PLASTICITY OF THE UNIT BALL
OF SOME $C(K)$ SPACES

MICHELLE PAKHOURY

Abstract. We show that if $K$ is a compact metrizable space with finitely many accumulation points, then the closed unit ball of $C(K)$ is a plastic metric space, which means that any non-expansive bijection from $B_{C(K)}$ onto itself is in fact an isometry. We also show that if $K$ is a zero-dimensional compact Hausdorff space with a dense set of isolated points, then any non-expansive homeomorphism of $B_{C(K)}$ is an isometry.

1. Introduction

Following [9], a metric space $(M, \rho)$ is said to be plastic if every non-expansive bijection $F : M \to M$ is in fact an isometry (“non-expansive” means “1-Lipschitz”). For example, it is a well known classical fact that any compact metric space is plastic; more generally, any totally bounded metric space is plastic (see [9] for a proof and for historical references, and see [1] for an extension to the setting of uniform spaces). On the other hand, every non-trivial normed space is non-plastic, as shown by $F(x) = \frac{1}{2}x$. Several interesting examples can be found in [9].

In recent years, the following intriguing problem was considered by a number of authors: is it true that for any real Banach space $X$, the closed unit ball $B_X$ is plastic? As the formulation suggests, no counterexample is known. On the other hand, several positive results have been obtained. For example, the following Banach spaces have a plastic unit ball (we state the results more or less in chronological order):

- Any finite-dimensional normed space $X$, since in this case $B_X$ is compact.
- Any strictly convex space ([3]).
- The space $\ell_1$ ([3]).
- Any Banach space whose unit sphere is the union of its finite-dimensional polyhedral extreme subsets ([1]).
- Any $\ell_1$-direct sum of strictly convex spaces ([3]).
- The space $c_0$ if one weakens the definition of plasticity by considering only non-expansive bijections with a continuous inverse ([3]).
- Any $\ell_\infty$-direct sum of two strictly convex spaces ([3]).
- The space $\ell_1 \oplus_2 \mathbb{R}$ ([3]).

2020 Mathematics Subject Classification. 46B20, 46B25, 51F30.
Key words and phrases. Plasticity; non-expansive bijections; isometries; $C(K)$ spaces.

The author would like to thank Etienne Matheron for his help regarding this paper.

This work was supported in part by the project FRONT of the French National Research Agency (grant ANR-17-CE40-0021).
Obvious “missing examples” are the spaces $L_1 = L_1(0, 1)$ or $C(K)$, where $K$ is a compact Hausdorff space (here, of course, $C(K)$ is the Banach space of all real-valued continuous functions on $K$).

Note that the space $c$ mentioned above is a $C(K)$ space, in fact the simplest infinite-dimensional such space: indeed, $c$ is isometric to $C(K)$ for any compact metric space $K$ with exactly 1 accumulation point. In this paper, we enlarge a little bit the list of Banach spaces with a plastic unit ball by proving the following result.

**Theorem 1.1.** If $K$ is a compact metrizable space with finitely many accumulation points, then $B_{C(K)}$ is plastic.

The assumptions made on the space $K$ in Theorem 1.1 are, of course, extremely strong. However, it will turn out that in most parts of the proof they can be relaxed; and this will give the following result, whose conclusion is weaker but which applies to a much more general class of compact Hausdorff spaces. Recall that a topological space is said to be zero-dimensional if it has a basis consisting of clopen sets.

**Theorem 1.2.** If $K$ is a zero-dimensional compact Hausdorff space with a dense set of isolated points, then any non-expansive homeomorphism $F : B_{C(K)} \to B_{C(K)}$ is an isometry.

**Corollary 1.3.** If $K$ is a countable compact Hausdorff space, then any non-expansive homeomorphism of $B_{C(K)}$ is an isometry.

**Proof.** This is obvious by Theorem 1.2.

**Corollary 1.4.** Any non-expansive homeomorphism of $B_{\ell_\infty}$ is an isometry.

**Proof.** The space $\ell_\infty$ is isometric to $C(\beta N)$, where $\beta N$ is the Stone-Čech compactification of $N$. The space $\beta N$ is zero-dimensional with a dense set of isolated points (namely $N$), so we may apply Theorem 1.2.

It would be nice to remove the assumption that $F^{-1}$ is continuous in Theorem 1.2 since this would say that $B_{C(K)}$ is plastic. Actually, it seems quite plausible that for an arbitrary Banach space $X$, any non-expansive bijection $F : B_X \to B_X$ is a homeomorphism; but we have been unable to prove that.

The paper is organized as follows. In Section 2 we collect some preliminary facts. In Section 3, which is the longest part of the paper, we prove Theorem 1.1. As may be guessed, several ideas are borrowed from the clever proof given by N. Leo [7] in the case of $c$. However, exactly as in the cases of $\ell_1$-sums of strictly convex spaces in [6] and of $\ell_\infty$-sums of two strictly convex spaces in [3], a number of technicalities appear, which makes the whole proof rather long. (What is in fact missing is a general result concerning direct sums of Banach spaces with plastic unit balls, which is yet to be found.) In Section 4 we prove Theorem 1.2 by slightly modifying the proof of Theorem 1.1. Finally, in Section 5 we prove some additional results in the spirit of Theorem 1.2.
2. Preliminary facts

In this short section, we collect a few results that will be needed for the proof of Theorem 1.1.

The following general theorem was proved in [3].

**Theorem 2.1.** Let $X$ be a normed space. Denote by $S_X$ and $B_X$ the unit sphere and the closed unit ball of $X$ respectively, and by $\text{ext}(B_X)$ the set of extreme points of $B_X$.

The following facts hold true for any non-expansive bijection $F : B_X \to B_X$:

1) $F(0) = 0$, 
2) if $x \in S_X$, then $F^{-1}(x) \in S_X$, 
3) if $x \in \text{ext}(B_X)$, then $F^{-1}(x) \in \text{ext}(B_X)$ and $F^{-1}(\alpha x) = \alpha F^{-1}(x)$ for every $\alpha \in [-1, 1]$.

Next, we state Mankiewicz’s local version of the classical Mazur-Ulam Theorem ([8], see also [2] Theorem 14.1]).

**Theorem 2.2.** Let $X$ and $Y$ be normed spaces and let $U$ be a subset of $X$ and $V$ be a subset of $Y$. If $U$ and $V$ are convex with non-empty interior, then any isometric bijection $\Psi : U \to V$ extends to an affine isometric bijection $\widetilde{\Psi} : X \to Y$.

The following lemma is the natural generalization of [4] Lemma 4.1]. The proof is very similar.

**Lemma 2.3.** Let $K$ be a compact Hausdorff space, and let $f$ and $g$ be two non-zero elements of $B_{C(K)}$. Then, the closed balls $\overline{B}(f, 1)$ and $\overline{B}(g, 1)$ cover $B_{C(K)}$ if, and only if, there exist an isolated point $u$ of $K$ and real numbers $c, d$ such that $f = c1_{\{u\}}, g = d1_{\{u\}}$ and $cd < 0$.

**Proof.** Suppose that $B_{C(K)} \subset \overline{B}(f, 1) \cup \overline{B}(g, 1)$.

Since $f \neq 0$, we may choose $u \in K$ such that $f(u) \neq 0$.

Suppose that there exists $v \neq u \in K$ such that $g(v) \neq 0$. By the Tietze extension Theorem, one can find a function $h \in B_{C(K)}$ such that $h(u) = -(\text{sgn}(f(u)))$ and $h(v) = -\text{sgn}(g(v))$. Then, $|h(u) - f(u)| > 1$ and $|h(v) - g(v)| > 1$; so $h \notin \overline{B}(f, 1) \cup \overline{B}(g, 1)$, which is a contradiction.

Hence, we have $g(x) = 0$, for all $x \neq u$. Since $g \neq 0$, it follows that $g = d1_{\{u\}}$ for some real number $d \neq 0$. Note that this implies that $u$ is an isolated point of $K$ because $g$ is continuous.

Now, if we repeat the same argument, we get $f(x) = 0$, for all $x \neq u$. Hence, $f = c1_{\{u\}}$ for some real number $c \neq 0$.

Finally we have $cd < 0$. Indeed, if $cd > 0$, then the function $-\text{sgn}(c)1_{\{u\}} = -\text{sgn}(d)1_{\{u\}} \in B_{C(K)}$ does not belong to $\overline{B}(f, 1) \cup \overline{B}(g, 1)$, which is a contradiction.

For the converse implication, suppose that there exists an isolated point $u$ of $K$ such that $f = c1_{\{u\}}, g = d1_{\{u\}}$ and $cd < 0$.

Let $h \in B_{C(K)}$ be arbitrary. If $h(u) = 0$, then $h \in \overline{B}(f, 1) \cap \overline{B}(g, 1)$. If $h(u) \neq 0$, then either $\text{sgn}(h(u)) = \text{sgn}(c)$, in which case $h \in \overline{B}(f, 1)$, or $\text{sgn}(h(u)) = \text{sgn}(d)$, in which case $h \in \overline{B}(g, 1)$. Hence, we see that $B_{C(K)} \subset \overline{B}(f, 1) \cup \overline{B}(g, 1)$. □
Finally, we will use a few times the following well known fact.

**Remark 2.4.** Let $K$ be a zero-dimensional compact Hausdorff space. If $A \subset \mathbb{R}$ has empty interior, then the set $D = \{ f \in C(K); \forall x \in K, f(x) \notin A \}$ is dense in $C(K)$.

**Proof.** Since $K$ is zero-dimensional, the set of all $\varphi \in C(K)$ taking only finitely many values is dense in $C(K)$ by the Stone-Weierstrass Theorem. So it is enough to show that any such function $\varphi$ can be approximated by some $f \in D$; which is clear since $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$. $\square$

3. Proof of Theorem 1.1

In what follows, we consider a compact Hausdorff space $K$ with a finite number of accumulation points. We denote by $K'$ the set of all accumulation points of $K$. We also assume that $K$ is infinite (so $K' \neq \emptyset$), since otherwise $C(K)$ is finite-dimensional and we already know that $B_{C(K)}$ is plastic.

Note that $K$ is necessarily zero-dimensional. Moreover, $K \setminus K'$ is dense in $K$, a fact that will be used repeatedly in the proof of Theorem 1.1.

For simplicity, we write $B$ instead of $B_{C(K)}$, and we denote by $\overline{B}(f,r)$ the closed ball with center $f \in C(K)$ and radius $r$.

From now on, we fix a non-expansive bijection $F : B \to B$. Our aim is to show that $F$ is an isometry. At some point we will need to assume that $K$ is metrizable; but for the beginning of the proof this is not necessary.

3.1. A reduction. The following lemma will allow us to reduce the proof that $F$ is an isometry to a more tractable job, namely proving that a suitably defined non-expansive bijection from $B$ onto itself with some additional properties is the identity map.

**Lemma 3.1.** There exist a homeomorphism $\sigma : K \to K$ and a continuous function $\alpha : K \to \{-1,1\}$ such that, for every $f \in B$ and $a \in K \setminus K'$, we have the following:

1) if $f(a) = 0$, then $F(f)(\sigma(a)) = 0$,
2) if $f(a)\alpha(a) < 0$, then $F(f)(\sigma(a)) \leq 0$,
3) if $f(a)\alpha(a) > 0$, then $F(f)(\sigma(a)) \geq 0$.

**Proof.** As suggested above, in this proof we will not assume that the space $K$ is metrizable; so we will use nets rather than sequences.

We will first define a map $\sigma_0 : K \setminus K' \to K \setminus K'$ and a continuous function $\alpha : K \to \{-1,1\}$, then show that $\sigma_0$ can be extended to a homeomorphism $\sigma : K \to K$, and finally prove 1), 2) and 3).

**Step 1.** For any $a \in K \setminus K'$, there exist a point $\sigma_0(a) \in K \setminus K'$ and a continuous and strictly monotonic function $c_a : [-1,1] \to [-1,1]$ with $c_a(0) = 0$ (so $c_a(-1)c_a(1) < 0$) such that

$$F(t\mathbb{1}_{\{a\}}) = c_a(t)\mathbb{1}_{\{\sigma_0(a)\}} \quad \text{for all } t \in [-1,1].$$

**Proof.** Let $a \in K \setminus K'$.

Lemma 2.3 implies that $B \subset \overline{B}(\mathbb{1}_{\{a\}},1) \cup \overline{B}(-\mathbb{1}_{\{a\}},1)$. As $F$ is non-expansive and surjective, it follows that $B \subset \overline{B}(F(\mathbb{1}_{\{a\}}),1) \cup \overline{B}(F(-\mathbb{1}_{\{a\}}),1).$ Moreover, Theorem 2.1 (item 1) implies that $F(\mathbb{1}_{\{a\}})$ and $F(-\mathbb{1}_{\{a\}})$ are non-zero. By Lemma 2.3 one can find
\[ \sigma_0(a) \in K \setminus K' \text{ such that } F(\mathbb{1}_{\{a\}}) \text{ and } F(-\mathbb{1}_{\{a\}}) \text{ are non-zero multiples of } \mathbb{1}_{\{\sigma_0(a)\}} \text{ and } F(\mathbb{1}_{\{a\}})(\sigma_0(a)) F(-\mathbb{1}_{\{a\}})(\sigma_0(a)) < 0. \]

Let us show that for any \( t \in [-1, 1] \), we have \( F(t\mathbb{1}_{\{a\}}) = c_a(t)\mathbb{1}_{\{\sigma_0(a)\}} \) for some real number \( c_a(t) \in [-1, 1] \).

Let \( t \in [-1, 1] \) and let \( f = t\mathbb{1}_{\{a\}} \). If \( t = 0 \), then \( f = 0 \), and Theorem 2.4 implies that \( F(f) = 0 \). If \( t > 0 \), then Lemma 2.3 implies that \( B \subset \overline{B}(f, 1) \cup \overline{B}(-\mathbb{1}_{\{a\}}, 1) \). As \( F \) is non-expansive and surjective, it follows that \( B \subset \overline{B}(F(f), 1) \cup \overline{B}(F(-\mathbb{1}_{\{a\}}), 1) \).

Since \( F(f) \) and \( F(-\mathbb{1}_{\{a\}}) \) are non-zero by Theorem 2.4 (item 1), Lemma 2.3 and the definition of \( \sigma_0(a) \) imply that \( F(f) \) is a non-zero multiple of \( \mathbb{1}_{\{\sigma_0(a)\}} \). The case \( t < 0 \) is analogous. So we have shown that \( F(t\mathbb{1}_{\{a\}}) = c_a(t)\mathbb{1}_{\{\sigma_0(a)\}} \) for some real number \( c_a(t) \), and \( c_a(t) \in [-1, 1] \) since \( F(t\mathbb{1}_{\{a\}}) \in B \).

We have \( c_a(t) = F(t\mathbb{1}_{\{a\}})(\sigma_0(a)) \) for all \( t \in [-1, 1] \). Since \( F \) is continuous, this shows that the function \( c_a : [-1, 1] \to [-1, 1] \) is continuous. Moreover, \( c_a \) is also injective because \( F \) is injective. Hence, \( c_a \) is strictly monotonic.

Let us now define the continuous function \( \alpha := F^{-1}(\mathbb{1}) \), where \( \mathbb{1} \) is the constant function.

Since \( \mathbb{1} \) is an extreme point of \( B \), Theorem 2.1 (item 3) implies that \( \alpha \) is also an extreme point of \( B \), so \( |\alpha| \equiv 1 \).

The following fact will be used several times in the remaining of the proof.

**Fact 3.2.** If \( g \) is any extreme point of \( B \) and \( f = F^{-1}(g) \), then

\[ g(\sigma_0(a)) = \alpha(a)f(a) \quad \text{for all } a \in K \setminus K'. \]

**Proof.** Let \( a \in K \setminus K' \). Since \( g \) is an extreme point of \( B \), Theorem 2.1 (item 3) implies that \( f \) is an extreme point of \( B \), i.e. \( |g| \equiv |f| \equiv 1 \), and it also implies that \( F(f(a)f) = f(a)g \). We have \( \|\mathbb{1}_{\{a\}} - f(a)f\| = 1 \). Since \( F \) is non-expansive and \( F(f(a)f) = f(a)g \), it follows that \( \|F(\mathbb{1}_{\{a\}}) - f(a)g\| \leq 1 \). This implies in particular that \( f(a)g(\sigma_0(a)) \) and \( F(\mathbb{1}_{\{a\}})(\sigma_0(a)) \) have the same sign because \( |f(a)g(\sigma_0(a))| = 1 \) and \( F(\mathbb{1}_{\{a\}})(\sigma_0(a)) \neq 0 \), so we get \( f(a)g(\sigma_0(a)) = \text{sgn}(F(\mathbb{1}_{\{a\}})(\sigma_0(a))). \)

Applying that to \( g = \mathbb{1} \) and (hence) \( f = \alpha \), we see that \( \alpha(a) = \text{sgn}(F(\mathbb{1}_{\{a\}})(\sigma_0(a))). \)

Altogether, for an arbitrary extreme point \( g \) and \( f = F^{-1}(g) \), we obtain \( f(a)g(\sigma_0(a)) = \alpha(a) \), i.e. \( g(\sigma_0(a)) = \alpha(a)f(a) \). \( \square \)

**Step 2.** The map \( \sigma_0 : K \setminus K' \to K \setminus K' \) is bijective.

**Proof.** Let us first show that \( \sigma_0 \) is injective. Let \( a, b \in K \setminus K' \) be such that \( \sigma_0(a) = \sigma_0(b) \).

With the notation of Step 1, since \( c_a \) and \( c_b \) are continuous and since \( c_a(-1)c_b(1) < 0 \) and \( c_b(-1)c_b(1) < 0 \), it follows that \( c_a([−1, 1]) = [u_a, v_a] \) and \( c_b([−1, 1]) = [u_b, v_b] \), where \( u_a < 0 < v_a \) and \( u_b < 0 < v_b \). Hence, there exist \( t_1, t_2 \in [−1, 1] \setminus \{0\} \) such that \( c_a(t_1) = c_b(t_2) \). And since \( \sigma_0(a) = \sigma_0(b) \), we get \( F(t_1\mathbb{1}_{\{a\}}) = F(t_2\mathbb{1}_{\{b\}}) \), and this implies that \( t_1\mathbb{1}_{\{a\}} = t_2\mathbb{1}_{\{b\}} \) since \( F \) is injective. Therefore, \( t_1 = t_2 \) and \( a = b \) since \( t_1 \) and \( t_2 \) are non-zero.
Now, let us show that $\sigma_0$ is surjective. Towards a contradiction, suppose that there exists $y \in K \setminus K'$ such that $y \neq \sigma_0(x)$ for all $x \in K \setminus K'$.
Let $h$ be the following extreme point of $B$:
$$h(x) = \begin{cases} 
1 & \text{if } x \neq y, \\
-1 & \text{if } x = y.
\end{cases}$$

By Fact 3.2, we have for every $a \in K \setminus K'$:
$$F^{-1}(h)(a) = \alpha(a)h(\sigma_0(a)) = \alpha(a) \quad \text{because } \sigma_0(a) \neq y.$$  

Since $K \setminus K'$ is dense in $K$, we get $F^{-1}(h) = \alpha = F^{-1}(1)$. Hence, $h = 1$, which is a contradiction.

\[ \square \]

**Step 3.** The map $\sigma_0 : K \setminus K' \to K \setminus K'$ extends to a continuous map $\sigma : K \to K$.

**Proof.** Since $K \setminus K'$ is dense in $K$, it is enough to prove the following: if $z \in K'$ and if $(a_d)_{d \in D}$ is any net in $K \setminus K'$ such that $a_d \to z$, then the net $(\sigma_0(a_d))$ has at most one cluster point. Indeed, since $K$ is compact we may then define $\sigma(z) = \lim \sigma_0(a_d)$ for any net in $K \setminus K'$ such that $a_d \to z$ (the limit will not depend on $(a_d)$), and the map $\sigma : K \to K$ will be continuous.

So let us fix $z \in K'$ and a net $(a_d)_{d \in D} \subset K \setminus K'$ such that $a_d \to z$.

Suppose that the net $(\sigma_0(a_d))$ admits two different cluster points $u$ and $v$, i.e. there exist two subnets $(a_{d_i})_{i \in I}$ and $(a_{d_j})_{j \in J}$ of $(a_d)$ such that
$$\sigma_0(a_{d_i}) \to u \quad \text{and} \quad \sigma_0(a_{d_j}) \to v.$$  

Since $K$ is zero-dimensional, one can find a clopen neighbourhood $U$ of $u$ such that $v \notin U$. Let $g$ be the following extreme point of $B$:
$$g(x) = \begin{cases} 
1 & \text{if } x \in U, \\
-1 & \text{otherwise}.
\end{cases}$$  

By the definition of $g$, we have
$$g(\sigma_0(a_{d_i})) = 1 \quad \text{and} \quad g(\sigma_0(a_{d_j})) = -1 \quad \text{for all large enough } i, j.$$  

Denoting $F^{-1}(g)$ by $f$, we obtain from Fact 3.2 that $f(a_{d_i}) = \alpha(a_{d_i})$ and $f(a_{d_j}) = -\alpha(a_{d_j})$ for all large enough $i, j$. Since $f$ and $\alpha$ are continuous and $a_d \to z$, it follows that $\alpha(z) = f(z) = -\alpha(z)$, which is a contradiction since $|\alpha(z)| = 1$. Thus, we have shown that $\sigma_0$ extends to a continuous map $\sigma : K \to K$.  

Now, let us state the following consequence of Fact 3.2.

**Fact 3.3.** If $g$ is any extreme point of $B$, then
$$F^{-1}(g)(x) = \alpha(x)g(\sigma(x)) \quad \text{for all } x \in K.$$  

**Proof.** Fact 3.2 says that $F^{-1}(g)(x) = \alpha(x)g(\sigma(x))$, for all $x \in K \setminus K'$, and this completes the proof since $\alpha$ and $\sigma$ are continuous and $K \setminus K'$ is dense in $K$.  

\[ \square \]

**Step 4.** The map $\sigma : K \to K$ is a homeomorphism.
Proof. The map $\sigma : K \to K$ is surjective because $\sigma(K)$ is a compact set containing $\sigma_0(K \setminus K') = K \setminus K'$ and $K \setminus K'$ is dense in $K$.

It remains to prove that $\sigma$ is injective.
First, we observe that $\sigma(K') \subset K'$. Indeed, let $b \in K'$. Since $K \setminus K'$ is dense in $K$, any neighbourhood of $b$ contains infinitely points of $K \setminus K'$. Since $\sigma$ is continuous and $\sigma|_{K \setminus K'} = \sigma_0$ is injective, it follows that any neighbourhood of $\sigma(b)$ contains infinitely points of $K$; and hence $\sigma(b) \in K'$.

So the map $\sigma : K \to K$ is surjective, it maps $K \setminus K'$ bijectively onto itself by Step 2, and $\sigma(K') \subset K'$. Since $K'$ is a finite set, it follows that $\sigma$ is injective.

Therefore, $\sigma : K \to K$ is a continuous bijection, hence a homeomorphism since $K$ is compact.  \( \square \)

**Step 5.** 1), 2) and 3) hold true for every $f \in D := \{ g \in B; \forall z \in K', g(z) \neq 0 \}$.

**Proof.** Let $f \in D$ and let

$$S = \{ x \in K; f(x) \neq 0 \}.$$ 

Since $f \in D$, we see that $S$ is an open set containing $K'$, and hence a clopen subset of $K$ (with finite complement).

Since $\sigma$ is a homeomorphism and $f \in D$, we may define the following extreme point of $B$:

$$\psi(\sigma(x)) = \begin{cases} 
\alpha(x) \text{sgn}(f(x)) & \text{if } x \in S, \\
1 & \text{otherwise.}
\end{cases}$$

By Fact 3.3 we have

$$F^{-1}(\psi)(x) = \begin{cases} 
\text{sgn}(f(x)) & \text{if } x \in S, \\
\alpha(x) & \text{otherwise.}
\end{cases}$$

Hence, $||F^{-1}(\psi) - f|| \leq 1$. Since $F$ is non-expansive, it follows that $||\psi - F(f)|| \leq 1$, and this implies 2) and 3).

To prove 1), consider $y \in K \setminus K'$ such that $f(y) = 0$ (then $y \notin S$).

We know that $||\psi - F(f)|| \leq 1$, so $F(f)(\sigma(y)) \geq 0$ since $\psi(\sigma(y)) = 1$.

Now, let us define the following extreme point of $B$:

$$\phi(x) = \begin{cases} 
\psi(x) & \text{if } x \neq \sigma(y), \\
-1 & \text{if } x = \sigma(y).
\end{cases}$$

We have, as before, that $||\phi - F(f)|| \leq 1$, which implies that $F(f)(\sigma(y)) \leq 0$.

Therefore, $F(f)(\sigma(y)) = 0$. \( \square \)

We have just proved 1), 2) and 3) for all $f \in D$. Since $D$ is dense in $B$ by Remark 2.4 and $F$ is continuous, it follows immediately that 2) and 3) hold true for every $f \in B$. Finally, if $f \in B$ and if $f(a) = 0$ for some $a \in K \setminus K'$, then $f$ can be approximated by elements $g$ of $D$ such that $g(a) = 0$. So 1) holds true as well for any $f \in B$. \( \square \)
Remark. In the proof of Lemma 3.1 the fact that $K'$ is a finite set was used just once, at the end of the proof of Step 4, to ensure that the map $\sigma$ is injective; and this assumption will not be needed anywhere else in the proof of Theorem 1.1.

However, note that if $K$ has infinitely many accumulation points, then it may be possible to construct a surjective continuous map $\sigma : K \to K$ such that $\sigma$ maps bijectively $K \setminus K'$ onto itself, $\sigma(K') = K'$ and yet $\sigma$ is not injective. Consider for example the simplest compact space $K$ with infinitely many accumulation points, i.e.

$$K = \{x_{n,m}; n, m \in \mathbb{N}\} \cup \{x_{n,\infty}; n \in \mathbb{N}\} \cup \{x_{\infty,\infty}\}$$

where all points are distinct, $x_{n,m} \to x_{n,\infty}$ as $m \to \infty$ for each $n \in \mathbb{N}$, and $x_{n,\infty} \to x_{\infty,\infty}$ as $n \to \infty$. Define $\sigma : K \to K$ as follows:

- $\sigma(x_{\infty,\infty}) = x_{\infty,\infty}$,
- $\sigma(x_{n,\infty}) = x_{(n-1),\infty}$ for all $n \geq 2$, and $\sigma(x_{1,\infty}) = x_{1,\infty}$,
- $\sigma(x_{n,m}) = x_{(n-1),m}$ for all $n \geq 3$ and $m \in \mathbb{N}$,
- $\sigma(x_{2,m}) = x_{1,(2m-1)}$ and $\sigma(x_{1,m}) = x_{1,2m}$ for all $m \in \mathbb{N}$.

Then, $\sigma$ has the properties stated above.

With the notation of Lemma 3.1 let us consider the linear isometric bijection $J : C(K) \to C(K)$ defined as follows: for every $f \in C(K)$ and $x \in K$,

$$Jf(\sigma(x)) = \alpha(x)f(x).$$

In other words,

$$J = M_{\alpha\sigma^{-1}}C_{\sigma^{-1}},$$

where $M_{\alpha\sigma^{-1}}$ is the isometric multiplication operator defined by the unimodular function $\alpha \circ \sigma^{-1}$ and $C_{\sigma^{-1}}$ is the isometric composition operator defined by the homeomorphism $\sigma^{-1}$.

Considering $J_B$ as an isometric bijection of $B$ onto itself, the proof of Theorem 1.1 will be complete if we can show that $J_B = F$. In other words, setting

$$\Phi = (J_B)^{-1} \circ F,$$

our aim is now to show that $\Phi = Id_B$. Note that by definition of $\Phi$, we have $\Phi = M_{\alpha}C_{\sigma} \circ F$, i.e.

$$\Phi(f)(x) = \alpha(x)F(f)(\sigma(x))$$

for every $f \in B$ and $x \in K$. The next lemma summarizes all what we need to know about $\Phi$ in what follows.

**Lemma 3.4.** The map $\Phi$ is a non-expansive bijection of $B$ onto itself. Moreover, for any $f, g \in B$ and every $x \in K \setminus K'$, the following facts hold true.

1) If $f(x) = 0$, then $\Phi(f)(x) = 0$.
2) If $f(x) < 0$, then $\Phi(f)(x) \leq 0$.
3) If $f(x) > 0$, then $\Phi(f)(x) \geq 0$.
4) If $g(x) < 0$, then $\Phi^{-1}(g)(x) < 0$.
5) If $g(x) > 0$, then $\Phi^{-1}(g)(x) > 0$.
6) If $g$ is an extreme point of $B$, then $\Phi(g) = g$. 
Proof. Properties 1), 2) and 3) are immediate consequences of (3.1) and Lemma 3.1. If \( g(x) < 0 \) then, applying [1] and [3] to \( f := \Phi^{-1}(g) \), we see that we cannot have \( \Phi^{-1}(g)(x) \geq 0 \); which proves [4]. The proof of [5] is the same.

Let us now prove (6). Assume that \( g \) is an extreme point of \( B \). With the notation of Lemma 3.1, for every \( x \in K \), we have

\[
(3.2) \quad \mathcal{J} g(\sigma(x)) = \alpha(x)g(x).
\]

Since \( g \) is an extreme point of \( B \), (3.2) implies that \( \mathcal{J} g \) is also an extreme point of \( B \). Hence, by Fact 3.3, we get \( F^{-1}(\mathcal{J} g)(x) = g(x) \), for every \( x \in K \). This implies that \( \Phi^{-1}(g) = g \), which is equivalent to \( \Phi(g) = g \). □

To summarize this subsection: we have defined a non-expansive bijection \( \Phi : B \to B \) satisfying the properties stated in Lemma 3.4, and Theorem 1.1 will be proved if we can show that \( \Phi = Id_B \).

From now on, we will assume that \( K \) is metrizable. This assumption will be needed to ensure that the closed set \( K' \) is a "zero set", i.e. there is a continuous function \( \chi : K \to \mathbb{R} \) such that \( K' = \{ \chi = 0 \} \). And this will be used only in the proof of Fact 3.14 below.

On the other hand, the fact that \( K' \) is a finite set will not be needed in the remaining of the proof.

3.2. The main lemma. The fact that \( \Phi = Id_B \) will follow easily from Lemma 3.5 below. Let us first introduce some notation. Given \( f \in B \), we define

\[
u_{0,f} = \min \{|f(z)|; z \in K'\}
\]

and for any \( n \in \mathbb{Z}_+ \), we set

\[
u_{n,f} = \frac{n + \nu_{0,f}}{n + 1}
\]

Lemma 3.5. For each \( n \in \mathbb{Z}_+ \), we have the following: for each \( f \in B \) such that \( \nu_{0,f} \notin \{0, 1\} \) and \( x \in K \setminus K' \),

1) if \( |f(x)| < \nu_{n,f} \), then \( \Phi(f)(x) = f(x) \),
2) if \( f(x) \geq \nu_{n,f} \), then \( \Phi(f)(x) \in [\nu_{n,f}, f(x)] \),
3) if \( f(x) \leq -\nu_{n,f} \), then \( \Phi(f)(x) \in [f(x), -\nu_{n,f}] \).

Assume that we have been able to prove Lemma 3.5. Then we easily deduce

Corollary 3.6. For each \( f \in B \) such that \( \nu_{0,f} \notin \{0, 1\} \), we have \( \Phi(f) = f \).

Proof. Let \( f \in B \) be such that \( \nu_{0,f} \notin \{0, 1\} \) and let \( x \in K \setminus K' \). If \( |f(x)| < 1 \) then, since \( \nu_{n,f} \to 1 \) as \( n \to \infty \), there exists \( n \in \mathbb{Z}_+ \) such that \( |f(x)| < \nu_{n,f} \), so \( \Phi(f)(x) = f(x) \) by Lemma 3.5 (item 1). If \( f(x) = 1 \), then \( f(x) \geq \nu_{n,f} \) for all \( n \in \mathbb{Z}_+ \), and hence \( \Phi(f)(x) = 1 = f(x) \) by Lemma 3.5 (item 2). Similarly, if \( f(x) = -1 \) then \( \Phi(f)(x) = -1 = f(x) \). Hence, for every \( x \in K \setminus K' \), we have \( \Phi(f)(x) = f(x) \), and this implies that \( \Phi(f) = f \) since \( K \setminus K' \) is dense in \( K \). □

Using Corollary 3.6, it is very easy to show that \( \Phi = Id_B \), and hence to conclude the proof of Theorem 1.1. Let \( D = \{ f \in B; \forall z \in K', f(z) \notin \{0, 1, -1\} \} \).
The set \( D \) is contained in \( \{ f \in B; u_{0,f} \notin \{0,1\} \} \); so, by Corollary 3.6, we have \( \Phi(f) = f \) for all \( f \in D \). Since \( \Phi \) is continuous and \( D \) is dense in \( B \) by Remark 2.4, it follows that \( \Phi = \text{Id}_B \).

In view of the previous discussion, our only task is now to prove Lemma 3.5, which will occupy us for a while.

### 3.3. Proof of the main lemma: \( n = 0 \)

The case \( n = 0 \) of Lemma 3.5 will be deduced from the next two lemmas.

**Lemma 3.7.** For any \( f, g \in B \) and every \( x \in K \setminus K' \), the following implications hold true.

1. If \( f(x) < 0 \), then \( \Phi(f)(x) \in [f(x), 0] \).
2. If \( f(x) > 0 \), then \( \Phi(f)(x) \in [0, f(x)] \).
3. If \( g(x) < 0 \), then \( \Phi^{-1}(g)(x) \in [-1, g(x)] \).
4. If \( g(x) > 0 \), then \( \Phi^{-1}(g)(x) \in [g(x), 1] \).
5. If \( g(x) = \pm 1 \), then \( \Phi^{-1}(g)(x) = g(x) \).

**Proof.** To prove 1), let \( f \in B \) and let \( x \in K \setminus K' \) be such that \( f(x) < 0 \); we want to show that \( \Phi(f)(x) \in [f(x), 0] \). Note that by Lemma 3.4, we already know that \( \Phi(f)(x) \leq 0 \); so we only need to show that \( \Phi(f)(x) \geq f(x) \).

Let us first assume that \( f(z) \neq 0 \) for every \( z \in K \). Then, we can define \( g \in B \) as follows:

\[
g(y) = \begin{cases} 
1 & \text{if } y = x, \\
\text{sgn}(f(y)) & \text{if } y \neq x.
\end{cases}
\]

We have \( ||f - g|| = 1 - f(x) \). Since \( g \) is an extreme point of \( B \), Lemma 3.4 implies that \( \Phi(g) = g \). Hence, \( ||\Phi(f) - g|| = ||\Phi(f) - \Phi(g)|| \leq 1 - f(x) \), because \( \Phi \) is non-expansive. Then,

\[
|\Phi(f)(x) - g(x)| \leq 1 - f(x) \implies 1 - \Phi(f)(x) \leq 1 - f(x) \implies \Phi(f)(x) \geq f(x).
\]

Now consider any \( f \in B \) such that \( f(x) < 0 \). One can find a sequence \( (f_n)_{n \in \mathbb{N}} \subset B \) such that \( f_n \rightharpoonup f \), \( f_n(x) = f(x) \) for all \( n \) and the functions \( f_n \) never vanish on \( K \). Then, \( \Phi(f_n)(x) \in [0, f_n(x)] \) for all \( n \), and hence \( \Phi(f_n)(x) \in [f(x), 0] \) because \( \Phi \) is continuous.

The proof of 2) is the same. (First consider \( f \in B \) such that \( f(x) > 0 \) and \( f(z) \neq 0 \) for every \( z \in K \). Define \( g \) as above but with \( g(x) = -1 \). Then conclude by approximation.)

Let us prove 3) and 4). If \( g \in B \) and \( x \in K \setminus K' \), then

\[
g(x) < 0 \implies \Phi^{-1}(g)(x) < 0 \quad \text{by Lemma 3.4} \implies \Phi(\Phi^{-1}(g))(x) \in [\Phi^{-1}(g)(x), 0] \quad \text{by (1)} \implies -1 \leq \Phi^{-1}(g)(x) \leq g(x).
\]

The case \( g(x) > 0 \) is similar.

Finally, 5) follows from 3) and 4). \( \square \)

**Lemma 3.8.** If \( f \in B \) is such that the set \( S_f = \{ x \in K; f(x) \notin \{-1, 1\} \} \) is finite, then \( \Phi(f) = f \).
Proof. We first observe that for any $f \in B$ such that $S_f$ is finite, we have
$$K' \cap S_f = \emptyset.$$  
Indeed, suppose that there exists $x_0 \in K' \cap S_f$; so any neighbourhood of $x_0$ contains infinitely many points, and $|f(x_0)| < 1$. Since $f$ is continuous, it follows that there exist infinitely many $x \in K$ such that $f(x) \notin \{-1, 1\}$, which is a contradiction.

To prove the lemma, we proceed by induction on the number of elements of $S_f$. If $S_f$ is empty, then $f$ is an extreme point of $B$, and Lemma 5.4 implies that $\Phi(f) = f$. Now, let $M$ be a non-negative integer and suppose that the lemma has been proved for all $f \in B$ such that $\# S_f \leq M$. Let us show that the claim also holds for all $f \in B$ such that $\# S_f = M + 1$.

Let $f \in B$ be such that $\# S_f = M + 1$. We need to show that $\Phi(f) = f$, i.e. $\Phi^{-1}(f) = f$.

Lemma 5.7 implies that
\begin{equation}
\Phi^{-1}(f)(x) = f(x) \quad \text{for all } x \in K \setminus (K' \cup S_f).
\end{equation}
Moreover, if $x \in K'$, then there exists a net $(x_d)_{d \in D}$ in $K \setminus (K' \cup S_f)$ such that $x_d \to x$ (because $K \setminus K'$ is dense in $K$ and $S_f$ is finite). The continuity of $\Phi^{-1}(f)$ implies that $\Phi^{-1}(f)(x_d) \to \Phi^{-1}(f)(x)$, i.e. $f(x_d) \to f(x)$ by (3.3). Since $f$ is continuous, it follows that $\Phi^{-1}(f)(x) = f(x)$.

Hence, we have shown that $\Phi^{-1}(f)(x) = f(x)$ for all $x \in K \setminus S_f$. It remains to show that $\Phi^{-1}(f)(x) = f(x)$ for all $x \in S_f$.

Let $x \in S_f$ (then $x \in K \setminus K'$ since $S_f \cap K' = \emptyset$). We define the following two functions $g$ and $h$ from $K$ to $\mathbb{R}$:
$$g(x) = 1, \quad h(x) = -1 \quad \text{and} \quad g(y) = h(y) = \Phi^{-1}(f)(y) \quad \text{for all } y \neq x.$$

Since $x \in K \setminus K'$, these functions $g$ and $h$ are continuous, and hence $g, h \in B$.

Note that $S_g = \{x \in K; g(x) \notin \{-1, 1\}\}$ and $S_h = \{x \in K; h(x) \notin \{-1, 1\}\}$ are contained in the set $S_f \setminus \{x\}$. Since $\#(S_f \setminus \{x\}) = M$ (because $\# S_f = M + 1$ and $x \in S_f$), it follows that $\# S_h \leq M$ and $\# S_g \leq M$. Therefore, we can apply the induction hypothesis to $g$ and $h$ to obtain $\Phi(g) = g$ and $\Phi(h) = h$.

Since $g, h$ and $\Phi^{-1}(f)$ coincide on $K \setminus \{x\}$, we have
$$||\Phi^{-1}(f) - g|| = ||\Phi^{-1}(f)(x) - g(x)||,$$
and
$$||\Phi^{-1}(f) - h|| = ||\Phi^{-1}(f)(x) - h(x)||.$$
Since $\Phi$ is non-expansive, $\Phi(g) = g$ and $\Phi(h) = h$, it follows that
\begin{equation}
||f(x) - g(x)|| \leq ||f - g|| \leq ||\Phi^{-1}(f) - g|| = ||\Phi^{-1}(f)(x) - g(x)||,
\end{equation}
and
\begin{equation}
||f(x) - h(x)|| \leq ||f - h|| \leq ||\Phi^{-1}(f) - h|| = ||\Phi^{-1}(f)(x) - h(x)||.
\end{equation}
Hence,
\begin{equation}
1 - f(x) \leq 1 - \Phi^{-1}(f)(x) \quad \Rightarrow \quad f(x) \geq \Phi^{-1}(f)(x),
\end{equation}
and
\begin{equation}
f(x) + 1 \leq \Phi^{-1}(f)(x) + 1 \quad \Rightarrow \quad f(x) \leq \Phi^{-1}(f)(x).
\end{equation}
Therefore, \( \Phi^{-1}(f)(x) = f(x) \).

We are now in position to prove the case \( n = 0 \) of Lemma 3.5. We prove in fact a slightly more general result since we do not assume that \( u_{0,f} \neq 1 \).

**Lemma 3.9.** Let \( f \in B \) be such that \( h = u_{0,f} = \min \{ |f(z)|; z \in K' \} \neq 0 \). For any \( x \in K \setminus K' \), we have the following implications:

1. if \( |f(x)| < h \), then \( \Phi(f)(x) = f(x) \),
2. if \( f(x) \geq h \), then \( \Phi(f)(x) \in [h, f(x)] \),
3. if \( f(x) \leq -h \), then \( \Phi(f)(x) \in [f(x), -h] \).

**Proof.** 1) Let \( x \in K \setminus K' \) be such that \( |f(x)| < h \).

If \( f(x) = 0 \), then Lemma 3.4 implies that \( \Phi(f)(x) = 0 \); so we assume that \( f(x) \neq 0 \). In what follows, we set \( \varepsilon = h - |f(x)| > 0 \).

Since \( f : K \to \mathbb{R} \) is continuous, \( u_{0,f} \neq 0 \) and \( K \) is compact and zero-dimensional, one can find clopen sets \( V_1, \ldots, V_N \subset K \) such that

- \( V_i \cap K' \neq \emptyset \) for all \( i \) and \( \bigcup_{i=1}^N V_i \supset K' \),
- the \( V_i \) are pairwise disjoint and do not contain \( x \),
- \( f \) has constant sign \( \varepsilon_i \) on each \( V_i \),
- \( \text{diam}(f(V_i)) < \varepsilon \) for \( i = 1, \ldots, N \).

Now, we define \( g \in B \) as follows:

\[
g(y) = \begin{cases} 
\text{sgn}(f(x)) & \text{if } y = x, \\
\varepsilon_i & \text{if } y \in V_i \text{ for some } i \in \{1, \ldots, N\}, \\
f(y) & \text{if } y \notin \bigcup_{i=1}^N V_i \cup \{x\}.
\end{cases}
\]

Since \( K \setminus \bigcup_{i=1}^N V_i \) is a closed subset of \( K \) that does not contain accumulation points, and since \( K \) is compact, we see that \( K \setminus \bigcup_{i=1}^N V_i \) is finite. Hence, we can apply Lemma 3.8 to \( g \), and we obtain

\( \Phi(g) = g \).

Let us show that \( \|f - g\| \leq 1 - |f(x)| \), i.e., \( |f(y) - g(y)| \leq 1 - |f(x)| \) for all \( y \in K \).

- If \( y \notin \left( \bigcup_{i=1}^N V_i \cup \{x\} \right) \), then \( |f(y) - g(y)| = 0 \).
- If \( y = x \), then \( |f(y) - g(y)| = 1 - |f(x)| \).
- If \( y \in V_i \) for some \( i \), choose \( z \in K' \cap V_i \). Then, \( |f(y) - f(z)| < \varepsilon \) and \( \text{sgn}(f(y)) = \varepsilon_i \).

So, by the definition of \( g \) and \( h \), we get

\[
|f(y) - g(y)| = 1 - |f(y)| < 1 - |f(z)| + \varepsilon \\
\leq 1 - h + \varepsilon = 1 - |f(x)|.
\]

So we have indeed \( \|f - g\| = 1 - |f(x)| \).

Since \( \Phi \) is non-expansive and \( \Phi(g) = g \), it follows that \( \|\Phi(f) - g\| \leq 1 - |f(x)| \), and in particular

\[
|\Phi(f)(x) - g(x)| \leq 1 - |f(x)|. 
\]
If \( f(x) < 0 \), then \( g(x) = -1 \) and hence
\[
\text{(3.6)} \quad \implies \quad 1 + \Phi(f)(x) \leq 1 + f(x) \implies \Phi(f)(x) \leq f(x).
\]
Moreover, Lemma 3.7 implies that \( \Phi(f)(x) \geq f(x) \). So we get \( \Phi(f)(x) = f(x) \).

Similarly, if \( f(x) > 0 \) then \( g(x) = 1 \); so, using (3.6), we obtain \( \Phi(f)(x) \geq f(x) \), and hence \( \Phi(f)(x) = f(x) \) by Lemma 3.7.

2) Let \( x \in K \setminus K' \) be such that \( f(x) \geq h \).
Since \( f(x) \geq h > 0 \), Lemma 3.7 implies that \( \Phi(f)(x) \leq f(x) \). So it remains to show that \( \Phi(f)(x) \geq h \).
Let \( \varepsilon > 0 \) be arbitrary.
Let us choose \( V_1, \ldots, V_N \) as in the proof of 1), and define \( g \in B \) as in the proof of 1), so that \( \Phi(g) = g \).
Let us show that \( \|f - g\| \leq 1 - h + \varepsilon \).
- If \( y \notin \left( \bigcup_{i=1}^{N} V_i \right) \cup \{x\} \), then \( |f(y) - g(y)| = 0 \).
- If \( y = x \), then \( |f(y) - g(y)| = 1 - f(x) \leq 1 - h \).
- If \( y \in V_i \) for some \( i \), take \( z \in V_i \cap K' \). Then, \( |f(y) - f(z)| < \varepsilon \) and \( \text{sgn}(f(y)) = \text{sgn}(f(z)) = \varepsilon_i \), so we get as above \( |f(y) - g(y)| < 1 - h + \varepsilon \). This shows that \( \|f - g\| \leq 1 - h + \varepsilon \).

Since \( f(g) = 1 \) and \( \Phi \) is non-expansive, we get
\[
\|\Phi(f) - g\| = \|\Phi(f) - \Phi(g)\| \leq 1 - h + \varepsilon \implies |\Phi(f)(x) - g(x)| \leq 1 - h + \varepsilon \\
\implies 1 - \Phi(f)(x) \leq 1 - h + \varepsilon \\
\implies \Phi(f)(x) \geq h - \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that \( \Phi(f)(x) \geq h \).

3) The proof is similar to that of 2). \( \square \)

For future use, we state two consequences of Lemma 3.4.

**Corollary 3.10.** For each \( f \in B \), we have \( u_{0, f} = u_{0, \Phi(f)} \).

**Proof.** Let us first show that if \( u_{0, f} = 0 \) then \( u_{0, \Phi(f)} = 0 \).
Let \( z_0 \in K' \) be such that \( f(z_0) = 0 \). Lemma 3.4 (item 1) and Lemma 3.7 (items 1 and 2) imply that \( \Phi(f)(z_0) = 0 \). Then, \( u_{0, \Phi(f)} = 0 \).
Now, let us show that if \( u_{0, f} \neq 0 \) then \( u_{0, \Phi(f)} = u_{0, f} \).
On the one hand, let \( z_0 \in K' \) be such that \( u_{0, f} = |f(z_0)| \). Lemma 3.9 (items 1, 2 and 3) implies that \( \Phi(f)(z_0) = f(z_0) \). Hence, \( u_{0, \Phi(f)} \leq u_{0, f} \).
On the other hand, for any \( z \in K' \), we have \( |f(z)| \geq u_{0, f} \). If \( |f(z)| = u_{0, f} \), then Lemma 3.9 (items 1, 2 and 3) implies that \( \Phi(f)(z) = f(z) \), whereas if \( |f(z)| > u_{0, f} \), then Lemma 3.9 (items 2 and 3) implies that \( |\Phi(f)(z)| \geq u_{0, f} \). Hence, we have \( |\Phi(f)(z)| \geq u_{0, f} \) for all \( z \in K' \), so that \( u_{0, f} \leq u_{0, \Phi(f)} \).
Altogether, we obtain \( u_{0, f} = u_{0, \Phi(f)} \). \( \square \)

**Corollary 3.11.** Let \( g \in B \) be such that \( u_{0, g} \neq 0 \) and \( x \in K \setminus K' \). If \( |g(x)| < u_{0, g} \), then \( \Phi^{-1}(g)(x) = g(x) \).
Moreover, since \( J \) is an isometric bijection. Then, \( J \in \mathbb{B} \) and \( \{ −g \} \). Lemma 3.12. Let \( u, f \Rightarrow \). Remark. Similarly, if \( f \leq −u, f \), then Lemma 3.13 gives a contradiction. So, we have \( |f(x)| < u, f \). Hence, by Lemma 3.14 again, \( f(x) = \Phi(f)(x) \), i.e. \( \Phi^{-1}(g)(x) = g(x) \). \( \square \)

Remark. If we knew that \( \Phi^{-1} \) is continuous then, in Corollary 3.11 it would be enough to assume that \( |g(x)| \leq u, g \). Indeed, for any \( n \in \mathbb{N} \), the function \( g_n \in B \) defined by \( g_n(y) = (1 - \frac{1}{n})g(y) \) and \( g_n(y) = g(y) \) for \( y \neq x \) satisfies \( |g_n(x)| < u, g = u, g_n \), and \( g_n \rightarrow g \) as \( n \rightarrow \infty \). So, applying Corollary 3.11 to \( g_n \), we get \( \Phi^{-1}(g)(x) = g(x) \).

3.4. A digression. In this subsection, we obtain some information on \( \Phi(f) \) when \( f \in B \) takes only a finite number of values. This will be needed for the proof of Lemma 3.13.

Let us fix some notation.

We denote by \( \mathcal{V} \) the family of all finite sequences \( \mathcal{V} = (V_1, \ldots, V_N) \) of clopen subsets of \( K \) such that the \( V_i \) are pairwise disjoint, \( V_i \cap K' \neq \emptyset \) for all \( i \) and \( \bigcup_{i=1}^N V_i \supset K' \).

If \( \mathcal{V} = (V_1, \ldots, V_N) \in \mathcal{V} \) and \( h = (h_1, \ldots, h_N) \in [−1, 1]^N \), we define
\[
B_{\mathcal{V}, h} = \{ f \in B; f|_{V_i} = h_i, \forall i \in \{1, \ldots, N\} \}.
\]

We are going to prove the following lemma.

Lemma 3.12. Let \( \mathcal{V} = (V_1, \ldots, V_N) \in \mathcal{V} \), and let \( h \in (0, 1) \). For any \( h = (h_1, \ldots, h_N) \in \{-h, h\}^N \), we have \( \Phi|_{B_{\mathcal{V}, h}} = Id \).

Proof. This will follow from the next two facts.

Fact 3.13. Let \( \mathcal{V} = (V_1, \ldots, V_N) \in \mathcal{V} \), \( h = (h_1, \ldots, h_N) \in [−1, 1]^N \), and assume that \( h = \min \{ |h_i|; i \in \{1, \ldots, N\} \} \neq 0 \). If \( \Phi(B_{\mathcal{V}, h}) = B_{\mathcal{V}, h} \), then \( \Phi|_{B_{\mathcal{V}, h}} = Id \).

Proof of Fact 3.13. For any \( f \in B_{\mathcal{V}, h} \), let us define a function \( J(f) : K \rightarrow \mathbb{R} \) as follows:
\[
J(f)(x) = f(x) \quad \text{for all } x \notin \bigcup_{i=1}^N V_i,
\]
and
\[
J(f)|_{V_i} = 0 \quad \text{for } i = 1, \ldots, N.
\]
Then, \( J(f) \in B_{\mathcal{V}, 0} \) where \( 0 = (0, \ldots, 0) \). Moreover, it is clear that \( J : B_{\mathcal{V}, h} \rightarrow B_{\mathcal{V}, 0} \) is an isometric bijection.

Since \( \Phi : B_{\mathcal{V}, h} \rightarrow B_{\mathcal{V}, h} \) is a bijection, we can define a bijective map \( \Psi_0 : B_{\mathcal{V}, 0} \rightarrow B_{\mathcal{V}, 0} \) as follows:
\[
\Psi_0(J(f)) = J(\Phi(f)) \quad \text{for all } f \in B_{\mathcal{V}, h}.
\]
Moreover, since \( J \) is an isometry and \( \Phi \) is non-expansive, \( \Psi_0 \) is non-expansive. Now, \( B_{\mathcal{V}, 0} \) is the unit ball of the finite-dimensional space
\[
X_{\mathcal{V}, 0} = \{ f \in C(K); f|_{V_i} = 0, \forall i \in \{1, \ldots, N\} \}.
\]
so $B_{\mathbf{V},0}$ is plastic. Therefore, $\Psi_0$ is in fact an isometry. Hence, by Theorem 2.2, $\Psi_0$ extends to a linear isometric bijection $\tilde{\Psi}_0 : X_{\mathbf{V},0} \to X_{\mathbf{V},0}$.

Let

$$M = \left\{ x \in K \setminus K'; \ x \notin \bigcup_{i=1}^N V_i \right\}.$$ 

Since $K' \subset \bigcup_{i=1}^N V_i$, the set $M$ is finite. We write $M = \{x_1, \ldots, x_n\}$.

Now, let $f \in B_{\mathbf{V},h}$, and let us show that $\Phi(f) = f$.

Since $h \neq 0$, for each $j \in \{1, \ldots, n\}$ we may write

$$f(x_j) = \alpha_j h \text{ where } \alpha_j \in \mathbb{R}.$$ 

Then,

$$J(f) = \sum_{j=1}^n \alpha_j h \mathbb{1}_{\{x_j\}}.$$ 

For each $j \in \{1, \ldots, n\}$, we define $f_j \in B_{\mathbf{V},h}$ as follows:

$$\begin{cases} 
  f_{j_i} \equiv h_i & \text{for } i = 1, \ldots, N, \\
  f_j(y) = 0 & \text{for all } y \notin \left( \bigcup_{i=1}^N V_i \cup \{x_j\} \right), \\
  f_j(x_j) = h.
\end{cases}$$ 

Note that $J(f_j) = h \mathbb{1}_{\{x_j\}}$, for each $j \in \{1, \ldots, n\}$. Moreover, by Lemma 3.9 and since we are assuming that $\Phi(B_{\mathbf{V},h}) = B_{\mathbf{V},h}$, we see that $\Phi(f_j) = f_j$ for all $j \in \{1, \ldots, n\}$: indeed, we have $\Phi(f_j) \equiv h_i \equiv f_j$ on each set $V_i$, and $\Phi(f_j)(x) = f_j(x)$ for all $x \in K \setminus \bigcup_{i=1}^N V_i$ by Lemma 3.9. Hence, using the linearity of $\tilde{\Psi}_0$, we get

$$\Psi_0(J(f)) = \tilde{\Psi}_0 \left( \sum_{j=1}^n \alpha_j h \mathbb{1}_{\{x_j\}} \right)$$

$$= \sum_{j=1}^n \alpha_j \Psi_0(h \mathbb{1}_{\{x_j\}})$$

$$= \sum_{j=1}^n \alpha_j J(\Phi(f_j))$$

$$= \sum_{j=1}^n \alpha_j J(f_j)$$

$$= \sum_{j=1}^n \alpha_j h \mathbb{1}_{\{x_j\}} = J(f).$$ 

It follows that $J(\Phi(f)) = J(f)$ and this implies that $\Phi(f) = f$. So we have shown that $\Phi|_{B_{\mathbf{V},h}} = Id$. □

Fact 3.14. Let $\mathbf{V} = (V_1, \ldots, V_N) \in \mathcal{V}$ and $h \in (0, 1)$. For any $h = (h_1, \ldots, h_N) \in \{-h, h\}^N$, we have $\Phi^{-1}(B_{\mathbf{V},h}) \subset B_{\mathbf{V},h}.$
Proof of Fact 3.14. Let $g \in B_{V,h}$. Our goal is to show that $\Phi^{-1}(g) \in B_{V,h}$, i.e. $\Phi^{-1}(g)|_{V_i} \equiv h_i$, for every $i \in \{1, \ldots, N\}$.

Let $i_0 \in \{1, \ldots, N\}$ and let $x \in K \setminus K'$ be such that $x \in V_{i_0}$. We want to show that $\Phi^{-1}(g)(x) = h_{i_0}$.

Suppose for example that $h_{i_0} > 0$, i.e. $h_{i_0} = h$.

Since $x \in V_{i_0}$, we have $g(x) = h$. Moreover, since $h > 0$, Lemma 3.7 implies that $\Phi^{-1}(g)(x) \geq h$. So, it remains to show that $\Phi^{-1}(g)(x) \leq h$. For the sake of contradiction, suppose that $\Phi^{-1}(g)(x) > h$.

Since $K$ is metrizable, the closed set $K'$ is also $G_\delta$, and hence it is a “zero set”; so one can choose a continuous function $\chi : K \to \mathbb{R}$ such that

$$0 \leq \chi(z) < \frac{1}{2} \quad \text{for all } z \in K \quad \text{and} \quad K' = \{z \in K; \chi(z) = 0\}.$$

Finally, let us denote $1 - h$ by $v$.

We define $l \in B$ as follows:

$$l(y) = \begin{cases} 1 & \text{if } y = x, \\ \text{sgn}(h_i)(h + \frac{v}{2}(1 - \chi(y))) & \text{if } y \in V_i \text{ for some } i \text{ and } y \neq x, \\ g(y) & \text{if } |g(y)| < h, \\ \text{sgn}(g(y))(h + \frac{v}{2}) & \text{if } y \notin \bigcup_{i=1}^{N} V_i \text{ and } |g(y)| \geq h. \end{cases}$$

The function $l$ is indeed continuous since the $V_i$ are clopen and $\bigcup_{i=1}^{N} V_i \supset K'$.

For each $i \in \{1, \ldots, N\}$ and $y \in K \setminus (K' \cup \{x\})$ such that $y \in V_i$, we have:

$$0 < \chi(y) < \frac{1}{2} \implies h + \frac{v}{4} < h + \frac{v}{2}(1 - \chi(y)) < h + \frac{v}{2}.$$ 

So we see that $|l(y)| < h + \frac{v}{2}$ for every $y \in K \setminus (K' \cup \{x\})$. Since $|l(z)| = h + \frac{v}{2}$ for all $z \in K'$, Corollary 3.11 then implies that $\Phi^{-1}(l)(y) = l(y)$ for all $y \in K \setminus (K' \cup \{x\})$.

Since $K \setminus K'$ is dense in $K$, it follows that $\Phi^{-1}(l)(y) = l(y)$ for all $y \neq x$. Moreover $l(x) = 1$, so Lemma 3.7 implies that $\Phi^{-1}(l)(x) = 1$. Thus, we have proved that $\Phi^{-1}(l) = l$.

Now, we are going to show that

$$\|\Phi^{-1}(g) - l\| < v,$$

from which we will easily get a contradiction. Denote $\Phi^{-1}(g)$ by $g'$.

First, recall that we are assuming that $g'(x) > h$, so $|g'(x) - l(x)| = 1 - g'(x) < 1 - h = v$.

Next, let us show that $|g'(y) - l(y)| \leq \frac{1}{2}v$ for every $y \in K \setminus (K' \cup \{x\})$.

Let $y \in K \setminus (K' \cup \{x\})$.

- If $|g(y)| < h$, then Corollary 3.11 implies that $g'(y) = g(y)$. Hence, $|g'(y) - l(y)| = 0$.
- Assume that $y \in V_i$ for some $i$ and $g_{|V_i} \equiv h$. Then $g(y) = h$, and Lemma 3.7 implies that $g'(y) \in [h, 1]$. Moreover, $h + \frac{v}{2} < l(y) < h + \frac{v}{2}$. So, we see that if $g'(y) \leq l(y)$, then $|g'(y) - l(y)| = l(y) - g'(y) \leq h + \frac{v}{2} - h = \frac{v}{2}$; whereas if $g'(y) > l(y)$, then $|g'(y) - l(y)| = g'(y) - l(y) \leq 1 - h - \frac{v}{4} = \frac{3}{4}v$. 


- Similarly, one gets $|g'(y) - l(y)| \leq \frac{3}{4}v$ when $y \in V_i$ and $g_{V_i} \equiv -h$.
- Assume that $y \notin \bigcup_{i=1}^{N} V_i$ and $g(y) \leq -h$. Then, $g'(y) \in [-1, -h]$ (from Lemma 3.7) and $l(y) = -h - \frac{3}{4}$. So, if $g'(y) \geq l(y)$ then $|g'(y) - l(y)| = g'(y) - l(y) \leq -h + h + \frac{3}{4} = \frac{3}{4}$; and if $g'(y) < l(y)$ then $|g'(y) - l(y)| = l(y) - g'(y) \leq -h - \frac{3}{4} + 1 = \frac{3}{4}v$.
- Finally, one gets in the same way that $|g'(y) - l(y)| \leq \frac{3}{4}v$ when $y \notin \bigcup_{i=1}^{N} V_i$ and $g(y) \geq h$.

Hence, we have shown that $|g'(y) - l(y)| \leq \frac{3}{4}v$ for every $y \in K \setminus (K' \cup \{x\})$. Since $K \setminus K'$ is dense in $K$, it follows that $|g'(y) - l(y)| \leq \frac{3}{4}v$ for all $y \neq x$; and hence we get $\|g' - l\| < v$ since we observed above that $|g'(x) - l(x)| < v$.

It follows that

$$v = 1 - h = l(x) - g(x) \leq \|l - g\| \leq \|\Phi^{-1}(l) - \Phi^{-1}(g)\| = \|l - \Phi^{-1}(g)\| < v,$$

which is a contradiction.

The case $h_0 < 0$ is analogous.

So, we have shown that $\Phi^{-1}(g_i) = h_i$ for all $i \in \{1, \ldots, N\}$ and every $x \in (K \setminus K') \cap V_i$.

And since $K \setminus K'$ is dense in $K$, it follows that $\Phi^{-1}(g_{V_i}) = h_i$, for all $i \in \{1, \ldots, N\}$. \(\square\)

It is now easy to conclude the proof of Lemma 3.12. On the one hand, since $|h_i| = h$ for all $i \in \{1, \ldots, N\}$, Lemma 3.9 (together with the density of $K \setminus K'$) implies that $\Phi(B_{V,h}) \subset B_{V,h}$. On the other hand, Fact 3.14 implies that $\Phi^{-1}(B_{V,h}) \subset B_{V,h}$. So we have $\Phi(B_{V,h}) = B_{V,h}$; and hence $\Phi_{|_{B_{V,h}}} = Id$ by Fact 3.13. \(\square\)

3.5. Proof of the main lemma: $n = 1$. In this subsection, we prove Lemma 3.5 in the case $n = 1$. (We need to treat this case separately because our proof of the inductive step from $n$ to $n + 1$ works only if $n \geq 1$.) Let us recall the statement.

**Lemma 3.15.** Let $f \in B$ be such that $u_{0,f} \notin \{0,1\}$. For any $x \in K \setminus K'$,

1. if $|f(x)| < u_{1,f}$, then $\Phi(f)(x) = f(x)$,
2. if $f(x) \geq u_{1,f}$, then $\Phi(f)(x) \in [u_{1,f}, f(x)]$,
3. if $f(x) \leq -u_{1,f}$, then $\Phi(f)(x) \in [f(x), -u_{1,f}]$.

**Proof.** 1) Let us first assume that $f \in B_{V,h}$ for some $V = (V_1, \ldots, V_N) \in \mathcal{V}$ and some $h = (h_1, \ldots, h_N) \in [-1, 1]^N$ such that $h = \min \{|h_i|; \ i \in \{1, \ldots, N\}\} \notin \{0, 1\}$. Note that $h = u_{0,f}$.

Let $x \in K \setminus K'$ be such that $|f(x)| < u_{1,f}$.

If $|f(x)| \leq u_{0,f}$, then Lemma 3.9 implies that $\Phi(f)(x) = f(x)$. So we may suppose that $|f(x)| > u_{0,f}$.

Let us define $g \in B$ as follows:

$$g(y) = \begin{cases} 
\text{sgn}(f(x)) & \text{if } y = x, \\
\text{sgn}(h_i) |f(x)| & \text{if } y \in V_i \text{ for some } i \text{ and } y \neq x, \\
f(y) & \text{if } y \notin \bigcup_{i=1}^{N} V_i \cup \{x\}.
\end{cases}$$

Lemma 3.12 implies that $\Phi(g) = g$. 
Let us show that $\|f - g\| = 1 - |f(x)|$.

First, we see that $\|g(x) - f(x)\| = 1 - |f(x)|$.

Now, let $y \in K \setminus \{x\}$. If $y \notin \bigcup_{i=1}^{N} V_i$, then $|f(y) - g(y)| = 0$. If $y \in V_i$ for some $i$, two cases may occur: $|f(x)| < |h_i|$ or $|f(x)| \geq |h_i|$.

- If $|f(x)| < |h_i|$, then $|f(y) - g(y)| = |h_i| - |f(x)| \leq 1 - |f(x)|$.
- If $|f(x)| \geq |h_i|$, then $|f(y) - g(y)| = |f(x)| - |h_i| \leq |f(x)| - u_{0,f}$. Moreover, since $|f(x)| < u_{1,f} = \frac{1 + u_{0,f}}{2}$, we have $|f(x)| - u_{0,f} < 1 - |f(x)|$. Hence, $|f(y) - g(y)| \leq 1 - |f(x)|$.

So we have indeed shown that $\|f - g\| = 1 - |f(x)|$. Hence,

$$\text{(3.7)} \quad |\Phi(f)(x) - \Phi(g)(x)| \leq \|\Phi(f) - \Phi(g)\| \leq \|f - g\| = 1 - |f(x)|.$$  

Since $\text{sgn}(\Phi(f)(x)) = \text{sgn}(f(x))$ (from Lemma 3.9) and $\Phi(g) = g$, $\text{(3.7)} \quad \implies 1 - |\Phi(f)(x)| \leq 1 - |f(x)| \implies |\Phi(f)(x)| \geq |f(x)|$.

Moreover, Lemma 3.7 implies that $|\Phi(f)(x)| \leq |f(x)|$.

So we have $|\Phi(f)(x)| = |f(x)|$ and $\text{sgn}(\Phi(f)(x)) = \text{sgn}(f(x))$, i.e. $\Phi(f)(x) = f(x)$.

Now, we can easily prove 1) in the general case, approximating $f$ by functions taking only finitely many values. Let us fix $x \in K \setminus K'$ such that $|f(x)| < u_{1,f}$.

Since $K$ is zero-dimensional, for any $m \in \mathbb{N}$ one can find $g_m \in B$ taking only finitely many values such that $\|f - g_m\| \leq \frac{1}{m}$, $f(x) = g_m(x)$ and $u_{0,g_m} = u_{0,f}$.

Then $u_{1,g_m} = u_{1,f}$ and hence $|g_m(x)| = |f(x)| < u_{1,g_m}$. So $\Phi(g_m)(x) = g_m(x) = f(x)$, by the special case we have already treated (since $g_m$ takes only finitely many values, it belongs to $B_{V,h}$ for some pair $(V,h)$).

Hence, for any $m \in \mathbb{N}$, $|\Phi(f)(x) - \Phi(g_m)(x)| \leq |\Phi(f) - \Phi(g_m)| \leq \|f - g_m\| \leq \frac{1}{m}$. It follows that for any $m \in \mathbb{N}$, $|\Phi(f)(x) - f(x)| \leq \frac{1}{m}$; so we obtain $\Phi(f)(x) = f(x)$ by letting $m$ tend to infinity.

2) Since $f(x) \geq u_{1,f} > 0$, Lemma 3.7 implies that $\Phi(f)(x) \leq f(x)$. So it remains to show that $\Phi(f)(x) \geq u_{1,f}$.

Let $\varepsilon > 0$ be arbitrary, and let us choose clopen sets $V_1, \ldots, V_N$ as in the proof of Lemma 3.9 that is,

- $V_i \cap K' \neq \emptyset$ for all $i$ and $\bigcup_{i=1}^{N} V_i \supset K'$,
- the $V_i$ are pairwise disjoint and do not contain $x$,
- $f$ has constant sign $\varepsilon_i$ on each $V_i$,
- $\text{diam}(f(V_i)) < \varepsilon$ for $i = 1, \ldots, N$.

We define $g \in B$ as follows:

$$g(y) = \begin{cases} 
1 & \text{if } y = x, \\
\varepsilon_i u_{1,f} & \text{if } y \in V_i \text{ for some } i \in \{1, \ldots, N\}, \\
f(y) & \text{if } y \notin \bigcup_{i=1}^{N} V_i \cup \{x\}.
\end{cases}$$

Lemma 3.12 implies that $\Phi(g) = g$.

Let us show that $\|f - g\| \leq 1 - u_{1,f} + \varepsilon$.

First, we have $|f(x) - g(x)| = 1 - f(x) \leq 1 - u_{1,f}$.

Now, let $y \in K \setminus \{x\}$. 

If \( y \notin \bigcup_{i=1}^{N} V_i \), then \(|f(y) - g(y)| = 0\).
If \( y \in V_i \) for some \( i \), choose \( z \in V_i \cap K' \). Then \(|f(y) - f(z)| < \varepsilon \) and \( \text{sgn}(f(z)) = \varepsilon_i \).
Hence,
\[
|f(y) - g(y)| \leq |f(y) - f(z)| + |f(z) - g(y)| < \varepsilon + |f(z) - g(y)|.
\]
It follows that if \(|f(z)| < u_{1,f}\), then \(|f(z) - g(y)| = u_{1,f} - |f(z)| \leq u_{1,f} - u_{0,f} = 1 - u_{1,f}\); whereas if \(|f(z)| \geq u_{1,f}\), then \(|f(z) - g(y)| = |f(z)| - u_{1,f} \leq 1 - u_{1,f}\). In either case, we have \(|f(y) - g(y)| < 1 - u_{1,f} + \varepsilon\).
So, we have shown that \(|f - g| \leq 1 - u_{1,f} + \varepsilon\).
Since \( \Phi \) is non-expansive, we obtain
\[
\|\Phi(f) - g\| = \|\Phi(f) - \Phi(g)\| \leq 1 - u_{1,f} + \varepsilon \Rightarrow |\Phi(f)(x) - g(x)| \leq 1 - u_{1,f} + \varepsilon \Rightarrow 1 - \Phi(f)(x) \leq 1 - u_{1,f} + \varepsilon \Rightarrow \Phi(f)(x) \geq u_{1,f} - \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that \( \Phi(f)(x) \geq u_{1,f} \).

3) The proof is similar to that of 2). \( \square \)

3.6. Proof of the main lemma: general case. In this subsection, we prove Lemma 3.5 by induction on \( n \in \mathbb{Z}_+ \).

We already know that the result holds true for \( n = 0 \) and \( n = 1 \). Let us now fix \( n \geq 1 \). Assume that Lemma 3.5 has been proved for \( n \), and let us prove it for \( n + 1 \).

**Step 1.** For each \( g \in B \) such that \( u_{0,g} \notin \{0, 1\} \) and \( x \in K \setminus K' \), if \(|g(x)| < u_{n,g}\), then \( \Phi^{-1}(g)(x) = g(x) \).

**Proof.** Let \( g \in B \) be such that \( u_{0,g} \notin \{0, 1\} \), and let \( x \in K \setminus K' \) be such that \(|g(x)| < u_{n,g}\).
Denote \( \Phi^{-1}(g) \) by \( f \).
Since \( u_{0,f} = u_{0,g} \) (from Corollary 3.10), we have \( u_{0,f} \notin \{0, 1\} \) and \( u_{n,f} = u_{n,g} \).
Three cases may occur: \(|f(x)| < u_{n,f}, f(x) \geq u_{n,f} \) or \( f(x) \leq -u_{n,f} \).
If \( f(x) \geq u_{n,f} \), then the induction hypothesis implies that \( g(x) \geq u_{n,f} = u_{n,g} \), which is a contradiction. Similarly, if \( f(x) \leq -u_{n,f} \) then \( g(x) \leq -u_{n,f} = -u_{n,g} \), which is again a contradiction. Hence, \(|f(x)| < u_{n,f} \). By the induction hypothesis, it follows that \( f(x) = g(x) \), i.e. \( \Phi^{-1}(g)(x) = g(x) \). \( \square \)

**Step 2.** Let \( V = (V_1, \ldots, V_N) \in V \) and let \( h = (h_1, \ldots, h_N) \in [-1, 1]^N \). Assume that \( h = \min \{ |h_i|; \ i \in \{1, \ldots, N\} \} \notin \{0, 1\} \), and that \(|h_i| \leq \frac{n+h}{n+1} \) for all \( i \in \{1, \ldots, N\} \).
Then, \( \Phi^{-1}(BV,h) \subset BV,h \).

**Proof.** Let \( g \in BV,h \). Our goal is to show that \( \Phi^{-1}(g)|_{V_i} \equiv h_i \), for every \( i \in \{1, \ldots, N\} \).
And since \( K \setminus K' \) is dense in \( K \), it is in fact enough to show that \( \Phi^{-1}(g)(x) = h_i \) for all \( i \) and every \( x \in (K \setminus K') \cap V_i \).
Note that
\[
u_{0,g} = h \quad \text{and} \quad u_{n,g} = \frac{n+h}{n+1} \]
Let $i_0 \in \{1, \ldots, N\}$ and let $x \in (K \setminus K') \cap V_{i_0}$. Then, $g(x) = h_{i_0}$. We want to show that $\Phi^{-1}(g)(x) = h_{i_0}$.

If $|h_{i_0}| < \frac{n+h}{n+1} = u_{n,g}$, then Step 1 implies that $\Phi^{-1}(g)(x) = g(x) = h_{i_0}$.

If $|h_{i_0}| = \frac{n+h}{n+1}$, suppose for example that $h_{i_0} > 0$, i.e., $g(x) = h_{i_0} = \frac{n+h}{n+1}$.

Since $g(x) > 0$, Lemma 3.7 implies that $\Phi^{-1}(g)(x) \geq g(x) = \frac{n+h}{n+1}$. So, it remains to show that $\Phi^{-1}(g)(x) \leq \frac{n+h}{n+1}$.

For the sake of contradiction, suppose that $\Phi^{-1}(g)(x) > \frac{n+h}{n+1}$.

Let us set $u = \frac{n+h}{n+1}$ and $v = 1 - u$.

We note that

$$
(3.8) \quad \frac{(2n+1)h + 1}{2(n+1)} < u.
$$

Indeed,

$$
\frac{(2n+1)h + 1}{2(n+1)} - u = \frac{(2n+1)h + 1}{2(n+1)} - \frac{n+h}{n+1} = \frac{(2n-1)(h-1)}{2(n+1)} < 0.
$$

Moreover, since $h < 1$, we have $h < \frac{(2n+1)h + 1}{2(n+1)}$.

Now, let us define $l \in B$ as follows:

$$
l(y) = \begin{cases} 
1 & \text{if } y = x, \\
g(y) & \text{if } |g(y)| < h \\
\text{sgn}(g(y)) \cdot \frac{(2n+1)h + 1}{2(n+1)} & \text{if } h \leq |g(y)| < u, \\
\text{sgn}(g(y)) \left( u + \frac{v}{2(n+1)} \right) & \text{if } |g(y)| \geq u \text{ and } y \neq x.
\end{cases}
$$

Note that $l$ is constant on $V_{i_0} \setminus \{x\}$ and on each set $V_i$ for $i \neq i_0$, since $g \in B_{\mathbb{V},h}$. If we choose $z_i \in V_i \setminus \{x\}$ for $i = 1, \ldots, N$, then $l \in B_{\mathbb{V}',h'}$ where $\mathbb{V}' = (V'_1, \ldots, V'_N)$, $V'_i = V_i$ for all $i \neq i_0$, $V'_{i_0} = V_{i_0} \setminus \{x\}$ and $h' = (l(z_1), \ldots, l(z_N))$.

Note also that $v_{0,i} = \frac{(2n+1)h + 1}{2(n+1)}$. Hence, $u_{n,i} = \frac{n+u_{0,i}}{n+1} = \frac{2n^2+2n+2nh+h+1}{2(n+1)}$.

Moreover, $u + \frac{v}{2(n+1)} = u + \frac{1-u}{2(n+1)} = \frac{2n^2+2n+2nh+h+1}{2(n+1)^2}$. So, we get

$$
u_{n,i} = u + \frac{v}{2(n+1)}.
$$

Now, we have $|l(y)| < u_{n,i}$, for all $y \neq x$. Then, Step 1 implies that $\Phi^{-1}(l)(y) = l(y)$, for all $y \in K \setminus (K' \cup \{x\})$. And since $K \setminus K'$ is dense in $K$, it follows that $\Phi^{-1}(l)(y) = l(y)$ for all $y \neq x$.

Moreover, since $l(x) = 1$, Lemma 3.7 implies that $\Phi^{-1}(l)(x) = 1$. So we have shown that $\Phi^{-1}(l) = l$.

Now, we are going to show that $\|\Phi^{-1}(g) - l\| < v$, from which we will obtain a contradiction. Denote $\Phi^{-1}(g)$ by $g'$.

First, note that according to our assumption, we have $g'(x) > u$; so $|g'(x) - l(x)| = 1 - g'(x) < 1 - u = v$.
Since \( g(y) < h \) or \( \frac{(2n+1)h+1}{2(n+1)} \leq |g(y)| < u \), then \( |g(y)| < u = u_n,g \) and Step 1 implies that \( g'(y) = g(y) \). Hence, \( |g'(y) - l(y)| = 0 \).

- If \( h \leq |g(y)| < \frac{(2n+1)h+1}{2(n+1)} \), then (3.8) implies that \( |g(y)| < u = u_{n,g} \). Hence, Step 1 implies that \( g'(y) = g(y) \). So we get

\[
|l(y) - g'(y)| = |l(y) - g(y)| = \frac{(2n+1)h+1}{2(n+1)} - |g(y)| \\
\leq \frac{(2n+1)h+1}{2(n+1)} - h = \frac{1-h}{2(n+1)}.
\]

Since \( v = 1 - \frac{n+h}{n+1} = \frac{1-h}{n+1} \), it follows that \( |l(y) - g'(y)| \leq \frac{v}{7} \).

- If \( g(y) \geq u \), then Lemma 3.7 implies that \( g'(y) \in [u, 1] \) and \( l(y) = u + \frac{3n+2}{3(n+1)} \). So, if \( g'(y) \leq l(y) \), then we obtain \( |g'(y) - l(y)| = l(y) - g'(y) \leq u + \frac{3n+2}{3(n+1)} - u = \frac{3n+2}{3(n+1)} \); whereas if \( g'(y) > l(y) \), then \( |g'(y) - l(y)| = g'(y) - l(y) \leq 1 - u = \frac{3n+2}{3(n+1)} \). In either case, we see that \( |g'(y) - l(y)| \leq \frac{3n+2}{3(n+1)} v \).

- Finally, one shows in the same way that if \( g(y) \leq -u \) then \( |g'(y) - l(y)| \leq \frac{3n+2}{3(n+1)} v \).

Hence, we have \( |g'(y) - l(y)| \leq \frac{3n+2}{3(n+1)} v \) for all \( y \in K \setminus (K' \cup \{x\}) \); and it follows that \( |g'(y) - l(y)| \leq \frac{3n+2}{3(n+1)} v \) for all \( y \neq x \). Since \( |g'(x) - l(x)| < v \) (as observed above), we conclude that \( \|g' - l\| < v \).

Therefore,

\[
v = 1 - u = l(x) - g(x) \leq \|l - g\| \leq \|\Phi^{-1}(l) - \Phi^{-1}(g)\| = \|l - \Phi^{-1}(g)\| < v,
\]

which is a contradiction.

The case \( h_{i_0} < 0 \) is analogous. \( \Box \)

**Step 3.** Let \( V = (V_1, \ldots, V_N) \in \mathcal{V} \) and let \( h = (h_1, \ldots, h_N) \in [-1, 1]^N \). Assume that \( h = \min \{ |h_i|; \ i \in \{1, \ldots, N\} \} \notin \{0, 1\} \), and that \( |h_i| \leq \frac{n+h}{n+1} \) for all \( i \in \{1, \ldots, N\} \).

Then, \( \Phi_{|B_{V,h}} \equiv \text{Id} \).

**Proof.** The induction hypothesis (together with the density of \( K \setminus K' \)) implies that \( \Phi(B_{V,h}) \subset B_{V,h} \). Moreover, Step 2 implies that \( \Phi^{-1}(B_{V,h}) \subset B_{V,h} \). So we have \( \Phi(B_{V,h}) = B_{V,h} \); and hence Fact 3.13 implies that \( \Phi_{|B_{V,h}} \equiv \text{Id} \). \( \Box \)

**Step 4.** For each \( f \in B \) such that \( u_{0,f} \notin \{0, 1\} \) and \( x \in K \setminus K' \), if \( |f(x)| < u_{(n+1),f} \), then \( \Phi(f)(x) = f(x) \).

**Proof.** Let us first assume that \( f \in B_{V,h} \) for some \( V = (V_1, \ldots, V_N) \in \mathcal{V} \) and some \( h = (h_1, \ldots, h_N) \in [-1, 1]^N \) such that \( h = \min \{ |h_i|; \ i \in \{1, \ldots, N\} \} \notin \{0, 1\} \).

Note that \( h = u_{0,f} \).
Let $x \in K \setminus K'$ be such that $|f(x)| < u_{(n+1),f}$.
If $|f(x)| \leq u_{n,f}$, then the induction hypothesis implies that $\Phi(f)(x) = f(x)$.

So we consider the case where $|f(x)| > u_{n,f}$.

Since $|f(x)| > u_{n,f} \geq u_{1,f}$ (because $n \geq 1$) and since $u_{1,f} = \frac{1 + u_{0,f}}{2}$, it follows that $u_{0,f} + 1 - |f(x)| < |f(x)|$.

Now, we define $g \in B$ as follows:

$$
g(y) = \begin{cases} 
\text{sgn}(f(x)) & \text{if } y = x, \\
|f(x)| & \text{if } |f(x)| < u_{0,f}, \\
\text{sgn}(f(y))(u_{0,f} + 1 - |f(x)|) & \text{if } u_{0,f} \leq |f(y)| < u_{0,f} + 1 - |f(x)|, \\
\text{sgn}(f(y))|f(x)| & \text{if } |f(y)| \geq |f(x)| \text{ and } y \neq x.
\end{cases}
$$

Note that $g$ is constant on each set $V_i \setminus \{x\}$ since $f \in B_{V_i,h'}$. If we choose $z_i \in V_i \setminus \{x\}$ for $i = 1, \ldots, N$, then $g \in B_{V',h''}$ where $V' = (V_1 \setminus \{x\}, \ldots, V_N \setminus \{x\})$ and $h'' = (g(z_1), \ldots, g(z_N))$. Note also that $|g(y)| \leq |f(x)|$ for all $y \neq x$.

We can see that $u_{n,g} = u_{0,f} + 1 - |f(x)|$. Hence, $u_{n,g} = \frac{n + u_{0,f} + 1 - |f(x)|}{n+1}$.

Next, we observe that $u_{n,g} > |f(x)|$.

Indeed, since $|f(x)| < u_{(n+1),f} = \frac{n+1+u_{0,f}}{n+2}$, we have $(n+2)|f(x)| < n+1 + u_{0,f}$, so we get

$$u_{n,g} - |f(x)| = \frac{n + u_{0,f} + 1 - (n+2)|f(x)|}{n+1} > 0.$$  

Since $g \in B_{V',h''}$, $u_{0,g} \notin \{0,1\}$ and $|g(z_i)| \leq |f(x)| < u_{n,g}$, for all $i \in \{1, \ldots, N\}$, Step 3 implies that $\Phi(g) = g$.

Let us show that $\|g - f\| = 1 - |f(x)|$.

First, we have $|g(x) - f(x)| = 1 - |f(x)|$.

Now, let $y \in K \setminus \{x\}$.

- If $|f(y)| < u_{0,f}$ for $u_{0,f} + 1 - |f(x)| < |f(y)| < |f(x)|$, then $|f(y) - g(y)| = 0$.
- If $u_{0,f} \leq |f(y)| < u_{0,f} + 1 - |f(x)|$, then

$$|f(y) - g(y)| = |g(y)| - |f(y)| \\ \leq u_{0,f} + 1 - |f(x)| - u_{0,f} \\ = 1 - |f(x)|.$$  

- If $|f(y)| \geq |f(x)|$, then

$$|f(y) - g(y)| = |f(y)| - |g(y)| \\ = |f(y)| - |f(x)| \\ \leq 1 - |f(x)|.$$  

So we have proved that $\|g - f\| = 1 - |f(x)|$. Hence,  

$$|\Phi(f)(x) - \Phi(g)(x)| \leq \|\Phi(f) - \Phi(g)\| \leq \|f - g\| = 1 - |f(x)|.$$  

(3.9)
Since $\text{sgn}(\Phi(f)(x)) = \text{sgn}(f(x))$ (from Lemma 3.9) and $\Phi(g) = g$, 
\[ 1 - |\Phi(f)(x)| \leq 1 - |f(x)| \implies |\Phi(f)(x)| \geq |f(x)|. \]
Moreover, Lemma 3.7 implies that $|\Phi(f)(x)| \leq |f(x)|.
So we have $|\Phi(f)(x)| = |f(x)|$ and $\text{sgn}(\Phi(f)(x)) = \text{sgn}(f(x))$, i.e. $\Phi(f)(x) = f(x)$.

Now, we can prove Step 4 by an approximation argument.
Let $f \in B$ be such that $u_{0,f} \notin \{0,1\}$ and let $x \in K \setminus K'$ be such that $|f(x)| < u_{(n+1),f}$. For any $m \in \mathbb{N}$, one can find $g_m \in B$ taking finitely many values such that $|f - g_m| \leq \frac{1}{m}$, $f(x) = g_m(x)$ and $u_{0,g_m} = u_{0,f}$. Since $u_{0,g_m} = u_{0,f}$, it follows that $u_{(n+1),g_m} = u_{(n+1),f}$. So we have $|g_m(x)| = |f(x)| < u_{(n+1),g_m}$, hence $\Phi(g_m)(x) = g_m(x) = f(x)$ (we have used the fact that $g_m \in B_{V,h}$ for some pair $(V,h)$). Consequently, $|\Phi(f)(x) - \Phi(g_m)(x)| \leq |f - g_m| \leq \frac{1}{m}$ for all $m \in \mathbb{N}$. It follows that $|\Phi(f)(x) - f(x)| \leq \frac{1}{m}$ for all $m \in \mathbb{N}$; so we obtain $\Phi(f)(x) = f(x)$ by letting $m$ tend to infinity.

**Step 5.** For each $f \in B$ such that $u_{0,f} \notin \{0,1\}$ and $x \in K \setminus K'$, if $f(x) \geq u_{(n+1),f}$, then $\Phi(f)(x) \in [u_{(n+1),f}, f(x)]$.

**Proof.** Let us first assume that $f \in B_{V,h}$ for some $V = (V_1, \ldots, V_N) \in V$ and some $h = (h_1, \ldots, h_N) \in [-1,1]^N$ such that $h = \min \{|h_i|: i \in \{1, \ldots, N\}\} \notin \{0,1\}$. Note that $h = u_{0,f}$.

Let $x \in K \setminus K'$ be such that $f(x) \geq u_{(n+1),f}$.

Since $u_{(n+1),f} > u_{1,f}$ (because $n \geq 1$), we have $u_{0,f} + 1 - u_{(n+1),f} < u_{(n+1),f}$.

Indeed,
\[
u_{1,f} = \frac{1 + u_{0,f}}{2} \implies 1 + u_{0,f} = 2u_{1,f} < 2u_{(n+1),f} \]
\[
\implies 1 + u_{0,f} - u_{(n+1),f} < u_{(n+1),f}.
\]

Now, we define $g \in B$ as follows:
\[
\begin{align*}
g(y) &= \begin{cases} 
1 & \text{if } y = x, \\
\text{sgn}(f(y))(u_{0,f} + 1 - u_{(n+1),f}) & \text{if } |f(y)| < u_{0,f} \\
\text{sgn}(f(y))u_{(n+1),f} & \text{or } u_{0,f} + 1 - u_{(n+1),f} \leq |f(y)| < u_{(n+1),f}, \\
\end{cases}
\end{align*}
\]

If we choose $z_i \in V_i \setminus \{x\}$ for $i = 1, \ldots, N$, then $g \in B_{V',h'}$ where $V' = (V_1 \setminus \{x\}, \ldots, V_N \setminus \{x\})$ and $h' = (g(z_1), \ldots, g(z_N))$.

Since $u_{0,g} = u_{0,f} + 1 - u_{(n+1),f}$, it follows that $u_{n,g} = \frac{n + u_{0,f} + 1 - u_{(n+1),f}}{n+1}$. And since $u_{(n+1),f} = \frac{n + u_{0,f}}{n+2}$, we get $u_{n,g} = u_{(n+1),f}$.

Hence, $g \in B_{V',h'}$, $u_{0,g} \notin \{0,1\}$ and $|g(z_i)| \leq u_{(n+1),f} = u_{n,g}$, for all $i \in \{1, \ldots, N\}$.

Step 3 then implies that $\Phi(g) = g$.

Let us show that $\|f - g\| \leq 1 - u_{(n+1),f}$.
First, we have $|f(x) - g(x)| = 1 - f(x) \leq 1 - u_{(n+1),f}$. 

Now, let \( y \in K \setminus \{x\} \).
- If \( |f(y)| < u_{0,f} \) or \( u_{0,f} + 1 - u_{(n+1),f} < |f(y)| < u_{(n+1),f} \), then \( |f(y) - g(y)| = 0 \).
- If \( u_{0,f} \leq |f(y)| \leq u_{0,f} + 1 - u_{(n+1),f} \), then
  \[
  |f(y) - g(y)| = |g(y)| - |f(y)| 
  \leq u_{0,f} + 1 - u_{(n+1),f} - u_{0,f} 
  = 1 - u_{(n+1),f}.
  \]
- If \( |f(y)| \geq u_{(n+1),f} \), then
  \[
  |f(y) - g(y)| = |f(y)| - |g(y)| 
  = |f(y)| - u_{(n+1),f} 
  \leq 1 - u_{(n+1),f}.
  \]
So we have shown that \( \|f - g\| \leq 1 - u_{(n+1),f} \). Hence,
\[
|\Phi(f)(x) - \Phi(g)(x)| \leq \|\Phi(f) - \Phi(g)\| \leq \|f - g\| \leq 1 - u_{(n+1),f}.
\]
Since \( \Phi(g) = g \),
\[
(3.10) \quad 1 - \Phi(f)(x) \leq 1 - u_{(n+1),f} \quad \Rightarrow \quad \Phi(f)(x) \geq u_{(n+1),f}.
\]

Now, we can prove Step 5, again by an approximation argument.
Let \( f \in B \) be such that \( u_{0,f} \notin \{0,1\} \) and let \( x \in K \setminus K' \) be such that \( f(x) \geq u_{(n+1),f} \). Since \( f(x) \geq u_{(n+1),f} > 0 \), it follows from Lemma 3.7 that \( \Phi(f)(x) \leq f(x) \). So, it remains to show that \( \Phi(f)(x) \geq u_{(n+1),f} \).

For any \( m \in \mathbb{N} \), one can find \( g_m \in B \) taking finitely many values such that \( \|f - g_m\| \leq \frac{1}{m} \), \( f(x) = g_m(x) \) and \( u_{0,g_m} = u_{0,f} \). Then, we have \( u_{(n+1),g_m} = u_{(n+1),f} \). So \( g_m(x) = f(x) \geq u_{(n+1),g_m} \), and hence \( \Phi(g_m)(x) \geq u_{(n+1),g_m} = u_{(n+1),f} \) for all \( m \in \mathbb{N} \). Since \( |\Phi(f)(x) - \Phi(g_m)(x)| \leq \|\Phi(f) - \Phi(g_m)\| \leq \|f - g_m\| \leq \frac{1}{m} \), it follows that \( \Phi(f)(x) \geq u_{(n+1),f} \).

**Step 6.** For each \( f \in B \) such that \( u_{0,f} \notin \{0,1\} \) and \( x \in K \setminus K' \), if \( f(x) \leq -u_{(n+1),f} \), then \( \Phi(f)(x) \in [f(x), -u_{(n+1),f}] \).

**Proof.** The proof is analogous to that of Step 5. \( \square \)

At this point, we have proved Lemma 3.5 by induction. So the proof of Theorem 1.1 is finally complete.

### 4. Proof of Theorem 1.2

In this section, \( K \) is a zero-dimensional compact Hausdorff space such that \( K \setminus K' \) is dense in \( K \), and we denote \( B_{C(K)} \) by \( B \). We fix a non-expansive homeomorphism \( F : B \to B \), and we want to show that \( F \) is an isometry.

Looking back at the proof of Theorem 1.1 we see that we only need two things:
- that Lemma 3.1 holds as stated, without assuming that \( K' \) is a finite set,
- that Fact 3.14 holds as stated, without assuming that \( K \) is metrizable.
Concerning Lemma 3.1, the only place in the proof where we used that \( K' \) is finite is the end of the proof of Step 4, to get that the map \( \sigma \) is injective.

Let us now show that \( \sigma \) is injective by using the continuity of \( F^{-1} \). For this we need the following three facts (which are true without assuming that \( F^{-1} \) is continuous). Recall that \( \alpha = F^{-1}(1) \).

**Fact 4.1.** Let \( f \in B \) be such that \( f = \alpha \cdot g \) where \( g(x) \geq 0 \) for all \( x \in K \). Then, \( F(f)(x) \geq 0 \) for all \( x \in K \). Moreover, if \( a \in K \setminus K' \) is such that \( f(a) = 0 \), then \( F(f)(\sigma(a)) = 0 \).

**Proof.** For every \( x \in K \), we have \( |f(x) - \alpha(x)| = |\alpha(x)g(x) - \alpha(x)| = |1 - g(x)| \leq 1 \) because \( g(x) \geq 0 \), hence \( \|f - \alpha\| \leq 1 \). Since \( F \) is non-expansive, it follows that
\[
\|F(f) - F(\alpha)\| \leq 1 \implies \|F(f) - 1\| \leq 1
\]
\[
\implies F(f)(x) \geq 0, \text{ for all } x \in K.
\]
Now, let \( a \in K \setminus K' \) be such that \( f(a) = 0 \). We want to show that \( F(f)(\sigma(a)) = 0 \).

We know that \( F(f)(\sigma(a)) \geq 0 \), so it remains to show that \( F(f)(\sigma(a)) \leq 0 \).

Let \( h \) be the following extreme point of \( B \):
\[
h(x) = \begin{cases} 1 & \text{if } x \neq \sigma(a), \\ -1 & \text{if } x = \sigma(a). \end{cases}
\]
Since \( h \) is an extreme point of \( B \), Fact 3.3 implies that
\[
F^{-1}(h)(a) = h(\sigma(a))\alpha(a) = -\alpha(a),
\]
and that for every \( x \neq a \), we have
\[
F^{-1}(h)(x) = h(\sigma(x))\alpha(x) = \alpha(x),
\]
since \( \sigma(x) \neq \sigma(a) \) (because \( \sigma_0 \) is injective and \( \sigma(K') \subset K' \)).

Hence,
\[
\|F^{-1}(h) - f\| \leq 1 \implies \|h - F(f)\| \leq 1
\]
\[
\implies |h(\sigma(a)) - F(f)(\sigma(a))| \leq 1
\]
\[
\implies 1 + F(f)(\sigma(a)) \leq 1
\]
\[
\implies F(f)(\sigma(a)) \leq 0.
\]
\[\square\]

**Fact 4.2.** Let \( f \in B \) be such that \( f = \alpha \cdot g \) where \( g(x) \geq 0 \) for all \( x \in K \). Then, for every \( a \in K \setminus K' \), we have
\[
0 \leq F(f)(\sigma(a)) \leq g(a).
\]

**Proof.** Let \( a \in K \setminus K' \).

If \( f(a) = 0 \), then Fact 4.1 implies that \( F(f)(\sigma(a)) = 0 \).

Now, assume that \( f(a) \neq 0 \). We know from Fact 4.1 that \( F(f)(\sigma(a)) \geq 0 \), so it remains to show that \( F(f)(\sigma(a)) \leq g(a) \).

We take \( h \) as in the proof of Fact 4.1, hence \( F^{-1}(h)(x) = \alpha(x) \) for all \( x \neq a \), and
\(F^{-1}(h)(a) = -\alpha(a)\). So, we obtain that \(|F^{-1}(h)(x) - f(x)| \leq 1\) for all \(x \neq a\), and \(|F^{-1}(h)(a) - f(a)| = 1 + g(a)\). Hence,

\[
\|F^{-1}(h) - f\| = 1 + g(a) \implies \|h - F(f)\| \leq 1 + g(a)
\]

\[
\implies |h(\sigma(a)) - F(f)(\sigma(a))| \leq 1 + g(a)
\]

\[
\implies 1 + F(f)(\sigma(a)) \leq 1 + g(a)
\]

\[
\implies F(f)(\sigma(a)) \leq g(a).
\]

\[\square\]

**Fact 4.3.** Let \(U\) be a clopen subset of \(K\) and let \(f \in B\) be such that \(f = \alpha \cdot g\) where

\[
g(x) = \begin{cases} 
  a & \text{if } x \in U, \\
  b & \text{otherwise},
\end{cases}
\]

and \(0 \leq b \leq a \leq 1\).

(i) If \(a = b\), then \(F(f) = a\mathbb{1}\).

(ii) If \(a > b\), then

- for every \(x \in K \setminus U\), \(F(f)(\sigma(x)) = b\),
- for every \(x \in U\), \(b \leq F(f)(\sigma(x)) \leq a\).

**Proof.** (i) By Theorem 2.1 (item 3), we have \(F^{-1}(a\mathbb{1}) = aF^{-1}(\mathbb{1}) = a\alpha = f\). Hence, \(F(f) = a\mathbb{1}\).

(ii) Define \(l = \alpha \in B\) where \(c = \frac{a+b}{2}\). By (i), we have \(F(l) = c\mathbb{1}\).

For every \(x \in U\),

\[
|l(x) - f(x)| = |\alpha(x) - a\alpha(x)| = a - c,
\]

and for every \(x \in K \setminus U\),

\[
|l(x) - f(x)| = |\alpha(x) - b\alpha(x)| = c - b.
\]

Hence, \(\|l - f\| = c - b = a - c\) because \(c = \frac{a+b}{2}\). Since \(F\) is non-expansive, we get \(\|F(l) - F(f)\| \leq c - b\).

For every \(x \in K\), we have

\[
|F(l)(\sigma(x)) - F(f)(\sigma(x))| \leq c - b \implies F(l)(\sigma(x)) - F(f)(\sigma(x)) \leq c - b
\]

\[
\implies c - F(f)(\sigma(x)) \leq c - b
\]

\[
\implies F(f)(\sigma(x)) \geq b.
\]

Hence, by Fact 4.2 we get \(F(f)(\sigma(x)) = b\) for all \(x \in K \setminus (K' \cup U)\), and \(b \leq F(f)(\sigma(x)) \leq a\) for all \(x \in (K \setminus K') \cap U\).

Since \(U\) is a clopen subset of \(K\), \(K \setminus K'\) is dense in \(K\) and \(\sigma\) and \(F(f)\) are continuous, the proof of (ii) is complete. \(\square\)

Now we are ready to show that \(\sigma\) is injective. Towards a contradiction, suppose that there exist \(u, v \in K\) such that \(u \neq v\) and \(\sigma(u) = \sigma(v) := z\).

Since \(\sigma(K') \subset K'\) and \(\sigma_0\) is injective, we get \(u, v \in K'\).

Since \(K\) is zero-dimensional, one can find a clopen neighbourhood \(U\) of \(u\) such that
Now, let us define \( f \in B \) as follows:

\[
f(x) = \begin{cases} 
  a\alpha(x) & \text{if } x \in U, \\
  b\alpha(x) & \text{otherwise},
\end{cases}
\]

where \( 0 \leq b < a \leq 1 \).

Fact \( 4.3 \) implies that \( F(f)(\sigma(x)) = b \) for all \( x \in K \setminus U \), and \( b < F(f)(\sigma(x)) \leq a \) for all \( x \in U \).

In particular, we have \( F(f)(y) = F(f)(\sigma(v)) = b \) since \( v \in K \setminus U \).

Fact \( 4.3 \) implies that for every \( d \), we have \( b < F(f)(\sigma(a_d)) \leq a \).

Now, for every \( d \), we define \( g_d \in B \) as follows:

\[
g_d(x) = \begin{cases} 
  c\alpha(x) & \text{if } x \neq a_d, \\
  a\alpha(x) & \text{if } x = a_d,
\end{cases}
\]

where \( c = \frac{a + b}{2} \).

Since \( \{a_d\} \) is a clopen subset of \( K \) (because \( a_d \in K \setminus K' \)), Fact \( 4.3 \) implies that \( F(g_d)(\sigma(x)) = c \) for every \( x \neq a_d \), and \( c \leq F(g_d)(\sigma(a_d)) \leq a \). This implies that \( F(g_d)(y) = c \), for all \( y \neq \sigma(a_d) \). Indeed, let \( y \in K \setminus (K' \cup \{a_d\}) \).

Since \( g_0 \) is bijective, there exists \( x \in K \setminus (K' \cup \{a_d\}) \) such that \( \sigma(x) = y \). Hence, \( F(g_d)(y) = c \). It follows that \( F(g_d)(y) = c \), for all \( y \neq \sigma(a_d) \), since \( K \setminus K' \) is dense in \( K \).

For every \( d \), we have

\[
|g_d(x) - f(x)| = \begin{cases} 
  |a\alpha(a_d) - a\alpha(a_d)| = 0 & \text{if } x = a_d, \\
  |c\alpha(x) - a\alpha(x)| = a - c & \text{if } x \in U \setminus \{a_d\}, \\
  |c\alpha(x) - b\alpha(x)| = c - b & \text{if } x \in K \setminus U,
\end{cases}
\]

because \( a_d \in U \).

Since \( a - c = c - b \), it follows that \( \|g_d - f\| = c - b \), for every \( d \). Hence, \( \|F(g_d) - F(f)\| \leq c - b \) and in particular \( F(g_d)(\sigma(a_d)) - F(f)(\sigma(a_d)) \leq c - b \), for every \( d \).

Since we know that \( F(f)(\sigma(a_d)) \rightarrow b \) and \( F(g_d)(\sigma(a_d)) \geq c \) for every \( d \), it follows that \( F(g_d)(\sigma(a_d)) \rightarrow c \).

Now, let us define \( l \in B \) such that \( l = c\alpha \). Fact \( 4.3 \) implies that \( F(l) = c1 \).

We have

\[
\|F(g_d) - F(l)\| = |F(g_d)(\sigma(a_d)) - F(l)(\sigma(a_d))| \rightarrow 0.
\]

Hence, \( F(g_d) \rightarrow F(l) \) and the continuity of \( F^{-1} \) implies that \( g_d \rightarrow l \), which is a contradiction since \( \|g_d - l\| = |g_d(a_d) - l(a_d)| = a - c \) for every \( d \).

Let us turn to Fact \( 3.14 \). Keeping the notation of the proof we have given for it assuming the metrizability of \( K \), one can modify the definition of the function \( l \) by setting \( l(y) = \text{sgn}(h_i)(h + \frac{1}{2}) \) if \( y \in V_i \) and \( y \neq x \). In this way, the function \( \chi \) does not appear any more and hence the metrizability assumption is no longer used. The definition of \( l \) shows that \( |l(y)| \leq u_{0,l} \) for all \( y \in K \setminus (K' \cup \{x\}) \); and since \( F^{-1} \) is continuous (because \( F^{-1} \) is continuous), we can apply Corollary \( 3.11 \) and the remark
following it to conclude that $\Phi^{-1}(l)(y) = l(y)$ for all $y \in K \setminus (K' \cup \{x\})$. Once this is known, the remaining of the proof works without any change.

5. Additional results

In this section, we prove the following theorem.

**Theorem 5.1.** Let $K$ be a zero-dimensional compact Hausdorff space with a dense set of isolated points such that $K'$ is a $G_δ$ subset of $K$, and let $F : B_{C(K)} \to B_{C(K)}$ be a non-expansive bijection. Under any of the following additional assumptions, one can conclude that $F$ is an isometry.

(i) $F$ has the following property $(\ast)$: for any $u, v, z \in K$ with $u \neq v$ and for any sign $\omega \in \{-1, 1\}$, one can find $f \in B_{C(K)}$ such that $f(u)f(v)$ is non-zero with sign $\omega$ and $|F(f)| \equiv 1$ in a neighbourhood of $z$.

(ii) $\inf \{\|F(\omega 1_a(\omega))\| : a \in K \setminus K', \ \omega = \pm 1\} > 1/2.$

**Proof.** Looking back at the proof of Theorem 1.1, we see that in fact, we only need to have

$\omega \in \{-1, 1\}$, one can find $f \in B_{C(K)}$ such that $f(u)f(v)$ is non-zero with sign $\omega$ and $|F(f)| \equiv 1$ in a neighbourhood of $z$.

The only trouble with the proof we have given for Lemma 3.1 is at the end of the proof of Step 4, where we cannot conclude immediately that the map $\sigma$ is injective. So we have to find another way of showing that $\sigma$ is indeed injective, without assuming that $K'$ is a finite set.

Before showing that $\sigma$ is injective, let us state the following fact: for every $a \in K \setminus K'$, we have

$$\tag{5.1} \text{sgn}\left(F(\omega 1_a(\omega))(\sigma(a))\right) = \omega \alpha(a) \quad \text{for } \omega = \pm 1.$$ 

Indeed, we have shown at the end of the proof of Fact 3.2 that for every $a \in K \setminus K'$, we have $\alpha(a) = \text{sgn}(F(1_a(\omega))(\sigma(a)))$. This implies $(5.1)$ since $F(1_a(\omega))(\sigma(a))$ and $F(-1_a(\omega))(\sigma(a))$ have opposite signs by Step 1 of the proof of Lemma 3.1.

Towards a contradiction, assume that $\sigma$ is not injective, so one can find $u \neq v \in K$ such that $\sigma(u) = \sigma(v) := z$.

Since $\sigma(K') \subset K'$ and $\sigma_0$ is injective, we have $u, v \in K'$, so let us fix two nets $(a_i), (b_j)$ in $K \setminus K'$ such that $a_i \to u$ and $b_j \to v$.

Assume that $F$ satisfies property $(\ast)$. Then, one can find $f \in B$ such that $\alpha(u)f(u) < 0 < \alpha(v)f(v)$ and $|F(f)| \equiv 1$ in a neighbourhood of $z$. By continuity of $\alpha$ and $f$, we have $\alpha(a_i)f(a_i) < 0 < \alpha(b_j)f(b_j)$ for all large enough $i, j$, so that $\|f + \alpha(a_i)1_{\{a_i\}}\| \leq 1$ and $\|f - \alpha(b_j)1_{\{b_j\}}\| \leq 1$. Since $F$ is non-expansive, this implies in particular that $|F(f)\sigma(a_i)) - F(-\alpha(a_i)1_{\{a_i\}})(\sigma(a_i))| \leq 1$ and $|F(f)\sigma(b_j)) - F(-\alpha(b_j)1_{\{b_j\}})(\sigma(b_j))| \leq 1$ for all large enough $i, j$. However, since $\sigma(a_i) \to z$ and $\sigma(b_j) \to z$ (because $\sigma$ is continuous), we also have $|F(f)\sigma(a_i))| = 1 = |F(f)\sigma(b_j))|$ for all large enough $i, j$. Since we know by $(5.1)$ that $F(-\alpha(a_i)1_{\{a_i\}})(\sigma(a_i)) < 0 < F(\alpha(b_j)1_{\{b_j\}})(\sigma(b_j))$, it follows
that $F(f)(\sigma(a_i)) = -1$ and $F(f)(\sigma(b_j)) = 1$ for all large enough $i, j$. Hence $F(f)(z)$ is both equal to $-1$ and $1$, which is the required contradiction.

Assume now that $c := \inf \{|F(\omega I_{\{a_i\}})|; \ a \in K \setminus K', \ \omega = \pm 1\} > 1/2$. Choose a function $f \in B$ with $\|f\| = 1/2$ such that $f - \alpha \equiv -1/2$ in a neighbourhood of $u$ and $f - \alpha \equiv 1/2$ in a neighbourhood of $v$. Then, $f(a_i)\alpha(a_i) = -1/2$ and $f(b_j)\alpha(b_j) = 1/2$ for all large enough $i, j$, which implies that $\|f + \alpha(a_i)I_{\{a_i\}}\| = 1/2$ and $\|f - \alpha(b_j)I_{\{b_j\}}\| = 1/2$. So we get $|F(f)(\sigma(a_i)) - F(-\alpha(a_i)I_{\{a_i\}})(\sigma(a_i))| \leq 1/2$ and $|F(f)(\sigma(b_j)) - F(\alpha(b_j)I_{\{b_j\}})(\sigma(b_j))| \leq 1/2$ for all large enough $i, j$. But $F(-\alpha(a_i)I_{\{a_i\}})(\sigma(a_i)) \leq -c$ and $F(\alpha(b_j)I_{\{b_j\}})(\sigma(b_j)) \geq c$, by (5.1) and the definition of $c$. So we see that $F(f)(\sigma(a_i)) \leq -c + 1/2$ and $F(f)(\sigma(b_j)) \geq c - 1/2$ for all large enough $i, j$. Hence, $F(f)(z) \leq -c + 1/2 < 0 < c - 1/2 \leq F(f)(z)$, which is again a contradiction. 

Corollary 5.2. If $K$ is a countable compact Hausdorff space, then any non-expansive bijection $F : B_{C(K)} \to B_{C(K)}$ either sending extreme points to extreme points or such that $\inf \{|F(f)|; \ \|f\| = 1\} > 1/2$ is an isometry.

Proof. The space $K$ is zero-dimensional with a dense set of isolated points, and it is also metrizable (so that the closed set $K'$ is $G_\delta$). Hence, the result follows from Theorem 5.1 (i) and (ii). 

Corollary 5.3. Any non-expansive bijection $F : B_{\ell_\infty} \to B_{\ell_\infty}$ either sending extreme points to extreme points or such that $\inf \{|F(f)|; \ \|f\| = 1\} > 1/2$ is an isometry.

Proof. The space $\ell_\infty$ is isometric to $C(\beta N)$, where $\beta N$ is the Stone-Čech compactification of $N$. The space $K = \beta N$ is zero-dimensional with a dense set of isolated points, and $K' = \beta N \setminus N$ is a $G_\delta$ subset of $\beta N$. So we may apply Theorem 5.1 (i) and (ii). 

References

[1] C. Angosto, V. Kadets and O. Zavarzina, Non-expansive bijections, uniformities and polyhedral faces. J. Math. Anal Appl. 471 (2019), 38–52.
[2] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. American Mathematical Society Colloquium Publications 48 (2000).
[3] B. Cascales, V. Kadets, J. Orihuela and E. J. Wingler, Plasticity of the unit ball of a strictly convex Banach space. RACSAM 110 (2016), 723–727.
[4] R. Haller, N. Leo and O. Zavarzina, Two new examples of Banach spaces with a plastic unit ball. Acta Comment. Univ. Tartu. Math. 26 (2022), 89–101.
[5] V. Kadets and O. Zavarzina, Plasticity of the unit ball of $\ell_1$. Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh. 83 (2017), 4–9.
[6] V. Kadets and O. Zavarzina, Nonexpansive bijections to the unit ball of the $\ell_1$-sum of strictly convex Banach spaces. Bull. Aust. Math. Soc. 97 (2018), 285–292.
[7] N. Leo, Plasticity of the unit ball of $c$ and $c_0$. J. Math. Anal Appl. 507 (2022).
[8] P. Mankiewicz, On extension of isometries in normed linear spaces. Bull. Acad. Polon. Sci. 20 (1972), 367–371.
[9] S. A. Naimpally, Z. Piotrowski and E. J. Wingler, Plasticity in metric spaces. J. Math. Anal Appl. 313 (2006), 38–48.

Univ. Artois, UR 2462, Laboratoire de Mathématiques de Lens (LML), F-62300 Lens, France

Email address: micheline.fakhoury@univ-artois.fr