REPRESENTATIONS OF POINTED HOPF ALGEBRAS
AND THEIR DRINFEL’D QUANTUM DOUBLES

LEONID KROP AND DAVID E. RADFORD

Abstract. We study representations of nilpotent type nontrivial
liftings of quantum linear spaces and their Drinfel’d quantum dou-
bles. We construct a family of Verma-type modules in both cases
and prove a parametrization theorem for simple modules. We com-
pute the Loewy and socle series of Verma modules under a mild
restriction on the datum of a lifting. We find bases and dimensions
of simple modules.

Introduction

Let $H$ be a finite-dimensional Hopf algebra over a field $k$. The goal of
this paper is two-fold. First, we want to describe structure of a family of
Verma-type $H$-modules, when $H$ is a certain lifting of a quantum linear
space, which entails determination of all simple $H$-modules. In the
second place we carry out a similar program for the Drinfel’d quantum
double of $H$.

We survey some previous related work. There has been significant
interest in recent years in representation theory of nonsemisimple Hopf
algebras and their quantum doubles $[2, 33, 9, 15, 18, 19, 10, 29]$. In
the general setting of such algebra the primary focus is on classifying
all simple $H$-modules in terms of simple modules of the coradical $H_0$
of $H$ $[28, 19]$. When $H_0 = kG$ where $G = G(H)$ is the group of
grouplike elements of $H$, and $G$ is abelian with $k$ a splitting field for
$G$ of characteristic zero, the simple $H_0$-modules are given by the elements
of the dual group $\hat{G}$. The problem of establishing a bijection between
$\hat{G}$ and the set of isomorphism classes of simple $H$-modules will be
called parametrization of simple $H$-modules. A typical example of
pointed Hopf algebra with abelian group of grouplike is provided by
the fundamental classification result of Andruskiewitsch and Schneider

Date: February 2, 2022.

Research by the first author partially supported by a grant from the College of
Liberal Arts and Sciences at DePaul University.

Research by the second author partially supported by NSA Grant H98230-04-1-
0061.
Recently Radford and Schneider \[29\] proved the parametrization property for every algebra of the form \(u(D, \lambda)\) from the classification. This result generalizes an earlier theorem of Lusztig involving his small quantum groups \[21\]. The aforementioned theorems invariably involve a Hopf algebra \(A\) with a triangular decomposition

\[
A = A^- \otimes A_0 \otimes A^+
\]

satisfying the following conditions.

\[
A_0 \text{ is a subHopf algebra, } A^- \text{ and } A^+ \text{ are subalgebras stable under } \text{ad}_\ell \text{ and } \text{ad}_r \text{ action of } A_0, \text{ and}
\]

\[
A^- = k \oplus J^-, \quad A^+ = k \oplus J^+
\]

where \(J^-, J^+\) are nilpotent ideals of \(A^-, A^+,\) respectively.

Let \(H_i\) denote the \(i\)th term of the coradical filtration of \(H\). By e.g. \[18\] \(H_1\) is a free \(H_0\)-module on a basis \(\{1, x_1, \ldots, x_n\}\) where \(x_i\) are some skew-primitive elements. When \(H\) is generated by \(H_1\) we say that \(n\) is the rank of \(H\). The simplest family of Hopf algebras within the class of pointed Hopf algebras \(H\) with abelian \(G(H)\) are liftings of quantum linear spaces completely described in \[3\]. These Hopf algebras are natural generalizations of Lusztig’s small quantum group \(u_q(sl_2)\) associated to the simple Lie algebra of rank 1. The concept of (Hopf) rank is applicable to them, and in small ranks \(n = 1, 2\) the regular representation of \(H\) is described in \[18, 10\]. For special cases of rank 2 liftings the entire finite-dimensional representation theory has been obtained in \[9, 15\].

We give an outline of the results. Let \(H\) be a lifting of a quantum linear space, and \(x_1, \ldots, x_n\) be skew-primitive generators of \(H\). We say that \(x_i\) and \(x_j\) are linked if \(x_ix_j - q_{ij}x_jx_i \neq 0\) for a certain root of unity \(q_{ij}\). Let \(\Gamma\) be the simple graph on the vertex set \(\{1, 2, \ldots, n\}\) with the edge set made up by pairs \((i, j)\) such that \(x_i, x_j\) are linked. We say that \(\Gamma\) is simply linked if every two vertices are connected by at most one edge. A lifting \(H\) is called of nilpotent type if all \(x_i\) are nilpotent.

The first part of the paper is concerned with algebras satisfying conditions \((0.1)-(0.3)\), especially nilpotent simply linked lifting \(H\) of a quantum linear space. In the latter case \(H_0 = kG(H)\). By analogy with the representation theory of \(u_q(sl_2)\) we can use either subHopfalgebra \(H_{\geq 0} := H_0H^+\) or \(H_{\leq 0} := H_0H^-\) to construct Verma-type modules \(Z(\gamma)\) where \(\gamma\) runs over \(\hat{G}\). From the general Theorem \[3,3\] we have that \(Z(\gamma)\) has a unique maximal submodule \(R(\gamma)\), the radical of \(Z(\gamma)\). Iterating this procedure gives the radical filtration \(\{R^m(\gamma)\}\) of \(Z(\gamma)\). The first major result of the paper is a description of \(R^m(\gamma)\) and the
(Loewy) layers of the filtration. We also show that the socle filtration of $Z(\gamma)$ coincides with the radical filtration.

In the second part of the paper we study representations of the Drinfel’d quantum double $D(H)$ of algebras $H$ as in the first part. The doubling procedure yields a new class of Hopf algebras beyond a generalization of quantum groups in [3, 4]. For one thing $D(H)$ is not pointed, and for another it does not have decomposition (0.1). Nevertheless, $D(H)$ retains enough good features for developing a sort of Lie theory for a Verma-type modules $I(\lambda)$ where $\lambda$ runs over the characters of $G \times \hat{G}$. As in part one, but for different reasons, each $I(\lambda)$ has a unique maximal submodule $R(\lambda)$. This enables us to show that the set of simple $D(H)$-modules has the parametrization property. We proceed to describe the radical filtration of $I(\lambda)$ under a mild restriction on the datum for $H$, which is void whenever the certain structure constants $q_i$ of $H$ have odd orders. The ‘odd order’ condition is one way to have all weight spaces of $I(\lambda)$ one-dimensional. When this is the case, the lattice $\Lambda(I(\lambda))$ of $D(H)$-submodules is distributive. This is a very strong property which implies that every submodule of $I(\lambda)$ is a unique sum of some local submodules. Thus the lattice $\Lambda(I(\lambda))$ can be recovered from the partially ordered set $\mathcal{J}$ of local submodules. We close with a classification of elements of $\mathcal{J}$.

A more detailed description of material by sections is as follows.

In section 1 we review the construction of liftings of quantum linear spaces and we develop formulas for skew-derivations associated to linked liftings. The main result of the section is the construction of an iterated Ore extension corresponding to the datum of a lifting. This result complements Ore extensions considered in [7] and gives an alternate proof of the basis theorem [3, Prop. 5.2].

In section 2 we construct the dual basis to the basis of $H$ obtained above. It transpires that the algebra structure of $H^*$ is that of a nilpotent lifting of the quantum linear space with the grouplike and characters switched around. However, $H^*$ is not pointed. Its coradical is computed in §3.3. For related material see [6].

In section 3 we first establish a general parametrization theorem for simple $A$-modules where $A$ satisfies (1.1)-(1.3). We then turn to nilpotent type liftings and determine the structure of the radical filtration of induced modules $Z(\gamma)$, $\gamma \in \hat{G}$, in Theorem 3.7.

We take up the harder case of Drinfel’d double of $H$ in section 4. Our calculation of multiplication in $D(H)$ is informed by Lemma 4.1 which says that multiplication in the double of a Hopf algebra generated by grouplike and skew-primitive elements is expressed in terms
of automorphisms and skew-derivations associated to skew-primitives. In the case at hand we compute explicitly those skew-derivations in a series of lemmas in §4.1. As a first step toward parametrization theorem for simple $D(H)$-modules we find a basic subalgebra in the sense of representation theory of algebras. We then construct a family of induced modules $I(\lambda)$ parametrized by $\lambda \in \Gamma$ where $\Gamma = G \times \hat{G}$. The action of $\Gamma$ splits up $I(\lambda)$ into a direct sum of weight subspaces. These are made explicit in Lemma 4.14 leading up to the parametrization Theorem 4.15.

The problem of determining the Loewy filtration of $I(\lambda)$ is finer and it is there that we impose a restriction on datum in Definition 4.16. The key step of our analyses consists in showing that generators of $D(H)$ act as raising and lowering operators on the weight basis of $I(\lambda)$. From this we derive the Loewy structure of $I(\lambda)$ in Theorem 4.28. The distributive case is handled in Theorem 4.31.

1. Preliminaries

1.1. Liftings of $V$. We fix some notation. Below $k$ is a field of characteristic 0 containing all roots of 1 and $k^\bullet = k \setminus \{0\}$. We denote a finite abelian group by $G$, let $\hat{G} := \text{Hom}(G, k^\bullet)$ denote the dual group, and we let $kG$ stand for the group algebra of $G$ over $k$. The order of $g \in G$ of a group $G$, is denoted by $|g|$. In particular, for a root of unity $q \in k^\bullet$, $|q|$ denotes the order of $q$. We set $\underline{n} = \{1, 2, \ldots, n\}$ and $[n] := \{0, 1, \ldots, n - 1\}$. For a vector space $V$ and a subset $X$ of $V$ we denote the span of $X$ by $(X)$. The unsubscribed ‘$\otimes$’ means ‘$\otimes_k$’. For all $n, m \in \mathbb{Z}$ with $m \geq 0$, $\binom{n}{m}_q$ denotes the Gaussian $q$-binomial coefficient [17], $\binom{n}{m}_q = \binom{n}{m}_q$ and $\binom{n}{m}_q! = (1)_q \cdots (n)_q$.

We review construction of the underlying algebras of this paper following [3]-[4]. They belong in the class of Hopf algebras parametrized by some elements of $G \times \hat{G}$. The starting point of their construction is a left-left Yetter-Drinfel’d finite-dimensional module $V$ over $kG$, or a $YD$-module, for short. This means [4] that $V$ is a left $kG$-module and a left $kG$-comodule with the $G$-action preserving $G$-grading. Let’s denote by $\omega : V \to kG \otimes V$ the comodule structure map and by $\cdot$ the $G$ action. By [4, Section 1.2] $V$ has a basis $\{v_i | i \in \underline{n}\}$, where $n = \dim V$, of $G$- and $\hat{G}$-eigenvectors, namely there are $a_i \in G$ and $\chi_i \in \hat{G}$, $i \in \underline{n}$ such that

\begin{align}
(1.1) & \quad g.v_i = \chi_i(g)v_i \\
(1.2) & \quad \omega(v_i) = a_i \otimes v_i
\end{align}
for all $g \in G$. We set
\[ \overline{D} = (G, (a_i), (\chi_i) | i \in n) \]
and call this tuple a linear datum associated with $V$. We denote by $\mathcal{G}_{YD}$ the category of all $YD$-modules over $kG$.

The $YD$-module structure on $V$ extends to a $YD$-structure on $V^\otimes m$ for every integer $m \geq 0$ by using the diagonal action and the codiagonal coaction of $kG$ on a tensor product. Explicitly, this means that
\begin{equation}
\tag{1.3}
g(v_1 \cdots v_{im}) = g \cdot v_1 \cdots g \cdot v_{im},
\end{equation}
\begin{equation}
\tag{1.4}
\omega(v_1 \cdots v_{im}) = a_{i_1} \cdots a_{i_m} \otimes v_{i_1} \cdots v_{i_m},
\end{equation}
for all $g$ and $1 \leq i_1, \ldots, i_m \leq n$. Let $F(V)$ be a free associative algebra generated by $V$. As $F(V) = \bigoplus_{m \geq 0} V^\otimes m$, $F(V)$ becomes a graded $YD$-module and it follows easily from the formulas (1.3), (1.4) that $F(V)$ is an algebra in $\mathcal{G}_{YD}$. Moreover, $F(V)$ is endowed with a special structure of Hopf algebra in $\mathcal{G}_{YD}$ [4, Section 2.1]. This is done as follows. First, it is well known [30, 4] that the $\mathcal{G}_{YD}$ is a braided tensor category with the tensor product just defined and the braiding given by the formula
\[ c(u \otimes v) = u_{(-1)} \cdot v \otimes u_{(0)}, \]
where we write $\omega(u) = u_{(-1)} \otimes u_{(0)}$ for all $u \in F(V)$. Using this braiding we define the multiplication $\cdot$ in $F(V) \otimes F(V)$ by
\begin{equation}
\tag{1.5}
(x \otimes y) \cdot (u \otimes v) = x(y_{(-1)}, u) \otimes y_{(0)} v \]
Second, by a straightforward verification (see also [25, §10.5]) the definition (1.5) turns $F(V) \otimes F(V)$ into an algebra in $\mathcal{G}_{YD}$ denoted by $F(V) \otimes F(V)$. Since $F(V)$ is a free algebra there is an algebra homomorphism
\[ \delta : F(V) \rightarrow F(V) \otimes F(V) \]
defined on the generators by $\delta(v_i) = 1 \otimes v_i + v_i \otimes 1$ for $i \in n$. Another verification shows that $\delta$ is $G$-linear and $G$-colinear. All in all we see that $F(V)$ is a bialgebra in $\mathcal{G}_{YD}$. Further by [32, 11.0.10] the coalgebra $F(V)$ has the coradical $k$, and then an argument of Takeuchi [25, 5.2.10] proves existence of the antipode. Thus $F(V)$ is indeed a Hopf algebra in $\mathcal{G}_{YD}$.

Multiplication law (1.3) can be elucidated as follows. The formulas (1.3)-(1.4) allow us to associate with a monomial $v = v_{i_1} \cdots v_{i_m}$ the bidegree $(\chi_v, g_v)$ where $\chi_v = \chi_{i_1} \cdots \chi_{i_m}$ and $g_v = a_{i_1} \cdots a_{i_m}$. The set \{\(u \otimes v\)\} forms a basis of $F(V) \otimes F(V)$ in which the definition (1.5) takes on the form
\begin{equation}
\tag{1.6}
(x \otimes y) \cdot (u \otimes v) = \chi_u(g_y) x \cdot u \otimes y \cdot v \]
\]
Equation (1.6) shows that the definition of $F(V)$ is analogous to Lusztig’s definition of algebra $f$ [20]. Moreover, when $G$ is generated by the $a_i$’s and the mapping $a_i \mapsto \chi_i$ is a homomorphism $G \to \hat{G}$, $F(V)$ is exactly the Lusztig’s type algebra associated to the bilinear form $(,)_{G \times G \to \mathbb{k}}$ defined on the generators by $(a_i, a_j) = \chi_i(a_j)$, for all $i, j \in n$.

We can now define a fundamental object of the theory. Let $\mathcal{F}(V) = F(V) \otimes \mathbb{k}G$ be the vector space made into a Hopf algebra by the smash product and smash coproduct constructions. By [26, Theorem 1], $\mathcal{F}(V)$, denoted by $F(V) \# \mathbb{k}G$, is indeed an ordinary Hopf algebra whose bialgebra structure is described by

\begin{align*}
(1.7) & \quad (u \# g)(v \# h) = u(g.v) \# gh \\
(1.8) & \quad \Delta(u \# g) = u^{(1)} \# (u^{(2)})_{(-1)} g \otimes (u^{(2)})_{(0)} \# g
\end{align*}

where we write coproduct of $F(V)$ by $\delta(u) = u^{(1)} \otimes u^{(2)}$.

The algebras of interest to us are tied to a special kind of $YD$-module.

**Definition 1.1.** (3) A $YD$-module $V$ with datum $\overline{D} = ((a_i), (\chi_i) | i \in n)$ is called a quantized linear space if

\begin{equation}
(1.9) \quad \chi_i(a_j)\chi_j(a_i) = 1 \quad \text{for all } i \neq j
\end{equation}

From now on we assume that $V$ is a quantum linear space. We let $q_{ij} = \chi_j(a_i)$ for $i \neq j$, $q_i = \chi_i(a_i)$ and $m_i = |q_i|$. A datum (or compatible datum [3]) $D$ for $V$ is a triple

$D = (\overline{D}, (\mu_i), (\lambda_{ij}))$

composed of the linear datum $\overline{D}$ of $V$ and two sets of scalars $(\mu_i)_{i \in n}$ and $(\lambda_{i,j})$ with $i \neq j$, $i, j \in n$ such that

\begin{align*}
(1.10) & \quad \mu_i = 0 \text{ if } a_i^{m_i} = 1 \text{ or } \chi_i^{m_i} \neq \epsilon \\
(1.11) & \quad \lambda_{ij} = 0 \text{ if } a_i a_j = 1 \text{ or } \chi_i \chi_j \neq \epsilon \\
(1.12) & \quad \lambda_{ji} = -q_{ji} \lambda_{ij}
\end{align*}

We will identify $v_i$ with $v_i \# 1$. For a datum $D$ we define the elements $p_i$ and $r_{ij}$ by

\begin{align*}
p_i & = v_i^{m_i} - \mu_i(a_i^{m_i} - 1) \quad \text{for all } i \in n \\
r_{ij} & = v_i v_j - q_{ij} v_j v_i - \lambda_{ij}(a_i a_j - 1), \quad \text{for all } 1 \leq i \neq j \leq n
\end{align*}

We let $I(D)$ be the ideal of $\mathcal{F}(V)$ generated by $p_i, r_{ij}$ for $i, j \in n$ and we set

$H(D) = \mathcal{F}(V)/I(D)$.

We remark that formula (1.8) implies readily that $\Delta(v_i) = v_i \otimes 1 + a_i \otimes v_i$, thus $S(v_i) = -a_i^{-1} v_i$, where $S$ is the antipode of $\mathcal{F}(V)$. A direct verification yields that $p_i$ is a primi-(a_i^{m_i}, 1)-primitive (3, p.23) and likewise $r_{ij}$ is...
(a_i a_j, 1)-primitive, thanks to \((1.9)\). In addition, a routine calculation gives \(S(p_i) = -a_i^{-m_i} p_i\) \([3\text{ p.24}]\) and also \(S(r_{ij}) = -a_i a_j r_{ij}\). Consequently \(I(D)\) is a Hopf ideal, hence \(H(D)\) is a Hopf algebra associated to \(D\).

As a point of terminology we recall the meaning of lifting of a Hopf algebra \([4]\). A pointed Hopf algebra \(H\) is a lifting of a Hopf algebra \(K\) if there is a Hopf algebra isomorphism

\[
\text{gr} \, H \simeq K,
\]

where \(\text{gr} \, H\) is the graded Hopf algebra associated to the coradical filtration of \(H\). Setting the parameters \(\mu_i\) and \(\lambda_{ij}\) of \(D\) to zero results in a linear datum \(\mathcal{D}\). The Hopf algebra \(H(D)\) has a special place in the theory. It is a biproduct of the braided Hopf algebra \(R = F(V)/I(\mathcal{D})\) and \(kG\) and by \([3, 5.3]\) \(\text{gr} \, H(D) \simeq H(\mathcal{D})\). As \(V\) determines \(H(D)\) we call \(H(D)\) a lifting of \(V\). We note that one of the main results of \([3]\) is that every lifting of \(H(D)\) has the form \(H(D)\) for a datum \(D\) on a quantum linear space \(V\).

1.2. Skew-derivations. We begin by recalling the concept of left and right skew-derivation. Let \(A\) be an algebra, \(a\) an element of \(A\) and \(\phi\) be an algebra endomorphism of \(A\). The assignments

\[
\begin{align*}
    b &\mapsto ab - \phi(b)a, \text{ for all } b \in A \\
    b &\mapsto ba - a\phi(b), \text{ for all } b \in A
\end{align*}
\]

define two linear mappings denoted by \(\phi[a, b]\) and \([a, b]\phi\) and called the left and right \(\phi\)-commutators, respectively. They are a left and right \(\phi\)-derivations, respectively, in the sense of having the property

\[
\begin{align*}
    \phi[a, uv] &= \phi[a, u]v + \phi(u)\phi[a, v] \quad \text{for all } u, v \in A \\
    [a, uv]\phi &= [a, u]\phi(v) + u[a, v]\phi \quad \text{for all } u, v \in A
\end{align*}
\]

In all applications the endomorphism \(\phi\) is the inner automorphism \(\iota_g : h \mapsto ghg^{-1}, h \in H\) induced by an invertible element \(g \in H\). We shall use a shorter notation \(\iota[a, b]\) and \([a, b]_g\) for a left/right \(\iota_g\)-commutators.

We shall need a commutation formula for powers of generators. Let \(A\) be an algebra, \(a, b, x, y \in A\) and \(\lambda, q \in \mathbb{k}^*\), where \(a, b\) are invertible. Suppose

\[
gxg^{-1} = qx, \quad gyg^{-1} = q^{-1}y \quad \text{for } g = a, b \quad \text{and} \quad \iota[y, x] = \lambda(ab - 1)
\]

Lemma 1.2. For every natural \(m \geq 1\) the following hold

\[
\begin{align*}
    \iota^m[y^m, x] &= \lambda(m)_q (q^{m-1}ab - 1)y^{m-1} \\
    \iota[y, x^m] &= \lambda(m)_q x^{m-1}(q^{m-1}ab - 1)
\end{align*}
\]
Proof: (1) The formula holds for \( m = 1 \) by definition. We induct on \( m \) assuming the formula holds for a given \( m \). We begin by noting that 

\[
[y^m, x] = [x, y^m]_{a^{-1}}.
\]

Therefore we can apply (1.15) to carry out the induction step. This gives

\[
[x, y^{m+1}]_{a^{-1}} = y^m[x, y]_{a^{-1}} + [x, y^m]_{a^{-1}}qy
\]

(which by the basis of induction and the induction hypothesis)

\[
= \lambda[y^m(ab - 1) + q(m)q(q^{m-1}ab - 1)y^m].
\]

Since \( y^mab = q^{2m}aby^m \) the right hand side equals

\[
\lambda(q^{2m}ab - 1 + q(m)q(q^{m-1}ab - 1))y^m
\]

which gives the desired formula.

Part (2) is proven by a similar (and simpler) argument. \( \square \)

We proceed to the general case. The formula below is a generalization of the Kac’s formula [13, (1.3.1)].

**Lemma 1.3.** For all integers \( j \) and \( k \)

\[
b_j [y^j, x^k] = \sum_{i=1}^{\min(j,k)} x^{k-i} f_i^{j,k} y^{j-i}
\]

holds, where \( f_i^{j,k} = \lambda^i \binom{j}{i}_q \binom{k}{i}_q (i)_q q^{(k-i)(j-i)} \prod_{m=1}^{i} (q^{i+k-m-i}ab - 1) \)

Proof: We can assume \( \lambda = 1 \) by rescaling \( x \) via \( x' = x/\lambda \) and return back to \( x \) by multiplying the formula by \( \lambda^k \). The assertion holds for every \( j \geq 1 \) and \( k = 1 \) by the preceding lemma. We induct on \( k \) assuming the lemma holds for every \( j \geq 1 \) for a given \( k \). By (1.15) we carry out the induction step as follows

\[
\begin{align*}
\nu[y^j, x^{k+1}] & = \nu[y^j, x^k]x + b^j x^k b^{-j} \nu[y^j, x] \\
& = \nu[y^j, x^k]x + q^{kj} x^k \nu[y^j, x].
\end{align*}
\]

By the preceding lemma and the induction hypothesis the right hand side of (1.16) equals to

\[
\begin{align*}
(\sum_{i=1}^{\min(j,k)} x^{k-i} f_i^{j,k} y^{j-i})x & + q^{kj} x^k (j)_q (q^{j-1}ab - 1)y^{j-1} \\
& = (\sum_{i=1}^{\min(j,k)} x^{k-i} f_i^{j,k} y^{j-i})x + q^{kj} x^k (j)_q (q^{j-1}ab - 1)y^{j-1}
\end{align*}
\]

We apply Lemma 1.2 to \( y^{j-i}x \) for every \( 1 \leq i < j \)

\[
y^{j-i}x = q^{j-i} xy^{j-i} + (j-i)_q (q^{j-i-1}ab - 1)y^{j-i-1}
\]
and use \((ab)x = q^2 x(ab)\) to rewrite \(f^{j,k}_i x = x \tilde{f}^{j,k}_i\) where
\[
\tilde{f}^{j,k}_i = \left( \frac{j}{i} \right)_q \left( \frac{k}{j} \right)_q (i)_q! q^{(k-i)(j-i)} \prod_{m=1}^{i} (q^{i+k+2-m-ia} - 1).
\]
This allows us to obtain
\[
(1.18) \quad x^{k-i} f^{j,k}_i y^{j-i} x = q^{j-i} x^{k+1-i} \tilde{f}^{j,k}_i y^{j-i} + (j-i) x^{k+1-(i+1)} f^{j,k}_i (q^{j-i-1} ab - 1) y^{j-i-1}
\]
It follows that
\[
\omega [y^i, x^{k+1}] = \sum_r x^{k+1-r} f^{j,k+1}_r y^{j-r}
\]
with \(1 \leq r \leq k + 1\), if \(k < j\), and \(1 \leq r \leq j\), otherwise, thus showing that \(1 \leq r \leq \min\{j, k + 1\}\). Moreover, by (1.18) \(f^{j,k+1}_r\) satisfy the recurrence relation
\[
\begin{align*}
f^{j,k+1}_i &= q^{kj} (j)_q (q^{j-1} ab - 1) + q^{j-1} \tilde{f}^{j,k}_i \\
f^{j,k+1}_r &= (j-r+1) q f^{j,k}_{r-1} (q^{j-r} ab - 1) + q^{j-r} \tilde{f}^{j,k}_r \quad \text{for all } 2 \leq r \leq k \\
f^{j,k+1}_{k+1} &= (j-k) q f^{j,k}_k (q^{j-k-1} ab - 1) \quad \text{if } k < j
\end{align*}
\]
We will show that \(f^{j,k+1}_r, 2 \leq r \leq k\) has the desired form leaving verification of the other cases to the reader. To this end we note that
\[
\left( \frac{j}{r} \right)_q (r)_q! = \left( \frac{j}{r-1} \right)_q (r-1)_q! (j-r+1)_q, \quad \left( \frac{k}{r} \right)_q = \left( \frac{k}{r-1} \right)_q \frac{(k-r+1)_q}{(r)_q}
\]
and
\[
\prod_{m=1}^{r} (q^{j+k+2-r-m ab} - 1) = \prod_{m=1}^{r-1} (q^{j+k+1-r-m ab} - 1) (q^{j+k+1-r-1 ab} - 1).
\]
Therefore
\[
f^{j,k+1}_r = \left( \frac{j}{r-1} \right)_q \left( \frac{k}{r-1} \right)_q (r-1)_q \prod_{m=1}^{r-1} (q^{j+k+1-r-m ab} - 1) \phi
\]
where \(\phi\) can be written in the form \(\phi = (j-r+1)_q q^{(k+1-r)(j-r)} \psi\) with
\[
\psi = q^{k+1-r} (q^{j-r} ab - 1) + \frac{(k-r+1)_q}{(r)_q} (q^{j+k+1-r ab} - 1).
\]
It is a straightforward calculation to deduce that
\[
\psi = (q^{k+j+1-2r} + 1) \frac{(k+1)_q}{(r)_q}, \quad \text{which in turn implies that}
\]
\[
\phi = (j-r+1)_q q^{(k+1-r)(j-r)} \frac{(k+1)_q}{(r)_q} (q^{k+j+1-2r ab} - 1)
\]
and this completes the proof.

\[\Box\]

1.3. **A basis for** $H(D)$. Our next goal is to give a simple proof of the basis theorem [34 Prop. 5.2]. We will make use of a connection between our algebras and a construction of ring theory known as Ore extension [16]. In the present setting this connection was studied in [7].

**Theorem 1.4.** The following set

\[(1.19) \quad \{v_1^{i_1} \cdots v_n^{i_n} g | g \in G, 0 \leq i_j \leq m_j - 1, j \in \mathbb{N}\}\]

is a basis of $H(D)$. This is the standard basis of $H$

**Proof:** For generalities on Ore extensions we refer to [16] and we adopt its notation. We induct on $n$ starting with $n = 1$. We put $R_0 = kG$ and define an automorphism $\alpha$ of $R_0$ by setting $\alpha(g) = \chi^{-1}(g)g$ and extending it to $R_0$ by linearity. Next we form a left Ore extension $R = \bigoplus_{n \geq 0} R_0 x^n$ of $R_0$ with the automorphism $\alpha$ and the $\alpha$-derivation $\delta = 0$. Thus $R$ is a free left $R_0$-module with basis $\{x^i | i = 0, 1, \ldots\}$ whose multiplication is generated by the relations

$$x^i x^j = x^{i+j} \text{ and } xr = \alpha(r)x.$$ 

Let $D = \{a, \chi, \mu\}$ be a datum on the set $\{1\}$. Let $I = I(D)$ be the ideal of $R$ generated by $x^m - \mu(a^m - 1)$, where $m = |\chi(a)|$. It is immediate that $R/I$ is a free left and right $R_0$-module with basis $\{gx^i | 0 \leq i < m\}$ and satisfies the algebra defining relations of $H(D)$. As these relations imply that $H(D)$ is a span of the set $\{gx^i | 0 \leq i < m, g \in G\}$, the theorem holds for $n = 1$.

Let $\mathfrak{H} = \mathfrak{H} \{1\}$ and denote by $D'$ the restriction of $D$ to $\mathfrak{H}$. Similarly we let $V' = (v_i | i \in \mathfrak{H})$, $F' = F(V')$, $I' = I(D')$ and $H' = H(D') = F'/I'$. We assume that $H'$ is a free span of $\{gv_2^{i_2} \cdots v_n^{i_n} | g \in G, 0 \leq i_j \leq m_j, j \in \mathfrak{H}\}$. We want to show that $H'$ has an automorphism $\alpha$ and a left $\alpha$-derivation $\delta$ such that

$$\alpha(gv_2^{i_2} \cdots v_n^{i_n}) = \chi_1^{-1}(g)\chi_2^{i_2} \cdots \chi_n^{i_n}(a_1)gv_2^{i_2} \cdots v_n^{i_n} \text{ and}$$

$$\delta(v_j) = \lambda_1(a_1 a_j - 1) \text{ for all } j \in \mathfrak{H}.$$ 

We note that were this true we could form a left Ore extension $R = H'[x; \alpha, \delta]$ and then pass on to $\overline{R} = R/(x^{m_1} - \mu_1(a^{m_1} - 1))$. This $\overline{R}$ satisfies all algebra relations of $H(D)$ and has the right dimension $|G|m_1 \cdots m_n$, which completes the induction step by the argument used for $n = 1$. 
To prove the statement about \( \alpha, \delta \) we introduce \( \mathbb{k} \)-algebra \( \mathbb{F} \) freely generated by \( \{ v_i | i \in \hat{\mathbb{A}} \} \) and the set \( \{ x_g | g \in G \} \). We define the mappings \( \alpha, \delta : \mathbb{F} \to \mathbb{F} \) by setting their values on the generators via

\[
\alpha(x_g) = \chi_1^{-1}(g)x_g \quad \text{and} \quad \alpha(v_i) = \chi_i(a_1)v_i
\]

\[
\delta(x_g) = 0 \quad \text{and} \quad \delta(v_i) = \lambda_{1i}(x_{a_1a_i} - 1)
\]

and extending them to \( \mathbb{F} \) by requiring \( \alpha \) and \( \delta \) to be an automorphism and left \( \alpha \)-derivation, respectively. We can form a left Ore extension \( \mathbb{F}[x; \alpha, \delta] \). Let \( J \) be the ideal of \( \mathbb{F} \) generated by the elements \( r_{g,h} = x_gx_h - x_{gh}, r_{g,i} = x gv_i - \chi_i(g)v_ix_g \) and the analogs of \( p_i \) and \( r_{ij} \) in which every appearance of a group element \( g \) is replaced by \( x_g \). Clearly \( \mathbb{F}/J \cong H' \) as algebras. We claim that \( \alpha \) and \( \delta \) factor through to \( H' \). This boils down to showing that \( J \) is invariant under \( \alpha \) and \( \delta \).

We begin with inclusion \( \alpha(J) \subseteq J \). It is trivial to see that \( \alpha(r_{g,h}) = \chi_1^{-1}(gh)r_{g,h} \) and \( \alpha(r_{g,i}) = \chi_1^{-1}(g)\chi_i(a_1)r_{g,i} \). Next we have

\[
\alpha(p_i) = \chi_i^{m_i}(a_1)v_i^{m_i} - \mu_i(\chi_1^{-1}(a_i)x_i^{m_i} - 1).
\]

If \( \mu_i \neq 0 \), then \( \chi_i^{m_i} = \epsilon \) by (1.10), and, since \( \chi_i^{-m_i}(a_i) = \chi_i^{m_i}(a_1) \) by (1.9) we obtain \( \alpha(p_i) = \chi_i^{m_i}(a_1)p_i \). Next, a simple calculation gives \( \alpha(r_{ij}) = \chi_i\chi_j(a_1)(v_i v_j - q_iq_j v_i) - \lambda_{ij}(\chi_1^{-1}(a_1a_j)x_{a_1a_j} - 1) \). If \( \lambda_{ij} \neq 0 \) then, first, \( \chi_i\chi_j = \epsilon \), and, second, \( \chi_1^{-1}(a_1a_j) = \chi_i\chi_j(a_1) \) by (1.11) and (1.9), respectively. It follows that \( \alpha(r_{ij}) = \chi_i\chi_j(a_1)r_{ij} \).

Moving on to the inclusion \( \delta(J) \subseteq J \) we note that \( \delta(r_{g,h}) = 0 = \delta(r_{g,i}) \). The first of these equalities is obvious, and the second follows from \( \delta(r_{g,i}) = (\chi_1^{-1}(g) - \chi_i(g))\lambda_{1i}x_{ag}x_{a_1a_i} - 1 \) together with the fact that \( \lambda_{1i} \neq 0 \) implies \( \chi_1^{-1} = \chi_i \) by (1.11). In case of \( p_i \) we have \( \delta(p_i) = \delta(v_i^{m_i}) = \alpha[x, v_i^{m_i}] \) in the notation of (1.2) and the latter is zero mod \( J \) by Lemma (1.2) (2).

It remains to compute \( \delta(r_{ij}) \). A direct calculation gives

\[
(*) \quad \delta(r_{ij}) = \lambda_{1i}[(x_{a_1a_i} - 1)v_j \chi_j(a_1)v_j(x_{a_1a_i} - 1)] + \lambda_{1j}[\chi_i(a_1)v_i(x_{a_1a_j} - 1) - \chi_j(a_1)(x_{a_1a_j} - 1)v_i]
\]

Using the relation \( x_gv_k \equiv \chi_k(g)v_kx_g \mod J \) we rewrite (*) as follows

\[
\delta(r_{ij}) \equiv \lambda_{1i}v_j(\chi_j(a_1a_i) - 1) + \lambda_{1j}v_i(\chi_j(a_i) - \chi_i(a_1)) \mod J
\]

The proof is completed by noting that if \( \lambda_{1i} \neq 0 \), then the equalities \( \chi_j(a_1) = \chi_1^{-1}(a_1) = \chi_i(a_1) \) on account of (1.9) and (1.11), respectively, give \( \chi_j(a_1a_i) = \chi_i(a_j)\chi_j(a_1) = 1 \) by (1.9) again. Furthermore, if \( \lambda_{1j} \neq 0 \), then \( \chi_i(a_1) = \chi_1^{-1}(a_1) \) by (1.9) again, as \( \chi_1^{-1} = \chi_j \) by (1.11). □
2. Algebra structure of $H^*$

2.1. A basis for $H^*$. We begin by fixing some vector notation. For an $n$-tuple $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$ and any $n$ noncommuting variables $v_1, \ldots, v_n$ we put $v^\mathbf{i} := v_1^{i_1} \cdots v_n^{i_n}$. We write $\delta_{\mathbf{i}, \mathbf{j}} = \delta_{i_1, j_1} \cdots \delta_{i_n, j_n}$, $\mathbf{i}! = \prod_{k=1}^n (i_k)_{q_k}$, and $\binom{\mathbf{i}}{\mathbf{j}} = \prod_{k=1}^n \binom{i_k}{j_k}$. There $\binom{n}{m}_{q}$ denotes the Gaussian $q$-binomial coefficient [17]. We let $u_k$ stand for the $k$th unit vector $(0 \cdots 1 \cdots 0)(k\text{th}1)$. For two vectors $\mathbf{i}$ and $\mathbf{j}$ we write $\mathbf{i} \leq \mathbf{j}$ if $i_k \leq j_k$ for all $k \in \mathbb{N}$.

Every $\gamma \in \hat{G}$ gives rise to a functional $\tilde{\gamma} : H \to k$ defined by
\begin{equation}
\tilde{\gamma}(v^\mathbf{g}) = \delta_{\mathbf{0}, \gamma}(g)
\end{equation}
The mapping $\gamma \to \tilde{\gamma}$ is a group embedding $\hat{G} \to H^*$, but not a coalgebra map, if the set $\hat{G}$ is given the group-like coalgebra structure. Below we identify $\gamma$ with $\tilde{\gamma}$ via that embedding.

For every $g \in G$ we associate a minimal idempotent
\begin{equation}
\epsilon_g = \frac{1}{|G|} \sum_{\gamma \in \hat{G}} \gamma(g^{-1})\gamma
\end{equation}
of $k\hat{G}$. The natural pairing
\begin{equation}
G \times \hat{G} \to k^*, \langle g, \gamma \rangle \mapsto \gamma(g)
\end{equation}
induces the canonical isomorphism $G \simeq \hat{G}$. It follows that the set $
\{\epsilon_g | g \in G\}$ forms a basis of $k\hat{G}$ dual to the standard basis $\{g | g \in G\}$ of $kG$.

We will find useful to have a formula for straightening out certain products. For $\mathbf{m} \leq \mathbf{i}$ we define the scalars
\begin{equation}
\phi(\mathbf{m}, \mathbf{i}) = \prod_{p=2}^n \chi_p^{m_p}(a_1^{i_1-1} \cdots a_p^{i_p-1-m_p-1})
\end{equation}

\textbf{Lemma 2.1.} In the foregoing notation, for every $g \in G$

\begin{equation}
v_1^{m_1} a_1^{i_1-1} \cdots v_n^{m_n} a_n^{i_n-1} g = \phi(\mathbf{m}, \mathbf{i}) v^{\mathbf{m}} v^\mathbf{i} g
\end{equation}

\textbf{Proof:} The formula follows immediately from relation [17].

We define the functionals $\xi_i$, $i \in \mathbb{N}$ by the rule
\begin{equation}
\xi_i(v^\mathbf{g}) = \delta_{u_i, \mathbf{g}}
\end{equation}

\textbf{Lemma 2.2.} For every $c < m_k$

(i) $\xi_c(v^\mathbf{g}) = (c)_n! \delta_{cu_k, \mathbf{g}}$. 


(ii) $\xi_k^{m_k} = 0$ for all $k \in \mathbf{n}$.

**Proof:** We begin by noting that in view of 
$(a_k \otimes v_k)(v_k \otimes 1) = q_k(v_k \otimes 1)(a_k \otimes v_k)$ the quantum binomial formula \[17\] gives

$$\Delta(v_k^i) = \sum_{m_k=0}^{i_k} \binom{i_k}{m_k} q_k^{i_k-m_k} \otimes v_k^{i_k-m_k}$$

It follows from this together with Lemma 2.1 that

$$\Delta(v^{i_k}g) = \sum_{m,l} \binom{i_m}{l} \phi(m,l)v^{m}\otimes v^{-m}g.$$  

Now (i) holds for $c = 1$ by the definition of $\xi_k$. Assuming it holds for $c$, the induction step is as follows.

$$\xi^{c+1}(v^{i_k}g) = \langle \xi^c \otimes \xi_k, \Delta(v^{i_k}g) \rangle$$

$$= \sum \binom{i_m}{l} \phi(m,l)\xi^c(v^{m}\otimes v^{-m}g)\xi_k(v^{-m}g).$$

By the induction hypothesis and the basis of induction 
$\xi_k(v^{m}\otimes v^{-m}g) = (c)_q! \delta_{cu_k,m}$, and $\xi_k(v^{-m}g) = \delta_{u_k,m}\delta_{0,l}$. It follows readily that the nonzero terms in the right side of (2.5) satisfy $m = cu_k, i - m = u_k$. Thus $i = (c + 1)u_k$. Therefore the sum in (2.5) equals $(c+1)^{u_k}! \phi(cu_k, (c + 1)u_k)$. It remains to note that $(c+1)^{u_k}! = (c + 1)q_k!$ and $\phi(cu_k, (c + 1)u_k) = \chi_k(a_0^0 \cdots a_0^{c_k}) = 1$.

(ii) follows from (i) as $(m_k)_q = 0$. \[\square\]

We can give a formula for the dual basis to the standard basis of $H$. For related results see [6].

**Proposition 2.3.** For every $c$ there holds

1. $\xi(v^g) = (c)_q! \delta_{c^g}.$
2. The set

$$\{(c)_q!^{-1} \xi(v^g) | 0 \leq c_k < m_k \text{ for all } k \in \mathbf{n} \text{ and } g \in \mathbf{G}\}$$

is the dual basis to the standard basis of $H$.

**Proof:** We begin with (1). We induct on $n$, referring to the preceding lemma for the case $n = 1$. Assuming the formula holds for all $c$ with $c_n = 0$, take $c$ with $c_n \neq 0$, and set $\xi' = (c_1, \ldots, c_{n-1}, 0)$. As $\xi = \xi^c c_n$ formula (2.4) gives

$$\xi(v^g) = \sum \binom{i_m}{l} \phi(m,l)\xi(v^{m}\otimes v^{-m}g)\xi_n^c(v^{-m}g).$$
By the induction hypothesis and Lemma 2.2 the sum equals
\[(c')! (c_n) q_n \left( \frac{i}{m} \right) \phi(m_i)\]
where \(m = c'\) and \(i - m = c_n u_n\). Thus \(i = c' + c_n u_n = c\) and it is easy to check that \(\left( \frac{i}{m} \right) = 1 = \phi(m_i)\). This completes the proof of (1).

We move to (2). Using formula (2.4) we compute
\[\xi^c \epsilon_h(v^i g) = \sum \left( \frac{i}{m} \right) \phi(m_i) \xi^c(v^{m_i} v^{i-m} g) \epsilon_h(v^{i-m} g).\]
By part (1) and definition (2.1) the value \(v\) of the above sum is \(v = (c')! \epsilon_h(g)\), provided \(i = m = c\), and zero, otherwise, as needed. □

**Proposition 2.4.**

1. For every \(\gamma \in \hat{G}\) and \(1 \leq k \leq n\)
   \[\gamma \xi_k = \gamma(a_k) \xi_k \gamma\]

2. For all \(s, t \in \mathbb{N}\) there holds
   \[\xi_s \xi_t = \chi_s(a_t) \xi_t \xi_s\]

**Proof:** To show (1) we compare the values of \(\gamma \xi_k (v^i g)\) and \(\xi_k \gamma(v^i g)\). Using (2.4), the definition of \(\xi_k\) and (2.1), the first scalar equals \(\gamma(a_k) g\) while the second is \(\gamma(g)\), provided \(i = u_k\), and zero, otherwise.

We proceed to a proof of (2). For \(s < t\) the preceding proposition gives \(\xi_s \xi_t (v^i g) = \delta_{u_s + u_t} \phi_{g_j} \). In the opposite order using (2.4) one can compute easily that \(\xi_t \xi_s (v^i g) = \delta_{u_s + u_t} \phi_{g_j} \). Noting that \(\phi(u_t, u_s + u_t) = \chi_t(a_s)\) by the definition of \(\phi\) we conclude that \(\xi_t \xi_s = \chi_t(a_s) \xi_s \xi_t\). But then (1) holds for \(\xi_s \xi_t\) as well, because \(\chi_t^{-1}(a_s) = \chi_s(a_t)\) by (1.9).

The last two propositions yield a short alternate proof to [3 Corollary 5.3].

**Corollary 2.5.** (3) Let \(H\) be a lifting of a quantum linear space and \(H_r\) be the \(r\)th term of the coradical filtration of \(H\). There holds
\[H_r = (v^{i_1} \cdots v^{i_n} g | \sum i_j \leq r \text{ and } g \in G).\]

**Proof:** The ideal \(J\) of \(H^*\) generated by \(\xi_i, i \in \mathbb{N}\), is nilpotent, thanks to Propositions 2.3(2), 2.4. \(J\) is the radical of \(H^*\) as \(H^*/J \simeq \mathbb{k}\hat{G}\) by Proposition 2.3. The assertion follows from [25 5.2.9]. □
3. Triangulated Algebras

3.1. A general theorem. In this section we prove a general form of parametrization property for algebras with certain triangular decomposition. Our proof is similar to [19, Thm. 1].

We begin with several preliminary remarks. Let $A$ be a Hopf algebra satisfying the conditions set forth in (0.1)-(0.3). Let's call a subalgebra of $A$ normal if it is stable under both adjoint actions of $A_0$. The restriction of counit $\epsilon$ to $A^+$ has kernel $J^+$. Since $\epsilon$ is $A_0$-linear with respect to both adjoint actions, $J^+$ is a normal subalgebra. The identities $ax = \sum (\text{ad}_a)(x)a_2$ and $xa = \sum a_1(\text{ad}_x)a_2$ with $a \in A_0, x \in A^+$ show that $A_0A^+ = A^+A_0$, hence $A^{\geq 0} := A_0A^+$ is a subalgebra of $A$. The splitting $A^+ = k \oplus J^+$ implies that $A^{\geq 0} = A_0 \oplus A_0J^+$. Since $J^+$ is a normal subalgebra, $A_0J^+ = J^+A_0$ and therefore $A_0J^+$ is a nilpotent ideal of $A^{\geq 0}$. It follows that all simple $A^{\geq 0}$-modules are pullbacks of simple $A_0$-modules along $A^{\geq 0} \rightarrow A_0$. For every simple (left) $A_0$-module $V$ we define $A$-module $Z(V)$ by the formula

\[(3.1)\quad Z(V) = A \otimes_{A^{\geq 0}} V\]

For an $A$-module $M$ we denote by $M_0$ the socle of its restriction to $A^{\geq 0}$. We need two auxiliary observations.

**Lemma 3.1.** For every $A_0$-module there holds

\[Z(V) = V \oplus J^{-} V.\]

**Proof:** Condition (0.1) implies readily the decomposition

\[A = A^{\geq 0} \oplus J^{-} A^{\geq 0}.\]

Tensoring this direct sum by $V$ over $A^{\geq 0}$ gives the desired formula. □

**Lemma 3.2.** For every simple $A_0$-module $V$ the induced module $Z(V)$ has a unique maximal $A$-submodule contained in $J^{-} V$.

**Proof:** By the preceding lemma and since $A_0J^- = J^- A_0$ the subspace $J^- V$ is a maximal $A_0$-submodule of $Z(V)$. Suppose $M$ is a proper $A$-submodule of $Z(V)$ not contained in $J^- V$. Then $M + J^- V = Z(V)$ and since $J^-$ is nilpotent, the argument of the Nakayama’s lemma gives $M = Z(V)$, a contradiction. Now set $R$ equal to the sum of all proper $A$-submodules of $Z(V)$. □

We denote the maximal submodule of the above lemma by $R(V)$. We define a family of simple $A$-modules by

\[L(V) = Z(V)/R(V).\]
Theorem 3.3. The mapping \( V \mapsto L(V) \) sets up a bijection between the isomorphism classes of simple \( A_0 \)-modules and the isomorphism classes of simple \( A \)-modules.

**Proof:** Let \( M \) be a simple \( A \)-module. Select a simple left \( A^\geq 0 \)-submodule \( V \) of \( M \). We observe that an \( A^\geq 0 \)-map \( \iota_V : V \to Z(V) \), \( \iota_V(v) = 1 \otimes v \) is universal among all \( A^\geq 0 \)-maps of \( V \) in \( A \)-modules. Namely, every \( f : V \to M \), \( M \) is an \( A \)-module, can be uniquely extended to \( f_* : V \to M \) satisfying the equality \( f_* = f \) via \( f_*(a \otimes v) = af(v) \). It follows that \( M \cong L(V) \) for some simple \( A_0 \)-module \( V \). It remains to show that \( L(V) \cong L(U) \) for two simple \( A_0 \)-modules \( V \) and \( U \) if and only if \( V \cong U \). To this end it suffices to show that \( L(V)_0 = V \).

Let \( \nu : Z(V) \to L(V) \) be the natural epimorphism. Set \( \overline{V} = \nu(V) \) and notice that since \( \text{Ker } \nu = R(V) \subset J^\perp V \) we have an isomorphism of \( A^\geq 0 \)-modules \( \overline{V} \cong V \) as well as the decomposition \( L(V) = \overline{V} \oplus J^\perp \overline{V} \).

Let \( \pi \) be the \( A_0 \)-projection of \( L(V) \) on \( J^\perp \overline{V} \). Suppose there is a simple \( A^\geq 0 \)-submodule \( U \) of \( L(V) \) distinct from \( \overline{V} \). Set \( U' = \pi(U) \) and notice that \( U' \) is a simple \( A_0 \)-submodule of \( J^\perp \overline{V} \). Evidently \( U' \subset U + \overline{V} \), hence \( J^+ U' = 0 \). Therefore by simplicity of \( L(V) \) we have \( L(V) = AU' = U' + J^+ U' \). It follows that \( L(V) = J^\perp \overline{V} \) hence \( L(V) = J^\perp L(V) \) forcing \( L(V)_0 = 0 \), a contradiction. \( \square \)

3.2. Representations of \( H \). Let \( \mathcal{D} = (G, (a_i), (\chi_i), (\mu_i), (\lambda_{ij}), i, j \in \mathbb{N}) \) be a datum on a quantum linear \( N \)-dimensional space. Following [29, Section 4.1]) we associate to \( \mathcal{D} \) its linking graph \( \Gamma(\mathcal{D}) \) which is a simple graph with the vertex set \( \mathbb{N} \) and the edge set of all \( (i, j) \) such that \( \lambda_{ij} \neq 0 \). As usual in graph theory the degree of a vertex \( i \) is the number \( d(i) \) of all \( j \) such that \( \lambda_{ij} \neq 0 \). We say that \( \mathcal{D} \) is simply linked datum if \( d(i) \leq 1 \) for all \( i \). The simplicity condition is not very severe. For by remark 33 Section 5] \( d(i) \leq 1 \) whenever \( |q_i| \geq 3 \). The vertices of degree zero give rise to generators of \( H(\mathcal{D}) \) lying in the radical of \( H(\mathcal{D}) \). For our purposes we can assume that \( \Gamma(\mathcal{D}) \) does not have such vertices. We call \( \mathcal{D} \) and \( H(\mathcal{D}) \) of nilpotent type if \( \mu_i = 0 \) for every vertex \( i \). From now on \( \mathcal{D} \) is a simply linked datum of nilpotent type with every vertex of degree 1. Clearly, the number of vertices \( N \) is even, so we set \( n = N/2 \). Renumbering vertices, if necessary we can assume that the edge set of \( \Gamma(\mathcal{D}) \) is \( \{(i, i + n) | i \in \mathbb{N} \} \). It will be convenient to modify notation. We put \( b_i = a_i+n \), \( x_i = v_i \) and \( y_i = v_{i+n} \) for every \( i \in \mathbb{N} \). Rescaling \( x_i \) we will assume \( \lambda_{i,i+n} = 1 \). Thus \( \mathcal{D} \) has the form

\[ \mathcal{D} = \{G, (a_i), (b_i), (\chi_i), \lambda_{ij}, n | i, j \in \mathbb{N} \} \]
Now (1.11) implies $a_i b_i \neq 0$ and $\chi_{i+n} = \chi_i^{-1}$ for all $i \in \mathbb{n}$ which in turn gives the following conditions:

(D0) $a_i b_i \neq 0$ for all $i$

(D1) $\chi_j(a_i) = \chi_i(b_j)$ for all $i, j$

(D2) $\chi_i(a_j) \chi_j(a_i) = 1$ for all $i \neq j$

(D3) $\chi_i(b_j) \chi_j(b_i) = 1$ for all $i \neq j$

The Hopf algebra $H = H(D)$ attached to $D$ is explicitly described as follows. $H$ is generated by $G$ and $2n$ symbols $\{x_i, y_i, i \in \mathbb{n}\}$ subject to the relations of $G$ and the following relations:

(R1) $g x_i = \chi_i(g) x_i g$ for all $g \in G$.

(R2) $g y_i = \chi^{-1}(g) y_i g$ for all $g \in G$.

(R3) $x_i x_j - q_{ij} x_j x_i = 0$ for $i \neq j$.

(R4) $y_i y_j - q_{ij} y_j y_i = 0$ for $i \neq j$.

(R5) $x_i y_j - q_{ij}^{-1} y_j x_i = \delta_{ij} (a_i b_i - 1)$ for all $i$.

(R6) $x_i^{m_i} = 0 = y_i^{m_i}$ for all $i$.

(R7) $\Delta x_i = a_i \otimes x_i + x_i \otimes 1$ for all $i$.

(R8) $\Delta y_i = b_i \otimes y_i + y_i \otimes 1$ for all $i$.

We proceed now to a classification of simple $H$-modules. Our approach is a generalization of [10]. Let $Y$ be the subHopf algebra of $H$ generated by $G$ and $y_i, i \in \mathbb{n}$. For every $\gamma \in \hat{G}$ we make $k$ a $Y$-module denoted by $k_\gamma$ by setting

$$g.1_\gamma = \gamma(g) \text{ for all } g \in G$$

$$y_i.1_\gamma = 0 \text{ for all } i$$

where $1_\gamma$ is identified with $1 \in k$. We define the $H$-module $Z(\gamma)$ by inducing from $Y$ to $H$, viz.

$$Z(\gamma) = H \otimes_Y k_\gamma.$$ 

Since $H$ is a free $Y$ module on a basis $\{ x_\lambda \}$ we see that the set $\{ x_\lambda \otimes 1_\gamma \}$ forms a basis for $Z(\gamma)$.

Let $M$ be an $H$-module. We say that $0 \neq m \in M$ is a weight element of weight $\gamma \in \hat{G}$ if $g.m = \gamma(g)m$ for all $g \in G$. A weight element is
called primitive if $y_{i} m = 0$ for all $i$. For $\gamma \in \hat{G}$ we define $S(\gamma)$ to be the subset of all $j \in \mathfrak{m}$ such that

\begin{equation}
\gamma(a_{j} b_{j}) = q_{j}^{-e_{j}} \text{ for some } 0 \leq e_{j} \leq m_{j} - 2.
\end{equation}

We denote by $e_{j}(\gamma)$ the above integer, dropping $\gamma$ whenever it is clear from the context. We say that elements $x, y$ of an algebra skew commute if $xy = qyx$ for some non-zero $q \in k$.

**Lemma 3.4.** A monomial $x^{i} \otimes 1_{\gamma}$ is primitive if and only if $i_{j} = 0, e_{j} + 1$ for every $j \in S(\gamma)$, and $i_{k} = 0$ for all $k \notin S(\gamma)$.

**Proof:** Since $y_{j}$ skew commutes with every $x_{i}, i \neq j$, there is $c \in k^{\ast}$ such that

\[ y_{j} x^{i} \otimes 1_{\gamma} = cx^{i_{1}} \cdots x^{i_{j-1}} y_{j} x^{i_{j}} \cdots x^{i_{n}} \otimes 1_{\gamma}. \]

By Lemma [1.2] $y_{j} x^{i_{j}} \otimes 1_{\gamma} = -q_{j}(i_{j}) q_{j} x^{i_{j}-1} (q_{j}^{-1} a_{j} b_{j} - 1)$, provided $i_{j} \neq 0$, and 0, otherwise. Further, for every $k \neq j$ the condition (D1) implies that

\begin{equation}
\chi_{k}(a_{j} b_{j}) = \chi_{k}(a_{j}) \chi_{k}(b_{j}) = \chi_{k}(a_{j}) \chi_{j}(a_{k}) = 1.
\end{equation}

Therefore $a_{j} b_{j}$ commutes with every $x_{k}, k \neq j$. It follows that $y_{j} x^{i} \otimes 1_{\gamma} = 0$ if and only if $i_{j} = 0, e_{j} + 1$, the last possibility occurring for $j \in S(\gamma)$ only.

By Theorem 3.3 each $Z(\gamma)$ has a unique maximal submodule $R(\gamma)$, possibly zero. We associate a simple $H$-module

\[ L(\gamma) = Z(\gamma)/R(\gamma) \]

to every $\gamma \in \hat{G}$. The next result explicitly describes $R(\gamma)$.

**Proposition 3.5.** In the foregoing notation

(1) The family $\{ L(\gamma) | \gamma \in \hat{G} \}$ is a full set of representatives of simple $H$-modules.

(2) $R(\gamma)$ is the sum of all submodules generated by $x^{e_{j} + 1} \otimes 1_{\gamma}, j \in S(\gamma)$.

**Proof:** (1) is a particular case of Theorem 3.3.

(2) On the one hand each primitive vector $x^{i} \otimes 1_{\gamma}$ generates a submodule spanned by all $x^{i} \otimes 1_{\gamma}$ with $i \geq i_{j}$, hence a proper one.

Conversely, suppose $v = \sum_{j} c_{j}^{i} x^{i} \otimes 1_{\gamma}, c_{j} \in k^{\ast}$, generates a proper submodule. If $i_{j} \geq e_{j} + 1$ for at least one $j \in S(\gamma)$, then $x^{i} \otimes 1_{\gamma}$ lies in $H.(x^{e_{j} + 1} \otimes 1_{\gamma})$, hence is contained by $R(\gamma)$ by the opening remark.
Suppose \( v \) involves a monomial \( x^\gamma \otimes 1 \), with \( i_j \leq e_j \) for all \( j \in S(\gamma) \). By Lemma (L3),

\[
y_j^{i_j} x_j^{i_j} \otimes 1_\gamma = -q_j(i_j)q_j! \prod_{m=1}^{i_j} (q_j^{i_j-m} \gamma_a(jb_j) - 1) \otimes 1_\gamma.
\]

It follows that

\[
y_1^{i_1} \cdots y_n^{i_n} v = c_d \otimes 1_\gamma + \sum_{m \neq 0} \kappa_m x^m \otimes 1_\gamma
\]

with \( d \neq 0, \kappa_m \in k \). But the latter element generates \( Z(\gamma) \) since \( x_i \) is nilpotent for all \( i \). This completes the proof.

We derive from the previous proposition dimension of \( L(\gamma) \).

**Corollary 3.6.** \( \dim L(\gamma) = \prod_{k \notin S(\gamma)} m_k \prod_{j \in S(\gamma)} (e_j + 1) \).

We proceed to calculation of the radical and socle series of \( Z(\gamma) \). First we recall [22] that for a finite-dimensional algebra \( A \) and a left \( A \)-module \( M \) the radical \( R(M) \) of \( M \) is the smallest submodule such that \( M/R(M) \) is semisimple. It is easy to see that \( R(M) = JM \) where \( J \) is the radical of \( A \). Dually the socle \( \Sigma(M) \) of \( M \) is the largest semisimple submodule of \( M \). The radical or Loewy series \( \{ R^m(M) \} \) of \( M \) is defined recursively by \( R^0(M) = M \) and \( R^m(M) = R(R^{m-1}(M)) \) for \( m \geq 1 \). Similarly, the socle series \( \{ \Sigma_m(M) \} \) is defined by \( \Sigma_0(M) = 0 \) and \( \Sigma_m(M) \) is the preimage in \( M \) of \( \Sigma(M/\Sigma_{m-1}(M)) \) for \( m \geq 1 \). We note that the numbers \( \min \{ m | R^m(M) = 0 \} \) and \( \min \{ m | \Sigma_m(M) = M \} \) coincide. The common value is known as the Loewy length of \( M \), denoted by \( \ell(M) \). Moreover, the two series are related by inclusion \( R^m(M) \subseteq \Sigma_{\ell(M) - m}(M) \) for all \( m \).

For \( A = H \) and \( M = Z(\gamma) \) we write \( \ell(\gamma) \) and \( R^m(\gamma), \Sigma_m(\gamma) \) for the Loewy length and the terms of the Loewy and the socle series, respectively. We define the rank of monomial \( x^\gamma \otimes 1 \), as the number \( \text{rk}(x^\gamma \otimes 1) \) of all \( j \in S(\gamma) \) such that \( i_j \geq e_j(\gamma) + 1 \).

**Theorem 3.7.** (1) For every \( m < \ell(\gamma) \) \( R^m(\gamma) \) is generated by the primitive vectors of rank \( m \).

(2) The radical and the socle series coincide.

(3) The Loewy layers \( \mathbb{L}^m(\gamma) := R^m(\gamma)/R^{m+1}(\gamma) \) are given by

\[
\mathbb{L}^m \simeq \oplus \{ L(\eta) | \eta \text{ is weight of primitive basis vector of rank } m \}.
\]

(4) \( \ell(\gamma) = |S(\gamma)| + 1 \)

**Proof:** (1)-(4). We often drop \( \gamma \) when it is clear from the context. We induct on \( m \). The assertion holds for \( m = 0 \) as \( Z(\gamma) \) is generated by \( 1 \otimes 1 \). Suppose it is true for \( R^m \). Pick a primitive vector \( v_\gamma := x^\gamma \otimes 1 \).
Clearly \( \eta = \gamma \chi \) is weight of \( v_j \). By the universal property of induced modules there is an epimorphism \( \phi : Z(\eta) \to H, v_\perp \) sending \( 1 \otimes 1_\eta \mapsto v_\perp \).

It follows that \( R(H, v_\perp) = \phi(R(\eta)) \) which by the previous proposition equals \( \sum H\phi(w) \), where \( w \) runs over all primitive monomials of \( Z(\eta) \) of rank 1. We next compute \( S(\eta) \). We have \( \eta(a_kb_k) = \gamma \chi_k^v(a_kb_k) \), because \( \chi_l(a_kb_k) = 1 \) for \( l \neq k \), as in the proof of Lemma 3.2.4. Noticing that \( i_k = 0 \) for \( k \notin S(\gamma) \) and \( i_k = e_k(\gamma) + 1 \), otherwise, we arrive at \( \eta(a_kb_k) = \gamma(a_kb_k) \) if \( k \notin S(\gamma) \) and \( \eta(a_kb_k) = q_k^{-e_k(\gamma)}q_k^{-(m_k-e_k(\gamma)-2)} \), otherwise. It follows that \( S(\eta) = S(\gamma) \) with \( e_k(\eta) = e_k(\gamma) \) or \( e_k(\eta) = m_k - e_k(\gamma) - 2 \) for all \( k \in S(\gamma) \). Furthermore, as \( w = x_j^{e_j(\eta)+1} \otimes 1_\eta \) for some \( j \in S(\gamma) \) we get \( \phi(w) = x_j^{e_j(\eta)+1}v_j \). Therefore if \( i_j \neq 0 \), then in view of \( m_j = e_j(\eta) + e_j(\gamma) + 2 \) and \( x_j^{m_j} = 0 \), we have \( \phi(w) = 0 \).

Otherwise, \( e_j(\eta) = e_j(\gamma) \), hence \( \phi(w) = x_j^{e_j(\gamma)+1}v_j \). It follows that every non-zero \( \phi(w) \) has rank \( m + 1 \). Moreover, every primitive monomial of rank \( m + 1 \) has the form \( \phi(w) \) for a choice of \( v_j \) and \( j \). Noting that \( R^{m+1}(\gamma) = \sum R(H, v_j) \) where \( v_j \) runs over all primitive monomials, the induction step is complete. This proves (1) from which (4) is an obvious consequence.

(2) Since \( R^\ell = \Sigma_0 \) we may assume by the reverse induction on \( m \) that part (2) holds for all \( k > m \). Set \( M = Z(\gamma)/R^{m+1}(\gamma) \). By the induction hypotheses \( R^{m+1} = \Sigma_{\ell-m-1} \), hence \( \Sigma(M) = \Sigma_{\ell-m}/R^{m+1} \).

Therefore, were \( R^m \) a proper submodule of \( \Sigma_{\ell-m} \) there would be a simple \( H \)-module \( L \) of \( M \) not contained in \( R^m/R^{m+1} \). Let \( k \) be the largest integer such that \( L \subset R^k/R^{m+1} \). We write \( v_\perp \) for the image of \( v_\perp \) in \( M \) and define \( \text{rk}(v_\perp) \) as \( \text{rk}(v_\perp) \). We claim that \( M \) is the span of the images of monomials. For this is true of \( H, v_\perp \) for a primitive \( v_\perp \) because the latter is the span of \( x^jv_\perp \) where \( x^j \) runs over the standard basis of \( H \). By part (1) same holds for \( R^m \) for every \( m_i \) hence for \( M \). In fact we have

\[
R^m = (v_i | \text{rk}(v_\perp) \geq m)
\]

Let \( u \) be a generator of \( L \) written as

\[
u = \sum c_i v_i, \quad 0 \neq c_i \in k
\]

By the choice of \( k \) the sum \( u_k \) of terms of \( (*) \) of rank \( k \) is nonzero. Let’s call the number of terms for \( u_k \) in the sum \( (*) \) the length of \( u_k \). We pick a generator \( u \) with \( u_k \) of the smallest length. As \( k < m \leq \ell - 1 \) for each \( \overline{v_\perp} \) of rank \( k \) there is \( j \in S(\gamma) \) with \( i_j < e_j + 1 \) where \( e_j = e_j(\gamma) \). Set \( u' = x_j^{e_j+1-i_j}u \) and observe that \( u' \neq 0 \), because distinct terms \( v_\perp \) remain distinct or zero upon multiplication by \( x_j^{e_j+1-i_j} \) and \( x_j^{e_j+1-i_j} \overline{v_\perp} \neq 0 \),
since it has rank $k + 1 \leq m$. However, $u'_k$ has length smaller than $u_k$, a contradiction.

(3) By part (1)

\[(**)
\mathbb{L}^m = \sum H.\overline{v}_j
\]

where $v_j$ runs over all primitive basis vectors of rank $m$. For each $i$, the set $\tilde{B} = \{x^j\overline{v}_k | \text{rk}(x^j\overline{v}_k) = m\}$ is a basis of $H.\overline{v}_j$. The proof follows immediately, once if we show that $\overline{v}_i$ is the only primitive vector of $H.\overline{v}_j$ within a scalar multiple.

Suppose $u$ is a primitive vector of $H.\overline{v}_i$. Write out $u$ in basis $B$:

\[u = \sum c_jx^j\overline{v}_i, \text{ } 0 \neq c_j \in \mathbb{k}.
\]

Since $\overline{v}_i$ is primitive, the argument of Lemma 3.4 shows that

\[(***)
\gamma_kx^j\overline{v}_i = d_jx^{j_u}a^{(j_k-1}\eta(a_kb_k) - 1)\overline{v}_i, d_j \in \mathbb{k}^*.
\]

where $\eta$ is the weight of $\overline{v}_i$. Monomials $x^{j_u}\overline{v}_i$ are distinct elements of $B$, which implies that $\gamma_ku = 0$ if and only if $\gamma_kx^j\overline{v}_i = 0$. This condition must hold for all $k$ and therefore, by equation (***) it is equivalent to $x^j \otimes 1_\eta$ is primitive in $Z(\eta)$. From the proof of part (1) we have that for every $k \notin S(\gamma), j_k = 0$, and for $k \in S(\gamma)$, either $j_k = 0$, or $j_k = m_k - e_k(\gamma) - 1, e_k(\gamma) + 1$. Assuming $j_k \neq 0$, in the first case $x^j\overline{v}_i = 0$, and in the second $\text{rk}(x^j\overline{v}_i) \geq m + 1$, hence $x^j\overline{v}_i = 0$ again. Thus $j = 0$, so that $u = c\overline{v}_i$ for some $c \in \mathbb{k}$. It follows that every $H.\overline{v}_i$ is a simple module and the sum (**) is direct. For otherwise, some primitive $\overline{v}_i$ would be a linear combination of other primitive monomials of $\mathbb{L}^m$, a contradiction. \(\square\)

3.3. The coradical of $H^*$. We denote by $\rightarrow$ and $\leftarrow$ two standard actions of $H$ and $H^*$ on each other [32, Chapter 5]. For every $\gamma \in \hat{G}$ we define subcoalgebra $C(\gamma)$ by $C(\gamma) = H \rightarrow \gamma \leftarrow H$.

**Proposition 3.8.** The family $\{C(\gamma) | \gamma \in \hat{G}\}$ contains every simple subcoalgebra of $H^*$. Thus

\[\text{corad}(H^*) = \sum_{\gamma \in \hat{G}} C(\gamma).
\]

**Proof:** It suffices to show that $H \rightarrow \gamma \simeq L(\gamma)$. To this end we observe $g \rightarrow \gamma = \gamma(g)\gamma$ and $y_k \leftarrow \gamma = 0$ for all $k$. The first of these equalities is obvious. For the second we compute

\[(y_k \leftarrow \gamma)(x^jz^k) = \gamma(x^jy^kgy_k) = \chi_k^{-1}(g)\gamma(x^jy^kgy_k) = 0
\]
by the definition of $\gamma$. We see that $\gamma$ is a primitive vector of weight $\gamma$. Therefore $H \to \gamma$ is the image of $Z(\gamma)$ under $\phi : 1 \otimes 1, \gamma \mapsto \gamma$. It remains to show that $\phi(R(\gamma)) = 0$. By Proposition 3.5 this is equivalent to the equality $x_k^{e_k+1} \to \gamma = 0$ for every $k \in S(\gamma)$.

Pick an integer $m$. By definition of the left action $v = (x_k^m \to \gamma)(x_j^g y^l) = \gamma(x_j^g y^l x_k^m)$. Since every $g \in G$ and every $y_j, j \neq k$, skew commute with $x_k$ we can reduce $v$ to the form

$$v = c \chi_k^m(g) \gamma(x_j^g y^l (y_j^j) x_k^m y_j^m), \quad c \in k^*,$$

where $j' = (j - 1, \ldots, j_{k-1}), j'' = (j_{k+1}, \ldots, j_n)$. Using Lemma 1.3 we see readily that $v = 0$ unless $j = 0, j = mu_k$. When these conditions hold $v = \chi_k^m(g) \gamma(f_m^m g)(g)$, where $f_m^m = (m)^{q_k} \prod_{p=1}^{m} (q_k^{m-p} a_k b_k - 1)$ again by Lemma 1.3. As $\prod_{p=1}^{m} (q_k^{m-p} \gamma(a_k b_k) - 1) = 0$ for every $k \in S(\gamma)$ and $m \geq e_k + 1$, we conclude that every $C(\gamma)$ is simple coalgebra.

On the other hand every simple $H$-module is isomorphic to $L(\gamma)$ by Proposition 3.5, which completes the proof.

The functions

$$c_k^m : \hat{G} \to k, \quad c_k^m(\gamma) = \prod_{p=1}^{m} (q_k^{m-p} \gamma(a_k b_k) - 1)$$

will play a rôle below.

4. The Drinfel’d double

4.1. Multiplication in $D(H)$. The original definition of the Drinfel’d double $D(H)$ [14] of a Hopf algebra is rather technical. For an intrinsic definition of $D(H)$ via the double crossproduct construction see [23-24]. We will follow, though, a more transparent description of $D(H)$ due to Doi-Takeuchi [12].

We recall that $D(H)$ is $H^* \otimes H$ as a vector space and $H^{\text{cop}} \otimes H$ as a coalgebra with the tensor product coalgebra structure. Note that if $S$ is the antipode of $H$, then $(S^{-1})^*$ is the antipode of $H^{\text{cop}}$. There is a natural bilinear form

$$\tau : H^{\text{cop}} \otimes H \to k, \quad \tau(\alpha, h) = \alpha(h), \quad \text{for } \alpha \in H^*, h \in H.$$ 

$\tau$ is an invertible bilinear form in the convolution algebra $\text{Hom}_k(H^{\text{cop}} \otimes H, k)$ with the inverse $\tau^{-1}(\alpha, h) = \tau((S^{-1})^*(\alpha), h)$. Using $\tau$ the algebra structure on $D(H)$ is given with product

$$(\alpha \otimes h)(\beta \otimes k) = \alpha \tau(\beta_3, h_1)\beta_2 h_2 \tau^{-1}(\beta_1, h_3)k$$

where $\Delta^2_H(\beta) = \beta_1 \otimes \beta_2 \otimes \beta_3$ and $\Delta^2_H(h) = h_1 \otimes h_2 \otimes h_3$. In what follows we will drop the “$\otimes$”-sign and write $\alpha h$. The essential part of
definition (4.1) is
\[(4.2) \quad h\beta = \tau(\beta_3, h_1)\beta_2h_2\tau^{-1}(\beta_1, h_3) \]
\[= \beta_3(h_1)\beta_2h_2\beta_1(S^{-1}(h_3)) \]
Inverting (4.2) gives the equivalent identity
\[(4.3) \quad \beta h = \tau^{-1}(\beta_3, h_1)h_2\beta_2\tau(\beta_1, h_3) \]
\[= \beta_3(S^{-1}(h_1)h_2\beta_2\beta_1(h_3)) \]
It is convenient to rewrite identities (4.2) and (4.3) in terms of actions \(\rightarrow\) and \(\leftarrow\). An immediate verification gives
\[(4.4) \quad h\beta = (h_1 \rightarrow \beta \leftarrow S^{-1}(h_3))h_2 \]
\[= \beta_2((S^{-1})^*(\beta_1) \rightarrow h \leftarrow \beta_3) \quad \text{and} \]
\[(4.5) \quad \beta h = h_2(S^{-1}(h_1) \rightarrow \beta \leftarrow h_3) \]
\[= (\beta_1 \rightarrow h \leftarrow (S^{-1})^*(\beta_3))\beta_2 \]
We note that formulas (4.4) and (4.5) were obtained in [27] and [31], respectively.

One consequence of (4.4) is the formula
\[g\alpha g^{-1} = g \rightarrow \alpha \leftarrow g^{-1}. \]
It shows that \(H^*\) is invariant under the action by \(G\) by conjugation. Therefore we have a Hopf subalgebra \(\tilde{H}^* := H^*#kG\) in \(D(H)\). Next, suppose \(x\) is a \((a,1)\)-primitive element of \(H\) satisfying \(gx = \chi_x(g)xg\) for \(\chi_x \in \hat{G}\). We associate with \(x\) two mappings \(\phi_x, \delta_x : \tilde{H}^* \rightarrow \tilde{H}^*\) defined as follows
\[\phi_x(\alpha g) = (a^{-1} \rightarrow \alpha)\chi_x(g)g\]
\[\delta_x(\alpha) = (\alpha \leftarrow xa^{-1})a - xa^{-1} \rightarrow \alpha \quad \text{and} \]
\[\delta_x(\alpha g) = \delta_x(\alpha)\chi_x(g) \]
for all \(\alpha \in H^*\) and \(g \in G\).

Lemma 4.1. (1) \(\phi_x\) is algebra automorphism and \(\delta_x\) is a right \(\phi_x\)-derivation of \(\tilde{H}^*\).
(2) \([x^s, \alpha]_{\phi_x} = [x^{s-1}, \alpha]_{\phi_x^{-1}}x + x^{s-1}[x, \phi_x^{-1}(\alpha)]\phi\) for every \(\alpha \in H^*\) and \(s \geq 1\).

Proof: (1) The claim about \(\phi_x\) is obvious. For the rest it suffices to show that
\[\alpha x = x\phi_x(\alpha) + \delta_x(\alpha) \quad \text{for all} \quad \alpha \in H^*.\]
Note the equalities 
\[ \Delta^{(2)}(x) = a \otimes a \otimes x + a \otimes x \otimes 1 + x \otimes 1 \otimes 1 \]
and 
\[ S^{-1}(x) = -xa^{-1}. \]
Therefore we have from (4.5)
\[ \alpha x = a(a^{-1} \mapsto \alpha \leftarrow x) + x(a^{-1} \mapsto \alpha) - xa^{-1} \mapsto \alpha. \]
As 
\[ a(a^{-1} \mapsto \alpha \leftarrow x) = (\alpha \rightarrow xa^{-1})a \]
by (4.4) and 
\[ gx = \chi_x(g)xg, \]
the proof of (1) is complete.

(2) We recall that 
\[ [x^s,\alpha]_{\phi^s_x} \]
stands for the right derivation \( \alpha \mapsto \alpha x^s - x^s \phi^s_x(\alpha) \) as defined in Section 1.2. The proof of the identity is by a direct verification.

For the sequel we must modify the standard basis of \( H \) and the generators of \( H^*. \) Since every \( g \in \hat{G} \) skew commutes with all \( x_i \) and \( y_i \)
the set \( \{x^i y^j | 0 \leq i, j < m_k \text{ and } g \in G \} \) is another basis of \( H. \) We define the functionals \( \gamma, \xi_k, \eta_k \) for all \( k \in \mathbb{n} \) by setting
\[ \gamma(x^i y^j) = \delta_{0,i} \delta_{0,j} \gamma(g) \]
(4.6)
\[ \xi_k(x^i y^j) = \delta_{0,i} \delta_{0,j} \xi_k \] for every \( g \in G. \)
(4.7)
\[ \eta_k(x^i y^j) = \delta_{0,i} \delta_{0,j} \eta_k \] for every \( g \in G. \)
(4.8)

By an argument almost identical to one for Lemma 2.2 one can show

Lemma 4.2. The formulas
\[ \xi^c_k \gamma(x^i y^j) = (c)_{q_k} \delta_{0,i} \delta_{0,j} \gamma(g) \] and
\[ \eta^c_k \gamma(x^i y^j) = (c)_{q_k} \delta_{0,i} \delta_{0,j} \gamma(g). \]
hold for all \( \gamma \in \hat{G}, k \in \mathbb{n} \text{ and } 0 \leq c \leq m_k. \)

We also record an analog of Proposition 2.4.

Proposition 4.3. (1) For every \( \gamma \in \hat{G} \) and \( 1 \leq k \leq n \)
\[ \gamma \xi_k = \gamma(a_k) \xi_k \gamma \text{ and } \gamma \eta_k = \gamma(b_k) \eta_k \gamma. \]
(2) For all \( s,t \in \mathbb{n} \) the equalities
\[ \xi_s \xi_t = \chi_s(a_t) \xi_t \xi_s, \]
\[ \eta_s \eta_t = \chi_s(b_t) \eta_t \eta_s, \]
\[ \xi_s \eta_t = \eta_t \xi_s \]
hold.

We move on to an explicit description of multiplication in \( D(H). \) We start off with the conjugation action of \( G. \)
Lemma 4.4. For all \( g \in G, \gamma \in \hat{G}, 1 \leq k \leq n \) the identities
\[
(4.9) \quad gg' = \gamma g \\
(4.10) \quad g\xi_k = \chi_k^{-1}(g)\xi_k g \\
(4.11) \quad g\eta_k = \chi_k(g)\eta_k g
\]
hold.

Proof: By (4.12) \( g\alpha g^{-1} = g \mapsto \alpha \mapsto g^{-1} \). From the definition of \( \gamma, \xi_k, \eta_k \) the equations
\[
(4.12) \quad g \mapsto \gamma = \gamma(g)\gamma \quad \text{and} \quad \gamma \mapsto g = \gamma(g)\gamma \\
(4.13) \quad g \mapsto \xi_k = \xi_k \quad \text{and} \quad \xi_k \mapsto g = \chi_k(g)\xi_k \\
(4.14) \quad g \mapsto \eta_k = \chi_k(g)\eta_k \quad \text{and} \quad \eta_k \mapsto g = \eta_k
\]
follow which complete the proof. \( \square \)

We need a technical lemma. Below we use the convention that for any set of variables \( v_j, v^j = 0 \) if \( i_k < 0 \) for at least one \( k \).

Lemma 4.5. There are scalars \( c, c', d, d' \in k^* \) depending on \( k, i, j \) and \( g, h \in G \) such that
\[
(4.15) \quad (x^i y^j)_k = cx^i y^j_k + c' y^i_k x^j + (q_k^{-1} a_k b_k - 1) y^i_k y^j_k \\
(4.16) \quad (y_k h)(x^i y^j)_k = dy^i_k y^j_k h + d' y^i_k y^j_k h (q_k^{-1} a_k b_k - 1) y^i_k y^j_k
\]

Proof: Since elements of \( G \) and \( y_l, l \neq k \), skew commute with \( x_k \),
\((x^i y^j)_k = ax^i y^j_k \) with \( j' = (j_1, \ldots, j_{k-1}) \) and \( j'' = (j_{k+1}, \ldots, j_n) \). Next we use Lemma 4.2(1) according to which
\[
y^i_k x_k = q_k^i x_k y^i_k - q_k(j_k) q_k(j_k) (q_k^{-1} a_k b_k - 1) y^i_k.
\]
This formula and the fact that \( x_k \) skew commutes with \( x_l, l \neq k \), completes the proof of (4.15).

The proof of (4.16) is almost identical. One must use Lemma 4.2(2) together with the observation that \( x_l, l \neq k \), commutes with \( a_k b_k \). \( \square \)

The next three lemmas completely determine the algebra structure of \( D(H) \).

Lemma 4.6. For every \( \gamma \in \hat{G} \) and \( 1 \leq k \leq n \)
\[
(4.17) \quad \gamma x_k = \gamma(a_k^{-1}) x_k \gamma + \gamma(a_k^{-1}) q_k(\gamma(a_k b_k) - 1) \eta_k \gamma \\
(4.18) \quad \gamma y_k = \gamma(b_k^{-1}) y_k \gamma - \gamma(b_k^{-1})(\gamma(a_k b_k) - 1) \xi_k \gamma b_k
\]

Proof: By Lemma 4.1 we need only compute \( \delta_{x_k}(\gamma) \). By definition this involves finding \( \gamma \mapsto x_k a_k^{-1} \) and \( x_k a_k^{-1} \mapsto \gamma \). First we show that
\[
(4.19) \quad \gamma \mapsto x_k a_k^{-1} = 0
\]
For by definition \((\gamma \leftarrow x_k a_k^{-1})(x^i g y^j) = \gamma(x_k a_k^{-1} x^i g y^j)\) and the latter is zero because \(x_k a_k^{-1} x^i g y^j = c x^{i+u_k} g a_k^{-1} y^j, c \in k^*,\) with \(i + u_k \neq 0\) for all \(i, j, g.\)

Next we compute
\[
v := (x_k a_k^{-1} \leftarrow \gamma)(x^i g y^j) = \gamma(x^i g y^j x_k a_k^{-1}).
\]

Using (4.15) we express \(v = v_1 + v_2\) where \(v_1 = c \gamma(x^{i+u_k} g a_k^{-1} y^j)\) and \(v_2 = c' \gamma(x^i g a_k^{-1}(q_k^{-1} a_k b_k - 1) y^j).\) As in the proof of (4.19) \(v_1 = 0\) for all basis elements. The definition of \(\gamma\) makes it clear that \(v_2 = 0,\) unless \(i = 0\) and \(j = u_k.\) In the latter case \(c' = -q_k,\) hence \(v_2 = -q_k \gamma(g a_k^{-1}(a_k b_k - 1)).\) That is to say
\[
(4.20) \quad x_k a_k^{-1} \leftarrow \gamma = -q_k \gamma(a_k^{-1}) \gamma(a_k b_k - 1) \eta_k \gamma
\]

by Lemma 4.2 and this completes the proof of (4.17).

The proof of (4.18) is almost identical. The main steps are the equalities
\[
(4.21) \quad y_k b_k^{-1} \rightarrow \gamma = 0 \quad \text{and} \quad \gamma \leftarrow y_k b_k^{-1} = -\gamma(b_k^{-1}) \gamma(a_k b_k - 1) \xi_k. \quad \Box
\]

**Lemma 4.7.** For all \(k, l \in n\)
\[
(4.22) \quad \xi_l x_k = x_k \xi_l + \delta_{k,l}(a_l - \chi_k)
\]
\[
(4.23) \quad \xi_l y_k = y_k \xi_l - \delta_{k,l} q_k^{-1}(q_k - 1) \xi_l \delta_{k,l}
\]

**Proof:** We use again Lemma 4.1 Since \(a_k^{-1} \leftarrow \xi_l = \xi_l\) by (4.10), it remains to find \(\delta_{x_k}(\xi_l).\) First, we claim that
\[
(4.24) \quad \xi_l \leftarrow x_k a_k^{-1} = \delta_{k,l} \epsilon
\]

This is the matter of showing that \(v = \xi_l(x_k a_k^{-1} x^i g y^j) = \delta_{k,l} \delta g \delta_{j,g}.\) We observe that \(v = c \chi_k^{-1} (a_k^{-1}) \xi_l(x^{i+u_k} g y^j)\) for some \(h \in G\) and \(0 \neq c \in k^{*}\) with \(c = 1\) when \(i = 0 = j,\) which yields the claim by the definition of \(\xi_l.\)

Next we show the identity
\[
(4.25) \quad x_k a_k^{-1} \rightarrow \xi_l = \delta_{k,l} \chi_k
\]

Now we must compute \(v' = \xi_l(x^i g y^j x_k a_k^{-1}).\) Applying the straightening out formula (4.15) we write \(v' = v_1 + v_2\) where \(v_1 = c \xi_l(x^{i+u_k} g a_k^{-1} y^j)\) and \(v_2 = c' \xi_l(x^i g a_k^{-1}(q_k^{-1} a_k b_k - 1) y^j).\) We note that \(v_2\) is zero for all \(i, g, j.\) For, if \(j - u_k \neq 0,\) then surely \(v_2 = 0.\) Else, \(v_2 = c' \xi_l(x^i(s-t))\) for some \(s, t \in G,\) which is again zero.

As for \(v_1,\) if \(l \neq k,\) then \(v_1 = 0,\) because \(i + u_k \neq u_l\) for all \(i, l.\) Suppose \(l = k.\) Then \(v_1 \neq 0\) if and only if \(i = 0 = j.\) If so, \(c = \chi_k(g),\) hence \(v_1 = \chi_k(g) \delta_{u} \delta_{u}.\) which gives (4.25).
For the proof of the second part we need two observations. First off, the equality \( y_kb_k^{-1} \to \xi_l = 0 \) for all \( k, l \) is self-evident. In the second place we claim the identity

\[
\xi_l \leftarrow y_kb_k^{-1} = -q_k^{-1}(q_k - 1)\xi_k^2
\]

Again the proof boils down to finding \( v = \xi_l(y_kb_k^{-1}x^i\ell y_j^l) \) which by (4.16) splits up as \( v = v_1 + v_2 \) with \( v_1 = d\xi_l(x^i\ell y_j^l y^j_k a_k b_k - 1)g_j \) and \( v_2 = d'\xi_l(x^i\ell u_k b_k^{-1}(q_k^{-1}a_k b_k - 1)y_j^l) \). Now \( v_1 \) is always zero, because \( j + u_k \neq u_l \) for all \( j \). If \( l \neq k \), then \( j - u_k \neq u_l \) for all \( j \) forces \( v_2 = 0 \) for all choices of \( i, j, k \). Take the case \( k = l \). Now \( v_2 = 0 \) for all \( l, j \) such that \( i - u_k \neq u_k \) or \( j \neq 0 \). In the remaining case, i.e. \( i = 2u_k \) and \( j = 0 \), we have from Lemma 4.2 the identity

\[
y_kb_k^{-1}x_k^2 = q_k^{-2}y_k x_k b_k^{-1} = (x_k^2y_k - q_k^{-1}(2)_q k x_k(q_k a_k b_k - 1))b_k^{-1}
\]

which gives \( d' = -q_k^{-1}(2)_q k \). Noting that \( \xi k(x_k(q_k s - t)) = q_k - 1 \) for all \( s, t \in G \) we arrive at \( v_2 = -q_k^{-1}(2)_q k (q_k - 1)\delta_{l,2u_k}^{-1}b_k^{-1} \). In view of Lemma 4.2 we obtain the desired formula.

Lemma 4.8. For all \( k, l \in \mathbb{N} \)

\[
\eta_l x_k = q_k^{-1}x_k\eta_l + \delta_{k,l}(q_k - 1)\eta_k^2
\]

\[
\eta_l y_k = q_k^{-1}y_k\eta_l + \delta_{k,l}q_k^{-1}(\chi_k b_k - \epsilon)
\]

Proof: The proof follows from the following equations

\[
\eta_l \leftarrow x_k a_k^{-1} = 0 \text{ for all } l, k, \text{ and } x_k a_k \to \eta_l = \delta_{k,l}(q_k - 1)\eta_k^2
\]

\[
y_k b_k^{-1} \to \eta_l = \delta_{k,l}q_k^{-1}\epsilon \text{ and } \eta_l \leftarrow y_k b_k^{-1} = \delta_{k,l}q_k^{-1}\chi_k
\]

A verification of these equations follows the proof of the preceding lemma and is left for the reader.

In keeping with our convention we put \( \bar{m} = (m_1, \ldots, m_n) \). We let \( \mathbb{Z}(m) \) denote \([m_1] \times \cdots \times [m_n]\). As a consequence of the last three lemmas we have the fact that the set of all products of \( x^i, y^j, \ell_k, \eta_l \) with \( i, j, k, l \in \mathbb{Z}(m) \) in any prescribed order forms a basis for \( D(H) \).

4.2. Parametrization of Simple \( D(H) \)-Modules. We denote by \( \Gamma \) the group \( G \times \hat{G} \). We recall that \( J(A) \) denotes the radical of algebra \( A \). We consider the subalgebra \( A = A(\Gamma, y, \ell) \) of \( D(H) \) generated by \( \Gamma, y_i, \ell_i \) for all \( i \in \mathbb{N} \). Relations (4.18) and (4.24) imply that \( A \) is a free span of the set \( \{x^i\ell_k g_j g_\gamma \mid 0 \leq i_k, j_k \leq m_k - 1, g_\gamma \in \Gamma \} \). We set \( |i| = \sum |i_k| \) for an \( n \)-tuple \( i \).
Lemma 4.9. \( A \) is a right coideal of \( D(H) \) and a basic algebra in the sense that
\[
A = \mathbb{k}\Gamma \oplus J(A).
\]

Proof: We let \( B \) denote the subalgebra of \( A \) generated by \( \Gamma \) and all \( \xi_k \). Clearly \( B = \mathbb{k}\Gamma \oplus J(B) \), where \( J(B) \) is the span of all \( \xi_jg\gamma \) with \( |j| > 0 \). Since the elements of \( \Gamma \) skew-commute with every \( \xi_k \), \( J(B) \) is nilpotent. Let \( N \) be the largest integer with \( J(B)^N \neq 0 \). Set \( I \) to be the span of all \( y^i\xi_jg\gamma \) with \( |i| + |j| > 0 \). Evidently \( I \) is a complement of \( \mathbb{k}\Gamma \) in \( A \), and
\[
I = \sum_{|i| > 0} y^iB.
\]
We lift the radical filtration
\[
B \supset J(B) \supset \cdots \supset J^N(B) \supset 0
\]
to a filtration
\[
I = I_0 \supset I_1 \supset \cdots \supset I_N \supset 0
\]
where \( I_k \) is defined by \( I_k = \sum_{|i| > 0} y^iJ^k(B) \). Thanks to the relations (4.18) and (4.23) the elements of \( \Gamma \) skew commute with all \( y^i z \), \( z \in J^k(B) \) modulo \( I_k + 1 \). Therefore the \( I_k \) form an ideal filtration with nilpotent quotients \( I_k/I_k + 1 \) because the \( y_k \) generate a nilpotent subalgebra. Thus \( I = J(A) \).

We take up the first claim. The comultiplication of \( y^i g \) (cf. (2.4)) makes it clear that it suffices to show that the subalgebra \( B' \) of \( H^* \) generated by \( \hat{G} \) and the \( \xi_k \) is a right coideal. Since \( \Delta_D = \Delta_{H^\text{cop}} \) on \( H^* \) this is equivalent to \( B' \) is a left coideal of \( H^* \), or a right \( H \)-submodule with respect to the ‘\( \leftarrow \)'-action. As \( H^* \) is an \( H \)-module algebra we need only to establish inclusion \( z \leftarrow h \in B' \) for \( z \) and \( h \) running over the generators of \( B' \) and \( H \), respectively. Now using (4.12), (4.13), and (4.19), (4.21) we have
\[
\gamma \leftarrow g = \gamma(g)\gamma, \quad \xi_k \leftarrow g = \chi_k(g)\xi_k \quad \text{for all} \quad g \in G, \quad \text{and}
\gamma \leftarrow x_k = 0, \quad \text{and} \quad \gamma \leftarrow y_k = -(\gamma(a_k b_k) - 1)\xi_k \gamma.
\]

A reference to (4.24) and (4.26) completes the proof.

The above lemma makes it obvious that every simple \( A \)-module is 1-dimensional. Pick \( \lambda \in \hat{\Gamma} \) and let \( \mathbb{k}\lambda \) denote the \( A \)-module \( \mathbb{k} \) with the \( A \)-action defined by
\[
J(A).1_k = 0 \quad \text{and} \quad g\gamma.1_k = \lambda(g\gamma).
\]
We write \( 1_\lambda \) for the element \( 1_k \) of \( \mathbb{k}\lambda \). We define a family of \( D \)-modules
\[
I(\lambda), \lambda \in \hat{\Gamma}
\]
by setting
\[
I(\lambda) = D \otimes_A \mathbb{k}\lambda,
\]
where we write \( D = D(H) \). As we mentioned earlier the set
\[
\{x^i\eta^j l^k \xi_j, j, k, l \in \mathbb{Z}(m)\}
\]
is a basis for \( D(H) \). Therefore, directly
from the definition we obtain that $I(\lambda)$ is a free span of the set \( \{ x^i_\lambda \otimes 1_\lambda | i, j \in \mathbb{Z}(m) \} \). This is the standard basis of $I(\lambda)$.

**Proposition 4.10.** For every $\lambda \in \hat{\Gamma}$ the $D$-module $I(\lambda)$ has a unique maximal submodule.

**Proof:** Let $\Pi$ be the hyperplane of $I(\lambda)$ spanned by all \( \{ x^i_\lambda \otimes 1_\lambda \} \) with $|i| + |j| > 0$. We claim that every proper $D$-submodule $M$ of $I(\lambda)$ lies in $\Pi$. If not, then

\[
1 \otimes 1_\lambda + \sum_{|i| + |j| > 0} c^i_j x^i_\lambda \otimes 1_\lambda \in M \text{ for some } c^i_j \in \mathbb{k}
\]

By an argument verbatim to one used in the proof of Lemma 4.9 the $x_i, \eta_j$ generate a nilpotent subalgebra of $D$. Therefore

\[
z = \sum_{|i| + |j| > 0} c^i_j x^i_\lambda \otimes 1_\lambda \text{ is nilpotent, hence } 1 \otimes 1_\lambda \in M,
\]

a contradiction. \(\square\)

Let $R(\lambda)$ denote the radical of $I(\lambda)$, i.e. the sum of all proper submodules or zero, if $I(\lambda)$ is simple. We set

\[
L(\lambda) = I(\lambda)/R(\lambda).
\]

Let $M$ be a $D$-module. An element $0 \neq v \in M$ is called weight element of weight $\mu \in \hat{\Gamma}$ if

\[
g_\gamma.v = \mu(g_\gamma)v \text{ holds for all } g_\gamma \in \Gamma.
\]

We say that $v \neq 0$ is primitive if $J(A).v = 0$. We note that every $D$-module $M$ contains primitive weight elements, in fact, a simple submodule of the $A$-socle of $M$ is spanned by a primitive element.

The subalgebra of $D(H)$ generated by $H^*$ and $x_k$ is an Ore extension with the automorphism $\phi_{x_k} : \alpha \mapsto (a_k^{-1} \mapsto \alpha), \alpha \in H^*$, and a right $\phi_{x_k}$-derivation $\delta_{x_k}$ determined on generators of $H^*$ by relations (4.17) and (4.27). Therefore for every $\alpha \in H^*$, $\alpha x^s = \sum_{i=0}^{s} x^i_k \alpha_i$ for some $\alpha_i \in H^*$. The next two lemmas give a more precise form of these identities.

**Lemma 4.11.** There are polynomials $h_i^{(s)}(t)$ for $s = 1, 2, \ldots, i = 1, 2, \ldots, s$ such that

\[
[x_k^s, \eta_k]_{\phi_{x_k}^s} = \sum_{i=1}^{s} h_i^{(s)}(q_k)x_k^{s-i} \eta_k^{i+1}
\]

**Proof:** To simplify notation we drop the subscript $k$ and set $\phi = \phi_{x_k}$. We induct on $s$ noting that the case $s = 1$ holds by (4.27). First we
have \( \phi^*(\eta) = a^{-s} \rightarrow \eta = q^{-s}\eta \) by (4.14). Now the induction hypothesis and Lemma 4.1(2) give the identity

\[
[x^{s+1}, \eta]_{\phi^{s+1}} = q^{-s}x^s[x, \eta]_\phi + \left( \sum_{i=1}^{s} h_i^{(s)}(q)x^{s-i}\eta^{i+1}\right)x
\]

It remains to pass \( \eta^{i+1} \) over \( x \). We have by Lemma 4.1(1) that \( \eta^{i+1}x = q^{-(i+1)}x\eta^{i+1} + \delta_x(\eta^{i+1}) \). There \( \delta_x(\eta) = (q - 1)\eta^2 \) by (4.27). We claim that for every \( m \)

\[
\delta_x(\eta^m) = (q - 1)(m)_{q-1}\eta^{m+1}
\]

For by (1.15) \( \delta_x(\eta^{m+1}) = \delta_x(\eta^m)q^{-1}\eta + \eta^m(q - 1)\eta^2 \), and assuming the formula for \( m \) we get \( \delta_x(\eta^{m+1}) = (q - 1)(q^{-1}(m)_{q-1} + 1)\eta^{m+2} \) as asserted.

**Lemma 4.12.** There are polynomials \( g_i^{(s)}(t) \) for \( s = 1, 2, \ldots, i = 1, 2, \ldots s \) such that

\[
[x_k^s, \gamma]_{\phi^{k+1}} = \gamma(a_k^{-s})(x_k^s) + \sum_{i=1}^{s} g_i^{(s)}(q_k)c_i^{(s)}(\gamma)x_k^{s-i}\eta_i^1\gamma
\]

where \( c_k^i(\gamma) \) are functions defined in (3.4).

**PROOF:** We suppress the subscript \( k \) as in the preceding lemma. We argue by induction on \( s \) starting with (4.17). Since \( \phi^*(\gamma) := a^{-s} \rightarrow \gamma = \gamma(a^{-s})\gamma \), Lemma 4.1 gives

\[
[x^{s+1}, \gamma]_{\phi^{s+1}} = x^s\gamma(a^{-s})[x, \gamma]_\phi + [x^s, \gamma]_{\phi^s}x
\]

\[
= \gamma(a^{-s+1})qc_1^i(\gamma)x^s\eta\gamma + \gamma(a^{-s})(\sum_{i=1}^{s} g_i^i(q)c_i^i(\gamma)x^{s-i}\eta^i\gamma)x
\]

the last equality by (4.17) and the induction hypothesis. The proof will be completed if we show that

\[
(\eta^i\gamma)x = \gamma(a^{-1})(\kappa_i x\eta^i\gamma + \kappa_2(q^i\gamma(ab) - 1)\eta^{i+1}\gamma),
\]

where \( \kappa_i \) are some polynomials of \( q \). This equality is derived as follows. First, Lemma 4.1(1) lets us write

\[
(\eta^i\gamma)x = x(a^{-1} \rightarrow \eta^i\gamma) + \delta_x(\eta^i\gamma). \]

Next, the \( \rightarrow \) action is an algebra homomorphism, hence \( a^{-1} \rightarrow \eta^i\gamma = q^{-i}\gamma(a^{-1})\eta^i\gamma \) with the help of Lemma 4.1. Lastly, recalling that \( \delta_x(\eta^i\gamma) \) is a right \( \phi \)-derivation we compute

\[
\delta_x(\eta^i\gamma) = \delta_x(\eta^i)\phi(\gamma) + \eta^i\delta_x(\gamma)
\]

\[
= (q - 1)(i)_{q-1}\eta^{i+1}\gamma(a^{-1}\gamma + \gamma(a^{-1})q^i\eta^{i+1}\gamma
\]

\[
= \gamma(a^{-1})(q - 1)(i)_{q-1} + q(\gamma(ab) - 1)\eta^{i+1}\gamma
\]
where the second line is written by \( \text{(1.32)} \) and the basic relation \( \text{(1.17)} \). It remains to note that the expression in the square brackets equals 
\[
q^{-i(i-1)}(q^i \gamma(ab) - 1).
\]

We record for the future reference
\[
\delta_{x_k}(\eta_k^i \gamma) = q_k^{-i(i-1)}(a_k^{-1} q_k^i \gamma(a_k b_k) - 1)\eta_k^{i+1} \gamma.
\]

We move on to the general case of the preceding lemma. For an \( n \)-tuple \( \mathbf{i} \) we write \( c_1 = c_1^{i_1} \cdots c_n^{i_n} \). We recall that \( a_\mathbf{i} \) stands for \( a_1^{i_1} \cdots a_n^{i_n} \).

**Lemma 4.13.** For every pair \( (s,t) \) with \( s, t \in \mathbb{Z}(m) \) there are polynomials \( g_{\mathbf{i}} = g_{\mathbf{i}}(q_1, \ldots, q_n), i \leq s \) such that
\[
\gamma(x^{d_i} t^j) \otimes 1_\lambda = \lambda(\gamma) \gamma(a^{-d_i} b^j)(x^{d_i} t^j \otimes 1_\lambda + \sum_{t \leq s} q_{\mathbf{i}} c_{\mathbf{i}}(\gamma) x^{s_i} t^{s_j} \otimes 1_\lambda)
\]

**Proof:** We derive from the previous lemma that \( \gamma(x^{d_i} t^j) \otimes 1_\lambda \) is the sum of monomials \( m_{\mathbf{i}} = \gamma(a^{-d_i} g_1^{i_1} \cdots g_n^{i_n} c_\mathbf{i}(\gamma)x_1^{d_1} \cdots x_{-d_n} \cdots x_{-d_n} \eta_1^{i_1} \cdots \eta_n^{i_n} \gamma \).

Now observe that \( \eta_i \) skew commutes with every \( x_j, j \neq i, \) all \( \eta_j, \) and \( \gamma \eta_i = (\gamma \cdot b^i) \eta_i \) by Proposition \( \text{(4.33)}(1) \). Thus \( m_{\mathbf{i}} \) can be rewritten as \( q_1^{p_1} \cdots q_n^{p_n} x^{s_i + i} \eta_i \) for a suitable integers \( p_i \) and the lemma follows. \( \square \)

We begin to build the weight space decomposition of \( I(\lambda) \). For \( g \in G \) we denote by \( \hat{g} \) the character of \( \hat{G} \) sending \( \gamma \) to \( \gamma(g) \) for every \( \gamma \in \hat{G} \).

For \( s, t \in (\mathbb{Z})^n \) and \( \lambda \in \Gamma \) we define the character \( \lambda_{s,t} \) by
\[
\lambda_{s,t} = \lambda a^{-d_i} b^j = \lambda s_i t^j.
\]

Recall the idempotent \( e_\lambda = |\Gamma|^{-1} \sum_{g_\lambda} \lambda^{-1}(g_\lambda g)g_\lambda \) associated to \( \lambda \in \Gamma \). We denote by \( e_{s,t} \) the idempotent corresponding to \( \lambda_{s,t} \). We define vector \( v_{s,t} \) by the formula \( v_{s,t} = e_{s,t} x^{d_i} t^j \otimes 1_\lambda \). In the same spirit we let \( I_{s,t}(\lambda) \) denote the weight space \( e_{s,t} I(\lambda) \). We put an equivalence relation on the set \( \mathbb{Z}(m) \times \mathbb{Z}(m) \) by declaring \( (s,t) \sim (s',t') \) if and only if \( \lambda_{s,t} = \lambda_{s',t'} \). We let \( [s,t] \) denote the equivalence class of \( (s,t) \).

**Lemma 4.14.**

1. The set
\[
\{v_{s,t} | s,t \in \mathbb{Z}(m)\}
\]
is a basis for \( I(\lambda) \).

2. The set
\[
\{v_{s',t'} | (s',t') \in [s,t]\}
\]
is a basis for \( I_{s,t}(\lambda) \).

3. \( I(\lambda) = \bigoplus \{I_{s,t} | s,t \in \mathbb{Z}(m)\} \)
Proof: (1) The defining relations of $H$ and (4.11) make it clear that $x^{\pm \eta} \otimes 1_{\lambda}$ has $G$-weight $\lambda|_{\mathfrak{g}} \lambda$. This observation combined with the previous lemma shows that

\begin{equation}
(4.36) \quad v_{s,t} = x^{\pm \eta} \otimes 1_{\lambda} + \sum_{i<s} g_{s,t}^{\pm i} x^{\pm i} \eta^{i} \otimes 1_{\lambda}
\end{equation}

where $\overline{c} := |G|^{-1} \sum_{\gamma \in \mathfrak{g}} c_{\gamma}^G(\gamma)$ is the average value of $c_{\gamma}^G$ over $\Gamma$. We see that $v_{s,t}$ has the leading term $x^{\pm \eta} \otimes 1_{\lambda}$. We claim that the set in part (1) is linearly independent. Else, there is a linear relation (*) $\sum \kappa_{s,t} v_{s,t} = 0$, $0 \neq \kappa_{s,t} \in \mathbb{k}$. Pick $(s',t')$ such that $s'$ is the largest among all $(s,t)$ involved in (*) and $t'$ is the smallest among all $(s',t')$ involved in (*) in the ordering of $\S$2.1. Then $x^{s'} \eta^{t'} \otimes 1_{\lambda}$ cannot get cancelled, a contradiction. As $\dim I(\lambda) = \text{card } |\mathbb{Z}(\underline{m})|$, the assertion holds.

(2) and (3). We note that every $v_{s',t'}$ with $(s',t') \in [s,t]$ lies in $I_{s,t}(\lambda)$ by the definition of the latter, hence $\dim I_{s,t}(\lambda) \geq |[s,t]|$. However, the sum of cardinalities of the distinct sets $[s,t]$ equals $\dim I(\lambda)$. On the other hand, sum of $I_{s,t}(\lambda)$ is direct of dimension no greater than $\dim I(\lambda)$. This proves (2) and (3).

The next theorem is the main result of this section. We note that the theorem and its proof resemble theorems of Curtis [11] and Lusztig [21] on parametrization of simple modules.

**Theorem 4.15.** The simple modules $L(\lambda)$, $\lambda \in \hat{\Gamma}$, are a full set of representatives of simple $D$-modules.

Proof: As every simple $D$-module $L$ is generated by a primitive weight element, $L$ is the image of $I(\lambda)$ for a suitable $\lambda$. It remains to show that $L(\lambda) \simeq L(\mu)$ implies $\lambda = \mu$. Let $\Pi$ be the hyperplane in $I(\lambda)$ of Proposition 4.11, and denote by $P$ its image in $L(\lambda)$. Were $L(\lambda) \simeq L(\mu)$, there would be a primitive vector $v$ of weight $\mu$ in $L(\lambda)$.

We note that $P$ is a proper subspace of $L(\lambda)$. For, a linear relation

\[ 1 \otimes 1_{\lambda} + \sum c_{s,t} \otimes \eta \otimes 1_{\lambda} \equiv 0 \mod R(\lambda) \]

implies $1 \otimes 1_{\lambda} \in R(\lambda)$, because the $x_i, \eta_k$ generate a nilpotent subalgebra, a contradiction. As $\lambda \neq \mu$ by assumption, $v$ is a linear combination of the images of $v_{s,t}$ with $(s,t) \neq (0,0)$. By (4.36) $v \in P$, hence, as $v$ is primitive, $L(\lambda)$ is the span of all $x^{s'} \eta^{t'} v$. Refering to Lemma 4.11 we see that $P$ is invariant under multiplication by $\eta_k$ and, of course, $x_i$. Thus $L(\lambda) \subset P$, a contradiction.
4.3. The Loewy Filtration of $I(\lambda)$.

4.3.1. Action of generators on standard basis. Our ultimate goal is to describe action of generators of $D$ on the weight basis of $I(\lambda)$. The next proposition is a key step toward this goal. It runs smoothly under a certain restriction on the datum for $H$. We don’t know whether this restriction is essential for our results.

For an even $m$ we put $m' = m/2$.

Definition 4.16. We say that a simply linked datum $D$ is half-clean if $G$ does not have nontrivial relations $r = 1$ of the form

$$r = \prod_{i=1}^{n} (a_i b_i)^{t_i}$$

with $t_i = 0$ or $m'_i$.

The above definition has the following implication for the set of relations of $G$. $G$ does not have relators $r = \prod_{i=1}^{n} (a_i b_i)^{t_i}$ with $t_i \neq 1$ for all $i$. For, since $\chi_j(a_i b_i) = 1$ for every $j \neq i$, $\chi_i(r) = \chi_i(a_i b_i)^{t_i} = q^{2t_i}$. Therefore $r = 1$ implies $2t_i \equiv 0 \mod m_i$. Thus $t_i = 0$ if $m_i$ is odd, and $t_i = 0$ or $m'_i$, otherwise.

Proposition 4.17. Suppose $D$ is a half-clean datum, $s \in \mathbb{Z}(m)$ and $t \in \mathbb{Z}^n$.

1. For every $i$, $0 < i \leq s$,

$$e_s x s^{-i} \eta_1^{t+i} \otimes 1_\lambda = 0.$$

2. For every $k \in n$ and every $i$, $0 \leq i \leq s$,

$$e_{s+k+1} x s^{-i} \eta_1^{t+i} \otimes 1_\lambda = 0.$$

3. For every $i$, $0 \leq i \leq s_k - 1$,

$$e_{s-k-1} x s^{-i-1} \eta_1^{t+i} \otimes 1_\lambda = 0.$$

Proof: As a preliminary to the proof we point out that for every abelian group $G$, $\sum_{\gamma \in \hat{G}} \gamma(g) = |G| \delta_{1,g}$.

1. Put $v = x s^{-i} \eta_1^{t+i} \otimes 1_\lambda$. By Lemma 4.13 and (4.11) $v$ acquires weight $\lambda s^{-i} \eta_1^{t+i} \otimes 1_\lambda$ upon multiplication by $g\gamma$. One can see readily that $\lambda s^{-i} \eta_1^{t+i} = (ab)^{i}$. Consequently Lemma 4.13 gives the equality

$$e_s x s^{-i} \eta_1^{t+i} \otimes 1_\lambda = \sum_{j \leq s-1} g_j \{ |G|^{-1} \sum_{\gamma \in \hat{G}} \gamma((ab)^{i}) c_{j}(\gamma) \} x s^{-j} \eta_1^{t+i} \otimes 1_\lambda$$

(*)
We begin to work out the inner sum in (*). First, for a positive integer \( m \) and every \( k \in \mathbb{N} \) we have the expansion
\[
c_m^k(\gamma) = \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} q_k^{(l)} \gamma((a_k b_k)^l)
\]
by the Gauss' binomial formula [17].

Replacing \( m \) by \( j k \) and \( l \) by \( l k \) and letting \( k \) run over \( n \) we see that the inner sum in (*) equals \( |G|^{-1} \sum_{i \leq l} \gamma((ab)^{i+l}) \). Since \( i + l \leq s \) our assumption on \( D \) imply that \( (ab)^{i+l+1} \neq 1 \) for every \( l \), whence the sum vanishes by the opening remark, and this proves (1).

(2) We set \( v = x^{s-i} \eta^l u_k + 1 \otimes 1 \). A simple verification gives
\[
\lambda^{-1} \gamma^l \lambda_{a-b}^{i+l} u_k = (ab)^{i+l}.
\]
Repeating the argument leading up to the equality (*) we derive that
\[
e_{s+i+l} v = \sum_{l \leq j} g_j |G|^{-1} \sum_{\gamma \in \mathcal{G}} \gamma((ab)^{i+l} c_j(\gamma)) x^{s-i} \eta^l u_k + 1 \otimes 1 \lambda.
\]
Applying the Gauss' binomial formula to \( c_j(\gamma) \) we have
\[
\sum_{\gamma \in \mathcal{G}} \gamma((ab)^{i+l} c_j) = \sum_{l \leq j} \kappa l \sigma_i l, \kappa l \in k, \quad \text{where}
\]
\[
\sigma_i = \sum_{\gamma \in \mathcal{G}} \gamma((ab)^{i+l} c_j)
\]
Further we note that, as \( i + l \leq s \), if \( s_k < m_k - 1 \), we have \( 0 < i + u_k + l < m_k \), hence the sum (**) is zero for all \( l \) and therefore the sum (**) is zero for all \( j \). Same conclusion holds if \( s_k = m_k - 1 \), but \( j_k < s_k \). What remains to be considered is the case when \( l_k = j_k = m_k - 1 - i_k \). Now \( l_k + i_k + 1 = m_k \), hence the exponent on \( \eta_k \) in \( r_k^{i+l} u_k + 1 \) is \( \geq m_k \), and therefore this term vanishes. Thus the sum (**) is always zero, and (2) is done.

(3) can be reduced to (1) by replacing \( t \) with \( t - u_k \). \( \square \)

From now on we assume that \( D \) is half-clean. We need one more commutation formula.

**Lemma 4.18.** For every \( s \geq 1 \) and every \( k \in \mathbb{N} \) there are functions \( r_i^s \) of \( q_k \), \( 1 \leq i \leq s \) such that
\[
[x_k^s, \xi_k]_{\phi_k^s} = (s) q_k x_k^{s-1}(a_k - q_k^{-(s-1)} \chi_k) + \sum_{i=1}^{s-1} r_i^s x_k^{s-1-i} \eta_k \chi_k
\]

**Proof:** As before we drop the subscript \( k \) throughout. We induct on \( s \) noting that the formula holds for \( s = 1 \) by (4.22). To carry out the
induction step we use Lemma 4.11(2), namely
\[ [x^{s+1}, \xi]\phi^{s+1} = x^s[x, a^{-s} \to \xi]\phi + [x^s, \xi]\phi^s x \]
(which by (4.3), (4.22), and the induction hypothesis equals)
\[ x^s(a - \chi) + (s)q x^{s-1}(a - q^{-(s-1)}\chi)x + \sum_{i=1}^{s-1} r_i x^{s-1-i}\eta^i\chi x \]
For every \( m = s - i, i > 0 \) the coefficient of \( x^m \) is of the form \( x^{s+1}\eta^{i+1}\chi \) by (4.34) as needed. It remains to compute the coefficient of \( x^s \). Using (4.17), we have \( \chi x = q^{-1}x\chi + (q^2 - 1)\eta\chi \), hence \( (a - q^{s-1})\chi x = x(qa - q^{-s}\chi) + \kappa\eta\chi \) for some \( \kappa \in k \). Thus this coefficient equals \( (a - \chi) + (s)q(qa - q^{-s}\chi) = (s + 1)q(a - q^{-s}\chi) \) and the proof is complete. \( \square \)

In what follows we set \( v_{\pm k} = 0 \) if \( s_k \) or \( t_k \) is zero for some \( k \in \mathbb{N} \). We derive now action of \( \eta_k, \xi_k \) on \( I(\lambda) \).

**Proposition 4.19.** For each \( k \in \mathbb{N} \) and \( s, t \in \mathbb{Z}(m) \) there are roots of unity \( \theta, \theta' \) such that
\[
\begin{align*}
(1) & \quad \eta_k v_{\pm k} = \theta v_{\pm k + uk} \\
(2) & \quad \xi_k v_{\pm k} = \theta'(s_k)q_k(\lambda(a_k\chi_k^{-1}) - q_k^{-(s_k-1)})v_{\pm -u_kk} \\
\end{align*}
\]

**Proof:** (1) We start off by deriving action of \( \eta_k, \xi_k \) on idempotents \( e_{\pm k} \). Pick \( \mu \in \Gamma \). We have
\[
\eta_k e_{\mu} = |\Gamma|^{-1} \sum \mu((g\gamma)^{-1})\eta_k g\gamma = |\Gamma|^{-1} \sum \mu((g\gamma)^{-1})\chi_k(g^{-1})\hat{b}_k(\gamma^{-1}) = e_\nu \eta_k \quad \text{where} \quad \nu = \hat{\mu} b_k \chi_k
\]
Setting \( \mu = \lambda_{\pm k} \) we obtain the first identity, namely
\[ \eta_k e_{\pm k} = e_{\pm k + uk} \eta_k \]
For \( \xi_k \) we can show similarly that
\[ \xi_k e_{\pm k} = e_{\pm -u_kk} \xi_k \]
We continue with part (1). In order to apply (4.38) we must expand \( \eta_k x^s\gamma \otimes 1_\lambda \) in the standard basis of \( I(\lambda) \). By (4.27) \( \eta_k \) skew commute with \( x_l \) for all \( l \neq k \). For \( l = k \) we use Lemma 4.11 to pass \( \eta_k \) over \( x^s_k \). Noting that \( \eta_k \) skew commutes with every \( \eta_l \) we arrive at the equality
\[ \eta_k x^s\gamma \otimes 1_\lambda = \theta x^s\gamma \otimes 1_\lambda + \sum_{i=1}^{s} h_i x^{s-iuk}\eta^{i+(i+1)uk} \otimes 1_\lambda \]
where \( h_i \) are some elements of \( k \). By part (1) of the preceding proposition \( e_{\pm k + uk} \) annihilates \( x^{s-iuk}\gamma^{i+(i+1)uk} \otimes 1_\lambda \) for every \( i \geq 1 \), which gives (1).
(2) We begin by working out an expansion of $\xi_k x^i \eta^l \otimes 1_\lambda$ in the standard basis of $I(\lambda)$. Set $s' = (s_1, \ldots, s_{k-1}, 0, \ldots, 0)$ and $s'' = (0, \ldots, 0, s_{k+1}, \ldots, s_n)$. Noting that $\xi_k$ commutes with $x_l$ for every $l \neq k$ and commutes with all $\eta_l$, as well as the equality $\xi_k 1_\lambda = 0$, we have with the help of Lemma 4.18 the first equality, viz.

\[(*) \quad \xi_k x^i \eta^l \otimes 1_\lambda = (s_k)_{q_k} w_0 + \sum_{i=1}^{s_k-1} \kappa_i w_i \]

where $\kappa_i \in \mathbb{k}$, $w_0 = x^{s'}_k x_{s_k}^{-1} (a_k - q_k^{-s_k-1}) x^{s''}_k \eta^l \otimes 1_\lambda$ and $w_i = x^{s'}_k x_{s_k}^{-i-1} \eta_{k} \chi_k x^{s''}_k \eta^l \otimes 1_\lambda$.

The rest of the proof will be carried out in steps. (i) By (3.3) and (4.17) $\chi_k x_l = q_k^{-1} x_l \chi_k$, and also $a_k x_l = q_k^{-1} x_l a_k$ for $l \neq k$. Further $a_k$ skew commutes with $\eta^l$ with scalar $\chi_l(a_k)$ and $\chi_k$ skew commutes with $\eta^l$ with scalar $\chi_k(b^l_k)$. As $\chi_l(a_k) = \chi_k(b^l_k)$ by condition (D1), it follows that $w_0 = \theta'(\lambda(a_k \chi_k^{-1} - q^{-s_k-1}) x^{s-uk} \eta^l \otimes 1_\lambda$.

(ii) We claim that $\eta_{k} \chi_k$ skew commutes with $x_l$ for every $l \neq k$, or, equivalently, $\delta_{x_l}(\eta_{k} \chi_k) = 0$. Indeed, $\delta_{x_l}(\eta_{k}) = 0$ holds by (4.27) and $\delta_{x_l}(\chi_k) = 0$ follows from the first line of proof of (i). Since $\delta_{x_l}$ is a skew derivation, the claim follows. As $\eta_{k}, \chi_k$ skew commute with all $\eta_l$ it becomes clear that $w_i = \kappa_i x^{s-(i+1) uk} \eta^{l+iu_k} \otimes 1_\lambda$ for some $\kappa \in \mathbb{k}$.

From (i) and (ii) we have the expansion

\[
\xi_k x^i \eta^l \otimes 1_\lambda = \theta'(s_k)_{q_k} \lambda(a_k \chi_k^{-1} - q_k^{-s_k-1}) x^{s-uk} \eta^l \otimes 1_\lambda \\
+ \sum_{i=1}^{s_k-1} \kappa_i x^{s-(i+1) uk} \eta^{l+iu_k} \otimes 1_\lambda
\]

(4.41)

Applying $e_{-uk}^l$ to equality (4.41) we arrive by Proposition 4.17(1) at the desired result.

The next lemma gives a commutation relation between some generators of $H$ and primitive idempotents.

**Lemma 4.20.** For every $\eta \in \mathfrak{n}$ and $s, t \in \mathbb{Z}(m)$

1. $x_k e_{\pm u_k} = e_{\pm u_k} x_k + q_k (e_{\pm u_k} - e_{\pm u_k} \eta_k)$
2. $y_k e_{\pm u_k} = e_{\pm u_k} y_k + q_k (e_{\pm u_k} - e_{\pm u_k} \eta_k \xi_k)$

**Proof:** Combining Lemmas 4.1, 4.4, 4.6 and Proposition 4.3 we compute

\[
x_k g \gamma = \chi_k (g^{-1}) \gamma(a_k) g \gamma x_k - q_k \chi_k (g^{-1}) (\gamma(a_k) - \gamma(b_k^{-1})) g \gamma \eta_k
\]

\[
y_k g \gamma = \chi_k (g) \gamma(b_k) g \gamma y_k + q_k \chi_k (g) (\gamma(b_k) - \gamma(a_k^{-1})) g \gamma b_k \xi_k
\]
It follows that for every \( \mu \in \widehat{\Gamma} \) \( x_k e_\mu \) splits up into the sum 
\[ x_k e_\mu = e_{\mu'} x_k - q_k (e_{\mu'} - e_{\mu''}) \eta_k \] 
where \( \mu' = \mu \chi_k a_k^{-1} \) and \( \mu'' = \mu \chi_k b_k \).

Similarly \( y_k e_\mu = e_{\mu'} y_k + q_k (e_{\mu'} - e_{\mu''}) b_k \xi_k \) where \( \mu' = \mu \chi_k^{-1} b_k^{-1} \) and \( \mu'' = \mu \chi_k^{-1} a_k \). Setting \( \mu = \lambda \frac{1}{2} \lambda \) we obtain the lemma.

We can now describe action of \( x_k \) on \( I(\lambda) \). Briefly, \( x_k \) acts as a raising operator, however, not literally, because the set of weights is not ordered.

**Proposition 4.21.** For every \( k \in \mathbb{N} \) and \( s, t \in \mathbb{Z}(m) \) there are roots of unity \( \theta, \theta' \) such that
\[ x_k v_{\lambda + u_k} = \theta v_{\lambda + u_k} + \theta' v_{\lambda + t + u_k}. \]

**Proof:** Since \( x_k \) skew commutes with every \( x_l, l \neq k \), \( x_k (x^2 \eta \xi) \otimes 1_{\lambda} = \theta x^{2 + u_k} \eta \xi \otimes 1_{\lambda} \). By the previous lemma the latter monomial contributes \( v_{\lambda + u_k} \) to \( x_k v_{\lambda + u_k} \). We move on to \( w := \eta x^2 \eta \xi \otimes 1_{\lambda} \). From the expansion (4.40) and Proposition 4.17(1) we see that \( e_{\lambda + u_k} w = \theta' v_{\lambda + t + u_k} \). On the other hand part (2) of that proposition gives \( e_{\lambda + u_k} w = 0 \), and this completes the proof.

We isolate one step of the next proposition in-

**Lemma 4.22.** For every \( k \in \mathbb{N} \) and \( t < m_k \) there holds 
\[ y_k \eta_k = q_k \eta_k y_k - (t) q_k \eta^{t-1} (q_k \chi_k b_k - \epsilon) \]

**Proof:** We use induction on \( t \), the case \( t = 1 \) is covered by Lemma 4.8. Let \( \phi : H^* \to H^* \) be the automorphism \( \phi(\alpha) = b_k \to \alpha, \alpha \in H^* \). Below we drop the subscript \( k \) on \( y_k, \eta_k \). For the induction step we make use of (1.14) which allows us to write
\[ \phi[y, \eta^{t+1}] = \phi[y, \eta^t] \eta + q^t q' \phi[y, \eta] \]
\[ = -(t) q^{t-1} (q^t \chi b - \epsilon) \eta - q^t \eta^{t-1} (\chi b - \epsilon) \]
where for the last line we used the induction hypothesis and the basis of induction. One can check easily the equality \( \chi b \eta = q^t \eta \chi b \) which leads to the equality \( \phi[y, \eta^{t+1}] = -\eta^t (q^t ((t) q + 1) \chi b - ((t) q + q^t) \epsilon) = -\eta^t (t + 1) q (q^t \chi b - \epsilon) \).

We finish this section by showing that \( y_k \) acts as a lowering operator. More precisely

**Proposition 4.23.** For every \( k \in \mathbb{N} \) and \( s, t \in \mathbb{Z}(m) \) there are roots of unity \( \theta, \theta' \) such that
\[ y_k v_{\lambda + t} = \theta (t_k) q_k (\lambda (\chi_k b_k) - q_k^{-1} (t_k + 1)) v_{\lambda + u_k} + \theta' (s_k) q_k (\lambda (a_k \chi_k^{-1}) - q_k^{-1} (s_k + 1)) v_{\lambda + u_k} \]
Proof: In view of Lemma 4.20 we want to express $y_k x^s \kappa \otimes 1_\lambda$ and $\xi_k x^s \eta_t \otimes 1_\lambda$ in the standard basis of $I(\lambda)$. For the second of those vectors the expansion is given by (4.41). Multiplying (4.41) by $e_{-u_k}^s$ and $e_{-u_k}^s$ in turn, we get the second summand of (4.42) and zero, respectively, by Proposition 4.17(1),(3).

We turn now to $w = y_k x^s \eta_t \otimes 1_\lambda$. $y_k$ skew commutes with $x_l$ and $\eta_l$ for $l \neq k$. For $l = k$ the product $y_k x_k^s$ can be straightened out by Lemma 1.2(2). In addition $a_k b_k$ commutes with $x_l$ for $l \neq k$ and skew commutes with all $\eta_l$. It follows that

$$w = \kappa x^s y_k \eta_t \otimes 1_\lambda + \mu x^s \eta_t \otimes 1_\lambda$$

where $\kappa, \mu \in k$ with $\kappa$ a root of unity. Since $e_{-u_k}^s$ annihilates the second monomial in $w$ by Proposition 4.17(3), we turn to the first summand in $w$. Using the preceding lemma and the fact that $y_k 1_\lambda = 0$ $w$ reduces to the form $\theta(t_k) q_k^s (\lambda(b_k) = q_k^{-(t_k-1)} x^s \eta_t \otimes 1_\lambda$ which, upon multiplication by $e_{-u_k}^s$, becomes the first summand of (4.42).

4.3.2. The Main Theorems. In this section we will show that representation theory of $D(H)$ in $I(\lambda)$ follows the pattern established for $H$ in Theorem 3.7.

Definition 4.24. For $\lambda \in \hat{\Gamma}$ we let $S(\lambda)$ stand for all $j \in \underline{n}$ satisfying the condition

$$\lambda(a_j \chi_j^{-1}) = q_j^{-e_j} \text{ or } (4.43)$$

$$\lambda(b_j \chi_j) = q_j^{-e_j'} \text{ for some } 0 \leq e_j, e_j' \leq m_j - 2$$

Those constants depend on $\lambda$. We write them as $e_j(\lambda), e_j'(\lambda)$ when this dependance must be emphasized.

For an $n$-tuple $\underline{a} \in \mathbb{Z}^n$ we define the rank of $\underline{a}$ as the number $\text{rk}(\underline{a})$ of all $j \in S(\lambda)$ satisfying $a_j \geq e_j + 1$. We define the rank of $x^s \eta_t \otimes 1_\lambda$ by $\text{rk}(x^s \eta_t \otimes 1_\lambda) = \text{rk}(s) + \text{rk}(t)$.

Lemma 4.25. A weight vector $v_{s,t}$ is primitive if and only if $s_j = 0, e_j + 1$ or $t_j = 0, e_j' + 1$ for every $j \in S(\lambda)$, and $s_k = 0, t_k = 0$ for every $k \notin S(\lambda)$.

Proof: This follows immediately from Proposition 4.19 and Proposition 4.23.

Let’s denote the $D$-module generated by a set $X$ by $\langle X \rangle$. 
Proposition 4.26. $R(\lambda)$ is generated by the primitive elements of rank 1.

Proof: Suppose $v$ is primitive. From definition of primitivity we have that $\langle v \rangle$ is the span of all $x^a y^b v$. Let now $v$ has rank 1. Then $v$ has the form $w_j = v_{(e_j+1)u_j}$ or $w'_j = v_{0(e'_j+1)u_j}$. From Propositions 4.19 and 4.27 we see that $\langle v \rangle$ is the span of either all $v_{s\xi d}$ with $c \geq (e_j + 1)u_j$ and $d \geq 0$ or $v_{s\xi d}$ with $c \geq 0$ and $d \geq (e'_j + 1)u_j$. Therefore were $R(\lambda) \neq \sum_{j \in S(\lambda)} \langle w_j \rangle + \langle w'_j \rangle$ there would be a weight vector $v \in R(\lambda)$ involving $v_{s\xi d}$ with $s_j \leq e_j$ and $t_j \leq e'_j$ for all $j \in S(\lambda)$. We may assume that $v_{s\xi d}$ is the minimal such in the sense that $R(\lambda)$ doesn’t contain weight vectors whose expansion in the standard basis involves $v_{s\xi d}$ with either $s < s$ and $d \leq t$ or $s \leq s$ and $d < t$. Let

$$v = v_{s\xi d} + \sum c_{s'\xi' d'} v_{s'\xi' d'}; c_{s'\xi' d'} \in k$$

be the expansion of $v$ in the standard basis of $I(\lambda)$. If $s = 0 = t$, then $1 \otimes 1 + \sum_{s\xi d \neq (0,0)} v_{s\xi d} \in I(\lambda)$, hence, as $x_i, \eta_i$ generate a nilpotent subalgebra, $1 \otimes 1 \in R(\lambda)$, a contradiction. Else, set $v' = \xi_k v$ and $v'' = y_k v$. Then, either $s_k \neq 0$ for some $k$, hence $v'$ involves a weight vector smaller than $v_{s\xi d}$ by Proposition 4.19 or $s = 0$, but $t_k \neq 0$, and then the same holds for $v''$ by Proposition 4.23. In either case we arrive at a contradiction.

For the dimension formula we introduce some subsets of $S(\lambda)$. We let $S^{(1)}(\lambda), S^{(2)}(\lambda)$ and $S^{(3)}(\lambda)$ be defined by the conditions

$$S^{(1)}(\lambda) = \{ j \in S(\lambda) | j \text{ satisfies only (4.43)} \}$$
$$S^{(2)}(\lambda) = \{ j \in S(\lambda) | j \text{ satisfies only (4.44)} \}$$
$$S^{(3)}(\lambda) = S^{(1)}(\lambda) \cap S^{(2)}(\lambda)$$

Corollary 4.27. For every $\lambda \in \widehat{\Gamma}$

$$\dim L(\lambda) = \prod_{j \notin S(\lambda)} m_j^2 \prod_{j \in S^{(1)}(\lambda)} (e_j+1)m_j \prod_{j \in S^{(2)}(\lambda)} (e'_j+1)m_j \prod_{j \in S^{(3)}(\lambda)} (e_j+1)(e'_j+1).$$

Proof: By definition $L(\lambda) = I(\lambda)/R(\lambda)$. By the above proposition $L(\lambda)$ is the span of all $v_{s\xi d}$ with $s_j \leq e_j$ and $t_j \leq e'_j$ for $j \in S^{(1)}(\lambda)$ or $j \in S^{(2)}(\lambda)$, respectively, and arbitrary integers within $[0, m_j - 1]$, otherwise. We refer to section 3.2 for a discussion of the Loewy and socle series. We denote by $\ell(\lambda)$ the Loewy length of $I(\lambda)$.


Theorem 4.28. (1) For every $m < \ell(\lambda)$, $R^m(\lambda)$ is generated by the primitive vectors of rank $m$.

(2) The radical and the socle series coincide.

(3) The Loewy layers $L^m := R^m(\lambda)/R^{m+1}(\lambda)$ are given by the formula

$$L^m \simeq \oplus \{ L(\mu) | \mu \text{ is weight of primitive basis vector of rank } m \}.$$ 

(4) $\ell(\lambda) = |S^{(1)}(\lambda)| + |S^{(2)}(\lambda)| + 1$

Proof: (1)-(4). We abbreviate $R(\lambda)$ to $R$. We induct on $m$. The fact that $I(\lambda)$ is generated by $1 \otimes 1_\lambda$ gives the basis of induction. Suppose (1) holds for $R^m$. The induction step will be carried out in steps.

(i) Let $\nu_{2\lambda}$ be a primitive vector of rank $m$, and $\mu = \lambda_{2\lambda}$ be its weight. We claim that $S^{(i)}(\mu) = S^{(i)}(\lambda)$ for $i = 1, 2, 3$. Namely, $e_j(\mu) = e_j(\lambda)$ if $s_j = 0$, and $e_j(\mu) = m_j - e_j(\lambda) - 2$, otherwise. Similarly, $e'_j(\mu) = e'_j(\lambda)$ if $t_j = 0$, and $e'_j(\mu) = m_j - e'_j(\lambda) - 2$, otherwise. This follows from the calculation

$$\chi_{a-k}^{\frac{1}{2}}(a_k \chi_k^{-1}) = \chi_{a-k}^{\frac{1}{2}}(a_k a_{-1})$$

$$\sum_{m \neq k} \chi_m(a_k) \chi_m(a_k) \chi_k(a_k)[\prod_{m \neq k} \chi_j(a_m) \chi_k(a_m)]^t [\prod_{m \neq k} \chi_m(a_m) \chi_k(a_m)]^t$$

with the last equality by the datum conditions (D1-D2). Similarly $\chi_{a-k}^{\frac{1}{2}}(b_k \chi_k) = \chi_{b-k}^{2}(b_k)$. Therefore $\mu(a_k \chi_k^{-1}) = \lambda(a_k \chi_k^{-1}) \chi_{k}^{2k}(a_k)$ and $\mu(b_k \chi_k) = \lambda(b_k \chi_k) \chi_{b-k}^{2k}(b_k)$. It follows that if $s_k = 0$ or $t_k = 0$, then the value of $\mu$ and $\lambda$ at $a_k \chi_k^{-1}$ or $b_k \chi_k$ coincide. Else, $s_k = e_k(\lambda) + 1$ or $t_k = e'_k(\lambda) + 1$, whence $\mu(a_k \chi_k^{-1}) = \lambda(a_k \chi_k^{-1}) q_k^{2(e_k(\lambda)+1)} q_k^{-m_k-e_k(\lambda)-2}$, or similarly $\mu(b_k \chi_k) = q_k^{-m_k-e'_k(\lambda)-2}$. This proves our claim.

(ii) Since $\nu_{2\lambda}$ is primitive of weight $\mu$ there is a $D$-epimorphism $\phi: I(\mu) \rightarrow D\nu_{2\lambda}$ determined on the generator by $\phi: 1 \otimes 1_\mu \mapsto \nu_{2\lambda}$. It follows that $R(D\nu_{2\lambda}) = \phi(R(\mu))$, hence, by the preceding proposition, we have that $R(D\nu_{2\lambda}) = \sum D\phi(w)$ where $w$ runs over the primitive vectors of $I(\mu)$ of rank 1. Put $f_j = e_j(\mu) + 1$ and $f'_j = e'_j(\mu)$ + 1 for brevity. We know that $w$ is either $w_j = e_\nu(x_j^{f_j} \otimes 1_\mu)$ or $w'_j = e_\nu(y_j^{f'_j} \otimes 1_\mu)$ for some $j \in S(\mu)$ where $\nu = \mu_{f_j u_j, 0}$ and $\nu' = \mu_{f'_j u_j}$ are weights of $w_j$ and $w'_j$, respectively. We consider two cases.
(a) Suppose \( w = w_j \). Since \( \mu = \lambda_{\frac{\mu}{c}} \), we have
\[
\nu = \lambda_{\frac{\mu}{c}} a^{-f_1 u_j} \chi^{f_1 u_j} = \lambda_{\frac{\mu}{c} + f_1 u_j} \nu.
\]
Therefore \( \phi(w) = e_\nu x_j^{f_j} v_{\frac{\mu}{c}} \) which by Lemma 4.21 is seen to be
\[
e_\nu \left( \sum_{l+k=f_j} c_{l,k} v_{\frac{\mu}{c}+lu_j+ku_j} \right) \text{ for some } c_{l,k} \in \mathbb{k}.
\]
Furthermore, the equality of characters \( \lambda_{\frac{\mu}{c} + lu_j + ku_j} = \nu \) is equivalent to \( a_j^{\nu} b_j^k = a_j^{f_j} \) which, in view of \( f_j = l + k \), reduces to \( (a_j b_j)^k = 1 \). By our assumption on datum the last condition holds only for \( k = 0 \).

Thus \( e_\nu x_j^{f_j} v_{\frac{\mu}{c}} = cv_{\frac{\mu}{c} + f_1 u_j}, c \in \mathbb{k}, \) and \( c \neq 0 \) by another application of Lemma 4.21. The last formula and part (i) make it clear that if \( j \in S^{(\mu)} = S^{(\lambda)} \) and \( s_j = 0 \), then \( \phi(w) \) is a primitive vector of rank \( m+1 \). If \( s_j = e_j(\lambda) + 1 \), then \( f_j = e_j(\mu) + 1 = m_j - e_j(\lambda) - 1 \) again by part (i), hence \( f_j + s_j = m_j \), whence \( \phi(w) = 0 \).

(b) We let \( w = w'_j \). Proposition 4.19(1) and Lemma 4.4 show that \( \eta_j f_j \otimes 1_\mu \) has weight \( \nu' \), hence \( w'_j = \eta_j f_j \otimes 1_\mu \). By Proposition 4.19(1) \( \phi(w) = cv_{\frac{\mu}{c} + f'_j u_j}, c \neq 0 \). If \( j \in S^{(\mu)} = S^{(\lambda)} \) and \( t_j = 0 \), then \( f'_j = e'_j(\lambda) + 1 \), hence \( \phi(w) \) is a primitive vector of rank \( m + 1 \). Otherwise, by an argument as above, \( f'_j + t_j = m_j \), which gives \( \phi(w) = 0 \). It remains to note that the primitive vectors constructed in (a) and (b) account for all primitive vectors of rank \( m+1 \).

For (4) we note that every element of \( S \setminus S^{(3)} \) contributes a one and every element of \( S^{(3)} \) contributes a two to the rank function. Thus the largest value the rank function can have is \( |S \setminus S^{(3)}| + 2|S^{(2)}| = |S^{(1)}| + |S^{(2)}| \).

(2) We begin the argument as in the same part of Theorem 3.7. We assume by the reverse induction on \( m \) that \( R^{m+1} = \Sigma_{l-m-1} \) and we let \( M = I(\lambda)/R^{m+1} \). We want to prove the equality \( R^m/R^{m+1} = \Sigma(M) \).

Assuming it does not hold, there is a simple \( D \)-module \( L \) in \( \Sigma(M) \) not contained in \( R^m/R^{m+1} \). Let \( k \) be the largest integer such that \( L \subset R^k/R^{m+1} \). We denote by \( \varphi \) the image of \( v \in I(\lambda) \) in \( M \). We define \( \text{rk}(v_{\frac{\mu}{c} + lu_j}) = \text{rk}(v_{\frac{\mu}{c} + lu_j}) \).

Our first step is to show that \( M \) is the free span of all \( v_{\frac{\mu}{c} + lu_j} \) with \( \text{rk}(v_{\frac{\mu}{c} + lu_j}) < m \). Indeed, a \( v_{\frac{\mu}{c} + lu_j} \) of rank \( \geq m + 1 \) is characterized by the property that \( s_j \geq e_j + 1 \) or \( t_j \geq e'_j + 1 \) for some \( j, j' \in S(\lambda) \) of the total number greater than \( m \). Therefore there is a primitive \( v_{\frac{\mu}{c} + lu_j} \) such that \( s_j = c_j + k_j \) and \( t_j = d_j + l_j \) for some positive integers \( k_j \) and \( l_j \). Now a glance at Propositions 4.19 and 4.21 leads to conclusion that
$v_{\xi,\lambda}$ lies in $x^{j}l_{j}v_{\xi,d}$. Since by part (1) $R^{m+1}$ is generated by primitive vectors of rank $m+1$, the assertion follows.

Let $u$ be a generator of $L$ written in basis $B$ as

(*)

$$u = \sum c_{\xi,\lambda}v_{\xi,\lambda}, \quad 0 \neq c_{\xi,\lambda} \in \mathbb{k}.$$  

The sum $u_{k}$ of all $v_{\xi,\lambda}$ of rank $k$ occurring in (*) is nonzero. Let's call the number of terms in the sum (*) for $u_{k}$ the length of $u_{k}$. We pick a generator $u$ with $u_{k}$ of the smallest length. Fix one $v_{\xi,\lambda}$ involved in $u_{k}$. Assuming $k < m$ we can find $j \in S(\lambda)$ such that $c_{j} < e_{j}(\lambda) + 1$ or $d_{j} < e'_{j}(\lambda) + 1$. Suppose $c_{j} < e_{j}(\lambda) + 1$. Set $l_{j} = e_{j}(\lambda) + 1 - c_{j}$ and $\nu = \lambda_{\xi,\lambda}$. By the argument used in the proof of (1,(ii)) for every vector $v_{\xi,\lambda} \in \mathbb{k}$, we see that $e_{j}x_{j}u$ is a nonzero generator with a lesser number of basis monomials of rank $k$, a contradiction.

Assuming $d_{j} < e'_{j}(\lambda) + 1$, set $l'_{j} = e'_{j}(\lambda) + 1 - d_{j}$. Using the argument of part (1,(ii)) we deduce that $\eta u_{k}$ is a nonzero element of $L$ with the lesser number of basis monomials of rank $k$. This completes the proof of (2).

(3) We keep notation of part (2). By part (1)

(**)

$$\mathcal{L}^{m} = \sum D.v_{\xi,\lambda}$$

where $v_{\xi,\lambda}$ runs over all primitive basis vectors of rank $m$. Fix one $v_{\xi,\lambda}$. By Propositions 1.19 and 1.21 $D.v_{\xi,\lambda}$ is the span of the set $B_{\xi,\lambda} = \{v_{\xi,\lambda} | \text{rk}(v_{\xi,\lambda}) = m \text{ and } c \geq \xi, \lambda \geq t \}$. The condition $\text{rk}(v_{\xi,\lambda}) = m$ forces $0 \leq c_{j} \leq e_{j}$ if $j \in S(\lambda)$ and $s_{j} = 0$, and, likewise, $0 \leq d_{j} \leq e'_{j}$ if $j \in S(\lambda)$ with $t_{j} = 0$. If $j \in S(\lambda)$ or $j \in S(\lambda)$ with $s_{j} = e_{j} + 1$ or $t_{j} = e'_{j} + 1$, then $0 \leq c_{j} \leq m_{j} - e_{j} - 2$ or $0 \leq d_{j} \leq m_{j} - e'_{j} - 2$, respectively. For $j \notin S(\lambda)$, $c_{j}, d_{j}$ take on every value in $[m_{j}]$. Therefore by part 1(i) and the dimension formula of Corollary 1.27 $|B_{\xi,\lambda}| = \dim L(\lambda_{\xi,\lambda})$. As $D.v_{\xi,\lambda} \subset L(\lambda_{\xi,\lambda})$ we obtain the equality $D.v_{\xi,\lambda} = L(\lambda_{\xi,\lambda})$. Now, were sum (**) not direct, some $v_{\xi,\lambda}$ would be a linear combination of elements of other $\lambda_{\xi,\lambda}$. Since $v_{\xi,\lambda}$ is a basis element, it would lie in some $D.v_{\xi,\lambda}'$. However, from Theorem 4.15 one sees that every simple $D$-module has a unique line of primitive elements, a contradiction.

The (neo)classical quantum groups of Drinfel’d, Jimbo and Lusztig have the group of grouplike equal to the direct sum of cyclic subgroups generated by the grouplike associated to the ‘positive’ or ‘negative’ half of skew primitive generators. In the finite-dimensional case (see e.g. [21]) the orders of all $q_{i}$ are odd. In general those two conditions on datum are independent of each other. We call datum $D$ classical.
if either all $|q_i|$ are odd, or the elements $\{a_i, b_i\}_{i \in \mathbb{N}}$ are independent in the sense that they generate subgroup equal to the direct sum of cyclic subgroups generated by $a_i$ and $b_i$. We turn to liftings $H$ with classical data. We will give a complete description of the lattice of submodules of $I(\lambda)$ for every $\lambda \in \hat{\Gamma}$. This is possible because the lattice of $D$-submodules turns out to be distributive, a consequence of the next lemma.

**Lemma 4.29.** Suppose $D$ is classical. Then $D$ is half-clean and the weights $\lambda, s, t \in \mathbb{Z}(m)$ are distinct.

**Proof:** The first assertion holds by definition if all $|q_i|$ are odd. Else, suppose the set $\{a_i, b_i\}_{i \in \mathbb{N}}$ is independent. Then $\prod_{i=1}^{n} (a_i b_i)^i = 1$ implies $a_i^t = 1$ for all $i$. Since $\chi_i(a_i) = q_i$ and $q_i$ has order $m_i$, $s$ is divisible by $m_i$, which proves that $D$ is half-clean.

Moving on to the second claim we must show that $s = s'$ and $t = t'$ whenever $\hat{a} - s b t \chi_{s} + t = \hat{a} - s' b t' \chi_{s'} + t'$. This equation is equivalent to

\[
\begin{align*}
\chi_{s} & = \chi_{s'} \\
\chi_{s} + t & = \chi_{s'} + t'
\end{align*}
\]

Set $p_i = s_i - s'_i$ and $r_i = t_i - t'_i$ for all $i$. We rewrite the above two equations as

\[
\begin{align*}
(*) & \quad a^{-p_i} b^{r_i} = 1 \\
(**) & \quad \chi^{p_i} = \chi^{s_i}
\end{align*}
\]

Suppose $\{a_i, b_i\}_{i \in \mathbb{N}}$ are independent. The equation $(*)$ implies equalities $a_i^{p_i} = 1$ and $b_i^{r_i} = 1$ for all $i$. As $\chi_i(a_i) = \chi_i(b_i) = q_i$ and the latter has order $m_i$, we see that $p_i$ and $r_i$ are divisible by $m_i$. Since $-m_i < p_i, r_i < m_i$ we conclude that $p_i = 0 = r_i$, and this holds for all $i$.

Next assume that all $m_i$ are odd. We induct on $n$ assuming the lemma holds for every datum on $< n$ points. Since for $n = \emptyset$ the claim is vacuously true we proceed to the induction step. Applying $\chi_1$ to the equality $(*)$ gives

\[
(*) \quad \chi_1(a_1)^{-p_1} \chi_1(b_1)^{r_1} \prod_{i=2}^{n} \chi_1(a_i)^{-p_i} \chi_1(b_i)^{r_i} = 1
\]

Using datum conditions (D1)-(D2) we have $\chi_1(a_1) = \chi_1(b_1) = q_1$, $\chi_1(a_i)^{-1} = \chi_i(a_1)$ and $\chi_1(b_i) = \chi_i(a_1)$. Therefore equality $(*)$ takes on the form

\[
q_1^{r_1-p_1} \prod_{i=2}^{n} \chi_i(a_1)^{p_i+r_i} = 1.
\]
Further, evaluating the left side of (**) at $a_1$ we get the equality

$$q_1^{p_1 + r_1} \prod_{i=2}^{n} \chi_i(a_1)^{p_i + r_i} = 1.$$ 

It follows that $r_1 - p_1 \equiv r_1 + p_1 \mod m_i$. In addition taking the value of the left side of (**) at $b_1$ we have

$$q_1^{p_1 + r_1} \prod_{i=2}^{n} \chi_i(b_1)^{p_i + r_i} = q_1^{p_1 + r_1} \prod_{i=2}^{n} \chi_i(a_1)^{- (p_i + r_i)} = 1.$$ 

Comparing the last two equalities we see that $q_1^{2(p_1 + r_1)} = 1$ whence $p_1 + r_1 \equiv 0 \mod m_i$, as $m_i$ is odd. It follows that both $p_i$ and $r_i$ are divisible by $m_i$, hence $p_1 = 0 = r_1$, and the proof is complete. □

The above lemma makes it clear that all vectors $v_{s,t}, s,t \in \mathbb{Z}(m)$ have distinct weights. Therefore every weight subspace $I_{\mu}(\lambda) := e_{\mu}I(\lambda)$ is one-dimensional. Since $I(\lambda)$ is a semisimple $k\hat{\Gamma}$-module $I(\lambda) \simeq \oplus_{\mu \in \hat{\Gamma}} m_{\mu}L(\mu)$ where $m_{\mu}$ is the multiplicity of $L(\mu)$ in $I(\lambda)$. As $L(\mu)$ contains a $\mu$-weight vector, $m_{\mu} \leq 1$ for all $\mu$. Thus $I(\lambda)$ is a multiplicity-free module for all $\lambda \in \hat{\Gamma}$. We digress briefly into a general theory of such modules. For an alternate treatment see [1].

Let $A$ be an algebra and $M$ a left $A$-module of finite length with every simple $A$-module occurring at most once in a composition series of $M$. Let $\Lambda$ be the submodule lattice of $M$. $\Lambda$ is distributive by a standard criterion [8 II.13]. An element $J \neq 0$ will be called local (or join-irreducible [8]) if $A \subset J$ and $B \subset J$ imply $A + B \subset J$. Clearly the radical $R(J)$ is a unique maximal submodule of $J$. We let $R(J) = 0$ if $J$ is simple. Let $J = J(M)$ denote the poset (partially ordered set) of local submodules ordered by inclusion. By [8 Cor. III.3] $J$ forms a distinguished basis for $\Lambda$ in the sense that every $X \in \Lambda$ has a unique representation as the sum of an irredundant set of local submodules. Let $S$ be the set of all composition factors of $M$. It turns out that $J$ also determines $S$ and the way composition factors are ‘stuck’ together. The precise statement is-

**Proposition 4.30.** In the foregoing notation

1. Let $N = J_1 + \cdots + J_k$ be an irredundant sum of local submodules of $M$. Then each $J_i$ is a maximal submodule of $N$,

$$R(N) = \sum_{i=1}^{k} R(J_i) \text{ and } N/R(N) = \bigoplus_{i=1}^{k} J_i/R(J_i).$$

2. The mapping $J \mapsto J/R(J)$ sets up a bijection between $J$ and $S$. 
(3) The composition length of $M$ equals $|J|$.  

(4) For $L \in S$ let $J(L)$ be the preimage of $L$ under the map in (2).  
For any two simple modules $L$ and $L'$, $L$ occurs before $L'$ in a composition series of $M$ if and only if $J(L)$ and $J(L')$ are either incomparable in $J$ or $J(L) \supset J(L')$.

PROOF: (1) Suppose $N$ is as in (1).  Let $K = \sum_{i=1}^{k} R(J_i)$. Then $J_i \cap K = R(J_i) + \sum_{k \neq i} J_i \cap R(J_k) = R(J_i)$ because $J_i \cap R(J_k) \subset J_i \cap J_k \subset R(J_i)$. Therefore $N/K \cong \bigoplus_{i=1}^{k} J_i/R(J_i)$, hence $K \supset R(N)$. On the other hand, $J_i + R(N)/R(N)$ is semisimple for all $i$, hence $R(N) \cap J_i \subset R(J_i)$, whence $R(N) \subset K$.

(2) Taking filtration $M \supset R(M) \supset R^2(M) \supset \cdots \supset 0$ we see by part (1) that every composition factor of $M$ is of the form $J/R(J)$ for some $J \in J$. Thus the mapping $J \to S$, $J \mapsto J/R(J)$ is onto. Were $J/R(J) \cong J'/R(J')$ for some $J \neq J'$, the multiplicity $m$ of $J/R(J)$ in $M$ would be $\geq 2$. For, if $J \supset J'$, then refining $J \supset J' \supset 0$ we get $m \geq 2$. Else, we refine $J + J' \supset J > 0$, and get $m \geq 2$, again. By our assumption that $M$ is multiplicity free, the assertion follows.

(3) is [8 Lemma III.2]. It also follows immediately from part (2) as the composition length is $|S| = |J|$.

(4) Suppose there is a composition series of $M$ with $L$ preceding $L'$. Say $L = A/B$ and $L' = B/C$. By part (1) $J(L) \subset A$ and $J(L') \subset B$. Since $J(L) \not\subset B$, lest we have the multiplicity of $L \geq 2$, we see that $J(L) \not\subset J(L')$. Suppose $J(L)$ and $J(L')$ are incomparable. The refining $M \supset J(L) + J(L') \supset J(L') \supset 0$ we get $L$ before $L'$, and refining $M \supset J(L) + J(L') \supset J(L) \supset 0$ reverses the order of their appearance.

We return to modules $I(\lambda)$ assuming the datum to be classical. For every primitive vector $v_{\lambda, t}$ we define pair of sets $(S'(v_{\lambda, t}), S''(v_{\lambda, t}))$ by $S'(v_{\lambda, t}) = \{j \in S(\lambda) | s_j = e_j + 1 \}$ and $S''(v_{\lambda, t}) = \{j \in S(\lambda) | t_j = e_j + 1 \}$. We let $P = P(\lambda)$ denote the set of all pairs $(S', S'')$ with $S' \subset S(\lambda)$ and $S'' \subset S(\lambda)$. We turn $P$ into a poset by defining an ordering $(S', S'') \preceq (T', T'')$ if and only if $S' \subset T'$ and $S'' \subset T''$. The main result is

**Theorem 4.31.** In the foregoing notations with $H = H(D)$ and $D = D(H)$

(1) A submodule $J$ of $I(\lambda)$ is local if and only if $J$ is generated by a primitive weight vector.

(2) The poset $J$ of local submodules of $I(\lambda)$ is isomorphic to $P$.

(3) The composition length of $I(\lambda)$ equals $2^{e(\lambda) - 1}$
Proof: (1) Since all \(v_{s,t}\) have distinct weights every submodule \(M\) of \(I(\lambda)\) is the span of \(M \cap B\) where \(B = \{v_{s,t} | s, t \in \mathbb{Z}(m)\}\). Therefore, if \(J\) is generated by a single basis vector \(v \in B\), then were \(J = K + M\), \(v\) would lie in \(K\) or \(N\), hence \(J\) is local. Conversely, if \(J\) is local, then, as \(J = \sum_{v \in J \cap B} Dv\), we have \(J = Dv\) for some \(v\).

Let \(v = v_{s,t}\) be a generator of \(J\). If \(s_k \neq 0\) for some \(k \notin S(\lambda)\) or \(s_k \neq e_k + 1\) for some \(k \in S(\lambda)\), then by Propositions 4.19 and 4.21 \(\xi_k v_{s,t}\) also generates \(J\). Thus we can assume that \(s_k = 0\) for every \(k \notin S(\lambda)\) and \(s_k = 0, e_k + 1\) for every \(k \in S(\lambda)\). We come to a similar conclusion about every \(t_k\), viz. \(t_k = 0\) if \(k \notin S(\lambda)\) and \(t_k = 0, e'_k + 1\) for every \(k \in S(\lambda)\) by using Propositions 4.19 and 4.23. This proves (1)

(2) Let \(\psi : J \rightarrow P\) be the mapping sending \(J\) to \(\psi(v) := (S'(v), S''(v))\) where \(v\) is a primitive generator of \(J\). By primitivity of \(v\), \(J\) is the span of all \(x^i y^j v\). Therefore Propositions 4.19(2) and 4.21 give the relation \(\psi(w) \leq \psi(v)\) for every primitive weight vector \(w \in J\). This shows first that if \(v\) and \(w\) generate \(J\) then \(\psi(w) = \psi(v)\), so that \(\psi\) is well-defined, and second that \(\psi\) is isotonic, which yields (2)

(3) By part (3) of the preceding proposition the composition length of \(I(\lambda)\) equals \(|J|\), hence \(|P|\). The latter is \(2^{|S(1)|+|S(2)|}\) which yields the claim by Theorem 4.28(4).

\[ \square \]

References

[1] J. Alperin, Diagrams for modules, J. Pure and Appl. Algebra 16 (1980), 111-119.
[2] N. Andruskiewitsch and M. Beattie, Irreducible representations of liftings of quantum planes, Lie theory and its applications in physics V, 414–423, World Sci. Publ., River Edge, NJ, 2004.
[3] N. Andruskiewitsch and H.-J.ürgen Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order \(p^3\), J. Algebra 209 (1998), 658-691.
[4] N. Andruskiewitsch and H.-J.ürgen Schneider, Pointed Hopf algebras. New directions in Hopf algebras, 1-68, Math. Sci. Res. Ins. Publ. 43, Cambridge Univ. Press, Cambridge, 2002.
[5] N. Andruskiewitsch and H.-J.ürgen Schneider, On the classification of finite-dimensional pointed Hopf algebras, to appear Ann. Math.
[6] M. Beattie, Duals of pointed Hopf algebras, J. Algebra 262 (2003), 54-76.
[7] M. Beattie, S. Dăscălescu, L. Grünefelder, Constructing Pointed Hopf Algebras by Ore Extensions, J. Algebra 225 (2000), 743-770.
[8] G. Birkhoff, Lattice Theory, 3rd edition, Coll. Publ., XXV, American Math. Society, Providence, R.I., 1967.
[9] H.-X. Chen, Finite-dimensional representations of a quantum double, J. Algebra 251 (2002), 751-789.
[10] W. Chin, L. Krop, Representation theory of liftings of quantum planes, To appear.
[11] C. Curtis, Representations of Lie algebras of classical type with applications to linear groups, *Journal of Mathematics and Mechanics* 9 (1960), 307-326.
[12] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycle—the quantum version, *Comm. Algebra* 22 (1994), 5715-5732.
[13] C. De Concini and V. G. Kac, Representations of quantum groups at roots of 1, in Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, (Paris 1989), Birkhäuser, Boston, 471-506.
[14] V. G. Drinfel’d, Quantum groups, in “Proceedings, International Congress of Mathematicians, Berkeley, CA 1987”, pp. 798-820.
[15] K. Erdmann, E.L. Green, N. Snashall and R. Taillefer, Representation theory of the Drinfeld doubles of a family of Hopf algebras, *J. Pure Appl. Algebra* 204 (2006), no. 2, 413–454.
[16] K. Goodearl and R. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, London Math. Soc. Student Texts 16 (Cambridge Univ. Press, Cambridge, 1989).
[17] V. Kac and P. Cheung, Quantum Calculus, Universitext, Springer-Verlag, 2002.
[18] L. Krop and D. Radford, Finite-dimensional Hopf algebras of rank one in characteristic zero, *J. Algebra* 302 (2006), no. 1, 214–230.
[19] L. Krop, D. Radford, Simple modules for the Drinfel’d double of a class of Hopf algebras, *Contemporary Mathematics* 420 (2006), 429-435.
[20] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, 1993.
[21] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, *J. Amer. Math. Soc.* 3 (1990), 257-296.
[22] P. Landrock, Finite Group algebras and Their Modules, London Math. Soc. Lecture Note Ser., 84, Cambridge Univ. Press, Cambridge, 1983.
[23] S. Majid, Quasitriangular Hopf algebras and Yang-Baxter equations, *Internat. J. Modern Physics A* 5 (1990), 1-91.
[24] S. Majid, Physics for Algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. *J. Algebra* 130 (1990), 17-64.
[25] S. Montgomery, Hopf Algebras and their Actions on Rings, in: *CMBS Reg. Conf. Ser. Math.*, 82, AMS, 1993.
[26] D. Radford, Hopf algebras with projection, *J. Algebra* 92 (1985), 322-347.
[27] D. Radford, Minimal quasitriangular Hopf algebras, *J. Algebra* 157 (1993), 285-315.
[28] D. Radford, On oriented quantum algebras derived from representations of the quantum double of a finite-dimensional Hopf algebra, *J. Algebra* 270 (2003), 670-695.
[29] David Radford and Hans-Jürgen Schneider, On the Simple Representations of Generalized Quantum Groups and Quantum Doubles, *J. Algebra*, To Appear
[30] D.E. Radford and J. Towber, Yetter-Drinfel’d categories associated to an arbitrary bialgebra, *J. Pure and Appl. Algebra* 87 (1993), 259-279.
[31] M. Rosso, An analogue of PBW theorem and the Universal R-matrix for $U_q sl(N + 1)$, *Commun. Math. Phys.* 124 (1989), 307-318.
[32] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[33] S. Yang, Representations of simple pointed Hopf algebras, *J. Algebra and Appl.* 3 (2004), 91-104.
DePaul University, Department of Mathematical Sciences, 2320 N. Kenmore, Chicago, IL 60614
E-mail address: lkrop@depaul.edu

University of Illinois at Chicago, Department of Mathematics, Statistics and Computer Science, 801 South Morgan Street, Chicago, IL 60607-7045
E-mail address: radford@uic.edu