Transitive and Co–Transitive caps

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1 Introduction

Let $PG(r,q)$ be the projective space of dimension $r$ over $GF(q)$. A $k$–cap $\bar{K}$ in $PG(r,q)$ is a set of $k$ points, no three of which are collinear, and a $k$–cap is said to be complete if it is maximal with respect to set–theoretic inclusion. The maximum value of $k$ for which there is known to exist a $k$–cap in $PG(r,q)$ is denoted by $m_2(r,q)$. Some known bounds for $m_2(r,q)$ are given below.

Suppose that $\bar{K}$ is a cap in $PG(r,q)$ with automorphism group $\bar{G}_0 \leq P\Gamma L(r+1,q)$. Then $\bar{K}$ is said to be transitive if $\bar{G}_0$ acts transitively on $\bar{K}$, and co-transitive if $\bar{G}_0$ acts transitively on $PG(r,q) - \bar{K}$.

Our main result is the following theorem.

Theorem 1 Suppose $\bar{K}$ is a transitive, co–transitive cap in $PG(r,q)$. Then one of the following occurs:

1. $\bar{K}$ is an elliptic quadric in $PG(3,q)$ and $q$ is a square when $q$ is odd;
2. $\bar{K}$ is the Suzuki–Tits ovoid in $PG(3,q)$ and $q = 2^h$, with $h$ odd and $\geq 3$;
3. $\bar{K}$ is a hyperoval in $PG(2,4)$;
4. $\bar{K}$ is an 11–cap in $PG(4,3)$ and $\bar{G}_0 \simeq M_{11}$;
5. $\bar{K}$ is the complement of a hyperplane in $PG(r,2)$;
6. $\bar{K}$ is a union of Singer orbits in $PG(r,q)$ and $G_0 \leq \Gamma L(1,p^d) \leq GL(d,p)$.

In each of 1–5 $\bar{K}$ is indeed a transitive co–transitive cap.

Our conclusion is that transitive, co–transitive caps are rare with the possible exception of unions of Singer cyclic orbits.
The origin of this problem are papers by Hill [8], [7], in which he studies such caps whose automorphism group acts 2–transitively on the cap. [As he notes [8, Theorem 1], it is trivial to show that if $\bar{K}$ is a subset of $PG(r, q)$ lying in no proper subspace and admitting a 3–transitive group then $\bar{K}$ must be a cap.] Hill gives a short list of possibilities (omitting Suzuki–Tits ovoids) but excludes caps in $PG(r, q)$ for $q > 2$ and $r \geq 13$. We find no new caps but show that any other transitive, co–transitive cap is a union of Singer cyclic orbits.

The known upper bounds on cap sizes are summarised in the following Result.

**Result 2** [10, Theorem 27.3.1]

- $m_2(2, q) = q + 1$ (for $q$ odd);
- $m_2(2, q) = q + 2$ (for $q$ even);
- $m_2(3, q) = q^2 + 1$ for $q > 2$;
- $m_2(r, 2) = 2^r$; and
- $m_2(r, q) \leq q^r - 1$ for $q > 2$ and $r \geq 4$.

We begin by showing that as a consequence of Result 2, a cap must be smaller than its complement (with one exception). It then follows that in considering subgroups of $P\Gamma L(r + 1, q)$ having two orbits, we need only consider the smaller orbit when looking for transitive, co-transitive caps.

**Lemma 3** Suppose that $\bar{K}$ is a cap in $PG(r, q)$. Then either $|\bar{K}| < (q^{r+1} - 1)/2(q - 1)$, or $q = 2$ and $\bar{K}$ is the complement of a hyperplane.

**Proof.** It is easy to deduce from Result 2, that the result holds when $q \neq 2$. Thus suppose now that $q = 2$ and that $|\bar{K}| \geq (2^{r+1} - 1)/2$. The only possibility is that $|\bar{K}| = 2^2$. Let $x \in \bar{K}$. For each $y \in \bar{K} - \{x\}$ there is a line through $x$ and $y$ and the $2^r - 1$ such lines must be distinct since $\bar{K}$ is a cap. However $x$ lies on exactly $2^r - 1$ lines in $PG(r, 2)$ and so every line in $PG(r, 2)$ through $x$ meets $\bar{K}$ in two points and $PG(r, q) - \bar{K}$ in one point. Therefore any line meeting $PG(r, q) - \bar{K}$ in at least two points is contained in $PG(r, q) - \bar{K}$. This shows that $PG(r, q) - \bar{K}$ is a subspace of $PG(r, 2)$ and its size shows that it is a hyperplane.

Using Result 2 we shall know orbit lengths when looking at candidates for transitive, co-transitive caps. Lemma 3 below helps in eliminating a number of possibilities.

**Definition 4** Suppose that $\bar{K}$ is a cap in $PG(r, q)$. For any $x \in PG(r, q)$, the chord-number of $x$ is the number of chords (2-secants) of $\bar{K}$ passing through $x$. 


Lemma 5 Suppose that \( \bar{K} \) is a transitive, co-transitive cap in \( PG(r, q) \) and suppose that \( x \in PG(r, q) - \bar{K} \). Let \( k = |\bar{K}| \) and \( m = |PG(r, q) - \bar{K}| \). Then the chord-number, \( c \), of \( x \) is given by
\[
c = \frac{k(k-1)(q-1)}{2m}.
\]

In particular the expression for \( c \) always yields an integer.

Proof. We count combinations of chords and points of \( PG(r, q) - \bar{K} \) in two ways. Firstly there are \( k(k-1)/2 \) chords of \( \bar{K} \) and each has \( q-1 \) points not in \( \bar{K} \). There is a subgroup \( G_0 \) of \( \Gamma L(r+1, q) \) acting transitively on \( PG(r, q) - \bar{K} \), so each of these \( m \) points has the same chord-number \( c \) and a second count gives \( mc \) chord-point combinations. Thus \( mc = k(k-1)(q-1)/2 \) leading to the required expression for \( c \).

The main tool in our investigation is the substantial result by M.W. Liebeck [12], where the affine permutation groups of rank three are classified.

Result 6 [12] Let \( G \) be a finite primitive affine permutation group of rank three and of degree \( n = p^d \), with socle \( V \), where \( V \simeq (\mathbb{Z}_p^d) \) for some prime \( p \), and let \( G_0 \) be the stabilizer of the zero vector in \( V \). Then \( G_0 \) belongs to one of the following families:

(A) 11 Infinite classes;

(B) Extraspecial classes with \( G_0 \leq N_{\Gamma L(d,p)}(R) \), where \( R \) is a 2–group or 3–group irreducible on \( V \);

(C) Exceptional classes. Here the socle \( L \) of \( G_0/Z(G_0) \) is simple (where \( Z(G_0) \) denotes the centre of \( G_0 \)).

We shall recall the details of the groups belonging to the classes in (A), (B) and (C) as we need them.

Suppose \( \bar{K} \) is a cap in \( PG(r, q) \) such that a subgroup \( \bar{G}_0 \) of \( \Gamma L(r+1, q) \) acts transitively on each of \( \bar{K} \) and its complement. Then \( \bar{G}_0 \) corresponds to a subgroup \( G_0 \) of \( GL(d, p) \) having three orbits on the vectors of \( V(d, p) \), where \( p \) is prime and \( p^d = q^{r+1} \). Moreover \( G_0 \) will contain matrices corresponding to scalar multiplication by elements of \( GF(q)^* \). As we demonstrate shortly, with one exception, \( V(d, p) \cdot G_0 \) is primitive as a permutation group, so Liebeck’s theorem may be applied. Notice that since we are interested in groups \( G_0 \) containing \( GF(q)^* \) we avoid the possibility of two orbits of vectors in \( V(d, p) \) giving rise to a single orbit of points in \( PG(r, q) \).

Clearly \( G_0 \) may be embedded in \( \Gamma L(r+1, q) \). At the beginning of Section 1 of [12], Liebeck notes that in his result \( G_0 \leq GL(d, p) \) is embedded in \( \Gamma L(a, p^{d/a}) \) with \( a \) minimal. Thus \( r + 1 \geq a \) i.e. \( q \leq p^{d/a} \). Moreover in
almost all cases it is clear that the groups he identifies have orbits that are unions of 1-dimensional subspaces of $V(a, p^{d/a})$ (excluding the zero vector). If a 1-dimensional subspace over $GF(p^{d/a})$ does contain vectors $u, v$ that are linearly independent over $GF(q)$, then $u, v$ and $u + v$ correspond to three collinear points in $PG(r, q)$ and the orbit in $PG(r, q)$ cannot be a cap. Thus in our setting we usually have $q = p^{d/a}$: there is just one exception, the class A1, although we have to justify $q = p^{d/a}$ for the class A2.

Lemma 7 Suppose $\bar{K}$ is a transitive, co-transitive cap in $PG(r, q)$ with $G_0 \leq P\Gamma L(r+1, q)$ acting transitively on each of $\bar{K}$ and $PG(r, q) - \bar{K}$ and suppose that $G_0$ is the pre-image of $G_0$ in $GL(d, p)$. Let $H = V(d, p) \cdot G_0$. Then $H$ is imprimitive on $V = V(d, p)$ if and only if $q = 2$ and $\bar{K}$ is the complement of a hyperplane.

Proof. Suppose that $H$ is imprimitive on $V$. Let $\Omega$ be a block containing 0. Then the two orbits of non-zero vectors of $G_0$ are $\Omega \setminus 0$ and $V \setminus \Omega$. Let $u$ and $v$ be any two vectors in $\Omega$, then $\Omega + v$ is a block containing $0 + v$ and $u + v$ so $\Omega + v = \Omega$. In other words $u + v$ is in $\Omega$ and so $\Omega$ is a $GF(p)$-subspace of $V$. More than this $G_0$ contains the scalars in $GF(q)^*$ and so $\Omega$ is actually a $GF(q)$-subspace. Thus $\Omega$ cannot correspond to a cap. In $PG(r, q)$ our two orbits consist of points in a subspace and the complement. A line not in the subspace meets the subspace in at most one point so the complement cannot form a cap except perhaps when $p = q = 2$ and the subspace has projective dimension $r - 1$. Conversely, as is well known, the complement of a hyperplane is indeed a cap in $PG(r, 2)$ and it is the only way in which the complement of a subspace is a cap. It is easy to see that this cap is transitive and co-transitive. \(\square\)

We recall for the reader that the socle of a finite group is the product of its minimal normal subgroups. In our setting $V(d, p) \cdot G_0$ has $V$ as its unique minimal normal subgroup.

Liebeck’s theorem tells us the possibilities for $G_0$ and gives two orbits of $G_0$ on the non-zero vectors of $V(d, p)$. We denote these by $K_1$ and $K_2$, and the corresponding sets of points in $PG(r, q)$ by $\bar{K}_1$ and $\bar{K}_2$. We assume that neither $K_1$ nor $K_2$ lies in a subspace of $V(r + 1, q)$; given $GF(q)^* \leq G_0$ this means that neither $K_1$ nor $K_2$ lies in a subspace of $V(d, p)$. We may henceforth assume that $V(d, p) \cdot G_0$ is a finite primitive affine permutation group of rank 3 and degree $p^d$, so we may apply Result 6.

We begin with the case by case analysis. In many cases we use data from Result 6 and apply Lemmas 2, 5, but there are occasions when we need to look at the structure of orbits in detail; there are also occasions when using the structure of the orbits is more illuminating and yet no less efficient than the bound and chord-number arguments.
2 The infinite classes A

2.1 The class A1

In this case $G_0$ is a subgroup of $\Gamma L(1, p^d)$ containing $GF(q)^*$. Such a subgroup is generated by $\omega^N$ and $\omega^e\alpha^s$, for some $N, e, s$ where $\omega$ is a primitive element of $GF(p^d)$ and $\alpha$ is the generating automorphism $x \mapsto x^p$ of $GF(p^d)$; if we write $p^d = q^a$, then $N$ divides $(q^a - 1)/(q - 1)$. Let $H_0$ be the subgroup of $\Gamma L(1, p^d)$ generated by $\omega^N$. Then $H_0$ is a Singer subgroup of $GL(1, p^d)$ and the orbits of $H_0$ in $PG(r, q)$ are called Singer orbits. Clearly if $G_0$ has two orbits in $PG(r, q)$, then each orbit is the union of Singer orbits. If the smaller orbit is to be a cap, then each Singer orbit must itself be a cap. A precise criterion for deciding when Singer orbits are caps in $PG(r, q)$ is given by Szönyi [14, Proposition 1].

Precise criteria for there to be two orbits for $G_0$ on non-zero vectors of $V(d, p)$ are given by Foulser and Kallaher [5]. These involve numbers $m_1$ and $v$ such that the primes of $m_1$ divide $p^s - 1$, $v$ is a prime $\neq 2$ and $\text{ord}_v p^{sm_1} = v - 1$, $(e, m_1) = 1$, $m_1(s(v - 1)/d, N = vm_1$. The orbit lengths are $m_1(p^d - 1)/N$ and $(v - 1)m_1(p^d - 1)/N$. Notice that when $p = 2$ the smaller orbit has odd size. Hill [3] suggests the possibility of transitive, co–transitive caps of size 78 in $PG(5, 4)$ and 430 in $PG(6, 4)$. It is now clear that these cannot be caps from class A1 and our main theorem then shows that they cannot be caps at all.

2.2 The class A2

$G_0$ preserves a direct sum $V_1 \oplus V_2$, where $V_1, V_2$ are subspaces of $V(d, p)$. One orbit must be $K_1 = (V_1 \cup V_2) - \{0\}$ and the other $K_2 = \{v_1 + v_2 : 0 \neq v_1 \in V_1, 0 \neq v_2 \in V_2\}$. We first show that $V_1, V_2$ are subspaces over $GF(q)$. Observe that for any $\lambda \in GF(q)^* \leq G_0$, $\lambda V_1 = V_1$ or $V_2$ and let $F = \{\lambda \in GF(q)^* : \lambda V_1 = V_1\} \cap \{0\}$. Then $F$ is a subfield of $GF(q)$ having order greater than $q/2$ so must be $GF(q)$. It is now clear that $V_1, V_2$ are subspaces of $V(r + 1, q)$ of dimension $t = (r + 1)/2$. Given that $r \geq 2$, we must have $t \geq 2$, so $K_1$ contains lines of $PG(r, q)$ and cannot be a cap. Moreover $|K_1| = 2(q^t - 1)/(q - 1) < (q^{t+1} - 1)/2$ so $K_1$ is the smaller orbit and therefore $K_2$ cannot be a cap.

2.3 The class A3

$G_0$ preserves a tensor product $V_1 \otimes V_2$ over $GF(q)$, with $V_1$ having dimension 2 over $GF(q)$. One orbit must be $K_1 = \{v_1 \otimes v_2 : 0 \neq v_1 \in V_1, 0 \neq v_2 \in V_2\}$ and the other $K_2 = V - (K_1 \cup \{0\})$.

Consider the $GF(q)$–subspace $V_1 \otimes v_2$ of $V$ for some $0 \neq v_2 \in V_2$. It has dimension 2 in $V(r + 1, q)$ so corresponds to a line in $PG(r, q)$ inside $K_1$. 

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Hence $K_1$ is not a cap.

Let $b$ be the dimension of $V_2$ over $GF(q)$. Then $r + 1 = 2b$ and $|K_1| = (q+1)(q^b-1)/(q-1)$ (Table 12) so $|K_2| = q(q^b-1)(q^{b-1}-1)/(q-1) > |K_1|$ except when $q = 2, b = 2$ (i.e., $r + 1 = d = 4$). Thus there is only one case in which $K_2$ can possibly be a cap.

Suppose that $q = p = 2$ and $d = 4$, i.e. we are reduced to considering caps in $PG(3, 2)$. In $PG(3, 2)$, we see that $|K_1| = 9$ and $|K_2| = 6$. Thus here $K_1$ is too big and for $K_2$ it is simplest to note that $(6.5.1)/(2.9) \notin \mathbb{Z}$, so neither is a cap (by Lemmas 2 and 3).

### 2.4 The class A4

$G_0 \geq SL(a, s)$ and $p^d = s^{2a}$. Here $q = s^2$ and $p^d = q^a$ with $SL(a, s)$ embedded in $GL(d, p)$ as a subgroup of $SL(a, q)$: let $e_1, e_2, ..., e_a$ be a basis for $V$ over $GF(q)$ then with respect to this basis $SL(a, s)$ consists of the matrices in $SL(a, q)$ having every entry in $GF(s)$. If $G_0$ has two orbits on non-zero vectors of $V$ then those orbits must be $K_1 = \{ \gamma \sum \lambda_i e_i (\lambda_i \in GF(s), \text{not all } 0), 0 \neq \gamma \in GF(q) \}$ and $K_2$ the set of all remaining non-zero vectors. In other words $G_0$ preserves a subgeometry of $PG(r, q)$. We have $r > 1$ so that $a \geq 3$. Thus three collinear points of $PG(r, s)$ are still three collinear points in $PG(r, q)$ and so $K_1$ is not a cap.

Let us turn to $K_2$. As noted above, $r > 1$ so $a \geq 3$. Let $u = e_1 + \sigma e_2, v = e_2 + \sigma e_3$, where $\sigma \in GF(q) \setminus GF(s)$. Then $u, v$ and $u + v = e_1 + (\sigma + 1)e_2 + \sigma e_3 \in K_2$ correspond to collinear points of $PG(r, q)$, all in $K_2$. Hence $K_2$ is not a cap.

### 2.5 The class A5

$G_0 \geq SL(2, s)$ and $p^d = s^6$. Here $q = s^3$ and $p^d = q^2$ with $SL(2, s)$ embedded in $GL(d, p)$ as a subgroup of $SL(2, q)$. However $r = 1$ in this case so it does not concern us.

### 2.6 The class A6

$G_0 \geq SU(a, q')$ and $p^d = ((q')^2)^a$. In this case $q = (q')^2$. Here one orbit $K_1$ consists of the non-zero isotropic vectors and the other orbit $K_2$ consists of the non-isotropic vectors with respect to an appropriate non-degenerate hermitian form. Each orbit is a union of 1–dimensional subspaces of $V(a, q)$ (excluding the zero vector). To begin with, a non–isotropic line of $PG(r, q)$ contains at least three isotropic points, i.e., three points of $K_1$. Therefore $K_1$ cannot be a cap.

Now consider $K_2$. Given $a \geq 3$, consider a line of $PG(r, q)$ that is isotropic.
but not totally isotropic, then it contains one point of $\bar{K}_1$ and $q \geq 4$ points of $\bar{K}_2$. Hence $\bar{K}_2$ is not a cap.

### 2.7 The class A7

$G_0 \geq \Gamma^\pm(a, q)$ and $p^d = (q)^a$ with $a$ even (and if $q$ is odd, $G_0$ contains an automorphism interchanging the two orbits of $\Omega^\pm(a, q)$ on non-singular 1-spaces). The arguments here are similar to the Unitary case. $K_1$ consists of the non-zero singular vectors and $K_2$ consists of the non-singular vectors. Let $b$ be the Witt index of the appropriate quadratic form on $V(a, q)$ i.e., the dimension of a maximal totally singular subspace. Then $a$ is one of $2b, 2b + 2$. Any totally singular line would be a line of $PG(r, q)$ lying inside $\bar{K}_1$. Given that $a \geq 3$, it follows that the only possibility for $\bar{K}_1$ being a cap is when $\bar{K}_1$ is an elliptic quadric in $PG(3, q)$. In passing we note that for odd $q$, the necessary automorphism is contained in $G_0$ only when $q$ is square; in this case and in the case $q$ even, the elliptic quadric gives a well known cap.

Let us turn to $\bar{K}_2$. Any anisotropic line of $PG(r, q)$ lies inside $\bar{K}_2$ so $\bar{K}_2$ can never be a cap.

### 2.8 The class A8

$G_0 \geq SL(5, q)$ and $p^d = (q)^a$ (from the action of $SL(5, q)$ on the skew square $\bigwedge^2 V(5, q)$). Here one orbit of non-zero vectors must be $\bar{K}_1 = \{0 \neq u \bigwedge v : u, v \in V(5, q)\}$ with the other non-zero vectors belonging to $K_2$. One can argue in a similar manner to the case of the tensor product. However it is quicker here to note that the orbits of $G_0$ on $PG(r, q)$ have sizes $k = (q^5 - 1)(q^2 + 1)/(q - 1)$ and $m = q^2(q^5 - 1)(q^3 - 1)/(q - 1)$ ([12, Table12]) with $k < m$ for all values of $q$. The chord-number is then given by $c = k(k - 1)(q - 1)/2m$ by Lemma [3] i.e., $c = (q^2 + 1)(q^3 + q + 1)/2q \notin \mathbb{Z}$. Hence neither $\bar{K}_1$ nor $\bar{K}_2$ is a cap.

### 2.9 The class A9

$G_0/Z(G_0) \geq \Gamma(7, q) \cdot Z(2, q - 1)$ and $p^d = q^5$ (from the action of $B_3(q)$ on a spin module) [4, 23]. The study of Clifford algebras leads to the construction of “spin modules” for $P\Gamma(m, q)$. When $m = 8$ this leads to the triality automorphism of $P\Omega^+(8, q)$. One finds that it is possible (via this automorphism) to embed $\Omega(7, q) \cong P\Omega(7, q)$ inside $P\Omega^+(8, q)$ as an irreducible subgroup. The important thing from our point of view is that two non–trivial orbits of $G_0$ must be the set of all non–zero singular vectors and the set of all non–singular vectors with respect to a non–degenerate quadratic form on $V(8, q)$. In this setting the arguments employed for class A7 apply: neither orbit can be a
cap.

2.10 The class A10

\( G_0/Z(G_0) \simeq P\Omega^+(10, q) \) and \( p^d = q^{16} \) (from the action of \( D_5(q) \) on a spin module) \([12], [14]\). Once again we have a spin representation, this time of \( P\Omega^+(10, q) \) on \( PG(15, q) \). On this occasion it is quickest to work from the orbit lengths.

The orbits of \( G_0 \) on \( PG(r, q) \) have sizes \( k = (q^6 - 1)(q^3 + 1)/(q - 1) \) and \( m = q^3(q^5 - 1)(q^5 - 1)/(q - 1) \) \([12] \text{ Table12}\) with \( k < m \) for all values of \( q \).

The chord-number is then given by \( c = k(k - 1)(q - 1)/2m \) by Lemma 3 i.e., \( c = (q^3 + 1)(q^5 + q^2 + 1)/2q^3 \notin \mathbb{Z} \). Hence neither \( K_1 \) nor \( K_2 \) is a cap.

2.11 The class A11

\( G_0 \simeq Sz(q) \) and \( p^d = (q)^4 \), with \( q \geq 8 \) an odd power of \( 2 \) (from the embedding \( Sz(q) \leq Sp(4, q) \)). Here the smaller orbit \( K_1 \) on \( PG(3, q) \) is a Suzuki–Tits ovoid containing \( q^2 + 1 \) points and this is indeed a cap \([13], [16] \text{ 16.4.5}\].

3 The Extraspecial classes

In most cases here \( G_0 \leq M \) where \( M \) is the normalizer in \( \Gamma L(a, q) \) of a 2–group \( R \), where \( p^d = (q)^a \) and \( a = 2m \) for some \( m \geq 1 \); either \( R \) is an extraspecial group \( 2^{1+2m} \) or \( R \) is isomorphic to \( Z_4 \circ 2^{1+2m} \). In all cases here \( p \) is odd. There are two types of extraspecial group \( 2^{1+2m} \), denoted \( R_m^1 \) and \( R_m^2 \); the first of these has the structure \( D_8 \circ D_8 \circ \ldots \circ D_8 \) \( (m \text{ copies}) \) and the second \( D_8 \circ \ldots \circ D_8 \circ Q_8 \) \( (m - 1 \text{ copies of } D_8) \), where \( D_8 \) and \( Q_8 \) are respectively the dihedral and quaternion groups of order 8, and ’o’ indicates a central product. The group \( Z_4 \circ 2^{1+2m} \) is again a central product, this time \( Z_4 \circ D_8 \circ D_8 \circ \ldots \circ D_8 \) \( (m \text{ copies of } D_8) \) and is denoted by \( R_m^3 \). Notice that \( R \) modulo its centre is an elementary abelian 2–group, i.e. a \( 2m \)–dimensional vector space over \( GF(2) \) and in fact \( M/RZ \) \( (Z \text{ being the centre of } \Gamma L(a, q)) \) may be embedded in \( GSp(2m, 2) \). In just one case \( G_0 \leq M \) with \( M \) the normalizer in \( \Gamma L(3, 4) \) of a 3–group of order 27. We record from \([12] \text{ Table 13}\) that in this case the non–trivial orbit sizes of \( G_0 \) on \( V(3, 4) \) are 27 and 36, i.e. the point orbit sizes in \( PG(2, 4) \) are 9 and 12, but the largest possible size of a cap (here better termed an arc) in \( PG(2, 4) \) is 6. Hence there are no caps here and we may henceforth assume that \( R \) is a 2–group, with \( p \) odd.

There are sixteen instances where \( G_0 \) has two non–trivial orbits on \( V(d, p) \simeq V(a, q) \), but ten of these have \( a = 2 \) \( (\text{i.e. } m = 1) \) and so refer to action on a projective line, i.e. \( r < 2 \); note that two of these cases have \( q > p \). Thus we concentrate on the remaining six cases. In each of these cases \( q = p \) and in
all but the last case the vector space is $V(4, p)$. In the last case the vector space is $V(8, 3)$. Four cases follow immediately from known bounds - they are listed in the table below.

| $p=q$ | $r$ | $R$ | smaller orbit size | max. cap size |
|-------|-----|-----|-------------------|---------------|
| 3     | 3   | $R_1^2$ | 16               | 10            |
| 5     | 3   | $R_2^2$ | 60               | 26            |
| 5     | 3   | $R_3^2$ | 60               | 26            |
| 7     | 3   | $R_2^3$ | 80               | 50            |

**The case $p = q = 3$, $r = 7$, $R = R_3^3$.**

In this case smaller orbit of $\bar{G}_0$ on $PG(7, 3)$ has size 720, while the maximum size for a cap in $PG(7, 3)$ is only known to be $\leq 729$. Instead we use Lemma 3: the larger orbit has size 2560 and $\frac{720}{2560} \notin \mathbb{Z}$.

**The case $p = q = 3$, $r = 3$, $R = R_2^3$.**

Here Liebeck notes that $R$ has five orbits of size 16 on $V(4, 3)$ and $M$ permutes these orbits acting as $S_5$, the symmetric group of degree 5. Thus there are a number of possibilities for $G_0$ having two non-trivial orbits on $V(4, 3)$. However it is straightforward to construct generating matrices for $R$ and we see immediately that one orbit of size 16 on $V(4, 3)$ cannot correspond to a cap in $PG(3, 3)$. Therefore none of the orbits of size 16 can correspond to a cap and hence no possible choices of $G_0$ can give rise to a cap.

### 4 The Exceptional classes

Finally we turn to the exceptional classes where the socle $L$ of $G_0/Z(G_0)$ is simple. There are just thirteen different possibilities for $L$, although on occasion more than one possibility for $G_0$ corresponds to a given $L$. For example for $L = A_5$ there are seven different possibilities for $G_0$ (one of which leads to a single orbit in $PG(d - 1, p)$); however all of these lead to $r < 2$ and so do not concern us.

We employ a variety of techniques to tackle these cases. Liebeck [12, Table 14] gives the orbit sizes in $V(d, p)$ and sometimes we can use these to rule out the possibility of caps. On other occasions we can use the fact that the chord-number is an integer. On two occasions, neither of these approaches works and we have to investigate the known structure of the smaller orbit. There remain two cases where a cap does occur.

**The cases where caps occur.**
When $L = A_6$ and $(d, p) = (6, 2)$, $L$ admits an embedding in $PSL(3, 4)$ (so here $q = 4$) and $G_0$ has an orbit of size 6. In fact this in a hyperoval in $PG(2, 4)$ [2, 3] so we do have a cap.

When $L = M_{11}$ and $(d, p) = (5, 3)$ there is a representation of $L$ in which one orbit has size 11 and in fact this is a cap. In passing we note that this cap arises as an orbit of a Singer cyclic subgroup of $PG(4, 3)$ [4]; moreover $PG(4, 3)$ is partitioned into eleven 11–caps (the eleven orbits of the Singer cyclic subgroup). Note also that there is a second representation of $L = M_{11}$ on $PG(4, 3)$ (see below). In fact both representations appear in the context of the ternary Golay code [1, Ch. 6].

Cases where known bounds rule out caps.
In each of the following cases the smaller orbit is larger than the known upper bound for a cap size, so cannot be a cap. In the table $k$ is the smaller orbit size.

| $L$    | $(d, p)$ | $r$ | $q$ | $k$  | Max. cap size |
|--------|-----------|-----|-----|------|---------------|
| $A_6$  | (4, 5)    | 3   | 5   | 36   | 26            |
| $A_7$  | (4, 7)    | 3   | 7   | 120  | 50            |
| $M_{11}$ | (5, 3)   | 4   | 3   | 55   | $\leq 27$    |
| $J_2$  | (6, 5)    | 5   | 5   | 1890 | $\leq 625$   |
| $J_2$  | (12, 2)   | 5   | 4   | 525  | $\leq 256$   |

Cases where $c$ an integer rules out caps.
In each of the following cases a calculation $c = k(k - 1)(q - 1)/2m$ yields a non-integer and so by Lemma 3, the smaller orbit does not correspond to a cap. In the table $k$ is the smaller orbit size and $m$ the larger orbit size.

| $L$    | $(d, p)$ | $r$ | $q$ | $k$  | $m$  |
|--------|-----------|-----|-----|------|------|
| $A_9$  | (8, 2)    | 7   | 2   | 120  | 135  |
| $A_{10}$ | (8, 2)   | 7   | 2   | 45   | 210  |
| $L_{2}(17)$ | (8, 2) | 7   | 2   | 102  | 153  |
| $M_{22}$ | (11, 2)  | 10  | 2   | 276  | 1771 |
| $M_{24}$ | (11, 2)  | 10  | 2   | 759  | 1288 |
| Suz or $J_4$ | (12, 3) | 11  | 2   | 65520| 465920|

The case $L = A_7$ and $(d, p) = (8, 2)$.
Here $L$ is embedded in $PSL(4, 4)$ (so $q = 4$). In fact $L$ may actually be embedded in $A_8 \simeq PSL(4, 2) \leq PSL(4, 4)$. The group $A_8$ and therefore $A_7$
preserve a subgeometry whose 15 points form the smaller orbit. There are numerous examples of three points on a line in the subgeometry. Thus we have no caps.

**The case $L = \text{PSU}(4, 2)$ and $(d, p) = (4, 7)$.

The vectors in the smaller orbit are given by Liebeck [12, Lemma 3.4]:

$$(\theta; 0, 0, 0), \quad (0; \theta, 0, 0), \quad (0; \omega^a, \omega^b, \omega^c), \quad (\omega^a; 0, \omega^b, -\omega^c),$$

(together with all scalar multiples) where $\theta = \omega = 2; a, b, c$ take any integral values; and the last three coordinates may be permuted cyclically. It suffices here to observe that $(1; 0, 0, 0), (1; 0, 1, 6)$ and $(2; 0, 1, 6)$ all lie in this orbit and give three collinear points in $PG(3, 7)$. So no cap arises here.

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