CYLINDRICAL LATTICE PATHS AND THE
LOEHR-WARRINGTON 10ⁿ CONJECTURE

JONAS SJÖSTRAND

Abstract. The following special case of a conjecture by Loehr and Warrington
was proved recently by Ekhad, Vatter, and Zeilberger:

There are 10ⁿ zero-sum words of length 5n in the alphabet {+3, −2} such that
no zero-sum consecutive subword that starts with +3 may be followed immedi-
ately by −2.

We give a simple bijective proof of the conjecture in its original and more
general setting. To do this we reformulate the problem in terms of cylindrical
lattice paths.

1. Introduction
Let a and b be positive integers. Given a word w in the alphabet {+a, −b}, a zero-
sum consecutive subword of w is said to be illegal if it starts with +a, and −b comes
immediately after the subword in w. Example:

\[ w = -2 + 3 \underbrace{3 - 2 - 2 - 2 + 3}_{\text{illegal subword}} - 2 - 2 + 3 \]

We will prove the following:

**Theorem 1.1.** If a and b are relatively prime, there are \( \binom{a+b}{a}^{2n} \) zero-sum words of
length \((a+b)n\) in the alphabet \{+a, −b\} without illegal subwords.

Example: If \( a = 2 \), \( b = 1 \) and \( n = 2 \), the \( \binom{2+1}{2}^{2} = 9 \) words counted in the theorem are

\[ +2 - 1 - 1 + 2 - 1 - 1 \]
\[ -1 + 2 - 1 + 2 - 1 - 1 \]
\[ -1 - 1 + 2 + 2 - 1 - 1 \]
\[ -1 - 1 - 1 + 2 + 2 - 1 \]
\[ -1 + 2 - 1 - 1 + 2 - 1 \]
\[ -1 - 1 + 2 - 1 + 2 - 1 \]
\[ -1 - 1 - 1 + 2 - 1 + 2 \]
\[ -1 - 1 - 1 + 2 + 2 \]
\[ -1 - 1 + 2 - 1 - 1 + 2 \]

The theorem was conjectured by Nick Loehr and Greg Warrington. In a recent
paper [1] Shalosh Ekhad, Vince Vatter, and Doron Zeilberger proved the special
case \( a = 3 \), \( b = 2 \), using a computer. Inspired by their proof, Loehr and Warrington,
together with Bruce Sagan [1], found a computer-free proof of the more general case

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when \( b = 2 \) and \( a \) is any odd positive integer. We greatly admire the automatic approach of Ekhad, but we feel that a beautiful problem like this ought to have a beautiful solution. And indeed it has!

First we will present a geometrical construction where the words in the alphabet \( \{+a, -b\} \) are interpreted as paths on a cylinder graph. Then we will give bijections between these paths, certain weight functions on the edges of the graph, and ordered sequences of lap cycles. The latter ones are easy to count. Finally, in the last section we examine what happens if \( a \) and \( b \) have a common factor.

2. THE GEOMETRICAL CONSTRUCTION

In the following we let \( a \) and \( b \) be any positive integers.

After thinking about the Loehr-Warrington conjecture for a while, most people will probably discover the following natural reformulation:

You live in a skyscraper \( \mathbb{Z} \). In the morning you get your exercises by climbing out through the window, following \((a + b)n\) one-way ladders, and climbing into your apartment again. At each level there is one ladder going \( a \) levels up and another ladder going \( b \) levels down. Once you have climbed up from a level you never climb down from that level anymore that morning. In how many ways can you perform your exercises?

Now here is the key observation: Since \( a \equiv -b \pmod{a + b} \), after \( x \) ladders we are at a level \( y \) such that \( y \equiv ax \pmod{a + b} \). We define a directed graph \( G_{a,b} \) whose vertex set is the subset of the infinite cylinder \( \mathbb{Z}_{a+b} \times \mathbb{Z} \) consisting of all points \((x, y)\) such that \( y \equiv ax \pmod{a + b} \). From every vertex point \((x, y)\) there is an up-edge \((x, y) \rightarrow (x + 1, y + a)\) and a down-edge \((x, y) \rightarrow (x + 1, y - b)\). If \( a \) and \( b \) are relatively prime, no two points in \( G_{a,b} \) have the same \( y \)-coordinate\(^1\). We have mapped the ladders to the cylinder such that no ladders intersect!

Figure 1 shows a graphical representation of \( G_{3,2} \). It is an infinite vertical strip whose borders are welded together. The points with \( x = 0 \) constitute the weld and are called weld points.

As far as we know, no one has studied this graph before. The closest related research we could find is two papers about nonintersecting lattice paths on the cylinder, one by Peter Forrester \([2]\) and one by Markus Fulmek \([3]\). Curiously, their paths essentially go along the axis of the cylinder while ours essentially go around it!

Let us fix the following graph terminology: A path is an ordered sequence of vertices \( v_0v_1, \ldots, v_m \) such that there is an edge from \( v_{i-1} \) to \( v_i \) for \( i = 1, 2, \ldots, m \). (Repeated vertices and edges are allowed.) The integer \( m \) is the length of the path. If \( v_0 = v_m \) the path is called a cycle.

A path on \( G_{a,b} \) may leave a certain vertex several times, sometimes going down, sometimes going up. If for each vertex all downs come before all ups, the path is said to be downs-first. Now Theorem 1.1 can be reformulated:

\[(1) \quad \text{There are } \binom{a+b}{a}^n \text{ downs-first cycles on } G_{a,b} \text{ of length } (a+b)n \text{ starting at the origin.}\]

We will prove (1) for any positive integers \( a \) and \( b \). (Note that this does not imply that Theorem 1.1 is true if \( a \) and \( b \) have a common factor, see footnote 1.)

\(^1\)This is the only time we use the assumption in Theorem 1.1 that \( a \) and \( b \) are relatively prime.
A path of length $a + b$ that starts and ends on the weld is called a \textit{lap}, and a lap that is a cycle is called a \textit{lap cycle}. Obviously, there are \( \binom{a+b}{a} \) different lap cycles starting at the origin. Our proof will be a bijection that maps downs-first cycles to a sequence of lap cycles. The following lemma is crucial.

\textbf{Lemma 2.1.} A downs-first cycle beginning at a weld point never visits higher weld points.

\textit{Proof.} Suppose the downs-first cycle, starting at a weld point $p$, visits a weld point $q$ higher than $p$. Let $pq$ be the path along the cycle from its starting point $p$ to $q$ (if the cycle visits $q$ several times, choose any visit), and let $qp$ be the remaining path along the cycle from $q$ to the finish point $p$. Obviously, $pq$ and $qp$ must intersect somewhere$^2$. Specifically they must intersect at a point where $pq$ goes up and $qp$ goes down. But this contradicts the assumption that the cycle is downs-first. \hfill \Box

We conclude that the weld points above the origin, and hence the points above the highest lap cycle from the origin, can never be reached by the downs-first cycles counted in \textbf{1}. Let $H_{a,b}$ be the resulting graph when these points are removed from $G_{a,b}$. Figure \textbf{2} shows an example.

Now \textbf{1} can be slightly reformulated:

\begin{equation}
\text{There are } \binom{a+b}{a}^n \text{ downs-first cycles on } H_{a,b} \text{ of length } (a+b)n \text{ starting at the origin.}
\end{equation}

Before proving \textbf{2}, and hence our main theorem, we need some more definitions.

A \textit{weight function} on $H_{a,b}$ is an assignment of a nonnegative integer to every edge in $H_{a,b}$. The \textit{in-weight} and \textit{out-weight} of a vertex in $H_{a,b}$ is the sum of the weights of the edges going in to and out from the vertex, respectively. A weight function is

$^2$If your cylindrical intuition fails you, think like this: The path $pq$ must make at least one lap starting at $p$ or below and ending at a weld point above $p$. Similarly, the path $qp$ must make at least one lap starting at a weld point above $p$ and ending at $p$ or below. Clearly these laps intersect and cross.
said to be balanced if at each vertex the in-weight and out-weight are equal. A path in $H_{a,b}$ is said to be covered by the weight function if every edge is used by the path at most as many times as its weight. The weight function is origin-connected if for every vertex with positive out-weight there is a covered path from the origin to the vertex.

For an example of a balanced origin-connected weight function, see Figure 3.

3. The bijections

Please keep the cylinder graph $H_{a,b}$ in your mind throughout the paper.

Since the number of lap cycles beginning at the origin is $\binom{a+b}{a}$, the formulation of our main theorem follows from the result in this section:

**Theorem 3.1.** There are bijections between the following three sets:

1. downs-first cycles of length $(a + b)n$ beginning at the origin,
2. balanced origin-connected weight functions with total weight sum $(a + b)n$,
3. ordered sequences of $n$ lap cycles beginning at the origin.

**Proof.** We will define four functions, $f_{1,2} : 1 \to 2$, $f_{2,1} : 2 \to 1$, $f_{3,2} : 3 \to 2$, and $f_{2,3} : 2 \to 3$. It should be apparent from the presentation below that $f_{1,2} \circ f_{2,1}$, $f_{2,1} \circ f_{1,2}$, $f_{3,2} \circ f_{2,3}$, and $f_{2,3} \circ f_{3,2}$ are all identity functions. Figure 3 gives an example of the bijections.

1 $\to$ 2: Given a downs-first cycle beginning at the origin, to every edge of $H_{a,b}$ we assign a weight that is the number of times the edge is used by the cycle. This weight function is obviously balanced and origin-connected. Furthermore, it is the only such function with total sum $(a + b)n$ that covers the cycle.

2 $\to$ 1: Given a balanced and origin-connected weight function we construct a downs-first cycle as follows: Start at the origin. At each point, go down if that edge has positive weight, otherwise go up. Decrease the weight of the followed edge by one. Continue until you come to a point with zero out-weight. This must be the origin so we have created a downs-first cycle $C$. 
We must show that the length of $C$ is $(a+b)n$. Suppose not. Then there remain some positive weights. Since the original weight function was origin-connected there exists a point $p$ on $C$ with positive out-weight. Since the remaining weight function is still balanced it covers some cycle $C'$ that contains $p$. Now start at $p$ and follow $C$ and $C'$ in parallel until $C$ reaches the origin. Since the origin has no remaining in- or out-weight, $C'$ must have reached a point on the weld below the origin (the other weld points were removed when we constructed $H_{a,b}$). This implies that $C$ and $C'$ intersect at a point where $C$ goes up and $C'$ goes down. But that is impossible by the construction of $C$.

Thus $C$ has length $(a+b)n$, and it is easy to see that among all downs-first cycles of that length starting at the origin, $C$ is the only one that is covered by the given weight function.
3 → 2: Given a sequence of lap cycles $C_1, C_2, \ldots, C_n$ starting at the origin, translate $C_1, C_2, \ldots, C_{n-1}$ downwards so that, for $1 \leq i \leq n-1$, $C_i$ intersects $C_{i+1}$ in at least one point but otherwise goes below it. Observe that there is a unique way of “packing” the cycles like that.

Now let the weight of each edge in $H_{a,b}$ be the number of times the edge is used by the cycles. The result is obviously a balanced origin-connected weight function.

2 → 3: Given a balanced origin-connected weight function, by iteration of the following procedure we construct $n$ lap cycles. At the beginning of each iteration the weight function is always balanced.

Start at the lowest weld point $p$ with a positive out-weight and create a down-first cycle from there like this: In each step, go down if that edge has positive weight, otherwise go up. Decrease the weight of the followed edge by one. Stop as soon as you reach $p$ again. By Lemma 2.1 this down-first cycle never visits another weld point than $p$, which implies that it is a lap cycle. (Remember that a path must visit the weld every $(a+b)$-th step.)

After $n$ iterations we have consumed all weights and produced a packed sequence of lap cycles $C_1, C_2, \ldots, C_n$. Simply translate the lap cycles so that they all start at the origin.

4. What if $a$ and $b$ have a common factor?

The condition that $a$ and $b$ should be relatively prime is not an essential restriction, as the following corollary to Theorem 1.1 shows.

**Corollary 4.1.** Let $a$ and $b$ be any positive integers, and put $c = \gcd(a,b)$. The number of zero-sum words in the alphabet $\{+a, -b\}$ of length $(a+b)n$ without illegal subwords is

$$\left(\frac{(a+b)c}{a/c}\right)^n.$$

**Proof.** Let $A = a/c$ and $B = b/c$. Clearly the words counted in the corollary are in one-to-one correspondence with the zero-sum words in the alphabet $\{+A, -B\}$ of length $(A+B)cn$ without illegal subwords. According to Theorem 1.1 there are $(A+B)^{cn}$ such words.

However, Theorem 1.1 can be generalized in a less trivial way:

**Theorem 4.2.** Let $a$ and $b$ be any positive integers. There are $\binom{a+b}{a}^n$ zero-sum words in the alphabet $\{+a, -b\}$ of length $(a+b)n$ without illegal subwords whose length is a multiple of $a+b$.

**Proof.** In the proof of Theorem 1.1 the only time we make use of the fact that $a$ and $b$ are relatively prime is when we conclude that no two points in $G_{a,b}$ have the same $y$-coordinate. We need this in order to show that the vertices in $G_{a,b}$ are in one-to-one correspondence with the levels in the skyscraper.

However, even if $a$ and $b$ have a common factor, the condition that the word should have no illegal subword whose length is a multiple of $a+b$ is equivalent to the down-first condition on the corresponding path on $G_{a,b}$. (It just happens that the phrase “whose length is a multiple of $a+b$” is unnecessary if $a$ and $b$ are relatively prime.) Thus we can simply bypass the skyscraper nonsense and go directly to the proposition 1.1 which is still valid.
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Department of Mathematics, Royal Institute of Technology, SE-100 44 Stockholm, Sweden

E-mail address: jonass@kth.se