Analytic Semi-device-independent Entanglement Quantification for Bipartite
Quantum States

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We define a property called nondegeneracy for Bell inequalities, which describes the situation that in a Bell setting, if a Bell inequality and involved local measurements are chosen and fixed, any quantum state with a given dimension and its orthogonal quantum state cannot violate the inequality remarkably at the same time. We prove that for an arbitrary quantum dimension, based on the measurement statistics only, we can give an analytic lower bound for the entanglement of formation of the unknown bipartite quantum state by choosing a proper nondegenerate Bell inequality, making the whole process semi-device-independent. We provide specific examples to demonstrate the existence of nondegeneracy and applications of our approach.

I. INTRODUCTION

As a major computational resource in quantum information processing and quantum communication tasks, certifying entanglement for a unknown quantum system is a fundamental and important problem. For small quantum systems tomography is a possible solution, but as the problem size grows, the cost of tomography goes up exponentially, making this approach infeasible. In this case, one can instead use the idea of entanglement witness to detect entanglement, but one drawback of this approach is that the knowledge on quantum dimension and the accurate measurement implementations must be given, which are often unpractical, otherwise the results may not be reliable.

To overcome this problem, it turns out that the approach of device-independence, a method that was first introduced in the area of quantum key distribution and self-testing, is very helpful. In this approach, all involved quantum devices are regarded as black boxes and quantum tasks like entanglement certification are usually accomplished by checking the existence of Bell nonlocality, i.e., a violation to some Bell inequality that any classical systems cannot make. Particularly, this approach has been utilized extensively to certify the existence of genuine multipartite entanglement. Since nontrivial and reliable conclusions can be drawn from limited measurement data only, device-independence is highly valuable in realizing quantum schemes physically. Moreover, for the situations that partial reliable information on the target quantum system is known, people add some modest assumptions to fully device-independent quantum models, resulting in measurement-device-independent and semi-device-independent scenarios.

A further step of entanglement certification is the quantification of entanglement, on which several device-independent schemes have also been proposed. Inspired by the NPA method, a device-independent method to lower bound the negativity was provided in. Using the concept of semiquantum nonlocal games introduced in, a measurement-device-independent approach to quantify negative-partial-transposition entanglement has also been reported. Usually, this kind of works face two inevitable difficulties. First, nonlocality and entanglement are known as two different resources for quantum information processing, profoundly making quantifying entanglement in a device-independent way challenging. Second, the mathematical structures of sets of quantum correlations is very complicated, for example accurate Tsirelson bounds are often notoriously hard to find out, which makes it quite hard to study most device-independent quantum tasks in an analytical way, especially when the dimension is high. As a consequence, in most cases of device-independent quantum tasks one needs to perform costly numerical calculations. Therefore, despite these encouraging progresses, in order to gain deeper understanding for the fundamental relations between nonlocality and entanglement measures, especially those standard entanglement measures with clear operational meanings, direct analytical results for general cases of Bell experiments are highly demanded.

In this paper, for a general unknown bipartite quantum state, we provide an analytic method to quantify the entanglement of formation, one of the most well-known standard entanglement measures, in a semi-device-independent manner, where besides the measurement statistics data, the only assumption we make is quantum dimension. The main idea behind our approach is a new property we define for Bell inequalities, which describes the situation in a Bell setting, if any quantum state with a given dimension and its orthogonal quantum state cannot violate the inequality remarkably at the same time by using the same set of local measurements, we say the Bell inequality is nondegenerate. This property seems quite common among Bell inequalities. By choosing proper Bell inequalities, we prove that a relation between Bell inequality violations and the entanglement of formation can be built, eventually giving the analytic result we

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want. We demonstrate the applications of our method by specific examples.

II. NONDEGENERATE BELL INEQUALITIES

In a two-party Bell experiment, Alice and Bob located at different places share a physical system, and perform local measurements on their own subsystems without communications. Specifically, Alice (Bob) has a set of measurement apparatus labelled by a finite set \( X (Y) \), and the set of possible measurement outcomes are labelled by a finite set \( A (B) \). When the experiment begins, they choose random apparatuses to measure the system and repeat the whole process many times. By recording the frequency of every choices and corresponding outcomes, they calculate the joint probability distribution \( p(ab|xy) \), indicating the probability of obtaining outcomes \( a \in A \) and \( b \in B \) when choosing measurement apparatuses \( x \in X \) and \( y \in Y \). The collection of all \( |A \times B \times X \times Y| \) joint probability distributions can be written as a vector \( p := \{p(ab|xy)\} \), called a correlation.

The set of correlations depends heavily on the physical laws that the system that Alice and Bob share obeys. If the experiment is purely classical, all the correlations they are able to produce are local correlations, which can be explained by sharing a public randomness before the experiment begins, and then generating local distributions with respect to the distribution of the public randomness, which is called a local hidden variable (LHV) model. On the other hand, if what they share beforehand is a quantum state \( \rho \), the correlation is called quantum and can be written as,

\[
p(ab|xy) = \text{Tr}(\rho (M^a_x \otimes M^b_y)),
\]

where \( M^a_x \) and \( M^b_y \) are the measurement operators of the measurement apparatuses \( x \) and \( y \).

A major discovery of quantum mechanics is that there exist quantum correlations that cannot be produced with LHV models, which can be explained by the concept of Bell inequalities [10]. A typical Bell inequality can be expressed as

\[
I := \sum_{abxy} s_{abxy} p(ab|xy) \leq C_l,
\]

where \( s_{abxy} \)'s are normally real coefficients, and \( C_l \) is the maximal value of the Bell expression \( I \) that local correlations achieve. It turns out that in some cases the maximal value of \( I \) that quantum correlations achieve, called Tsirelson bound and denoted \( C_q \), can be strictly larger than \( C_l \), revealing the profound discovery we just mentioned.

In this paper, we define and focus on a special case of Bell inequalities called nondegenerate. We will show that this property makes it possible to obtain analytic results on the entanglement of formation of unknown bipartite quantum states by utilizing the measurement statistics data only, assuming the quantum dimension is known.

For convenience, we denote the Bell expression of the correlation generated by measuring a quantum state \( \rho \) acting on Hilbert space \( \mathcal{H}^d \otimes \mathcal{H}^d \) with measurements \( \{M^a_x\} \) and \( \{M^b_y\} \) as \( I(\rho, M^a_x, M^b_y) \). Then we say a Bell inequality \( I \leq C_l \) nondegenerate if there exists two real number \( 0 \leq \epsilon_1 < \epsilon_2 \leq C_q \), such that for any pure state \( |\psi \rangle \) acting on \( \mathcal{H}^d \otimes \mathcal{H}^d \) and any measurements \( \{M^a_x\} \) and \( \{M^b_y\} \),

\[
I(|\psi \rangle \langle \psi |, M^a_x, M^b_y) \geq C_q - \epsilon_1
\]

always implies that

\[
I(|\psi^\perp \rangle \langle \psi^\perp |, M^a_x, M^b_y) \leq C_q - \epsilon_2,
\]

where \( |\psi^\perp \rangle \) is any pure state orthogonal to \( |\psi \rangle \). Intuitively, this means that if a quantum state makes a large violation to the Bell inequality, any orthogonal quantum state cannot with the involved measurements unchanged.

A few remarks on this definition are in order. First, nondegeneracy is meaningful only when the dimension is given, as any Bell inequality cannot satisfy the definition if extra dimensions can be introduced freely in the form of ancillary subsystems. Second, note that in some device-independent quantum tasks like self-testing [2, 11, 27, 28], a crucial issue is whether the maximal violation to a Bell inequality is achieved by multiple pure quantum states, where the involved measurement sets can be essentially different. For convenience in this case we say this Bell inequality enjoys the uniqueness property. We stress that the nondegeneracy property we define is much weaker than the uniqueness property. After all, in principle it is possible that two close but essentially different quantum pure states achieve the maximal violation at the same time, but they are using different measurements, thus still satisfy the definition of nondegeneracy. Usually it is notoriously hard to determine whether or not a given Bell inequality has the uniqueness property. Therefore, a looser requirement in the definition may make it much easier to certify the nondegeneracy property, which potentially results in wider applications of this new definition. Actually, we conjecture that the majority of Bell inequalities are nondegenerate with dimension restricted, and later we will show some examples. Third, another issue worth pointing out is that, though for simplicity we mainly focus on linear forms of Bell inequalities in this paper, nondegeneracy can be defined on general Bell inequalities of any forms.

III. PRINCIPAL COMPONENT ANALYSIS

Suppose in a Bell experiment, a quantum correlation \( p(ab|xy) \) is produced by measuring a bipartite quantum state \( \rho \) of dimension \( d \times d \), where the involved measurements are \( \{M^a_x\} \) and \( \{M^b_y\} \). Then the main task in this
paper is, how can we quantify the amount of entanglement of $\rho$ in an analytic way? In this paper, we choose the entanglement measure as the entanglement of formation $E_f(\rho)$, one of the most well-known standard entanglement measure. We suppose there exists a nondegenerate Bell inequality $I \leq C_l$ with parameters $\epsilon_1$ and $\epsilon_2$ such that the Bell expression given by $p(ab|xy)$ is larger than $C_q - \epsilon_1$, that is,

$$I(\rho, M^a_x, M^b_y) \geq C_q - \epsilon_1. \quad (3)$$

To estimate the entanglement of formation of $\rho$, the first step of our semi-device-independent entanglement quantification for $\rho$ is testing its principal component, i.e., the component in an orthogonal decomposition with the largest violation to the Bell inequality. Let an orthogonal decomposition of $\rho$ be $\rho = \sum_{i=1}^r a_i |\psi_i\rangle\langle\psi_i|$. Since for fixed local measurements the Bell expression is linear in the shared quantum state, there must be a $|\psi_i\rangle$ such that $I(|\psi_i\rangle\langle\psi_i|, M^a_x, M^b_y) \geq C_q - \epsilon_1$. Without loss of generality, we suppose $i = 1$. Then it holds that

$$I(\rho, M^a_x, M^b_y) = \sum_{i=1}^r a_i \cdot I(|\psi_i\rangle\langle\psi_i|, M^a_x, M^b_y) \leq a_1 \cdot \log S(\rho) \leq a_1 \cdot (C_q - \epsilon_2) \leq a_1 \cdot C_q + (1 - a_1) (C_q - \epsilon_2),$$

where we have used the definition of nondegenerate Bell inequality and the fact that the maximal Bell expression for quantum correlations is $C_q$.

Combining the above inequality with Eq. (3), we immediately have that $a_1 \geq 1 - \epsilon_1/\epsilon_2$. Intuitively, this means that if $\epsilon_1/\epsilon_2 \ll 1$, violating the Bell inequality almost maximally means that this quantum state must be close to pure, as the purity of $\rho$ can be lower bounded by $\Tr(\rho^2) = \sum_{i=1}^r a_i^2 \geq a_1^2 \geq (1 - \epsilon_1/\epsilon_2)^2$. In this case, the principal component of $\rho$ we define is also the component of $\rho$ with the largest weight in the orthogonal decomposition.



IV. LOWER BOUNDING THE ENTANGLEMENT

We now use the principal component analysis above to quantify the entanglement of formation for $\rho$. Actually, [34] introduces an approach to lower bound $E_f(\rho)$ based on the purity of $\rho$ and the measurement statistics data $p(ab|xy)$, where the idea is that one first lower bounds $E_f(|\psi_1\rangle\langle\psi_1|)$ based on the measurement statistics and the weight of $|\psi_1\rangle$, then uses the continuous property of the entanglement of formation to bound the gap between $E_f(|\psi_1\rangle\langle\psi_1|)$ and $E_f(\rho)$. In this paper, we will use a modified version of the approach in [34] to estimate $E_f(\rho)$.

Before beginning our discussion, we would like to stress that the differences of the current work from [34] are essential. First, the lower bound for $E_f(\rho)$ given in [34] has to be based on the assumption that the purity of $\rho$ is known, thus the estimation there is far from device-independent, which restricts its application seriously. In the current work, however, this assumption is dropped completely due to the introduction of nondegenerate Bell inequalities. We believe the semi-device-independence of the current work will make its applications much wider and easier. Second, the lower bound itself is also strengthened dramatically. As a result, the bound in [34] can be meaningful only when $\rho$ is close to achieve the Tsirelson bound, while the result in the current work can be utilized for much more quantum states.

Meanwhile, recall that the trace distance and the fidelity between two quantum state $\rho$ and $\sigma$ are defined as $D(\rho, \sigma) = \frac{1}{2} \Tr|\rho - \sigma|$ and $F(\rho, \sigma) = \Tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ respectively.

Specifically, suppose $\rho_A = \Tr_B (\rho)$ and $\rho_{A1} = \Tr_B (|\psi_1\rangle\langle\psi_1|)$, that is, $\rho_A$ and $\rho_{A1}$ are the reduced density matrices of $\rho$ and $|\psi_1\rangle\langle\psi_1|$ on Alice’s side respectively. Again, here $|\psi_1\rangle$ is the principal component of $\rho$. Then according to [33], the von Neumann entropy of $\rho_{A1}$, denoted $S(\rho_{A1})$, can be lower bounded as

$$S(\rho_{A1}) \geq - \log_2 (\min f_1(p), f_2(p)) + 2 \log_2 (1 - \frac{\epsilon_1}{\epsilon_2}), \quad (4)$$

where

$$f_1(p) = \min_{x_1, x_2} \left( \sum_{a_1, a_2} \min_x \sqrt{p(ab|x_1)p(ab|x_2)} \right), \quad (5)$$

and

$$f_2(p) = \min_{x_1, x_2} \left( \sum_{a_1, a_2} \min_y \sqrt{p(ab|x_1)p(ab|x_2)} \right). \quad (6)$$

Note that $S(\rho_{A1})$ is exactly the entanglement of formation of $|\psi_1\rangle$. Then to estimate $E_f(\rho)$, a nature way is to consider $|E_f(\rho) - E_f(|\psi_1\rangle\langle\psi_1|)|$ via the continuous property for the entanglement of formation. In [34] this property is described by a result proved by Nielsen in [35], which shows that

$$E_f(\rho_{AB}) - E_f(\sigma_{AB}) \leq 18D' \cdot \log_2 d - 4D' \cdot \log_2 (2D'),$$

where $D' = \sqrt{1 - F(\rho_{AB}, \sigma_{AB})^2}$.

Nielsen’s result was recently strengthened largely by Winter [36]. Suppose $\eta = D(\rho_{AB}, \sigma_{AB})$, and $\delta = \sqrt{\eta(2 - \eta)}$, then Winter proved that

$$E_f(\rho_{AB}) - E_f(\sigma_{AB}) \leq \delta \cdot \log_2 d + (1 + \delta) \cdot H\left( \frac{\delta}{1 + \delta} \right), \quad (7)$$

where $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ is the Shannon entropy. This improvement makes us able to get a better way to estimate $E_f(\rho)$ for our problem.

Actually, since $a_1 \geq 1 - \epsilon_1/\epsilon_2$, we have that

$$D(\rho, |\psi_1\rangle\langle\psi_1|) = 1 - a_1 \leq \frac{\epsilon_1}{\epsilon_2}. \quad (8)$$
Putting this into Eq. (7) and combining it with Eq. (3), we eventually obtain the main result of the current paper as follows.

**Theorem.** Suppose \( p(ab|xy) \) is a quantum correlation generated by a \( d \times d \) bipartite quantum state \( \rho \), and \( I \leq C_1 \) is a nondegenerate Bell inequality with parameters \( C_q, \epsilon_1 \) and \( \epsilon_2 \) defined as above. If the Bell expression of \( p(ab|xy) \) is larger than \( C_q - \epsilon_1 \), then it holds that

\[
E_f(\rho) \geq -\log_2(\min f_1(p), f_2(p)) + 2\log_2(1 - \frac{\epsilon_1}{\epsilon_2}) - \delta \cdot \log_2 d - (1 + \delta) \cdot H(\frac{\delta}{1 + \delta}),
\]

where \( f_1(p) \) and \( f_2(p) \) are given in Eqs. (3), (4), and \( \delta = \sqrt{\frac{\epsilon_1(2 - \epsilon_1)}{\epsilon_2}} \).

**V. TWO EXAMPLES**

As mentioned above, we believe that nondegeneracy is quite common in Bell inequalities. We now give some examples to show that nondegenerate Bell inequalities do exist, and then demonstrate its applications. For future work, to find more settings that nondegeneracy can be certified will be an interesting problem.

We first consider the simplest case of Bell inequalities, the Clauser-Horne-Shimony-Holt (CHSH) inequality. In this setting, \( X = Y = \{0, 1\} \), and \( A = B = \{-1, 1\} \). Let \( \langle a_x b_y \rangle = \sum_{ab} ab \cdot p(ab|xy) \), then the CHSH inequality can be expressed as

\[
I_{CHSH} = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2.
\]

It has been well-known that the CHSH inequality can be violated by quantum systems and the maximal value of \( I \) is \( 2\sqrt{2} \), that is, \( C_q = 2\sqrt{2} \). Furthermore, according to [38] if the CHSH inequality is violated by a two-qubit pure state \( |\psi\rangle \) and choices of measurements \( \{M^a_x\} \) and \( \{M^b_y\} \), and if \( I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) > 0 \), it holds that

\[
I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) = I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) + I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) \leq 8.
\]

This means that a large \( I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) \) must result in a small \( I(|\psi\rangle \langle \psi|, M^a_x, M^b_y) \), implying that the CHSH inequality is nondegenerate. In this case, \( \epsilon_1 \) can be chosen as any value in the interval \( [0, 2(\sqrt{2} - 1)] \), and \( \epsilon_2 \) as \( 2\sqrt{2} - \sqrt{4\sqrt{2} - 4} \).

Let us see a set of concrete data. Suppose there is a qubit-qubit quantum state \( \rho \) producing the following correlation,

\[
p(ab|xy) = \begin{cases} 
(2 + \sqrt{2})/8, & \text{if } a \oplus b = xy, \\
(2 - \sqrt{2})/8, & \text{if } a \oplus b \neq xy.
\end{cases}
\]

It can be seen that \( I_{CHSH} = 2\sqrt{2} \), then one can choose \( \epsilon_1 = 0 \) and \( \epsilon_2 = 2\sqrt{2} \). Substituting these parameters into Eq. (9), we have that \( E_f(\rho) \geq 1 \). This can only be achieved by maximal entangled states, implying that the result in Eq. (10) can be tight.

On the other hand, if the data of \( \{p(ab|xy)\} \) makes \( I_{CHSH} = 2\sqrt{2} - 10^{-3} \), one can choose \( \epsilon_1 = 10^{-3} \) and \( \epsilon_2 \) accordingly. Then Eq. (9) is approximately

\[
E_f(\rho) \geq -\log_2(\min f_1(p), f_2(p)) - 0.208.
\]

It should be pointed out that for the case of qubit-qubit quantum states, analytic device-independent results for lower bounding the entanglement of formation have been reported [34] (see also [40]). However, our new method applies to general case of bipartite quantum states, thus we now turn to another case with a larger dimension.

In [41], a new Bell inequality \( I_{3,2} \leq (1 + 3\sqrt{3})/2 \) is proposed and studied, where \( A = B = \{0, 1, 2\} \), \( X = Y = \{1, 2\} \), and the Tsirelson bound \( C_q = 4 \). An interesting fact on this Bell inequality is that the only qutrit-qutrit state that achieves \( C_q \) can be proved to be \( |\psi\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3} \). What is more, Fig.1 of [41] implies that any quantum state orthogonal to \( |\psi\rangle \) has Bell expression smaller than 3.79, which means that \( I_{3,2} \leq (1 + 3\sqrt{3})/2 \) is nondegenerate, and one choice for the parameters is \( \epsilon_1 = 0 \), and \( \epsilon_2 = 0.21 \). Actually, they can be taken from an interval respectively determined by the relation between fidelity and Bell expression given in Fig.1 of [41]. In this way, for a given quantum correlation \( p(ab|xy) \) produced by a qutrit-qutrit quantum state, we can choose \( I_{3,2} \leq (1 + 3\sqrt{3})/2 \) with proper parameters to lower bound the entanglement of formation.

**VI. MULTIPARTITE CASE**

The approach above can be generalized to multipartite case, as the concept of nondegeneracy can also be defined naturally on multipartite Bell inequalities. But two major issues are raised in multipartite case and have to be addressed. First, because of the existence of Schmidt decompositions, entanglement quantification for bipartite pure states based on measurement statistics data is not a difficult task, but Schmidt decompositions do not always exist for multipartite pure quantum states, thus this part has to be redeveloped carefully. Second, the continuity property of the entanglement of formation needs to be build for multipartite case as well. We hope these two problems will be discussed in future work.

**VII. CONCLUSIONS**

We define a concept called nondegeneracy for Bell inequalities, which provides us a new insight to study this fundamental tool. Based on this new concept, we propose an approach to quantify the entanglement of formation of shared quantum state in a semi-device-independent manner, which is analytic and does not rely on any numerical calculations, unlike most results on device-independent
quantum tasks. Here the reason why the information on dimension is needed is twofold. First, we have to utilize the continuity property of the entanglement of formation, which involves the dimension directly. Second, the definition of nondegeneracy also needs the dimension to be restricted. Considering the fundamental role that the entanglement plays in quantum information processing, we hope our results have more nontrivial applications in problems and quantum schemes based on quantum nonlocality.

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