An invariant of states on Cuntz algebras

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Abstract

For an arbitrary state $\omega$ on a Cuntz algebra, we define a number $1 \leq \kappa(\omega) \leq \infty$ such that if the GNS representations of $\omega$ and $\omega'$ are unitarily equivalent, then $\kappa(\omega) = \kappa(\omega')$. By using $\kappa$, we define minimal states and it is shown that the classification problem of states is reduced to that of minimal states. By using results of Dutkay, Haussermann, and Jorgensen, we give a sufficient condition of the minimality of a state. Properties of $\kappa$ and examples are shown. As an application, a new invariant of a certain class of endomorphisms of $\mathcal{B}(\mathcal{H})$ is given.

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1 Introduction

The most different aspect in operator algebra from other mathematics is the treatment of non-type I $C^*$-algebras [22]. By definition, a non-type I $C^*$-algebra is characterized by its representations. Hence the study of representations of non-type I $C^*$-algebras is a core component of operator algebra. For example, Cuntz algebras are non-type I. The aim of this paper is to classify states on Cuntz algebras by using a new invariant. In this section, we introduce the invariant and show its properties. In §1.2, we will state our main results. In §1.3 the significance and advantages of the new invariant will be explained.

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1.1 Invariant

1.1.1 Definition

For \( 2 \leq n \leq \infty \), let \( \mathcal{O}_n \) denote the Cuntz algebra with Cuntz generators \( s_1, \ldots, s_n \) \[14\], that is, \( \mathcal{O}_n \) is a \( C^* \)-algebra which is universally generated by a (finite or infinite) sequence \( s_1, \ldots, s_n \) satisfying \( s_i^* s_j = \delta_{ij} I \) for \( i, j = 1, \ldots, n \) and

\[
\sum_{i=1}^{n} s_i s_i^* = I \quad \text{when} \quad n < \infty, \quad \sum_{i=1}^{k} s_i s_i^* \leq I, \quad k = 1, 2, \ldots \text{ when } n = \infty
\]  

(1.1)

where \( I \) denotes the unit of \( \mathcal{O}_n \). The Cuntz algebra \( \mathcal{O}_n \) is an infinite dimensional, noncommutative simple \( C^* \)-algebra with unit.

Let \( S(\mathcal{O}_n) \) denote the set of all states on \( \mathcal{O}_n \). For \( \omega, \omega' \in S(\mathcal{O}_n) \), we write \( \omega \sim \omega' \) when their Gel’fand-Naimark-Segal (GNS) representations are unitarily equivalent. The problem is to classify elements in \( S(\mathcal{O}_n) \) by the equivalence relation \( \sim \). For \( \omega \in S(\mathcal{O}_n) \) with GNS representation \( (\mathcal{H}, \pi, \Omega) \), define the nonzero closed subspace \( K(\omega) \) of \( \mathcal{H} \) \[8, 9, 19\] by

\[
K(\omega) := \operatorname{Lin}\langle \{\pi(s_J)^* \Omega : J \in \mathcal{I}_n\} \rangle
\]

(1.2)

where \( \mathcal{I}_n := \bigcup_{l \geq 0} \{1, \ldots, n\}^l, \{1, \ldots, n\}^0 := \{\emptyset\}, s_J := s_{j_1} \cdots s_{j_l} \) for \( J = (j_1, \ldots, j_l) \), and \( s_{\emptyset} := I \). When \( n = \infty \), replace \( \{1, \ldots, n\}^l \) with \( \{1, 2, \ldots\}^l \).

Define \( \text{cdim} \omega \) and \( \kappa(\omega) \) by

\[
\text{cdim} \omega := \dim K(\omega), \quad \kappa(\omega) := \min\{\text{cdim} \omega' : \omega' \in S(\mathcal{O}_n), \omega' \sim \omega\},
\]

(1.3)

A state \( \omega \) on \( \mathcal{O}_n \) is said to be minimal if \( \text{cdim} \omega = \kappa(\omega) \). By definition, the following hold immediately.

**Theorem 1.1**

(i) For any \( \omega \in S(\mathcal{O}_n) \), there exists a minimal state \( \omega' \) on \( \mathcal{O}_n \) which is equivalent to \( \omega \). We call such \( \omega' \) a minimal model of \( \omega \).

(ii) For \( \omega, \omega' \in S(\mathcal{O}_n) \), if \( \omega \sim \omega' \), then \( \kappa(\omega) = \kappa(\omega') \).

**Proof.**

(i) Since \( \{\text{cdim} \omega' : \omega' \sim \omega\} \) is a subset of \( \{1, 2, \ldots, \infty\} \), it always has the smallest element with respect to the standard linear ordering where \( \infty \) means the countably infinite cardinality. Hence there always exists a minimal state \( \omega' \) which is equivalent to \( \omega \).

(ii) Assume \( \omega \sim \omega' \). Then their minimal models are also equivalent. By definitions of \( \kappa(\omega) \) and \( \kappa(\omega') \), the statement holds.
From Theorem 1.1 the classification problem of (pure) states on $\mathcal{O}_n$ is reduced to that of minimal (pure) states on $\mathcal{O}_n$ with $\text{cdim} = d$ for each number $1 \leq d \leq \infty$. Remark that $\kappa(\omega)$ is an invariant of $\omega$, but $\text{cdim}\omega$ is not (Proposition 3.20). For a given $1 \leq d \leq \infty$, there exist continuously many minimal pure states with $\kappa(\omega) = d$ (Theorem 3.24). A minimal model of a state is not unique in general (Proposition 3.7). The symbol “$\text{cdim}\omega$” originates in our old terminology, “the correlation dimension of $\omega$” (see Definition 1.7(i)).

For $2 \leq n \leq \infty$, we write $U(n)$ when $n < \infty$, as the group of all unitaries on $\ell^2 := \{ (z_j) : \sum_{j \geq 1} |z_j|^2 = 1 \}$ when $n = \infty$. We show properties of $\kappa$ with respect to the unitary group action as follows.

**Proposition 1.2** ($U(n)$ invariance) Let $\alpha$ denote the standard $U(n)$-action on $\mathcal{O}_n$, that is, $\alpha_g(s_i) := \sum_{j=1}^{n} g_{ji}s_j$ for $i = 1, \ldots, n$ and $g = (g_{ij}) \in U(n)$. Let $\omega \in S(\mathcal{O}_n)$.

(i) For any $g \in U(n)$, $\text{cdim}(\omega \circ \alpha_g) = \text{cdim}\omega$.

(ii) For any $g \in U(n)$, $\kappa(\omega \circ \alpha_g) = \kappa(\omega)$.

**Proof.** (i) Let $(\mathcal{H}, \pi, \Omega)$ denote the GNS representation of $\omega$. Since $\omega = \langle \Omega|\pi(\cdot)|\Omega \rangle$, we obtain $\omega \circ \alpha_g = \langle \Omega|\pi(\alpha_g(\cdot))|\Omega \rangle$. Hence we identify the GNS representation of $\omega \circ \alpha_g$ with $(\mathcal{H}, \pi \circ \alpha_g, \Omega)$ (see 4.5.3 Proposition of [23]). Then $\mathcal{K}(\omega \circ \alpha_g)$ is spanned by the set $\{ \pi(\alpha_g(s))|\Omega \} : J \in I_n \}$. This is contained in $\mathcal{K}(\omega)$ by the definition of $\alpha_g$. From this, $\mathcal{K}(\omega \circ \alpha_g) \subset \mathcal{K}(\omega)$. By replacing $(\omega, g)$ with $(\omega \circ \alpha_g, g^*)$, we obtain $\mathcal{K}(\omega) = \mathcal{K}((\omega \circ \alpha_g) \circ \alpha_g^*) \subset \mathcal{K}(\omega \circ \alpha_g)$. Hence the statement holds.

(ii) Remark that $\omega \sim \omega'$ if and only if $\omega \circ \alpha_g \sim \omega' \circ \alpha_g$. From this and (i), the statement holds.

From Proposition 1.2 and Theorem 1.1, $\kappa(\omega)$ can be regarded as an invariant of a $U(n)$-orbit in the set of all unitary equivalence classes of (pure) states on $\mathcal{O}_n$.

**Corollary 1.3** For $\omega, \omega' \in S(\mathcal{O}_n)$, if $\omega' \sim \omega \circ \alpha_g$ for some $g \in U(n)$, then $\kappa(\omega) = \kappa(\omega')$.

**Proof.** From Proposition 1.2(ii) and Theorem 1.1(ii), the statement holds.
1.1.2 Cuntz states as the case of \( \kappa = 1 \)

We review well-known results about Cuntz states by using cdim and \( \kappa \). Let \((\mathbb{C}^n)_1 := \{z \in \mathbb{C}^n : \|z\| = 1\}\). For any \( z = (z_1, \ldots, z_n) \in (\mathbb{C}^n)_1 \), a state \( \omega_z \) on \( \mathcal{O}_n \) which satisfies

\[
\omega_z(s_j) = \overline{z_j} \quad \text{for all } j = 1, \ldots, n, 
\]

exists uniquely and is pure, where \( \overline{z_j} \) denotes the complex conjugate of \( z_j \). When \( n = \infty \), replace \( \mathbb{C}^n \) with \( \ell^2 \). The state \( \omega_z \) is called the Cuntz state [5, 6, 7, 9, 10, 11, 15, 25, 26].

**Theorem 1.4** ([25], Appendix B) For \( z, y \in (\mathbb{C}^n)_1 \), \( \omega_z \sim \omega_y \) if and only if \( z = y \).

From Theorem 1.4, \((\mathbb{C}^n)_1\) is the complete set of invariants of Cuntz states on \( \mathcal{O}_n \).

**Fact 1.5** Assume \( 2 \leq n \leq \infty \).

(i) For \( \omega \in \mathcal{S}(\mathcal{O}_n) \), cdim \( \omega = 1 \) if and only if \( \omega \) is a Cuntz state. Especially, any Cuntz state is minimal.

(ii) For \( \omega \in \mathcal{S}(\mathcal{O}_n) \), \( \kappa(\omega) = 1 \) if and only if \( \omega \) is equivalent to a Cuntz state.

(iii) For \( \omega, \omega' \in \mathcal{S}(\mathcal{O}_n) \), if \( \kappa(\omega) = 1 = \kappa(\omega') \), then \( \omega' \sim \omega \circ \alpha_g \) for some \( g \in U(n) \).

**Proof.** (i) Let \((\mathcal{H}, \pi, \Omega)\) denote the GNS representation of \( \omega \). We see that (1.4) is equivalent that \( \pi(s_j)\Omega = z_j\Omega \) for all \( j \). From this and the definition of cdim, the statement holds.

(ii) By definition, \( \kappa(\omega) = 1 \) if and only if \( \omega \) is equivalent to \( \omega' \) such that cdim \( \omega' = 1 \). This is equivalent to the statement that \( \omega \) is equivalent to a Cuntz state from (i).

(iii) From (ii), there exist Cuntz states \( \omega_1 \) and \( \omega'_1 \) such that \( \omega \sim \omega_1 \) and \( \omega' \sim \omega'_1 \). Since \( \omega_1 \circ \alpha_g = \omega'_1 \) for some \( g \in U(n) \), the statement holds.

By combining Fact 1.5(ii) and Theorem 1.4 the case of \( \kappa = 1 \) is completely classified. Remark that \( \kappa = 1 \) implies the purity of a state automatically because any Cuntz state is pure. Fact 1.5(iii) does not hold when \( \kappa(\omega) = \kappa(\omega') \geq 2 \) (Example 3.8).
Remark that any Cuntz state is completely defined by only a parameter \( z \in (\mathbb{C}^n)_1 \). This is stated as the “uniqueness” of \( \omega_z \) in (1.1). In other words, it is not necessary to define the value \( \omega_z(s_J s_K^*) \) for all \( J, K \in \mathcal{I}_n \). Thanks to the uniqueness, one can describe Cuntz states very concisely. This type uniqueness holds for various other states in § 3.

1.2 Main theorems

We state our main theorems in this subsection. Since their proofs require some lemmas, we will prove theorems in § 2.2.

1.2.1 Minimality of a state

Let \( \text{cdim} \omega \) and \( \kappa(\omega) \) be as in (1.3). In order to make use of our new invariant \( \kappa \), we must be able to compute \( \kappa(\omega) \). If we know that \( \omega \) is minimal, then \( \text{cdim} \omega = \kappa(\omega) \). Since the computation of \( \text{cdim} \omega \) is easier than that of \( \kappa(\omega) \), the determination of its minimality makes sense.

Let \( \mathcal{I}_n \) be as in (1.2). Define

\[
\mathcal{O}_n^+ := \overline{\text{Lin}\{s_J : J \in \mathcal{I}_n, J \neq \emptyset\}} \subset \mathcal{O}_n.
\]

(1.5)

Then \( \mathcal{O}_n^+ \) is a nonunital, non-selfadjoint subalgebra of \( \mathcal{O}_n \). We obtain a sufficient condition that a given state is minimal.

**Theorem 1.6** For \( \omega \in S(\mathcal{O}_n) \), if there exists an isometry \( u \) in \( \mathcal{O}_n^+ \) (that is, \( u^* u = I \)) such that \( \omega(u) = 1 \), then \( \omega \) is minimal.

By using Theorem 1.6 we will show examples of minimal state in § 3.1. For a state, its minimality is neither necessary nor sufficient for its purity in general (Proposition 3.6).

1.2.2 Properly infinite correlation of a state

**Definition 1.7**

(i) (i) A state \( \omega \) on \( \mathcal{O}_n \) is said to be infinitely correlated if \( \text{cdim} \omega = \infty \). Otherwise, \( \omega \) is said to be finitely correlated.

(ii) A state \( \omega \) on \( \mathcal{O}_n \) is said to be properly infinitely correlated if \( \kappa(\omega) = \infty \). Otherwise, \( \omega \) is said to be essentially finitely correlated.

By definition, a state is either properly infinitely correlated or essentially finitely correlated. Any finitely correlated state is essentially finitely correlated, but the converse is not true (Proposition 3.20). If \( \omega \) is pure and
finitely correlated, then any vector state of the GNS representation space of $\omega$ is essentially finitely correlated. From Theorem 1.1(ii), the following holds.

**Fact 1.8** For $\omega, \omega' \in \mathcal{S}(\mathcal{O}_n)$, assume $\omega \sim \omega'$. If $\omega$ is properly infinitely correlated, then so is $\omega'$. Otherwise, $\omega'$ is essentially finitely correlated.

We give a sufficient condition such that a given state is properly infinitely correlated.

**Theorem 1.9** Let $\mathcal{O}_n^+$ be as in (1.5). For $\omega \in \mathcal{S}(\mathcal{O}_n)$, assume that there exists a sequence $(a_i)_{i \geq 1}$ of isometries in $\mathcal{O}_n^+$ which satisfies

$$\omega(a_1 \cdots a_l a_k^* \cdots a_1^*) = \delta_{lk} \quad \text{for all } l, k \geq 1.$$  

(1.6)

Then $\omega$ is properly infinitely correlated.

By using Theorem 1.9, we will show examples of properly infinitely correlated state in §3.2.

### 1.3 Summary of results

We summarize the significance and advantages of $\kappa$.

(i) **Refinement of definitions**: According to [6], there exist two classes of states on $\mathcal{O}_n$, that is, finitely correlated states and infinitely correlated states (Definition 1.7(i)). From Proposition 3.20, it becomes clear that this classification is incompatible with the unitary equivalence of states. For example, Figure 1 in [25] is exceedingly inappropriate. Instead of these notions, essentially finitely/properly infinitely correlated states are established by using $\kappa$ (Fact 1.8).

(ii) **New classification method**: As a classification theory of representations of $\mathbb{C}^*$-algebras, the Murray-von Neumann-Connes classification [32, 13] is well known, that is, for a given factor representation, its type is determined by the type of the von Neumann algebra generated by its image. Unfortunately, this classification is no use for the classification of irreducible representations of $\mathcal{O}_n$ because the type of any irreducible representation of $\mathcal{O}_n$ is type $I_\infty$. As a finer classification of irreducible representations of $\mathcal{O}_n$, $\kappa$ is essentially new ($\S$4.1).

(iii) **Invariant for arbitrary states**: Until now, there exist several small subclasses of states or representations of $\mathcal{O}_n$ which are often completely
classified \[1, 3, 4, 7, 17, 18, 20, 24, 25, 26\]. They are often parameterized by their complete sets of invariants. On the other hand, \(\kappa\) can be defined on the whole of states (see also (4.2)).

(iv) **Reduction**: The new notion “minimal state” reduces the classification problem of states to that of minimal states from Theorem 1.1(i).

(v) **Many examples**: The existence of many examples firmly establishes that the theory of \(\kappa\) is not vacuous. In \(\S\) 1.1.2 and \(\S\) 3, examples are shown and their values of \(\kappa\) are computed. Especially, we will show that the cardinality of the set of mutually inequivalent pure states with same invariant number are uncountable in Theorem 3.24.

(vi) **Generalization of symbolic dynamical system**: In Remark 3.23, we will show that \(\kappa\) is a generalization of period length in the full one-sided shift. This implies a naturality of \(\kappa\).

(vii) **Applications**: In \(\S\) 4, we will show that \(\kappa\) can be defined on both arbitrary irreducible representations of \(\mathcal{O}_n\) and arbitrary ergodic endomorphisms of \(\mathcal{B}(\mathcal{H})\) as their invariants.

The paper is organized as follows. In \(\S\) 2 we will prove Theorem 1.6 and Theorem 1.9. In \(\S\) 3 we will show examples. In \(\S\) 4 we will show applications.

## 2 Proofs of main theorems

In this section, we prove Theorem 1.6 and Theorem 1.9. For this purpose, we prove lemmas needed later.

### 2.1 Dutkay-Haussermann-Jorgensen theory and its generalization

We review a part of the work by Dutkay, Haussermann, and Jorgensen ([19], \(\S\) 3.1) which shows a kind of structure theorem of a representation space of \(\mathcal{O}_n\) in a general setting. Our analysis is dependent on their results to a great extent. We write “a representation of \(\mathcal{O}_n\)” to denote a unital \(*\)-representation of \(\mathcal{O}_n\) in this paper.

#### 2.1.1 Dutkay-Haussermann-Jorgensen decomposition

Let \(\mathcal{K}(\omega)\) and \(\mathcal{I}_n\) be as in (1.2).
Definition 2.1 Let \((\mathcal{H}, \pi)\) be a representation on \(\mathcal{O}_n\) and \(M\) a subspace.

(i) \((\text{[8], \S 1})\) \(M\) is said to be \(s_i^*\)-invariant if \(\pi(s_i)^* M \subset M\) for all \(i\).

(ii) \((\text{[19], Definition 2.4})\) \(M\) is said to be cyclic if \(\{\pi(s_J s_K^*) x : J, K \in \mathcal{I}_n, x \in M\}\) spans \(\mathcal{H}\).

Both \(\{0\}\) and \(\mathcal{H}\) are trivial \(s_i^*\)-invariant subspaces of \(\mathcal{H}\). Therefore any nonzero representation of \(\mathcal{O}_n\) contains a nonzero \(s_i^*\)-invariant subspace at least. If \(M\) contains a cyclic vector, then \(M\) is cyclic. If \((\mathcal{H}, \pi)\) is irreducible, then any nonzero subspace of \(\mathcal{H}\) is cyclic. For any state \(\omega\), \(K(\omega)\) in (1.2) is a closed cyclic \(s_i^*\)-invariant subspace of the GNS representation space of \(\omega\).

Remark 2.2 Let \(\mathcal{R}_n \subset \mathcal{O}_n\) denote the algebra generated by \(s_1^*, \ldots, s_n^*\) over \(\mathbb{C}\), that is,

\[
\mathcal{R}_n := \mathbb{C}\langle s_1^*, \ldots, s_n^* \rangle \tag{2.1}
\]

which consists of all noncommutative polynomials in \(s_1^*, \ldots, s_n^*\) over \(\mathbb{C}\). We see that \(\mathcal{R}_n\) is the free algebra generated by \(s_1^*, \ldots, s_n^*\) over \(\mathbb{C}\) \((\text{[12], \S 6.2})\). Remark that \(\mathcal{R}_n\) is not a self-adjoint algebra because \(s_1, \ldots, s_n \notin \mathcal{R}_n\). Clearly, the standard terminology of “\(s_i^*\)-invariant subspace” is just “left \(\mathcal{R}_n\)-module.” We use the conventional word “\(s_i^*\)-invariant” in this paper.

Theorem 2.3 (Dutkay-Haussermann-Jorgensen \([19]\)) Let \((\mathcal{H}, \pi)\) be a representation of \(\mathcal{O}_n\). If \(M\) is a closed cyclic \(s_i^*\)-invariant subspace of \(\mathcal{H}\), then there exists a unique orthogonal decomposition of \(\mathcal{H}\),

\[
\mathcal{H} = \bigoplus_{l \geq 0} \mathcal{H}_l \tag{2.2}
\]

such that \(\{\pi(s_J) v : J \in \{1, \ldots, n\}^k, v \in M\}\) spans \(\bigoplus_{l=0}^k \mathcal{H}_l\) for all \(k \geq 0\). We call (2.2) the Dutkay-Haussermann-Jorgensen (=DHJ) decomposition of \((\mathcal{H}, \pi)\) by \(M\).

Proof. The existence is proved in \(\S 3.1\) of \([19]\). The uniqueness holds from the properties of subspaces \(\bigoplus_{l=0}^k \mathcal{H}_l\) for \(k \geq 0\). \(\blacksquare\)

Let \(\mathcal{O}_n^+\) be as in (1.5). By definition, \(\mathcal{H}_0 = M\) and \(\pi(s_i)^* \mathcal{H}_l \subset \mathcal{H}_{l-1}\) and \(\pi(s_i) \mathcal{H}_l \subset \mathcal{H}_{l+1}\) for all \(i\) when \(l \geq 1\). From this, any \(x \in \mathcal{O}_n^+\) satisfies

\[
\pi(x^*) \mathcal{H}_l \subset \bigoplus_{k=0}^{l-1} \mathcal{H}_k \quad (l \geq 1). \tag{2.3}
\]

Theorem 2.3 indicates that an essential part of a representation of \(\mathcal{O}_n\) is its \(s_i^*\)-invariant subspace. In Theorem 2.3, \(\pi(s_i) \mathcal{H}_0\) is not a subspace of \(\mathcal{H}_1\) in general (Proposition 3.2). If \(M = \mathcal{H}\), then \(\mathcal{H}_l = \{0\}\) for all \(l \geq 1\).
2.1.2 Lemmas

The following are slight generalizations of Theorem 3.2 and Lemma 3.4 in [19].

**Lemma 2.4** Let $O^+_n$ be as in [19]. Assume that $O_n$ acts on a Hilbert space $\mathcal{H}$, $M$ is a closed cyclic $s^+_i$-invariant subspace of $\mathcal{H}$, and $a = (a_i)_{i \geq 1}$ is a sequence of isometries in $O^+_n$. Let $a[l] := a_1 \cdots a_l$ for $l \geq 1$.

(i) For any $\varepsilon > 0$ and $v \in \mathcal{H}$, there exists $l_0 \geq 1$ such that

$$\|P_M a[l]^* v - a[l]^* v\| < \varepsilon \quad \text{for all } l \geq l_0$$

where $P_M$ denotes the projection from $\mathcal{H}$ onto $M$.

(ii) Define the projection $T := \bigwedge_{l \geq 1} a[l] a[l]^*$ on $\mathcal{H}$. If $\dim M < \infty$, then $T \mathcal{H} \subset M$.

**Proof.** (i) Let $\mathcal{H} = \bigoplus_{l \geq 0} \mathcal{H}_l$ denote the DHJ decomposition by $M$ (Theorem 2.3). From (2.3), $a_i^* \mathcal{H}_l \subset \bigoplus_{k=0}^{l-1} \mathcal{H}_k$ for all $i \geq 1$ when $l \geq 1$. We can find $l_0 \geq 1$ and vectors $v_1, v_2 \in \mathcal{H}$ such that $v = v_1 + v_2$, $v_1 \in \bigoplus_{l=0}^{l_0} \mathcal{H}_l$, $v_2 \in \bigoplus_{l > l_0} \mathcal{H}_l$, and $\|v_2\| < \varepsilon$. Then $\|v_2\| < \varepsilon$ and $a[l]^* v_1 \in \mathcal{H}_0 = M$ for all $l \geq l_0$. Hence $(P_M - I) a[l]^* v_1 = 0$ for all $l \geq l_0$. From this,

$$P_M a[l]^* v - a[l]^* v = (P_M - I) a[l]^* v = (P_M - I) a[l]^* (v_1 + v_2) = (P_M - I) a[l]^* v_2.$$  \hspace{1cm} (2.5)

Therefore

$$\|P_M a[l]^* v - a[l]^* v\| \leq \|P_M - I\| \|a[l]^*\| \|v_2\| \leq \|v_2\| < \varepsilon.$$  \hspace{1cm} (2.6)

(ii) Remark that $a[l]^* a[l] = I$ for all $l$ because $a[l]$ is a product of isometries. Since $\{a[l]^* a[l] : l \geq 1\}$ is a decreasing sequence of projections on $\mathcal{H}$, $T$ is well defined. It is sufficient to show the case of $T \neq 0$. Assume $T \neq 0$. We prove $(T \mathcal{H} \cap M) \perp T \mathcal{H} = \{0\}$. By definitions of $T$ and $a[l]$, we see that $a[l]^*$ is a unitary on $T \mathcal{H}$ for all $l \geq 1$. Since $a[l]^* (T \mathcal{H} \cap M) \subset T \mathcal{H} \cap M$ and $\dim M < \infty$, we obtain $a[l]^* (T \mathcal{H} \cap M) = T \mathcal{H} \cap M$. This implies $a[l]^* \{(T \mathcal{H} \cap M) \perp T \mathcal{H}\} = (T \mathcal{H} \cap M) \perp T \mathcal{H}$. Hence $a[l]^*$ is also a unitary from $(T \mathcal{H} \cap M) \perp T \mathcal{H}$ onto $(T \mathcal{H} \cap M) \perp T \mathcal{H}$.

Assume $v \in (T \mathcal{H} \cap M) \perp T \mathcal{H}$. Then

$$a[l]^* v \in (T \mathcal{H} \cap M) \perp T \mathcal{H} \quad (l \geq 1).$$  \hspace{1cm} (2.7)
From (i),
\[ \| a[l] v - P_{M} a[l] v \| \to 0 \quad \text{when } l \to \infty. \]  
(2.8)

Since \( \dim M < \infty \), the sequence \( \{ P_{M} a[l] v : l \geq 1 \} \) has a convergent subsequence \( \{ P_{M} a[l] v : i \geq 1 \} \) in the compact subset \( M' := \{ x \in M : \| x \| \leq \| v \| \} \) of \( M \). Let
\[ v_{\infty} := \lim_{i \to \infty} P_{M} a[l] v \in P_{M} \mathcal{H} = M. \]  
(2.9)

From this, (2.8), and (2.7), we obtain
\[ v_{\infty} = \lim_{i \to \infty} a[l] v \in (M \cap \mathcal{T} \mathcal{H}) ^{\perp} \cap \mathcal{TH}. \]  
From this and (2.9),
\[ v_{\infty} \in M \cap (M \cap \mathcal{T} \mathcal{H}) ^{\perp} \cap \mathcal{TH} = (M \cap \mathcal{T} \mathcal{H}) ^{\perp} \cap (M \cap \mathcal{T} \mathcal{H}) = \{ 0 \}. \]  
(2.10)

Hence \( v_{\infty} = 0 \). On the other hand, since \( a[l] ^{*} \) is a unitary on \( (M \cap \mathcal{T} \mathcal{H}) ^{\perp} \cap \mathcal{TH} \), we obtain \( 0 = \| v_{\infty} \| = \lim_{i \to \infty} \| a[l] v \| = \| v \| \). Hence \( v = 0 \). Therefore
\[ (\mathcal{T} \mathcal{H} \cap M)^ {\perp} \cap \mathcal{TH} = \{ 0 \}. \]

In (2.6), if \( M = \mathcal{H} \), then \( \| P_{M} - I \| = 0 \). Hence \( \| a[l] ^{*} \| \| v_{2} \| \leq \| v_{2} \| \) can not be replaced with \( \| P_{M} - I \| \| a[l] ^{*} \| \| v_{2} \| = \| v_{2} \| \) in general.

**Lemma 2.5** Assume that \( \mathcal{O}_{n} \) acts on a Hilbert space \( \mathcal{H} \) and \( M \) is a finite-dimensional cyclic \( s_{i} ^{*} \)-invariant subspace of \( \mathcal{H} \). If \( \Omega \in \mathcal{H} \) satisfies \( u \Omega = \Omega \) for some isometry \( u \) in \( \mathcal{O}_{n} ^{+} \), then \( \Omega \in M \).

**Proof.** In Lemma 2.3 let \( a_{i} := u \) for all \( i \geq 1 \). By assumption, we obtain \( T \Omega = \Omega \). From this and Lemma 2.4(ii), \( \Omega = T \Omega \in \mathcal{T} \mathcal{H} \subset M \).

### 2.2 Proofs of theorems

Recall \( \mathcal{K}(\omega) \), \( \text{cdim} \omega \) and \( \kappa(\omega) \) in §1.1.1.

**Proof of Theorem 1.6** We prove the equality \( \kappa(\omega) = \text{cdim} \omega \). This is equivalent to the following statement:
\[ \text{cdim} \omega \leq \text{cdim} \omega' \text{ for any } \omega' \in \mathcal{S}(\mathcal{O}_{n}) \text{ such that } \omega' \sim \omega. \]  
(2.11)

We prove (2.11) as follows. Let \( (\mathcal{H}, \pi, \Omega) \) denote the GNS representation of \( \omega \). By the assumption of \( \omega(u) = 1 \), we obtain \( \pi(u) \Omega = \Omega \). Assume that a state \( \omega' \) on \( \mathcal{O}_{n} \) satisfies \( \omega' \sim \omega \). Since \( \omega \sim \omega' \), we can identify \( M := \mathcal{K}(\omega') \) with a subspace of \( \mathcal{H} \). If \( \dim M = \infty \), then \( \text{cdim} \omega \leq \infty = \text{cdim} \omega' \). Hence (2.11) holds. If \( \dim M < \infty \), then \( M \) and \( \Omega \) satisfy the assumption in
Lemma 2.5. Hence $\Omega \in M$. This implies $K(\omega) \subset M = K(\omega')$. Hence $\text{cdim} \omega \leq \text{cdim} \omega'$.

From the proof of Theorem 1.6, the following holds.

**Corollary 2.6** For $\omega \in S(O_n)$ with GNS representation space $H$, assume that $\omega$ satisfies the assumption in Theorem 1.6 and $\text{cdim} \omega < \infty$. Then $K(\omega)$ is smallest in the sense that any nonzero finite-dimensional cyclic $s^*_i$-invariant subspace of $H$ contains $K(\omega)$ as a subspace.

**Proof of Theorem 1.9.** We prove $\kappa(\omega) = \infty$. This is equivalent to the following statement:

$$\text{cdim} \omega' = \infty \text{ for any } \omega' \in S(O_n) \text{ such that } \omega' \sim \omega. \quad (2.12)$$

Let $(H, \pi, \Omega)$ denote the GNS representation of $\omega$. For $l \geq 1$, let $a[l] := a_1 \cdots a_l$ and $v_l := \pi(a[l])^* \Omega$. From (1.6), $X := \{v_l : l \geq 1\}$ is an orthonormal system in $H$ and

$$\pi(a[l]a[l]^*)\Omega = \Omega \quad \text{for all } l \geq 1. \quad (2.13)$$

Since $X \subset K(\omega)$, $\text{cdim} \omega = \infty$. From (2.13), we obtain $T\Omega = \Omega$ where $T := \bigwedge_{l \geq 1} \pi(a[l]a[l]^*)$.

Assume that $\omega' \in S(O_n)$ satisfies $\omega' \sim \omega$. We identify $M := K(\omega')$ with a subspace of $H$. If $\text{cdim} \omega' < \infty$, then $M$ satisfies assumptions in Lemma 2.4(ii). Hence $\Omega = T\Omega \in TH \subset M$. From this, $K(\omega) \subset M = K(\omega')$. Hence $\infty = \text{cdim} \omega \leq \text{cdim} \omega' < \infty$. This is a contradiction. Hence $\text{cdim} \omega' = \infty$. Therefore (2.12) is proved.

**3 Examples**

In this section, we show examples of minimal states. For this purpose, we review properties of known states.

**3.1 Minimal states**

**3.1.1 Extensions of Cuntz states**

Recall from §1.2.1 the definition of a Cuntz state. For a unital C*-algebra $A$, a unital C*-subalgebra $B$ of $A$, and a state $\omega$ on $B$, $\omega'$ is an extension of $\omega$ to $A$ if $\omega'$ is a state on $A$ which satisfies $\omega'|_B = \omega$. The following is a corollary of Theorem 1.6.
Corollary 3.1 Assume $2 \leq n \leq m \leq \infty$. Let $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ denote Cuntz generators of $O_m$ and $O_n$, respectively. Assume that there exists a unital embedding $f$ of $O_m$ into $O_n$ (this requires $n \leq m$). We identify $O_m$ with $f(O_m) \subset O_n$. If $f(t_i) \in O_n^+$ for all $i$, then any extension of a Cuntz state on $O_m$ to $O_n$ is minimal.

Proof. Let $\omega$ be the Cuntz state on $O_m$ by $z = (z_1, \ldots, z_m) \in (C^m)_1$ and assume that $\omega'$ is an extension of $\omega$ to $O_n$. Let $t(z) := z_1 t_1 + \cdots + z_m t_m$ and $u := f(t(z))$. Then $u \in O_n^+$ and $u^* u = I$. By assumption, $\omega'(u) = \omega(t(z)) = 1$. From Theorem 1.6, the statement holds. When $m = \infty$, replace $C^m$ with $l^2$. Then the statement holds in a similar fashion.

Proposition 3.2 There exists a representation $(H, \pi)$ of $O_n$ with a closed cyclic $s_i^*$-invariant subspace $M$ of $H$ such that $\pi(s_i) H_0$ is not a subspace of $H$ for some $i \in \{1, \ldots, n\}$ where $H = \bigoplus_{i \geq 0} H_i$ denotes the DHJ decomposition by $M$.

Proof. Let $\omega$ be the Cuntz state on $O_n$ by $z = (1, 0, \ldots, 0)$ in $§ 1.1.2$. Then $K(\omega)$ equals $C\Omega$ for the GNS representation $(H, \pi, \Omega)$ of $\omega$, and it is a closed cyclic $s_i^*$-invariant subspace of $H$. For the DHJ decomposition by $K(\omega)$, $H_0 = C\Omega$. Since $\omega(s_1) = 1$, we see $\pi(s_1) \Omega = \Omega$. This implies $\pi(s_1) H_0 = H_0 \not\subset H_1$.

3.1.2 Sub-Cuntz states

Sub-Cuntz states were introduced by Bratteli and Jorgensen (7) as extensions of Cuntz states. We review results in [25]. For $1 \leq m < \infty$, let $V_{n,m}$ denote the complex Hilbert space with the orthonormal basis $\{e_J : J \in \{1, \ldots, n\}^m\}$, that is, $V_{n,m} = l^2(\{1, \ldots, n\}^m) \cong \mathbb{C}^{nm}$. Let $(V_{n,m})_1 := \{z \in V_{n,m} : \|z\| = 1\}$. When $n = \infty$, let $V_{\infty,m} := l^2(\{1, 2, \ldots\}^m)$.

Definition 3.3 For $z = \sum z_J e_J \in (V_{n,m})_1$, $\omega$ is a sub-Cuntz state on $O_n$ by $z$ if $\omega$ is a state on $O_n$ which satisfies the following equations:

$$\omega(s_J) = \overline{z_J} \quad \text{for all } J \in \{1, \ldots, n\}^m$$

(3.1)

where $s_J := s_{j_1} \cdots s_{j_m}$ when $J = (j_1, \ldots, j_m)$, and $\overline{z_J}$ denotes the complex conjugate of $z_J$. In this case, $\omega$ is called a sub-Cuntz state of order $m$. 

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When $n = \infty$, replace $V_{n,m}$ with $V_{\infty,m}$. A sub-Cuntz state $\omega$ of order 1 is just a Cuntz state. We identify $V_{n,m}$ with $(V_{n,1})^\otimes m$ by the correspondence between bases $e_J \mapsto e_{j_1} \otimes \cdots \otimes e_{j_m}$ for $J = (j_1, \ldots, j_m) \in \{1, \ldots, n\}^m$. From this identification, we obtain $V_{n,m} \otimes V_{n,l} = V_{n,m+l}$ for any $m, l \geq 1$. Then the following hold.

**Theorem 3.4**

(i) ([25], Fact 1.3) For any $z \in (V_{n,m})_1$, a sub-Cuntz state on $O_n$ by $z$ exists.

(ii) ([25], Theorem 1.4) For a sub-Cuntz state $\omega$ on $O_n$ by $z \in (V_{n,m})_1$, $\omega$ is unique if and only if $z$ is nonperiodic, that is, $z = x^\otimes p$ for some $x$ implies $p = 1$. In this case, $\omega$ is pure and we write it as $\tilde{\omega}_z$.

(iii) ([25], Theorem 1.5) Let $p \geq 2$ and $z = x^\otimes p$ for a nonperiodic element $x \in (V_{n,m})_1$. If $\omega$ is a sub-Cuntz state on $O_n$ by $z$, then $\omega$ is a convex hull of sub-Cuntz states by $e^{2\pi j\sqrt{-1}/p}x$ for $j = 1, \ldots, p$.

(iv) ([25], Theorem 1.7) For $z, y \in \bigcup_{m \geq 1} (V_{n,m})_1$, assume that both $z$ and $y$ are nonperiodic. Then the following are equivalent:

(a) GNS representations of $\tilde{\omega}_z$ and $\tilde{\omega}_y$ are unitarily equivalent.

(b) $z$ and $y$ are conjugate, that is, $z = y$, or $z = x_1 \otimes x_2$ and $y = x_2 \otimes x_1$ for some $x_1, x_2 \in \bigcup_{m \geq 1} (V_{n,m})_1$. 

When $n < \infty$, any sub-Cuntz state on $O_n$ is finitely correlated ([25], Lemma 2.4(i)). Furthermore, the following holds.

**Proposition 3.5** Any sub-Cuntz state is minimal.

**Proof.** For $z = \sum_J z_J e_J \in (V_{n,m})_1$, let $\omega$ be a sub-Cuntz state on $O_n$ by $z$ and let $u := \sum_J z_J s_J \in O_n^+$. Then $u^* u = I$. From (3.1), $\omega(u) = 1$. By Theorem 1.6 $\omega$ is minimal. When $n = \infty$, replace $V_{n,m}$ by $V_{\infty,m}$. Then the statement is verified in a similar way.

Proposition 3.5 can be also proved by using Corollary 3.1 ([26], § 2.2).

**Proposition 3.6**

(i) There exists a pure state on $O_n$ which is not minimal.

(ii) There exists a minimal state on $O_n$ which is not pure.
Proof. (i) Let $\omega$ be the Cuntz state by $z = (1, 0, \ldots, 0)$ with GNS representation $(\mathcal{H}, \pi, \Omega)$. Let $\omega' := \omega(s_2^*(\cdot)s_2)$. We identify $\mathcal{K}(\omega)$ with a subspace of $\mathcal{H}$. Then $\mathcal{K}(\omega')$ is spanned by the orthonormal set $\{\Omega, \pi(s_2)\Omega\}$. Hence $\text{cdim}\, \omega' = 2$. Since $\omega' \sim \omega$ and $\text{cdim}\, \omega = 1$, $\omega'$ is pure but not minimal.

(ii) Let $\omega_{\pm}$ denote the the Cuntz state by $(\pm 1, 0, \ldots, 0) \in (\mathbb{C}^n)_1$, respectively. Define $\omega'' = (\omega_+ + \omega_-)/2$. Since $\omega_+ \not\sim \omega_-$ from Theorem 1.4, $\omega''$ is not pure. On the other hand, we can prove $\omega''(s_2s_1) = 1$. Hence $\omega''$ is a sub-Cuntz state on $O_n$ by $z = e_1^\otimes 2 \in (V_n, 2)_1$. Therefore it is minimal from Proposition 3.5.

Proposition 3.7 A minimal model of a state is not unique in general.

Proof. Let $\omega$ and $\omega'$ be states on $O_n$ such that $\omega(s_1s_2) = 1 = \omega'(s_2s_1)$. Then they are pure sub-Cuntz states which exist uniquely, and $\omega \sim \omega'$ from Theorem 3.4(i)~(iv). From Proposition 3.5, they are minimal. From $\omega'(s_1s_2) = \omega'(s_2s_1)^*s_1s_2 = 0$. Therefore $\omega \not\sim \omega'$.

If $\kappa(\omega) = 1$, then a minimal model of $\omega$ is unique from Fact 1.5 and Theorem 1.4.

Example 3.8 Let $\omega$ and $\omega'$ be states on $O_n$ which satisfy $\omega(s_1s_2) = 1$ and $\omega'(s_1s_1 + s_1s_2) = \sqrt{2}$. Then such states are pure sub-Cuntz states from Theorem 3.4(ii). From Proposition 3.5, they are minimal and we can prove $\kappa(\omega) = \kappa(\omega') = 2$, but $\omega' \not\sim \omega \circ \alpha_g$ for any $g \in U(n)$ (see also Theorem 4.1(iv) in [25]).

3.1.3 Geometric progression states

Geometric progression states were introduced in [26] as extensions of Cunts states with respect to different embeddings of Cunt algebras from the case of sub-Cuntz states. Assume $2 \leq n < \infty$ in this section.

Definition 3.9 Let $\omega \in S(O_n)$ and $m := (n - 1)k + 1$ for $k \geq 2$.

(i) $\omega$ is a geometric progression state by $z = (z_1, \ldots, z_m) \in (\mathbb{C}^m)_1 := \{y \in \mathbb{C}^m : \|y\| = 1\}$ if $\omega$ satisfies

$$
\begin{align*}
\omega(s_n^r s_i) &= \frac{1}{\pi[(n-1)r+i]} \quad (r = 0, 1, \ldots, k-1, i = 1, \ldots, n-1), \\
\omega(s_n^k) &= \frac{1}{z_m}.
\end{align*}
$$

(3.2)
(ii) \( \omega \) is a geometric progression state by \( z = (z_1, z_2, \ldots) \in \ell_1^2 := \{ y \in \ell^2 : \|y\| = 1 \} \) if \( \omega \) satisfies

\[
\omega(s^n r_i) = z^{(n-1)r+i} (r \geq 0, i = 1, \ldots, n-1). \tag{3.3}
\]

**Theorem 3.10** \((\text{[26]})\)

(i) For \( k \geq 2 \), let \( m = (n-1)k + 1 \). For \( z = (z_1, \ldots, z_m) \in (\mathbb{C}^m)_1 \), a geometric progression state on \( O_n \) by \( z \) is unique if and only if \( |z_m| < 1 \).

In this case, it is pure. We write this \( \omega'_z \).

(ii) For any \( z \in \ell_1^2 \), a geometric progression state on \( O_n \) by \( z \) is unique and pure. We write this \( \omega'_z \).

In Theorem 3.10(i), if \( k = 1 \), then \( m = n \) and \( \omega \) is just a Cuntz state.

**Theorem 3.11** \((\text{[26], Theorem 1.8})\)

Let \( \omega'_z \) be as in Theorem 3.10.

(i) For \( m = (n-1)k + 1 \) with \( k \geq 2 \), let \( W_m := \{ (w_1, \ldots, w_m) \in (\mathbb{C}^m)_1 : |w_m| < 1 \} \). For \( z, y \in W_m \), \( \omega'_z \sim \omega'_y \) if and only if \( z = y \).

(ii) For \( z, y \in \ell_1^2 \), \( \omega'_z \sim \omega'_y \) if and only if \( z = y \).

**Theorem 3.12** \((\text{[26], Theorem 1.9(i)})\)

Let \( \{e_i\} \) denote the standard basis of \( \ell^2 \) and let \( z \in \ell_1^2 \). For \( y = (y_1, \ldots, y_n) \in (\mathbb{C}^n)_1 \), let \( \omega_y \) be as in (1.4). Then \( \omega'_z \sim \omega'_y \) if and only if \( |y_n| < 1 \) and \( z = \tilde{y} \) where \( \tilde{y} \in \ell_1^2 \) is defined as

\[
\tilde{y} := \sum_{r=0}^{n-1} \sum_{i=1} y_{ni} e_{(n-1)r+i}. \tag{3.4}
\]

**Theorem 3.13** \((\text{[26], Theorem 1.10(iv)})\)

Assume \( m = (n-1)k + 1 \) and \( k \geq 2 \). Let \( z \in W_m \) and \( y = (y_1, \ldots, y_n) \in (\mathbb{C}^n)_1 \). Let \( \omega_y \) be as in (1.4). Then \( \omega'_z \sim \omega'_y \) if and only if \( |y_n| < 1 \) and \( z = \hat{y} \) where \( \hat{y} \in (\mathbb{C}^m)_1 \) is defined as

\[
\hat{y} := \sum_{r=0}^{k-1} \sum_{j=1}^{n-1} y_{nj} e_{(n-1)r+j} + y_{nk} e_m \tag{3.5}
\]

where \( \{e_j\} \) denotes the standard basis of \( \mathbb{C}^m \).

**Theorem 3.14** \((\text{[26], Theorem 1.11})\)

For \( n < \infty \), any geometric progression state \( \omega \) on \( O_n \) of order \( k < \infty \) satisfies \( \dim K(\omega) \leq k \). Especially, \( \omega \) is finitely correlated.
Corollary 3.15  

(i) For \( z \in \ell_2^1 \), \( \kappa(\omega'_z) \geq 2 \) if and only if \( z \) can not be written as \( \tilde{y} \) in (3.4).

(ii) For any \( z \in W'_{(n-1)k+1} \), \( \kappa(\omega'_z) \leq k \). In addition, \( 2 \leq \kappa(\omega'_z) \leq k \) if and only if \( z \) can not be written as \( \hat{y} \) in (3.5).

Proof. (i) From Theorem 3.12 and Fact 1.5(ii), the statement holds.

(ii) Recall that \( \kappa(\omega) \leq \operatorname{cdim}\omega = \dim K(\omega) \) for any \( \omega \in S(O_n) \). From Theorem 3.14, the former statement holds. From Theorem 3.13, Fact 1.5(ii), and the former, the latter holds.

Proposition 3.16  Any geometric progression state is minimal.

Proof. Assume \( m = (n-1)k + 1 \). For \( z = (z_1, \ldots, z_m) \in (\mathbb{C}^m)_1 \), let \( \omega \) be a geometric progression state on \( O_n \) by \( z \). Let \( u_r := \sum_{i=1}^{n-1} z_{(n-1)r+i} s_i + z_m s_n^k \in O_n^+ \). Then \( u^*u = I \). By (3.2) and (3.3), \( \omega(u) = 1 \). From Theorem 1.9, \( \omega \) is minimal.

When \( m = \infty \), let \( u := \sum_{r \geq 0} \sum_{i=1}^{n-1} z_{(n-1)r+i} s_i s_i \in O_n^+ \). Then the statement holds as in the previous case.

Proposition 3.16 can be also proved by using Corollary 3.1 ([26], § 1.2.3).

3.2 Properly infinitely correlated states

In this subsection, we show examples of properly infinitely correlated states. Let \( \mathbb{N} := \{1, 2, \ldots\} \).

3.2.1 States associated with permutative representations

Let \( \{e_{k,m} : (k, m) \in \mathbb{N} \times \mathbb{Z}\} \) denote the standard basis of \( \ell^2(\mathbb{N} \times \mathbb{Z}) \). For \( 2 \leq n < \infty \), define the representation \( \pi \) of \( O_n \) on \( \ell^2(\mathbb{N} \times \mathbb{Z}) \) by

\[
\pi(s_i) e_{k,m} := e_{n(k-1)+i, m+1} \quad ((k, m) \in \mathbb{N} \times \mathbb{Z}, i = 1, \ldots, n).
\]

By definition, \( \pi(s_1^m) e_{1,0} = e_{1,-m} \) for any \( m \geq 1 \). Define \( \omega := \langle e_{1,0} | \pi(\cdot) e_{1,0} \rangle \).

Let \( a_i := s_i \in O_n^+ \) for all \( i \geq 1 \). Then

\[
\omega(a[k]a[l]^*) = \omega(s_1^k s_1^l) = \langle e_{1,-k} | e_{1,-l} \rangle = \delta_{k,l} \quad (l, k \geq 1).
\]

From Theorem 1.9, \( \omega \) is properly infinitely correlated.

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3.2.2 Induced product states

In this subsection, we review induced product representations [11, 3, 4] and introduce induced product states. We give a parametrization of induced product states by one-sided infinite sequences of unit complex vectors.

Let \((\mathbb{C}_1^n) := \{z \in \mathbb{C}^n : \|z\| = 1\}\). For a sequence \(z \in (\mathbb{C}_1^n)_{\infty} := \{(z^{(i)})_{i \geq 1} : z^{(i)} \in (\mathbb{C}_1^n)_{\infty}\}\) and \(J = (j_1, \ldots, j_m) \in \{1, \ldots, n\}^m\), define \(z_J := z_{j_1}^{(1)} \cdots z_{j_m}^{(m)}\) for \(m \geq 1\) and \(z_0 := 1\).

**Definition 3.17** For \(z = (z^{(i)}) \in (\mathbb{C}_1^n)_{\infty}\), define the state \(\omega_z\) on \(\mathcal{O}_n\) as

\[
\omega_z(s_J s_K^*) := \begin{cases} 
\overline{z_J} z_K & \text{when } |J| = |K|, \\
0 & \text{(otherwise)}
\end{cases}
\]

(3.8)

for \(J, K \in \mathcal{I}_n\) where \(|J|\) denotes the length of a word \(J\). We call \(\omega_z\) the induced product state by \(z\).

**Theorem 3.18** (i) For any \(z \in (\mathbb{C}_1^n)_{\infty}\), \(\omega_z\) exists uniquely.

(ii) For \(z, y \in (\mathbb{C}_1^n)_{\infty}\), \(\omega_z \sim \omega_y\) if and only if there exists \(k \geq 0\) such that \(\sum_{i=1}^{\infty} (1 - |\langle z^{(i)}| y^{(l+k)} \rangle|) < \infty\) or \(\sum_{i=1}^{\infty} (1 - |\langle z^{(i)}| y^{(l+k)} \rangle|) < \infty\).

(iii) For \(z \in (\mathbb{C}_1^n)_{\infty}\), \(\omega_z\) is pure if and only if \(\sum_{i=1}^{\infty} (1 - |\langle z^{(i)}| z^{(l+k)} \rangle|) = \infty\) for any \(k \geq 1\). In this case, \(z\) is said to be aperiodic (36).

**Proof.** (i) Let \(\gamma\) denote the \(U(1)\)-gauge action on \(\mathcal{O}_n\), that is, \(\gamma_z(s_i) := zs_i\) for all \(i = 1, \ldots, n\) and \(z \in U(1)\). Let \(P\) denote the conditional expectation from \(\mathcal{O}_n\) to \(\mathcal{O}^{U(1)}_n := \{x \in \mathcal{O}_n : \gamma_z(x) = x\}\) for all \(z \in U(1)\) \(\cong UHF_n\). Let \(E_{JK} := s_J s_K^*\) for \(J, K \in \mathcal{I}_n\). Then \(\mathcal{O}^{U(1)}_n = \text{Lin}(\{E_{JK} : J, K \in \mathcal{I}_n, |J| = |K|\})\).

For \(z = (z^{(i)}) \in (\mathbb{C}_1^n)_{\infty}\), define the state \(F_z\) on \(\mathcal{O}^{U(1)}_n\) by

\[
F_z(E_{JK}) := \overline{z_J} z_K \quad (J, K \in \{1, \ldots, n\}^l, l \geq 1).
\]

(3.9)

By the natural identification \(\text{Lin}(\{E_{JK} : J, K \in \{1, \ldots, n\}^l\}) \cong M_n(\mathbb{C})^\otimes l\), \(F_z\) is identified with the product state \(\bigotimes_{i \geq 1} \langle z^{(i)}| z^{(i)} \rangle\) on \(M_n(\mathbb{C})^\otimes \infty\). Then we can verify \(\omega_z = F_z \circ P\). Hence the statement holds.

(ii) See Theorem 3.16 of [3].

(iii) See Corollary 3.17 of [3].
Proposition 3.19 Any induced product state is properly infinitely correlated.

Proof. Let \( z = (z^{(i)}) \in (\mathbb{C}^n)_{1}^{\infty} \) and \( z^{(i)} = (z_{1}^{(i)}, \ldots, z_{n}^{(i)}) \). For \( i \geq 1 \), let \( a_{i} := \sum_{j=1}^{n} z_{j}^{(i)} s_{j} \in \mathcal{O}_{n}^{+} \). Then \( a_{i}^{*} a_{i} = I \) for all \( i \) and

\[
\omega_{z}(a_{i}^{*} a_{i}^{(i)*}) = \sum_{|J|=l, |K|=k} z_{J}^{(i)} \omega_{z}(s_{J}^{(i)} s_{K}^{*}) = \delta_{l,k} \sum_{|J|=l, |K|} |z_{J}|^{2} |z_{K}|^{2} = \delta_{l,k}
\]

for \( l, k \geq 1 \). From Theorem 1.9, \( \omega_{z} \) is properly infinitely correlated. \( \blacksquare \)

3.3 Example of an essentially finitely correlated state which is not finitely correlated

Any finitely correlated state is essentially finitely correlated, but the converse is not true.

Proposition 3.20 For any \( 2 \leq n \leq \infty \), there exists an essentially finitely correlated state on \( \mathcal{O}_{n} \) which is not finitely correlated.

Proof. Let \( \omega \) be the Cuntz state on \( \mathcal{O}_{n} \) such that \( \omega(s_{1}) = 1 \). Then the following state \( \omega' \) on \( \mathcal{O}_{n} \) is essentially finitely correlated, but not finitely correlated:

\[
\omega'(x) := \sum_{l \geq 1} 2^{-l} \omega(A_{l}^{*} x A_{l}) \quad (x \in \mathcal{O}_{n})
\]

(3.11)

where \( A_{l} := s_{l}^{-1} s_{1} s_{l} \in \mathcal{O}_{n} \) for \( l \geq 1 \). In order to show this, we prove

\[
\kappa(\omega') = 1 \text{ and } \cdim \omega' = \infty.
\]

(3.12)

Since \( ||A_{l}|| = 1 \) for all \( l \) and \( \omega(s_{1}) = 1 \), we see that \( \omega(A_{l}^{*} \cdot A_{l}) \) is also a state on \( \mathcal{O}_{n} \) and it is equivalent to \( \omega \) for all \( l \). Therefore \( \omega' \) is also equivalent to \( \omega \). Hence we obtain \( \kappa(\omega') = 1 \) because \( \kappa(\omega) = 1 \). Since \( A_{l}^{*} A_{l} = \delta_{l,1} I \), we see \( \omega'(x) = \omega(A_{l}^{*} x A_{l}) \) for \( x \in \mathcal{O}_{n} \) where \( A := \sum_{l \geq 1} 2^{-l/2} A_{l} \in \mathcal{O}_{n} \). Let \( (\mathcal{H}, \pi, \Omega) \) denote the GNS representation of \( \omega \). For \( x \in \mathcal{O}_{n} \), we write \( \pi(x) \) as \( x \) for short. Then we can write \( \omega' = \langle A \Omega | \cdot | A \Omega \rangle \) and

\[
\mathcal{K}(\omega') = \lim \{ \{ s_{J}^{*} A \Omega : J \in \mathcal{I}_{n} \} \}.
\]

(3.13)

For \( l \geq 1 \), define \( v_{l} \in \mathcal{K}(\omega') \) by \( v_{l} := (s_{l}^{-1} s_{1})^{\ast} A \Omega \). Then \( v_{l} = 2^{-l/2} s_{l} \Omega \neq 0 \) for all \( l \), and \( \langle v_{l} | v_{l} \rangle = 2^{-(l+l)/2} \omega((s_{l}^{\ast})^{\ast} s_{l}^{\ast}) = \delta_{l,l}/2^{l} \) because \( \omega(s_{l}^{\ast}) = 0 \) for all \( l \geq 1 \). Therefore \( \{ v_{l} : l \geq 1 \} \) is an infinite orthogonal system in \( \mathcal{K}(\omega') \). This implies \( \cdim \omega' = \dim \mathcal{K}(\omega') = \infty. \)

\( \blacksquare \)
3.4 Shift representation

We review the shift representation of $O_n$ [7, 27]. Fix $2 \leq n \leq \infty$. Define $\Lambda := \{1, \ldots, n\}^{\infty}$ when $2 \leq n < \infty$, and $\Lambda := \{1, 2, \ldots\}^{\infty}$ when $n = \infty$. Let $\mathcal{H} := l^2(\Lambda)$ and define the representation $\Pi$ of $O_n$ on $\mathcal{H}$ by

$$\Pi(s_i)e_x := e_{ix} \quad (i = 1, \ldots, n, x \in \Lambda)$$

(3.14)

where $\{e_x : x \in \Lambda\}$ denotes the standard basis of $\mathcal{H}$ and $ix$ denotes the concatenation of two words $i$ and $x$ [31]. The data $(\mathcal{H}, \Pi)$ is called the shift representation of $O_n$ [7]. Let $\sim$ denote the tail equivalence in $\Lambda$ [7], that is, for $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \Lambda$, we write $x \sim y$ if there exist $p, q \geq 1$ such that $x_{k+p} = y_{k+q}$ for all $k \geq 1$. For $x \in \Lambda$, $x$ is said to be eventually periodic if there exist $i_0, p \geq 1$ such that $x_{i+p} = x_i$ for all $i \geq i_0$. Otherwise, $x$ is said to be non-eventually periodic. Define $\hat{\Lambda} := \Lambda/\sim$. For $x \in \Lambda$, we write $[x] := \{y \in \Lambda : y \sim x\} \in \hat{\Lambda}$. Then the following is known.

**Proposition 3.21**

(i) The following irreducible decomposition holds:

$$\mathcal{H} = \bigoplus_{[x] \in \hat{\Lambda}} \mathcal{H}_{[x]}$$

(3.15)

where $\mathcal{H}_{[x]}$ denotes the closed subspace of $\mathcal{H}$ generated by the set $\{e_y : y \in [x]\}$.

(ii) For $x \in \Lambda$, let $\Pi_{[x]}$ denote the subrepresentation of $\Pi$ associated with the subspace $\mathcal{H}_{[x]}$. Then $\Pi_{[x]}$ and $\Pi_{[y]}$ are unitarily equivalent if and only if $x \sim y$. Especially, (3.15) is multiplicity free.

**Proof.** See chapter 6 of [7] and Proposition 2.5 of [27].

In addition to Proposition 3.21, we show the following.

**Proposition 3.22**

(i) For $x \in \Lambda$, define $\omega_x := \langle e_x | \Pi(\cdot) e_x \rangle$ (see also [15], 3.1 Proposition). Then $\omega_x$ is a pure state on $O_n$ and the following hold:

(a) If $x \in \Lambda$ is non-eventually periodic, then $\omega_x$ is properly infinitely correlated, that is, $\kappa(\omega_x) = \infty$.

(b) If $x \in \Lambda$ is eventually periodic with the period length $d$, then $\kappa(\omega_x) = d$.

(ii) Let $\omega_x$ be as in (i). If $x \in \Lambda$ is eventually periodic, then the following are equivalent:
(a) \( \omega_x \) is minimal.

(b) \( x = (x_1, x_2, \ldots) \) is purely periodic, that is, there exists \( d \geq 1 \) such that \( x_{i+d} = x_i \) for all \( i \geq 1 \).

**Proof.** (i) The purity of \( \omega_x \) holds from the irreducibility of \( \Pi_{[x]} \).

(a) Assume that \( x = (x_1, x_2, \ldots) \in \Lambda \) is non-eventually periodic. Define \( a_i := s_{x_i} \in O_n^+ \). Then \( a_i^*a_i = I \) for all \( i \). Since \( x \) is non-eventually periodic, we see that \( \omega_x(a[k]a[l]^*) = \delta_{kl} \) for all \( k, l \geq 1 \). From Theorem 1.9, \( \omega_x \) is properly infinitely correlated.

(b) Assume that \( x \in \Lambda \) has a minimal repeating block \( x' \in \{1, \ldots, n\}^d \). Then there exists \( x'' \in \{1, \ldots, n\}^e \) such that \( x = x''x'x'' \cdots \). Remark that \( x' \) is not periodic by definition. Define \( \hat{x}' := x'x'x'' \cdots \in \Lambda \). Then \( \omega_x \sim \omega_{\hat{x}} \) because \( x \sim \hat{x} \) and Proposition 3.21(ii). Let \( (\mathcal{H}', \pi', \Omega) \) denote the GNS representation of \( \omega_{\hat{x}} \). When \( x' = (j_1, \ldots, j_d) \), let \( v_k := \pi'(s_{j_k} \cdots s_{j_d}) \Omega \) for \( k = 1, \ldots, d \). Then we can verify that \( \{v_k : k = 1, \ldots, d\} \) is an orthonormal basis of \( K(\omega_{\hat{x}}) \) because \( x' \) is not periodic. Therefore \( \text{cdim} \omega_{\hat{x}} = d \). Let \( u := s_{j_1} \cdots s_{j_d} \in O_n^+ \). Then \( u^*u = I \) and \( \omega_{\hat{x}}(u) = 1 \). Therefore \( \omega_{\hat{x}} \) is minimal from Theorem 1.6. From this, \( \kappa(\omega_x) = \kappa(\omega_{\hat{x}}) = \text{cdim} \omega_{\hat{x}} = d \).

(ii) Let \( d \) be the period length of \( x \). Let \( x = (x_1, x_2, \ldots) \) and define \( x^{(i)} := (x_i, x_{i+1}, \ldots) \in \Lambda \) for \( i \geq 1 \). From (i)(b), \( \kappa(\omega_x) = d \) and \( x^{(i+d)} = x^{(i)} \) for \( i \geq i_0 \) for some \( i_0 \geq 1 \). From Proposition 3.21(i) and (ii), we can identify \( K(\omega_x) \) with a subspace of \( \mathcal{H}[x] \) generated by \( X_x := \{\Pi(s_j)^*e_x : J \in \mathcal{I}_n\} \setminus \{0\} \). From (3.14), we see \( X_x = \{e_{x^{(i)}} : i \geq 1\} \). From this and \( \langle e_{x^{(i)}}, e_{x^{(j)}} \rangle = \delta_{x^{(i)}, x^{(j)}} \), we obtain

\[
\text{cdim} \omega_x = \#X_x. \tag{3.16}
\]

(a)\(\Rightarrow\)(b). Assume that \( \omega_x \) is minimal. From this and (3.16), \( \#X_x = \text{cdim} \omega_x = \kappa(\omega_x) = d \). Therefore \( X_x = \{e_{x^{(1)}}, \ldots, e_{x^{(d)}}\} \). Hence \( x \) is purely periodic.

(b)\(\Rightarrow\)(a). Assume that \( x \) is purely periodic. Then \( X_x = \{e_{x^{(1)}}, \ldots, e_{x^{(d)}}\} \). From (3.16), \( \text{cdim} \omega_x = \#X_x = d = \kappa(\omega_x) \). Hence \( \omega_x \) is minimal.

**Remark 3.23** We give an interpretation of the invariant \( \kappa \) as the theory of symbolic dynamical systems from Proposition 3.22. For an eventually periodic element \( x \in \Lambda \), let \( d(x) \) denote the period length of \( x \), that is, the length of a minimal repeating block of \( x \). For a non-eventually periodic element \( x \in \Lambda \), we define \( d(x) := \infty \). Then the map

\[
d : \Lambda \to \{1, 2, \ldots, \infty\} \tag{3.17}
\]
is surjective, and if \( x \sim y \), then \( d(x) = d(y) \), that is, \( d \) is an invariant of elements in the orbit space \( \Lambda \). By using \( \kappa \), we can write

\[
\kappa(\omega_x) = d(x) \quad (x \in \Lambda)
\]  

(3.18)

where \( \omega_x \) denotes the state in Proposition 3.22(i). Therefore the invariant \( \kappa(\omega) \) of a state \( \omega \) can be regarded as a generalization of the period length of an orbit of the full one-sided shift on \( \Lambda \) [28]. This perspective is natural in a sense that a Cuntz algebra is a special Cuntz-Krieger algebra [16] and Cuntz-Krieger algebras were introduced as a class of C\(^*\)-algebra associated with topological Markov chains. From Theorem 3.22(ii), the minimality of a state is also interpreted as the pure periodicity of an element in \( \Lambda \).

### 3.5 Cardinality of minimal pure states

**Theorem 3.24** For any \( 2 \leq n \leq \infty \) and \( 1 \leq d \leq \infty \), there exist continuously many mutually inequivalent pure states \( \omega \) on \( \mathcal{O}_n \) which are minimal and \( \kappa(\omega) = d \).

We prove Theorem 3.24 as follows.

#### 3.5.1 \( d < \infty \)

Fix \( 1 \leq d < \infty \).

Assume \( n < \infty \). Let \( \mathcal{V}_{n,m} \) be as in § 3.1.2. We identify \( \mathcal{V}_{n,m} \) with \( (\mathcal{V}_{n,1})^\otimes m = \text{Lin}\{e_{i_1} \otimes \cdots \otimes e_{i_m} : i_1, \ldots, i_m = 1, \ldots, n\} \). For \( c \in U(1) := \{ c \in \mathbb{C} : |c| = 1 \} \), let \( \rho_c \) denote the sub-Cuntz state on \( \mathcal{O}_n \) by \( z = \mathcal{C} e_2^{d-1} \otimes e_1 \in (\mathcal{V}_{n,d})_1 \). From Theorem 3.24(ii), \( \rho_c \) is uniquely defined as a state which satisfies

\[
\rho_c(s_2^{d-1}s_1) = c,
\]  

(3.19)

and it is pure. Let \( (\mathcal{H}, \pi, \Omega) \) denote the GNS representation of \( \rho_c \). For \( x \in \mathcal{O}_n \), we write \( \pi(x) \) as \( x \) for short. From (3.19), we obtain \( s_2^{d-1}s_1\Omega = c\Omega \). Let \( v_i := s_2^{i-1}s_1\Omega \in \mathcal{H} \) for \( i = 1, \ldots, d \). Then we can verify that \( \{v_1, \ldots, v_d\} \) is an orthonormal basis of \( \mathcal{K}(\rho_c) \). Hence we obtain \( \text{cdim} \rho_c = d \). From Proposition 3.5 \( \kappa(\rho_c) = \text{cdim} \rho_c = d \). From Theorem 3.4(iv), \( \rho_c \sim \rho_{c'} \) if and only if \( c = c' \).

When \( n = \infty \), replace \( \mathcal{V}_{n,m} \) with \( \mathcal{V}_{\infty,m} \). Then the statement is verified in a similar way.

Hence Theorem 3.24 holds when \( d < \infty \).
3.5.2 \( d = \infty \)

Let \( \Lambda, \sim, \hat{\Lambda}, [x] \) be as in §3.4. We write \( \aleph_0 \) and \( \aleph_1 \) for the cardinalities of \( \mathbb{N} \) and \( \mathbb{R} \), respectively.

**Lemma 3.25**

(i) For any \( x \in \Lambda \), \( \#[x] = \aleph_0 \).

(ii) \( \#\hat{\Lambda} = \aleph_1 \).

(iii) Let \( \hat{\Lambda}_{\text{nep}} := \{ [x] \in \hat{\Lambda} : x \text{ is non-eventually periodic} \} \). Then \( \#\hat{\Lambda}_{\text{nep}} = \aleph_1 \).

**Proof.** Define \( A^+ := \bigcup_{l \geq 1} \{1, \ldots, n\}^l \). When \( n = \infty \), replace \( \{1, \ldots, n\}^l \) with \( \{1, 2, \ldots\}^l \) for each \( l \geq 1 \).

(i) For \( x \in \Lambda \), if \( y \in [x] \), then \( y = y_1 x_2 \) for some \( y_1, x_1 \in A^+ \) and \( x_2 \in \Lambda \) such that \( x = x_1 x_2 \). From this, \( y \) is determined only by \( y_1 \). Hence \( [x] \cong A^+ \) as a set. Therefore \( \#[x] = \#A^+ = \aleph_0 \).

(ii) Since \( \#\Lambda = \aleph_1 \), the statement holds from (i).

(iii) Let \( \Lambda_{\text{ep}} := \{ [x] \in \hat{\Lambda} : x \text{ is eventually periodic} \} \). Then \( \hat{\Lambda} = \hat{\Lambda}_{\text{ep}} \sqcup \hat{\Lambda}_{\text{nep}} \).

Any \( [x] \in \Lambda_{\text{ep}} \) has a minimal repeating block \( x' \in A^+ \). Hence \( \hat{\Lambda}_{\text{ep}} \cong A^+ \) as a set. Therefore \( \#\Lambda_{\text{ep}} = \#A^+ = \aleph_0 \). Hence \( \#\hat{\Lambda}_{\text{nep}} = \#(\hat{\Lambda} \setminus \hat{\Lambda}_{\text{ep}}) = \aleph_1 \) from (ii).

From Proposition 3.24(ii) and Proposition 3.22(i), \( \{ \omega_x : [x] \in \hat{\Lambda} \} \) is a set of mutually inequivalent pure states on \( \mathcal{O}_n \). From this and Proposition 3.22(i)(a), \( \Xi := \{ \omega_x : [x] \in \hat{\Lambda}_{\text{nep}} \} \) is a set of mutually inequivalent properly infinitely correlated pure states on \( \mathcal{O}_n \). Since a properly infinitely correlated state \( \omega \) satisfies \( \infty = \kappa(\omega) \leq \operatorname{cdim} \omega \leq \infty \), it is minimal. From this, \( \Xi \) is a set of mutually inequivalent minimal pure states on \( \mathcal{O}_n \) with \( \kappa = \infty \). From this and Lemma 3.25(iii), \( \#\Xi = \#\hat{\Lambda}_{\text{nep}} = \aleph_1 \). Hence the case of \( d = \infty \) in Theorem 3.24 is proved.

4 Applications

4.1 Invariant of irreducible representations of \( \mathcal{O}_n \)

Let \( \operatorname{Irr} \mathcal{O}_n \) denote the class of all irreducible representations of \( \mathcal{O}_n \). For \( \pi, \pi' \in \operatorname{Irr} \mathcal{O}_n \), we write \( \pi \sim \pi' \) when they are unitarily equivalent. As an application of \( \kappa \) in (1.3), we introduce an invariant of arbitrary irreducible representations of \( \mathcal{O}_n \) with respect to \( \sim \).

For \( (\mathcal{H}, \pi) \in \operatorname{Irr} \mathcal{O}_n \) and \( x \in \mathcal{H}_1 := \{ y \in \mathcal{H} : \|y\| = 1 \} \), define

\[ \kappa(\pi) := \kappa(\omega_x \circ \pi) \quad (4.1) \]
where $\omega_x := \langle x|\cdot\rangle x$. Then $\kappa(\pi)$ is independent in the choice of $x$ because $\omega_x \circ \pi \sim \omega_y \circ \pi$ for any $y \in \mathcal{H}_1$. From Theorem 1.1(i) and Proposition 1.2(ii), the following hold.

**Proposition 4.1**  
(i) For $\pi, \pi' \in \text{Irr} \mathcal{O}_n$ if $\pi \sim \pi'$, then $\kappa(\pi) = \kappa(\pi')$.

(ii) For any $\pi \in \text{Irr} \mathcal{O}_n$ and $g \in U(n)$, $\kappa(\pi \circ \alpha_g) = \kappa(\pi)$.

By combining Proposition 1.1(i) and (ii), if $\pi, \pi' \in \text{Irr} \mathcal{O}_n$ satisfy $\pi \sim \pi' \circ \alpha_g$ for some $g \in U(n)$, then $\kappa(\pi) = \kappa(\pi')$.

For example, $\Pi_{[x]}$ in Proposition 3.21(ii) and $d(x)$ in Remark 3.23 satisfy $\kappa(\Pi_{[x]}) = d(x)$ for all $x \in \Lambda$.

Let $\hat{\mathcal{O}}_n$ denote the **spectrum of $\mathcal{O}_n$** [33], that is, the set of all unitary equivalence classes of irreducible representations of $\mathcal{O}_n$. Then we obtain the following decomposition by using $\kappa$:

$$
\hat{\mathcal{O}}_n = \bigsqcup_{d=1}^{\infty} \hat{\mathcal{O}}_n(d), \quad \hat{\mathcal{O}}_n(d) := \{[\pi] \in \hat{\mathcal{O}}_n : \kappa(\pi) = d\} \quad (1 \leq d \leq \infty) \quad (4.2)
$$

where $[\pi] := \{\pi' \in \text{Irr} \mathcal{O}_n : \pi' \sim \pi\}$. From Theorem 3.23, $\#\hat{\mathcal{O}}_n(d) = \aleph_1$ for all $2 \leq n \leq \infty$ and $1 \leq d \leq \infty$. Especially, $\hat{\mathcal{O}}_n(1) \cong \text{“the set of all Cuntz states on $\mathcal{O}_n$” by Fact 1.5}$.

### 4.2 New invariant of ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$

For $\mathcal{H} := \ell^2$, let $\text{End} \mathcal{B}(\mathcal{H})$ denote the set of all unital endomorphisms of $\mathcal{B}(\mathcal{H})$. For $\varphi_1, \varphi_2 \in \text{End} \mathcal{B}(\mathcal{H})$, $\varphi_1$ and $\varphi_2$ are said to be **conjugate** if there exists an automorphism $\gamma$ of $\mathcal{B}(\mathcal{H})$ such that $\varphi_2 = \gamma \circ \varphi_1 \circ \gamma^{-1}$. In this case, we write $\varphi_1 \sim \varphi_2$. The classification problem of elements in $\text{End} \mathcal{B}(\mathcal{H})$ by $\sim$ has been considered in [2] [6] [9] [21] [29] [35]. As an invariant of elements in $\text{End} \mathcal{B}(\mathcal{H})$, the Powers index is well known [31]. We introduce a new invariant for a special subset of $\text{End} \mathcal{B}(\mathcal{H})$.

**Theorem 4.2** ([2] [29], see also § 3 of [9]) For $2 \leq n \leq \infty$, let $s_1, \ldots, s_n$ denote Cuntz generators of $\mathcal{O}_n$ and let $\mathcal{O}_1 := C(T)$. For $\mathcal{O}_1$, we define $s_1$ as a unitary which generates $\mathcal{O}_1$. Let $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ denote the set of all unital representations of $\mathcal{O}_n$ on $\mathcal{H}$. For any $\varphi \in \text{End} \mathcal{B}(\mathcal{H})$, there exist $1 \leq n \leq \infty$ and $\pi \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$ such that $\varphi = \sum_{i=1}^{n} \pi(s_i)(\cdot)\pi(s_i)^*$. The number $n$ is called the **Powers index** of $\varphi$. We write $\text{Ind} \varphi$ as $n$.

Assume $2 \leq n \leq \infty$. Let

$$
\text{End}_n \mathcal{B}(\mathcal{H}) := \{\varphi \in \text{End} \mathcal{B}(\mathcal{H}) : \text{Ind} \varphi = n\}. \quad (4.3)
$$
For \( \pi \in \text{Rep}(O_n, \mathcal{H}) \), define \( \varphi_\pi \in \text{End}_B(\mathcal{H}) \) by
\[
\varphi_\pi := \sum_{i=1}^{n} \pi(s_i)(\cdot)\pi(s_i)^*.
\] (4.4)

From this and Theorem 4.2, the map
\[
\text{Rep}(O_n, \mathcal{H}) \ni \pi \mapsto \varphi_\pi \in \text{End}_B(\mathcal{H})
\] (4.5)
is surjective. In other words, we can write \( \text{End}_B(\mathcal{H}) = \{\varphi_\pi : \pi \in \text{Rep}(O_n, \mathcal{H})\} \).

About this map, the following holds.

Theorem 4.3 ([2, 29], see also § 3 of [9]) Assume \( 2 \leq n, m \leq \infty \). For \( \pi_1 \in \text{Rep}(O_n, \mathcal{H}) \) and \( \pi_2 \in \text{Rep}(O_m, \mathcal{H}) \), the following are equivalent:

(i) \( \varphi_{\pi_1} \sim \varphi_{\pi_2} \).

(ii) \( n = m \) and \( \pi_1 \sim \pi_2 \circ \alpha_g \) for some \( g \in U(n) \).

Especially, \( \varphi_{\pi_1} = \varphi_{\pi_2} \) if and only if \( n = m \) and \( \pi_1 = \pi_2 \circ \alpha_g \) for some \( g \in U(n) \).

Remark that \( \text{End}_B(\mathcal{H}) \) is in one-to-one correspondence with the \( U(n) \)-orbit space \( \text{Irr}(O_n, \mathcal{H})/U(n) := \{\langle \pi \rangle : \pi \in \text{Irr}(O_n, \mathcal{H})\} \) where \( \langle \pi \rangle := \{\pi \circ \alpha_g : g \in U(n)\} \).

Theorem 4.4 ([9]) Assume \( 2 \leq n \leq \infty \). For \( \pi \in \text{Rep}(O_n, \mathcal{H}) \), the following are equivalent:

(i) \( \varphi_\pi \) is ergodic, that is, \( \{X \in B(\mathcal{H}) : \varphi_\pi(X) = X\} = CI \).

(ii) \( \pi \) is irreducible.

By combining Theorem 4.4, Theorem 4.3, and (4.1), we can define a number \( \kappa(\varphi_\pi) \) for an ergodic endomorphism \( \varphi_\pi \) by
\[
\kappa(\varphi_\pi) := \kappa(\pi)
\] (4.6)
where \( \kappa(\pi) \) is as in (4.1). From Proposition 4.1(ii), \( \kappa(\varphi_\pi) \) is well defined. From Proposition 4.1(i), \( \varphi_\pi \sim \varphi_{\pi'} \) implies \( \kappa(\varphi_\pi) = \kappa(\varphi_{\pi'}) \), that is, we obtain an invariant of ergodic endomorphisms:
\[
\kappa : \text{Erg}_n B(\mathcal{H}) \to \{1, 2, \ldots, \infty\}
\] (4.7)
where \( \text{Erg}_n B(\mathcal{H}) := \{ \varphi \in \text{End}_n B(\mathcal{H}) : \varphi \text{ is ergodic} \} \). From examples in §3, we can construct an ergodic endomorphism \( \varphi \) with \( \text{Ind} \varphi = n \) and \( \kappa(\varphi) = d \) for any \( 2 \leq n \leq \infty \) and \( 1 \leq d \leq \infty \).

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