Higher order moments of the estimated tangency portfolio weights

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**Online Supplementary Materials**

In Lemma 0.1, we present a stochastic representation for \(\mathbf{l}^T \mathbf{A}^{-1} \mathbf{z}\) obtained in [2], that plays a fundamental role in the derivations of the higher moments for \(\mathbf{l}^T \mathbf{A}^{-1} \mathbf{z}\). Below, the symbol \(\mathcal{F}(d_1, d_2, s)\) stands for the non-central \(\mathcal{F}\)-distribution with \(d_1\) and \(d_2\) degrees of freedom and the non-centrality parameter \(s\), while the symbol \(\sim\) stands for the equality in distribution.

**Lemma 0.1.** Let \(\mathbf{A} \sim \mathcal{W}_k(n, \Sigma)\), \(n > k\) and \(\mathbf{z} \sim \mathcal{N}_k(\mu, \lambda \Sigma)\) with \(\lambda > 0\) and positive definite \(\Sigma\). Furthermore, let \(\mathbf{A}\) and \(\mathbf{z}\) be independent and \(\mathbf{l}\) be a \(k\)-dimensional vector of constants. Then the stochastic representation of \(\mathbf{l}^T \mathbf{A}^{-1} \mathbf{z}\) is given by

\[
\mathbf{l}^T \mathbf{A}^{-1} \mathbf{z} \sim \frac{1}{u_1} \left( \mathbf{l}^T \Sigma^{-1} \mu + \sqrt{\left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right)} \right) \mathbf{l}^T \Sigma^{-1} \mathbf{l} u_2
\]

where \(u_1 \sim \chi^2_{n-k+1}\), \(u_2 \sim \mathcal{N}(0, 1)\) and \(u_3 \sim \mathcal{F}(k-1, n-k+2, s)\) with \(s = \mu^T \mathbf{R}_1 \mu / \lambda\) and \(\mathbf{R}_1 = \Sigma^{-1} - \mu \Sigma^{-1} \mu^T \Sigma^{-1} / \mathbf{l}^T \Sigma^{-1} \mathbf{l}\). The random variables \(u_1\), \(u_2\) and \(u_3\) are mutually independently distributed.

From Lemma 0.1, note that the stochastic representation of \(\mathbf{l}^T \mathbf{A}^{-1} \mathbf{z}\) is given in terms of independently distributed \(\chi^2\), a standard normal and a non-central \(\mathcal{F}\) random variables.

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Proof of Theorem 4.1: From Lemma 0.1 we obtain that

\[
E \left[ (l^TA^{-1}z)^r \right] = E \left[ \left( \frac{1}{u_1} \left( t^T \Sigma^{-1} \mu + \sqrt{\left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right) l^T \Sigma^{-1} u_2} \right) \right)^r \right]
\]

\[
= E \left[ \left( \frac{1}{u_1} \right)^r \right] E \left[ \left( l^T \Sigma^{-1} \mu + \sqrt{\left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right) l^T \Sigma^{-1} u_2} \right)^r \right]
\]

where the last equality follows from the fact that \( u_1 \) is independent of \( u_2 \) and \( u_3 \).

Since \( u_1 \sim \chi^2_{n-k+1} \), we get that \( 1/u_1 \sim \text{Inv-\chi}^2_{n-k+1} \). From [3, p. 18] it follows that

\[
E \left[ \left( \frac{1}{u_1} \right)^r \right] = \frac{\Gamma \left( \frac{n-k+1}{2} - r \right)}{2^r \Gamma \left( \frac{n-k+1}{2} \right)} = \frac{\Gamma \left( \frac{n-k+1}{2} - r \right)}{2^r (\frac{n-k+1}{2} - 1)...(\frac{n-k+1}{2} - 2r+1) \Gamma \left( \frac{n-k+1}{2} - r \right)}
\]

\[
= \frac{1}{(n-k-1)...(n-k-2r+1)}, \quad n-k+1 > 2r.
\]

This result follows from the property \( \Gamma(x) = (x-1)\Gamma(x-1) \).

Using the well-known binomial formula (see [1, p. 129]) and the fact that \( u_2 \) and \( u_3 \) are independent, we obtain that

\[
E \left[ \left( \frac{1}{u_1} \right)^r \right] E\left[ \left( l^T \Sigma^{-1} \mu + \sqrt{\left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right) l^T \Sigma^{-1} u_2} \right)^r \right] = \sum_{i=0}^{r} \binom{r}{i} \left( l^T \Sigma^{-1} \mu \right)^{r-i} \left( \sqrt{\left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right) l^T \Sigma^{-1} u_2} \right)^{i/2} E \left[ u_2^{j/2} \right] E \left[ \left( \lambda + \frac{\lambda(k-1)}{n-k+2} u_3 \right)^{i/2} \right]
\]

Let us note that the odd moments of the standard normal distribution are equal to zero, i.e. \( E \left[ u_2^{2j+1} \right] = 0 \) for \( j \in \{0, 1, 2, \ldots \} \), while the even moments are given by

\[
E[u_2^{2j}] = \frac{(2j)!}{2^j j!} \quad \text{for} \quad j \geq 1,
\]

c.f. [4, Chapter 34.2]). It leads us to
\[
\begin{align*}
E & \left[ (l^T \Sigma^{-1} \mu + \sqrt{\frac{\lambda + \lambda(k-1)}{n-k+2}u_3} l^T \Sigma^{-1}u_2)^r \right] \\
& = \sum_{i=0}^{r} \binom{r}{i} (l^T \Sigma^{-1} \mu)^{r-i} (\lambda l^T \Sigma^{-1} l)^{i/2} E \left[ u_2^i \right] E \left[ \left( 1 + \frac{k-1}{n-k+2}u_3 \right)^{i/2} \right] \\
& = (l^T \Sigma^{-1} \mu)^r + \sum_{j=1}^{\lfloor r/2 \rfloor} \binom{r}{2j} (l^T \Sigma^{-1} \mu)^r - 2j (\lambda l^T \Sigma^{-1} l)^j E \left[ u_2^{2j} \right] E \left[ \left( 1 + \frac{k-1}{n-k+2}u_3 \right)^j \right].
\end{align*}
\]

Applying the binomial formula again we get
\[
E \left[ \left( 1 + \frac{k-1}{n-k+2}u_3 \right)^j \right] = E \left[ \sum_{m=0}^{j} \binom{j}{m} \left( \frac{k-1}{n-k+2}u_3 \right)^m \right] = \sum_{m=0}^{j} \binom{j}{m} \left( \frac{k-1}{n-k+2} \right)^m E [u_3^m].
\]

For \( m \geq 1 \) it holds that (see [4, Chapter 32.2])
\[
\begin{align*}
c_m & := \left( \frac{k-1}{n-k+2} \right)^m E [u_3^m] \\
& = \frac{\Gamma \left( \frac{n-k+2}{2} - m \right) \Gamma \left( \frac{k-1}{2} + m \right)}{\Gamma \left( \frac{n-k+2}{2} \right) \Gamma \left( \frac{k-1}{2} \right)} e^{-\frac{s^2}{2}} F_1 \left( m + \frac{k-1}{2} ; \frac{k-1}{2} ; \frac{s}{2} \right) \\
& = \frac{(k-1+2(m-1)) \ldots (k-1)}{(n-k-2(m-1)) \ldots (n-k)} e^{-\frac{s^2}{2}} F_1 \left( m + \frac{k-1}{2} ; \frac{k-1}{2} ; \frac{s}{2} \right),
\end{align*}
\]

Finally, putting all the terms together we get the statement of the theorem.

\( \square \)

**Proof of Corollary 4.2:** From Theorem 4.1 it follows that
\[
\kappa_1 := E \left[ l^T A^{-1} z \right] = \frac{1}{n-k-1} l^T \Sigma^{-1} \mu.
\]

Using the binomial formula and properties of the mathematical expectation, we
obtain that
\[
E \left[ (l^T A^{-1} z - \kappa_1)^r \right] = E \left[ \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-\kappa_1)^{r-i} (l^T A^{-1} z)^i \right] = (-\kappa_1)^r + \sum_{i=1}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \kappa_1^{r-i} E \left[ (l^T A^{-1} z)^i \right]. \tag{3}
\]

Finally, applying Theorem 4.1 we get the statement of the corollary.

\[\square\]

**Proof of Corollary 4.3:** From Theorem 4.1 and [4, Chapter 32.2] we obtain that
\[
c_1 = \frac{k - 1}{n - k + 2} E[u_3] = \frac{s + k - 1}{n - k},
\]
\[
c_2 = \left( \frac{k - 1}{n - k + 2} \right)^2 E[u_3^2] = \frac{s^2 + (2s + k - 1)(k + 1)}{(n - k)(n - k - 2)}.
\]

Using Corollary 4.2, we get that the second order central moment of $l^T A^{-1} z$ is given by
\[
E[(l^T A^{-1} z - E[l^T A^{-1} z])^2] = (-\kappa_1)^2 + \sum_{i=1}^{2} \left( \begin{array}{c} 2 \\ i \end{array} \right) \frac{(-\kappa_1)^{2-i}}{(n - k - 1) \ldots (n - k - 2i + 1)}
\]
\[
\times \left[ (l^T \Sigma^{-1} \mu)^i + \sum_{j=1}^{\lfloor i/2 \rfloor} \left( \begin{array}{c} i \\ 2j \end{array} \right) \frac{(2j)!}{2^j j!} (l^T \Sigma^{-1} \mu)^{i-2j} (\lambda l^T \Sigma^{-1} l)^j \left( 1 + \sum_{m=1}^{j} \left( \begin{array}{c} j \\ m \end{array} \right) c_m \right) \right]
\]
\[
= \kappa_1^2 - \frac{2\kappa_1}{n - k - 1} l^T \Sigma^{-1} \mu + \frac{1}{(n - k - 1)(n - k - 3)}
\]
\[
\times \left[ (l^T \Sigma^{-1} \mu)^2 + \lambda \left( 1 + \frac{s + k - 1}{n - k} \right) l^T \Sigma^{-1} l \right]
\]
\[
= \left[ \frac{1}{(n - k - 1)(n - k - 3)} - \frac{1}{(n - k - 1)^2} \right] (l^T \Sigma^{-1} \mu)^2
\]
\[
+ \frac{\lambda}{(n - k - 1)(n - k - 3)} \left( 1 + \frac{s + k - 1}{n - k} \right) l^T \Sigma^{-1} l
\]
\[
= d_1^{(0)} (l^T \Sigma^{-1} \mu)^2 + d_2^{(0)} l^T \Sigma^{-1} l
\]
with $d_1^{(0)}$ and $d_2^{(0)}$ as defined in the formulation of the corollary.
In order to derive the third order central moment of $\mathbf{1}^T \mathbf{A}^{-1} \mathbf{z}$, it holds that

$$
\mathbb{E}
\left[
\left(\mathbf{1}^T \mathbf{A}^{-1} \mathbf{z} - \mathbb{E}[\mathbf{1}^T \mathbf{A}^{-1} \mathbf{z}]\right)^3
\right]
= (-\kappa_1)^3 + \sum_{i=1}^{3} \binom{3}{i} \frac{(-\kappa_1)^{3-i}}{(n-k-1) \ldots (n-k-2i+1)}
\times
\left[
\left(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^i + \sum_{j=1}^{\lfloor i/2 \rfloor} \binom{i}{2j} \frac{(2j)!}{2^j j!} \left(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^{i-2j} \left(\lambda \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^j \left(1 + \sum_{m=1}^{j} \binom{j}{m} c_m\right)
\right]
= -\kappa_1^3 + \frac{3\kappa_1^2}{n-k-1} \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu - \frac{3\kappa_1}{(n-k-1)(n-k-3)} \left[\left(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^2 + \lambda (1 + c_1) \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu \cdot \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right]
+ \frac{1}{(n-k-1)(n-k-3)(n-k-5)} \left[\left(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^3 + 3\lambda (1 + c_1) \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu \cdot \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right]
\times \left(1 + \frac{s+k-1}{n-k}\right) \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu \cdot \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu
\right]
= d_1^{(1)} \left(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mu\right)^3 + d_2^{(1)} \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu \cdot \mathbf{1}^T \mathbf{\Sigma}^{-1} \mu
\right]
$$

with $d_1^{(1)}$ and $d_2^{(1)}$ which are defined in the statement of the corollary.

Finally, we derive the fourth order central moment of $\mathbf{1}^T \mathbf{A}^{-1} \mathbf{z}$. From Corollary 4.2,
we have

$$
E \left[ (1^T A^{-1} z - E[1^T A^{-1} z])^4 \right] = (-\kappa_1)^4 + \frac{4}{(n-k-1)...(n-k-2i+1)} \sum_{i=1}^{4} \binom{4}{i} (-\kappa_1)^{4-i} \\
\times \left[ (1^T \Sigma^{-1} \mu)^i + \sum_{j=1}^{[i/2]} \binom{i}{2j} (\frac{1}{2^j})^j (1^T \Sigma^{-1} \mu)^{i-2j} \left( \lambda (1^T \Sigma^{-1} \mu)^2 \right)^{j} \left( 1 + \sum_{m=1}^{j} \binom{j}{m} c_m \right) \right] \\
= \frac{1}{(n-k-1)^4} (1^T \Sigma^{-1} \mu)^4 - \frac{4\kappa_1^3}{n-k-1} 1^T \Sigma^{-1} \mu + \frac{6\kappa_1^2}{(n-k-1)(n-k-3)} (1^T \Sigma^{-1} \mu)^2 \\
+ \frac{6\lambda(1 + c_1)\kappa_1^2}{(n-k-1)(n-k-3)} (1^T \Sigma^{-1} \mu)^3 - \frac{4\kappa_1}{(n-k-1)(n-k-3)(n-k-5)} (1^T \Sigma^{-1} \mu)^4 \\
+ \frac{6\lambda(1 + c_1)}{(n-k-1)...(n-k-7)} (1^T \Sigma^{-1} \mu)^4 + \frac{6\lambda(1 + c_1)}{(n-k-1)...(n-k-7)} (1^T \Sigma^{-1} \mu)^2 1^T \Sigma^{-1} \mu \\
+ \frac{6\lambda(1 + c_1)}{(n-k-1)...(n-k-7)} (1^T \Sigma^{-1} \mu)^3 \\
+ \frac{3\lambda^2(1 + 2c_1 + c_2)}{(n-k-1)...(n-k-7)} (1^T \Sigma^{-1} \mu)^4 \\
= \left[ \frac{1}{(n-k-1)^3(n-k-3)} - \frac{3}{(n-k-1)^4} - \frac{4}{(n-k-1)^2(n-k-3)(n-k-5)} \right] (1^T \Sigma^{-1} \mu)^4 \\
+ \left[ \frac{6\lambda(1 + c_1)}{(n-k-1)^3(n-k-3)} - \frac{12\lambda(1 + c_1)}{(n-k-1)^2(n-k-3)(n-k-5)} \right] (1^T \Sigma^{-1} \mu)^3 \\
+ \left[ \frac{6\lambda(1 + c_1)}{(n-k-1)...(n-k-7)} \right] (1^T \Sigma^{-1} \mu)^2 1^T \Sigma^{-1} \mu \\
+ \left[ \frac{3\lambda^2(1 + 2c_1 + c_2)}{(n-k-1)...(n-k-7)} \right] (1^T \Sigma^{-1} \mu)^2 \\
= d_1^{(2)} (1^T \Sigma^{-1} \mu)^4 + d_2^{(2)} (1^T \Sigma^{-1} \mu)^2 1^T \Sigma^{-1} \mu + d_3^{(2)} (1^T \Sigma^{-1} \mu)^2
$$

with $d_1^{(2)}$, $d_2^{(2)}$, and $d_3^{(2)}$ as defined in the formulation of the corollary. It completes the proof of the corollary.

\[\square\]
References

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[4] Walck, C. (1996). Hand-book on statistical distributions for experimentalists. Internal Report SUF–PFY/96–01, Stockholm University.
Table 1. Mean, variance, skewness and kurtosis of the estimated TP weights. The returns are assumed to be independently multivariate normally and $t$-distributed. $k$ is taken to be 10, and $l = 1_k$.

| Risk Aversion | Moments | $n = 30$ | $n = 60$ | $n = 120$ |
|---------------|---------|----------|----------|-----------|
|               | $N_10(\mu, \Sigma)$ | $t_{10}(5, \mu, 0.6\Sigma)$ | $t_{10}(10, \mu, 0.8\Sigma)$ | $N_10(\mu, \Sigma)$ | $t_{10}(5, \mu, 0.6\Sigma)$ | $t_{10}(10, \mu, 0.8\Sigma)$ |
| $\alpha = 3$ | Mean    | 0.864776 | 1.110531 | 0.947911 | 0.659764 | 0.761651 | 0.701336 | 0.591427 | 0.651834 | 0.611435 |
|               | Variance | 5.567486 | 11.583640 | 7.539621 | 1.168022 | 2.292749 | 1.532996 | 0.412463 | 0.797585 | 0.539417 |
|               | Skewness | 0.312385 | 0.340557 | 0.326781 | 0.165592 | 0.193511 | 0.183743 | 0.105690 | 0.123817 | 0.126428 |
|               | Kurtosis | 5.839568 | 5.987360 | 5.883379 | 3.760057 | 3.921065 | 3.777257 | 3.307217 | 3.303203 | 3.384185 |
| $\alpha = 5$ | Mean    | 0.518866 | 0.668637 | 0.569284 | 0.395859 | 0.463346 | 0.417420 | 0.354856 | 0.391722 | 0.364384 |
|               | Variance | 2.004295 | 4.130472 | 2.696112 | 0.420488 | 0.826535 | 0.552009 | 0.148487 | 0.288967 | 0.193801 |
|               | Skewness | 0.312385 | 0.293890 | 0.302154 | 0.165592 | 0.197716 | 0.189284 | 0.105690 | 0.110918 | 0.119119 |
|               | Kurtosis | 5.839568 | 5.912035 | 5.583926 | 3.760057 | 3.897097 | 3.808134 | 3.307217 | 3.317417 | 3.320139 |
| $\alpha = 10$ | Mean    | 0.259433 | 0.330666 | 0.289738 | 0.197929 | 0.230357 | 0.210714 | 0.177428 | 0.196028 | 0.183325 |
|                | Variance | 0.501074 | 1.039429 | 0.672494 | 0.105122 | 0.206214 | 0.139646 | 0.037122 | 0.072717 | 0.047920 |
|               | Skewness | 0.312385 | 0.345765 | 0.320551 | 0.165592 | 0.175258 | 0.193042 | 0.105690 | 0.115015 | 0.102036 |
|               | Kurtosis | 5.839568 | 6.592671 | 5.900996 | 3.760057 | 3.828284 | 3.867597 | 3.307217 | 3.330776 | 3.327171 |
| $\alpha = 50$ | Mean×10 | 0.518866 | 0.668637 | 0.569284 | 0.395859 | 0.463346 | 0.417420 | 0.354856 | 0.391722 | 0.364384 |
|                | Variance×10 | 2.004295 | 4.130472 | 2.696112 | 0.420488 | 0.826535 | 0.552009 | 0.148487 | 0.288967 | 0.193801 |
|               | Skewness | 0.312385 | 0.293890 | 0.302154 | 0.165592 | 0.197716 | 0.189284 | 0.105690 | 0.110918 | 0.119119 |
|               | Kurtosis | 5.839568 | 5.912035 | 5.583926 | 3.760057 | 3.897097 | 3.808134 | 3.307217 | 3.317417 | 3.320139 |
| $\alpha = 100$ | Mean×10 | 0.259433 | 0.330666 | 0.289738 | 0.197929 | 0.230357 | 0.210714 | 0.177428 | 0.196028 | 0.183325 |
|                | Variance×10 | 0.501074 | 1.039429 | 0.672494 | 0.105122 | 0.206214 | 0.139646 | 0.037122 | 0.072717 | 0.047920 |
|               | Skewness | 0.312385 | 0.345765 | 0.320551 | 0.165592 | 0.175258 | 0.193042 | 0.105690 | 0.115015 | 0.102036 |
|               | Kurtosis | 5.839568 | 6.592671 | 5.900996 | 3.760057 | 3.828284 | 3.867597 | 3.307217 | 3.330776 | 3.327171 |
Table 2. Mean, variance, skewness and kurtosis of the estimated TP weights. The returns are assumed to be independently multivariate normally and t-distributed. $k$ is taken to be 15, and $l = 1_k$.

| Risk Aversion | Moments | $n = 30$ | $n = 60$ | $n = 120$ |
|---------------|---------|----------|----------|-----------|
| $\alpha = 3$  | Mean    | 0.547237 | 0.336593 | 0.283420 |
|               | Variance| 27.687910| 5.960050 | 0.856055 |
|               | Skewness| 0.138575 | 0.060116 | 0.036902 |
|               | Kurtosis| 7.962276 | 3.842378 | 3.314540 |
| $\alpha = 5$  | Mean    | 0.328342 | 0.201956 | 0.170052 |
|               | Variance| 9.967647 | 1.067152 | 0.311618 |
|               | Skewness| 0.138575 | 0.060116 | 0.036902 |
|               | Kurtosis| 7.962276 | 3.842378 | 3.314540 |
| $\alpha = 10$ | Mean    | 0.164171 | 0.100978 | 0.085026 |
|               | Variance| 2.491912 | 0.266788 | 0.077904 |
|               | Skewness| 0.138575 | 0.060116 | 0.036902 |
|               | Kurtosis| 7.962276 | 3.842378 | 3.314540 |
| $\alpha = 50$ | Mean×10 | 0.328342 | 0.201956 | 0.170052 |
|               | Variance×10| 9.967647 | 1.067152 | 0.311618 |
|               | Skewness×10| 0.138575 | 0.060116 | 0.036902 |
|               | Kurtosis×10| 7.962276 | 3.842378 | 3.314540 |
| $\alpha = 100$| Mean×10 | 0.164171 | 0.100978 | 0.085026 |
|               | Variance×10| 2.491912 | 0.266788 | 0.077904 |
|               | Skewness×10| 0.138575 | 0.060116 | 0.036902 |
|               | Kurtosis×10| 7.962276 | 3.842378 | 3.314540 |
Table 3. Bias and MSE of the estimated TP weights. The returns are assumed to be independently multivariate normally and $t$-distributed. $k$ is taken to be 5, and $l = 1/k$.

| Risk Aversion | Measures | $n = 30$ | | | $n = 60$ | | | $n = 120$ | | |
|---------------|----------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\alpha = 3$  | Bias     | 0.143391| 0.273864| 0.189988| 0.062226| 0.127956| 0.0842447| 0.029186| 0.067652| 0.0410798|
|               | MSE      | 0.462743| 0.875565| 0.597934| 0.150241| 0.276909| 0.191387| 0.061858| 0.115833| 0.077792|
| $\alpha = 5$  | Bias     | 0.086035| 0.162429| 0.111631| 0.037336| 0.078528| 0.050852| 0.017512| 0.039863| 0.022786|
|               | MSE      | 0.166587| 0.317996| 0.211986| 0.054087| 0.100505| 0.068980| 0.022269| 0.041823| 0.027900|
| $\alpha = 10$ | Bias×10  | 0.430174| 0.822092| 0.572502| 0.186679| 0.391421| 0.247014| 0.087558| 0.197323| 0.120782|
|               | MSE×10   | 0.04164684| 0.08003997| 0.05369736| 0.01352169| 0.02513045| 0.01714567| 0.005567204| 0.01045792| 0.007018445|
| $\alpha = 50$ | Bias×10  | 0.086035| 0.162498| 0.110764| 0.037336| 0.078350| 0.050550| 0.017512| 0.039870| 0.024039|
|               | MSE×10²  | 0.166587| 0.316099| 0.213209| 0.054087| 0.101500| 0.068313| 0.022269| 0.041733| 0.027994|
| $\alpha = 100$| Bias×10² | 0.430174| 0.813215| 0.566921| 0.186679| 0.387199| 0.246360| 0.087558| 0.200873| 0.115619|
|               | MSE×10³  | 0.416468| 0.792642| 0.539376| 0.135217| 0.252014| 0.170761| 0.556720| 0.104761| 0.701831|
Table 4. Bias and MSE of the estimated TP weights. The returns are assumed to be independently multivariate normally and \( t \)-distributed. \( k \) is taken to be 10, and \( l = 1_k \).

As it can been seen that, the estimator shows some biases for small \( n \) but with the increase in sample size \( n \) and number of assets \( k \), it starts reducing. It is interesting to point out here that relatively large Bias and MSE are observed for small \( n \) and large \( k \), which further diminishes with the increase in \( \alpha \).

| Risk Aversion | Measures | \( n = 30 \) | \( n = 60 \) | \( n = 120 \) |
|---------------|----------|-------------|-------------|-------------|
| \( \alpha = 3 \) | Bias     | 0.328018    | 0.123007    | 0.054670    |
|               | MSE      | 5.675082    | 1.183152    | 0.415452    |
| \( \alpha = 5 \) | Bias     | 0.196811    | 0.073804    | 0.032802    |
|               | MSE      | 2.043029    | 0.425935    | 0.149563    |
| \( \alpha = 10 \) | Bias    | 0.098406    | 0.036902    | 0.016401    |
|                | MSE     | 0.510757    | 0.106484    | 0.037391    |
| \( \alpha = 50 \) | Bias  | 0.196811    | 0.073804    | 0.032802    |
|                | MSE    | 0.204303    | 0.042594    | 0.014956    |
| \( \alpha = 100 \) | Bias | 0.098406    | 0.036902    | 0.016401    |
|                  | MSE   | 0.510757    | 0.106484    | 0.037391    |

Note: The \( t \)-distribution parameters are adjusted for different risk aversion levels: \( \mu = 5, \mu = 10, \mu = 50, \mu = 100 \), and \( \Sigma = 0.6 \).
Table 5. Bias and MSE of the estimated TP weights. The returns are assumed to be independently multivariate normally and \(t\)-distributed. \(k\) is taken to be 15, and \(l = 1_k\).

Here the estimator shows relatively smaller bias compared to the case presented earlier. It is interesting to point out here that relatively large Bias and MSE are observed for small \(n\) and large \(k\), which further diminishes with the increase in \(\alpha\).

| Risk Aversion | Measures | \(n = 30\) | \(n = 60\) | \(n = 120\) |
|---------------|----------|------------|------------|------------|
|               | \(N_{15}(\mu, \Sigma)\) | \(t_{15}(5, \mu, 0.6\Sigma)\) | \(t_{15}(10, \mu, 0.8\Sigma)\) | \(N_{15}(\mu, \Sigma)\) | \(t_{15}(5, \mu, 0.6\Sigma)\) | \(t_{15}(10, \mu, 0.8\Sigma)\) |
| \(\alpha = 3\) | Bias | 0.301924 | 0.508598 | 0.397823 | 0.091279 | 0.157649 | 0.112649 | 0.038107 | 0.072682 | 0.048728 |
| | MSE | 27.779070 | 61.88950 | 38.818540 | 2.972642 | 5.984903 | 3.992272 | 0.867057 | 1.701807 | 1.140734 |
| \(\alpha = 5\) | Bias | 0.181154 | 0.291950 | 0.229171 | 0.054768 | 0.103522 | 0.074764 | 0.022864 | 0.046490 | 0.029294 |
| | MSE | 10.000460 | 22.102220 | 14.035850 | 1.070151 | 2.152747 | 1.429632 | 0.312140 | 0.616901 | 0.404402 |
| \(\alpha = 10\) | Bias | 0.090577 | 0.157973 | 0.113672 | 0.027384 | 0.045524 | 0.034965 | 0.011432 | 0.022334 | 0.015459 |
| | MSE | 2.500116 | 5.625195 | 3.486134 | 0.267538 | 0.536525 | 0.354263 | 0.078035 | 0.152945 | 0.101585 |
| \(\alpha = 50\) | Bias | 0.181154 | 0.307832 | 0.221892 | 0.054768 | 0.103480 | 0.066069 | 0.022864 | 0.043727 | 0.032967 |
| | MSE | 1.000005 | 2.20673 | 1.38010 | 0.010702 | 0.021713 | 0.014337 | 0.003121 | 0.006119 | 0.004109 |
| \(\alpha = 100\) | Bias | 0.090577 | 0.163285 | 0.121813 | 0.027384 | 0.054628 | 0.035939 | 0.011432 | 0.023690 | 0.013484 |
| | MSE | 2.500116 | 5.56487 | 3.51213 | 0.026754 | 0.053872 | 0.035620 | 0.007804 | 0.015352 | 0.010141 |
Table 6. Confidence intervals of the estimated TP weights. We provide three different types of CI, i.e., approximated (or asymptotic) CI evaluated via first two exact moments, the CI obtained via stochastic representation obtained by [2], and the CI obtained via stochastic representations under the assumption of \( t \)-distributed data, which are further evaluated at 5 degrees of freedom. The confidence level is 95%. The returns are assumed to be independently multivariate normally and \( t \)-distributed. \( \hat{k} \) is taken to be 5, and \( l = \frac{k}{k} \).

| Risk Aversion | Limits       | \( n = 30 \)                      | \( n = 60 \)                      | \( n = 120 \)                     |
|---------------|--------------|-----------------------------------|-----------------------------------|-----------------------------------|
|               | \( \chi^2 \) \( [\mu, \Sigma] \) | \( \chi^2 \) \( t [15, \mu, 0.8 \Sigma] \) | \( \chi^2 \) \( t [15, \mu, 0.8 \Sigma] \) | \( \chi^2 \) \( t [15, \mu, 0.8 \Sigma] \) |
| \( \alpha = 3 \) | Low Limit    | -0.610254                         | -0.573308                         | -0.143866                         | 0.094753                         |
|               | Upper Limit  | 1.996370                          | 2.160804                          | 1.780636                          | 1.559077                         |
| \( \alpha = 5 \) | Low Limit    | -0.366152                         | -0.345320                         | -0.081375                         | 0.056852                         |
|               | Upper Limit  | 1.197822                          | 1.444679                          | 1.071927                          | 0.938136                         |
| \( \alpha = 10 \) | Low Limit   | -0.183076                         | -0.170182                         | -0.043104                         | 0.031507                         |
|               | Upper Limit  | 0.509811                          | 0.864512                          | 0.535793                          | 0.466585                         |
| \( \alpha = 50 \) | Low Limit   | -0.306215                         | -0.346264                         | -0.045905                         | 0.056852                         |
|               | Upper Limit  | 0.119782                          | 0.144493                          | 0.107914                          | 0.093135                         |
| \( \alpha = 100 \) | Low Limit  | -0.183076                         | -0.170182                         | -0.043104                         | 0.031507                         |
|               | Upper Limit  | 0.509811                          | 0.864512                          | 0.535793                          | 0.466585                         |
Table 7. Confidence intervals of the estimated TP weights where we provide three different types of CI, i.e., approximated (or asymptotic) CI evaluated via first two exact moments, the CI obtained via stochastic representation obtained by [2], and the CI obtained via stochastic representations under the assumption of \( t \)-distributed data, which are further evaluated at 10 degrees of freedom. It can be seen that for small \( n \) the asymptotic CI and CI based on stochastic representation obtained via MC simulations from normal distribution are not similar, but with the increase in \( n \) the estimates become more consistent. Similarly, the CI based on MC simulations from \( t \)-distribution converges to normal with the increase in degrees of freedom as per the theory says. For large sample size, \( n \), the estimates converge to all in one. Here, it is interesting to note that the length of CI decreases with the increase in risk aversion parameter \( \alpha \). The confidence level is 95%. The returns are assumed to be independently multivariate normally and \( t \)-distributed. \( k \) is taken to be 10, and \( l = \frac{k}{k} \).

| Risk Aversion | Limits | \( n = 30 \) | \( n = 60 \) | \( n = 120 \) |
|---------------|--------|-------------|-------------|-------------|
| \( \alpha = 3 \) | Low Limit | -3.759861 | -3.695323 | -3.695323 | -5.392500 | -4.340719 | -1.458468 | -1.416387 | -2.142870 | -1.664340 | -0.667327 | -0.651245 | -1.066886 | -0.806229 |
| Upper Limit   | 2.777996 | 2.876329 | 3.909600 | 3.260189 | 1.850181 | 1.897353 | 2.467225 | 2.103798 |
| \( \alpha = 5 \) | Low Limit | -2.255917 | -2.185177 | -2.185177 | -3.236593 | -2.572920 | -1.085038 | -1.013131 | -0.400936 | -0.387532 | -0.64039 | -0.477428 |
| Upper Limit   | 3.293818 | 3.548385 | 5.045671 | 4.072855 | 1.667978 | 1.725642 | 2.399321 | 1.957184 | 1.101089 | 1.132573 | 1.480275 | 1.258643 |
| \( \alpha = 10 \) | Low Limit | -1.127958 | -1.092018 | -1.092018 | -1.427360 | -1.271304 | -0.432594 | -0.406259 | -0.200198 | -0.191904 | -0.323172 | -0.242320 |
| Upper Limit   | 1.640824 | 1.741153 | 2.501313 | 2.038984 | 0.833399 | 0.869044 | 1.172196 | 0.984178 | 0.555054 | 0.567932 | 0.740275 | 0.628078 |
| \( \alpha = 50 \) | Low Limit | -0.225592 | -0.222586 | -0.222586 | -0.258794 | -0.057068 | -0.065381 | -0.128393 | -0.101194 | -0.040401 | -0.03850 | -0.064075 | -0.048679 |
| Upper Limit   | 0.329865 | 0.352950 | 0.504080 | 0.405009 | 0.166690 | 0.173931 | 0.231844 | 0.194239 | 0.131101 | 0.113893 | 0.148280 | 0.125301 |
| \( \alpha = 100 \) | Low Limit | -0.112796 | -0.112154 | -0.112154 | -0.127279 | -0.043754 | -0.043050 | -0.063991 | -0.049749 | -0.020620 | -0.019502 | -0.0323172 | -0.024417 |
| Upper Limit   | 0.164082 | 0.176414 | 0.250189 | 0.201104 | 0.083340 | 0.086869 | 0.116531 | 0.097198 | 0.055505 | 0.056390 | 0.074297 | 0.062810 |
Table 8. Confidence intervals of the estimated TP weights where we provide three different types of CI, i.e., approximated (or asymptotic) CI evaluated via first two exact moments, the CI obtained via stochastic representation obtained by [2], and the CI obtained via stochastic representations under the assumption of \( t \)-distributed data, which are further evaluated at 15 degrees of freedom. The confidence level is 95%. The returns are assumed to be independently multivariate normally and \( t \)-distributed. \( k \) is taken to be 15, and \( l = 15 \).

| Risk Aversion | Limits | \( n = 30 \) | \( n = 60 \) | \( n = 120 \) |
|---------------|--------|-------------|-------------|-------------|
| \( \alpha = 3 \) | Low limit | -9.765966 | -9.41929 | -9.175559 |
|               | Upper limit | 10.869331 | 11.403907 | 11.765420 |
| \( \alpha = 5 \) | Low limit | -5.859574 | -5.904934 | -5.904934 |
|               | Upper limit | 6.516258 | 6.788271 | 10.135339 |
| \( \alpha = 10 \) | Low limit | -2.929787 | -2.98060 | -3.062362 |
|               | Upper limit | 3.258129 | 3.426047 | 5.076556 |
| \( \alpha = 50 \) | Low limit | -0.585697 | -0.605093 | -0.675586 |
|               | Upper limit | 0.651626 | 0.679716 | 1.015747 |
| \( \alpha = 100 \) | Low limit | -0.292979 | -0.295542 | -0.352876 |
|               | Upper limit | 0.325813 | 0.342677 | 0.513268 |