A fundamental ingredient in wormhole physics is the presence of exotic matter, which involves the violation of the null energy condition. In this context, we investigate the possibility that wormholes could be supported by quark matter at extreme densities. Theoretical and experimental investigations of the structure of baryons show that strange quark matter, consisting of the u, d and s quarks, is the most energetically favorable state of baryonic matter. Moreover, at ultra-high densities, quark matter may exist in a variety of superconducting states, namely, the Color-Flavor-Locked (CFL) phase. Motivated by these theoretical models, we explore the conditions under which wormhole geometries may be supported by the equations of state considered in the theoretical investigations of quark-gluon interactions. For the description of the normal quark matter we adopt the Massachusetts Institute of Technology (MIT) bag model equation of state, while the color superconducting quark phases are described by a first order approximation of the free energy. By assuming specific forms for the bag and gap functions, several wormhole models are obtained for both normal and superconducting quark matter. The effects of the presence of an electrical charge are also taken into account.

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I. INTRODUCTION

A fundamental property in wormhole physics, in the context of classical general relativity, is that these exotic geometries are supported by “exotic matter” [1], which involves a stress-energy tensor $T_{\mu\nu}$ that violates the null energy condition (NEC), i.e., has $T_{\mu\nu}k^{\mu}k^{\nu} < 0$ at the wormhole throat and its neighbourhood, where $k^{\mu}$ is any null vector [1, 2]. A wide variety of solutions have been obtained since the seminal Morris-Thorne paper [1], ranging from dynamic wormhole geometries [2], rotating solutions [3], thin-shell wormholes constructed using the cut-and-paste technique [4], observational signatures using thin accretion disks [5], solutions in conformal symmetry, which presents a more systematic approach in searching for exact wormhole solutions [6], wormhole geometries in the semi-classical regime [8], and more recently in modified theories of gravity [9, 10].

In the modified gravity context, it was shown that the normal matter threading the wormhole can be constrained to satisfy the null energy condition, and it is the higher order curvature terms, interpreted as a gravitational fluid, that sustain these non-standard wormhole geometries, fundamentally different from their counterparts in general relativity. It has also been argued that wormhole solutions can be supported by several dark energy models responsible for the late-time cosmic acceleration [11], by imposing specific equations of state. In this work, we explore the possibility that wormholes could be supported by quark matter at extreme densities.

This approach is motivated by theoretical and experimental investigations of baryonic structure showing that strange quark matter, consisting of the u (up), d (down) and s (strange) quarks is the most energetically favorable state of baryon matter. The idea of the existence of stars made of quarks was initially introduced in [12] and [13]. Two ways of formation of stellar strange matter have been proposed in [14] and [15]: the quark-hadron phase transition in the early universe, and the conversion of neutron stars into strange ones at ultrahigh densities. In the theories of strong interactions the quark bag models suppose that the breaking of physical vacuum takes place inside hadrons. As a result the vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure $B$ on a bag wall equilibrates the pressure of quarks thus stabilizing the system [15].

The structure of a realistic strange star is very complicated but its basic properties can be described as follows [15]. Beta-equilibrated strange quark-star matter consists of an approximately equal mixture of u, d and s quarks, with a slight deficit of the latter. The Fermi gas of $3A$ quarks constitutes a single color-singlet baryon with baryon number $A$. This structure of the quarks leads to a net positive charge inside the star. Since stars in their lowest energy state are supposed to be charge neutral, electrons must balance the net positive quark charge in strange matter stars [15].

However, the electrons, being bound to the quark mat-
ter by the electromagnetic interaction only (and not by the strong force), are able to displace freely across the quark surface. But they cannot move to infinity because of the electrostatic interaction with quarks. The electron distribution extends up to $10^3$ fm above the quark surface. The Coulomb barrier at the quark surface of a hot strange star could represent a powerful source of electron-positron ($e^+e^-$) pairs \[10\], which are created in the extremely strong electric field of the barrier. At surface temperatures of around $10^{11}$ K, the luminosity of the quark star surface may be of the order $10^{51}$ ergs$^{-1}$ \[17\]. Moreover, due to both photon emission and $e^+e^-$ pair production, for about $8.6 \times 10^8$ s for normal quark matter and for up to around $3 \times 10^9$ s for superconducting quark matter, the thermal luminosity from the quark star surface may be orders of magnitude higher than the Eddington limit \[18\].

The existence of a large variety of color superconducting states of quark matter at ultra-high densities has also been suggested and intensively investigated \[19-22\]. At very high densities, matter is expected to form a degenerate Fermi gas of quarks in which the quark Cooper pairs with very high binding energy condense near the Fermi surface. This phase of the quark matter is called a color superconductor. Such a state is significantly more bound than ordinary quark matter. This implies that at extremely high density the ground state of quark matter is the superconducting Color-Flavor-Locked (CFL) phase, and that this phase of matter rather than nuclear matter may be the ground state of hadronic matter \[22\]. The existence of the CFL phase can enhance the possibility of the existence of a pure stable quark star \[22\].

In this context, the possibility that stellar mass black holes, with masses in the range of $3.8M_\odot$ and $6M_\odot$, respectively, could be in fact quark stars in the CFL phase was considered in \[23\]. Depending on the value of the gap parameter, rapidly rotating CFL quark stars can achieve much higher masses than standard neutron stars, thus making them possible stellar mass black hole candidates. Moreover, quark stars have a very low luminosity and a completely absorbing surface – the infalling matter on the surface of the quark star is converted into quark matter.

It is the purpose of the present paper to investigate the possibility that wormhole geometries can be realized by using quark matter, in both normal and superconducting phases. To describe quark matter we adopt the Massachusetts Institute of Technology (MIT) bag model equation of state, while for the investigation of the superconducting quark matter we consider the equation of state obtained in a first order expansion of the free energy of the system. Generally the equations of state depend on several parameters, of which the most important are the bag and the gap constant. The bag constant forces the quarks to remain confined inside the baryons, while the gap constant describes the superconducting properties of the quark matter. However, in high density systems, which can be achieved, for example, in the interior of neutron stars, both the bag and the gap constants, as well as the quark masses, become effective, density dependent functions. It is exactly this property of strongly interacting systems in dense media we will exploit in order to obtain wormhole solutions of the static, spherically symmetric gravitational field equations in the presence of quark matter. By appropriately choosing the forms of the bag and gap functions several wormhole type solutions of the gravitational field equations are obtained, with the matter source represented by normal and superconducting quark matter, respectively.

The present paper is organized as follows. In Section II the quark matter equations of state are presented. In Section III we explore the conditions under which wormhole geometries may be supported by the equations of state considered in the theoretical investigations of quark-gluon interactions. We discuss and conclude our results in Section IV.

II. QUARK MATTER EQUATIONS OF STATE

The state of matter at extreme densities represents one of the most important subjects of study in present day physics. The problem is complicated, not only from the theoretical point of view, but also by the fact that laboratory experiments cannot provide the necessary data for a full understanding of the question. In order to test our understanding of the relevant physics we need to turn to astrophysics, and the dynamics of compact general relativistic objects. In fact, “neutron stars” represent unique laboratories of such extreme physics. With core densities reaching about one order of magnitude beyond nuclear saturation, they are likely to contain exotic states of matter like hyperon phases with net strangeness and/or deconfined quarks \[24\].

The theory of the equation of state of quark matter is directly based on the fundamental Quantum Chromodynamics (QCD) Lagrangian, given by \[25\]

$$L_{QCD} = \frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi} \left( i\gamma^\mu \partial_\mu - g\gamma^\mu A^a_\mu \frac{\lambda^a}{2} - m_f \right) \psi,$$  \[1\]

where the subscript $f$ denotes the various quark flavors $u, d, s, c$ etc., $g$ is the coupling constant, and $A^a_\mu$ is the vector potential taking values in the Lie algebra with generators $\lambda^a$. The nonlinear gluon field strength $F_{\mu\nu}^a$ is given by

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu.$$  \[2\]

QCD predicts a weakening of the quark-quark interaction at short distances (or high momenta $Q^2$), because the one-loop series for the gluon propagator yields a running coupling constant \[25\]

$$\alpha_s(Q^2) = \frac{16\pi^2}{(11 - 2N_f/3)\ln|Q^2/\Lambda^2|}.$$  \[3\]
where $N_f$ is the number of active quark flavors and the QCD scale parameter $\Lambda \approx 200$ MeV. The coupling constant $g^2 (Q^2)$ vanishes for high momenta $Q^2$, and tends to infinity for $N_f \to 33/2$.

A. The MIT bag model equation of state

Assuming that interactions of quarks and gluons are sufficiently small, the energy density $\varepsilon$ and pressure $P$ of a quark-gluon plasma at temperature $T$ and chemical potential $\mu$ can be calculated by thermal theory. Neglecting quark masses in first order perturbation theory, the equation of state is [25]

$$\varepsilon = \left(1 - \frac{15}{4\pi} \alpha_s \right) \frac{8\pi^2}{15} T^4 + N_f \left(1 - \frac{50}{21\pi} \alpha_s \right) \frac{7\pi^2}{10} T^4 + \sum_f 3 \left(1 - 2\frac{\alpha_s}{\pi}\right) \left(\pi^2 T^2 + \frac{\mu_f^2}{2}\right) \frac{\mu_f^2}{\pi^2} + B,$$

or

$$\varepsilon = \sum_{i=u,d,s,c,e-} \varepsilon_i + B,$$

where $\alpha_s$ is the strong interaction coupling constant, and $B$ is the difference between the energy density of the perturbative and non-perturbative QCD vacuum (the bag constant). The thermodynamic parameters of the quark-gluon plasma are related by the equation of state of the quark matter, given by

$$P = \frac{1}{3} (\varepsilon - 4B),$$

or

$$P + B = \sum_{i=u,d,s,c,e-} p_i.$$

The entropy density of the quark-gluon plasma is given by $s = (\partial P/\partial T)_\mu$. Equation (6) is essentially the equation of state of a gas of massless particles with corrections due to the QCD trace anomaly and perturbative interactions. These are always negative, and when $\alpha_s = 0.5$ they reduce the energy density at a given temperature by about a factor of two [23].

Most of the investigations of the stellar quark-gluon plasma have been done under the assumption of the electric charge neutrality of the quark-gluon plasma that reads $\sum_{i=u,d,s,c,e-} q_i n_i = 0$. In the case of a star formed from massless $u$, $d$ and $s$ quarks the charge neutrality condition can be explicitly formulated as $2n_u/3 = (n_d + n_s)/3$ [15].

More sophisticated investigations of quark-gluon interactions have shown that Eq. (4) represents a limiting case of more general equations of state. For example, MIT bag models with massive strange quarks and lowest order QCD interactions lead to some correction terms in the equation of state of quark matter. Models incorporating restoration of chiral quark masses at high densities and giving absolutely stable strange matter can no longer be accurately described by using Eq. (6). If the quark interaction is described by a colour-Debye-screened inter-quark vector potential, originating from gluon exchange, and by a density-dependent scalar potential, which restores chiral symmetry at high density (in the limit of massless quarks) the resulting EOS has asymptotic freedom built in, shows confinement at zero baryon density, and deconfinement at high density. This density-dependent scalar potential arises from the density dependence of the in-medium effective quark masses $m_q$, which are assumed to depend on the baryon number density $n_B$ [26].

On the other hand, in these types of models the equation of state $P = P(\varepsilon)$ can be well approximated by a linear function in the energy density $\varepsilon$ [27]. The linear approximation of the equation of state was studied in [28], and all the parameters of the EOS have been obtained as polynomial functions of the strange quark mass, QCD coupling constant and bag constant.

B. Color Flavor Locked quark matter

It is generally agreed today that the Color-Flavor-Locked state is likely to be the ground state of matter, at least for asymptotic densities, and even if the quark masses are unequal [19–22, 29]. Moreover, the equal number of flavors is enforced by symmetry, and electrons are absent, since the mixture is automatically neutral. The properties of the CFL quark matter depends strongly on the values of the deconfinement phase transition density and the CFL gap parameter, which are poorly known from both a theoretical and experimental point of view. The free energy density $\Omega_{\text{CFL}}$ for quark matter in the CFL phase is given by [30]

$$\Omega_{\text{CFL}}(\mu, \mu_e) = \Omega_{\text{CFL}}^{\text{quarks}}(\mu) + \Omega_{\text{CFL}}^{GB}(\mu, \mu_e) + \Omega_{\text{CFL}}^{\text{electrons}}(\mu_e),$$

where $\Omega_{\text{CFL}}^{GB}$ is the contribution from the Goldstone bosons arising due to the breaking of chiral symmetry in the CFL phase. By assuming that the mass $m_s$ of the $s$ quark is not large compared to the chemical potential $\mu$, the thermodynamical potential of the quark matter in the CFL phase can be approximated as [31]

$$\Omega_{\text{CFL}} = -\frac{3\mu^4}{4\pi^2} + \frac{3m_s^2}{4\pi^2} - \frac{1 - 12\ln (m_s/2\mu)}{32\pi^2} m_s^4,$$

and $\Omega_{\text{CFL}} = -\frac{3}{\pi^2} \Delta^2 \mu^2 + B,$

where $\Delta$ is the gap energy. With the use of this expression the pressure $P$ of the quark matter in the CFL phase can be obtained as an explicit function of the energy den-
density $\varepsilon$ in the form \[31\]

$$P = \frac{1}{3} (\varepsilon - 4B) + \frac{2\Delta^2 \delta^2}{\pi^2} - \frac{m_s^2 \delta^2}{2\pi^2},$$

(10)

where

$$\delta^2 = -\alpha + \sqrt{\alpha^2 + \frac{4}{9} \pi^2 (\varepsilon - B)},$$

(11)

and

$$\alpha = -\frac{m_s^2}{6} + \frac{2\Delta^2}{3}.$$  

(12)

III. FIELD EQUATIONS FOR STATIC AND SPHERICALLY SYMMETRIC WORMHOLES

In this work, motivated by the proposal that at high densities the phases of quark matter are described either by the equation of state of the MIT bag model, or by the CFL phase equation of state, we consider the possibility that wormhole geometries can be supported by quark matter, in both normal and superconducting states.

In the following we assume that the wormhole metric takes the form \[1\]

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(13)

where the metric function $\Phi(r)$ is denoted the redshift function and $b(r)$ the shape function \[1\]. The redshift function $\Phi(r)$ must be finite everywhere to avoid the presence of event horizons \[1\]. In order to have a wormhole geometry, the shape function $b(r)$ must obey the flaring out condition of the throat, which translates as $(b - b'(r))/b' > 0$ \[1\]. At the throat, we have $b(r_0) = r = r_0$, and taking into account the flaring-out condition the inequality $b'(r_0) < 1$ is imposed.

In classical general relativity, taking into account the above-mentioned flaring-out condition, and through the Einstein field equation one deduces that the matter threading the wormhole throat violates the null energy condition (NEC). More specifically, the NEC imposes that $T_{\mu\nu} k^\mu k^\nu \geq 0$, where $k^\mu$ is any null vector. Thus, a fundamental ingredient in wormhole physics, in classical general relativity, is the violation of the NEC, i.e., $T_{\mu\nu} k^\mu k^\nu < 0$ somewhere more specifically, at the wormhole throat ans its vicinity. Matter satisfying the latter condition is denoted as exotic matter.

The field equations are given by the following stress-energy scenario

$$\varepsilon(r) = \frac{1}{8\pi} \frac{b'}{r^2},$$

(14)

$$p_r(r) = \frac{1}{8\pi} \left[ 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} - \frac{b}{r^2} \right],$$

(15)

$$p_t(r) = \frac{1}{8\pi} \left[ \frac{b}{r} \left( \Phi'' + (\Phi')^2 - \frac{b'}{r} + \frac{b'}{r^2} \Phi' - \frac{b'}{2r(r - b)} \right) \right].$$

(16)

where the prime denotes the derivative with respect to the radial coordinate $r$, $\varepsilon(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the tangential pressure, measured in the orthogonal direction to the radial direction, respectively.

Using the conservation of the stress-energy tensor, $T^\mu\nu;\nu = 0$, we obtain the following equation

$$p'_r = \frac{2}{r} (p_t - p_r) - (\varepsilon + p_r) \Phi',$$

(17)

which can be interpreted as the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole.

Note that now we have three independent equations, Eqs. (14)-(16), with five unknown functions of the radial coordinate $r$, i.e., $\Phi(r)$, $b(r)$, $\varepsilon(r)$, $p_r(r)$ and $p_t(r)$. To solve the system, different strategies have been adopted in the literature. For instance, one may model an appropriate spacetime geometry by considering a specific equation of state and impose one of the functions $\Phi(r)$ or $b(r)$, thus closing the system of the coupled differential equations. One may also impose the form of the functions $b(r)$ and $\Phi(r)$ by hand and consequently determine the stress-energy tensor components. Conversely, one could construct a suitable source for the spacetime geometry by imposing the stress-energy components, and consequently determine the metric fields.

In this work, we consider a variant of the the first approach, by choosing one of the quark model equations of state, and exploring specific functions of the radial coordinate that appear in the resulting differential equations, to find specific wormhole solutions.

IV. SPECIFIC SOLUTIONS: WORMHOLE GEOMETRIES SUPPORTED BY THE MIT BAG MODEL EQUATION OF STATE

Despite the fact that the MIT bag model equation of state represents an isotropic pressure, in the context of quark compact spheres, instability inhomogeneities may form as a result of density perturbations. Therefore, the pressure in the MIT bag model equation of state may be regarded a radial pressure, and the tangential pressure is determined through the Einstein field equations.

Thus, taking into account the MIT bag model equation of state, given in the form

$$p_r = \frac{1}{3} (\varepsilon - 4B),$$

(18)

and using the Eqs. (14)-(15), we deduce the following differential equation

$$\Phi'(r) = \frac{r}{2(r - b)} \left[ \frac{b(r)}{r^3} + \frac{b'(r)}{3r^2} - \frac{32\pi}{3} B(r) \right].$$

(19)
Due to the high energy density regime considered in the Introduction, we assume that the factor $B$, which is the difference between the energy density of the perturbative and non-perturbative QCD vacuum, is a function of the radial coordinate, i.e., $B = B(r)$.

A. Constant MIT bag parameter

1. Constant redshift function, $\Phi'(r) = 0$

In this section, we consider a constant MIT bag parameter, i.e., $B = B_0$, in order to gain some insight into the physics involved. First, consider a constant redshift function, $\Phi'(r) = 0$, so that the differential equation, Eq. (19) yields the following solution for the shape function

$$b(r) = \frac{16}{3}\pi Br^3 \left[1 - \left(\frac{r_0}{r}\right)^6\right] + r_0 \left(\frac{r_0}{r}\right)^3 \quad (20)$$

Note that this solution is not asymptotically flat, so that it needs to be matched to an exterior vacuum solution. For instance, consider that the exterior solution is the Schwarzschild spacetime, given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (21)$$

In this case the spacetimes given by the metrics Eqs. (13) and (21) are matched at $a$, and one has a thin shell surrounding the wormhole. Using the the Darmois-Israel formalism [3], the surface stresses are given by

$$\sigma = -\frac{1}{4\pi a} \left(\frac{1}{\sqrt{1 - \frac{2M}{a}}} - \frac{1}{\sqrt{1 - \frac{b(a)}{a}}}\right), \quad (22)$$

$$\mathcal{P} = \frac{1}{8\pi a} \left(\frac{1}{\sqrt{1 - \frac{2M}{a}}} - [1 + a\Phi'(a)]\sqrt{1 - \frac{b(a)}{a}}\right) \quad (23)$$

where $\sigma$ is the surface energy density and $\mathcal{P}$ the surface pressure.

Taking into account the flaring-out condition at the throat, $b'(r_0) < 1$, one arrives at the restriction $32\pi Br_0^2 < 3$, which places an upper bound on the wormhole throat

$$r_0^2 < \frac{3}{32\pi B}. \quad (24)$$

2. Isotropic pressure: $p_t(r) = p_t(r)$

Consider the case of isotropic pressure, $p_t(r) = p_t(r) = p(r)$, so that the conservation equation reduces to $p'(r) = -(e(r) + p(r))\Phi'(r)$, yields the solution

$$p(r) = -B + Ce^{-2\Phi(r)}. \quad (25)$$

The integration constant $C$ is given by $C = (p_0 + B)e^{4\Phi_0}$, where $p_0$ and $\Phi_0$ are the the values of the pressure and redshift function evaluated at the throat. Thus, the isotropic pressure is finally given by

$$p(r) = -B + (p_0 + B)e^{-4\Phi(r) - \Phi_0}. \quad (26)$$

From the field equation (14), one deduces the relationship

$$\Phi(r) = -\frac{1}{4}\ln\left[\frac{b(r)}{8\pi r^2} - \frac{8B}{(p_0 + B)e^{4\Phi_0}}\right]. \quad (27)$$

From this relationship one verifies that for $\Phi(r)$ to be finite, then $b(r) \propto r^3$, so that generically one cannot construct asymptotically flat traversable wormholes with isotropic pressures. Nevertheless, one may match the interior wormhole solution to an exterior vacuum spacetime at a finite junction surface, as mentioned above.

From Eq. (27), one may obtain following generic restriction $b'(r)/(8\pi r^2) - 8B/[(p_0 + B)e^{4\Phi_0}] > 0$, which at the throat reduces to

$$b_0' > \frac{64\pi r_0^2 B}{(p_0 + B)e^{4\Phi_0}}. \quad (28)$$

From the flaring-out condition at the throat, $b'(r_0) < 1$, one obtains the upper bound on the wormhole throat

$$r_0^2 < \frac{64\pi B}{(p_0 + B)e^{4\Phi_0}}. \quad (29)$$

Note that taking into account the violation of the NEC the throat, $(e+p)|_{r_0} < 0$ and the MIT bag model equation of state, one readily deduces the condition $p_0 + B < 0$. However, from the inequality (29), one verifies that the Bag parameter is necessarily negative, in order to have consistent isotropic pressure solutions.

B. Shape function dependent bag function

A careful analysis of the solutions to the differential equation (19), shows several problematic issues related to wormhole physics. First, considering a specific shape function, $b(r)$, one immediately verifies that solving Eq. (19) for $\Phi(r)$ produces solutions with event horizons, i.e., $\Phi(r) \propto \ln(1 - b(r)/r)$, rendering the wormhole non-traversable. This difficulty arises due to the factor $(1 - b(r)/r)$ in the denominator in Eq. (19).

Now, in order to avoid the presence of event horizons, one may choose a suitable bag function $B(r)$ of the form

$$\frac{32\pi}{3}B(r) = \frac{b(r)}{r^3} + \frac{b'(r)}{3r^2} - \left[1 - \frac{b(r)}{r}\right] \frac{C_0}{r_0^n} \left(\frac{r_0}{r}\right)^n. \quad (30)$$

By substituting this choice into the differential equation Eq. (19), one finds the following solution for the redshift function

$$\Phi(r) = -\frac{C_0}{2(n - 2)} \left(\frac{r_0}{r}\right)^{n-2}, \quad (31)$$
where $C_0$ is an arbitrary constant. The redshift function is finite for all $r$, and for $n > 2$ falls off to zero as $r \to \infty$.

We emphasize that this solution has the feature that one could leave the function $b(r)$ generic, and one may suitably model a wormhole geometry by specifically choosing the shape function, which consequently also specifies the bag function given by Eq. (33). Note that the introduction of a radial-dependent Bag function, introduces a new unknown function so that we are left with a new degree of freedom, so for instance, we may choose a specific shape function.

In this context, consider the particular choice of the form function $b(r) = r_0(r/r_0)^{a}$, with $0 < a < 1$. For this case we readily verify that $b'(r) = a(r/r_0)^{a-1}$, so that at the throat $b'(r_0) = a < 1$, and that for $r \to \infty$ we have $b(r)/r = (r_0/r)^{1-a} \to 0$. In addition to this choice for the shape function, consider the redshift function given above by Eq. (31), but rewritten as $\Phi(r) = \Phi_0(r_0/r)^{\beta}$, with $\beta = n - 2 > 0$ and $\Phi_0 = C_0/(2n - 2)$. Note that this choice of the redshift function is finite everywhere, so that no event horizons are present.

Thus, Eq. (18) provides the following Bag function

$$B(r) = \frac{3}{32\pi} \left( 1 - \frac{\alpha}{3} \right) \left( \frac{r_0}{r} \right)^{1-\alpha} + 2\beta\Phi_0 \left( \frac{r_0}{r} \right)^{\beta} \left[ 1 - \left( \frac{r_0}{r} \right)^{1-\alpha} \right] \right),$$

which reduces to $B(r_0) = B_0 = 3(1 - \alpha/3)/32\pi$ at the wormhole throat and $B \to 0$ for $r \to \infty$.

It is also interesting to consider the “volume integral quantifier,” which provides information on the total amount of matter violating the averaged null energy condition in the spacetime. This is defined by $I_V = \int \rho(r) + p_t(r) dV$ (see Ref. [34] for details), and with a cut-off of the stress-energy at $a$ is given by

$$I_V = \int_0^a (r-b) \left[ \ln \left( \frac{e^{2\Phi}}{1-b/r} \right) \right]' dr.$$

Taking into account the shape and redshift functions provided above, the “volume integral quantifier” given by Eq. (33) provides the following solution

$$I_V = \frac{r_0}{\alpha(\beta - \alpha)(1 - \beta)} \times \left\{ \left( \frac{a}{r_0} \right)^{\alpha} - 1 \right\} (\alpha - \beta) \times [1 + \alpha\beta - (\alpha + \beta)] + 2\alpha\beta\Phi_0 \times \left[ 1 - \alpha + (\alpha - \beta) \left( \frac{a}{r_0} \right)^{\alpha} + (\beta - 1) \left( \frac{a}{r_0} \right)^{\alpha - \beta} \right].$$

Now taking the limit $a \to r_0$, one verifies that $I_V \to 0$. Therefore, as in the examples presented in Refs. [11, 33], one verifies that, in principle, one may construct wormhole geometries with vanishingly small amounts of quark matter violating the averaged null energy condition.

An interesting constraint on the wormhole’s dimensions, in particular, on the throat radius may be inferred from the tidal acceleration restrictions [1]. The latter constraints as measured by a traveler moving radially through the wormhole, are given by the following inequalities

$$\left| \left( 1 - \frac{b'}{r} \right) \left[ \Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(b - r)} \Phi' \right] \right| \eta' \leq g_\oplus,$$ (35)

$$\left| \frac{\gamma^2}{2r^2} \left[ v^2 \left( b' - \frac{b}{r} \right) + 2(r - b)\Phi' \right] \right| \eta'' \leq g_\oplus,$$ (36)

where $\eta'$ is the separation between two arbitrary parts of his body measured in the traveler’s reference frame. We shall consider $|\eta'| = |\eta|$, for simplicity. We refer the reader to Ref. [1] for details. The radial tidal constraint, inequality (35), constrains the redshift function; and the lateral tidal constraint, inequality (36), constrains the velocity with which observers traverse the wormhole. These inequalities are particularly simple at the throat, $r_0$,

$$|\Phi'(r_0)| \leq \frac{2g_\oplus r_0}{(1 - b')|\eta|}, \quad \gamma^2 v^2 \leq \frac{2g_\oplus r_0^2}{(1 - b')|\eta|},$$ (37)

One may also consider that there exist two space stations positioned outside the junction radius, $a$, at $l = -l_1$ and $l = l_2$, respectively, where $dl = (1 - b/r)^{-1/2} dr$ is the proper radial distance. Now, the traversal time as measured by an observer traversing through the wormhole and for the observers that remain at rest at space stations are given by

$$\Delta \tau = \int_{-l_1}^{+l_2} \frac{dl}{v\gamma}, \quad \Delta t = \int_{-l_1}^{+l_2} \frac{dl}{v\Phi'}.$$ (38)

respectively.

Consider now the wormhole geometry constructed in this section. In addition to this, assume a constant non-relativistic, $\gamma \approx 1$, traversal velocity, and considering the equality cases of (37), we obtain the following relationships

$$r_0 \approx \frac{\beta\Phi_0(1 - \alpha)|\eta|}{2g_\oplus}, \quad v \approx r_0 \sqrt{\frac{2g_\oplus}{(1 - \alpha)|\eta|}}.$$ (39)

From the second restriction of (39), taking into account $\alpha = 1/2$, and imposing that the wormhole throat is given by $r_0 = 2m$, then one obtains $v \approx 4 \times 10^5 m/s$ for the traversal velocity. If one considers that the junction radius is given by $a \approx 10^3 m$, then from the traversal times $\Delta \tau \approx \Delta t \approx 2a/v$ (assuming for simplicity that $\Phi \ll 1$), one obtains $\Delta \tau \approx \Delta t \approx 50$ s.
C. Specific radial coordinate-dependent bag function

A particular solution may also be deduced by considering a constant redshift function, i.e., \( \Phi' = 0 \), and specifying the following choice for the bag function

\[
\frac{32\pi}{3} B(r) = \frac{1}{r_0^2} \left( \frac{r_0}{r} \right)^n.
\]

By taking into account the differential equation \[19\], one finally ends up with the following shape function

\[
b(r) = \frac{r_0}{6 - n} \left[ 3 \left( \frac{r_0}{r} \right)^{n-3} + (3 - n) \left( \frac{r_0}{r} \right)^3 \right],
\]

which is always positive and asymptotically flat for \( 2 \leq n \leq 3 \). This solution satisfies the condition \( b(r_0) = 0 \). The flaring out condition \( [b(r) - rb'(r)] / b^2 > 0 \) gives the condition

\[
\frac{4(n - 3) - 3(n - 2)}{n - 6} \left( \frac{r}{r_0} \right)\left( \frac{r}{r_0} \right)^{n-6} > 0.
\]

For a constant redshift function \( \Phi' = 0 \) from the field equations Eqs. \[14\], \[15\] it follows that \( \varepsilon + p_r \) can be written as

\[
\varepsilon + p_r = \frac{1}{8\pi} \frac{b'}{r^2} \frac{b}{r^3} = -\frac{1}{8\pi r^3} (b - rb') < 0,
\]

thus satisfying the flaring out condition.

D. Wormhole geometries supported by a MIT bag model equation of state with electric charge

The total stress-energy tensor \( T^\mu_\nu \) inside the wormhole is assumed to be the sum of two parts \( M^\mu_\nu \) for the quark matter and \( E^\mu_\nu \) for an electromagnetic contribution, respectively: \( T^\mu_\nu = M^\mu_\nu + E^\mu_\nu \).

The stress-energy tensor for an anisotropic distribution of quark matter is provided by

\[
M^\mu_\nu = (\varepsilon + p_r) u_\mu u_\nu + p_t \delta^\mu_\nu + (p_r - p_t) \chi_\mu \chi_\nu,
\]

where \( u_\mu = \delta_0^\mu e^{-\Phi} \) is the four-velocity satisfying the condition \( u_\mu u^\mu = -1 \); \( \chi^\mu \) is the unit spacelike vector in the radial direction, i.e., \( \chi^\mu = \sqrt{1 - b(r)/r} \delta^\mu_r \); with \( p_r \) and \( \varepsilon \) related by the bag model equation of state \[18\].

The electromagnetic contribution is given by

\[
E^\mu_\nu = \frac{1}{4\pi} \left( F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta^\mu_\mu F_{\alpha\beta} F^{\alpha\beta} \right),
\]

where \( F_{\mu\nu} \) is the electromagnetic field tensor defined in terms of the four-potential \( A_\mu \) as

\[
F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu},
\]

where a comma denotes the derivative with respect to the coordinates. For the electromagnetic field we shall adopt the gauge \( A_\mu = (\varphi(r), 0, 0, 0) \).

The Maxwell equations describing the interior of a charged quark wormhole can be expressed as

\[
F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0, \quad \nabla_\nu F^{\mu\nu} = -\frac{j^\mu}{2}. \tag{47}
\]

where \( j^\mu = \bar{\rho}_e u^\mu \) is the four-current density and \( \bar{\rho}_e \) the proper charge density. The second equation can be rewritten as:

\[
\frac{d}{dr} (r^2 E) = \frac{1}{2} \bar{\rho}_e r^2. \tag{48}
\]

In Eq. \[48\] \( E \) is the usual electric field intensity defined as \( E^2 = -F_{01} F^{01} \) and \( E(r) = \sqrt{\phi - b(r)/r} \phi'(r) \), with \( \phi'(r) = F_{01} \). The charge density \( \rho_e \) in Eq. \[48\] is related to the proper charge density \( \bar{\rho}_e \) by \( \rho_e = \bar{\rho}_e / \sqrt{1 - b(r)/r} \). By integrating Eq. \[48\], we obtain

\[
E(r) = \frac{q(r)}{r^2}, \tag{49}
\]

where

\[
q(r) = \frac{1}{2} \int_0^r \bar{\rho}_e r^2 dr = \frac{1}{2} \int_0^r \bar{\rho}_e r^2 dr / \sqrt{1 - b(r)/r}, \tag{50}
\]

is the charge within radius \( r \).

In the presence of an electric field the gravitational field equations are given by the following relationships

\[
\varepsilon(r) = \frac{1}{8\pi} \frac{b'}{r^2} - E^2, \tag{51}
\]

\[
p_r(r) = \frac{1}{8\pi} \left[ 2 \left( \frac{1}{r} - \frac{b}{r^3} \right) \phi'(r) - \frac{b}{r^3} \right] + E^2, \tag{52}
\]

\[
p_t(r) = \frac{1}{8\pi} \left( \frac{1}{b(r)} - \frac{b}{r} \right) \left( \varepsilon(r) + \phi'(r) \right) - \frac{b}{2r^2} \phi' + \frac{\phi'}{r} - E^2. \tag{53}
\]

Using the MIT bag model with the equation of state \[18\], from Eqs. \[51\] and \[52\] one arrives at the following differential equation

\[
\phi' = \frac{r}{2(1 - b/r)} \left[ \frac{b}{r^3} + \frac{b'}{3r^2} - \frac{32\pi}{3} B - \frac{32\pi}{3} \frac{q^2}{r^4} \right]. \tag{54}
\]

As in the previous example, in order to avoid the presence of event horizons, we consider the bag function \( B(r) \) given by

\[
\frac{32\pi}{3} B(r) = \frac{b(r)}{r^3} + \frac{b'(r)}{3r^2} - \frac{r_0}{r^3} \left[ 1 - \frac{b(r)}{r} \right]. \tag{55}
\]
Analogously, the charge distribution \( q^2(r) \) is taken as \( q^2 \propto (1 - b/r) \). Taking into account the following choice
\[
q^2(r) = q_0 r_0^2 \left[ 1 - \frac{b(r)}{r} \right] \left( \frac{r_0}{r} \right)^n ,
\]
with \( n > 0 \) and with \( q_0 \) an arbitrary constant, the solution for the redshift function is given by
\[
\Phi(r) = -\frac{r_0}{2r} + \frac{16\pi q_0}{3(n+2)} \left( \frac{r_0}{r} \right)^{n+2} ,
\]
which is finite throughout the radial coordinate range, i.e., \( r_0 \leq r < \infty \).

The general solution of Eq. (55) can be obtained as
\[
b(r) = \frac{e^{3r_0/r}}{r^3} \left\{ \int \left( 32\pi r_3 B(r) + 3r_0 \right) r^2 e^{-3r_0/r} dr + C_2 \right\} ,
\]
where \( C_2 \) is an arbitrary constant of integration.

A second class of charged quark matter wormhole solutions can be obtained by imposing the condition
\[
\frac{32\pi}{3} \left( B + \frac{q^2}{r^2} \right) = \frac{b}{r^2} + \frac{b'}{3r^2} - 2 \left[ 1 - \frac{b}{r} \right] h(r) ,
\]
where \( h(r) \) is an arbitrary function of the radial coordinate \( r \). The redshift function \( \Phi \) can be obtained in this case as
\[
\Phi(r) = \int r h(r) dr ,
\]
where we have taken an arbitrary integration constant as zero. Now, one may simply model the wormhole geometry by conveniently choosing the function \( h(r) \).

\section{Wormhole Geometries Supported by Color Flavor Locked Superconducting Quark Matter}

\subsection{Radial coordinate-dependent bag and gap function}

Consider the equation of state given by Eq. (10), with the use of Eqs. (14) and (15) we obtain the following differential equation
\[
2 \left[ 1 - \frac{b(r)}{r} \right] \frac{b'(r)}{r^3} - \frac{b(r)}{r^3} - \frac{b'(r)}{3r^2} + \frac{32\pi}{3} B(r)
+ \frac{24\pi}{3} \alpha \left\{ \alpha - \sqrt{\alpha^2 + \frac{4}{9} \pi^2 [\varepsilon - B(r)]} \right\} = 0.
\]

We note that the differential equation (61) can be solved by separating terms. For instance, consider the following simplifying condition
\[
2 \left[ 1 - \frac{b(r)}{r} \right] \frac{b'(r)}{r^3} - \frac{b(r)}{r^3} - \frac{b'(r)}{3r^2} + \frac{32\pi}{3} B(r)
+ \frac{24\pi}{3} \alpha^2 = 0 ,
\]
which can be rewritten as
\[
\Phi'(r) = \frac{r}{2} \left[ 1 - \frac{b(r)}{r} \right]^{-1} \left[ \frac{b(r)}{r^3} + \frac{b'(r)}{3r^2} + \frac{32\pi}{3} B(r) - \frac{24\pi}{3} \alpha^2 \right] .
\]

Let us consider the bag function as
\[
\frac{32\pi}{3} B(r) = -\frac{24\pi}{3} \alpha^2 + \frac{b(r)}{r^3} + \frac{b'(r)}{3r^2}
- \left[ 1 - \frac{b(r)}{r} \right] \frac{C_1}{r_0^3} \left( \frac{r_0}{r} \right)^n ,
\]
where \( C_1 \) is an arbitrary constant. The solution for \( \Phi \) is given by
\[
\Phi(r) = -\frac{C_1}{2(n-2)} \left( \frac{r_0}{r} \right)^{n-2} ,
\]
which is finite for \( n > 2 \).

The differential equation (61) also imposes the condition
\[
-\frac{24\pi}{3} \alpha \times \sqrt{\alpha^2 + \frac{4}{9} \pi^2 [\varepsilon - B(r)]} = 0. \tag{66}
\]

Assuming that \( \alpha \neq 0 \), then the term within the square root is zero. Thus from Eq. (66) we obtain
\[
\alpha^2 + \frac{4}{9} \pi^2 [\varepsilon - B(r)] = 0. \tag{67}
\]

Now, substituting the bag function given by Eq. (64) into Eq. (67), yields the following differential equation
\[
\frac{b'(r)}{r^2} - \frac{b(r)}{r^3} + \left[ 1 - \frac{b(r)}{r} \right] \frac{C_1}{r_0^3} \left( \frac{r_0}{r} \right)^n + \frac{48\pi}{3} \alpha^2 = 0. \tag{68}
\]

Note that one now has a certain freedom in choosing a suitable \( \Delta \) function. We consider the following choice
\[
\frac{48\pi}{3} \alpha^2 = \frac{b(r)}{r^3} - \left[ 1 - \frac{b(r)}{r} \right] \frac{C_1}{r_0^3} \left( \frac{r_0}{r} \right)^n + \frac{k}{r_0^3} \left( \frac{r_0}{r} \right)^{k+3} ,
\]
which provides the solution for the shape function given by
\[
b(r) = r_0 \left( \frac{r_0}{r} \right)^k ,
\]
with \( k > 0 \).
For simplicity, consider \( k = 0 \), and after inserting the functional form of \( \alpha \) given by Eq. (12), can be rearranged to give

\[
\Delta(r) = \frac{1}{2} \left( m_s^2 \pm \sqrt{\frac{3\pi}{4} \frac{b(r)}{r^3} - \left[ 1 - \frac{b(r)}{r} \right] C_1 \left( \frac{r_0}{r} \right)^n} \right).
\]

(71)

Inserting this choice into the differential equation Eq. (68), one immediately obtains the solution for a constant shape function, \( b(r) = r_0 \).

### B. Density dependent bag function CFL wormhole models

A new set of wormhole solutions supported by quark matter in the CFL phase can be obtained by assuming in Eq. (61) the condition

\[
24 \pi \alpha \left\{ \alpha - \sqrt{\alpha^2 + \frac{4}{9} \pi^2 [\varepsilon - B(r)]} \right\} = \gamma_1(r)\varepsilon(r),
\]

where \( \gamma_1(r) \) is an arbitrary function of the radial coordinate. From the above equation we can obtain \( \alpha \) as

\[
\alpha = \pm \frac{\sqrt{\pi} \gamma_1(r)\varepsilon/24}{\left( (4\pi/9)(\varepsilon - B) + \gamma_1(r)\varepsilon/12 \right)^{1/2}}.
\]

(73)

By using the definition of \( \alpha \) given by Eq. (12) we obtain for \( \Delta \) the expression

\[
\Delta^2 = \frac{3}{2} \left[ \frac{m_s^2}{6} \pm \frac{\sqrt{\pi} \gamma_1(r)\varepsilon/24}{\left( (4\pi/9)(\varepsilon - B) + \gamma_1(r)\varepsilon/12 \right)^{1/2}} \right].
\]

(74)

In the following we assume that the condition \((16\pi/3)(\varepsilon - B) + \gamma_1(r)\varepsilon > 0\) holds for all values of the physical parameters and of the function \( \gamma_1(r) \). As a second choice we fix the bag function as

\[
32\pi B = \frac{8\pi}{3} \gamma_2(r)\varepsilon + \frac{b}{r^3} = \gamma_2(r)\left( \frac{b(r)}{3r^2} + \frac{b}{3r^3} \right).
\]

(75)

By inserting this form of the bag function into Eq. (61) we obtain

\[
2 \left( \frac{1 - b}{r} \right) \Phi' + \left[ \frac{\gamma_2(r) - 1}{3} + \frac{\gamma_1(r)}{8\pi} \right] \frac{b'(r)}{r^2} = 0.
\]

(76)

By choosing

\[
\frac{\gamma_2(r) - 1}{3} + \frac{\gamma_1(r)}{8\pi} = \left[ 1 - \frac{b(r)}{r} \right] \gamma_3(r),
\]

(77)

we obtain

\[
2r\Phi' + \gamma_3(r)b' = 0.
\]

(78)

We may now find a wide variety of wormhole geometries by choosing suitably either the shape/redshift functions or the \( \gamma_3 \) function.

### VI. DISCUSSIONS AND FINAL REMARKS

Even though direct observations of quarks are still missing, the quark structure of baryonic matter is the central paradigm of the present-day elementary particle physics. At very high densities, which can be achieved in the interior of neutron stars, a deconfinement transition can break the baryons into their constitutive components, the quarks, thus leading to the formation of the quark-gluon plasma. Moreover, the strange quark matter, consisting of a mixture of up, down and strange quarks, may be the most energetically favorable state of matter. At high densities quark matter may also undergo a phase transition to a color superconducting state. The thermodynamic properties of the quark matter are well-known from a theoretical point of view, and several equations of state of the dense quark-gluon plasma have been proposed in the framework of a Quantum Chromodynamical approach, such as the MIT bag model equation of state and the equations of state of the superconducting Color-Flavor-Locked phase.

Motivated by these theoretical models, in the present paper we have explored the conditions under which wormhole geometries may be supported by the equations of state considered in the theoretical investigations of quark-gluon interactions. Since quark-gluon plasma can exist only at very high densities, the existence of the quark-gluon wormholes requires quark matter at extremely high densities. In these systems the basic physical parameters describing the properties of the QCD quark-gluon plasma (bag constant, gap energy, quark masses) become effective, density and interaction dependent quantities. It is this specific property of the strong interactions we have used to generate specific mathematical functional forms of the bag function and of the gap function that could make possible the existence of a wormhole geometry supported by a strongly gravitationally confined normal or superconducting quark-gluon plasma.

In the case of the normal quark-gluon plasma, wormhole solutions can be obtained by assuming either a specific dependence of \( B \) on the shape function \( b \), or some simple functional representations of \( B \). In both cases in the limit of large \( r \) the bag function tends to zero, \( \lim_{r \to \infty} B = 0 \), and in this limit the equation of state of the quark matter becomes the radiation type equation of the normal baryonic matter, \( p = \varepsilon/3 \). Therefore, once the density of the quark matter increases after a deconfinement transition, a density (radial coordinate) dependent bag function could lead to the violation of the null energy condition, with the subsequent generation of a wormhole supported by the quark-gluon plasma. A high intensity electric field with a shape function dependent charge distribution could also play a significant role in the formation of the wormhole.

In the case of the superconducting quark matter the gravitational field equations can be solved by assuming that both the bag function and the gap function are shape
function and $s$ quark mass dependent quantities. However, in the large $r$ limit, in order to reobtain the standard baryonic matter equation of state, the condition of the vanishing of the mass of the $s$ quark is also required, \( \lim_{r \to \infty} m_s = 0 \). The assumption of a zero asymptotic $u$, $d$ and $s$ quark mass is also frequently used in the study of quark star models \[13\].

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