Braided Premonoidal Coherence

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1 Introduction

The constraints of a monoidal category are too strong for some applications in physics. Although the natural isomorphism property is founded on strong physical principles (see Joyce [2]) there are no conclusive grounds for requiring that the pentagon and triangle diagrams hold. For example in Joyce [4,5] recouplings embodying the Pauli exclusion principle are natural isomorphisms that do not satisfy the pentagon diagram for representations of the group $SU(n)$ where $n > 2$. This provides motivation for investigating weakened monoidal structures. A partial exploration of the problem is given in Joyce [3] and does not consider braids, nor violations of the triangle diagram. An alternative approach using $n$–categories is given by Yanofsky [10]

Given a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ one may form “words” by iterating the bifunctor. Thus one may ask what is the relationship between different words. We initially investigate structures with only a natural isomorphism $\alpha : \otimes(\otimes \times 1) \rightarrow \otimes(1 \times \otimes)$ for associativity. Any extra conditions such as the usual pentagonal constraint of a monoidal category is a luxury and not necessarily a fundamental requirement. This minimum structure we call a premonoidal structure. We introduce the groupoid of coupling trees which characterises this structure functorially. Further we consider weaker constraints than the pentagonal con-
straint and prove suitable coherence theorems. Ultimately we ar-
rive at the Mac Lane coherence theorem \[8\] without a unit.

Next we consider two extensions of these structures. First to a
category with a unit structure where the triangle diagram is not
necessarily required to hold. The relevant groupoid is of coupling
trees with nodules. The second extension is to braids where the
hexagonal diagrams are not necessarily required to hold. This
requires the exploded groupoid of braids. The case of braided
monoidal coherence is elegantly addressed by Joyal and Street
\[11\]. The present paper concludes by exhibiting a diagram calculus
representing all commutative diagrams in \(C\).

We collect some elementary, but useful results on functor cate-
gories. As a general reference we recommend Mac Lane’s book
\[9\].

**Proposition 1** Let \(\tau : F \to G\) be a natural transformation be-
tween two functors \(F, G : C \to D\).

1. Given a functor \(H : \mathcal{B} \to \mathcal{C}\) then \(\tau H : FH \to GH\) defined
by \(b \mapsto \tau_{Ha} : FHb \to GHb\) is a natural transformation.
Furthermore, if \(\tau\) is a natural isomorphism, then so is \(\tau H\).
2. Given a functor \(K : \mathcal{D} \to \mathcal{E}\) then \(K\tau : KF \to KG\) defined
by \(c \mapsto K\tau_c : KFc \to KGc\) is a natural transformation.
Furthermore, if \(\tau\) is a natural isomorphism, then so is \(K\tau\).
3. Given another natural transformation \(\sigma : H \to K\) then \(\tau \times
\sigma : F \times H \to G \times K\) is a natural transformation defined by
\((c, b) \mapsto (\tau_c, \sigma_b)\).

The proof is straightforward and given by checking the naturality
condition holds using the functorial properties of \(H\) and \(K\).

2 Premonoidal Structure of a Category

A category \(\mathcal{C}\) with a functor \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) does not have
sufficient properties to be of any interest. However, including a
natural isomorphism for associativity leads to a surprisingly rich structure. Moreover, this section and the next is a warm up for the richer structures of later sections.

**Definition 2** A premonoidal structure for a category \( \mathcal{C} \) is a doublet \((\otimes, a)\) where \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) is a functor called tensor product and \( a : \otimes(\otimes \times 1) \to \otimes(1 \times \otimes) \) is a natural isomorphism called the recoupling for associativity.

A premonoidal structure is not necessarily monoidal because the pentagonal constraint does not hold and there is no account taken of a unit. Despite the lack of a pentagonal constraint we can nevertheless measure the degree to which the pentagonal constraint is deformed as defined in the next definition.

**Definition 3** The recoupling for deformativity is the natural automorphism \( q : \otimes(\otimes \times \otimes) \to \otimes(\otimes \times \otimes) \) defined by

\[
q_{a,b,c,d} = a_{a,b,c,d}^{-1} (1_a \otimes a_{b,c,d}) a_{a,b\otimes c,d} (a_{a,b,c} \otimes 1_d) a_{a\otimes b,c,d}^{-1}
\]

for all objects \( a, b, c, d \) of \( \mathcal{C} \), as depicted in the following diagram.

Note that if \( q_{a,b,c,d} = 1_{(a\otimes b)(c\otimes d)} \) for all objects \( a, b, c, d \) then the pentagonal constraint holds. This is then a monoidal structure without unit. Next we define the pseudo–monoidal structure first introduced in Joyce [3].

**Definition 4** A pseudo–monoidal structure for a category \( \mathcal{C} \) is
a premonoidal structure \((\otimes, a)\) such that the following two do-decagon diagrams commute.

\[
\begin{align*}
((a \otimes b) \otimes c) \otimes (d \otimes f) & \xrightarrow{q_{a \otimes b, c, d, f}} ((a \otimes b) \otimes c) \otimes (d \otimes f) \\
(a \otimes (b \otimes c)) \otimes (d \otimes f) & \xrightarrow{q_{a, b \otimes c, d, f}} (a \otimes (b \otimes c)) \otimes (d \otimes f)
\end{align*}
\]

\[
\begin{align*}
(a \otimes b) \otimes ((c \otimes d) \otimes f) & \xrightarrow{q_{a, b, c, d, f}} (a \otimes b) \otimes ((c \otimes d) \otimes f) \\
1_{a \otimes f} \otimes a_{c, d, f} & \xrightarrow{} 1_{a \otimes f} \otimes a_{c, d, f}
\end{align*}
\]

for all objects \(a, b, c, d, f\) of \(C\).

3 Coherence of Premonoidal Structures

We begin with some preliminary definitions. Let \([n] = \{1, 2, \ldots, n\}\).

**Definition 5** A coupling tree \(t\) of length \(n\) is a planar binary rooted tree with \(n\) leaves, together with a linear ordering of its vertices subject to the condition that any connected loop-free sequence of vertices from the root to a leaf is (strictly) increasing. Hence all but the null coupling tree, are uniquely characterised by a bijection \(t : [n - 1] \rightarrow [n - 1]\) giving the order in which the branch point levels occur. The length of the tree, denoted \(|t|\), is the number of its leaves \(n\).

Note that the null coupling tree is represented formally by \(0 : [-1] \rightarrow [-1]\), where as the empty map \(1 : [0] \rightarrow [0]\) represents the (unique) coupling tree of length one. We denote the groupoid of coupling trees of length \(n\) by \(\text{Cptr}_n\) where there is a unique arrow between each tree called a recoupling. Thus the groupoid of coupling trees is given by
\[ Cptr = \coprod_{n \in \mathbb{W}} Cptr_n \quad \text{(2)} \]

where we define \( \mathbb{W} = \mathbb{N} \cup \{0\} \). A coupling tree of length \( n \) is equally well represented by a linear ordering of the elements \([n-1]\) in the following obvious way. Writing the level at the bottom of each region between adjacent leaves from left to right in a sequence gives a linear ordering. For example, the coupling tree \((1243)(5) : [5] \rightarrow [5]\) of length 6 has the linear ordering

\[
\begin{array}{cccccc}
2 & 4 & 3 & 1 & 5 \\
\downarrow & & & & \\
24315
\end{array}
\]

This defines an injective functor from \( \text{LO} \) into \( Cptr \) where \( \text{LO} \) denotes the groupoid of linear orderings and

\[ \text{LO} = \coprod_{n \in \mathbb{W}} \text{LO}_n \quad \text{(3)} \]

where \( \text{LO}_n \) is the full subgroupoid generated by the linear orderings of length \( n \). The functor between \( \text{LO} \) and \( Cptr \) is invertible if we extend \( \text{LO} \) to \( \text{LO} \cup \{\ast\} \) where \( \ast \) is a discrete object mapping to the null tree. The recoupplings between two coupling trees (of the same length) are represented by permutations. In what follows we do not distinguish between the two groupoids using the linear ordering to denote the coupling trees and recoupplings.

If we cut a coupling tree \( t \) at its root then we obtain two coupling trees. We denote the left coupling tree by \( Lt \) and the right by \( Rt \). We may also split recoupplings (or permutations) about the root vertex. Let \( \pi : s \rightarrow t \) be an arrow of \( Cptr \) that leaves the root fixed. Furthermore, if \( r \) is the position of the root in \( s \), \( n = |s| - 1 \) and \([r-1]\) and \( r + [n-r] \equiv \{r+1, \ldots, n\} \) are closed under \( \pi \), then we define \( L\pi : Ls \rightarrow Lt \) to be the unique recoupling from
\( Ls \) to \( Lt \), similarly for \( R\pi : Rs \to Rt \).

We now have the following coherence result where we use the notation

\[
C^S \equiv \prod_{s \in S} C^s
\]

for any \( S \subset \mathcal{W} \) with \( C^0 \) defined to be the one arrow category.

**Theorem 6** Given a category \( \mathcal{C} \) with premonoidal structure \((\otimes, a)\) there is a unique functor \( \Gamma : \mathbf{Cptr} \to \mathbf{Funct}(C^\mathcal{W}, \mathcal{C}) \) satisfying:

1. \[
\Gamma(t) = \otimes (\Gamma(Lt) \times \Gamma(Rt))
\]
2. \[
\Gamma(\emptyset) = 1
\]

for all objects \( t \) of \( \mathbf{Cptr} \).

(2) Let \( (ij) : s \to t \) be a transposition interchanging \( k \) in the \( i \)th position with \( k + 1 \) in the \( j \)th position of \( s \) such that the position of level 1 is not between \( i \) and \( j \), then for \( k = 1 \)

\[
\Gamma(ij) = \begin{cases} 
    a (\Gamma(LLs) \times \Gamma(RLs) \times \Gamma(Rs)) & : i > j \\
    a^{-1} (\Gamma(Ls) \times \Gamma(LRs) \times \Gamma(RRs)) & : i < j
\end{cases}
\]

for \( k > 1 \) and \( s^{-1}(1) < \min\{i, j\} \)

\[
\Gamma(ij) = \otimes (\Gamma L(ij) \times 1_{\Gamma(Rs)})
\]

and for \( k > 1 \) and \( s^{-1} > \max\{i, j\} \)

\[
\Gamma(ij) = \otimes (1_{\Gamma(Ls)} \times \Gamma R(ij))
\]

Before giving a proof we make some remarks and introduce some convenient notation. The deformed pentagon diagram in \( \mathbf{Cptr} \) is given by
and maps under the functor $\Gamma$ to the following diagram in $\text{Funct}(C^4, C)$.

Next we introduce some important operations that may be performed on coupling trees. Let $t$ be a coupling tree of length $n + 1$. We define $\{t\} = [n]$. Every tree defines a partial ordering on $\{t\}$ where $i \leq_t j$ if level $i$ is connected to level $j$ without passing through levels less than $i$. In particular $1 \leq_t i$ for all levels $i \in \{t\}$.

We define the cut operations to be maps $\land_i, \lor_j : (\text{Cptr})_0 \to (\text{Cptr})_0$ where $i, j \in \mathbb{N}$ defined as follows. Given a coupling tree
t, if one cuts the tree at the branch at level \( i \) (if it exists) then one obtains two trees. The upper tree gives rise to a coupling tree denoted \( \vee_i t \) while the bottom gives rise to a coupling tree denoted \( \wedge_i t \). If no branch exists at level \( i \) then \( \vee_i t = 0 \) is the null tree, and \( \wedge_i t = t \).

**Lemma 7** The cut operations satisfy the following properties:

1. \( \wedge_1 t = 1 \) and \( \wedge_i t = t \) for all \( i \geq |t| \) and all coupling trees \( t \).
2. \( \vee_1 t = t, \, |\vee_{|t|} t| = 2, \, \vee_i t = 0 \) for all \( i \geq |t| \) and all coupling trees \( t \).
3. \( \wedge_i = \wedge_i \wedge_j \) whenever \( i \leq t \). In particular \( \wedge_i \) is idempotent.
4. Given a coupling tree \( t \) then \( i \leq t \) implies \( \vee_k \vee_i t = \vee_j t \) for some \( k \leq |i - j| \).

These properties are easily demonstrated.

The reattachment operation \( p(n) : t \rightarrow t' \) at the \( n \)th level is an arrow satisfying \( n \leq t_n + 1 \) and \( n \leq t_{n+1} + 1 \) that only interchanges levels \( n \) and \( n + 1 \). It is called a reattachment to the left (resp. right) if \( t^{-1}n < t^{-1}(n + 1) \) (resp. \( t^{-1}n > t^{-1}(n + 1) \)). Hence \( \vee_{n+1} t = \vee_{n+1} t' \) and \( \wedge_n t = \wedge_n t' \). For a left reattachment \( R \wedge_n t = \wedge_{n+1} t \), and for a right reattachment \( L \wedge_n t = \wedge_{n+1} t \).

**Definition 8** An arrow of \( C_{\text{ptr}} \) is called primitive if it is an identity or corresponds to a single reattachment operation.

**Proof of Theorem 6** First let \( \rho(m) : t \rightarrow t' \) be a reattachment arrow at the \( m \)th level. If \( \rho(m) \) reattaches to the right we define

\[
\Gamma \rho(m) = \Gamma \wedge_m t \left( 1^p \times a(\Gamma LL \vee_m t \times \Gamma RL \wedge_m t \times \Gamma R \vee_m t) \times 1^q \right)
\]

where \( p + 1 = \min \{t^{-1} n : m \leq t, n \} \) and \( q = |t| - p - |\vee_m t| \). Otherwise it reattaches to the left and we define

\[
\Gamma \rho(m) = \Gamma \wedge_m t \left( 1^p \times a^{-1}(\Gamma L \vee_m t \times \Gamma LR \wedge_m t \times \Gamma RR \vee_m t) \times 1^q \right)
\]
For any arrow $f$ of $\textbf{Cptr}$ there is a (directed) sequence of primitive reattachment arrows with $f = \rho_n \cdots \rho_1$. Then we define $\Gamma(f) = (\Gamma\rho_n) \cdots (\Gamma\rho_1)$. It only remains to show that $\Gamma$ is well-defined and a functor. Equivalently we show that any commutative polygonal diagram of primitives in $\textbf{Cptr}$ maps to a commutative diagram in $\textbf{Funct}(\mathcal{C}^W, \mathcal{C})$. Proof is by induction on coupling tree length $n$. It is easy to show for $n = 1, 2, 3, 4$. Consider a polygonal diagram in $\textbf{Cptr}$ with vertices $t_0, \ldots, t_{N-1}$ of length $n + 1 > 4$, where we take $t_0 = t_N$, and all arrows are primitive. We show that the commutativity of this diagram is equivalent to the commutativity of a diagram in which the highest level $q = n - 1$ is maintained in a fixed region for all of its vertices. Such a diagram is commutative by the induction hypothesis. Let $r = \max\{t_k^{-1} q : k \in [N]\}$ be the right-most region containing the level $q$ in some vertex. Suppose $t_k \to \cdots \to t_l$ is a section of the diagram where $t_{k-1}^{-1} q, t_{l+1}^{-1} q < r$ and $t_i^{-1} q = r$ whenever $k \leq i \leq l$. We replace this section with an alternative section $t_{k-1} \to s_1 \to \cdots \to s_d \to t_{l+1}$ such that the enclosed region commutes. Iterating this procedure until level $q$ is in a fixed region will complete the proof. The arrows $t_{k-1} \to t_k$ and $t_l \to t_{l+1}$ are primitives at the $q$th level. There exists a primitive sequence $t_k \to u_1 \to \cdots \to u_d \to t_l$ keeping the levels $q$ and $q - 1$ fixed. Moreover the enclosed diagram under $\Gamma$ commutes by the induction hypothesis since level $q$ is kept fixed. Next construct the same sequence of operations starting with $t_{k-1}$ giving a sequence $t_{k-1} \to s_1 \to \cdots \to s_d \to t_{k+1}$. This encloses a diagram with the previous sequence that commutes under $\Gamma$ because it is a ladder of natural squares. This is the desired replacement sequence completing the proof.

For a premonoidal structure the primitive reattachment operations are restricted to adjacent levels. In the pseudo–monoidal situation the adjacent restriction is lifted. Thus a primitive arrow $\rho(n) : t \to t'$ for (pseudo–monoidal) reattachment at the $n$th level satisfies $q \equiv \min\{m : n <_t m\} = \min\{m : n <_{t'} m\}$ and only interchanges levels $n$ and $q$. We say an arrow $\pi : s \to t$ is
split about the \( m \) level if \( \{ l : m \leq s \} = \{ l : m \leq t \} \). Thus we can write \( \pi = \sigma \tau = \tau \sigma \) where \( \sigma \) permutes only the levels in \( \{ l : m < s \} \) and \( \tau \) only the levels in \( \{ l : m \not\leq m \} \). We have the following coherence result from Joyce \[3\].

**Theorem 9** If \((\otimes, a)\) is a pseudo–monoidal structure then \( \Gamma : \text{Cptr} \to \text{Funct}(\mathcal{C}^\omega, \mathcal{C}) \) of theorem \[8\] satisfies

\[
\Gamma(\sigma \tau) = \otimes(\Gamma(\sigma) \times \Gamma(\tau)) \\
\Gamma(1) = 1
\]

for any arrow of \( \text{Cptr} \) split about level 2 into \( \sigma \) and \( \tau \).

**Proof** The proof mirrors that of theorem \[8\] except we have a different procedure for calculating the alternative sequence \( t_{k-1} \to s_1 \to \cdots \to s_d \to t_{l+1} \) in the induction step. Suppose \( t_{k-1} \to t_k \) is a reattachment at the \( m \)th level. We replace this arrow with the following sequence. Suppose \( t^{-1}(m + 1) < t^{-1}m \) (the other case is shown similarly and left to the reader) and consider the following diagram.
The top arrow is $t_{k-1} \rightarrow t_k$. The sides of the top diagram are parallel operations keeping the subtree with root $m$ fixed. The bottom diagram is a $q$–square. Hence substituting for $t_{k-1} \rightarrow t_k$ using the outside sequence we re–identify the maximal sequence with $t_{k-1} \rightarrow t_k$ reattaching at level $m + 1$. Inductively we are led to a maximal sequence with $t_{k-1} \rightarrow t_k$ reattaching at level $q - 1$. Similarly we are led to $t_l \rightarrow t_{l+1}$ reattaching about level $q - 1$. We can construct an alternative sequence from $t_k$ to $t_l$ keeping $q$ and $q - 1$ fixed. The diagram enclosed commutes by the induction hypothesis. If we apply the same sequence of operations from $t_{k-1}$ to $t_{l+1}$ we enclose a ladder of natural squares. Moreover, this is the desired replacement sequence completing the proof.

4 Coupling Trees and Monoidal Structures without Unit

Underlying every coupling tree is a planar rooted binary tree, or bracketing, given by simply forgetting the levels. This allows us to define an equivalence relation where $s \sim t$ if and only if both trees give the same bracketing under forgetting levels. Thus we have a full forgetful functor $U : \text{Cptr} \rightarrow \text{Cptr}/ \sim$ onto the quotient category $\text{Cptr}/ \sim$.

**Proposition 10** The forgetful functor $U : \text{Cptr} \rightarrow \text{Cptr}/ \sim$ given by forgetting levels has a right adjoint.

Let $[t] = \{s : s \sim t\}$ and let $\phi : (\text{Cptr})_0/ \sim \rightarrow (\text{Cptr})_0$ be a choice function choosing an isotypical member for each equivalence class. Thus $[\phi[t]] = [t]$. Define the faithful functor $M : \text{Cptr}/ \sim \rightarrow \text{Cptr}$ given by $\phi$ on objects assigning the unique arrow between any two objects of the same length. Clearly $UM = 1$ so the counit is strict and the unit $\eta : MU \rightarrow 1$ is given by assigning $\eta_t$ to be the unique arrow $t \rightarrow \phi[t]$.

When making a choice of isotypical objects without invoking the axiom of choice one needs a criterion. For example, define the
following order on coupling trees. Given $s$ and $t$ we define $s < t$ if $|s| < |t|$. If $|s| = |t|$ then $s < t$ if there is some $k$ such that $s_j = t_j$ for all $j < k$ and $s_k < t_k$. Thus $\phi[t]$ can be chosen to be the maximal (alternatively minimal) member of $[t]$.

There is no canonical choice of tensor product on $\mathbf{Cptr}$, however, there is on $\mathbf{Cptr}/\sim$. Given two bracketings $b_1$ and $b_2$ there is a unique bracketing $b_3$ such that $ULM b_3 = b_1$ and $URM b_3 = b_2$. We then define $b_1 \otimes b_2 = b_3$ which extends to a unique bifunctor $\otimes : \mathbf{Cptr}/\sim \times \mathbf{Cptr}/\sim \to \mathbf{Cptr}/\sim$. This defines a unique monoidal structure on $\mathbf{Cptr}/\sim$. The adjunction $U \dashv M$ of proposition 10 can be used to lift this bifunctor to $\mathbf{Cptr}$ by defining $\otimes_M \equiv M \otimes (U \times U) : \mathbf{Cptr} \times \mathbf{Cptr} \to \mathbf{Cptr}$. Thus each $\otimes_M$ admits a unique monoidal structure $(\otimes_M, a_M^M \equiv Ma(U \times U \times U))$ on $\mathbf{Cptr}$. Unlike for $\otimes$ on $\mathbf{Cptr}/\sim$, no single bifunctor $\otimes_M$ generates all objects of $\mathbf{Cptr}$ from a single generator.

**Proposition 11** Let $\mathcal{C}$ be a category with premonoidal structure $(\otimes, a)$. The functor $\Gamma : \mathbf{Cptr} \to \mathbf{Funct}(\mathcal{C}^W, \mathcal{C})$ of theorem 6 is such that, for all coupling trees $s, t, u$ we can find $p : (s \otimes_M t) \otimes_M u \to M[(s \otimes_M t) \otimes_M u]$ and $q : s \otimes_M (t \otimes_M u) \to M[s \otimes_M (t \otimes_M u)]$ such that the following square commutes.

$$
\begin{array}{ccc}
\Gamma((s \otimes_M t) \otimes_M u) & \xrightarrow{\Gamma a_M^{s,t,u}} & \Gamma(s \otimes_M (t \otimes_M u)) \\
\downarrow \Gamma_p & & \downarrow \Gamma_q \\
\otimes(\otimes \times 1)(\Gamma s \times \Gamma t \times \Gamma u) & \xrightarrow{a(\Gamma s \times \Gamma t \times \Gamma u)} & \otimes(\Gamma s \times \Gamma t \times \Gamma u)
\end{array}
$$

The notion of a premonoidal functor is that of a monoidal functor without the properties pertaining to the unit.

**Definition 12** Given categories $\mathcal{C}$ and $\mathcal{C}'$ with premonoidal structures $(\otimes, a)$ and $(\otimes', a')$ respectively, a premonoidal functor is a pair $(F, \phi)$ where $F : \mathcal{C} \to \mathcal{C}'$ is a functor and $\phi : \otimes'(F \times F) \to F \otimes$ is a natural transformation satisfying
A premonoidal functor \((F, \phi)\) is called strong (resp. strict) if \(\phi\) is an isomorphism (resp. identity).

We have a (weakened) restatement of Mac Lane’s coherence theorem \([8]\) for monoidal categories without a unit.

**Corollary 13** If \(C\) is a category with a monoidal structure without unit \((\otimes, a)\) then \(\Gamma M : \text{Cptr}/ \sim \to \text{Funct}(\text{C}^\text{W}, C)\) is a strong premonoidal functor.

This follows from noting that \(M : \text{Cptr}/ \sim \to \text{Cptr}\) is a strong monoidal functor and applying proposition \([11]\) where \(\Gamma p = 1\) and \(\Gamma q = 1\) for a monoidal structure.

5 Premonoidal Structures with Unit

In this section we add a unit without introducing triangle constraints.

**Definition 14** A premonoidal (resp. pseudo-monoidal) structure with unit for a category \(C\) is a pentuple \((\otimes, a, l, r, e)\) where \((\otimes, a)\) is a premonoidal (resp. pseudo-monoidal) structure for \(C\), \(e\) is an object of \(C\) called the unit object, and \(l : \otimes(I \times 1) \to \pi_2\) and \(r : \otimes(1 \times I) \to \pi_2\) are natural isomorphisms called respectively the recouplings for left unit and right unit.

The functor \(I : C \to C\) is defined by \(If = 1_e\) for all arrows \(f\) and may be called the unit functor. The functors \(\pi_1, \pi_2 : C^2 \to C\) are given by the universal projections of the Cartesian product.
That is \( \pi_k(f_1, f_2) = f_k \) for all arrows \((f_1, f_2)\) of \( \mathcal{C}^2 \) and \( k = 1, 2 \). Thus for any arrow \((f, g) : (a, b) \to (c, d)\) the recouplings for left unit and right unit satisfy the natural squares

The labeling of the left and right unit recouplings are redundant so often we identify \( l_b \equiv l_{e,b} \) and \( r_a \equiv r_{a,e} \). In fact this identification gives the usual form of the left and right unit recouplings as \( l : e \otimes - \to 1 \) and \( r : - \otimes e \to 1 \).

The triangle diagrams do not hold so we measure their non-commutativity by defining ghost natural automorphisms.

**Definition 15** We define the ghosts for associativity to be the natural automorphisms \( g(12), g(23), g(13) : \otimes \to \otimes \) defined by

\[
\begin{align*}
g(23)(\pi_2 \times 1) &= (l(1 \times \otimes))(a(I \times 1 \times 1))(\otimes(I^{-1} \times 1)) \quad (14) \\
g(13)(\pi_1 \times 1) &= (\otimes(1 \times I))(a(1 \times I \times 1))(\otimes(r^{-1} \times 1)) \quad (15) \\
g(12)(1 \times \pi_1) &= (\otimes(1 \times r))(a(1 \times 1 \times I))(r^{-1}(\otimes \times 1)) \quad (16)
\end{align*}
\]

The ghost associativity natural automorphisms satisfy (and are defined by) the following ghostly triangle diagrams.
for all objects \(a, b, c\) of \(C\).

Note that the associative structure with unit is monoidal whenever the deformativity and ghost natural automorphisms are identities. The ghostly triangle diagram for \(g(13)\) becomes the triangle constraint, and together with the Pentagonal constraint imply the other triangle constraints \[6\]. In the monoidal situation, Mac Lane [8] has proved a well–known coherence result.

Similarly we can define ghosts for deformativity \(g(234), g(134) : \otimes(1 \times \otimes) \to \otimes(1 \times \otimes)\) and \(g(124), g(123) : \otimes(\otimes \times 1) \to \otimes(\otimes \times 1)\) according to

\[
\begin{align*}
\mathcal{g}(234)(\pi_2 \times 1 \times 1) &= (\otimes(I \times \otimes)) (q(I \times 1 \times 1 \times 1)) (\otimes(I^{-1} \times \otimes)) \\
\mathcal{g}(134)(\pi_1 \times 1 \times 1) &= (\otimes(r \times \otimes)) (q(1 \times I \times 1 \times 1)) (\otimes(r^{-1} \times \otimes)) \\
\mathcal{g}(124)(1 \times 1 \times \pi_2) &= (\otimes(\otimes \times I)) (q(1 \times 1 \times I \times 1)) (\otimes(\otimes \times I^{-1})) \\
\mathcal{g}(123)(1 \times 1 \times \pi_1) &= (\otimes(\otimes \times r)) (q(1 \times 1 \times I \times 1)) (\otimes(\otimes \times r^{-1}))
\end{align*}
\]

We leave it to the reader to write down diagrams.

6 Coherence of Premonoidal Structures with Unit

We extend the groupoid of coupling trees by attaching two types of nodules on leaves, called unit and ghost nodules. Given a finite set \(U\) we construct a groupoid \(\mathcal{N}(U)\) called the nodule groupoid over \(U\). The objects are pairs \((u, v)\) where \(u, v \subset U\), \(u \cap v = \emptyset\) and \(v \neq U\). We say the object \((u, v)\) represents \(|u|\) unit nodules and \(|v|\) ghost nodules. There is at most one arrow between two objects given by the condition: \((u, v) \to (u', v')\) is an arrow if and only if \(u \cup v = u' \cup v'\). In other words the arrows interchange nodule type. We see that

\[
\mathcal{N}(U) = \prod_{k=0}^{|U|} \mathcal{N}_k(U)
\]

(17)
where \( N_k(U) \) is the full subgroupoid whose objects are given by \((u, v)\) such that \(|u| + |v| = k\). The groupoid of coupling trees with nodules is given by

\[
\text{NCptr} = \coprod_{n \in \mathbb{W}} \text{Cptr}_n \times N([n])
\]

\[
= \coprod_{n \in \mathbb{W}} \coprod_{k=0}^n \text{Cptr}_n \times N_k([n])
\]

A noduled coupling tree \((t, u, v)\) has \(|t|\) leaves, with unit nodules (open circles) in positions \(i\) for all \(i \in u\), and ghost nodules (closed circles) in positions \(j\) for all \(j \in v\). For example \((514632, \{3\}, \{5, 6\})\) is given by

The left and right coupling tree operations are extended to noduled coupling trees by

\[
L(t, u, v) = (Lt, u \cap [m], v \cap [m])
\]

\[
R(t, u, v) = (Rt, u \setminus [m] - m, v \setminus [m] - m)
\]

provided \(v \cap [m] \neq [m]\) and \((v \setminus [m] - m) \cap [|Rt|] \neq [|Rt|]\) where \(m = |Lt|\) and we have defined \(U + k \equiv \{i + k : i \in U\}\) for \(U \subset \mathbb{Z}\) and \(k \in \mathbb{Z}\).

We make a few convenient definitions. Define \(1^k = 1 \times 1 \times \cdots \times 1 : C^k \rightarrow C^k\) and \(\pi_i^k : C^k \rightarrow C\), the latter taking \((f_1, \ldots, f_k) \mapsto f_i\). Also given a set \(U \subset \mathbb{N}\) and a coupling tree \(t\) we define \(C_U t\) to be the unique coupling tree obtained by contracting out those leaves whose positions are in the set \(U\).

We define an equivalence relation on \(\text{NCptr}\) as follows. We write
\((s, u, v) \sim (t, w, x)\) if and only if \(s \sim t\), \(C_v s = C_x t\), \(u = w\) and \(v = x\). This equivalence extends uniquely to arrows. This defines a forgetful functor \(U : \mathbf{NCptr} \to \mathbf{NCptr}/\sim\) determined by mapping each arrow to its equivalence class. As in proposition 10, \(U\) has right adjoint sections \(M\) given by choosing a representative member of each equivalence class of objects. Each equivalence class may be thought of as the coupling tree without levels and the edges attached to ghost nodules omitted as in the following example.

\[
\begin{align*}
\bullet & \quad \circ & \quad \circ & \quad \bullet \\
\circ & \quad \bullet & \quad \circ \\
\circ & \quad \bullet & \quad \circ \\
\end{align*}
\equiv
\begin{align*}
\bullet & \quad \circ & \quad \circ \\
\circ & \quad \bullet & \quad \circ \\
\circ & \quad \bullet & \quad \circ \\
\end{align*}
\]

The category \(\mathbf{NCptr}/\sim\) is a monoidal category with bifunctor \(\otimes\) given by concatenation and joining the roots to the leaves of the tree 1. This can be lifted to a bifunctor on \(\mathbf{NCptr}\) as \(\otimes_M = M \otimes (U \times U)\).

**Definition 16** Given categories \(\mathcal{C}\) and \(\mathcal{C}'\) with monoidal structures \((\otimes, \alpha, \iota, r, e)\) and \((\otimes', \alpha', \iota', r', e')\) respectively, a monoidal functor is a triplet \((F, \phi, \psi)\) where \((F, \phi)\) is a premonoidal functor and \(\psi : FI \to I'F\) a natural transformation satisfying

\[
\begin{align*}
\otimes'(I'F \times F) & \quad \otimes'(F \times F) & \quad F\pi_2 & \quad F\pi_1 \\
\otimes'(\psi \times F) & \quad FI & \quad F\pi' & \quad F\iota \\
\otimes'(F \times I'F) & \quad \otimes'(F \times \psi) & \quad F\otimes(I \times 1) & \quad \otimes'(F \times FI) \quad F\otimes(1 \times I) \\
\phi(I \times 1) & \quad \phi(1 \times I) & & & &
\end{align*}
\]

A monoidal functor \((F, \phi, \psi)\) is called strong (resp. strict) if \(\phi\) and \(\psi\) are isomorphisms (resp. identities).

We extend the functor of theorem 6 to \(\mathbf{NCptr}\) giving the following coherence result.
Theorem 17 Given a category $\mathcal{C}$ and a premonoidal structure with unit $(\otimes, a, l, r, e)$ there is an extension of $\Gamma : \mathcal{C}_{ptr} \to \mathcal{C}$ to $\mathcal{N}\mathcal{C}_{ptr}$ such that the arrows $(1, \{1\}, \emptyset) \to (1, \emptyset, \{1\})$ and $(1, \{2\}, \emptyset) \to (1, \emptyset, \{2\})$ map under $\Gamma$ to $l$ and $r$ respectively. Furthermore, if $(\otimes, a)$ is pseudo–monoidal and all the ghosts vanish then $\Gamma M : \mathcal{N}\mathcal{C}_{ptr}/\sim \to \mathcal{C}$ is a monoidal functor for all $\otimes_M$.

Proof $\Gamma$ is characterised inductively on objects as follows. We define

\begin{align*}
\Gamma(t, \emptyset, \emptyset) &= \Gamma t \quad (22) \\
\Gamma(\emptyset, \{1\}, \emptyset) &= e \quad (23)
\end{align*}

Whenever $\Gamma L(t, u, v)$ and $\Gamma R(t, u, v)$ are defined then

\[ \Gamma(t, u, v) = \otimes(\Gamma L(t, u, v) \times \Gamma R(t, u, v)) \] ,

(24)

If $|\Gamma Lt| = 2$ then

\[ \Gamma(t, u, v) = \Gamma R(t, u, v)(\pi_i \times 1^{|Rt|}) \] ,

(25)

whenever $\{i\} = v$, and if $|Rt| = 2$ and then

\[ \Gamma(t, u, v) = \Gamma L(t, u, v)(1^{|Lt|} \times \pi_i) \] ,

(26)

whenever $\{i\} = v$. We take the primitive arrows $s \to t$ to be either reattachment arrows at the $n$th level where each of $L \lor_{n+1}s$, $R \lor_{n+1}s$, $L \lor ns$ and $R \lor ns$ contain a ghost nodule free leaf, or to be nodule change arrows where a single nodule type is changed. The image of a reattachment arrow under $\Gamma$ is given by theorem 6. Let $\rho(m) : t \to t'$ be a nodule change arrow converting a unit nodule into a ghost nodule at the $m$th level (a nodule in position $t^{-1}m$ is changed). Let $p + 1 = \min\{t^{-1}n : m \leq_t n\}$ and $q = p + |\lor_m t|$. If $t^{-1}m = p + 1$ we define

\[ \Gamma \rho(m) = (\Gamma \land_t m)(1^p \times l(I \times \Gamma R \lor_m t) \times 1^q) \]
Otherwise $t^{-1}m = q - 1$ and we define

$$\Gamma \rho(m) = (\Gamma \wedge_m t) (1^p \times r(\Gamma L \vee_m t \times I) \times 1^q)$$

For any arrow $f$ of $\textbf{NCptr}$ there is a (directed) sequence of primitive reattachment arrows with $f = \rho_n \cdots \rho_1$. We then define $\Gamma(f) = (\Gamma \rho_n) \cdots (\Gamma \rho_1)$. It only remains to show that $\Gamma$ is well-defined and a functor. Equivalently we show that any commutative diagram of primitives in $\textbf{NCptr}$ maps to a commutative diagram in $\textbf{Funct} (\mathcal{C}^W, \mathcal{C})$. Specifically we show how to remove ghost nodules in the $i$th position. Then all ghost nodules can be removed and the result follows from theorem 6. It is not hard to see that the primitive arrows for changing ghost nodules into unit nodules commute with all other primitive arrows. Hence the sections of the diagram with ghost nodules in the $i$th position can be replaced by an alternative sequence without ghost nodules enclosing a ladder of natural squares.

7 Braidings

We introduce a braid structure on a category requiring only that it possess a premonoidal structure.

**Definition 18** A prebraid structure $(\otimes, a, c)$ for a category $\mathcal{C}$ is a premonoidal structure $(a, \otimes)$ and a natural isomorphism $c : \otimes \to \otimes \tau_{(12)}$ where $\tau_{(12)}$ is the switch match. This structure is called braid premonoidal if the following three diagrams

![Braid Diagram](image-url)
commute for all objects $a, b, c, d$ of $\mathcal{C}$. If in addition $(\otimes, a)$ is pseudo–monoidal and the following square diagram

$\begin{array}{cccc}
(a \otimes b) \otimes (c \otimes d) & q_{a,b,c,d} & (a \otimes b) \otimes (c \otimes d) \\
\downarrow c_{a \otimes b,c \otimes d} & & \downarrow c_{a \otimes b,c \otimes d} \\
(c \otimes d) \otimes (a \otimes b) & q_{c,d,a,b} & (c \otimes d) \otimes (a \otimes b)
\end{array}$

commutes for all objects $a, b, c, d$ of $\mathcal{C}$ then the structure is called braid pseudo–monoidal. Finally, whenever $c^{-1} = c_{\tau(12)}$ the braid is called a symmetry.

More generally the switch map extends to an action $\tau : S_n \to \text{End}(\mathcal{C}^n)$ where $\pi \mapsto \tau_\pi$ is given by $\tau_\pi(c_1, \ldots, c_n) = (c_{\pi 1}, \ldots, c_{\pi n})$.

**Definition 19** Given categories $\mathcal{C}$ and $\mathcal{C}'$ with prebraid structures $(\otimes, a, c)$ and $(\otimes', a', c')$ respectively, a braid premonoidal functor is a premonoidal functor $(F, \phi)$ satisfying

$\begin{array}{cccc}
\otimes'(F \times F) & \phi(F \times F) & \otimes'(F \times F) \tau_{(12)} \\
\downarrow & & \downarrow \phi \tau_{(12)} & \downarrow \phi \tau_{(12)} \\
F \otimes & F \tau & F \otimes \tau_{(12)}
\end{array}$

A braid premonoidal functor $(F, \phi)$ is called strong (resp. strict) if $\phi$ is an isomorphism (resp. identity).

Coherence will be described with respect to the Artin braid groups $B_n$ where $n \in \mathbb{W}$. These are groupoids on one object. The group
$B_n$ is generated by $\tau_1, \ldots, \tau_{n-1}$ satisfying the conditions

\begin{align*}
\tau_i \tau_j &= \tau_j \tau_i \quad (27) \\
\tau_{i+1} \tau_i \tau_{i+1} &= \tau_i \tau_i+1 \tau_i \quad (28)
\end{align*}

for all $i, j = 1, 2, \ldots, n-1$ satisfying $|i - j| > 1$. Let $V : B_n \to S_n$ be the forgetful functor forgetting the order in which braids cross. This functor is completely determined on generators by $V(\tau_i) = (i i+1)$. We define the exploded $n$th braid groupoid $xB_n$ to be given by the formal collection of arrows $\tau : \pi \to V(\tau)\pi$ where $\pi \in S_n$ and $\tau \in B_n$. Composition is inherited from $B_n$ whenever the source and target match. The objects are given by $S_n$ and the hom–sets by $xB_n(\pi, \sigma) = \{\tau \in B_n : V(\tau)\pi = \sigma\}$.

We define the braid groupoid of coupling trees by

$$
BCptr = \coprod_{n \in \mathbb{W}} \text{Cptr}_n \times xB_n \quad (29)
$$

We can now state the main coherence result.

**Theorem 20** Given a braided premonoidal structure for $\mathcal{C}$ there is an extension of theorem 6 to $\Gamma : BCptr \to \text{Funct}(\mathcal{C}^\mathbb{W}, \mathcal{C})$ where $\Gamma \tau_1 = c$ on $\text{Funct}(\mathcal{C}^2, \mathcal{C})$.

We note the following lemma.

**Lemma 21** Given a category $\mathcal{C}$ with a premonoidal structure $(\otimes, a, c)$, the following quasi–Yang–Baxter diagram commutes.
Proof Reading left to right the third row corresponds to \( \otimes (c \times 1) (\otimes \times 1) \), as does the fourth row. The square formed is natural. The hexagonal diagrams formed above and below are those of the definition. Hence the entire diagram commutes.

Proof of Theorem 20 The primitive arrows for interchange (about the region \( i \)) are of the form \((1, \tau_i) : (s, \pi) \to (s, (i i + 1)\pi)\) such that \( |\bigvee_{s_i} s| = 2 \). In other words this corresponds to the interchange of two attached (adjacent) leaves. Every interchange arrow \( \tau : (s, \pi) \to (t, (i i + 1)\pi) \) may be written as a sequence of primitive arrows \( p_1 \cdots p_m \) with precisely one corresponding to a primitive interchange arrow. We define \( \Gamma \tau = (\Gamma p_1) \cdots (\Gamma p_n) \), where for an arrow \((f, 1) : (s, \pi) \to (t, \pi)\) we have \( \Gamma(s, \pi) = (\Gamma s)\tau_\pi \) and \( \Gamma(f, 1) = (\Gamma f)\tau_\pi \), and for any primitive interchange \((1, \tau_i)\)

\[
\Gamma(1, \tau_i) = (1^{i-1} \times c \times 1^{s|i\pi-i+2})\tau_\pi
\]

The proof is completed by showing that this definition is well-defined and that conditions (27) and (28) hold whenever composition is allowed.

We show it is well-defined in two steps. Firstly that there is a sequence of primitive arrows with precisely one primitive inter-
change arrow \((1, \tau_i) : (s', \pi) \to (t', (i + 1)\pi)\) with \(s'^{-1}(|s| - 1) = i\). Secondly, that any two alternative such sequences form a commutative diagram in \(C\). Consider the following diagram.

The sequence of arrows running along the top is \(p_1 \to \cdots \to p_r\) with the interchange about the \(k\)th level given by the centre arrow. We suppose that the level \(k + 1\) is to the right of \(k\) in the source and target trees of this arrow. We construct parallel sequences of reattachments (vertically downward on diagram) maintaining the position of the levels greater than \(k + 1\), into a form containing the subtree indicated. The diagram is enclosed with the interchange arrow forming a ladder of natural diagrams under \(\Gamma\). Next we apply the sequence of four arrows corresponding to interchanging the levels \(k\) and \(k + 1\) completing the region by the relevant diagram of the definition of braid structure. Finally we can complete the sequence around the bottom with
primitive reattachment arrows. By theorem 6 the left and right side diagrams commute. Hence the sequence around the bottom composes to give the same arrow under $\Gamma$ as the top sequence. Applying this argument inductively we arrive at a desired sequence where the interchange occurs about the maximal level $|s| - 1$.

Given an alternative such primitive sequence $q_1 \to \cdots \to q_r'$ we can suppose that the primitive interchange occurs about the $(|s| - 1)$th level. Hence we can construct two parallel sequences of reattachment arrows between the sources and between the targets of the two interchange arrows preserving the position of level $|s| - 1$. The enclosed diagram commutes under $\Gamma$ because it is a ladder of natural diagrams. Also the remaining two regions enclosed, one containing the source of $p_1$ and $q_1$, the other the target of $p_r$ and $q_r'$, commute under $\Gamma$ by theorem 6. Hence the two sequences give the same arrow under $\Gamma$ and the definition is well-defined.

Condition (27) holds because we can suppose that the source and target trees of the interchange arrow in a sequence of primitives composing to give $\tau_i$ and $\tau_j$ are identical. The result follows by naturality. Similarly condition (28) holds by lemma 21.

The coherence of the related cases for prebraid and braid pseudo-monoidal structures are by now a variation on a theme. We make the following remarks.

**Remark 22** Theorem 20 may be weakened to a prebraid structure where the primitive adjacent interchange arrows are taken as those interchanging two leaves. The hexagon diagrams define the adjacent interchange of three leaves. Thus interchanges involving more than three adjacent leaves are given by iterating the hexagon diagrams.

**Remark 23** Alternatively theorem 20 may be strengthened to the braided pseudo-monoidal situation where primitive arrows are not restricted by the requirement that levels are adjacent.
Finally we bring everything together in the following definition.

**Definition 24** A braid premonoidal structure with unit \((\otimes, a, c, l, r, e)\) for a category \(C\) is a premonoidal structure with unit \((\otimes, a, l, r, e)\) and a braid premonoidal structure \((\otimes, a, c)\).

Similar definitions hold for prebraid structure with unit and braided pseudo–monoidal structure with unit. We define

\[
BNC_{\text{ptr}} = \coprod_{n \in \mathbb{N}} C_{\text{ptr}} n \times xB_n \times \mathcal{N}([n])
\]  

We are now in a position to state the expected coherence result.

**Theorem 25** If \((\otimes, a, c, l, r, e)\) is a braided premonoidal structure with unit for \(C\) then there is an extension of theorems 17 and 20 to a functor \(\Gamma : BNC_{\text{ptr}} \rightarrow \text{Funct}(C^W, C)\).

The proof of this is very straightforward as are the analogous results for prebraid and braided pseudo–monoidal structures with unit.

9 Diagram Calculi

Thus far coherence is a functor \(\Gamma\) between some groupoid \(\text{Cohr}\), taken as \(C_{\text{ptr}}, N_{\text{Cptr}}, B_{\text{Cptr}}\) or \(B_{\text{NCptr}}\), and \(\text{Funct}(C^W, C)\). Ultimately, coherence concerns the commutativity of certain diagrams in \(C\). Thus we introduce an evaluation functor

\[
ev \equiv \coprod_{n \in \mathbb{N}} \text{ev}_n : \prod_{n \in \mathbb{N}} \text{Funct}(C^n, C) \times C^n \rightarrow C
\]  

given by mapping the arrows \((\tau, f) : (F, a) \rightarrow (G, b)\) to \(\text{ev}_n(\tau, f) = (Gf)\tau a\) which by the natural property of \(\tau\) is also given by \((\tau b)Ff\). Next we define precisely what we mean by a diagram
Definition 26 A collection of arrows \( D \) for a category \( \mathcal{C} \) is called a (commutative) diagram if given any two composable sequences of arrows \( f_1, \ldots, f_m \) and \( g_1, \ldots, g_n \) from \( D \) with matching source \( (sf_1 = sg_1) \) and target \( (tf_m = tg_n) \) then we have \( f_m \cdots f_1 = g_n \cdots g_1 \).

Clearly if \( E \subset D \) and \( D \) is a diagram then so is \( E \). A diagram is a labeled directed graph and so inherits the notion of connectedness. Furthermore, every diagram is the disjoint union of connected diagrams.

Definition 27 A functor \( \Gamma : \mathcal{C} \to \mathcal{D} \) is called coherent if for every diagram \( D \) of \( \mathcal{D} \) there is a diagram \( C \) of \( \mathcal{C} \) such that \( \Gamma C = D \).

Remark 28 The converse of definition 27 clearly holds because \( \Gamma \) is a functor.

We define the canonical functor by

\[
\text{can} \equiv \prod_{n \in \mathbb{W}} \text{can}_n : \prod_{n \in \mathbb{W}} \text{Cohr}_n \times \mathcal{C}^n \to \mathcal{C}
\]  

(33)

where \( \text{can}_n = \text{ev}_n(\Gamma \times 1_{\mathcal{C}^n}) \). We can now state the self evident coherence result.

Theorem 29 The functor \( \text{can} \) is coherent.

We illustrate the diagram calculus for \( \text{BCptr} \). An object \( (s, \pi, a) \) of \( \text{Cptr}_n \times \text{xB}_n \times \mathcal{C}^n \) consists of a coupling tree \( s \), of length \( n \) say, with leaves labeled from left to right by the \( n \)-tuple of objects \( a = (a_1, \ldots, a_n) \) from \( \mathcal{C}^n \), and \( \pi \in S_n \). An arrow \( (\sigma, \tau, f) : (s, \pi, a) \to (t, \phi, b) \) consists of an \( n \)-tuple of arrows \( f = (f_1, \ldots, f_n) \) with \( f_i \in \mathcal{C}(a_i, b_i) \), a permutation \( \sigma \in S_{n-1} \), and a braid \( \tau \in \text{xB}_n(\pi, \phi) \).

We represent an arrow by a labeled box on a string. Boxes are free to slide along strings (naturality) and the identity arrow is simply given by a string. Composition is given by combining vertically
aligned consecutive boxes as depicted in the following diagram for the composition of

$$( (14)(45), \tau_5 \tau_4^{-1} \tau_3 \tau_1, f_1, ..., f_6 ) : (01234, 1, a) \to (23014, (23)(56), b)$$

with

$$( (13)(24), \tau_5^{-1} \tau_2, g_1, ..., g_6 ) : (23014, (23)(56), b) \to (14023, (12643), c)$$

giving

$$( g_1 f_1, ..., g_6 f_6, \tau, (13452) ) : (01234, 1, a) \to (14023, (12643), c)$$

where $\tau = \tau_5 \tau_4^{-1} \tau_5^{-1} \tau_3 \tau_1 \tau_2$.

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