Existence of positive solutions for $p$-Laplacian boundary value problems of fractional differential equations

Farid Chabane¹, Maamar Benbachir², Mohammed Hachama³ and Mohammad Esmael Samei⁴*

Abstract
In this paper, we study the existence and multiplicity of $p$-concave positive solutions for a $p$-Laplacian boundary value problem of two-sided fractional differential equations involving generalized-Caputo fractional derivatives. Using Guo–Krasnoselskii fixed point theorem and under some additional assumptions, we prove some important results and obtain the existence of at least three solutions. To establish the results, Green functions are used to transform the considered two-sided generalized Katugampola and Caputo fractional derivatives. Finally, applications with illustrative examples are presented to show the validity and correctness of the obtained results.

MSC: 34A08; 34B15

Keywords: Fractional differential equations; $p$-Laplacian; Generalized Caputo fractional derivative; Concave; Positive solutions

1 Introduction
Last decades witnessed an increased number of theoretical studies and practical applications of fractional differential equations in science, engineering, biology, etc. [1–10]. In particular, fractional $p$-Laplacian has been used in modeling different problems [11–17].

In 2007, Su et al. studied the existence of positive solution for a nonlinear four-point singular boundary value problem

\[
\begin{align*}
(\phi_p(q'))' + h(\tau)\phi(q(\tau)) &= 0, & 0 < \tau < 1, \\
\eta_1\phi_p(q(0)) - \eta_2\phi_p(q'(\xi)) &= 0, \\
\eta_3\phi_p(q(1)) + \eta_4\phi_p(q'(\lambda)) &= 0,
\end{align*}
\]

by using the fixed point index theory, where $\eta_1, \eta_3 > 0$, $\eta_2, \eta_4 \geq 0$, $0 < \xi < \lambda < 1$, and $h : (0,1) \to [0,\infty)$ [15]. Also, they applied the theory to study the existence of positive
solutions for the nonlinear third-order two-point singular boundary value problem

\[
\begin{align*}
(\phi_p(q^{(n-1)}))(\tau) + h(\tau)\varphi(q(\tau)) &= 0, \quad 0 < \tau < 1, \\
q(0) = q'(0) = \cdots = q^{(n-3)}(0) = q^{(n-1)}(0) &= 0, \\
q(1) &= \sum_{i=1}^{m-2} \eta_i q(\lambda_i),
\end{align*}
\]

where

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-2} < 1, \quad \eta_i > 0,
\]

with \(\sum_{i=1}^{m-2} \eta_i \lambda_i^{n-2} < 1\) [18]. Chai in [19], considered the nonlinear fractional boundary value problem

\[
\begin{align*}
0^+ G_{RL}^{\sigma_1}(\phi_p(0^+ G_{RL}^{\sigma_2} q))((\sigma_1, 0^+ G_{RL}^{\sigma_2} q) = 0, \quad 0 < \tau < 1, \\
q(0) = 0^+ G_{RL}^{\sigma_1} q(0) = q(1) + \eta_1 0^+ G_{RL}^{\sigma_1} q(1) = 0,
\end{align*}
\]

on a cone and obtained some results and positive solutions, where \(1 < \sigma_2 \leq 2, 0 < \sigma_1, \sigma_3 \leq 1, 0 \leq \sigma_2 - \sigma_3 - 1, \eta > 0, \) and \(p\)-Laplacian operator is defined as \(\phi_p(\xi) = |\xi|^{p-2} \xi, p > 1\). Based on the coincidence degree theory, Chen et al. gave new results about the problem

\[
\begin{align*}
0^+ C^{\sigma_1}_{C}(\phi_p(0^+ C^{\sigma_2}_{C} q))((\sigma_1, 0^+ C^{\sigma_2}_{C} q) = \varphi(\tau, q(\tau)), \quad \tau \in [0, 1], \\
0^+ C^{\sigma_2}_{C} q(0) = 0^+ C^{\sigma_2}_{C} q(1) = 0,
\end{align*}
\]

where \(0 < \sigma_1, \sigma_2 \leq 1 (1 < \sigma_1 + \sigma_2 \leq 2)\) [20]. In 2018, Bai used the Guo–Krasnoselskii fixed point theorem and the Banach contraction mapping principle to prove the existence and uniqueness of positive solutions for the following fractional boundary value problem:

\[
\begin{align*}
(\phi_p(0^+ C^{\sigma_1}_{RL} q))'(\tau) + \varphi(\tau, q(\tau)) &= 0, \quad 0 < \tau < 1, \\
q(0) = 0^+ C^{\sigma_1}_{RL} q(0) = 0^+ C^{\sigma_2}_{C} q(0) = 0^+ C^{\sigma_2}_{C} q(1) = 0,
\end{align*}
\]

where \(0 < \sigma_2 \leq 1, 2 < \sigma_1 < 2 + \sigma_2, 0^+ C^{\sigma_1}_{RL} \) and \(0^+ C^{\sigma_2}_{C}\) are the Riemann–Liouville and Caputo fractional derivatives of orders \(\sigma_1, \sigma_2\), respectively, \(p > 1\), and \(\varphi : [\tau_1, \tau_2] \times \mathbb{R} \to \mathbb{R}\) is a continuous function [21]. Using the coincidence degree theory, Tang et al. gave a new result on the existence of positive solutions to the fractional boundary value problem

\[
\begin{align*}
0^+ C^{\sigma_1}_{C}(\phi_p 0^+ C^{\sigma_2}_{C} q)(\tau) = \varphi(\tau, q(\tau)), \quad 0 < \sigma_1, \sigma_2 \leq 1, \\
q(0) = 0^+ C^{\sigma_1}_{C} q(0) = 0^+ C^{\sigma_2}_{C} q(1),
\end{align*}
\]

where \(1 < \sigma_1 + \sigma_2 \leq 2\) and \(0^+ C^{\sigma_i}_{C} (i = 1, 2)\) denotes the Caputo fractional derivatives [13]. Torres studied the existence and multiplicity for a mixed-order three-point boundary value problem of fractional differential equation involving Caputo’s differential operator and the boundary conditions with integer order derivatives

\[
\begin{align*}
(\phi_p(0^+ C^{\sigma_1}_{C} q))'(\tau) + h(\tau)\varphi(q(\tau)) &= 0, \quad 0 < \tau < 1, \\
0^+ C^{\sigma_2}_{C} q(0) = q(0) = q''(0) = 0, \quad q'(1) = \eta q'(\lambda),
\end{align*}
\]
where $\eta, \lambda \in (0, 1)$, $\sigma \in (2, 3)$ [12]. In 2022, Alkhazzan et al. proved the existence and uniqueness as well as the Hyers–Ulam stability for the following general system of nonlinear hybrid fractional differential equations under $p$-Laplacian operator:

$$
\begin{align*}
\mathcal{T}_{12}(\tau, q_2(\tau))_{|_{\tau=0}} &= 0, \\
\mathcal{T}_{22}(\tau, q_1(\tau))_{|_{\tau=0}} &= 0,
\end{align*}
$$

for $i \in \mathbb{R}_0^{m-1} \setminus \{1\}$, under the conditions

$$
\begin{align*}
q_1(i)_{|_{\tau=0}} &= q_1(i)_{|_{\tau=1}} = 0, \\
q_2(i)_{|_{\tau=0}} &= q_2(i)_{|_{\tau=1}} = 0,
\end{align*}
$$

for $i \in \mathbb{R}_0^{m-1}$, and

$$
\begin{align*}
q_1(1) - \frac{1}{(m-1)!} q_1^{(m-1)}(0) &= 0, \\
q_2(1) - \frac{1}{(m-1)!} q_2^{(m-1)}(0) &= 0,
\end{align*}
$$

where $D_{C}^{\sigma_0,i,j}_C$, $i, j = 1, 2$, are the Caputo fractional derivatives with $m - 1 < \sigma_0 \leq m$ and $m$ is a nonnegative integer number, $\mathcal{T}_{ij}$ is a continuous function and belongs to $L[0, 1]$, $\phi_p(\tau) = |\tau|^{p-2}\tau$ is a $p$-Laplacian operator, where $\phi_q = \phi_p^{-1}$ and $\frac{1}{\sigma_0} + \frac{1}{\sigma_1} = 1$ [14]. For more recent works of the models, we refer to [22–34].

In this work, we study the following $p$-Laplacian fractional boundary value problem:

$$
\begin{align*}
\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}(\phi_p(\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}q))(\tau) + h(\tau)\phi(q(\tau)) &= 0, \\
q(\hat{\alpha}) - F_1(\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}q(\hat{\alpha})) &= 0, \\
\delta_{\rho_1}^{1} q(\hat{\alpha}) &= 0, \\
\delta_{\rho_1}^{1} q(i) &= \delta_{\rho_1}^{1} q(\eta) + \lambda, \\
\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}q(i) &= -\delta_{\rho_2}^{1} \left[ \phi_p(\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}q) \right](\hat{\alpha}) \\
&= \delta_{\rho_2}^{1} \left[ \phi_p(\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}q) \right](\bar{i}) = 0,
\end{align*}
$$

where $\rho_1^{1/\sigma_1}C^{\sigma_1}_{C,k}$ and $\rho_2^{1/\sigma_2}C^{\sigma_2}_{C,k}$, $(\rho_1, \rho_2 \in \mathbb{R} \setminus \{1\})$ are the right- and left-sided Caputo–Katugampola fractional derivatives, $2 < \sigma_1, \sigma_2 \leq 3$, $\phi_p$ is the $p$-Laplacian operator, i.e., $\phi_p(\hat{\xi}) = |\hat{\xi}|^{p-2}\hat{\xi}$, $p > 1$,

$$
\delta_{\rho}^{1} = \left( \frac{d}{d\tau} \right)^{\rho},
$$

$F_1$ is a continuous even function, $q$, $h$ are continuous and positive functions. $\eta \in (\hat{\alpha}, \bar{i})$, $0 \leq \mu < 1$, and $\lambda \geq 0$. In this paper, we obtain some sufficient conditions ensuring the.
existence of at least one, two, and three positive solutions for fractional boundary value problem (9). These results can be extended in some works such as [35–37].

The rest of the paper is organized as follows. Section 2 presents some basic definitions, lemmas, and preliminary results. In Sect. 3, we derive some conditions on the parameter \( \lambda \) to obtain the existence of at least one positive solution. We derive an interval for \( \lambda \), which ensures the existence of \( \rho \)-concave positive solutions of the fractional boundary value problem in Sect. 4. In Sect. 5, we discuss the existence of multiple positive solutions. Finally, we give some illustrative examples in Sect. 6.

2 Preliminaries and background material

In addition to the notations introduced with problem (9), let \( J = [\alpha, \iota] \subset (0, \infty) \), and \( \rho > 0 \),

1: \( C(J) \) denotes the Banach space of continuous functions \( q \) on \( J \) endowed with the norm \( \| q \|_C = \max_{\tau \in J} |q(\tau)| \), and

\[
C^*(J) = \{ q \in C(J) : q(\tau) \geq 0 \ \forall \tau \in J \}.
\]

2: \( AC(J) \) and \( C^n(J) \) denote the spaces of absolutely continuous and \( n \) times continuously differentiable functions on \( J \) respectively.

3: \( L^p(\alpha, \iota) \) denotes the space of Lebesgue integrable functions on \( (\alpha, \iota) \).

4: \( C^n_{\rho}(J) \) is the Banach space of \( n \) continuously differentiable functions on \( J \) with respect to \( \delta_{\rho} \):

\[
C^n_{\rho}(J) = \{ q \in C(J) : \delta_{\rho}^k q \in C(J), k = 0, 1, \ldots, n \},
\]

endowed with the norm

\[
\| q \|_{C^n_{\rho}} = \sum_{k=0}^{n} \| \delta_{\rho}^k q \|_{C}.
\]

5: \( [\sigma] \) is the largest integer less than or equal to \( \sigma \). Throughout the paper, we use \( n = [\sigma] \) if \( \sigma \) is an integer and \( n = [\sigma] + 1 \) otherwise.

2.1 Fractional calculus

We present basic definitions and lemmas from fractional calculus theory [1, 2, 5–7].

Definition 2.1 (Function space) For \( r \in \mathbb{R} \), consider the Banach space

\[
\mathcal{M}^p_{\rho}(\alpha, \iota) = \left\{ q : J \to \mathbb{R} : \| q \|_{\mathcal{M}^p_{\rho}} = \left( \int_\alpha^\iota |q(\tau)|^p \frac{d\tau}{\tau} \right)^{1/p} < +\infty \right\}.
\]

Remark 2.1 If \( r \in \mathbb{R}_+^* \) and \( \iota \leq (pr)^{1/pr} \), then \( C(J) \hookrightarrow \mathcal{M}^p_{\rho}(J) \) and \( \| q \|_{\mathcal{M}^p_{\rho}} \leq \| q \|_C \) for each \( q \in C(J) \).

Now, we recall the Katugampola and Caputo–Katugampola fractional integrals and derivatives [38].
Definition 2.2 The Katugampola left-sided $^{\rho} L^\alpha_K$ and right-sided $^{\rho} L^\alpha_K$ fractional integrals of noninteger order $\alpha > 0$ of a function $q \in M_c^\rho ([a, i])$ are defined by

$$^{\rho} L^\alpha_K q(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{t^\rho - \xi^\rho}{t^\rho - \xi^\rho} \right)^{\alpha-1} q(t) \, dt,$$

$\tau > \hat{a}$,

$$^{\rho} L^\alpha_K q(t) = \frac{1}{\Gamma(\alpha)} \int_t^i \left( \xi^\rho - t^\rho \right)^{\alpha-1} q(t) \, dt,$$

$\tau < i$.

The Katugampola fractional derivatives of $q$ are defined by

$$^{\rho} D^\alpha_K q(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left( t^\rho - \xi^\rho \right)^{n-\alpha} q(t) \, dt,$$

$\rho$ \begin{itemize}
\item (i) \begin{itemize}
\item $^{\rho} L^\alpha_K q(t) \in M_c^\rho ([a, i])$;
\end{itemize}
\item (ii) $^{\rho} L^\alpha_K$ and $^{\rho} D^\alpha_K$ are linear;
\item (iii) $^{\rho} L^\alpha_K q(t) = I_{\rho^\alpha} p^{\rho^\alpha} q(t)$ when $\sigma_2 \geq \sigma_1$;
\item (iv) $^{\rho} L^\alpha_K q(t) = p^{\rho^\alpha} I_{\rho^\alpha} q(t)$.
\end{itemize}

Definition 2.4 The Caputo–Katugampola fractional derivatives of a function $q \in C_c^\rho ((\hat{a}, [i])$ (or $\in AC_c^\rho ([\hat{a}, i])$) are defined by

$$^{\rho} C^\sigma_{\rho} q(t) = \frac{1}{\Gamma(\sigma)} \int_{\hat{a}}^t \left( \frac{t^\rho - \xi^\rho}{t^\rho - \xi^\rho} \right)^{\sigma-1} q(t) \, dt,$$

and

$$^{\rho} C^\sigma_{\rho} q(t) = (-1)^n \left( \frac{t^\rho - \xi^\rho}{t^\rho - \xi^\rho} \right)^{\sigma-1} q(t) \, dt.$$
Lemma 2.6 ([38]) Let $\sigma_2 > \sigma_1 > 0$, $q \in M^p_{c_0}(a, \bar{i})$, $q \in AC^n_\sigma(J)$, or $C^n_\sigma(J)$. Then we have

$$p^\sigma q C \{p^\sigma D q\} = p^\sigma D q,$$

and for some real constants $N_k$ and $M_k$,

$$p^\sigma D q C \{p^\sigma D q\} = q - N_k \left( \frac{\tau^p}{\rho} \right)^k,$$

$$p^\sigma D q C \{p^\sigma D q\} = q - M_k \left( \frac{\rho^p}{\tau^p} \right)^k.$$

Lemma 2.7 ([2]) If $p^\sigma D q C \{p^\sigma D q\} \in C(J)$, then $q \in C^n_{p^{-1}}(J)$.

2.2 Fixed point theorems

Let $E$ be a real Banach function space, endowed with the infinity norm. A nonempty closed convex set $K \subset E$ is called a cone

(i) if for each $q \in K$ and for all $\lambda > 0$: $\lambda q \in K$;

(ii) if for all $q \in K$, if $-q \in K$, then $q = 0$.

A continuous operator is called completely continuous operator if it maps bounded sets into precompact sets. Let $K$ be a cone, $\ell > 0$,

$$\Omega_\ell = \{ q \in K : \|q\| < \ell \},$$

and $i$ is the fixed point index function.

Theorem 2.8 ([39, 40]) Let $L : K \cap \Omega_\ell \rightarrow K$ be a completely continuous operator such that $Lq \neq q$, $\forall q \in \partial \Omega_\ell$. Then

(i) if $\|Lq\| \leq \|q\|$ for all $q \in \partial \Omega_\ell$, then $i(L, \Omega_\ell, K) = 1$;

(ii) if $\|Lq\| \geq \|q\|$ for all $q \in \partial \Omega_\ell$, then $i(L, \Omega_\ell, K) = 0$.

Theorem 2.9 (Guo–Krasnoselskii [1]) Assume that $\Omega_1$ and $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, and $\overline{\Omega_2} \subset \Omega_2$. Let $L : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator. Consider

(D1) $\|Lq\| \leq \|q\|$ for all $q \in K \cap \partial \Omega_1$ and $\|Lq\| \geq \|q\|$ for all $q \in K \cap \partial \Omega_2$;

(D2) $\|Lq\| \leq \|q\|$, $\forall q \in K \cap \partial \Omega_2$ and $\|Lq\| \geq \|q\|$, $\forall q \in K \cap \partial \Omega_1$.

If (D1) or (D2) holds, then $L$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2.3 Convexity

Let $q : J \rightarrow (0, \infty)$ be continuous.

Definition 2.10 ([41, 42]) We say that $q$ is $p$-convex if

$$q \left( \left[ (1 - \eta)\tau^p + \eta \tilde{\tau}^p \right]^{\frac{1}{p}} \right) \leq (1 - \eta)q(\tau) + \eta q(\tilde{\tau})$$

for each $\tau, \tilde{\tau} \in J$, and $\eta \in [0, 1]$. $q$ is called $p$-concave if $(-q)$ is $p$-convex.
Remark 2.2 ([41, 42])
1. \( q \) is \( p \)-convex (concave) if and only if \( \varphi(\varphi^{-1}) \) is convex (concave), where \( \varphi(\tau) = \tau^{\rho} \).
2. \( \varphi \) is \( p \)-convex (concave) if and only if \( \delta_p \varphi(q) \) is increasing (decreasing).

The following technical hypotheses will be used later.

(H1) \( h \) does not vanish identically on any closed subinterval of \( (\hat{a}, \hat{t}) \).
(H2) \( F_\tau \) is even and continuous on \( \mathbb{R}^+ \), and there exist \( A, B > 0 \):

\[
-B^{p^{-1}} \leq F_\tau(v) \leq A v^{p^{-1}} \quad (v \in \mathbb{R}^+).
\]

3 Main results

We present some important lemmas which assist in proving our main results. Consider the linear generalized fractional boundary value problem associated with (9)

\[
\begin{align*}
\rho_1 \hat{a}^{\rho_1} & G_{C^}\text{K} q(\tau) + w(\tau) = 0, \quad \hat{a} < \tau < \hat{t}, \\
q(\hat{a}) - F_\tau(\rho_1 \hat{a}^{\rho_1} G_{C^}\text{K} q(\hat{a})) = 0, \\
\delta_\rho \hat{a} q(\hat{a}) = 0, \quad \delta_\rho \hat{t} q(\hat{t}) - B \delta_\rho \hat{a} q(\eta) = \lambda.
\end{align*}
\]

(14)

Lemma 3.1 For \( w \in C(f) \), the integral solution of (14) is given by

\[
q(\tau) = \int_\hat{a}^\tau G_1(\tau, \xi) w(\xi) \, d\xi + \mu \left( \frac{\rho_1 - \hat{a}^{\rho_1}}{\rho_1 (1 - \mu)} \right) \int_\hat{a}^\tau G_2(\tau, \xi) w(\xi) \, d\xi
\]

\[
+ \lambda \left( \frac{\rho_1 - \hat{a}^{\rho_1}}{\rho_1 (1 - \mu)} \right) + F_\tau(w(\hat{a}))
\]

(15)

for \( \tau, \xi \in J \), where

\[
G_1(\tau, \xi) = \begin{cases}
\frac{1}{\Gamma(\sigma_1)} \left( \frac{\rho_1}{\hat{a}^{\rho_1}} \right)^{\rho_1 - 2 \xi} \eta, & \xi \leq \tau, \\
\frac{1}{\Gamma(\sigma_1)} \left( \frac{\rho_1}{\hat{a}^{\rho_1}} \right)^{\rho_1 - 1 - 2 \xi} \eta^{\rho_1 - 1}, & \xi \leq \tau,
\end{cases}
\]

(16)

and

\[
G_2(\tau, \xi) = \begin{cases}
\frac{1}{\Gamma(\sigma_1)} \left( \frac{\rho_1}{\hat{a}^{\rho_1}} \right)^{\rho_1 - 2 \xi} \eta, & \xi \leq \tau, \\
\frac{1}{\Gamma(\sigma_1)} \left( \frac{\rho_1}{\hat{a}^{\rho_1}} \right)^{\rho_1 - 1 - 2 \xi} \eta^{\rho_1 - 1}, & \tau \leq \xi.
\end{cases}
\]

(17)

Proof By applying (12), equation (14) becomes

\[
q(\tau) = -l_0 - l_1 \left( \frac{\rho_1 - \hat{a}^{\rho_1}}{\rho_1} \right) - l_2 \left( \frac{\rho_1 - \hat{a}^{\rho_1}}{\rho_1} \right)^2
\]

\[
- \frac{\rho_1 - \sigma_1}{\Gamma(\sigma_1)} \int_{\hat{a}}^{\tau} (\hat{a}^{\rho_1} - \xi^{\rho_1})^{\rho_1 - 1 - 2 \xi} \eta^{\rho_1 - 1} \xi^{\rho_1 - 1} w(\xi) \, d\xi
\]
for some arbitrary constants $l_0, l_1, l_2 \in \mathbb{R}$. From the boundary conditions of (14) we get

$$q(\tau) = F_0(w(\hat{a})) + \lambda \left( \frac{\tau^{p_1} - \hat{a}^{p_1}}{p_1(1 - \mu)} \right)$$

$$- \frac{1}{\Gamma(\sigma_1)} \int_0^\tau \left( \frac{\tau^{p_1} - \xi^{p_1}}{p_1} \right)^{\sigma_1 - 1} \xi^{p_1 - 1} w(\xi) \, d\xi$$

$$+ \int_0^\tau \left( \frac{\tau^{p_1} - \xi^{p_1}}{p_1} \right)^{\sigma_1 - 1} \xi^{p_1 - 1} w(\xi) \, d\xi$$

The converse follows by direct computation. The proof is completed.
Now, consider the generalized $p$-Laplacian fractional boundary value problem associated with (9)

\[
\begin{align*}
\rho^2 \int_0^\tau \mathcal{C}_{\mathbb{C}^\ast}^{p_2}(\phi_p(p_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau))) &= w(\tau), \quad \hat{\alpha} < \tau < \hat{\lambda}, \\
q(\hat{\alpha}) - F_p(\rho_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\hat{\alpha})) &= 0, \\
\delta_{p_1}^2 q(\hat{\alpha}) &= 0, \\
\delta_{p_1}^1 q(\hat{\alpha}) - \rho \delta_{p_1}^1 q(\eta) &= \lambda, \\
\rho_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau) &= \delta_{p_2}^1 \left[ \phi_p(p_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau)) \right](\hat{\alpha}) \\
&= \delta_{p_2}^1 \left[ \phi_p(p_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau)) \right](\hat{\alpha}) = 0.
\end{align*}
\]

**Lemma 3.2** For $w(\tau) \in C^1(I)$, fractional boundary value problem (18) has a unique solution

\[
q(\tau) = \int_a^\tau \mathcal{G}_1(\tau, \xi)\phi_p \left( \int_a^\tau \mathcal{H}(\xi, s)w(s) \, ds \right) \, d\xi
\]

\[
+ \mu \left( \frac{\tau - \hat{\alpha}}{\rho_1 - \mu \rho_1} \right) \int_a^\tau \mathcal{G}_2(\tau, \xi)\phi_p \left( \int_a^\tau \mathcal{H}(\xi, s)w(s) \, ds \right) \, d\xi
\]

\[
+ \lambda \left( \frac{\tau - \hat{\alpha}}{\rho_1 - \mu \rho_1} \right) + F_p \left( \phi_p \left( \int_a^\tau \mathcal{H}(\tau, \xi)w(\xi) \, d\xi \right) \right),
\]

where

\[
\mathcal{H}(\tau, \xi) = \begin{cases} 
\frac{1}{\Gamma(\rho_2 - 1)} \frac{(\xi - \tau)^{\rho_2 - 2}}{\rho_2} \left( \frac{\xi - \tau}{\rho_2} \right)^{\rho_2 - 2} \xi^{\rho_2 - 1} & \tau < \xi, \\
\frac{1}{\Gamma(\rho_2)} \left( \frac{\xi - \tau}{\rho_2} \right)^{\rho_2 - 1} \xi^{\rho_2 - 1} & \tau < \xi, \\
\frac{1}{\Gamma(\rho_2 - 1)} \frac{(\xi - \tau)^{\rho_2 - 2}}{\rho_2} \left( \frac{\xi - \tau}{\rho_2} \right)^{\rho_2 - 2} \xi^{\rho_2 - 1} & \xi < \tau,
\end{cases}
\]

$\mathcal{G}_1(\tau, \xi), \mathcal{G}_2(\tau, \xi)$ are defined in Lemma 3.1 and $\tilde{p} = \frac{p}{p - 1}$.

**Proof** From Lemma 2.6, equation (18) is equivalent to the equation

\[
\phi_p(p_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau)) = -l_0 - l_1 \left( \frac{\xi^2 - \tau^2}{\rho_2} \right) - l_2 \left( \frac{\xi^2 - \tau^2}{\rho_2} \right)^2
\]

\[
+ \rho_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} w(\tau)
\]

for some constants $l_0, l_1, l_2 \in \mathbb{R}$. Using the second boundary condition, we get

\[
\phi_p(p_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau)) = \rho_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} \mathcal{G}(\tau, \xi)w(\xi) \, d\xi
\]

\[
- \left( \frac{\xi^2 - \tau^2}{\rho_2} \right) \frac{1}{\Gamma(\sigma_2 - 1)} \int_a^\tau \left( \frac{\xi^2 - \tau^2}{\rho_2} \right)^{\sigma_2 - 2} \xi^{\rho_2 - 1}w(\xi) \, d\xi
\]

\[
= - \int_a^\tau \mathcal{H}(\tau, \xi)w(\xi) \, d\xi.
\]

Consequently,

\[
\rho_1^\ast \mathcal{C}_{\mathbb{C}^\ast}^{p_1} q(\tau) = -\phi_p \left( \int_a^\tau \mathcal{H}(\tau, \xi)w(\xi) \, d\xi \right).
\]
Thus, problem (18) can be written as
\[
\begin{align*}
\left\{\begin{array}{l}
p_1^{a_1^p}C^{C_{\text{gK}}}_{x}(\tau) + \phi_{\text{gK}}(\tau)H(\tau, \xi)w(\xi) d\xi = 0, \quad \tau \in (\hat{a}, \hat{i}), \\
q(\hat{a}) - F(\tau_1^a C^{C_{\text{gK}}}_{x}(\hat{a})) = 0, \\
\delta^1_{\rho_1}q(\hat{a}) = 0, \quad \delta^1_{\rho_1}q(i) = \mu \delta^1_{\rho_1}q(\eta) + \lambda,
\end{array}\right.
\end{align*}
\]
which, according to Lemma 3.1, has a unique solution of the form (19).

\textbf{Lemma 3.3} The functions \(G_1, G_2,\) and \(H,\) equations (16), (17), and (20) satisfy the following:

(i) \(G_1(\tau, \xi), G_2(\tau, \xi),\) and \(H(\tau, \xi)\) are continuous on \([\hat{a}, i] \times [\hat{a}, i].\)

(ii) For all \((\tau, \xi) \in [\hat{a}, i] \times [\hat{a}, i],\)

\[
G_1(\tau, \xi) \leq \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} \int_{\hat{a}}^{\tau} G_1(\tau, \xi) d\xi
\]

\[
= \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} \int_{\hat{a}}^{\tau} \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} d\tau
\]

\[
G_2(\tau, \xi) \leq \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2} \int_{\hat{a}}^{\tau} G_2(\tau, \xi) d\xi
\]

\[
= \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2} \int_{\hat{a}}^{\tau} \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} d\tau
\]

\[
H(\tau, \xi) \leq \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} \int_{\hat{a}}^{\tau} H(\tau, \xi) d\xi
\]

\[
= \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} \int_{\hat{a}}^{\tau} \left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 1} d\tau
\]

(iii) For all \((\tau, \xi) \in [\hat{a}, i]^2 : G_1(\tau, \xi) \geq 0, G_2(\tau, \xi) \geq 0, H(\tau, \xi) \geq 0.\)

(iv) For all \(\xi \in J,\) the function \(\tau \rightarrow G_1(\tau, \xi)\) is increasing and \(\tau \rightarrow H(\tau, \xi)\) is decreasing.

In addition, \(\forall (\tau, \xi) \in (\hat{a}, i)^2\) we have

\[
\left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{i_1^a - \hat{\tau}^{\rho_1}} \right)^{\sigma_1 - 1} G_1(\hat{a}, \xi) \leq G_1(\tau, \xi)
\]

and

\[
\left( \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{i_1^a - \hat{\tau}^{\rho_1}} \right)^{\sigma_1 - 1} H(\hat{\tau}, \xi) \leq H(\tau, \xi).
\]

(v) For all \((\tau, \xi) \in (\hat{a}, i)^2,\) we have

\[
\frac{\tau_1^a - \hat{\tau}^{\rho_1}}{i_1^a - \hat{\tau}^{\rho_1}} \left[ 1 - \left( \frac{1}{\tau_1^a} \right)^{\rho_1(\sigma_1 - 2)} \right] G_1(\hat{a}, \xi)
\]

\[
\leq G_1(\tau, \xi) \leq \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\tau_1^a - \hat{\tau}^{\rho_1}}{i_1^a - \hat{\tau}^{\rho_1}} G_1(\hat{a}, \xi).
\]
Proof. Using the definitions of $G_1$, $G_2$, and $H$, (i) and (ii) are obtained straightforwardly. For property (iii), we only consider the case $\xi \leq \tau$ as the other case is straightforward. When $\xi \leq \tau$, we have

$$G_1(\tau, \xi) \geq \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right) \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 1} \hat{a}^{\rho_1 - 1}$$

$$- \frac{1}{\Gamma(\sigma_1)} \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 1} \hat{a}^{\rho_1 - 1}$$

$$\geq \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 1} \hat{a}^{\rho_1 - 1} \left[ \frac{1}{\Gamma(\sigma_1 - 1)} - \frac{1}{\Gamma(\sigma_1)} \right] \geq 0,$$

because $\Gamma(\sigma_1 - 1) \leq \Gamma(\sigma_1)$ for $2 < \sigma_1 \leq 3$. Similarly, we can easily prove that $G_2(\tau, \xi) \geq 0$ and $H(\tau, \xi) \geq 0, \forall (\tau, \xi) \in J$. Now, for property (iv), we first check that $G_1(\tau, \xi)$ is nondecreasing w.r.t. $\tau \in J$.

$$\frac{\partial G_1}{\partial \tau}(\tau, \xi) = \begin{cases} \frac{\tau^{\rho_1 - 1}}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 1} \xi^{\rho_1 - 1} & \tau \leq \xi, \\ - \frac{\tau^{\rho_1 - 1}}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^{\rho_1} - \xi^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 1} \xi^{\rho_1 - 1} & \xi \leq \tau, \end{cases} \quad (22)$$

Thus, $G_1(\tau, \xi)$ is increasing with respect to $\tau \in J$, and therefore $G_1(\tau, \xi) \leq G_1(\tau, \xi)$ for $\hat{a} \leq \tau$, $\xi \leq \xi$. Furthermore, for $\tau \leq \xi$, we have

$$\frac{\partial H(\tau, \xi)}{\partial \tau} = -\frac{\tau^{\rho_2 - 1}}{\Gamma(\sigma_2 - 1)} \left( \frac{\tau^{\rho_2} - \hat{a}^{\rho_2}}{\rho_2} \right)^{\alpha_2 - 1} \xi^{\rho_2 - 1}$$

$$+ \left( \frac{\tau^{\rho_2 - 1}}{\Gamma(\sigma_2 - 1)} \left( \frac{\tau^{\rho_2} - \hat{a}^{\rho_2}}{\rho_2} \right)^{\alpha_2 - 1} \right)$$

$$= \frac{\tau^{\rho_2 - 1}}{\Gamma(\sigma_2 - 1)} \xi^{\rho_2 - 1} \left[ \left( \frac{\tau^{\rho_2} - \hat{a}^{\rho_2}}{\rho_2} \right)^{\alpha_2 - 1} \right]$$

and for $\xi \leq \tau$, we have

$$\frac{H(\tau, \xi)}{\partial \tau} = -\frac{\tau^{\rho_2 - 1}}{\Gamma(\sigma_2 - 1)} \left( \frac{\tau^{\rho_2} - \hat{a}^{\rho_2}}{\rho_2} \right)^{\alpha_2 - 1} \xi^{\rho_2 - 1} \leq 0.$$
As

\[
\left( \frac{\tau^\sigma - \xi^\sigma}{\tau^\sigma - \bar{a}^\sigma} \right)^\alpha \leq \left( \frac{i^\sigma - \xi^\sigma}{i^\sigma - \bar{a}^\sigma} \right)^\alpha,
\]

for \( \sigma > 0 \), we obtain

\[
\frac{G_1(\tau, \xi)}{G_1(i, \xi)} \geq \frac{1}{(\sigma_1 - 1)(i^\sigma - \bar{a}^\sigma)(i^\sigma - \xi^\sigma)^{\sigma_1-2} - (i^\sigma - \xi^\sigma)^{\sigma_1-1}} \\
\times \left[ \left( \sigma_1 - 1 \right) \left( \tau^\sigma - \bar{a}^\sigma \right) \left( i^\sigma - \xi^\sigma \right)^{\sigma_1-2} \right] \\
- \left( \tau^\sigma - \xi^\sigma \right)^{\sigma_1-1} \left( i^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}
\]

\[
\geq \frac{\left( \tau^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}}{\left( \tau^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}} \\
\times \left[ \left( \sigma_1 - 1 \right) \left( i^\sigma - \bar{a}^\sigma \right) \left( i^\sigma - \xi^\sigma \right)^{\sigma_1-2} - (i^\sigma - \xi^\sigma)^{\sigma_1-1}} \right] \\
\times \left( \tau^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1} \left( i^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1} \\
\times \left( \tau^\sigma - \xi^\sigma \right)^{\sigma_1-1} \left( i^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1} \\
\geq \frac{\left( \tau^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}}{\left( \tau^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}} \cdot
\]

For \( \tau \leq \xi \), we have

\[
\frac{G_1(\tau, \xi)}{(\tau^\sigma - \bar{a}^\sigma)^{\sigma_1-1}} = \frac{\rho_1^{\sigma_1-1} \xi^\sigma \rho_1}{\Gamma(\sigma_1 - 1)} \left( i^\sigma - \xi^\sigma \right)^{\sigma_1-2} - (i^\sigma - \xi^\sigma)^{\sigma_1-1}} \\
\times \left( \frac{i^\sigma - \bar{a}^\sigma}{\tau^\sigma - \bar{a}^\sigma} \right)^{\sigma_2-2} \left( \sigma_1 - 1 \right) \left( i^\sigma - \bar{a}^\sigma \right) \left( i^\sigma - \xi^\sigma \right)^{\sigma_1-2} \\
- \left( i^\sigma - \xi^\sigma \right)^{\sigma_1-1} \left( i^\sigma - \bar{a}^\sigma \right)^{\sigma_1-1}
\]

which is a nonincreasing function as \( \sigma_1 \geq 0 \). Consequently,

\[
\frac{G_1(\tau, \xi)}{(\tau^\sigma - \bar{a}^\sigma)^{\sigma_1-1}} \geq \frac{\bar{G}(i, \xi)}{(i^\sigma - \bar{a}^\sigma)^{\sigma_1-1}},
\]

which implies

\[
G_1(\tau, \xi) \geq \left( \frac{\tau^\sigma - \bar{a}^\sigma}{i^\sigma - \bar{a}^\sigma} \right)^{\sigma_1-1} \bar{G}(i, \xi).
\]
Thus, the proof is completed. Consequently, for \( A \leq \xi, \tau < i \). Therefore (iv) of Lemma 3.3 holds. Finally, for property (v), we can consider two cases. Nevertheless, we prove the results for the case \( \xi \leq \tau \) only. The simpler case \( \xi \leq \tau \) can be treated with similar arguments. When \( \xi \leq \tau \), we have

\[
\mathcal{G}_1'(\tau, \xi) \frac{(i^{p_1} - 1)}{i^{p_1} p_1(\sigma_1 - 1)} \geq \frac{(i^{p_1} - \xi)^{\sigma_1 - 2} - (\tau p_1 - \xi)^{\sigma_1 - 2}}{(\sigma_1 - 1)(i^{p_1} - \xi)^{\sigma_1 - 2} - \frac{(i^{p_1} - \xi)^{\sigma_1 - 1}}{[\xi^{p_1}_1 - \xi^{p_1}_1]^1}}.
\]

Consequently,

\[
\mathcal{G}_1'(\tau, \xi) \frac{(i^{p_1} - 1)}{i^{p_1} p_1(\sigma_1 - 1)} \geq \frac{(i^{p_1} - \xi)^{\sigma_1 - 2} - (\tau p_1 - \xi)^{\sigma_1 - 2}}{(\sigma_1 - 1)(i^{p_1} - \xi)^{\sigma_1 - 2} - \frac{(i^{p_1} - \xi)^{\sigma_1 - 1}}{[\xi^{p_1}_1 - \xi^{p_1}_1]^1}}
\]

On the other hand,

\[
\mathcal{G}_1'(\tau, \xi) \frac{(i^{p_1} - 1)}{i^{p_1} p_1(\sigma_1 - 1)} \geq \frac{(\sigma_1 - 1)((i^{p_1} - \xi)^{\sigma_1 - 2} - (\tau p_1 - \xi)^{\sigma_1 - 2})}{(\sigma_1 - 1)(i^{p_1} - \xi)^{\sigma_1 - 2} - \frac{(i^{p_1} - \xi)^{\sigma_1 - 1}}{[\xi^{p_1}_1 - \xi^{p_1}_1]^1}}
\]

Thus, the proof is completed.

Now, consider the Banach space \( E = C^{p_1}_{\rho_1}(J) \). Suppose that \( p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau] \) is continuous on \( J \) for all \( q \in \mathcal{E}_j \), then from Definition 2.6 and Lemma 2.4 we can define the norm on \( E \) as follows:

\[
||q|| = \left\{ \begin{array}{ll}
\max \{1, \max_{\tau \in J} |p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau]|), & 2 < \sigma_1 < 3, \\
\max \{1, \max_{\tau \in J} |p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau]|), & \sigma_1 = 3,
\end{array} \right.
\]

in which

\[
\tilde{M}_1 = \max \left\{ \max_{\tau \in J} |q(\tau)|, \max_{\tau \in J} |p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau]|, \max_{\tau \in J} |p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau]|, \max_{\tau \in J} |p_1^{i^{p_1} \xi^{p_1}_1} C^{p_1}_{\rho_1}[\tau]| \right\},
\]
and the cone

\[ K = \{ q \in \mathbb{E} : q \text{ is nonnegative, increasing, and } \rho_1 \text{-concave} \}. \]

**Lemma 3.4** Assume (H2) and let \( q \) be the unique solution of fractional boundary value problem (18) associated with given \( w(\tau) \in C^* (J) \). Then \( q \in K \) and the following inequalities hold for \( \tau \in [\hat{a}, i_0] \subset (\hat{a}, i) \):

\[
\max_{\tau \in J} |q(\tau)| \leq \left( \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}} \right)^{\sigma_1 - 1} \left( \frac{i^{\rho_2} - i^{\rho_2}}{i^{\rho_2} - i^{\rho_2}} \right)^{\sigma_1 - 1} \right)^{-1} q(\tau), \tag{23}
\]

\[
\max_{\tau \in J} |\delta_1^1 q(\tau)| \leq \frac{\sigma_1 - 1}{\sigma_1 - 2} \frac{\rho_1}{i^{\rho_1} - \hat{a}^{\rho_1}} \max_{\tau \in J} |q(\tau)|, \tag{24}
\]

\[
\max_{\tau \in J} |\delta_2^1 q(\tau)| \leq \left( \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}} \right)^{\sigma_1 - 1} Z(\iota_0) \int_a^i G_t(\iota, \xi) d\xi \right)^{-1} \times \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1} \right)^{\sigma_1 - 2} \max_{\tau \in J} |q(\tau)|, \tag{25}
\]

\[
\max_{\tau \in J} |\rho_1 \hat{a}_1^1 C_{\cdot}^1 q(\tau)| \leq \left( \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}} \right)^{\sigma_1 - 1} Z(\iota_0) \int_a^i G_t(\iota, \xi) d\xi \right)^{-1} \times \max_{\tau \in J} |q(\tau)|, \quad \forall \sigma_1 \in (2, 3), \tag{26}
\]

\[
\min_{\tau \in [\mu, \iota)]} q(\tau) \geq \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}} \right)^{\sigma_1 - 1} \tilde{M}_2 \| q \|, \tag{27}
\]

where

\[
Z(\tau) = \phi_{\rho} \left( \left( \frac{i^{\rho_2} - i^{\rho_2}}{i^{\rho_2} - i^{\rho_2}} \right)^{\sigma_2 - 1} \right)
\]

and

\[
\tilde{M}_2 = \min \left\{ 1, \frac{\sigma_1 - 2}{\sigma_1 - 1} \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1} \right), \right. \right.
\left. \min \left\{ \Gamma(\sigma_1 - 1) \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1} \right)^{2 - \sigma_1}, 1 \right\} \right. \right.
\left. \left. \times \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}} \right)^{\sigma_1 - 1} \times Z(\iota_0) \int_a^i G_t(\iota, \xi) d\xi \right\}. \tag{28}
\]

**Proof** From Lemma 3.2, we have

\[
q(\tau) = \int_a^i G_t(\tau, \xi) \phi_{\rho} \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi
\]

\[
\quad + \mu \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i G_t(\tau, \xi) \phi_{\rho} \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi
\]

\[
\quad + \lambda \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) + F_0 \left( \phi_{\rho} \left( \int_a^i \mathcal{H}(\hat{a}, \xi) w(\xi) \, d\xi \right) \right).
\]
(1) The functions $G_1$, $G_2$, and $H$ are nonnegative (Lemma 3.3(iii)). In addition, $F_0(v)$ is nonnegative for $v \geq 0$ (thanks to (H2)). Thus, $q$ is also nonnegative. Furthermore, as $G_1$ is increasing w.r.t. $\tau$ (Lemma 3.3(iv)), so it is the function $q$. To prove that $q$ is $\rho_1$-concave, we need to show that $\delta^1_{\rho_1} q(\tau)$ is decreasing on $J$ (Remark 2.2), which can be obtained from the negativity of the derivative

\[
(\delta^1_{\rho_1} q(\tau))' = - \frac{\tau^{\rho_1 - 1}}{\Gamma(\sigma_1 - 2)} \int_{\tau}^{\tau^1} \left( \frac{\tau^{\rho_1 - 1} - \xi^{\rho_1 - 1}}{\rho_1} \right)^{\sigma_1 - 3} \xi^{\rho_1 - 1} 
\times \phi_p \left( \int_{\xi}^{\xi^1} H(\xi, s) w(s) \, ds \right) \, d\xi \leq 0.
\]

(2) As $q$ is nonnegative and increasing, we have

\[
\max_{\tau \in J} |q(\tau)| = q(i)
\]

\[
= \int_{J} G_1(\xi) \phi_p \left( \int_{\xi}^{\xi^1} H(\xi, s) w(s) \, ds \right) \, d\xi
\]

\[
+ \mu \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right) \int_{\tau}^{\tau^1} G_2(\tau, \xi) \phi_p \left( \int_{\xi}^{\xi^1} H(\xi, s) w(s) \, ds \right) \, d\xi
\]

\[
+ \lambda \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right) + F_0 \left( \phi_p \left( \int_{\xi}^{\xi^1} H(\hat{a}, \xi) w(\xi) \, d\xi \right) \right).
\]

For $\tau \in [\hat{a}, i]$, using (iv) of Lemma 3.3 and the fact that

\[
\left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{i}^{\rho_1} - \hat{a}^{\rho_1}} \right) < 1,
\]

we get

\[
q(\tau) \geq \int_{J} G_1(\xi) \phi_p \left( \int_{\xi}^{\xi^1} H(\xi, s) w(s) \, ds \right) \, d\xi
\]

\[
+ \mu \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{i}^{\rho_1} - \hat{a}^{\rho_1}} \right) \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right) 
\times \int_{J} G_2(\tau, \xi) \phi_p \left( \int_{\xi}^{\xi^1} H(\xi, s) w(s) \, ds \right) \, d\xi
\]

\[
+ \lambda \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{i}^{\rho_1} - \hat{a}^{\rho_1}} \right) \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right) 
+ \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{i}^{\rho_1} - \hat{a}^{\rho_1}} \right) \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right)
\]

\[
\times F_0 \left( \phi_p \left( \int_{\xi}^{\xi^1} H(\hat{a}, \xi) w(\xi) \, d\xi \right) \right) \right).
\]

Consequently,

\[
q(\tau) \geq \left( \frac{\hat{a}^{\rho_1} - \hat{a}^{\rho_1}}{\hat{i}^{\rho_1} - \hat{a}^{\rho_1}} \right) \left( \frac{\tau^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu} \right) \max_{\tau \in J} |q(\tau)|,
\]

and thus (23) holds.
(3) We have

$$\delta_{\rho_1}^1 q(\tau) = \tau^{1-\rho_1} \int_a^\tau G_1'(\tau, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi$$

$$+ \frac{\mu}{(1-\mu)} \int_a^i G_2(\tau, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi + \frac{\lambda}{(1-\mu)}.$$

From Lemma 3.3 ((iii) and (v)), we can deduce that \(\delta_{\rho_1}^1 q(\tau) \geq 0\) and

$$\delta_{\rho_1}^1 q(\tau) \leq \frac{\sigma_1 - 1}{\sigma_1 - 2 \alpha^1 - a^1} \left[ \int_a^i G_1(\xi, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi$$

$$+ \frac{\mu}{(1-\mu)} \int_a^i G_2(\tau, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi + \frac{\lambda}{(1-\mu)} \right]$$

$$\leq \frac{\sigma_1 - 1}{\sigma_1 - 2 \alpha^1 - a^1} \left[ \int_a^i G_1(\xi, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi$$

$$+ \frac{\mu}{(1-\mu)} \int_a^i G_2(\tau, \xi) \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi$$

$$+ \frac{\lambda}{(1-\mu)} \right]$$

Thus, we obtain (24).

(4) A straightforward calculus gives

$$\delta_{\rho_1}^2 q(\tau) = -\frac{1}{\Gamma(\sigma_1 - 2)} \int_a^\tau \left( \frac{\tau^{\rho_1 - 1} - \xi^{\rho_1 - 1}}{\rho_1} \right)^{\alpha_1 - 3} \xi^{\rho_1 - 1}$$

$$\times \phi_\beta \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) d\xi.$$

Then we get

$$|\delta_{\rho_1}^2 q(\tau)| \leq \phi_\beta \left( \int_a^i \mathcal{H}(\xi, \xi) w(\xi) \, d\xi \right)$$

$$\times \frac{1}{\Gamma(\sigma_1 - 2)} \int_a^\tau \left( \frac{\tau^{\rho_1 - 1} - \xi^{\rho_1 - 1}}{\rho_1} \right)^{\alpha_1 - 3} \xi^{\rho_1 - 1} \, d\xi$$

$$\leq \phi_\beta \left( \int_a^i \mathcal{H}(\xi, \xi) w(\xi) \, d\xi \right) \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^{\rho_1} - \hat{\alpha}^{\rho_1}}{\rho_1} \right)^{\alpha_1 - 2}.$$
Thus,

$$\max_{\tau \in J} |\delta_{p_1}^2 q(\tau)| \leq \phi_p \left( \int_a^i \mathcal{H}(\tilde{\alpha}, \xi) w(\xi) \, d\xi \right) \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2}.$$ 

By multiplying both sides of the previous inequality by

$$\phi_p \left( \left( \frac{\tau^\rho_2 - \hat{\alpha}^\rho_2}{\rho_2} \right)^{\rho_2 - 1} \right),$$

we get

$$\phi_p \left( \left( \frac{\tau^\rho_2 - \hat{\alpha}^\rho_2}{\rho_2} \right)^{\rho_2 - 1} \right) \max_{\tau \in J} |\delta_{p_1}^2 q(\tau)|$$

$$\leq \phi_p \left( \int_a^i \mathcal{H}(\tilde{\alpha}, \xi) w(\xi) \, d\xi \right) \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2}.$$  

(29)

Multiplying both sides by $G_i(\tau, \xi)$ and integrating over $\int_a^i \xi \, d\xi$, we get

$$\max_{\tau \in J} |\delta_{p_1}^2 q(\tau)| \int_a^i G_i(\tau, \xi) \phi_p \left( \left( \frac{\tau^\rho_2 - \hat{\alpha}^\rho_2}{\rho_2} \right)^{\rho_2 - 1} \right) \, d\xi$$

$$\leq \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2} \int_a^i G_i(\tau, \xi)$$

$$\times \phi_p \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) \, d\xi$$

$$\leq \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2} \left[ \int_a^i G_i(\tau, \xi) \right.$$

$$\times \phi_p \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) \, d\xi + \lambda \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1 - \mu \rho_1} \right)$$

$$+ \mu \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1 - \mu \rho_1} \right) \int_a^i G_2(\tau, \xi) \phi_p \left( \int_a^i \mathcal{H}(\xi, s) w(s) \, ds \right) \, d\xi$$

$$+ F_q \left( \phi_p \left( \int_a^i \mathcal{H}(\tilde{\alpha}, \xi) w(\xi) \, d\xi \right) \right)$$

$$= \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2} q(\tau)$$

$$\leq \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{\tau^\rho_1 - \hat{\alpha}^\rho_1}{\rho_1} \right)^{\sigma_1 - 2} \max_{\tau \in J} |q(\tau)|.$$
Furthermore, for \( \tau \in [\hat{a}_0, i_0] \),
\[
\int_{\hat{a}}^{i} G_1(\tau, \xi) \phi_p \left( \frac{\tau^{p_1} - \xi^{p_1}}{i^{p_2} - \hat{a}^{p_2}} \right)^{\alpha_2 - 1} \, d\xi \geq \left( \frac{\tau^{p_1} - \hat{a}^{p_1}}{i^{p_1} - \hat{a}^{p_1}} \right)^{\alpha_1 - 1} Z(\hat{a}) \int_{\hat{a}}^{i} G_1(\hat{a}, \xi) \, d\xi
\]
and
\[
\max_{\tau \in J} |\delta_1^2 q(\tau)| \int_{\hat{a}}^{i} G_1(\tau, \xi) \phi_p \left( \frac{\tau^{p_2} - \xi^{p_2}}{i^{p_2} - \hat{a}^{p_2}} \right)^{\alpha_1 - 1} \, d\xi \geq \left( \frac{\tau^{p_1} - \hat{a}^{p_1}}{i^{p_1} - \hat{a}^{p_1}} \right)^{\alpha_1 - 1} Z(\hat{a}) \int_{\hat{a}}^{i} G_1(\hat{a}, \xi) \, d\xi \max_{\tau \in J} |\delta_2^1 q(\tau)|.
\]

Thus, we obtain (25).

(5) From the first equation in (21), one can see that
\[
\rho_1 \hat{a}^{\tau} C_{C_{k}} q(\tau) = -\phi_p \left( \int_{\hat{a}}^{i} H(\tau, \xi) w(\xi) \, d\xi \right) \quad (2 < \sigma_1 \leq 3).
\]

Thus,
\[
\max_{\tau \in J} |\rho_1 \hat{a}^{\tau} C_{C_{k}} q(\tau)| \leq \phi_p \left( \int_{\hat{a}}^{i} H(\hat{a}, \xi) w(\xi) \, d\xi \right) \quad (2 < \sigma_1 \leq 3).
\]

As in (2), we can deduce (26).

(6) Equation (27) is a direct consequence of the previous results.

Then, for given \([\hat{a}_0, i_0] \subset (\hat{a}, i)\), we define the cone
\[
\Upsilon = \left\{ q \in K : \min_{\tau \in [\hat{a}_0, i_0]} q(\tau) \geq \left( \frac{\hat{a}^{p_1} - \hat{a}^{p_1}}{i^{p_1} - \hat{a}^{p_1}} \right)^{\alpha_1 - 1} M_2 \|q\| \right\},
\]
and the integral operator \( \mathcal{N}_\hat{a} : \Upsilon \to E \) is defined for \( \tau \in [\hat{a}_0, i_0] \) by
\[
\mathcal{N}_\hat{a}(q)(\tau) = \int_{\hat{a}}^{i} G_1(\tau, \xi) \phi_p \left( \int_{\hat{a}}^{i} H(\xi, s) h(s) \, d\xi \right) \, d\xi + \mu \left( \frac{\tau^{p_1} - \hat{a}^{p_1}}{i^{p_1} - \hat{a}^{p_1}} \right) \int_{\hat{a}}^{i} G_2(\tau, \xi) \times \phi_p \left( \int_{\hat{a}}^{i} H(\xi, s) h(s) \, d\xi \right) \, d\xi + \lambda \left( \frac{\tau^{p_1} - \hat{a}^{p_1}}{i^{p_1} - \hat{a}^{p_1}} \right) \int_{\hat{a}}^{i} G_3(\tau, \xi) \times \phi_p \left( \int_{\hat{a}}^{i} H(\hat{a}, \xi) h(\xi) \, d\xi \right) \, d\xi.
\]

When (H2) holds, we have \( \mathcal{N}_\hat{a}(\Upsilon) \subset \Upsilon \), and the fixed points of \( \mathcal{N}_\hat{a} \) are the solutions of (9).

To use some fixed point theorems, we need to show that \( \mathcal{N}_\hat{a} \) is completely continuous.

**Lemma 3.5** ([19]) *Let \( c, s > 0 \). For any \( x, y \in [0, c] \), the following propositions hold:*

...
(1) If \( s > 1 \), then \( |x^s - y^s| \leq sc^{s-1}|x - y| \);

(2) If \( 0 < s \leq 1 \), then \( |x^s - y^s| \leq |x - y|^s \).

**Lemma 3.6** Assume (H2) is true. Then \( \mathcal{N}_\lambda : \Upsilon \to \Upsilon \) is continuous and compact.

**Proof** The continuity of \( \mathcal{N}_\lambda \) is a consequence of the continuity and positiveness of \( G_1, G_2, H_1, \) and \( g \). To prove that \( \mathcal{N}_\lambda \) is compact, let us consider a bounded subset \( \Omega \subset \Upsilon \). Then there exists \( L > 0 \) such that for any \( q \in \Omega \) we have \( |\Phi(q(\tau))| \leq L \). For any \( q \in \Omega \), as \( \mathcal{N}_\lambda q \) is positive and \( G_1 \) is increasing w.r.t. \( \tau \), we have

\[
\max_{\tau \in J} |N_\lambda(q(\tau))| = N_\lambda(q(\bar{\tau})).
\]

Consequently, using the previous inequality and hypothesis (H2), we get

\[
\max_{\tau \in J} |N_\lambda(q(\tau))| \leq \int_\alpha^\beta G_1(\bar{\tau}, \xi) \phi \left( \int_\alpha^\beta H(\bar{\tau}, s) L \, ds \right) d\xi
+ \mu \left( \frac{i^\rho_1 - i^\rho_1}{\rho_1 - \mu \rho_1} \right) \int_\alpha^\beta G_2(\tau, \xi) \phi \left( \int_\alpha^\beta H(\bar{\tau}, s) L \, ds \right) d\xi
+ \lambda \left( \frac{i^\rho_1 - i^\rho_1}{\rho_1 - \mu \rho_1} + A \int_\alpha^\beta H(\bar{\tau}, \xi) L \, d\xi \right) =: \bar{L}.
\]

Then, as in Lemma 3.4, we obtain \( \|\mathcal{N}_\lambda q\| \leq \tilde{M}_3 \bar{L} \), where

\[
\tilde{M}_3 = \max \left\{ 1, \frac{\sigma_1 - 1}{\sigma_1 - 2} \left( \frac{\rho_1}{i^\rho_1 - i^\rho_1} \right), \right. \\
\left. \max \left\{ 1, \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{i^\rho_1 - i^\rho_1}{\rho_1} \right)^{\sigma_1 - 2} , 1 \right\} \right. \\
\left. \times \left[ \frac{i^\rho_1 - i^\rho_1}{\rho_1 - \mu \rho_1} \right]^{\sigma_1 - 1} Z(\bar{\tau}) \int_\alpha^\beta G_1(\bar{\tau}, \xi) \, d\xi \right\}.
\]

Hence, \( \mathcal{N}_\lambda(\Omega) \) is uniformly bounded. Furthermore, by using Lemmas (3.2), (3.5), (3.3), and the Lebesgue dominated convergence theorem, we deduce the equicontinuity of \( \mathcal{N}_\lambda(\Omega) \). Therefore, \( \mathcal{N}_\lambda \) is completely continuous by the Arzelà–Ascoli theorem. \( \square \)

### 4 Existence of solutions in a cone

In this section, we derive an interval for \( \lambda \), which ensures the existence of \( \rho_1 \)-concave positive solutions of the fractional boundary value problem.

**Theorem 4.1** Assume that all conditions (H1) and (H2) hold, and that there exist \( 0 < \ell_1 < \ell_2 \) and

\[
m_1 \in (0, \tilde{M}_4), \quad m_2 \in (\Lambda_6, \infty),
\]

where \( \tilde{M}_4 = \min\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5\} \) such that
(H3) For all \( q \in [0, \ell_1] \), we have \( \varphi(q) \leq \min(\phi_p(m_1 \ell_1), m_1 \ell_1) \);

(H4) For all \( q \in [\gamma \ell_2, \ell_2] \), we have \( \varphi(q) \geq \phi_p(m_2 \ell_2) \).

Then fractional boundary value problem (9) has at least one \( p_1 \)-concave positive solution for \( \lambda > 0 \) small enough, where

\[
\gamma := \left( \frac{\hat{a}^{p_1} - \hat{a}^{p_1}}{1^{p_1} - \hat{a}^{p_1}} \right)^{\sigma_1-1} \tilde{M}_2
\]

and

\[
\Lambda_1 := \left[ A \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right]^{-1},
\]

\[
\Lambda_2 := \left[ \left( \int_a^1 G_1(\hat{a}, \xi) \, d\xi + \mu \left( \frac{1^{p_1} - \hat{a}^{p_1}}{p_1 - \rho_1 \mu} \right) \int_a^1 G_2(\hat{a}, \xi) \, d\xi \right) \phi_p \left( \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right) \right]^{-1},
\]

\[
\Lambda_3 := \left[ \frac{\sigma_1 - 1}{\sigma_1 - 2^{p_1} - \hat{a}^{p_1}} \left( \int_a^1 G_1(\hat{a}, \xi) \, d\xi \right) \phi_p \left( \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right) \right]^{-1},
\]

\[
\Lambda_4 := \left[ \frac{1}{\Gamma(\sigma_1 - 1)} \left( \frac{1^{p_1} - \hat{a}^{p_1}}{p_1} \right)^{\sigma_1-2} \phi_p \left( \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right) \right]^{-1},
\]

\[
\Lambda_5 := \left[ \phi_p \left( \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right) \right]^{-1},
\]

\[
\Lambda_6 := \gamma \left( \frac{\hat{a}^{p_1} - \hat{a}^{p_1}}{1^{p_1} - \hat{a}^{p_1}} \right)^{\sigma_1-1} Z(\hat{a}) \left( \int_a^1 G_1(\hat{a}, \xi) \, d\xi \right) \phi_p \left( \int_a^1 H(\hat{a}, \xi) h(\xi) \, d\xi \right) \right]^{-1}.
\]

**Proof** Let \( \Omega_{\ell_1} = \{ q \in K : \|q\| \leq \ell_1 \} \) and \( \lambda \) satisfy

\[
0 < \lambda \leq \frac{1}{2} (1 - \mu) \ell_1 \min \left\{ 1, \frac{p_1}{1^{p_1} - \hat{a}^{p_1}} \right\},
\]

so that

\[
2\lambda \left( \frac{1^{p_1} - \hat{a}^{p_1}}{p_1 - \mu p_1} \right) \leq \ell_1,
\]

and \( 2\lambda \leq \ell_1 (1 - \mu) \). Let \( q \in K \cap \partial \Omega_{\ell_1} \), i.e., \( \|q\| = \ell_1 \). From (H2) and (H3), we get

\[
F_0 \left( \phi_p \left( \int_a^1 h(\hat{a}, \xi) h(\xi) \varphi(q(\xi)) \, d\xi \right) \right) \leq A \int_a^1 h(\hat{a}, \xi) h(\xi) \varphi(q(\xi)) \, d\xi
\]

\[
\leq m_1 \ell_1 A \int_a^1 h(\hat{a}, \xi) h(\xi) \, d\xi,
\]

\[
\phi_p \left( \int_a^1 h(\tau, \xi) h(\xi) \varphi(q(\xi)) \, d\xi \right) \leq m_1 \ell_1 \phi_p \left( \int_a^1 h(\tau, \xi) h(\xi) \, d\xi \tau \right).
\]
However,
\[
\max_{\tau \in \mathcal{J}} |\mathcal{N}_\lambda(q(\tau))| = \mathcal{N}_\lambda(q) \\
= \int_a^i G_1(\xi) \phi_p \left( \int_a^i \mathcal{H}(\xi, s) \lambda(\mathbf{y}(s)) \mathbf{d}s \right) d\xi \\
+ \mu \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) \int_a^i G_2(\tau, \xi) \phi_p \left( \int_a^i \mathcal{H}(\xi, s) \lambda(\mathbf{y}(s)) \mathbf{d}s \right) d\xi \\
+ \lambda \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) + F \left( \phi_p \left( \int_a^i \mathcal{H}(\xi) \lambda(\mathbf{y}(\xi)) \mathbf{d}x \right) \right). 
\]

Then
\[
\max_{\tau \in \mathcal{J}} |\mathcal{N}_\lambda(q(\tau))| \leq \frac{\Lambda_1 \ell_1}{4} \left[ \left( \int_a^i G_1(\xi) d\xi + \mu \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) \int_a^i G_2(\tau, \xi) d\xi \right) \right] \right] + \frac{\ell_1}{2} + \frac{\Lambda_1 \ell_1 A}{4} \int_a^i \mathcal{H}(\xi) \mathbf{d}x.
\]

Consequently,
\[
\max_{\tau \in \mathcal{J}} |\mathcal{N}_\lambda(q(\tau))| \leq \frac{\ell_1}{4} + \frac{\ell_1}{2} + \frac{\ell_1}{4} = \|q\|.
\]

Similarly, we obtain
\[
\max_{k \in [1, 2]} \max_{\tau \in \mathcal{J}} |\mathcal{N}_\lambda(q(\tau))| \leq \|q\|.
\]

Therefore, we conclude that \(\|\mathcal{N}_\lambda q\| \leq \|q\|\) for all \(q \in K \cap \partial \Omega_{\ell_1}\). Then Theorem 2.8 implies that
\[
i(\mathcal{N}_\lambda, \Omega_{\ell_1}, K) = 1. \quad (37)
\]

On the other hand, let us consider \(\Omega_{\ell_2} = \{q \in K : \|q\| \leq \ell_2\}\). Then, for any \(q \in K \cap \partial \Omega_{\ell_2}\), by Lemma 3.4 one has \(\ell_2 \geq \min_{\tau \in \mathcal{J}} |q(\tau)| \geq \gamma \ell_2\). Using hypothesis (H4), we get
\[
\mathcal{N}_\lambda(q) \geq \left( \frac{\hat{a}^p_1 - \tilde{a}^p_1}{\tilde{a}^p_1 - \tilde{a}^p_1} \right)^{-1} \int_a^i G_1(\xi) \phi_p \left( \int_a^i \mathcal{H}(\xi, s) \lambda(\mathbf{y}(s)) \mathbf{d}s \right) d\xi \\
+ \mu \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) \int_a^i G_2(\tau, \xi) \phi_p \left( \int_a^i \mathcal{H}(\xi, s) \lambda(\mathbf{y}(s)) \mathbf{d}s \right) d\xi \\
+ \lambda \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) + F \left( \phi_p \left( \int_a^i \mathcal{H}(\xi) \lambda(\mathbf{y}(\xi)) \mathbf{d}x \right) \right) \\
\geq \left( \frac{\hat{a}^p_1 - \tilde{a}^p_1}{\tilde{a}^p_1 - \tilde{a}^p_1} \right)^{-1} m_2 \ell_2 \gamma Z(\xi) \left( \int_a^i G_1(\xi) \right) \\
+ \mu \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) \int_a^i G_2(\tau, \xi) \phi_p \left( \int_a^i \mathcal{H}(\xi) \lambda(\mathbf{y}(\xi)) \mathbf{d}x \right) \right) \\
\geq \left( \frac{\hat{a}^p_1 - \tilde{a}^p_1}{\tilde{a}^p_1 - \tilde{a}^p_1} \right)^{-1} m_2 \ell_2 \gamma Z(\xi) \left( \int_a^i G_1(\xi) \right) \\
+ \mu \left( \frac{i^p_1 - \tilde{a}^p_1}{p_1 - \mu p_1} \right) \int_a^i G_2(\tau, \xi) \phi_p \left( \int_a^i \mathcal{H}(\xi) \lambda(\mathbf{y}(\xi)) \mathbf{d}x \right) \right)
\]
\[
\geq \left( \frac{\hat{a}^{\sigma_1}_1 - \hat{a}^{\rho_1}}{\hat{\rho}_1 - \hat{\sigma}_1} \right)^{\sigma_1-1} A_{\emptyset \ell_2 Y Z(\delta)} \left( \int_{\bar{u}}^i G_1(\hat{u}, \xi) \right.
+ \mu \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) G_2(\tau, \xi) d\xi \bigg) \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\hat{\bar{u}}, \xi) h(\xi) d\xi \right) := \ell_2 = \|q\|,
\]

which implies that \(\|\mathcal{N}_q\| \geq \|q\|\) for any \(q \in K \cap \partial \Omega_{l_2}\). Hence Theorem 2.8 implies that

\[
i(\mathcal{N}_q, \Omega_{l_2}, K) = 0. \tag{38}
\]

Therefore, by equations (37), (38) and \(\ell_1 < \ell_2\), we have

\[
i(\mathcal{N}_q, \overline{\Omega_{l_2}} \setminus \Omega_{l_1}, K) = 1.
\]

By employing Theorem 2.9, one can see that the operator \(\mathcal{N}_q\) has at least one fixed point \(q \in K \cap \overline{\Omega_{l_2}} \setminus \Omega_{l_1}\), which is a \(p_1\)-concave positive solution of fractional boundary value problem (9). \(\square\)

**Theorem 4.2** Assume that all conditions (H1), (H2), and (H4) hold. Then FBVP (9) has no \(p_1\)-concave positive solution for \(\lambda\) large enough.

**Proof** Suppose that \(\exists \mathbb{N} \in \mathbb{N}\) and \((\lambda_j)\) such that \(\lim_{j \to -\infty} \lambda_j = +\infty\) and fractional boundary value problem (9) has \(p_1\)-concave positive solution \(q_j (j \geq \mathbb{N})\), i.e.,

\[
q_j(\tau) = \int_{\bar{u}}^i G_1(\tau, \xi) \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\xi, s) h(s) \phi(q(s)) ds \right) d\xi
+ \mu \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\bar{u}}^i G_2(\tau, \xi) \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\xi, s) h(s) \phi(q(s)) ds \right) d\xi
+ \lambda_j \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) + F_{\emptyset} \left( \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\hat{\bar{u}}, \xi) h(\xi) \phi(u(\xi)) d\xi \right) \right).
\]

Thus,

\[
q_j(\bar{u}) \geq \left( \frac{\hat{a}^{\sigma_1}_1 - \hat{a}^{\rho_1}}{\hat{\rho}_1 - \hat{\sigma}_1} \right)^{\sigma_1-1} \left[ \int_{\bar{u}}^i G_1(\hat{\bar{u}}, \xi) \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\xi, s) h(s) \phi(q(s)) ds \right) ds \right] d\xi
+ \mu \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) \int_{\bar{u}}^i G_2(\tau, \xi) \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\xi, s) h(s) \phi(q(s)) ds \right) d\xi
+ \lambda_j \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right) + F_{\emptyset} \left( \phi_p \left( \int_{\bar{u}}^i \mathcal{H}(\hat{\bar{u}}, \xi) h(\xi) \phi(u(\xi)) d\xi \right) \right).
\]

Consequently,

\[
q_j(\bar{u}) \geq \left( \frac{\hat{a}^{\sigma_1}_1 - \hat{a}^{\rho_1}}{\hat{\rho}_1 - \hat{\sigma}_1} \right)^{\sigma_1-1} \lambda_j \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right).
\]

Without loss of generality, we can suppose that \(\mathbb{N}\) is large enough to get, for \(j \geq \mathbb{N}\),

\[
\lambda_j > j \left( \frac{\rho_1 - \mu \rho_1}{\hat{\rho}_1 - \hat{\sigma}_1} \right) \left( \frac{\hat{a}^{\sigma_1}_1 - \hat{a}^{\rho_1}}{\hat{\rho}_1 - \hat{\sigma}_1} \right)^{\sigma_1-1} \left( \frac{i^{\rho_1} - \hat{a}^{\rho_1}}{\rho_1 - \mu \rho_1} \right).
\tag{39}
\]
Then we have \( q_j(i) > j \). Consequently, \( \lim_{j \to +\infty} \| q_j \| = +\infty \). Using (H4), we deduce that there exist \( m_2 > \Lambda_6 \) and \( \ell_2 > 0 \) such that \( \varphi(q) \geq \phi_p(m_2 \ell_2) \) for all \( q \in [\gamma \ell_2, \ell_2] \). Again, we can choose \( \bar{N} \) large enough to get \( \| q_j \| \geq \ell_2, \forall j \geq \bar{N} \). By writing \( m_2 = \Lambda_6 + \sigma \), where \( \sigma > 0 \), we get

\[
\| q_j \| \geq q_j(\bar{i}) \\
\geq \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right)^{\sigma_1 - 1} \left( \int_a^l G_1(i, \xi) \phi_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi \\
+ \mu \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) \left( \int_a^l \phi_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi \\
+ \lambda \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) F_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(\xi)) \, d\xi \right) \right) \right) \\
\geq (\Lambda_6 + \sigma) \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right)^{\sigma_1 - 1} Z(\bar{i}, \xi) \\
+ \mu \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) \left( \int_a^l \phi_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(\xi)) \, d\xi \right) \, d\xi \\
+ \lambda \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) F_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(\xi)) \, d\xi \right) \right) \right) \\
\geq \| q_j \|(\Lambda_6 + \sigma)^\gamma \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right)^{\sigma_1 - 1} Z(\bar{i}, \xi) \\
+ \mu \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) \left( \int_a^l \phi_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(\xi)) \, d\xi \right) \, d\xi \\
+ \lambda \left( \frac{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}}{\hat{a}_{\alpha_1} - \hat{a}_{\beta_1}} \right) F_p \left( \int_a^i H(\xi, s) h(s) \varphi(q(\xi)) \, d\xi \right) \right) \right) \\
= \| q_j \|(1 + \sigma \Lambda_6^{-1}),
\]

which leads to a contradiction \( \| q_j \| \sigma \Lambda_6^{-1} \leq 0 \). The proof is completed. \( \square \)

**Remark 4.1** Let

\[
\phi_0 := \lim_{q \to 0} \frac{\varphi(q)}{\min \{ \phi_p(q), q \}}, \quad \phi_\infty := \lim_{q \to \infty} \frac{\varphi(q)}{\phi_p(q)},
\]

(40)

If \( \phi_0 = 0 \) and \( \phi_\infty = \infty \) hold, then conditions (H3) and (H4) hold respectively. Moreover, if the functions \( \varphi \) and \( F_\sigma \) are nondecreasing, the following theorem holds.

**Theorem 4.3** Assume that the hypotheses of Theorem 4.1 hold and that \( \varphi \) and \( F_\sigma \) are nondecreasing. Then there exists \( \lambda^* > 0 \) such that fractional boundary value problem (9) has at least one \( \rho \)-concave positive solution for \( \lambda \in (0, \lambda^*) \) and has no \( \rho_1 \)-concave positive solution for \( \lambda \in (\lambda^*, \infty) \).

**Proof** Let \( \hat{\Gamma} \subset \mathbb{R}^*_+ \) be the set of all \( \lambda \) such that fractional boundary value problem (9) has at least one \( \rho_1 \)-concave positive solution and \( \lambda^* = \sup \hat{\Gamma} \). It follows from Theorem 4.1 that \( \hat{\Gamma} \neq \emptyset \), and thus \( \lambda^* \) exists. We denote by \( q_0 \) the solution of fractional boundary value problem (9) associated with \( \lambda_0 \) and

\[
\mathcal{K}(q_0) = \{ q \in K : q(\tau) < q_0(\tau), \forall \tau \in J \}.
\]
Let $\lambda \in (0, \lambda_0)$ and $q \in \mathcal{K}(q_0)$. It follows from the definition of $N_{\lambda}$ (31) and the monotonicity of $f$ that, for any $\tau \in J$,

$$N_{\lambda}(q(\tau)) \leq N_{\lambda}(q_0(\tau)) = q_0(\tau).$$

Thus $N_{\lambda}(\mathcal{K}(q_0)) \subseteq \mathcal{K}(q_0)$. Now, Schauder’s fixed point theorem implies that there exists a fixed point $q \in \mathcal{K}(q_0)$ such that it is a positive solution of (9). The proof is completed. □

**Theorem 4.4** Suppose that conditions (H1) and (H2) hold. Assume that $\varphi$ also satisfies:

(H5) $\varphi_0 = \sigma_1 \in [0, \min\{k^{p-1}, k\})$, $k = \frac{1}{\sqrt{2}}M_4$;

(H6) $\varphi_{\infty} = \sigma_2 \in ((\frac{2\Lambda_6}{\gamma})^{p-1}, \infty)$.

Then fractional boundary value problem (9) has at least one $\rho_1$-concave positive solution for $\lambda$ small enough.

**Proof** Firstly, from the definition of $\varphi_0$, for all $\epsilon > 0$, there exists an adequate small positive number $\bar{\delta}(\epsilon)$ such that

$$\varphi(q) \leq (\epsilon + \sigma_1) \min\{q^{p-1}, q\} \leq (\epsilon + \sigma_1) \min\{\bar{\delta}^{p-1}, \bar{\delta}\},$$

$\forall q \in [0, \bar{\delta}(\epsilon)]$. Then, for $\epsilon = \min\{k^{p-1}, k\} - \sigma_1$, we have

$$\varphi(q) \leq \min\{k^{p-1}, k\} \min\{\bar{\delta}(\epsilon)^{p-1}, \bar{\delta}(\epsilon)\}
\leq \min\{k^{p-1}\bar{\delta}(\epsilon)^{p-1}, k\bar{\delta}(\epsilon)\}
\leq \min\{(2k\bar{\delta}(\epsilon))^{p-1}, 2k\bar{\delta}(\epsilon)\}.$$

It is enough to take $\ell_1 = \bar{\delta}(\epsilon)$ and $m_1 = 2k \in (0, \bar{M}_4)$, i.e., condition (H3) holds. Next, since (H6) holds, then for every $\epsilon > 0$ there exists an adequate big positive number $\ell_2 \neq \ell_1$ such that

$$\varphi(q) \geq (\sigma_2 - \epsilon)q^{p-1} \geq (\sigma_2 - \epsilon)(\gamma \ell_2)^{p-1} \quad (q \geq \gamma \ell_2).$$

Hence, for $\epsilon = \sigma_2 - (\frac{2\Lambda_6}{\gamma})^{p-1}$, we get

$$\varphi(q) \geq \left(\frac{2\Lambda_6}{\gamma}\right)^{p-1} (\gamma \ell_2)^{p-1} = (2\Lambda_6 \ell_2)^{p-1}. \quad (41)$$

By considering $m_2 = 2\Lambda_6 > \Lambda_6$, condition (H4) holds by Theorem 4.1, we complete the proof. □

**5 Several solutions in a cone**

In order to show the existence of multiple solutions, we will use the Leggett–Williams fixed point theorem [43]. For this, we define the following subsets of a cone $\mathcal{K}$:

$$\Omega_c = \{q \in \mathcal{K} : \|q\| < c\},$$

$$\Omega_c(b, d) = \{q \in \mathcal{K} : b \leq \varphi(q), \|q\| \leq d\}.$$
A map $\Pi : K \to [0, \infty)$ is said to be a nonnegative continuous concave functional on a cone $K$ of a real Banach space $\mathcal{E}$, if it is continuous and

$$\Pi(\lambda q + (1 - \lambda)\tilde{q}) \geq \lambda \Pi(q) + (1 - \lambda)\Pi(\tilde{q})$$

for all $q, \tilde{q} \in K$ and $\lambda \in [0, 1]$.

**Theorem 5.1** ([43]) Let $\mathcal{T} : \Omega_\infty \to \Omega_\infty$ be a completely continuous operator and $\varphi$ be a nonnegative continuous concave functional on $K$ such that $\varphi(q) \leq \|q\|$ for all $q \in \Omega_\infty$. Suppose that there exist constants $0 < a < b < d < c$ such that

(D3) $\{q \in \Omega_\varphi(b, d) : \varphi(q) > b\} \neq \emptyset$ and $\varphi(\mathcal{T}q) > b$ if $q \in K_\varphi(b, d)$;

(D4) $\|\mathcal{T}q\| < a$ if $q \in \Omega_\varphi$;

(D5) $\varphi(\mathcal{T}q) > b$ for $q \in \Omega_\varphi(b, c)$ with $\|\mathcal{T}q\| > d$.

Then $\mathcal{T}$ has at least three fixed points $q_1, q_2$, and $q_3$ such that $\|q_1\| < a, b < \varphi(q_2)$, and $\|q_3\| > a$ with $\varphi(q_3) < b$.

**Theorem 5.2** Suppose that conditions (H1) and (H2) hold, if there exist $\bar{a}, b, c$ with $0 < \bar{a} < \gamma b < c$ such that

(H7) $\varphi(q(\tau)) < \min\{\varphi_p(m_1\lambda), m_1\lambda\}$ for $(\tau, q) \in J \times [0, \bar{a}]$;

(H8) $\varphi(q(\tau)) \geq \varphi_p(m_2\gamma b)$ for $(\tau, q) \in [\bar{a}, c] \times [\gamma b, c]$;

(H9) $\varphi(q(\tau)) \leq \min\{b, c\}$ for $(\tau, q) \in J \times [0, c]$;

(H10) $0 < \lambda < \frac{\min\{\|q\|, d\}}{\|q\|c}$

where the constants $m_2$ and $m_1$ are defined in (33). Then fractional boundary value problem (9) has at least three positive $\varphi_\mathcal{I}$-concave solutions $q_1, q_2$, and $q_3$ satisfying $\|q_1\| < \bar{a}, \gamma b < \varphi(q_2)$, and $\|q_3\| > a$ with $\varphi(q_3) < b$ for $\lambda$ small enough.

**Proof** We prove that fractional boundary value problem (9) has at least three positive $\varphi_\mathcal{I}$-concave solutions for $\lambda > 0$ small enough. By Lemma 3.6, $\mathcal{X}_\lambda : \mathcal{Y} \to \mathcal{Y}$ is completely continuous. Let $\varphi(q) = \min_{t \in [a, b]} q(t)$. Obviously, $\varphi(q)$ is a nonnegative, continuous, and concave functional on $K$ with $\varphi(q) \leq \|q\|$ for $q \in \Omega_\infty$. Now we will show that all conditions of Theorem 5.1 are satisfied. Suppose that $q \in \Omega_\infty$, that is, $\|q\| < c$. For $\tau \in J$, by equation (31), Lemmas 3.4, 3.5, we acquire

$$\max_{\tau \in J}|\mathcal{N}_\lambda(q(\tau))| = \int_a^\tau \mathcal{G}_1(\lambda, \xi)\varphi_p(q(\tau))\left(\int_a^\tau \mathcal{H}(\xi, s)\varphi(q(s))\,ds\right)\,d\xi$$

$$+ \mu\left(\frac{\rho_1 - \rho_1}{\rho_1 - \rho_1} + \frac{\rho_1}{\rho_1 - \rho_1}\right) \int_a^\tau \mathcal{G}_2(\tau, \xi)\varphi_p\left(\int_a^\tau \mathcal{H}(\xi, s)\varphi(q(s))\,ds\right)\,d\xi$$

$$+ \lambda\left(\frac{\rho_1 - \rho_1}{\rho_1 - \rho_1} + \frac{\rho_1}{\rho_1 - \rho_1}\right) + \mathcal{F}_\lambda\left(\int_a^\tau \mathcal{H}(\xi, s)\varphi(q(s))\,d\xi\right).$$

From (H2), (H9), and (H10), we get

$$\max_{\tau \in J}|\mathcal{N}_\lambda(q(\tau))| \leq \int_a^\tau \mathcal{G}_1(\lambda, \xi)\varphi_p\left(\int_a^\tau \mathcal{H}(\xi, s)\varphi(q(s))\,ds\right)\,d\xi$$

$$+ \mu\left(\frac{\rho_1 - \rho_1}{\rho_1 - \rho_1} + \frac{\rho_1}{\rho_1 - \rho_1}\right) \int_a^\tau \mathcal{G}_2(\tau, \xi)\varphi_p\left(\int_a^\tau \mathcal{H}(\xi, s)\varphi(q(s))\,ds\right)\,d\xi$$

$$+ \lambda\left(\frac{\rho_1 - \rho_1}{\rho_1 - \rho_1} + \frac{\rho_1}{\rho_1 - \rho_1}\right).$$
Thus, in view of (31), Lemmas 3.3, 3.4, 3.5, and (H8), we have

\[
\varphi(N_q) = \min_{t \in [i_b, i_u]} \left[ \int_a^i \mathcal{G}_1(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi 
\right.
\]

\[
+ \mu \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \mathcal{G}_2(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi 
\]

\[
+ \lambda \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \Phi_p \left( \int_a^i \mathcal{H}(\xi, \tilde{\xi}) \varphi(q(\tilde{\xi})) \, d\tilde{\xi} \right) \, d\xi 
\]

\[
\leq m_1 c \left[ \left( \int_a^i \mathcal{G}_1(\tilde{\xi}) \, d\tilde{\xi} \right) + \mu \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \mathcal{G}_2(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\tilde{\xi}, \xi) h(\xi) \, d\xi \right) \right] + \frac{c}{2} 
\]

\[
\leq \frac{\Lambda_2 c}{4} \left[ \left( \int_a^i \mathcal{G}_1(\tilde{\xi}) \, d\tilde{\xi} \right) + \mu \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \mathcal{G}_2(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\tilde{\xi}, \xi) h(\xi) \, d\xi \right) \right] + \frac{c}{2} 
\]

\[
= \frac{c}{4} + \frac{c}{4} + \frac{c}{2} = c 
\]

and

\[
\max \left\{ \max_{\xi \in [1, 2]} |\alpha^b_{\rho_1} \mathcal{N}_1(q(\tau))|, \max_{\xi \in [1, 2]} |\alpha^{i-\sigma}_{\rho_1} \mathcal{G}_{\alpha} \mathcal{N}_1(q(\tau))| \right\} \leq \|q\|. 
\]

Therefore, we have

\[
\| \mathcal{N}_1 q(\tau) \| \leq c \quad (\forall q \in \mathcal{O}_\rho).
\]

This implies that \( \mathcal{N}_1 : \mathcal{O}_\rho \rightarrow \mathcal{O}_\rho \). By the same method, if \( q \in \overline{\mathcal{O}_\rho} \), then we can get \( \| \mathcal{N}_1 q(\tau) \| < \tilde{a} \), therefore (D4) has been checked. Next, we assert that

\[
\{ q \in \mathcal{O}_\rho(yb, b) : \varphi(q) > yb \} \neq \emptyset
\]

and \( \varphi(N_1 \rho(q)) > yb \) for all \( q \in \mathcal{O}_\rho(yb, b) \). In fact, the constant function \( \frac{yb}{2} \in \mathcal{O}_\rho(yb, b) \) and \( \varphi\left( \frac{yb}{2} \right) > yb \). On the other hand, for \( q \in \mathcal{O}_\rho(yb, b) \), we have

\[
yb \leq \varphi(q) = \min \{ q(\tau) \leq \|q\| = b \quad (\forall t \in [\tilde{a}_b, i_\rho]) \}
\]

Thus, in view of (31), Lemmas 3.3, 3.4, 3.5, and (H8), we have

\[
\varphi(N_1 q) = \min_{\xi \in [i_b, i_u]} \left[ \int_a^i \mathcal{G}_1(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi 
\right.
\]

\[
+ \mu \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \mathcal{G}_2(\tau, \xi) \Phi_p \left( \int_a^i \mathcal{H}(\xi, s) h(s) \varphi(q(s)) \, ds \right) \, d\xi 
\]

\[
+ \lambda \left( \frac{\tau^{p_1} - \dot{a}^{p_1}}{\rho_1 - \mu \rho_1} \right) \int_a^i \Phi_p \left( \int_a^i \mathcal{H}(\tilde{\xi}, \xi) \varphi(q(\tilde{\xi})) \, d\tilde{\xi} \right) \, d\xi 
\]
Thus, (D3) has been verified. Finally, we need to show that if \( q \in \Omega_{\gamma b, b} \) with \( \| N_{\gamma} q \| > b \), then \( \| N_{\gamma} q \| > \gamma b \). In fact, to see this, suppose that \( q \in \Omega_{\gamma b, b} \) with \( \| N_{\gamma} q \| > b \), then through Lemma 3.4 we have

\[
\varphi(N_{\gamma} q) = \min_{\bar{t}_c \leq t \leq \bar{t}_o} (N_{\gamma} q)(t) \geq \gamma \| N_{\gamma} q \| > \gamma b.
\]

Thus (D5) is satisfied. Hence, an application of Theorem 5.1 completes the proof. \( \square \)

**Corollary 5.1** Suppose that conditions (H1) and (H2) hold. If there exist constants

\[
0 < r_1 < b_1 < \gamma b_1 \leq r_2 < b_2 < \gamma b_2 \leq \cdots \leq r_n
\]

for \( 1 \leq j \leq n - 1 \) and the following conditions are satisfied:

(H11) \( \varphi(q(t)) < \min\{\phi_{p_0}(m_1 r_j), m_1 r_j\} \) for \( (t, q) \in J \times [0, r_j] \);

(H12) \( \varphi(q(t)) > \phi_{p_0}(m_2 b_j) \) for \( (t, q) \in [\bar{a}_o, \bar{i}_o] \times [\gamma b_j, b_j] \);

(H13) \( 0 < \lambda < \frac{1 - \alpha^j}{2} \max\{1, \frac{p_0}{p_1 - \alpha p_j}\} \).

Then fractional boundary value problem (9) has at least \( 2n - 1 \) positive \( p_1 \)-concave solutions.

**Proof** By the induction method, we get the proof. \( \square \)

### 6 Applications

In this section, we give some examples to illustrate the usefulness of our main results.
Example 6.1 Let us consider the following $p$-Laplacian fractional boundary value problem:

$$
\begin{align*}
\rho_2 e^2 \mathcal{C}^{5/2}_{CK}(p_2 (p_1 e^2 \mathcal{C}^{5/2}_{CK} q))(\tau) \\
+ \frac{q^{3/2}(\tau)}{\sqrt{(2.2 - \ln(\tau))([\ln(\tau)] - 0.9)}} = 0, & \quad e < \tau < e^2, \\
q(e) - \frac{\sqrt{|q|}}{\sqrt{|q|}} = 0, & \\
\delta_0^2, q(e) = 0, & \quad \delta_1^1, q(e) = \frac{1}{\tau} \delta_0^1, q(e) + \lambda, \\
\rho_1 e^2 \mathcal{C}^{5/2}_{CK} q(e^2) = -\delta_0^1, (p_2 (p_1 e^2 \mathcal{C}^{5/2}_{CK} q))(\tau) \\
= \delta_0^2, (p_2 (p_1 e^2 \mathcal{C}^{5/2}_{CK} q))(e^2) = 0.
\end{align*}
$$

Here, $J = [e, e^2]$, $\sigma_1 = \sigma_2 = \frac{5}{2} \in (2, 3]$, $\mu = \frac{1}{2} \in (0, 1)$, $\eta = e^{\frac{6}{5}} \in J$, $[a_0, \iota_0] = [e^{3/2}, e^{7/4}] \subset J$.

We put

$$
\rho_1 = 0.5 \in \mathbb{R} \setminus \{1\}, \quad \rho_2 = 1.3 \in \mathbb{R} \setminus \{1\}, \quad p = \frac{3}{2},
$$

and so $\tilde{p} = 3$, $A = \frac{3}{2}$, $B = \frac{1}{2}$, $\rho_1 e^2 \mathcal{C}^{5/2}_{CK}$ and $\rho_2 e^2 \mathcal{C}^{5/2}_{CK}$ are the left- and right-sided Caputo–Katugampola fractional derivatives, $F_0(v) = \sqrt{|v|}$ and

$$
h(\tau) = \frac{1}{\sqrt{(2.2 - \ln(\tau))([\ln(\tau)] - 0.9)}}.
$$

We can easily show that (H1), (H2) hold, and from (40) we get $\varphi(q(\tau)) = (q(\tau))^{3/2}$ satisfies

$$
\varphi_0 = \lim_{q \to 0^+} \frac{\varphi(q)}{\min\{\varphi(q), q\}} = \lim_{q \to 0^+} \frac{q^{3/2}}{\min\{q^{3/2}, q\}} = 0, \\
\varphi_{\infty} = \lim_{q \to \infty} \frac{\varphi(q)}{\varphi(q)} = \lim_{q \to \infty} \frac{q^{3/2}}{q^{3/2}} = \lim_{q \to \infty} \frac{q^{1/2}}{|q|^{3/2}} = \infty.
$$

Then, obviously, $Z(\iota_0) = 0.05549$,

$$
\hat{M}_4 = \min\left\{ \Lambda_1, \Lambda_2, \frac{\Lambda_3}{4}, \Lambda_4, \Lambda_5 \right\} \simeq 0.00007, \Lambda_6 \simeq 0.000007.
$$

Tables 1 and 2 show the numerical results (for getting the technique, see Algorithm 1).

So, by assuming that $\lambda = 1.5$ and $\ell_1 = 12$, all conditions of Theorem 4.1 hold, then we can choose $\ell_2 > \ell_1$ and $\lambda$ satisfying

$$
0 < \lambda \leq \frac{1}{2}(1 - \mu) \min\left\{ 1, \frac{\rho_1}{|p_1^{\rho_1} - \lambda p_1^{\rho_1}|} \right\} = 2.4542789 < \ell_2, \\
2\lambda \left( \frac{|p_1^{\rho_1} - \lambda p_1^{\rho_1}|}{\rho_1 - \mu \rho_1} \right) = 3.42259 \leq 12 = \ell_1,
$$

respectively.
Table 1 Numerical values of \( \int_a^\iota G_1(\iota, \xi) \, d\xi \), \( \tilde{\iota}_0^2 \), \( \gamma \), \( \int_a^\iota G_2(\iota, \xi) \, d\xi \), \( \int_a^\iota H(\iota, \xi) \theta(\xi) \, d\xi \), and \( \Delta = \phi(\iota) H(\iota, \xi) \theta(\xi) \) in Example 6.1 for \( \tau \in J \)

| \( \tau \) | \( \int_a^\iota G_1(\iota, \xi) \, d\xi \) | \( \tilde{\iota}_0^2 \) | \( \gamma \) | \( \int_a^\iota G_2(\iota, \xi) \, d\xi \) | \( \int_a^\iota H(\iota, \xi) \theta(\xi) \, d\xi \) | \( \Delta \) |
|---|---|---|---|---|---|---|
| 2.7183 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.0377 | 0.2357 | 0.0039 | 0.0011 | 0.0346 | 3.7320 | 13.9275 |
| 3.3947 | 0.5121 | 0.0085 | 0.0025 | 0.6944 | 10.0873 | 339.0940 |
| 3.7937 | 0.8293 | 0.0138 | 0.0040 | 1.0416 | 28.8166 | 830.3948 |
| 4.2395 | 1.1850 | 0.0197 | 0.0057 | 1.3835 | 18.4145 | 339.0948 |
| 4.7377 | 1.5738 | 0.0261 | 0.0076 | 1.7148 | 41.5311 | 1724.8351 |
| 5.2945 | 1.9852 | 0.0329 | 0.0095 | 2.0276 | 56.8307 | 5617.7976 |
| 5.9167 | 2.3994 | 0.0387 | 0.0115 | 2.3105 | 74.9520 | 9213.7513 |
| 6.6120 | 2.7785 | 0.0461 | 0.0133 | 2.5445 | 95.9883 | 14,326.5251 |

Table 2 Numerical values of \( \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6, \) and \( \tilde{\iota}_4 \) in Example 6.1 for \( \tau \in J \)

| \( \tau \) | \( \Lambda_1 \) | \( \Lambda_2 \) | \( \Lambda_3 \) | \( \Lambda_4 \) | \( \Lambda_5 \) | \( \Lambda_6 \) | \( \tilde{\iota}_4 \) |
|---|---|---|---|---|---|---|---|
| 2.7183 | inf | inf | inf | inf | inf | inf | inf |
| 3.0377 | 0.178637 | 0.585071 | 0.179564 | 0.043506 | 53.867322 | 0.071800 | 0.043506 |
| 3.3947 | 0.066090 | 0.045238 | 0.011463 | 0.005955 | 1.916555 | 0.009828 | 0.005731 |
| 3.7937 | 0.036203 | 0.009195 | 0.002150 | 0.001787 | 0.240558 | 0.002949 | 0.001075 |
| 4.2395 | 0.023135 | 0.000621 | 0.000021 | 0.000730 | 0.051977 | 0.001204 | 0.000311 |
| 4.7377 | 0.016052 | 0.000104 | 0.000002 | 0.000062 | 0.005456 | 0.000310 | 0.000049 |
| 5.2945 | 0.011731 | 0.000499 | 0.000097 | 0.000188 | 0.005456 | 0.000310 | 0.000049 |
| 5.9167 | 0.008895 | 0.000252 | 0.000047 | 0.000108 | 0.000227 | 0.000178 | 0.000023 |
| 6.6120 | 0.006945 | 0.000140 | 0.000025 | 0.000066 | 0.000109 | 0.000019 | 0.000012 |
| 7.3891 | 0.005570 | 0.000085 | 0.000015 | 0.000042 | 0.000061 | 0.000007 | 0.000007 |

Figure 1 2D-graph of \( \tilde{\iota}_2 \) for \( \tau \in [e, e^2] \) in Example 6.1

and \( 2\lambda \leq \ell_1(1 - \mu) = 10.5 \) such that

\[
\Omega_1 = \{ q \in K : \| q \| < \ell_1 \}, \quad \Omega_2 = \{ q \in K : \| q \| < \ell_2 \}.
\]

Figures 1, 2, and 3 show a graphical representation of the variables. As shown in Fig. 1, \( \tilde{\iota}_2 \) is directly related to \( \tau \in [e, e^2] \) and increases with increasing \( \tau \). It can be seen in Fig. 2(a) that all values of \( \Lambda_i \) for \( i = 1, 2, 3, 4, 5 \) are inversely proportional to \( \tau \). Also, \( \tilde{\iota}_4 \) has the
The same behavior for $\tau \in J$, which can be seen in Fig. 2(b). Finally, the trend of variable $\Lambda_6$ with respect to $\tau$ is shown in Fig. 3. Then we can show that fractional boundary value problem (42) has at least a positive solution $q \in K \cap (\Omega_2 \setminus \Omega_1)$ for $\lambda$ small enough.

**Example 6.2** Let us consider the following $p$-Laplacian fractional boundary value problem:

\[
\begin{align*}
\rho \tau^{\alpha/2} C^{5/2}_{CK}(\phi_p(1/2\tau^{5/2} C^{5/2}_{CK} q))(\tau) \\
+ \frac{5\tau^{5/4}}{4} \ln(\tau) \phi(q(\tau)) = 0, & \quad 1 < \tau < e, \\
q(1) - F_1(1/2\tau^{5/2} C^{5/2}_{CK} q(1)) = 0, & \\
\delta_1^2 q(1) = 0, & \quad \delta_1^1 q(e) = \frac{1}{2} \delta_1^1 q(\sqrt{e}) + \lambda, \\
1/2\tau^{5/2} C^{5/2}_{CK} q(e) = -\delta_0^1 [\phi_p(1/2\tau^{5/2} C^{5/2}_{CK} q)](1) & = \delta_0^2 [\phi_p(1/2\tau^{5/2} C^{5/2}_{CK} q)](e) = 0.
\end{align*}
\]  

(43)
Here, \( J = [1, e] \), \( \sigma_1 = \sigma_2 = \frac{3}{2} \in (2, 3] \),
\[
\mu = \frac{1}{2} \in (0, 1), \quad \eta = \sqrt{e} \in J, \quad [\sqrt{e}, \sqrt[3]{e}] \subset J.
\]

We put
\[
\rho_1 = 0.5 \in \mathbb{R} \setminus \{1\}, \quad \rho_2 = 2 \in \mathbb{R} \setminus \{1\}, \quad p = \frac{3}{2},
\]
and so \( \bar{p} = 3, A = \frac{3}{2}, B = \frac{1}{2} \). \( p_1 \frac{\Gamma(1/2)}{\Gamma(5/2)} \) and \( p_2 \frac{\Gamma(1/2)}{\Gamma(5/2)} \) are the left- and right-sided Caputo–Katugampola fractional derivatives, \( f_\sigma(v) = \sqrt{|v|} \) and
\[
h(\tau) = \frac{5\sqrt{\pi}}{4} \ln(\tau),
\]
and
\[
\varphi(q) = \begin{cases} 6q^2, & q \leq 1, \\ 5 + q^{1/4}, & q > 1. \end{cases}
\]

Through a simple calculation, we have
\[
\int_1^e H(e, \xi) h(\xi) \, d\xi = 12.5716,
\]
\[
\gamma = \left( \frac{\bar{\alpha}_{\delta}^{p_1} - \bar{\alpha}^{p_1}}{\bar{\alpha}^{p_1} - \bar{\alpha}_{\delta}^{p_1}} \right)^{\sigma_1 - 1}, \quad \bar{M}_2 = \left( \frac{e^{0.25} - 1}{e^{0.5} - 1} \right)^{3/2} \times 0.4325 = 0.1253.
\]

Tables 3 and 4 show the numerical results (for getting the technique, see Algorithm 2).
\[
\bar{M}_4 = \min \left\{ \frac{\Lambda_1}{4}, \frac{\Lambda_2}{4}, \frac{\Lambda_3}{2}, \Lambda_4, \Lambda_5 \right\} \simeq 0.001499,
\]
and \( \Lambda_6 \simeq 1.583636 \). Figures 4, 5, and 6 show a graphical representation of the variables. As shown in Fig. 4, \( \bar{M}_2 \) is directly related to \( \tau \in [1, e] \) and increases with increasing \( \tau \). It can be seen in Fig. 5(a) that all values of \( \Lambda_i \) for \( i = 1, 2, 3, 4, 5 \) are inversely proportional to \( \tau \). Also, \( \bar{M}_4 \) has the same behavior for \( \tau \in J \), which can be seen in Fig. 5(b). Finally, the

### Table 3

| \( \tau \) | \( \int_0^\tau H(\bar{\eta}, \xi) \) d\( \xi \) | \( \bar{M}_2 \) | \( \gamma \) | \( \int_0^\tau G_1(\bar{\eta}, \xi) \) d\( \xi \) | \( \int_0^\tau H(\bar{\eta}, \xi) h(\xi) \) d\( \xi \) | \( \Delta \) |
|---|---|---|---|---|---|---|
| 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.1052 | 0.0602 | 0.1059 | 0.0307 | 0.0384 | 0.0119 | 0.001 |
| 1.2214 | 0.1299 | 0.2284 | 0.0662 | 0.0771 | 0.0793 | 0.0063 |
| 1.3499 | 0.2091 | 0.3677 | 0.1065 | 0.1157 | 0.2572 | 0.0661 |
| 1.4918 | 0.2975 | 0.4325 | 0.1253 | 0.1539 | 0.6196 | 0.3839 |
| 1.6487 | 0.3942 | 0.4325 | 0.1253 | 0.1913 | 1.2649 | 1.5999 |
| 1.8221 | 0.4975 | 0.4325 | 0.1253 | 0.2273 | 2.3154 | 5.3612 |
| 2.0138 | 0.6046 | 0.4325 | 0.1253 | 0.2610 | 3.9084 | 15.2758 |
| 2.2255 | 0.7105 | 0.4325 | 0.1253 | 0.2913 | 6.1656 | 38.0151 |
| 2.4596 | 0.8056 | 0.4325 | 0.1253 | 0.3162 | 9.1216 | 83.2040 |
| 2.7183 | 0.8654 | 0.4325 | 0.1253 | 0.3307 | 12.5716 | 158.0444 |
Table 4 Numerical values of $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, $\Lambda_4$, $\Lambda_5$, and $\tilde{M}_4$ in Example 6.2 for $\tau \in J$

| $\tau$ | $\Lambda_1$ | $\Lambda_2$ | $\Lambda_3$ | $\Lambda_4$ | $\Lambda_5$ | $\Lambda_6$ | $\tilde{M}_4$ |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1.0000 | Inf         | Inf         | Inf         | Inf         | Inf         | Inf         | Inf         |
| 1.1052 | 56.016486   | 16.599423   | 46.45330726 | 5493.069851 | 1132.69598  | 7060.155197 | 4.149856    |
| 1.2214 | 8.405999    | 7.690366    | 487.818288  | 123.697821  | 243.285192  | 158.986839  | 1.922592    |
| 1.3499 | 2.592400    | 4.749445    | 28.980434   | 11.764874   | 93.355843   | 15.121205   | 0.648100    |
| 1.4918 | 1.075949    | 3.242105    | 3.526105    | 2.026594    | 54.177100   | 2.604749    | 0.268987    |
| 1.6487 | 0.527061    | 2.217810    | 0.641322    | 0.486300    | 37.060645   | 0.625034    | 0.131765    |
| 1.8221 | 0.287924    | 1.382433    | 0.152208    | 0.145124    | 23.101100   | 0.186526    | 0.071981    |
| 2.0138 | 0.170572    | 0.744218    | 0.044104    | 0.050933    | 12.436232   | 0.065463    | 0.022052    |
| 2.2255 | 0.108126    | 0.361888    | 0.015127    | 0.020467    | 6.047314    | 0.026305    | 0.007563    |
| 2.4596 | 0.073086    | 0.175974    | 0.006109    | 0.009351    | 2.940615    | 0.012019    | 0.003055    |
| 2.7183 | 0.053030    | 0.094769    | 0.002998    | 0.004923    | 1.583636    | 0.006327    | 0.001499    |

![Figure 4](image-url) 2D-graph of $\tilde{M}_4$ for $\tau \in [1,e]$ in Example 6.2

![Figure 5](image-url) Graphical representation of $\Lambda_i$ ($i = 1, 2, 3, 4, 5$) and $\tilde{M}_4$ for $\tau \in J$ in Example 6.2
trend of variable $\Lambda_6$ with respect to $\tau$ is shown in Fig. 6. Choosing $\hat{a} = 10^{-2}, b = \frac{11}{10}, c = 10^5, m_1 = 0.001 \in (0, \tilde{M}_4), m_2 = 13 \in (\Lambda_6, \infty) = (1.583636, \infty)$, we get

$$\varphi(q) < \varphi(10^{-2}) = 6 \times 10^{-4} < \min\{\varphi_{\beta}(\hat{a}m_1), \hat{a}m_1\}$$

$$= \hat{a}m_1 = 8 \times 10^{-4} \in [0, 10^{-2}],$$

$$\varphi(q(\tau)) > 5 + (m_2 \gamma b)^{1/4} = 5 + \left(0.1253 \times 13 \times \frac{11}{10}\right)^{1/4} \simeq 1.1570 > \varphi(b \gamma m_2)$$

$$\simeq 0.739467251 \in \left[\frac{11}{10} \gamma, \frac{11}{10}\right],$$

$$\varphi(q(\tau)) < \varphi(10^4) = 15 \times \min\{\varphi_{\beta}(cm_1), cm_1\}$$

$$= \varphi_{\beta}(cm_1) = \sqrt{8000.001} q \in [0, 10^4],$$

$$0 < \lambda \leq \frac{(1 - \mu)\hat{a}}{2} = 2.5 \times 10^{-3}.$$

Then, conditions (H7), (H8), and (H9) are satisfied. Therefore, it follows from Theorem 5.2 that fractional boundary value problem (43) has at least three $\frac{1}{2}$-concave positive solutions $q_1, q_2,$ and $q_3$ such that

$$\|q_1\| < 10^{-2}, \quad \frac{11}{10} \gamma < \varphi(q_2), \quad \|q_3\| > 10^{-2},$$

with $\varphi(q_3) < \frac{11}{10} \gamma$.

7 Conclusion

The paper presents a new $p$-Laplacian boundary value problem of two-sided fractional differential equations involving generalized Caputo fractional derivatives, and we investigate the existence and multiplicity of $p$-concave positive solutions of it. We made some
additional assumptions to prove some important results and obtain the existence of at least three solutions by using some fixed point theorems.

Appendix

Algorithm 1 (MATLAB lines for calculation of variable values in Example 6.1)
Algorithm 2 (MATLAB lines for calculation of variable values in Example 6.2)

1 clear;
2 format short;
3 sym s t r a x s b v e;
4 uprho_1=0.5; uprho_2=2;
5 sigma_1= 5/2; sigma_2= 5/2;
6 ga=1; gi=exp(1) ; gacirc=sqrt(exp(1)) ; gicirc=exp(1)^(1/4) ;
7 eta=exp(1)^8/5; mu=1/8;
8 p = 3/2;
9 barp=p/(p–1);
10 A= 3/2; B = 1/2;
11 hslash=5 * sqrt(pi)* log(x)/4;
12 fcirc=sqrt (abs (v));
13 Zedgi=upphi(barp , gi);
14 Zedgicirc=upphi(barp,gicirc);
15 mathringa=10^( –2);
16 column=1;
17 row=1;
18 tau=ga ;
19 while tau<=gi
20 RMatrix(row, column)=row;
21 RMatrix(row, column+1)=tau ;
22 xi= (( gi^uprho_2 – gicirc^uprho_2)...
23 /( gi^uprho_2 –ga^uprho_2))^(sigma_2 –1);
24 Zedgicirc=upphi(barp,xi);
25 RMatrix(row, column+2)=Zedgicirc ;
26 if tau<=gi
27 G1=1/gamma(s–1)*((b^r –x^r)/r)*((x^r –a^r)/r)^(s–2)*x^(r–1)...
28 –1/gamma(s)*((x^r –t^r)/r)^(s–1)*x^(r–1);
29 else
30 G1=1/gamma(s–1)*((b^r –x^r)/r)*((x^r –a^r)/r)^(s–2)*x^(r–1);
31 end
32 intG1=int(subs(G1, {t , s , r , a , b} , {gi , sigma_1 , uprho_1 , ga , gi }) , x, ga ,
33 tau);
34 breveM_2 = min(min(1, (sigma_1 –2)/(sigma_1 –1)...
35 * ((gacirc^uprho_1 –ga^uprho_1)...
36 /uprho_1)^2),gamma(s–1)*((b^r –x^r)/r)*((x^r –a^r)/r)^(s–2)*x^(r–1)...
37 –1/gamma(s)*((x^r –t^r)/r)^(s–1)*x^(r–1));
38 RMatrix(row, column+3)=breveM_2;
39 if tau<=gi
40 RMatrix(row, column+4)=vargamma;
41 else
42 RMatrix(row, column+4)=1/gamma(s–1)*((b^r –x^r)/r)*((x^r –a^r)/r)^(s–2)*x^(r–1)...
43 –1/gamma(s)*((x^r –t^r)/r)^(s–1)*x^(r–1);
44 end
45 intGHhslash=int (hslash *subs(GH, {t , s , r , a , b} , {ga , sigma_2 , uprho_2 ,
46 ga , gi } ), x, ga , tau);
47 RMatrix(row, column+6)=intGHhslash ;
48 Lambda1=1/(A * intGHhslash);
49 RMatrix(row, column+7)=Lambda1;
50 if tau<=gi
51 G2=1/gamma(s–1)*((b^r –x^r)/r)*((x^r –a^r)/r)^(s–2)*x^(r–1)...
52 –1/gamma(s)*((x^r –t^r)/r)^(s–1)*x^(r–1);
53 end

\begin{verbatim}
else
  G2 = 1 / \Gamma(s - 1) * \left( \frac{b - r - a}{r} \right)^{(s - 2)} * \frac{x^{s - 2}}{r} * \left( \frac{b - r - x}{r} \right)^{(s - 2)} * \frac{x^{s - 2}}{r};
end;
intG2 = int(subs(G2, \{ t, s, r, a, b \}, \{ gi, \sigma_1, upho_1, ga, gi \}), x, ga, tau);
RMatrix(row, column + 8) = intG2;
upphiintGH = upphi(barp, intGHSlash);
RMatrix(row, column + 9) = upphiintGH;
temp = intGI + \sigma_1 * \left( \frac{g_i^{upho_1 - ga^{upho_1}}}{upho_1} \right) * \left( \frac{g_i^{upho_1 - ga^{upho_1}}}{upho_1} \right) * \left( \frac{g_i^{upho_1 - ga^{upho_1}}}{upho_1} \right) * \left( \frac{g_i^{upho_1 - ga^{upho_1}}}{upho_1} \right) * \left( \frac{g_i^{upho_1 - ga^{upho_1}}}{upho_1} \right); 
Lambda2 = 1 / temp;
Lambda3 = 1 / \left( \frac{\sigma_1 - 1}{\sigma_1 - 2} \right) * \frac{gi^{upho_1 - ga^{upho_1}}}{upho_1};
Lambda4 = 1 / \left( \frac{\sigma_1 - 1}{\sigma_1 - 2} \right) * \frac{gi^{upho_1 - ga^{upho_1}}}{upho_1};
Lambda5 = 1 / \upphiintGH;
Lambda6 = 1 / \left( \frac{\sigma_1 - 1}{\sigma_1 - 2} \right) * \frac{gi^{upho_1 - ga^{upho_1}}}{upho_1};
end;
\end{verbatim}

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Author contribution
FC: Actualization, methodology, formal analysis, validation, investigation, initial draft, and major contribution in writing the manuscript. MB: Methodology, formal analysis, validation, investigation, and initial draft. MH: Actualization, methodology, formal analysis, validation, investigation, initial draft, and major contribution in writing the manuscript. MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft, and major contribution in writing the manuscript. All authors read and approved the final manuscript.

Author details
1Laboratory of Mathematics and Applied Sciences, University of Ghardaia, Ghardaia 47000, Algeria. 2Faculty of Sciences, Saad Dahlab University, Bida, Algeria. 3Laboratory of Pure and Applied Mathematics, University Med Boudiaf, Box 166, 2800, M'sila, Algeria. 4Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)

2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. A Wiley-Inter Science Publication. Wiley, New York (1993)

3. Etemad, S., Rezapour, S., Samei, M.E.: On a fractional Caputo–Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint. Math. Methods Appl. Sci. 43(17), 9719–9734 (2021). https://doi.org/10.1002/mma.6644

4. Samei, M.E., Matar, M.M., Etemad, S., Rezapour, S.: On the generalized fractional snap boundary problems via $q$-Caputo operators: existence and stability analysis. Adv. Differ. Equ. 2021, 498 (2021). https://doi.org/10.1186/s13662-021-03654-9

5. Oldham, K.B., Spanier, J. Fractional Calculus. Academic Press, New York (1974)

6. Rezapour, S., Mohammadi, H., Samei, M.E.: SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order. Adv. Differ. Equ. 2020, 490 (2021). https://doi.org/10.1186/s13662-020-02952-y

7. Podlubny, I.: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. 5, 367–386 (2002)

8. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to mumps virus with optimal control. Chaos Solitons Fractals 144, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668

9. Samei, M.E., Hedayati, V., Rezapour, S.: Existence results for a fraction hybrid differential inclusion with Caputo–Hadamard type fractional derivative. Adv. Differ. Equ. 2019, 163 (2019). https://doi.org/10.1186/s13662-019-2090-8

10. Hedayati, V., Samei, M.E.: Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions. Bound. Value Probl. 2019, 141 (2019). https://doi.org/10.1186/s13661-019-1251-8

11. Elmoataz, A., Desquesnes, X., Lezoray, O.: Non-local morphological PDEs and $p$-Laplacian equation on graphs with applications in image processing and machine learning. IEEE J. Sel. Top. Signal Process. 6(7), 764–779 (2012)

12. Torres, F.: Positive solutions for a mixed-order three-point boundary value problem for p-Laplacian, abstract and applied analysis. J. Math. Anal. Appl. 2013, Article ID 912576 (2013). https://doi.org/10.1155/2013/912576

13. Tang, X., Yan, C., Liu, Q.: Existence of solutions of two point boundary value problems for fractional p-Laplacian differential equations at resonance. J. Appl. Math. Comput. 41, 119–131 (2013). https://doi.org/10.1007/s12190-012-0598-0

14. Alkhazzan, A., Al-Sadi, W., Wattanajeekom, V., Khan, H.: A new study on the existence and stability to a system of coupled higher-order nonlinear BVP of hybrid FDEs under the p-Laplacian operator. AIMS Math. 7(8), 14187–14207 (2022). https://doi.org/10.3934/math.2022782

15. Su, H., Wei, Z., Wang, B.: The existence of positive solutions for a nonlinear four-point singular boundary value problem with a p-Laplacian operator. Nonlinear Anal., Theory Methods Appl. 66, 2204–2217 (2007). https://doi.org/10.1016/j.na.2005.10.039

16. Rezapour, S., Abbas, M.I., Etemad, S., Dienie, N.M.: On a multi-point p-Laplacian fractional differential equation with generalized fractional derivatives. Mathematics (2022). https://doi.org/10.1002/mma.8801

17. Qiu, J., Li, Y., Liu, L., Mu, X.: On the existence and uniqueness of solutions for fractional boundary value problems of fractional order. Adv. Differ. Equ. 2020, 272 (2020). https://doi.org/10.1186/s13662-020-02544-w

18. Su, H., Wei, Z., Wang, B.: Positive solutions for m-order p-Laplacian operator singular boundary value problem. Appl. Math. Comput. 199, 122–132 (2008). https://doi.org/10.1016/j.amc.2007.09.043

19. Chai, G.: Positive solutions for boundary value problems of fractional differential equation with p-Laplacian. Bound. Value Probl. 2012, 18 (2012). https://doi.org/10.1186/1687-2770-2012-18

20. Chen, T., Liu, W., Hu, Z.: A boundary value problem for fractional differential equation with p-Laplacian operator at resonance. Bound. Value Probl. 75(6), 3210–3217 (2012). https://doi.org/10.1186/2048-7689-75-62020

21. Bai, C.: Existence and uniqueness of solutions for fractional boundary value problems with p-Laplacian operator. Adv. Differ. Equ. 2018, 4 (2018). https://doi.org/10.1186/s13662-017-1460-3

22. Nalini, H., Etemad, S., Patanaranpeelert, N., Asmaoah, J.K.K., Rezapour, S., Sithiwiiratatham, T.: A study on dynamics of CD4+ T-cells under the effect of HIV-1 infection based on a mathematical fractal-fractional model via the Adams-Bashforth scheme and Newton polynomials. Mathematics 10(9), 1366 (2022). https://doi.org/10.3390/math10091366

23. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of CD4+ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. 2020, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w

24. Aydogan, M.S., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. Bound. Value Probl. 2018, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9

25. Baleanu, D., Etemad, S., Pourraz, S., Rezapour, S.: On the new fractional hybrid boundary value problems with three-point integral hybrid conditions. Adv. Differ. Equ. 2019, 473 (2019). https://doi.org/10.1186/s13662-019-2407-7

26. Baleanu, D., Rezapour, S., Saberinia, Z.: On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivative. Bound. Value Probl. 2019, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0

27. Abdeljawad, T., Samei, M.E.: Applying quantum calculus for the existence of solution of $q$-integro-differential equations with three criteria. Discrete Contin. Dyn. Syst., Ser. S 14(10), 3351–3386 (2021). https://doi.org/10.3390/math10091366

28. Alizadeh, S., Baleanu, D., Rezapour, S.: Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. Adv. Differ. Equ. 2020, 55 (2020). https://doi.org/10.1186/s13662-020-2527-0
29. Matar, M.M., Abbas, M.I., Alzabut, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the $p$-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. 2021, 68 (2021). https://doi.org/10.1186/s13662-021-03228-9

30. Thabet, S.T.M., Etemad, S., Rezapour, S.: On a coupled Caputo conformable system of pantograph problems. Turk. J. Math. 45(1), 496–519 (2021). https://doi.org/10.3906/mat-2010-70

31. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. 2020, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0

32. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. 59(5), 3019–3027 (2020). https://doi.org/10.1016/j.aej.2020.04.053

33. Baleanu, D., Hedayaati, H., Rezapour, S., Mohamed Al Qurashi, M.: On two fractional differential inclusions. SpringerPlus 2016, 882 (2016). https://doi.org/10.1186/s40064-016-3564-z

34. Hajiseyedazizi, S.N., Samei, M.E., Alzabut, J., Chu, Y.: On multi-step methods for singular fractional $q$-integro-differential equations. Open Math. 19, 1378–1405 (2021). https://doi.org/10.1515/math-2021-0093

35. Samei, M.E., Rezapour, S.: On a system of fractional $q$-differential inclusions via sum of two multi-term functions on a time scale. Bound. Value Probl. 2020, 135 (2020). https://doi.org/10.1186/s13661-020-01433-1

36. Samei, M.E., Yang, W.: Existence of solutions for $k$-dimensional system of multi-term fractional $q$-integro-differential equations under anti-periodic boundary conditions via quantum calculus. Math. Methods Appl. Sci. 43(7), 4360–4382 (2020). https://doi.org/10.1002/mma.6198

37. Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional $q$-integro-differential equation. Bound. Value Probl. 2020, 38 (2020). https://doi.org/10.1186/s13661-020-01342-3

38. Katugmpola, U.N.: A new approach to generalized fractional derivatives. Bull. Math. Anal. Appl. 6(4), 1–15 (2014)

39. Guo, D., Lakshmikantham, V., Liu, X.: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic, Dordrecht (1996)

40. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press, San Diego (1988). https://doi.org/10.1016/j.matpur.2013.01.005

41. Zhang, K.S., Wan, J.P.: $p$-Convex functions and their properties. Pure Appl. Math. 23(1), 130–133 (2007)

42. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Generalized convexity and inequalities. J. Math. Anal. Appl. 335(2), 1294–1308 (2007)

43. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. 28, 673–688 (1979). https://doi.org/10.1512/iumj.1979.28.28046