Supersymmetric $CP^N$ Sigma Model on Noncommutative Superspace

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Abstract

We construct a closed form of the action of the supersymmetric $CP^N$ sigma model on noncommutative superspace in four dimensions. We show that this model has $\mathcal{N} = \frac{1}{2}$ supersymmetry and that the transformation law is not modified. The supersymmetric $CP^N$ sigma model on noncommutative superspace in two dimensions is obtained by dimensionally reducing the model in four dimensions.

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1 Introduction

Noncommutative geometry [1] appears in M-theory, string theory and condensed matter physics. Noncommutative field theories are known to describe the effective theory of string in a constant NS-NS $B$ field [2]. (2+1)-dimensional noncommutative field theories have been applied to the quantum Hall effect.

In supersymmetric field theories, there are a few alternatives in introducing non(anti)commutativity of the supercoordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$\(^1\). In particular, supersymmetric Yang-Mills theory on noncommutative superspace [4, 5, 6] describes the effective field theory of string in a constant selfdual graviphoton background [6, 7, 8]. In the field theoretical viewpoint, these theories keep $\mathcal{N} = \frac{1}{2}$ supersymmetry and have some interesting properties.

In this letter, we construct the supersymmetric nonlinear sigma model whose target space is $CP^N$ ($CP^N$ SNLSM) on noncommutative superspace in four and two dimensions. Low-dimensional SNLSMs on ordinary superspace have interesting properties. In two dimensions, the $CP^N$ SNLSM is integrable, i.e., it has infinitely many conservation laws. It shares important properties with four-dimensional supersymmetric gauge theories, such as asymptotic freedom and dynamical mass gap. In three dimensions, the $CP^N$ SNLSM has been investigated using the large-$N$ expansion [9]. As we will see below, since the Kähler potential of SNLSM is generally non-polynomial, the action of SNLSM on noncommutative superspace has infinitely many terms [10]. It is difficult to study the properties of this model either perturbatively or non-perturbatively. We introduce an auxiliary vector superfield to linearize the $CP^N$ SNLSM, mimicking the commutative case [11]. Once introducing the vector superfield, we can eliminate all auxiliary fields and obtain a closed form of the action.

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\(^1\) We follow the notation of [3].
2 Noncommutative Superspace

2.1 Noncommutative Superspace

We recapitulate noncommutative superspace, closely following Seiberg [6]. We consider four-dimensional $\mathcal{N} = 1$ supersymmetric field theories on the noncommutative superspace. The non(anti)commutativity is introduced by

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0,$$

$$[y^\mu, y^{\nu}] = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^{\dot{\alpha}}] = 0,$$

where $y^\mu$ is the chiral coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}.$$  \hspace{1cm} (2)

The product of functions of $\theta$ is Weyl ordered by using the Moyal product, which is defined by

$$f(\theta) \ast g(\theta) = f(\theta) \exp \left( -\frac{1}{2} C^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \right) g(\theta)$$

$$= f(\theta) \left[ 1 - \frac{1}{2} C^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} - \det C \frac{\partial}{\partial (\theta\theta)} \frac{\partial}{\partial (\theta\theta)} \right] g(\theta).$$ \hspace{1cm} (3)

The supercovariant derivatives are defined by

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial y^\mu}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}. \hspace{1cm} (4)$$

Since $D_\alpha$ and $\bar{D}_{\dot{\alpha}}$ do not contain $\theta$, their anticommutation relations are same as those on the commutative superspace.

$$\{D_\alpha, D_\beta\} = 0, \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \quad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu}. \hspace{1cm} (5)$$

The supercharges are defined by

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha}, \quad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + 2i\theta^{\alpha} \sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu}. \hspace{1cm} (6)$$
Since $\bar{Q}_\alpha$ contains $\theta$, the anticommutation relations are modified as follows.

\[
\{Q_\alpha, Q_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = +2i\sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu},
\]

\[
\{\bar{Q}_\alpha, \bar{Q}_\beta\} = -4C^{\alpha\beta} \sigma^\mu_{\alpha\dot{\alpha}} \sigma^{\nu}_{\beta\dot{\beta}} \frac{\partial^2}{\partial y^\mu \partial y^\nu}.
\]

Furthermore, $\bar{Q}_\alpha$ does not act as derivations on the Moyal product of fields

\[
\bar{Q}_\alpha(f * g) \neq (\bar{Q}_\alpha f) * g + f * (\bar{Q}_\alpha g).
\]

Then $\bar{Q}_\alpha$ is not a symmetry of the theory in general, hence we have $\mathcal{N} = \frac{1}{2}$ supersymmetry.

### 2.2 Superfields

The chiral superfield is defined by $\bar{D}_\dot{\alpha}\Phi = 0$, and hence, $\Phi = \Phi(y, \theta)$. In terms of the component fields, it is given by

\[
\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),
\]

where $\theta\theta = -\theta^1\theta^2 + \theta^2\theta^1$ and is Weyl ordered.

The antichiral superfield is defined by $D_\alpha\bar{\Phi} = 0$, and hence, $\bar{\Phi} = \bar{\Phi}(\bar{y}, \bar{\theta})$, where $\bar{y}^\mu$ is given by

\[
\bar{y}^\mu = y^\mu - 2i\theta\sigma^\mu\bar{\theta}, \quad [\bar{y}^\mu, \bar{y}^\nu] = 4\bar{\theta}\theta C^{\mu\nu}, \quad C^{\mu\nu} = C^{\alpha\beta}\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_{\gamma}.
\]

In the component fields, it is convenient to express the antichiral superfield in terms of $y$ and $\theta$ and to Weyl order the $\theta$s

\[
\bar{\Phi}(y - 2i\theta\sigma\bar{\theta}, \bar{\theta}) = \bar{\phi}(y) + \sqrt{2}\bar{\theta}\bar{\psi}(y) - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi}(y)
\]

\[
+ \bar{\theta}\bar{\theta}\left[\bar{F}(y) + \sqrt{2}i\theta\sigma^\mu\partial_\mu\bar{\psi}(y) + \theta\theta\partial^2\bar{\phi}(y)\right].
\]
We also need the U(1) vector superfield in constructing the $CP^N$ SNLSM later. The vector superfield is written in the Wess-Zumino gauge as

$$V(y, \theta, \bar{\theta}) = -\theta \sigma^\mu \overline{\theta} A_\mu(y) + i \theta \overline{\theta} \overline{\lambda}(y) - i \overline{\theta} \theta \alpha \left[ \lambda_\alpha(y) + \frac{1}{4} \epsilon_{\alpha \beta} C^{\beta \gamma} \sigma^\mu_{\gamma \epsilon} \{ \overline{\lambda}^\epsilon, A_\mu \} \right]$$

$$+ \frac{1}{2} \theta \overline{\theta} \overline{\theta} \left[ D(y) - i \partial_\mu A^\mu(y) \right].$$

(14)

The $C$-deformed part in the $\overline{\theta} \theta \overline{\theta}$ term is introduced in order that the component fields transform canonically under the gauge transformation. The powers of $V$ are obtained by

$$V^2 = \overline{\theta} \left[ -\frac{1}{2} \theta \theta A_\mu A^\mu - \frac{1}{2} C^{\mu \nu} A_\mu A_\nu \right.$$  

$$+ \frac{i}{2} \theta_\alpha C^{\alpha \beta} \sigma^\mu_{\beta \alpha} [A_\mu, \overline{\lambda}^\alpha] - \frac{1}{8} |C|^2 \overline{\lambda} \lambda \right] ,$$

(15)

$$V^3 = 0,$$  

(16)

where $|C|^2 = C^{\mu \nu} C_{\mu \nu} = 4 \det C$.

3 Supersymmetric $CP^N$ Sigma Model on Noncommutative Superspace

The Lagrangian of four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear sigma model (SNLSM) is written using the Kähler potential $K$ as

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \ K(\Phi, \bar{\Phi}).$$

(17)

The same expression can be used for $\mathcal{N} = 2$ SNLSM in two dimensions. The results derived below for the four-dimensional case hold true for the two-dimensional case after slight modification. See the argument of dimensional reduction to two dimensions in the end of this section.
In the case of a single pair of chiral and antichiral superfields, the Berezin integration in eq. (17) with the noncommutativity (1) was calculated in [10]. It is given by

\[ \mathcal{L} = \mathcal{L}(C = 0) + \sum_{n=1}^{\infty} (\det C)^n [A_n F^{2n-1} + B_n F^{2n}], \] (18)

where \( A_n \) and \( B_n \) are functions of the component fields. Eq. (18) contains infinitely many terms since generally \( K \) is not a polynomial and the powers of \( \theta \) are nonzero. We have not found a good way to analyze this model as it stands.

Some SNLSMs are expressed as supersymmetric gauge theories. Such SNLSMs contain the model whose target space is a Hermitian symmetric space [12] (e.g. \( CP^N \) and Grassmannian \( G_{N,M} \), \( T^*CP^N \)) and \( T^*G_{N,M} \). We construct the \( CP^N \) SNLSM on noncommutative superspace as the noncommutative extension of [11] using the result of [13]. We start from the following Lagrangian

\[ \mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \left[ \bar{\Phi}^i \ast e^V \ast \Phi^i - V \right], \] (19)

where \( i = 1, 2, \ldots, N + 1 \). \( V \) is the U(1) vector superfield. It is written modulo total derivatives in terms of component fields as

\[ \mathcal{L} = \bar{\Phi}^i F^i - i \bar{\psi}^i \sigma^\mu D_\mu \psi^i - D_\mu \bar{\phi}^i D^\mu \phi^i + \frac{1}{2} \bar{\phi}^i (\bar{\phi}^i \lambda \psi^i - \bar{\psi}^i \bar{\lambda} \phi^i) \]
\[ + \frac{i}{2} C^{\mu \nu} \bar{\phi}^i F_{\mu \nu} F^i - \frac{1}{16} C^2 \bar{\phi}^i \lambda \bar{\lambda} F^i - \frac{1}{\sqrt{2}} C^{\alpha \beta} (D_\mu \bar{\phi}^i) \sigma^\mu_{\beta \alpha} \bar{\lambda} \psi^i \]
\[ - \frac{1}{2} D. \] (20)

Here \( D_\mu \) is the gauge covariant derivative defined by

\[ D_\mu \phi^i = \partial_\mu \phi^i + \frac{i}{2} A_\mu \phi^i, \quad D_\mu \psi^i = \partial_\mu \psi^i + \frac{i}{2} A_\mu \psi^i. \] (21)

\( T^*CP^N \) denotes the cotangent bundle of \( CP^N \). \( T^*G_{N,M} \) is similar.
Following [13], we redefine the antichiral superfields $\bar{\Phi}^i$ in the Lagrangian (20) as
\[
\bar{\Phi}^i(y, \bar{\theta}) = \bar{\phi}^i(y) + \sqrt{2} \bar{\theta} \bar{\psi}^i(y) + \bar{\theta} \left[ \bar{F}^i(y) + iC^\mu \partial_\mu (\bar{\phi}^i)A_\nu(y) - \frac{1}{4} C^\mu \bar{\phi}^i A_\mu A_\nu(y) \right],
\]
so that the component fields transform canonically under the gauge transformation.

Eq. (20) contains the auxiliary fields $F^i, \bar{F}^i, D, \lambda, \bar{\lambda}$ and $A_\mu$. They have the role of imposing constraints on the fields as follows.

$\bar{F}^i : F^i = 0,$  
$F^i : \bar{F}^i + \frac{i}{2} C^\mu \bar{\phi}^i F_\mu = \frac{1}{16} |C|^2 \bar{\phi}^i \lambda \bar{\lambda} = 0,$  
$D : \bar{\phi}^i \dot{\phi}^i = 1,$  
$\lambda^\alpha : \bar{\phi}^i \dot{\psi}^i_\alpha = 0,$  
$\bar{\lambda}^\dot{\alpha} : \frac{i}{\sqrt{2}} \bar{\psi}^i_\dot{\alpha} \dot{\phi}^i - \frac{1}{8} |C|^2 \bar{\phi}^i \bar{\lambda}_\dot{\alpha} F^i - \frac{1}{\sqrt{2}} C^{\alpha \beta} (D_\mu \bar{\phi}^i) \sigma^\mu_{\beta \dot{\alpha}} \psi^i_\alpha = 0,$  
$A_\mu : \frac{1}{2} \bar{\psi}^i \sigma^\mu \psi^i + \frac{i}{2} (\bar{\phi}^i \partial^\mu \phi^i - \partial^\mu \bar{\phi}^i \cdot \phi^i) - \frac{1}{2} (\bar{\phi}^i \dot{\phi}^i) A_\mu$
\[
+ iC^\nu \partial_\nu (\bar{\phi}^i F^i) - \frac{i}{2 \sqrt{2}} C^{\alpha \beta} \sigma^\mu_{\beta \dot{\alpha}} \bar{\lambda}^\dot{\alpha} \bar{\phi}^i \psi^i_\alpha = 0. 
\]

After eliminating $F^i$ and $\bar{F}^i$, the Lagrangian (20) takes a simple form
\[
\mathcal{L} = -D_\mu \bar{\phi}^i D^\mu \phi^i - i \bar{\psi}^i \sigma^\mu D_\mu \psi^i, 
\]
with the constraints
\[
\bar{\phi}^i \dot{\phi}^i = 1, 
\]
\[
\bar{\phi}^i \dot{\psi}^i_\alpha = 0, 
\]
\[
\bar{\psi}^i_\dot{\alpha} \phi^i + iC^{\alpha \beta} \sigma^\mu_{\beta \dot{\alpha}} (D_\mu \bar{\phi}^i) \psi^i_\alpha = 0, 
\]
\[
A_\mu = i (\bar{\phi}^i \partial_\mu \phi^i - \partial_\mu \bar{\phi}^i \cdot \phi^i) + \bar{\psi}^i \sigma^\mu \psi^i. 
\]
The constraints (30-32) are solved as follows

\[
\phi^i = \frac{1}{\sqrt{1 + \varphi}} \left( \frac{\varphi^a}{1} \right), \quad \bar{\phi}^i = \frac{1}{\sqrt{1 + \bar{\varphi}}} \left( \frac{\bar{\varphi}^\alpha}{1} \right),
\]

(34)

\[
\psi^{ij}_a = \frac{1}{\sqrt{1 + \varphi}} P^{ij} \chi^{j}_a, \quad \chi^{i}_a = \left( \begin{array}{c} \lambda^{a}_\alpha \ 
\end{array} \right),
\]

(35)

\[
\bar{\psi}^{\dot{j}}_\dot{a} = \frac{1}{\sqrt{1 + \bar{\varphi}}} \left[ \bar{\chi}^{\dot{j}}_\dot{a} P^{\dot{j}i} - i C^{\alpha\beta} \sigma^{\mu}_\beta \bar{\phi}(\partial_\mu \bar{\phi}) P^{\dot{j}k} \chi^{k}_\alpha \right], \quad \bar{\chi}^{\dot{i}}_\dot{a} = \left( \begin{array}{c} \bar{\lambda}^{\dot{a}}_\dot{\alpha} \ 
\end{array} \right),
\]

(36)

where \(a, \bar{a} = 1, 2, \ldots, N\). \(P^{ij} = \delta^{ij} - \phi^i \bar{\phi}^j\) is a projection operator which satisfies

\[
P^2 = P, \quad \bar{\phi}^i P^{ij} = P^{ij} \bar{\phi}^j = 0.
\]

(37)

Substituting eqs. (33-36) into eq. (29), the Lagrangian becomes

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_C,
\]

(38)

\[
\mathcal{L}_0 = -g^{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b - ig^{ab} \bar{\chi}^{\dot{b}} \sigma^\mu D_\mu \chi^a - \frac{1}{4} R_{abcd}(\chi^a \chi^c)(\bar{\chi}^{\dot{b}} \bar{\chi}^{\dot{d}}),
\]

(39)

\[
\mathcal{L}_C = 2g^{ab}g_{cd} C^{\alpha\beta}(\sigma^{\mu\nu})_\beta^\gamma \chi^a \chi^c(\partial_\mu \varphi^b)(\partial_\nu \bar{\varphi}^\gamma),
\]

(40)

where \(g_{ab}, D_\mu \chi^a\) and \(R_{abcd}\) are given by

\[
g_{ab} = \frac{(1 + \varphi \bar{\varphi}) \delta_{ab} - \bar{\varphi}^a \varphi^b}{(1 + \varphi \bar{\varphi})^2}, \quad D_\mu \chi^a = \partial_\mu \chi^a + \Gamma^a_\mu \chi^c, \quad \Gamma^a_\mu = g^{ad} \partial_\mu g_{cd}.
\]

(41)

\[
R_{bc} = -g_{ab} \partial_\mu (g^{cd} \partial_\mu g_{fb}) = g_{ab} g_{cd} + g_{ad} g_{cb}.
\]

(42)

\(g_{ab}\) is the Fubini-Study metric of \(CP^N\). \(\Gamma^a_\mu\) and \(R_{abcd}\) are the Christoffel symbol and the Riemann curvature tensor respectively. In the \(CP^1\) case, the \(C\)-deformed part \(\mathcal{L}_C\) vanishes.

\[
\mathcal{L}_C^{(CP^1)} = 2(1 + \varphi \bar{\varphi})^{-4} C^{\alpha\beta}(\sigma^{\mu\nu})_\beta^\gamma \chi^a \chi^c(\partial_\mu \varphi^b)(\partial_\nu \bar{\varphi}^\gamma) = 0.
\]

(43)

We study supersymmetry of the Lagrangian (38). In the \(C = 0\) case, the \(\mathcal{N} = 1\) supersymmetry transformation is generated by \(Q_\alpha\) and \(\bar{Q}_{\dot{\alpha}}\). \(Q_\alpha\)
generates the transformation
\[
\begin{align*}
\delta \varphi^a &= \sqrt{2} \xi \chi^a, \\
\delta \bar{\varphi}^\alpha &= 0, \\
\delta \chi^a_\alpha &= -\sqrt{2} \Gamma^b_{ac}(\xi \chi^b) \chi^c_\alpha, \\
\delta \bar{\chi}^{\bar{\alpha}} &= -\sqrt{2} i (\bar{\sigma}^\mu \xi)_\alpha \partial_\mu \bar{\varphi}^\alpha.
\end{align*}
\] (44)

In the present case with \( C \neq 0 \), the same transformation (44) and (45) give
\[
\begin{align*}
\delta g_{ab} &= \partial_c g_{ab} \delta \varphi^c + \partial_c g_{ab} \delta \chi^c_\alpha = \sqrt{2} g_{bc} \Gamma^b_{ac}(\xi \chi^c) \chi^a_\alpha - \sqrt{2} \Gamma^b_{bc}(\xi \chi^b) \chi^c_\alpha g_{ab} \partial_\mu \bar{\varphi}^\alpha = 0. \\
\delta (g_{ab} \chi^a_\alpha \partial_\mu \varphi^b) &= \left[ \sqrt{2} g_{bc} \Gamma^b_{ac}(\xi \chi^c) \chi^a_\alpha - \sqrt{2} \Gamma^b_{bc}(\xi \chi^b) \chi^c_\alpha g_{ab} \right] \partial_\mu \varphi^b = 0. \\
\end{align*}
\] (47)

We then obtain
\[
\delta L_C = \delta \left[ 2 C^{\alpha\bar{\beta}} \epsilon^{\alpha\beta}(g_{ab} \chi^a_\alpha \partial_\mu \varphi^b)(g_{cd} \chi^c_\gamma \partial_\nu \varphi^{\bar{d}}) \right] = 0.
\] (48)

We have shown that the Lagrangian (38) is invariant under the \( \mathcal{N} = 1/2 \) supersymmetry transformation (44) and (45).

Using dimensional reduction, we obtain the Lagrangian of \( CP^N \) SNLSM on noncommutative superspace in two dimensions.
\[
\begin{align*}
\mathcal{L}_{2D} &= \frac{1}{2} g_{AB} \partial_\mu \varphi^A \partial_\nu \varphi^B + i g_{ab} \left( \chi^b_L D^a_L \chi^a_L + \chi^b_R D^a_R \chi^a_R \right) + R_{abcd} \chi^a_L \chi^b_L \chi^c_R \chi^d_R \\
&\quad + 2 g_{ab} g_{cd} (C^{11} \chi^a_L \chi^b_L - C^{22} \chi^a_R \chi^b_R) \epsilon^{\mu\nu}(\partial_\mu \varphi^b)(\partial_\nu \varphi^{\bar{d}}),
\end{align*}
\] (49)

where
\[
\varphi^A = (\varphi^a, \bar{\varphi}^\alpha), \quad g_{AB} = \begin{pmatrix} 0 & g_{ab} \\ g_{\bar{a}b} & 0 \end{pmatrix}, \quad \chi^a_\alpha = \begin{pmatrix} \chi^a_L \\ \chi^a_R \end{pmatrix}, \quad \bar{\chi}^{\bar{\alpha}} = \begin{pmatrix} \bar{\chi}^{\bar{\alpha}}_L \\ \bar{\chi}^{\bar{\alpha}}_R \end{pmatrix}.
\] (50)

4 Discussion

In this letter, we have studied the supersymmetric \( CP^N \) sigma model on noncommutative superspace. We have constructed a closed form of the Lagrangian of the model (38). We have found that the \( \mathcal{N} = 1/2 \) supersymmetry transformation law of the model is not modified.

In two dimensions ordinary NLSMs with extended supersymmetry have a few remarkable properties.
i) Models are integrable (at least at classical level).

ii) They have good UV divergence properties, i.e., finite to certain loops for $\mathcal{N} = 2$ and finite to all loops for $\mathcal{N} = 4$.

iii) They possess instantons.

It is interesting to see whether these nice properties hold true for two-dimensional SNLSM on noncommutative superspace.

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