Regularized Weighted Discrete Least Squares Approximation by Orthogonal Polynomials

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Abstract

We consider polynomial approximation over the interval \([-1, 1]\) by a class of regularized weighted discrete least squares methods with \(\ell_2\)-regularization and \(\ell_1\)-regularization terms, respectively. It is merited to choose classical orthogonal polynomials as basis sets of polynomial space with degree at most \(L\). As to the node sets we use zeros of orthogonal polynomials such as Chebyshev points of the first kind, Legendre points. The number of nodes, say \(N + 1\), is chosen to ensure \(L \leq 2N + 1\). With the aid of Gauss quadrature, we obtain approximation polynomials of degree \(L\) in closed form without solving linear algebra or optimization problem. As a matter of fact, these approximation polynomials can be expressed in the form of barycentric interpolation formula \([4, 30]\) when the interpolation condition is satisfied. We then study the approximation quality of \(\ell_2\)-regularization approximation polynomial in terms of the Lebesgue constants, and the sparsity of \(\ell_1\)-regularization approximation polynomial, respectively. Finally, we give numerical examples to illustrate these theoretical results and show that well-chosen regularization parameter can provide good performance approximation, with or without contaminated data.

Keywords. regularized least squares approximation, orthogonal polynomials, Lebesgue constant, sparsity, barycentric interpolation

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1 Introduction

In this paper, we are interested in finding orthogonal polynomials for approximating or recovering functions (possibly noisy) over the interval \([-1, 1]\). These orthogonal polynomials arise as minimizers of \(\ell_2\)- or \(\ell_1\)-regularized least squares approximation problems as follows.

We will consider the \(\ell_2\)-regularized approximation problem

\[
\min_{p \in \mathbb{P}_L} \left\{ \sum_{j=0}^{N} \omega_j (p(x_j) - f(x_j))^2 + \lambda \sum_{j=0}^{N} (\mathcal{R}_L p(x_j))^2 \right\}, \quad \lambda > 0,
\]

(1.1)

and the \(\ell_1\)-regularized approximation problem appearing in the similar form of

\[
\min_{p \in \mathbb{P}_L} \left\{ \sum_{j=0}^{N} \omega_j (p(x_j) - f(x_j))^2 + \lambda \sum_{j=0}^{N} |\mathcal{R}_L p(x_j)| \right\}, \quad \lambda > 0,
\]

(1.2)

where \(f\) is a given continuous function with values (possibly noisy) taken at \(N + 1\) distinct points \(X_{N+1} = \{x_j\}_{j=0}^{N}\) over the interval \([-1, 1]\); \(\mathbb{P}_L\) is a linear space of polynomials of degree at most \(L\) with \(L \leq 2N + 1\); \(\omega = [\omega_0, \omega_1, \ldots, \omega_N]^T\) is a vector of positive Gauss quadrature weights \([12]\); the regularization operator \(\mathcal{R}_L : \mathbb{P}_L \rightarrow \mathbb{P}_L\) is a linear operator; \([\mu_{\ell}]_{\ell=0}^{L}\) are nonnegative parameters called penalization parameters; and \(\lambda > 0\) is the regularization parameter.

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It is known that approximation schemes (1.1) and (1.2) are special cases of the classical penalized least squares methods, see [12]. Some optimization methods or iterative algorithms are presented to find minimizers. However, we will concentrate on aspects of constructing minimizer to (1.1) and (1.2) by nature properties of orthogonal polynomials over \([-1,1]\), which give rise to closed-form solutions to these problems. As is known to all, Gauss quadrature rule goes hand in hand with the theory and computation of orthogonal polynomials, see [30] and references therein. It is merited and desirable by choosing \(X_{N+1}\) to be Gauss (quadrature) points set [12, 28].

**Definition 1.1** A distinct points set \(X_{N+1} = \{x_0, x_1, \ldots, x_N\} \subset [-1,1]\) is called Gauss (quadrature) points set if it satisfies

\[
\int_{-1}^1 p(x) d\omega(x) = \sum_{j=0}^N \omega_j p(x_j) \quad \forall p \in P_{2N+1},
\]

(1.3)

where \(d\omega(x)\) denotes positive measure, and \(\omega_j > 0\) for \(j = 0, 1, \ldots, N\) are called the Gauss quadrature weights of the Gauss quadrature rule.

We employ classical orthogonal polynomials over \([-1,1]\) to transform problem (1.1) and (1.2) into finding coefficients \(\{\tilde{\beta}_\ell\}_{\ell=0}^L\) such that

\[
p(x) = \sum_{\ell=0}^L \tilde{\beta}_\ell \tilde{\Phi}_\ell(x), \quad x \in [-1,1],
\]

(1.4)

where \(\{\tilde{\Phi}_\ell(x)\}_{\ell=0}^L\) is a class of normalized orthogonal polynomials [12, 28]. In this paper we consider the regularization operator \(R_L\) acting as \(R_L p(x) = \sum_{\ell=0}^L \mu_\ell \beta_\ell \tilde{\Phi}_\ell(x)\). The orthogonality is with respect to the \(L_2\) inner product

\[
(f,g)_{L_2} := \int_{-1}^1 f(x) g(x) d\omega(x).
\]

Given a continuous function \(f\) defined on \([-1,1]\), sampling on \(X_{N+1}\) generates

\[
f := f(X_{N+1}) = [f(x_0), f(x_1), \ldots, f(x_N)]^T \in \mathbb{R}^{N+1}.
\]

Let \(A_L := A_L(X_{N+1}) \in \mathbb{R}^{(N+1) \times (L+1)}\) be a matrix of orthogonal polynomials evaluated at the points of \(X_{N+1}\):

\[
A_L = [\tilde{\Phi}_\ell(x_j)] \in \mathbb{R}^{(N+1) \times (L+1)}, \quad j = 0, 1, \ldots, N, \quad \ell = 0, 1, \ldots, L.
\]

Subtracting (1.3) into (1.1), the problem (1.1) transforms into the following problem

\[
\min_{\beta \in \mathbb{R}^{L+1}} \|A_L \beta - f\|_2^2 + \lambda \|R_L \beta\|_2^2, \quad \lambda > 0,
\]

(1.5)

where

\[
\Lambda = \text{diag}(\omega_0, \omega_1, \ldots, \omega_N) \in \mathbb{R}^{(N+1) \times (N+1)},
\]

and diagonal matrix \(R_L := \text{diag}(\mu_0, \mu_1, \ldots, \mu_N) \in \mathbb{R}^{(L+1) \times (L+1)}\) is a semi-definite positive matrix.

With the same basis and weight vector as \(\ell_2\)-regularized approximation problem above, the problem (1.2) transforms into

\[
\min_{\beta \in \mathbb{R}^{L+1}} \|A_L^T \beta - f\|_2^2 + \lambda \|R_L \beta\|_1, \quad \lambda > 0.
\]

(1.6)

Now the next step is to fix \(\beta = [\beta_0, \beta_1, \ldots, \beta_L]^T \in \mathbb{R}^{L+1}\).

The goal of this paper is to explore the properties of the minimizers of problems (1.1) and (1.2). Clearly, the solution to problem (1.1) converges to the solution to continuous \(\ell_2\)-regularized approximation problem, see Theorem 2.2. In contrast, the solution to \(\ell_1\)-regularized approximation problem (1.2) does not converge to its continuous case problem, see Remark 2.2.
The orthogonal polynomials occurs in a wide range of applications and acts a remarkable role in pure and applied mathematics. The Chebyshov and Legendre polynomials are two excellent factors of the family of orthogonal polynomials. Almost every polynomial approximation textbooks introduces the fruitful results of Chebyshov and Legendre polynomials \cite{13, 22, 30, 28}. In particular, we take these two orthogonal polynomials (Chebyshov and Legendre) as representative examples in the choices of basis and corresponding Gauss (quadrature) points set.

In the next section, we introduce some necessary notations and terminologies. The construction of $\ell_2$– and $\ell_1$–regularized minimizers to problems \eqref{1.1} and \eqref{1.2} are presented, respectively. The crucial fact is that both $\ell_2$– and $\ell_1$–regularized minimizers could be presented as the barycentric form of the interpolation formula. It is worth noting that the Wang-Xiang formula \cite{33} is a special case of the minimizer to problem \eqref{1.1} when we setting the Legendre polynomials as the basis, see Section \ref{2.3}. In Section \ref{4} we study the quality of approximating polynomials from problem \eqref{1.1} in terms of the Lebesgue constant. We illustrate the Lebesgue constant decays when the regularization parameter increases. Section \ref{5} makes analysis on $\ell_1$–regularized problem \eqref{1.1} in the view of sparsity. Especially, we present the nonzero elements distribution of the solution to problem \eqref{1.1}. We consider, in Section \ref{6} numerical experiments containing approximation with exact and contaminated data.

All numerical results in this paper are carried out by using MATLAB R2017A on a desktop (8.00 GB RAM, Intel(R) Processor 5Y70 at 1.10 GHz and 1.30 GHz) with Windows 10 operating system. All codes are available at \cite{2}.

\section{2 Regularized weighted least squares approximation}

The construction of minimizers to problems \eqref{1.1} and \eqref{1.2} is presented in this section.

\subsection{2.1 $\ell_2$–regularized approximation problem}

We first consider the $\ell_2$–regularized weighted discrete least squares approximation problem (the $\ell_2$–regularized approximation problem) \eqref{1.1}. The problem can be transformed into a convex and differential optimization problem \eqref{1.5} somehow.

Taking the first derivative of objective function in problem \eqref{1.5} with respect to $\beta$ leads to the first order condition

$$
(A_T^T A) \beta = A_T^T Af,
$$

One may solve \eqref{2.1} by numerical linear algebra method; however, in this paper we concentrate on how to obtain solution of \eqref{2.1} in closed form.

\begin{lemma}
Let $\{\tilde{\Phi}_j\}_{j=0}^L$ be a class of normalized orthogonal polynomials with weight function $\omega(x)$ on $[-1, 1]$, and $\mathcal{X}_{N+1} = \{x_0, x_1, \ldots, x_N\}$ be the zero set of $\tilde{\Phi}_{N+1}$. Assume $L \leq 2N + 1$ and $w$ is the vector of weights satisfying Gauss quadrature rule \eqref{1.3}. Then

$$
H_L := A_L^T A A_L = I_L \in \mathbb{R}^{(L+1) \times (L+1)},
$$

where $I_L$ is the $(L+1) \times (L+1)$ identity matrix.

\begin{proof}

The $(N+1)$–point Gauss quadrature rule \eqref{1.3} is exact for $p \in \mathbb{P}_{2N+1}$ (see \cite{12, 30}). When $\mathcal{X}_{N+1}$ is the zero set of $\tilde{\Phi}_{N+1}$, also that of $\tilde{\Phi}_{N+1}$. For any $\tilde{\Phi}_L$, $\tilde{\Phi}_L$ for $L, L' = 0, 1, \ldots, L$, we have

$$
(H_L)_{L, L'} = \sum_{j=0}^N \omega_j \tilde{\Phi}_{L-1}(x_j) \tilde{\Phi}_{L-1}(x_j) = \int_{-1}^1 \tilde{\Phi}_{L-1}(x) \tilde{\Phi}_{L-1}(x) d\omega(x) = \delta_{L-1, L'-1},
$$

where $\delta_{L-1, L'-1}$ is the Kronecker delta.

\end{proof}

\begin{theorem}
Under the condition of Lemma \ref{2.1}, the optimal solution of problem \eqref{1.5} can be expressed by

$$
\beta_L = \frac{1}{1 + \lambda \rho^2} \sum_{j=0}^N \omega_j \tilde{\Phi}_L(x_j) f(x_j), \quad \ell = 0, 1, \ldots, L, \quad \lambda > 0,
$$

\end{theorem}
or equivalently in matrix form,
\[ \beta = D_L A_T^T A f, \]
where \( D_L = (I_L + \lambda R_L^T R_L)^{-1} \) is a diagonal matrix. Consequently, the minimizer of the \( \ell_2 \)-regularized approximation problem (1.1) is
\[ p_{L,N+1}(x) = \sum_{\ell=0}^{L} \frac{\hat{\Phi}_\ell(x)}{1 + \lambda \mu_\ell^2} \sum_{j=0}^{N} \omega_j \hat{\Phi}_\ell(x_j) f(x_j) = \beta^T \Phi_L(x), \]
where \( \Phi_L(x) = [\hat{\Phi}_0(x), \hat{\Phi}_1(x), \ldots, \hat{\Phi}_L(x)]^T. \)

Proof. This is immediately obtained from (2.1) and Lemma 2.1.

In the limiting case \( N \rightarrow \infty \), we obtain the following simple but interesting result.

**Theorem 2.2** Adopt the notation and assumptions of Lemma 2.1. Let \( f \in C([-1,1]) \), and let \( L \geq 0 \) be given. Then the unique minimizer \( p_{L,N+1} \in P_L \) of (1.1) has the uniform limit \( p_L \) as \( N \rightarrow \infty \), that is
\[ \lim_{N \rightarrow \infty} \| p_{L,N+1} - p_L \|_\infty = 0, \]
where \( p_L \in P_L \) denotes the unique minimizer of the continuous \( \ell_2 \)-regularized approximation problem
\[ \min_{p \in P_L} \| f - p \|^2_{L^2} + \lambda \| R_L p \|^2_{L^2}, \quad \lambda > 0. \]

Proof. The minimizer of problem (2.3) is in a similar way given by
\[ p_L(x) = \sum_{\ell=0}^{L} \frac{\hat{\Phi}_\ell(x)}{1 + \lambda \mu_\ell^2} \int_{-1}^{1} \hat{\Phi}_\ell(x) f(x) d\omega(x). \]
Since the sums over (2.2) and (2.4) are finite, to prove the theorem, it is sufficient to prove that for \( 0 \leq \ell \leq L, \)
\[ \lim_{N \rightarrow \infty} \sum_{j=0}^{N} \omega_j \hat{\Phi}_\ell(x_j) f(x_j) = \int_{-1}^{1} \hat{\Phi}_\ell(x) f(x) d\omega(x). \]
It is known that the Gauss quadrature \( \sum_{j=0}^{N} \omega_j \hat{\Phi}_\ell(x_j) f(x_j) \) converges for continuous function \( \hat{\Phi}_\ell f \) for all \( \ell = 0, 1, \ldots, L \) [13, Section 3.2.3]. Hence (2.5) holds, proving the whole theorem.

### 2.2 \( \ell_1 \)-regularized approximation problem

Now we are starting to discuss the \( \ell_1 \)-regularized approximation problem (1.2), but we convert to solve the problem (1.6) in matrix form. To solve this problem, we first define the soft threshold operator \( S_k(a) \).

**Definition 2.1** ([9]) The soft threshold operator, denoted as \( S_k(a) \), is defined by
\[ S_k(a) = \max(0, a - k) + \min(0, a + k). \]
And the soft threshold operator in vector form is expressed by
\[ S_k(a[i]) = \max(0, a[i] - b[i]) + \min(0, a[i] + b[i]), \]
where \( a, b \in \mathbb{R}^n, i = 1, 2, \ldots, n. \)
Theorem 2.3  Adopt the notation and assumptions of Lemma 2.1. Then the problem (1.6) has the unique closed-form solution

\[ \beta_\ell = \frac{1}{2} S_{\lambda \mu}(2\alpha_\ell), \quad \ell = 0, 1, \ldots, L, \]  

where \( \alpha_\ell = \sum_{j=0}^{N} \omega_j \bar{\Phi}_\ell(x_j) f(x_j) \), or in equivalence,

\[ \beta = \frac{1}{2} S_{\lambda \mu}(2A^T_L Af), \]

where \( \mu = [\mu_0, \mu_1, \ldots, \mu_L]^T \). Consequently, the minimizer of the \( \ell_1 \)-regularized approximation problem is

\[ p_{L,N+1}(x) = \frac{1}{2} \sum_{\ell=0}^{N} S_{\lambda \mu}(2\alpha_\ell) \bar{\Phi}_\ell(x). \]  

Proof. Since \( H_L \) is non-singular, for problem (1.6), we have

\[ 0 \in 2H_L \beta - 2A^T_L Af + \lambda \partial(\|R_L \beta\|_1), \]  

where \( \partial(\cdot) \) denotes as the subgradient [8]. Since \( H_L = I_L \) is an identity matrix and \( R_L \) is a diagonal matrix, \( \beta \) is the solution if and only if

\[ 0 \in 2\beta_\ell - 2\alpha_\ell + \lambda \mu_\ell \partial|\beta_\ell|, \quad \ell = 0, 1, \ldots, L, \]

and \(-1 \leq \partial|\beta_\ell| \leq 1\). We denote \( \beta_\ell^* \) as the best solution, then

\[ \beta_\ell^* = \frac{1}{2} (2\alpha_\ell - \lambda \mu_\ell \partial|\beta_\ell^*|), \quad \ell = 0, 1, \ldots, L. \]

Then three cases are considered:

1. If \( 2\alpha_\ell > \lambda \mu_\ell \), then \( 2\alpha_\ell - \lambda \mu_\ell \partial|\beta_\ell^*| > 0 \), thus \( \beta_\ell^* > 0 \), yielding \( \partial|\beta_\ell^*| = 1 \). Then

\[ \beta_\ell^* = \frac{1}{2} (2\alpha_\ell - \lambda \mu_\ell) > 0. \]

2. If \( 2\alpha_\ell < -\lambda \mu_\ell \), then \( 2\alpha_\ell + \lambda \mu_\ell \partial|\beta_\ell^*| < 0 \), thus \( \beta_\ell^* < 0 \), giving \( \partial|\beta_\ell^*| = -1 \). Then

\[ \beta_\ell^* = \frac{1}{2} (2\alpha_\ell + \lambda \mu_\ell) < 0. \]

3. If \( -\lambda \mu_\ell \leq 2\alpha_\ell \leq \lambda \mu_\ell \), then \( \beta_\ell^* > 0 \) leads to \( \partial|\beta_\ell| = 1 \), and thus \( \beta_\ell^* \leq 0 \); \( \beta_\ell^* < 0 \) produces \( \partial|\beta_\ell| = -1 \), and hence \( \beta_\ell^* = 0 \). Thus

\[ \beta_\ell^* = 0. \]

As what we have hoped, with the aid of soft threshold operator, we obtain

\[ \beta_\ell^* = \frac{1}{2} (\max(0, 2\alpha_\ell - \lambda \mu_\ell) + \min(0, 2\alpha_\ell + \lambda \mu_\ell)) \]

\[ = \frac{1}{2} S_{\lambda \mu}(2\alpha_\ell). \]

\[ \square \]

Remark. In the limiting case \( N \to \infty \), the minimizer \( p_{L,N+1} \in \mathbb{P}_L \) of (1.2) does not converge to the minimizer \( p_L \) of the continuous \( \ell_1 \)-regularized approximation problem

\[ \min_{p \in \mathbb{P}_L} \| f - p \|^2_L + \lambda \| R_L p \|_1, \quad \lambda > 0, \]  

(2.9)
where \( \|f(x)\|_{L_1} = \int_{-1}^{1} |f(x)| \, dx \). Suppose \( p_L(x) = \sum_{\ell=0}^{L} \beta_{\ell} \Phi_{\ell}(x) \), and the problem (2.9) converts into

\[
\min_{\beta \in \mathbb{R}^{L+1}} \left\{ \int_{-1}^{1} \left( \sum_{\ell=0}^{L} \beta_{\ell} \Phi_{\ell}(x) - f(x) \right)^2 \, dx + \lambda \int_{-1}^{1} |\mu \beta_{\ell} \Phi_{\ell}(x)| \, dx \right\}, \quad \lambda > 0. \tag{2.10}
\]

The solution of problem (2.10) is

\[
\beta_{\ell} = \begin{cases} 
\int_{-1}^{1} \Phi_{\ell}(x)f(x) \, dx - \lambda \mu_{\ell} \int_{-1}^{1} |\Phi_{\ell}(x)| \, dx, & \beta_{\ell} > 0, \\
0, & 0 \leq \ell \leq L.
\end{cases}
\]

\[
|\beta_{\ell} - \tilde{\beta}_{\ell}| = 0 \text{ only except } \tilde{\beta}_{\ell} = \beta_{\ell} = 0. \text{ If they have the same sign and do not equal to zero, with the aid of (2.8),}
\]

\[
|\beta_{\ell} - \tilde{\beta}_{\ell}| = \frac{1}{2} \lambda \mu_{\ell} \left| 1 - \int_{-1}^{1} \Phi_{\ell}(x) \, dx \right| = \frac{1}{2} \lambda \mu_{\ell} \left| 1 - \left( \Phi_{\ell}(x), 1 \right)_{L_2} \right| \neq 0.
\]

Hence the minimizer \( p_{L,N+1} \in \mathbb{P}_L \) of (1.2) does not converge to the minimizer \( p_L \) of the continuous \( \ell_1 \)-regularized approximation problem. This result is quite different from the case of \( \ell_2 \)-regularized approximation problem, see Theorem 2.2.

### 2.3 Regularized barycentric interpolation formulae

In this subsection, we focus on the condition of \( L = N \) and interpolation condition \( p(x_j) = f(x_j) \) for \( j = 0, 1, \ldots, N \), where \( p(x) \) is the interpolant of \( f(x) \). Under these conditions, we induce the \( \ell_2 \)- and \( \ell_1 \)-regularized minimizers to problems (1.1) and (1.2) in barycentric form [13]

\[
p(x) = \frac{\sum_{j=0}^{N} \Omega_j \cdot f(x_j)}{\sum_{j=0}^{N} \Omega_j \cdot x - x_j},
\]

respectively. The study of the barycentric weights \( \{\Omega_j\}_{j=0}^{N} \) for roots and extrema of the classical orthogonal polynomials is well developed, see [4, 24, 25, 32, 33].

We first derive the \( \ell_2 \)-regularized barycentric interpolation formula. Recall

\[
p_{L,N+1}(x) = \sum_{\ell=0}^{N} \sum_{j=0}^{N} \omega_j \Phi_{\ell}(x_j) f(x_j) \Phi_{\ell}(x) = \sum_{j=0}^{N} \omega_j f(x_j) \sum_{\ell=0}^{N} \Phi_{\ell}(x_j) \Phi_{\ell}(x) \left( 1 + \lambda \mu_{\ell}^2 \right) = \frac{\Phi_0(x) \Phi_0(0)}{1 + \lambda \mu_0^2} = \frac{1}{1 + \lambda \mu_0^2}.
\]

From the orthogonality of \( \Phi_{\ell}(x) \), \( \ell = 0, 1, \ldots, N \), we have

\[
\sum_{j=0}^{N} \omega_j \sum_{\ell=0}^{N} \Phi_{\ell}(x_j) \Phi_{\ell}(x) = \sum_{\ell=0}^{N} \sum_{j=0}^{N} \omega_j \Phi_{\ell}(x_j) \cdot 1 \left( 1 + \lambda \mu_{\ell}^2 \right) = \frac{\Phi_0(x) \Phi_0(0)}{1 + \lambda \mu_0^2} = \frac{1}{1 + \lambda \mu_0^2}.
\]

Then the \( \ell_2 \)-regularized minimizer (2.11) can be expressed as

\[
p_{L,N+1}(x) = \frac{\sum_{j=0}^{N} \omega_j \sum_{\ell=0}^{N} \Phi_{\ell}(x_j) \Phi_{\ell}(x) f(x_j)}{\left( 1 + \lambda \mu_0^2 \right) \sum_{j=0}^{N} \omega_j \sum_{\ell=0}^{N} \Phi_{\ell}(x_j) \Phi_{\ell}(x)}.
\]
Without loss of generality, suppose \( \mu_\ell = 1 \) for \( \ell \geq N + 1 \). Note that \( \left\{ \frac{\Phi_\ell(x)}{\sqrt{1 + \lambda \mu_\ell^2}} \right\}_{\ell \in \mathbb{N}} \) is still a sequence of orthogonal polynomials. By Christoffel-Darboux formula [12, Section 1.3.3],

\[
\sum_{\ell=0}^{N} \Phi_\ell(x) \Phi_\ell(x_j) = \frac{k_N}{h_N k_{N+1}} \frac{\Phi_{N+1}(x)}{1 + \lambda \mu_{N+1}^2} - \frac{\Phi_N(x)}{1 + \lambda \mu_N^2} - \frac{\Phi_{N+1}(x_j)}{1 + \lambda \mu_{N+1}^2} \frac{\Phi_N(x)}{1 + \lambda \mu_N^2} (x - x_j),
\]

where \( k_\ell \) and \( h_\ell \) denote the leading coefficient and \( L_2 \) norm of \( \frac{\Phi_\ell(x)}{\sqrt{1 + \lambda \mu_\ell^2}} \), respectively.

Combine this with (2.12) and cancel \( \frac{k_N}{h_N k_{N+1}} \frac{\Phi_{N+1}(x)}{1 + \lambda \mu_{N+1}^2} \) from both numerator and denominator. Hence we obtain solution with barycentric form to \( \ell_2 \)-regularized approximation problem under the condition of \( L = N \), and we name it the \( \ell_2 \)-regularized barycentric interpolation formula:

\[
p_{\text{bary}}^{\ell_2}(x) = \sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} f(x_j),
\]

where \( \Omega_j = \omega_j \Phi_N(x_j) \) is the corresponding barycentric weight at \( x_j \). The relation between barycentric weights and Gauss quadrature weights is revealed by Wang, Huybrechs and Vandewalle [32]; however, it does not lead to fast computation since it still requires evaluating the orthogonal polynomials on \( \mathcal{X}_{N+1} \). From the relation they also find the explicit barycentric weights for all classical orthogonal polynomials.

Furthermore, we induce the \( \ell_1 \)-regularized barycentric interpolation formula. From (2.7), \( \ell_1 \)-regularized minimizer (of problem 2.9) can be expressed as the sum of two terms:

\[
p_{\lambda, N+1}(x) = \sum_{\ell=0}^{N} S_{\lambda \mu_\ell} \left( 2 \sum_{j=0}^{N} \omega_j \Phi_\ell(x_j) f(x_j) \right) \Phi_\ell(x) = \sum_{\ell=0}^{N} \left( \sum_{j=0}^{N} \omega_j \Phi_\ell(x_j) f(x_j) \right) \Phi_\ell(x) + \sum_{\ell=0}^{N} c_\ell \Phi_\ell(x),
\]

where

\[
c_\ell = \frac{S_{\lambda \mu_\ell}(2 \alpha_\ell)}{2} - \alpha_\ell, \quad \ell = 0, 1, \ldots, N.
\]

The first term in (2.14) can be written as barycentric form directly by letting \( \lambda = 0 \) in \( \ell_2 \)-regularized barycentric least squares formula. Let the basis \( \{ \Phi_\ell \}_{\ell=0}^{N} \) transform into Lagrange polynomials \( \{ \ell_j(x) \}_{j=0}^{N} \). By the basis-transformation relation between orthogonal polynomials and Lagrange polynomials [11], we have

\[
\sum_{\ell=0}^{N} c_\ell \Phi_\ell(x) = \sum_{j=0}^{N} \left( \sum_{\ell=0}^{N} c_\ell \Phi_\ell(x_j) \right) \ell_j(x).
\]

With the same procedure of obtaining barycentric formula from classical Lagrange interpolation formula in [11], we have

\[
\sum_{\ell=0}^{N} c_\ell \Phi_\ell(x) = \frac{\sum_{j=0}^{N} \Omega_j}{\sum_{j=0}^{N} x - x_j} f(\sum_{j=0}^{N} \Omega_j \ell_j(x_j)) \sum_{j=0}^{N} \Omega_j (x - x_j).
\]
Together with (2.14) and (2.16), we obtain the $\ell_1$-regularized barycentric interpolation formula:

$$p_{bary}^{\ell_1} (x) = \frac{\sum_{j=0}^{N} \frac{\Omega_j}{x - x_j} \left( f(x_j) + \sum_{\ell=0}^{N} c_{\ell} \Phi_\ell(x_j) \right)}{\sum_{j=0}^{N} \frac{\Omega_j}{x - x_j}}.$$  \hspace{1cm} (2.17)

Inspired by the work of Higham [16], we will take numerical study on both regularized barycentric interpolation formulae (2.13) and (2.17), such as numerical stability, see the next paper [3].

3 Quality of $\ell_2$-regularized weighted least squares approximation

In this section, we study the quality of $\ell_2$-regularized weighted least squares approximation in terms of Lebesgue constants.

3.1 Lebesgue constants with the basis of Chebyshev polynomials of the first kind

We mimic the discuss of least squares without regularization in [23, Section 2.4]. We shall treat the cases of normalized Chebyshev polynomials of the first kind as the basis for $P_L$. Similar results are also available in [21]. Consider a weighted Fourier series of a given continuous function $g(\theta)$ over $[-\pi, \pi]$:

$$q_L(\theta) = \frac{\rho_{0,L}}{2} a_0 + \sum_{\ell=1}^{L} \rho_{\ell,L} (a_{\ell} \cos \ell \theta + b_{\ell} \sin \ell \theta),$$

where $a_0, a_1, \ldots, a_L, b_1, \ldots, b_L$ are the Fourier coefficients defined as

$$a_{\ell} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos \ell t dt, \quad \ell = 0, 1, \ldots, L,$$

$$b_{\ell} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin \ell t dt, \quad \ell = 1, 2, \ldots, L,$$

and weights $\rho_{\ell,L} = 1/(1 + \lambda \mu_\ell^2)$, $\ell = 0, 1, \ldots, L$.

Lemma 3.1 ([23]) If $g(\theta)$ is continuous on $[-\pi, \pi]$ with period $2\pi$, then

$$q_L(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t + \theta) u_L(t) dt,$$

where

$$u_L(t) = \frac{\rho_{0,L}}{2} + \sum_{\ell=1}^{L} \rho_{\ell,L} \cos \ell t.$$

Definition 3.1 The Lebesgue constants for $\ell_2$-regularized least squares approximation using Chebyshev polynomials of the first kind are defined as

$$\Lambda_L := \frac{1}{\pi} \int_{-\pi}^{\pi} |u_L(t)| dt.$$ 

The case of $\lambda = 0$ leads to the Lebesgue constants for Fourier series (without regularization) [23, Section 2.4] in the form of

$$\Lambda_L = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{b(L + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt.$$
Lemma 3.2 For the estimation of integral of absolute Dirichlet kernel \([26]\)

\[
D_n(x) = \sum_{k=-n}^{n} e^{ikx} = 1 + 2 \sum_{k=1}^{n} \cos(kx) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}},
\]

we have for all \(n \geq 2\)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)|\,dx \leq \frac{4}{\pi^2} \log(n - 1) + \eta, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)|\,dx \to \frac{4}{\pi^2} \log n \text{ as } n \to \infty,
\]

where \(\eta = \frac{1}{\lambda} + \frac{2}{\pi}(1 + \int_{0}^{\pi} \sin \frac{x}{2}\,dx) = 2.220884 \ldots.\)

The proof of Lemma 3.2 is given in Appendix: Proof of Lemma 3.2.

Theorem 3.1 Suppose \(f\) is continuous on \([-1,1]\), and the normalized Chebyshev polynomials constitute the basis for \(P_L\). Then the Lebesgue constants \(\Lambda_L\) for \(\ell_2\)–regularized least squares approximation of degree \(L\) \((L \geq 2)\) on \([-1,1]\) satisfy

\[
\Lambda_L \leq \frac{4 \log(L - 1)/\pi^2 + \eta}{1 + \tilde{\lambda}L^2} \quad \text{and} \quad \Lambda_L \to \frac{4 \log L/\pi^2}{1 + \tilde{\lambda}^2} \text{ as } L \to \infty,
\]

where \(\tilde{\lambda} = \min\{\mu, \mu_1, \ldots, \mu_L\}, \min\{\mu_0, \mu_1, \ldots, \mu_L\} \leq \tilde{\mu} \leq \max\{\mu_0, \mu_1, \ldots, \mu_L\}\) and \(\eta = 2.220884 \ldots\).

Proof. Since \(f\) is continuous on \([-1,1]\), then \(g(\theta) = f(\cos \theta)\) is continuous on \([0,\pi]\). If \(g(-\theta) = g(\theta)\), then \(g\) is continuous on \([-\pi,\pi]\). The even function \(g\) gives \(b_{-\ell} = 0\) for all \(\ell = 1, \ldots, L\), and then

\[
q_L(\theta) = \frac{\rho_0 L}{2} a_0 + \sum_{\ell=1}^{L} \rho_{\ell,L} a_{\ell} \cos \theta = \frac{\rho_0 L}{2} a_0 + \sum_{\ell=1}^{L} \rho_{\ell,L} a_{\ell} T_{\ell}(x)
\]

\[
= \sqrt{\frac{\pi}{2}} \rho_0 L a_0 T_0 + \sum_{\ell=1}^{L} \sqrt{\frac{\pi}{2}} \rho_{\ell,L} a_{\ell} T_{\ell}(x),
\]

which reveals that this is \(\ell_2\)–regularized least squares approximation of degree \(L\) with the basis for \(P_L\) being the normalized Chebyshev polynomials of the first kind. Since \(g(\theta)\) is continuous on \([-\pi,\pi]\) with period \(2\pi\), there must exist \(M \geq 0\) such that

\[
|g(t + \theta)| \leq M, \quad t, \theta \in [-\pi,\pi].
\]

By Lemma 3.1 we have

\[
\max_{\theta \in [-\pi,\pi]} |q_L(\theta)| \leq M \Lambda_L.
\]

When \(\lambda = 0\), one may easily verify that

\[
u_L(t) = \frac{1}{2} + \sum_{\ell=0}^{L} \cos \ell t = \frac{\sin(L + \frac{1}{2})t}{2 \sin \frac{L}{2}}.
\]

then by Lemma 3.2

\[
\Lambda_L = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\sin(L + \frac{1}{2})t|}{\sin \frac{L}{2}} \,dt \leq \frac{4}{\pi^2} \log(L - 1) + \eta,
\]

and

\[
\Lambda_L \to \frac{4}{\pi^2} \log L \text{ as } L \to \infty.
\]

If \(\lambda \neq 0\), let \(\bar{\mu} = \min\{\mu_0, \mu_1, \ldots, \mu_L\}, \bar{\pi} = \max\{\mu_0, \mu_1, \ldots, \mu_L\}\) and \(\underline{\mu} \leq \bar{\mu} \leq \bar{\pi}\), then

\[
u_L(t) \leq \frac{1}{1 + \lambda \bar{\mu}^2} \left(\frac{1}{2} + \sum_{\ell=0}^{L} \cos \ell t\right) = \frac{1}{1 + \lambda \bar{\mu}^2} \frac{\sin(L + \frac{1}{2})t}{2 \sin \frac{L}{2}}.
\]
and
\[ u_L(t) = \frac{1}{1 + \lambda \hat{\mu}^2} \left( \frac{1}{2} + \sum_{\ell=0}^{L} \cos \ell t \right) = \frac{1}{1 + \lambda \hat{\mu}^2} \frac{\sin(L + \frac{1}{2})t}{2 \sin \frac{t}{2}}, \]
which gives the bounds and asymptotic results (3.1). □

**Remark.** From the proof we know that the case of \( \lambda = 0 \) is reduced into the bounds of Lebesgue constants for Chebyshev projection given in [23, Section 2.4].

Take the family of normalized Chebyshev polynomials of the first kind \( \{ \tilde{T}_\ell(x) \}_{\ell=0}^{L} \) as the basis for \( \mathbb{P}_L \) and the zero set of \( \tilde{T}_{L+1}(x) \) as the node set. With degree \( L \) of approximation polynomial equaling to \( N \) and \( \lambda = 10^{-1} \), Fig. 1 illustrates the Lebesgue constant with respect to different choices of regularization parameter \( \lambda \).

![Figure 1](image_url)

Figure 1: The Lebesgue constant of \( \ell_2 \)-regularized approximation with \( L = N \) and \( \mu_\ell = 1 \) for \( \ell = 0, 1, \ldots, L \) using Chebyshev polynomials of the first kind

### 3.2 Lebesgue constants with the basis of Legendre polynomials

A rather concise proof will be given for asymptotic bounds for the Lebesgue constants of \( \ell_2 \)-regularized approximation by using Legendre polynomials. Without loss of generality, we use the classical Legendre polynomials for analysis here. Recall that the reason why we choose normalized polynomials is to obtain closed-form solutions. Consider the kernel

\[ K_L(x, y) = \sum_{\ell=0}^{L} \frac{2\ell + 1}{1 + \lambda \mu_\ell^2} P_\ell(x)P_\ell(y), \]

where \( P_\ell(\cdot) \) is the Legendre polynomial of degree \( \ell \). The case of \( \lambda = 0 \) gives a rather simple kernel

\[ T_L(x, y) = \sum_{\ell=0}^{L} (2\ell + 1)P_\ell(x)P_\ell(y) = (L + 1) \frac{P_L(x)P_{L+1}(y) - P_{L+1}(x)P_L(y)}{y - x}. \]

Thus

\[ K_L(x) \overset{\Delta}{=} K_L(x, 1) := \sum_{\ell=0}^{L} \frac{2\ell + 1}{1 + \lambda \mu_\ell^2} P_\ell(x), \]

\[ T_L(x) \overset{\Delta}{=} T_L(x, 1) := \sum_{\ell=0}^{L} (2\ell + 1)P_\ell(x) = (L + 1) \frac{P_L(x) - P_{L+1}(x)}{1 - x}. \]
where the rightmost equality is due to Christoffel-Darboux formula [12, Section 1.3.3], and obviously,
\[ |K_L(x)| \leq \frac{1}{1 + \lambda^2} |T_L(x)|. \] (3.2)

Definition 3.2 The Lebesgue constants for \( \ell_2 \)-regularized approximation using Legendre polynomials are defined as
\[ \Lambda_L := \frac{1}{2} \int_{-1}^{1} |K_L(x)| dx. \]
The case of \( \lambda = 0 \) leads to
\[ \Theta_L := \frac{1}{2} \int_{-1}^{1} |T_L(x)| dx = \frac{L}{2} + \frac{1}{2} \int_{-1}^{1} \frac{P_L(x) - P_{L+1}(x)}{1 - x} \, dx, \] (3.3)
which is the definition of Lebesgue constant of Legendre truncation of degree \( L \) [15].

Lemma 3.3 ([15, 27]) Let \( \Theta_L \) be define by (3.3). Then
\[ \lim_{L \to \infty} \frac{\Theta_L}{\sqrt{L}} = 2 \sqrt{\frac{2}{\pi}}. \] (3.4)
Combining (3.3) with (3.2), we obtain the estimation on \( \Lambda_L \) in the case of Legendre polynomials.

Theorem 3.2 Suppose \( f \) is continuous on \([-1, 1]\), and the normalized Legendre polynomials constitute the basis for \( \mathbb{P}_L \). Then the Lebesgue constants \( \Lambda_L \) for \( \ell_2 \)-regularized least squares approximation of degree \( L \) (\( L \geq 2 \)) on \([-1, 1]\) satisfy
\[ \Lambda_L \leq \frac{1}{1 + \lambda\mu} \left( \frac{2^{3/2}}{\sqrt{\pi}} L^{1/2} + o(L^{1/2}) \right), \]
where \( \mu = \min\{\mu_0, \mu_1, \ldots, \mu_L\} \).

The proof for Theorem 3.2 is based on the above discussion.

4 Sparsity of solution for \( \ell_1 \)-regularized approximation problem

Some real-world problems such as signal processing often have sparse solutions with considerable evidence proving it. One may seek the sparsest solution of a problem, that is, the solution containing zero elements at most. However, a vector of real data would rarely contains many strict zeros. One may introduce other measure of sparsity, such as \( \min_x \|x\|_p \), where \( \|x\|_p = \left( \sum_i |x_i|^p \right)^{1/p} \), \( 0 < p < 1 \). Nevertheless, optimization problems mentioned above are nonconvex and nondifferentiable [6, 8]. Regularized methods, especially \( \ell_1 \)-regularized cases, also produce sparse solutions, according to our examples. One may find a relatively sparse solution by minimizing \( \ell_1 \) norm, because such an optimization problem is a convex optimization problem and the closest one to the sparsest solution. For topics on sparsity, we refer to [6]. We consider the sparsity of the solution \( \beta \) of \( \ell_1 \)-regularized approximation problem [14, 6]. The sparsity is measured by the number of nonzero elements of \( \beta \), denoted as \( \|\beta\|_0 \), also known as the zero norm” (it is not a norm actually) of \( \beta \) [6].

Before discussing upper bound for \( \|\beta\|_0 \), we just offer a quick glimpse of zero elements distribution of \( \ell_1 \)-regularized approximation solution.

Proposition 4.1 (zero elements distribution of \( \ell_1 \)-regularized approximation solution)
Adopt the notation and assumptions of Lemma 3.1. If \( \mu_\ell \) satisfies
\[ -\lambda\mu_\ell \leq 2 \sum_{j=0}^{N} \omega_j \Phi(x_j) f(x_j) \leq \lambda\mu_\ell, \] (4.1)
then its corresponding \( \beta_\ell \) is zero, \( \ell = 0, 1, \ldots, L \).
If \( \lambda > 0 \), \( \|A_L^T Af\|_0 \) becomes an upper bound for the number of nonzero elements of \( \beta \). Furthermore, we obtain the exact number of nonzero elements of \( \beta \) with the help of information of \( \beta \).

**Theorem 4.1** Let \( \beta = [\beta_0, \beta_1, \ldots, \beta_L]^T \) be the solution of \( \ell_1 \)-regularized problem (1.6). If \( \lambda > 0 \), then the number of nonzero elements of \( \beta \) satisfies

\[
\|\beta\|_0 \leq \|A_L^T Af\|_0, \tag{4.2}
\]

and

\[
\|\beta\|_0 = \|A_L^T Af\|_0 - \# \{ \text{occurrences of } \beta_\ell = 0 \text{ but } \alpha_\ell \neq 0 \}, \tag{4.3}
\]

where \( \# \{ \text{occurrences of } \beta_\ell = 0 \text{ but } \alpha_\ell \neq 0 \} \) denotes the number of occurrences of \( \beta_\ell = 0 \) but \( \alpha_\ell \neq 0 \) for \( \ell = 0, 1, \ldots, L \).

**Proof.** By (2.8), we have

\[
\beta \in A_L^T Af - \lambda \partial(\|R_L \beta\|_1) \frac{1}{2}.
\]

To obtain a solution, there must exist an \( L + 1 \) vector \( h = [h_0, h_1, \ldots, h_L]^T \in \partial(\|R_L \beta\|_1) \) such that

\[
\beta = A_L^T Af - \lambda h \frac{1}{2}. \tag{4.4}
\]

Recall \( \mu_\ell > 0 \) for all \( \ell = 0, 1, \ldots, L \), and the subgradient of \( \| \cdot \|_1 \),

\[
h_\ell = \begin{cases} 
\mu_\ell, & \mu_\ell \beta_\ell > 0, \ i.e., \beta_\ell > 0 \\
-\mu_\ell, & \mu_\ell \beta_\ell < 0, \ i.e., \beta_\ell < 0 \\
r_{\mu_\ell} \forall r \in [-1, 1], & \mu_\ell \beta_\ell = 0, \ i.e., \beta_\ell = 0,
\end{cases}
\]

yielding \( \|h\|_0 \geq \|\beta\|_0 \). Equation (4.4) gives

\[
\left\| \beta + \frac{\lambda h}{2} \right\|_0 = \left\| A_L^T Af \right\|_0.
\]

If \( \beta_\ell > (or <) 0 \), then \( h_\ell > (or <) 0 \). If \( \beta_\ell = 0 \), whereas \( h_\ell \) may not be zero. Thus

\[
\left\| \beta + \frac{\lambda h}{2} \right\|_0 = \|h\|_0.
\]

Hence

\[
\|\beta\|_0 \leq \|h\|_0 = \|A_L^T Af\|_0.
\]

We denote \( \beta^*_\ell \) as the best solution. With the aid of closed-form solution of \( \ell_1 \)-regularized approximation problem, equation (4.4) gives birth to

\[
\frac{h_\ell}{\mu_\ell} = \frac{2}{\lambda \mu_\ell} (\alpha_\ell - \beta^*_\ell) = \begin{cases} 
1, & \beta^*_\ell > 0, \\
-1, & \beta^*_\ell < 0, \\
\frac{2\alpha_\ell}{\lambda \mu_\ell}, & \beta^*_\ell = 0.
\end{cases}
\]

Due to \( \|h\|_0 = \left\| \frac{h}{\mu} \right\|_0 \), where \( \frac{h}{\mu} \) denotes the pointwise division between \( h \) and \( \mu \), the difference between \( \|h\|_0 \) and \( \|\beta\|_0 \) is expressed by

\[
\|h\|_0 - \|\beta\|_0 = \left\| \frac{h}{\mu} \right\|_0 - \|\beta\|_0 = \# \{ \text{occurrences of } \beta_\ell = 0 \text{ but } \alpha_\ell \neq 0 \}. \tag{4.5}
\]

Together with \( \|h\|_0 = \|A_L^T Af\|_0 \) and (4.3), we obtain (4.3). \( \Box \)
Corollary 4.1 If $\lambda = 0$, then the number of nonzero elements of $\beta$ satisfies

$$\|\beta\|_0 = \|A_f^T A_f\|_0. \quad (4.6)$$

Remark. Together with Theorem 4.1 and Corollary 4.1, it states that regularized minimization is better than unregularized minimization in terms of sparsity.

Let the basis for $P_L$ be the family of normalized Chebyshev polynomials of the first kind $\{\tilde{T}_\ell(x)\}_{\ell=0}^L$ and the node set be the zero set of $\tilde{T}_{N+1}(x)$. With degree $L$ of approximation polynomial ranging from 1 to 60, $\lambda$ evaluated $10^{-1}$ and $\mu\ell$ evaluated 1 for all $\ell = 0, 1, \ldots, L$. Fig. 2 gives four examples on bounds and estimations given above.

Figure 2: Examples on bounding the number of nonzero elements, where # denotes $\#\{\text{occurrences of } \beta_\ell = 0 \text{ but } \alpha_\ell \neq 0\}$

5 Numerical experiments

In this section, we report numerical results to illustrate the theoretical results derived above and test the efficiency of the $\ell_2$– and $\ell_1$–regularized approximation model (1.1) and (1.2). The choice of basis for $P_L$ and point set $X_{N+1}$ is primary when using both models. We choose Chebyshev polynomials of the first kind and the corresponding Chebyshev points. Certainly, choosing other orthogonal polynomials such as Legendre polynomials is also practicable. All computations were performed in MATLAB in double precision arithmetic. Some related commands, for instance, obtaining quadrature points and weights, are included in Chebfun 5.7.0 [31].

To test the efficiency of approximation, we define the uniform error and the $L_2$ error to measure the approximation error:

- The uniform error of the approximation is estimated by

$$\|f - p_{L,N+1}\|_\infty := \max_{x \in [-1, 1]} |f(x) - p_{L,N+1}(x)| \simeq \max_{x \in X} |f(x) - p_{L,N+1}(x)|,$$

where $X$ is a large but finite set of well distributed points (for example, clustered grids, see [29 Chapter 5]) over the interval $[-1, 1]$. 


The $L_2$ error of the approximation is estimated by

$$\|f - p_{L,N+1}\|_{L_2} := \left( \int_{-1}^{1} (f(x) - p_{L,N+1}(x))^2 d\omega(x) \right)^{1/2}$$

$$\simeq \left( \sum_{j=0}^{N} \omega_j (f(x_j) - p_{L,N+1}(x_j))^2 \right)^{1/2}. \quad (5.1)$$

The set \{\(x_0, x_1, \ldots, x_N\)\} can be \(N + 1\) zeros of the orthogonal polynomial of degree \(N + 1\), corresponding to the weight function \(\omega(x)\) chosen in (5.1).

5.1 Regularized approximation models for exact data

The fact should always stick in readers’ mind that regularization is introduced to solve ill-posed problems or to prevent overfitting. When approximation applies to functions without noise, regularization parameter \(\lambda = 0\) (no regularization) contributes to the best choice of approximating.

Fig. 3 reports the efficiency and errors for approximating function

$$f(x) = \tanh(20 \sin(12x)) + 0.02e^{3x} \sin(300x),$$

with or without regularization over \([-1,1]\). The test function is given in [30]. Let \(N = 600\), \(L = 200\), \(\lambda = 10^{-1}\) and \(\mu = 1\) for all \(\ell = 0, 1, \ldots, L\).

Figure 3: Smooth approximation \(f(x) = \tanh(20 \sin(12x)) + 0.02e^{3x} \sin(300x)\)

Fig. 3 illustrates that regularization is beyond use in well-posed approximation problem, and \(\ell_2\)–regularization is better than \(\ell_1\)–regularization in approximating smooth functions.

5.2 Regularized approximation models for contaminated data

We consider

$$f(x) = \begin{cases} \frac{5 \sin(5\pi x)}{5\pi x}, & x \neq 0, \\ 5, & x = 0, \end{cases} \quad (5.2)$$

which is the Fourier transform of the gate signal

$$g(t) = \begin{cases} 1, & |t| \leq 5/2, \\ 0, & |t| > 5/2, \end{cases}$$
see [5]. We use regularized least squares models to reduce Gaussian white noise added to the spectral density with the signal-noise ratio (SNR) 10 dB. The choice of $\lambda$ is critical in these models, so we first consider the relation between $\lambda$ and approximation errors to choose the optimal $\lambda$. We take polynomial degree $L = 30$ and point set $X_{100+1}$, and $\lambda = 10^{-15}, 10^{-14.5}, 10^{-14}, \ldots, 10^{-5}, 10^5$ to choose the best regularization parameter. Here we choose $\lambda = 10^{-1}$. More advanced methods to choose the parameter $\lambda$, we refer to read [18, 20] for a further discussion.

Fig. 4 shows that the $\ell_2$- and $\ell_1$-regularized approximation models with $\lambda = 10^{-1}$ is effective in recovering the noisy function. In the case we let

$$\mu_\ell = \frac{1}{F(\ell/L)}, \quad \ell = 0, 1, \ldots, L,$$

where the filter function $F$ is defined as [1]

$$F(x) = \begin{cases} 1, & x \in [0, 1/2] \\ \sin^2 \pi x, & x \in [1/2, 1] \\ 0, & x \in [1, +\infty]. \end{cases}$$

![Figure 4: Regularized approximation models with $L = 30$ and $N = 100$ to recover the spectral density of unit gate function from contaminated data](image)

The results in Fig. 5 illustrate that $\ell_1$-regularized approximation model is the best choice when recovering a contaminated function, which chords with the known fact [19]. It also shows that regularized approximation models shall lead to stable computations. Both Fig. 4 and 5 shows that regularized models are better that those without regularization ($\lambda = 0$).

Besides, Consider highly oscillatory function $f(x) = \text{Airy}(40x)$ on $[-1, 1]$ with 12dB Gauss white noise added (noisy function is shown in Fig. 6. We use the regularized barycentric formulae (2.13) and (2.17) to conduct this experiment. Let $L = N = 500$ and $\{\mu_\ell\}_{\ell=0}^L$ be the same as above. Different values of $\lambda$, say $10^{-1}, 10^{-2}, 10^{-5}$, lead to different results, see Fig. 6. This experiment indicates that one could apply a simple formula to denoise function, rather than employ an iterative scheme.

These numerical examples illustrate that under some conditions, $\ell_2$-regularization also can be better than $\ell_1$-regularization. For example, $\lambda = 10^{-1}$ suit $\ell_2$-regularization, but almost straighten the function by $\ell_1$-regularization. Besides, we can see that $\ell_2$-regularization is blessed with lower sensitivity than $\ell_1$-regularization.
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Figure 5: Errors for regularized approximation model to recover the spectral density of unit gate function with fixed $N = 100$

Figure 6: Denoising by regularized barycentric formulae with $L = N = 500$: function $f(x) = \text{Airy}(40x)$ with 12dB Gauss white noise added
6 Concluding remarks

In this paper, we have investigated minimizers to \(\ell_2\)– and \(\ell_1\)–regularized least squares approximation problems with the aid of properties of orthogonal polynomials on \([-1, 1]\). Based on these explicit constructed approximation polynomials (minimizers to problems (1.1) and (1.2)), the barycentric interpolation formulae have been derived immediately. In addition, quality of \(\ell_2\)–regularized approximation, the Lebesgue constant, is studied in the case of normalized Legendre polynomials and normalized Chebyshev polynomials of the first kind. Bound for sparsity of \(\ell_1\)–regularized approximation is obtained by the refinement of subgradient. Numerical results indicates that both \(\ell_2\)– and \(\ell_1\)–regularized approximation are practicable and efficient. These results provide some new insight into \(\ell_2\)– and \(\ell_1\)–regularized approximation, and can be adaptable to some practical applications such as noise reduction by using spectral interpolation on the Jacobi nodes.

Appendix: Proof of Lemma 3.2

Proof. The proof is divided into two parts: upper bound and Asymptotic result.

- (Upper bound) Rivlin [23, Section 2.4] gives a preliminary bound
  \[
  \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\sin(n + \frac{1}{2})x|}{\sin \frac{x}{2}} dx < \frac{4}{\pi^2} \log n + 3,
  \]
  we mimic his proof but bring up a sharp bound for it. Suppose \(n \geq 2\), then
  \[
  \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\sin(n + \frac{1}{2})x|}{\sin \frac{x}{2}} dx = \frac{1}{\pi} \int_{0}^{\pi} \frac{|\sin(n + \frac{1}{2})x|}{\sin \frac{x}{2}} dx
  = \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin nx}{\tan \frac{x}{2}} + \cos nx dx
  \leq \frac{1}{\pi} \int_{0}^{\pi} \frac{|\sin nx|}{\tan \frac{x}{2}} dx + \frac{1}{\pi} \int_{0}^{\pi} \cos nx dx.
  \]
  \[
  (6.1)
  \]
  \[\diamond\] Also we have
  \[
  \int_{0}^{\pi} \cos nx dx = \frac{1}{n} \int_{0}^{n\pi} \cos x dx.
  \]
  Let \(I_n = \int_{0}^{\pi} \cos nx dx\), then
  \[
  I_{k+1} - I_k = \int_{k\pi}^{(k+1)\pi} \cos nx dx = \int_{0}^{\pi} \cos(x + k\pi) dx = I_1 = 2,
  \]
  and then \(I_n = 2n\). Thus
  \[
  \frac{1}{\pi} \int_{0}^{\pi} \cos nx dx = \frac{2}{\pi}.
  \]
  \[
  (6.2)
  \]
  \[\diamond\] The fact that \(\tan x \geq x\) for \(0 \leq x \leq \frac{\pi}{2}\) leads to
  \[
  \int_{0}^{\pi} \frac{|\sin nx|}{\tan \frac{x}{2}} dx \leq 2 \int_{0}^{\pi} \frac{|\sin nx|}{x} dx.
  \]
  Since
  \[
  \int_{0}^{\pi} \frac{|\sin nx|}{x} dx = \int_{0}^{x} \frac{|\sin x|}{x} dx = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx
  = \sum_{k=0}^{n-1} \int_{0}^{\pi} \frac{|\sin(x + k\pi)|}{x + k\pi} dx = \int_{0}^{\pi} \sin x \sum_{k=0}^{n-1} \frac{1}{x + k\pi} dx,
  \]
we have
\[
\int_0^\pi \left| \frac{\sin nx}{x} \right| dx = \int_0^\pi \frac{\sin x}{x} dx + \int_0^\pi \sin x \sum_{k=1}^{n-1} \frac{1}{x + k\pi} dx.
\]

In \(0 \leq x \leq \pi\),
\[
\sum_{k=1}^{n-1} \frac{1}{x + k\pi} \leq \frac{1}{\pi} \sum_{k=1}^{n-1} -\frac{1}{k},
\]
and
\[
\sum_{k=1}^{n-1} \frac{1}{k} \leq 1 + \int_1^{n-1} \frac{1}{x} dx = 1 + \log(n - 1),
\]
the inequality becomes an equality when \(n = 2\). Therefore
\[
\int_0^\pi \sin x \sum_{k=1}^{n-1} \frac{1}{x + k\pi} dx \leq \frac{2}{\pi} [1 + \log(n - 1)],
\]
and
\[
\int_0^\pi \left| \frac{\sin nx}{\tan \frac{x}{2}} \right| dx \leq 2 \int_0^\pi \frac{\sin x}{x} dx + \frac{4}{\pi} [1 + \log(n - 1)].
\]

Finally, together with (6.1), (6.2) and (6.3), we have
\[
\frac{1}{2\pi} \int_{-\pi}^\pi \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} dx \leq \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} dx + \frac{4}{\pi^2} [1 + \log(n - 1)] + \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} dx,
\]
where \(4/\pi^2 + 2/\pi^2 + 2 \int_0^\pi \frac{\sin x}{x} dx / \pi = \eta = 2.220884 \ldots\).

(Asymptotic result) The asymptotic result is a quite well-known result, given in [10]. It is even an exercise in Stein and Shakarchi’s Fourier Analysis [26, Section 2.7.2].

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