The second pinching theorem for hypersurfaces with constant mean curvature in a sphere

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Abstract We generalize the second pinching theorem for minimal hypersurfaces in a sphere due to Peng–Terng, Wei–Xu, Zhang, and Ding–Xin to the case of hypersurfaces with small constant mean curvature. Let \( M^n \) be a compact hypersurface with constant mean curvature \( H \) in \( S^{n+1} \). Denote by \( S \) the squared norm of the second fundamental form of \( M \). We prove that there exist two positive constants \( \gamma(n) \) and \( \delta(n) \) depending only on \( n \) such that if \( |H| \leq \gamma(n) \) and \( \beta(n, H) \leq S \leq \beta(n, H) + \delta(n) \), then \( S \equiv \beta(n, H) \) and \( M \) is one of the following cases: (i) \( S^k \left( \frac{k}{n} \right) \times S^{n-k} \left( \frac{n-k}{n} \right) \), \( 1 \leq k \leq n-1 \); (ii) \( S^1 \left( \frac{1}{\sqrt{1+\mu^2}} \right) \times S^{n-1} \left( \frac{\mu}{\sqrt{1+\mu^2}} \right) \). Here \( \beta(n, H) = n + \frac{n^3}{2(n-1)} H^2 + \frac{n(n-2)}{2(n-1)} n^2 H^4 + 4(n-1) H^2 \) and \( \mu = \frac{|H| + \sqrt{n^2 H^4 + 4(n-1) H^2}}{2} \).

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1 Introduction

Let \( M^n \) be an \( n \)-dimensional compact hypersurface with constant mean curvature \( H \) in an \((n+1)\)-dimensional unit sphere \( S^{n+1} \). Denote by \( S \) the squared length of the second fundamental form of \( M \) and \( R \) its scalar curvature. Then \( R = n(n-1) + n^2 H^2 - S \).
When $H = 0$, the famous pinching theorem due to Simons [12], Lawson [8], and Chern, do Carmo and Kobayashi ([2]) says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$, i.e., $M$ must be the great sphere $S^n$ or the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n - 1$. Further discussions have been carried out by many other authors (see [6,9,13,16,17,22,23], etc.). In 1970s, Chern proposed the following conjectures.

**Chern Conjecture I.** Let $M$ be a compact minimal hypersurface with constant scalar curvature in $S^{n+1}$. Then the possible values of $S$ a discrete set. In particular, if $n \leq S \leq 2n$, then $S \equiv n$, or $S \equiv 2n$.

**Chern Conjecture II.** Let $M$ be a compact minimal hypersurface in $S^{n+1}$. If $n \leq S \leq 2n$, then $S \equiv n$, or $S \equiv 2n$.

In 1983, Peng and Terng made breakthrough on the Chern conjectures I and II. They [10] proved that if $M$ is a compact minimal hypersurface with constant scalar curvature in the unit sphere $S^{n+1}$, and if $n \leq S \leq n + \frac{1}{12n}$, then $S = n$. Moreover, Peng and Terng [11] proved that if $M$ is a compact minimal hypersurface in the unit sphere $S^{n+1}$, and if $n \leq 5$ and $n \leq S \leq n + \tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on $n$, then $S \equiv n$. During the past three decades, there have been some important progress on these aspects (see [1,4,5,7,14,15,24,25], etc.). In 1993, Chang [1] solved Chern Conjecture I for the case of dimension 3. In [4,24], Cheng, Ishikawa and Yang obtained some interesting results on the Chern conjectures.

In 2007, Suh–Yang and Wei–Xu made some progress on Chern Conjectures, respectively. Suh and Yang [14] proved that if $M$ is a compact minimal hypersurface with constant scalar curvature in $S^{n+1}$, and if $n \leq S \leq n + \frac{3}{7}n$, then $S = n$ and $M$ is a minimal Clifford torus. Meanwhile, Wei and Xu [15] proved that if $M$ is a compact minimal hypersurface in $S^{n+1}$, $n = 6, 7$, and if $n \leq S \leq n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on $n$, then $S \equiv n$ and $M$ is a minimal Clifford torus. Later, Zhang [25] extended the second pinching theorem due to Peng–Terng [11] and Wei–Xu [15] to 8-dimensional compact minimal hypersurfaces in a unit sphere. Recently Ding and Xin [7] obtained the following pinching theorem for $n$-dimensional minimal hypersurfaces in a sphere.

**Theorem A** Let $M$ be an $n$-dimensional compact minimal hypersurface in a unit sphere $S^{n+1}$, and $S$ the squared length of the second fundamental form of $M$. Then there exists a positive constant $\tau(n)$ depending only on $n$ such that if $n \leq S \leq n + \tau(n)$, then $S \equiv n$, i.e., $M$ is a Clifford torus.

The pinching phenomenon for hypersurfaces of constant mean curvature in spheres is much more complicated than the minimal hypersurface case (see [16,18]). In [16], Xu proved the following pinching theorem for submanifolds with parallel mean curvature in a sphere.

**Theorem B** Let $M$ be an $n$-dimensional compact submanifold with parallel mean curvature vector ($H \neq 0$) in an $(n+p)$-dimensional unit sphere $S^{n+p}$. If $S \leq \alpha(n, H)$, then either $M$ is pseudo-umbilical, or $S \equiv \alpha(n, H)$ and $M$ is the isoparametric hypersurface $S^{n-1}(\frac{1}{\sqrt{1+k^2}}) \times S^{1}(\frac{\lambda}{\sqrt{1+\lambda^2}})$ in a great sphere $S^{n+1}$. In particular, if $M$ is a compact hypersurface with constant mean curvature $H(\neq 0)$ in $S^{n+1}$, then $M$ is either
a totally umbilical sphere \( S^n(\frac{1}{\sqrt{1+H^2}}) \), or a Clifford hypersurface \( S^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times S^1(\frac{\lambda}{\sqrt{1+\lambda^2}}) \). Here \( \alpha(n, H) = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)} \) and \( \lambda = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)} \).

In [19], Xu and Tian generalized Suh–Yang’s pinching theorem [14] to the case where \( M \) is a compact hypersurface with constant scalar curvature and small constant mean curvature in \( S^{n+1} \). The following second pinching theorem for hypersurfaces with small constant mean curvature was proved for \( n \leq 7 \) by Cheng et al. [3] and Xu–Zhao [20] respectively, and for \( n = 8 \) by Xu [21].

**Theorem C** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant mean curvature \( H(\neq 0) \) in a unit sphere \( S^{n+1} \), \( n \leq 8 \). Then there exist two positive constants \( \gamma_0(n) \) and \( \delta_0(n) \) depending only on \( n \) such that if \( |H| \leq \gamma_0(n) \), and \( \beta(n, H) \leq S < \beta(n, H) + \delta_0(n) \), then \( S \equiv \beta(n, H) \) and \( M = S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}}) \). Here \( \beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}H^2 \) and \( \mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2} \).

In this paper, we prove the second pinching theorem for \( n \)-dimensional hypersurfaces with constant mean curvature, which is a generalization of Theorems A and C.

**Main Theorem.** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant mean curvature \( H \) in a unit sphere \( S^{n+1} \). Then there exist two positive constants \( \gamma(n) \) and \( \delta(n) \) depending only on \( n \) such that if \( |H| \leq \gamma(n) \), and \( \beta(n, H) \leq S \leq \beta(n, H) + \delta(n) \), then \( S \equiv \beta(n, H) \) and \( M \) is one of the following cases: (i) \( S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}), 1 \leq k \leq n-1 \); (ii) \( S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}}) \). Here \( \beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}H^2 \) and \( \mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2} \).

**2 Preliminaries**

Let \( M^n \) be an \( n \)-dimensional compact hypersurface with constant mean curvature in a unit sphere \( S^{n+1} \). We shall make use of the following convention on the range of indices.

\[
1 \leq A, B, C, \ldots, \leq n+1, \quad 1 \leq i, j, k, \ldots, \leq n.
\]

For an arbitrary fixed point \( x \in M \subset S^{n+1} \), we choose an orthonormal local frame field \( \{e_A\} \) in \( S^{n+1} \) such that \( e_i \)’s are tangent to \( M \). Let \( \{\omega_A\} \) be the dual frame fields of \( \{e_A\} \) and \( \{\omega_{AB}\} \) the connection 1-forms of \( S^{n+1} \). Restricting to \( M \), we have

\[
\omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\]
Let $h$ be the second fundamental form of $M$. Denote by $R$, $H$ and $S$ the scalar curvature, mean curvature and squared length of the second fundamental form of $M$, respectively. Then we have

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad (2)$$

$$S = \sum_{i,j} h_{ij}^2, \quad H = \frac{1}{n} \sum_i h_{ii}, \quad (3)$$

$$R = n(n-1) + n^2 H^2 - S. \quad (4)$$

We choose $e_{n+1}$ such that $H = \frac{1}{n} \sum_i h_{ii} \geq 0$. Denote by $h_{ijk}$, $h_{ijkl}$ and $h_{ijklm}$ the first, second and third covariant derivatives of the second fundamental tensor $h_{ij}$, respectively. Then we have

$$\nabla h = \sum_{i,j,k} h_{ijk} \omega_i \otimes \omega_j \otimes \omega_k, \quad h_{ijk} = h_{ikj}, \quad (5)$$

$$h_{ijkl} = h_{ijk} + \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \quad (6)$$

$$h_{ijklm} = h_{ikml} + \sum_r h_{rjk} R_{rilm} + \sum_r h_{rkm} R_{rjlm} + \sum_r h_{irk} R_{rklm}. \quad (7)$$

At each fixed point $x \in M$, we take orthonormal frames $\{e_i\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ for all $i$, $j$. Then $\sum_i \lambda_i = nH$ and $\sum_i \lambda_i^2 = S$. By a direct computation, we have

$$\frac{1}{2} \Delta S = S(n-S) - n^2 H^2 + nHf_3 + |\nabla h|^2, \quad (8)$$

$$\frac{1}{2} \Delta |\nabla h|^2 = (2n + 3 - S)|\nabla h|^2 - \frac{3}{2} |\nabla S|^2 + |\nabla^2 h|^2 + \sum_{i,j,k,l,m} (6h_{ijk}h_{ilm}h_{jil}h_{km} - 3h_{ijk}h_{ijl}h_{km}h_{mi})$$

$$+ 3nH \sum_{i,j,k,l} h_{ijk}h_{ijk}h_{ii}$$

$$= (2n + 3 - S)|\nabla h|^2 - \frac{3}{2} |\nabla S|^2 + |\nabla^2 h|^2 + 3(2B - A) + 3nHC, \quad (9)$$

where

$$f_k = \sum_i \lambda_i^k, \quad A = \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j, \quad B = \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j \lambda_k, \quad C = \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j \lambda_k.$$

Using a similar method as in [10], we obtain

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The second pinching theorem

\[ h_{ijij} = h_{jiji} + t_{ij}, \]  
\[ |\nabla^2 h|^2 \geq \frac{3}{4} \sum_{i \neq j} t_{ij}^2 = \frac{3}{4} \sum_{i,j} t_{ij}^2, \]  
(10)  
(11)

and

\[ 3(A - 2B) \leq a |\nabla h|^2, \]  
(12)

where \( t_{ij} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j) \) and \( a = \sqrt{\frac{17}{2}} + 1. \) From (11), we have

\[ |\nabla^2 h|^2 \geq \frac{3}{2} \left[ Sf_4 - f_3^2 - S^2 - S(S - n) - n^2 H^2 + 2n H f_3 \right]. \]  
(13)

By a computation, we obtain

\[
\frac{1}{3} \sum_{i,j} h_{ij}(f_3)_{ij} = \frac{1}{3} \sum_k \lambda_k (f_3)_{kk} \\
= \sum_k \lambda_k \left( \sum_i h_{iikk} \lambda_i^2 + 2 \sum_{i,j} h_{ijk}^2 \lambda_i \right) \\
= \sum_{i,k} h_{iikk} \lambda_k \lambda_i^2 + 2 \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_k \\
= \sum_{i,k} [h_{kkii} + (\lambda_i - \lambda_k)(1 + \lambda_i \lambda_k)] \lambda_k \lambda_i^2 + 2B \\
= \sum_i \left( \frac{S_{ii}}{2} - \sum_{j,k} h_{ijk}^2 \right) \lambda_i^2 + \sum_{i,k} \lambda_i^2 \lambda_k (\lambda_i - \lambda_k)(1 + \lambda_i \lambda_k) + 2B \\
= \sum_{i,j,k} \frac{h_{ijk} h_{kj}}{2} S_{ij} + n H f_3 - S^2 - f_3^2 + Sf_4 - (A - 2B). \]  
(14)

Since \( \int_M \sum_{i,j} h_{ij}(f_3)_{ij} dM = 0, \) we drive the following integral formula.

\[
\int_M (A - 2B) dM = \int_M \left( n H f_3 - S^2 - f_3^2 + Sf_4 + \sum_{i,j,k} \frac{h_{ijk} h_{kj}}{2} S_{ij} \right) dM \\
= \int_M \left( n H f_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} (h_{ijk} h_{kj}) j S_{ij} \right) dM \\
= \int_M \left( n H f_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ijk} h_{kj} \frac{S_{ij}}{2} \right) dM.
\]
\[-\sum_{i,j,k} h_{ik} h_{kj} \frac{S_i}{2} \] 
\[= \int_M \left( n H f_3 - S^2 - f_3^2 + S f_4 - \sum_{i,j,k} h_{ik} h_{kj} \frac{S_i}{2} \right) dM \]
\[= \int_M \left( n H f_3 - S^2 - f_3^2 + S f_4 - |\nabla S|^2 \right) dM. \quad (15)\]

3 Proof of Main Theorem

The key to the proof of Main Theorem is to establish some integral equalities and inequalities on the second fundamental form of \(M\) and its covariant derivatives by the parameter method.

To simplify the computation, we introduce the tracefree second fundamental form \(\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j\), where \(\phi_{ij} = h_{ij} - H \delta_{ij}\). If \(h_{ij} = \lambda_i \delta_{ij}\), then \(\phi_{ij} = \mu_i \delta_{ij}\), where \(\mu_i = \lambda_i - H\). Putting \(\Phi = |\phi|^2\) and \(\bar{f}_k = \sum_i \mu_k^i\), we get \(\Phi = S - n H^2\), \(f_3 = \bar{f}_3 + 3H\Phi + n H^3\) and \(f_4 = \bar{f}_4 + 4H \bar{f}_3 + 6H^2\Phi + n H^4\). From (8), we obtain

\[\frac{1}{2} \Delta \Phi = S(n - S) - n^2 H^2 + n H f_3 + |\nabla h|^2 \]
\[= -\Phi^2 + n \Phi + n H \bar{f}_3 + n H^2 \Phi + |\nabla \phi|^2 \]
\[= -F(\Phi) + |\nabla \phi|^2, \quad (16)\]

where \(F(\Phi) = \Phi^2 - n \Phi - n H^2 \Phi - n H \bar{f}_3\). Therefore, we have

\[|\nabla \phi|^2 = \frac{1}{2} \Delta \Phi^2 - \Phi \Delta \Phi = \frac{1}{2} \Delta \Phi^2 + 2 \Phi F(\Phi) - 2 \Phi |\nabla \phi|^2, \quad (17)\]

and

\[\int_M F(\Phi) dM = \int_M |\nabla \phi|^2 dM. \quad (18)\]

Lemma 1 (See [16]) Let \(a_1, a_2, \ldots, a_n\) be real numbers satisfying \(\sum_i a_i = 0\) and \(\sum_i a_i^2 = a\). Then

\[\left| \sum_i a_i^3 \right| \leq \frac{n - 2}{\sqrt{n(n - 1)}} a^\frac{3}{2}, \]

and the equality holds if and only if at least \(n - 1\) numbers of \(a_i\)’s are same with each other.

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From Lemma 1, we get

\[ F(\Phi) \geq \Phi^2 - n\Phi - nH^2\Phi - \frac{n(n-2)H\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \]

\[ = \Phi \left[ \Phi - \frac{n(n-2)H\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1 + H^2) \right] \]

\[ \geq 0, \]  

(19)

provided

\[ \Phi \geq \beta_0(n, H) := n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2 - nH^2}. \]

Moreover, \( F(\Phi) = 0 \) if and only if \( \Phi = \beta_0(n, H) \).

Set

\[ G = \sum_{i,j} (\lambda_i - \lambda_j)^2(1 + \lambda_i\lambda_j)^2. \]

Then we have

\[ G = 2[Sf_4 - f_3^2 - S^2 - S(S - n) + 2nf_3 - n^2H^2]. \]  

(20)

This together with (8) and (15) implies

\[ \frac{1}{2} \int_M GdM = \int_M \left[ (A - 2B) - |\nabla h|^2 + \frac{1}{4}|\nabla S|^2 \right] dM. \]  

(21)

**Lemma 2**  
Let \( M \) be an \( n (\geq 4) \)-dimensional compact hypersurface with constant mean curvature in \( S^{n+1} \). If \( S \geq \beta(n, H) \), then we have

\[ 3(A - 2B) \leq 2S|\nabla h|^2 + C_1(n)|\nabla h|^2G^{\frac{1}{2}}, \]

where \( C_1(n) = (\sqrt{17} - 3)[6(\sqrt{17} + 1)]^{-\frac{1}{2}}(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{\sqrt{17}} - \frac{1}{n})^{-\frac{3}{2}}. \)

**Proof** We derive the estimate above at each fixed point \( x \in M \). If \( \lambda_i^2 - 4\lambda_i\lambda_j \leq 2S \) for all \( i \neq j \), then we get the desired estimate immediately. Otherwise, we assume that there exist \( i \neq j \), such that \( \lambda_i^2 - 4\lambda_i\lambda_j = tS > 2S \).

We get

\[ S \geq \lambda_i^2 + \lambda_j^2 = \left( \frac{tS - \lambda_j^2}{4\lambda_j} \right)^2 + \lambda_j^2. \]  

(22)
Then
\[
\lambda_j^2 \leq \frac{1}{17} \left( t + 8 + 4\sqrt{4 + t - t^2} \right) S, \quad 2 < t \leq \frac{\sqrt{17} + 1}{2},
\]  
(23)
which implies
\[
-\lambda_i \lambda_j \geq \frac{1}{17} \left( 4t - 2 - \sqrt{4 + t - t^2} \right) S \geq 0.26S > \frac{S}{n} \geq 1.
\]  
(24)
On the other hand, we have
\[
(\lambda_i - \lambda_j)^2 = \left( \frac{\lambda_j}{2} + \lambda_i \right)^2 + \frac{3}{4} \left( \lambda_j^2 - 4\lambda_i \lambda_j \right) \geq \frac{3t^2}{4S}.
\]  
(25)
By the definition of \( G \), we get
\[
G \geq 2(\lambda_i - \lambda_j)^2(1 + \lambda_i \lambda_j)^2 \\
\geq \frac{3t^2}{2} S(1 + \lambda_i \lambda_j)^2 \\
\geq \frac{3t^2}{2} S \left( -\lambda_i \lambda_j - \frac{S}{n} \right)^2 \\
\geq \frac{3t^2}{2} \left[ \frac{1}{17} \left( 4t - 2 - \sqrt{4 + t - t^2} \right) - \frac{1}{n} \right]^2 S^3.
\]  
(26)
We define an auxiliary function
\[
\zeta(t) = \frac{t}{(t - 2)^3} \left[ \frac{1}{17} \left( 4t - 2 - \sqrt{4 + t - t^2} \right) - \frac{1}{n} \right]^2, \quad 2 < t \leq \frac{\sqrt{17} + 1}{2}.
\]
Then we have
\[
\zeta(t) \geq \frac{t}{(t - 2)^3} \left[ \frac{1}{17} \left( 4t - 2 - \sqrt{2} \right) - \frac{1}{n} \right]^2 \\
\geq \frac{t}{(t - 2)^3} \left[ \frac{1}{17} \left( 4t - 2 - \sqrt{2} \right) - \frac{1}{n} \right]^2 \\
= \frac{4(\sqrt{17} + 1)}{(\sqrt{17} - 3)^3} \left( \frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n} \right)^2.
\]  
(27)
Hence
\[
(\lambda_j^2 - 4\lambda_i \lambda_j - 2S)^3 = (t - 2)^3 S^3 \\
\leq \frac{2G}{3\zeta(t)}.
\]
This implies
\[
3(A - 2B) \leq \sum_{i,j,k \text{ distinct}} \left[ 2 \left( \lambda_i^2 + \lambda_j^2 + \lambda_k^2 \right) - \left( \lambda_i + \lambda_j + \lambda_k \right)^2 \right] h_{ijk}^2 + 3 \sum_{i \neq j} (\lambda_j^2 - 4\lambda_i\lambda_j) h_{ijj}^2 \\
\leq 2S \sum_{i,j,k \text{ distinct}} h_{ijk}^2 + 3 \sum_{i \neq j} h_{ijj}^2 \left( 2S + C_1(n)G^{\frac{1}{3}} \right) \\
\leq 2S|\nabla^2 h|^2 + C_1(n)|\nabla h|^2 G^{\frac{1}{3}}. \tag{29}
\]

\[\square\]

**Proof of Main Theorem**

(i) When \( H = 0 \), the assertion follows from Theorem A.

(ii) When \( H \neq 0 \), the assertion for lower dimensional cases \((n \leq 8)\) was verified in [3,20,21]. We consider the case for \( n \geq 4 \). From (10) and (11), we see that \( G = \sum_i h_{ii}^2 \) and \(|\nabla^2 h|^2 \geq \frac{3}{4} G \). Letting \( 0 < \theta < 1 \), we have

\[
\int_M |\nabla^2 h|^2 dM \geq \left[ \frac{3(1 - \theta)}{4} + \frac{3\theta}{4} \right] \int_M G dM. \tag{30}
\]

From (9), (21), Lemma 2 and Young’s inequality, we drive the following inequality.

\[
\frac{3(1 - \theta)}{4} \int_M G dM \\
\leq \int_M \left[ (S - 2n - 3)|\nabla h|^2 + \frac{3}{2} |\nabla S|^2 + 3(A - 2B) - 3nHC - \frac{3\theta}{4} G \right] dM \\
= \int_M \left( S - 2n - 3 + \frac{3\theta}{2} \right) |\nabla h|^2 dM + \left( 3 - \frac{3\theta}{2} \right) \int_M (A - 2B) dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM - 3nH \int_M C dM \\
\leq \int_M \left( S - 2n - 3 + \frac{3\theta}{2} \right) |\nabla h|^2 dM + \left( 1 - \frac{\theta}{2} \right) \int_M (2S|\nabla h|^2 \\
+ C_1(n)|\nabla h|^2 G^{\frac{1}{3}}) dM + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM - 3nH \int_M C dM
\]
\[
\leq \int_M \left[ (3 - \theta)S - 2n - 3 + \frac{3\theta}{2} \right] |\nabla h|^2 dM + \frac{3(1 - \theta)}{4} \int_M G dM \\
+ C_2(n, \theta) \int_M |\nabla h|^3 dM + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM \\
- 3nH \int M C dM, \tag{31}
\]

where \( C_2(n, \theta) = \frac{4}{9} C_1(n)^{\frac{3}{2}} (1 - \frac{\theta}{2})^{\frac{3}{2}} (1 - \theta)^{-\frac{1}{2}} \).

Letting \( \epsilon > 0 \), from (16), we get

\[
\int_M |\nabla h|^3 dM = \int_M |\nabla \phi|^3 dM \\
= \int_M |\nabla \phi| \left( F(\Phi) + \frac{1}{2} \Delta \Phi \right) \, dM \\
= \int_M F(\Phi) |\nabla \phi| \, dM - \frac{1}{2} \int_M \nabla |\nabla \phi| \cdot \nabla \Phi \, dM \\
\leq \int_M F(\Phi) |\nabla \phi| \, dM + \epsilon \int_M |\nabla^2 \phi|^2 \, dM + \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 \, dM. \tag{32}
\]

Since

\[
|C| \leq \sqrt{3} |\nabla h|^2, \tag{33}
\]

we have

\[
0 \leq \int_M \left[ (3 + 3\sqrt{nH} - \theta)(\Phi + nH^2) - 2n - 3 + \frac{3\theta}{2} \right] |\nabla \phi|^2 \, dM \\
+ C_2(n, \theta) \left[ \int_M F(\Phi) |\nabla \phi| \, dM + \epsilon \int_M |\nabla^2 \phi|^2 \, dM + \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 \, dM \right] + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla \Phi|^2 \, dM. \tag{34}
\]
Substituting (12) and (33) into (9), we have

\[ \int_M |\nabla^2 \phi|^2 dM = \int_M |\nabla^2 h|^2 dM \]

\[ \leq \int_M \left[ (S-2n-3)|\nabla h|^2 + \frac{3}{2} |\nabla S|^2 + aS|\nabla h|^2 - 3nHC \right] dM \]

\[ \leq \int_M [ (a+1+3\sqrt{nH})S-2n-3]|\nabla \phi|^2 dM + \frac{3}{2} \int_M |\nabla S|^2 dM. \]

(35)

Combining (16) and (17), we have

\[ \int_M \frac{1}{2} |\nabla \Phi|^2 dM = \int_M \Phi F(\Phi) dM - \int_M \Phi |\nabla \phi|^2 dM + \beta_0(n, H) \int_M |\nabla \phi|^2 dM \]

\[ - \beta_0(n, H) \int_M F(\Phi) dM \]

\[ = \int_M (\Phi - \beta_0(n, H)) F(\Phi) dM + \int_M (\beta_0(n, H) - \Phi) |\nabla \phi|^2 dM. \]

(36)

Hence

\[ 0 \leq \int_M \left\{ 3 + 3\sqrt{nH} - \theta + \epsilon C_2(n, \theta)(a + 1 + 3\sqrt{nH}) \right\} (\Phi - \beta_0(n, H)) \]

\[ + \beta(n, H) \left[ 3 + 3\sqrt{nH} - \theta + \epsilon C_2(n, \theta)(a + 1 + 3\sqrt{nH}) \right] \]

\[ - 2 \left( \frac{3}{2} - \frac{3\theta}{8} + \frac{C_2(n, \theta)}{16\epsilon} + \frac{3\epsilon C_2(n, \theta)}{2} \right) (\Phi - \beta_0(n, H)) \]

\[ - 2n - 3 + \frac{3\theta}{2} - \epsilon C_2(n, \theta)(2n + 3) \} |\nabla \phi|^2 dM \]

\[ + 2 \left( \frac{3}{2} - \frac{3\theta}{8} + \frac{C_2(n, \theta)}{16\epsilon} + \frac{3\epsilon C_2(n, \theta)}{2} \right) \int_M (\Phi - \beta_0(n, H)) F(\Phi) dM \]

\[ + C_2(n, \theta) \int_M F(\Phi)|\nabla \phi|dM \]

\[ = \int_M \left\{ D(n, H) \left[ 3 + 3\sqrt{nH} - \theta + \epsilon C_2(n, \theta)(a + 1 + 3\sqrt{nH}) \right] \]

\[ + (1-\theta)n - 3 + \frac{3\theta}{2} + 3n^2H + \epsilon C_2(n, \theta)(an + 3n^2H - n - 3) \} |\nabla \phi|^2 dM \]
\[-\left(\frac{\theta}{4} + \frac{C_2(n, \theta)}{8\epsilon} - 3\sqrt{nH}\right) + \epsilon C_2(n, \theta) \left(2 - a - 3\sqrt{nH}\right)\int_M (\Phi - \beta_0(n, H))|\nabla \phi|^2 dM \]
\[+ \left(3 - \frac{3\theta}{4} + \frac{C_2(n, \theta)}{8\epsilon} + 3\epsilon C_2(n, \theta)\right)\int_M (\Phi - \beta_0(n, H))F(\Phi) dM \]
\[+ C_2(n, \theta) \int_M F(\Phi)|\nabla \phi| dM,\]  

(37)

where \(\beta(n, H) = \beta_0(n, H) + nH^2\) and \(D(n, H) = \beta(n, H) - n\).

Note that
\[\frac{\theta}{4} + \frac{C_2(n, \theta)}{8\epsilon} - 3\sqrt{nH} + \epsilon C_2(n, \theta) \left(2 - a - 3\sqrt{nH}\right) \geq 0,\]  

(38)

for all \(\epsilon \in (0, \epsilon_1]\), where \(\epsilon_1\) is some positive constant. When \(\beta(n, H) \leq S \leq \beta(n, H) + \epsilon^2\), we obtain
\[0 \leq \int_M \left[ (1 - \theta)n - 3 + \frac{3\theta}{2} + 3n^2 H + D(n, H)(3 + 3\sqrt{nH} - \theta) \right. \]
\[\left. + O(\epsilon, \theta, H) \right]|\nabla \phi|^2 dM + C_2(n, \theta) \int_M F(\Phi)|\nabla \phi| dM,\]  

(39)

where
\[O(\epsilon, \theta, H) = \epsilon D(n, H)C_2(n, \theta) \left(a + 1 + 3\sqrt{nH}\right) \]
\[+ \epsilon C_2(n, \theta)(an + 3n^2 H - n - 3) \]
\[+ \epsilon^2 \left(3 - \frac{3\theta}{4} + \frac{C_2(n, \theta)}{8\epsilon} + 3\epsilon C_2(n, \theta)\right).\]

On the other hand, we have
\[C_2(n, \theta) \int_M F(\Phi)|\nabla \phi| dM \]
\[\leq \frac{3}{8} \int_M F(\Phi) dM + \frac{2C_2(n, \theta)^2}{3} \int_M F(\Phi)|\nabla \phi|^2 dM.\]  

(40)
Using Lemma 1, we drive an upper bound for $F(\Phi)$.

\[
F(\Phi) \leq \Phi^2 - n\Phi - nH^2\Phi + \frac{n(n - 2)H\Phi^\frac{1}{2}}{\sqrt{n(n - 1)}}
\]

\[
= \Phi \left[ \Phi + \frac{n(n - 2)H\Phi^\frac{1}{2}}{\sqrt{n(n - 1)}} - n(1 + H^2) \right]
\]

\[
= \Phi \left( \Phi^\frac{1}{2} + \beta_0(n, H)^\frac{1}{2} \right) (\Phi - \alpha_0(n, H))
\]

\[
\Phi^\frac{1}{2} + \alpha_0(n, H)^\frac{1}{2},
\]

(41)

where $\alpha_0(n, H) = \left[ \frac{-n(n-2) + \sqrt{n^2H^2 + 4n - 4}}{2\sqrt{n(n-1)}} \right]^2$.

When $\delta(n) \leq \epsilon^2$ and $\epsilon \leq 1$, we choose positive constant $\gamma_1(n)$ such that $n \leq \Phi \leq 2n$ and $\beta_0(n, H) \leq 2n - 1$ for all $H \leq \gamma_1(n)$. We obtain

\[
F(\Phi) \leq 8n(\Phi - \alpha_0(n, H)) \leq 8n \left( \epsilon^2 + \frac{n(n - 2)}{(n - 1)} \sqrt{n^2H^4 + 4(n - 1)H^2} \right).
\]

(42)

Let $\theta = \theta(n) = 1 - \frac{1}{8\sqrt{n}}$. We choose positive constants $\gamma_2(n)$ and $\gamma_3(n)$ such that $3n^2H + D(n, H)(3 + 3\sqrt{nH}) \leq \frac{1}{8}$ for all $H \leq \gamma_2(n)$, and

\[
\frac{16n^2(n-2)}{(n-1)} \sqrt{n^2\gamma_3(n)^4 + 4(n - 1)\gamma_3(n)^2} \leq \frac{9}{16C_2(n, \theta(n))^2}.
\]

Take $\epsilon_2(n) = \left[ \frac{n(n-2)}{(n-1)^2} \sqrt{n^2\gamma_3(n)^4 + 4(n - 1)\gamma_3(n)^2} \right]^\frac{1}{2} > 0$. Combining (39), (40) and (42), we obtain

\[
\int_M \left[ -\frac{1}{2} + O(\epsilon, \theta(n), H) \right] |\nabla \Theta|^2 d\mathcal{M} \geq 0,
\]

(43)

for all $H \leq \gamma(n) = \min\{\gamma_1(n), \gamma_2(n), \gamma_3(n)\}$ and $\epsilon \leq \min\{1, \epsilon_1, \epsilon_2(n)\}$. For $\epsilon \leq 1$, we have

\[
O(\epsilon, \theta(n), H) \leq \epsilon D(n, \gamma(n))C_2(n, \theta(n))(a + 1 + 3\sqrt{n}\gamma(n))
\]

\[
+ \epsilon C_2(n, \theta(n))(an + 3n^2\gamma(n))
\]

\[
+ \epsilon \left( 3 - \frac{3\theta(n)}{4} + \frac{C_2(n, \theta(n))}{8} + 3C_2(n, \theta(n)) \right)
\]

\[
:= \epsilon \eta(n),
\]

(44)

where $a = \frac{\sqrt{17} + 1}{2}$.

For $\epsilon \leq \epsilon_1(n)$, where $\epsilon_1(n) = \frac{C_2(n, \theta(n))}{8[3\sqrt{n}\gamma(n) + C_2(n, \theta(n))(a + 3\sqrt{n}\gamma(n) - 2)]} > 0$, $a = \frac{\sqrt{17} + 1}{2}$, we have

\[
\frac{C_2(n, \theta(n))}{8\epsilon} \geq 3\sqrt{n}\gamma(n) + C_2(n, \theta(n))(a + 3\sqrt{n}\gamma(n) - 2) - \frac{\theta(n)}{4}.
\]

(45)
So

\[ \frac{\theta(n)}{4} + \frac{C_2(n, \theta(n))}{8\epsilon} - 3\sqrt{n}H + \epsilon C_2(n, \theta(n)) \left( 2 - a - 3\sqrt{n}H \right) \geq 0. \]

Taking \( \delta(n) = \epsilon(n)^2 \), where \( \epsilon(n) = \min\{1, \epsilon_1(n), \epsilon_2(n), \epsilon_3(n)\} \) and \( \epsilon_3(n) = \frac{1}{2n(n)} \), we have \( \delta(n) > 0 \). From (43) and the assumption that \( \beta(n, H) \leq S \leq \beta(n, H) + \delta(n) \), we obtain \( \nabla \phi = 0 \). This implies \( F(\Phi) = 0 \) and \( \Phi = \beta_0(n, H) \).

By Lemma 1, we have

\[ \lambda_1 = \cdots = \lambda_{n-1} = H - \sqrt{\frac{\beta(n, H) - nH^2}{n(n-1)}}, \]

\[ \lambda_n = H + \sqrt{\frac{(n-1)(\beta(n, H) - nH^2)}{n}}. \]

Therefore \( M \) is the Clifford hypersurface

\[ S^1 \left( \frac{1}{\sqrt{1 + \mu^2}} \right) \times S^{n-1} \left( \frac{\mu}{\sqrt{1 + \mu^2}} \right) \]

in \( S^{n+1} \), where \( \mu = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2} \). This completes the proof of Main Theorem.

Finally we would like to propose the following problems.

**Open Problem A** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant mean curvature \( H \) in the unit sphere \( S^{n+1} \). Does there exist a positive constant \( \delta(n) \) depending only on \( n \) such that if \( \beta(n, H) \leq S \leq \beta(n, H) + \delta(n) \), then \( S \equiv \beta(n, H) \)?

**Open Problem B** For an \( n \)-dimensional compact hypersurface \( M^n \) with constant mean curvature \( H \) in \( S^{n+1} \), set \( \mu_k = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2k} \). Suppose that \( \alpha(n, H) \leq S \leq \beta(n, H) \). Is it possible to prove that \( M \) must be the isoparametric hypersurface

\[ S^k \left( \frac{1}{\sqrt{1 + \mu_k^2}} \right) \times S^{n-k} \left( \frac{\mu_k}{\sqrt{1 + \mu_k^2}} \right), \]

\( k = 1, 2, \ldots, n-1 \)?

When \( H = 0 \), the rigidity theorem due to Lawson [8], Chern, do Carmo and Kobayashi [2] provides an affirmative answer for Open Problem B.

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