The Kirchhoff Index of Enhanced Hypercubes

Ping Xu, Qiongxiang Huang

College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830046, P.R.China

Abstract Let \( \{e_1, \ldots, e_n\} \) be the standard basis of abelian group \( \mathbb{Z}_2^n \), which can be also viewed as a linear space of dimension \( n \) over the Galois field \( \mathbb{F}_2 \), and \( \varepsilon_k = e_k + e_{k+1} + \cdots + e_n \) for some \( 1 \leq k \leq n - 1 \). It is well known that the so called enhanced hypercube \( Q_{n,k} (1 \leq k \leq n - 1) \) is just the Cayley graph \( \text{Cay}(\mathbb{Z}_2^n, S) \) where \( S = \{e_1, \ldots, e_n, \varepsilon_k\} \). In this paper, we obtain the spectrum of \( Q_{n,k} \), from which we give an exact formula of the Kirchhoff index of the enhanced hypercube \( Q_{n,k} \). Furthermore, we prove that, for a given \( n \), \( Kf(Q_{n,k}) \) is increased with the increase of \( k \). Finally, we get \( \lim_{n \to \infty} \frac{Kf(Q_{n,k})}{n} = 1 \).

Keywords: Enhanced hypercube; Irreducible characters; Kirchhoff index

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1 Introduction

An interconnection network is always represented by a graph \( \Gamma = (V, E) \), where \( V \) denotes the node set and \( E \) denotes the edge set. Now various interconnections are proposed. The hypercubes network obtained considerable attention in virtue of its perfect properties, such as symmetry, regular structure, strong connectivity, and small diameter [2,7]. An \( n \)-dimensional hypercube denoted by \( Q_n \) has \( 2^n \) vertices, and the vertex set is \( V(Q_n) = \{x_1x_2\cdots x_n \mid x_i = 0 \ or \ 1, i = 1, 2, \ldots, n\} \). Two vertices \( X = x_1x_2\cdots x_n \) and \( Y = y_1y_2\cdots y_n \) are adjacent if and only if there exists \( 1 \leq i \leq n \), such that \( x_i = \overline{y}_i \), where \( \overline{y}_i \) denoted the complement of binary digit \( y_i \), and \( x_j = y_j \) for all \( j \neq i \). As the importance of the hypercubes networks, many variants of it were presented, among which, for instance, are enhanced hypercube, augmented hypercube, folded hypercube [4,6,20]. The \( n \)-dimensional enhanced hypercube is one of the important variants of hypercube introduced by Tzeng in [20] which is defined as follows.

Definition 1.1. For \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \), the enhanced hypercube \( Q_{n,k} = (V, E) \) is an undirected simple graph with vertex set \( V = \{x_1x_2\cdots x_n \mid x_i = 0 \ or \ 1, i = 1, 2, \ldots, n\} \). Two vertices \( X = x_1x_2\cdots x_n \) and \( Y = y_1y_2\cdots y_n \) are adjacent if \( Y \) satisfies one of the following two conditions:

1. \( Y = x_1x_2\cdots x_{i-1}\overline{x}_i x_{i+1}\cdots x_n \) for some \( 1 \leq i \leq n \);

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†Corresponding author. Email: huangqx@xju.edu.cn, huangqxmmath@163.com
(2) \( Y = x_1x_2 \cdots x_{k-1}x_kx_{k+1} \cdots x_n \).

According to the above definition, we can see that \( Q_{n,k} \) contains \( Q_n \) as its subgraph. In fact, \( Q_{n,k} \) is a \((n+1)\)-regular graph with \( 2^n \) vertices and \((n+1)2^{n-1} \) edges. Its special case of \( k = 1 \) is the well-known folded hypercube denoted by \( FQ_n \). As a variant of the hypercube, the \( n \)-dimensional folded hypercube \( FQ_n \), proposed first by El-Amawy and Latifi [6], is a graph obtained from the hypercube \( Q_n \) by adding some edges, called a complementary edges, between vertices \( X = x_1x_2 \cdots x_n \) and \( \bar{X} = \bar{x}_1\bar{x}_2 \cdots \bar{x}_n \). The enhanced hypercube is superior to the hypercube in many aspects. For example, the diameter of the enhanced hypercube is almost half of the hypercube. The hypercube is \( n \)-regular and \( n \)-connected, whereas the enhanced hypercube is \((n+1)\)-regular and \((n+1)\)-connected [12].

Let \( G \) be a finite group, and let \( S \) be a symmetric subset of \( G \) such that \( 1 \notin S \) and \( S^{-1} = \{s^{-1} \mid s \in S\} = S \). The Cayley graph on \( G \) with respect to a symmetric subset \( S \) of \( G \), denoted by \( Cay(G, S) \), is the undirected graph with vertex set \( G \) and with an edge \((g, h)\) connecting \( g \) and \( h \) if \( hg^{-1} \in S \), or equivalently \( gh^{-1} \in S \). It is well known that \( Cay(G, S) \) is connected if and only if \( S \) generates \( G \). Particularly, the group \( Z^n_2 = Z_2 \times Z_2 \times \cdots \times Z_2 \) can be viewed as a vector space of dimension \( n \) over the Galois filed \( F_2 \). Suppose that \( \{e_1, \ldots, e_n\} \) is the standard basis of \( Z^n_2 \) and \( e_i \) is the vector with first \( k - 1 \) entries equal to 0 and other entries equal to 1 for some \( 1 \leq k \leq n \). Let \( S = \{e_1, \ldots, e_n, e_k\} \) be the subset of \( Z^n_2 \). It is clear that the so called enhanced hypercube \( Q_{n,k} \) is just the Cayley graph \( Cay(Z^n_2, S) \).

The concept of resistance distance of a graph \( \Gamma = (V, E) \) was first introduced by Klein and Randić [10]. Let \( \Gamma \) be a connected graph. The resistance distance between vertices \( v_i \) and \( v_j \) of \( \Gamma \), denoted by \( r_{ij} \), is defined to be the effective resistance between the nodes \( v_i \) and \( v_j \) as computed with ohm’s law when all the edges of \( \Gamma \) are considered to be unit resistors. The traditional distance between vertices \( v_i \) and \( v_j \), denoted by \( d_{ij} \), is the length of a shortest path connecting them. The Wiener index \( W(\Gamma) \) was given by \( W(\Gamma) = \sum_{i \leq j} d_{ij} \) in [21]. As an analogue to the Wiener index, the sum \( Kf(\Gamma) = \sum_{i < j} r_{ij} \) was proposed in [10]. later called the Kirchhoff index of \( \Gamma \) in [11]. Klein and Randić [10] proved that \( r_{ij} \leq d_{ij} \) and thus \( Kf(\Gamma) \leq W(\Gamma) \) with equality if and only if \( \Gamma \) is a tree.

The Kirchhoff index has wide applications in physical interpretations, electric circuit, graph theory, and chemistry. For example, Gutman and Mohar [8] and Zhu, Klein et al. [24] proved that the Kirchhoff index of a connected graph \( \Gamma \) with \( n(n \geq 2) \) vertices is the sum of reciprocal nonzero Laplacian eigenvalues of the graph multiplied by the number of the vertices. Like the Wiener index, the Kirchhoff index is a structure descriptor [22]. The Kirchhoff index has been computed for cycles [11, 16], complete graphs [10], geodetic graphs [17], distance transitive graphs [17], and so on. The Kirchhoff index of certain composite operations between two graphs was studied, such as product, lexicographic product [23] and join, corona, cluster [25].

In [13], we can see that the exact formula of the Kirchhoff index of the hypercubes networks \( Q_n \) and related complex networks had been provided. Motivated by previous results, in this paper we present the formulae for the Kirchhoff index of the enhanced hypercube \( Q_{n,k} \) and such formulae are brief for \( Q_{n,1} \) and \( Q_{n,n-1} \). Moreover, we prove that \( Kf(Q_{n,k}) \) is increased as \( k \) increases for a given \( n \). Additionally, the bounds of \( Kf(Q_{n,k}) \) and its Limit function is obtained, that is, \( \lim_{n \to \infty} \frac{Kf(Q_{n,k})}{n^2} = 1 \).
2 Preliminaries

Let $\Gamma$ be a simple graph with vertex set $V$ and edge set $E$. The adjacency matrix $A(\Gamma)$ of $\Gamma$ is the $n \times n$ matrix with the $(i, j)$-entry equal to 1 if vertices $v_i$ and $v_j$ are adjacent and 0 otherwise. For $v_i \in V$, let $N(v_i)$ denote the set of neighbours of $v_i$, that is, $N(v_i) = \{v_j \in V \mid v_j \sim v_i\}$. The size of $N(v_i)$ is called the degree of $v_i$, denoted by $d_i$. Let $D(\Gamma)$ be the diagonal matrix with $i$-th diagonal entry equal to $d_i$. The Laplacian matrix of $\Gamma$ is defined by $L(\Gamma) = D(\Gamma) - A(\Gamma)$. The eigenvalues of $A(\Gamma)$ and $L(\Gamma)$ are called the adjacency eigenvalues and Laplacian eigenvalues of $\Gamma$, respectively. The multiset of adjacency (resp. Laplacian) eigenvalues together with their multiplicities is called the adjacency (resp. Laplacian) spectrum of $\Gamma$, denoted by $\text{Spec}_A(\Gamma)$ (resp. $\text{Spec}_L(\Gamma)$).

In [3], Babai derived an expression for the spectrum of the Cayley graph $\text{Cay}(G, S)$ in terms of irreducible characters of $G$. Here, we need only to know the eigenvalues of the Cayley graph of an abelian group.

**Lemma 2.1 ([19]).** Let $G = \{g_1, g_2, \ldots, g_n\}$ be an abelian group and $S \subseteq G$ is a symmetric set. Let $\chi_1, \ldots, \chi_n$ be the irreducible characters of $G$ and $A$ be the adjacency matrix of the Cayley graph of $G$ with respect to $S$. Then the eigenvalues of the adjacency matrix $A$ are the real numbers

$$\lambda_i = \sum_{s \in S} \chi_i(s)$$

where $1 \leq i \leq n$.

**Lemma 2.2 ([19]).** Let $G_1, G_2$ be abelian groups and suppose that $\chi_1, \ldots, \chi_m$ and $\varphi_1, \ldots, \varphi_n$ are the irreducible representations of $G_1, G_2$, respectively. In particular, $m = |G_1|$ and $n = |G_2|$. Then the functions $\alpha_{ij} : G_1 \times G_2 \rightarrow \mathbb{C}$ with $1 \leq i \leq m, 1 \leq j \leq n$ given by

$$\alpha_{ij}(g_1, g_2) = \chi_i(g_1)\varphi_j(g_2)$$

form a complete set of irreducible representations of $G_1 \times G_2$.

It is well known that $Z_2$ has two irreducible characters $\chi_0(a) = 1(\forall a \in Z_2)$ and $\chi_1(a) = (-1)^a(\forall a \in Z_2)$. As a direct consequence of Lemma 2.2, the irreducible characters $\chi_{i_1, i_2, \ldots, i_n}$ of $Z_2^n$ are given by

$$\chi_{i_1, \ldots, i_n}(a_1, \ldots, a_n) = (-1)^{i_1a_1 + \cdots + i_na_n}$$  \hspace{1cm} (1)

where $(a_1, \ldots, a_n) \in Z_2^n$ and $i_j \in \{0, 1\}$ for $1 \leq j \leq n$. From (1), Lemma 2.1 and Lemma 2.2 we get the eigenvalues of $\text{Cay}(Z_2^n, S)$.

**Theorem 2.3.** Let $S$ be a subset of $Z_2^n$. Then the adjacency eigenvalues of the Cayley graph $\text{Cay}(Z_2^n, S)$ are given by

$$\lambda_{i_1, i_2, \ldots, i_n} = \sum_{(s_1, s_2, \ldots, s_n) \in S} (-1)^{i_1s_1 + \cdots + i_ns_n}$$

where $(i_1, \ldots, i_n) \in Z_2^n$. 
Lemma 2.4 ([8,24]). Let $\Gamma$ be a connected graph with $n \geq 2$ vertices. Then

$$Kf(\Gamma) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i},$$

where $0 = \mu_0 < \mu_1 \leq \cdots \leq \mu_{n-1}$ are the Laplacian eigenvalues of $\Gamma$.

Lemma 2.5 ([9]). Let $A$ be the adjacency matrix of a graph $\Gamma$, and let $\rho$ be its spectral radius. Then the following are equivalent:

1. $\Gamma$ is bipartite.
2. The spectrum of $A$ is symmetric about the origin, i.e., for any $\lambda$, the multiplicities of $\lambda$ and $-\lambda$ as eigenvalues of $A$ are the same.
3. $-\rho$ is an eigenvalue of $A$.

3 The eigenvalues of enhanced hypercubes

In this section, we focus on the eigenvalues of enhanced hypercubes. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $Z^n_2$, and let $e_k = e_k + e_{k+1} + \cdots + e_n$ for some $1 \leq k \leq n - 1$. The $n$-dimensional enhanced hypercube $Q_{n,k}$ is just the Cayley graph $Cay(Z^n_2)$, where $Z^n_2 = Z_2 \times Z_2 \times \cdots \times Z_2$ and $S = \{e_1, \ldots, e_n, e_k\}$.

Lemma 3.1. For $1 \leq k \leq n - 1$, the eigenvalues of $Q_{n,k}$ are given by $\xi_t = n - 2t - 1$ with multiplicity $\alpha_t = \sum_{j=0}^{n-1} \binom{n-k+1}{2j+1} \binom{k-1}{t-2j+1}$ for $t = 1, 2, \ldots, n$, and $\xi_{n-t} = n - 2t + 1$ with multiplicity $\beta_t = \sum_{j=0}^{n-1} \binom{n-k+1}{2j} \binom{k-1}{t-2j}$ for $t = 0, 1, 2, \ldots, n$.

Proof. Since $Q_{n,k} \cong Cay(Z^n_2)$, we need only to calculate the eigenvalues of $Cay(Z^n_2)$. For $v = (v_1, v_2, \ldots, v_n)^T \in Z^n_2$, according to (1), the value of the irreducible character $\chi_v$ at $a = (a_1, a_2, \ldots, a_n)^T \in Z^n_2$ is $\chi_v(a) = (-1)^{v_1 a_1 + v_2 a_2 + \cdots + v_n a_n}$. Thus

$$\begin{align*}
\chi_v(e_i) &= (-1)^{v_i} \\
\chi_v(e_k) &= (-1)^{v_1 + v_2 + \cdots + v_k}
\end{align*}$$

From (2) and Theorem 2.3, the eigenvalue corresponding to $v = (v_1, v_2, \ldots, v_n)^T \in Z^n_2$ is

$$\lambda_v = \sum_{s \in S} \chi_v(s) = (-1)^{v_1} + (-1)^{v_2} + \cdots + (-1)^{v_n} + (-1)^{v_1 + v_2 + \cdots + v_n}$$

(3)

Given $0 \leq r \leq t \leq n$, let $\Lambda(t, r) = \{v \in Z^n_2 \mid v_1 + v_2 + \cdots + v_n = t, v_k + v_{k+1} + \cdots + v_n = r\}$. It is easy to see that $|\Lambda(t, r)| = \binom{n-k+1}{t-r} \binom{k-1}{t-r}$. From (3), for any $v \in \Lambda(t, r)$, we have

$$\lambda_v = \sum_{s \in S} \chi_v(s) = n - 2t + (-1)^r$$

(4)
Clearly, $Z_2^n = \bigcup_{0 \leq r \leq n} \Lambda(t, r)$ is a partition of $Z_2^n$. By taking $r = 2j - 1$ for $1 \leq j \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, from (4) we get the eigenvalue $\zeta_t = n - 2t - 1$, where $t = 1, 2, \ldots, n$, with multiplicity $|\Lambda(t, r)| = \sum_{j=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (n-k+1)\binom{k-1}{2j-1} - (t-2j+1)$ which equals $\alpha_t = \sum_{j=0}^{n} (n-k+1)\binom{k-1}{2j} (t-2j+1)$; by taking $r = 2j$ for $0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, we get the eigenvalue $\xi_t = n - 2t + 1$, where $t = 0, 1, 2, \ldots, n$, with multiplicity $|\Lambda(t, r)| = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (n-k+1)\binom{k-1}{2j}$ which equals $\beta_t = \sum_{j=0}^{n} (n-k+1)\binom{k-1}{2j}$.

This completes the proof. □

Note that some of the two kind of eigenvalues described in Lemma 3.1 may be equal, in fact, $\zeta_t = \xi_{t+1}$ for $t = 1, 2, \ldots, n - 1$. Thus we can give the spectrum of $Q_{n,k}$ from Lemma 3.1

**Theorem 3.2.** Let $1 \leq k \leq n - 1$, the spectrum of $Q_{n,k}$ is given by

$$\text{Spec}_A(Q_{n,k}) = \{n + 1, [n - 1]^{k-1}, [n - 2t - 1]^\gamma, [-n - 1]^{\gamma_n} \mid t = 1, 2, \ldots, n - 1\},$$

where $\gamma_t = \sum_{j=0}^{n} \binom{n-k+2}{2j} \binom{k-1}{t+2j}$, for $1 \leq t \leq n - 1$ and $\gamma_n = \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{n-2j+1}$.

**Proof.** For convenience, denote by $\lambda_0 > \lambda_1 > \lambda_2 > \cdots$ the distinct eigenvalues of $Q_{n,k}$. By Lemma 3.1, we know that $\lambda_0 = \xi_0 = n + 1$ with multiplicity $\beta_0 = \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{0-2j} = 1$. $\lambda_1 = \xi_1 = n - 1$ with multiplicity $\beta_1 = \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{1-2j} = \binom{k-1}{1} = k - 1$. In addition, we see that $\lambda_{t+1} = \xi_t = \xi_{t+1} = n - 2t - 1$ for $t = 1, 2, \ldots, n - 1$, so the multiplicity of the eigenvalue $n - 2t - 1$ is $\alpha_t + \beta_{t+1}$, where $\gamma_t = \alpha_t + \beta_{t+1}$ can be simplified as

$$\gamma_t = \alpha_t + \beta_{t+1} = \sum_{j=1}^{n} \binom{n-k+1}{2j-1} \binom{k-1}{t-2j+1} + \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{t-2j+1}$$

$$= \sum_{j=1}^{n} \left(\binom{n-k+1}{2j-1} + \binom{n-k+1}{2j} \right) \binom{k-1}{t-2j+1} + \binom{k-1}{t+1}$$

$$= \sum_{j=0}^{n} \binom{n-k+2}{2j} \binom{k-1}{t+2j}.$$

At last, $\lambda_{n+1} = \zeta_n = -n - 1$ with multiplicity $\gamma_n = \alpha_n = \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{n-2j+1}$.

This completes the proof. □

Since the enhanced hypercube $Q_{n,k}$ is regular, the Laplacian spectrum of $Q_{n,k}$ can be reduced by its A-spectrum. From Theorem 3.2, we get the following result.

**Theorem 3.3.** Let $1 \leq k \leq n - 1$, the Laplacian spectrum of $Q_{n,k}$ is given by

$$\text{Spec}_L(Q_{n,k}) = \{0, [2]^{k-1}, [2t + 2]^{\gamma}, [2n + 2]^{\gamma_n} \mid t = 1, 2, \ldots, n - 1\},$$

where $\gamma_t = \sum_{j=0}^{n} \binom{n-k+2}{2j} \binom{k-1}{t+2j}$, for $1 \leq t \leq n - 1$ and $\gamma_n = \sum_{j=0}^{n} \binom{n-k+1}{2j} \binom{k-1}{n-2j+1}$. 
Proof. Since $Q_{n,k}$ is $n + 1$ regular, we have $L(Q_{n,k}) = (n + 1)I - A(Q_{n,k})$. Thus every eigenvector of $A(Q_{n,k})$ with eigenvalue $\lambda$ is an eigenvector of $L(Q_{n,k})$ with eigenvalue $(n + 1) - \lambda$. By Theorem 3.2, the result yields immediately. \hfill \Box

In general, the summations of $\alpha_i$ and $\beta_i$ can not be simply calculated. However, we can find the simple expression of the spectrum for some special $k$. By Theorem 3.2 and Theorem 3.3, we give the $A$-spectrum and Laplacian spectrum of $Q_{n,k}$ for $k = 1$.

**Corollary 3.4.** For $k = 1$ and $n \geq 2$, the $A$-spectrum and Laplacian spectrum of $Q_{n,1}$ is given by

1. $\text{Spec}_A(Q_{n,1}) = \begin{cases} 
{n + 1, [n - 3]^{(w_1)}, \ldots, [-n + 3]^{(w_1)}, -n - 1} & \text{if } n \text{ is odd} \\
{n + 1, [n - 3]^{(w_1)}, \ldots, [-n + 5]^{(w_1)}, [-n + 1]^{n+1}} & \text{if } n \text{ is even.}
\end{cases}$

2. $\text{Spec}_L(Q_{n,1}) = \begin{cases} 
{0, [4]^{(w_1)}, [8]^{(w_1)}, \ldots, [2n - 2]^{(w_1)}, 2n + 2} & \text{if } n \text{ is odd} \\
{0, [4]^{(w_1)}, [8]^{(w_1)}, \ldots, [2n - 4]^{(w_1)}, [2n]^{n+1}} & \text{if } n \text{ is even.}
\end{cases}$

**Remark 3.5.** The folded hypercube $FQ_n$ introduced in [5] is just the enhanced hypercube $Q_{n,k}$ for $k = 1$, i.e., $FQ_n = Q_{n,1}$. Thus Corollary 3.4 gives the $A$-spectrum and Laplacian spectrum of the folded hypercube $FQ_n$.

We give the following two Corollaries at the last of this section, which can be regarded as an application of Theorem 3.2. Denote the adjacency eigenvalues of $\Gamma$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $\lambda_1 - \lambda_2$ is the so-called spectral gap of the adjacency matrix, which is also an important spectral parameter related to the expander property [18].

**Corollary 3.6.** For $2 \leq k \leq n - 1$, the second largest eigenvalue of $Q_{n,k}$ is $n - 1$, and so the adjacency spectral gap of $Q_{n,k}$ is $2$. Especially, the second largest eigenvalue of $Q_{n,1}$ is $n - 3$, the corresponding adjacency spectral gap of $Q_{n,1}$ is $4$.

**Proof.** From Theorem 3.2 we known that $n + 1$ is the first largest eigenvalue of $Q_{n,k}$. Moreover, for $k \geq 2$, the second largest eigenvalue is $n - 1$, for $k = 1$, the second largest eigenvalue of $Q_{n,1}$ is $n - 3$. \hfill \Box

H. M. Liu in [14] characterized the bipartite graphs among enhanced hypercubes $Q_{n,k}$. Here we give it another simple proof by the spectrum of $Q_{n,k}$.

**Corollary 3.7.** $Q_{n,k}$ is a bipartite graph if and only if $n$ and $k$ have the same parity for $1 \leq k \leq n - 1$.

**Proof.** By Lemma 2.4, $Q_{n,k}$ is a bipartite graph if and only if $-n - 1$ is an eigenvalue with multiplicity one, by theorem 2.3 if and only if $\gamma_n = \sum_{j=1}^{n} \binom{n-k+1}{2j-1} \binom{k-1}{n-2j+1} = 1$. Note that the valid term in the summation of $\gamma_n$ must satisfy $n-k+1 \geq 2j-1$ and $k-1 \geq n-2j+1$. Hence $n-k+1 = 2j-1$, that is, $j = \frac{n-k+2}{2}$. So we have $\gamma_n = \sum_{j=1}^{n} \binom{n-k+1}{2j-1} \binom{k-1}{n-2j+1} = \binom{n-k+1}{k-1} = 1$ if $n$ and $k$ have the same parity and $\gamma_n = 0$ otherwise. It immediately follows our result. \hfill \Box
4 The Kirchhoff index of enhanced hypercubes

In this section, we focus on our main results. First we give the formula of the Kirchhoff index of the enhanced hypercube $Q_{n,k}$ where $1 \leq k \leq n - 1$. Next we will show the monotonicity of $Kf(Q_{n,k})$. At last, we get a limiting function for $Kf(Q_{n,k})$.

**Theorem 4.1.** Let $1 \leq k \leq n - 1$. The Kirchhoff index of the enhanced hypercube $Q_{n,k}$ is given by

$$Kf(Q_{n,k}) = \begin{cases} 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right) & \text{if } n \equiv k \pmod{2} \\ 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right) & \text{if } n \not\equiv k \pmod{2}. \end{cases}$$

**Proof.** We denote the Laplacian eigenvalues of $Q_{n,k}$ by $\mu_i$ for $i = 0, 1, \ldots, 2^n - 1$. By Lemma 2.4 and Theorem 3.3 we have

$$Kf(Q_{n,k}) = 2^n \sum_{i=1}^{2^{n-1}} \frac{1}{\mu_i} = 2^n \sum_{\mu \in \text{Spec}_i(Q_{n,k})} \frac{1}{\mu}$$

$$= 2^n \left(\frac{k-1}{2} + \frac{\gamma_n}{2n+2} + \sum_{j=1}^{n-1} \frac{\gamma_j}{2n+2}\right)$$

$$= 2^{n-1} \left(k - 1 + \frac{\gamma_n}{n+1} + \sum_{j=1}^{n-1} \frac{\gamma_j}{n+1}\right).$$

By Corollary 3.7, we known that $\gamma_n = 1$ if $n$ and $k$ have the same parity, and $\gamma_n = 0$ otherwise. By Theorem 3.2 we have $\gamma_i = \sum_{j=0}^{n} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right)$ for $t = 1, 2, \ldots, n-1$. Therefore, we have

$$Kf(Q_{n,k}) = \begin{cases} 2^{n-1} \left(k - 1 + \frac{1}{n+1} + \sum_{i=1}^{n-1} \sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right)\right) & \text{if } n \equiv k \pmod{2} \\ 2^{n-1} \left(k - 1 + \sum_{i=1}^{n-1} \sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right)\right) & \text{if } n \not\equiv k \pmod{2}. \end{cases} \quad (5)$$

It is easy to verify that $\sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right) = k - 1$ if $t = 0$. On the other hand, if $t = n$, we have $\sum_{j=0}^{n} \frac{1}{n+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{i+1-2j}\right) = \frac{1}{n+1} \sum_{j=0}^{n} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{n+1-2j}\right) = \frac{1}{n+1}$ for $n \equiv k \pmod{2}$. Thus (5) becomes

$$Kf(Q_{n,k}) = \begin{cases} 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{n+1-2j}\right) & \text{if } n \equiv k \pmod{2} \\ 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left(\frac{n-k+2}{2j}\right) \left(\frac{k-1}{n+1-2j}\right) & \text{if } n \not\equiv k \pmod{2}. \end{cases}$$

This completes the proof. \qed

Theorem 4.1 gives an exact formula of the Kirchhoff index of $Q_{n,k}$. However, its representation has some complicated. Now we give a lemma that will be used to simplify the Kirchhoff index of $Q_{n,k}$ for some $k$. 


Lemma 4.2. For a positive integer \( n \), we have \[ \sum_{i=1}^{n} \frac{1}{2i} \binom{n+1}{2i} = \sum_{i=1}^{\frac{n-1}{2i-1}} \frac{2^{i-1}}{2i}. \]

**Proof.** By the fundamental identity of combination \( \binom{n-1}{2i-1} + \binom{n-1}{2i} = \binom{n}{2i} \) and \( \frac{1}{2i} \binom{n}{2i} = \frac{1}{n} \binom{n}{2i} \), we get

\[
\sum_{i=1}^{n} \frac{1}{2i} \binom{n+1}{2i} = \sum_{i=1}^{n} \left( \frac{1}{n+1} \binom{n+1}{2i} + \frac{1}{2i} \binom{n}{2i} \right) = \sum_{i=1}^{n} \left( \frac{1}{n+1} \binom{n+1}{2i-1} + \frac{1}{2i} \binom{n}{2i} \right)
= \sum_{i=1}^{n} \left( \frac{1}{n+1} \binom{n+1}{2i-1} + \frac{n}{2i} \binom{n}{2i-1} + \frac{1}{2i} \binom{n}{2i} \right)
= \sum_{i=1}^{n} \left[ \frac{1}{n+1} \binom{n+1}{2i-1} + \frac{1}{2i} \binom{n}{2i-1} + \cdots + \frac{1}{n} \binom{n}{2i-1-n-2i} + \frac{1}{2i} \binom{n}{2i-1-n-2i-1} \right]
= \sum_{i=1}^{n} \frac{1}{n+1-i} \sum_{r=0}^{n-i} \binom{n+1-r}{2i} = \sum_{i=1}^{n} \frac{1}{n+1-i} \sum_{r=0}^{n} \binom{n+1-r}{2i} = \sum_{i=1}^{n} \frac{1}{n+1-i} (2^{n-r} - 1) = \sum_{i=1}^{n} \frac{2^{i-1}}{i+1}.
\]

This completes the proof. \( \square \)

By Lemma 4.2, we arrive at the following result, which is a special case of Theorem 4.1. In fact, the formula of Kirchhoff index of the folded hypercube \( FQ_n \) have been presented in [15]. Corollary 4.3 simplifies the formula.

**Corollary 4.3.** The Kirchhoff index of the enhanced hypercube \( Q_{n,1} \) is given by

\[ Kf(Q_{n,1}) = 2^{n-1} \sum_{i=1}^{2^{i-1}}. \]

**Proof.** By Theorem 4.1, the Kirchhoff index of \( Q_{n,1} \) can be presented by

\[ Kf(Q_{n,1}) = \begin{cases} 
2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \binom{n+1}{2j} \binom{0}{i+1-2j} & \text{if } n \text{ is odd} \\
2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \binom{n+1}{2j} \binom{0}{i+1-2j} & \text{if } n \text{ is even}.
\end{cases} \]

It is easy to check that

\[
\sum_{i=0}^{n} \frac{1}{i+1} \binom{n+1}{2j} \binom{0}{i+1-2j} = \sum_{j=1}^{n} \frac{1}{2j} \binom{n+1}{2j} = \sum_{j=1}^{n} \frac{1}{2j} \binom{n}{2j}.
\]

and

\[
\sum_{i=0}^{n} \frac{1}{j+1} \binom{n+1}{2j} \binom{0}{i+1-2j-1} = \sum_{j=1}^{n} \frac{1}{2j} \binom{n+1}{2j} = \sum_{j=1}^{n} \frac{1}{2j} \binom{n}{2j}.
\]

Combined with the above two situations, we get

\[ Kf(Q_{n,1}) = 2^{n-1} \sum_{j=1}^{n} \frac{1}{2j} \binom{n+1}{2j}. \]

By lemma 4.2, we have

\[ Kf(Q_{n,1}) = 2^{n-1} \sum_{j=1}^{n} \frac{1}{2j} \binom{n+1}{2j} = 2^{n-1} \sum_{i=1}^{n} \frac{2^{i-1}}{i+1}. \]

This completes the proof. \( \square \)
Corollary 4.4. The Kirchhoff index of the enhanced hypercube $Q_{n,n-1}$ is given by

$$Kf(Q_{n,n-1}) = 2^{n-1} \left( \sum_{i=1}^{n-2} \frac{2^{i-1}}{i} + 3 \frac{(n-2)^{n-1} + 1}{n(n-1)} \right).$$

Proof. Let $k = n - 1$, then $n$ and $k$ have the different parity. By Theorem 4.1, we have

$$Kf(Q_{n,n-1}) = 2^{n-1} \sum_{t=0}^{n-1} \sum_{j=0}^{t+1} \binom{3}{t+1} \binom{n-2}{t+1}.$$ 

By Lemma 4.2, it follows that

$$\sum_{t=0}^{n-1} \frac{1}{t+1}(n-2)^{t+1} = \sum_{r=1}^{n} \frac{1}{r} (n-2)^{r} = \sum_{s=1}^{n-2} \frac{2^{s-1}}{s}. $$

Besides,

$$\sum_{t=0}^{n-1} \frac{1}{t+1} = \sum_{t=0}^{n-1} \frac{1}{t+1} \left( \binom{n-1}{t} - \binom{n-2}{t} \right) = \sum_{t=0}^{n-1} \frac{1}{t+1} \binom{n-1}{t} - \sum_{t=0}^{n-1} \frac{1}{t+1} \binom{n-2}{t}$$

$$= \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{t+1} - \frac{1}{n} \sum_{t=0}^{n-1} \binom{n-1}{t+1} = \frac{1}{n(n-1)}[(n-2)2^{n-1} + 1]$$

Hence we obtain that

$$Kf(Q_{n,n-1}) = 2^{n-1} \left( \sum_{i=1}^{n-2} \frac{2^{i-1}}{i} + 3 \frac{(n-2)^{n-1} + 1}{n(n-1)} \right).$$

This completes the proof. □

| $Kf(Q_{n,k})$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| n            | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
| 2            | 3     | –     | –     | –     | –     | –     | –     | –     | –     |
| 3            | 13    | 14    | –     | –     | –     | –     | –     | –     | –     |
| 4            | 50    | 51.6  | 54    | –     | –     | –     | –     | –     | –     |
| 5            | 182.7 | 185.3 | 189.1 | 194.9 | –     | –     | –     | –     | –     |
| 6            | 653.3 | 657.9 | 664   | 672.8 | 687.5 | –     | –     | –     | –     |
| 7            | 2322.7| 2330.7| 2341.0| 2355.0| 2376.3| 2413.6| –     | –     | –     |
| 8            | 8272  | 8286.2| 8304  | 8327.4| 8360.4| 8412.4| 8509.4| –     | –     |
| 9            | 29626 | 29651 | 29682 | 29722 | 29776 | 29854 | 29984 | 30242 | –     |
| 10           | 106870| 106910| 106970| 107040| 107130| 107250| 107440| 107770| 108480 |

We calculate the Kirchhoff index of $Q_{n,k}$ for some $n$ and $k$ in Tabla 1 from which we see that the Kirchhoff index of $Q(n,k)$ is increased with the increase of $k$. Fortunately, it is true in general. To prove this property, we lead-in the following lemma.
Lemma 4.5. Let $1 \leq k \leq n - 1$. We have $\sum_{r=0}^{n} \sum_{i=0}^{n} \frac{1}{r+1} \left[ \left( \begin{array}{c} n-k \\ 2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j-1 \end{array} \right) - \left( \begin{array}{c} n-k \\ 2j-2 \\ 2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j-1 \\ 2j \end{array} \right) \right] \geq 0$.

Proof. Since $1 \leq k \leq n - 1$, we have $1 \leq n-k \leq n-1$ and $0 \leq k-1 \leq n-2$. It is clear that $F(n, k) = \sum_{r=0}^{n} \sum_{t=0}^{n} \frac{1}{r+1} \left[ \left( \begin{array}{c} n-k \\ 2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j-1 \end{array} \right) - \left( \begin{array}{c} n-k \\ 2j-2 \\ 2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j-1 \\ 2j \end{array} \right) \right] = \sum_{r=0}^{n} \sum_{t=0}^{n} \left( -1 \right)^{r} \left( \begin{array}{c} n-k \\ r \\ (t-1)-r \end{array} \right) \left( \begin{array}{c} k-1 \\ (t-1)-r \end{array} \right)$, from which we define a function $F_{n,k}(x) = \sum_{r=0}^{n} \sum_{t=0}^{n} \left( -1 \right)^{r} \left( \begin{array}{c} n-k \\ r \\ (t-1)-r \end{array} \right) x^{t}$. Obviously, $F_{n,k}(1) = F(n, k)$.

Now we introduce another function,

$$G(x) = x(1-x)^{n-k}(1+x)^{k-1} = x \sum_{i=0}^{n-k} (-1)^{i} \left( \begin{array}{c} n-k \\ i \end{array} \right) x^{i} \sum_{j=0}^{k-1} \left( \begin{array}{c} k-1 \\ j \end{array} \right) x^{j}$$

$$= x \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ j \end{array} \right) x^{i+j} = x \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ (k-1)-j \end{array} \right) x^{j}$$

$$= \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ (k-1)-j \end{array} \right) x^{j}.$$  

Then we have

$$H(x) = \int_{0}^{x} G(u) \, du = \int_{0}^{x} u(1-u)^{n-k}(1+u)^{k-1} \, du$$

$$= \int_{0}^{x} \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ (k-1)-j \end{array} \right) u^{i} \, du$$

$$= \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ (k-1)-j \end{array} \right) \int_{0}^{x} u^{i} \, du$$

$$= \sum_{i=0}^{n-k} \sum_{j=0}^{k-1} (-1)^{i} \left( \begin{array}{c} n-k \\ i \\ (k-1)-j \end{array} \right) \frac{1}{i+1} u^{i+1}.$$  

Hence, $H(1) = F_{n,k}(1) = F(n, k)$. On the other hand,

$$H(1) = \int_{0}^{1} G(u) \, du = \int_{0}^{1} u(1-u)^{n-k}(1+u)^{k-1} \, du.$$  

Since $G(u) = u(1-u)^{n-k}(1+u)^{k-1} \geq 0$ while $0 \leq u \leq 1$, we have

$$F(n, k) = \int_{0}^{1} G(u) \, du = \int_{0}^{1} u(1-u)^{n-k}(1+u)^{k-1} \, du \geq 0.$$  

This completes the proof. \qed

Theorem 4.6. Let $1 \leq k \leq n - 1$. For a given $n$, the Kirchhoff index of the enhanced hypercube $Q_{n,k}$ is increased with the increase of $k$.

Proof. Denote by $\nabla_k = Kf(Q_{n,k+1}) - Kf(Q_{n,k})$ for $1 \leq k < n - 1$. We will show that $\nabla_k > 0$ by induction on $k$. By Theorem 4.1, if $n \neq k \pmod{2}$, we have

$$\nabla_k = 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n-k+1 \\ 2j \end{array} \right) \left( \begin{array}{c} k \\ 2j \end{array} \right) - 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n-k+2 \\ 2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j \end{array} \right)$$

$$= 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n-k+1 \\ 2j-2j \end{array} \right) \left( \begin{array}{c} k \\ 2j-2j \end{array} \right) - 2^{n-1} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n-k+2 \\ 2j-2j \end{array} \right) \left( \begin{array}{c} k-1 \\ 2j-2j \end{array} \right).$$  

The second equality holds because $\sum_{j=0}^{n} \frac{1}{i+1} \left( \begin{array}{c} n-k+2 \\ 2j-2j \end{array} \right) \left( \begin{array}{c} k \end{array} \right) = 0$ if $t = n$ (in fact, by $n-k+2 \geq 2j$ and $k-1 \geq n-2j$, we obtain that $n-k = 2j-2$, which contradict to $n \neq k \pmod{2}$).
Thus (6) can be simplified as

\[ \nabla_k = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+1}{t+1} - \binom{n-k+2}{t+2} \right] \]

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+1}{t+2} - \binom{n-k+2}{t+2} \right] \] (7)

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+1} - \binom{n-k+2}{t+1} \right) - \left( \binom{n-k+1}{t+2} + \binom{n-k+2}{t+2} \right) \right] \]

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+2} - \binom{n-k+1}{t+1} \right) \right] \] (8)

If \( n \equiv k \mod 2 \), we have

\[ \nabla_k = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+1}{t+2} \right] - 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+2}{t+2} \right] \]

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+1} - \binom{n-k+2}{t+1} \right) \right] \]

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+1} - \binom{n-k+1}{t+2} \right) - \left( \binom{n-k+1}{t+2} + \binom{n-k+2}{t+2} \right) \right] \]

\[ = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+2} - \binom{n-k+1}{t+1} \right) \right] \] (8)

The second equality holds because \( \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+1}{t+2} \right] = 0 \) if \( t = n \) (in fact, by \( n-k+1 \geq 2j \) and \( k \geq n-2j+1 \), we can imply that \( n-k = 2j-1 \), which contradict to \( n \equiv k \mod 2 \)).

The last representations of \( \nabla_k \) in (7) and (8) are the same. Hence we need not to distinguish the above two cases in what follows.

For \( k = 1 \), from (7) we have

\[ \nabla_1 = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n}{t+2} - \binom{n}{t+1} \right) \right] \]

\[ = 2^{n-1} \left( \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n}{t+2} \right] - \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n}{t+1} \right] \right) \]

\[ = 2^{n-1} \left( \frac{n}{2} - \sum_{j=1}^{n} \frac{1}{j} \right) \]

\[ = 2^{n-1} \left( \frac{n+1}{2} + \sum_{j=1}^{n} \frac{1}{j} \right) = 2^{n-1} \frac{n+1}{n} > 0. \] (9)

Now we assume that \( \nabla_k = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \left( \binom{n-k+1}{t+2} - \binom{n-k+1}{t+1} \right) \right] > 0 \) holds for \( 1 \leq k < n-2 \). Next, we will prove that \( \nabla_{k+1} > 0 \). By regarding \( k \) as \( k+1 \), from (7) we get

\[ \nabla_{k+1} = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{n} \frac{1}{t+1} \left[ \binom{n-k+1}{t+2} - \binom{n-k+1}{t+1} \right] \] (10)

Notice that the general term of (10) can be simplified as

\[
\begin{align*}
\binom{n-k+1}{2j} & - \binom{n-k+1}{2j-1} \\
\binom{n-k+1}{2j} & - \binom{n-k+1}{2j-1} + \binom{k-1}{2j-1} - \binom{k-1}{2j} - \binom{k-1}{2j-1} + \binom{k-1}{2j} + \binom{k-1}{2j-1} \\
\binom{n-k+1}{2j} & - \binom{n-k+1}{2j-1} + \binom{k-1}{2j-1} - \binom{k-1}{2j} - \binom{k-1}{2j-1} + \binom{k-1}{2j} + \binom{k-1}{2j-1} \\
\binom{n-k+1}{2j} & - \binom{n-k+1}{2j-1} + \binom{k-1}{2j-1} - \binom{k-1}{2j} - \binom{k-1}{2j-1} + \binom{k-1}{2j} + \binom{k-1}{2j-1} - \binom{k-1}{2j} \\
\binom{n-k+1}{2j} & - \binom{n-k+1}{2j-1} + \binom{k-1}{2j-1} - \binom{k-1}{2j} - \binom{k-1}{2j-1} + \binom{k-1}{2j} + \binom{k-1}{2j-1} - 2\binom{k-1}{2j} \\
\end{align*}
\]
We can rewrite $\nabla_{k+1}$ as follows.

$$
\nabla_{k+1} = 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{t} \frac{1}{t+1} \left[ \binom{n-k}{2j} \binom{k}{t-2j} - \binom{n-k}{2j+1} \binom{k}{t-2j+1} \right] 
= 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{t} \frac{1}{t+1} \left[ \binom{n-k+1}{2j} \binom{k-1}{t-2j-1} - \binom{n-k+1}{2j+1} \binom{k-1}{t-2j-1+1} \right] 
+ 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{t+1} \frac{1}{t+1} \left[ \binom{n-k}{2j} \binom{k-1}{t-2j-1} + \binom{n-k}{2j+1} \binom{k-1}{t-2j-1+1} - 2 \binom{n-k}{2j} \binom{k-1}{t-2j} \right] 
= \nabla_k + 2^{n-1} \sum_{t=0}^{n} \sum_{j=0}^{t} \frac{1}{t+1} \left[ \binom{n-k}{2j} \binom{k-1}{t-2j-1} + \binom{n-k}{2j+1} \binom{k-1}{t-2j-1+1} - 2 \binom{n-k}{2j} \binom{k-1}{t-2j} \right] 
= \nabla_k + 2^{n} \sum_{t=0}^{n} \sum_{j=0}^{t} \frac{1}{t+1} \left[ \binom{n-k}{2j} \binom{k-1}{t-2j-1} + \binom{n-k}{2j+1} \binom{k-1}{t-2j-1+1} - 2 \binom{n-k}{2j} \binom{k-1}{t-2j} \right] 
= \nabla_k + 2^{n} \sum_{t=0}^{n} \sum_{j=0}^{t} \frac{1}{t+1} \left[ \binom{n-k}{2j} \binom{k-1}{t-2j-1} - \binom{n-k}{2j+1} \binom{k-1}{t-2j-1+1} \right].
$$

By the induction hypothesis, we have $\nabla_k > 0$. From Lemma 4.5, we know that the above last term are no less than zero. It implies that $\nabla_{k+1} = Kf(Q_{n,k+2}) - Kf(Q_{n,k+1}) > 0$, i.e., $Kf(Q_{n,k+1}) > Kf(Q_{n,k})$ for any $1 \leq k < n - 1$.

This completes the proof. \[ \Box \]

**Remark 4.7.** From Theorem 4.6 we have an interesting observation that $Q_{n,1}, Q_{n,2}, \ldots, Q_{n,n-1}$ have different Kirchhoff indexes. It implies that they have different spectra and so they are not isomorphic from each other.

By Theorem 4.6, we know that the Kirchhoff index of $Q_{n,k}$ is increased as $k$ increases. It follows that $Kf(Q_{n,1}) \leq Kf(Q_{n,k}) \leq Kf(Q_{n,n-1})$ for $1 \leq k \leq n - 1$. Thus we obtain the lower and upper bounds of $Kf(Q_{n,k})$ below.

**Corollary 4.8.** $\sum_{t=1}^{n} \sum_{j=0}^{t} \frac{2^{j-1}}{t+1} \leq Kf(Q_{n,k}) \leq \sum_{t=1}^{n-2} \sum_{j=0}^{t} \frac{2^{j-1}}{t+1} + \frac{3(n-2)2^{n-1}+1}{n(n-1)}$.

From the above Corollary we have the following limit function for $Kf(Q_{n,k})$.

**Theorem 4.9.** $\lim_{{n \to \infty}} \frac{Kf(Q_{n,k})}{2^{n+1}} = 1$.

**Proof.** By Corollary 4.8, we have $\sum_{t=1}^{n} \sum_{j=0}^{t} \frac{2^{j-1}}{t+1} \leq Kf(Q_{n,k}) \leq \sum_{t=1}^{n-2} \sum_{j=0}^{t} \frac{2^{j-1}}{t+1} + \frac{3(n-2)2^{n-1}+1}{n(n-1)}$.

Hence

$$
\sum_{t=1}^{n} \frac{2^{j-1}}{t+1} \leq Kf(Q_{n,k}) \leq \frac{\sum_{t=1}^{n-2} \frac{2^{j-1}}{t+1} + 3\left(\frac{n-2)2^{n-1}+1}{n(n-1)}\right)}{n(n-1)}.
$$

Denote by $A_n = \sum_{t=1}^{n} \frac{2^{j-1}}{t+1}$ and $B_n = \sum_{t=1}^{n-2} \frac{2^{j-1}}{t+1} + 3\left(\frac{n-2)2^{n-1}+1}{n(n-1)}\right)$. In what follows, we will show that $\lim A_n = \lim B_n = 1$. Let $x_n = \sum_{t=1}^{n} \frac{2^{j-1}}{t+1} = \frac{1}{2} + \frac{3}{4} + \ldots + \frac{2^{n-1}}{n+1}$ and $y_n = \frac{2^{n+1}}{n+1}$. We have

$$
\lim \frac{x_n - y_n}{y_n - x_n} = \lim \frac{\frac{2^{j-1}}{t+1}}{\frac{2^{j+1}}{t+1}} = \lim \frac{n}{n-1} - \lim \frac{n}{n-1}/(n+1)^2 = 1 - 0 = 1. \tag{11}
$$
By the Stolz-Cesàro Theorem, we get
\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{X_n}{Y_n} = 1.
\]

Let \( x'_1 = x'_2 = 0 \), and for \( n \geq 3 \) let \( x'_n = \sum_{t=1}^{n-2} \frac{2^t-1}{t} = 1 + \frac{3}{2} + \frac{7}{3} + \cdots + \frac{2^{n-2}-1}{n-2} \), \( y'_n = \frac{2^{n+1}}{n+1} \). As similar as (11), we have
\[
\lim_{n \to \infty} \frac{\sum_{t=1}^{n-2} \frac{2^t-1}{t}}{y'_n-y'_{n-1}} = \lim_{n \to \infty} \frac{n^2+n}{4(n^4-3n^2+2)} - \lim_{n \to \infty} \frac{n(n+1)}{(n-2)(n-1)^2} = \frac{1}{4}.
\]

Thus
\[
\lim_{n \to \infty} B_n = \lim_{n \to \infty} \left( \frac{\sum_{t=1}^{n-2} \frac{2^t-1}{t}}{y'_n-y'_{n-1}} + 3\left[ \frac{n^2-n-2}{4n^4-4n} + \frac{n+1}{n(n-1)^2} \right] \right)
= \lim_{n \to \infty} \frac{\sum_{t=1}^{n-2} \frac{2^t-1}{t}}{y'_n-y'_{n-1}} + 3 \lim_{n \to \infty} \frac{n^2-n-2}{4n^4-4n} + 3 \lim_{n \to \infty} \frac{n+1}{n(n-1)^2}
= \frac{1}{4} + \frac{3}{4} - 0 = 1.
\]

By the Squeeze Theorem, we get
\[
\lim_{n \to \infty} \frac{Kf(Q_{n,k})}{\frac{2^{n+1}}{n+1}} = \lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = 1.
\]

This completes the proof. \( \square \)

**Remark 4.10.** Although we get explicit formula of \( Kf(Q_{n,k}) \) when \( k = 1 \) or \( n-1 \) in Corollary 4.3 and Corollary 4.4, respectively. The calculation of the general representation of \( Kf(Q_{n,k}) \) given in Theorem 4.1 is more complex. Fortunately, Theorem 4.9 provides a simple uniform approximation function for \( Kf(Q_{n,k}) \) which is also independent of \( k \), it means that \( Kf(Q_{n,k}) \) can be replaced with \( \frac{2^{n+1}}{n+1} \) if \( n \) is large enough.

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