Riemann zeta factorial function

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Abstract. This paper focuses on extending the theory of Riemann zeta function to Riemann zeta factorial function for infinite series using inverse principle of forward difference operator. This newly introduced series is generated by replacing polynomial by polynomial factorial in Riemann zeta function. Several formulae on higher order Riemann zeta factorial functions have been derived. Suitable examples are inserted to validate the above findings.

Keywords: Difference operator, factorial function, Infinite series, Inverse principle, Zeta function.

AMS Classification: 39A70, 11R42, 11S40.

1. Introduction
The Riemann zeta function \( \zeta(s) \) has been studied in many different forms for centuries. The harmonic series \( \zeta(1) \) was proven to be divergent as far back as the 14th century [11]. Leonhard Euler, a Swiss mathematician, discovered a closed form expression in 18th century for the sum of the reciprocals of the squared integers i.e. \( \zeta(2) \). He also generalized this result and found a closed form expression for \( \zeta(2n) \) for \( n \in \mathbb{N} \) [12]. In the 19th century, the German mathematician Bernhard Riemann considered \( \zeta \) as a complex function. He published his work in the 1859 paper “On the Number of Primes Less Than a Given Magnitude”, which is one of the most influential works of modern mathematics [5]. The classical definition of Riemann zeta function is \( \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \) [2, 8, 9]. In this paper, the Riemann zeta factorial function, denoted as \( \zeta_\ell(k, s) \), is generated by replacing the corresponding polynomial \( k^s \) into polynomial factorial \( k_\ell^s \) in the expression of zeta function. Several properties of Riemann zeta factorial functions are derived by applying the difference operator having shift value \( \ell \). Some applications of difference operator and its equation can be found in several real world phenomena [1, 3, 4, 5, 6].

2. Preliminaries and basic definitions
In this section, we introduce Riemann zeta factorial function which is an extension of Riemann zeta function and obtain certain results on it using difference operator.

Definition 2.1. [7] Let \( u(k) \) be a real valued function defined on \( (-\infty, \infty) \) and \( \ell > 0 \). The \( \ell \)-difference operator, denoted as \( \Delta_\ell \), on \( u(k) \) is defined by

\[ \Delta_\ell u(k) = u(k + \ell) - u(k). \] (1)

If there exists a real value function \( v(k) \) such that \( \Delta_\ell v(k) = u(k) \), then \( v(k) \) is said to be the inverse difference of \( u(k) \) and is denoted as \( v(k) = \Delta_\ell^{-1} u(k) \).
Remark 2.2. Through out this paper, we denote that $$\Delta^{-1}_t u(k) \big|_k^\infty = \Delta^{-1}_t u(\infty) - \Delta^{-1}_t u(k) = v(\infty) - v(k)$$.

Definition 2.3. [7, 10] Let $$k \in (-\infty, \infty), \ell$$ is any fixed real number and $$n$$ is a positive integer. Then, the polynomial factorial having shift value $$\ell$$ is defined by

$$k^n_\ell = k(k-\ell)(k-2\ell) \cdots (k-(n-1)\ell) \quad \text{and} \quad k^0_\ell = 1. \quad (2)$$

The following lemma gives the inverse principle of $$\Delta_t$$ on $$u(k)$$.

Lemma 2.4. [10] If $$\ell > 0$$, and $$\sum_{r=0}^{\infty} u(k + r\ell)$$ is convergent, then we have

$$\Delta^{-1}_t u(t) \big|_k^\infty = \sum_{r=0}^{\infty} u(k + r\ell) \quad (3)$$

and hence

$$\Delta^{-1}_t u(k) = -\sum_{r=0}^{\infty} u(k + r\ell). \quad (4)$$

Proof. (4) follows from $$\Delta^{-1}_t u(\infty) = 0$$ and $$\Delta^{-1}_t u(\infty) - \Delta^{-1}_t u(k) = \sum_{r=0}^{\infty} u(k + r\ell). \quad \square$$

We give the following example which will be used in the subsequent discussion.

Example 2.5. Let $$s > 1$$ and $$k^{(s)}_\ell \neq 0$$. Since $$\Delta_t \left\{ -\frac{1}{\ell(s-1)(k-\ell)^{(s-1)}_\ell} \right\} = \frac{1}{k^{(s)}_\ell}$$, from the definition 2.1, we have

$$\Delta^{-1}_t \frac{1}{k^{(s)}_\ell} = -\frac{1}{\ell(s-1)(k-\ell)^{(s-1)}_\ell}. \quad (5)$$

Since $$\Delta^{-1}_t \frac{1}{t^{(s)}_\ell} = 0$$ as $$t \to \infty$$, taking limits on both sides of (5) yields

$$\Delta^{-1}_t \frac{1}{t^{(s)}_\ell} \big|_k^\infty = -\frac{1}{\ell(s-1)(k-\ell)^{(s-1)}_\ell}. \quad (6)$$

Applying (3) on (6) gives $$\sum_{r=0}^{\infty} \frac{1}{(k + r\ell)^{(s)}_\ell} = \frac{1}{\ell(s-1)(k-\ell)^{(s-1)}_\ell}$$. 

Definition 2.6. For $$\ell > 0$$, $$s \in N(2) = \{2, 3, 4, \ldots\}$$ and $$(k + r\ell)^{(s)}_\ell \neq 0$$, the Riemann zeta factorial function, denoted as $$\zeta_\ell(k, s)$$, is defined as

$$\zeta_\ell(k, s) = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)^{(s)}_\ell}. \quad (7)$$

Example 2.7. For $$s \in N(2)$$, $$\zeta_1(s, s) = \sum_{r=0}^{\infty} \frac{1}{(s + r)^{(s)}_1} = \frac{1}{s^{(s)}_1} + \frac{1}{(s+1)^{(s)}_1} + \frac{1}{(s+2)^{(s)}_1} + \ldots$$

Theorem 2.8. If $$s \in N(2)$$, $$\ell > 0$$ and $$(k - 1)^{(s-1)}_\ell \neq 0$$, then we have

$$\zeta_\ell(k, s) = \frac{1}{((s-1)\ell(k-\ell))^{(s-1)}_\ell} = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)^{(s)}_\ell}. \quad (8)$$
Proof. The proof follows by taking \( u(t) = \frac{1}{\ell^k} \) in (3), Definition (2.6) and (6).

Remark 2.9. Similarly, replacing the polynomial \( k^n \) by polynomial factorial \( k^{(n)}_{\ell} \) in the expression of exponential function, one may define extorial function, denoted as \( e^{(k)}_{\ell} \), as
\[
e^{(k)}_{\ell} = \frac{k^{(0)}_{\ell}}{0!} + \frac{k^{(1)}_{\ell}}{1!} + \frac{k^{(2)}_{\ell}}{2!} + \frac{k^{(3)}_{\ell}}{3!} + \cdots
\]
and can obtain several applications in the field of difference and integral calculas.

3. Higher order zeta factorial functions

In this section, we define the higher order zeta factorial function and derive several identities involving infinite series using inverse principle given in Lemma 2.4.

Definition 3.1. For \( m \in N(m+1) \), the \( m^{th} \) order Riemann zeta factorial function is defined by
\[
\zeta^m_{\ell}(k, s) = \Delta^{-1}_{\ell} \zeta_{\ell}(t, s)\Big|_k, s > m \text{ and } k^{(s)}_{\ell} \neq 0.
\]

Theorem 3.2. If \( s \in N(3) \), \( \ell > 0 \) and \( (k - 2\ell)_{\ell}^{(s-2)} \neq 0 \), then we have
\[
\zeta^2_{\ell}(k, s) = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)_{\ell}^{(s)}} = \frac{1}{\ell^2(s-1)(k - 2\ell)_{\ell}^{(s-2)}}.
\]

Proof. From the Theorem 2.8, and taking \( \Delta^{-1}_{\ell} \) on (8), we obtain
\[
\Delta^{-1}_{\ell} \zeta_{\ell}(k, s) = \Delta^{-1}_{\ell} \frac{1}{(s-1)\ell(k - 2\ell)_{\ell}^{(s-1)}} = \Delta^{-1}_{\ell} \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)_{\ell}^{(s)}}.
\]

Since \( \Delta^{-1}_{\ell} \) is linear, replacing \( k \) by \( k - \ell \) and \( s \) by \( s - 1 \) in (5), we arrive
\[
\Delta^{-1}_{\ell} \zeta_{\ell}(k, s) = \Delta^{-1}_{\ell} \frac{1}{(s-1)\ell(k - 2\ell)_{\ell}^{(s-1)}} = \frac{-1}{(s-1)(s-2)\ell^2(k - 2\ell)_{\ell}^{(s-2)}},
\]
which yields
\[
\Delta^{-1}_{\ell} \zeta_{\ell}(t, s)\Big|_k = \frac{1}{\ell^2(s-1)(s-2)(k - 2\ell)_{\ell}^{(s-2)}}.
\]

By taking \( u(k) = \zeta_{\ell}(k, s) \) in (4), we find that
\[-\sum_{r=0}^{\infty} \zeta_{\ell}(k + r\ell, s) = \frac{-1}{\ell^2(s-1)(s-2)(k - 2\ell)_{\ell}^{(s-2)}},
\]
which yields, from the definition of the higher order Zeta factorial function,
\[
\zeta^2_{\ell}(k, s) = \frac{1}{\ell^2(s-1)(s-2)(k - 2\ell)_{\ell}^{(s-2)}} = \sum_{r=0}^{\infty} \zeta_{\ell}(k + r\ell, s).
\]

Since \( \zeta_{\ell}(k, s) = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)_{\ell}^{(s)}} \), we have \( \zeta_{\ell}(k + r\ell, s) = \sum_{p=0}^{\infty} \frac{1}{(k + r\ell + p\ell)_{\ell}^{(s)}} \) and (14) becomes
\[
\zeta^2_{\ell}(k, s) = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{(k + (r + p)\ell)_{\ell}^{(s)}}.
\]
By expanding the terms, we arrive
\[ \zeta^2(k, s) = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)^{\ell}_s} + \sum_{r=0}^{\infty} \frac{1}{(k + (r + 1)\ell)^{\ell}_s} + \sum_{r=0}^{\infty} \frac{1}{(k + (r + 2)\ell)^{\ell}_s} + \ldots \]
\[ = \frac{1}{k^s_{\ell}} + \frac{1}{(k + \ell)^{\ell}_s} + \frac{1}{(k + 2\ell)^{\ell}_s} + \frac{1}{(k + 3\ell)^{\ell}_s} + \frac{1}{(k + 4\ell)^{\ell}_s} + \ldots \]
\[ + \frac{1}{(k + \ell)^{\ell}_s} + \frac{1}{(k + 2\ell)^{\ell}_s} + \frac{1}{(k + 3\ell)^{\ell}_s} + \frac{1}{(k + 4\ell)^{\ell}_s} + \ldots \]
\[ + \frac{1}{(k + 2\ell)^{\ell}_s} + \frac{1}{(k + 3\ell)^{\ell}_s} + \ldots \]
By rearranging the terms and applying (14) and (15), we get (10).

**Theorem 3.3.** If \( s \geq m + 1 \), \( m \) and \( s \) are positive integers, \( \ell > 0 \) and \( (k + (r - 1)\ell)^{\ell}_s \neq 0 \), then
\[ \zeta^m_{\ell}(k, s) = \sum_{r=0}^{\infty} \frac{(r + (m - 1))_{\ell}^{(m-1)}}{(m - 1)!} (k + r\ell)^{\ell}_s = \frac{1}{\ell^m(s-1)^{(m)}_{\ell} (k - m\ell)^{\ell}_s^{(s-m)}}. \] \( \ell \)

**Proof.** By applying \( \Delta_{\ell}^{-1} \) on (10), we find
\[ \Delta_{\ell}^{-1} \zeta^2_{\ell}(k, s) = \Delta_{\ell}^{-1} \sum_{r=0}^{\infty} \frac{(r + 1)_{\ell}^{(1)}}{(k + r\ell)^{\ell}_s^{(s-2)}} = \Delta_{\ell}^{-1} \left( \frac{1}{\ell^2(s-1)^{(2)}_{\ell}(k-2\ell)^{\ell}_s^{(s-2)}} \right) \]
Putting \( m = 3 \) in (9) gives
\[ \zeta^3_{\ell}(k, s) = \Delta_{\ell}^{-1} \sum_{r=0}^{\infty} \frac{(r + 1)_{\ell}^{(1)}}{(t + r\ell)^{\ell}_s^{(s)}} = \Delta_{\ell}^{-1} \left\{ \frac{1}{\ell^2(s-1)^{(2)}_{\ell}(t-2\ell)^{\ell}_s^{(s-2)}} \right\}^{\infty}_{k}. \]
Since \( \Delta_{\ell}^{-1} \) is linear, from (5) and (9), we arrive
\[ \zeta^3_{\ell}(k, s) = \frac{1}{\ell^3(s-1)^{(3)}_{\ell}(k - 3\ell)^{\ell}_s} = \Delta_{\ell}^{-1} \sum_{r=0}^{\infty} \frac{(r + 1)_{\ell}^{(1)}}{(t + r\ell)^{\ell}_s^{(s)}} \Bigg|_{k}. \]
\[ \zeta^2_{\ell}(k, s) = \sum_{r=0}^{\infty} \sum_{p=0}^{p+1} \frac{1}{(k + (r + p)\ell)^{\ell}_s} \]
\[ = \sum_{r=0}^{\infty} \left[ \frac{1}{(k + r\ell)^{\ell}_s} + \frac{2}{(k + r\ell)^{\ell}_s} + \frac{3}{(k + r\ell)^{\ell}_s} + \ldots \right] \]
\[ = \frac{1}{k^s_{\ell}} + \frac{1}{(k + \ell)^{\ell}_s} + \frac{1}{(k + 2\ell)^{\ell}_s} + \frac{1}{(k + 3\ell)^{\ell}_s} + \frac{1}{(k + 4\ell)^{\ell}_s} + \ldots \]
\[ + \frac{2}{(k + \ell)^{\ell}_s} + \frac{2}{(k + 2\ell)^{\ell}_s} + \frac{2}{(k + 3\ell)^{\ell}_s} + \frac{2}{(k + 4\ell)^{\ell}_s} + \ldots \]
\[ + \frac{3}{(k + 2\ell)^{\ell}_s} + \frac{3}{(k + 3\ell)^{\ell}_s} + \frac{3}{(k + 4\ell)^{\ell}_s} + \ldots \]
\[ + \frac{1 + 2}{(k + \ell)^{\ell}_s} + \frac{1 + 2 + 3}{(k + 2\ell)^{\ell}_s} + \ldots \]
Since \( 1 + 2 + 3 + \ldots + p = \frac{(p + 1)_2}{2} \), we get
\[ \zeta^2_{\ell}(k, s) = \frac{2^2_{\ell}}{2k^s_{\ell}} + \frac{3^2_{\ell}}{2(k + \ell)^{\ell}_s} + \frac{4^2_{\ell}}{2(k + 2\ell)^{\ell}_s} + \ldots \frac{(p + 1)_{\ell}^{(2)}}{2(k + p\ell)^{\ell}_s} + \ldots \]
Now, (16) follows by induction on \( m \). □
The following example illustrates the Theorem 3.3.

**Example 3.4.** Taking \( m = 3, s = 5, k = 7, \ell = 1 \) in the equation (16) gives

\[
\zeta_5^4(7, 5) = \sum_{r=0}^{\infty} \frac{(r + 1)^{(2)}}{(2)!/(7 + (r))^{(5)}} = \frac{1}{(4)^{(3)}(4)^{(2)}} = \frac{1}{192}.
\]

**Corollary 3.5.** If \( (k - 2\ell)^{(s-2)} \neq 0 \) and the integer \( s \geq 3 \), we have

(i) \( \zeta_2^2(k, s) - \zeta_2^3(k + 5s, s) - 5\zeta_2^2(k + 2\ell, s) + 5 \sum_{r=1}^{3} \frac{r}{(k + (r + 1)\ell)^{(s)}} = \frac{r}{(k + (r + 2)\ell)^{(s)}} \)

(ii) \( \zeta_2^2(k, s) - \zeta_2^3(k + 7\ell, s) - 7\zeta_2^2(k + 3\ell, s) + 7 \sum_{r=1}^{4} \frac{r}{(k + (r + 2)\ell)^{(s)}} = \frac{r}{(k + (r + 2)\ell)^{(s)}} \)

(iii) \( \zeta_2^2(k, s) - \zeta_2^3(k + (2m + 1)\ell, s) - (2m + 1)\zeta_2^2(k + m\ell, s) + (2m + 1) \sum_{r=1}^{m+1} \frac{r}{(k + (r + (m - 1))\ell)^{(s)}} = \sum_{r=0}^{2m} \frac{(r + 2)^{(2)}}{(k + r\ell)^{(s)}} .\)

**Proof.** The proof of this corollary follows by expanding and rearranging the terms in LHS of (i), (ii) and (iii) and applying (16).

4. **Conclusion**

This innovative methodology of extension of Riemann zeta function into Riemann zeta factorial function has given birth to a new set of profound series. The Riemann zeta factorial functions and the above findings are validated with its numerical verification. This has numerous applications in the field of calculus and thus has many real time applications.

**Compliance with ethical standards**

**Conflict of interest** No potential conflict of interest from the authors.

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