On the oscillation of second-order half-linear functional differential equations with mixed neutral term

Ercan Tunç and Orhan Özdemir
Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey

ABSTRACT
In this article, the authors establish new sufficient conditions for the oscillation of solutions to a class of second-order half-linear functional differential equations with mixed neutral term. The results obtained improve and complement some known results in the relevant literature. Examples illustrating the results are included.

1. Introduction
This paper deals with the oscillatory behaviour of solutions to a class of second order half-linear functional differential equations with mixed neutral term of the form

\[ (r(t)(z'(t))^α)' + q(t)x^α(h(t)) = 0, \quad t ≥ t_0 > 0, \]

where \( z(t) = x(t) + p_1(t)x(g_1(t)) + p_2(t)x(g_2(t)), \) \( α \) is a quotient of odd positive integers, and the following conditions are always assumed to hold:

(C1) \( r, q : [t_0, ∞) → (0, ∞) \) are real valued continuous functions with

\[ \int_{t_0}^{∞} r^{-1/α}(s) \, ds = ∞; \]

(C2) \( g_1, g_2, h : [t_0, ∞) → \mathbb{R} \) are real valued continuous functions such that \( g_1(t) < r, g_2(t) > t, g_1 \) and \( g_2 \) are strictly increasing, and \( \lim_{t→∞} g_1 (t) = \lim_{t→∞} g_2 (t) = \lim_{t→∞} h (t) = ∞; \)

(C3) \( p_1, p_2 : [t_0, ∞) → \mathbb{R} \) are real valued continuous functions with \( p_1(t) ≥ 0, p_2(t) ≥ 1, \) and \( p_2(t) \neq 1 \) eventually; or

(C3) \( p_1, p_2 : [t_0, ∞) → \mathbb{R} \) are real valued continuous functions with \( p_2(t) ≥ 0, p_1(t) ≥ 1, \) and \( p_1(t) \neq 1 \) eventually.

By a solution of equation (1) we mean a function \( x : [t_0, ∞) → \mathbb{R}, \) \( t_0 ≥ t_0 \), such that \( z ∈ C^1([t_0, ∞), \mathbb{R}), \)

\( r(z')^α ∈ C^1([t_0, ∞), \mathbb{R}) \), and which satisfies (1) on \([t_0, ∞).\)

We consider only those solutions \( x(t) \) of (1) that satisfy \( \sup \{ |x(t)| : t ≥ T \} > 0 \) for all \( T ≥ t_0; \) moreover, we tacitly assume that (1) possesses such solutions. Such a solution \( x(t) \) of (1) is said to be oscillatory if it has arbitrarily large zeros on \([t_0, ∞); \) otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Oscillation and asymptotic behaviour of solutions to various classes of delay and advanced neutral differential and dynamic equations have been widely discussed in the literature; see, for example, [1–19], and the references contained therein.

However, oscillation results for mixed neutral differential and dynamic equations are relatively scarce in the literature; some results can be found, for example, in [20–32], and the references cited therein. We would like to point out that the results obtained in [20–32] require both of \( p_1 \) and \( p_2 \) to be constants or bounded functions, and hence, the results established in these papers cannot be applied to the cases where \( \lim_{t→∞} p_1 (t) = ∞ \) and/or \( \lim_{t→∞} p_2 (t) = ∞. \) In view of the observations above, we wish to develop new sufficient conditions which can be applied to the cases where \( \lim_{t→∞} p_1 (t) = ∞ \) and/or \( \lim_{t→∞} p_2 (t) = ∞. \) In this connection, the results obtained in the present paper are new, improve and complement some existing results in the relevant literature. Furthermore, the results in this paper can easily be extended to more general second-order mixed neutral differential and dynamic equations to obtain more general oscillation
results. It is therefore hoped that the present paper will contribute significantly to the study of oscillation of solutions of second-order mixed neutral differential equations.

2. Some preliminary lemmas

In this section, we present some lemmas that will play an important role in establishing our main results. For notational purposes, we let, for any continuous function \(d\),

\[
d_+(t) := \max \{0, d(t)\}, \quad R(t, t_1) := \int_{t_1}^{t} r^{-1/\alpha}(s) \, ds,
\]

and throughout this paper, we define

\[
\xi(t) := \frac{1}{p_2(g_2^{-1}(t))} \left[ 1 - \frac{1}{p_2(g_2^{-1}(g_1^{-1}(t)))} \right],
\]

\[
\varphi(t) := \frac{1}{p_1(g_1^{-1}(t))} \left[ 1 - \frac{1}{p_1(g_1^{-1}(g_1^{-1}(t)))} \right],
\]

for all sufficiently large \(t\), where \(g_1^{-1}\) and \(g_2^{-1}\) denote the inverse functions of \(g_1\) and \(g_2\), respectively, and \(m\) is a function to be specified later.

Lemma 2.1 ([33]): If \(D\) and \(E\) are nonnegative and \(\lambda > 1\), then

\[
\lambda D E^{\lambda-1} - D^{\lambda} \leq (\lambda - 1) E^\lambda,
\]

where equality holds if and only if \(D = E\).

Lemma 2.2: Assume that conditions (C1)–(C3g) (or (C1), (C2), and (C3g) hold, and let \(x(t)\) be an eventually positive solution of (1). Then there exists \(t_1 \geq t_0\) such that, for \(t \geq t_1\),

\[
(r(t)(z'(t))^\alpha)' < 0, \quad z'(t) > 0, \quad \text{and} \quad z(t) > 0.
\]

Proof: The proof is standard and we omit the details of its proof. ■

Lemma 2.3: In addition to conditions (C1)–(C3g) (or (C1), (C2), and (C3g)), assume that \(x(t)\) is an eventually positive solution of (1) such that (2) holds. If there exist a positive function \(m \in C^1([t_0, \infty), \mathbb{R})\) and \(t_2 \in [t_1, \infty)\) such that

\[
\frac{m(t)}{r^{1/\alpha} R(t, t_1)} - m'(t) \leq 0 \quad \text{for} \quad t \geq t_2,
\]

then

\[
(z(t)/m(t))' \leq 0 \quad \text{for} \quad t \geq t_2.
\]

Proof: Since \(r(t)(z'(t))^\alpha\) is decreasing on \([t_1, \infty)\), we obtain

\[
z(t) = z(t_1) + \int_{t_1}^{t} \frac{(r(s)(z'(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)} \, ds \geq r^{1/\alpha}(t) R(t, t_1) z'(t).
\]

Thus, in view of (3) and (5), we see that

\[
\left(\frac{z(t)}{m(t)}\right)' = \frac{z'(t)m(t) - z(t)m'(t)}{m^2(t)} \leq \frac{z(t)}{m^2(t)} \left(\frac{m(t)}{r^{1/\alpha}(t) R(t, t_1)} - m'(t)\right) \leq 0,
\]

for \(t \geq t_2\) and for some \(t_2 \in [t_1, \infty)\). This completes the proof of the lemma. ■

Lemma 2.4: Let conditions (C1)–(C3g) hold and \(\xi(t) > 0\). If \(x(t)\) is an eventually positive solution of (1) such that (2) holds, then \(z(t)\) satisfies the inequality

\[
(r(t)(z'(t))^\alpha)' + q(t)\xi(t)(h(t)) z''(g_2^{-1}(h(t))) \leq 0
\]

for sufficiently large \(t\).

Proof: Let \(x(t)\) be an eventually positive solution of (1) such that \(x(t) > 0, x(g_1^{-1}(t)) > 0, x(g_2^{-1}(t)) > 0, x(h(t)) > 0, \xi(t) > 0\) and \(z(t)\) satisfies (2) for \(t \geq t_1\) and for some \(t_1 \in [t_0, \infty)\). From the definition of \(z(t)\), (see also (8.6) in [1]), we obtain

\[
x(t) = \frac{1}{p_2(g_2^{-1}(t))} \left( z(g_2^{-1}(t)) - x(g_2^{-1}(t)) \right) - p_1(g_2^{-1}(t)) x(g_1^{-1}(t)))
\]

\[
= \frac{z(g_2^{-1}(t))}{p_2(g_2^{-1}(t))} \left[ \frac{z(g_2^{-1}(g_1^{-1}(t)))) - x(g_2^{-1}(g_1^{-1}(t))))}{p_1(g_2^{-1}(t)) x(g_1^{-1}(g_1^{-1}(t)))} \right] - p_1(g_2^{-1}(t)) x(g_1^{-1}(g_1^{-1}(t)))
\]

\[
= \frac{z(g_2^{-1}(g_1^{-1}(t)))) - x(g_2^{-1}(g_1^{-1}(t))))}{p_2(g_2^{-1}(g_1^{-1}(t)))}
\]

\[
= \frac{p_1(g_2^{-1}(g_1^{-1}(t)))) - x(g_2^{-1}(g_1^{-1}(t))))}{p_2(g_2^{-1}(g_1^{-1}(t)))}
\]

\[
= \frac{p_1(g_2^{-1}(g_1^{-1}(t)))) - x(g_2^{-1}(g_1^{-1}(t))))}{p_2(g_2^{-1}(g_1^{-1}(t)))}
\]

\[
= \frac{p_1(g_2^{-1}(g_1^{-1}(t)))) - x(g_2^{-1}(g_1^{-1}(t))))}{p_2(g_2^{-1}(g_1^{-1}(t)))}
\]
Since (3) holds, we again have (4) holds, i.e., eventually positive solution of \( z \) increases, and noting that \( g_1(t) < t < g_2(t) \), we get
\[
z(g_2^{-1}(g_2^{-1}(t))) < z(g_2^{-1}(t)) \tag{9}
\]
and
\[
z(g_2^{-1}(g_1(g_2^{-1}(t)))) < z(g_2^{-1}(t)). \tag{10}
\]
Using (9) and (10) in (8) gives
\[
x(t) \geq \frac{1}{p_2(g_2^{-1}(t))} \left[ 1 - \frac{1}{p_2(g_2^{-1}(g_2^{-1}(t)))} \right] z(g_2^{-1}(t)) \tag{11}
\]
for \( t \geq t_1 \). Since \( \lim_{t \to \infty} h(t) = \infty \), we can choose \( t_2 \geq t_1 \) such that \( h(t) \geq t_1 \) for all \( t \geq t_2 \). Thus, from (11) we have
\[
x(h(t)) \geq \xi(h(t)) z(g_2^{-1}(h(t))) \quad \text{for } t \geq t_2. \tag{12}
\]
Substituting (12) into (1) gives (7) and completes the proof.

**Lemma 2.5:** Let conditions (C1), (C2), and (C3x) hold. Assume further that there exists a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that \( \varphi(t) > 0 \) and (3) hold. If \( x(t) \) is an eventually positive solution of (1) such that (2) holds, then \( z(t) \) satisfies the inequality
\[
(r(t)(z'(t))^\alpha + q(t) \psi^\alpha(h(t)) z^\alpha(g_1^{-1}(h(t)))) \leq 0 \quad \text{for sufficiently large } t.
\]

**Proof:** Let \( x(t) \) be an eventually positive solution of (1) such that \( x(t) > 0, x(g_1(t)) > 0, x(g_2(t)) > 0, x(h(t)) > 0, \varphi(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \) and for some \( t_1 \in [t_0, \infty) \). Following a similar argument as in the proof of Lemma 2.4, we obtain
\[
x(t) \geq \frac{z(g_1^{-1}(t)) - z(g_1^{-1}(g_1^{-1}(t)))}{p_1(g_1^{-1}(t))} \frac{1}{p_1(g_1^{-1}(g_1^{-1}(t)))} + \frac{p_1(g_1^{-1}(t)) z(g_1^{-1}(g_2^{-1}(t)))}{p_1(g_1^{-1}(t)) p_1(g_1^{-1}(g_1^{-1}(t)))} \tag{14}
\]
Using the fact that \( g_1 \) and \( g_2 \) are strictly increasing, and noting that \( g_1(t) < t < g_2(t) \), we get
\[
g_1^{-1}(g_1^{-1}(t)) > g_1^{-1}(t) \tag{15}
\]
and
\[
g_1^{-1}(g_2(g_1^{-1}(t))) > g_1^{-1}(t). \tag{16}
\]
Since (3) holds, we again have (4) holds, i.e., \( z/m \) is nonincreasing on \([t_2, \infty) \subseteq [t_1, \infty) \). Thus, we deduce from (15) and (16) that
\[
\frac{m(g_1^{-1}(g_1^{-1}(t))) z(g_1^{-1}(t))}{m(g_1^{-1}(t))} \geq z(g_1^{-1}(g_1^{-1}(t))) \quad \text{for } t \geq t_2, \tag{17}
\]
and
\[
\frac{m(g_1^{-1}(g_2(g_1^{-1}(t)))) z(g_1^{-1}(t))}{m(g_1^{-1}(t))} \geq z(g_1^{-1}(g_2(g_1^{-1}(t)))) \quad \text{for } t \geq t_2, \tag{18}
\]
respectively.

Using (17) and (18) in (14) yields
\[
x(t) \geq \frac{1}{p_1(g_1^{-1}(t))} \left[ 1 - \frac{1}{p_1(g_1^{-1}(g_1^{-1}(t)))} \right] m(g_1^{-1}(g_1^{-1}(t))) \tag{19}
\]
for \( t \geq t_2 \). Since \( \lim_{t \to \infty} h(t) = \infty \), we can choose \( t_3 \geq t_2 \) such that \( h(t) \geq t_2 \) for all \( t \geq t_3 \). Thus, from (19) we have
\[
x(h(t)) \geq \psi(h(t)) z(g_1^{-1}(h(t))) \quad \text{for } t \geq t_3. \tag{20}
\]
Substituting (20) into (1) gives (13) and completes the proof.

**3. Main results**

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (1). We begin with the following result.

**Theorem 3.1:** Assume that (C1)–(C3x) hold, \( \xi(t) > 0 \) and \( g_2(t) \geq h(t) \). Assume further that there exists a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that (3) holds. If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in (t_1, \infty) \),
\[
\lim_{t \to \infty} \int_{t}^{T} \frac{\chi_1(s) - \frac{\eta'(s)}{\eta(s)} r(s)(m'(s))^\alpha}{m^\alpha(s)} ds = \infty,
\]
where
\[
\chi_1(t) = \eta(t) q(t) \xi^\alpha(h(t)) \frac{m^\alpha(g_2^{-1}(h(t)))}{m^\alpha(t)},
\]
then every solution of equation (1) is oscillatory.

**Proof:** Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists \( t_1 \in [t_0, \infty) \) such that \( x(t) > 0, x(g_1(t)) > 0, x(g_2(t)) > 0, x(h(t)) > 0 \).
0, \( x(h(t)) > 0 \), \( \xi(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \). (The proof if \( x(t) \) is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper). Proceeding as in the proofs of Lemmas 2.3 and 2.4, we see that (4), (5) and (7) hold on \([t_2, \infty) \subseteq [t_1, \infty)\). Define the Riccati substitution

\[
\omega(t) = \eta(t) \frac{r(t) (\zeta(t))^\alpha}{z^\alpha(t)} \quad \text{for } t \geq t_1.
\]

(22)

Clearly \( \omega(t) > 0 \) for \( t \geq t_1 \), and from (7) we obtain

\[
\omega'(t) \leq \eta_+/(t) \frac{r(t) (\zeta(t))^\alpha}{z^\alpha(t)} - \eta(t) q(t) \xi^\alpha(h(t)) \frac{z^\alpha(g^{-1}_2(h(t)))}{z^\alpha(t)} \]

\[
- \alpha \eta(t) r(t) \left( \frac{\zeta(t)}{z(t)} \right)^{\alpha+1} \quad \text{for } t \geq t_2.
\]

(23)

From \( g_2(t) \geq h(t) \) and the fact that \( g_2 \) is strictly increasing, we see that \( t \geq g_2^{-1}(h(t)) \). Hence, by (4) we get

\[
\frac{z(g_2^{-1}(h(t)))}{z(t)} \geq \frac{m(g_2^{-1}(h(t)))}{m(t)} \quad \text{for } t \geq t_2.
\]

(24)

Substituting (24) into (23) gives

\[
\omega'(t) \leq \eta_+(t) \frac{r(t) (\zeta'(t))^\alpha}{z^\alpha(t)} - \eta(t) q(t) \xi^\alpha(h(t)) \frac{m^\alpha(g_2^{-1}(h(t)))}{m^\alpha(t)} \]

\[
- \alpha \eta(t) r(t) \left( \frac{\zeta'(t)}{z(t)} \right)^{\alpha+1} \quad \text{for } t \geq t_2.
\]

(25)

In view of (3) and (5), it is easy to see that

\[
m'(t) \geq \frac{1}{\rho(t)} \frac{1}{R(h(t))} \geq \frac{z'(t)}{z(t)}.
\]

(26)

From (26), \( z(t) > 0 \), and \( z'(t) > 0 \), (25) yields

\[
\omega'(t) \leq \eta_+(t) \frac{r(t) (m'(t))^\alpha}{m^\alpha(t)} - \eta(t) q(t) \xi^\alpha(h(t)) \frac{m^\alpha(g_2^{-1}(h(t)))}{m^\alpha(t)} \]

\[
- \alpha \eta(t) r(t) \left( \frac{\zeta'(t)}{z(t)} \right)^{\alpha+1} \quad \text{for } t \geq t_2.
\]

(27)

for \( t \geq t_2 \). An integration of (27) from \( t_2 \) to \( t \) gives

\[
\int_{t_2}^{t} \left( \chi_1(s) - \frac{\eta_+(s) r(s) (m'(s))^\alpha}{m^\alpha(s)} \right) ds \leq \omega(t_2),
\]

which contradicts condition (21). This completes the proof.

**Theorem 3.2:** Assume that (C1)–(C3) hold, \( \bar{\xi}(t) > 0 \) and \( g_2(t) \geq h(t) \). Assume further that there exists a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that (3) holds. If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in (t_1, \infty) \),

\[
\limsup_{t \to \infty} \int_{T}^{t} \left( \chi_1(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(s) (\eta_+(s))^{\alpha+1}}{\eta^\alpha(s)} \right) ds = \infty,
\]

(28)

where \( \chi_1(t) \) is as in Theorem 3.1, then every solution of (1) oscillates.

**Proof:** Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exist \( t_1 \in [t_0, \infty) \) such that \( x(t) > 0, x(g_1(t)) > 0, x(g_2(t)) > 0, x(h(t)) > 0, \bar{\xi}(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \). Proceeding as in the proof of Theorem 3.1, we again arrive at (25) for \( t \geq t_2 \). From (22) and the definition of \( \chi_1(t) \), inequality (25) can be written as

\[
\omega'(t) \leq \eta_+(t) \frac{r(t) \omega(t) - \chi_1(t)}{\eta(t)} - \frac{\alpha}{(\eta(t) r(t))^{1/\alpha}} \omega^{(\alpha+1)/\alpha} \quad \text{for } t \geq t_2.
\]

(29)

Applying Lemma 2.1 with \( \lambda := (\alpha+1)/\alpha \),

\[
D = \frac{\alpha^{1/\lambda}}{[(\eta(t) r(t))^{1/\lambda}]^{1/\lambda}} \omega(t), \quad \text{and}
\]

\[
E = \left[ \frac{\alpha}{\alpha+1} \frac{[(\eta(t) r(t))^{1/\lambda}]^{1/\lambda}}{\eta_+(t)} \right]^{\alpha}
\]

we obtain

\[
\frac{\eta_+(t)}{\eta(t)} \omega(t) = \frac{\alpha}{(\eta(t) r(t))^{1/\lambda}} \omega^{(\alpha+1)/\alpha} \leq \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(t) (\eta_+(t))^{\alpha+1}}{\eta^\alpha(t)}.
\]

(30)

Substituting (30) into (29) gives

\[
\omega'(t) \leq \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(t) (\eta_+(t))^{\alpha+1}}{\eta^\alpha(t)} - \chi_1(t).
\]

Integrating the last inequality from \( t_2 \) to \( t \) yields

\[
\int_{t_2}^{t} \left( \chi_1(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(s) (\eta_+(s))^{\alpha+1}}{\eta^\alpha(s)} \right) ds \leq \omega(t_2),
\]

which is a contradiction to our assumption (28). This proves the theorem.

**Theorem 3.3:** Let \( \alpha \geq 1 \). Assume that (C1)–(C3) hold, \( \xi(t) > 0 \) and \( g_2(t) \geq h(t) \). Assume further that there exists
a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that (3) holds. If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in (t_1, \infty) \),

\[
\limsup_{t \to \infty} \int_T^t \left( \chi_1(s) - \frac{r^{1/\alpha}(s)}{4\alpha\eta(s) R(s,t_1)^{1/\alpha}} \right)^2 \, ds = \infty,
\]

(31)

where \( \chi_1(t) \) is as in Theorem 3.1, then every solution of equation (1) oscillates.

**Proof:** Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists \( t_1 \in [t_0, \infty) \) such that \( x(t) > 0, x(g_1(t)) > 0, x(g_2(t)) > 0, x(h(t)) > 0, \xi(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \). Proceeding as in the proof of Theorem 3.2, we again arrive at (29) which can be written as

\[
\omega'(t) \leq \frac{\eta(t)}{\eta(t)} \omega(t) - \frac{\alpha \omega(t) - 1}{(\eta(t) r(t))^{1/\alpha} \omega^2(t)} \quad \text{for } t \geq t_2.
\]

(32)

From (22) and (5), we obtain

\[
(\omega(t))^{1/\alpha - 1} = (\eta(t) r(t))^{1/\alpha - 1} \left( \frac{z'(t)}{z(t)} \right)^{1-\alpha}
\]

\[
= (\eta(t) r(t))^{1/\alpha - 1} \left( \frac{z(t)}{z'(t)} \right)^{\alpha - 1}
\]

\[
\geq (\eta(t) r(t))^{1/\alpha - 1} \left( r^{1/\alpha}(t) R(t, t_1) \right)^{\alpha - 1}
\]

\[
= (\eta(t))^{1/\alpha - 1} (R(t, t_1))^{\alpha - 1}.
\]

(33)

Using (33) in (32), we conclude that

\[
\omega'(t) \leq \frac{\eta(t)}{\eta(t)} \omega(t) - \chi_1(t)
\]

\[
- \frac{\alpha \left( R(t, t_1) \right)^{\alpha - 1}}{(\eta(t) r(t))^{1/\alpha} \omega^2(t)} \quad \text{for } t \geq t_2.
\]

(34)

Completing the square with respect to \( \omega \), it follows from (34) that

\[
\omega'(t) \leq -\chi_1(t) + \frac{r^{1/\alpha}(t)}{4\alpha (R(t, t_1))^{\alpha - 1}} \left( \frac{\eta(t)}{\eta(t)} \right)^2 \omega(t)
\]

(35)

Integrating (35) from \( t_2 \) to \( t \) gives

\[
\int_{t_2}^{t} \left( \chi_1(s) - \frac{r^{1/\alpha}(s)}{4\alpha\eta(s) (R(s,t_1))^{1/\alpha}} \right)^2 \, ds \leq \omega(t_2),
\]

which contradicts condition (31). This completes the proof of the theorem. \[\blacksquare\]

**Theorem 3.4:** Assume that (C1)-(C3c) hold, \( \xi(t) > 0 \) and \( g_2(t) \leq h(t) \). If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in (t_1, \infty) \),

\[
\limsup_{t \to \infty} \int_T^t \left( \chi_2(s) - \frac{\eta^2(s)}{R(s,t_1)^{1/\alpha}} \right) ds = \infty,
\]

(36)

where

\[
\chi_2(t) = \eta(t) q(t) \xi(t)(h(t)),
\]

then every solution of equation (1) is oscillatory.

**Proof:** Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists \( t_1 \in [t_0, \infty) \) such that \( x(t) > 0, x(g_1(t)) > 0, x(g_2(t)) > 0, x(h(t)) > 0, \xi(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \). Proceeding exactly as in the proof of Theorem 3.1, we again arrive at (23) for \( t \geq t_2 \). From \( g_2(t) \leq h(t) \) and the fact that \( g_2 \) is strictly increasing, we see that

\[
t \leq g_2^{-1}(h(t)),
\]

(37)

and so

\[
\frac{z \left( g_2^{-1}(h(t)) \right)}{z(t)} \geq 1.
\]

(38)

Using (38) in (23) yields

\[
\omega'(t) \leq \eta(t) \left( \frac{r(t)}{z(t)} \right)^{\alpha - 1} - \frac{\xi(t)}{z(t)} \frac{z(t)}{z(t)} \quad \text{for } t \geq t_2.
\]

(39)

Taking into account that (5) holds, and using the fact that \( z'(t) > 0 \), inequality (39) takes the form

\[
\omega'(t) \leq \frac{\eta(t)}{R(t, t_1)} - \chi_2(t) \quad \text{for } t \geq t_2.
\]

(40)

The remainder of the proof is similar to that of Theorem 3.1, and so the details are omitted. \[\blacksquare\]

**Theorem 3.5:** Assume that (C1)-(C3c) hold, \( \xi(t) > 0 \) and \( g_2(t) \leq h(t) \). If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in (t_1, \infty) \),

\[
\limsup_{t \to \infty} \int_T^t \left( \chi_2(s) - \frac{1}{\alpha+1} \frac{r(s)}{\eta^2(s)} \right) ds = \infty,
\]

(40)

where \( \chi_2(t) \) is as in Theorem 3.4, then equation (1) is oscillatory.

**Proof:** The proof follows from (22), (30), (39), and Theorem 3.2. \[\blacksquare\]
Theorem 3.6: Let $\alpha \geq 1$. Assume that $(C1)-(C3_{\xi})$ hold, $\xi(t) > 0$ and $g_2(t) \leq h(t)$. If there exists a positive function $\eta(t) \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $t_1 \in [t_0, \infty)$ and for some $T \in (t_1, \infty),$

$$\limsup_{t \to \infty} \int_T^t \chi_2(s) \left( \frac{r(t)}{\eta_-^2(s)} \frac{(\eta_+'(s))^2}{m(s)} \right) \, ds = \infty,$$

where $\chi_2(t)$ is as in Theorem 3.4, then equation (1) is oscillatory.

Proof: The proof follows from (22), (33), (39), and Theorem 3.3.

Theorem 3.7: Assume that conditions $(C1)$, $(C2)$, and $(C3_\psi)$ hold and $g_1(t) \geq h(t)$. Assume further that there exists a positive function $m \in C^1([t_0, \infty), \mathbb{R})$ such that $\psi(t) > 0$ and (3) hold. If there exists a positive function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $t_1 \in [t_0, \infty)$ and for some $T \in [t_2, \infty) \subseteq (t_1, \infty),$

$$\limsup_{t \to \infty} \int_T^t \chi_3(s) - \frac{1}{\eta_+^2(s)} \frac{(\eta_+')^2}{\eta_-^2(s)} \, ds = \infty,$$

where $\chi_3(t)$ is as in Theorem 3.7, then every solution of equation (1) oscillates.

Proof: The proof follows from (22), (30), (45), and Theorem 3.2.

Theorem 3.8: Assume that conditions $(C1)$, $(C2)$, and $(C3_\psi)$ hold and $g_1(t) \geq h(t)$. Assume further that there exists a positive function $m \in C^1([t_0, \infty), \mathbb{R})$ such that $\psi(t) > 0$ and (3) hold. If there exists a positive function $\eta(t) \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $t_1 \in [t_0, \infty)$ and for some $T \in [t_2, \infty) \subseteq (t_1, \infty),$

$$\limsup_{t \to \infty} \int_T^t \Omega_4(s) - \frac{1}{\eta_+^2(s)} \frac{(\eta_+')^2}{\eta_-^2(s)} \, ds = \infty,$$

where $\Omega_4(t)$ is as in Theorem 3.7, then every solution of equation (1) oscillates.

Proof: The proof follows from (22), (33), (45), and Theorem 3.3.

Theorem 3.9: Let $\alpha \geq 1$. Assume that conditions $(C1)$, $(C2)$, and $(C3_\psi)$ hold and $g_1(t) \geq h(t)$. Assume further that there exists a positive function $m \in C^1([t_0, \infty), \mathbb{R})$ such that $\psi(t) > 0$ and (3) hold. If there exists a positive function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $t_1 \in [t_0, \infty)$ and for some $T \in [t_2, \infty) \subseteq (t_1, \infty),$

$$\limsup_{t \to \infty} \int_T^t \chi_4(s) - \frac{1}{\eta_+^2(s)} \frac{(\eta_+')^2}{\eta_-^2(s)} \, ds = \infty,$$

where $\chi_4(t)$ is as in Theorem 3.7, then every solution of equation (1) oscillates.

Proof: The proof follows from (22), (33), (45), and Theorem 3.3.
Proof: Let \( x(t) \) be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists \( t_1 \in [t_0, \infty) \) such that \( x(t) > 0 \), \( x(g_1(t)) > 0 \), \( x(g_2(t)) > 0 \), \( x(h(t)) > 0 \), \( \psi(t) > 0 \) and \( z(t) \) satisfies (2) for \( t \geq t_1 \). Proceeding as in the proof of Theorem 3.7, we again arrive at (43). From \( g_1(t) \leq h(t) \) and the fact that \( g_1 \) is strictly increasing, we see that

\[
 t \leq g_1^{-1}(h(t)),
\]

and so, from the fact that \( z'(t) > 0 \), we have

\[
 z \left( \frac{g_1^{-1}(h(t))}{z(t)} \right) > 1.
\]

Using (50) in (43) gives

\[
 \omega'(t) \leq \frac{\eta(t) t \left( z'(t) \right)^{p} \psi(z(t))}{z(t)} - \eta(t) q(t) \psi(z(t)) - \alpha \omega(t) \frac{z'(t)}{z(t)} \cdot
\]

Taking into account that (5) holds, and from the fact that \( z'(t) > 0 \), inequality (51) takes the form

\[
 \omega'(t) \leq \frac{\eta(t) t \left( z'(t) \right)^{p} \psi(z(t))}{z(t)} - \eta(t) q(t) \psi(z(t)) (h(t)).
\]

The remainder of the proof is similar to that of Theorem 3.1, and so the details are omitted. ■

Theorem 3.11: Assume that conditions (C1), (C2), and (C3) hold and \( g_1(t) \leq h(t) \). Assume further that there exists a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that \( \psi(t) > 0 \) and (3) hold. If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in [t_2, \infty) \subseteq (t_1, \infty) \),

\[
 \limsup_{t \to \infty} \int_{T}^{t} \left( x_4(s) - \frac{1}{(\alpha + 1)^{p + 1}} \frac{r(s) \eta_+(s)^{\alpha + 1}}{\eta(s)} \right) ds = \infty,
\]

where \( x_4(t) \) is as in Theorem 3.10, then equation (1) is oscillatory.

Proof: The proof follows from (22), (30), (51), and Theorem 3.2. ■

Theorem 3.12: Let \( \alpha \geq 1 \). Assume that conditions (C1), (C2), and (C3) hold and \( g_1(t) \leq h(t) \). Assume further that there exists a positive function \( m \in C^1([t_0, \infty), \mathbb{R}) \) such that \( \psi(t) > 0 \) and (3) hold. If there exists a positive function \( \eta \in C^1([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( t_1 \in [t_0, \infty) \) and for some \( T \in [t_2, \infty) \subseteq (t_1, \infty) \),

\[
 \limsup_{t \to \infty} \int_{T}^{t} \left( x_4(s) - \frac{r(s)^{\alpha} (\eta_+(s))^{\alpha + 1}}{4 \alpha \eta(s) (R(s, t_1))^{\alpha + 1}} \right) ds = \infty,
\]

where \( x_4(t) \) is as in Theorem 3.10, then equation (1) is oscillatory.

Proof: The proof follows from (22), (33), (51), and Theorem 3.3. ■

We conclude this paper with the following examples to illustrate the above results. First example is concerned with the case where \( p_2(t) \to \infty \) as \( t \to \infty \), second example is concerned with the case where \( p_1 \) and \( p_2 \) are constants or bounded functions, third example is concerned with the case where \( p_1(t) \to \infty \) and \( p_2(t) \to \infty \) as \( t \to \infty \), and fourth example is concerned with the case where \( p_1(t) \to \infty \) as \( t \to \infty \).

Example 3.13: Consider the neutral differential equation

\[
 \left( \left( x(t) + x \left( \frac{t}{2} \right) + tx(2t) \right)^{3} \right)' + \left( \frac{t^4 + 1}{t^2} \right) x^3 (2t - 1) = 0, \quad t \geq 13.
\]

Here we have \( \alpha = 3 \), \( r(t) = p_1(t) = 1 \), \( p_2(t) = t \), \( q(t) = t^4 + 1 \), \( g_1(t) = t/2 \), \( g_2(t) = 2t \) and \( h(t) = 2t - 1 \). It is clear that conditions (C1) – (C3) hold, \( g_2(t) \geq h(t) \), and

\[
 \xi(t) = \frac{1}{t^2} \left[ 1 - \frac{1}{t^4} - \frac{1}{t^8} \right] = \frac{2t - 24}{t^2} > 0.
\]

On the other hand, if we choose \( m(t) = R(t, t_1) \), we see that

\[
 m(t) = \int_{t_1}^{t} \frac{1}{t^{1/\alpha} (s)} ds = \int_{13}^{t} ds = t - 13,
\]

and so (3) holds. With \( \eta(t) = t \), condition (21) with \( T > 13 \) becomes

\[
 \limsup_{t \to \infty} \int_{T}^{t} \left( x_1(s) - \frac{r(s)^{\alpha} (\eta_+(s)^{\alpha + 1}}{m^\alpha (s)} \right) ds
\]

\[
 = \limsup_{t \to \infty} \int_{T}^{t} \left( s^{4 + 1} \left( \frac{4s - 26}{(2s - 1)^2} \right) \left( \frac{s - 27/2}{s - 13} \right)^3 \right) ds
\]

\[
 > \frac{1}{(s - 13)^3} ds = \infty,
\]

due to

\[
 \limsup_{t \to \infty} \int_{T}^{t} s^{4 + 1} \left( \frac{4s - 26}{(2s - 1)^2} \right)^3 \left( \frac{s - 27/2}{s - 13} \right)^3 ds = \infty
\]

and

\[
 \limsup_{t \to \infty} \int_{T}^{t} \frac{1}{(s - 13)^3} ds < \infty.
\]

Hence, by Theorem 3.1, every solution of (54) is oscillatory.
Example 3.14: Consider the neutral differential equation

\[
\left( \frac{1}{t^{1/3}} \left( \left( x(t) + 2x \left( \frac{t}{3} \right) + 8x(4t) \right)^3 \right) \right)' + (t^2 + t)x^{1/3}(5t) = 0, \quad t \geq 2. \tag{56}
\]

Here we have \( \alpha = 1/3, r(t) = 1/t^{1/3}, p_1(t) = 2, p_2(t) = 8, q(t) = t^2 + t, g_1(t) = t/3, g_2(t) = 4t \) and \( h(t) = 5t \). It is clear that conditions (C1)–(C3) hold, and \( g_2(t) \leq h(t) \), and

\[
\xi(t) = 1/8 \left[ 1 - 1 - 8/2 \right] = 5/64 > 0. \tag{57}
\]

On the other hand, if we choose \( m(t) = R(t, t_1) \), we see that

\[
m(t) = \int_{t_1}^{t} \frac{1}{t^{1/3}(s)} \, ds = \int_{t_2}^{t} s \, ds = \frac{t^2 - 4}{2},
\]

and (3) holds. With \( \eta(t) = c > 0 \) is a constant, condition (36) with \( T > 2 \) becomes

\[
\lim_{t \to \infty} \int_{t}^{T} \left( x_2(s) - \frac{\eta'_+(s)}{R^2(s, t)} \right) \, ds = \lim_{t \to \infty} \int_{t}^{T} c(s^2 + s)^{1/3} \, ds = \infty,
\]

i.e. condition (36) holds. Hence, by Theorem 3.4, every solution of (56) is oscillatory.

Example 3.15: Consider the neutral differential equation

\[
\left( \left( \left( x(t) + 3tx \left( \frac{t}{3} \right) + tx(2t) \right)^3 \right)' \right) + (t^5 + t)x^{3/4}(t) = 0, \quad t \geq 2. \tag{58}
\]

Here we have \( \alpha = 3, r(t) = 1, p_1(t) = 3t, p_2(t) = t, q(t) = t^5 + t, g_1(t) = t/3, g_2(t) = 2t \) and \( h(t) = t/4 \). It is clear that conditions (C1)–(C3) hold and \( g_1(t) \geq h(t) \). If we choose \( m(t) = R(t, t_1) \), we see that \( m(t) = t - 2, (3) \) holds, and

\[
\varphi(t) = \frac{1}{9t} \left[ 1 - \frac{1}{27} \frac{9t - 2}{3t} - \frac{3t}{54} \frac{18t - 2}{3t} \right]
= \frac{108t^2 - 120t + 4}{1458t^3 - 972t^2} > 0. \tag{59}
\]

With \( \eta(t) = t^2 \), we see that condition (42) holds for \( T > 2 \). Hence, by Theorem 3.7, every solution of (58) is oscillatory.

Example 3.16: Consider the neutral differential equation

\[
\left( x(t) + \epsilon x(t - 2\tau) + x(t + \pi) \right)'' + 2\epsilon^2 x \left( t - \frac{\pi}{2} \right) = 0, \quad t \geq 5. \tag{60}
\]

Here we have \( \alpha = 1, r(t) = p_2(t) = 1, p_1(t) = \epsilon^2, q(t) = 2\epsilon^2, g_1(t) = t - 2\pi, g_2(t) = t + \pi \) and \( h(t) = t - \pi/2 \).

It is clear that conditions (C1), (C2), and (C3) hold, and \( g_1(t) \leq h(t) \). On the other hand, if we choose \( m(t) = R(t, t_1) \), we see that \( m(t) = t - 5 \), and so \( \varphi(t) > 0 \) and (3) hold. With \( \eta(t) = c \), it is easy to see that condition (48) holds. Hence, by Theorem 3.10, every solution of (60) is oscillatory. In fact, \( x(t) = \sin t \) is such a solution.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Ercan Tunc @ http://orcid.org/0000-0001-8860-608X
Orhan Özdemir @ http://orcid.org/0000-0003-1294-5346

References

[1] Agarwal RP, Grace SR, O’Regan D. The oscillation of certain higher-order functional differential equations. Math Comput Model. 2003;37:705–728.
[2] Agarwal RP, Bohner M, Li T, et al. Oscillation of second-order Emden–Fowler neutral delay differential equations. Ann Math Pura Appl. 2014;193:1861–1875.
[3] Baculíková B, Dzurina J. Oscillation theorems for second order neutral differential equations. Comput Math Appl. 2011;61:94–99.
[4] Bohner M, Grace SR, Jadlovská I. Oscillation criteria for second-order neutral delay differential equations. Electron J Qual Theory Differ Equ. 2017;2017(60):1–12.
[5] Chen D-X. Oscillation of second-order Emden-Fowler neutral delay dynamic equations on time scales. Math Comput Model. 2010;51:1221–1229.
[6] Han Z, Li T, Sun S, et al. Remarks on the paper [Appl Math Comput. 2009;207:388–396]. Appl Math Comput. 2010;215:3998–4007.
[7] Graef JR, Tunc E, Grace SR. Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation. Opuscula Math. 2017;37(6):839–852.
[8] Kubiaczyk I, Saker SH, Sikorska-Nowak A. Oscillation criteria for nonlinear neutral functional dynamic equations on time scales. Math Slovaca. 2013;63:263–290.
[9] Li T, Agarwal RP, Bohner M. Some oscillation results for second-order neutral dynamic equations. Hacet J Math Stat. 2012;41:715–721.
[10] Li T, Rogovchenko YV. Oscillatory behavior of second-order neutral nonlinear differential equations. Abstr Appl Anal. 2014;2014:Article ID:143614:8 pages.
[11] Li T, Rogovchenko YV. Oscillation of second-order neutral differential equations. Math Nachr. 2015;268:1150–1162.
[12] Li T, Rogovchenko YV. Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations. Monatsh Math. 2017;184:489–500.
[13] Li T, Thandapani E, Graef JR, et al. Oscillation of second-order Emden–Fowler neutral differential equations. Nonlinear Stud. 2013;20(1):1–8.
[14] Liu H, Meng F, Liu P. Oscillation and asymptotic analysis on a new generalized Emden-Fowler equation. Appl Math Comput. 2012;219:2739–2748.
[15] Saker SH, Graef JR. Oscillation of third-order nonlinear neutral functional dynamic equations on time scales. Dyn Syst Appl. 2012;21:583–606.
[16] Shi Y, Han Z, Sun Y. Oscillation criteria for a generalized Emden–Fowler dynamic equation on time scales. Adv Differ Equ. 2016;2016:Article ID:3:1–12.
[17] Tunç E. Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. Electron J Differ Equ. 2017;2017 (16):1–12.

[18] Tunç E, Özdemir O. On the asymptotic and oscillatory behavior of solutions of third-order neutral dynamic equations on time scales. Adv Difference Equ. 2017;2017: Article ID:127:1–13.

[19] Wang R, Li Q. Oscillation and asymptotic properties of a class of second-order Emden–Fowler neutral differential equations. Springer Plus. 2016;5:1956, 15 pages.

[20] Agwa HA, Khodier AMM, Arafa HM. Oscillation of second-order nonlinear neutral dynamic equations with mixed arguments on time scales. J Basic Appl Res Int. 2016;17:49–66.

[21] Arul R, Shobha VS. Oscillation of second order nonlinear neutral differential equations with mixed neutral term. J Appl Math Phys. 2015;3:1080–1089.

[22] Han Z, Li T, Zhang C, et al. Oscillation criteria for certain second-order nonlinear neutral differential equations of mixed type. Abstr Appl Anal. 2011;2011:Article ID:387483:1–9.

[23] Grace SR. Oscillations of mixed neutral functional differential equations. Appl Math Comput. 1995;68:1–13.

[24] Ji T, Tang S, Thandapani E. Oscillation of second-order neutral dynamic equations with mixed arguments. Appl Math Inf Sci. 2014;8:2225–2228.

[25] Li T. Comparison theorems for second-order neutral differential equations of mixed type. Electron J Differ Equ. 2010;2010(167):1–7.

[26] Li T, Baculíková B, Dzurina J. Oscillation results for second-order neutral differential equations of mixed type. Tatra Mt Math Publ. 2011;48:101–116.

[27] Li T, Senel MT, Zhang C. Oscillation of solutions to second-order half-linear differential equations with neutral terms. Electron J Differ Equ. 2013;2013(229):1–7.

[28] Qi Y, Yu J. Oscillation of second order nonlinear mixed neutral differential equations with distributed deviating arguments. Bull Malays Math Sci Soc. 2015;38:543–560.

[29] Thandapani E, Padmavathi S, Pinelas P. Oscillation criteria for even-order nonlinear neutral differential equations of mixed type. Bull Math Anal Appl. 2014;6(1):9–22.

[30] Thandapani E, Selvarangam S, Vijaya M, et al. Oscillation results for second order nonlinear differential equation with delay and advanced arguments. Kyungpook Math J. 2016;56:137–146.

[31] Yan J. Oscillations of higher order neutral differential equations of mixed type. Israel J Math. 2000;115:125–136.

[32] Zhang C, Baculíková B, Dzurina J, et al. Oscillation results for second-order mixed neutral differential equations with distributed deviating arguments. Math Slovaca. 2016;66(3):615–626.

[33] Hardy GH, Littlewood JE, Polya G. Inequalities. Reprint of the 1952 ed. Cambridge: Cambridge University Press; 1988.