Distribution of Records Defined on Ordered Words Representing Lattice Paths

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Abstract: Problem statement: Let the sequence i_1, i_2, ..., i_m denoted by S^m_n be an increasing ordered word of length n taken from the set of the n positive integers S = {1,2,...,n}, i, j = 1,2,...,n, the distribution of a single weak record as well as the joint distribution of weak records were found before. Results: By defining the notion of strong records on the sequence {L_j}i, the distribution of a single strong record was found for m=n. In another aspect, it can be shown that the lattice path in the plane from (0,1) to (m, n), consisting of unit segments up and to the right, can be represented by a sequence i_1, i_2,...,i_n where 1 <= i_1 <= i_2 <= ... <= i_n <= n, m,n ∈ N^* m ≥ n. That is, such lattice paths can be represented, in one to one correspondence, by ordered increasing words of length m taken from the set S. Conclusion/Recommendations: In this article, we are going to extend the notion of weak and strong records to these sequences representing lattice paths for m>n and obtain their distributions. This result allows us to study lattice paths via ordered words of non negative integers.

Key words: Lattice paths, increasing ordered words, random variables, probability distribution

INTRODUCTION

An ordered word of length m taken from the set S = {1,2,...,n} Alahmadi (2009), is a set of m integers i_1, i_2,...,i_m such that 1 <= i_1 <= i_2 <= ... <= i_m <= n. For m=n, the element 1, i, j = 1,2,...,n is denoted by the weak record L_i the probability distribution of a single L_i as well as the joint probability distribution of L_i, L_j,...,L_k, were 1 <= i <= j <= ... <= k <= n, were given by the following theorem, for other results on ordered sequences El-Faheem and Mahmoud (2010), Khidir and Radwan (2000).

Theorem 1: Let P(L_i = j) be the probability that the integer j falls in the j^{th} place in the sequence j_1,..., j_n, i = 1,2,...,n, then we have Eqn. 1:

P(L_i = j) = \binom{n-1}{j-1} \binom{2(n-i)}{n-j}, j=1,2,...,n \tag{1}

Similarly, if P(L_n = j_1, L_n = j_2,..., L_n = j_k) be the probability that the numbers j_1, j_2,..., j_k falling in the locations i_1, i_2,...,i_k, respectively, then Eq. 2:

P(L_n = j_1, L_n = j_2,..., L_n = j_k) = \binom{k+k-2}{k-1} \binom{2k-2-j}{j-1} \binom{k-k+i-1}{i-1} \binom{n-k}{n-k} \tag{2}

In the case m>n, similar expressins are given for P(L_n = j_1, L_n = j_2,..., L_n = j_k) <= j_1 <= ... <= j_k <= n, sk<=m.

We notice that 1 <= i <= L_1 <= ... <= L_n <= n, thus they form some kind of weak records, the strong record on the sequence j_1j_2...j_n or alternately on L_1,L_2,...,L_n was introduced as a parallel to record values in the continuous case. For records from continuous distributions, Ahsanullah and Ragab (2006) Ahsanullah (1995) for discrete records, Dembinska (2007).

MATERIALS AND METHODS

Definition 1: For m>=n.

Let R(1)=1 and R(i+1)= \min{k>R(i): L_k > X_{R(i)}} when L_k exists, otherwise R(i+1)= R(i), i=1,2,..., m-i }, then X_{R(1)}, X_{R(2)},...,X_{R(m)} are called the strong record values on L_1, L_2,...,L_m and P(X_{R(1)} = k) is given by the following theorem Eqn. 3.
Theorem 2:

\[ P(X_{R(i)} = k) = \binom{k-1}{i-1} \left( \frac{2n-k-1}{n-i} \right) + \sum_{r=0}^{i-2} \binom{m-1}{r} \binom{m-i-r}{n-i} \]

(3)

In this article, we generalize Theorem 2 for \( m>n \). to get:

Theorem 3: For \( m>n \), the distribution of the record value \( X_{R(i)} = k \) is equal to Eqn. 4:

\[ P(X_{R(i)} = k) = \binom{k-1}{i-1} \left( \frac{n+m-k-1}{n-i} \right) + \sum_{r=0}^{i-2} \binom{m-1}{r} \binom{m-i-r}{n-i} \]

(4)

The case \( m<n \), can be obtained from (3) by interchanging \( m \) and \( n \) in (2).

Proof of Theorem 3: We call the sequence of random variables \( L_1 L_2 \ldots L_m \) (alternatively the ordered word \( j_1 j_2 \ldots j_m \)) the original sequence and its corresponding sequence of the record values, \( X_{R(1)} X_{R(2)} \ldots X_{R(m)} \) the reduced sequences. For example, for \( n = 6, m = 7 \), the sequence of random variables 2234666 is an original sequence and its corresponding records 2346666 is the corresponding reduced sequence. We notice that the value of the \( i^{th} \) record \( X_{R(i)} \) is equal to the \( i^{th} \) entry in the reduced sequence, \( I = 1, 2, \ldots, m \). Therefore, to obtain the number of original sequences \( L_1 L_2 \ldots L_m \), or alternatively \( j_1 j_2 \ldots j_m \), contributing to the event \( X_{R(i)} = k \), \( k=1,2,\ldots,n \). as in (1-2), we have to look for the original sequences that give these reduced sequences when applying definition 3.

For the reduced sequences in (2.1), we can construct the original sequences that give them as follows: From the set of integers \( \{1,2,\ldots,(k-1)\} \), we select \( r \) integers without replacement in \( \binom{k-1}{r} \) ways, denote these \( r \) integers \( j_1, j_2, \ldots, j_r \), \( 0 \leq r \leq (i-1) \). From the \( m \) places reserved for the sequence \( L_1 L_2 \ldots L_m \), we save the last location for the integer \( k \) and choose from the remaining \( (n-1) \) places, \( r \) places for the previously selected \( r \) integers in \( \binom{n-1}{r} \) ways. Place these \( r \) integers in the \( r \) selected places to get the following arrangement:

\[ j_1 j_2 \ldots j_r k \]

Now we fill in the empty places by the integer succeeding it to get the following arrangement:

\[ j_1 j_2 \ldots j_i j_{i+1} \ldots j_m k k \ldots k \]

Clearly, the above arrangement, as original sequence, gives a reduced sequence as in (1). The number of such original sequences, for particular \( r \) is

\[ \binom{(n-1)}{r} \binom{m-1}{s} \binom{m-i-s}{n-i} \]

(5)

When \( r = 0 \), (5), reduces to the sequence \( k \ldots \ldots k \).

Type 2, sequences with the first \( (i-1) \) entries are formed of strictly increasing words from the set \( \{1,2,\ldots,(k-1)\} \), the \( i^{th} \) entry equals \( k \) and the remaining \( (m-i) \) entries are formed of strictly increasing ordered word formed from the set \( \{k,(k+1),\ldots,n\} \). These sequences are of the form Eqn. 6:

\[ j_1 j_2 \ldots j_{i-1} k_{i+1} \ldots j_m \]

(6)

After determining the reduced sequences that contribute to the event \( X_{R(i)} = k, i>1, k=1,2,\ldots,n \). in (1-2), we have to look for the original sequences that give these reduced sequences when applying definition 3.
For the reduced sequences in (8), we can see that the original sequences that can give the reduced sequences of type 2, as in (8), are of the form as below:

\[ j_1 \ldots j_1 j_2 \ldots j_{i+1} \ldots j_{i+1} k j_{i+1} \ldots j_m \]

Where:

\[ 1 \leq j_i < j_1 < \ldots < j_{i+1} \leq (k-1), j_i = k, \]

\[ k \leq j_{i+1} \leq \ldots \leq j_n \leq n \]  

(8)

The number of the arrangements (2.4) can be obtained as follows: chose \((i-1)\) strictly increasing positive integers, \(1 \leq j_1 < j_2 < \ldots < j_{i-1} \leq (k-1)\), out the set \(\{1,2,...,k\}\) ways. Then we take the first \(s\) places, \(1 \leq s \leq m\), we save the \(s^{th}\) place for the integer \(k\) and choose from the remaining \((s-1)\) places, \((i-1)\) places for the \((i-1)\) integers, \(1 \leq s \leq j_{i+1} \leq (k-1)\) and fill in the empty \((s-i)\) places, as above, by the integers succeeding each. This gives us the first part of the scheme (9). To obtain the arrangement (9), we fill in the remaining \((m-s)\) places by ordered words of length \((m-s)\) formed from the set \(\{k,k+1,\ldots,n\}\).

The total number of original sequences of type 2 will be equal to:

\[ \binom{k-1}{i-1} \sum_{s=i}^{m} \binom{s-1}{i-1} \binom{m-s-k-1}{m-s} \]  

(9)

The sum in (10) is equal to:

\[ \binom{m-s-k-1}{m-i} \]

thus (5) equals to:

\[ \binom{k-1}{i-1} \binom{m-s-k-1}{m-i} \]  

(10)

From (3) and (6), the theorem follows.

RESULTS

Theorem 2 is an extension to theorem 1 and the following Table 1 can be used to justify the theorem.

| k/l | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| 1   | 210 | 84  | 28  | 7   | 1   |
| 2   | 1   | 127 | 113 | 64  | 25  |
| 3   | 1   | 7   | 83  | 124 | 115 |
| 4   | 1   | 7   | 28  | 99  | 195 |
| 5   | 1   | 7   | 28  | 84  | 210 |
| 6   | 1   | 7   | 28  | 84  | 210 |
| 7   | 1   | 7   | 28  | 84  | 210 |

DISCUSSION

It is interesting to proof a similar theorem to theorem 1 and 2, in case the sequence \(1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq n\) represents the order statistics from the discrete uniform distribution on \(S = \{1,2,\ldots,n\}\). The result if obtained, can be used together with current result in testing hypotheses concerning records.

CONCLUSION

The study of lattice paths is closely related to the study of increasing ordered words of non negative integers, this relation can facilitate the study of either via the other.

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