Finite groups in which every maximal subgroup is nilpotent or normal or has $p'$-order

Jiangtao Shi$^a$, $^*$, Na Li$^b$, Rulin Shen$^c$

$^a$ School of Mathematics and Information Sciences, Yantai University, Yantai 264005, P.R. China
$^b$ Department of Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China
$^c$ Department of Mathematics, Hubei Minzu University, Enshi 445000, P.R. China

Abstract

Let $G$ be a finite group and $p$ a fixed prime divisor of $|G|$. Combining the nilpotence, the normality and the order of groups together, we prove that if every maximal subgroup of $G$ is nilpotent or normal or has $p'$-order, then (1) $G$ is solvable; (2) $G$ has a Sylow tower; (3) There exists at most one prime divisor $q$ of $|G|$ such that $G$ is neither $q$-nilpotent nor $q$-closed, where $q \neq p$.

Keywords: maximal subgroup; nilpotent; normal; $p'$-order; $q$-nilpotent; $q$-closed

MSC(2010): 20D10

1 Introduction

In this paper all groups are assumed to be finite. It is known that a group $G$ is nilpotent or minimal non-nilpotent if every maximal subgroup of $G$ is nilpotent, and a group $G$ is nilpotent if every maximal subgroup of $G$ is normal, see [5, Theorem 9.1.9] and [5, Theorem 5.2.4], respectively. As a generalization, combining the nilpotence and the normality of groups together, Li and Guo [3, Theorem 1.2] proved that if all non-normal maximal subgroups of a group $G$ are nilpotent then $G$ is solvable and $G$ is $p$-nilpotent for some prime $p$, that is, $G$ has a normal $p$-complement. It is clear that the hypothesis that all non-normal maximal subgroups of a group $G$ are nilpotent is equivalent to the hypothesis that every maximal subgroup of $G$ is nilpotent or normal. Lu, Pang and Zhong [4, Theorems 1.3 and 3.5] proved that if every maximal subgroup of a group $G$ is nilpotent or normal then $G$ is solvable and $G$ is $p$-nilpotent and $q$-closed for some primes $p$ and $q$, that is, $G$ has a normal $p$-complement and the Sylow $q$-subgroup of $G$ is normal. In [2, Theorem 1.1] we gave an elementary proof of the solvability of a group in which every maximal subgroup is nilpotent or normal. Moreover, the first author of this paper [8, Theorem 5] proved that such a group has a Sylow tower.

$^*$J. Shi was supported by Shandong Provincial Natural Science Foundation, China (ZR2017MA022 and ZR2020MA044) and NSFC (11761079). R. Shen was supported by NSFC (12161035).

$^*$ Corresponding author.
E-mail addresses: jiangtaoshi@126.com (J. Shi), ln18865550588@163.com (N. Li), shenrulin@hotmail.com (R. Shen).
The order of subgroups or groups play an important role in characterizing the solvability of groups. Feit-Thompson theorem shows that a group having odd order is solvable. Thompson [5, Theorem 10.4.2] implies that a group having a nilpotent maximal subgroup of odd order is also solvable. Moreover, it is easy to see that a group $G$ satisfying $(|G|, 15) = 1$ is solvable by the classification of minimal non-abelian simple groups.

Note that the nilpotence, the normality and the order are three distinct characteristic properties of groups. In this paper, we combine the nilpotence, the normality and the order of groups together to give a complete characterization of the structure of the group in which every maximal subgroup is nilpotent or normal or has $p'$-order for a fixed prime divisor $p$ of its order, which extends and generalizes the researches in [3, Theorem 1.2], [4, Theorems 1.3 and 3.5] and [8, Theorem 5].

First we obtain a basic structural property of the non-solvable groups, see Theorem 1.1, whose proof is given in Section 2.

**Theorem 1.1** Let $G$ be a non-solvable group and $p$ a fixed prime divisor of $|G|$, then $G$ has non-nilpotent maximal subgroups of order divisible by $p$.

**Remark 1.2** Note that 2 is a prime divisor of the order of the alternating group $A_5$ but $A_5$ has no non-nilpotent maximal subgroups of $2'$-order. This example shows that a non-solvable group might not have non-nilpotent maximal subgroups of $p'$-order for some fixed prime divisor $p$ of its order.

Our main results are the following Theorems 1.3, 1.4 and 1.7, whose proofs are given in Sections 3, 4 and 5 respectively.

**Theorem 1.3** Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If every maximal subgroup of $G$ is nilpotent or normal or has $p'$-order, then $G$ is solvable.

**Theorem 1.4** Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If every maximal subgroup of $G$ is nilpotent or normal or has $p'$-order, then $G$ has a Sylow tower.

**Remark 1.5** Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If we assume that every maximal subgroup of $G$ is nilpotent or normal or has order divisible by $p$, we cannot get that $G$ has a Sylow tower. For example, every maximal subgroup of the alternating group $A_5$ has order divisible by 2 then $A_5$ naturally satisfies the hypothesis. But $A_5$ is a non-solvable group which has no Sylow tower.

**Remark 1.6** The alternating group $A_4$ implies that the Sylow tower of the group $G$ in Theorem 1.4 might not be supersolvable type.

**Theorem 1.7** Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If every maximal subgroup of $G$ is nilpotent or normal or has $p'$-order, then there exists at most one prime divisor $q$ of $|G|$ such that $G$ is neither $q$-nilpotent nor $q$-closed, where $q \neq p$.

Comparing with Theorem 1.7, we have the following three results, all of whose proofs are given in Section 6.
Theorem 1.8 Let $G$ be a group in which every maximal subgroup is nilpotent or normal, then $G$ is either $q$-nilpotent or $q$-closed for each prime divisor $q$ of $|G|$.

Theorem 1.9 Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If every maximal subgroup of $G$ is nilpotent or has $p'$-order, then $G$ is either $q$-nilpotent or $q$-closed for each prime divisor $q$ of $|G|$.

Theorem 1.10 Let $G$ be a group and $p$ a fixed prime divisor of $|G|$. If every maximal subgroup of $G$ is normal or has $p'$-order, then $G$ is either $q$-nilpotent or $q$-closed for each prime divisor $q$ of $|G|$.

2 Proof of Theorem 1.1

Proof. Let $G$ be a counterexample of minimal order. Then every maximal subgroup of $G$ is nilpotent or has $p'$-order. Note that $p$ is a prime divisor of $|G|$, one has that $G$ must have maximal subgroups of order divisible by $p$ and such maximal subgroups are nilpotent by the hypothesis. Since $G$ is non-solvable and both a nilpotent group and a minimal non-nilpotent group are solvable by [5, Theorem 9.1.9], $G$ must have non-nilpotent maximal subgroups which have $p'$-order.

Let $P \in \text{Syl}_p(G)$. First suppose that there exists a nontrivial subgroup $U$ of $P$ such that $U \trianglelefteq G$. Considering the quotient group $G/U$. (1) Assume $U < P$. Then $G/U$ is a non-solvable group of order divisible by $p$ since $G$ is non-solvable and $U$ is solvable. By the minimality of $G$, one has that $G/U$ has a non-nilpotent maximal subgroup $H/U$ of order divisible by $p$. It follows that $H$ is a non-nilpotent maximal subgroup of $G$ of order divisible by $p$, a contradiction. (2) Assume $U = P$. Then $U$ is a Sylow $p$-subgroup of $G$. Let $M$ be a non-nilpotent maximal subgroup of $G$ of $p'$-order. It is clear that $U \ntriangleleft M$ and $U \cap M = 1$. Then $G = U \rtimes M$, the semidirect product of $U$ and $M$. For every maximal subgroup $M_0$ of $M$, one has that $U \rtimes M_0$ is a maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $U \rtimes M_0$ must be nilpotent. It follows that every maximal subgroup of $M$ is nilpotent, which implies that $M$ is a nilpotent group or a minimal non-nilpotent group. One has that $M$ is solvable. Then $G = U \rtimes M$ is solvable, this contradicts that $G$ is non-solvable.

Next suppose that every nontrivial subgroup $U$ of $P$ is not normal in $G$, that is, $N_G(U) < G$. Since the order of $N_G(U)$ is divisible by $p$, $N_G(U)$ can only be contained in a nilpotent maximal subgroup of $G$ of order divisible by $p$. Then $N_G(U)$ is nilpotent. By [11, IV, Theorem 5.8(b)], one has that $G$ is $p$-nilpotent. There exists a normal subgroup $K$ of $G$ such that $G = K \rtimes P$. Let $L$ be a maximal subgroup of $G$ of $p'$-order. Assume $K \nleq L$. Then $KL > L$. One has $G = KL$, which implies that $G$ has $p'$-order, a contradiction. Thus $K \leq L$. Since $K$ is a $p'$-Hall subgroup of $G$ and $L$ has $p'$-order, one has that $K = L$ is a maximal subgroup of $G$. It follows that $P \cong G/K$ is a cyclic group of order $p$. Note
that $P < G$ and $P$ can only be contained in a nilpotent maximal subgroup of $G$. (1) Assume $p = 2$. By [1] IV, Theorem 7.4], one has that $G$ is solvable, a contradiction. (2) Assume $p > 2$. (i) Suppose $Z(G) \neq 1$, that is, the center of $G$ is not equal to 1. Considering the quotient group $G/Z(G)$. If $p \nmid |G/Z(G)|$, then $P \leq Z(G)$. It follows that $G = K \times P$. Note that $K \cong G/P$ is non-solvable, one has that $K$ has a non-nilpotent maximal subgroup $K_0$. Then $K_0 \times P$ is a non-nilpotent maximal subgroup of $G$ of order divisible by $p$, a contradiction. If $p \mid |G/Z(G)|$, then $G/Z(G)$ is a non-solvable group of order divisible by $p$. By the minimality of $G$, $G/Z(G)$ has a non-nilpotent maximal subgroup $R/Z(G)$ of order divisible by $p$. It follows that $R$ is a non-nilpotent maximal subgroup of $G$ of order divisible by $p$, a contradiction, too. (ii) Suppose $Z(G) = 1$. By [6, Theorem 1], every nilpotent maximal of $G$ is a Sylow 2-subgroup of $G$. It implies that $P$ cannot be contained in any nilpotent maximal subgroup of $G$ since $p > 2$, this contradicts that $P$ can only be contained in a nilpotent maximal subgroup of $G$.

Thus the counterexample of minimal order does not exist and so $G$ has non-nilpotent maximal subgroups of order divisible by $p$. \hfill \Box

3 Proof of Theorem 1.3

Proof. Let $G$ be a counterexample of minimal order. Since $G$ is non-solvable, $G$ has non-nilpotent maximal subgroups of order divisible by $p$ by Theorem 1.1. Then by the hypothesis, the group $G$ is in particular not simple.

Let $N$ be a minimal normal subgroup of $G$. We will show that $N$ is non-solvable. (1) Suppose that $G/N$ has $p'$-order. Then $N$ contains the Sylow $p$-subgroup of $G$. For every non-nilpotent maximal subgroup $L/N$ of $G/N$, $L$ is a non-nilpotent maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $L \leq G$. It follows that $G/N$ is a group in which every maximal subgroup is nilpotent or normal. By [2, Theorem 1.1], $G/N$ is solvable. Thus $N$ is non-solvable since $G$ is non-solvable. (2) Suppose that $G/N$ has order divisible by $p$. Since the hypothesis of the theorem holds for $G/N$ and $|G/N| < |G|$, one has that $G/N$ is solvable by the minimality of $G$. Thus we also have that $N$ is non-solvable.

It follows that $\Phi(G) = 1$ and $Z(G) = 1$, that is, both the Frattini subgroup of $G$ and the center of $G$ are equal to 1.

Let $R$ be any non-nilpotent maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $R \leq G$. Assume $N \not\leq R$. Then $G = NR$. Note that $N \cap R \leq N$ and $N \cap R \leq R$, one has $N \cap R \leq NR = G$. It follows that $N \cap R = 1$ since $N \cap R < N$. Then $G = N \times R$, which implies that $N \cong G/R$ is a cyclic group of prime order. This contradicts that $N$ is non-solvable. Thus $N \leq R$.

Claim that $|N|$ is divisible by $p$. Otherwise, assume that $N$ has $p'$-order. Since $G$ is non-solvable, one has that the intersection of all non-nilpotent maximal subgroups of $G$ is
equal to \( \Phi(G) \) by \cite{7} Theorem 1. By above argument, the intersection of all non-nilpotent maximal subgroups of \( G \) is equal to 1. Since every non-nilpotent maximal subgroup of \( G \) of order divisible by \( p \) contains \( N \), there exists a non-nilpotent maximal subgroup \( E \) of \( G \) of \( p' \)-order such that \( N \not\leq E \). One has \( G = NE \). It follows that \( |G| = \frac{|N||E|}{|N \cap E|} \) has \( p' \)-order, a contradiction. Thus \( |N| \) is divisible by \( p \).

Since \( Z(G) = 1 \), one has that all nilpotent maximal subgroups of \( G \) are Sylow 2-subgroups of \( G \) by \cite{6} Theorem 1. Then the maximal subgroups of \( G \) may only be: Sylow 2-subgroups, non-nilpotent maximal subgroups of order divisible by \( p \) or non-nilpotent maximal subgroups of \( \Phi(G) \)-order. Assume that \( q \) is a prime divisor of \( |N| \) such that \( q \neq p \). Let \( P \in \text{Syl}_p(N) \) and \( Q \in \text{Syl}_q(N) \). By Frattini argument, one has \( G = NN_G(P) = NN_G(Q) \). Since \( N \) is a minimal normal subgroup of \( G \) and \( N \) is non-solvable, \( N_G(P) < G \) and \( N_G(Q) < G \).

It is obvious that both \( N_G(P) \) and \( N_G(Q) \) cannot be contained in any non-nilpotent maximal subgroup of \( G \) of order divisible by \( p \) since every non-nilpotent maximal subgroup of \( G \) of order divisible by \( p \) contains \( N \). (1) Assume \( p = 2 \). Then \( q \) is an odd prime. One has that \( N_G(Q) \) can only be contained in a non-nilpotent maximal subgroup \( H \) of \( G \) of \( 2' \)-order. Then \( G = NN_G(Q) = NH \). And \( N_G(P) \) can only be contained in a Sylow 2-subgroup \( K \) of \( G \). It follows that \( G = NN_G(P) = NK \). One has \( |G| = |NH| = |NK| \).

Then \( \frac{|H|}{|H \cap N|} = \frac{|K|}{|K \cap N|} \). Note that \( \frac{|H|}{|H \cap N|} > 1 \) is a 2'-number, but \( \frac{|K|}{|K \cap N|} > 1 \) is a 2-power, a contradiction. (2) Assume \( p > 2 \). One has that \( N_G(P) \) cannot be contained in any Sylow 2-subgroup of \( G \) and \( N_G(Q) \) cannot be contained in any non-nilpotent maximal subgroup of \( G \) of \( p' \)-order, either, a contradiction.

Hence the counterexample of minimal order does not exist, then \( G \) is solvable. \( \square \)

4 Proof of Theorem 1.4

**Proof.** First we show that \( G \) has a normal Sylow subgroup. Let \( G \) be a counterexample of minimal order.

Claim \( \Phi(G) = 1 \). Otherwise, assume \( \Phi(G) \neq 1 \). Since \(|G/\Phi(G)| \) and \(|G| \) have the same prime divisors, \( G/\Phi(G) \) is also a group of order divisible by \( p \) in which every maximal subgroup is nilpotent or normal or has \( p' \)-order. By the minimality of \( G \), one has that \( G/\Phi(G) \) has a normal Sylow subgroup \( P\Phi(G)/\Phi(G) \), where \( P \) is a Sylow subgroup of \( G \). Using Frattini argument, one gets that \( G \) has a normal Sylow subgroup \( P \), a contradiction. Thus \( \Phi(G) = 1 \).

Note that \( G \) is solvable by Theorem 1.3. Let \( N \) be a minimal normal subgroup of \( G \), then \( N \) is an elementary abelian group of prime power order. One has that there exists a maximal subgroup \( M \) of \( G \) such that \( N \not\leq M \) since \( \Phi(G) = 1 \). It follows that \( G = NM \).

First consider the case when \( M \) is a non-nilpotent maximal subgroup of \( G \) of order
divisible by \( p \). By the hypothesis, one has \( M \trianglelefteq G \). Note that \( N \cap M \) is normal in \( M \) and \( N \cap M \) is also normal in \( N \) as \( N \) is abelian. Then \( N \cap M \) is normal in \( NM = G \). It follows that \( N \cap M = 1 \) since \( N \) is a minimal normal subgroup of \( G \) and \( N \cap M < N \). One has \( G = N \times M \), which implies that \( M \cong G/N \). (1) Suppose that \( G/N \) has \( p' \)-order. Then \( N \) is a normal Sylow \( p \)-subgroup of \( G \), a contradiction. (2) Suppose that \( G/N \) has order divisible by \( p \). One has that \( G/N \) has a normal Sylow subgroup since the hypothesis of the theorem holds for \( G/N \) and \( |G/N| < |G| \). It follows that \( M \) has a normal Sylow subgroup \( Q \) since \( M \cong G/N \). (i) Assume \((|N|, |Q|) \neq 1\). Then \( N \times Q \) is a normal Sylow subgroup of \( G \), a contradiction. (ii) Assume \((|N|, |Q|) = 1\). Then \( Q \) is a normal Sylow subgroup of \( G \), a contradiction, too.

Next consider the case when \( M \) is a non-nilpotent maximal subgroup of \( G \) of \( p' \)-order or a nilpotent maximal subgroup of \( G \) of \( p' \)-order. Note that \( G = NM \) is a group of order divisible by \( p \) and \( N \) is an elementary abelian group of prime power order. It follows that \( N \) is a Sylow \( p \)-subgroup of \( G \), which is also a normal Sylow subgroup of \( G \), a contradiction.

Finally consider the case when \( M \) is a nilpotent maximal subgroup of \( G \) of order divisible by \( p \). Assume \(|N| = q^m\) for some prime \( q \) and some positive integer \( m \geq 1 \). Let \( Q \in \text{Syl}_q(M) \). Then \( NP \in \text{Syl}_q(G) \). One has \( NP \trianglelefteq NM = G \), a contradiction.

By above arguments, the counterexample of minimal order does not exist and so \( G \) has a normal Sylow subgroup.

In the following we prove that \( G \) has a Sylow tower.

Let \( P_1 \) be a normal Sylow \( p_1 \)-subgroup of \( G \). (1) Suppose \( p_1 \neq p \). Observe that \( G/P_1 \) is also a group of order divisible by \( p \) in which every maximal subgroup is nilpotent or normal or has \( p' \)-order, arguing as above, one gets that \( G/P_1 \) has a normal Sylow \( p_2 \)-subgroup \( P_1P_2/P_1 \), where \( P_2 \in \text{Syl}_{p_2}(G) \). (2) Suppose \( p_1 = p \). Let \( M/P_1 \) be a non-nilpotent maximal subgroup of \( G/P_1 \), where \( M \) is a non-nilpotent maximal subgroup of \( G \) of order divisible by \( p \). By the hypothesis, one has \( M \trianglelefteq G \), then \( G/P_1 \) is a group in which every maximal subgroup is nilpotent or normal. One can also get that \( G/P_1 \) has a normal Sylow \( p_2 \)-subgroup \( P_1P_2/P_1 \) by [8] Theorem 5], where \( P_2 \in \text{Syl}_{p_2}(G) \).

Similarly, considering the quotient group \( G/P_1P_2 \), arguing as above we can get that \( G/P_1P_2 \) has a normal Sylow \( p_3 \)-subgroup \( P_1P_2P_3/P_1P_2 \), where \( P_3 \in \text{Syl}_{p_3}(G) \). And so on, we can obtain a normal subgroups series:

\[
P_1 \trianglelefteq P_1P_2 \trianglelefteq P_1P_2P_3 \trianglelefteq \cdots \trianglelefteq P_1P_2 \cdots P_s = G
\]

(1)

where \( P_i \in \text{Syl}_{p_i}(G) \) for \( 1 \leq i \leq s \), which implies that \( G \) has a Sylow tower. \( \Box \)

## 5 Proof of Theorem 1.7

**Proof.** Assume that \( G \) is nilpotent, then \( G \) is \( q \)-nilpotent and \( q \)-closed for each prime divisor \( q \) of \(|G|\). In the following we assume that \( G \) is non-nilpotent. Since \( G \) has Sylow
tower by Theorem 1.4, \( G \) can be written as \( G = (P_1 \times P_2 \times \cdots \times P_s) \rtimes (Q_1 Q_2 \cdots Q_t) \), where \( P_i \in \text{Syl}_{p_i}(G) \) and \( P_i \leq G \) for \( 1 \leq i \leq s \), \( Q_j \in \text{Syl}_{q_j}(G) \) and \( Q_j \) is not normal in \( G \) for \( 1 \leq j \leq t \). Let \( M = P_1 \times P_2 \times \cdots \times P_s \) and \( N = Q_1 Q_2 \cdots Q_t \). Then \( G = M \rtimes N \).

Considering the case when the Sylow \( p \)-subgroup of \( G \) is normal in \( G \). Assume \( P_1 \in \text{Syl}_{p_1}(G) \), that is, \( p_1 = p \).

1. Suppose that \( N \) is nilpotent. Then \( G \) is obviously \( p_i \)-closed for every \( 1 \leq i \leq s \) and \( q_j \)-nilpotent for every \( 1 \leq j \leq t \).

2. Suppose that \( N \) is non-nilpotent. Then there exists a maximal subgroup \( N_0 \) of \( N \) such that \( N_0 \) is not normal in \( N \). For the maximal subgroup \( MN_0 \) of \( G \) of order divisible by \( p \), \( MN_0 \) is not normal in \( G \). By the hypothesis, one has that \( MN_0 \) is nilpotent. Note that \( |N : N_0| \) is a prime power by the solvability of \( G \). We can assume \( Q_j \nleq N_0 \) and \( Q_j \leq N_0 \) for every \( 2 \leq j \leq t \). Then \( MN_0 = P_1 \times P_2 \times \cdots \times P_s \times Q_1 \times Q_2 \times \cdots \times Q_t \), where \( Q_1 \leq Q_j \). For any non-nilpotent maximal subgroup \( H \) of \( N \), one has that \( MH \) is a non-nilpotent maximal subgroup of \( G \) of order divisible by \( p \). By the hypothesis, \( MH \leq G \). It follows that \( H \leq N \). That is, every maximal subgroup of \( N \) is nilpotent or normal, one has that \( N \) has normal Sylow subgroups by [\$\text{Theorem 5}\$]. Assume \( Q_j \leq N \) for some \( 2 \leq j \leq t \), then \( N \leq N_G(Q_j) \). It follows that \( G = N_G(Q_j) \) since \( M \leq N_G(Q_j) \), which contradicts that \( Q_j \) is not normal in \( G \) for every \( 2 \leq j \leq t \). Thus \( Q_j \) is not normal in \( N \) for every \( 2 \leq j \leq t \). One has \( Q_j \leq N \). Then \( G = (P_1 \times P_2 \times \cdots \times P_s) \rtimes (Q_1 \times (Q_2 \times Q_3 \times \cdots \times Q_t)) \). It is easy to see that \( G \) is \( p_i \)-closed for every \( 1 \leq i \leq s \), and \( G \) is \( q_j \)-nilpotent for every \( 2 \leq j \leq t \). For \( q_1 \), \( G \) is neither \( q_1 \)-nilpotent nor \( q_1 \)-closed.

Next consider the case when the Sylow \( p \)-subgroup of \( G \) is not normal in \( G \). Assume \( Q_1 \in \text{Syl}_{p_1}(G) \), that is, \( q_1 = p \).

Here \( G/M \cong N \) is a group of order divisible by \( p \). Claim that \( G/M \) is nilpotent. Otherwise, assume that \( G/M \) is non-nilpotent. Since \( G/M \) is also a group of order divisible by \( p \) in which every maximal subgroup is nilpotent or normal or has \( p' \)-order, \( G/M \) has a normal Sylow subgroup by Theorem 1.4, which implies that \( N \) has a normal Sylow subgroup. Assume \( Q_j \leq N \) for some \( 1 \leq j \leq t \). Then \( N \leq N_G(Q_j) \). It follows that \( N_G(Q_j) \) has order divisible by \( p \). Since \( Q_j \) is not normal in \( G \) and all non-nilpotent maximal subgroups of \( G \) of order divisible by \( p \) are normal, \( N_G(Q_j) \) can only be contained in some nilpotent maximal subgroup of \( G \) of order divisible by \( p \). It implies that \( N \) is nilpotent, this contradicts that \( N \cong G/M \) is non-nilpotent. Hence \( N \cong G/M \) is nilpotent. Then \( G \) is \( p_i \)-closed for every \( 1 \leq i \leq s \) and \( q_j \)-nilpotent for every \( 1 \leq j \leq t \).

\[\square\]

### 6 Proofs of Theorems 1.8, 1.9 and 1.10

**Proof of Theorem 1.8.** Let \( G \) be a group in which every maximal subgroup is nilpotent or normal. Suppose that \( G \) is nilpotent, then \( G \) is \( q \)-nilpotent and \( q \)-closed for each prime
divisor $q$ of $|G|$. 

Next we suppose that $G$ is non-nilpotent. Note that $G$ has Sylow tower by [5, Theorem 5], one has $G = (P_1 \times P_2 \times \cdots \times P_s) \rtimes (Q_1 Q_2 \cdots Q_t)$, where $P_i \in \mathrm{Syl}_{p_i}(G)$ and $P_i \unlhd G$ for $1 \leq i \leq s$, $Q_j \in \mathrm{Syl}_{q_j}(G)$ and $Q_j$ is not normal in $G$ for $1 \leq j \leq t$. Let $K = P_1 \times P_2 \times \cdots \times P_s$ and $L = Q_1 Q_2 \cdots Q_t$. Then $G = K \rtimes L$.

Claim that $L$ is nilpotent. Otherwise, assume that $L$ is non-nilpotent. For every non-nilpotent maximal subgroup $L_0$ of $L$, one has that $KL_0$ is a non-nilpotent maximal subgroup of $G$. By the hypothesis, one has $KL_0 \unlhd G$. It follows that $L_0 \unlhd L$. That is, $L$ is a group in which every maximal subgroup is nilpotent or normal. By [8, Theorem 5], $L$ has a normal Sylow subgroup. Assume $Q_1 \unlhd L$. Then $L \unlhd N_G(Q_1) < G$. Since $L$ is non-nilpotent, $N_G(Q_1)$ can only be contained in a non-nilpotent maximal subgroup $H$ of $G$. By the hypothesis, $H \unlhd G$. Then by Frattini argument, one has $Q_1 \unlhd G$, a contradiction. Thus $L$ is nilpotent. It follows that $G$ is $p_i$-closed for every $1 \leq i \leq s$ and $q_j$-nilpotent for every $1 \leq j \leq t$. □

**Proof of Theorem 1.9.** If $G$ is nilpotent, it is obvious that $G$ is $q$-nilpotent and $q$-closed for each prime divisor $q$ of $|G|$. In the following assume that $G$ is non-nilpotent. Arguing as in proof of Theorem 1.7, one has $G = (P_1 \times P_2 \times \cdots \times P_s) \rtimes (Q_1 Q_2 \cdots Q_t)$, where $P_i \in \mathrm{Syl}_{p_i}(G)$ and $P_i \unlhd G$ for $1 \leq i \leq s$, $Q_j \in \mathrm{Syl}_{q_j}(G)$ and $Q_j$ is not normal in $G$ for $1 \leq j \leq t$. Assume $M = P_1 \times P_2 \times \cdots \times P_s$ and $N = Q_1 Q_2 \cdots Q_t$. That is, $G = M \rtimes N$.

We will show that $N$ is nilpotent.

First assume $p_1 = p$, that is, the Sylow $p$-subgroup of $G$ is normal and $P_1 \in \mathrm{Syl}_p(G)$. Let $N_0$ be any maximal subgroup of $N$. Then $M \rtimes N_0$ is a maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $M \rtimes N_0$ is nilpotent. It follows that $N_0$ is nilpotent. Then $N$ is nilpotent or minimal non-nilpotent. Claim that $N$ cannot be minimal non-nilpotent. Otherwise, assume that $N$ is a minimal non-nilpotent group. We can assume $N = Q_1 \times Q_2$ by [5, Theorem 9.1.9], where $Q_1 \unlhd N$. Let $Q_2$ be any maximal subgroup of $Q_2$. Then $M \rtimes (Q_1 \times Q_2)$ is a maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $M \rtimes (Q_1 \times Q_2)$ is nilpotent. It follows that $M \unlhd N_G(Q_1)$. Note that $N \unlhd N_G(Q_1)$. Then $N_G(Q_1) = MN = G$, which implies that $Q_1 \unlhd G$, a contradiction. Thus $N$ is nilpotent.

Second assume $q_1 = p$, that is, the Sylow $p$-subgroup of $G$ is not normal and $Q_1 \in \mathrm{Syl}_p(G)$. It is obvious that $N < G$ and $N$ has order divisible by $p$. Then there exists a maximal subgroup $L$ of $G$ of order divisible by $p$ such that $N \unlhd L$. By the hypothesis, $L$ is nilpotent. It follows that $N$ is also nilpotent.

By the nilpotence of $N$, one has that $G$ is $p_i$-closed for every $1 \leq i \leq s$ and $q_j$-nilpotent for every $1 \leq j \leq t$. □

**Proof of Theorem 1.10.** It is clear that the result holds if $G$ is nilpotent. Considering the case that $G$ is non-nilpotent. Arguing as above, let $G = (P_1 \times P_2 \times \cdots \times P_s) \rtimes \cdots \rtimes \cdots \times (Q_1 Q_2 \cdots Q_t)$.
Claim that the Sylow $p$-subgroup of $G$ is normal. Otherwise, assume that the Sylow $p$-subgroup of $G$ is not normal. Let $Q_1 \in \text{Syl}_p(G)$. Note that $N_G(Q_1) < G$ and $N_G(Q_1)$ has order divisible by $p$. Then there exists a maximal subgroup $R$ of $G$ of order divisible by $p$ such that $N_G(Q_1) \leq R$. By the hypothesis, $R \trianglelefteq G$. It follows that $Q_1 \trianglelefteq G$ by Frattini argument, a contradiction.

Thus the Sylow $p$-subgroup of $G$ is normal. Let $P_i \in \text{Syl}_p(G)$. For every maximal subgroup $N_0$ of $N$, $MN_0$ is a maximal subgroup of $G$ of order divisible by $p$. By the hypothesis, $MN_0 \trianglelefteq G$. It follows that $N_0 \trianglelefteq N$ and then $N$ is nilpotent. Thus $G$ is $p_i$-closed for every $1 \leq i \leq s$ and $q_j$-nilpotent for every $1 \leq j \leq t$.

Acknowledgements

The authors are thankful to everyone who provides valuable suggestions and helpful comments for improving our paper.

References

[1] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin-Heidelberg, 1967.

[2] Na Li and Jiangtao Shi, A note on a finite group with all non-nilpotent maximal subgroups being normal, Italian Journal of Pure and Applied Mathematics 42 (2019) 700-702.

[3] Qianlu Li and Xiuyun Guo, On $p$-nilpotence and solubility of groups, Archiv der Mathematik 96(1) (2011) 1-7.

[4] Jiakuan Lu, Linna Pang and Xianggui Zhong, Finite groups with non-nilpotent maximal subgroups, Monatshefte für Mathematik 171 (2013) 425-431.

[5] D.J.S. Robinson, A Course in the Theory of Groups (Second Edition), Springer-Verlag, New York, 1996.

[6] J.S. Rose, On finite insoluble groups with nilpotent maximal subgroups, Journal of Algebra 48(1) (1977) 182-196.

[7] Jiangtao Shi, Cui Zhang and Songtao Guo, A note on theorem of Shlyk (Chinese), Journal of Guangxi Normal University (Natural Science Edition) 30(1) (2012) 22-24.

[8] Jiangtao Shi, A finite group in which all non-nilpotent maximal subgroups are normal has a Sylow tower, Hokkaido Mathematical Journal 48(2) (2019) 309-312.