CONCENTRATION OF SOLUTIONS FOR A FOURTH ORDER ELLIPTIC EQUATION IN $\mathbb{R}^N$

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ABSTRACT. In this paper, we study the following fourth order elliptic problem

$$\Delta^2 u = (1 + \varepsilon K(x))u^{2^* - 1}, \quad x \in \mathbb{R}^N,$$

where $2^* = \frac{2N}{N-4}, N \geq 5, \varepsilon > 0$. We prove that the existence of two peaks solutions for the above problem, if $K(x)$ has two critical points satisfying certain conditions, provided $\varepsilon$ is small enough.

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1. Introduction and main results

In this paper, we study the following nonlinear fourth order elliptic equation

$$\begin{cases}
\Delta^2 u = (1 + \varepsilon K(x))u^{2^* - 1}, & u > 0, \quad x \in \mathbb{R}^N \\
u \in D^{2,2} (\mathbb{R}^N)
\end{cases}$$

(1.1)

where $N \geq 5, \varepsilon > 0, K(x) \in C^1 (\mathbb{R}^N) \cap L^\infty (\mathbb{R}^N)$ and $D^{2,2} (\mathbb{R}^N)$ be the completion of $C^\infty_0 (\mathbb{R}^N)$ with the respect to the norm

$$\|u\|^2 = \langle u, u \rangle, \quad \text{where} \quad \langle u, v \rangle = \int_{\mathbb{R}^N} \Delta u \Delta v.$$

$2^* = \frac{2N}{N-4}$ is the critical exponent of the embedding $D^{2,2} (\mathbb{R}^N) \hookrightarrow L^{2^*} (\mathbb{R}^N)$. A Fourier transformation argument yields $D^{2,2} (\mathbb{R}^N) = D^{2,2}_0 (\mathbb{R}^N)$.

In the past few years, there has been many study on concentration of solutions for second-order elliptic equations with critical Sobolev exponent; See e.g. [1, 2, 3, 7, 10, 11, 14, 16, 18, 19, 21] and the references therein. Recently, some researches have been developed on the existence of peak solutions of fourth order elliptic equations involving critical exponent, see for example [3, 6, 8, 9, 13, 17].

Define the Euler-Lagrange functional $I$ corresponding to (1.1) as follows

$$I_\varepsilon (u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} (1 + \varepsilon K(x)) |u|^{2^*}.$$ 

One of the main features of problem (1.1) is the lack of compactness, i.e. the functional $I_\varepsilon$ does not satisfy the Palais-Smale condition. Such a fact follows from the noncompactness of the embedding of $D^{2,2} (\mathbb{R}^N) \hookrightarrow L^{2^*} (\mathbb{R}^N)$ and the unboundness of the domain $\mathbb{R}^N$. In
In this article, we use a construction method to obtain peak solutions for (1.1). Precisely, we extend the argument employed in [10] to the framework of such higher order equations. To do so, we first take advantage of a type of Lyapunov-Schmidt reduction to transform the problem of finding critical points for the functional $I_\varepsilon$ into one of finding critical points of a functional defined on finite dimensional domain. Then we construct a suitable bounded domain with finite dimension in which the associated variational problem, by topological degree argument, can have a critical point. In our proof, to obtain a fine analysis on the energy of the functional $I_\varepsilon$, we perform a careful expansion on $I_\varepsilon$ by making full use of the precise computation of the contribution of function $K(x)$ to its critical points. Moreover, we have to prove the positivity of the critical points obtained by our process. It is well-known that such a proof, in general, is quite difficult for higher order equations.

To state the main result, we need to introduce some notations and assumptions.

Consider the equation
\[ \Delta^2 u = |u|^{2^* - 2} u, \quad u > 0, \quad x \in \mathbb{R}^N, \quad u \in \mathcal{D}^{2,2} (\mathbb{R}^N). \] (1.2)

It has been proved in [15] that the following function, for $y \in \mathbb{R}^N$ and $\lambda > 0$,
\[ U_{y,\lambda}(x) = C_N \frac{\lambda^{N-4}}{(1 + \lambda^2 |x - y|^2)^{\frac{N-4}{2}}}, \quad C_N = [(N - 4)(N - 2)N(N + 2)]^{-\frac{N+4}{8}} \]
solves (1.2) on $\mathbb{R}^N$.

Let $\Sigma$ denote the set consisting of all the critical points $z$ of $K(x)$, satisfying (after a suitable rotation of the coordinate system depending on $z$),
\[ K(x) = K(z) + \sum_{i=1}^N a_i |x_i - z_i|^\beta + O(|x - z|^{\beta + \sigma}) \]
for $x$ close to $z$, where $a_i, \beta$ and $\sigma$ are some constants depending only on $z$, $a_i \neq 0$ for $i = 1, 2, \cdots, N$, $\sum_{i=1}^N a_i < 0$, $\beta \in (1, N - 4)$ and $\sigma \in (0, 1)$.

We now state our main result of the paper:

**Theorem 1.1.** Assume that $\Sigma$ contains at least two points. Then for each $z^1, z^2 \in \Sigma, z^1 \neq z^2$, there exists an $\varepsilon_0 > 0$ such that (1.1) has a solution of the form
\[ u_\varepsilon = \sum_{j=1}^2 \alpha_{j,\varepsilon} U_{y_j,\lambda_j,\varepsilon} + v_\varepsilon \]
if $\varepsilon \in (0, \varepsilon_0)$ with $\alpha_{j,\varepsilon} \to 1, y_j \to z^j, \lambda_j \to +\infty$ and $\int_{\mathbb{R}^N} |\Delta v_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$.

Remark 1.2. If the set $\Sigma$ contains $k (k \geq 2)$ points, it is easy to see that, by theorem 1.1, there exist $\binom{k}{2}$ solutions for problem (1.1).

Remark 1.3. Let $z^1, z^2, \cdots, z^k$ be $k$ local maximum points of $K(x)$ satisfying
\[ K(x) = K(z^i) + \sum_{j=1}^N a_{ij} |x_j - z^i_j|^\beta + O(|x - z^i|^{\beta + \sigma}), \quad x \in B_\delta(z^i), \]
where $\beta \in (1, N-4)$, $a_{ij} \neq 0$ and $\sum_{j=1}^{N} a_{ij} < 0$ and $\sigma > 0$. Using the technique in the proof of Theorem 1.3 of [21], we can construct a $k$-peaked solution for (1.1), such that there is exactly one local maximum point near each $z_i$, $i = 1, 2, \cdots, k$.

This paper is organized as follows: we first introduce some notations and perform a finite-dimensional reduction in section 2, and then use topological degree argument to prove theorem 1.1 in section 3. In order that we can give a clear line of our framework, we list all the proofs of the needed estimates in the appendix.

2. NOTATIONS AND THE FINITE-DIMENSIONAL REDUCTION

For $y = (y^1, y^2, \cdots, y^k) \in \mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N$, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in \mathbb{R}^k$, define

$$E^k_{y, \lambda} = \{ v \in D^2,2(\mathbb{R}^N) | \langle U_{y^j, \lambda_j}, v \rangle = \left\langle \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_j}, v \right\rangle = \left\langle \frac{\partial U_{y^j, \lambda_j}}{\partial y_i}, v \right\rangle = 0 \text{ for } j = 1, 2, \cdots, k; i = 1, 2, \cdots, N \}$$

(2.1)

We look for solutions $u_\varepsilon$ of (1.1) of the following form

$$u_\varepsilon = \alpha_1,\varepsilon U_{y^1, \lambda_1, \varepsilon} + \alpha_2,\varepsilon U_{y^2, \lambda_2, \varepsilon} + v_\varepsilon$$

(2.2)

with $\alpha_\varepsilon = (\alpha_1,\varepsilon, \alpha_2,\varepsilon) \to (1, 1), y_\varepsilon = (y^1_\varepsilon, y^2_\varepsilon) \to (z^1, z^2)$, $\lambda_{j, \varepsilon} \to +\infty$ for $j = 1, 2$ and $v_\varepsilon \in E^2_{y_\varepsilon, \lambda_\varepsilon}$ satisfying $\|v_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

For each $z^1, z^2 \in \Sigma, z^1 \neq z^2$ and $\mu > 0$, define

$$D_\mu = \left\{ (y, \lambda) \big| y = (y^1, y^2) \in B_\mu(z^1) \times B_\mu(z^2), \lambda = (\lambda_1, \lambda_2) \in \left(\frac{1}{\mu}, +\infty\right) \times \left(1, \frac{1}{\mu}, +\infty\right) \right\} .$$

Define

$$M_\mu = \{(\alpha, y, \lambda, v) | \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+, (y, \lambda) \in D_\mu, v \in E^2_{y, \lambda}, \|v\| \leq \mu \}$$

and

$$J_\varepsilon(\alpha, y, \lambda, v) = I_\varepsilon \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} + v \right) .$$

It is well known now that for $\mu > 0$ sufficiently small if $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of $J_\varepsilon$ in $M_\mu$, then $u = \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} + v$ is a critical point of $I_\varepsilon$ in $D^2,2(\mathbb{R}^N)$. The fact that $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of $J_\varepsilon$ in $M_\mu$ is equivalent to the fact that the following
equations are satisfied
\[
\frac{\partial J_\varepsilon}{\partial \alpha_j} = 0, \; j = 1, 2, \tag{2.3}
\]
\[
\frac{\partial J_\varepsilon}{\partial v} = 2 \sum_{j=1}^2 A_j U_{y^j, \lambda_j} + 2 \sum_{j=1}^2 B_j \frac{\partial U_{y^j, \lambda_j}}{\partial \alpha_j} + \sum_{j=2}^2 \sum_{i=1}^N C_{ji} \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^j}, \tag{2.4}
\]
\[
\frac{\partial J_\varepsilon}{\partial y_i^j} = B_j \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial \alpha_j \partial y_i^j}, v \right\rangle + \sum_{l=1}^N C_{jl} \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial y_i^j \partial y_l^j}, v \right\rangle, i = 1, 2, \cdots, N, j = 1, 2, \tag{2.5}
\]
\[
\frac{\partial J_\varepsilon}{\partial \lambda_j} = B_j \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial \alpha_j^2}, v \right\rangle + \sum_{l=1}^N C_{jl} \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial y_i^j \partial \lambda_j}, v \right\rangle, j = 1, 2, \tag{2.6}
\]
for some \(A_j, B_j, C_{ji} \in \mathbb{R}, j = 1, 2, i = 1, 2, \cdots, N.\)

As in [10], we first reduce the problem of finding a solution for (1.1) to that of finding a critical point for a function defined in a finite dimensional domain. Then we use topological degree argument to solve the latter problem. We next establish some preliminary result. Throughout this paper we will let \(\varepsilon_{12} = \frac{1}{(\lambda_1 \lambda_2)^{\frac{N+1}{2}}}.\)

**Proposition 2.1.** Suppose that \(z^1, z^2 \in \Sigma\) and \((y, \lambda) \in D_\mu.\) Then there exist \(\varepsilon_0 > 0\) and \(\mu_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\) and \(\mu \in (0, \mu_0),\) there is a unique \(C^1\) map \((y, \lambda) \in D_\mu \rightarrow (\alpha_\varepsilon(y, \lambda), v_\varepsilon(y, \lambda)) \in \mathbb{R}^2 \times D^{2, 2}(\mathbb{R}^N)\) such that \(v_\varepsilon \in E^2_{y^\lambda}, (\alpha_\varepsilon, y, \lambda, v_\varepsilon)\) satisfies (2.3), (2.6). Furthermore, \(\alpha_\varepsilon = (\alpha_{1, \varepsilon}, \alpha_{2, \varepsilon})\) and \(v_\varepsilon\) satisfy the following estimate as \(\varepsilon \to 0\)
\[
\sum_{j=1}^2 |\alpha_{j, \varepsilon} - (1 + \varepsilon K(z^j))^{-\frac{N+4}{2}}| + ||v_\varepsilon|| = O \left(\varepsilon \sum_{j=1}^2 \left(|y^j - z^j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}}\right) + \varepsilon^{\frac{1}{12} + \tau}\right) \tag{2.7}
\]
where \(\theta_j = \inf\{\beta_j, \frac{N+4}{2}\}, \tau > 0\) is some constant.

**Proof.** Let \(\hat{\alpha} = ((1 + \varepsilon K(z^1))^{-\frac{N+4}{2}}, (1 + \varepsilon K(z^2))^{-\frac{N+4}{2}}), \hat{\alpha} = \alpha - \hat{\alpha}, \omega = (\hat{\alpha}, v) \in \mathbb{R}^2 \times E^2_{y^\lambda}.\)

As in [10] (see also [17]) we expand \(\hat{J}_\varepsilon(y, \lambda, \omega) = J_\varepsilon(\alpha, y, \lambda, v)\) at \(\omega = 0\) and obtain
\[
\hat{J}_\varepsilon(y, \lambda, \omega) = \hat{J}_\varepsilon(y, \lambda, 0) + \left\langle f_\varepsilon, \omega \right\rangle + \frac{1}{2} \left\langle Q_\varepsilon, \omega, \omega \right\rangle + R_\varepsilon(\omega), \tag{2.8}
\]
where \(f_\varepsilon \in \mathbb{R}^2 \times E^2_{y^\lambda}\) is a linear form given by
\[
\left\langle f_\varepsilon, \omega \right\rangle = -\int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2^*-1} v
\]
\[
+ \sum_{k=1}^2 \hat{\alpha}_k \left\langle \sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j}, U_{y^k, \lambda_k} \right\rangle \tag{2.9}
\]
\[
- \int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2^*-1} U_{y^k, \lambda_k},
\]
$Q_\varepsilon$ is a quadratic form on $\mathbb{R}^2 \times E^2_{y,\lambda}$ given by

$$
\langle Q_\varepsilon \omega, \omega \rangle = \sum_{k=1}^{2} \sum_{i=1}^{2} \bar{\alpha}_i \alpha_k \left( \langle U_{y^i,\lambda_i} U_{y^k,\lambda_k} \rangle \right)
$$

$$
- (2^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j,\lambda_j} \right)^{2^* - 2} U_{y^i,\lambda_i} U_{y^k,\lambda_k}
$$

$$
+ \|v\|^2 - (2^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j,\lambda_j} \right)^{2^* - 2} v^2
$$

$$
- 2(2^* - 1) \sum_{k=1}^{2} \bar{\alpha}_k \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j,\lambda_j} \right)^{2^* - 2} U_{y^k,\lambda_k} v
$$

and $R_\varepsilon$ is the higher order term satisfying

$$
D^{(i)} R_\varepsilon(\omega) = O(\|\omega\|^{2^* + \theta - i}), \ i = 0, 1, 2
$$

where $\theta > 0$ is some constant.

To show the existence of $(\alpha_\varepsilon(y, \lambda), v_\varepsilon(y, \lambda)) \in \mathbb{R}^2 \times E^2_{y,\lambda}$ so that $(\alpha_\varepsilon, y, \lambda, v_\varepsilon)$ satisfy (2.4) and (2.5), it suffices to obtain $\omega(y, \lambda) \in \mathbb{N}^2 \times E^2_{y,\lambda}$ such that $D\hat{J}_\varepsilon(y, \lambda, \omega) = 0$ for each fixed $(y, \lambda) \in D_\mu$, where $D$ stands for the derivative with respect to $\omega$. $D\hat{J}_\varepsilon$ is equivalent to

$$
f_\varepsilon + Q_\varepsilon + DR_\varepsilon(\omega) = 0.
$$

As in [10], it is not difficult by using Lemmas A.4-A.6 to check that if $\mu > 0, \varepsilon > 0$ are small enough, then for each $(y, \lambda) \in D_\mu, Q_\varepsilon$ is invertible and there exists $C > 0$, independent of $(y, \lambda)$, such that $\|Q_\varepsilon^{-1}\| \leq C$. So by the implicit function theorem, we can prove that there is a $C^1-$map $\omega(y, \lambda)$ satisfying (2.12). Furthermore

$$
\|\omega\| \leq C \|f_\varepsilon\|. \tag{2.13}
$$

Applying Lemmas A.1-A.3 in Appendix A to (2.9) we get

$$
|\langle f_\varepsilon, \omega \rangle| = O \left( \varepsilon \sum_{j=1}^{2} \left| y^j - z^j \right|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} \right) \|\omega\|,
$$

and consequently we obtain (2.7). So we have completed the proof of the proposition. \(\square\)

It is worthwhile to point out that from Lagrange multiplier theorem, there are $A_j, B_j, C_{ji}(j = 1, 2, i = 1, 2, \cdots, N)$ in $\mathbb{R}$ such that $(\alpha, y, \lambda, v)$ satisfy (2.3) and (2.4). So we only need to solve finite dimensional problem (2.5) and (2.6).
3. Proof of main result

In this section, we prove that for the \( A_j, B_j, C_{ji} \in \mathbb{R} \) obtained in the above section satisfying (2.3) and (2.4) there exists \((\tilde{y}, \tilde{\lambda}) \in D_\mu\) such that (2.5) and (2.6) are satisfied by \((\tilde{\alpha}, \tilde{y}, \tilde{\lambda}, \tilde{v})\). Firstly, we give some estimates.

**Lemma 3.1.** Let \((y, \lambda) \in D_\mu\), \((\alpha, v)\) be obtained as in Proposition 2.1. For \(\mu > 0\) and \(\varepsilon > 0\) small enough, we have for \(k = 1, 2\)

\[
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial \lambda_k} = - \frac{C_{N, \beta_k} \varepsilon}{\lambda_k^{\beta_k+1}} \sum_{i=1}^{N} a_i^k + \frac{C_0 \varepsilon_12}{\lambda_k |z^1 - z^2|^{N-4}} + O \left( \frac{\varepsilon \varepsilon_12}{\lambda_k} \right) + O \left( \frac{\varepsilon^{1+\tau_1}}{\lambda_k} \right) + O \left( \frac{\varepsilon}{\lambda_k} \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\beta_j + \sigma}} + |y^j - z^j|^{\beta_j + \sigma} \right) \right),
\]

where \(C_{N, \beta_k}\) is a positive constant depending only on \(N\) and \(\beta_k\), \(C_0\) is a positive constant, \(\tau_1 = \min \{ \tau, \frac{1}{N-4} \}\).

**Proof.** Without loss of generality, we take \(k = 1\). Direct computations yield

\[
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial \lambda_1} = \alpha_1 \left[ \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} + v \right) \frac{\partial U_{y_1, \lambda_1}}{\partial \lambda_1} \right] \\
- \int_{\mathbb{R}^N} \left( 1 + \varepsilon K \right) \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} + v \right) \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} + v \right) \frac{\partial U_{y_1, \lambda_1}}{\partial \lambda_1}.
\]

It is easy to check that

\[
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial \lambda_1} = \alpha_1 \left[ \alpha_2 \int_{\mathbb{R}^N} U^{2* - 1}_{y^1, \lambda_2} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} \right] \\
- \int_{\mathbb{R}^N} \left( 1 + \varepsilon K \right) \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2* - 1} + (2* - 1) (\alpha_1 U_{y^1, \lambda_1})^{2* - 2} (\alpha_2 U_{y^2, \lambda_2}) \right] \\
- \alpha_1 (2* - 1) \int_{\mathbb{R}^N} \left( 1 + \varepsilon K \right) \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2* - 2} v + O \left( \frac{\varepsilon^{1+\tau_1}}{\lambda_1} \right) + \left( \frac{\|v\|^2}{\lambda} \right) \\
\triangleq I_1 - I_2 + O \left( \frac{\varepsilon^{1+\tau_1}}{\lambda_1} \right) + \left( \frac{\|v\|^2}{\lambda} \right).
\]
By Lemmas A.7, B.1-B.2, we can obtain

\[ I_1 = \alpha_1(\alpha_2 - \alpha_2^2) \int_{\mathbb{R}^N} U_{y^2, y_2} \frac{\partial U_{y^2, y_2}}{\partial \lambda_1} - (2^* - 1)\alpha_1 \alpha_2 U_{y^1, y_1} U_{y^2, y_2} \]

\[ - \alpha_1^2 \int_{\mathbb{R}^N} K \frac{\partial U_{y^1, y_1}}{\partial \lambda_1} U_{y^2, y_1} + O \left( \frac{\varepsilon}{\lambda_1} \right) \]

\[ = -\alpha_1^2 \int_{\mathbb{R}^N} K \frac{\partial U_{y^1, y_1}}{\partial \lambda_1} U_{y^2, y_1} - (2^* - 1)\alpha_1 \alpha_2 U_{y^1, y_1} U_{y^2, y_2} + O \left( \frac{\varepsilon}{\lambda_1} \right) \]

\[ = -\alpha_1^2 \sum_{i=1}^N a_i^1 + O \left( \frac{\varepsilon}{\lambda_1} |y^1 - z^1| \right) + O \left( \frac{\varepsilon}{\lambda_1 \beta_1 + 1 + \sigma} \right) + O \left( \frac{\varepsilon}{\lambda_1} |y^1 - z^1| \right) \]

and

\[ I_2 = O \left( \frac{\varepsilon^{1+\tau}}{\lambda_1} + \varepsilon \sum_{j=1}^2 \left( \frac{1}{\lambda_j^{\beta_j + \sigma}} + |y^j - z^j|^{\beta_j + \sigma} \right) \right) \|v\|. \]

Combining the above equalities, we can derive the conclusion. \( \square \)

**Lemma 3.2.** Under the same assumption as in Lemma3.1, we have \((k, l = 1, 2, k \neq l)\)

\[ \frac{\partial J_\varepsilon}{\partial y_i^k} = -D_{N, \beta_k} a_i^k \frac{\varepsilon}{\lambda_k^{\beta_k - 2}} (y_i^k - z_i^k) - C_1 \frac{(y_i^k - y_i^l)}{(\lambda_1 \lambda_2)^{N/2}} + O \left( \lambda_k \varepsilon^{1+\tau_1} \right) \]

\[ + O \left( \frac{\varepsilon}{\lambda_k^{\beta_k - 1}} y_i^k - z_i^k \right) + O \left( \varepsilon \lambda_k \varepsilon_{12} \right) \]

\[ + O \left( \varepsilon \lambda_k \sum_{j=1}^2 \left( \frac{1}{\lambda_j^{\beta_j + \sigma}} + |y^j - z^j|^{\beta_j + \sigma} \right) \right) + O \left( \frac{N-2}{\varepsilon_{12}^{N-4}} \right), \]

where \(D_{N, \beta_k}\) is a positive constant depending only on \(N\) and \(\beta_k\), \(C_1 > 0\) is a constant. \( \tau_1 \) and \( \sigma \) are the same as in Lemma3.1.

**Proof.** By using Hölder inequality and lemmas A.7, B.3-B.4 in the Appendix, the calculation is similar to the proof of Lemma3.1 we omit the detail. \( \square \)
Lemma 3.3. For \((y, \lambda) \in D_\mu\), let \((\alpha, v) \in \mathbb{R}^2 \times E_{y,\lambda}^2\) be obtained in Proposition 2.1. Then

\[
B_k = O \left( \lambda_k^2 \sum_{j=1}^2 \left( \frac{\varepsilon}{\lambda_j^{u+1}} + \varepsilon |y^j - z^j|_{\beta_j+1} \right) \right) + O(\lambda_k \varepsilon),
\]

\[
C_{ki} = O \left( \frac{\varepsilon}{\lambda_k} \left( \lambda_k |y^k - z^k| + \varepsilon \right) \right)
+ O \left( \frac{1}{\lambda_k} \sum_{j=1}^2 \left( \frac{\varepsilon}{\lambda_j} - \varepsilon |y^j - z^j|_{\beta_j+1} \right) \right),
\]

where \(\sigma > 0\) is some constant.

Proof. Let \(\varphi \in D_0^{2,2}(\mathbb{R}^N)\), then

\[
\left\langle \frac{\partial J_\varepsilon}{\partial v}, \varphi \right\rangle = \sum_{j=1}^2 A_j \left\langle U_{y^j,\lambda_j}, \varphi \right\rangle + \sum_{j=1}^2 B_j \left\langle \frac{\partial U_{y^j,\lambda_j}}{\partial \lambda_j}, \varphi \right\rangle
+ \sum_{j=1}^2 \sum_{i=1}^N C_{ji} \left\langle \frac{\partial U_{y^j,\lambda_j}}{\partial y^i_h}, \varphi \right\rangle.
\]

Taking \(\varphi = U_{y^k,\lambda_k}, \frac{\partial U_{y^k,\lambda_k}}{\partial y^k_h}, \lambda_k\), \(k = 1, 2, h = 1, 2, \cdots, N\) respectively and noting that

\[
\frac{\partial J_\varepsilon}{\partial \lambda_k} = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial U_{y^k,\lambda_k}}{\partial \lambda_k} \right\rangle, \quad \frac{\partial J_\varepsilon}{\partial y^h_k} = \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial U_{y^k,\lambda_k}}{\partial y^h_k} \right\rangle,
\]

we obtain a quasi-diagonal linear system of equations of \(A_j, B_j\) and \(C_{ji}\), whose coefficients are given by

\[
\left\langle U_{y^j,\lambda_j}, U_{y^k,\lambda_k} \right\rangle = \begin{cases} E, & j = k, \\ O(\varepsilon_{jk}), & j \neq k, \end{cases}
\]

\[
\left\langle U_{y^j,\lambda_j}, \frac{\partial U_{y^k,\lambda_k}}{\partial \lambda_k} \right\rangle = \begin{cases} 0, & j = k, \\ O(\frac{\varepsilon_{jk}}{y^k_h}), & j \neq k, \end{cases}
\]

\[
\left\langle \frac{\partial U_{y^j,\lambda_j}}{\partial \lambda_j}, U_{y^k,\lambda_k} \right\rangle = \begin{cases} 0, & j = k, \\ O(\lambda_k \varepsilon_{jk}), & j \neq k, \end{cases}
\]

\[
\left\langle \frac{\partial U_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial U_{y^k,\lambda_k}}{\partial \lambda_k} \right\rangle = \begin{cases} F_{j}, & j = k, \\ O(\varepsilon_{jk} \lambda_j \lambda_k), & j \neq k, \end{cases}
\]

\[
\left\langle \frac{\partial U_{y^j,\lambda_j}}{\partial \lambda_j}, \frac{\partial U_{y^k,\lambda_k}}{\partial y^h_k} \right\rangle = \begin{cases} 0, & j = k, \\ O(\lambda_k \varepsilon_{jk}), & j \neq k, \end{cases}
\]
\[
\langle \frac{\partial U_{y^i, \lambda_j}}{y_i}, \frac{\partial U_{y^k, \lambda_k}}{y_k} \rangle = \begin{cases} 
G \lambda_j^2 \delta_{hl}, & j = k, \\
O(\lambda_j \lambda_k \varepsilon_{12}), & j \neq k,
\end{cases}
\]
where \(E, F, G\) are strictly positive constants depending only on \(N\) and \(\delta_{hl}\) is the Kronecker symbol.

Using the estimates of \(\frac{\partial J}{\partial \lambda_k}, \frac{\partial J}{\partial y_k}\) in Lemmas 3.1 and 3.2, we can obtain the estimates of \(B_k\) and \(C_{ki}\).

**Proof of Theorem 1.1.** Let \(L_\varepsilon = \varepsilon^{\beta_1 \beta_2 / [\beta_1 \beta_2 - N / 2(\beta_1 + \beta_2)]}\). To obtain the existence of solution \(u_\varepsilon\) of the form (2.2), we only need to show that (2.5) and (2.6) are satisfied by some \((y, \lambda) \in D_\mu\). We will show that for some suitable \(\delta > 0, \gamma_1 > 0\) small and \(\gamma_2 > 0\) large, there exists \((y, \lambda)\) such that \((\lambda_1, \lambda_2) \in [\gamma_1 L_\varepsilon^{\beta_1 - 1}, \gamma_2 L_\varepsilon^{\beta_1 - 1}] \times [\gamma_1 L_\varepsilon^{\beta_2 - 1}, \gamma_2 L_\varepsilon^{\beta_2 - 1}]\), \(y = (y^1, y^2) \in B_{\lambda_1}(z^1) \times B_{\lambda_2}(z^2)\) together with the \((\alpha(y, \lambda), y, \lambda, v(y, \lambda))\) satisfy (2.5) and (2.6).

From Proposition 2.1 and Lemmas 3.1-3.2, we get the following equivalent form of (2.5) and (2.6):

\[
e^\lambda_k \lambda_k (y^k_i - z^k_i) = O \left( \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j} + |y^j - z^j|^{\beta_j + \sigma} \right) \right) + O \left( \frac{\varepsilon_{12}}{\lambda_k} \right), \quad k = 1, 2, i = 1, 2, \ldots, N, \tag{3.1}
\]

\[
\frac{\varepsilon}{\lambda_1^2} \sum_{i=1}^{N} a_i^k + \frac{d_k}{(\lambda_1 \lambda_2)^{N/2}} = O(\varepsilon \varepsilon_{12}) + O(\varepsilon_{12}^{1+2r_1}) + O \left( \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j} + |y^j - z^j|^{\beta_j + \sigma} \right) \right) \tag{3.2}
\]

where \(d_k, r_1\) are some positive constant.

Let

\[
\lambda_1 = t_1 L_\varepsilon^{\beta_1 - 1}, \lambda_2 = t_2 L_\varepsilon^{\beta_2 - 1}, t_1, t_2 \in [\gamma_1, \gamma_2],
\]

\(y^1 - z^1 = \lambda_1^{-1} x^1, y^2 - z^2 = \lambda_2^{-1} x^2, x^1, x^2 \in B_\delta(0)\).

Then (3.1) and (3.2) can be rewritten in the following equivalent way

\[
x^k = o_\varepsilon(1) \quad k = 1, 2, \tag{3.3}
\]

\[
t_k^{-\beta_k} \left( \sum_{i=1}^{N} a_i^k \right)^{-1} d_k (t_1 t_2)^{-N/2} = o_\varepsilon(1) \quad k = 1, 2. \tag{3.4}
\]
Set
\[ f(x^1, x^2) = (x^1, x^2), \quad (x^1, x^2) \in \Omega_1 = \mathbb{B}_0(0) \times \mathbb{B}_0(0) \]
\[ g(t_1, t_2) = (g_1(t_1, t_2), g_2(t_1, t_2)), \quad (t_1, t_2) \in \Omega_2 = [\gamma_1, \gamma_2] \times [\gamma_1, \gamma_2] \]
\[ g_k(t_1, t_2) = \frac{1}{t_k} - \frac{m_k}{(t_1t_2)^{\frac{N-4}{2}}}, \quad \text{where} \quad m_k = -\frac{d_k}{\sum_{i=1}^N a_i^k} > 0. \]

Then
\[ \deg(f, \Omega_1, 0) = 1. \quad (3.5) \]

On the other hand, it is easy to see that \( g = 0 \) has a unique solution \((t_1^*, t_2^*)\) in \([\gamma_1, \gamma_2]\) if \( \gamma_1 \) is small and \( \gamma_2 > 0 \) is large. Furthermore,
\[ \frac{\partial g_1}{\partial t_1}(t_1^*, t_2^*) = \frac{1}{t_1^*} \left( -\beta_1 + \frac{N - 4}{2} \right) \frac{m_1}{(t_1^* t_2^*)^{\frac{N-4}{2}}}, \]
\[ \frac{\partial g_1}{\partial t_2}(t_1^*, t_2^*) = \frac{(N - 4)m_1}{2t_2^*(t_1^* t_2^*)^{\frac{N-4}{2}}}, \]

and
\[ \frac{\partial g_2}{\partial t_1}(t_1^*, t_2^*) = \frac{(N - 4)m_2}{2t_1^*(t_1^* t_2^*)^{\frac{N-4}{2}}}, \]
\[ \frac{\partial g_2}{\partial t_2}(t_1^*, t_2^*) = \frac{1}{t_2^*} \left( -\beta_2 + \frac{N - 4}{2} \right) \frac{m_2}{(t_1^* t_2^*)^{\frac{N-4}{2}}}. \]

Then
\[ \text{Jac } g|(t_1^*, t_2^*) = \left( \beta_1 \beta_2 - (\beta_1 + \beta_2) \frac{N - 4}{2} \right) \frac{m_1 m_2}{(t_1^* t_2^*)^{(N-3)}} < 0. \]

So \( \deg(g, \Omega_2, 0) = -1 \). As a consequence
\[ \deg((f, g), \Omega_1 \times \Omega_2, 0) = \deg(f, \Omega_1, 0) \times \deg(g, \Omega_2, 0) = -1. \]

Thus, \((f, g), (t_1^*, t_2^*)\) has a solution and \( u_\varepsilon = \alpha_1,\varepsilon U_{y_1, \lambda_1, \varepsilon} + \alpha_2,\varepsilon U_{y_2, \lambda_2, \varepsilon} + v_\varepsilon \) is a critical point of \( I_\varepsilon \). By Proposition 3.4 we know \( u_\varepsilon > 0 \) for \( \varepsilon > 0 \) sufficiently small. Thus the result follows. \( \square \)

**Proposition 3.4.** Assume that \( u_\varepsilon = \alpha_1,\varepsilon U_{y_1, \lambda_1, \varepsilon} + \alpha_2,\varepsilon U_{y_2, \lambda_2, \varepsilon} + v_\varepsilon \) is a critical point of \( I_\varepsilon \) and \( v_\varepsilon \) satisfies \((2.7)\). Then for \( \varepsilon > 0 \) sufficiently small, \( u_\varepsilon > 0 \).

**Proof.** we follow the idea in \([6], [17]\) to prove the proposition. Set \( u_\varepsilon^- = \max\{-u_\varepsilon, 0\} \).

Let us introduce \( \omega \) satisfying
\[ \Delta^2 \omega = -(1 + \varepsilon K) \left( u_\varepsilon^- \right)^{\frac{N+4}{N-4}} \text{ in } \mathbb{R}^N, \]
and note that \( c_N|x|^{4-N} \) is the Green function of the operator \( \Delta^2 \) in \( D_0^{2,2}(\mathbb{R}^N) \).

Thus, \( \omega \) can be written as
\[ \omega = -c_N \int_{\mathbb{R}^N} |x - y|^{4-N} (1 + \varepsilon K) \left( u_\varepsilon^- \right)^{\frac{N+4}{N-4}} dy. \]
It is easy to see that $\omega \leq 0$ and

$$
\|\omega\|^2 = -\int_{\mathbb{R}^N} (1 + \varepsilon K) (u^-_\varepsilon)^{\frac{N+4}{2}} \omega \leq C\|\omega\|\|u^-_\varepsilon\|_L^{N+2}.
$$

Assume that $\|\omega\| \neq 0$, then

$$
\|\omega\| \leq C|u^-_\varepsilon|_L^{N+2}.
$$

On the other hand

$$
\|\omega\|^2 = -\int_{\mathbb{R}^N} (1 + \varepsilon K) (u^-_\varepsilon)^{\frac{N+4}{2}} \omega \\
\geq -\int_{\{u \leq 0\}} (1 + \varepsilon K) (u^-_\varepsilon)^{\frac{N+4}{2}} \omega + \int_{\{u \geq 0\}} (1 + \varepsilon K) (u^+_\varepsilon)^{\frac{N+4}{2}} \omega \\
= \int_{\mathbb{R}^N} (1 + \varepsilon K) |u^-_\varepsilon|^\frac{N+4}{2} u^-_\varepsilon \omega = \int_{\mathbb{R}^N} \omega \Delta^2 u^-_\varepsilon = \int_{\mathbb{R}^N} u^-_\varepsilon \Delta^2 \omega \\
= -\int_{\mathbb{R}^N} (1 + \varepsilon K) (u^-_\varepsilon)^{\frac{N+4}{2}} u^-_\varepsilon = \int_{\mathbb{R}^N} (1 + \varepsilon K)(u^-_\varepsilon)^{2^*} \\
\geq c'|u^-_\varepsilon|_L^{2^*}.
$$

Thus,

$$
c'|u^-_\varepsilon|_L^{2^*} \leq \|\omega\|^2 \leq c|u^-_\varepsilon|_L^{\frac{2(N+4)}{N+4}}.
$$

It is obvious that $|u^-_\varepsilon|_L^{2^*} \leq |v|_L^{2^*}$, So for $\varepsilon$ sufficiently small, $u^-_\varepsilon \equiv 0$, which implies that $\omega \equiv 0$ and we get a contradiction. As a result, $\omega \equiv 0$ and $u^-_\varepsilon \equiv 0$. Therefore, $u_\varepsilon > 0$ and we complete the proof. \qed

**Appendix A.**

In this appendix, we give some estimates and result that used in Section 2.

**Lemma A.1.** For any $(y, \lambda) \in D_\mu$ and $v \in E^2_{y, \lambda}$, there exists $\tau > 0$ such that

$$
\int_{\mathbb{R}^N} K\left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j}\right)^{2^*-1} v = O\left(\sum_{j=1}^2 \left(\|y^j - z^j\|^{\beta_j} + \frac{1}{\theta_j}\right) + \varepsilon \frac{\tau}{2}\right) \|v\|.
$$

where $\tau > 0$ and $\theta_j = \inf\{\beta_j, \frac{N+4}{2}\}$.

**Proof.** Use the inequality

$$
\|a + b\|^p - a^p - b^p \leq \begin{cases} 
Ca^{p/2}b^{p/2}, & 1 < p < 2, \\
Ca^p - 1 + Ca^{p-1}b, & p \geq 2.
\end{cases}
$$

(A.1)
We have
\[
\int_{\mathbb{R}^N} K \left( \sum_{j=1}^{2^*} \alpha_j U_{y_j, \lambda_j} \right)^{2^*-1} v = \int_{\mathbb{R}^N} K \sum_{j=1}^{2^*} (\alpha_j U_{y_j, \lambda_j})^{2^*-1} v = \int_{\mathbb{R}^N} \sum_{j=1}^{2^*} \alpha_j U_{y_j, \lambda_j}^{2^*-1} v + \left\{ \begin{array}{ll}
O \left( \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i, \lambda_i}^{2^*-1} |v| \right), & 2 < 2^* < 3, \\
O \left( \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i, \lambda_i}^{2^*-2} |v| \right), & 2^* \geq 3
\end{array} \right.
\]

Lemma A.2. There exists \( \tau > 0 \) such that for any \((y, \lambda) \in D_\mu \) and \( v \in E_{y, \lambda}^2 \), we have
\[
\int_{\mathbb{R}^N} \left( \sum_{j=1}^{2^*} \alpha_j U_{y_j, \lambda_j} \right)^{2^*-1} v = O \left( \varepsilon_{12}^{\frac{1}{2} + \tau} \|v\| \right).
\]

Proof. By inequality (A.1), we get
\[
\int_{\mathbb{R}^N} \left( \sum_{j=1}^{2^*} \alpha_j U_{y_j, \lambda_j} \right)^{2^*-1} v = \int_{\mathbb{R}^N} \left( \sum_{j=1}^{2^*} \alpha_j U_{y_j, \lambda_j} \right)^{2^*-1} v + \left\{ \begin{array}{ll}
O \left( \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i, \lambda_i}^{2^*-1} U_{y_j, \lambda_j}^{2^*-1} |v| \right), & 2 < 2^* < 3, \\
O \left( \sum_{i \neq j} \int_{\mathbb{R}^N} U_{y_i, \lambda_i}^{2^*-2} U_{y_j, \lambda_j}^{2^*-2} |v| \right), & 2^* \geq 3
\end{array} \right.
\]

\[
= O \left( \varepsilon_{12}^{\frac{1}{2} + \tau} \|v\| \right).
\]
since $\mu$ is small and $\lambda_1, \lambda_2 \geq \frac{1}{\mu}$.

**Lemma A.3.** Suppose $(y, \lambda) \in D_\mu$. Then for $\mu$ small enough we have

$$\left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y', \lambda_j}, U_{y^k, \lambda_k} \right) - \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2^*-1} U_{y^k, \lambda_k}$$

$$= O \left( \varepsilon \left( \frac{1}{\lambda_{\beta_k}^k} + \|y^k - z^k\|_{\beta_k} \right) + O(\varepsilon_{12}) \right).$$

**Proof.**

$$\left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j, \lambda_j}, U_{y^k, \lambda_k} \right) - \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2^*-1} U_{y^k, \lambda_k}$$

$$= \hat{\alpha}_k \left( \int_{\mathbb{R}^N} U_{y^k, \lambda_k}^2 - \int_{\mathbb{R}^N} \frac{1 + \varepsilon K}{1 + \varepsilon K(z^k)} U_{y^k, \lambda_k}^2 \right) + O(\varepsilon_{12})$$

$$= \frac{\varepsilon}{(1 + \varepsilon K(z^k))^{N+4}} \int_{\mathbb{R}^N} \left( K(x) - K(z^k) \right) \frac{1}{(1 + |z|^2)^N} + O(\varepsilon_{12})$$

$$= O(\varepsilon) \int_{\mathbb{R}^N} \left| K \left( \frac{z}{\lambda_k} + y^k \right) - K(z^k) \right| + O(\varepsilon_{12})$$

$$= O \left( \varepsilon \left( \frac{1}{\lambda_{\beta_k}^k} + \|y^k - z^k\|_{\beta_k} \right) + O(\varepsilon_{12}) \right).$$
for all $v \in E^2_{y, \lambda}$. Notice that

$$\left| \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y_j, \lambda_j} \right)^{2^* - 2} v^2 - \int_{\mathbb{R}^N} \left( \sum_{j=1}^{2} \hat{\alpha}_j U_{y_j, \lambda_j} \right)^{2^* - 2} v^2 \right|$$

$$\leq O(\varepsilon) \|v\|^2 + O\left( \int_{\mathbb{R}^N} U_{y_1, \lambda_1}^{2^*-2} U_{y_2, \lambda_2}^{2^*-2} v^2 \right)$$

$$= O\left( \varepsilon + (\lambda_1 \lambda_2)^{-\hat{\sigma}} \right) \|v\|^2$$

for some $\hat{\sigma} > 0$ and Lemma A.4 follows.

\[ \Box \]

**Lemma A.5.** Suppose $(y, \lambda) \in D_\mu$. Then for $\mu$ and $\varepsilon$ small enough we have

$$\langle U_{y^l, \lambda_l} U_{y^k, \lambda_k} \rangle - (2^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y_j, \lambda_j} \right)^{2^* - 2} U_{y^l, \lambda_l} U_{y^k, \lambda_k}$$

$$= \begin{cases} 
(1 - (2^* - 1)\alpha_i^{2^* - 2}) A + O(\varepsilon_1^r) + O(\varepsilon), & \text{if } k = l = 1, 2, \\
O(\varepsilon_1^r) + O(\varepsilon), & \text{if } k \neq l, k, l = 1, 2,
\end{cases}$$

for some $r > 0, A > 0$.

**Proof.**

$$\langle U_{y^l, \lambda_l} U_{y^k, \lambda_k} \rangle - (2^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y_j, \lambda_j} \right)^{2^* - 2} U_{y^l, \lambda_l} U_{y^k, \lambda_k}$$

$$= \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2^* - 1} U_{y^k, \lambda_k} - (2^* - 1)\alpha_i^{2^* - 2} \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2^* - 1} U_{y^k, \lambda_k}$$

$$+ O\left( \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2^* - 2} U_{y^2, \lambda_2} + U_{y^2, \lambda_2}^{2^* - 1} U_{y^l, \lambda_l} U_{y^2, \lambda_2}^{2^* - 2} U_{y^2, \lambda_2} U_{y^l, \lambda_l} \right) + O(\varepsilon)$$

$$= [1 - (2^* - 1)\alpha_i^{2^* - 2}] \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2^* - 1} U_{y^k, \lambda_k} + O(\varepsilon_1^r) + O(\varepsilon).$$

Let $A = \int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2^*} = \int_{\mathbb{R}^N} U_{0, 1}^{2^*}$, we get the conclusion of Lemma A.5.

\[ \Box \]

**Lemma A.6.** Suppose $(y, \lambda) \in D_\mu, v \in E^2_{y, \lambda}$. If $\mu$ and $\varepsilon$ small, then

$$\int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y_j, \lambda_j} \right)^{2^* - 2} U_{y^k, \lambda_k} v$$

$$= O\left( \varepsilon_{1^r}^{*+\tau} + \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\theta_j}} + |y^j - z^j|^{|\beta_j|} \right) \right) \|v\|$$

where $\tau > 0$ and $\theta_j = \inf\{\beta_j, \frac{N+1}{2}\}$. 
Proof. By using Hölder inequality and our assumption on $K(x)$, we have

$$
\int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2^* - 2} U_{y^k, \lambda_k} v
$$

$$
= \alpha_k^{2^* - 2} \int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2^* - 1} v + \alpha_k^{2^* - 2} \varepsilon \int_{\mathbb{R}^N} K(x) U_{y^k, \lambda_k}^{2^* - 1} v
$$

$$
+ \int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2^* - 2} - (\alpha_k U_{y^k, \lambda_k})^{2^* - 2} \right) U_{y^k, \lambda_k} v
$$

$$
= O \left( \varepsilon^{\frac{1}{12}} + \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) \right) \|v\|.
$$

\[ \square \]

Lemma A.7. Suppose $(y, \lambda) \in D_{\mu}, v \in E^{2}_{y, \lambda}$. If $\mu$ and $\varepsilon$ small, then

$$
\int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2^* - 2} \frac{\partial U_{y^k, \lambda_k} v}{\partial \lambda_k}
$$

$$
= O \left( \varepsilon^{\frac{1}{12}} + \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) \right) \|v\|
$$

and

$$
\int_{\mathbb{R}^N} (1 + \varepsilon K) \left( \sum_{j=1}^{2} \alpha_j U_{y^j, \lambda_j} \right)^{2^* - 2} \frac{\partial U_{y^k, \lambda_k} v}{\partial y^k}
$$

$$
= O \left( \lambda_k \varepsilon^{\frac{1}{12}} + \lambda_k \varepsilon \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) \right) \|v\|
$$

where $\tau > 0$ and $\theta_j = \inf \{ \beta_j, \frac{N+4}{2} \}$.

Proof. The proof of this lemma is similar to that of Lemma A.6. \[ \square \]

Appendix B.

The estimates given in this appendix will play the key role in proving our main result in Section 3. The computation here is very similar the one performed in [4], so we give a sketch proof.
Lemma B.1. Suppose \((y, \lambda) \in D_\mu\). Then for \(\mu\) small, \(k = 1, 2\), we have
\[
\int_{\mathbb{R}^N} KU^{2^*-1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} = C_{N, \beta_k} \sum_{i=1}^{N} a_i^k + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right)
\]
where \(C_{N, \beta_k}\) is a positive constant depending only on \(N\) and \(\beta_k\).

Proof.
\[
\int_{\mathbb{R}^N} KU^{2^*-1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k}
= \int_{\{|x-z^k| \leq r_0\}} \left( \sum_{i=1}^{N} a_i^k |x_i - z_i^k|^{\beta_k} + O \left( |x - z^k|^{\beta_k+\sigma} \right) \right) U^{2^*-1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} + O \left( \frac{1}{\lambda_k^{N+1}} \right)
= C'_{N, \beta_k+1} \int_{\{|x-z_k(z^k)| < \lambda_k r_0\}} \sum_{i=1}^{N} a_i^k |x_i - \lambda_k (y^k_i - z^k_i)|^{\beta_k} \left( \frac{1}{1 + |x|^2} \right)^{N+1}
+ O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right)
= C''_{N, \beta_k+1} \sum_{i=1}^{N} a_i^k |x_i|^{\beta_k} \left( \frac{1}{1 + |x|^2} \right)^{N+1}
+ O \left( \frac{1}{\lambda_k^\sigma} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right)
\]
\[
= C''_{N, \beta_k+1} \sum_{i=1}^{N} a_i^k + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1}} \right) + O \left( \frac{1}{\lambda_k^{\beta_k+1+\sigma}} \right).
\]

Lemma B.2. Suppose \((y, \lambda) \in D_\mu, \mu\) small, we have for \(k \neq l, k, l = 1, 2\),
\[
\int_{\mathbb{R}^N} U^{2^*-2} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} U_{y^l, \lambda_l} = -\frac{N-4}{2} C_0 \frac{\varepsilon_{12}}{\lambda_k} z^1_2 - z^2_2 |^{N-4} + O \left( \frac{\varepsilon_{12}}{\lambda_k} \right)
\]
where \(C_0 > 0\).
Proof.

\[
\int_{\mathbb{R}^N} U^{2^*-2} \frac{\partial U_{y^k,\lambda_k}}{\partial \lambda_k} U_{y^l,\lambda_l} = \frac{1}{2^*-1} \int_{\mathbb{R}^N} \frac{\partial U^{2^*-1}}{\partial \lambda_k} U_{y^l,\lambda_l}
\]
\[
= \frac{1}{2^*-1} \int_{\mathbb{R}^N} \Delta \frac{\partial U_{y^k,\lambda_k}}{\partial \lambda_k} U_{y^l,\lambda_l} = \frac{1}{2^*-1} \int_{\mathbb{R}^N} \frac{\partial U^{2^*-1}}{\partial \lambda_k} U_{y^l,\lambda_l}
\]
\[
= -\frac{N-4}{2} C_0 \frac{\varepsilon_{12}}{\lambda_k |z_1 - z_2|^N - 4} + O \left( \frac{\varepsilon_{N-4}}{\lambda_k} \right)
\]

(Follow the exact same line in proving estimate (F16) in [4]). \qed

Lemma B.3. Suppose that \((y, \lambda) \in D_\mu\) and \(\mu\) small. Then

\[
\int_{\mathbb{R}^N} K U^{2^*-1} \frac{\partial U_{y^k,\lambda_k}}{\partial y^k_i} = D_{N,\beta_k} a^k \frac{1}{\lambda_{k-1}} \lambda_k (y^k_i - z^k_i) + O \left( \frac{1}{\lambda_k^{N-1}} \right) + O \left( \lambda_k |y^k - z^k|^{\beta_k + \sigma} \right)
\]

where \(D_{N,\beta_k}\) is a positive constant depending only on \(N\) and \(\beta_k\).

Proof.

\[
\int_{\mathbb{R}^N} K U^{2^*-1} \frac{\partial U_{y^k,\lambda_k}}{\partial y^k_i} = (N-4) D_N \int_{\mathbb{R}^N} K(x) U^{2^*} \frac{\lambda_k^2 (x_i - y^k_i)}{1 + \lambda_k^2 |x - y^k|^2}
\]
\[
= (N-4) D_N \int_{\{|x - z^k| \leq r_0\}} \left( \sum_{h=1}^N a^k_h |x_h - z^k_h|^{\beta_h} + O(|x - z^k|^{\beta_k + \sigma}) \right)
\]
\[
\times U^{2^*} \frac{\lambda_k^2 (x_i - y^k_i)}{1 + \lambda_k^2 |x - y^k|^2} + O \left( \frac{1}{\lambda_k^{N-1}} \right)
\]
\[
= \frac{D'_N}{\lambda_k^{\beta_k - 1}} \int_{\mathbb{R}^N} \sum_{h=1}^N a^k_h \left[ |x|^\beta_k + \beta_k |x|^{\beta_k - 1} x_h \lambda_k (y^k_h - z^k_h) \right] \frac{U^{2^*}_{0,1} x_i}{1 + |x|^2}
\]
\[
+ O \left( \frac{1}{\lambda_k^{\beta_k - 1}} \lambda_k^{2 |y^k - z_k|^2} \right) + O \left( \frac{1}{\lambda_k^{\beta_k - 1 + \sigma}} \right) + O \left( \lambda_k |y^k - z^k|^{\beta_k + \sigma} \right)
\]
\[ D'_N,\beta^k a^k_i \int_{\mathbb{R}^N} \frac{|x|^{\beta_k}}{(1 + |x|)^{N+1}} \cdot \lambda_k (y^k_i - z^k_i) + O \left( \frac{1}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z_k|^2 \right) \\
+ O \left( \frac{1}{\lambda_k^{\beta_k-1+\sigma}} \right) + O \left( \lambda_k |y^k - z^k|^{\beta_k+\sigma} \right) \\
\triangleq D_{N,\beta^k a^k_i} \int_{\mathbb{R}^N} \frac{1}{\lambda_k^{\beta_k-1}} \lambda_k (y^k_i - z^k_i) + O \left( \frac{1}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z_k|^2 \right) \\
+ O \left( \frac{1}{\lambda_k^{\beta_k-1+\sigma}} \right) + O \left( \lambda_k |y^k - z^k|^{\beta_k+\sigma} \right). \]

Lemma B.4. Suppose that \((y, \lambda) \in D_\mu\) and \(\mu\) small, Then we have for \(k \neq l, k, l = 1, 2,\)

\[
\int_{\mathbb{R}^N} U^2 - \frac{\partial U}{\partial y_i^k \lambda_k} U^{y_i^l, \lambda_l} = C_1 \lambda_1 \lambda_2 (y^k_i - y^l_i) \frac{N-2}{12} + O \left( \frac{N-2}{12} \right)
\]

where \(C_1 > 0\) is a constant depending only on \(N\).

Proof. Follow the exact same procedure in proving estimate (F20) in [4].

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