Likelihood Degenerations

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(with an appendix by Thomas Lam)

Abstract
Computing all critical points of a monomial on a very affine variety is a fundamental task in algebraic statistics, particle physics and other fields. The number of critical points is known as the maximum likelihood (ML) degree. When the variety is smooth, it coincides with the Euler characteristic. We introduce degeneration techniques that are inspired by the soft limits in CEGM theory, and we answer several questions raised in the physics literature. These pertain to bounded regions in discriminantal arrangements and to moduli spaces of point configurations. We present theory and practise, connecting complex geometry, tropical combinatorics, and numerical nonlinear algebra.

1 Introduction

A very affine variety $X$ is a closed subvariety of an algebraic torus $(\mathbb{C}^*)^n$. For any integer vector $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, the Laurent monomial $z^u = z_1^{u_1} \cdots z_n^{u_n}$ is a regular function on $(\mathbb{C}^*)^n$, and we are interested in the set of critical points of $z^u$ on $X$. The natural approach is via the gradient of the log-likelihood function $\log(z^u) = \sum_{i=1}^n u_i \log(z_i)$. This makes sense for any complex vector $u \in \mathbb{C}^n$. The coordinates of $\nabla \log(z^u)$ are rational functions, and we seek points $z \in X$ at which that gradient vector lies in the normal space. This leads to a system of rational function equations whose solutions are the critical points. Their number is independent of $u$, provided $u$ is generic. This is an invariant of $X$, denoted $\text{MLdegree}(X)$, and known as the maximum likelihood degree; see [11, 16, 17, 18]. Whenever $X$ is smooth, we know from [17, Theorem 1] that it coincides with the signed Euler characteristic of $X$:

$$\text{MLdegree}(X) = (-1)^{\dim X} \cdot \chi(X). \quad (1)$$

The term likelihood comes from statistics [32]. A discrete statistical model on $n-1$ states is a subset $M$ of the probability simplex $\Delta_{n-2} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{R}_{\geq 0}^{n-1} : z_1 + \cdots + z_{n-1} = 1\}$. In algebraic statistics, the model $M$ is a semialgebraic set, and one replaces $M$ by its Zariski closure $X$. That closure is taken in the torus $(\mathbb{C}^*)^n$, where $z_n = z_1 + \cdots + z_{n-1}$. After collecting data, we write $u_i \in \mathbb{N}$ for the number of samples in state $i$. The monomial $z^u$ is the likelihood function. The goal of likelihood inference is to maximize $z^u$ over $M$. Thus, statisticians seek those critical points of $\log(z^u)$ that are real and positive. The ML degree of $X$ is an algebraic complexity measure for performing likelihood inference with the model $M$. 


In particle physics, very affine varieties arise from scattering equations [8, 9, 10, 33]. These equations are central to the Cachazo–He–Yuan (CHY) formulas for biadjoint scalar amplitudes. Their solutions are the critical points of \( \log(z^u) \) on the moduli space \( \mathcal{M}_{0,m} = \text{Gr}(2,m)/\mathbb{C}^m \) of genus zero curves with \( m \) marked points. In the more general context of Cachazo–Early–Guevara–Mizera (CEGM) amplitudes [8], one considers the very affine varieties \( X(k,m) = \text{Gr}(k,m)/\mathbb{C}^m \) with \( k \geq 2 \). These are moduli spaces of \( m \) points in \( \mathbb{P}^{k-1} \) in linearly general position, natural generalizations of \( \mathcal{M}_{0,m} \). While the maximum likelihood degree \( \text{MLdegree}(X(2,m)) \) is known to be \( (m-3)! \), much less is known for \( k \geq 3 \). The connection between maximum likelihood and scattering equations was developed in [33].

The term degeneration comes from algebraic geometry. It represents the idea of studying the properties of a general object \( X_t \) for \( t \neq 0 \) by letting it degenerate to a more special object \( X_0 \), which is often easier to understand. This corresponds to finding a nice compactification of the variety \( X^0 = \bigcup_{t \neq 0} X_t \) to a variety \( X \), by adding the special fiber \( X_0 \). There are many possibilities for such constructions. Most relevant in our context are the tropical compactifications in [22, §6.4] and their connection to likelihood inference in [34].

Cachazo, Umbert and Zhang [10] introduced a class of degenerations called soft limits. The present article arose from our desire to gain a mathematical understanding of that construction from physics. We succeeded in reaching that understanding, and we here share it from multiple perspectives: algebraic geometry, combinatorics and numerical mathematics.

We now give an overview of our contributions. We start in Section 2 with a discussion of the moduli space \( X(k,m) \), and how it fits into the framework of very affine varieties. The soft limits in [10] are special instances of likelihood degenerations that are well adapted to the geometry of configurations. We explain how these are related to the deletion maps

\[
\pi_{k,m} : X(k, m+1) \to X(k, m). \tag{2}
\]

These maps are shown to be stratified fibrations. We discuss both the strata and the fibers. This sets the stage for the computation of Euler characteristics by combinatorial methods.

The fibers of (2) are complements of discriminantal hyperplane arrangements. By Varchenko’s Theorem [11, Theorem 3], the ML degree is the number of bounded regions. In Section 3 we focus on the generic fiber. This arises from the \( \binom{m}{k-1} \) hyperplanes spanned by \( m \) generic points in \( \mathbb{P}^{k-1} \). We show that, for fixed \( k \), the number of bounded regions is a polynomial in \( m \) of degree \( (k-1)^2 \). This polynomial was denoted \( \text{Soft}_{k,m} \) in [10]. We display it explicitly for \( k \leq 7 \). Our result extends to all coefficients of the characteristic polynomial (Theorem 3.3). Its proof rests on constructions of Koizumi, Numata and Takemura in [20].

In Section 4 we turn from fibers to strata in the base of (2). These can be modeled as strict realization spaces of matroids. Such matroid strata are very affine varieties. In Theorem 4.2 we furnish a comprehensive study for small matroids of rank \( k \) on \( m \) elements. Note that the uniform matroid corresponds to \( X(k,m) \). We compute the ML degrees for all matroids in the range \( k = 3, m \leq 9 \) and \( k = 4, m = 8 \). For larger matroids this would become infeasible by Mnëv’s Universality Theorem [3, §6.3]. Our result is achieved by integrating software tools from computer algebra [1], combinatorics [23], and certified numerics [5, 6].

In Section 5 we examine the space \( X(3,m) \) of \( m \) points in general position in the projective plane \( \mathbb{P}^2 \). Cachazo, Umbert and Zhang [10] report that the ML degree of \( X(3,m) \) equals 26.
for \( m = 6 \), 1 272 for \( m = 7 \), and 188 112 for \( m = 8 \). These numbers are denoted \( N^{(3)}_m \) in [10]. Thomas Lam (Appendix A) derived them using finite field methods, and he also computed

\[
N^{(3)}_9 = \text{MLdegree}(X(3, 9)) = 74 570 400.
\]

We present a topological proof of these results, and we prove the conjecture in [10, §6]. This involves a careful study of the stratified fibration [2]. The combinatorics we develop along the way, such as posets of strata and Möbius functions, should be of independent interest.

Section 6 is dedicated to configurations of eight points in projective 3-space. Based on our computational results, we predict that the ML degree of \( X(4, 8) \) is equal to 5 211 816. This is the number of solutions to the likelihood equations, found numerically by the software HomotopyContinuation.jl [6]. We present a detailed analysis of the tropical geometry of soft limits in this case. This confirms the combinatorial predictions made in [10, Table 2], and offers a blueprint for future research that connects tropical geometry and numerical analysis.

In Section 7 we turn to algebraic statistics, and we follow up on earlier work on likelihood degenerations due to Gross and Rodriguez [16]. We introduce the tropical version of maximum likelihood estimation for discrete statistical models. This arises by replacing the real numbers \( \mathbb{R} \) by the Puiseux series field \( \mathbb{R}\{\{t\}\} \), in both the data and the solutions. Our main result (Theorem 7.1) characterizes the tropical MLE for linear discrete models. This involves the intersection of two Bergman fans, corresponding to a dual pair of matroids.

In Section 8 we present numerical methods for likelihood degenerations, with an emphasis on recovering the description of tropical curves from floating point coordinates. This extends the tropical MLE approach in Section 7 from linear models to other very affine varieties, and it allows us to find tropical solutions to scattering equations in particle physics.

The paper concludes with Appendix A written by Thomas Lam, which gives the computation of ML degrees for \( k = 3 \) using finite field methods, based on the Weil conjectures.

The code used in this paper together with computational results are available at

https://mathrepo.mis.mpg.de/LikelihoodDegenerations.

2 Very affine varieties of point configurations

This work revolves around a very affine variety arising in particle physics [8] [9] [10], namely the moduli space \( X(k, m) \) of \( m \) points in \( \mathbb{P}^{k-1} \) in linearly general position. More explicitly, this moduli space parametrizes \( m \)-tuples \([P] = [P_1, \ldots, P_m]\) of points \( P_i \in \mathbb{P}^{k-1} \) such that no \( k \) of them lie on a hyperplane. Moreover, the \( m \)-tuples are considered only up to the action of \( \text{GL}(k, \mathbb{C}) \). The dimension of \( X(k, m) \) equals \((k - 1)(m - k - 1)\). Observe that \( X(2, m) \) is the familiar \((m - 3)\)-dimensional moduli space \( \mathcal{M}_{0,m} \) of \( m \) distinct labeled points on \( \mathbb{P}^1 \).

We next present a parametrization which shows that \( X(k, m) \) is very affine. Taking homogeneous coordinates for the points \( P_i \), any \( m \)-tuple \([P]\) as above can be represented by a complex \( k \times m \) matrix whose \( k \times k \) minors are all nonzero. Two such representations are equivalent if and only if they differ by left multiplication by \( \text{GL}(k, \mathbb{C}) \), or by a rescaling of
the columns by an element in the torus \((\mathbb{C}^*)^m\). Hence, this identifies \(X(k, m)\) as the quotient
\[
X(k, m) = \text{Gr}(k, m)^c / (\mathbb{C}^*)^m,
\]
where \(\text{Gr}(k, m)\) is the Grassmannian of \(k\)-dimensional subspaces in \(\mathbb{C}^m\) and \(\text{Gr}(k, m)^c\) is the open cell where all Plücker coordinates \(p_{i_1...i_k}\) are nonzero. We can uniquely write the homogeneous coordinates of an \(m\)-tuple \([P] \in X(k, m)\) as the columns of the \(k \times m\) matrix
\[
M_{k,m} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & (-1)^k & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & (-1)^{k-1} & 0 & 1 & x_{1,1} & x_{1,2} & \cdots & x_{1,m-k-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & 1 & x_{k-3,1} & x_{k-3,2} & \cdots & x_{k-3,m-k-1} \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & x_{k-2,1} & x_{k-2,2} & \cdots & x_{k-2,m-k-1} \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & x_{k-1,1} & x_{k-1,2} & \cdots & x_{k-1,m-k-1}
\end{bmatrix},
\]
provided all maximal minors \(p_{i_1...i_k}\) of this matrix are nonzero. The antidiagonal matrix in the left \(k \times k\) block was chosen so that each unknown \(x_{i,j}\) equals such a minor for \(i_1 < \cdots < i_k\).

This identifies \(X(k, m)\) as an open subset of the torus \((\mathbb{C}^*)^{(k-1)(m-k-1)}\) with coordinates \(x_{i,j}\). The Plücker embedding realizes \(X(k, m)\) as a closed subvariety of a high-dimensional torus:
\[
X(k, m) \hookrightarrow (\mathbb{C}^*)^{(m)} , \quad M_{k,m} \mapsto (p_{i_1...i_k}) .
\]
The corresponding log-likelihood function, known as the potential function in physics, is
\[
\mathcal{L}_{k,m} = \sum_{i_1...i_k} u_{i_1...i_k} \cdot \log(p_{i_1...i_k}).
\]
The critical point equations, known as scattering equations in physics, are given by
\[
\frac{\partial \mathcal{L}_{k,m}}{\partial x_{i,j}} = 0 \quad \text{for} \quad 1 \leq i \leq k-1 \quad \text{and} \quad 1 \leq j \leq m-k-1.
\]
This is a system of rational function equations. One way to compute the ML degree of \(X(k, m)\) is by counting the number of solutions for general values of the parameters \(u_{i_1...i_k}\). This can be done by finding explicit solutions to (7), notably by the numerical approach described in [33, §3] which rests on the software HomotopyContinuation.jl [6].

**Remark 2.1.** Likelihood degenerations arise when the coefficients \(u_{i_1...i_k}\) depend on a parameter \(t\), and one studies the behavior of the solutions in the limit \(t \to 0\). A special instance are the soft limits of Cachazo et al. [10]. This likelihood degeneration is presented in (21). We explain in Section 6 how soft limits are related to the methods in this section. Sections 7 and 8 explore arbitrary likelihood degenerations, with a view towards statistics and numerics.

Another approach to computing the ML degree is topological, through the Euler characteristic. Indeed, since the variety \(X(k, m)\) is smooth, the formula (1) applies and gives
\[
\text{MLdegree}(X(k, m)) = (-1)^{(k-1)(m-k-1)} \cdot \chi(X(k, m)).
\]
Our strategy to compute the Euler characteristic is to exploit the deletion map in (2):

\[ \pi_{k,m} : X(k, m + 1) \to X(k, m), \quad \{P_1, \ldots, P_m, P_{m+1}\} \mapsto \{P_1, \ldots, P_m\}. \]

This approach relies on the fact that the Euler characteristic is multiplicative along fibrations.

**Example 2.2** \((k = 2)\). The space \(X(2, 3)\) is a single point, so \(\chi(X(2, 3)) = 1\). For \(m \geq 3\) we consider the map \(\pi_{2,m} : X(2, m + 1) \to X(2, m)\). A point in \(X(2, m)\) is identified with an \(m\)-tuple \([P_1, \ldots, P_m]\), where \(P_i \in \mathbb{P}^1\) are pairwise distinct, and \(P_1 = (0 : -1), P_2 = (1 : 0), P_3 = (1 : 1)\) are fixed. The fiber over this point equals \(F = \mathbb{P}^1 \setminus \{P_1, \ldots, P_m\}\). This has Euler characteristic \(2 - m\). All fibers are homeomorphic to \(F\), so the map \(\pi_{2,m}\) is a fibration. Multiplicativity of the Euler characteristic along fibrations implies \(\chi(X(2, m + 1)) = \chi(F) \cdot \chi(X(2, m)) = (2 - m) \cdot \chi(X(2, m))\). By induction on \(m\), we conclude that \(\chi(X(2, m)) = (-1)^{m-3}(m-3)!\). \(\triangle\)

Our main difficulty for \(k \geq 3\) is that the deletion map \(\pi_{k,m}\) is generally not a fibration. But it is a **stratified fibration**, that is, it is a map \(f : X \to Y\) of complex algebraic varieties such that \(Y\) has a stratification \(\mathcal{S} = \{S \subseteq Y\}\) by finitely many closed strata, with the property that, over each open stratum \(S^o = S \setminus \bigcup_{S' \supseteq S} S'\), the map is a fibration with fiber \(F_S\).

The set \(\mathcal{S}\) of all strata \(S\) in a stratified fibration \(f : X \to Y\) is naturally a poset, ordered by inclusion. We can use this combinatorial structure to compute the Euler characteristic of \(X\). The following result is standard but we include it here for the sake of reference.

**Lemma 2.3.** Let \(f : X \to Y\) be a stratified fibration and \(\mu\) the Möbius function of \(\mathcal{S}\). Then

\[
\chi(X) = \sum_{S \in \mathcal{S}} \chi(S) \cdot \sum_{S' \in \mathcal{S}, S' \supseteq S} \mu(S, S') \chi(F_{S'})
\]

\[
= \chi(Y) \cdot \chi(F_Y) + \sum_{S \in \mathcal{S}} \chi(S) \cdot \sum_{S' \in \mathcal{S}, S' \supseteq S} \mu(S, S') \cdot (\chi(F_{S'}) - \chi(F_Y)).
\]

**Proof.** By the excision property of the Euler characteristic, together with the multiplicativity along fibrations, we know that \(\chi(X) = \sum_{S' \in \mathcal{S}} \chi(S^o) \cdot \chi(F_{S'})\). We can rewrite this as follows: for any closed stratum \(S' \in \mathcal{S}\) we have \(\chi(S') = \sum_{S' \supseteq S} \chi(S^o)\). The Möbius inversion formula yields \(\chi(S^o) = \sum_{S' \supseteq S} \mu(S, S') \chi(S)\). Plugging this into the previous formula, we obtain the first equality in Lemma 2.3. The second equality comes from the definition of the Möbius function, which stipulates that \(\sum_{S' \supseteq S} \mu(S, S') = 0\), for a fixed stratum \(S \in \mathcal{S} \setminus \{Y\}\). \(\square\)

Lemma 2.3 can be used to compute the ML degree of a very affine variety \(X\) that is smooth. If we are given a stratified fibration \(X \to Y\), then the computation reduces to the topological task of computing the Euler characteristics of the fibers and of the strata, together with the combinatorial task of computing the Möbius function of the poset.

We shall apply this method to the deletion map in (2). For this, we need to argue that this map is a stratified fibration, which requires us to identify the strata and the fibers. The fibers are complements of discriminantal arrangements, to be discussed thoroughly in Section 3. The strata are given by a certain matroidal stratification, see Section 4. We describe the general setting here, and we will give explicit computations for the case \(k = 3\) in Section 5.
First, let us consider the fibers. Given a representative $[P] = [P_1, \ldots, P_m]$ for an element of $X(k, m)$, the corresponding discriminantal arrangement is the set $\mathcal{B}(P)$ of all hyperplanes linearly spanned by any set of $k - 1$ points $P_{i_1}, \ldots, P_{i_{k-1}}$. Observe that these span a hyperplane because of the requirement that the points are in linearly general position. The fiber of $\pi_{k,m}$ over $[P]$ is the complement of this hyperplane arrangement:

$$\pi_{k,m}^{-1}([P]) \cong \mathbb{P}^{k-1} \setminus \bigcup_{H \in \mathcal{B}(P)} H.$$  \hspace{1cm} (9)

The Euler characteristic of such a complement is found with the methods in Section 3. One subtle aspect of this computation is that all objects are taken up to the action of $GL(k, \mathbb{C})$.

This description also indicates the appropriate stratification of $X(k, m)$: it depends on the linear dependencies satisfied by the $\binom{m}{k-1}$ hyperplanes in the arrangement $\mathcal{B}(P)$. This comes from a certain matroid stratification. Let us give an example for the case $k = 3$:

**Example 2.4.** Consider $\pi_{3,6} : X(3, 7) \to X(3, 6)$. To any $[P]$ in $X(3, 6)$ we associate the arrangement of 15 lines $P_iP_j$. Move one of them to the line at infinity in $\mathbb{P}^2$. For generic $[P]$, the number of bounded regions in the resulting arrangement of 14 lines is the Euler characteristic of the fiber $\pi_{3,6}^{-1}([P])$. This number is 42, as we shall see in Section 3. For special fibers the Euler characteristic drops. Consider $[P]$ where the lines $P_1P_2$, $P_3P_4$, $P_5P_6$ are concurrent. These three lines are not in general position [3 Figure 4-3]. The Euler characteristic of the fiber of $\pi_{3,6}$ over $[P]$ is 41. Our stratification of $X(3, 6)$ must account for this. \triangle

In general, the situation is as follows. Each point of $X(k, m)$ is represented by a $k \times m$ matrix $M_{k,m}$ as in [4]. We consider the $(k - 1)$st exterior power $\wedge_{k-1} M_{k,m}$ of that matrix. That new matrix has $k$ rows and $\binom{m}{k-1}$ columns, and its entries are the signed $(k-1) \times (k-1)$ minors of $M_{k,m}$. Each column of $\wedge_{k-1} M_{k,m}$ is the normal vector to the hyperplane spanned by $k-1$ points $P_{i_1}, \ldots, P_{i_{k-1}}$. Here we allow for the possibility that the column vector is zero, which means that $P_{i_1}, \ldots, P_{i_{k-1}}$ lie on a $(k-2)$-plane in $\mathbb{P}^{k-1}$. Taking the exterior power is equivariant with respect to the action of $GL(k, \mathbb{C})$, so we obtain a map of Grassmannians

$$\text{Gr}(k, m) \to \text{Gr}(k, \binom{m}{k-1}).$$  \hspace{1cm} (10)

We now consider the matroid stratification [3 §4.4] of the big Grassmannian $\text{Gr}(k, \binom{m}{k-1})$. The strata correspond to arrangements of $\binom{m}{k-1}$ hyperplanes in $\mathbb{P}^{k-1}$ that have a fixed intersection lattice. We next consider the pullback of this matroid stratification under the map (10). This is a very fine stratification of $\text{Gr}(k, m)$. Its strata correspond to point configurations whose discriminantal hyperplane arrangement has a fixed intersection lattice. This stratification is much finer than the matroid stratification of $\text{Gr}(k, m)$. In particular, it defines a highly nontrivial stratification of the open cell $\text{Gr}(k, m)^\circ$. This stratification of $\text{Gr}(k, m)^\circ$ is compatible with the action of the torus $(\mathbb{C}^*)^m$, given that all our constructions are based on projective geometry. Passing to the quotient in (3), let $S_{k,m}$ denote the induced stratification of $X(k, m)$. We call this the discriminantal stratification of the space $X(k, m)$.

**Proposition 2.5.** The deletion map $\pi_{k,m} : X(k, m + 1) \to X(k, m)$ defines a stratified fibration with respect to the discriminantal stratification on the very affine variety $X(k, m)$. 

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Proof and discussion. Each fiber of $\pi_{k,m}$ is the complement of a discriminantal hyperplane arrangement. These arrangements have fixed intersection lattice as the base point $[P]$ ranges over an open stratum of $S_{k,m}$. This implies that the Euler characteristic is constant on each fiber. This is what we need to be a fibration on each stratum. It is possible that the homotopy type varies across such fibers [28]. This would require further subdivisions into constructible sets. But, all we need here is for the Euler characteristic to be constant.

In conclusion, the topological approach to computing the ML degree of $X(k, m)$ consists of two parts: computing the Euler characteristic of the fibers, which is done in Section 3 and computing the Euler characteristic of the matroid strata, which is done in Section 4.

3 Discriminantal hyperplane arrangements

This section concerns the discriminantal hyperplane arrangements that form the fibers (9). Our main focus lies on the generic fibers. Such arrangements have been studied for decades, e.g. in [12, 14, 20, 27]. Our general reference on the relevant combinatorics is [31].

We work with two models in real affine space. First, fix $m$ general points in $\mathbb{R}^{k-1}$. We write $A(k, m)$ for the arrangement of $\binom{m}{k-1}$ hyperplanes spanned any $k - 1$ of these points. Second, we consider the set $\tilde{B}(k, m)$ of $\binom{m}{k-1}$ hyperplanes through the origin in $\mathbb{R}^k$ that are spanned by any $k - 1$ of the columns of the matrix $M_{k,m}$ in [4], where the $x_{i,j}$ are generic. Let $B(k, m)$ be the restriction of this hyperplane arrangement to $\mathbb{R}^{k-1} \simeq \{x_1 = 1\}$. Thus $B(k, m)$ is an affine arrangement of $\binom{m}{k-1} - 1$ hyperplanes in $\mathbb{R}^{k-1}$. As pointed out by Falk in [14], the combinatorics of the arrangement $A(k, m)$ is not independent of the choice of the $m$ general points as for some choices the dependencies among those points admit “second-order” dependencies beyond the Grassmann-Plücker relations. Following Bayer and Brandt [4] we assume that the $m$ points are very generic in the sense that such dependencies do not appear. All such very generic choices yield combinatorially equivalent arrangements $A(k, m)$, see also [29] for a discussion of non-very generic discriminantal arrangements.

In Section 5 we will explore non-very generic degenerations of these discriminatal arrangements. Both $A(k, m)$ and $B(k, m)$ are induced by the open cell in the large Grassmannian under an embedding of the form (10). But their combinatorial structures are different.

Figure 1: The discriminantal line arrangements $A(3, 4)$ (left) and $B(3, 4)$ (right).
Example 3.1 $(k = 3, m = 4)$. The arrangement $A(3,4)$ consists of the six lines that are spanned by four general points in the affine plane $\mathbb{R}^2$. Its complement in $\mathbb{R}^2$ consists of 18 regions, of which six are bounded. The arrangement $B(3,4)$ is obtained from $A(3,4)$ by moving one of the six lines to infinity. Thus $B(3,4)$ consists of five lines, which divide $\mathbb{R}^2$ into 12 regions, of which two are bounded. The two arrangements are shown in Figure 1. Their characteristic polynomials \( \chi_{A(3,4)}(t) = t^2 - 6t + 11 \) and $\chi_{B(3,4)}(t) = t^2 - 5t + 6$. △

| $k$ | $m = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|--------|---|---|---|---|---|---|---|
| 2   | 0      | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3   | 0      | 2 | 13| 42| 101| 205| 372|
| 4   | 0      | 0 | 6 | 192| 1858| 10644| 44595|
| 5   | 0      | 0 | 24| 5388| 204117| 3458276|
| 6   | 0      | 0 | 120| 255180| 46545915|
| 7   | 0      | 0 | 720| 18699210|

Table 1: The number of bounded regions of $B(k,m)$ for various $k$ and $m$. See Remark 3.2.

It is the second arrangement, $B(k,m)$, which plays the center stage for our application. We shall reduce its analysis to that of $A(k,m)$, so we can use results of Koizumi et al. [20].

Remark 3.2. The generic fiber of the map $\pi_{k,m}$ is the complement of a hyperplane arrangement in complex projective space $\mathbb{P}^{k-1}$ that is combinatorially isomorphic to $B(k,m)$. Hence the Euler characteristic of the generic fiber equals the number of bounded regions of $B(k,m)$.

We begin with some basics from [31]. Given an arrangement $A$ of hyperplanes $H_1, \ldots, H_N$ in real affine space $\mathbb{R}^d$, we write $F_I = \bigcap_{i \in I} H_i$ for any subset $I$ of $[N] := \{1, \ldots, N\}$. The collection of nonempty $F_I$ forms a poset $L(A)$ by reverse inclusion. This is called the intersection poset, and it is graded by the codimension of $F_I$. The characteristic polynomial is

\[
\chi_A(t) = \sum_{F_I \in L(A)} \mu(F_I) \cdot t^{d-\text{codim}(F_I)} =: \sum_{i=0}^{d} (-1)^i b_i(A) \cdot t^{d-i} \quad (11)
\]

where $\mu(F_I) = \mu(\mathbb{R}^d, F_I)$ is the Möbius function of $L(A)$. The coefficients $\{b_i(A)\}_{i=0}^{d}$ are the Betti numbers of $A$. They are nonnegative integers. The components of the complement of the arrangement in $\mathbb{R}^d$ are called the regions of $A$. Zaslavsky [35] showed that the regions are counted, up to sign, by $\chi_A(-1)$. The bounded regions of $A$ are counted by $|\chi_A(1)|$.

We are interested in the characteristic polynomials of the generic discriminantal arrangements $A(k,m)$ and $B(k,m)$ defined above. The following is our main result in this section.

Theorem 3.3. For fixed $k$, the Betti numbers of $A(k,m)$ and $B(k,m)$ are polynomials in the parameter $m$. Here the $i$-th Betti number is a polynomial of degree $i(k-1)$. The number of bounded regions of either arrangement is given by a polynomial in $m$ of degree $(k-1)^2$.

In light of Remark 3.2, we are primarily interested in the number of bounded regions of $B(k,m)$. Cachazo et al. [10, §3] wrote these as polynomials of degree 4 and 9 for $k = 3, 4$. This led to the polynomiality conjecture that is proved here. We list our polynomials for $k \leq 7$:
Corollary 3.6. Deleting the constant term and dividing by \( t \) coefficients of their characteristic polynomials, i.e., the Betti numbers, are polynomials in \( B \).

The discriminantal arrangement \( B(k,m) \) satisfies

\[
\begin{align*}
\chi_{B(k,m)}(t) & = t^3 - 6t^2 + 11t - 6.
\end{align*}
\]

The characteristic polynomials in Example 3.1 are derived from this. We obtain \( \chi_{A(3,4)}(t) \) by deleting the constant term and dividing by \( t \). We obtain \( \chi_{B(3,4)}(t) \) by dividing by \( t - 1 \). △

Corollary 3.6. The number of bounded regions of \( B(k,m) \) is given, up to a sign, by

\[
\chi_{B(k,m)}(1) = \frac{d}{dt}\chi_A(k,m)(1) + \chi_A(k,m)(1).
\]

Koizumi et al. [20] provide formulas for \( \chi_A(k,m) \) for \( 1 \leq k \leq 7 \) and general \( m \). The coefficients of their characteristic polynomials, i.e., the Betti numbers, are polynomials in \( m \). We show that this trend continues. The extension to \( B(k,m) \) follows from Lemma 3.4.
Lemma 3.7. The Betti numbers of both $A(k, m)$ and $B(k, m)$ are polynomials in $m$.

Proof. We shall use [20] to derive the result for $A(k, m)$. We set $d = k - 1$ and write $L(d, m)$ for the intersection poset of $A(k, m)$, with Möbius function $\mu_{d, m}$. Note that $\mu_{d, m}(\emptyset)$ is the number of bounded regions in $A(k, m)$. In [20], the elements $F$ of $L(d, m)$ are grouped according to partitions $\gamma = (\gamma_1, \ldots, \gamma_t)$, called the type of $F$. The number of elements of type $\gamma$ is denoted $\lambda_{d, m}(\gamma)$. A main result of [20] is that the Möbius function applied to $F$ depends only on the type of $F$. Hence, the characteristic polynomial of $L(d, m)$ equals

$$
\chi_{d, m}(t) = \sum_{i=0}^{d} \sum_{\gamma \vdash i} \lambda_{d, m}(\gamma) \mu_{d, m}(\gamma) t^{d-i}.
$$

(12)

We will show that $\lambda_{d, m}(\gamma)$ and $\mu_{d, m}(\gamma)$ are both polynomials in $m$ for fixed $d$. Then (12) implies that the coefficients of the characteristic polynomial are polynomials in $m$ as well. Proposition 4.1 of [20] gives a recursive definition for $\mu_{d, m}(\gamma)$ in terms of $\mu_{d', m'}(\emptyset)$ for strictly smaller $d'$ and $m'$. Since these are the numbers of bounded regions of $A(k', m')$ for strictly smaller $k'$ and $m'$, their polynomiality follows inductively, proving the result.

Proposition 4.1 of [20] gives a recursive definition for $\mu_{d, m}(\gamma)$ in terms of $\mu_{d', m'}(\emptyset)$ for strictly smaller $d'$ and $m'$. Since these are the numbers of bounded regions of $A(k', m')$ for strictly smaller $k'$ and $m'$, their polynomiality follows inductively, proving the result.

Proof of Theorem 3.3. The Betti number $b_i$ of interest is the coefficient of $t^{d-i}$ in $\chi_{d, m}(t)$. We know that this is a polynomial in $m$. We first argue that the degree of this polynomial is at most $d \cdot i$, and then we show that it has degree at least $d \cdot i$. For the upper bound, we note that the $i$-th Betti number of a generic arrangement with $\binom{m}{d}$ hyperplanes in $\mathbb{R}^d$ is a polynomial in $m$ of degree $d \cdot i$. Any other arrangement with the same number of hyperplanes, including discriminantal ones, must have Betti numbers bounded by these generic Betti numbers.

To derive the lower bound, we recall that, by definition of the Betti numbers in [11],

$$
b_i = \sum_{F \in L(d, m), \text{codim}(F) = i} \mu_{d, m}(F).
$$

(14)
All terms in this sum have the same sign. Suppose $m > d \cdot i$ and consider all $T_{e_1}, \ldots, T_{e_i} \subset [m]$ where each $T_{e_j}$ consists of $d$ points and $T_{e_j} \cap T_{e_k} = \emptyset$ for $1 \leq j < k \leq i$. These collections correspond to the codimension $i$ flats $\bigcap_{j=1}^{i} H_j$ where $H_j$ is the hyperplane spanned by the points indexed by $T_{e_j}$. These are special flats in our arrangement. Their number is

$$\frac{1}{d!} \left( \begin{array}{c} m \\ d \end{array} \right) \cdot \left( \begin{array}{c} m-d \\ d \end{array} \right) \cdots \left( \begin{array}{c} m-id+d \\ d \end{array} \right).$$

This product of $i$ binomial coefficients is a polynomial in $m$ of degree $d \cdot i$. Therefore, the degree of $b_i$ as polynomial in $m$ is at least $d \cdot i$. This completes the proof of Theorem 3.3.

**Corollary 3.8.** The ML degree of the generic fiber of $\pi_{k,m}$ is a polynomial of degree $(k-1)^2$.

We presented formulas for these ML degrees for $k \leq 7$ and arbitrary $m$. These play a major role in the stratified fibration approach of Section 2. However, in addition to the generic fiber, we also need to know the Euler characteristic for the fiber over each stratum in the base space $X(k,m)$. All of these fibers are discriminantal arrangements, arising from lower-dimensional matroid strata in the large Grassmannian on the right hand side of (10). In other words, we need to compute $\chi_A(1)$ for many large hyperplane arrangements $A$.

In practice, this task can now be accomplished easily, thanks to the software recently presented in [7]. This implementation was essential to us in getting this project started, and in validating the polynomial formulas for $B(k,m)$ displayed above. We expect it to be useful for a wide range of applications, not just in mathematics, but also in physics and statistics.

## 4 Matroid strata

In our topological approach to computing the ML degree of $X(k,m)$, we encountered special strata of points and lines in $\mathbb{P}^{k-1}$ with prescribed incidence conditions. Such strata can be modeled as matroid strata. From now on, all matroids are assumed to be simple, so the term “matroid” will mean “simple matroid”. We shall comprehensively study all small matroids.

For a matroid $M$ of rank $k$ on $m$ elements, we consider the points in $\text{Gr}(k,m)$ whose nonzero Plücker coordinates are precisely those indexed by the bases of $M$. Let $X(M)$ be the quotient of that constructible set modulo the action of $(\mathbb{C}^*)^m$. One represents $X(M)$ by a $k \times m$ matrix with at least one entry 1 per column and some unknown entries that satisfy equations and inequations of degree $\leq k-1$ arising from nonbases and bases. This encoding shows that $X(M)$ is a very affine variety. If $M$ is the uniform matroid then $X(M) = X(k,m)$. The aim of this section is to compute the ML degree of $X(M)$ for every matroid of small size.

**Example 4.1** $(k = 3, m = 9)$. The *Pappus matroid* $M$ (shown in Figure 2a) has the nonbases

123 148 159 247 269 357 368 456 789.

These are precisely the triples that index the vanishing $3 \times 3$ minors of the matrix

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & y & 0 & 1 & y \\
0 & 1 & x & 0 & 1 & y & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1/(1-x) & x/(y(x-1)) & 1
\end{bmatrix}.$$ (15)
Indeed, the stratum \( X(M) \) for the Pappus matroid is the set of pairs \( (x, y) \in (\mathbb{C}^*)^2 \) such that all \( \binom{9}{3} - 9 = 75 \) other maximal minors are nonzero. The log-likelihood function equals
\[
u_1 \log(x) + u_2 \log(y) + u_3 \log(1-x)+u_4 \log(1-y)+u_5 \log(1-x-y)+u_6 \log(1-xy)+u_7 \log(xy-x-y).
\]
The two partial derivatives of this expression are rational functions in \( x \) and \( y \). By equating these to zero, we obtain a system of equations that has precisely eight solutions in \((\mathbb{C}^*)^2\), provided \( u_1, u_2, \ldots, u_7 \) are general enough. Hence the stratum of the Pappus matroid \( M \) is a very affine variety \( X(M) \) of dimension 2 whose ML degree is equal to 8. It is one of the 12 matroids of rank 3 on 9 elements with these invariants, marked \( s_{12} \) in Theorem 4.2. \( \triangle \)

We now present the result of our computations for matroids of rank \( k \) on \( m \) elements. For fixed \( k, m \), we list the matroid strata by dimension, and we list all occurring ML degrees together with their multiplicity of occurrences. For instance, the string \( s_5[1, 24] \) accounts for five strata of dimension 2: four have ML degree 2 and one has ML degree 1. If the variety \( X(M) \) is reducible, then we list the ML degree for each component, e.g. in the format \( (3, 3)^2 \).

Below we follow the convention in the applied algebraic geometry literature (e.g. [26]) to assign a star to a theorem obtained by a numerical computation which was not fully certified.

**Theorem** 4.2. The strata \( X(M) \) are smooth for all matroids \( M \) with \( k = 2 \) or \( (k = 3 \text{ and } m \leq 9) \) or \( (k = 4 \text{ and } m = 8) \). Their ML degrees are given in the following lists.

For \( k = 3, m = 5 \) there are 4 matroids, up to permuting labels, and all are realizable over \( \mathbb{C} \):

\[
\begin{bmatrix} 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

For \( k = 3, m = 6 \) there are 9 matroids, up to permuting labels, and all are realizable over \( \mathbb{C} \):

\[
\begin{bmatrix} 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 6 \end{bmatrix} \quad \begin{bmatrix} 2 & 1, 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

For \( k = 3, m = 7 \) there are 22 orbits of \( \mathbb{C} \)-realizable matroids:

\[
\begin{bmatrix} 6 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 19 \end{bmatrix} \quad \begin{bmatrix} 4 & 24, 28 \end{bmatrix} \quad \begin{bmatrix} 3 & 7 \end{bmatrix} \quad \begin{bmatrix} 2 & 1, 6, 10, 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 11, 2, 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

For \( k = 3, m = 8 \) there are 66 orbits of \( \mathbb{C} \)-realizable matroids:

\[
\begin{bmatrix} 8 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 13 \end{bmatrix} \quad \begin{bmatrix} 6 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

For \( k = 3, m = 9 \) there are 368 orbits of \( \mathbb{C} \)-realizable matroids:

\[
\begin{bmatrix} 10 & 1 \end{bmatrix} \quad \begin{bmatrix} 9 & 6 \end{bmatrix} \quad \begin{bmatrix} 8 & 5 \end{bmatrix} \quad \begin{bmatrix} 7 & 4 \end{bmatrix} \quad \begin{bmatrix} 6 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]
For $k = 4$, $m = 8$ there are 554 orbits of $\mathbb{C}$-realizable matroids:

\begin{align*}
9_1[5211816] & \quad 8_1[5167631] & \quad 7_5[212402, 46392, 52392, 63040_1] \\
6_{14}[12722, 15602, 21361, 29762, 51301, 69762, 76091, 94241, 103682] & \\
5_{10}[26, 1204, 1924, 2646, 3683, 4881, 5202, 5682, 6921, 7081, 7701, 9361, 9501, 10809, 12941, 12962, 13221, 14001, 17101, 17681, 18122] & \\
4_{89}[0_{41}, 41, 62, 2414, 261, 381, 551, 568, 723, 803, 884, 1041, 1152, 1206, 1221, 1342, 1361, 1543, 1622, 1809, 1811, 2161, 2201, 2241, 2261, 2321, 2362, 2721, 2821, 2882, 3082, 3842, 4101] & \\
3_{153}[11, 23, 38, 6301, 1015, 125, 1613, 175, 1810, 249, 255, 272, 281, 371, 341, 353, 365, 381, 392, 401, 451, 462, 481, 506, 511, 523, 532, 551, 562, 582, 642, 662, 671, 722, 1041] & \\
2_{153}[18, 235, 429, 51, 618, 710, 816, 92, 106, 116, 128, 133, 143, 153, 182, 191, 242] & \\
1_{81}[0_{14}, 120, 236, 39, 49, 52, (2, 2_1)] & 0_{18}[115, 23] & 
\end{align*}

Proof. This was obtained by exhaustive computations. The matroids were taken from the database [23] that is described in [24]. The GAP packages alcove [21] and ZariskiFrames [1] were used to obtain a representing $k \times m$ matrix, such as (15), along with further equations for nonbases and inequations for bases, whenever needed. The details of the underlying algorithm are described in [2]. From this matrix, together with the defining equations, we compute the ML degree as the number of critical points of the log-likelihood function. This is illustrated in Example 4.1. These computations were performed using the Julia package HomotopyContinuation.jl [6]. See also [33]. The ML degrees of the uniform matroids $U_{3,7}$, $U_{3,8}$, $U_{3,9}$, $U_{4,8}$ are derived and discussed in Sections 5 and 6. 

Remark 4.3. As the method of homotopy continuation is numerical, it is inherently subject to rounding errors. Hence, the numbers in Theorem 4.2 come with this disclaimer as well. For those numbers which are not underlined, we successfully performed the certification method described in [5], using the implementation conveniently available in [6]. Such a certified result delivers a mathematical proof that our number bounds the true ML degree from below. Of the six underlined entries, two are the uniform matroids $U_{3,9}$ and $U_{4,8}$ (discussed in Sections 5 and 6) and another is a matroid whose ML degree is computed in Example 4.1. The certificates proving the correctness of our results can be found at https://zenodo.org/record/7454826#.Y6DCdezMKdY.

We now briefly discuss some special matroids that appear in our lists, shown in Figure 2. 

Example 4.4. The Pappus matroid was seen in Example 4.1. The non-Fano matroid has $k = 3$ and $m = 7$. It is projectively unique, so its stratum has dimension 0 and ML degree 1. The affine geometry $AG(2, 3)$ has $k = 3$ and $m = 9$. It is also known as the dual Hessian configuration. Its stratum is 0-dimensional of degree 2. Hence, its ML degree is 2. The table for $k = 3, m = 9$ lists nine $\mathbb{C}$-realizable matroids with ML degree 0. The one with the fewest nonbases has 6 nonbases; its stratum is 4-dimensional and has degree 4. A geometric representation of this matroid has the form of $3 \times 3$ grid and is depicted in Figure 2.

5 Fibrations of points and lines in the plane

We now study the moduli space $X(3, m)$ of $m$ labeled points in $\mathbb{P}^2$ in linearly general position. This is very affine of dimension $2m - 8$. Here is what we know about its Euler characteristic.
Figure 2: Geometric representations of the matroids discussed in Examples 4.1 and 4.4.

**Theorem 5.1.** The ML degree of $X(3, m)$ is given by the following table for $m \leq 9$:

| $m$ | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|
| $\chi(X(3, m))$ | 1  | 2  | 26 | 1272 | 188112 | 74570400 |

These numbers are easy to prove for $m \leq 6$. A computational proof for $m = 6$ appeared in [8, Appendix C]. The numbers for $m = 7, 8$ were derived with the soft limit argument in [10]. This was a proof in the sense of physics but perhaps not in the sense of mathematics. The verification by numerical computation was presented in [33, Proposition 5]. Thomas Lam had derived and proved all numbers, including the $m = 9$ case, using finite field methods. This work was mentioned in the introduction of [10], but Lam’s finite fields proof had been unpublished so far. It now appears along with this article, namely in Appendix A below.

The aim of this section is to solve this problem for general $m$ using the stratification approach in Section 2. Our techniques will be of independent interest. In particular, they can be adapted to give a geometric proof for each of the ML degrees of matroids in Theorem 4.2. As a warm-up, here are geometric proofs for the first three numbers in Theorem 5.1.

**Example 5.2** $(m = 4, 5, 6)$. The space $X(3, 4)$ is just a point, hence $\chi(X(3, 4)) = 1$. The very affine surface $X(3, 5)$ is the complement of the arrangement $B(3, 4)$ in Example 3.1. We have $\chi(X(3, 5)) = 2$, by Varchenko’s Theorem, as there are two bounded regions on the right in Figure 1. For $m = 6$, we consider the deletion map $\pi_{3,6} : X(3, 6) \to X(3, 5)$. The discriminantal arrangement for any five points in $\mathbb{P}^2$, with no three on a line, is isomorphic to $B(3, 5)$. All fibers of $\pi_{3,6}$ are homotopy equivalent to the complement of $B(3, 5)$. Hence $\pi_{3,6}$ is a fibration, where each fiber has Euler characteristic 13, by Table 1. The base $X(3, 5)$ has Euler characteristic 2. Hence their product 26 is the Euler characteristic of $X(3, 6)$. △

We now consider $m \geq 6$. Following Section 2, we study the stratification $S_{3,m}$ of $X(3, m)$. The codimension one strata are all combinatorially equivalent: they are the loci of configurations $[P] = [P_1, \ldots, P_m]$ where three lines $P_i P_j, P_k P_l, P_r P_s$ meet in a new point in $\mathbb{P}^2$. We write $S_{(ij)(kl)(rs)}$ for this stratum in $X(3, m)$. All other strata are intersections of those, hence they can be denoted by a collection of triples $(ij)(kl)(rs)$. For $m = 6$, there are 15 distinct codimension one strata $S_{(ij)(kl)(rs)}$, one for each tripartition of $\{1, \ldots, 6\}$. All other strata in $S_{3,6}$ are obtained by intersecting these 15 divisors. We found that $S_{3,6}$ has two combinatorially distinct codimension two strata, two codimension three strata and two codimension...
| Type | Codim | Representatives for the divisors \((i,j)(k,l)(r,s)\) that intersect in the stratum |
|------|-------|------------------------------------------------------------------|
| I    | 1     | \((12)(34)(56)\)                                               |
| II   | 2     | \((12)(34)(56),(12)(35)(46)\)                                    |
| III  | 2     | \((12)(34)(56),(15)(23)(46),(14)(26)(35)\)                      |
| IV   | 3     | \((12)(34)(56),(15)(23)(46),(14)(26)(35),(15)(26)(34)\)         |
| V    | 3     | \((12)(34)(56),(12)(35)(46),(13)(26)(45),(14)(25)(36),(15)(24)(36),(16)(23)(45)\) |
| VI   | 4     | \((12)(34)(56),(12)(35)(46),(14)(23)(56),(14)(26)(35),(15)(23)(46),(15)(26)(34)\) |
| VII  | 4     | \((12)(34)(56),(12)(35)(46),(13)(24)(56),(13)(26)(45),(14)(25)(36)，(14)(26)(35),(15)(23)(46),(15)(24)(36),(16)(23)(45),(16)(25)(36)\) |

Table 2: All types of strata in the 4-dimensional very affine variety \(X(3,6)\).

Figure 3: Geometric representation of the strata of Type I, II, and IV in Table 2

four strata. In Table 2 we list all strata explicitly, up to combinatorial equivalence. Figure 3 shows point configurations \([P_1,\ldots,P_6]\) for three among the seven strata in our list.

For our proof of Theorem 5.1 we shall use Lemma 2.3, here rewritten in the specific form

\[
\chi(X(3, m+1)) = \chi(X(3, m)) \cdot \chi(F_{X(3,m)}) - \sum_{S \in \mathcal{S}} \chi(S) \cdot \rho(S),
\]

where we define

\[
\rho(S) := \sum_{S' \subseteq S, S' \supseteq S} \mu(S,S') \cdot (\chi(F_{X(3,m)}) - \chi(F_{S'})).
\]

For a fixed stratum \(S\), the sum in \(17\) is over all \(S'\) that contain \(S\). These form a subposet \(\mathcal{P}_S\) (actually, a filter) of the poset \(\mathcal{S}_{3,m}\). It turns out that for many strata \(S\), the factor \(\rho(S)\) is zero. This drastically simplifies the computation. Let us illustrate this with an example.

**Example 5.3.** Consider the map \(\pi_{3,6} : X(3,7) \to X(3,6)\). Figure 4 shows the subposets for the two codimension three strata, of type IV and V, together with the Möbius function values at each node. The poset is ranked by codimension: zero for the top stratum, and three for the bottom stratum. Our aim is to compute \(\rho(S)\). Let us look at the left poset \(\mathcal{P}_S\). The
codimension zero stratum $S'$ has $\chi(F_{S'}) = \chi(F_X(3,6))$ by definition. The codimension one strata have $\chi(F_X(3,6)) - \chi(F_{S'}) = 1$, by Example 2.4. Looking at the strata of codimension two, we find that those of type III have $\chi(F_X(3,6)) = 3$, whereas those of type II have $\chi(F_X(3,6)) - \chi(F_{S'}) = 2$. Finally, the stratum of type IV has $\chi(F_X(3,6)) - \chi(F_{S'}) = 4$. With this, one computes $\rho(S) = -2 \cdot 0 + 3 \cdot (1 \cdot 1) + 2 \cdot 1 + (-1) \cdot 3 + 3 \cdot (-1 \cdot 2) + 1 \cdot 4 = 0$. \(\triangle\)

Figure 4: The posets $P_S$ for the codimension three strata in $X(3,6)$. The numbers in black are the values of the Möbius function. The types of the strata are shown in blue.

This phenomenon generalizes to other strata and to all spaces $X(3,m)$. Indeed, the only strata $S \in S_{3,m}$ for which $\rho(S) \neq 0$ are those whose lines meet in only one extra special point apart from the original ones. This is illustrated in [10, Figure 8]. More precisely, for each stratum $S$, let $[P] = [P_1, \ldots, P_m]$ be a general element in $S$, and for each $h \geq 3$ denote by $n_h(S)$ the number of points, apart from $P_1, \ldots, P_m$, where exactly $h$ lines of the discriminantal arrangement $B(P)$ meet. The next result proves the conjecture in [10, §6]:

**Theorem 5.4.** Every stratum $S \in S_{3,m}$ satisfies $\rho(S) \in \{-1, 0, +1\}$. More precisely:

(a) If $n_h(S) = 1$ for some $h$ and $n_i(S) = 0$ for all $i \neq h$, then $\rho(S) = (-1)^{h-1}$.

(b) In all other cases, we have $\rho(S) = 0$.

Item (b) is the punchline: a stratum $S$ does not contribute to the Euler characteristic unless it looks like [10, Figure 8]. To prove this, we introduce some further notation. For any stratum $S \in S_{3,m}$ let $\sigma(S) := \chi(F_X(3,m)) - \chi(F_S)$. When the discriminantal arrangement in the fiber $F_S$ over $S$ is realizable over $\mathbb{R}$, this denotes the difference between the numbers of bounded regions in $B(3,m)$ and $F_S$. Applying the Möbius inversion formula to (17) yields

$$\sigma(S) = \rho(S) + \sum_{S' \in P_S \setminus \{S\}} \rho(S'). \quad (18)$$

We start by computing the function $\sigma(S)$ in terms of the counting function $n_h(S)$:

**Lemma 5.5.** Every stratum $S \in S_{3,m}$ satisfies $\sigma(S) = \sum_{h=3}^{\infty} \binom{h-1}{2} \cdot n_h(S)$. 
Proof. Let $\mathcal{B}$ be the discriminantal arrangement associated to the fiber $F_S$. Let $\mathcal{B}(3, m)$ be as in Section 3. Its characteristic polynomial is $\chi_{\mathcal{B}(3, m)} = t^2 - \left(\begin{pmatrix} m \\ 2 \end{pmatrix} - 1 \right) t + b_2(\mathcal{B}(3, m))$ where $b_2(\mathcal{B}(3, m))$ is the second Betti number of $\mathcal{B}(3, m)$. Using (11), the characteristic polynomial of the special arrangement $\mathcal{B}$ equals

$$
\chi_{\mathcal{B}}(t) = t^2 - \left(\begin{pmatrix} m \\ 2 \end{pmatrix} - 1 \right) t + b_2(\mathcal{B}(3, m)) - \sum_{h=3}^{\infty} n_h(S)\begin{pmatrix} h \\ 2 \end{pmatrix} + \sum_{h=3}^{\infty} n_h(S)(h - 1),
$$

since every additional intersection point of $h$ lines in $\mathcal{B}$ removes $\begin{pmatrix} h \\ 2 \end{pmatrix}$ many generic codimension two elements of $L(\mathcal{B}(3, m))$ and adds a summand with Möbius value $h - 1$. Evaluating these two polynomials at $t = 1$ and taking their difference yields the claim. \qed

We now show that $\sigma$ is monotonic with respect to the poset $S$ in the following sense.

Lemma 5.6. Let $S', S \in S_{3, m}$ be two strata such that $S' \supseteq S$. Then $\sigma(S') < \sigma(S)$.

Proof. The assertion is equivalent to $\chi(F_{S'}) = \chi_{\mathcal{B}_{S'}}(1) > \chi(F_S) = \chi_{\mathcal{B}_S}(1)$ where $\mathcal{B}_S$ and $\mathcal{B}_{S'}$ are the discriminantal arrangements corresponding to the fibers $F_S$ and $F_{S'}$, respectively. Since each arrangement has the same number of lines, it suffices to prove $b_2(\mathcal{B}_{S'}) > b_2(\mathcal{B}_S)$.

We will establish a general statement, namely a strict inequality between the Betti numbers of two arrangements where one is a strict weak image of the other in the sense of matroid theory. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ and $\mathcal{A}' = \{H'_1, \ldots, H'_n\}$ be two affine arrangements of lines in $\mathbb{C}^2$ with $n \geq 3$. We claim that $b_2(\mathcal{A}) < b_2(\mathcal{A}')$ holds under the following two assumptions:

(i) If for $I \subseteq [n]$ the lines $\{H'_i\}_{i \in I}$ meet in $\mathcal{A}'$ then the lines $\{H_i\}_{i \in I}$ also meet in $\mathcal{A}$,

(ii) The lines $H_1, H_2, H_3$ intersect but the lines $H'_1, H'_2, H'_3$ do not intersect.

We prove this claim by induction on $n$. If $n = 3$ then the statement is trivial since (ii) forces $\mathcal{A}'$ to consist of three generic lines and $\mathcal{A}$ of three concurrent lines, so $b_2(\mathcal{A}) = 2 < 3 = b_2(\mathcal{A}')$.

Now fix some $n > 3$. The deletion–restriction relation for the characteristic polynomial (cf. [27, Corollary 2.57]) implies $b_2(\mathcal{A}) = b_2(\mathcal{A}' \setminus \{H_n\}) + |\mathcal{A}^{H_n}|$ where $\mathcal{A}^{H_n}$ is the restriction of $\mathcal{A}$ to $H_n$. The analogous relation holds for $\mathcal{A}'$. Since removing the last hyperplane preserves both assumptions (i) and (ii), we have $b_2(\mathcal{A} \setminus \{H_n\}) < b_2(\mathcal{A}' \setminus \{H'_n\})$ by induction. Moreover, assumption (i) implies $|\mathcal{A}^{H_n}| \leq |\mathcal{A}'^{H'_n}|$. So in total this proves $b_2(\mathcal{A}) < b_2(\mathcal{A}')$. \qed

Proof of Theorem 5.4. Note that $\rho(X(3, m)) = 0$ by definition. Let $S$ be a stratum with the property that $n_j(S) = 1$ and all other are zero. Observe that the only $S' \subsetneq X(3, m)$ strictly containing $S$ are the $\begin{pmatrix} h \\ j \end{pmatrix}$ strata $S'$ with $n_j(S') = 1$ and all others zero for $j = 3, \ldots, h - 1$. If $h = 3$ then $S$ is a codimension one stratum and is only strictly contained in $X(3, m)$, so Lemma 5.5 and formula (18) show that $\rho(S) = 1$. We now rewrite (18) via Lemma 5.5 as

$$
\begin{pmatrix} h - 1 \\ 2 \end{pmatrix} = \rho(S) + \begin{pmatrix} h \\ 3 \end{pmatrix} - \begin{pmatrix} h \\ 4 \end{pmatrix} + \cdots + (-1)^h\begin{pmatrix} h \\ h - 1 \end{pmatrix}
$$

by induction on $h$. This shows that $\rho(S) = (-1)^{h-1}$. 

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We prove part (b) by induction on $\sigma(S)$. The base case $\sigma(S) = 0$ corresponds to the stratum $X(3, m)$, for which we know already that $\rho(S) = 0$. For the induction step, let $S \in S_{3,m}$ be another stratum that is not one of those considered in part (a). For all strata $S' \in P_S \setminus \{S\}$ we have $\sigma(S') < \sigma(S)$ by Lemma 5.6. The induction hypothesis shows that $\rho(S') = 0$ for all strata that are not of the type of part (a). Thus, we have

$$\sigma(S) = \sum_{h=3}^\infty \binom{h-1}{2} \cdot n_h(S) = \rho(S) + \sum_{h=3}^\infty N_h(S) \cdot (-1)^{h-1}$$

where $N_h(S)$ is the number of strata $S' \in P_S \setminus \{S\}$ like those in part (a). This number is given directly by $N_h(S) = \sum_{k=3}^\infty n_k(S) \cdot \binom{k}{h}$. We conclude that

$$\sum_{h=3}^\infty \binom{h-1}{2} \cdot n_h(S) = \rho(S) + \sum_{k=3}^\infty n_k(S) \cdot \sum_{h=3}^\infty (-1)^{h-1} \binom{k}{h} = \rho(S) + \sum_{k=3}^\infty n_k(S) \cdot \binom{k-1}{2}.$$ 

This implies $\rho(S) = 0$ for all strata $S$ other than those in case (a).

Equipped with Theorem 5.4 we can prove Theorem 5.1 with a few calculations.

**Proof of Theorem 5.1.** We want to use the formula (16). By Theorem 5.4 we need to consider only those strata $S \in S_{3,m}$ where $n_h(S) = 1$ for a certain $h$ and $n_i(S) = 0$ for all $i \neq h$. The number of these strata in $X(3, m)$ is $\frac{1}{h!} \binom{m}{2} \cdot \binom{m-2}{2} \ldots \binom{m-2h+2}{2}$. Furthermore, all these strata are combinatorially equivalent, so we write $\chi(3, m; h)$ for the Euler characteristic of any of them. Using Theorem 5.4 we rewrite the formula (16) as follows:

$$\chi(X(3, m+1)) = \chi(X(3, m)) \cdot \chi(F_{X(3,m)}) + \sum_{h=3}^\infty \frac{(-1)^h}{h!} \binom{m}{2} \cdot \binom{m-2}{2} \ldots \binom{m-2h+2}{2} \cdot \chi(3, m; h).$$

In particular, when $m = 6, 7$ the formula becomes

$$\chi(X(3, 7)) = \chi(X(3, 6)) \cdot \chi(F_{X(3,6)}) - 15 \cdot \chi(3, 6; 3),$$

$$\chi(X(3, 8)) = \chi(X(3, 7)) \cdot \chi(F_{X(3,7)}) - 105 \cdot \chi(3, 7; 3).$$

The Euler characteristic $\chi(X(3, 6)) = 26$ is obtained in Example 2.4 whereas the bounded chamber counts $\chi(F_{X(3,6)}) = 42$ and $\chi(F_{X(3,7)}) = 101$ are taken from Table 1. To compute $\chi(3, m; 3)$, we note that corresponding stratum in $X(3, m)$ is isomorphic to a matroid stratum of codimension $3$ in $X(3, m+1)$, obtained by requiring that the new point $P_{m+1}$ lies on the three special lines. Those matroid strata have Euler characteristic $\chi(3, 6; 3) = -12$ for $m = 6$ and $\chi(3, 7; 3) = -568$. This is proved either geometrically, or by computer algebra. Note that these strata are identified uniquely in Theorem 4.2 namely for $m = 6$ under $3_5[\ldots, 12]$, and for $m = 7$ under $5_7[\ldots, 568]$. In conclusion, we can write $\chi(X(3, 7)) = 26 \cdot 42 + 15 \cdot 12 = 1272$ and $\chi(X(3, 8)) = 1272 \cdot 101 + 105 \cdot 568 = 188112$. If instead $m = 8$ the formula becomes

$$\chi(X(3, 9)) = \chi(X(3, 8)) \cdot \chi(F_{X(3,8)}) - 420 \cdot \chi(3, 8; 3) + 105 \cdot \chi(3, 8; 4).$$
Table 1 shows that $\chi(F_{X(3,8)}) = 205$. The numbers $\chi(3, 8; 3) = -81040$ and $\chi(3, 8; 4) = 18768$ can be computed using HomotopyContinuation.jl [6] as the ML degrees of the corresponding strata. Note that 18768 appears in Theorem 4.2 as the ML degree of the rank 3 matroid on 9 elements corresponding to four lines meeting in a point. The number 81040 does not appear in this theorem. However, 81040 = 99808 − 18768 where 99808 is the ML degree of the matroid on 9 elements where three lines meet in a point. Thus, the contribution above stems from this matroid strata where one needs to remove the locus of four concurrent lines. In conclusion, we have $\chi(X(3, 9)) = 188112 \cdot 205 + 420 \cdot 81040 + 105 \cdot 18768 = 74570400$.

As a proof of concept, we show how this story extends to nonuniform matroids.

**Example 5.7.** Let $M$ be the matroid of rank 3 on 9 elements with one nonbasis 789. The map $X(M) \to X(3, 8)$ which forgets the ninth point is a surjection. The generic fiber is no longer the complement of $B(3, 8)$, but rather its restriction to the line $P_7P_8$. That restricted arrangement has 15 bounded regions. The nongeneric fiber over $S_{ij}(kl)(rs)$ has 14 bounded regions provided $(78) \in \{(ij), (kl), (rs)\}$, and 15 otherwise. There are 45 strata of the form $S_{ij}(kl)(78)$. Similarly, the fiber over a stratum of four concurrent lines has 13 bounded regions provided $P_7P_8$ is one of those lines. There are 15 such codimension 2 strata. This gives

$$\chi(X(M)) = 188112 \cdot 15 + 45 \cdot 81040 + 15 \cdot 18768 = 6750000,$$

as a decomposition of 6750000, verifying the entry $9_1[6750000_1]$ in Theorem 4.2.

6 Eight points in 3-space

We now turn to configurations in $\mathbb{P}^3$. For $k = 4$ and $m \leq 7$, we can apply Grassmann duality, which shows that $X(4, m) = X(m - 4, m)$. Hence the ML degrees for few points in $\mathbb{P}^3$ are

$$|\chi(X(4, 6))| = |\chi(X(2, 6))| = 6 \quad \text{and} \quad \chi(X(4, 7)) = \chi(X(3, 7)) = 1272.$$

This section is devoted to the 9-dimensional very affine variety $X(4, 8)$. Here is our result:

**Theorem** 6.1. The Euler characteristic of $X(4, 8)$ equals $5211816$.

The number 5211816 is new. Unlike the ML degrees in Theorem 5.1, it did not yet appear in the physics literature. Also, Lam’s finite field method (Appendix A) did not yield this number. Thus far, the (4, 8) case was out of reach for all available techniques, in spite of the progress in [10, §6]. Our result proves the conjecture stated tacitly in [10, Table 2].

Our derivation of Theorem 6.1 rests on numerical computations with the software HomotopyContinuation.jl [6]. However, the methodology is closely related to the topological approach seen in previous sections. We fully exploit a specific likelihood degeneration, namely the soft limits [10], to analyze $X(4, 8)$ in a way that is similar to Lemma 2.3.

We start by describing our computational setup. For each quadruple $(i, j, k, \ell)$, where $1 \leq i < j < k < \ell \leq 8$, let $p_{ijkl}$ be the determinant of the corresponding $4 \times 4$ submatrix of

$$M_{4,8} = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 & 1 & x_1 & x_2 & x_3 \\
0 & 1 & 0 & 0 & 1 & y_1 & y_2 & y_3 \\
-1 & 0 & 0 & 0 & 1 & z_1 & z_2 & z_3
\end{bmatrix}. \quad (19)$$
Let $\mathcal{I}$ be the set of all such quadruples with $k \geq 4$ and $\ell \geq 6$. We write $(x, y, z)$ for the nine variables appearing in [19]. Note that $X(4, 8)$ is the complement of the union of hypersurfaces $\mathcal{H} = \bigcup_{(i,j,k,\ell) \in \mathcal{I}} V(p_{ijkl}) \subset \mathbb{C}^9$. This is a very affine variety in $(\mathbb{C}^*)^{62}$ with parametrization $\mathbb{C}^9 \setminus \mathcal{H} \to (\mathbb{C}^*)^{62}$ given by $(x, y, z) \mapsto (p_{ijkl}(x, y, z))_{(i,j,k,\ell) \in \mathcal{I}}$. By [17, Theorem 1], the quantity $|\chi(X(4, 8))|$ is the number of critical points of the log-likelihood function

$$L_{4,8}(x, y, z) = \sum_{(i,j,k,\ell) \in \mathcal{I}} u_{ijkl} \cdot \log(p_{ijkl}(x, y, z)).$$

Here we count critical points in $\mathbb{C}^9 \setminus \mathcal{H}$, for generic data $u_{ijkl} \in \mathbb{C}$. In other words, we seek the number of solutions to the following system of 9 rational function equations in 9 unknowns:

$$\frac{\partial L_{4,8}}{\partial x_i} = \frac{\partial L_{4,8}}{\partial y_i} = \frac{\partial L_{4,8}}{\partial z_i} = 0 \quad \text{for} \quad i = 1, 2, 3. \quad (20)$$

In particle physics, these are the scattering equations for $X(4, 8)$ in the CEGM model [8]. The projection of the likelihood correspondence [18, 34] to the space of data is a branched covering of degree $|\chi(X(4, 8))|$. For generic complex numbers $u_{ijkl}$, the fiber can be computed using the command monodromy_solve in HomotopyContinuation.jl, as explained in [33, §3]. In principle, we can use the certify command [5] to give a proof of the inequality

$$|\chi(X(4, 8))| \geq 5211816.$$ 

In practise, we missed 218 solutions. In our run, the method quickly certified that 5211598 paths correspond to distinct true solutions, giving a proof that $|\chi(X(4, 8))| \geq 5211598$.

The main idea in this section is to follow up this brute force monodromy computation by a likelihood degeneration of $\mathcal{H}_{\lambda}$ to study $X(4, 8)$ in a more structured way. In particular, the degeneration will help us to decompose the number $|\chi(X(4, 8))|$ into positive summands, much like what we did in the Section [5]. We keep assuming that the data $u_{ijkl}$ are generic.

We introduce a parameter $t$ into the log-likelihood function $L_{4,8}$ by setting

$$\tilde{L}_{4,8}(x, y, z, t) = \sum_{(i,j,k,\ell) \in \mathcal{I}} u_{ijkl} \cdot \log(p_{ijkl}(x, y, z)) + \sum_{(i,j,k,8) \in \mathcal{I}} u_{ijk8} \cdot t \cdot \log(p_{ijk8}(x, y, z)). \quad (21)$$

The limit for $t \to 0$ is the soft limit in [10]. Taking partial derivatives, we obtain rational function equations in the unknowns $x, y, z$ with coefficients in the rational function field $\mathbb{C}(t)$:

$$\frac{\partial \tilde{L}_{4,8}(x, y, z; t)}{\partial x_i} = \frac{\partial \tilde{L}_{4,8}(x, y, z; t)}{\partial y_i} = \frac{\partial \tilde{L}_{4,8}(x, y, z; t)}{\partial z_i} = 0 \quad \text{for} \quad i = 1, 2, 3. \quad (22)$$

There are $|\chi(X(4, 8))|$ solutions $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$ over the field of Puiseux series $K = \mathbb{C}\{\{t\}\}$. We are interested in computing the valuations of the Plücker coordinates for these solutions. In other words, suppose we knew these solutions, and suppose we were to substitute them into the $4 \times 8$ matrix [19]. Each of its $4 \times 4$ minors is then a Puiseux series $\hat{p}_\bullet(t)$, and
proof.

Proposition 6.2. The ray $\hat{\rho} = \hat{\rho}(t)$, denoted $\text{val}_i(\hat{\rho})$. The result of this process would be the tropical Plücker vector

$$\hat{q} = \left( \text{val}_i(\rho_{ijk}(\hat{x}(t), \hat{y}(t), \hat{z}(t))) \right)_{1 \leq i < j < k \leq 8} \in \mathbb{Q}^{70}. \quad (23)$$

Recall from [3] that $X(4, 8)$ is the quotient of $\text{Gr}(4, 8)^c$ by the action of the torus $(K^*)^8$. On the tropical side, where the vector $\hat{q}$ lives, this corresponds to an additive action of $\mathbb{R}^8$ on $\mathbb{R}^{70}$. We take the quotient of this additive action by setting to 0 the eight coordinates not in $\mathcal{I}$. Thus our choice of 62 coordinates is compatible with tropicalization (cf. [22]), and we obtain $\text{trop}(X(4, 8))$ as a 9-dimensional pointed fan in $\mathbb{R}^{62}$, with coordinates indexed by $\mathcal{I}$.

Now, here is the punchline: we cannot compute solutions over $K$, and we have no access to the Puiseux series in the argument of $\text{val}_i$ in (23). Instead, we carry out floating point computations over $\mathbb{R}$. This will give us enough information to identify the coordinates of $\hat{q}$. Indeed, from the point of view of complex geometry, the equations (22) define an affine curve

$$C \subset X(4, 8) \times \mathbb{C} \subset (\mathbb{C}^*)^{62} \times \mathbb{C},$$

where the second factor is the line with coordinate $t$. The vectors $\hat{q}$ in (23), which will be called tropical critical points in Section 7, span the rays in the partial tropical curve

$$\text{trop}(C) \subset \text{trop}(X(4, 8)) \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^{62} \times \mathbb{R}_{\geq 0}.$$ 

The solution $\hat{q} = 0$ represents classical solutions $(x(t), y(t), z(t))$ which, in the soft limit $t \to 0$, converge in $X(4, 8)$. The corresponding ray in $\text{trop}(C)$ is $\rho_0 = \mathbb{R}_{\geq 0} \cdot (0, \ldots, 0, 1)$.

**Proposition 6.2.** The ray $\rho_0 = \mathbb{R}_{\geq 0} \cdot (0, \ldots, 0, 1)$ has multiplicity 2 363 376 in $\text{trop}(C)$. \hfill \Box

**Proof.** The multiplicity of $\rho_0$ is the number of classical solutions converging in $X(4, 8)$ for $t \to 0$. In this limit, the equations for $i = 1, 2$ in [22] are the likelihood equations for $X(4, 7)$. Plugging any solution of these equations into those for $i = 3$, we find, up to division by $t$, the likelihood equations for the complement in $\mathbb{C}^3$ of the discriminantal arrangement $\mathcal{B}(4, 7)$. The ML degree of that arrangement complement is 1 858, as seen in Table 1. This gives the formula $\chi(X(4, 7)) \cdot |\chi(\mathbb{C}^3 \setminus \mathcal{B}(4, 8))| = 1 272 \cdot 1 858$ for the multiplicity of $\rho_0$. \hfill \Box

The number 2 363 376 is the number of regular solutions in [10]. In the spirit of Sections 2 and 3 it is the contribution to $\chi(X(4, 8))$ coming from the dense stratum in $X(4, 7)$. The nonzero tropical critical points $\hat{q}$ correspond to the singular solutions in [10]. For $t \to 0$, these curves move to the boundary of $X(4, 8)$. The following result verifies a conjecture in [10].

**Theorem** 6.3. There are 3 150 distinct nonzero tropical critical points $\hat{q}$. All of them are given by $\{0, 1\}$-vectors in $\mathbb{R}^{70}$ and they come in 7 combinatorial types, summarized in Table 3.

We here assume that the $u_{ijk}$ are generic complex numbers. The result is derived by computations with HomotopyContinuation.j1. The $t$-adic order of each numerical solution was found using Algorithm 1. The nonzero tropical critical points $\hat{q}$ span the rays in $\text{trop}(C)$, other than $\rho_0$, whose generator has a positive last coordinate. The ML degree equals

$$|\chi(X(4, 8))| = 2 363 376 + A_1 \cdot B_1 + A_{11} \cdot B_{11} + \cdots + A_{71} \cdot B_{71}, \quad (24)$$
| Type | representative \( ((i, j, k, \ell) \text{ for which } \text{val}_t(p_{ijk\ell}) = 1) \) | \# configurations |
|------|-------------------------------------------------|-------------------|
| I    | \((1,4,7,8),(2,5,7,8),(3,6,7,8)\)               | 105               |
| II   | \((1,2,3,8),(3,4,5,8),(5,6,1,8),(2,4,6,8)\)     | 210               |
| III  | \((1,2,3,8),(3,4,5,8),(5,6,7,8),(2,4,6,8)\)     | 1260              |
| IV   | \((1,4,7,8),(2,5,7,8),(3,6,7,8),(1,2,3,8),(4,5,6,8)\) | 420               |
| V    | \((1,2,3,8),(1,4,5,8),(1,6,7,8),(2,4,6,8),(2,5,7,8)\) | 630               |
| VI   | \((1,2,3,8),(1,4,5,8),(1,6,7,8),(2,4,6,8),(2,5,7,8),(3,4,7,8)\) | 210               |
| VII  | \((1,2,3,8),(1,2,4,8),(1,2,5,8),(1,2,6,8),(1,2,7,8),(3,4,5,8),(3,6,7,8)\) | 315               |

Table 3: Combinatorial types contributing singular solutions to (22). The second column shows a representative for each type indicating which Plücker coordinates have valuation 1. The third column shows the cardinality of the orbit of the action of \( S_7 \) on the first 7 indices.

where \( A_I = 105 \) is the number of configurations of type I, \( B_I \) is the multiplicity of each \( A_I \) ray in \( \text{trop}(C) \), and likewise for the other types. In other words, \( B_I \) is the number of Puiseux series solutions to (22) whose \( t \)-adic valuation is the representative of type I in Table 3.

We use numerical computation to obtain the multiplicities \( B_I, \ldots, B_{VII} \). This is done without any prior knowledge about \( \text{trop}(X(4, 8)) \) or the fibration \( \pi_{4,8} \). Approximate solutions of (20) are tracked numerically along the soft limit degeneration to learn the tropical critical points (23). We explain the details of this computation in a more general context in Section 8.

To learn the multiplicities \( B_\bullet \) in (24), we record, for each tropical solution \( \hat{q} \) of type \( \bullet \), the number of classical solutions that were found to have valuation \( \hat{q} \). The multiplicities are

\[
B_I = 5680, \quad B_{II} = 1704, \quad B_{III} = 988, \quad B_{IV} = 832, \quad B_{V} = 308, \quad B_{VI} = 240, \quad B_{VII} = 72. \quad (25)
\]

Plugging these values into (24) leads to Theorem* 6.1. Additional strong support arises from the fact that the number 5211 816 is also the number of approximate solutions found in a stand-alone run of monodromy_solve, although not all of them could be certified via [5].

Remark 6.4. The decomposition (24) is strongly related to the sum in Lemma 2.3. For instance, the formula \( \chi(X(3, 7)) = 26\cdot42+15\cdot12 \) from the proof of Theorem 5.1 partitions the 1272 critical points of the log-likelihood function into 1092 solutions that converge in \( X(3, 7) \) and 180 solutions that move to the boundary in the soft limit. These boundary solutions escape from the open variety \( X(3, 7) \) in 15 groups of 12. Here \( A_I = 15 \) is a combinatorial number, and \( B_I = 12 \) is the ML degree of a stratum in \( X(3, 6) \). This is isomorphic to the codimension 3 matroid stratum in \( X(3, 7) \) with ML degree 12 in Theorem 4.2. A similar interpretation holds for \( X(3, 8) \) and \( X(3, 9) \). In the case of \( X(4, 8) \), however, not all ray multiplicities \( B_\bullet \) are equal to the ML degree of a corresponding stratum in \( X(4, 7) \). A notable difference to the \( k = 3 \) case is that the matroid stratum of type I, seen for \( k = 4, m = 8 \) in Table 3, differs in dimension from its corresponding stratum in \( X(4, 7) \).
7 Statistical models and their tropicalization

We consider maximum likelihood estimation (MLE) for discrete statistical models \[18, 32\]. Our conventions and notation will be as in \[33\]. The given model is a $d$-dimensional subvariety $X$ of the projective space $\mathbb{P}^n$, which is assumed to intersect the simplex $\Delta_n \subset \mathbb{R}P^n$ of positive points. We seek to compute the critical points on $X$ of the log-likelihood function

$$L_u = u_0 \cdot \log(p_0) + u_1 \cdot \log(p_1) + \cdots + u_n \cdot \log(p_n) - (u_0 + u_1 + \cdots + u_n) \cdot \log(p_0 + p_1 + \cdots + p_n).$$

For any positive constants $u_0, u_1, \ldots, u_n$, representing data in statistics, this is a well-defined function on $\Delta_n$. The aim of likelihood inference is to maximize $L_u$ over all points $p$ in the model $X \cap \Delta_n$. The number of all complex critical points, for generic $u$, is the ML degree of the model $X$. If $X$ is smooth then this equals the signed Euler characteristic of the open variety $X^\circ$, which is the complement of the divisor in $X$ defined by $p_0 p_1 \cdots p_n (\sum_{i=0}^n p_i) = 0$.

Likelihood degenerations were first introduced in the setting of algebraic statistics by Gross and Rodriguez in [16], who studied the behavior of the MLE when some of the $u_i$ approach zero. They distinguish between model zeros, structural zeros and sampling zeros. These statistical concepts can serve as a guide for interpreting likelihood degenerations.

We draw samples independently from some unknown distribution that is in $X \cap \Delta_n$. The probabilities in the following definitions refer to that sampling distribution. The data are summarized in a vector $u \in \mathbb{N}^{n+1}$, where $u_i$ denotes the number of observations found to be in state $i$. Suppose that state $i$ was never observed in our sample. The entry $u_i = 0$ is called:

- a **structural zero** if the probability of it being zero is equal to one;
- a **sampling zero** if the probability of it being zero is less than one;
- a **model zero** if the maximizer of $L_u$ over $X \cap \Delta_n$ is a critical point of the restriction of $L_u$ to the hyperplane section $X \cap \{p_i = 0\}$.

Structural zeros may mean that the wrong model was chosen, so we exclude this possibility. What remains is a consideration of sampling zeros and model zeros. These statistical concepts led Huh and Sturmfels to propose the following formula in [18, Conjecture 3.19]:

$$\text{MLdegree}(X) = \text{MLdegree}(X \cap \{p_j = 0\}) + \text{MLdegree}(X|_{u_j=0}). \quad (26)$$

The last summand counts critical points of $L_u$ on $X$ when $u_j = 0$ and the other $u_i$ are generic. The identity \[26\] holds under suitable smoothness and transversality assumptions. They ensure that the ML degrees are signed Euler characteristics of very affine varieties, obtained by removing the arrangement $\mathcal{H} = \{p_0 p_1 \cdots p_n (\sum_{i=0}^n p_i) = 0\}$ of $n + 2$ hyperplanes from $\mathbb{P}^n$. For the last summand we remove only $n + 1$ hyperplanes. The Euler characteristic is additive relative to the additional hyperplane $\{p_j = 0\}$. The sum becomes a minus for the signed Euler characteristic, as the dimensions differ by one, so the identity \[26\] follows.

Familiar combinatorics arises when $X$ is a linear space. In this case, $X^\circ$ is the complement of $n + 1$ hyperplanes in affine $d$-space. The number of bounded regions can be computed by deletion-restriction. This is precisely the formula in \[26\]. Moreover, all critical points are real, and there is one critical point per bounded region. An example from \[33\] is the space
\( X = X(2, m) \) of \( m \) points on the line \( \mathbb{P}^1 \), modulo projective transformations. Here \( d = m - 3 \), \( n = m(m - 3)/2 \), and the ML degree equals \( (m - 3)! \). The formula (26) is essentially that for soft limits in [8, 10]. We count solutions in (26) as singular solutions plus regular solutions.

In this section, we introduce a vast generalization of soft limits, namely tropical degenerations. We examine MLE for discrete statistical models through the lens of tropical geometry. In what follows, the real numbers \( \mathbb{R} \) are replaced by the real Puiseux series \( R = \mathbb{R}\{\{t\}\} \). This is a real closed field, and it comes with the \( t \)-adic valuation. The uniformizer \( t \) is positive and infinitesimal. A scalar \( u \) in \( R \) can be viewed as the germ of a function \( u(t) \) near \( t \to 0 \).

We are now given \( u_0, \ldots, u_n \in R \), with valuations \( w_i = \text{val}_t(u_i) \). We call \( w = (w_0, \ldots, w_n) \) the tropical data vector. Each critical point \( p \) of \( \mathcal{L}_u \) has its coordinates \( \hat{p}_i \) in the algebraic closure \( K = \mathbb{C}\{(t)\} \) of the ordered field \( R \). We set \( \hat{q}_i = \text{val}_t(\hat{p}_i) \), and we refer to \( \hat{q} = (\hat{q}_0, \ldots, \hat{q}_n) \) as a tropical critical point. Given any model \( X \), we would like to describe the multivalued map that takes a tropical data vector \( w \) to the set of its tropical critical points \( \hat{q} \).

The following theorem accomplishes this goal for the class of linear models [32, §4.2]. We augment the homogeneous linear forms defining \( X \) by the equation \( p_0 + p_1 + \cdots + p_n = 1 \), and we identify \( X \) with the resulting \( d \)-dimensional affine-linear subspace in \( \mathbb{R}^{n+1} \). We write \( X^\perp \) for the linear subspace of \( \mathbb{R}^{n+1} \) that consists of all vectors perpendicular to \( X \), with respect to the usual dot product. Thus \( X^\perp \) is a vector space of dimension \( n - d + 1 \) in \( \mathbb{R}^{n+1} \).

The tropical affine space \( \text{trop}(X) \) is a pointed cone of dimension \( d \) in \( \mathbb{R}^{n+1} \). Combinatorially, this is the Bergman fan [22, §4.2] of the rank \( d + 1 \) matroid on \( n + 2 \) elements defined by \( X \). Here the matroid is associated with the hyperplane arrangement \( X \setminus X^0 \). The tropical linear space \( \text{trop}(X^\perp) \) has dimension \( n - d + 1 \). It is a fan with \( 1 \)-dimensional lineality space spanned by \((1, 1, \ldots, 1) \). Combinatorially, it is the Bergman fan of the rank \( n - d + 1 \) matroid on \( n + 1 \) elements defined by \( X^\perp \). Here the matroid of \( X^\perp \) is the dual of a one-element contraction of the matroid of \( X \). The contracted element corresponds to the hyperplane at infinity, namely \( \{p_0 + p_1 + \cdots + p_n = 0\} \). It is very important to distinguish this element.

**Theorem 7.1.** If the tropical data vector \( w \) is sufficiently generic then there are exactly \( \text{MLdegree}(X) \) many distinct tropical critical points. They are given by the intersection

\[
\hat{q} \in \text{trop}(X) \cap (w - \text{trop}(X^\perp)).
\] (27)

We call the subspace \( X \) general if both of the above matroids are uniform. In that case we have \( \text{MLdegree}(X) = \binom{n}{d} \); see [33, Example 4]. The matroid of \( X \) is the uniform matroid \( U_{d+1,n+2} \), and the matroid of \( X^\perp \) is the uniform matroid \( U_{n-d+1,n+1} \). We abbreviate \( e_{n+1} = -e_0 - e_1 - \cdots - e_n \), the negated sum of all unit vectors in \( \mathbb{R}^{n+1} \). The tropical affine space \( \text{trop}(X) \) is the union of all \( \binom{n+2}{d} \) cones \( \text{pos}(e_i : i \in I) \) where \( I \) runs over \( d \)-element subsets of \{0, 1, \ldots, n, n + 1\}. The tropical linear space \( \text{trop}(X^\perp) \) is the union of all \( \binom{n+1}{n-d} \) cones \( \Re e_{n+1} + \text{pos}(e_j : j \in J) \), where \( J \) runs over \( (n - d) \)-element subsets of \{0, 1, \ldots, n\}. Let \( w \in \mathbb{R}^{n+1} \) and suppose \( w_0 \) is its smallest coordinate. Then \( w + w_0 e_{n+1} \) is a nonnegative vector with first coordinate 0. Replacing \( w \) by \( w + w_0 e_{n+1} \) does not change tropical critical points.

**Corollary 7.2.** If \( X \) is general then (27) consists of the \( \binom{n}{d} \) points \( \hat{q} = \sum_{i \in I} (w_i - w_0) e_i \), where \( I \) runs over all \( d \)-element subsets of \{1, 2, \ldots, n\}. These are the tropical critical points.
Proof. By construction, the vector $\hat{q}$ lies in $\text{trop}(X)$. We also have
\[
w - \hat{q} = w - \sum_{i \in I} w_i e_i + \sum_{i \notin I} w_0 e_i = \sum_{j \notin I} (w_j - w_0) e_j - w_0 e_{n+1}.
\]
This vector is visibly in $\text{trop}(X^\perp)$. We have thus constructed $\binom{n}{d}$ distinct vectors in the intersection $[27]$. The corollary hence follows from Theorem 7.1, applied to general $X$. □

Example 7.3 $(n = 3, d = 2)$. We consider the 2-parameter linear model $X$ for the state space $\{A, C, G, T\}$ discussed in [32, Example 1.1]. This model is defined by one homogeneous linear constraint $c_A p_A + c_C p_C + c_G p_G + c_T p_T = 0$. Its coefficients $c_i$ are nonzero real numbers. Consider the data vector $u = (t^{w_A}, t^{w_C}, t^{w_G}, t^{w_T})$. The four exponents are nonnegative integers, and we assume that $w_A$ is the smallest among them. The log-likelihood function has three critical points $\hat{p} = (\hat{p}_A, \hat{p}_C, \hat{p}_G, \hat{p}_T)$, one for each bounded polygon. These are functions in $t$, and we seek their behavior for $t \to 0$. This is given by the three tropical critical points:
\[
\hat{q} = (0, w_C - w_A, w_G - w_A, 0), (0, w_C - w_A, 0, w_T - w_A) \text{ or } (0, 0, w_G - w_A, w_T - w_A).
\]
We change the model by setting $c_A = 0$, so $X$ is no longer general. The number of bounded regions drops from 3 to 2. The matroid of $X$ changes, and so does $\text{trop}(X)$. We now find $\hat{q} = (0, w_C - w_A, w_G - w_A, 0 - w_A)$ or $(0, w_C - w_A, w_G - w_A, w_C - w_A)$ provided $w_C < \min(w_G, w_T)$. □

Proof of Theorem 7.1. We use the formulation of the likelihood correspondence given in [18, Proposition 1.19]. This states, in our notation, that the critical points $\hat{p}$ are the elements of
\[
X \cap (u^{-1} \ast X^\perp)^{-1}.
\]
Here, $\ast$ is the Hadamard product, and $u^{-1}$ is the coordinatewise reciprocal of the vector $u$. The intersection in [28] commutes with tropicalization, provided $w = \text{val}_i(u)$ is generic:
\[
\text{trop}(X \cap (u^{-1} \ast X^\perp)^{-1}) = \text{trop}(X) \cap -\text{trop}(u^{-1} \ast X^\perp) = \text{trop}(X) \cap (w - \text{trop}(X^\perp)).
\]
Indeed, the left expression is contained in the middle expression, and they are equal in the sense of stable intersection. This is the content of [22, Theorem 3.6.1]. In [29] we intersect polyhedral spaces of dimensions $d$ and $n+1-d$ in $\mathbb{R}^{n+1}$. The second equation follows from [22, Proposition 5.5.11]. Since $w$ is generic, the intersection is transverse at any intersection point and each intersection point is isolated. Lemma 7.4 below shows that the multiplicity of every tropical intersection is 1, even in the more general case of a nonrealizable matroid. □

Fix a matroid $X$ of rank $d+1$ on the elements $\{0, 1, \ldots, n+1\}$. Let $X^\perp = (X/(n+1))^\ast$ be the dual of the contraction of $X$ by the element $n+1$. Furthermore, let $\mathcal{F} := \{F_1 \subset \cdots \subset F_d\}$ and $\mathcal{F}^\perp := \{F^\perp_1 \subset \cdots \subset F^\perp_{n-d}\}$ be flags of flats of $X$ and $X^\perp$, respectively with $\text{rank}(F_i) = i$ and $\text{rank}(F^\perp_j) = j$ for all $i, j$. Since $X$ has rank $d+1$, we can assume $F_d \subset \{0, \ldots, n\}$.

Lemma 7.4. Each intersection in [29] has multiplicity 1. It is the signed determinant of an $(n+1) \times (n+1)$ matrix whose columns are indicator vectors of flats in flags as above:
\[
M_{\mathcal{F}, \mathcal{F}^\perp} := \begin{pmatrix}
\vdots & e_{F_1} & \cdots & e_{F_d} & e_{F^\perp_1} & \cdots & e_{F^\perp_{n-d}} & e_{\{0, \ldots, n\}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}.
\]
Such a matrix $M_{\mathcal{F},\mathcal{F}^\perp}$ has determinant $0$ or $\pm 1$. Moreover, if $M_{\mathcal{F},\mathcal{F}^\perp}$ is invertible, there exist complementary bases of the matroids $X/(n+1)$ and $X^\perp$ generating the flags $\mathcal{F}$ and $\mathcal{F}^\perp$.

**Proof.** We proceed along the following four steps.

1. We can assume that $\{0, \ldots, d-1, n+1\}$ is a basis of $X$, and

$$
M_{\mathcal{F},\mathcal{F}^\perp} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & \ddots & 1 \\
0 & 0 & 0 & 1 \\
* & * & * & *
\end{pmatrix},
$$

where the left block has $d$ columns and the symbols ‘*’ represent arbitrary $\{0, 1\}$-entries. The definition of flag ensures that within each column block, a 1-entry is followed by only 1’s in the same row. Therefore, rows in each column block are uniquely determined by their number of 1-entries.

2. Let $I_j$ be the set of row indices corresponding to rows having precisely $j$ 1-entries in the left block. For each row index $i$, let $\gamma(i)$ be the smallest column index in the right block such that the entry $(i, \gamma(i))$ is 1. By permuting rows of $M_{\mathcal{F},\mathcal{F}^\perp}$ we can ensure that all rows indexed by $I_j$ are sorted by increasing number of 1’s in the right block. Since a 1-entry in the right block is followed by 1’s in the same row, we have that

$$
\text{if } i, i' \in I_j \text{ are such that } i' < i, \text{ then } \gamma(i) \leq \gamma(i').
$$

If equality holds for some $i \in I_j \setminus \{i'\}$, then two rows are equal and $M_{\mathcal{F},\mathcal{F}^\perp}$ has determinant 0, in which case we are done. Hence, in what follows we assume that the last inequality in (31) is strict for all distinct $i, i' \in I_j$.

3. We perform the following elementary row operations. Each row $r_i$ indexed by $i > d$ is replaced by $r_i - r_{i'}$, where $i \in I_{i'}$. After this operation, the lower left block in $M_{\mathcal{F},\mathcal{F}^\perp}$ is zero, all entries are still 0 or 1 and the function $\gamma$ is unchanged.

4. It remains to show that the determinant of the lower right block matrix is $\pm 1$. Since the matrix $M_{\mathcal{F},\mathcal{F}^\perp}$ is still of the form (30), the elements $\{0, \ldots, d-1, n+1\}$ are still a basis of $X$ (after the above permutations). Therefore by definition of $X^\perp$, the set $\{d, \ldots, n\}$ is a basis of $X^\perp$. This implies that, up to a permutation of rows, the lower right block is an upper triangular matrix with 1-entries on the diagonal.

We now apply Theorem 7.1 to the CHY model $X^\circ = X(2, m)$, where $n = m(m-3)/2$. The tropical linear space $\text{trop}(X)$ consists of ultrametrics on $m-1$ points [22, Lemma 4.3.9]. The matroid of $X$ is the graphic matroid of the complete graph $K_{m-1}$; see [22, Example 4.2.14]. Using [4] with $k = 2$, the vertices of $K_{m-1}$ are labeled by $2, 3, \ldots, m$. The special edge $e = \{2, 3\}$ corresponds to the hyperplane at infinity. The matroid of $X^\perp$ is the cographic...
matroid of the graph $K_{m-1}/e$ obtained by contracting $e$. This is dual to the graphic matroid of $K_{m-1}/e$. Vectors in $\text{trop}(X)$ have the minimum attained twice on every circuit of $K_{m-1}$, where the edge $e$ has weight 0. Vectors in $\text{trop}(X^\perp)$ attain their minimum twice on every cocircuit of $K_{m-1}/e$. Thus tropical MLE amounts to writing the vector $w$ as a sum of two such minimum-attained-twice vectors. The number of such decompositions equals $(m-3)!$.

**Example 7.5** ($m = 6$). Following [33, Example 2], we consider the CHY model $X(2,6)$. This corresponds to an arrangement of 9 planes in $\mathbb{R}^3$, with six bounded regions, namely the tetrahedron of a triangulated 3-cube. We coordinatize this model by the matrix (4), so that $e = \{2, 3\}$ is the special edge of $K_5$. Our data are the Mandelstam invariants

$$u_{24} = t^{12}, \ u_{25} = t^6, \ u_{26} = t^9, \ u_{34} = t^{12}, \ u_{35} = t^5, \ u_{36} = t, \ u_{45} = t^{10}, \ u_{46} = t^{11}, \ u_{56} = t^3.$$  

Hence the tropical data vector equals $w = (w_{24}, w_{25}, \ldots, w_{45}) = (12, 6, 9, 12, 5, 1, 10, 11, 3)$. Our task is to find all additive decompositions $w = \hat{q} + (w - \hat{q})$ where the two summands lie in the respective Bergman fans $\text{trop}(X)$ and $\text{trop}(X^\perp)$. One such decomposition equals

$$w = \hat{q} + (w - \hat{q}) = (7, 5, 2, 0, 0, 0, 5, 2, 2) + (5, 1, 7, 12, 5, 1, 5, 9, 1). \quad (32)$$

This solution is verified in Figure 5: the minimum over each circuit is attained at least twice.

Figure 5: Contracting the edge 23 in the green graph $K_5$ yields a planar graph whose dual is the blue graph on the right. The orange edge weights are the entries in (32). The key property is that the minimum is attained at least twice in each cycle of the respective graph.

We find that the intersection (27) consists of six points. The tropical critical points $\hat{q}$ are

$$(0, 0, 8, 4, 2, 0, 2, 0, 0), \quad (0, 5, 2, 2, 0, 0, 0, 0, 2), \quad (1, 0, 8, 0, 2, 0, 0, 1, 0),$$  

$$ (2, 5, 2, 0, 0, 0, 2, 3, 2), \quad (7, 5, 2, 0, 0, 0, 5, 2, 2), \quad (9, 0, 8, 0, 2, 0, 0, 8, 0).$$

Each $\hat{q}$ gives a decomposition as in (32), where the two summands are compatible with the cycles of the two graphs in Figure 5. These solutions specify small arcs that lie in the six tetrahedra. These arcs converge for $t \to 0$ with the given rates to vertices of the arrangement.
What we have outlined here is the combinatorial theory of tropical \textit{CHY scattering}. This works for all $m \geq 5$. The soft limits of \cite{10} arise as the very special case when the tropical Mandelstam invariants $w$ satisfy $w_{im} = 1$ and $w_{ij} = 0$ for $i \leq j \leq m - 1$. The key player in tropical CHY scattering is the space $	ext{trop}(X)$ of ultrametric phylogenetic trees. In the case $m = 5$, this space is a cone over the Petersen graph. See \cite{22} §4.3 for details.

In this section we presented the theory of tropical MLE for linear models. This raises the question of what happens when the model $X$ is an arbitrary (nonlinear) projective variety in $\mathbb{P}^n$. A partial answer based on homotopy techniques is given in the next section. The role of \cite{28} is now played by the likelihood correspondence. According to \cite{18} Theorem 1.6, this is an $n$-dimensional subvariety in $\mathbb{P}^n \times \mathbb{P}^n$. An ambitious goal is to determine the tropicalization of that subvariety. The desired pairs $(w, \hat{q})$ are points in that tropical likelihood correspondence. This leads to very interesting geometry and combinatorics. For instance, it is closely related to the Bernstein-Sato slopes studied by van der Veer and Sattelberger \cite{34}.

8 Learning valuations numerically

Previous sections developed two techniques for obtaining the ML degree of a very affine variety: Euler characteristics via stratified fibrations, and tropical geometry. Each method leads to a meaningful combinatorial description of the ML degree, but it requires significant combinatorial efforts. In this section we propose a numerical method which computes the tropical solutions $\hat{q}$ discussed in Sections 6 and 7 directly, while avoiding any combinatorial overhead. A decomposition of the ML degree as a positive sum of integers naturally emerges as a byproduct. We used this method to verify the multiplicities (25) as detailed in Section 6.

Let $K = \mathbb{C}\{(t)\}$ and consider a very affine variety $X \subset (K^*)^n$ that is defined over $\mathbb{C}$. Fix a data vector $u \in K^n$. The problem of tropical MLE is to compute the coordinate-wise valuations $\hat{q} = \text{val}_t(\hat{p})$ of the critical points $\hat{p}$ of the log-likelihood function $L_u$ restricted to $X$. Note that each $\hat{q}$ is a vector in $\mathbb{Q}^n$, so the output can be written in exact arithmetic.

In this section, we think of $t$ as a coordinate for $\mathbb{C}^*$. We assume for simplicity that $u \in \mathbb{C}[t, t^{-1}]^n$ is given by Laurent polynomials. The following incidence variety is a curve:

$$C = \{(\hat{p}, t) \in X \times \mathbb{C}^* \mid \hat{p} \text{ is a critical point of } (L_u(t))_X\}.$$ 

The generalization to the case where the $u_i$ are Laurent series, convergent in the punctured disk $\{t \in \mathbb{C}^* \mid |t| < 1\}$, is straightforward. We assume that $u$ is sufficiently generic, so that:

- the projection map $\pi_{\mathbb{C}^*} : C \to \mathbb{C}^*$ is an MLdegree($X$)-to-one branched covering,
- the half-open real line segment $(0, 1] \subset \mathbb{C}^*$ avoids the branch locus of $\pi_{\mathbb{C}^*}$.

Any point $(\hat{p}(1), 1)$ of the curve $C$ lies on a unique path in $\pi_{\mathbb{C}^*}^{-1}((0, 1])$ which corresponds to a critical point $\hat{p}(t) = (\hat{p}_1(t), \ldots, \hat{p}_n(t)) \in K^n$. Each coordinate of $\hat{p}(t)$ is a Puiseux series

$$\hat{p}_i(t) = c_i t^{\hat{q}_i} + \text{ higher order terms},$$

where $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)$ is its corresponding tropical critical point.
For any real constant $t' \in (0, 1]$ with $t' \ll 1$, the following approximation holds:

$$
\log |\hat{p}_i(t')| \approx \log |c_i| + \hat{q}_i \cdot \log t'.
$$

(34)

Hence, for all values of $t'$ that are small enough, the points $(a_{t'}, b_{t'}) = (\log t', \log |\hat{p}_i(t')|) \in \mathbb{R}^2$ lie approximately on a line with slope $\hat{q}_i = \text{val}_i(\hat{p}_i(t))$. We wish to learn that slope.

Using standard numerical predictor-corrector techniques on the critical point equations, one can compute $(\hat{p}(t'), t') \in \pi_{\mathbb{C}^*}^{-1}(t')$ for any $t' \in [0, 1]$. This amounts to evaluating the solution $\hat{p}(t)$ at $t'$. In our project, we used HomotopyContinuation.jl [6] for this computation.

By evaluating $\hat{p}(t)$ at many $t'$ near 0, we may approximate the $i$th coordinate $\hat{q}_i$ by fitting a line through the points $(a_{t'}, b_{t'})$. See (34). We find $\hat{q}$ by doing this for $i = 1, 2, \ldots, n$. This discussion is summarized in Algorithm 1 for computing the tropical critical point $\hat{q}$.

**Algorithm 1: Numerical computation of tropical critical points**

**Input:** Data $u$ and a critical point $\hat{p}(1) \in (\mathbb{C}^*)^n$ of the log-likelihood function $L_u(1)$.

**Output:** The tropical critical point $\hat{q}$ associated to the solution $\hat{p}(t)$, together with some measure $\rho$ of trust in this result.

1. With $\hat{p}(1)$ as the start solution, use numerical predictor-corrector techniques to compute $\hat{p}(t')$ for a finite set $T \subset (0, 1]$ of values $t'$, all sufficiently close to 0.

2. Fit lines $\ell_i$ through the data points $\{(\log t, \log |\hat{p}_i(t)|) \mid t \in T\}$ for each coordinate $\hat{p}_i$, using standard regression techniques.

3. Using any rationalization heuristic, such as the built-in function rationalize in Julia, round the slope of the line $\ell_i$ to obtain the rational number $\hat{q}_i$.

4. As a measure of trust, record the regression error

$$
\rho := \sum_{t' \in T} (\log |\hat{p}_i(t')| - (\log |c_i| + \hat{q}_i \cdot \log t'))^2.
$$

(35)

5. return $(\hat{q}, \rho)$

**Example 8.1 (Soft limits).** We apply this to the CEGM model $X(3, 7)$. It is parametrized by the $3 \times 3$ minors $p_{ijk}$ of the matrix $M_{3, 7}$ given in (4). Consider the log-likelihood function

$$
L_u(t) = \hat{L}_{3, 7}(x, t) = \sum_{1 \leq i < j < k \leq 6} u_{ijk} \cdot \log p_{ijk} + \sum_{1 \leq i < j \leq 6} t \cdot u_{ij7} \cdot \log p_{ij7}
$$

(36)

for generic complex parameters $u_{ijk}$. The numerical computation in [33, §4] provides 1272 start solutions $\hat{p}(1)$. Each of these now serves as the input to Algorithm 1. That algorithm performs the likelihood degeneration (soft limit) purely numerically for $t \to 0$.

The 1272 runs of Algorithm 1 lead to only 16 distinct tropical critical points $\hat{q}$. This means that different start solutions $\hat{p}(1)$ yield the same output. The multiplicity of $\hat{q}$ in the tropical curve $\text{trop}(C)$ is the number of runs that yield output $\hat{q}$. The coordinates and multiplicities of all tropical critical points are shown in the columns of Table 4. We stress that this table was computed blindly, without any prior knowledge about the model $X(3, 7)$.
Remarkably, one learns the geometry of the stratified fibration $\pi_{3,7}$ from the output in Table 4. The first column corresponds to the 1092 regular solutions, i.e., those which converge in $X(3,7)$. The others correspond to 15 groups of 12 boundary solutions, whose limit for $t \to 0$ lies on the boundary in the tropical compactification of $X(3,7)$. Thus, by using Algorithm 1, we discover the ML degrees in Theorem 5.1 in a purely automatic manner.

Recall that the action of $(\mathbb{C}^*)^7$ on the Plücker coordinates tropicalizes to an additive action of $\mathbb{R}^7$ on $\mathbb{R}^{35}$. Modulo this $\mathbb{R}^7$-action, each column of Table 4 except the first one, can be represented by a $\{0, 1\}$ vector that has precisely three nonzero entries. For instance, the second column has its entries 1 in the rows $p_{137}, p_{267}, p_{457}$, so it identifies the divisor $S_{(13),(26),(45)}$ in the stratification of $X(3,6)$. This corresponds to a codimension 3 matroid stratum for $k = 3, m = 7$. In this manner, one automatically learns the 15 strata of type I in Table 2. On average, the regression error (35) for all 1272 paths was $\sim 0.0045$, the largest one being $\sim 0.162$. The computation took no more than a couple of minutes. △

Extending Example 8.1 to a more challenging scenario, we applied Algorithm 1 to each of the 5211816 solutions to the maximum likelihood equations (20) for $k = 4, m = 8$. This led us to the multiplicities (25) and the derivation of Theorem* 6.3 as discussed in Section 6.

A Finite field methods (Appendix by Thomas Lam)

In this appendix we present a proof of Theorem 5.1 that is based on point counting over finite fields and the Weil conjectures. This proof predates the work reported in this article.
We compute the Euler characteristic of $X(3, m) = \operatorname{Gr}(3, m)^o/(\mathbb{C}^*)^m$ for small values of $m$.

**Theorem A.1.**  The Euler characteristics of the configuration spaces $X(3, 6), X(3, 7), X(3, 8)$ and $X(3, 9)$ are 26, 1272, 188112 and 74570400 respectively.

**Proof.** The variety $X(k, m)$ can be defined over a finite field $\mathbb{F}_q$. In sufficiently large characteristic, $X(k, m)_{\mathbb{F}_q}$ is smooth. The number of points in $X(3, 6)_{\mathbb{F}_q}$, $X(3, 7)_{\mathbb{F}_q}$, $X(3, 8)_{\mathbb{F}_q}$, $X(3, 9)_{\mathbb{F}_q}$ was computed by Glynn [15], and Iampolskaia, Skorobogatov, Sorokin [19], and can be found in the paper of Skorobogatov [30], §5. For $q = p^n$ with $p > 3$, they are

$$
\#X(3,6)_{\mathbb{F}_q} = (q - 2)(q - 3)(q^2 - 9q + 21)
$$
$$
\#X(3,7)_{\mathbb{F}_q} = (q - 3)(q - 5)(q^4 - 20q^3 + 148q^2 - 468q + 498)
$$
$$
\#X(3,8)_{\mathbb{F}_q} = (q - 5)(q^7 - 43q^6 + 788q^5 - 7937q^4 + 47097q^3 - 162834q^2
$$
$$
+ 299280q - 222960) + 840b(q)
$$
$$
\#X(3,9)_{\mathbb{F}_q} = q^{10} - 75q^9 + 2530q^8 - 50466q^7 + 657739q^6 - 5835825q^5 + 35563770q^4
$$
$$
- 146288034q^3 + 386490120q^2 - 588513120q + 389442480
$$
$$
+ 840(9q^2 - 243q + 1684)b(q) + 30240(9d(q) + 2e(q)).
$$

We have omitted the terms involving $a(q)$ and $c(q)$ in [30] which vanish when $p > 3$. The functions $b(q), d(q), e(q)$ count the number of solutions of certain quadratic equations in $\mathbb{F}_q$:

$$
b(q) = \# \{ x \in \mathbb{F}_q \mid x^2 + x + 1 = 0 \},
$$
$$
d(q) = \# \{ x \in \mathbb{F}_q \mid x^2 + x - 1 = 0 \},
$$
$$
e(q) = \# \{ x \in \mathbb{F}_q \mid x^2 + 1 = 0 \}.
$$

We shall show that the Euler characteristic of $X(k, m)$ is obtained by setting $b(q) = d(q) = e(q) = 2$ and then $q = 1$ in (37). Note that $b(q) = d(q) = e(q) = 2$ holds if

$$
p \equiv 1 \mod 3, \quad p \equiv \pm 1 \mod 5, \quad p \equiv 1 \mod 4.
$$

By Dirichlet’s Theorem, such primes exist. Fix $q = p^n$ such that $X(k, m)$ is smooth in characteristic $p$, and such that (38) holds. Consider the zeta function

$$
Z(q) := \exp \left( \sum_{n=1}^{\infty} \frac{\#X(k, m)_{\mathbb{F}_q^n}}{n} q^{-ns} \right).
$$

It follows from the Grothendieck-Lefschetz trace formula [13] that $Z(q)$ is a rational function in the variable $q^{-s}$. Moreover, it can be written in the form $Z = P/Q$, where

$$
\deg(P) = \dim \, H^\text{odd}_c(X(k, m)_{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell}), \quad \deg(Q) = \dim \, H^\text{even}_c(X(k, m)_{\mathbb{F}_q}, \overline{\mathbb{Q}_\ell})
$$

are the total dimensions of odd (resp., even) compactly supported étale ($\ell$-adic) cohomology.

By a standard argument in algebraic geometry (see for example [21, Theorem 20.5 and Theorem 21.1]), the étale cohomology of $X(k, m)_{\mathbb{F}_q}$ is isomorphic to the étale cohomology
of the complex algebraic variety $X(k,m)$, which is in turn isomorphic to the singular cohomology of the complex manifold $X(k,m)_{\mathbb{C}}$. Since $X(k,m)$ is smooth and even-dimensional as a real manifold, its compactly supported Euler characteristic is the same as its Euler characteristic. We conclude that the Euler characteristic is $\chi(X(k,m)) = -\deg(Z(q))$.

On the other hand, $Z(q)$ is easy to write down when $\#X(k,m)_{\mathbb{F}_q}$ is a polynomial in $q$, as is the case in (37). Each term $+q^a$ (resp. $-q^a$) in $\#X(k,m)_{\mathbb{F}_q}$ contributes a linear factor to the denominator (resp. numerator) of $Z(q)$. Thus $-\deg(Z(q))$ is equal to the evaluation of the polynomial $\#X(k,m)_{\mathbb{F}_q}$ at $q = 1$. By substituting $q = 1$ into (37) along with $b(q) = d(q) = e(q) = 2$, one obtains the positive integers stated in Theorem A.1. □

Remark A.2. We point out that each of the numbers $26, 1272, 188112, 74570400$ has one large prime factor, $13, 53, 3919, 10357$. This seems unrelated to the geometry in Section 5.

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