Greedy and Dynamic Programming Algorithms for Scheduling Deadline-Sensitive Parallel Tasks

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ABSTRACT

Scheduling parallel tasks is a fundamental problem for many applications such as cloud computing. We consider the problem of scheduling a set of n deadline-sensitive parallel tasks on C machines. Each task is specified by a value, a workload, a deadline and a parallelism bound. The objective is to maximize the sum of values of jobs completed by their deadlines. For this problem, a greedy algorithm GreedyRTL (Jain et al., ACM SPAA 2012) was previously proposed and analyzed based on the dual fitting technique, achieving a performance guarantee \( \frac{C}{C - k + 1} \), where \( C \) and \( s \) are parameters specific to the tasks.

In this paper, without recourse to the dual fitting technique, we propose a novel analysis technique for the greedy algorithms in the problem above. This technique enables improving the performance guarantee of GreedyRTL to \( \frac{C}{C - k + 1} \). We then point out that \( \frac{C}{C - k + 1} \) is the best performance guarantee that a general greedy algorithm can achieve. Based on the proposed analysis technique, we further derive the following algorithmic results: (i) an improved greedy algorithm achieving the performance guarantee \( \frac{C}{C - k + 1} \); (ii) the first exact dynamic programming algorithm in the case where the set of possible task deadlines is finite, and (iii) an exact algorithm for the machine minimization problem with the objective of minimizing the number of machines needed to schedule a set of tasks. The machine minimization problem is considered here for the first time under the model of this paper. The proposed analysis technique may have more applications.

Categories and Subject Descriptors

F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—scheduling and scheduling; K.6.2 [Management of Computing and Information Systems]: Installation Management—pricing and resource allocation

1. INTRODUCTION

1.1 Background and Motivation

Cloud computing has become the norm for a wide range of applications. In particular, with the emergence of big data analytics, many applications require the execution on large computing clusters of batch jobs, i.e., non-real-time jobs with some flexibility in the execution period. While cloud providers typically rent virtual machines (i.e., computing power) by the hour, what really matters for tenants is completion of their jobs within a set of associated constraints (e.g., deadlines), regardless of the time of execution and computing power used. This gap between providers offer and tenants goal has generated recent research efforts aiming to allow tenants to describe more precisely the characteristics of their jobs [1, 2]. Such technical progress raises new algorithmic challenges on how to optimally schedule jobs knowing their flexibility and constraints. It has motivated the recent study of the fundamental problems of scheduling parallel tasks with deadlines to maximize social welfare or resource utilization [3, 4, 5, 6, 7, 8].

In particular, Jain et al. [3, 4] consider a model where each job is specified by a demand (or workload), a parallelism constraint, a deadline and a value. The objective is to maximize the social welfare (i.e., the sum of values of jobs fully completed by the deadline). Jain et al. [4] propose a greedy algorithm GreedyRTL for this problem and show that it achieves a performance guarantee of \( \frac{C - k}{C - k + 1} \). Here, \( C \) is the total number of machines; \( k \) is the parallelism bound (intuitively, \( k \) sets the maximal number of machines that can be utilized by a task simultaneously); and \( s \) (\( \geq 1 \)) is the slackness which characterizes the degree of the resource allocation flexibility (e.g., \( s = 1 \) means that \( k \) machines have to be allocated to the task at every time slot until its deadline to ensure full completion). The GreedyRTL algorithm considers the tasks in the non-increasing order of the marginal value, namely, the ratio of the value of a task to its demand; when a task being considered satisfies a specified allocation condition, it is fully allocated in a certain way.

GreedyRTL is significant and has a number of advantages...
due to its greedy nature. It is computationally efficient and the performance guarantee is good in certain situations (when all jobs are very flexible and the number of machines is large in contrast to parallelism bound) [4]. The design of the GreedyRTL algorithm, however, is tightly linked to its analysis using the dual fitting method. In the authors own terms, “the algorithm is specifically designed to maintain unique properties, which are realized under clever dual fitting arguments.”

The dual fitting technique (as well as the primal dual technique) is often used to analyze and design algorithms with a greedy nature [11][12][13]. It formulates the problem as an integer programming problem and gives the dual of the relaxed linear programming problem. Due to the weak duality, the value obtained by a feasible solution to the dual will be an upper bound of the value obtained by the optimal solution. Here, the feasible solution to the dual is constructed in a certain way according to the algorithm. Although the dual fitting technique provides a general operable way of bounding the performance guarantee, its abstract nature also makes it difficult in certain scenarios to keep sight of the functions of related parameters in a scheduling problem that could be important for better algorithm design and analysis. Here for example, it does not permit to understand finely the structure of the resource allocation that would benefit the problem of scheduling deadline-sensitive parallel tasks with some flexibility. As a result, it may be difficult to obtain more insights from the analysis of the greedy algorithm by dual fitting that could be used to design other types of algorithms.

In this paper, without recourse to the dual fitting technique, we identify some key concepts related to the resource allocation and propose a novel analysis technique to obtain a relatively complete understanding of the application of the greedy and dynamic programming approaches to our problem. The related results and their motivations are detailed in the next section.

1.2 Our Results

1.2.1 Greedy algorithms

Greedy algorithms are often the first algorithms one considers for many optimization problems. In terms of the maximization problem, the general form of a greedy algorithm is as follows: it tries to build a solution by iteratively executing the following steps until no item remains to be considered in a set of items: (1) selection standard: in a greedy way, choose and consider an item that is locally optimal according to a simple criterion at the current stage; (2) feasibility condition: for the item being considered, accept it if it satisfies a certain condition such that this item constitutes a feasible solution together with the tasks that have been accepted so far under the constraints of this problem, and reject it otherwise [13][15]. Here, an item that has been considered and rejected will never be considered again. The selection criterion is related to the objective function and constraints, and is usually the ratio of ‘advantage’ to ‘cost’, measuring the efficiency of an item [16]. In the problem of this paper, the constraint comes from the capacity to hold the chosen tasks and the objective is to maximize the social welfare; therefore, the selection criterion here is the ratio of the value of a task to its demand.

Result (i): an algorithm analysis technique. Without recourse to the dual-fitting technique, we propose a novel analysis technique for the greedy algorithms in the problem of this paper.

In this analysis, we strive to understand the underlying principle of designing a greedy algorithm for the problem of scheduling parallel tasks with deadlines by answering the following questions: (1) what resource allocation structure is good and how can it be achieved? (2) which parameters are the key to performance guarantee so that we can make an effort to improve them? and (3) what features of the resource allocation structure should be maintained so that the bounds of those parameters can imply the performance guarantee? As a result, from the resource utilization perspective, we identify a definition measuring the optimality of resource allocation to a set of deadline-sensitive parallel tasks in a particular time slot interval (Section 2.3) and define two features of the resource allocation structure (Section 5.2). Then, our analysis highlights the role of resource utilization in greedy algorithms that treat jobs in decreasing order of marginal value and points out that, if one of the features is maintained (optimal resource allocation in a certain time slot interval), we can always infer the performance guarantee of a greedy algorithm from the other feature (the resource utilization bound).

Result (ii): a tighter analysis of GreedyRTL. We next analyze how the operations in GreedyRTL can achieve the two features above and this allows us to improve its previous performance guarantee \( \frac{2}{\alpha} \cdot \frac{3}{\alpha} + 1 \) to \( \frac{2}{\alpha} \cdot \frac{3}{\alpha} + 1 \).

For the GreedyRTL algorithm, we still have the following concerns. It specifies an allocation condition as a sufficient feasibility condition. This condition depends on the parallelism bound and the current available machines by the deadline; the latter being determined by the allocation to the tasks that have been accepted so far. The allocation of tasks previously accepted is assumed fixed when a new task is considered; hence the specified condition might be too strict: it could prevent a locally optimal task \( T_i \) from being accepted although it could have been accepted if it was allowed to adjust the previous allocation to adapt to the need of fully allocating \( T_i \). In a greedy sense, we cannot judge a priori whether this would degrade the optimal performance of a greedy algorithm since in certain steps the algorithm should have been able to accept the more efficient task (with higher marginal value) than the subsequent accepted tasks. Motivated by such considerations, we define a class GREEDY of greedy algorithms of the general form as described at the beginning of this subsection:

**Definition 1.** A greedy algorithm belongs to the class GREEDY if it operates as follows:

1. the algorithm considers tasks in the non-increasing order of the marginal value;
2. let \( \mathcal{A} \) denote the set of the tasks that have been accepted so far, and, for a task \( T_i \) being considered, \( T_i \) is fully allocated if and only if there exists a feasible schedule for \( \mathcal{A} \cup \{ T_i \} \).

Unlike GreedyRTL, the acceptance of a task being considered here does not rely on any specified allocation condition.

Result (iii): the best possible greedy algorithm. We prove that \( \frac{2}{\alpha} \cdot \frac{3}{\alpha} + 1 \) is the best performance guarantee a greedy algorithm in GREEDY can achieve. Based on Results (i) and
(ii), we also propose an improved greedy algorithm GreedyRLM and show that it achieves this best performance guarantee \( \frac{2}{3} \) in GREEDY.

Result (iii) shows that we have no way to design a better greedy algorithm in GREEDY, so we can close the endeavor in this direction. Here, it is worth noticing that GreedyRLM uses the same allocation condition as GreedyRTL so that our result also shows that the specification of such an allocation condition does not hinder the optimal design of a greedy algorithm. The time complexity of GreedyRTL and GreedyRLM is \( O(nDT \max\{T, n\}) \), where \( T \) is the maximal deadline of tasks and \( D \) is the maximal workload of tasks.

1.2.2 Dynamic programming algorithm

When the problem of this paper was first considered in \([3, 4]\), it was pointed out that with the relaxation of the parallelism bound, the model here coincides with the problem of scheduling preemptive tasks on a single machine \([18]\). For this, Lawler \([18]\) gives an exact algorithm in pseudo-polynomial time via dynamic programming. However, \([3, 4]\) also indicates that the algorithm in \([18]\) cannot be extended to our problem with parallelism bound directly. In the course of constructing a dynamic programming algorithm, the primary concern is characterizing the structure of an optimal solution \([13, 19]\). As far as scheduling parallel tasks is concerned, ignoring the objective of our problem, we first need to identify a quantifiable state in which the multiple machines can be said to be optimally utilized by a set of parallel tasks with deadlines, and propose a scheduling algorithm that can achieve this optimal state. Then, based on this, we can define a boundary condition such that there exists a feasible schedule for a set of tasks if and only if it satisfies this condition.

To design a dynamic programming algorithm for our problem, the above considerations are the main difficulties, in contrast to the problem of scheduling on a single machine in \([18]\) where the famous earliest deadline first rule can directly achieve the optimal resource utilization. Fortunately, our previous understanding of greedy algorithms, where we have introduced the notion of optimality of resource utilization in a particular time interval and showed how to achieve the optimality in this interval, has laid the foundation for addressing those difficulties.

Result (iv): the first exact algorithm. For scheduling parallel tasks with deadlines, we derive a boundary condition such that there exists a feasible schedule for a set of tasks if and only if it satisfies this condition. Then, to find the optimal solution, we only need to find a subset of tasks satisfying this boundary condition. We assume that all the deadlines of the tasks belong to a set \( \{\tau_0, \ldots, \tau_L\} \) of cardinal \( L \), where \( 0 = \tau_0 < \tau_1 < \cdots < \tau_L \). The boundary condition can be expressed by a \( L \)-dimensional vector, each coordinate representing a capacity constraint in time interval \( [\tau_{i-1} + 1, \tau_i] \) \( (1 \leq i \leq L) \). We therefore identify a dominant condition and propose a dynamic programming algorithm for our problem (Section 1.2.3). This is the first exact algorithm with a time complexity \( O(\max\{nT^L C^L, nDT \max\{T, n\}\}) \).

1.2.3 A machine minimization problem

Resource utilization can be an indicator of system performance (e.g., high utilization is one of the main goals in the cloud computing field \([4, 5]\)), as well as a significant parameter guiding algorithm design as we saw above. On the other hand, the boundary condition mentioned in the previous subsection in fact identifies a condition under which we can say the \( C \) machines are optimally utilized by a set of parallel tasks with deadlines. Hence, apart from the main goal of this paper, we also incidentally consider a machine minimization problem \([17]\) under the model of this paper. The goal of this problem is to minimize the total number of machines needed to produce a feasible schedule for a given set of tasks.

Result (v): an exact algorithm. We propose an exact algorithm via a binary search with a time complexity \( O(\max\{\ln kn, nDT \max\{T, n\}\}) \) for the machine minimization problem (Section 1.2.3). To the best of our knowledge, this problem is considered for the first time under the model of this paper.

Throughout the paper, the details of omitted proofs can be found in appendix.

1.3 Related Work

The linear programming approaches to designing and analyzing algorithms for our problem \([3, 4]\) and its variants \([5, 6]\) have been well studied. We refer the reader to \([19, 20]\) for more details on the general techniques to design scheduling algorithms. All the works in \([3, 4, 5, 6]\) consider scheduling malleable parallel tasks with the same objective as ours. For a malleable task, the number of machines allocated to it can be changed during its execution. The works \([3, 4]\) consider the same model as ours. In \([3, 5]\), Jain et al. consider the general user valuation functions and propose an algorithm with an approximation factor of \( (1 + \frac{1}{e}) \) via deterministic rounding of linear programming. Subsequently, in \([3]\), Jain et al. propose a greedy algorithm GreedyRTL and use the dual-fitting technique to derive an approximation ratio \( \frac{2}{3} \). In \([6]\), Bodik et al. consider an extension of the model of \([3, 4]\) in which DAG-structured parallel tasks are to be scheduled and, based on randomized rounding of linear programming, they propose an algorithm with an expected approximation ratio of \( \alpha(\lambda) \) for every \( \lambda > 0 \), where \( \alpha(\lambda) = \frac{1}{2} e^{-\frac{1}{2\lambda}} \cdot \frac{1}{1 - e^{-\frac{2\lambda}{2\lambda}}} \cdot \frac{1}{\ln(1 - \frac{1}{\lambda})} \). The online version of our problem is considered in \([4]\) and, again based on the dual-fitting technique, a weighted greedy algorithm is proposed with a competitive ratio of \( 3 + O\left(\frac{1}{(s-1)^2}\right) \) in the case \( 1 < s < 2 \), and \( 2 + O\left(\frac{1}{s}\right) \) in the case \( s \geq 2 \).

Several works have also considered scheduling non-malleable tasks with the same objective as ours \([9, 10]\); however these works assume that all tasks have the same deadline. In \([9]\), Anderson et al. use a direct application of the knapsack problem to their scenario in order to maximize the value. In \([10]\), Jansen and Zhang consider rigid tasks (i.e., tasks that require a fixed number of machines to be executed) and propose an \( \frac{2}{3} \)-approximation algorithm.

2. PRELIMINARIES

2.1 Problem Description and Models

There are \( C \) identical machines (or processors) \([\dagger]\) and a set of tasks \( T = \{T_1, T_2, \ldots, T_n\} \). Each task \( T_i \) is specified by several characteristics: (1) value \( v_i \), (2) demand (or workload) \( D_i \) (3) deadline \( d_i \), and (4) parallelism bound \( k_i \). Time is discrete and we assume that the time horizon is divided \( \dagger \)We use the terms machine and processor interchangeably.
into $T$ time slots: $\{1, 2, \cdots, T\}$, where $T = \max_{t \in T} d_t$. A task $T_i$ can only utilize the machines located in time slot interval $[1, d_i]$. The allocation of machines to a task $T_i$ is a function $y_i: [1, d_i] \rightarrow \{0, 1, 2, \cdots, k_i\}$, where $y_i(t)$ is the number of processors allocated to task $T_i$ at a time slot $t \in [1, d_i]$. The value $v_i$ is obtained only if it is fully allocated by the deadline, i.e., $\sum_{t \leq d_i} y_i(t) \geq D_i$. This means that partial execution of a task yields no value.

The parallelism bound $k_i$ imposes that, at any time slot $t$, $T_i$ can be executed on at most $k_i$ processors simultaneously. Let $k = \max_{t \in T} k_i$ be the maximum parallelism bound. For the system of $C$ machines, denote by $W(t) = \sum_{i=1}^{n} y_i(t)$ the workload of the system at time slot $t$; and by $\overline{W}(t) = C - W(t)$ its complementary, i.e., the amount of available resource at time $t$. We call time $t$ saturated (resp. fully utilized) if $\overline{W}(t) < k$ (resp. $\overline{W}(t) = 0$); and unsaturated otherwise, i.e., if $\overline{W}(t) \geq k$.

We assume that the number $n$ of tasks is so large that $C$ machines cannot process all tasks with their respective requirements satisfied. The objective is to select a subset $S$ of $T$ and produce a feasible schedule for $S$ to maximize the social welfare $\sum_{i \in S} v_i$, that is, the total value of the tasks completed by their deadlines. Here, a feasible schedule means: (1) for a selected task, it is fully allocated and the constraints from the deadline and the parallelism bound are not violated, and (2) the use of the resource is within its capacity $C$ at every time slot, i.e., $W(t) \leq C$ for all $t$.

The following concepts will facilitate the algorithm analysis. Let $len_i = \lfloor D_i/k_i \rfloor$ denote the minimal length of execution time of $T_i$. Denote by $s_i = \frac{d_i}{\overline{W}(d_i)}$ the slackness of $T_i$, measuring the time flexibility of machine allocation (e.g., $s_i = 1$ may mean that $T_i$ should be allocated the maximal amount of machines $k_i$ at every $t \in [1, d_i]$) and let $s = \min_{T_i \in T} s_j$ be the slackness of the least flexible task ($s \geq 1$). Denote by $v'_i = \frac{\overline{W}(d_i)}{s_i}$ the marginal value, i.e., the value obtained by the system per unit of demand executed of the task $T_i$. Finally, let $D = \max_{T_i \in T} \{D_i\}$ be the demand of the largest task; and denote by $[l]$ and $[l]^+$ the sets $\{0, 1, \cdots, l\}$ and $\{1, 2, \cdots, l\}$.

### 2.2 Connection with Knapsack Problem

The knapsack problem captures the essence of many resource allocation problems and has an extensive applications. In this problem, there is a knapsack of capacity $B$ and a set $I$ of $n$ items. With abuse of notation, each item $I_i$ is specified by a size $D_i \geq 0$ and a value $v_i$. Given a subset of items $A \subseteq I$, define $s(A) = \sum_{i \in A} D_i$ and $v(A) = \sum_{i \in A} v_i$. The goal is to choose a subset of items $A$ so that $s(A) \leq B$ and $v(A)$ is maximized. The knapsack problem can be seen as a simplification of our problem with $B = CT$ in the case where $d_i = T$ and $k_i = C$ for all $T_i \in T$. In fact, most of the results in this paper also build on our understanding of the most basic cases of the problem and especially of the case where $d_i = T$; however, we do not present these cases separately.

#### 2.2.1 Greedy algorithm

Assume that $D_i \leq \epsilon B$ for all $I_i \in I$, and that all the items are sorted in non-increasing order of the ratio of their value to their size. Let $l$ be an integer such that $\sum_{i=1}^{l} D_i \leq B$ and $\sum_{i=1}^{l+1} D_i > B$. Then, the solution of selecting the first $l$ items to be packed into the knapsack is a $(1 - \epsilon)$-approximation to the optimal solution. The main idea here is that those $l$ tasks are of the maximal marginal value and then $(1 - \epsilon)$, which can be viewed as resource utilization, will naturally become the performance guarantee since $\sum_{i=1}^{l+1} \frac{v_i}{B(1-\epsilon)} \geq \frac{OPT}{B}$, where $OPT$ is the value achieved by an optimal solution. This seemingly simple idea is one of the key that leads us to obtain a thorough understanding of the greedy algorithms in our problem.

#### 2.2.2 Dynamic programming algorithm

Assume now that the sizes and values of all items are integer. The dynamic programming algorithm constructs a list $A(j)$ for $j = 1, \cdots, n$. Each $A(j)$ is a list of pairs $(b, v)$ in which $(b, v)$ indicates that there is a subset $S$ of $\{I_1, \cdots, I_j\}$ that uses space exactly $b \leq B$ and has value exactly $v$. It starts with $A(1) = \{(0, 0), (D_1, v_1)\}$. For each $j = 2, \cdots, n$, it first sets $A(j) \leftarrow A(j - 1)$, and, for each $(b, v) \in A(j - 1)$, adds the pair $(b + D_j, v + v_j)$ to the list $A(j)$ if $b + D_j \leq B$. To reduce the size of $A(j)$, if there are two pairs $(b', v'), (b'', v'') \in A(j)$ with $b' \leq b''$ and $v' \geq v''$, remove $(b'', v'')$ from $A(j)$. The pair $(b', v')$ is said to be dominated by $(b'', v'')$. Finally, $A(n)$ contains all the non-dominated pairs and the pair $(b, v)$ with the maximal $v$ corresponds to an optimal solution. The computational complexity of this algorithm is $O(n \min(V, B))$, where $V = \sum_{i=1}^{n} v_i$. We refer the reader to [20] for more details. As we will see, in our problem, the constraints from the deadlines and parallelism bounds forces us to derive a more general form of dynamic programming algorithm.

### 2.3 Our Core Definition

For the tasks in our model, we emphasize that the deadline decides the latest time slot $d_i$ in which a task can utilize the machines and the parallelism bound imposes the restriction that $T_i$ can only utilize at most $k_i$ machines at every time slot in $[1, d_i]$. Recall that $T$ denotes the maximal deadline of a set of tasks and let $T'$ be a time slot earlier than $T$, i.e., $T' < T$. Then, we introduce the following definition to identify a sufficient condition for a time interval $[T', T]$ to be optimally utilized by $T$, i.e., to be such that the maximal amount of the total demand of $T$ that could be executed over $[T', T]$ is executed:

**Definition 2.** The interval $[T', T]$ is optimally utilized by $T$ if, for all tasks $T_i \in T$ with $d_i \geq T'$, the following two conditions are satisfied:

1. if $len_i \leq d_i - T' + 1$, then $\sum_{t=T'}^{T'} y_i(t) = D_i$;
2. if $len_i > d_i - T' + 1$, then $y_i(t) = k_i$ for all $t \in [T', d_i]$.

In particular, if there exists no task $T_i \in T$ such that $d_i \geq T'$, $[T', T]$ is optimally utilized by $T$ trivially.

In fact, this definition answers the question of what resource allocation structure for a set of tasks is good. It is also the cornerstone of enabling the applications of the ideas in the knapsack problem to our problem of scheduling parallel tasks with deadlines.

### 3. GREEDY ALGORITHMS

In this section, we give a novel technique to analyze a greedy algorithm in the problem of this paper, answering the questions listed in Section 1.2. Based on this technique, we show that GreedyRTL has a tighter bound of
performance guarantee \( \min \{ \frac{c_m}{t_m^h}, \frac{1}{C} \} \) than the bound \( \frac{c_{m}}{t_{m}^{h}} \) proved in \( \text{[1]} \). We point out that \( \frac{1}{C} \) is the best performance guarantee in a class of greedy algorithms of the general form. The proposed analysis technique also provides an operable method for us to propose an improved greedy algorithm achieving the best performance guarantee \( \frac{1}{C} \).

3.1 Notation

The general form of a greedy algorithm is as follows: (1) consider the tasks in the non-increasing order of the marginal value; and (2) for a task \( T_i \) being considered, accept it and fully allocate it if and only if it satisfies a certain allocation condition. To describe the resource allocation process of a greedy algorithm, we define the sets of consecutive accepted (i.e., fully allocated) and rejected tasks \( A_1, R_1, A_2, \cdots \). Specifically, let \( A_m = \{ T_m, T_{m+1}, \cdots , T_{m+1-1} \} \) be the \( m \)-th set of all the adjacent tasks that are fully allocated after the task \( T_{m-1} \), where \( T_m \) is the first rejected task following the set \( A_m \). Correspondingly, \( R_m = \{ T_m, \cdots , T_{m+1-1} \} \) is the \( m \)-th set of all the adjacent rejected tasks following the set \( A_m \), where \( m \in [K]^{+} \) for some integer \( L \) and \( i_1 = 1 \). Integer \( K \) represents the last step: in the \( K \)-th step, \( A_2 \neq \emptyset \) and \( R_K \) can be empty or non-empty. We also define \( c_m = \max_{T_i \in R_2 \cup \cdots \cup R_m} \{ d_i \} \) and \( c_m' = \max_{T_i \in A_2 \cup \cdots \cup A_m} \{ d_i \} \). In the following, we refer to this generic greedy algorithm as Greedy. While the tasks in \( A_m \cup R_m \) are being considered, we refer to Greedy as being in the \( m \)-th phase. Before the execution of Greedy, we refer to it as being in the 0-th phase.

In the \( m \)-th phase, upon completion of the resource allocation to a task \( T_i \in A_m \cup R_m \), we define \( D_m^{t_1,t_2} = \sum_{t_1}^{t_2} y_i(t) \) to describe the current total allocation to \( T_i \) over \( [t_1, t_2] \). After the completion of \( T_i \), we define \( D_{K+1,i}^{t_1,t_2} = \sum_{t_1}^{t_2} y_i(t) \) to describe the final total allocation to \( T_i \) over \( [t_1, t_2] \). We further define \( T_{K+1,i}^{t_1,t_2} \) as an imaginary task with characteristics \( \{ v_{K+1,i}^{t_1,t_2}, D_{K+1,i}^{t_1,t_2}, d_{K+1,i}^{t_1,t_2}, k_i \} \), where \( v_{K+1,i}^{t_1,t_2} = v_i D_{K+1,i}^{t_1,t_2}/D_i, d_{K+1,i}^{t_1,t_2} = \min \{ t_2, d_i \} \).

3.2 Features of Resource Allocation

In this section, we define two features of the resource allocation structure related to the accepted tasks. We will show that if Greedy can achieve a resource allocation structure satisfying those two features, its performance guarantee can be deduced immediately.

We first introduce an additional notation. Upon completion of the \( m \)-th phase of Greedy, we define the threshold parameter \( t_m^{th} \) as follows. If \( c_m \geq c_m' \), then set \( t_m^{th} = c_m \). If \( c_m < c_m' \), then set \( t_m^{th} \) to a certain time slot in \( [c_m, c_m'] \). We emphasize here that \( d_i \leq t_m^{th} \) for all \( T_i \in \cup_{j=1}^{m} R_j \) hence the allocation to the tasks of \( \cup_{j=1}^{m} R_j \) in \( [t_m^{th} + 1, T] \) is ineffective and yields no value due to the constraint from the deadline. For ease of exposition, we let \( t_0^{th} = 0 \) and \( t_{K+1}^{th} = T \). With this notation, we define the following two features that we will want the resource allocation to satisfy for all \( m \in [K]^{+} \):

**Feature 1.** The resource utilization achieved by the set of tasks \( \cup_{j=1}^{m} A_j \) in \([1, t_m^{th}]\) is at least \( r \), i.e., \( \sum_{T_i \in \cup_{j=1}^{m} A_j} D_{K+1,i}^{t_1,t_2}/(C \cdot t_m^{th}) \geq r \).

Viewing \( T_{K+1,i}^{t_1,t_2} \) as a real task with the same allocation done by Greedy as that of \( T_i \) in \([1, t_m^{th}]\), we define the second feature as:

**Feature 2.** \([t_m^{th} + 1, t_{m+1}^{th}]\) is optimally utilized by \( \{ T_{K+1,i}^{t_1,t_2} \mid T_i \in \cup_{j=1}^{m} A_j \} \).

For ease of the subsequent exposition, we add a dummy time slot 0 but the task \( T_i \in T \) can not get any resource there, that is, \( y_i(0) = 0 \) forever. We also let \( A_0 = R_0 = A_{K+1} = R_{K+1} = \emptyset \).

3.3.3 Theorem: A Novel Analysis Technique

In this section, we prove the following theorem:

**Theorem 1.** If Greedy achieves a resource allocation structure that satisfies Feature \( \text{[2]} \) and Feature \( \text{[3]} \) it gives an \( r \)-approximation to the optimal social welfare.

3.3.1 Scheduling tasks with relaxed constraints

Our analysis is retrospective. We will in this section treat \( T_{K+1,i}^{t_1,t_2} \) as a real task, and consider the welfare maximization problem through scheduling the following tasks:

\[
\begin{align*}
(1) \quad & T_{m+1}^{t_1,t_2} = \{ T_{K+1,i}^{t_1,t_2} \mid T_i \in \cup_{j=1}^{m} A_j \}, \\
(2) \quad & F_{m+1} = \{ T_{K+1,i}^{t_1,t_2} \mid T_i \in \cup_{j=m+1}^{K} A_j \}, \\
(3) \quad & \mathcal{N} = \cup_{j=0}^{K} R_j.
\end{align*}
\]

Here, \( m \in [K] \) and we relax several restrictions on the tasks. Specifically, partial execution can yield linearly proportional value, that is, if \( T_i \) is allocated \( \sum_{t_j} y_i(t) < D_i \) resources by its deadline, a value \( \sum_{t_j} y_i(t) \cdot v_i \) will be added to the social welfare. The parallelism bound \( k_i \) of the tasks \( T_i \) in \( F_{m+1} \cup \mathcal{N} \) is reset to be \( C \).

Let \( T_{m+1} = T_{m}^{t_1,m+1} \cup F_{m+1} \cup \mathcal{N} \). Denote by \( O P T^{[1,t_{m+1}]} \) and \( O P T^{[t_m,t_{m+1}]} \) the optimal social welfare by scheduling \( T_{m+1} \) in the segmented timescales \([1, t_{m+1}]\) and \([t_m, t_{m+1}]\). The connection of the problem above with our original problem is that with the relaxation of some constraints of tasks \( O P T^{[1,t_{m+1}]} \) is an upper bound of the optimal social welfare in our original problem. In the following, we will bound \( O P T^{[1,t_{m+1}]} \) by bounding \( O P T^{[1,t_{m}]} \) and \( O P T^{[t_m,t_{m+1}]} \), \( m \in [K]^{+} \). In the special case where \( t_m = t_{m+1}^{th} \), \( O P T^{[1,t_{m+1}]} \geq O P T^{[1,t_{m}]} \), since \( T_{m+1} = T_{m} \), \( T_{m+1} \cup T_{m}^{t_{m+1}+1} \leq T_{m}^{t_{m+1}+1} \), and the tasks in \( F_{m+1} \) are more relaxed than \( T_{m+1}^{t_{m+1}+1} \); hence we can directly bound \( O P T^{[1,t_{m}]} \) in order to bound \( O P T^{[1,t_{m+1}]} \). We therefore assume that \( t_{m+1} > t_m \) subsequently. The lemma below shows that the bound on \( O P T^{[1,t_{K+1}]} \) can be obtained through bounding each \( O P T^{[t_{m}+1,t_{m+1}]} \) \((m \in [K])\):

**Lemma 2.** \( O P T^{[1,t_{m+1}]} \) \( \leq \) \( O P T^{[1,t_{m}]} + O P T^{[t_{m}+1,t_{m+1}]} \) \((m \in [K]^{+})\).

**Proof.** Consider an optimal schedule achieving \( O P T^{[0,t_{m}]} \). If a task \( T_{K+1,i}^{t_1,t_2} \in T_{m+1}^{t_1,t_2} \cup F_{m+1} \) is allocated more than \( D_{K+1,i}^{t_1,t_2} \) resources in the time slot \([1, t_{m}^{th}]\), we transfer the part larger than \( D_{K+1,i}^{t_1,t_2} \) (at most \( D_{K+1,i}^{t_1,t_2} \)) to
The lemma holds. By Feature 2, the lemma holds.

A schedule can be transformed into a feasible schedule of $T_{m+1}$ plus a feasible schedule of $T_{m+1}$ in $[t^m, t^{m+1}]$. The lemma holds. □

3.3.2 Bound of time slot intervals

In this section, we consider the following schedule of $T_{m+1}$ ($m \in [K]$). Whenever $T_{m+1}$ is concerned, the allocation to the tasks of $T_{m+1}$ at every time slot $t \in [1, t^{m+1}]$ is $T_{m+1} \times \{ t^m \} = \{ t^m \} \frac{t^m}{y_t(t)}$, as is done by Greedy in $[1, t^{m+1}]$ for the set of tasks $T$. Note that the tasks in $N$ are all rejected. We will study how to bound $OPT_{m+1}$ using the above allocation of $T_{m+1}$.

We first observe, in the next two lemmas, what schedule can achieve $OPT_{m+1}$.

Lemma 3. $OPT_{m+1} = \sum_{T_i \in \mathcal{A}_m} A_j \cup_{K+1} \mathcal{A}_{m+1}$.

Proof. When $m = K$, $d_t \leq \frac{t^m}{t^{m+1}}$ for all $T_i \in \mathcal{A}_m$, and $F_{m+1} = \emptyset$. The allocation of $N_{m+1}$ in $[t^m, t^{m+1}, t^{m+1}]$ yields no value. By Feature 2 the lemma holds. □

For ease of exposition, let $\mathcal{F}_{m+1} = \{ T_i \in \mathcal{A}_{m+1} \}$, and $\mathcal{F}_{m+1} = \{ T_i \in \mathcal{A}_{m+1} \} \cup_{K+1} \mathcal{A}_{m+1}$. We also let $N_{m+1} = \cup_{j=0}^{m+1} R_j$, and $N_{m+1} = \cup_{j=0}^{m+1} R_j$. Here, $\mathcal{F}_{m+1} \cup \mathcal{F}_{m+1}$ and $N_{m+1} = N_{m+1} \cup N_{m+1}$.

Lemma 4. With the relaxed constraints of tasks, we have for all $m \in [K-1]$ that $OPT_{m+1}$ can be achieved by the following schedule:

1. $D_{m+1}$ resources are allocated to every task $T_{m+1}$ in $T_{m+1}$;

2. For the unused resources in $[t^{m+1}, t^{m+1}]$, we execute the following loop until there is no available resources or $N_{m+1} \cup N_{m+1}$ is empty; select a task in $\mathcal{F}_{m+1} \cup \mathcal{F}_{m+1}$ with the maximal marginal value and allocate as many resources as possible with the constraint of deadline; delete this task from $\mathcal{F}_{m+1} \cup \mathcal{F}_{m+1}$.

Proof. With the constraint of deadlines, the allocation of $N_{m+1}$ in $[t^{m+1}, t^{m+1}]$ yields no value due to $d_t \leq t^{m+1}$. As a result of the order that Greedy considers tasks, the tasks of $T_{m+1}$ will have higher marginal values than the tasks in $N_{m+1} \cup \mathcal{F}_{m+1}$. By Feature 2 the lemma holds. □

We now bound $OPT_{m+1}$ ($m \in [K-1]$) using the allocation of $T_{m+1}$ specified above. With abuse of notation, for a set of accepted tasks $A$, let $A$ denote both the area occupied by the allocation of the tasks of $A$ in some time slot interval and the size of this area.

Lemma 5. There exist $K$ areas $C_1, C_2, \ldots, C_K$ from $T_{m+1}$ such that

1. the size of each $C_{m+1}$ is $r \cdot C_{i-m}(t^{m+1})$ ($m \in [K-1]$);

2. every $C_{m+1}$ is obtained from the area $T_{m+1}$ in $[t^{m+1}, t^{m+1}] \cup \mathcal{F}_{m+1}$, where $C_{m+1} = \emptyset$, and is the part of that area with the maximal marginal value.

Proof. When $m = 0$, by Feature 2 $\sum_{T_i \in \mathcal{A}_m} D_{m+1} \geq r \cdot C_{i-m}(t^{m+1})$ and the lemma holds. Assume that when $m \leq 0$ ($l \leq 0$), the lemma holds. Now, we prove the lemma holds when $m = l + 1$. By Feature 2 $T_{m+1} \cup \mathcal{F}_{m+1} \geq r \cdot C_{i-m+1}$.

We emphasize here that the tasks in $A_1 \cup R_1 \cup A_2 \cup \cdots \cup R_K$ have been considered and sequenced in the non-increasing order of the marginal value. Recall the composition of $T_{m+1}$ and the allocation to $T_{m+1}$ in fact corresponds to the allocation of $T_{m+1}$ in $[t^{m+1}, t^{m+1}]$. Let $V_{m+1}$ be the total value associated with the area $C_{m+1}$. Since $T_{m+1} \cup \mathcal{F}_{m+1} = \sum_{j=0}^{m+1} C_j \geq 0$ and $T_{m+1} \cup \mathcal{F}_{m+1}$ is obtained $T_{m+1}$, the way that we obtain $C_{m+1}$ in Lemma 4 and the optimal schedule that achieves $OPT_{m+1}$ in Lemma 2 we have that

$$\frac{V_{m+1}}{r \cdot C_{i-m+1}} \geq \frac{OPT_{m+1}}{C_{i-m+1}}.$$ 

The conclusion above in fact shows that the average marginal value of $C_{m+1}$ is no less than the one in an optimal schedule, which uses the same principle as the greedy algorithm in the knapsack problem in Section 2.2.1. Finally, we have that $OPT_{m+1} \leq \frac{V_{m+1}}{r}$ for all $m \in [K-1]$.

By Lemma 2 $OPT_{m+1} \leq \sum_{j=1}^{K} V_{j}/r$. By Lemma 3 we further have that

$$OPT_{m+1} \leq \sum_{j=1}^{K} \frac{V_{j} + \sum_{T_i \in \mathcal{A}_m} A_j}{r} \leq \sum_{T_i \in \mathcal{A}_m} A_j.$$ 

Hence, Theorem 1 holds.

3.4 A Tighter Analysis of GreedyRTL

The GreedyRTL algorithm is presented as Algorithm 1 in the Appendix. Here, we explain its execution process using the new technique introduced above and prove a tighter performance guarantee. Upon the completion of its $m$-th phase ($m \in [K]$), the threshold parameter $t^{m+1}$ is defined as follows:

- if $c_m > c_m$, then set $t^{m+1} = c_m$;
- if $c_m < c_m$, then set $t^{m+1}$ to time slot just before the first unsaturated time slot after $c_m$ or to $c_m$ if there is no unsaturated time slot in $[c_m, c_m]$. The allocation condition $D_t \leq \sum_{l=0}^{K} \min \{ l, t^{m+1}, \frac{t^{m+1}}{l} \}$ decides whether a task $T$ being considered will be accepted by GreedyRTL.
Proposition 6. Upon the completion of GreedyRTL, Feature \( \square \) holds in which \( r = \min \left( \frac{C}{2}, \frac{s}{2} \right) \).

Proof. Upon completion of the \( m \)-th phase of GreedyRTL, consider a task \( T_i \in \bigcup_{j=1}^{m} A_j \) such that \( d_i = c_m \). Since \( T_i \) is not accepted when being considered, it means that \( D_i \leq \sum_{t \leq C_i} \min \{ k_i, W(t) \} \) at that time and there are at most \( \text{len}_i - 1 = \left\lceil \frac{W(t)}{4} \right\rceil - 1 \) time slots \( t \) with \( W(t) \geq k_i \) in \([1, c_m]\). We assume that the number of the current time slots \( t \) with \( W(t) \geq k_i \) is \( \mu \). Since \( T_i \) cannot be fully allocated, we have the current resource utilization in \([1, c_m]\) is at least

\[
\frac{C \cdot d_i - \mu C - (D_i - \mu k_i)}{C \cdot d_i} \\
\geq \frac{C d_i - D_i - (\text{len}_i - 1)(C - k_i)}{C \cdot d_i} \\
\geq \frac{C(d_i - \text{len}_i) + (C - k_i) + (\text{len}_i k_i - D_i)}{C \cdot d_i} \\
\geq \frac{s - 1}{s} \geq r.
\]

We assume that \( T_i \in R_k \) for some \( h \in [m]^+ \). Due to the function of lines 6-8 of AllocateRTL(\( \cdot \)), after the completion of the \( h \)-th phase of GreedyRTL, the subsequent call to AllocateRTL(\( \cdot \)) will never change the current allocation of \( \bigcup_{j=1}^{m} A_j \) in \([1, c_m]\). Hence, if \( t_m = c_m \), the lemma holds with regard to \( \bigcup_{j=1}^{m} A_j \) \( (m \geq h) \); if \( t_m > c_m \), since each time slot in \([c_m + 1, t_m]\) is saturated, the resource utilization in \([c_m + 1, t_m]\) is \( \frac{C - k_i}{C} \) and the final resource utilization will also be at least \( r \). \( \square \)

Proposition 7. Upon the completion of GreedyRTL, Feature \( \square \) holds.

The proposition below follows directly from Theorem 1, Proposition 6 and Proposition 7.

Proposition 8. GreedyRTL gives an \( \min \{ C, \frac{C - k_i + 1}{2} \} \)-approximation to the optimal social welfare.

3.5 Best Possible Greedy Algorithm

Recall that we defined in Definition 1 (Section 1.2.1) a class GREEDY of greedy algorithms of the general form.

Proposition 9. The best performance guarantee that a greedy algorithm in GREEDY can achieve is \( \frac{C}{4} \).

Inspired by Theorem 1, we now focus on the problem of how to increase the parameter \( r \) of Feature \( \square \) in a greedy algorithm to \( \frac{C}{2} \) while Feature \( \square \) still holds, in order to achieve the above best possible performance guarantee \( \frac{C}{4} \).

In Proposition 6, we can see that, to make \( r \) independent of the parameter \( \frac{C}{2} \), we only need to redefine the threshold parameter \( t^h_m \) as follows. Initially set \( t^0_m = 0 \). Upon completion of the \( m \)-th phase of the greedy algorithm for every \( m \in [K]^+ \), define \( t_m \) as follows: if \( c_m \geq c'_m \), then set \( t^h_m = c_m \); if \( c_m < c'_m \), then set \( t^h_m \) to time slot just before the first time slot \( t \) with \( W(t) > 0 \) after \( c_m \) or to \( c'_m \) if there is no time slot \( t \) with \( W(t) > 0 \) in \([c_m, c'_m]\). On the other hand, we also need to make sure that we keep the resource allocation structure satisfying Feature \( \square \) in order to obtain the performance guarantee \( \frac{C}{4} \). For the purposes above, we propose a resource allocation algorithm Allocate-A(\( i \)) for a single task \( T_i \), presented as Algorithm 2. The final algorithm GreedyRLM is presented in Algorithm 1. We refer the reader to the appendix for more details.

Proposition 10. GreedyRLM gives an \( \frac{1}{2} \)-approximation to the optimal social welfare.

Algorithm 1: GreedyRLM

Input: \( n \) jobs with \( \text{type}_i = \{ v_i, d_i, D_i, k_i \} \)

Output: A feasible allocation of resources to jobs

1. initialize: \( y_i(t) \leftarrow 0 \) for all \( T_i \in T \) and \( 1 \leq t \leq T \), \( m = 0 \), \( t^0_m = 0 \);
2. sort jobs in the non-increasing order of the marginal values: \( v'_1 \geq v'_2 \geq \cdots \geq v'_n \);
3. \( i \leftarrow 1 \);
4. while \( i \leq n \) do
5. \( \quad \text{if } \sum_{t \leq C_i} \min \{ W(t), k_i \} \geq D_i \text{ then} \)
6. \( \quad \quad \text{Allocate-A}(i); \quad \text{// in the } (m + 1)-\text{th phase} \)
7. \( \quad \text{else} \)
8. \( \quad \quad \text{Allocate-A}(i); \quad \text{// in the } (m + 1)-\text{th phase} \)
9. \( \quad \text{if } t_{i-1} \text{ has ever been accepted then} \)
10. \( \quad \quad m \leftarrow m + 1; \quad \text{// in the } m-\text{th phase, the} \)
11. \( \quad \quad \text{// allocation to } A_m \text{ was completed; the} \)
12. \( \quad \quad \text{// first rejected task is } T_{j_m} = T_i; \quad \text{end} \)
13. \( \quad \quad \text{while } \sum_{t \leq C_i} \min \{ W(t), k_i \} < D_i \text{ do} \)
14. \( \quad \quad \quad i \leftarrow i + 1; \quad \text{end} \)
15. \( \quad \quad \text{// the last rejected task is } T_{m+1}; \quad \text{end} \)
16. \( \quad \text{if } c_m \geq c'_m \text{ then} \)
17. \( \quad \quad t^0_m \leftarrow c_m; \quad \text{end} \)
18. \( \quad \text{else} \)
19. \( \quad \quad t^0_m \text{ to time slot just before the first time} \)
20. \( \quad \quad \text{slot } t \text{ with } W(t) > 0 \text{ after } c_m \text{ or to } c'_m \text{ if} \)
21. \( \quad \quad \text{there is no time slot } t \text{ with } W(t) > 0 \text{ in} \)
22. \( \quad \quad \text{[c_m, c'_m]}; \quad \text{end} \)
23. \( \quad i \leftarrow i + 1; \quad \text{end} \)

Algorithm 2: Allocate-A(\( i \))

1. Fully-Utilize(\( i \))
2. AllocateRLM(\( i, 0 \))

3.6 Discussion

A greedy algorithm considers the tasks in the non-increasing order of the marginal value. As in the knapsack problem, it is possible that the utilization can imply the performance guarantee of a greedy algorithm with the good design of resource allocation structure. However, the deadline of a task decides the range in which the resource can be utilized. In a greedy algorithm, there inevitably exist cases where some tasks with slightly larger marginal values but smaller deadlines are considered first and then the other tasks with larger deadlines cannot be fully allocated. This leads to a large amount of resources wasted, and the upper bound of the performance guarantee of a greedy algorithm. Hence, to
Algorithm 3: Fully-Utilize($i$)
1 for $t \leftarrow d_i$ to 1 do
2 \hspace{1em} $y_i(t) \leftarrow \min\{k_i, D_i - \sum_{l=t+1}^{d_i} y_i(l), \overline{W}(t)\}$
3 end

Algorithm 4: AllocateRLM($i$, $\eta_1$)
1 $t \leftarrow d_i$
2 while $\sum_{i=1}^{t-1} y_i(\overline{\tau}) > 0$ do
3 \hspace{1em} $\Delta \leftarrow \min\{k_i - y_i(\overline{\tau}), \sum_{l=1}^{t-1} y_i(\overline{\tau})\}$
4 \hspace{1em} Routine($\Delta$, $\eta_1$, 1)
5 \hspace{1em} $\theta \leftarrow \overline{W}(t), y_i(t) \leftarrow (t + \overline{W}(t))$
6 \hspace{1em} let $t''$ be such a time slot that $\sum_{i=1}^{t''-1} y_i(\overline{\tau}) < \theta$ and $\sum_{i=1}^{t''-1} y_i(\overline{\tau}) \geq \theta$
7 \hspace{1em} $\theta \leftarrow \theta - \sum_{i=1}^{t''-1} y_i(\overline{\tau}), y_i(t'') \leftarrow y_i(t'') - \theta$
8 \hspace{1em} for $l \leftarrow 1$ to $t - t''$ do
9 \hspace{2em} $y_i(\overline{\tau}) \leftarrow 0$
10 end
11 $t \leftarrow t - 1$
12 end

Algorithm 5: Routine($\Delta$, $\eta_1$, $\eta_2$)
1 while $\overline{W}(t) < \Delta$ do
2 \hspace{1em} $t' \leftarrow$ the time slot earlier than and closest to $t$ so that $\overline{W}(t') > 0$
3 \hspace{1em} if $\eta_1 = 1$ then \hspace{2em} break
4 \hspace{1em} else \hspace{1em} if $t' \leq t^m_m$, or there exists no such $t'$ then \hspace{2em} break
5 \hspace{1em} else \hspace{1em} let $i'$ be such a job such that $y_i'(t) < y_i'(t')$
6 \hspace{2em} $y_i'(t) \leftarrow y_i'(t) - 1, y_i'(t') \leftarrow y_i'(t') + 1$
8 end

4. MORE APPLICATIONS

In this section, we show that the understanding and techniques described in Section 3 enable the extension of the dynamic programming algorithm of Section 2.2.2 to our problem. We also incidentally give an exact algorithm for the machine minimization problem.

4.1 Notation

Given a set $C$ of machines, the deadlines $d_i$ of all the tasks $T_i \in C$ constitute a finite set $\{\tau_1, \tau_2, \ldots, \tau_L\}$, where $0 = \tau_0 < \tau_1 < \cdots < \tau_L$. Given $L \in \mathbb{N}$, we denote the set of tasks with deadline $\tau_i$ ($i \in [L]$). Let $D_i$ denote the set of tasks with deadline $\tau_i$ and the minimal execution time in $(\tau_i - \tau_{i-1}, \tau_i - \tau_{i-j})$. Assume that the demand of each task is an integer. For a set of tasks $S$, we use its capital $S$ to denote the total demand of the tasks of $S$.

4.2 Optimal Resource Utilization

In this section, we identify a boundary condition such that there exists an efficient feasible schedule for a set of tasks if and only if it satisfies this boundary condition (as indicated in Lemma 12 and Proposition 13 below).

4.2.1 Optimality and boundary condition

Let $S$ be a subset of $C$. Denote $S_i = S \cap D_i$ and $S_{i,j} = S \cap \{D_i \cup D_j \mid i \in [L], j \in [i]^{+}\}$. The following notions are introduced according to Definition 2. Let $\lambda_m(S) = \sum_{m=1}^{L-m+1} \left(\sum_{l=1}^{m} S_{l} + \sum_{j=L-m+1}^{m} \sum_{i=S_{i,j}} k_i(\tau_i - \tau_{L-m})\right)$ for all $m \in [L]$. Here, we emphasize that a task $T_i \in S_{i,j}$ can be fully allocated in $[\tau_L - m, T]$ and a task $T_i \in S_{i,j} (j \in \{L + m + 1, \ldots, L\})$ can utilize at most $k_i(\tau_L - \tau_{L-m})$ resources in $[\tau_L - m + 1, T]$ with the constraints of the deadline and parallelism bound. Hence, ignoring the capacity constraint, $\lambda_m(S)$ represents the maximal resource that $S$ could utilize in time slot interval $[\tau_L - m + 1, T]$. We further define $\lambda_m^C(S) = \lambda_m^C(S) + \min\{\lambda_m(S) - \lambda_m^C(S), C(\tau_L - L-m+1 - \tau_{L-m})\}$, where $\lambda_0^C(S) = 0$ and $m \in [L]$. $\lambda_m^C(S)$ is the maximal (optimal) resource that a set of tasks $S$ could utilize in time slot interval $[\tau_L - m + 1, T]$ on $C$ machines, $m \in [L]$. Let $\mu_m^C(S) = S - \lambda_m^C(S)$ denote the remaining demand that needs to be executed after $S$ has optimally utilized the resource on $C$ machines in $[\tau_m + 1, T]$. The GreedyRLM algorithm in Section 4.3 achieves the optimality of resource utilization in a particular time slot interval. Based on such understanding, we propose Algorithm 1. The objective of this algorithm is to maximize the resource utilization; therefore, we consider the tasks in the non-increasing order of the deadlines.

PROPOSITION 13. SchedulingRTL($S$) gives a feasible schedule for a set of tasks $S$ that satisfies the boundary condition.

LEMMA 14. The time complexities of GreedyRLM, GreedyRLM and SchedulingRTL($S$) are $O(nTD \max\{T, n\})$.

4.3 Welfare Maximization Problem

LEMMA 15. The set of tasks in an optimal solution satisfies the boundary condition.
By Lemma 16 to solve our problem, we only need consider the following problem: If we are given $C$ machines, how can we choose a subset $S$ of tasks in $D_1 \cup \cdots \cup D_L$ such that (1) this subset satisfies the boundary condition, and (2) no other subset of selected tasks achieve a better social welfare?

### 4.3.1 Dynamic programming: choosing the optimal tasks

As in the knapsack problem, one key here is to find a dominant condition to reduce the search space. However, due to the constraints from the deadline and parallelism bound, a new dominant condition is required. Let $F$ be a subset of tasks. From our understanding of the role of parameter $\lambda^j(F)$, we define a $L$-dimensional vector $H(F) = (\lambda^1(F) - \lambda^0(F), \lambda^2(F) - \lambda^1(F), \cdots, \lambda^L(F) - \lambda^L-1(F))$ to imply the optimal resource utilization of a set of tasks in the segmented timescale. We also denote by $v(F)$ the total value of the tasks in $F$. The following definition is then introduced:

**Definition 3 (Dominant Condition).** We say that a pair $(F, v(F))$ dominates another $(F', v(F'))$ if $H(F) = H(F')$ and $v(F) \geq v(F')$.

Once we can identify the structure of an optimal solution implied in Lemma 15 and the corresponding dominant condition, the construction of a dynamic programming procedure to find the optimal subset of tasks is similar to what is done in Section 2.2.2 for the knapsack problem. The specific procedure is proposed in $DP(T)$, presented as Algorithm 12 in the Appendix. Here, we iteratively construct the lists $A(j)$ for all $j \in [n]^+$. Each $A(j)$ is a list of pairs $(F, v(F))$, in which $F$ is a subset of $\{T_1, T_2, \cdots, T_l\}$ satisfying the boundary condition and $v(F)$ is the total value of the tasks in $F$. Each list only maintains all the dominant pairs. Specifically, we start with $A(1) = \{(\emptyset, 0), (\{T_1\}, v_1)\}$. For each $j = 2, \cdots, n$, we first set $A(j) \leftarrow A(j-1)$, and for each $(F, v(F)) \in A(j-1)$, we add $(F, v(F) + \sum_{i \in D_i} y_i(t))$ to the list $A(j)$ if $F \cup \{T_j\}$ satisfies the boundary condition. We finally remove from $A(j)$ all the dominated pairs. $DP(T)$ will select a subset $F$ of $T$ from all pairs $(F, v(F)) \in A(n)$ so that $v(F)$ is maximal.

**Proposition 16.** $DP(T)$ outputs a subset $S$ of $T = \{T_1, \cdots, T_n\}$ such that $v(S)$ is the maximal value subject to the condition that $S$ satisfies the boundary condition. The time complexity of $DP(T)$ is $O(nT^L C^L)$.

### 4.3.2 Exact Algorithm

**Algorithm 8: Fully-Allocate(i)**

```plaintext
1 $t \leftarrow d_i, \Omega \leftarrow D_i - \sum \gamma_i(\bar{t})$
2 while $\Omega > 0$
3 $\Delta \leftarrow \min(k_i - y_i(t), \Omega)$
4 Routine($\Delta, 1, 0$)
5 $y_i(t) \leftarrow y_i(t) + \bar{\gamma}(t), \Omega \leftarrow \Omega - \bar{\gamma}(t)$
6 $t \leftarrow t - 1$
7 end
```

**Algorithm 9: ExactAlgo**

**Input:** A set of tasks $T = \{T_1, T_2, \cdots, T_n\}$

**Output:** A feasible schedule for a subset of $T$

1 $S \leftarrow DP(T)$
2 SchedulingRTL(S)

**Proposition 17.** $ExactAlgo$ gives an optimal solution to the welfare maximization problem with a time complexity $O(\max\{nT^L C^L, nTD \max\{T, n\}\})$.

### 4.3.3 Discussion

As in the knapsack problem, to construct a dynamic programming algorithm, we need to maintain the pairs of the possible state of resource utilization and the corresponding best social welfare. However, we have to use a $L$-dimensional vector to indicate the resource utilization state here. This seems to imply that we cannot make the time complexity of a dynamic programming algorithm polynomial in $L$. Regardless of the theoretical importance of Algorithm 9 as the first exact algorithm for our problem, one may be interested in its implementation aspect. In scenarios like the ones in $[8, 6]$, the tasks are often scheduled periodically, e.g., on an hourly or daily basis, and many tasks have a relatively soft deadline (e.g., finishing after four hours instead of three will not trigger a financial penalty). Then, the scheduler can negotiate with the tasks and select an appropriate set of deadlines $\{\tau_1, \tau_2, \cdots, \tau_L\}$, thereafter rounding the deadline of a task down to the closest $\tau_i$ ($i \in [L]$). By reducing $L$, this could permit to use Algorithm 9 rather than Algorithm 11 which is better if the slackness $s$ is close to one.

### 4.4 Machine Minimization Problem

The boundary condition introduced above identifies the optimal resource utilization state for a set of parallel tasks with deadlines. Apart from the main goal of this paper, the results in Section 4.2 also imply an exact algorithm for a machine minimization problem under the task and machine model of this paper, whose goal is to minimize the total number of machines needed to produce a feasible schedule for a given set of tasks.

**Lemma 18.** For a set of tasks $T$, the minimal number of machines needed to produce a feasible schedule of $T$ is exactly the minimal value of $C$ such that the boundary condition is satisfied.
Proposition 19. There exists an exact algorithm of the time complexity \(O(\max\{\ln kn, nDT \max\{T, n\}\})\) for the machine minimization problem.

Proof. We know if we have \(C = kn\) machines, there must exist a feasible schedule for \(T\) and \(kn\) is an upper bound of the minimal \(C\) in Lemma 15. The binary search can be used to find the minimal \(C\) so that the boundary condition is satisfied. The time complexity of this binary search is \(O(\ln kn)\). With Lemma 18, the lemma holds.

5. CONCLUDING REMARKS

In this paper, we consider the problem of scheduling deadline-sensitive parallel tasks with the objective of maximizing the social welfare. We first propose a novel analysis technique for the greedy algorithm and point out that \(\frac{\gamma - 1}{\gamma}
\) is the best performance guarantee a greedy algorithm of general form can achieve. Using this technique, we obtain several progressive results. We improve the performance guarantee of a previous greedy algorithm from \(\frac{C - 1}{C + 1}
\) to \(\min\{\frac{C - 1}{C + 1}, \frac{1}{\gamma}\}\) and propose an improved algorithm achieving the performance guarantee \(\frac{\gamma - 1}{\gamma}\). We further propose an algorithm that achieves the optimal resource utilization. This brings two results. One is an exact algorithm with the time complexity \(O(\max\{\ln kn, nDT \max\{T, n\}\})\) for the machine minimization problem. Another is the first exact dynamic programming algorithm with a time complexity \(O(\max\{nT^L-C_k, nDT \max\{T, n\}\})\) for our problem, where \(L\) is the number of different deadlines of the tasks.

We are currently extending the analysis technique proposed here to the online version and to the machine minimization problem in an extended model, each task being associated with a release time. Another interesting direction is the extension of the analysis technique and related algorithms here to the extended model in [6], where each task consists of a set of subtasks with precedence constraints.

6. REFERENCES

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A. PRELIMINARIES

A.1 Fully-Utilize(i)

The responsibility of Fully-Utilize(i) is to make the task T_i fully utilize the current available machines at the time slots closest to its deadline with the constraint of parallelism bound. During the execution of Fully-Utilize(i), the allocation to T_i at each time slot t is done from the deadline towards earlier time slots, and T_i is allocated \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t')\} machines at t. \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t')\}, \overline{W}(t) (i) is the maximal amount of machines it can or need to utilize at time slot t with the constraint of parallelism bound and the current available machines. In the algorithm GreedyRLM, since the condition that T_i is accepted to be scheduled is \sum_{t=1}^{T} \min\{k_i, \overline{W}(t)\} \geq D_i, we can ensure that T_i is fully allocated by Fully-Utilize(i).

Lemma 20. Upon completion of Fully-Utilize(i), if \overline{W}(t) > 0, Fully-Utilize(i) will allocate \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t')\} machines to T_i. Further, if D_i - \sum_{t'=t+1}^{d_i} y_i(t') > 0, we have y_i(t) = k_i.

Proof. By contradiction. If T_i is not allocated \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t')\} machines at t with \overline{W}(t) > 0, it should have utilized some \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t')\} - y_i(t), \overline{W}(t) (i) machines when being allocated at t. Further, if we also have D_i - \sum_{t'=t+1}^{d_i} y_i(t') > 0, and T_i is not allocated k_i machines at t, T_i should also have been allocated some more \min\{k_i, D_i - \sum_{t'=t+1}^{d_i} y_i(t'), \overline{W}(t)\}, k_i - y_i(t) machines at t. □

A.2 Routine(\Delta, \eta_1, \eta_2)

Routine(\Delta, \eta_1, \eta_2) mainly focuses on the resource allocation at a single time slot t, and aims to make the number of available machines \overline{W}(t) at t become \Delta by transferring the allocation of other tasks to an earlier time slot that is closest to t but not fully utilized. Specifically, for each loop iteration of Routine(\cdot), let t' be the time slot earlier than but closest to t such that \overline{W}(t') > 0. If such t' exists, there also exists a task T_{t'} such that y_{t'}(t) > y_{t'}(t'). We will subsequently show the existence of such T_{t'} when it is applied to the specific algorithm. Then, it decreases the allocation y_{t'}(t) of T_{t'} at t by 1 and increases its allocation y_{t'}(t') at t' by 1. This operation does not change the total allocation to T_{t'} and violate the parallelism bound k_i of T_{t'} since the current y_{t'}(t') is no more than the initial y_{t'}(t).

Routine(\cdot) will be called in Fully-Allocate(i) and AllocateRLM(i, \eta_1); the latter is further called in Allocate-A(i) and Allocate-B(i). In Fully-Allocate(i), the loop of Routine(\cdot), where \eta_1 = 1 and \eta_2 = 0, will stop whenever the following two conditions is satisfied:

1. the number of current available machines \overline{W}(t) is \Delta;
2. there exists no such t' in the loop.

In AllocateRLM(i, \eta_1), the loop will stop whenever the current state satisfies one of the two conditions above or the condition below that:

(A) in Allocate-A(i), there exists such t' but either \sum_{t'=1}^{t-1} y_i(t') \leq \overline{W}(t) or t' \leq t_i', where T_i \in A_m, \eta_1 = 0 and \eta_2 = 1;

(B) in Allocate-B(i), there exists such t' but \sum_{t'=1}^{t-1} y_i(t') \leq \overline{W}(t), where \eta_1 = 1 and \eta_2 = 1.

A.3 Fully-Allocate(i)

In the algorithm SchedulingRLT(S), when a task T_i is to be allocated by Allocate-B(i), we can not guarantee that Fully-Utilize(i) can fully allocate D_i resources to T_i since we do not have the allocation condition \sum_{t'=1}^{d_i} \min\{k_i, \overline{W}(t')\} \geq D_i here. Hence, we need to use Fully-Allocate(i) as a support to guarantee this. This will be shown subsequently. We now explain the executing process of Fully-Allocate(i).

As for a task T_i, Fully-Allocate(i) also deals with the time slots from its deadline towards earlier time slots one by one. Let \Omega = D_i - \sum_{t'=t+1}^{d_i} y_{t'}(i) denote the partial demand of T_i that remains to be allocated more resources for the full completion of T_i. For a time slot being considered t, its function is first to check whether or not \Omega > 0 and T_i can be allocated more machines at this time slot, namely k_i - y_{t'}(i) > 0. Then, let \Delta = \min\{k_i - y_{t'}(i), \Omega\} and it tries to make the number of available machines at t become \Delta by calling Routine(\Delta, 1, 0). Subsequently, the algorithm allocates the current available machines \overline{W}(t) at t to T_i. Here, upon completion of the loop iteration of Fully-Allocate(i) at t, \overline{W}(t) = 0 if Fully-Allocate(i) has increased the allocation of T_i at t; we will also see that \overline{W}(t) = 0 just before the execution of this loop iteration at t in this case.

Lemma 21. Fully-Allocate(i) will never decrease the allocation y_{t'}(i) of T_i at every time slot t done by Fully-Utilize(i);

If \overline{W}(t) > 0 upon completion of Fully-Allocate(i), we also have that \overline{W}(t) > 0 before the execution of every loop iteration of Fully-Allocate(i).

Proof. The only operations of changing the allocation of tasks occur in line 5 of Fully-Allocate(i) and line 17 of Routine(\cdot). In those processes, there is no operation that will decrease the allocation of T_i. In line 17 of Routine(\cdot), the allocation to T_{t'} at t' will be increased and the allocation to T_{t'} at t is reduced. \overline{W}(t') is increased and \overline{W}(t) is reduced. However, in line 5 of Fully-Allocate(i), \overline{W}(t) will becomes zero and \overline{W}(t) = \overline{W}(i). Hence, the allocation \overline{W}(t) at t is also not decreased upon completion of a loop iteration of Fully-Allocate(i). The lemma holds. □

According to Lemma 21, we further make the following observation. At the beginning of every loop iteration of Fully-Allocate(i), if \Delta > 0, we have that \overline{W}(t) = 0 since the current allocation of T_i at t is still the one done by Fully-Utilize(i) and \Omega > 0; otherwise, it should have been allocated some more machines at t. If there exists a t' such
that $\overline{W}(t') > 0$ in the loop of Routine$(\cdot)$, since the allocation of $T_i$ at $t'$ now is also still the one done by Fully-Utilize$(i)$ and $\Omega > 0$, we can know that $y_i(t') = k_i$. Then, we have that $W(t) - y_i(t) > W(t') - y_i(t')$ and there exists a task $T_i$ such that $y_i(t') < y_i(t)$; otherwise, we will not have that inequality.

In the subsequent execution of the loop of Routine$(\cdot)$, $\overline{W}(t)$ becomes greater than 0 but $\overline{W}(t) < \Delta \leq k_i - y_i(t)$. We still have $W(t) - y_i(t) = C - \overline{W}(t) - y_i(t) > W(t') - k_i = W(t') - y_i(t')$ and such $T_i$ can still be found.

Let $\omega$ denote the last time slot in which Fully-Allocate$(i)$ will increase the allocation of $T_i$. In other words, the allocation of $T_i$ at every time slot in $[1, \omega - 1]$ is still the one achieved by Fully-Utilize$(i)$.

**Lemma 22.** Upon completion of Fully-Allocate$(i)$, $y_i(t) = k_i$ for all $t \in [\omega + 1, d_i]$.

**Proof.** Suppose there exists a time slot $t \in [\omega + 1, d_i]$ at which the allocation of $T_i$ is less than $k_i$. By the definition of $\omega$, we have that $\Omega = D_i - \sum_{j=1}^{d_i} y_i(t) > 0$ and there also exists a time slot $t' \in [1, \omega - 1]$ that is not fully utilized upon completion of the loop iteration of Fully-Allocate$(i)$ at $t$. This contradicts with the stop condition of Routine$(\cdot)$ described in Section [A.3]. The lemma holds.

### A.4 AllocateRLM$(i, \eta_i)$

Without changing the total allocation $\sum_{t=1}^{d_i} y_i(t)$ of $T_i$ in $[1, d_i]$, AllocateRLM$(i, \eta_i)$ takes the responsibility to make the time slots closest to $d_i$ fully utilized by $T_i$ and the other fully allocated tasks with the constraint of parallelism bound, namely, the Right time slots being Loaded Most.

To that end, AllocateRLM$(i, \eta_i)$ considers the time slots $t$ from the deadline of $T_i$ towards earlier time slots one by one. For the current $t$ being considered, if the total allocation $\sum_{t=1}^{d_i} y_i(t)$ of $T_i$ in $[1, t-1]$ is greater than 0, we enter the loop of AllocateRLM$(\cdot)$. Let $\Delta = \min\{k_i - y_i(t), \sum_{t=1}^{d_i} y_i(t)\}$ and $\Delta$ is the maximum extra machines that $T_i$ can utilize at $t$. If $\Delta > 0$, we enter the loop of Routine$(\cdot)$. Here, $\Delta > 0$ also means $y_i(t) < k_i$. When Routine$(\cdot)$ stops, we have that the number of available machines $\overline{W}(t)$ at $t$ is no more than $\Delta$. Let $u_i$ be the last time slot $t'$ in the loop of Routine$(\cdot)$ for $t$ such that it satisfies the condition in line 2 of Routine$(\cdot)$ and passes the verification of lines 3-15. In a different case than the current state here, AllocateRLM$(\cdot)$ does nothing and take no effect on the allocation of $T_i$ at $t$; then set $u_i = u_{i+1}$ if $t < d_i$ and $u_i = d_i$ if $t = d_i$.

Then, AllocateRLM$(i, \eta_i)$ decreases the current allocation of $T_i$ at the earliest time slots in $[1, u_i - 1]$ to 0, and accordingly increase the allocation of $T_i$ at $t$ by $\overline{W}(t)$. Here, upon completion of the loop iteration of AllocateRLM$(\cdot)$ at $t$, $\overline{W}(t) = 0$ if AllocateRLM$(\cdot)$ has taken an effect on the allocation of $T_i$ at $t$; we will also see that $\overline{W}(t) = 0$ just before the execution of this loop iteration at $t$ in this case.

We next show some lemmas to help us identify the resource allocation state. This will further show the existence of $T_i$ in line 16 of Routine$(\cdot)$. We first define a parameter. Upon completion of the loop iteration in AllocateRLM$(\cdot)$ at $t$, if the allocation of $T_i$ at $t$ has ever been changed, let $v_i$ denote the time slot $t''$ in line 6 of AllocateRLM$(\cdot)$, where $\sum_{t=1}^{d_i} y_i(t) = 0$ and $v_i$ is the farthest time slot in which the allocation of $T_i$ is decreased. If $y_i(t)$ is not changed by AllocateRLM$(\cdot)$, set $v_i = v_i + 1$ if $t < d_i$ and $v_i = 1$ if $t = d_i$.

**Lemma 23.** Upon completion of the loop iteration of AllocateRLM$(\cdot)$ at $t$, the allocation at every time slot in $[v_i + 1, T]$ is never decreased since the execution of AllocateRLM$(\cdot)$.

**Proof.** The only operations of decreasing the allocation of tasks occur in lines 6-10 of AllocateRLM$(\cdot)$ and line 17 of Routine$(\cdot)$. The operations in lines 6-10 of AllocateRLM$(\cdot)$ does not affect the allocation in $[v_i + 1, T]$ by the definition of $v_i$. Although the allocation at $t$ is decreased in line 17 of Routine$(\cdot)$, it will finally become $C$ in line 5 of AllocateRLM$(\cdot)$.

**Lemma 24.** While the loop iteration in AllocateRLM$(\cdot)$ for $t$ begins until its completion, if there exists a time slot $t'' \in [v_i + 1, t - 1]$ such that $\overline{W}(t'') > 0$, we have that

1. $\psi_d = \cdots = \psi_t < \psi_t < \cdots < \psi_{u_i}$;

2. $t''$ has never been fully utilized since the execution of AllocateRLM$(\cdot)$;

3. the allocation of $T_i$ at every time slot in $[1, t - 1]$ has never been increased since the execution of AllocateRLM$(\cdot)$.

**Proof.** By the definition of $v_i$ and the way that AllocateRLM$(\cdot)$ decreases the allocation of $T_i$ (lines 6-10), we have $\psi_d = \cdots = \psi_{u_i}$. Further, according to the stop condition 3 of the loop of Routine$(\cdot)$, we always have that $\sum_{t=1}^{d_i} y_i(t) \leq \overline{W}(t)$ and further conclude that $v_i < u_i$ since the allocation at $t'$ cannot be decreased by lines 6-10 of AllocateRLM$(\cdot)$. The allocation at $t'' \in [v_i + 1, t - 1]$ is not decreased by AllocateRLM$(\cdot)$ at any moment since $\psi_d = \cdots = \psi_t < \psi_{t''}$ and $t'' < t$. Lemma 24(2) holds. By Lemma 24(2), the loop iteration for $t$ is also not fully utilized in the previous loop iteration. Hence, $u_i \leq u_i$ for a $T \in [t + 1, d_i]$. Lemma 24(3) holds. So far, AllocateRLM$(\cdot)$ only attempts to increase the allocation of $T_i$ in $[t, d_i]$ and there is no operation to increase its allocation in $[1, t - 1]$. Lemma 24(3) holds.

The lemma below follows directly from Lemma 23(1):
Lemma 23. We also have that $\sum_{t=1}^{\omega} y_i(\overline{t}) > 0$ when AllocateRLM() is in its loop iteration for $t$ and is also in the loop of Routine().

In Allocate-A(i), we can conclude that $\overline{W}(t') > 0$ upon completion of Fully-Utilize() since $t' \geq u_2 > v_1$ and by Lemma 23. We also have that $\sum_{t=1}^{\omega} y_i(\overline{t}) > 0$ upon completion of Fully-Utilize() by Lemma 24(3) and $y_i(t') = k_i$ by Lemma 20. $y_i(t')$ is still $k_i$ in the loop iteration of AllocateRLM() for $t$ since $t' \geq v_2$. By Lemma 24(3) and Lemma 23, we also have $\overline{W}(t) = 0$ currently from $\sum_{t=1}^{\omega} y_i(\overline{t}) > 0$. In Allocate-B(i), one more function Fully-Allocate(i) is called and if such $t'$ exists when Routine(i) is called in AllocateRLM(), we have $t \leq \Delta$ when $\Delta > 0$ and AllocateR-LM() takes an effect on $y_i(t)$ by Lemma 24. We can come to the same conclusion in Allocate-B(i) that $y_i(t') = k_i$ and $\overline{W}(t) = 0$ using an additional Lemma 21.

Finally, in both Allocate-A(i) and Allocate-B(i) we have the same observation as we have made in Fully-Allocate(i): $W(t) - y_i(t) > W(t') - y_i(t')$ and there must exist such a task $T_i$ that $y_i(t') < y_i(t)$; otherwise, we can not have that inequality. In the subsequent loop iterations of Routine(), $\overline{W}(t)$ becomes greater than 0 but $\overline{W}(t) < \Delta \leq k_i - y_i(t)$. We still have $W(t) - y_i(t) = C - \overline{W}(t) - y_i(t) > W(t') - k_i = W(t') - y_i(t')$. Such $T_i$ can still be found.

A.5 Allocate-X(i)

Let $A$ denote the set of the tasks that have been fully allocated so far excluding $T_i$.

Lemma 26. Upon every completion of the allocation algorithm Allocate-A(i) or Allocate-B(i), the workload $W(t)$ at every time slot is not decreased in contrast to the one just before the execution of this allocation algorithm. In other words, if $\overline{W}(t) > 0$ upon its completion, $\overline{W}(t) > 0$ just before its execution.

Proof. We observe the resource allocation state on the whole. Fully-Utilize(i) never change the allocation of any $T_i \in A$ at every time slot. To further prove Lemma 26, we only need to show that, in the subsequent execution of Allocate-A(i) or Allocate-B(i), the total allocation at every time slot $t$ is no less than the total allocation of $A$ at $t$ upon completion of Fully-Utilize(i), i.e., $\sum_{T_i \in A} y_i(t)$.

In Allocate-A(i), when AllocateRLM(i, $\eta_1$) is dealing with a time slot $t$, it does not decrease the allocation at every time slot in $[u_d + 1, T]$ by Lemma 23 where $v_t \leq u_d$. The only operations of decreasing the workload at $t \in [1, v_t]$ come form lines 6-10 of AllocateRLM(i) and they only decrease and change the allocation of $T_i$. The allocation of $T_i \in A$ at every time slot in $[1, v_t]$ is still the one upon completion of Fully-Utilize(i) since we always have $t' \geq u_2 > v_1$ in Routine(i) by Lemma 24(1). Hence, the final workload of $A$ at $t \in [1, v_t]$ is at least the same as the one upon completion of Fully-Utilize(i). The lemma holds in Allocate-A(i).

In Allocate-B(i), we need to additionally consider a call to Fully-Allocate(i). By Lemma 24, the lemma holds upon completion of Fully-Allocate(i). Since $y_i(t) = k_i$ for all $t \in [\omega + 1, d_i]$ by Lemma 24, we have that the subsequent call to AllocateRLM() will take no effect on the workload at $t \in [\omega + 1, d_i]$ and the lemma holds in $[\omega + 1, d_i]$ upon completion of Allocate-A(i). Since $T_i$ is allocated some more $\overline{W}(t)$ machines at $t$ in the loop iteration of Fully-Allocate(i) for $\omega$, $\overline{W}(t)$ becomes 0 and will also be 0 upon completion of AllocateRLM() as we described in its executing process. Upon completion of Fully-Allocate(i), the allocation of $T_i$ in $[1, \omega - 1]$ is still the one done by Fully-Utilize(i) by the definition of $\omega$ and the total allocation of $A$ at every time slot in $[1, \omega - 1]$ is no less than the one upon completion of Fully-Utilize(i). Then, AllocateRLM() will change the allocation in $[1, \omega - 1]$ in the same way as it does in Allocate-A(i) and the lemma holds in $[1, \omega - 1]$. Hence, the lemma holds in Allocate-B(i).

Lemma 27. Upon completion of Allocate-A(i) or Allocate-B(i), if there exists a time slot $t''$ that satisfies: $t'' \in [t''_m - 1, 1]$ (here assume that $T_i \in A_m$) such that $\overline{W}(t'') > 0$ in Allocate-A(i), or $t'' \in [t, 1]$ such that $\overline{W}(t'') > 0$ in Allocate-B(i), we have that

1. in the case that $\eta_{n_1} < d_i - t'' + 1$, $\sum_{i=1}^{\omega} y_i(\overline{t}) = 0$;
2. in the case that $\eta_{n_1} \leq d_i - t'' + 1$, $T_i$ is allocated $k_i$ machines at each time slot $t \in [t'', d_i]$;
3. the total allocation $\sum_{i=1}^{\omega} y_i(\overline{t})$ for every $T_i \in A$ in $[t'', d_i]$ is still the same as the one just before the execution of Allocate-A(i) or Allocate-B(i).

Proof. We observe the stop state of AllocateRLM(). In the case that $\eta_{n_1} < d_i - t'' + 1$, if $\sum_{i=1}^{\omega} y_i(\overline{t}) > 0$, there exists a time slot $t$ such that $y_i(t) < k_i$. The loop of Routine($\Delta, \eta_1, \eta_2$) for $t$ would not stop with the current state by the condition given in Section A.2. Lemma 27(1) holds. In the case that $\eta_{n_1} \geq d_i - t'' + 1$, we have that $\sum_{i=1}^{\omega} y_i(\overline{t}) > 0$ and there is no time slot $t$ such that $y_i(t) < k_i$; otherwise, the loop of Routine($\Delta, \eta_1, \eta_2$) for $t$ would not stop with the current state. Lemma 27(2) holds.

Now, we prove Lemma 27(3). In Allocate-A(i), we discuss two cases on $t''$ just before the execution of the loop iteration of AllocateRLM(i, $\eta_1$) at every $t \in [t''_m - 1, 1]$: $\overline{W}(t'') = 0$ and $\overline{W}(t'') > 0$. We will prove $t'' \leq t'$ in the loop of Routine() since the change of the allocation of $A$ only occurs between $t$ and $t'$ for the task $T_i$.

If there exists a certain loop iteration of AllocateRLM() at $t \in [t''_m - 1, 1]$ such that $\overline{W}(t'') = 0$ initially but $\overline{W}(t'') > 0$ upon completion of this loop iteration, this shows that there exist some operations that decrease the allocation at $t''$. Such operations only occur in lines 6-10 of AllocateRLM() and we have that $t'' < v_t$. Since $v_t < \eta_{n_1} + 1 \leq \cdots \leq u_d$, we have that $t'' \leq t'$ in the loop iteration of Routine() for every $t \in [t''_m - 1, 1]$ by Lemma 24. Here, we also have that AllocateRLM() will do nothing in its loop iteration at $t''$ since $\sum_{i=1}^{\omega} y_i(\overline{t}) = 0$. Hence, Lemma 27(3) holds in this case. In the other case, $\overline{W}(t'') > 0$ just before the execution of every loop iteration of AllocateRLM() for every $t \in [t''_m - 1, 1]$, and we have that $t' < t''$. Just before the execution of AllocateRLM(), we also have that either $y_i(t'') = k_i$ or $\sum_{i=1}^{\omega} y_i(\overline{t}) = 0$ by Lemma 23. Then, the loop iteration of AllocateRLM() will take no effect on the allocation at $t''$. Hence, Lemma 27(3) also holds in this case.

In Allocate-B(i), the additional function Fully-Allocate(i) will be called and we will discuss the positions of $t''$ in $[\omega + 1, d_i]$. If $t'' \in [\omega + 1, d_i]$, AllocateRLM() will take no effect
on the allocation at every time slot in $[t', d_i]$ and we have that $\bar{W}(t') > 0$ upon completion of Fully-Allocate($i$). By Lemma 21 and Lemma 20 we have either $y(t') = k_i$ or $\sum_{t=1}^{t'-1} y(t) = 0$ upon completion of Fully-Utilize($i$). In the latter case, the call to Fully-Allocate($i$) will not cause any effect on the allocation at every time slot in $[1, d_i]$ since $\Omega = \emptyset$ is there. In the former case, we always have $t' \geq t''$ in the loop iteration of Fully-Allocate($i$) for every $t \in [t''+1, d_i]$ and Fully-Allocate($i$) cannot change the allocation of any task at its loop iteration for $t''$. Hence, Lemma 27(3) holds when $t'' \in [\omega + 1, d_i]$. If $t'' = \omega$, AllocateRLM($i$) does not decrease the allocation at $\omega$. We also have that $\bar{W}(t'') > 0$ upon completion of Fully-Allocate($i$), and then either $y(t'') = k_i$ or $\sum_{t=1}^{t''-1} y(t) = 0$. AllocateRLM($i$) will also do nothing at $t''$ (i.e., $\omega$) and Lemma 27(3) holds when $t'' = \omega$.

If $t'' \in [1, \omega - 1]$, we first observe the effect of Fully-Allocate($i$). In the case that $\bar{W}(t'') = 0$ upon completion of Fully-Allocate($i$) but $\bar{W}(t'') > 0$ upon completion of AllocateRLM($i$), we have that the allocation in $[1, t'']$ have ever been decreased and there exists a slot $t'$ such that $\bar{W}(t') > 0$ when AllocateRLM($i$) is in its loop iteration at a certain $t \in [1, \omega]$, where $t'' < u_i \leq t < \omega$. By Lemma 23 this $u_i$ is also not fully utilized upon completion of Fully-Allocate($i$). Further, by Lemma 23 upon completion of every loop iteration of Fully-Allocate($i$) at $t \in [1, d_i]$, $u_i$ is not fully utilized and Lemma 23(3) holds in this case. In the other case, $\bar{W}(t'') > 0$ upon completion of Fully-Allocate($i$). By Lemma 23, $t''$ is not fully utilized upon completion of every iteration at $t \in [d_i, \omega]$, $\bar{W}(t'') > 0$ and the exchange of allocation of $A$ only occurs in two time slots $t$ and $t'$ in Routine($i$) that are in $[t'', d_i]$. Hence, Lemma 23(3) holds in this case. Further, Lemma 23(3) holds upon completion of Fully-Allocate($i$). As we analyze in Allocate-A($i$), Lemma 23(3) still holds upon completion of AllocateRLM($i$).

### B. GREEDY ALGORITHMS

#### B.1 GreedyRTL

In this section, we make some explanation for GreedyRTL and give the omitted proofs in Section 5.3. We first characterize the executing process of GreedyRTL as follows:

1. considers the tasks in the non-increasing order of the marginal value;
2. for a task $T_i$ being considered, if it does not satisfy the allocation condition $\sum_{t \leq d_i} \min\{\bar{W}(t), k_i\} \geq D_i$, set $t^{th}$ as the threshold parameter in the $m$-th phase as we do in GreedyRTL;
3. for a task $T_i$ being considered, if it satisfies the allocation condition, call AllocateRTL($i$) to make $T_i$ fully allocated. AllocateRTL($i$) will allocate the machines to $T_i$ at every time slot from the deadline towards the earlier ones. When $T_i$ is being allocated at $t \in [t^{th} + 1, d_i]$, if there exists such $t'$ that $t' > t^{th}$ and $\bar{W}(t') > 0$, AllocateRTL($i$) tries to make the current $\bar{W}(t)$ become $\Delta = \min\{k_i, D_i - \sum_{t=t^{th}+1}^{t'} y_i(t)\}$ by transferring the allocation of other tasks here to the earlier time slot $t'$, and allocate $\Delta$ machines to $T_i$. If there exist no such $t'$ or $t = t^{th}$, from now on, just allocate $\bar{W}(t)$ machines to $T_i$ for every $t \in [1, t]$ from $t$ towards earlier time slots until $T_i$ is fully allocated.

**Algorithm 10: GreedyRTL**

```
Input : n jobs with type $i = \{v_i, d_i, D_i, k_i\}$
Output: A feasible allocation of resources to jobs
1 initialize: $y_t(i) \leftarrow 0$ for all $T_i \in T$ and $1 \leq t \leq T$,
2 sort jobs in the non-increasing order of the marginal values: $v'_1 \geq v'_2 \geq \cdots \geq v'_n$;
3 $i \leftarrow 1$;
4 while $i \leq n$ do
5     if $\sum_{t \leq d_i} \min\{\bar{W}(t), k_i\} \geq D_i$ then
6         AllocateRTL($i$); // in the ($m + 1$)-th phase
7     else
8         if $T_{i-1}$ has ever been accepted then
9             $m \leftarrow m + 1$; // in the $m$-th phase, the allocation to $A_m$ has been completed
10        and the first rejected task is
11            $T_{m} = T_i$;
12        while $\sum_{t \leq d_i} \min\{\bar{W}(t), k_i\} < D_i$ do
13            $i \leftarrow i + 1$;
14        end
15        /* in the $m$-th phase, the last rejected task is $T_{m+1} = T_i$ and
16        $R_m = \{T_{m+1}, \cdots, T_{m+1}\}$ */
17        if $c_m \leq c'_m$ then
18            $t^{th} \leftarrow c_m$;
19        else
20            set $t^{th}$ to time slot just before the first unsaturated time slot after $c_m$ or to $c'_m$ if there is no unsaturated time slot in $[c_m, c'_m]$;
21        end
22        end
23 $i \leftarrow i + 1$;
```

**Lemma 28.** Upon completion of every loop iteration of AllocateRTL($i$) at a certain $t \in [1, d_i]$ for all $T_i \in A_m$, the time slots $t^{th} + 1, \cdots, t^{th} + 1$ are unsaturated.

**Proof.** We only need to show that the allocation at every time slot is never decreased upon completion of every loop iteration of AllocateRTL($i$) at a certain $t \in [1, d_i]$ for all $T_i \in \cup_{m \in M} A_m$. Then, since the time slot $t^{th} + 1 (m \leq h \leq K)$ is unsaturated upon completion of the $h$-th phase of GreedyRTL by the definition of $t^{th}$, the lemma holds. The only operation of decreasing the allocation of a task occurs in line 11 of AllocateRTL($i$). However, this time slot will become fully utilized once the operation in line 14 or line 18 of AllocateRTL($i$) is done. The allocation at every time slot in $[1, T]$ is never decreased upon completion of every loop iteration.

**Lemma 29.** Upon completion of AllocateRTL($i$) for a task $T_i \in A_m$, $[t^{th} + 1, T]$ is optimally utilized by $T_i$ for all $m \leq j \leq K$.

**Proof.** By Definition 22 we only need show by contradiction that either $\sum_{t=t^{th}+1}^{T} y(t) = D_i$ or $y(t) = k_i$ for all $t \in [t^{th} + 1, T]$. Otherwise, we have $D_i - \sum_{t=t^{th}+1}^{T} y(t) > 0$
Algorithm 11: AllocateRTL($i$)

Input: Task $T_i$ to be allocated
1 $t \leftarrow d_i$
2 while $T_i$ has not been fully allocated do
3 $\Delta \leftarrow \min\{k_i, D_i - \sum_{t'=t+1}^{t_h} y_i(t')\}$
4 while $\overline{W}(t) < \Delta$ do
5 $t' \leftarrow$ closest unsaturated time slot earlier than $t$
6 if $t' \leq t_h$ or no such $t'$ exists then
7 jump to next while
8 end
9 let $i'$ be a job such that $y_i(t) > y_{i'}(t')$
10 while $\overline{W}(t) < \Delta \land y_i(t) > y_{i'}(t')$ do
11 Dec. $y_i(t)$ and Inc. $y_{i'}(t')$ simultaneously
12 end
13 end
14 $y_i(t) \leftarrow \Delta (= \overline{W}(t))$
15 $t \leftarrow t - 1$
16 end
17 while $T_i$ has not been fully allocated do
18 $y_i(t) \leftarrow \overline{W}(t)$
19 $t \leftarrow t - 1$
20 end

and there exists a time slot $t \in [t_h^{i'} + 1, d_i]$ such that $y_i(t) < k_i$. However, upon completion of the loop iteration at $t$ of AllocateRTL($i$), we have $\overline{W}(t_h^{i'} + 1) > k_i$ by Lemma 28 and this contradicts with the condition that the loop iteration at $t$ then stopped with the state above. Hence the lemma holds. □

Lemma 30. $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{m,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]}$ for all $T_i \in A_m$ and $m \leq j \leq K$.

Proof. We only need to show that upon completion of AllocateRTL($i$), the subsequent execution of AllocateRTL($l$) for a task $T_l \in A_{m} \cup \cdots \cup A_{K}$ will not change the total allocation of $T_i$ in $[t_h^{i'} + 1, t_h^{i'} + 1]$. Assume that $T_l \in A_l$, where $m \leq l \leq K$. During the execution of AllocateRTL($i$), upon completion of every loop iteration at $t \in [1, d_l]$, $t_h^{i'} + 1, \ldots, t_h^{i'} + 1$ are unsaturated by Lemma 28. $T_l$ is directly allocated $\Delta$ machines at those time slots and the loop iteration at those time slots will not transfer the allocation here to the earlier time slots since $\overline{W}(t_h^{i'} + 1), \ldots, \overline{W}(t_h^{i'} + 1) > 0$ upon completion of the corresponding loop iterations and we have that $\overline{W}(t) > \Delta$ in line 3 of AllocateRTL($l$) at the very beginning of these loop iterations. We now deal with the case that $t \in (t_h^{i'} + 1, t_h^{i'} + 1]$, where $h \leq f \leq K$. Upon completion of the loop iteration at $t$, $t_h^{i'} + 1$ is unsaturated and we know the allocation only occurs between $t$ and a time slot $t'$ that is no earlier than $t_h^{i'} + 1$. Hence the total allocation of $T_i$ in $[t_h^{i'} + 1, t_h^{i'} + 1]$ never changes. For the case that $t \in [t_h^{i'} + 1, t_h^{i'} + 1]$, due to the function of the threshold parameter $t_h^{i'} + 1$ (line 6 of AllocateRTL($l$)), no allocation at $t$ will be transferred to the time slot that is earlier than $t_h^{i'} + 1$. Hence, the total allocation of $T_i$ in $[t_h^{i'} + 1, t_h^{i'} + 1]$ is not changed. Also due to the function of the threshold parameter, the allocation of $T_i$ at every time slot in $[1, t_h^{i'} + 1]$ is never changed by AllocateRTL($l$). Hence, the lemma holds. □

Proof of Proposition 7. We observe the allocation of a task $T_j \in \bigcup_{j=1}^{m} A_j$ after the completion of GreedyRLY by Lemma 20 and Lemma 21. If $len_i \leq d_i - t_h^{i'} + 1$, we have that $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = 0$. If $d_i - t_h^{i'} + 1 < len_i \leq d_i - t_h^{i'}$, we have that $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{K+1,i}^{[1]} + d_i - t_h^{i'} + 1$ and $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{K+1,i}^{[1]} + d_i - t_h^{i'} + 1$. If $d_i - t_h^{i'} < len_i$, we have that $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{K+1,i}^{[1]} + d_i - t_h^{i'} + 1$ and $D_{K+1,i}^{[t_h^{i'} + 1, t_h^{i'} + 1]} = D_{K+1,i}^{[1]} + d_i - t_h^{i'} + 1$. Hence the lemma holds by Definition 22. □

B.2 Upper Bound of Greedy Algorithms

Proof of Proposition 9. Let us consider a special instance:
1. Let $D = \{d_1', d_2'\}$ be the set of deadlines, where $d_2', d_1' \in \mathbb{Z}^+$ and $d_2' > d_1'$. Let $D_i = \{T_i \in T | d_i = d_i'\}$ (1 ≤ i ≤ 2).
2. Let $\epsilon \in (0, 1)$ be small enough. For all $T_i \in D_1$, (a) $\epsilon d_1' = 1 + \epsilon$ and $k_1 = 1$; (b) There are $C \cdot d_1'$ tasks $T_i$ with $D_i = 1$.
3. $\epsilon d_1' = 1, k_1 = 1$ and $D_i = d_2' - d_1' + 1$ for all $T_i \in D_2$. GREEDY will always fully allocate resource to the tasks in $D_1$, with all the tasks in $D_2$ rejected to be allocated any resource. The performance guarantee of GREEDY will be no more than $\frac{C \cdot d_1'}{\epsilon d_1' - 1 + C \cdot d_2'}$. Further, with $\epsilon \to 0$, this performance guarantee approaches $\frac{d_1'}{d_2'}$. In this instance, $s = \frac{d_2'}{d_2' - 1}$ and $\frac{1}{s} = \frac{d_2'}{d_2'}$. When $d_2' \to +\infty$, $\frac{1}{s} = \frac{1}{s}$.
Hence, this proposition holds. □

B.3 GreedyRLM

This section is supplementary to Section 4.9. We will prove that Feature 4 and Feature 5 hold in GreedyRLM, in which two functions are to be called: Fully-Utilize($i$) and AllocateRLM($i, 0$). The way to show this is similar to our proofs in GreedyRTL.

Proposition 31. Upon completion of GreedyRLM, Feature 4 holds in which $r = \frac{1}{s+1}$.

Proof. Upon completion of the $m$-th phase of GreedyRLM, consider a task $T_i \in \bigcup_{j=1}^{m} R_j$ such that $d_i = c_m$. Since $T_i$ is not accepted when being considered, it means that $D_i \leq \sum_{t=1}^{d_i} \min\{k_i, \overline{W}(t)\}$ at that time and there are at most $len_i - 1 = \left\lceil \frac{c_m}{s+1} \right\rceil - 1$ time slots $t$ with $\overline{W}(t) \geq k_i$ in $[1, c_m]$. We assume that the number of the current time slots $t$ with $\overline{W}(t) \geq k_i$ is $\mu$. Since $T_i$ cannot be fully allocated, we have the current resource utilization in $[1, c_m]$ is at least

\[
C \cdot d_i' - \mu C - (D_i - \mu k_1) > C \cdot d_i' - (\mu - 1) C - k_1
\]

\[
> C d_i' - (len_i - 1)(C - k_1)
\]

\[
> C d_i' - len_i + (C - k_1) + (len_i k_1 - D_i)
\]

\[
> \frac{s - 1}{s} \geq r.
\]

We assume that $T_i \in R_h$ for some $h \in [m]$. AllocateA($j$) consists of two functions: Fully-Utilize($j$) and AllocateRLM($j, 0$). Fully-Utilize($j$) will not change the allocation to
the previous accepted tasks at every time slot. In AllocateRLM(j, 0), the operations of changing the allocation to other tasks happen in its call to Routine(Δ, 0, 1). Due to the function of lines 9-11 of Routine(Δ, 0, 1), after the completion of the h-th phase of GreedyRLM, the subsequent call to Allocate-A(j) will never change the current allocation of \( \cup_{j=1}^{m}A_j \) in \([1, c_m]\). Hence, if \( t_{lm}^h = c_m \), the lemma holds with regard to \( \cup_{j=1}^{m}A_j \) (\( m \geq h \)); if \( t_{lm}^h > c_m \), since each time slot in \([c_m + 1, t_{lm}^h]\) is fully utilized, the resource utilization in \([c_m + 1, t_{lm}^h]\) is 1 and the final resource utilization will also be at least 1.

**Lemma 32.** Due to the function of the threshold \( t_{lm}^h \) and its definition, we have for all \( T_i \in A_m \) that

1. \( [t_j^h + 1, 1] \) is optimally utilized by \( T_j \) upon completion of Allocate-A(\( i \)), where \( m \leq j \leq K \);

2. \( D_{K+1}^{[t_j^h, t_{lm}^h+1]} = D_{m+1}^{[t_j^h, t_{lm}^h+1]} \).

**Proof.** The time slots \( t_{hm}^h+1, \ldots, t_{K}^h+1 \) are not fully utilized upon completion of Allocate-A(\( i \)) by Lemma 29 and the definition of \( t_{lm}^h \). By Lemma 27(1) and (2), \( T_i \) fully utilizes the time slots in \([t_j^h + 1, 1] \), namely, either \( \sum_{t_m = t_j^h + 1}^{t_{lm}^h + 1} y(t) \) or \( y_i(t) = k_i \) for all \( t \in [t_j^h + 1, d_i] \) upon completion of Allocate-A(\( i \)).

In the subsequent execution of Allocate-A(\( i \)) for \( T_i \in A_h \), where \( m \leq h \leq K \), due to the fact that \( t_{lm}^h + 1, \ldots, t_{K}^h + 1 \) are not fully utilized upon completion of Allocate-A(\( i \)), the total allocation of \( T_i \) in \([t_j^h + 1, 1] \) keeps invariant by Lemma 27(3) for \( h \leq j \leq K \). Due to the function of the threshold parameter \( t_{k}^h \) in the h-th phase of GreedyRLM (line 9 of Routine(\( j \))), when AllocateRLM(\( i, 0 \)) is dealing with a time slot \( t \in [t_{kh}^h + 1, d_i] \), the allocation change of other tasks can only occur in \([t_{kh}^h + 1, t] \). Hence, we have that the total allocation of \( T_i \) in \([t_{kh}^h + 1, 1] \) keeps invariant. We can therefore conclude that the total allocation of \( T_i \) in each \([t_{kh}^h + 1, t_{lm}^h] \) keeps invariant upon every completion of Allocate-A(\( i \)). Hence the lemma holds.

**Proposition 33.** \([t_{lm}^h + 1, t_{lm}^h + 1] \) is optimally utilized by \( \{T_{K+1}^{[t_j^h, t_{lm}^h+1]} | T_i \in \cup_{m=n}^{m=1}A_i \} \).

**Proof.** The proof of this proposition is the same as the one in Proposition 4 given Lemma 29.

**C. MORE APPLICATIONS**

**Proof of Lemma 11** By induction. When \( m = 0 \), the lemma holds trivially. Assume that this lemma holds when \( m = l \). If \( \lambda_{l+1}(S) - \lambda_l(S) < C(\tau_{l-1} - \tau_{l-1}) \), it means that with the capacity constraint in \([\tau_{l-1} - 1, 1] \), \( S \) can still utilize at most \( \lambda_l(S) \) resources in \([\tau_{l-1} + 1, 1] \) and this lemma holds; otherwise, after \( S \) has utilized the maximal amount of \( \lambda_l(S) \) resources in \([\tau_{l-1} - 1, 1] \), it can only utilize at most \( C(\tau_{l-1} - \tau_{l-1}) \) resources in \([\tau_{l-1} + 1, 1] \). The lemma holds when \( m = l + 1 \).

**Proof of Lemma 12** By the implication of the parameter \( \lambda_{m}^{C_m}(S) \) in Lemma 10, after \( S \) has optimally utilized the machines in \([\tau_m + 1, 1] \), if there exists a feasible schedule for \( S \), the total amount of the remaining demands in \( S \) should be no less than the capacity \( C \tau_m \) in \([1, \tau_m] \).

In the following, we let \( A \) always denote the set of tasks that have been fully allocated so far excluding \( T_i \) and we will prove Proposition 13. We first give the following lemma:

**Lemma 34.** Let \( m \in [L]^+ \). Suppose \( T_i \in S_{L-m+1} \) is about to be allocated. If we have the relation that \( \bar{W}(1) \geq \bar{W}(2) \geq \cdots \geq \bar{W}(T_{L-m+1}) \) before the execution of Allocate-B(\( i \)), such relation on the available machines at each time slot still holds after the completion of Allocate-B(\( i \)).

**Proof.** We observe the executing process of Allocate-B(\( i \)). Allocate-B(\( i \)) will call three functions Fully-Utilize(\( i \)), Fully-Allocate(\( i \)) and AllocateRLM(\( i, 1 \)). In every call to those functions, the time slots \( t \) will be considered from the deadline of \( T_i \) towards earlier time slots. During the execution of Fully-Utilize(\( i \)), the allocation to \( T_i \) at \( t \) is \( y_i(t) = \min\{k_i, D_i - \sum_{j=1}^{d_i-1} y_i(t_j)\} \). Before time slots \( t \) and \( t + 1 \) are considered, we have \( \bar{W}(t) \geq \bar{W}(t + 1) \). Then, after those two time slots are considered, we still have \( \bar{W}(t) \geq \bar{W}(t + 1) \).

During the execution of Allocate-B(\( i \)), let \( t \) always denote the current time slot such that \( \bar{W}(t') \geq \bar{W}(0) \) and \( \bar{W}(t') = 0 \) if such time slot exists. \( t \) is also unique when the relation on the available machines holds. By Lemma 29 if \( \Omega > 0 \) at the very beginning of the execution of Fully-Allocate(\( i \)), we have \( y_i(t) = \cdots = y_i(t') = k_i \) upon completion of Fully-Utilize(\( i \)) and the allocation of \( T_i \) at every time slot in \([1, t'] \) will not be changed by Fully-Allocate(\( i \)) since \( \Delta = 0 \) then. When Fully-Allocate(\( i \)) is considering a time slot \( t (t > t') \), it will transfer partial allocation of \( T_i \) at \( t \) to the time slot \( t' \). If \( t' \) becomes fully utilized, \( t' - 1 \) becomes the current \( t \) and Routine(\( ) \) will make time slots fully utilized one by one from \( t' \) towards earlier time slots. In addition, the time slot \( t' \) will again become fully utilized by line 5 of Fully-Allocate(\( i \)), i.e., \( \bar{W}(t') = 0 \). The allocation at every time slot in \([1, t' - 1] \) are still the one upon completion of Fully-Utilize(\( i \)) and the allocation at \( t' \) is never decreased. Hence, the relation on the available machines still holds upon completion of every loop iteration of Fully-Allocate(\( i \)) for \( t \in [1, d_i] \).

From the above, we have the following facts upon completion of Fully-Allocate(\( i \)) and the allocation of \( A \) in \([1, t'] \) is still the one upon completion of Fully-Utilize(\( i \)). The allocation of \( A \) in \([1, t' - 1] \) is still the one just before the execution of Allocate-B(\( i \)), and (3) the allocation of \( A \) at \( t' \) is not decreased in contrast to the one just before the execution of Allocate-B(\( i \)). Hence, we have that \( C - \sum_{T_i \in A} y_i(t) \geq C - \sum_{T_i \in A} y_i(t') \).

Upon completion of every loop iteration of AllocateRLM(\( ) \) at \( t \in [t' + 1, d_i] \), \( u_t \equiv t' \) or \( t' + 1 \). When AllocateRLM(\( ) \) is considering \( t \), the time slots from \( t' \) towards earlier time slots will become fully utilized in the same way as Fully-Allocate(\( i \)). Hence, the relation on the number of available machines holds obviously in \([t' + 1, d_i] \) since every time slot there is fully utilized. Since \( v_{t'} < u \) by Lemma 24 the allocation of \( A \) in \([1, v_{t'}] \) has not been changed since the execution of AllocateRLM(\( ) \), and we still have that the relation on the number of available machines in holds in \([1, v_{t'}] \) in both the case where \( v_t < t' \) and the case where \( v_t = t' \). Here, if \( v_t < t' \), the allocation at \( t' \) is never decreased by AllocateRLM(\( ) \) and further if \( t' - 1 \geq v_t + 1 \), the allocation at every \( t \in [v_t + 1, t' - 1] \) has also not been changed so far by AllocateRLM(\( ) \). Hence, we can conclude that Lemma 33 holds upon completion of the loop iteration at \( t' + 1 \). Then,
if \( t' = v_{t'+1} \), AllocateRLM(\cdot) will not take an effect on the allocation in \([1, t']\) and the lemma holds naturally. Otherwise, \( t' > v_{t'+1} \) and the allocation of \( T_i \) in \([v_{t'+1} + 1, t']\) is still the one done by Fully-Utilize(\cdot) and the allocation in \([v_{t'+1} + 1, t']\) is also not decreased in contrast to the one upon completion of Fully-Utilize(\cdot) (i.e., \( W'(t) > 0 \) then). By Lemma 20, we conclude that AllocateRLM(\cdot) will not take an effect on the allocation in \([1, t']\) and the lemma holds upon completion of AllocateRLM(\cdot).

**Lemma 35.** Upon completion of Allocate-B(i) for a task \( T_i \in \mathcal{S}_{L = m+1} \ (m \in [L]) \), we have that

1. Let \( t_1, t_2 \in [1, \tau_{L = m+1}] \) and \( t_1 < t_2 \); then, \( W(t_1) > W(t_2) \);
2. \( T_i \) is fully allocated;
3. \( \lambda^c_j(\mathcal{A} \cup \{T_i\}) \) resources in \([\tau_{L = j} + 1, \tau_L]\) have been allocated to \( \mathcal{A} \cup \{T_i\} \) (1 \( \leq j \leq L \).

**Proof.** By induction. Initially, \( \mathcal{A} = \emptyset \). When the first task \( T_i = T_{j\ 1} \) in \( \mathcal{S}_{j} \) is being allocated, Lemma 35(1) holds by Lemma 24. Since the lines 1-3 of Algorithm 5 will allocate \( \min\{k, D_l - 2v_{l+1}, y_l(t), W(t)\} \) machines to \( T_i \) from its deadline towards earlier time slots, and the single task can be fully allocated definitely, the lemma holds. We assume that when the first \( l \) tasks in \( \mathcal{S}_i \) have been fully allocated, this lemma holds. Assume that this lemma holds just before the execution of Allocate-B(i) for a task \( T_i \in \mathcal{S}_{L = m+1} \). We now show that this lemma also holds upon completion of Allocate-B(i). By Lemma 34, Lemma 35(1) holds upon completion of Allocate-B(i). Allocate-B(i) makes no change to the allocation of \( \mathcal{A} \) in \([\tau_{L = m+1} + 1, \tau_L]\) due to the deadline \( d_l \) and Lemma 35(3) holds in the case that \( j \in [m - 1]^+ \) by the assumption. Here, if \( m = 1 \), the conclusion above holds trivially. Let \( t' \) always denote the current time slot such that \( W(t') > 0 \) and \( W(t' + 1) = 0 \) if such time slot exists. If such time slot does not exist upon completion of Allocate-B(i), \( T_i \) has been fully allocated since \( \mathcal{S} \) satisfies the boundary condition. Now, we discuss the case that \( W(1) > 0 \) upon completion of Allocate-B(i). By Lemma 24, we know that \( T_i \) has also been fully allocated, and Lemma 35(2) holds upon completion of Allocate-B(i). Assume that \( t' \in (\tau_{L = l'} + 1, \tau_{L = l'} + 1) \). By the definition of \( t' \), Lemma 35(3) holds in the case that \( m \leq j \leq l' - 1 \) obviously. By Lemma 27, \( T_i \) has already optimally utilized the resource in \([\tau_{L = j} + 1, \tau_L]\) for all \( l' \leq l \leq L \) and so has the set \( \mathcal{A} \) together with the assumption. Lemma 35(3) holds.

**Proof of Proposition 11** By Lemma 24, the proposition holds when all the tasks in \( \mathcal{S} \) have been considered in the algorithm SchedulingRTL(S).

**Proof of Lemma 14** The time complexity of Allocate-A(\cdot) comes from AllocateRLM(i, \( \eta_i \)) and the time complexity of Allocate-B(i) comes from Fully-Allocate(\cdot) or AllocateRLM(\cdot). In Allocate-B(i), in the worst case, Fully-Allocate(\cdot) and AllocateRLM(\cdot) have the same time complexity from the execution of Routine(\cdot) for every time slot \( t \in [1, d_i] \). In the call to AllocateRTL(i) or AllocateRLM(\cdot) for every task \( T_i \in \mathcal{L} \), the loop iteration there for all \( t \in [1, d_i] \) needs to seek for the time slot \( t' \) and the task \( T_{i'} \) up to at most \( D_i \) times. The time complexity of seeking for \( t' \) is \( O(T) \); the time complexity of seeking for \( T_{i'} \) is \( O(n) \). Since \( |\mathcal{T}| = n \), \( d_i \leq T \), and \( D_i \leq D \), we have that the time complexities of those algorithms are \( O(nDT \max\{T, n\}) \).

**Algorithm 12: DP(\mathcal{F})**

1. \( \mathcal{F} \leftarrow \{T_1\} \);
2. \( A(1) \leftarrow (\emptyset, 0, (\mathcal{F}, v(\mathcal{F}))) \);
3. for \( j \leftarrow 2 \) to \( n \) do
4. \( A(j) \leftarrow A(j-1) \);
5. for each \( (\mathcal{F}, v(\mathcal{F})) \in A(j-1) \) do
6. if \( \{T_j\} \cup \mathcal{F} \) satisfies the boundary condition then
7. if there exist a pair \((\mathcal{F}', v(\mathcal{F}')) \in A(j)\) such that (1) \( H(\mathcal{F}') = H(\mathcal{F} \cup \{T_j\}) \), and (2) \( v(\mathcal{F}') \geq v(\mathcal{F} \cup \{T_j\}) \) then
8. Add \((\{T_j\} \cup \mathcal{F}, v(\{T_j\} \cup \mathcal{F}))\) to \( A(j) \);
9. Remove the dominated pair \((\mathcal{F}', v(\mathcal{F}'))\) from \( A(j) \);
end
else
10. Add \((\{T_j\} \cup \mathcal{F}, v(\{T_j\} \cup \mathcal{F}))\) to \( A(j) \);
end
end
end
end
return arg \( \max_{(\mathcal{F}, v(\mathcal{F})) \in A(n)} v(\mathcal{F}) \);

**Proof of Lemma 15** The lemma follows directly from Lemma 12.

**Proof of Proposition 14** The proof is very similar to the one for knapsack problem in [20]. By induction, we need to prove that \( A(j) \) contains all the non-dominated pairs corresponding to feasible sets \( \mathcal{F} \in \{T_1, \ldots, T_j\} \). When \( j = 1 \), the proposition holds obviously. Now suppose it hold for \( A(j - 1) \). Let \( \mathcal{F}' \subseteq \{T_1, \ldots, T_j\} \) and \( H(\mathcal{F}') \) satisfies the boundary condition. We claim that there is some pair \((\mathcal{F}', v(\mathcal{F}')) \in A(j)\) such that \( H(\mathcal{F}') = H(\mathcal{F}) \) and \( v(\mathcal{F}') \geq v(\mathcal{F}) \). First, suppose that \( T_j \notin \mathcal{F}' \). Then, the claim follows by the induction hypothesis and by the fact that we initially set \( A(j) \) to \( A(j - 1) \) and removed dominated pairs. Now suppose that \( T_j \in \mathcal{F}' \) and let \( \mathcal{F}' = \mathcal{F}' - \{T_j\} \). By the induction hypothesis there is some \((\mathcal{F}_1, v(\mathcal{F}_1)) \in A(j - 1)\) that dominates \((\mathcal{F}, v(\mathcal{F}))\). Then, the algorithm will add the pair \((\mathcal{F}_1 \cup \{T_j\}, v(\mathcal{F}_1 \cup \{T_j\}))\) to \( A(j) \). Thus, there will be some pair \((\mathcal{F}', v(\mathcal{F}')) \in A(j)\) that dominates \((\mathcal{F}', v(\mathcal{F}'))\). Since the size of the space of \( H(\mathcal{F}) \) is no more than \( C^d T^L \), the time complexity of DP(\( \mathcal{F} \)) is \( nC^d T^L \).

**Proof of Proposition 14** This proposition follows from Proposition 10, Lemma 15, Proposition 14, and Lemma 13.

**Proof of Lemma 14** This proposition follows directly from Lemma 12 and Proposition 14.

**D. THE ALGORITHM DP(\( \mathcal{T} \))**