Nonclassicality of single quantum excitation of a thermal field in thermal environments

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The nonclassicality of single photon-added thermal states in the thermal channel is investigated by exploring the volume of the negative part of the Wigner function. The Wigner functions become positive when the decay time exceeds a threshold value \( \gamma t_c \), which only depends on the effective temperature or mean thermal photon number of the thermal channel, but not depends on the effective temperature of the initial thermal state. This phenomenon is similar with the case of single quantum excitation of classical coherent states in thermal channel. Furthermore, we firstly demonstrate \( \gamma t_c \) is the same for arbitrary pure or mixed nonclassical optical fields with zero population in vacuum state.

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The preparations of nonclassical non-gaussian optical fields have attracted much attention, which have many applications in quantum information processing \[1\]. Usually, the nonclassicality manifests itself in specific properties of quantum statistics \[2, 3, 4, 5\], in which the partial negative Wigner distribution \[6\] is indeed a good indication of the highly nonclassical character of the optical fields. Reconstruction of the Wigner distribution in experiments with quantum tomography \[6, 7, 8\] have demonstrated the appearance of the negative values, which does not have any classical counterparts. A variety of nonclassical states has recently been characterized by means of the negativity of their Wigner function \[9, 10, 11, 12, 13\], which is a sufficient and not necessary condition for nonclassicality \[14\].

The photon-added thermal states (PATSS) were introduced by Agarwal and Tara \[15\], which do not exhibit squeezing, sub-Poissonian statistics, and any coherence. The single photon-added thermal state (SPA TS) has been experimentally prepared by Zavatta et al. and its nonclassical properties have been detected by homodyne tomography technology \[16\]. For the SPATS, their nonclassical properties have been investigated by several authors \[15, 16, 17, 18\]. Furthermore, Parigi et al. have experimentally investigated quantum commutation rules by addition and subtraction of single photons to or from a light field initially in the thermal state \[19\]. Ordinarily, the interaction between the nonclassical optical fields and their surrounding thermal environment may deteriorate the degree of nonclassicality. Thus, to study the dynamical behaviors of the partial negativity of Wigner distribution and understand how long a nonclassical field preserves its partial negative Wigner distribution in thermal channel may be very desirable for experimentally quantifying the variation of nonclassicality. Here, the nonclassicality of photon-added thermal states in the thermal channel is investigated by exploring the partial negative Wigner distribution. The exact expression of the time evolution Wigner function is given out and the dynamical behavior of the volume of the negative part of the Wigner function is analytically derived. The threshold values \( \gamma t \) of the decay time corresponding to the transition of the Wigner distribution from partial negative to completely positive are derived. For SPATSs in thermal channel, it is shown that the threshold value of the decay time is independent of the mean thermal photon number of initial thermal state seed. Furthermore, we firstly demonstrate \( \gamma t_c \) is the same for arbitrary pure or mixed nonclassical optical fields with zero population in vacuum state, and is given by \( \gamma t_c = \ln(\frac{n+2}{n+1}) \), where \( n \) is the mean thermal photon number of the thermal channel.

Let us first briefly recall the definition of the single photon-added thermal states (SPATS) \[15\]. The SPATSs are defined by

\[
\hat{\rho} = \frac{1}{\bar{n}(\bar{n} + 1)} \sum_{\ell=0}^{\infty} \frac{\bar{n}^\ell}{(\ell + \bar{n})!} |\ell\rangle \langle \ell|,
\]

where \( |\ell\rangle \) is the Fock state and \( \bar{n} \) is the mean photon number of the thermal state seed. When the SPATS evolves in the thermal channel, the evolution of the density matrix can be described by \[20\]

\[
\frac{d\hat{\rho}}{dt} = \frac{\gamma(n + 1)}{2} (2a\hat{\rho}a^\dagger - a^\dagger a\hat{\rho} - \hat{\rho}a^\dagger a) + \frac{\gamma n}{2} (2a^\dagger \hat{\rho}a - aa^\dagger \hat{\rho} - \hat{\rho}aa^\dagger),
\]

where \( \gamma \) represents dissipative coefficient and \( n \) denotes the mean thermal photon number of the thermal channel. \( a^\dagger (a) \) is the creation operator (annihilation operator) of the optical mode. When \( n = 0 \), the Eq.(2) reduces to the master equation describing the photon-loss channel.

For an optical field in the state \( \hat{\rho} \), its Wigner function, the Fourier transformation of characteristics function \[21\] of the state \( \hat{\rho} \) can be derived by \[22, 23\]

\[
W(\beta) = \frac{2}{\pi} \text{Tr}[(\hat{O}_e - \hat{O}_o)\hat{D}(\beta)\hat{D}^\dagger(\beta)],
\]

where \( \hat{O}_e \equiv \sum_{k=0}^{\infty} |2k\rangle \langle 2k| \) and \( \hat{O}_o \equiv \sum_{k=0}^{\infty} |2k+1\rangle \langle 2k+1| \) are the even and odd parity operators respectively. In the thermal channel described by the master Eq.(2),
the time evolution Wigner function satisfies the following Fokker-Planck equation [24]

\[
\frac{\partial}{\partial t} W(q, p, t) = \frac{\gamma}{2} \left( \frac{\partial}{\partial q} q + \frac{\partial}{\partial p} p \right) W(q, p, t) \\
+ \frac{\gamma (2n+1)}{8} \left( \frac{\partial^2}{\partial q^2} \frac{\partial^2}{\partial p^2} \right) W(q, p, t) \tag{4}
\]

where \( q \) and \( p \) represent the real part and imaginary part of \( \beta \), respectively. The time evolution Wigner function can be derived as following:

\[
W(q, p, \gamma t) = \exp(\gamma t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_T(x, y) \exp\left( -\frac{2(x^2 + y^2)}{1 + 2n} \right) \, dx \, dy
\]

where

\[
W_T(x, y) = \frac{2}{\pi(1 + 2n)} \exp\left( -\frac{2(x^2 + y^2)}{1 + 2n} \right) \tag{6}
\]

is the Wigner function of the thermal state with mean photon number \( n \). Substituting the initial Wigner function of the SPA TS [13]

\[
W^S(q, p, 0) = \frac{2}{\pi} \frac{4(1 + \bar{n})(q^2 + p^2)}{(1 + 2n)^3 - 1} e^{-\frac{1}{2}(q^2 + p^2)} \tag{7}
\]

into the Eq.(5), it is easy to obtain the corresponding time evolution Wigner function as follows:

\[
W^S(q, p, \gamma t) = \frac{\kappa + 8(1 + \bar{n})e^{\gamma t}(q^2 + p^2) \exp\left(-\frac{2\gamma t}{\xi + \zeta} \right)}{\pi \xi^3} \\
\xi = 2(\bar{n} - n) + (1 + 2n)e^{\gamma t}, \\
\zeta = 2(\bar{n} - n)\gamma t + (1 + 2n)\gamma te^{\gamma t}, \\
\kappa = -8(\bar{n} - n)(1 + n) + 2(1 + 2n)^2e^{2\gamma t} \\
+ 4[\bar{n}(1 + 2n) - (1 + 2n)^2]|e^{\gamma t}| \tag{8}
\]

In Fig.1, the Wigner function of the SPATS with \( \bar{n} = 1 \) in the thermal channel with \( n = 0.5 \) is plotted for three different values of decay time. It is shown that the phase space Wigner distribution of the SPATS exhibits partial negativity around the origin, and the region of the negative part in phase space is a circle. The absolute value of negative minimum of the Wigner distribution decreases as \( \gamma t \) increases, and the thermal noise causes the disappearance of the partial negativity of the Wigner function if the decay time exceeds a threshold value. The ringlike wings in the distribution gradually disappear and the distribution becomes more and more similar to the Gaussian typical of a thermal state.

Recently, the volume \( P_{NW} \) of the negative part of Wigner distribution function has been suggested as a good choice for quantifying the nonclassicality [25, 26, 27, 28, 29]. \( P_{NW} \) is defined by

\[
P_{NW} = \int_{\Omega} W(q, p) dq dp, \tag{9}
\]

where \( \Omega \) is the negative Wigner distribution region. In Ref. [28], we have investigated \( P_{NW} \) of photon-added coherent states in the photon-loss channel. It was shown that \( P_{NW} \) and entanglement potential defined in Ref. [30] exhibit the consistent behaviors in short decay time.

Now, we bring our attention to the influence of thermal noise on the nonclassicality of the quantum excitation of classical non-coherent optical thermal fields by calculating \( P_{NW} \). Substituting the time evolving Wigner function in Eq.(8) into Eq.(9), we could obtain

\[
P_{NW} = \frac{\left[ \frac{\kappa}{2} + 2(1 + \bar{n})(1 - e^{\frac{\gamma t}{\xi}}) \right] e^{-\gamma t} e^{\gamma t} \xi}{\xi} \tag{10}
\]
optical fields are very fragile against the thermal noise. \( e^{-\gamma t_c(n) + 2n} \frac{1}{1 + 2n} \),

where \( \gamma t_c(0) \) is the threshold decay time in the photon loss channel.

The similarity between the amplitude-independence of the threshold decay time \( \gamma t_c \) corresponding to the disappearance of partial negativity of the Wigner function of SPACSs and SPATSs in the thermal channel implies there may exist a universal relation about \( \gamma t_c \) for arbitrary single quantum excitation of classical gaussian states. Here, we demonstrate that all of the nonclassical states whose density operators \( \rho \) satisfy \( \langle 0|\rho|0 \rangle = 0 \), where \( |0 \rangle \) is the vacuum state, completely lose the negativity of their Wigner functions at the threshold decay time \( \gamma t = \ln \frac{2 + 2n}{1 + 2n} \). The strict proof will be presented elsewhere. Here, we briefly outline the proof procedures. Firstly, considering the case of photon-loss channel, i.e. \( n = 0 \). In this situation, based on Eq.(5), for arbitrary initial nonclassical pure or mixed states whose vacuum state population is zero, we can derive

\[
W(q, p, \gamma t)|_{\gamma t = \ln 2} \geq 0
\]

for any values of \( q \) and \( p \), and

\[
W(0, 0, \gamma t)|_{\gamma t = \ln 2} = c|0|\rho|0 \rangle = 0
\]

where \( c \) is a constant. Eqs.(13-14) imply that the threshold decay time \( \gamma t_c(0) \) in Eq.(12) is \( \ln 2 \) for these states. Substituting it into Eq.(12), we can complete this proof. In the derivation of Eq.(13), we have used the relation between the Wigner distribution function and the \( Q \) function. The \( Q \) function gives the probability distribution for finding the coherent state \( |\alpha \rangle \) in the state \( \hat{\rho} \) since \( Q(q, p) = \frac{1}{\pi} |\alpha| |\rho| \alpha \rangle \), where \( \alpha = q + ip \). The \( Q \) function is always non-negative. The \( Q \) function is a particular case of a class of non-negative quantum distribution functions, the Husimi functions, obtained by smoothing the Wigner distribution function with a minimum uncertainty squeezed Gaussian function \[31\]. The \( Q \) function can be obtained when the Wigner function is smoothed by a coherent state wave packet. From Eq.(5), we can find, for any quantum fields with initial state \( \hat{\rho}_0 \) in photon loss channel, their time evolution Wigner functions at decay time \( \gamma t = \ln 2 \) can be rewritten as

\[
W(q, p, \ln 2) \propto \langle \sqrt{2a}|\hat{\rho}_0|\sqrt{2a} \rangle = \pi Q_0(\sqrt{2q}, \sqrt{2p})
\]

Therefore, we have the Eq.(13) by referencing the characteristics of the \( Q \) function.

In summary, we have investigated the nonclassicality of single photon excitation of thermal optical field in the thermal channel by exploring the partial negativity of the Wigner function. The total volume of the negative part defined by the absolute value of the integral of the Wigner function over the negative distribution region is analytically calculated. For the case of SPATSs in thermal channel, the exact threshold value of the decay time

\[
\gamma t_c = \ln \frac{2 + 2n}{1 + 2n},
\]
beyond which the evolving Wigner function becomes positive is given as $\gamma_{t_c} = \ln \left( \frac{1+2n}{1+2n} \right)$, which is the same as the one in the case of single quantum excitation of the classical coherent field. For all of the nonclassical states whose density operators $\rho$ satisfy $\langle 0 | \rho | 0 \rangle = 0$, where $| 0 \rangle$ is the vacuum state, it is demonstrated that the threshold decay times are the same and given by $\gamma_{t_c} = \ln \left( \frac{1+2n}{1+2n} \right)$. These results clearly imply Wigner distributions of any photon-added thermal states are partial negative before the threshold decay time $\gamma_{t_c}$ even if their initial thermal state seeds are macroscopic with arbitrary large $n$ but finite. Obviously, any photon-added classical gaussian states including photon-added thermal states, photon-added coherent states, and photon-added displaced thermal states belong to the class of states satisfying $\langle 0 | \rho | 0 \rangle = 0$. Therefore, the above conclusions can be generalized to: In thermal channel with mean thermal photon number $n$, all nonclassical pure or mixed states $\rho$ with zero population in vacuum state have partial negative Wigner distribution before the threshold decay time $\gamma_{t_c} = \ln \left( \frac{1+2n}{1+2n} \right)$ if $\text{Tr}(\rho a^\dagger a)$ is finite.

The above results can be regarded as benchmark to investigate the robustness of other indicators of nonclassicality such as squeezing, antibunch, and entanglement potential of nonclassical optical fields in thermal channel if compared with the partial negative Wigner distribution. Recently, the physical realization of controlled phase gate based on the single-photon-added coherent states has also been proposed [29]. Our studies in this report may find some applications in these quantum information processes in which photon-added states are involved.

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