Critical phenomena of static charged AdS black holes in conformal gravity

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Abstract

The extended thermodynamics of static charged AdS black holes in conformal gravity is analyzed. The $P - V$ criticality of these black holes has some unusual features. There exists a single critical point with critical temperature $T_c$ and critical pressure $P_c$. At fixed $T > T_c$ (or at fixed $P > P_c$), there are two zeroth order phase transition points but no first order phase transition points. The systems favors large pressure states at constant $T$, or high temperature states at constant $P$.

1 Introduction

Recently the study of thermodynamics of black holes in AdS spacetime has been generalized to the extended phase space, in which the cosmological constant is identified as a thermodynamic pressure and its variations are included in the first law of black hole thermodynamics [1–5]. Such studies are mainly motivated by the geometric derivation of the Smarr relation [1,3,5,6], in which a term consisting of the pressure and its conjugate “thermodynamic volume” appears. Once one takes the cosmological constant as thermodynamic pressure in the first law, the black hole mass should be explained as enthalpy rather than internal energy of the system. Besides, the criticality associated
with the pressure was discussed in the extended phase space [7–25]. With the extended structure of the thermodynamic phase space, one can often find phase transitions and critical points of various kinds. For instance, an interesting phenomena of black hole reentrant phase transition is found recently in four-dimensional Born-Infeld-AdS black hole [8], higher dimensional rotating AdS black holes [16], Kerr AdS black holes [19], GB-AdS black hole [21] and the third order Lovelock AdS black holes [26], which is observed previously in multi-component fluids [27]. It is also possible to take other parameters as novel dimensions of the thermodynamic phase space, e.g. the Born-Infeld parameter in the Born-Infeld-AdS black holes [8] and the GB parameter in GB-AdS black holes [25].

The reentrant phase transition includes both first and zeroth order phase transitions. However, the origin of zeroth order phase transition is still unclear. This makes it interesting to consider black hole systems containing only the zeroth order phase transition. The thermodynamics of the AdS black holes in conformal gravity [28] to be studied in this work provides precisely such an example.

On the other hand, although taking the cosmological constant as a thermodynamic pressure is nowadays a common practice, such operations implicitly assume that gravitational theories which differ only in the values of the cosmological constants are considered to fall in the “same class”, with unified thermodynamic relations. The common excuse for doing this is that the classical theory of gravity may be an effective theory which follows from a yet unknown fundamental theory, in which all the presently “physical constants” are actually moduli parameters that can run from place to place in the moduli space of the fundamental theory. Given that the fundamental theory is yet unknown, it is preferable to consider the extended thermodynamics of gravitational theories involving only a single action, which requires that all variables included in the thermodynamical relations must be integration constants. It so happens that, in conformal gravity, the cosmological constant comes as an integration constant rather than as a bare parameter which appear explicitly in the classical action.

In this paper, we consider the $P - V$ criticality of the static charged AdS Black hole in conformal gravity. There exists a single critical temperature, above which there are two zeroth order phase transition points. In the next section, we first revisit the thermodynamics of static charged AdS black hole in conformal gravity in the extended phase space. Section 3 is devoted to the $P - V$ criticality, particular attention is paid
toward the appearance of critical points and the zeroth order phase transitions. Finally, some concluding remarks are given in the last section.

2 Thermodynamics of static charged AdS Black hole in conformal gravity revisited

We consider the static charged AdS Black hole in conformal gravity. The action of conformal gravity reads

\[ S = \alpha \int d^4x \sqrt{-g} \left( \frac{1}{2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{1}{3} F^{\mu\nu} F_{\mu\nu} \right), \tag{1} \]

where the unusual sign in front of the Maxwell term is inspired by critical gravity [29] and is required by requiring that the Einstein gravity emerges from conformal gravity in the infrared limit [30].

The static black hole solution with AdS asymptotics for conformal gravity is found in [28], which takes the form

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,\epsilon}^2, \tag{2} \]

where

\[ f(r) = -\frac{1}{3} \Lambda r^2 + c_1 r + c_0 + \frac{d}{r}, \tag{3} \]

and

\[ A = -\frac{Q}{r} dt \tag{4} \]

is the corresponding Maxwell field. \( d\Omega_{2,\epsilon}^2 \) represents the line element of a 2d maximally symmetric Einstein space with constant curvature \( 2\epsilon \), with \( \epsilon = 1, 0 \) and \(-1\), respectively. There are six parameters \( Q, c_0, c_1, d, \Lambda, \epsilon \) in the solution, five of which are integration constants, and the last one determines the spacial sectional geometry of the horizon. These parameters obey a constraint

\[ 3c_1 d + \epsilon^2 + Q^2 = c_0^2, \tag{5} \]

so there are actually only 5 independent parameters (\( c_0 \) is to be considered to be determined by other parameters as in (5)). Except the discrete parameter \( \epsilon \), the rest
4 parameters $Q, c_1, d, \Lambda$ are related to conserved charges: electric charge, charge of massive spin-2 hair, enthalpy and pressure, respectively. However, at fixed charges $Q, c_1, d, \Lambda$ and $\epsilon$, there still exist a discrete freedom in choosing the integration constant $c_0$:

$$c_0 = \pm \sqrt{3c_1d + \epsilon^2 + Q^2}.$$ 

Under the point of view of taking the cosmological constant $\Lambda$ as a thermodynamic pressure

$$P = -\frac{\Lambda}{8\pi},$$

the energy calculated by employing the Noether charge associated with the time-like Killing vector [31] should be identified with the enthalpy $H$ of the gravitational system. It reads

$$H = \frac{\alpha (c_0 - \epsilon)(\Lambda r_0^2 - 3c_0)}{72\pi r_0} + \frac{\alpha(2\Lambda r_0^2 - c_0 + \epsilon)d}{24\pi r_0^2},$$

$$= \frac{\alpha(c_1c_0 - c_1\epsilon - 16\pi Pd)}{24\pi},$$

where $r_0 > 0$ denotes the largest real root of $f(r)$ which corresponds to the event horizon black hole$^1$, $\alpha$ is overall coupling which is present in the action (1). Because of the double-valuedness of the integration constant $c_0$, one immediately sees a double-valued behavior of the enthalpy. Such behaviors are also present in the four dimensional charged rotating black hole [32] and six dimensional static black holes [33] of conformal gravity. One can expect that the temperature and Gibbs free energy may also be double-valued. These double-valued variables must be all considered in order to have a holistic look at the thermodynamics in the extended phase space of the black hole.

The thermodynamical conjugate of the pressure, i.e. the “thermodynamic volume”, is given by

$$V = \left( \frac{\partial H}{\partial P} \right)_{S,Q,\Xi} = -\frac{\alpha d}{3}.$$ 

It can be seen that the sign of $V$ is determined by the sign of the parameter $d$. Besides $P$ and $V$, all the other thermodynamic quantities are given in [28]. The temperature is

$$T = \frac{8\pi P r_0^3 - 3c_0r_0 - 6d}{12\pi r_0^2},$$

$^1$We take $\Lambda < 0$, and so there is no cosmological horizon in the solution.
and its conjugate, i.e. the entropy is

$$S = \frac{\alpha (\epsilon r_0 - c_0 r_0 - 3 d)}{6r_0}. \quad (10)$$

The electric charge and the conjugate potential are respectively

$$Q_e = \frac{\alpha Q}{12\pi}, \quad (11)$$
$$\Phi = -\frac{Q}{r_0}. \quad (12)$$

The parameter $c_1$ is a massive spin-2 hair which is now taken as a novel dimension in the thermodynamic phase space. We label this novel dimension and its conjugate as $\Xi, \Psi$:

$$\Xi = c_1, \quad (13)$$
$$\Psi = \frac{\alpha (c_0 - \epsilon)}{24\pi}. \quad (14)$$

Throughout this work, we will take the normalization $\alpha = 2$. It can be checked that the first law of thermodynamics

$$dH = TdS + \Phi dQ_e + \Psi d\Xi + V dP \quad (15)$$

and the Smarr relation

$$H = 2PV + \Psi \Xi \quad (16)$$

hold in the extended thermodynamic phase space [28]. The absence of $TS$ and $\Phi Q$ terms in the Smarr relation can be explained by scaling arguments. The enthalpy can be viewed as a homogeneous function of the extensive variables $S, Q_e, \Xi, P$, i.e.

$$H = H(S, Q_e, \Xi, P).$$

Assuming each extensive variable has a scaling dimension which is denoted $d_S, d_Q, d_\Xi, d_P$ respectively. If the enthalpy $H$ itself has scaling dimension $d_H$, then after a rescaling of the extensive variables we get

$$\lambda^{d_H} H = H(\lambda^{d_S} S, \lambda^{d_Q} Q_e, \lambda^{d_\Xi} \Xi, \lambda^{d_P} P).$$

Taking the first derivative with respect to $\lambda$ and then setting $\lambda = 1$, we get

$$d_H H = d_S TS + d_Q \Phi Q_e + d_\Xi \Psi \Xi + d_P VP.$$
After a little calculation, one can find that $c_1$ scales as $[\text{length}]^{-1}$; $c_0$ scales as $[\text{length}]^0$; $\Lambda$ scales as $[\text{length}]^{-2}$ which is the same with $P$; $d$ scales as $[\text{length}]^1$; $Q$ scales as $[\text{length}]^0$, which is the same with $Q_e$; $S$ scales as $[\text{length}]^0$; $H$ scales as $[\text{length}]^{-1}$; $d$ scales as $[\text{length}]^1$, which results in the Smarr relation (16), as expected.

While considering critical behaviors, the Gibbs free energy will play an important role. It is given as follows:

$$G = H - TS$$
$$= -\frac{\alpha Pd}{3} - \frac{\alpha}{24\pi r_0^3} \left( (c_0 r_0 - \epsilon r_0 + 3d)(c_0 r_0 + 2d) + (c_0 - \epsilon)(c_0 r_0 + d) r_0 \right). \quad (17)$$

On the other hand, the Helmholtz free-energy can be obtained from the Euclideanized action:

$$F = \frac{\alpha}{24\pi r_0^2} \left( 2(c_0 - \epsilon)\epsilon r_0 + (3\epsilon + 8\pi P r_0^2) d \right). \quad (18)$$

A direct check yields

$$F = H - TS - \Phi Q_e = G - \Phi Q_e. \quad (19)$$

This in turn justifies the explanation of $H$ as thermodynamic enthalpy.

Before proceeding, let us reveal a natural constraint of the black hole solutions. From $f(r_0) = 0$ we get $P = -\frac{24\pi (r_0^2 c_1 + c_0 r_0 + d)}{r_0^3}$. On the other hand, from eq. (5) we get $d = -\frac{\epsilon^2 + Q^2 - 3c_0^2}{c_1}$. Inserting both into (9), one can obtain

$$T = \frac{\epsilon^2 + Q^2}{4c_1 \pi r_0^2} - \frac{(c_0 + r_0 c_1)^2}{4c_1 \pi r_0^2},$$

which leads to the constraint on $T$ and $r_0$

$$(Q^2 + \epsilon^2) - 4c_1 \pi r_0^2 T \geq 0. \quad (20)$$

This constraint implies an upper bound of $T$ at fixed horizon radius $r_0$, or an upper bound of $r_0$ at fixed temperature $T$. 
3 $P−V$ criticality

3.1 Critical points

To consider the $P−V$ criticality of the black hole, we should begin with the equation of state (EOS) in $P−V$ plane at fixed conserved charges $Q$ and $c_1$. The EOS arises from the expression (9) for the temperature $T$. However, to use (9) as a reasonable EOS, we need to eliminate the parameters $c_0$ and $d$. Assuming that both of these parameters are nonzero. Then the condition $f(r_0) = 0$ yields

$$c_0 = -\frac{8}{3}\pi Pr_0^2 - c_1 r_0 - \frac{d}{r_0}.$$  \hfill (21)

Inserting (21) into eqs.(5) and (9) respectively, one gets

$$\frac{64}{9}\pi^2 P^2 r_0^4 + \frac{16}{3}\pi Pr_0 \left( c_1 r_0^2 + d \right) + c_1^2 r_0^2 + \frac{d^2}{r_0^2} - Q^2 - \epsilon^2 - c_1 d = 0$$  \hfill (22)

$$T = \frac{4}{3} Pr_0 + \frac{c_1}{4\pi} - \frac{d}{4\pi r_0^2}.$$  \hfill (23)

Then from eq.(23), we find

$$d = \frac{16}{3}\pi Pr_0^3 - 4 T\pi r_0^2 + c_1 r_0^2.$$  

Inserting this into eq.(22), we get the EOS

$$64\pi^2 r_0^4 P^2 - 16\pi \left( 4T - c_1 \right) r_0^3 P + 16\pi^2 r_0^4 T^2 - 4\pi c_1 r_0^3 T + c_1^2 r_0^2 T + (Q^2 + \epsilon^2) = 0.$$  

Solving this equation for $P$ we get

$$P = \frac{T}{2r_0} - \frac{c_1 r_0 \pm \sqrt{(Q^2 + \epsilon^2) - 4 c_1 \pi r_0^2 T}}{8\pi r_0^2},$$

and thanks to the condition (20), both branches of solutions should be considered physical. Comparing the above expressions for the pressure with the van der Waals equation

$$P = \frac{T}{v - b} - \frac{a}{v^2} \simeq \frac{T}{v} + \frac{bT}{v^2} - \frac{a}{v^2} + O(v^{-3}),$$

we are tempted to use the variable

$$v = 2r_0$$  \hfill (24)
as an effective specific volume for the black hole system under consideration. Thus we rewrite the full EOS as

\[ 4\pi^2 v^4 P^2 - 2(4\pi T - c_1)\pi v^3 P + 4\pi^2 v^2 T^2 - c_1\pi v^2 T + \frac{c_1^2 v^2}{4} - (Q^2 + \epsilon^2) = 0. \]  

(25)

The critical points \((P_c, v_c, T_c)\) results from the conditions

\[ \frac{\partial P}{\partial v} |_{v=v_c,T=T_c} = 0, \]  

(26)

\[ \frac{\partial^2 P}{\partial v^2} |_{v=v_c,T=T_c} = 0, \]  

(27)

\[ P_c = P |_{v=v_c,T=T_c}. \]  

(28)

The partial derivative in (26) and (27) can be evaluated directly using (25), which read

\[ \frac{\partial P}{\partial v} = -\frac{2P}{v} + \frac{4T\pi - c_1}{4\pi v^2} + \frac{c_1 T}{v^2 (4T\pi - c_1 - 4\pi P v)}, \]  

(29)

\[ \frac{\partial^2 P}{\partial v^2} = \frac{6P}{v^2} - \frac{4T\pi - c_1}{v^3\pi} - \frac{3c_1 T}{(4T\pi - c_1 - 4\pi P v) v^3} + \frac{4\pi T^2 c_1^2}{(4T\pi - c_1 - 4\pi P v)^3 v^5}. \]  

(30)

Solving these two equations and substituting into (28), we get two sets of critical point parameters as follows.

1. When \(c_1 > 0\),

\[ v_c = \sqrt{X\left(Q^2 + \epsilon^2\right)} \div c_1, \quad T_c = \frac{(3X - 8)c_1}{4\pi (27X - 8)}, \quad P_c = \frac{-c_1^2 (3X - 8)}{16\pi \sqrt{X\left(Q^2 + \epsilon^2\right)}X}; \]  

2. When \(c_1 < 0\),

\[ v_c = \sqrt{X\left(Q^2 + \epsilon^2\right)} \div -c_1, \quad T_c = \frac{(3X - 8)c_1}{4\pi (27X - 8)}, \quad P_c = \frac{c_1^2 (3X - 8)}{16\pi \sqrt{X\left(Q^2 + \epsilon^2\right)}X}. \]  

In the above, \(X\) can take two discrete values \(X_1\) or \(X_2\):

\[ X_1 = \frac{40}{3} + \frac{16}{3}\sqrt{6} \simeq 26.3973, \]  

(31)

\[ X_2 = \frac{40}{3} - \frac{16}{3}\sqrt{6} \simeq 0.2694. \]  

(32)

However, the physical critical point must obey the constraints \(P_c > 0\), \(r_c > 0\) and \(T_c > 0\), which lead to

\[ X < \frac{8}{27}, \quad c_1 > 0. \]  

(33)
This exclude the choice \( c_1 < 0 \) and \( X = X_1 \), and we are left with a single critical point which is characterized by the parameters

\[
v_c = \frac{\sqrt{X_2 (Q^2 + \epsilon^2)}}{c_1}, \quad T_c = \frac{(8 - 3X_2) c_1}{4\pi (8 - 27X_2)}, \quad P_c = \frac{c_1^2 (8 - 3X_2)}{16\pi \sqrt{X_2 (Q^2 + \epsilon^2)X_2}},
\]

(34)

in which \( c_1 > 0 \). These parameters satisfy the relation

\[
\frac{P_cv_c}{T_c} = \frac{8 - 27X_2}{4X_2} \simeq 0.6742,
\]

(35)

which gives a pure number and is independent of all parameters. Actually, this relation is very similar to the one of the van der Waals system, which behaviors as \( \frac{P_cv_c}{T_c} = \frac{3}{8} \) at its critical point. However, if one replaces \( v \) in Eq.(35) by the thermodynamic volume \( V \), the result of this relation will be no longer independent of parameters. In this sense, it is more natural to consider \( v \) as the specific volume instead of the thermodynamic volume \( V \). Note that the existence of critical point also exclude the possibility of \( c_1 = 0 \). Also, the square sum of the charge \( Q \) and the signature \( \epsilon \) must also be nonzero.

Therefore, there is neither need to distinguish the case \( Q = 0 \) from the \( Q \neq 0 \) cases (as long as \( \epsilon \neq 0 \)) nor need to distinguish the \( \epsilon = 0 \) case from the \( \epsilon = \pm 1 \) cases (as long as \( Q \neq 0 \)).

One may also be curious about the cases with \( d = 0 \) (the BPS black hole) or \( c_0 = 0 \), both of which will result in an EOS of ideal gas after considering eq. (9) directly. Thus they are all out of our discussion. Another degenerated case \( c_1 = 0, Q = 0 \) corresponds to the Schwarzschild-AdS black hole. In this case one can never find physical critical points as is known in [7].

The isothermal plots at generic parameters \( Q, c_1 \) are depicted in Fig.1 (the right plot is a magnification of a single isotherm at the temperature \( T = 1.5T_c \)). While creating the plots, we take \( \frac{\sqrt{Q^2 + \epsilon^2}}{c_1}, c_1 \) and \( \frac{c_1^2}{\sqrt{Q^2 + \epsilon^2}} \) respectively as units for \( v_c, T_c \) and \( P_c \). The pressure corresponding to the extremal specific volume is denoted \( P_1 \).

It can be seen that on each isotherm there is an upper bound for the specific volume (black hole radius) \( v_{ex} \). For \( v < v_{ex} \), the isotherm can be subdivided into two segments, i.e. the lower branch and the upper branch, which reflect the double-valuedness of the pressure. The difference between the \( T < T_c \) and \( T > T_c \) curves lies in that, each branch of the isotherm in the former case is monotonic with respect to the special volume \( v \), while the lower branch in the latter case is non-monotonic. Consequently phase transitions will occur only in the \( T > T_c \) regime.
Another way of subdividing the isotherms is according to the sign of \((\frac{\partial P}{\partial v})_T\), which is inversely proportional to the isothermal compressibility

\[ \alpha \equiv \frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_T. \]

According to the sign of the isothermal compressibility, each isotherm with \(T > T_c\) can be subdivided into 4 segments, two with positive isothermal compressibility and two with negative isothermal compressibility.

Let us take a closer look at the magnified plot given on the right diagram in Fig.1. This is a curve corresponding to isotherm with \(T = 1.5T_c\). The lower and upper branches of the curve is joined together at the point D which corresponds to the extremal specific volume \(v_{ex}\) and the pressure \(P = P_1\). On the lower branch (plotted in solid line) one can see that there is a local maximum \(P_2\) and local minimum \(P_0\) for \(P\), the corresponding points on the isotherm are marked with B and C respectively. The 4 segments of the isotherm are \(IB, BC, CD\) and \(DJ\) respectively. Among these, \(IB\) and \(CD\) have negative isothermal compressibilities which imply that black hole states falling in these segments may be unstable.

![Figure 1: The isotherms: on the left plot, at the intersection of each isotherms with the horizontal axes, the temperature decreases from left to right. The specific volume on each isotherm (horizon radius) has an upper bound. The dashed curve corresponds to critical temperature.](image)

The right plots in Fig.1 is typical for all \(T > T_c\). At such temperatures, we can subdivide the range of the pressure into 4 regimes: \(P < P_0, P_0 \leq P < P_1, P_1 \leq P < P_2\)
and $P \geq P_2$. When $P < P_0$, there is a single unstable black hole phase represented by the segment $\widehat{IM}$. Due to its unstable nature, a perturbative increase in the pressure would result in an increase of the black hole radius until it reaches the state $M$, then through a phase transition it will enter the second regime for the pressure, $P_0 \leq P < P_1$. In this second regime for the pressure, there are 3 different black hole states for a single pressure value, among these, only the one with intermediate sized specific volume (i.e. the state lying on the segment $\widehat{KC}$) is stable. Therefore, as the pressure increases, the black hole state will evolve from the state $C$ until the state $K$ is reached and then the pressure enters the next regime, $P_1 \leq P < P_2$. There are still 3 black hole states at each pressure values in this regime, however, two of these have positive isothermal compressibility (i.e. the states lying on the $\widehat{KB}$ and $\widehat{DN}$ segments), so it is hard to tell which is more stable by looking at the EOS alone. If the pressure enters the fourth regime, $P \geq P_2$, then there is only a single stable phase which corresponds to states on the segment $\widehat{NJ}$. From the above analysis, it is clear that there are two possible phase transition points, the first one occurs at $P = P_0$, where the black hole will most probably transit from the state $M$ to the state $C$ which is more stable upon perturbation. The second phase transition point occurs either at $P = P_1$ (if the Gibbs free energy on the segment $\widehat{KB}$ is higher than that on $\widehat{DN}$) or at $P = P_2$ (if the Gibbs free energy on the segment $\widehat{KB}$ is lower than that on $\widehat{DN}$). In the next subsection, it will be clear that the Gibbs free energy on the segment $\widehat{KB}$ is always lower than it is on $\widehat{DN}$, so the second phase transition point occurs at $P = P_2$, where the black hole picks the state $B$ instead of $N$, because the state $B$ has lower Gibbs free energy than the state $N$.

### 3.2 Gibbs free energy and the zeroth order phase transitions

In order to have a further look at the critical points, we need to plot the Gibbs free energy versus pressure at fixed temperature. The Gibbs free energy $G$ as a function of $T$ and $P$ is quite complicated and cannot be given explicitly with ease. So we will try to present the $G - P$ relationship in terms of a pair of parametric equations.

First we can invert the relation (23) to get $P$ as a function of $T$ and other parameters. Eliminating $c_0$ in the resulting expression by use of (21), we get

$$P(v, d) = \frac{3d}{2\pi v^3} + \frac{3(4\pi T - c_1)}{8\pi v}, \quad (36)$$
where we have also replaced \( r_0 \) by \( v/2 \). Inserting this equation together with (21) into (17), we get

\[
G(v, d) = \frac{(4\pi T - c_1) d}{12v\pi} - \frac{(4\pi T + c_1)^2 v}{144\pi} - \frac{\epsilon (4\pi T + c_1)}{12\pi} - \frac{2(Q^2 + \epsilon^2)}{9v\pi}.
\] (37)

In (1) and (37), \( d \) cannot be taken as a free parameter, because there is an extra constraint (22), which can be rewritten as

\[
\frac{9d^2}{v^2} + \frac{3d}{2} (4\pi T - c_1) + \frac{v^2}{16} (4\pi T + c_1)^2 - (Q^2 + \epsilon^2) = 0.
\] (38)

This quadratic algebraic equation gives two solutions for \( d \) as a function of \( T, v \). Picking each solution and substituting into (36) and (37), the resulting pair of equations can be taken as parametric definition for the function \( G(P, T) \) for constant \( T \). So, we can easily plot the \( G - P \) diagram at constant \( T \), which is presented in Fig. 2.

Figure 2: The Gibbs free energy \( G \) versus \( P \) at constant temperatures. On the intersections with vertical axes, the temperature decreases from bottom to top, and the temperatures for each curve are in one to one correspondence to the isotherms given in Fig. 1. The right plot is a magnification of the lower-left corner of the (red) line with the highest temperature in the left plot. The marked points are the same as in the right plot of Fig. 1. The global minimum of the Gibbs free energy in the right plot is highlighted by the thick line.

It can be seen in Fig. 2 that, for all \( T > T_c \), a downcast swallow tail appears on each \( G - P \) curve. From \( T = T_c \) and downwards, the swallow tail disappears, with
\( T = T_c \) corresponding to the critical point. This is different from that of a Van der Waals liquid-gas system, where \( T < T_c \) is required for a phase equilibrium. The same phenomenon has also been observed in the study of criticality associated with the GB coupling constant [25] in Gauss-Bonnet gravity.

The downcast swallow tail is different from the usual upcast one, which corresponds to first order phase transitions [7]. On the magnified plot given on the right plot of Fig.2, it is clear that the Gibbs free energy on the segment \( \text{CKB} \) is lower than that on \( \text{CDN} \) and \( \text{BLM} \). At \( P = P_0 \) and \( P = P_2 \), there exist discontinuities for the Gibbs free energy, indicating that there are zeroth order phase transitions at these two particular pressures.

If one follows the red curve given in the left plot of Fig.2, it would be clear that at sufficiently high pressure, the Gibbs free energy on the dotted segment would eventually become lower than it is on the solid segment. This implies that the thermodynamics of the system favors large pressure states. On the other hand, if one looks at the Gibbs free energy versus temperature plots at fixed pressure (Fig.3), one would see the similar downcast swallow tail at \( P > P_c \). Moreover, following a single \( G - T \) curve reveals that the system favors high temperature states if the pressure is kept fixed.

![Figure 3: The Gibbs free energy versus temperature at constant pressure: the curves on the upper-right has higher pressure](image)
4 Concluding remarks

In this paper, we considered the $P - V$ criticality of static charged AdS black holes in conformal gravity. Unlike the cases of Einstein gravity, the cosmological constant arises as an integration constant in conformal gravity, making the analysis for $P - V$ criticality more self-contained, e.g. without need to consider systems with different actions.

The thermodynamics in the extended phase space for black hole in conformal gravity possesses several unusual features:

- there exists only one critical point but there are two phase transition points, both of which corresponds to zeroth order phase transitions;
- there is no first order phase transition in the system;
- the phase transition can occur only when $T > T_c$ (or $P > P_c$) but not the other way round;
- at fixed $T > T_c$, the system favors large pressure states, whilst at fixed $P > P_c$, the system favors high temperature states.

We do not know of any other black hole or ordinary matter systems which exhibit similar thermodynamic behaviors. It would be interesting to find other examples which yield similar behaviors, because otherwise the system under study would seem to be too bizarre to understand.

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References

[1] D. Kastor, S. Ray and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” Class. Quant. Grav. 26, 195011 (2009) [arXiv:0904.2765].
[2] B. P. Dolan, “The cosmological constant and the black hole equation of state,” Class. Quant. Grav. 28, 125020 (2011) [arXiv:1008.5023].

[3] B. P. Dolan, “Pressure and volume in the first law of black hole thermodynamics,” Class. Quant. Grav. 28, 235017 (2011) [arXiv:1106.6260].

[4] B. P. Dolan, “Compressibility of rotating black holes,” Phys. Rev. D 84, 127503 (2011) [arXiv:1109.0198].

[5] M. Cvetkovic, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume,” Phys. Rev. D 84, 024037 (2011) [arXiv:1012.2888].

[6] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann and J. Traschen, “Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes,” Phys. Rev. D 87, no. 10, 104017 (2013) [arXiv:1301.5926].

[7] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” JHEP 1207, 033 (2012) [arXiv:1205.0559].

[8] S. Gunasekaran, R. B. Mann and D. Kubiznak, “Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization,” JHEP 1211, 110 (2012) [arXiv:1208.6251].

[9] B. P. Dolan, “Where is the PdV term in the fist law of black hole thermodynamics?,” [arXiv:1209.1272].

[10] A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, “On Thermodynamics of AdS Black Holes in Arbitrary Dimensions,” Chin. Phys. Lett. 29, 100401 (2012) [arXiv:1210.4617].

[11] S. H. Hendi and M. H. Vahidinia, “P-V criticality of higher dimensional black holes with nonlinear source,” Phys. Rev. D 88, 084045 (2013) [arXiv:1212.6128].

[12] S. Chen, X. Liu, C. Liu and J. Jing, “P − V criticality of AdS black hole in f(R) gravity,” Chin. Phys. Lett. 30, 060401 (2013) [arXiv:1301.3234].

[13] R. Zhao, H. -H. Zhao, M. -S. Ma and L. -C. Zhang, “On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes,” Eur. Phys. J. C 73, 2645 (2013) [arXiv:1305.3725].

[14] A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, “Critical Behaviors of 3D Black Holes with a Scalar Hair,” [arXiv:1306.2518].

[15] M. B. J. Poshteh, B. Mirza and Z. Sherkatghanad, “Phase transition, critical behavior, and critical exponents of Myers-Perry black holes,” Phys. Rev. D 88, no. 2, 024005 (2013) [arXiv:1306.4516].

[16] N. Altamirano, D. Kubiznak and R. B. Mann, “Reentrant Phase Transitions in Rotating AdS Black Holes,” Phys. Rev. D 88, 101502 (2013) [arXiv:1306.5756].

[17] R. -G. Cai, L. -M. Cao, L. Li and R. -Q. Yang, “P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space,” JHEP 1309, 005 (2013) [arXiv:1306.6233].
[18] A. Belhaj, M. Chabab, H. E. Mounni, L. Medari and M. B. Sedra, “The Thermodynamical Behaviors of Kerr-Newman AdS Black Holes,” Chin. Phys. Lett. 30, 090402 (2013) [arXiv:1307.7421].

[19] N. Altamirano, D. Kubizk, R. B. Mann and Z. Sherkatghanad, “Kerr-AdS analogue of triple point and solid/liquid/gas phase transition,” Class. Quant. Grav. 31, 042001 (2014) [arXiv:1308.2672].

[20] N. Altamirano, D. Kubiznak, R. B. Mann and Z. Sherkatghanad, “Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume,” Galaxies 2, 89 (2014) [arXiv:1401.2586].

[21] S. -W. Wei and Y. -X. Liu, “Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space,” [arXiv:1402.2837].

[22] D. Kubiznak and R. B. Mann, “Black Hole Chemistry,” [arXiv:1404.2126].

[23] D. -C. Zou, S. -J. Zhang and B. Wang, “Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics,” Phys. Rev. D 89, 044002 (2014) [arXiv:1311.7299].

[24] D. -C. Zou, Y. Liu and B. Wang, “Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble,” [arXiv:1404.5194].

[25] W. Xu, H. Xu and L. Zhao, “Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality,” [arXiv:1311.3053].

[26] H. Xu, W. Xu and L. Zhao, “Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions,” [arXiv:1405.4143].

[27] T. Narayanan and A. Kumar, Reentrant phase transitions in multicomponent liquid mixtures, Physics Reports 249 (1994) 135218.

[28] J. Li, H. -S. Liu, H. Lu and Z. -L. Wang, “Fermi Surfaces and Analytic Green’s Functions from Conformal Gravity,” JHEP 1302, 109 (2013) [arXiv:1210.5000].

[29] H. Lu and C. N. Pope, “Critical Gravity in Four Dimensions,” Phys. Rev. Lett. 106, 181302 (2011) [arXiv:1101.1971].

[30] J. Maldacena, “Einstein Gravity from Conformal Gravity,” [arXiv:1105.5632].

[31] H. Lu, Y. Pang, C. N. Pope and J. F. Vazquez-Poritz, “AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities,” Phys. Rev. D 86, 044011 (2012) [arXiv:1204.1062].

[32] H. -S. Liu and H. Lu, “Charged Rotating AdS Black Hole and Its Thermodynamics in Conformal Gravity,” JHEP 1302, 139 (2013) [arXiv:1212.6264].

[33] H. Lu, Y. Pang and C. N. Pope, “Black Holes in Six-dimensional Conformal Gravity,” Phys. Rev. D 87, 104013 (2013) [arXiv:1301.7083].