Simulating planet migration in globally evolving disks.

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\section*{ABSTRACT}

\textbf{Context.} Numerical simulations of planet-disk interactions are usually performed with hydro-codes that – because they consider only an annulus of the disk, over a 2D grid – can not take into account the global evolution of the disk. However, the latter governs planetary migration of type II, so that the accuracy of the planetary evolution can be questioned.

\textbf{Aims.} To develop an algorithm that models the local planet-disk interactions together with the global viscous evolution of the disk.

\textbf{Methods.} We surround the usual 2D grid with a 1D grid ranging over the real extension of the disk. The 1D and 2D grids are coupled at their common boundaries via ghost rings, paying particular attention to the fluxes at the interface, especially the flux of angular momentum carried by waves. The computation is done in the frame centered on the center of mass to ensure angular momentum conservation.

\textbf{Results.} The global evolution of the disk and the local planet-disk interactions are both well described and the feedback of one on the other can be studied with this algorithm, for a negligible additional computing cost with respect to usual algorithms.

\textbf{Key words.} Methods: numerical ; Accretion, accretion disks ; Solar system: formation

\section{1. Introduction}

Planetary formation occurs in disks of gas and dust around protostars. In particular, giant planets – whose mass is mostly made of hydrogen and helium – must have formed before the dissipation of the gas disk. Consequently, they must have exerted tidal forces on the gas and their orbits must have evolved in response to the gas.

In particular, the presence of a planet on a circular orbit leads to the formation of a spiral density wake in the disk. Goldreich & Tremaine \cite{1980ApJ...238..857G} and Lin \& Papaloizou \cite{1979ApJ...233..371L} showed that, through the wake, the planet gives angular momentum to the part of the disk exterior to its orbit, while it takes angular momentum from the inner part. For small mass planets, Ward \cite{1997ApJ...476..762W} showed that the net result is a loss of angular momentum for the planet, which makes its orbit decay on a short timescale. This is usually referred to as \textit{type I migration}.

As the planet gives angular momentum to the outer part of the disk, it repels it outward; symmetrically it repels the inner part inward. If the planet is massive enough (the threshold mass depending on the disk’s viscosity and scale height; see Crida et al. \cite{2006arXiv06080660C} and references therein) a clear gap opens around the planet’s orbit, effectively splitting the disk into an internal and an external part. In this situation, the planet gets locked in the middle of the gap, because both the outer and the inner part of the disk are repellent for it. Thus, the migration of the planet has to go in parallel with the migration of the gap \cite{1986ApJ...309..755L,2003ApJ...586..566W}, which in turn has to follow the viscous evolution of the disk (characterized by a radial spreading of the disk as the gas is accreted onto the central star \cite{1974MNRAS.168..603L}). This is usually referred to as \textit{type II migration}.

Although analytic theories have brought a great deal of understanding of the fundamentals of planet-disk interactions, it has become increasingly evident in the last years that numerical simulations are an essential tool of investigation (see Papaloizou et al. \cite{2005ssd..conf..259P} for the most recent review). Numerical simulations, however, are difficult and time-consuming, and they inevitably involve some simplifications, which might turn out to produce artifacts. In this paper we are concerned with the simulation of type II migration.

As we have seen above, a proper simulation of type II migration requires not only the correct calculation of planet-disk interactions – which are essentially \textit{local} – but also of the \textit{global} evolution of the disk. Unfortunately, planet migration is typically simulated with hydro-dynamical codes using a 2 dimensional polar grid which, for numerical reasons, is truncated at an inner and an outer radius. This enables to describe well the...
Indeed, the boundary conditions at the extreme rings of the polar grid can not take into account what happens in the whole disk, outside of the grid. For instance, the use of open boundary conditions allows the gas to leave the region covered by the grid; this makes the studied disk annulus behave as if it were surrounded by vacuum, so that it empties very rapidly, which is obviously not realistic. In the opposite extreme case, boundary conditions which impose that the mean density on the extreme rings remains constant with time, disable the accretion and spreading of the gas. Some other prescriptions between these two extremes may be used, (for instance allowing inflow, or imposing an outflow given by an analytical model, or setting the flow on the last ring equal to the viscous flow measured in the neighboring rings, etc.). However, it is very difficult to adapt these prescriptions to the changing behavior of the disk with time, in particular when the disk undergoes perturbation from the planets. In any case these prescriptions are rather arbitrary and, thus they may introduce possible artifacts in the planetary evolution.

In this paper we present a novel idea for the correct calculation of the global evolution of the disk. It consists in surrounding the 2D polar grid with a 1D radial grid. The 1D grid extends from the real inner edge of the gas disk (e.g. the X-wind truncation radius at a few tenths of AU) to the real outer edge (e.g. the photo-dissociation radius at hundreds of AU). This 1D grid has open boundaries at the inner and outer edges, and exchanges information with the 2D grid for the definition of realistic, time-dependent boundary conditions of the latter. Our algorithm for the interfacing between the 1D and 2D grids is driven by the requirement that the angular momentum of the global system (the disk in the 2D section, plus the disk in the 1D section plus the planet-star system) is conserved. We will describe it in detail in Sect. 3. First, however, we need to revisit the algorithm used for modeling the gas evolution and planet-disk interactions on the 2D grid, to ensure that it also conserves angular momentum, which is often not the case in standard implementations. We discuss this issue in Sect. 2. In Sect. 4 the results of our new algorithm will be discussed, in terms of CPU time and robustness with respect to the positioning of the interfaces between the two grids. Finally in Sect. 5 we will describe some interesting astrophysical applications of this new hydrodynamical code.

2. Conservation of angular momentum in 2 dimension hydrodynamical algorithms

Consider a simulation of the planet-disk interactions, with the disk represented on a 2D polar grid with open boundary conditions. A necessary requirement for the simulation to be correct is that the sum of the angular momenta of the disk, of the star-planet(s) system and of the gas outflowing through the boundaries, remains constant over time.

We have tested if this is the case, using the code FARGO (Masset 2000).

Our simulation accounts for a Jupiter mass planet on an initially circular orbit. Here and in the rest of the paper we adopt the following units: the solar mass, the initial semi-major axis of the planet and its orbital frequency, so that the gravitational constant \( G = 1 \) and an orbit lasts \( 2 \pi \) time units at \( r = 1 \). The grid used to represent the disk extends from \( r = 0.25 \) to \( r = 3 \) with open boundaries. It is equally divided in \( N_r = 165 \) elementary rings, and \( N_\phi = 320 \) sectors. The disk aspect ratio (\( H/r \)) is set uniformly to 5%. It is assumed to be constant in time, hence the disk is locally isothermal. The equation of state used in FARGO is: \( P = c_s^2 \Sigma \), with \( c_s = H \Omega \) as usual, where \( P \) is the pressure, \( \Sigma \) the density, \( c_s \) the sound speed, \( H \) the disk scale height, and \( \Omega \) the local angular velocity. The gas kinematic viscosity is \( \nu = 10^{-5.5} \) in our normalized units (which corresponds to the viscosity at the location of the planet for \( \alpha = 1.25 \times 10^{-3} \) in a Shakura & Sunyaev (1973) prescription). Its mean density is \( \Sigma = 3.10^{-4} \), which is a bit less than the Hayashi Minimal Mass Solar Nebula at Jupiter (Hayashi 1981); the initial density profile can be seen in Fig. 5 and corresponds to a disk that evolved for some time under the effect of its own viscosity; it can be very well approximated by \( \Sigma(r) = 0.000306 \exp(-r^2/52.8) \).

The evolution of the total mass and angular momentum of the whole system (including the outflow, e.g. the cumulated mass and momentum advected outside of the region spanned by the 2D grid by the gas outflowing through the boundaries) is presented in Fig. 1. As one sees, while the total mass is conserved at the level of numerical errors (top panel), the conservation of the angular momentum is quite poor (middle panel).

The bottom panel shows that the gain of angular momentum of the whole system (plain line), which is the error of the simulation, amounts to 10% of what the planet exchanges with the gas in its migration process (dashed line). Thus we can expect that the migration of the planet measured in the simulation is correct only at the 10% level.

This shows the need to improve the algorithm in order to achieve a much better conservation of angular momentum. We...
describe a procedure that considerably improves angular momentum conservation below.

2.1. Choice of reference frame

Need of a frame centered on the center of mass. Most, if not all, the published numerical simulations of disk-planet interaction that we are aware of used a non-inertial frame centered on the primary (Crida et al. 2006; Kley 1999; Masset 2002; Masset et al. 2006; Nelson et al. 2000; Nelson & Benz 2003; Varnière et al. 2004, 2005). This is done for practical reasons, and because it is thought to be better adapted to describe the motion of the inner part of the disk. This reference frame is not inertial, so that indirect forces are taken into account for the simulation to be realistic. It is well known that the presence of these forces destroys the conservation of the angular momentum measured in the non-inertial frame. In principle however, one can make at any time a change of coordinates to compute position and velocities in the inertial frame (centered on the center of mass), and the angular momentum computed from these coordinates should remain conserved. This is for instance what happens in N-body codes, when the simulation is done in heliocentric coordinates (except from truncation and round-off errors). This is not the case here. The transport algorithm for the gas – in which mass, angular momentum, linear momentum etc. advect from a cell to its neighbors (sub-step 5 in Sect. 2.2) – imposes the conservation of each of these quantities. This is correct for the mass, but not for the momenta, because they are not supposed to be conserved in the adopted frame. As a consequence of this imposed, unphysical conservation, the conservation of the momenta in the inertial frame is corrupted. The solution out of this problem is the use of the frame centered on the center of mass throughout the algorithm.

Implementation. This is not trivial to implement. A first possibility is to suppress all indirect forces in the algorithm and, whenever the position and the velocity of the star are needed, to compute them imposing that the position of the center of mass of the whole system (star, planets and disk) is at (0,0). This is, for instance, what is usually done in N-body barycentric simulations. In this case, however, the situation is more complicated. While in N-body codes all interactions are treated simultaneously, planet-disk interaction simulators involve a purely gravitational part and an hydrodynamical part, and treat the planetary system and the disk separately (see 2.2 for the sequence of integration sub-steps). When the disk is advected, the star is moved to assure the position of the center of mass, but not the planet. Thus, at the next step, the planet sees a star in a different location in phase-space with respect to the end of the previous step. This has dramatic effect for the orbital stability of the planet and the overall conservation properties. A second, more advantageous possibility is to consider the star exactly like a planet during all the stages of the computation. This ensures a symmetry in the treatment of the star and of the planet(s). When the gas is advected, the planetary system (now including the star) is translated in phase-space to ensure that the center of mass of the whole system remains motionless at $r = 0$. This little correction does not perturb the relative planetary motion. It corrupts, in principle, the conservation of the total angular momentum, but the errors introduced have negligible consequences, as we will test and discuss below.

Caveats. Because the grid is centered on the center of mass, a good numerical accuracy can be achieved only if the motion of the gas is approximated by a rotation around the center of mass. This is the case as long as the central object is close to the center of mass, namely if the total mass of the planets is negligible with respect to the one of the star (which is the common case) or – in case of a large stellar companion – if one considers a distant circumbinary disk. The choice of a frame centered on the center of mass is obviously not adapted for studying a disk around a star with a massive companion or a circumplanetary disk.

In the case of an axisymmetric problem (accretion disk with no planet), the choice of such a frame may lead to numerical instabilities because small deviations from axial symmetry in the disk due to numerical errors cause a shift of the star relative to the center of mass, which in turn can enhance the disk’s asymmetries. Thus, if the planetary system is made of the sole star, we recommend to keep it fixed at the origin of the frame.

Finally, we remark that in many applications, the equations of motion are implemented in a rotating frame in order to keep the planet at a constant position. This can still be done in a frame centered on the center of mass.

2.2. Sub-steps sequence

As we anticipated above, the integration algorithm treats separately the pure gravitational part and the hydrodynamical part. To ensure a good preservation of the conserved quantities, it is necessary to respect as much as possible the action-reaction principle in the planet-disk interactions. This requires that the sequence of integration sub-steps is taken in a specific order.

Here is the sequence that we adopt:

1. The gravitational potential of the planets and star is computed and stored.
2. The velocities of the planets and star are updated using the gravitational influence of the disk.
3. The velocities of the gas are changed in each cell according to the non advective part of the Navier-Stokes equations (the external forces due to the gradients of pressure and gravity field and the viscous stress); the gravitational potential computed at sub-step 1 is used.
4. The planet-star system is advanced under its own gravitational interactions using a N-body algorithm (specifically a 5th order Runge–Kutta integrator).
5. The advective part of the Navier-Stokes equations is performed: using the disk velocities computed at sub-step 3, the mass, angular and radial momentum are advected from each cell to the neighboring cells. The new velocities of the gas are then computed in each cell from the new momenta and masses.
6. The conservation of the center of mass is ensured by translating in space and velocity the center of mass of the planets-star system.

As one sees, in the above algorithm, every time that an action is made on the planetary system, the equivalent action is immediately applied on the disk. Moreover, there is no difference in treatment between planets and star, which is essential for the reasons discussed above. However, particular care has to be paid in the implementation of the algorithm to ensure the effective symmetry of these actions; this is discussed in next subsection.

2.3. Action-reaction symmetry

The action-reaction principle has to be perfectly fulfilled in the computation of sub-steps 2 and 3. Notice that it is not the case, for instance, if one computes the action of a planet on the disk from the gradient of its potential on each cell, and the action of the gas on the planet from the sum of the elementary gravitational forces that it feels from each cell. In principle, both calculations should be equivalent, giving total forces of the same strength and opposite directions. However, if the gradient is computed by finite differences (typical in grid calculations), this introduces a difference with respect to the force exerted by the corresponding cell on the planet.

In order to impose action-reaction symmetry, we proceed as follows. After sub-step 1, we measure the total change of angular momentum that the gas will have at sub-step 3 due to the potential of each planet. Then, in sub-step 2, we compute the change of azimuthal velocity of each planet according to the change of angular momentum that it causes to the disk. This ensures angular momentum conservation between sub-step 2 and sub-step 3. Similarly, in sub-step 2 the planet’s radial velocity change is computed by measuring the total change of linear momentum in this direction caused to the gas disk by the same planet.

2.4. Results and discussion

All these precautions allow us to improve greatly the angular momentum conservation of the whole system. The same simulation of Fig. 1 is computed with our modified algorithm. We denote by $\delta H$ the variation of angular momentum, in the inertial frame centered on the center of mass, of the whole system (the sum of the momenta of the gas in the 2D grid, of the planet-star system, and of the outflow which, we remind, is the cumulative momentum carried by the gas that flew out of the grid). It should be zero if the code were perfectly conservative. Thus, $\delta H$ is a measure of the error of the integration scheme. It is compared with three relevant quantities in Fig. 2. The bottom curve (solid line) corresponds to the logarithm of the relative variation of angular momentum of the system: $\delta H/H_0$, where $H_0$ is the initial total angular momentum (it corresponds to the middle panel of Fig. 1); on a long timescale ($16000$ time units, $\geq 2500$ orbits at $r = 1$), this normalized error does not exceed $10^{-5.3}$. The middle curve (long dashed line) corresponds to the logarithm of $\delta H/H_p$, where $H_p$ is the angular momentum of the planet: the error in total angular momentum is about 4.5 orders of magnitude smaller than the angular momentum of the planet. Thus, it should not affect its migration. Indeed, the top curve (short dashed line) shows the logarithm of the ratio between $\delta H$ and $\delta H_p$, where $\delta H_p$ is the variation of angular momentum of the planet: this ratio is smaller than $10^{-3}$.

Although highly satisfactory for our purposes, the conservation is admittedly not perfect, in particular when compared to the mass conservation (which is in the new simulation as good as in the example of Fig. 1). This is due to two sources of error.

The most obvious one is that the algorithm used for the advancement of the planetary system, which is in our case a Runge-Kutta algorithm, is not symplectic. However, we checked that in our simulation, the amount of angular momentum artificially introduced in the system in this way is 3 orders of magnitude smaller than the total global error.

The second source of error is in sub-step 6, due to the translation of the center of mass of the planet-star system. This translation occurs for two reasons. (i) It is required to compensate the center of mass motion due to errors introduced by the discretization of the grid and to the second order (in time) advection of the disk. Thus, the amplitude of this effect decreases with increasing disk resolution and decreasing time-step. (ii) Gas continuously leaves the grid through its inner or outer boundary. This gas is deleted from the simulation, while its total mass and momentum are recorded as outflow. The problem is that the outflowing gas is in general not axisymmetric. Therefore, its elimination from the simulation causes a shift (in position and velocity) of the center of mass of the whole system. The shift is artificial, in the sense that it would not exist if the gas were modeled with an infinite 2D grid, and it requires, for compensation, the translation of the planetary system. This issue is a conceptual one. Thus this source of error cannot be reduced by tuning the grid resolution or time-step. We think that there is no way around it: it is the price to pay to work with 2D
grids that are not as extended as the physical disk. However, as we have seen in Fig. 2 this noneliminable error is so small that for all practical purposes we can safely ignore it.

3. Coupling of the 2D grid with a 1D grid

As we discussed in the introduction, for a correct simulation of type II migration the conservation of angular momentum is a necessary, but not sufficient condition. It is also necessary that the global evolution of the disk is correctly reproduced, which is not the case if only a portion of the disk (the annulus represented by the 2D grid) is considered.

For this reason, we consider the whole disk, from its real inner to outer radius, corresponding respectively to the ‘X’ radius at the inner edge (Shu 2002), and to a photo-evaporation radius at the outer edge (see Hollenbach & Adams 2004). Because the representation of the whole disk with a 2D grid would be computationally prohibitive, outside of the radial distance range covered by the usual 2D grid we study the evolution of the extended disk in one dimension, with the aid of a 1D grid representation. In practise, the 1D grid is made of two parts: one extending from the real inner edge of the disk to the inner edge of the 2D grid, and the other one extending from the outer edge of the 2D grid to the real outer edge of the disk.

In this section, we first present the equations for the 1D disk. Then we detail how we couple the 1D and 2D portions of the disk in a way that conserves mass and angular momentum, and that ensures a proper simulation of the disk’s global evolution.

3.1. 1D equations

The Navier-Stokes equation and the advection of the gas are computed in the 1D grid exactly like in the 2D grid, but with \( N_z = 1 \), so that each cell can be considered as an elementary ring. Consequently, there are no azimuthal components for the various gradients, neither an advective transport phase in the azimuthal direction. The gravitational potential due to the planets and the star is computed for the 1D grid as a function of \( r \) only: the whole planetary system is considered as a single central mass, the mass of which is the sum of all the bodies inside the considered radius. Denoting by \( \mathcal{H} \) the Heaviside function equal to 1 for positive arguments and to 0 for negative ones, the gravitational potential felt by a cell of the 1D grid located at a radius \( r \) from the center of mass reads:

\[
\phi^{1D}(r) = \sum_p \frac{G M_p}{r} H(r - r_p)
\]

where the index \( p \) goes through the whole planetary system (including the star). This means that the potential felt is equal to \(-G/r\) multiplied by the mass of all the celestial bodies that lie inside the ring. Thus the position of the star with respect to the center of mass has no influence; it would be the same if the 1D rings were centered on the star.

We assume that the gas disk in the 1D grid is axisymmetric, so that in principle there is no torque in the interaction between the planets and the 1D grid. For more realism, however, one can assume that each 1D ring feels a gravitational torque due to each planet, using the formula of Goldreich & Tremaine (1980) or Lin & Papaloizou (1979):

\[
\delta T_g(r) \approx 0.4 \left( \frac{M_p}{M} \right) 2 r_p^3 \Omega_p^2 r^{-1} \left( \frac{r_p}{\Delta} \right)^4 \left( 2 \pi \Sigma \delta r \right)
\]

where \( \Delta = r - r_p, \Omega_p \) is the angular velocity of the planet, and \( \delta r \) is the width of the ring. Of course, for angular momentum conservation, the opposite of this torque exerted on the 1D grid has to be exerted on the planet.

Because of the assumption of axial symmetry, what leaves the 1D grid through its boundaries does not impose any re-adjustment of the star-planet(s) center of mass. Thus, the conceptual problem discussed at the end of Sect. 2.3 is not relevant here. Consequently, the evolution on the 1D grid always shows a perfect angular momentum conservation, at the level of numerical errors.

Despite of this perfect conservation, however, for the results to be correct it is necessary that the assumption of axial symmetry is a good approximation for the real disk. Moreover, because Eq. (1) is an approximation, it is necessary that the planetary torque is small for the error to be small in absolute value. Both requirements are fulfilled if the interfaces between the 1D and 2D portions of the disk are placed sufficiently far from the planet(s). We will detail on this issue in Sect. 4 with quantitative tests.

3.2. Ghost rings

The 1D grid and the 2D grid communicate with each other through a system of ghost rings. This technique is inspired from that used for the parallelization of a hydro-code on a distributed memory architecture. We describe it here for the inner interface between the 1D grid and the 2D grid, that is the interface at the inner edge of the 2D grid; the 1D grid is inside, extending from the inner to outer edge of the disk (the truncation radius or the surface of the star) to the 2D grid; the 2D grid lies outside this interface, as sketched in Fig. 3.

Outside the interface, a number \( N_G \) of 1D rings are added, which superpose to the first \( N_G \) rings of the 2D grid; they are the ghosts for the inner 1D grid (see Fig. 3). At the beginning of every time-step, the density and the velocity in the 1D ghost rings are set as follows. The density of gas (\( \Sigma \)), radial momentum (\( \Sigma v_r \)), and angular momentum (\( \Sigma r \Omega \)) in the ghost ring are set to the azimuthal average of the respective quantities in the corresponding ring of the 2D grid. Denoting with the superscript ‘1D’ the quantities in 1D rings and by \( h \) the specific an-
gular momentum \((h = rv_\theta)\), this leads to the following equations:

\[
\Sigma_{1D} = \frac{1}{2\pi r} \int_0^{2\pi} \Sigma r d\theta ,
\]

\[
\Sigma_{1D}v_{r1D} = \frac{1}{2\pi r} \int_0^{2\pi} \Sigma v_r r d\theta ,
\]

\[
\Sigma_{1D}h_{1D} = \frac{1}{2\pi r} \int_0^{2\pi} \Sigma h r d\theta .
\]

As \(\Sigma_{1D}\) is given by Eq. (2), \(v_{r1D}\) and \(v_\theta1D\) are easily computed from Eqs. (3) and (4).

Then, during the time step, the computation of all stages is performed in the 1D grid as well as in the ghost rings. The 1D grid ‘feels’ the ghosts via the Navier-Stokes equations. As the 1D ghost rings have been filled with quantities inherited from the 2D grid, the 1D grid behaves as if it felt the 2D grid outside the interface.

Symmetrically, inside the interface, a \(N_G\) number of 2D rings are added: the 2D grid ghosts. They are treated as normal 2D rings, except that at the beginning of every time step, the azimuthal means of surface density, of radial momentum and angular momentum density are set equal to the respective values measured in the corresponding 1D rings, as sketched with arrows in Fig. 3. So, the 2D grid effectively ‘feels’ the 1D grid inside the interface. Filling the 2D ghosts is a bit more elaborated than in the 1D ghost rings case. Indeed, for a given 2D ghost ring, one has to proceed in four steps: (i) store for each cell the surface density, the radial momentum, and the angular momentum; (ii) compute the azimuthal means of these quantities; (iii) subtract in each cell, for each of the three considered quantities, the difference between its azimuthal mean and its value in the corresponding 1D ring; (iv) find the velocity in each cell by dividing the new momenta by the new surface density.

The minimal number of ghost rings needed, \(N_G\), depends on the numerical scheme: it is the number of rings that causally affects a given ring during a time-step, or equivalently the number of rings on which the information contained in a ring propagates during a time-step; it is often called the kernel. For instance in FARGO, \(N_G = 5\).

This way of coupling the 1D and 2D grids ensures a smooth transition for each of the quantities computed in the code. In particular, if there is no planetary perturbation, the global disk (2D + 1D parts) behaves exactly as predicted by the \textit{Lynden-Bell & Pringle (1974)} equations. This can be seen in Fig. 4 which corresponds to Fig. 4 in the aforementioned paper: it shows the evolution of the density distribution at four times, for \(\Sigma_{1D}(r) = \delta(r-1)\), with \(\delta\) the Dirac distribution. Like in \textit{Lynden-Bell & Pringle (1974)}, \(T*\) denotes the viscous time: \(T* = 6\nu/\sigma\) in our units. The little discrepancy between the theoretical profile (thin) and the numerical one (bold) at \(T* = 0.004\) most likely comes from the fact that the initial distribution is not exactly a \(\delta\) function; this little difference vanishes with time.

### 3.3. Fluxes at the interface

The fluxes of mass and angular momentum through the interfaces have to be perfectly conservative. What leaves the 2D grid has to enter the surrounding 1D grid, and vice-versa.

#### 3.3.1. Mass flux

The definition of \(v_{r1D}\) in Eq. (3) makes the mass flux computed in 1D be the same as the one computed in 2D. Indeed their theoretical expressions write:

\[
F_{m2D} = \int_0^{2\pi} \Sigma v_r r d\theta,
\]

\[
F_{m1D} = 2\pi r \Sigma_{1D} v_{r1D}.
\]

One can see that Eq. (5) is equivalent to \(F_{m1D} = F_{m2D}\). However, in a numerical scheme with finite time steps and discrete grid, the flux is most often not computed so straightforwardly. Care has to be taken that the numerical expressions of the flux coincide at the interface between the 1D and the 2D grid. In staggered mesh codes, the density and the velocity are not defined at the same locations in the cells of the grid; typically, the density is given at the center, while the radial component of the velocity is given on the middle point of the inner edge and the tangential component on the middle point of the left edge (see for instance Fig. 3 of \textit{Stone & Norman 1993}). Thus, the fluxes are most often computed as:

\[
F_{m1\text{computed}} = 2\pi r \Sigma_{1D} v_{r1D}.
\]

\[
F_{m2\text{computed}} = \sum_j \Sigma \{ [j] v_r [j] r d\theta.
\]

where the index \(j\) goes through all the cells of the ring, \(\delta\theta\) is the angular size of a cell, \(r\) is the radius of the interface, and \(\Sigma\) and
\(\Sigma^{1D}\) are the values of the densities \(\Sigma\) and \(\Sigma^{1D}\) at the interface after half a time-step (Stone & Norman 1993, paragraph 4.4).

To ensure the mass conservation, we are led to solve the following equation instead of (4) for the ring next to the interface:

\[
F_m^{1D\text{ computed}} = F_m^{2D\text{ computed}}.
\]

This equation is only slightly different from Eq. (5). It is an implicit equation in \(v_{r,1D}\), because \(\Sigma^{1D}\) depends on \(v_{r,1D}\). Nevertheless, it can be solved by iteration.

### 3.3.2 Angular momentum flux

Denoting with a prime sign the non axisymmetric components of the quantities \((h' = h - h^{1D}, v'_r = v_r - v^{1D})\), using Eq. (2)–(4), the angular momentum fluxes in 1D and in 2D write:

\[
\begin{align*}
F_{h,1D} &= 2\pi \Sigma h^{1D} v_{r,1D} \\
F_{h,2D} &= \int_0^{2\pi} \Sigma h' v'_r r d\theta = F_{h,1D} + \int_0^{2\pi} \Sigma h' v'_r r d\theta
\end{align*}
\]

Independently of the numerical scheme and of the modification of \(v_{r,1D}\) seen before, it appears that it is impossible for the 1D flux to equal the 2D flux. While \(F_{h,1D}\) corresponds to the angular momentum carried by the gas flowing through the interface, the term \(F_{h,2D}' = \int_0^{2\pi} \Sigma h' v'_r r d\theta\) corresponds to a flux of angular momentum that is not due to advection by the axisymmetric flow; it comes from the azimuthal perturbations of \(v_r\) and \(v_\theta\), that represents the angular momentum carried by a wave. The wave-carried propagation of angular momentum is well-known in planet-disk interactions (see for instance Appendix C of Crida et al. 2006, Goodman & Rafikov 2001). However, at the interfaces, the shear gives an azimuthal viscosity and scale-height (an assumption partially justified in Goodman & Rafikov 2001). The coupling of the 1D and the 2D grid described in the two subsections above ensures a smooth, conservative evolution of the disk inside a given radius \(r_0\) (or, better, \(\delta F_h = F_{h,2D}' - F_{h,1D}\approx F_{h,2D}'_\text{computed} = F_{h,1D}\approx F_{h,2D}'_\text{computed} = F_{h,1D}\approx F_{h,2D}'_\text{computed}\), which might be slightly different from the former according to the numerical scheme). Then, over a time-step \(\delta t\), we spread in the 1D grid the amount of angular momentum \(\delta F_h\). A prescription for the deposition of the flux can be found in Goodman & Rafikov (2001) as a function of the distance from the planet. This could be used when there is only one planet, but not if there are several planets, because it is difficult to know which fraction of \(\delta F_h\) is due to each planet. Thus, we adopt an exponential function, because it is scale free so that, once \(\delta F_h\) is known at the interface, its deposition does not depend on the position of the planet(s) and is a function of the distance from the interface only. In practice, we assume that, over a time-step \(\delta t\), the angular momentum deposited by the wave in a ring of width \(dr\) located at a distance \(d\) from the interface between the 1D and 2D grids is:

\[
\delta h = \delta F_h \delta t \lambda \exp(-\frac{d}{\lambda}) dr,
\]

where \(\lambda\) is the damping length-scale of the flux. This assumption will be supported in Sect. 4.1, where the evolution of the disk in the 1D grid is found to be insensitive to the position of the interface with respect to the planet.

We assume that \(\lambda = 0.5\) in natural units, which is the value obtained by fitting with an exponential the decay of the wave-carried flux in appendix C of Crida et al. (2006). For simplicity, we assume that \(\lambda\) does not depend on the disk’s viscosity and scale-height (an assumption partially justified in Goodman & Rafikov 2001).

The deposit of the quantity \(\delta h\) of angular momentum in a ring is simulated by applying a suitable torque, namely by adding to \(v_{\theta,2D}'\) the quantity \(\delta v_{\theta,1D}' = \delta h/r\), where \(r\) is the radial distance of the ring from the star.

Notice that the integral of (6) from \(\delta = 0\) to infinity is equal to the total angular momentum \(\delta F_h\) carried by the wave at the grid interface. However, the 1D disk is not infinite in radial extent. Thus, a fraction of the angular momentum carried by the wave will not be deposited in the disk, but will outflow from the system. This outflowing momentum, as well as the angular momentum and mass advected through the inner and outer radius of the 1D grid are recorded, in analogy with what was done in Sect. 2 for the 2D grid alone.

### 3.3.3 Mean viscous torque

The coupling of the 1D and the 2D grid described in the two subsections above ensures a smooth, conservative evolution of the disk. The gas is free to accrete and spread from the star to its physical outer edge, through the interfaces between the grids. However, at the interfaces, the shear gives an azimuthal viscous stress. This appears as a torque exerted on a grid by its ghosts (and reciprocally). The torque exerted by the part of the disk inside a given radius \(r_0\) on the part of the disk outside \(r_0\) reads, from the Navier-Stokes equations:

\[
t_r(r_0) = \int_0^{2\pi} T_{r\theta} r_0 d\theta,
\]
where \( \hat{T} = \begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} \) is the stress tensor, \( \tilde{T} = 2\Sigma v \left( \tilde{D} - \frac{1}{2} \nabla \ln \tilde{D} \right) \), with \( \tilde{D} \) the strain tensor and \( \tilde{I} \) the identity matrix. Thus, \( T_{\theta\theta} = T_{\theta r} = \Sigma v \left( \frac{\partial u^{\theta}}{\partial r} + \frac{\partial u^{r}}{\partial \theta} \right) \). In a staggered mesh scheme, \( T_{\theta\theta} \) is defined at the inner edge of every cell.

For angular momentum conservation, the torque felt by the 1D grid from its ghosts must equal the one exerted by the 2D grid on its ghosts. This requires \( T_{\theta r}^{1D}(r_{\text{interface}}) = \frac{1}{r_{\text{interface}}} \sum_j T_{\theta r}(j)(r_{\text{interface}}) \), where the index \( j \) goes through the \( N_j \) cells of the 2D ring next to the interface. However, given that the expression of \( T_{\theta r} \) implies the product of velocity gradients by the density, its average is not equal to the product of the averages, and the required equality is not necessarily true. Thus, one has to replace the value of \( T_{\theta r}^{1D} \) at the interface by the average of \( T_{\theta r} \) on the interface in the 2D grid.

3.4. Results and discussion

We consider the final state of the simulation of Sect. 2, a Jupiter mass planet initially at \( r_p = 1 \) evolves in a gas disk represented by a 2D grid extending from \( r = 0.25 \) to 3 for 16000 time units (\( \approx 2500 \) orbits). The final density profile of the gas disk is shown in Fig. 5 as a dashed line. One can see that the planet has opened a gap and migrated inward (the planet is located in the middle of the gap). Moreover, the surface density of the gas over the considered range is strongly reduced relative to its initial value (dot-dashed line in Fig. 5). We then compute another simulation with the same planetary system and the same initial value (dot-dashed line in Fig. 5). We then compute the evolution of the mass and angular momentum of the planetary system, plus those of the gas in the two grids, plus those advected through the inner and outer radius of the 1D grid and recorded as outflow) is presented in Fig. 6 and compared to the conservation obtained in the code using the 2D grid only. One can see that the use of the extended 1D disk does not change the excellent result obtained in Sect. 2. Thus, the coupling between the two grids that we described above is correct in terms of conservation properties. Actually, the error in angular momentum is slightly larger in the case with the 1D grid because the density at the inner interface of the 2D grid does not vanish, as can be seen in Fig. 5, so, the major source of error (the non axisymmetric outflow from the 2D grid, discussed at the end of Sect. 2.4) does not disappear, while it does for the simulation using the 2D grid alone.

4. Performance of the code

In this section, we check the accuracy of the results, as a function of the size of the 2D grid, and we discuss the CPU cost.
of this hybrid scheme. We show that the coupling of a 1D grid increases by far the realism of the results of the hydro-code, for a low additional computation cost.

4.1. Position of the interfaces with respect to the planet.

As already mentioned in sec. 3.1, the interface between the 2D and the 1D grids has to be sufficiently far from every planet, so that the flux of angular momentum carried by the wave at the grid interface and the azimuthal dependence of the disk surface density are both small enough. An obvious test of our code is to check the dependence of the results on the positions of the interfaces between the two grids.

We ran a bench of simulations like the one described in sec. 3.4, with different inner radii for the 2D grid, but keeping constant the grid resolutions. Figure 7 shows the density profiles of the inner disk obtained at the end of the computations (16000 time units ~ 2500 orbits). It appears that the differences among the results are small, even in the cases where the planet is not very far from the interface ($r_p \approx 0.75$). Only the cases with a transition radius bigger than 0.5 show not negligible differences with the three other runs, which are remarkably similar to each other, in particular for what concerns the density profile of the inner edge of the gap. The case with interface at 0.5833 is clearly not accurate, but note that it is not unrealistic while the 1D grid starts only at 4 Hill radii from the planet.

All cases, however, give a quite consistent representation of the surface density profile of the inner disk, which is – on the contrary – very different from that obtained by using the 2D grid alone (compare with the dashed curve in Fig. 5).

This would not be the case if we had not implemented in the 1D disk the exponential damping of the angular momentum carried by the density wave launched by the planet. For instance, Fig. 8 shows 3 of the simulations, recomputed by switching off the calculation of the pressure torque studied in Crida et al. (2006).

The evolution of the disk has a strong influence on the type II migration of the planet. Figure 10 shows the different migration rates in the simulations of Fig. 7 and in the simulation obtained with the 2D grid alone. Once again, the two first cases ($R_{\text{interface}} = 0.3333$ and 0.4333) are remarkably similar, while the other ones show a slightly slower inward motion of the planet. However, the simulations done with the sole 2D grid, gives a migration that is sensibly faster, as expected due to the disappearance of the inner disk.

In conclusion, the results presented in this section show that our method for coupling the 2D and 1D calculations, not only ensures the conservation of the angular momentum, but also allows a robust (i.e. weakly dependent on the grid interface position) modeling of the disk’s structure and evolution.
Thus, it is obvious that the computation of the disk evolution at every time-step. They are the filling of the ghost rings, the operations required to couple the grids, which also need to be done in the 2D grid. In addition, one has to consider the three operations in the 1D grid is negligible with respect to the computation in the 2D grid. In most codes, the length of the time-step is adapted to the conditions imposed by the required resolution and the existing perturbations. In general, it is determined by the Courant-Friedrichs-Lewy (CFL) condition \cite{StoneNorman1993}. The limit most likely comes from the cells with smallest radii, where the angular velocity is the largest. Indeed, to avoid numerical instabilities, the time-step length $\delta t$ is set in order to avoid that information propagates more than one cell over one time-step; thus one has $\delta t \Omega < 2 \pi / N_s$, where $\Omega$ is the angular velocity, which is about Keplerian in a gas disk. Consequently, denoting by $R_{\text{min}}$ the inner boundary radius of the 2D grid, one has $\delta t \propto R_{\text{min}}^{-3/2}$. This shows that extending the 2D grid toward the star shortens the time-step and slows down the simulation significantly. The use of the FARGO algorithm \cite{Masset2000} enables one to get rid of the mean angular velocity and to consider only the perturbed motion and the shear, but this leads to a similar conclusion, although the scaling of $\delta t$ with $R_{\text{min}}$ is generally different.

The CFL condition in a 1D grid is much less constraining. Consequently, the addition of a 1D grid inside the inner edge of the 2D grid has no influence on the time step length, still determined by what happens in the 2D grid. It is thus important to increase $R_{\text{interface}}$ as much as possible, to speed up the computation. Some benchmarking confirmed this reasoning \cite{Subsect. 4.1}, so that an acceptable trade-off can be found.

5. Astrophysical applications

As mentioned in the introduction, the viscous evolution of the protoplanetary disk is thought to govern type II migration. Our code, which has been designed in order to correctly reproduce this evolution, while resolving the planet-disk interactions, is therefore very useful for studying any problem related to type II migration and the feedback exerted by the presence of planets onto the evolution of the disk.

We present two problems for which this hybrid scheme is the tool of choice.

5.1. Outward migrating planets

A demonstration of how type II migration depends on the evolution of the disk has been provided by \cite{VerasArmitage2004}. A disk evolving under its own viscosity accretes onto the central star, while spreading outward under the constraint of angular momentum conservation. Thus, at any instant in time there is a boundary in the disk, within which the radial motion of the gas in negative, and beyond which it is positive – see the right panel of Fig. \ref{fig:outward_migration} and \cite{Lynden-BellPringle1974}. If a giant planet is located beyond this boundary it should move outward, as its migration has to follow the local evolution of the star shortens the time-step and slows down the simulation. Consequently, the addition of a 1D grid inside the inner edge of the 2D grid has no influence on the time step length, still determined by what happens in the 2D grid. It is thus important to increase $R_{\text{interface}}$ as much as possible, to speed up the computation. Some benchmarking confirmed this reasoning \cite{Subsect. 4.1} for more detailed results). The accuracy loss of the computation with the increase of $R_{\text{interface}}$ has been discussed in Subsect. \ref{sec:accuracy_loss} so that an acceptable trade-off can be found.
If the migration of a giant planet is strongly influenced by the negative torque exerted in the planet-disk interaction had the form \( \omega_p \), and the effect of waves carrying an angular momentum flux was neglected.

Using our code, we perform more precise simulations of this process. The interaction between the planet and the disk is simulated in 2D, and the effect of waves is taken into account. This implies that the gap opened by the planet in the disk is less wide and deep than in \cite{Veras2004} representation \cite{Crida2006}. Moreover, the planet also feels a corotation torque, which is otherwise neglected in the analytic expression of Eq. \( \omega_p \).

Indeed we find that the situation is not as simple as illustrated by \cite{Veras2004}. For instance, the left panel of Fig. \ref{fig:11} shows the result of a simulation in which the planet is initially put at \( r_p = 1.4 \), which is deeply in the outward spreading zone as shown by the right panel. Nevertheless, the planet migrates inward, in what seems to be a runaway type III migration \cite{Masset2003}. This is because the spreading disk forces enough gas to pass through the coorbital region of the planet so that the planet feels a strong corotation torque \cite{Masset2001} and decouples from the disk evolution.

We do obtain outward planet migrations, but only in specific cases. For instance when we place the planet initially outside of the spreading disk (which is unrealistic), or if we hold it for a sufficiently large number of orbits, so that it can open a deep gap that effectively truncates the disk at its inner edge (which, in practice, places again the planet outside of the spreading disk).

A detailed description of this mechanism and the comprehensive exploration of the parameter space are beyond the scope of this paper. However, we think that this example is significant, because it shows that in reality the evolution of a Jovian mass planet is not an ideal type II migration. Because the gap is not extremely clean, the planet is not fully locked in the disk’s evolution. The planet feels the global motion of the disk (accretion or spreading) but at the same time it feels also a non-negligible corotation torque. Only a code like the one that we have developed in this paper can simulate the two effects correctly and hence allows a quantitative study of the planet’s evolution.

5.2. Cavity opening

If the migration of a giant planet is strongly influenced by the evolution of the disk, the presence of the planet also influences the evolution of the disk, with a non negligible feedback on its own migration.

Perhaps the best example is that of the formation of an inner cavity in the disk. Once a planet has opened a wide and deep gap, the accretion time of the inner disk onto the star is smaller than the time scale of planet’s migration, because of the negative torque exerted by the planet \cite{Varniere2005}. This leads to the disappearance of the inner disk. This in turn enhances the imbalance between the inner and outer torques felt by the planets, and accelerates the planet’s migration. The presence of cavities are deduced in some protoplanetary disks, because the spectral energy distribution (SED) presents a lack of emissivity at the wavelengths corresponding to hot gas in the vicinity of the star (see for instance \cite{Calvet2005}). \cite{Rice2003} suggest the SED of GM Aur could be explained by a cavity maintained by a two Jupiter mass planet.

Our algorithm clearly enables a more quantitative analysis of the cavity opening process than ever done before, for a low computational cost. As we have shown in Fig. \ref{fig:5} when the 1D grid extends down to the real inner edge of the disk, the evolution of the surface density of the inner disk is found to be slower than in a classical code with a truncated 2D grid, which makes the appearance of a deep cavity more difficult. Therefore, we think that the estimates of the planet mass required to achieve cavities of given depth, and the estimates of the lifetimes of these cavities, need to be revised. A detailed study of the cavity opening mechanism as a function of planetary mass, disk viscosity and aspect ratio is in progress and will be addressed in a forthcoming article.

6. Conclusion and perspectives

It is well known that the migration of giant planets (i.e. planets massive enough to open significant gaps in the disk’s density distribution) is governed – or at least strongly affected – by the global evolution of the disk under its own viscosity.

Usual simulation algorithms solve the hydrodynamical equations over a 2 dimensional polar grid that is truncated at an inner and outer radius to keep the computing time within reasonable limits. Thus, they cannot reproduce correctly the evolution of the disk, and no more the planet’s migration.

In this paper, we have shown how the use of a 1D grid surrounding a classical 2D grid allows us to simulate the global evolution of the disk, while resolving the local planet-disk interactions with the accuracy of the usual algorithms. Coupling the two grids via a system of ghost rings, with special attention paid to the conservation of the angular momentum, leads to a smooth evolution of the disk from its inner radius (the truncation radius or the surface of the star) to its outer edge, several
hundreds of AU away. This increases by far the accuracy of the simulation with essentially no additional CPU cost.

Consequently, this algorithm is a tool of choice to properly simulate type II migration, and related problems. More generally, it also enables the study of the effect of the presence of planets on the disk’s global evolution, which in turn affects the migration of the planets in a feedback effect.

In our simulations, we observe that the disks slowly disappear through the open boundaries of the 1D grid. However, disks are believed to disappear rapidly after only a few million years, under the action of photo-evaporation. This phenomenon depends on time and location. This will open the possibility to simulate the evolution of planets in a globally evolving gas disk, over the disk’s lifetime.

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Appendix A: Open boundary condition

The “open” boundary condition, that allows outflow from the grid but not inflow is implemented as follows at the inner edge of the 1D grid:

In the innermost ring (number 0), the radial velocity is always 0 and the density is the one of the neighboring ring. In the latter (ring number 1), the radial velocity is set equal to the one of the following ring (number 2) if and only if it corresponds to outflow from the grid (negative radial velocity), and to 0 otherwise. This gives:

- \( \Sigma(0) \) is set to \( \Sigma(1) \).
- Whenever \( v_r(2) \geq 0 \), \( v_r(1) \) is set to 0.
- Whenever \( v_r(2) < 0 \), \( v_r(1) \) is set to \( v_r(2) \).

So, the first three rings are used for this computation, which explains the artifact observed in Figs. 4, 5, 7, and 8.

Appendix B: Benchmarking

We ran 3 simulations of a Jupiter mass planet initially placed at \( r_p = 1 \), evolving in a gas disk for 1000 time units on an Intel Xenon 2.66 GHz processor. The results, summed up in Table B.1 confirm our reasoning of Sect. 4.2.

First, we see that with or without the 1D grid, the computing cost is about the same.

Second, it is obvious that the computing time is not directly proportional to the number of cells, and is strongly increased by the higher shear in the vicinity of the star, otherwise the third simulation would not have been more than twice as long as the first one.

Table B.1. Benchmarking results

| parameters of the 2D grid | 2D grid CPU time of process |
|---------------------------|----------------------------|
| extension \( N_r \) | \( N_r \) | extension \( N_r \) | 1D grid extension | CPU time of process |
| 0.35 - 3.0 | 165 | 320 | none | 6167 s. |
| 0.35 - 3.0 | 165 | 320 | 0.1167 - 20 | 6008 s. |
| 0.1167 - 5.0 | 294 | 320 | none | 99494 s. |

Third, the second simulation was slightly shorter than the first one because with the 1D grid, the density at the boundaries of the 2D grid does not tend to zero, the profile is smoother and consequently the CFL condition is less constraining there.

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