Classical Symmetries of Some
Two-Dimensional Models Coupled to Gravity

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Abstract

This paper is a sequel to one in which we examined the affine symmetry
algebras of arbitrary classical principal chiral models and symmetric space models
in two dimensions. It examines the extension of those results in the presence of
gravity. The main result is that even though the symmetry transformations of
the fields depend on the gravitational background, the symmetry algebras of
these classical theories are completely unchanged by the presence of arbitrary
gravitational backgrounds. On the other hand, we are unable to generalize the
Virasoro symmetries of the flat-space theories to theories with gravity.

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1 Introduction

String theories possess large discrete symmetry groups – called dualities. The first class of these to be understood were “T dualities” \cite{1} – symmetries associated with compactification, which are realized order-by-order in string perturbation theory. Subsequently, a class of symmetries called “S dualities,” \cite{2} a string generalization of the electric-magnetic duality of certain field theories \cite{3}, was conjectured. While they look quite similar to T duality from the viewpoint of low-energy effective field theory, they are much more speculative than T dualities. The reason is that they are non-perturbative symmetries of theories that are only known perturbatively! While this means that we are really not yet in a position to prove that they are true (though supporting “evidence” has been obtained \cite{4}), it also makes them extremely interesting as a window into the deeper non-perturbative workings of string theories. In certain cases, such as toroidal compactification of type II strings to four dimensions or the heterotic string to three dimensions, the S and T dualities are subgroups of a larger group of symmetries, known as U dualities \cite{5, 6}. This is another indication that S and T duality are not really so different. Also, it has been conjectured that certain string theories have dual formulations in which the roles of S and T duality are interchanged – so that S becomes perturbative and T becomes nonperturbative. This phenomenon is called “duality of dualities” \cite{7}.

A remarkable possibility suggested by recent works is that there is just one superstring theory, and that the type I, type II, and heterotic theories, compactified in various ways, are interrelated by a fascinating web of non-perturbative transformations \cite{4, 8}. It is even conceivable that all solutions could be part of a smoothly connected moduli space \cite{9}. This suggests that someday it may be possible to identify a unique quantum ground state of a unique theory! This has been my dream for more than 20 years, though developments during the period 1985-88 made it appear very unlikely. Today the outlook is much brighter.

The viewpoint that has motivated me recently is that string theory has a large (as yet unknown) group of duality symmetries, subgroups of which become visible for various compactifications. If this picture is correct, determining the group would constitute an important step towards understanding the theory. Since the visible groups become larger when more dimensions are compactified, that seems to be a fruitful direction to go. In classical effective field theories, the duality groups are non-compact Lie groups, realized nonlinearly by scalar fields on a symmetric space. Quantum effects break the symmetry to a discrete subgroup, which can often be described as the restriction to matrices with integer entries. A first step, however, is to identify the continuous classical symmetry groups. At the level of effective field theory, it appears that generically they are finite-dimensional Lie groups for $D \geq 3$, affine Lie algebras for $D = 2$, and hyperbolic Lie algebras for $D = 1$. Understanding the hyperbolic symmetries in $D = 1$ could be very interesting. As a more modest beginning, I have been exploring the affine algebras for $D = 2$. As discussed in my previous paper \cite{10}, this is a subject with a
long history, so much of the relevant work has already been done\cite{1,2,3,4}. (A more complete list of references to earlier work is given in \cite{10}.) Related discussions of string theory dualities in two dimensions have been given recently by a number of authors\cite{15,16,17,18}. Sen’s work is the most far-reaching, as he described the discrete duality subgroup of affine O(8,24) that occurs for the heterotic string toroidally compactified to two dimensions.

The first paper explored the affine Lie algebra symmetries of principal chiral models (PCM’s) and symmetric space models (SSM’s) in flat two-dimensional space-time. The purpose of this paper is to extend those results to include coupling to gravity, which is a necessary step for eventual application to string theory. Many of the results described here have been obtained previously in the cited references. However, our derivation of the symmetry algebra is made more transparent than previous ones (in my opinion) by the use of a convenient contour integral representation. Focusing on field transformations rather than Poisson brackets also simplifies the study of the algebra, though it prevents us from deriving the classical central extension of the algebra found by previous authors. The only specific point (already discussed in \cite{10}) on which there seems to be a disagreement with prior work concerns the precise description of the affine algebra that occurs in symmetric space models.

The paper is organized as follows. Section 2 summarizes the results obtained previously in flat space-time. Section 3 describes the generalization of the PCM results when gravity is included. The main conclusion is that even though the symmetry transformations depend on the gravitational background (which is described by a solution of the two-dimensional wave equation), the affine symmetry algebra remains unchanged from flat space. However, we are unable to generalize the Virasoro symmetry of the flat space theory to the theory with gravity. Section 4 describes SSM’s coupled to gravity. There, too, we find the same symmetry algebra as in flat space-time.

2 Summary of Previous Results

2.1 Principal Chiral Models

Principal chiral models (PCM’s) are based on fields $g(x)$ that map space-time into a group manifold $G$, which we may assume to be compact. Even though these models are not directly relevant to the string theory and supergravity applications that we have in mind, they serve as a good warm-up exercise, as well as being of some interest in their own right. Symmetric space models, which are relevant, share many of the same features, but are a little more complicated. They will be described in Section 2.3.

The classical theory of PCM’s, in any dimension, is defined by the lagrangian

$$\mathcal{L} = \eta^{\mu\nu} \text{tr}(A_\mu A_\nu),$$  \hspace{1cm} (2.1)
where the connection $A_\mu$ is defined in terms of the group variables by

$$A_\mu = g^{-1} \partial_\mu g = \sum A^i_\mu T_i. \quad (2.2)$$

Here $\eta^{\mu\nu}$ denotes the Minkowski metric for flat space-time, and the $T_i$ are the generators of the Lie algebra,

$$[T_i, T_j] = f^{ij}_k T_k. \quad (2.3)$$

They may be taken to be matrices in any convenient representation. The classical equation of motion is derived by letting $\delta g$ be an arbitrary infinitesimal variation of $g$ for which $\eta = g^{-1} \delta g$ belongs to the Lie algebra $G$. Under this variation

$$\delta A_\mu = D_\mu \eta = \partial_\mu \eta + [A_\mu, \eta], \quad (2.4)$$

and the classical equation of motion is

$$\partial_\mu A^\mu = 0, \quad (2.5)$$

as is well-known. Since $A_\mu$ is pure gauge, the Bianchi identity is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \quad (2.6)$$

The PCM in any dimension has manifest global $G \times G$ symmetry corresponding to left and right group multiplication. Remarkably, in two dimensions this is just a small subgroup of a much larger group of “hidden” symmetries. To describe how they arise, it is convenient to introduce light-cone coordinates

$$x^\pm = x^0 \pm x^1, \quad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1). \quad (2.7)$$

Expressed in terms of these coordinates, the equation of motion and Bianchi identity take the forms

$$\partial_\mu A^\mu = \partial_+ A_- + \partial_- A_+ = 0 \quad (2.8)$$

$$F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \quad (2.9)$$

A standard technique (sometimes called the “inverse scattering method”) for discovering the “hidden symmetries” of integrable models, such as a PCM in two dimensions, begins by considering a pair of linear differential equations, known as a Lax pair. In the present context the appropriate equations are

$$(\partial_+ + \alpha_+ A_+) X = 0 \quad \text{and} \quad (\partial_- + \alpha_- A_-) X = 0, \quad (2.10)$$

where $\alpha_\pm$ are constants. These equations are compatible, as a consequence of eqs. (2.8) and (2.9), provided that

$$\alpha_+ + \alpha_- = 2\alpha_+ \alpha_- \quad (2.11)$$
It is convenient to write the solutions to this equation in terms of a “spectral parameter” \( t \) in the form
\[
\alpha_+ = \frac{t}{t - 1}, \quad \alpha_- = \frac{t}{t + 1}.
\] (2.12)

The variable \( X \) in eq. (2.10) is a group-valued function of the space-time coordinate, as well as the spectral parameter. The integration constant can be fixed by requiring that \( X \) reduces to the identity element of the group at a “base point” \( x_0^\mu \). A formal solution to eqs. (2.10) is then given by a path-ordered exponential
\[
X(x, t) = P\exp\left\{-\int_{x_0}^{x} (\alpha_+ A_+ dy^+ + \alpha_- A_- dy^-)\right\},
\] (2.13)
where the path ordering has \( x \) on the left and \( x_0 \) on the right. The integral is independent of the contour provided the space-time is simply connected. This is the case, since we are assuming a flat Minkowski space-time. If one were to choose a circular spatial dimension instead, the multivaluedness of \( X \) would raise new issues, which we will not consider here. Note that \( X \) is group-valued for any real \( t \), except for the singular points \( t = \pm 1 \).

The next step is to consider the variation
\[
g^{-1} \delta g = \eta(\epsilon, t) = X(t)\epsilon X(t)^{-1},
\] (2.14)
where \( \epsilon = \sum \epsilon^i T_i \) and \( \epsilon^i \) are infinitesimal constants. The claim is that the variation \( \delta(\epsilon, t)g = g\eta \) preserves the equation of motion \( \partial \cdot A = 0 \) and, therefore, describes symmetries of the classical theory. To show this, one simply notes that the Lax pair implies that
\[
\delta A_\pm = D_\pm \eta = \partial_\pm \eta + [A_\pm, \eta] = \pm \frac{1}{t} \partial_\pm \eta,
\] (2.15)
and, therefore, \( \partial \cdot (\delta A) = 0 \) as required.

To compute the commutator of two infinitesimal symmetry transformations the key identity that we require is
\[
\delta_1 X_2 = \frac{t_2}{t_1 - t_2} (\eta_1 X_2 - X_2 \epsilon_1),
\] (2.16)
where \( \delta_i = \delta(\epsilon_i, t_i) \), \( \eta_i = \eta(\epsilon_i, t_i) \), and \( X_i = X(t_i) \). Identities such as this are used frequently in this work. The method of proof is to show that both sides of the equation satisfy the same pair of linear differential equations (obtained by varying the Lax pair) and the boundary condition that \( \delta X \) vanishes at \( x_0 \). Using eqs. (2.16) and (2.14), it is easy to compute \([\delta_1, \delta_2]g\) and derive
\[
[\delta(\epsilon_1, t_1), \delta(\epsilon_2, t_2)] = \frac{t_1 \delta(\epsilon_1, t_1) - t_2 \delta(\epsilon_2, t_2)}{t_1 - t_2},
\] (2.17)
where
\[
\epsilon_{12} = [\epsilon_1, \epsilon_2] = f_{ij}^k \epsilon_1^i \epsilon_2^j T_k.
\] (2.18)
In order to understand the relationship between the algebra (2.17) and the affine algebra associated with the group $G$, we need to extract some sort of mode expansion from the dependence on the parameter $t$. The standard approach in the literature is to do a power series expansion in $t$, $\delta(\epsilon, t) = \sum_{n=0}^{\infty} \delta_n(\epsilon)t^n$, identifying the $\delta_n(\epsilon)$ as distinct symmetry transformations. This gives half of an affine Lie algebra:

$$[\delta_m(\epsilon_1), \delta_n(\epsilon_2)] = \delta_{m+n}(\epsilon_{12}) \quad m, n \geq 0.$$  

(2.19)

Actually, $\delta(\epsilon, t)$ contains more information than is extracted in this way, and in Ref. [10] I found a nice way to reveal it. The idea is to define variations $\Delta_n(\epsilon) g$ for all integers $n$ by the contour integral

$$\Delta_n(\epsilon) g = \int_C \frac{dt}{2\pi i} t^{-n-1} \delta(\epsilon, t) g \quad n \in \mathbb{Z},$$  

(2.20)

where the contour $C = C_+ + C_-$ and $C_\pm$ are small clockwise circles about $t = \pm 1$. By distorting contours it is easy to show that $\Delta_n(\epsilon) = \delta_n(\epsilon)$ for $n > 0$, a result that arises entirely from the pole at $t = 0$. The negative integers $n$ are given entirely by the pole at $t = \infty$. (Explicitly, $\Delta_0(\epsilon) g = [g, \epsilon]$.) Because $g^{-1}\Delta_n g$ can be related to such series expansions, it is clear that it is Lie-algebra valued. [3]

Using the definition (2.20) and the commutator (2.17), it is an easy application of Cauchy’s theorem to deduce the affine Lie algebra (without center)

$$[\Delta_m(\epsilon_1), \Delta_n(\epsilon_2)] = \Delta_{m+n}(\epsilon_{12}) \quad m, n \in \mathbb{Z}.$$  

(2.21)

Equivalently, in terms of charges, we have

$$[J^i_m, J^j_n] = f^{ij}_{\ k} J^k_{m+n}.$$  

(2.22)

### 2.2 Virasoro symmetries

Having found affine Lie algebra symmetries for classical PCM’s, it is plausible that they should also have Virasoro symmetries. Modulo an interesting detail, this is indeed the case. Since the infinitesimal parameter in this case is not Lie-algebra valued, it can be omitted without ambiguity. With this understanding, the Virasoro transformation is

$$\delta^V(t) g = g((t^2 - 1)\dot{X}(t)X(t)^{-1} + I),$$  

(2.23)

where the dot denotes a $t$ derivative and

$$I = \dot{X}(0) = \int_{t_0}^{t} (A_+ dy^+ - A_- dy^-).$$  

(2.24)

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3If one tried to define further symmetries corresponding to the contours $C_\pm$ separately or by allowing $n$ to be non-integer, the transformations defined in this way would also appear to preserve $\partial \cdot A = 0$. However, these could fail to be honest symmetries because $g^{-1}\delta g$ might not be Lie-algebra valued.
This is also an invariance of the equation of motion $\partial \cdot A = 0$. We can extract modes $\delta^V_n$, for all integers $n$, by the same contour integral definition used above

$$\delta^V_n g = \int_C \frac{dt}{2\pi i} t^{-n-1} \delta^V(t) g.$$ \hfill (2.25)

Again, contour deformations give pole contributions at $t = 0$ and $t = \infty$ only, and therefore, one sees that $g^{-1}\delta^V_n g$ is Lie-algebra valued.

The analysis of the algebra proceeds in the same way as for the affine symmetry algebra, though the formulas are quite a bit more complicated. For example, commuting a Virasoro symmetry transformation with an affine algebra symmetry transformation gives

$$[\delta^V(t_1), \delta(\epsilon, t_2)] g = \left( \frac{1}{t_2} (\delta(\epsilon, 0) - \delta(\epsilon, t_2)) + \frac{t_2(t_1^2 - 1)}{(t_1 - t_2)^2} (\delta(\epsilon, t_1) - \delta(\epsilon, t_2)) \right) + \frac{t_1(1 - t_2^2)}{t_1 - t_2} \frac{\partial}{\partial t_2} \delta(\epsilon, t_2) g.$$ \hfill (2.26)

Using this equation and the contour integral definitions, one finds after an integration by parts and use of Cauchy’s theorem that

$$[\delta^V_m, \Delta_n(\epsilon)] g = n \int_C \frac{dt}{2\pi i} t^{-m-n-2} (t^2 - 1) \delta(\epsilon, t) g.$$ \hfill (2.27)

Now let us re-express the algebra in terms of charges $J^i_n$ (as before) and $K_m$ (corresponding to $\delta^V_m$). In this notation, eq. (2.27) becomes

$$[K_m, J^i_n] = n(J^i_{m+n-1} - J^i_{m+n+1}).$$ \hfill (2.28)

This formula is to be contrasted with what one would expect for the usual Virasoro generators $L_n$

$$[L_m, J^i_n] = -nJ^i_{m+n}.$$ \hfill (2.29)

Comparing equations, we see that we can make contact with the usual (centerless) Virasoro algebra if we identify

$$K_n = L_{n+1} - L_{n-1}.$$ \hfill (2.30)

However, it should be stressed that we have only defined the differences $K_n$ and not the individual $L_n$’s. Still, this identification is useful since it tells us that

$$[K_m, K_n] = (m - n)(K_{m+n+1} - K_{m+n-1}).$$ \hfill (2.31)

Let us see what happens if we try to construct the $L_n$’s. The easiest approach is to define $K(\sigma) = \sum_{-\infty}^{\infty} K_n e^{in\sigma}$ and $T(\sigma) = \sum_{-\infty}^{\infty} L_n e^{in\sigma}$. Then eq. (2.30) implies that

$$T(\sigma) = \frac{i}{2} \frac{K(\sigma)}{\sin \sigma}.$$ \hfill (2.32)

The remarkable fact is that $K(\sigma)$ does not vanish at $\sigma = 0$ and $\sigma = \pi$. Therefore, $T(\sigma)$ diverges at these points and the individual $L_n$’s do not exist. The integrals that would define them are logarithmically divergent.
2.3 Symmetric Space Models

An interesting class of integrable two-dimensional models consists of theories whose fields map the space-time into a symmetric space. Let $G$ be a simple group and $H$ a subgroup of $G$. Then the Lie algebra $\mathcal{G}$ can be decomposed into the Lie algebra $\mathcal{H}$ and its orthogonal complement $\mathcal{K}$, which contains the generators of the coset $G/H$. The coset space $G/H$ is called a symmetric space if $[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}$, in other words the commutators of elements of $\mathcal{K}$ belong to $\mathcal{H}$. The examples that arise in string theory and supergravity are non-compact symmetric space models (SSM’s). For such models, $G$ is a non-compact Lie group and $H$ is its maximal compact subgroup. The generators of $\mathcal{H}$ are antihermitian and those of $\mathcal{K}$ are hermitian. Therefore, since the commutator of two hermitian matrices is antihermitian, $[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}$ and $G/H$ is a (non-compact) symmetric space.

Symmetric space models can be formulated starting with arbitrary $G$-valued fields, $g(x)$, like those of PCM’s. To construct an SSM, we associate local $H$ symmetry with left multiplication and global $G$ symmetry with right multiplication. Thus, we require invariance under infinitesimal transformations of the form

$$\delta g = -h(x)g + ge \quad h \in \mathcal{H}, \quad e \in \mathcal{G}. \quad (2.33)$$

The local symmetry effectively removes $H$ degrees of freedom so that only those of the coset remain. The next step is to define

$$A_\mu = M^{-1} \partial_\mu M, \quad (2.34)$$

where

$$M = g^\dagger g. \quad (2.35)$$

Note that $g$ and $M$ are analogous to a vierbein and metric in general relativity. $M$, which is invariant under local $H$ transformations, parametrizes the symmetric space $G/H$ without extra degrees of freedom. In the case of a compact SSM the factor $g^\dagger$ in the definition of $M$ must be generalized to a quantity $\tilde{g}$, which is described in Ref. [10]. Since $A_\mu$ is pure gauge, its field strength vanishes

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \quad (2.36)$$

The lagrangian is $\mathcal{L} = \text{tr}(A^\mu A_\mu)$ and the classical equation of motion is

$$\partial^\mu A_\mu = 0. \quad (2.37)$$

These formulas look the same as for PCM’s, but $A_\mu$ is given in terms of $g(x)$ by a completely different formula (eqs. (2.34) and (2.35) instead of eq. (2.2)).

In two dimensions we once again have the Bianchi identity $F_{+-} = 0$ and the equation of motion $\partial_+ A_- + \partial_- A_+ = 0$. Therefore, it is natural to investigate whether
the formulas that gave rise to symmetries of PCM’s also gives rise to symmetries in this case. With this motivation, we once again form the Lax pair of equations
\[(\partial_{\pm} + \alpha_{\pm}A_{\pm})X = 0,\]  
and note that they are compatible if we write \(\alpha_{\pm}\) in terms of a spectral parameter \(t\) as in eq. (2.12). Then the solution is given by the contour-independent integral
\[X(t) = P\exp\left(-\int_{x_0}^{x}(\alpha_+ A_+ dy^+ + \alpha_- A_- dy^-)\right),\]  
as before. The obvious guess is that, just as for PCM’s, the hidden symmetry is described by
\[\delta g = gX(t)\epsilon X(t)^{-1}.\]  
This turns out to be correct. Under an arbitrary infinitesimal variation \(g^{-1}\delta g = \eta(x) \in \mathcal{G}\), we have
\[\delta M = \eta^\dagger M + M\eta,\]  
which implies that
\[\delta A_\mu = D_\mu\eta + D_\mu(M^{-1}\eta^\dagger M).\]  
The first term is the same as for a PCM, but the second one is new. The symmetry requires that \(\partial^\mu(\delta A_\mu) = 0\), when we substitute \(\eta = X\epsilon X^{-1}\). The vanishing of the contribution from the first term in eq. (2.42) is identical to the PCM case. The second term in eq. (2.42) also has a vanishing divergence (for \(\eta = X\epsilon X^{-1}\)).

Next, we wish to study the algebra of these symmetry transformations. The first step is to derive a suitable generalization of eq. (2.16), which is
\[\delta_1 X_2 = \frac{t_2}{t_1 - t_2}(\eta_1 X_2 - X_2 \epsilon_1) + \frac{t_1 t_2}{1 - t_1 t_2}(M^{-1}\eta_1^\dagger MX_2 - X_2 M_0^{-1} \epsilon_1^\dagger M_0),\]  
where \(M_0 = M(x_0)\). The first term is the one we found for PCM’s. The second term, which is new, is required to compensate for the extra piece of \(\delta A_\mu\) in eq. (2.42) that occurs for SSM’s. The commutator is then found to be
\[[\delta(\epsilon_1, t_1), \delta(\epsilon_2, t_2)]g = \frac{t_1 \delta(\epsilon_{12}, t_1) - t_2 \delta(\epsilon_{12}, t_2)}{t_1 - t_2}g + \delta'g + \delta''g,\]  
where the first term is the same as we found for PCM’s, but there are two additional pieces. The \(\delta'g\) term is a local \(\mathcal{H}\) transformation of the form \(h_{12}(x)g\), with
\[h_{12}(x) = \frac{t_1 t_2}{1 - t_1 t_2}\left[(g^\dagger)^{-1}\eta_1^\dagger M\eta_2 g^{-1} + g\eta_1 M^{-1}\eta_2^\dagger g^\dagger\right] - \text{h.c.},\]  
which is a symmetry of the theory. It is trivial in its action on \(M = g^\dagger g\), which is all that appears in \(\mathcal{L}\). The \(\delta''g\) term is given by
\[\delta''g = \frac{t_1 t_2}{1 - t_1 t_2}(\delta(\epsilon'_{12}, t_1) - \delta(\epsilon'_{12}, t_2))g,\]  
where \(\delta(\epsilon'_{12}, t) = \frac{1}{2}(\delta(\epsilon_{12}, t) - \delta(\epsilon_{21}, t))\).
where
\[ \epsilon_{12} = M^{-1}_0 \epsilon_1^\dagger M_0 \epsilon_2 - \epsilon_2 M^{-1}_0 \epsilon_1^\dagger M_0. \] (2.47)

As in the PCM, we define modes by contour integrals of the form given in eq. (2.20), and associate charges \( J^i_n \) to the transformation \( \Delta_n(\epsilon) \). These can be converted to “currents” \( J^i(\sigma) = \sum e^{in\sigma} J^i_n \). In the case of an SSM, there are two distinct classes of currents, those belonging to \( \mathcal{H} \) and those belonging to \( \mathcal{K} \). As Ref. [10] shows in detail, the significance of the \( \delta''g \) term in eq. (2.44) is that the \( \mathcal{H} \) currents satisfy Neumann boundary conditions at the ends of the interval \( 0 \leq \sigma \leq \pi \), while the \( \mathcal{K} \) currents satisfy Dirichlet boundary conditions at the two ends
\[ J^i(0) = J^i(\pi) = 0 \quad \text{for} \quad J^i \in \mathcal{H}. \] (2.48)
\[ J^i(0) = J^i(\pi) = 0 \quad \text{for} \quad J^i \in \mathcal{K}. \] (2.49)

As a result, \( J^i_n = J^{-i}_n \) for \( \mathcal{H} \) charges and \( J^i_n = -J^{-i}_n \) for \( \mathcal{K} \) charges. In terms of the modes, the affine symmetry algebra on the line segment \( 0 \leq \sigma \leq \pi \) then implies that
\[ [J^i_m, J^j_n] = f^{ij}_{\ k}(J^k_{m+n} + J^k_{m-n}) \quad \text{for} \quad J^i_n \in \mathcal{H} \] (2.50)
\[ [J^i_m, J^j_n] = f^{ij}_{\ k}(J^k_{m+n} - J^k_{m-n}) \quad \text{for} \quad J^i_n \in \mathcal{K}. \] (2.51)

I propose to call this kind of an affine Lie algebra \( \hat{G}_H \).

The Virasoro symmetries of PCM’s also generalize to SSM’s. The natural guess is that, just as for the affine algebra symmetry, the same formula will describe the symmetry in this case, namely
\[ \delta V(t)g = g \left( (t^2 - 1)\dot{X}(t)X(t)^{-1} + I \right). \] (2.52)

This turns out to be correct, but once again the algebra differs from that of PCM’s. We find that
\[ [\delta V(t_1), \delta(\epsilon, t_2)]g = \delta g + \delta' g + \delta'' g, \] (2.53)
where \( \delta g \) is the PCM result given in eq. (2.26). The \( \delta' g \) is a local \( \mathcal{H} \) transformation and \( \delta'' g \) contains new terms. (The formulas are given in Ref. [10].) The crucial question becomes what \( \delta'' g \) contributes to \([\delta V, \delta n(\epsilon)]g\), when we insert it into the appropriate contour integrals, or, equivalently, what it contributes to \([K_m, J^i_n]\). The result is
\[ [K_m, J^i_n] = n(J^i_{m+n-1} - J^i_{m+n+1} - J^i_{n-m+1} + J^i_{n-m-1}). \] (2.54)

The first two terms are the PCM result of eq. (2.28), while the last two terms are the new contribution arising from \( \delta'' g \).

After our experience with the affine Lie algebra symmetry, the interpretation of the result (2.54) is evident. The generators \( K_m \) satisfy the restrictions \( K_m = K_{-m} \), just like the \( \mathcal{H} \) currents. In other words, \( K(\sigma) \) satisfies Neumann boundary conditions at the ends of the interval \( 0 \leq \sigma \leq \pi \). Just as for PCM’s, one can define a stress tensor by eq. (2.32), which satisfies the standard stress tensor algebra. As before, it is singular at \( \sigma = 0 \) and \( \sigma = \pi \), so that modes \( L_m \) do not exist.
3 Principal Chiral Models Coupled to Gravity

3.1 Formulation of the theory

Coupling a principal chiral model to gravity is completely straightforward at the classical level for $D > 2$. One simply generalizes the lagrangian in (2.1) to

$$\mathcal{L} = \sqrt{-h} (R(h) + h^\mu{}^\nu \text{tr}(A_\mu A_\nu)).$$

(3.1)

We use $h_{\mu\nu}$ for the space-time metric, since the symbol $g$ has already been used for the group variable. As before, $A_\mu = g^{-1}\partial_\mu g$, and $R(h)$ is the scalar curvature, of course. Again, normalization factors are omitted, since our considerations are classical. Our main interest is in two dimensions, and it is important that we choose the right theory in that case — the one that is relevant to string theory. The above formula in $D = 2$ is not what we want. Instead, the theory that we shall consider is obtained by starting with eq. (3.1) in three dimensions and doing a dimensional reduction to two dimensions. This means simply dropping the dependence of the fields on one of the spatial coordinates, which is a consistent truncation.

To do the dimensional reduction from three dimensions to two dimensions, it is convenient to decompose the dreibein $\hat{e}_a^\mu$ in terms of a zweibein $e_\mu^a$, a vector $B_\mu$, and a scalar $\rho$ as follows

$$\hat{e}_a^\mu = \left( e_\mu^a \sqrt{\rho} B_\mu \right).$$

(3.2)

The zeros are obtained by gauge fixing the local Lorentz transformations between the dimension we are dropping and the two we are keeping. In terms of the metric, this formula corresponds to

$$\hat{h} = \hat{\eta} \hat{e}^T = \left( \begin{array}{cc} h_{\mu\nu} + \rho B_\mu B_\nu & \rho B_\mu \\ \rho B_\nu & \rho \end{array} \right).$$

(3.3)

Making these substitutions in the lagrangian, dropping all derivatives in the $x^2$ direction, and defining $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, leads to

$$\mathcal{L}^{(0)}_2 = \sqrt{-\hat{h}\rho} (R(h) - \frac{1}{2} \rho^2 G_{\mu\nu} G^{\mu\nu} + h^\mu{}^\nu \text{tr}(A_\mu A_\nu)),$$

(3.4)

a result that has been obtained by many authors. By considering the $B_\mu$ field equation it is easy to convince oneself that this field has no effect on the theory. The $G^2$ term can simply be dropped, which is what I will do. (In two dimensions it is possible to set $G_{\mu\nu}$ proportional to $\epsilon_{\mu\nu}$, which gives rise to a cosmological constant. We will not do that, however.)

Because of the work on string theory, a great deal is known about two-dimensional gravity theories. One lesson is that the story is particularly simple when there is
conformal symmetry. Indeed that is an important requirement in string theory. Since
the theory being considered here is regarded as a target-space theory, rather than a
world-sheet theory, there is no reason that we should require conformal symmetry.
Indeed we do not have it. Specifically, if we rescale the metric by \( h^{\mu\nu} \rightarrow e^\phi h^{\mu\nu} \), one
finds that (up to a total derivative)
\[
L_2^{(0)} \rightarrow L_2 = L_2^{(0)} + \sqrt{-h} h^{\mu\nu} \partial_\mu \rho \partial_\nu \phi.
\]
We can take \( \phi \) to be an additional independent field and define the theory to be given
by \( L_2 \). Then we do have local Weyl invariance under
\[
h_{\mu\nu} \rightarrow e^\lambda h_{\mu\nu}, \quad \phi \rightarrow \phi - \lambda.
\]
This has the practical consequence that the energy–momentum tensor \( T_{\mu\nu} \) is traceless.

Let us now form all the equations of motion. After forming them we can use the
diffeomorphism and Weyl symmetries to choose the gauge \( h_{\mu\nu} = \eta_{\mu\nu} \). The equations
of motion in this gauge become
\[
\partial_+(\rho A_-) + \partial_- (\rho A_+) = 0
\]
\[
\partial_+ \partial_- \rho = 0
\]
\[
\partial_+ \partial_- \phi = \text{tr}(A_+ A_-)
\]
\[
T_{++} = \rho \text{tr}(A_+ A_+) + \partial_+ \phi \partial_+ \rho + c \partial_-^2 \rho = 0
\]
\[
T_{--} = \rho \text{tr}(A_- A_-) + \partial_- \phi \partial_- \rho + c \partial_+^2 \rho = 0,
\]
where \( c \) is a constant. Equation (3.7) is sometimes called the Ernst equation. We begin
by solving the 2D wave equation for \( \rho \):
\[
\rho(x) = \rho^+(x^+) + \rho^-(x^-).
\]
From now on \( \rho^+(x^+) \) and \( \rho^-(x^-) \) will be regarded as arbitrary given functions that
describe a “fixed gravitational background.” One could redefine coordinates by setting
\( \rho^+(x^+) = x^+ \) and \( \rho^-(x^-) = x^- \), for example, without loss of generality. However,
there is no need to do this, and I prefer to keep \( \rho^+ \) and \( \rho^- \) arbitrary. The constant \( c \),
which appears in \( T_{++} \) and \( T_{--} \), is of no consequence, because it can be absorbed in a
redefinition
\[
\phi \rightarrow \phi + c \log(\partial_+ \rho \partial_- \rho).
\]
Therefore, we will set it to zero.

The last three of the equations of motion, the ones that involve \( \phi \), can be solved
explicitly by giving a formula for \( \phi \) provided one uses the other two equations. To see
this, let us first solve \( T_{++} = 0 \) for \( \partial_+ \phi \):
\[
\partial_+ \phi = -\frac{\rho}{\partial_+ \rho} \text{tr}(A_+ A_+).
\]
Before attempting to integrate this, let us differentiate it with respect $x^-$, using the identity

$$\partial_- A_+ = \frac{1}{2} [A_+, A_-] - \frac{1}{2 \rho} (\partial_+ \rho A_- + \partial_- \rho A_+), \quad (3.15)$$

which is obtained by combining $F_{+-} = 0$ and $\partial_\mu (\rho A^\mu) = 0$. This immediately leads to

$$\partial_+ \partial_- \phi = \text{tr}(A_+ A_-), \quad (3.16)$$

which is therefore not an independent equation. The same result is obtained starting from $T_{--} = 0$. Therefore, the $T_{++}$ and $T_{--}$ equations are compatible and can be integrated to give

$$\phi(x) = \phi(x_0) - \int_{x_0}^x \rho \left[ \frac{\text{tr}(A_+ A_+)}{\partial_+ \rho} dy^+ + \frac{\text{tr}(A_- A_-)}{\partial_- \rho} dy^- \right], \quad (3.17)$$

which is contour independent, when the equation of motion is used. Thus, aside from its value at one point, $\phi$ is given in terms of $g$ and $\rho$.

### 3.2 The symmetry transformations

Our goal is to generalize the hidden symmetries that we found for PCM’s in 2D flat space to PCM’s coupled to 2D gravity. To do this, we follow the same steps as before. First we seek a Lax pair

$$(\partial_\pm + \alpha_\pm A_\pm)X = 0, \quad (3.18)$$

whose consistency follows from the equation of motion $\partial_\mu (\rho A^\mu) = 0$ and the Bianchi identity $F_{+-} = 0$. As before, a necessary condition is that $\frac{1}{\alpha_+} + \frac{1}{\alpha_-} = 2$, so once again we write

$$\alpha_+ = \frac{\tau}{\tau - 1}, \quad \alpha_- = \frac{\tau}{\tau + 1}. \quad (3.19)$$

We now use the symbol $\tau$ in place of $t$, which we used earlier, because $\tau$ will turn out to be $x^\mu$ dependent, and we want to reserve the symbol $t$ for a constant that will be defined later.

Requiring $[\partial_+ + \alpha_+ A_+, \partial_- + \alpha_- A_-] = 0$, and using $\partial_\mu (\rho A^\mu) = 0$ and $F_{+-} = 0$, gives the differential equations

$$2 \rho \partial_+ \alpha_- = (\alpha_- - \alpha_+) \partial_+ \rho \quad \text{and} \quad 2 \rho \partial_- \alpha_+ = (\alpha_+ - \alpha_-) \partial_- \rho \quad (3.20)$$

Suppose we now let

$$\tau = \frac{1 - R}{1 + R}, \quad (3.21)$$

so that

$$\alpha_+ = \frac{1}{2} \left(1 - \frac{1}{R}\right), \quad \alpha_- = \frac{1}{2} (1 - R). \quad (3.22)$$
Substituting these expressions then gives the equations

\[ R^{-1} \partial_+ R = \left(1 - R^{-2}\right) \frac{\partial_+ \rho}{2\rho} \quad \text{and} \quad R^{-1} \partial_- R = \left(R^2 - 1\right) \frac{\partial_- \rho}{2\rho}. \]  

(3.23)

The general solution of this pair of equations is

\[ R = \left(1 - \kappa \rho^+\right) \left(1 + \kappa \rho^-\right)^{1/2}, \]  

(3.24)

where \( \kappa \) is an integration constant. To define the branch of the square root unambiguously, we restrict attention to the neighborhood of \( \kappa = 0 \), avoiding branch points, with the understanding that \( R \to +1 \) as \( \kappa \to 0 \). The function

\[ \tau(\kappa, x) = \frac{\sqrt{1 + \kappa \rho^-} - \sqrt{1 - \kappa \rho^+}}{\sqrt{1 + \kappa \rho^-} + \sqrt{1 - \kappa \rho^+}}, \]  

(3.25)

also has a series expansion in \( \kappa \), whose first term is \( \tau(\kappa, x) \sim \frac{1}{4} \kappa \rho \). Notice that changing the choice of sheet corresponds to the transformation \( \tau \to 1/\tau \).

Now we can generalize the symmetry transformations of \( g(x) \) that were obtained in the flat-space case:

\[ \delta(\epsilon, \kappa) g(x) = F(\kappa, x) g(x) \eta(\epsilon, \kappa, x), \]  

(3.26)

where \( \eta \) is defined exactly as before

\[ \eta(\epsilon, \kappa, x) = X(\kappa, x) \epsilon X(\kappa, x)^{-1} \]  

(3.27)

\[ X(\kappa, x) = P \exp\left\{-\int_{x_0}^x \left(\alpha_+ A_+ dy^+ \right) + \alpha_- A_- dy^-\right\}. \]  

(3.28)

There are two significant differences from the flat-space case. First, \( \alpha^\pm \), as defined by eqs. (3.19) and (3.23), are functions of \( \gamma^\mu \) and \( \kappa \). Second, we allow for an extra factor \( F(\kappa, x) \) in the variation. This factor is required to be a scalar function, unlike \( g \) and \( \eta \) which are matrices. Thus, so long as \( F \) is real, \( g^{-1} \delta g \) is Lie-algebra valued.

The question now is whether there is a choice of \( F(\kappa, x) \) such that the equation of motion \( \partial_\mu (\rho A^\mu) = 0 \) is preserved under the variation \( \delta g \) given above. To examine this, we first note that

\[ \delta(\epsilon, \kappa) A_\pm = D_\pm (F \eta) = \frac{F}{\tau} \partial_\pm \eta + \eta \partial_\pm F. \]  

(3.29)

Requiring that \( \partial_\mu (\rho \delta A^\mu) = 0 \) then gives the conditions

\[ \rho \partial_\pm F = \pm \partial_\pm (F \rho / \tau). \]  

(3.30)

These equations imply that, up to a multiplicative factor, \( F \) is given by

\[ f(\kappa, x) = \left[(1 - \kappa \rho^+)(1 + \kappa \rho^-)\right]^{-1/2} \]  

(3.31)
which is again unambiguous in the neighborhood of \( \kappa = 0 \). The choice of normalization that will turn out to be most convenient is

\[
F(\kappa, x) = \frac{f(\kappa, x)}{f(\kappa, x_0)}.
\]

(3.32)

Specifically, it follows that

\[
\delta A_\pm = D_\pm(F \eta) = \pm \frac{1}{\rho} \partial_\pm(F \rho \eta/\tau),
\]

(3.33)

so that the conserved charges are given by

\[
Q(\epsilon, t) = \int_{-\infty}^{\infty} \rho D_0(F \eta) dx = \left( \frac{\rho F}{\tau} X_\epsilon X^{-1} \right) \bigg|_{-\infty}^{\infty}.
\]

(3.34)

### 3.3 The symmetry algebra

To derive the commutator of two transformations \([\delta_1, \delta_2] = [\delta(\epsilon_1, \kappa_1), \delta(\epsilon_2, \kappa_2)]\), we need to generalize the formula for \(\delta_1 X_2\) in the flat-space theory. Guided by what we found in eq. (2.16) in that case, let us try

\[
\delta_1 X_2 = \zeta_{12} \eta_1 X_2 + \lambda_{12} X_2 \epsilon_1,
\]

(3.35)

where \(\zeta_{12}\) and \(\lambda_{12}\) are unknown functions to be determined. The important point is that, like \(F\), they are not matrices. The required conditions are obtained by varying the equations \((\partial_\pm + \alpha_\pm A_\pm)X = 0\). The resulting equations for \(\zeta_{12}\) and \(\lambda_{12}\) tell us that \(\lambda_{12}\) is a constant and

\[
\zeta_{12} = \frac{\tau_2 F_1}{\tau_1 - \tau_2}.
\]

(3.36)

They also require that

\[
\partial_\pm \zeta_{12} + \frac{\tau_2}{\tau_2 \mp 1} \partial_\pm F_1 = 0,
\]

(3.37)

which is true for this choice of \(\zeta_{12}\). The constant \(\lambda_{12}\) is determined by noting that since \(X_2 \to 1\) as \(x \to x_0\), it is necessary that \(\delta_1 X_2(x_0) = 0\). This then implies that

\[
\lambda_{12} = -\zeta_{12}(x_0) = \frac{t_2}{t_2 - t_1},
\]

(3.38)

where we have defined

\[
t_i = \tau(\kappa_i, x_0).
\]

(3.39)

Henceforth we use this formula to replace \(\kappa_i\) by \(t_i\). Note that for small \(\kappa_i, t_i \sim \frac{1}{4} \kappa_i \rho(x_0)\).

The algebra is now easy to derive. The formula for \(\delta_1 X_2\) implies that

\[
\delta_1 \eta_2 = [\delta_1 X_2 \cdot X_2^{-1}, \eta_2] = \zeta_{12}[\eta_1, \eta_2] + \lambda_{12} X_2 \epsilon_{12} X_2^{-1},
\]

(3.40)
where $\epsilon_{12} = [\epsilon_1, \epsilon_2]$, as before. Now the commutator of two symmetry transformations can be evaluated using

$$g^{-1}[\delta_1, \delta_2]g = F_1 F_2[\eta_1, \eta_2] + F_2 \delta_1 \eta_2 - F_1 \delta_2 \eta_1$$

(3.41)

and the identity

$$F_1 F_2 + F_2 \zeta_{12} + F_1 \zeta_{21} = 0.$$  

(3.42)

Then one finds precisely the same formula as in the flat-space case, namely

$$[\delta(\epsilon_1, t_1), \delta(\epsilon_2, t_2)]g = \frac{t_1 \delta(\epsilon_{12}, t_1) - t_2 \delta(\epsilon_{12}, t_2)}{t_1 - t_2}g.$$  

(3.43)

We may expand

$$\delta(\epsilon, t) = \sum_{n=0}^{\infty} \delta_n(\epsilon) t^n$$

(3.44)

and obtain half of an affine algebra

$$[\delta_m(\epsilon_1), \delta_n(\epsilon_2)]g = \delta_{m+n}(\epsilon_{12})g.$$  

(3.45)

However, by using a contour integral representation for the modes, instead, a complete affine algebra can be obtained. We shall do that shortly, but let us first consider the field $\phi(x)$. Equation (3.17) implies that $\phi(x) - \phi(x_0)$ satisfies the same algebra as $g(x)$:

$$[\delta(\epsilon_1, t_1), \delta(\epsilon_2, t_2)](\phi(x) - \phi(x_0)) = \frac{t_1 \delta(\epsilon_{12}, t_1) - t_2 \delta(\epsilon_{12}, t_2)}{t_1 - t_2}((\phi(x) - \phi(x_0))).$$  

(3.46)

This still leaves the possibility that the commutator also gives rise to a constant translation of $\phi(x)$, which would correspond to a global conformal rescaling of the two-dimensional metric. Such a central term was found to occur in refs. [12, 13, 14]. The fact that it is undetermined here may indicate a limitation of our approach, which is based on classical field transformations rather than Poisson brackets.

The contour integral construction of $\Delta_n$ presented in Section 2.1 can be generalized to the case of a 2D PCM coupled to gravity. The key step is to understand the analytic structure of $X(t)$. Whereas it had isolated singularities at $t = \pm 1$ in the flat space theory, it has branch cuts in the theory with gravity. To see this, it is useful to express the function $\tau(x)$ in terms of the parameter $t$ by eliminating the parameter $\kappa$. A little algebra gives

$$\kappa = \frac{4\tau}{\rho(1 + \tau^2) + 2\tilde{\rho}\tau} = \frac{4t}{\rho_0(1 + t^2) + 2\tilde{\rho}_0 t},$$

(3.47)

where we have defined $\rho_0 = \rho(x_0)$ and

$$\tilde{\rho}(x) = \rho^+(x^+) - \rho^-(x^-).$$  

(3.48)
From this it follows that
\[ R = \frac{1 - \tau}{1 + \tau} = \left( \frac{\rho_0 (1 + t^2) + 2\tilde{\rho}_0 t - 4t\rho^+}{\rho_0 (1 + t^2) + 2\rho_0 t + 4t\rho^-} \right)^{1/2}. \]
(3.49)

Coordinate dependence enters this formula through the functions \( \rho^\pm(x^\pm) \).

The quantities \( \alpha_+ = \frac{1}{2} (1 - R^{-1}) \) and \( \alpha_- = \frac{1}{2} (1 - R) \), which appear inside the integral defining \( X(t) \), are singular whenever the numerator or the denominator of \( R \) vanishes. This defines branch cuts connecting branch points at the locations where \( \alpha_\pm(x_0) \) and \( \alpha_\pm(x) \) are singular. The roots of the expression in the numerator of \( R \) are given by
\[ n_\pm(x^+) = \frac{2\rho^+ - \tilde{\rho}_0 \pm 2[(\rho^+ + \rho_0)(\rho^+ - \rho_0^+)]^{1/2}}{\rho_0} \]
(3.50)
and those of the denominator are
\[ d_\pm(x^-) = -\frac{2\rho^- - \tilde{\rho}_0 \pm 2[(\rho^- + \rho_0^+)(\rho^- - \rho_0^-)]^{1/2}}{\rho_0}. \]
(3.51)
At the base point \( x_0 \) one finds that \( n_\pm(x_0) = 1 \) and \( d_\pm(x_0) = -1 \). Therefore, there are branch cuts connecting \( t = 1 \) to \( t = n_\pm(x) \) and branch cuts connecting \( t = -1 \) to \( t = d_\pm(x) \).

Now we can again define
\[ \Delta_n(\epsilon)g = \int_{C} \frac{dt}{2\pi i} t^{-n-1}\delta(\epsilon, t)g, \]
(3.52)
with \( C = C_+ + C_- \). The new feature is that now \( C_+ \) must enclose the branch cuts that connect \( t = 1 \) to \( t = n_\pm(x) \) and \( C_- \) must enclose the branch cuts that connect \( t = -1 \) to \( t = d_\pm(x) \). Once this is done, \( \Delta_n \) becomes a well-defined finite expression. The only thing that needs to be checked is that the contours never get pinched against the points \( t = 0 \) or \( t = \infty \). In other words we must examine when \( n_\pm(x) \) and \( d_\pm(x) \) can vanish or diverge. A little algebra shows that the only way this can happen is if \( \rho^+(x^+) \) or \( \rho^-(-) \) becomes infinite. However, this is not allowed, at least in the finite plane, and therefore should not be a problem. It can now be demonstrated that [\( \Delta_m(\epsilon_1), \Delta_n(\epsilon_2) \] = \( \Delta_{m+n}(\epsilon_{12}) \) follows from eq. (3.43) exactly as in the flat space theory.

### 3.4 Virasoro symmetries

Having found a generalization of the flat space affine algebra symmetries, it is plausible that it should be possible to do the same for the Virasoro symmetries. In fact, despite considerable effort, I have been unable to find the desired formulas. It is unclear to me whether this failure reflects a fundamental obstruction or a lack of ingenuity. The discussion that follows shows what has been achieved and the difficulties that were encountered.
Equation (2.23), the flat space Virasoro symmetry formula, contains two terms. Let us begin by presenting a curved space generalization of the first term:

\[ g^{-1} \delta^{(a)} g = (t^2 - 1) F \dot{X}(t) X(t)^{-1}, \quad (3.53) \]

where \( t \) is the constant spectral parameter defined in eq. (3.39), \( F \) is the function defined in eqs. (3.31) and (3.32), and the dot represents a derivative with respect to \( t \). This term gives

\[ \delta^{(a)} A_{\pm} = (t^2 - 1) D_{\pm} (F \dot{X} X^{-1}) \]

\[ = (t^2 - 1) \left\{ (\partial_{\pm} F) \dot{X} X^{-1} \pm \frac{1}{\tau} F \partial_{\pm} (\dot{X} X^{-1}) - \frac{\dot{\alpha}_{\pm}}{\alpha_{\pm}} F A_{\pm} \right\} \]

\[ = (t^2 - 1) \left\{ \pm \frac{1}{\rho} \partial_{\pm} \left( \frac{\rho F}{\tau} \dot{X} X^{-1} \right) - \frac{\dot{\alpha}_{\pm}}{\alpha_{\pm}} F A_{\pm} \right\}, \quad (3.54) \]

where the last step uses eq. (3.30). The first term in the last expression does not contribute to \( \partial^{\mu}(\rho \delta A_{\mu}) \), so we can ignore it, and focus on the second term.

The expression

\[ \delta'(A_{\pm}) = (1 - t^2) \frac{\dot{\alpha}_{\pm}}{\alpha_{\pm}} F A_{\pm}, \quad (3.55) \]

can be re-expressed using the identities

\[ \frac{\dot{\alpha}_{\pm}}{\alpha_{\pm}} = \frac{\dot{\tau}}{\tau (1 \mp \tau)}, \quad (3.56) \]

and

\[ \dot{\tau} = \frac{\tau}{t} F, \quad (3.57) \]

in the form

\[ \delta'A_{\pm} = \frac{1 - t^2}{t} \frac{1}{1 \mp \tau} F^2 A_{\pm}. \quad (3.58) \]

In flat space \( \tau = t \) and \( F = 1 \), so this reduces to \( (t^{-1} \pm 1) A_{\pm} \), which has divergence \( \partial_- A_+ - \partial_+ A_- = [A_+, A_-] \). This is cancelled by the \( I \) term in eq. (2.23), since \( D_{\pm} I = \pm A_{\pm} + [A_{\pm}, I] \) has divergence \( -[A_+, A_-] \). Unfortunately, this does not seem to generalize to curved space.

The equations \( \partial^{\mu}(\rho A_{\mu}) = 0 \) and \( F_{+\mp} = 0 \) can be solved to give

\[ \partial_+ A_- = -\frac{1}{2} [A_+, A_-] - \frac{1}{2} \left( \frac{\partial_+ \rho}{\rho} A_- + \frac{\partial_- \rho}{\rho} A_+ \right) \]

\[ \partial_- A_+ = \frac{1}{2} [A_+, A_-] - \frac{1}{2} \left( \frac{\partial_- \rho}{\rho} A_- + \frac{\partial_+ \rho}{\rho} A_+ \right). \quad (3.59) \]

Using these relations, the divergence of \( \delta'A_{\mu} \) contains three types of terms proportional to \( [A_+, A_-], A_+ \partial_- \rho, \) and \( A_- \partial_+ \rho \). Let us examine the \( [A_+, A_-] \) term. We find

\[ \partial_-(\rho \delta'A_+) + \partial_+(\rho \delta'A_-) = \frac{1 - t^2}{t} \frac{\tau}{1 - \tau^2} F^2 [A_+, A_-] + \ldots, \quad (3.60) \]
where the dots represent the $A_+ \partial_- \rho$ and $A_- \partial_+ \rho$ terms. The coefficient of $[A_+, A_-]$ reduces to one in flat space, of course, but it is a complicated function of $t$ and $x$ in curved space. If one attempts to add a variation $\delta^{(b)} g$ where such a factor multiplies $I$, generalizing the second term in eq. (2.23), we can cancel the $[A_+, A_-]$ part of the divergence. However, the $x$ dependence of this coefficient results in various additional pieces in $\partial^\mu (\rho \delta^{(b)} A_\mu)$. Thus, cancelling one term generates new ones. I have not succeeded in finding a procedure to eliminate all of them.

4 Symmetric Space Models Coupled to Gravity

The coupling of 2D SSM’s to 2D gravity is given by formulas identical to those for PCM’s given in the preceding section. In particular, the equations of motion are given by eqs. (3.7) – (3.11). The only difference is that, just as for flat space, the connection $A_\mu$ is given by $A_\mu = M^{-1} \partial_\mu M$ and $M = g^\dagger g$ rather than $A_\mu = g^{-1} \partial_\mu g$.

4.1 The symmetry transformations

In view of our previous experience it is natural to conjecture that, just as in flat space, the infinitesimal symmetry transformations that preserve the equations $\partial^\mu (\rho A_\mu) = 0$ and $F_{\pm} = 0$ for an SSM are given by the same formulas as for PCM’s. These we found to be $g^{-1} \delta g = F \eta$, where $F$ is given by eqs. (3.31) and (3.32) and $\eta$ is given by eqs. (3.27) and (3.28). These result in the variation

$$\delta A_\mu = D_\mu (F \eta) + D_\mu (F M^{-1} \eta^\dagger M).$$

The first term is identical to the PCM case and has already been shown to give a vanishing contribution to $\partial^\mu (\rho \delta A_\mu)$. Thus, only the contribution of the second term needs to be checked. Of course, we know from Section 2.3 that it gives a vanishing contribution in the flat space ($\rho = \text{constant}$) limit. The generalization to curved space works because of the identity

$$D_{\pm} (M^{-1} \eta^\dagger MF) = \pm \frac{1}{\rho} \partial_{\pm} (\tau \rho FM^{-1} \eta^\dagger M).$$

The derivation of this formula depends on the relation

$$\rho \partial_{\pm} F = \pm \partial_{\pm} (\tau \rho F).$$

This formula differs from eq. (3.30) in that $\tau$ appears now in the numerator rather than the denominator. Remarkably, both formulas are true because $\tau \rho F$ and $\rho F / \tau$ differ by a constant. Specifically,

$$\rho f \tau - \frac{\rho f}{\tau} = -\frac{4}{\kappa}.$$

Thus, $g^{-1} \delta g = F \eta$ is indeed a classical symmetry of SSM’s coupled to gravity.
4.2 The symmetry algebra

The crucial formula required for commuting two symmetry transformations is $\delta_1 X_2$. As usual, it is found by solving

$$(\partial_\pm + \alpha_{2\pm} A_\pm)\delta_1 X_2 + \alpha_{2\pm}(\delta_1 A_\pm) X_2 = 0, \quad (4.5)$$

subject to the boundary condition that the variation vanishes at $x = x_0$. The result has the structure

$$(\delta_1 X_2)_\text{SSM} = (\delta_1 X_2)_\text{PCM} + (\delta_1 X_2)_\text{extra}, \quad (4.6)$$

where $(\delta_1 X_2)_\text{PCM}$ refers to eq. (3.35), the result found in Section 3.3. This term is attributable to the term $D_\mu(F\eta)$ in eq. (4.1). Thus, the rest of the answer requires solving the differential equations with the source $\delta_1 A_\pm = D_\pm(F_1 M^{-1}\eta_1^\dagger M)$. The answer that results is

$$(\delta_1 X_2)_\text{extra} = \tau_1 \tau_2 F_1 M^{-1}\eta_1^\dagger M X_2 + \frac{t_1 t_2}{t_1 t_2 - 1} X_2 M_0^{-1} \epsilon_1^\dagger M_0. \quad (4.7)$$

The first term solves the inhomogeneous differential equation, and the second one is a solution of the homogeneous equation chosen to ensure that the variation vanishes at the base point $x_0$. In verifying this result, one finds the formula by similar algebra to the flat space case. However, an additional identity

$$\partial_\pm \left( \frac{\tau_1 \tau_2}{1 - \tau_1 \tau_2} F_1 \right) + \alpha_{2\pm} \partial_\pm F_1 = 0, \quad (4.8)$$

which is trivial in flat space, also needs to be satisfied. Even though it was expected to work, it still seems remarkable that it is true.

Now we can compute the commutator of two symmetry transformations. As in the flat space case, we find that

$$[\delta_1, \delta_2] g = \frac{t_1 \delta(\epsilon_{12}, t_1) - t_2 \delta(\epsilon_{12}, t_2)}{t_1 - t_2} g + \delta' g + \delta'' g. \quad (4.9)$$

The term shown explicitly is the PCM result, unchanged from flat space. The $\delta'' g$ term is also unchanged from flat space

$$\delta'' g = \frac{t_1 t_2}{1 - t_1 t_2} (\delta(\epsilon_{21}, t_1) - \delta(\epsilon_{12}, t_2)) g, \quad (4.10)$$

The $\delta' g$ term is again a local $H$ transformation ($\delta' g = h_{12} g$). However, the formula for $h_{12}$ is modified from the flat space one given in eq. (2.45)

$$h_{12} = \frac{\tau_1 \tau_2}{1 - \tau_1 \tau_2} F_1 F_2 [(g^\dagger)^{-1}\eta_1^\dagger M \eta_2 g^{-1} + g\eta_1 M^{-1}\eta_2^\dagger g^\dagger] - \text{h.c.}. \quad (4.11)$$

As usual, it can be ignored.

Now modes $\Delta_n(\epsilon) g$ can be defined by the usual contour integral. The integral has the same singularity structure as in the PCM case, and therefore, the contours are taken to enclose the branch cuts as in that case. Just as for PCM’s, the algebra is unchanged from the flat space case (aside from irrelevant local $H$ terms). Thus, it remains the same as described in Section 2.3.
5 Conclusion

In this paper we have shown that the affine Lie algebra symmetries of classical PCM’s and SSM’s in flat two-dimensional space-time remain symmetries in arbitrary gravitational backgrounds. On the other hand, we were unable to demonstrate that the Virasoro symmetries of the flat-space theories survive in gravitational backgrounds. Perhaps it will become clearer how this issue is resolved when the analysis is extended to the quantum theory, and a Sugawara construction of the internal stress tensor that generates the Virasoro symmetries can be attempted. However, since the quantum theory is only expected to contain a discrete subgroup of duality symmetries, which would not be accessible in terms of generators, it may be necessary to understand finite group transformations first. Another interesting direction to explore is the extension of our analysis to the case of a circular spatial dimension. This is an essential preliminary to the study of the hyperbolic symmetry algebras that are expected to appear after compactification of all spatial dimensions.

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