Dynamic Black Holes in a FRW background: Lemaitre transformations

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Since the conformal transformations of metric do not change its causal structure, we use these transformations to embed the Lemaitre metrics into the FRW background. In our approach, conformal transformation is in agreement with the universe expansion regimes. Indeed, we use the Lemaitre metrics because the horizon singularity is eliminated in these metrics. For our solutions, there is an event horizon while its physical radii is increasing as a function of the universe expansion provided suitable metrics for investigating the effects of the universe expansion on the Black Holes. In addition, the physical and mathematical properties of the introduced metrics have well-defined behavior on the event horizon. Moreover, some physical and mathematical properties of the introduced metrics have been addressed.

I. INTRODUCTION

The universe expansion is described by Friedmann-Robertson-Walker (FRW) spacetime,

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \]

where, \( d\Omega \) is the line element on the unit 2-sphere hypersurface and \( k \) is the background curvature which points to flat, open and close universe for \( 0, -1 \) and \( 1 \), respectively. According to the standard cosmology, universe can be expanded either as a power law \( a(t) = At^{\omega} \) for \( \omega > -1 \) or exponentially \( a(t) = A \exp(\Omega t) \) for \( \omega = -1 \) (dark energy), depending on the equation of the state parameter \( \omega = \frac{k}{2} \). Furthermore, for \( \omega < -1 \), known as Phantom dark energy, the scale factor increases as \( a(t) = A(\omega r_t - t)^{-\frac{1}{\omega + 1}} \). In the above statement, \( H(\equiv \frac{\dot{a}}{a}) \) and \( \omega_r \) are the Hubble parameter and the Big Rip singularity time, when the expansion of universe ends catastrophically and everything will ultimately decompose into its elementary constituents, respectively [1]. Some aspects of the conformal form of the FRW metric has been studied in Refs. [2, 3]. In this metric \( \xi = a(t)r \) is the physical radii in this metric and \( r \) is called co-moving radius. It be mentioned that observations support the flat universe \( (k = 0) \) [1].

In order to find the effects of the universe expansion on local physics, McVittie derived a metric explaining a point-like chargeless particle in the FRW background [3]. Considering the perfect fluid concept together with the isotropic form of the FRW metric, the Mcvittie spacetime can be generalized to arbitrary dimensions and charged objects [6, 7]. It is easy to show that the curvature scalars diverge at the redshift singularity and energy conditions are not satisfied [8]. Also, the radii of the redshift singularity depends on the background curvature \( (k) \) and the mass and the charge are decreased by the expansion [7]. These failures are the unsatisfactory parts of these attempts and therefore, they may not provide a suitable metric for studying dynamical Black Holes (BHs) [8, 13].

Using perfect fluid concept in a non-static spherically symmetric background and considering the universe expansion, one can find solutions including constant mass, charge and cosmological constant [14]. On one hand, the curvature scalars diverge on the redshift singularity as the Mcvittie spacetime. On the other hand, the redshift singularity is independent of the background curvature contrary to the Mcvittie solution and its generalizations. In addition, The first law of thermodynamics is satisfied on hypersurface with radii equal to the redshift singularity. In continue and just same as the Mcvittie spacetime, energy conditions are not satisfied near this hypersurface. Finally, like the Mcvittie spacetime and its generalizations, it seems that these solutions are not suitable for studying dynamical BHs. Indeed, these two class of solutions may contain naked singularity which attracted more attempts to itself, since it was shown that the naked singularity can be considered as a possible source for gravitational lensing in astrophysical observations [14, 16]. More studies in which the perfect fluid concept is used to derive the dynamic spherically symmetric solutions can be found in Refs. [15, 18].

In another approach, Einstein et al. tried to connect the Schwarzschild spacetime to the FRW spacetime on a timelike hypersurface as a boundary [19]. In this model the Schwarzschild spacetime explains the inner spacetime where the FRW metric is used to describe the outer spacetime while junction conditions should be satisfied on the boundary [20, 21]. Since the inner spacetime is illustrated by the Schwarzschild metric, these solutions do not include dynamical BHs. Moreover, these solutions satisfy energy conditions and are classified into a more general class of solutions named Swiss-Cheese models [8].

It is shown that the conformal transformation does not change the causal structure of the spacetime [27]. Therefore, it arouses considerable interests in order to find dynamical BHs. Inspired by this property of the conformal transformation, some authors have considered the

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Schwarzschild and Reissner-Nordstrom metrics in various coordinates and then they tried to lay these BHs into the FRW background by using conformal transformation while conformal factor explains the universe expansion. Indeed, this technique helps us find new metrics. Thakurta spacetime and its generalization to charged BHs, which include conformal Schwarzschild and Reissner-Nordstrom metrics in their original coordinates, can be classified into the one class of solutions that has gravitational lagrangian (Ricci scalar) conformal to FRW’s. Indeed, these solutions can be again classified into the more general class of solutions with special symmetry and including conformal de-Sitter solutions. For these solutions, curvature scalars diverge at the redshift singularities which are null hypersurface and they are expanding with time. The generalization of the Thakurta spacetime to charged BHs does not satisfy energy condition where the Thakurta spacetime satisfies the energy conditions out of the redshift singularity. Since, the Thakurta spacetime includes the conformal transformation of the Schwarzschild spacetime and satisfies energy conditions out of the redshift singularity, it is now accepted that this redshift singularity may point to a dynamical BH. It was also shown that the first law of thermodynamics is satisfied on the horizon of the Thakurta spacetime. Apparent horizons and its properties including temperature and the amount of energy which is confined to it can be found in Ref. [31].

There is also a solution introduced by Sultana et al., which includes conformal transformation of the Eddington-Finkelstein form of the Schwarzschild metric. This solution does not satisfy energy conditions everywhere but the redshift singularity, but curvature scalars do not diverge at the redshift singularity which make good sense about the nature of this singularity. Because of the behavior of the curvature scalars and since this spacetime is just a conformal transformation of the Eddington-Finkelstein form of the Schwarzschild metric, it is now accepted that this spacetime may include dynamical BH.

Another solution has been introduced by M’Clure et al., which includes conformal transformation of the isotropic form of Schwarzschild satisfying the energy conditions everywhere. The generalization of this solution to the charged spacetime has also been investigated. It is easy to check that the curvature scalars diverge at the redshift singularity, where is a null expanding hypersurface, which it seems unsatisfactory. Since these objects are conformal transformations of isotropic form of the Schwarzschild and Reissner-Nordstrom metrics, they may point to dynamical BHs.

In this paper, we firstly review some properties of Lemaitre metrics. Then, in order to study the effects of the universe expansion on a BH, we introduce conformal incoming Lemaitre spacetime and verify its physical and mathematical properties including redshift, the behavior of the curvature scalars and energy conditions, in section II. In section III, we use the conformal incoming Lemaitre spacetime to investigate the effects of the universe expansion on the horizon. In addition, we investigate its physical and mathematical properties. Section IV is devoted to summary and conclusion.

Throughout this paper, the metric signature we shall adopt is (+ − − −), and we take $\hbar = c = G = 1$ for simplicity.

II. CONFORMATIONAL INWARD LEMAITRE SPACETIME

The Schwarzschild line element as the stationary vacuum solution of the Einstein field equations with spherically symmetry can be written as

$$dS^2_S = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $r_S = 2M$ are the line element on the unit 2-sphere and the Schwarzschild radius, respectively. In addition, $M$ is the mass enclosed by radii $r_S$, and Schwarzschild metric is only valid for $r > r_S$ while it is also undefined at horizon ($r = r_S$). In order to eliminate the metric singularity at the Schwarzschild radius one can apply the following coordinate transformations on the Schwarzschild coordinates $(r, t)$:

$$\bar{t} = \pm t \pm \int \frac{\sqrt{\bar{r}^2 - r_S^2}}{(1 - \bar{r}/r)} dr, \quad (3)$$

$$\bar{r} = t + \int \frac{\sqrt{\bar{r}/r^2}}{(1 - \bar{r}/r)} dr.$$

These transformations are called the Lemaitre transformations eliminating the metric singularity at the Schwarzschild radii and therefore, the Lemaitre metrics are regular at the event horizon. It means that these metrics can be used to explain a particle moving freely in a given field. Choosing the positive sign in (3), the Lemaitre element for the freely falling particles becomes

$$dS^2_L = dt^2 - \frac{r_S}{r} dr^2 - r^2 d\Omega^2, \quad (4)$$

where

$$r = \left(\frac{3}{2}(\bar{r} - \bar{t})\right)^{\frac{2}{3}} r_S^{\frac{1}{3}}. \quad (5)$$

We should note that the metric is non-stationary while the coordinates $\bar{r}$ and $\bar{t}$ are everywhere spacelike and timelike respectively. It is apparent from Eq. (4) that $\bar{r} - \bar{t} > 0$ must be valid everywhere. $\bar{t}$ is the proper time for particles which are at rest in the Lemaitre coordinate system. In these coordinates, the singularity on the Schwarzschild coordinates corresponds on the equality $r_S = \frac{3}{2}(\bar{r} - \bar{t})$, and particles cannot remain at rest for $r < r_S$. Calculations for the Kretschmann scalar lead to $R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} = \frac{64}{27(\bar{r} - \bar{t})}$, showing that there remains an essential curvature singularity.
at the origin (where $\tilde{r} - \tilde{t} = 0$) like the Schwarzschild coordinates [33, 37]. After some algebra we have,

$$1 + z = \frac{\lambda_O}{\lambda_E} = \frac{1 - \left(\frac{2r}{3(\tilde{r} - \tilde{t})_O}\right)^{2/3}}{1 - \left(\frac{2r}{3(\tilde{r} - \tilde{t})_E}\right)^{2/3}}, \quad (6)$$
as the redshift of incoming photon at radii $r_0 = (\frac{3}{2}(\tilde{r} - \tilde{t})_O)\tilde{r}_S$ emitted at $r_E = (\frac{3}{2}(\tilde{r} - \tilde{t})_E)\tilde{r}_S$. Redshift diverges at $r_E = r_S$ due to this fact that there is an event horizon at this radii. There is also another singularity at $(\tilde{r} - \tilde{t})_E = 0$ in accordance with the coordinate singularity of metric at the origin ($r = 0$).

By defining the conformal time $\tilde{t}$ as

$$\tilde{t} = \int_0^t \frac{dt'}{a(t')}, \quad \quad (7)$$
one can rewrite Eq. (1) as

$$ds^2 = a(\tilde{t})^2(-dt^2 + \left[\frac{dr^2}{1 - kr^2} + r^2d\Omega^2\right]), \quad \quad (8)$$
where $a(\tilde{t}) = \frac{4\tilde{t}}{\tilde{t}}$ and $a(\tilde{t}) = \frac{4\tilde{t}^2}{\tilde{t}}$ for $\omega = \frac{1}{3}$ and the matter domination era ($\omega = -1$). Indeed, the speed of light, as the traversed distance in the unit conformal time, is one in this new coordinates. Therefore, the universe expansion eras can also be modeled by using conformal scale factor $a(\tilde{t})$ instead of $a(t)$.

Considering the conformal scale factor $a(\tilde{t})$, we introduce metric

$$dS^2_{CL} = a(\tilde{t})^2\left(dt^2 - \frac{r_S}{r}dr^2 - r^2d\Omega^2\right), \quad \quad (9)$$
which is the conformal transformation of metric [1] and therefore, its corresponding causal structure is the same as that of [22] which can be found in [36]. This metric suffers generally from two singularities at $a(\tilde{t}) = 0$ (big bang) and $r = 0$ (origin). Simple calculation for the redshift leads to

$$1 + z = \frac{\lambda_O}{\lambda_E} = \frac{a(\tilde{t}_O)}{a(\tilde{t}_E)} \times \sqrt{\frac{1 - \left(\frac{2r}{3(\tilde{r} - \tilde{t})_O}\right)^{2/3}}{1 - \left(\frac{2r}{3(\tilde{r} - \tilde{t})_E}\right)^{2/3}}}, \quad \quad (10)$$
Redshift diverges at $\frac{3}{2}(\tilde{r} - \tilde{t})_E = r_S$ and $\frac{3}{2}(\tilde{r} - \tilde{t})_E = 0$ the same as the Lemaître metric. In addition, the FRW result is covered in the $r_S = 0$ limit as a desired result.

**Curvature scalars and energy conditions**

Let us calculate the surface area of a hypersurface with constant radii $r$ at constant time $\tilde{t}$

$$A = \int a(\tilde{t})^2r^2d\Omega = 4\pi a(\tilde{t})^2 r^2, \quad \quad (11)$$
where $r$ was defined in Eq. (5). Therefore, we have $A = 0$ for $r = 0$ and $A \neq 0$ for $r = r_S$ signalling that there is a curvature singularity at $r = 0$. For the normal to the hypersurface, $\phi = (\frac{3}{2}(\tilde{r} - \tilde{t}))\tilde{r}_S^2 - C = 0$, we have

$$n_\alpha = (-1, 1, 0, 0)(\tilde{r} - \tilde{t})^{-1/3}r_S^2. \quad \quad (12)$$
leading to

$$n_\alpha n^\alpha = (\tilde{r} - \tilde{t})^{-2}r_S^2a(\tilde{t})^{-2}(1 - \frac{r}{r_S}). \quad \quad (13)$$
Therefore, $r = r_S$ is a null hypersurface which is the same as that of [4]. In order to clarify the nature of the redshift singularities, we use the curvature scalars. Calculations lead to

$$R = 6\left(\frac{\dot{a} - \bar{a}}{\bar{a}(\tilde{r} - \tilde{t})}\right), \quad \quad \quad \quad (14)$$
and

$$R_{\mu\nu}R^\mu\nu = \frac{12\dot{a}^2}{a^2} - \frac{\bar{a}(12\bar{a}(\tilde{r} - \tilde{t}) + 16\bar{a}^3)}{a^2(\tilde{r} - \tilde{t})^2} + \frac{a^2(12\bar{a}(\tilde{r} - \tilde{t})^2 - 4\bar{a}^3(\tilde{r} - \tilde{t}) + 12\bar{a}^3)}{a^2(\tilde{r} - \tilde{t})^4}, \quad \quad (15)$$
for the Ricci scalar and square. Furthermore, for the Riemann and Weyl squares we get

$$K = \frac{24\dot{a}^4}{a^3} + \frac{12\dot{a}^2 - 8\bar{a}^3(\tilde{r} - \tilde{t}) + 12a^3(\tilde{r} - \tilde{t})^2}{a(\tilde{r} - \tilde{t})^2} - \frac{a^3(\tilde{r} - \tilde{t})^4}{2\bar{a}^4}\frac{1}{a(\tilde{r} - \tilde{t})^4}, \quad \quad (16)$$
and

$$W = \frac{64}{27a^4(\tilde{r} - \tilde{t})^4}. \quad \quad (17)$$
It can be seen that, unlike $r = 0$, none of them diverge at $r = r_S$. Therefore, $r = 0$ is a naked singularity while $r = r_S$ indicates that we face with an event horizon at this radii. Here, we can conclude that $r = r_S$ is the co-moving radii of the event horizon while its physical radii ($\xi$) is $\xi = a(\tilde{t})r_S$ in accordance with the FRW background. Therefore this event horizon covers the origin singularity ($r = 0$). Finally, we should note that the FRW results are accessible in the appropriate limit ($r \gg 1$). Bearing the Einstein field equation in mind, the energy-momentum tensor supporting the conformal Lemaître metric [9] is as follows,

$$T^\mu_\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & p_{22} & 0 \\ 0 & 0 & 0 & p_{33} \end{pmatrix}, \quad \quad (18)$$
where $\rho$ is the energy density and $p_{11}$ and $p_{22} (= p_{33})$ are the pressure components. The energy density and pressure components are obtained as follows

$$\rho = \frac{3\dot{a}^2(\tilde{r} - \tilde{t}) - 2\bar{a}\ddot{a}}{a^2(\tilde{r} - \tilde{t})}, \quad \quad (19)$$
It is be mentioned that, in the \( r > 1 \) limit \((\vec{r} - \bar{t}) \gg 1\), these relations converge to those of FRW as a desirable expectation. The weak (WEC) and strong (SEC) energy conditions state that

\[
\rho \geq 0 \quad \rho + p_{ii} \geq 0, \tag{22}
\]

and

\[
\rho + p_{ii} \geq 0 \quad \rho + \sum_{i=1}^{3} p_{ii} \geq 0, \tag{23}
\]

respectively. Combining Eqs. (5) and (19) and using WEC, we see that the \( \rho \geq 0 \) is valid when radii meets the \( r \geq (\frac{4}{9})^{\frac{1}{3}} \equiv r_{\rho} \) condition. It should be noted that \( h \equiv \frac{\dot{a}(\bar{t})}{a(\bar{t})} \) and \( \dot{a} \) denotes derivative with respect \( \bar{t} \). In order to satisfy \( \rho + p_{22} \geq 0 \), by combining Eqs. (5) and (21), we get \( r \geq (\frac{(2\bar{t})^{\frac{1}{2}}}{(h - h_{\rho})})^{\frac{1}{3}} \equiv r_{r} \). Since \( a(\bar{t}) \sim t \) and \( \dot{a}(\bar{t}) \sim t^{2} \) for the radiation (RDA) and matter (MDA) dominated eras respectively, \( h > 0, \dot{h} < 0 \) and thus \( h^{2} - \dot{h} > h^{2} \) yielding \( r \rho > r_{r} \). Therefore, when \( r \) meets the \( r \geq r_{\rho} \) condition, the \( \rho \geq 0 \) and \( \rho + p_{22} \geq 0 \) conditions are satisfied, in these regimes, simultaneously. The condition \( \rho + p_{11} \geq 0 \) leads to \( \frac{h^{2}}{r_{3}^{2}}(h^{2} - \dot{h}) + \frac{h}{r_{3}} \geq 0 \) which is valid in the RDA and MDA regimes \((\dot{h} < 0)\). Moreover, the \( \rho + \sum_{i=1}^{3} p_{ii} \geq 0 \) condition leads to \( \frac{\dot{a}}{r_{3}^{2}}(-\dot{h}) + \frac{\dot{a}}{r_{3}} \geq 0 \) which is again valid in the RDA and MDA regimes. Therefore, independent of \( \bar{t} \), the energy conditions are satisfied when \( r \) meets the \( r \geq r_{\rho} \) condition in the the RDA and MDA eras. Finally, we should note that since in the FRW limit \( r_{S} \to 0 \), \( r_{\rho} \to 0 \) and therefore, the energy conditions will be fully satisfied.

Using the conformal scale factor for the mysterious dark energy era introduced in previous section, since \( H \) is constant in the standard cosmology including a fluid with non-negative density and \( \omega = -1 \) for describing dark energy called cosmological constant \([1, 4]\), we have \( \dot{h} = h^{2} > 0 \). Therefore, \( \rho \geq 0 \) yields \( r \geq r_{\rho} \). In this era, \( \rho + p_{22} \geq 0 \) leads to \( -2\dot{h} > 0 \) indicating that this condition is not valid in this regime. In addition, \( \rho + p_{11} \geq 0 \) leads to \( \dot{h} \geq 0 \) which is valid independent of \( r \) and \( \bar{t} \). If \( r \) meet the \( r \leq (\frac{2\bar{t}}{3})^{\frac{1}{2}} = \frac{1}{4t} \) condition, then the \( \rho + \sum_{i=1}^{3} p_{ii} \geq 0 \) condition will be also valid. Therefore, for \( r \geq r_{\rho} \), only the \( \rho \geq 0 \) and \( \rho + p_{11} \geq 0 \) conditions are satisfied. Here, we must note that since the values of \( \rho \) and \( p_{ii} \) converge to those of FRW in the \((\vec{r} - \bar{t}) \gg 1 \) limit, the energy conditions are asymptotically converged to their values in the FRW metric where \( \rho > 0, \rho + p_{ii} = 0 \) and \( \rho + \sum_{i=1}^{3} p_{ii} \leq 0 \) are valid for a de-Sitter universe \((\omega = -1)\) \([1, 4]\). The same result is valid in the \( r_{S} \to 0 \) limit.

### III. CONFORMALLY OUTWARD LEMAITRE SPACETIME

In the previous section, we studied the mathematical and physical features of introduced metric which is conformal to the Lemaître metric for a system in which the particles freely falling into a gravitational field due to a point-like mass \( M \). Here, we extend our analysis to the metric which is conformal to the Lemaître metric describing a system where particles trajectories move outward from the singularity at \( r = 0 \). To do this, we should choose minus sign in (5), and we will get metric (6). But, here, we have

\[
r = (\frac{3}{2}(\vec{r} + \bar{t}))^{\frac{1}{2}} r_{S}^{\frac{3}{2}}. \tag{24}
\]

Using conformal transformation and after some algebra, we get:

\[
1 + z = \frac{\lambda_{0}}{\lambda_{E}} = \frac{a(\bar{t}_{0})}{a(\bar{t})} \times \frac{1 - \left(\frac{2r}{3(h - 1)_{0}}\right)^{2/3}}{1 - \left(\frac{2r}{3(h - 1)}\right)^{2/3}}. \tag{25}
\]

Again, we see that redshift diverges at \((\vec{r} + \bar{t}) = 0 \) and \((\vec{r} + \bar{t}) = r_{S} \) while the surface area at this radius are the same as those of obtained in the previous section. In addition, for a hypersurface with \( r = C \) we find

\[
n_{\alpha}n^{\alpha} = (\vec{r} + \bar{t})^{-\frac{5}{2}} r_{S}^{\frac{3}{2}} a(\bar{t})^{-2}(1 - \frac{r}{r_{S}}), \tag{26}
\]

indicating that we have a null hypersurface at \( r = r_{S} \) the same as the previous case. We have also

\[
R = -\frac{6(\dot{a} + a(\vec{r} + \bar{t}))}{a(\vec{r} + \bar{t})^{3}}, \tag{27}
\]

and

\[
R_{\mu\nu}R_{\mu\nu} = \frac{12a^{2}}{a^{2}} + \frac{\ddot{a}(16a - 12a(\vec{r} + \bar{t}))}{a^{2}(\vec{r} + \bar{t})} + \frac{a^{2}(12a^{2}(\vec{r} + \bar{t})^{2} + 4a(\dot{a}(\vec{r} + \bar{t}) + 12a^{2})}{a^{2}(\vec{r} + \bar{t})^{2}}, \tag{28}
\]

for the Ricci scalar and square. Moreover,

\[
K = \frac{24a^{4}}{a^{2}} + \frac{12a^{2} + 4a(\dot{a}(\vec{r} + \bar{t}) + 12a^{2}(\vec{r} + \bar{t})^{2}}{a^{2}(\vec{r} + \bar{t})^{2}} + \frac{1}{27a^{2}(\vec{r} + \bar{t})^{2}}, \tag{29}
\]

and

\[
W = \frac{64}{27a^{4}(\vec{r} + \bar{t})^{4}}. \tag{30}
\]
are the Riemann and Weyl squares respectively. As again, none of them diverge at $r = r_S$. In addition, we note that since our spacetime is conformal to the Lemaitre, its causal structure is the same that of Lemaitre. Therefore, exactly the same as the previous case, $r = 0$ and $r = r_S$ point to a naked singularity and the co-moving radii of an event horizon respectively, while for the physical radii we have $\xi = a(t) r$. This spacetime implies
\[
\rho = \frac{3a^2(\bar{r} + \bar{t}) + 2a\dot{a}}{a^4(\bar{r} + t)},
\]
\[
p_{11} = \frac{\dot{a}^2(\bar{r} + \bar{t}) + \frac{8}{3}a\dot{a} - 2a\ddot{a}(\bar{r} + \bar{t})}{a^4(\bar{r} + t)},
\]
\[
p_{22} = p_{33} = \frac{\dot{a}^2(\bar{r} + \bar{t}) + \frac{2}{3}a\dot{a} - 2a\ddot{a}(\bar{r} + \bar{t})}{a^4(\bar{r} + t)}.
\]

It can be found that in the $(\bar{r} + \bar{t}) \gg 1$ limit, the results of the FRW metric are obtainable. In addition, $\rho \geq 0$ is valid everywhere and it is independent of the conformal scale factor ($a(t)$). $p + p_{11} \geq 0$ leads to $3(\bar{r} + \bar{t})[h^2 - \bar{h}] + 7h \geq 0$ which, independent of the expansion regime, it is valid. For $p + p_{22} \geq 0$, we get $3(\bar{r} + \bar{t})[h^2 - \bar{h}] + 4h \geq 0$ indicating that this condition is again valid in the RDA and MDA regimes as well as the dark energy era. Finally, we can show that the $p + p_{11} + 2p_{22} \geq 0 \text{ condition leads to } 3(\bar{r} + \bar{t})[-\bar{h}] + h \geq 0$. The latter is valid in the RDA as well as MDA regimes, since $\bar{h} < 0$. In order to satisfy $\rho + \frac{3}{i = 1} p_{ii} \geq 0$ in the dark energy era, $r$ should meet the $r \leq (\frac{\rho_{CR}}{M^3})^{\frac{1}{3}}$ condition. In summary, the validity of the energy conditions in the RDA as well as MDA regimes is independent of $r_S$, while it depends on the mass enclosed by the event horizon located at $r_S$ in the dark energy era.

Loosely speaking, the energy conditions are not valid in the dark energy era for $r > (\frac{\rho_{CR}}{M^3})^{\frac{1}{3}}$.

IV. CONCLUSIONS

Since the Lemaitre transformations of the Schwarzschild metric provides the suitable metrics for studying the freely moving particles in a gravitational field due to the massive motionless object ($M$), we used this metric to encounter the effects of the universe expansion on the horizon. Indeed, the singularity of the Schwarzschild metric at the horizon radii is eliminated by Lemaitre transformations leading to this fact that if one embedded this metric into the FRW background, using the conformal transformation, then the resultant metric and its physical and mathematical properties will be well-defined at this radii. We should note again that since we have conformal transformed the Lemaitre metrics, the causal structure of the resultant metrics are the same as the primary metrics. In addition, the physical radii of the horizon is increased as a function of the universe expansion. Energy conditions are investigated for the conformal incoming and outgoing Lemaitre metrics. Therefore, unlike other works mentioned in the introduction, the conformal transformation of the Lemaitre metrics can provide suitable spacetimes for investigating the effects of the universe expansion on the local physics such as the BHs and thus the dynamic Black Holes.

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