Quasi-Normal Modes of Brane-Localised Standard Model Fields

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We present here a detailed study of the quasi-normal spectrum of brane-localised Standard Model fields in the vicinity of $D$-dimensional black holes. A variety of such backgrounds (Schwarzschild, Reissner-Nordström and Schwarzschild-(Anti) de Sitter) are investigated. The dependence of the quasi-normal spectra on the dimensionality $D$, spin of the field $s$, and multipole number $\ell$ is analyzed. Analytical formulae are obtained for a number of limiting cases: in the limit of large multipole number for Schwarzschild, Schwarzschild-de Sitter and Reissner-Nordström black holes, in the extremal limit of the Schwarzschild-de Sitter black hole, and in the limit of small horizon radius in the case of Schwarzschild-Anti de Sitter black holes. We show that an increase in the number of hidden, extra dimensions results in the faster damping of all fields living on the brane, and that the localization of fields on a brane affects the QN spectrum in a number of additional ways, both direct and indirect.

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I. INTRODUCTION

Upon an external perturbation of a black hole background, realized either through the addition of a field or by perturbing the metric itself, the gravitational system enters a phase of damping oscillations, or quasi-normal ringing \[1, 2\] as it is alternatively called. During this phase, the frequency of the field consists of a real part \(\omega_{\text{Re}}\), that drives the field oscillations, and of an imaginary part \(\omega_{\text{Im}}\), that causes the simultaneous damping of these oscillations. The smaller \(\omega_{\text{Im}}\) is, the longer the damping time, therefore certain quasi-normal modes (QNMs) can dominate the spectrum at very late times after the initial perturbation, thus governing the dynamical evolution of the black hole.

The spectrum of quasi-normal modes has been the subject of an intensive study over the years, not only for their theoretical interest, but also due to the fact that their potential experimental detection could lead to the discovery of black holes. In a 4-dimensional context, they have been both numerically and analytically studied for a variety of black-hole backgrounds \[3\]. Among the different species of fields, the quasi-normal modes associated to gravitons, generated by metric perturbations, were particularly looked at, for the simple reason that gravitational QNMs originating from astrophysical black holes could potentially be observed with the help of gravitational wave detectors \[1, 2\]. Unfortunately, such an experimental confirmation has not been obtained up to now.

A few years ago, the landscape in gravitational physics changed with the formulation of theories postulating the existence of additional spacelike dimensions in nature \[4, 5\]. According to these theories, all Standard Model (SM) particles (scalars, fermions and gauge bosons) are restricted to live on a (3+1)-dimensional hypersurface – a brane – embedded in the higher-dimensional ‘bulk’. Gravitons can propagate both on and off the brane, with the same being true for other particles, like scalars, that carry no charges under the Standard Model gauge group. This geometrical set-up protects the accurately observed properties of SM fields from being altered due to the presence of extra dimensions while opening the way for the study of new gravitational backgrounds.

The theory with Large Extra Dimensions \[4\] predicts the existence of $d$ additional spacelike compact dimensions, all having – in the simplest case – the same size $L$. Black holes produced by the gravitational collapse of matter on the brane would naturally extend off the brane, being gravitational objects. Whereas macroscopic astrophysical black holes extend mainly along the usual three, non-compact spatial dimensions, thus being effectively 4-dimensional objects, microscopic black holes with a horizon radius $r_H \ll L$ would virtually live in a non-compact spacetime with $D = 4 + d$ dimensions in total. Higher-dimensional generalizations of 4-dimensional black hole solutions \[5, 6\], derived previously, came back in the spotlight, and the study of the QNMs associated with these higher-dimensional backgrounds soon became the subject of a renewed research activity \[7\].

One of the most exciting predictions of the theory with Large Extra Dimensions \[4\] is that such microscopic black holes may be created on ground-based accelerators during the collision of highly energetic particles with center-of-mass energy $\sqrt{s} > M_c$ \[4, 8\]. The energy scale

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$M_*$ denotes the fundamental Planck scale of the higher-dimensional gravitational theory, that becomes effective at distances $r < L$. This scale can be much lower than the 4-dimensional Planck scale $M_{Pl}$, even as low as 1 TeV, therefore trans-Planckian collisions can be easily realised at next-generation particle colliders. What is of the outmost importance is the fact that these tiny black holes, contrary to what happens in the case of macroscopic black holes, will be created in our neighbourhood in a controlled experiment inside a laboratory; therefore, their detection, either through the emission of Hawking radiation or the detection of their QNMs spectrum, will be substantially more favoured. 

In the latter case, the spectrum of QN modes associated with SM fields living in our brane will be the most important of all due to the well-developed techniques for the detection of fermions and gauge bosons, compared to the ones for the up-to-now elusive gravitons.

In this work, we attempt to fill the gap in the existing literature by presenting a comprehensive study of the QN modes associated with the brane-localized SM fields. We will be assuming that the black hole is characterized by a mass $M_{BH}$ that is at least a few orders of magnitude larger than the fundamental scale of gravity $M_*$, so that quantum corrections can be safely ignored. Also, in order to avoid a hierarchy problem, the brane self-energy can be naturally assumed to be of the order of the fundamental Planck scale $M_*$ and thus much smaller than $M_{BH}$, therefore its effect on the gravitational background can also be ignored.

In section II, we present the theoretical framework for our analysis and the equations of motion for the SM fields propagating in the brane background. Section III focuses on the associated QNM spectrum in the case of an induced-on-the-brane $D$-dimensional Schwarzschild black hole, and investigates the dependence of the spectrum on the dimensionality of spacetime, spin of the particle and multipole number. In Section IV we proceed to consider the QN modes of SM particles propagating on a brane embedded in a charged $D$-dimensional Reissner-Nordström black hole, and the effect of the black-hole charge on the QN spectrum is examined in detail. The spectra of QNMs for brane-localised SM fields in the background of a $D$-dimensional Schwarzschild-de Sitter and Schwarzschild-Anti de Sitter black hole are derived in sections V and VI, respectively, and the role of the bulk cosmological constant is investigated. We present our conclusions in section VII.

II. MASTER EQUATION FOR PROPAGATION OF FIELDS ON THE BRANE

As mentioned above, in this work we will concentrate on spherically-symmetric higher-dimensional black-hole backgrounds arising under the assumption of the existence of $d = D - 4$ additional, compact spacelike dimensions in nature. A large variety of such backgrounds may be described by a unique line-element of the form

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2_{d+2},$$

where $d\Omega^2_{d+2}$ is the area of the $(d + 2)$-dimensional unit sphere given by

$$d\Omega^2_{d+2} = d\theta^2_{d+1} + \sin^2 \theta_{d+1} \left(d\theta^2_d + \sin^2 \theta_d \left(... \right)\right),$$

with $0 < \varphi < 2\pi$ and $0 < \theta_i < \pi$, for $i = 1, ..., d + 1$. The radial-dependent metric function $h(r)$ may now be assumed to take the form

$$h(r) = 1 - \frac{\mu}{r^{D-3}},$$

describing a higher-dimensional, neutral black hole formed in a flat, empty space [6], with the parameter $\mu$ related to the ADM mass of the black hole through the relation

$$\mu = \frac{k_D^3 M_{BH}}{(D-2) \pi^{(D-1)/2} \Gamma \left[\frac{D-1}{2}\right]}.$$}

Alternatively, it may be taken to be

$$h(r) = 1 - \frac{\mu}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}},$$

in the case of a higher-dimensional, charged black hole [7] with $Q^2$ the electromagnetic charge. Finally, it may be written as

$$h(r) = 1 - \frac{\mu}{r^{D-3}} - \frac{2\kappa_D^2 \Lambda r^2}{(D-1)(D-2)}$$

to describe a Schwarzschild-(Anti) de Sitter black hole formed in a $(d + 4)$-dimensional spacetime with the presence of a (negative) positive cosmological constant $\Lambda$ [8]. In the above expressions, $\kappa_D^2 = 8\pi G = 8\pi M_{Pl}^{D-2}$ stands for the higher-dimensional Newton’s constant.

In what follows, we will study all three types of black-hole backgrounds and derive exact numerical results for the associated quasi-normal modes of all species of Standard Model (SM) fields: scalars, fermions and gauge bosons. According to the assumptions of the model [4], all SM fields are localised to a brane embedded in the bulk [10]. Since a potential observer also lives on the brane, the study of the QNMs of brane-localised fields...
is the most phenomenologically interesting one. To this end, we need first to determine the line-element of the background in which the brane-localized modes propagate. This can be found by fixing the values of the additional angular coordinates, describing the compact $d$ dimensions, to $\theta_1 = \pi/2$ [10, 11]. This results in the projection of the higher-dimensional background onto the brane, and in the induced-on-the-brane line-element

$$ds^2 = -h(r) dt^2 + \frac{dr^2}{h(r)} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right). \tag{7}$$

As the projection clearly affects only the angular part of the higher-dimensional line element given in Eq. (2), the metric function $h(r)$ remains unaffected and is still given by Eq. (3), (5) or (6). Note that, although the additional, spacelike dimensions have been projected out, the induced-on-the-brane line-element carries an explicit dependence on the number and content of extra dimensions through the metric function $h(r)$.

The next step in our analysis is the derivation of the equations of motion of the various species of fields propagating on the brane. This task was performed in [12] (see also [13]), for fields with spin $s = 0, 1/2$ and 1, by using the Newman-Penrose method [14, 15]. For a field with spin $s$, the following standard factorization was employed

$$\Psi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} \Delta^{-s} P_s(r) S_s(\theta), \tag{8}$$

where $\Delta = hr^2$, and $S_s(\theta)$ are the spin-weighted spherical harmonics [10]. As it was shown in [12], the radial and angular parts of each equation are decoupled, and take the ‘master’ form

$$\Delta^s \frac{d}{dr} \left( \Delta^{1-s} \frac{dP_s}{dr} \right) + \left( \frac{\omega^2 r^2}{h} + 2is \omega r - \frac{i\omega r^2 h'}{h} - \lambda \right) P_s = 0, \tag{9}$$

and

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_s}{d\theta} \right) + \left[ -2ms \cot \theta \frac{\sin \theta}{\sin^2 \theta} + s - s^2 \cot^2 \theta + \lambda \right] S_s = 0, \tag{10}$$

respectively. Following [17], these Teukolsky-like equations hold for the upper component $s = |s|$ of all fields with spin $s = 0, 1/2$ and 1. The constant $\lambda$ appearing in the radial equation is related to the angular eigenvalue $\lambda$ as follows

$$\hat{\lambda} \equiv \lambda + 2s = \ell(\ell + 1) - s(s - 1). \tag{11}$$

By choosing the desired expression for the metric function $h(r)$ from the set of Eqs. (3), (5) and (6), Eqs. (9) and (10), then, describe the motion of a particle with spin $s$ localized on a brane embedded in a higher-dimensional Schwarzschild, Reissner-Nordström and Schwarzschild-(Anti) de Sitter background, respectively.

### A. Alternative form: One-Dimensional wave equation

For the purpose of calculating the QNMs of brane-localized fields living in a general, spherically-symmetric background of the form (12), studying the radial equation (13) will be adequate. In what follows, we will rewrite this equation in a more convenient form, namely in the form of a one-dimensional Schrödinger (or wave-like) equation, and derive the form of the corresponding effective potential. To this end, we follow the analysis performed in [18], and define a new radial function and a new (“tortoise”) coordinate according to

$$P_s = r^{2(s-1/2)} Y_s, \quad \frac{dr_s}{dr} = \frac{1}{h}. \tag{12}$$

Then, Eq. (13) takes the form

$$\left( \frac{d^2}{dr^2} + \omega^2 \right) Y_s + \mathcal{P} \left( \frac{d}{dr_s} + i\omega \right) Y_s - Q Y_s = 0, \tag{13}$$

where we have defined

$$\mathcal{P} = s \left( \frac{4h}{r} - \frac{\Delta'}{r^2} \right), \tag{14}$$

and

$$Q = \frac{h}{r^2} \left\{ \lambda - (2s - 1)(s - 1) \left( 2h - \frac{\Delta'}{r} \right) \right\}. \tag{15}$$

By defining further

$$Y_s = hZ_s + 2i\omega \left( \frac{d}{dr_s} - i\omega \right) Z_s, \tag{16}$$

eq (19)

Eq. (19) can now be written as a one-dimensional Schrödinger equation

$$\left( \frac{d^2}{dr_s^2} + \omega^2 \right) Z_s = V_s Z_s. \tag{17}$$

The effective potential $V_s$ is found to be spin-dependent and given by the expression

$$V_{s=1} = \frac{h}{r^2} \ell(\ell + 1), \tag{18}$$

for spin-1 particles, where we have used Eq. (14), and

$$V_{s=1/2} = h k \left[ \frac{k}{r^2} + \frac{d}{dr} \left( \frac{\sqrt{h}}{r} \right) \right], \tag{19}$$

for spin-1/2 particles, where

$$k = \sqrt{\ell(\ell + 1) + 1/4} = 1, 2, 3, \ldots \tag{20}$$

For spin-0 particles, the quantity $\mathcal{P}$ defined in Eq. (14) vanishes trivially, and the equation for $Y_s$ (13) automatically takes a wave-like form, with potential

$$V_{s=0} = h \left[ \frac{\ell(\ell + 1)}{r^2} + \frac{h'}{r} \right]. \tag{21}$$
The above results hold for any $D$-dimensional, spherically-symmetric black hole (Schwarzschild, Reissner-Nordström and Schwarzschild-dS/AdS) projected onto the brane, upon choosing the correct value of the metric function $h(r)$.

III. NEUTRAL NON-ROTATING BLACK HOLE: WKB VALUES OF BRANE QNMS

For a $D$-dimensional Schwarzschild black hole background, the effective potentials $V_s$, seen by scalars, fermion and gauge bosons propagating on the embedded brane, have the form of a positive-definite potential barrier that attains its maximum value close to the black hole while it vanishes at the event horizon ($r_s = -\infty$) and spatial infinity ($r_s = +\infty$). According to Eq. (17) then, the solution for all types of fields in these two asymptotic regimes can be written in terms of incoming and outgoing plane waves. If we write the quasi-normal frequency as $\omega = \omega_{Re} - i\omega_{Im}$, under the choice of the positive sign for $\omega_{Re}$, the QNMs for all Standard Model particles propagating on the brane satisfy the boundary conditions [1, 2].

\[ \Psi_s(r_s) \approx C_{\pm} \exp(\pm \omega r_s) \quad \text{as} \quad r_s \rightarrow \pm \infty, \quad (22) \]

corresponding to purely in-going waves at the event horizon and purely out-going waves at spatial infinity.

The fact that the effective potentials $V_s$ have the form of a positive-definite potential barrier also allows us to use the well-known WKB method in order to find the various QNMs with a significantly good accuracy. The WKB formula for QN frequencies has the form:

\[ i \frac{\omega^2 - V_0}{\sqrt{-2V_0}} - L_2 - L_3 - L_4 - L_5 - L_6 = n + \frac{1}{2}, \quad (23) \]

where $V_0$ is the height of the potential at its maximum point, located at $r_0$, and $V''_0$ its second derivative with respect to the tortoise coordinate at the same point. The lower terms $L_2$ and $L_3$ can be found in [13], while the higher ones $L_4$, $L_5$ and $L_6$ in [20]. Finally, the symbol $n$ in the above formula denotes the various overtones.

By using the above formula, we have calculated the QNMs for all species of SM fields for various values of the dimensionality $D$ of the spacetime, multipole moment $\ell$ and overtone number $n$. In Tables I, II and III we display, as illustrative examples, the QNMs for scalars, fermions and gauge bosons, respectively, for $D = 5$ and $D = 6$, and for $n \leq \ell$ up to $\ell = 3$. The only numerical results available in the literature for QN modes of brane-localised fields are the ones presented in [21], where the case $D = 5$ was studied. Comparison with their numerical data shows that there is a very good agreement between our results obtained by using the WKB method and the exact numerical ones, and that the higher the WKB order is, the better the accuracy. It is known that the WKB method is accurate for modes with $\ell > n$, therefore within the lower overtones, the worst (but still reasonable) precision will be for the fundamental scalar mode ($\ell = n = 0$): we have found $0.265391 - 0.497380i$ in 3rd WKB order, and $0.256429 - 0.391293i$ in 6th WKB order, while the numerical value is $0.27339 - 0.41091$, i.e. the relative error is about 4% for the real part and 2% for the imaginary part. For the fermionic ($k = 1, n = 0$) and electromagnetic ($\ell = 1, n = 0$) fundamental mode, the WKB method is expected to be rather accurate since $k, \ell > n$; indeed, for the electromagnetic mode, for example, we have found $0.545243 - 0.331346i$ in 3rd WKB order, and $0.576508 - 0.313923i$ in 6th WKB order, while the numerical value is $0.57667 - 0.31749i$; therefore, the relative error is less than 0.01% for the real part, and about 0.2% for the imaginary part. For higher multipoles, a much better WKB accuracy is expected. As the reader will note, the two 6-th order WKB values for the modes $k = n = 1$ and $\ell = n = 1$ for $D = 6$ are absent from Tables II and III: this is due to the fact that, for particular modes, the WKB method begins to show bad convergence as $D$ starts taking large values.

One can find an analytical expression for QNMs in the limit of large multipole number $\ell$, by making use of the formula [28] within first order and expanding in terms of $1/\ell$. In the eikonal approximation – as this approximation is called, the following formula for scalar and electromagnetic perturbations is valid:

\[ \omega = \sqrt{\frac{r_0^{D-3} - r_H^{D-3}}{r_0^{D-4}}} \left[ \ell + \frac{1}{2} - i\sqrt{D-3} \left( n + \frac{1}{2} \right) \right], \quad (24) \]

where $r_H = \mu^{1/(D-3)}$ is the black hole horizon radius, and $r_0$, as mentioned above, the value of the radial coordinate where the effective potential attains its maximum value. For scalar, Dirac, and electromagnetic perturbations, this
Table I: WKB values of the quasi-normal frequencies for scalar field perturbations in the 3rd and 6th order beyond the eikonal approximation with $\ell \geq n$, for a $D$-dimensional Schwarzschild black hole projected on the brane.

| $D = 5$ | WKB3 | WKB6 | $D = 6$ | WKB3 | WKB6 |
|---------|-------|-------|---------|-------|-------|
| $\ell = 0$; $n=0$; | 0.265391 - 0.497380 i | 0.256429 - 0.391293 i | $\ell = 0$; $n=0$; | 0.249904 - 0.793263 i | 0.146960 - 0.600541 i |
| $\ell = 1$; $n=0$; | 0.728209 - 0.358853 i | 0.748461 - 0.371463 i | $\ell = 1$; $n=0$; | 0.705383 - 0.527453 i | 0.834846 - 0.512377 i |
| $\ell = 1$; $n=1$; | 0.535200 - 1.234129 i | 0.568185 - 1.214473 i | $\ell = 1$; $n=1$; | 0.355372 - 1.918880 i | 0.392642 - 1.711230 i |
| $\ell = 2$; $n=0$; | 1.243914 - 0.358055 i | 1.249839 - 0.358066 i | $\ell = 2$; $n=0$; | 1.376420 - 0.499199 i | 1.393460 - 0.507541 i |
| $\ell = 2$; $n=1$; | 1.099791 - 1.142086 i | 1.116151 - 1.121364 i | $\ell = 2$; $n=1$; | 1.039578 - 1.918880 i | 1.088113 - 1.656560 i |
| $\ell = 3$; $n=0$; | 1.747954 - 0.355789 i | 1.750048 - 0.355587 i | $\ell = 3$; $n=0$; | 1.971510 - 0.497366 i | 1.978550 - 0.496292 i |
| $\ell = 3$; $n=1$; | 1.642277 - 1.091913 i | 1.648191 - 1.090268 i | $\ell = 3$; $n=1$; | 1.733260 - 1.551881 i | 1.752046 - 1.543178 i |
| $\ell = 3$; $n=2$; | 1.468478 - 1.870707 i | 1.458460 - 1.895964 i | $\ell = 3$; $n=2$; | 1.347778 - 2.703307 i | 1.317857 - 2.782190 i |
| $\ell = 3$; $n=3$; | 1.246835 - 2.677609 i | 1.219431 - 2.814159 i | $\ell = 3$; $n=3$; | 0.851976 - 3.917659 i | 0.742520 - 4.384462 i |

Table II: WKB values of the quasi-normal frequencies for Dirac field perturbations in the 3rd and 6th order beyond the eikonal approximation with $k \geq n$, for a $D$-dimensional Schwarzschild black hole projected on the brane.

| $D = 5$ | WKB3 | WKB6 | $D = 6$ | WKB3 | WKB6 |
|---------|-------|-------|---------|-------|-------|
| $k=1$; $n=0$; | 0.373020 - 0.400822 i | 0.427813 - 0.324042 i | $k=1$; $n=0$; | 0.296822 - 0.633767 i | 0.389272 - 0.404875 i |
| $k=1$; $n=1$; | 0.236229 - 1.299335 i | 0.199685 - 1.251460 i | $k=1$; $n=1$; | 0.086481 - 1.975351 i | – |
| $k=2$; $n=0$; | 0.945819 - 0.355480 i | 0.973434 - 0.351309 i | $k=2$; $n=0$; | 1.008933 - 0.499552 i | 1.122648 - 0.454934 i |
| $k=2$; $n=1$; | 0.771922 - 1.163379 i | 0.806766 - 1.153473 i | $k=2$; $n=1$; | 0.664685 - 1.765862 i | 0.716437 - 1.523422 i |
| $k=2$; $n=2$; | 0.559579 - 2.029365 i | 0.580093 - 2.075602 i | $k=2$; $n=2$; | 0.280211 - 3.100330 i | 0.329359 - 3.257230 i |
| $k=3$; $n=0$; | 1.471906 - 0.355680 i | 1.479067 - 0.354610 i | $k=3$; $n=0$; | 1.641846 - 0.490566 i | 1.668661 - 0.502019 i |
| $k=3$; $n=1$; | 1.341192 - 1.101171 i | 1.359844 - 1.095934 i | $k=3$; $n=1$; | 1.334248 - 1.595761 i | 1.417715 - 1.561658 i |
| $k=3$; $n=2$; | 1.143877 - 1.905989 i | 1.152074 - 1.932798 i | $k=3$; $n=2$; | 0.913705 - 2.836118 i | 0.932631 - 2.876656 i |
| $k=3$; $n=3$; | 0.898975 - 2.738309 i | 0.906727 - 2.920396 i | $k=3$; $n=3$; | 0.392077 - 4.126275 i | 0.331583 - 4.842823 i |

One may easily note that the above formulae, valid for large multipole numbers, work well even for relatively small $\ell$, being a good approximation already at $\ell = 4$.

Note that, for electromagnetic perturbations, the above equation is an exact one, valid for any multipole number $\ell$. An expression similar to Eq. (24) can be found also for Dirac perturbations, and has the form

$$\omega = \sqrt{\frac{r_D - 3 - i r_D - 3 H}{r_0 - 1}} \left[ k - i \sqrt{D - 3 \left(n + \frac{1}{2}\right)} \right].$$

(26)

By looking at our data, displayed in Tables I-III, one may draw important conclusions for the dependence of the QN frequencies on the dimensionality of spacetime $D$, spin $s$ of the particle and multipole moment $\ell$. Although the various SM fields are restricted to live on a 4-dimensional brane, their QN behaviour is significantly different from the one in a purely 4-dimensional Schwarzschild background, and resembles more the one of higher-dimensional fields living in the bulk. As in the case of gravitons propagating in the higher-dimensional spacetime [26], the imaginary part of the QN frequencies for all types of brane-localised fields increase, as $D$ also increases. As a result, the greater the number of hidden extra dimensions is, the greater the damping rate, and thus the shorter-lived the quasi-normal ringing phase on the brane. The behaviour of the real part of the QN frequencies is however less monotonic, and seems to crucially depend on the values of the multipole moment $\ell$. 

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(26)
Table III: WKB values of the quasi-normal frequencies for electromagnetic field perturbations in the 3rd and 6th order beyond the eikonal approximation with \( \ell \geq n \), for a \( D \)-dimensional Schwarzschild black hole projected on the brane.

| \( D = 5 \) | \( D = 6 \) |
|-----|-----|
| \( \ell = 1; n = 0; \) | 0.545243 - 0.331346 i | 0.482332 - 0.459270 i |
| \( \ell = 1; n = 1; \) | 0.317482 - 1.135031 i | 0.339969 - 1.081171 i |
| \( \ell = 2; n = 0; \) | 1.142703 - 0.343905 i | 1.244327 - 0.472984 i |
| \( \ell = 2; n = 1; \) | 0.992174 - 1.082547 i | 0.101222 - 1.074150 i |
| \( \ell = 2; n = 2; \) | 0.767162 - 1.877775 i | 0.437738 - 2.732445 i |
| \( \ell = 3; n = 0; \) | 1.675991 - 0.348490 i | 1.67917 - 0.348103 i |
| \( \ell = 3; n = 1; \) | 1.567818 - 1.070305 i | 1.571663 - 1.067999 i |
| \( \ell = 3; n = 2; \) | 1.389311 - 1.835125 i | 1.376476 - 1.860555 i |
| \( \ell = 3; n = 3; \) | 1.160459 - 2.628529 i | 1.126499 - 2.770443 i |

and overtone number \( n \). For \( \ell, k > 2 \), the fundamental \( (n = 0) \) and first \( (n = 1) \) overtones are characterized by an increasing real oscillation frequency, as the value of \( D \) increases, in agreement with similar results derived for gravitational QNMs in the bulk \cite{26}. We expect this behaviour to hold for arbitrarily large multipole number where the eikonal approximation becomes valid: Fig. 1 depicts the dependence of \( \omega_{\Re e} \) and \( \omega_{\Im m} \) as a function of \( D \) in the eikonal regime, with the increase in both parts being obvious. For \( \ell, k \leq 2 \), or \( n \approx \ell, k \) though, our data seem to indicate that the real part of the QN frequency tends to decrease with the total number of dimensions, with the only exceptions to this rule being the scalar and electromagnetic fundamental mode \((n = 0)\) for \( \ell = 1 \). As nevertheless noted previously, the accuracy of our calculation decreases exactly in this regime, therefore this result should be taken with caution and confirmed by numerical analysis.

Turning to the dependence of the QNM frequencies on the spin of the particle, let us consider the fundamental modes of all three types of SM fields that correspond to the lowest possible value of the multipole number in each case. Within the 6th order WKB approximation, we obtained: \( \omega_{\Re e} = 0.25643 - 0.39129i \) for scalar \((\ell = 0)\), \( \omega_{\Re e} = 0.42781 - 0.32404i \) for Dirac \((k = 1)\), and \( \omega_{\Re e} = 0.57651 - 0.31392i \) for gauge fields \((\ell = 1)\). Therefore, among the lowest fundamental modes, the greater the spin of the field is, the greater the \( \omega_{\Re e} \), and the smaller the \( \omega_{\Im m} \). As a result, fields with higher spin will decay slower and will dominate during the “final ringdown” stage. To this respect, the brane-localised fields behave in a similar way to purely 4-dimensional ones \cite{3}. In contrast, in studying gravitational perturbations in the bulk, it was found that an increasing spin-weight led again to the enhancement of the real frequency but to the increase of the damping rate as well \cite{20}. Among the lowest three fundamental modes mentioned above, the field with the higher spin \((s = 1)\) has also the largest ratio \( |\omega_{\Re e}/\omega_{\Im m}| \), known as the quality factor of the oscillator, a result that renders this mode the best oscillator of all three in the vicinity of the projected-on-the-brane black hole. However, if we look at the fundamental modes corresponding to higher values of the multipole number, one may see that, for fixed \( \ell \), it is instead the scalar field that is the better oscillator of all three species.

Finally, for fixed dimensionality of spacetime and spin of the particle, the various QN frequencies still depend on the multipole number \( \ell \). A simple inspection of our data shows that, for brane-localised fields, the real oscillatory part increases with \( \ell \), for all species of fields and values of \( D \). The imaginary part of the frequency is decreasing for scalar fields and increasing for fermions and gauge fields, however in all cases it soon approaches a constant value. This is in agreement with Eqs. \(21\) and \(24\), from where we may see that, in the high frequency approximation \((\ell \to \infty)\), the real part of each mode is proportional to the multipole number \( \ell \), while the damping rate approaches a constant value that depends solely on the fundamental parameters of the gravitational system such as \( r_H, r_0 \) and \( D \).

IV. CHARGED NON-ROTATING BLACK HOLE: WKB VALUES OF BRANE QNMS

In this case, the metric function \( h(r) \), that describes the projected-on-the-brane Reissner-Nordstr"om black-hole line-element, is given by Eq. \(15\). The corresponding effective potentials \( V_s \) for the various SM fields propagating on the brane are again determined by the formulae in Eqs. \(16-21\), and, as before, have the form of positive-definite potential barriers. As a result, the WKB method can be consistently applied also in this case.

The quasi-normal modes of a 4-dimensional Reissner-Nordstr"om black hole were investigated in \cite{21} within the WKB approximation, and in \cite{27} with the help of the Frobenius method. Quasi-normal modes of more general, or alternative, solutions for charged black holes were also considered in the literature: \((D > 4)\)-dimensional Reissner-Nordstr"om black holes \cite{20}, dilaton black holes \cite{27}, and Born-Infeld black holes \cite{23}. In all the aforementioned works, one common feature was found to emerge concerning the QNMs of charged black holes: the damping rate was monotonically increasing as a function of the charge \( Q \), until a critical value \( Q \approx 0.7 - 0.8Q_{\text{crit}} \) was reached; after this point, the behaviour of the damping
rate changed into sharp decreasing. Thereby, $\omega_{\text{Im}}$ as a function of $Q$ had a maximum somewhere near the value $Q \approx 0.7 - 0.8Q_{\text{ext}}$, where $Q_{\text{ext}} = \mu/2$ the value of the charge that corresponds to an extreme charged black hole with a degenerate horizon.

In Figs. 2 & 3 and Table IV, we display an indicative sample of our results, derived by using the 6th order beyond the eikonal approximation WKB method; these correspond to the fundamental modes ($n = 0$) for scalars, fermions and gauge fields, for the multipole number values $\ell = 2$ and $\ell = 1$, respectively. For simplicity, the mass parameter and therefore the extremal value of the charge have been fixed to the values $\mu = 1$ and $Q_{\text{ext}} = 0.5$, respectively. For a given value of the charge $Q$, the dependence of the QNMs on $D$ and spin $s$ remains the same as in the neutral case, therefore we will not comment further on this dependence here. We will instead focus our attention to the dependence of the QN frequency solely on the charge of the black hole.

According to our results, the quasi-normal modes of particles living in a charged black-hole background projected on the brane have one, very distinctive feature compared to the previously studied Reissner-Nordström cases: the characteristic maximum of the damping rate as a function of the charge is absent. Indeed, as one may see in Fig. 2, the damping rate of the $\ell = 2$ fundamental modes of all species of fields is a monotonically decreasing function of the charge $Q$. The same behaviour was found for the fundamental modes of all species corresponding to higher multipole numbers, $\ell > 2$. The absence of the aforementioned maximum is not a feature unseen before: it has been observed recently for $D$-dimensional black holes in Gauss-Bonnet theory [20]. The real oscillatory part of the QN frequency on the other hand was found to be monotonically increasing, as a function of $Q$, for all species of particles, and its exact dependence for the case $\ell = 2$ is depicted in Fig. 3.

As in the previous subsection, the fundamental modes of some species of fields for the multipole number $\ell = 1$ offer an exception to the aforementioned rule. In Table IV, we display these modes for all species of fields of the Standard Model. While the $\ell = 1$ fundamental mode of scalar fields follows the behaviour displayed in Figs. 2 and 3 both for the imaginary and real part of the QN frequency, the one of fermions and gauge bosons deviate by exhibiting a non-monotonic behaviour. In the case of fermions, the imaginary part of the QN frequency does initially decrease with the charge $Q$, as the rule dictates, however for higher values of $Q$, it goes through a maximum value before decreasing again; similarly, its real part starts increasing as expected but again, after a critical point is reached, it starts decreasing. In the case of gauge bosons, the real part of the $\ell = 1$ fundamental mode does indeed follow the pattern shown in Fig. 3; however, its imaginary part seems to follow instead the traditional Reissner-Nordström frequency behaviour by initially increasing with $Q$ and then decreasing. The behaviour described above holds both for $D = 5$ and $D = 6$, with the increase in the dimensionality of spacetime enhancing the height of the various maxima points.

Let us note at this point that, throughout this analysis, we consider minimally-coupled fields propagating in a given black-hole background projected on the brane. As a result, we do not take into account the interaction of the spin-1 propagating particle with the electromagnetic field of the black-hole background. In a more realistic description, the electromagnetic perturbations will induce gravitational ones and vice versa, and the quasi-normal spectrum of vector perturbations may be considerably different from the one described here.

As in the case of a neutral Schwarzschild black hole, an expression may be derived that describes the QN frequency in the eikonal approximation, i.e. in the limit of large multipole number. By making use of the first order WKB approximation, one can obtain, in the eikonal approximation.
regime $\ell \to \infty$, the following formula

$$\omega = \sqrt{\frac{Q^2 - \mu r_0^{D-3} + r_0^{D-6}}{r_0^{D-4}}} \times \left[ \ell + \frac{1}{2} - iC \left( n + \frac{1}{2} \right) \right],$$

(27)

for scalar and vector perturbations, and

$$\omega = \sqrt{\frac{Q^2 - \mu r_0^{D-3} + r_0^{D-6}}{r_0^{D-4}}} \left[ k - iC \left( n + \frac{1}{2} \right) \right],$$

(28)

for Dirac perturbations. As before, $r_0$ denotes the value of the radial coordinate that corresponds to the maximum of the effective potential, which is now given by

$$r_0 = \left[ \frac{(D - 1)\mu + \sqrt{(D - 1)^2\mu^2 - 16(D - 2)Q^2}}{4} \right]^{1/3},$$

(29)

also in the limit $\ell \to \infty$. In the above equations, $C$ is a rather cumbersome constant depending on the black hole parameters $\mu$ and $Q$ and on the spacetime dimensionality $D$. When $Q = 0$, the above formulae reduce to Eqs. (24)-(26) for a neutral black hole. One may clearly see that, even for a non-vanishing value of $Q$, the real part of the QN frequency increases proportionally to the multipole number, while the imaginary part, and thus the damping rate, is determined through $C$ by the fundamental parameters of the problem.

Table IV: WKB values in 6th order beyond the eikonal approximation for quasi-normal frequencies of brane-localised fields, for $D = 5$ and $D = 6$ Reissner-Nordström black holes; $\ell = 1, n = 0$.

| $s$ | $Q$ | $D = 5$ | $D = 6$ |
|-----|-----|---------|---------|
| 0   | 0.05| 0.749152 - 0.371256 i | 0.835852 - 0.511642 i |
| 0   | 0.15| 0.754757 - 0.369605 i | 0.843518 - 0.506642 i |
| 0   | 0.25| 0.766514 - 0.366119 i | 0.858076 - 0.499811 i |
| 0   | 0.35| 0.786714 - 0.358488 i | 0.887353 - 0.484023 i |
| 0   | 0.45| 0.818792 - 0.334299 i | 0.934330 - 0.425248 i |
| 0   | 0.499 | 0.830394 - 0.313416 i | 0.935737 - 0.396232 i |
| 1/2 | 0.05| 0.428119 - 0.323953 i | 0.389386 - 0.403475 i |
| 1/2 | 0.15| 0.430879 - 0.322740 i | 0.392510 - 0.388419 i |
| 1/2 | 0.25| 0.438019 - 0.318018 i | 0.409388 - 0.346311 i |
| 1/2 | 0.35| 0.447564 - 0.314646 i | 0.390580 - 0.382414 i |
| 1/2 | 0.45| 0.458170 - 0.320913 i | 0.419673 - 0.437986 i |
| 1/2 | 0.499 | 0.450377 - 0.290551 i | 0.355861 - 0.374075 i |
| 1   | 0.05| 0.577071 - 0.314252 i | 0.625394 - 0.305570 i |
| 1   | 0.15| 0.581787 - 0.316804 i | 0.632436 - 0.321188 i |
| 1   | 0.25| 0.592555 - 0.321297 i | 0.655043 - 0.345745 i |
| 1   | 0.35| 0.613317 - 0.324651 i | 0.712217 - 0.358226 i |
| 1   | 0.45| 0.652107 - 0.311345 i | 0.811198 - 0.319470 i |
| 1   | 0.499 | 0.668226 - 0.289308 i | 0.832820 - 0.291253 i |

V. SCHWARZSCHILD-DE SITTER BLACK HOLE: WKB VALUES OF BRANE QNMS

We now turn to the case of a $D$-dimensional Schwarzschild-de Sitter (SdS) black hole projected onto a 4-dimensional brane. The corresponding line-element is defined in terms of the metric function $h(r)$ given in Eq. (4) with a positive cosmological constant, $\Lambda > 0$. This gravitational background is characterized by two horizons, the black hole event horizon $r_H$ and the cosmological horizon $r_c$. The effective potentials [13]-[21], again, take the form of a potential barrier that vanishes at both horizons. We can, therefore, once again apply consistently the WKB method.

In Table V, we display our results for the QNM frequencies for scalar, Dirac and electromagnetic perturbations propagating in the vicinity of a projected-on-the-brane SdS black hole. The data correspond to the fundamental modes for the value $\ell = 1$ of the multipole number, and for the dimensionalities $D = 5$ and $D = 6$. The value of the cosmological constant is parametrised in terms of its extreme value $\Lambda_{ext}$, that corresponds to the degenerate case where the black hole and cosmological horizon coincide. This maximal value of $\Lambda$ is defined through the relation:

$$\mu^2 = \frac{4(D - 3)^{D-3}}{(D - 1)^{D-1}} \left( \frac{(D - 1)(D - 2)}{2k^2D\Lambda_{ext}} \right)^{D-3}.$$  

(30)

The behaviour of the QN frequencies of the various species of brane-localised fields does not yield any surprises in regard with their dependence on the cosmological constant. Similarly to the behaviour exhibited by fields in the vicinity of a purely 4-dimensional SdS spacetime [3] or by gravitational fields living in a $D$-dimensional SdS spacetime [20], both the real and imaginary parts of the QN frequency are decreasing, as the value of the cosmological constant increases. This decrease is observed, and has the same magnitude, for all species of fields: as $\Lambda$ ranges from zero to its maximum value $\Lambda_{ext}$, all fields undergo a suppression of their QN frequency by approximately 90%. For any given value of $\Lambda$, an increase in the dimensionality of spacetime results, as in the case of QNMs of bulk gravitons, to an increase of the value of both $\omega_{Re}$ and $\omega_{Im}$, nevertheless the relative suppression of $\omega$ with $\Lambda$ remains the same.

As it is well known, in the regime of near extremal values of $\Lambda$, the effective potential approaches the Poschl-Teller potential [32] for which there is an exact analytical solution. Therefore, the exact formula for QNMs in the extremal limit is:

$$\omega = \frac{\ell(\ell + 1)(r_c - r_H)}{2k_Hr_H^2} - \frac{1}{4} - i \left( n + \frac{1}{2} \right),$$

(31)

for scalar and electromagnetic perturbations, and

$$\omega = \frac{k^2(r_c - r_H)}{2k_Hr_H^2} - \frac{1}{4} - i \left( n + \frac{1}{2} \right),$$

(32)
for Dirac perturbations. Here, the radii of the event and cosmological horizon are approximated by the formulae:

\[
r_H \approx \sqrt{\frac{(D - 1)(D - 2)}{2\kappa^2_D \Lambda}} \left( \frac{D - 3}{D - 1} - \frac{\delta}{D - 1} \right),
\]

\[
r_c \approx \sqrt{\frac{(D - 1)(D - 2)}{2\kappa^2_D \Lambda}} \left( \frac{D - 3}{D - 1} + \frac{\delta}{D - 1} \right),
\]

respectively, where

\[
\delta = \frac{\mu^2(D - 1)^{D - 1}}{4 \left( \frac{(D - 1)(D - 2)}{2\kappa^2_D \Lambda} \right)^{D - 3} (D - 3)^{D - 3}}.
\]

Finally, the surface gravity \( k_H \) at the event horizon is given by the formula

\[
k_H = \frac{1}{2} \frac{dh(r)}{dr} \bigg|_{r=r_H}.
\]

As we can see from the above formulae, in the extremal limit, fields of different spin decay with exactly the same rate. Indeed, from the entries of Table V, one can see that, despite its initially different values, \( \omega_{\text{Im}} \) reduces to almost the same value for all species of fields when \( \Lambda = 0.99\Lambda_{\text{ext}} \). Note that the same behaviour is observed in the QNM spectrum for ordinary 4-dimensional SdS black holes, and is therefore not an unexpected property of the QNM spectrum of brane-localised fields.

In the regime of large multipole numbers \( \ell \), the value of the radial coordinate \( r = r_0 \), at which the effective potential has a maximum, is again expressed by Eq. (25) for fields of any spin. Using the first order WKB formula,

one can find

\[
\omega = \sqrt{\frac{1}{r_0^2} - \frac{\mu}{r_0^{D-1}} - \frac{2\kappa^2_D \Lambda}{(D-1)(D-2)} \left( \ell + \frac{1}{2} + iC(n + \frac{1}{2}) \right)},
\]

for scalar and electromagnetic perturbations, and

\[
\omega = \sqrt{\frac{1}{r_0^2} - \frac{\mu}{r_0^{D-1}} - \frac{2\kappa^2_D \Lambda}{(D-1)(D-2)} \left( k + iC(n + \frac{1}{2}) \right)},
\]

for Dirac perturbations. When \( D = 4 \), the above formulae go over to the ones for SdS black holes presented in \[32\].

VI. SCHWARZSCHILD-ANTI DE SITTER BLACK HOLE: WKB VALUES OF BRANE QNMS

A great interest in quasi-normal modes for asymptotically Anti de Sitter (AdS) black holes was stipulated the last 5 years after the observation was made, first in \[31\], that quasi-normal modes of classical \((d + 1)-\)dimensional asymptotically AdS black holes should coincide with poles of the retarded Green function in the dual conformal field theory (CFT) at finite temperature in \( d \) dimensions in the limit of strong coupling. In this context, the black hole temperature corresponds to the temperature in the dual thermal field theory. Therefore, by making use of this correspondence, one can investigate different thermal phenomena in the limit of strong coupling in CFT, such as dispersion relations and the hydrodynamic limit of CFT \[34\].

Assuming that the background in the bulk coincides with the one of a Schwarzschild-Anti de Sitter (SAdS) black hole, the projected-on-the-brane line-element will be described once again by Eq. (7), where \( h(r) \) is given in Eq. (9) with the cosmological constant now taking negative values, \( \Lambda < 0 \). For later use, we define the “effective” anti-de Sitter radius \( R \) as

\[
R = \sqrt{\frac{(D-1)(D-2)}{2\kappa^2_D |\Lambda|}}.
\]

In the case of an SAdS background, the effective potentials \[15-21\] vanish at the black hole horizon but are divergent at infinity, and therefore Dirichlet boundary conditions, demanding the vanishing of the fields themselves at infinity, are physically motivated. For perturbations of scalar fields, the application of the Dirichlet boundary conditions poses no problem, as these are also required by the AdS/CFT correspondence. However, for fields of higher spin, one needs to impose the boundary conditions not on the wave function \( \Psi \) of the field, but on some alternative function that has the correct interpretation in the context of the dual CFT \[54\]. The form of
the correct boundary conditions for higher-spin fields in an asymptotically AdS spacetime is a controversial, and still open, question. Since our interest in the AdS case is motivated mainly by the AdS/CFT interpretation of the quasi-normal modes, here we limit our discussion to the case of scalar fields for which the correct boundary conditions are known.

In order to find the quasi-normal modes for scalar field perturbations in the projected-on-the-brane SAdS background, we shall apply the Horowitz-Hubeny method.

Here, \( r_H \) is the largest of the zeros of the metric function \( h(r) \), that corresponds to the black hole horizon radius. On the other hand, at infinity, the Dirichlet boundary conditions, that we will use here, dictate that

\[
|\Psi(r = +\infty)| \equiv |\Psi(x = 0)| = 0 . \tag{41}
\]

The method amounts to solving the above equation, whose roots will then give us the corresponding quasi-normal frequencies \( \omega \). To this end, we need to truncate the sum \( \sum_{m=0}^{\infty} a_m (x - x_H)^m \) at some large \( m = N \), and make sure that the roots of Eq. \( \sum_{m=0}^{N} a_m (x - x_H)^m = 0 \) still converge for higher values of \( m \). Note that the larger the dimensionality \( D \) is, the larger the truncation number \( N \) needs to be. In addition, for small black holes and high overtones we need a very large \( N \) in order to find the QN modes with a good accuracy. Thus, for instance, for \( r_H \approx 0.6 \) and \( D = 6 \), \( N \approx 10^4 \).

The properties of the quasi-normal spectrum for asymptotically AdS black holes strongly depend on the size of the black hole relative to the AdS radius. Thus, QN spectra of large (\( r_H \gg R \)), intermediate (\( r_H \approx R \)) and small (\( r_H \ll R \)) black holes are totally different. QNMs of large AdS black holes are proportional to the black hole radius and therefore proportional to the temperature of the black hole. In the regime of intermediate black holes, this proportionality is broken, and, for small AdS black holes, QNMs reduce to the normal modes of the AdS spacetime when a black hole radius goes to zero. At high overtones, the quasi-normal spectrum becomes equidistant, with a spacing which is independent of the multipole number and spin of the field being perturbed. Below, we shall show that the above properties also hold for a higher-dimensional, asymptotically AdS black hole projected on the brane.

The fundamental quasi-normal frequencies for scalar field perturbations \( (\ell = 0, n = 0) \) in the vicinity of an SAdS black hole projected on the brane, are shown in Table VI, for the cases \( D = 5 \) and 7. The black hole horizon radius is taken to cover the regime from a fraction of the AdS radius to a hundred times larger than \( R \). From the displayed data, one may easily see that the QN frequency increases as the black-hole horizon increases, and that, for \( r_H \gg R \), \( \omega \) is indeed proportional to \( r_H \).

The asymptotic regime of high damping begins at relatively small overtone numbers, i.e. at \( n \approx 10 \). In Table VII, we display the first 9 overtones for a projected-on-

| Table VI: Fundamental quasi-normal frequencies \((\ell = 0, n = 0)\) for scalar field perturbations for a Schwarzschild-Anti de Sitter black hole; \( D = 5 \), 6, and 7. |
|---|---|---|
| \( r_H \) | \( D = 5 \) | \( D = 6 \) | \( D = 7 \) |
| 100 \( R \) | 262.02934 - 230.20913 i | 299.44402 - 200.49039 i | 320.32124 - 179.05396 i |
| 50 \( R \) | 131.03212 - 115.10340 i | 149.74331 - 100.24219 i | 160.18552 - 89.52188 i |
| 10 \( R \) | 26.33589 - 23.00814 i | 30.08465 - 20.02929 i | 32.17411 - 17.87798 i |
| 5 \( R \) | 13.36830 - 11.48460 i | 15.25271 - 9.95099 i | 16.30117 - 8.90885 i |
| 1 \( R \) | 3.7449 - 2.1871 i | 4.17065 - 1.84684 i | 4.39542 - 1.62536 i |
| 0.8 \( R \) | 3.4051 - 1.7056 i | 3.76193 - 1.42408 i | 3.94602 - 1.24788 i |
| 0.6 \( R \) | 3.1239 - 1.2168 i | 3.41142 - 0.99847 | 3.55367 - 0.87043 i |

| Table VII: Higher overtones for scalar field perturbations of large \((r_H = 100R)\) SAdS black hole; \( D = 5 \), 6; \( \ell = 0 \) |
|---|---|---|
| \( n \) | \( D = 5 \) | \( D = 6 \) |
| 1 | 465.01779 - 431.64399 i | 541.12945 - 374.61983 i |
| 2 | 665.99215 - 632.02215 i | 780.16118 - 547.73831 i |
| 3 | 866.46020 - 832.18457 i | 1018.54515 - 720.49533 i |
| 4 | 1066.73187 - 1032.27206 i | 1256.67421 - 893.49563 i |
| 5 | 1266.90874 - 1232.32573 i | 1494.67836 - 1066.30165 i |
| 6 | 1467.03327 - 1432.36158 i | 1732.61261 - 1239.08844 i |
| 7 | 1667.12614 - 1632.38699 i | 1970.50406 - 1411.86365 i |
| 8 | 1867.1851 - 1832.40582 i | 2208.36754 - 1584.63141 i |
| 9 | 2067.25695 - 2032.42026 i | 2446.21180 - 1757.39410 i |
the brane SAdS black hole with \( r_H = 100R \), from where we may see that the quasi-normal spectrum very quickly becomes equidistant with a spacing that depends only on the dimensionality of spacetime \( D \) and the black-hole horizon \( r_H \). From the numerical data of Table VII, we may deduce the spacing for both the real and imaginary part of the QN frequency, that come out to be

\[
\omega_{n+1} - \omega_n \approx 2(1 + i) r_H, \quad D = 5,
\]

\[
\omega_{n+1} - \omega_n \approx (2.4 + 1.7i) r_H, \quad D = 6.
\]

Let us consider next the limit of very small black-hole radius. From Ref. 22 one can learn, that the QNMs of an asymptotically AdS black hole approaches its pure AdS values (normal modes). The normal modes for scalar fields in an AdS spacetime can be found analytically. For the case of a field propagating on a brane, that is embedded in a pure AdS spacetime, the metric function takes the form

\[
h(r) = 1 - \frac{2\kappa_s^2 \Lambda r^2}{(D-1)(D-2)}, \quad \Lambda < 0,
\]

and describes an “effective” 4-dimensional Anti de Sitter spacetime with a re-scaled cosmological constant: \( \Lambda \to 6\Lambda/(D-1)(D-2) \). Therefore, the normal modes for brane-localised scalar field perturbations can follow immediately from the corresponding analysis of the purely 4-dimensional case 23, and are given by:

\[
\omega_{\text{scalar}} = \sqrt{\frac{2\kappa_s^2 |\Lambda|}{(D-1)(D-2)}} (2n + \ell + 3).
\]

For instance, for the particular case of \( n = 0 \) and \( D = 5 \), the above formula leads to the result: \( \omega_{n=0} \approx 3 \). Then, from the entries of Table VI, one may clearly see that, as the black-hole radius decreases, the QN frequency asymptotes to the same value. Thus, the QN modes of scalar fields, in the vicinity of a very small asymptotically AdS black hole projected on the brane, approach indeed the normal modes of the pure AdS spacetime.

In Fig. 4, we depict the dependence of the QNM frequency spectrum on the multipole number \( \ell \). In contrast to what happens in the asymptotically flat case, where the imaginary part of the QN frequency was approaching a constant value in the limit \( \ell \to \infty \), in the case of a SAdS black hole, the damping rate is decreasing when \( \ell \) is growing, therefore higher multipoles should decay more slowly. Yet, as was mentioned in 30, this does not pose a problem for AdS/CFT correspondence since a decomposition according to multipole numbers can be done in the dual CFT as well.

Finally, another deviation – from the behaviour noted in the previously studied cases – is observed here, this time in respect to the dependence of the QN spectrum on the dimensionality of spacetime. According to our results, in the case of a projected-on-the-brane SAdS black hole, when the number of hidden, transverse dimensions increases, the imaginary part of the QN frequencies is decreasing, while the real part of \( \omega \) is increasing. We remind the reader that in the case of asymptotically flat or de Sitter black holes, both \( \omega_{\text{Re}} \) and \( \omega_{\text{Im}} \) grow when \( D \) is increasing (see, for example, Fig. 1).

\[\text{Figure 4: Dependence of the real (diamond) and imaginary (star) part of QNMs on multipole number } \ell, \text{ for } n = 0, r_H = 1 \text{ and } D = 6. \text{ Scalar field perturbations of SAdS black hole.}\]

\[\text{VII. CONCLUSIONS}\]

The theories postulating the existence of extra, compact spacelike dimensions have opened the way for low-scale gravitational theories and the observation of strong gravitational phenomena such as the creation of tiny, higher-dimensional black holes in the controlled environment of a ground-based particle accelerator. Standard Model fields, that are restricted to live on the 4-dimensional brane and happen to propagate in the vicinity of such a black hole, will feel only the projected-on-the-brane black-hole background. In such a situation, the observation of their QN frequencies may be highly more likely to take place than the one for the diligently pursued, but up to now elusive, gravitons. Such a detection will provide not only evidence for the existence of the black holes themselves, but also information on the fundamental parameters of the higher-dimensional theory.

In this work, we have investigated a variety of spherically-symmetric \( D \)-dimensional black-hole backgrounds, that are then projected onto the 4-dimensional brane. The induced-on-the-brane backgrounds depend on a single metric function \( h(r) \), that carries a signature of the parameters of the \( D \)-dimensional theory, such as the dimensionality \( D \), charge \( Q \) and cosmological constant \( \Lambda \). The same function characterizes the form of the effective potentials felt by the scalars, fermions and gauge bosons propagating in the brane background. In 3 of the cases studied (Schwarzschild, Reissner-Nordström and Schwarzschild-de Sitter), the effective potentials had the form of positive-definite barriers, a result that allowed
us to use the WKB method to derive the QN spectrum in the 6th order beyond the eikonal approximation. In the 4th case (Schwarzschild-Anti de Sitter), the effective potential diverged at infinity, and the Horowitz-Hubeny method was used instead.

In the case of a Schwarzschild black-hole background, the QN spectrum of all SM fields was computed for various values of the dimensionality of spacetime $D$ and multipole number $\ell$. It was found that, as the number of hidden, extra dimensions increased, the imaginary part of the QN frequency for all species increased as well, thus making the ring-down phase on the brane shorter. The real part of the QN frequency was also found to be $D$-dependent and to predominantly increase with $D$, although particular modes with $\ell \approx n$ may deviate from this behaviour. In respect to the dependence on the spin, fields with higher spin were found to damp with a slower rate and thus to survive longer.

If a charge $Q$ is present in the bulk background, the QN frequency spectrum is found to significantly deviate from the one of purely 4-dimensional ones, and to resemble more the one in the vicinity of a $D$-dimensional Gauss-Bonnet black hole: an increase in the charge of the black hole was found to lead to a monotonically decreasing behaviour for the imaginary part of the QN frequency – and thus to a longer ring-down phase – and to a monotonically increasing behaviour for the real part, rendering all SM fields much better oscillators than in the neutral case.

When a positive cosmological constant is turned on in the bulk, the resulting QN spectrum of the SM fields on the brane does not yield any surprises. As $\Lambda$ increases, both the real and imaginary part of the QN frequency are suppressed, and for $\Lambda = \Lambda_{cri}$, the suppression reaches the magnitude of 90%. The number of dimensions enhances again the individual values of $\omega_{Re}$ and $\omega_{Im}$, however the relative suppression as $\Lambda$ varies comes out to be only mildly $D$-dependent. In the extreme limit, the feature of fields with different spin decaying with almost the same rate is observed also in the present case of brane-localised fields in an effective SdS background.

In the case of a negative cosmological constant being present in the bulk, the spectrum of QN frequencies was shown to depend on the ratio of the black-hole horizon to the AdS radius, similarly to the case of purely 4-dimensional or $D$-dimensional SAdS backgrounds. For large black holes, the QN spectrum comes out to be proportional to $r_H$, and to become equidistant for the higher overtones. In the opposite limit of a very small black hole, the QN frequencies approach the normal modes of the projected-on-the-brane AdS spacetime.

Summarizing our results, we may say that the existence of extra dimensions affects the QNM spectrum of fields living on the brane both in a direct and an indirect way. In all the cases studied here, an increase in the number of transverse-to-the-brane dimensions causes a significant increase in the imaginary part of all fields, thus directly affecting the damping rate of the field perturbations on the brane. Additional features, such as the distance in the frequency spacing between successive quasi-normal modes in the SAdS spacetime, or asymptotic values of QN frequencies in various limits and backgrounds, also depend explicitly on the total number of dimensions $D$. In an indirect way, the localization of fields on a brane embedded in a higher-dimensional black-hole background leads to a deviation from the behaviour observed either in the purely 4-dimensional or in a purely $D$-dimensional case: the monotonic behaviour of the QN spectrum as a function of the charge $Q$, for the majority of the modes, is an indicative example of this.

In this work, we have restricted our analysis to the study of spherically-symmetric $D$-dimensional black-hole backgrounds projected on the brane. The study of axially-symmetric, rotating black-hole backgrounds has already been initiated and we hope to report our results soon in a follow-up article. In addition, here we were limited by the consideration of quasi-normal spectra for only massless fields, yet, as was shown in [3], for scalars, and in [30] for vector fields, the mass term can change the lower modes of the spectrum considerably, leaving unaffected the high damping limit of the spectrum – the study of the effect of the mass term on the QN spectrum of brane-localized fields in various backgrounds is also among our future plans.

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[1] H. P. Nollert, Class. Quantum Grav. 16, R159 (1999).
[2] K. D. Kokkotas and B. G. Schmidt, Living Rev. Rel. 2, 2 (1999) gr-qc/9909058.
[3] P. R. Brady, C. M. Chambers, W. G. Laarakkers and E. Poisson, Phys. Rev. D 60, 064003 (1999) gr-qc/9902010.
B. Wang, E. Abdalla and R. B. Mann, Phys. Rev. D 65, 084006 (2002) hep-th/0107243.

V. Suneeta, Phys. Rev. D 68, 024020 (2003) gr-qc/0303113.
A. Zhidenko gr-qc/0510039.
J. x. Tian, Y. x. Gui, G. h. Guo, Y. Lv, S. h. Zhang and W. Wang, Gen. Rel. Grav. 35, 1473 (2003) gr-qc/0304009.
C. Molina, D. Giugno, E. Abdalla and A. Saa, Phys. Rev. D 69, 104013 (2004) gr-qc/0309079.
