CLASSIFICATION OF SPHERICAL ALGEBRAIC SUBALGEBRAS OF REAL SIMPLE LIE ALGEBRAS OF RANK 1

LISA KNAUSS AND CHRISTIAN MIEBACH

Abstract. We determine all spherical algebraic subalgebras in any simple Lie algebra of real rank 1.

1. Introduction

Let $U$ be a complex reductive group with maximal compact subgroup $U$. It has been shown in [Bri87] (see also [HW90]) that smooth compact complex spherical $U^\mathbb{C}$-varieties $Z$ may be characterized by the fact that a moment map $\mu: Z \to u^*$ separates the $U$-orbits in $Z$.

In [HS07] it has been shown that the so-called gradient maps are the right analogue for moment maps when one is interested in actions of a real reductive group $G = K \exp(p)$. Spherical gradient manifolds have been introduced in [MS10] in order to carry over Brion’s theorem to the real reductive case. To be more precise, we call a $G$-gradient manifold $X \subset Z$ with gradient map $\mu_p: X \to p$ spherical if a minimal parabolic subgroup of $G$ has an open orbit in $X$. If $G$ is connected complex reductive, then a minimal parabolic subgroup is the same as a Borel subgroup of $G$, so that there is no ambiguity in this definition. The main result of [MS10] states that $X$ is spherical if and only if $\mu_p$ almost separates the $K$-orbits in $X$.

Very recently, real spherical manifolds have attracted attention from the representation theoretical viewpoint (see [KO13] and [KS13b]) as well as from a geometric one (see [KS13a] and [KKS13]). In [KS13a], the authors have shown that, given a homogeneous real spherical manifold $X = G/H$, any minimal parabolic subgroup of $G$ has only finitely many orbits in $X$. Moreover, the same paper contains the list of all reductive spherical subalgebras of $g = \mathfrak{so}(n, 1)$. In [Ma74] the author has found all decompositions of $\mathfrak{so}(n, 1)$ as the sum of two subalgebras.

In this paper we classify all spherical algebraic subalgebras of $g$ where $g$ is a simple Lie algebra of real rank 1 by methods in the spirit of [MS10]. Although a subalgebra $h$ of $g$ is spherical whenever $h_\mathbb{C}$ is a complex spherical subalgebra of $g_\mathbb{C}$, we would like to stress the fact that the converse is not true (see the example in Section 3). In particular one cannot reduce the question to the complex classification. The tables containing all spherical algebraic subalgebras are given in Theorems 5.3 and 5.4 for $g = \mathfrak{so}(n, 1)$, in Theorems 6.4 and 6.5 for $g = \mathfrak{su}(n, 1)$, in Theorems 7.3 and 7.4 for $g = \mathfrak{sp}(n, 1)$, and in Theorems 8.3 and 8.6 for the exceptional Lie algebra $g = \mathfrak{f}_4 = \mathfrak{f}_4(-20)$.

Let us outline the main steps of the proof as well as the organization of this paper. In Section 2 we show that the homogeneous manifold $X = G/H$ admits a $G$-gradient map if and only if $H$ is an algebraic subgroup of $G$. This is the reason why we classify spherical algebraic subalgebras. In Section 3 we characterize reductive and non-reductive spherical algebraic subgroups $H$ of $G$ by the fact that a maximal compact subgroup of $H$ acts transitively on the spheres in a certain representation related to the inclusion $H \hookrightarrow G$. More precisely our starting point is the following
Proposition 1.1. Let $G = K \exp(p)$ be a connected simple Lie group of real rank 1.

1. Let $H = K_H \exp(p_H)$ be a reductive algebraic subgroup of $G$. Then $H$ is spherical if and only if $K_H$ acts transitively on the connected components of the spheres in $p_H$.

2. Let $H = M_H A_H N_H$ be a non-reductive algebraic subgroup with $\dim n_H^1 \geq 2$. Then $H$ is spherical if and only if $A_H = A$ and $M_H$ acts transitively on the spheres in $n_H^1$.

The main technical work consists in reducing considerations to a normal form of the Lie algebra of $H$ where we can carry out explicit calculations. Then we make essential use of Onishchik’s classification of transitive actions on spheres in order to single out those normal forms that are spherical. More details on this general scheme are given in Section 4. In the remaining Sections 5 to 8 we carry out the necessary calculations case by case for $so(n,1)$, $su(n,1)$, $sp(n,1)$ and $f_4$.

After this paper was finished, we learned that Kimelfeld has considered the classification problem of algebraic spherical subgroups in real simple Lie groups $G = K \exp(p)$ of rank 1, too. In [Ki87] he obtains Proposition 1.1 by differential geometric arguments based on the Karpelevich compactification of the hyperbolic space $G/K$ and then gives a general description of all spherical algebraic subgroups of $G$. However, he does not provide an explicit list of all spherical algebraic subalgebras of $g$.

Acknowledgments. We would like to thank Peter Heinzner and Valdemar Tsanov for helpful discussions on the subject presented here. We are much obliged to Friedrich Knop for informing us about Kimelfeld’s paper [Ki87] as well as about an inaccuracy in the statement of Theorem 5.3.

Notation. We denote Lie groups by upper case roman letters while their Lie algebras are denoted by the corresponding lower case gothic letters, i.e., $g = \text{Lie}(G)$.

2. (Homogeneous) Gradient Manifolds

In this section, $G$ denotes a connected semisimple Lie group that embeds as a closed subgroup into its universal complexification $G^C$. Let $U$ be a compact real form of $G^C$ such that $G$ is stable under the corresponding Cartan involution of $G^C = U^C$. Then we obtain the Cartan decomposition $K \times p \to G$, $(k, \xi) \mapsto k \exp(\xi)$, where $K := G \cap U$ is a maximal compact subgroup of $G$ and where $p := g \cap iu$.

By a $G$-gradient manifold we mean the following. Let $(Z, \omega)$ be a Kähler manifold endowed with a holomorphic $G^C$-action such that there exists a $U$-equivariant moment map $\mu : Z \to u^*$ for the $U$-action on $(Z, \omega)$. We call such a $Z$ a Hamiltonian $G^C$-manifold. Basic examples are given by $G^C$-stable complex submanifolds of some projective space $\mathbb{P}(V)$ where $V$ is a finite-dimensional complex $G^C$-representation space. In particular, homogeneous algebraic $G^C$-varieties are Hamiltonian. Identifying $u^*$ with $iu$ and composing $\mu$ with the orthogonal projection to $p \subset iu$ with respect to a $U$-invariant inner product, we obtain the $K$-equivariant gradient map $\mu_p : Z \to p$. Any $G$-stable closed real submanifold $X$ of $Z$ is called a $G$-gradient manifold with gradient map $\mu_p|_X$. For more details and the basic properties of gradient maps we refer the reader to [HS07] and [HSS08].

Proposition 2.1. Let $G$ be a connected semisimple Lie group and let $H$ be a closed subgroup of $G$. Then $X = G/H$ is a $G$-gradient manifold if and only if $H$ is algebraic, i.e., if $H \cap G = H$ holds for the Zariski closure $\overline{H}$ of $H$ in $G^C$.

Proof. Suppose first that $\overline{H} \cap G = H$ holds. It follows that $X = G/H$ is a real submanifold of the homogeneous space $G^C/\overline{H}$. Since the latter is quasi-projective, it is in particular Kähler, and since $G^C$ is semisimple, there exists a unique $U$-equivariant moment map on $G^C/\overline{H}$. Then the construction described above yields a $G$-gradient map on $X = G/H$. 
If \( X = G/H \) is a gradient manifold, then by definition there exists a \( G \)-equivariant diffeomorphism \( G/H \cong G \cdot z \subset Z \) where \( Z \) is a Hamiltonian \( G^\mathbb{C} \)-manifold. From this we obtain \( G/H \hookrightarrow G^\mathbb{C}/(G^\mathbb{C})_z \), and \( G^\mathbb{C}/(G^\mathbb{C})_z \) is again K"ahler. Therefore \( (G^\mathbb{C})_z \) is an algebraic subgroup of \( G^\mathbb{C} \), see [GMO11] Corollary 4.12, hence contains \( \mathbb{H} \). This implies \( \overline{H} \cap G \subset (G^\mathbb{C})_z \cap G = G_z = H \), as was to be shown. □

3. Characterization of spherical homogeneous gradient manifolds

As in the previous section let \( G = K \exp(p) \) be a connected semisimple Lie group. Let \( H \subset G \) be a closed subgroup such that \( X = G/H \) is a \( G \)-gradient manifold with gradient map \( \mu_p : X \to p \). We say that \( X = G/H \) is spherical if a minimal parabolic subgroup of \( G \) has an open orbit in \( X \). In this case we call \( H \) a spherical subgroup of \( G \) and \( \mathfrak{h} \) a spherical subalgebra of \( \mathfrak{g} \). As shown in [MS10] this is equivalent to the fact that \( \mu_p \) almost separates the \( K \)-orbits in \( X \), i.e., that the map \( X/K \to p/K \) induced by \( \mu_p : X \to p \) has discrete fibers.

For the rest of this paper we assume that \( G \) is simple and has real rank 1. Then \( \mathfrak{g} \) is isomorphic to either \( \mathfrak{so}(n,1) \) (\( n \geq 3 \)) or \( \mathfrak{su}(n,1) \) (\( n \geq 1 \)) or \( \mathfrak{sp}(n,1) \) (\( n \geq 2 \)) or the exceptional Lie algebra \( f_4 = f_4(-20) \), see [Kna02, Ch. VI.11].

As our main result we will classify the algebraic spherical subalgebras of \( \mathfrak{g} \) up to conjugation by an element of \( G \), i.e., those subalgebras \( \mathfrak{h} \) for which there exists a closed subgroup \( H \subset G \) having \( \mathfrak{h} \) as Lie algebra such that \( X = G/H \) is a spherical gradient manifold.

Remark. Any symmetric subalgebra of \( \mathfrak{g} \) is spherical in \( \mathfrak{g} \), see [MS10] §6.1. It is not hard to see that, if \( \mathfrak{h}^\mathbb{C} \) is spherical in \( \mathfrak{g}^\mathbb{C} \) in the usual sense, then \( \mathfrak{h} \) is spherical in \( \mathfrak{g} \). However, as the following examples show, the converse does not hold, i.e., there are more spherical subalgebras of \( \mathfrak{g} \) than just real forms of complex spherical subalgebras of \( \mathfrak{g}^\mathbb{C} \).

Example. The unipotent radical \( N \) of a minimal parabolic subgroup of \( G \) is always spherical in \( G \). However, in general \( N^\mathbb{C} \) is not a spherical subgroup of \( G^\mathbb{C} \). As a concrete example one may take \( G = \text{SO}^\mathbb{C}(5,1) \).

A semisimple example is given by the spherical subgroup \( H = \text{Sp}(1,1) \) of \( G = \text{Sp}(2,1) \) (see Section 7) since \( H^\mathbb{C} = \text{Sp}(2,\mathbb{C}) \) is not spherical in \( G^\mathbb{C} = \text{Sp}(3,\mathbb{C}) \), see [Kra79].

For the classification we distinguish the cases that \( H \) is reductive or not. If \( H \) is reductive, then by definition we may assume that, after conjugation by an element of \( G \), we have a Cartan decomposition \( H = K_H \exp(p_H) \) where \( K_H := H \cap K \) is a maximal compact subgroup of \( H \) and where \( p_H := \mathfrak{h} \cap p \) is a \( K_H \)-invariant subspace of \( p \) with \( [p_H, p_H] \subset \mathfrak{k}_H \). We write \( p = p_H \oplus p_H^\perp \) with respect to the \( K \)-invariant inner product on \( p \) that comes from the \( U \)-invariant inner product on \( i\mathfrak{u} \). In this situation, sphericity of \( X = G/H \) has been characterized in [MS10] Proposition 6.1. Under the additional assumption \( \text{rk}_G H = 1 \) we can make the following more precise statement.

Proposition 3.1. Let \( G = K \exp(p) \) be a connected simple Lie group of real rank 1 and let \( H = K_H \exp(p_H) \subset G \) be a closed reductive subgroup. Then \( X = G/H \) is spherical if and only if \( K_H \) has an open orbit in every sphere in \( p_H^\perp \subset p \).

Remark. Except for \( \text{codim} p_H = 1 \) Proposition 3.1 says that \( X = G/H \) is spherical if and only if \( K_H \) acts transitively on the spheres in \( p_H^\perp \). In particular, \( X \) can only be spherical if the \( K_H \)-representation on \( p_H^\perp \) is irreducible.

Proof of Proposition 3.1. By [MS10] Theorem 1.1] the homogeneous gradient manifold \( X = G/H \) is spherical if and only if any gradient map on it almost separates the \( K \)-orbits.
The Mostow decomposition (see [HS07] for a proof using gradient maps) exhibits $X$ as $K$-equivariantly isomorphic to the twisted bundle $K \times_{K_H} \mathfrak{p}_H$. A particular gradient map is given by $\mu_k[k, \xi] = \text{Ad}(k)\xi$ for $k \in K$ and $\xi \in \mathfrak{p}_H$. Since the $K$-orbits in $K \times_{K_H} \mathfrak{p}_H$ correspond to the $K_H$-orbits in $\mathfrak{p}_H$, this map $\mu_k$ separates the $K$-orbits if and only if the map $\mathfrak{p}_H \times_{K_H} p/K$ that is induced by the inclusion $\mathfrak{p}_H \to \mathfrak{p}$, has discrete fibers. Since these fibers are precisely the $K_H$-orbits in the compact sets $(K \cdot \xi) \cap \mathfrak{p}_H$, we note in particular that the fibers have to be finite.

In the case $\text{rk}_K G = 1$, the $K$-orbits in $\mathfrak{p}$ are spheres, so their intersections with any subspace of $\mathfrak{p}$ are again spheres and in particular connected (unless the subspace is a line). Thus, $X$ is spherical if and only if $K_H$ has an open orbit in every sphere $(K \cdot \xi) \cap \mathfrak{p}_H$, hence in any sphere in $\mathfrak{p}_H$.

In the rest of this section we give a similar criterion for non-reductive $H$. For this we fix a minimal parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{a}$ is a maximal Abelian subspace (i.e., a line) of $\mathfrak{p}$. The corresponding group is $G_0 = MAN$ where $M := Z_K(\mathfrak{a})$. Let $\mathfrak{h}$ be a non-reductive algebraic subalgebra of $\mathfrak{g}$ and let $H$ be a corresponding subgroup of $G$. As is shown in [KS13a, Lemma 3.1], after conjugation by an element of $G$ we may assume that $\mathfrak{h} = l_H \oplus \mathfrak{n}_H$ where $l_H \subseteq \mathfrak{m} \oplus \mathfrak{a}$ is reductive in $\mathfrak{g}$ and $\{0\} \neq \mathfrak{n}_H \subset \mathfrak{n}$ is a nilpotent ideal of $\mathfrak{h}$.

On the group level we have $H \cong L_H \ltimes N_H$ with a reductive group $L_H = M_H A_H \subset MA$ and $N_H \subset N$. Note that $L_H$ acts by conjugation on $N$ and stabilizes $N_H$, hence acts on $N/N_H$. On the Lie algebra level we have the decomposition $\mathfrak{n} = \mathfrak{n}_H \oplus \mathfrak{n}_H$ of $\mathfrak{n}$ as an $M_H$-module.

**Proposition 3.2.** Let $X = G/H$ be a $G$-gradient manifold such that $H = M_H A_H N_H$ is non-reductive. Then $X$ is spherical if and only if either

1. $N_H = N$ and $L_H = M_H A_H$ is arbitrary, or
2. $\dim N/N_H = 1$ and $A_H = A$ and $M_H \subset N_M(\mathfrak{n}_H)$ is arbitrary, or
3. $\dim N/N_H \geq 2$ and $A_H = A$ and $M_H \subset N_M(\mathfrak{n}_H)$ acts transitively on the spheres in $\mathfrak{n}_H$.

**Remark.** Every algebraic subgroup $H \subset G$ that contains $N$ is spherical. Moreover, the proof of Proposition 3.2 will show that every $H$ containing $AN_H$ with $\dim N/N_H = 1$ is spherical. Therefore, we will concentrate on the case $\dim \mathfrak{n}_H^2 \geq 2$.

**Proof of Proposition 3.2.** Let $H$ be a non-reductive algebraic subgroup of $G$ of the form $H = M_H A_H N_H \subset Q_0 = MAN$. Since we have the $G$-equivariant fiber bundle $X = G/H \to G/Q_0$, we see that $X$ is $G$-equivariantly diffeomorphic to the twisted product $G \times_{Q_0} (Q_0/H)$.

By [He01], Cor. IX.1.8] the unique open orbit of the opposite minimal parabolic subgroup $Q_0^\ominus = MAN^-$ in $G/Q_0$ is $Q_0^\ominus \cdot eQ_0$. This implies that $X = G/H$ is spherical if and only if there is some $xH \in Q_0/H$ such that $Q_0^\ominus \cdot [e, xH]$ is open in $G \times_{Q_0} (Q_0/H)$. The latter is the case if and only if $MA = Q_0^\ominus \cap Q_0$ has an open orbit in $Q_0/H$. Using the fact that $N_H$ is normal in $H$, we see that $Q_0/H$ is $Q_0$-equivariantly diffeomorphic to $(MA) \times_{L_H} (N/N_H)$ where $L_H$ acts by conjugation on $N/N_H$. This proves that $X = G/H$ is spherical if and only if $L_H$ has an open orbit in $N/N_H$.

Suppose now that $\dim \mathfrak{n}_H^2 \geq 1$. Then $L_H$ can have an open orbit in $\mathfrak{n}_H^2$ only if it contains $A$, i.e., if $A_H = A$. If $\dim \mathfrak{n}_H^2 = 1$, this condition is also sufficient. Since every $A$-orbit in $\mathfrak{n}_H^2$ intersects any sphere in $\mathfrak{n}_H^2$ precisely once, the claim follows.

As we have seen, in order to classify non-reductive spherical algebraic subalgebras $\mathfrak{h} \subset \mathfrak{g}$ we may assume without loss of generality $\dim \mathfrak{n}_H^2 \geq 2$. In particular, $M_H$ must act irreducibly.

---

4If $H$ is a subgroup of $G$ and $Y$ is a set on which $H$ acts, then the twisted bundle $G \times_H Y$ is defined as the quotient of $G \times Y$ with respect to the diagonal $H$-action $h \cdot (g, y) := (gh^{-1}, h \cdot y)$. 
on $n^\perp_H$ if $X = G/H$ is spherical. In closing this section, we will exploit this observation a little further.

We have $n = g_\alpha \oplus g_{2\alpha}$ according to the restricted root space decomposition of $g$ with respect to $\alpha$ where $\alpha$ is a simple restricted root. If $2\alpha$ is not a restricted root, we set $g_{2\alpha} = 0$. (Note that this is only the case for $g = \mathfrak{so}(n,1)$.) The group $M$ acts irreducibly on every root space, see e.g. [Wol84] Ch. 8.13. Suppose that $X = G/H$ is spherical with $\dim n_H \geq 2$. Since $A \subset L_H$ acts with two different weights on $g_\alpha$ and $g_{2\alpha}$, we obtain

$$n^\perp_H = (n^\perp_H \cap g_\alpha) \oplus (n^\perp_H \cap g_{2\alpha}).$$

Consequently, $M_H$ acts irreducibly on $n^\perp_H$ only if $n^\perp_H$ contains $g_\alpha$ or $g_{2\alpha}$. This proves the following

**Proposition 3.3.** Let $H = L_H N_H$ be a spherical non-reductive algebraic subgroup of $G$. Then $n_H$ contains $g_\alpha$ or $g_{2\alpha}$.

**Remark 3.4.** Arguing case by case we will see that $[g_\alpha, g_\alpha] = g_{2\alpha}$ holds for every simple Lie algebra $g$ of real rank 1. This implies that $n_H$ always contains $g_{2\alpha}$.

4. **Strategy of the classification**

Let us outline the principal steps that will lead to the classification result. Recall that $G = K \exp(p)$ is a connected simple Lie group of real rank 1 that embeds into $G^\mathbb{C}$. Let $h \subset g$ be an algebraic subalgebra which we assume first to be reductive. After conjugation by an element of $G$ we have $h = \mathfrak{k}_H \oplus p_H$. The first step in the classification consists in analyzing the $K$-action on the Grassmannians $Gr_k(p)$ of real $k$-planes in $p$ for classical $G$ so that we can conjugate $p_H$ by an element of $K$ into a suitable normal form.

One verifies directly that for classical $G$ the $K$-representation on $p$ is given as follows. If $G = SO^\circ(n,1)$, then we have the defining representation of $K \cong SO(n)$ on $p \cong \mathbb{R}^n$. For $G = SU(n,1)$ we have $K \cong U(n)$ and may identify $p$ with $\mathbb{C}^n$ due to the hermitian structure on $G/K$. Under these identifications we obtain the defining representation of $U(n)$ on $\mathbb{C}^n$. If $G = Sp(n,1)$, then $K \cong Sp(n) \times Sp(1)$ and $p \cong \mathbb{H}^n$. On $\mathbb{H}^n$ we consider the quaternionic inner product $q(x,y) = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$ where $\overline{y}$ is the quaternionic conjugate of $y \in \mathbb{H}$. By definition, $Sp(n)$ is the isometry group of $q$. The action of $Sp(n) \times Sp(1)$ on $\mathbb{H}^n$ is given by $(A,b) \cdot x = Axb^{-1}$. The map

$$\varphi : \mathbb{H}^n \to \mathbb{C}^{2n} , \quad v = a + ib + jc + kd \mapsto \begin{pmatrix} a+ib \\ c-id \end{pmatrix},$$

is a complex-linear isomorphism and induces the map $\Phi : \mathbb{H}^{n \times n} \to \mathbb{C}^{2n \times 2n}$,

$$M = A + iB + jC + kD \mapsto \begin{pmatrix} A + iD & C - iD \\ C + iD & A - iD \end{pmatrix}.$$ 

Since $q$ corresponds under $\varphi$ to the pair of the standard hermitian form and the standard complex symplectic form on $\mathbb{C}^{2n}$, we obtain $\Phi(\text{Sp}(n)) = \text{Sp}(\mathbb{C}^{2n}) \cap U(\mathbb{C}^{2n})$. In the following, we will usually identify $(z,w) \in \mathbb{C}^{2n}$ with the matrix $\begin{pmatrix} z & \overline{w} \\ w & -\overline{z} \end{pmatrix} \in \mathbb{C}^{2n \times 2n}$. Under this identification, the representation of $K \cong \text{Sp}(n) \times SU(2)$ on $p$ is given by

$$(A,B) \cdot \begin{pmatrix} z & \overline{w} \\ w & -\overline{z} \end{pmatrix} := A \begin{pmatrix} z & \overline{w} \\ w & -\overline{z} \end{pmatrix} B^{-1}.$$ 

**Lemma 4.1.** Let $G = K \exp(p)$ be a connected classical simple Lie group of real rank 1. The orbit structure of $G$ on the Grassmannian $Gr_k(p)$ of real $k$-planes in $p$ is as follows.

(i) If $G = SO^\circ(n,1)$, then $K \cong SO(n)$ acts transitively on $Gr_k(p)$ for any $k$.

(ii) If $G = SU(n,1)$, then two real $k$-planes in $p$ lie in the same orbit of $K \cong U(n)$ if and only if their maximal complex subspaces have the same dimension.
(iii) If \( G = \text{Sp}(n, 1) \), then any real subspace of \( \mathfrak{p} \) is after conjugation with an element of \( \text{Sp}(n) \subset K \cong \text{Sp}(n) \times \text{SU}(2) \) equal to
\[
V_{k,l,m,p,ξ} := \{ v \in \mathbb{H}^n : v_1, \ldots, v_k = 0, v_{k+1}, \ldots, v_{k+l} \in \mathbb{R}, \\
v_{k+l+j} \in (\mathbb{R} + ξ_j \mathbb{R}) \text{ for all } 1 \leq j \leq m, \\
v_{k+l+m+1}, \ldots, v_{k+l+m+p} \in \text{Im}(\mathbb{H}) \},
\]
for unique \( k, l, m, p \in \mathbb{N} \) and \( ξ_j \in \text{Im}(\mathbb{H}) \) with \( \| ξ_j \| = 1 \) for \( 1 \leq j \leq m \).

Proof. Let us first prove the claim for the case \( G = \text{SU}(n, 1) \). Let \( V \) be a real subspace of \( \mathbb{C}^n \) and \( W \) be a maximal complex subspace of \( V \) with orthonormal basis \( (v_1, \ldots, v_m) \). Let \( (u_1, \ldots, u_l) \) be an orthonormal basis of \( W^\perp \subset V \). Then the matrix \((\text{id}, u_1, \ldots, u_l, v_1, \ldots, v_m)\) is in \( \text{SU}(n) \) and maps \( \{0\}^{n-l-m} \times \mathbb{R}^l \times \mathbb{C}^m \) to \( V \). Analogously one proves the claim for \( G = \text{SO}(n, 1) \).

Let us now consider the \( \text{Sp}(n) \)-action on real \( k \)-planes in \( \mathbb{H}^n \). Suppose first that \( n = 1 \). Since \( \text{Sp}(1) \) acts transitively on \( S^3 \subset \mathbb{H} \), any one- or three-dimensional real subspace of \( \mathbb{H} \) can be mapped onto \( \mathbb{R} \) or \( \text{Im}(\mathbb{H}) \), respectively, by a suitable element of \( \text{Sp}(1) \). Any real plane can be mapped onto \( \mathbb{C}_ξ := \mathbb{R} \oplus ξ \mathbb{C} \) with \( ξ \in S^3 \cap \text{Im}(\mathbb{H}) \). Note that there is no element in \( \text{Sp}(1) \) that maps \( \mathbb{C}_ξ \) onto \( \mathbb{C}_ξ \), if \( ξ \neq ξ' \).

In the general case, we will show that any real \( k \)-dimensional subspace \( V \subset \mathbb{H}^n \) contains a basis of its quaternionic span \( \langle V \rangle_\mathbb{H} \) over \( \mathbb{H} \) that is orthonormal with respect to \( q \). Consequently, there exists an element in \( \text{Sp}(n) \) that maps this basis to a subset of the standard basis of \( \mathbb{H}^n \). Re-ordering if necessary, and acting with elements of \( \text{Sp}(1)^n \subset \text{Sp}(n) \) we obtain therefore the following normal form given by
\[
V_{k,l,m,p,ξ} := \{ v \in \mathbb{H}^n : v_1, \ldots, v_k = 0, v_{k+1}, \ldots, v_{k+l} \in \mathbb{R}, \\
v_{k+l+j} \in (\mathbb{R} + ξ_j \mathbb{R}) \text{ for all } 1 \leq j \leq m, \\
v_{k+l+m+1}, \ldots, v_{k+l+m+p} \in \text{Im}(\mathbb{H}) \},
\]
where \( ξ_j \in \text{Im}(\mathbb{H}) \) with \( \| ξ_j \| = 1 \) for \( 1 \leq j \leq m \).

Without loss of generality we may assume \( \langle V \rangle_\mathbb{H} = \mathbb{H}^n \). Let \( V_\mathbb{H} \) be the maximal quaternionic subspace of \( V \). Since \( V = V_\mathbb{H} \oplus (V_\mathbb{H})^\perp \), we may assume \( V_\mathbb{H} = \{0\} \). We decompose \( V \) into the sum \( V = V_1 \oplus V_2 \oplus V_3 \), where
\[
V_1 := \{ v \in V : \lambda \cdot v \notin V \text{ for all } \lambda \in \text{Im}(\mathbb{H}) \} \\
V_2 := \{ v \in V : \exists \lambda \in \text{Im}(\mathbb{H}) \} \setminus \{0\} \text{ s.t. } \lambda \cdot v \in V \text{ and } \mu \cdot v \notin V \forall \mu \in \text{Im}(\mathbb{H}) \setminus \mathbb{R} \cdot \lambda \\
V_3 := \{ v \in V : \exists \lambda, \mu \in \text{Im}(\mathbb{H}) \} \setminus \{0\} \text{ s.t. } \mu \notin \mathbb{R} \cdot \lambda \text{ and } \lambda \cdot v, \mu \cdot v \in V \}.
\]
Choosing a \( q \)-orthonormal basis of \( V \) compatible with this decomposition we proved the existence of the adapted orthonormal basis. \( \square \)

In the next step we conjugate \( H \) by an element of \( K \) such that \( \mathfrak{p}_H \) is of the normal form defined by the help of Lemma 3.1. In order to understand all possibilities for \( K_H \) we determine \( \mathcal{N}_f(\mathfrak{p}_H) \) and \( [\mathfrak{p}_H, \mathfrak{p}_H] \) since we know that
\[
(4.1) \quad [\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{h}_H \subset \mathcal{N}_f(\mathfrak{p}_H)
\]
must hold.

Due to Proposition 3.1 the subalgebra \( \mathfrak{h} \) is spherical if and only if \( K_H \) acts transitively on the (connected components of the) spheres in \( \mathfrak{p}_H^\perp \). Therefore, in the third and last step we single out those groups \( K_H \) fulfilling (4.1) which do indeed act transitively on the spheres in \( \mathfrak{p}_H \). In this step we will make essential use of Onishchik’s classification of transitive actions on spheres. We recall his result in the following theorem where we consider the defining representations of \( O(n) \) on \( \mathbb{R}^n \), of \( U(n) \) on \( \mathbb{C}^n \) and of \( \text{Sp}(n) \) on \( \mathbb{H}^n \).
Theorem 4.2. Let $K$ be either $O(n)$ or $U(n)$ or $Sp(n)$ and let $V$ denote the respective defining representation of $K$. The following table lists all connected proper subgroups $L$ of $K$ that act transitively on the spheres in $V$, up to conjugation in $K$. Note that $K = Sp(n)$ does not contain such a subgroup.

| $K$           | $L$                                    |
|---------------|----------------------------------------|
| $U(n), n \geq 2$ | $SU(n)$                               |
| $U(2n), n \geq 2$ | $Sp(n)$                               |
| $U(2n), n \geq 2$ | $Sp(n) \times S^4$                   |
| $O(n), n \geq 2$ | $SO(n)$                               |
| $O(2n), n \geq 3$ | $SU(n)$                               |
| $O(4n), n \geq 2$ | $Sp(n)$                               |
| $O(16)$       | $Spin(9)$                             |
| $O(8)$        | $Spin(7)$                             |
| $O(4)$        | $G_2$                                 |
|               | $p(\text{Sp}(1) \times L_2)$ or $p(L_2 \times \text{Sp}(1))$ where $p: \text{Sp}(1) \times \text{Sp}(1) \to SO(4)$ is the universal covering and $L_2 \subset \text{Sp}(1)$ is an arbitrary connected subgroup |

Proof. Since a connected subgroup of $O(n)$ lies in $SO(n)$ and since $SO(n)$ ($n \geq 3, n \neq 4$) and $Sp(n)$ are simple, the result follows directly from [Oni94, Table 8] for these groups.

Let us discuss the case $K = O(4)$. Identifying $\mathbb{R}^4$ with $\mathbb{H}$ one sees that $so(4) \cong sp(1) \oplus sp(1)$. The isotropy algebra $so(4)_{e_1}$ of $e_1 \in S^3 \subset \mathbb{R}^4$ corresponds to the diagonal $\Delta_{sp(1)}$ in $sp(1) \oplus sp(1)$. Therefore we have to find all subalgebras $l$ of $sp(1) \oplus sp(1)$ (up to conjugation) that verify

$$l + \Delta_{sp(1)} = sp(1) \oplus sp(1).$$

Consider the projections

$$sp(1) \oplus sp(1) \xrightarrow{\pi_1} sp(1) \xrightarrow{\pi_2} sp(1).$$

Let $l$ be a subalgebra of $sp(1) \oplus sp(1)$ verifying (4.2). This implies

$$3 \leq \dim l = \dim \pi_1(l) + \dim \ker(\pi_1|l)$$

where $\ker(\pi_1|l) = l \cap \{0\} \oplus sp(1)$ is an ideal in $l$.

If $\dim \pi_1(l) = 0$, then $l = \{0\} \oplus sp(1)$. If $\dim \pi_1(l) = 1$, then $\pi_1(l) =: t$ is a maximal torus in $sp(1)$ and $\dim l \cap \{0\} \oplus sp(1) \geq 2$, hence $l = t \oplus sp(1)$.

Finally, suppose that $\dim \pi_1(l) = 3$ and note that $\dim l \cap \{0\} \oplus sp(1) \in \{0, 1, 3\}$. If $\dim l \cap \{0\} \oplus sp(1) = 3$, then $l = sp(1) \oplus sp(1)$. If $\dim l \cap \{0\} \oplus sp(1) = 0$, then $\dim l = 3$ and thus $l$ is simple. Therefore $\dim l \cap \{0\} \oplus sp(1)$ is either $sp(1) \oplus \{0\}$ (and then $l = sp(1) \oplus \{0\}$) or $\{0\}$. In the latter case $\pi_1|l$ and $\pi_2|l$ are isomorphisms onto $sp(1)$ and $l$ coincides with the graph of $\varphi := (\pi_2|l) \circ (\pi_1|l)^{-1} \in \text{Aut}(sp(1))$. Since we can identify $l \cap \Delta_{sp(1)}$ with the space of fixed points $sp(1)^\varphi$ in this case and since $\dim sp(1)^\varphi \geq 1$, we obtain $\dim l + \Delta_{sp(1)} \leq 5$, hence $l$ cannot verify (4.2). Finally consider the case $\dim l \cap \{0\} \oplus sp(1) = 1$. Then $\dim l = 4$ and
\[ I \cap (\{0\} \oplus \mathfrak{sp}(1)) \text{ is a one-dimensional ideal in } I \text{ which implies } I \cap (\{0\} \oplus \mathfrak{sp}(1)) = \mathfrak{z}(I). \text{ Thus } I = \mathfrak{sp}(1) \oplus \mathfrak{z}(I). \]

In the unitary case we use \[\text{Oni94} \text{ Theorem 1.5.1} \] in order to reduce the classification to \( \text{SU}(n) \). We denote the isotropy algebra \( \mathfrak{u}(n)_{e_1} \) of \( e_1 \in S^{2n-1} \subset \mathbb{C}^n \) by \( \mathfrak{u}(n-1) \). As above we look for subalgebras \( I \) of \( \mathfrak{u}(n) \) that verify \( I + \mathfrak{u}(n-1) = \mathfrak{u}(n) \). According to \[\text{Oni94} \text{ Theorem 1.5.1} \] this holds if and only if \( I' + \mathfrak{su}(n-1) = \mathfrak{su}(n) \) (where \( I' \) is the derived algebra of \( I \) and \( 3(\mathfrak{u}(n)) = \pi(3(I) + 3(\mathfrak{u}(n-1))) \) where \( \pi \) is the projection of \( \mathfrak{u}(n) \) onto its center with kernel \( \mathfrak{su}(n) \). Note that the second condition is automatically verified since already \( \pi(3(\mathfrak{u}(n-1))) = 3(\mathfrak{u}(n)) \). Consequently, the result follows from the classification of subgroups of \( \text{SU}(n) \) that act transitively on \( S^{2n-1} \subset \mathbb{C}^n \) given in \[\text{Oni94} \text{ Table 8}. \]

The non-reductive case will be treated along the same lines as follows. As we have seen above, after conjugating by an element of \( G \) we may suppose that any non-reductive algebraic subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) is of the form \( \mathfrak{h} = \mathfrak{m}_H \oplus \mathfrak{a}_H \oplus \mathfrak{n}_H \). We always assume that \( \dim \mathfrak{n}_H \geq 1 \) so that we have \( \mathfrak{a}_H = \mathfrak{a} \). Then \( H \) is spherical if and only if \( \mathfrak{M}_H \) acts transitively on the connected components of the spheres in \( \mathfrak{n}_H \). As above we start by analyzing the \( M \)-representation on \( \mathfrak{n} \) in order to find a normal form for \( \mathfrak{n}_H \). Afterwards we determine \( \mathcal{N}_M(\mathfrak{n}_H) \) for this normal form since this normalization must contain \( \mathfrak{M}_H \). Finally, we apply again Onishchik’s result Theorem 4.2 in order to single out those \( \mathfrak{M}_H \) that do act transitively on the spheres in \( \mathfrak{n}_H \).

5. \( G = \text{SO}^\circ(n, 1) \)

5.1. \textbf{The reductive case.} Let \( \mathfrak{h} \) be a reductive algebraic subalgebra of \( \mathfrak{g} = \mathfrak{so}(n, 1) \). To simplify the notation we write \( \mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q} \) for its Cartan decomposition. On the group level we have \( H = L \exp(\mathfrak{q}) \). It follows from Lemma 4.1 that \( K = \text{SO}(n) \) acts transitively on \( \text{Gr}_{\dim \mathfrak{q}}(\mathfrak{p}) \). Hence, we may assume
\[
\mathfrak{q} = \mathfrak{q}_k := \left\{ \left( \begin{array}{c} 0 \\ x^t \\ 0 \end{array} \right) : x_1 = \cdots = x_k = 0 \right\},
\]
for some \( 0 \leq k \leq n \).

\textbf{Lemma 5.1.} For \( 0 \leq k \leq n \) we have
\[
[q_k, q_k] = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{array} \right) : B \in \mathfrak{so}(n-k) \right\},
\]
and
\[
\mathcal{N}_k(q_k) = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ A & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) : A \in \mathfrak{so}(k), B \in \mathfrak{so}(n-k) \right\}.
\]

\textbf{Proof.} Since \([q_k, q_k] \) is the vector space generated over the reals by elements \([\xi, \eta] \), where \( \xi, \eta \) are in \( q_k \), it suffices to calculate the bracket for basis elements of \( q_k \). Let \((e_1, \ldots, e_n)\) be the standard basis of \( \mathbb{R}^n \). Then
\[
\left[ \left( \begin{array}{ccc} 0 & e_i & 0 \\ e_i & 0 & 0 \\ 0 & e_j & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & e_j & 0 \\ 0 & e_i & 0 \end{array} \right) \right] = \left( \begin{array}{ccc} e_i e_j & 0 & 0 \\ 0 & e_j e_i & 0 \\ 0 & 0 & e_j e_i \end{array} \right) = \left( \begin{array}{ccc} (E_{ij} - E_{ji}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\]
Hence \([q_k, q_k] \) is the vector space generated over the reals by elements \([A, B] \in \mathfrak{so}(n-k) \) for \( 0 \leq k \leq n \). The second claim follows from direct calculation. \[\square\]

Since \([q_k, q_k] \oplus q_k \cong \mathfrak{so}(n-k, 1) \) (with \( \mathfrak{so}(0, 1) := \{0\} \)), we have shown the following

\textbf{Proposition 5.2.} Any reductive subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) is conjugate under \( G \) to one of the form \( \mathfrak{b} \oplus \mathfrak{so}(n-k, 1) \), where \( \mathfrak{b} \) is any subalgebra of \( \mathfrak{so}(k) \) and \( 0 \leq k \leq n \).
Note that the action of $N_K(q_k)$ on $q_k^+$ is given by
\[
\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ Bx \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}^k.
\]

Hence, combining this result with Proposition 3.3 and Theorem 4.2 we obtain the following list of all spherical reductive algebraic subalgebras of $\mathfrak{so}(n,1)$ up to conjugacy in $G$. This list has been found in [KS13a, Remark 5.1].

**Theorem 5.3.** All spherical reductive algebraic subalgebras $\mathfrak{h}$ of $\mathfrak{so}(n,1)$ are (up to conjugation in $G$) one of the following, where $\mathfrak{so}(0,1) := \{0\}$ and $\mathfrak{l}_2 \subseteq \mathfrak{sp}(1)$ is arbitrary.

| Subalgebra | Condition |
|------------|-----------|
| $\mathfrak{so}(k) \oplus \mathfrak{so}(n-k,1)$ | $0 \leq k \leq n$ |
| $\mathfrak{su}(m) \oplus \mathfrak{so}(n-k,1)$ | $0 \leq k \leq n$, $k = 2m$, $m \geq 4$ |
| $\mathfrak{sp}(m) \oplus \mathfrak{so}(n-k,1)$ | $0 \leq k \leq n$, $k = 4m$, $m \geq 2$ |
| $\mathfrak{spin}(9) \oplus \mathfrak{so}(n-16,1)$ | |
| $\mathfrak{spin}(7) \oplus \mathfrak{so}(n-8,1)$ | |
| $\mathfrak{g}_2 \oplus \mathfrak{so}(n-7,1)$ | |
| $\mathfrak{sp}(1) \oplus \mathfrak{l}_2 \oplus \mathfrak{so}(n-4,1)$ | |
| $\mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(n-4,1)$ | |

According to [Ber57, Table 2], the symmetric subalgebras are $\mathfrak{so}(n) \oplus \mathfrak{so}(1)$ and $\mathfrak{so}(k) \oplus \mathfrak{so}(n-k,1)$ for $2 < k \leq n-1$.

**Remark.** The only subalgebras $\mathfrak{h}$ of $\mathfrak{so}(n,1)$ in the above list that might be conjugate to each other by an element of $G$ are $\mathfrak{sp}(1) \oplus \mathfrak{l}_2 \oplus \mathfrak{so}(n-4,1)$ and $\mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(n-4,1)$. Note that these subalgebras are indeed conjugate by an element of $G$ if and only if $n > 4$.

### 5.2. The non-reductive case.

In this subsection we classify the spherical non-reductive algebraic subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{so}(n,1)$. Due to Proposition 3.2 we can assume $H = M_H AN_H$, where $1 \leq \dim n_H^+ < n - 1$ and $M_H \subset N_M(n_H)$. After understanding the $M$-action on $n$ we are going to find a normal form of $n_H$ and of $N_M(n_H)$. We then only have to classify the subgroups $M_H \subset N_M(n_H)$ that act transitively on the connected components of the spheres in $n_H^+$.

First we determine the structure of a minimal parabolic subalgebra of $\mathfrak{g} = \mathfrak{so}(n,1)$. Choosing $\mathfrak{a} = \mathbb{R} \begin{pmatrix} 0 & c \end{pmatrix}$ as maximal Abelian subspace of $\mathfrak{p}$, we obtain $\mathfrak{q}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with
\[
\mathfrak{n} = \left\{ \begin{pmatrix} 0 & -v & 0 \\ 0 & 0 & 0 \\ -v & 0 & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\}.
\]

Moreover, we have
\[
M = Z_K(\mathfrak{a}) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K \right\} \cong \text{SO}(n-1).
\]

Identifying $\mathfrak{n}$ with $\mathbb{R}^{n-1}$ via $\begin{pmatrix} 0 & v & -v \\ v & 0 & 0 \\ -v & 0 & 0 \end{pmatrix} \mapsto v$ we see that the $M$-representation on $\mathfrak{n}$ is isomorphic to the defining representation of $\text{SO}(n-1)$ on $\mathbb{R}^{n-1}$. Thus, Lemma 4.1 applies to show that any subspace $\mathfrak{n}_H$ of $\mathfrak{n}$ is $M$-conjugate to
\[
\mathfrak{n}_k := \left\{ \begin{pmatrix} 0 & -v & 0 \\ -v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \{0\}^k \times \mathbb{R}^{n-k-1} \right\}.
\]
where \( k \) is the dimension of \( n^+_H \) in \( n \). The normalizer of \( n_k \) in \( M \) is given by

\[
N_M(n_k) = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M : A \in O(k), B \in O(n - k - 1) \right\}
\]

\( \cong S(O(k) \times O(n - k - 1)) \),

and its action on \( n_k \) is \( \left\{ \begin{pmatrix} 0 & v & -v \\ -v & 0 & 0 \\ -v^t & 0 & 0 \end{pmatrix} : v \in \mathbb{R}^k \times \{0\}^{n-k-1} \right\} \cong \mathbb{R}^k \) is \( O(k) \)-equivariantly isomorphic to the standard \( O(k) \)-action on \( \mathbb{R}^k \).

Combining these observations with Proposition 3.2 and Theorem 4.2 we obtain the following

**Theorem 5.4.** Every spherical non-reductive algebraic subalgebra of \( \mathfrak{so}(n,1) \) is \( G \)-conjugate to one in the following list where \( c_k \subset \mathfrak{so}(n-k-1) \) and \( l_2 \subset \mathfrak{sp}(1) \) are arbitrary (under the condition that the maximal compact subalgebra is a Lie algebra, see Remark 5.5 below).

| \( l_H \oplus n \) | \( l_H \subset m \oplus a \) arbitrary |
|------------------|------------------|
| \( \mathfrak{so}(k) \oplus c_k \oplus a \oplus n_k \) | \( 1 \leq k \leq n - 2, \) |
| \( \mathfrak{su}(m) \oplus c_k \oplus a \oplus n_k \) | \( 1 \leq k \leq n - 2, k = 2m, m \geq 4 \) |
| \( \mathfrak{sp}(m) \oplus c_k \oplus a \oplus n_k \) | \( 1 \leq k \leq n - 2, k = 4m, m \geq 2 \) |
| \( \mathfrak{spin}(9) \oplus c_{16} \oplus a \oplus n_{16} \) | \( \mathfrak{spin}(7) \oplus c_8 \oplus a \oplus n_8 \) |
| \( \mathfrak{g}_2 \oplus c_7 \oplus a \oplus n_7 \) | \( \mathfrak{sp}(1) \oplus l_2 \oplus c_4 \oplus a \oplus n_4 \) |
| \( l_2 \oplus \mathfrak{sp}(1) \oplus c_4 \oplus a \oplus n_4 \) | \( l_2 \oplus \mathfrak{sp}(1) \oplus c_4 \oplus a \oplus n_4 \) |

**Remark 5.5.** Let us explain the notation in Theorem 5.4. Let \( \pi : \mathfrak{so}(k) \oplus \mathfrak{so}(n - k) \to \mathfrak{so}(k) \) denote the projection onto the first factor. An arbitrary subalgebra \( \mathfrak{k} \subset \mathfrak{so}(k) \oplus \mathfrak{so}(n - k) \) is of the form

\[
\mathfrak{k} = \{(\xi, \varphi(\xi)) : \xi \in \pi(\mathfrak{k})\} \oplus (\mathfrak{k} \cap \{0\} \oplus \mathfrak{so}(n - k)),
\]

where \( \varphi : \pi(\mathfrak{k}) \to \mathfrak{so}(n - k) \) is a homomorphism of Lie algebras. In order to simplify the notation, we write here and in the rest of this paper \( \mathfrak{k} = \pi(\mathfrak{k}) \oplus \mathfrak{c} \) where \( \mathfrak{c} = \mathfrak{k} \cap \{0\} \oplus \mathfrak{so}(n - k) \). The condition that \( \pi(\mathfrak{k}) \oplus \mathfrak{c} \) is a subalgebra means that \( \mathfrak{c} \) is contained in the centralizer of \( \varphi(\pi(\mathfrak{k})) \) in \( \mathfrak{so}(n - k) \). Note that, if \( \pi(\mathfrak{k}) \) is simple, then \( \varphi \) is either identically zero or injective.

6. \( G = SU(n,1) \)

6.1. The reductive case. Let \( \mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q} \) be a reductive algebraic subalgebra of \( \mathfrak{g} \). Due to Lemma 4.1 two real subspaces of \( \mathfrak{p} \) are conjugate to each other by an element of \( K \) if and only if they have the same real dimension and their maximal complex subspaces have the same dimension. Hence, we may assume

\[
\mathfrak{q} = \mathfrak{q}_{k,l} := \left\{ \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} : z \in \{0\}^k \times \mathbb{R}^l \times \mathbb{C}^{n-k-l} \subset \mathbb{C}^n \right\},
\]

where \( 2k + l \) is the real dimension of \( \mathfrak{q}^\perp \) and \( k \) the complex dimension of the maximal complex subspace in \( \mathfrak{q}^\perp \).
Lemma 6.1. For $0 \leq k + l \leq n$ we have

$$[q_{k,l}, q_{k,l}] = \begin{cases} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & ix \end{array} \right) & \in \mathfrak{t} : A, C \in \mathfrak{su}(n - k - l), ix = -Tr(A) - Tr(C) \\ \in \mathfrak{s}(\mathfrak{u}(n - k) \times \mathfrak{u}(1)), \end{cases}$$

$$\mathcal{N}_q(q_{k,l}) = \left\{ \begin{array}{cccc} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ix \end{array} \right\} \in \mathfrak{t} : A \in \mathfrak{u}(k), C \in \mathfrak{u}(n - k - l), B - ixid \in \mathfrak{so}(l), x \in \mathbb{R} \right\}.$$

Proof. The action of $u$ on $q$ using 

$$q_{k,l} = (\mathbf{1} - k + l)(\mathbf{1} + k + l) - k - l.$$  

Note that 

$$q_{k,l} ⊆ \mathfrak{so}(1, n) - \mathfrak{k}.$$ 

Hence, combining these results with Proposition 3.1 and Theorem 4.2 we can prove the following list of all spherical reductive algebraic subalgebras of $\mathfrak{su}(n, 1)$ up to conjugacy in $G$.

**Theorem 6.4.** All spherical, reductive subalgebras $\mathfrak{h}$ of $\mathfrak{su}(n, 1)$ are (up to conjugation in $G$) one of the following:

| Subalgebra | Conditions |
|------------|------------|
| $\mathfrak{u}(n)$ | |
| $\mathfrak{su}(n)$ | |
| $\mathfrak{sp}(m)$ | $m = 2n$ and $m \geq 2$ |
| $\mathfrak{sp}(m) \oplus \mathfrak{s}^1$ | $m = 2n$ and $m \geq 2$ |
| $\mathfrak{so}(n, 1)$ | $m = 2n$ and $m \geq 2$ |
| $\mathfrak{su}(k) \oplus \mathfrak{su}(n - k, 1)$ | $0 \leq k < n$ |
| $\mathfrak{so}(n, 1)$ | $0 \leq k < n$ |
| $\mathfrak{sp}(m) \oplus \mathfrak{u}(n - k, 1)$ | $0 \leq k < n, k = 2m, m \geq 2$ |
| $\mathfrak{so}(n, 1)$ | $0 \leq k < n, k = 2m, m \geq 2$ |
According to [Ber57, Table 2] the symmetric subalgebras are \(\mathfrak{su}(1) \oplus \mathfrak{su}(n) \oplus \mathfrak{su}(1)\) and \(\mathfrak{su}(k) \oplus \mathfrak{su}(n-k, 1) \oplus \mathfrak{su}(1)\) and \(\mathfrak{su}(k, 1) \oplus \mathfrak{su}(n-k-1) \oplus \mathfrak{su}(1)\) and \(\mathfrak{so}(n, 1)\).

6.2. The non-reductive case. Let \(H\) now be a non-reductive algebraic subgroup of \(G\). Due to Proposition 3.2 we may assume that it is of the form \(H = M_H AN_H\), where \(1 \leq \dim_{\mathbb{R}}(n_H^1) < n - 1\) and \(M_H \subset N_M(n_H)\). After understanding the \(M\)-action on \(n\) we are going to find a normal form of \(n_H\) and of \(N_M(n_H)\). We then only have to classify the subgroups \(M_H \subset N_M(n_H)\) that act transitively on the connected components of the spheres in \(n_H^1\).

Choosing \(a = \mathbb{R} \cdot \left(\begin{smallmatrix} 0 & e_n \\ e_n^t & 0 \end{smallmatrix}\right)\) as a maximal Abelian subspace of \(p\), we obtain \(g = t \oplus a \oplus n\) with

\[
n = \left\{ \begin{pmatrix} 0 & v & -v \\ -v & 0 & ix \\ -ix & ix & 0 \end{pmatrix} : v \in \mathbb{C}^{n-1}, x \in \mathbb{R} \right\}.
\]

Moreover, we have

\[
M = Z_K(a) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} : A \in SU(n-1), \lambda^2 \det A = 1 \right\} \cong U(n-1).
\]

The map \(n \to \mathbb{C}^{n-1} \times i\mathbb{R}\) with \(\begin{pmatrix} 0 & v & -v \\ -v & 0 & ix \\ -ix & ix & 0 \end{pmatrix} \mapsto (v, ix)\) is a \(U(n-1)\)-equivariant isomorphism with respect to the adjoint action of \(M\) on \(n\) and the \(U(n-1)\)-action on \(\mathbb{C}^{n-1} \times i\mathbb{R}\) given by \(A \cdot (z, ix) = (Az, iz)\). Since \([g_a, g_a] = g_{2a}\) holds, Lemma 4.1 implies that any subspace \(n_H\) of \(n\) is \(M\)-conjugate to

\[
n_{k, l} := \left\{ \begin{pmatrix} 0 & v & -v \\ -v & 0 & ix \\ -ix & ix & 0 \end{pmatrix} : v \in \{0\}^k \times \mathbb{R}^l \times \mathbb{C}^{n-1-k-l}, x \in \mathbb{R} \right\},
\]

where \(2k + l\) is the real dimension of \(n_H^1\) in \(n\) and \(k\) the complex dimension of the maximal complex subspace of \(n_H^1\) in \(n\). The normalizer of \(n_{k, l}\) in \(M\) is given by

\[
N_M(n_{k, l}) = \left\{ \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda B & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \end{pmatrix} : A \in U(k), B \in O(l), \lambda \in S^1 \right\},
\]

and its action on \(n_{k, l}^1 = \left\{ \begin{pmatrix} 0 & v & -v \\ -v & 0 & ix \\ -ix & ix & 0 \end{pmatrix} : v \in \mathbb{C}^k \times (i\mathbb{R})^l \times \{0\}^{n-1-k-l} \right\} \cong \mathbb{C}^k \times (i\mathbb{R})^l\) is given by \((A, B, C, \lambda) \cdot (z, iy) = (\lambda Az, iBy)\). Note that this action is reducible if \(k\) and \(l\) are both nonzero.

Combining these observations with Proposition 3.2 and Theorem 4.2 we obtain the following

**Theorem 6.5.** Every spherical non-reductive algebraic subalgebra of \(\mathfrak{su}(n, 1)\) is \(G\)-conjugate to one in the following list where \(b_i \subset u(n - 1 - i)\), \(c \subset u(1)\) and \(l_2 \subset \mathfrak{sp}(1)\) are arbitrary
(under the condition that the maximal compact subalgebra is a Lie algebra, see Remark 5.2).

| $I_H \oplus n$       | $I_H \subset m \oplus a$ arbitrary |
|-----------------------|-------------------------------------|
| $s(so(l) \oplus b_l \oplus c) \oplus a \oplus n_{0,l}$ | $1 \leq l \leq n - 1,$ |
| $s(su(m) \oplus b_l \oplus c) \oplus a \oplus n_{0,l}$ | $2 \leq l \leq n - 1, l = 2m, m \geq 4$ |
| $s(spin(9) \oplus b_{16} \oplus c) \oplus a \oplus n_{0,16}$ | $2 \leq l \leq n - 1, l = 4m, m \geq 2$ |
| $s(spin(7) \oplus b_8 \oplus c) \oplus a \oplus n_{0,8}$ |                                             |
| $s(g_2 \oplus b_7 \oplus c) \oplus a \oplus n_{0,7}$ |                                             |
| $s(sp(1) \oplus l_2 \oplus b_4 \oplus c) \oplus a \oplus n_{0,4}$ |                                             |
| $s(l_2 \oplus sp(1) \oplus b_4 \oplus c) \oplus a \oplus n_{0,4}$ |                                             |
| $s(u(k) \oplus b_k \oplus c) \oplus a \oplus n_{k,0}$ | $1 \leq k \leq n - 1,$ |
| $s(su(k) \oplus b_k \oplus c) \oplus a \oplus n_{k,0}$ | $1 \leq k \leq n - 1$ |
| $s(sp(m) \oplus b_k \oplus c) \oplus a \oplus n_{k,0}$ | $1 \leq k \leq n - 1, k = 2m, m \geq 2$ |
| $s(sp(m) \oplus s^1 \oplus b_k \oplus c) \oplus a \oplus n_{k,0}$ | $1 \leq k \leq n - 1, k = 2m, m \geq 2$ |
| $s(c \oplus b_1 \oplus u(1)) \oplus a \oplus n_{1,0}$ |                                             |

7. $G = Sp(n, 1)$

Since $sp(1, 1) \cong so(4, 1)$ [Hel01, Ch. X §6.4] we assume $n > 1$. The Lie algebras $g$, $t$ and $p$ have the following form

$$
\mathfrak{g} = \left\{ \begin{pmatrix}
  A & -B & 0 \\
  z & e & 0 \\
  B & w & 0
\end{pmatrix} : A, B \in \mathbb{C}^{n \times n}, z, w \in \mathbb{C}^n, f \in \mathbb{C}, e \in i\mathbb{R}, \overline{A} = -A, B^t = B \right\}
$$

$$
\mathfrak{t} = \left\{ \begin{pmatrix}
  A & -B & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} : A, B \in \mathbb{C}^{n \times n}, f \in \mathbb{C}, e \in i\mathbb{R}, \overline{A} = -A, B^t = B \right\}
$$

$$
\mathfrak{p} = \left\{ \begin{pmatrix}
  0 & z & 0 \\
  \overline{w} & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} : z, w \in \mathbb{C}^n \right\} \cong \left\{ \begin{pmatrix}
  z & \overline{w} \\
  w & z
\end{pmatrix} : z, w \in \mathbb{C}^n \right\}
$$

$$
K = \left\{ \begin{pmatrix}
  A & -B & 0 \\
  0 & 0 & -5 \\
  0 & 0 & 0
\end{pmatrix} : (A \overline{B}) \in \text{Sp}(n), \left( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) \in \text{Sp}(1) \right\}
$$

7.1. The reductive case. Let $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$ be a reductive algebraic subalgebra of $sp(n, 1)$. It follows from Lemma 6.1 that after conjugation with an element in $K$, we may assume that $\mathfrak{q}$ equals

$$
\mathfrak{q}_{k,l,m,p,\xi} := \left\{ \begin{pmatrix}
  z & -\overline{w} \\
  w & \overline{z}
\end{pmatrix} : z \in \{0\}^k \times \mathbb{R}^l \times \mathbb{C}^{n-k-l}, w \in \{0\}^{k+l} \times \mathbb{C}^m \times \mathbb{R}^p \times \mathbb{C}^{n-k-l-m-p},
\right\}
$$

$$
i \cdot \text{Im}(z_{k+l+i}) + j \cdot w_{k+l+i} \in \mathbb{R} \cdot \xi_i \text{ for } 1 \leq i \leq m,$$

where $k, l, m, p \in \mathbb{N}$, $\xi = \text{Im}(\mathbb{H}) \cap S^3 \mathbb{R}^m$ and $(1, i, j, k)$ is the standard real basis of the quaternions.
Theorem 7.3. All spherical reductive algebraic subalgebras $G = L$ and the action of $\Lambda$.

Lemma 7.1. For $0 \leq k + l + m + p \leq n$ we have

$$\{q_{k,l,m,p,ξ}, q_{k,l,m,p,ξ} \}$$

$$\left\{ \left( \begin{array}{c} A \ \\ B \ \\ A \end{array} \right), \left( \begin{array}{c} e \ \\ f \ \\ e \end{array} \right) \right\} \in A = \left( \begin{array}{c} k \ l \ m \ p \ \\ 0 \ * \ * \ * \ 0 \ \\ 0 \ * \ * \ * \ 0 \ \\ 0 \ * \ * \ * \ 0 \ \end{array} \right), \ B = \left( \begin{array}{c} k \ l \ m \ p \ \\ 0 \ * \ * \ * \ 0 \ \\ 0 \ * \ * \ * \ 0 \ \end{array} \right)$$

* only has real entries, $0 \neq C_1 = C_1(ξ_1, ..., ξ_m)$.

$$N_l(q_{k,l,m,p,ξ}) \subset$$

$$\left\{ \left( \begin{array}{c} A \ \\ B \ \\ A \end{array} \right), \left( \begin{array}{c} e \ \\ f \ \\ e \end{array} \right) \right\} \in A = \left( \begin{array}{c} k \ l \ m \ p \ \\ 0 \ C \ 0 \ 0 \ 0 \ \\ 0 \ * \ * \ * \ 0 \ \\ 0 \ * \ * \ * \ 0 \ \end{array} \right), \ C = \left( \begin{array}{c} k \ l \ m \ p \ \\ 0 \ -f \ id \ 0 \ 0 \ \\ 0 \ 0 \ * \ 0 \ 0 \ \\ 0 \ 0 \ 0 \ 7 id 0 \ \end{array} \right)$$

$$(C - e id) \in so(l), (D + e id) \in so(p) \}.

Proof. The claim follows from direct calculation.

Since $\{q_{k,l,m,p,ξ}, q_{k,l,m,p,ξ} \} \subset \in N_l(q_{k,l,m,p,ξ})$ has to hold, we have shown the following

Proposition 7.2. If $\mathfrak{h}$ is a reductive algebraic subalgebra of $\mathfrak{g}$ with $\mathfrak{g} = q_{k,l,m,p,ξ}$, then either $l = m = p = 0$ (i.e., $\mathfrak{g} = (0)^k \times \mathbb{H}^n-k \subset \mathbb{H}^n$), $n - k - l = p = m = 0$ (i.e., $\mathfrak{g} = (0)^k \times \mathbb{R}^n-k \subset \mathbb{R}^n$) or $n - k - m = p = l = 0$ (i.e., $\mathfrak{g} = (0)^k \times (\mathbb{C}_{ξ_1} \times ... \times \mathbb{C}_{ξ_{n-k}}) \subset \mathbb{H}^n$) has to hold.

Considering these cases separately we prove the following result.

Theorem 7.3. All spherical reductive algebraic subalgebras $\mathfrak{h}$ of $\mathfrak{sp}(n,1)$ are (up to conjugation in $G$) one of the following

| $\mathfrak{sp}(n)$ | $0 \leq k < n$ |
|---------------------|-----------------|
| $\mathfrak{sp}(n) \oplus \mathfrak{s}^1$ | |
| $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ | |
| $\mathfrak{sp}(n-1,1)$ | |
| $\mathfrak{s}^1 \oplus \mathfrak{sp}(n-1,1)$ | |
| $\mathfrak{sp}(k) \oplus \mathfrak{sp}(n-k,1)$ | |
| $\mathfrak{su}(n,1)$ | |
| $\mathfrak{su}(n,1) \oplus \mathfrak{s}^1$ | |

According to [Ber57, Table 2] the symmetric subalgebras are $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ and $\mathfrak{sp}(k,1) \oplus \mathfrak{sp}(n-k)$ and $\mathfrak{sp}(k) \oplus \mathfrak{sp}(n-k,1) \oplus \mathfrak{su}(n,1) \oplus \mathfrak{s}^1$.

Proof. If $l = m = p = 0$ and $k = n$ then $\mathfrak{h} = l$ is compact. Therefore Proposition 3 and Theorem 4 yield the first three cases in the list.

If $l = m = p = 0$ and $0 \leq k < n$ we have

$$\mathfrak{sp}(n-k,1) \cong [q_{k,0,0,0}, q_{k,0,0,0}] \cong q_{k,0,0,0} \subset \mathfrak{h} \subset N_{K}(q_{k,0,0,0}) \cong \mathfrak{sp}(k) \oplus \mathfrak{sp}(n-k,1)$$

and the action of $N_{K}(q_{k,0,0,0}) \cong Sp(k) \times Sp(n-k) \times Sp(1)$ on

$$q_{k,0,0,0} = \{ (z, w) : z \in \mathbb{C}^k \times \{0\}^{n-k}, w \in \mathbb{C}^k \times \{0\}^{n-k} \} \cong \mathbb{C}^{2k}$$
We denote \( \xi \). Calculation shows that the action of \( N \) on \( k \) we just considered. We therefore additionally assume \( k \neq n \). The action of

\[
N_K(q_{k,n-k,0,0}) = \left\{ \left( \begin{array}{c} A - B \frac{e}{\tau} \\
A \end{array} \right), \left( \begin{array}{c} a - f \frac{e}{\tau} \\
b \end{array} \right) \right\} \in K : aA + bBz + bA^z = 0
\]

for all \( z \in \{0\}^k \times \mathbb{R}^{n-k} \)

on

\[
q_{k,n-k,0,0} = \left\{ \left( \begin{array}{c} z - \frac{e}{\tau} \\
w \end{array} \right) : z \in \mathbb{C}^k \times (i\mathbb{R})^{n-k}, w \in \mathbb{C}^n \right\}
\]
is reducible if \( k \neq 0 \). If \( k = 0 \) we can decompose \( N_K(q_{0,n,0,0}) \) into

\[
N_K(q_{0,n,0,0}) = \left\{ \left( \begin{array}{c} A - B \frac{e}{\tau} \\
A \end{array} \right), \left( \begin{array}{c} a - f \frac{e}{\tau} \\
b \end{array} \right) \right\} \in K : A = B, A = 0 \implies O(n) \times Sp(1).
\]

Calculation shows that the action of \( N_K(q_{0,n,0,0}) \) on \( q_{0,n,0,0}^+ \) is not transitive on the spheres in \( q_{0,n,0,0}^+ \).

Let us now assume \( n - k - m = p = l = 0 \). If \( m = n - k = 0 \) we are again in the first case. We therefore assume \( k \neq n \) and obtain

\[
q_{k,0,n-k,0,0} = \left\{ \left( \begin{array}{c} z - \frac{e}{\tau} \\
w \end{array} \right) : z \in \{0\}^k \times \mathbb{C}^{n-k}, w \in \{0\}^k \times \mathbb{C}^{n-k},
\]

\[
i \cdot \text{Im}(z_{k+i}) + j \cdot w_{k+i} \in \mathbb{R} : \xi_i \text{ for all } 1 \leq i \leq n - k \right\}.
\]

The action of

\[
N_K(q_{k,0,n-k,0,0}) \subset \left\{ \left( \begin{array}{c} A - B \frac{e}{\tau} \\
A \end{array} \right), \left( \begin{array}{c} a - f \frac{e}{\tau} \\
b \end{array} \right) \right\} \in K : A = \left( \begin{array}{c} A_1 \ 0
\\nA_2 \end{array} \right), C = \left( \begin{array}{c} C_1 \ 0
\\n0 \ C_2 \end{array} \right)
\]

on \( q_{k,0,n-k,0,0}^+ \) is reducible if \( k \neq 0 \). We therefore assume \( k = 0 \). For dimensional reasons \( N_K(q_{0,n,0,0}) \) can only act transitively on the spheres in \( q_{0,n,0,0}^+ \) if \( \xi_l \in \mathbb{R} \cdot \xi_1 \) for all \( 1 \leq l \leq n \).

We denote \( \xi_1 = i\alpha_1 + j\beta_1 + k\gamma_1 \), where \( \alpha_1, \beta_1, \gamma_1 \in \mathbb{R} \). Recall that \( \|\xi_1\| = 1 \).

If \( \beta_1 = \gamma_1 = 0 \) we obtain

\[
\mathfrak{su}(n, 1) = [q_{0,0,0,0,0}, q_{0,0,0,0,0}] \oplus q_{0,0,0,0,0} \subset \mathfrak{n} \subset N_K(q_{0,0,0,0,0}) \oplus q_{0,0,0,0,0} = \mathfrak{u}(n, 1).
\]

If \( \alpha_1 = 0 \) we obtain

\[
N_K(q_{0,0,0,0,0}) \oplus q_{0,0,0,0,0} = \left\{ \left( \begin{array}{c} A - B \frac{e}{\tau} \\
A \end{array} \right), \left( \begin{array}{c} a - f \frac{e}{\tau} \\
b \end{array} \right) \right\} \in K : A \in \mathbb{R}^{n \times n}, z \in \mathbb{R}^n, e = 0,
\]

\[
B \in (\beta_1 - i\gamma_1)\mathbb{R}^{n \times n}, w \in (\beta_1 - i\gamma_1)\mathbb{R}^n, f \in (\beta_1 - i\gamma_1)\mathbb{R}
\]

Conjugation with \( \left( \begin{array}{c} z \text{id} \\
0 \text{id} \end{array} \right) \in K \), where \( z = \sqrt{(\beta_1 - i\gamma_1)} \) maps this Lie algebra to

\[
\left\{ \left( \begin{array}{c} A - B \frac{e}{\tau} \\
A \end{array} \right), \left( \begin{array}{c} a - f \frac{e}{\tau} \\
b \end{array} \right) \right\} \in \mathfrak{g} : A, B \in \mathbb{R}^{n \times n}, z, w \in \mathbb{R}^n, e = 0, f \in \mathbb{R}
\]
Conjugation with $\frac{1}{\sqrt{2}} \begin{pmatrix} \text{id} & \text{id} \\ \text{id} & \text{id} \end{pmatrix} \in K$ maps it then to $u(n, 1)$ while the algebra $[q_{0,0,n,0,\xi}, q_{0,0,n,0,\xi}] \oplus q_{0,0,n,0,\xi}$ is mapped to $su(n, 1)$.

If $\alpha_1$ and $\beta_1 - i\gamma_1$ are non zero we obtain

$$\mathcal{N}(q_{0,0,n,0,\xi}) \oplus q_{0,0,n,0,\xi} = \left\{ \begin{pmatrix} A & z - \frac{\beta_1 - i\gamma_1}{\alpha_1} \\ e & w \\ e & w \\ -w^t f & z^t \end{pmatrix} \right\} \in g : B = \frac{\beta_1 - i\gamma_1}{\alpha_1} \text{Im}(A), w = \frac{\beta_1 - i\gamma_1}{\alpha_1} \text{Im}(z), f = \frac{\beta_1 - i\gamma_1}{\alpha_1} \text{Im}(e) \right\}.$$ 

Conjugation with $\left( \begin{pmatrix} z & 0 \\ 0 & \text{id} \end{pmatrix} \right) \in K$, where $z = \sqrt{\frac{\alpha_1(\beta_1 - i\gamma_1)^{-1}}{\alpha_1(\beta_1 - i\gamma_1)^{-1}}}$ maps this Lie algebra to

$$\left\{ \begin{pmatrix} A & z - \frac{\beta_1 - i\gamma_1}{\alpha_1} \\ e & w \\ e & w \\ -w^t f & z^t \end{pmatrix} \right\} \in g : B = \text{Im}(A), w = \text{Im}(z), f = \text{Im}(e) \right\}.$$ 

Conjugation with $\left( \begin{pmatrix} z & 0 \\ 0 & -\text{id} \end{pmatrix} \right) \in K$, where $z = \sqrt{1 - \frac{\sqrt{2}}{4} i}$, $w = \sqrt{1 + \frac{\sqrt{2}}{4}}$ maps it then to

$$\left\{ \begin{pmatrix} A & z - \frac{\beta_1 - i\gamma_1}{\alpha_1} \\ e & w \\ e & w \\ -w^t f & z^t \end{pmatrix} \right\} \in g : A, B \in \mathbb{R}^{n \times n}, z, w \in \mathbb{R}^n, e = 0, f \in \mathbb{R} \right\}.$$ 

Hence we obtain again

$$\mathcal{N}(q_{0,0,n,0,\xi}) \oplus q_{0,0,n,0,\xi} \cong u(n, 1), \quad [q_{0,0,n,0,\xi}, q_{0,0,n,0,\xi}] \oplus q_{0,0,n,0,\xi} \cong su(n, 1).$$

The action of

$$\mathcal{N}(q_{0,0,n,0,\xi}) \cong \left\{ \left( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right) \in K \right\} \cup \left\{ \left( \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \right) \right) \in K \right\}$$

on

$$q_{0,0,n,0,\xi}^{\perp} \cong \left\{ \left( \begin{pmatrix} z & -w \end{pmatrix} : z = 0, w \in \mathbb{C}^n \right) \right\}$$

is given by

$$\left( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right) \cdot \left( \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \right) \right) = \left( \begin{pmatrix} 0 & -A\xi \eta \\ A\xi \eta & 0 \end{pmatrix}, \right),$$

$$\left( \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \right) \right) \cdot \left( \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \left( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \right) \right) = \left( \begin{pmatrix} 0 & -B_{wz} \\ B_{wz} & 0 \end{pmatrix}, \right).$$

Since the compact part of $SU(n, 1)$ already acts transitively on the spheres in $q_{0,0,n,0,\xi}^{\perp} \cong \mathbb{C}^n$, Proposition 3.1 and Theorem 4.2 yield the last two cases in the list. \qed
7.2. The non-reductive case. Choosing $a = \mathbb{R} \begin{pmatrix} 0 & e_1 & 0 & 0 \\ e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \end{pmatrix}$ we obtain the following subalgebras of $\mathfrak{g}$.

$$m = \mathfrak{z}_\mathfrak{t}(a) = \begin{cases} A 0 -f_0 0 \\ 0 0 -f_0 \\ B 0 A \\ 0 0 0 \end{cases} \in \mathfrak{k}: A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\cong \mathfrak{sp}(n-1) \times \mathfrak{sp}(1)$

$$\mathfrak{g}_\alpha = \begin{cases} A z -B \mathcal{M}_0 -f_0 \\ \mathcal{M}_0 B 0 \mathcal{M}_0 0 \\ -w^t 0 z^t \end{cases} \in \mathfrak{g} : w_1 = 0, z_1 = 0, A = \begin{pmatrix} 0 & \mathcal{M}_2 & \cdots & \mathcal{M}_n \\ -z_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -z_n & 0 & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -w_2 & \cdots & -w_n \\ -w_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -w_n & 0 & \cdots & 0 \end{pmatrix}$$

$\cong \begin{cases} \left( \begin{pmatrix} z' & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix}, z', w' \in \mathbb{C}^{n-1} \right) \end{cases}$

$$\mathfrak{g}_{2\alpha} = \begin{cases} A z -B \mathcal{M}_0 -f_0 \\ \mathcal{M}_0 B 0 \mathcal{M}_0 0 \\ -w^t 0 z^t \end{cases} \in \mathfrak{g} : z = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, w = \begin{pmatrix} f \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} -f & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\cong \begin{cases} \left( \begin{pmatrix} e & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix}, f \in \mathbb{C}, e \in i\mathbb{R} \right) \end{cases}$

$$n = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$$

Moreover, we have

$$M = \begin{cases} \left( \begin{pmatrix} 0 & -f_0 & 0 \\ 0 & 0 & 0 \\ A -B \mathcal{M}_0 B 0 \mathcal{M}_0 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \in \mathfrak{sp}(n), \left( \begin{pmatrix} a & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix} \right) \in \mathfrak{sp}(1), A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right) \in \mathfrak{sp}(1) \right) \end{cases} \cong \begin{cases} \left( \begin{pmatrix} A' & -B' \mathcal{M}_0 B 0 \mathcal{M}_0 0 \end{pmatrix} \right) \in \mathfrak{sp}(n-1) \right) \times \left( \begin{pmatrix} a & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix} \right) \in \mathfrak{sp}(1) \right) \end{cases} \cong \mathfrak{sp}(n-1) \times \mathfrak{sp}(1) \end{cases}$$

The identification of $n$ with $\mathbb{C}^{2n \times 2} \times \mathfrak{su}(2)$ indicated above is a $M$-equivariant isomorphism with respect to the adjoint $M$-action on $n$ and the $\mathfrak{sp}(n-1) \times \mathfrak{sp}(1)$-action defined by

$$(A, B) \cdot \left( \begin{pmatrix} z' & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix}, \begin{pmatrix} e & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix} \right) = \left( A \begin{pmatrix} z' & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix} B^{-1}, B \begin{pmatrix} e & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix} B^{-1} \right)$$

for all $A \in \mathfrak{sp}(n-1), B \in \mathfrak{sp}(1)$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$ holds, Lemma 4.1 shows that we may assume that $n_H$ equals

$$n_{k,l,m,p,\xi} := \mathfrak{g}_{2\alpha} \oplus \left( \begin{pmatrix} z' & -\mathcal{M}_0 \\ \mathcal{M}_0 & 0 \end{pmatrix}, z' \in \{0\}^k \times \mathbb{R}^l \times \mathbb{C}^{n-1-k-l}, w' \in \{0\}^k \times \mathbb{R}^l \times \mathbb{C}^{n-1-k-l-m-p}, i \cdot \text{Im}(z_{k+i}) + j \cdot w_{k+i} \in \mathbb{R} \cdot \xi_i \text{ for } 1 \leq i \leq m \right),$$

where $k, l, m, p \in \mathbb{N}$ and $\xi \in (\text{Im}(\mathbb{H}) \cap S^3)^m$. 
As before we have

$$N_m(n_{k,l,m,p}, \xi) \subset \left\{ \left( \begin{array}{c} A' - f \frac{\xi}{A} \\ B' \end{array} \right), \left( \begin{array}{c} z \frac{-7}{w} \\ T \end{array} \right) \right\} \in M : A' = \left( \begin{array}{c} * \ 0 \ 0 \ 0 \\ 0 * \ 0 \ 0 \\ 0 0 * \ 0 \\ 0 0 0 * \end{array} \right), B' = \left( \begin{array}{c} * \ 0 \ 0 \ 0 \\ 0 * \ 0 \ 0 \\ 0 0 * \ 0 \\ 0 0 0 * \end{array} \right), (C - e \text{id}) \in \mathfrak{so}(l), (D + e \text{id}) \in \mathfrak{so}(p) \right\}.$$

Since the exponential mapping does not destroy the block structure, we conclude that the action of $N^\pm_M(n_{k,l,m,p}, \xi)$ on $n_{k,l,m,p}^\pm$ is reducible if more than one of the indices $k, l, m, p$ is non-zero.

Considering each of these cases separately and using Proposition 6.2 and Theorem 4.2 we prove the following result.

**Theorem 7.4.** Every spherical non-reductive algebraic subalgebra of $\mathfrak{sp}(n, 1)$ is $G$-conjugate to one in the following list, where $b_k \subset \mathfrak{sp}(n - 1 - k)$, $c \subset \mathfrak{sp}(1)$, $d \subset s^1$ and $l_2 \subseteq \mathfrak{sp}(1)$ are arbitrary (under the condition that the maximal compact subalgebra is a Lie algebra, see Remark 5.5).

| $I_H \oplus n$ | $I_H \subset m \oplus a$ arbitrary |
|-----------------|----------------------------------|
| $\mathfrak{sp}(k) \oplus b_k \oplus c \oplus a \oplus n_{k,0,0,0}$ | $1 \leq k \leq n - 1$ |
| $c \oplus b_1 \oplus \mathfrak{sp}(1) \oplus a \oplus n_{1,0,0,0}$ | $1 \leq m \leq n - 1$ |
| $\mathfrak{sp}(1) \oplus b_1 \oplus a \oplus n_{0,1,0,0}$ | $1 \leq m \leq n - 1$ |
| $u(m) \oplus b_m \oplus d \oplus a \oplus n_{0,0,m,0}$ | $1 \leq m \leq n - 1, m = 2j, j \geq 2$ |
| $su(m) \oplus b_m \oplus d \oplus a \oplus n_{0,0,m,0}$ | $1 \leq m \leq n - 1, m = 2j, j \geq 2$ |
| $\mathfrak{sp}(j) \oplus s^1 \oplus b_m \oplus d \oplus a \oplus n_{0,0,m,0}$ | $1 \leq p \leq n - 1$ |
| $\mathfrak{sp}(j) \oplus s^1 \oplus b_m \oplus d \oplus a \oplus n_{0,0,m,0}$ | $1 \leq p \leq n - 1, p = 2j, j \geq 4$ |
| $\mathfrak{so}(p) \oplus b_p \oplus c \oplus a \oplus n_{0,0,0,p}$ | $1 \leq m \leq n - 1, m = 2j, j \geq 2$ |
| $\mathfrak{su}(j) \oplus b_p \oplus c \oplus a \oplus n_{0,0,0,p}$ | $1 \leq m \leq n - 1, m = 2j, j \geq 2$ |
| $\mathfrak{sp}(j) \oplus b_p \oplus c \oplus a \oplus n_{0,0,0,p}$ | $1 \leq p \leq n - 1$ |
| $\mathfrak{sp}(j) \oplus b_p \oplus c \oplus a \oplus n_{0,0,0,p}$ | $1 \leq p \leq n - 1, p = 4j, j \geq 2$ |
| $\mathfrak{spin}(9) \oplus b_{16} \oplus c \oplus a \oplus n_{0,0,0,16}$ | $1 \leq m \leq n - 1$ |
| $\mathfrak{spin}(7) \oplus b_8 \oplus c \oplus a \oplus n_{0,0,0,8}$ | $1 \leq m \leq n - 1$ |
| $\mathfrak{g}_2 \oplus b_7 \oplus c \oplus a \oplus n_{0,0,0,7}$ | $1 \leq p \leq n - 1$ |
| $\mathfrak{sp}(1) \oplus l_2 \oplus b_{14} \oplus c \oplus a \oplus n_{0,0,0,4}$ | $1 \leq m \leq n - 1$ |
| $l_2 \oplus \mathfrak{sp}(1) \oplus b_{14} \oplus c \oplus a \oplus n_{0,0,0,4}$ | $1 \leq p \leq n - 1, p = 4j, j \geq 2$ |

**Proof.** If $l = m = p = 0$ the action of $N_M(n_{k,0,0,0}) = \text{Sp}(k) \times \text{Sp}(n-k-1) \times \text{Sp}(1)$ on $n_{k,0,0,0}^\pm \cong \mathbb{C}^{2k}$ is given by $(A, B, C) \cdot (z, w) = A \cdot (\frac{z}{w} - \overline{w}) \cdot C^{-1}$ for all $A \in \text{Sp}(k), B \in \text{Sp}(n-k-1), C \in \text{Sp}(1)$ and $z, w \in \mathbb{C}^k$.

The action of

$$N_M(n_{0,1,0,0}) \cong \left\{ \left( \begin{array}{c} A' - f \frac{\xi}{A} \\ B' \end{array} \right), \left( \begin{array}{c} a \frac{0}{w} \\ b \frac{-7}{w} \\ T \end{array} \right) \right\} \in M : A' = \left( \begin{array}{c} A_1 \ 0 \\ 0 \ A_2 \end{array} \right), B' = \left( \begin{array}{c} 0 \ 0 \\ B_1 \ 0 \ B_2 \end{array} \right), \pi A_1 \in O(l) \right\}$$

$$\cup \left\{ \left( \begin{array}{c} A' - \overline{f} \frac{\xi}{A} \\ B' \end{array} \right), \left( \begin{array}{c} a \frac{0}{w} \\ b \frac{-7}{w} \\ T \end{array} \right) \right\} \in M : A' = \left( \begin{array}{c} A_1 \ 0 \\ 0 \ A_2 \end{array} \right), B' = \left( \begin{array}{c} B_1 \ 0 \\ 0 \ B_2 \end{array} \right), b \neq 0,$$

$$A_1 = \frac{0}{b} \overline{f} \frac{1}{b} \overline{f} \frac{1}{b}, \overline{b} \overline{f} \frac{1}{b} \overline{f} \frac{1}{b} \in O(l)$$

$$\cong O(l) \times \text{Sp}(n - 1 - l) \times \text{Sp}(1),$$
Spherical Algebraic Subalgebras

8. The Exceptional Group $G = F_4$

In this section let $G = K \exp(p)$ be the connected simple group $F_4$ of real rank 1. The maximal compact subgroup $K$ is isomorphic to Spin(9) while $\dim p = 16$, see [Nis13, §9].

8.1. The reductive case. We want to classify the spherical reductive algebraic subalgebras $h = l \oplus q$ of $g = f_4$. Using the classification of maximal reductive subalgebras of $g$, Krötz and Schlichtkrull have shown that a spherical reductive subalgebra of $g$ must be contained in a symmetric one, see [KS13a, Lemma 6.2]. According to [Ber57], the symmetric subalgebras of $f_4$ are $so(9)$, $so(8,1)$ and $sp(2,1) \oplus su(2)$. The next lemma determines $q$.

Lemma 8.1. Let $G^\sigma = K^\sigma \exp(p^\sigma)$ be a symmetric subgroup of $G$ (defined as the fixed point set of an involution $\sigma$ that commutes with the Cartan involution) and $H = L \exp(q)$ a subgroup of $G^\sigma$. The adjoint $L$-action on $p$ can only be transitive on the spheres in $q^\perp \subset p$ if $q = p^\sigma$.

Proof. Since $h$ is a subalgebra of $g^\sigma$, $q$ lies in $p^\sigma$, which implies that $q^\perp \subset p = p^\sigma \oplus p^{-\sigma}$ is the direct sum of $p^{-\sigma}$ and $q^\perp \cap p^\sigma$. Since the adjoint $t^\sigma$ action on $p$ stabilizes $p^\sigma$ and $p^{-\sigma}$, $L$ can only act irreducibly on $q^\perp \subset p$ if $q = p^\sigma$.

If $g^\sigma$ is simple and non-compact, then we have $t^\sigma = [p^\sigma, p^\sigma]$. These observations are sufficient to handle the case that $h$ is a subalgebra of $so(9)$ or $so(8,1)$. For the case that $h$ is a subalgebra of $sp(2,1) \oplus su(2)$ we also need the following technical lemma.
Lemma 8.2. Let $G = G_1 \times G_2$ be compact and $V$ be an irreducible representation of $G$. Then $V$ is $G$-equivariantly isomorphic to $U \otimes_D W$, where $D \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$ and $U, W$ are $D$-vector spaces, $U$ is an irreducible $G_1$-representation and $W$ is an irreducible $G_2$-representation.

Proof. We decompose $V$ into isotypical components with respect to $G_2$, i.e. $V = V_1 \oplus \ldots \oplus V_k$. The group $G_1$ leaves each $V_i$ invariant since the $G_1$-action on $V$ commutes with the $G_2$-action on $V$. Therefore each $V_i$ is $G$-invariant. Since $V$ is assumed to be irreducible, $V$ is just one isotypical component. Therefore we can write $V = W^m$, where $W$ is an irreducible representation of $G_2$.

We define $D := \text{End}_{G_2}(W)$. Note that $D$ is a finite-dimensional associative division algebra over the reals and therefore (by Frobenius-Theorem) equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

We define the bilinear map $\Phi : \text{Hom}_{G_2}(W, V) \times D \to V$ by $(\varphi, w) \mapsto \varphi(w)$. Note that $V$ is a $D$-vector space with respect to multiplication from the right. Therefore $V = W^m$ is also a $D$-vector space. This induces a $D$-vector space structure on $\text{Hom}_{G_2}(W, V)$ by $(d \cdot \varphi)(w) = \varphi(dw)$ for all $w \in W$, $d \in D$ and $\varphi \in \text{Hom}_{G_2}(W, V)$. The map $\Phi$ is $D$-bilinear. Hence it induces a map $\text{End}_{G_2}(W, V) \otimes_D W \to V$ that we also denote by $\Phi$.

We define a $G_1 \times G_2$-action on $\text{End}_{G_2}(W, V) \otimes_D W$ by $(g_1, g_2) \cdot (\varphi \otimes w) = (g_1 \cdot \varphi) \otimes (g_2 \cdot w)$, where the action of $G_2$ on $W$ is the given one and the action of $G_1$ on $\text{End}_{G_2}(W, V)$ is induced by the $G_1$-action on $V$ via $(g_1 \cdot \varphi)(w) = g_1(\varphi(w))$. The map $\Phi$ is $G$-equivariant with respect to the given actions. Since the $G$-action on $V$ is irreducible (and $\Phi$ is not the zero map) $\Phi$ has to be surjective. Since the dimension of $\text{End}_{G_2}(W, V) \otimes_D W$ equals the dimension of $V$, it is bijective and therefore a $G$-equivariant isomorphism. Note that the $G_1$-action on $\text{End}_{G_2}(W, V)$ has to be irreducible in order for the $G$-action on $V$ to be irreducible. \(\square\)

Now we are in position to prove the main result of this subsection.

Theorem 8.3. Up to conjugation by an element of $G$, the following tables exhaust the spherical, reductive subalgebras $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$ of $\mathfrak{f}_4$.

| spin(9) | spin(8, 1) | $\mathfrak{sp}(2, 1)$ | $\mathfrak{sp}(2, 1) \oplus \mathfrak{s}^1$ | $\mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$ |

According to [Ber57 Table 2], the symmetric ones are spin(9) and spin(8, 1) and $\mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$.

Proof. Let us first consider the case that $\mathfrak{g}^\sigma = \mathfrak{so}(8, 1)$. Since $\mathfrak{q} = \mathfrak{p}^\sigma$ (Lemma 8.1) and $\mathfrak{g}^\sigma$ is simple non-compact, $\mathfrak{h} \subset \mathfrak{g}^\sigma$ has to contain $\mathfrak{so}(8, 1)$. Hence $\mathfrak{h} = \mathfrak{so}(8, 1)$ is the only spherical subalgebra of $\mathfrak{g}$ that lies in $\mathfrak{so}(8, 1)$.

Let us now consider the case that $\mathfrak{h} \subset \mathfrak{so}(9)$. In this case $\mathfrak{l} = \mathfrak{l}$ and $\mathfrak{q}^\perp = \mathfrak{p} \cong \mathbb{R}^{16}$. We are therefore looking for connected subgroups of Spin(9) that act transitively on the spheres in $\mathbb{R}^{16} \cong \mathfrak{p}$ with respect to the adjoint action. Due to Theorem 7.2 there are none except for Spin(9) itself. This gives $\mathfrak{h} = \text{spin}(9)$.

If $\mathfrak{g}^\sigma = \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$ then $\mathfrak{t}^\sigma = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{su}(2)$ and $\dim_{\mathbb{R}}(\mathfrak{p}^\sigma) = 8$. Since $\mathfrak{q} = \mathfrak{p}^\sigma$ (Lemma 8.1) and $\mathfrak{sp}(2, 1)$ is simple non-compact, $\mathfrak{h} \subset \mathfrak{g}^\sigma$ has to contain $\mathfrak{sp}(2, 1)$. Hence $\mathfrak{h} = \mathfrak{sp}(2, 1) \oplus \mathfrak{b}$, where $\mathfrak{b}$ is a subalgebra of $\mathfrak{su}(2)$. Note that we have $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \subset \mathfrak{l} \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{su}(2) = \mathfrak{t}^\sigma$.

The condition for $H$ to be spherical in $G$ is that $L$ acts transitively on the spheres in $\mathfrak{q}^\perp \cong \mathfrak{p}^\perp \cong \mathbb{R}^8$. 
Proposition 8.4. The $K^\sigma$-representation on $p^\sigma$ is irreducible. Lemma \[1\] implies that $p^\sigma$ decomposes into a tensor product of three $D$-vector spaces that are irreducible representations for $Sp(2)$, $Sp(1)$ and $Sp(1)$ respectively, where $D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Since the adjoint $K^\sigma$-representation on $p^\sigma$ is not only irreducible but transitive on the spheres this tensor product has two factors that are equal to $D$ (since simple tensors cannot be mapped to non-simple tensors by this action). Using again the fact that $K^\sigma$ acts transitively on the sphere $S^7 \subset \mathbb{R}^8 \cong p^\sigma$, and the fact that $\dim_{\mathbb{R}}(Sp(1)) = 3 < 7 = \dim_{\mathbb{R}}(S^7)$ we obtain the following decomposition

$$p^\sigma \cong \begin{cases} \mathbb{R}^8 \otimes \mathbb{R} \otimes \mathbb{R} \\ \mathbb{C}^4 \otimes \mathbb{C} \otimes \mathbb{C} \\ \mathbb{H}^2 \otimes \mathbb{H} \otimes \mathbb{H} \end{cases}$$

Note that in each case $Sp(2)$ acts by the standard action on $p^\sigma$.

8.2. The non-reductive case. We start by analyzing the structure of a minimal parabolic subgroup $Q_0 = MAN$ of $G = E_7$. According to [Nis12, Lemma 7.3] we have that $M = \text{Spin}(7)$ and that $N$ is not Abelian. Since the $M$-representations on $g_\alpha$ and $g_{2\alpha}$ are irreducible, see [Wol84, Ch. 13], we obtain $[g_\alpha, g_\alpha] = g_{2\alpha}$ where $n = g_\alpha \oplus g_{2\alpha}$ is the restricted root space decomposition. It is shown in [Nis13] that $\dim g_\alpha = 8$ and $\dim g_{2\alpha} = 7$.

First we are going to identify these two irreducible $M$-representations.

Proposition 8.4. The $M$-representation on $g_\alpha$ is induced by the embedding $\varphi : \text{spin}(7) \hookrightarrow \mathfrak{so}(8)$ given by

$$\begin{pmatrix} 0 & -a & -b & -c & -d & -e & -f \\ a & 0 & -g & -h & -i & -j & -k \\ b & g & 0 & -l & -m & -n & -p \\ c & h & l & 0 & -q & -r & -s \\ d & i & m & q & 0 & -t & -u \\ e & j & n & r & t & 0 & -v \\ f & k & p & s & u & v & 0 \end{pmatrix} \mapsto \begin{pmatrix} a-s+t & 0 & -a+s-t & -b-r-u & -c-k+n & -d+j+p & -e-i-l & -f+h-m & g+q+v \\ 0 & a-s+t & 0 & -g+q-v & f-h-m & -e-i+l & d+j-p & -c-k-n & -b+r+u \\ b+r+u & g-q+v & 0 & -a-s-t & -b+r-u & a-s-t & d+j-p & c-k+n & -f+h-m \\ c+k-n & -f+h+m & e+i-l & 0 & g-q-v & -b+r+u & a-s-t & -d-j-p & c-k+n \\ d+j-p & e+i-l & f+h+m & g+q+v & 0 & -a-s-t & -b+r-u & a-s+t & 0 \\ e+i+l & d+j-p & -d+j-p & -a+s+t & -b+r-u & a+s+t & 0 & -g-q-v & f+h-m \\ f-h-m & c+k+n & -d+j-p & -a+s+t & -b+r-u & b+r+u & g+q+v & 0 & -e-i+l \end{pmatrix}$$

while the $M$-representation on $g_{2\alpha}$ is the defining representation of $SO(7)$ up to the covering $M \to SO(7)$.

Proof of Proposition 8.4. Firstly, we give the construction of the irreducible 8-dimensional $\text{spin}(7)$-representation described above.

The embedding from $\text{Spin}(7)$ into $SO(8)$ is described in [Oni91, Ch.1 §5.3] as follows. Let $1 \leq i < j \leq 7$ then $(-E_{ij} + E_{ji})$ forms a basis of $\mathfrak{so}(7)$, where $E_{ij}$ is the matrix with a one in line $i$ and column $j$ and zero everywhere else. We take the standard orthogonal basis
\{e_0, e_1, \ldots, e_7\} of octonions over \mathbb{R} with the following multiplication table.

\[
\begin{array}{cccccccc}
&e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\text{X} & & & & & & & & \\
e_0 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_2 & e_1 & e_0 & e_1 & e_4 & -e_3 & e_5 & e_6 & e_7 \\
e_3 & e_2 & e_1 & e_2 & -e_0 & e_5 & e_4 & e_6 & e_7 \\
e_4 & e_3 & e_2 & -e_1 & e_6 & e_0 & e_5 & e_4 & e_7 \\
e_5 & e_4 & e_3 & e_1 & e_7 & e_1 & e_0 & e_5 & e_4 \\
e_6 & e_5 & e_4 & e_2 & e_7 & e_2 & e_1 & e_0 & e_6 \\
e_7 & e_6 & e_5 & e_3 & e_7 & e_7 & e_2 & e_1 & e_0 \\
\end{array}
\]

Note that the space of pure imaginary octonions \(V\) is spanned by \(\{e_1, \ldots, e_8\}\). The Lie algebra \(\text{spin}(7)\) is spanned by elements \(e_ie_j\) with \(1 \leq i < j \leq 7\). The map \(2(-E_{ij} + E_{ji}) \mapsto e_ie_j\) for all \(1 \leq i < j \leq 7\) is an isomorphism from \(\mathfrak{so}(7)\) to \(\mathfrak{spin}(7)\). The map \(\lambda : e_ie_j \mapsto (x \mapsto e_i(e_jx))\) is an embedding of \(\mathfrak{spin}(7)\) into \(\mathfrak{so}(8)\), where we choose the basis \(\{e_1, \ldots, e_7, e_0\}\) of \(\mathbb{R}^8\). It is explicitly given by the following map as calculation with the multiplication table shows.

Note that the induced action of \(\text{Spin}(7)\) on \(\mathbb{R}^8\) is transitive on the spheres. This follows since \(\lambda(\text{spin}(7))(e_0) = T_{e_0}(\text{Spin}(7) \cdot e_0)\) coincides (by the multiplication table) with the purely imaginary octonions, which equals \(T_{e_0}(S^7)\). Therefore the orbit of \(\text{Spin}(7)\) is open in \(S^7\). Since \(\text{Spin}(7)\) and hence also its orbits are compact, it coincides with \(S^7\). In particular, this representation is irreducible.

In order to finish the proof we will show that the irreducible \(\text{spin}(7)\)-representations in dimensions 7 and 8 are unique up to isomorphism.

To see this, let \(\mathfrak{m}^C := \text{spin}(7, \mathbb{C}) = \mathfrak{so}(7, \mathbb{C})\). Then \(\mathfrak{m}^C\) is a complex simple Lie algebra. We apply the Weyl Dimension Formula (see e.g. [Kna02, Ch.V §6]) in order to determine the possible dimensions of irreducible \(\mathfrak{m}^C\)-representations. To do this we use the information on \(\mathfrak{so}(7, \mathbb{C})\) given in [Kna02] Appendix C §1. Then elementary calculations show that there are unique irreducible representations in complex dimensions 7 and 8 and none in complex dimensions 14 and 16. This proves the statement about real representations of \(\mathfrak{m}\).

Proposition [5.4] allows us to give a normal form for the subspace \(\mathfrak{n}_H \subset \mathfrak{n}\). Since we already know from Remark [5.4] that \(\mathfrak{g}_{2\alpha} \subset \mathfrak{n}_H\), it is sufficient to give a normal form for subspaces of \(\mathfrak{g}_{2\alpha}\) under the action of \(M\).

**Lemma 8.5.** Let \(\text{Spin}(7)\) act irreducibly on \(\mathbb{R}^8\) and \(V\) be a \(n\)-dimensional subspace. Then \(V\) is \(\text{Spin}(7)\)-conjugate to the span of \(e_1, \ldots, e_n\) or to the span of \(e_1, e_2, e_3, e_4 + e_5\), where \(c \in \mathbb{R}\).

**Proof.** The proof of the last Proposition shows that \(\text{Spin}(7)\) acts transitively on the spheres in \(\mathbb{R}^8\). The stabilizer of \(\text{Spin}(7)\) in \(e_0\) is equal to \(G_2\) ([Oni94, Ch. 1 §5]). Due to Onischik's
classification ([On94, Table 8]) the group $G_2$ acts transitively on the spheres in $\mathbb{R}^7$ and has stabilizer equal to $SU(3)$. The group $SU(3)$ acts transitively on the sphere $S^5 \subset \mathbb{C}^3 \cong \mathbb{R}^6$ but maps complex subspaces to complex subspaces.

Now let $e_1, \ldots, e_n$ be an orthonormal basis of $V$. If $n \leq 3$ we can map $v_1$ to $e_1$ with $\text{Spin}(7)$, $v_2$ to $e_2$ with $G_2$ and $v_3$ to $e_3$ with $SU(3)$. If $n > 4$ the orthogonal complement of $V$ has dimension less or equal to three. By the previous argument we can map the orthogonal complement of $V$ to $(0)^n \times \mathbb{R}^{8-n}$, thus mapping $V$ to the span of $e_1, \ldots, e_n$.

Now let $n = 4$. As before we use $\text{Spin}(7)$, $G_2$ and $SU(3)$ to map $v_1$ to $e_1$, $v_2$ to $e_2$ and $v_3$ to $e_3$. The isotropy $SU(2)$ of $SU(3)$ acts on the remaining $\mathbb{R}^5 \cong i \mathbb{R} \times \mathbb{C}^2$ by $A \cdot (ix, z) = (ix, Az)$, where the $SU(2)$-action on $\mathbb{C}^2$ is the standard one. If $v_4$ is an element of $i \mathbb{R} \subset i \mathbb{R} \times \mathbb{C}^2$ then $V$ is $\text{Spin}(7)$-conjugate to the the span of $e_1, \ldots, e_4$. If $v_4$ is in $i \mathbb{R} \times \mathbb{C}^2 \setminus (i \mathbb{R} \times \{0\})$ we can use the described $SU(2)$ action to map it to $xe_4 + ye_5$, where $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$. The assumption follows since $\frac{1}{y}(xe_4 + ye_5) = \frac{x}{y}e_4 + e_5 =: ce_4 + e_5$ holds.

Now we arrive at the main result of this section.

**Theorem 8.6.** Every non-reductive spherical algebraic subalgebra of $\mathfrak{f}_4$ is $G$-conjugate to one in the following table where $\mathfrak{b}_k \subset \mathfrak{so}(k)$ and $\mathfrak{l}_k \subset \mathfrak{sp}(1)$ are arbitrary (under the condition that the maximal compact subalgebra is a Lie algebra, see Remark 5.3).

$$
\begin{array}{l|l}
| l_H \oplus n & l_H \subset m \oplus a \text{ arbitrary} \\
| m' \oplus a \oplus n_1 & m' \subset g_2 \text{ arbitrary} \\
| \text{spin}(7) \oplus a \oplus n_0 & \\
| g_2 \oplus a \oplus n_1 & \\
| \mathfrak{so}(4) \oplus a \oplus n_4 & \\
| \varphi^{-1}(\mathfrak{b}_4 \oplus \mathfrak{sp}(1) \oplus l_2) \oplus a \oplus n_4 & \\
| \varphi^{-1}(\mathfrak{b}_4 \oplus l_2 \oplus \mathfrak{sp}(1)) \oplus a \oplus n_4 & \\
| \varphi^{-1}(\mathfrak{b}_k \oplus \mathfrak{so}(8 - k)) \oplus a \oplus n_k & k = 5, 6 \\
| \mathfrak{so}(4) \oplus a \oplus n_c & \mathfrak{so}(4) \text{ acts via } \rho_c \\
| \mathfrak{sp}(1) \oplus l_2 \oplus a \oplus n_c & \mathfrak{so}(4) \text{ acts via } \rho_c \\
| l_2 \oplus \mathfrak{sp}(1) \oplus a \oplus n_c & \mathfrak{so}(4) \text{ acts via } \rho_c \\
\end{array}
$$

**Proof.** We assume $n_H = V \oplus g_2a$. Lemma 8.5 shows that $V$ may (after conjugation in $M$) be chosen as $\mathbb{R}^k \times \{0\}^{8-k}$ or as the real span of $e_1, e_2, e_3, ce_4 + e_5$ where $k$ is the real dimension of $V$ and $c \in \mathbb{R}$. The case that $V$ is the real span of $e_1, e_2, e_3, ce_4 + e_5$ will be considered at the end of this proof.

If $V = \mathbb{R}^k \times \{0\}^{8-k}$ we have

$$n_H = n_k := (\mathbb{R}^k \times \{0\}^{8-k}) \oplus g_2a,$$

where $0 \leq k < 7$ is the dimension of $n_H \cap g_a$. Direct calculation yields

$$\mathcal{N}_M(n_k) = \left\{ A \in \text{Spin}(7) : \varphi(A) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A_1 \in \mathbb{R}^{k \times k}, A_2 \in \mathbb{R}^{(8-k) \times (8-k)} \right\}.
$$

Moreover, the action of $\mathcal{N}_M(n_k)$ on $n_k^+ = \{0\}^k \times \mathbb{R}^{8-k}$ is given by

$$\varphi(A) \cdot (0) = \left( \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right) \cdot (0) = \left( \begin{pmatrix} 0 \\ A_2(0) \end{pmatrix} \right) .$$

Onishchik’s classification Theorem 4.2 shows that the subgroup $M_H$ of $\mathcal{N}_M(n_k)$ can only act transitively on the spheres in $n_k^+$ if the projection of $\varphi(m_H) \subset \mathfrak{so}(k) \oplus \mathfrak{so}(8-k)$ onto $\mathfrak{so}(8-k)$
is one of the following Lie algebras, where \( l_2 \subseteq \mathfrak{sp}(1) \) is arbitrary.

| \( k \) | \( \varphi(\mathfrak{m}_k) \subseteq \varphi(\mathfrak{N}_m(\mathfrak{n}_k)) \) |
|---|---|
| 0 | \( \mathfrak{so}(8), \mathfrak{spin}(7), \mathfrak{su}(4), \mathfrak{sp}(2) \) |
| 1 | \( \mathfrak{so}(7), \mathfrak{g}_2 \) |
| 2 | \( \mathfrak{so}(6), \mathfrak{su}(3) \) |
| 3 | \( \mathfrak{so}(5) \) |
| 4 | \( \mathfrak{so}(4), \mathfrak{sp}(1) \oplus l_2, l_2 \oplus \mathfrak{sp}(1) \) |
| 5 | \( \mathfrak{so}(3) \) |
| 6 | \( \mathfrak{so}(2) \) |

Since \( \varphi(\mathfrak{m}_k) \subseteq \varphi(\mathfrak{N}_m(\mathfrak{n}_k)) \) holds, our next step is to take each of the Lie algebras in the table above and check if it is contained in the projection onto \( \mathfrak{so}(8-k) \) of

\[
\varphi(\mathfrak{N}_m(\mathfrak{n}_k)) = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \varphi(\mathfrak{spin}(7)) : A_1 \in \mathbb{R}^{k \times k}, A_2 \in \mathbb{R}^{(8-k) \times (8-k)} \right\}.
\]

We start with \( k = 0 \). Note that \( \mathfrak{N}_m(\mathfrak{n}_k) = \mathfrak{m} \) in this case. It is clear that \( \mathfrak{so}(8) \) is not contained in \( \varphi(\mathfrak{spin}(7)) \), since the dimension of \( \mathfrak{spin}(7) \) is smaller than the dimension of \( \mathfrak{so}(8) \). Due to [Oni94] Ch. 1 §2.8 and 5.3 the representation \( \varphi : \mathfrak{Spin}(7) \to \mathfrak{SO}(8) \) is injective. Therefore \( \mathfrak{Spin}(7) \) is a subgroup of \( \varphi(\mathfrak{N}_m(\mathfrak{n}_0)) \) that acts transitively on the spheres in \( \mathfrak{n} \) (with respect to the action defined by \( \varphi \)).

Note that \( \mathfrak{sp}(2) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} : A, B \in \mathbb{C}^{2 \times 2}, \overline{A} = -A, B^t = B \right\} \) is naturally embedded in \( \mathfrak{su}(4) \). And \( \mathfrak{su}(4) \) is embedded in \( \mathfrak{so}(8) \) via:

\[
\begin{pmatrix}
  iw & a_1+ia_2 & b_1+ib_2 & c_1+ic_2 \\
  -a_1+ia_2 & iz & d_1+id_2 & e_1+ie_2 \\
  -b_1+ib_2 & -d_1+id_2 & iy & f_1+if_2 \\
  -c_1+ic_2 & -e_1+ie_2 & -f_1+if_2 & iz
\end{pmatrix}
\mapsto
\begin{pmatrix}
  0 & -w & -a_1 & -a_2 & b_1 & -b_2 & c_1 & -c_2 \\
  -w & 0 & a_1 & a_2 & -b_1 & b_2 & c_1 & c_2 \\
  -a_1 & -a_2 & 0 & -x & d_1 & -d_2 & e_1 & -e_2 \\
  -a_2 & -a_1 & x & 0 & -d_1 & d_2 & -e_1 & e_2 \\
  -b_1 & -b_2 & -d_1 & -d_2 & 0 & -y & f_1 & -f_2 \\
  b_2 & -b_1 & d_2 & -d_1 & y & 0 & f_2 & f_1 \\
  -c_1 & -c_2 & -e_2 & -e_1 & -f_1 & -f_2 & 0 & -z \\
  i & c_2 & -c_1 & -e_2 & -f_1 & -f_2 & f_2 & -f_1 & z
\end{pmatrix}
\]

Calculation shows that the intersection of \( \mathfrak{sp}(2) \) with \( \varphi(\mathfrak{so}(7)) \) does not contain \( \mathfrak{sp}(2) \). Hence \( \mathfrak{Sp}(2) \) is not contained in \( \varphi(\mathfrak{N}_m(\mathfrak{n}_0)) = \varphi(\mathfrak{M}) \). Therefore the same holds for \( \mathfrak{su}(4) \). Now let \( k = 1 \). Since \( G_2 = (\mathfrak{Spin}(7))_{c_0} \) (with respect to the action defined by \( \varphi \)) it is clear that \( G_2 \) is contained in \( \varphi(\mathfrak{N}_m(\mathfrak{n}_1)) \). Calculation shows that the projection of \( \varphi(\mathfrak{N}_m(\mathfrak{n}_1)) \) onto \( \mathfrak{so}(7) \) does not contain \( \mathfrak{so}(7) \). Therefore the projection of \( \varphi(\mathfrak{so}(7)) \) onto \( \mathfrak{so}(7) \) can not contain \( \mathfrak{so}(7) \).

If \( k = 2 \) calculation shows that the projection of \( \mathfrak{N}_m(\mathfrak{n}_2) \) onto \( \mathfrak{so}(6) \) does not contain \( f(\mathfrak{su}(3)) \), where \( f \) is the natural embedding of \( \mathfrak{su}(3) \) into \( \mathfrak{so}(6) \). Since \( \mathfrak{su}(3) \) is a subalgebra of \( \mathfrak{so}(6) \) the same holds for \( \mathfrak{so}(6) \).

If \( k = 3 \) calculation shows that the projection of \( \mathfrak{N}_m(\mathfrak{n}_3) \) onto \( \mathfrak{so}(5) \) does not contain \( \mathfrak{so}(5) \).

If \( k = 4 \) then the projection of

\[
\varphi(\mathfrak{N}_m(\mathfrak{n}_4)) = \left\{ \begin{pmatrix}
  0 & -t & -u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  t & 0 & -v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  u & v & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  -n & m & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & t & 0 & -v & -m & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & u & v & 0 & i & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -n & m & -i & 0 & 0 & 0
\end{pmatrix} : i, m, n, t, u, v \in \mathbb{R} \right\} \cong \mathfrak{so}(4)
\]
onto $\mathfrak{so}(4)$ is surjective. Furthermore $\varphi(\mathfrak{m}_H) = \varphi(\mathcal{N}_m(\mathfrak{n}_4))$ in this case (since $\mathfrak{so}(4) \subset \varphi(\mathfrak{m}_H) \subset \mathcal{N}_m(\mathfrak{n}_4) \cong \mathfrak{so}(4)$ has to hold). Therefore $M_H = \mathcal{N}_M(\mathfrak{n}_4)$ in this case, where

$$\mathcal{N}_m(\mathfrak{n}_4) = \left\{ \begin{pmatrix} 0 & -t & -u & 0 & 0 & i & m \\ t & 0 & v & 0 & -i & 0 & n \\ u & -v & 0 & 0 & -m & -n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & m & 0 & 0 & -l & -v \\ -i & 0 & n & 0 & l & 0 & v \\ -m & -n & 0 & u & v & 0 & 0 \end{pmatrix} : i, m, n, t, u, v \in \mathbb{R} \right\} \cong \mathfrak{so}(4).$$

Furthermore this shows that the projection of $\varphi(\mathfrak{m}_H)$ onto $\mathfrak{so}(4)$ also contains $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. If $k = 5$ then the projection of $\varphi(\mathcal{N}_m(\mathfrak{n}_5))$ onto $\mathfrak{so}(3)$ is surjective. If $k = 6$ then the projection of $\varphi(\mathcal{N}_m(\mathfrak{n}_6))$ onto $\mathfrak{so}(6)$ is surjective.

To summarize, we have seen that $\mathfrak{m}_H$ has to be one of the following if $\mathfrak{h}$ is a spherical algebraic subalgebra of $\mathfrak{g}$.

| $k$ | $\mathfrak{m}_H$ |
|-----|------------------|
| 0   | $\mathfrak{sp}(7)$ |
| 1   | $\mathfrak{g}_2$, $\varphi^{-1}(\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : B \in \mathfrak{so}(4) \})$ |
| 4   | $\varphi^{-1}(\mathfrak{l}' \oplus \mathfrak{sp}(1) \oplus \mathfrak{l}_2)$ where $\mathfrak{l}' \subset \mathfrak{so}(4), \mathfrak{l}_2 \subset \mathfrak{sp}(1)$ are arbitrary, $\varphi^{-1}(\mathfrak{l}' \oplus \mathfrak{l}_2 \oplus \mathfrak{sp}(1))$ where $\mathfrak{l}' \subset \mathfrak{so}(4), \mathfrak{l}_2 \subset \mathfrak{sp}(1)$ are arbitrary, $\varphi^{-1}(\mathfrak{l}' \oplus \mathfrak{so}(3))$ where $\mathfrak{l}' \subset \mathfrak{so}(5)$ is arbitrary |
| 5   | $\varphi^{-1}(\mathfrak{l}' \oplus \mathfrak{sp}(2))$ where $\mathfrak{l}' \subset \mathfrak{so}(6)$ is arbitrary |

Now let us consider the case that $V$ is the real span of $e_1, e_2, e_3$ and $ce_4 + e_5$. In this case we denote $\mathfrak{n}_H = V \oplus \mathfrak{g}_{2n}$ by $\mathfrak{n}_c$. Then $\mathcal{N}_m(\mathfrak{n}_c)$ equals

$$\{ A \in \mathfrak{spin}(7) : \varphi(A)V \subset V \} = \left\{ A \in \mathfrak{spin}(7) : \varphi(A) = \begin{pmatrix} * & * & * & v_1 & w_1 & 0 & 0 & 0 \\ * & * & * & v_2 & w_2 & 0 & 0 & 0 \\ * & * & * & v_3 & w_3 & 0 & 0 & 0 \\ v_4 & v_5 & v_6 & 0 & 0 & 0 & 1 & 2 & 3 \\ w_4 & w_5 & w_6 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & x_4 & y_4 & * & * & * & * \\ 0 & 0 & 0 & x_5 & y_5 & * & * & * & * \\ 0 & 0 & 0 & x_6 & y_6 & * & * & * & * \end{pmatrix} : v_i = cw_i, -cx_i = y_i \text{ for all } 1 \leq i \leq 6 \right\} = \left\{ A \in \mathfrak{spin}(7) : \varphi(A) = \begin{pmatrix} 0 & s+rc & -r+sc & pc & p & 0 & 0 & 0 \\ -s-rc & 0 & -v & lc & l & 0 & 0 & 0 \\ r-sc & v & 0 & -hc & -h & 0 & 0 & 0 \\ -pc & -lc & hc & 0 & 0 & r & s & p \\ -p & -l & h & 0 & 0 & rc & sc & pc \\ 0 & 0 & 0 & r & -rc & 0 & -v & h+l \\ 0 & 0 & 0 & s & -sc & v & 0 & -hc+l \\ 0 & 0 & 0 & p & -pc & -hc & hc-l & 0 \end{pmatrix} : h, l, p, r, s, v \in \mathbb{R} \right\}$$

which is as Lie algebra isomorphic to $\mathfrak{so}(4)$ and will be denoted by $\mathfrak{so}(4)_c$. We will denote the induced representation of $\mathfrak{so}(4)_c$ on $\mathfrak{n}_c^\perp$ by $\rho_c$. We have to show that $\mathcal{N}_M(\mathfrak{n}_c)$ acts transitively on the spheres in $\mathfrak{n}_c^\perp$ which can be identified with the real span $W$ of $e_4 - ce_5, e_6, e_7, e_8$. Note that $e_4 - ce_5 = w$ lies in the sphere $S^3 \subset W \cong \mathbb{R}^4$ and that

$$\mathcal{N}_m(\mathfrak{n}_c) : e_4 = \left\{ (1 + c^2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : r, s, p \in \mathbb{R} \right\} = T_wS^3.$$

Hence the orbit of $\mathcal{N}_M(\mathfrak{n}_c)$ through $w$ is open in the sphere in $\mathfrak{n}_c^\perp$. Since $\mathcal{N}_M(\mathfrak{n}_H)$ is compact its orbit through $w$ is equal to the sphere. Note that our calculations also show that the only proper subalgebras of $\mathcal{N}_m(V)$ that act transitively on the spheres are $(\mathfrak{sp}(1) \oplus \mathfrak{l}_2) \subset \mathfrak{so}(4)$ and $(\mathfrak{l}_2 \oplus \mathfrak{sp}(1)) \subset \mathfrak{so}(4)$, where $\mathfrak{l}_2 \subset \mathfrak{sp}(1)$ is arbitrary. □
References

[Ber57] Marcel Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

[Bri87] Michel Brion, *Sur l’image de l’application moment*, Séminaire d’algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), Lecture Notes in Math., vol. 1296, Springer, Berlin, 1987, pp. 177–192.

[GMO11] Bruce Gilligan, Christian Miebach, and Karl Oeljeklaus, *Homogeneous Kähler and Hamiltonian manifolds*, Math. Ann. **349** (2011), no. 4, 889–901.

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.

[HS07] Peter Heinzner and Gerald W. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), no. 1, 197–232.

[HS08] Peter Heinzner, Gerald W. Schwarz, and Henrik Stötzel, *Stratifications with respect to actions of real reductive groups*, Compos. Math. **144** (2008), no. 1, 163–185.

[HW90] A. T. Huckleberry and T. Wurzbacher, *Multiplicity-free complex manifolds*, Math. Ann. **286** (1990), no. 1-3, 261–280.

[Ki87] Boris Kimelfeld, *Homogeneous domains on flag manifolds*, J. Math. Anal. Appl. **121** (1987), no. 2, 506–588.

[KKS13] Friedrich Knop, Bernhard Krötz, and Henrik Schlichtkrull, *The local structure theorem for real spherical varieties*, arXiv:1310.6290v1 (2013).

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[KO13] Toshiyuki Kobayashi and Toshio Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math. **248** (2013), 921–944.

[Krä79] Manfred Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math. **38** (1979), no. 2, 129–153.

[KS13a] Bernhard Krötz and Henrik Schlichtkrull, *Finite orbit decomposition of real flag manifolds*, arXiv:1307.2375v1 (2013).

[KS13b] Bernhard Krötz and Henrik Schlichtkrull, *Multiplicity bounds and the subrepresentation theorem for real spherical spaces*, arXiv:1309.0930v1 (2013).

[Ma74] F. M. Malyshev, *Local decompositions of the pseudo-orthogonal group*, Mat. Zametki **16** (1974), 633–643.

[MS10] Christian Miebach and Henrik Stötzel, *Spherical gradient manifolds*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 6, 2235–2260.

[Nis12] Akihiro Nishio, *The classification of orbits on certain exceptional Jordan algebra under the automorphism group*, arXiv:1011.0789v4 (2012).

[Nis13] Akihiro Nishio, *The Iwasawa decomposition and the Bruhat decomposition of the automorphism group on certain exceptional Jordan algebra*, Tsukuba J. Math. **37** (2013), no. 1, 85–119.

[Oni94] Arkadi L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.

[Wol84] Joseph A. Wolf, *Spaces of constant curvature*, fifth ed., Publish or Perish Inc., Houston, TX, 1984.

E-mail address: lisa.knauss@ruhr-uni-bochum.de

E-mail address: miebach@lmpa.univ-littoral.fr