Branes and Categorifying Integrable Lattice Models

Meer Ashwinkumar*, Meng-Chwan Tan†, and Qin Zhao‡

Department of Physics
National University of Singapore
2 Science Drive 3, Singapore 117551

Abstract

We elucidate how integrable lattice models described by Costello’s 4d Chern-Simons theory can be realized via a stack of D4-branes ending on an NS5-brane in type IIA string theory, with D0-branes on the D4-brane worldvolume sourcing a holomorphic RR 1-form, and fundamental strings forming the lattice. This provides us with a nonperturbative integration cycle for the 4d Chern-Simons theory, and by applying T- and S-duality, we show how the R-matrix, the Yang-Baxter equation and the Yangian can be categorified, that is, obtained via the Hilbert space of a 6d gauge theory.

*E-mail: meerashwinkumar@u.nus.edu
†E-mail: mctan@nus.edu.sg
‡E-mail: phyzhq@nus.edu.sg
1 Introduction and summary

The Yang-Baxter equation with spectral parameter was recently found to arise from a 4d variant of Chern-Simons gauge theory devised by Costello [1, 2, 3], with the action

\[ S = \frac{1}{\hbar} \int_{Y \times \Sigma} C \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \]  

(1.1)

where \( A \) is a complex-valued gauge field, \( Y \) is a framed 2-manifold, and \( \Sigma \) is a complex Riemann surface endowed with a holomorphic one-form \( C = C(z) dz \), which can have poles but no zeros. Within the realm of perturbation theory, this gauge theory encapsulates the underlying structure of integrable lattice models of two-dimensional classical statistical mechanics.

Notice that the action

\[ S = \frac{1}{\hbar} \int_M C \wedge \text{Tr} \left( \mathcal{F} \wedge \mathcal{F} \right) \]  

(1.2)

where \( Y \times \Sigma = \partial M \) for some 5-manifold, \( M \), can give us (1.1) via Stoke’s theorem. This will allow us to categorify the Yang-Baxter equation, as an analogous recasting of the 3d Chern-Simons action leads to the categorification of knot polynomials in terms of Khovanov homology via string theory [4].

Outside of perturbation theory, the 4d Chern-Simons theory (1.1) is not well-understood (its path integral is exponentially divergent), and it was suggested in [5] that a nonperturbative definition of the theory should arise from the D4-NS5 brane system of type IIA string theory, in a manner similar to how analytically-continued 3d Chern-Simons theory can be given a nonperturbative definition via the D3-NS5 system.

Our aim in this work is to firstly verify the suggestion in [5], and to derive the integration cycle which allows (1.1) to be well-defined beyond perturbation theory; this shall be done in Section 2.
Secondly, we will categorify the elements of the R-matrix which solves the aforementioned Yang-Baxter equation, thereby categorifying the Yang-Baxter equation itself, and we shall furthermore categorify the Yangian; this shall be done in Section 3, where we apply T- and S-duality to arrive at an NS5-D5 system in type IIB string theory.

A brief summary of our results is as follows. We shall first show that the D4-NS5 system with a holomorphic RR 1-form is equivalent to Costello’s theory (2.25) with nonperturbative integration cycle defined by (2.26). Using T- and S-duality, we arrive at the NS5-D5 brane system, the supersymmetric Hilbert space of which is defined by the Floer cohomology of the 6d equations (3.3) which interpolate solutions of the 5d equations (3.4). Via these dualities, we are able to express each R-matrix element in terms of a trace over this Hilbert space in (3.6) (thereby categorifying each R-matrix element), and we are also able to categorify the Yang-Baxter equation, as shown in (3.8), as well as the Yangian.

2 4d Chern-Simons theory from 5d topologically twisted MSYM coupled to holomorphic RR 1-form

2.1 D4-brane worldvolume theory

We concern ourselves with the worldvolume theory of $N$ D4-branes wrapping a flat (Euclidean) manifold, $M$, in this section. This theory has $U(N)$ gauge symmetry as well as $\mathcal{N} = 2$ supersymmetry. In addition, it has $SO(5)$ as its rotation symmetry group as well as its R-symmetry group, and we shall use $SO_E(5)$ and $SO_R(5)$ to denote them, respectively. The worldvolume theory has the action

$$S = \frac{1}{g_5^2} \int_M d^5x \; \text{Tr} \left( \frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \phi_X D^M \phi_Y + \frac{1}{4} [\phi_X, \phi_Y] [\phi_X, \phi_Y] - \overline{\rho}^{\alpha j} (\Gamma^M)_a^b D_M \rho_{\alpha j} 
- \overline{\rho}^{aj} (\Gamma^M)_j^k [\phi_X, \rho_{ak}] \right),$$

(2.1)

where $SO_E(5)$ spinor indices are denoted $(a, b, \ldots)$ and $SO_R(5)$ spinor indices are denoted $(j, k, \ldots)$.

We shall twist this worldvolume theory (as first shown by Geyer and M"ulsch [6]) by redefining the rotation group to be $SO_E(5)' \subset SO_E(5) \times SO_R(5)$, where $SO_E(5)'$ is the diagonal subgroup. This is equivalent to the identification of the indices $(j, k, \ldots)$ and $(a, b, \ldots)$, and the indices $(X, Y, \ldots)$ and $(M, N, \ldots)$. The twisted action is given by

$$S = \frac{1}{g_5^2} \int_M d^5x \; \text{Tr} \left( \frac{1}{4} \overline{F}_{MN} \overline{F}^{MN} + \frac{1}{2} D_M \phi^M D_N \phi^N - i \overline{\rho}^{ac} (\Gamma^M)_a^b D_M \rho_{bc} 
- \overline{\rho}^{ab} (\Gamma^M)_b^c [\phi_M, \rho_{ac}] \right),$$

(2.2)

where $\phi_M$ has been used to complexify the gauge field $A_M$, i.e.,

$$A_M = A_M + i \phi_M, \quad \overline{A}_M = A_M - i \phi_M,$$

(2.3)

with the corresponding field strengths denoted as $\overline{F}_{MN}$ and $\overline{F}^{MN}$. In addition, we can perform
the decomposition
\[ \rho_{ab} = \frac{1}{2 \sqrt{2}} \left( \frac{1}{2} (\Gamma^{MN})_{ab} \chi_{MN} + (\Gamma^M)_{ab} \psi_M - \Omega_{ab} \eta \right), \quad \rho^{ab} = \Omega^{ac} \Omega^{bd} \rho_{cd}, \] (2.4)

where \( \eta \) is a scalar field, \( \psi_M \) a vector field, and \( \chi_{MN} \) an antisymmetric tensor field, and where \( \Omega^{ab} \) is a real, antisymmetric tensor used to lower and raise \( Sp(4) \) indices. This leads us to the twisted action in the form
\[ S = \frac{1}{g^2_5} \int_M d^5x \ Tr \left( \frac{1}{4} \mathcal{F}_{MN} \mathcal{F}^{MN} + \frac{1}{2} D_M \phi^M D_N \phi^N - i \chi^M \mathcal{D}_N \psi - i \psi^M \overline{\mathcal{D}}_M \eta \right) - \frac{i}{8} \epsilon^{MNPQR} \chi_{QR} \overline{\mathcal{D}}_P \chi_{MN}, \] (2.5)

with the complex covariant derivatives defined as
\[ \mathcal{D}_M \equiv \partial_M + [\mathcal{A}_M, \cdot], \]
\[ \overline{\mathcal{D}}_M \equiv \partial_M + [\overline{\mathcal{A}}_M, \cdot]. \] (2.6)

We can utilize an auxiliary field, \( d \), to obtain
\[ S = \frac{1}{g^2_5} \int_M d^5x \ Tr \left( \frac{1}{4} \mathcal{F}_{MN} \mathcal{F}^{MN} - \frac{1}{2} d^2 - dD_M \phi^M - i \chi^M \mathcal{D}_N \psi - i \psi^M \overline{\mathcal{D}}_M \eta \right) - \frac{i}{8} \epsilon^{MNPQR} \chi_{QR} \overline{\mathcal{D}}_P \chi_{MN}, \] (2.7)

which is invariant under the following supersymmetry transformations:
\[ \mathcal{Q} \mathcal{A}_M = \psi_M, \]
\[ \mathcal{Q} \psi_M = 0, \]
\[ \mathcal{Q} \overline{\mathcal{A}}_M = 0, \]
\[ \mathcal{Q} \chi_{MN} = -i \mathcal{F}_{MN}, \]
\[ \mathcal{Q} \eta = d, \]
\[ \mathcal{Q} d = 0, \] (2.8)

where the scalar supercharge, \( \mathcal{Q} \), satisfies \( \mathcal{Q}^2 = 0 \). Then, we can write the action in terms of a \( \mathcal{Q} \)-exact part and a topological part, i.e.,
\[ S = \mathcal{Q} \Psi - \frac{1}{g^2_5} \int_M d^5x \ Tr \left( \frac{i}{8} \epsilon^{MNPQR} \chi_{QR} \overline{\mathcal{D}}_P \chi_{MN} \right), \] (2.9)

where \( \Psi \) is given by
\[ \Psi = \frac{1}{g^2_5} \int_M d^5x \ Tr \left( \frac{i}{4} \chi^M \mathcal{F}_{MN} - \frac{1}{2} \eta d - \eta D_M \phi^M \right). \] (2.10)

It then becomes clear that the action is \( \mathcal{Q} \)-invariant, using the Bianchi identity
\[ \epsilon^{MNPQR} \overline{\mathcal{D}}_P \mathcal{F}_{QR} = 0, \] (2.11)
as well as the nilpotency of $Q$. In addition, upon examining the supersymmetry transformations (2.8), we find that the path integral of the twisted 5d theory localizes to the space of solutions of $\mathcal{F}_{MN} = 0$ (flat complexified connections) and $d = -D_M \phi^M = 0$, which are the bosonic fixed points of the transformations.

We can also include a topological term in the action, which couples to the RR 1-form, $C$, sourced by D0-branes in the worldvolume, i.e.,

$$S_{\text{top}} = \frac{1}{g_5^2} \int_M C \wedge \text{Tr} (F \wedge F).$$

(2.12)

As long as the worldvolume has no boundaries, this term is invariant under general variations, including supersymmetry variations.

In the following sections, we shall take the worldvolume to contain a Riemann surface $\Sigma$, and (with a choice of complex structure on $\Sigma$) we shall consider only D0-branes which are charged such that $C$ has holomorphic dependence on $\Sigma$, i.e., $C = C(z) dz$.

### 2.2 NS5-brane boundary conditions

The 5d worldvolume theory admits boundary conditions corresponding to the D4-branes ending on an NS5-brane. Let the flat worldvolume contain the half-line $\mathbb{R}_+$, denoted by the coordinate $x^5 = w$. Without the topological term (2.12), the bosonic boundary conditions for the untwisted action (2.1) are

$$F_{\mu w} |_{w=0} = 0, \quad D_w \phi^{1,2} |_{w=0} = 0, \quad \phi^{3,4,5} |_{w=0} = 0,$$

(2.13)

while the fermionic fields obey projection conditions (here, $\mu$ is a 4d index). The twisting of the worldvolume theory is unaffected by the presence of boundaries. However, in order to put the twisted action in the form (2.9), we have to perform integration by parts on the term proportional to $\psi_M \nabla_M \eta$ in (2.5), which gives rise to a boundary term, and which vanishes via the boundary condition $\eta = 0$.

The inclusion of the topological term (2.12) will require additional boundary interactions in order to preserve supersymmetry. The boundary interactions take the form,

$$S_{\text{boundary}} = \frac{1}{g_5^2} \int_{\partial M} d^4 x \text{Tr} (-C_\mu \epsilon^{\mu \nu \rho \sigma} 2i \phi_\nu F_{\rho \sigma} - C_\mu \epsilon^{\mu \nu \rho \sigma} 2\phi_\nu D_\rho \phi_\sigma)$$

(2.14)

The supersymmetry invariance of the sum of the topological term and these interaction terms is obvious since $C$ is a closed form, because in this case, the topological term (2.12) can be substituted by the boundary term

$$\frac{1}{g_5^2} \int_{\partial M} C \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

(2.15)

whereby the sum of (2.15) and (2.14) can be written as the boundary term

$$\frac{1}{g_5^2} \int_{\partial M} C \wedge \text{Tr} \left( \overline{A} \wedge d\overline{A} + \frac{2}{3} \overline{A} \wedge \overline{A} \wedge \overline{A} \right),$$

(2.16)

which is $Q$-invariant since $Q \overline{A} = 0$.

In summary, the $Q$-invariant topological action in the presence of boundaries takes the form (2.9) + (2.16), and the boundary conditions on the fields are $\eta = 0$ and those implied by (2.13).
2.3 $M = Y \times \Sigma \times \mathbb{R}_+$ with holomorphic $C$-field

We shall now specialize to the case where $M = Y \times \Sigma \times \mathbb{R}_+$, where $Y$ is a framed 2-manifold, the most important examples (in the context of the Yang-Baxter equation) that we will consider being $T^2$ and $\mathbb{R}^2$, and where $\Sigma$ is a Riemann surface which is either $\mathbb{C}$, $\mathbb{C}^\times$ or $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. These three choices of Riemann surfaces will eventually correspond to rational, trigonometric and elliptic integrable lattice models. With the manifold $M$ and one-form $C = C(z)dz$, we investigate the functional integral of the topological theory, where the bosonic boundary conditions (2.13) now read

$$F_{\mu\nu}|_{w=0} = 0, \quad D_w\phi_{x,y}|_{w=0} = 0, \quad \phi_{z,\overline{z}}|_{w=0} = 0,$$

(2.17)

having labelled the local coordinates on $Y$ as $(x,y)$ and the local complex coordinates on $\Sigma$ as $(z,\overline{z})$. Since this functional integral localizes to the fixed points of the fermionic symmetry, it will be convenient to evaluate it by expanding in perturbation theory around these fixed points. This shall be done by rescaling the $Q$-exact part of the action by a real parameter $l$, which we shall take to be very large, while expanding all fields in terms of their fixed point values plus fluctuations around these fixed points multiplied by some inverse power of $l$.

However, before doing so, we shall reexpress our action in a convenient form by noting that the topological twist discussed in the previous subsection is equivalent (in the case of $M = Y \times \Sigma \times \mathbb{R}_+$) to a topological twist along $Y \times \Sigma$ only, since $\mathbb{R}_+$ is flat. This is a twist of the $SO(4)$ subgroup of the rotation group corresponding to local rotations along $V = Y \times \Sigma$, and ought to result in fields transforming as representations of a twisted rotation group $SO(4)'$.

Let us further elucidate this twist, starting with the twist of the scalar fields. These fields transform in the 5 of the $SO(5)$ R-symmetry group. We shall pick the $SO(4)$ R-symmetry group of $\phi^1$, $\phi^2$, $\phi^3$ and $\phi^4$ for twisting. These fields form a 4 under $SO(4)_R$, and performing the twist gives us the field $\phi_\mu$ which transforms as the 4 of the $SO(4)'$ rotation group of $V = Y \times \Sigma$. In addition, the fermions of the 5d MSYM transform as $(4,4)$ under $SO(5)_E \times SO(5)_R$. The aforementioned breaking to $SO(4)_E \times SO(4)_R$, results in each 4 reducing to $2 \oplus \overline{2}$. Twisting amounts to taking the tensor product

$$(2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}) = 1 \oplus 3 \oplus 4 \oplus 1 \oplus 3 \oplus 4.$$  

(2.18)

Thus, we can write

$$\overline{A}_5 = A_5 - iB, \quad A_5 = A_5 + iB, \quad \chi_{\mu5} = \overline{\psi}_\mu, \quad \psi_5 = \overline{\eta}$$

(2.19)

to obtain fields in the aforementioned representations of $SO(4)'$ (in particular, the fermionic field $\chi_{\mu\nu}$ transforms as $3 \oplus \overline{3}$, i.e., the antisymmetric tensor representation, while the remaining fermions
Since the theory is topological, we can also rescale the inverse of metric as

$$g^{55} \rightarrow i \psi^5 g^{55} - i \eta \partial_5 \eta g^{55} - i \partial_5 \eta \eta g^{55}$$

(2.24)

Taking $l \rightarrow \infty$, the total action becomes

$$S = \frac{1}{g_5^2} \int_M d^5 x \text{Tr} (\bar{F}_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (\bar{D}_\mu (A_5 - i B) - \partial_5 \bar{A}_\mu) (D^\mu (A_5 + i B) - \partial_5 A^\mu) g^{55} - d \eta g^{55}$$

(2.20)

and the SUSY transformations that leave it invariant are

$$Q A_\mu = \psi_\mu,$$

$$Q \bar{A}_\mu = 0,$$

$$Q (A_5 + i B) = \bar{\eta},$$

$$Q (A_5 - i B) = 0,$$

$$Q \psi_\mu = 0,$$

$$Q \bar{\psi}_\mu = -i (\bar{D}_\mu (A_5 - i B) - \partial_5 \bar{A}_\mu),$$

$$Q \chi_{\mu \nu} = -i F_{\mu \nu},$$

$$Q \eta = d,$$

$$Q \bar{\eta} = 0,$$

$$Q d = 0.$$
and we can therefore perform the functional integrals over the fluctuations to obtain a constant that is independent of $\mathcal{A}_0$, which can be absorbed into the measure for $\mathcal{A}_0$.\(^1\)

Hence, we are left with the functional integral

$$
\int_{\Gamma} \mathcal{D}\mathcal{A}_0 e^{-\frac{i}{g^2} \int_{\partial M} C \wedge \text{Tr} \left( \mathcal{A}_0 \wedge d\mathcal{A}_0 + \frac{2}{3} \mathcal{A}_0 \wedge \mathcal{A}_0 \wedge \mathcal{A}_0 \right)}
$$

(2.25)

This is the partition function of Costello’s theory with gauge group $U(N)_C = GL(N, \mathbb{C})$ (where the Planck constant $\hbar = g^2$), with the functional integral performed over a nonperturbative integration cycle $\Gamma$ in field space corresponding to the boundary values of the solutions of

$$
\begin{align*}
F_{0MN} &= 0 \\
D_{0M} f_0^M &= 0
\end{align*}
$$

(2.26)

(which have been written from the point of view of the equivalent fully twisted 5d theory), with boundary conditions given in (2.17).

### 2.4 Wilson lines

Now, it is a fact that $Q\mathcal{A} = 0$ also allows us to define supersymmetric Wilson lines, i.e.,

$$
W = \text{Tr}(P(e^{\int \mathcal{A}})),
$$

(2.27)

as observables of the 5d theory. From the point of view of string theory, these Wilson lines arise from the worldsheet boundaries of fundamental strings ending on the D4-brane worldvolume. Now, in [2], a general class of Wilson lines were considered when $Y = \mathbb{R}^2$, i.e., not only those associated with representations of $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$, but also representations of $\mathfrak{g}[[z]]$. These Wilson lines associated with representations of the latter algebra (the Lie algebra of polynomial loops of $\mathfrak{g}$) are necessary in deriving the Yangian algebra associated with rational integrable lattice models. Such a Wilson line associated with $\mathfrak{g}[[z]]$ is constructed by starting with a Wilson line along $Y$ in representations of $\mathfrak{g}$, and giving holomorphic dependence on $\Sigma$ to the gauge field in the operator, while also removing the trace (gauge invariance is maintained by insisting that the gauge field vanishes at infinity along $\mathbb{R}^2$). In the string picture, such a Wilson line ought to be realized by an identical modification to the Wilson line along the boundary of a fundamental string worldsheet, i.e., choosing a background gauge field with holomorphic $z$-dependence, and which vanishes at infinity along $\mathbb{R}^2$.

Let us consider the functional integral of the topological theory with Wilson lines associated with representations of $\mathfrak{g}$ along $Y \subset \partial M$ forming a lattice. Localization of the path integral along the lines of the previous subsection leads us to Costello’s 4d Chern-Simons theory with Wilson line insertions in the path integral, i.e.,

$$
\int_{\Gamma} \mathcal{D}\mathcal{A}_0 \prod_i \text{Tr}(P(e^{\int_{\mathcal{L}_i} \mathcal{A}_0})) e^{-\frac{i}{g^2} \int_{\partial M} C \wedge \text{Tr} \left( \mathcal{A}_0 \wedge d\mathcal{A}_0 + \frac{2}{3} \mathcal{A}_0 \wedge \mathcal{A}_0 \wedge \mathcal{A}_0 \right)}
$$

(2.28)

\(^1\)We have assumed here that there are no fermionic zero modes. This is a reasonable assumption, given that the NS5 boundary conditions consist of Dirichlet and Neumann boundary conditions which are both elliptic, and the index of the fermionic operators ought to be proportional to $\chi(M)\dim G$, which vanishes for our present choice of $M$. 

8
and this correlation function is identified with the partition function of an integrable lattice model. This is because contracting R-matrices (which correspond to intersections of Wilson lines) and taking a trace gives a transfer matrix, and similarly contracting transfer matrices and taking a trace gives us the partition function.

3 Categorification of R-matrix elements

3.1 T-duality as a lift to 6d

To categorify the R-matrix elements and the Yang-Baxter equation with spectral parameter, we need to lift our 5d twisted theory to a 6d one. Our 5d theory was a twist of a theory with 16 supercharges, and therefore we expect the 6d lift to also have 16 supercharges. However, the R-symmetry group for 6d \( \mathcal{N} = (1, 1) \) SYM is \( SO(4) \) and thus, the maximal twist is with a \( SO(4) \) subgroup of the \( SO(6) \) rotation group. This lift is consistent at the level of topological field theories as the 5d theory in the previous subsection was also twisted by \( SO(4) \) only.

To carry out the lift, we note firstly that the 6d \( \mathcal{N} = (1, 1) \) theory has four scalar fields, which transform as the \( 4 \) of \( SO(4)_R \). Twisting these fields with the \( SO(4)_V \) subgroup of \( SO(6)_E \), we obtain the field \( \phi_\mu \) which transforms as a \( 4 \) of \( SO(4)' \). The fermionic fields of the \( \mathcal{N} = (1, 1) \) theory are in the representations \( (\bar{4}, 2) \oplus (\bar{4}, \bar{2}) \) of \( SO(6)_E \times SO(4)_R \). Breaking \( SO(6)_E \) to \( SO(4) \), \( 4 \) and \( \bar{4} \) each reduce to \( \bar{2} \oplus 2 \). Twisting these fields then gives us the representations \( 1 \oplus 3 \oplus 4 \oplus 1 \oplus \bar{3} \oplus 4 \), which is identical to those on the RHS of (2.18). Hence, the field content of the 5d and 6d twisted theories are almost the same, except that in going from the former to the latter, the field \( B \) ought to be replaced by the gauge field component \( A_6 \). We may then directly lift the theory from 5d to 6d, by converting the gauge field components \( A_6 \) to covariant derivatives \( D_6 \). This gives us the action

\[
S_{6d} = \frac{1}{g_5^2(2\pi r)} \int_{S^1 \times M} d^6x \text{Tr}(\mathcal{F}_{\mu\nu} F_{\mu\nu} + \mathcal{F}_{5\nu} F_{5\nu} + \mathcal{F}_{6\nu} F_{6\nu} + i\mathcal{F}_{5\nu} F_{6\nu} - i\mathcal{F}_{6\nu} F_{5\nu} - \frac{1}{2} d(D) - \frac{1}{2} d^2 \\
+ i\bar{\psi}^\mu (D_5 + iD_6) \psi^\mu - i\bar{\psi}^\mu D_\mu \eta - i\psi^\mu D_\mu \bar{\eta} = \int_{S^1 \times M} d\theta \wedge C \wedge \text{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})
\]

where \( x^6 = \theta \) is the coordinate along \( S^1 \), which has radius \( r \). This can be put in \( Q \)-exact form plus two topological terms

\[
S_{6d} = \frac{1}{g_5^2(2\pi r)} \int d^6x \left\{ Q \left( \chi_{\mu\nu} F^{\mu\nu} + 2\chi_{\mu5} [D^\mu, D_5 + iD_6] - \frac{i}{2} \eta [D_\mu, D^\mu] - \eta [D_5, D_6] - \frac{1}{2} \eta d \right) \\
- \frac{i}{2} \epsilon^{\mu\nu\rho\gamma} \chi_{\rho\gamma} [D_\nu \psi^\mu, \bar{D}_\nu \bar{\psi}^\mu] - \frac{i}{8} \epsilon^{\mu\nu\rho\gamma} \chi_{\rho\gamma} (D_5 - iD_6) \chi_{\mu\nu} \right\} \\
+ \frac{1}{g_5^2(2\pi r)} \int_{S^1 \times M} d\theta \wedge C \wedge \text{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})
\]
where the equation of motion of $d$ is $d = -\frac{i}{2}[D_\mu, \overline{T}^\mu] - [D_5, D_6]$, and the $Q$-transformations which leave the action invariant are

$$
\begin{align*}
Q A_\mu &= \psi_\mu, \\
Q \overline{A}_\mu &= 0, \\
Q (A_5 + A_6) &= \tilde{\eta}, \\
Q (A_5 - A_6) &= 0, \\
Q \psi_\mu &= 0, \\
Q \tilde{\psi}_\mu &= -i(\partial_\mu A_5 - D_5 \overline{A}_\mu) - (\partial_\mu A_6 - D_6 \overline{A}_\mu), \\
Q \chi_{\mu\nu} &= -i \mathcal{F}_{\mu\nu}, \\
Q \eta &= d, \\
Q \tilde{\eta} &= 0, \\
Q d &= 0.
\end{align*}
$$

(3.2)

From the perspective of type IIA string theory, we are studying D4-branes wrapping the $V \times \mathbb{R}$ subspace of $S^1 \times T^* V \times \mathbb{R}$, with an NS5-brane at the origin of $\mathbb{R}$; the twist follows essentially because $V$ is the zero section of the cotangent bundle $T^* V$, and therefore ‘coordinates’ normal to $V$ in $T^* V$ must be components of one-forms [7]. The dimensional lifting we have just discussed is understood as T-duality along $S^1$ to obtain the D5-NS5 brane system in type IIB theory, where the factor of $2\pi r$ which multiplies $g_5^2$ is identified with $2\pi \sqrt{\alpha'}$, where $\sqrt{\alpha'}$ is the string scale, i.e., $g_6^2 = g_5^2 2\pi \sqrt{\alpha'}$.

### 3.2 S-duality

We now go one step further, and study the S-dual of this system, whereby D5 and NS5-branes are exchanged. We note that both D5-branes and NS5-branes in Type IIB string theory are described at low energy by $\mathcal{N} = (1, 1)$ Super Yang-Mills. Hence, the dual theory is also a twist of $\mathcal{N} = (1, 1)$ Super Yang-Mills, but one which lives on a stack of NS5-branes ending on a D5-brane. In addition, the F-strings which give rise to Wilson lines become D1-branes which are represented by ’t Hooft lines. Next, we note that before S-duality, the coupling of the 6d theory on the D5-brane is $g_5^2 (2\pi \sqrt{\alpha'}) = (2\pi)^3 g \alpha'$, where $g$ is the string coupling, and thus after S-duality, the coupling of the 6d theory on the NS5-brane is $(2\pi)^3 \alpha'/g$. In other words, the action of the S-dual theory is multiplied by $g / (2\pi)^3 \alpha' = g_5^2 / (2\pi)^3 (\alpha')^{3/2}$. Finally, the RR 2-form $d\theta \wedge C$ in the D5-brane worldvolume is identified with an NS-NS 2-form in the NS5-brane worldvolume theory.

We shall use this S-dual 6d theory to categorify the R-matrix elements, thereby providing a categorification of the Yang-Baxter equation with spectral parameter. In other words, we want to be able to describe the R-matrix elements in terms of the Hilbert space of a 6d theory. The reason for using the S-dual theory is that the lattice on $Y$ is now described by intersecting ’t Hooft lines, which will allow us to easily write the path integral as a trace over a Hilbert space without explicit operator insertions. This theory is described by the twisted 6d theory studied in the previous subsection (with the transformations mentioned in the previous paragraph), but with singular field configurations for the complexified gauged fields, represented (in the case of two perpendicularly intersecting ’t Hooft lines) by replacing the field strength $F$ in the action by
\( \mathcal{F}' = \mathcal{F} - 2\pi \alpha_1 \delta_1 - 2\pi \alpha_2 \delta_2 \) (where \( \delta_1 \) and \( \delta_2 \) are proportional to Dirac delta two-forms which characterize the singularities along the 't Hooft lines, and \( \alpha_1 \) and \( \alpha_2 \) are valued in the Cartan of the representations dual to that of the Wilson lines), as well as the corresponding replacement for the gauge field \( A \).

### 3.3 Hilbert Space of 6d Theory and categorification

The BPS equations to which this theory localizes will be ‘ramified’ versions of those of the original 6d theory on the D5-branes, which can be read off from (3.2), i.e.,

\[
\begin{align*}
-i(\partial_\mu A_5 - D_5 A_6) - (\partial_\mu A_6 - D_6 A_5) &= 0 \\
\mathcal{F}'_{\mu\nu} &= 0 \\
\frac{i}{2}[\mathcal{D}'_\mu, \mathcal{D}'_\nu] + [D_5, D_6] &= 0
\end{align*}
\]

(3.3)

Let us now use these equations to describe the Hilbert space of the S-dual theory, taking the sixth dimension \( S^1 \) to be the Euclidean time dimension. We shall first find the space of classical ground states, which in the present case is a time-independent classical solution of the six-dimensional equations given in (3.3). That is, a classical ground state would solve the 5d equations

\[
\begin{align*}
-i(\partial_\mu A_5 - D_5 A_6) - (\partial_\mu A_6 - D_6 A_5) &= 0 \\
\mathcal{F}'_{\mu\nu} &= 0 \\
D'_\mu \phi^\mu + D_5 A_6 &= 0
\end{align*}
\]

(3.4)

which are the familiar BPS equations from our study of the D4-brane worldvolume theory. We shall assume for simplicity in what follows that these equations have a finite set of solutions, and that they are all nondegenerate (i.e., when expanding around a given solution, there are no bosonic zero modes). The latter condition implies that expansion around a such a solution gives a single quantum state of zero energy, at least perturbatively.

We shall now take into account quantum corrections to the classical spectrum. Firstly, it is known that nonzero energy eigenstates of a supersymmetric Hamiltonian always occur in pairs. In perturbation theory, the supersymmetric spectrum is unaffected, because we always expand around a single approximate ground state, which is obtained by quantizing the corresponding classical solution, and hence because in perturbation theory only this single state is accessible, it is impossible for it to pair up with another state to leave the spectrum of supersymmetric ground states.

However, nonperturbatively, quantum tunnelling via ‘instantons’ between such approximate ground states can lift a pair of ground states to a pair of excited states. The ‘instantons’ in our case are solutions to the 6d equations (3.3), and interpolate the 5d solutions of (3.4). An approximate ground state \( \psi_i \), i.e., which perturbatively obeys \( Q \psi_i = 0 \), would now obey

\[
Q \psi_i = \sum_{j \in S^5} m_{ij} \psi_j,
\]

(3.5)
(where $S_5$ is the space of solutions of (3.3) in the full quantum theory, where $m_{ij}$ is obtained by summing contributions from all ‘instantons’ that interpolate between the solutions labelled by $i$ and $j$. Hence, the quantum Hilbert space, $\mathcal{H}$, of the 6d theory will be given by the cohomology of the operator $Q$, which is the Floer cohomology corresponding to the equations (3.3).

Moreover, since we are considering ‘ramified’ BPS equations due to the presence of the ‘t Hooft lines, each such Hilbert space will be labelled by the data of the ‘t Hooft lines, i.e., representations which are S-dual to those of the Wilson lines, as well as their positions on the Riemann surface $\Sigma$. Since the path integral of the 6d theory at hand, which, via the aforementioned dualities, is equal to an $R$-matrix element, we can write

\[ R_{ij,kl}(z_1, z_2) = \text{Tr}_{\mathcal{H}_{i',j',k',l'}^{Y \times \Sigma}} ((-1)^F e^{iP(z_1, z_2)}) \]  

where $z_1$ and $z_2$ are the positions of the Wilson lines on $\Sigma$ and correspond to the spectral parameters which label the $R$-matrix, $i', j', k', l'$ are basis elements of the S-dual representations of the ‘t Hooft lines, $F$ is the fermion number operator, and where the operator

\[ P(z_1, z_2) = \int_M C \wedge \text{Tr} (F' \wedge F') + \int_M \text{Tr} \left( \frac{i}{8} \chi \wedge \nabla \chi \right) \]  

as obtained from the topological terms of the S-dual action at a point in time.\(^2\)

In other words, we have identified a vector space with each $R$-matrix element, whereby the vector space is the Floer cohomology of the set of 6d ‘ramified’ partial differential equations (3.3). We may compute each $R$-matrix element by counting solutions of these equations, which are associated with the Hilbert space of the 6d theory (we shall leave this for future work). This may be viewed as an indirect categorification of the Yang-Baxter equation, but we may also categorify directly. This involves three ‘t Hooft lines labelled by spectral parameters $z_1, z_2$ and $z_3$, whereby the topological invariance along $Y$ allows us to reproduce the diagrammatic form of the Yang-Baxter equation by moving one of the ‘t Hooft lines. With the Hilbert spaces before and after this move denoted as $\mathcal{H}$ and $\tilde{\mathcal{H}}$ respectively, we may represent the Yang-Baxter equation as

\[ \text{Tr}_{\mathcal{H}_{i',j',k',l'}^{Y \times \Sigma \times R_+}} ((-1)^F e^{iP(z_1, z_2, z_3)}) = \text{Tr}_{\mathcal{H}_{i',j',k',l'}^{Y \times \Sigma \times R_+}} ((-1)^F e^{iP(z_1, z_2, z_3)}) \]  

Moreover, we may, in a similar vein, categorify the Yangian algebra associated with rational integrable lattice models, in the form of the RTT relation. The latter is realized in the 4d Chern-Simons theory using three Wilson lines as well, but with one of them associated with a representation of $\mathfrak{g}[[z]]$ instead of $\mathfrak{g}$. Moving this Wilson line using the topological invariance along $Y$ gives rise to the RTT relation, and therefore an expression of the form (3.3) categorifies the Yangian algebra as well. It is expected that affine and elliptic quantum algebras can be categorified in an analogous manner.

\(^2\)Here we have taken the liberty to set $\alpha'$ to a convenient constant, as we are viewing the system at a fixed length scale.
3.4 S-dual 4d Chern-Simons theory

The 6d theory is topological, and hence also scale invariant, and at low energies it can effectively be regarded as the 5d theory obtained via dimensional reduction. Subsequently, we may localize the path integral of the 5d theory we obtain, in an analogous manner to how we localized the path integral of the 5d theory in Sections 2.2 and 2.3. Doing so, we obtain

\[
\int_{\Gamma'} D\mathcal{A}_0 e^{-\hbar \int_{\partial M} C \wedge \text{Tr} (\mathcal{A}_0 \wedge d \mathcal{A}_0 + \frac{2}{3} \mathcal{A}_0 \wedge \mathcal{A}_0 \wedge \mathcal{A}_0)},
\]

(3.9)

where \( \Gamma' \) is the cycle defined by the equations \( F'_{0MN} = 0 \) and \( D'_{0M} \phi'^M = 0 \), and this is S-dual to (2.28). Hence, we have an ‘S-dual’ of Costello’s 4d Chern-Simons theory where the coupling is inverted as

\[
\frac{1}{\hbar} \to \hbar.
\]

(3.10)

This may be compared with the S-duality of analytically-continued 3d Chern-Simons theory, studied in [8, 9]. S-duality of this Chern-Simons theory inverts the coupling and exchanges the gauge group with its Langlands dual, and was argued to arise as a consequence of the S-duality of 4d \( \mathcal{N} = 4 \) SYM. Similarly, in our case, the S-duality can be understood to arise from the S-duality of the D5-NS5 system.

Acknowledgements

We would like to thank Junya Yagi for discussions. This work is supported in part by the NUS FRC Tier 1 grant R-144-000-377-114.

References

[1] K. Costello, Supersymmetric gauge theory and the Yangian, ArXiV High-Energy Physics-Theory e-prints (March, 2013) [arXiv:1303.2632]
[2] K. Costello, E. Witten, M. Yamazaki, Gauge Theory and Integrability, I, ArXiV High-Energy Physics-Theory e-prints (September, 2017) [arXiv:1709.09993]
[3] K. Costello, E. Witten, M. Yamazaki, Gauge Theory and Integrability, II, ArXiV High-Energy Physics-Theory e-prints (February, 2018) [arXiv:1802.01579]
[4] E. Witten, Fivebranes and knots, ArXiV High-Energy Physics-Theory e-prints (January, 2011) [arXiv:1101.3216]
[5] E. Witten, Integrable Lattice Models From Gauge Theory, ArXiV High-Energy Physics-Theory e-prints (November, 2016) [arXiv:1611.00592]
[6] B. Geyer, D. Mülsch, Higher-dimensional analogue of the BlauThompson model and \( NT = 8, D= 2 \) Hodge-type cohomological gauge theories, Nuclear Physics B 662 (3) (2003) 531-553 [arXiv:hep-th/0211061]
[7] M. Bershadsky, C. Vafa, V. Sadov, \textit{D-branes and topological field theories}, \textit{Nuclear Physics B} \textbf{463} (2-3) (1996) 420-434 \texttt{arXiv:hep-th/9511222}

[8] Y. Terashima, M. Yamazaki, \textit{SL(2,\mathbb{R}) Chern-Simons, Liouville, and Gauge Theory on Duality Walls}, \textit{Journal of High Energy Physics} (2011) (8), 135 \texttt{arXiv:1103.5748}

[9] T. Dimofte, S. Gukov, \textit{Chern-Simons theory and S-duality}, \textit{Journal of High Energy Physics} (2013) (5), 109 \texttt{arXiv:1106.4550}