Bipartite entanglement and hypergraph states

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Abstract We investigate some properties of multipartite entanglement of hypergraph states in purely hypergraph theoretical terms. We first introduce an approach for computing the concurrence between two specific qubits of a hypergraph state by using the so-called Hamming weights of several special subhypergraphs of the corresponding hypergraph. Then, we quantify and characterize bipartite entanglement between each qubit pair of several special hypergraph states in terms of the concurrence obtained by using the above approach. Our main result includes that a graph $g$ has a component with the vertex set $\{i, j\}$ if and only if the qubit pair labeled by $\{i, j\}$ of the graph state $|g\rangle$ is entangled.

Keywords Hypergraph states · Bipartite entanglement · Concurrence

1 Introduction

Entanglement is one of the most extraordinary features of quantum theory. It lies at the very heart of quantum information theory [2] and is now regarded as a key physical resource in realizing many quantum information tasks. While the bipartite
entanglement is well understood, the ultimate goal to cope with the properties of multipartite entanglement [1] of arbitrary multipartite states is far from being reached. Therefore, several special classes of entangled states have been introduced and identified to be useful for certain tasks. It is well known that graph states [3–6] are an example of these classes. Any graph state can be constructed on the basis of a (simple and undirected) graph. Although graph states can describe a large family of entangled states including cluster states [7], GHZ states, stabilizer states [8,9], it is clear that they cannot represent all entangled states. To go beyond graph states and still keep the appealing connection to graphs, Ref. [10] introduces an axiomatic framework for mapping graphs to quantum states of a suitable physical system and extends this framework to directed graphs and weighted graphs. Several classes of multipartite entangled states, such as qudit graph states [11], Gaussian cluster states [12], projected entangled pair states [13–15], and quantum random networks [16], emerge from the axiomatic framework. Moreover, we generalize the above axiomatic framework to encoding hypergraphs into so-called quantum hypergraph states [17,18]. In [19], we also present an approach for mapping weighted hypergraphs into (up to local unitary transformations) locally maximally entanglable states [20].

The main aim of this work is to investigate some properties of multipartite entanglement of hypergraph states in purely hypergraph theoretical terms. Several literatures have shown some approaches for this issue. For graph states, Ref. [3] presents various upper and lower bounds to the Schmidt measure [21] in graph theoretical terms. For hypergraph states, similar work is done in [17,18]. Moreover, Ref. [17,18] qualitatively studies the entanglement structure of hypergraph states in purely hypergraph theoretical terms. Reference [22] introduces an approach for computing local entropic measure on qubit $t$ of a hypergraph state by using the Hamming weight of the so-called $t$-adjacent subhypergraph. Moreover, Ref. [23] shows that some properties of the entanglement of four-qubit hypergraph states. In this paper, we will use the concurrence [24] to quantify and characterize the bipartite entanglement between two specific qubits $\{i, j\}$ of a hypergraph state $|g\rangle$ in purely hypergraph theoretical terms. For this, we will present an approach for computing the concurrence between the qubit pair $\{i, j\}$ of the state $|g\rangle$ by using the so-called Hamming weights [22] of several special subhypergraphs of the corresponding hypergraph $g$. Then, we will investigate some properties of the entanglement of several special hypergraph states in terms of the concurrence obtained by using the above approach. We will give a sufficiency and necessary condition of two qubits of a graph state being entangled in purely graph theoretical terms.

This paper is organized as follows. In Sect. 2, we recall notations of hypergraphs, hypergraph states, etc. In Sect. 3, we present an approach for computing the concurrence between two specific qubits of a hypergraph state by means of the Hamming weights of some special subhypergraphs. In Sect. 4, we investigate some properties of the entanglement of several special hypergraph states by means of the concurrence. Section 5 contains our conclusions.
2 Preliminaries

Formally, a hypergraph is a pair \((V, E)\), where \(V\) is the set of vertices, \(E \subseteq \wp(V)\) is the set of hyperedges and \(\wp(S)\) denotes the power set of the set \(S\). The set of all hypergraphs of \(n\) vertices is denoted by \(\Theta_n\). The empty hypergraph is defined as \((V, \emptyset)\). If a hypergraph only contains the empty hyperedge \(\emptyset\) or one-vertex hyperedges (called loops), it is trivial. The rank of a hypergraph \(g\), denoted by \(\text{ran}(g)\), is the maximum cardinality of a hyperedge in \(g\). Moreover, a hypergraph can be depicted by the visual form as shown in Fig. 1. Each vertex is represented as a dot, while each hyperedge is represented as a closed curve which encloses the dots corresponding to vertices incident with the hyperedge.

A hypergraph \((V', E')\) is called a subhypergraph of \((V, E)\) if \(V' \subseteq V\) and \(E' \subseteq E\). Let \(g = (V, E)\) be a hypergraph. For a vertex \(t \in V\), we define the \(t\)-adjacent subhypergraph \(g_t\) of \(g\) as \(g_t = (V_t, E_t)\) where \(V_t = V - \{t\}\) and \(E_t = \{e - \{t\}\} | t \in e, e \in E\). For any two different vertices \(i, j \in V\), the \((i, j)\)-adjacent subhypergraph \(g_{\{i,j\}}\) and the \((i, j)\)-adjacent subhypergraph \(g_{(i,j)}\) of \(g\) are respectively defined as follows: \(g_{\{i,j\}} = (V_{\{i,j\}}, E_{\{i,j\}})\) where \(V_{\{i,j\}} = V - \{i, j\}\) and \(E_{\{i,j\}} = \{e - \{i, j\}\} | i, j \in e, e \in E\); and \(g_{(i,j)} = (V_{(i,j)}, E_{(i,j)})\) where \(V_{(i,j)} = V - \{i, j\}\) and \(E_{(i,j)} = \{e - \{i\}| i \in e, j \notin e, e \in E\}\).

The vertices incident with the same hyperedge are referred to as being adjacent (neighbor). A sequence of vertices \(v_1, v_2, \ldots, v_p\) such that \(v_k\) and \(v_{k+1}\) are adjacent for all \(1 \leq k \leq p - 1\) is called a path joining \(v_1\) to \(v_p\). A hypergraph is connected if any two vertices are joined by a path. Otherwise, it is disconnected. A component of a hypergraph \(g\) is a connected subhypergraph contained in no other connected subhypergraph. Moreover, we define the sum of \(g = (V, E)\) and \(g' = (V', E')\) as \(g \Delta g' = (V \cup V', E \Delta E')\) where \(E \Delta E'\) denotes the symmetric difference of \(E\) and \(E'\), that is, \(E \Delta E' = E \cup E' - E \cap E'\).
Denote the Pauli matrices by
\[
I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\] (1)

Let \( Z_k \) be the \( 2^k \times 2^k \) diagonal matrix which satisfies
\[
(Z_k)_{jj} = \begin{cases} 
-1 & j = 2^k \\
1 & \text{otherwise}
\end{cases}
\] (2)

where \( k \) is a nonnegative integer. Suppose that \( V = [n] \equiv \{1, 2, \ldots, n\} \) and \( e \subseteq V \). Then, the \( n \)-qubit hyperedge gate \( Z_e \) is defined as \( Z_e \otimes I^{\otimes n-|e|} \) which means that \( Z_e \) acts on the qubits in \( e \), while the identity \( I \) acts on the rest. An \( n \)-qubit hypergraph state \( |g\rangle \) can be constructed by \( g = (V, E) \) as follows. Each vertex labels a qubit (associated with a Hilbert space \( \mathbb{C}^2 \) initialized in \( |\phi\rangle = |+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \)). The state \( |g\rangle \) is obtained from the initial state \(|+\rangle^{\otimes n}\) by applying the hyperedge gate \( Z_e \) for each hyperedge \( e \in E \), that is,
\[
|g\rangle = \prod_{e \in E} Z_e |+\rangle^{\otimes n}.
\] (3)

Thus, hypergraph states of \( n \) qubits are corresponding to \((\mathbb{C}^2, |+\rangle, \{Z_k |0 \leq k \leq n\})\) by the axiomatic approach shown in [17,18], while graph states are related with \((\mathbb{C}^2, |+\rangle, Z_2)\) [10,17,18]. It is known that real equally weighted states [26] are equivalent to hypergraph states [17,18]. In fact, let \( V = [n] \) and define a mapping \( c \) on \( \wp (V) \) as
\[
\forall e \subseteq V, \ c (e) = \begin{cases} 
1 & e = \emptyset \\
\prod_{k \in e} x_k & e \neq \emptyset
\end{cases}.
\] (4)

where \( x_1, x_2, \ldots, x_n \) are \( n \) Boolean variables. Then, we can construct a \( 1 \rightarrow 1 \) mapping \( u \) between hypergraphs and Boolean functions which satisfies \( \forall g = (V, E), \)
\[
u(g)(x_1, x_2, \ldots, x_n) = \bigoplus_{e \in E} c(e).
\] (5)

where \( \bigoplus \) denotes the addition operator over \( Z_2 \). Thus we have
\[
|g\rangle = \prod_{e \in E} Z_e |+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{c(e)} |x\rangle \equiv |\psi_{u(g)}\rangle
\] (6)

where \( |\psi_{u(g)}\rangle \) is just the real equally weighted state associate with the Boolean function \( u(g) \). Moreover, it is clear that \( \forall g, g' \in \Theta_n, \)
\[
u(g \triangle u(g')) = u(g) \oplus u(g').
\] (7)
It is known that the Hamming weight of a Boolean function \( f \) is defined as \( |f^{-1}(1)| \) where \( |S| \) denotes the cardinality of the set \( S \). By (5), we also can define the Hamming weight of a hypergraph \( g \) with \( n \) vertices as

\[
hw(g) \equiv \left| f^{-1}(1) \right| \tag{8}
\]

where \( f(x_1, x_2, \ldots, x_n) = u(g)(x_1, x_2, \ldots, x_n) \). Reference [22] introduces an approach for calculating the Hamming weight of \( g \) in purely hypergraph theoretical terms.

3 Concurrence and hypergraph states

Concurrence is a famous bipartite entanglement measure. Let \(|\phi\rangle\) be a pure state of \( n \) qubits. The reduced density matrix \( \rho_{ij} \) on two different qubits \{\( i, j \)\} of \(|\phi\rangle\) is defined as \( \rho_{ij} \equiv \text{Tr}_{\text{all but } \{i, j\}}(|\phi\rangle\langle\phi|) \). One can evaluate the so-called spin-flipped operator defined as

\[
\tilde{\rho}_{ij} = (\sigma_y \otimes \sigma_y) \rho_{ij}^* (\sigma_y \otimes \sigma_y) \tag{9}
\]

where a star denotes a complex conjugation. Let \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) be eigenvalues of the matrix \( \rho_{ij} \tilde{\rho}_{ij} \) in decreasing order. The concurrence \( C_{ij} \) between two qubits \{\( i, j \)\} is defined as

\[
C_{ij} \equiv \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\} \tag{10}
\]

Moreover, it is known that \( \rho_{ij} \) is separable or disentangled if and only if \( C_{ij} = 0 \) [24].

Now let us show how to compute the concurrence between two specific qubits of an \( n \)-qubit hypergraph state. Let \( g = ([n], E) \) be a hypergraph. By (6), the reduced density matrix on two different qubits \{\( i, j \)\} of the corresponding hypergraph state \(|g\rangle\) can be written into

\[
\rho_{ij} = \text{Tr}_{\text{all but } \{i, j\}}(|g\rangle\langle g|) = [a_{rs}]_{4 \times 4} \tag{11}
\]

where for any \( r, s \in \{0, 1, 2, 3\} \)

\[
a_{rs} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{u(g)(r,y) \oplus u(g)(s,y)} \tag{12}
\]

and

\[
u(g)(z, y) \equiv u(g)(x_i, x_j, y) \equiv u(g)(x_1, x_2, \ldots, x_n). \tag{13}
\]

Note that \( z = x_i x_j \), that is,

\[
z = \begin{cases} 
0 & x_i = 0, x_j = 0 \\
1 & x_i = 0, x_j = 1 \\
2 & x_i = 1, x_j = 0 \\
3 & x_i = 1, x_j = 1 
\end{cases}
\]
It is similar for \( y \). Moreover, it is known that there are four \((n - 2)\)-valuable Boolean functions \( v, v', v'', \text{ and } w \) such that

\[
  u(g)(x_1, x_2, \ldots, x_n) = x_i x_j v(y) \oplus x_i v'(y) \oplus x_j v''(y) \oplus w(y).
\]  

(14)

Then, we can obtain that

\[
  \forall r \in \{0, 1, 2, 3\}, \quad a_{rr} = \frac{1}{4},
\]

\[
  a_{01} = a_{10} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v''(y)}, \quad a_{02} = a_{20} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v'(y)}
\]

\[
  a_{03} = a_{30} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v(y) \oplus v'(y) \oplus v''(y)}, \quad a_{12} = a_{21} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v'(y) \oplus v''(y)}
\]

\[
  a_{13} = a_{31} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v(y) \oplus v'(y)}, \quad a_{23} = a_{32} = \frac{1}{2^n} \sum_{y=0}^{2^{n-2}-1} (-1)^{v'(y) \oplus v''(y)}.
\]

(15)

By the definitions of the \((i, j)\)-adjacent and \((i, j)\)-adjacent subhypergraphs, (5) and (14), it is clear that

\[
  v(y) = u(g_{i,j}), \quad v'(y) = u(g_{i,j}), \quad \text{and} \quad v''(y) = u(g_{j,i}).
\]

(16)

From (7), (8), (15), and (16), we can obtain

\[
  a_{01} = a_{10} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j}) , \quad a_{02} = a_{20} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j}) ,
\]

\[
  a_{03} = a_{30} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j} \Delta g_{i,j} \Delta g_{j,i}) ,
\]

\[
  a_{12} = a_{21} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j} \Delta g_{j,i}) ,
\]

\[
  a_{13} = a_{31} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j} \Delta g_{i,j}) ,
\]

\[
  a_{23} = a_{32} = \frac{1}{4} - \frac{1}{2^{n-1}} h w(g_{i,j} \Delta g_{j,i}) .
\]

(17)

Thus, it is important for obtaining the reduced density matrix \( \rho_{i,j} \) to calculate the Hamming weights of the subhypergraphs which occur in (17). It is known that the Hamming weight of a hypergraph can be evaluated by using the approach in [22]. Thus, we can obtain the reduced density matrix \( \rho_{i,j} \) of \( |g\rangle \) in purely hypergraph theoretical terms.

In the following, we show how to compute the concurrence of the reduced density matrix \( \rho_{i,j} \) of the hypergraph state \( |g\rangle \). Since all elements of \( \rho_{i,j} \) in (11) are real,
the operator in (9) is equal to $\tilde{\rho}_{(i,j)} = (\sigma_y \otimes \sigma_y) \rho_{(i,j)} (\sigma_y \otimes \sigma_y)$. Clearly, $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ are corresponding to the squares of eigenvalues of the matrix $\rho_{(i,j)} (\sigma_y \otimes \sigma_y)$ in decreasing order since $\rho_{(i,j)} \tilde{\rho}_{(i,j)} = [\rho_{(i,j)} (\sigma_y \otimes \sigma_y)]^2$. Thus we can obtain the concurrence of $\rho_{(i,j)}$ according to (10).

4 Several special hypergraph states

In this section, we discuss some properties of the entanglement of the hypergraph states corresponding to several special hypergraphs by means of the concurrence. These hypergraphs include the hypergraph whose rank is equal to two, the hypergraph $g^* = ([n], \{[n]\})$, and so on.

4.1 The hypergraph whose rank equals to two

If a hypergraph $g$ is trivial, the concurrence between any two qubits of the hypergraph state $|g\rangle$ is zero since the state $|g\rangle$ is disentangled or fully separable [17,18]. In the following, we calculate the concurrence between two specific qubits of the hypergraph state corresponding to a hypergraph whose rank equals to two. For convenience, we first define a special function $\varepsilon : \Theta_n \to \{-1, 0, 1\}$ as

$$\forall g = (V, E) \in \Theta_n, \varepsilon (g) = \begin{cases} 1 & E = \emptyset \\ 0 & \text{ran} (g) \geq 1 \\ -1 & E = \{\emptyset\} \end{cases}. \quad (18)$$

Let $g = ([n], E)$ be a hypergraph and $\text{ran} (g) = 2$. Then, we can obtain the reduced density matrix $\rho_{(i,j)}$ on two different qubits $i, j \in [n]$ of the hypergraph state $|g\rangle$ as follows.

**Proposition 1**

(i) If $\varepsilon (g_{(i,j)}) = 1$, then

$$\rho_{(i,j)} = \frac{1}{4} \left[ I \otimes I + \varepsilon (g_{(i,j)}) \sigma_x \otimes I + \varepsilon (g_{(j,i)}) I \otimes \sigma_x + \varepsilon (g_{(i,j)}) \Delta g_{(j,i)} \sigma_x \otimes \sigma_x \right]. \quad (19)$$

(ii) If $\varepsilon (g_{(i,j)}) = -1$, then

$$\rho_{(i,j)} = \frac{1}{4} \left[ I \otimes I + \varepsilon (g_{(i,j)}) \sigma_x \otimes \sigma_z + \varepsilon (g_{(j,i)}) \sigma_z \otimes \sigma_x + \varepsilon (g_{(i,j)}) \Delta g_{(j,i)} \sigma_y \otimes \sigma_y \right]. \quad (20)$$

The above proposition can be described as follows.
Table 1  All possible values of the concurrence $C_{ij}$ between two specific qubits $i, j \in [n]$ of the hypergraph state $|g\rangle$ where $\text{ran}(g) = 2$ and $\varepsilon \{g_{(i,j)}\} = 1$

| No. | $\varepsilon \{g_{(i,j)}\}$ | $\varepsilon \{g_{(j,i)}\}$ | $\varepsilon \{g_{(i,j)} \Delta g_{(j,i)}\}$ | $\rho_{ij}$ | $C_{ij}$ |
|------|----------------|----------------|----------------|-------------|--------|
| 1    | 0              | 0              | 0              | $\frac{1}{4} (I \otimes I)$ | 0      |
| 2    | 0              | 0              | 1              | $\frac{1}{4} (I \otimes I + \sigma_x \otimes \sigma_x)$ | 0      |
| 3    | 0              | 0              | $-1$           | $\frac{1}{4} (I \otimes I - \sigma_x \otimes \sigma_x)$ | 0      |
| 4    | 0              | 1              | 0              | $\frac{1}{4} (I \otimes I + \sigma_x \otimes I)$ | 0      |
| 5    | 0              | $-1$           | 0              | $\frac{1}{4} (I \otimes I - \sigma_x \otimes I)$ | 0      |
| 6    | 1              | 0              | 0              | $\frac{1}{4} (I \otimes I + \sigma_x \otimes I)$ | 0      |
| 7    | $-1$           | 0              | 0              | $\frac{1}{4} (I \otimes I - \sigma_x \otimes I)$ | 0      |
| 8    | 1              | 1              | 1              | $\frac{1}{4} (I + \sigma_x) \otimes (I + \sigma_x)$ | 0      |
| 9    | 1              | $-1$           | $-1$           | $\frac{1}{4} (I + \sigma_x) \otimes (I - \sigma_x)$ | 0      |
| 10   | $-1$           | 1              | $-1$           | $\frac{1}{4} (I - \sigma_x) \otimes (I + \sigma_x)$ | 0      |
| 11   | $-1$           | $-1$           | 1              | $\frac{1}{4} (I - \sigma_x) \otimes (I - \sigma_x)$ | 0      |

Proposition 1' The reduced density matrix $\rho_{(i,j)}$ satisfies

$$
\rho_{(i,j)} = \frac{1}{4} \left\{ I \otimes I + \varepsilon \{g_{(i,j)}\} \sigma_x \otimes \sigma_z^{\delta_{s,t}(g_{(i,j)})} + \varepsilon \{g_{(j,i)}\} \delta_{s,t}(g_{(i,j)}) \otimes \sigma_x + \varepsilon \{g_{(i,j)} \Delta g_{(j,i)}\} \sigma_x \otimes \sigma_z^{\delta_{s,t}(g_{(i,j)})} \right\}. \quad (21)
$$

where $\delta_{s,t} = 1$ if $s = t$; otherwise, $\delta_{s,t} = 0$.

Proof It is easy to obtain the reduced density $\rho_{(i,j)}$ according to Sect. 3 and the properties of the Hamming weights of hypergraphs shown in [22]. For instance, we consider how to obtain $\rho_{(i,j)}$ of $|g\rangle$ when $g_{(i,j)} = ([n] - \{i, j\}, \mathcal{O})$, $\text{ran}(g_{(i,j)}) = 1$ and $g_{(j,i)} = ([n] - \{i, j\}, \mathcal{O})$. By (18), it is clear that $\varepsilon \{g_{(i,j)}\} = -1$, $\varepsilon \{g_{(j,i)}\} = 0$ and $\varepsilon \{g_{(i,j)} \Delta g_{(j,i)}\} = 1$. Since $\text{ran}(g_{(i,j)}) = 1$ and $g_{(j,i)} = ([n] - \{i, j\}, \mathcal{O})$, it is known that $\text{ran}(g_{(i,j)} \Delta g_{(j,i)}) = 1$. This implies that $\varepsilon \{g_{(i,j)} \Delta g_{(j,i)}\} = 0$ by (18). Similarly, we can obtain that $\text{ran}(g_{(i,j)} \Delta g_{(j,i)}) = 1$, $\varepsilon \{g_{(i,j)} \Delta g_{(j,i)}\} = -1$, and $\text{ran}(g_{(i,j)} \Delta g_{(j,i)} \Delta g_{(j,i)}) = 1$. According to the proposition 4 in [22] and (17), we can obtain

$$
\rho_{(i,j)} = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & -\frac{1}{4} \\
0 & 0 & -\frac{1}{4} & \frac{1}{4}
\end{bmatrix} = \frac{1}{4} (I \otimes I + \sigma_z \otimes \sigma_x) \quad (22)
$$

which is just the reduced density matrix corresponding to No. 4 shown in the Table 2. Moreover, all possible cases of $\rho_{(i,j)}$ of the hypergraph state $|g\rangle$ are shown in the Tables 1 and 2. This implies that (19–21) are true. \(\square\)
Table 2. All possible values of the concurrence \( C_{ij} \) between two specific qubits \( i, j \in [n] \) of the hypergraph state \( |g\rangle \) when \( \text{ran}(g) = 2 \) and \( \varepsilon(g_{i,i,j}) = -1 \)

| No. | \( \varepsilon(g_{i,j}) \) | \( \varepsilon(g_{j,i}) \) | \( \varepsilon(g_{i,j}) \Delta g_{i,j} \) | \( \rho_{ij} \) | \( C_{ij} \) |
|-----|----------------|----------------|--------------------------------|----------------|----------------|
| 1   | 0              | 0              | 0                             | \( \frac{1}{2} (I \otimes I) \) | 0              |
| 2   | 0              | 0              | 1                             | \( \frac{1}{2} (I \otimes I + \sigma_y \otimes \sigma_y) \) | 0              |
| 3   | 0              | 0              | -1                            | \( \frac{1}{2} (I \otimes I - \sigma_y \otimes \sigma_y) \) | 0              |
| 4   | 0              | 1              | 0                             | \( \frac{1}{2} (I \otimes I + \sigma_z \otimes \sigma_z) \) | 0              |
| 5   | 0              | -1             | 0                             | \( \frac{1}{2} (I \otimes I - \sigma_z \otimes \sigma_z) \) | 0              |
| 6   | 1              | 0              | 0                             | \( \frac{1}{2} (I \otimes I + \sigma_x \otimes \sigma_z) \) | 0              |
| 7   | -1             | 0              | 0                             | \( \frac{1}{2} (I \otimes I - \sigma_x \otimes \sigma_z) \) | 0              |
| 8   | 1              | 1              | 1                             | \( \frac{1}{2} (I \otimes I + \sigma_z \otimes \sigma_x) (I \otimes I + \sigma_x \otimes \sigma_z) \) | 1              |
| 9   | 1              | -1             | -1                            | \( \frac{1}{2} (I \otimes I - \sigma_z \otimes \sigma_x) (I \otimes I + \sigma_x \otimes \sigma_z) \) | 1              |
| 10  | -1             | 1              | -1                            | \( \frac{1}{2} (I \otimes I + \sigma_z \otimes \sigma_x) (I \otimes I - \sigma_x \otimes \sigma_z) \) | 1              |
| 11  | -1             | -1             | 1                             | \( \frac{1}{2} (I \otimes I - \sigma_z \otimes \sigma_x) (I \otimes I - \sigma_x \otimes \sigma_z) \) | 1              |

According to Sect. 3, we can obtain the concurrence \( C_{ij} \) by calculating the eigenvalues of \( \rho_{[i,j]} \) \( (\sigma_y \otimes \sigma_z) \). All possible values of \( C_{ij} \) of the hypergraph state \( |g\rangle \) are also shown in the Tables 1 and 2. Moreover, we can obtain the following proposition by these two tables. By the proof of proposition 1, we can obtain the following proposition for a general hypergraph \( g \) (whose rank need not be 2).

**Proposition 2** If a hypergraph \( g \) has a component whose vertex set is \( \{i, j\} \), then \( \rho_{[i,j]} \) of the hypergraph state \( |g\rangle \) is entangled.

Note the converse of the above proposition is not true since each qubit pair of the hypergraph state \( |g^a_n\rangle \) is entangled, which is shown in Sect. 4 (B). Now let us discuss some properties of the entanglement of graph states by means of the concurrence. By the above proposition, we can give a sufficiency and necessary condition for two qubits of a graph state being entangled as follows.

**Corollary 3** A graph \( g \) has a component with the vertex set \( \{i, j\} \) if and only if \( \rho_{[i,j]} \) of the corresponding graph state \( |g\rangle \) is entangled.

From the above corollary, we can also obtain the following corollary.

**Corollary 4** Suppose that \( g = ([n], E) \) is a connected graph and \( n \geq 3 \). Then, for any two different vertices \( i, j \in [n] \), the reduced density matrix \( \rho_{[i,j]} \) of the graph state \( |g\rangle \) is separable.

The above corollary has been proved in [27] by using a different approach. It is known that many entanglement criteria (which are shown in [27] and its references) use only bipartite correlations for the entanglement detection. Thus, these criteria must fail to recognize in graph states of three or more qubits [27].

Reference [25] introduces a concept of an entangled graph such that each qubit of a multipartite system is associated with a vertex, while a bipartite entanglement between two specific qubits is represented by an edge between these vertices. For an
n-qubit state, its entangled graph can visually show how a bipartite entanglement is “distributed” in n qubits. By the corollaries 3 and 4, we can obtain the following proposition.

**Proposition 5** Suppose that $g = ([n], E)$ is a graph. Then, the entangled graph $G = ([n], E_G)$ of the graph state $| g \rangle$ satisfies that each vertex is adjacent with at most one vertex, that is, for each $i \in [n]$

$$|[\{j | [i, j] \in E_G\}]| \leq 1. \quad (23)$$

By the above proposition, it is easy to draw all entangled graphs of $n$-qubit graph states. Entangled graphs of three-qubit graph states have been shown in [28]. All entangled graphs of four-qubit graph states are shown in Fig. 2.

4.2 The hypergraph $g_n^* = ([n], \{[n]\})$

In the following, we calculate the concurrence of the reduced density matrix $\rho_{[i,j]}$ on two specific qubits $i, j \in [n]$ of the hypergraph state $| g_n^* \rangle$. According to the proposition 1 in [22] and (17), we can obtain that

$$\rho_{[i,j]} = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \frac{1}{2^{n-1}} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \frac{1}{2^{n-1}} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \frac{1}{2^{n-1}} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \frac{1}{2^{n-1}} \\
\frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} 
\end{bmatrix}. \quad (24)$$

Then, we can get

$$\rho_{[i,j]} (\sigma_y \otimes \sigma_y) = \begin{bmatrix}
\frac{1}{2^{n-1}} - \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{2^{n-1}} - \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{2^{n-1}} - \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{2^{n-1}} - \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} & \frac{1}{4} - \frac{1}{2^{n-1}} - \frac{1}{4}
\end{bmatrix}. \quad (25)$$
Let us calculate four eigenvalues of $\rho_{\{i, j\}} (\sigma_y \otimes \sigma_y)$. This means that we should solve the equation $\det \left[ \rho_{\{i, j\}} (\sigma_y \otimes \sigma_y) - \lambda I \otimes I \right] = 0$. By computing, we can obtain

$$\det \left[ \rho_{\{i, j\}} (\sigma_y \otimes \sigma_y) - \lambda I \otimes I \right] = \lambda^2 \left( \lambda - \frac{1}{2^{n-1}} - \frac{1}{2^{n/2}} \right) \left( \lambda - \frac{1}{2^{n-1}} + \frac{1}{2^{n/2}} \right)$$

(26)

Thus, it is known that four eigenvalues of $\rho_{\{i, j\}} (\sigma_y \otimes \sigma_y)$ are respectively $\frac{1}{2^{n-1}} + \frac{1}{2^{n/2}}, \frac{1}{2^{n-1}} - \frac{1}{2^{n/2}}, 0$ and $0$. According to Sect. 3, the eigenvalues of $\rho_{\{i, j\}} \tilde{\rho}_{\{i, j\}}$, in decreasing order, are $\lambda_1 = \left( \frac{1}{2^{n-1}} + \frac{1}{2^{n/2}} \right)^2, \lambda_2 = \left( \frac{1}{2^{n-1}} - \frac{1}{2^{n/2}} \right)^2$ and $\lambda_3 = \lambda_4 = 0$. From (10), we can obtain that

$$C_{ij} = \frac{2}{2^{n/2}} \neq 0$$

(27)

Therefore, we can get the following proposition.

**Proposition 6** Each qubit pair of the hypergraph state $|g_n^a\rangle$ is entangled.

## 5 Conclusions

We first use the Hamming weight of several special subhypergraphs to calculate the concurrence between two specific qubits of a hypergraph state. Then, we discuss the properties of the bipartite entanglement of several special hypergraph states by using the concurrence. Our research reveals that the sufficiency and necessary condition of a qubit pair $\{i, j\}$ of a graph state being entangled is that the corresponding graph has a component with the vertex set $\{i, j\}$. The result is important because it shows that a notation (i.e., the component with the vertex set $\{i, j\}$) in hypergraph theorem is related with one (i.e., the entangled qubit pair $\{i, j\}$) in quantum theory. Thus, it implies that hypergraph states supply an approach to construct a bridge between hypergraph theory and quantum theory: (1) one can study quantum properties (for instance, multipartite entanglement) of hypergraph states by means of hypergraph theory terms; (2) Conversely, one might solve several hard problems in hypergraph theory by some properties of hypergraph states.

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