Non-factorisable metrics and Gauss–Bonnet terms in higher dimensions

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Abstract

An iterative construction of higher order Einstein tensors for a maximally Gauss-Bonnet extended gravitational Lagrangian was introduced in a previous paper. Here the formalism is extended to non-factorisable metrics in arbitrary $(d + 1)$ dimensions in the presence of superposed Gauss-Bonnet terms. Such a generalisation turns out to be remarkably convenient and elegant. Having thus obtained the variational equations we first construct bulk solutions, with nonzero and zero cosmological constant. It is also

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pointed out that in the absence of Gauss-Bonnet terms a Schwarzschild type solution can be obtained in the non-factorisable case. Two positive tension branes are then inserted and their tensions are obtained in terms of parameters in the warp factor. Relations to recent studies of several authors are pointed out.
1 Introduction

A particularly helpful formulation was presented in [1] for explicit construction of static spherically symmetric metrics (generalizing Schwarzschild and de-Sitter type solutions) in \((d - 1)\) space dimensions in presence of higher derivative Gauss–Bonnet (GB) terms in the gravitational Lagrangian. The key feature was an iterative construction of the Riemann, Ricci and Einstein tensors to incorporate the contributions of GB terms of successive orders. This provides a powerful and elegant approach systematizing the combinatorics involved when the maximal number of GB terms are present. The solutions can be displayed systematically.

In four dimensional theories the GB term is topological, namely a total divergence, and hence contributes nothing to the classical equations of motion. However, in the context of higher dimensional theories the GB invariants are non-trivial and their presence is required for obtaining a ghost-free theory of gravity. For example, in string theory, the GB combination appears naturally in the tree-level effective action of heterotic superstring at the next-to-leading order in the \(\alpha'\) (string-tension) expansion [2].

In the last years, considerable effort has been devoted to the study of higher dimensional space-times in the presence of non-factorizable metrics [3, 4, 5]. The effect of the GB invariant has also been considered in the recent literature [6, 7, 8]. In this article, we study the consequences of the presence of maximal GB terms in the context of Randall–Sundrum type metrics in higher dimensions, utilizing the techniques developed in [1]. More specifically, we show how the bigravity model with positive tension branes of Ref. [4] generalises to arbitrary dimensions in the presence of maximal number of GB terms.

In Sec. 2 we start with the following metric in \(d + 1\) dimensions

\[
\begin{align*}
\text{ds}^2 &= f(y) \left\{ \mp (1 - L(r)) \, dt^2 + \frac{1}{(1 - L(r))} \, dr^2 + r^2 \, d\Omega_{d-2}^2 \right\} + dy^2 \\
\end{align*}
\]

where \(f(y)\) and \(L(r)\) are, to start with, unknown functions. We show how the formalism of [1] smoothly generalises to yield the Einstein tensors in the presence of the maximal number of GB terms superposed to give the generalised Lagrangian. Then in Sec. 3 bulk solutions are presented with

\[
\begin{align*}
L(r) &= b r^2 \\
f(y) &= c_0 + c_1 e^{2ky} + c_2 e^{-2ky},
\end{align*}
\]

the constant parameters satisfying

\[
\begin{align*}
c_0^2 &= 4 c_1 c_2 , \quad b = -2 c_0 k^2 .
\end{align*}
\]

Now, with (1.2) and (1.3), (1.1) solves the generalised gravitational equation

\[
\begin{align*}
G^a_b &= \sum_{p=1}^{P} \kappa_{(p)} G_{(p)}^a b = \Lambda \, \delta^a_b , \quad 2P \leq d + 1 ,
\end{align*}
\]
where $\kappa(p)$ is the coupling strength of the $2p$-th order Ricci scalar in the Lagrangian

$$
\mathcal{L} = \sum_{p=1}^{P} \frac{1}{2p} \sqrt{g} \kappa(p) R(p)
$$

(1.5)

and $\Lambda$ is the bulk cosmological constant. Eqn. (1.4) is satisfied provided that \((k^2)\) satisfies the polynomial equation

$$
\frac{1}{2} \kappa(1) d(d - 1)(k^2) - \frac{1}{8} \kappa(2) d(d - 1)(d - 2)(d - 3)(k^2)^2 + \ldots + \frac{(-1)^{P+1}}{2^P P!} d(d - 1) \ldots + (d - 2P + 1)(k^2)^P = \Lambda
$$

(1.6)

Thus, as more GB terms are superposed (with more nonzero $\kappa(p)$) the form of the metric (1.1) is conserved but \((k^2)\) has to satisfy a correspondingly higher order polynomial equation (1.6).

This crucial constraint on the warp factor $f(y)$ (1.2) is extracted in a remarkably compact and convenient fashion in our formalism. Further comments on (1.6) can be found in Sec. 2.

We consider both cases

$$
\Lambda \neq 0 \quad \text{and} \quad \Lambda = 0.
$$

(1.7)

We also point out a possibility usually ignored. For $P = 1$ (i.e. in the absence of GB terms) one can generalise $L(r)$ in (1.2) to

$$
L(r) = \frac{c}{r^{d-3}} + br^2.
$$

(1.8)

One thus obtains a Schwarzschild type black hole in the non-factorisable case.

In Sec. 4 we insert the \((d-1)\)-branes by changing, to start with, $y$ to $|y|$ in (1.2) and writing $f(y)$

$$
f(y) = (a(y))^2 = \left( \frac{\cosh k(y_0 - |y|)}{\cosh ky_0} \right)^2.
$$

(1.9)

This corresponds, along with $|y|$ for $y$, to

$$
(c_0 , c_1 , c_2) = (e^{ky_0} + e^{-ky_0})^{-2} \left( 2, e^{-2ky_0}, e^{2ky_0} \right),
$$

(1.10)

satisfying the constraint

$$
c_0^2 = 4c_1c_2,
$$

in (1.3). The close relation to Ref. [4] is now evident, and the brane tensions (of the two positive tension branes at $y = 0$ and at $y = Y$, say) are obtained in a similar fashion.
It is shown, somewhat analogously to (1.6), that the effect of the GB terms in the resulting brane tensions appears in the polynomial factors

\[ K(P) = \sum_{p=1}^{P} \kappa(p) \frac{(-1)^{p-1}}{2^{p-1}(p-1)!} (d-1) \cdots (d-2p+1) (k^2)^{p-1} \]  

(1.11)

in the delta function terms. In our formalism this crucial feature is also remarkably easily obtained. We defer other references and comments to the concluding section.

2 Ansatz for a nonfactorizable metric and construction of the corresponding GB terms

We start with a metric containing two unknown functions (to be indicated below) and construct explicitly and systematically the Riemann, Ricci and Einstein tensor and their GB generalizations (involving the above-mentioned functions and their derivatives in specific fashions). They will serve as inputs for the bulk Lagrangian to be defined in the following section leading to exact bulk solutions of the variational equations. Consequences of insertions of branes will be considered next.

The coordinates will be denoted as

\[ (t, r, \theta_1, \theta_2, \ldots, \theta_{d-2}, y) \]  

(2.1)

where \((\theta_1, \theta_2, \ldots, \theta_{d-2})\) are the angular coordinates. The dimension is \(D = d + 1\).

Let

\[ ds^2 = f(y) \left\{ \mp (1 - L(r)) dt^2 + \frac{1}{(1 - L(r))} dr^2 + r^2 d\Omega_{d-2}^2 \right\} + dy^2 \]  

(2.2)

where

\[ \Omega_{d-2}^2 = \sum_{i=1}^{d-2} P_i^2 d\theta_i \]  

(2.3)

and \(P_1 = 1, P_i = \prod_{k=1}^{i-1} \sin \theta_k, i = 2, \ldots, d - 2\). Choices for \(f(y)\) and \(L(r)\) will be specified in the next section. We will usually suppress the arguments \(y\) and \(r\) in solutions below. Tangent plane vectors (vielbeins) are the 1-forms

\[ e^t = \sqrt{f(1 - L)} dt, \quad e^r = \sqrt{\frac{f}{1 - L}} dr, \quad e^i = \sqrt{f r P_i} d\theta_i, \quad e^y = dy, \quad i = 1, \ldots, d - 2 \]  

(2.4)
Tangent plane indices \((a, b, \ldots)\) will be continued to be denoted by \((t, r, i_1, \ldots, i_{d-2}, y)\) instead of, say, by \((\hat{t}, \hat{r}, \ldots)\). For our diagonal metric there will be no confusion. The indices \((a, b, \ldots)\) namely \((t, r, \ldots)\) as tangent plane ones will be raised and lowered by \(\eta^a_b\) rather than \(g^\mu_\nu\).

The metric being diagonal, the spin connection 1–forms are

\[
\omega^{ab} = -\omega^{ba} = \frac{1}{\sqrt{g_{aa} g_{bb}}} \left\{ (\partial_b \sqrt{g_{aa}}) - (\partial_a \sqrt{g_{bb}}) \right\}, \ a, b \not= \text{not summed} \tag{2.5}
\]

the Riemann tensor 2–forms are

\[
R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} \tag{2.6}
\]

Define,

\[
h_1(y) = \frac{1}{4f^2} \left( \frac{df}{dy} \right)^2 \tag{2.7}
\]

\[
h_2(y) = \frac{1}{4f^2} \left( 2f \frac{d^2f}{dy^2} - \left( \frac{df}{dy} \right)^2 \right) \tag{2.8}
\]

and

\[
M(r, y) = \frac{1}{f(y)} \left( L(r) - r^2 f(y) h_1(y) \right) \tag{2.9}
\]

Suppressing arguments will usually write

\[
M = \frac{1}{f} \left( L - r^2 f h_1 \right) \tag{2.10}
\]

and \(M' = \frac{\partial M(r, y)}{\partial r}, \ M'' = \frac{\partial^2 M(r, y)}{\partial r^2} \).

From (2.4), (2.5) and (2.6) one obtains the nonvanishing components of (2.6) (with \(i = 1, \ldots, d - 2\)) as

\[
R^{rr} = \frac{1}{2} M'' e^r \wedge e^r = \frac{1}{2} \left( \frac{r}{\sqrt{\frac{r}{2}}} \right) \left( \frac{M}{r^2} \right) e^r \wedge e^r \tag{2.11}
\]

\[
R^{ti} = \frac{1}{2r} M' e^t \wedge e^i = \frac{1}{2} \left( \frac{r}{\sqrt{\frac{r}{2}}} \right) \left( \frac{M}{r^2} \right) e^t \wedge e^i
\]

\[
R^{ri} = \frac{1}{2} \left( \frac{r}{\sqrt{\frac{r}{2}}} \right) \left( \frac{M}{r^2} \right) e^r \wedge e^i
\]

\[
R^{ij} = \left( \frac{M}{r^2} \right) e^i \wedge e^j
\]

\[
R^{ty} = -h_2 e^t \wedge e^y
\]

\[
R^{ry} = -h_2 e^r \wedge e^y
\]

\[
R^{iy} = -h_2 e^i \wedge e^y
\]
As compared to \([1]\), in \((R^r, R^i, R^r)\) \(M\) replaces \(L\) and there are the extra components \((R^y, R^v, R^y)\) each proportional to \(h^2\). Each \(R^{ab}\) still has only one “diagonal” component \(R^{ab}_{ab}\). Hence the Ricci-tensor components are easily obtained as follows. The nonzero (all diagonal) terms (adding a subscript \((1)\) in view of subsequent generalization to higher order GB terms) are

\[
R_{(1)t} = \sum_a R_{ta} = R_{tr} + \sum_{i=1}^{d-2} R_{ti} + R_{ty}^y = \frac{1}{2} \left( r \frac{d}{dr} + 2 \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right) - h^2
\]

\[
R_{(1)r} = R_{(1)t} 2.11
\]

\[
R_{(1)i} = R_{ti} + R_{ri}^i + \sum_{j \neq i} R_{ij} + R_{iy}^y = \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right) - h^2
\]

\[
R_{(1)y} = -d h^2
\]

The Ricci scalar is

\[
R_{(1)} = R_{(1)t} + R_{(1)r} + \sum_{i=1}^{d-2} R_{(1)i} + R_{(1)y} = \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right) - 2 d h^2 \quad (2.13)
\]

The nonzero components of the corresponding Einstein tensor

\[
G_{(1)a}^b = R_{(1)b}^a - \frac{1}{2} \eta^a_b R_{(1)}
\]

are obtained as

\[
G_{(1)t} = -\frac{1}{2} (d-2) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right) + (d-1) h^2
\]

\[
G_{(1)r} = G_{(1)t}
\]

\[
G_{(1)i} = -\frac{1}{2} \left( r \frac{d}{dr} + (d-2) \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right) + (d-1) h^2 \quad (2.14)
\]

\[
G_{(1)y} = -\frac{1}{2} \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)
\]

Now we show how the recursion formula used in \([1]\) simplifies here also the construction of GB terms of order \(2p (p = 2, \ldots, P, 2P \leq D)\). (For \(p = 1\) one has the standard \(R^{ab}\) obtained above.) Totally antisymmetrized \(2p\)-forms are obtained as follows. We continue to use exclusively tangent plane indices.

For \(p = 2\),

\[
R^{abcd} = R^{ab} \wedge R^{cd} + R^{ad} \wedge R^{bc} + R^{ac} \wedge R^{db} \quad (2.15)
\]

\(^1\)Note that for \(L = cr^2, f = cy^2\), one obtains \(M = 0 = h^2\) and hence a flat space.
the indices \((b, c, d)\) being circularly permuted. For arbitrary \(p\)

\[ R^{a_1a_2\ldots a_{2p}} = R^{a_1a_2} \wedge R^{a_3\ldots a_{2p}} + (\text{circ. perm. of } a_2, \ldots, a_{2p}) \]  

(2.16)

giving \((1 \cdot 3 \cdot 5 \ldots \cdot (2p - 1))\) terms of the type

\[ R^{a_1a_2} \wedge R^{a_3a_4} \wedge \ldots \wedge R^{a_{2p-1}a_{2p}}. \]

For \(p = 2\), implementing (2.11) in (2.15),

\[ R^{trij} = R^{tr} \wedge R^{ij} + r^{tij} \wedge R^{ri} + R^{ti} \wedge R^{jr} \]

\[ = \frac{1}{4} \left( r \frac{d}{dr} + 4 \right) \left( r \frac{d}{dr} + 3 \right) \left( \frac{M}{r^2} \right)^2 e^t \wedge e^r \wedge e^i \wedge e^j. \]  

(2.17)

Proceeding similarly for the other members of \(R^{abcd}\), the nonzero (all diagonal) components are obtained as

\[ R^{trij}_{trij} = \frac{1}{4} \left( r \frac{d}{dr} + 4 \right) \left( r \frac{d}{dr} + 3 \right) \left( \frac{M}{r^2} \right)^2 \]

\[ R^{tijk}_{tijk} = R^{tij}_{tij} = \frac{3}{4} \left( r \frac{d}{dr} + 4 \right) \left( \frac{M}{r^2} \right)^2 \]

\[ R^{ijkl}_{ijkl} = 3 \left( \frac{M}{r^2} \right)^2 \]  

(2.18)

(2.19)

(Simultaneous permutations of the same top and bottom indices leave the values unchanged.)

Now the nonzero components of the \(p = 2\) Ricci tensor components leave the values unchanged.

In the last step there is no sum over \(i, j, \ldots\). The values of \(R^{trij}_{trij}\) for different \((ij)\) being the same \(\sum_{i, j}\) is replaced by \(\binom{d-2}{2}\). Similarly for other terms.
Thus
\[
R_{(2)i}^t = \frac{1}{8} (d-2)(d-3) \left( r \frac{d}{dr} + 4 \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^2 \\
- \frac{1}{2} h_2 (d-2) \left( r \frac{d}{dr} + 3 \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)
\] (2.20)

Proceeding similarly
\[
R_{(2)r}^i = R_{(2)i}^t
\] (2.21)

and
\[
R_{(2)i}^i = \sum_j R_{ijrj}^i + \sum_{j,k} (R_{ijjk}^i + R_{irjk}^i) + \sum_{j,k,l} R_{ijkl}^i + R_{iiry}^i \\
+ \sum_j \left( R_{ijjy}^i + R_{irjy}^i \right) + \sum_{j,k} R_{ijky}^i
\] (2.22)

Here, \( i \) being fixed, the sums over \( (j), (j,k) \) and \( (j,k,l) \) are replaced respectively by \( \binom{d-3}{1} \), \( \binom{d-3}{2} \) and \( \binom{d-3}{3} \).

Finally,
\[
R_{(2)i}^i = \frac{1}{4} (d-3) \left( r \frac{d}{dr} + 2(d-2) \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^2 \\
- \frac{1}{2} h_2 \left( r \frac{d}{dr} + 3(d-2) \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)
\] (2.23)

Similarly,
\[
R_{(2)y}^y = \sum_i R_{iyri}^y + \sum_{i,j} \left( R_{iijy}^y + R_{irij}^y \right) + \sum_{i,j,k} R_{ijjk}^y \\
= - \frac{1}{2} (d-2) h_2 \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)
\] (2.24)

The \( p = 2 \) Ricci scalar is
\[
R_{(2)} = R_{(2)t}^t + R_{(2)r}^r + \sum_i R_{(2)i}^i + R_{(2)y}^y \\
= \frac{1}{2} (d-2)(d-3) \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^2 \\
- 2 h_2 (d-2) \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)
\] (2.25)
The Einstein tensor for arbitrary $p$ is (since $\eta^a_b = 1$ for our indices)

$$G_{(p)}^a_b = R_{(p)}^a_b - \frac{1}{2p} R^{(p)}$$ \hfill (2.26)

Hence for $p = 2$,

$$G_{(2)}^t_t = R_{(2)}^t_t - \frac{1}{4} R^{(2)}$$ \hfill (2.27)

$$G_{(2)}^i_i = R_{(2)}^i_i - \frac{1}{4} R^{(2)}$$ \hfill (2.28)

$$G_{(2)}^y_y = R_{(2)}^y_y - \frac{1}{4} R^{(2)}$$ \hfill (2.29)

Successive iterations, using (2.16), can be shown to lead for arbitrary $p$ ($2p \leq d$) to the nonzero components $R_{(p)}^a_b$ given below. For $p > 2$, one obtains

$$R_{(p)}^t_t = R_{(p)}^r_r = \frac{1}{2^p p!} (d - 2)(d - 3) \ldots (d - 2p + 1) \left( r \frac{d}{dr} + (d - 1) \right) \left( r \frac{d}{dr} + 2p \right) \left( r \frac{d}{dr} + (d - 1) \right) \left( \frac{M}{r^2} \right)^p$$

$$- \frac{1}{2^{p-1}(p-1)!} h_2(d - 2) \ldots (d - 2p + 2) \times \left( r \frac{d}{dr} + (p - 1) \right) \left( r \frac{d}{dr} + (d - 1) \right) \left( \frac{M}{r^2} \right)^{p-1}$$ \hfill (2.30)

$$R_{(p)}^i_i = \frac{1}{2^p p!} (d - 3) \ldots (d - 2p + 1) \left( (p - 1)r \frac{d}{dr} + p(d - 2) \right) \left( r \frac{d}{dr} + (d - 1) \right) \left( \frac{M}{r^2} \right)^p$$

$$- \frac{1}{2^{p-1}(p-1)!} h_2(d - 3) \ldots (d - 2p + 2)$$
\begin{align*}
\times \left( (2p-3) \frac{d}{dr} + (2p-1)(d-2) \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^{p-1} \\
R_{(p)}^y &= -\frac{1}{2^{p-1}(p-1)!} h_2 (d-2) \ldots (d-2p+2) \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^{p-1}
\end{align*}

Hence,
\begin{align*}
R_{(p)} &= R_{(p)}^t + R_{(p)}^r + \sum_i R_{(p)}^i + R_{(p)}^y \\
&= \frac{1}{2^{p-1}(p-1)!} (d-2) \ldots (d-2p+1) \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^{p} \\
&\quad - \frac{1}{2^{p-1}(p-1)!} h_2 (d-2) \ldots (d-2p+2) \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^{p-1}
\end{align*}

Now using (2.26), namely
\begin{align*}
G_{(p)}^a &= R_{(p)}^a - \frac{1}{2p} R_{(p)} \\
G_{(p)}^t &= G_{(p)}^r \\
&= -\frac{1}{2p} (d-2) \ldots (d-2p+1) \\
&\times \left( r \frac{d}{dr} + (d-1) \right) \left\{ (d-2p) \left( \frac{M}{r^2} \right) - 2ph_2 \right\} \left( \frac{M}{r^2} \right)^{p-1} \\
G_{(p)}^i &= -\frac{1}{2p} (d-3) \ldots (d-2p+1) \\
&\times \left( r \frac{d}{dr} + (d-2) \right) \left( r \frac{d}{dr} + (d-1) \right) \left\{ (d-2p) \left( \frac{M}{r^2} \right) - 2ph_2 \right\} \left( \frac{M}{r^2} \right)^{p-1} \\
G_{(p)}^y &= -\frac{1}{2p} (d-2) \ldots (d-2p+1) \times \left( r \frac{d}{dr} + d \right) \left( r \frac{d}{dr} + (d-1) \right) \left( \frac{M}{r^2} \right)^{p}
\end{align*}

These results should be compared with the corresponding results in [1] (Eqns (4.1),(4.2),(4.3) of [1] give the summed up contribution of all $G_{(p)}$).

3 The bulk Lagrangian and solutions

With suitable choices of units and sign conventions, the bulk Lagrangian with a maximal number of GB terms is defined to be
\begin{align*}
\mathcal{L} = \sum_{p=1}^{P} \frac{1}{2p} \sqrt{g} \kappa_{(p)} R_{(p)} \quad (2P \leq D)
\end{align*}
(For odd \(d\), the GB term with \(2P = D = d + 1\), does not contribute to the equations of motion but has a topological significance. For \(2p > (d + 1)\), \(R^{a_1a_2\ldots a_{2p}}\) and hence \(R_{(p)}\) is identically zero due to total antisymmetry.)

The constants \(\kappa_{(p)}\) are arbitrary, but we will usually assume

\[
\kappa_{(1)} \gg \kappa_{(p)} \quad (p > 1)
\]

and possibly

\[
\kappa_{(1)} \gg \kappa_{(2)} \gg \kappa_{(3)} \gg \ldots
\]

and so on. The variational equations (in the absence of sources and matter fields)

\[
G^a_b = \sum_{p=1}^{P} \kappa_{(p)} G_{(p)}^a_b = \Lambda \delta^a_b
\]

(with tangent plane indices) where \(\Lambda\) is some bulk cosmological constant in suitable units.

### 3.1 Bulk solutions for superposed GB terms

It was shown in [1] how to take into account the effect of a term \(\sim r^2\) in \(L(r)\) (see Sec. VII of [1]). Here we show, if \(L(r)\) has only this AdS (or dS) type term, how \(f(y)\) can be chosen to provide exact solutions.

Set

\[
L = b r^2
\]

\[
f = c_0 + c_1 e^{2ky} + c_2 e^{-2ky}
\]

Then from (2.7), (2.8) and (2.9)

\[
h_1 = k^2 \left(1 - \frac{2c_0}{f} + \frac{c^2 - 4c_1c_2}{f^2}\right)
\]

\[
h_2 = k^2 \left(1 + \frac{c^2 - 4c_1c_2}{f^2}\right)
\]

\[
(M/r^2) = -k^2 + \frac{b + 2c_0 k^2}{f} - k^2 \frac{c^2 - 4c_1c_2}{f^2}
\]

Hence for

\[
c_0^2 = 4c_1c_2, \quad b = -2c_0 k^2
\]
one obtains, along with \((1 - L(r)) = 1 + 2c_0k^2r^2\) in the metric

\[
h_2 = -\frac{M}{r^2} = k^2 .
\]  
(3.9)

(Note that (3.8) is invariant under the exchange of \(c_1, c_2\), namely for \(k \rightarrow -k\) in (3.5). The equations below will involve \(k^2\).

Now, from (2.34), (2.35) and (2.36), the nonzero components of \(G^a_{\beta} \mid p\) all coincide yielding

\[
G^t_{\ t} = G^r_{\ r} = G^y_{\ y} = (-1)^{(p+1)} \frac{1}{2^p p!} d(d-1) \ldots (d-2p+1)(k^2)^p ,
\]  
(3.10)

whence (3.4) reduces to

\[
\frac{1}{2} \kappa_{(1)} d(d-1)(k^2) - \frac{1}{8} \kappa_{(2)} d(d-1)(d-2)(d-3)(k^2)^2 + \ldots + (-1)^{p+1} \frac{\kappa_{(p)}}{2^p p!} d(d-1) \ldots + (d-2P+1)(k^2)^P = \Lambda .
\]  
(3.11)

Thus it seen that in the context of (3.5), (3.6) and (3.8), the effect of the BG terms reduce to the fact that \(k^2\) is constrained to satisfy a polynomial equation of degree \(P\). Substituting this \(k^2\) in (3.8), for \(c_0 > 0\) (i.e. \((1 - L) = 1 + 2c_0k^2r^2\), one has an AdS type metric.

For \(P = 1\), one has (in the absence of GB terms),

\[
k^2 = \frac{2\Lambda}{\kappa_{(1)} d(d-1)}
\]  
(3.12)

For \(P = 2\), one obtains

\[
\frac{1}{2} \kappa_{(1)} d(d-1)(k^2) - \frac{1}{8} \kappa_{(2)} d(d-1)(d-3)(k^2)^2 = \Lambda
\]  
(3.13)

or

\[
(k^2) = \frac{2\kappa_{(1)}}{\kappa_{(2)}(d-2)(d-3)} \left\{ 1 - \left( 1 - 2\Lambda \frac{\kappa_{(2)}(d-2)(d-3)}{\kappa_{(1)}^2 d(d-1)} \right)^{\frac{1}{2}} \right\}
\]  
(3.14)

The sign before the square root has been chosen to obtain (3.12) as the leading term on expanding the square root in powers of \(\kappa_{(1)}\).

For \(P > 2\), if (3.3) and (3.8) are maintained, one has to select a solution satisfying similarly consistency (developing in powers of \(\kappa_{(2)}, \kappa_{(2)}^2, \kappa_{(2)}\kappa_{(3)}, \ldots\)) with the lower degree cases.

For \(d = 10\), keeping all possible \(p\)'s \((p = 1, 2, 3, 4, 5)\) one obtains \(k^2\) in terms of elliptic functions. For \(d > 10\), hyperelliptic functions are needed. (Compare the discussion of Section V of [1], where such functions arise for \((L/r^2)\) and the horizon \(r_H\).)
3.2 Special cases

(1) $\Lambda = 0$

For zero bulk cosmological constant and for $P > 1$ one still obtains nontrivial solutions ($k^2 \neq 0$), satisfying a polynomial of degree $(P - 1)$ in $k^2$. Thus for example, (3.13) reduces to

$$k^2 = \frac{2\kappa(1)}{\kappa(2) (d - 2) (d - 3)}$$  \hspace{1cm} (3.15)

Note that for (3.2) one has the large ratio $(\kappa(1)/\kappa(2))$.

(2) Schwarzschild-type bulk black hole in the absence of GB terms ($P = 1$):

In (3.5) we set

$$L = b r^2$$

But for $P = 1$, when $G_{(1)}^a$ are given by (2.14) one can set

$$\frac{L}{r^2} = \frac{c}{r^{d-1}} + b$$  \hspace{1cm} (3.16)

the term $cr^{-(d-1)}$ is annihilated by the factor

$$\left( r \frac{d}{dr} + (d - 1) \right)$$  \hspace{1cm} (3.17)

present in each $G_{(1)}^a$ acting on $(\frac{M}{r^2})$ and hence on $\frac{L}{r^2}$. The operator (3.17) is present for all $p$. But it acts on $(\frac{M}{r^2})^p$ and $(\frac{M}{r^2})^{(p-1)}$ and hence $c$ is not eliminated.

In [1] a polynomial equation was solved for $\frac{L}{r^2}$ in the presence of GB terms (as we do for $k^2$ here). But the presence of $y$ and $f(y)$ no longer permit that.

Since $c$ in (3.16) is eliminated (for $P = 1$) to start with, in the equations of motion one can proceed as before with (3.10) replacing (3.3) and maintaining (3.6),(3.7), (3.8) and (3.12). One thus obtains a Schwarzschild type black hole in the nonfactorizable case in the absence of GB terms. The lapse fraction $f(y)(1 - L(r))$ now depends on $y$. But the horizon ($L(r) = 1$) still depends only on $r$, and neither on $t$ nor on $y$.

4 Insertion of branes

Having presented our bulk solutions, we now consider the modifications necessary for inserting two positive tension branes in a fashion quite analogous to the treatment of [4]. In the notation of [4] our $f(y)$ is $a(z)^2$ where

$$a(y) = \frac{\cosh(k (y_0 - |y|))}{\cosh(k y_0)} = \cosh k|y| - (\tanh k y_0) \sinh k|y|.$$  \hspace{1cm} (4.1)
We now set, introducing orbifold symmetry permitting branes,\[ f(y) = a^2(y) = \frac{2 + e^{-2k|y|} + 2k|y| + e^{2k|y|} - 2k|y|}{(e^{ky} + e^{-ky})^2} \] (4.2)

Comparing to (3.6) (along with $|y|$ for $y$) one has\[ \{c_0, c_1, c_2\} = (e^{ky} + e^{-ky})^{-2} \{2, e^{-2ky}, e^{2ky}\} \] (4.3)
satisfying (3.8).\[ 1 - L(r) = 1 + \frac{4k^2}{(e^{ky} + e^{-ky})^2} r^2 \] (4.4)

where $k^2$ is determined from (3.11).

But now one must match coefficients of $\delta$-functions in derivatives of $a(y)$ to those contributed by the brane tensions.

Let us suppose the two branes are inserted respectively at
\[ y = 0 \quad \text{and} \quad y = Y. \]

Note, that, from (2.7) and (2.8), using (4.2),\[ h_1 = \left(a^{-1} a_y\right)^2, \quad h_2 = a^{-1} a_{yy}, \quad \left(a_y = \frac{da}{dy}\right). \] (4.5)

Singular terms will arise from $h_2$ only which appears linearly in all $G_{(p)}^a (a = t, r, i)$ and not in $G_{(q)}^y$. One obtains, in a fashion analogous to [4],\[ a_{yy} = k^2 a + \lambda_1 \delta(y) + \lambda_1 \delta(y - Y) \] (4.6)

where\[ \lambda_1 = 2a_y(0^+) \quad \lambda_2 = -2a_y(Y^-). \]

Hence\[ h_2 = k^2 + \frac{2a_y(0^+)}{a(0)} \delta(y) - \frac{2a_y(Y^-)}{a(Y)} \delta(y - Y) \] (4.7)

$\hat{h}_2$ denoting the singular part.

\[ ^3 \text{Note that for } b = 0, \ i.e. \ L = 0, \ \text{one can only have } a(y) \sim e^{\pm k|y|}. \]
In (2.34), (2.35) and (2.36) we now set, as in (3.9)
\[ \frac{M}{r^2} = -k^2 \]
but \( h_2 = k^2 + \hat{h}_2 \) has now an additional singular part as in (4.7). Thus one obtains, with (3.10) denoting “bulk” values,
\[ G(p)_a = (G(p)_a)^{\text{bulk}} + (a = t, r, i_1, \ldots, i_{d-2}) \] (4.8)
\[ G(p)_y = (G(p)_y)^{\text{bulk}}. \]
The bulk terms contribute to match the bulk \( \Lambda \delta_0^a \) as in (3.4). But now (3.4) is modified to match the brane tensions (\( \Lambda_1 \) and \( \Lambda_2 \), say, for the branes at \( y = 0 \) and \( y = Y \) respectively) by the contributions induced by the terms proportional to \( \hat{h}_2 \) in (4.8). This gives
\[ \left\{ \sum_{p=1}^{P} \kappa(p) \frac{(-1)^{p-1}}{2^{p-1} (p-1)!} (d-1) \ldots (d-2p+1)(k^2)^{p-1} \right\} (a(y))^{-1} (\lambda_1 \delta(y) + \lambda_2 \delta(y - Y)) = \Lambda_1 \delta(y) + \Lambda_2 \delta(y - Y) \] (4.9)
Setting,
\[ K(p) = \left\{ \sum_{p=1}^{P} \kappa(p) \frac{(-1)^{p-1}}{2^{p-1} (p-1)!} (d-1) \ldots (d-2p+1)(k^2)^{p-1} \right\} \] (4.10)
and using (4.6) and (4.7), one obtains
\[ \Lambda_1 = K(p) 2k \tanh ky_0 \] (4.11)
\[ \Lambda_2 = K(p) 2k \frac{\sinh (y_0 - Y)}{\cosh ky_0} . \] (4.12)
For \( y_0 > Y > 0 \) and \( \kappa(1) \gg \kappa(p) \), \( (p = 2, 3, \ldots) \), \( \Lambda_1 \) and \( \Lambda_2 \) are both positive. The results remain formally similar to the corresponding ones of [4] for the following reason. The linearity of each \( G(p)_a \) \( (a = t, r, i) \) of (4.8) in \( h_2 \) (and hence \( \hat{h}_2 \)) assure terms linear in the \( \delta \)'s consistently on both sides of (4.9). Furthermore the cumulative effect of all the superposed GB terms, is accounted for by the factor \( K(p) \), (4.10).
5 Remarks

In the previous sections we have proceeded in successive steps: a particularly compact and convenient construction of the generalised Einstein tensors, then the construction of bulk solutions and finally the insertion of branes.

The two functions $h_2(y)$ and $M(r, y)$ introduced in (2.8) and (2.9) respectively, played a crucial role in our construction. On writing the warp function $f(y)$ as

$$f(y) = a^2(y)$$

these functions are expressed as

$$h_2(y) = a^{-1}(y) \frac{d^2 a(y)}{dy^2}$$

(5.13)

$$M(r, y) = a^{-2}(y) \left( L(r) - r^2 \left( \frac{da(y)}{dy} \right)^2 \right).$$

(5.14)

It was shown how simply and systematically the results of [1] (where the coordinate $y$ was absent and the warp factor was unity) can be generalised to the present case by simply replacing the role of the function $\left( \frac{L(r)}{r^2} \right)$ in [1] with the function $\left( \frac{M(r)}{r^2} \right)$ here, as well as adding a term linear in $h_2(y)$ in $G_i^i, G_r^r, G_i^r \ (i = 1, 2, ..., d - 2)$. There is now, of course, an additional component of the Einstein tensor $G_y^y$ which has simply depends only on $\left( \frac{M(r)}{r^2} \right)$ but not on $h_2(y)$. (See Eqns. (2.27), (2.28) and (2.29).) Our formalism directly selects out these crucial functions.

One can now envisage the following steps. Linear perturbations of the metric will reveal the consequences, in our context, of the GB terms on the zero and Kaluza-Klein modes. In particular, how the relevant results of [1] are generalised.

Also, rather than seeking possible localisation of gravity in $d$-dimensions, one can introduce intersecting branes [9], leading to suitably lower dimensional intersections. These aspects will be studied elsewhere.

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