Convergence Rate of Distributed Optimization Algorithms Based on Gradient Tracking

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ABSTRACT
We study distributed, strongly convex and nonconvex, multiagent optimization over (directed, time-varying) graphs. We consider the minimization of the sum of a smooth (possibly nonconvex) function—the agent’s sum-utility plus a nonsmooth convex one, subject to convex constraints. In a companion paper, we introduced SONATA, the first algorithmic framework applicable to such a general class of composite minimization, and we studied its convergence when the smooth part of the objective function is nonconvex. The algorithm combines successive convex approximation techniques with a perturbed push-sum consensus mechanism that aims to track locally the gradient of the (smooth part of the) sum-utility. This paper studies the convergence rate of SONATA. When the smooth part of the objective function is strongly convex, SONATA is proved to converge at a linear rate whereas sublinear rate is proved when the objective function is nonconvex. To our knowledge, this is the first work proving a convergence rate (in particular, linear rate) for distributed algorithms applicable to such a general class of composite, constrained optimization problems over graphs.

1. Introduction

The paper studies distributed multiagent convex and nonconvex optimization over networks. We consider the following general formulation

\[ \min_x U(x) \triangleq \sum_{i=1}^{I} f_i(x) + G(x) \quad \text{(P)} \]

\[ \text{s.t. } x \in \mathcal{K}, \]

where \(f_i : \mathbb{R}^n \to \mathbb{R}\) is the cost function of agent \(i\), assumed to be smooth (possibly nonconvex) and known only to agent \(i\); \(G : \mathbb{R}^n \to \mathbb{R}\) is a nonsmooth convex function; and \(\mathcal{K} \subseteq \mathbb{R}^n\) represents the set of common constraints, assumed to be closed and convex. Agents are connected through a communication network, modeled as a graph, possibly directed and/or time-varying. No specific topology is assumed for the graph (such as star or hierarchical structure), but some long term connectivity (cf. Sec.2). In this setting, agents seek to cooperatively solve Problem (P) by exchanging iteratively information with their immediate neighbors.

Distributed optimization in the form (P) has found a wide range of applications in

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several areas, including network information processing, telecommunications, multi-agent control, and machine learning—see, e.g., [33] and reference therein.

In the companion paper [34], a general distributed algorithmic framework has been proposed for Problem (P) with nonconvex \( F \), and termed SONATA (Successive cON- vex Approximation algorithm over Time-varying digraPhs). The algorithm builds on the idea of Successive Convex Approximation (SCA) techniques [5, 31–33], coupled with a judiciously designed perturbed consensus mechanism that aims to track locally the gradient of \( F \), an information that is not available at the agents’ side. Quite interestingly, SONATA contains as special cases a gamut of distributed algorithms based on gradient tracking and applicable to special instances of (P) [33, Ch. 3.4.3.1]. In [34] we studied convergence of SONATA. This work complements [34] and derives its convergence rate, in the case of strongly convex and nonconvex \( F \). When \( F \) is strongly convex, we prove that the sequence \( \{U(x^\nu_i)\}_{\nu \in \mathbb{N}, i = 1, \ldots, I} \), generated by SONATA converges to the optimal value \( U^* \) at an \( R \)-linear rate, where \( x^\nu_i \) is agent \( i \)’s estimate at iteration \( \nu \) of the optimal solution of (P). This is achieved over possibly time-varying and/or directed graphs. We are not aware of any algorithm in the literature that achieves linear rate in the same setting (in particular in the presence of the nonsmooth convex function \( G \) and/or constraints)–see Table 1. When \( F \) is nonconvex, we prove that a suitably defined measure of stationarity and consensus disagreement goes below the accuracy \( \epsilon \) in \( \mathcal{O}(1/\epsilon) \) iterations, resulting in the first rate analysis of a distributed algorithm applicable to (P)–see Table 2.

1.1. Related works

We review prior art grouping relevant works in two categories, namely: distributed methods for strongly convex \( F \) and nonconvex \( F \). Since the focus of this work is on the convergence rate of SONATA, we will bias this literature review towards solution methods with provable rate in the two cases above.

1.1.1. Strongly convex \( F \)

Early works on distributed optimization aimed at decentralizing the (sub)gradient algorithm. The Distributed Gradient Descent (DGD) was introduced in [21] for unconstrained instances of (P) and in [14] for least squares; both schemes are implementable over undirected graphs, with [21] handling also time-varying topologies. A refined convergence rate analysis of DGD [21] can be found in [51]. Subsequent variants and extensions of DGD include: i) the projected (sub)gradient algorithm [22] and its stochastic extension [28], and the proximal consensus scheme [15] (see also references therein): [22, 28] are applicable to convex constrained problems while [15] can handle also private convex constraints; ii) the push-sum gradient consensus algorithm [20], implementable over (possibly time varying) digraphs, and its stochastic extension [23]; and iii) some Nesterov-based accelerated instances of DGD [11]. While different, the updates of the agents’ variables in the above algorithms can be abstracted as a combination of one (or multiple) consensus step(s) (weighted average with neighbors variables) and a local (sub)gradient descent step, controlled by a step-size (in some schemes, followed by a proximal operation). Even if all \( f_i \)’s are smooth and strongly convex, these schemes need to use a diminishing step-size to reach exact consensus on the solution, converging thus at a sublinear rate. For instance, DGD-based schemes with diminishing step-size generate iterates that converge to the exact solution at a rate of \( \mathcal{O}((\log \nu)/\nu) \) [11], where \( \nu \) is the iteration index. Furthermore, convergence analysis of the aforementioned algorithms is carried out under the assumption of bounded
(sub)gradients—unbounded (sub)gradients can potentially cause algorithm divergence. With a fixed step-size $\alpha$, DGD in the setting above can be faster–linear rate of the iterates is achievable—but it can only converge to a $O(\alpha)$-neighborhood of the solution.

Several subsequent attempts have been proposed to cope with this speed-accuracy dilemma, leading to algorithms converging to the exact solution while employing a constant step-size. Based upon the mechanism put forth to cancel the steady state error in the individual gradient direction, existing proposals can be roughly organized in three groups, namely: i) primal-based distributed methods leveraging the idea of gradient tracking; ii) distributed schemes using ad-hoc corrections of the local optimization direction; and iii) primal-dual-based methods.

- Gradient-tracking-based methods: In these schemes, each agent updates its own variables along a surrogate direction that tracks the gradient average $\nabla F$. This idea was proposed independently in the NEXT algorithm for Problem (P) and in AUG-DGM for strongly convex, smooth, unconstrained optimization. The works introduced SONATA, extending NEXT over (time-varying) digraphs. A convergence rate analysis of was later developed in , with considering also (time-varying) digraphs. Other algorithms based on the idea of gradient tracking and implementable over digraphs are ADD-OPT and . Subsequent schemes improved on earlier works along the following directions: i) the Push-Pull and the algorithms relaxed previous conditions on the mixing matrices used in the consensus and gradient tracking steps over digraphs, which neither need to be row- nor column-stochastic; and ii) and the $\mathcal{AB}m$ algorithm introduced acceleration, combining the idea of gradient tracking with Nesterov acceleration and the heavy-ball method, respectively, with and $\mathcal{AB}m$ applicable to digraphs.

Notice that all the above schemes but NEXT and SONATA are applicable only to smooth, unconstrained distributed optimization, with each $f_i$ strongly convex. In this setting, the schemes converge to the exact minimizer of $F$ at an R-linear rate while employing a fixed step-size. However, it is unknown whether linear rate is still achievable when the more general composite, constrained formulation is considered. Also, the assumption that each $f_i$’s is strongly convex (rather than just $F$) is quite restrictive; for instance, in several machine learning applications with data distributed across the agents, each $f_i$ is generally not strongly convex but so is $F$.

- Ad-hoc gradient correction-based methods: These methods developed specific corrections of the plain DGD direction. Specifically, EXTRA and its variant over digraphs, EXTRA-PUSH, introduce two different weight matrices for any two consecutive iterations, as opposed to a single weight matrix as well as leverage history of gradient information. They are applicable only to smooth, unconstrained problems; when each $f_i$ is strongly convex, they generate iterates that converge linearly
Table 2. Existing distributed methods for special nonconvex instances of Problem (P). SONATA is the only algorithm applicable to (P) with a provable convergence rate.

| Algorithms       | NEXT | [4] | P-Proc-PDA | P-Proc-PDA | DeFW [42] | SONATA |
|------------------|------|-----|------------|------------|-----------|--------|
| Problem \(\text{I}\) |      |     |            |            |           |        |
| \(G\) (nonsmooth) | ✓    | ✓   | ✓          | ✓          | ✓         |        |
| Constraints \(K\) | ✓    | ✓   | compact    | compact    |           | ✓      |
| Unbounded gradient | ✓    |     | ✓          |            |           |        |
| Network \(\text{II}\) |     |     |            |            |           |        |
| Time-varying digraph | ✓    | ✓   | ✓          |            |           |        |
| Convergence rate   | ✓    | ✓   | ✓          | ✓          | ✓         | ✓      |

to the minimizer of \(F\). To deal with an additive (possibly extended-valued) convex nonsmooth term in the objective, \cite{36} proposed PG-EXTRA, the proximal-gradient variant of EXTRA. PG-EXTRA is thus applicable to Problem (P) over undirected graphs. However, it is unknown whether it can achieve linear rate when \(f_i\)'s (or \(F\)) are strongly convex and \(G\) \(\neq 0\). A different approach to achieve exact linear convergence with fixed step-size is to use increasing number of consensus steps (linear increase with iteration number), as studied in \cite{1} for unconstrained minimization of smooth, strongly convex \(f_i\)'s over undirected graphs.

- **Primal-dual methods**: A common theme of these schemes is employing a primal-dual reformulation of the original multiagent problem whereby dual variables associated to a properly defined (augmented) Lagrangian function serve the purpose of correcting the plain DGD local direction. Examples of such algorithms include: i) distributed ADMM methods \cite{12, 37} and their inexact implementations \cite{13, 17, 19}; ii) distributed Augmented Lagrangian-based methods with randomized primal variable updates \cite{9}; iii) dual accelerated schemes \cite{30}, which apply the Nesterov’s acceleration gradient descent to the dual optimization problem formulated in \cite{9}; and iv) a distributed dual ascent method employing tracking of the average of the primal variable \cite{16}. Finally, it is worth mentioning that some of the primal methods discussed above have their equivalent primal-dual formulation; for instance, \cite{18} showed that EXTRA \cite{35} is a primal-dual gradient-like method while \cite{10} extended this primal-dual connection to the gradient-tracking-based scheme \cite{27}.

The primal-dual-based schemes discussed above are all applicable to smooth, unconstrained optimization over undirected graphs, with \cite{16} handling time-varying graphs. When all \(f_i\)'s are strongly convex, \cite{9, 16, 30, 37} are proved to achieve linear rate, with \cite{17} requiring only \(F\) being strongly convex. The extension of these methods to digraphs seems not straightforward, because it is not clear how to enforce consensus via constraints on directed networks.

To summarize, the above literature review shows that currently there exists no distributed algorithm for the general formulation (P) that provably converges at linear rate to the exact solution, when \(F\) (but not necessarily each \(f_i\)) is strongly convex and in the presence of a nonsmooth function \(G\) or constraints—see Table 1.

### 1.1.2. Nonconvex \(F\)

Distributed algorithms for nonconvex instances of Problem (P) are scarce; we group them in primal \cite{2, 4, 40, 42} and dual-based methods \cite{7, 8, 53}, and discuss their main features next—see also Table 2.

- **Primal methods**: The scheme in \cite{2} combines the distributed stochastic projection algorithm, employing a diminishing step-size, with the random gossip protocol. It can
handle smooth objective functions over undirected static graphs; the convergence rate of the scheme is unknown. In [40], the authors showed that the (randomly perturbed) push-sum gradient algorithm with diminishing step-size [20] converges also when applied to nonconvex smooth unconstrained problems. A sublinear convergence rate was proved (under the assumption that the set of stationary points of $U$ is finite). To our knowledge, the first distributed algorithm able to deal with the general formulation $\mathcal{P}$ over undirected (time-varying) graphs is NEXT [4]; convergence was proved but no rate analysis is available. In [42], the authors studied DeFW, a decentralization of the Frank-Wolfe algorithm coupled with the gradient tracking mechanism introduced in NEXT [4]; under a diminishing step-size (and further technical assumptions on the set of stationary solutions), a sublinear convergence rate is proved for the minimization of a smooth (possibly nonconvex) $U$ over undirected static graphs.

Notice that all the above algorithms require that the (sub)gradient of $U$ is bounded on $\mathcal{K}$ (or $\mathbb{R}^m$). This is a key assumption to prove convergence: in the analysis of descent, it permits to treat the optimization and consensus steps separately, with the consensus error being a summable perturbation.

- **Dual-based methods**: In [53] a distributed approximate dual subgradient algorithm, coupled with a consensus scheme (using double-stochastic weight matrices), is introduced to solve $\mathcal{P}$ over time-varying graphs. Assuming zero-duality gap, the algorithm is proved to asymptotically find a pair of primal-dual solutions of an auxiliary problem, which however might not be stationary for the original problem; also, consensus is not guaranteed. No rate analysis is provided. In [8], a proximal primal-dual algorithm is proposed to solve an unconstrained, smooth instance of $\mathcal{P}$ over undirected static graphs. The algorithm, termed Prox-PDA, employs either a constant or increasing penalty parameter (which plays the role of the step-size); a sublinear convergence rate of a suitably defined primal-dual gap is proved. A perturbed version of Prox-PDA, P-Prox-PDA, was introduced in [7], which is applicable to $\mathcal{P}$ over undirected, fixed graphs. P-Prox-PDA was proved to converge to an $\epsilon$-solution of $\mathcal{P}$ (and thus inexact consensus), under the assumptions that i) the subgradient of $G$ is bounded; ii) $\mathcal{K}$ is compact; and iii) the step-size and penalty parameter are chosen according to a suitable rule that depends on $\epsilon$. Sublinear rate is also proved.

In summary, the above literature review shows that there exists no distributed algorithm for the nonconvex formulation $\mathcal{P}$ over (undirected or directed, time-varying) graphs with provable convergence rate (or complexity analysis)–see Table 2.

### 1.2. Summary of the contributions

The paper provides the first linear convergence rate analysis of a distributed algorithm applicable to strongly convex composite, constrained optimization problems over (possibly directed, time-varying) graphs; and the first complexity result of the same algorithm when applied to nonconvex instances of the problem. This enlarges the class of optimization problems and network topology to which distributed algorithms with provable rates can be applied to, including, e.g., applications in machine learning, signal processing, and data analytic applications; see, e.g., [53]. On the technical side, our contributions are the following.

**Strongly convex $F$:** Our convergence proof improves upon existing primal and primal-dual based techniques, which fail proving linear rate in the presence of nonsmooth (convex) $G$ and/or constraints. The main difficulty in extending current primal-based techniques is establishing a tight connection between the (inexact) optimization
direction and some suitably defined optimality gap. On the other hand, convergence analyses based on primal-dual reformulations call for a bound between (suitably defined) primal and dual optimality gaps, which in the distributed setting is currently known only when the primal optimization step is smooth and unconstrained.

**Non convex** $F$: Our complexity analysis of SONATA leverages a Lyapunov-like function, which suitably combines the objective function $U$ evaluated on the agents’ average iterates with the consensus disagreement. These two terms alone do not “sufficiently” decrease along the iterates as local optimization and consensus steps might act as competing forces. The Lyapunov function is introduced just to show that a proper linear combination of these two terms is in fact monotonically decreasing over the iterations. Convergence of the Lyapunov function is the key step to establish the desired complexity bounds. This complements the convergence results in [34].

**Beyond gradient methods:** We remark that a further complication in our analysis is the use of surrogate functions in the local optimization problems, which replace the more classical first order approximation of the agents’ objective function. The use of such surrogate permits to better exploit some favorable structure in the objective functions—this is a common feature in several applications—leading to distributed schemes with faster practical convergence; see, e.g., [33].

1.3. **Paper organization**

Sec. 2 introduces the SONATA algorithm in the setting of undirected graphs–liner convergence for strongly convex functions $F$ is proved in Sec. 2.2, while the case of nonconvex $F$ is studied in Sec. 2.3. The case of time-varying, possibly directed, graphs is considered in Sec. 3.

2. **Distributed optimization over undirected graphs**

In this section we consider the case where agents solve Problem (P) over undirected fixed graphs. We made the following standard assumptions on Problem (P) and the graph topology.

**Assumption A** (On Problem (P)).

A1 The set $\emptyset \neq \mathcal{K} \subseteq \mathbb{R}^n$ is closed and convex;
A2 Each $f_i : \mathcal{O} \rightarrow \mathbb{R}$ is $C^1$, where $\mathcal{O} \supseteq \mathcal{K}$ is open; and $\nabla f_i$ is $L_i$-Lipschitz continuous on $\mathcal{K}$;
A3 $G : \mathcal{K} \rightarrow \mathbb{R}$ is convex possibly nonsmooth;
A4 $U$ is lower bounded on $\mathcal{K}$.

We will prove linear rate under the following extra assumption.

**Assumption B** (Strong convexity of $F$). $F$ is $\mu$-strongly convex on $\mathcal{K}$.

When $F$ is nonconvex, we cope with the nonconvexity leveraging SCA techniques [5, 31, 33]. We will use the concept of SCA surrogate, as defined next.

**Definition 2.1** (SCA surrogate). Given a $C^1$ function $f : \mathcal{C} \rightarrow \mathbb{R}$, with $\emptyset \neq \mathcal{C} \subseteq \mathbb{R}^n$ closed and convex. A function function $\tilde{f} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ is an $\tilde{L}$-smooth, $\tilde{\mu}$-strongly convex SCA surrogate of $f$ if $\tilde{f}$ is $C^1$ with respect to its first argument and satisfies the following conditions:

(i) $\nabla \tilde{f}(\mathbf{x}; \mathbf{x}) = \nabla f(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{C}$;
(ii) \( \nabla \tilde{f}(x; \bullet) \) is \( \tilde{L} \)-Lipschitz continuous;

(iii) \( \tilde{f}(\bullet; y) \) is \( \tilde{\mu} \)-strongly convex on \( K \);

where \( \nabla \tilde{f}(x; y) \) denotes the partial gradient of \( \tilde{f} \) with respect to the first argument, evaluated at \((x, y)\).

The communication network of the agent is modeled as a fixed (undirected) graph \( G \triangleq (V, E) \), with the vertex set \( V \triangleq \{1, \ldots, I\} \) and \( E \triangleq \{(i, j) \mid i, j \in V\} \) representing the set of agents and the communication links, respectively. Specifically, \((i, j) \in E\) iff there exists a communication link between agent \( i \) and \( j \). We make the following standard assumption on the graph connectivity.

**Assumption C** (On the network). The graph \( G \) is connected.

### 2.1. The SONATA/NEXT algorithm

We recall here the SONATA/NEXT algorithm \[4, 34\], customized to the above setting. Each agent \( i \) maintains and updates iteratively a local copy \( x_i \in \mathbb{R}^n \) of the global variable \( x \), along with the auxiliary variable \( y_i \in \mathbb{R}^n \), which is instrumental to locally estimate the gradient of \( F \). Denoting by \( x_i^\nu \) (resp. \( y_i^\nu \)) the values of \( x_i \) (resp. \( y_i \)) at iteration \( \nu \in \mathbb{N}_+ \), the update of these variables reads: for all \( i \in V \),

\[
\begin{align*}
\tilde{x}_i^\nu & \triangleq \arg\min_{x_i \in \mathcal{K}} \tilde{f}_i(x_i; x_i^\nu) + (I \cdot y_i^\nu - \nabla f_i(x_i^\nu))^\top (x_i - x_i^\nu) + G(x_i) \quad (1a) \\
x_i^{\nu+\frac{1}{2}} & = x_i^\nu + \alpha \cdot d_i^\nu, \quad d_i^\nu \triangleq \tilde{x}_i^\nu - x_i^\nu, \quad (1b) \\
x_i^{\nu+1} & = \sum_{j=1}^I w_{ij} x_j^{\nu+\frac{1}{2}}, \quad (1c) \\
y_i^{\nu+1} & = \sum_{j=1}^I w_{ij} (y_j^\nu + \nabla f_j(x_j^{\nu+1}) - \nabla f_j(x_j^\nu)), \quad (1d)
\end{align*}
\]

with initial conditions: \( x_i^0 \in \mathcal{K} \) (arbitrary) and \( y_i^0 = \nabla f_i(x_i^0) \), for all \( i = 1, \ldots, I \). In \((1a)\), \( \tilde{f}_i(\bullet; x_i^\nu) \) is chosen according to the following assumption.

**Assumption D.** Each \( \tilde{f}_i : \mathcal{K} \times \mathcal{K} \to \mathbb{R} \) is an \( \tilde{L}_i \)-smooth \( \tilde{\mu}_i \)-strongly convex SCA surrogate of \( f_i \) (cf. Definition \[2.1\]).

Roughly speaking, \( \tilde{f}_i \) is a convex approximation of \( f_i \) at the current iterate \( x_i^\nu \). Several examples of such approximations exploiting a potentially favorable structure in \( f_i \) are discussed in \[1, 31, 33, 34\]. The auxiliary variable \( y_i^\nu \) in \((1a)\) can be regarded as an estimate of the gradient average \((1/I) \nabla F(x_i^\nu)\) (see discussion below). Hence, the linear term in \((1a)\) is a proxy of the sum-gradient of the other agents’ functions.

After solving its own strongly convex problem \((1a)\), each agent \( i \) updates its own \( x_i^\nu \) along the local direction \( d_i^\nu \) (cf. \((1b)\)), using the step-size \( \alpha \in (0, 1] \) (to be properly chosen); the resulting point \( x_i^{\nu+1/2} \) is broadcast to its neighbors. The update \( x_i^{\nu+1/2} \to x_i^{\nu+1} \) is obtained via the consensus step \((1c)\), where the mixing weights are chosen according to the following standard assumption.

**Assumption E.** The weight matrix \( W \triangleq (w_{ij})_{i,j=1}^I \) has a sparsity pattern compliant
with $G$, that is

$E1$ $w_{ii} > 0$, for all $i = 1, \ldots, I$;

$E2$ $w_{ij} > 0$, if $(i, j) \in E$; and $w_{ij} = 0$ otherwise.

Furthermore, $W$ is doubly stochastic, that is, $1^T W = 1^T$ and $W1 = 1$.

Several rules have been proposed in the literature that satisfies Assumption $E$; examples include the Laplacian, the Metropolis-Hasting, and the maximum-degree weights rules $[46]$. Note that, since $W$ has the same sparsity pattern of the graph $G$, the consensus step (1c) is implementable in a distributed way, as each agent only needs to collect information from its immediate neighbors.

Finally the $y$-variables are updated via the perturbed consensus (1d), aiming at tracking $(1/I) \nabla F(x_i^\nu)$. To see this, sum (1d) over $i = 1, \ldots, I$ and invoke the doubly stochasticity of $W$; we have the following dynamics for the average process:

$$y_i^{\nu+1} = \bar{y}_i^\nu + \nabla f_i(x_i^{\nu+1}) - \nabla f_i(x_i^\nu),$$

where

$$\bar{y}_i^\nu \triangleq \frac{1}{I} \sum_{i=1}^I y_i^\nu \quad \text{and} \quad \nabla f_i(x_i^\nu) \triangleq \frac{1}{I} \sum_{i=1}^I \nabla f_i(x_i^\nu).$$

Applying the telescopic cancellation to (2) and using the initial condition $y_i^0 = \nabla f_i(x_i^0), i = 1, \ldots, I$, we have

$$\bar{y}_i^\nu = \nabla f_i(x_i^\nu), \quad \forall \nu \in \mathbb{N}_+.$$ (4)

That is, the average of all the $y_i^\nu$'s in the network is equal to that of the $\nabla f_i(x_i^\nu)$'s, at every iteration $\nu$. Assuming that consensus on $x_i^\nu$'s and $y_i^\nu$'s is asymptotically achieved, that is, $\|x_i^\nu - x_j^\nu\| \to 0$ and $\|y_i^\nu - y_j^\nu\| \to 0$, $i \neq j$, (4) would imply

$$\lim_{\nu \to \infty} \left\| \left( I \cdot y_i^\nu - \nabla f_i(x_i^\nu) \right) - \sum_{j \neq i} \nabla f_j(x_j^\nu) \right\| = 0,$$

which shows the desired tracking property employed by the $y$-variables.

Finally, notice that, as for the consensus step (1c), also the tracking update (1d) is implementable using only local information.

We conclude this section introducing some quantities that will be used in the rest of the paper. Whenever $F$ is assumed strongly convex, we denote by $x^*$ the unique solution of Problem (P), and define the optimality gap as

$$p^\nu \triangleq \sum_{i=1}^I \left( U(x_i^\nu) - U(x^*) \right).$$

Let $\bar{x}^\nu \triangleq (1/I) \cdot \sum_{i=1}^I x_i^\nu$; stacking the $x, y$-variables in the vectors $x^\nu \triangleq [x_1^\nu, \ldots, x_I^\nu]^T$ and $y^\nu \triangleq [y_1^\nu, \ldots, y_I^\nu]^T$, the consensus disagreements on $x_i^\nu$'s and $y_i^\nu$'s are

$$x_i^\perp \triangleq x_i^\nu - 1_I \otimes \bar{x}^\nu \quad \text{and} \quad y_i^\perp \triangleq y_i^\nu - 1_I \otimes \bar{y}_i^\nu,$$ (6)

respectively. Also, we introduce the gradient tracking error, defined as

$$\delta^\nu \triangleq [\delta_1^\nu, \ldots, \delta_I^\nu]^T, \quad \text{with} \quad \delta_i^\nu \triangleq \nabla F(x_i^\nu) - I \cdot y_i^\nu, \quad i = 1, \ldots, I.$$ (7)
Finally, recalling $L_i$, $\bar{L}_i$, and $\bar{\mu}_i$ as given in Assumptions [A] and [D] we introduce the following problem-dependent parameters

$$L_{\text{max}} \triangleq \max_{i=1,\ldots,l} L_i \quad \text{and} \quad L \triangleq \sum_{i=1}^{l} L_i$$

and algorithm-depended parameters

$$\bar{L}_{\text{max}} \triangleq \max_{i=1,\ldots,l} \bar{L}_i \quad \text{and} \quad \bar{\mu}_{\text{min}} \triangleq \min_{i=1,\ldots,l} \bar{\mu}_i.$$  \hspace{1cm} (9)

We study next the convergence rate of SONATA/NEXT, distinguishing the cases of strongly convex $F$ (cf. Sec. 2.2) and nonconvex $F$ (cf. Sec. 2.3).

2.2. Strongly convex $F$: Linear convergence rate

In this section, we consider Problem (P), under Assumptions [A] and [B]. Our proof of linear rate of SONATA/NEXT passes through the following steps. 

**Step 1:** We begin showing that the optimality gap $p^\nu$ converges linearly up to an error of the order of $O(\|x^\delta_i\|^2 + \|y^\nu_i\|^2)$, see Proposition 2.5. 

**Step 2** proves that $\|x^\nu_i\|$ and $\|y^\nu_i\|$ are also linearly convergent up to an error $O(\|d^\nu\|)$, see Proposition 2.6. 

**Step 3** we close the loop establishing $\|d^\nu\| = O(\sqrt{\nu} + \|y^\nu_i\|)$, see Proposition 2.7. 

Finally, in **Step 4**, we properly chain together the above inequality (cf. Proposition 2.9), so that linear rate is proved for the sequences $\{p^\nu\}_{\nu \in \mathbb{N}^+}$, $\{\|x^\nu_i\|^2\}_{\nu \in \mathbb{N}^+}$, $\{\|y^\nu_i\|^2\}_{\nu \in \mathbb{N}^+}$, and $\{|d^\nu|\|\}_{\nu \in \mathbb{N}^+}$—see Theorems 2.10 and 2.11.

2.2.1. Step 1: $p^\nu$ converges linearly up $O(\|x^\delta_i\|^2 + \|y^\nu_i\|^2)$

We begin regarding the local optimization (1a)-(1b) as a perturbed descent step on the objective function, whose perturbation is due to the tracking error $\delta^\nu_i$. Consider (1a); for notational simplicity define therein

$$
\bar{F}_i(x_i; x^\nu_i) \triangleq \bar{f}_i(x_i; x^\nu_i) + (I \cdot y^\nu_i - \nabla f_i(x^\nu_i))^\top (x_i - x^\nu_i),
$$

$$
\bar{U}_i(x_i; x^\nu_i) \triangleq \bar{F}_i(x_i; x^\nu_i) + G(x_i).
$$

Notice that i) $d^\nu_i$ is a descent direction of $\bar{U}(\bullet; x^\nu_i)$ at $x^\nu_i$; and ii) if the tracking error $\delta^\nu_i$ (cf. (7)) was zero, $\bar{F}_i$ would be a SCA surrogate of $F$ (cf. Defintion 2.1). These two facts suggest that, for sufficiently small $\alpha$, the local update (1b) will decrease the objective value $U$ up to some error, related to the $\delta^\nu_i$. This is made formal below.

**Lemma 2.2.** Consider Problem (P) under Assumption [A] and the SONATA/NEXT algorithm (1a)-(1d), under Assumptions [B] and [D]. Then, there holds

$$U(x^\nu_i + \hat{\delta}_i) \leq U(x^\nu_i) - \alpha \left(\bar{\mu}_i - \alpha \frac{L}{2}\right) \|d^\nu_i\|^2 + \alpha \|d^\nu_i\| \|\delta^\nu_i\|,$$

with $\delta^\nu_i$ and $L$ defined in (7) and (8), respectively.

**Proof.** By the first order optimality condition of $\hat{x}^\nu_i$:

$$(-d^\nu_i)^\top (\nabla \bar{f}_i(\hat{x}^\nu_i; x^\nu_i) + I \cdot y^\nu_i - \nabla f_i(x^\nu_i)) + G(x^\nu_i) - G(\hat{x}^\nu_i) \geq 0,$$
and properties (i) and (iii) in Definition 2.1, it follows
\[
(d_i^{\nu'})^T (I \cdot y_i^{\nu'}) \leq -\bar{\mu}_i \|d_i^{\nu'}\|^2 + G(x_i^{\nu'}) - G(\hat{x}_i^{\nu'}). \tag{13}
\]

Applying the descent lemma to $F$ and using (13), yields
\[
F(x_i^{\nu' + \frac{1}{2}}) \leq F(x_i^{\nu'}) + (\nabla F(x_i^{\nu'}) \pm (I \cdot y_i^{\nu'}))^T (\alpha d_i^{\nu'}) + \frac{L}{2} \|\alpha d_i^{\nu'}\|^2
\leq F(x_i^{\nu'}) + \alpha \|d_i^{\nu'}\| \|\delta_i^{\nu'}\| - \alpha \bar{\mu}_i \|d_i^{\nu'}\|^2 + \frac{L}{2} \alpha^2 \|d_i^{\nu'}\|^2 + \alpha G(x_i^{\nu'}) - \alpha G(\hat{x}_i^{\nu'})
\leq F(x_i^{\nu'}) + \alpha \|d_i^{\nu'}\| \|\delta_i^{\nu'}\| - \alpha \bar{\mu}_i \|d_i^{\nu'}\|^2 + \frac{L}{2} \alpha^2 \|d_i^{\nu'}\|^2 + G(x_i^{\nu'}) - G(x_i^{\nu' + \frac{1}{2}})
\]
where in the last inequality we used $G(x_i^{\nu' + \frac{1}{2}}) \leq (1 - \alpha) G(x_i^{\nu'}) + \alpha G(\hat{x}_i^{\nu'})$, due to the convexity of $G$. This proves (12).

As next step, let us connect the individual decreases in (12) with that of the optimality gap $p^{\nu}$, defined in (5). Notice that
\[
\sum_{i=1}^{I} U(x_i^{\nu'+1}) \leq \sum_{i=1}^{I} \sum_{j=1}^{I} w_{ij} U \left(x_j^{\nu' + \frac{1}{2}} \right) = \sum_{i=1}^{I} U(x_i^{\nu' + \frac{1}{2}}), \tag{14}
\]
due to the convexity of $U$ and doubly stochasticity of $W$. Summing (12) over $i = 1, \ldots, I$ and using (14), we obtain
\[
p^{\nu+1} \leq p^{\nu} + \sum_{i=1}^{I} \left\{ \alpha \|d_i^{\nu'}\| \|\delta_i^{\nu'}\| - \alpha \bar{\mu}_i \|d_i^{\nu'}\|^2 + \frac{L}{2} \alpha^2 \|d_i^{\nu'}\|^2 \right\}
\overset{(a)}{\leq} p^{\nu} - \left( \bar{\mu}_{\min} - \frac{\alpha L}{2} - \frac{1}{2} \epsilon_{opt} \right) \alpha \|d^{\nu'}\|^2 + \frac{1}{2} \epsilon_{opt} \alpha \|\delta^{\nu'}\|^2,
\tag{15}
\]
where in (a) we used the Young’s inequality, with $\epsilon_{opt} > 0$ satisfying
\[
\bar{\mu}_{\min} - \frac{\alpha L}{2} - \frac{1}{2} \epsilon_{opt} > 0. \tag{16}
\]

We now lower bound $\|d^{\nu'}\|$ by some quantity related to the optimality gap.

**Lemma 2.3.** In the setting of Lemma 2.2 and under Assumption 1.2, there holds:
\[
\alpha \|d^{\nu'}\|^2 \geq \frac{\mu}{6(L^2 + F^2_{\max})} \left( p^{\nu+1} - (1 - \alpha)p^{\nu} - \frac{\alpha}{\mu} \|\delta^{\nu'}\|^2 \right). \tag{17}
\]

**Proof.** Invoking the optimality condition of $\hat{x}_i^{\nu'}$, yields
\[
G(x^*) - G(\hat{x}_i^{\nu'}) \geq -(x^* - \hat{x}_i^{\nu'})^T \left( \nabla f_i(\hat{x}_i^{\nu'}; x_i^{\nu'}) + I \cdot y_i^{\nu'} - \nabla f_i(x_i^{\nu'}) \right), \tag{18}
\]
Using the $\mu$-strong convexity of $F$, we can write

\[
\begin{align*}
U(x^*) &\geq U(\hat{x}^\nu) + G(x^*) - G(\hat{x}^\nu) + \nabla F(\hat{x}^\nu)^\top (x^* - \hat{x}^\nu) + \frac{\mu}{2} \|x^* - \hat{x}^\nu\|^2 \\
&\geq U(\hat{x}^\nu) + \left(\nabla F(\hat{x}^\nu) - \nabla \bar{f}_i(\hat{x}^\nu; \tilde{x}^\nu_i) - (I \cdot y_i^\nu - \nabla f_i(x_i^\nu))\right)^\top (x^* - \hat{x}^\nu) + \frac{\mu}{2} \|x^* - \hat{x}^\nu\|^2 \\
&= U(\hat{x}^\nu) + \frac{\mu}{2} \|x^* - \hat{x}^\nu\|^2 + \frac{1}{\mu} \left(\nabla F(\hat{x}^\nu) - \nabla \bar{f}_i(\hat{x}^\nu; \tilde{x}^\nu_i) - (I \cdot y_i^\nu - \nabla f_i(x_i^\nu))\right)^2 \\
&\quad - \frac{1}{2\mu} \|(\nabla F(\hat{x}^\nu) - \nabla \bar{f}_i(\hat{x}^\nu; \tilde{x}^\nu_i) - (I \cdot y_i^\nu - \nabla f_i(x_i^\nu))\|^2 \\
&\geq U(\hat{x}^\nu) - \frac{1}{2\mu} \|\nabla F(\hat{x}^\nu) - \nabla F(\hat{x}^\nu) + \nabla f_i(x_i^\nu) - \nabla f_i(x_i^\nu)\|^2 - \frac{1}{\mu} \|\delta_i^\nu\|^2 \\
&\geq U(\hat{x}^\nu) - \frac{6}{\mu} \left(L^2 + \bar{L}^2\right) \|d_i^\nu\|^2 - \frac{1}{\mu} \|\delta_i^\nu\|^2.
\end{align*}
\]

Rearranging the terms and summing over $i = 1, \ldots, I$, yields

\[
\|d^\nu\|^2 \geq \frac{\mu}{6(I^2 + \bar{L}^2)_{\text{max}}} \left(\sum_{i=1}^I (U(\hat{x}^\nu_i) - U(x^*)) - \frac{1}{\mu} \|\delta_i^\nu\|^2\right). \tag{19}
\]

Using (14) in conjunction with $U(x_i^{\nu + \frac{1}{2}}) \leq \alpha U(\hat{x}^\nu_i) + (1 - \alpha)U(x_i^\nu)$ leads to the following lower bound

\[
\alpha \sum_{i=1}^I (U(\hat{x}^\nu_i) - U(x^*)) \geq p^{\nu+1} - (1 - \alpha)p^\nu. \tag{20}
\]

Combining (19) with (20) yields the desired result (17).

As last step, we upper bound $\|\delta_i^\nu\|^2$ in (15) in terms of the consensus errors $\|x_i^\nu\|^2$ and $\|y_i^\nu\|^2$.

**Lemma 2.4.** The total tracking error $\|\delta_i^\nu\|^2$ can be bounded as

\[
\|\delta_i^\nu\|^2 \leq 4I^2L_{\text{max}}^2 \|x_i^\nu\|^2 + 2I^2 \|y_i^\nu\|^2, \tag{21}
\]

where $L_{\text{max}}$ is defined in (8).
Proof.

\[
\|\delta^\nu\|^2 \equiv \sum_{i=1}^I \| \nabla F(x^\nu_i) \pm I \cdot \gamma^\nu_i - I \cdot y^\nu_i \|^2 \\
\equiv \sum_{i=1}^I \left\| \sum_{j=1}^I \nabla f_j(x^\nu_i) - \sum_{j=1}^I \nabla f_j(x^\nu_i) + I \cdot \gamma^\nu_i - I \cdot y^\nu_i \right\|^2 \\
\leq \sum_{i=1}^I \left( 2I \sum_{j=1}^I L^2_{\text{mx}} \| x^\nu_i - x^\nu_j \|^2 + 2I^2 \| \gamma^\nu_i - y^\nu_i \|^2 \right) = 4I^2 L^2_{\text{mx}} \| x^\nu_i \|^2 + 2I^2 \| y^\nu_i \|^2.
\]

We are ready to prove the linear convergence of the optimality gap up to consensus errors. The results is summarized in Proposition 2.5 below. The proof follows readily multiplying (15) and (17) by \( \tilde{\mu}_{\text{min}} - \frac{\mu}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}} \) and \( 6(L^2_2 + L^2_{\text{mx}}) / \mu \), respectively, adding them together to cancel out \( \| d^\nu \| \), and using (21) to bound \( \| \delta^\nu \|^2 \).

**Proposition 2.5.** Under Assumptions [4, 7, 9] \( p^\nu \) [cf. (5)] satisfies

\[
p^{\nu+1} \leq \sigma(\alpha) \cdot p^\nu + \eta(\alpha) \cdot (4I^2 L^2_{\text{mx}} \| x^\nu \|^2 + 2I^2 \| y^\nu \|^2),
\]

where \( \sigma(\alpha) \in (0, 1) \) and \( \eta(\alpha) > 0 \) are coefficients given by

\[
\sigma(\alpha) \equiv 1 - \alpha \cdot \frac{\bar{\mu}_{\text{min}} - \frac{\mu}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}}}{6(L^2_2 + L^2_{\text{mx}}) / \mu} + (\bar{\mu}_{\text{min}} - \frac{\mu}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}}),
\]

\[
\eta(\alpha) \equiv \frac{\mu}{2} \epsilon_{\text{opt}} \alpha \cdot \frac{6(L^2_2 + L^2_{\text{mx}}) / \mu}{\mu} + \frac{6(L^2_2 + L^2_{\text{mx}}) / \mu}{\mu} - \frac{\mu}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}}.
\]

with \( \epsilon_{\text{opt}} \) satisfying (16); and \( L, L_{\text{mx}} \) and \( \bar{\mu}_{\text{min}} \) defined in (3) and (4).

2.2.2. Step 2: \( \| x^\nu \| \) and \( \| y^\nu \| \) linearly converge up to \( \mathcal{O}(\| d^\nu \|) \)

We upper bound \( \| x^\nu \| \) and \( \| y^\nu \| \) in terms of \( \| d^\nu \| \). Introducing

\[
\hat{W} \triangleq W \otimes I_n, \quad \mathbf{J} \triangleq \frac{1}{I} \mathbf{I}_I \otimes I_n, \quad \nabla f^\nu \triangleq [\nabla f_1(x^\nu_1)^T, \ldots, \nabla f_I(x^\nu_I)^T]^T,
\]

the SONATA/NEXT algorithm (1a)-(1d) can be written in compact form as

\[
x^{\nu+1} = \hat{W}(x^\nu + \alpha d^\nu) \tag{26a}
\]

\[
y^{\nu+1} = \hat{W}(y^\nu + \nabla f^{\nu+1} - \nabla f^\nu). \tag{26b}
\]

Noting that \( x^\nu_i \equiv (I - J)x^\nu \) [similarly, \( y^\nu_i \equiv (I - J)y^\nu \) and \( (I - J)\hat{W} = \hat{W} - J \) (due to the doubly stochasticity of \( W \)), it follows from (26) that

\[
x_i^{\nu+1} = (\hat{W} - J)(x^\nu_i + \alpha d^\nu) \tag{27}
\]

\[
y_i^{\nu+1} = (\hat{W} - J)(y^\nu_i + \nabla f^{\nu+1} - \nabla f^\nu). \tag{28}
\]
Under Assumptions [C] and [E] it is well known that (see, e.g., [41])
\[ \rho \triangleq \sigma(\hat{W} - J) < 1, \]  
(29)
where \( \sigma(\bullet) \) denotes the largest singular value of its argument. Using (27)-(28), Proposition 2.6 below establishes linear convergence of the consensus errors \( x^\nu_\perp \) and \( y^\nu_\perp \), up to a perturbation.

Proposition 2.6. Under Assumptions [A], [C] and [E], there holds:
\[
\|x^{\nu+1}_\perp\| \leq \rho \|x^\nu_\perp\| + \alpha \rho \|d^\nu\|, \tag{30a}
\]
\[
\|y^{\nu+1}_\perp\| \leq \rho \|y^\nu_\perp\| + 2L_{mx}\rho \|x^\nu_\perp\| + \alpha L_{mx} \rho \|d^\nu\|, \tag{30b}
\]
with \( \rho \) and \( L_{mx} \) defined in (29) and (8), respectively.

Proof. We prove next (30b); (30a) follows readily from (27). Using (26a), (28), and the Lipschitz continuity of \( \nabla f_i \) (Assumption [A]), we can bound \( \|y^{\nu+1}_\perp\| \) as
\[
\|y^{\nu+1}_\perp\| \leq \rho \|y^\nu_\perp\| + \rho \|\nabla f^{\nu+1}_i - \nabla f^\nu_i\| \\
\leq \rho \|y^\nu_\perp\| + L_{mx}\rho \|\hat{W} - I\| x^\nu_\perp \\
= (\hat{W} - I) x^\nu_\perp  \\
\leq \rho \|y^\nu_\perp\| + 2L_{mx}\rho \|x^\nu_\perp\| + \alpha L_{mx} \rho \|d^\nu\|,
\]
where in the last inequality we used \( \|W\| \leq 1 \).

2.2.3. Step 3: \( \|d^\nu\| = O(\sqrt{r^\nu} + \|y^\nu_\perp\|) \) (closing the loop)

Given the inequalities in Propositions [2.5] and [2.6] to close the loop, one needs to link \( \|d^\nu\| \) to the quantities in the aforementioned inequalities, which is done next.

Proposition 2.7. Under Assumptions [A], [E], \( \|d^\nu\| \) can be bounded as
\[
\|d^\nu\| \leq \sqrt{\frac{2I}{\mu} \left( \frac{L_{mx}}{\mu_{\min}} + \frac{IL_{mx}}{\bar{p}_{\min}} + 1 \right) \sqrt{p^\nu} + \frac{I}{\bar{p}_{\min}} \|y^\nu_\perp\|}, \tag{31}
\]
where \( L_{mx}, \bar{L}_{mx} \) and \( \bar{p}_{\min} \) are defined in (8) and (9).

Proof. We begin leveraging the \( \bar{\mu}_1 \)-strong convexity of \( \tilde{U}_i^\nu \) [cf. (11)]. We have
\[
\tilde{U}_i(\tilde{x}_i^\nu; x_i^\nu) \geq \tilde{U}_i(x^*; x_i^\nu) + \nabla F(x^*; x_i^\nu)^T (\tilde{x}_i^\nu - x^*) + \frac{\bar{\mu}_1}{2} \|\tilde{x}_i^\nu - x^*\|^2 + G(\tilde{x}_i^\nu) - G(x^*)
\]
\[ \begin{aligned}
&\geq \tilde{U}_i(x^*; x_i^*) + \left( \nabla \tilde{F}(x^*; x_i^*) - \nabla F(x^*) \right) \top (\tilde{x}_i^* - x^*) + \frac{\tilde{\mu}_i}{2} \| \tilde{x}_i^* - x^* \|^2 \\
&\geq \tilde{U}_i(x^*; x_i^*) - \| \nabla \tilde{f}_i(x^*; x_i^*) + I \cdot y_i^\nu - \nabla f_i(x_i^*) - \nabla F(x^*) \| \cdot \| \tilde{x}_i^* - x^* \|^2 \\
&\quad + \frac{\tilde{\mu}_i}{2} \| \tilde{x}_i^* - x^* \|^2 \end{aligned} \]
\[ \geq \tilde{U}_i(x^*; x_i^*) + \frac{\tilde{\mu}_i}{2} \| \tilde{x}_i^* - x^* \|^2 - \| I \cdot \tilde{y}^\nu - I \cdot y_i^\nu \| \cdot \| \tilde{x}_i^* - x^* \| \]
\[ - \| \nabla \tilde{f}_i(x^*; x_i^*) - \nabla f_i(x_i^*) \| \cdot \| \tilde{x}_i^* - x^* \| - \| I \cdot \tilde{y}^\nu - I \cdot y_i^\nu \| \cdot \| \tilde{x}_i^* - x^* \| \]
\[ \geq \tilde{U}_i(x^*; x_i^*) + \frac{\tilde{\mu}_i}{2} \| \tilde{x}_i^* - x^* \|^2 - \sum_{j \neq i} L_j \| x_j^\nu - x^* \| \cdot \| \tilde{x}_i^* - x^* \| \]
\[ - \tilde{L}_i \| x_i^\nu - x^* \| \cdot \| \tilde{x}_i^* - x^* \| - \| I \cdot \tilde{y}^\nu - I \cdot y_i^\nu \| \cdot \| \tilde{x}_i^* - x^* \|, \]
\end{aligned} \]

where in (a) we used \(G(\tilde{x}_i^\nu) - G(x^*) \geq - \nabla F(x^*) \top (\tilde{x}_i^\nu - x^*)\).

Since \(\tilde{U}_i^\nu\) is \(\tilde{\mu}_i\)-strongly convex and \(\tilde{x}_i^\nu\) is its unique minimizer, there holds
\[ \tilde{U}_i(x^*; x_i^*) \geq \tilde{U}_i(\tilde{x}_i^\nu; x_i^*) + \frac{\tilde{\mu}_i}{2} \| \tilde{x}_i^\nu - x^* \|^2. \]

(33)

Putting together (32) and (33) and using the reverse triangle inequality, yields
\[ \tilde{L}_i \| x_i^\nu - x^* \| + \sum_{j=1}^I L_j \| x_j^\nu - x^* \| + I \cdot \| \tilde{y}^\nu - y_i^\nu \| \]
\[ \geq \tilde{\mu}_i \| \tilde{x}_i^\nu - x^* \| \geq \tilde{\mu}_i (\| d_i^\nu \| - \| x_i^\nu - x^* \|). \]

(34)

Rearranging terms and summing over \(i = 1, \ldots, I\), we can bound \(\| d^\nu \|\) as
\[ \| d^\nu \| \leq \sum_{i=1}^I \left( \left( \frac{\tilde{L}_i}{\mu_i} + 1 \right) \| x_i^\nu - x^* \| + \frac{1}{\tilde{\mu}_i} \left( \sum_{j=1}^I L_j \| x_j^\nu - x^* \| \right) + \frac{I}{\tilde{\mu}_i} \| \tilde{y}^\nu - y_i^\nu \| \right) \]
\[ \leq \sum_{i=1}^I \left( \left( \frac{\tilde{L}_i}{\mu_i} + 1 \right) \sqrt{\frac{\tilde{T}}{\mu}} \cdot \sqrt{p_i^\nu} + \frac{1}{\tilde{\mu}_i} \left( \sum_{j=1}^I L_j \sqrt{\frac{\tilde{T}}{\mu}} \cdot \sqrt{p_j^\nu} \right) + \frac{I}{\tilde{\mu}_i} \sqrt{T} \| y_i^\nu \| \right) \]
\[ \leq \sum_{i=1}^I \left( \left( \frac{\tilde{L}_i}{\mu_{\min}} + 1 \right) \sqrt{\frac{\tilde{T}}{\mu}} \cdot \sqrt{p_i^\nu} + \frac{1}{\mu_{\min}} \left( \sum_{j=1}^I L_j \sqrt{\frac{\tilde{T}}{\mu}} \cdot \sqrt{p_j^\nu} \right) + \frac{I}{\mu_{\min}} \sqrt{T} \| y_i^\nu \| \right) \]
\[ \leq \sqrt{\frac{\tilde{T}}{\mu}} \left( \frac{\tilde{I}_{\text{max}}}{\mu_{\min}} + 1 \right) \sqrt{p^\nu} + \frac{1}{\mu_{\min}} \sqrt{\frac{\tilde{T}}{\mu}} \sqrt{I L_{\text{max}} \sqrt{T} \| y^\nu \|} + \frac{I}{\mu_{\min}} \| y^\nu \| \]
\[ \leq \sqrt{\frac{2 I}{\mu}} \left( \frac{\tilde{I}_{\text{max}}}{\mu_{\min}} + \frac{I L_{\text{max}}}{\mu_{\min}} + 1 \right) \sqrt{p^\nu} + \frac{I}{\mu_{\min}} \| y^\nu \|. \]
2.2.4. Step 4: Proof of the linear rate (chaining the inequalities)

We are now ready to prove linear rate of the SONATA/NEXT algorithm. We build on the following intermediate result, introduced in [24].

**Lemma 2.8.** Given the sequence \((s^\nu)_{\nu \in \mathbb{N}_+}\), define the transformations

\[
S^K(z) \triangleq \max_{\nu=0, \ldots, K} |s^\nu| z^{-\nu} \quad \text{and} \quad S(z) \triangleq \sup_{\nu \in \mathbb{N}} |s^\nu| z^{-\nu},
\]

for \(z \in (0, 1)\). If \(S(z)\) is bounded, then \(|s^\nu| = O(z^\nu)\).

We show next how to chain the inequalities \([22], [30] \text{ and } [31]\) so that Lemma 2.8 can be applied to the sequences \(\{p^\nu\}_{\nu \in \mathbb{N}_+}, \{\|x^\nu\| \} \_{\nu \in \mathbb{N}_+}, \{\|y^\nu\| \} \_{\nu \in \mathbb{N}_+}, \{\|z^\nu\| \} \_{\nu \in \mathbb{N}_+}, \text{ and } \{\|d^\nu\|^2 \} \_{\nu \in \mathbb{N}_+}\), establishing thus their linear convergence.

**Proposition 2.9.** Let \(P^K(z), X^K_\perp(z), Y^K_\perp(z) \text{ and } D^K(z)\) denote the transformation \((36)\) applied to the sequences \(\{p^\nu\}_{\nu \in \mathbb{N}_+}, \{\|x^\nu\| \} \_{\nu \in \mathbb{N}_+}, \{\|y^\nu\| \} \_{\nu \in \mathbb{N}_+}, \{\|z^\nu\| \} \_{\nu \in \mathbb{N}_+}, \text{ and } \{\|d^\nu\|^2 \} \_{\nu \in \mathbb{N}_+}\), respectively. Given the constants \(\sigma(\alpha)\) and \(\eta(\alpha)\) defined in Proposition 2.5 and the free parameters \(\epsilon_x, \epsilon_y > 0\), the following holds

\[
P^K(z) \leq G_P(\alpha, z) \cdot (4I^2 L^2_{mx} X^K_\perp(z) + 2I^2 Y^K_\perp(z)) + \omega_p,
\]

\[
X^K_\perp(z) \leq G_X(z) \cdot \rho^2 \alpha^2 D^K(z) + \omega_x,
\]

\[
Y^K_\perp(z) \leq G_Y(z) \cdot 8I^2 L^2_{mx} \\rho^2 X^K_\perp(z) + G_Y(z) \cdot 2I^2 L^2_{mx} \rho^2 \alpha^2 D^K(z) + \omega_y,
\]

\[
D^K(z) \leq C_1 \cdot P^K(z) + C_2 \cdot Y^K_\perp(z),
\]

for all

\[
z \in \left(\max\{\sigma(\alpha), \rho^2(1 + \epsilon_x), \rho^2(1 + \epsilon_y)\}, 1\right),
\]

where

\[
G_P(\alpha, z) \triangleq \frac{\eta(\alpha)}{z - \sigma(\alpha)}, \quad \omega_p \triangleq \frac{z}{z - \sigma(\alpha)} \cdot p^0,
\]

\[
G_X(z) \triangleq \frac{(1 + \epsilon_x^{-1})}{z - \rho^2(1 + \epsilon_x)}, \quad \omega_x \triangleq \frac{z}{z - \rho^2(1 + \epsilon_x)} \cdot \|x^0\|^2,
\]

\[
G_Y(z) \triangleq \frac{(1 + \epsilon_y^{-1})}{z - \rho^2(1 + \epsilon_y)}, \quad \omega_y \triangleq \frac{z}{z - \rho^2(1 + \epsilon_y)} \cdot \|y^0\|^2,
\]

\[
C_1 \triangleq \frac{4I^2}{\mu} \left(\frac{\bar{L}_{mx}}{\mu_{\min}} + \frac{I L_{mx}}{\mu_{\min}} + 1\right)^2, \quad C_2 \triangleq 2 \left(\frac{I \sqrt{T}}{\mu_{\min}}\right)^2.
\]

**Proof.** Squaring \((30)\) and using the Young’s inequality yield

\[
\|x^{\nu+1}_\perp\|^2 \leq \rho^2 (1 + \epsilon_x) \|x^\nu\|^2 + \rho^2 (1 + \epsilon_x^{-1}) \alpha^2 \|d^\nu\|^2,
\]

\[
\|y^{\nu+1}_\perp\|^2 \leq \rho^2 (1 + \epsilon_y) \|y^\nu\|^2 + \rho^2 (1 + \epsilon_y^{-1}) \left(8L^2_{mx} \|x^\nu_\perp\|^2 + 2\alpha^2 L^2_{mx} \|d^\nu\|^2\right),
\]

where \(\epsilon_x\) and \(\epsilon_y\) are positive constants. The proof is completed by taking the maximum of both sides of the inequalities \([22], [31], \text{ and } [10]\) over \(\nu = 0, \ldots, K\) and using the fact
Figure 1. Chain of the inequalities in Proposition 2.9 leading to (11).

The proof is organized in two steps, namely:

1) We first consider the “marginal” stable case by letting $z = 1$, and show that there exists some $\tilde{\alpha} > 0$ so that $\mathcal{P}(\alpha, 1) < 1$, for all $\alpha \in (0, \tilde{\alpha})$; 2) Then, invoking the continuity of $\mathcal{P}(\alpha, z)$, we argue that, for any $\alpha \in (0, \tilde{\alpha})$, one can find some $\tilde{\epsilon}(\alpha) < 1$ such that $\mathcal{P}(\alpha, z(\alpha)) < 1$. This implies the boundedness of $D^K(\bar{z}(\alpha))$, and thus $\|d^\nu\|^2 = O(\tilde{z}(\alpha)^\nu)$ (cf. Lemma 2.8).

Proof. Let $\{s^\nu\}_{\nu \in \mathbb{N}}$ be a sequence such that $\max_{\nu = 0, \ldots, K} s^{\nu + 1} z^{-\nu} \geq z \cdot \max_{\nu = 0, \ldots, K} s^{\nu} z^{-\nu} - z \cdot s^0$.

Chaining the inequalities in Proposition 2.9 in the way shown in Fig. 2.2.4, we can bound $D^K(z)$ as (see Appendix A for the proof)

$$D^K(z) \leq \mathcal{P}(\alpha, z) \cdot D^K(z) + \mathcal{R}(\alpha, z)$$

(41)

where $\mathcal{P}(\alpha, z)$ is defined as

$$\mathcal{P}(\alpha, z) \triangleq G_P(\alpha, z) \cdot G_X(z) \cdot C_1 \cdot 4I^2L^2_{mx} \cdot \rho^2 \cdot \alpha^2 + (G_P(\alpha, z) \cdot 2I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 2L^2_{mx}\rho^2 \cdot \alpha^2 + (G_P(\alpha, z) \cdot 2I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 8L^2_{mx}\rho^2 \cdot G_X(z) \cdot \rho^2 \cdot \alpha^2.$$ 

and $\mathcal{R}(\alpha, z)$ is a remainder, which is bounded under (38).

Therefore, as long as $\mathcal{P}(\alpha, z) < 1$, (41) implies

$$D^K(z) \leq \frac{\mathcal{R}(\alpha, z)}{1 - \mathcal{P}(\alpha, z)} \leq B < +\infty$$

(43)

with $B$ being a constant independent of $K$. Letting $K \to \infty$ we have $D(z) \leq B$ and thus $\{\|d^\nu\|^2\}_{\nu \in \mathbb{N}}$ converges R-linearly (cf. Lemma 2.8). Applying the same argument to the other inequalities in Proposition 2.9, one can conclude that also the sequences $\{p^\nu\}_{\nu \in \mathbb{N}}$, $\{\|x^\nu\|^2\}_{\nu \in \mathbb{N}}$ and $\{\|y^\nu\|^2\}_{\nu \in \mathbb{N}}$ converge R-linearly to zero.

The last step is then to show that there exist a sufficiently small step-size $\alpha \in (0, 1]$ and $z \in (0, 1)$ satisfying (38), such that $\mathcal{P}(\alpha, z) < 1$. This is proved in the Theorem 2.10 below. Recall therein the definition of problem parameters $L_i, L_{mx}, L$, and $\mu$ as given in Assumptions A, B and C; the algorithm parameters $\tilde{L}_i, L_{mx}, \tilde{\mu}_i, \tilde{\mu}_{min}$ as given in Assumption D and E; and network parameter $\rho$, defined in (29).

**Theorem 2.10.** Consider Problem (P) under Assumptions A-C and the SONATA/NEXT algorithm (1a)-(1d), under Assumptions D and E. Then, there exists a sufficiently small step-size $\tilde{\alpha} \in (0, 1]$ [see the proof for its expression] such that for all $\alpha < \tilde{\alpha}$, $\{U(x^\nu_i)\}_{\nu \in \mathbb{N}}$ converges to $U^*$ at an R-linear rate, $i = 1, \ldots, I$.

**Proof.** The proof is organized in two steps, namely: 1) We first consider the “marginal” stable case by letting $z = 1$, and show that there exists some $\tilde{\alpha} > 0$ so that $\mathcal{P}(\alpha, 1) < 1$, for all $\alpha \in (0, \tilde{\alpha})$; 2) Then, invoking the continuity of $\mathcal{P}(\alpha, z)$, we argue that, for any $\alpha \in (0, \tilde{\alpha})$, one can find some $\tilde{\epsilon}(\alpha) < 1$ such that $\mathcal{P}(\alpha, z(\alpha)) < 1$. This implies the boundedness of $D^K(\bar{z}(\alpha))$, and thus $\|d^\nu\|^2 = O(\tilde{z}(\alpha)^\nu)$ (cf. Lemma 2.8).

1) We begin optimizing the free parameters $\epsilon_x, \epsilon_y$, and $\epsilon_{opt}$. Since the goal is to find the largest $\tilde{\alpha}$ so that $\mathcal{P}(\alpha, 1) < 1$, for all $\alpha \in (0, \tilde{\alpha})$, the optimal choice of $\epsilon_x, \epsilon_y$, and
\[ \epsilon_\text{opt} \] is the one that minimizes \( \mathcal{P}(\alpha, 1) \), that is,

\[
\epsilon^* = \arg\min_{\epsilon > 0} \frac{1 + \epsilon^{-1}}{1 - \rho^2(1 + \epsilon)} = \frac{1 - \rho}{\rho}. \quad (44)
\]

We then set \( \epsilon_x = \epsilon_y = \epsilon^* \), and proceed to optimize \( \epsilon_\text{opt} \), which appears in \( \eta(\alpha) \) and \( \sigma(\alpha) \). Recalling the definition of \( \eta(\alpha) \) and \( \sigma(\alpha) \) (cf. Proposition 2.5) and the constraint (16), the problem boils down to minimize

\[
G_P(\alpha, 1) = \frac{\eta(\alpha)}{1 - \sigma(\alpha)} = \frac{1}{2} \epsilon_\text{opt}^{-1} \cdot \frac{\mu_\text{opt} + \frac{L}{\mu_\text{opt}}}{\mu_\text{min} - \frac{L}{2} \alpha - \frac{\epsilon_\text{opt}}{2}},
\]

subject to \( \epsilon_\text{opt} \in (0, 2\mu_\text{min} - \alpha L) \). Note that, in order to have a nonempty feasible set, we require \( \alpha < 2\mu_\text{min}/L \). Setting the derivative of \( G_P(\alpha, 1) \) with respect to \( \epsilon_\text{opt} \) to zero, gives \( \epsilon_\text{opt}^* = \mu_\text{min} - \alpha \cdot L/2 \), which is strictly feasible, and thus the solution.

Let \( \mathcal{P}^*(\alpha, z) \) denote the value of \( \mathcal{P}(\alpha, z) \) corresponding to the optimal choice of the above parameters. The expression of \( \mathcal{P}^*(\alpha, 1) \) reads

\[
\mathcal{P}^*(\alpha, 1) \triangleq G_P^*(\alpha) \cdot C_1 \cdot 4I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^2}{(1 - \rho)^2} \cdot \alpha^2
\]

\[
+ \left( G_P^*(\alpha) \cdot 2I^2 \cdot C_1 + C_2 \right) \cdot 2I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^2}{(1 - \rho)^2} \cdot \alpha^2
\]

\[
+ \left( G_P^*(\alpha) \cdot 2I^2 \cdot C_1 + C_2 \right) \cdot 8I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^4}{(1 - \rho)^4} \cdot \alpha^2,
\]

where

\[
G_P^*(\alpha) \triangleq \frac{\mu_\text{opt} + \frac{L}{\mu_\text{opt}}}{\mu_\text{min} - \frac{L}{2} \alpha - \frac{\epsilon_\text{opt}}{2}}. \quad (46)
\]

2) Since \( \mathcal{P}^*(\cdot, 1) \) is continuous and monotonically increasing on \((0, 2\mu_\text{min}/L)\), with \( \mathcal{P}^*(0, 1) = 0 \), there must exist a sufficiently small \( \bar{\alpha} < 2\mu_\text{min}/L \) such that \( \mathcal{P}^*(\bar{\alpha}, 1) < 1 \), for all \( \alpha \in (0, \bar{\alpha}) \). One can verify that, for any \( \alpha \in (0, 2\mu_\text{min}/L) \), \( \mathcal{P}^*(\alpha, z) \) is continuous at \( z = 1 \). Therefore, for any fixed \( \alpha \in (0, \bar{\alpha}) \), \( \mathcal{P}^*(\alpha, 1) < 1 \) implies the existence of some \( \bar{z}(\alpha) < 1 \) so that \( \mathcal{P}^*(\alpha, \bar{z}(\alpha)) < 1 \).

We conclude the proof providing the expression of a valid \( \bar{\alpha} \). Restricting \( \alpha \leq \bar{\mu}_\text{min}/L \), we can upper bound \( G_P^*(\alpha) \) by \( G_P^*(\bar{\mu}_\text{min}/L) \). Using for \( G_P^*(\alpha) \) this upper bound in (45) and solving the resulting \( \mathcal{P}^*(\alpha, 1) < 1 \) for \( \alpha \), yield

\[
\alpha < \alpha_1 \triangleq \left( G_P^*(\bar{\mu}_\text{min}/L) \cdot C_1 \cdot 4I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^2}{(1 - \rho)^2} \right.
\]

\[
+ \left( G_P^*(\bar{\mu}_\text{min}/L) \cdot 2I^2 \cdot C_1 + C_2 \right) \cdot 2I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^2}{(1 - \rho)^2} \n\]

\[
+ \left( G_P^*(\bar{\mu}_\text{min}/L) \cdot 2I^2 \cdot C_1 + C_2 \right) \cdot 8I_2^2L_{\text{mx}}^2 \cdot \frac{\rho^4}{(1 - \rho)^4} \right)^{-\frac{1}{4}}.
\]

Therefore, a valid \( \bar{\alpha} \) is \( \bar{\alpha} = \min\{\bar{\mu}_\text{min}/L, \alpha_1\}. \)

The next theorem provides an explicit expression of the convergence rate in Theorem 2.10 in terms of the step-size \( \alpha \); the constants \( J, A, \), and \( \alpha^* \) therein are defined in (B7), (B5) with \( \theta = 1/2 \), and (B9), respectively.
Theorem 2.11. In the setting of Theorem 2.10, suppose that the step-size $\alpha$ satisfies $\alpha \in (0, \alpha_{\text{mx}})$, with $\alpha_{\text{mx}} \triangleq \min\{(1 - \rho)^2/A_{\frac{1}{2}}, \mu_{\text{min}}/L\}$. Then, $\{U(x_i^\nu)\}_{\nu \in \mathbb{N}_+}$ converges to $U^*$ at an $R$-linear rate $O(\nu)$, for all $i = 1, \ldots, I$, where

$$z = \begin{cases} 
1 - J \cdot \alpha & \text{for } \alpha \in (0, \min\{\alpha^*, \alpha_{\text{mx}}\}), \\
\left(\rho + \sqrt{\alpha A_{\frac{1}{2}}}\right)^2 & \text{for } \alpha \in [\min\{\alpha^*, \alpha_{\text{mx}}\}, \alpha_{\text{mx}}). 
\end{cases}$$

(48)

Proof. See Appendix B.

Theorem 2.11 reveals an interesting phenomenon on the algorithm’s behavior. When the step-size $\alpha$ is small, the local optimization progresses “slow enough” to dominate the rate of the consensus system and determine the overall rate. However, as the step-size increases, the optimization rate improves and at certain point $\min\{\alpha^*, \alpha_{\text{mx}}\}$ a switch to the other regime happens wherein the consensus dominates the optimization system. In this regime, the overall rate is determined by the consensus rate $\rho$ plus an off-set term contributed by both the consensus and optimization system. Of course it is also possible that $\alpha^*$ is larger than $\alpha_{\text{mx}}$, which corresponds to the scenario where the optimization rate is always worse than consensus, for all admissible choices of step-size $\alpha \in (0, \alpha_{\text{mx}})$ such that the system is stable.

2.3. Nonconvex $F$

We now consider Problem (P) with $F$ possibly nonconvex. We introduce a merit function $J^\nu$ [cf. (54)] that measures consensus disagreement and distance from stationarity of the agents’ average iterates, and prove that it goes below an arbitrary accuracy $\epsilon > 0$ in $O(1/\epsilon)$ iterations [cf. Theorem 2.10]. Our proof is organized in the following steps: Step 1: We begin by showing that $U(\bar{x}^\nu)$ descends along $(1/I) \sum_{i=1}^I \tilde{d}_i^\nu$ within an error of the order of $O(||x_i^\nu||^2 + ||y_i^\nu||^2)$, where $\tilde{d}_i^\nu \triangleq \bar{x}_i^\nu - \bar{x}^\nu$; see Proposition 2.12. In Step 2 we show that $K_x||x_1^\nu||^2 + K_y||y_1^\nu||^2$ (with arbitrary $K_x, K_y > 0$) decreases within an error of $O(||d_i^\nu||^2)$, with $d_i^\nu \triangleq (d_i^\nu)^I$; see Proposition 2.13. Since no “enough” descent can be established on $U(\bar{x}^\nu)$ or $K_x||x_1^\nu||^2 + K_y||y_1^\nu||^2$ separately, the idea employed in Step 3 is to combine these two terms in a single function, $\Phi^\nu \triangleq U(\bar{x}^\nu) + K_x||x_1^\nu||^2 + K_y||y_1^\nu||^2$, and prove that $\Phi^\nu$ decreases monotonically; see Proposition 2.14. As a consequence, we also have $||x_1^\nu||, ||y_1^\nu|| \to 0$ as $\nu \to \infty$. Finally, in Step 4, we establish the bound $\Phi^\nu - \Phi^\nu+1 \geq C_\Phi J^\nu$, with some $C_\Phi > 0$, which leads to the sublinear convergence rate of $\{J^\nu\}_{\nu \in \mathbb{N}_+}$.

2.3.1. Step 1: Decrease of $U(\bar{x}^\nu)$ up to $O(||x_1^\nu||^2 + ||y_1^\nu||^2)$

We prove the following decrease properties of $U(\bar{x}^\nu)$.

Proposition 2.12. Consider Problem (P) under Assumption A and the SONATA /NEXT algorithm (l.a)-(l.d), under Assumptions (l) and (l). Then, there holds:

$$U(\bar{x}^{\nu+1}) - U(\bar{x}^\nu) \leq -\frac{\alpha}{I} \left(\tilde{\mu}_{\text{min}} - \frac{L\alpha}{2} - 3\epsilon_d\right) ||\tilde{d}^\nu||^2$$

$$+ \frac{\alpha}{I\epsilon_d} \left(I\epsilon_d \mu_{\text{mx}}^2 + (\tilde{L}_{\text{mx}} + L_{\text{mx}})^2\right) ||x_1^\nu||^2 + \frac{I\alpha}{\epsilon_d} ||y_1^\nu||^2,$$

(49)
for all $\nu \in \mathbb{N}_+$, where $\tilde{d}^\nu = [\tilde{d}_1^\nu, \ldots, \tilde{d}_I^\nu]^\top$, with $\tilde{d}_i^\nu = \tilde{x}_i^\nu - \bar{x}^\nu$; and $\epsilon_d > 0$ is an arbitrary constant (to be properly chosen).

**Proof.** See Appendix $\square$

2.3.2. Step 2: Inexact decrease of $K_x\|x_{\perp}^\nu\|^2 + K_y\|y_{\perp}^\nu\|^2$

**Proposition 2.13.** Under Assumptions $[C]$ and $[E]$ there holds:

\[
(K_x\|x_{\perp}^{\nu+1}\|^2 + K_y\|y_{\perp}^{\nu+1}\|^2) - (K_x\|x_{\perp}^\nu\|^2 + K_y\|y_{\perp}^\nu\|^2) \\
\leq -\left(K_x(1 - \tilde{\rho}_x) - 3K_yL_{mx}^2\rho^2\left(1 + \epsilon_y^{-1}\right)(4 + \alpha^2)\right)\|x_{\perp}^\nu\|^2 \\
+ \alpha^2\rho^2\left(K_x(1 + \epsilon_x^{-1}) + 3K_yL_{mx}^2\left(1 + \epsilon_y^{-1}\right)\right)\|\tilde{d}^\nu\|^2,
\]

(50)

for all $\alpha \in [0, 1]$, $\nu \in \mathbb{N}_+$ and $\epsilon_x, \epsilon_y, K_x, K_y > 0$; $\rho$ and $L_{mx}$ are defined in (29) and (8), respectively; and $\tilde{\rho}_x \triangleq (1 + \epsilon_x)(1 - \alpha)^2\rho^2$, and $\tilde{\rho}_y \triangleq (1 + \epsilon_y)\rho^2$.

**Proof.** (50) is the weighted sum of the following two inequalities on the consensus errors, whose proof follows the same path established for that of Proposition 2.6 and thus is omitted:

\[
\|x_{\perp}^{\nu+1}\|^2 \leq \tilde{\rho}_x\|x_{\perp}^\nu\|^2 + \alpha^2\rho^2\left(1 + \epsilon_x^{-1}\right)\|\tilde{d}^\nu\|^2,
\]

\[
\|y_{\perp}^{\nu+1}\|^2 \leq \tilde{\rho}_y\|y_{\perp}^\nu\|^2 + 3L_{mx}^2\rho^2\left(1 + \epsilon_y^{-1}\right)(4 + \alpha^2)\|x_{\perp}^\nu\|^2 + 3L_{mx}^2\rho^2\left(1 + \epsilon_y^{-1}\right)\alpha^2\|\tilde{d}^\nu\|^2.
\]

2.3.3. Step 3: Lyapunov-like function $\Phi^\nu$

Building on the bounds established in the previous two steps, we prove that $\Phi^\nu = U(\tilde{x}^\nu) + K_x\|x_{\perp}^\nu\|^2 + K_y\|y_{\perp}^\nu\|^2$ is monotonically decreasing. Combining (19) and (50), we have

\[
\Phi^{\nu+1} - \Phi^\nu \\
\leq -\left(K_x(1 - \tilde{\rho}_x) - 3K_yL_{mx}^2\rho^2\left(1 + \epsilon_y^{-1}\right)(4 + \alpha^2) - \frac{\alpha}{L\epsilon_d}\left(I_{mx}L^2 + (\bar{I}_{mx} + L_{mx})^2\right)\right)\|x_{\perp}^\nu\|^2 \\
+ \frac{\alpha}{L}\left(\rho^2\left(K_x(1 + \epsilon_x^{-1}) + 3K_yL_{mx}^2\left(1 + \epsilon_y^{-1}\right)\right)\right)\|\tilde{d}^\nu\|^2.
\]

(51)

In order to have descent on $\Phi^\nu$, one needs to choose the free parameters $\epsilon_d, \epsilon_x, \epsilon_y, K_x$ and $K_y$ above so that the coefficients multiplying $\|x_{\perp}^\nu\|^2$, $\|y_{\perp}^\nu\|^2$, $\|\tilde{d}^\nu\|^2$ are all negative. In Appendix $[D]$ we provide a convenient choice of the above parameters towards this goal. The final result is summarized in the next proposition.

**Proposition 2.14.** Under the setting of Propositions 2.12 and 2.13, there exists a sufficiently small $\alpha^* > 0$ (see (123) in Appendix $[D]$ for its explicit expression) and some $K_x, K_y > 0$ (see (111) and (114) in Appendix $[D]$) such that $\Phi^\nu$ satisfies:

\[
\Phi^{\nu+1} - \Phi^\nu \leq -\frac{6IL_{mx}}{\mu_{\min}}(\alpha^* - \alpha)\|x_{\perp}^\nu\|^2 - \frac{6L}{\mu_{\min}}(\alpha^* - \alpha)\|y_{\perp}^\nu\|^2 - \frac{L\alpha}{2L}(\alpha^* - \alpha)\|\tilde{d}^\nu\|^2,
\]

(52)

19
for all $0 < \alpha < \alpha^*$ and $\nu \in \mathbb{N}_+$. Furthermore, $\sum_{\nu=0}^{\infty} \|x_i^\nu\|^2 < \infty$, $\sum_{\nu=0}^{\infty} \|y_i^\nu\|^2 < \infty$, and $\sum_{\nu=0}^{\infty} \|\hat{d}^\nu\|^2 < \infty$.

**Remark 1.** By the expression of $\alpha^*$ (see (D23) in Appendix D), when $\rho$ tends to zero (i.e., the graph is fully connected), $\alpha^*$ approaches $\overline{\mu}_{\min}/L$, which is in consistent with the upper bound on the step-size of centralized SCA algorithms [32].

### 2.3.4. Step 4: proof of the sublinear rate

First let us define a valid measure of stationarity and consensus. Given $z \in \mathcal{K}$, define for $i \in \mathcal{I}$ the vector function $\hat{x}_i$:

$$
\hat{x}_i(z) \triangleq \arg\min_{u \in \mathcal{K}} \overline{f}_i(u; z) + \sum_{\nu \neq j \neq i} \langle \nabla f_i(z), u - z \rangle + G(u). \quad (53)
$$

Note that, $\|\hat{x}_i(z) - z\|^2$ is a valid measure of stationarity, in the sense that it is continuous and $\|\hat{x}_i(z) - z\| = 0$ if and only if $z$ is a stationary solution of Problem (P) [32]. Using $\|\hat{x}_i(z) - z\|^2$, we introduce the following optimality-consensus merit

$$
J^\nu = M(\bar{x}^\nu) + C_x \|\bar{x}_i^\nu\|^2, \quad \text{with} \quad M(\bar{x}^\nu) \triangleq \langle 1/I \rangle \cdot \sum_{i \in \mathcal{V}} \|\hat{x}_i^\nu(\bar{x}^\nu) - \bar{x}^\nu\|^2, \quad (54)
$$

where $C_x$ is an arbitrary positive parameter.

Recall that $\sum_{\nu=0}^{\infty} \|\nu_i^\nu\|^2 < \infty$, $\sum_{\nu=0}^{\infty} \|y_i^\nu\|^2 < \infty$ and $\sum_{\nu=0}^{\infty} \|\hat{d}^\nu\|^2 < \infty$ (cf. Proposition 2.14). Next we upper bound $J^\nu$ by these vanishing terms. To do so, we first introduce the following intermediate result (proved in the Appendix D).

**Lemma 2.15.** The following bound holds for $\bar{x}_i$ [cf. (1a)] and $\hat{x}_i$ [cf. (53)]:

$$
\|\bar{x}_i^\nu(\bar{x}^\nu) - \bar{x}_i^\nu\| \leq \frac{L_{\max} \sqrt{T}}{\mu_i} \|\bar{x}^\nu\| + \frac{\overline{L}_i + L_i}{\overline{\mu}_i} \|\bar{x}^\nu - x_i^\nu\| + \frac{I}{\mu_i} \|y_i^\nu - \bar{y}^\nu\|. \quad (55)
$$

We now bound $M(\bar{x}^\nu)$ as follows:

$$
M(\bar{x}^\nu) \leq 2 \sum_{i=1}^{I} \|\hat{x}_i^\nu(\bar{x}^\nu) - \hat{x}_i^\nu\|^2 + 2\|\hat{d}^\nu\|^2 \\
\leq \sum_{i=1}^{I} \frac{6I^2}{\mu_i} \|y_i^\nu - \bar{y}^\nu\|^2 + \sum_{i=1}^{I} \frac{6L_{\max} I}{\mu_i} \|\bar{x}_i^\nu\|^2 + \sum_{i=1}^{I} \frac{6(\overline{L}_i + L_i)^2}{\mu_i^2} \|\bar{x}^\nu - x_i^\nu\|^2 + 2\|\hat{d}^\nu\|^2, \quad (56)
$$

where in (a) we used Lemma 2.15. This together with (54) yields

$$
J^\nu \leq \left[ C_x + \frac{12I}{\overline{\mu}_{\min}} \left( \overline{L}_{\max} + L_{\max} \right)^2 \right] \|\bar{x}_i^\nu\|^2 + \frac{6I}{\overline{\mu}_{\min}} \|y_i^\nu\|^2 + \frac{2}{I} \|\hat{d}^\nu\|^2. \quad (56)
$$

Combining (56) with (52) leads to

$$
\Phi^\nu - \Phi^{\nu+1} \geq c_f(\alpha^* - \alpha)\overline{\mu}_{\min}J^\nu, \quad \text{with} \quad c_f \triangleq \min \left( \frac{6I \left( \frac{L_{\max}}{\overline{\mu}_{\min}} \right)^2}{C_x + 12I \left( \frac{L_{\max} + L_{\max}}{\overline{\mu}_{\min}} \right)^2}, \frac{L \alpha}{4\overline{\mu}_{\min}} \right). \quad (57)
$$

20
Given $\epsilon > 0$, let $T_\epsilon \triangleq \min \{ \nu \in \mathbb{N}_+ : J^\nu \leq \epsilon \}$. We have

$$T_\epsilon \epsilon \leq \sum_{t=0}^{T_\epsilon-1} J^t \leq C(\alpha, \rho) \triangleq \frac{\Phi_0}{\mu_{\min}(\alpha^\star - \alpha)c_J},$$

(58)

The above complexity result is summarized in the following theorem.

**Theorem 2.16.** Consider Problem $\mathbb{P}$ under Assumption $A$ and the SONATA /NEXT algorithm (1a)-(1d), under Assumptions $D$ and $E$, and steps-size satisfying $\alpha \in (0, \alpha^\star)$ [with $\alpha^\star$ defined in (D23)]. For any $\epsilon > 0$, we have $T_\epsilon \leq C(\alpha, \rho)/\epsilon$.

To our knowledge, this is the first complexity result for a distributed algorithm applicable to a nonconvex instance of Problem $\mathbb{P}$.

### 3. Distributed optimization over directed time-varying graphs

In this section we extend SONATA/NEXT and its convergence analysis to the case where agents solve Problem $\mathbb{P}$ over *directed time-varying graphs*. More specifically, the communication network is now a time-varying digraph $G^\nu = (V, E^\nu)$, where the set of edges $E^\nu$ represents the agents’ communication links: $(i, j) \in E^\nu$ if at iteration $\nu$ there is a link going from agent $i$ to agent $j$. We will prove convergence under the following standard “long-term” connectivity property of the graphs.

**Assumption F (On graph connectivity).** The graph sequence $\{G^\nu\}_{\nu \in \mathbb{N}_+}$ is $B$-strongly connected, i.e., there exists a finite integer $B > 0$ such that the graph with edge set $\bigcup_{t=\nu B}^{(\nu+1)B-1} E^t$ is strongly connected, for all $\nu \in \mathbb{N}_+$.

#### 3.1. The SONATA algorithm

The SONATA/NEXT scheme (1a)-(1d) is not readily applicable in this more general setting, as constructing a doubly stochastic weight matrix compliant with a directed graph is generally infeasible or computationally costly—see e.g. [6]. Conditions on the weight matrices can be relaxed if the consensus and tracking schemes (1c)-(1d) are properly changed to deal with the lack of doubly stochasticity. Here, we consider the proposal in the companion paper [34] (but in the Adapt-Then-Combine (ATC) form), which builds on the perturbed push-sum protocol. The resulting distributed algorithm, still termed SONATA, reads

$$\hat{x}_i^\nu \triangleq \arg\min_{x_i \in \mathcal{K}} \bar{f}_i(x_i; x_i^\nu) + (I \cdot y_i^\nu - \nabla f_i(x_i^\nu))^\top (x_i - x_i^\nu) + G(x_i),$$

(59a)

$$x_i^{\nu+\frac{1}{2}} = x_i^\nu + \alpha \cdot d_i^\nu, \quad d_i^\nu = \hat{x}_i^\nu - x_i^\nu,$$

(59b)

$$\phi_i^{\nu+1} = \sum_{j=1}^{I} c_{ij}^\nu \phi_j^\nu, \quad x_i^{\nu+1} = \frac{1}{\phi_i^{\nu+1}} \sum_{j=1}^{I} c_{ij}^\nu \phi_j^\nu x_j^{\nu+\frac{1}{2}}$$

(59c)

$$y_i^{\nu+1} = \frac{1}{\phi_i^{\nu+1}} \sum_{j=1}^{I} c_{ij}^\nu \left( \phi_j^\nu y_j^\nu + \nabla f_j(x_j^{\nu+1}) - \nabla f_j(x_j^\nu) \right),$$

(59d)
with initialization: $x_i^0 \in \mathcal{K}$, $y_i^0 = \nabla f_i(x_i^0)$, and $\phi_i^0 = 1$, for all $i = 1, \ldots, I$. In the perturbed push-sum protocols (59c)-(59d), $C' \triangleq (c_{ij}')_{i,j=1}^I$ is only column-stochastic.

**Assumption G.** For each $\nu \geq 0$, the weight matrix $C' \triangleq (c_{ij}')_{i,j=1}^I$ has a sparsity pattern compliant with $\mathcal{G}'$, i.e., there exists a constant $\kappa > 0$ such that, for all $\nu \in \mathbb{N}_+$

G1 $c_{ij}' \geq \kappa > 0$, for all $i = 1, \ldots, I$;
G2 $c_{ij}' \geq \kappa > 0$, if $(j, i) \in \mathcal{E}'$; and $c_{ij}' = 0$ otherwise.

Moreover, $C'$ is column stochastic, i.e., $1^T C' = 1^T$ for all $\nu \in \mathbb{N}_+$.

We conclude this section stating the counterparts of the definitions introduced in Sec. 2 for the SONATA/NEXT algorithm, adjusted to this new setting. Using the column stochasticity of $C'$ and (59d), one can see that in contrast to (2) the average gradient is now preserved on the weighted average of the $y_i$’s:

$$
\frac{1}{I} \sum_{i=1}^I \phi_i^{\nu+1} y_i^{\nu+1} = \frac{1}{I} \sum_{i=1}^I \phi_i^{\nu} y_i^{\nu} + \frac{\nabla f_i^{\nu+1}}{\phi_i^{\nu+1}} - \nabla f_i^{\nu}.
$$

This suggests to decompose $y^{\nu}$ into its weighted average and the consensus error, defined respectively as

$$
y_{\phi}^{\nu} \triangleq \frac{1}{I} \sum_{i=1}^I \phi_i^{\nu} y_i^{\nu} \quad \text{and} \quad y_{\phi, \perp}^{\nu} \triangleq y^{\nu} - 1_I \otimes y_{\phi}^{\nu}.
$$

Accordingly, we define the weighted average of $x^{\nu}$ and the consensus error as

$$
x_{\phi}^{\nu} \triangleq \frac{1}{I} \sum_{i=1}^I \phi_i^{\nu} x_i^{\nu} \quad \text{and} \quad x_{\phi, \perp}^{\nu} \triangleq x^{\nu} - 1_I \otimes x_{\phi}^{\nu}.
$$

In addition, we also generalize the definition of the optimality gap as

$$
p_{\phi}^{\nu} \triangleq \sum_{i=1}^I \phi_i^{\nu} p_i, \quad \text{with} \quad p_i^{\nu} \triangleq (U(x_i^{\nu}) - U^*).
$$

Finally, apart from the problem parameters $L_i$, $L_{mx}$, $L$, $\mu$ [cf. (5) Assumption H] and and algorithm parameters $\tilde{L}_i$, $\tilde{L}_{mx}$, $\tilde{\mu}$, $\tilde{\mu}_{mx}$ [cf. (9)], we introduce the following network parameters, borrowed by [34, Prop. 1]:

$$
\phi_{lb} \triangleq \kappa^{2(I-1)B}, \quad \phi_{ub} \triangleq I - \kappa^{2(I-1)B}, \quad \rho_B \triangleq 2c_0 I \rho \kappa^{-(I-1)B},
$$

with $\kappa$ and $B$ given in Assumption G and F respectively;

$$
c_0 \triangleq 2 \left(1 + \kappa^{-2(I-1)B}\right), \quad \rho \triangleq 1 - \tilde{\kappa}^{(I-1)B}, \quad \tilde{\kappa} \triangleq \kappa^{2(I-1)B+1}/I;
$$

and $\tilde{B}$ being a sufficiently large integer so that $\rho_{\tilde{B}} < 1$. Furthermore, we will use the following lower and upper bounds of $\phi_i^{\nu}$ [34, Prop. 1]

$$
\phi_{lb} \leq \phi_i^{\nu} \leq \phi_{ub}, \quad \text{for all} \quad i = 1, \ldots, I, \quad \nu \in \mathbb{N}_+.
$$

We study next the convergence rate of SONATA/NEXT in the case of strongly convex $F$. Because of space limitation, we omit the study for the nonconvex case, which follows the same line of analysis introduced in Sec. 2.3 for undirected graphs.
3.2. **Strongly convex \( F \): Linear convergence rate**

The proof of linear convergence follows the same path of the one developed in Sec. 2.2 for the case of undirected graphs. Hence, we omit similar derivations and highlight only the main results.

### 3.2.1. Step 1: \( p^\nu_F \) converges linearly up to \( O(\|x^\nu_{\phi,\perp}\|^2 + \|y^\nu_{\phi,\perp}\|^2) \)

This is counterpart of Proposition 2.5 (cf. Sec. 2.2), and stated as follows.

**Proposition 3.1.** Consider Problem (\( P \)) under Assumptions A, B, F, and D, and the SONATA algorithm (59), under Assumptions D and G. Then, \( p^\nu_F \) satisfies:

\[
p^\nu_{\phi} + 1 \leq \sigma(\alpha) \cdot p^\nu_{\phi} + \eta(\alpha) \cdot \phi_{ub} \cdot \left(8I^2 L^2_{mx} \|x^\nu_{\phi,\perp}\|^2 + 2I^2 \|y^\nu_{\phi,\perp}\|^2\right),
\]

where the constants \( L, L^2_{mx} \) and \( \bar{\mu}_{min} \) are defined in (8) and (9), respectively; and \( \sigma(\alpha) \in (0, 1) \) and \( \eta(\alpha) > 0 \) are defined in (23).

**Proof.** The proof follows closely that of Proposition 2.5 and thus is omitted. Here, we only notice that, instead of (14), we used: \( \sum_{i=1}^{I} \phi^\nu_{i} U(x^\nu_{i}) \leq \sum_{i=1}^{I} \phi^\nu_{i} U(x^\nu_{i+1}) \), where we used \( \sum_{j=0}^{2\nu} \phi^\nu_{j} / \phi^\nu_{j+1} = 1 \), for all \( i = 1, \ldots, I \). \( \square \)

### 3.2.2. Step 2: Decay of \( \|x^\nu_{\phi,\perp}\| \) and \( \|y^\nu_{\phi,\perp}\| \)

The following result is a consequence of [34], Lemma 11\(^1\) and the Young’ s inequality; hence, its proof is omitted.

**Lemma 3.2.** Under Assumptions A, C, D, and E there holds:

\[
\|x^\nu_{\phi,\perp}\|^2 \leq \rho_B^2 (1 + B\epsilon_x) \|x^\nu_{\phi,\perp}\|^2 + 2I^3 \alpha^2 (B + \epsilon_x) \sum_{t=0}^{B-1} \|d^\nu_{\epsilon+t}\|^2,
\]

\[
\|y^\nu_{\phi,\perp}\|^2 \leq \rho_B^2 (1 + B\epsilon_y) \|y^\nu_{\phi,\perp}\|^2 + 4I^4 \phi_{lb}^{-2} L^2_{mx} (B + \epsilon_y) \sum_{t=0}^{B-1} \left(4 \|x^\nu_{\phi,\perp}\|^2 + \alpha^2 \|d^\nu_{\epsilon+t}\|^2\right),
\]

where \( \bar{B}, \rho_B \) are constants defined in (64).

### 3.2.3. Step 3: \( \|d^\nu\| = O(\sqrt{p^\nu_{\phi} + \|y^\nu_{\phi,\perp}\|}) \)

**Proposition 3.3.** Under Assumptions A, D, E, and F, \( \|d^\nu\| \) is bounded as

\[
\|d^\nu\| \leq \sqrt{\frac{2I}{\mu_{lb}} \left(\frac{\tilde{L}_{mx}}{\bar{\mu}_{min}} + \frac{L_{mx}}{\mu_{min}} + 1\right)} \sqrt{p^\nu_{\phi} + \frac{I}{\bar{\mu}_{min}} \|y^\nu_{\phi,\perp}\|},
\]

where \( L_{mx}, \tilde{L}_{mx}, \bar{\mu}_{min} \) and \( \phi_{lb} \) are defined in (8), (9) and (64).

\(^1\) Note that some constants in Lemma 3.2 differ from those in [34], Lemma 11, as here we considered SONATA in the ATC form, rather than CTA form as in [34].
3.3. Establishing linear rate

We can now prove linear rate following the path introduced in Sec. 2.2; for sake of simplicity, we will use the same notation as in Sec. 2.2. We begin applying the transformation (36) to the sequences \{p'_\phi\}_\nu \in \mathbb{N}^+, \{\|x'_{\phi,\perp}\|^2\}_\nu \in \mathbb{N}^+, \{\|y'_{\phi,\perp}\|^2\}_\nu \in \mathbb{N}^+, and \{\|d''\|^2\}_\nu \in \mathbb{N}^+, satisfying the inequalities (66), (67a), (67c), and (68), respectively.

**Proposition 3.4.** Let \(P^K_\phi(z), D^K_\phi(z), X^K_{\phi,\perp}(z),\) and \(Y^K_{\phi,\perp}(z)\) denote the transformation (36) of the sequences \{p'_\phi\}_\nu \in \mathbb{N}^+, \{\|x'_{\phi,\perp}\|^2\}_\nu \in \mathbb{N}^+, \{\|y'_{\phi,\perp}\|^2\}_\nu \in \mathbb{N}^+, and \{\|y''_{\phi,\perp}\|^2\}_\nu \in \mathbb{N}^+. Given the constants \(\sigma(\alpha)\) and \(\eta(\alpha)\), defined in Proposition 3.1, and the free parameters \(\epsilon_x, \epsilon_y > 0\), the following holds:

\[
P^K_\phi(z) \leq G_P(\alpha, z) \cdot (8\phi_{ub}I^2L_{mx}^2X^K_{\phi,\perp}(z) + 2\phi_{ub}I^2Y^K_{\phi,\perp}(z)) + \omega_p \tag{69}
\]
\[
X^K_{\phi,\perp}(z) \leq G_X(z) \cdot 2I^3\alpha^2D^K(z) + \omega_x \tag{70}
\]
\[
Y^K_{\phi,\perp}(z) \leq G_Y(z) \cdot 4I^4\phi_{ib}^{-2}L_{mx}^2(4X^K_{\phi,\perp}(z) + \alpha^2D^K(z)) + \omega_y \tag{71}
\]
\[
D^K(z) \leq C_1 \cdot P^K_\phi(z) + C_2 \cdot Y^K_{\phi,\perp}(z), \tag{72}
\]

for all
\[
z \in \left(\max \left\{ \sigma(\alpha), \sqrt{\frac{\rho_B^2(1 + B\epsilon_x)}{\eta(\alpha)}}, \sqrt{\frac{\rho_B^2(1 + B\epsilon_y)}{\eta(\alpha)}} \right\}, 1 \right), \tag{73}
\]

where

\[
G_P(\alpha, z) \equiv \eta(\alpha) \cdot \frac{z}{z - \sigma(\alpha)}, \quad \omega_p \equiv \frac{z}{z - \sigma(\alpha)} \cdot \rho_\phi \tag{74}
\]
\[
G_X(z) \equiv \frac{B(\tilde{B} + \epsilon_x^{-1})}{z^B - \rho_B^2(1 + B\epsilon_x)}, \quad \omega_x \equiv \frac{z^B}{z^B - \rho_B^2(1 + B\epsilon_x)} \sum_{t=0}^{B-1} \frac{|||x'_{\phi,\perp}||^2}{z^t} \tag{75}
\]
\[
G_Y(z) \equiv \frac{B(\tilde{B} + \epsilon_y^{-1})}{z^B - \rho_B^2(1 + B\epsilon_y)}, \quad \omega_y \equiv \frac{z^B}{z^B - \rho_B^2(1 + B\epsilon_y)} \sum_{t=0}^{B-1} \frac{|||y'_{\phi,\perp}||^2}{z^t} \tag{76}
\]
\[
C_1 \equiv \frac{4I}{\mu_{\phi ib}} \left( \frac{\bar{I}_{max}}{\bar{I}_{min}} + \frac{\bar{I}_{max}}{\bar{I}_{min}} + 1 \right)^2, \quad C_2 \equiv \frac{2}{\mu_{\phi ib}} \left( \frac{\bar{I}_{max}}{\bar{I}_{min}} \right)^2, \tag{77}
\]

**Proof.** See Appendix F. \(\square\)

Chaining the inequalities in Proposition 3.4 as done in Sec. 2.2 for (37) (cf. Fig. 2.2.4), we can bound \(D^K(z)\) as

\[
D^K(z) \leq \mathcal{P}(\alpha, z) \cdot D^K(z) + \mathcal{R}(\alpha, z), \tag{78}
\]

where \(\mathcal{P}(\alpha, z)\) is defined as

\[
\mathcal{P}(\alpha, z) \equiv G_P(\alpha, z) \cdot G_X(z) \cdot C_1 \cdot 8\phi_{ub}I^2L_{mx}^2 \cdot 2I^3 \cdot \alpha^2
\]
\[
+ (G_P(\alpha, z) \cdot 2\phi_{ub}I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 4I^4\phi_{ib}^{-2}L_{mx}^2 \cdot \alpha^2
\]
\[
+ (G_P(\alpha, z) \cdot 2\phi_{ub}I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 4I^4\phi_{ib}^{-2}L_{mx}^2 \cdot G_X(z) \cdot 8I^3 \cdot \alpha^2 \tag{79}
\]

and \(\mathcal{R}(\alpha, z)\) is a bounded remainder term.
Recall the definition of problem parameters $L_i$, $L_{\text{max}}$, $L$, and $\mu$ given in Assumption A, B, and C, and algorithm parameters $\bar{L}_i$, $\bar{L}_{\text{max}}$, $\bar{\mu}_i$, $\bar{\mu}_{\text{min}}$ given in Assumption D and H. The following theorem proves R-linear rate of SONATA.

**Theorem 3.5.** Consider Problem D under Assumptions A, B, and C, and the SONATA algorithm 59, under Assumptions D and G. Then, there exists a sufficiently small step-size $\alpha \in (0, 1]$ such that for all $\alpha < \bar{\alpha}$, $\{U(x^\nu_i)\}_{\nu \in \mathbb{N}^+}$ converges to $U^*$ at an R-linear rate, for all $i = 1, \ldots, I$.

**Proof.** See Appendix C. \(\square\)

The expression of the rate in Theorem 3.5 can be obtained along the same line of that stated in Theorem 2.10, in the case of undirected graphs—see Theorem G.1 in Appendix G.1.

**Appendix A. Proof of (41)**

Chaining the inequalities in (37) as shown in Fig. 2.2.4, we have

\[
D^K(z) \leq C_1 \cdot P^K(z) + C_2 \cdot Y^K(z)
\]

\[
\leq C_1 \cdot \left( G_P(\alpha, z) \cdot \left( 4I^2L_{\text{max}}^2X^K_\perp(z) + 2I^2Y^K_\perp(z) \right) + \omega_p \right) + C_2 \cdot Y^K(z)
\]

\[
= C_1 \cdot G_P(\alpha, z) \cdot 4I^2L_{\text{max}}^2X^K_\perp(z) + (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2)Y^K_\perp(z) + C_1 \cdot \omega_p
\]

\[
\leq C_1 \cdot G_P(\alpha, z) \cdot 4I^2L_{\text{max}}^2 \cdot G_X(z) \cdot \rho^2 \alpha^2 D^K(z)
\]

\[
+ (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot G_Y(z) \cdot 8L_{\text{max}}^2 \rho^2 X^K_\perp(z)
\]

\[
+ (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot G_Y(z) \cdot 2I^2 \rho^2 \alpha^2 D^K(z)
\]

\[
+ C_1 \cdot \omega_p + (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot \omega_y + C_1 \cdot G_P(\alpha, z) \cdot 4I^2L_{\text{max}}^2 \cdot \omega_x
\]

\[
\leq C_1 \cdot G_P(\alpha, z) \cdot 4I^2L_{\text{max}}^2 \cdot G_X(z) \cdot \rho^2 \alpha^2 D^K(z)
\]

\[
+ (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot G_Y(z) \cdot 8L_{\text{max}}^2 \rho^2 \cdot G_X(z) \cdot \rho^2 \alpha^2 D^K(z)
\]

\[
+ (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot G_Y(z) \cdot 2I^2 \rho^2 \alpha^2 D^K(z)
\]

\[
+ C_1 \cdot \omega_p + (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot \omega_y + C_1 \cdot G_P(\alpha, z) \cdot 4I^2L_{\text{max}}^2 \cdot \omega_x
\]

\[
+ (C_1 \cdot G_P(\alpha, z) \cdot 2I^2 + C_2) \cdot G_Y(z) \cdot 8L_{\text{max}}^2 \rho^2 \cdot \omega_x.
\]

Notice that, under (38), $G_P(\alpha, z)$, $G_X(z)$, $G_Y(z)$, and $\omega_p$, $\omega_x$, $\omega_y$ are all bounded, which implies that the reminder $R(\alpha, z)$ in (37) is bounded as well. \(\square\)

**Appendix B. Proof of Theorem 2.11**

We find the smallest $z$ satisfying (38) such that $P(\alpha, z) < 1$, for $\alpha \in (0, \alpha_{\text{max}})$, with $\alpha_{\text{max}} \in (0, 1)$ to be determined.

Let us begin considering the condition $z > \sigma(\alpha)$ in (38). To simplify the analysis, we impose instead the following stronger version

\[
z \geq \sigma(\alpha) + \frac{(\theta \cdot \alpha) \cdot (\bar{\mu}_{\text{min}} - \frac{L}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}})}{\alpha (L^2 + L_{\text{max}}^2) \mu} + (\bar{\mu}_{\text{min}} - \frac{L}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}})
\]  

(B1)
for some $\theta \in (0, 1)$, which will be chosen to tighten the bound. Notice that the RHS of [B1] is strictly larger than $\sigma(\alpha)$ but still strictly less than one, for any $\alpha \in (0, 2\bar{\mu}_{\min} - \epsilon_{\text{opt}})$, with given $\epsilon_{\text{opt}} \in (0, 2\bar{\mu}_{\min})$.

Observe that in the expression of $\mathcal{P}(\alpha, z)$, the only coefficient multiplying $\alpha^2$ that depends on $\alpha$ is the optimization gain $G_P(\alpha, z) \triangleq \eta(\alpha)/(z - \sigma(\alpha))$. Using [B1], $G_P(\alpha, z)$ can be upper bounded as

\[
G_P(\alpha, z) \leq \inf_{\epsilon_{\text{opt}} \in (0, 2\bar{\mu}_{\min} - \alpha L)} \frac{1}{2} \epsilon_{\text{opt}}^{-1} \cdot \frac{6(L^2 + L_{\text{mx}}^2)}{\mu} + \frac{1}{\mu} \left( \bar{\mu}_{\min} - \frac{L}{2} \alpha - \frac{1}{2} \epsilon_{\text{opt}} \right) \cdot \theta^{-1} 
\]

(B2)

where the minimum is attained at $\epsilon_{\text{opt}} \equiv \bar{\mu}_{\min} - \alpha L/2$, provided that $\alpha \in (0, 2\bar{\mu}_{\min}/L)$; and $G_P^*(\alpha)$ is defined in [B2]. Substituting the upper bound [B2] in $\mathcal{P}(\alpha, z)$ and setting therein $\epsilon_{\text{opt}} = \epsilon^*$, we get the following sufficient condition for $\mathcal{P}(\alpha, z) < 1$:

\[
G_P^*(\alpha) \cdot \theta^{-1} \cdot \left( C_1 \cdot 4I^2L_{\text{mx}}^2 \cdot G_X(z) \cdot \rho^2 \cdot \alpha^2 + (G_P^*(\alpha) \cdot \theta^{-1} \cdot 2I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 2I_{\text{mx}}^2 \rho^2 \cdot \alpha^2 + (G_P^*(\alpha) \cdot \theta^{-1} \cdot 2I^2 \cdot C_1 + C_2) \cdot G_Y(z) \cdot 8I_{\text{mx}}^2 \rho^2 \cdot G_X(z) \cdot \rho^2 \cdot \alpha^2 \cdot \alpha^2 < 1. \]  

(B3)

To minimize the left hand side, we set $\epsilon_x = \epsilon_y = (\sqrt{z} - \rho)/\rho$. Furthermore, using the fact that $G_P^*(\alpha)$ is monotonically increasing on $\alpha \in (0, 2\bar{\mu}_{\min}/L)$, and restricting $\alpha \in (0, \bar{\mu}_{\min}/L]$, a sufficient condition for [B3] is

\[
\alpha \leq \alpha(z) \triangleq \left( A_{1,\theta} \frac{1}{(\sqrt{z} - \rho)^2} + A_{2,\theta} \frac{1}{(\sqrt{z} - \rho)^2} + A_{3,\theta} \frac{1}{(\sqrt{z} - \rho)^4} \right)^{-1/2}, \]

(B4)

where $A_{1,\theta}$, $A_{2,\theta}$ and $A_{3,\theta}$ are constants defined as

\[
A_{1,\theta} \equiv G_P^*(\bar{\mu}_{\min}/L) \cdot \theta^{-1} \cdot C_1 \cdot 4I^2L_{\text{mx}}^2 \cdot \rho^2 \\
A_{2,\theta} \equiv (G_P^*(\bar{\mu}_{\min}/L) \cdot \theta^{-1} \cdot 2I^2 \cdot C_1 + C_2) \cdot 2I_{\text{mx}}^2 \rho^2 \\
A_{3,\theta} \equiv (G_P^*(\bar{\mu}_{\min}/L) \cdot \theta^{-1} \cdot 2I^2 \cdot C_1 + C_2) \cdot 8I_{\text{mx}}^2 \rho^2.
\]

To find an explicit expression of a lower bound of $z$ in terms of $\alpha$, instead of $\alpha(z)$, we use in [B4] the lower bound $\alpha(z) \geq (\sqrt{z} - \rho)^2/A_{\theta}$ with $A_{\theta}$ defined in [B5] below. We obtain

\[
z \geq \left( \rho + \sqrt{A_{\theta} \alpha} \right)^2, \quad \text{with} \quad A_{\theta} \equiv \sqrt{A_{1,\theta} + A_{2,\theta} + A_{3,\theta}}. \]

(B5)

Notice that, under $\epsilon_x = \epsilon_y = (\sqrt{z} - \rho)/\rho$, [B5] is sufficient for $z > \rho^2(1 + \epsilon_x) = \rho^2(1 + \epsilon_x) = \rho^2 \sqrt{z}$, which are the other two conditions on $z$ in [B3]. Therefore, overall, $z$ must satisfy [B1] and [B5]. Letting $\epsilon_{\text{opt}} = \epsilon^*$ in [B1], the condition reduces to

\[
z \geq 1 - \frac{\bar{\mu}_{\min} - \frac{L}{2} \alpha}{12(L^2 + L_{\text{mx}}^2) / \mu + (\bar{\mu}_{\min} - \frac{L}{2} \alpha)} \cdot (1 - \theta) \alpha.
\]

Therefore, the overall convergence rate can be upper bounded by $O(\bar{z})$, where

\[
\bar{z} = \inf_{\theta \in (0, 1)} \max \left\{ \left( \rho + \sqrt{A_{\theta} \alpha} \right)^2, 1 - \frac{\bar{\mu}_{\min} - \frac{L}{2} \alpha}{12(L^2 + L_{\text{mx}}^2) / \mu + (\bar{\mu}_{\min} - \frac{L}{2} \alpha)} \cdot (1 - \theta) \alpha \right\}. \]

(B6)
Finally, we further simplify (B6). Letting $\theta = 1/2$ and using $\alpha \in (0, \tilde{\mu}_{\min}/L]$, the second term in (B6) can be upper bounded by

$$1 - \frac{\tilde{\mu}_{\min}\mu}{24(L^2 + L_{\max}^2) + \tilde{\mu}_{\min}\mu} \cdot \frac{1}{2} \alpha. \quad (B7)$$

The condition $\tilde{z} < 1$ imposes the following upper bound on $\alpha$: $\alpha < \alpha_{\max} = \min\{(1 - \rho^2)/A_\hat{\alpha}, \tilde{\mu}_{\min}/L\}$. Eq. (B6) then simplifies to

$$\tilde{z} = \max\left\{\left(\rho + \sqrt{\alpha A_\hat{\alpha}}\right)^2, 1 - J\alpha\right\}. \quad (B8)$$

Note that as $\alpha$ increases from 0, the first term in the max operator above is monotonically increasing from $\rho^2 < 1$ while the second term is monotonically decreasing from 1. Therefore, there must exist some $\alpha^*$ so that the two terms are equal, which is

$$\alpha^* = \left(-\rho\sqrt{A_\hat{\alpha}} + \sqrt{A_\hat{\alpha} + J(1 - \rho^2)}\right)^2 / A_\hat{\alpha} + J. \quad (B9)$$

To conclude, given the step-size satisfying $\alpha \in (0, \alpha_{\max})$, the sequence $\{\|d^\nu\|^2\}_{\nu \in \mathbb{N}^+}$ converges at rate $\mathcal{O}(z^\nu)$, with $z$ given in (B8). \qed

Appendix C. Proof of Proposition 2.12

We begin introducing the following property of the mapping $\hat{x}^\nu$ defined in (1a), whose proof follows readily from the optimality of $\hat{x}^\nu$ and the properties of $\tilde{f}_i$ (cf. Assumption A). 

Lemma C.1. The mapping $\hat{x}^\nu$ defined in (1a) satisfies

$$\left\langle Iy_i^\nu, \hat{d}_i^\nu \right\rangle + G(\hat{x}^\nu) - G(\hat{x}^\nu) \leq -\tilde{\mu}_i \|\hat{d}_i^\nu\|^2 + (\tilde{L}_i + L_i) \|x_i^\nu - \hat{x}^\nu\| \|\hat{d}_i^\nu\|.$$ 

We can now prove Proposition 2.12. By the $L$-Lipschitz continuity of $\nabla F$, convexity of $G$ and the updating rule (26a), it follows

$$U(x^{\nu+1}) - U(x^\nu) \overset{(a)}{\leq} \alpha \left(\nabla F(x^\nu), \frac{1}{I} \sum_{i=1}^I \hat{d}_i^\nu\right) + \frac{L_\alpha^2}{2I} \sum_{i=1}^I \|\hat{d}_i^\nu\|^2 + \frac{\alpha}{I} \sum_{i=1}^I \left(G(x^\nu) - G(x^\nu)\right)$$

$$= \frac{\alpha}{I} \sum_{i=1}^I \left(\nabla F(x^\nu) - Iy_i^\nu, \hat{d}_i^\nu\right) + \frac{L_\alpha^2}{2I} \sum_{i=1}^I \|\hat{d}_i^\nu\|^2 + \frac{\alpha}{I} \sum_{i=1}^I \left(\left\langle Iy_i^\nu, \hat{d}_i^\nu \right\rangle + G(\hat{x}^\nu) - G(\hat{x}^\nu)\right)$$

$$\leq \frac{\alpha}{I} \sum_{i=1}^I \left(\nabla F(x^\nu) - Iy_i^\nu, \hat{d}_i^\nu\right) + \frac{L_\alpha^2}{2I} \sum_{i=1}^I \|\hat{d}_i^\nu\|^2$$

$$- \frac{\alpha \tilde{\mu}_{\min}}{I} \sum_{i=1}^I \|\hat{d}_i^\nu\|^2 + \frac{\tilde{L}_{\max} + L_{\max}}{I} \alpha \sum_{i=1}^I \|x_i^\nu - \hat{x}^\nu\| \|\hat{d}_i^\nu\|, \quad (C1)$$

27
where in (a) we used the descent lemma and convexity of $G$; and in (b) we used Lemma C.1. The final inequality (49) follows readily from (C1) using $(\forall a, b \in \mathbb{R}, \forall \epsilon_d \in \mathbb{R}_+)$ $ab \leq \epsilon_d a^2 + (1/\epsilon_d) b^2$, where $\epsilon_d > 0$ is an arbitrary constant.

### Appendix D. Proof of Proposition 2.14

We begin finding the range of values of the step-size $\alpha > 0$ and free positive parameters $\epsilon_d$, $\epsilon_x$, $\epsilon_y$, $K_x$ and $K_y$ so that the coefficients multiplying $\|x^\nu_+\|^2$, $\|y^\nu_+\|^2$, $\|\dot{d}^\nu\|^2$ in (51) are all negative. This leads to the following conditions

\[
K_x(1 - \tilde{\rho}_x) - 3K_yL^2_{mx}\rho^2(1 + \epsilon_y^{-1})(4 + \alpha^2) - \frac{\alpha}{I\epsilon_d} \left( I^2 + (I_{mx} + L_{mx})^2 \right) > 0, \\
(D1)
\]

\[
K_y(1 - \tilde{\rho}_y) - \frac{I\alpha}{\epsilon_d} > 0, \\
(D2)
\]

\[
\frac{\alpha I}{\tilde{\mu}_{\min}} - \frac{L\alpha}{2} - 3\epsilon_d \frac{\rho^2}{(K_x(1 + \epsilon_x^{-1}) + 3K_yL^2_{mx}(1 + \epsilon_y^{-1}))} > 0. \\
(D3)
\]

Under (D1)-(D3), since $\lim \inf_{\nu} \Phi^\nu > -\infty$ (cf. Assumption A4), (51) implies convergence of $\{\Phi^\nu\}_{\nu \in \mathbb{N}_+}$, and thus $\sum_{\nu=0}^{\infty} \|x^\nu_+\|^2 < \infty$, $\sum_{\nu=0}^{\infty} \|y^\nu_+\|^2 < \infty$, and $\sum_{\nu=0}^{\infty} \|\dot{d}^\nu\|^2 < \infty$.

Next, we show that the system of inequalities (D1)-(D3) has in fact a solution and we derive an explicit bound for the step-size $\alpha$. Setting $\epsilon_d = \tilde{\mu}_{\min}/6$, (D1)-(D3) become

\[
\frac{6\alpha}{I\tilde{\mu}_{\min}} \left( I^2 + (I_{mx} + L_{mx})^2 \right) < K_x(1 - \tilde{\rho}_x) - 3K_yL^2_{mx}\rho^2(1 + \epsilon_y^{-1})(4 + \alpha^2), \\
(D4)
\]

\[
\alpha < \frac{\tilde{\mu}_{\min}}{6I} K_y(1 - \tilde{\rho}_y), \\
(D5)
\]

\[
\alpha < \frac{\tilde{\mu}_{\min}}{2I\rho^2} \left( K_x(1 + \epsilon_x^{-1}) + 3K_yL^2_{mx}(1 + \epsilon_y^{-1}) + \frac{L}{2I\rho^2} \right). \\
(D6)
\]

Note that (D4) and (D5) can possibly be satisfied only if $1 - \tilde{\rho}_x > 0$ and $1 - \tilde{\rho}_y > 0$. A sufficient condition for that is (using $0 \leq \alpha \leq 1$):

\[
\epsilon_x < 1 - \frac{\rho^2}{\rho^2}, \quad \epsilon_y < 1 - \frac{\rho^2}{\rho^2}, \quad \text{with} \quad \epsilon_x = \epsilon_y = \theta \frac{1 - \rho^2}{\rho^2}, \\
(D7)
\]

for some $\theta \in (0, 1)$. Using (D7) and $0 \leq \alpha \leq 1$, we obtain the following sufficient conditions for (D4)-(D6):

\[
\frac{12I\alpha}{\tilde{\mu}_{\min}} (I_{mx} + L_{mx})^2 < K_x(1 - \theta)(1 - \rho^2) - 15K_yL^2_{mx}\rho^2 \frac{1 + \frac{1 - \theta}{\rho^2}}{1 - \rho^2}, \\
(D8)
\]

\[
\alpha < \frac{\tilde{\mu}_{\min}}{6I} K_y(1 - \theta)(1 - \rho^2), \\
(D9)
\]

\[
\alpha < \frac{\tilde{\mu}_{\min}}{2I\rho^2} \left( (K_x + 3K_yL^2_{mx}) \frac{1 + \frac{1 - \theta}{\rho^2}}{1 - \rho^2} + \frac{1}{2I\rho^2} \right). \\
(D10)
\]

28
Next we choose $K_x$ and $K_y$ such that the RHS of (D8)-(D10) are equal. The RHS of (D8) and (D9) are so if

$$K_x = q_0(\theta) \cdot K_y,$$

with $q_0(\theta) \triangleq 2(\bar{L}_{mx} + L_{mx})^2 + \frac{15L_{mx}^2\rho^2}{(1-\rho^2)^2} \frac{1 + \frac{\theta}{\rho^2}}{1-\theta}. \quad (D11)$$

Using (D11), the RHS of (D9) and (D10) are equal if

$$\frac{\rho^2(1-\rho^2)(1-\theta)K_y}{3} = \frac{1}{q_1(\theta)K_y} + \frac{L}{2I\rho^2}, \quad (D12)$$

where

$$q_1(\theta) \triangleq (q_0(\theta) + 3L_{mx}^2) \frac{1 + \frac{\theta}{\rho^2}}{1-\rho^2}. \quad (D13)$$

Equation (D12) has the positive solution given by

$$K_y = \bar{K}_y(\theta) \triangleq \sqrt{\frac{L^2 + \frac{48L_{mx}^2\rho^2 q_1(\theta)}{(1-\rho^2)(1-\theta)} - L}{4I\rho^2 q_1(\theta)}}. \quad (D14)$$

Using the above choices, (D8)-(D10) reduce to

$$\alpha < \bar{\alpha}(\theta) \triangleq \frac{\bar{\mu}_{\text{min}}}{6I} \bar{K}_y(\theta)(1-\theta)(1-\theta). \quad (D15)$$

Now let us choose $\theta \in (0, 1)$ in order to maximize $\bar{\alpha}(\theta)$ in (D15). This is equivalent to maximize the following function:

$$M(\theta) \triangleq \sqrt{L^2 Q(\theta)^2 + s_0 Q(\theta)} - L Q(\theta), \quad \text{with} \quad Q(\theta) \triangleq \frac{1 - \theta}{q_1(\theta)}, \quad s_0 \triangleq \frac{48L_{mx}^2\rho^2}{1-\rho^2}. \quad (D16)$$

Rewrite $q_1(\theta)$ in (D13) as

$$q_1(\theta) = \left[ s_1 + s_2 \frac{1 + \frac{\theta}{\rho^2}}{1-\theta} \right] \left( 1 + \frac{1 - \theta}{\theta} \frac{\rho^2}{1-\rho^2} \right), \quad (D17)$$

with

$$s_1 \triangleq \frac{3L_{mx}^2}{1-\rho^2} + 2 \left( L_{mx} + \bar{L}_{mx} \right)^2, \quad s_2 \triangleq \frac{15L_{mx}^2\rho^2}{(1-\rho^2)^2}. \quad (D18)$$

Since $\lim_{\theta \to 0} M(\theta) = \lim_{\theta \to 1} M(\theta) = 0$ and $M(\bullet)$ is differentiable and positive on $(0, 1)$, the maximizer of $M(\theta)$ is a solutions of $M'(\theta) = 0$, that is

$$Q'(\theta) \left[ \frac{2L^2 Q(\theta) + s_0}{2\sqrt{L^2 Q(\theta)^2 + s_0 Q(\theta)}} - L \right] = 0. \quad (D19)$$
Note that, since \(L^2Q(\theta)^2 + s_0Q(\theta) \neq 0\) and \(s_0 \neq 0\), it must be

\[
\frac{2L^2Q(\theta) + s_0}{2\sqrt{L^2Q(\theta)^2 + s_0Q(\theta)}} - L \neq 0.
\]

Hence, (D19) reduces to \(Q'(\theta) = 0\), which has the following four solutions:

\[
\rho \pm 1, \quad s_1 + 2s_2(1 - \rho^2) \pm \sqrt{(s_1 + 2s_2(1 - \rho^2)^2)(s_1 + 2s_2(1 + \rho)^2^2)},
\]

(D20)

The only eligible solution (residing in \((0, 1)\)) is

\[
\theta^* \triangleq \frac{\rho}{1 + \rho},
\]

(D21)

which gives

\[
M(\theta^*) = \frac{\sqrt{L^2 + s_0h} - L}{h}, \quad \text{with } h \triangleq (1 + \rho)^2(s_1 + s_2(1 + \rho^2)).
\]

(D22)

Thus the maximum of \(\bar{\alpha}(\theta)\) is

\[
\alpha^* \triangleq \bar{\alpha}(\theta^*) = \frac{\bar{\mu}_{\text{min}}(1 - \rho^2)}{24I^2\rho^2}M(\theta^*).
\]

(D23)

Note that by (D22) and (D23) it is not difficult to check that

\[
\lim_{\rho \to 0} \alpha^* = \frac{\bar{\mu}_{\text{min}}}{L}.
\]

(D24)

Finally, using above choice of parameters, (51) reads

\[
\Phi^{\nu+1} - \Phi^{\nu} \leq - \frac{6}{\bar{\mu}_{\text{min}}} \left( L_m^2 I^2 + (I_m + L_m)^2 \right)(\alpha^* - \alpha)\|x^*_\perp\|^2 - \frac{6I}{\bar{\mu}_{\text{min}}} (\alpha^* - \alpha)\|y^*_\perp\|^2
\]

\[
\quad - \alpha \rho^2 \left( K_y(\theta^*) \left( q_0(\theta^*) + 3L_m \right) \frac{1 + \frac{1 - \theta^*}{\theta'}}{1 - \rho^2} + \frac{L}{2I\rho^2} \right) (\alpha^* - \alpha)\|\hat{d}^\nu\|^2,
\]

(D25)

which trivially leads to (52).

\[\square\]

Appendix E. Proof of Lemma 2.15

By the first order optimality of \(\hat{x}_i^\nu\) and \(\hat{x}_i(\hat{x}^\nu)\), we have

\[
\left\langle \nabla \tilde{f}_i(\hat{x}_i^\nu; x_i^\nu) + Iy_i^\nu - \nabla f_i(x_i^\nu), \hat{x}_i(\hat{x}^\nu) - \hat{x}_i^\nu \right\rangle + G(\hat{x}_i(\hat{x}^\nu)) - G(\hat{x}_i^\nu) \geq 0,
\]

\[
\left\langle \nabla \tilde{f}_i(\hat{x}_i(\hat{x}^\nu); \hat{x}^\nu) + \sum_{j \neq i} \nabla f_j(\hat{x}^\nu), \hat{x}_i^\nu - \hat{x}_i(\hat{x}^\nu) \right\rangle + G(\hat{x}_i^\nu) - G(\hat{x}_i(\hat{x}^\nu)) \geq 0.
\]

(E1)
Summing up (E1) and using Assumptions D1 and D2 and Definition 2.1, yield
\[
\bar{\mu}_i \| \hat{x}_i (\bar{x}^\nu) - \bar{x}^\nu \|^2 \\
\leq \langle I y^\nu - \nabla F (\bar{x}^\nu), \hat{x}_i (\bar{x}^\nu) - \bar{x}^\nu \rangle \\
+ \left\langle \nabla \bar{f}_i (\hat{x}_i (\bar{x}^\nu) ; x^\nu) - \nabla \bar{f}_i (\hat{x}_i (\bar{x}^\nu) ; \bar{x}^\nu), \hat{x}_i (\bar{x}^\nu) - \bar{x}^\nu \right\rangle \\
+ \langle \nabla f_i (\bar{x}^\nu) - \nabla f_i (x^\nu), \hat{x}_i (\bar{x}^\nu) - \bar{x}^\nu \rangle
\]
where in (a) we used \( \bar{y}^\nu = \bar{g}^\nu \). Using Assumption D3, inequality (E2) yields
\[
\| \hat{x}_i (\bar{x}^\nu) - \bar{x}^\nu \|^2 \\
\leq \frac{I}{\bar{\mu}_i} \| y^\nu - \bar{y}^\nu \| + \frac{I}{\bar{\mu}_i} \left\| \bar{g}^\nu - \frac{1}{I} \nabla F (\bar{x}^\nu) \right\| + \frac{\bar{L}_i + L_i}{\bar{\mu}_i} \| \bar{x}^\nu - x^\nu \| (E3)
\]
\[
\leq \frac{I}{\bar{\mu}_i} \| y^\nu - \bar{y}^\nu \| + \frac{L_{\max} \sqrt{I}}{\bar{\mu}_i} \| \bar{x}^\nu \| + \frac{\bar{L}_i + L_i}{\bar{\mu}_i} \| \bar{x}^\nu - x^\nu \| .
\]

\[\square\]

Appendix F. Proof of Proposition 3.4

The proof of the first two inequalities (69) and (72) follows the same steps as those used to prove Proposition 2.9. Hence, next we prove only (70) and (71).

Consider (67a); dividing both sides by \( z^\nu + B \) yields
\[
\frac{\| x_{\phi,\perp}^\nu + \bar{B} \|}{z^\nu + B} \leq \rho_B^2 (1 + \bar{B} \epsilon_x) z^{-\bar{B}} \frac{\| x_{\phi,\perp}^\nu \|}{z^\nu} + 2 I^3 \alpha^2 \left( \bar{B} + \frac{1}{\epsilon_x} \right) \sum_{t=0}^{\bar{B}-1} z^{t-B} \frac{\| d^{\nu+t} \|}{z^{\nu+t}} .
\]
Taking the maximum over \( \nu = 0, 1, \ldots, K - \bar{B} \) while using
\[
\frac{\| x_{\phi,\perp}^\nu \|}{z^\nu} \leq \sum_{t=0}^{\bar{B}-1} \frac{\| x_{\phi,\perp}^t \|}{z^t} , \quad \nu = 0, \ldots, \bar{B} - 1,
\]
yields
\[
X_{\phi,\perp}^K (z) \leq \rho_B^2 (1 + \bar{B} \epsilon_x) z^{-\bar{B}} X_{\phi,\perp}^{K-B} (z) + 2 I^3 \alpha^2 (\bar{B} + \epsilon_x^{-1}) \sum_{t=0}^{\bar{B}-1} z^{t-B} D^{K-B+t} (z) + \sum_{t=0}^{\bar{B}-1} \frac{\| x_{\phi,\perp}^t \|}{z^t}
\]
\[
\leq \rho_B^2 (1 + \bar{B} \epsilon_x) z^{-\bar{B}} X_{\phi,\perp}^K (z) + 2 I^3 \alpha^2 (\bar{B} + \epsilon_x^{-1}) \sum_{t=0}^{\bar{B}-1} z^{t-B} D^K (z) + \sum_{t=0}^{\bar{B}-1} \frac{\| x_{\phi,\perp}^t \|}{z^t} .
\]
Rearranging the terms and using the inequality \((1 - z^B)/(1 - z) < \bar{B}\) leads to (70). The proof of (71) follows the same rationale and thus it is omitted. □

Appendix G. Proof of Theorem 3.5

We follow the same roadmap as in the proof of Theorem 2.10. We minimize \(\mathcal{P}(\alpha, 1)\) defined in (79) with respect to \(\epsilon_x\) and \(\epsilon_y\). The optimal \(\epsilon_x\) and \(\epsilon_y\) are

\[
\epsilon^* = \arg\min_{\epsilon} \frac{\bar{B} + \epsilon^{-1}}{1 - \rho_B^2 (1 + B\epsilon)} = \frac{1 - \rho_B}{\rho_B \cdot \bar{B}}.
\]  

(G1)

Following the same steps as in the proof of Theorem 2.10, we obtain the following expression for the optimal \(\epsilon_{opt}\) appearing in \(\eta(\alpha)\) and \(\sigma(\alpha)\):

\[
\epsilon^*_{opt} = \bar{\mu}_{min} - \frac{L}{2} \alpha,
\]  

(G2)

where \(\alpha\) must satisfy

\[
\alpha < 2\bar{\mu}_{min}/L.
\]  

(G3)

Setting \(\epsilon_x = \epsilon_y = \epsilon^*\), \(\epsilon_{opt} = \epsilon^*_{opt}\), and denoting the corresponding \(\mathcal{P}(\alpha, z)\) as \(\mathcal{P}^*(\alpha, z)\), the expression of \(\mathcal{P}^*(\alpha, 1)\) reads

\[
\mathcal{P}^*(\alpha, 1) \triangleq G^*_P(\alpha) \cdot C_1 \cdot 8\phi_{ub} I^2 L_{mx}^2 \cdot 2I^3 \cdot \frac{B}{(1 - \rho_B)^2} \alpha^2 \\
+ (G^*_P(\alpha) \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2) \cdot 4I^4 \phi_{ub}^{-2} L_{mx}^2 \cdot \frac{B}{(1 - \rho_B)^2} \alpha^2 \\
+ (G^*_P(\alpha) \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2) \cdot 4I^4 \phi_{ub}^{-2} L_{mx}^2 \cdot 8I^3 \cdot \frac{B^2}{(1 - \rho_B)^4} \alpha^2,
\]  

(G4)

where

\[
G^*_P(\alpha) \triangleq \frac{6(L^2 + L_{mx}^2)}{\mu} + \frac{1}{\mu} \cdot (\bar{\mu}_{min} - \frac{L}{2} \alpha)^2.
\]  

(G5)

Since \(\mathcal{P}^*(\alpha, 1)\) is continuous and monotonically increasing with respect to \(\alpha \in (0, 2\bar{\mu}_{min}/L)\) and \(\mathcal{P}^*(0, 1) = 0\), there must exist a sufficiently small \(\bar{\alpha} < 2\bar{\mu}_{min}/L\) so that \(\mathcal{P}^*(\alpha, 1) < 1\), for all \(\alpha \in (0, \bar{\alpha})\). One can verify that for any \(\alpha \in (0, 2\bar{\mu}_{min}/L)\), \(\mathcal{P}^*(\alpha, z)\) is continuous at \(z = 1\). Therefore, for any fixed \(\alpha \in (0, \bar{\alpha})\), \(\mathcal{P}^*(\alpha, 1) < 1\) implies the existence of some \(\bar{z}(\alpha) < 1\) so that \(\mathcal{P}^*(\alpha, \bar{z}(\alpha)) < 1\). Next, we provide an expression of such an \(\bar{\alpha}\).

By imposing \(\alpha \leq \bar{\mu}_{min}/L\) we can upper bound \(G^*_P(\alpha)\) as

\[
G^*_P(\alpha) \leq G^*_P (\bar{\mu}_{min}/L) = \frac{1}{\mu} + \frac{24(L^2 + L_{mx}^2)}{\mu \cdot \bar{\mu}_{min}^2}.
\]  

(G6)
Therefore, for $\mathcal{P}^*(\alpha, 1) < 1$, it suffices to require

\[
\alpha < \alpha_2 \triangleq \left( G^*_P(\bar{\mu}_{\text{min}}/L) \cdot C_1 \cdot 8\phi_{ub}I^2L_{\text{mx}}^2 \cdot 2I^3 \cdot \frac{\bar{B}}{(1 - \rho_B)^2} \right) \tag{G7}
\]

\[
+ \left( G^*_P(\bar{\mu}_{\text{min}}/L) \cdot 2\phi_{ub}I^2 \cdot C_1 + C_2 \right) \cdot 4I^4\phi_{\#l}^{-2}L_{\text{mx}}^2 \cdot \frac{\bar{B}}{(1 - \rho_B)^2} \tag{G8}
\]

\[
+ \left( G^*_P(\bar{\mu}_{\text{min}}/L) \cdot 2\phi_{ub}I^2 \cdot C_1 + C_2 \right) \cdot 4I^4\phi_{\#l}^{-2}L_{\text{mx}}^2 \cdot 8I^3 \cdot \frac{\bar{B}^2}{(1 - \rho_B)^4} \right)^{-1/2}, \tag{G9}
\]

which together with $\alpha \leq \bar{\mu}_{\text{min}}/L$ lead to a $\bar{\alpha}$ given by $\bar{\alpha} = \min\{\bar{\mu}_{\text{min}}/L, \alpha_2\}$. □

### G.1. Expression of the rate in Theorem 3.5

The following theorem provides an explicit expression of the convergence rate in Theorem 3.5 in terms of the step-size $\alpha$; the constants $J$ and $A_{\#}$ therein are defined in \((G20)\) and \((G17)\) with $\theta = 1/2$, respectively.

**Theorem G.1.** In the setting of Theorem 3.5, suppose that the step-size $\alpha$ satisfies $\alpha \in (0, \alpha_{\text{mx}})$, with $\alpha_{\text{mx}} \triangleq \min\{(1 - \rho_B)^2/A_{\#}, \bar{\mu}_{\text{min}}/L\}$. Then $\{U(x^*_\nu)\}_{\nu \in \mathbb{N}_+}$ converges to $U^*$ at the R-linear rate $O(z^*)$, for all $i = 1, \ldots, I$, where

\[
z = \begin{cases} 
1 - J \cdot \alpha, & \text{if } \alpha \in (0, \min\{\alpha^*, \alpha_{\text{mx}}\}), \\
\left( \rho_B + \sqrt{A_{\#} \alpha} \right)^2, & \text{if } \alpha \in [\min\{\alpha^*, \alpha_{\text{mx}}\}, \alpha_{\text{mx}}],
\end{cases} \tag{G10}
\]

and

\[
\alpha^* = \arg\min_{\alpha \in [0, \alpha_{\text{mx}}]} \left\{ \left( \rho_B + \sqrt{A_{\#} \alpha} \right)^2, 1 - J \alpha \right\}. \tag{G11}
\]

Upper and lower bounds of $\alpha^*$ are: for $\bar{B} > 1$,

\[
\min \left[ \alpha_{\text{mx}}, \left( \frac{\sqrt{A_{\#} + 2(1 - \rho_B)JB - \sqrt{A_{\#}}}}{\sqrt{JB}} \right)^2 \right] \leq \alpha^* \leq \min \left( \alpha_{\text{mx}}, \frac{1 - \rho_B}{J} \right). \tag{G12}
\]

**Proof.** The proof follows similar steps as the proof of Theorem 2.4. For sake of simplicity, we used the same notation as therein. We find the smallest $z$ satisfying \((73)\) such that $\mathcal{P}(\alpha, z) < 1$, for $\alpha \in (0, \alpha_{\text{mx}})$, and $\alpha_{\text{mx}} \in (0, 1)$ to be determined [recall that $\mathcal{P}(\alpha, z)$ is defined in \((79)\)].

Let us begin considering the condition $z > \sigma(\alpha)$ in \((73)\), with $\sigma(\alpha)$ defined in \((23)\). To simplify the analysis, we impose instead the following stronger version

\[
z \geq \sigma(\alpha) + \frac{(\theta \cdot \alpha) \cdot (\bar{\mu}_{\text{min}} - \frac{L}{\mu} \alpha - \frac{L}{\mu} \epsilon_{\text{opt}})}{6I^2L_{\text{mx}}^2} + (\bar{\mu}_{\text{min}} - \frac{L}{\mu} \alpha - \frac{L}{\mu} \epsilon_{\text{opt}}), \tag{G13}
\]

\[33\]
for some \( \theta \in (0, 1) \), which will be chosen to tighten the bound. Notice that the RHS of (B1) is strictly larger than \( \sigma(\alpha) \) but still strictly less than one, for any \( \alpha \in (0, 2\mu_{\min} - \epsilon_{opt}) \), with given \( \epsilon_{opt} \in (0, 2\mu_{\min}) \).

Observe that in the expression of \( \mathcal{P}(\alpha, z) \) in (79), the only coefficient multiplying \( \alpha^2 \) that depends on \( \alpha \) is the optimization gain \( G_P(\alpha, z) \equiv \eta(\alpha)/(z - \sigma(\alpha)) \). Using (G13), \( G_P(\alpha, z) \) in (74) can be upper bounded as

\[
G_P(\alpha, z) \leq \inf_{\epsilon_{opt} \in (0, 2\mu_{\min} - \alpha L)} \frac{1}{\epsilon_{opt}} \cdot \frac{6(L^2 + \bar{L}^2)}{\mu} + \frac{1}{\mu} \left( \mu_{\min} - \frac{1}{2} \alpha - \frac{1}{2} \epsilon_{opt} \right) \cdot \theta^{-1}
\]

where the minimum is attained at \( \epsilon_{opt} \equiv \bar{\mu}_{\min} - \alpha L/2 \), provided that \( \alpha \in (0, 2\mu_{\min}/L) \); and \( G_P^*(\alpha) \) is defined in (G5). Using in \( \mathcal{P}(\alpha, z) \) the upper bound (G14) and letting \( \epsilon_{opt} = \epsilon_{opt} \), we get the following sufficient condition for \( \mathcal{P}(\alpha, z) < 1 \):

\[
G_P^*(\alpha) \cdot \theta^{-1} \cdot G_X(z) \cdot C_1 \cdot 8\phi_{ub} I^2 L_{\text{max}}^2 \cdot 2 I^3 \cdot \alpha^2 + \left( G_P^*(\alpha) \cdot \theta^{-1} \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2 \right) \cdot G_Y(z) \cdot 4I^4 \phi_{lb}^{-2} L_{\text{max}}^2 \cdot \alpha^2 + \left( G_P^*(\alpha) \cdot \theta^{-1} \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2 \right) \cdot G_Y(z) \cdot 4I^4 \phi_{lb}^{-2} L_{\text{max}}^2 \cdot G_X(z) \cdot 8I^3 \cdot \alpha^2 < 1.
\]

To minimize the left hand side, we set \( \epsilon_x = \epsilon_y = (\sqrt{z^B} - \rho_B)/(\rho_B \cdot \bar{B}) \). Furthermore, using the fact that \( G_P^*(\alpha) \) is monotonically increasing on \( \alpha \in (0, 2\mu_{\min}/L) \), and restricting \( \alpha \in (0, \mu_{\min}/L) \), a sufficient condition for (G15) is

\[
\alpha \leq \alpha(z) \equiv \left( A_{1,\theta} \frac{1}{(\sqrt{z^B} - \rho_B)^2} + A_{2,\theta} \frac{1}{(\sqrt{z^B} - \rho_B)^2} + A_{3,\theta} \frac{1}{(\sqrt{z^B} - \rho_B)^4} \right)^{-1/2},
\]

where \( A_{1,\theta}, A_{2,\theta} \) and \( A_{3,\theta} \) are constants defined as

\[
A_{1,\theta} \equiv G_P^*(\mu_{\min}/L) \cdot \theta^{-1} \cdot C_1 \cdot 8\phi_{ub} I^2 L_{\text{max}}^2 \cdot 2I^3 \bar{B} \]
\[
A_{2,\theta} \equiv \left( G_P^*(\mu_{\min}/L) \cdot \theta^{-1} \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2 \right) \cdot 4I^4 \phi_{lb}^{-2} L_{\text{max}}^2 \bar{B} \]
\[
A_{3,\theta} \equiv \left( G_P^*(\mu_{\min}/L) \cdot \theta^{-1} \cdot 2\phi_{ub} I^2 \cdot C_1 + C_2 \right) \cdot 4I^4 \phi_{lb}^{-2} L_{\text{max}}^2 \cdot 8I^3 \bar{B}^2.
\]

To find an explicit expression of a lower bound of \( z \) in terms of \( \alpha \), instead of \( \alpha(z) \), we use in (G16) the lower bound \( \alpha(z) \geq (\sqrt{z^B} - \rho_B)^2/A_\theta \), with \( A_\theta \) defined in (G17) below. We obtain

\[
z \geq \left( \rho_B + \sqrt{A_\theta \alpha} \right) \bar{\mu}, \quad \text{with} \quad A_\theta \equiv \sqrt{A_{1,\theta} + A_{2,\theta} + A_{3,\theta}}.
\]

Notice that, under \( \epsilon_x = \epsilon_y = (\sqrt{z^B} - \rho_B)/(\rho_B \cdot \bar{B}) \), (G17) is sufficient for \( z > \sqrt{\rho_B^2 (1 + B\epsilon_x)} \) and \( z > \sqrt{\rho_B^2 (1 + B\epsilon_y)} \), which are the two conditions in (73). Therefore, overall, \( z \) must satisfy (G13) and (G17). Letting \( \epsilon_{opt} = \epsilon_{opt} \) in (G13), the
condition reduces to
\[ z \geq 1 - \frac{\mu_{\min} - \frac{L}{2} \alpha}{12(L^2 + L_{\max}^2)\mu} + (\mu_{\min} - \frac{L}{2} \alpha) \cdot (1 - \theta) \alpha. \]  
(G18)

Therefore, the overall convergence rate can be upper bounded by \( O(\bar{z}^\nu) \), where
\[ \bar{z} = \inf_{\theta \in (0,1)} \max \left\{ \left( \rho_B + \sqrt{A_\theta \alpha} \right) \frac{\bar{z}}{\pi}, 1 - \frac{\mu_{\min} - \frac{L}{2} \alpha}{12(L^2 + L_{\max}^2)\mu} + (\mu_{\min} - \frac{L}{2} \alpha) \cdot (1 - \theta) \alpha \right\}, \]
with \( A_\theta \) defined in (G17).

Finally, we further simplify (G19). Letting \( \theta = 1/2 \) and using \( \alpha \in (0, \mu_{\min}/L] \), the second term in the max of (G19) can be upper bounded by
\[ 1 - \left( \rho_B^2 \bar{B} / \mu + \sqrt{A_{1/2} \alpha} \right) \frac{\mu_{\min} \mu}{\Delta J} \cdot \frac{1}{2} \alpha. \]  
(G20)

The condition \( \bar{z} < 1 \) imposes the following upper bound on \( \alpha \): \( \alpha < \alpha_{\text{mx}} = \min\{(1 - \rho_B^2)/(A_{1/2} \bar{B}), \mu_{\min}/L\} \). Eq. (G19) then simplifies to (G11). Note that such an \( \alpha^* \) exists; in fact, as \( \alpha \) increases from 0, the first term in the max in (G11) is monotonically increasing from \( \rho_B^2 \), while the second term is monotonically decreasing from 1.

Now let us establish lower and upper bounds of \( \alpha^* \) given in (G12). The upper bound can be established by observing that \( 1 - J\alpha \) is non-increasing and \( (\rho_B + \sqrt{A_{1/2} \alpha}) \bar{B} \) is non-decreasing; therefore, there holds \( \alpha^* \leq \min(\bar{\alpha}, \alpha_{\text{mx}}) \), where \( \bar{\alpha} \) satisfies \( 1 - J\bar{\alpha} = \rho_B^2 \bar{B} \).

The lower bound can be obtained as follows: considering (G11), \( \alpha^* \) is the smallest \( \alpha > 0 \) satisfying
\[ (1 - J\alpha) \frac{\bar{B}}{\pi} = \rho_B + \sqrt{A_{1/2} \alpha}. \]  
(G21)

Note that the LHS of (G21) is a convex function in \( \alpha \) (when \( \bar{B} > 1 \)) and thus it is lower bounded by \( 1 - \frac{J\bar{B}}{2} \). The RHS is non-decreasing, thus the solution of
\[ 1 - \frac{J\bar{B}}{2} = \rho_B + \sqrt{A_{1/2} \alpha} \]  
(G22)

is smaller than the one of (G21). Therefore, the minimum between solution of (G22) and \( \alpha_{\text{mx}} \) gives the lower bound in (G12).

To conclude, given the step-size satisfying \( \alpha \in (0, \alpha_{\text{mx}}) \), by Lemma 2.8, the sequence \( \{||d^\nu||^2\}_{\nu \in \mathbb{N}_+} \) converges at rate \( O(z^\nu) \), with \( z \) defined in (G10). 

\[ \square \]
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