A Lefschetz formula for higher rank

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Abstract

In this paper a Lefschetz formula is proved for the geodesic flow of a compact locally symmetric space. The flow is described in terms of actions of split tori of various dimensions and the geometric side of the Lefschetz formula is a sum over closed geodesics which correspond to a given torus. The cohomological side is given in terms of Lie algebra cohomology.

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Introduction

In this paper we prove a Lefschetz formula for the geodesic flow of a compact locally symmetric space. The first case of such a formula appears in [21], where it is proved for compact quotients of symmetric spaces of rank one. In higher rank, the geodesic flow extends to an action of a higher dimensional torus on the sphere bundle. The sphere bundle itself carries a stratification according to the orbit type and the Lefschetz formula is formulated for each stratum separately.

In the first section we fix notation and collect some prerequisites, most of them going back to Harish-Chandra’s work. In section 2 we construct Euler-Poincaré functions in a more general setting as in the literature. The Selberg trace formula is given in section 3. Finally, in section 4 we formulate and prove the Lefschetz formula which connects geometric information on closed orbits with spectral data from the action of the flow on Lie algebra cohomology.

This formula can be used to show that Zeta functions of Selberg type have an analytic continuation if the torus is one dimensional (within a higher rank group). But the Lefschetz formula gives valuable information for each dimension. A weaker version of the highest dimensional case has been used in [21] to derive asymptotic formulae for the length distribution of closed geodesics.
1 Prerequisites

In this section we collect some facts from literature which will be needed in the sequel. Proofs will only be given by sketches or references.

1.1 Notations

We denote Lie groups by upper case roman letters $G, H, K, \ldots$ and the corresponding real Lie algebras by lower case German letters with index 0, that is: $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{k}_0, \ldots$. The complexified Lie algebras will be denoted by $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \ldots$, so, for example: $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$.

For a Lie group $L$ with Lie algebra $\mathfrak{l}_0$ let $\text{Ad} : L \rightarrow \text{GL}(\mathfrak{l}_0)$ be the adjoint representation ([5] III.3.12). By definition, $\text{Ad}(g)$ is the differential of the map $x \mapsto gxg^{-1}$ at the point $x = e$. Then $\text{Ad}(g)$ is a Lie algebra automorphism of $\mathfrak{g}_0$. A Lie group $L$ is said to be of inner type if $\text{Ad}(L)$ lies in the complex adjoint group of the Lie algebra $\mathfrak{l}$.

A real Lie group $G$ is said to be a real reductive group if there is a linear algebraic group $G$ defined over $\mathbb{R}$ which is reductive as an algebraic group and a morphism $\alpha : G \rightarrow G(\mathbb{R})$ with finite kernel and cokernel (Since we do not insist that the image of $\alpha$ is normal the latter condition means that $\text{im}(\alpha)$ has finite index in $G(\mathbb{R})$). This implies in particular that $G$ has only finitely many connected components ([3] 24.6.c).

A real reductive group $G$ has a maximal compact subgroup $K$ which meets every connected component. The group $G$ is of inner type if and only if $\text{Ad}(K)$ lies in the complex adjoint group of $\mathfrak{g}$.

Note that any real reductive group $G$ of inner type is of Harish-Chandra class, i.e., $G$ is of inner type, the Lie algebra $\mathfrak{g}_0$ of $G$ is reductive, $G$ has finitely many connected components and the connected subgroup $G^0_{\text{der}}$ corresponding to the Lie subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ has finite center.

The following are of importance:

- a connected semisimple Lie group with finite center is a real reductive group of inner type ([31] 2.1.3),
- if $G$ is a real reductive group of inner type and $P = MAN$ is the Langlands decomposition ([31] 2.2.7) of a parabolic subgroup then the groups $M$ and $AM$ are real reductive of inner type ([31] 2.2.8).

The usual terminology of algebraic groups carries over to real reductive groups, for example a torus (or a Cartan subgroup) in $G$ is the inverse image of (the real points of) a torus or a Cartan subgroup in $G(\mathbb{R})$. An element of $G$ will be
called *semisimple* if it lies in some torus of $G$. The *split component* of $G$ is the identity component of the greatest split torus in the center of $G$. Note that for a real reductive group the Cartan subgroups are precisely the maximal tori.

Let $G$ be a real reductive group then there exists a *Cartan involution* i.e., an automorphism $\theta$ of $G$ satisfying $\theta^2 = Id$ whose fixed point set is a maximal compact subgroup $K$ and which is the inverse ($a \mapsto a^{-1}$) on the split component of $G$. All Cartan involutions are conjugate under automorphisms of $G$.

Fix a Cartan involution $\theta$ with fixed point set equal to the maximal compact subgroup $K$ and let $k_0$ be the Lie algebra of $K$. The group $K$ acts on $g_0$ via the adjoint representation and there is a $K$-stable decomposition $g_0 = k_0 \oplus p_0$, where $p_0$ is the eigenspace of (the differential of) $\theta$ to the eigenvalue $-1$. Write $g = \mathfrak{t} \oplus \mathfrak{p}$ for the complexification. This is called the *Cartan decomposition*.

**Lemma 1.1** There is a symmetric bilinear form $B : g_0 \times g_0 \to \mathbb{R}$ such that

- $B$ is invariant, that is $B(\text{Ad}(g)X, \text{Ad}(g)Y) = B(X, Y)$ for all $g \in G$ and all $X, Y \in g_0$ and
- $B$ is negative definite on $\mathfrak{k}_0$ and positive definite on its orthocomplement $\mathfrak{p}_0 = \mathfrak{k}_0^\perp \subset g_0$.

**Proof:** For $X \in g_0$ let $\text{ad}(X) : g_0 \to g_0$ be the adjoint defined by $\text{ad}(X)Y = [X, Y]$. Since for $g \in G$ the map $\text{Ad}(g)$ is a Lie algebra homomorphism we infer that $\text{ad}(\text{Ad}(g)X) = \text{Ad}(g)\text{ad}(X)\text{Ad}(g)^{-1}$ and therefore the *Killing form*

$$B_K(X, Y) = \text{tr}(\text{ad}(X)\text{ad}(Y))$$

is invariant. It is known that if $B_K$ is nondegenerate, i.e., $g_0$ is semisimple, then $B = B_K$ satisfies the claims of the lemma. In the general case we have $g_0 = \mathfrak{a}_0 \oplus \mathfrak{c}_0 \oplus g_0'$, where $\mathfrak{a}_0 \oplus \mathfrak{c}_0$ is the center of $g_0$ and $g_0'$ is its derived algebra, which is semisimple and so $B_K|_{g_0'}$ is nondegenerate, whereas $B_K|_{\mathfrak{a}_0 \oplus \mathfrak{c}_0} = 0$. Further $\mathfrak{c}_0$ is the eigenspace of $\theta$ in the center of $g_0$ corresponding to the eigenvalue 1 whereas $\mathfrak{a}_0$ is the eigenspace of $-1$. For $g \in G$ the adjoint $\text{Ad}(g)$ is easily seen to preserve $\mathfrak{a}_0$ and $\mathfrak{c}_0$, so we get a representation $\rho : G \to GL(\mathfrak{a}_0) \times GL(\mathfrak{c}_0)$. This representation is trivial on the connected component $G^0$ of $G$ hence it factors over the finite group $G/G^0$. Therefore there is a positive definite symmetric bilinear form $B_\alpha$ on $\mathfrak{a}_0$ which is invariant under $\text{Ad}_\alpha$ and similarly a negative definite symmetric bilinear form $B_\epsilon$ on $\mathfrak{c}_0$ which is invariant. Let

$$B = B_\alpha \oplus B_\epsilon \oplus B_K|_{g_0'},$$

Then $B$ satisfies the claims of the lemma. $\square$

Let $U(g_0)$ be the universal enveloping algebra of $g_0$. It can be constructed as the quotient of the tensorial algebra

$$T(g_0) = \mathbb{R} \oplus g_0 \oplus (g_0 \otimes g_0) \oplus \ldots$$
by the two-sided ideal generated by all elements of the form $X \otimes Y - Y \otimes X - [X,Y]$, where $X,Y \in \mathfrak{g}_0$.

The algebra $U(\mathfrak{g}_0)$ can be identified with the $\mathbb{R}$-algebra of all left invariant differential operators on $G$. Let $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}_0$ and $U(\mathfrak{g})$ be its universal enveloping algebra which is the same as the complexification of $U(\mathfrak{g}_0)$. Then $\mathfrak{g}$ is a subspace of $U(\mathfrak{g})$ which generates $U(\mathfrak{g})$ as an algebra and any Lie algebra representation of $\mathfrak{g}$ extends uniquely to a representation of the associative algebra $U(\mathfrak{g})$.

The form $B$ gives an identification of $\mathfrak{g}_0$ with its dual space $\mathfrak{g}_0^*$. On the other hand $B$ defines an element in $\mathfrak{g}_0^* \otimes \mathfrak{g}_0^*$. Thus we get a natural element in $\mathfrak{g}_0 \otimes \mathfrak{g}_0 \subset T(\mathfrak{g}_0)$. The image $C$ of this element in $U(\mathfrak{g}_0)$ is called the Casimir operator attached to $B$. It is a differential operator of order two and it lies in the center of $U(\mathfrak{g}_0)$. In an more concrete way the Casimir operator can be described as follows: Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{g}_0$ and let $Y_1, \ldots, Y_m$ be the dual basis with respect to the form $B$ then the Casimir operator is given by

$$C = X_1Y_1 + \cdots + X_mY_m.$$ 

We have

**Lemma 1.2** For any $g \in G$ the Casimir operator is invariant under $\text{Ad}(g)$, that is $\text{Ad}(g)C = C$.

**Proof:** This follows from the invariance of $B$.

Let $X$ denote the quotient manifold $G/K$. The tangent space at $eK$ identifies with $\mathfrak{p}_0$ and the form $B$ gives a $K$-invariant positive definite inner product on this space. Translating this by elements of $G$ defines a $G$-invariant Riemannian metric on $X$. This makes $X$ the most general globally symmetric space of the noncompact type [19].

Let $\hat{G}$ denote the unitary dual of $G$, i.e., $\hat{G}$ is the set of isomorphism classes of irreducible unitary representations of $G$.

Let $(\pi, V_{\pi})$ be a continuous representation of $G$ on some Banach space $V_{\pi}$. The subspace $V_{\pi}^\infty$ of smooth vectors is defined to be the subspace of $V_{\pi}$ consisting of all $v \in V_{\pi}$ such that the map $g \mapsto \pi(g)v$ is smooth. The universal enveloping algebra $U(\mathfrak{g})$ operates on $V_{\pi}^\infty$ via

$$\pi(X) : v \mapsto X_g(\pi(g)v) \mid_{g=e}$$

for $X$ in $\mathfrak{g}$.

A $(\mathfrak{g}, K)$-module is by definition a complex vector space $V$ which is a $K$-module such that for each $v \in V$ the space spanned by the orbit $K.v$ is finite dimensional. Further $V$ is supposed to be a $\mathfrak{g}$-module and the following compatibility conditions should be satisfied:
for $Y \in \mathfrak{t} \subset \mathfrak{g}$ and $v \in V$ it holds
\[ Y.v = \left. \frac{d}{dt} \right|_{t=0} \exp(tY).v, \]

- for $k \in K$, $X \in \mathfrak{g}$ and $v \in V$ we have
\[ k.X.v = \text{Ad}(k)X.k.v. \]

A $(\mathfrak{g}, K)$-module $V$ is called irreducible or simple if it has no proper submodules and it is called of finite length if there is a finite filtration
\[ 0 = V_0 \subset \cdots \subset V_n = V \]
of submodules such that each quotient $V_j/V_{j-1}$ is irreducible. Further $V$ is called admissible if for each $\tau \in \hat{K}$ the space $\text{Hom}_K(\tau, V)$ is finite dimensional.

An admissible $(\mathfrak{g}, K)$-module of finite length is called a Harish-Chandra module.

Now let $(\pi, V_\pi)$ again be a Banach representation of $G$. For each $\tau \in \hat{K}$ let $V_\pi(\tau)$ denote the isotypical component of $\tau$, i.e., $V_\pi(\tau)$ is the image of the map
\[ \text{Hom}_K(V_\tau, V_\pi) \otimes V_\tau \rightarrow V_\pi, \]
\[ (\alpha, v) \mapsto \alpha(v). \]

Let $V_{\pi, K}$ be the subspace of $V_\pi$ consisting of all vectors $v \in V_\pi$ such that the $K$-orbit $\pi(K)v$ spans a finite dimensional space. Then $V_{\pi, K}$ is called the space of $K$-finite vectors in $V_\pi$. The space $V_{\pi, K}$ is no longer a $G$-module but remains a $K$-module. Further the space $V_{\pi, K}^\infty = V_{\pi, K} \cap V_\pi^\infty$ is dense in $V_{\pi, K}$ and is stable under $K$, so $V_{\pi, K}^\infty$ is a $(\mathfrak{g}, K)$-module. By abuse of notation we will often write $\pi$ instead of $V_\pi$ and $\pi_K^\infty$ instead of $V_{\pi, K}^\infty$. The representation $\pi$ is called admissible if $\pi_K^\infty$ is. In that case we have $\pi_K^\infty = \pi_K$ since a dense subspace of a finite dimensional space equals the entire space. So then $\pi_K$ is a $(\mathfrak{g}, K)$-module. Further, the representation $\pi$ is called a Harish-Chandra representation if $\pi_K$ is a Harish-Chandra module.

**Lemma 1.3** Let $(\pi, V_\pi)$ be an irreducible admissible representation of $G$ then the Casimir operator $C$ acts on $V_\pi^\infty$ by a scalar denoted $\pi(C)$.

**Proof:** By the formula $\pi(g)\pi(C)\pi(g)^{-1} = \pi(\text{Ad}(g)C)$ and Lemma 1.2 we infer that $\pi(C)$ commutes with $\pi(g)$ for every $g \in G$. Therefore the claim follows from the Lemma of Schur (31 3.3.2).

Any $f \in L^1(G)$ will define a continuous operator
\[ \pi(f) = \int_G f(x)\pi(x)dx \]
on $V_\pi$.

Let $N$ be a natural number and let $L^1_{2N}(G)$ be the set of all $f \in C^{2N}(G)$ which satisfy $Df \in L^1(G)$ for any $D \in U(g)$ with $\deg(D) \leq 2N$.

**Lemma 1.4** Let $N$ be an integer $> \dim G$. Let $f \in L^1_{2N}(G)$ then for any irreducible unitary representation $\pi$ of $G$ the operator $\pi(f)$ will be of trace class.

**Proof:** Let $C$ denote the Casimir operator of $G$ and let $C_K$ be the Casimir operator of $K$. Let $\Delta = -C + 2C_K \in U(g)$ the group Laplacian. It is known that for some $a > 0$ the operator $\pi(\Delta + a)$ is positive and $\pi(\Delta + a)^{-N}$ is of trace class. Let $g = (\Delta + a)^N f$ then $g \in L^1(G)$, so $\pi(g)$ is defined and gives a continuous linear operator on $V_\pi$. We infer that $\pi(f) = \pi(\Delta + a)^{-N} \pi(g)$ is of trace class. $\square$

Finally we need some more notation. The form $\langle X, Y \rangle = -B(X, \theta(Y))$ is positive definite on $g_0$ and therefore induces a positive definite left invariant top differential form $\omega_L$ on any closed subgroup $L$ of $G$. If $L$ is compact we set

$$v(L) = \int_L \omega_L.$$ 

Let $H = AB$ be a $\theta$-stable Cartan subgroup where $A$ is the connected split component of $H$ and $B$ is compact. The double use of the letter $B$ here will not cause any confusion. Then $B \subset K$. Let $\Phi$ denote the root system of $(g, h)$, where $g$ and $h$ are the complexified Lie algebras of $G$ and $H$. Let $g = h \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the root space decomposition. Let $x \rightarrow x^c$ denote the complex conjugation on $g$ with respect to the real form $g_0 = Lie(G)$. A root $\alpha$ is called *imaginary* if $\alpha^c = -\alpha$ and it is called *real* if $\alpha^c = \alpha$. Every root space $g_\alpha$, is one dimensional and has a generator $X_\alpha$ satisfying:

$$[X_\alpha, X_{-\alpha}] = Y_\alpha \quad \text{with} \quad \alpha(.) = B(Y_\alpha, .)$$

$$B(X_\alpha, X_{-\alpha}) = 1$$

and $X_\alpha^c = X_{-\alpha}$ if $\alpha$ is non-imaginary and $X_\alpha^c = \pm X_{-\alpha}$ if $\alpha$ is imaginary. An imaginary root $\alpha$ is called *compact* if $X_\alpha^c = -X_{-\alpha}$ and *noncompact* otherwise. Let $\Phi_\nu$ be the set of noncompact imaginary roots and choose a set $\Phi^+$ of positive roots such that for $\alpha \in \Phi^+$ nonimaginary we have that $\alpha^c \in \Phi^+$. Let $W = W(G, H)$ be the Weyl group of $(G, H)$, that is

$$W = \frac{\text{normalizer}(H)}{\text{centralizer}(H)}.$$ 

Let $\rho_\mathbb{R}(G)$ be the dimension of a maximal $\mathbb{R}$-split torus in $G$ and let $\nu = \dim G/K - \rho_\mathbb{R}(G)$. We define the *Harish-Chandra constant* of $G$ by

$$c_G = (-1)^{\left|\Phi_\nu^+\right|} (2\pi)^{\left|\Phi^+\right|/2} \frac{v(T)}{v(K)} |W|.$$
1.2 Normalization of Haar measures

Although the results will not depend on normalizations we will need to normalize Haar measures for the computations along the way. First for any compact subgroup \( C \subset G \) we normalize its Haar measure so that it has total mass one, i.e., \( \text{vol}(C) = 1 \). Next let \( H \subset G \) be a reductive subgroup, and let \( \theta_H \) be a Cartan involution on \( H \) with fixed point set \( K_H \). The same way as for \( G \) itself the form \( B \) restricted to the Lie algebra of \( H \) induces a Riemannian metric on the manifold \( X_H = H/K_H \). Let \( dx \) denote the volume element of that metric. We get a Haar measure on \( H \) by defining

\[
\int_H f(h) dh = \int_{X_H} \int_{K_H} f(xk) dk dx
\]

for any continuous function of compact support \( f \) on \( H \).

Let \( P \subset G \) be a parabolic subgroup of \( G \) (see [31] 2.2). Let \( P = MAN \) be the Langlands decomposition of \( P \). Then \( M \) and \( A \) are reductive, so there Haar measures can be normalized as above. Since \( G = PK = MANK \) there is a unique Haar measure \( dn \) on the unipotent radical \( N \) such that for any constant function \( f \) of compact support on \( G \) it holds:

\[
\int_G f(x) dx = \int_M \int_A \int_N \int_K f(mank) dk dndadm.
\]

Note that these normalizations coincide for Levi subgroups with the ones met by Harish-Chandra in (15 sect. 7).

1.3 Invariant distributions

In this section we shall throughout assume that \( G \) is a real reductive group of inner type. A distribution \( T \) on \( G \), i.e., a continuous linear functional \( T : C_\infty^c(G) \rightarrow \mathbb{C} \) is called invariant if for any \( f \in C_\infty^c(G) \) and any \( y \in G \) it holds:

\( T(f^y) = T(f) \), where \( f^y(x) = f(yxy^{-1}) \). Examples are:

- orbital integrals: \( f \mapsto \mathcal{O}_y(f) = \int_{G_y \setminus G} f(x^{-1}yx) dx \) and
- traces: \( f \mapsto \text{tr} \pi(f) \) for \( \pi \in \hat{G} \).

These two examples can each be expressed in terms of the other. Firstly, Harish-Chandra proved that for any \( \pi \in \hat{G} \) there exists a conjugation invariant locally integrable function \( \Theta_\pi \) on \( G \) such that for any \( f \in C_\infty^c(G) \)

\[
\text{tr} \pi(f) = \int_G f(x) \Theta_\pi(x) dx.
\]
Recall the Weyl integration formula which says that for any integrable function \( \varphi \) on \( G \) we have

\[
\int_G \varphi(x)dx = \sum_{j=1}^r \frac{1}{|W(G, H_j)|} \int_{H_j} \int_{G/H_j} \varphi(xh^{-1}) |\det(1 - h|g/h_j)| dxdh,
\]

where \( H_1, \ldots, H_r \) is a maximal set of nonconjugate Cartan subgroups in \( G \) and for each Cartan subgroup \( H \) we let \( W(G, H) \) denote its Weyl group, i.e., the quotient of the normalizer of \( H \) in \( G \) by its centralizer.

An element \( x \) of \( G \) is called regular if its centralizer is a Cartan subgroup. The set of regular elements \( G_{\text{reg}} \) is open and dense in \( G \) with complement of measure zero. Therefore the integral above can be taken over \( G_{\text{reg}} \) only. Letting \( H_{\text{reg}} := H_j \cap G_{\text{reg}} \) we get

**Proposition 1.5** Let \( N \) be a natural number bigger than \( \frac{\dim G}{2} \), then for any \( f \in L^1_{2N}(G) \) and any \( \pi \in \hat{G} \) we have

\[
\text{tr} \pi(f) = \sum_{j=1}^r \frac{1}{|W(G, H_j)|} \int_{H_j^{\text{reg}}} O_h(f) \Theta_{\pi}(h) |\det(1 - h|g/h_j)| dh.
\]

That is, we have expressed the trace distribution in terms of orbital integrals. In the other direction it is also possible to express semisimple orbital integrals in terms of traces.

At first let \( H \) be a \( \theta \)-stable Cartan subgroup of \( G \). Let \( h \) be its complex Lie algebra and let \( \Phi = \Phi(g, h) \) be the set of roots. Let \( x \to x^c \) denote the complex conjugation on \( g \) with respect to the real form \( g_0 = \text{Lie}_\mathbb{R}(G) \). Choose an ordering \( \Phi^+ \subset \Phi \) and let \( \Phi^+_I \) be the set of positive imaginary roots. To any root \( \alpha \in \Phi \) let

\[
H \to \mathbb{C}^\times, \quad h \mapsto h^\alpha
\]

be its character, that is, for \( X \in g_0 \) the root space to \( \alpha \) and any \( h \in H \) we have \( \text{Ad}(h)X = h^\alpha X \). Now put

\[
\Delta_I(h) = \prod_{\alpha \in \Phi^+_I} (1 - h^{-\alpha}).
\]

Let \( H = AT \) where \( A \) is the connected split component and \( T \) is compact. An element \( at \in AT = H \) is called split regular if the centralizer of \( a \) in \( G \) equals the centralizer of \( A \) in \( G \). The split regular elements form a dense open subset containing the regular elements of \( H \). Choose a parabolic \( P \) with split component \( A \), so \( P \) has Langlands decomposition \( P = MAN \). For \( at \in AT = H \)
Let
\[
\Delta_+(at) = \left| \det((1 - \text{Ad}((at)^{-1}))|_{\mathfrak{g}\oplus\mathfrak{m}}) \right|^{\frac{1}{2}}
\]
\[
= \left| \det((1 - \text{Ad}((at)^{-1}))|_{\mathfrak{n}}) \right| a^{\rho_P}
\]
\[
= \prod_{\alpha \in \Phi^{+} - \Phi_{0}^{+}} (1 - (at)^{-\alpha}) a^{\rho_P},
\]
where \(\rho_P\) is the half of the sum of the roots in \(\Phi(P,A)\), i.e., \(a^{2\rho_P} = \det(\mathfrak{a}|\mathfrak{n})\). We will also write \(h^{\rho_P}\) instead of \(a^{\rho_P}\).

For any \(h \in H^{\text{reg}} = H \cap G^{\text{reg}}\) let
\[
'F^H_f(h) = F_f(h) = \Delta_f(h) \int_{G/A} f(xhx^{-1})dx.
\]
It then follows directly from the definitions that for \(h \in H^{\text{reg}}\) it holds
\[
\mathcal{O}_h(f) = \frac{F_f(h)}{h^{\rho_P} \det(1 - h^{-1}|(\mathfrak{g}/\mathfrak{h})^+)}.
\]
where \((\mathfrak{g}/\mathfrak{h})^+\) is the sum of the root spaces attached to positive roots. There is an extension of this identity to nonregular elements as follows: For \(h \in H\) let \(G_h\) denote its centralizer in \(G\). Let \(\Phi^{+}(\mathfrak{g}_h,\mathfrak{h})\) be the set of positive roots of \((\mathfrak{g}_h,\mathfrak{h})\). Let
\[
\varpi_h = \prod_{\alpha \in \Phi^{+}(\mathfrak{g}_h,\mathfrak{h})} Y_\alpha,
\]
then \(\varpi_h\) defines a left invariant differential operator on \(G\).

**Lemma 1.6** For any \(f \in L^1_{2\mathbb{N}}(G)\) and any \(h \in H\) we have
\[
\mathcal{O}_h(f) = \frac{\varpi'_h F_f(h)}{c_h h^{\rho_P} \det(1 - h^{-1}|(\mathfrak{g}/\mathfrak{g}_h)^+)}.
\]

**Proof:** This is proven in section 17 of [10]. \(\square\)

Our aim is to express orbital integrals in terms of traces of representations. By the above lemma it is enough to express \(F_f(h)\) it terms of traces of \(f\) when \(h \in H^{\text{reg}}\). For this let \(H_1 = A_1 T_1\) be another \(\theta\)-stable Cartan subgroup of \(G\) and let \(P_1 = M_1 A_1 N_1\) be a parabolic with split component \(A_1\). Let \(K_1 = K \cap M_1\). Since \(G\) is connected the compact group \(T_1\) is an abelian torus and its unitary dual \(\widehat{T_1}\) is a lattice. The Weyl group \(W = W(M_1, T_1)\) acts on \(\widehat{T_1}\) and \(\hat{t}_1 \in \widehat{T_1}\) is called *regular* if its stabilizer \(W(\hat{t}_1)\) in \(W\) is trivial. The regular set
\( \tilde{T}^{\text{reg}}_1 \) modulo the action of \( \tilde{W}(K_1,T_1) \subset \tilde{W}(M_1,T_1) \) parameterizes the discrete series representations of \( M_1 \) (see [23]). For \( \tilde{t}_1 \in \tilde{T}_1 \), Harish-Chandra [17] defined a distribution \( \Theta_{\tilde{t}_1} \) on \( G \) which happens to be the trace of the discrete series representation \( \pi_{\tilde{t}_1} \) attached to \( \tilde{t}_1 \) when \( \tilde{t}_1 \) is regular. When \( \tilde{t}_1 \) is not regular the distribution \( \Theta_{\tilde{t}_1} \) can be expressed as a linear combination of traces as follows.

Choose an ordering of the roots of \( (M_1,T_1) \) and let \( \Omega \) be the product of all positive roots. For any \( \nu, \pi \) parabolically from \( \tilde{t}_1 \) attached to \( \tilde{t}_1 \) we get \( \Theta_{\tilde{t}_1} = \frac{1}{|\tilde{W}(\tilde{t}_1)|} \sum_{w \in \tilde{W}(\tilde{t}_1)} \epsilon(w) \Theta_{\pi,\tilde{t}_1} \), where \( \Theta'_{\pi,\tilde{t}_1} \) is the character of an irreducible representation \( \pi_{\tilde{t}_1} \) called a limit of discrete series representation. We will write \( \pi_{\tilde{t}_1} \) for the virtual representation

\[
\frac{1}{|\tilde{W}(\tilde{t}_1)|} \sum_{w \in \tilde{W}(\tilde{t}_1)} \epsilon(w) \pi_{\tilde{t}_1}
\]

Let \( \nu : a \mapsto a^\nu \) be a unitary character of \( A_1 \) then \( \tilde{h}_1 = (\nu, \tilde{t}_1) \) is a character of \( H_1 = A_1 T_1 \). Let \( \Theta_{\tilde{h}_1} \) be the character of the representation \( \pi_{\tilde{h}_1} \) induced parabolically from \( (\nu, \pi_{\tilde{t}_1}) \). Harish-Chandra has proven

**Theorem 1.7** Let \( H_1, \ldots, H_r \) be maximal a set of nonconjugate \( \theta \)-stable Cartan subgroups with split components \( A_1, \ldots, A_r \). Let \( H = H_j \) for some \( j \) with split component \( A \). Then for each \( j \) there exists a continuous function \( \Phi_{H|H_j} \) on \( H^{\text{reg}} \times \hat{H}_j \) such that for \( h \in H^{\text{reg}} \) it holds

\[
\sum_{j=1}^r \int_{\hat{H}_j} \Phi_{H|H_j}(h, \hat{h}_j) \text{ tr } \pi_{\hat{h}_j}(f) \, d\hat{h}_j.
\]

Further \( \Phi_{H|H_j} = 0 \) unless there is \( g \in G \) such that \( gA_1 g^{-1} \subset A_j \). Finally for \( H_j = H \) the function can be given explicitly as

\[
\Phi_{H|H}(h, \hat{h}) = \frac{1}{|\tilde{W}(G,H)|} \sum_{w \in \tilde{W}(G,H)} \epsilon(w) T(h) \langle \hat{w}h, h \rangle
\]

where \( '\triangle = \triangle_+ ' \triangle_- \).

**Proof:** [17].

### 1.4 Smoothness of induced functions

Let \( M \) be a smooth (i.e. \( C^\infty \)) manifold and let \( D \subset M \) be an open subset. Let \( S = M \backslash D \) be its complement. A real or complex valued function \( f \) on \( D \) is said to vanish to order at least \( k \in \mathbb{Z} \) at a point \( s \in S \), if there exists an open neighbourhood \( U \) of \( s \) in \( M \) such that the function

\[
u \mapsto \frac{|f(u)|}{d(u,s)^k}
\]
is bounded above on \( U \cap D \). Here \( d(x, y) \) is the distance function attached to a Riemannian metric on \( M \). By its local nature, this notion does not depend on the choice of the metric. Likewise, we say that \( f \) vanishes to order at most \( k \) at \( s \) if the function \( u \mapsto \frac{|f(u)|}{d(u, s)^k} \) is bounded away from zero on \( U \cap D \). If both conditions hold with the same \( k \in \mathbb{Z} \), we say that \( f \) vanishes to order \( k \) or that \( f \) has order \( k \) at \( s \). If \( k \) is negative, we then also say that \( f \) has a pole of order \( -k \) at \( s \). Finally, we say that the function \( f \) vanishes to order at most/at least \( k \) on \( S \) if it does so for every \( s \in S \).

Let \( M' \) be another smooth manifold and let \( F: M \to M' \) be a smooth map. Let \( E \to M \) and \( E' \to M' \) be smooth vector bundles and \( \tilde{F}: E \to E' \) be a smooth linear lift of \( F \), i.e., \( \tilde{F} \) is smooth and maps the vector space \( E_m \) linearly to the vector space \( E'_{\tilde{F}(m)} \) for every \( m \in M \). Then we say that \( \tilde{F} \) vanishes to order at least/at most \( k \) at \( S \) if for every two sections \( \sigma \) of \( E \) and \( \alpha \) of \( (E')^* \) the function

\[
m \mapsto \alpha(\tilde{F}(\sigma(m)))
\]

does so. We speak of any such function as an entry of \( \tilde{F} \). An example of a smooth linear lift is the differential,

\[
TF: TM \to TM'.
\]

An open subset \( C \) of \( D \) is called full, if the boundary of \( C \) is contained in \( S \). This is equivalent to saying that \( C \) is a union of connected components of \( D \).

For \( x \in \mathbb{R} \) let \([x]\) be the largest integer with \([x] \leq x\).

**Proposition 1.8** Let \( C \) be a full subset of \( D \). Let \( F: M \to M' \) be smooth and assume that \( F \) restricted to \( C \) is a diffeomorphism with open image and that the boundary of \( F(C) \) is contained in \( F(S) \). Assume further that \( \det(TF) \) vanishes to order at most \( k \in \mathbb{N} \) at \( S \). Let \( f \) be a real or complex valued function on \( C \) that vanishes to order at least \( j \in \mathbb{N} \) at \( S \) and is \( j \)-times continuously differentiable inside \( C \). Extend \( f \) to a continuous function on \( M \) by setting \( f \equiv 0 \) outside \( C \). Assume that \( f \) factors over \( F \), so \( f \) induces a function \( f' \) on \( M' \) which vanishes outside \( F(C) \).

Then the function \( f' \) is at least \( r \)-times continuously differentiable on \( M' \), where

\[
r = \left\lfloor \frac{j - 1}{r + 1} \right\rfloor.
\]

**Proof:** Let \( m \in M \). Taking local co-ordinates on \( M \) and \( M' \) the tangential \( TF \) can be viewed as a Jacobi matrix \( JF \). The chain rule implies that on the open set \( F(C) \) one has

\[
T(F^{-1}) = (TF)^{-1} = \frac{1}{\det(JF)}(JF)^{\#},
\]

where \( (JF)^{\#} \) denotes the pseudoinverse of \( JF \).
where for a matrix $A$ we write $A^#$ for its complementary matrix. Note that the entries of $A^#$ are polynomials in the entries of $A$. Since $\det(TF)$ vanishes to order at most $k$ at $S$, it follows that $T(F^{-1})$, which is defined on $F(C)$, has a pole of order at most $k$ at the boundary of $F(C)$. On the open set $F(C)$ we can write $f' = f \circ F^{-1}$, so $Tf' = Tf \circ TF^{-1}$. Since $f$ vanishes at $S$ to order at least $j$ it follows that $Tf$ vanishes at $S$ to order at least $j - 1$, so $Tf'$ vanishes at $S'$ to order at least $j - 1 - k$.

We now finish the proof of Proposition 1.8 by induction of $j$. First assume $j \leq r + 1$. Then, since $j \geq 1$, the function $f'$ extends to a continuous function on $M'$ and so $f' \in C^0(M')$ as claimed.

For the induction step assume $j > r + 1$. Then $Tf'$ vanishes on $S'$ to order at least $j - 1 - k \in \mathbb{N}$. This means that every entry of $Tf'$ vanishes at least to this order. We pick an entry $g'$ and consider the function $g = g' \circ F$ on $M$. Then $g$ vanishes at least to order $j - 1 - k$ at $S$ and by induction hypothesis the function $g'$ is continuously differentiable up to order $\left\lfloor \frac{j - 1 - k - 1}{k + 1} \right\rfloor = \left\lfloor \frac{j - 1}{k + 1} \right\rfloor - 1$. Since $g'$ is an arbitrary entry of $Tf'$, it follows that $f'$ is of class $C^r$ with $r = \left\lfloor \frac{j - 1}{k + 1} \right\rfloor$. \hfill \Box

## 2 Euler-Poincaré functions

In this section we generalize the construction of pseudo-coefficients and Euler-Poincaré functions [7, 24] to non-connected groups. Here $G$ will be a real reductive group. It will be assumed that $G$ has a compact Cartan subgroup. It then follows that $G$ has compact center.

### 2.1 Existence

Fix a maximal compact subgroup $K$ of $G$ and a Cartan $T$ of $G$ which lies inside $K$. The group $G$ is called orientation preserving if $G$ acts by orientation preserving diffeomorphisms on the manifold $X = G/K$. For example, the group $G = SL_2(\mathbb{R})$ is orientation preserving but the group $PGL_2(\mathbb{R})$ is not. Recall the Cartan decomposition $g_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$. Note that $G$ is orientation preserving if and only if its maximal compact subgroup $K$ preserves orientations on $\mathfrak{p}_0$.

**Lemma 2.1** The following holds:

- Any connected group is orientation preserving.
- If $X$ carries the structure of a complex manifold which is left stable by $G$, then $G$ is orientation preserving.
**Proof:** The first is clear. The second follows from the fact that biholomorphic maps are orientation preserving.

Let $t$ be the complexified Lie algebra of the Cartan subgroup $T$. We choose an ordering of the roots $\Phi(g, t)$ of the pair $(g, t)$. This choice induces a decomposition $p = p_+ \oplus p_-$, where $p_\pm$ is the sum of the positive/negative root spaces which lie in $p$. As usual denote by $\rho$ the half sum of the positive roots. The chosen ordering induces an ordering of the compact roots $\Phi_k(t, t)$ which form a subset of the set of all roots $\Phi(g, t)$. Let $\rho_K$ denote the half sum of the positive compact roots. Recall that a function $f$ on $G$ is called $K$-central if $f(kek^{-1}) = f(x)$ for all $x \in G, k \in K$. For any $K$-representation $(\rho, V)$ let $V^K$ denote the space of $K$-fixed vectors, i.e.,

$$V^K = \{ v \in V | \rho(k)v = v \; \forall k \in K \}.$$  

Let $(\tau, V_\tau)$ be a representation of $K$ on a finite dimensional complex vector space $V_\tau$. Let $V_\tau^*$ be the dual space then there is a representation $\tilde{\tau}$ on $V_\tau^* := V_\tau^*$ given by

$$\tilde{\tau}(k)\alpha(v) = \alpha(\tau(k^{-1})v),$$

for $k \in K$, $\alpha \in V_\tau^*$ and $v \in V_\tau$. This representation is called the contragredient or dual representation. The restriction from $G$ to $K$ gives a ring homomorphism:

$$res^G_K : \text{Rep}(G) \to \text{Rep}(K).$$

**Theorem 2.2** (Euler-Poincaré functions) Let $(\tau, V_\tau)$ a finite dimensional representation of $K$. If $G$ is orientation preserving or $\tau$ lies in the image of $res^G_K$, then there is a compactly supported smooth $K$-central $K$-finite function $f_\tau$ on $G$ such that for every admissible representation $(\pi, V_\pi)$ of $G$ we have

$$\text{tr } \pi(f_\tau) = \sum_{p=0}^{\dim(p)} (-1)^p \dim(V_\pi \otimes \wedge^p p \otimes V_\tau)^K.$$  

We call $f_\tau$ an Euler-Poincaré function for $\tau$. Note that, since $f$ is $K$-finite and $\pi$ is admissible, the operator $\pi(f)$ has finite rank, so the trace exists.

If $G$ is orientation preserving and $K$ leaves invariant the decomposition $p = p_+ \oplus p_-$ then there is a compactly supported smooth $K$-central function $g_\tau$ on $G$ such that for every admissible representation $(\pi, V_\pi)$ we have

$$\text{tr } \pi(g_\tau) = \sum_{p=0}^{\dim(p_+)} (-1)^p \dim(V_\pi \otimes \wedge^p p_- \otimes V_\tau)^K.$$  

If the representation $\tau$ lies in the image of $res^G_K$ and the group $G$ is connected then the theorem is well known, [7], [24].
Proof: We start with the case when $G$ is orientation preserving. Without loss of generality assume $\tau$ irreducible. Suppose given a function $f$ which satisfies the claims of the theorem except that it is not necessarily $K$-central, then the function
\[ x \mapsto \int_K f(kxk^{-1})dk \]
will satisfy all claims of the theorem. Thus one only needs to construct a function having the claimed traces.

If $G$ is orientation preserving the adjoint action gives a homomorphism $K \to SO(p)$. If this homomorphism happens to lift to the double cover $Spin(p)$, we let $\tilde{G} = G$ and $\tilde{K} = K$. In the other case we apply the

Lemma 2.3 If the homomorphism $K \to SO(p)$ does not factor over the spin group $Spin(p)$ then there is a double covering $\tilde{G} \to G$ such that with $\tilde{K}$ denoting the inverse image of $K$ the induced homomorphism $\tilde{K} \to SO(p)$ factors over $Spin(p) \to SO(p)$. Moreover the kernel of the map $\tilde{G} \to G$ lies in the center of $\tilde{G}$.

Proof: At first $\tilde{K}$ is given by the pullback diagram:
\[
\begin{array}{ccc}
\tilde{K} & \to & Spin(p) \\
\downarrow & & \downarrow \alpha \\
K & \to & SO(p)
\end{array}
\]

that is, $\tilde{K}$ is given as the set of all $(k, g) \in K \times Spin(p)$ such that $\text{Ad}(k) = \alpha(g)$. Then $\tilde{K}$ is a double cover of $K$.

Next we use the fact that $K$ is a retract of $G$ to show that the covering $\tilde{K} \to K$ lifts to $G$. Explicitly let $P = \text{exp}(p_0)$ then the map $K \times P \to G, (k, p) \mapsto kp$ is a diffeomorphism. Let $g \mapsto (k(g), p(g))$ be its inverse map. We let $\tilde{G} = \tilde{K} \times P$ then the covering $\tilde{K} \to K$ defines a double covering $\beta : \tilde{G} \to G$. We have to install a group structure on $\tilde{G}$ which makes $\beta$ a homomorphism and reduces to the known one on $\tilde{K}$. Now let $k, k' \in K$ and $p, p' \in P$ then by
\[ k'p'kp = k'k^{-1}p'kp \]
it follows that there are unique maps $a_K : P \times P \to K$ and $a_P : P \times P \to P$ such that
\[ k(k'p'kp) = k'ka_K(k^{-1}p'k, p) \]
\[ p(k'p'kp) = a_P(k^{-1}p'k, p). \]
Since $P$ is simply connected the map $a_K$ lifts to a map $\tilde{a}_K : P \times P \to \tilde{K}$. Since $P$ is connected there is exactly one such lifting with $\tilde{a}_K(1, 1) = 1$. For $k \in \tilde{K}$
let $\bar{k}$ be its image in $K$. Now the map 

$$(\tilde{K} \times P) \times (\tilde{K} \times P) \to \tilde{K} \times P$$

$$(k', p'), (k, p) \mapsto (kk'\tilde{a}_K(\bar{k}^{-1}p'\bar{k}, p), \alpha_p(\bar{k}^{-1}p'\bar{k}, p))$$

defines a multiplication on $\tilde{G} = \tilde{K} \times P$ with the desired properties.

Finally $\ker(\beta)$ will automatically be central because it is a normal subgroup of order two. □

Let $S$ be the spin representation of $\text{Spin}(p)$ (see [25], p.36). It splits as a direct sum of two distinct irreducible representations

$$S = S^+ \oplus S^-.$$ 

We will make use of the following properties of the spin representation.

- The virtual representation 

$$(S^+ - S^-) \otimes (S^+ - S^-)$$

is isomorphic to the adjoint representation on $\wedge^{\text{even}} p - \wedge^{\text{odd}} p$ (see [24], p. 36).

- If $K$ leaves invariant the spaces $p_-$ and $p_+$, as is the case when $X$ carries a holomorphic structure fixed by $G$, then there is a one dimensional representation $\epsilon$ of $\tilde{K}$ such that 

$$(S^+ - S^-) \otimes \epsilon \cong \wedge^{\text{even}} p_- - \wedge^{\text{odd}} p_-.$$ 

The proof of this latter property will be given in section 2.2.

**Theorem 2.4** (Pseudo-coefficients) Assume that the group $G$ is orientation preserving. Then for any finite dimensional representation $(\tau, V_{\tau})$ of $\tilde{K}$ there is a compactly supported smooth function $h_{\tau}$ on $\tilde{G}$ such that for every admissible representation $(\pi, V_\pi)$ of $\tilde{G}$,

$$\text{tr} \pi(h_{\tau}) = \dim(V_{\pi} \otimes S^+ \otimes V_{\tau})^{\tilde{K}} - \dim(V_{\pi} \otimes S^- \otimes V_{\tau})^{\tilde{K}}.$$ 

The functions given in this theorem are also known as pseudo-coefficients [24]. This result generalizes the one in [24] in several ways. First, the group $G$ needn’t be connected and secondly the representation $\tau$ needn’t be spinorial. The proof of this theorem relies on the following lemma.

**Lemma 2.5** Let $(\pi, V_\pi)$ be an irreducible admissible representation of $\tilde{G}$ and assume 

$$\dim(V_{\pi} \otimes S^+ \otimes V_{\tau})^{\tilde{K}} - \dim(V_{\pi} \otimes S^- \otimes V_{\tau})^{\tilde{K}} \neq 0,$$ 

then the Casimir eigenvalue satisfies $\pi(C) = \tilde{\tau}(C_{\tilde{K}}) - B(\rho) + B(\rho_{\tilde{K}}).$
Proof: Let the $\hat{K}$-invariant operator

$$d_\pm : V_\pi \otimes S^\pm \to V_\pi \otimes S^\mp$$

be defined by

$$d_\pm : v \otimes s \mapsto \sum_i \pi(X_i)v \otimes c(X_i)s,$$

where $(X_i)$ is an orthonormal base of $p$. The formula of Parthasarathy, [1], p. 55 now says

$$d_-d_+ = d_+d_- = \pi \otimes s^\pm(C_K) - \pi(C) \otimes 1 - B(\rho) + B(\rho_K),$$

where $s^\pm$ is the representation on $S^\pm$. Our assumption leads to $\ker(d_-d_-) \cap \pi \otimes S(\tau) \neq 0$, where $\pi \otimes S(\tau)$ is the $\tau$ $K$-type of $\pi \otimes S$. But on this space the $K$-Casimir $C_K$ acts by the scalar $\tau(C_K)$, so that we get $0 = \tau(C_K) - \pi(C) - B(\rho) + B(\rho_K)$. \hfill \Box

For the proof of Theorem 2.4 let $(\tau, V_\tau)$ a finite dimensional irreducible unitary representation of $\hat{K}$ and write $E_\tau$ for the $\hat{G}$-homogeneous vector bundle over $X = \hat{G}/\hat{K}$ defined by $\tau$. The space of smooth sections $\Gamma^\infty(E_\tau)$ may be written as $\Gamma^\infty(E_\tau) = (C^\infty(\hat{G}) \otimes V_\tau)^{\hat{K}}$, where $\hat{K}$ acts on $C^\infty(\hat{G})$ by right translations. The Casimir operator $C$ of $\hat{G}$ acts on this space and defines a second order differential operator $C_\tau$ on $E_\tau$. On the space of $L^2$-sections $L^2(X, E_\tau) = (L^2(\hat{G}) \otimes V_\tau)^{\hat{K}}$ this operator is formally selfadjoint with domain, say, the compactly supported smooth functions and extends to a selfadjoint operator. Let $g$ be a Paley-Wiener function on $\mathbb{R}$, i.e., $g$ is the Fourier-transform of a smooth function of compact support. Then $g$ extends to a holomorphic function on $\mathbb{C}$. Assume that $g$ is even, i.e., $g(z) = g(-z)$. Then the power series of $g(z)$ around zero contains only even powers of $z$. So there is an entire function $f$ such that $g(z) = f(z^2)$. Then $f|_{\mathbb{R}}$ is a Schwartz-Bruhat function. In [S] Lemma 2.9 and Lemma 2.11 it is shown that there exists a smooth function of compact support $\hat{f}_\pi$ on $\hat{G}$ such that for every irreducible unitary representation $\pi$ of $\hat{G}$:

$$\pi(\hat{f}_\pi) = f(\pi(C))\text{Pr}_\pi,$$

where $\text{Pr}_\pi$ is the orthogonal projection to the $K$-type $\hat{\tau}$, and $\pi(C)$ denotes the Casimir eigenvalue on $\pi$. This formula holds as well for $\pi$ being an irreducible admissible representation, as is seen as follows. First let $\pi = \pi_{\xi, \lambda}$ be a representation induced from a minimal parabolic $P = MAN$, where $\xi \in \hat{M}$ and $\lambda \in \mathfrak{a}^*$. If $\lambda$ is imaginary and generic, then $\pi$ is irreducible unitary, so one has

$$\pi_{\xi, \lambda}(\hat{f}_\pi) = f(\pi_{\xi, \lambda}(C))\text{Pr}_\pi.$$

Both sides of this equality are holomorphic functions in $\lambda$ with values in the finite dimensional space $\text{End}(\pi(\hat{\tau}))$, hence the result holds for any $\lambda$ by the identity theorem for holomorphic functions. Finally, any irreducible admissible
\(\pi\) is a sub-representation of an induced representation, hence the formula \(1\) holds for every irreducible admissible representation \(\pi\).

For \(\tau = \tau_1 \oplus \tau_2\) let \(\tilde{f}_\tau = \tilde{f}_{\tau_1} + \tilde{f}_{\tau_2}\), so \(\tilde{f}_\tau\) is defined for all finite dimensional representations of \(K\). Next for a virtual representation \(\tau = \tau_1 - \tau_2\) we let \(\tilde{f}_\tau = \tilde{f}_{\tau_1} - \tilde{f}_{\tau_2}\).

Choose \(f\) as above such that \(f(\tilde{\tau}(C_K) - B(\rho) + B(\rho_K)) = 1\). Such an \(f\) clearly exists. Let \(\tau \in \hat{K}\) and let \(\gamma\) be the virtual representation of \(\tilde{K}\) on the space

\[V_\gamma = (S^+ - S^-) \otimes V_{\tilde{\tau}},\]

then set \(h_\tau = \tilde{f}_\gamma\). Then for an irreducible admissible representation \(\pi\) of \(G\) one has

\[\text{tr} \pi(h_\tau) = f(\pi(C)) \left(\dim(V_\pi \otimes S^+ \otimes V_{\tilde{\tau}})^K - \dim(V_\pi \otimes S^- \otimes V_{\tilde{\tau}})^{\tilde{K}}\right).
\]

By Lemma 2.5 and the choice of \(f\) this gives

\[\text{tr} \pi(h_\tau) = \left(\dim(V_\pi \otimes S^+ \otimes V_{\tilde{\tau}})^K - \dim(V_\pi \otimes S^- \otimes V_{\tilde{\tau}})^{\tilde{K}}\right).
\]

So the function \(h_\tau\) has the property claimed in Theorem 2.4 for irreducible \(\pi\). It immediately follows for direct sums of irreducibles. Since the assertion only involves the trace of \(\pi(h_\tau)\), it follows for arbitrary \(\pi\) since it is valid for the semisimplification of \(\pi\).

To get the first part of Theorem 2.2 from Theorem 2.4 one replaces \(\tau\) in the proposition by the virtual representation on \((S^+ - S^-) \otimes V_{\tilde{\tau}}\) and \((S^+ - S^-) \otimes (S^+ - S^-)\) is as \(\tilde{K}\) module isomorphic to \(\wedge^p p\) we get the desired function, say \(j\) on the group \(\tilde{G}\). Now if \(\tilde{G} \neq G\) let \(z\) be the nontrivial element in the kernel of the isogeny \(G \to \tilde{G}\), then the function

\[f(x) = \frac{1}{2}(j(x) + j(zx))\]

factors over \(G\) and satisfies the claim.

To get the second part of the theorem one proceeds similarly replacing \(\tau\) by \(\epsilon \otimes \tau\).

It remains to consider the case when \(G\) is not orientation preserving, but \(\tau\) lies in the image of the restriction map. For this it suffices to show the claim in the case when \(\tau\) is replaced by a finite dimensional irreducible representation \((\sigma, V_{\sigma})\) of \(G\). Then one proceeds as in the proof of Theorem 2.4 except that the role of Lemma 2.5 is taken up by the following lemma.

**Lemma 2.6** Let \((\sigma, V_{\sigma})\) be an irreducible finite dimensional representation of \(G\). Let \((\pi, V_\pi)\) be an irreducible unitary representation of \(G\) and assume

\[
\sum_{p=0}^{\dim(p)} (-1)^p \dim(V_\pi \otimes \wedge^p p \otimes V_{\sigma})^K \neq 0,
\]
then the Casimir eigenvalues satisfy
\[ \pi(C) = \sigma(C). \]

**Proof:** Recall that the Killing form of \( G \) defines a \( K \)-isomorphism between \( p \) and its dual \( p^* \), hence in the assumption of the lemma we may replace \( p \) by \( p^* \). Let \( \pi_K \) denote the \((g, K)\)-module of \( K \)-finite vectors in \( V_\pi \) and let \( C^q(\pi_K \otimes V_\sigma) = \text{Hom}_R(\wedge^q p, \pi_K \otimes V_\sigma) = (\wedge^q p^* \otimes \pi_K \otimes V_\sigma) \) the standard complex for the relative Lie algebra cohomology \( H^q(g, \mathfrak{t}, \pi_K \otimes V_\sigma) \). Further \((\wedge^q p^* \otimes \pi_K \otimes V_\sigma)^K_{\mathfrak{m}}\) forms the standard complex for the relative \((g, K)\)-cohomology \( H^q(g, K, \pi_K \otimes V_\sigma) \). In \cite{4}, p.28 it is shown that \( H^q(g, K, \pi_K \otimes V_\sigma) = H^q(g, \mathfrak{t}, \pi_K \otimes V_\sigma)_{K/K_0} \). Our assumption implies \( \sum_q (-1)^q \dim H^q(g, K, \pi_K \otimes V_\sigma) \neq 0 \), therefore there is a \( q \) with \( 0 \neq H^q(g, K, \pi_K \otimes V_\sigma) = H^q(g, \mathfrak{t}, \pi_K \otimes V_\sigma)_{K/K_0} \), hence \( H^q(g, \mathfrak{t}, \pi_K \otimes V_\sigma) = 0 \) for all \( q \). The claim follows. \( \square \)

### 2.2 Clifford algebras and Spin groups

This section is solely given to provide a proof of the properties of the spin representation used in the last section. We will therefore not strive for the utmost generality but plainly state things in the form needed. For more details the reader is referred to \cite{25}.

Let \( V \) be a finite dimensional complex vector space and let \( q : V \to \mathbb{C} \) be a non-degenerate quadratic form. We use the same letter for the symmetric bilinear form:
\[ q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)). \]

Let \( SO(q) \subset GL(V) \) be the special orthogonal group of \( q \). The **Clifford algebra** \( Cl(q) \) will be the quotient of the tensorial algebra
\[ TV = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \ldots \]
by the two-sided ideal generated by all elements of the form \( v \otimes v + q(v) \), where \( v \in V \).

This ideal is not homogeneous with respect to the natural \( \mathbb{Z} \)-grading of \( TV \), but it is homogeneous with respect to the induced \( \mathbb{Z}/2\mathbb{Z} \)-grading given by the even and odd degrees. Hence the latter is inherited by \( Cl(q) \):
\[ Cl(q) = Cl^0(q) \oplus Cl^1(q). \]

For any \( v \in V \) we have in \( Cl(q) \) that \( v^2 = -q(v) \) and therefore \( v \) is invertible in \( Cl(q) \) if \( q(v) \neq 0 \). Let \( Cl(q)^\times \) be the group of invertible elements in \( Cl(q) \).
The algebra $Cl(q)$ has the following universal property: For any linear map $\varphi : V \to A$ to a $C$-algebra $A$ such that $\varphi(v)^2 = -q(v)$ for all $v \in V$ there is a unique algebra homomorphism $Cl(v) \to A$ extending $\varphi$.

Let $Pin(q)$ be the subgroup of the group $Cl(q)^\times$ generated by all elements $v$ of $V$ with $q(v) = \pm 1$. Let the complex spin group be defined by

$$Spin(q) = Pin(q) \cap Cl^0(q),$$

i.e., the subgroup of $Pin(q)$ of those elements which are representable by an even number of factors of the form $v$ or $v^{-1}$ with $v \in V$. Then $Spin(q)$ acts on $V$ by $x.v = xvx^{-1}$ and this gives a fourfold covering: $Spin(q) \to SO(q)$.

Assume the dimension of $V$ is even and let

$$V = V^+ \oplus V^-$$

be a polarization, that is $q(V^+) = q(V^-) = 0$. Over $\mathbb{C}$ polarizations always exist for even dimensional spaces. By the nondegeneracy of $q$ it follows that to any $v \in V^+$ there is a $\tilde{v} \in V^-$ such that $q(v, \tilde{v}) = -1$. Further, let $V^-^v$ be the space of all $w \in V^-$ such that $q(v, w) = 0$, then

$$V^- = \mathbb{C}\tilde{v} \oplus V^-^v.$$ Let

$$S = \wedge^* V^- = \mathbb{C} \oplus V^- \oplus \wedge^2 V^- \oplus \cdots \oplus \wedge^{top} V^-,$$

then we define an action of $Cl(q)$ on $S$ in the following way:

- for $v \in V^-$ and $s \in S$ let
  $$v.s = v \wedge s,$$
- for $v \in V^+$ and $s \in \wedge^* V^-^v$ let
  $$v.s = 0,$$
- and for $v \in V^+$ and $s \in S$ of the form $s = \tilde{v} \wedge s'$ with $s' \in \wedge^* V^-^v$ let
  $$v.s = s'.$$

By the universal property of $Cl(V)$ this extends to an action of $Cl(q)$. The module $S$ is called the spin module. The induced action of $Spin(q)$ leaves invariant the subspaces

$$S^+ = \wedge^{even} V^-,$$
$$S^- = \wedge^{odd} V^-,$$

the representation of $Spin(q)$ on these spaces are called the half spin representations. Let $SO(q)^+$ the subgroup of all elements in $SO(q)$ that leave stable the decomposition $V = V^+ \oplus V^-$. This is a connected reductive group isomorphic to $GL(V^+)$, since, let $g \in GL(V^+)$ and define $\hat{g} \in GL(V^-)$ to be the inverse of
the transpose of \( g \) by the pairing induced by \( q \) then the map \( GL(v) \rightarrow SO(q)^+ \)
given by \( g \mapsto (g, \hat{g}) \) is an isomorphism. In other words, choosing a basis on \( V^+ \)
and a the dual basis on \( V^- \) we get that \( q \) is given in that basis by \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
Then \( SO(q)^+ \) is the image of the embedding
\[
GL(V^-) \rightarrow SO(q)
\]
\[
A \mapsto \begin{pmatrix}
A & 0 \\
0 & tA^{-1}
\end{pmatrix}.
\]

Let \( Spin(q)^+ \) be the inverse image of \( SO(q)^+ \) in \( Spin(q) \). Then the covering \( Spin(q)^+ \rightarrow SO(q)^+ \cong GL(V^-) \) is the “square root of the determinant”, i.e.,
it is isomorphic to the covering \( GL(V^-) \rightarrow GL(V^-) \) given by the pullback diagram of linear algebraic groups:
\[
\begin{array}{ccc}
\hat{G}L(V^-) & \rightarrow & GL(1) \\
\downarrow & & \downarrow \\
GL(V^-) & \rightarrow & GL(1),
\end{array}
\]
\[
\begin{array}{c}
\downarrow \\
\det
\end{array}
\]

As a set, \( \hat{G}L(V^-) \) is given as the set of all pairs \((g, z) \in GL(V^-) \times GL(1)\) such that \( \det(g) = z^2 \) and the maps to \( GL(V^-) \) and \( GL(1) \) are the respective projections.

**Lemma 2.7** There is a one dimensional representation \( \epsilon \) of \( Spin(q)^+ \) such that
\[
S^\pm \otimes \epsilon \cong \wedge^\pm V^+
\]
as \( Spin(q)^+ \)-modules, where \( \wedge^\pm \) means the even or odd powers respectively.

**Proof:** Since \( Spin(q)^+ \) is a connected reductive group over \( \mathbb{C} \) we can apply
highest weight theory. If the weights of the representation of \( Spin(q)^+ \) on \( V \) are
given by \( \pm \mu_1, \ldots, \pm \mu_m \), then the weights of the half spin representations are given by
\[
\frac{1}{2}(\pm \mu_1 \pm \cdots \pm \mu_m)
\]
with an even number of minus signs in the one and an odd number in the other case. Let \( \epsilon = \frac{1}{2}(\mu_1 + \cdots + \mu_m) \) then \( \epsilon \) is a weight for \( Spin(q)^+ \) and \( 2\epsilon \) is the weight of, say, the one dimensional representation on \( \wedge^{top} V^+ \). By Weyl’s dimension formula this means that \( 2\epsilon \) is invariant under the Weyl group and therefore \( \epsilon \) is. Again by Weyl’s dimension formula it follows that the representation with highest weight \( \epsilon \) is one dimensional. Now it follows that \( S^+ \otimes \epsilon \) has the same weights as the representation on \( \wedge^+ V^+ \), hence must be isomorphic to the latter. The case of the minus sign is analogous. \( \square \)
2.3 Orbital integrals

It now will be shown that $\text{tr} \pi(\tau) \text{ vanishes for a principal series representation } \pi$. To this end let $P = MAN$ be a nontrivial parabolic subgroup with $A \subset \exp(p_0)$. Let $(\xi, V_\xi)$ denote an irreducible unitary representation of $M$ and $e^{\nu}$ a quasicharacter of $A$. Let $\pi_{\xi, \nu} := \text{Ind}^G_P(\xi \otimes e^{\nu} \otimes 1)$.

**Lemma 2.8** We have $\text{tr} \pi_{\xi, \nu}(f_\tau) = 0$.

**Proof:** By Frobenius reciprocity we have for any irreducible unitary representation $\gamma$ of $K$:

$$\text{Hom}_K(\gamma, \pi_{\xi, \nu}|_K) \cong \text{Hom}_{K_M}(\gamma|_{K_M}, \xi),$$

where $K_M := K \cap M$. This implies that $\text{tr} \pi_{\xi, \nu}(f_\tau)$ does not depend on $\nu$. On the other hand $\text{tr} \pi_{\xi, \nu}(f_\tau) \neq 0$ for some $\nu$ would imply $\pi_{\xi, \nu}(C) = \hat{\tau}(C_K) - B(\rho) + B(\rho_K)$ which only can hold for $\nu$ in a set of measure zero. \[\square\]

Recall that an element $g$ of $G$ is called elliptic if it lies in a compact Cartan subgroup. Since the following relies on results of Harish-Chandra which were proven under the assumption that $G$ is of inner type, we will from now on assume this.

**Theorem 2.9** Assume that $G$ is of inner type. Let $g$ be a semisimple element of the group $G$. If $g$ is not elliptic, then the orbital integral $O_g(f_\tau)$ vanishes. If $g$ is elliptic we may assume $g \in T$, where $T$ is a Cartan in $K$ and then we have

$$O_g(f_\tau) = \text{tr} \tau(g) c_g^{-1}|W(t, g)| \prod_{\alpha \in \Phi^+_G} (\rho_g, \alpha),$$

where $c_g$ is Harish-Chandra’s constant, it does only depend on the centralizer $G_g$ of $g$. Its value is given in [14].

**Proof:** The vanishing of $O_g(f_\tau)$ for nonelliptic semisimple $g$ is immediate by the lemma above and Theorem 1.7. So consider $g \in T \cap G'$, where $G'$ denotes the set of regular elements. Note that for regular $g$ the claim is $O_g(f_\tau) = \text{tr} \tau(g)$. Assume the claim proven for regular elements, then the general result follows by standard considerations as in [10], p.32 ff. where however different Haar-measure normalizations are used that produce a factor $[G_g : G^0_g]$, therefore these standard considerations are now explained. Fix $g \in T$ not necessarily
regular. Let \( y \in T^0 \) be such that \( gy \) is regular. Then

\[
\text{tr } \tau(gy) = \int_{T/G} f_\tau(x^{-1}gyx)dx
\]

\[
= \int_{T^0 \setminus G} f_\tau(x^{-1}gyx)dx
\]

\[
= \int_{G_0 \setminus G} \int_{T^0 \setminus G_y} f_\tau(x^{-1}z^{-1}gyzx)dz \, dx
\]

\[
= \int_{G_0 \setminus G} \sum_{G_y : G_0} \frac{1}{[G_y : G_0]} \int_{T^0 \setminus G_y} f_\tau(x^{-1}y^{-1}zyx)dz \, dx.
\]

The factor \( \frac{1}{[G_y : G_0]} \) comes in by the Haar-measure normalizations. On \( G^0_0 \) consider the function \( h(y) = f(x^{-1}y^{-1}ygx) \).

Now apply Harish-Chandra’s operator \( \omega_{G_y} \) to \( h \) then for the connected group \( G^0_0 \) it holds

\[
h(1) = \lim_{y \to 1} c_{G_0}^{-1} \omega_{G_0} F_{h^G}(y),
\]

where \( F_h \) is Harish-Chandra’s invariant integral \([14]\). When \( y \) tends to 1 the \( \eta \)-conjugation drops out and the claim follows.

So in order to prove the proposition one only has to consider the regular orbital integrals. Next the proof will be reduced to the case when the compact Cartan \( T \) meets all connected components of \( G \). For this let \( G^+ = TG^0 \) and assume the claim proven for \( G^+ \). Let \( x \in G \) then \( xTx^{-1} \) again is a compact Cartan subgroup. Since \( G^0 \) acts transitively on all compact Cartan subalgebras it follows that \( G^0 \) acts transitively on the set of all compact Cartan subgroups of \( G \). It follows that there is a \( y \in G^0 \) such that \( xTy^{-1} \subset TG^0 = G^+ \), which implies that \( G^+ \) is normal in \( G \).

Let \( \tau^+ = \tau|_{G^+ \cap K} \) and \( f_{\tau^+} \) the corresponding Euler-Poincaré function on \( G^+ \).

**Lemma 2.10** \( f_{\tau^+} = f_{\tau}|_{G^+} \)

Since the Euler-Poincaré function is not uniquely determined the claim reads that the right hand side is a EP-function for \( G^+ \).

**Proof:** Let \( \tau^+ = \tau|_{K^+} \), where \( K^+ = TK^0 = K \cap G^+ \). Let \( \varphi^+ \in (C_c^\infty(G^+) \otimes V_\tau)^{K^+} \), which may be viewed as a function \( \varphi^+ : G^+ \to V_\tau \) with \( \varphi^+(xk) = \tau(k^{-1})\varphi^+(x) \) for \( x \in G^+ , k \in K^+ \). Extend \( \varphi^+ \) to \( \varphi : G \to V_\tau \) by \( \varphi(xk) = \tau(k^{-1})\varphi^+(x) \) for \( x \in G^+ , k \in K \). This defines an element of \( (C_c^\infty(G) \otimes V_\tau)^K \) with \( \varphi|_{G^+} = \varphi^+ \). Since \( C_\tau \) is a differential operator it follows \( f(C_\tau)\varphi|_{G^+} = f(C_\tau)\varphi^+ \), so

\[
(\varphi \ast \tilde{f}_\tau)|_{G^+} = \varphi^+ \ast \tilde{f}_{\tau^+}.
\]
Considering the normalizations of Haar measures gives the lemma. □

For \( g \in T' \) we compute

\[
O_g(f_\tau) = \int_{T \backslash G} f_\tau(x^{-1}gx) \, dx
= \sum_{y \in G/G^+} \frac{1}{[G : G^+]} \int_{y^{-1}T_g \backslash G^+} f_\tau(x^{-1}y^{-1}gyx) \, dx,
\]

where the factor \( \frac{1}{[G : G^+]} \) stems from normalization of Haar measures and we have used the fact that \( G^+ \) is normal. The latter equals

\[
\frac{1}{[G : G^+]} \sum_{y \in G/G^+} O_{y^{-1}gy}^G(f_\tau) = \frac{1}{[G : G^+]} \sum_{y \in G/G^+} O_{y^{-1}gy}^{G^+}(f^+_\tau).
\]

Assuming the theorem proven for \( G^+ \), this is

\[
\frac{1}{[G : G^+]} \sum_{y \in G/G^+} \text{tr} \tau(y^{-1}gy) = \text{tr} \tau(g).
\]

From now on one thus may assume that the compact Cartan \( T \) meets all connected components of \( G \). Let \((\pi, V_\pi) \in \hat{G}\). Harish-Chandra has shown that for any \( \varphi \in C_c^\infty(G) \) the operator \( \pi(\varphi) \) is of trace class and there is a locally integrable conjugation invariant function \( \Theta_\pi \) on \( G \), smooth on the regular set such that

\[
\text{tr} \pi(\varphi) = \int_G \varphi(x) \Theta_\pi(x) \, dx.
\]

For any \( \psi \in C^\infty(K) \) let \( \pi|_K(\psi) = \int_K \psi(k) \pi(k) \, dk \).

**Lemma 2.11** Assume \( T \) meets all components of \( G \). For any \( \psi \in C^\infty(K) \) the operator \( \pi|_K(\psi) \) is of trace class and for \( \psi \) supported in the regular set \( K' = K \cap G' \) we have

\[
\text{tr} \pi|_K(\psi) = \int_K \psi(k) \Theta_\pi(k) \, dk.
\]

(For \( G \) connected this assertion is in [1] p.16.)

**Proof:** Let \( V_\pi = \bigoplus \pi(i) \) be the decomposition of \( V_\pi \) into \( K \)-types. This is stable under \( \pi|_K(\psi) \). Harish-Chandra has proven \([\pi|_K : \tau] \leq \dim \tau\) for any \( \tau \in \hat{K} \). Let \( \psi = \sum_j \psi_j \) be the decomposition of \( \psi \) into \( K \)-bitypes. Since \( \psi \) is smooth the sequence \( \| \psi_j \|_1 \) is rapidly decreasing for any enumeration of the \( K \)-bitypes. Here \( \| \psi \|_1 \) is the \( L^1 \)-norm on \( K \). It follows that the sum \( \sum_j \text{tr} (\pi|_K(\psi)|V_\pi(i)) \) converges absolutely, hence \( \pi|_K(\psi) \) is of trace class.
Now let $S = \exp(p_0)$ then $S$ is a smooth set of representatives of $G/K$. Let $G.K = \cup_{g \in G} gKg^{-1} = \cup_{s \in S} sKs^{-1}$, then, since $G$ has a compact Cartan, the set $G.K$ has non-empty interior. Applying the Weyl integration formula to $G$ and backwards to $K$ gives the existence of a smooth measure $\mu$ on $S$ and a function $D$ with $D(k) > 0$ on the regular set such that

$$\int_{G.K} \varphi(x)dx = \int_S \int_K \varphi(sks^{-1})D(k)dkd\mu(s)$$

for $\varphi \in L^1(G.K)$. Now suppose $\varphi \in C_c^\infty(G)$ with support in the regular set. Then

$$\text{tr} \pi(\varphi) = \int_{G.K} \varphi(x)\Theta_\pi(x)dx$$

$$= \int_S \int_K \varphi(sks^{-1})D(k)\Theta_\pi(k)d\mu(s)$$

$$= \int_K \int_S \varphi^s(k)d\mu(s)D(k)\Theta_\pi(k)dk,$$

where we have written $\varphi^s(k) = \varphi(sks^{-1})$. On the other hand

$$\text{tr} \pi(\varphi) = \text{tr} \int_{G.K} \varphi(x)\pi(x)dx$$

$$= \text{tr} \int_S \int_K \varphi(sks^{-1})D(k)\pi(sks^{-1})dkd\mu(s)$$

$$= \text{tr} \int_S \pi(s)\pi|_K(\varphi^sD)\pi(s)^{-1}d\mu(s)$$

$$= \int_S \text{tr} \pi|_K(\varphi^sD)d\mu(s)$$

$$= \text{tr} \pi|_K \left( \int_S \varphi^s d\mu(s) D \right).$$

This implies the claim for all functions $\psi \in C_c^\infty(K)$ which are of the form

$$\psi(k) = \int_S \varphi(sks^{-1})d\mu(s)D(k)$$

for some $\varphi \in C_c^\infty(G)$ with support in the regular set. Consider the map

$$F : S \times K' \to G.K'$$

$$(s,k) \mapsto sks^{-1}$$

Then the differential of $F$ is an isomorphism at any point and by the inverse function theorem $F$ locally is a diffeomorphism. So let $U \subset S$ and $W \subset K'$ be open sets such that $F|_{U \times W}$ is a diffeomorphism. Then let $\alpha \in C_c^\infty(U)$ and $\beta \in C_c^\infty(W)$, then define

$$\phi(sks^{-1}) = \alpha(s)\beta(k) \quad \text{if } s \in U, \ k \in W$$
and \( \varphi(g) = 0 \) if \( g \) is not in \( F(U \times W) \). We can choose the function \( \alpha \) such that 
\[
\int_S \alpha(s) d\mu(s) = 1.
\]
Then
\[
\int_S \varphi(sks^{-1}) d\mu(s) D(k) = \beta(k) D(k).
\]
Since \( \beta \) was arbitrary and \( D(k) > 0 \) on \( K' \) the lemma follows. \( \square \)

Let \( W \) denote the virtual \( K \)-representation on \( \bigwedge^{\text{even}} p \otimes V_{\tau} - \bigwedge^{\text{odd}} p \otimes V_{\tau} \) and write \( \chi_W \) for its character.

**Lemma 2.12** Assume \( T \) meets all components of \( G \), then for any \( \pi \in \hat{G} \) the function \( \Theta_{\pi} \chi_W \) on \( K' = K \cap G' \) equals a finite integer linear combination of \( K \)-characters.

**Proof:** It suffices to show the assertion for \( \tau = 1 \). Let \( \varphi \) be the homomorphism \( K \to O(p) \) induced by the adjoint representation, where the orthogonal group is formed with respect to the Killing form. We claim that \( \varphi(K) \subset SO(p) \), the subgroup of elements of determinant one. Since we assume \( K = K^0 T \) it suffices to show \( \varphi(T) \subset SO(p) \). For this let \( t \in T \). Since \( t \) centralizes \( t \) it fixes the decomposition \( p = \oplus_{\alpha} p_{\alpha} \) into one dimensional root spaces. So \( t \) acts by a scalar, say \( c \) on \( p_{\alpha} \) and by \( d \) on \( p_{-\alpha} \). There is \( X \in p_{\alpha} \) and \( Y \in p_{-\alpha} \) such that \( B(X, Y) = 1 \). By the invariance of the Killing form \( B \) we get
\[
1 = B(X, Y) = B(\text{Ad}(t)X, \text{Ad}(t)Y) = cdB(X, Y) = cd.
\]
So on each pair of root spaces \( \text{Ad}(t) \) has determinant one hence also on \( p \).

Replacing \( G \) by a double cover if necessary, which doesn’t effect the claim of the lemma, we may assume that \( \varphi \) lifts to the spin group \( \text{Spin}(p) \). Let \( p = p^+ \oplus p^- \) be the decomposition according to an ordering of \( \varphi(t, g) \). This decomposition is a polarization of the quadratic space \( p \) and hence the spin group acts on \( S^+ = \bigwedge^{\text{even}} p^+ \) and \( S^- = \bigwedge^{\text{odd}} p^+ \) in a way that the virtual module \( (S^+ - S^-) \otimes (S^+ - S^-) \) becomes isomorphic to \( W \). For \( K \) connected the claim now follows from [1] (4.5). An inspection shows however that the proof of (4.5) in [1], which is located in the appendix (A.12), already applies when we only assume that the homomorphism \( \varphi \) factors over the spin group. \( \square \)

We continue the proof of the theorem. Let \( \hat{T} \) denote the set of all unitary characters of \( T \). Any regular element \( t \in \hat{T} \) gives rise to a discrete series representation \( (\omega, V_{\omega}) \) of \( G \). Let \( \Theta_{\omega} = \Theta_{\omega} \) be its character which, due to Harish-Chandra, is known to be a function on \( G \). Harish-Chandra’s construction gives a bijection between the set of discrete series representations of \( G \) and the set of \( W(G, T) = W(K, T) \)-orbits of regular characters of \( T \).
Let $\Phi^+$ denote the set of positive roots of $(g, t)$ and let $\Phi^+_c, \Phi^+_n$ denote the subsets of compact and noncompact positive roots. For each root $\alpha$ let $t \mapsto \tau^\alpha$ denote the corresponding character on $T$. Define

$$\Delta_c(t) = \prod_{\alpha \in \Phi^+_c} (1 - t^{-\alpha})$$

$$\Delta_n(t) = \prod_{\alpha \in \Phi^+_n} (1 - t^{-\alpha})$$

and $\Delta = \Delta_c \Delta_n$. If $\hat{t} \in \hat{T}$ is singular, Harish-Chandra has also constructed an invariant distribution $\Theta_{\hat{t}}$ which is a virtual character on $G$. For $\hat{t}$ singular let $W(\hat{t}) \subset W(g, t)$ be the isotropy group. One has $\Theta_{\hat{t}} = \sum_{w \in W(\hat{t})} \epsilon(w) \Theta_w(t)$, with $\Theta_w(t)$ the character of an induced representation acting on some Hilbert space $V_w$ and $\epsilon(w) \in \{\pm 1\}$. Let $E_2(G)$ denote the set of discrete series representations of $G$ and $E_2^s(G)$ the set of $W(G, T)$-orbits of singular characters.

By Theorem 1.7 the current theorem will follow from the

**Lemma 2.13** For $t \in T$ regular we have

$$\text{tr} \tau(t) = \frac{1}{|W(G, T)|} \sum_{i \in \mathcal{F}} \Theta_i(f_T) \Theta_i(t).$$

**Proof:** Let $\gamma$ denote the virtual $K$-representation on $(\wedge^{\text{even}} p - \wedge^{\text{odd}} p) \otimes V_T$. Harish-Chandra has shown ([17] Theorem 12) that for any $\hat{t} \in \hat{T}$ there is an irreducible unitary representation $\pi^0_{\hat{t}}$ such that $\Theta_{\hat{t}}$ coincides up to sign with the character of $\pi^0_{\hat{t}}$ on the set of elliptic elements of $G$ and $\pi^0_{\hat{t}} = \pi^0_{\hat{t}'}$ if and only if there is a $w \in W(G, T) = W(K, T)$ such that $\hat{t}' = wt$.

Further ([17], Theorem 14) Harish-Chandra has shown that the family

$$\left( \frac{\Delta(t) \Theta_i(t)}{\sqrt{|W(G, T)|}} \right)_{i \in \hat{T}/W(G, T)}$$

forms an orthonormal basis of $L^2(T)$. Here we identify $\hat{T}/W(G, T)$ to a set of representatives in $\hat{T}$ to make $\Theta_i$ well defined.

Consider the function $g(t) = \frac{\text{tr} \gamma(t) \Delta(t)}{\Delta_n(t)} = \text{tr} \tau(t) \Delta(t)$. Its coefficients with respect to the above orthonormal basis are

$$\langle g, \frac{\Delta \Theta_i}{\sqrt{|W(G, T)|}} \rangle = \frac{1}{\sqrt{|W(G, T)|}} \int_T \text{tr} \gamma(t) \Delta(t) \Theta_i(t) dt = \frac{1}{\sqrt{|W(G, T)|}} \int_K \text{tr} \gamma(k) \Theta_i(k) dk.$$
where we have used the Weyl integration formula for the group $K$ and the fact that $W(G,T) = W(K,T)$. Next by Lemma 2.12 this equals
\[ \sqrt{|W(G,T)|} \dim((\wedge^{\text{even}} p - \wedge^{\text{odd}} p) \otimes \bar{\tau} \otimes \pi_\tau^0)^K = \sqrt{|W(G,T)|} \Theta^\wedge_t(f_\tau). \]
Hence
\[ g(t) = \text{tr} \tau(t)' \triangle(t) = \sum_{i \in \hat{T}/W(G,T)} \Theta_i(f_\tau)' \triangle(t) \Theta_i(t) = \frac{1}{|W(G,T)|} \sum_{i \in \hat{T}} \Theta_i(f_\tau)' \triangle(t) \Theta_i(t). \]
The lemma and the theorem are proven.

**Corollary 2.14** If $\tilde{g} \in \hat{G}$ is semisimple and not elliptic then $O_{\tilde{g}}(g_\tau) = 0$. If $\tilde{g}$ is elliptic regular then
\[ O_{\tilde{g}}(g_\tau) = \frac{\text{tr} \tau(\tilde{g})}{\text{tr}(g_\tau S^+ - S^-)}. \]

**Proof:** Same as for the last proposition with $g_\tau$ replacing $f_\tau$. □

**Proposition 2.15** Assume that $\tau$ extends to a representation of the group $G$ on the same space. For the function $f_\tau$ we have for any $\pi \in \hat{G}$:
\[ \text{tr} \pi(f_\tau) = \sum_{p=0}^{\dim \mathfrak{g}/\mathfrak{t}} (-1)^p \dim \operatorname{Ext}_{\mathfrak{g},K}^p(V_\tau, V_\pi), \]
i.e., $f_\tau$ gives the Euler-Poincaré numbers of the $(\mathfrak{g},K)$-modules $(V_\tau, V_\pi)$, this justifies the name Euler-Poincaré function.

**Proof:** By definition it is clear that
\[ \text{tr} \pi(f_\tau) = \sum_{p=0}^{\dim \mathfrak{g}/\mathfrak{t}} (-1)^p \dim H^p(\mathfrak{g},K,V_\tau \otimes V_\pi). \]
The claim now follows from [4], p. 52. □

### 3 The Selberg trace formula

In this section we will fix the basic notation and set up the trace formula. For compactly supported functions this formula is easily deduced. In the sequel we however will need it for functions with noncompact support and therefore will have to show more general versions of the trace formula.

Let $G$ denote a real reductive group.
3.1 The trace formula

Let $\Gamma \subset G$ be a discrete subgroup such that the quotient manifold $\Gamma \backslash G$ is compact. We say that $\Gamma$ is \textit{cocompact} in $G$. Examples are given by nonisotropic arithmetic groups [26]. Since $G$ is unimodular the Haar measure on $G$ induces a $G$-invariant measure on $\Gamma \backslash G$, so we can form the Hilbert space $L^2(\Gamma \backslash G)$ of square integrable measurable functions on $\Gamma \backslash G$ modulo null functions. More generally, let $(\omega, V_\omega)$ be a finite dimensional unitary representation of $\Gamma$ and let $L^2(\Gamma \backslash G, \omega)$ be the Hilbert space of all measurable functions $f : G \to V_\omega$ such that $f(\gamma x) = \omega(\gamma) f(x)$ for all $\gamma \in \Gamma$ and all $x \in G$ and such that

$$\int_{\Gamma \backslash G} \| f(x) \|^2 \, dx < \infty$$

modulo null functions. The scalar product of $f, g \in L^2(\Gamma \backslash G, \omega)$ is

$$(f, g) = \int_{\Gamma \backslash G} \langle f(x), g(x) \rangle \, dx,$$

where $\langle ., \rangle$ is the scalar product on $V_\omega$. Let $C^\infty(\Gamma \backslash G, \omega)$ be the subspace consisting of smooth functions.

The group $G$ acts unitarily on $L^2(\Gamma \backslash G, \omega)$ by

$$R(y) \varphi(x) = \varphi(xy)$$

for $x, y \in G$ and $\varphi \in L^2(\Gamma \backslash G, \omega)$. Let $\pi$ be any unitary representation of $G$, any $f \in L^1(G)$ defines a bounded operator

$$\pi(f) = \int_G f(x) \pi(x) \, dx$$

on the space of $\pi$. We apply this to the case $\pi = R$. Let $C^\infty_c(G)$ be the space of all smooth functions of compact support on $G$. Let $f \in C^\infty_c(G), \varphi \in L^2(\Gamma \backslash G, \omega)$. Fix a fundamental domain $\mathcal{F} \subset G$ for the $\Gamma$-action on $G$ and compute formally at first:

$$R(f) \varphi(x) = \int_G f(y) \varphi(xy) \, dy$$

$$= \int_G f(x^{-1}y) \varphi(y) \, dy$$

$$= \sum_{\gamma \in \Gamma} \int_{\gamma \mathcal{F}} f(x^{-1}y) \varphi(y) \, dy$$

$$= \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} f(x^{-1}y) \omega(\gamma) \varphi(y) \, dy$$

$$= \int_{\Gamma \backslash G} k_f(x, y) \varphi(y) \, dy,$$
where \( k_f(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y) \omega(\gamma) \). Since \( f \) has compact support the latter sum is locally finite and therefore defines a smooth Schwartz kernel on the compact manifold \( \Gamma \setminus G \). This implies that the operator \( R(f) \) is a smoothing operator and hence of trace class.

Since the convolution algebra \( C_c^\infty \) contains an approximate identity we can infer that the unitary \( G \)-representation \( R \) on \( L^2(\Gamma \setminus G, \omega) \) decomposes into a direct sum of irreducibles, i.e.,

\[
L^2(\Gamma \setminus G, \omega) \cong \bigoplus_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \pi
\]

with finite multiplicities \( N_{\Gamma, \omega}(\pi) \in \mathbb{N}_0 \). Moreover, the trace of \( R(f) \) is given by the integral over the diagonal, so

\[
\text{tr } R(f) = \int_{\Gamma \setminus G} \text{tr } k_f(x, x) dx,
\]

where the trace on the right hand side is the trace in \( \text{End}(V_\omega) \). We plug in the sum for \( k_f \) and rearrange that sum in that we first sum over all conjugacy classes in the group \( \Gamma \). We write \( \Gamma_\gamma \) and \( G_\gamma \) for the centralizers of \( \gamma \in \Gamma \) in \( \Gamma \) and in \( G \) resp.

\[
\text{tr } R(f) = \sum_{\gamma \in \Gamma} \text{tr } \omega(\gamma) \int_{\Gamma_\gamma} f(x^{-1} \gamma x) dx
\]

\[
= \sum_{[\gamma]} \sum_{\sigma \in \Gamma / \Gamma_\gamma} \text{tr } \omega(\gamma) \int_{\Gamma_\gamma} f((\sigma x)^{-1} \gamma \sigma x) dx
\]

\[
= \sum_{[\gamma]} \text{tr } \omega(\gamma) \int_{\Gamma_\gamma} f(x^{-1} \gamma x) dx
\]

\[
= \sum_{[\gamma]} \text{tr } \omega(\gamma) \text{vol}(\Gamma_\gamma \setminus G_\gamma) \mathcal{O}_\gamma(f),
\]

where \( \mathcal{O}_\gamma(f) \) is the orbital integral. We have proved the following proposition.

**Proposition 3.1** For \( f \in C_c^\infty(G) \) we have

\[
\sum_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \text{tr } \pi(f) = \sum_{[\gamma]} \text{tr } \omega(\gamma) \text{vol}(\Gamma_\gamma \setminus G_\gamma) \mathcal{O}_\gamma(f),
\]

where all sums and integrals converge absolutely. □

All this is classical and may be found at various places. For our applications we will need to extend the range of functions \( f \) to be put into the trace formula. For this sake we prove the proposition below.

Let \( k, l \in \mathbb{N} \) and define \( L^1_{k,l}(G) \) to be the set of all functions \( f \) on \( G \) which are \( \max k, l \)-times continuously differentiable and satisfy
• $|Df|$ is integrable on $G$ for every $D \in U(g)$ of degree $\leq k$, and
• $|Df|$ is bounded on $G$ for every $D \in U(g)$ of degree $\leq l$.

Every $D \in U(g)$ of degree $\leq k$ induces a seminorm on $L^1_{k,l}(G)$ by

$$\sigma_D(f) = \int_G |Df(x)| \, dx.$$

Further, every $D \in U(g)$ of degree $\leq l$ induces a seminorm on $L^1_{k,l}(G)$ by

$$s_D(f) = \sup_{x \in G} |Df(x)|.$$

where $\| \cdot \|_1$ is the $L^1$-norm. We equip $L^1_{k,l}(G)$ with the topology given by these seminorms.

**Proposition 3.2** Suppose $f \in L^1_{2N,1}(G)$ with $N > \frac{\dim G}{2}$. Then the trace formula is valid for $f$ and either side of the trace formula defines a continuous linear functional on $L^1_{2N,1}(G)$.

**Proof:** We consider $U(g)$ as the algebra of all left invariant differential operators on $G$. Choose a left invariant Riemannian metric on $G$ and let $\triangle$ denote the corresponding Laplace operator. Then $\triangle \in U(g)$ and thus it makes sense to write $R(\triangle)$, which is an elliptic differential operator of order 2 on the compact manifold $\Gamma \setminus G$, essentially selfadjoint and non-negative. The theory of pseudodifferential operators implies that $R(\triangle + 1)^{-N}$ has a $C^1$-kernel and thus is of trace class. Let $g = (\triangle + 1)^{-N}f$ then $g \in L^1(G)$, so $R(g)$ is defined and gives a continuous linear operator on the Hilbert space $L^2(\Gamma \setminus G, \phi)$. We infer that $R(f) = R(\triangle + 1)^{-N}R(g)$ is of trace class.

Let $\chi : [0, \infty] \to [0, 1]$ be a monotonic $C^{2N}$-function with compact support, $\chi \equiv 1$ on $[0, 1]$ and $|\chi^{(k)}(t)| \leq 1$ for $k = 1, \ldots, 2N$. Let $h_n(x) := \chi^{\frac{\text{dist}(x,e)}{n}}$ for $x \in G, n \in \mathbb{N}$. Then $|Dh_n(x)| \leq \frac{C_n}{n}$ for any $D \in gU(g)$. Let $f_n = h_nf$ then $f_n \to f$ locally uniformly.

**Claim.** For the $L^1$-norm on $G$ we have

$$\|Df_n - Df\|_1 \to 0$$

as $n \to \infty$ for any $D \in U(g)$ of degree $\leq 2N$.

Proof of the claim: By the Poincaré-Birkhoff-Witt theorem $D(f_n) = D(h_nf)$ is a sum of expressions of the type $D_1(h_n)D_2(f)$ and $D_1$ can be chosen to be the identity operator or in $gU(g)$. The first case gives the summand $h_nD(f)$ and it is clear that $\|Df - h_nDf\|_1$ tends to zero as $n$ tends to infinity. For the rest assume $D_1 \in gU(g)$. Then $|D_1(h_n)(x)| \leq \frac{C_n}{n}$ hence $\|D_1(h_n)D_2(f)\|_1$ tends to zero because $D_2(f)$ is in $L^1(G)$. The claim follows.
To prove the lemma we estimate the operator norm as 
\[ \| R((\Delta + 1)^N f_n) - R((\Delta + 1)^N f) \| \leq \| (\Delta + 1)^N f_n - (\Delta + 1)^N f \| \]
the latter tends to zero according to the claim proven above. Denoting the trace norm by \( \| \cdot \|_{tr} \) we infer 
\[ \| R(f_n) - R(f) \|_{tr} \]
\[ = \| R((\Delta + 1)^{-N} (R((\Delta + 1)^N f_n) - R((\Delta + 1)^N f))) \|_{tr} \]
\[ \leq \| R((\Delta + 1)^{-N} \|_{tr} \| R((\Delta + 1)^N f_n) - R((\Delta + 1)^N f) \| \]
which tends to zero. Therefore \( \text{tr} R(f_n) \) tends to \( \text{tr} R(f) \) as \( n \to \infty \). It follows
\[ \sum_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \text{tr} \pi(f) = \text{tr} R(f) \]
\[ = \lim_{n} \text{tr} R(f_n) \]
\[ = \lim_{n} \sum_{[\gamma]} \text{tr} \omega(\gamma) \text{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f_n). \]

Now suppose as an additional condition that \( f \geq 0 \) and \( \omega = 1 \). Then we are allowed to interchange the limit and the sum by monotone convergence and thus in this case
\[ \sum_{\pi \in \hat{G}} N_{\Gamma, 1}(\pi) \text{tr} \pi(f) = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f). \]
In particular, the right hand side is finite. For general \( f \) and \( \omega \), we use the boundedness of \( \text{tr} \omega \) and we can justify the same interchange by dominated convergence if we show that there is \( \tilde{f} \in L^2_{2N,1}(G) \) with \( \tilde{f} \geq |f| \), because then the trace formula is valid for \( \tilde{f} \) and thus
\[ \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(\tilde{f}) < \infty. \]

It remains to show the existence of \( \tilde{f} \). For this let \( U, V \) be small neighbourhoods of the unit in \( G \) with \( U \subset V \). Let \( \alpha \in C^\infty(G) \) with values in \([0, 1]\), support in \( V \) and such that \( \alpha \equiv 1 \) on \( U \). Since \( Xf \) is bounded for every \( X \in \mathfrak{g} \) it follows that there is \( C > 0 \) such that for small \( t > 0 \) and every \( x \in G \) one has
\[ \| f(x \exp(tX)) - |f(x)|| \leq |f(x \exp(tX)) - f(x)| \leq Ct. \]
One has
\[ |f| * \alpha(x) = \int_G |f(y)| \alpha(y^{-1}x) \, dy \]
\[ = \int_G |f(xy^{-1})| \alpha(y) \, dy \]
\[ \geq \int_U |f(xy^{-1})| \, dy. \]
It follows that there exists $\varepsilon > 0$ with
\[ |f| * \alpha(x) \geq \varepsilon |f(x)| \]
for every $x \in G$. Set $\tilde{f} = \frac{1}{\varepsilon} |f| * \alpha$, then $\tilde{f}$ lies in $L^1_{2N}(G)$ and $\tilde{f} \geq |f|$. By the above the trace formula is valid for $f$. Finally, the norm-estimates also imply the claimed continuity of the linear functional $f \mapsto \text{tr}(R(f))$ on $L^1_{2N}(G)$. \qed

4 The Lefschetz formula

In this section $G$ will be a connected semisimple Lie group with finite center.

4.1 Euler characteristics

Let $L$ be a real reductive group and suppose there is a finite subgroup $E$ of the center of $L$ and a reductive and Zariski-connected linear group $L$ over $\mathbb{R}$ such that $L/E$ is isomorphic to a subgroup of $L(\mathbb{R})$ of finite index. Note that these conditions are satisfied whenever $L$ is a Levi component of a connected semisimple group $G$ with finite center. Let $K_L$ be a maximal compact subgroup of $L$ and let $\Gamma$ be a cocompact discrete subgroup of $L$. Fix a nondegenerate invariant bilinear form on the Lie algebra $\mathfrak{l}_0$ of $L$ such that $B$ is negative definite on the Lie algebra of $K_L$ and positive definite on its orthocomplement. Let $\theta$ be the Cartan involution fixing $K_L$ pointwise then the form $-B(X, \theta(Y))$ is positive definite and thus defines a left invariant metric on $L$. For any closed subgroup $Q$ we get a left invariant metric on $Q$. The volume element of that metric gives a Haar measure, called the standard volume with respect to $B$ on $Q$.

Let $\Gamma \subset G$ denote a cocompact discrete subgroup. If $\Gamma$ is torsion free, it acts fixed point free on the contractible space $X$ and hence $\Gamma$ is the fundamental group of the Riemannian manifold
\[ X_\Gamma = \Gamma \backslash X = \Gamma \backslash G/K \]
it follows that we have a canonical bijection of the homotopy classes of loops:
\[ [S^1 : X_\Gamma] \to \Gamma/\text{conjugacy}. \]
For a given class $[\gamma]$ let $X_\gamma$ denote the union of all closed geodesics in the corresponding class in $[S^1 : X_\Gamma]$. Then $X_\gamma$ is a smooth submanifold of $X_{\Gamma_H}$ \cite{[12]}, indeed, it follows that
\[ X_\gamma \cong \Gamma_\gamma \backslash G_\gamma / K_\gamma, \]
where $G_\gamma$ and $\Gamma_\gamma$ are the centralizers of $\gamma$ in $G$ and $\Gamma$ and $K_\gamma$ is a maximal compact subgroup of $G_\gamma$. Further all closed geodesics in the class $[\gamma]$ have the same length $l_\gamma$. 


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Let $r \in \mathbb{N}_0$. If $\Gamma$ is torsion-free, we define the $r$-th Euler characteristic of $\Gamma$ by

$$
\chi_r(\Gamma) = \chi_r(X_\Gamma) = \dim_{X_\Gamma} \sum_{j=0}^{\dim X_\Gamma} (-1)^{j+r} \binom{j}{r} b^j(X_\Gamma),
$$

where $b^j(X_\Gamma)$ is the $j$-th Betti number of $X_\Gamma$. We want to extend this notion to groups $\Gamma$ which are not necessarily torsion-free.

Let now $H$ be a $\theta$-stable Cartan subgroup, then $H = AB$, where $A$ is a connected split torus and $B \subset K_L$ is a Cartan of $K_L$. On the space $L/H$ we have a pseudo-Riemannian structure given by the form $B$. The Gauss-Bonnet construction (sect. 24 or see below) generalizes to Pseudo-Riemannian structures to give an Euler-Poincaré measure $\eta$ on $L/H$. Define a (signed) Haar-measure by

$$
\mu_{EP} = \eta \otimes \text{(normalized Haar measure on } H\text{)}.\n$$

Let $W = W(L, H)$ denote the Weyl group and let $W_C = W(L_C, H_C)$ be the Weyl group of the complexifications. Let the generic Euler characteristic be defined by

$$
\chi_{gen}(\Gamma\backslash L/K_L) = \frac{\mu_{EP}(\Gamma\backslash L)}{|W|}.\n$$

Write $X_\Gamma = \Gamma\backslash L/K$.

**Lemma 4.1** If $H$ is compact and $\Gamma$ is torsion-free, then the generic Euler characteristic equals the ordinary Euler characteristic, i.e., $\chi_{gen}(X_\Gamma) = \chi(X_\Gamma)$.

**Proof:** In this case we have $H = B$. The Gauss-Bonnet Theorem tells us

$$
\eta(\Gamma\backslash L/B) = \chi(\Gamma\backslash L/B).
$$

Now $\Gamma\backslash L/B \to \Gamma\backslash L/K_L$ is a fiber bundle with fiber $K_L/B$, therefore we get

$$
\chi(\Gamma\backslash L/B) = \chi(\Gamma\backslash L/K_L)\chi(K_L/B).
$$

Finally the Hopf-Samelson formula says $\chi(K_L/B) = |W|$.

For the next proposition assume that $A$ is central in $L$, $L = AL_1$ where $L_1$ has compact center. Let $C$ denote the center of $L$, then $A \subset C$ and $C = ABC$, where $B_C = B \cap C$. Let $L'$ be the derived group of $L$ and let $\Gamma' = L' \cap \Gamma C$ and $\Gamma_C = \Gamma \cap C$ then by Lemma 3.3 in [34] we infer that $\Gamma_C$ is a cocompact subgroup of $C$ and $\Gamma'$ is a cocompact discrete subgroup of $L'$. Let $\Gamma_A = A \cap \Gamma C B_C$ the projection of $\Gamma_C$ to $A$. Then $\Gamma_A$ is discrete and cocompact in $A$.

**Proposition 4.2** Assume $\Gamma$ is torsion-free and $A$ is central in $L$ of dimension $r$, and $\Gamma' \subset \Gamma$ is a subgroup of finite index. Then the group $A/\Gamma_A$ acts freely on $X_\Gamma$ and $\chi_{gen}(X_\Gamma) = \chi_r(\Gamma)\text{vol}(A/\Gamma_A)$. It follows that

$$
\chi_r(\Gamma) = \chi_r(\Gamma') \frac{|\Gamma_A : \Gamma'|}{|\Gamma : \Gamma'|}.\n$$
**Proof:** The group $A_{\Gamma} = A/\Gamma_A$ acts on $\Gamma \backslash L/B$ by multiplication from the right. We claim that this action is free, i.e., that it defines a fiber bundle

$$A_{\Gamma} \rightarrow \Gamma \backslash L/B \rightarrow \Gamma \backslash L/H.$$ 

To see this let $\gamma aB = \gamma xB$ for some $a \in A$ and $x \in L$, then $a = x^{-1}\gamma xb$ for some $\gamma \in \Gamma$ and $b \in B$. Writing $\gamma$ as $\gamma' a_\gamma b_\gamma$ we conclude that $a_\gamma \in A_A$ and $a = a_\gamma$, whence the claim.

In the same way we see that we get a fiber bundle

$$A_{\Gamma} \rightarrow \Gamma \backslash L/L \rightarrow A_{\Gamma} \backslash L/K_L.$$ 

We now apply the Gauss-Bonnet theorem to conclude

$$\chi_{\text{gen}}(X_{\Gamma}) = \frac{\eta(\Gamma \backslash L/H)}{|W|} \chi(\Gamma \backslash L/H)$$

$$= \frac{\chi(A_{\Gamma} \backslash L/B)}{\chi(K_L/B)}$$

$$= \frac{\chi(A_{\Gamma} \backslash L/K_L)}{\chi(K_L/B)}.$$ 

It remains to show that $\chi(A \backslash X_{\Gamma}) = \chi_{\Gamma}(\Gamma)$. For this let $a_0$ be the real Lie algebra of $A$ and let $l_0^{\text{der}}$ be the Lie algebra of the derived group $L^{\text{der}}$. The Lie algebra $l_0$ of $L$ can be written as

$$l_0 = a_0 \oplus l_0^{\text{der}} \oplus \zeta_0,$$

where $\zeta_0$ is central in $l_0$. Let $X_1, \ldots, X_r$ be a basis of $a_0$. We consider $X_j$ as a vector filed on $\Gamma \backslash L/K_L$ by means of the left translation. Let $\omega_1, \ldots, \omega_r$ be the dual basis of $a_0^*$. Via the above decomposition we can view each $\omega_j$ as an element of $l_0$; thus as a 1-form on $\Gamma \backslash L/K_L$. Since $A \cong \mathbb{R}^r$ and the $\omega_j$ are the differential forms given by a set of co-ordinates, the forms $\omega_1, \ldots, \omega_r$ are all closed. The group $A_{\Gamma} = A/\Gamma_A$ is connected and compact, therefore the cohomology of the deRham complex of $\Gamma \backslash L/K_L$ coincides with the cohomology of the subcomplex of $A_{\Gamma}$-invariants $\Omega(X_{\Gamma})^{A_{\Gamma}}$. Using local triviality of the bundles one sees that

$$\Omega(X_{\Gamma})^{A_{\Gamma}} = \bigoplus_{I \subseteq \{1, \ldots, r\}} \pi^* \Omega(A \backslash X_{\Gamma}) \wedge \omega_I,$$

where $\pi$ is the projection $X_{\Gamma} = \Gamma \backslash L/K_L \rightarrow A_{\Gamma} \backslash L/K_L = A \backslash X_{\Gamma}$ and

$$\omega_{\{i_1, \ldots, i_k\}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$$

for $i_1 < i_2 < \cdots < i_k$. Since the $\omega_j$ are closed, it follows for the real valued cohomology that

$$H^*(X_{\Gamma}) \cong \bigoplus_{I \subseteq \{1, \ldots, r\}} H^{*-|I|}(A \backslash X_{\Gamma}).$$
So we compute
\[
\begin{align*}
\chi_r(\Gamma) &= \dim X_\Gamma \sum_{j=r}^{\dim X_\Gamma} (-1)^{j+r} \binom{j}{r} b^j(X_\Gamma) \\
&= \dim X_\Gamma \sum_{j=r}^{\dim X_\Gamma} (-1)^{j+r} \binom{j}{r} \sum_{I \subset \{1, \ldots, r\}} b^{j-|I|}(A \setminus X_\Gamma) \\
&= \dim X_\Gamma \sum_{j=r}^{\dim A \setminus X_\Gamma} (-1)^{j+r} \binom{j}{r} \sum_{k=0}^{r} \binom{r}{k} b^{j-k}(A \setminus X_\Gamma) \\
&= \dim A \setminus X_\Gamma \sum_{p=0}^{r+p} b^p(A \setminus X_\Gamma) \sum_{j=r}^{\dim X_\Gamma} (-1)^{j+r} \binom{j}{r} \binom{r}{j-p} \\
&= \chi(A \setminus X_\Gamma).
\end{align*}
\]

The last step uses the combinatorial identity
\[
\sum_{j=r}^{r+p} (-1)^{j+r} \binom{j}{r} \binom{r}{j-p} = (-1)^p.
\]

Define the $r$-th Euler-number of $\Gamma$ by
\[
\chi_r(\Gamma) = \chi_r(\Gamma') \frac{[\Gamma_A : \Gamma_A']}{[\Gamma : \Gamma']},
\]
where $\Gamma' \subset \Gamma$ is a torsion-free subgroup of finite index (which always exists by Selberg’s Lemma \cite{Selberg}). Proposition 4.2 shows that $\chi_r(\Gamma)$ does not depend on the choice of $\Gamma'$. Further, this definition allows us to extend Proposition 4.2 to arbitrary cocompact $\Gamma$:
\[
\chi_{\text{gen}}(X_\Gamma) = \chi_r(\Gamma) \text{ vol}(A/\Gamma_A).
\]

We will compute $\chi_{\text{gen}}(X_\Gamma)$ in terms of root systems. Let $\Phi$ denote the root system of $(L, B)$, where $L$ and $B$ are the complexified Lie algebras of $L$ and $H$. Let $\Phi_\nu$ be the set of noncompact imaginary roots and choose a set $\Phi_+^*$ of positive roots such that for $\alpha \in \Phi_+^*$ nonimaginary we have that $\alpha^c \in \Phi^+$. Let $\nu = \dim L/K_L - \text{rank} L/K_L$, let $\rho$ denote the half of the sum of all positive roots. For any compact subgroup $U$ of $L$ let $v(U)$ denote the standard volume.

**Theorem 4.3** The generic Euler number satisfies
\[
\begin{align*}
\chi_{\text{gen}}(X_\Gamma) &= \frac{(-1)^{|\Phi_\nu^*|}|W|_C \prod_{\alpha \in \Phi_+^*} (\rho, \alpha) v(K_L)}{(2\pi)^{|\Phi_\nu^*|}|W|^{2\nu/2} v(B)} \text{ vol}(\Gamma \setminus L) \\
&= c^{-1}_L |W|_C \prod_{\alpha \in \Phi_+^*} (\rho, \alpha) \text{ vol}(\Gamma \setminus L),
\end{align*}
\]
where \( c_L \) is Harish-Chandra’s constant, i.e.,

\[
c_L = (-1)^{|\Phi^+|}|(2\pi)|^{v/2} \frac{v(B)}{v(K_L)} |W|.
\]

So, especially in the case when \( A \) is central we get

\[
\chi_r(\Gamma) = \frac{|W| \prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle}{c_L \text{vol}(A/\Gamma)} \text{vol}(\Gamma\setminus L).
\]

**Proof:** On the manifold \( L/H \) the form \( B \) gives the structure of a pseudo-Riemannian manifold. Let \( P \) denote the corresponding \( SO_{p,q} \) fiber bundle, where \((p,q)\) is the signature of \( B \) on \( \mathfrak{g}_0 \). Let \( \varphi : H \to SO_{p,q} \) denote the homomorphism induced by the adjoint representation. We have \( P = L \times \varphi \). A connection on \( P \) is given by the \( L \)-invariant connection 1-form

\[
\omega \left( A + \sum_{\alpha} c_{\alpha} X_{\alpha} \right) = A, \quad A \in \mathfrak{so}(p,q),
\]

where we have used \( B_{\gamma} P \cong \mathfrak{so}(p,q) \oplus (\oplus \mathfrak{g}_\alpha) \cap \mathfrak{g}_0 \). By an inspection in local charts one finds the following formula for the \( L \)-invariant 2-form \( d\omega \):

\[
d\omega(e) \left( A + \sum_{\alpha} c_{\alpha} X_{\alpha}, A' + \sum_{\alpha} c'_{\alpha} X_{\alpha} \right) = [A,A'] - \sum_{\alpha} c_{\alpha} c'_{\alpha} \varphi_{+} H_{\alpha}.
\]

Let \( \Omega = d\omega \) and let \( Pf \) be the Pfaffian as in 24.46.10. Let \( J \) be the diagonal matrix having the diagonal entries \((1,\ldots,1,-1,\ldots,-1)\) with \( p \) ones and \( q \) minus ones and let \( P(X) = Pf(JX) \). Write this as

\[
P(X) = (2\pi)^{-m} \sum_{h,k} \chi_{h,k} X_{h_1,k_1} \cdots X_{h_m,k_m},
\]

where \( m = \frac{1}{2}(p+q) \in \mathbb{N} \). Let

\[
F(\Omega) = (2\pi)^{-m} \sum_{h,k} \chi_{h,k} \Omega_{h_1,k_1} \wedge \cdots \wedge \Omega_{h_m,k_m}.
\]

There is a unique \( G \)-invariant form \( F_B(\Omega) \) on \( L/H \) which at the origin is the pullback of \( F(\Omega) \) with respect to a section \( s : L/H \to P \). Writing \( \Omega = \Omega_1 - \Omega_2 \) with

\[
\Omega_1 \left( A + \sum_{\alpha} c_{\alpha} X_{\alpha}, A' + \sum_{\alpha} c'_{\alpha} X_{\alpha} \right) = [A,A']
\]

we get

\[
F_B(\Omega) = F_B(-\Omega_2) = (-1)^m F_B(\Omega_2).
\]

We call this form \( \eta \). On the space \( \mathbb{C} X_{\alpha} + \mathbb{C} X_{-\alpha} \) we have that

\[
\varphi_{+} Y = -\alpha(Y)i \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]
with respect to the basis \((X_\alpha, X_{-\alpha})\). Let \(\alpha_1, \ldots, \alpha_m\) be an enumeration of \(\Phi^+\), then
\[
\eta = (2\pi)^{-m} \sum_{\sigma \in \text{Per}(\Phi)} \sum_{h,k} \chi_{h,k}((\varphi_* H_{\sigma(h)}(\alpha_1))_{h,1,k_1} \cdots (\varphi_* H_{\sigma(h)}(\alpha_m))_{h,m,k_n}) \tilde{\omega}_\sigma,
\]
where
\[
\tilde{\omega}_\sigma = dX_{\sigma(h)} \wedge dX_{\sigma(h_1)} \wedge \cdots \wedge dX_{\sigma(h_m)} \wedge dX_{\sigma(h_1)}.
\]
We end up with
\[
\eta = (2\pi)^m (-1)^m \sum_{\alpha \in \Phi^+, \sigma \in \text{Per}(\Phi^+)} (\alpha, \sigma(\alpha)) \tilde{\omega}_\sigma
\]
\[
= (2\pi)^m (-1)^m |W_C| \prod_{\alpha \in \Phi^+} (\rho, \alpha) \tilde{\omega}_\sigma.
\]
Comparing \(\eta\) with the standard measure given by the form \(B\) gives the claim. \(\square\)

4.2 Setting up the formula

Let \(K \subset G\) be a maximal compact subgroup with Cartan involution \(\theta\). Let \(X = G/K\) denote the symmetric space attached to \(G\). Let \(H \subset G\) be a Cartan subgroup. Modulo conjugation we may assume that \(H\) is stable under \(\theta\). Then \(H = AB\), where \(A\) is a connected split torus and \(B\) is a subgroup of \(K\). The double use of the letter \(B\) for the group and a bilinear form on the Lie algebra will not cause any confusion. Fix a parabolic subgroup \(P\) of \(G\) with split component \(A\). Then \(P = MAN\) where \(N\) is the unipotent radical of \(P\) and \(M\) is reductive with compact center and finite component group. The choice of the parabolic \(P\) amounts to the same as a choice of a set of positive roots \(\Phi^+ = \Phi^+(g,a)\) in the root system \(\Phi(g,a)\) such that for the Lie algebra \(\mathfrak{n} = \text{Lie}(N)\) we have \(\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha\). Let \(\bar{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}\), \(n_0 = \bar{n} \cap \mathfrak{g}_0\) and \(\bar{N} = \text{exp}(n_0)\) then \(\bar{P} = MAN\) is the parabolic opposite to \(P\). The root space decomposition then writes as \(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bar{n} \oplus \bar{\bar{n}}\). Let \(\rho_P\) be the half of the sum of the positive roots, each weighted with its multiplicity, i.e., for \(a \in A\) we have \(a^{\rho_P} = \det(a|\mathfrak{n})\). Let \(A^- \subset A\) denote the negative Weyl chamber corresponding to that ordering, i.e., \(A^-\) consists of all \(a \in A\) which act contractingly on the Lie algebra \(\mathfrak{n}\). Further let \(\overline{A^-}\) be the closure of \(A^-\) in \(G\), this is a manifold with boundary. Let \(K_M\) be a maximal compact subgroup of \(M\). We may suppose that \(K_M = M \cap K\) and that \(K_M\) contains \(B\). Fix an irreducible unitary representation \((\tau, V_\tau)\) of \(K_M\).

Let \(\mathcal{E}_P(\Gamma)\) denote the set of all conjugacy classes \([\gamma]\) in \(\Gamma\) such that \(\gamma\) is in \(G\) conjugate to an element \(a_\gamma b_\gamma\) of \(A^- B\).

Take a class \([\gamma]\) in \(\mathcal{E}_P(\Gamma)\). Then there is a conjugate \(H_\gamma = A_\gamma B_\gamma\) of \(H\) that contains \(\gamma\). Then the centralizer \(\Gamma_\gamma\) projects to a lattice \(\Gamma_{A,\gamma}\) in the split part \(A_\gamma\). Let \(\lambda_\gamma\) be the covolume of this lattice.
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Let \((\omega, V\omega)\) be a finite dimensional unitary representation of \(\Gamma\).

For any module \(V\) of the Lie algebra \(n\) let \(H^q(n, V)\), \(q = 0, \ldots, \dim n\) denote the Lie algebra cohomology \([4]\). If \(\pi \in \hat{G}\) then \(H^q(n, \pi K)\) is an admissible \((a \oplus m, K_M)\)-module of finite length \([18]\).

For \(\mu \in a^* \cong \text{Hom}(A, C^\times)\) and \(j \in \mathbb{N}\) let \(C^{\mu,j}(A^-)\) denote the space of all functions on \(A\) which

- are \(j\)-times continuously differentiable on \(A\),
- are zero outside \(A^-\),
- satisfy \(|D\varphi| \ll |a^\mu|\) for every invariant differential operator \(D\) on \(A\) of degree \(\leq j\).

This space can be topologized with the seminorms

\[ N_D(\varphi) = \sup_{a \in A} |a^{-\mu} D\varphi(a)|, \]

\(D \in U(a), \deg(D) \leq j\). Since the space of operators \(D\) as above is finite dimensional, one can choose a basis \(D_1, \ldots, D_n\) and set

\[ \| \varphi \| = N_{D_1}(\varphi) + \cdots + N_{D_n}(\varphi). \]

The topology of \(C^{\mu,j}(A^-)\) is given by this norm and thus \(C^{\mu,j}(A^-)\) is a Banach space.

**Theorem 4.4 (Lefschetz formula)** Assume that \(M\) is orientation preserving or that \(\tau\) lies in the image of the restriction map \(\text{res}^M_{K_M}\). There is \(\mu \in a^*\) and \(j \in \mathbb{N}\) such that for every \(\varphi \in C^{\mu,j}(A^-)\) the expression

\[ \sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) \sum_{p,q} (-1)^{p+q} \int_{A^-} \varphi(a) \text{tr} \left( a | (H^q(n, \pi_K) \otimes \wedge^p p_M \otimes V_\tau) K_M^M \right) da, \]

henceforth referred to as the global side, equals

\[ (-1)^{\dim A} \sum_{[\gamma] \in \mathcal{E}_F(\Gamma)} \lambda_\gamma \chi_{r}(\Gamma_\gamma) \text{tr} \omega(\gamma) \frac{\varphi(a_\gamma) \text{tr} \tau(b_\gamma)}{\det(1 - a_\gamma b_\gamma |n)}, \]

called the local side, where \(r = \dim A\). Either side defines a continuous linear functional on the Banach space \(C^{\mu,j}(A^-)\).

The proof will be given in section 4.3.

We will give a reformulation of the theorem and for this we need the following notation. Let \(V\) be a complex vector space on which \(A\) acts linearly. For each
$\lambda \in a^*$ let $V^\lambda$ denote the generalized $\lambda$-eigenspace in $V$, i.e., $V^\lambda$ consists of all $v \in V$ such that there is $n \in \mathbb{N}$ with
\[(a - a^\lambda)^n v = 0\]
for every $a \in A$. The theorem above implies the following.

**Theorem 4.5** Assume that $M$ is orientation preserving or that $\tau$ lies in the image of the restriction map $\text{res}^M_{K_M}$. Then we have the following identity of distributions on $A^-$.
\[
\sum_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \sum_{\lambda \in a^*} m_\lambda(\pi)(\cdot)^\lambda = \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} c_\gamma \delta_{a_\gamma}.
\]
Here $(\cdot)^\lambda$ means the function $a \mapsto a^\lambda$ and $m_\lambda(\pi)$ equals
\[
\sum_{p, q} (-1)^{p+q+\dim N} \dim \left( H^q(n, \pi_K)^\lambda \otimes \Lambda^p p_M \otimes V_\tau \right)_{K_M}.
\]
The sum indeed is finite for each $\lambda \in a^*$. Further, for $[\gamma] \in \mathcal{E}_P(\Gamma)$ we set
\[
c_\gamma = \lambda_\gamma \chi_r(\Gamma_\gamma) \tr \omega(\gamma) \frac{\tr \tau(b_\gamma)}{\det(1 - a_\gamma b_\gamma | n)}.
\]

**Corollary 4.6** Assume that $M$ is orientation preserving or that $\tau$ lies in the image of the restriction map $\text{res}^M_{K_M}$. There is $\mu \in a^*$ and $j \in \mathbb{N}$ such that for every $\varphi \in C^{\mu,j}(A^-)$ the expression
\[
\sum_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \sum_{p, q, r} (-1)^{p+q+r+\dim N} \int_{A^-} \varphi(a) \tr (\cdot)(H^q(n, \pi_K)^\lambda \otimes \Lambda^p p_M \otimes \Lambda^r n \otimes V_\tau)_{K_M} da,
\]
equals
\[
\sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \lambda_\gamma \chi_r(\Gamma_\gamma) \tr \omega(\gamma) \varphi(a_\gamma) \tr \tau(b_\gamma).
\]

**Proof:** Let $\Lambda^r n = \bigoplus_{j \in J_r} V_j$ be the decomposition of the adjoint action of the group $M$ into irreducible representations. On $V_j$ the torus $A$ acts by a character $\lambda_j$. We apply the theorem to $\tau$ replaced by $\tau \oplus V_j|K_M$ and $\varphi(a)$ replaced by $\varphi(a)\lambda_j(a)$. We sum these over $j$ and take the alternating sum with respect to $r$. On the local side we apply the identity
\[
\sum_{r=0}^{\dim N} (-1)^r \tr (a_\gamma b_\gamma | \Lambda^r n) = \det(1 - a_\gamma b_\gamma | n)
\]
to get exactly the local side of the corollary. On the global side we need to recall that the $K_M$-module $\Lambda^r n$ is self dual. The corollary follows. \qed
4.3 Proof of the Lefschetz formula

The notations are as in section 4.2. Let $G$ act on itself by conjugation, write $g.x = gxg^{-1}$, write $G.x$ for the orbit, so $G.x = \{gxg^{-1} | g \in G\}$ as well as $G.S = \{gs^{-1}g | s \in S, g \in G\}$ for any subset $S$ of $G$. We are going to consider functions that are supported in the set $G$. By Theorem 2.3 there exists an Euler-Poincaré function $f^M_\tau \in C^\infty_c(M)$ to the representation $\tau$.

For a finite dimensional complex vector space $V$ and $T \in \text{GL}(V)$ let $E(T)$ be the set of eigenvalues of $T$. Let $\lambda_{\text{min}}(T) := \min\{|\lambda| : \lambda \in E(T)\}$ and $\lambda_{\text{max}}(T) := \max\{|\lambda| : \lambda \in E(T)\}$. We are particularly interested in the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. So for $g \in G$ and $V_a$ a $g$-invariant subspace of $\mathfrak{g}$ we write $g|_V$ for the induced element of $\text{GL}(V)$.

For $am \in AM$ define $\lambda(am) := \frac{\lambda_{\text{min}}(a|\vec{n})}{\lambda_{\text{max}}(m|\mathfrak{g})}$. Note that $\lambda_{\text{max}}(m|\mathfrak{g})$ is always $\geq 1$ and that $\lambda_{\text{max}}(m|\mathfrak{g})\lambda_{\text{min}}(m|\mathfrak{g}) = 1$.

We will consider the set

$$(AM)^\sim := \{am \in AM | \lambda(am) > 1\}.$$ 

Let $M_{\text{ell}}$ denote the set of elliptic elements in $M$.

**Lemma 4.7** The set $(AM)^\sim$ has the following properties:

1. $A^- M_{\text{ell}} \subset (AM)^\sim$
2. $am \in (AM)^\sim$ $\Rightarrow$ $a \in A^-$
3. $am, a'm' \in (AM)^\sim, g \in G$ with $a'm' = gamg^{-1}$ $\Rightarrow$ $a = a', g \in AM$.

**Proof:** The first two are immediate. For the third let $am, a'm' \in (AM)^\sim$ and $g \in G$ with $a'm' = gamg^{-1}$. Observe that by the definition of $(AM)^\sim$ we have

$$\lambda_{\text{min}}(am|\vec{n}) \geq \lambda_{\text{min}}(a|\vec{n})\lambda_{\text{min}}(m|\mathfrak{g})$$

$$> \lambda_{\text{max}}(m|\mathfrak{g})^2 \lambda_{\text{min}}(m|\mathfrak{g})$$

$$= \lambda_{\text{max}}(m|\mathfrak{g})$$

$$\geq \lambda_{\text{max}}(m|a + m + n)$$

$$\geq \lambda_{\text{max}}(am|a + m + n)$$

that is, any eigenvalue of $am$ on $\vec{n}$ is strictly bigger than any eigenvalue on $a + m + n$. Since $g = a + m + n + \vec{n}$ and the same holds for $a'm'$, which has the same eigenvalues as $am$, we infer that $\text{Ad}(g)\vec{n} = \vec{n}$. So $g$ lies in the normalizer of
\(\bar{n}\), which is \(\bar{P} = MA\bar{N} = \bar{N}AM\). Now suppose \(g = nm_1a_1\) and \(\hat{m} = m_1mm_1^{-1}\) then
\[
gam g^{-1} = na\hat{m}^{-1} = a\hat{m}\ (a\hat{m})^{-1}n(a\hat{m})\ n^{-1}.
\]
Since this lies in \(AM\) we have \((a\hat{m})^{-1}n(a\hat{m}) = n\) which since \(am \in (AM)\) implies \(n = 1\). The lemma is proven. \(\square\)

Let \(C \subset M\) be a compact subset. In our application, \(C\) will be the support of the Euler-Poincaré function \(f^M_\tau\). Let \(\{\alpha_1, \ldots, \alpha_r\}\) be the set of simple roots in \(\phi^-(g, a)\). Then \(\alpha_1, \ldots, \alpha_r\) is a basis of \(\mathfrak{a}^*\) and \(\mathfrak{a}_R\) is the set of all \(X \in \mathfrak{a}\) with \(\alpha_j(X) > 0\) for \(j = 1, \ldots, r\). For \(a \in A\) we write \(a_j = \alpha_j(\log a)\) and thus we get global co-ordinates on \(A\) such that \(a \in A \iff a_j > 0 \ \forall j\). For \(T > 0\) set \(A_T = \{a \in A : a_j \geq T \ \forall j\}\). Then there exists \(T > 0\) such that the set closed \(A_T \cdot C\) is contained in \((AM)\).

The boundary \(S\) of \((AM)\) in \(AM\) decomposes into two disjoint subsets \(S = S_1 \cup S_2\), where
\[
S_1 = \{am \in S : a \neq 1\}
\]
\[
S_2 = \{am \in S : a = 1\}.
\]

We want to construct a smooth function \(\chi : (AM) \to [0, 1]\) such that

- \(\chi\) vanishes to infinite order at every point of \(S_1\).
- \(\chi\) is invariant under conjugation by elements of \(m\).
- \(\chi(am) = 1\) if \(m\) is elliptic.
- If \(a_j \geq T\) for some \(j\) and \(m \in C\), then
\[
\frac{\partial}{\partial a_j}\chi(am) = 0.
\]

Note that the last condition implies that \(\chi\) is constant on \(A_T \cdot C\).

In order to construct conjugation invariant functions one considers the geometric quotient \(M/\text{conj}\) which is an affine variety as \(M\) is reductive. Note that on the complex valued points the map \(M_C \to M/\text{conj}(C)\) is open as a consequence of the Kempf-Ness Theorem [22]. Embed \(M/\text{conj}\) into affine space \(\mathbb{A}^n\), then \(M/\text{conj}(C) \to \mathbb{A}^n(C) \cong \mathbb{C}^n\). We use the isomorphism of \(A\) with \(\mathfrak{a}_R\), so we embed \(A\) into \(\mathbb{C}^r\). Thus we get a map
\[
\alpha : AM \to AM/\text{conj} = a \times (M/\text{conj}) \to C^{n+r}.
\]
Let \((A_C M_C)\) be the set of all \(am \in A_C M_C\) with \(\lambda(am) > 1\). Then \((A_C M_C)\) is a complex neighbourhood of \((AM)\) and there is an open subset \(U\) of \(\mathbb{C}^{n+r}\) such that \(((A_C M_C)) = \alpha^{-1}(U \cap \alpha(A_C M_C))\). It follows that for each compact subset
C of $\mathbb{C}^n$ there exists $T > 0$ such that for every $z \in \mathbb{C}^r$ with $\text{Re}(z_j) \geq T \forall j$ one has $C \times \{z\} \subset U$. The task of constructing $\chi$ now boils down to constructing a function on $U$ with the indicated properties which is easily established.

Extend $\chi$ from $(AM)^\sim$ to all of $AM$ by setting

$$\chi(am) = 0, \quad \text{if } am \notin (AM)^\sim.$$ 

Fix a smooth function $\eta$ on $N$ which has compact support, is positive, invariant under $K_M$ and satisfies $\int_N \eta(n)dn = 1$. Given these data let $f = f_{\eta,\tau,\varphi} : G \to \mathbb{C}$ be defined by

$$f(knma(kn)^{-1}) := \eta(n)f^M_m(m)\frac{\varphi(a)\chi(am)}{\det(1 - (ma)|n)},$$

for $k \in K, n \in N, m \in M, a \in A$. Further $f(x) = 0$ if $x$ is not in $G.(AM)$. Note that indeed, $f$ is supported in the closure of $G.(AM)^\sim$.

**Lemma 4.8** The function $\frac{\varphi(a)\chi(am)}{\det(1 - am|n)}$ is $j$-times continuously differentiable on $(AM)^\sim$ and vanishes on the boundary $\partial(AM)^\sim$ in $AM$ to order at least $j - \dim n$.

**Proof:** Let $a_0m_0$ be a boundary point of $(AM)^\sim$. If $a_0 \in A^-$, then $\chi$ vanishes at $a_0m_0$ to infinite order, and so does $\frac{\varphi(a)\chi(am)}{\det(1 - am|n)}$. If $a_0$ lies on the boundary of $A^-$, then the vanishing order is determined by $\frac{\varphi(a)}{\det(1 - am|n)}$. □

**Lemma 4.9** The function $f$ is well defined and for given $N \in \mathbb{N}$ there are $\mu, j$ such that the map $C^{\mu,j}(A^-) \to L^2_{2N}(G); \varphi \mapsto f_{\eta,\tau,\varphi}$ is continuous.

**Proof:** By the decomposition $G = KP = KNMA$ every element $x \in G.(AM)^\sim$ can be written in the form $knma(kn)^{-1}$. Now suppose two such representations coincide, that is

$$knma(kn)^{-1} = k'n'm'a'(k'n')^{-1},$$

then by Lemma 4.7 we get $(n')^{-1}(k')^{-1}kn \in MA$, or $(k')^{-1}k \in n'MAn^{-1} \subset MAN$, hence $(k')^{-1}k \in K \cap MAN = K \cap M = K_M$. Write $(k')^{-1}k = k_M$ and $n'' = k_Mnk_M^{-1}$, then it follows

$$n''k_Mmk_M^{-1}a(n'')^{-1} = n'm'a'(n')^{-1}.$$

Again by Lemma 4.7 we conclude $(n')^{-1}n'' \in MA$, hence $n' = n''$ and so

$$k_Mmk_M^{-1}a = m'a',$$

which implies the well-definedness of $f$. 

Let \( N \in \mathbb{N} \). We will show that for \( \text{Re}(\mu), j \) and \( l \) sufficiently large the function \( f \) lies in \( L^1_{2N,1}(G) \).

Let the group \( K_M = K \cap M \) act from the right on \( K \times N \times M \times A \) by

\[
(k, n, m, a)k_M = (kk_M, k_M^{-1}nk_M, k_M^{-1}mk_M, a).
\]

Let \( (K \times N \times M \times A)/K_M \) denote the quotient, then the projection

\[
K \times N \times M \times A \to (K \times N \times M \times A)/K_M
\]

is a principal \( K_M \)-fibre bundle.

Consider the map

\[
F : (K \times N \times M \times A)/K_M \to G
\]

\[
[k, n, m, a] \mapsto knam(n^{-1}).
\]

Then \( f \) is a \( j-1-\dim(n) \)-times continuously differentiable function on \( Z = (K \times N \times M \times A)/K_M \) which factors over \( F \). Now set \( D = (K \times N \times (AM)^{-1})/K_M \subset Z \).

Then \( D \) is an open subset of \( Z \). Set \( S = Z \setminus D \). The first part of this proof shows that \( F \) is a diffeomorphism on \( D \) with open image.

Thus to compute the order of differentiability of \( f \) we can apply Proposition 1.8. To compute the order of differentiability of \( f \) as a function on \( G \) we have to take into count the zeroes of the differential of \( F \). So we compute the differential \( TF \) of \( F \). Let at first \( X \in \mathfrak{k} \), then

\[
TF(X)f(knam(n^{-1})^{-1}) = \frac{d}{dt} |_{t=0} f(k \exp(tX)namn^{-1} \exp(-tX)k^{-1}),
\]

which implies the equality

\[
TF(X)_x = (\text{Ad}(k))(\text{Ad}(n(amn^{-1})^{-1}n^{-1}) - 1)X_x,
\]

when \( x \) equals \( knam(n^{-1})^{-1} \). Note that for \( X \in \mathfrak{k} \), unless \( X \in \mathfrak{k}_M \), we have \( (n\text{Ad}(am)n^{-1} - 1)(X) \neq 0 \). Similarly for \( X \in \mathfrak{n} \) we get that

\[
TF(X)_x = (\text{Ad}(kn))(\text{Ad}((am^{-1})^{-1}) - 1)X_x
\]

and for \( X \in \mathfrak{a} \oplus \mathfrak{m} \) we finally have \( TF(X)_x = (\text{Ad}(kn))X_x \). From this it becomes clear that \( F \), regular on \( K \times N \times M \times A \), may on the boundary have vanishing differential of order at most \( \dim(n) + \dim(\mathfrak{k}) \). In order to apply Proposition 1.8 we next need to determine the vanishing order of \( f \circ F \) at \( S \).

Applying Proposition 1.8 we get that \( f \) is \( \left[j-\dim(n)-2 \atop \dim(n)+\dim(\mathfrak{k})\right] \)-times continuously differentiable on \( G \). So we assume \( j \geq 2 \dim(n) + \dim(\mathfrak{k}) + 2 \) from now on. We have to show that \( Df \in L^1(G) \) for any \( D \in U(g) \) of degree \( \leq 2N \). The same computation will also show the boundedness of \( Df \) for \( \deg(D) \leq 1 \). For this we
recall the map \( F \) and our computation of its differential. Let \( q \subset \mathfrak{t} \) be a complementary space to \( \mathfrak{t}_M \). On the regular set \( TF \) is bijective. Fix \( x = knam(kn)^{-1} \) in the regular set and let \( TF_x^{-1} \) denote the inverse map of \( TF_x \) which maps to \( q \oplus n \oplus a \oplus m \). Introducing norms on the Lie algebras we get an operator norm for \( TF_x^{-1} \) and the above calculations show that \( \|TF_x^{-1}\| \leq P(am) \), where \( P \) is a class function on \( AM \), which, restricted to any Cartan \( H = AB \) of \( AM \) is a linear combination of quasi-characters. Supposing \( j \) and Re(\( \mu \)) large enough we get for \( D \in U(g) \) with \( \deg(D) \leq 2N \):

\[
|Df(knam(kn)^{-1})| \leq \sum_{D_1} P_{D_1}(am)|D_1f(k,n,a,m)|,
\]

where the sum runs over a finite set of \( D_1 \in U(\mathfrak{t} \oplus n \oplus a \oplus m) \) of degree \( \leq 2N \) and \( P_{D_1} \) is a function of the type of \( P \). On the right hand side we have considered \( f \) as a function on \( K \times N \times A \times M \). This discussion uses the facts that \( K \) is compact, \( N \) is unipotent, and \( \det(\text{Ad}(n(am)^{-1}n^{-1}) - 1) = \det(\text{Ad}(am)^{-1} - 1) \). Finiteness of the sum in the inequality above follows from the Poincaré-Birkhoff-Witt Theorem. Integrating, it becomes clear that for \( \mu \) and \( j \) sufficiently large the map \( \varphi \mapsto f \) indeed is continuous.

We will plug \( f \) into the trace formula. For the geometric side let \( \gamma \in \Gamma \). We have to calculate the orbital integral:

\[
\mathcal{O}_\gamma(f) = \int_{G \gamma \backslash G} f(x^{-1}\gamma x)dx.
\]

by the definition of \( f \) it follows that \( \mathcal{O}_\gamma(f) = 0 \) if \( \gamma \notin G.(AM)^\gamma \). It remains to compute \( \mathcal{O}_{am}(f) \) for \( am \in (AM)^\gamma \). Again by the definition of \( f \) it follows

\[
\mathcal{O}_{am}(f) = \mathcal{O}_m^M(f_M) \frac{\varphi(am)}{\det(1 - ma|n)}.
\]

where \( \mathcal{O}_m^M \) denotes the orbital integral in the group \( M \).

Since only elliptic elements have nonvanishing orbital integrals at \( f_M \) it follows that only those conjugacy classes \( [\gamma] \) contribute for which \( \gamma \) is in \( G \) conjugate to \( a_{\gamma}b_{\gamma} \in A \backslash B \). Recall that Theorem 1.83 says

\[
\text{vol}(\Gamma \gamma \backslash G_\gamma) = \chi_r(\Gamma_\gamma) \lambda_\gamma \frac{c_{G_\gamma}}{|W_{\gamma,c}| \prod_{\alpha \in \Phi_\gamma^+} (\rho_{\gamma}, \alpha)}.
\]

By Theorem 2.49 we on the other hand get

\[
\mathcal{O}_\gamma(f) = \frac{|W_{\gamma,c}| \prod_{\alpha \in \Phi_\gamma^+} (\rho_{\gamma}, \alpha)}{c_{G_\gamma}} \text{tr} \tau(b_\gamma) \frac{\varphi(a_\gamma)}{\det(1 - a_\gamma b_\gamma |n)},
\]

so that

\[
\text{vol}(\Gamma \gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) = \chi_r(\Gamma_\gamma) \lambda_\gamma \text{tr} \tau(b_\gamma) \frac{\varphi(a_\gamma)}{\det(1 - a_\gamma b_\gamma |n)}.
\]
It follows that the geometric side of the trace formula coincides with the geometric side of the Lefschetz formula.

Now for the spectral side let $\pi \in \hat{G}$. We want to compute $\text{tr} \, \pi(f)$. Let $\Theta^G_\pi$ be the locally integrable conjugation invariant function on $G$ such that

$$\text{tr} \, \pi(f) = \int_G f(x) \Theta^G_\pi(x) \, dx.$$ 

To evaluate $\text{tr} \, \pi(f)$ we will employ the Hecht-Schmid character formula \[18\]. For this let

$$(AM)^- = \text{interior of } MA \text{ of the set}$$

$$\{g \in MA \mid \det(1 - ga|n) \geq 0 \text{ for all } a \in A^- \}.$$ 

Note that $(AM)^-$ is a subset of $(AM)^- \cap G^{reg}$. The main result of \[18\] is that for $ma \in (AM)^- \cap G^{reg}$, the regular set, we have

$$\Theta^G_\pi(ma) = \sum_{p=0}^{\dim N} (-1)^p \Theta^{AM}_{H_p(n, \pi_K)}(am) \frac{\det(1 - am|\bar{n})}{\det(1 - ma|n)}.$$ 

Let $h$ be supported on $G.(AM)^-$, then the Weyl integration formula implies that

$$\int_G f(x) \, dx = \int_{G/MA} \int_{MA^-} h(gma^{-1}) |\det(1 - ma|n \oplus \bar{n})| \, dadm.$$ 

So that for $\pi \in \hat{G}$:

$$\text{tr} \, \pi(f) = \int_G \Theta^G_\pi(x) f(x) \, dx = \int_{MA^-} f^M(m) \hat{\varphi}(am) |\det(1 - ma|\bar{n})| \, dadm = \int_{MA^-} f^M(m) \sum_{p=0}^{\dim N} (-1)^p \Theta^{AM}_{H_p(n, \pi_K)}(am)$$

$$\times \frac{|\det(1 - am|\bar{n})|}{\det(1 - am|n)} \hat{\varphi}(am) \, dadm.$$ 

Now we find that

$$|\det(1 - am|\bar{n})| = (-1)^{\dim N} \det(1 - am|\bar{n}) = (-1)^{\dim N} a^{-2\rho P} \det(a^{-1} - m|\bar{n}) = (-1)^{\dim N} a^{-2\rho P} \det((am)^{-1} - 1|\bar{n}) = a^{-2\rho P} \det(1 - (am)^{-1}|\bar{n}) = a^{-2\rho P} \det(1 - am|n).$$
so that
\[
\text{tr } \pi(f) = \int_{MA^-} f^M_\tau(m) \prod_{p=0}^{\dim N} (-1)^p \Theta^A_M H^p(n,\pi_K)(am) a^{-2p} \bar{\varphi}(am) \text{dadm}.
\]

We have an isomorphism of \((a \oplus m, K_M)\)-modules
\[
H^p(n, \pi_K) \cong H^{\dim N - p}(n, \pi_K) \otimes \Lambda^\text{top} n.
\]
This implies
\[
\sum_{p=0}^{\dim N} (-1)^p \Theta^A_M H^p(n,\pi_K)(am) a^{-2p} = (-1)^{\dim N} \sum_{p=0}^{\dim N} (-1)^p \Theta^A_M H^p(n,\pi_K)(am).
\]
And so
\[
\text{tr } \pi(f) = \int_{MA^-} f^M_\tau(m) \prod_{p=0}^{\dim N} (-1)^{p+\dim N} \Theta^A_M H^p(n,\pi_K)(am) \bar{\varphi}(am) \text{dadm}.
\]

Let \(B = H_1, \ldots, H_n\) be the conjugacy classes of Cartan subgroups in \(M\). By the Weyl integration formula the integral over \(M\) is a sum of expressions of the form
\[
\int_{H_j} \int_{M/H_j} f^M_\tau(x^{-1}hx) \Theta^A_M H^p(n,\pi_K)(ha) \bar{\varphi}(ha) \text{dxdh} = \int_{H_j} \int_{M/H_j} f^M_\tau(x^{-1}hx) \Theta^A_M H^p(n,\pi_K)(ha) \bar{\varphi}(ha) \text{dxdh},
\]
where we have used the conjugacy invariance of \(\Theta^A_M H^p(n,\pi_K)\) and \(\bar{\varphi}\). The orbital integrals \(O^M_h(f^M_\tau)\) are nonvanishing only for \(h\) elliptic, so only the summand with \(H_j = H_1 = B\) survives. In this term we may replace \(\bar{\varphi}(ha)\) by \(\varphi(a)\) so that we get
\[
\text{tr } \pi(f) = \int_{MA^-} f^M_\tau(m) \prod_{p=0}^{\dim N} (-1)^{p+\dim N} \Theta^A_M H^p(n,\pi_K)(am) \varphi(a) \text{dadm}.
\]
\[
= \sum_{p,q \geq 0} (-1)^{p+q+\dim N} \int_{A^-} \varphi(a) \text{tr } a(H^p(n,\pi_K) \otimes \Lambda^p p_M \otimes V_\tau)^{K_M} \text{da}.
\]

The theorem follows. \(\square\)
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