A THEOREM ON CENTRAL VELOCITY DISPERSIONS

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ABSTRACT

It is shown that, if the tracer population is supported by a spherical dark halo with a core or a cusp diverging more slowly than that of a singular isothermal sphere (SIS), the logarithmic cusp slope $\gamma$ of the tracers must be given exactly by $\gamma = 2\beta$, where $\beta$ is their velocity anisotropy parameter at the center unless the same tracers are dynamically cold at the center. If the halo cusp diverges faster than that of the SIS, the velocity dispersion of the tracers must diverge at the center too. In particular, if the logarithmic halo cusp slope is larger than two, the diverging velocity dispersion also traces the behavior of the potential. The implication of our theorem on projected quantities is also discussed. We argue that our theorem should be understood as a warning against interpreting results based on simplifying assumptions such as isotropy and spherical symmetry.

Key words: dark matter – galaxies: halos – galaxies: kinematics and dynamics – stellar dynamics

1. INTRODUCTION

Gravitationally interacting collisionless $N$-body systems are a good model for both dark matter halos and stellar systems, and therefore have been a subject of many studies. Early on, it was realized that the evolution of such systems is governed by what is now known as the collisionless Boltzmann equation (Eddington 1915; Jeans 1915). However, since it deals with the distribution function, which is generally inaccessible in all but very extensive and nearly complete data sets, its utility is somewhat limited in reality. An alternative to dealing directly with the distribution function is to focus on the relations among the statistical moments of it. The set of their governing equations is obtained by taking velocity moment integrals on the Boltzmann equation. However, since Equation (1) is linear in $\nu$, it is still valid for any constant multiple of $\nu$, such as the mass or the luminosity density, provided that the tracer population has homogeneous properties.

The Jeans equations under the spherical-symmetry and the steady-state assumptions reduce to

$$\frac{d}{dr} (r^2 \sigma^2) + 2\beta \frac{\nu \sigma^2}{r} = -\nu \frac{d\Psi}{dr},$$

(1)

where $\nu = \nu(r)$ and $\sigma_r = \sigma_\theta(r)$ are the density profile and the radial velocity dispersion of the tracer population. The density here is nominally the number density, which follows the derivation of the Jeans equations from the collisionless Boltzmann equation. However, since Equation (1) is linear in $\nu$, it is still valid for any constant multiple of $\nu$, such as the mass or the luminosity density, provided that the tracer population has homogeneous properties.

The velocity anisotropy parameter $\beta = \beta(r)$ is defined such that

$$\beta = 1 - \frac{\sigma_\theta^2}{\sigma_r^2},$$

(2)

where $\sigma_\theta(r)$ is the one-dimensional tangential velocity dispersion of the same tracers (Binney & Tremaine 2008). The luminous tracers are moving in a gravitational potential $\Psi(r)$, which, through the spherical Poisson equation,

$$4\pi G \rho = \frac{1}{r^2} \frac{d}{dr} \left( r^2 d\Psi / dr \right),$$

(3)

is related to the density profile $\rho(r)$ of the dark halo. Strictly speaking, $\rho$ in Equation (3) is the total mass density that includes all gravitating masses. Here, it will be simply referred to be a dark halo, which is basically a label that signifies that we do not demand that the potential should be self-consistently generated.
by the density profile of the tracers. Although physical solutions are still subject to constraints that they must be non-negative and the tracer mass density may not be greater than that of “dark halo” anywhere, these will not be considered explicitly in this paper.

3. POWER-LAW SOLUTIONS TO JEANS EQUATIONS

We begin our study with an analysis of power-law solutions to the Jeans equations with constant anisotropy parameter. These are often good approximations locally, and have the advantage that they are analytically tractable.

The spherical Jeans Equation (1) is always formally integrable such that

$$Q v \sigma_r^2 = - \int dr Qv \frac{d\Psi}{dr},$$

where $Q = Q(r)$ is the integrating factor

$$\ln Q = \int dr \frac{2B}{r_r} .$$

If $v$ and $\Psi$ behave locally like a power law,

$$v \simeq Ar^{-\gamma}; \quad \frac{d\Psi}{dr} \simeq B r_B^{-\gamma},$$

where $A, B > 0$ and $\delta \leq 1$, we can find solutions to the Jeans equations once the behavior for the anisotropy is prescribed. If the potential is self-consistently generated, the power indices and normalization constants are related to each other such that $\gamma = \delta + 2$ and $4\pi GA = B(1 - \delta)$ (see Equation (3)).

The easiest assumption to make regarding the behavior of $\beta$ is that it is constant. In reality, this probably is not true, but here we are interested in the generic behavior of solutions for given local power-law assumptions on $v$ and $\Psi$. Therefore, assuming constant $\beta$ is valid provided that its variation is much slower than that of the density and the potential. Notably, Hansen & Moore (2006) found that the logarithmic density slope and the anisotropy parameter are linearly related in simulated dark halos. Although their detailed finding appears not to be always true (see, e.g., Navarro et al. 2008), the general idea seems to be still valid. That is to say, the spatial variation of the anisotropy parameter is sufficiently slow so that it can be considered to be locally constant while the density profile is approximated as a power law. The situation is believed not to be much different in stellar systems, for which no evidence to the contrary is obvious.

Under the assumption of the constancy of $\beta$, we have $Q = r^\beta$, which results in

$$\sigma_r^2 \simeq \begin{cases} \frac{B}{\gamma - 2\beta + \delta} r^{-\delta} + Cr^{\gamma - 2\beta} & \text{if } \delta \neq 2\beta - \gamma; \\ Cr^{\gamma - 2\beta} (B \ln r^{-1} + C) & \text{if } \delta = 2\beta - \gamma, \end{cases}$$

where $C$ is an integration constant to be determined from the boundary condition.

While the solution in Equation (6) is always valid for given assumptions, it is somewhat easier to follow its behavior if we consider a finitely deep and an infinitely deep potential well separately. In the next two subsections, we investigate each case in detail, and find the constraints on the behavior of the radial velocity dispersion provided by the Jeans equations.

3.1. Case 1: Finite Central Potential Wells

If the dark halo diverges like a singular isothermal sphere (which behaves as $r^{-2}$; henceforth SIS), the resulting potential is logarithmically divergent and so $\delta = 0$. Thus, given the power-law assumption, for any cusped halo diverging slower than a SIS, we can limit $\delta < 0$. Letting $p = -\delta > 0$ (that is, the potential behaves like $\Psi \simeq \Psi_0 + Br^p/p$), the leading term for $\sigma_r^2$ as $r \to 0$ becomes

$$\sigma_r^2 \simeq \begin{cases} \frac{B}{\gamma - 2\beta - p} r^p & \text{if } p < \gamma - 2\beta; \\ r^p (B \ln r^{-1} + C) & \text{if } p = \gamma - 2\beta; \\ Cr^{\gamma - 2\beta} & \text{if } \gamma - 2\beta < p. \end{cases}$$

For $0 < p < \gamma - 2\beta$, we have $\sigma_r^2 \sim r^p \to 0$ as $r \to 0$. If $p = \gamma - 2\beta > 0$, then $\sigma_r^2 \sim r^p \ln r^{-1}$ and again $\lim_{r \to 0} \sigma_r^2 = 0$. Moreover, the logarithmic slope of $\sigma_r^2$ still tends to $p > 0$ in the limit of $r \to 0$. Finally, if $\gamma - 2\beta < p$, then $\sigma_r^2 \sim r^{\gamma - 2\beta}$. However, for this last case, if $\gamma < 2\beta$, then $r^{\gamma - 2\beta} \to \infty$ as $r \to 0$. Since any finitely deep central potential well is unable to support tracer populations with divergent velocity dispersions, the physical possibilities are limited to be $\gamma \geq 2\beta$ (cf., An & Evans 2006).

In conclusion, with a finite central potential well, the possibilities are either (1) $\sigma_r^2 \to 0$ as $r \to 0$ (here, the logarithmic slope of $\sigma_r^2$ tends to $\min[p, \gamma - 2\beta] > 0$) or (2) $\gamma = 2\beta$ with finite and nonzero $\sigma_r^2$ at $r = 0$ (see also Evans et al. 2009).

3.2. Case 2: Centrally Divergent Potentials

Next, we consider a centrally diverging potential, for which $0 \leq \delta \leq 1$. Here, the $\delta = 0$ case corresponds to a logarithmic potential and a SIS-like dark halo cusp, whereas a point mass potential is represented by $\delta = 1$.

For these cases, the theorem of An & Evans (2006) provides us with the constraint that $\gamma \geq \frac{1}{2}\delta + \beta(2 - \delta)$, and so that $\delta - 2\beta + \gamma \geq (\frac{1}{2} - \beta)\delta$. Since $\beta \leq 1$, we finally find that $\delta - 2\beta + \gamma \geq 0$. Here, this is strictly larger than zero if $\delta \neq 0$. Consequently, the leading term for $\sigma_r^2$ as $r \to 0$ with a divergent potential is

$$\sigma_r^2 \simeq \begin{cases} \frac{B}{\gamma - 2\beta} + Cr^{\gamma - 2\beta} & \text{if } \delta = 0 \text{ and } \gamma > 2\beta; \\ B \ln r^{-1} + C & \text{if } \delta = 2\beta - \gamma = 0, \end{cases}$$

or

$$\sigma_r^2 \simeq \frac{B}{\gamma - 2\beta + \delta} r^\delta = \frac{\delta}{\gamma - 2\beta + \delta} |\Psi|$$

if $0 < \delta \leq 1$, for which $\gamma - 2\beta + \delta > 0$. Here, note that $\Psi \simeq B \ln r^{-1}$ if $\delta = 0$ and $\Psi \simeq -(B/\delta) r^{-\delta}$ for $0 < \delta \leq 1$. In addition, $\rho \propto r^{-(2+\delta)}$ for $0 \leq \delta < 1$ if the potential is generated by $\rho$, whereas $B = GM_\bullet$ and $\delta = 1$ if a black hole of mass $M_\bullet$ dominates the potential.

In conclusion, $\sigma_r^2$ in a divergent potential well traces the potential except for the case of a logarithmically divergent potential with $\gamma > 2\beta$, for which it is nonzero and finite.

4. PROPERTIES OF SOLUTIONS TO JEANS EQUATIONS AT CENTER

The preceding results are interesting, but the arguments leading to them are restricted to power-law solutions of the Jeans
equations with constant anisotropy parameter. In fact, the results hold good more generally albeit in a weaker form. In the following, we shall derive a general theorem that extends the preceding results. In this section, we will set up the framework and proceed to the case concerning the halo with a core or a cusp that is shallower than the SIS, which is likely to encompass most astrophysically interesting models. We will extend the theorem to all physically allowed models of halos and resulting potentials including those dominated by a central point mass in Section 5.

The theorem is deduced by analyzing the Jeans Equation (1) in the limit of \( r \to 0 \) without reference to constancy of the anisotropy parameter or power-law behaviors. The resulting constraints however are only strictly applicable to the central limiting values.

### 4.1. Preliminaries

First, let us recast the Jeans equation into a more useful form. We begin by integrating Equation (3), which leads us to

\[
\int r^2 d\Psi = GM(r) = GM_\bullet + 4\pi G \int_0^r d\tilde{r} \tilde{r}^2 \rho(\tilde{r}),
\]

where \( M(r) \) is the enclosed mass within the radius of \( r \), and the integration constant \( M_\bullet \) represents a central point mass (e.g., a supermassive black hole). However, we postpone detailed consideration of the \( M_\bullet \neq 0 \) case until Section 5.

Equation (1) is then equivalent, with the enclosed mass, to

\[
\frac{GM(r)}{r} = \sigma_0^2(\gamma - 2\beta - \alpha).
\]

Also we have introduced the logarithmic slopes of the tracer density and the velocity dispersion, namely (note the signs)

\[
\gamma = -\frac{d \ln \nu}{d \ln r} = -\frac{r}{v} \frac{dv}{dr}; \quad \alpha = \frac{d \ln \sigma_0^2}{d \ln r} = \frac{r}{\sigma_0^2} \frac{d \sigma_0^2}{dr}.
\]

In the following, we consider the behavior of the system at the center, as indicated by the limit of Equation (11) as \( r \to 0 \). All the subsequent arguments operate under the assumption that every quantity considered here is well behaved, continuous and smooth.

### 4.2. Systems with Vanishing M/r at the Center

Here, we basically repeat the argument found in Section 5 of Evans et al. (2009) with a slight refinement. The result will form a part of the theorem to be proven in Section 5, and highlights the astrophysically relevant information.

The condition for the left-hand side of Equation (11) to vanish in the limit \( r \to 0 \) is given by \( \lim_{r \to 0} \frac{GM(r)}{r} = M_\bullet = 0 \) and, from l'Hôpital's rule, \( dM/dr |_{r=0} = 0 \). The last bit is equivalent to \( \lim_{r \to 0} \rho r^2 = 0 \) – that is to say, \( \lim_{r \to 0} \rho \) diverges (i.e., a cored profile) or \( \rho \) diverges at the center slower than a SIS. Hence, assuming \( M_\bullet = 0 \) (i.e., no central point mass), if \( \lim_{r \to 0} \rho r^2 = 0 \), then the right-hand side of Equation (11) should also vanish as \( r \to 0 \). This is possible only if (1) \( \sigma_0^2 = 0 \), or (2) \( \alpha_0 = \gamma_0 = 2\beta_0 \). Here and throughout, the subscript “0” is used to indicate the limiting value at the center.

Suppose that (2) is the case. Here, if \( \gamma_0 < 2\beta_0 \), then it would be that \( \sigma^2 \sim r^{-2(\beta_0-\gamma_0)} \to \infty \) as \( r \to 0 \). However, the SIS, for which \( \lim_{r \to 0} \rho r^2 \) is nonzero and finite, can only generate a potential diverging as fast as logarithmic. Thus, the velocity dispersion that diverges like a power law cannot be supported by the potential generated by the density cusp that is shallower than that of the SIS. Consequently, the corresponding case, \( \gamma_0 < 2\beta_0 \), is unphysical and not allowed. On the other hand, if \( \gamma_0 > 2\beta_0 \), then \( \sigma^2 \sim r^{(\gamma_0-2\beta_0)} \to 0 \), which reduces to the case (1). In conclusion, a spherical dark halo with a core or a milder cusp than that of a SIS (i.e., \( \lim_{r \to 0} \rho r^2 = 0 \), can only permit tracer populations satisfying the constraint \( \beta_0 = \gamma_0/2 \) or those with \( \sigma_{r,0}^2 = 0 \).

### 4.3. Vanishing Central Velocity Dispersions

The preceding discussion indicates that \( \sigma_{r,0}^2 = 0 \) is necessary for \( \lim_{r \to 0} \rho r^2 = 0 \) if \( \gamma_0 \neq 2\beta_0 \). The initial impression of Equation (11) notwithstanding, \( \sigma_{r,0}^2 = 0 \) alone, however, is not sufficient for vanishing \( M/r \), either. Formally, this is because the behavior of Equation (11) cannot be specified for tracers for which \( \sigma_{r,0}^2 = 0 \) and \( \beta_0 = -\infty \) without reference to the speed of each approach to its limiting value.

From its definition in Equation (2), \( \beta = -\infty \) if \( \sigma^2 \neq 0 \) unless \( \sigma_0^2 = 0 \). Here, \( \beta_0 = -\infty \) indicates a tracer population with purely circular orbits toward the center. Note that in principle it is always possible to construct any spherical model with purely circular orbits (although such models are subject to resonant overstabilities; see Palmer et al. 1989). The nonzero tangential velocity dispersion here is the result of the random orientations of the orbital planes while the circular speed is uniquely specified by the enclosed mass. That is to say, the local dispersion of the speed of the tracers is actually zero although the tangential velocity dispersion may be nonzero.

On the other hand, for all finite values of \( \beta_0 \), then matters are simpler. Since

\[
\sigma_{r,0}^2 = \sigma_0^2(1 - \beta_0)\sigma_{r,0}^2 = 0,
\]

we easily find that \( \sigma_{r,0}^2 = 0 \) indicates that the total three-dimensional velocity dispersion at the center also vanishes. That is, the system must be dynamically cold at the center.

The conclusion of Section 4.2 therefore may be rephrased as follows: in a spherical potential well generated by a halo that is cored or cusped less severe than a SIS, the only allowed populations of tracers are those either consisting of purely circular orbits toward the center or exhibiting vanishing central velocity dispersions unless the limiting values of the cusp power index \( \gamma_0 \) and the anisotropy parameter \( \beta_0 \) at the center are constrained such that \( \gamma_0 = 2\beta_0 \).

### 4.4. Discussion

At first glance, the result of Section 4.2 may appear to be counterintuitive as though it seems to suggest no pressure support at the center whereas the Jeans equation is supposed to balance the force. This reasoning is faulty because the actual “pressure” in this case is given by \( \rho \sigma^2 \), not \( \sigma^2 \). Given the density cusp, the system possesses nonvanishing kinematic pressure at the center even though it is dynamically cold. This is obvious in the power-law solutions to the Jeans equation of Section 3, which show that \( \sigma^2 \sim r^{\min(p-\gamma,-2b)} \). For a system that is radially biased or isotropic toward the center (i.e., \( \beta \geq 0 \)), for which \( \gamma \geq 0 \) from An & Evans 2006 and so tracers with a hole at the center are not allowed), the central “pressure” is therefore always nonvanishing as \( r \to 0 \) (it would be actually divergent unless \( \beta = 0 < p - \gamma \)). The “pressure” of the tangentially biased system (\( \beta < 0 \)) on the other hand can be vanishing if \( p > \gamma \). However, in this last scenario, it can be understood...
that the tangentially biased system is preferentially rotationally supported toward the center.

If \( \sigma_r^2 \) indeed vanishes at the center, this implies that there are no radial orbits in the model and that the distribution function has the property \( \lim_{r \to 0} f(L^2, E) = 0 \). This seems unusual, but there are mechanisms known that can depopulate the radial orbits—for example, scattering by the central cusp (Gerhard & Binney 1985) or the radial orbit instability (Palmer & Papaloizou 1987)—albeit at the cost of driving the system away from sphericity.

The result can also be applied to a self-consistent system (or equivalently interpreted as a constraint on the central velocity dispersion of the dark halo itself). In a spherical dark halo that has a core or a cusp less severe than a SIS, the central limiting value of the anisotropy parameter must be exactly half of the numerical value of the cusp slope unless the central velocity dispersion vanishes. In particular, a cored halo must have an isotropic velocity dispersion at the center if it is not dynamically cold there.

5. THE GENERAL THEOREM

So far, we have investigated some astrophysically important subcases. Here, we derive and prove the general theorem that makes precise the interlocking constraints between the central limiting values of the density and velocity dispersion of the tracers and the potential. Those who are primarily interested in the result should skip to Section 5.2. The next subsection provides a rigorous mathematical analysis of Equation (11) that leads to our result.

5.1. A Derivation of the Theorem

We start by giving the binary relation "\( \sim \)" its precise mathematical meaning. In the following, it is understood to be the short-hand notation such that \( a \sim b \) as \( r \to 0 \) if and only if both \( \lim_{r \to 0} (a/b) \) and \( \lim_{r \to 0} (b/a) \) are finite. We also define the binary relations \( \lesssim \) and \( \gtrsim \) similarly. That is to say, \( a \lesssim b \) (or \( a \gtrsim b \)) as \( r \to 0 \) if and only if \( \lim_{r \to 0} (a/b) \) is divergent (or zero).

In addition, it is also to be understood that the limit is taken to be always \( r \to 0 \) unless the explicit reference to the limit is given to override.

Next, we consider the behavior of the left-hand side of Equation (11) in relation to the mass density profile that generates the potential. From the argument of Section 4.2, we find that \( M/r \) decays to zero in the limit \( r \to 0 \) if \( M_* = 0 \) and \( \rho \lesssim r^{-2} \). The corresponding potential is either finite at the center or diverges strictly slower than logarithmic (i.e., \( \Psi \gtrsim \ln r \)). On the other hand, it attains a nonzero finite limiting value at \( r \to 0 \) if and only if \( M(r) \sim r \). This is equivalent to \( \rho \sim r^{-2} \) and also \( \Psi \sim \ln r \). Finally, \( M/r \) diverges as \( r \to 0 \) if \( M_* \) is nonzero (implying the presence of a central point mass) or \( \rho \gtrsim r^{-2} \) in the same limit. The corresponding potential also diverges but strictly faster than logarithmic (i.e., \( \Psi \gtrsim \ln r \)) in the same limit.

The behaviors of \( M/r \) and \( \Psi \) in relation to each other and their respective logarithmic slopes may be explored in further detail. Let us first define \( p \), the logarithmic slope of \( M/r \), i.e.,

\[
p = \frac{d \ln(M/r)}{d \ln r} = \frac{d}{d \Psi/ dr} \frac{d}{dr} \left( \frac{GM}{r} \right).
\]

Note that in the limit of \( r \to 0 \), the logarithmic slope of the potential (difference) necessarily tends to the same value as \( p_0 \). In particular, if \( \Psi(0) = \Psi_0 \) is finite, it naturally follows that \( GM/r = r(d\Psi/dr) \to 0 \) and so

\[
\lim_{r \to 0} \frac{d \ln(\Psi - \Psi_0)}{d \ln r} = \lim_{r \to 0} \frac{GM/r}{\Psi - \Psi_0} = p_0 \geq 0
\]

using l'Hôpital's rule. If \( \lim_{r \to 0} \Psi = -\infty \) on the other hand, we also find that

\[
\lim_{r \to 0} \frac{d \ln|\Psi|}{d \ln r} = \lim_{r \to 0} \frac{GM/r}{\Psi} = p_0 \leq 0.
\]

Although the usual conditions for l'Hôpital's rule for this case are only strictly met if \( M/r \) diverges, l'Hôpital's rule can in fact be proven only assuming a divergent denominator. Thus, the result holds even though \( M/r \) tends to a finite value (including zero) as \( r \to 0 \). Moreover, it is obvious that \( p_0 = 0 \) if \( \Psi \to \infty \) and \( M/r \) is finite as \( r \to 0 \). Next, if \( p_0 \neq 0 \), then \( M/r \sim |\Delta\Psi| \) where \( \Delta\Psi = \Psi \) for \( \lim_{r \to 0} \Psi = -\infty \) or \( \Delta\Psi = |\Psi - \Psi_0| \) for \( \Psi(0) = \Psi_0 \) being finite. By contrast, that \( p_0 = 0 \) indicates that \( M/r \lesssim |\Delta\Psi| \). If the central potential is additionally finite (e.g., \( \Psi - \Psi_0 \sim |\ln r|^{-1} \)), for which \( M/r \sim |\ln r|^{-2} \), then \( M/r \rightarrow 0 \). The behavior of \( M/r \) as \( r \to 0 \) for a divergent central potential depends on how fast \( \Psi \) diverges relative to logarithmic divergence (in \( r^{-1} \))—e.g., for \( |\Psi| \sim \ln |\ln r|^{-1} \), \( |\Psi| \sim |\ln r|^{-1} \), and \( |\Psi| \sim \frac{1}{2}|\ln r|^2 \), we have that \( M/r \sim |\ln r|^{-1} \), \( M \sim r \), and \( M/r \sim |\ln r|^{-1} \), respectively.

Next, we proceed to analyzing the right-hand side of Equation (11). Here, we do not consider the \( \beta = -\infty \) case, which represents the formal possibility of building the system with purely circular orbits. Then, since \( M \) and \( \sigma_r^2 \) must be non-negative, \( \gamma_0 - 2\beta_0 = \alpha_0 \geq 0 \). Moreover, \( \gamma_0 \leq 3 \) from the constraint that the central mass concentration must be finite. Together with the assumption that \( p_0 \) is finite, we find that \( \gamma_0 - 2\beta_0 - \alpha_0 \) is divergent only if \( \alpha_0 \) diverges to negative infinity. However, then \( \sigma_r^2 \) diverges faster than any power law to an essential singularity (e.g., \( r^{1/\gamma} \)). This is physically impossible, because no real potential diverges faster than \( 1/r \) nor is thus able to support such steeply diverging velocity dispersions. Therefore, we limit \( \gamma_0 - 2\beta_0 - \alpha_0 \) to be finite.

If \( \gamma_0 - 2\beta_0 - \alpha_0 > 0 \), it is clear that \( \sigma_r^2 \sim GM/r \), which also indicates that \( \alpha_0 = p_0 \). If \( \alpha_0 = p_0 \neq 0 \), then \( \sigma_r^2 \sim |\Delta\Psi| \), too. If \( \alpha_0 = p_0 = 0 \) on the other hand, the behavior of \( \sigma_r^2 \) still traces that of \( M/r \), but \( \sigma_r^2 \sim M/r \gtrsim |\Delta\Psi| \).

If \( \gamma_0 - 2\beta_0 = \alpha_0 \), then \( \sigma_r^2 \) \( \sim M/r \) and so \( \alpha_0 = \gamma_0 = -2\beta_0 < p_0 \), from the constraint of An & Evans (2006). For a divergent potential (introducing \( \delta_0 = -p_0 \), for which \( 0 \leq \delta_0 \leq 1 \)), the constraint of An & Evans (2006), \( \gamma_0 \geq \frac{1}{2} \delta_0 + p_0/(2 - \delta_0) \), indicates that \( \delta_0 \geq -2 \beta_0 \leq \gamma_0 - 2\beta_0 < \alpha_0 = p_0 = -\delta_0 \). Now, if \( \delta_0 > 0 \), this would imply \( p_0 \geq \frac{3}{2} \). This is obviously impossible, and therefore \( \delta_0 = p_0 = \alpha_0 = \gamma_0 - 2\beta_0 = 0 \). In addition, if \( M \gtrsim r \), it is clear that \( \sigma_r^2 \rightarrow \infty \). Furthermore, once Equation (11) is recast to be

\[
r d\Psi = (\gamma - 2\beta)\sigma_r^2 dr - r \frac{d\sigma_r^2}{dr},
\]

we can find for sufficiently fast-decaying \( \gamma - 2\beta \) that \( \sigma_r^2 \sim |\Psi| \).

This essentially implies that \( \sigma_r^2 \) cannot diverge faster than \( \Psi \).

5.2. The Statement of the Theorem

In summary, the spherical Jeans equations permit only restricted physical possibilities regarding the limiting behaviors
at the center. In particular, the central limiting value of the velocity anisotropy ($\beta_0$; Equation (2)) and those of the logarithmic slopes of the luminous tracer density ($\gamma_0$; Equation (12)), the radial velocity dispersion ($\sigma_0$; Equation (12)) and the potential ($\rho_0$; Equation (13)) must meet one, and only one, of the following list of choices:

1. $\rho_0 = \alpha_0 < \gamma_0 - 2\beta_0$ and $\sigma_0^2 \sim M/r$.
2. $\rho_0 \geq \alpha_0 = \gamma_0 - 2\beta_0 \geq 0$ and $\Psi_0$ is finite.
3. $\rho_0 = \alpha_0 = \gamma_0 - 2\beta_0 = 0$ and $\lim_{r \to 0} \Psi = -\infty$.
4. $\beta_0 = -\infty$.

Focusing on the behavior of the velocity dispersion, the result with the proviso $\beta > -\infty$ is summarized as

$$
\lim_{r \to 0} \frac{d \ln \sigma_r^2}{d \ln r} = \begin{cases} 
\min(2 - \Gamma_0, \gamma_0 - 2\beta_0) \geq 0 \ (\Gamma_0 < 2) \\
-(\Gamma_0 - 2) \leq \gamma_0 - 2\beta_0 \ (\Gamma_0 \geq 2)
\end{cases}
$$

(14)

although this does not include all the information encompassed in the above choices. Here,

$$
\Gamma = -\frac{d \ln \rho}{d \ln r}
$$

so that $\rho = 2 - \Gamma$, and extending to include the central point mass by setting $\Gamma_0 = 3$.

For a prescribed behavior of $M/r$ or $\Psi$, the above list returns the natural extension and generalization of our earlier results. If $M/r \to 0$, for example, then either (a) $\sigma_0^2 \to 0$ with $\sigma_0^2 \sim M/r$ or $\alpha_0 = \gamma_0 - 2\beta_0 > 0$, or (b) $\alpha_0 = \gamma_0 - 2\beta_0 = 0$. Consequently, we recover the conclusion of Section 4.2. The implication of the list, however, is more detailed. First, if $\gamma_0 - 2\beta_0 > \rho_0 \geq 0$ ($\rho_0 > 0$ is necessary for vanishing $M/r$), then only the case (1) is possible and so $\sigma_0^2 \sim M/r$ and $\alpha_0 = \rho_0$. If $\rho_0 > 0$ additionally, then $\sigma_0^2 \sim |\Psi - \Psi_0|$ with a finite $\Psi_0$ whereas $\sigma_0^2 \sim |\Delta \Psi|$ for $\rho_0 = 0$. On the other hand, with $\rho_0 \geq \gamma_0 - 2\beta_0 > 0$, we have $\alpha_0 = \gamma_0 - 2\beta_0 > 0$ and so $\sigma_0^2 \to 0$ (and $\rho_0 > 0$ similarly indicating that $M/r \sim |\Psi - \Psi_0|$). The remaining physical possibility, $\rho_0 \geq \gamma_0 - 2\beta_0 = 0$, implies that $\alpha_0 = 0$, which can lead to a nonzero finite limit for $\lim_{r \to 0} \sigma_0^2$.

If $\Psi_0$ is finite, then $\sigma_0^2$ must not diverge, but if $\Psi$ is divergent (but not faster than logarithmic) as $r \to 0$, then the exact limiting behavior of the corresponding $\sigma_0^2$ should be inferred from the particular solution to the Jeans equations.

By contrast, if $M/r$ diverges (for which $\Psi \to -\infty$), then case (1) indicates that $\sigma_0^2 \sim M/r \to \infty$ whereas case (3) requires $\sigma_0^2 \gtrsim M/r$ and so $\sigma_0^2 \to \infty$ (but $\alpha_0 = \rho_0 = 0$). In other words, $\sigma_0^2$ necessarily diverges if $M/r \to \infty$. Furthermore, $\sigma_0^2$ must be divergent as fast as $M/r$ (note that if $\delta_0 > 0$, then $\sigma_0^2 \sim M/r \sim |\Psi|$, but $\sigma_0^2 \sim M/r \gtrsim |\Psi|$ for $\delta_0 = 0$, where $\delta_0$ is the negative logarithmic slope of $M/r$ or the potential) unless $\delta_0 = \gamma_0 - 2\beta_0 = 0$ for which $\sigma_0^2$ diverges faster than $M/r$ but not faster than $|\Psi|$.

For $M \sim r$ (and $\Psi \sim \ln r$), the result is basically that of Equation (8); case (1) yielding the possibility of a finite limiting value of $\sigma_0^2$ whereas case (3) is consistent with $\sigma_0^2$ diverging at most logarithmically or slower.

5.3. Infinite Velocity Dispersions?

In the framework of classical Newtonian mechanics upon which the Jeans equations and the collisionless Boltzmann equation are ultimately based, the divergence of $\sigma_0^2$ when $M/r$ and the corresponding potential also diverge is in principle physically acceptable despite its mathematical quirk. However, it is clear that the arguments given in this paper eventually break down as $\sigma_0$ approaches the speed of light. Moreover, in the corresponding halo, $M/r$ should be divergent as $r \to 0$, and therefore there exists a radius below which $GM(r)/c^2 > r$. Consequently, the central cusp, if it ever were present, must collapse to a singularity. In other words, one would expect that the formal infinity of the velocity dispersion can be always circumvented through the presence of a central black hole. The proper examination of physical behaviors of the tracers and the halo under these conditions would require consideration of relativistic physics, which is out of the scope of the current paper. Of course, in reality, it is more likely that other various physical complexities in the system intervene to prevent the spherical Jeans equations to be applied uncritically all the way down to the center even before any relativistic effects become important.

6. PROJECTED QUANTITIES

The direct measurement of radial and tangential velocity dispersions of stellar tracers is limited to nearby populations. More generally, the true observables are limited to the line-of-sight velocity dispersion—either the “aperture-averaged” value or its profile for a subset. The implication of the theorem on the behavior of the line-of-sight velocity dispersion is therefore of a great practical interest. However, we shall see that the integral transformation involved in the line-of-sight velocity dispersion weakens the theorem’s practical constraints.

It is usually assumed that the observed line-of-sight velocity dispersion follows the luminosity-weighted integration of the velocity dispersions along the line-of-sight direction. The latter $\sigma_1$ is mathematically well-defined quantity such that

$$
\sigma_1^2(R) = \frac{2}{I} \int_R^\infty \left(1 - \frac{\beta R^2}{r^2}\right) v \sigma_r^2 r dr,
$$

(15)

where

$$
I(R) = 2 \int_R^\infty \frac{vr dr}{\sqrt{r^2 - R^2}}
$$

is the surface density of the tracers. If $\beta \sigma_2^2$ is nondivergent, the leading term of $I \sigma_1^2$ as $R \to 0$ cannot be dominant over that of the surface density $I(R)$. Consequently, the leading term of $\sigma_1^2$ in the central limit is largely dictated by the tracer density profile.

In particular, if the density profiles of the tracers are approximated as power law like, we find the behavior of the leading terms for the surface density for (J. H. An & H. S. Zhao 2009, in preparation)

$$
v \simeq Ar^{-\gamma} \rightarrow \begin{cases} 
I \sim R^{-(\gamma-1)} \ (\gamma > 1) \\
I \sim \ln R^{-1} \ (\gamma = 1) \\
I \sim I_0 - I_1 R^{1-\gamma} \ (0 < \gamma < 1)
\end{cases}
$$

$$
v \simeq v_0 - Ar^q \rightarrow \begin{cases} 
I \sim I_0 - I_1 R^{1+q} \ (0 < q < 1) \\
I \sim I_0 - I_1 R^{2q} \ln R^{-1} \ (q = 1) \\
I \sim I_0 - I_1 R^{2q} \ (q > 1)
\end{cases}
$$

where $I_0$ is the finite central surface density, and $A$ and $I_1$ are some positive constants. Assuming $v \sim r^{-\gamma}$ ($\gamma = 0$ if cored), $\sigma_1^2 \sim r^\alpha (\alpha > 0)$, and $\beta_0 > -\infty$, the corresponding behavior
for Equation (15) is similarly found to be

\[
I\sigma^2_i \sim \begin{cases} 
R^{-(γ-\alpha-1)} & (γ > \alpha + 1) \\
\ln R^{-1} & (γ = \alpha + 1) \\
C_0 + C_1 R^{α+γ} & (α - 1 < γ < α + 1) \\
C_0 + C_1 R^2 \ln R^{-1} & (γ = \alpha - 1) \\
C_0 + C_1 R^2 & (γ < \alpha - 1) 
\end{cases}
\]

with \(C_0\) and \(C_1\) being some nonzero constants. Given the implication of the theorem for tracers with \(γ_0 = 2β_0\) in a nondivergent potential, i.e., \(σ_0 \to 0\) as \(r \to 0\) and so \(α > 0\), we surmise that \(σ_0 \to 0\) (as \(R^{min(γ-1,α)}\)) as \(R \to 0\) if \(γ \geq 1\) whereas it attains a finite limiting value (and typically increasing outward) if \(γ < 1\). If on the other hand \(σ_0^2,0\) is finite (for which \(γ_0 = 2β_0\) according to the theorem) or \(β_0 = -∞\) (and \(σ_{0,0}^2 = σ_{ϕ,0}^2\) is nonzero), the leading term behavior of \(Iσ^2_i\) is similar to that of \(I\) alone, and thus \(σ^2,0\) is finite.

These essentially imply that the behavior of \(σ^2_γ\) cannot in general be directly inferred from the leading term approximation of \(σ^2_γ\) alone, and that the strict constraint from the theorem is somewhat lost by going through the integral transformation. Although one may deal with \(Iσ^2_i\) instead of \(σ^2_γ\) or can in principle invert the integral equation for \(σ^2_γ\) to yield \(σ^2_i\) (assuming some particular \(β\)), this still indicates that inferring \(σ^2_γ\) from \(σ^2_i\) involves analyzing higher-order behaviors of the latter and thus requires high-precision measurements. Furthermore, this is independent of the well known degeneracy of \(σ^2_γ\) and \(β\) in the inversion of \(σ^2_i\) in a sense that even though one possesses perfect a priori information on \(β\), the uncertainties in the recovered \(σ^2_i\) are always amplified by inverting \(σ^2_γ\).

6.1. A Central Black Hole

The preceding discussion presumes the finite central potential well, which is appropriate for the potential dominated by the halo that is cored or cusped but not so steep as the SIS. If the potential, however, is dominated by the central point mass, the theorem indicates that \(σ^2_i \sim |Ψ| \sim 1/r\) and so \(Iσ^2_i \sim R^{−γ}\) for \(γ > 0\) (i.e., cusped tracer populations) or \(Iσ^2_i \sim \ln R^{-1}\) for \(γ = 0\) (i.e., cored tracer populations). That is to say, the line-of-sight velocity dispersion of the population tracing the Keplerian potential is necessarily divergent with its logarithmic slope, \(\frac{d}{dr} \ln σ^2_i / d \ln R\) being equal to min(1, γ) where γ is the three-dimensional density (negative) logarithmic slope of the same tracers unless the orbits of tracers are completely circularized toward the center. Nevertheless, the direct application of this inference to the observational results warrants caution since an assumption of “infinite” resolution is implicit in the argument. That is to say, the result is strictly relevant only if the observation can resolve the so-called sphere of influence of the central point mass.

7. CONCLUSIONS

In this paper, we have established a general theorem—stated in Section 5.2—that makes precise the relationship between the central limiting values of the density and velocity dispersions of a stellar population, together with the potential. Our theorem gives all the mutually exclusive possibilities that can occur in a stellar system. We note that our theorem has straightforward applications to a number of astrophysical problems, including the kinematic modeling of the stellar populations in dwarf spheroidal galaxies and elliptical galaxies.

In Evans et al. (2009), we presented a simplified version of the theorem and argued that it is the consequence of the spherical-symmetry assumption. However, after the appearance of the preprint version of Evans et al. (2009), S. Tremaine (2008, private communication) convinced us that the theorem is due to the “nonanalytic” point at the center. The spherical symmetry is of secondary importance and only indirectly responsible for the theorem by requiring a coordinate singularity at the center. The theorem in this respect might be understood as an incomplete boundary condition imposed on the Jeans equations at the center resulting from the consideration of one-sided regularity.

Applying to the real astrophysical problems, the true moral of our theorem is the urging of caution against interpreting results based on simplifying assumptions. For instance, if one were to reconstruct the dark halo density from the observations of the surface density and the line-of-sight velocity dispersion profile of a tracer population, the seemingly benign assumptions of spherical symmetry and isotropy combined with a cored luminosity profile already severely restrict the possible halo density (it cannot be cusped). Such idealized reconstructions are limited by the straitjacket imposed by the theorem, yet the restrictions may be nonexistent in reality—not unlike assuming a spherical cow.

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