Finite-Size Scaling of Correlation Function

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We propose the finite-size scaling of correlation function in a finite system near its critical point. At a distance $r$ in the finite system with size $L$, the correlation function can be written as the product of $|r|^{(d-2+\eta)}$ and its finite-size scaling function of variables $r/L$ and $tL^{1/\nu}$, where $t = (T - T_c)/T_c$. The directional dependence of correlation function is nonnegligible only when $|r|$ becomes comparable with $L$. This finite-size scaling of correlation function has been confirmed by correlation functions of the Ising model and the bond percolation in two-dimensional lattices, which are calculated by Monte Carlo simulation. We can use the finite-size scaling of correlation function to determine the critical point and the critical exponent $\eta$.

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keywords: critical phenomena, finite-size scaling, correlation function, lattice model

I. INTRODUCTION

The concept of finite-size scaling has played an important role in the investigation of finite-size effects near critical point over last decades\textsuperscript{4}. The free-energy density $f(t, L)$ at the reduced temperature $t = (T - T_c)/T_c$ and vanishing external field in a $d$-dimensional cubic geometry of volume $L^d$ with periodic boundary conditions (PBC) can be decomposed as

$$f(t, L) = f_s(t, L) + f_0(t), \quad (1)$$

where $f_s(t, L)$ denotes the singular part of $f$ and $f_0(t)$ is the regular part. It was asserted by Privman and Fisher\textsuperscript{2} that, below the upper critical dimension $d = 4$, $f_s(t, L)$ has the asymptotic finite-size scaling structure

$$f_s(t, L) = L^{-d}F(tL^{1/\nu}), \quad (2)$$

where $F(x)$ is a universal scaling function.

For the free energy density in the bulk limit $L \rightarrow \infty$, it was believed that there is the two-scale-factor universality in general\textsuperscript{1}. It has been demonstrated that the two-scale-factor universality is valid for isotropic systems but not for anisotropic systems with noncubic symmetry\textsuperscript{5}. The finite-size scaling function of the free energy density $F$ depends on the normalized anisotropic matrix $A$ of anisotropic systems\textsuperscript{6}. The Binder cumulant ratio $U$ was introduced\textsuperscript{7} to determine the transition point and compute the critical exponent of the correlation length. Its value at the transition point $U(T_c)$ was believed to be universal and can indicate the the universality class together with critical exponents\textsuperscript{1}. It has been proved in Ref.\textsuperscript{8} that $U(T_c)$ depends on $A$ also.

Using the Monte Carlo simulations of two-dimensional anisotropic Ising models, the dependence of $U(T_c)$ on $A$ has been confirmed\textsuperscript{2,6}. The experimental investigations of the finite-size scaling have been done extensively with $^4$He at the superfluid transition and are reviewed in Refs.\textsuperscript{6,8}.

A similar finite-size scaling ansatz of the correlation length $\xi(t, L)$ was made by Privman and Fisher\textsuperscript{2} as

$$\xi(t, L) = LX(tL^{1/\nu}), \quad (3)$$

with the universal scaling function $X(x)$. In the three-dimensional Ising model, the finite-size scaling analysis of the correlation length with Monte Carlo simulations was reported\textsuperscript{9}. Above the upper critical dimension, the finite-size scaling of the correlation length was studied in the five-dimensional Ising model\textsuperscript{10}. As the central quantity in a statistical system, the correlation length in ionic fluids was investigated by integral equation theory\textsuperscript{11}. The finite-size scaling can be related to the large distance behavior of bulk order-parameter correlation function\textsuperscript{12}.

In this paper, we study the correlation function $g(r, t, L)$ of finite system with size $L$. In Sec. II, the finite-size scaling structure of correlation function $g(r, t, L)$ is introduced and discussed. With the correlation function of Ising model in Sec. III and the correlation function of bond percolation in Sec. IV, the finite-size scaling of correlation function is confirmed by Monte Carlo simulation. In Sec. V, we draw some conclusions.

II. FINITE-SIZE SCALING OF CORRELATION FUNCTION

Correlation functions describe how microscopic variables at different positions are related. In an infinite system with a lattice spacing $\tilde{a}$, the correlation function

\[ E_{\text{corr}}(r) = \int d^d x \int d^d y \rho(x) \rho(y) \delta(r - |x - y|) \]

remains finite near $r = 0$. This is different from the correlation function in a finite system, which is not well-defined at $r = 0$. The finite-size scaling ansatz of the correlation function is

\[ g(r, t, L) = g_s(r, t, L) + g_0(t), \quad (1) \]

where $g_s(r, t, L)$ denotes the singular part of $g$ and $g_0(t)$ is the regular part. The directional dependence of correlation function is nonnegligible only when $|r|$ becomes comparable with $L$. This finite-size scaling of correlation function has been confirmed by correlation functions of the Ising model and the bond percolation in two-dimensional lattices, which are calculated by Monte Carlo simulation. We can use the finite-size scaling of correlation function to determine the critical point and the critical exponent $\eta$.

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where the averages are done over all configurations with weight $p(\{S_i\})$. Using $C_{ij}$ as its elements, a $N \times N$ correlation matrix $C$ can be obtained. $C$ has $N$ eigenvectors and eigenvalues. For eigenvector $b_n$ of eigenvalue $\lambda_n$, there is the relation

$$C b_n = \lambda_n b_n, \quad n = 1, \ldots, N,$$

where

$$b_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{Nn} \end{bmatrix}.$$

The normalized eigenvectors are orthogonal to each other and satisfy the conditions

$$b_n \cdot b_l = \sum_j b_{jn} b_{jl} = \delta_{nl},$$

where $\delta_{nl}$ is the Kronecker delta.

From the fluctuations of $N$ spins, we can define $N$ principal fluctuation modes

$$\delta S_n = \sum_{j=1}^N b_{jn} \delta S_j, \quad n = 1, \ldots, N.$$

Using the orthogonal conditions in Eq. (8) the correlation between principal fluctuation modes can be calculated as

$$\tilde{C}_{nl} \equiv \langle \delta S_n \delta S_l \rangle = \lambda_n \delta_{nl}.$$

There is no correlation between different principal fluctuation modes. The mean square of principal fluctuation mode $\delta S_n$ is equal to $\lambda_n$.

We can express the correlation $C_{ij}$ between spin $i$ and spin $j$ by eigenvectors and eigenvalues as

$$C_{ij} = \sum_{n=1}^N b_{in} b_{jn} \lambda_n.$$

We introduce $C^{(n)}_{ij} = b_{in} b_{jn}$, which can be understood as the correlation between spin $i$ and spin $j$ in the $n$-th principal fluctuation mode. The total correlation between spin $i$ and spin $j$ is obtained by summing correlations of all principal fluctuation modes as

$$C_{ij} = \sum_{n=1}^N \lambda_n C^{(n)}_{ij}.$$

In Ref. [13], it has been proposed and confirmed by the two-dimensional Ising model that eigenvalues satisfy the following finite-size scaling form

$$\lambda_n(t, L) = L^{2-\eta} f_n(t L^{1/\nu}),$$

where $\eta$ is related to the critical exponent $\gamma$ of susceptibility by the hyperscaling relation $2 - \eta = \gamma / \nu$. Using position vectors $r_i$ of spins, eigenvector $b_n$ can be represented by a spatial function $b_n(r_i)$. In a finite system of volume $L^d$ and periodic boundary conditions, the spatial function can be expressed as

$$b_n(r_i) = \frac{1}{\sqrt{N}} \sum_k \hat{b}_n(k) \exp(ikr_i)$$

with the Fourier component

$$\hat{b}_n(k) = \frac{1}{\sqrt{N}} \sum_{r_i} b_n(r_i) \exp(-ikr_i),$$

where $k$ has components $k_j = 2\pi m_j / L$, $m_j = 0, \pm 1, \pm 2, \ldots$, $j = 1, 2, \ldots, d$ in the range $-\pi \leq k_j < \pi$. Fourier components follow the relation $\hat{b}_n^*(k) = \hat{b}_n(-k)$ according to Eq. (15) The correlation function of distance can be calculated as

$$g(r, t, L) = \frac{1}{N} \sum_{r_i} \sum_{r_j} b_n(r_i) b_n(r_i + r) \lambda_n.$$

By using the orthogonal condition

$$\frac{1}{N} \sum_{r_i} \exp(ikr_i) \exp(i k' r_i) = \delta_{k+k',0}$$

and the finite-size scaling form of eigenvalues in Eq. (13), the correlation function of distance can be expressed as

$$g(r, t, L) = L^{-d+2-\eta} \sum_n \sum_k f_n(t L^{1/\nu}) \left| \hat{b}_n(k) \right|^2 \exp(-ikr),$$

where $k = 2\pi m / L$.

From this result, we propose that the correlation function of distance follows the finite-size scaling form

$$g(r, t, L) = A \left| r \right|^{-d+2-\eta} G \left( r/L, t L^{1/\nu} \right),$$

for $|t| \ll 1$ and $L \gg \bar{a}$. It is expected that the directional dependence of the universal scaling function $G(r/L, t L^{1/\nu})$ can be neglected for $|r| \ll L$ and becomes unnegligible when $|r|$ is comparable to $L$. 

\[ g(r, t, \infty) \] reads [1, 2] for $|t| \ll 1$ and $|r| \gg \bar{a}$, 

$$g(r, t, \infty) = A |r|^{-d+2-\eta} \Phi(|r|/\xi)$$

with a universal scaling function $\Phi$, a nonuniversal amplitude $A$, and with $\xi = \xi_0 |t|^{-\nu \prime}$, apart from correlations to scaling.

In a finite system with $N$ spins, the correlation $C_{ij}$ between spin $i$ positioned at $r_i$ and spin $j$ positioned at $r_j$ can be calculated as [13]

$$C_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle,$$

where the averages are done over all configurations with weight $p(\{S_i\})$. Using $C_{ij}$ as its elements, a $N \times N$ correlation matrix $C$ can be obtained. $C$ has $N$ eigenvectors and eigenvalues. For eigenvector $b_n$ of eigenvalue $\lambda_n$, there is the relation

$$C b_n = \lambda_n b_n, \quad n = 1, \ldots, N,$$

where

$$b_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{Nn} \end{bmatrix}.$$
From Eq. (19), we can get the finite-size scaling of the second moment correlation length

$$\xi(t, L) = \sqrt{\frac{1}{2d} \sum_r |r|^2 g(r, t, L)} = LX(tL^{1/\nu})$$  \hspace{1cm} (20)

in agreement with Eq. (19). Further, we can obtain the finite-size scaling of susceptibility

$$\chi(t, L) = L^{\gamma/\nu} f(tL^{1/\nu})$$  \hspace{1cm} (21)

with $\gamma/\nu = 2 - \eta$.

From correlation functions of different system sizes $L$, we choose the distances $x_\lambda = \lambda L$ along the $x$-direction. The logarithm of correlation function at distance $x_\lambda$ can be written as

$$\ln g(x_\lambda, t, L) = (-d + 2 - \eta) \ln L + \ln G(\lambda, tL^{1/\nu}) + C,$$

where $C = \ln \left(A \lambda^{-d+2-\nu}\right)$.

At critical point with $t = 0$, $\ln g(x_\lambda, 0, L)$ depends on $\ln L$ linearly with slope $-d + 2 - \eta$. The deviation from this linear dependence of $\ln g(x_\lambda, t, L)$ appears when $t \neq 0$. We can use this property to fix the critical point and critical exponent $\eta$ of a system.

Using the correlation function at distances $x, 2x, 4x$ along the $x$-direction in the lattice with sizes $L, 2L, 4L$ respectively, we can define a ratio

$$R = \frac{g(x, t, L)g(4x, t, 4L)}{g(2x, t, 2L)^2}.$$  \hspace{1cm} (23)

This ratio can be written into a scaling form as

$$R(\lambda, tL^{1/\nu}) = \frac{G(\lambda, tL^{1/\nu})G(\lambda, 2^{2/\nu}tL^{1/\nu})}{G(\lambda, 2^{1/\nu}tL^{1/\nu})^2},$$  \hspace{1cm} (24)

where $\lambda = x/L$. At critical point $t = 0$, we have $R(\lambda, 0) = 1$, which is independent of $\lambda$ and $L$. At $t \neq 0$, the ratio $R$ depends on both $\lambda$ and $L$. We can determine the critical point of a system from the fixed point of $R$ as a function of $T$ for different $\lambda$ and $L$.

To verify the finite-size scaling structure of correlation function in Eq. (19), we investigate correlation functions of the Ising model and the bond percolation in two-dimensional lattice with PBC using the Monte Carlo simulation. The lattice sizes $L = 32, 64,$ and $128$ are taken in our simulations.

### III. Correlation Function of Ising Model

At zero external field, the Ising model has the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i S_j,$$  \hspace{1cm} (25)

where interactions are restricted to the nearest neighbors and spins can take two values, $S_i \in \{-1, +1\}$.

A configuration of system is characterized by $\{S_i\} = \{S_1, S_2, ..., S_N\}$ with $N = L \times L$. It has a probability $p\{\{S_i\}\} = e^{-H/k_BT}/Z$ with $Z = \sum_{\{S_i\}} e^{-H/k_BT}$, where the summation is done for all samples of configuration. The Wolff algorithm is used to simulate configurations of the Ising model.

In the average of Eq. (8), only configurations with the positive total magnetization are used. If the total magnetization $M$ is negative after a Monte Carlo simulation step, we make a flip $S_i \rightarrow -S_i$ to all spins so that $M$ becomes positive again. For $N$ spins in the lattice, there are $N(N_1)/2$ correlations between spins. The correlation function $g(\vec{r}, t, L)$ is calculated by the average of correlations $C_{ij}$ with $\vec{r}_i - \vec{r}_j = \vec{r}$ as

$$g(\vec{r}, t, L) = \frac{\sum_{i,j} C_{ij} \delta[\vec{r} - (\vec{r}_i - \vec{r}_j)]}{\sum_{i,j} \delta[\vec{r} - (\vec{r}_i - \vec{r}_j)]}.$$  \hspace{1cm} (26)

It has been found that the two-dimensional Ising model in square lattice has the critical point at $k_BT_c/J = 2/\ln(1 + \sqrt{2}) \approx 2.269$, and the critical exponents $\nu = 1$ and $\eta = \frac{1}{2}$. To verify the finite-size scaling structure of correlation function, we simulate the Ising model of sizes $L = 32, 64, 128$ around the critical point with $tL^{1/\nu} = -2, 0, 2$.

**FIG. 1.** Left: correlation function along $x$-axis of the Ising model at $T_c$. Right: its finite-size scaling function.

In Fig. 1 (a), the correlation function $g(x, t, L)$ along the $x$-direction is plotted at the critical point and for sizes $L = 32, 64, 128$. The scaled correlation function $g(x, t, L)x^\eta$ is shown with respect to the scaled distance $x/L$ in Fig. 1 (b), where the curves of different $L$ collapse into one curve. This confirms the finite-size scaling structure of correlation function in Eq. (19).

Below the critical point $T_c$, the Ising model is simulated at sizes $L = 32, 64, 128$ and corresponding temperatures with $tL^{1/\nu} = -2$. The results are presented in Fig 2. The finite-size scaling structure of correlation function is confirmed also at temperatures below $T_c$.

Above the critical point $T_c$, the correlation functions of the Ising model are simulated at sizes $L = 32, 64, 128$ and corresponding temperatures with $tL^{1/\nu} = 2$. They are shown in Fig 3 and confirm the finite-size scaling structure of correlation function for temperatures above $T_c$.

At the critical point, the correlation functions along the diagonal direction of lattice with $\vec{r} = (\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})$ are
FIG. 2. Left: correlation function along x-axis of the Ising model below $T_c$. Right: its finite-size scaling function.

FIG. 3. Left: correlation function along x-axis of the Ising model above $T_c$. Right: its finite-size scaling function.

FIG. 4. Left: correlation function along diagonal direction of the Ising model at critical point. Right: its finite-size scaling function in comparison with that along x-axis.

FIG. 5. Left: correlation function along diagonal direction of the Ising model below $T_c$. Right: its finite-size scaling function in comparison with that along x-axis.

The finite-size scaling structure of correlation function is further confirmed. We can see that the finite-size scaling function at distance comparable with $L/2$ becomes dependent on direction and its value in the diagonal direction is different from that in the x-direction. Below $T_c$, with $tL^{1/\nu} = -2$, the correlation functions along the diagonal direction are shown on the left and their finite-size scaling function on the right of Fig. 5. The correlation functions and their finite-size scaling function above $T_c$ with $tL^{1/\nu} = 2$ are plotted in Fig. 6.

The log-log plot of $g(x, t, L)$ with respect to $L$ is shown in Fig. 7 for $x_{\lambda} = \lambda L$ with $\lambda = 1/8, 3/16, 1/4$, respectively. At $T < T_c$, the correlation function has nonzero curvature at first. With the increase of temperature, it becomes a straight line at $T_c$ and is curved again at $T > T_c$.

In Fig. 5, the ratio $R$ is presented as a function of temperature $T$ at $\lambda = 1/8, 3/16, 1/4$ and for $L = 32$. The different curves of $R$ at different $\lambda$ cross at $T_c$ with $R(\lambda, 0) = 1$.

IV. CORRELATION FUNCTION OF BOND PERCOLATION

In a two-dimensional lattice, bonds are added randomly to connect any two sites in neighbourhood. With the increase of bond number $N_p$, the largest cluster in the lattice becomes larger and larger. When the reduced bond number $p = N_p/N$ reach its critical value $p_c = 0.5$, the size of the largest cluster becomes comparable with system size and there is a bond percolation phase transition with critical exponents $\nu = 4/3$ and $\eta = 24/10$. In one- and two-dimensional lattices, the criticality of networks with long-range connections has been investigated. For any configuration in the two-dimensional lattice, two sites $i$ and $j$ are considered to be connected if they belong to the same cluster except for the largest one. The correlation $p_{ij}$ between $i$ and $j$ in a configuration is equal to 1 when they are connected and otherwise is equal to 0. With the average over all configurations, the correlation between sites $i$ and $j$ is calculated as $18, 19$

$$C_{ij} = \langle p_{ij} \rangle . \quad (27)$$

Using the definition in Eq. 20, we can calculate the correlation function $g(r, t, L)$ of two-dimensional bond percolation from $C_{ij}$. The results at $T_c$ are shown in Fig. 8. At the left side, the correlation functions along diagonal direction of different $L$ are shown. Their finite-size scaling functions are presented at the right side and are compared with that along x-direction. Further, the correlation functions of bond percolation below and above $p_c$ are demonstrated in Figs. 10 and 11, respectively. The finite-size structure of correlation function in Eq. 19 has presented in Fig. 4. The finite-size scaling function above $T_c$ with $tL^{1/\nu} = 2$ are shown in Fig. 9. The size of the largest cluster becomes comparable with system size and there is a bond percolation phase transition with critical exponents $\nu = 4/3$ and $\eta = 24/10$. In one- and two-dimensional lattices, the criticality of networks with long-range connections has been investigated.

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been also confirmed by the correlation function of bond percolation.

To verify the finite-size structure in Eq. 22, the log-log plot of $g(x, t, L)$ with respect to $L$ is shown in Fig. 12 with $t = (p - p_c)/p_c$. With $\lambda = 1/8, 3/16, 1/4$ respectively, the correlation functions at $x = \lambda L$ are plotted for $p = 0.498, 0.499, 0.500, 0.501, 0.502$. With the increase of $p$, $\ln g(x, t, L)$ as a function of $\ln L$ is curved downward at the beginning and becomes curved upward finally. It is at $p_c = 0.500$ that $\ln g(x, t, L)$ depends on $\ln L$ linearly. In Fig. 13 the ratio $R$ defined in Eq. 23 is shown with respect to $p$ for $\lambda = 1/8, 3/16, 1/4$. It is found that $R$ is independent of $\lambda$ and equal to 1 at $p_c$. Therefore, the finite-size scaling of correlation function has been confirmed by the bond percolation in two-dimensional lattice.

V. CONCLUSIONS

We propose here the finite-size scaling of correlation function in a finite system near its critical point. For the finite system with size $L$ and at the reduced temperature $t = (T - T_c)/T_c$, its correlation function at distance $r$ can be scaled as $g(r, t, L) = A|r|^{-(d-2+\eta)}G(r/L, tL^{1/\nu})$.

\[ G(r/L, tL^{1/\nu}) \] is independent of the direction of $r$. When the distance becomes comparable with $L$, the directional dependence of $g(r/L, tL^{1/\nu})$ is nonnegligible. From the finite-size scaling of correlation function, we can obtain the finite-scalings of susceptibility [1] and of the second moment correlation length [2].

Using Monte Carlo simulation, the correlation functions of two-dimensional Ising model and bond percolation in square lattices are calculated. These results of both Ising model and bond percolation verify the finite-size scaling of correlation function proposed above.

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FIG. 7. Log-log plot of correlation function of the Ising model along $x$-axis $g(\lambda L, t, L)$ with respect to system size $L$.

FIG. 8. Ratio $R(\lambda, tL^{1/\nu})$ of the Ising model with respect to temperature at $L = 32$ and different $\lambda$.

FIG. 9. Left: correlation function along diagonal direction for bond percolation at $p = p_c$. Right: its finite-size scaling function in comparison with that along $x$-axis.
FIG. 10. Left: correlation function along diagonal direction for bond percolation at $p < p_c$. Right: its finite-size scaling function in comparison with that along $x$-axis.

FIG. 11. Left: correlation function along diagonal direction of bond percolation at $p > p_c$. Right: its finite-size scaling function in comparison with that along $x$-axis.
FIG. 12. Log-log plot of correlation function along $x$-axis for bond percolation $g(\lambda L, t, L)$ with respect to system size $L$.

FIG. 13. Ratio $R(\lambda, tL^{1/\nu})$ of bond percolation with respect to $p$ at $L = 32$ and different $\lambda$. 