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\[ \text{When is } \text{ch}(\mathcal{K})^{(m \cdot w)} = m - 1? \]
Abstract

Let $n_m$ be the smallest integer $n$ such that $ch(K_{m,n}) = m - 1$, where $ch(G)$ denotes the choice (list chromatic) number of the graph $G$. We prove that there is an infinite sequence of integers $S$, such that if $m \in S$, then $n_m \leq 0.4643(m - 2)^{m-2}$. If $m \to \infty$, then $n_m$ is asymptotically at most $0.474(m - 2)^{m-2}$. 
1 Introduction

A list assignment of a graph $G = (V, E)$ from a family of sets (color lists) $L$ is an assignment to each vertex $v \in V(G)$ of a list $L(v)$ of colors. A $k$-list assignment is a list assignment that satisfies $|L(v)| = k$ for every $v \in V(G)$. An $L$-coloring is a function $c : V(G) \to \bigcup_{v \in V(G)} L(v)$ that assigns each vertex $v$ a color $c(v) \in L(v)$. A proper $L$-coloring is an $L$-coloring such that the neighbors of each vertex $v$ are colored in a different color than that of $v$. A graph $G$ for which there is a proper $L$-list coloring is called $L$-list choosable. The choice number $ch(G)$ is the minimum number $k$ such that for every $k$-list assignment $L$ of $G$, there is a proper $L$-coloring of $G$. The concept of choosability was introduced by Vizing in 1976 [6] and independently by Erdős, Rubin and Taylor in 1979 [1]. It is also shown in [1] that the choice number of the complete bipartite graph $K_{n,n}$ satisfies $ch(K_{n,n}) = (1 + o(1)) \log_2 n$.

In [2], Gazit and Krivelevich calculate the asymptotic value of the choice number of complete multi-partite graphs where the sizes of the different parts are not too far apart, i.e. of graphs of the form $K_{n_0,...,n_s}$, $n_0 \leq n_1 \leq \ldots \leq n_s$, where $n_0$ is not too small compared to $n_s$.

In particular, for the bi-partite case, Gazit and Krivelevich prove:

(*) Let $2 \leq n_0 \leq n_1$ be integers, and let $n_0 = (\log n_1)^{o(1)}$. Denote $k = \frac{\log n_1}{\log n_0}$. Let $x_0$ be the unique root of the equation $x - 1 - \frac{x}{\log n_0} = 0$ in the interval $[1, \infty)$. Then $\text{ch}(K_{n_0,n_1}) = (1 + o(1)) \frac{\log n_1}{\log x_0}$.

While the above-mentioned result deals with bipartite graphs $K_{m,n}$ in which $m$ is not too small compared to $n$, in this paper we consider bipartite graphs $K_{m,n}$ in which $m$ is very small compared to $n$.

It is trivial to see that $\text{ch}(K_{m,n}) = m + 1$ if $n \geq m^m$. To see this, let
(M, N) (|M| = m, |N| = m^m) be a bipartition of K_{m,m^m}. Assign m pairwise-disjoint lists of colors to M, and assign all m^m colorings of these m lists as color lists to the vertices of N. Clearly, K_{m,m^m} is not choosable from these lists. This shows that ch(K_{m,m^m}) > m (and therefore also ch(K_{m,n}) > m for n ≥ m^m). But ch(K_{m,n}) ≤ m + 1 for every n, since for every (m + 1)-list assignment L, every coloring of the vertices on M uses at most m different colors, leaving at least one color d(v) ∈ L(v) in the color list of every v ∈ N which has not been used - assigning d(v) to every v ∈ N, completes a proper L-coloring.

Hoffman and Johnson [3] proved that ch(K_{m,n}) = m if and only if (m − 1)^{m−1} − (m − 2)^{m−1} ≤ n < m^m.

Let n_m be the smallest integer n such that ch(K_{m,n}) = m − 1. In this paper we aim to find an upper bound on n_m. We will show:

**Theorem 1** If n_m = p(m − 2)^{m−2}, and m′ − 2 = k(m − 2), with k integer, then n_m′ ≤ p(m′ − 2)^{m′−2}.

**Theorem 2** There is an infinite sequence of integers S, such that if m ∈ S, then n_m ≤ 0.4643(m − 2)^{m−2}.

**Theorem 3** If m → ∞, then n_m is asymptotically at most 0.474(m − 2)^{m−2}.
2 Definitions and Preliminary Observations

We begin by giving several definitions. Note that in this paper all sets are of finite cardinality.

A hypergraph $H$ is a pair $H = (V, E)$, where $V$ is a set of elements called vertices, and $E$ is a set of subsets of $V$ called hyperedges.

A transversal of a family of sets $S$ is a set $S_t$ such that $s \cap S_t \neq \emptyset$ for all $s \in S$.

The transversal hypergraph of a family of sets $S$, which will be denoted by $T(S)$ is the hypergraph whose vertices are the union of the sets in $S$, and whose edges are all the transversals of cardinality $\leq |S|$ of $S$. The transversal hypergraph of the set of edges of a hypergraph $H$ will be denoted by $T(H)$.

Given an ordered family of sets $L$ of cardinality $m$, a track is an ordered $m$-tuple, created by choosing element $c_i$ from set $L_i$, $1 \leq i \leq m$.

Given a family of sets $S$ and a transversal $e$ of $S$, a track $t$ belongs to $e$ if the set of distinct elements of $t$ is $e$. We say that the transversal $e$ represents the track $t$.

Let $R = (V_R, E_R)$, $H = (V_H, E_H)$ be hypergraphs. An $R$-cover of $H$ is a sub-hypergraph $C$ of $R$ such that every hyperedge of $H$ contains at least one hyperedge $c \in E_C$.

A $k$-cover of a hypergraph $H$ is an $R$-cover of $H$, $R$ being all the subsets of size $k$ of the vertices of $H$ (though the subsets can also be taken from a larger set).

A minimum $R$-cover is an $R$-cover whose edge set has the least cardinality among those of all $R$-covers. The cardinality of a minimum $R$-cover of $H$ will be called $\text{cov}_R(H)$. If $H$ allows no $R$-cover, $\text{cov}_R(H) = \infty$.

A minimal $R$-cover is an $R$-cover which does not contain any other $R$-cover.
Clearly, a minimum $R$-cover is also a minimal $R$-cover. It is easy to see that the following lemma holds.

**Lemma 2.1** Let $R'$ be a minimal $R$-cover of a hypergraph $H$ with $E_H \neq \emptyset$. If $e_R \in R'$, then there is at least one edge $e_H \in E_H$ such that $e_R$ is a unique edge of $R'$ which is a subset of $e_H$.

A **minimal edge** of a hypergraph $H = (V, E)$ is an edge that does not contain any other edge in $H$.

Since the edges are of finite cardinality, every edge $e$ of $H$ contains a minimal edge (for example, an edge $h$ of $H$ of minimum cardinality contained in $e$, is obviously a minimal edge).

Removing edges from a hypergraph cannot turn a minimal edge into a non-minimal edge. Removing non-minimal edges from a hypergraph cannot turn a non-minimal edge to a minimal edge, since every non-minimal edge contains a minimal edge.

Therefore removing non-minimal edges from a hypergraph does not change the set of minimal edges of the hypergraph.

A **minimal hypergraph** is a hypergraph in which every edge is minimal (i.e. a graph in which no edge is contained in any other edge).

The **cover hypergraph** of a hypergraph $H$ is the sub-hypergraph of $H$ whose edge set is all the minimal edges of $H$.

The cover hypergraph is a minimal hypergraph, since removing edges from a hypergraph can not change a minimal edge to a non-minimal one.

Vertices $a$ and $b$, $a \neq b$ of a hypergraph $H$ are called **min-equal** if:

**Definition 1** for every minimal edge $e \in E_H$, if $a \in e$ then there is a minimal edge $e' \in E_H$ such that $a \notin e'$, $b \in e'$, and $v \in e' \Rightarrow v \in e$, and likewise if $b \in e$. 

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Definition 2 for every minimal edge \( e \in E_H \), if \( a \in e \) then \( e' = e \setminus \{a\} \cup \{b\} \) is a minimal edge of \( H \), and likewise if \( b \in e \).

It is easy to see the two definitions are equivalent.

A vertex \( a \) is min-equal to itself.

It is easy to see that min-equality is an equivalence relation.

Lemma 2.2 If \( a \) and \( b \) are min-equal vertices of \( H \), \( a \neq b \), then no minimal edge contains both \( a \) and \( b \).

Proof. If \( e \in H \) is minimal and \( a \in e \), there is a minimal edge \( e' \in E_H \) such that \( a \notin e' \), \( b \in e' \), and \( v \in e' \Rightarrow v \in e \). But then if \( b \in e \), \( e' \subseteq e \) (\( a \in e \) but \( a \notin e' \)) in contradiction to the minimality of \( e \). \( \blacksquare \)

Vertices \( a \) and \( b \) of a hypergraph \( H \) are called \( H \)-equal, if for every edge \( e \in E_H \), if \( a \in e \) and \( b \notin e \), then \( e' = e \setminus \{a\} \cup \{b\} \) is also a hyperedge of \( H \), and likewise if \( b \in e \).

3 Proof of Theorem 1

Theorem 1 Let integers \( m, m' \) satisfy \( m' - 2 = k(m - 2) \) for some integer \( k \). If \( n_m = p(m - 2)^{m-2} \) for some real \( p \), then \( n_{m'} \leq p(m' - 2)^{m'-2} \).

To prove this theorem, we shall first restate the problem of finding \( n_m \) in terms of finding minimum \( R \)-covers of hypergraphs.

Hoffman and Johnson [3] prove the following simple lemma:

Lemma 3.1 If \( L \) is a list assignment to \( K_{m,n} \), with bipartition \( (M,N) \) (\( |M| = m, |N| = n \)), then there is no proper \( L \)-coloring of \( K_{m,n} \) if and only if each transversal of the sets \( L(v), v \in M \), contains one of the sets \( L(u), u \in N \).
Corollary 3.2  Let $R_m$ be the hypergraph whose vertices are $S = \{1, 2, \ldots, m(m-2)\}$ and whose edges are all subsets of cardinality $m-2$ of $S$. Then $n_m$ is equal to the minimum over all sub-hypergraphs $R'$ of $R_m$ with $m$ edges, of $\text{cov}_{R_m}T(R')$.

Proof. Let $\min_\alpha\text{cov}(m)$ be the minimum over all sub-hypergraphs $R'$ of $R_m$ with $m$ edges, of $\text{cov}_{R_m}T(R')$. Let us take a sub-hypergraph with $m$ edges $R'$ and a minimum $R_m$-cover $R''$ of its transversal hypergraph $T(R')$, such that $|R''| = \min_\alpha\text{cov}(m)$. Then assigning the edges of $R'$ (a family of $m$ sets of size $m-2$) to the vertices of $M$, and the edges of $R''$ (a family of $\min_\alpha\text{cov}(m)$ sets of cardinality $m-2$) to the vertices of $N$, gives, by Lemma 3.1, an $(m-2)$-list assignment $L$ of $K_{m,\min_\alpha\text{cov}(m)}$ such that there is no proper $L$-coloring of $K_{m,\min_\alpha\text{cov}(m)}$, implying that $K_{m,\min_\alpha\text{cov}(m)}$ is not $(m-2)$-choosable, and $n_m \leq \min_\alpha\text{cov}(m)$.

By the definition of $n_m$, there is an $(m-2)$-list assignment $L$ to $K_{m,n_m}$ such that there is no proper $L$-coloring of $K_{m,n_m}$. The color lists assigned to $M$ in $L$ are $m$ lists of cardinality $m-2$, so there are at most $m(m-2)$ colors in their union $\bigcup_{v \in M} L(v)$, and we may label them WLOG as $\{1, 2, \ldots, m(m-2)\}$ (therefore they make up a sub-hypergraph $R'$ of $R_m$ with $m$ edges). If any of the lists $L(u), u \in N$ contains a color other than $\{1, 2, \ldots, m(m-2)\}$, then no transversal of the sets $L(v), v \in M$ contains $L(v)$. Therefore if we remove $v$ (and $L(v)$) from $N$ to get $N'$, $|N'| = n_m - 1$, Lemma 3.1 gives us $\text{ch}(K_{m,n_m-1}) > m - 2$, in contradiction to the minimality of $n_m$.

Therefore the color lists assigned to $N$ are $n_m$ subsets of cardinality $m-2$ of $\{1, 2, \ldots, m(m-2)\}$ - i.e. a sub-hypergraph $R''$ of $R_m$ of cardinality $n_m$. By Lemma 5.1 each transversal of the sets $L(v), v \in M$, contains one of the sets $L(u), u \in N$, i.e. $R''$ is an $R_m$-cover of $T(R')$, therefore $n_m \geq \text{cov}_{R_m}T(R') \geq \min_\alpha\text{cov}(m)$.
Now we shall show that when looking for a minimum $R$-cover of a hypergraph, it suffices to look for a minimum $R$-cover of its cover hypergraph.

It is easy to see that the following lemma holds:

**Lemma 3.3** Let $H, H'$ and $R$ be hypergraphs. If every $R$-cover of $H'$ is an $R$-cover of $H$, and if $R'$ is a minimum (minimal) $R$-cover of $H$ and it is also an $R$-cover of $H'$, then $R'$ is a minimum (minimal) $R$-cover of $H'$.

**Corollary 3.4** Let $H, H'$ and $R$ be hypergraphs. If for every sub-hypergraph $R'$ of $R$, $R'$ is an $R$-cover of $H$ if and only if $R'$ is an $R$-cover of $H'$, then $R'$ is a minimum (minimal) $R$-cover of $H$ if and only if $R'$ is a minimum (minimal) $R$-cover of $H'$.

**Lemma 3.5** Given a hypergraph $H$, let $H_{cov}$ be the cover hypergraph of $H$. Let $R$ be another hypergraph. Then $R'$ is an $R$-cover of $H$ if and only if $R'$ is an $R$-cover of $H_{cov}$. Therefore, by Corollary 3.4, $R'$ is a minimum (minimal) $R$-cover of $H$ if and only if $R'$ is a minimum (minimal) $R$-cover of $H_{cov}$.

**Proof.** If $R'$ is an $R$-cover of $H$ it is also an $R$-cover of $H_{cov}$, since $H_{cov}$ is a sub-hypergraph of $H$. Now let $R'$ be an $R$-cover of $H_{cov}$. We will show that for every $e \in H$ there is an $r \in R'$ which $e$ contains. If $e$ is a minimal edge then $e \in H_{cov}$, and since $R'$ is an $R$-cover of $H_{cov}$, there is an $r \in R'$ which $e$ contains. If $e$ is a non-minimal edge then $e$ contains a minimal edge $e'$. Since $e'$ is minimal it is in $H_{cov}$, therefore there is an $r' \in R'$ so that $r' \subseteq e' \subseteq e$. 

**Proposition 3.6** Let $R$ and $H$ be hypergraphs. Let $\{V_{\alpha}\}, \alpha \in A$, be a partition of $H$ into disjoint sets of min-equal, $R$-equal vertices. For every $\alpha \in A$ let us choose a representative $v_\alpha \in V_{\alpha}$, and attach a weight $w(v_\alpha) = |V_{\alpha}|$ to it.
Let $H_{\text{cov}}(\{v_\alpha\}_{\alpha \in A})$ be the sub-hypergraph of $H$ whose edges are the minimal edges of $H$ contained in $\{v_\alpha\}_{\alpha \in A}$. Then $\text{cov}_R(H)$ is equal to the minimum over all $R$-covers $R'$ of $H_{\text{cov}}(\{v_\alpha\}_{\alpha \in A})$ of $\sum_{e \in R'} \prod_{v \in e} w(v)$.

**Proof.** By Lemma 3.5, $R'$ is a minimum $R$-cover of $H$ if and only if $R'$ is a minimum $R$-cover of $H_{\text{cov}}$.

**Claim 3.7** Let $R'$ be a minimum $R$-cover of $H_{\text{cov}}$, and let $\alpha \in A$. Then $R'$ partitions the edges of $H_{\text{cov}}$ into the following $|V_\alpha| + 1$ sets:

- $W_\emptyset$ - edges that contain an $e_R \in R'$ such that $V_\alpha \cap e_R = \emptyset$;
- For each $v_\beta \in V_\alpha$, a set $W_\beta$ such that if $e_H \in W_\beta$, then $v_\beta \in e_H$, and $V_\alpha \cap e_R = \{v_\beta\}$ for every $e_R \in R'$ s.t. $e_R \subseteq e_H$.

In this partition, for every $v_\beta, v_\beta' \in V_\alpha$, $e_H \in W_\beta \Rightarrow e_H \setminus \{v_\beta\} \cup \{v_\beta'\} \in W_{\beta'}$.

**Proof.** Let $e_H$ be an edge in $H_{\text{cov}}$. Then since $e_H$ is minimal and the vertices in $V_\alpha$ are min-equal, by Lemma 2.2 either $V_\alpha \cap e_H = \emptyset$ or there is a $v_\beta \in V_\alpha$ such that $V_\alpha \cap e_H = \{v_\beta\}$. If $V_\alpha \cap e_H = \emptyset$ then $e_H \in W_\emptyset$ ($R'$ is a cover).

Suppose $V_\alpha \cap e_H = \{v_\beta\}$. If there is an $e_R \in R'$, $e_R \subseteq e_H$ such that $V_\alpha \cap e_R = \emptyset$, then $e_H \in W_\emptyset$. Otherwise, $e_H \in W_\beta$. This gives us the partition. $e_H \in W_\beta \Rightarrow e_H \setminus \{v_\beta\} \cup \{v_\beta'\} \in W_{\beta'}$ because since $v_\beta$ and $v_\beta'$ are min-equal, if $e_H \in H_{\text{cov}}$, then $e'_H = e_H \setminus \{v_\beta\} \cup \{v_\beta'\} \in H_{\text{cov}}$. Therefore either $e'_H \in W_\emptyset$ or $e'_H \in W_{\beta'}$.

But if there is an $e_R \in R'$, $e_R \subseteq e'_H$ such that $V_\alpha \cap e_R = \emptyset$, then since $v_\beta' \notin e_R$ and $e_R \subseteq e'_H$, $e_R \subseteq e_H$, in contradiction to the assumption that $e_H$ contains no such $e_R$. ■

**Claim 3.8** Let $v_\beta \in V_\alpha$. Let $R_\beta$ be the edges of $R$ that contain $v_\beta$ and do not contain any other $v_\beta' \in V_\alpha$. Then $R'$ induces a minimum $R_\beta \text{-cover } R'_\beta \ast$ on $W_\beta$, i.e. the set of $e_R \in R'$ s.t. $e_R \subseteq e_H$ for some $e_H \in W_\beta$ is a minimum $R_\beta \text{-cover of } W_\beta$. 10
Proof. The edges in \( R' \) used to cover \( W_\beta \) (i.e. those contained in edges of \( W_\beta \)) all contain \( v_\beta \) and not any other \( v_{\beta'} \in V_\alpha \), so the induced cover is an \( R_\beta \)-cover. If it is not a minimum \( R_\beta \)-cover, then we can replace the edges used to cover \( W_\beta \) with those of a minimum \( R_\beta \)-cover, and get an \( R \)-cover of smaller cardinality of \( H_{\text{cov}} \) (every edge not in \( W_\beta \) contains an edge in \( R' \) that does not contain \( v_\beta \) - so this is still an \( R \)-cover), in contradiction to the fact that \( R' \) is a minimum \( R \)-cover.

Replacing the \( R_\beta \)-cover on \( W_\beta \) with another minimum \( R_\beta \)-cover without touching the \( r \in R' \) which do not contain \( v_\beta \) still gives a minimum \( R \)-cover (since it is still a cover - we use only edges in \( R \), and it is of the same cardinality as the previous \( R \)-cover).

If \( v_\beta \) and \( v_{\beta'} \) are \( R \)-equal, then the edges in \( R_\beta \) are the edges in \( R'_\beta \) with \( v_\beta \) changed to \( v_{\beta'} \). Therefore, since there is an isomorphism between \( W_\beta \) and \( W_{\beta'} \) and between \( R_\beta \) and \( R_{\beta'} \) in which \( v_\beta \to v_{\beta'} \), and the other elements stay the same, the minimum \( R_\beta \)-covers are also isomorphic, and therefore are of the same cardinality. Thus, if we take the \( R_\beta \)-cover of \( W_\beta \) used in \( R' \), then for every \( v_{\beta'} \in V_\alpha \), replacing \( v_\beta \) in every edge of the \( R_\beta \)-cover by \( v_{\beta'} \) gives a minimum \( R_{\beta'} \)-cover of \( W_{\beta'} \), and thus we get a new minimum \( R \)-cover \( R'' \).

If \( e_H \in W_\emptyset \) and there is an \( e_R \in R'' \) such that \( e_R \subseteq e_H \) and \( e_R \cap V_\alpha = \{ v_\beta \} \), then there is an \( e'_{H} \in W_\beta \) such that \( e_R \subseteq e'_{H} \) (otherwise we can remove \( e_R \) from \( R'' \) and still get a cover). Therefore, for every \( v_\beta \in V_\alpha \), the cardinality of \( R'' \) is equal to \(|\{e_R \in R''| e_R \cap V_\alpha = \emptyset \}| + |V_\alpha|\{e_R \in R''| e_R \cap V_\alpha = \{ v_\beta \} \}|.

This is because according to the previous statement, for every \( v_{\beta'} \in V_\alpha \), if \( e'_{R} \in R'' \), then \( e'_{R} \cap V_\alpha = \{ v_{\beta'} \} \) if and only if there is an \( e_H \in W_{\beta'} \) such that \( e'_{R} \subseteq e_H \), i.e. if and only if \( e'_{R} \) belongs to the \( R_{\beta'} \)-cover of \( W_{\beta'} \) induced by \( R'' \). So \(|\{e_R \in R''| e_R \cap V_\alpha = \{ v_{\beta'} \} \}| is equal to the cardinality of the \( R_{\beta'} \)-cover of \( W_{\beta'} \) induced by \( R'' \), and these are all equal to the cardinality of the \( R_{\beta'} \)-cover.

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of $W_\beta$ induced by $R''$, which is equal to $|\{ e_R \in R'' | e_R \cap V_\alpha = \{ v_\beta \} \}|$.

Let us define $R_\emptyset$ to be the sub-hypergraph of $R$ whose edges satisfy $e_R \cap V_\alpha = \emptyset$. Given a $v_\beta \in V_\alpha$, let us define $H_{\text{cov}}\{ v_\beta \}$ to be the sub-hypergraph of $H_{\text{cov}}$ composed of all edges $e_H \in H_{\text{cov}}$ such that $e_H \cap V_\alpha \subseteq \{ v_\beta \}$.

Let $R''\{ v_\beta \}$ be the $\{ R_\emptyset \cup R_\beta \}$-cover that $R''$ induces on $H_{\text{cov}}\{ v_\beta \}$.

Observe that $\{ e_R \in R'' | e_R \cap V_\alpha = \emptyset \} = \{ e_R \in R'' | e_R \cap V_\alpha = \emptyset \}$. This is because if $e_R \in R''$ and $e_R \cap V_\alpha = \emptyset$, then since $R''$ is minimal, there is an $e_H \in W_\emptyset$ such that $e_R \subseteq e_H$. If $e_H \in H_{\text{cov}}\{ v_\beta \}$ then $e_R \in R''\{ v_\beta \}$. Otherwise, $e_H \cap V_\alpha = \{ v_\beta' \}$, and since $e_R \cap V_\alpha = \emptyset$, $e_R \subseteq e_H \setminus \{ v_\beta' \} \cup \{ v_\beta \} \in H_{\text{cov}}\{ v_\beta \}$, so $e_R \in R''\{ v_\beta \}$.

Also $\{ e_R \in R''\{ v_\beta \} | e_R \cap V_\alpha = \{ v_\beta \} \} = \{ e_R \in R'' | e_R \cap V_\alpha = \{ v_\beta \} \}$. This is because if $e_R \in R''$ and $e_R \cap V_\alpha = \{ v_\beta \}$ then since $R''$ is minimal there is an $e_H$ such that $e_R \subseteq e_H$, but then $e_H \in H_{\text{cov}}\{ v_\beta \}$ and $e_R \in R_\beta$, so $e_R \in R''\{ v_\beta \}$.

Therefore, there is an $\{ R_\emptyset \cup R_\beta \}$-cover $R^*$ of $H_{\text{cov}}\{ v_\beta \}$ such that $\text{cov}_R(H) = |\{ r \in R^* | v_\beta \notin r \}| + |V_\alpha| \times |\{ r \in R^* | v_\beta \in r \}|$.

In order to find a minimum $R$-cover of $H_{\text{cov}}$, it suffices to find an $\{ R_\emptyset \cup R_\beta \}$-cover $R^{**}$ of $H_{\text{cov}}\{ v_\beta \}$, in which $|\{ r \in R^{**} | v_\beta \notin r \}| + |V_\alpha| \times |\{ r \in R^{**} | v_\beta \in r \}|$ is minimal. This is equivalent to putting a weight of $w(v_\beta) = |V_\alpha|$ on $v_\beta$ and 1 on all other vertices in $H$, and finding a minimum weighted cover - i.e. each $r \in R$ is given a value of $\prod_{i \in r} w(i)$, and we want to find a cover for which the sum of the values of the edges in the cover is minimal.

To get an $R$-cover of $H_{\text{cov}}$ from $R^{**}$, we simply take the edges in $R^{**}$ that contain $v_\beta$ and add edges in which it is replaced by $v_\beta'$, for every $v_\beta' \in V_\alpha$. This is an $R$-cover of $H_{\text{cov}}$ because if $e_H \in \{ H_{\text{cov}} \setminus H_{\text{cov}}\{ v_\beta \} \}$, then $e_H \cap V_\alpha = \{ v_\beta \}$, so $e'_H = e_H \setminus \{ v_\beta \} \cup \{ v_\beta \} \in H_{\text{cov}}\{ v_\beta \}$. If there is an edge $e_R$ in $R^{**}$, $e_R \subseteq e_H$ such that $e_R \cap V_\alpha = \emptyset$, then $e_R \subseteq e'_H$. Otherwise there is an edge $e_R$ in $R^{**}$, $e_R \subseteq e_H$ such that $e_R \cap V_\alpha = v_\beta$, so $e_R \setminus \{ v_\beta \} \cup \{ v_\beta \} \subseteq e'_H$. 

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The cardinality of this $R$-cover will be $|\{r \in R^* | v_\beta \notin r\}| + |V_\alpha| \times |\{r \in R^* | v_\beta \in r\}| \leq |\{r \in R^* | v_\beta \notin r\}| + |V_\alpha| \times |\{r \in R^* | v_\beta \in r\}| = \text{cov}_R(H)$, and therefore it is a minimum $R$-cover.

Now given $v_\beta \in V_\alpha$ we have a problem of finding a minimum weighted $\{R_\emptyset \cup R_\beta\}$-cover $R'$ of $H_{\text{cov}}\{v_\beta\}$. Given $V_\alpha'$, we can repeat this process. Given a minimum $\{R_\emptyset \cup R_\beta\}$-cover $R'$, we divide the edges of $H_{\text{cov}}\{v_\beta\}$ into those that contain $r \in R'$ that are disjoint from $V_\alpha'$ - some of which contain only $r \in R'$ that intersect $V_\alpha$, and those that do not contain any such $r \in R'$, which are divided into $|V_\alpha'|$ isomorphic sets $W'_\beta$ that contain only one $v_\beta' \in V_\alpha'$ (and the isomorphism changes only the $v_\beta$’s). The same proof shows that this problem is identical to that in which for one $v_\beta' \in V_\alpha'$ we look only at edges $e \in H_{\text{cov}}\{v_\beta\}$ such that $e \cap V_\alpha' \subseteq \{v_\beta'\}$, put a weight on $v_\beta'$ of $|V_\alpha'|$, and look for a minimum weighted cover.

If we continue this process with all the $V_\alpha$’s, we get that to find a minimum $R$-cover of $H$, it is enough to choose one representative $v_\alpha$ for each $V_\alpha$, and look only at the sub-hypergraph $H' = H_{\text{cov}}(\{v_\alpha\}_{\alpha \in A})$ of $H$ which is composed of all the minimal edges of $H$ contained in $\{v_\alpha\}_{\alpha \in A}$. Each representative $v_\alpha$ is assigned a weight of $w(v_\alpha) = |V_\alpha|$, and we look for an $R$-cover $R'$ of $H'$ which minimizes the expression $\sum_{r \in R'} \prod_{v \in r} w(v)$.

This ends the proof of Proposition 3.6.

Now we have only two small lemmas left to prove before we reach our theorem.

**Lemma 3.9** Let $H$ be the transversal hypergraph $T(L)$ of a family of lists $L$. Let $v_1$ be a vertex in $H$ that appears in a subset $L'$ of the family of lists $L$. Then every minimal transversal in $H$ containing $v_1$ represents at least one track in which $v_1$ is chosen out of every $l \in L'$. 

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Proof. If a transversal $e$ contains $v_1$, then in every track that $e$ represents, $v_1$ is chosen out of some list $l \in L'$. Let us take such a track $t$, and create a track $t'$ by changing the element chosen out of every list $l \in L'$ to $v_1$. The transversal that represents $t'$ is contained in $e$ (we did not add elements to the transversal), so since the transversal $e$ is minimal, $t'$ belongs to $e$. \qed

Lemma 3.10 Let $H = T(L)$. Let $v_1, v_2$ be vertices in $H$. Assume that for every list $l \in L$, $v_1 \in l$ if and only if $v_2 \in l$. Then $v_1$ and $v_2$ are min-equal in $H$.

Proof. Let $L'$ be the subset of $L$ of lists that contain $v_1$ and $v_2$. Let $e \in H$ be a minimal edge in $H$, such that, WLOG, $v_1 \in e$. Then by Lemma 3.9 there is a track $t$ that belongs to $e$ in which $v_1$ is chosen out of every $l \in L'$. Let us take the track $t'$ in which we choose the same color we chose in $t$ out of every $l \in L \setminus L'$, and choose $v_2$ out of every $l \in L'$. Then the transversal of the new track is $e' = e \setminus \{v_1\} \cup \{v_2\}$.

Suppose $e'$ is not minimal. If $e'' \subseteq e'$, then $v_2 \in e''$ (otherwise $e'' \subsetneq e$). Take a minimal edge $e'' \subseteq e'$. Since $e'' \neq e'$, there is a $v \in e'$, $v \neq v_2$ (thus $v \in e$), s.t. $v \notin e''$. Since $v_2 \in e''$, by Lemma 3.9 $e''$ represents at least one track $t''$ in which $v_2$ is chosen out of every $l \in L'$. Creating a new track which is the same as $t''$ except we choose $v_1$ out of every $l \in L'$, gives a transversal which is contained in $e$ but not equal to it, in contradiction to $e$ being minimal. \qed

We are now ready to prove Theorem 1. First let us re-formulate it.

Theorem 1 Let integers $m, m'$ satisfy $m' - 2 = k(m - 2)$ for some integer $k$. Then, for every family $L_0$ of $m$ $(m - 2)$-tuples (i.e. a sub-hypergraph of
cardinality $m$ of $R_m$) such that the cardinality of a minimum $(m - 2)$-cover of $T(L_0)$ is $s(m - 2)^{m-2}$ (for $s$ real), there is a family $L'_0$ of $m'$ $(m' - 2)$-tuples such that the cardinality of a minimum $(m' - 2)$-cover of $T(L'_0)$ is at most $s(m' - 2)^{m'-2}$. Therefore, if $n_m = p(m - 2)^{m-2}$ for some real $p$, then $n_{m'} \leq p(m' - 2)^{m'-2}$.

Proof. Given a family $L_0$ of $m$ $(m - 2)$-tuples $l_i, 1 \leq i \leq m$, we create a family $L'_0$ of $m'$ $(m' - 2)$-tuples $l_i, 1 \leq i \leq m$, as follows - we take $k$ families of $(m - 2)$-tuples $L_j (1 \leq j \leq k)$ isomorphic to $L_0$, each in colors which have not been used by the previous families. Let us denote the $m$ members of $L_j$ by $l_{ji}, 1 \leq i \leq m$. We put the lists $L_j$ side by side as the first $m$ lists (i.e. the new list in the $i$-th place is $\bigcup_{1 \leq j \leq k} l_{ji}$), and add $m' - m$ lists disjoint from all others.

Let $T = T(L_0)$ (the transversal hypergraph of $L_0$).

Let $\{V_\alpha\}, \alpha \in A$, be a partition of the colors in $L_0$ into disjoint sets of colors that appear only together in the lists of $L_0$ (i.e. if $v_1, v_2 \in V_\alpha$ then for every $l \in L$, $v_1 \in l$ if and only if $v_2 \in l$). Then by Lemma 3.10 the vertices in each $V_\alpha$ are min-equal. When speaking of an $R_m$-cover, all vertices are $R_m$-equal (since the sets in $R_m$ are all subsets of size $m - 2$ of a set $S$).

Let us take a representative $v_\alpha$ from every $V_\alpha$. Each $v_\alpha$ is given a weight of $w(v_\alpha) = |V_\alpha|$.

According to Proposition 3.6 there is an $(m - 2)$-cover $R^* = T_{cov}(\{v_\alpha\}_{\alpha \in A})$ (the sub-hypergraph whose edges are the minimal edges of $T$ contained in $\{v_\alpha\}_{\alpha \in A}$) such that the cardinality of a minimum $(m - 2)$-cover of $T$, $s(m - 2)^{m-2}$, equals $\sum_{r \in R^*} \{\prod_{v \in r} w(v)\}$.

Let us use the partition $\{V_\alpha\}, \alpha \in A$, to build a partition $\{V'_\alpha\}, \alpha \in A \cup \{1, \ldots, m' - m\}$, of the colors in $L'_0$ to sets of colors that appear only together: We partition the colors in the first $m$ lists to sets
\{V'_\alpha\}, \alpha \in A$, where $V'_\alpha$ is composed of the $k$ copies of $V_\alpha$ in the isomorphisms between $L_0$ and $L_1, \ldots, L_k$ - then the colors in $V'_\alpha$ appear exactly where $V_\alpha$ appeared before (i.e. if $v \in V_\alpha$, $v \in l_i$, then the $k$ copies of $v$ are in $l'_i$; and if $v \notin l_i$, none of the copies are in $l'_i$), therefore this is a set of colors that appear only together in the lists of $L'_0$, now of cardinality $k|V_\alpha|$. Since the original $V_\alpha$'s were disjoint, so are the $V'_\alpha$'s. Our last step is to add the $m' - m$ lists $l'_i$, $m < i \leq m'$, as $m' - m$ sets in $\{V'_\alpha\}$.

According to Lemma 3.10 the vertices in each $\{V'_\alpha\}$ are min-equal. Let us take a representative $v'_\alpha$ from every $\{V'_\alpha\}$. If we let one of the copies of $L_0$ be an exact copy (for example, if $l_{i_1} = l_i$ for $1 \leq i \leq m$) then we can simply take $v_\alpha$ to represent $\{V'_\alpha\}$ in the first $m$ lists.

Let $H$ be the transversal hypergraph of $L'_0$. Let $H' = H_{\text{cov}}(\{v'_\alpha\}_{\alpha' \in A'})$ (the sub-hypergraph of $H$ composed of all minimal edges $e \in H$ such that $e \cap V'_\alpha \subseteq \{v'_\alpha\}$ for every $\alpha \in A'$).

Claim 3.11 $e' \in H'$ if and only if $e' = e \cup \bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\}$ for some $e \in T'$ (i.e. the edges of $H'$ are the edges of $T'$ to which are added the representatives of all the lists from $m + 1$ on).

Proof. Every edge in $H$ contains exactly one element of each of the lists $l'_i$, $i > m$, so an edge whose intersection with $V'_\alpha$ is a subset of $\{v'_\alpha\}$, necessarily contains $v'_\alpha$. So the set of edges from which we need to choose minimal ones is the set of all edges $e \in H$ such that $e \cap V'_\alpha \subseteq \{v'_\alpha\}$ for every $\alpha \in A$, to which are added the representatives of all the lists from $m + 1$ on. But such an edge is minimal if and only if the induced edge on the first $m$ lists is minimal (the edge sets are isomorphic). ■

Each representative $v'_\alpha$ is given a weight $w(v'_\alpha) = |V'_\alpha|$. If $v'_\alpha$ is contained in the first $m$ lists (i.e. it belongs to a copy of $V_\alpha$), then $|V'_\alpha| = k|V_\alpha|$. Otherwise
\[ |V_a'| = m' - 2 \] \((V_a'\) is then a list of the colors in \(l'_i\) for some \(i > m\)).

According to Proposition 3.6, in order to show that the cardinality of a minimum \((m' - 2)\)-cover of \(T(L'_0)\) is at most \(s(m' - 2)^{m'-2}\) it is enough to find an \((m' - 2)\)-cover \(R'\) of \(H'\) for which \(\sum_{r' \in R'} \{\prod_{v \in r'} w(v)\} = s(m' - 2)^{m'-2}\).

Let us use the \((m - 2)\)-cover \(R^*\) of \(T'\) to build such an \((m' - 2)\)-cover \(R'\) of \(H'\): \(r' \in R'\) if and only if \(r' = r \cup \bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\}\) for some \(r \in R^*\) (i.e. we add the representatives of the last \(m' - m\) lists to every \(r \in R^*\)).

This is a cover because if \(e' \in H'\) then by Claim 3.11 there is an \(e \in T'\) such that \(e' = e \cup \bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\}\). Since \(e \in T'\) there is an \(r \in R^*\) such that \(r \subseteq e\), and since \(\bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\} \subseteq e', \ r' = r \cup \bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\} \subseteq e'\).

For every \(r' \in R'\), since \(r' = r \cup \bigcup_{a \in \{1, \ldots, m'-m\}} \{v'_a\} \subseteq e'\) for some \(r \in R^*\), \(\prod_{v \in r'} w(v) = (m' - 2)^{m'-m} \prod_{v \in r} kw(v)\). Therefore, \(\sum_{r' \in R'} \{\prod_{v \in r'} w(v)\} = (m' - 2)^{m'-m} \sum_{r \in R^*} \{\prod_{v \in r} w(v)\} = (m' - 2)^{m'-m} \frac{(m'-2)^{m'-2}}{s(m'-2)^{m'-2}} = s(m' - 2)^{m'-2}\).

This ends the proof of Theorem 1.

Two immediate conclusions from Theorem 1 are:

1. For every even \(m\), \(n_m \leq \frac{1}{2}(m - 2)^{m-2}\). This is because \(n_4 = 2\) (\(ch(K_{4,2}) = 3\) and \(ch(K_{4,1}) = 2\)).

2. For every \(m\) such that \(3 | (m - 2)\), i.e., \(m \mod 3 = 2\), \(n_m \leq \frac{13}{27}(m - 2)^{m-2}\). This is because \(n_5 = 13\), as shown by Füredi, Shende and Tesman in 4 (the configuration \(\{1, 6, 7\}, \{1, 8, 9\}, \{2, 8, 7\}, \{2, 6, 9\}, \{3, 4, 5\}\) with a minimum 3-cover of its transversal graph shows \(n_5 \leq 13\)).
4 Proof of Theorem 2

We have found a list of six 4-tuples with a cover of size 123, which shows 

\[ n_6 \leq 123. \]

This list \( L_6 \) is: \( \{\{1, 3, 5, 13\}, \{1, 4, 6, 14\}, \{2, 3, 7, 15\}, \{2, 4, 8, 16\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\). Let us generalize this structure of lists and apply it to all values of \( m \), to show:

**Theorem 4.1** There is an infinite sequence of integers \( S \), such that if 

\[ m \in S, \text{ then } n_m \leq 0.4643(m - 2)^{m-2}. \]

**Proof.** Let \( k_1, k_2, k_3, k_4, \) and \( k_5 \) be integers such that \( \sum_{1 \leq i \leq 5} k_i = m - 2 \), \n
\[ 4k_4 \leq m - 2, \quad 0 \leq k_i \leq m - 2 \text{ for every } 1 \leq i \leq 5. \]

Let \( L \) be a set of \( m \) \((m - 2)\)-tuples, divided into five lists which will be described next, and \n
\( m - 5 \) lists that are disjoint from all other lists. The first five lists \( l_1, \ldots, l_5 \) are built as follows: the intersection of every threesome of these lists is empty, 

\[ |l_1 \cap l_2| = |l_3 \cap l_4| = k_1, \quad |l_1 \cap l_3| = |l_2 \cap l_4| = k_2, \quad |l_1 \cap l_4| = |l_2 \cap l_3| = k_3, \quad \text{and} \]

\[ |l_1 \cap l_5| = |l_2 \cap l_5| = |l_3 \cap l_5| = |l_4 \cap l_5| = k_4. \]

We can partition the colors of \( L \) into disjoint sets according to the lists in which they appear together as follows: 

\[ V_1 = \{l_1 \cap l_2\}, \quad V_2 = \{l_3 \cap l_4\}, \]

\[ V_3 = \{l_1 \cap l_3\}, \quad V_4 = \{l_2 \cap l_4\}, \quad V_5 = \{l_1 \cap l_4\}, \quad V_6 = \{l_2 \cap l_3\}, \quad V_7 = \{l_1 \cap l_5\}, \]

\[ V_8 = \{l_2 \cap l_5\}, \quad V_9 = \{l_3 \cap l_5\}, \quad V_{10} = \{l_4 \cap l_5\}, \]

\[ V_{10+i} = l_i \setminus \bigcup_{1 \leq j \leq 5, j \neq i} \{l_i \cap l_j\} \quad (|V_{10+i}| = k_5) \text{ for } 1 \leq i \leq 4, \]

\[ V_{15} = l_5 \setminus \bigcup_{1 \leq j \leq 4} \{l_5 \cap l_j\} \quad (|V_{15}| = m - 2 - 4k_4), \quad \text{and} \]

\[ V_{15+i} = l_{5+i} \quad (|V_{15+i}| = m - 2) \text{ for } 1 \leq i \leq m - 5. \]

According to Proposition 3.6, the cardinality of the smallest \((m - 2)\)-cover of \( T(L) \) is equal to the the value of a minimum weighted \((m - 2)\)-cover of the
transversal hypergraph of a set of lists in which we take one representative
$v_i$ from each of the $V_i$'s, and give $v_i$ weight $|V_i|$.

Taking one representative from each $V_i$, gives us a set of $m$ lists $L'$ with
the following structure:

1 3 5 7 11
1 4 6 8 12
2 3 6 9 13
2 4 5 10 14
7 8 9 10 15
16
17
...

with weights $w(1) = w(2) = k_1$, $w(3) = w(4) = k_2$, $w(5) = w(6) = k_3$,
$w(7) = w(8) = w(9) = w(10) = k_4$, $w(11) = w(12) = w(13) = w(14) = k_5$,
$w(15) = m - 2 - 4k_5$, and $w(16) = w(17) = ... = m - 2$ (there are $m - 5$
vertices of this last type).

If we set $\alpha_i = \frac{k_i}{m - 2}$, then the weight of any $(m - 2)$-tuple, divided
by $(m - 2)^{m-2}$, is a function of the $\alpha_i$'s alone - let us from now on omit
$(m - 2)^{m-2}$.

Let us describe a cover for the above hypergraph (it is possible to prove
that this is a minimal cover of this structure, but we shall not prove it in
this paper): Start with the set $T$ of all tracks of $L'$. At step $i$ take all the
transversals of cardinality $m - 2$ that belong to at least one track in $T$ as
edges in the cover. Remove the tracks that belong to these transversals from
the set $T$. Now all the tracks in $T$ belong to transversals of cardinality $m - 1$
or more, so we remove the last coordinate from every track in $T$, and continue
to step $i + 1$. We end the process when $T$ is empty (after two steps all the
tracks left have $m - 2$ vertices).

A straightforward calculation shows that the value (sum of weights) of this $(m-2)$-cover is: $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2(\alpha_4^2 + 2\alpha_4\alpha_5 + (1 - 2\alpha_4)\alpha_5^2)(\alpha_1 + \alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_5(1 - 3\alpha_4))(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 4\alpha_1\alpha_2\alpha_3(1 - 3\alpha_4) + ((\alpha_4 + \alpha_5)^4 - \alpha_5^4) + \alpha_5^4(1 - 4\alpha_4)$.

Putting in $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$, $\alpha_4 = \alpha_5 = 0$ gives $\frac{13}{27}$, which we know is the value for $m = 5$.

Putting in $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \frac{1}{3}$, $\alpha_3 = 0$ gives $\frac{123}{256}$, which is the value we calculated for $m = 6$ (this is better than Eaton’s result of $\frac{125}{256}$, mentioned in Tuza’s survey paper [5]).

Now we wish to see when this is brought to a minimum as a function of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. For a given $m$ we can only take $\alpha_i$’s such that $\alpha_i(m - 2)$ is an integer for $1 \leq i \leq 5$ - the result we get will be applicable only to such $m$’s.

Minimizing by setting $\alpha_3 = \alpha_1 + \varepsilon$ and equating the derivative of the expression as a function of $\varepsilon$ to 0, gives $\alpha_1 = \alpha_2 = \alpha_3$ or $\alpha_1 = \alpha_2$, $\alpha_3 = 0$.

In the case $\alpha_1 = \alpha_2 = \alpha_3$ we get that the minimum cover is of cardinality $\frac{13}{27} - \frac{6}{27}\alpha_4 + \frac{13}{9}\alpha_4^2 - \frac{58}{27}\alpha_4^3 + \frac{13}{9}\alpha_4^4 + \alpha_5(\frac{2}{9} - \frac{22}{9}\alpha_4 + \frac{26}{9}\alpha_4^2 + \frac{4}{3}\alpha_4^3) + \alpha_5^2(\frac{1}{9} - \frac{2}{9}\alpha_4 + \frac{10}{9}\alpha_4^2) + \alpha_5^3(-\frac{22}{27} + \frac{40}{27}\alpha_4) + \alpha_5^4(1 - 4\alpha_4)$.

Numerical optimization gives a value of 0.4642..., for $\alpha_4 = 0.1969...$, $\alpha_5 = 0.2123...$.

\[ \blacksquare \]
5 Proof of Theorem 3

Theorem 5.1 If $m \to \infty$, then $n_m$ is asymptotically at most $0.474(m - 2)^{m-2}$.

Proof. Let us take the structure of $L$ in which $|l_1 \cap l_2 \cap l_3| = k \geq 2$, $(|l_1 \cap l_2 \cap l_3 \cap l_i| = 0$ for $4 \leq i \leq m - k + 1)$, and $|l_1 \cap l_i| = 1$ for $4 \leq i \leq m - k + 2$. Also $|(l_2 \cap l_3) \setminus l_1| = l$. All the rest of the colors are different:

1 2 3... 4 5... 6 7...
1 2 3... 8 9... 10 11...
1 2 3... 8 9... 12 13...
4 14...
5 15...
...
6 17...
7 18...
...
19...
...

The cover in this case is: All $(m - 2)$-tuples that represent tracks in which the same color $c$ in $l_1 \cap l_2 \cap l_3$ is chosen from $l_1, l_2, l_3$ - there are $k(m - 2)^{m-3}$ such $(m - 2)$-tuples; All $(m - 2)$-tuples that represent tracks in which the same color $j \in (l_2 \cap l_3) \setminus l_1$ is chosen out of lists $l_2$ and $l_3$, and the color chosen out of $l_1$ is chosen again out of the other list in $l_4$, ... which contains it - there are $l((m - 2)^{m-2-k} - (m - 3)^{m-2-k}) (m - 2)^{k-1}$ such tracks; All the tracks that have not been covered yet belong to transversals of cardinality $m - 1$
- so we remove the last coordinate in every track, and take all the minimal transversals that are left - each belongs to exactly one track in which the color chosen out of \( l_1 \) is also chosen by the other list that contains it - there are \((m - 2 - k - l)^2((m - 2)^{m-2-k} - (m - 3)^{m-2-k})(m - 2)^{k-2}\) such tracks.

All in all the cover has cardinality
\[
k(m - 2)^{m-3} + l((m - 2)^{m-2-k} - (m - 3)^{m-2-k})(m - 2)^{k-1} + (m - 2 - k - l)^2((m - 2)^{m-2-k} - (m - 3)^{m-2-k})(m - 2)^{k-2} = k(m - 2)^{m-3} + (m - 2)^{k-2}[(m - 2)^{m-2-k} - (m - 3)^{m-2-k}][(m - k - 2)^2 + l^2 + 2lk - l(m - 2))].
\]
and this expression is minimal as a function of \( l \) when \( l = \frac{m - 3 - 2k}{2} \) (since \( l \) must be an integer, if \( m \) is odd we will take \( l = \frac{m - 3 - 2k}{2} \), and the rest of the proof is similar). Putting this value of \( l \) back into the expression gives
\[
k(m - 2)^{m-3} + (m - 2)^{k-2}[(m - 2)^{m-2-k} - (m - 3)^{m-2-k}][\frac{3}{4}(m - 2)^2 - k(m - 2)].
\]
When \( m \to \infty \), the expression tends to
\[
k(m - 2)^{m-3} + (m - 2)^{k-2}[(m - 2)^{m-2-k}(1 - \frac{1}{e})][\frac{3}{4}(m - 2)^2 - k(m - 2)] = (\frac{k}{e(m-2)} + \frac{3}{4}(1 - \frac{1}{e}))(m - 2)^{m-2}.
\]
This expression is minimal when \( k \) is as small as possible - in this case when \( k = 2 \). If this is the case we get that the cardinality of the cover tends to \((\frac{3}{4}(1 - \frac{1}{e}))(m - 2)^{m-2} = 0.474(m - 2)^{m-2}\).
6 Conclusion and Open Problems

In this paper we used certain structures of the family of lists assigned to $M$ ($|M| = m$) to calculate upper bounds on $n_m$, the smallest integer $n$ such that $ch(K_{m,n}) = m - 1$. These are not, of course, all the possible structures of a family of lists on $M$ with all transversals of cardinality at least $m - 2$ (the transversal hypergraph of a family of $m$ lists which contain a transversal of cardinality less than $m - 2$ does not allow an $(m - 2)$-cover and therefore does not need to be considered), and the remaining structures still need to be analyzed. The method we have devised for finding a minimum cover of a hypergraph by another hypergraph, of solving an equivalent problem of finding a minimum weighted cover, can also be used to calculate upper bounds on the smallest integer $n$ such that $ch(K_{m,n}) = m - k$ for other small $k$’s.
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