A Certainty Equivalence Result in Team-Optimal Control of Mean-Field Coupled Markov Chains

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Abstract—This paper studies a large number of homogeneous Markov decision processes where the transition probabilities and costs are coupled in the empirical distribution of states (also called mean-field). The state of each process is not known to others, which means that the information structure is fully decentralized. The objective is to minimize the average cost, defined as the empirical mean of individual costs, for which a sub-optimal solution is proposed. This solution does not depend on the number of processes, yet it converges to the optimal solution of the so-called mean-field sharing as the number of processes tends to infinity. Under some mild conditions, it is shown that the convergence rate of the proposed decentralized solution is proportional to the square root of the inverse of the number of processes. Finding this sub-optimal solution involves a non-smooth non-convex optimization problem over an uncountable set, in general. To overcome this drawback, a combinatorial optimization problem is introduced that achieves the same rate of convergence.

I. INTRODUCTION

A. Motivation

Team-optimal control of Markov chains have recently attracted much attention due to their potential applications in emerging areas such as smart grids [1], social networks [2], swarm robotics [3], and transportation networks [4]. These applications normally involve many interconnected decision makers, wishing to collaborate in order to minimize a common cost function [5].

When the decision makers are modeled as controlled Markov chains and joint state is known to all, the optimal solution is identified by the celebrated dynamic programming [6]. The computational complexity of solving this dynamic program is exponential in the number of decision makers, in general. In addition, at each time instant, the joint state (a vector of the same size as the number of decision makers) must be communicated among all decision makers. In practice, however, each decision maker has limited computation and communication resources. Due to such practical limitations, mean-field models have received much attention recently for the scalablity of their solution. Inspired by statistical mechanics and classical physics, mean-field models have received much attention recently for the scalability of their solution. In practice, however, each decision maker has limited computation and communication resources. Due to such practical limitations, mean-field models have received much attention recently for the scalability of their solution. Inspired by statistical mechanics and classical physics, mean-field models have received much attention recently for the scalability of their solution.

In mean-field games, the solution concept is Nash strategy and the term mean-field refers to the empirical distribution of infinite population of players. When the population is large, the effect of a single player on other players becomes negligible. Using this observation, an approximate Nash strategy is derived such that the approximation error converges to zero as the size of population goes to infinity.

In the context of team theory, mean-field teams were first introduced in [19] and the early results were presented in [20]–[23]. In mean-field teams, the solution concept is a team-optimal strategy and the term mean-field refers to the empirical distribution of finite population. In [20], a dynamic programming decomposition is derived to obtain a globally optimal solution, irrespective of the size of population (not necessarily large population), under mean-field sharing information structure. To implement the mean-field sharing, the communication network of agents must be connected. In practice, however, having a connected network may not be practically feasible or economically viable, specially when the population is large. Therefore, a completely decentralized strategy is desirable in this type of problem.

In [24], a solution approach of mean-field games is adopted to find an approximate person-by-person optimal strategy for the finite-horizon case. The strategy is identified by a dynamic program, and the approximation error is shown to go to zero at the rate $1/\sqrt{n}$ as $n$ increases, under some Lipschitz conditions on the dynamics, cost, and the strategy. In [25], the existence of an approximate person-by-person optimal strategy for the discounted cost infinite-horizon case is established. The strategy is identified by an irregular Hamilton-Jacobi-Bellman equation for which the solution is not necessarily the viscosity solution. Under some Lipschitz conditions, the approximation error of such a strategy is shown to converge to zero in distribution as $n$ increases.

In this paper, it is desired to find a completely decentralized strategy whose performance is sufficiently close to that obtained by the mean-field sharing strategy. Finding such a strategy is conceptually challenging because every agent has a different perspective (i.e., information) of the system and any such discrepancy would make it difficult to establish cooperation among agents. In contrast to [24], we use the dynamic program of mean-field teams that is fundamentally different from that of mean-field games; in addition, we do not impose any assumption on the strategy.

In contrast to [25], the convergence here is in the sense of

Note that verifying any assumption on the strategy is typically very difficult because finding the strategy itself is an open problem, in general.
almost surely. In contrast to both papers, we consider global optimality rather than person-by-person optimality.

The rest of this paper is organized as follows. In Section II the problem is formulated in the context of controlled mean-field coupled Markov chains. Then, the main results are presented in Section III followed by concluding remarks given in Section IV.

B. Notation

Throughout the paper, \( \mathbb{N} \), \( \mathbb{R}_{>0} \), and \( \mathbb{R}_{>0} \) refer to natural numbers, non-negative real numbers, and positive real numbers, respectively. The finite set of integers \( \{1, \ldots, k\} \) is denoted by \( \mathbb{N}_k \). Moreover, \( P(\cdot) \) is the probability of a random variable; \( \mathbb{E}[\cdot] \) represents the expectation of an event; \( \mathbb{I}(\cdot) \) is the indicator function of a set; \( ||\cdot||\) represents the infinity norm of a vector, and \( |\cdot| \) denotes the absolute value of a real number or the cardinality of a set. The short-hand notation \( x_{i,t} \) is used to denote vector \( \text{vec}(x_1, \ldots, x_t) \). Given \( n \in \mathbb{N} \) and a finite set \( \mathcal{X} \), the following spaces are defined.

| Table I | TABLE OF THE SPACES USED IN THIS PAPER. |
|---------|----------------------------------------|
| Space of probability measures | \( \Delta(\mathcal{X}) = \{ (p_1, \ldots, p_{|\mathcal{X}|}) | p_k \in [0,1], k \in \mathbb{N}_{|\mathcal{X}|}, \sum_{k=1}^{X} p_k = 1 \} \) |
| Space of empirical distributions (mean-field) | \( \mathcal{M}_n = \{ (p_1, \ldots, p_{|\mathcal{X}|}) | p_k \in \{0, \frac{1}{n}, \ldots, 1\}, k \in \mathbb{N}_{|\mathcal{X}|}, \sum_{k=1}^{X} p_k = 1 \} \) |
| Product space of unit intervals | \( \mathcal{I}(\mathcal{X}) = \{ (p_1, \ldots, p_{|\mathcal{X}|}) | p_k \in [0,1], k \in \mathbb{N}_{|\mathcal{X}|} \} \) |
| Uniformly quantized by \( \frac{1}{n} \) | \( \mathcal{Q}_n = \{ (p_1, \ldots, p_{|\mathcal{X}|}) | p_k \in \{0, \frac{1}{n}, \ldots, 1\}, k \in \mathbb{N}_{|\mathcal{X}|} \} \) |

The following relationships hold between above spaces:
\[ \mathcal{M}_n \subset \Delta(\mathcal{X}) \subset \mathcal{I}(\mathcal{X}) \quad \text{and} \quad \mathcal{M}_n \subset \mathcal{Q}_n \subset \mathcal{I}(\mathcal{X}). \]

II. Problem Formulation

Consider a dynamical system consisting of \( n \in \mathbb{N} \) homogeneous agents (decision makers or controlled Markov chains) operating over a fixed finite horizon \( T \in \mathbb{N} \). Let \( x_{i,t} \in \mathcal{X} \) denote the state of agent \( i \in \mathbb{N}_n \) at time \( t = NT \) and \( u_{i,t} \in \mathcal{U} \) represent its control action. Let also \( m_t \) be the empirical distribution of states at time \( t \), i.e.,
\[ m_t(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(x_{i,t} = x), \quad x \in \mathcal{X}, \]
where \( m_t \in \mathcal{M}_n \). At time \( t \in \mathbb{N}_T \), the state of agent \( i \in \mathbb{N}_n \) evolves as follows:
\[ x_{i,t+1} = f_t(x_{i,t}, u_{i,t}, w_{i,t}, m_t), \]
where \( w_{i,t} \in \mathcal{W} \) is the local noise of agent \( i \) at time \( t \). The spaces \( \mathcal{X}, \mathcal{U}, \) and \( \mathcal{W} \) are finite-valued and it is assumed that

1 For ease of reference, we only use term agent in the sequel.

2 The main results of this paper hold for any measurable set \( \mathcal{W} \) as long as the variance of the noise process is uniformly bounded.
Problem 1 Let $J^*$ denote the optimal performance under mean-field sharing information structure \( G \). It is desired to find a sub-optimal strategy \( g \), given by \( \mathcal{G} \), under which the system performance \( J(g) \) is guaranteed to be within \( \epsilon(n) \)-neighborhood of \( J^* \), i.e.,

\[
|J(g) - J^*| \leq \epsilon(n). \tag{8}
\]

III. MAIN RESULTS

In this section, a completely decentralized strategy is proposed, as a sub-optimal alternative to the mean-field sharing solution, as noted in the previous section. In particular, it is shown that the optimality gap, given by (8), converges to zero at the rate \( 1/\sqrt{n} \) as \( n \) increases. To this end, the following assumption is made.

Assumption 1 The transition probabilities and per-step costs are Lipschitz functions in mean-field. More precisely, there exist constants \( K_1^\gamma, K_2^\gamma \in \mathbb{R}_{>0}, \in \mathbb{N}_T, \) such that for every \( x, y \in \mathcal{X}, u \in \mathcal{U}, z_1, z_2 \in \mathcal{I}(\mathcal{X}), \)

\[
\left| P \left( y|x, u, z_1 \right) - P \left( y|x, u, z_2 \right) \right| \leq K_1^\gamma \| z_1 - z_2 \|_\infty,
\]

\[
|\ell_t(x, u, z_1) - \ell_t(x, u, z_2)| \leq K_2^\gamma \| z_1 - z_2 \|_\infty.
\]

Remark 1 It is to be noted that every polynomial function of mean-field is a Lipschitz function because mean-field is confined to the bounded interval \( \mathcal{I}(\mathcal{X}) \) [28, Corollary 12.2]. It is worth highlighting that, according to Weierstrass Approximation Theorem [29], any continuous function can be uniformly approximated as closely as desired by polynomial functions.

Let \( \gamma_t : \mathcal{X} \to \mathcal{U} \) be the local map from state space \( \mathcal{X} \) to action space \( \mathcal{U} \) at time \( t \in \mathbb{N}_T \), i.e., from (7)

\[
\gamma_t(\cdot) := g_t(\cdot, g_{t; t-1}). \tag{9}
\]

According to (7) and (9),

\[
u_t = \gamma_t(x_t).
\]

Denote by \( \mathcal{G} \) the set of all mappings \( \gamma : \mathcal{X} \to \mathcal{U} \) and note that \( \mathcal{G} \) is a finite set of size \( |\mathcal{G}| = |\mathcal{U}|^{|\mathcal{X}|} \). For every \( \gamma \in \mathcal{G} \) and \( z, \gamma \in \mathcal{I}(\mathcal{X}) \), define

\[
\bar{f}_t(z, \gamma)(\cdot) := \sum_{x \in \mathcal{X}} x(z) P(\cdot|x, \gamma(x), z), \tag{10}
\]

and

\[
\bar{c}_t(z, \gamma) := \sum_{x \in \mathcal{X}} z(x) \ell_t(x, \gamma(x), z). \tag{11}
\]

Lemma 1 Let Assumption (7) hold. Then, there exist constants \( K_1^\gamma, K_2^\gamma \in \mathbb{R}_{>0}, \in \mathbb{N}_T, \) such that for every \( \gamma \in \mathcal{G} \) and \( z_1, z_2 \in \mathcal{I}(\mathcal{X}), \)

\[
\| \bar{f}_t(z_1, \gamma) - \bar{f}_t(z_2, \gamma) \|_\infty \leq K_1^\gamma \| z_1 - z_2 \|_\infty,
\]

\[
|\bar{c}_t(z_1, \gamma) - \bar{c}_t(z_2, \gamma)| \leq K_2^\gamma \| z_1 - z_2 \|_\infty.
\]

Proof According to [28, Theorem 12.1] and [28, Theorem 12.4] any linear combination or product of Lipschitz functions is a Lipschitz function as well. Hence, function \( \bar{f}_t(z, \gamma) \) given by (10) is Lipschitz because it is a linear combination of the product of two Lipschitz functions \( z \) and \( \ell_t(x, \gamma(x), z) \).

Lemma 2 Consider \( n \) i.i.d. random variables \( W^i \in \mathcal{W} \) with common probability mass function \( P(W) \). Then, for every realization \( w \in \mathcal{W} \), one has

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(W^i = w) - P(W = w) \right] \leq O\left( \frac{1}{\sqrt{n}} \right).
\]

Proof Let \( b \) be a random variable on \( \mathbb{R} \) and \( n \in \mathbb{R}_{>0} \), then, as the first step, the following inequality is established

\[
\mathbb{E}\left[ |b| \right] \leq \frac{\sqrt{n}}{2} \mathbb{E}\left[ b^2 \right] + \frac{1}{2\sqrt{n}}. \tag{12}
\]

This follows immediately from the inequality \( 0 \leq (b + \frac{1}{\sqrt{n}})^2 \), after rewriting it in the following form: \( -\frac{1}{\sqrt{n}}\mathbb{E}[b] \leq \frac{1}{2\sqrt{n}} \) and noting that the expectation operator is monotone.

In the second step, one has

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(W^i = w) - P(W = w) \right] = \frac{1}{n^2} \mathbb{E}\left[ \sum_{i=1}^{n} \left( \mathbb{1}(W^i = w) - P(W = w) \right) \right] = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}\left[ \left( \mathbb{1}(W^i = w) - P(W = w) \right)^2 \right] \leq O\left( \frac{1}{n} \right), \tag{13}
\]

where (a) follows from the fact that the random variables \( (\mathbb{1}(W^i = w) - P(W = w)) \) have zero-mean \( \mathbb{E}[\mathbb{1}(W^i = w) - P(W = w)] = 0 \) and are mutually independent too, which implies that the cross-terms are equal to zero; (b) follows from the inequality \( \mathbb{E}\left[ (\mathbb{1}(W^i = w) - P(W = w))^2 \right] \leq 1, \forall i \in \mathbb{N}_n \).

The proof is now complete by virtue of inequalities (12) and (13).
One can then write
\[
P(m_{t+1}(y)|m_t, \gamma_t) = \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} 1(x^t_i = y) \right)
\]
\[
\geq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} 1(f_t(x^t_i, \gamma(x^t_i), m_t, w^t_i) = y) \right)
\]
\[
= \mathbb{P}\left( \sum_{x \in X} \sum_{w \in W} \frac{1}{n} \sum_{i=1}^{n} 1(f_t(x, \gamma(x), m_t, w) = y) \right)
\]
\[
= \mathbb{P}\left( \sum_{x \in X} \sum_{w \in W} \mathbb{1}(f_t(x, \gamma(x), m_t, w) = y) \right).
\]

Lemma 4: For every \( m \in \mathcal{M}_n \) and \( \gamma \in \mathcal{G} \),
\[
\mathbb{E}\left[ \|\hat{f}_t(m, \gamma, w) - \hat{f}_t(m, \gamma)\|_\infty \right] \leq O\left( \frac{1}{\sqrt{n}} \right),
\]
where the expectation is taken with respect to \( w \in W^n \).

**Proof:** For every \( y \in \mathcal{X} \), one has
\[
\mathbb{E}\left[ \|f_t(m, \gamma, w_t)(y) - f_t(m, \gamma)(y)\| \right]
\]
\[
\leq \mathbb{E}\left[ \sum_{x \in \mathcal{X}} \sum_{w \in W} m(x) \mathbb{1}(f_t(x, \gamma(x), m, w) = y) \right]
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(w^t_i = w) - \mathbb{P}(w)
\]
\[
\leq \sum_{x \in \mathcal{X}} \sum_{w \in W} m(x) \mathbb{1}(f_t(x, \gamma(x), m, w) = y)
\]
\[
\cdot \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(w^t_i = w) - \mathbb{P}(w) \right] \leq O\left( \frac{1}{\sqrt{n}} \right),
\]
where (a) follows from (4), (10), and (14); (b) follows from the triangle inequality and monotonicity of the expectation operator, and (c) follows from Lemma 2 the fact that \( \mathcal{X} \) and \( W \) are finite-valued spaces, and noting that \( m(x) \leq 1 \). Since the above result holds for every \( y \in \mathcal{X} \), it also holds in the infinity norm.

**Lemma 5:** Let Assumption 4 be satisfied. Then, given \( \gamma_t \in \mathcal{G} \), \( m_t \in \mathcal{M}_n \), and \( z_t \in \Delta(\mathcal{X}) \), the following inequality holds for \( t \in \mathbb{N}_t \),
\[
\mathbb{E}\left[ \|m_{t+1} - z_{t+1}\|_\infty \right] \leq K_t^3 \|m_t - z_t\|_\infty + O\left( \frac{1}{\sqrt{n}} \right).
\]

**Proof:** It is straightforward to show that
\[
\mathbb{E}\left[ \|m_{t+1} - z_{t+1}\|_\infty \right] \leq K_t^3 \|m_t - z_t\|_\infty + O\left( \frac{1}{\sqrt{n}} \right),
\]
where (a) follows from Lemma 4 and (10), (b) follows from Lemma 4, and (c) follows from Lemmas 4 and 4.

**Theorem 1:** Let Assumption 4 hold. Let also \( \psi_t(z_t) \) be any argmin of the right-hand side of (16) at time \( t \in \mathbb{N}_T \). Define fully decentralized strategy \( g := \{g_t\}_{t=1}^{T} \) for Problem 1 such that
\[
g_t(x) := \psi_t(z_t)(x), \quad x \in \mathcal{X}, t \in \mathbb{N}_T.
\]
Then,
\[
|J(g) - J^*| \leq \epsilon(n) \in O\left( \frac{1}{\sqrt{n}} \right).
\]

**Proof:** From the triangle inequality, one has
\[
|J(g) - J^*| \leq |J^* - \hat{V}_1(z_1)| + |J(g) - \hat{V}_1(z_1)|.
\]
It is desired now to show both terms on the right-hand side of (21) are \( O\left( \frac{1}{\sqrt{n}} \right) \).

**Step 1:** In this step, we consider the first term of (21). From (16) and (19) and noting that \( J^* = \mathbb{E}[\hat{V}_1(m_1)] \), we
have

\[ |J^* - \hat{V}_1(z_1)| = |E[V_1(m_1)] - \hat{V}_1(z_1)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} K_t^3 \|m_t - z_t\|_\infty + O\left(\frac{1}{\sqrt{n}}\right), \]

where (a) follows from the fact that \( \hat{V}_1(z_1) \) is deterministic; (b) follows from the monotonicity of the expectation operator, which implies that \( E[y] \leq E[|y|] \) for every random variable \( y \); (c) follows from Lemma 6 and (d) follows from (11), \( z_1 = \mathbb{P}(x_1) \), Lemma 2 and the fact that the initial states are assumed to be i.i.d. random variables.

**Step 2:** In this step, we consider the second term of (21). Let \( \hat{m}_t \) denote the empirical distribution of \( n \) agents when they use strategy \( g \), given by Theorem 1, i.e., \( \gamma_t = \psi_t(z_t) \). Therefore,

\[
|J(g) - \hat{V}_1(z_1)| = \left| \sum_{t=1}^{T} K_t^4 \|\hat{m}_t - z_t\|_\infty \right| \leq \sum_{t=1}^{T} K_t^4 \|\hat{m}_t - z_t\|_\infty + O\left(\frac{1}{\sqrt{n}}\right),
\]

(22)

where (a) follows from (5) and (16) (where \( \min \) becomes equality under \( \psi_t(z_t) \)) and (b) follows from Lemma 1. Note that \( t = 1, \hat{m}_1 = m_1 \), and for \( t \in \mathbb{N}_{T-1} \), the evolution of the mean-field is given by Lemma 3 as follows:

\[
\hat{m}_{t+1} = \tilde{f}_t(\hat{m}_t, \psi_t(z_t), w_t).
\]

(23)

Since both \( \hat{m}_{t+1} \) that evolves according to (23) and \( z_{t+1} \) (that evolves according to (17)) use identical strategy \( \gamma_t = \psi_t(z_t) \), we can use Lemma 5 to compute the expected difference. In particular, given \( \hat{m}_t \) and \( z_t \),

\[
E[\|\hat{m}_{t+1} - z_{t+1}\|_\infty] \leq K_t^3 \|\hat{m}_t - z_t\|_\infty + O\left(\frac{1}{\sqrt{n}}\right).
\]

(24)

Now, it results from recursively using (24) in (22) and from the monotonicity of the expectation operator that there exist constants \( K_6, K_7 \in \mathbb{R}_{>0} \) such that

\[
|J(g) - \hat{V}_1(z_1)| \leq K_6 \|m_1 - z_1\|_\infty + K_7 O\left(\frac{1}{\sqrt{n}}\right),
\]

(25)

where (a) follows from the fact that \( \hat{m}_1 = m_1 \) and (d) follows from Lemma 2 on noting that the initial states are i.i.d. random variables.

**Remark 2** An important feature of Theorem 1 is that its proposed solution is independent of the number of agents \( n \) because the functions \( f_t(\cdot) \) and \( \hat{c}_t(\cdot) \), given respectively by (10) and (4), are independent of \( n \).

**Remark 3** Note that (15), (16) and (17) do not depend on the information of agents; hence, they may be solved offline. More precisely, every agent can independently compute \( \{\psi_t(z_t)\}_{t=1}^{T} \) in a distributed manner with no communication required. In the case of multiple solutions, agents can make sure they all compute the same solution while using argmin by agreeing upon a deterministic rule to break a tie. Thus, strategy (20) can be implemented based on a completely decentralized information structure (4).

Initially, every agent locally computes the argmin of (16) for all \( t \in \mathbb{N}_{T} \), i.e., \( \{\psi_t(z_t)\}_{t=1}^{T} \). Then, when the system is operating, agent \( i \) makes a decision based on the local state \( x^i_t \), i.e.,

\[
u^i_t = \psi_t(z_t)(x^i_t).
\]

According to Theorem 1, the decision of each agent \( i \in \mathbb{N}_{n} \) at time \( t \in \mathbb{N}_{T} \) is determined by three factors: (a) strategy \( \psi_t \) that depends on the model of the system; (b) variable \( z_t \) that is common knowledge among all agents and evolves to \( z_{t+1} \) according to (17), and (c) local state \( x^i_t \) that is only known to agent \( i \).

**Proposition 1** Let Assumption 7 hold. Then, \( \hat{V}_t(z_t) \) is a Lipschitz function, i.e., there exists a constant \( K_t \in \mathbb{R}_{>0} \), \( t \in \mathbb{N}_{T} \), such that for every \( z_1, z_2 \in \mathcal{I}(x) \),

\[
|\hat{V}_t(z_1) - \hat{V}_t(z_2)| \leq K_t \|z_1 - z_2\|_\infty.
\]

**Proof** The proof is given in Appendix IV-B.

The recursion introduced in (15)–(17) is computationally intractable, in general, because \( \mathcal{I}(x) \) is an uncountable set. According to Lemma 1 and Proposition 1, \( f(z) \) and \( \hat{V}(z) \) are Lipschitz continuous, respectively. Therefore, one could quantize the infinite-set optimization of Theorem 1 into a finite-set one such that the quantization error is upper-bounded by some Lipschitz function. Using this idea, it is shown in Corollary 1 that the rate of convergence in Theorem 1 is preserved under a uniform quantization.

**Corollary 1** Let Assumption 7 hold and function \( Q : \Delta(\mathcal{X}) \rightarrow Q_n \) map every point \( z \in \Delta(\mathcal{X}) \) to its nearest point \( \hat{z} \) in \( Q_n \), i.e.,

\[
Q(z) = \arg\min_{\hat{z} \in Q_n} \|z - \hat{z}\|_\infty.
\]

Define \( \hat{V}_{T+1}(\hat{z}_{T+1}) := 0 \), \( \hat{z}_{T+1} \in Q_n \), and for \( t = T, \ldots, 1 \), \( \hat{z}_t \in Q_n \),

\[
\hat{V}_t(\hat{z}_t) := \min_{\gamma_t \in V} \left( \hat{c}_t(\hat{z}_t, \gamma_t) + \hat{V}_{t+1}(\hat{z}_{t+1}) \right),
\]

(25)

where \( \hat{z}_1 = Q(\mathbb{P}(x_1), \mathbb{P}(x_1)) \in \Delta(\mathcal{X}) \), and \( \hat{z}_t \) evolves deterministically as follows

\[
\hat{z}_{t+1} = Q(\tilde{f}_t(\hat{z}_t, \gamma_t)).
\]

Let \( \psi_t(\hat{z}_t) \) be any argmin of the right-hand side of (25) and define \( g_t := \{g_t\}_{t=1}^{T} \), where

\[
g_t(x) := \psi_t(\hat{z}_t)(x), \quad x \in \mathcal{X}, t \in \mathbb{N}_T.
\]
Then, strategy \( g \) is a solution to Problem [6] such that

\[
|J(g) - J^*| \leq \epsilon(n) \in O\left(\frac{1}{\sqrt{n}}\right).
\]

**Proof** The error associated with quantizing \( z \in \Delta(X) \) into \( \hat{z} \in Q_n \) is bounded by

\[
\sup_{z \in \Delta(X)} \|z - Q(z)\|_\infty \leq \frac{1}{n}.
\]

According to Lemma 1 and Proposition 1, functions \( f_t \) and \( \hat{V}_t \) are Lipschitz in \( z \). Therefore, this quantization error will be upper-bounded by \( O\left(\frac{1}{\sqrt{n}}\right) \) over a fixed finite horizon \( T \). The proof is complete on nothing that \( O\left(\frac{1}{\sqrt{n}}\right) \) is dominated by \( O\left(\frac{1}{n}\right) \), when \( n \) is large.

### IV. Conclusions

In this paper, team-optimal control of a large number of homogeneous agents modeled as mean-field coupled Markov chains is considered. Every agent observes only its own local state, i.e., the strategy has a fully decentralized information structure. A sub-optimal strategy, independent of the number of agents \( n \), is proposed whose performance converges to that of the optimal mean-field sharing strategy at the rate \( 1/\sqrt{n} \).

To establish this result, it is assumed that the transition probabilities and costs are Lipschitz continuous in the mean-field (i.e., no assumption is imposed on the strategy). To find the sub-optimal strategy, it is required to solve an infinite-set optimization problem, in general. To address this concern, a novel idea is proposed to quantize the infinite-set optimization problem into a finite-set one. In particular, it is shown that under uniform quantization with the step-size of \( 1/n \), the convergence rate \( 1/\sqrt{n} \) is preserved.

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where \((a)\) follows from the triangle inequality, per-step costs being non-negative (by definition), and the monotonicity of the minimum operator; \((b)\) follows from Lemma 8 and the monotonicity of minimum operator; and \((c)\) follows from (16). Therefore, \(K_T^5 := K_T^4\), i.e.,

\[
|V_T(m_T) - \hat{V}_T(z_T)| \leq K_T^5 \|m_T - z_T\|_\infty.
\]

Suppose now that the inequality holds at time \(t + 1\), i.e.,

\[
|V_{t+1}(m_{t+1}) - \hat{V}_{t+1}(z_{t+1})| \leq K_{t+1}^5 \|m_{t+1} - z_{t+1}\|_\infty + O\left(\frac{1}{\sqrt{n}}\right).
\]

It is desired to prove the result for time \(t\). It is deduced from (19) that

\[
V_t(m) = \min_{\gamma_t \in \Psi} \left( \hat{c}_t(m_t, \gamma_t) + E[V_{t+1}(m_{t+1})|m_t, \gamma_t] \right)
\]

\[
= \min_{\gamma_t \in \Psi} \left( \hat{c}_t(m_t, \gamma_t) - \hat{c}_t(z_t, \gamma_t) 
+ E[V_{t+1}(m_{t+1})|m_t, \gamma_t] - \hat{V}_{t+1}(z_{t+1}) 
+ \hat{c}_t(z_t, \gamma_t) + \hat{V}_{t+1}(z_{t+1}) \right)
\]

\[
\leq \min_{\gamma_t \in \Psi} \left( \hat{c}_t(m_t, \gamma_t) - \hat{c}_t(z_t, \gamma_t) 
+ E[V_{t+1}(m_{t+1}) - \hat{V}_{t+1}(z_{t+1})|m_t, z_t, \gamma_t] 
+ \hat{c}_t(z_t, \gamma_t) + \hat{V}_{t+1}(z_{t+1}) \right)
\]

\[
\leq K_t^4 \|m_t - z_t\|_\infty
+ K_{t+1}^5 E[\|m_{t+1} - z_{t+1}\|_\infty|m_t, z_t, \gamma_t] + O\left(\frac{1}{\sqrt{n}}\right) 
+ \hat{c}_t(z_t, \gamma_t) + \hat{V}_{t+1}(z_{t+1})
\]

\[
\leq (K_t^4 + K_{t+1}^5) \|m_t - z_t\|_\infty + (1 + K_{t+1}^5)O\left(\frac{1}{\sqrt{n}}\right) 
+ \min_{\gamma_t \in \Psi} \left( \hat{c}_t(z_t, \gamma_t) + \hat{V}_{t+1}(z_{t+1}) \right)
\]

\[
\leq (K_t^4 + K_{t+1}^5) \|m_t - z_t\|_\infty + O\left(\frac{1}{\sqrt{n}}\right) + \hat{V}_t(z_t),
\]

where \((d)\) follows from the triangle inequality, per-step costs being non-negative (by definition), and the monotonicity of minimum and expectation operators; \((e)\) follows from Lemma 8 and (26); \((f)\) follows from Lemma 5 and \((g)\) follows from (16). Therefore, there exists a constant \(K_T^5 := K_t^4 + K_{t+1}^5\) such that

\[
|V_t(m_t) - \hat{V}_t(z_t)| \leq K_t \|m_t - z_t\|_\infty + O\left(\frac{1}{\sqrt{n}}\right).
\]

### B. Proof of Proposition 7

We use backward induction. At \(t = T\),

\[
\hat{V}_T(z_T) = \min_{\gamma_T} \hat{c}_T(z_T, \gamma_T) 
= \min_{\gamma_T} \left( \hat{c}_T(z_T, \gamma_T) - \hat{c}_T(z_T, \gamma_T) + \hat{c}_T(z_T, \gamma_T) \right)
\]

\[
\leq \min_{\gamma_T} \left( \hat{c}_T(z_T, \gamma_T) - \hat{c}_T(z_T, \gamma_T) + \hat{c}_T(z_T, \gamma_T) \right)
\]

\[
\leq \min_{\gamma_T} \left( K_T^4 \|z_T - z_T\|_\infty + \hat{c}_T(z_T, \gamma_T) \right)
\]

\[
= K_T^4 \|z_T - z_T\|_\infty + \min_{\gamma_T} \hat{c}_T(z_T, \gamma_T)
\]

\[
\leq (K_T^4 + K_{T+1}^5) \|z_T - z_T\|_\infty + \hat{V}_T(z_T),
\]

where \((a)\) follows from the triangle inequality, the fact that \(\hat{c}_T(\cdot) \in \mathbb{R}_{\geq 0}\), and the monotonicity of the minimum operator; \((b)\) follows from Lemma 8 and the monotonicity of the minimum operator; \((c)\) follows from (16). Suppose now that the inequality holds at time \(t + 1\), i.e.,

\[
|\hat{V}_{t+1}(z_{t+1}) - \hat{V}_{t+1}(z_{t+1})| \leq K_{t+1} \|z_{t+1} - z_{t+1}\|_\infty.
\]

It is desired to prove the the result for time \(t\). One can write

\[
\hat{V}_t(z_t) = \min_{\gamma_t} \left( \hat{c}_t(z_t, \gamma_t) + \hat{V}_{t+1}(\hat{f}_t(z_t, \gamma_t)) \right)
\]

\[
= \min_{\gamma_t} \left( \hat{c}_t(z_t, \gamma_t) - \hat{c}_t(z_t, \gamma_t) + \hat{c}_t(z_t, \gamma_t) \right)
+ \hat{V}_{t+1}(\hat{f}_t(z_t, \gamma_t)) + \hat{V}_{t+1}(\hat{f}_t(z_t, \gamma_t))
\]

\[
\leq \min_{\gamma_t} \left( \hat{c}_t(z_t, \gamma_t) - \hat{c}_t(z_t, \gamma_t) \right)
+ \hat{V}_{t+1}(\hat{f}_t(z_t, \gamma_t)) + \hat{V}_{t+1}(\hat{f}_t(z_t, \gamma_t))
\]

\[
\leq (K_t^4 + K_{t+1}^5) \|z_t - z_t\|_\infty + \hat{V}_t(z_t),
\]

where \((d)\) follows from the triangle inequality, the fact that \(\hat{c}_t(\cdot) \in \mathbb{R}_{\geq 0}\), and the monotonicity of the minimum operator; \((e)\) follows from Lemma 8 and (27), and \((f)\) follows from (16).