1, x_2, \ldots, x_{N_k}),$$

where $q(x_1, \ldots, x_{N_k}) \in H^0 \left( X^{N_k}, (k - l_1)\pi_1^*(-K_X) + \cdots + (k - l_{N_k})\pi_{N_k}^*(-K_X) \right)$ is not divisible by the irreducible divisor $s(x_i)$ on $X^{N_k}$ for any $i$. The integer $l_1$ coincides with the order of vanishing of $\det S^{(k)}$ along $\{ s(x_1) = 0 \} \subset X^{N_k}$, for generic $(x_2, \ldots, x_{N_k})$ and likewise for $i \neq 1$. Since $\det S^{(k)}(x_1, \ldots, x_{N_k})$ is totally antisymmetric it follows that $l_1 = l_i$ for all $i$. This means that there exists $l \leq k$ such that

$$\det S^{(k)}(x_1, \ldots, x_{N_k}) = s(x_1)^{\otimes l} s(x_2)^{\otimes l} \cdots q(x_1, x_2, \ldots, x_{N_k}),$$

where $q(x_1, \ldots, x_{N_k}) \in H^0(X^{N_k}, -(k - l)K_{X^{N_k}})$. The assumption $N_k > 1$ forces $l < k$. Indeed, otherwise, $q \in H^0(X^{N_k})$ and thus $q$ is constant, $q = C$, giving $\det S^{(k)}(x_1, \ldots, x_{N_k}) = C s(x_1)^{\otimes k} s(x_2)^{\otimes k} \cdots$. Since $\det S^{(k)}(x_1, \ldots, x_{N_k})$ is totally anti-symmetric this can only happen if $N_k = 1$. Hence,

$$\left| \det S^{(k)} \right|^{-2\gamma/k} \left| s^\otimes N_k \right|^{-2(1-\gamma)} = |q|^{-2\gamma/k} |s(x_1)|^{-2(1-\gamma + l/k)} \cdots |s(x_{N_k})|^{-2(1-\gamma + l/k)}$$

By construction, for generic $x_2, \ldots, x_{N_k}$ the section $x \mapsto q(x, x_2, \ldots, x_{N_k})$ does not vanish identically along $\{ s(x_1) = 0 \} \subset X$ and likewise for $i \neq 1$. Note that since $l < k$ the exponent $(1 - \gamma + l/k)$ is strictly smaller than 1. As a consequence, we also have

$$k^{-1} = (k - l)^{-1} (k - l - 1)^{-1}$$

and thus (since we may, without loss of generality, assume that $|q(x_1, \ldots, x_{N_k})| \leq 1$)

$$\left| \det S^{(k)} \right|^{-2\gamma/k} \left| s^\otimes N_k \right|^{-2(1-\gamma)} \leq \left( |q(x_1, \ldots, x_{N_k})|^{-2(k-l)} \right)^\gamma \left| s(x_1) \right|^{-2(1-\gamma)} \cdots \left| s(x_{N_k}) \right|^{-2(1-\gamma)}$$

for some $\delta \in [0, 1]$. Next we will apply inversion of adjunction \[21\] Thm 7.5 in the following form: let $B$ be a divisor on $X$ such that the support of $S$ is not contained in the support of $B$, then

$$B_{|S} \text{ klt close to } S \implies B + (1 - \delta)S \text{ klt close to } S$$

for any $\delta \in [0, 1]$ (in the case when $X$ and $S$ are non-singular this is a direct consequence of the local Ohmura-Takegoshi extension theorem; see \[21\] Section 7). Now set $B = \gamma Q$, where $Q$ is the anti-canonical divisor on $X$ defined as $(k - l)^{-1}$ times the divisor cut out by the holomorphic section $x \mapsto q(x, x_2, \ldots, x_{N_k})$ of $-(k - l)K_X$, for a fixed generic $(x_2, \ldots, x_{N_k}) \in X^{N_k-1}$. Then

$$\gamma < \alpha(S, -K_X|S) \implies \gamma Q_{|S} \text{ klt close to } S,$$
where \(-K_X|S\) denotes the restriction to \(S\) of \(-K_X\) and \(\text{lct}(Y, L)\) denotes the \textit{global log canonical threshold} of a given line bundle \(L\) over a variety \(Y\) with log terminal singularities (which, by \([15]\), coincides with Tian’s \textit{alpha-invariant} of \(L \to Y\)):

\[
\text{lct}(L) := \sup_{t>0} \{t : tk^{-1}(q_k) = 0\} \text{ is klt} \ \forall q_k \in H^0(Y, kL), \forall k \in \mathbb{Z}_+ \}
\]

where \((q_k = 0)\) denotes the divisor in \(Y\) defined cut-out by \(q_k\), including multiplicities. On any open set \(U\) in the complement of the zero-locus of \(S\) in \(X\) we trivially have that

\[
\gamma < \text{lct}(X, -K_X) \implies \gamma Q + (1 - \delta)S \text{ klt on } U.
\]

Hence,

\[
\gamma < \min \{\text{lct}(X, -K_X), \text{lct}(S, -K_X|S)\} \implies \int_X \left|\det S^{(k)}\right|^{-2\gamma/k} \left|s^{\otimes N_k}\right|^{-2(1-\gamma)} < \infty.
\]

It then follows from the compactness of \(X\) and the semi-continuity of integrability thresholds (just as in the appendix of \([4]\)) that there exists a constant \(C_\gamma\) only depending on \(\gamma\) such for any fixed generic \((x_2, ..., x_{N_k}) \in X^{N_k-1}\)

\[
\int_{x \in X} \left|\det S^{(k)}(x, x_2, ..., x_{N_k})\right|^{-2\gamma/k} \left|s(x)\right|^{-2(1-\gamma)} \leq C_\gamma \left(\sup_{x \in X} \left|\det S^{(k)}(x, x_2, ..., x_{N_k})\right|\right)^{-2\gamma/k}
\]

for a given fixed smooth metric \(\|\cdot\|\) on \(-K_X\). As a consequence, the inequality, in fact holds for any fixed \((x_2, ..., x_{N_k}) \in X^{N_k-1}\). Iterating this inequality \(N_k-1\) times (replacing the index \(i = 1\) with any index \(i\)) thus gives

\[
\int_{X^{N_k}} \left|\det S^{(k)}\right|^{-2\gamma/k} \left|s^{\otimes N_k}\right|^{-2(1-\gamma)} \leq C_\gamma \left(\sup_{X^{N_k}} \left|\det S^{(k)}\right|\right)^{-2\gamma/k} < \infty,
\]

which concludes the proof of the theorem, using that \(\text{lct}(Y, L) > 0\) if \(Y\) has log terminal singularities and \(\dim H^0(Y, kL) \geq 1\) for some positive integer \(k\). Indeed, if \(q_k\) is a non-trivial element in \(H^0(Y, kL)\), then the log pair \((Y, tk^{-1}(q_k = 0))\) is klt when \(t\) is a sufficiently small positive number, since, \((Y, 0)\) is assumed klt and the klt condition is stable under small perturbations.

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