SPIN NETWORKS IN QUANTUM GRAVITY

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Abstract. This is a review paper about one of the approaches to unify Quantum Mechanics and the theory of General Relativity. Starting from the pioneer work of Regge and Penrose other scientists have constructed state sum models, as Feynmann path integrals, that are topological invariant on the triangulated Riemannian surfaces, and that in the continuous limit become the Hilbert-Einstein action.

1. Introduction

This is a review talk to present the main ideas of some line of research in quantum gravity, namely the spin foam approach, that has been explored by a great number of physicists and mathematicians and has attracted much attention. The three main lines of research in quantum gravity are denoted as “canonical”, “covariant” or “sum over histories”.

The canonical line of research is a theory in which the Hilbert space carries a representation of the quantum operators corresponding to the full metric without background metric to be fixed. It can be considered as a quantum field theory on a differentiable manifold. The basis of the Hilbert space are cilindrical functions defined on a graph (Wilson loops) depending on Ashtehar variables. A very important result of this approach was the discrete eigenvalues for the area and volumen operators.

The covariant line of research is the attempt to built the theory as a quantum field theory of the fluctuations of the metric over a flat Minkowski space, or some other background metric space. The theory has been proved to be renormalizable and finite order by order.

The sum over histories line of research uses the Feynmann path integral to quantize the Einstein Hilbert action. There exist a duality between this model and group field theories. The sum over spin foam can be generated as the Feynmann perturbative expansion of the group field theories. Each space-time appears as the
Feynmann graph of the auxiliary groups field theory. Our presentation is going to be concentrated on this third line of research, namely, the spin network and the spin foam models, from an historical point of view.

2. Regge calculus (1961)

The Regge’s paper was a pionnier work in the discretization of GR, that was motivated by the need to avoid coordinates, because the physical prediction of the theory was coordinates independent. For that purpose he discretizes a continuous manifold by \( n \)-simplices, that are glued together by identification of their \((n-1)\)-simplices. The curvature lies on the \((n-2)\)-dimensional subspaces, known as hinges or bones. From pedagogical reasons we take a triangulation of a 2-dimensional surface. When a collection of triangle meeting at a vertex is flattened there will be a gap or deficit angle \( \epsilon \), indicating the presence of curvature. Using the Gauss-Bonet formula we can calculate the excess angle by \( \epsilon = KA \), where \( K \) is the curvature at that vertex and \( A \) the area of the triangles around the vertex. If the number of vertices increases we can take \( K = \epsilon \rho \), where \( \rho \) is the density of vertices in the triangulation (= number of vertices by unit area). This method is easily enlarged to higher dimensions.

In order to have connection with GR we translate into the triangulated surface (the skeleton) the Hilbert-Einstein action \( L = \frac{1}{8\pi} \int K d^4x \sqrt{-g} \) where \( K \) is the scalar curvature in 4-dimensions. The discrete version for a 4-dimensional skeleton is given in terms of the deficit angle in each bone where the curvature \( K \) is calculated and some measure function \( L \) is defined:

\[
L = \sum_{n=1}^{N} \epsilon_n L_n
\]

here the summation extends to all the bones in the skeleton. In the continuous case the Einstein’s equations are derived from a stationary action, varying \( L \) with respect to the metric. In the discrete version one derives the action with respect to the edge lengths, because in the simplicial decomposition all the properties can be derived from these edges. Using Schlaefli differential identity one finds

\[
\delta L = \frac{1}{8\pi} \sum_{n=1}^{N} \epsilon_n \delta L_n = 0 \Rightarrow \sum_{n=1}^{N} \epsilon_n \frac{\partial L_n}{\partial l_p} = 0
\]

which is the discrete version of Einstein’s equations.
3. The Ponzano-Regge model (1968)

Some years later Ponzano and Regge [17] made use of \{6j\} symbols attached to the tetrahedra decomposition in order to calculate the state sum and were able to connect it to the Feynmann integral corresponding to the Hilbert-Einstein action.

The \{6j\} Wigner symbols is a generalization of the Clebsch-Gordan coefficients that appear in the coupling of two angular momenta \(J = J_1 + J_2\).

The new basis is given in term of the old basis:

\[
|j_1j_2jm\rangle = \sum \langle j_1j_2m_1m_2| j_1j_2jm\rangle |j_1j_2m_1m_2\rangle
\]

If we couple \(J\) with a new angular momentum \(J_3\) we have two possibilities

\( (J_1 + J_2) + J_3 = J \quad \text{or} \quad J_1 + (J_2 + J_3) = J \)

In the first case the new basis is given (in obvious notation)

\[
|j_1j_2j_3j_12jm\rangle = \sum \langle j_1j_2j_3m_1m_2m_3| j_1j_2j_3j_12jm\rangle |j_1j_2j_3m_1m_2m_3\rangle
\]

In the second case the new basis is given by

\[
|j_1j_2j_3j_23jm\rangle = \sum \langle j_1j_2j_3m_1m_2m_3| j_1j_2j_3j_23jm\rangle |j_1j_2j_3m_1m_2m_3\rangle
\]

The transforming matrix between the two basis is given precisely by the \{6j\} symbols, namely,

\[
U(j_1j_2j_3) = (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23} 
\end{array} \right\}
\]

Given a tetrahedra decomposition of a 3-dimensional surface we can attach a \{6j\} symbol to each tetrahedra, the edges of which have the length equal to the numerical values of the angular momenta appearing in the 6j-symbol.

This choice is consistent with the inequalities

\( j_{12} < j_1 + j_2 \quad \text{and} \quad j_{23} < j_2 + j_3 \)
and the equalities
\[ j_1 + j_2 + j_{12} \subset N, \quad j_2 + j_3 + j_{23} \subset N \]
The \{6j\} symbols are proportional to the Racah polynomials\[^{14}\]
\[ (-1)^{j_1+j_2+j_{12}} \sqrt{(2j_{12}+1)(2j_{23}+1)} \left\{ \begin{array}{ccc}
  j_1 & j_2 & j_{12} \\
  j_3 & j & j_{23}
\end{array} \right\} = \frac{\sqrt{\rho(x)}}{d_n} U^{(\alpha,\beta)}_n(x,a,b) \]
From this equality and the asymptotic properties of Racah polynomials one can derive very important limit
\[ \left\{ \begin{array}{ccc}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{array} \right\} \rightarrow \{6j\} \sum_{i=1}^6 \left( j_i + \frac{1}{2} \right) \vartheta_i + \frac{\pi}{4} \]
where \( V \) is the volume of the tetrahedra and \( \vartheta_i \) the exterior dihedral angle adjoint to the edge \( j_i \). In order to see the connection between \{6j\} symbols and quantum gravity we take a tetrahedra decomposition and take external edges \( l_i \) of the bounding surface and internal edges \( x_i \). Then Ponzano and Regge construct the state sum as follows
\[ S(l_i) = \sum_{x_i} \prod_{\text{tetrahedra}} \{6j\} (-1)^X \prod_{\text{edges}} (2x_i + 1) \]
where \( X \) is a phase factor. Substituting the \{6j\} symbols by their asymptotic values and the function cosine by the Euler expression we arrive at
\[ S(l_i) = \sum_{x_i} \prod_{i=1}^{\text{tetrahedra}} (2x_i + 1) \exp \left\{ i \left[ \sum_{\text{tetrahedra},k} \vartheta_i - \pi p_i + 2\pi \right] x_i \right\} \]
We may replace the summation with an integral. Then the most important contribution comes from the stationary phase that is to say when \( \sum_{\text{tetrahedra}} \left( \pi - \vartheta_i^k \right) = 2\pi \)
Introducing this value in the state sum we obtain\[^{19}\]
\[ S(l_i) = \int \prod_{x_i} (2x_i + 1) dx_i \exp \left( i \sum j_i \epsilon_i \right) \]
where \( \epsilon_i = 2\pi - \sum_k \left( \pi - \theta_i^k \right) \)
The summation $\sum j_l \epsilon_l$ approaches the Hilbert-Einstein action that was given in the Regge calculus, therefore in the limit the state sum strongly resembles a Feynmann summation over histories with density of Lagrangian $\mathcal{L} = \frac{1}{8\pi} \int Ra^4 x \sqrt{-g}$ namely

$$S = \int d\mu(x_i)e^{i\mathcal{L}}$$

4. Penrose’s spin networks (1971)

Penrose was interested in the interpretation of space-time [16] by purely combinatorial properties of some elementary units that are connected among themselves by some interactions that follow the angular momentum quantum rules, and form some network of elementary units with assigned spins. Soon it was realized that the spin network was analog to simplicial quantum gravity, in particular the Ponzano-Regge model [12]. His networks had trivalent vertices and the edges were labeled with spin, satisfying the standard conditions at the vertices. The model was generalized to any group different from the rotation group. Formally a spin network ia a triple $(\gamma, \rho, \iota)$ where

- i) $\gamma$ is a graph with a finite set of edges $e$ and a finite set of vertices, $v$,
- ii) to each edge $e$ we attach an irrep of a group $G$, $\rho_e$
- iii) to each vertex $v$ we attach an intertwiner.

When we take the dual of an spin network we obtain a triangulated figure, which, after embedding in a 3-dimensional manifold becomes the triangulation of Regge calculus.

5. The Turaev-Viro state-sum invariant (1992)

They defined a state sum for triangulated 3-manifold (as in the Ponzano-Regge model) that was independent os the triangulation and finite [22]. For this purpose they assign a value from the set $I_r = (0, 1/2, 1, (r - 2)/2)$, integer, to each edge of the triangulation, subject to the condition that the “coloring” of the three edges forming a triangle should satisfy the triangle inequalities and their sum should be and integer less than or equal to $r - 2$. Define the quantum object

$$|M|_{\phi} = \omega^{-2\rho} \prod_{\text{tetra} k} |T_k^\phi| \prod_{\text{edge} j} \omega_j^2$$
where \( \varphi \) is an admisible coloring of the edge \( j \),

\[
\omega_j = (-i)^j [2j + 1]^{1/2}, \quad \omega^2 = \sum_{j \in I} \omega_j^4
\]

and \( |T^R_k| \) is the quantum 6j-simbol corresponding to the tetrahedron \( k \) with coloring \( \varphi \), such that

\[
|i j k| l m n = (-1)^{i+j+k+l+m+n} \left\{ \begin{array}{ccc}
[i] & [j] & [k] \\
[l] & [m] & [n]
\end{array} \right\}
\]

where \([n]\) is the quantum number satisfying \([n]\) \to \( n \). Summing \( |M|_\varphi \) over all admisible coloring we obtain an expresion in the limit \( q \to 1 \) or \( r \to \infty \) that becomes identical to the Ponzano-Regge state sum. Turaev and Viro proved that their expression is manifold invariant (or independent of triangulation) under Alexander moves, and also finite.

6. The 3-dimensional Boulatov model (1992)

The Ponzano-Regge state sum and the Turaev-Viro model are defined over 3-dimensional manifold. To enlarge the model to four dimensions it vas necessary to increase the Wigner symbols to \( 3nj \). The key to this approach was given by Boulatov [9] by the use of topological lattice gauge theories, taking group elements as variables (matrix models). The basic objects is a set of real functions of 3 variables \( \phi(x,y,z) \) (where \( x,y,z \in SU(2) \)) invariant under simultaneous right shift of all variables by \( u \in SU(2) \) and also by cyclic permutation of \( x,y,z \). This function \( \phi \) can be expanded, by Peter-Weyl theorem, in terms of representations of \( SU(2) \) and 3j-symbols. An action of interest can be constructed with those functions as follows

\[
S = \frac{1}{2} \int dx dy dz \phi^2(x,y,z) - \lambda \frac{1}{4!} \int dx dy dz du dv dw \phi(x,y,z) \phi(x,u,v) \phi(y,v,w) \phi(z,w,u)
\]

If we attach the variable to the edges, the first term (the kinematical term) represent two glued triangle and the second one (the interacting term) four triangles forming a tetrahedron. Substituting the Fourier expansion of funcion \( \phi \), and integrating out group variables we obtain an action depending on the Fourier coefficients and 6j-simbols. From this result we calculate the partition function as a
Feynman path integral with respect to the Fourier coefficients

\[ Z = \int D\phi e^{-S} = \sum_{\{C\}} \lambda^{N_3} \sum_j \prod_l (2j_l + 1) \prod_T \{6j\} \]

where the products extend to all tetrahedra T, all edges l, and the summation extend to all the representations \(\{j\}\), all the simplicial complexes \(\{C\}\) and \(N_3\) is the number of tetrahedra in complex \(C\). This partition function is equivalent, up to renormalization, to Ponzano-Regge state sum applied to some triangulation of 3-dimensional manifold. The underlying mathematical structure is a topological lattice gauge theory, it has the advantage that is topological invariant. In order to prove it, Boulatov used the Alexander moves, by which one complex, and the corresponding partition function is topological invariant.

7. The 4-dimensional Ooguri’s model (1992)

The 3-dimensional Boulatov model paved the way to the Ooguri’s model in four dimensions. Let \(\phi\) be a real valued function of four variables \(\phi(g_1, g_2, g_3, g_4)\) on \(G\) \((g_i \in G)\) a compact group. For simplicity we take \(G = SU(2)\). We require \(\phi\) to be invariant under the right action of \(G\), and by cyclic permutation of these variables. The Peter-Weyl theorem, we can expand \(\phi\) in terms of these representations and the 3j-symbols. We define the action

\[ S = \frac{1}{2} \int \prod d g_i \phi^2 (g_1 g_2 g_3 g_4) + \lambda \int \prod_{i=1}^{10} d g_i \phi (g_1 g_2 g_3 g_4) \times \phi (g_4 g_5 g_6 g_7) \phi (g_7 g_8 g_9 g_10) \phi (g_9 g_6 g_2 g_1) \phi (g_10 g_8 g_5 g_1) \]

where the first term (the kinematical term) represents the coupling of a tetrahedron with itself because each element \(g_i\) is associated to each face of the tetrahedron, and the second term (the interacting term) represents gluing faces of five tetrahedra to make a four-simplex. Substituting the Fourier expansion into the action we can integrate out the group variable, and then the action can be used to calculate a partition function as a Feynman path integral with respect to this action:

\[ Z = \int D M e^{-S(M)} = \sum_{C} \lambda^{N_4} \sum_{\{j\}} \prod_t (2j_t + 1) \prod_T \{6j\} \prod_S \{15j\} \]

where the integral is defined in terms of the Fourier coefficients \(M\), appearing in the action and in the measure, the first sum is over all complexes \(C\) (four-dimensional combinatorial manifolds), \(N_4(C)\) is the number of 4-simplices in
the second summation is over all irreducible representations of \(SU(2)\) with angular momentum \(j\); \(t, T\) and \(S\) are the triangles, tetrahedra and 4-simplexes respectively appearing in the complex. Ooguri also proved that the partition function is topological invariant under the Alexander moves. As in the Boulatov model two complexes are combinatorially equivalent if and only if they are connected by a sequence of transformations called Alexander moves.

8. The Barrett-Crane model (1998)

A more abstract approach was taken by Barrett and Crane \cite{5} generalizing Penrose’s spin networks to four dimensions. The novelty of this model consists in the association of representations of \(SO(4)\) group to the faces of the tetrahedra, instead of the edges. They decompose a triangulation of a 4-dimensional manifold into 4-simplices, the geometrical properties of which are characterized in terms of bivectors.

A geometric 4-simplex in Euclidean space is given by the embedding of an ordered set of 5 points in \(\mathbb{R}^4(0, x, y, z, t)\) which is required to be non-degenerate (the points should not lie in any hyperplane). Each triangle in it determines a bivector constructed out of the vectors for the edges. Barrett and Crane proved that classically, a geometric 4-simplex in Euclidean space is completely characterized (up to parallel translation an inversion through the origin) by a set of 10 bivectors \(b_i\), each corresponding to a triangle in the 4-simplex and satisfying the following properties:

i) the bivector changes sign if the orientation of the triangle is changed;

ii) each bivector is simple, i.e. is given by the wedge product of two vectors for the edges;

iii) if two triangles share a common edge, the sum of the two bivector is simple;

iv) the sum (considering orientation) of the 4 bivectors corresponding to the faces of a tetrahedron is zero;

v) for six triangles sharing the same vertex, the six corresponding bivectors are linearly independent;

vi) the bivectors (thought of as operators) corresponding to triangles meeting at a vertex of a tetrahedron satisfy \(\text{tr } b_1 [b_2, b_3] > 0\) i.e. the tetrahedron has non-zero volume.
Then Barrett and Crane define the quantum 4-simplex with the help of bivectors thought as elements of the Lie algebra $SO(4)$, associating a representation to each triangle and a tensor to each tetrahedron. The representations chosen should satisfy the following conditions corresponding to the geometrical ones:

i) different orientations of a triangle correspond to dual representations;

ii) the representations of the triangles are “simple” representations of $SO(4)$, i.e. $j_1 = j_2$;

iii) given two triangles, if we decompose the pair of representations into its Clebsch-Gordan series, the tensor for the tetrahedron is decomposed into summands which are non-zero only for simple representations;

iv) the tensor for the tetrahedron is invariant under $SO(4)$.

Now it is easy to construct an amplitude for the quantum 4-simplex. The graph for a relativistic spin network is the 1-complex, dual to the boundary of the 4-simplex, having five 4-valent vertices (corresponding to the five tetrahedra), with each of the ten edges connecting two different vertices (corresponding to the ten triangles of the 4-simplex each shared by two tetrahedra). Now we associate to each triangle (the dual of which is an edge) a simple representation of the algebra $SO(4)$ and to each tetrahedra (the dual of which is a vertex) a intertwiner; and to a 4-simplex the product of the five intertwiner with the indices suitable contracted, and the sum for all possible representations. The proposed state sum suitable for quantum gravity for a given triangulation (decomposed into 4-simplices) is

$$Z_{BC} = \sum_J \prod_{\text{triang.}} A_{\text{tr}} \prod_{\text{tetrahedra}} A_{\text{tetr.}} \prod_{\text{4-simplices}} A_{\text{simp.}}$$

where the sum extends to all possible values of the representations $J$. 

![Diagram of a 4-simplex](attachment:4-simplex.png)
In order to know the representation attached to each triangle of the tesselation, we take the unitary representation of SO(4) in terms of Euler angles.

\[ U(\varphi, \theta, \tau, \alpha, \beta, \gamma) = R_3(\varphi) R_2(\theta) S_3(\tau) R_3(\alpha) R_2(\beta) R_3(\gamma) \]

where \( R_2 \) is the rotation matrix in the \((x_1x_2)\) plane, \( R_3 \) the rotation matrix in the \((x_1x_3)\) plane and \( S_3 \) the rotation (“boost”) in the \((x_3x_4)\) plane. In the angular momentum basis, the action of \( S_3 \) is follows

\[ S_3(\tau) \psi_{jm} = \sum_j d_{jj'lm}^{ij,jm}(\tau) \psi_{j'm} \]

where

\[ d_{jj'lm}^{ij,jm}(\tau) = \sum_{m_1+m_2=m} \langle j_1j_2m_1m_2 | jm \rangle e^{-i(m_1-m_2)\tau} \langle j_1j_2m_1m_2 | j'm \rangle \]

is the Biedenharn-Dolginov function \[8\].

9. Evaluation of the state sum for the 4-dimensional spin network

In order to evaluate the state sum for a particular triangulation of the total \( \mathbb{R}^4 \) space by 4-simplices, we assign an element \( h_k \in SU(2) \) to each tetrahedron of the 4-simplex \((k = 1, 2, 3, 4, 5)\) and a representation \( \rho_{kl} \) of \( SO(4) \) to each triangle shared by two tetrahedra. From this triangulation we obtain an 2-complex by the dual graph where one vertex corresponds to a tetrahedron and an edge corresponds to triangle, with the ends of the edges identified with the vertices. Then we attach a representation of \( SU(2) \), \( \rho(h_k) \) and \( \rho(h_l) \) to the vertices \( k \) and \( l \) and contract both representation along the edges \((k,l) \equiv e\), giving

\[ Tr \rho(h_k) \rho(h_l^{-1}) = Tr \rho_{kl}(h_k h_l^{-1}) \]

where \( \rho_{kl} \) is the representation of \( SO(4) \) corresponding to the product \( h_k h_l^{-1} \), the left and right components of the \( SO(4) \) group. The state sum for the 2-dimensional complex (the Feymann graph of the model) is obtained by taking the product for all the edges of the graph and integrating for all the copies of \( SU(2) \)

\[ I = \int_{h \in SU(2)^5} \prod h_k h_l^{-1} \]

Due to the trace condition this expression is invariant under left and right multiplication of some elements of \( SU(2) \). \[3\]
For the representation $\rho_{kl}$ we choose the spherical function with respect to the identity representation. Given a completely irreducible representation of the group $G : g \to T_g$ on the space $R$, we define the spherical function with respect to the finite irrepr. of the subgroup $K$

$$f_k(g) = Tr \left\{ E_k T_g \right\}$$

where $E_k$ is a projector of $R$ onto the space $R_k$ of $K$

We take for $G \equiv SO(4)$ the simple representation $(j_1 = j_2)$ and for the subgroup $SU(2)$ the identity representation $k = 0$. Since $f_k$ is invariant under $K$ we can restrict the unitary representations to those of the boost $S_3(\tau)$. With the help of the Biedenharn-Dolginov function it can be proved

$$f_0(\tau) = Tr \left\{ E^0 S_3(\tau) \right\} = \frac{\sin(2j_1 + 1) \tau}{\sin \tau}$$

With this formula it is still possible to give a geometrical interpretation of the probability amplitude encompassed in the trace. In fact the spin dependent factor appearing in the exponential of the spherical function

$$e^{i(2j_{kl} + 1) \tau_{kl}}$$

corresponding to the two tetrahedra $k, l$ intersecting the triangle $kl$, can be interpreted as the product of the angle between the two vectors $h_k, h_l$, perpendicular to the triangle, and the area $A_{kl} = 2j_{kl} + 1$ of the intersecting triangle, $j_{kl}$ being the spin corresponding to the representation $\rho_{kl}$ associated to the triangle $k, l$.

Substituting this value in the state sum, we obtain

$$I = \prod_{h \in SU(2)} \frac{1}{\sin \tau_{kl}} \exp \left( i \sum_{\text{triangle } kl} A_{kl} \tau_{kl} \right)$$

where the product extend to all tetrahedra with the vector $h$ perpendicular to the subspace where the tetrahedra is embedded, and summation is extended to all the triangle $k, l$ intersected by two tetrahedra $k$ and $l$. The exponential term correspond to the Regge action, that in the asymptotic limit becomes the Hilbert-Einstein action [6].

Because we are interested in the physical and mathematical properties of the Barrett-Crane model, we mention some recent work about this model combined with the matrix model approach of Boulatov and Ooguri [11]. In this work the 2-dimensional quantum space-time emerges as a Feymann graph, in the manner of the 4 dimensional matrix models. In this way a spin foam model is connected to the Feyman diagram of quantum gravity.
10. The Lorentzian spin foam model

Now we apply the same technique to calculate the state sum invariant under the Lorentz group that we have used in the case of the SO(4) group for the Barrett-Crane model.

The unitary irreducible representation of the SL(2,C) group for the principal series is given by the formula [7]

\[
(T^{|m,\rho|}_{g}\psi)(z) = (\beta z + \delta)^m \psi(z) e^{i\rho z}
\]

where \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2,C) \), \( m \), integer, \( \rho \) real \( \psi(z) \in L^2(\mathbb{C}) \). The numbers \( m, \rho \) determine the eigenvalues of the representation

\[
C_1 = -\frac{m^2 - \rho^2 - 4}{2} , \quad C_2 = m\rho
\]

In order to calculate the state sum we need the spherical functions of the irrep of SL(2,C). These are given in terms of the Biedenharn-Dolginov function that corresponds to the boost operator

\[
d^{[m,\rho]}_{J,J'}(\tau) = \int_{-\infty}^{\infty} d^{J-1}_{J-M} P^{(M-m,M+m)}_{J-M} (\lambda, \rho) e^{-i\tau \lambda} \times \\
\times d^{J-1}_{J'-M} P^{(M-m,M+m)}_{J'-M} (\lambda, \rho) \omega(\lambda) d\lambda
\]

where \( J, M \) are the angular momentum eigenvalues, \( d_n^{(\alpha,\beta)} \) are some normalization constant, and \( P_n^{(\alpha,\beta)} \) are the Hahn polynomials of imaginary argument [14]. Given the unitary representation \( T_g \) of the group SL(2,C) and the identity representation of SU(2), the spherical functions is defined as in the case of SO(4)

\[
f_0(\tau) = Tr \{ E^0 T_g \} = d^{00}_{000} (\tau) = \frac{1}{\rho} \frac{\sin \rho \tau}{\rho \sinh \tau}
\]

where the last step has been calculated with the residue theorem.

11. A SO(3,1) invariant for the state sum of spin foam model

As in the case of euclidean SO(4) invariant model, we take a non degenerate finite triangulation of a 4-manifold. We consider the 4-simplices in the homogeneous space \( SO(3,1)/SO(3) \sim H_3 \), the hyperboloid \( \{ x | x \cdot x = 1, x^0 > 0 \} \) and
define the bivectors $b$ on each face of the 4-simplex, that can be space-like, null or timelike ($\langle b, b \rangle > 0, = 0, < 0$, respectively). \[4\]

In order to quantize the bivectors, we take the isomorphism $b = * L \left( b^{abcd} L^e_{abcd} \right)$ with $g$ a Minkowski metric.

The condition for $b$ to be a simple bivector $\langle b, * b \rangle = 0$, gives $C_2 = \langle L, * L \rangle = \vec{J} \cdot \vec{K} = m \rho = 0$

We have two cases:

1) $\rho = 0$, $C_1 = \langle L, L \rangle = \vec{J}^2 - \vec{K}^2 = m^2 - 1 > 0$; $L$, space-like, $b$ time-like,

2) $m = 0$, $C_1 = \vec{J}^2 - \vec{K}^2 = - \rho^2 - 1 < 0$; $L$, time like, $b$ space like (remember, the Hodge operator $*$ changes the signature)

In case 2) $b$ is space-like, $\langle b, b \rangle > 0$. Expanding this expression in terms of space like vectors, $x, y$,

\[ b_{\mu \nu} b^{\mu \nu} = (x_{\mu} y_{\nu} - x_{\nu} y_{\mu}) (x^{\mu} y^{\nu} - x^{\nu} y^{\mu}) = \|x\|^2 \|y\|^2 - \|x\|^2 \|y\|^2 \cos^2 \eta (x, y) = \|x\|^2 \|y\|^2 \sin^2 \eta (x, y) \]

where $\eta (x, y)$ is the Lorentzian space-like angle between $x$ and $y$; this result gives a geometric interpretation between the parameter $\rho$ and the area expanded by the bivector $b = x \wedge y$, namely, $\langle b, b \rangle = \text{area} \{ x, y \} = \langle * L, * L \rangle \cong \rho^2$. (This result is equivalent to that obtained in the euclidean case where the area of the triangle expanded by the bivector was proportional to the value $(2j + 1)$, $j$ being the spin of the representation).

In order to construct the Lorentz invariant state sum we take a non-degenerate finite triangulation in a 4-dimensional simplices in such a way that all 3-dimensional and 2-dimensional subsimplices have space-like edge vectors which span space-like subspace. We attach to each 2-dimensional face a simple irrep. of $SO(3, 1)$ characterized by the parameter $[0, \rho]$.

The state sum is given by the expression \[10\]

\[ Z = \int_{\rho=0}^{\infty} d\rho \prod_{\text{triang}} \rho^2 \prod_{\text{tetra}} \Theta_4 \left( \rho'_1, \cdots, \rho'_4 \right) \prod_{\text{4-simplex}} I_{10} \left( \rho''_1, \cdots, \rho''_{10} \right) \]

where $\rho$ refers to all the faces in the triangulation, $\rho'$ corresponds to the simple irrep attached to 4 triangles in the tetrahedra and $\rho''$ corresponds to the simple irrep attached to the 10 triangles in the 4-simplices. The functions $\Theta_4$ and $I_{10}$ are defined as traces of recombination diagrams for the simple representations. The traces are explicitly given as multiple integrals on the upper sheet $H$ of the 2-sheeted hyperboloid in Minkowski space. For the integrand we take the spherical
function
\[ f_p(x, y) = \frac{1}{\rho} \frac{\sin \rho \tau(x, y)}{\sin h \tau(x, y)} \]

Where \( \tau(x, y) \) is the hyperbolic distance between \( x \) and \( y \).

The trace of a recombination diagram is given by a multiple integral of products of spherical functions.

For a tetrahedron we have
\[ \Theta_4(\rho'_1, \cdots, \rho'_4) = \frac{1}{2\pi^2} \int_H f_{\rho_1} (x, y) \cdots f_{\rho_4} (x, y) \, dy \]

where we have dropped one integral for the sake of normalization without losing Lorentz symmetry.

For a 4-simplex we have
\[ I_{10}(\rho'_1, \cdots, \rho'_4) = \frac{1}{2\pi^2} \int_{H^4} \prod_{i<j<1,5} f_{\rho_{ij}}(x_i, x_j) \, dx_1 dx_2 dx_3 dx_4 \]

The last 4 equation defines the state sum completely, that has been proved to be finite \[6\].

The asymptotic properties of the spherical functions are related to the Einstein-Hilbert action giving a connection of the model with the theory of general relativity. \[13\]

**Acknowledgements**

The author wants to express gratitude to the organizers of the Workshop in particular, to Professors Odzijewicz and Golinsk for the invitation to give this review talk. This work was supported partially by M.E.C. (Spain) through a grant BFM 2003-00313/FIS.
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