Sumsets of the distance set in $\mathbb{F}_q^d$

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Abstract

Let $\mathbb{F}_q$ be a finite field of order $q$, where $q$ is large odd prime power. In this paper, we improve some recent results on the additive energy of the distance set, and on sumsets of the distance set due to Shparlinski (2016). More precisely, we prove that for $E \subseteq \mathbb{F}_q^d$, if $d = 2$ and $q^{1+\frac{1}{d-1}} = o(|E|)$ then we have $|k\Delta_{\mathbb{F}_q}(E)| = (1-o(1))q$; if $d \geq 3$ and $q^{d+\frac{1}{d}} = o(|E|)$ then we have $|k\Delta_{\mathbb{F}_q}(E)| = (1-o(1))q$, where $k\Delta_{\mathbb{F}_q}(E) := \Delta_{\mathbb{F}_q}(E) + \cdots + \Delta_{\mathbb{F}_q}(E)$ ($k$ times).

1 Introduction

The Erdős distance problem asks for the minimal number of distinct distances determined by a finite point set of $n$ points in the plane $\mathbb{R}^2$. In 1946, Erdős [5] showed that a $\sqrt{n} \times \sqrt{n}$ integer lattice determines $\Theta(n/\sqrt{\log n})$ distinct distances. From this construction, he conjectured that any set of $n$ points in $\mathbb{R}^2$ determines at least $n^{1-o(1)}$ distinct distances. This conjecture has recently been solved by Guth and Katz [7] in 2010. They showed that a set of $n$ points in $\mathbb{R}^2$ has at least $cn/\log n$ distinct distances. For the latest developments on the Erdős distance problem in higher dimensions and variants, see [14, 17, 6], and the references contained therein.

Let $\mathbb{F}_q$ be a finite field of order $q$, where $q$ is large odd prime power. The distance function between two points $x$ and $y$ in $\mathbb{F}_q^d$, denoted by $||x-y||$, is defined as

$$||x-y|| = (x_1-y_1)^2 + \cdots + (x_d-y_d)^2.$$ 

Although it is not a norm, the function $||x-y||$ has properties similar to the Euclidean norm, for example, it is invariant under orthogonal matrices and translations. For $E \subseteq \mathbb{F}_q^d$, we define the set of distances determined by points in $E$ as

$$\Delta_{\mathbb{F}_q}(E) = \{||x-y|| : x, y \in E\} \subseteq \mathbb{F}_q.$$ 

Bourgain, Katz, and Tao [1] made the first investigation to the prime field analogue of the Erdős distinct distance problem. More precisely, they proved that for any set $E \subseteq \mathbb{F}_p^2$
with \( |\mathcal{E}| = p^\alpha \), \( 0 < \alpha < 2 \), the distance set satisfies \( |\Delta_{p^\alpha}(\mathcal{E})| \geq |\mathcal{E}|^{\frac{1}{2}+\epsilon} \) for some \( \epsilon > 0 \) depending on \( \alpha \). In the case \( |\mathcal{E}| \ll p^{15/11} \), Stevens and de Zeeuw \([20]\) improved this exponent to \( |\mathcal{E}|^{8/15} \). This is the current best bound in the literature. \(^1\)

For the case of large sets over arbitrary finite fields, the first explicit exponent for \( |\Delta_{p^\alpha}(\mathcal{E})| \) was given by Iosevich and Rudnev \([11]\) in 2007 by using Fourier analytic methods.

**Theorem 1.1 (Iosevich-Rudnev, \([11]\)).** For \( \mathcal{E} \subseteq \mathbb{F}_q^d \) with \( |\mathcal{E}| \gg q^4 \), we have

\[
|\Delta_{p^\alpha}(\mathcal{E})| \geq \min \left\{ q, \frac{|\mathcal{E}|}{q^{(d-1)/2}} \right\}.
\]

This result implies that if \( |\mathcal{E}| \gg q^{(d+1)/2} \), then \( |\Delta_{p^\alpha}(\mathcal{E})| \gg q \). Hart, Iosevich, Koh, Rudnev \([8]\) indicated that the threshold \( q^{d+1}/2 \) is the best possible in odd dimensions, at least in general fields. The interested reader can find further results in \([2, 3, 4, 9, 12, 19]\).

Recently Shparlinski \([18]\) used character sum techniques to discover more properties of the distance sets. In particular, he studied properties of the additive energy of the distance sets, where the additive energy of the distance set corresponding to \( \mathcal{E} \) and \( \mathcal{F} \) in \( \mathbb{F}_q^d \), which is denoted by \( E_k^+(\mathcal{E}, \mathcal{F}) \), is defined as the cardinality of

\[
\{(x_i, y_i)_{i=1}^{2k} \in (\mathcal{E} \times \mathcal{F})^{2k} : ||x_1 - y_1|| + \cdots + ||x_k - y_k|| = ||x_{k+1} - y_{k+1}|| + \cdots + ||x_{2k} - y_{2k}||\}.
\]

When \( \mathcal{E} = \mathcal{F} \), we will use the notation \( E_k^+(\mathcal{E}) \) instead of \( E_k^+(\mathcal{E}, \mathcal{F}) \). The first result in \([18]\) is the following theorem.

**Theorem 1.2 (Shparlinski, \([18]\)).** For \( \mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d \), we have

\[
\left| E_k^+(\mathcal{E}, \mathcal{F}) - \frac{|\mathcal{E}|^4 |\mathcal{F}|^4}{q^{3d}} \right| \leq q^{d-1} |\mathcal{E}|^3 |\mathcal{F}|^3 + q^{2d} |\mathcal{E}|^2 |\mathcal{F}|^2.
\]

As a consequence of Theorem 1.2, the author of \([18]\) obtained the following result on a sumset of the distance set.

**Theorem 1.3 (Shparlinski, \([18]\)).** For \( \mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d \), we have

\[
|\Delta_{p^\alpha}(\mathcal{E}, \mathcal{F}) + \Delta_{p^\alpha}(\mathcal{E}, \mathcal{F})| \geq \frac{1}{3} \min \left\{ q, \frac{|\mathcal{E}|^2 |\mathcal{F}|^2}{q^{3d/2}}, \frac{|\mathcal{E}| |\mathcal{F}|}{q^{d-1}} \right\},
\]

where \( \Delta_{p^\alpha}(\mathcal{E}, \mathcal{F}) = \{||x - y|| : x \in \mathcal{E}, y \in \mathcal{F}\} \).

**Corollary 1.4 (Shparlinski, \([18]\)).** Let \( \mathcal{E} \) be a set in \( \mathbb{F}_q^d \). Suppose that \( q^{d+1/2} = o(|\mathcal{E}|) \), then we have

\[
|\Delta_{p^\alpha}(\mathcal{E}) + \Delta_{p^\alpha}(\mathcal{E})| = (1 - o(1))q.
\]

\(^1\)Here and throughout, \( X \gg Y \) means that there exists \( C > 0 \) such that \( X \geq CY \), \( X = o(Y) \) means that \( X/Y \to 0 \) as \( q \to \infty \), where \( X, Y \) are viewed as functions in \( q \).
Note that the additive energy of sets is closely related to their combinatorial properties, for example, see \[13, 21, 22, 23, 24\]. Moreover, some additive character sums can also be estimated via the additive energy, for instance, see \[15\] for more details.

The main purpose of this paper is to give improvements of Theorems 1.2 and 1.3 by using methods from spectral graph theory. For the sake of simplicity of this paper, we will consider the case \( E = F \). We will give some discussions at the end of Section 3 for the case \( E \neq F \). Our first result is the following.

**Theorem 1.5.** Let \( \mathbb{F}_q \) be a finite field of order \( q \) with \( q \equiv 3 \mod 4 \). Let \( k \geq 2 \) be an integer, and \( E \) be a set in \( \mathbb{F}_q^2 \) with \( |E| \gg q \). We have

\[
\left| E_k^k(E) - \frac{|E|^{4k}}{q} \right| \ll q^{2k-1 - 1/2}|E|^{2k+1/2}.
\]

Our next theorem is a result on sumsets of the distance set.

**Theorem 1.6.** Let \( \mathbb{F}_q \) be a finite field of order \( q \) with \( q \equiv 3 \mod 4 \). Let \( k \geq 2 \) be an integer, and \( E \) be a set in \( \mathbb{F}_q^2 \). Suppose that \( q^{1+1/4k} = o(|E|) \), then we have

\[
|k\Delta_{\mathbb{F}_q}(E)| = (1 - o(1))q.
\]

As consequences of Theorem 1.5 and Theorem 1.6, we are able to improve Theorem 1.2 and Corollary 1.4 in the case \( d = 2 \).

**Corollary 1.7.** Let \( \mathbb{F}_q \) be a finite field of order \( q \) with \( q \equiv 3 \mod 4 \). Let \( E \) be a set in \( \mathbb{F}_q^2 \). Suppose that \( |E| \gg q \), then we have

\[
\left| E_2^2(E) - \frac{|E|^8}{q} \right| \ll q^3|E|^{9/2}.
\]

**Corollary 1.8.** Let \( \mathbb{F}_q \) be a finite field of order \( q \) with \( q \equiv 3 \mod 4 \). Let \( E \) be a set in \( \mathbb{F}_q^2 \). Suppose that \( q^{8/7} = o(|E|) \), then we have

\[
|\Delta_{\mathbb{F}_q}(E) + \Delta_{\mathbb{F}_q}(E)| = (1 - o(1))q.
\]

When \( E \) is a subset in \( \mathbb{F}_q^d \) with \( d \geq 3 \), by using the same techniques, we obtain a similar result as follows.

**Theorem 1.9.** Let \( \mathbb{F}_q \) be a finite field of order \( q \). Let \( k \geq 2 \) be an integer, and \( E \) be a set in \( \mathbb{F}_q^d \) with \( d \geq 3 \). We have the following

\[
\left| E_k^k(E) - \frac{|E|^{4k}}{q} \right| \ll q^{dk-1/2}|E|^{2k}.
\]

As an application of Theorem 1.9 we are able to improve Corollary 1.4 in the case \( d = 3 \).

**Theorem 1.10.** Let \( \mathbb{F}_q \) be a finite field of order \( q \). Let \( k \geq 2 \) be an integer, and \( E \) be a set in \( \mathbb{F}_q^d \) with \( d \geq 3 \). Suppose that \( q^{d+1/4k} = o(|E|) \), then we have

\[
|k\Delta_{\mathbb{F}_q}(E)| = (1 - o(1))q.
\]

The rest of this paper is organized as follows: in Section 2, we recall some graph-theoretic tools; proofs of Theorems 1.5, 1.6, 1.9, and 1.10 are given in Section 3.
2 Graph-theoretic tools

2.1 Expander mixing lemma

For a graph $G$ of order $n$, let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be the eigenvalues of its adjacency matrix. The quantity $\gamma(G) = \max\{\gamma_2, -\gamma_n\}$ is called the second eigenvalue of $G$. A graph $G = (V, E)$ is called an $(n, d, \gamma)$-graph if it is $d$-regular, has $n$ vertices, and the second eigenvalue of $G$ is at most $\gamma$.

Suppose that $B$ and $C$ are two multi-sets of vertices in an $(n, d, \gamma)$-graph. Let $m_X(x)$ denote the multiplicity of $x$ in $X$, and $e_m(B, C)$ be the number of edges with multiplicity between $B$ and $C$ in $G$, by multiplicity we mean that if there is an edge between $b \in B$ and $c \in C$, then this edge will be counted $m_B(b) \cdot m_C(c)$ times in $e_m(B, C)$. Recently, Hanson et al. [9] gave the following estimate on $e_m(B, C)$ in an $(n, d, \gamma)$-graph.

Lemma 2.1 ([9]). Let $G = (V, E)$ be an $(n, d, \gamma)$-graph. The number of edges between two multi-sets of vertices $B$ and $C$ in $G$ satisfies:

$$|e_m(B, C) - \frac{d}{n} \left( \sum_{b \in B} m_B(b) \right) \left( \sum_{c \in C} m_C(c) \right)| \leq \sqrt{\sum_{b \in B} m_B(b)^2} \sqrt{\sum_{c \in C} m_C(c)^2} \gamma,$$

where $m_X(x)$ is the multiplicity of $x$ in $X$.

2.2 Sum-product graphs

The sum-product graph $SP_{q,d}$ is defined as follows. The vertex set of the sum-product graph $SP_{q,d}$ is the set $V(SP_{q,d}) = \mathbb{F}_q^d \times \mathbb{F}_q$. Two vertices $U = (a, b)$ and $V = (c, d) \in V(SP_{q,d})$ are connected by an edge, $(U, V) \in E(SP_{q,d})$, if and only if $a \cdot c = b + d$. Vinh [25] proved the following lemma on the $(n, d, \gamma)$ form of $SP_{q,d}$.

Lemma 2.2 (Vinh, [25]). For any $d \geq 1$, the sum-product graph $SP_{q,d}$ is an $(q^{d+1}, q^d, \sqrt{2q^d}) -$ graph.

3 Proofs of Theorems 1.5, 1.6, 1.9, and 1.10

For $\mathcal{E} \subseteq \mathbb{F}_q^d$ and $\lambda \in \mathbb{F}_q$, we define

$$\nu_\mathcal{E}(\lambda) = |\{(x, y) \in \mathcal{E} \times \mathcal{E} : ||x - y|| = \lambda\}|.$$

In order to prove Theorems 1.5, 1.6, 1.9, and 1.10 we need the following lemmas, where the first one follows from the proof of [12, Theorem 3.5].

Lemma 3.1 (Koh-Sun, [12]). Let $\mathbb{F}_q$ be a finite field of order $q$ with $q \equiv 3 \mod 4$. Let $\mathcal{E}$ be a set in $\mathbb{F}_q^2$ with $|\mathcal{E}| \gg q$. Then we have

$$E_+^1(\mathcal{E}) = \sum_{\lambda \in \mathbb{F}_q} \nu_\mathcal{E}(\lambda)^2 \leq \frac{|\mathcal{E}|^4}{q} + (1 + \sqrt{3})q|\mathcal{E}|^{5/2}.$$
Lemma 3.2 (Koh-Sun, [12]). Let \( \mathcal{E} \) be a set in \( \mathbb{F}_q^d \) with \( d \geq 3 \). Then we have

\[
E_+^1(\mathcal{E}) = \sum_{\lambda \in \mathbb{F}_q} \nu_{\mathcal{E}}(\lambda)^2 \leq \frac{|\mathcal{E}|^4}{q} + q^d |\mathcal{E}|^2.
\]

We will use the following lemma to prove Theorem 1.5 and Theorem 1.9.

Lemma 3.3. Let \( k \geq 2 \) be an integer, and \( \mathcal{E} \) be a set in \( \mathbb{F}_q^d \). We have

\[
\left| E_+^k(\mathcal{E}) - \frac{|\mathcal{E}|^{4k}}{q} \right| \ll q^d |\mathcal{E}|^2 E_+^{k-1}(\mathcal{E}).
\]

Proof. We first define two multi-sets of vertices in the sum-product graph \( SP_{q,2d} \) as follows:

\[
\mathcal{B} := \{(\mathbf{x}_1, -\mathbf{x}_2, -||\mathbf{x}_1|| - ||\mathbf{x}_2|| - ||\mathbf{x}_3 - \mathbf{y}_3|| - \cdots - ||\mathbf{x}_k - \mathbf{y}_k|| + ||\mathbf{x}_{k+1} - \mathbf{y}_{k+1}||) : x_i, y_i \in \mathcal{E} \},
\]

\[
\mathcal{C} := \{(\mathbf{y}_1, \mathbf{y}_2, -||\mathbf{y}_1|| - ||\mathbf{y}_2|| + ||\mathbf{x}_{k+2} - \mathbf{y}_{k+2}|| + \cdots + ||\mathbf{x}_{2k} - \mathbf{y}_{2k}||) : x_i, y_i \in \mathcal{E} \}.
\]

For \((\mathbf{x}_i, \mathbf{y}_i)_{i=1}^{2k} \in (\mathcal{E} \times \mathcal{E})^{2k}\), if we have

\[
||\mathbf{x}_1 - \mathbf{y}_1|| + \cdots + ||\mathbf{x}_k - \mathbf{y}_k|| = ||\mathbf{x}_{k+1} - \mathbf{y}_{k+1}|| + \cdots + ||\mathbf{x}_{2k} - \mathbf{y}_{2k}||,
\]

then there is an edge between

\[
(\mathbf{x}_1, -\mathbf{x}_2, -||\mathbf{x}_1|| - ||\mathbf{x}_2|| - ||\mathbf{x}_3 - \mathbf{y}_3|| - \cdots - ||\mathbf{x}_k - \mathbf{y}_k|| + ||\mathbf{x}_{k+1} - \mathbf{y}_{k+1}||) \in \mathcal{B}
\]

and

\[
(\mathbf{y}_1, \mathbf{y}_2, -||\mathbf{y}_1|| - ||\mathbf{y}_2|| + ||\mathbf{x}_{k+2} - \mathbf{y}_{k+2}|| + \cdots + ||\mathbf{x}_{2k} - \mathbf{y}_{2k}||) \in \mathcal{C}
\]

in the sum-product graph \( SP_{q,2d} \). Therefore \( E_+^k(\mathcal{E}) \) is equal to the number of edges between \( \mathcal{B} \) and \( \mathcal{C} \) in \( SP_{q,2d} \). In order to apply Lemma 2.1, we need to estimate upper bounds of \( \sum_{b \in \mathcal{B}} m_B(b)^2 \) and \( \sum_{c \in \mathcal{C}} m_C(c)^2 \). One can check that

\[
\sum_{b \in \mathcal{B}} m_B(b)^2 \leq |\mathcal{E}|^2 E_+^{k-1}(\mathcal{E}), \quad \sum_{c \in \mathcal{C}} m_C(c)^2 \leq |\mathcal{E}|^2 E_+^{k-1}(\mathcal{E}), \quad \text{and} \quad |\mathcal{B}| = |\mathcal{C}| = |\mathcal{E}|^{2k}.
\]

It follows from Lemmas 2.1 and 2.2 that the number of edges between \( \mathcal{B} \) and \( \mathcal{C} \) in the sum-product graph \( SP_{q,2d} \) satisfies

\[
\left| E_+^k(\mathcal{E}) - \frac{|\mathcal{E}|^{4k}}{q} \right| \ll q^d |\mathcal{E}|^2 E_+^{k-1}(\mathcal{E}),
\]

which concludes the proof of the lemma. \( \square \)
Proof of Theorem 1.5: The proof proceeds by induction on $k$. The base case $k = 2$ follows from Lemma 3.1 and Lemma 3.3 with $d = 2$. Suppose that the claim holds for $k - 1 \geq 2$, we show that it also holds for $k$. Indeed, it follows from Lemma 3.3 with $d = 2$ that

$$|E_k^+(\mathcal{E}) - |\mathcal{E}|^{4k}/q| \ll q^2|\mathcal{E}|^2E_{k-1}^+(\mathcal{E}). \quad (3.1)$$

By induction hypothesis, we have

$$E_{k-1}^+(\mathcal{E}) \ll |\mathcal{E}|^{4(k-1)/q} + q^{2(k-1) - 1}|\mathcal{E}|^{2(k-1) + \frac{1}{2}}. \quad (3.2)$$

Putting (3.1) and (3.2) together gives us

$$|E_k^+(\mathcal{E}) - |\mathcal{E}|^{4k}/q| \ll q^{2k-1}|\mathcal{E}|^{2k + \frac{1}{2}},$$

which ends the proof of the theorem. □

Proof of Theorem 1.6: For each $\lambda \in \mathbb{F}_q$, let $N_{\lambda}$ be the number of tuples $(x_1, y_1, \ldots, x_k, y_k)$ in $\mathcal{E}^{2k}$ satisfying $||x_1 - y_1|| + ||x_2 - y_2|| + \cdots + ||x_k - y_k|| = \lambda$. We have $\sum_{\lambda \in \mathbb{F}_q} N_{\lambda} = |\mathcal{E}|^{2k}$. It is easy to check that $\sum_{\lambda \in \mathbb{F}_q} N_{\lambda}^2 = E_k^+(\mathcal{E})$. By applying the Cauchy-Schwarz inequality, we obtain the following

$$\sum_{\lambda \in \mathbb{F}_q} N_{\lambda} \leq \sqrt{|k\Delta_{\mathbb{F}_q}(\mathcal{E})|} (E_k^+(\mathcal{E}))^{1/2}.$$

This implies that

$$|k\Delta_{\mathbb{F}_q}(\mathcal{E})| \geq \frac{|\mathcal{E}|^{4k}}{E_k^+(\mathcal{E})}.$$

Thus the theorem follows immediately from Theorem 1.5. □

Proof of Theorem 1.9: The proof of Theorem 1.9 is as similar as that of Theorem 1.5 except that we use Lemma 3.2 instead of Lemma 3.1. □

Proof of Theorem 1.10: The proof of Theorem 1.10 is as similar as that of Theorem 1.6 except that we use Theorem 1.9 instead of Theorem 1.5. □

Remarks: We conclude this paper with some discussions on $E_k^+(\mathcal{E}, \mathcal{F})$ for $\mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d$ satisfying $|\mathcal{E}| < |\mathcal{F}|$. The main steps in our approach are Lemma 3.3 and upper bounds of $E_k^+(\mathcal{E}, \mathcal{F})$. For two sets $\mathcal{E}$ and $\mathcal{F}$ in $\mathbb{F}_q^2$ with $q \equiv 3 \mod 4$, it has been shown in [12] that

$$E_k^+(\mathcal{E}, \mathcal{F}) \ll \frac{|\mathcal{E}|^2|\mathcal{F}|^2}{q} + q|\mathcal{E}|^{3/2}|\mathcal{F}| \text{ for } d = 2, \quad (3.3)$$

and

$$E_k^+(\mathcal{E}, \mathcal{F}) \ll \frac{|\mathcal{E}|^2|\mathcal{F}|^2}{q} + q^{d-1}|\mathcal{E}|^{3/2}|\mathcal{F}| \text{ for odd } d \geq 3. \quad (3.4)$$
For \( \mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d \), one can follow the proof of Lemma 3.3 to obtain the following
\[
\left| E_+^k(\mathcal{E}, \mathcal{F}) - \frac{|\mathcal{E}|^{2k} |\mathcal{F}|^{2k}}{q} \right| \ll q^d |\mathcal{E}||\mathcal{F}| E_+^{k-1}(\mathcal{E}, \mathcal{F}). \tag{3.5}
\]
If we put (3.3), (3.4), and (3.5) together, then we have
\[
\left| E_+^2(\mathcal{E}, \mathcal{F}) - \frac{|\mathcal{E}|^4 |\mathcal{F}|^4}{q} \right| \leq q^d |\mathcal{E}|^3 |\mathcal{F}|^3 + q^3 |\mathcal{E}|^{\frac{5}{2}} |\mathcal{F}|^2 \quad \text{for } d = 2,
\]
\[
\left| E_+^2(\mathcal{E}, \mathcal{F}) - \frac{|\mathcal{E}|^4 |\mathcal{F}|^4}{q} \right| \leq q^{d-1} |\mathcal{E}|^3 |\mathcal{F}|^3 + q^{\frac{3d-1}{2}} |\mathcal{E}|^3 |\mathcal{F}|^2 \quad \text{for odd } d \geq 3.
\]
These results are also improvements of Theorem 1.2.

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References

[1] J. Bourgain, N. Katz, T. Tao, A sum-product estimate in finite fields, and applications, Geom. Funct. Anal. 14 (2004), 27–57.

[2] M. Bennett, D. Hart, A. Iosevich, J. Pakianathan, M. Rudnev, Group actions and geometric combinatorics in \( \mathbb{F}_q^d \), to appear in Forum Mathematicum 2016.

[3] J. Chapman, M.B. Erdogan, D. Hart, A. Iosevich, D. Koh, Pinned distance sets, k-simplices, Wolff’s exponent in finite fields and sum-product estimates, Math. Z. 271 (2012) 63–93.

[4] D. Covert, D. Koh, Y. Pi, The k-resultant modulus set problem on algebraic varieties over finite fields, arXiv: 1508.02688 (2015).

[5] P. Erdős, On sets of distances of \( n \) points, Amer. Math. Monthly 53 (1946), 248–150.

[6] J. Fox, J. Pach, A. Suk, More distinct distances under local conditions, accepted in Combinatorica, 2016.

[7] L. Guth, N. Katz, On the Erdős distinct distances problem in the plane, Annals of Mathematics, 181(1) (2015), 155–190.
[8] D. Hart, A. Iosevich, D. Koh, M. Rudnev, *Averages over hyperplanes, sum-product theory in finite fields, and the Erdős–Falconer distance conjecture*, Trans. Am. Math. Soc. **363** (2011), 3255–3275.

[9] B. Hanson, B. Lund, and O. Roche-Newton, *On distinct perpendicular bisectors and pinned distances in finite fields*, Finite Fields and Their Applications, **37** (2016), 240-264.

[10] D. Hart, A. Iosevich, J. Solymosi, *Sum-product estimates in finite fields via Kloosterman sums*, Int. Math. Res. Not. no. 5, (2007) Art. ID rnm007.

[11] A. Iosevich, M. Rudnev, *Erdős distance problem in vector spaces over finite fields*, Trans. Am. Math. Soc. **359** (2007), 6127–6142.

[12] D. Koh and H. Sun, *Distance sets of two subsets of vector spaces over finite fields*, Proceedings of the American Mathematical Society, **143**(4) (2015), 1679–1692.

[13] O. Roche-Newton, M. Rudnev, I.D. Shkredov, *New sum-product type estimates over finite fields*, Adv. Math. **293** (2016) 589–605.

[14] N. H. Katz and G. Tardos, *A new entropy inequality for the Erdős distance problem*, Contemp. Math. 342, Towards a theory of geometric graphs, 119–126, Amer. Math. Soc., Providence, RI (2004).

[15] S.V. Konyagin, *Bounds of exponential sums over subgroups and Gauss sums*, in: Proc. 4th Intern. Conf. Modern Problems of Number Theory and Its Applications, Moscow Lomonosov State Univ., Moscow, 2002, pp. 86–114 (in Russian).

[16] G. Mockenhaupt, T. Tao, *Restriction and Kakeya phenomena for finite fields*, Duke Math. J. **121** (2004), no. 1, 35–74.

[17] J. Solymosi, V. Vu, *Near optimal bounds for the number of distinct distances in high dimensions*, Combinatorica, (2005).

[18] I. E. Shparlinski, *On the additive energy of the distance set in finite fields*, Finite Fields and Their Applications, **42** (2016), 187–199.

[19] I.E. Shparlinski, *On some generalisations of the Erdős distance problem over finite fields*, Bull. Aust. Math. Soc. **73** (2006), 285–292.

[20] S. Stevens, F. de Zeeuw, *An improved point-line incidence bound over arbitrary fields*, arXiv:1609.06284 2016.

[21] I.D. Shkredov, *Some new inequalities in additive combinatorics*, Mosc. J. Comb. Number Theory **3** (2013) 425–475.

[22] I.D. Shkredov, *Energies and structure of additive sets*, Electron. J. Comb. **21** (2014) 1–53, P3.44.
[23] I.D. Shkredov, *An introduction to higher energies and sumsets*, preprint, arxiv. 1512.00627, 2015.

[24] T. Tao, V. Vu, *Additive Combinatorics*, Cambridge Univ. Press, Cambridge, 2006.

[25] L. A. Vinh, *The solvability of norm, bilinear and quadratic equations over finite fields via spectra of graphs*, Forum Mathematicum, Vol. 26 (2014), No. 1, pp. 141–175.