A sufficient condition for a locally compact almost simple group to have open monolith

Colin D. Reid

February 19, 2019

Abstract

We obtain a sufficient condition, given a totally disconnected, locally compact group $G$ with a topologically simple monolith $S$, to ensure that $S$ is open in $G$ and abstractly simple.

Acknowledgement  The impetus for writing this note was a question of Waltraud Lederle, as well as some questions arising from [1] and from an ongoing project with Alejandra Garrido and David Robertson. I thank Waltraud Lederle and my collaborators for their insightful questions and comments.

We recall some definitions from [1].

Definition. Let $G$ be a totally disconnected, locally compact (t.d.l.c.) group. We say $G$ is expansive if there is a neighbourhood $U$ of the identity in $G$ such that $\bigcap_{g \in G} gUg^{-1} = \{1\}$. The group $G$ is regionally expansive if there is a compactly generated open subgroup $O$ of $G$ such that $O$ is expansive; equivalently, every open subgroup containing $O$ is expansive.

A topological group $G$ is monolithic if there is a unique smallest nontrivial closed normal subgroup of $G$, called the monolith $\text{Mon}(G)$ of $G$. A t.d.l.c. group $G$ is robustly monolithic if it is monolithic and the monolith is nondiscrete, regionally expansive, and topologically simple.

Note that every topologically simple t.d.l.c. group $S$ is expansive, and hence if $S$ is compactly generated, then it is regionally expansive. Thus in the context of topologically simple groups, ‘regionally expansive’ should be considered a generalization of ‘compactly generated’. The definition of ‘robustly monolithic’ allows us to consider a more general situation, where the regionally expansive topologically simple group $S$ is embedded as a closed normal subgroup in some larger t.d.l.c. group $G$, such that $C_G(S) = \{1\}$. It is then natural to ask how complex the quotient $G/S$ can be as a topological group.

Here is a sufficient condition for $G/S$ to be discrete, in other words, for $S$ to be open in $G$; in the situation described, in fact $S$ is abstractly simple.

Theorem 1. Let $G$ be a robustly monolithic t.d.l.c. group. Suppose that $G$ has an open subgroup of the form $K \times L$ where $K$ and $L$ are nontrivial closed subgroups of $G$. Then for every nontrivial subgroup $H$ of $G$ such that $\text{Mon}(G) \leq N_G(H)$, without assuming
that $H$ is closed, it follows that $H$ is open in $G$ and contains $\text{Mon}(G)$. In particular, $\text{Mon}(G)$ itself is abstractly simple and open in $G$.

The proof is based on the local structure theory developed in [2], [3] and [1]; we briefly recall the necessary background.

**Definition.** We define the quasi-centralizer $QC_G(H)$ of a subgroup $H$ of a topological group $G$ to be the set of elements $g \in G$ such that $g$ commutes with some open subgroup of $H$, and write $QZ(G) := QC_G(G)$. We say $G$ is $[A]$-semisimple if $QZ(G) = \{1\}$, and whenever $A$ is an abelian subgroup of $G$ with open normalizer, then $A = \{1\}$.

Given an $[A]$-semisimple t.d.l.c. group $G$, the (globally defined) centralizer lattice of $G$ is the set

$$LC(G) = \{C_G(K) \mid K \leq G, N_G(K) \text{ is open in } G\},$$

equipped with the partial order of inclusion of subsets of $G$. Within $LC(G)$, the (globally defined) decomposition lattice $LD(G)$ consists of those $K \in LC(G)$ such that $KC_G(K)$ is open in $G$.

By [1, Proposition 5.1.2], every robustly monolithic t.d.l.c. group $G$ is $[A]$-semisimple, so the definitions above apply. Note that if $G$ is $[A]$-semisimple, then so is every open subgroup of $G$.

By construction, given $K \in LC(G)$, then $K$ is closed in $G$ and $N_G(K)$ is open in $G$. It is shown in [2] that $LC(G)$ is a Boolean algebra, on which the map $K \mapsto C_G(K)$ is the complementation map. The centralizer lattice is a local invariant of $G$, in the sense that if $O$ is any open subgroup of $G$, then $LC(O)$ is $O$-equivariantly isomorphic to $LC(O)$ via the map $K \mapsto K \cap O$. The decomposition lattice is a local invariant of $G$ in the same manner; in particular, it accounts for all direct factors of open subgroups of $G$. The decomposition lattice has the following additional property:

($\ast$) Given $A_1, \ldots, A_n \in LD(G)$ with least upper bound $A$ in $LD(G)$, and given open subgroups $B_i$ of $A_i$, then as a subset of $A$, the product $B_1B_2\ldots B_n$ is a neighbourhood of the identity.

(To see why ($\ast$) holds, note that we can choose compact open subgroups $B_i$ of $A_i$ that normalize each other, so that $B = B_1B_2\ldots B_n$ is a compact subgroup of $G$; the fact that $B$ is an open subgroup of an element of $LD(G)$ then follows by [2, Theorem 4.5].)

There is a natural action of $G$ on $LC(G)$ by conjugation, which preserves the partial order and hence the Boolean algebra structure; note also that the stabilizers of this action are open. There is then a corresponding continuous action of $G$ by homeomorphisms on the Stone space $\mathcal{S}(LC(G))$ of $LC(G)$, which is a compact zero-dimensional Hausdorff space. The latter action has useful dynamical properties. Given a group $G$ acting on a topological space $X$, we say the action is **minimal** if every orbit is dense, and **compressible** if there is a nonempty open subset $Y$ such that for every nonempty open subset $Z$ of $X$, there is $g \in G$ such that $gY \subseteq Z$.

**Lemma 2.** Let $G$ be a robustly monolithic t.d.l.c. group and let $A$ be a $G$-invariant subalgebra of $LC(G)$. Then the $G$-action on $\mathcal{S}(A)$ is continuous and the $\text{Mon}(G)$-action is minimal and compressible.
Proof. The action is continuous because stabilizers of elements of $\mathcal{A}$ are open. By [1, Theorem 7.3.3], the action of $\text{Mon}(G)$ on $\mathfrak{S}(\text{LC}(G))$ is minimal and compressible; these properties pass to any quotient $\text{Mon}(G)$-space.

We can now prove the theorem.

Proof of Theorem 1. As noted above, $G$ is $[A]$-semisimple, and hence the centralizer lattice $\text{LC}(G)$ is a Boolean algebra, with $G$-invariant subalgebra $\text{LD}(G)$. The condition that $G$ has an open subgroup that splits nontrivially as a direct product then amounts to the condition that $|\text{LD}(G)| > 2$. Applying Stone duality, the $G$-space $X_C := \mathfrak{S}(\text{LC}(G))$ admits a $G$-equivariant quotient space $X_D := \mathfrak{S}(\text{LD}(G))$ with $|X_D| > 1$.

Let $S = \text{Mon}(G)$. By Lemma 2, the action of $G$ on $X_D$ is continuous and the action of $S$ is minimal and compressible. In particular, $S$ acts nontrivially on $X_D$. Since $S$ is topologically simple and the action is continuous, in fact $S$ acts faithfully on $X_D$, so the action of $G$ on $X_D$ is also faithful.

Claim: Let $H$ be a nontrivial subgroup of $G$, such that $N_G(H)$ is open in $G$ and contains $S$ (do not assume that $H$ is closed). Then $H$ is open in $G$ and $S \leq H$.

We begin the proof of the claim with some reductions. By [1, Lemma 5.1.4], the normalizer $N_G(H)$ is robustly monolithic with monolith $S$. Since $N_G(H)$ is open in $G$, it also has an open subgroup that splits nontrivially as a direct product. Thus we may assume without loss of generality that $G = N_G(H)$. Since $H$ and $S$ are normal, we have $[H, S] \leq H \cap S$. Since $C_G(S) = \{1\}$, it follows that $H$ and $S$ have nontrivial intersection; thus we may replace $H$ with $H \cap S$ and assume that $H$ is a normal subgroup of $S$. Since $S$ is topologically simple, it follows that $H$ is dense in $S$. By continuity, we then see that the action of $H$ on $X_D$ is minimal and compressible.

Since $|X_D| > 1$, $X_D$ is zero-dimensional, and the action is compressible, there is a nonempty clopen subset $\alpha$ of $X$ and $h \in H$ such that $h\alpha$ is properly contained in $\alpha$. Correspondingly, there is $K \in \text{LD}(G)$ and $h \in H$, such that $hKh^{-1}$ is a subgroup of $K$ that is closed but not open in $K$. It then follows that $K$ has an open subgroup of the form $hKh^{-1} \times L$ where $L = C_K(hKh^{-1})$; in turn, $L$ is a nontrivial element of $\text{LD}(G)$. By [4, Proposition 5.1], we have $\text{con}_G(h) \leq H$, where

$$\text{con}_G(h) := \{g \in G \mid h^ngh^{-n} \to 1 \text{ as } n \to +\infty\}.$$ 

By [3, Proposition 6.14], the intersection $L^* = \text{con}_G(h) \cap L$ is open in $L$. Let $\beta$ be the clopen subset of $X_D$ corresponding to $L \in \text{LD}(G)$. Since $X_D$ is compact and the action of $H$ on $X_D$ is minimal, there are $h_1, \ldots, h_n \in H$ such that $X_D = \bigcup_{i=1}^n h_i\beta$. By (*) it follows that $\langle h_iL^*h_i^{-1} \mid 1 \leq i \leq n \rangle$ is open in $G$; thus $H$ is open in $G$. In particular, $H$ is closed in $S$; since $H$ is dense in $S$, it follows that $H = S$. This completes the proof of the claim.

The claim applies in particular when $S = H$, so $S$ is open in $G$. Thus the assumption that $N_G(H)$ is open follows automatically from assuming that $N_G(H)$ contains $S$. We have therefore shown that given a nontrivial subgroup $H$ of $G$ such that $S \leq N_G(H)$, then $H$ is open in $G$ and $S \leq H$. As this conclusion applies in particular to any nontrivial normal subgroup of $S$, we see that $S$ is abstractly simple. □
References

[1] P.-E. Caprace, C. D. Reid and P. R. Wesolek, Approximating Simple Locally Compact Groups by Their Dense Locally Compact Subgroups. Int. Math. Res. Not. (2019), rny298, https://doi.org/10.1093/imrn/rny298

[2] P.-E. Caprace, C. D. Reid and G. A. Willis, Locally normal subgroups of totally disconnected groups; Part I: General theory. Forum Math. Sigma 5 (2017), e11, 76pp.

[3] P.-E. Caprace, C. D. Reid and G. A. Willis, Locally normal subgroups of totally disconnected groups; Part II: Compactly generated simple groups. Forum Math. Sigma 5 (2017), e12, 89pp.

[4] P.-E. Caprace, C. D. Reid and G. A. Willis, Limits of contraction groups and the Tits core. J. Lie Theory 24 (2014), 957–967.