**CAUCHY–SZEGÖ COMMUTATORS ON WEIGHTED MORREY SPACES**

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**Abstract.** In the setting of quaternionic Heisenberg group \( \mathbb{H}^{n-1} \), we characterize the boundedness and compactness of commutator \([b, C]\) for the Cauchy–Szegö operator \( C \) on the weighted Morrey space \( L^{p,n}_\omega(\mathbb{H}^{n-1}) \) with \( p \in (1, \infty) \), \( \kappa \in (0, 1) \) and \( \omega \in A_p(\mathbb{H}^{n-1}) \). More precisely, we prove that \([b, C]\) is bounded on \( L^{p,n}_\omega(\mathbb{H}^{n-1}) \) if and only if \( b \in \text{BMO}(\mathbb{H}^{n-1}) \). And \([b, C]\) is compact on \( L^{p,n}_\omega(\mathbb{H}^{n-1}) \) if and only if \( b \in \text{VMO}(\mathbb{H}^{n-1}) \).

1. Introduction and statement of main results

Since 1980s, it is an active direction to develop a theory for quaternionic regular functions of several variables instead of holomorphic functions on \( \mathbb{C}^n \). Let \( \mathbb{H} \) be the algebra of quaternion numbers and let \( \text{Re} \) and \( \text{Im} \) denote the real part and imaginary part of \( x \) respectively. Then \( \text{Re} x = x_1 \) and \( \text{Im} x = x_2 i + x_3 j + x_4 k \). The \( n \)-dimensional quaternionic space \( \mathbb{H}^n \) is the collection of \( n \)-tuples \((q_1, \ldots, q_n)\). For \( l \)-th coordinate of a point \( q = (q_1, \ldots, q_n) \in \mathbb{H}^n \) we write \( q_l = x_{4l-3} + x_{4l-2}i + x_{4l-1}j + x_{4l}k \). An \( \mathbb{H} \)-valued function \( f : \Omega \to \mathbb{H} \) over a domain \( \Omega \subset \mathbb{H}^n \) is called regular if \( \bar{x}f(q) = 0 \), where \( l = 1, \ldots, n \), and

\[
\bar{q}_l = \frac{\partial}{\partial x_{4l-3}} + i \frac{\partial}{\partial x_{4l-2}} + j \frac{\partial}{\partial x_{4l-1}} + k \frac{\partial}{\partial x_{4l}}.
\]

So far several fundamental results have been established for the quaternionic counterparts, e.g., Hartogs phenomenon, \( k \)-Cauchy–Fueter complexes, quaternionic Monge–Ampere equations, etc. (see for example \[12, 10, 30, 32\] and the references therein). Since the quaternions \( \mathbb{H} \) is non-commutative, the behavior of quaternionic regular functions is quite different from holomorphic functions, e.g. the product of two such functions is not regular in general. Hence, proofs and even statements of results are completely different from the standard setting of complex variables.

It is natural to consider the Hardy space of regular functions over a bounded domain in \( \mathbb{H}^n \), in particular, over the unit ball. By quaternionic Cayley transformation, it is equivalent to consider the Hardy space over the Siegel upper half domain

\[
U_n := \left\{ q = (q_1, \ldots, q_n) = (q_1, q') \in \mathbb{H}^n \mid \text{Re} q_1 > |q'|^2 \right\},
\]

where \( q' = (q_2, \ldots, q_n) \in \mathbb{H}^{n-1} \), whose boundary \( \partial U_n := \{(q_1, q') \in \mathbb{H}^n \mid \text{Re} q_1 = |q'|^2 \} \) is a quadratic hypersurface, which can be identified with the quaternionic Heisenberg group \( \mathbb{H}^{n-1} \).

For any function \( F : U_n \to \mathbb{H} \), we write \( F_\varepsilon \) for its “vertical translate”, where the vertical direction is given by the positive direction of \( \text{Re} q_1 \). If \( \varepsilon > 0 \), then \( F_\varepsilon \) is defined in a neighborhood of \( \partial U_n \). In particular, \( F_\varepsilon \) is defined on \( \partial U_n \). The Hardy space \( \mathcal{H}^2(U_n) \) consists of all regular functions

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The quaternionic Heisenberg group \( H^{n-1} \) plays the fundamental role in quaternionic analysis and geometry \([9, 17, 31, 26]\). Its analytic and geometric behaviours are different from the usual Heisenberg group in many aspects, e.g., there does not exist nontrivial quasiconformal mappings between the quaternionic Heisenberg groups \([25]\) while quasiconformal mappings between Heisenberg groups are abundant \([21, 22]\).

The Cauchy–Szegő projection operator \( \mathcal{C} \) can be defined via the “vertical translate” from Cauchy–Szegő kernel for \( \mathcal{U}_n \) by

\[
(\mathcal{C} f)(q) = \lim_{\varepsilon \to 0} \int_{\partial \mathcal{U}_n} S(q + \varepsilon \cdot e, p) f(p) d\beta(p), \quad \forall f \in \mathcal{L}^2(\partial \mathcal{U}_n), \quad q \in \partial \mathcal{U}_n,
\]

where the limit exists in the \( \mathcal{L}^2(\partial \mathcal{U}_n) \) norm and \( \mathcal{C}(f) \) is the boundary limit of some function in \( \mathcal{H}^2(\mathcal{U}_n) \).

In view of the action of the quaternionic Heisenberg group, the operator \( \mathcal{C} \) can be explicitly described as a convolution operator on this group:

\[
(\mathcal{C} f)(g) = (f * K)(g) = p.v. \int_{\mathcal{H}^{n-1}} K(h^{-1} \cdot g) f(h) dh,
\]

where the kernel \( K(g) \) is defined in Section 2 below. We can write

\[
(\mathcal{C} f)(g) = p.v. \int_{\mathcal{H}^{n-1}} K(g, h) f(h) dh,
\]

where \( K(g, h) = K(h^{-1} \cdot g) \) for \( g \neq h \). Note that \(1.4\) holds whenever \( f \) is an \( \mathcal{L}^2 \) function supported in a compact set, for every \( g \) outside the support of \( f \).

In \([3]\), Chang et al. verify that the kernel \( K(g) \) is a standard Calderón–Zygmund kernel with respect to the quasi-metric \( \rho \) (defined in Section 2), that is, it satisfies the standard size and smoothness conditions in terms of \( \rho \).
Theorem B ([3]). Suppose $j = 1, \ldots, 4n - 4$, and we denote $Y_j$ the standard left-invariant vector fields on $\mathcal{H}^{n-1}$ (defined as in [2,3] in Section 2). Then we have

\begin{equation}
|Y_j K(g)| \lesssim \frac{1}{\rho(g,0)^{Q+1}}, \quad g \in \mathcal{H}^{n-1} \setminus \{0\},
\end{equation}

where 0 is the neutral element of $\mathcal{H}^{n-1}$ and $Q = 4n + 2$ is the homogeneous dimension of $\mathcal{H}^{n-1}$.

Then we further have the Cauchy–Szegő kernel $K(g, h)$ on $\mathcal{H}^{n-1}$ ($g \neq h$) satisfies the following conditions.

(i) $|K(g, h)| \lesssim \frac{1}{\rho(g, h)^{Q+1}}$;

(ii) $|K(g, h) - K(g_0, h)| \lesssim \frac{\rho(g, g_0)}{\rho(g_0, h)^{Q+1}}$, if $\rho(g_0, h) \geq c\rho(g, g_0)$;

(iii) $|K(g, h) - K(g, h_0)| \lesssim \frac{\rho(h, h_0)}{\rho(g, h_0)^{Q+1}}$, if $\rho(g, h_0) \geq c\rho(h, h_0)$

for some constant $c > 0$, where $\rho$ is defined in Section 2.

Theorem C ([3]). The Cauchy–Szegő kernel $K(\cdot, \cdot)$ on $\mathcal{H}^{n-1}$ satisfies the following pointwise lower bound: there exist a large positive constant $r_0$ and a positive constant $C$ such that for every $g \in \mathcal{H}^{n-1}$, there exists a “twisted truncated sector” $S_g \subset \mathcal{H}^{n-1}$ such that

$$
\inf_{g \in S_g} \rho(g, h) = r_0
$$

and that for every $g_1 \in B(g, 1)$ and $g_2 \in S_g$ we have

$$
|K(g_1, g_2)| \geq \frac{C}{\rho(g_1, g_2)^Q}.
$$

Moreover, this sector $S_g$ is regular in the sense that $|S_g| = \infty$ and that for every $R_2 > R_1 > 2r_0$

$$
|\{B(g, R_2) \setminus B(g, R_1)\} \cap S_g| \approx |B(g, R_2) \setminus B(g, R_1)|
$$

with the implicit constants independent of $g$ and $R_1, R_2$.

Using the above two theorems, the authors established the characterization of the BMO space and the VMO space via the commutator $[b, C]$ in [3]. It is well-known that the boundedness and compactness of Calderón–Zygmund operator commutators on certain function spaces and their characterizations play an important role in various area, such as harmonic analysis, complex analysis, (nonlinear) PDE, etc. Recently, equivalent characterizations of the boundedness and compactness of commutators were further extended to Morrey spaces over the Euclidean space by Di Fazio and Ragusa [12] and Chen et al. [6], and by Tao et al. [23, 24] for the Cauchy integral and Beurling-Ahlfors transformation commutator, respectively. Komori and Shirai [20] proved the boundedness of Calderón-Zygmund operator commutators with BMO functions over weighted Morrey spaces. In this article, we consider the boundedness and compactness characterizations of Cauchy–Szegő operator commutator $[b, C]$ on the weighted Morrey spaces over the quaternionic Heisenberg group.

Let $p \in (1, \infty)$. A non-negative function $w \in L^1_{\text{loc}}(\mathcal{H}^{n-1})$ is in $A_p(\mathcal{H}^{n-1})$ if

$$
[w]_{A_p(\mathcal{H}^{n-1})} := \sup_{B \subseteq \mathcal{H}^{n-1}} \left( \frac{1}{|B|} \int_B w(g)dg \right) \left( \frac{1}{|B|} \int_B w(g)^{-1/(p-1)}dg \right)^{p-1} < \infty,
$$

where the supremum is taken over all balls $B$ in $\mathcal{H}^{n-1}$. A non-negative function $w \in L^1_{\text{loc}}(\mathcal{H}^{n-1})$ is in $A_1(\mathcal{H}^{n-1})$ if there exists a constant $C$ such that for all balls $B \subseteq \mathcal{H}^{n-1}$,

$$
\frac{1}{|B|} \int_B w(g)dg \leq C \text{essinf}_{x \in B} w(g).
$$
For $p = \infty$, we define

$$A_{\infty}(\mathcal{H}^{n-1}) = \bigcup_{1 \leq p < \infty} A_p(\mathcal{H}^{n-1}).$$

Let $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $w \in A_p(\mathcal{H}^{n-1})$. The weighted Morrey space $L^p_w(\mathcal{H}^{n-1})$ (c.f. [19]) is defined by

$$L^p_w(\mathcal{H}^{n-1}) := \left\{ f \in L^p_{\text{loc}}(\mathcal{H}^{n-1}) : \|f\|_{L^p_w(\mathcal{H}^{n-1})} < \infty \right\}$$

with

$$\|f\|_{L^p_w(\mathcal{H}^{n-1})} := \sup_B \left\{ \frac{1}{|w(B)|^{\kappa}} \int_B |f(h)|^p w(h) \, dh \right\}^{1/p}.$$  

We get the boundedness characterization of Cauchy–Szegő operator commutator.

**Theorem 1.1.** Let $p \in (1, \infty)$, $\kappa \in (0, 1)$, $w \in A_p(\mathcal{H}^{n-1})$ and $b \in L^1_{\text{loc}}(\mathcal{H}^{n-1})$. Then the Cauchy–Szegő operator commutator $[b, C]$ has the following boundedness characterization:

(i) If $b \in \text{BMO}(\mathcal{H}^{n-1})$, then $[b, C]$ is bounded on $L^p_w(\mathcal{H}^{n-1})$.

(ii) If $b$ is real-valued and $[b, C]$ is bounded on $L^p_w(\mathcal{H}^{n-1})$, then $b \in \text{BMO}(\mathcal{H}^{n-1})$.

Based on Theorem 1.1, we further obtain the compactness characterization of Cauchy–Szegő operator commutator.

**Theorem 1.2.** Let $p \in (1, \infty)$, $\kappa \in (0, 1)$, $w \in A_p(\mathcal{H}^{n-1})$ and $b \in \text{BMO}(\mathcal{H}^{n-1})$. Then the Cauchy–Szegő operator commutator $[b, C]$ has the following compactness characterization:

(i) If $b \in \text{VMO}(\mathcal{H}^{n-1})$, then $[b, C]$ is compact on $L^p_w(\mathcal{H}^{n-1})$.

(ii) If $b$ is real-valued and $[b, C]$ is compact on $L^p_w(\mathcal{H}^{n-1})$, then $b \in \text{VMO}(\mathcal{H}^{n-1})$.

This paper is organized as follows. In Section 2 we recall some necessary preliminaries on quaternionic Heisenberg groups. In Section 3 we give the proof of Theorem 1.1. The proof of Theorem 1.2 will be provided in Section 4.

**Notation:** Throughout this paper, $C$ will denote positive constant which is independent of the main parameters, but it may vary from line to line. By $f \lesssim g$, we shall mean $f \leq Cg$ for some positive constant $C$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \approx g$.

## 2. Preliminaries

Recall that the space $\mathbb{H}$ of quaternion numbers forms a division algebra with respect to the coordinate addition and the quaternion multiplication

$$xx' = (x_1 + x_2i + x_3j + x_4k)(x'_1 + x'_2i + x'_3j + x'_4k)$$

$$= x_1x'_1 - x_2x'_2 - x_3x'_3 - x_4x'_4 + (x_1x'_2 + x_2x'_1 + x_3x'_4 - x_4x'_3) i$$

$$+ (x_1x'_3 - x_2x'_4 + x_3x'_1 + x_4x'_2) j + (x_1x'_4 + x_2x'_3 - x_3x'_2 + x_4x'_1) k,$$

for any $x = x_1 + x_2i + x_3j + x_4k$, $x' = x'_1 + x'_2i + x'_3j + x'_4k \in \mathbb{H}$. The conjugate $\bar{x}$ is defined by

$$\bar{x} = x_1 - x_2i - x_3j - x_4k,$$

and the modulus $|x|$ is defined by

$$|x|^2 = xx = \sum_{j=1}^{4} x_j^2.$$
The conjugation inverses the product of quaternion number in the following sense $\bar{q} \cdot \bar{\sigma} = \bar{\sigma} \cdot \bar{q}$ for any $q, \sigma \in \mathbb{H}$. It is clear that

$$\text{Im}(\bar{x}x') = \text{Im}\{(x_1 - x_2i - x_3j - x_4k)(x_1' + x_2i + x_3j + x_4'k)\}$$

$$= (x_1x_2' - x_2x_1' - x_3x_4' + x_4x_3') i + (x_1x_3' + x_2x_4' - x_3x_1' - x_4x_2') j$$

$$+ (x_1x_4' - x_2x_3' + x_3x_2' - x_4x_1') k$$

\[ (2.1) \]

where $i_1 = i, i_2 = j, i_3 = k$, and $b_{k,j}^n$ is the $(k, j)$ th entry of the following matrices $b^n$:

$$b^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad b^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad b^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.\]$$

The underlying vector space of the quaternion space $\mathbb{H}^n$ is $\mathbb{R}^{4n}$ and that of the pure imaginary $\text{Im}\mathbb{H}$ is $\mathbb{R}^3$.

The quaternionic Heisenberg group $\mathcal{H}^{n-1}$ is the space $\mathbb{R}^{4n-1} = \mathbb{R}^3 \times \mathbb{R}^{4(n-1)}$, which is the underlying vector space of $\text{Im}\mathbb{H} \times \mathbb{H}^{n-1}$, endowed with the non-commutative multiplication

\[ (t, y) \cdot (t', y') = (t + t' + 2\text{Im}(y, y'), y + y'), \]

where $t = t_1i + t_2j + t_3k$, $t' = t'_1i + t'_2j + t'_3k \in \text{Im}\mathbb{H}$, $y, y' \in \mathbb{H}^{n-1}$, and $\langle \cdot, \cdot \rangle$ is the inner product defined by

\[ \langle y, y' \rangle = \sum_{l=1}^{n-1} \bar{y}_l y'_l, \quad y = (y_1, \cdots, y_{n-1}), \quad y' = (y'_1, \cdots, y'_{n-1}) \in \mathbb{H}^{n-1}.\]

It is easy to check that the identity of $\mathcal{H}^{n-1}$ is the origin $0 := (0, 0)$, and the inverse of $(t, y)$ is given by $(-t, -y)$.

The boundary of quaternionic Siegel upper half-space $\partial \mathcal{U}_n$ can be identified with the quaternionic Heisenberg group $\mathcal{H}^{n-1}$ via the projection

\[ \pi : \partial \mathcal{U}_n \longrightarrow \text{Im}\mathbb{H} \times \mathbb{H}^{n-1}, \]

\[ (|q|^2 + x_2i + x_3j + x_4k, q') \longmapsto (x_2i + x_3j + x_4k, q'). \]

Let $d\beta$ be the Lebesgue measure on $\partial \mathcal{U}_n$ obtained by pulling back the Haar measure on the group $\mathcal{H}^{n-1}$ by the projection $\pi$.

By (2.1), the multiplication of the quaternionic Heisenberg group in terms of real variables can be written as (cf. [27])

\[ (t, y) \cdot (t', y') = \left( t + t' + 2 \sum_{l=0}^{n-1} \sum_{k=1}^{4} b_{k,j}^n y_{l+k} y_{l+j}, y + y' \right), \]

\[ (2.4) \]

where $t = (t_1, t_2, t_3)$, $t' = (t'_1, t'_2, t'_3) \in \mathbb{R}^3$, $t = (y_1, y_2, \cdots, y_{4n-4})$, $y' = (y'_1, y'_2, \cdots, y'_{4n-4}) \in \mathbb{R}^{4n-4}$.

The following vector fields are left invariant on the quaternionic Heisenberg group by the multiplication laws of the quaternionic Heisenberg group in (2.4):

\[ Y_{l+j} = \frac{\partial}{\partial y_{l+j}} + \sum_{k=1}^{4} b_{k,j}^n y_{l+k} \frac{\partial}{\partial t_k}, \]

\[ (2.5) \]
for \( l, l' = 0, \cdots, n - 2 \), \( j, k = 1, \cdots, 4 \). Then the horizontal tangent space at \( g \in \mathscr{H}_n \), denoted by \( g \), is spanned by the left invariant vectors \( Y_1(g), \cdots, Y_{4n-4}(g) \). For each \( g \in \mathscr{H}_n \), we fix a quadratic form \((\cdot, \cdot)_H\) on \( H \) with respect to which the vectors \( Y_1(g), \cdots, Y_{4n-4}(g) \) are orthonormal.

For any \( p = (t, y) \in \mathscr{H}_n \), we can associate the automorphism \( \tau_p \) of \( \mathcal{U}_n \):

\[
(2.6) \quad \tau_p : (q_1, q') \mapsto (q_1 + \| y \|^2 + t + 2(y, q') + y).
\]

It is obviously extended to the boundary \( \partial \mathcal{U}_n \). It is easy to see that the action on \( \partial \mathcal{U}_n \) is transitive. In particular, we have

\[
\tau_p : (0, 0) \mapsto (\| y \|^2 + t, y).
\]

And we can write each \( q \in \partial \mathcal{U}_n \) as \( q = \tau_0(0) \) for a unique \( g \in \mathscr{H}_n \). In this correspondence we have that \( d\beta(q) = dg \), the invariant measure on \( \mathscr{H}_n \). Similarly, we write \( p \in \partial \mathcal{U}_n \) in the form \( p = \tau_0(0) \). Then from (2.6) we can see that

\[
S(q + \varepsilon e, p) = S(\tau_n(0 + \varepsilon e), \tau_n(0)) = S(\tau_n(0 + \varepsilon e, 0) = S(\tau_n(0 + \varepsilon e, 0).
\]

Take \( K(\varepsilon) = S(\tau_n(0 + \varepsilon e, 0), \tau_n(0)) = (\mathcal{C}f)(\varepsilon)(0) = (\mathcal{C}f)(\varepsilon)(0) \) by abuse of notations. Then

\[
(2.7) \quad (\mathcal{C}f)(\varepsilon) = \lim_{\varepsilon \to 0} \int_{\mathscr{H}_n \setminus 1} K(\varepsilon)(h^{-1} \cdot g) f(h) \, dh, \quad f \in L^2(\mathscr{H}_n),
\]

where the limit is taken in \( L^2(\mathscr{H}_n) \).

Recall that the convolution on \( \mathscr{H}_n \) is defined as

\[
(f * \tilde{f})(\varepsilon) = \int_{\mathscr{H}_n \setminus 1} f(h) \tilde{f}(h^{-1} \cdot g) \, dh.
\]

Therefore, (2.7) can be formally rewritten as

\[
(\mathcal{C}f)(\varepsilon) = (f * K)(\varepsilon),
\]

where \( K \) is the distribution given by \( \lim_{\varepsilon \to 0} K(\varepsilon) \). Thus, if \( g = (t, y) \in \mathscr{H}_n \) with \( t = t_1 i + t_2 j + t_3 k \), then

\[
(2.8) \quad K(g) = \lim_{\varepsilon \to 0} K(\varepsilon)(g) = s(\| y \|^2 + t).
\]

For any \( g = (t, y) \in \mathscr{H}_n \), the homogeneous norm of \( g \) is defined by

\[
\| g \| = \left( \| y \|^4 + \sum_{j=1}^{3} |t_j|^2 \right)^{\frac{1}{4}}.
\]

Obviously, \( \| g^{-1} \| = \| - g \| = \| g \| \) and \( \| \delta_r(g) \| = r \| g \| \), where \( \delta_r, r > 0 \), is the dilation on \( \mathscr{H}_n \), which is defined as

\[
\delta_r(t, y) = (r^2 t, ry).
\]

On \( \mathscr{H}_n \), we define the quasi-distance

\[
\rho(h, g) = \| g^{-1} \cdot h \|.
\]

This is a standard definition in general stratified Lie group, see for example [13 Chapter 1]. It is clear that \( \rho \) is symmetric and satisfies the generalized triangle inequality

\[
(2.9) \quad \rho(h, g) \leq C_\rho(\rho(h, w) + \rho(w, g)),
\]
for any $h, g, w \in \mathcal{H}^{n-1}$ and some $C_p > 0$. Using $\rho$, we define the balls $B(g, r)$ in $\mathcal{H}^{n-1}$ by $B(g, r) = \{ h : \rho(h, g) < r \}$. Then
\[ |B(g, r)| \approx r^Q, \]
where $Q = 4n + 2$ is the homogeneous dimension of $\mathcal{H}^{n-1}$.

We now recall the BMO and VMO spaces. Note that $\mathcal{H}^{n-1}$ falls into the scope of homogeneous group, and hence we have the natural BMO space in this setting due to Folland and Stein [13]. To be self-enclosed, we recall the definition of the BMO space.

\[ \text{BMO}(\mathcal{H}^{n-1}) = \{ b \in L^1_{\text{loc}}(\mathcal{H}^{n-1}) : \| b \|_{\text{BMO}(\mathcal{H}^{n-1})} < \infty \}, \]

where
\[ \| b \|_{\text{BMO}(\mathcal{H}^{n-1})} = \sup_B \frac{1}{|B|} \int_B |b(g) - b_B| dg, \]
where the supremum is taken over all balls $B \subset \mathcal{H}^{n-1}$ and $b_B = \frac{1}{|B|} \int_B b(g) dg$. According to [11, Lemma 5.2],
\[ \| b \|_{\text{BMO}(\mathcal{H}^{n-1})} \approx \| b \|^p_{\text{BMO}^p(\mathcal{H}^{n-1})}, \]
for $1 \leq p < \infty$, where
\[ \| b \|^p_{\text{BMO}^p(\mathcal{H}^{n-1})} = \sup_B \left( \frac{1}{|B|} \int_B |b(g) - b_B|^p dg \right)^{\frac{1}{p}}. \]

Similarly we also have the VMO space on $\mathcal{H}^{n-1}$, which is the closure of $C_c(\mathcal{H}^{n-1})$ under the norm of $\| \cdot \|_{\text{BMO}(\mathcal{H}^{n-1})}$, see [7] for more details of the definition and properties of this VMO space in the more general setting, the stratified Lie group.

3. Boundedness characterization of Cauchy–Szegő commutators

In this section, we will give the proof of Theorem [14]. Here and hereafter, let
\[ M(f; B) := \frac{1}{|B|} \int_B |f(g) - f_B| \, dg \quad \text{with} \quad f_B = \frac{1}{|B|} \int_B f(g) \, dg. \]

We first recall the median value $\alpha_B(f)$ (c.f. [7]). For any real-valued function $f \in L^1_{\text{loc}}(\mathcal{H}^{n-1})$ and ball $B \subset \mathcal{H}^{n-1}$, let $\alpha_B(f)$ be a real number such that
\[ \inf_{c \in \mathbb{R}} \frac{1}{|B|} \int_B |f(g) - c| \, dg \]
is attained. Moreover, it is known that $\alpha_B(f)$ satisfies that
\[ |\{ g \in B : f(g) > \alpha_B(f) \}| \leq \frac{|B|}{2} \]
and
\[ |\{ g \in B : f(g) < \alpha_B(f) \}| \leq \frac{|B|}{2}. \]

Recall that an absolutely continuous curve $\gamma : [0, 1] \to \mathcal{H}^{n-1}$ is horizontal if its tangent vectors $\dot{\gamma}(t), t \in [0, 1]$, lie in the horizontal tangent space $H_{\gamma(t)}$. By [3], any given two points $p, q \in \mathcal{H}^{n-1}$ can be connected by a horizontal curve.

The Carnot–Carathéodory metric on $\mathcal{H}^{n-1}$ as follows. For $g, h \in \mathcal{H}^{n-1}$,
\[ d_{cc}(g, h) := \inf_\gamma \int_0^1 \| \dot{\gamma}(t) \|_H^{\frac{1}{2}} \, dt, \]
where $\gamma : [0, 1] \to \mathcal{H}^{n-1}$ is a horizontal Lipschitz curve with $\gamma(0) = g$, $\gamma(1) = h$. It is known that the Carnot–Carathéodory metric $d_{cc}$ is left-invariant, and it is equivalent to
the homogeneous metric $\rho$ in the sense that: there exist $C_d, \tilde{C}_d > 0$ such that for any $g, h \in \mathcal{H}^{n-1}$ (see \cite{18} (1.21)),

\begin{equation}
C_d \rho(g, h) \leq d_{cc}(g, h) \leq \tilde{C}_d \rho(g, h).
\end{equation}

In order to prove Theorem \ref{thm1} we need the following lemmas. Denote $w(B) := \int_B w(g) \, dg$.

**Lemma 3.1.** Let $w \in A_p(\mathcal{H}^{n-1}), p \geq 1$. Then there exist constants $\hat{C}_1, \hat{C}_2, \hat{C} > 0$ and $\sigma \in (0, 1)$ such that

\[ \hat{C}_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq \hat{C}_2 \left( \frac{|E|}{|B|} \right)^\sigma \]

for any measurable subset $E$ of a ball $B$. Especially, for any $\lambda > 1$,

\[ w(B(g_0, \lambda R)) \leq \hat{C}_\lambda w(B(g_0, R)). \]

**Lemma 3.2 (\cite{14}).** Let $b \in \text{BMO}(\mathcal{H}^{n-1})$ and $T$ be a Calderón-Zygmund operator. If $1 < p < \infty$ and $w \in A_p(\mathcal{H}^{n-1})$, then $[b, T]$ is bounded on $L^p_w(\mathcal{H}^{n-1})$.

**Lemma 3.3 (\cite{3}).** Since $K(g, h)$ is $\mathbb{H}$-valued, we write

\[ K(g, h) = K_1(g, h) + K_2(g, h)i + K_3(g, h)j + K_4(g, h)k, \]

where each $K_i(g, h)$ is real-valued, $i = 1, 2, 3, 4$. Then there is at least one of the $K_i$ above satisfies the following argument:

There exist positive constants $3 \leq A_1 \leq A_2$ such that for any ball $B := B(g_0, r) \subset \mathcal{H}^{n-1}$, there exist another ball $\hat{B} := B(h_0, r) \subset \mathcal{H}^{n-1}$ such that $A_1 r \leq d_{cc}(g_0, h_0) \leq A_2 r$, and for all $(g, h) \in (B \times \hat{B})$, $K_i(g, h)$ does not change sign and

\begin{equation}
|K_i(g, h)| \geq \frac{C}{p(g, h)\sigma}.
\end{equation}

**Proof of Theorem \ref{thm1} (i).** Let $1 < p < \infty$. It is sufficient to prove that

\[ \left\{ \frac{1}{[w(B)]^\alpha} \int_B ||b, C|| w(g) \, dg \right\}^{1/p} \lesssim ||b||_{\text{BMO}(\mathcal{H}^{n-1})} \|f\|_{L^p_w(\mathcal{H}^{n-1})}, \]

holds for any ball $B$.

Now fix a ball $B = B(g_0, r)$ and decompose $f = f_{2B} + f_{\mathcal{H}^{n-1}\setminus 2B} =: f_1 + f_2$. Then

\[ \frac{1}{w(B)^\alpha} \int_B ||b, C|| f_1 g \, w(g) \, dg \]

\[ \lesssim \left\{ \frac{1}{w(B)^\alpha} \int_B ||b, C|| f_1 g \, w(g) \, dg + \frac{1}{w(B)^\alpha} \int_B ||b, C|| f_2 g \, w(g) \, dg \right\}^{1/p} \]

\[ =: I + II. \]

For the term $I$, by Lemma \ref{lem2}, we can obtain

\[ \frac{1}{w(B)^\alpha} \int_B ||b, C|| f_1 g \, w(g) \, dg \leq \frac{1}{w(B)^\alpha} \int_{\mathcal{H}^{n-1}} ||b, C|| f_1 g \, w(g) \, dg \]

\[ \lesssim ||b||_{\text{BMO}(\mathcal{H}^{n-1})} \frac{1}{w(B)^\alpha} \int_{2B} |f(g)| w(g) \, dg \]

\[ \lesssim ||b||_{\text{BMO}(\mathcal{H}^{n-1})} ||f||_{L^p_w(\mathcal{H}^{n-1})}. \]

Thus, we have

\[ ||[b, C]f_1||_{L^p_w(\mathcal{H}^{n-1})} \lesssim ||b||_{\text{BMO}(\mathcal{H}^{n-1})} ||f||_{L^p_w(\mathcal{H}^{n-1})}. \]

For the term $II$, observe that for $g \in B$, by Theorem B, we have

\[ ||[b, C]f_2||^p \]
\[
\begin{align*}
&\leq \left( \int_{\mathbb{R}^n} |b(g) - b(u)| |K(g, u)| f_2(u) |du| \right)^p \\
&\lesssim \left( \int_{\mathbb{R}^n} \frac{|b(g) - b(u)|}{\rho(g, u)^Q} |f(u)| |du| \right)^p \\
&\lesssim \left( \int_{\mathbb{R}^n} \frac{|f(u)|}{\rho(g_0, u)^Q} \{|b(g) - b_{B,w}| + |b_{B,w} - b(u)|\} |du| \right)^p \\
&\lesssim \left( \int_{\mathbb{R}^n} \frac{|f(u)|}{\rho(g_0, u)^Q} |du| \right)^p |b(g) - b_{B,w}|^p + \left( \int_{\mathbb{R}^n} \frac{|f(u)|}{\rho(g_0, u)^Q} |b_{B,w} - b(u)| |du| \right)^p,
\end{align*}
\]

where \( b_{B,w} = \frac{1}{w(B)} \int_B b(g) w(g) dg \). Then we have

\[
\begin{align*}
\frac{1}{w(B)^{\kappa}} \int_B \|b, C\| f_2(g)^p w(g) dg & \\
&\lesssim \frac{1}{w(B)^{\kappa}} \left( \int_{\mathbb{R}^n} \frac{|f(u)|}{\rho(g_0, u)^Q} |du| \right)^p \int_B |b(g) - b_{B,w}|^p w(g) dg \\
&\quad + \left( \int_{\mathbb{R}^n} \frac{|f(u)|}{\rho(g_0, u)^Q} |b_{B,w} - b(u)| |du| \right)^p w(B)^{1-\kappa} \\
&=: III + IV.
\end{align*}
\]

For \( III \), by the Hölder inequality, Theorem 3.5 in [15] and Lemma 3.1, we have

\[
III \lesssim \|f\|^p_{L^p_{\infty}(\mathbb{R}^n)} \left( \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{1-p}} \right)^p \int_B |b(g) - b_{B,w}|^p w(g) dg \\
\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|^p_{L^p_{\infty}(\mathbb{R}^n)} \left( \sum_{j=1}^{\infty} \frac{w(B) \frac{1-p}{p}}{w(2^{j+1}B)^{1-p}} \right)^p \\
\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|^p_{L^p_{\infty}(\mathbb{R}^n)}.
\]

For \( IV \), by the Hölder inequality, we can get

\[
IV \lesssim \left( \sum_{j=1}^{\infty} \frac{1}{2^{j}B} \int_{2^{j+1}B} |f(u)||b(u) - b_{B,w}| |du| \right)^p w(B)^{1-\kappa} \\
\lesssim \left\{ \sum_{j=1}^{\infty} \frac{1}{2^{j}B} \left( \int_{2^{j+1}B} |f(u)|^p w(u) |du| \right)^{\frac{1}{p}} \right\} \left( \int_{2^{j+1}B} |b(u) - b_{B,w}|^{p'} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}} \\
\lesssim \|f\|^p_{L^p_{\infty}(\mathbb{R}^n)} \left\{ \sum_{j=1}^{\infty} \frac{w(2^{j+1}B)^{\frac{p}{p'}}}{2^{j}B} \left( \int_{2^{j+1}B} |b(u) - b_{B,w}|^{p'} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}} \right\} w(B)^{1-\kappa}.
\]

Observe that

\[
\left( \int_{2^{j+1}B} |b(u) - b_{B,w}|^{p'} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}} \\
\leq \left( \int_{2^{j+1}B} \left\{ |b(u) - b_{2^{j+1}B,w}^{1-p'}| + |b_{2^{j+1}B,w}^{1-p'} - b_{B,w}| \right\}^{p'} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}} \\
\leq \left( \int_{2^{j+1}B} |b(u) - b_{2^{j+1}B,w}^{1-p'}|^{p'} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}} + |b_{2^{j+1}B,w}^{1-p'} - b_{B,w}| \left( \int_{2^{j+1}B} w(u)^{-1-p'} |du| \right)^{\frac{1}{p'}}.
\]
exist \[\text{This completes the proof.}\]

Similarly, we have

Together with Lemma 3.1, we have

Since \(b \in \text{BMO}(\mathcal{H}^{n-1})\), by the John-Nirenberg inequality (c.f. [23, Proposition 6]), there exist \(C_1 > 0\) and \(C_2 > 0\) such that for any ball \(B\) and \(\alpha > 0\),

\[
\{ (g \in B : |b(g) - b_B| > \alpha) \} \leq C_1 |B| e^{-\frac{C_2 \alpha}{|B| \text{BMO}(\mathcal{H}^{n-1})}},
\]

Then by Lemma 3.1 we have

\[
w \{ (g \in B : |b(g) - b_B| > \alpha) \} \leq C_1 w(B) e^{-\frac{C_2 \alpha}{|B| \text{BMO}(\mathcal{H}^{n-1})}},
\]

for some \(\sigma \in (0, 1)\). Therefore,

\[
\int_B |b(u) - b_B| w(u) du = \int_0^\infty w \{ (g \in B : |b(g) - b_B| > \alpha) \} d\alpha \\
\lesssim w(B) \int_0^\infty e^{-\frac{C_2 \alpha}{|B| \text{BMO}(\mathcal{H}^{n-1})}} d\alpha \\
\lesssim w(B) \| b \|_{\text{BMO}(\mathcal{H}^{n-1})}.
\]

Similarly, we have

\[
\left( \int_{2j+1B} |b(u) - b_{2j+1B}| w(u)^{1-p'} du \right)^{\frac{1}{p'}} \lesssim (j + 1) \| b \|_{\text{BMO}(\mathcal{H}^{n-1})} w^{1-p'}(2j+1B)^{1/p'}.
\]

Together with Lemma 3.1 we have

\[
IV \lesssim \| f \|_{L_p^c(\mathcal{H}^{n-1})}^p \| b \|_{\text{BMO}(\mathcal{H}^{n-1})}^p \left[ \sum_{j=1}^{\infty} \frac{w(2j+1B)^{1-p}}{2jB} \left( j + 1 \right) w^{1-p'}(2j+1B)^{1/p'} \right]^p w(B)^{1-k}\]

\[
\lesssim \| f \|_{L_p^c(\mathcal{H}^{n-1})}^p \| b \|_{\text{BMO}(\mathcal{H}^{n-1})}^p \left[ \sum_{j=1}^{\infty} \frac{(j + 1) w(B)^{1-k}}{w(2j+1B)^{1-n}} \right]^p
\]

\[
\lesssim \| f \|_{L_p^c(\mathcal{H}^{n-1})}^p \| b \|_{\text{BMO}(\mathcal{H}^{n-1})}^p \left[ \sum_{j=1}^{\infty} (j + 1) 2^{-(2j+1)(1-n)/p} \right]^p
\]

As a consequence, we have

\[
\| [b, C] f_2 \|_{L_p^c(\mathcal{H}^{n-1})} \lesssim \| f \|_{L_p^c(\mathcal{H}^{n-1})} \| b \|_{\text{BMO}(\mathcal{H}^{n-1})}.
\]

This completes the proof.
To prove that $b \in \text{BMO}(\mathcal{H}^{n-1})$, it suffices to show that, for any ball $B \subset \mathcal{H}^{n-1}$, $M(b; B) \lesssim 1$. Let $B = B(g_0, r)$ be a ball in $\mathcal{H}^{n-1}$. Let $\tilde{B} := B(h_0, r) \subset \mathcal{H}^{n-1}$ be the ball in Lemma 3.3. Let

$$E_1 := \{ g \in B : b(g) \geq \alpha_{\tilde{B}}(b) \} \quad \text{and} \quad E_2 := \{ g \in B : b(g) < \alpha_{\tilde{B}}(b) \};$$

$$F_1 = \{ u \in \tilde{B} : b(u) \leq \alpha_{\tilde{B}}(b) \} \quad \text{and} \quad F_2 = \{ u \in \tilde{B} : b(u) \geq \alpha_{\tilde{B}}(b) \}.$$

From (3.1) and (3.2) we can see $|F_1| \geq \frac{1}{2} |\tilde{B}| = \frac{1}{2} |B|$ and $|F_2| \geq \frac{1}{2} |\tilde{B}| = \frac{1}{2} |B|$. For any $(g, u) \in E_j \times F_j$, $j \in \{1, 2\}$,

$$|b(g) - b(u)| = |b(g) - \alpha_{\tilde{B}}(b)| + |\alpha_{\tilde{B}}(b) - b(u)| \geq |b(g) - \alpha_{\tilde{B}}(b)|.$$

Since $b$ is real-valued, from Lemma 3.3 the Hölder inequality and the boundedness of $[b, C]$ on $L^p_w(\mathcal{H}^{n-1})$, we deduce that

$$M(b; B) \lesssim \frac{1}{|B|} \int_B |b(g) - \alpha_{\tilde{B}}(b)| \, dg \approx \sum_{j=1}^2 \frac{1}{|B|} \int_{E_j} |b(g) - \alpha_{\tilde{B}}(b)| \, dg \approx \sum_{j=1}^2 \frac{1}{|B|} \int_{F_j} \frac{|b(g) - \alpha_{\tilde{B}}(b)|}{\rho(g, u)^Q} \, du \, dg \approx \sum_{j=1}^2 \frac{1}{|B|} \int_{E_j} \frac{\int_{F_j} [b(g) - b(u)]K_1(g, u) \, du}{\rho(g, u)^Q} \, du \, dg \approx \sum_{j=1}^2 \frac{1}{|B|} \int_{E_j} \| [b, C] \chi_{F_j}(g) \|_{L^p_w(\mathcal{H}^{n-1})} \, dg \lesssim \sum_{j=1}^2 \frac{1}{|B|} \| [b, C] \chi_{F_j} \|_{L^p_w(\mathcal{H}^{n-1})} \| w(B) \|_{L^{\infty}_w(\mathcal{H}^{n-1})} \| w(B) \|_{L^{\infty}_w(\mathcal{H}^{n-1})} \| w(B) \|_{L^{\infty}_w(\mathcal{H}^{n-1})} \| w(B) \|_{L^{\infty}_w(\mathcal{H}^{n-1})}.$$

This finishes the proof of Theorem 1.1.

4. Compactness characterization of Cauchy–Szegő commutators

In this section, we will give the proof of Theorem 1.2.

4.1. Proof of Theorem 1.2(i). We first give a sufficient condition for subsets of weighted Morrey spaces to be relatively compact. Recall that a subset $\mathcal{F}$ of $L^p_w(\mathcal{H}^{n-1})$ is said to be totally bounded (or relatively compact) if the $L^p_w(\mathcal{H}^{n-1})$-closure of $\mathcal{F}$ is compact.

Lemma 4.1. For any $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $w \in A_p(\mathcal{H}^{n-1})$, a subset $\mathcal{F}$ of $L^p_w(\mathcal{H}^{n-1})$ is totally bounded (or relatively compact) if the set $\mathcal{F}$ satisfies the following three conditions:

(i) $\mathcal{F}$ is bounded, namely,

$$\sup_{f \in \mathcal{F}} \| f \|_{L^p_w(\mathcal{H}^{n-1})} < \infty;$$

(ii) $\mathcal{F}$ is relatively compact.

(iii) $\mathcal{F}$ is totally bounded.

Proof. The proof follows from the definitions and properties of weighted Morrey spaces.
There exists a positive constant $M$ such that, for any $f \in F$,

$$\|f\chi\{g\in \mathcal{H}^{n-1}: \|g\|>M\}\|_{L^{p,n}_{loc}(\mathcal{H}^{n-1})} < \epsilon;$$

(iii) $F$ is uniformly equicontinuous, namely, for any $\epsilon \in (0, \infty)$, there exists some positive constant $\beta$ such that, for any $f \in F$ and $\xi \in \mathcal{H}^{n-1}$ with $\|\xi\| \in [0, \beta]$,

$$\|f(\xi) - f(\cdot)\|_{L^{p,n}_{loc}(\mathcal{H}^{n-1})} < \epsilon.$$ 

The proof of this lemma follows from [24, Theorem 1.5] by a minor modification from Euclidean setting to quaternionic Heisenberg group, since it only requires the following key elements of the underlying space: metric and doubling measure.

Before we give the proof of Theorem 1.2, we first need to establish the boundedness of the maximal operator $C_{\eta}$ of a family of smooth truncated Cauchy–Szegö transforms $\{C_{\eta}\}_{\eta \in (0, \infty)}$ as follows. For $\eta \in (0, \infty)$, let

$$C_{\eta}f(g) := \int_{\mathcal{H}^{n-1}} K_{\eta}(g, u)f(u) \, du,$$

where the kernel $K_{\eta}(g, u) := K(g, u)\varphi\left(\frac{\|g, u\|}{\eta}\right)$ with $\varphi \in C_{c}^{\infty}(\mathbb{R})$ satisfying that

$$\phi_{\pm}(g) = \begin{cases} 
\varphi(t) = 0 & \text{if } t \in (-\infty, \frac{1}{2}), \\
\varphi(t) \in [0, 1] & \text{if } t \in [\frac{1}{2}, 1], \\
\varphi(t) = 1 & \text{if } t \in (1, \infty).
\end{cases}$$

Let

$$[b, C_{\eta}]f(g) := \int_{\mathcal{H}^{n-1}} [b(g) - b(u)]K_{\eta}(g, u)f(u) \, du.$$

The maximal operator $C_{\eta}$ is defined by setting, for any suitable function $f$ and $g \in \mathcal{H}^{n-1}$,

$$C_{\eta}f(g) := \sup_{\eta \in (0, \infty)} \left| \int_{\mathcal{H}^{n-1}} K_{\eta}(g, u)f(u) \, du \right|.$$ 

Recall that the Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(g) := \sup_{B \ni g} \frac{1}{|B|} \int_{B} |f(u)| \, du,$$

for any $f \in L^{1}_{loc}(\mathcal{H}^{n-1})$ and $g \in \mathcal{H}^{n-1}$, where the supremum is taken over all the balls $B$ of $\mathcal{H}^{n-1}$ that contain $g$.

We now recall the mean value theorem on homogeneous groups (c.f. [13, Theorem 1.41]), which covers the quaternionic Heisenberg group.

**Lemma 4.2.** There exist $C > 0$ and $\gamma > 0$ such that for any $f \in C^{1}(\mathcal{H}^{n-1})$ and $g, u \in \mathcal{H}^{n-1}$,

$$|f(g \cdot u) - f(g)| \leq C\|u\| \sup_{\|\xi\| \leq \gamma\|u\|, 1 \leq j \leq 4n-4} |Y_{j}f(g \cdot \xi)|.$$ 

Denote $\nabla_{H} = (Y_{1}, \ldots, Y_{4n-4}).$ We have the following conclusions.

**Lemma 4.3.** There exists a positive constant $C$ such that, for any $b \in C_{c}^{\infty}(\mathcal{H}^{n-1})$, $\eta \in (0, \infty)$, $f \in L^{1}_{loc}(\mathcal{H}^{n-1})$ and $g \in \mathcal{H}^{n-1}$,

$$|[b, C_{\eta}]f(g) - [b, C]f(g)| \leq C\eta \|\nabla_{H}b\|_{L^{\infty}(\mathcal{H}^{n-1})} Mf(g).$$
Proof. Let \( f \in L^1_{\text{loc}}(\mathcal{H}^{n-1}) \). For any \( g \in \mathcal{H}^{n-1} \), we have

\[
\| b, C_n \| f(g) - [b, C] f(g) \|
\]

\[
= \int_{\eta/2 < \rho(g, u) \leq \eta} |b(g) - b(u)| K(g, u) f(u) \, du - \int_{\rho(g, u) \leq \eta} |b(g) - b(u)| K(g, u) f(u) \, du
\]

\[
\lesssim \int_{\rho(g, u) \leq \eta} |b(g) - b(u)| \| K(g, u) \| |f(u)| \, du.
\]

From the smoothness of \( b \), Lemma 4.2 and Theorem B, we deduce that

\[
\int_{\rho(g, u) \leq \eta} |b(g) - b(u)| \| K(g, u) \| |f(u)| \, du
\]

\[
\lesssim \| \nabla b \|_{L^\infty(\mathcal{H}^{n-1})} \sum_{j=0}^{\infty} \int_{\frac{\eta}{2^j} < \rho(g, u) \leq \frac{\rho(g, u)}{2^j}} \rho(g, u)^p |f(u)| \, du
\]

\[
\lesssim \eta \| \nabla b \|_{L^\infty(\mathcal{H}^{n-1})} \mathcal{M} f(g),
\]

which completes the proof of Lemma 4.4. \( \square \)

Lemma 4.4. Let \( p \in (1, \infty) \), \( \kappa \in (0, 1) \) and \( w \in A_p(\mathcal{H}^{n-1}) \). Then there exists a positive constant \( C \) such that, for any \( f \in L^p_w(\mathcal{H}^{n-1}) \),

\[
\| \mathcal{C}_* f \|_{L^p_w(\mathcal{H}^{n-1})} + \| \mathcal{M} f \|_{L^p_w(\mathcal{H}^{n-1})} \leq C \| f \|_{L^p_w(\mathcal{H}^{n-1})}.
\]

Proof. For the boundedness of \( \mathcal{M} \) on \( L^p_w(\mathcal{H}^{n-1}) \) one can refer to [2]. We only consider the boundedness of \( \mathcal{C}_* \). For any fixed ball \( B \subset \mathcal{H}^{n-1} \) and \( f \in L^p_w(\mathcal{H}^{n-1}) \), we write

\[
f := f_1 + f_2 := f \chi_{2B} + f \chi_{\mathcal{H}^{n-1}\setminus2B}.
\]

Observe \( f_1 \in L^p_w(\mathcal{H}^{n-1}) \). Then, from the boundedness of \( \mathcal{C}_* \) on \( L^p_w(\mathcal{H}^{n-1}) \) (see, for example, [10, Theorem 1.1]), the Hölder inequality, Theorem B, we deduce that

\[
\left( \int_B |\mathcal{C}_* f(g)|^p w(g) \, dg \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( \int_B |\mathcal{C}_* f_1(g)|^p w(g) \, dg \right)^{\frac{1}{p}} + \sum_{k=1}^{\infty} \left( \int_B \left( \int_{2k+1B \setminus 2kB} |f(u)| \frac{|w(u)|}{\rho(g, u)^p} \, du \right)^p w(g) \, dg \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( \int_{2B} |f(g)|^p w(g) \, dg \right)^{\frac{1}{p}} + \sum_{k=1}^{\infty} \left( \frac{w(B)}{2^kB} \left( \int_{2k+1B \setminus 2kB} |f(u)| \frac{|w(u)|}{\rho(g, u)^p} \, du \right)^p \right)^{\frac{1}{p}}
\]

\[
\lesssim \| f \|_{L^p_w(\mathcal{H}^{n-1})}^{\frac{\kappa}{\kappa-1}} |w(B)|^{\frac{1}{p}} + \sum_{k=1}^{\infty} \left( \frac{w(B)}{2^kB} \| f \|_{L^p_w(\mathcal{H}^{n-1})}^{\kappa-1} \right)^{\frac{1}{p}}
\]

\[
\lesssim \| f \|_{L^p_w(\mathcal{H}^{n-1})}^{\frac{\kappa}{\kappa-1}} |w(B)|^{\frac{1}{p}} + \sum_{k=1}^{\infty} \left( \frac{w(B)}{2^kB} \| f \|_{L^p_w(\mathcal{H}^{n-1})}^{\kappa-1} \right)^{\frac{1}{p}}
\]

\[
\lesssim \| f \|_{L^p_w(\mathcal{H}^{n-1})}^{\frac{\kappa}{\kappa-1}} |w(B)|^{\frac{1}{p}},
\]

where, in the fourth inequality, we used Lemma 3.1 with some \( \sigma \in (0, 1) \). This finishes the proof of Lemma 4.4. \( \square \)
Proof of Theorem 4.2(i). When \( b \in \text{VMO}(\mathcal{H}^{n-1}) \), for any \( \varepsilon \in (0, \infty) \), there exists \( b^{(\varepsilon)} \in C_c^\infty(\mathcal{H}^{n-1}) \) such that \( \|b - b^{(\varepsilon)}\|_{\text{BMO}(\mathcal{H}^{n-1})} < \varepsilon \). Then, from the boundedness of the commutator \([b, C] f\) on \( L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1}) \), we obtain

\[
\left\| [b, C] f - [b^{(\varepsilon)}, C] f \right\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})} = \left\| [b - b^{(\varepsilon)}, C] f \right\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})} \lesssim \left\| b - b^{(\varepsilon)} \right\|_{\text{BMO}(\mathcal{H}^{n-1})} \|f\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})} < \varepsilon \|f\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})}.
\]

Moreover, by using Lemmas 4.3 and 4.4, we get

\[
\lim_{\eta \to 0} \| [b, C_{\eta}] - [b, C] \|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})} = 0.
\]

Now it suffices to show that, for any \( b \in C_c^\infty(\mathcal{H}^{n-1}) \) and \( \eta \in (0, \infty) \) small enough, \([b, C_{\eta}]\) is a compact operator on \( L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1}) \), which is equivalent to show that, for any bounded subset \( F \subset L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1}) \), \([b, C_{\eta}]F\) is relatively compact. That is, we need to verify \([b, C_{\eta}]F\) satisfies the conditions (i) through (iii) of Lemma 4.4.

We first point out that, by \([20, \text{Theorem 3.4}]\) and the fact that \( b \in \text{BMO}(\mathcal{H}^{n-1}) \), we know that \([b, C_{\eta}]\) is bounded on \( L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1}) \) for the given \( p \in (1, \infty) \), \( \kappa \in (0, 1) \) and \( w \in A_p(\mathcal{H}^{n-1}) \), which implies that \([b, C_{\eta}]F\) satisfies condition (i) of Lemma 4.4.

Next, since \( b \in C_c^\infty(\mathcal{H}^{n-1}) \), we may further assume \( \|b\|_{L^\infty(\mathcal{H}^{n-1})} + \|\nabla b\|_{L^\infty(\mathcal{H}^{n-1})} = 1 \). Observe that there exists a positive constant \( R_0 \) such that \( \text{supp}(b) \subset B(0, R_0) \). Let \( M \in (10R_0, \infty) \). Thus, for any \( u \in B(0, R_0) \) and \( g \in \mathcal{H}^{n-1} \) with \( \|g\| \in (M, \infty) \), we have \( \rho(g, u) \sim \|g\| \). Then, for \( g \in \mathcal{H}^{n-1} \) with \( \|g\| > M \), by Theorem B and the Hölder inequality, we conclude that

\[
\left\| [b, C_{\eta}] f(g) \right\| \leq \int_{\mathcal{H}^{n-1}} |b(g) - b(u)| |K_{\eta}(g, u)| |f(u)| \, du \lesssim |b||_{L^\infty(\mathcal{H}^{n-1})} \int_{B(0, R_0)} \frac{|f(u)|}{\rho(g, u)^{\frac{1}{q}}} \, du \lesssim \frac{1}{\|g\|^q} \left( \int_{B(0, R_0)} |f(u)|^p w(u) \, du \right)^{\frac{1}{p}} \int_{B(0, R_0)} \left\{ \left| w(u) \right| \right\}^{\frac{q-1}{p}} \, du \lesssim \frac{1}{\|g\|^q} \left\| f \right\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})} \left| w(B(0, R_0)) \right|^{\frac{q-1}{p}} |B(0, R_0)|.
\]

Therefore, for any fixed ball \( B := B(\bar{g}, \bar{r}) \subset \mathcal{H}^{n-1} \), by Lemma 5.11 we have

\[
\frac{1}{|w(B)|^c} \int_{B \cap \{g \in \mathcal{H}^{n-1} : \|g\| > M\}} \left\| [b, C_{\eta}] f(g) \right\|^p w(g) \, dg \lesssim \frac{\left\| f \right\|_{L^{p, \kappa}_{\omega}(\mathcal{H}^{n-1})}^p \left| w(B(0, R_0)) \right|^{\kappa-1} |B(0, R_0)|^p}{\left| w(B) \right|^c} \sum_{j=0}^{\infty} \left[ w(B(0, 2^j M)) \right]^{1-\kappa} \frac{\left| w(B(0, 2^j M)) \right|^{1-\kappa} |B(0, R_0)|^p \left( \frac{w(B(0, 2^j M))}{|2^j M|^{1+p}} \right)^{\frac{q-1}{p}} \left[ \frac{w(B(0, M))}{M^{1+p}} \right]^{1-\kappa} \sum_{j=0}^{\infty} \frac{2^{2j+1} (1-\kappa)}{2^{2j} p}}.
\]
Thus, we conclude that
\[
\|\langle b, C_\eta f \rangle \chi_{\{g \in \mathcal{H}^{n-1}: \|g\| > M\}}\|_{L_{b,t}^p \mathcal{H}^{n-1}} \lesssim \left(\frac{R_0}{M}\right)^{kQ} \|f\|_{L_{b,t}^p \mathcal{H}^{n-1}}.
\]

Therefore, condition (ii) of Lemma 4.1 holds for \(b, C_\eta \mathcal{F}\) with \(M\) large enough.

It remains to prove that \([b, C_\eta] \mathcal{F}\) also satisfies condition (iii) of Lemma 4.1. Let \(\eta\) be a fixed positive constant small enough and \(\xi \in \mathcal{H}^{n-1}\) with \(\|\xi\| \in (0, \eta/8(1 + C_\rho))\). Then, for any \(g \in \mathcal{H}^{n-1}\), we have
\[
[b, C_\eta] f(g) - [b, C_\eta] f(g \cdot \xi) = [b(g) - b(g \cdot \xi)] \int_{\mathcal{H}^{n-1}} K_\eta(g, u) f(u) \, du + \int_{\mathcal{H}^{n-1}} [K_\eta(g, u) - K_\eta(g \cdot \xi, u)] [b(g \cdot \xi) - b(u)] f(u) \, du
\]
\[
=: L_1(g) + L_2(g).
\]

Since \(b \in C_c^\infty(\mathcal{H}^{n-1})\), by Lemma 4.2 it follows that, for any \(g \in \mathcal{H}^{n-1}\),
\[
|L_1(g)| = |b(g) - b(g \cdot \xi)| \int_{\mathcal{H}^{n-1}} K_\eta(g, u) f(u) \, du \lesssim \|\xi\| \|\nabla \mathcal{F}\|_{L_{b,t}^\infty(\mathcal{H}^{n-1})} C_s(f)(g).
\]

Then Lemma 4.4 implies \(\|L_1\|_{L_{b,t}^p \mathcal{H}^{n-1}} \lesssim \|\xi\| \|f\|_{L_{b,t}^p \mathcal{H}^{n-1}}\).

To estimate \(L_2(g)\), we first observe that \(K_\eta(g, u) = 0, K_\eta(g \cdot \xi, u) = 0\) for any \(g, u, \xi \in \mathcal{H}^{n-1}\) with \(\rho(g, u) \in (0, \eta/4(1 + C_\rho))\) and \(\|\xi\| \in (0, \eta/8(1 + C_\rho))\). Moreover, by the definition of \(K_\eta(g, u)\) and Theorem B, we know that, for any \(g, u \in \mathcal{H}^{n-1}\) with \(\rho(g, u) \in [\eta/4(1 + C_\rho), \infty)\),
\[
|K_\eta(g, u) - K_\eta(g \cdot \xi, u)| \lesssim \frac{\|\xi\|}{\rho(g, u)^{Q+1}}.
\]

This in turn implies that, for any \(g \in \mathcal{H}^{n-1}\),
\[
|L_2(g)| \lesssim \|\xi\| \int_{\rho(g, u) > \eta/4(1 + C_\rho)} \frac{|f(u)|}{\rho(g, u)^{Q+1}} \, du
\]
\[
\lesssim \sum_{k=0}^\infty \frac{\|\xi\|}{(2^k \eta)^{Q+1}} \int_{2^k \eta/4(1 + C_\rho) < \rho(g, u) \leq 2^{k+1} \eta/4(1 + C_\rho)} |f(u)| \, du
\]
\[
\lesssim \sum_{k=0}^\infty \frac{\|\xi\|}{(2^k \eta)^{Q+1}} \int_{B(g, 2^{k+1} \eta/4(1 + C_\rho))} |f(u)| \, du \lesssim \frac{\|\xi\|}{\eta} \mathcal{M} f(g).
\]

Then, by the boundedness of \(\mathcal{M}\) on \(L_{b,t}^p \mathcal{H}^{n-1}\), we obtain
\[
\|L_2\|_{L_{b,t}^p \mathcal{H}^{n-1}} \lesssim \frac{\|\xi\|}{\eta} \|f\|_{L_{b,t}^p \mathcal{H}^{n-1}}.
\]

Consequently, \([b, C_\eta] \mathcal{F}\) satisfies condition (iii) of Lemma 4.1. Thus, \([b, C_\eta]\) is a compact operator for any \(b \in C_c^\infty(\mathcal{H}^{n-1})\). This finishes the proof of Theorem 1.2. \(\square\)

4.2. Proof of Theorem 1.2(ii). We begin with recalling an equivalent characterization of \(VMO(\mathcal{H}^{n-1})\) from [7] Theorem 4.4.
Lemma 4.5. Let \( f \in \text{BMO}(\mathcal{H}^{n-1}) \). Then \( f \in \text{VMO}(\mathcal{H}^{n-1}) \) if and only if \( f \) satisfies the following three conditions:

(i) \( \lim_{a \to 0} \sup_{r_B \equiv a} M(f; B) = 0 \);
(ii) \( \lim_{a \to \infty} \sup_{r_B \equiv a} M(f; B) = 0 \);
(iii) \( \lim_{r \to \infty} \sup_{B \subset \mathcal{H}^{n-1} \setminus B(0, r)} M(f; B) = 0 \),

where \( r_B \) is the radius of the ball \( B \).

Next, we establish a lemma for the upper and the lower bounds of integrals of \([b, C]f_j\) on certain balls \( B_j \) in \( \mathcal{H}^{n-1} \) for any \( j \in \mathbb{N} \). It is easy to show that, for any \( f \in L^1_{\text{loc}}(\mathcal{H}^{n-1}) \) and ball \( B \subset \mathcal{H}^{n-1} \),

\[
M(f; B) \sim \frac{1}{|B|} \int_B |f(g) - \alpha_B(f)| \, dg
\]

with the equivalent positive constants independent of \( f \) and \( B \).

Lemma 4.6. Let \( p \in (1, \infty) \), \( \kappa \in (0, 1) \) and \( w \in A_p(\mathcal{H}^{n-1}) \). Suppose that \( b \in \text{BMO}(\mathcal{H}^{n-1}) \) is a real-valued function with \( \|b\|_{\text{BMO}(\mathcal{H}^{n-1})} = 1 \) and there exist \( \delta \in (0, \infty) \) and a ball \( B_0 = B(g_0, r_0) \subset \mathcal{H}^{n-1} \) with \( r_0 > 0 \), such that

\[
M(b; B_0) > \delta.
\]

Then there exist a real-valued function \( f_0 \in L^p_0(\mathcal{H}^{n-1}) \), positive constants \( K_0 \) large enough, \( C_0 \), \( \tilde{C}_1 \) and \( \tilde{C}_2 \), which are independent of \( g_0 \) and \( r_0 \), such that, for any integer \( k \geq K_0 \), \( \|f_0\|_{L^p_0(\mathcal{H}^{n-1})} \leq \tilde{C}_0 \),

\[
\int_{B^k_0} |[b, C] f_0(g)|^p w(g) \, dg \geq \tilde{C}_1 \frac{\delta^p}{A_2^{kpq}} [w(B_0)]^{\kappa - 1} w\left( A_2^k B_0 \right),
\]

where \( B^k_0 := A_2^{k-1} B_0 \) is the ball associates with \( A_2^k B_0 \) in Lemma 3.3. and

\[
\int_{A_2^{k+1} B_0 \setminus A_2^k B_0} |[b, C] f_0(g)|^p w(g) \, dg \leq \tilde{C}_2 \frac{1}{A_2^{kpq}} [w(B_0)]^{\kappa - 1} w\left( A_2^k B_0 \right).
\]

Proof. Set \( B_{0,1} = \{ g \in B_0 : b(g) > \alpha_{B_0}(b) \} \), \( B_{0,2} = \{ g \in B_0 : b(g) < \alpha_{B_0}(b) \} \).

We define function \( f_0 \) as follows:

\[
f_0^{(1)} := \chi_{B_{0,1}} - \chi_{B_{0,2}}, \quad f_0^{(2)} := a_0 \chi_{B_0}
\]

and

\[
f_0 := [w(B_0)]^{\frac{1}{p - 1}} \left( f_0^{(1)} - f_0^{(2)} \right),
\]

where \( B_0 \) is as in the assumption of Lemma 4.6 and \( a_0 \in \mathbb{R} \) is a constant such that

\[
\int_{\mathcal{H}^{n-1}} f_0(g) \, dg = 0.
\]

Then, by the definition of \( a_0 \), (3.1) and (3.2), we have \( |a_0| \leq 1/2 \), \( \text{supp} (f_0) \subset B_0 \) and, for any \( g \in B_0 \),

\[
f_0(g) (b(g) - \alpha_{B_0}(b)) \geq 0.
\]

Moreover, since \( |a_0| \leq 1/2 \), we can obtain that, for any \( g \in (B_{0,1} \cup B_{0,2}) \),

\[
|f_0(g)| \sim [w(B_0)]^{\frac{1}{p - 1}}.
\]
and hence
\[\|f_0\|_{L^{p_{\infty}}(\mathcal{H}^{n-1})} \lesssim \sup_{B \subset \mathcal{H}^{n-1}} \left\{ \frac{w(B \cap B_0)}{[w(B)]^{\kappa}} \right\}^{\frac{1}{p}} \left[ w(B_0) \right]^{\frac{s-1}{p}} \lesssim \sup_{B \subset \mathcal{H}^{n-1}} \left[ w(B \cap B_0) \right]^{\frac{1}{p}} \left[ w(B_0) \right]^{\frac{s-1}{p}} \lesssim 1.\]

Observe that, for any \( k \in \mathbb{N} \), we have
\[A^{k-1}_2 B_0 \subset (A_2 + 1) B_0 \subset A^{k+1}_2 B_0\]
and hence
\[(4.8) \quad w \left( B_0^k \right) \sim w \left( A^{k}_2 B_0 \right).\]

Note that
\[(4.10) \quad [b, C] f = [b - \alpha B_0(b)] C(f) - C \left( [b - \alpha B_0(b)] f \right).\]

By Theorem B, (4.5), (4.7) and the fact that \( \rho \) and hence \( \rho \) and hence \( (4.13) \), we deduce that there exists a positive constant \( B \) holds for any ball \( (4.12) \).

Observe that, for any \( k \in \mathbb{N} \), we have
\[(4.9) \quad \int_{A^{k+1}_2 B_0} |b(g) - \alpha B_0(b)| C(f_0)(g) = \int_{B_0} |K(g, \xi) - K(g, g_0)| f_0(\xi) d\xi \]
\[\lesssim \frac{|b(g) - \alpha B_0(b)|}{\rho(g, g_0)^p} \| f_0 \|_{L^1_{\infty}} \lesssim \alpha \| f_0 \|_{L^1_{\infty}} \lesssim k.\]

Since \( \|b\|_{\text{BMO}(\mathcal{H}^{n-1})} = 1 \), by (2.10), for each \( k \in \mathbb{N} \) and ball \( B \subset \mathcal{H}^{n-1} \), we have
\[(4.11) \quad \int_{A^{k+1}_2 B_0} |b(g) - \alpha B_0(b)|^{p} \ dg \lesssim k^p \left| A^{k}_2 B \right|,\]

where the last inequality is due to the fact that
\[\left| \alpha_{A^{k+1}_2 B_0} - \alpha_B \right| \lesssim \left| \alpha_{A^{k+1}_2 B_0} - b_{A^{k+1}_2 B_0} \right| + \left| b_{A^{k+1}_2 B} - b_B \right| + \left| b_B - \alpha_B \right| \lesssim k.\]

Since \( w \in A_p(\mathcal{H}^{n-1}) \), there exists \( \epsilon \in (0, \infty) \) such that the reverse Hölder inequality
\[\left[ \frac{1}{|B|} \int_B w(g)^{1+\epsilon} \ dg \right]^{\frac{1}{1+\epsilon}} \lesssim \frac{1}{|B|} \int_B w(g) \ dg\]
holds for any ball \( B \subset \mathcal{H}^{n-1} \). Then by the Hölder inequality, (4.12), (4.8) and (4.11), we can deduce that there exists a positive constant \( C \) such that, for any \( k \in \mathbb{N} \),
\[(4.13) \quad \int_{B^k_0} |b(g) - \alpha B_0(b)| C(f_0)(g) w(g) \ dg \]
Then together with \(4.9\), we obtain that there exists a positive constant \(\tilde{C}_2\) such that, for any integer \(k \geq 0\), we have

\[
\int_{B^k_n} |b(g) - \alpha_{B_0}(b)|^p w(g) dg \leq \frac{\tilde{C}_2}{A_2^{kpQ}} |w(B_0)|^{\kappa - 1} w \left( A_2^k B_0 \right).
\]

By Lemma 3.3, (4.6), (4.7), (4.1) and (4.2), for any \(g \in B^k_0\), we have

\[
|C (\{b - \alpha_{B_0}(b)\}) f_0 (g)| = \int_{B_1 \cup B_2} K(g, \xi) |b(\xi) - \alpha_{B_0}(b)| f_0(\xi) d\xi
\]

\[
\geq \int_{B_1 \cup B_2} \frac{|b(\xi) - \alpha_{B_0}(b)| f_0(\xi)}{\rho(g, \xi)} d\xi
\]

\[
\geq \frac{1}{\rho(g, \xi)} \int_{B_0} |b(\xi) - \alpha_{B_0}(b)| d\xi
\]

\[
\geq \frac{\delta}{A_2^{kpQ}} |w(B_0)|^{\kappa - 1}.
\]

Then together with \(4.13\), we obtain that there exists a positive constant \(\tilde{C}_4\) such that

\[
\int_{B^k_n} |C (\{b - \alpha_{B_0}(b)\}) f_0 (g)|^p w(g) dg \geq \frac{\tilde{C}_4}{A_2^{kpQ}} |w(B_0)|^{\kappa - 1} w \left( A_2^k B_0 \right).
\]

Take \(K_0 \in (0, \infty)\) large enough such that, for any integer \(k \geq K_0\),

\[
\tilde{C}_4 \frac{\delta^p}{2^{p-1}} - \tilde{C}_3 \frac{k^p}{A_2^{kpQ}} \geq \tilde{C}_4 \frac{\delta^p}{2^p}.
\]

From this, (4.10), (4.13) and (4.15), we have

\[
\int_{B^k_n} |[b, C] f_0(z)|^p w(g) dg
\]

\[
\geq \frac{1}{2^{p-1}} \int_{B^k_n} |C (\{b - \alpha_{B_0}(b)\}) f_0 (g)|^p w(g) dg - \int_{B^k_n} |b(g) - \alpha_{B_0}(b)| C(f_0)(g)|^p w(g) dg
\]

\[
\geq \left( \tilde{C}_4 \frac{\delta^p}{2^{p-1}} - \tilde{C}_3 \frac{k^p}{A_2^{kpQ}} \right) \frac{1}{A_2^{kpQ}} |w(B_0)|^{\kappa - 1} w \left( A_2^k B_0 \right)
\]

\[
\geq \tilde{C}_4 \frac{\delta^p}{2^p A_2^{kpQ}} |w(B_0)|^{\kappa - 1} w \left( A_2^k B_0 \right),
\]

which imply \(4.3\).
On the other hand, since supp $(f_0) \subset B_0$, by Theorem B, (4.17), (4.11) and $\|b\|_{\text{BMO}(\mathcal{H}^n-1)} = 1$, we can obtain that, for any $g \in A_2^{k+1}B_0 \setminus A_2^k B_0$,

$$|C((b - \alpha B_0) f_0)(g)| \lesssim |w(B_0)|^{-\frac{1}{p}} \int_{B_0} \frac{|b(\xi) - \alpha B_0(b)|}{p(g, \xi)^{\frac{1}{2}}} \, d\xi \lesssim |w(B_0)|^{-\frac{1}{p}} \frac{1}{A_2^{k \nu}}.$$  

Therefore, by (4.13) with $B_0^1$ replaced by $A_2^{k+1}B_0 \setminus A_2^k B_0$, we can deduce that, for any integer $k \geq K_0$,

$$\int_{A_2^{k+1}B_0 \setminus A_2^k B_0} |[b, C] f_0(g)|^p \, w(g) \, dg \lesssim \int_{A_2^{k+1}B_0 \setminus A_2^k B_0} |C((b - \alpha B_0) f_0)(g)|^p \, w(g) \, dg + \int_{A_2^{k+1}B_0 \setminus A_2^k B_0} |b(g) - \alpha B_0(b)| \, C(f_0)(g)^p \, w(g) \, dg \lesssim |w(B_0)|^{k-1} \frac{1}{A_2^{k \nu}} w(A_2^{k+1}B_0) + \frac{k^p}{A_2^{k \nu(Q+1)}} w(B_0)^{k-1} w(A_2^k B_0) \lesssim |w(B_0)|^{k-1} \frac{1}{A_2^{k \nu}} w(A_2^k B_0).$$

This finishes the proof of Lemma 4.6.

**Remark 4.7.** The reality of $b$ is used in the construction of the sets $B_{0,1}$, $B_{0,2}$ and the important inequality (4.10) in the above proof. Therefore, we require $b$ to be real in Theorem 1.1-1.2, while such condition is also required in the main results of [28, 29]. It is an interesting question to see whether Theorem 1.1-1.2 in this article hold for quaternionic valued $b$.

We also need the following technical result to handle the weighted estimate for the necessity of the compactness of the commutators.

**Lemma 4.8.** Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p(\mathcal{H}^{n-1})$, $b \in \text{BMO}(\mathcal{H}^{n-1})$, $\delta, K_0 > 0$. Assume that $B_j := B(g_j, r_j)$, $j \in \mathbb{N}$, satisfies (4.12) and the following two conditions:

(i) For any $\ell, m \in \mathbb{N}$ and $\ell \neq m$,

$$A_2 C_1 B_{\ell} \bigcap A_2 C_1 B_m = \emptyset,$$

where $C_1 := A_2^{K_1}$ and $K_2 := A_2^{K_0}$ for some $K_1 \in \mathbb{N}$ large enough.

(ii) $\{r_j\}_{j \in \mathbb{N}}$ is either non-increasing or non-decreasing in $j$, or there exist positive constants $C_{\min}$ and $C_{\max}$ such that, for any $j \in \mathbb{N}$,

$$C_{\min} \leq r_j \leq C_{\max}.$$

Let $f_j$, $j \in \mathbb{N}$, be the function constructed in Lemma 4.6 for $B_0$ to be $B_j$. Then there exists a positive constant $C$ such that, for any $j, m \in \mathbb{N}$,

$$\|b, C| f_j - [b, C] f_{j+m}\|_{L^p_w(\mathcal{H}^{n-1})} \geq C.$$

**Proof.** Without loss of generality, we may assume that $\|b\|_{\text{BMO}(\mathcal{H}^{n-1})} = 1$ and $\{r_j\}_{j \in \mathbb{N}}$ is non-increasing. Let $\tilde{C}_1, \tilde{C}_2$ be as in Lemma 4.6 such that (4.13) and (4.14) hold for $B_j, f_j$ uniformly associated with $\{B_j\}_{j \in \mathbb{N}}$. Recall that, for any $w \in A_p(\mathcal{H}^{n-1})$ with $p \in (1, \infty)$, there exists $p_0 \in (1, p)$ such that $w \in A_{p_0}(\mathcal{H}^{n-1})$. By (4.3), (4.19), Lemma 3.3 with $w \in A_{p_0}(\mathcal{H}^{n-1})$, we find that, for any $j \in \mathbb{N}$,

$$\left( \int_{C_1 B_j} |[b, C] f_j(g)|^p w(g) \, dg \right)^{1/p} \geq [w(C_1 B_j)]^{\kappa/p}.$$
follows that, for any 

\[ (4.19) \]

\[
\geq \left[w(C_1 B_j)\right]^{-\kappa/p} \left\{ \int_{B_j^{K_0-1}} |\langle b, \mathcal{C} f_j(g) \rangle|^p w(g) \, dg \right\}^{1/p}
\]

\[
\geq \left[w(C_1 B_j)\right]^{-\kappa/p} \left\{ \tilde{C}_1 \delta^p \frac{|w(B_j)|^{\kappa-1} w(A_2^{K_0-1} B_j)}{A_2^{p(K_0-1)Q}} \right\}^{1/p}
\]

\[
\geq \left[w(C_1 B_j)\right]^{-\kappa/p} \left\{ \delta^p \frac{|w(B_j)|^{\kappa}}{A_2^{Q(p-\sigma)(K_0-1)}} \right\}^{1/p}
\]

\[
\geq C_3 C_1^{-\frac{Qp}{p-\sigma}} \left| w(B_j) \right|^{-\kappa/p} \delta \left[w(B_j)\right]^{\kappa/p} = C_3 \delta C_1^{-\frac{Qp}{p-\sigma}}
\]

for some positive constant \( C_3 \) independent of \( \delta \) and \( C_1 \). We next prove that, for any \( j, m \in \mathbb{N} \),

\[ (4.18) \]

\[
\left[ \int_{C_1 B_j} \left| \langle b, \mathcal{C} f_j+m(g) \rangle \right|^p w(g) \, dg \right]^{1/p} \left[w(C_1 B_j)\right]^{-\kappa/p} \leq \frac{1}{2} C_3 \delta C_1^{-\frac{Qp}{p-\sigma}}.
\]

Indeed, since \( \text{supp} (f_j+m) \subset B_{j+m} \), from (4.16), (4.11), (4.10) and \( \|b\|_{\text{BMO}(\mathbb{R}^n)} = 1 \), it follows that, for any \( g \in C_1 B_j \),

\[
|\mathcal{C}(\langle b - \alpha_{B_{j+m}}(b) \rangle f_j+m)(g)| \lesssim \left[w(B_{j+m})\right]^{-\frac{1}{p} \frac{m}{p}} \int_{B_{j+m}} \left| K(g, \xi) \right| |b(g) - \alpha_{B_{j+m}}(b)| \, d\xi
\]

\[
\lesssim \left[w(B_{j+m})\right]^{-\frac{1}{p} \frac{m}{p}} \frac{r_{j+m}^Q}{\rho(g_j, g_{j+m})^Q}
\]

and hence

\[ (4.19) \]

\[
\left[ \int_{C_1 B_j} \left| \mathcal{C}(\langle b - \alpha_{B_{j+m}}(b) \rangle f_j+m)(g) \right|^p w(g) \, dg \right]^{1/p} \left[w(C_1 B_j)\right]^{-\kappa/p}
\]

\[
\lesssim \left[w(B_{j+m})\right]^{-\frac{1}{p} \frac{m}{p}} \frac{r_{j+m}^Q}{\rho(g_j, g_{j+m})^Q} \left[w(C_1 B_j)\right]^{-\frac{1}{p} \frac{m}{p}}
\]

\[
\lesssim \left[w(B_{j+m})\right]^{-\frac{1}{p} \frac{m}{p}} \frac{r_{j+m}^Q}{\rho(g_j, g_{j+m})^Q} \left[w\left(\frac{\rho(g_j, g_{j+m})}{r_{j+m}} B_{j+m}\right)\right]^{-\frac{1}{p} \frac{m}{p}}
\]

\[
\lesssim \frac{r_{j+m}^Q}{\rho(g_j, g_{j+m})^Q} \left(\frac{\rho(g_j, g_{j+m})}{r_{j+m}}\right)^{Q \frac{1}{p} \frac{m}{Q \frac{1}{p} \frac{m}{p} + Q \frac{1}{p} \frac{m}{p} - Q}} \sim \left(\frac{\rho(g_j, g_{j+m})}{r_{j+m}}\right)^{-\frac{Qp}{p-\sigma}}
\]

Moreover, from Theorem B and (4.7), we deduce that, for any \( g \in C_1 B_j \),

\[ (4.20) \]

\[
|\mathcal{C}(f_j+m)(g)| \leq \int_{B_{j+m}} \left| K(g, \xi) - K(g, g_{j+m})\right| |f_j+m(\xi)| \, d\xi
\]

\[
\lesssim \int_{B_{j+m}} \frac{r_{j+m}}{\rho(g_j, g_{j+m})^{Q+1}} |f_j+m(\xi)| \, d\xi
\]

\[
\lesssim \left[w(B_{j+m})\right]^{-\frac{1}{p} \frac{m}{p}} \frac{r_{j+m}^{Q+1}}{\rho(g_j, g_{j+m})^{Q+1}}
\]

Then, by (4.20), the fact that \( \{r_j\}_{j \in \mathbb{N}} \) is non-increasing in \( j \), the Hölder and the reverse Hölder inequalities, we conclude that

\[ (4.21) \]

\[
\left[ \int_{C_1 B_j} \left| \langle b(g) - \alpha_{B_{j+m}}(b) \rangle \mathcal{C}(f_j+m)(g) \right|^p w(g) \, dg \right]^{1/p} \left[w(C_1 B_j)\right]^{-\kappa/p}
\]
and hence

\[ \| b \|_{\text{BMO}(\mathbb{R}^{n-1})} = 1. \]

To show \( b \in \text{VMO}(\mathbb{R}^{n-1}) \), noticing that \( b \in \text{BMO}(\mathbb{R}^{n-1}) \) is a real-valued function, we can use a contradiction argument via Lemmas 4.5, 4.6 and 4.8. Now observe that, if

\[ \int_{C_1B_j} |[b, C]_{(f_j+m)}(g)|^p w(g) \, dg \]

This finishes the proof of Lemma 4.8. \( \square \)

Proof of Theorem 1.2 (ii). Without loss of generality, we may assume that \( \|b\|_{\text{BMO}(\mathbb{R}^{n-1})} = 1 \). To show \( b \in \text{VMO}(\mathbb{R}^{n-1}) \), noticing that \( b \in \text{BMO}(\mathbb{R}^{n-1}) \) is a real-valued function, we can use a contradiction argument via Lemmas 4.5, 4.6 and 4.8. Now observe that, if
\( b \notin \text{VMO}(\mathcal{H}^{n-1}) \), then \( b \) does not satisfy at least one of (i) through (iii) of Lemma 4.5. We show that \([b, C]\) is not compact on \( L_w^{p,n}(\mathcal{H}^{n-1})\) in any of the following three cases.

**Case 1** \( b \) does not satisfy Lemma 4.5(i). Then there exist \( \delta \in (0, \infty) \) and a sequence
\[
\{B_j^{(1)}\}_{j \in \mathbb{N}} := \{B(g_j^{(1)}, r_j^{(1)})\}_{j \in \mathbb{N}}
\]
of balls in \( \mathcal{H}^{n-1} \) satisfying (4.22) and that \( r_j^{(1)} \to 0 \) as \( j \to \infty \). We further consider the following two subcases.

**Subcase (i)** There exists a positive constant \( M \) such that \( 0 \leq \|g_j^{(1)}\| < M \) for all \( g_j^{(1)} \), \( j \in \mathbb{N} \). That is, \( g_j^{(1)} \in B_0 := B(0, M), \forall j \in \mathbb{N} \). Let \( \{f_j\}_{j \in \mathbb{N}} \) be associated with \( \{B_j\}_{j \in \mathbb{N}}, \tilde{C}_1, \tilde{C}_2, K_0 \) and \( C_2 \) be as in Lemmas 4.6 and 4.8. Let \( p_0 \in (1, p) \) be such that \( w \in A_{p_0}(\mathcal{H}^{n-1}) \) and \( C_4 := A_2^{K_2} > C_2 = A_2^{K_0} \) for \( K_2 \in \mathbb{N} \) large enough such that
\[
(4.23) \quad C_5 := \tilde{C}_1 \tilde{C}_2 \delta^p A_2^{QK_0(\sigma-p)} > 2 \frac{\tilde{C}_2}{1 - A_2^{Q(p_0-p)}} \frac{\hat{C}}{A_2^{QK_2(p-p_0)}},
\]
where \( \hat{C}_2 \) and \( \hat{C} \) are as in Lemma 3.1. Since \( |r_j^{(1)}| \to 0 \) as \( j \to \infty \) and \( \{g_j^{(1)}\}_{j \in \mathbb{N}} \subset B_0 \), we may choose a subsequence
of \( \{B_j^{(1)}\}_{j \in \mathbb{N}} \), for simplicity, we still denote it by \( \{B_j^{(1)}\}_{j \in \mathbb{N}} \), such that, for any \( j \in \mathbb{N} \),
\[
(4.24) \quad \frac{|B_{j+1}^{(1)}|}{|B_j^{(1)}|} < \frac{1}{C_4^2} \quad \text{and} \quad w(B_j^{(1)}) \leq w(B_j^{(1)}).
\]
For fixed \( \ell, m \in \mathbb{N} \), let
\[
\mathcal{J} := C_{4}B_{\ell}^{(1)} \setminus C_{2}B_{\ell}^{(1)}, \quad \mathcal{J}_1 := \mathcal{J} \setminus C_{4}B_{\ell+m}^{(1)} \quad \text{and} \quad \mathcal{J}_2 := \mathcal{H}^{n-1} \setminus C_{4}B_{\ell+m}^{(1)}.
\]
Notice that
\[
\mathcal{J}_1 \subset \left( C_{4}B_{\ell}^{(1)} \cap \mathcal{J}_2 \right) \quad \text{and} \quad \mathcal{J}_1 = \mathcal{J} \cap \mathcal{J}_2.
\]
We then have
\[
(4.25) \quad \left\{ \int_{C_{4}B_{\ell}^{(1)}} \|b, C\| (f_\ell)(g) - \|b, C\| (f_{\ell+m})(g) \|w(g)\| dg \right\}^{1/p} \geq \left\{ \int_\mathcal{J}_1 \|b, C\| (f_\ell)(g) - \|b, C\| (f_{\ell+m})(g) \|w(g)\| dg \right\}^{1/p}
= \left\{ \int_\mathcal{J}_1 \|b, C\| (f_\ell)(g) \|w(g)\| dg \right\}^{1/p} \quad - \left\{ \int_\mathcal{J}_1 \|b, C\| (f_{\ell+m})(g) \|w(g)\| dg \right\}^{1/p}
=: \mathcal{F}_1 - \mathcal{F}_2.
\]
We first consider the term \( \mathcal{F}_1 \). Assume that \( E_{\ell+m} := \mathcal{J} \setminus \mathcal{J}_2 \not= \emptyset \). Then \( E_{\ell} \subset C_{4}B_{\ell+m}^{(1)} \). Thus, by (4.22), we have
\[
(4.26) \quad |E_{\ell}| \leq C_4^Q |B_{\ell+m}^{(1)}| < |B_{\ell}^{(1)}|.
\]
Now let
\[
B_{(1)_{\ell, k}} := A_2^{K_2-1}B_{\ell}^{(1)},
\]
be the ball associates with $A_{k_2} B^{(1)}_\ell$ in Lemma 3.3. Then, from (4.26), we have
\[
|B^{(1)}_{\ell,k}| = A_2^{Q(k-1)} |B^{(1)}_{\ell}| > |E_{\ell}|.
\]
By this, we further know that there exist finite $\{B^{(1)}_{\ell,k}\}_{k = K_0}^{K_2-2}$ intersecting $E_{\ell}$. By (4.3) and Lemma 3.1, we conclude that
\[
(4.27) \quad F_1^p \geq \sum_{k = K_0, B^{(1)}_{\ell,k} \cap E_{\ell} = \emptyset} \int_{B^{(1)}_{\ell,k}} |[b, C](f_{\ell})(g)|^p w(g) dg
\]
\[
\geq \tilde{C}_1 \delta^p \sum_{k = K_0, B^{(1)}_{\ell,k} \cap E_{\ell} = \emptyset} \frac{|w(B^{(1)}_{\ell})|^{\kappa-1} w(A_2^k B^{(1)}_{\ell})}{A_2^k}
\]
\[
\geq \tilde{C}_1 \tilde{C}_2 \delta^p \sum_{k = K_0, B^{(1)}_{\ell,k} \cap E_{\ell} = \emptyset} \frac{|w(B^{(1)}_{\ell})|^\kappa}{A_2^k}
\]
\[
\geq \tilde{C}_1 \tilde{C}_2 \delta^p A_2^{K_0(\sigma-p)} \left[ w(B^{(1)}_{\ell}) \right]^\kappa = C_5 \left[ w(B^{(1)}_{\ell}) \right]^\kappa.
\]
If $E_{\ell} := \mathcal{J} \setminus \mathcal{J}_2 = \emptyset$, the inequality above is still true.
Moreover, from (4.24), Lemma 3.1 (4.23), and (4.24), we deduce that
\[
(4.28) \quad F_2^p \leq \sum_{k = K_2}^{\infty} \int_{A_{k+1} B^{(1)}_{\ell,m} \setminus A_k B^{(1)}_{\ell,m}} |[b, C](f_{\ell+m})(g)|^p w(g) dg
\]
\[
\leq \tilde{C}_2 \sum_{k = K_2}^{\infty} \frac{|w(B^{(1)}_{\ell+m})|^{\kappa-1} w(A_2^k B^{(1)}_{\ell+m})}{A_2^{k(p-\sigma)}}
\]
\[
\leq \tilde{C}_2 \sum_{k = K_2}^{\infty} \frac{\hat{C}}{A_2^{k(p-\sigma)}} \left[ w(B^{(1)}_{\ell+m}) \right]^\kappa
\]
\[
\leq \frac{\tilde{C}_2}{1 - A_2^{Q(p-\sigma)}} \hat{C} A_2^{QK_2(p-\sigma)} \left[ w(B^{(1)}_{\ell+m}) \right]^\kappa
\]
\[
\leq \frac{C_5}{2} \left[ w(B^{(1)}_{\ell+m}) \right]^\kappa \leq \frac{C_5}{2} \left[ w(B^{(1)}_{\ell}) \right]^\kappa.
\]
By (4.25), (4.27) and (4.28), we obtain
\[
\left\{ \int_{C_{4, B^{(1)}_{\ell}}} |[b, C](f_{\ell})(g)|^p w(g) dg \right\}^{1/p}
\]
\[
\geq C_5^{1/p} \left[ w(B^{(1)}_{\ell}) \right]^{\kappa/p} - \left( \frac{C_5}{2} \right)^{1/p} \left[ w(B^{(1)}_{\ell}) \right]^{\kappa/p} \geq \left[ w(B^{(1)}_{\ell}) \right]^{\kappa/p}.
\]
Thus, $\{[b, C] f_{\ell}\}_{\ell \in \mathbb{N}}$ is not relatively compact in $L_w^{\kappa} (\mathcal{H}^{n-1})$, which implies that $[b, C]$ is not compact on $L_w^{\kappa} (\mathcal{H}^{n-1})$. Therefore, $b$ satisfies condition (i) of Lemma 4.3.

**Subcase (ii)** There exists a subsequence of $\{B^{(1)}_{\ell}\}_{\ell \in \mathbb{N}} = \{B(g^{(1)}_j, r^{(1)}_j)\}_{j \in \mathbb{N}}$, for simplicity, we still denote it by $\{B^{(1)}_{j}\}_{j \in \mathbb{N}}$, such that $|g^{(1)}_j| \to \infty$ as $j \to \infty$. In this subcase, by $|B^{(1)}_{j}| \to 0$ as $j \to \infty$, we can take a mutually disjoint subsequence of $\{B^{(1)}_{j}\}_{j \in \mathbb{N}}$, still denoted by $\{B^{(1)}_{j}\}_{j \in \mathbb{N}}$, satisfying (4.10) as well. This, via Lemma 4.8, implies that $[b, C]$ is
not compact on $L_w^{p,\kappa}(\mathcal{X}^{n-1})$, which is a contradiction to our assumption. Thus, $b$ satisfies condition (i) of Lemma 4.5.

**Case ii**) If $b$ does not satisfy condition (ii) of Lemma 4.5. In this case, there exist $\delta \in (0, \infty)$ and a sequence $\{B_j^{(2)}\}_{j \in \mathbb{N}}$ of balls in $\mathcal{X}^{n-1}$ satisfying (4.32) and that $|r_{B_j^{(2)}}| \to \infty$ as $j \to \infty$. We further consider the following two subcases as well.

**Subcase (i)**

There exists an infinite subsequence of $\{B_j^{(2)}\}_{j \in \mathbb{N}}$, for simplicity, we still denote it by $\{B_j^{(2)}\}_{j \in \mathbb{N}}$, and a point $g_0 \in \mathcal{X}^{n-1}$ such that, for any $j \in \mathbb{N}$, $g_0 \in A_2 C_1 B_j^{(2)}$. Since $|r_{B_j^{(2)}}| \to \infty$ as $j \to \infty$, it follows that there exists a subsequence, still denoted by $\{B_j^{(2)}\}_{j \in \mathbb{N}}$, such that, for any $j \in \mathbb{N}$,

$$|B_j^{(2)}| \leq \frac{CQ}{C_4^2}.$$  

Observe that $2A_2 C_1 B_j^{(2)} \subset 2A_2 C_1 B_{j+1}^{(2)}$ for any $j \in \mathbb{N}$ and hence

$$w\left(2A_2 C_1 B_{j+1}^{(2)}\right) \geq w\left(2A_2 C_1 B_j^{(2)}\right), \quad M(b; 2A_2 C_1 B_{j+1}) > \frac{\delta}{8A_2^2 C_4^2}.$$  

We can use a similar method as that used in Subcase (i) of Case i) and redefine our sets in a reversed order. That is, for any fixed $\ell, k \in \mathbb{N}$, let

$$\tilde{J} : = 2A_2 C_4 C_1 B_{\ell+k+1}^{(2)} \setminus 2A_2 C_2 C_1 B_{\ell+k}^{(2)},$$

$$\tilde{J}_1 : = \tilde{J} \setminus 2A_2 C_4 C_1 B_{\ell}^{(2)},$$

$$\tilde{J}_2 : = \mathcal{X}^{n-1} \setminus 2A_2 C_4 C_1 B_{\ell}^{(2)}.$$  

As in Case i), by Lemma 4.6 (4.29) and (4.30), we conclude that the commutator $[b, \mathcal{C}]$ is not compact on $L_w^{p,\kappa}(\mathcal{X}^{n-1})$. This contradiction implies that $b$ satisfies condition (ii) of Lemma 4.5.

**Subcase (ii)** For any $A_2 \in \mathcal{X}^{n-1}$, the number of $\{A_2 C_1 B_j^{(2)}\}_{j \in \mathbb{N}}$ containing $g_0$ is finite. In this subcase, for each ball $B_{j_0}^{(2)} \in \{B_j^{(2)}\}_{j \in \mathbb{N}}$, the number of $A_2 C_1 B_j^{(2)}$ intersecting $A_2 C_1 B_{j_0}^{(2)}$ is finite. Then we take a mutually disjoint subsequence of $\{B_j^{(2)}\}_{j \in \mathbb{N}}$ satisfying (4.12) and (4.13). From Lemma 4.8 we deduce that $[b, \mathcal{C}]$ is not compact on $L_w^{p,\kappa}(\mathcal{X}^{n-1})$. Thus, $b$ satisfies condition (ii) of Lemma 4.5.

**Case iii**) Condition (iii) of Lemma 4.5 does not hold for $b$. Then there exists $\delta > 0$ such that for any $r > 0$, there exists $B \subset \mathcal{X}^{n-1} \setminus B(0, r)$ with $M(b, B) > \delta$. As in [7, p. 1661], for the $\delta$ above, there exists a sequence $\{B_j^{(3)}\}_j$ of balls such that for any $j$,

$$M(b, B_j^{(3)}) > \delta,$$

and for any $i \neq m$,

$$\gamma_1 B_i^{(3)} \cap \gamma_1 B_m^{(3)} = \emptyset,$$

for sufficiently large $\gamma_1$.

Since, by Case i) and ii), $\{B_j^{(3)}\}_{j \in \mathbb{N}}$ satisfies the conditions (i) and (ii) of Lemma 4.5, it follows that there exist positive constants $C_{\min}$ and $C_{\max}$ such that

$$C_{\min} \leq r_j \leq C_{\max}, \quad \forall j \in \mathbb{N}.$$
By this and Lemma 4.8 we conclude that, if $[b, C]$ is compact on $L^{p, r}_w(H^{n-1})$, then $b$ also satisfies condition (iii) of Lemma 4.5. This finishes the proof of Theorem 1.2(ii) and hence of Theorem 1.2. □

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