Magnetic Moments of $\Delta^{++}$ and $\Omega^{-}$ from QCD Sum Rules

Frank X. Lee

Nuclear Physics Laboratory, Department of Physics, University of Colorado,
Boulder, CO 80309-0446

(March 26, 2022)

Abstract

QCD sum rules for the magnetic moments of $\Delta^{++}$ and $\Omega^{-}$ are derived using the external field method. They are analyzed by a Monte-Carlo based procedure, using realistic estimates of the QCD input parameters. The results are consistent with the measured values, despite relatively large errors that can be attributed mostly to the poorly-known vacuum susceptibility $\chi$. It is shown that a 30% level accuracy can be achieved in the derived sum rules, provided the QCD input parameters are improved to the 10% level.

PACS numbers: 13.40.Em, 12.38.Lga, 11.55.Hx, 14.20.G, 02.70.Lg
The QCD sum rule method [1] is a powerful tool in revealing the deep connection between hadron phenomenology and QCD vacuum structure via a few condensate parameters. The method has been successfully applied to a variety of problems to gain a field-theoretical understanding into the structure of hadrons. Calculations of the nucleon magnetic moments in the approach were first carried out in Refs. [2] and [3]. They were later refined and extended to the entire baryon octet in Refs. [4–7]. On the other hand, the magnetic moments of decuplet baryons were less well studied within the same approach. There were previous, unpublished reports in Ref. [8] on $\Delta^{++}$ and $\Omega^{-}$ magnetic moments. The magnetic form factor of $\Delta^{++}$ in the low $Q^2$ region was calculated based on a rather different technique [9]. In recent years, the magnetic moment of $\Omega^{-}$ has been measured with remarkable accuracy [10]: $\mu_{\Omega^{-}} = -2.02 \pm 0.05 \mu_N$. The magnetic moment of $\Delta^{++}$ has also been extracted from pion bremsstrahlung [11]: $\mu_{\Delta^{++}} = 4.5 \pm 1.0 \mu_N$. The experimental information provides new incentives for theoretical scrutiny of these observables.

In this letter, we present an independent calculation of the magnetic moments of $\Delta^{++}$ and $\Omega^{-}$ in the QCD sum rule approach. The goal is two-fold. First, we want to find out the applicability of the method as an alternative way of understanding the measured $\mu_{\Delta^{++}}$ and $\mu_{\Omega^{-}}$, and hope to gain some insights into the internal structure of these baryons from a nonperturbative-QCD perspective. Second, we want to achieve some realistic understanding of the uncertainties involved in such a determination by employing a Monte-Carlo based analysis procedure. This will help find possible ways for improvement. The calculation is algebraically more involved than the octet case since one has to deal with the more complex spin structure of spin-3/2 particles, but presents no conceptual difficulties.

Consider the two-point correlation function in the QCD vacuum in the presence of a constant background electromagnetic field $F_{\mu\nu}$:

$$\Pi_{\alpha\beta}(p) = i \int d^4 x \ e^{ip\cdot x} \langle 0| T\{ \eta_{\alpha}(x) \bar{\eta}_{\beta}(0) \}|0\rangle_F, \quad (1)$$

where $\eta_{\alpha}$ is the interpolating field for the propagating baryon. The subscript $F$ means
that the correlation function is to be evaluated with an electromagnetic interaction term: \( \mathcal{L}_I = -A_\mu J^\mu \), added to the QCD Lagrangian. Here \( A_\mu \) is the external electromagnetic potential and is given by \( A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x^\nu \) in the fixed-point gauge, and \( J^\mu = e_q \bar{q} \gamma^\mu q \) the quark electromagnetic current. The magnetic moments can be obtained by considering the \textit{linear response} of the correlation function to the external field. The action of the external field is two-fold: it couples directly to the quarks in the baryon interpolating fields, and it also polarizes the QCD vacuum. The latter can be described by the introduction of vacuum susceptibilities.

The interpolating field is constructed from quark fields, and has the quantum numbers of the baryon in question. For \( \Delta^{++} \) and \( \Omega^- \), they are given by

\[
\begin{align*}
\eta_\alpha^{\Delta^{++}}(x) &= \epsilon^{abc} \left( u^a T^T(x) C \gamma^\alpha u^b(x) \right) u^c(x), \\
\eta_\alpha^{\Omega^-}(x) &= \epsilon^{abc} \left( s^a T^T(x) C \gamma^\alpha s^b(x) \right) s^c(x),
\end{align*}
\]

where \( C \) is the charge conjugation operator, and the superscript \( T \) means transpose. The ability of an interpolating field to annihilate the \textit{ground state} baryon into the QCD vacuum is described by a phenomenological parameter \( \lambda_B \) (called current coupling or pole residue), defined by the overlap

\[
\langle 0 | \eta_\alpha | B ps \rangle = \lambda_B u_\alpha(p, s),
\]

where \( u_\alpha \) is the Rarita-Schwinger spin-vector.

The QCD sum rules are derived by calculating the correlator in (1) using Operator-Product-Expansion (OPE), on the one hand, and matching it to a phenomenological representation, on the other. The calculation is similar in spirit to the octet case (see Ref. [4], for example), but with the added complexity of spin-3/2 structures. A direct evaluation led to numerous tensor structures, not all of them independent. The dependencies can be removed by ordering the gamma matrices in a specific order (we chose \( \hat{p} \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta \) where the
hat notation means \( \hat{p} \equiv p^\mu \gamma_\mu \). After a lengthy calculation, a sum rule involving only the magnetic moment can be isolated at one of the structures: \( \gamma_\alpha F_{\mu\nu} \sigma_{\mu\nu} p_\beta \). Here we give the final results, for \( \Delta^{++} \):

\[
\begin{align*}
\frac{9}{28} e_u L^{4/27} E_1 M^4 &+ \frac{3}{56} e_u b L^{4/27} - \frac{6}{7} e_u \alpha^2 L^{12/27} - \frac{4}{7} e_u \kappa \alpha^2 L^{28/27} \frac{1}{M^2} \\
- \frac{1}{14} e_u (4\kappa + \xi) a^2 L^{28/27} \frac{1}{M^2} &+ \frac{1}{4} e_u \alpha^2 L^{-2/27} \frac{1}{M^2} + \frac{1}{12} e_u m_0^2 a^2 L^{14/27} \frac{1}{M^4} \\
&= \tilde{\lambda}_B^2 \left( \frac{\mu_B}{M^2} + A \right) e^{-M_B^2/M^2},
\end{align*}
\]

and for \( \Omega^- \):

\[
\begin{align*}
\frac{9}{28} e_u L^{4/27} E_1 M^4 &- \frac{15}{7} e_u f \phi m_\chi a L^{-12/27} E_0 M^2 + \frac{3}{56} e_u b L^{4/27} - \frac{18}{7} e_u f m_\alpha L^{4/27} \\
- \frac{9}{28} e_u f^2 \phi (2\kappa + \xi) a L^{4/27} &- \frac{6}{7} e_u f^2 \phi \alpha^2 L^{12/27} - \frac{4}{7} e_u f^2 \kappa \alpha^2 L^{28/27} \frac{1}{M^2} \\
- \frac{1}{14} e_u f^2 (4\kappa + \xi) a^2 L^{28/27} \frac{1}{M^2} &+ \frac{1}{4} e_u f^2 \phi m_0^2 a^2 L^{-2/27} \frac{1}{M^2} \\
- \frac{9}{28} e_u f m_\alpha m_0^2 a L^{-10/27} \frac{1}{M^2} &+ \frac{1}{12} e_u f^2 m_0^2 a^2 L^{14/27} \frac{1}{M^4} \\
&= \tilde{\lambda}_B^2 \left( \frac{\mu_B}{M^2} + A \right) e^{-M_B^2/M^2}.
\end{align*}
\]

In these equations, the magnetic moment \( \mu_B \) is given in particle’s natural magnetons. The various symbols are defined as follows. The condensates are represented by \( a = -(2\pi)^2 \langle \bar{u}u \rangle \), \( b = \langle g_s^2 G^2 \rangle, \langle \bar{u}g_{\nu}\sigma \cdot Gu \rangle = -m_0^2 \langle \bar{u}u \rangle \), and the coupling \( \tilde{\lambda}_B = (2\pi)^2 \lambda_B \). The factors \( e_u = 2/3 \) and \( e_s = -1/3 \) are quark charges in units of electric charge. The vacuum susceptibilities \( \chi, \kappa \) and \( \xi \) are defined by \( \langle \bar{q}\sigma_{\mu\nu}q \rangle_F \equiv e_q \chi \langle \bar{q}q \rangle F_{\mu\nu} \), \( \langle \bar{q}g_{\nu}\sigma \cdot Gu \rangle \equiv e_q \kappa \langle \bar{q}q \rangle F_{\mu\nu} \), and \( \langle \bar{q}g_{\nu}\epsilon_{\mu\nu\rho\lambda}G^{\rho\lambda}q \rangle_F \equiv e_q \xi \langle \bar{q}q \rangle F_{\mu\nu} \). The parameters \( f = \langle \bar{s}s \rangle/\langle \bar{u}u \rangle = \langle \bar{s}g_{\nu}\sigma \cdot Gs \rangle/\langle \bar{u}g_{\nu}\sigma \cdot Gu \rangle \) and \( \phi = \chi_s/\chi = \kappa_s/\kappa = \xi_s/\xi \) account for flavor symmetry breaking of the strange quark. Possible violation of the four-quark condensate is considered by the parameter \( \kappa_v \) as defined in \( \langle \bar{u}u\bar{u}u \rangle = \kappa_v \langle \bar{u}u \rangle^2 \). The anomalous dimension corrections of the various operators are taken into account via the factor \( L = [\alpha_s(\mu^2)/\alpha_s(M^2)] = \left[ \ln(M^2/\Lambda_{QCD}^2) / \ln(\mu^2/\Lambda_{QCD}^2) \right] \), where \( \mu = 500 \text{ MeV} \) is the renormalization scale and \( \Lambda_{QCD} \) is the QCD scale parameter. As usual, the pure excited state contributions are modeled using terms on the OPE side surviving \( M^2 \to \infty \) under the assumption of duality, and are represented by the factors.
\( E_n(x) = 1 - e^{-x} \sum_n x^n/n! \) with \( x = w^2/M_B^2 \) and \( w \) an effective continuum threshold. The parameter \( A \) accounts for all contributions from the transitions caused by the external field between the ground state and the excited states. Such contributions are \textit{not} exponentially suppressed relative to the ground state double pole and must be included. The presence of such contributions is a general feature of the external field technique.

Let us note in passing that since \( \Omega^- \) and \( \Delta^{++} \) are simply related by the interchange of quark flavors \( u \leftrightarrow s \), one can verify that sum rule (6) reduces to sum rule (5) if one sets \( e_s \rightarrow e_u \), \( m_s = 0 \), \( f = 1 \), and \( \phi = 1 \).

To analyze the sum rules, we use a Monte-Carlo based procedure first developed in Ref. [12]. The basic steps are as follows. First, the uncertainties in the QCD input parameters are assigned. Then, randomly-selected, Gaussianly-distributed sets are generated, from which an uncertainty distribution in the OPE can be constructed. Next, a \( \chi^2 \) minimization is applied to the sum rule by adjusting the phenomenological fit parameters. This is done for each QCD parameter set, resulting in distributions for phenomenological fit parameters, from which errors are derived. Usually, 100 such configurations are sufficient for getting stable results. We generally select 1000 sets which help resolve more subtle correlations among the QCD parameters and the phenomenological fit parameters.

The Borel window over which the two sides of a sum rule are matched is determined by the following two criteria. First, \textit{OPE convergence}: the highest-dimension-operators contribute no more than 10\% to the QCD side. Second, \textit{ground-state dominance}: excited state contributions should not exceed more than 50\% of the phenomenological side. The first criterion effectively establishes a lower limit, the second an upper limit. Those sum rules which do not have a valid Borel window under these criteria are considered unreliable and therefore discarded.

The QCD input parameters and their uncertainty assignments are given as follows [13]. The condensates are taken as \( a = 0.52\pm0.05 \) GeV\(^3\), \( b = 1.2\pm0.6 \) GeV\(^4\), and \( m_0^2 = 0.72\pm0.08 \) GeV\(^2\). For the factorization violation parameter, we use \( \kappa_v = 2 \pm 1 \) and \( 1 \leq \kappa_v \leq 4 \). The
QCD scale parameter is restricted to $\Lambda_{QCD}=0.15\pm0.04$ GeV. The vacuum susceptibilities have been estimated in studies of nucleon magnetic moments [2–4], but the values vary in a wide range depending on the method used. Here we take their median values with 50% uncertainties: $\chi = -6.0 \pm 3.0$ GeV$^{-2}$, $\kappa = 0.75 \pm 0.38$, and $\xi = -1.5 \pm 0.75$. Note that $\chi$ is almost an order of magnitude larger than $\kappa$ and $\xi$, and is the most important of the three. The strange quark parameters are placed at $m_s = 0.15 \pm 0.02$ GeV, $f = 0.83 \pm 0.05$ and $\phi = 0.60 \pm 0.05$ [5]. These uncertainties are assigned conservatively and in accord with the state-of-the-art in the literature. While some may argue that some values are better known, others may find that the errors are underestimated. In any event, one will learn how the uncertainties in the QCD parameters are mapped into uncertainties in the phenomenological fit parameters.

To illustrate how well a sum rule works, we first cast it into the subtracted form, $\Pi_S = \tilde{\lambda}_B^2 B e^{-M_B^2/M^2}$, then plot the logarithm of the absolute value of the two sides against the inverse of $M^2$. In this way, the right-hand side will appear as a straight line whose slope is $-M_B^2$ and whose intercept with the y-axis gives some measure of the coupling strength and the magnetic moment. The linearity (or deviation from it) of the left-hand side gives an indication of OPE convergence, and information on the continuum model and the transitions.

To extract the magnetic moments, a two-stage fit was performed. First, the corresponding chiral-odd mass sum rule, as obtained previously in Ref. [13], was fitted to get the mass $M_B$, the coupling $\tilde{\lambda}_B^2$ and the continuum threshold $w_1$. Then, $M_B$ and $\tilde{\lambda}_B^2$ were used in the magnetic moment sum rule for a three-parameter fit: the transition strength $A$, the continuum threshold $w_2$, and the magnetic moment $\mu_B$. Note that $w_1$ and $w_2$ are not necessarily the same. We impose a physical constraint on both $w_1$ and $w_2$ requiring that they are larger than the mass, and discard QCD parameter sets that do not satisfy this condition. The above procedure is repeated for each QCD parameter set until a certain number of sets are reached. In the actual analysis of sum rules (5) and (6), however, we found that such a full search was unsuccessful: the search algorithm consistently returned $w_2$ either zero or
smaller than $M_B$. This signals insufficient information in the OPE to completely resolve the spectral parameters. To proceed, we fixed $w_2$ at $w_1$, which is a reasonable and commonly adopted choice in the literature, and searched for $A$ and $\mu_B$.

Fig. 1 shows the match for the sum rules (5) and (6). The extracted results are given in Table I. Relatively large errors in the magnetic moments are found, approaching 100%.

FIG. 1. Monte-Carlo fits of the magnetic moment sum rules (5) and (6). Each sum rule is searched independently. The solid line corresponds to the ground state contribution, the dotted line the rest of the contributions (OPE minus continuum minus transition). The error bars are only shown at the two ends for clarity.

But the sign and order of magnitudes are unambiguous when compared to the measured values. The situation is consistent with a previous finding on $g_A$ regarding three-point functions. It is interesting to observe that the distribution of errors is not uniform throughout the Borel window, with the largest errors at the lower end where the power corrections are expected to become more important. The quality of the match deteriorates for $\Omega^-$ in this region, signaling probably insufficient convergence of the OPE. The match for $\Delta^{++}$ is good in the entire Borel region, despite the large uncertainties. To gain some idea on how the
TABLE I. Monte-Carlo analysis of the QCD sum rules for the magnetic moments of $\Delta^{++}$ and $\Omega^-$. The third column represents the percentage contribution of the excited states and transitions to the phenomenological side at the lower end of the Borel region (it increases to 50% at the upper end). The second row for each sum rule corresponds to results with reduced, uniform 10% errors assigned to all the QCD input parameters. The uncertainties were obtained from consideration of 1000 QCD parameter sets. In the event that the resultant distribution is not Gaussian, the median and asymmetric deviations are reported.

| Sum Rule | Region       | Cont (%) | $w$ (GeV) | $A$ (GeV$^{-2}$) | $\mu_B$ ($\mu_N$) |
|----------|--------------|----------|-----------|-----------------|-------------------|
| $\Delta^{++}$ | 0.765 to 1.47 | 9        | 1.65      | $0.53 \pm 0.77$ | 3.60$^{+3.68}_{-3.55}$ |
|           | 0.765 to 1.47 | 8        | 1.65      | $0.35 \pm 0.37$ | 4.13$^{+1.40}_{-1.18}$ |
| $\Omega^-$ | 0.747 to 1.66 | 7        | 2.30      | $-0.15 \pm 0.14$ | $-1.25^{+1.12}_{-1.17}$ |
|           | 0.747 to 1.66 | 6        | 2.30      | $-0.10 \pm 0.05$ | $-1.49^{+0.40}_{-0.49}$ |

Uncertainties change with the input, we also analyzed the sum rules by adjusting the error estimates individually. We found large sensitivities to the susceptibility $\chi$. In fact, most of the errors came from the uncertainty in $\chi$. We also tried with reduced error estimates on all the QCD input parameters: 10% relative errors uniformly. The results are given in Table I as a second entry. It leads to about 30% accuracy on the magnetic moments. Further improvement of the accuracy by reducing the input errors is beyond the capability of these sum rules as it will lead to unacceptably large $\chi^2/N_{DF}$ [12]. For that purpose, one would have to resort to finding sum rules that depend less critically on $\chi$ and have better convergence properties.

A comparison with those from other calculations and the experimental data is compiled in Table II. The results with 10% errors from the QCD sum rule method are used in the comparison. They are consistent with data within errors, although the central value for $\Omega^-$ is somewhat underestimated. The result for $\Delta^{++}$ is consistent with other calculations, while
TABLE II. Comparisons of magnetic moments from various calculations: this work (QCDSR), lattice QCD (Latt) [15], chiral perturbation theory (χPT) [16], light-cone relativistic constituent quark model (RQM) [17], simple non-relativistic constituent quark model (NQM), chiral quark-soliton model (χQSM) [18]. All results are expressed in units of nuclear magnetons.

| Baryon | Exp.   | QCDSR     | Latt     | χPT     | RQM   | NQM   | χQSM |
|--------|--------|-----------|----------|---------|-------|-------|-------|
| Δ^{++} | 4.5 ± 1.0 | 4.13 ± 1.30 | 4.91 ± 0.61 | 4.0 ± 0.4 | 4.76 | 5.56 | 4.73 |
| Ω^-   | -2.024 ± 0.056 | -1.49 ± 0.45 | -1.40 ± 0.10 | —       | -2.48 | -1.84 | -2.27 |

for Ω^- it is closer to lattice QCD calculations.

In conclusion, we have demonstrated that the magnetic moments of Δ^{++} and Ω^- can be understood from the QCD sum rule approach, despite large errors that can be traced to the uncertainties the QCD input parameters. A 30% accuracy can be achieved in the derived sum rules with improved estimates of the QCD input parameters, preferably on the 10% accuracy level. In particular, large sensitivities to the vacuum susceptibility χ are found. Better estimate of this parameter is needed. Extension of the calculations to other members of the decuplet appears straightforward, and is under way [19]. There we hope to address the issues involved in greater detail.

It is a pleasure to thank D.B. Leinweber for providing an original version of his Monte-Carlo analysis program and for helpful discussions. This work was supported in part by U.S. DOE under Grant DE-FG03-93DR-40774.
REFERENCES

[1] M.A. Shifman, A.I. Vainshtein and Z.I. Zakharov, Nucl. Phys. B147, 385, 448 (1979).

[2] B.L. Ioffe and A.V. Smilga, Phys. Lett. B133, 436 (1983); Nucl. Phys. B232, 109 (1984).

[3] B.L. Ioffe and A.V. Smilga, Phys. Lett. B129, 328 (1983).

[4] C.B. Chiu, J. Pasupathy, S.L. Wilson, Phys. Rev. D33, 1961 (1986).

[5] J. Pasupathy, J.P. Singh, S.L. Wilson, and C.B. Chiu, Phys. Rev. D36, 1442 (1986).

[6] S.L. Wilson, J. Pasupathy, C.B. Chiu, Phys. Rev. D36, 1451 (1987).

[7] C.B. Chiu, S.L. Wilson, J. Pasupathy, and J.P. Singh, Phys. Rev. D36, 1553 (1987).

[8] V.M. Belyaev, preprint ITEP-118 (1984); ITEP report (1992), unpublished.

[9] V.M. Belyaev, preprint CEBAF-TH-93-02, hep-ph/9301257.

[10] N.B. Wallace et al., Phys. Rev. Lett. 74, 3732 (1995).

[11] A. Bosshard et al., Phys. Rev. D44, 1962 (1991).

[12] D.B. Leinweber, Ann. of Phys. (N.Y.) 254, 328 (1997).

[13] F.X. Lee, preprint CU-NPL-1147, hep-ph/9707332.

[14] F.X. Lee, D.B. Leinweber, and X. Jin, Phys. Rev. D55, 4066 (1997).

[15] D.B. Leinweber, T. Draper, and R.M. Woloshyn, Phys. Rev. D46, 3067 (1992).

[16] M.N. Butler, M.J. Savage, and R.P. Springer, Phys. Rev. D49, 3459 (1994).

[17] F. Schlumpf, Phys. Rev. D48, 4478 (1993).

[18] H.C. Kim, M. Praszalowicz, and K. Goeke, hep-ph/9706531.

[19] F.X. Lee, in preparation.