Quasi-energy spectral series and the Aharonov–Anandan phase for the nonlocal Gross–Pitaevsky equation

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Abstract

For the nonlocal \( T \)-periodic Gross–Pitaevsky operator, formal solutions of the Floquet problem asymptotic in small parameter \( \hbar \), \( \hbar \to 0 \), up to \( O(\hbar^{3/2}) \) have been constructed. The quasi-energy spectral series found correspond to the closed phase trajectories of the Hamilton–Ehrenfest system which are stable in the linear approximation. The monodromy operator of this equation has been constructed to within \( \hat{O}(\hbar^{3/2}) \) in the class of trajectory-concentrated functions. The Aharonov–Anandan phases have been calculated for the quasi-energy states.

1 Introduction

The global properties of the solutions of mathematical physics equations which simulate a physical system are determined by the nontrivial geometry and topology of the system. A basic problem in investigating the behavior of a system is to find mathematical structures which would describe the topology of the system and make possible its efficient analysis. For quantum systems subjected to an external action cyclically varying with time, these structures are well known as topological or geometric phases (GP’s) of the wave function. Some manifestations of GP’s were also observed in polarization optics and mechanics. Theoretical and experimental studies of geometric phases in quantum mechanics have been performed since the late 20s of the last century. The work by Berry [1] has substantially extended the area of application of GP’s and has led to the interpretation of this notion in terms of gauge symmetry and an effective gauge field or, in geometric formulation, in terms of a Hilbert bundle with a finite-dimensional base. A detailed description of this problem can be found, for instance, in [2–4].

The theory of GP’s in quantum mechanics is based on the linearity of the Schrödinger equation. For nonlinear equations, the notion of GP can also be introduced (see, e.g., [2]), but the GP’s in nonlinear systems are less well understood. Difficulties arise not only for lack of the principle of superposition of solutions. In case of a nonlinear equation, the expression for the GF’s is constructed by analogy with the linear theory (see, e.g., [2]), and it is not fully obvious that the expression constructed is governed only by the geometry of the system and that it does not involve the dynamic contribution due to nonlinearity. The nontrivial topology of a system can be determined by the boundary conditions as well as by the external fields. The latter appear in the equation as variable factors. Few, if any methods are available to construct exact solutions for equations of this type. Therefore, it is natural to study GP’s in nonlinear systems by using adequate approximation methods.

In this connection, the analysis of the GP’s for the nonlocal Gross–Pitaevsky equation given below would be of interest. The evolution of the initial state of the nonlinear equation (1.1) in the semiclassical approximation is determined in fact by a set of associated linear Schrödinger equations. This allows one not only to calculate in full measure the GP for the nonlinear equation in explicit form, but also to interpret it in terms of “an induced gauge field” (see [2]), which in turn proves its geometric origin.

A class of solutions asymptotic in small parameter \( \hbar \to 0 \), which are localized in a neighborhood of some phase curve, has been constructed [5–9] for the nonlocal Gross–Pitaevsky equation (NGPE) with an external field (variable coefficients) and a nonlocal nonlinearity

\[
\{-i\hbar \partial_t + \hat{H}(t) + \hat{V}(t, \Psi)\} \Psi = 0, \quad \Psi \in L_2(\mathbb{R}^n).
\]
Here, the operators
\[ \hat{H}(t) = \mathcal{H}(\hat{z}, t), \] (1.2)
\[ \hat{V}(t, \Psi) = \int_{\mathbb{R}^n} d\hat{y} \Psi^* (\hat{g}, t) V(\hat{z}, \hat{w}, t) \Psi(\hat{g}, t) \] (1.3)
are functions of the noncommuting and Weyl-ordered operators
\[ \hat{z} = \left( -i\hbar \frac{\partial}{\partial \hat{x}}, \hat{x} \right), \quad \hat{w} = \left( -i\hbar \frac{\partial}{\partial \hat{y}}, \hat{y} \right), \quad \hat{x}, \hat{y} \in \mathbb{R}^n, \]
the function \( \Psi^* \) is complex conjugate to \( \Psi \), \( \varkappa \) is a real parameter, and \( \hbar \) is a “small parameter”, \( \hbar \in [0, 1] \). For the operators \( \hat{z} \) and \( \hat{w} \) the following commutation relations are valid:
\[ [\hat{z}_k, \hat{z}_j] = [\hat{w}_k, \hat{w}_j] = i\hbar J_{k j}, \quad [\hat{z}_k, \hat{w}_j] = 0, \quad k, j = 0, 2n, \] (1.4)
where \( J = ||J_{k j}||_{2n \times 2n} \) is a symplectic unit matrix
\[ J = \begin{pmatrix} 0 & -I \n \n I & 0 \end{pmatrix}_{2n \times 2n}. \]

The goal of this work is to construct explicit expressions for the quasi-energy spectral series corresponding to the closed phase trajectories of the Hamilton–Ehrenfest system which are stable in the linear approximation and GP’s in the class of asymptotic solutions of equation (1.1)-(1.3), constructed in \([5–9]\), for the case where the operators \( \hat{H}(t) \) and \( \hat{V}(t, \Psi) \) are \( T \)-periodic functions of time:
\[ \hat{H}(t + T) = \hat{H}(t), \quad \hat{V}(t + T, \Psi) = \hat{V}(t, \Psi). \] (1.5)

For the linear Schrödinger equation \( \varkappa = 0 \) with a \( T \)-periodic Hamiltonian \( \hat{H}(t) \), Zeldovich \([10]\) and Ritus \([11]\) were first to introduce an important class of solutions — quasi-energy states \( \Psi_{\mathcal{E}}(\vec{x}, t, h) \), which possess the property
\[ \Psi_{\mathcal{E}}(\vec{x}, t + T, h) = e^{-i\mathcal{E} T/\hbar} \Psi_{\mathcal{E}}(\vec{x}, t, h), \] (1.6)
The quantity \( \mathcal{E} \), involved in (1.6), is called quasi-energy, and it is defined modulo \( \hbar \omega \) \( (\omega = 2\pi/T) \), i.e. \( \mathcal{E}' = \mathcal{E} + m\hbar \omega, m \in \mathbb{Z} \). States of this type play the key part in describing quantum-mechanical systems under strong periodic external actions for which the conventional methods of nonstationary perturbation theory are not applicable.

For the nonlocal Gross–Pitaevsky equation (1.1)-(1.3), we shall set the problem of finding a quasi-energy spectrum in the form (1.6).
The quasi-energy states (1.6) are a particular case of the cyclic states introduced by Aharonov and Anandan \([12]\) (see also \([2]\)). By cyclic evolution of a quantum system on a time interval \([0, T]\) it is implied the following: the state vector \( \Psi(t) \) has the form
\[ \Psi(t) = e^{if(t)} \varphi(t), \quad t \in [0, T], \] (1.7)
where
\[ f(T) - f(0) = \phi(\text{mod } 2\pi), \quad \varphi(T) = \varphi(0). \] (1.8)
The total phase \( \phi \) of the function (1.7) is subdivided into two terms: the dynamic phase
\[ \delta = -\frac{1}{\hbar} \int_0^T dt \frac{\langle \Psi(t) | [\hat{H}(t) + \varkappa \hat{V}(t, \Psi(t))] | \Psi(t) \rangle}{\langle \Psi(t) | \Psi(t) \rangle} \] (1.9)
and the Aharonov–Anandan geometric phase
\[ \gamma = i \int_0^T dt \frac{\langle \varphi(t) | \dot{\varphi}(t) \rangle}{\langle \varphi(t) | \varphi(t) \rangle}. \] (1.10)
Comparing (1.6) and (1.7), we obtain that for the quasi-energy states the function \( f(t) \) is given by
\[ f(t) = -\mathcal{E} t / \hbar, \] (1.11)
while for the total phase $\phi$, according to (1.8), we have

$$\phi = -\frac{\mathcal{E} t}{\hbar} \pmod{2\pi}. \quad (1.12)$$

By virtue of (1.9)–(1.12), the Aharonov–Anandan phase $\gamma_\mathcal{E}$ corresponding to a given quasi-energy state $\Psi_\mathcal{E}(\vec{x}, t, \hbar)$ can be determined by formula (mod $2\pi$):

$$\gamma_\mathcal{E} = -\frac{\mathcal{E} t}{\hbar} + \frac{1}{\hbar} \int_0^T dt \frac{\langle \Psi_\mathcal{E} | [\hat{H}(t) + \sqrt{\mathcal{E}} \hat{V}(t, \Psi(t))] | \Psi_\mathcal{E} \rangle}{\langle \Psi_\mathcal{E} | \Psi_\mathcal{E} \rangle}. \quad (1.13)$$

2 Statement of the problem in the class of semiclassically concentrated functions

We shall construct asymptotic solutions for the nonlocal Gross–Pitaevsky equation with the following assumption for the Weyl symbols of the operators $\mathcal{H}(\tilde{z}, t)$ and $V(\tilde{z}, \tilde{w}, t)$ in (1.1)–(1.3):

Assumption 1 The functions $\mathcal{H}(z, t)$ and $V(z, w, t)$ are $C^\infty$-smooth functions and for any multiindices $\alpha$ and $\beta$, $(\alpha, \beta, z, w) \in \mathbb{Z}_+^{2n}$ and $T > 0$, there exist constants $\kappa > 0$, $C_\alpha(T)$, and $C_{\alpha\beta}(T)$, such that

$$\left| \frac{\partial^{\alpha|\beta}\mathcal{H}(z, t)}{\partial z^{\alpha}} \right| \leq C_\alpha(T)(1 + |z|^\kappa), \quad \left| \frac{\partial^{\alpha+\beta}V(z, w, t)}{\partial z^{\alpha}w^{\beta}} \right| \leq C_{\alpha\beta}(T)(1 + |z| + |w|)^\kappa, \quad z, w \in \mathbb{R}^{2n}, \quad 0 \leq t \leq T.$$

Here,

$$\frac{\partial^{\alpha|\beta}V(z)}{\partial z^{\alpha}} = \frac{\partial^{\alpha|\beta}V(z)}{\partial z_1^{\alpha_1}z_2^{\alpha_2} \ldots z_{2n}^{\alpha_{2n}}}, \quad \alpha_j \in \mathbb{Z}_+, \quad j = 1, 2n, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n}), \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{2n}, \quad z^{\alpha} = z_1^{\alpha_1}z_2^{\alpha_2} \ldots z_{2n}^{\alpha_{2n}}.$$

We now turn to the description of the class of functions in which we shall seek localized asymptotic solutions of the equation (1.1)–(1.3). The functions of this class, which singularly depend on a small parameter $\hbar$, represent a generalization of a solitary wave. They depend on an arbitrary phase trajectory $z = Z(t, h) \in \mathbb{R}^{2n}_{p, x}$, $t \in \mathbb{R}^1$ and on a real-valued function $S(t, h)$ (analog of a classical action for a linear case with $\mathcal{E} = 0$). In the limit $h \to 0$, the functions of this class are concentrated in a neighborhood of a point moving along a given phase curve $z = Z(t, 0)$. Functions of this type are well known in quantum mechanics. In particular, among these are the coherent and “squeezed” states of quantum systems with a quadratic Hamiltonian [13, 14].

Let us denote this class of functions by $\mathcal{P}_h^l(Z(t, h), S(t, h))$ and define it as follows:

$$\mathcal{P}_h^l = \mathcal{P}_h^l(Z(t, h), S(t, h)) = \left\{ \Phi : \Phi(\tilde{x}, t, h) = \varphi \left( \frac{\Delta \tilde{x}}{\sqrt{h}} \right) t, \sqrt{h} \exp \left[ \frac{i}{h}(S(t, h) + \langle \hat{F}(t, h), \Delta \tilde{x} \rangle) \right] \right\}. \quad (2.1)$$

where the complex-valued function $\varphi(\xi, t, \sqrt{h})$ belongs to the Schwartz space $\mathbb{S}$ in the variable $\xi \in \mathbb{R}^n$, smoothly depends on $t$ and regularly depends on $\sqrt{h}$ as $h \to 0$. Here, $\Delta \tilde{x} = \tilde{x} - \tilde{X}(t, h)$, and the real function $S(t, h)$ and 2$n$-dimensional vector function $Z(t, h) = (\hat{F}(t, h), \tilde{X}(t, h))$, which characterize the class $\mathcal{P}_h^l(Z(t, h), S(t, h))$, regularly depend on $\sqrt{h}$ in a neighborhood of $h = 0$, such that $S(0, h) = 0$, $Z(0, h) = z_0 = (\hat{p}_0, \tilde{x}_0)$ is an arbitrary point of the phase space $\mathbb{R}^{2n}_{p, x}$, and are to be determined for $t > 0$.

The functions belonging to the class $\mathcal{P}_h^l$, at any fixed point in time $t \in \mathbb{R}^1$, are concentrated, in the limit $h \to 0$, in a neighborhood of a point lying on the phase curve $z = Z(t, 0)$, $t \in \mathbb{R}^1$ (the exact sense of this property is specified below by formulas (2.6), (2.7)). Therefore, it is natural to give the functions of the class $\mathcal{P}_h^l$ the name trajectory-concentrated functions as $h \to 0$. The definition of the class of trajectory-concentrated functions involves a phase trajectory $Z(t, h)$ and a scalar function $S(t, h)$ as “free” parameters. They turned out that these “parameters” can be uniquely determined by the Hamilton–Ehrenfest system (see Sect. 3), which corresponds to the nonlinear ($\mathcal{E} \neq 0$) Hamiltonian of equation (1.1)–(1.3). Note that for a Schrödinger type equation, in the linear case ($\mathcal{E} = 0$), the vector function $Z(t, 0)$ — the principal term of the expansion in $h \to 0$ — determines the phase trajectory of a Hamilton system with a classical Hamiltonian $\mathcal{H}(\vec{p}, \vec{x}, t)$, and the function $S(t, 0)$ is a classical action along this trajectory. In particular, in this case, the class $\mathcal{P}_h^l$ involves the well-known dynamic (squeezed) states of quantum systems with quadratic Hamiltonians.

Let us consider the basic properties of functions of the class $\mathcal{P}_h^l(Z(t, h), S(t, h))$ (see [5, 6, 15]).
1 For functions of the class \( P_h^1(Z(t, \hbar), S(t, \hbar)) \), the following asymptotic estimates are valid for the centered moments \( \Delta_\alpha(t, \hbar) \) of order \(|\alpha|\), \( \alpha \in \mathbb{Z}_+^{2n} \):

\[
\Delta_\alpha(t, \hbar) = \Delta_\alpha^\Phi(t, \hbar) = \frac{\langle \Phi | \Delta \hat{\varepsilon} \rangle^\alpha |\Phi\rangle}{||\Phi||^2} = O(\hbar^{\alpha/2}), \quad \hbar \to 0, \quad |\alpha| \neq 0;
\]

\[
\Delta_\alpha(t, \hbar) = 1, \quad |\alpha| = 0.
\]

Here \( \{\Delta \hat{\varepsilon}\}^\alpha \) denotes an operator with the Weyl symbol \( (\Delta \hat{\varepsilon})^\alpha \),

\[
\Delta \hat{\varepsilon} = z - Z(t, \hbar) = (\Delta \hat{p}, \Delta \hat{x}), \quad \Delta \hat{p} = \hat{p} - \tilde{P}(t, \hbar), \quad \Delta \hat{x} = \hat{x} - \tilde{X}(t, \hbar).
\]

Denote by the symbol \( \hat{O}(\hbar^k), \quad k \geq 0 \), an operator \( \hat{F} \), such that for any function \( \Phi \in P_h^1(Z(t, \hbar), S(t, \hbar)) \) the following asymptotic estimate is valid:

\[
\frac{||\hat{F}\Phi||}{||\Phi||} = O(\hbar^k), \quad \hbar \to 0.
\]

2 For functions of the class \( P_h^1(Z(t, \hbar), S(t, \hbar)) \), the following asymptotic estimates are valid:

\[
\{ -i\hbar \partial_t - \hat{S}(t, \hbar) + \langle \hat{F}(t, \hbar), \hat{X}(t, \hbar) \rangle + \langle \hat{Z}(t, \hbar), J \Delta \hat{\varepsilon} \rangle \} = \hat{O}(\hbar),
\]

\[
\{ \Delta \hat{\varepsilon}\}^\alpha = \hat{O}(\hbar^{\alpha/2}), \quad \alpha \in \mathbb{Z}_+^{2n}, \quad \hbar \to 0,
\]

and, in particular,

\[
\Delta \hat{e}_k = \hat{O}(\sqrt{\hbar}), \quad \Delta \hat{p}_j = \hat{O}(\sqrt{\hbar}), \quad k, j = 1, n.
\]

The following property gives an exact sense to the notion of the concentration, as \( \hbar \to 0 \), in a neighborhood of a point on the phase trajectory for functions of the class \( P_h^1 \).

3 For any function \( \Phi(\hat{x}, t, \hbar) \in P_h^1(Z(t, \hbar), S(t, \hbar)) \), the following limiting relations are valid:

\[
\lim_{\hbar \to 0} \frac{1}{||\Phi||^2} |\Phi(\hat{x}, t, \hbar)|^2 = \delta(\hat{x} - \tilde{X}(t, 0)),
\]

\[
\lim_{\hbar \to 0} \frac{1}{||\Phi||^2} |\Phi(\hat{p}, t, \hbar)|^2 = \delta(\hat{p} - \tilde{P}(t, 0)),
\]

where \( \hat{\Phi}(\hat{p}, t, \hbar) = F_{\hat{x} \to \hat{p}} \Phi(\hat{x}, t, \hbar), \quad F_{\hat{x} \to \hat{p}} \) is a Fourier \( \hbar \)-transform [16].

4 Denote by \( \langle \hat{L}(t) \rangle \) the mean value of an operator \( \hat{L}(t) \), \( t \in \mathbb{R}^1 \) selfadjoint in \( L^2(\mathbb{R}^n) \) that is calculated by the function \( \Phi(\hat{x}, t, \hbar) \in P_h^1 \). Then, for any function \( \Phi(\hat{x}, t, \hbar) \in P_h^1(Z(t, \hbar), S(t, \hbar)) \) and any operator \( \hat{A}(t, \hbar) \) whose Weyl symbol \( A(z, t, \hbar) \) satisfies Assumption 2, the following equality is valid:

\[
\lim_{\hbar \to 0} \langle \hat{A}(t, \hbar) \rangle = \lim_{\hbar \to 0} \frac{1}{||\Phi||^2} \langle \Phi(\hat{x}, t, \hbar)|\hat{A}(t, \hbar)|\Phi(\hat{x}, t, \hbar) \rangle = A(Z(t, 0), t, 0).
\]

By analogy with the linear theory [15], we give the following definition.

**Definition 1** A solution \( \Phi(\hat{x}, t, \hbar) \) of equation (1.1)–(1.3) in the class of functions \( P_h^1 \) is called semiclassically concentrated as \( \hbar \to 0 \) on the phase trajectory \( Z(t, \hbar) \).

The limiting character of the conditions \( (2.4) \)–\( (2.7) \) and the asymptotic character of the estimates \( (2.2) \)–\( (2.5) \), valid in the class of trajectory-concentrated functions, make it possible to construct the semiclassically concentrated solutions of the nonlocal Gross–Pitaevskiy equation not exactly, but approximately. In this case, the \( L_2 \)-norm of the error is of order \( \hbar^\alpha \), \( \alpha > 1 \), as \( \hbar \to 0 \) on any finite time interval \( [0, T] \).

Denote such an approximate solution by \( \Psi_{as} = \Psi_{as}(x, t, \hbar) \). It satisfies the following problem:

\[
[-i\hbar \frac{\partial}{\partial t} + \tilde{H}(t) + \varepsilon \tilde{V}(t, \Psi_{as})] \Psi_{as} = O(\hbar^\alpha),
\]

\[
\Psi_{as} \in P_h^1(Z(t, \hbar), S(t, \hbar), \hbar), \quad t \in [0, T],
\]

where \( O(\hbar^\alpha) \) denotes the function \( g^{(\alpha)}(\hat{x}, t, \hbar) \) with respect to the error for equation (1.1)–(1.3) and obeys the estimate in the norm \( L_2 \):

\[
\max_{0 \leq t \leq T} \| g^{(\alpha)}(\hat{x}, t, \hbar) \| = O(\hbar^\alpha), \quad \hbar \to 0.
\]
We shall also call the function \( \Psi_{as}(x, t, h) \), satisfying the problem (2.11)–(2.14), a semiclassically concentrated solution (mod \( \hbar^N \), \( h \to 0 \)) of the nonlocal Gross–Pitaevsky equation (1.1)–(1.3).

Thus, the semiclassically concentrated solutions \( \Psi_{as}(x, t, h) \) of the nonlocal Gross–Pitaevsky equation approximately describe the evolution of the initial state \( \psi_0(x, h) \) if it is chosen in the class of the trajectory-concentrated functions \( \mathcal{P}^0_h \):

\[
\Psi(x, t, h) = \psi(0, x, h), \quad \psi \in \mathcal{P}^0_h(z_0, 0).
\]

The functions of the class \( \mathcal{P}^0_h \) have the form

\[
\psi_0(x, h) = \exp\left\{ \frac{i}{\hbar} \left[ \hat{H}_0(x, \bar{x}_0) \right] \right\} \varphi_0 \left( \frac{x - \bar{x}_0}{\sqrt{\hbar}} \right), \quad \varphi_0(\xi, \sqrt{\hbar}) \in \mathcal{S} (\mathbb{R}_x^n),
\]

where \( z_0 = (\bar{p}_0, \bar{x}_0) \) is an arbitrary point of the phase space \( \mathbb{R}^{2n} \).

As in the linear case [17], among the solutions of equation (1.1) that satisfy the quasi-periodicity condition (1.0), in the class of solutions concentrated in a neighborhood of some phase trajectory, we can isolate a set of semiclassical asymptotics \( \Psi_{E^r} \) which possess the following properties:

1) \( \Psi_{E^r}(x, t, h) \) are approximate in mod \( \mathcal{O}(h^{5/2}) \) solutions of equation (1.1). This implies that

\[
\left[ -i\hbar \partial_t + \hat{H}(t) + \sqrt{\nu}(t, \Psi(t)) \right] \Psi_{E^r}(x, t, h) = v_r(x, t, h),
\]

\[
\max_{t \in [0, T]} \| v_r(x, t, h) \|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{5/2}).
\]

2) The functions \( \Psi_{E^r} \) have the form of wave packets concentrated at any time \( t \) in a neighborhood of a given \( T \)-periodic trajectory \( z = Z(t, h) \). We shall call these states, if any, the quasi-energy trajectory-coherent states (TCS’s) of the nonlocal Gross–Pitaevsky equation.

3 System of Hamilton–Ehrenfest equations

The symbols \( \mathcal{H}(z, t) \) and \( V(z, w, t) \) satisfy the conditions of Assumption [11]. Therefore, the operator \( \mathcal{H}(\hat{z}, t) \) in (1.1)–(1.3) is selfadjoint with respect to the scalar product \( \langle \Psi|\Phi \rangle \) in the space \( L_2(\mathbb{R}^n_x) \) and the operator \( \mathcal{V}(\hat{z}, \hat{w}, t) \) with respect to the scalar product \( L_2(\mathbb{R}^n_x\times \mathbb{R}^n_x) \): \( \langle \Psi(t)|\Phi(t) \rangle_{\mathbb{R}^{2n}} = \int d\bar{x}d\bar{y} \Psi^*(\bar{x}, \bar{y}, t) \Phi(\bar{x}, \bar{y}, t, h) \).

Hence, the squared norm of the exact solutions of equation (1.1)–(1.3) is reserved: \( \| \Psi(t) \|^2 = \| \Psi(0) \|^2 \), and for the mean values of the operator \( \hat{A}(t) = A(z, t) \) calculated for these solutions

\[
\frac{d}{dt} \langle \hat{A}(t) \rangle = \left\langle \frac{\partial \hat{A}(t)}{\partial t} \right\rangle + \frac{i}{\hbar} \langle [\hat{H}, \hat{A}(t)] \rangle + \frac{i
u}{\hbar} \int d\bar{y} \Psi^*(\bar{y}, t, h)[V(\hat{z}, \hat{w}, t), \hat{A}(t)]\Psi(\bar{y}, t, h),
\]

where \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \) is the commutator of the operators \( \hat{A} \) and \( \hat{B} \). By analogy with a linear \( (\nu = 0) \) Schrödinger equation in quantum mechanics, we call equality (3.1) the Ehrenfest equation for the mean values of the operator \( \hat{A}(t) \) which corresponds to the nonlocal Gross–Pitaevsky equation.

We suppose that for the nonlocal Gross–Pitaevsky equation (1.1)–(1.3) there exist exact (or differing by \( \mathcal{O}(h^\infty) \) from exact) solutions in the class of trajectory-concentrated functions. Let us write the Ehrenfest equations (3.1) for the mean values of the operators \( \hat{z}, \{ \hat{\Delta}_z \}^n \), which are calculated by this type (trajectory-concentrated) solutions of equation (1.1)–(1.3), with the use of the composition rules for the Weyl symbols [16]:

\[
C(z) = A \left( \hat{z} - \frac{i\hbar}{2} J \frac{\partial}{\partial z} \right) B(z) = B \left( \hat{z} - \frac{i\hbar}{2} J \frac{\partial}{\partial z} \right) A(z),
\]

where \( C(z) \) is the symbol of the operator \( \hat{C} = \hat{A}\hat{B} \) and the numbers 1 and 2 above an operator indicate the order of its action (recall that \( \hat{z} = (\hat{p}, \hat{x}) \), \( Z(t, h) = (\hat{P}(t, h), \hat{X}(t, h)) \), \( \hat{\Delta}_z = \hat{z} - Z(t, h) \)). Then, after calculations similar to that in the linear case \( \nu = 0 \) (see for details [15, 18]), restricting ourselves to the second-order moments, we obtain the following system of ordinary differential equations:

\[
\hat{z} = J\partial_z \left( 1 + \frac{1}{2}(\partial_z, \Delta_z) \right) (\mathcal{H}(z, t) + \sqrt{\nu} V(z, w, t)|_{w=z}),
\]

\[
\hat{\Delta}_z = J\partial_z (z, t) \Delta_z - \Delta_z \partial_z (z, t) J_z,
\]

where \( \Delta_z = (\Delta_z^{\alpha \beta}) \) is a \( (2n \times 2n) \) matrix of “variances”, the \( (2n \times 2n) \) matrix \( \partial_{zz}(z, t) \) is defined by

\[
\partial_{zz}(z, t) = \left[ \mathcal{H}(z, t) + \sqrt{\nu} V(z, w, t) \right]|_{w=z}.
\]
moments implies nonnegative determinacy of the matrix

\[ \langle \psi_0(\vec{x},\hbar) \rangle^2, \psi_0(\vec{x},\hbar) \] denotes the initial condition of the Cauchy problem \((2.1)\).

System \((3.2)\) can be written in an equivalent, setting in the second equation

\[ \Delta_2(t) = A(t)\Delta_2(0)A^+(t), \]

which then takes the form

\[ \dot{A} = J\delta_{zz}(z,t)A \quad A(0) = \mathbb{I}. \] \,(3.3)

We shall call the system of equations \((3.2), (3.3)\) the second order Hamilton–Ehrenfest (HE) system (here \(M = 2\) is the order of the greatest moment taken into account) corresponding to the Gross–Pitaevsky equation \((1.1)–(1.3)\).

**Theorem 1** Let functions \(\Psi(\vec{x},t,\hbar)\) and \(\Psi(\vec{x},t,\hbar)\) be, respectively, an asymptotic, to within \(O(\hbar^{3/2})\), and an exact solution of equation \((2.1)\), which coincide at some point in time \(t = t_0\). Then the second order Hamilton–Ehrenfest systems \((3.2)\), constructed for the functions \(\Psi(\vec{x},t,\hbar)\) and \(\Psi(\vec{x},t,\hbar)\), coincide.

**Theorem 2** Let functions \(\Psi(\vec{x},t,\hbar)\) and \(\Psi(\vec{x},t,\hbar)\) be, respectively, an asymptotic, to within \(O(\hbar^{3/2})\), and an exact solution of equation \((2.1)\), which coincide at some point in time \(t = t_0\). Then

\[ \Delta_2 \Psi(t,\hbar) = \Delta_2 \Psi(t,\hbar) + O(\hbar^{3/2}) = \Delta_2(t,\hbar) + O(\hbar^{3/2}), \]

\[ z_\Psi(t,\hbar) = z_\Psi(t,\hbar) + O(\hbar^{3/2}) = Z^2(t,\hbar) + O(\hbar^{3/2}). \] \,(3.4)

Let us consider the HE system \((3.2)\) as an abstract system of ordinary differential equations with arbitrary initial conditions. Obviously, not all solutions of the HE system \((3.2)\) can be obtained by averaging the corresponding operators over the solutions of the Gross–Pitaevsky equation \((1.1)–(1.3)\). For instance, the mean values should satisfy the Schrödinger–Robertson uncertainty relation \([19]\), which, for the second order moments, implies nonnegative determinacy of the matrix

\[ \Delta_2(t) = \frac{i\hbar}{2} J. \] \,(3.5)

In the one-dimensional case, condition \((3.5)\) is equivalent to the well-known Schrödinger uncertainty relation \([20]\)

\[ \sigma_{pp}\sigma_{xx} - \sigma_2^2 \geq \frac{\hbar^2}{4}. \] \,(3.6)

Denote by \(g(t,\mathcal{C})\) the general solution of the system of Hamilton–Ehrenfest equations \((3.2)\):

\[ g(t,\mathcal{C}) = (\vec{P}(t,\hbar,\mathcal{C}), \vec{X}(t,\hbar,\mathcal{C}), \Delta_{11}(t,\hbar,\mathcal{C}), \Delta_{12}(t,\hbar,\mathcal{C}), \ldots, \Delta_{2n2n}(t,\hbar,\mathcal{C}))^T \] \,(3.7)

and by \(\hat{g}\) the operator column

\[ \hat{g} = (\hat{\vec{P}}, \hat{\vec{X}}, (\Delta_\hat{P}_1)^2, \Delta_\hat{P}_1 \Delta_\hat{P}_2, \ldots, (\Delta_\hat{x}_n)^2)^T. \] \,(3.8)

Here,

\[ \mathcal{C} = (C_1, \ldots, C_N)^T \in \mathbb{R}^{3n+2n^2} \] \,(3.9)

are arbitrary constants, and \(B^T\) denotes the transpose to the matrix \(B\).

These constants \(\mathcal{C}\), Eq. \((3.7)\) specify the trajectory of a point in the phase space \(\mathcal{M}^N\).

**Lemma 1** Let \(\Psi(\vec{x},t)\) be a particular solution of equation \((2.1)\) with the initial condition \(\Psi(\vec{x},t)|_{t=0} = \psi(\vec{x})\). Determine the constants \(\mathcal{C}(\Psi(t))\) from the system

\[ g(t,\mathcal{C}) = \frac{1}{||\Psi||^2}(\langle \Psi(t)|\hat{g}|\Psi(t)\rangle) + O(\hbar^{3/2}), \] \,(3.10)

and the constants \(\mathcal{C}(\psi)\) from the system

\[ g(0,\mathcal{C}) = \frac{1}{||\psi||^2}(\langle \psi|\hat{g}|\psi\rangle) + O(\hbar^{3/2}). \] \,(3.11)

Then

\[ \mathcal{C}(\Psi(t)) = \mathcal{C}(\psi) + O(\hbar^{3/2}); \] \,(3.12)

that is, \(\mathcal{C}(\Psi(t))\) are asymptotic, to within \(O(\hbar^{3/2})\), integrals of motion for equation \((2.1)\).
**Proof.** By construction, the vector

$$g(t) = \frac{1}{\|\Psi\|^2} \langle \Psi(t) | \hat{g} | \Psi(t) \rangle = g(t, \mathcal{C}(\Psi(t)))$$  \hspace{1cm} (3.13)

is a particular solution (to within $O(h^{3/2})$) of system (3.2), which coincides with $g(t, \mathcal{C}(\psi))$ at $t = 0$. By virtue of the uniqueness of the Cauchy problem for system (3.2), the relation

$$g(t, \mathcal{C}(\psi)) = g(t, \mathcal{C}(\Psi(t))) + O(h^{3/2})$$  \hspace{1cm} (3.14)

holds true.

By virtue of the estimates (2.2), the Hamilton–Ehrenfest system can be solved not exactly, but approximately:

$$Z(t) = Z_0(t) + \hbar Z_1(t) + O(\hbar^2),$$

$$\Delta_2(t) = \Delta_2(0) + O(\hbar^2).$$  \hspace{1cm} (3.15)

Substituting (3.15) to system (3.2), we obtain, to within $O(h^{3/2})$, a system of equation for $z_0 = Z_0$, $z_1 = Z_1$, $\Delta_2 = \Delta_2(0)(t)$.

The first equation of (3.16) is a generalization of the system of Hamilton equation to the case of a self-action.

If a solution $z_0 = Z_0(t)$ of this system is known and also known is the set of solutions of two systems in variations (for this to be the case, it is sufficient that the matrix $V_{zw}(z, z, t)$ be symmetric)

$$\dot{a}_k = J\delta_{zz}(t) a_k, \quad k = 1, n, \quad \delta_{zz}(t) = \delta_{zz}(Z_0(t), t)$$  \hspace{1cm} (3.17)

normalized by the condition

$$\{a_k(t), a_l(t)\} = \{a^*_k(t), a^*_l(t)\} = 0, \quad \{a_k(t), a^*_l(t)\} = 2i\delta_{kl},$$  \hspace{1cm} (3.18)

where $\{a, b\}$ is a skew-scalar product:

$$\{a, b\} = \langle Ja, b \rangle = \langle \tilde{W}_a, \tilde{Z}_b \rangle - \langle \tilde{Z}_a, \tilde{W}_b \rangle,$$  \hspace{1cm} (3.19)

$$a = \begin{pmatrix} \tilde{W}_a \\ \tilde{Z}_a \end{pmatrix}, \quad b = \begin{pmatrix} \tilde{W}_b \\ \tilde{Z}_b \end{pmatrix},$$

and of the system

$$\dot{a}_k = J\tilde{\delta}_{zz}(t) a_k, \quad k = 1, n, \quad \tilde{\delta}_{zz}(t) = \delta_{zz}(Z_0(t), t) + \tilde{\delta}V_{zw}(Z_0(t), Z_0(t), t),$$  \hspace{1cm} (3.20)

also normalized by (3.18). Then the general solution of the two last equations of (3.16) has the form

$$Z_1(t) = \sum_{k=1}^{n} [b_k(t)a_k(t) + b^*_k(t)a^*_k(t)],$$  \hspace{1cm} (3.21)

$$\Delta_2(t) = A(t) \mathcal{D} \mathcal{A}^T(t),$$  \hspace{1cm} (3.22)

where

$$b_k(t) = \frac{1}{2\hbar} \int_0^t \{ F(t), a^*_k(t) \} dt + \mathcal{B}_k,$$  \hspace{1cm} (3.23)

$$F(t) = \frac{1}{2\hbar} J\partial_z \mathrm{Sp} \left\{ [H_{zz}(z, t) + \tilde{\delta}V_{zz}(z, w; t) + \tilde{\delta}V_{ww}(z, w; t)] \Delta_2 \right\} \bigg|_{w = z = Z_0(t)},$$

$$A(t) = (a_1(t), a_2(t), \ldots, a_n(t), a^*_1(t), a^*_2(t), \ldots, a^*_n(t)).$$

Here, $\mathcal{B}_k$ are integration constants and $\mathcal{D}$ is an arbitrary constant matrix. Thus, in this approximation, the solution of the problem is completely determined by the solution of the generalized Hamilton system and the system in variations.
The quasi-periodicity condition \(1.1\) leads to the following constraint on the solutions of the Hamilton–Ehrenfest system:

\[
Z_0(t + T) = Z_0(t), \quad Z_1(t + T) = Z_1(t), \quad \Delta_2(t + T) = \Delta_2(t)
\] \tag{3.24}

We shall denote by \(C_T\) the values of the constants \(C\) at which condition \(3.23\) is fulfilled.

In this case, the systems in variations \(3.17, 3.20\) are systems of ordinary differential equations with periodic coefficients

\[
\mathcal{H}_{zz}(t + T) = \mathcal{H}_{zz}(t), \quad \mathcal{J}_{zz}(t + T) = \mathcal{J}_{zz}(t).
\]

State for the systems in variations \(3.17, 3.20\) the Floquet problems:

\[
a_k(t + T) = e^{i\Omega k} a_k(t), \quad a_k(t + T) = e^{i\Omega k} a_k(t).
\] \tag{3.25}

Suppose that each Floquet problem of \(3.17, 3.20, 3.25\) has \(n, n = \dim \mathbb{R}_n^D\), linearly independent Floquet solutions \(a_k(t) = (\hat{W}_k(t), \hat{Z}_k(t))', \hat{a}_k(t) = (\hat{W}_k(t), \hat{Z}_k(t))', \) with pure imaginary Floquet indices \(i\Omega_n, i\Omega_n\), which satisfy the orthogonality and normalization condition \(3.18\).

It should be stressed that in the context of the Floquet theory for linear Hamilton systems with periodic coefficients, the conditions \(\text{Im} \Omega_k = 0\) imply stability of the phase trajectory \(z = Z_0(t)\) in the linear approximation. Recall that the \(2n\)-dimensional vectors \(a_k(t)\) and \(\hat{a}_k(t)\), \(k = 1, n\), constitute a simplectic germ on \(Z(t, h)\) \([21, 22]\).

Determine the integration constants \(B_k\) from the condition of periodicity of the function \(hZ_1(t)\) in time:

\[
Z_1(t + T) = Z_1(t).
\] \tag{3.26}

Like for \(3.21\), we put

\[
b_k(t) = \frac{1}{2i} \int_0^t d\tau \{F(\tau), \hat{a}_k(\tau)\} + B_k.
\] \tag{3.27}

Then conditions \(3.24, 3.26\) imply that

\[
b_k(t + T) = b_k(t) e^{-i\Omega k T}.
\] \tag{3.28}

Note that

\[
b_k(t + T) = \frac{1}{2i} \int_0^{t+T} d\tau \{F(\tau), \hat{a}_k(\tau)\} + B_k = b_k(T) + \frac{1}{2i} \int_T^{t+T} d\tau \{F(\tau), \hat{a}_k(\tau)\} =
\]

\[
= b_k(T) + \frac{1}{2i} \int_0^t d\tau \{F(\tau + T), \hat{a}_k(\tau + T)\} = b_k(T) + e^{-i\Omega k T}(b_k(t) - B_k).
\]

Hence,

\[
B_k = b_k(T) e^{i\Omega k T}.
\] \tag{3.29}

4 The linear associated Schrödinger equation

The linearization of the nonlocal Gross–Pitaevsky equation in the class of trajectory-concentrated functions is the central point in our approach.

Introduce the notation (see \(2.2\))

\[
\mathfrak{g}_\Psi(t) = \frac{1}{||\Psi||^2} \langle \Psi(t)|\hat{g}|\Psi(t)\rangle
\] \tag{4.1}

and expand the “kernel” of the operator \(\hat{V}(t, \Psi)\) in a Taylor series in powers of the operators \(\Delta \hat{w} = \hat{w} - z\Psi(t, h)\). Substituting this expansion in equations \(1.11-1.13\), for the functions \(\Psi \in \mathcal{P}_h^1\), in view of the estimate \(2.2\), we obtain

\[
\dot{L}^{(2)}(t, \Psi)\Psi = \left\{-i\hbar \partial_t + \mathcal{J}(\hat{z}, t) + \hat{z}V(\hat{z}, w, t) + \frac{\hat{z}}{2} \text{Sp}[V_{ww}(\hat{z}, w, t)\Delta z\Psi(t, h)]\right\}
\]

\[
|_{w = z\Psi(t, h)} \Psi = \hat{O}(\hbar^{3/2}).\] \tag{4.2}
Let us associate the nonlinear equation (4.4) with the linear equation that is obtained from (4.2) by substituting the corresponding solutions of the HE system \( g(t, C_T) \) for \( g_\Phi(t) \), the mean values of the operators of coordinates, momenta, and second order centered moments. As a result, we obtain the following equation:

\[
\tilde{L}^{(2)}(t, C_T)\Phi = 0, \quad \Phi \in P^1_h, \tag{4.3}
\]

\[
\tilde{L}^{(2)}(t, C_T) = -i\hbar \partial_t + \mathcal{H}(\hat{z}, t) + \bar{z}V(\hat{z}, w, t) \bigg|_{w=Z(t, \xi_T)} + \frac{\bar{z}}{2} \text{Sp} \left[ V_{ww}(\hat{z}, w, t) \bigg|_{w=Z(t, \xi_T)} \Delta_2(t, C_T) \right]. \tag{4.4}
\]

By virtue of (3.24), equations (4.3) are equations with periodic coefficients. The quasi-periodicity condition (1.0) reduces, for equation (4.4), the Floquet problem

\[
\Phi_{E}(\hat{x}, t + T, h) = e^{-iE T/\hbar}\Phi_{E}(\hat{x}, t, h). \tag{4.5}
\]

Thus, the change of the quantum means of the operators \( \hat{w} \) and \( \{\Delta \hat{w}\}^\alpha \) by the solutions of the second order Hamilton–Ehrenfest system in equation (4.2) linearizes the nonlocal Gross–Pitaevsky equation (1.1)–(1.3) to within \( O(\hbar^{3/2}) \). Hence, to construct the semiclassically concentrated states mod \( \hbar^{3/2} \) of the nonlocal Gross–Pitaevsky equation (1.1)–(1.3), it suffices to construct, with the same accuracy, an asymptotics of the solution of a Schrödinger type linear equation.

**Definition 2** We call equations of the form (4.3) parametrized by constants \( C \) (2.12) the set of associated linear Schrödinger equations for the nonlocal Gross–Pitaevsky equation (1.1)–(1.3), and denote by \( \Phi = \Phi(\tilde{x}, t, h, C) \) its (arbitrary) solution in the class of functions \( P^1_h \).

The following statement is obvious.

**Statement 1** If a function \( \Phi^{(2)}(\tilde{x}, t, h, C) \in P^1_h \) — an asymptotic (to within \( O(\hbar^{3/2}) \), \( \hbar \to 0 \)) solution of equation (4.3) — satisfies the initial condition (4.5):

\[
\Phi^{(2)}(\tilde{x}, t, h, C(\psi_0))|_{t=0} = \psi_0, \tag{4.6}
\]

then the function \( \Psi^{(2)}(\tilde{x}, t, h) = \Phi^{(2)}(\tilde{x}, t, h, C(\psi_0)) \) is an asymptotic (to within \( O(\hbar^{3/2}) \), \( \hbar \to 0 \)) solution of the Cauchy problem for the nonlocal Gross–Pitaevsky equation (1.1). Constants \( C(\psi_0) \) are determined by the equation (3.11).

To construct the function \( \Phi^{(2)}(\tilde{x}, t, h, \psi_0) \in P^1_h \) (1.1) of the semiclassically concentrated solution of equation (4.3) — “parameter” \( Z(t, h) \) entering into the definition of the class \( P^1_h \) — as a natural projection of the solution \( \psi^{(2)}(t, h) \) of the Hamilton–Ehrenfest system (3.2) onto the phase space, i.e. we set \( Z(t, h) = Z^2(t, h) = (\hat{P}^2(t, h), \hat{X}^2(t, h)) \). Define the function \( S(t, h) \) (second “parameter” of the class \( P^1_h \)) as an analog of the classical action along this (\( z = Z^2(t, h), t \in [0, T] \)) phase trajectory by a standard formula in a classical Hamiltonian corresponding not to the main, but to the total symbol \( 16 \mathcal{H}_\alpha^2(z, t) \) of the quantum Hamiltonian in (4.4), which, in view of the estimates \( \Delta_2^\alpha(t, h) = O(\hbar^{\alpha/2}) \), has the form

\[
\mathcal{H}_\alpha^2(z, t) = \mathcal{H}(z, t) + \bar{z}V(z, w, t) \bigg|_{w=Z(t, \xi_T)} + \frac{\bar{z}}{2} \text{Sp} \left[ V_{ww}(z, w, t) \bigg|_{w=Z(t, \xi_T)} \Delta_2(t, \xi_T) \right]. \tag{4.7}
\]

As a result, we have

\[
S(t, h) = S^{(2)}(t, h) = \int_0^t \left\{ \langle \hat{P}^{(2)}(\tau, h), \hat{X}^{(2)}(\tau, h) \rangle - \mathcal{H}_\alpha^{(2)}(Z^{(2)}(\tau, h), \tau) \right\} d\tau. \tag{4.8}
\]

Introduce the notation

\[
S^{(2)}(\tilde{x}, t, h) = S^{(2)}(t, h) + \langle \hat{P}^{(2)}(t, h), \tilde{x} - \hat{X}^{(2)}(t, h) \rangle.
\]

Expand the operators \( \mathcal{H}(\hat{z}, t) \) and \( \frac{\partial^{\alpha}|}{\partial w^\alpha} V(\hat{z}, w, t) \bigg|_{w=Z^{(2)}(t, h)} \) in Taylor series of order two in powers of the operator \( \Delta \hat{z} = \hat{z} - Z^{(2)}(t, h) \) with remainder terms \( \hat{R}^\alpha_3 \) and \( \hat{R}^V_3 \), respectively, and represent the operator \( -i\hbar \partial_\tau \partial t \) in the form

\[
-i\hbar \partial_\tau = \hat{A} + \hat{B},
\]

thereby stressing the dependence of the solution of equation (4.3) (via the coefficients of the operator \( \tilde{L}^{(2)}(t, C) \)) on the solution \( g(t, h, C) \) of the Hamilton–Ehrenfest equation (3.2).
\[ \hat{B} = -\langle \hat{P}^{(2)}(t, \hbar), \hat{X}^{(2)}(t, \hbar) \rangle + \hat{S}^{(2)}(t, \hbar) - \langle \hat{Z}^{(2)}(t, \hbar), J \Delta \hat{z} \rangle, \]  
\[ \hat{A} = -i\hbar \partial_t - \hat{S}^{(2)}(t, \hbar) + \langle \hat{P}^{(2)}(t, \hbar), \hat{X}^{(2)}(t, \hbar) \rangle + \langle \hat{Z}^{(2)}(t, \hbar), J \Delta \hat{z} \rangle = \hat{O}(\hbar). \]  

Substituting the resulting expressions in equation (1.3), in view of the estimates in \( \hbar \to 0 \), for the operators \( \hat{R}^{\psi}_{\alpha} \) and \( \hat{R}^{\psi}_{\beta} \) applied to functions of the class \( \mathcal{P}^\psi \), we obtain

\[ \{-i\hbar \partial_t + \hat{S}_0(t, \mathcal{C}_T)\} \Phi = O(\hbar^{3/2}), \]  
where

\[ \hat{S}_0(t, \mathcal{C}_T) = -\hat{S}^{(2)}(t, \hbar) + \langle \hat{P}^{(2)}(t, \hbar), \hat{X}^{(2)}(t, \hbar) \rangle + \langle \hat{Z}^{(2)}(t, \hbar), J \Delta \hat{z} \rangle + \frac{1}{2} \langle \Delta \hat{z}, \hat{S}_{zz}(t, \mathcal{C}_T) \Delta \hat{z} \rangle, \]

(4.11)

\[ \hat{S}_{zz}(t, \mathcal{C}_T) = [\mathcal{H}_{zz}(z, t) + \mathcal{Z}_{zz}(z, w, t)] \bigg|_{z=w=Z^{(2)}(t, \hbar)} \]  
(4.12)

This is a Schrödinger equation with the Hamiltonian quadratic in the operators \( \hat{p} \) and \( \hat{x} \).

5 Trajectory-coherent states of the nonlocal Gross–Pitaevsky equation

The solution of the linear Schrödinger equation with a quadratic Hamiltonian is well known. For our purposes, it is convenient to take for the basis of solutions of equation (4.10) the semiclassical trajectory-coherent states (TCS’s) of this equation [23, 24]. By Statement 1 these states are asymptotic (mod \( \hbar^{3/2} \)) solutions of the problem (2.14)–(2.12) if the function \( \psi_0(x, \hbar) \) coincides with the TCS at the time zero. We shall also call these solutions trajectory-coherent states of the nonlocal Gross–Pitaevsky equation (1.1)–(1.3). We now give their explicit form and some properties, which will be used below to solve the problem (2.14)–(2.12) with an arbitrary initial condition of the class of functions \( \mathcal{P}^\psi \).

Let us use the “momentum” and “coordinate” components of the solution of the Floquet problem (3.17), (3.25) to compose \( n \times n \) matrices

\[ a(t, \mathcal{C}_T) = (\hat{W}(t, \mathcal{C}_T)), \hat{Z}(t, \mathcal{C}_T)) \]  

(5.1)

The linear Hamilton system (3.17) is known [25] to conserve the standard symplectic structure \( i\hbar \partial_t + \mathcal{S} \) of the nonlocal Gross–Pitaevsky equation (2.1)–(2.2). We shall call these solutions trajectory-coherent states of the nonlocal Gross–Pitaevsky equation (1.1)–(1.3). We now give their explicit form and some properties, which will be used below to solve the problem (2.14)–(2.12) with an arbitrary initial condition of the class of functions \( \mathcal{P}^\psi \).

Let us use the “momentum” and “coordinate” components of the solution of the Floquet problem (3.17), (3.25) to compose \( n \times n \) matrices

\[ a(t, \mathcal{C}_T) = (\hat{W}(t, \mathcal{C}_T)), \hat{Z}(t, \mathcal{C}_T)) \]  

(5.1)

The linear Hamilton system (3.17) is known [25] to conserve the standard symplectic structure \( i\hbar \partial_t + \mathcal{S} \) of the phase space \( \mathbb{R}^3_{p,x} \) and, hence, the skew-scalor product \( \langle J a, b \rangle = \omega(a, b) \) of any two complex solutions \( a(t), b(t) \in \mathbb{C}_w \) of system (3.17) does not depend on time. (Here, \( \mathbb{C}_w \) denotes the complexification \( \mathbb{R}^3_{w} \) with complex coordinates \( w \in \mathbb{C}^n \).) From this reasoning, in view of (5.18), we obtain

\[ C^\dagger(t) B(t) - B^\dagger C(t) = \frac{1}{2} \| (a_i(t), a^*_j(t)) \|_{n \times n} = 0, \]  
(5.2)

\[ \frac{1}{2i} (C^* B(t) - B^* C(t)) = \frac{1}{2i} \| (a_i(t), a^*_j(t)) \|_{n \times n} = 0, \]  
(5.3)

Conventional reasoning (see, e.g., [15]), in view of (5.3), lead to the statement that the matrix \( C(t) \) is nondegenerate, \( \det C(t) \neq 0 \), and the imaginary part of the matrix

\[ Q(t) = B(t) C^{-1}(t) \]  
(5.4)

is positive for \( t \in [0, T] \), \( T > 0 \). Moreover, by virtue of (5.2), the matrix \( Q(t) \) is symmetric, \( Q^\dagger(t) = Q(t) \).

Let us now fix the continuous branch of the root of \( \det C(t), t \in [0, T] \), assuming, for instance, that \( \text{Arg} \sqrt{\det C(0)} = 0 \), and determine the function

\[ \Phi_0^{(2)}(\vec{x}, \hbar) = |0, t, \mathcal{C}_T) = \frac{N_0(\hbar)}{\sqrt{\det C(t)}} \exp \left\{ \frac{i}{\hbar} \left[ S^{(2)}(t, \hbar) + \langle \hat{P}^{(2)}(t, \hbar), \Delta \hat{x} \rangle + \frac{1}{2} \langle \Delta \hat{x}, Q(t) \Delta \hat{x} \rangle \right] \right\}, \]  
(5.5)

where \( \Delta \hat{x} = \vec{x} - \bar{X}^{(2)}(t, \hbar), (\hat{P}^{(2)}(t, \hbar), \bar{X}^{(2)}(t, \hbar)) = Z^{(2)}(t, \hbar) \), \( t \in [0, T] \) is the phase trajectory by virtue of the second order Hamilton–Ehrenfest system (4.17); \( S^{(2)}(t, \hbar) \) is defined in (4.17), and \( N_0(\hbar) = (\pi \hbar)^{-n/4} \) is a normalizing constant: \( \langle \hbar, 0 | 0, \hbar \rangle = 1 \).

Theorem 3 The function \( \Phi_0^{(2)}(\vec{x}, \hbar) \) is an (exact) solution of the Cauchy problem for equation (4.10) with an initial condition of the form

\[ \Phi_0^{(2)}(\vec{x}, \hbar)|_{t=0} = N_0(\hbar) e^{-\frac{i}{\hbar} \langle \vec{p}_0, \vec{x} - \vec{x}_0 \rangle + \frac{1}{2} \langle \vec{x} - \vec{x}_0, B(0) C^{-1}(0)(\vec{x} - \vec{x}_0) \rangle}, \]  
(5.6)

where \( \vec{p}_0, \vec{x}_0 \in \mathbb{R}^3_n \) : the \( (n \times n) \) complex matrices \( B(t) \) and \( C(t) \) are defined in (5.1).
Proof. We seek a solution of the linear equation (4.11) in the class of functions \( P_h^*(S^{(2)}(t, \hbar), Z^{(2)}(t, \hbar)) \) in the form of a Gaussian packet

\[
\Phi(\vec{x}, t, \hbar) = \exp \left\{ \frac{i}{\hbar} [S^{(2)}(t, \hbar) + (\bar{B}(2)(t, \hbar), \Delta \vec{x})] \right\} \exp \left\{ \frac{i}{\hbar} \langle \Delta \vec{x}, Q(t) \Delta \vec{x} \rangle \right\} \varphi(t), \tag{5.7}
\]

where the complex \((n \times n)\) matrix \(Q(t)\) is symmetric and \(\text{Im} \ Q(t) > 0\). Substituting (5.7) in (4.10) and equating the coefficients of the powers of the operator \(\Delta \vec{x}^k, \ k = 0, 2,\) to zero, we obtain a linear equation in the function \(\varphi(t)\) and a (matrix) Riccati equation in the matrix \(Q(t)\), respectively:

\[
\dot{\varphi} + \frac{1}{2} \text{Sp}[\delta_{px}(t, \mathcal{C}_T)] + \delta_{pp}(t, \mathcal{C}_T)Q(t)]\varphi = 0, \tag{5.8}
\]

\[
\dot{\mathcal{C}} + \delta_{zz}(t, \mathcal{C}_T) + Q(t)\delta_{px}(t, \mathcal{C}_T) + \delta_{xp}(t, \mathcal{C}_T)Q(t) + Q(t)\delta_{pp}(t, \mathcal{C}_T)Q(t) = 0. \tag{5.9}
\]

The ordinary change of variables \(Q(t) = B(t)C^{-1}(t)\) (see, e.g., [15]), provided that \(Q(0) = B(0)C^{-1}(0)\), reduces the problem of constructing the desired complex solution of the equation (5.9) to the problem

\[
\left( \begin{array}{c} \dot{B} \\ \dot{C} \end{array} \right) = J\delta_{zz}(t, \mathcal{C}_T) \left( \begin{array}{c} B \\ C \end{array} \right) \quad \iff \quad \dot{B} = -\delta_{xp}(t, \mathcal{C}_T)B - \delta_{xx}(t, \mathcal{C}_T)C, \tag{5.10}
\]

\[
\dot{C} = \delta_{pp}(t, \mathcal{C}_T)B + \delta_{px}(t, \mathcal{C}_T)C,
\]

which, by virtue of (5.1), is reduced to system (4.17). By virtue of the second equation of (5.10), we have

\[
\dot{C} = [\delta_{pz}(t, \mathcal{C}_T) + \delta_{pp}(t, \mathcal{C}_T)] Q(t)]C, \tag{5.11}
\]

where \(Q(t)\) is a solution of (5.9). Thus, by virtue of Liouville’s lemma, we find

\[
\det C(t) = \exp \int_0^t \text{Sp}[\delta_{px}(\tau, \mathcal{C}_T)] + \delta_{pp}(\tau, \mathcal{C}_T)] Q(\tau) d\tau,
\]

and, hence, by equation (5.8), we have \(\varphi(t) = (\det C(t))^{-1/2}\).

Let us now construct the Fock basis of solutions for equation (4.10). To do this, we first find for this equation the symmetry operators \(\tilde{a}(t, \mathcal{C}_T)\), linear in operators \(\Delta \vec{z}\), in the form \(\tilde{a}(t, \mathcal{C}_T) = N_a(b(t, \mathcal{C}_T), \Delta \vec{z})\), where \(N_a\) is a constant, \(b(t) = b(t, \mathcal{C}_T)\) is the complex 2n vector to be determined. By the equation

\[
-\hbar \frac{\partial \tilde{a}(t)}{\partial t} + [\tilde{S}_0(t, \mathcal{C}_T)], \tilde{a}(t)] = 0,
\]

which determines the operator \(\tilde{a}(t)\), in view of the explicit form of \(\tilde{S}_0(t, \mathcal{C}_T)\), we obtain

\[
-\hbar \{\tilde{b}(t), \Delta \vec{z}\} + \hbar \{\tilde{b}(t), \Delta \vec{z}\} + \left\{ -\tilde{S}^{(2)}(t, \hbar) + (\bar{B}^{(2)}(t, \hbar), \bar{X}^{(2)}(t, \hbar)) + \frac{1}{2} (\Delta \vec{z}, \delta_{zz}(t, \mathcal{C}_T)\Delta \vec{z}) \right\} = 0.
\]

From this equation, by virtue of the commutation relations \(\{\Delta \vec{z}_j, \Delta \vec{z}_k\} = i \hbar J_{jk}, \ j, k = \overline{1, 2n}\), we find

\[
-\hbar \{\tilde{b}(t), \Delta \vec{z}\} + \hbar \{\Delta \vec{z}, \delta_{zz}(t, \mathcal{C}_T)b(t)\} = 0,
\]

and, hence, \(\tilde{b}(t) = \delta_{zz}(t, \mathcal{C}_T)b(t)\). Using the notation \(b(t) = -Ja(t)\), we obtain for the vector \(a(t)\) the system in variations (3.17). Thus, the operator

\[
\tilde{a}(t) = \tilde{a}(t, \mathcal{C}_T) = N_a(b(t), \Delta \vec{z}) = N_a(a(t), J\Delta \vec{z}) \tag{5.11}
\]

is the symmetry operator for equation (4.10) if the vector \(a(t) = a(t, \mathcal{C}_T)\) is a solution of the system in variations (3.17). Let \(\tilde{a}(t)\) and \(\tilde{b}(t)\) be symmetry operators corresponding to two solutions of the system in variations, \(a(t)\) and \(b(t)\), respectively. It is easy to check that

\[
[\tilde{a}(t), \tilde{b}(t)] = i \hbar N_aN_b\{a(t), b(t)\} = i \hbar N_aN_b\{a(0), b(0)\}, \tag{5.12}
\]

the last equation being a corollary of the Hamiltonian character of system (3.17).

By formula (5.11), associate the vectors \(a_j^+(t)\) with the “creation” operators \(\tilde{a}_j^+(t)\) and the vectors \(a_j(t)\) with the “annihilation” operators \(\tilde{a}_j(t)\), setting \(N_a = (2\hbar)^{-1/2}\). Then, by virtue of formulas (5.12), the operators \(\tilde{a}_j^+(t), \tilde{a}_j(t), \ j = \overline{1, n}\), the following commutation relations, canonical for boson operators, are valid:

\[
[\tilde{a}_j(t), \tilde{a}_k(t)] = [\tilde{a}_j^+(t), \tilde{a}_k^+(t)] = 0, \quad [\tilde{a}_j(t), \tilde{a}_k^+(t)] = \delta_{jk}, \quad j, k = \overline{1, n}. \tag{5.13}
\]
Statement 2 The function $\Phi_0^{(2)}(x, t, h) = |0, t, \mathcal{E}_T\rangle$ is a “vacuum” trajectory-coherent state:

$$\hat{a}_j(t)|0, t, \mathcal{E}_T\rangle = 0, \quad j = 1, n. \quad (5.14)$$

Proof. Applying the annihilation operator $\hat{a}_j(t)$ to the function $|0, t\rangle$, we obtain that $\hat{a}_j(0, t)|0, t\rangle = |0, t\rangle[\hat{\mathcal{Z}}_j(t), \hat{Q}(t)\hat{\Delta}x] - (\hat{W}_j(t), \hat{\Delta}x)]$. From this (5.14) immediately follows since, by definition and in view of the properties of the matrix $Q(t)$, we have $Q(t)\hat{Z}_j(t) = B(t)C^{-1}(t)\hat{Z}_j(t) = \hat{W}_j(t)$.

Let us now define a countable set of states $|\nu, t, \mathcal{E}_T\rangle$ (exact solutions of equation (4.10)) as the result of the action of the birth operators on the vacuum state $|0, t, \mathcal{E}_T\rangle$ (5.5):

$$\Phi_\nu^{(2)}(\vec{x}, t, h) = |\nu, t, \mathcal{E}_T\rangle = \frac{1}{\nu!}(\hat{a}_+^\dagger(t, \mathcal{E}_T)\nu)|0, t, \mathcal{E}_T\rangle = \prod_{k=1}^{n} \frac{1}{\nu_k!}(\hat{a}_k^\dagger(t, \mathcal{E}_T))^{\nu_k}|0, t, \mathcal{E}_T\rangle. \quad (5.15)$$

The functions $\Phi^{(2)}_\nu(\vec{x}, t, h)$, $\nu \in \mathbb{Z}_n^+$ constitute the Fock basis of solutions of the (linear) equation (4.10) by formulas (5.14), (5.13), standard calculations check whether this set of functions is orthonormalized, $(\Phi^{(2)}_\nu|\Phi^{(2)}_{\nu'}) = \delta_{\nu\nu'}$, $\nu, \nu' \in \mathbb{Z}_n^+$, and the proof of its completeness follows, for example, from the results presented in [26]. Thus, from this reasoning and Statement 1 we arrive at the following theorem.

Theorem 4 Let the symbols of the operators $\hat{\mathcal{H}}(t)$ and $\hat{\mathcal{V}}(t, \Psi)$ in (1.1)–(1.3) satisfy the conditions of Assumption 1 and the conditions of Theorem 1 be fulfilled. Then, for any $\nu \in \mathbb{Z}_n^+$, the function

$$\Psi_\nu(\vec{x}, h) = \Phi_\nu(\vec{x}, h, \mathcal{E}_T) \quad (5.16)$$

is an asymptotic (to within $O(h^{3/2})$, $h \to 0$) solution of the nonlocal Gross–Pitaevskiy equation (1.1)–(1.3) with a quasiperiodicity condition (1.6), where

$$\mathcal{E}_\nu^\omega(\mod \omega) = -\frac{1}{T}S(T, h) + h \sum_{k=1}^{n} \Omega_k \left(\nu_k + \frac{1}{2}\right). \quad (5.17)$$

The constants $\mathcal{E}_\nu^\sigma$ are determined by the equation

$$g(t, \mathcal{E}_\nu^\sigma)_{t=s} = g_0(\psi_0) + O(h^{3/2}) = \frac{1}{\|\psi_0\|^2} \langle \psi_0 | g | \psi_0 \rangle + O(h^{3/2}), \quad (5.18)$$

where

$$\psi_0(\vec{x}, h) = \Phi^{(2)}_\nu(\vec{x}, 0, h), \quad (5.19)$$

and $\Phi^{(2)}_\nu(\vec{x}, t, h)$ are defined in (5.15).

By the periodicity condition (5.24) for the solutions of the Hamilton–Ehrenfest system and quasiperiodicity condition (5.25) for the solutions of the system in variations it follows that

$$\hat{a}_k(t + T) = e^{i\Omega_k T} \hat{a}_k(t), \quad \hat{a}(t) = \frac{1}{\sqrt{2h}}\langle a(t), J \hat{\Delta}z \rangle \quad k = 1, n, \quad (5.20)$$

$$\left[\hat{a}_+^\dagger(t + T)\right]^{\nu} = \exp \left[ -i \sum_{k=1}^{n} T \Omega_k \nu_k \right] \left[\hat{a}_+^\dagger(t)\right]^{\nu}, \quad (5.21)$$

$$\det C(t + T) = \exp \left[ i \sum_{k=1}^{n} T \Omega_k \nu_k \right] \det C(t), \quad Q(t + T) = Q(t). \quad (5.22)$$

From the relations

$$\left|0, t\right| = \left|0, t, \mathcal{E}_T\right\rangle = \frac{1}{\sqrt{\det C(t)\sqrt{(nh)^n}}} \times$$

$$\times \exp \left\{ -\frac{i}{h} \left[ S(t, h) + \langle \hat{\mathcal{P}}(t, h), \Delta \vec{x} \rangle + \frac{1}{2} \langle \Delta \vec{x}, Q(t)\Delta \vec{x} \rangle \right] \right\}, \quad (5.23)$$

$$S(t, h) = \int_0^t dt \left\{ \langle \hat{\mathcal{P}}(t, h), \dot{\hat{X}}(t, h) \rangle - \mathcal{H}(z, t) - \hat{\mathcal{Z}}_{ww}(z, w, t) - \frac{\hat{\mathcal{Z}}}{2} \left[ V_{ww}(z, w, t) \Delta_2(t, \mathcal{E}_T) \right] \right\}_{z=w=Z(t, \mathcal{E}_T)}. \quad (5.24)$$
which determine the vacuum trajectory-coherent state, the definition of semiclassical TCS’s \[5\]

\[
\Phi_{\nu}(\vec{x}, t, h, \mathcal{C}_T) = |\nu, t\rangle = |\nu, t, \mathcal{C}_T\rangle = \frac{1}{\sqrt{\nu!}}(\hat{\Phi}(t, \mathcal{C}_T))^{\nu}|0, t, \mathcal{C}_T\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu_k!}}(\hat{\alpha}_k^+(t, \mathcal{C}_T))^{\nu_k}|0, t, \mathcal{C}_T\rangle.
\]

(5.25)

and the quasi-periodicity conditions \[5.21\] it follows that

\[
|\nu, t + T\rangle = e^{-i\mathcal{E}_\nu^{(2)} T/\hbar}|\nu, t\rangle,
\]

(5.26)

where \(\mathcal{E}_\nu^{(2)}\) is defined in \[5.17\]. Hence, the functions \(\Psi_{\nu}(\vec{x}, t, h)\) \[5.10\] also satisfy the quasi-periodicity condition \[5.26\] and constitute the quasi-energy spectral series \(\{|\Psi_{\nu}(\vec{x}, t, h), \mathcal{E}_\nu^{(2)}\}\) of equation \[1.1\] which correspond to the phase curve \(z = Z_0(t)\).

### 6 Semiclassically concentrated solutions of the nonlocal Gross–Pitaevsky equation (the principal term of the asymptotics)

The asymptotic solutions of the Cauchy problem \[2.14\]–\[2.12\] (where \(\psi_0\) has the form of \[6.19\]) that have been constructed in the previous section are a special case of the semiclassically concentrated (\(mod\ \hbar^{3/2}\)) solutions of equation \[1.1\]–\[1.3\]. In case of arbitrary initial conditions \(\psi_0(\vec{x}, h)\) \[2.12\] belonging to the class \(\mathcal{P}_h^4(Z^{(2)}(t, h), S^{(2)}(t, h))\), \(t = s\), the principal term of the asymptotics \(\Psi_0^{(2)}(\vec{x}, t, h)\) of the nonlocal Gross–Pitaevsky equation \[1.1\]–\[1.3\] is determined by expanding the solutions of equation \[4.10\] in a series over the Fock basis \(|\nu, t, \mathcal{C}_T\rangle\). From the phase curve \(z = Z_0(t)\) and the quasi-periodicity conditions \[5.21\] it follows that

\[
|\nu, t + T\rangle = e^{-i\mathcal{E}_\nu^{(2)} T/\hbar}|\nu, t\rangle,
\]

(6.3)

where the operator \(\mathcal{H}_0\) is defined in \[6.11\]. Denote by \(\lambda_k(t, \mathcal{C}_T)\), \(k = 1, 2, 3, 4\), the \(n \times n\) matrices that are blocks of the fundamental matrix of the system in variations \[6.10\]:

\[
\Phi(t, \mathcal{C}_T) = \begin{pmatrix} \lambda_1(t, \mathcal{C}_T) & \lambda_2(t, \mathcal{C}_T) \\ \lambda_3(t, \mathcal{C}_T) & \lambda_4(t, \mathcal{C}_T) \end{pmatrix}, \quad \Phi(0, \mathcal{C}_T) = I_{2n \times 2n}.
\]

(6.2)

Let the following conditions be fulfilled:

\[
\det \mathcal{H}_{pp}(t, \mathcal{C}_T) \neq 0, \quad \det \lambda_3 (t - s, \mathcal{C}_T) \neq 0, \quad s, t \in [0, T],
\]

(6.3)

Then the Green’s function \(G^{(2)}_0(x, y, t, s, \mathcal{C}_T)\) has the form

\[
G^{(2)}_0(\vec{x}, \vec{y}, t, s, \mathcal{C}_T) = \frac{1}{\sqrt{\det(-i2\pi\hbar\lambda_3(\Delta t, \mathcal{C}_T))}} \exp \left\{ \frac{i}{\hbar} \left[ S^{(2)}(t, h) - S^{(2)}(s, h) + + \langle \vec{F}^{(2)}(t, h), \Delta \vec{x} \rangle - \langle \vec{p}_0, (\vec{y} - \vec{x}_0) \rangle - \frac{1}{2}\langle (\vec{y} - \vec{x}_0), \lambda_1(\Delta t, \mathcal{C}_T) \lambda_3^{-1}(\Delta t, \mathcal{C}_T)(\vec{y} - \vec{x}_0) \rangle + + \langle \Delta \vec{x}, \lambda_3^{-1}(\Delta t, \mathcal{C}_T)(\vec{y} - \vec{x}_0) \rangle - \frac{1}{2}\langle \Delta \vec{x}, \lambda_3^{-1}(\Delta t, \mathcal{C}_T) \lambda_4(\Delta t, \mathcal{C}_T) \Delta \vec{x} \rangle \right] \right\}
\]

Here, \(\Delta t = t - s\), \(Z^{(2)}(t, h) = \langle \vec{F}^{(2)}(t, h), \Delta \vec{x}^{(2)}(t, h) \rangle\) is the projection of the solution of the second order Hamilton–Ehrenfest system onto \(\mathbb{R}^{2n}_t\).
Remark 1 If condition (6.3) is not fulfilled, the solution of the problem (6.1) has a somewhat different form than (6.4) (see, e.g., [27]).

From this reasoning and Statement 1 we arrive at the following theorem.

**Theorem 5** Let the symbols of the operators $\hat{H}(t)$ and $\hat{V}(t, \Psi)$ in (1.1)–(1.3) satisfy the conditions of Assumption 1 and let the conditions of Theorem 3 and conditions (6.3) be fulfilled. Then the function

$$
\psi_0^{(2)}(\vec{x}, t, \hbar) = \hat{U}_0^{(2)}(t, 0)\psi_0, \quad t \in [0, T],
$$

(6.5)

where $\hat{U}_0^{(2)}(t, 0)$ is the evolution operator of the zero order associated Schrödinger equation (4.10) with a kernel $\hat{G}_0^{(2)}(\vec{x}, \vec{y}, t, 0, \hat{\psi}(\psi_0))$ (6.4), is an asymptotic (to within $O(h^{3/2})$, $\hbar \to 0$) solution of the nonlocal Gross–Pitaevsky equation (1.1)–(1.3) with the initial condition

$$
\Psi|_{t=0} = \psi_0(\vec{x}).
$$

**Corollary 5.1** Operator $\hat{U}_0^{(2)}(T) = \hat{U}_0^{(2)}(T, 0)$ is the monodromy operator the Gross–Pitaevsky equation (1.1)–(1.3). Functions $\Psi_\nu(\vec{x}, t, \hbar)$ (5.16) are eigenfunctions of operator $\hat{U}_0^{(2)}(T)$:

$$
\hat{U}_0^{(2)}(T)\Psi_\nu(\vec{x}, t, \hbar) = e^{-i\mathcal{E}_\nu^{(2)}T/\hbar}\Psi_\nu(\vec{x}, t, \hbar).
$$

7 Geometric phases of trajectory-coherent states

We now turn to the calculation of the Aharonov–Anandan phase corresponding to the quasi-energy states (5.17). To do this, we use formula (1.13) by which, neglecting the terms of order $O(\hbar)$, we obtain the following expression of the desired phase if the phase curve is periodic:

$$
\gamma_{\nu} = \frac{\mathcal{E}_\nu^{(2)}T}{\hbar} + \frac{1}{\hbar} \int_0^T \{ \hat{H}(t) + \hat{V}(t) + \frac{1}{2} \text{Sp} \left( \hat{S} z z(t) \Delta_2 \right) \} dt + \frac{\hbar}{2} \text{Sp} \left( \hat{V}_{\omega}(Z(t, \hbar), \omega, t) \Delta_2 \right)|_{\omega = Z(t, \hbar)}.
$$

(7.1)

Using (6.22), we can readily show that

$$
\text{Sp} \left( \left[ \hat{H} z z(t) + \omega \hat{V}_{zz}(t) \right] \Delta_2(t) \right) = \frac{\hbar}{2} \text{Sp} \text{Re} \left[ \hat{C}(t) \hat{D}_\nu^{-1} B(t) - \hat{B}(t) \hat{D}_\nu^{-1} C(t) \right] =
$$

$$
= -\hbar \text{Re} \sum_{k=1}^{n} \left( \nu_k + \frac{1}{2} \right) \{ \hat{a}_k(t), a_k^*(t) \},
$$

(7.2)

where, in our case, $D_\nu^{-1} = \text{diag}(2\nu_1 + 1, \ldots, 2\nu_n + 1)$. Note that the quantity under the summation sign on the right of (7.2), by virtue of (6.18), is real. Hence, the sign Re can be omitted. Introduce the notation $a_0(t) = Z_0(t)$, where $Z(t, \hbar) = Z_0(t) + hZ_1(t) + O(h^{3/2})$ are defined in (5.13). Then, substituting the explicit expression for quasi-energies $\mathcal{E}_\nu^{(2)}$ (5.17) in (7.1) and using relation (7.2), we find

$$
\gamma_{\nu} = \frac{1}{h} \int_0^T dt \{ \hat{P}(t), \dot{\hat{X}}(t) \} - \int_0^T dt \{ \hat{Z}_0(t), Z_1(t) \} - \frac{1}{2} \sum_{k=1}^{n} \left( \nu_k + \frac{1}{2} \right) \int_0^T dt \{ \dot{a}_k(t), a_k^*(t) \}.
$$

(7.3)

If now we introduce, instead of the Floquet solutions, the $T$-periodic vector functions

$$
\hat{a}_k(t) = e^{-i\mathcal{E}_k t} a_k(t), \quad \hat{a}_k(t + T) = \hat{a}_k(t),
$$

(7.4)

then formula (7.3) takes the form

$$
\gamma_{\nu} = \frac{1}{h} \int_0^T dt \{ \hat{P}(t), \dot{\hat{X}}(t) \} - \int_0^T dt \{ a_0(t), Z_1(t) \} - \frac{1}{2} \sum_{k=1}^{n} \left( \nu_k + \frac{1}{2} \right) \int_0^T dt \{ \dot{a}_k(t), a_k^*(t) \}.
$$

(7.5)

Using (5.23), we obtain the following expression for the Aharonov–Anandan phase (7.5):

$$
\gamma_{\nu} = \frac{1}{h} \int_0^T dt \{ \hat{P}(t), \dot{\hat{X}}(t) \} - \frac{1}{2h} \sum_{k=1}^{n} \left( \nu_k + \frac{1}{2} \right) \int_0^T dt \{ \dot{a}_k, a_k^* \} -
$$

$$
- \text{Re} \int_0^T dt \left[ \frac{1}{2} \sum_{l=1}^{n} \{ a_0(t), a_l(t) \} \sum_{k=1}^{n} \left( \nu_k + \frac{1}{2} \right) \int_0^T dt \{ a_k^*(\tau), \partial_z \} \{ a_k^*(\tau), J [ \hat{H}_{zz}(z, \tau) + \right.
$$

$$
+ \hat{Z} V_{zz}(z, w, \tau) + \hat{Z} V_{ww}(z, \omega, \tau) \} \{ a_k(\tau) \} \right]_{z=w=Z_0(\tau)} + b_k(T) e^{i\hat{a}_k T}.
$$

(7.6)
Thus, the the Aharonov–Anandan phase $\gamma_{E_{\nu}}$ corresponding to the quasi-energy TCS's $\Psi_{E_{\nu}}$ (5.10) is completely determined by two geometric objects: the closed phase trajectory $Z_{0}(t)$ of the Hamilton–Ehrenfest system, stable in the linear approximation, and the complex germ $r^{n}(Z_{0}(t))$ composed of $n$ linearly independent Floquet solutions of the system in variations (3.17).

Now consider Eqs. (1.1)–(1.3) in a 3-dimensional space with operators $\hat{H}(t), \hat{V}(t, \Psi)$ of the form

$$\hat{H}(\tilde{z}, t) = \frac{1}{2m} \tilde{p}^{2} - e\langle \tilde{E}(t), \tilde{x} \rangle + \frac{k}{2} \tilde{x}^{2},$$

$$V(\tilde{z}, \tilde{w}, t) = V(\tilde{x} - \tilde{y}) = V_{0} \exp \left[ - \frac{(\tilde{x} - \tilde{y})^{2}}{2\gamma^{2}} \right].$$

(7.7)

(7.8)

The external field in the linear operator (7.7) is electric field $\tilde{E}(t) = (E \cos \omega t, E \sin \omega t, 0)$ periodic in time with frequency $\omega$, and the field of an isotropic oscillator with potential $k\tilde{x}^{2}/2, k > 0$.

The periodic solution (3.10) corresponding to (7.7), (7.8) accurate to $O(h^{3/2})$

$$Z_{0}(t, h) = (\tilde{P}_{0}(t, h), \tilde{X}_{0}(t, h)),$$

(7.9)

$$\tilde{P}_{0}(t, h) = \left( -m\omega \xi \sin \omega t, m\omega \xi \cos \omega t, 0 \right),$$

$$\tilde{X}_{0}(t, h) = (\xi \cos \omega t, \xi \sin \omega t, 0).$$

Here $\xi = eE/[m(\omega_{0}^{2} - \omega^{2})].$

The Floquet solutions of system (3.17) corresponding to (7.7), (7.8) accurate to $O(h^{3/2})$

$$a_{1}(t) = e^{i\omega_{s}t}(g_{s}, 0, 0, -\frac{i}{g_{s}} 0, 0)^{T},$$

$$a_{2}(t) = e^{i\omega_{s}t}(0, g_{s}, 0, 0, -\frac{i}{g_{s}} 0, 0)^{T},$$

$$a_{3}(t) = e^{i\omega_{s}t}(0, 0, g_{s}, 0, 0, -\frac{i}{g_{s}} 0, 0)^{T}. $$

(7.10)

Here

$$\omega_{s} = \sqrt{\omega_{0}^{2} - \eta \omega_{nl}^{2}}; \ g_{s} = \sqrt{m\omega_{0}}; \ \omega_{nl} = \sqrt{\frac{\mid \tilde{E}V_{0} \mid}{m\gamma^{2}}}; \ \omega_{0} = \sqrt{\frac{k}{m}}; \ \tilde{x} = \parallel \Psi \parallel^{2}; \ \eta = \text{sign} (\tilde{x}V_{0}).$$

The spectrum of quasi-energies $E_{\nu}^{(2)} (5.17)$ is given by the relation

$$E_{\nu}^{(2)} = -\frac{eE}{2} \xi + \tilde{x}V_{0} + h \left[ (\omega_{s} - \frac{\eta \omega_{nl}^{2}}{2\omega_{s}})(\nu_{1} + \frac{1}{2}) + (\omega_{s} - \frac{\eta \omega_{nl}^{2}}{2\omega_{s}})(\nu_{2} + \frac{1}{2}) + (\omega_{s} - \frac{\eta \omega_{nl}^{2}}{2\omega_{s}})(\nu_{3} + \frac{1}{2}) \right] + O(h^{3/2}).$$

(7.11)

The Aharonov–Anandan geometric phase (7.10) is:

$$\gamma_{E_{\nu}} = \frac{T\omega^{2}}{h} m\xi^{2} + O(h^{1/2}).$$

(7.12)

We have found the explicit expressions for the Aharonov–Anandan phase in semiclassical approximation accurate to $O(h^{1/2})$ for the quasi-energy states which are governed by the GPE and belong to the class of trajectory concentrated functions. It is of interest to consider the expressions (7.6) in the adiabatic limit $T \rightarrow \infty$ and to observe in what sense the Aharonov–Anandan phase transforms to the Berry phase in the background of formalism of the GPE. The fact is that the splitting of the total phase, gained by a wave function, into dynamic and geometric parts is carried out by analogy with the case of linear quantum mechanics. Then it is not obvious that the expressions obtained for the nonlinear GPE are determined only by geometry of the system and do not contain a dynamic contribution due to the nonlinear term in the equation.

In the limit $T \rightarrow \infty$ we can neglect the dependence of operators $\hat{H}(t)$ and $\hat{V}(t, \Psi)$ of the form (1.1) on time $t$. Therefore, the quasi-energy spectral series (5.10) grade into discrete spectral series of the nonlinear problem

$$[\hat{H} + \xi \hat{V}(\Psi)]\Psi = E \Psi$$

in the limit $T \rightarrow \infty$. These spectral series [28] are localized in a neighborhood of stationary solutions of the Hamilton–Ehrenfest system (3.2), (3.10). The geometrical phases (7.12) do not contain quantum corrections accurate to $O(h^{1/2})$ in the considered example and do not depend on nonlinear potential. Evidently, it is related with the fact that the nonlinearity and the quantum corrections with the given accuracy in $h$ do not change solutions (7.9) of the Hamilton system (3.10) and, as a result, the geometry of the system.
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