Asymptotic Average Number of Different Categories of Trapping Sets, Absorbing Sets and Stopping Sets in Random Regular and Irregular LDPC Code Ensembles

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**Abstract**

The performance of low-density parity-check (LDPC) codes in the error floor region is closely related to some combinatorial structures of the code’s Tanner graph, collectively referred to as *trapping sets (TS)*. In this paper, we study the asymptotic average number of different types of trapping sets such as *elementary TSs (ETS)*, *leafless ETSs (LETS)*, *absorbing sets (ABS)*, *elementary ABSs (EABS)*, and *stopping sets (SS)*, in random variable-regular and irregular LDPC code ensembles. We demonstrate that, regardless of the type of the TS, as the code’s length tends to infinity, the average number of a given structure tends to infinity, to a positive constant, or to zero, if the structure contains no cycle, only one cycle, or more than one cycle, respectively. For the case where the structure contains a single cycle, we obtain an estimate of the expected number of the structure through the available approximations for the average number of its constituent cycle. These estimates, which are independent of the block length and only depend on the code’s degree distributions, are shown to be accurate even for finite-length codes.

**Index Terms:** Low-density parity-check (LDPC) codes, random LDPC codes, trapping sets (TS), elementary trapping sets (ETS), leafless elementary trapping sets (LETS), absorbing sets (ABS), elementary absorbing sets (EABS), stopping sets (SS).

**I. INTRODUCTION**

The performance of low-density parity-check (LDPC) codes in the error floor region is closely related to some combinatorial structures of the code’s Tanner graph, here referred to collectively as *trapping sets (TS)*. A trapping set $S$ is often identified by its number of variable nodes
a, and the number of unsatisfied check nodes b in its subgraph. In this case, the set \( S \) is said to belong to the \((a, b)\) class. For a given LDPC code, the harmful trapping sets would depend on the degree distribution of the code, the channel model, the decoding algorithm, the quantization scheme, and the structure of the code’s Tanner graph. Some categorizations of TSs include stopping sets (SS), elementary TSs (ETS), leafless ETSs (LETS), absorbing sets (ABS), and elementary ABSs (EABS). Stopping sets are known to be the problematic structures of the belief propagation algorithm over the binary erasure channel (BEC) \([1]\). Leafless ETSs are relevant for soft-decision iterative decoding of variable-regular LDPC codes over the additive white Gaussian noise (AWGN) channel \([2]\), \([3]\), \([4]\). The broader category of ETSs are harmful sets for irregular codes over the AWGN channel \([2]\), \([5]\). Absorbing sets are the fixed points of bit-flipping decoding algorithms \([6]\), \([7]\), and are also shown to be relevant in the context of quantized decoders over the AWGN channel \([7]\).

Regardless of the type of the TS, the problem of counting and/or enumerating trapping sets in a given code is a hard problem. It was shown in \([8]\) that for a given \( a \), finding an \((a, b)\) TS with the smallest \( b \) is an NP-hard problem. Also, finding an \((a, b)\) ETS with the smallest \( a \) for a given \( b \) is a hard problem \([8]\). Furthermore, it was proved in \([9]\) that for a given Tanner graph \( G \) and a positive integer \( t \), determining whether \( G \) has a stopping set of size \( t \) is NP-complete. Despite the intrinsic difficulty of such combinatorial problems, many advances are made on characterization of trapping sets and the development of efficient search algorithms to find them \([10]\), \([11]\), \([12]\), \([13]\), \([14]\), \([2]\), \([15]\), \([3]\), \([16]\), \([4]\), \([5]\).

More related to the work presented in this paper, asymptotic results on the distribution of trapping sets in different ensembles of LDPC codes were established in \([17]\), \([18]\), \([19]\), \([20]\). The distribution of stopping sets in Tanner graph ensembles was first studied in \([17]\). In \([18]\), using tedious combinatorial methods, Orlitsky et al. studied the asymptotic average distribution of stopping sets and the stopping number (the size of the smallest nonempty stopping set in a code) in regular and irregular LDPC code ensembles, as the code’s block length \( n \) tends to infinity. In particular, they proved that for almost all codes with smallest variable degree greater than two, stopping number increases linearly with \( n \). Milenkovic et al. \([19]\) used random matrix enumeration techniques and large deviations theory to study the asymptotic distribution of SSs and TSs of size constant and linear in \( n \), in random regular and irregular LDPC code ensembles. Specifically, for regular ensembles with variable degree \( d_v \), they demonstrated that as \( n \) tends to infinity, for constant \( a \) and \( b \) values, the average number of ETSs in the \((a, b)\) class tends.
to $cn^{a + \frac{b - ad_v}{2}}$, where $c$ is a constant with respect to $n$, but depends on $a$, $b$, and $d_v$. Using a technique similar to that of [19], Amiri et al. [20] studied the asymptotic distribution of ABSs and fully ABSs (FABSs) of size linear in $n$, in random regular LDPC code ensembles. They also derived simplified formulas for enumerating the asymptotic average number of EABSs, and FEABSs in regular LDPC code ensembles with variable degrees 3 and 4.

In this paper, we focus on the case where $a$ and $b$ are constant values (with respect to $n$). This case is of particular practical interest, as in practice, the size of the most harmful trapping sets may not necessarily increase with the code’s block length. For the scenario of constant $a$ and $b$ values, all the existing results on the asymptotic average number of trapping sets, as discussed in the previous paragraph, are consistent with the results of our analysis. In addition, for constant $a$ and $b$ values, our analysis extends some of the results in the literature. For example, while the results in [19] are limited to TSs and ETSs, in this work, we also cover LETSs, ABSs and elementary ABSs. Moreover, compared to the existing literature [18], [19], [20], one major difference is that rather than focusing on a class of trapping sets, we focus on individual non-isomorphic structures within a class. We show that such structures, although in the same class, may demonstrate completely different asymptotic behavior. For example, the average number of one structure may tend to a constant as $n$ tends to infinity, while that of another structure, within the same class, may tend to infinity. Or the asymptotic average number of one structure may be zero, while that of another structure may be a non-zero constant. If one only considers the behavior of the whole class, in the former case, the average number tends to infinity and in the latter, to a constant. The granularity of our analysis is important since it is known that, in general, different non-isomorphic trapping sets in the same class can differ in their harmfulness [21]. It is thus important to know the average number of individual non-isomorphic TS structures within a code ensemble. Finally, although the analysis in [18], [19], [20], may be conceptually easy to understand, but the derivations are tedious and the final formulas are often complicated. In contrast, at the core of our analysis is a simple, yet general, asymptotic result that the expected number of an structure $\mathcal{S}$ is equal to $\Theta(n^{|V(\mathcal{S})| - |E(\mathcal{S})|})$ where $|V(\mathcal{S})|$ and $|E(\mathcal{S})|$ are the number of nodes and the number of edges of $\mathcal{S}$, respectively. This result is then easily translated to an asymptotic result on the average multiplicity of the structure based on the number of cycles

1We use the notation $f(x) = \Theta(g(x))$, if for sufficiently large values of $x$, we have $a \times g(x) \leq f(x) \leq b \times g(x)$, for some positive $a$ and $b$ values.
TABLE I
THE ASYMPTOTIC AVERAGE NUMBER OF DIFFERENT TRAPPI NG SET STRUCTURES WITHIN \((a, b)\) CLASSES OF V ARIABLE-REGULAR LDPC CODE ENSEMBLES WITH VARIABLE DEGREE \(d_v\).

| TS Type | \(b/a < d_v - 2\) | \(b/a = d_v - 2\) | \(b/a > d_v - 2\) |
|---------|-----------------|-----------------|-----------------|
|         | \(d_v \geq 2\)  | \(d_v = 2\)     | \(d_v = 3\)     | \(d_v \geq 4\)  |
| ETSs    | 0               | \(\Theta(1)\)   | \(\infty\)     |
| LETSs   | 0               | \(\sim N_{2a}\) | –               |
| EASs    | 0               | \(\sim N_{2a}\) | –               |
| SSs     | 0               | \(\sim N_{2a}\) | –               |

in the structure. For the number of cycles equal to zero, one, or more than one, the average multiplicity tends to infinity, a non-zero constant, or zero, respectively.

In an ensemble \(\mathcal{E}\) of LDPC codes, for given \(a\) and \(b\) values, we say that the \((a, b)\) class has a consistent behavior in \(\mathcal{E}\), if all the structures within the class demonstrate the same asymptotic behavior in \(\mathcal{E}\), i.e., the average number of every non-isomorphic structure in the class tends to the same value (zero, infinity or a non-zero constant) in \(\mathcal{E}\), as \(n\) tends to infinity. Otherwise, we say that the class has an inconsistent behavior in \(\mathcal{E}\). As an example of our results, we prove that in random variable-regular LDPC codes, every class of ETSs and every class of LETSs has a consistent behavior. In irregular LDPC codes, however, classes of ETSs and LETSs have inconsistent behavior. Another consequence of our analysis is to provide approximations for the average number of TS structures that contain only one cycle. By definition, such a cycle must be chordless (simple). We use asymptotic results on the multiplicity of cycles in random regular and irregular Tanner graphs [22] to approximate the average number of simple cycles and thus the average number of corresponding structures in the ensemble. We demonstrate that these approximations are rather accurate even for individual codes selected randomly from finite-length ensembles.

In Table I, we have summarized our results on the asymptotic average number of \((a, b)\) TSs in variable-regular ensembles of LDPC codes with variable degree \(d_v\), for different types of trapping sets, and for different values of \(a\), \(b\) and \(d_v\). In Table I, the notation \(N_{2a}\) denotes the asymptotic average number of cycles of length \(2a\) in the Tanner graphs of the codes. The entry “-” in the table is used to indicate that it is impossible to have trapping sets of a particular type in the corresponding classes.
The organization of the rest of the paper is as follows: In Section II, we present some definitions and notation. This is followed in Section III by our main result on the asymptotic average number of an arbitrary local structure in the Tanner graph of random LDPC codes. This is followed by the application of the general result to different types of trapping sets including ETSs, LETSs, ABSs, EABSs and SSs, for both variable-regular and irregular LDPC code ensembles. Section IV is devoted to numerical results. The paper is concluded with some remarks in Section V.

II. Definitions and Notations

For a graph $G$, we denote the node set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. In this work, we consider graphs with no loop or parallel edges. For $v \in V(G)$ and $S \subseteq V(G)$, notations $N(v)$ and $N(S)$ are used to denote the neighbor set of $v$, and the set of nodes of $G$ which has a neighbor in $S$, respectively. A path of length $c$ in $G$ is a sequence of distinct nodes $v_1, v_2, \ldots, v_{c+1}$ in $V(G)$, such that $\{v_i, v_{i+1}\} \in E(G)$, for $1 \leq i \leq c$. A cycle of length $c$ is a sequence of distinct nodes $v_1, v_2, \ldots, v_c$ in $V(G)$ such that $v_1, v_2, \ldots, v_c$ form a path of length $c-1$, and $\{v_c, v_1\} \in E(G)$. We may refer to a path or a cycle by the set of their nodes, or by the set of their edges, or by both. A chordless or simple cycle in a graph is a cycle such that no two nodes of the cycle are connected by an edge that does not itself belong to the cycle.

A graph $G = (V, E)$ is called bipartite, if the node set $V(G)$ can be partitioned into two disjoint subsets $U$ and $W$ (i.e., $V(G) = U \cup W$ and $U \cap W = \emptyset$), such that every edge in $E$ connects a node from $U$ to a node from $W$. The Tanner graph of a low-density parity-check (LDPC) code is a bipartite graph, in which $U$ and $W$ are referred to as variable nodes and check nodes, respectively. Parameters $n$ and $n'$ in this case are used to denote $|U|$ and $|W|$, respectively. Parameter $n$ is the length of the code, and the code rate $R$ satisfies $R \geq 1 - (n'/n)$.

The number of edges incident to a node $v$ is called the degree of $v$, and is denoted by $d_v$ or $\deg(v)$. Also, we denote the maximum degree and the minimum degree of $G$ by $\Delta(G)$ and $\delta(G)$, respectively. A bipartite graph $G = (U \cup W, E)$ is called bi-regular, if all the nodes on the same side of the given bipartition have the same degree, i.e., if all the nodes in $U$ have the same degree $d_u$ and all the nodes in $W$ have the same degree $d_w$. It is clear that, for a bi-regular graph, $|U|d_u = |W|d_w = |E(G)|$. A bipartite graph that is not bi-regular is called irregular. A Tanner graph $G = (U \cup W, E)$ is called variable-regular with variable degree $d_v$, if for each variable node $u_i \in U$, $d_{u_i} = d_v$. Also, a $(d_v, d_c)$-regular Tanner graph is a variable-regular graph with variable degree $d_v$, in which for each check node $w_i \in W$, $d_{w_i} = d_c$. The variable
and check node degree distributions of irregular LDPC codes are represented by polynomials
\[ \lambda(x) = \sum_{i=d_{v_{\min}}}^{d_{v_{\max}}} \lambda_i x^{i-1} \] and \[ \rho(x) = \sum_{i=d_{c_{\min}}}^{d_{c_{\max}}} \rho_i x^{i-1} \], where \( \lambda_i \) and \( \rho_i \) denote the fraction of the edges in the Tanner graph connected to degree-\( i \) variable and check nodes, respectively, and \( d_{v_{\min}}, d_{v_{\max}}, d_{c_{\min}} \) and \( d_{c_{\max}} \) are the minimum and maximum variable and check node degrees in the Tanner graph, respectively.

In a Tanner graph \( G = (U \cup W, E) \), for a subset \( S \) of \( U \), the induced subgraph of \( S \) in \( G \), denoted by \( G(S) \), is the graph with the set of nodes \( S \cup N(S) \), and the set of edges \( \{\{u_i, w_j\} : \{u_i, w_j\} \in E(G), u_i \in S, w_j \in N(S)\} \). The set of check nodes with odd and even degrees in \( G(S) \) are denoted by \( N_o(S) \) and \( N_e(S) \), respectively. Also, the terms unsatisfied check nodes and satisfied check nodes are used to refer to the check nodes in \( N_o(S) \) and \( N_e(S) \), respectively. Throughout this paper, the size of an induced subgraph \( G(S) \) is defined to be the number of its variable nodes (i.e., \(|S|\)).

For a given Tanner graph \( G \), a set \( S \subset U \), is said to be an \((a, b)\) trapping set (TS) if \(|S| = a\) and \(|N_o(S)| = b\). Alternatively, set \( S \) is said to belong to the class of \((a, b)\) TSs. A stopping set (SS) \( S \) is a TS for which the degree of each check node in \( G(S) \) is at least two. An elementary trapping set (ETS) is a TS for which all the check nodes in \( G(S) \) have degree one or two. A leafless ETS (LETS) \( S \) is an ETS for which each variable node in \( S \) is connected to at least two satisfied check nodes in \( G(S) \). An absorbing set (ABS) \( S \) is a TS for which all the variable nodes in \( S \) are connected to more nodes in \( N_e(S) \) than in \( N_o(S) \). Also, an elementary absorbing set (EABS) \( S \) is an ABS for which all the check nodes in \( G(S) \) have degree one or two. A fully absorbing set (FABS) \( S \) is an ABS with the extra constraint that each variable node in the Tanner graph has strictly fewer unsatisfied check nodes than satisfied check nodes in its neighborhood. Also, an elementary fully absorbing set (EFABS) \( S \) is an FABS such that each check node in \( G(S) \) has degree one or two.

In the asymptotic analysis presented in this work, for a given ensemble of LDPC codes (Tanner graphs), identified by certain degree distributions and the block length \( n \), we tend \( n \) to infinity. In such an asymptotic scenario, we say that a structure \( S \) is local in a Tanner graph \( G \), if the definition of \( S \) depends on a constant number (with respect to \( n \)) of nodes in \( G \). In particular, as we are interested in subgraphs \( S = G(S) \) induced in \( G \) by a set of variable nodes \( S \), if such subgraphs are local, we refer to them as being locally induced subgraphs. For example, if one considers a constant positive integer \( a \), and Tanner graphs whose maximum variable degree is a constant in \( n \), then \((a, b)\) ETSs, \((a, b)\) LETSs, \((a, b)\) ABSs, \((a, b)\) EABSs, and \((a, b)\) SSs are
all local structures. (Note that such sets can only exist for $b \leq a \times d_{v_{\text{max}}}$, and thus $b$ is also a constant in $n$.) On the other hand, the definition of FABSs depends on all the variable nodes in the Tanner graph, and thus FABSs are not local structures. In this work, our focus is on $(a, b)$ TSs with $a$ and $b$ being constants in $n$. We also consider Tanner graphs whose maximum variable and check node degrees are constant. Thus, in this work, all ETSs, LETSs, ABSs, EABSs and SSs are local structures.

We conclude this section with the definition of $k$-permutations that will be used in the proof of our main result. The $k$-permutations of $n$ objects are the different ordered arrangements of $k$-element subsets of the $n$ objects. The number of such permutations is equal to $P(n, k) = n(n-1) \cdots (n-k+1)$.

III. Asymptotic Average Number of an Arbitrary Local Structure in LDPC Code (Tanner Graph) Ensembles

A. Main Result

In this subsection, we find a lower and an upper bound on the average number of a locally induced subgraph of a random Tanner graph with a given degree distribution, in the asymptotic regime where the size of the graph tends to infinity.

**Theorem 1.** Let $\Delta$ be a fixed natural number, such that $\Delta \geq d_1 \geq d_2 \geq \ldots \geq d_n$, and $\Delta \geq d_1' \geq d_2' \geq \ldots \geq d_n'$, and that $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n'} d_i' = \eta$. Consider the probability space $\mathcal{G}$ of all Tanner graphs with node set $(U, W)$, where $U = \{u_1, u_2, \ldots, u_n\}$, $W = \{w_1, w_2, \ldots, w_{n'}\}$, the degree of the node $u_i$ is $d_i$, and the degree of the node $w_i$ is $d_i'$. Suppose that the degree sets $\{d_i\}$ and $\{d_i'\}$ follow the distributions $\lambda(x)$ and $\rho(x)$, respectively, and that the graphs in $\mathcal{G}$ are selected uniformly at random. For $G \in \mathcal{G}$, denote by $X_S(G)$ the number of subgraphs $S$, induced by a constant number of variable nodes, in $G$. Then, in the asymptotic regime of $n \to \infty$, we have

$$E(X_S) = \Theta(n^{|V(S)|-|E(S)|}),$$

where $E(X_S)$ is the expected value of $X_S(G)$, and $|V(S)|$ and $|E(S)|$ are the number of nodes and edges of $S$, respectively.

**Proof.** First, we introduce the method that we use to construct a random Tanner graph. Suppose that the variable nodes are labeled as $\{u_1, u_2, \ldots, u_n\}$, and check nodes as $\{w_1, w_2, \ldots, w_{n'}\}$.
For each node \( z \) with degree \( d(z) \), we assign a bin that contains \( d(z) \) cells, and consider a random perfect matching (bijection) to pair the cells on one side of the graph to the cells on the other side. From each matching, we construct a Tanner graph such that if there is an edge between two bins, then we place an edge between the corresponding nodes in the Tanner graph. The resulted Tanner graphs are thus represented as images of the so-called configurations that are obtained from the random perfect matchings. There are \( N(\eta) = \eta! \) configurations, where \( \eta \) is the number of edges in the graph. Also, note that given a set of \( \ell \) fixed edges, there are \( N_\ell(\eta) = (\eta - \ell)! \) configurations containing those edges. Since each bipartite graph (which has no parallel edges) corresponds to \((d_1!)(d_2!)(d'_1!)(d'_2!)(d''_n!)(d''_{n'}!)(d''_{n''}!)(d''_{n'''}!)\) matchings, a bipartite graph can be chosen uniformly at random by choosing a matching uniformly at random and rejecting the result if it has parallel edges. We thus consider perfect matchings with no parallel edges in their image that are selected uniformly at random. Given a set of \( N \) cells in \( \pi \), we obtain an upper bound on \( \eta \). Now, consider a bipartite structure of interest \( S = (X \cup Y, E') \), such that \( X = \{x_1, x_2, \ldots, x_r\} \), \( Y = \{y_1, y_2, \ldots, y_{r'}\} \), and \( r + r' = |V(S)| \). We say there is a copy of \( S \) in a configuration corresponding to a Tanner graph \( G = (U \cup W, E) \), if there is a set of \( |E(S)| = |E'| \) edges in the configuration, whose image in \( G \) corresponds to a subgraph which is isomorphic to \( S \) \((X \subset U, Y \subset W)\). We denote the number of copies of \( S \) in a configuration by \( C_S \). We then have:

\[
E(X_S) = C_S \frac{M_{E(S)}(\eta)}{M_0(\eta)} \sim C_S \frac{N_{E(S)}(\eta)}{N(\eta)} = C_S \frac{(\eta - |E(S)|)!}{\eta!} \sim \frac{C_S}{\eta^{|E(S)|}},
\]

where, the last equation is valid asymptotically since \( |E(S)| \) is a constant in \( n \), and thus in the number of edges \( \eta \) of the Tanner graph. In the following, we derive upper and lower bounds on \( C_S \), and subsequently on \( E(X_S) \).

**Upper bound on \( C_S \):** To form a copy of the structure \( S \), we choose an \( r \)-permutation of the \( n \) bins from \( U \) and an \( r' \)-permutation of the \( n' \) bins from \( W \) (note that some of the permutations may result in the same copy of \( S \). By considering all possible \( r \)-permutations and \( r' \)-permutations, we obtain an upper bound on \( C_S \). Next, after fixing the \( r \)-permutation and the \( r' \)-permutation, for each \( i, 1 \leq i \leq r \), for the \( i^{th} \) bin in the \( r \)-permutation, we select \( d(x_i) \) cells in order. We note that if the number of cells in the \( i^{th} \) bin is not equal to \( d(x_i) \), then a copy of \( S \) cannot be formed (by definition, the degree of a variable node in a subgraph induced by a set of variable nodes must be equal to the degree of that variable node in the Tanner graph). On the check side,
for each \( i, 1 \leq i \leq r' \), for the \( i^{th} \) bin in the \( r' \)-permutation, we select \( d(y_i) \) cells in order. If the number of cells in the \( i^{th} \) bin is less than \( d(y_i) \), then a copy of \( S \) cannot be formed. (Note that by considering all possible orderings for \( d(x_i) \) cells and \( d(y_i) \) cells, we find an upper bound on \( C_S \), since some of those orderings may result in the same copy of \( S \)). The number of possible choices of the cells on the variable side is thus upper bounded by

\[
\prod_{j=1}^{r} \prod_{i=0}^{d(x_j)-1} (d(x_j) - i)
\]

that can be further bounded from above by \( \Delta|E(S)| \). Similarly, the number of choices of the cells on the check side is upper bounded by \( \Delta|E(S)| \). We therefore have

\[
C_S \leq \Delta^{|E(S)|} \times P(n, r) \times P(n, r')
\]

\[
\leq \Delta^{|E(S)|} \times n^r \times n^{r'}
\]

\[
= \Delta^{|E(S)|} \times n^{|V(S)|}.
\]  

(3)

Lower bound on \( C_S \): If any of the variable degrees of \( S \) is missing in the variable degree distribution \( \lambda(x) \) of the Tanner graph \( G \), then, no copy of \( S \) can exist in \( G \). Otherwise, assume that there are \( \alpha \) different variable degrees \( d_1, \ldots, d_\alpha \), in \( S \), and denote the number of variable nodes with degree \( d_i \) by \( r_i \). We thus have \( r_1 + \cdots + r_\alpha = r \). Denote by \( U_i \) the set of variable nodes in \( G \) with degree \( d_i \), \( i = 1, \ldots, \alpha \). By the construction of the ensemble, we have \( |U_i| \geq c_i \times n \), for some constant values \( c_i \), \( i = 1, \ldots, \alpha \). On the check side, denote the set of all check nodes with degree at least equal to the largest check degree in \( S \) by \( W' \). Here also, \( |W'| \geq c' \times n \), for some constant \( c' \). It is then easy to establish the following lower bound on \( C_S \):

\[
C_S \geq \left( |U_1| / r_1 \right) \cdots \left( |U_\alpha| / r_\alpha \right) \left( |W'| / r' \right),
\]  

(4)

where \( \binom{a}{b} = a!/(b!(a-b)! \). Using the linear lower bounds on \( |U_i|, i = 1, \ldots, \alpha \), and \( |W'| \), followed by the well-known lower bound \( \binom{a}{b} \geq \left( \frac{a}{b} \right)^b \), we obtain

\[
C_S \geq \left( \frac{c_1 n}{r_1} \right)^{r_1} \cdots \left( \frac{c_\alpha n}{r_\alpha} \right)^{r_\alpha} \left( \frac{c' n}{r'} \right)^{r'}
\]

\[
\geq \min\left\{ c_1, \ldots, c_\alpha \right\} r^r \left( \frac{c' n}{r'} \right)^{r'}
\]

\[
\geq \frac{c \times n^{|V(S)|}}{|V(S)|^{|V(S)|}},
\]

(5)

where \( c = \min\{c', c_1, \ldots, c_\alpha \} \) is a constant, and we have used \( r + r' = |V(S)| \).
The proof is then completed by combining (3) and (5) with (2), and noting that \( d_{\text{min}} \times n \leq \eta \leq \Delta \times n \), where \( d_{\text{min}} \) is the minimum variable degree in \( \lambda(x) \).

Theorem 1 shows that depending on the relative values of \(|V(S)|\) and \(|E(S)|\), the expected number of a structure \( S \) can tend to zero, infinity or a non-zero constant, as \( n \) tends to infinity. The following lemma establishes a connection between the number of cycles in \( S \), and the value of \(|V(S)| - |E(S)|\).

**Lemma 1.** Consider a graph \( S \) with the set of nodes \( V(S) \), and the set of edges \( E(S) \). If \(|V(S)| > |E(S)|\), then \( S \) does not contain any cycle. Else, if \(|V(S)| < |E(S)|\), then \( S \) contains at least two cycles, and if \(|V(S)| = |E(S)|\), then \( S \) contains only one (simple) cycle.

Based on Theorem 1 and Lemma 1 we have the following corollary.

**Corollary 1.** Consider a random ensemble of Tanner graphs and a given subgraph \( S \) induced by a constant number of variable nodes in such Tanner graphs. Depending on whether \( S \) contains at least two cycles, only one cycle, or no cycle, the average number of structure \( S \) in the ensemble tends to zero, to a positive constant, or to infinity, as the size of the graphs (length of the code) tends to infinity.

In the following subsections, we apply the results of this subsection to different categories of trapping sets, i.e., ETSs, LETSs, ABSs, EABSs and SSs, respectively.

**B. Elementary Trapping Sets (ETS)**

In this subsection, based on the general result of Theorem 1 we study the asymptotic behavior of ETSs in both regular and irregular LDPC code ensembles.

The following results show that every \((a, b)\) class of ETSs has a consistent behavior in variable-regular LDPC codes, and that the behavior is fully determined by the ratio \( b/a \) and the variable degree \( d_v \).

**Proposition 1.** Consider an \((a, b)\) ETS \( S \) in a variable-regular Tanner graph with variable degree \( d_v \). We then have

\[
|V(S)| - |E(S)| = a + \frac{b - ad_v}{2}.
\]
Proof. Suppose that $S$ is induced in a Tanner graph $G$ by the set of variable nodes $S$. By counting the number of edges in $S = G(S)$ from the variable side, we have $|E(S)| = a \times d_v$. We also have

$$|V(S)| = |S| + |N_o(S)| + |N_e(S)| = a + b + \frac{a d_v - b}{2} = a + b$$

where the second equality follows from the fact that all the satisfied and unsatisfied check nodes have degree two and one, in the ETS, respectively. The proof is then completed by subtracting $|E(S)| = a \times d_v$ from (7).

The next theorem is resulted from Theorem 1 and proposition 1, and describes the asymptotic expected number of ETSs in different $(a, b)$ classes of variable-regular LDPC code ensembles.

Theorem 2. Consider a random variable-regular LDPC code $C$ with variable degree $d_v$ and length $n$. Denote by $N^{ETS}_{a, b}$ the number of $(a, b)$ elementary trapping sets in $C$. Then, for $a$ being a constant in $n$, in the asymptotic regime where $n \to \infty$, we have:

$$\mathbb{E}(N^{ETS}_{a, b}) = \Theta(n^{a + \frac{b - a d_v}{2}}).$$

Thus, depending on whether $b/a > d_v - 2$, $b/a < d_v - 2$, or $b/a = d_v - 2$, the expected value $\mathbb{E}(N^{ETS}_{a, b})$ tends to infinity, zero, or a positive constant value in $n$.

Example 1. Figure 1 shows three ETS structures in variable-regular Tanner graphs with $d_v = 3$, each satisfying one of the conditions in Theorem 2 (Variable nodes, satisfied and unsatisfied check nodes are shown by full circles, empty and full squares, respectively.)

Unlike variable-regular Tanner graphs, in irregular Tanner graphs, classes of ETSs demonstrate an inconsistent behavior, i.e., in general, in an $(a, b)$ class, one can find at least two structures whose expected numbers tend to different values (infinity, zero or a non-zero constant), as $n \to \infty$. This is explained in the following example for the $(4, 2)$ class.

Example 2. Figure 2 shows three ETS structures, all in the $(4, 2)$ class, but each with a different asymptotic behavior. While the asymptotic expected values of the leftmost and the rightmost structures are infinity and zero, respectively, for the middle structure, the asymptotic expected value is a positive constant (see, Corollary 1).
C. Leafless Elementary Trapping Sets (LETS)

A LETS is a special case of an ETS, and thus the general results presented in Proposition 1 and Theorem 2 are also applicable to LETS structures of random variable-regular Tanner graphs. For LETSs, however, from the three scenarios of $b/a > d_v - 2$, $b/a < d_v - 2$, and $b/a = d_v - 2$, only the last two can happen. This is proved in the following lemma.

**Lemma 2.** For any LETS structure in the $(a, b)$ class of a variable-regular LDPC code with variable degree $d_v$, we have $b/a \leq d_v - 2$.

**Proof.** Consider an $(a, b)$ LETS structure induced by the set of variable nodes $S$. Counting the number of edges in the subgraph $G(S)$ from the two sides of the graph, we have $a \times d_v = b + 2|N_e(S)|$. Since $G(S)$ is a LETS structure, each variable node is connected to at least 2 satisfied check nodes. This implies $2|N_e(S)| \geq 2a$, which together with the previous equation complete the proof.
Theorem 3. Consider a random variable-regular LDPC code $C$ with variable degree $d_v$ and length $n$. Denote by $N_{a,b}^{\text{LETS}}$ the number of $(a, b)$ leafless elementary trapping sets in $C$. Then, for $a$ being a constant in $n$, in the asymptotic regime where $n \to \infty$, we have:

$$E(N_{a,b}^{\text{LETS}}) = \Theta(n^{a+\frac{b-a_d}{2}}).$$

Thus, depending on whether $b/a < d_v - 2$, or $b/a = d_v - 2$, the expected value $E(N_{a,b}^{\text{LETS}})$ tends to zero, or a positive constant value in $n$.

Based on Theorem 3, it is clear that, in the asymptotic regime, only the LETS classes with $b/a = d_v - 2$ are non-empty. All the structures in such classes contain only one cycle, and we thus have the following result.

Corollary 2. In random variable-regular LDPC codes with variable degree $d_v$ and length $n$, as $n \to \infty$, the only non-empty classes of local $(a, b)$ LETS structures are those with $b/a = d_v - 2$. For each such class, all structures within the class correspond to a simple (chordless) cycle of length $2a$.

For irregular LDPC codes, the following result is in parallel with Lemma 2.

Lemma 3. For any LETS structure $S$ in irregular LDPC codes, we have $|V(S)| \leq |E(S)|$.

Proof. To prove the lemma, we show that any LETS structure $S$ has a cycle. In $S$, each variable node is connected to at least two satisfied check nodes. On the other hand, the degree of each satisfied check node is two. Now, if we consider the subgraph formed by variable nodes, satisfied check nodes and the edges in between, we have a graph with minimum degree two. It is well-known that a graph with the minimum degree two has at least one cycle. \hfill \blacksquare

Based on Lemma 3, for irregular LDPC codes also, the asymptotic expected number of LETS structures is either zero or a constant non-zero value. The classes of LETS structures in irregular codes also demonstrate an inconsistent behavior, and among all the structures within an $(a, b)$ class, only those that correspond to simple cycles of length $2a$ have an asymptotically non-zero expected value for their multiplicity. We thus have the following result.

Proposition 2. Consider random irregular LDPC codes with $d_{\text{min}} \geq 2$, and length $n$. As $n \to \infty$, for a constant value of $a$, the sum of the expected number of LETS structures in all the $(a, b)$
classes, for different values of \( b \), tends to the expected value of the number of simple cycles of length \( 2a \) in the code.

The cycle structure of random regular and irregular Tanner graphs was studied in [22]. In particular, it was shown in [22] that the asymptotic expected value of the number of cycles of length \( c \) in irregular graphs with variable degrees \( \{d_i\}_{i=1}^n \) and check node degrees \( \{d'_i\}_{i=1}^{n'} \), as the size of the graph tends to infinity, is given by

\[
E(N_c) \sim \frac{\left( \frac{2}{\eta} \sum_{i=1}^{n} \left( \begin{array}{c} d_i \\ 2 \end{array} \right) \right) \left( \frac{2}{\eta} \sum_{i=1}^{n'} \left( \begin{array}{c} d'_i \\ 2 \end{array} \right) \right)^{c/2}}{c},
\]

(8)

where \( \eta = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n'} d'_i \) is the number of edges in the graph. This result for variable-regular graphs with variable degree \( d_v \) reduces to

\[
E(N_c) \sim \frac{\left( (d_v - 1) \left( \frac{2}{\eta} \sum_{i=1}^{n'} \left( \begin{array}{c} d'_i \\ 2 \end{array} \right) \right) \right)^{c/2}}{c},
\]

(9)

and for \( (d_v, d_c) \) bi-regular graphs to

\[
E(N_c) \sim \frac{\left( (d_v - 1)(d_c - 1) \right)^{c/2}}{c}.
\]

(10)

Considering that in the asymptotic regime of \( n \to \infty \), by Corollary 1, the expected number of cycles with chords tends to zero, one can use the above approximations for chordless cycles, and use them along with Corollary 2 and Proposition 2 to obtain asymptotic estimates on the average number of LETS structures in different classes of regular and irregular graphs, respectively.

D. Absorbing Sets (ABS)

The following result relates the number of nodes and edges of an ABS.

**Lemma 4.** For any ABS structure \( S \) within a Tanner graph with \( d_{v_{\text{min}}} \geq 2 \), we have \( |V(S)| \leq |E(S)| \).

**Proof.** Let \( S = G(S) \) be the ABS induced in the Tanner graph \( G \) by the set of variable nodes \( S \). By the definition of an ABS, each variable node in \( S \) is connected to more nodes in \( N_v(S) \) than in \( N_o(S) \). Each node in \( S \) thus has at least two neighbors in \( N_v(S) \). On the other hand, the degree of each node in \( N_v(S) \) within \( G(S) \) is at least two. Thus, if we consider the subgraph of
$G(S)$ containing the nodes in $S$ and $N_e(S)$, and the edges in between, we obtain a graph with minimum degree two. Such a graph, thus, has a cycle. This completes the proof.

Based on Theorem 1 and Lemma 4, it is clear that the average number of any ABS structure tends to either zero or a positive constant (not to infinity) as the block length tends to infinity. The following theorem distinguishes between the two cases depending on the variable degrees of the Tanner graph.

**Theorem 4.** Consider random Tanner graphs with variable degree distribution $\lambda(x)$. If $d_{v_{\text{min}}} \geq 4$, then all the classes of local ABSs have zero multiplicity asymptotically. Otherwise, if $d_{v_{\text{min}}} = 3$, then all the local $(a,b)$ classes of ABSs with $a \geq 2$ and $b \neq a$, have zero multiplicity asymptotically. In this case ($d_{v_{\text{min}}} = 3$), the structures in the local $(a,a)$ class with asymptotically non-zero multiplicity all correspond to simple cycles of length $2a$ consisting only of degree-3 variable nodes. Finally, if $d_{v_{\text{min}}} = 2$, then only the local ABS structures whose variable degrees are only 2 or 3 can have non-zero multiplicity asymptotically. In the $(a,b)$ classes with a given $a$ and different values of $b \leq a$, such structures are all simple cycles of length $2a$.

**Proof.** If $d_{v_{\text{min}}} \geq 4$, then each variable node in the ABS must be connected to at least three satisfied check nodes. Now, consider the subgraph that consists of the variable nodes of the ABS, its satisfied check nodes and the edges in between. This subgraph has minimum degree two, and a node of degree at least three, and thus contains at least two cycles. This means that the ABS itself contains at least two cycles and thus, based on Corollary 1, its multiplicity is zero asymptotically. The same argument applies to any ABS structure that has at least one node with degree greater than or equal to four (for the cases with $d_{v_{\text{min}}} = 3$ or 2).

If $d_{v_{\text{min}}} = 3$, thus, all ABS structures have asymptotically zero multiplicity, except those whose variable nodes, all, have degree three. In this case, by the definition of an ABS, we must have $b \leq a$, as each degree-3 variable node must be connected to at least two satisfied check nodes. Now consider the case where $a \geq 2$ and $b < a$. For this case, we consider two scenarios and show that for both scenarios the ABS structures will have more than one cycle and thus their multiplicity is zero asymptotically: (1) all unsatisfied check nodes have degree one, (2) at least one unsatisfied check node $c$ has degree 3 or larger. In the first scenario, since $b < a$, there must exist a variable node $v$ in the ABS that has no connection to unsatisfied check nodes. Node $v$ is thus connected to three satisfied check nodes, and with the same argument presented before,
the ABS will have more than one cycle. In the second scenario, consider the subgraph of the ABS consisting of variable nodes, satisfied check nodes and the unsatisfied check node $c$ plus all the edges in between. This subgraph has minimum degree 2 and a node with degree at least 3, and thus has more than one cycle. Therefore, for the case of $d_{v_{\min}} = 3$, the only classes of ABSs with asymptotically non-zero multiplicity are $(a, a)$ classes. The only elements within the $(a, a)$ class whose multiplicity is non-zero asymptotically are simple cycles consisting of only degree-3 variable nodes, where each variable node is connected to one unsatisfied check node of degree one, and the degree of all satisfied check nodes are two.

For graphs with $d_{v_{\min}} = 2$, the proof of the statement of the proposition is similar. In this case, the only structures whose average multiplicity tends to a constant are those with variable degrees only 2 or 3, that contain only a simple cycle.

From Theorem 4, it can be seen that any class of ABSs has a consistent behavior in any ensemble of variable-regular or irregular LDPC codes with $d_{v_{\min}} \geq 4$. The same applies to any class of $(a, b)$ ABSs with $a \neq b$ in any ensemble of LDPC codes with $d_{v_{\min}} = 3$. One can, however, easily provide examples where two ABS structures within the same $(a, a)$ class have different asymptotic behavior in an ensemble with $d_{v_{\min}} = 3$. This is demonstrated in Fig. 3 for the $(4, 4)$ class. While the structure in Fig. 3(a) has only one cycle and thus has an average multiplicity tending to a non-zero constant, the structure in Fig. 3(b) contains more than one cycle and thus its multiplicity is zero asymptotically. For ensembles with $d_{v_{\min}} = 2$, it can be seen that, in general, an $(a, b)$ class of ABSs with $a \geq 2$ and $b \leq a$, has an inconsistent behavior.
Fig. 4. Two (4, 1) ABS structures with different asymptotic behaviors in Tanner graphs with \( d_{v_{\min}} = 2 \).

(see Fig. 4 for two structures with different asymptotic behaviors in the (4, 1) class).

E. Elementary Absorbing Sets (EABS)

Elementary ABSs are a special case of ABSs. All the results presented in the previous subsection are thus applicable to EABSs as well. On the other hand, EABSs, for codes with \( d_{v_{\min}} \geq 2 \), are a special case of LETSs. Therefore, the results presented in Subsection III-C are also applicable to EABSs. In fact, for the variable-regular graphs with \( d_v = 2 \) or \( d_v = 3 \), or irregular graphs with variable degrees only 2 and 3, the sets of EABSs and LETSs are identical. In addition to the asymptotic results of Theorem 4, the following finite-length result applies to EABSs of variable-regular LDPC codes with \( d_v \geq 4 \).

**Lemma 5.** There is no \((a, b)\) EABS structure with \( b/a = d_v - 2 \), in a variable-regular Tanner graph with variable degree \( d_v \geq 4 \).

**Proof.** We note that for an EABS \( S \), the condition \( b/a = d_v - 2 \), is equivalent to \( |V(S)| = |E(S)| \), which in turn means that \( S \) contains only a simple cycle. The condition \( b/a = d_v - 2 \) thus implies that every variable node in \( S \) is connected to two satisfied and \( d_v - 2 \) unsatisfied check nodes. By the definition of an ABS, however, each variable node in \( S \) must be connected to more satisfied than unsatisfied check nodes. This implies that \( d_v - 2 < 2 \) or \( d_v < 4 \).

In the asymptotic regime of \( n \to \infty \), for variable-regular Tanner graphs with \( d_v \geq 4 \), using either Lemma 5 or Theorem 4 one can see that all \((a, b)\) EABS classes are empty. For the cases of \( d_v = 2 \) and \( d_v = 3 \), based of Corollary 2 the only non-empty classes (asymptotically) are those with \( b/a = d_v - 2 \), whose members are simple cycles of length \( 2a \). The average multiplicity of such classes can then be approximated by (9) or (10).
In general, for an irregular Tanner graph, the following result shows that among ABS structures, only those that are elementary can possibly have non–zero multiplicity in the asymptotic regime of $n \to \infty$. This implies that for graphs with $d_{v_{min}} \geq 2$, the asymptotic results presented in Theorem 4 related to ABS structures with constant average multiplicity, applies directly to EABSs.

**Proposition 3.** Any non-elementary local ABS structure $S$ in a Tanner graph with $d_{v_{min}} \geq 2$ contains more than one cycle. The multiplicity of $S$ in a random Tanner graph thus tends to zero as the size of the graph tends to infinity.

**Proof.** The non-elementary ABS structure $S$ has either (a) a satisfied check node with degree 4 or larger, or (b) an unsatisfied check node of degree 3 or larger. In addition, each variable node in $S$ is connected to at least two satisfied check nodes. For Case (a), consider the subgraph of $S$ consisting of all the variable nodes in $S$, all the satisfied check nodes and the edges in between. This subgraph has minimum degree 2 and has a node with degree 4 or larger. It thus contains more than one cycle, and so does the ABS structure $S$, itself. For Case (b), consider the subgraph of $S$ containing all the variable nodes, all the satisfied check nodes and the unsatisfied check node with degree 3 or larger, as well as all the edges in between. This subgraph has minimum degree 2 and has a node with degree 3 or larger, and thus contains more than one cycle.

---

**F. Stopping sets (SS)**

In stopping sets, each check node is connected to at least two variable nodes. For Tanner graphs with $d_{v_{min}} \geq 2$, therefore, any SS structure has a minimum degree at least 2, and as a result, contains at least one cycle. If we, however, further limit the degree distribution of Tanner graphs to $d_{v_{min}} \geq 3$, any SS structure will have a minimum degree of two and at least one node with degree 3 or larger. This implies that the SS structure will contain more than one cycle.

**Lemma 6.** Any SS structure in a Tanner graph with $d_{v_{min}} \geq 2$ contains at least one cycle, i.e., $|V(S)| \leq |E(S)|$. For Tanner graphs with $d_{v_{min}} \geq 3$, however, any SS structure contains more than one cycle, i.e., $|V(S)| < |E(S)|$.

It is easy to see that, in general, the only SS structures that contain only one cycle are those with all variable nodes having degree 2. For such stopping sets, we have the following result.
Lemma 7. Consider a stopping set that belongs to the \((a, b)\) class, and in which all the variable nodes have degree 2. We then have \(b < a\). Moreover, among all such SS structures, only those with \(b = 0\) can contain only one cycle. Those in other classes all have more than one cycle.

Proof. Since in an SS, each check node has degree at least 2, the number of check nodes in the SS (with all variable nodes having degree 2) must be less than or equal to the number of variable nodes and thus \(b \leq a\). If \(b = a\), however, since the degree of an unsatisfied check node in the SS is at least 3, then there will be \(3b\) edges connected to unsatisfied check nodes. This, alone, will be larger than the total number of edges connected to variable nodes. Thus, we must have \(b < a\). For the second part of the lemma, we notice that any SS with \(b > 0\) has at least one check node of degree 3 or larger, and thus contains more than one cycle. 

Based on Lemma 7, the only SS structures with only one cycle are codewords \((b = 0)\). In such structures, the number of variable and check nodes is the same, and all variable nodes and check nodes have degree 2. In fact, these structures are all simple cycles of length \(2a\) formed by degree-2 variable nodes.

Theorem 5. Any \((a, b)\) class of local stopping sets has a consistent behavior in any ensemble of LDPC codes with \(d_{\text{vmin}} \geq 3\), with the asymptotic multiplicity equal to zero. Moreover, the same result applies to LDPC codes with \(d_{\text{vmin}} = 2\) and stopping set classes with \(b > 0\). In ensembles with \(d_{\text{vmin}} = 2\), \((a, 0)\) classes, in general, however, do not demonstrate a consistent behavior. The only stopping set structures in these classes whose average multiplicity tends to a non-zero constant by increasing the block length are simple cycles of length \(2a\) consisting of a degree-2 variable nodes. All the remaining structures have an asymptotic multiplicity of zero.

Fig. 5 shows two stopping sets, both having only variable nodes with degree 2, and both in the \((4, 0)\) class, but with different asymptotic behavior.
IV. NUMERICAL RESULTS

In this section, we present some numerical results in relation to the theoretical results presented in previous sections. In particular, we provide the multiplicities of different trapping set structures within randomly constructed LDPC codes and compare the results to the average values predicted by the asymptotic analysis. We focus on LETS structures, and use the exhaustive search algorithm of [4] to find the multiplicity of LETS structures within different classes.

TABLE II
BI-REGULAR LDPC CODES FROM [24] USED IN THE FIRST EXPERIMENT

| Code | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $n$  | 816   | 1008  | 4000  | 20000 | 50000 | 4000  | 8000  | 10000 | 20000 |
| $R$  | 0.5   | 0.5   | 0.5   | 0.5   | 0.5   | 0.5   | 0.5   | 0.5   | 0.5   |
| $d_v$| 3     | 3     | 3     | 3     | 3     | 4     | 4     | 4     | 4     |

For the first experiment, we consider random bi-regular $(3, 6)$ and $(4, 8)$ LDPC codes with different block lengths as listed in Table II. The multiplicities of LETS structures in different $(a, b)$ classes with $a \leq 12, b \leq 5$, for the $(3, 6)$ codes, and with $a \leq 10, b \leq 10$, for the $(4, 8)$ codes, are listed in Tables III and IV respectively. In both tables, we have also listed the asymptotic average number of LETS structures within each class obtained based on Corollary [2]. For $(3, 6)$ and $(4, 8)$ codes, the asymptotic average value is non-zero only for classes with $b/a = 1$ and $b/a = 2$, respectively. For these cases, the asymptotic average multiplicity of $(a, a)$ and $(a, 2a)$ classes is approximated by (10) as $10^a/(2a)$ and $21^a/(2a)$, respectively.

The results of Tables III and IV show that the non-zero expected values provide a good approximation for the multiplicity of LETSs in $(a, a)$ and $(a, 2a)$ classes, for $(3, 6)$ and $(4, 8)$ codes, respectively, even at relatively short block lengths. The results also show that the multiplicity of other classes decrease with increasing $n$ (and tend asymptotically to zero). The rate of decrease, however, depends on the class and is faster for some classes than others.

In the second experiment, we consider irregular LDPC codes with degree distributions $\lambda(x) = 0.4286x^2 + 0.5714x^3$, and $\rho(x) = x^6$, and construct random codes of block lengths $n = 500, 1000, 4000, 10000, 20000$. All the codes have rate 0.5 and girth 6. Based on Proposition [2] the sum of the asymptotic expected values of LETSs of a given size $a$ tends to the average number of simple cycles of length $2a$. We have used the approximation (8) for the latter
TABLE III
MULTIPLICITIES OF $(a, b)$ LETSs WITHIN THE RANGE $a \leq 12$, $b \leq 5$, FOR $(3, 6)$ REGULAR LDPC CODES OF TABLE II IN COMPARISON WITH THE ASYMPTOTIC EXPECTED VALUES OF COROLLARY II

| Code $(a, a)$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | Expected value |
|--------------|-------|-------|-------|-------|-------|----------------|
| (3,3)        | 132   | 165   | 171   | 161   | 178   | 166            |
| (4,4)        | 1491  | 1252  | 1219  | 1260  | 1268  | 1250           |
| (5,5)        | 9169  | 10019 | 9935  | 10046 | 10231 | 10000          |

| Code $(a, b)$, $b \neq a$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | Expected value |
|--------------------------|-------|-------|-------|-------|-------|----------------|
| (4,2)                    | 3     | 6     | 1     | 0     | 0     | 0              |
| (5,3)                    | 90    | 100   | 21    | 2     | 1     | 0              |
| (6,2)                    | 2     | 5     | 0     | 0     | 0     | 0              |
| (6,4)                    | 2463  | 1885  | 476   | 95    | 52    | 0              |
| (7,3)                    | 110   | 116   | 10    | 0     | 1     | 0              |
| (7,5)                    | 33406 | 29736 | 7661  | 1540  | 640   | 0              |
| (8,2)                    | 1     | 3     | 0     | 0     | 0     | 0              |
| (8,4)                    | 4199  | 2961  | 183   | 4     | 1     | 0              |
| (9,3)                    | 195   | 169   | 4     | 0     | 0     | 0              |
| (9,5)                    | 84378 | 63787 | 4001  | 167   | 21    | 0              |
| (10,2)                   | 15    | 4     | 0     | 0     | 0     | 0              |
| (10,4)                   | 7965  | 4869  | 74    | 1     | 0     | 0              |
| (11,3)                   | 290   | 219   | 1     | 0     | 0     | 0              |
| (11,5)                   | 211273| 134236| 2278  | 8     | 2     | 0              |
| (12,2)                   | 15    | 6     | 0     | 0     | 0     | 0              |
| (12,4)                   | 17838 | 9041  | 36    | 0     | 0     | 0              |

(i.e., $15.42^a/(2a)$), and have provided the values for different values of $a = 3, 4, 5, 6$, in the last column of Table V. The actual sum of the multiplicities of LETSs of a given size $a$ for the five random codes are also provided in the table. The comparison shows a good match between the theoretical asymptotic value and the finite-length numerical results even at relatively short block lengths.

V. CONCLUSION

In this paper, we studied the asymptotic behavior of local structures within the Tanner graph of randomly constructed regular or irregular LDPC codes, as the code’s block length tends to infinity. Examples of such structures are different categories of trapping sets, such as stopping
sets, elementary trapping sets, or absorbing sets, that are harmful in the error floor region of LDPC codes. We derived a simple asymptotic relationship for the expected number of such structures based on the difference between the number of nodes and the number of edges of the structure. This was then related to the number of cycles in the structure, where we demonstrated that depending on the structure having, zero, one, or more than one cycle, the asymptotic expected value is infinity, a constant non-zero value, or zero, respectively. This general result was then applied to different categories of trapping sets to derive more specific results on the asymptotic expected values of different structures and classes of structures. In particular, for the case where the asymptotic expected value of a structure is a constant, we used the asymptotic results on the average number of cycles to estimate the constant value. We also demonstrated through numerical results that such estimates are rather accurate even at finite block lengths.

An important aspect of this work was to demonstrate that different (non-isomorphic) structures

| Code | \(C_6\) | \(C_7\) | \(C_8\) | \(C_9\) | Expected value |
|------|--------|--------|--------|--------|--------------|
| (3,6) | 1563   | 1620   | 1598   | 1531   | 1543         |
| (4,8) | 24269  | 24107  | 24241  | 24368  | 24310        |
| (5,10)| 402513 | 406289 | 406754 | 407743 | 408410       |
| (a, b), b \(\neq 2a\) |
| (4,6) | 91     | 55     | 33     | 17     | 0            |
| (5,6) | 2      | 2      | 1      | 0      | 0            |
| (5,8) | 4640   | 2303   | 1910   | 934    | 0            |
| (6,8) | 588    | 196    | 107    | 22     | 0            |
| (6,10)| 185544 | 96525  | 76621  | 37075  | 0            |
| (7,8) | 53     | 10     | 7      | 0      | 0            |
| (7,10)| 39794  | 10698  | 6649   | 1682   | 0            |
| (8,8) | 5      | 0      | 0      | 0      | 0            |
| (8,10)| 7006   | 1116   | 567    | 60     | 0            |
| (9,8) | 1      | 0      | 0      | 0      | 0            |
| (9,10)| 1145   | 101    | 53     | 0      | 0            |
| (10,10)| 185   | 13     | 4      | 0      | 0            |
within the same class of trapping sets can, in general, behave differently, in the asymptotic regime of infinite block length. This was not investigated in previous studies where the focus had been on the asymptotic behavior of the whole class rather than its individual members.

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