Group Variables in Quantum Gravity

M. Chichikina
Moscow State University, Physical Department
Chair of Quantum Field Theory and High Energy Physics
MSU, Moscow, 119992, Russia
E-mail: chich@goa.bog.msu.ru

Abstract. In this presentation we offer scheme of quantization of boson fields on the classical background by means of the Bogoliubov group variables.

1. Introduction

In article we represent the theory of quantization of nonlinear boson fields. We suppose that field have nontrivial classical component. Bogoliubov group variables are used for quantization. This scheme is applicable for any systems with invariant symplectic structures of a form

\[ \omega(f(x), g(x)) = \int_D (f_n(x)g(x) - f(x)g_n(x))dx \]  

(1)

here \( D \) is some space-like surface and \( f_n(x) \) means normal derivative of \( f(x) \) in the surface \( D \).

(Provided that we know representation of symmetry group)

Any nonlinear boson system have nontrivial classical part. If this part is small, it is possible to use standard perturbation theory. However the neglect by a classical field leads to infrared divergences even in a quantum electrodynamics. Those difficulties are overcome behind frameworks of the perturbation theory. In some cases it is impossible to consider a classical field to be small (gravitation and extended particles). In this case the main effect is the separation of a classical boson field.

Quantization on a classical background meet two basic difficulties.

Firstly, — and this main — a problem of the conservation laws. Explicit separation of the classical components lead to violation of the conservation laws. Let’s write down explicit separation of a classical field in the form

\[ \hat{\phi}(x) = \phi_0(x)\hat{E} + \hat{\phi}_1(x). \]

Many systems have continuous spatial symmetry, and each theory should be relativistic invariant. Suppose that we know representation of symmetry group of system. The operator \( \hat{S} \) describes transformation of this group and the classical field does not change:

\[ \hat{S}\hat{\phi}(x)\hat{S}^+ = \phi_0(x) + \hat{S}\hat{\phi}_1(x)\hat{S}^+, \]
that is
\[ \hat{\phi}(x') = \phi_0(x) + \hat{\phi}_1(x'). \]
It is clear, that in such case the classical field can be considered as external field. It means that the quantum field varies, and classical remains former. We see, that the explicit separation of a classical field breaks invariancy of system. Hence the conservation laws are violated.

The second difficulty of quantization on a classical background is the zero-mode problem. To bypass these problems we propose to use idea of the Bogoliubov, which was proposed in work [I]. N.N.Bogoliubov has proposed to carry out quantization in the terms of new variables. These variable make sense of parameters of symmetry group of system and they are cyclic. The integrals of motion in terms of new variable turns out generators of group and, hence, commute with Hamiltonian. Explicit and exact performance of the conservation laws in any order of the perturbation theory thus is provided. Obvious separation of classical components removes a zero-mode problem.

The transformations of the Bogoliubov are widely used in a quantum field theory. However, if the group of invariancy of system includes transformation of time, there are difficulties. Hamiltonian looks like a generator of time translation after solution of equations of motion. For record of equations of motion in explicit form it is necessary to know expression for Hamiltonian. This difficulty limited a field of application of the Bogoliubov group variables, because it was not possible to consider non-stationary systems.

In this work we propose quantization, new a way. This scheme allows to use the Bogoliubov group variables for systems having arbitrary symmetry group. (Including non-stationary systems). Quantization on the classical background seems to be a very special case, but existence of classical component is an essential feature of gravitational field.

We consider exact solutions of Einstein equation to be classical background and quantize in the neighborhood of this exact solutions. Now we would like to represent the scheme of quantization of gravitational field on the classical background by means of Bogoliubov group variables.

2. Space-time description
We consider gravitational field in (3+1)-dimensioned formalism that has been proposed by Arnowitt-Deser-Misner (ADM).

Metrical tensor in this formalism looks like
\[ g_{\alpha\beta} = \begin{pmatrix} -a^2 + b_t b_t & b_t \\ b_t & \gamma_{st} \end{pmatrix}, \]
here \( \gamma_{st} \) is metric of 3D-space in 4D-manyfold.

Canonical momentum \( \pi^{st} \) is determined as usual:
\[ \pi^{st} = -\sqrt{\gamma} (K^{st} - \gamma^{st} K), \]
here \( \sqrt{\gamma} = \sqrt{\det \| \gamma_{st} \|}; K_{lp} = -a \delta_{0lp}, s_{lp} = \frac{1}{2} \gamma^{sp} u_lp; u_lp = \gamma_{pl,t} + \gamma_{pt,l} - \gamma_{tl,p}. \)

Denoting as usually
\[ R_{\kappa\lambda} = \sigma_{\kappa\lambda,\sigma} - \sigma_{\kappa\sigma,\lambda} + \sigma_{\kappa\lambda} \delta_{\rho\rho} - \sigma_{\kappa\rho} \delta_{\rho\sigma}, \]
we can represent the action of gravitational field
\[ S = \int d^3x \sqrt{g} g^{\kappa\lambda} R_{\kappa\lambda}. \]
in the following form:

\[ S = \int d^3 x \left( \pi^{st} \gamma_{st,0} - aH - b_s H^s \right), \]

here

\[ H = \frac{1}{\sqrt{\gamma}} \left( \pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} R, \]

\[ H^s = -2\pi^s_{,j}. \]

Suppose that 4D manifold with given metric permits to chose space-like hypersurface \( \Sigma \) and to set normals field on this hypersurface. Those normals are tangent to geodesic and determine time coordinate. Hence geometry of 4D manifold could be described via Gaussian coordinates:

\[ g_{\alpha\beta} = \begin{pmatrix} -a^2 & 0 \\ 0 & \gamma_{st} \end{pmatrix}, \]

We have to chose hypersurface \( \Sigma \) in according with the following principle:

Suppose that normal \( \vec{n} \) is given in the hypersurface \( \Sigma \) at the point \( X \). Let’s chose arbitrary closed line \( K \subset \Sigma \). (\( X \subset K \))

Normal vector \( \vec{n} \) have to coinside with it’s original direction after whole cycle motion along the line \( K \)- this is criterium of the hypersurface \( \Sigma \) choice. Choquet-Bruhat showed that this criterium could be realized via imposing of the Hamiltonian constraint \( H = 0 \) and momentum constraint \( H^s = 0 \)

General principles of canonical formalism for the systems with constraints leads us to the following statement (Lichnerovitch, Choquet-Bruhat, Dirac, Antrowitt-Deser-Misner):

If the evolutions equations

\[ \gamma_{st,0} = \frac{2a}{\sqrt{\gamma}} \left( \pi_{st} - \frac{1}{2} \gamma_{st} \pi \right) + b_{ket} + b_{lt}; \]

\[ \pi^{st}_{,0} = -a\sqrt{\gamma} \left( R^{st} - \frac{1}{2} \gamma^{st} R \right) + \frac{a}{2\sqrt{\gamma}} \left( \pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) \gamma^{st} + \]

\[ -\frac{a}{2\sqrt{\gamma}} \left( \pi^{st}_{,t} - \frac{1}{2} \pi^{st}_{,t} \pi \right) + \sqrt{\gamma} \left( \gamma^{st} c^t_{,j} - \gamma_{st} c^t_{,j} \right) + \]

\[ + \left( \pi^{st}_{,t} b^t_{,j} - \pi^{st}_{,t} b^l_{,j} - \pi^{st}_{,l} b^t_{,j}; \right. \]

\[ c^t = \gamma^{st} a^t_{,s}; \]

are performed on the 3D-space, then in 4D manifold the Einstein equations holds true:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \quad (R^n_{ijkl,m} + R^n_{ijkm,l} + R^n_{iljk,m} = 0). \]
3. Bogoliubov Group Variables

Let's variables $x'$ are connected with $x$ by the following group transformation:

$$x' = f(x, a), \quad x'' = f(f(x, a), b) = f(x, c), \quad c = \varphi(a, b).$$

Under variation of group parameters $a$ variation of coordinates reads:

$$(\delta x')^i = \xi^i(x') B^i_p(a) \delta a^p,$$

where $i = 0, 1, 2, 3$ -the number of coordinates, $p = 1, ..., r$, where $r$ is a quantity of group generators, $B^i_p(a)$ defines group properties.

Note that conservation laws performance in curved space-time is connected with Killing vectors existence, they are not straightforward sequence of system space-time transformation invariance.

In present case Bogoliubov transformation reconstruct invariance with respect to transformation group, that has been violated due to explicit extraction of classical field.

It means the following: if we have made quantization in some surface in definite moment, application of group variables permits to state that we can move this surface with according to group laws (including transformation that changes the time coordinate).

Let's consider couples of functions $f_{st}(x)$, $f_{st}^n(x)$ and define Bogoliubov transformation as following:

$$f_{st}(x) = g v_{st}(x') + u_{st}(x'), \quad f_{st}^n(x) = g v_{st}^n(x') + u_{st}^n(x'), \quad (2)$$

dimensionless parameter $g$ is assumed to be large, and group parameters $a^p$ are independent new variables.

The substitution $f_{st}(x) \to \{u_{st}(a)\}$ enlarge the number of independent variables on $r$, so problem is how to formulate invariant conditions, which we have to impose on functions $u_{st}(x')$ in order to equalize the number of independent variables in the both part of equation (2).

We consider systems in which there are invariant symplectic forms that looks like the following:

$$\omega(u_{st}, N^{stp}) = \int (u_{st}^n(x') N^{stp}_n(x') - u_{st}(x') N^{stp}_n(x')) d\sigma,$$

here $\sigma$ is some space-like surface. Everywhere in the article we mean summation with respect to all $s$ and $t$.

Additional conditions are:

$$\omega(u_{st}, N^{stp}) = 0. \quad (3)$$

We chose some functions $N^{stp}_n(x')$ ($p = 1,...,r$, the quantity functions is equal to the quantity of new independent variables)

It is possible to obtain equations, which define group variables as functional of $f_{st}(x)$ and $f_{st}^n(x)$ on the .

$A^p_q(a)$ denote the matrix inverse to $B^p_q(a): B^p_q(a) A^p_q(a) = \delta^p_q.$ Straightforward calculations give us $f_{st}(x)$ and $f_{st}^n(x)$ in the terms of new variables:

$$\frac{\delta}{\delta f_{st}(x)} = \frac{\delta}{\delta u_{st}(x')} + B^p_q(a) \frac{\delta a^q}{\delta f_{st}(x)} \left( S_p + A^p_q(a) \frac{\partial}{\partial a^q} \right), \quad (4)$$
Here \( S_p \) is defined as 

\[
\int d\sigma \xi_p(x') \left( u_{stl}(x') \frac{\delta}{\delta u_{st}(x')} + u_{stn}(x') \frac{\delta}{\delta u_{st}(x')} \right) = S_p.
\]

4. Secondary Quantization.

The operators \( \hat{q}_{st}(x) \) and \( \hat{p}^{st}(x) \):

\[
\hat{q}_{st}(x) = \frac{1}{\sqrt{2}} \left( f_{st}(x) + i \frac{\delta}{\delta f_{st}(x)} \right), \quad \hat{p}^{st}(x) = \frac{1}{\sqrt{2}} \left( f_{st}(x) - i \frac{\delta}{\delta f_{st}(x)} \right), \tag{5}
\]

are defined in the space \( L \) of functionals \( F \) where the scalar product defined as

\[
<F_1|F_2> = \int Df_{st} Df^{st}_{n} F_{1n}[f_{st}, f^{st}_{n}] F_{2n}[f_{st}, f^{st}_{n}].
\]

The operators (5) are self-conjugated in this space. They satisfy the formal commutation relation:

\[
[q_{st}(x), p^{st}(x')] = i\delta(x - x').
\]

So we can treat \( \hat{q}_{st}(x) \) and \( \hat{p}^{st}(x') \) as operators of coordinate and momentum of oscillators of field and we can develop the secondary quantization scheme. But straightforward use of this procedure leads us to the doubling of numbers of possible field states, because there is are self-conjugate operators.

We use the following scheme:

1. We use Bogoliubov’s transformation (2) and, in spite of appearance of exceed states, we will develop scheme of perturbation theory. Then reduction of states number will be made, so it will depend on dynamical system equations.

2. Cause \( g >> 1 \), operators \( \frac{\partial}{\partial \alpha} \) enter in the \( \hat{q}_{st}(x') \) and \( \hat{p}^{st}(x') \) in the order \( O \left( \frac{1}{g} \right) \). With an eye to increase the order of the velocity, it is necessary to make a canonical transformation. Let’s substitute state vector \( \psi \) on the vector: \( \psi \rightarrow e^{ig^2 J} \psi \), that accords to the substitution:

\[
-iA^r_p(a) \frac{\partial}{\partial \alpha^r} \rightarrow g^2 J_p - iA^r_p(a) \frac{\partial}{\partial \alpha^r}. \tag{6}
\]

After canonical transformation the operators \( \hat{p}^{st}(x) \) and \( \hat{q}_{st}(x) \) become the following series:

\[
\hat{q}_{st} = g \left( F_{st}(x') + \frac{1}{g} \hat{Q}_{st}(x') + \frac{1}{g^2} A_{st}(x') \right), \quad \hat{p}^{st} = g \left( F^{st}_{n}(x') + \frac{1}{g} \hat{P}^{st}(x') + \frac{1}{g^2} A^{st}_{n}(x') \right).
\]

Explicit expressions for the addends in the series are:

\[
F_{st}(x') = \frac{1}{\sqrt{2}} \left( u_{st}(x') + N^k_{st}(x') J_k \right), \quad F^{st}_{n}(x') = \frac{1}{\sqrt{2}} \left( u^{st}_{n}(x') + N^{st}_{nk}(x') J_k \right),
\]

\[
\hat{Q}_{st}(x') = \frac{1}{\sqrt{2}} \left( u_{st}(x') + i \frac{\delta}{\delta u_{st}(x')} - N^k_{st}(x') R_k \right),
\]

\[
\hat{P}^{st}_{n}(x') = \frac{1}{\sqrt{2}} \left( u^{st}_{n}(x') + i \frac{\delta}{\delta u^{st}_{n}(x')} - N^{st}_{nk}(x') R_k \right).
\]
\[ \hat{P}^{st}(x') = \frac{1}{\sqrt{2}} \left( u^{st}_{n}(x') - i \frac{\delta}{\delta u^{st}(x')} - N^{stk}_{n}(x') r_{k} \right), \]

\[ A^{st}(x') = \frac{\delta a^{p}}{\delta f^{st}_{n}(x)} \left( B^{r}_{p}(a) R^{k}_{r} r_{k} - i K_{p} \right), \]

\[ A^{st}_{n}(x') = \frac{\delta a^{p}}{\delta f^{st}_{n}(x)} \left( B^{r}_{p}(a) R^{k}_{r} r_{k} - i K_{p} \right), \]

here \( T_{c} = K_{c} + R^{a}_{c} a, \) \( K_{p} = B^{p}_{k}(a) S_{q} + \frac{\partial}{\partial a^{p}}, \) \( r_{k} = R^{p}_{k} J_{p}. \)

We define contravariant components of coordinate operator and covariant component of momentum operator taking into account that they have to satisfy the following relations:

\[ \hat{q}^{st}(x) \hat{q}_{st}(x) = \delta^{t}_{s}, \quad \hat{q}^{st}(x) \hat{p}_{st}(x) = \hat{p}^{st}(x) \hat{p}_{st}(x). \]

It is possible if the operators reads:

\[ \hat{q}^{st}(x) = g \left( F^{st}(x') - \frac{1}{g} \hat{Q}^{st}(x') + \frac{1}{g^{2}} B^{st}(x') \right), \quad \hat{p}_{st}(x) = g \left( F_{nst}(x') + \frac{1}{g} S_{st}(x') + \frac{1}{g^{2}} D_{st}(x') \right), \]

and the addends are defined by the following way:

\[ F^{st} F_{nst} = \delta^{t}_{s}, \quad \hat{Q}^{st} = F^{rt} \hat{Q}_{r} F^{st}, \quad S_{kl} = F^{bd} \left( \hat{Q}_{bt} F_{kl} + \hat{Q}_{bl} F_{kt} \right) + F_{kb} \hat{P}^{st} F_{kl}, \]

\[ B^{st} = F^{sk} \hat{Q}_{kl} F^{tr} F^{mr} - F^{st} A_{kl} F^{lr}, \quad F_{n}^{st} F_{st} = F^{st} F_{nst}, \]

\[ D_{pl} = A_{np} + F^{st}_{n} \left( F_{tp} A_{sl} + F_{st} A_{tp} \right) + F^{sm}_{n} \hat{Q}_{st} \hat{Q}_{mp} + 2 F_{tp} \hat{P}^{st} \hat{Q}_{st}. \]

Then action can be represented as series with respect to inverted powers of coupling constant:

\[ S = S_{0} + S_{1} + S_{2}. \]  

(7)

5. Perturbation Theory Construction

Now we can quantize and substitute \( u^{st}(x'), \) \( u^{st}_{n}(x') \) as following:

\[ u^{st}(x') \rightarrow \hat{q}^{st}(x) \quad u^{st}_{n}(x') \rightarrow \hat{p}^{st}(x). \]

In the series (7) operators \( S_{0} \) are \( C \)-numbers. Let’s consider the higher order. Operator \( S_{1} \) is linear with respect to \( u^{st}(x'), \) \( u^{st}_{n}(x'), \) \( \frac{\partial}{\partial u^{st}(x')}, \) \( \frac{\partial}{\partial u^{st}_{n}(x')} \). There are no any normalizable eigenvectors of these operators, so it is required to set them to zero for perturbation theory construction. \( S_{1} \) is equal zero if the following equations holds true on:

\[ F^{st}_{stn} = 2a \sqrt{F} \left( F^{st}_{stn} - \frac{1}{2} F^{st} F_{stn} \right) \]

\[ F^{st}_{nn} = \frac{a}{2 \sqrt{F}} \left( F_{kn} F^{kl}_{n} - \frac{1}{2} F^{2}_{n} \right) F^{st} - \frac{2a}{\sqrt{F}} \left( F^{st}_{n} F^{kl}_{stn} - \frac{1}{2} F_{n} F^{st} \right) \]

\[ - a \sqrt{F} \left( R^{st} - \frac{1}{2} F^{st} R \right) - \sqrt{F} \left( F^{st} c^{t}_{\beta} - F^{st} c^{t}_{\beta} \right), \]  

(8)
These equations could be treated as evolution equations. 
Herandafter we assume $F_{st}(x)$ to be solution of the equations (8), and $F_{st}(x')$ and $F_{st}^{n}(x')$ on are the solution of the Cauchy problem on, so we can state that on the 3D manifold the evolution equation holds true. The constraint equations

$$
\frac{1}{\sqrt{F}} \left( F_{st}^{n} F_{stn} - \frac{1}{2} F_{n}^{2} \right) - \sqrt{F} R(F) = 0, \quad F_{st}^{n} = 0
$$

we obtain as conditions on the choice of surface. (See (1))

So we can state that Einstein equations performance is necessary condition for the perturbation theory to be applicable. We would like to underline that Einstein equations has been obtained in the process of perturbation theory construction as a condition of validity, not as a sequence of variational principle.

6. System State Space construction.
Now the task is to find the expressions for the action with second order accuracy with respect to inverted power of coupling constant. For that purpose it is necessary to construct system state space, in which one can make reduction of field state number and finds expressions for action derivatives with respect to symmetry group generators.

The number of independent variables has been doubled due to one consider $F_{st}(x)$ and $F_{st}^{n}(x)$ as independent. Cause of additional condition (3), which connect $u_{st}(x)$ and $u_{st}^{n}(x)$, the number of independent variables (minus group variables) become equal $(2 \ast \infty - r)$. To reduce them to $(\infty - r)$, one needs also $r$ conditions (Remind that $r$ is the number of group parameters).

For that end let’s represent $u_{st}(x')$ in the form:

$$
u_{st}(x') = w_{st}(x') + N_{st}^{a}(x')r_{a}, \quad u_{st}^{n}(x') = w_{st}^{n}(x') + N_{st}^{n}(x')r_{a}\eqno(1).
$$

We consider those relationships to express $u_{st}(x')$, $u_{st}^{n}(x')$ in the terms of independent variables $r_{a}$ and new functions $w_{st}(x')$, $w_{st}^{n}(x')$. Operators $Q$, $P$ in the terms of new variables reads:

$$
\dot{Q}_{st}(x') = \dot{Q}_{st}(x') + q_{st}(x'), \quad \dot{P}_{st}(x') = \dot{P}_{st}(x') + p_{st}(x'),
$$

where

$$
Q_{st}(x') = \frac{1}{\sqrt{2}} \left( w_{st}(x') + i \frac{\delta}{\delta w_{st}(x')} \right), \quad P_{st}(x') = \frac{1}{\sqrt{2}} \left( w_{st}^{n}(x') - i \frac{\delta}{\delta w_{st}^{n}(x')} \right),
$$

and

$$
q_{st}(x') = i \sqrt{2} \frac{\delta r_{a}}{\delta w_{st}^{n}(x')} \frac{\partial}{\partial r_{a}}, \quad p_{st}(x') = -i \sqrt{2} \frac{\delta r_{a}}{\delta w_{st}(x')} \frac{\partial}{\partial r_{a}}.
$$

Necessary reduction of the states number can be made by the following way: let’s suppose that field condition is defined by functionals of $w_{st}(x')$ and $w_{st}^{n}(x')$, in which $Q(x')$ and $P(x')$ become following ones:

$$
Q(x') \rightarrow \frac{1}{\sqrt{2}} w_{st}(x'), \quad P(x') \rightarrow i \frac{1}{\sqrt{2}} \frac{\delta}{\delta w_{st}(x')}.
$$

The variables $r_{a}$ have not physical sense. They have appeared as a rest of the state space reduction in the terms of Bogoliubov group variables. We will show that separation of these variables is connected with integrals of motion structure in the zero-point order, so it is dynamic by nature.
7. Creation-Annihilation Operators

After representation of quantum addend in the form (1) the action at the second order with respect to \( g \) reads:

\[
H_2 = \int \alpha \frac{\partial}{\partial \tau} \alpha + H_{21} + H_{22},
\]

The operators \( H_2 \) act at the space like \( F[w, \psi] \), so \( H_{21} \) act at the space \( F[r] \) but operator \( H_{22} \) act at the space \( F[w, \psi] \). Those spaces are orthogonal. Representation of \( w_{st}(x) \) as a following series is correct at least at the order \( O(t^2) \):

\[
w_{st}(x) = \sum_m \phi_{stm}(\vec{x})c_m(t) \quad \frac{\delta}{\delta w_{st}(x)} = \sum_m \phi_{stm}(\vec{x}) \frac{\delta}{\delta c_m(t)},
\]

\( \phi_{stm}(x) \) are functions \( \int \phi_{stm}(\vec{x})\phi_{sn}^{\dagger}(\vec{x}) = \delta_{mn} \). They are the solutions of the equations:

\[
\phi_{stm}(\vec{x}) = \frac{2a}{\sqrt{F}} \left( \phi_{stm}(\vec{x}) - \frac{1}{2} F_{st} \phi_{sm}(\vec{x}) \right), \quad \omega_m^2 \phi_{stm}(\vec{x}) = a\sqrt{F} \left( R_{st} \phi_{sm}(\vec{x}) - R_{st}(\phi_{sm}(\vec{x})) \right).
\]

Creation-annihilation operators can be represented as following functions:

\[
a_s = \sqrt{s} c_s + \frac{1}{2\sqrt{s}} \frac{\partial}{\partial c_s}, \quad a_s^\dagger = \sqrt{s} c_s - \frac{1}{2\sqrt{s}} \frac{\partial}{\partial c_s}.
\]

Operators \( H_{22} \) contain only exceed variables \( r_a, \frac{\partial}{\partial r_a} \) and can be removes by means of appropriate choice of state vector.

After removing of exceed variables action in the zero-point order looks like:

\[
H_{03} = -in^l A_l^\mu(a) \frac{\partial}{\partial q^\mu}.
\]

Field operator \( \psi(x) \) reads:

\[
\psi_{st}(x) = g F_{st}(x') + \hat{\psi}_{st}(x') + \phi_{st}(x') \frac{\partial}{\partial r_a} + \frac{1}{g} A_{st}(x', \tau),
\]

here \( \hat{\psi}_{st}(x') \) is the solution of the evolution equation (8) with a boundary condition on the:\n
\[
\hat{\psi}_{st}(x') = \hat{Q}_{st}(x'), \quad \hat{\psi}_{st}(x') = \hat{P}_{st}(x').
\]

We applied Bogoliubov transformation to the quantization of gravitational field in the neighborhood of nontrivial classical component, that permitted us to avoid zero-mode problem.

Einstein equations for the classical component has been obtained as a necessary condition for the perturbation theory to be applicable, not as a sequence of variational principle.

We obtained expression for quantum corrections of the field operator and explicit view of state vector, that permits us to calculate quantum corrections to the observable like effective mass, energy spectrum and so on.

8. Interacting Fields

The scheme proposed above has been applied to the different systems. The most complicated example is interacting gravitational and scalar field. We obtain that classical part of scalar field has to be solution of the equation:

\[
n_{nn} = a\sqrt{F} \left( -F^{\lambda\mu} \lambda_\mu + m^2 \right).
\]
Here $F^{\lambda\mu}$ is classical part of gravitational field that should satisfy to the equations:

\[
F_{stn} = 2a \sqrt{F} \left( F_{stn} - \frac{1}{2} F_n F_{st} \right),
\]

\[
F_{stn}^2 = \frac{a}{2\sqrt{F}} \left( F_{kn}^l F_{nl}^k - \frac{1}{2} F_n^2 \right) F_{st} - \frac{2a}{\sqrt{F}} \left( F_{stn}^l F_{stn} - \frac{1}{2} F_{st}^2 \right) - a \sqrt{F} \left( R_{st} - \frac{1}{2} F_{st} R \right) - \sqrt{F} \left( F_{st} c_{ij} - F_{st} C_{ij} \right) +
\]

\[
+ a \sqrt{F} \left( \frac{1}{2} \left( -\frac{1}{a^2} \delta^{ps} + F_{sp} s p - m^2 s \right) F_{sp} - s + \frac{F_{st}}{F_n} \right) ,
\]

\[
\frac{1}{\sqrt{F}} \left( F_{n} F_{stn} - \frac{1}{2} F_n^2 \right) - \sqrt{F} R F = \frac{1}{2\sqrt{F}} \frac{2}{n} - \frac{\sqrt{F}}{2} \left( F_{st} s \phi - m^2 \right) ,
\]

\[
F_{st} = \frac{\sqrt{F}}{2} F_{stn} f t
\]

and that we can treat as metric.

Quantum corrections $w_{st}$ for gravity have to satisfy to the equations

\[
w_{st}(\vec{x}) = 2 \sqrt{F} \left( w_{st}(\vec{x}) - \frac{1}{2} F_{st} w(\vec{x}) \right),
\]

\[
\omega_{m} w_{st}(\vec{x}) = 2 \sqrt{F} \left( R w_{st}(\vec{x}) - \frac{1}{2} A_{m}^{ps} \varphi_{m}^{ps}(\vec{x}) \right) w_{st}(\vec{x}) -
\]

\[
-2 \sqrt{F} \left( R_{st} (w(\vec{x})) - \frac{1}{2} R(w(\vec{x})) F_{st} \right) w_{st}(\vec{x}) - Y(w(\vec{x})).
\]

and quantum part $\mu$ for scalar field is the solution of equation:

\[
\omega^2 \mu = -a \sqrt{F} (-F_{st} \mu_{st} + m^2 \mu).
\]

As example we consider quantization on the background on flat metric

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

then

\[
F_{st} = \delta_{st}.
\]

Quantum addend for gravity $w_{st}$ looks like:

\[
w_{st} = \begin{pmatrix} w_{22} & w_{23} \\
w_{23} & -w_{22} \end{pmatrix}
\]

Suppose quantum addend depends on one space coordinate (say, $x$) only. In this case classical part of scalar field is the solution of

\[
\nabla^2 + m^2 = 0,
\]
and equation quantum addend is the following:

\[ \nabla \psi + m^2 \mu = 0. \]

Equations of motion for quantum part of gravitational field looks like:

\[
\frac{\partial^2 w_{st}}{\partial t^2} = \frac{\partial^2 w_{st}}{\partial x^2}, \quad \frac{\partial^2 w_{st}}{\partial x^2} = \left( \frac{1}{n^2} - m^2 \right) w_{st}.
\]

It means that quantum part of gravitational field is doublepolarized waves. All the other important results it is possible to find in [3].

9. Discussion
So we can state that totality of the received results is the following in general case:
- Equation of motion of a classical part;
- Equation of motion of the quantum additive;
- Energy spectrum;
- Expressions for integrals of a motion;
- Heisenberg equations;
- Explicit form of the field operator in the terms of new variables;

allows to assert, that our theory gives the complete description of system of boson fields, quantized on classical background. While theory guarantees exact performance of the conservation laws in any order of perturbation theory and also allows to avoid a zero-mode problem.

References
1. N.N.Bogoliubov *the Ukrainian Mathematical Journal*, v.2, p. 3-24 (1950)
2. O.A.Khrustalev, M.V.Chichikina *Teor.Mat.Fiz*. T. 111. N2, p. 242-251 (1997).
3. O.D.Timofeevskaya O.A.Khrustalev, M.V.Chichikina *Moscow Universit Bulletin*, N4, p.3, N5 p.3, N6 .11-14 (2003).