The charged excitations in the system of the title are vortex-antivortex pairs in the spin-texture described in the theory by Yang et al which, in the commensurate phase, are bound together by a “string”. It is shown that their excitation energy drops as the string lengthens as the parallel magnetic field approaches the critical value, then goes up again in the incommensurate phase. This produces a sharp downward cusp at the critical point. An alternative description based on the role of disorder in the tunnelling and which appears not to produce a minimum in the excitation energy is also discussed. It is suggested that a similar transition could also occur in compressible Fermi-liquid-like states.
Recent experiments on double layer quantum Hall systems [1], consisting of two two-
dimensional electron gases (2DEGs) separated by only about $d \simeq 100$ Å in a double quantum
well structure, have clarified the phase diagram in a perpendicular magnetic field $B$ at
Landau level filling factor $\nu = 1$ and, surprisingly, showed evidence of an apparent phase
transition, in that the activation energy gap drops dramatically when a parallel magnetic
field is introduced, saturating at a constant value for higher $B_{\parallel}$. A theory for a phase
transition for such a system was proposed [2], but in that work the activation gap itself was
not discussed. The present paper is mainly concerned with this issue. The Indiana group has
independently reached some of the same conclusions in more recent work [3]. I also discuss
possible effects of disorder.

The theory [2, 3], which draws upon earlier work [1, 4, 5, 6, 7, 8], can be summarised as
follows. I will not discuss the SU(2) symmetry that is present only in the limit $d \to 0$
which is physically unrealizable in this context. All we need to know is that when the
total filling factor $\nu = n\Phi_0/B$ is of the form $1/q, q$ odd (and presumably also at some
hierarchical generalizations, in particular whenever the corresponding system with real spin
is spin polarized in the absence of Zeeman splitting) the system (at $B_{\parallel} = 0$) forms an
incompressible fluid exhibiting a quantum Hall plateau, at least for $d$ not too large (for
larger $d$, the layers can form essentially independent even denominator compressible states
[9]). The density of electrons in each layer, labelled $\uparrow, \downarrow$, is essentially the same, $= n/2$. I
denote the density difference $\varpi(\mathbf{r}) = \frac{1}{2}(n_{\uparrow}(\mathbf{r}) - n_{\downarrow}(\mathbf{r}))$ (pronounced $p\imath$). It turns out that, in
the states under discussion, there is long range order in the phase $\theta$ canonically conjugate
to $\varpi$, so that $[\theta(\mathbf{r}), \varpi(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}')$ or more accurately since $\theta$ is periodic we should use
$e^{i\theta}$ (with complex eigenvalues of modulus unity) and $[\varpi(\mathbf{r}), e^{i\theta}(\mathbf{r}')] = e^{i\theta}(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')$. This
operator can be viewed as $e^{i\theta}(\mathbf{r}) \simeq 2n^{-1}c_{\uparrow}^{\dagger}(\mathbf{r})c_{\downarrow}(\mathbf{r})$ which transfers an electron at $\mathbf{r} = (x, y)$
from the lower to the upper layer. The long range phase coherence of this operator is a consequence of interlayer correlations (two electrons avoid each other whether they are in the same layer or not) induced by interactions.

Since charge fluctuations (in \( n_\uparrow(\mathbf{r}) + n_\downarrow(\mathbf{r}) \)) are gapped, the Hamiltonian for the system including the tunnelling term with coefficient \( t \), and in the presence of the parallel magnetic field \( B_\parallel \) in the \( y \) direction, can be written in terms of the “isospin” variables only, as

\[
H = \int d^2r \left\{ \frac{1}{2} \chi^{-1} \nabla \theta \nabla \theta^\dagger - \frac{1}{2} \rho_s |\nabla \theta|^2 - t \cos(\theta - Qx) \right\}.
\]

The coefficients deserve comment: \( Q = 2\pi B_\parallel d/\Phi_0 \) \[^2\], \( t \) is related to Murphy et al’s tunnelling gap \( \Delta_{SAS} \) by \( t \approx \nu \Delta_{SAS}/4\pi\ell_B^2 \) and (for \( \nu = 1 \)) to Yang et al’s \( t \) by \( t \rightarrow t/2\pi\ell_B^2 \) (\( \ell_B \) is the magnetic length \( 2\pi\ell_B^2 = \Phi_0/B \)). \( \rho_s \) is the “isospin stiffness” resulting from the change in exchange energy when \( \theta \) varies, and \( \chi \) is a charge susceptibility related to the capacitance per unit area (this term is omitted in most equations in \[^2\], but appears as \( U_z \) in their eq. (4)). As \( d \rightarrow 0 \) (the “SU(2) limit”), \( \chi \) diverges as \( d^{-1} \), \( \rho_s \propto e^2/\epsilon\ell_B \) can be calculated exactly for \( \nu = 1 \) \[^7\], and \( t = \nu \Delta_{SAS}/4\pi\ell_B^2 \); all these quantities are modified at finite \( d \).

One other fact is important: I allow states where \( e^{i\theta} \) becomes undefined at isolated points, and winds around the unit circle once or more (in either sense) around such points, so there is a vortex at these points. The underlying theory predicts that a vortex carries a real electric charge corresponding to \( \pm \frac{1}{2q} \) of an electron for a \( 2\pi \) vortex in \( \theta \); the charge can take either sign, independent of the sign of the vortex. Therefore the Hamiltonian \[*\] is implicitly supplemented with a Coulomb interaction between the charges on the vortices. I will consider only the smallest vortices, of magnitude \( \pm 2\pi \). They have quantum mechanical dynamics which can be viewed (at zeroth approximation, neglecting changes in the densities \( n_\uparrow, n_\downarrow \) from their averages = \( \frac{1}{2} n \)) as that of charges \( \pm \frac{1}{2q} e \) in the same external field \( B \) and restricted to their lowest Landau level; thus they will drift perpendicular to any electric field
they experience. I specialize to $q = 1$ from now on.

To understand the phase transition as $B_{\parallel}$ (or $Q$) is increased, it is helpful to consider the imaginary time path integral representation of the partition function, as $\beta = 1/k_BT \to \infty$:

$$Z = \text{Tr} e^{-\beta H}$$

$$= \int D[\theta] e^{-S}$$

where

$$S = \int d^2r \int_0^\beta d\tau \left\{ \frac{1}{2} \chi \left( \frac{\partial \theta}{\partial \tau} \right)^2 + \frac{1}{2} \rho_s |\nabla \theta|^2 - t \cos(\theta - Qx) \right\}.$$  

As is common, the $T = 0$ quantum transition in $D = 2$ space dimensions is equivalent to a classical finite temperature transition in $D + 1 = 3$ dimensions (the temperature in the classical problem has been absorbed into the parameters). The classical problem has been well studied, and we can make use of existing results, reviewed in [10, 11].

A first step is to solve the problem classically (as $\hbar \to 0$) by simply minimizing $S$ or equivalently $H$ regarded as a classical Hamiltonian. It is convenient to define a new phase $\theta' = \theta - Qx$; this corresponds to a change of gauge in the original problem. In terms of $\theta'$, $S$ is

$$S = \int d^2r \int_0^\beta d\tau \left\{ \frac{1}{2} \chi \left( \frac{\partial \theta'}{\partial \tau} \right)^2 + \frac{1}{2} \rho_s \left[ \left( \frac{\partial \theta'}{\partial y} \right)^2 + \left( \frac{\partial \theta'}{\partial x} + Q \right)^2 \right] - t \cos \theta' \right\}.$$  

The minimum will have $\theta'$ independent of $\tau, y$. $Q$ enters through a total derivative which does not affect the equation obtained by varying $\theta'$ which is

$$\rho_s \frac{d^2 \theta'}{dx^2} = t \sin \theta'.$$

This equation has well known soliton solutions in which $\theta'$ changes by $2\pi$ at each domain wall of width $\sim (\rho_s/t)^{1/2}$. The role of $Q$ is to act as a chemical potential for domain walls. For $Q \leq Q_c = 8(t/\rho_s)^{1/2}/2\pi$ [12, 10] the minimum action $S$ is obtained by selecting the solution
\( \theta' = 0 \), with no domain walls. This is the commensurate phase, in which the tunnelling \( t \) overcomes the stiffness \( \rho_s \), because \( \partial \theta / \partial x \) is small. When the chemical potential exceeds the action per unit area (in the \( y, \tau \) plane) of a wall, i.e. when \( Q > Q_c \), it is advantageous to form walls. The walls repel each other exponentially due to the exponential relaxation of \( \theta' \) to 0 (mod 2\( \pi \)) outside the “core” region of each wall. Consequently the minimum action is obtained for a periodic solution (the “striped incommensurate solid”) in which the number of walls per unit length in the \( x \) direction, \( \bar{\ell}^{-1} \equiv \tilde{Q}/2\pi \), is \([12, 10]\)

\[
\tilde{Q} \sim 1/ \ln[1/(Q - Q_c)].
\]  

(7)

This wall spacing \( \bar{\ell} \) defines the correlation length \( \xi_x = \bar{\ell} \) in the incommensurate phase, and diverges only logarithmically as \( Q \searrow Q_c \), thus the critical exponent \( \nu \) (not to be confused with the filling factor) is zero. For illustrations of a wall configuration and of \( \tilde{Q} \) versus \( Q \), see Figs 11, 12 in \([10]\). A useful variable in the incommensurate phase is

\[
\bar{\theta} \equiv \theta' + \tilde{Q} x.
\]  

(8)

In the periodic array of domain walls, \( \bar{\theta} \) is then constant on length scales \( > 2\pi/\tilde{Q} \). As \( Q \rightarrow \infty, \tilde{Q} \rightarrow Q \) and so \( \bar{\theta} \rightarrow \theta \). In this limit, \( \theta \) has ceased to rotate to follow \( Qx \), even on short length scales; it is as if \( t \) went to zero.

Next we must consider the role of fluctuations. For the case \( D = 1 \), Pokrovsky and Talapov \([13, 10, 11]\) showed that quantum fluctuations (or thermal fluctuations in the equivalent 2 dimensional classical problem) lead to \( \tilde{Q} \sim (Q - Q_c)^{1/2} \) instead of (7). In general dimension \( D \), the walls are \( D \)-dimensional objects, and are effectively flat for \( D > 2 \), wander due to fluctuations for \( D < 2 \), and for \( D = 2 \) are only logarithmically rough. Fisher and Fisher \([14]\) argued that in the latter case, this roughening is unimportant and (7) still holds (the application to the present \( T = 0 D \)-dimensional quantum situation is mentioned in their
paper). It is also argued that walls that end on a vortex line are not important, and we expect that the same is true here, even though the action of the vortex lines differs due to their different (lowest Landau level) dynamics.

Now consider the case where the physical temperature $T$ is nonzero. Apart from some renormalization of parameters at low $T$ (which will be unimportant because fluctuations of the walls at $T = 0$ are unimportant), the problem is described by the same form of "action" (now classical Hamiltonian) given by (1) or (3) with the $\tau$ dependence of $\theta$ or $\theta'$ dropped and so $\int d\tau$ replaced by a factor $\beta$. As it stands, this model has a transition of the Pokrovsky-Talapov type. However, it is known that, because of vortices that terminate a domain wall and hence act as dislocations in the domain wall "solid", the finite $T$ transition is preempted by a Kosterlitz-Thouless (KT) transition to an incommensurate fluid as $Q_c$ is approached from above, because this is the $p = 1$ case of the more general theory \[15\]. Note that at finite $T$, $\langle e^{i\bar{\theta}(r)}e^{-i\bar{\theta}(0)} \rangle \sim r^{-\eta(T)}$ in the "solid" phase, but $\sim e^{-r/\xi}$ in the fluid, for some correlation length $\xi$. Also $\langle e^{i\theta'(r)}e^{-i\theta'(0)} \rangle \sim e^{iQx}e^{-r/\xi}$ in the incommensurate fluid, but as $Q$ is decreased further, there is in principle a boundary where $\bar{Q}$ reaches zero and stays there for smaller $Q$ (the "commensurate fluid"). This is not a true transition since no broken symmetries are involved, and $\xi$ remains finite as it is crossed. Further, its position may depend on details of its definition \[11\]. In the $(Q, T)$ plane, both this line and the KT transition line approach $Q = Q_c$ as $T \to 0$. In the present case of $p = 1$, there is no other finite $T$ transition between the $\bar{Q} \to 0$ line and the zero temperature axis. Note that the KT transition extends all the way in to $B_\parallel = 0$ when $t \to 0$; the transition temperature $T_c$ goes to a constant at large $Q \gg Q_c$ for finite $t$ which is the same as the value at all $Q$ when $t = 0$ \[16\].

After the above, somewhat lengthy, review of earlier work, we turn to the question of how the energy gap of the charged excitations, as seen experimentally in the thermally activated
longitudinal resistance, behaves as $Q$ increases and passes through $Q_c$. The basic excitations are vortices of charge $\pm \frac{1}{2}e$. Since $\theta$ winds by $2\pi$ around the vortex, for $t \neq 0$ a vortex is at the end of a domain wall in spacetime (or “string” in space), which when $Q$ is zero has an energy $\sim (\rho_s t)^{1/2}$ per unit length in the $x$-$y$ plane (I again begin by considering the classical $\hbar = 0, T = 0$ system, and taking $\tau$-independent configurations). This string tension means that the domain wall must end at another vortex of opposite vorticity in order to give finite free energy. Thus vortices will be “confined” by the linearly increasing potential between opposite vortices; no free vortices exist. We are interested in such vortex pairs with total charge $\pm e$, so the charges at the ends repel each other. (The other case, where the charges sum to zero, gives an excitation that should be identified with the usual gapped collective mode.) For nonzero $Q$, the string tension depends on the orientation of the wall and is smallest when it is parallel to the $y$ axis (and largest when it is antiparallel). As $Q$ increases, the string tension decreases, the string lengthens, and the minimum excitation energy $E_{\text{min}}$ decreases. At $Q = Q_c$ the string tension is zero for a string parallel to the $y$ axis, and is negative for $Q > Q_c$ so infinitely long domain walls condense spontaneously to form the “striped incommensurate solid”. The total energy of the bound vortex-antivortex pair for $Q < Q_c$, taking a straight string of length $R$ and of orientation angle $\phi$ relative to the $y$ axis, and including a “core energy” for each vortex, is thus

$$E(R, \phi) = 2E_{\text{core}} + \frac{e^2}{4\epsilon R} + 2\pi \rho_s R (Q_c - Q \cos \phi).$$

(9)

The minimum at fixed $\phi$ is

$$R(\phi) = \left[\frac{e^2}{8\pi \epsilon \rho_s (Q_c - Q \cos \phi)}\right]^{1/2}$$

(10)

$$E_{\text{min}}(\phi) = 2E_{\text{core}} + 2\left[2\pi \rho_s (Q_c - Q \cos \phi)e^2/4\epsilon\right]^{1/2}.$$  

(11)

The minimum is then $E_{\text{min}}(0)$. At $Q = Q_c$, $E_{\text{min}}(0) = 2E_{\text{core}}$. These expressions are of course
accurate when $R$ is large compared with the width of the wall, $\sim (\rho_s/t)^{1/2}$, that is when

$$\frac{e^2t}{8\pi\rho_s^2(Q_c - Q \cos \phi)} \gg O(1).$$

(12)

In the experimental context, the long wavelength forms (1), (4) are valid when $(\rho_s/t)^{1/2} \gg \ell_B$, which, since $\rho_s \sim e^2/\ell_B$ and assuming finite $d$ corrections are not too large, is equivalent to $\Delta_{SAS}/(e^2/\ell_B) \ll 1$, which is easily satisfied; however the criterion (12) becomes, for $Q = 0$, $\Delta_{SAS}/(e^2/\ell_B) \gg 1$ which is not compatible. However, it may still be that for small $Q$ there is a tendency for the charge $\pm e$ excitation to separate slightly into two lumps of charge $\pm \frac{1}{2}e$; again this is a topic for numerical study. On the other hand, as $Q$ increases the above considerations become more accurate. Once the string picture is applicable, the experiment will actually pick up contributions from all orientations (and lengths) of the string, weighted by a Boltzmann factor involving $E(R, \phi)$; this will be especially important at small $B_\parallel$ where the variation of the gap with $\phi$ becomes smaller than $k_B T$ for accessible $T$. Thus the measured “gap” will not be given simply by $E_{\min}(0)$, which varies linearly with $Q$ at small $Q/Q_c$, but will have a much flatter appearance, $\sim Q^2$. Near $Q = Q_c$, however, the form $E_{\min}(0)$ should dominate. Note that in this region there are still excited states of the bound pair, with constant density of states just above $E_{\min}(0)$; this will contribute a prefactor $\sim T$ to the exponentially activated temperature dependence.

The preceding paragraph was the classical picture, which will be useful when the temperature is higher than the quantum-mechanical energy splittings. For the quantum dynamics of the bound pair, within the straight string approximation, we should recall that each vortex behaves as a charge $\pm \frac{1}{2}e$ particle in the field $B$, restricted to its lowest Landau level. (The quantum mechanics of a similar, but alas not identical, system including a flexible string has been studied in [14].) The bound pair system then behaves as a charge $\pm e$ particle in the lowest Landau level with the potential $E(R, \phi)$. For a first look, we consider
the semiclassical approximation. The particle moves on the equipotentials of \( E(R, \phi) \) but is subject to the quantization condition that an orbit encloses an integral number of flux quanta. The \( Q = 0 \) case is simple due to its rotational invariance; the quantization condition is \( \pi R^2 = 2\pi ml_B^2 \), where the integer \( m \) is essentially the angular momentum. If this condition is satisfied with \( m = m^* \) at the minimum of \( E(R, \phi) \) at \( R^* = R(\phi) \) as given above, then using a quadratic approximation to the radial potential near \( R = R^* \) we find that the energy difference of the \( m = m^* \) and \( m = m^* \pm 1 \) states is

\[
\Delta E = \frac{e^2 \ell_B^4}{\epsilon R^{*5}}
\]

\[
= 2^{25/2} \ell_B^4 (\rho_s t)^{5/4} (\epsilon/e^2)^{3/2}.
\]

(13)

As \( R(\phi) \) varies (at \( Q = 0 \), by varying e.g. the density of electrons), levels will cross, since each level has different angular momentum, so the energy gap will actually exhibit quadratic minima separated by cusps, on top of the classical behaviour \( E_{\text{min}}(\phi) \). The scale of this variation will be given by \( 1/4 \) of (13). At \( Q \neq 0 \), angular momentum is no longer conserved. For small \( Q \) we may use perturbation theory around the \( Q = 0 \) limit to calculate \( E_{\text{min}} \). Since \( Q \) enters the Hamiltonian through \( R \cos \phi \), it changes angular momentum by \( \pm 1 \) so the first correction to \( E_{\text{min}} \) is of order \( Q^2 \) (this statement does not depend on the use of the string and semiclassical approximations). As \( Q \) varies, or as other parameters vary at nonzero \( Q \), the ground state energy will now exhibit variations similar to those at \( Q = 0 \), but level crossings will be avoided. However such structure will be washed out as \( Q \) increases toward \( Q_c \). Semiclassically, it is clear that as \( Q \) increases from zero, a point is reached at which the lowest allowed orbit changes topology from encircling the origin to not encircling the origin. For larger \( Q \), the variation of \( E_{\text{min}} \) with \( Q \) will have no oscillating component. Thus it is clear that quantum effects in \( E_{\text{min}} \) are not important as \( Q \to Q_c \).

Turning to the incommensurate phase, for \( Q \) not too much larger than \( Q_c \) so we can still
use the picture of well-separated domain walls, we might imagine that free vortices can exist, since domain walls have condensed anyway and can simply readjust to accommodate the wall ending at the vortex. However, this introduces distortion into the array of walls, equivalent to a dislocation. The elastic energy will cause the usual logarithmic divergence in the total energy of such a configuration. Consequently, vortices are again bound in pairs, but the attractive term is now logarithmic for large separation instead of linear. To calculate the behaviour of the gap, we need the elastic constants of the domain wall solid. I have been unable to find such a calculation for this $T = 0$ case in the literature, so it is worked out below.

Once again our main interest is in time-independent configurations, whose long-wave-length potential energy can be expressed in terms of $\bar{\theta}$ as (neglecting an additive constant)

$$E = \int d^2r \frac{1}{2} \left( K_x \left( \frac{\partial \bar{\theta}}{\partial x} \right)^2 + K_y \left( \frac{\partial \bar{\theta}}{\partial y} \right)^2 \right). \quad (14)$$

In terms of domain walls, it is evident that nonzero $\partial \bar{\theta}/\partial x$ represents a change in the separation of the walls, while nonzero $\partial \bar{\theta}/\partial y$ represents a rotation of the array away from its preferred orientation with walls parallel to the $y$ axis. To obtain the energies, we consider the energy of a periodic array of straight walls, described by $\ell = (\ell_x, \ell_y)$, the vector center-to-center separation of adjacent walls; the ground state will be $\ell = (\bar{\ell}, 0)$. On scales $\gg \ell = |\ell|$, we have $\nabla \bar{\theta} \approx -2\pi \bar{\ell}/\ell^2$ (note the minus sign due to our definitions in (5)) and hence for small deviations from the ground state $\nabla \bar{\theta} \simeq 2\pi \bar{\ell}^{-2}(\bar{\ell} - \ell_x, -\ell_y)$. Generalizing a formula from [10, 14], the energy density is

$$E/L^2 = -2\pi \rho_s Q \ell_x/\ell^2 + 2\pi \rho_s Q_c/\ell + W e^{-\kappa \ell}/\ell. \quad (15)$$

$W e^{-\kappa \ell}$ is the interaction energy per unit length of the array of walls; $\kappa$ is exactly $(t/\rho_s)^{1/2}$ in the classical approximation, and $W = 32(\rho_s t)^{1/2}$ [10]. The energy density is minimized for $\bar{\ell}$
given by

\[ \kappa \bar{\ell} e^{-\kappa \bar{\ell}} \sim 2\pi (Q - Q_c)/32\kappa \]

\[ \bar{\ell} \sim (\rho_s/t)^{1/2} \ln(32(t/\rho_s)^{1/2}/2\pi (Q - Q_c)), \] (16)

for \( \kappa \bar{\ell} \gg 1 \), as stated earlier. Expanding the energy density near its minimum at \((\bar{\ell},0)\) we obtain

\[ K_x = (\rho_s t)^{1/2} (Q - Q_c)\bar{\ell}^2/2\pi \]
\[ K_y = \rho_s Q_c \bar{\ell}/2\pi = 8(\rho_s t)^{1/2}\bar{\ell}/(2\pi)^2. \] (17)

Thus near the transition, the array of walls is much softer in the direction perpendicular to the walls. When the walls are no longer dilute, however, the constants \( K_x, K_y \) will \( \to \rho_s \) [16]. Note that from (15), (17) we can obtain the behaviour of the finite \( T \) KT transition temperature, \( T_c \sim (K_x K_y)^{1/2} \sim (Q - Q_c)^{1/2} \ln^{3/2}[1/(Q - Q_c)]. \)

For \( Q > Q_c \), the energy of a well-separated vortex-antivortex pair of total charge \( \pm 1 \) will be, for separation \( X, Y, R^2 = X^2 + Y^2 \),

\[ E(X,Y) = 2E_{\text{core}} + \frac{e^2}{4\epsilon R} + C'(K_x K_y)^{1/2} \ln \left\{ (K_y X^2 + K_x Y^2)/K_y \bar{\ell}^2 \right\}^{1/2} \] (18)

where \( C' \) is another constant, the additional core energy due to the domain wall solid is neglected. The logarithmic term was obtained by first rescaling the coordinates to make the energy isotropic, and using the wall separation as the lower cutoff on the \( X \) separation at which the logarithm is arbitrarily set to zero. The logarithm is meaningful only at larger separations, which for the \( Y \) direction means \( Y > (K_y/K_x)^{1/2} \bar{\ell} \). Minimizing once again, it is found that the energy is lowest when the pair is oriented in the \( y \) direction and is then given by \( E_{\text{min}} = 2E_{\text{core}} + C'(K_x K_y)^{1/2}(1 + \ln(e^2/4\epsilon C' K_y \bar{\ell})) \). However the corresponding value of \( Y \) is \( Y = e^2/(4\epsilon C'(K_x K_y)^{1/2}) \) which as \( Q \to Q_c \), is smaller than the condition for the validity
of the logarithmic interaction by a factor $\sim \ln^2[1/(Q - Q_c)]$. Therefore, the form (18) is not, in fact, appropriate.

To try to obtain the correct asymptotic form of the energy gap near $Q_c$, let us consider the opposite limit, where the separation of the vortex and antivortex is in the $y$ direction and is much less than the length $(K_y/K_x)^{1/2}\bar{\ell}$. There are two cases to consider: the sign of the separation is such that either (i) an additional short length of parallel string is inserted between the walls in the array, or (ii) a short length of a wall is removed from the array. Here “short” means compared with $(K_y/K_x)^{1/2}\bar{\ell}$, but not necessarily compared with $\bar{\ell}$. At such short separations, the distortion of the array may become small, that is the effective dipole moment of the pair, as determined by the $\bar{\theta}$ field far from the pair, may be much less than their actual moment, as defined by the vorticity times the separation. Even if the elastic contribution to the vortex-antivortex potential is not zero, there is still likely to be a linear piece, as will be described shortly, that dominates the potential, and is not screened out as it will be for large separations by elastic deformation of the array that produces the asymptotically logarithmic behaviour. For case (i), the energy of the pair is then

$$E(Y) = 2E_{\text{core}} + \frac{e^2}{4eY} + 2\pi\rho_s(Q_c - Q)Y + 2We^{-\kappa\bar{\ell}/2}Y \approx 2E_{\text{core}} + \frac{e^2}{4eY} + 2(2\pi\rho_sW(Q - Q_c)/(\kappa\bar{\ell}))^{1/2}Y$$

so the minimum is found to be

$$Y = \left[\frac{e^4\kappa\bar{\ell}}{128\pi^2e^2\rho_s^2W^2(Q - Q_c)}\right]^{1/4}$$

$$E_{\text{min}} = 2E_{\text{core}} + 2\left[2\pi\rho_sW(Q - Q_c)e^4/4\kappa\bar{\ell}e^2\right]^{1/4}.$$ (20)

For case (ii), the pair have the opposite orientation from case (i), but in terms of the magnitude $Y$ of the separation, the energy is now

$$E(Y) = 2E_{\text{core}} + \frac{e^2}{4eY} + 2\pi\rho_s(Q - Q_c)Y - 2We^{-\kappa\bar{\ell}}Y$$
\[ E \approx 2E_{\text{core}} + \frac{e^2}{4\epsilon Y} + 2\pi \rho_s (Q - Q_c) Y - 4\pi \rho_s (Q - Q_c) Y / \kappa \bar{\ell} \]  

(21)

Note the opposite signs from case (i), since a length of wall has been removed rather than added, and that the interaction term containing \( W \) has the same form as in (14), while in case (i) the argument of the exponential was halved because the inserted walls were only \( \bar{\ell}/2 \) away from the others. This term in (21) can now be neglected, and the form for the minimum is therefore similar to (11). Since the resulting \( E_{\text{min}} \) has a square-root dependence, this is asymptotically lower than (20), allowing us to write a single unified formula for our final result on either side of the transition,

\[
Y^* = \left( \frac{e^2}{8\pi \epsilon \rho_s |Q - Q_c|} \right)^{1/2}
\]

\[
E_{\text{min}} = 2E_{\text{core}} + (2\pi e^2 \rho_s |Q - Q_c| / \epsilon)^{1/2}.
\]

(22)

Thus there is a symmetrical downward square-root cusp in \( E_{\text{min}} \) at \( Q_c \), with no direct dependence on the tunnelling \( t \). The value of \( Y^* \) is smaller by a factor \( \bar{\ell}^{-1/2} \sim \ln^{-1/2} 1/(Q - Q_c) \) than \( (K_y/K_x)^{1/2} \bar{\ell} \) as \( Q \to Q_c \) on the incommensurate side, so the calculation should be asymptotically valid. Quantum corrections should be asymptotically negligible on both sides.

Since the binding of the vortices will be somewhat weaker than at \( Q = 0 \) even when \( Q \gg Q_c \), due to the effective removal of \( t \) from the action (4), it is to be expected that the activation gap saturates at a value below its \( B_{\parallel} = 0 \) value at large \( B_{\parallel} \) [16], as is observed very strikingly in the experiments [1].

So far the cusp in the activation gap has not been observed in experiments. While this might be due simply to large scale variations in the density and hence in the value of \( Q_c \), another possibility is that it is connected with the effects of small scale disorder. M.P.A. Fisher [18] has pointed out that randomness in the tunnelling \( t \) is a relevant perturbation in
the incommensurate phase, which can destroy the ordered state both at finite and at zero temperature. If the tunnelling varies from place to place, perhaps because of defects such as steps in the layer separation, then $t$ in eq. (1) varies with $x$, though it probably remains real. In the incommensurate phase, in the $\bar{\theta}$ variables that describe the array at scales large compared with the spacing, the randomness in $t$ causes deviations in the array of walls that can still be described in terms of a term like $-t(r)\cos(\bar{\theta}(r))$. It is well-known that such a term (which tends to pin the domain walls randomly) destroys the long-range order (or quasi-long-range order at $T \neq 0$) whenever the spacetime dimension is less than 4, no matter how weak the disorder. While the magnitude and correlation length of the disorder in the samples are uncertain, it is worthwhile working out the consequences of this effect for the theory of the activation gap. At large enough length scales the disorder will destroy the ordering, and this should affect the behaviour of the charged excitations which as we have seen become very large near the transition.

In the presence of disorder, I claim that a zero temperature phase transition can still occur as $B_\parallel$ (or $Q$) is increased, since the state at $Q = 0$ is essentially stable to weak disorder, which can at worst only introduce short lengths or loops of string into the ground state described for the pure system. At some $Q_c$ it becomes favourable to condense infinitely long domain walls, but they will now form a randomly pinned array that may contain dislocations (vortices) due to the quenched disorder. Since there is no long-range order, there can be no asymptotically logarithmic potential to confine the charged vortex excitations into pairs, and so the possibility of $\pm e/2$ excitations exists. Accordingly, it is plausible that the energy gap decreases as $Q$ increases to $Q_c$, and then saturates at an excitation energy $E_{\text{core}}$ immediately $Q > Q_c$.

The critical properties of the transition itself will of course also be affected by the destruct-
tion of the ordered state. Since the time-independent disorder distinguishes the imaginary
time direction from the two space directions, it is possible that correlations scale at criticality
with different powers of separation in each of the three directions $(x, y, \tau)$. It is also possible
that the disorder effectively restores the isotropy of the scaling in the two space directions;
an analogous phenomenon occurs in the pure system at the finite temperature KT transition.
This would leave a single dynamic exponent $z$ relating distances and times, $\tau \sim r^z$. All of
these properties of this quantum phase transition are completely open at present. It would
be interesting to study them by Monte Carlo simulation in the simple model consisting of
eq. (1) (on a lattice) plus disorder, neglecting the subtle effects of the lowest Landau level
dynamics of the vortices.

The theory presented above extends to other fractions $\nu$, so long as an incompressible
ground state that is fully polarized in the isospin variable $e^{i\theta}$ is expected at $B_\parallel = 0$. It is
also interesting to consider compressible states of the type discussed in $\square$. In particular,
for $\nu = 1/q$, $q$ even, a compressible Fermi-liquid-like state is possible at short separations $d$
$\square$, in which the Laughlin-Jastrow type correlation factor is independent of which layer the
electrons are in, just as in the incompressible odd $q$ states discussed here and in $\square$. $\square$. (Note
that this requires stronger inter-layer correlations than in the 331 state supposed responsible
for the observed $\nu = 1/2$ incompressible state $\square$..) If such a state approximates the ground
state when tunnelling $t$ is neglected, then it is an itinerant ferromagnet with fully polarized
isospin, analogous to a Stoner-type metallic ferromagnet, though the Coulomb interactions
are, as before, not fully SU(2) invariant, so the order can again be described simply by an
angle $\theta$. Bonesteel $\square$ has discussed the opposite case of a two layer system with the same
correlations but without spontaneous isospin polarization, in which there are Fermi surfaces
for both $\uparrow$ and $\downarrow$ spins. There is also the intermediate possibility of a partially polarized state
at $t = 0$. All systems having a spontaneous polarization at $t = 0$ have an isospin stiffness but there are no compact, well defined charged vortex excitations because the system has nonzero compressibility for changes in $n_\uparrow + n_\downarrow$. Due to the stiffness, a commensurate-incommensurate transition should occur in these cases also, which might be observable through its effect on the Fermi surface.

To conclude, observation of the downward cusp in the activation energy gap predicted here, as given in eq. (22), on going through the transition would confirm the general picture first presented in [2]. However, the alternative possibility of significant disorder effects should also be considered further.

I am grateful for discussions with J.P. Eisenstein, S.Q. Murphy, S.M. Girvin, K. Yang, M.P.A. Fisher, D. Huse and S. Sachdev, and for the hospitality of AT&T Bell Labs. This work was supported by NSF-DMR-9157484.

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