ESTIMATES FOR INVARIANT METRICS ON C-CONVEX DOMAINS

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Abstract. Geometric lower and upper estimates are obtained for invariant metrics on C-convex domains containing no complex lines.

1. Introduction and results

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. For a domain $D \subset \mathbb{C}^n$ the Carathéodory and Kobayashi (pseudo)metrics are defined in the following way (cf. [12]):

$$\gamma_D(z; X) = \sup\{|f'(z)X| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$
$$\kappa_D(z; X) = \inf\{\alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha \varphi'(0) = X\}.$$ 

It is clear that $\gamma_D \leq \kappa_D$.

Recall that a domain $D \subset \mathbb{C}^n$ is called C-convex if any non-empty intersection with a complex line is a simply connected domain (cf. [1, 10]). A consequence of the fundamental Lempert theorem is the equality $\gamma_D = \kappa_D$ for any convex domain and any bounded C-convex domain $D$ with $C^2$ boundary; for the last statement use that such a domain can be exhausted by smooth bounded strictly C-convex domains (see [11]).

A domain $D \subset \mathbb{C}^n$ is said to be linearly convex (respectively, weakly linearly convex) if for any $a \in \mathbb{C}^n \setminus D$ (for any $a \in \partial D$) there exists a complex hyperplane through $a$ which does not intersect $D$.

Recall that the following implications hold:

$C$-convexity $\Rightarrow$ linear convexity $\Rightarrow$ weak linear convexity.

Moreover, these three notions coincide in the case of $C^1$-smooth domains in dimension greater than 1 (cf. [1, 16]).

2000 Mathematics Subject Classification. 32F45, 32A25.

Key words and phrases. C-convex domain, Carathéodory, Kobayashi and Bergman metrics, Bergman kernel.

This paper was written during the stay of the first-named author at the Carl von Ossietzky Universität Oldenburg, (November-December 2008) supported by the DFG grant 436POL113/103/0-2. The third-named was supported by the research grant No. N N201 361436 of the Polish Ministry of Science and Higher Education.
For \( C \)-convex domains we shall prove the following results for the boundary behavior of the Carathéodory and Kobayashi metrics.

**Proposition 1.** Let \( D \) be a \( C \)-convex domain containing no complex line through \( z \in D \) in direction of \( X \). Then

\[
\frac{1}{4} \leq \gamma_D(z; X)d_D(z, X) \leq \kappa_D(z; X)d_D(z, X) \leq 1,
\]

where

\[
d_D(z, X) = \sup\{ r > 0 : z + \lambda X \in D \text{ if } |\lambda| < r \}
\]

is the distance from \( z \) to \( \partial D \) in direction \( X \).

The constant \( \frac{1}{4} \) can be replaced by \( \frac{1}{2} \) in the case of convex domains (see [2]). On the other hand, the constant \( \frac{1}{4} \) is the best one in the planar case as the image \( D = \mathbb{C} \setminus [1/4, \infty) \) of \( \mathbb{D} \) under the Koebe function \( \frac{z}{1+z^2} \) shows. It is clear that the upper constant 1 is attained if, for example, \( D = \mathbb{D} \).

**Corollary 2.** For any \( C \)-convex domain \( D \subset \mathbb{C}^n \) one has that \( \kappa_D \leq 4\gamma_D \).

Recall that if a \( C \)-convex domain \( D \subset \mathbb{C}^n \) contains a complex line, then it is linearly equivalent to the Cartesian product of \( \mathbb{C} \) and a \( C \)-convex domain in \( \mathbb{C}^{n-1} \).

For a boundary point \( a \) of a domain \( D \subset \mathbb{C}^n \) denote by \( L_a \) the set of all vectors \( X \in \mathbb{C}^n \) for which there exists \( \varepsilon > 0 \) such that \( \partial D \supset \Delta_X(a, \varepsilon) = \{ a + \lambda X : |\lambda| < \varepsilon \} \).

The following result is a consequence of Proposition 1.

**Proposition 3.** Let \( a \) be a boundary point of a \( C \)-convex domain \( D \subset \mathbb{C}^n \).

(i) Then

\[
\lim_{z \to a} \gamma_D(z; X) = \infty
\]

locally uniformly in \( X \not\in L_a \).

(ii) If \( \partial D \) is \( C^1 \)-smooth at \( a \), then \( L_a \) is a linear space. Moreover, for any non-tangential cone \( \Lambda \) with vertex at \( a \) there is a constant \( c > 0 \) such that

\[
\lim sup_{\Lambda \ni z \to a} \kappa_D(z; X) \leq c
\]

locally uniformly in the unit vectors \( X \in L_a \).

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1 This means that for any \( M > 0 \) there are neighborhoods \( U \) of \( a \) and \( V \) of \( X \) such that \( \gamma_D(z; Y) > M \) for any \( z \in D \cap U \) and \( Y \in V \setminus L_a \).
Next we shall discuss types related to a ($C^\infty$-)smooth boundary point $a$ of a domain $D \subset \mathbb{C}^n$ and a vector $X \in (\mathbb{C}^n)^*$. Denote by $m_a$ the (usual) type of $a$, i.e. the maximal order of contacts of non-trivial analytic discs through $a$ and $\partial D$ at the point $a$. Replacing analytic discs by complex lines, we define the linear type $l_a$ of $a$. We may also define $l_{a,X}$ as the order of contact of the line through $a$ in direction of $X$ and $\partial D$ at $a$. Then $m_a \geq l_a = \sup_X l_{a,X}$. Note that if $l_{a,X} < \infty$, then $X \not\in L_a$.

Proposition 4. Let $a$ be a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ and let $X \in (\mathbb{C}^n)^*$ with $l_{a,X} < \infty$. Denote by $n_a$ the inner normal to $\partial D$ at $a$. Then there exist a neighborhood $U$ of $a$ and a constant $c > 1$ such that
\[
c^{-1} d_D(z) \leq d_D(z, X)^{l_{a,X}} \leq c d_D(z), \quad z \in D \cap U \cap n_a,
\]
where $d_D$ is the distance to $\partial D$.

Combining Proposition [1] and [4] we immediately get an extension of the main result in [17] from the convex to the $\mathbb{C}$-convex case.

Corollary 5. Under the notations of Proposition [4], there is a constant $c > 0$ such that
\[
c^{-1} (d_D(z))^{-1/l_{a,X}} \leq \gamma_D(z; X) \leq \kappa_D(z; X) \leq c (d_D(z))^{-1/l_{a,X}}.
\]

The main result in [19] (see also [4]) states that $m_a = l_a$ for convex domains. The same remains true for a $\mathbb{C}$-convex domain.

Proposition 6. If $a$ is a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then $m_a = l_a$.

Remark. We like to mention that the proof in [4] immediately implies the above proposition in dimension 2. But we do not know if the criterion in [4] (for the equality $m_a = l_a$) holds for any $\mathbb{C}$-convex domain.

Moreover, in the case of infinite type we have the following result.

Proposition 7. If $a$ is a $C^1$-smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then $\partial D$ contains no nontrivial analytic disc through $a$ if only if $L_a = \{0\}$.

Remark. Some of the above propositions in (A) and (B) have local versions. In this connection recall that there is a localization principle for the Kobayashi metric of any hyperbolic domain (cf. [12]).

(C) Now we are going to discuss multitypes of boundary points. Recall that a smooth finite type pseudoconvex boundary point $a$ of a
domain $D \subset \mathbb{C}^n$ is said to be semiregular \cite{8} (or h-extendible \cite{30}) if its Catlin multitype $\mathcal{M}(a)$ coincides with its D’Angelo type $\Delta(a)$. Based on the fact that the usual type is equal to the line type in the case of convex domains, it is shown in \cite{29} that if $a$ is a smooth convex point (not necessarily of finite type), then $\mathcal{L}(a) = \mathcal{M}(a) = \Delta(a)$, where $\mathcal{L}(a)$ denotes the linear multitype of $a$.

We shall say that $a$ is a $\mathbb{C}$-convex boundary point of a domain $D \subset \mathbb{C}^n$ if there is a neighborhood $U$ of $D$ such that $D \cap U$ is a $\mathbb{C}$-convex domain.

**Proposition 8.** If $a$ is a smooth $\mathbb{C}$-convex boundary point of a domain $D \subset \mathbb{C}^n$, then $\mathcal{L}(a) = \mathcal{M}(a) = \Delta(a)$.

Then the main result in \cite{30} implies the following.

**Corollary 9.** Any smooth finite type $\mathbb{C}$-convex boundary point $a$ of a domain $D \subset \mathbb{C}^n$ is a local (holomorphic) peak point. Moreover, there is a neighborhood $U$ of $a$ and a domain $\mathbb{C}^n \supset G \supset D \cap U \setminus \{a\}$ such that $a \in \partial G$ is a peak point w.r.t. the algebra $A(G)$.

This corollary is also a direct consequence of the main result in \cite{9}, where local holomorphic support functions which depend smoothly on the boundary points are constructed.

We point out that the assumption of smoothness is essential as the domain $D = \mathbb{D} \setminus [0, 1)$ may show. It is easy to see that the points from the deleted interval are not peak points for $A(D)$.

On the other hand, in \cite{28}, the following result is claimed.

**Proposition 10.** Let $D \subset \mathbb{C}^n$ be a bounded convex domain. Then $a \in \partial D$ is a peak point w.r.t. $A(D)$ if and only if $L_a = \{0\}$.

For the convenience of the reader, we shall prove this result.

Note that there is a smooth convex bounded domain $D \subset \mathbb{C}^2$ containing no non-trivial analytic discs in the boundary but some of the boundary points (not of finite type) are not peak points w.r.t. $A^\alpha(D)$ for any $\alpha > 0$ (see \cite{27}).

Note also that main result in \cite{21} (see also \cite{31} and \cite{5}) and Proposition 8 give the following fact about the boundary behavior of invariant metrics (see also \cite{3, 18}).

**Corollary 11.** Let $a$ be a finite type $\mathbb{C}$-convex boundary point of a smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Let $\mathcal{M}(a) = (m_1, \ldots, m_n)$ be the Catlin multitype of $a$. $m_1 = 1$ and $m_2 \leq \cdots \leq m_n$ are even

\footnote{The same result may be found in \cite{7}; the proof there is related on good local coordinates and on the proof in \cite{29}, whereas our proof is based on the simple geometric Lemma 15 and on the proof in \cite{29}.}
numbers). Denote by \( n_a \) the inner normal to \( \partial D \) at \( a \). There is a basis \( \{ e_1, \ldots, e_n \} \) (\( e_1 \) is the complex normal vector and \( \{ e_2, \ldots, e_n \} \subset T^\mathbb{C}_a(\partial D) \)) and a constant \( c > 1 \) such that for any \( X = \sum_{j=1}^n X_j e_j \) we have

\[
\frac{1}{c} \leq \liminf_{n_a \ni z \to a} F_D(z; X) \left( \sum_{j=1}^n \frac{|X_j|}{(d_D(z))^{1/m_j}} \right)^{-1}
\]

\[
\leq \limsup_{n_a \ni z \to a} F_D(z; X) \left( \sum_{j=1}^n \frac{|X_j|}{(d_D(z))^{1/m_j}} \right)^{-1} \leq c.
\]

Here \( F_D \) is any of the Carathéodory, Kobayashi or Bergman metrics.

We point out that this corollary implies Proposition 4 in the finite type case, showing in addition that for any \( X \in (\mathbb{C}^n)^* \) there is \( j = 1, \ldots, n \) with \( l_{a,X} = m_j \).

(D) Finally, we turn to the main part in this paper, namely, the boundary behavior of the Bergman metric of \( \mathbb{C} \)-convex domains. Denote by \( L^2_h(D) \) the Hilbert space of all holomorphic functions \( f \) on a domain \( D \subset \mathbb{C}^n \) that are square-integrable and by \( \| f \|_D \) the \( L_2 \)-norm of \( f \). Let \( K_D \) be the restriction to the diagonal to the Bergman kernel function of \( D \). It is well-known that (cf. [12])

\[
K_D(a) = \sup\{|f(a)|^2 : f \in L^2_h(D), \| f \|_D \leq 1 \}.
\]

If \( K_D(z) > 0 \) for some point \( z \in D \), then the Bergman metric \( B_D(z; X), X \in \mathbb{C}^n \), is well-defined and can be given by the equality

\[
B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}},
\]

where \( M_D(z; X) = \sup\{|f'(z)X| : f \in L^2_h(D), \| f \|_D = 1, f(z) = 0 \} \). Recall that (cf. [12])

\[
\gamma_D \leq B_D.
\]

On the other hand, there exists a constant \( c_n > 0 \), depending only on \( n \) such that for any convex domain \( D \subset \mathbb{C}^n \), containing no complex line, the following inequality holds (see [24]):

\[
B_D \leq c_n \gamma_D.
\]

This fact extends to any \( \mathbb{C} \)-convex domain as the following theorem shows.

**Theorem 12.** There exists a constant \( c_n > 0 \), depending only on \( n \), such that for any \( \mathbb{C} \)-convex domain \( D \subset \mathbb{C}^n \), containing no complex
lines\(^3\) one has that
\[ B_D(z; X) d_D(z, X) \leq c_n, \quad z \in D, X \in \mathbb{C}^n. \]
In particular, by Proposition \(^4\)
\[ \frac{k_D}{4} \leq B_D \leq 4c_n \gamma_D. \]

To prove Theorem \(^12\) we shall need a lower geometrical estimates for the Bergman kernel. For this, similarly to the convex case (see \([24]\); see also \([13, 14, 7]\) and compare with \([6, 19, 20]\)), we introduce the following geometrical objects related to an arbitrary domain \(D \subset \mathbb{C}^n\), containing no complex lines.

For \(z^0 \in D =: D_0 \subset \mathbb{C}^n =: H_0\) define \(d_{1,D}(z^0) := \text{dist}(z^0, \partial D) = d_D(z^0)\). Fix an \(a^1 \in \partial D\) such that \(\|a^1 - z^0\| = d_{1,D}(z^0)\). Let \(l_1 = z^0 + V_1\) be the complex line passing through \(z^0\) and \(a^1\). Let \(H_1 := V_1^\perp\) be the \((n-1)\)-dimensional complex space orthogonal to \(V_1\).

Then fix a point \(a^2 \in \partial_{z^0 + H_1}(D_1)\) with \(\|a^2 - z^0\| = d_{2,D}(z^0)\). Denote by \(l_2 = z^0 + V_2\) the complex line through \(z^0\) and \(a^2\). Note that \(V_2 \subset V_1^\perp\).

Then the following result is a consequence of Proposition \(^1\) and Theorem \(^12\).

\textbf{Theorem 13.} Let \(D \subset \mathbb{C}^n\) be a \(\mathbb{C}\)-convex domain containing no complex lines. Then
\[ \frac{1}{(16\pi)^n} \leq K_D(z) p_D^2(z) \leq \frac{(2n)!}{(2\pi)^n}. \]

Recall that the constant 16 can be replaced by 4 in the case of convex domains (see \([24]\)).

The next result extends earlier ones treating convex domains of finite type (cf. \([6, 20]\) and the proof here is easier and pure geometrical. Take a vector \(X = z^0\). For any point \(z \in D\), decompose \(X\) w.r.t to the orthogonal basis mentioned above, i.e. \(X = (X_1(z), \ldots, X_n(z))\).

Then the following result is a consequence of Proposition \(^1\) and Theorem \(^12\).

\footnote{Under the given assumptions \(D\) is biholomorphic to a bounded domain (cf. \([26]\)) and hence \(B_D\) is well-defined.}
Proposition 14. There exists a constant $c_n > 1$, depending only on $n$, such that for any $C$-convex domain $D \subset \mathbb{C}^n$, containing no complex lines, one has that
\[
c_n^{-1} \leq F_D(z; X) \left( \sum_{j}^{n} \frac{|X_j(z)|}{d_{j,D}(z)} \right)^{-1} \leq c_n,
\]
where $F_D$ denotes any of the Carathéodory, Kobayashi or Bergman metrics.

This result is in the spirit of Corollary 11.

Remark. Proposition 3 and Corollary 5 hold for the Bergman metric, if the domain contains no complex lines. (In fact, then Proposition 3 transports the main result in [15] and a result in [23] from the convex to the $C$-convex case). Moreover, these and the other results for the Bergman kernel and metric have local versions on bounded pseudoconvex domains due to the localization principle for the Bergman invariants (cf. [12]).

2. Proofs

Proof of Proposition 7. The upper bound is trivial and holds for any domain $D$, since it contains the disc with center $z$ and radius $d_D(z, X)$ in direction $X$.

To prove the lower bound, we may assume that $||X|| = 1$. Denote by $l$ the complex line trough $z$ in direction $X$ and choose $a \in l \cap \partial D$ such that $||z - a|| = d_D(z, X)$. Consider a complex hyperplane $H$ through $a$ such that $D \cap H = \emptyset$ and denote by $G$ the projection of $D$ onto $l$ in direction $H$. Note that $G$ is a simply connected domain (cf. [11] [16]), $a \in \partial G$ and $d_G(z) = ||z - a||$. It remains to apply the Koebe theorem to get that
\[
\gamma_D(z; X) \geq \gamma_G(z; 1) \geq \frac{1}{4d_G(z)}.
\]

Many of the next proofs will be based on the following geometric property of weakly linearly convex domains (see also [33] and (for the finite type case) [7]).

Lemma 15. Assume that a weakly linearly convex domain $G \subset \mathbb{C}^n$ contains the unit disc $\mathbb{D}_j$ in the $j$-th complex coordinate line for any $j = 1, \ldots, n$. Then $G$ contains the convex hull of $\bigcup_{j=1}^{n} \mathbb{D}_j$, i.e.
\[
E := \{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| < 1 \} \subset G.
\]
Proof. For any \( \varepsilon \in (0, 1) \) there is \( \delta > 0 \) such that
\[
X_{\varepsilon} := \bigcup_{j=1}^{n} \left( \delta D \times \cdots \times \delta D \times \varepsilon D \times \delta D \times \cdots \times \delta D \right) \subset G.
\]
Recall that
\[
\hat{X}_\varepsilon \subset G,
\]
where \( \hat{X}_\varepsilon \) is the smallest linearly convex set containing \( X_\varepsilon \). Moreover,
\[
\hat{X}_\varepsilon = \left\{ z \in \mathbb{C}^n : \forall b \in \mathbb{C}^n : \langle z, b \rangle = 1 \exists a \in X_\varepsilon : \langle a, b \rangle = 1 \right\}.
\]
(cf. \([1, 16]\)). Then \( \hat{X}_\varepsilon \) is a balanced domain and, therefore, convex (see \([26]\)). Hence,
\[
E_\varepsilon := \left\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| < \varepsilon \right\} \subset \hat{X}_\varepsilon \subset G, \quad \varepsilon \in (0, 1),
\]
which proves Lemma 15. \( \square \)

Remark. The same argument implies that \( G \) contains the convex hull of any balanced domain lying in \( G \). In particular, the maximal balanced domain lying in \( G \) is convex (see also \([33]\)).

Proof of Proposition 3. (i) Assuming the contrary, we may find an \( r > 0 \) and sequences \( D \supset (z_j)_j, \, z_j \to a, \, \mathbb{C}^n \supset (X_j)_j, \, X_j \to X \not\in L_a \) such that \( \gamma_D(z_j; X_j) \leq \frac{1}{4r} \). Note that, by Proposition 1, \( d_D(z_j; X_j) \geq r \) (this is trivial if \( D \) contains the complex line through \( z_j \) in direction \( X_j \)). Then \( \Delta_{X_j}(a, r) \subset D_r = D \cap \mathbb{B}_a(a, 2r) \) for any large \( j \). Note that \( D_r \) is a (weakly) linearly convex open set. It is easy to see that \( D_r \) is taut, i.e. the family \( \mathcal{O}(D, D_r) \) is normal (cf. \([25]\)). Hence \( \Delta_X(a, r) \subset \partial D \); a contradiction.

(ii) Recall that \( \partial D \) is \( C^1 \)-smooth. Therefore, for any two linearly independent vectors \( X, Y \in L_a \), we may find a neighborhood \( U \) of \( a \) and a number \( \varepsilon > 0 \) such that \( \Delta_X(z, \varepsilon) \subset D \) and \( \Delta_Y(z, \varepsilon) \subset D \) for \( z \in D \cap U \cap \Lambda \). It follows by Lemma 15 that \( \Delta_{X+Y}(z, \varepsilon') \subset D \) for some \( \varepsilon' > 0 \). We get as in (i) that \( \Delta_{X+Y}(a, \varepsilon') \subset \partial D \). Therefore, \( L_a \) is a linear space.

Then, choosing a basis in \( L_a \) and applying Lemma 15 we see that there are a neighborhood \( U \) of \( a \) and a number \( c > 0 \) such that \( \Delta_X(z, c) \subset D \) for any \( z \in D \cap U \cap \Lambda \) and any unit vector \( X \in L_a \). Now the desired estimates follow by Proposition 1. \( \square \)

Proof of Proposition 4. We may assume that \( \text{Re}(z_1) < 0 \) is the inner normal direction to \( \partial D \) at \( a = 0 \). Let \( r(z) = \text{Re}(z_1) + o(|z_1|) + \rho'(z) \) be a smooth defining function of \( D \) near 0.
For any small $\delta > 0$ we have that $\delta = d_D(\delta_n)$, where $\delta_n = (-\delta^*,0)$. Set $L_\delta(\zeta) = -\delta_n + \zeta X$, $\zeta \in \mathbb{C}^n$.

We shall consider two cases.

1. $l_{a,X} = 1$. This means that $X_1 \neq 0$. Then $r(L_\delta(\zeta)) = -\delta + \text{Re}(\zeta X_1) + o(|\zeta|)$. It follows that $L_\delta(\zeta) \in D$ if $|\zeta| < \frac{\delta}{2|X_1|}$ and $\delta$ is small enough. This proves the left-hand side inequality.

The opposite inequality follows by the inequality $r(L_\delta(2\delta/X_1)) > 0$ which holds for any small $\delta > 0$.

2. $l_{a,X} \geq 2$. This means that $X_1 = 0$. Then $r(L_\delta(\zeta)) = -\delta + \rho(\zeta' X)$. Since $\rho(\zeta' X) \leq c|\zeta|^l$ for some $c > 0$, we conclude that $L_\delta(\zeta) \in D$ if $c|\zeta|^l < \delta$. This implies the left-hand side inequality.

To prove the opposite inequality, we have to find $c_1 > 0$ such that for any small $\delta > 0$ there is $\zeta$ with $|\zeta|^l = c_1^{-1}\delta$ and $\rho(\zeta' X) \geq \delta$. Since $D$ is (weakly) linearly convex, it follows that $\rho(\zeta' X) = h(\zeta) + o(|\zeta|^l) \geq 0$, where

$$h(\zeta) = \sum_{j+k=l} a_{jk} \zeta^j \zeta^k \not= 0.$$

Then the homogeneity of $h$ implies that $h \geq 0$. Moreover, since $h \not= 0$ we may find a $\zeta$ with $|\zeta| = 1$ and $h(\zeta) > c_1$ for some $c_1 > 0$. Then the constant $c_1$ does the job for any small $\delta > 0$. \hfill \Box

Proof of Proposition 6. The inequality $l_a \leq m_a$ is trivial. To prove the opposite one, we may assume that $l_a \leq m_a$. It follows from Propositions 1 and 4 that

$$\liminf\limits_{D \ni m_a \geq z \to a} \gamma_D(z; X) a^{1/l_a} \geq c_X > 0.$$ 

Hence, $m_a \leq l_a$ by Corollary 2 in [32] (in fact, lim sup instead of lim inf above is sufficient). \hfill \Box

Proof of Proposition 7. We shall use the same notations as in the proof of Proposition 4. It is enough to show that if $\varphi : \mathbb{D} \to \partial D$ is a non-trivial analytic disc with $\varphi(0) = 0$, then $L_a \not= \{0\}$. Since $\partial D$ is smooth near $a$, it follows that there is a $c > 0$ such that $\varphi_\delta(\zeta) = -\delta + \varphi(\zeta) \in D$ if $\delta < c$ and $|\zeta| < c$. Let $m = \text{ord}_0 \varphi$ and $X = \varphi^{(m)}(0)$. Denoting by $\kappa_D^{(m)}$ the Kobayashi metric of order $m$ (cf. [32] for this notion), it follows that $\kappa_D^{(m)}(\delta_a; X) \leq 1/c$. Since $\gamma_D \leq \kappa_D^{(m)}$, we get as in the proof of Proposition 3 (i) that $\Delta_X(a, c/4) \subset \partial D$. \hfill \Box

Proof of Proposition 8. The proof can be done following line by line the proofs in [27]. We only point out how the replace the arguments there that use convexity. We may assume that $D$ is a $\mathbb{C}$-convex domain and $a = 0$. Following the notation from Proposition 4, let $r(z) = \ldots$
Let \( X, Y \subset \mathbb{C}^{n-1} \) be such that \( \rho(\zeta X) \leq C|\zeta|^m \) and \( \rho(\zeta Y) \leq C|\zeta|^m \). We have to show that \( \rho(\zeta(X+Y)/2) \leq C|\zeta|^m \). For this, fix \( \zeta \neq 0 \) and take \( \delta = C|\zeta|^m \). Then \( \Delta_X(\delta_n, |\zeta|) \subset D, \Delta_Y(\delta_n, |\zeta|) \subset D \) and hence \( \Delta_{(X+Y)/2}(\delta_n, |\zeta|) \subset D \) by Lemma 15. This implies the desired inequality.

We may do the same to get the formula (2.13) on page 845.

Our Proposition 4 is an extension of Theorem C which is invoked on page 845.

It remains to show Proposition 2 on page 843. Let \( k_2, \ldots, k_n \) be even integers such that \( \rho(\zeta e_j) \leq C|\zeta|^{k_j} \) for any \( j = 2, \ldots, n \). It is enough to prove that \( D^aD^3\rho(0) = 0 \) for any \( n \)-tuples \( \alpha = (\alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_2, \ldots, \beta_n) \) of non-negative integers with \( w_{\alpha, \beta} = \sum_{j=2}^n \frac{\alpha_j + \beta_j}{k_j} < 1 \).

Since \( \Delta_{\delta_n}(\delta, \delta^{1/k_j}) \subset D \) for any \( \delta > 0 \), it follows by Lemma 11 that \( \rho^{(z/n)} < C\delta \) for any \( z \) with \( |z_j|^{k_j} \leq \delta \). In particular, if \( \rho_t(z) = \rho(t^{1/k_2^2}z_2, \ldots, t^{1/k_n^2}z_n), t > 0 \), then

\[
0 \leq \rho_t^{(z/n)} < Ct, \quad z \in \mathbb{D}^{n-1}.
\]

Let now \( s = \min\{w(\alpha, \beta) : D^aD^3\rho(0) \neq 0\} \). Then

\[
\lim_{t \to 0} t^{-s} \rho_t^{(z)} = \sum_{w(\alpha, \beta) = s} D^aD^3\rho(0)^{\alpha}z^{\beta} \]

locally uniformly in \( z \). Assuming \( s < 1 \), the inequality (1) implies that the last polynomial vanishes, a contradiction.

---

**Proof of Proposition 10.** Let first \( L_0 \neq \{0\} \). This means that \( \Delta_X(a, r) \subset \partial D \) for some \( r > 0 \) and \( X \in (\mathbb{C}^n)_s \). By convexity, \( \Delta_X(c, r/2) \subset D \) for any \( c = ta + (1-t)b \) if \( b \in D \) and \( t \in (0, 1/2] \). Now the maximum principle implies that \( a \) is not a peak point.

Let now \( L_0 = \{0\} \). We may assume that \( a = 0 \) and \( D \subset \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\} \). Then \( e^{z_1} \) is an entire weak peak function for \( \overline{D} \) at 0. Setting \( H = \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\} \), it follows that \( \text{supp} \mu \subset D_1 = \partial D \cap H \) for any representing measure \( \mu \) for \( 0 \) w.r.t. \( A(D) \). Since \( L_0 = \{0\} \), it follows that \( 0 \) is a boundary point of the convex set \( D_1 \). Then there exists an entire function which is a weak peak function for \( D_1 \) at 0 (we need such a function function to be in \( A(D) \)). We get as above that \( \text{supp} \mu \) is contained in some \((n-2)\) dimensional space. Repeating this procedure, it follows that \( \text{supp} \mu \subset \partial D \cap l \), where \( l \) is a complex line. Since 0 is a boundary point of the last convex set,
then there is an entire function which is a peak function for $\partial D \cap l$ at 0. So $\text{supp} \mu = \{0\}$, i.e. 0 is a peak point w.r.t. $A(D)$ (cf. [10]). □

Proof of Theorem 13. We first prove the lower bound. Fix $z^0 \in D$. Using a translation and then successive rotations we may assume (see the description of the numbers $d_{j,D}$) that $z^0 = 0$, $H_j = \{0\} \times \mathbb{C}^{n-j}$, $j = 1, \ldots, n-1$, and $a_j^i = (0, a_j^i, 0) \in \mathbb{C}^{j-1} \times \mathbb{C} \times \mathbb{C}^{n-j}$ with $d_{j,D}(z^0) = |a_j^i|$

Recall that $D$ is $\mathbb{C}$-convex. Therefore, there exist affine hyperplanes $a_j^i + W_j$ through $a_j^i$ which do not intersect $D$. Note that $W_1 \cap H_1$ is orthogonal to $a_2^1$, i.e. $W_1 \cap H_1 \subset \{0\} \times \mathbb{C}^{n-2}$. Hence $W_1$ is given by the equation $\alpha_2^1 z_1 + z_2 = 0$. Moreover, using a similar argument, the equations for $W_j$, $j = 1, \ldots, n-1$, are the following ones:

\[
\alpha_j^{i,1} z_1 + \cdots + \alpha_{j,j} z_j + z_{j+1} = 0.
\]

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be the linear mapping given by the matrix $A$ whose rows are given by the vectors $(\alpha_j^{i,1}, \ldots, \alpha_{j,j}, 1, 0, \ldots, 0)$, $j = 0, \ldots, n-1$. Define $G = F(D)$ and observe that $G$ is again $\mathbb{C}$-convex. Note that $K_D(0) = K_G(0)$ since $\det A = 1$. Finally, put $G_j := \pi_j(G)$, where $\pi_j$ is the projection onto the $j$-th coordinate axis. Then (see [11]) $G_j$ is a simply connected domain, $j = 1, \ldots, n$, and $G \subset G_1 \times \cdots \times G_n$. Hence

\[
K_D(0) \geq K_{G_1 \times \cdots \times G_n}(0) = K_{G_1}(0) \cdots K_{G_n}(0).
\]

Since $G_j$ is simply connected, using the Koebe theorem we get

\[
\sqrt{\pi K_{G_j}(0)} = \gamma_{G_j}(0; 1) \geq \frac{1}{4d_{G_j}(0)}.
\]

Note that $F(a_j^i) \in \partial G$, its $j$-th coordinate is $a_j^{i,j}$, and the affine hyperplane $\{z \in \mathbb{C}^n : z_j = a_j^i\}$ does not intersect $G$. Hence $a_j^i \in \partial G_j$; in particular, $d_{j,D}(z^0) = |a_j^i| \geq d_{G_j}(0)$, which finally gives the lower bound.

To show the upper bound, consider the dilatation of coordinates

\[
\Phi(z) = (z_1/d_{1,D}(z^0), \ldots, z_n/d_{n,D}(z^0))
\]

and set $\tilde{G} = \Phi(D)$. Hence

\[
K_{\tilde{G}}(z^0) = \frac{K_D(z^0)}{P_D(z^0)}.
\]

Then the upper bound follows from Lemma 15 and the following formula (cf. [12, 23]):

\[
K_{E}(0) = \frac{(2n)!}{(2\pi)^n}.
\]

□
Proof of Theorem 12. The proof can be done following line by line the proof of Theorem 2 in [24] and using Theorem 13 and Lemma 15. For convenience of the reader, we include a complete proof.

We shall use the geometric constellation in the proof of Theorem 13. Let $X \in (\mathbb{C}^n)_*$ and fix $k \in J := \{ j : X_j \neq 0 \}$. Then

$$
\Psi_k(z) := \left( z_1 - \frac{X_1}{X_k} z_k, \ldots, z_{k-1} - \frac{X_{k-1}}{X_k} z_k, z_k, z_{k+1} - \frac{X_{k+1}}{X_k} z_k, \ldots, z_n - \frac{X_n}{X_k} z_k \right)
$$

is a linear mapping with jacobian equal to 1 and $Y_k := \Psi_k(X) = (0, \ldots, 0, X_k, 0, \ldots, 0)$. Let $\Delta_j$ be the disc in the $j$-th coordinate plane with center at 0 and radius $d_j D(0)$ if $j \neq k$, and $d'_k := |X_k| d_D(0, X)$ if $j = k$. Then $\Delta_j \subset D_k := \Psi_k(D)$ and, by Lemma 15,

$$
D_k \supset E_k = \{ z \in \mathbb{C}^n : |z_k| + \sum_{j=1, j \neq k}^n \frac{|z_j|}{d'_j} < 1 \}.
$$

Hence

$$
M_{D}(0; X) = M_{D_k}(0; Y^k) \leq M_{E_k}(0; Y^k) = C \frac{d_{k,D}(0)}{|X_k| p_D(0) d^2_D(0, X)},
$$

where $C := M_{E}(0; e_1) = \sqrt{\frac{2(2(n+1)!)^2}{6(2\pi)^n}}$ (cf. [24]) and $e_1$ is the first basis vector. Applying the lower bound in Theorem 13 we obtain that

$$
B_D(0; X) = \frac{M_D(0; X)}{\sqrt{K_D(0)}} \leq \frac{c'_n d_{k,D}(0)}{|X_k| d^2_D(0, X)}, \quad 1 \leq k \leq n,
$$

where $c'_n = (4\sqrt{\pi})^n C_n = 2^n \sqrt{\frac{(2(n-1)!)!}{3}}$. It remains to apply Lemma 15 to get that

$$
\frac{1}{d_D(0, X)} \leq \sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)}
$$

and then to choose $c_n = nc'_n$.

□

Proof of Proposition 14. It follows by (2) and the inequality

$$
B_D(z; X) \geq \frac{1}{4 d_D(z, X)}
$$

that

$$
\frac{|X_j(z)|}{d_{j,D}(z)} \leq \frac{4c'_n}{d_D(z)}.
$$

Hence,

$$
\frac{1}{d_D(z, X)} \leq \sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)} \leq \frac{4c_n}{d_D(z, X)},
$$

\(\square\)
where \( c_n = nc'_n. \) Then (2) and (3) imply that

\[
(16c_n)^{-1} \leq F_D(z; X) \left( \sum_{j=1}^{n} \frac{|X_j(z)|}{d_{j,D}(z)} \right)^{-1} \leq c_n.
\]

Remarks. (a) In [6, 20], the numbers \( d_{j,D}(z) \) are replaced by other numbers \( \tilde{d}_{j,D}(z) \) in the finite type convex case. Note that \( \tilde{d}_{1,D}(z) = d_{1,D}(z) \) and \( \tilde{d}_{j,D}(z) \geq d_{j+1,D}(z), \) \( 2 \leq j \leq n-1 \) (in contrast to \( d_{j,D}(z) \leq d_{j+1,D}(z) \)). The inductive definition of \( d_{j,D}(z) \) is similar to that of \( d_{j,D}(z) \) but in any step \( j \geq 2 \) the number \( d_{j,D}(z) \) is the radius of the largest (not the smallest!) disc in the respective \( (n-j+1) \)-dimensional set. However, one can show the the respective supporting hyperplanes \( \tilde{W}_j \) have the same equations as \( W_j \) when \( z \) is near \( \partial D. \) Then the above approach allows us to get the same estimates as in Theorem 13 and Proposition 14 in terms of these numbers and the respective coordinates. In particular, it leads to (4) also for the \( \tilde{d}_{j,D} \)-situation.

(b) Assume that a domain \( D \subset \mathbb{C}^n \) is smooth and weakly linearly convex near a boundary point \( a \) of finite type \( m. \) Then \( d_{j,D}(z) \leq (d_D(z))^{1/m} \) for \( j = 1, \ldots, n \) (see [3]). Since \( a \) is a local peak point for \( D \) at \( a, \) it follows that there is a neighborhood \( U \) of \( a \) and a constant \( c > 0 \) such that

\[
\kappa_D(z; X) \geq \frac{c|X|}{(d_D(z))^{1/m}}, \quad z \in D \cap U;
\]

the same estimate holds for \( B_D \) if \( D \) is pseudoconvex (not necessary bounded - use e.g. localization results in [22]).

(c) Finally, note that there is a number \( c_n > 1 \) depending only on \( n \) such that

\[
c_n^{-1} \leq \frac{d_{j,D}(z)}{d_{n-j,D}(z)} \leq c_n, \quad z \in D, j = 1, \ldots, n,
\]

for any \( z \) near the boundary of any smooth convex domain \( D \subset \mathbb{C}^n \) containing no complex lines.

In fact, a more general statement is true. Let \( \{p_1, \ldots, p_n\} \) and \( \{q_1, \ldots, q_n\} \) be orthonormal bases in \( \mathbb{C}^n. \) Let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) be increasing sequences of positive numbers. Assume that there is \( c > 1 \)

\footnote{It will be interesting to have a pure geometric proof of this inequality and for (4) with best possible constants.}
such that

\[
    c^{-1} \leq \frac{\sum_{j=1}^{n} a_j |\langle X, p_j \rangle|}{\sum_{j=1}^{n} b_j |\langle X, q_j \rangle|} \leq c
\]

for any \( X \in (\mathbb{C}^n)^* \).

Then

\[
    c'^{-1} \leq \frac{a_j}{b_j} \leq c', \quad j = 1, \ldots, n,
\]

where \( c' = n!c \).

For this, observe that expanding the determinant of the matrix of the unitary transformation between the bases, it follows that

\[
    \prod_{j=1}^{n} |\langle p_j, q_{\sigma(j)} \rangle| \geq \frac{1}{n!}
\]

for some permutation \( \sigma \) of \( \{1, \ldots, n\} \). In particular, \( |\langle p_j, q_{\sigma(j)} \rangle| \geq 1/n! \).

Then the given condition implies that

\[
    c'^{-1} \leq \frac{a_j}{b_{\sigma(j)}} \leq c'.
\]

Assume now that \( a_k > c'b_k \) for some \( k \). Using the monotonicity and the inequality \( a_j \leq c'b_{\sigma(j)} \), it follows that \( \sigma(j) > k \) for any \( j \geq k \). Since \( \sigma \) is a permutation of \( \{1, \ldots, n\} \), we get a contradiction.

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