Abstract. We define the property of $\Pi_2$-compactness of a statement $\phi$ of set theory, meaning roughly that the hard core of the impact of $\phi$ on combinatorics of $\aleph_1$ can be isolated in a canonical model for the statement $\phi$. We show that the following statements are $\Pi_2$-compact: “dominating number $= \aleph_1$,” “cofinality of the meager ideal $= \aleph_1$,” “cofinality of the null ideal $= \aleph_1$,” “bounding number $= \aleph_1$”, existence of various types of Souslin trees and variations on uniformity of measure and category $= \aleph_1$. Several important new metamathematical patterns among classical statements of set theory are pointed out.
0. Introduction

One of the oldest enterprises in higher set theory is the study of combinatorics of the first uncountable cardinal. It appears that many phenomena under investigation in this area are \(\Sigma_2\) statements in the structure \(\langle H_{\aleph_2}, \in, \mathcal{J} \rangle\), where \(H_{\aleph_2}\) is the collection of sets of hereditary cardinality \(\aleph_1\) and \(\mathcal{J}\) is a predicate for nonstationary subsets of \(\omega_1\). For example:

1. the Continuum Hypothesis—or, “there exists an \(\omega_1\) sequence of reals such that every real appears on it” [Ca]
2. the negation of Souslin Hypothesis—or, “there exists an \(\omega_1\)-tree without an uncountable antichain” [So]
3. \(d = \aleph_1\)—or, “there is a collection of \(\aleph_1\) many functions in \(\omega\omega\) such that any other such function is pointwise dominated by one of them”
4. indeed, every equality \(x = \aleph_1\) for \(x\) a classical invariant of the continuum is a \(\Sigma_2\) statement—\(b = \aleph_1, s = \aleph_1, \text{additivity of measure} = \aleph_1 \ldots\) [BJ]
5. there is a partition \(h : [\omega_1]^2 \to \{0, 1\}\) without an uncountable homogeneous set [T2].
6. the nonstationary ideal is \(\aleph_1\)-dense [W2]

It appears that \(\Sigma_2\) statements generally assert that the combinatorics of \(\aleph_1\) is complex. Therefore, given a sentence \(\phi\) about sets, it is interesting to look for models where \(\phi\) and as few as possible \(\Sigma_2\) statements hold, in order to isolate the real impact of \(\phi\) to the combinatorics of \(\aleph_1\). The whole machinery of iterated forcing [S1] and more recently the \(P_{\max}\) method [W2] were developed explicitly for this purpose. This paper is devoted to constructing such canonical \(\Sigma_2\)-poor (or \(\Pi_2\)-rich) models for a number of classical statements \(\phi\).

We consider cases of \(\phi\) being \(d = \aleph_1\), cofinality of the meager ideal= \(\aleph_1\), cofinality of the null ideal= \(\aleph_1\), \(b = \aleph_1\), existence of some variations of Souslin trees, variations on uniformity of measure and category = \(\aleph_1\) and for all of these we find canonical models. It is also proved that \(\phi = \text{"reals can be covered by } \aleph_1 \text{ many meager sets"}\) does not have such a model. But let us first spell out exactly what makes our models canonical.

Fix a sentence \(\phi\). Following the \(P_{\max}\) method developed in [W2], we shall aim for a \(\sigma\)-closed forcing \(P_\phi\) definable in \(L(\mathbb{R})\) so that the following holds:

**Theorem Scheme 0.1.** Assume the Axiom of Determinacy in \(L(\mathbb{R})\). Then in \(L(\mathbb{R})^{P_\phi}\), the following holds:

1. ZFC, \(\varepsilon = \aleph_2\), the nonstationary ideal is saturated, \(\delta_2^1 = \aleph_2\)
2. \(\phi\)

**Theorem Scheme 0.2.** Assume that \(\psi\) is a \(\Pi_2\) statement for \(\langle H_{\aleph_2}, \in, \omega_1, \mathcal{J} \rangle\) and

1. the Axiom of Determinacy holds in \(L(\mathbb{R})\)
2. there is a Woodin cardinal with a measurable above it
3. \(\phi\) holds
4. \(\langle H_{\aleph_2}, \in, \omega_1, \mathcal{J} \rangle \models \psi\).

Then in \(L(\mathbb{R})^{P_\phi}\), \(\langle H_{\aleph_2}, \in, \omega_1, \mathcal{J} \rangle \models \psi\).

If these two theorems can be proved for \(\phi\), we say that \(\phi\) is \(\Pi_2\)-compact.

What exactly is going on? Recall that granted large cardinals, the theory of \(L(\mathbb{R})\) is invariant under forcing [W1] and so must be the theory of \(L(\mathbb{R})^{P_\phi}\). Now
varying the ZFC universe enveloping $L(\mathbb{R})$ so as to satisfy various $\Pi_2$ statements $\psi$, from Theorem Scheme 0.2 it follows that necessarily $L(\mathbb{R})^{P_\kappa}$ must realize all such $\Pi_2$ sentences ever achievable in conjunction with $\phi$ by forcing in presence of large cardinals. In particular, roughly if $\psi_1 : i \in I$ are $\Pi_2$-sentences one by one consistent with $\phi$ then even their conjunction is consistent with $\phi$. And $L(\mathbb{R})^{P_\kappa}$ is the model isolating the impact of $\phi$ on combinatorics of $\aleph_1$.

It is proved in [W2] that $\phi = \text{"true"}, \text{"the nonstationary ideal is } \aleph_1\text{-dense}"$ and others are $\Pi_2$-compact assertions. This paper provides many classical $\Sigma_2$ statements which are $\Pi_2$-compact as well as examples of natural noncompact statements. In general, our results appear to run parallel with certain intuitions related to iterated forcing. The $\Pi_2$-compact assertions often describe phenomena for which good preservation theorems [BJ, Chapter 6] are known. This is not surprising given that in many cases the $P_{\max}$ machinery can serve as a surrogate to the preservation theorems–see Theorem 1.15(5)–and that many local arguments in $P_{\max}$ use classical forcing techniques–see Lemma 4.4 or Theorem 5.6. There are many open questions left:

**Question 0.3.** Is it possible to define a similar notion of $\Pi_2$-compactness without reference to large cardinals?

**Question 0.4.** Is the Continuum Hypothesis $\Pi_2$-compact? Of course, (1) of Theorem Scheme 0.1 would have to be weakened to accommodate the Continuum Hypothesis.

The first section outlines the proof scheme using which all the $\Pi_2$ compactness results in this paper are demonstrated. The scheme works subject to verification of three combinatorial properties –Lemma schemes 1.10, 1.13, 1.16, of independent interest– of the statement $\phi$ in question, which is done in Sections 2-5. These sections can be read and understood without any knowledge of [W2]. The only indispensable–and truly crucial–reference to [W2] appears in the first line of the proof of Theorem 1.15. At the time this paper went into print, a draft version of [W2] could be obtained from its author. There were cosmetical differences in the presentation of $P_{\max}$ in this paper and in [W2].

Our notation follows the set-theoretical standard as set forth in [J2]. The letter $\mathcal{J}$ stands for the nonstationary ideal on $\omega_1$. A system $\alpha$ of countable sets is stationary if for every function $f : (\bigcup a)^{<\omega} \to \bigcup a$ there is some $x \in a$ closed under $f$. $H_\kappa$ denotes the collection of all sets of hereditary size $< \kappa$. By a “model” we always mean a model of ZFC if not explicitly said otherwise. The symbol $\mathcal{J}$ stands for the statement: there is a sequence $\langle A_\alpha : \alpha \in \omega_1 \rangle$ such that $A_\alpha \subset \alpha$ for each $\alpha \in \omega_1$ and for every $B \subset \omega_1$ the set $\{ \alpha \in \omega_1 : B \cap \alpha = A_\alpha \} \subset \omega_1$ is stationary. $\omega_1$-trees grow downward, are always infinitely branching, are considered to consist of functions from countable ordinals to $\omega$ ordered by reverse inclusion, and if $T \leq$ is such a tree then $T_\alpha = \{ t \in T : \text{ordertype of the set } \{ s \in T : t \leq s \} \text{ under } \geq \text{ is just } \alpha \}$ and $T_{<\alpha} = \bigcup_{\beta < \alpha} T_\beta$. For $t \in T$, $lev(t)$ is the unique ordinal $\alpha$ such that $t \in T_\alpha$. For trees $S$ of finite sequences, we write $[S]$ to mean the set $\{ x : \forall n \in \omega \cdot x \upharpoonright n \in S \}$. When we compare open sets of reals sitting in different models then we always mean to compare the open sets given by the respective definitions. $\delta^2_1$ is the supremum of lengths of boldface $\Delta^2_1$ prewellorderings of reals, $\Theta$ is the supremum of lengths of all prewellorderings of reals in $L(\mathbb{R})$. In forcing, the western convention of writing $q \leq p$ if $q$ is more informative than $p$ is utilized. $\prec$ denotes the relation of complete
embedding between complete Boolean algebras or partial orders. \( RO(P) \) is the complete Boolean algebra determined by a partially ordered set \( P \), and \( C_\kappa \) is the Cohen algebra on \( \kappa \) coordinates. The algebra \( \mathbb{C} = \mathbb{C}_\omega \) is construed as having a dense set \( \omega \) ordered by reverse inclusion. Lemma and Theorem “schemes” indicate that we shall attempt to prove some of their instances later.

The second author would like to thank Rutgers University for its hospitality in November 1995 and W. H. Woodin for many discussions concerning his \( P_{max} \) method.

1. General comments

This section sets up a framework in which all \( \Pi_2 \)-compactness results in this paper will be proved. Subsection 1.0 introduces a crucial notion of an iteration of a countable transitive model of ZFC. In Subsection 1.1, a uniform in a sentence \( \phi \) way of defining the forcing \( P_\phi \) and proving instances of Theorem schemes 0.1, 0.2 is provided. In this proof scheme there are three combinatorial lemmas–1.10, 1.13, 1.16–which must be demonstrated for each \( \phi \) separately, and that is done in the section of the paper dealing with that particular assertion \( \phi \). In Subsection 1.2 it is shown how subtle combinatorics of \( \phi \) can yield regularity properties of the forcing \( P_\phi \). And finally Subsection 1.3 gives some examples of failure of \( \Pi_2 \)-compactness.

1.0. Iterability.

The cornerstone of the \( P_{max} \) method is the possibility of finding generic elementary embeddings of the universe with critical point equal to \( \omega_1 \). This can be done in several ways from sufficiently large cardinals. Here is our choice:

**Definition 1.1.** [W1] Let \( \delta \) be a Woodin cardinal. The nonstationary tower forcing \( Q_{\prec \delta} \) is defined as the set \( \{ a \in V_\delta : \text{a is a stationary system of countable sets} \} \) ordered by \( b \leq a \) if for every \( x \in b, x \cap \bigcup a \in a \).

The important feature of this notion is the following. Whenever \( \delta \) is a Woodin cardinal and \( G \subset Q_{\prec \delta} \) is a generic filter then in \( V[G] \) there is an ultrapower embedding \( j : V \rightarrow M \) such that the critical point of \( j \) is \( \omega_1^V, j(\omega_1^V) = \delta \) and \( M \) is closed under \( \omega \) sequences; in particular \( M \) is wellfounded. All of this has been described and proved in [W1]. We shall be interested in iterations of this process.

**Definition 1.2.** [W2] Let \( M \) be a countable transitive model of ZFC, \( M \models \delta \) is a Woodin cardinal. An iteration of \( M \) of length \( \gamma \) based on \( \delta \) is a sequence \( \langle M_\alpha : \alpha \in \gamma \rangle \) together with commuting maps \( j_{\alpha \beta} : M_\alpha \rightarrow M_\beta : \alpha \in \beta \in \gamma \) so that:

1. \( M = M_0 \)
2. each \( M_\alpha \) is a model of ZFC, possibly not transitive. Moreover, \( j_{\alpha \beta} \) are elementary embeddings.
3. for each \( \alpha \) with \( \alpha + 1 \in \gamma \) there is a \( M_\alpha \) generic filter \( G_\alpha \subset (Q_{\prec j_{0\alpha}(\delta)})^{M_\alpha} \).
4. The model \( M_{\alpha+1} \) is the generic ultrapower of \( M_\alpha \) by \( G_\alpha \) and \( j_{\alpha \alpha+1} \) is the ultrapower embedding.
5. at limit ordinals \( \alpha \in \gamma \) a direct limit is taken.

**Convention 1.3.** If the models \( M_\alpha \) are wellfounded we replace them with their transitive isomorphs. Everywhere in this paper, in the context of one specific iteration we keep the indexation system as in the above definition. We write \( \theta_\alpha = \omega_1^{M_\alpha} \) and \( Q_\alpha = (Q_{\prec j_{0\alpha}(\delta)})^{M_\alpha} \).
Definition 1.4. [W2] An iteration $j$ of a model $M$ is called full if it is of length $\omega_1 + 1$ and for every pair $(x, \beta)$ with $x \in Q_\beta$ and $\beta \in \omega_1$ the set $\{\alpha \in \omega_1 : j_{\beta_\alpha}(x) \in G_{\alpha}\} \subseteq \omega_1$ is stationary.

If all models in an iteration $j : M \to N$ of length $\omega_1 + 1$ are wellfounded then $j$ can be thought of as stretching $\mathcal{P}(\omega_1)^M$ into a collection of subsets of the real $\omega_1$. The fullness of $j$ is then a simple bookkeeping requirement on it, making sure in particular that the model $N$ is correct about the nonstationary ideal, that is $\mathcal{J} \cap N = \mathcal{J}^N$.

Definition 1.5. [W2] A countable transitive model $M$ is said to be iterable with respect to its Woodin cardinal $\delta$ if all of its iterations based on $\delta$ produce only wellfounded models. $M$ is called stable iterable with respect to $\delta$ if all of its generic extensions by forcings of size $< \delta$ are iterable with respect to $\delta$.

It is not a priori clear whether iterability and stable iterability are two different notions. We shall often neglect the dependence of the above definition on the ordinal $\delta$.

Of course, a problem of great interest is to produce many rich iterable models. The following lemma and its two corollaries record the two methods of construction of such models used in this paper.

Lemma 1.6. [W2] Let $N$ be a transitive model of ZFC such that $\omega_1 = On \cap N$ and $N \models \"\delta < \kappa \text{ are a Woodin and an inaccessible cardinal respectively}\"$. Then $M = N \cap V_\kappa$ is stable iterable with respect to $\delta$.

Corollary 1.7. [W2] Suppose that the Axiom of Determinacy holds in $L(\mathbb{R})$. Then for every real $x$ there is a stable iterable model containing $x$.

Proof. The determinacy assumption provides a model $N$ as in the Lemma containing every real $x$ given beforehand [Sc]. Then $x \in N \cap V_\kappa = M$ is the desired countable stable iterable model. $\square$

Corollary 1.8. [W2] Suppose $\delta < \kappa$ are a Woodin and a measurable cardinal respectively. Then for every real $x$ there is a stable iterable model elementarily embeddable into $V_\kappa$ which contains $x$.

Proof. Fix a real $x$ and choose a countable elementary substructure $Z \preceq V_{\kappa+2}$ containing $x, \delta, \kappa$ and a measure $U$ on $\kappa$. Let $\pi : Z \to \bar{Z}$ be the transitive collapse. Then the model $\bar{Z}$ is iterable in Kunen’s sense [Ku] with respect to its measure $\pi(U)$, since its iterations lift those of the universe using the measure $U$. Let $N^*$ be the $\omega_1$-th iterate of $\bar{Z}$ using the measure $\pi(U)$, let $N = N^* \cap V_{\omega_1}$ and let $M = N \cap V_{\pi(\kappa)}$. The lemma applied to $N, \pi(\delta)$ and $\pi(\kappa)$ shows that the model $M$ is stable iterable; moreover, $M = \bar{Z} \cap V_{\pi(\kappa)}$ and so the map $\pi \upharpoonright M$ elementarily embeds $M$ into $V_\kappa$. Since $x \in M$, the proof is complete. $\square$

Proof of Lemma. We shall show that $M$ is iterable; the iterability of its small generic extensions $M[G]$ follows from an application of our proof to the model $N[G]$.

For contradiction, assume that there is an iteration $j : M \to M^*$ which yields an illfounded model. Since $j^{\omega_1}(On \cap M)$ is cofinal in the ordinals of $M^*$, there must be some $\beta \in M$ such that $j(\beta)$ is illfounded. Choose the iteration $j$ of the minimal possible length $\gamma_0$ and so that the least ordinal $\beta_0$ with $j(\beta_0)$ illfounded is smallest possible among all iterations of length $\gamma_0$. Note that $\gamma_0$ must be a successor of a countable limit ordinal.
Now $\gamma_0, \beta_0$ are definable in the model $N$ as the unique solutions to the formula $\psi(x,y,M) = "\text{for every large enough cardinal } \lambda, \text{Coll}(\omega, \lambda) \vDash \chi(\check{x}, \check{y}, M)"$, where $\chi(x,y,z)$ says “$x$ is the minimal length of a bad iteration of $z$ and $y$ is the minimal bad ordinal among such iterations of length $x$”. The point is that whenever $\kappa, \gamma, \beta < \lambda < \omega_1$ and $G \subset \text{Coll}(\omega, \lambda)$ is an $N$-generic filter, then in the model $N[G] \chi(\gamma_0, \beta_0, M)$ is a $\Sigma^1_2$ property of hereditarily countable objects and therefore evaluated correctly.

There must be ordinals $\gamma_1 < \gamma_0$ and $\beta_1 < j_{0, \gamma_1}(\beta_0)$ such that $j_{\gamma_1, \gamma_0}(\beta_1)$ is ill-founded. Since $\kappa$ is an inaccessible cardinal of $N$, the iteration $j_{0, \gamma_1}$ can be copied to an iteration of the whole model $N$ using the same nonstationary tower generic filters. Write $j_{0, \gamma_1} : N \rightarrow N'$ for this extended version of $j_{0, \gamma_1}$ again and note that $M_{\gamma_1} = j_{0, \gamma_1}(M) = N' \cap V_{j_{0, \gamma_1}(\kappa)}$. By elementarity, the ordinals $j_{0, \gamma_1}(\gamma_0), j_{0, \gamma_1}(\beta_0)$ are the unique solution to the formula $\psi(x, y, j_{0, \gamma_1}(M))$ in the model $N'$. However, an application of the previous paragraph to $N'$ shows that this cannot be, since $\gamma_0 - \gamma_1, \beta_1$ are better candidates for such a solution. Contradiction. \qed

1.1. The $P_{\text{max}}$ method.

In this subsection we present a proof scheme used in this paper to show that various $\Sigma^2_2$ sentences $\phi$ for the structure $\langle H_{\aleph_3}, \in, \omega_1, 3 \rangle$ are $\Pi^2$ compact. For the record, all statements $\phi$ considered here are consequences of $\Diamond$ and therefore easily found consistent with large cardinals.

**Definition 1.9.** The set $P_\phi$ is defined by induction on rank of its elements. $p \in P_\phi$ if $p = \langle M_p, w_p, \delta_p, H_p \rangle$ where if no confusion is possible we drop the subscript $p$ and

1. $M$ is a countable transitive model of ZFC iterable with respect to its Woodin cardinal $\delta$
2. $M \models w$ is a witness for $\phi$
3. $H \in M$ is the history of the condition $p$; it is a set (possibly empty) of pairs $\langle q, j \rangle$ where $q \in P_\phi$ and $j$ is in $M$ a full iteration of the model $M_q$ based on $\delta_q$ such that $j(w_q) = w$ and $j(H_q) \subseteq H$
4. if $\langle q, j \rangle, \langle q, k \rangle$ are both in $H$ then $j = k$.

The ordering on $P_\phi$ is defined by $q \leq p$ just in case $\langle p, j \rangle \in H_q$ for some $j$.

The notion of a witness for $\phi$ used above is the natural one; if $\phi = \exists x \forall y \chi(x,y)$ with $\chi$ a $\Sigma^0_1$ formula, then $x \in H_{\aleph_2}$ is a witness for $\phi$ whenever $\langle H_{\aleph_2}, \in, \omega_1, 3 \rangle \models \forall y \chi(x,y)$. However, for obviously equivalent versions of the sentence $\phi$ this notion can vary a little. A special care will always be taken as to what variation of $\phi$ we are working with.

The idea behind the definition of the forcing $P_\phi$ is to construct $H_{\aleph_2}$ of the resulting model as a sort of direct limit of its approximations in countable models taken under iterations—not which are recorded in the histories—and extensions.

The possibility of $\Pi^2$-compactness of $\phi$ depends on the validity of three combinatorial lemmas which show how witnesses for $\phi$ in countable transitive models can be stretched by iterations of these models into real witnesses for $\phi$. These lemmas are used in Theorem 1.15 for $\sigma$-closure and various density arguments about $P_\phi$.

The first combinatorial fact to be proved is:

**Lemma scheme 1.10.** (Simple Iteration Lemma) Suppose $\Diamond$ holds. If $M$ is a countable transitive model of ZFC iterable with respect to its Woodin cardinal $\delta$ and $M \models \"w is a witness for $\phi$\" then there is a full iteration $j$ based on $\delta$ of the model $M$ such that $j(w)$ is a witness for $\phi$. 

Certainly there is a need for some assumption of the order of $\emptyset$, since a priori $\phi$ does not have to hold at all and then $j(w)$ could not be a witness for it! Later we shall try to optimise this assumption to $(H_{\omega_1}, \in, \omega_1, 3) \models \phi$, the weakest possible.

For a detailed analysis of the forcing $P_\phi$ a more involved variant of this lemma will be necessary. Essentially, the iteration $j$ is to be built cooperatively by two players, one of whom attempts to make $j(w)$ into a witness for $\phi$. The other one stages various local obstacles to that goal. The relevant definitions:

**Definition 1.11.** A sequence $\vec{N}$ of models with a witness is a system $\langle w, N_i, \delta_i : i \in \omega \rangle$ where

1. $N_i$ are countable transitive models of ZFC+$\delta_i$ is a Woodin cardinal+$w$ is a witness for $\phi$; we set $Q_i = Q_{< \delta_i}$ as computed in $N_i$
2. $N_i \in N_{i+1}$ and $\omega_1^{N_i}$ is the same for all $i \in \omega$
3. if $N_i \models "a \in V_{\delta_i}\text{ is a stationary system of countable sets}"$ then $N_{i+1} \models "a\text{ is a stationary system of countable sets}"$; so $Q_0 \subset Q_1 \subset \ldots$
4. if $N_i \models "A \subset Q_i\text{ is a maximal antichain}"$ then $N_{i+1} \models "A \subset Q_{i+1}\text{ is a maximal antichain}"$

We say that the sequence begins with the triple $\langle N_0, w, \delta_0 \rangle$, set $Q_N = \bigcup_{i \in \omega} Q_i$ and $\omega_1^N = \omega_1^{N_0}$. A filter $G \subset Q_N$ is said to be $\vec{N}$-generic if it meets all maximal antichains of $Q_N$ which happen to belong to $\bigcup_{i \in \omega} N_i$.

This definition may seem a little artificial, an artifact of the machinery of [W2]. The really interesting information a sequence of models carries is the model $\bigcup_{i \in \omega} N_i$ with its first-order theory. This model can be viewed as a $w$-correct extension of $N_0$. It is important that

1. $\langle \bigcup_{i \in \omega} N_i, \in \rangle \models w$ is a witness for $\phi$
2. $\langle \bigcup_{i \in \omega} N_i, \in \rangle$ satisfies all $\Pi_2$-consequences of ZFC in the language $\in, \mathcal{G}$, where
   $\mathcal{G}$ is the predicate for stationary systems of countable sets
3. $N_0$ is in $\bigcup_{i \in \omega} N_i$ correct about stationary systems of countable sets in $V_{\delta_0}$ and their maximal antichains.

It should be noted that though $Q_N$ is not an element of $\bigcup_{i \in \omega} N_i$, it is a class in that model—the class of all stationary systems of countable sets. If a filter $G \subset Q_N$ is $\vec{N}$-generic then the filters $G \cap Q_i$ are $N_i$-generic by (3,4) of the above definition. However, not every $N_i$-generic filter on $Q_i$ can be extended into an $\vec{N}$-generic filter on $Q_N$.

**Definition 1.12.** $G_\phi$ is a two-person game of length $\omega_1$ between players Good and Bad. The rules are:

**Round 0:** The player Bad plays $M, w, \delta$ such that $M$ is a countable transitive model of ZFC iterable with respect to its Woodin cardinal $\delta$ and $M \models "w\text{ is a witness for }\phi"$

**Round $\alpha > 0$:** an ordinal $\gamma_\alpha$ and an iteration $j_{\gamma_\alpha} : M \rightarrow M_{\gamma_\alpha}$ of length $\gamma_\alpha + 1$ based on $\delta$ are given.

- Bad plays a sequence $\vec{N}$ of models beginning with $M_{\gamma_\alpha}, j_{\gamma_\alpha}(w), j_{\gamma_\alpha}(\delta)$ and a condition $p \in Q_N$.
- Good plays an $\vec{N}$-generic filter $G \subset Q_N$ with $p \in G$.
- Bad plays an ordinal $\gamma_{\alpha+1} > \gamma_\alpha$ and an iteration $j_{\gamma_{\alpha+1}}$ of $M$ of length $\gamma_{\alpha+1} + 1$ which prolongs the iteration $j_{\gamma_\alpha}$ and such that the $\gamma_\alpha$-th ultrapower on it is taken using the filter $G \cap Q_{\gamma_\alpha}$.
Here, $\gamma_1 = -1, j_1 = id$ and at limit $\alpha$’s, $j_{\gamma_\alpha}$ is the direct limit of the iterations played before $\alpha$.

In the end, let $\bar{j}$ be the direct limit of the iterations played. The player Good wins if either the player Bad cannot play at some stage or the iteration $\bar{j}$ is not full or $j(w)$ is a witness for $\phi$.

Thus the player Bad is responsible for the bookkeeping to make the iteration full and has a great freedom in prolonging the iteration on a nonstationary set of steps. The player Good has a limited access on a closed unbounded set of steps to steering $j(w)$ into a witness for $\phi$. In the real life, the player Bad can easily play all the way through $\omega_1$ and make the resulting iteration full.

We shall want to prove

**Lemma scheme 1.13.** *(Strategic Iteration Lemma)* Suppose $\Diamond$ holds. Then the player Good has a winning strategy in the game $G_\phi$.

Now suppose that the relevant instances of Lemma schemes 1.10, 1.13 are true for $\phi$. Then, granted the Axiom of Determinacy in $L(\mathbb{R})$, the model $L(\mathbb{R})^{P_\phi}$ can be completely analysed using the methods of [W2] to verify Theorem scheme 0.1. Let $G \subset P_\phi$ be a generic filter.

**Definition 1.14.** In $L(\mathbb{R})[G]$, for any $p \in G$ define

1. $k_p$ is the iteration of $M_p$ which is the direct limit of the system \( \{ j : \exists q \in G \langle p, j \rangle \in H_q \} \).
2. $W = k_p(w_p)$.

It is obvious from the definition of the poset $P_\phi$ that the system \( \{ j : \exists q \in G \langle p, j \rangle \in H_q \} \) is directed, and that the definition of $W$ does not depend on the particular choice of $p \in G$.

**Theorem 1.15.** Assume the Axiom of Determinacy in $L(\mathbb{R})$ and the relevant instances of Lemma schemes 1.10 and 1.13 hold. Then $P_\phi$ is a $\sigma$-closed notion of forcing and in $L(\mathbb{R})[G]$, the following hold:

1. ZFC
2. for every $X \in H_{\mathbb{R}_2}$ there are $p \in G$ and $x \in M_p$ such that $X = k_p(x)$.
3. $\mathcal{C} = \mathbb{N}_2$, the nonstationary ideal is saturated, $\delta_1^2 = \omega_2$
4. $\langle H_{\mathbb{R}_2}, \in, \omega_1, \emptyset \rangle \models \phi$ and $W$ is a witness for $\phi$
5. Suppose $\psi = \forall x \exists y \chi(x, y)$ for some $\Sigma_0$-formula $\chi$ is a $\Pi_2$-statement for $\langle H_{\mathbb{R}_2}, \in, \omega_1, \emptyset \rangle$. Suppose that $ZFC + \Diamond$ proves that for each $x \in H_{\mathbb{R}_2}$ there is a forcing $P$ preserving witnesses to $\phi$ and stationary subsets of $\omega_1$ such that $P \Vdash \exists y \chi(\bar{x}, y)$. Then $\langle H_{\mathbb{R}_2}, \in, \omega_1, \emptyset \rangle \models \psi$.

**Proof.** Parts (1,2,3) are straightforward generalizations of Section 4.3 in [W2]. Work in $L(\mathbb{R})[G]$ and prove (4).

First note that whenever $p \in G$ then $k''_p(\emptyset)^{M_p} = \emptyset \cap k''_p M_p$. To see it, suppose $p \in G, M_p \models \text{"s} \subset \omega_1$ is a stationary set" and fix a club $C \subset \omega_1$. By (2), there is a condition $q \in G$ and $c \in M_q$ so that $C = k_q(c)$. Let $r \in G$ be a common lower bound of $p, q$ with $\langle p, i \rangle, \langle q, j \rangle \in H_r$. Then

1. $M_r \models \langle c \rangle \subset \omega_1$ is a club
2. $M_r \models \langle s \rangle$ is stationary, since the iteration $i$ is full in $M_r$. 


Therefore $j(c) \cap i(s) \neq 0$. By absoluteness, $k_r j(c) \cap k_r i(s) \neq 0$ and since $k_r j = k_q$ and $k_r i = k_p$, we have $C \cap k_p(s) = 0$ and $k_p(s)$ is stationary.

Now suppose (4) fails; so $\phi = \exists x \forall y \chi(x, y)$ for some $\Sigma_0$ formula $\chi$, and $W$ is not a witness for $\phi$, and $\neg \chi(W, Y)$ for some $Y \in H_{K_2}$ By (2), there is a condition $p \in G$ and $y \in M_p$ such that $Y = k_p(y)$.

But now, $M_p \models \langle H_{K_2}, \in, \omega_1, \exists \rangle \models \chi(w_p, y)$, since $w_p$ is a witness for $\phi$ in the model $M_p$. By elementarity of $k_p$, absoluteness and the previous paragraph $\langle H_{K_2}, \in, \omega_1, \exists \rangle \models \chi(k_p(x), Y)$, contradiction.

To prove (5), fix the formula $\chi$ and note that by (2) it is enough to show that for $p \in P_\Phi$ and $x \in M_p$ there is $q \leq p$ which forces an existence of $Y$ such that $\langle H_{K_2}, \in, \omega_1, \exists \rangle \models \chi(k_p(x), Y)$. And indeed, using Corollary 1.7 choose a countable transitive stable iterable model $M$ with $M \models \Diamond + \delta$ is a Woodin cardinal" with $p \in M$. Apply the Iteration Lemma 1.10 in $M$ to get a full iteration $j$ of $M_p$ such that $j(w_p)$ is a witness for $\phi$ in $M$. By the assumptions on $\chi$ applied in $M \cap V_{k}$, where $k$ is the least inaccessible cardinal of $M$, there is a generic extension $M[K]$ of $M$ by a forcing of size $< \kappa$ preserving $j(w_p)$ and stationary subsets of $\omega_1$ such that there is $y \in M[K]$ with $\chi(j(x), y)$. Obviously, setting $q = \langle M[K], j(w_p), \delta, H \rangle$, where $H = j(H_p) \cup \{ \langle p, j \rangle \}$, we have $q \leq p$ and $q \models \chi(k_p(x), k_q(y))$ as desired. □

The rudimentary comparison of the cardinal structure of $L(\mathbb{R})$ and $L(\mathbb{R})[G]$ carries over literally from [W2]: namely $\mathcal{N}_1, \mathcal{N}_2$ are the same in these models, $\Theta = \mathcal{N}_3^{L(\mathbb{R})[G]}$ and all cardinals above $\Theta$ are preserved. This will not be used anywhere in this paper.

It should be remarked that under the assumptions of the Theorem, the model $K = L(\mathcal{P}(\omega_1))$ as evaluated in $L(\mathbb{R})[G]$ satisfies (1)-(5) and it can be argued that $K$ is the "canonical model" in view of its minimal form and Theorem 1.23. In fact, $L(\mathbb{R})[G]$ is a generic extension of $K$ by the poset $(\omega_2^{<\omega_2})^K$.

The final point in the analysis of the model $L(\mathbb{R})[G]$ is the proof of Theorem scheme 0.2 for $P_\Phi$. We know of only one approach for doing this, namely to prove

**Lemma scheme 1.16.** *(Optimal Iteration Lemma)* Suppose $\langle H_{K_2}, \in, \omega_1, \exists \rangle \models \phi$. Whenever $M$ is a countable transitive model iterable with respect to its Woodin cardinal $\delta$ and $M \models \Diamond \text{w is a witness for } \phi$ then there is a full iteration $j$ of $M$ based on $\delta$ such that $j(w)$ is a witness for $\phi$.

It is crucial that the assumption of this optimal iteration lemma is truly the weakest possible. Provided Lemma schemes 1.10, 1.13 and 1.16 are true for $\phi$, we can conclude

**Corollary 1.17.** Suppose instances of Iteration Lemmas 1.10, 1.13 and 1.16 for $\phi$ are true. Then $\phi$ is $\Pi_2$-compact.

**Proof.** We shall prove the relevant instance of Theorem Scheme 0.2. Assume that $\psi$ is a $\Pi_2$ sentence, $\psi = \forall x \exists y \chi(x, y)$ for some $\Sigma_0$ formula $\chi$. Assume that there is a Woodin cardinal $\delta$ with a measurable cardinal $\kappa$ above it, and $\langle H_{K_2}, \in, \omega_1, \exists \rangle \models \psi \land \phi$. It must be proved that $L(\mathbb{R})[P_\Phi] \models \langle H_{K_2}, \in, \omega_1, \exists \rangle \models \psi$.

For contradiction suppose that $p \in P_\Phi$ forces $\neg \psi = \exists x \forall y \neg \chi(x, y)$ holds in $\langle H_{K_2}, \in, \omega_1, \exists \rangle$ of the generic extension. By Theorem 1.15 (2), by eventually strengthening the condition $p$ we may assume that there is $x \in \langle H_{K_2}, \in, \omega_1, \exists \rangle^{M_p}$ so that $p \models \forall y \neg \chi(k_p(x), y)$ holds in $\langle H_{K_2}, \in, \omega_1, \exists \rangle$ of the generic extension.
Following Corollary 1.8, there is a countable transitive iterable model \( M \) elementarily embeddable into \( V_\kappa \) containing \( p \), which is a hereditarily countable object in \( M \). By the relevant instance of the Optimal Iteration Lemma applied within \( M \) there is a full iteration \( j \) of the model \( M_p \) such that \( j(w_p) \) is a witness for \( \phi \) in \( \langle H_{R_2}, \in, \omega_1, \mathcal{J} \rangle^M \). It follows that the quadruple \( q = \langle M, j(w_p), \delta, H \rangle \), where \( H = j(H_p) \cup \{ \langle p, j \rangle \} \) and \( \delta \) is a Woodin cardinal of \( M \), is a condition in \( P_\phi \) and \( q \leq p \). Since in \( M \), \( (H_{R_2}, \in, \omega_1, \mathcal{J}) \models \psi \), necessarily there is \( y \in \langle H_{R_2}, \in, \omega_1, \mathcal{J} \rangle^M \) such that \( (H_{R_2}, \in, \omega_1, \mathcal{J})^M \models \chi(j(x), y) \). It follows that \( q \models \chi(k_q j(x) = k_p(x), k_q(y)) \) in \( \langle H_{R_2}, \in, \omega_1, \mathcal{J} \rangle \) of the generic extension, a contradiction to our assumptions on \( p, x \).

Predicates other than \( \mathcal{J} \) can be added to the language of \( H_{R_2} \) keeping the amended version of Theorem scheme 0.2 valid. For example, if Iteration Lemmas 1.10, 1.13 and 1.16 hold for \( \phi \) then the relevant instance of Theorem scheme 0.2 can be shown to hold with the richer structure \( \langle H_{R_2}, \in, \mathcal{J}, X : X \subset \mathbb{R}, X \in L(\mathbb{R}) \rangle \); however, the proof is a little involved and we omit it. See [W2]. In certain cases a predicate for witnesses for \( \phi \) can be added keeping Theorem scheme 0.2 true for \( \phi \). This increases the expressive power of the language a little. Such a possibility will be discussed on a case-by-case basis.

Let us recapitulate what we proved in this subsection. Let \( \phi \) be a \( \Sigma_2 \) sentence for \( \langle H_{R_2}, \in, \omega_1, \mathcal{J} \rangle \), a consequence of \( \Box \). If instances of Lemma schemes 1.10 and 1.13 are shown to hold, then the model \( L(\mathbb{R})^{P_\phi} \) has the properties listed in Theorem 1.15 or Theorem scheme 0.1. And if the optimal iteration lemma 1.16 for \( \phi \) is proved then the relevant instance of Theorem scheme 0.2 is true and \( \phi \) is \( \Pi_2 \)-compact. It should be noted that Iteration Lemma 1.10 follows from both Lemma 1.13 and Lemma 1.16. We include it because it is frequently much easier to prove and because it is often the first indication that a \( \Pi_2 \)-compactness type of result can be proved.

1.2. Order of witnesses.

After an inspection of the proofs of iteration lemmas in the subsequent sections the following notion comes to light:

**Definition 1.18.** Let \( \phi \) be a \( \Sigma_2 \) sentence. For \( v, w \in H_{R_2} \) we set \( v \leq_\phi w \) if in every forcing extension of the universe whenever \( v \) is a witness to \( \phi \) then \( w \) is such a witness.

Of course, the formally impermissible consideration of all forcing extensions can be expressed as quantification over partially ordered sets. While restricting ourselves to just forcing extensions may seem to be somewhat artificial, it is logically the easiest way and the resulting notion fits all the needs of the present paper. It should be noted that \( \leq_\phi \) is sensitive to the exact definition of a witness as it was the case for \( P_\phi \). Obviously \( \leq_\phi \) is a quasiorder and the nonwitnesses form the \( \leq_\phi \) smallest \( \leq_\phi \)-equivalence class.

**Example 1.19.** For \( \phi = \text{“there is a Souslin tree”} \) and \( S, T \) such trees the relation \( T \leq_\phi S \) is equivalent to the assertion “for every \( s \in S \) there are \( s' \leq s, t \in T \) such that \( RO(S \upharpoonright s') \) can be completely embedded into \( RO(T \upharpoonright t) \)”. For then, preservation of the Souslinity of \( T \) implies the preservation of the Souslinity of \( S \). On the other hand, if the assertion fails, there must be \( s \in S \) such that \( T \models \text{“S \upharpoonright s is an Aronszajn tree”} \) because every cofinal branch through \( S \) is generic. By the c.c.c. productivity theorem then, the finite condition forcing specializing the tree \( S \upharpoonright s \) preserves the Souslinity of \( T \) and collapses the Souslinity of \( S \); ergo, \( T \not\leq_\phi S \).
Note that in the above example the relation $\leq_{\phi}$ was $\Sigma_1$ on the set of all witnesses. It is not clear whether this behavior is typical; the proofs of the iteration lemmas in this paper always use a $\Sigma_1$ phenomenon to guarantee the relation $\leq_{\phi}$ or the $\leq_{\phi}$-equivalence of two witnesses.

**Definition 1.20.** Suppose $\phi = \exists x \forall y \chi(x, y)$ for some $\Sigma_0$ formula $\chi$ and let $\psi(x_0, x_1)$ be $\Sigma_1$. We say that $\psi$ is a copying procedure for $\phi$ if $\text{ZFC} \vdash (H_{\aleph_2}, \in, \omega_1, \mathcal{F}) \models \forall x_0, x_1 \psi(x_0, x_1) \rightarrow (\forall y \chi(x_0, y) \leftrightarrow \forall y \chi(x_1, y))$. In other words, $\psi(x_0, x_1)$ guarantees that $x_1$ is a witness for $\phi$ iff $x_0$ is a witness for $\phi$.

**Example 1.21.** Let $\phi =$ “there is a nonmeager set of reals of size $\aleph_1$”. One possible copying procedure for $\phi$ is $\psi(x_0, x_1) =$ “there is a continuous category-preserving function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f''(x_0) = x_1$”. Note that this is really a statement about a code for $f$, which is essentially a real, and it can be cast in a $\Sigma_1$ form.

The following theorems, quoted without proof, are applications of the above concepts. The first implies that the forcings $P_\phi$ for sentences $\phi$ considered in this paper are all homogeneous and therefore the $\Sigma_n$ theory of $L(\mathbb{R})^{\mathcal{F}_\phi}$ is a (definable) element of $L(\mathbb{R})$ for every $n \in \omega$. The second shows that models in sections 2, 3 and 5 not only optimize the $\Sigma_2$-theory of $(H_{\aleph_2}, \in, \omega_1, \mathcal{F})$ but are in fact characterized by this property. The choice of copying procedures necessary for its proof will always be clear from the arguments in the section dealing with that particular $\phi$.

**Theorem 1.22.** Suppose the axiom of determinacy holds in $L(\mathbb{R})$, suppose $\phi$ is a $\Sigma_2$ sentence for which iteration lemmas 1.10, 1.13 hold and $\psi$ is a copying procedure for $\phi$ such that ZFC proves one of the following:

1. for every two witnesses $x_0, x_1$ for $\phi$ there is a forcing $P$ preserving stationary subsets of $\omega_1$ and witnesses for $\phi$ such that $P \models \psi(x_0, x_1)$

2. for every witness $x_0$ and every countable transitive iterable model $M$ such that $M \models \langle \forall x \psi(x_0, x) \rightarrow \forall x \psi(x_1, x) \rangle$ there is a full iteration $j$ of $M$ such that

Then $P_\phi$ is a homogeneous notion of forcing.

**Theorem 1.23.** Suppose the axiom of determinacy holds in $L(\mathbb{R})$, suppose $\phi$ is a $\Sigma_2$ sentence for which iteration lemmas 1.10, 1.13 hold and $\psi$ is a copying procedure for $\phi$ such that ZFC proves “for every two witnesses $x_0, x_1$ to $\phi$ there is a forcing $P$ preserving stationary subsets of $\omega_1$ and witnesses to $\phi$ such that $P \models \psi(x_0, x_1)$. If the $\Sigma_2$-theory of the structure $(H_{\aleph_2}, \in, \mathcal{F}, X : X \subset \mathbb{R}, X \in L(\mathbb{R}))$ is the same in $V$ as in $L(\mathbb{R})^{\mathcal{F}_\phi}$ then $P(\omega_1) = P(\omega_1) \cap L(\mathbb{R})[G]$ for some possibly external $L(\mathbb{R})$-generic filter $G \subset P_\phi$.

1.3. Limitations.

Of course by far not every $\Sigma_2$ sentence $\phi$ can be handled using the proof scheme outlined in Subsection 1.1. Each of the three iteration lemmas can prove to be a problem; in some cases, it is possible to show that the statement $\phi$ is not $\Pi_2$-compact by exhibiting $\Pi_2$ assertions $\psi_i : i \in I$ each of whom is consistent with $\phi$ yet $\bigwedge_{i \in I} \psi_i \vdash \neg \phi$.

**Example 1.24.** The simple iteration lemma for $\phi =$ “the Continuum Hypothesis” fails. The reason is that whenever $M$ is a countable transitive iterable model and $j$ is an iteration of $M$ then $j(\mathbb{R} \cap M) \neq \mathbb{R}$ –namely, the real coding the model $M$ is missing from $j(\mathbb{R} \cap M)$.
**Example 1.25.** The simple iteration lemma for \( \phi = \"\text{there is a maximal almost disjoint family (MAD) of sets of integers of size } \aleph_1 \text{\" fails. Note that if } A \subset P(\omega_1) \text{ is a MAD extending the set of all branches in } \omega_1 \omega \text{ then } A \text{ is collapsed as a MAD whenever a new real is added to the universe. Thus if } M \text{ is a countable transitive iterable model with } M \models \text{the Continuum Hypothesis holds and } A \text{ is a MAD as above} \text{ then no iteration } j \text{ of } M \text{ makes } j(A) \text{ into a MAD for the same reason in the previous example.} \)**

**Example 1.26.** The strategic iteration lemma for \( \phi = \"\text{there is a nonmeager set of reals of size } \aleph_1 \text{\" with the natural notion of witness fails. The reason is somewhat arcane and we omit it.} \)**

**Example 1.27.** The optimal iteration lemma for \( \phi = \"\text{the reals can be covered with } \aleph_1 \text{ many meager sets}\) cannot be proved. For suppose that \( M \) is a countable transitive iterable model and \( M \models \text{the Continuum Hypothesis holds and } \mathcal{C} = \{ X_f : f \in \omega_1 \omega \} \text{, where } X_f \subset \omega_1 \omega \text{ is the set of all reals pointwise dominated by the function } f \). So \( \mathcal{C} \text{ constitutes a covering of the real line by } \aleph_1 \text{ meager sets.} \) Also suppose that \( \phi \land b > \aleph_1 \) holds in the universe–this is consistent and happens after adding \( \aleph_2 \) Laver reals to a model of GCH \([Lv]\). Then no iteration \( j \) of the model \( M \) can make \( j(C) \) into a covering of the real line, because there always will be a function in \( \omega \omega \) eventually dominating all of \( j(\mathbb{R} \cap M) \). Note that in this case, \( \mathcal{C} \) should be thought of as a collection of Borel codes as opposed to a set of sets of reals.

The previous example suggests that \( \phi \) is not \( \Pi_2 \)-compact, and indeed, it is not. For consider \( \Pi_2 \) sentences

\[
\psi_0 = \"b > \aleph_1\", \\
\psi_1 = \"\text{for every bounded family } A \subset \omega_1 \text{ of size } \aleph_1 \\
\text{there is a function infinitely many times equal to every function in } A\".
\]

Now \( \phi \land \psi_0 \) holds after iterating Laver reals, \( \phi \land \psi_1 \) holds after iterating proper \( \omega \omega \)-bounding forcings \([S3, Proposition 2.10]\), and \( \psi_0 \land \psi_1 \vdash \neg \phi \) can be derived easily from the combinatorial characterization of \( \phi \) in \([Ba, BJ]\).

**Example 1.28.** The optimal iteration lemma for \( \phi = \"t = \aleph_1 \" \) cannot be proved. Recall that \( t \) is the minimal length of a tower and a tower is a decreasing sequence of infinite subsets of \( \omega \) without lower bound in the modulo finite inclusion ordering. To see the reason for the failure, suppose \( M \) is a countable transitive iterable model and \( M \models \text{the Continuum Hypothesis holds and } t \text{ is a tower of height } \omega_1 \text{ consisting of sets of asymptotic density one.} \) That such towers exist under CH has been pointed out to us by W. Hugh Woodin. Suppose that in the universe \( t = \aleph_1 \) holds and no towers consist of sets from a fixed Borel filter–such a situation can be attained by iterating Souslin c.c.c. forcings over a model of CH. Then no iteration \( j \) of \( M \) can make \( j(t) \) into a tower.

It seems that the two \( \Pi_2 \) assertions

\[
\psi_0 = \text{\"no towers consist of sets from a fixed Borel filter\"}
\]
\[
\psi_1 = \text{\"every tower consists of sets from some Borel filter\"}
\]

provide a witness for non-\( \Pi_2 \)-compactness of \( \phi \), however, the consistency of \( \phi \land \psi_1 \) seems to be a difficult open problem.
Example 1.29. The optimal iteration lemma for \( \phi = \text{"there is a Souslin tree"} \) cannot be proved. For suppose that \( M \) is a countable transitive iterable model with \( M \models \text{"}\phi\text{" and } T \text{ is a homogeneous Souslin tree"}. Suppose that in the universe there are Souslin trees but none of them are homogeneous—this was proved consistent in [AS]. Then no iteration \( j \) of \( M \) can make \( j(T) \) into a Souslin tree, since \( j(T) \) is necessarily homogeneous.

Again, the above example provides natural candidates to witness the non-\( \Pi^2_1 \)-compactness of \( \phi \). Let

\[
\begin{align*}
\psi_0 &= \text{"for every finite set } T_i : i \in I \text{ of Souslin trees there are } t_i \in T_i \\
&\text{such that } \prod T_i \upharpoonright t_i \text{ is c.c.c."}, \\
\psi_1 &= \text{"for every Souslin tree } T \text{ there are finitely many } t_i : i \in I \text{ in } T \\
&\text{such that } \prod T \upharpoonright t_i \text{ is nowhere c.c.c."}.
\end{align*}
\]

The sentence \( \phi \land \psi_0 \) was found consistent by [AS], but the consistency of \( \phi \land \psi_1 \) is an open problem. In view of the results of Subsection 4.0 the sentences \( \psi_0, \psi_1 \) are the only candidates for noncompactness of \( \phi \).

2. Dominating number

The proof of \( \Pi^2_1 \)-compactness of the sentence “there is a family of \( \aleph_1 \) many functions in \( \omega^\omega \) such that any function in \( \omega^\omega \) is modulo finite dominated by one in the family” or \( d = \aleph_1 \), is in some sense prototypical, and the argument will be adapted to other invariants in Section 3. The important concept we isolate to prove the iteration lemmas is that of subgenericity. It essentially states that the classical Hechler forcing is the optimal way for adding a dominating real. To our knowledge, this concept has not been explicitly defined before.

2.0. The combinatorics of \( d \).

There is a natural Souslin [JS] forcing associated to the order of eventual dominance on \( \omega^\omega \) designed to add a “large” function:

Definition 2.1. The Hechler forcing \( \mathbb{D} \) is the set \( \{ \langle a, A \rangle : \text{dom}(a) = n \text{ for some } n \in \omega, \text{ rng}(a) \subset \omega \text{ and } A \text{ is a finite subset of } \omega^\omega \} \). The order is defined by \( \langle a, A \rangle \leq \langle b, B \rangle \) if

1. \( b \subset a, B \subset A \)
2. \( \forall n \in \text{dom}(a \setminus b) \forall f \in B a(n) \geq f(n) \).

For a condition \( a \) the function \( \text{body}(p) \in \omega^\omega \) is defined as \( \text{body}(p)(n) = a(n) \) if \( n \in \text{dom}(a) \) and \( \text{body}(p)(n) = \max \{ f(n) : f \in A \} \) if \( n \notin \text{dom}(a) \).

If \( G \subseteq \mathbb{D} \) is a generic filter, the Hechler real \( d \) is defined as \( \bigcup \{ a : \langle a, 0 \rangle \in G \} \).

Below, we shall make use of restricted versions of \( \mathbb{D} \). Say \( f \in \omega^\omega \). Then define \( \mathbb{D}(f) \) to be the set of all \( p \in \mathbb{D} \) with \( \text{body}(p) \) pointwise dominated by the function \( f \), with the order inherited from \( \mathbb{D} \). Note that \( \mathbb{D} \) as defined above is not a separative poset.

Obviously, all forcings defined above are c.c.c. The important combinatorial fact about \( \mathbb{D} \) is that a Hechler real is in fact an “optimal” function eventually dominating every ground model function. This will be immediately made precise:
Lemma 2.3. Let $f$ be a function. If $D \subset D \cap M$ is a dense set in $M$, then $D \cap \mathbb{D}(f)$ is dense in $\mathbb{D}(f) \cap M$.

Note that $\mathbb{D} \cap M$ is $\mathbb{D}$ as computed in $M$; also $\mathbb{D}(f) \cap M \notin M$.

Proof. Fix a dense set $D \subset \mathbb{D} \cap M$ in $M$ and a condition $p = \langle a, A \rangle$ in $\mathbb{D}(f) \cap M$. We shall produce a condition $q \leq p$ in $D \cap \mathbb{D}(f)$. Working in $M$, it is easy to construct a sequence $\langle a_i, A_i \rangle : i \in \omega$ of conditions in $D$ with $\text{dom}(a_i) = n_i$ and

1. $\langle a_i, A_i \rangle \leq p$
2. $\langle a_{i+1}, A_{i+1} \rangle$ is any element of $D$ stronger than $\langle \text{body}(p) \upharpoonright n_i, A_i \rangle$.

Define a function $g \in \omega$ as follows: for $n \leq n_0$, let $g(n) = a(n)$. For $n \geq n_0$, find an integer $i \in \omega$ with $n_i \leq n < n_{i+1}$ and set $g(n) = \max\{a_{i+1}(n), h(n) : h \in A_i\}$. Since $g \in \omega \cap M$, the function $f$ dominates $g$ pointwise starting from some $n_i$. Then $q = \langle a_{i+1}, A_{i+1} \rangle \in D \cap \mathbb{D}(f)$ is the desired condition. \hfill \□

Corollary 2.4. (Subgenericity) Let $P$ be a forcing and $\dot{g}$ a $P$-name such that

1. $P \Vdash \dot{g} \in \omega \mathbb{D}$-dominates the ground model
2. for every $f \in \omega$ the boolean value $||\dot{f} \leq \dot{g}||_P$ is non-zero.

Then there is a complete embedding $RO(\mathbb{D}) < P \ast (\mathbb{D}(\dot{g}) \cap \text{the ground model}) = R$ so that $R \Vdash "\dot{d} \leq \dot{g} \text{ pointwise}"$, where $\dot{d}$ is the $\mathbb{D}$-generic real.

Thus under every $\mathbb{D}$-dominating real, a Hechler real is lurking behind the scenes.

Proof. The Hechler real $\dot{d}$ will be read off the second iterand in the natural way, and Lemma 2.3 will guarantee its genericity. By some Boolean algebra theory, that yields a complete embedding of $RO(\mathbb{D} \upharpoonright p)$ to $RO(R)$, for some $0 \neq p \in \mathbb{D}$. We must prove that $p = 1$. But fix an arbitrary $q = \langle a, A \rangle \in \mathbb{D}$ and let $g = \text{body}(q)$. Then $\langle ||\dot{g} \leq \dot{f}||_P, q \rangle \in R$ is a nonzero element of $RO(R)$ forcing that the $\mathbb{D}$-generic real meets the condition $q$. Thus with the given embedding, any condition in $\mathbb{D}$ can be met, consequently $p = 1$ and the proof is complete. \hfill \□

Corollary 2.5. Let $M$ be a countable transitive model of ZFC such that $M \models \text{"}P$ is a forcing adding a dominating function $\check{f}$ as in Corollary 2.4, $\text{"}$, let $p \in P$ and let $g \in \omega$ be any function, not necessarily in the model $M$. Then there is an $M$-generic filter $G \subset B$ containing $p$ such that $g$ is eventually dominated by $f/G$.

Proof. Apply the previous Corollary in the model $M$ and find the forcing $R$ and the relevant embeddings $P \prec R$, $\mathbb{D} \prec R$. Assume $P \subset R$ and set $x$ to be the projection of $r$ into $\mathbb{D}$ via the above embedding. Step out of the model $M$ and find a filter $H \subset \mathbb{D}$ such that

1. $x \in H$
2. $H$ meets every maximal antichain of $\mathbb{D}$ which is an element of $M$
3. the Hechler real $\check{e}$ derived from $H$ eventually dominates the function $g$.

This is easily done. Now the key point is that the model $M$ computes maximal antichains of $\mathbb{D}$ correctly: if $M \models \text{"}A \subset \mathbb{D}$ is a maximal antichain" then this is a $\Pi^1_1$ fact about $A$ under suitable coding and therefore $A$ really is a maximal antichain of $\mathbb{D}$. Consequently, the filter $H \cap M \subset \mathbb{D}^M$ is $M$-generic.

Choose an $M$-generic filter $K \subset R$ with $H \cap M \subset K$ under the embedding of $\mathbb{D}$ mentioned above. Let $G = K \cap P$. The filter $G \subset P$ is $M$-generic and has the
desired properties: the function $g$ is eventually dominated by $e$ which is pointwise smaller than $\hat{f}/G$. □

2.1. A model for $\mathfrak{d} = \aleph_1$.

A natural notion of a witness for $\mathfrak{d} = \aleph_1$ to be used in the definition of $P_{\mathfrak{d} = \aleph_1}$ is that of an eventual domination cofinal subset of $\omega^\omega$ of size $\aleph_1$. We like to consider an innocent strengthening of this notion in order to later ensure that assumptions of Corollary 2.5 are satisfied.

**Lemma 2.6.** The following are equivalent.

1. $\mathfrak{d} = \aleph_1$
2. there is a sequence $d : \omega_1 \rightarrow \omega^\omega$ increasing in the eventual domination order such that for every $f \in \omega^\omega$ the set $S_f = \{ \alpha \in \omega_1 : f \leq d(\alpha) \text{ pointwise} \}$ is stationary.

A sequence $d$ as in (2) will be called a *good* dominating sequence and will be used as a witness for $\mathfrak{d} = \aleph_1$.

**Proof.** Only (1)→(2) needs an argument. Choose an arbitrary eventual domination cofinal set $\{f_\alpha : \alpha \in \omega_1\} \subset \omega^\omega$ and a sequence $\langle S_{a,\alpha} : a \in \omega^\omega, \alpha \in \omega_1 \rangle$ of pairwise disjoint stationary subsets of $\omega_1$. By a straightforward induction on $\beta \in \omega_1$ it is easy to build the sequence $d : \omega_1 \rightarrow \omega^\omega$ so that

1. $d(\beta)$ eventually dominates all $f_\alpha : \alpha \in \beta$ and $d(\alpha) : \alpha \in \beta$
2. if $\beta \in S_{a,\alpha}$ then $d(\beta)$ pointwise dominates both $a$ and $f_\alpha$.

The sequence $d$ is as required. For choose $f \in \omega^\omega$. There are $a \in \omega^\omega, \alpha \in \omega_1$ so that $f$ is pointwise dominated by the function $g$ taking maxima of functional values of $a$ and $f_\alpha$. The set $S_f$ is then a superset of $S_{a,\alpha}$ and therefore stationary. □

Suppose now that $M$ is a countable transitive model of ZFC, $M \models \text{"} d : \omega_1 \rightarrow \omega^\omega \text{ is a good dominating sequence and } d \text{ is a Woodin cardinal". Working in } M \text{ a simple observation is that } Q<d, \omega_1^M \models \text{"}(j_Q)(\omega_1^M) \text{-dominates } M\text{"}, \text{ where } j_Q \text{ is the term for the generic nonstationary tower embedding. Moreover, by (2) above the pair } Q<d, (j_Q)(\omega_1^M) \text{ satisfies requirements of Corollary 2.5. Thus there is a generic ultrapower of } M \text{ lifting } (j_Q)(\omega_1^M) \text{ arbitrarily high in the eventual domination order in } V. \text{ Also, whenever } j \text{ is a full iteration of the model } M \text{ such that } j(d) \text{ is a dominating sequence, it is really a good dominating sequence.}

**Optimal Iteration Lemma 2.7.** Suppose $\mathfrak{d} = \aleph_1$. Whenever $M$ is a countable transitive model iterable with respect to its Woodin cardinal $\delta$ and $M \models \text{"} d \text{ is a good dominating sequence"} \text{ there is a full iteration } j \text{ of } M \text{ so that } j(d) \text{ is a good dominating sequence.}

**Proof.** Let $\{ f_\alpha : \alpha \in \omega_1 \}$ be an eventual domination cofinal family of functions. We shall produce a full iteration $j$ of the model $M$ based on $\delta$ with $\theta_\alpha = \omega_1^{M_\alpha}$ such that the function $jd(\theta_\alpha)$ eventually dominates the function $f_\alpha$, for every $\alpha \in \omega_1$. This will prove the lemma.

The iteration $j$ will be constructed by induction on $\alpha \in \omega_1$. First, fix a partition $\{ S_\xi : \xi \in \omega_1 \}$ of $\omega_1$ into pairwise disjoint stationary sets. By induction on $\alpha \in \omega_1$ build models $M_\alpha$ together with the elementary embeddings plus an enumeration $\{ (x_\xi, \beta_\xi) : \xi \in \omega_1 \}$ of all pairs $(x, \beta)$ with $x \in Q_\beta$. The induction hypotheses at $\alpha$ are:

1. the function $jd(\theta_\alpha)$ eventually dominates $f_\alpha$
(2) the initial segment \( \{ (x_\xi, \beta_\xi) : \xi \in \theta_\alpha \} \) of the enumeration under construction has been built and it enumerates all pairs \( \langle x, \beta \rangle \) with \( x \in Q_\beta, \beta \in \alpha \).

(3) for \( \gamma \in \alpha \) if \( \theta_\gamma \in S_\xi \) for some (unique) \( \xi \in \theta_\gamma \) then \( j_{\beta_\gamma, \gamma}(x_\xi) \in G_\gamma \).

The hypothesis (1) ensures that the resulting sequence \( j_d \) will be dominating. The enumeration together with (3) will imply the fullness of the iteration.

At limit stages, the direct limit of the previous models and the union of the enumerations constructed so far is taken. The successor step is handled easily using a version of Corollary 2.5 below the condition \( j_{\beta_\xi, \gamma}(x_\xi) \in Q_\alpha \) if \( \alpha \in S_\xi \) for some \( \xi \in \theta_\alpha \) and the observation just before the formulation of Lemma 2.7. Let \( j \) be the direct limit of the iteration system constructed in the induction process. □

This lemma could have been proved even without subgenericity, since after all, the forcing \( Q \) adds a Hechler real by design. With the sequences of models entering the stage though it is important to have a sort of a uniform term for this real.

**Strategic Iteration Lemma 2.8.** Suppose \( d = \kappa_1 \). The player Good has a winning strategy in the game \( G_{d = \kappa_1} \).

**Proof.** Let \( \vec{N} = \langle d, N_i, \delta_i : i \in \omega \rangle \) be a sequence of models with a good dominating sequence \( d \), let \( y_0 \in Q_\vec{N} \) and let \( f \in \omega^\omega \) be an arbitrary function. We shall show that there is an \( \vec{N} \)-generic filter \( G \subset Q_\vec{N} \) with \( y_0 \in G \) such that \( j_0(d)(\omega^\vec{N}) \) eventually dominates the function \( f \), where \( j_0 \) is the generic ultrapower embedding of the model \( N_0 \) using the filter \( G \cap Q_\vec{N}_{<\delta_0} \). With this fact a winning strategy in the game \( G_{d = \kappa_1} \) for the good player consists of an appropriate bookkeeping using a fixed dominating sequence as in the previous lemma.

First, let us fix some notation. Choose an integer \( i \in \omega \), work in \( N_i \) and set \( Q_i = Q_{N_i}^{<\delta_i} \). Consider the \( Q_i \)-term \( j_i \) for a \( Q_i \)-generic ultrapower embedding of the model \( N_i \). The function \( j_i(d)(\omega^\vec{N}) \) is forced to be represented by the function \( \alpha \mapsto j_i(d)(\alpha) \) and to \( D \)-dominate the model \( N_i \). Applying Corollary 2.4 in \( N_i \) to \( Q_i \) and \( j_i(d)(\omega^\vec{N}) \) it is possible to choose a particular dense subset \( R_i \) of the iteration found in that Corollary, namely \( R_i = \{ \langle y, a, A \rangle : y \in Q_i, \langle a, A \rangle \in D, A \subset \text{rng}(d) \) and for every \( x \in y \) the function \( d(x \cap \omega_1) \) pointwise dominates \( \text{body}(\langle a, A \rangle) \} \) ordered by \( (z, b, B) \leq (y, a, A) \) just in case \( z \leq y \) in \( Q_i \) and \( (b, B) \leq (a, A) \) in \( D \).

It is possible to restrict ourselves to sets \( A \subset \text{rng}(d) \) since \( \text{rng}(d) \) is an eventual domination cofinal family in \( N_i \). As in that corollary, the \( Q_i \)-generic will be read off the first coordinate and the \( D \)-generic real \( e \in \omega^\omega \) will be read off the other two, with \( j_i(d)(\omega_1) \) pointwise dominating the function \( e \). With this embedding of \( D \) into the poset \( R_i \), we can compute the projection \( pr_D(\langle y, a, A \rangle) = \Sigma_D \{ (b, B) \in D : (b, B) \leq (a, A), B \subset \text{rng}(d) \} \) and the system \( z = \{ x \in y : d(x \cap \omega_1) \) pointwise dominates the function \( \text{body}(\langle b, B \rangle) \} \) is stationary.

Now step out of the model \( N_i \). There are two key points, capturing the uniformity of the above definitions in \( i \in \omega \):

1. \( R_0 \subset R_1 \subset \ldots \)
2. \( \text{pr}_D(\langle y, a, A \rangle) \) computes the same value in \( D \) in all models \( N_i \) with \( y \in Q_i \).

Therefore we can write \( \text{pr}_D(\langle y, a, A \rangle) \) to mean the constant value in \( D \) of this expression without any danger of confusion. Another formulation of (2) is that \( N_0 \) computes a function from \( D \) into \( R_0 \) which constitutes a complete embedding of \( D \) into all \( R_i \) in the respective models \( N_i \). Note that \( R_0 \) is not a complete suborder of the \( R_i \)'s.
Now everything is ready to construct the filter \( G \subset Q_N \). First, let us choose a sufficiently generic filter \( H \subset D \). There are the following requirements on \( H \):

1. \( \text{pr}_D(\langle y_0, 0, 0 \rangle) \in H \)

2. \( H \) meets all maximal antichains of \( D \) which happen to be elements of \( \bigcup_i N_i \)

3. the Hechler real \( e \in \omega^\omega \) given by the filter \( H \) eventually dominates the function \( f \in \omega^\omega \).

This is easily arranged. It follows from (4) that \( H \cap N_i \) is a \( N_i \)-generic subset of \( D \) since the model \( N_i \) is \( \Sigma_1^1 \) correct and therefore computes maximal antichains of \( D \) correctly.

Let \( X_k : k \in \omega \) be an enumeration of all maximal antichains of \( Q_N \) which are elements of \( \bigcup_i N_i \). By induction on \( k \in \omega \) build a descending sequence \( y_0 \geq y_1 \geq \ldots \geq y_k \geq \ldots \) of conditions in \( Q_N \) so that

1. \( y_{k+1} \) has an element of \( X_k \) above it

2. \( \text{pr}_D(\langle y_{k+1}, 0, 0 \rangle) \in H \)

This is possible by the genericity of the filter \( H \). Suppose \( y_k \) is given. There is an integer \( i \in \omega \) such that \( y_k \in Q_i \) and \( X_k \in N_i \) is a maximal antichain in \( Q_i \). Now \( H \cap N_i \) is a Hechler \( N_i \)-generic filter and \( X_k = \{ \langle z, 0, 0 \rangle : z \in X_k \} \subset R_i \) is a maximal antichain in \( R_i \). Therefore, there must be a condition \( y_{k+1} \leq y_k \) as required in (6, 7).

In the end, let \( G \subset Q_N \) be the filter generated by the conditions \( y_k : k \in \omega \). It is an \( N \)-generic filter by (6) and the function \( j_0 d(\omega_1^N) \) pointwise dominates the Hechler function \( e \) by (7) and therefore—from (5)—eventually dominates the function \( f \in \omega^\omega \) as desired.

Conclusion 2.11. The sentence \( \phi = \exists = \aleph_1 \) is \( \Pi_2 \)-compact, moreover, in Theorem Scheme 0.2 we can add a predicate for dominating sequences of length \( \omega_1 \) to the language of \( \langle H_{\aleph_2}, \in, \omega_1, 3 \rangle \).

Proof. All the necessary iteration lemmas have been proved. To see that the dominating predicate \( \mathcal{D} \) can be added, go through the proof of Corollary 1.15 again and note that if \( j \) is a full iteration of a countable transitive iterable model \( M \) such that \( j(M \cap R) \) is cofinal in the eventual domination ordering then \( \mathcal{D} \cap M_{\omega_1} = \mathcal{D}^{M_{\omega_1}} \).

3. Other \( \mathcal{D} \)-like cardinal invariants

The behavior of the dominating number seems to be typical for a number of other cardinal invariants. We present here two cases which can be analysed completely. Recall that for an arbitrary ideal, the cofinality of that ideal is defined as the minimal size of a collection of small sets such that any small set is covered by one in that collection. This is an important cardinal characteristic of that ideal [BJ].

3.0. Cofinality of the meager ideal.

In this subsection we prove that the statement “the cofinality of the meager ideal is \( \aleph_1 \)” is \( \Pi_2 \)-compact. It is not difficult to see and will be proved below that it is enough to pay attention to the nowhere dense ideal. As in the previous section, there is a canonical forcing related to this ideal.

Definition 3.1.

1. NWD is the set of all perfect nowhere dense trees on \( \{0, 1\} \).
(2) for a tree $T \in \text{nwd}$ and a finite set $x \subset T$ the tree $T \upharpoonright x$ is defined as the set of all elements of $T \subset$-comparable with some element of $x$

(3) for a tree $S \in \text{nwd}$ and a sequence $\eta \in S$ the tree $S(\eta)$ is the set $\{\tau : \eta \cap \tau \in S\}$.

(4) $\text{um}$, the universal meager forcing [S, Definition 4.2], is the set $\{(n,S) : n \in \omega, S \in \text{nwd}\}$ ordered by $\langle n,S \rangle \leq \langle m,T \rangle$ if $n \geq m, T \subset S$ and $S \cap m^2 = T \cap m^2$.

(5) for a tree $U \in \text{nwd}$ we write $\text{um}(U) = \{(n,S) \in \text{um} : S \subset U\}$; this set has an order on it inherited from $\text{um}$.

Obviously, the collection $\{\langle T \rangle : T \in \text{nwd}\}$ is a base for the nowhere dense ideal. $\text{um}$ is a $\omega$-centered Souslin forcing designed so as to produce a very large nowhere dense tree: if $G \subset \text{um}$ is a generic filter, then this tree is $U_G = \bigcup\{\langle T \rangle : \langle 0,T \rangle \in G\}$; it is nowhere dense and it codes the generic filter. The following is the instrumental weakening of genericity:

**Definition 3.2.** Let $M$ be a transitive model of ZFC and $U \in \text{nwd}$. We say that the tree $U \text{um}$-dominates the model $M$ if there is some element $T \in M \cap \text{nwd}$ included in $U$ and for every $T \in M \cap \text{nwd}$ there is an integer $n \in \omega$ such that setting $x = T \cap U \cap n^2$, the inclusion $T \upharpoonright x \subset U \upharpoonright x$ holds.

This notion has certain obvious monotonicity properties. Suppose $S \subset T$ and $U$ are perfect nowhere dense trees and $n$ is an integer such that setting $x = T \cap U \cap n^2$, $T \upharpoonright x \subset U \upharpoonright x$ holds. Then with $y = S \cap U \cap n^2$ we have $S \upharpoonright y \subset U \upharpoonright y$ and for any integer $m > n$ and $z = T \cap U \cap m^2$ we have $T \upharpoonright z \subset U \upharpoonright z$.

It is immediate that if $U \in \text{nwd}$ is $\text{um}$-generic then it $\text{um}$-dominates the ground model. On the other hand, any $\text{um}$-dominating tree covers an $\text{um}$-generic tree:

**Lemma 3.3.** Let $M$ be a transitive model of ZFC and let $U \in \text{nwd}$ $\text{um}$-dominate the model $M$. If $D \subset \text{um} \cap M$ is a dense set in $M$ then $D \cap \text{um}(U)$ is dense in $\text{um}(U) \cap M$.

**Proof.** First, the set $\text{um}(U) \cap M$ is nonempty. Now let $\langle n,S \rangle \in \text{um}(U) \cap M$ and let $D \in M$ be a dense subset of $\text{um} \cap M$ which is an element of the model $M$. We shall produce a condition $p \in D \cap \text{um}(U), p \leq \langle n,S \rangle$, proving the lemma.

Work in $M$. By induction on $i \in \omega$, build conditions $\langle n_i,T_i \rangle, p_{x,i} \in \text{um}$ so that:

1. $\langle 0,T_0 \rangle = \langle n,S \rangle$ and $\langle n_{i+1},T_{i+1} \rangle \leq \langle n_i,T_i \rangle$
2. for every integer $i > 0$, for every sequence $\eta \in n^2$ there is a sequence $\tau \in n^2$ with $\eta \subset \tau$ and $\tau \notin T_i$
3. to produce $\langle n_{i+1},T_{i+1} \rangle$ from $\langle n_i,T_i \rangle$, for every nonempty set $x \subset n^2 \cap T_i$ find a condition $p_{x,i} = \langle n_{i,x},S_{x,i} \rangle \leq \langle n_i,T_i \upharpoonright x \rangle$ in the dense set $D$. Set $T_{i+1} = \bigcup S_{x,i}$ and let $n_{i+1}$ be arbitrary so that (2) is satisfied.

After this is done, let $T_\omega = \bigcup T_i$. The induction hypothesis (2) implies that $T_\omega \in \text{nwd} \cap M$ and therefore, there is an integer $i \in \omega$ such that setting $x = n^2 \cap T \cap U$ we get $T_\omega \upharpoonright x \subset U \upharpoonright x$. Note that the set $x$ is nonempty, because it includes $S \cap n^2$. Now $p_{x,i}$ is the desired condition. □

**Corollary 3.4.** (Subgenericity) Let $P$ be a forcing and $\dot{S}$ a $P$-name such that

1. $P \Vdash \text{the tree } \dot{S} \text{ um-dominates the ground model}$
2. for every $T \in \text{nwd}$ the boolean value $\|\dot{T} \subset \dot{S}\|_P$ is non-zero.
Then there is a complete embedding \( RO(\mathbb{U}M) \prec P^*(\mathbb{U}M(\dot{S}) \cap \text{the ground model}) = R \) such that \( R \models \langle \dot{U} \subset \dot{S} \rangle \), where \( \dot{U} \) is the \( \mathbb{U}M \)-generic tree.

**Corollary 3.5.** Let \( M \) be a countable transitive model of ZFC such that \( M \models \text{"}P \) is a forcing adding a \( \mathbb{U}M \)-dominating tree \( \dot{S} \) as in Corollary 3.4\textquotedblright, let \( p \in P \) and let \( T \in \text{NWD} \) be any tree, not necessarily in the model \( M \). Then there is an \( M \)-generic filter \( G \subset P \) containing \( p \) such that for some sequence \( \eta \in \dot{S}/G \) we have \( T \subset \dot{S}/G(\eta) \).

Set \( \phi = \text{"}cofinality of the meager ideal is } \aleph_1 \text{"} \). The analysis of the forcing \( P_\phi \) is now completely parallel to the treatment in Section 2.

**Definition 3.6.** A witness for \( \phi \) is an \( \omega_1 \)-sequence \( s \) of perfect nowhere dense trees such that

1. for every NWD tree \( T \) the set \( \{ \alpha \in \omega_1 : T \subset s(\alpha) \} \subset \omega_1 \) is stationary
2. for every NWD tree \( T \) the set \( C_T = \{ \alpha \in \omega_1 : \text{there is } n \in \omega \text{ such that if } x = n^2 \cap T \cap s(\alpha) \text{ then } T \restriction x \subset s(\alpha) \restriction x \subset \omega_1 \text{ contains a club} \)
3. there is a NWD tree \( T \) which is contained in all \( s(\alpha) : \alpha \in \omega_1 \).

Of course, it is important to verify that this notion deserves its name.

**Lemma 3.7.** The following are equivalent:

1. the cofinality of the meager ideal is \( \aleph_1 \)
2. the cofinality of the nowhere dense ideal is \( \aleph_1 \)
3. there is a witness for \( \phi \).

**Proof.** (1)\(\rightarrow\)(2): let \( \{ Y_\alpha : \alpha \in \omega_1 \} \) be a base for the meager ideal, \( Y_\alpha \subset \bigcup[\{ T^i_\alpha \} : i \in \omega] \) for some sequence \( T^i_\alpha : i \in \omega \) of NWD trees with \( T^i_\alpha \subset T^{i+1}_\alpha \). We shall show that the collection \( \{ [T^i_\alpha(\eta)] : \alpha \in \omega_1, i \in \omega, \eta \in T^i_\alpha \} \) is a base for the nowhere dense ideal, proving the lemma. Indeed, let \( S \) be a nowhere dense tree on \( \omega \). We shall produce \( \alpha, i, \eta \) so that \( S \subset T^i_\alpha(\eta) \) and therefore \( [S] \subset [T^i_\alpha(\eta)] \).

It is a matter of an easy surgery on the tree \( S \) to obtain a nowhere dense tree \( \dot{S} \) so that for every \( \eta \in \dot{S} \) there is \( \tau \in \dot{S} \) with \( \eta \subset \tau \) and \( \dot{S}(\tau) = S \). Choose a countable ordinal \( \alpha \) so that \( [\dot{S}] \subset Y_\alpha \) and attempt to build a descending sequence \( \eta_i : i \in \omega \) of elements of \( \dot{S} \) so that \( \eta_i \notin T^i_\alpha \). There must be an integer \( i \in \omega \) such that the construction cannot proceed past \( \eta_i \)–otherwise the branch \( \bigcup_{i \in \omega} \eta_i \in \dot{S} \) would lie outside of the set \( Y_\alpha \). But then, \( \dot{S}(\eta_i) \subset T^i_\alpha(\eta_i) \) and if \( \tau \in \dot{S} \) is such that \( \eta_i \subset \tau \) and \( \dot{S}(\tau) = S \) then necessarily \( S = \dot{S}(\tau) \subset T^i_\alpha(\tau) \).

(2)\(\rightarrow\)(3): let \( T_\xi : \xi \in \omega_1 \) be a \( \subset \)-cofinal family of NWD trees. Fix a partition \( S_\xi : \xi \in \omega_1 \) into disjoint stationary sets and a NWD tree \( T \). For each \( \alpha \in \omega_1 \) choose a filter \( G_\alpha \subset \mathbb{U}M \) such that

1. \( \langle 0, T \cup T_\xi \rangle \in G_\alpha \) whenever \( \alpha \in S_\xi \)
2. \( G_\alpha \) meets the dense sets \( D_\beta = \{ \langle n, U \rangle \in \mathbb{U}M : \text{setting } x = U \cap n^2 \text{ we have } T_\beta \restriction x \subset U \restriction x \text{ for all } \beta \in \alpha \} \).

Using the remarks after Definition 3.2 it is easy to see that the sequence \( s : \omega_1 \to \text{NWD} \) defined by \( s(\alpha) = \bigcup \{ U : \langle 0, U \rangle \in G_\alpha \} \) is the desired witness for \( \phi \).

(3)\(\rightarrow\)(1) let \( s : \omega_1 \to \text{NWD} \) be a witness for \( \phi \). Obviously, the family \( Y_\alpha = \bigcup_{\beta \in \alpha} [s(\beta)] : \alpha \in \omega_1 \) is \( \subset \)-cofinal in the nowhere dense ideal. \( \square \)

Now if \( M \) is a transitive model of ZFC with \( M \models \text{"}s \text{ is a witness for } \phi \text{ and } \delta \text{ is a Woodin cardinal} \text{"} \) then in \( M, Q_{<\delta} \models \text{"}js(\omega_1) \text{ is a NWD tree } \mathbb{U}M \text{-covering the} \).
model $M^\pi$, where $j$ is the term for the generic nonstationary tower embedding; also $M, Q_{\leq \delta}, j\phi(\omega_1)$ satisfy the assumptions of Corollary 3.5. The proof of $\Pi_2$-compactness of $\phi$ translates now literally from the previous section. We prove the strategic iteration lemma from optimal assumptions.

**Strategic Iteration Lemma 3.8.** Suppose that the cofinality of the meager ideal is equal to $\aleph_1$. The good player has a winning strategy in the game $G_\phi$.

**Proof.** Let $\vec{N} = \langle s, N_i, \delta_i : i \in \omega \rangle$ be a sequence of models with a witness for $\phi$, let $y_0 \in Q_{\vec{N}}$ and let $T$ be an arbitrary NWD tree. We shall show that there is an $\vec{N}$-generic filter $G \subseteq Q_{\vec{N}}$ such that letting $S = j_G s(\vec{N})$, where $j_G$ is the $Q_{\leq \delta_0}$-generic ultrapower embedding using the filter $G \cap Q_{\leq \delta_0}$, we have that for some $\eta \in S, T \subseteq S(\eta)$. With this fact in hand, the winning strategy for the good player consists just from an appropriate bookkeeping:

Since cofinality of the meager ideal is $\aleph_1$, it is possible to choose a $\subset$-cofinal family $T_\alpha : \alpha \in \omega_1$ of NWD trees. So the good player can easily play the game so that with the resulting embedding $j$ of the initial iterable model $M$, for every $\alpha \in \omega_1$ there is $\gamma \in \omega_1$ and a sequence $\eta$ in the tree $j_S(\gamma)$ such that $T_\alpha \subseteq j_S(\gamma)(\eta)$. It is immediate that if this is the case and the iteration $j$ is full, the sequence $j(s)$ is a witness for $\phi$ and the good player won the run of the game. For let $S \in \text{NWD}$ be an arbitrary tree. Then there are $\alpha, \gamma$ and $\eta$ such that $S \subseteq T_\alpha \subseteq j_S(\gamma)(\eta)$ and so

1. the set $\{ \beta \in \omega_1 : S \subseteq j_S(\beta) \}$ contains the set $\{ \beta \in \omega_1 : j_S(\gamma)(\eta) \subseteq j_S(\beta) \}$ which is in the target model of the iteration $j$, is stationary there from Definition 3.6(1) and so is stationary in $V$ by the fullness of the iteration
2. the set $\{ \beta \in \omega_1 : \text{for some } n \in \omega, S \models x \in j_S(\beta) \mid x \text{ holds with } x = n^2 \cap S \cap j_S(\beta) \}$ contains the set $\{ \beta \in \omega_1 : \text{for some } n \in \omega, j_S(\gamma)(\eta) \mid x \in j_S(\beta) \mid x \text{ holds with } x = n^2 \cap j_S(\gamma)(\eta) \cap j_S(\beta) \}$, which is in the target model of the iteration $j$ and contains a club by Definition 3.6(2).

Therefore Definition 3.6(1,2) are verified for $j(s)$ and (3) of that definition follows from elementarity of the embedding $j$. Thus $j_S$ is a witness for $\phi$ as desired.

The proof of the local fact about the sequence of models carries over from Lemma 2.8 with the following changes:

1. $d$ is replaced with $s$, $D$ is replaced with $\cup M$, the Hechler real $e$ is replaced with a NWD tree $U$
2. the step (5) of that proof is replaced with: there is a sequence $\eta \in U$ such that $T \subseteq U(\eta)$. Note that the set $\{ (n, S) \in \cup M : \exists \eta \in S, T \subseteq S(\eta) \}$ is dense in $\cup M$.
3. the ordering $R_i$ is defined as follows: $R_i = \{ (y, n, S) : y \in Q_i, (n, S) \in \cup M, S = s(\alpha) \mid z \text{ for some } \alpha \in \omega_1^N \text{ and finite set } z \text{ such that } \forall x \in y S \subseteq s(x \cap \omega_1) \}$. It is possible to restrict ourselves to the trees $S$ of the above form, since the sequence $s(\alpha) : \alpha \in \omega_1$ is $\subset$-cofinal in $\text{NWD} \cap N_i$.

□

**Conclusion 3.9.** The sentence $\phi = \text{cofinality of the meager ideal} = \aleph_1$ is $\Pi_2$-compact. Theorem Scheme 0.2 holds even with an extra predicate for $\omega_1$-sequences of meager sets cofinal in the ideal.
3.1. Cofinality of the null ideal.
In this subsection we shall show that “cofinality of the null ideal = \aleph_1” is a 
\Pi_2-compact statement. The following textbook equality will be used:

**Lemma 3.10.** Cofinality of the null ideal is equal to the cofinality of the poset of 
the open subsets of reals of finite measure ordered by inclusion.

Therefore we will really care about large open sets of finite measure.

**Definition 3.11.** The amoeba forcing \(\mathbb{A}\) is the set \(\{(O, \epsilon) : O \text{ is an open set of } \langle\mathbb{R}\rangle, \epsilon \text{ is a positive rational greater than } \mu(O)\}\) ordered by \(\langle\mathbb{P}, \delta\rangle \leq \langle\mathbb{P}, \epsilon\rangle\) if \(\mathbb{P} \subset O\) and \(\epsilon \leq \delta\). The restricted poset \(\mathbb{A}(O)\) for an open set \(O \subset \mathbb{R}\) is 
\(\{(\mathbb{P}, \epsilon) : A : \mathbb{P} \subset O\}\) with the inherited ordering.

It is not a priori clear why the different versions of the amoeba forcing should 
be isomorphic, see [Tr]. The amoeba poset is a \(\sigma\)-linked Souslin forcing notion 
designed to add a “large” open set of finite measure. If \(G \subset \mathbb{A}\) is generic then 
the set \(O_G = \bigcup\{\mathbb{P} : (\mathbb{P}, \epsilon) \in G\text{ for some } \epsilon\}\) is this open set and it determines the 
generic filter. Again, there is a natural weakening of the notion of genericity. Fix 
one and for all a sequence \(f_i : i \in \omega\) of measure-preserving functions from \(\mathbb{R}\) to \(\mathbb{R}\) 
so that the sets \(f_i/\mathbb{R}\) are pairwise disjoint and the sequence is arithmetical.

**Definition 3.12.** Let \(M\) be a transitive model of \(ZFC\) and \(O\) be an open set of 
reals. We say that \(\mathbb{A}\)-dominates the model \(M\) if for every open set \(\mathbb{P}\) of finite 
measure in the model \(M\) for all but finitely many integers \(m \in \omega\), \(f_m \mathbb{P} \subset O\).

Obviously, the amoeba generic open set does \(\mathbb{A}\)-dominate the ground model. We 
aim for the subgenericity theorems.

**Lemma 3.13.** Let \(M\) be a transitive model of \(ZFC\) and \(O\) dominate \(M\). For 
every dense set \(D \subset \mathbb{A} \cap M\) which is in the model \(M\) the set \(D \cap \mathbb{A}(O)\) is dense in 
\(M \cap \mathbb{A}(O)\).

**Proof.** Let \(M, O, D\) be as in the lemma and let \(\langle\mathbb{P}, \epsilon\rangle \in M \cap \mathbb{A}(O)\). We shall produce 
a condition \(p \in D \cap \mathbb{A}(O)\) below \(\langle\mathbb{P}, \epsilon\rangle\), proving the lemma.

Work in the model \(M\). By induction on \(i \in \omega\) build conditions \(\langle R_i, \delta_i \rangle \leq \langle\mathbb{P}, \epsilon_i\rangle\) 
so that:

1. \(\epsilon = \epsilon_0, \langle R_{i+1}, \delta_{i+1} \rangle \in D\)
2. for every integer \(i > 0\) the inequality \(\epsilon_i - \mu(\mathbb{P}) < 2^{-i}\) holds.

Let \(S \in M\) be any open set of finite measure which covers the set \(\bigcup_{i \in \omega} f_i(R_i \setminus \mathbb{P})\). 
Since \(O\)-\(\mathbb{A}\)-dominates the model \(M\), there must be an integer \(i \in \omega\) such that 
\(f_i^{-1}S \subset O\), and so \(R_i \subset O\). Then \(\langle R_i, \delta_i \rangle \leq \langle\mathbb{P}, \epsilon_i\rangle\) is the desired condition. \(\square\)

**Corollary 3.14.** (Subgenericity) Let \(\mathbb{P}\) be a forcing and \(\mathbb{O}\) a \(\mathbb{P}\)-name such that

1. \(\mathbb{P} \models \text{“the open set } \mathbb{O} \subset \mathbb{R} \text{ \(\mathbb{A}\)-dominates the ground model”}\)
2. for every open set \(\mathbb{P}\) of finite measure the boolean value \(\|\mathbb{P} \subset \mathbb{O}\|_{\mathbb{P}}\) is 
   nonzero.

Then there is a complete embedding \(RO(\mathbb{A}) \preceq \mathbb{P}*(\mathbb{A}(\mathbb{O}) \cap \text{the ground model}) = \mathbb{R}\) 
such that \(\mathbb{R} \models \text{“}\mathbb{R} \subset \mathbb{O}\”\), where \(\mathbb{R}\) is the name for the \(A\)-generic open set.

A witness for \(\phi = \text{“cofinality of the null ideal } = \aleph_1”\) is an \(\omega_1\)-sequence \(\alpha\) of open 
sets of finite measure such that

1. for every open set \(\mathbb{P}\) of finite measure the set \(\{\alpha \in \omega_1 : \mathbb{P} \subset o(\alpha)\} \subset \omega_1\) is 
   stationary.
(2) for every open set \( P \) of finite measure the set \( \{ \alpha \in \omega_1 : \text{for all but finitely many integers } m \in \omega \ f_m^{-1} P \subset o(\alpha) \} \) contains a club.

Again, it is very simple to prove using Lemma 3.10 that \( \phi \) is equivalent with the existence of a witness. The analysis of the forcing \( P_{\phi} \) almost literally translates from Section 2. We leave all of this to the reader.

**Conclusion 3.15.** The statement \( \phi = \text{“cofinality of the null ideal is } \aleph_1\) is \( \Pi_2 \)-compact. Theorem scheme 0.2 holds even with a predicate for cofinal families of null sets added to the language of \( \langle H_{\aleph_2}, \in, \omega_1, \mathcal{F} \rangle \).

4. **Souslin Trees**

The assertion “there is a Souslin tree” does not seem to be \( \Pi_2 \)-compact as outlined in Subsection 1.3; however, some of its variations are. A \( P_{\text{max}} \)-style model in which many Souslin trees exist was constructed in [W2] and in the course of the argument the following theorem, which implies the strategic iteration lemmas for all sentences considered in this section, was proved.

Let \( G_S \) be a two-person game played along the lines of \( G_{\phi} \)–defined in 1.12–with the following modifications:

1. in the 0-th move the player Bad specifies a collection \( S \) of Souslin trees in the model \( M \) instead of just one witness
2. in the \( \alpha \)-th step the player Bad must choose a sequence \( \langle N_i : i \in \omega \rangle \) of models such that \( j_{\gamma_\alpha}(S) \) consists of Souslin trees as seen from each \( N_i : i \in \omega \)
3. the player Good wins if \( j_{\omega_1}(S) \) is a collection of Souslin trees.

**Strategic Iteration Lemma 4.1.** [W2] Assume \( \diamondsuit \). Then the player Good has a winning strategy in the game \( G_S \).

**Proof.** Recall that \( \omega_1 \)-trees are by our convention sets of functions from countable ordinals to \( \omega \) with some special properties. Fix a diamond sequence \( \langle A_\beta : \beta \in \omega_1 \rangle \) guessing uncountable subsets of such trees. The player Good wins the game as follows. Suppose we are at \( \alpha \)-th stage of the play and let \( \beta = \omega_1^{M_\gamma} \) and \( S_\alpha = j_{\gamma, M_\gamma}(S) \). Suppose Bad played a sequence \( \bar{N} = \langle N_i, \delta_i : i \in \omega \rangle \) of models according to the rules–so \( N_0 = M_\gamma \) and some \( p \in Q_{\bar{N}} \). Let us call an \( \bar{N} \)-generic filter \( G \subset Q_{\bar{N}} \) good if setting \( j_0 \) to be the ultrapower embedding of \( N_0 \) derived from \( G \cap Q_0 \) we have: for every tree \( S \in S_\alpha \), if \( A_\beta \subset S \) is a maximal antichain then every node at \( \beta \)-th level of \( j_0(S) \) has an element of \( A_\beta \) above it in the tree ordering.

If the player Good succeeds in playing a good filter containing \( p \) at each stage \( \alpha \in \omega_1 \) of the game then he wins: every tree in the collection \( j_{\omega_1}(S) \) can then be shown Souslin by the usual diamond argument. Thus the following claim completes the proof.

**Claim 4.2.** At stage \( \alpha \) there is a good filter \( G \subset Q_{\bar{N}} \) containing \( p \).

**Proof.** Actually, any sufficiently generic filter is good. Note that every \( Q_0 \) name \( \dot{y} \in N_0 \) for a cofinal branch of any tree \( S \in S_\alpha \) is in fact a \( Q_{\bar{N}} \)-name for a generic subset of \( S \)–this follows from the fact that \( S \) is a Souslin tree in every model \( N_i : i \in \omega \). Thus if a filter \( G \subset Q_{\bar{N}} \) meets every dense set recursive in some fixed real coding \( \bar{N} \) and \( A_\beta \), necessarily the branch \( \dot{y}/G \) meets the set \( A_\beta \) if \( A_\beta \subset S \) is a maximal antichain. Consequently, such a filter is good, since every \( Q_0 \) name
4.0. Free Souslin trees.

The first $\Pi_2$-compact sentence considered in this section is $\phi =$ “there is a free tree” as clarified in the following definition:

**Definition 4.3.** A Souslin tree $S$ is free if for every finite collection $s_i : i \in I$ of distinct elements of the same level of $S$ the forcing $\prod_{i \in I} S \upharpoonright s_i$ is c.c.c.

Thus every finitely many pairwise distinct cofinal branches of a free tree are mutually generic. It is not difficult to prove that both of the classical methods for forcing a Souslin tree $[Te, J1]$ in fact provide free trees. It is an open problem whether existence of Souslin trees implies existence of free trees.

The following observation, pointed out to us by W. Hugh Woodin, greatly simplifies the proof of the optimal iteration lemma for $\phi$ : any sufficiently rich (external) collection of cofinal branches of a free tree determines a symmetric extension of the universe in the appropriate sense.

**Lemma 4.4.** Suppose that $M$ is a countable transitive model of a rich fragment of ZFC, $M \models "S$ is a free Souslin tree” and $B = \{b_i : i \in I\}$ is a countable collection of cofinal branches of $S$ such that $\bigcup B = S$. Then there is an enumeration $b_j : j \in \omega$ of $B$ such that the equations $b_j = c_j$ determine an $M$-generic filter on $P_S$.

Here, $P_S \in M$ is the finite support product of countably many copies of the tree $S$, with $c_j : j \in \omega$ being the canonical $P_S$-names for the added $\omega$ branches of $S$.

**Corollary 4.5.** Suppose $M, S$ and $B$ are as in the Lemma and suppose that $M \models "P$ is a forcing, $p \in P$ and $P \models \dot{C}$ is a collection of cofinal branches of the tree $S$ such that $\bigcup \dot{C} = \dot{S}"$. Then there is an $M$-generic filter $G \subset P$ with $p \in G$ and $C/G = B$.

**Proof.** Work in $M$. Without loss of generality we may assume that $p = 1$ and that the forcing $P$ collapses both $\kappa = (2^{\aleph_1})^+$ and $|\dot{C}|$ to $\aleph_0$. (Otherwise switch to $P \times Coll(\omega, \lambda)$ for some large enough ordinal $\lambda$.) There is a complete embedding of $RO(P_S)$ into $RO(P)$ such that $P \models \dot{C}$ is the canonical set of branches of $S$ added by $P_S$ under this embedding”. This follows from Lemma 4.4 applied in $M^P$ to $M \cap H_\kappa, S$ and $\dot{C}$. Another application of the Lemma to $M, S$ and $B$ gives an $M$-generic filter $H \subset P_S$ such that $B$ is the canonical set of branches of the tree $S$ added by $H$. Obviously, any $M$-generic filter $G \subset P$ extending $H$ via the above mentioned embedding is as desired.

**Proof of Lemma.** Say that the conditions in $P_S$ have the form of functions from some $n \in \omega$ to $S$ with the natural ordering. We shall show that for each injective $f : n \to B$ and every open dense set $O \subset P_S$ in the model $M$ there is an injection $g : m \to B$ extending $f$ and a condition $p \in O$ with $\text{dom}(p) = m$ and $\bigwedge_{k \in m} p(k) \in g(k)$. Granted that, a construction of the desired enumeration is straightforward by the obvious bookkeeping argument using the countability of both $M$ and $B$.

So fix $f$ and $O$ as above. There is an ordinal $\alpha \in \omega^M_\omega$ such that the branches $f(k) : k \in n$ pick pairwise distinct elements $s_k : k \in n$ from $\alpha$-th level of the tree $S$. Let $D = \{z \in \prod_{k \in n} S \mid s_k : \exists p \in O \text{ p | n = z} \in M \}$. Since $O \subset P_S$ is dense below the condition $(s_k : k \in n) \in P_S$, the set $D$ must be dense in $\prod_{k \in n} S \upharpoonright s_k$. Since this product is c.c.c. in the model $M$, the branches $f(k) : k \in n$ determine an
Optimal Iteration Lemma 4.6. Assume there is a free tree. Whenever $M$ is a countable transitive model of ZFC iterable with respect to its Woodin cardinal $\delta$ and $M \models "U$ is a free tree” there is a full iteration $j$ of $M$ such that $j(U)$ is a free tree.

Proof. Let $T$ be a free Souslin tree and let $M, U, \delta$ be as above; so $M \models "U$ is a free Souslin tree.” We shall produce a full iteration $j$ of $M$ such that there is a club $C \subset \omega_1$ and an isomorphism $\pi : T \restriction C \rightarrow j(U) \restriction C$. Then, since the trees $T, j(U)$ are isomorphic on a club, necessarily $j(U)$ is a free Souslin tree. This will finish the proof of the lemma.

The iteration will be constructed by induction on $\alpha \in \omega_1$ and we will have $\theta_\alpha = \omega_1^{M_\alpha}$ and $C = \{ \theta_\alpha : \alpha \in \omega_1 \}$. Also, we shall write $U_\alpha$ for the image of the tree $U$ under the embedding $j_{0, \alpha}$. This is not to be confused with the $\alpha$-th level of the tree $U$. In this proof, levels of trees are never indexed by the letter $\alpha$.

First, fix a partition $\{ S_\xi : \xi \in \omega_1 \}$ of the set of countable limit ordinals into disjoint stationary sets. By induction on $\alpha \in \omega_1$, build the models together with the elementary embeddings, plus an isomorphism $\pi : T \restriction C \rightarrow jU \restriction C$, plus an enumeration $\{ \langle x_\xi, \beta_\xi : \xi \in \omega_1 \}$ of all pairs $\langle x, \beta \rangle$ with $x \in Q_\beta$. The induction hypotheses at $\alpha \in \omega_1$ are:

1. the function $\pi \restriction T \restriction \{ \theta_\gamma : \gamma \in \alpha \}$ has been defined and it is an isomorphism of $T \restriction \{ \theta_\gamma : \gamma \in \alpha \}$ and $U_\alpha \restriction \{ \theta_\gamma : \gamma \in \alpha \}$
2. the initial segment $\{ \langle x_\xi, \beta_\xi : \xi \in \theta_\alpha \}$ has been constructed and every pair $\langle x, \beta \rangle$ with $\beta \in \alpha$ and $x \in Q_\beta$ appears on it
3. if $\gamma \in \alpha$ belongs to some–unique–set $S_\xi$ then $j_{\beta_\xi, \gamma}(x_\xi) \in G_\gamma$.

At limit steps, we just take direct limits and unions. At successor steps, given $M_\alpha, U_\alpha, \delta_\alpha$ and $\pi \restriction \{ \theta_\gamma : \gamma \in \alpha \}$, we must provide an $M_\alpha$-generic filter $G_\alpha \subset Q_\alpha$ such that setting $U_{\alpha+1} = j_0 U_\alpha$, where $j_0$ is the generic ultrapower of $M_\alpha$ by $G_\alpha$, it is possible to extend the isomorphism $\pi$ to $\theta_\alpha$-th levels of $T$ and $U_{\alpha+1}$.

First suppose $\alpha$ is a successor ordinal, $\alpha = \beta + 1$. Let $G_\alpha$ be an arbitrary $M_\beta$-generic filter; we claim that $G_\alpha$ works. Simply let for every $t \in T_{\delta_\beta} \pi \restriction (T \restriction t)_{\delta_\beta}$ to be a bijection of $(T \restriction t)_{\delta_\beta}$ and $(U_{\alpha+1} \restriction \pi(t))_{\delta_\beta}$. This is clearly possible since both of these sets are infinite and countable. Induction hypothesis (1) continues to hold, induction hypothesis (2) is easily arranged by extending the enumeration properly and (3) does not say anything about successor ordinals.

Finally, suppose $\alpha$ is a limit ordinal. In this case, the $\theta_\alpha$-th level of the tree $U_{\alpha+1}$ is determined by $\pi \restriction T \restriction \{ \theta_\gamma : \gamma \in \alpha \}$ and the necessity of extending the isomorphism $\pi$ to the $\theta_\alpha$-th level of the tree $T$. Namely we must have $\theta_\alpha$-th level of $U_{\alpha+1}$ equal to the set $D = \{ d_t : t \in T_{\delta_\alpha} \}$ where $d_t = \bigcup \{ \pi(r) : r \in T \restriction \{ \theta_\gamma : \gamma \in \alpha \}, t \subseteq r \}$. Corollary 4.5 applied to $M_\alpha, S_\alpha, Q_\alpha, (S_{\alpha+1})_{\theta_\alpha}$ and $D$ shows that an appropriate generic filter on $Q_\alpha$ can be found containing the condition $j_{\beta_\xi, \alpha}(x_\xi)$ if $\alpha \in S_\xi$. The isomorphism $\pi$ then extends in the obvious unique fashion mapping $t$ to $d_t$. □

Conclusion 4.7. The sentence $\phi = "there is a free tree”$ is $\Pi_2$-compact.
Another corollary to the proof of Lemma 4.6 is the fact that $\Sigma^1_1$ theory of free trees is complete and minimal in the following sense. Suppose $\psi$ is a $\Sigma^1_1$ property of trees $T$ which depends only on the Boolean algebra $RO(T)$, that is, $ZFC^+\vdash RO(S) = RO(T)$ implies $T \models \psi$ iff $S \models \psi$. Then, granted large cardinals, the sentence $\psi$ is either true on all free trees in all set-generic extensions of the universe or it fails on all such trees. Moreover, if $\psi$ fails on any $\omega_1$-tree in any set-generic extension then it fails on all free trees. Here, by $\Sigma^1_1$ property we mean a formula of the form $\exists A \subset T \chi$, where all quantifiers of $\chi$ range over the elements of $T$ only.

It should be noted that it is impossible to add a predicate $\mathcal{G}$ for free trees to the language of $\langle H_{\aleph_2}, \in, \omega_1, 3 \rangle$ and preserve the compactness result. For consider the following two $\Pi_2$ sentences for $\langle H_{\aleph_2}, \in, 3, \mathcal{G} \rangle$:

\[
\psi_0 = \forall S, T \in \mathcal{G} \exists s \in S, t \in T \; S \upharpoonright s \times T \upharpoonright t \text{ is c.c.c.}
\]

\[
\psi_1 = \text{ for every Aronszajn tree } T \text{ there is a tree } S \in \mathcal{G} \text{ such that } T \Vdash S \text{ is special.}
\]

It is immediate that $\psi_0$ and $\psi_1$ together imply that $\mathcal{G}$ is empty, i.e. $\neg \phi$. Meanwhile, $\psi_0 \land \phi$ was found consistent in $[AS]$–and in fact holds in our model–and $\psi_1 \land \phi$ holds after adding $\aleph_2$ Cohen reals to any model of GCH, owing to the following lemma:

**Lemma 4.8.** For every Aronszajn tree $T$, $C_{\aleph_1} \vDash \text{“there is a free tree which is specialized after forcing with } T\text{”}.$

Note that Cohen algebras preserve Souslin trees.

**Proof.** Let $T$ be an Aronszajn tree. Define a forcing $P$ as a set of pairs $p = \langle s_p, f_p \rangle$ where

1. $s_p$ is a finite tree on $\omega_1 \times \omega$ such that $\langle \alpha, n \rangle <_{s_p} \langle \beta, m \rangle$ implies $\beta < \alpha$...this is a finite piece of the tree $S$ under construction
2. $f_p$ is a finite function with domain contained in $T$ and each $f_p(t)$ a function from $\text{dom}(s) \cap \{\alpha \} \times \omega$ to $\omega$ where $\alpha = \text{lev}(t)$...this is a finite piece of the $S$-specializing $T$-name
3. For every $i <_{s_p} j$ and $t <_{T} u$ the inequality $f_p(i)(t) \neq f_p(u)(j)$ holds, if the relevant terms are defined...this is the specializing condition.

The ordering is defined by $q \leq p$ if $\text{dom}(s_p) \subset \text{dom}(s_q)$ and $s_q \cap \text{dom}(s_p) \times \text{dom}(s_p) = s_p$ and $f_q(t) \subset f_q(t)$ whenever $t \in \text{dom}(f_p)$.

Let $G \subset P$ be a generic filter and in $V[G]$ define a tree $S$ on $\omega_1 \times \omega$ as the unique tree extending all $s_p : p \in G$, and a function $\tau$ on the tree $T$ to be $\tau(t) = \bigcup_{p \in G} f_p(t)$. Obviously, $\tau$ represents a $T$-name for a specializing function on $S$ : if $b \subset T$ is a cofinal branch then the function $g : S \rightarrow \omega$, $g = \bigcup_{t \in b} \tau(t)$ specializes the tree $S$ due to the condition (3) in the definition of the forcing $P$. To complete the proof, we have to verify that $RO(P) = C_{\aleph_1}$ and that $P \Vdash S$ is a free Souslin tree. This is done in the following two claims.

**Claim 4.9.** $RO(P)$ is isomorphic to $C_{\aleph_1}$.

**Proof.** Obviously, $P$ has uniform density $\aleph_1$, therefore it is enough to prove that $P$ has a closed unbounded collection of regular subposets $[K]$. Let $\alpha \in \omega_1$ be a limit ordinal and let $P_\alpha = \{ p \in P : \text{dom}(s_p) \subset \alpha \times \omega \}$. It is easy to verify that all $P_\alpha$’s are regular subposets of $P$ and that they constitute an increasing continuous chain exhausting all of $P$, proving the lemma. $\Box$
Claim 4.10. $P \models S$ is a free Souslin tree.

Proof. Assume that $p \models " \dot{A} = \{ a_\alpha : \alpha \in \omega_1 \} "$ is a family of pairwise distinct elements of $\dot{S} \restriction i_0 \times \dot{S} \restriction i_1 \times \cdots \times \dot{S} \restriction i_n$ for some integer $n$ and pairwise incompatible elements $i_0 \ldots i_n$ of $\text{dom}(s_p)$. To prove the lemma, it is enough to produce a condition $q \leq p$ and ordinals $\alpha < \beta$ such that $q \models \dot{a}_\alpha$ and $\dot{a}_\beta$ are compatible.

Pick $p_\alpha, a_\alpha : \alpha \in \omega_1$ such that each $p_\alpha$ is a condition stronger than $p$ and it decides the value of the name $\dot{\alpha}$ of $\dot{\alpha}$.

By a repeated use of counting arguments and a $\Delta$-system lemma, thinning out the collection of $p_\alpha, a_\alpha$'s we may assume that

1. $\text{dom}(s_p)$ form a $\Delta$-system with root $r$ and $s_p \restriction r \times r$ is the same for all $\alpha$.
2. Even the sets $\text{lev}(s_p) = \{ \beta \in \omega : \text{dom}(s_p) \cap \{ \beta \} \times \omega \neq 0 \}$ form a $\Delta$-system with root $\text{lev}(r) = \{ \beta \in \omega_1 : r \cap \{ \beta \} \times \omega \neq 0 \}$.
3. $f_p \restriction T_\beta$ are the same for all $\alpha$, this for all $\beta \in \text{lev}(r)$.
4. $a_\alpha$ form a $\Delta$-system with root $b \subseteq r$.

Now let $x_\alpha = \text{dom}(f_p) \setminus \bigcup_{\beta \in \text{lev}(r)} T_\beta$. Thus $x_\alpha$ are pairwise disjoint finite subsets of the Aronszajn tree $T$, and it is possible to find countable ordinals $\alpha < \beta$ such that every $t \in x_\alpha$ is $T$-incompatible with every $u \in x_\beta$. It follows that any tree $s_q$ with $\text{dom}(s_q) = \text{dom}(s_p) \cup \text{dom}(s_{p_\beta})$, $s_q \restriction \text{dom}(s_p) \times \text{dom}(s_{p_\beta}) = s_p$ and $s_q \restriction \text{dom}(s_{p_\beta}) \times \text{dom}(s_{p_\beta}) = s_{p_\beta}$, together with the function $f_q = f_p \cup f_{p_\beta}$ give a condition $q = \langle s_q, f_q \rangle$ in the forcing $P$ which is stronger than both $p_\alpha$ and $p_\beta$. It is a matter of an easy surgery on $s_{p_\alpha}$ and $s_{p_\beta}$ to provide such a condition $q$ so that $a_\alpha, a_\beta$ are compatible in $s_q \restriction i_0 \times s_q \restriction i_1 \times \cdots \times s_q \restriction i_n$. Then $p \supseteq q \models " \dot{a}_\alpha$ and $\dot{a}_\beta$ are compatible elements of $\dot{A}"$ as desired. \qed

4.1. Strongly homogeneous Souslin trees.

In this subsection it is proved that the assertion $\phi = "\text{there is a strongly homogeneous Souslin tree}"$ is $\Pi_2$-compact, where

**Definition 4.11.** Let $T$ be an $\omega_1$-tree. A family $\{ h(s_0, s_1) : s_0, s_1 \in T \}$ are elements of the same level of $T$ is called coherent if:

1. $h(s_0, s_1)$ is a level-preserving isomorphism of $T \restriction s_0$ and $T \restriction s_1$; $h(s, s) = id$
2. (commutativity) let $s_0, s_1, s_2$ be elements of the same level of $T$ and $t_0 \leq s_0$. Then $h(s_1, s_2)h(s_0, s_1)(t_0) = h(s_0, s_2)(t_0)$
3. (coherence) let $s_0, s_1$ be elements of the same level of $T$ and $t_0 \leq s_0$. Let $t_1 = h(s_0, s_1)(t_0) \leq s_1$. Then $h(t_0, t_1) = h(s_0, s_1) \restriction T \restriction t_0$
4. (transitivity) if $\alpha$ is a limit ordinal and $t_0, t_1$ are two different elements at $\alpha$-th level of $T$ then there are $s_0, s_1 \in T^{<\alpha}$ such that $h(s_0, s_1)(t_0) = t_1$.

A tree is called strongly homogeneous if it has a coherent family of isomorphisms.

The existence of strongly homogeneous Souslin trees can be proved from $\dot{\phi}$ by a standard argument. Also, Todorcevic’s term for a Souslin tree in one Cohen real extension provides in fact for a strongly homogeneous tree:

**Theorem 4.12.** $\mathbb{C} \models "\text{there is a strongly homogeneous Souslin tree}"$.

Proof. An elaboration on Todorcevic’s argument [T1]. Let $T$ be a family of functions such that

1. every $f \in T$ is of the form $f : \alpha \to \omega$, finite-to-one for some countable ordinal $\alpha$
(2) for each \( \alpha \in \omega_1 \) there is \( f \in T \) with \( \alpha = \text{dom}(f) \)
(3) every two functions \( f, g \in T \) are modulo finite equal on the intersection of
their domains
(4) \( T \) is closed under finite changes of its elements.

Such a family is built as in [T1] and it can be understood as a special Aronszajn
tree under the reverse inclusion order. If \( c \in \omega \) is a Cohen real then [T1] the tree
\( T_c = \{ \text{cof} : f \in T \} \) ordered by reverse inclusion is a Souslin tree in \( V'[c] \).
To conclude the proof of the Theorem, we shall find a coherent family of isomorphisms of the tree
\( T \) which is easily seen to lift to the tree \( T_\alpha \). Namely, let \( f, g \in T, \text{dom}(f) = \text{dom}(g) \).
Define \( h(f, g)(e) = g \cup (e \setminus f) \) for \( e \in T \) with \( f \subset e \). By (3) and (4) above this is
a well-defined function and an isomorphism of the trees \( T \upharpoonright f \) and \( T \upharpoonright g \). The easy
proof that these isomorphisms form a coherent family on a tree \( T \) which lifts to the tree
\( T_\alpha \) is left to the reader. \( \square \)

Paul Larson proved that every strongly homogeneous Souslin tree contains a
regularly embedded free tree. In fact, every strongly homogeneous Souslin tree can
be written as a product of two free trees.

**Optimal Iteration Lemma 4.13.** Assume there is a strongly homogeneous Souslin
tree. Whenever \( M \) is a countable transitive model of ZFC iterable with respect to
its Woodin cardinal \( \delta \) with \( M \models \mathcal{U} \) is a strongly homogeneous Souslin tree" there
is a full iteration \( j \) of \( M \) such that \( j(U) \) is a strongly homogeneous Souslin tree.

**Proof.** Let \( T \) be a strongly homogeneous Souslin tree with a coherent family \( \{ g(t_0)(t_1) : t_0, t_1 \in T_\alpha \} \)
of isomorphisms and let \( M, U, \delta \) be as above and \( M \models \mathcal{U} \) is a strongly homogeneous Souslin tree as witnessed by a family \( h = \{ h(s_0)(s_1) : s_0, s_1 \in U_\xi \text{ for some } \xi \in \omega^M \} \). We shall produce a full iteration \( j \)
of \( M \) such that there is a club \( C \subset \omega_1 \) and an isomorphism \( \pi : T \upharpoonright C \to j(U) \upharpoonright C \)
which commutes with the internal isomorphisms of the trees: \( \pi g(t_0, t_1)(u) = jh(\pi t_0, \pi t_1)(\pi u) \) whenever the relevant terms are defined. Then, since the trees
\( T, j(U) \) are isomorphic on a club, necessarily \( j(U) \) is a Souslin tree, and it is
strongly homogeneous as witnessed by the family \( j(h) \). This will finish the proof of
the lemma. Again, below \( U_\alpha \) denotes the tree \( j_\alpha U \) and not the \( \alpha \)-th level of \( U \). Levels of trees are never indexed by \( \alpha \).

The iteration will be constructed by induction on \( \alpha \in \omega_1 \) and we will have
\( \theta_\alpha = \omega^{U_\alpha} \) and \( C = \{ \theta_\alpha : \alpha \in \omega_1 \} \). First, fix a partition \( \{ S_\xi : \xi \in \omega_1 \} \) of
the set of countable limit ordinals into disjoint stationary sets. By induction on \( \alpha \in \omega_1 \), we shall build the models together with the elementary embeddings, plus an
isomorphism \( \pi : T \upharpoonright C \to j(U) \upharpoonright C \), plus an enumeration \( \{ (x_\xi, \beta_\xi) : \xi \in \omega_1 \} \) of all
pairs \( (x, \beta) \) with \( x \in Q_\beta \). The induction hypotheses at \( \alpha \in \omega_1 \) are:

1. The function \( i \upharpoonright (T \upharpoonright \{ \theta_\gamma : \gamma \in \alpha \}) \) has been defined, it is an isomorphism of
   \( T \upharpoonright \{ \theta_\gamma : \gamma \in \alpha \} \) and \( U_\alpha \upharpoonright \{ \theta_\gamma : \gamma \in \alpha \} \) and it commutes with the internal
   isomorphisms of the trees, i.e. \( \pi g(t_0, t_1)(u) = j_\alpha h(\pi t_0, \pi t_1)(\pi u) \) whenever the
   relevant terms are defined
2. The initial segment \( \{ (x_\xi, \beta_\xi) : \xi \in \theta_\alpha \} \) has been constructed and every pair
   \( (x, \beta) \) with \( \beta \in \alpha \) and \( x \in Q_\beta \)
3. If \( \gamma \in \alpha \) belongs to some–unique–set \( S_\xi \) then \( j_{\beta_\xi, \gamma}(x_\xi) \in G_\gamma \).

As before, (1) is the crucial condition ensuring that the tree \( T \) is copied to \( j(U) \)
properly. (2,3) are just bookkeeping requirements for making the resulting iteration
full.
At limit steps, we just take direct limits and unions of the isomorphisms constructed so far. At successor steps, given \( M_\alpha \), we must produce a \( M_\alpha \)-generic filter \( G_\alpha \subset \mathbb{Q}_\alpha \) such that setting \( \langle M_{\alpha+1}, U_{\alpha+1}, \delta_{\alpha+1} \rangle \) to be the generic ultrapower of \( \langle M_\alpha, U_\alpha, \delta_\alpha \rangle \) by \( G_\alpha \), the isomorphism \( \pi \) can be extended to \( \theta_\alpha \)-th level of the trees \( T \) and \( U_{\alpha+1} \) preserving the induction hypothesis (1).

- **Case 1.** \( \alpha \) is a successor ordinal, \( \alpha = \beta + 1 \). Choose an arbitrary \( M_\alpha \)-generic filter \( G_\alpha \subset \mathbb{Q}_\alpha \). We shall show how the isomorphism \( \pi \) can be extended to the \( \theta_\alpha \)-th level of the trees \( T \) and \( U_{\alpha+1} \) preserving the induction hypothesis (1).

  Let \( t \in T_{\theta_\alpha} \) be arbitrary. The \( \theta_\beta \)-orbit of \( t \) is the set \( \{ u \in T_{\theta_\alpha} : \exists t_0, t_1 \in T_{\theta_\beta} \ u = g(t_0, t_1)(t) \} \). By the commutativity property of the isomorphisms \( g \), the \( \theta_\alpha \)-th level of the tree \( T \) partitions into countably many disjoint \( \gamma \)-orbits \( O_k : k \in \omega \). Also, for every \( u \in T_{\theta_\alpha} \) and integer \( k \in \omega \) there is a unique \( t \in O_k \) with \( t \prec T u \). The same analysis applies to the tree \( U_{\alpha+1} \) and isomorphisms \( h \). The \( \theta_\alpha \)-th level of the tree \( U_{\alpha+1} \) partitions into countably many disjoint \( \gamma \)-orbits \( N_k : k \in \omega \).

  Now it is easy to see that there is a unique way to extend the function \( \pi \) to \( T_{\theta_\alpha} \) so that \( \pi''O_k = N_k \) and \( \pi \) is order-preserving. Such an extended function will satisfy the induction hypothesis (1). The induction hypothesis (2) is easily managed and the induction hypothesis (3) does not say anything about successor ordinals \( \alpha \).

- **Case 2.** \( \alpha \) is a limit ordinal. In this case, \( \theta_\alpha \)-th level of the tree \( U_{\alpha+1} \) is already pre-determined by \( \pi \mid T \setminus \{ \theta_\gamma : \gamma \in \alpha \} \) and the necessity of extending \( \pi \). Namely, we must have \( (U_{\alpha+1})_{\theta_\alpha} = \{ u : \text{there is } t \in T_{\theta_\alpha} \text{ such that } u = \bigcup \{ \pi r : t \prec T r \} \} \). The challenge is to find an \( M_\alpha \)-generic filter \( G_\alpha \subset \mathbb{Q}_\alpha \) such that \( (U_{\alpha+1})_{\theta_\alpha} / G_\alpha \) is of the above-described form. Then necessarily the only possible order-preserving extension of \( \pi \) to \( T_{\theta_\alpha} \) will satisfy the induction hypothesis (1). We shall use the fact that it is enough to know one element of \( (U_{\alpha+1})_{\theta_\alpha} \) in order to determine the whole level–by transitivity, Definition 4.2(4).

  Work in \( M_\alpha \). Fix \( \dot{u} \), an arbitrary \( \mathbb{Q}_\alpha \)-name for an element of \( (U_{\alpha+1})_{\theta_\alpha} \), which will be identified with the cofinal–and therefore \( M_\alpha \)-generic–branch of the tree \( U_{\alpha+1} \) it determines. Let \( b_0 \in \mathbb{Q}_\alpha \) be defined as \( j_{\beta, \alpha}(\check{\gamma}) \) if \( \alpha \) belongs to some–unique–set \( S_\xi \) with \( \xi \in \theta_\alpha \), otherwise let \( b_0 = 1 \) in \( \mathbb{Q}_\alpha \). Let \( B \) be the complete subalgebra of \( RO(\mathbb{Q}_\alpha) \) generated by the name \( \dot{u} \). By some Boolean algebra theory, there must be \( b_1 \leq b_0 \) and \( s \in U_\alpha \) so that \( pr_1 b_1 = ([s] \cap [b] = b_2 \text{ and } B \cap b_2 \text{ is isomorphic to } B \cap b_2 \text{ by an isomorphism generated by the name } \dot{u} \). Without loss of generality \( lev(s) = \theta_\gamma \) for some \( \gamma \in \alpha \), since the set \( \{ \theta_\gamma : \gamma \in \alpha \} \) is cofinal in \( \theta_\alpha \).

  Now pick an element \( t \in T_{\theta_\alpha} \) such that \( t \leq_T \pi^{-1}s \). Then \( c = \bigcup \{ \pi(r) : t \leq_T r \} \) is a cofinal \( M_\alpha \)-generic branch through \( U_\alpha \) containing \( s \). Let \( H \subset B \) be the \( M_\alpha \)-generic filter determined by the equation \( c = \check{u} \) and let \( G_\alpha \subset RO(\mathbb{Q}_\alpha) \) be any \( M_\alpha \) generic filter with \( H \subset G_\alpha, b_1 \in G_\alpha \). We claim that \( G_\alpha \) works.

  Let \( \langle M_{\alpha+1}, U_{\alpha+1}, \delta_{\alpha+1} \rangle \) be the generic ultrapower of \( \langle M_\alpha, U_\alpha, \delta_\alpha \rangle \) using the filter \( G_\alpha \). Define \( \pi = j_{\theta_\alpha, \alpha+1} h(s_0, s_1) \) where \( r_0, r_1 \in T_{\theta_\alpha} \) for some \( \gamma \in \alpha \) and \( t \leq_T r_0, \pi(r_0) = s_0, \pi(r_1) = s_1 \) and \( t \) is the element of \( T_{\theta_\alpha} \) used to generate \( c \) in the previous paragraph. By the induction hypothesis (1) and coherence–Definition 4.2(4)–\( \pi \) is well-defined, and by transitivity applied to both \( T \) and \( U \) side \( \pi \mid T_{\theta_\alpha} : T_{\theta_\alpha} \to (U_{\alpha+1})_{\theta_\alpha} \) is a bijection. It is now readily checked that \( \pi \) commutes with the internal isomorphisms \( g, h \) and the induction hypothesis continues to hold at \( \alpha + 1 \). The hypothesis (2) is easily managed by suitably prolonging the enumeration, and the induction hypothesis (3) is maintained by the choice of \( b_0 \in G_\alpha \).
Conclusion 4.14. The sentence “there is a strongly homogeneous Souslin tree” is $\Pi_2$-compact.

Again, the proof of Lemma 4.13 shows that the $\Sigma_1^1$ theory of strongly homogeneous Souslin trees is complete in the same sense as explained in the previous Subsection. The first order theory of the model obtained in this Subsection has been independently studied by Paul Larson.

4.2. Other types of Souslin trees.

One can think of a great number of $\Sigma_1^1$ constraints on Souslin trees. With each of them, the first two iteration lemmas can be proved for $\phi = \text{“there is a Souslin tree with the given constraint”}$ owing to Lemma 4.1. However, the absoluteness properties of the resulting models as well as the status of $\Pi_2$-compactness of such sentences $\phi$ are unknown. Example:

Definition 4.15. A Souslin tree $T$ is self-specializing if $T \vDash \check{T} \setminus \check{b}$ is special, where $\check{b}$ is the generic branch.

A selfspecializing tree can be found under $\diamondsuit$ or after adding $\aleph_1$ Cohen reals. Such a tree is obviously neither free nor strongly homogeneous and no such a tree exists in the models from the previous two subsections.

5. The bounding number

In this section it will be proved that the sentence $b = \aleph_1$ is $\Pi_2$-compact even in the stronger sense with a predicate for unbounded sequences added to the language of $\langle H_{\aleph_2}, \in, \omega_1, I \rangle$. Everywhere below, by an unbounded sequence we mean a modulo finite increasing $\omega_1$-sequence of increasing functions in $\omega^\omega$ without an upper bound in the eventual domination ordering of $\omega^\omega$.

5.0. Combinatorics of $b$.

A subgenericity theorem similar to the one obtained in the dominating number section can be proved here too, in this case essentially saying that adding a Cohen real is an optimal way of adding an unbounded real. It is just a restatement of a familiar fact from recursion theory and is of limited use in what follows.

We abuse the notation a little writing $C = <\omega^\omega$ ordered by reverse extension and for a function $f \in \omega^\omega$, $C(f) = \{ \eta \in C : \eta$ is on its domain pointwise $\leq f \}$ ordered by reverse extension as well.

Lemma 5.1. Let $M$ be a transitive model of ZFC and $f \in \omega^\omega$ be an increasing function which is not bounded by any function in $M$. Whenever $D \in M$, $D \subset C$ is a dense set, then $D \cap C(f) \subset C(f)$ is dense.

Proof. Fix a dense subset $D \in M$ of $C$ and a condition $p \in C(f).$ We shall produce $q \leq p, q \in D \cap C(f)$, proving the lemma.

Work in the model $M$. By induction on $n \in \omega$ build a sequence $p_0, p_1, \ldots$, of conditions in $C$ so that

1. $p_0 = p,$ $\text{dom}(p_{n+1}) > \text{dom}(p_n)$
2. $p_{n+1}$ is any element of the set $D$ below the condition $r = p \upharpoonright (0, 0 \ldots 0)$ with as many zeros as necessary to get $\text{dom}(r) = m_n$.

Let $X \subset \omega$ be the set $\{ \text{dom}(p_n) : n \in \omega \}$ and let $g : X \to \omega$ be given by $g(m_n) = \text{max}(\text{rng}(p_{n+1})).$ Then $X, g \in M$ and since the function $f$ is increasing and
unbounded over the model $M$, there must be an integer $n$ such that $f(\text{dom}(p_n)) > g(\text{dom}(p_n))$. Then obviously $q = p_{n+1} \subseteq p$ is the desired condition in $D \cap \mathcal{C}(f)$. $\Box$

**Corollary 5.2. (Subgenericity)** Let $P$ be a forcing and $\dot{f}$ a $P$-name such that

1. $P \Vdash \text{"} \dot{f} \in {}^\omega \omega \text{ is an increasing unbounded function"}$
2. for every finite sequence $\eta$ of integers the boolean value $\|\dot{\eta}\|$ is bounded on its domain by $\dot{f}\|_P$ is nonzero.

Then there is a $P$-name $\dot{Q}$ and a complete embedding $\mathcal{C} \prec RO(P \ast \dot{Q})$ such that $P \ast \dot{Q} \Vdash \text{"} \dot{f} \text{ pointwise dominates } \dot{c}, \text{ the } \mathcal{C}\text{-generic function}.\text{"}$

**Proof.** Set $\dot{Q} = \mathcal{C}(\dot{f})$ and use the previous Lemma. $\Box$

It follows that a collection $A \subseteq {}^\omega \omega$ of increasing functions is unbounded just in case the set $X = \{f \in {}^\omega \omega : \text{ some } g \in A \text{ eventually dominates the function } f\}$ is nonmeager. For if $A$ is bounded by some $h \in {}^\omega \omega$ then $X \subseteq \{f \in {}^\omega \omega : h \text{ eventually dominates } f\}$ and the latter set is meager; on the other hand, if $A$ is unbounded then Lemma 5.1 provides sufficiently strong Cohen reals in the set $X$ to prove its nonmeagerness. A posteriori, a forcing preserving nonmeager sets preserves unbounded sequences as well.

A more important feature of the bounding number is that every two unbounded sequences can be made in some sense isomorphic. Recall the quasordering $\leq_b$ defined in Subsection 1.2.

**Definition 5.3.** For $b,c \in H_{\aleph_2}$ set $b \leq_b c$ if in every forcing extension of the universe $b$ is an unbounded sequence implies $c$ is an unbounded sequence.

While under suitable assumptions (for example the Continuum Hypothesis) the behavior of this quasordering is very complicated, in the model for $b = \aleph_1$ we will eventually build there will be exactly two classes of $\leq_b$-equivalence. The key point is the introduction of the following $\Sigma_1(\langle H_{\aleph_2}, \in, \omega_1, \exists \rangle)$ concept to ensure $\leq_b$-equivalence of two unbounded sequences.

**Definition 5.4.** Unbounded sequences $b,c$ are locked if there is an infinite set $x \subseteq \omega$ such that for every $\alpha \in \omega_1$ there is $\beta \in \omega_1$ with $b(\beta) \upharpoonright x$ eventually dominating $c(\alpha) \upharpoonright x$; and vice versa, for every $\alpha \in \omega_1$ there is $\beta \in \omega_1$ with $c(\beta) \upharpoonright x$ eventually dominating $b(\alpha) \upharpoonright x$.

It is immediate that locked sequences are $\leq_b$-equivalent. Note that any bound on an infinite set $x \subseteq \omega$ of a collection of increasing functions in $\omega^\omega$ easily yields a bound of that collection on the whole $\omega$.

Now it is possible to lock unbounded sequences using one of the standard tree forcings of [BJ]:

**Definition 5.5.** The Miller forcing $\mathcal{M}$ is the set of all nonempty trees $T \subseteq \omega^\omega$ consisting of increasing sequences for which

1. for every $t \in T$ there is a splitnode $s$ of $T$ which extends $t$
2. if a sequence $s$ is a splitnode of $T$ then $s$ has in fact infinitely many immediate successors in $T$.

$\mathcal{M}$ is ordered by inclusion.

The Miller forcing is proper, $\mathcal{M}$-friendly—see Definition 6.6—and as such preserves nonmeager sets of reals and unbounded sequences of functions by the argument following Corollary 5.2. If $G \subset \mathcal{M}$ is a generic filter then $f = \bigcup G \in \omega^\omega$ is an increasing function called a Miller real.
Theorem 5.6. \( M \models \text{"every two unbounded sequences from the ground model are locked".} \)

Corollary 5.7. It is consistent with ZFC that there are exactly two classes of \( \leq b \)-equivalence.

Proof. Start with a model of ZFC+GCH and iterate Miller forcing \( \omega_2 \) times with countable support. The resulting poset has \( \aleph_2 \)-c.c., it is proper and \( M \)-friendly [BJ], therefore it does not collapse unbounded sequences and forces \( b = \aleph_1 \). In the resulting model, there are exactly two classes of \( \leq b \)-equivalence: the objects which are not unbounded sequences and the unbounded sequences, which are pairwise locked by the above theorem and a chain condition argument. \( \square \)

It also follows from the Theorem that whenever there are two unbounded sequences, one of length \( \omega_1 \) and the other of length \( \omega_2 \), Miller forcing necessarily collapses \( \aleph_2 \) to \( \aleph_1 \).

Proof of Theorem 5.6. Let \( \dot{x} \) be an \( M \)-name for the range of the Miller real. We shall show that every two unbounded sequences \( b, c : \omega_1 \rightarrow \omega \) in the ground model are forced to be locked by \( \dot{x} \). To this end, given \( T \in M \) and \( \alpha \in \omega_1 \) a tree \( S \in M, S \subset T \) and an ordinal \( \beta \in \omega_1 \) will be produced such that \( S \Vdash \text{"}(c(\alpha) \upharpoonright \dot{x}) \text{ is eventually dominated by } b(\beta) \upharpoonright \dot{x} \". The theorem then follows by the obvious density and symmetricity arguments.

So fix \( b, c, T \) and \( \alpha \) as above. Let \( \beta \in \omega \) be an ordinal such that \( b(\beta) \) is not eventually dominated by any function recursive in \( c(\alpha) \) and \( T \). A tree \( S \in M, S \subset T \) with the same trunk \( t \) as \( T \) will be found such that

\[ s \in S, n \in \text{dom}(s) \setminus \text{dom}(t) \text{ implies } c(\alpha)(n) \leq b(\beta)(n). \]

This will complete the proof. Let \( S \) be defined by \( s \in S \) iff \( s \in T \) and if \( s' \) is the least splitnode of \( T \) above or equal to \( s \) then for every \( n \in \text{dom}(s') \setminus \text{dom}(t) \) it is the case that \( c(\alpha)(n) \leq b(\beta)(n) \).

Obviously \( t \in S \subset T \) and \( S \) has property (*), moreover \( S \) is closed under initial segment and if \( s \in S \) then the least splitnode of \( T \) above or equal to \( s \) belongs to \( S \) as well. We must show that \( S \in M \), and this will follow from the fact that if \( s \in S \) is a splitnode of the tree \( T \) then \( s \) has infinitely many immediate successors in \( S \). And indeed, let \( y \subset \omega \) be the infinite set of all integers \( n \in \omega \) with \( s^\infty(n) \in T \) and let \( g : y \rightarrow \omega \) be a function defined by \( g(n) = c(\alpha)(s'(m - 1)) \), where \( s' \) is the least splitnode of \( T \) above or equal to \( s^\infty(n) \) and \( m = \text{lth}(s') \). Then by the choice of the ordinal \( \beta \in \omega_1 \) the set \( z = \{ n \in y : g(n) \leq b(\beta)(n) \} \subset \omega \) is infinite and every sequence \( s^\infty(n) : n \in z \) belongs to the tree \( S \). \( \square \)

5.1. A model for \( b = \aleph_1 \).

The \( P_{\max} \) variant for \( b = \aleph_1 \) will be built using the following notion of a witness: \( b : \omega_1 \rightarrow \omega \) is a good unbounded sequence if it is unbounded and for every sequence \( \eta \in \omega \) the set \( \{ \alpha \in \omega_1 : b(\alpha) \text{ pointwise dominates } \eta \text{ on its domain} \} \subset \omega_1 \) is stationary. Note that if \( j \) is a full iteration of a model \( M, M \models \text{"} b \text{ is a good unbounded sequence"} \) and \( j(b) \) is unbounded then in fact \( j(b) \) is a good unbounded sequence. Also, whenever \( \delta \) is a Woodin cardinal of \( M \) and \( j_\delta \) is the \( Q_{<\delta} \)-term for the canonical ultrapower embedding of \( M \) then \( j_\delta b(\omega_1^M) \) is a name for an unbounded function.
Optimal Iteration Lemma 5.8. Assume \( b = \mathbb{R}_1 \). Whenever \( M \) is a countable transitive model of ZFC iterable with respect to its Woodin cardinal \( \delta \) and \( M \models "b \text{ is a good unbounded sequence}" \) there is a full iteration \( j \) of \( M \) based on \( \delta \) such that \( j(b) \) is an unbounded sequence.

Proof. Drawing on the assumption, choose an unbounded sequence \( c \) of length \( \omega_1 \) and fix an arbitrary iterable model \( M \) with \( M \models "b \text{ is an unbounded sequence and } \delta \text{ is a Woodin cardinal}" \). Two full iterations \( j_0, j_1 \) of the model \( M \) will be constructed simultaneously so that the function \( n \mapsto \max\{j_0b(\theta_{0,\alpha})(n), j_1b(\theta_{1,\alpha})(n)\} \) eventually dominates the function \( c(\alpha) \), this for every \( \alpha \in \omega_1 \). Here \( \theta_{0,\alpha} \) is \( \omega_1 \) in the sense of the \( \alpha \)-th model on the iteration \( j_0 \); similarly for \( \theta_{1,\alpha} \).

It follows immediately that one of the sequences \( j_0(b), j_1(b) \) must be unbounded, since if both were bounded—say by functions \( f_0, f_1 \) respectively—then the sequence \( c \) would be bounded as well by the function \( n \mapsto \max\{f_0(n), f_1(n)\} \), contrary to the choice of \( c \).

Now the iterations \( j_0, j_1 \) can be constructed easily using standard bookkeeping arguments and the following claim.

Claim 5.9. Let \( M_0, M_1 \) be countable transitive models of ZFC and let

1. \( M_0 \models P \) is a poset, \( p \in P \), and \( p \Vdash P \, \dot{x} \in {}^\omega \omega \) is an increasing function unbounded over \( M_0 \).
2. \( M_1 \models Q \) is a poset, \( q \in Q \), and \( q \Vdash Q \, \dot{y} \in {}^\omega \omega \) is an increasing function unbounded over \( M_1 \).

Suppose \( f \in {}^\omega \omega \) is an arbitrary function. Then there are \( M_0, M_1 \) (respectively) generic filters \( p \in G \subset P, q \in H \subset Q \) such that the function \( n \mapsto \max\{\dot{x}/G(n), \dot{y}/H(n)\} \) eventually dominates \( f \).

Proof. Without loss of generality assume that \( f \) is an increasing function. Let \( C_k : k \in \omega \) and \( D_k : k \in \omega \) enumerate all open dense subsets of \( P \) in the model \( M_0 \) and of \( Q \) in \( M_1 \) respectively. By induction on \( k \in \omega \) simultaneously build sequences \( p = p_0 \geq p_1 \geq \cdots \geq p_k \geq \cdots \) of conditions in \( P \), \( q = q_0 \geq q_1 \geq \cdots \geq q_k \geq \cdots \) of conditions in \( Q \) and integers \( 0 = n_0 = m_0, m_k \leq n_k < m_{k+1} \) so that

1. \( p_{k+1} \in C_k, q_{k+1} \in D_k \) for all \( k \in \omega \)
2. \( p_{k+1} \) decides \( \dot{x} \mid m_k, q_{k+1} \) decides \( \dot{y} \mid n_k \)
3. for \( k > 0 \), for infinitely many integers \( i \in \omega \) there is a condition \( r_i \leq p_k \) such that \( r_i \Vdash P \, \dot{x}(m_k) = i \); similarly, for infinitely many integers \( i \in \omega \) there is a condition \( s_i \leq q_k \) such that \( s_i \Vdash Q \, \dot{y}(n_k) = i \)
4. for \( k > 0 \), \( p_{k+1} \Vdash \dot{x}(m_k) > f(n_k) \) and \( q_{k+1} \Vdash \dot{y}(n_k) > f(m_{k+1}) \).

This is easily arranged—(3) is made possible by the fact that \( \dot{x}, \dot{y} \) are terms for unbounded functions. Let \( G \subset P, H \subset Q \) be the filters generated by the \( p_i \)'s or \( q_i \)'s respectively. These filters have the desired genericity properties by (1) above and the function \( n \mapsto \max\{\dot{x}/G(n), \dot{y}/H(n)\} \) pointwise dominates \( f \) from \( n_1 \) on. This can be argued from the induction hypothesis (4) and the fact that \( \dot{x}/G, \dot{y}/H \) and \( f \) are all increasing functions. \( \square \)

Strategic Iteration Lemma 5.10. Assume the Continuum Hypothesis. The good player has a winning strategy in the game \( G_{b=\mathbb{R}_1} \).

Proof. Let \( N = \langle b, N_i, \delta_i : i \in \omega \rangle \) be a sequence of models with a good unbounded sequence, let \( y_0 \in Q_N \) and let \( f \in {}^\omega \omega \) be an arbitrary function. We shall show that
there is an $\bar{N}$-generic filter $G \subset \mathcal{Q}_{\bar{N}}$ with $y_0 \in G$ such that the function $j_\mathcal{Q} b(\omega_{\bar{N}})$ is not eventually dominated by the function $f$, where $j_\mathcal{Q}$ is the $\mathcal{Q}_{\bar{N}}$-generic ultrapower of the model $N_0$ using the filter $G$. Then the winning strategy of the good player consists essentially only of a suitable bookkeeping using the Continuum Hypothesis.

Now the proof of the existence of such a filter $G$ is in fact very easy. We indicate a slightly inefficient though conceptual proof. Just use the subgenericity Corollary 5.2 and the proof of Lemma 2.8 with the following changes:

1. $d$ is replaced with $b$, the Hechler forcing is replaced with Cohen forcing and the real $e \in {}^\omega \omega$ will be taken sufficiently $\mathcal{C}$-generic.
2. step (5) in the proof of Lemma 2.8 is replaced by: $e$ is not eventually dominated by the function $f$.
3. the poset $R_i$ is now the set of all pairs $\langle y, \eta \rangle$ with $y \in \mathcal{Q}_i, \eta \in {}^{<\omega} \omega$ and for every $x \in y$ the sequence $\eta$ is everywhere on its domain dominated by the function $b(x \cap \omega_1)$.

$\Box$

**Conclusion 5.11.** The sentence $b = \mathcal{R}_1$ is $\Pi_2$-compact, even with the predicate $\mathfrak{B}$ for unbounded sequences added to the language of $\langle H_{\mathcal{R}_2}, \in, \omega_1, \mathfrak{B} \rangle$.

**Proof.** $\Pi_2$-compactness follows from Lemmas 5.8, 5.10. To show that $\Pi_2$ statements of $\langle H_{\mathcal{R}_2}, \in, \mathfrak{B} \rangle$ reflect to the model $L(\mathbb{R})^P_{\in^{\aleph_1}}$, proceed as in Corollary 1.17.

Suppose in $V$, a $\Pi_2$-sentence $\psi$-equal to $\forall x \exists y \chi(x, y)$ for some $\Sigma_0$ formula $\chi$ holds in $\langle H_{\mathcal{R}_2}, \in, \mathfrak{B} \rangle$ together with $b = \mathcal{R}_1$ and let $\delta < \kappa$ be a Woodin and a measurable cardinal respectively. For contradiction, suppose that $p \in P_{b=\mathcal{R}_1}$ forces $\neg \psi$ to hold; strengthening the condition $p$ if necessary we may assume that for some $x \in \mathcal{M}_P$, $p \Vdash \forall y \chi(k_p(x), y)$ where $k_p$ is the term for the canonical iteration of $\mathcal{M}_p$ as defined in 1.14. By Corollary 1.8, there is a countable transitive model $\mathcal{M}$ elementarily embeddable into $V_\xi$ such that $p \in \mathcal{M}$ and $\mathcal{M}$ and all of its generic extensions by posets of size $\leq \mathfrak{c}^\mathcal{M}$ are iterable. So $\mathcal{M} \models \langle b = \mathcal{R}_1 \rangle$ and $\langle H_{\mathcal{R}_2}, \in, \mathfrak{B} \rangle \models \psi$ and some cardinal $\bar{\delta}$ is Woodin". The optimal iteration lemma applied in $\mathcal{M}$ yields there a full iteration $j$ of the model $\mathcal{M}_P$ such that $j(b_p)$ is a good unbounded sequence. Let $N$ be an $\mathcal{M}$-generic extension of the model $\mathcal{M}_P$ and let $q = \langle N, j(b_p), \bar{\delta}, j(\mathcal{M}) \cup \{j(p)\} \rangle$.

First, $q \in P_{b=\mathcal{R}_1}$ is a condition strengthening $p$. To see that note that the model $N$ is iterable, the iteration $j$ is full in $N$, since the forcing $\mathcal{M}$ preserves stationary sets, and $j(b_p)$ is a good unbounded sequence in $N$ since $\mathcal{M}$ preserves such sequences.

Second, $q \models \forall \mathfrak{B}$ in the sense of the structure $k_q(H_{\mathcal{R}_2})^\mathcal{M}$ is just $\mathfrak{B} \cap k_q(H_{\mathcal{R}_2})^\mathcal{M}$. Observe that $N \models \langle \forall \mathfrak{B} \rangle$ (where there are unbounded are locked with $j(b_p)$"), by Theorem 5.6 applied in the model $\mathcal{M}$. Therefore $q$ forces even the following stronger statement, by the elementarity of the embedding $k_q$: "whenever $k_q(H_{\mathcal{R}_2})^\mathcal{M} \models \forall x \in \mathfrak{C} \chi(k_q(x), y)$, then $c$ and $k_q j(b_p)$ are locked; since $k_q j(b_p)$ is unbounded, the sequence $c$ must be unbounded as well".

Third, $q \models \exists y \chi(k_q j(x) = k_p(x), y)$", giving the final contradiction with our choice of $p$ and $x$. Since $\psi$ holds in the model $\mathcal{M}$, there must be some $y \in (H_{\mathcal{R}_2})^\mathcal{M}$ such that $\langle H_{\mathcal{R}_2}, \in, \mathfrak{B} \rangle^\mathcal{M} \models \chi(j(x), y)$. Then $q \models \langle H_{\mathcal{R}_2}, \in, \mathfrak{B} \rangle^\mathcal{M} \models \chi(k_q j(x), k_q(y))$ by the previous paragraph and absoluteness of $\Sigma_0$ formulas. $\Box$

6. **Uniformity of the meager ideal**

The sentence "there is a nonmeager set of reals of size $\mathcal{R}_1$" does not seem to be
compact, however, a similar a bit stronger assertion is.

**Definition 6.1.** A sequence \( \langle r_\alpha : \alpha \in \omega_1 \rangle \) of real numbers is called weakly Lusin if for every meager \( X \subseteq \mathbb{R} \) the set \( \{ \alpha \in \omega_1 : r_\alpha \in X \} \subseteq \omega_1 \) is nonstationary.

Thus the existence of a weakly Lusin sequence is a statement intermediate between a nonmeager set of size \( \aleph_1 \) and a Lusin set. It is equivalent to neither of them, as will be shown below.

### 6.0. A model for a weakly Lusin sequence.

We will prove the iteration lemmas necessary to conclude that the sentence \( \phi = \text{“there is a weak Lusin sequence”} \) is \( \Pi_2 \)-compact. Note that if \( (M, k, \delta) \) is a triple such that \( M \models \text{“} k \text{ is a weak Lusin sequence and } \delta \text{ is a Woodin cardinal} \text{”} \) then in \( M, Q_{<\delta} \models j_M(k_{\omega_1^M}) \) is a Cohen real over \( M \), and so it generates a natural Cohen subalgebra of \( Q_{<\delta} \). The following abstract copying lemma will be relevant:

**Lemma 6.2.** Let \( N \) be a countable transitive model of ZFC, \( N \models \text{“} P \text{ is a partially ordered set and } P \models \dot{r} \text{ is a Cohen real} \text{”} \). Suppose that \( p \in P \) and \( s \in {}^\omega \omega \) is a Cohen real over \( N \). Then there is an \( N \)-generic filter \( G \subseteq P \) so that \( p \in G \) and \( \dot{r}/G = s \mod \) finite.

**Proof.** In the model \( N \), let \( P = \mathbb{C} * \dot{Q} \) where \( \mathbb{C} \) is the cohen subalgebra of \( \mathbb{B} \) generated by the term \( r \). It follows from the assumptions and some boolean algebra theory in \( N \) that there are \( q \leq p \) in \( RO(P) \) and a finite sequence \( \eta \) such that \( \text{pr}_{\mathbb{C}}(q) = [[\eta \subseteq r]]_{\mathbb{C}} \). Let \( t \in {}^\omega \omega \) be the function defined by \( \eta \subseteq t \) and \( s(n) = t(n) \) for \( n \neq \text{dom}(\eta) \). Since the real \( s \) is Cohen over \( N \) and \( s = t \mod \) finite, even \( t \) is Cohen and the filter \( H \subseteq \mathbb{C} \) generated by the equation \( \dot{r} = t \) is \( N \)-generic. Finally, choose an \( N \)-generic filter \( G \subseteq P \) with \( H \subseteq G \) and \( q \in G \). This is possible since \( \text{pr}_{\mathbb{C}}(q) \in H \). The filter \( G \subseteq P \) is as desired. \( \square \)

**Optimal Iteration Lemma 6.3.** Assume that there is a weakly Lusin sequence. Whenever \( M \) is a countable transitive model iterable with respect to its Woodin cardinal \( \delta \) such that \( M \models \text{“} k \text{ is a weakly Lusin sequence} \text{”} \), then there is a full iteration \( j \) of \( M \) such that \( j(k) \) is a weakly Lusin sequence.

**Proof.** Fix a Lusin sequence \( J \) and \( M, k, \delta \) as in the Lemma. We shall produce a full iteration \( j \) of \( M \) such that there is a club \( C \subseteq \omega_1 \) with \( \forall \alpha \in C \ jk(\alpha) = J(\alpha) \mod \) finite. Then \( jk \) really is a Lusin sequence and the Lemma is proved.

The desired iteration \( j \) will be constructed by induction on \( \alpha \in \omega_1 \). Let \( S_x : \xi \in \omega_1 \) be a partition of \( \omega_1 \) into pairwise disjoint stationary sets. By induction on \( \alpha \in \omega_1 \), models \( M_\alpha \) together with the elementary embeddings will be built, plus an enumeration \( \{ \langle x_\xi, \beta_\xi : \xi \in \omega_1 \} \) of all pairs \( \langle x, \beta \rangle \) with \( x \in Q_\beta \). The induction hypotheses at \( \alpha \) are:

1. if \( \gamma < \alpha \) and \( J(\theta_\gamma) \) is a Cohen real over \( M_\gamma \) then \( k_{\gamma+1}(\theta_\gamma) = J(\theta_\gamma) \mod \) finite
2. if \( \gamma \leq \alpha \) then \( \{ \langle x_\xi, \beta_\xi : \xi \in \theta_\gamma \} \) enumerates all pairs \( \langle x, \beta \rangle \) with \( \beta < \gamma \)
3. if \( \gamma < \alpha \) and \( \theta_\gamma \in S_x \) for some \( \xi \in \theta_\gamma \) then \( j_{\beta_\xi, \gamma}(x_\xi) \in G_\gamma \).

As before, the hypothesis (1) makes sure that the sequence \( J \) gets copied onto \( jk \) properly and (2, 3) are just bookkeeping tools for making the resulting iteration full.
At limit ordinals just direct limits are taken and the new enumeration is the union of all old ones. The successor step is handled easily using the previous Lemma applied for \( N = M_\alpha, P = Q_\alpha, \dot{r} = j_\alpha k_\alpha(\theta_\alpha), s = J(\theta_\alpha) \) and \( b = j_\beta (\alpha, x_\xi) \), the last two in the case that \( s \) is Cohen over \( M_\alpha \) and \( \theta_\alpha \in S_\xi \) for some \( j \) (unique) \( \xi \in \theta_\alpha \).

To prove that the resulting iteration is as desired, note that it is full and that the set \( D = \{ \alpha \in \omega_1 : J(\theta_\alpha) \text{ is a Cohen real over } M_\alpha \} \) contains a closed unbounded set. For assume otherwise. Then the complement \( S \) of \( D \) is stationary and for every limit ordinal \( \alpha \in S \) there is a nowhere dense tree in some \( M_\beta, \beta \in \alpha \) such that the real \( J(\omega_1^{M_\beta}) \) is a branch of this tree–this is because a direct limit is taken at step \( \alpha \).

By a simple Fodor-style argument there is a nowhere dense tree and a stationary set \( T \subset S \) such that every \( J(\theta_\alpha) : \alpha \in T \) is a branch of this tree. This contradicts the assumption of \( J \) being a weakly Lusin sequence.

**Strategic Iteration Lemma 6.4.** Assume the Continuum Hypothesis. The good player has a winning strategy in the game \( G_\alpha \) connected with weakly Lusin sequences.

**Proof.** Given a sequence \( \hat{N} = \langle k, N_i, \delta_i : i \in \omega \rangle \) of models with a weakly Lusin sequence \( k \), a condition \( y_0 \in Q_{\hat{N}} \) and nowhere dense trees \( T_n : n \in \omega \), we shall show that there is an \( \hat{N} \)-generic filter \( G \subset Q_{\hat{N}} \) with \( y_0 \in G \) such that the real \( j_0 k(\omega_1^{\hat{N}}) \) is not a branch through any of the trees \( T_n \), where \( j_0 \) is the generic ultrapower embedding of the model \( N_0 \) using the filter \( G \cap Q_{\hat{N}} \). With this fact in hand, a winning strategy for the good player consists of just a suitable bookkeeping using the Continuum Hypothesis.

Again, we provide maybe a little too conceptual proof of the existence of the filter \( G \), using the ideas from Lemma 2.8. No subgenericity theorems are needed this time. Let \( i \in \omega \) and work in \( N_i \). Let \( Q_i = Q_{\delta_i} \) and \( j_i \) to be the \( Q_i \) term for the generic ultrapower embedding of the model \( N_i \). So the term \( j_i k(\omega_1^{\hat{N}}) \) is a term for a Cohen real, and it gives a complete embedding of \( \mathbb{C} \) into \( Q_i \). The key point is that with this embedding, the computation of the projection \( pr_C(y) \) gives the same value in \( \mathbb{C} \) in every model \( N_i \) with \( y \in Q_i \), namely \( \Sigma_{\mathbb{C}} \{ \eta \in <\omega : \text{the system } \{ x \in y : \eta \subset k(x \cap \omega_1) \text{ is stationary} \} \} \).

First, fix a suitably generic filter \( H \subset \mathbb{C} \). The requirements are:

1. \( pr_C(y_0) \in H \)
2. \( H \) meets all the maximal antichains that happen to belong to \( \bigcup_i N_i \)
3. the Cohen real \( c \in <\omega \) given by the filter \( H \) does not constitute a branch through any of the nowhere dense trees \( T_n \).

This is easily done. Now let \( X_n : n \in \omega \) be an enumeration of all maximal antichains of \( Q_{\hat{N}} \) in \( \bigcup N_i \) and by induction on \( n \in \omega \) build a decreasing sequence \( y_n : n \in \omega \) of conditions in \( Q_{\hat{N}} \) so that

1. \( pr_C(y_n) \in H \)
2. \( y_{n+1} \) has an element of \( X_n \) above it.

This can be done since the filter \( H \) is \( \mathbb{C} \)-generic over every model \( N_i \). In the end, let \( G \) be the filter on \( Q_{\hat{N}} \) generated by the conditions \( y_n : n \in \omega \). This filter is as desired. Note that \( c \) is the uniform value of \( j_i k(\omega_1^{\hat{N}}) \) as evaluated according to this filter.

**Conclusion 6.5.** The sentence “there is a weakly Lusin sequence of reals” is \( \Pi_2 \)-compact.
It is unclear whether it is possible to add a predicate for witnesses in this case.

### 6.1. Combinatorics of weakly Lusin sequences.

In this subsection it is proved that the existence of a Lusin set, weakly Lusin sequence and a nonmeager set of size \( \aleph_1 \) are nonequivalent assertions.

The following regularity property of forcings will be handy:

**Definition 6.6.** [BJ, 6.3.15] A forcing \( P \) is called \( \mathcal{M} \)-friendly if for every large enough regular cardinal \( \lambda \), every condition \( p \in P \), every countable elementary submodel \( M \) of \( H_\lambda \) with \( p, P \in M \) and every function \( h \in {}^{\omega} \omega \) Cohen-generic over \( M \) there is a condition \( q \leq p \) such that \( q \) is master for \( M \) and \( q \vdash \) “\( \hat{h} \) is Cohen-generic over \( M[G] \)”.

It is not difficult to prove that \( \mathcal{M} \)-friendly forcings preserve nonmeager sets. Moreover, \( \mathcal{M} \)-friendliness is preserved under countable support iterations [BJ, Section 6.3.C]. The following fact, pointed out to us by Tomek Bartoszyński, replaces our original more complicated argument.

**Lemma 6.7.**

1. The Miller forcing \( \mathbb{M} \)–see Definition 5.5–is \( \mathcal{M} \)-friendly.
2. \( \mathbb{M} \) destroys all weakly Lusin sequences from the ground model.

**Corollary 6.8.** It is consistent with ZFC that there is a nonmeager set of size \( \aleph_1 \) but no weakly Lusin sequences.

**Proof.** Iterate Miller forcing over a model of GCH \( \omega_2 \) times. □

**Proof of Lemma.** Suppose \( M \) is a countable transitive model of a rich fragment of ZFC, \( T \in \mathbb{M} \cap M \) and \( f \in {}^{\omega} \omega \) is a Cohen real over \( M \). We shall produce \( M \)-master conditions \( S_0, S_1 \subseteq T \) in \( \mathbb{M} \) such that

1. \( S_0 \Vdash \) \( f \) is a Cohen real over \( M \)
2. \( S_1 \Vdash \) \( f \) is eventually dominated by the Miller real.

This will finish the proof: (1) shows that \( \mathbb{M} \) is \( \mathcal{M} \)-friendly and (2) by a standard argument implies that for any weakly Lusin sequence \( \langle r_\alpha : \alpha \in \omega_1 \rangle \) of elements of \( {}^{\omega} \omega \) the set \( \{ \alpha \in \omega_1 : r_\alpha \) belongs to the meager set of all functions in \( {}^{\omega} \omega \) eventually dominated by the Miller real \} \( \subset \omega_1 \) is \( \mathcal{M} \)-forced to be stationary.

By a mutual genericity argument there is an \( \mathcal{M} \)-generic filter \( G \subseteq Coll(\omega, (2^\omega)^M) \) such that the function \( f \) is still Cohen generic over \( M[G] \). Work in \( M[G] \). Let \( \eta_k : k \in \omega \) enumerate the Cohen forcing \( {}^{<\omega} \omega \), let \( \hat{O}_k : k \in \omega \) enumerate all the \( \mathbb{M} \cap M \)-names for dense subsets of the Cohen forcing in \( M \) and let \( D_k : k \in \omega \) enumerate the open dense subsets of \( \mathbb{M} \cap M \) in \( M \). Build a fusion sequence \( T = T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots \) of trees in \( \mathbb{M} \cap M \) so that if \( s \) is a \( k \)-th level splitnode of \( T_k \) and \( \{ i_n : n \in \omega \} \) is an enumeration of the set of all integers \( i \) with \( s^<(i) \in T_k \) then for every \( n \in \omega \) the sequence \( s^<(i_n) \) belongs to \( T_{k+1} \). \( T_{k+1} \upharpoonright s^<(i_n) \in D_k \) and there is some extension \( \eta \) of \( \eta_n \) in the Cohen forcing \( {}^{<\omega} \omega \) such that \( T_{k+1} \Vdash \eta \in \hat{O}_k \). This is readily done. Let \( S = \bigsqcup T_i \subseteq \mathbb{M} \cap M[G] \). The tree \( S \) is an \( M \)-master condition in \( M \) and since \( f \in {}^{\omega} \omega \) is Cohen generic over the model \( M[G] \), the following two subtrees \( S_0, S_1 \) of \( S \) are still in \( \mathbb{M} \):

1. \( S_0 = \{ s \in S : \text{ whenever } s \) is a proper extension of a \( k \)-th level splitnode \( t \in T_k \) then \( T_{k+1} \upharpoonright s \Vdash \eta \in \hat{O}_k \) for some \( \eta \subset f \) \}
2. \( S_1 = \{ s \in S : \forall n \in dom(s) \text{ the trunk of } T \} f(n) < s(n) \).
It is easy to see that the trees $S_0, S_1$ are as desired. 

**Question 6.9.** Does the saturation of the nonstationary ideal plus the existence of a nonmeager set of reals of size $\aleph_1$ imply the existence of a weakly Lusin sequence?

Next it will be proved that the existence of a weakly Lusin sequence does not imply that of a Lusin set. A classical forcing argument can be tailored to fit this need; instead, we shall show that there are no Lusin sets in the model built in the previous Subsection. This follows from Theorem 1.15(2) and a simple density argument using the following fact:

**Lemma 6.10.** ($AD^{L(R)}$) Let $M$ be a countable transitive iterable model, $M \models \text{"K is a weakly Lusin sequence and } X \text{ is a Lusin set".}$ Then there is a countable transitive iterable model $N$ and an iteration $j \in N$ such that $N \models \text{"j is a full iteration of } M, j(K) \text{ is a weakly Lusin sequence and } j(X) \text{ is not a Lusin set"}.$

**Proof.** Fix $M, K$ and $X$ and for notational reasons assume that $X$ is really an injective function from $\omega_1^M$ to $\omega$ enumerating that Lusin set. Working in the model $M$, if $\delta$ is a Woodin cardinal, $Q_{<\delta}$ is the nonstationary tower and $j_\emptyset$ is the $Q_{<\delta}$-name for the canonical embedding of $M$ then both $j_\emptyset K(\omega_1^M)$ and $j_\emptyset X(\omega_1^M)$ are $Q_{<\delta}$ terms for Cohen reals.

Using the determinacy assumption, choose a countable transitive model $N_0$ such that $M \in N_0$ is countable there and $N_0$ and all of its generic extensions by forcings of size $\aleph_1^{N_0}$ are iterable. Work in $N_0$. Force two $\omega_1$-sequences $(c_\beta : \beta \in \omega_1), (d_\beta : \beta \in \omega_1)$ of Cohen reals—elements of $\omega^\omega$—and a function $e \in \omega^\omega$ eventually dominating every $d_\beta : \beta \in \omega_1$ with finite conditions.

Set $N = N_0[(c_\beta : \beta \in \omega_1), (d_\beta : \beta \in \omega_1), e]$ In the model $N$, the reals $(c_\beta : \beta \in \omega_1)$ constitute a weakly Lusin sequence, indeed a Lusin set, because their sequence is Cohen generic over the model $N_0[(d_\beta : \beta \in \omega_1), e]$. In the model $N_0[(c_\beta : \beta \in \omega_1), (d_\beta : \beta \in \omega_1)]$ build a full iteration $j$ of $M$ so that

1. if $\alpha \in \omega_1$ is limit then whenever made possible by the model $M_\alpha$ we have $jK(\theta_\alpha) = c_{\theta_\alpha}$ modulo finite
2. if $\alpha \in \omega_1$ is successor then $jX(\theta_\alpha)$ is equal to one of the reals $d_\beta : \beta \in \omega_1$ modulo finite.

By the arguments from the previous subsection and the fact that $(c_\beta : \beta \in \omega_1), (d_\beta : \beta \in \omega_1)$ are mutually generic sequences of Cohen reals over $N_0$, (2) is always possible to fulfill and the set $\{\beta \in \omega_1 : jK(\beta) = c_\beta \text{ modulo finite} \} \subset \omega_1$ will contain a club.

Now $N, j$ are as desired. In the model $N$, the iteration $j$ is full since $N$ is a c.c.c. extension of $N_0[(c_\beta : \beta \in \omega_1), (d_\beta : \beta \in \omega_1)]$ in which $j$ was constructed to be full; the sequence $jK$ is on a club modulo finite equal to a weakly Lusin sequence $\langle c_\beta : \beta \in \omega_1 \rangle$ and so is weakly Lusin itself; and the set $\{jX(\theta_\alpha) : \alpha \in \omega_1 \text{ successor} \}$ is an uncountable subset of $\text{rng}(jX)$ contained in the meager set of all reals eventually dominated by $e \in \omega^\omega$, consequently $\text{rng}(jX)$ is not a Lusin set.  

**6.2. The null ideal.**

The methods of this paper can be adapted to give a parallel result about the null ideal.

**Definition 6.11.** A sequence $\langle r_\alpha : \alpha \in \omega_1 \rangle$ of real numbers is called a weakly Sierpiński sequence if for every null set $S$, the set $\{\alpha \in \omega : r_\alpha \in S\}$ is nonstationary.
Theorem 6.12. The sentence “there is a weakly Sierpiński sequence” is $\Pi_2$-compact.

By an argument parallel to 6.7(2) it can be proved that existence of a nonnull set of size $\aleph_1$ and of a weakly Sierpiński sequence are nonequivalent statements.

REFERENCES

[AS] U. Abraham and S. Shelah, A $\Delta^2_5$ wellorder of the reals and incompactness of $L(Q^{MM})$, Ann. Pure Appl. Logic 59 (1993), 1–32.
[Ba] T. Bartoszyński, Combinatorial aspects of measure and category, Fund. Math. 127 (1987), 225–239.
[BJ] T. Bartoszyński and H. Judah, Set theory. On the structure of the real line, A K Peters, Wellesley, MA, 1995.
[BJZ] B. Balcar, T. Jech and J. Zapletal, Generalizations of Cohen algebras, preprint (1995).
[J1] T. J. Jech, Nonprovability of Souslin’s Hypothesis, Comment. Math. Univ. Carolinae 8 (1967), 291–305.
[J2] , Set Theory, Academic Press, New York, 1978.
[JS] H. Judah and S. Shelah, Suslin forcing, J. Symbolic Logic 53 (1988), 1188–1207.
[K] S. Koppelberg, Characterization of Cohen algebras, Paper on General Topology and Applications (S. Andima, R. Koppenman, P. R. Misra and A. R. Todd, eds.), Annals of the New York Academy of Sciences 704, 1993, pp. 227–237.
[L] R. Laver, On the consistency of Borel’s conjecture, Acta Math. 137 (1976), 151–169.
[S1] S. Shelah, Proper forcing, Lecture Notes in Math. 940, Springer Verlag, New York, 1982.
[S2] , Can you take Solovay inaccessible away?, Israel J. Math. 48 (1984), 1–47.
[S3] , Vive la difference I, Set Theory of the continuum (H. Judah, W. Just and W. H. Woodin, eds.), 1992, pp. 357–405.
[SZ] S. Shelah and J. Zapletal, Embeddings of Cohen algebras, Adv. Math. (to appear).
[So] M. Souslin, Problème 3, Fund. Math. 1 (1920), 223.
[T1] S. Tennenbaum, Souslin’s problem, Proc. Nat. Acad. Sci. USA 59 (1968), 60–63.
[T2] S. Todorcevic, Partitioning pairs of countable ordinals, Acta Math. 159 (1987), 261–294.
[T3] , Partition problems in topology, Cont. Math. 84, Amer. Math. Soc., Providence, 1989.
[Te] J. K. Truss, Connections between different amoeba algebras, Fund. Math. 130 (1988), 137–155.
[W1] W. H. Woodin, Supercompact cardinals, sets of reals, and weakly homogeneous trees, Proc. Natl. Acad. Sci. USA 88 (1988), 6587–6591.
[W2] , The axiom of determinacy, forcing axioms, and the nonstationary ideal (to appear).